MINIMAL IDEMPOTENT ULTRAFILTERS AND THE AUSLANDER-ELLIS THEOREM

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Abstract. We characterize the existence of minimal idempotent ultrafilters (on \( \mathbb{N} \)) in the style of reverse mathematics and higher-order reverse mathematics using the Auslander-Ellis theorem and variant thereof.

We obtain that the existence of minimal idempotent ultrafilters restricted to countable algebras of sets is equivalent to the Auslander-Ellis theorem (\( \text{AET} \)) and that the existence of minimal idempotent ultrafilters as higher-order objects is \( \Pi^1_2 \)-conservative over a refinement of \( \text{AET} \).

We characterize the existence of minimal idempotent ultrafilters (on \( \mathbb{N} \)) in the style of reverse mathematics and higher-order reverse mathematics using the Auslander-Ellis theorem and variant thereof.

The set of all ultrafilters on \( \mathbb{N} \) can be identified with the Stone-Čech compactification \( \beta \mathbb{N} \) of \( \mathbb{N} \). The natural number are represented in \( \beta \mathbb{N} \) as the principal ultrafilters via the embedding

\[
 n \mapsto \{ X \subseteq \mathbb{N} \mid n \in X \} \in \beta \mathbb{N}.
\]

One can show that one can extend the addition of \( \mathbb{N} \) to \( \beta \mathbb{N} \) in the following way:

\[
\mathcal{U} + \mathcal{V} = \{ X \subseteq \mathbb{N} \mid \{ n \in \mathbb{N} \mid X - n \in \mathcal{V} \} \in \mathcal{U} \}.
\]

An ultrafilter \( \mathcal{U} \) is called minimal if one of the following equivalent conditions holds:

1. \( \mathcal{U} \) belongs to a minimal ideal of \( (\beta \mathbb{N}, +) \),
2. \( (\mathcal{U} + \beta \mathbb{N}, \sigma) \) with \( \sigma: \mathcal{U} \mapsto \mathcal{U} + 1 \) is a minimal dynamical system (with respect to the usual topology on \( \beta \mathbb{N} \) generate by the basic open sets \( B(X) := \{ V \in \beta \mathbb{N} \mid X \in V \} \)),
3. For each \( X \in \mathcal{U} \) the set \( \{ n \in \mathbb{N} \mid X - n \in \mathcal{V} \} \) is syndetic. (Recall that a set \( X \subseteq \mathbb{N} \) is called syndetic if there is an \( m \) such that for each \( x \) we have \( X \cap [x, x + m] \neq \emptyset \).)

(see Hindman, Strauss [11]).

We will use [3] as definition for minimal ultrafilters, since it does not refer to any subsets of the Stone-Čech compactification and can, therefore, be expressed with the lowest quantifier complexity.

An ultrafilter \( \mathcal{U} \) is called idempotent if \( \mathcal{U} = \mathcal{U} + \mathcal{U} \) and minimal idempotent if it is minimal and idempotent.

Our interest in minimal idempotent ultrafilters stems from the fact that they are widely used in ergodic theory and combinatorics, see e.g. Carlson, Simpson [7], Carlson [6], Gowers [10], and Bergelson [3].

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First we will restrict our attention to countable collections of sets as Hirst did in his analysis of idempotent ultrafilters in [12]. Here, we will show that the statement that for a countable collection of sets a minimal idempotent ultrafilter restricted to this collection exists is equivalent to the Auslander-Ellis theorem (AET) over RCA_0.

Then we will analyze the existence of minimal idempotent ultrafilter in a higher-order system (RCA_0^ω) building on previous work in [15, 14]. We will see that the Π^1_2-consequences are equivalent to refinements of the Auslander-Ellis theorem (eAET, eAET', depending on the precise formulation of the existence of the ultrafilter). Beside of this the strength of eAET, eAET' remains unknown.

The paper is organized in the following way. In Section 1 we will recall the Auslander-Ellis theorem and define eAET and eAET', in Section 2 we will define and analyze the statement that a minimal idempotent ultrafilter restricted to a countable algebra exists, and in Section 3 we will analyze the general case in the higher-order setting.

1. The Auslander-Ellis theorem

Definition 1.

- Let (X, d) be a compact metric space and let T: X → X be a continuous mapping. Then we call (X, T) a compact topological dynamical system.
- A point x in (X, T) is called uniformly recurrent if for each ε > 0 the set \{n ∈ N | d(T^n x, x) < ε\} is syndetic.
- A pair of points x, y in (X, T) is called proximal if for each ε > 0 there are infinitely many n with d(T^n x, T^n y) < ε.

Definition 2. The Auslander-Ellis theorem (AET) is the statement that for each compact topological dynamical system (X, T) and each point x ∈ X there exists a uniformly recurrent point y such that x, y are proximal.

Recall that Hindman’s theorem (HT) is the statement that for each coloring c: N → 2 of the natural numbers there exists an infinite set X = \{x_0, x_1, \ldots\} such that the set of finite sum of X

FS(X) = FS((x_i)_i) := \{x_i_1 + \cdots + x_i_k | i_1 < i_2 < \cdots < i_k\}

is homogeneous for c. The iterated Hindman’s theorem (IHT) is the statement that for each sequence of colorings c_k: N → 2 there exists a strictly ascending sequence (x_i)_{i∈N} such that for each k the set FS((x_i)_{i<k}) is homogeneous for c_k. It is known that HT, IHT are provable in ACA^+_0 and imply ACA_0. However it is open where between these systems HT and IHT lie, and whether they are equivalent.

Blass, Hirst, and Simpson showed in [5] that AET is provable in ACA_0 + IHT and thus in ACA^+_0. In fact, AET is equivalent to IHT over RCA_0, see Corollary 7 below.

We will write

IP-lim (n→FS((n_i))) \ x_n = x,

for the statement for each ε there exists a k such that for all finite sums n ∈ FS((n_i)_{i≥k}) we have d(x_n, x) < ε.

Proposition 3 (RCA_0). For (X, T), x, y as in the Auslander-Ellis theorem the system RCA_0 proves that there exists an increasing sequence (n_i)_{i∈N} such that

IP-lim (n→FS((n_i))) \ T^n x = y.
To prove this proposition we need the following lemmata. For this fix $(X, T)$.

We will denote by $\overline{\text{Orb}}(y)$ the orbit closure of $y$, i.e. the set

$$\{ x \in X \mid \forall \varepsilon > 0 \exists n \ d(T^n y, x) < \varepsilon \}.$$  

**Lemma 4 (RCA$_0$).** If $y$ is uniform recurrent, then for each $z \in \overline{\text{Orb}}(y)$ we have $\overline{\text{Orb}}(y) = \overline{\text{Orb}}(z)$. In other words, $\overline{\text{Orb}}(y)$ is minimal.

**Proof.** It is clear that $\overline{\text{Orb}}(z) \subseteq \overline{\text{Orb}}(y)$. Suppose that $\overline{\text{Orb}}(z) \neq \overline{\text{Orb}}(y)$. Then there exists an $\varepsilon$ such that each point in $\overline{\text{Orb}}(z)$ is more than $3\varepsilon$ apart from $y$. Since $y$ is uniformly recurrent we have an $m$ such that in $(T^n(y))$ at least every $m$-th element is $\varepsilon$-close to $y$. However, there is also a sequence $(n_k)$ such that $T^{n_k} y \to z$. By continuity of $T$, we can find an $\varepsilon'$ such that for each $z'$ that is $\varepsilon'$-close to $z$ the iterates $T_{z'}, T^2_{z'}, \ldots, T^m_{z'}$ are all $\varepsilon$-close to $T_z$ resp. $T^2_z, \ldots, T^m_z$. Since $T_z, T^2_z, \ldots, T^m_z \in \overline{\text{Orb}}(z)$ these elements have a distance of $3\varepsilon$ to $y$ and therefore $T_{z'}, T^2_{z'}, \ldots, T^m_{z'}$ are at least $2\varepsilon$ apart from $y$. Now choosing a $k$ such that $d(T^{n_k} y, z') < \varepsilon'$ yields that $T^{1+n_k} y, T^{2+n_k} y, T^{m+n_k} y$ are all $2\varepsilon$ apart from $y$ contradicting the fact that $y$ is uniformly recurrent. □

**Lemma 5 (RCA$_0$).** If $y$ is uniformly recurrent then for each open set $U$ in $\overline{\text{Orb}}(y)$ there is an $m$ such that

$$\bigcup_{n=0}^{m} T^{-n}(U) \supseteq \overline{\text{Orb}}(y).$$

**Proof.** By Lemma 4 we known that the orbit of each $z \in \overline{\text{Orb}}(y)$ meets $U$. Therefore,

$$\bigcup_{n=0}^{\infty} T^{-n}(U) \supseteq \overline{\text{Orb}}(y).$$

In other words the sets $T^{-n}(U)$ form an open covering of the compact set $\overline{\text{Orb}}(y)$. Using WKL we can find a finite sub-covering and thus an $m$ such that (1) holds (see [16], IV.1).

Now (1) is equivalent to arithmetical the statement $\forall k T^k y \in \bigcup_{n=0}^{m} T^{-n}(U)$. Thus, the statement of the whole lemma is $\Pi^1_1$ and therefore by Corollary IX.2.6 of [16] provable without WKL. □

**Proof of Proposition 3.** We claim that for each open neighborhood $U$ of $y$ there is a $p$ such that $T^p x, T^p y \in U$.

Let $\varepsilon$ be such that $B(y, 2\varepsilon)$ is an open $2\varepsilon$-ball around $y$ contained in $U$. By Lemma 5 we can find an $m$ with

$$\bigcup_{n=0}^{m} T^{-n}(B(y, \varepsilon)) \supseteq \overline{\text{Orb}}(y).$$

Now by continuity of $T$ we can find a $\delta > 0$ such that for all $x', x''$ satisfying $d(x', x'') < \delta$ we have $\forall n \in [1, m] \ d(T^n x', T^n x'') < \varepsilon$. By proximality there is a $r$ such that $d(T^r x, T^r y) < \delta$ and thus $d(T^{n+r} x, T^{n+r} y) < \varepsilon$ for all $n \in [1, m]$. By 2 there is an $n \geq m$ with $d(T^{n+r} y, y) < \varepsilon$ and thus $d(T^{n+r} x, y) < 2\varepsilon$ and in particular $T^{n+r} x, T^{m+r} y \in U$.

Using this claim we will construct recursively a sequence $(n_i)$ and a sequence of neighborhoods $(U_i)$ of $y$ with the following properties:

- $U_{i+1} \subseteq U_i$, $T^n U_{i+1} \subseteq U_i$,
\[ T^{n_1}x, T^{n_1}y \in U_{i+1}, \]
\[ U_i \subseteq B(y, 2^{-i}). \]

For this set \( U_1 := B(y, 2^{-1}) \) and let \( n_1 \) be such that \( T^{n_1}x, T^{n_1}y \in U_1 \). Set \( U_{i+1} := B(y, 2^{-(i+1)}) \cap U_i \cap T^{-n_i}U_i \). This set is by induction hypothesis a neighborhood of \( y \). Let \( n_{i+1} \) be such it satisfies the claim for \( U_{i+1} \).

For a finite sum given by \( n_1, \ldots, n_k \) with \( i_1 < \cdots < i_k \) we then have
\[
T^{n_{i_k} + \cdots + n_{i_1}}(x) \in T^{n_{i_k-1} + \cdots + n_{i_1}}(U_{i_k}) \\
\subseteq T^{n_{i_k-2} + \cdots + n_{i_1}}(U_{i_k-1}) \subseteq \cdots \subseteq U_{i_1} \subseteq B(y, 2^{-i_1})
\]
or in other words that \( \text{IP-lim}_{n \to \infty} T^n x = y \).

This proof is based on Proposition 8.10 of [9].

**Remark 6.** Proposition 8 should be compared with Lemma 5.3 of [5] which states that IHT proves (and is in fact equivalent) to the statement that for each sequence \( (x_n)_n \) in a compact space there exists an infinite set \( N \) with \( \text{IP-lim}_{n \to \infty} T^n x_n \). In the context of AET, IHT is used to show that \( \text{IP-lim}_{n \to \infty} T^n x = y \) exists. One can show that this property implies that \( x, y \) are proximal, see [5, Lemma 5.2]. However, it is in general not the case that \( y \) is uniformly recurrent.

As immediate consequence we get the following corollary.

**Corollary 7** ([5], Folklore). \( \text{RCA}_0 \vdash \text{AET} \iff \text{IHT} \).

**Sketch of proof.** The left-to-right direction follows from the proof of AET in [5]. (Theorem 4.13 in [5] is IHT).

The right-to-left direction follows from Proposition 8 by viewing a coloring in \( c_i : \mathbb{N} \to 2 \) as a point in the dynamical system \((2^\mathbb{N}, T)\) where \( T \) is the left shift and considering the \( N \)-fold product of this to be able to deal with all colorings. AET together with Proposition 8 yields then the desired IP-set. See Theorem 11 below. \( \square \)

**Definition 8.** The extension Auslander-Ellis theorem (\( \text{eAET} \)) is the statement that given

1. a compact topological dynamical system \((X_1, T_1)\) with points \( x_1, y_1 \in X_1 \) satisfying the conclusion of AET, and
2. a second compact topological dynamical system \((X_2, T_2)\) with a point \( x_2 \in X_2 \)

then one can find a point \( y_2 \) extending the solution to AET to the product system \((X_1 \times X_2, T_1 \times T_2)\), i.e. \( \left( y_1 \right. \left/ x_1 \right) \) is uniformly recurrent and \( \left( x_1 \right. \left/ y_2 \right) \) are proximal.

\( \text{eAET} \) follows from an easy adaption of any of the classical proofs of the Auslander-Ellis theorem like the original one by Ellis or Auslander (see [9] and [5] for a reference), a different proof by Auslander [2], or the proof using minimal idempotent ultrafilters (see [4] and Chapter 19 of [11]). However, it does not seem to be possible to adapt the proof of Blass, Hirst, Simpson to \( \text{eAET} \) even though is based on the original proof as presented in [9].

We will also use the following consequence of \( \text{eAET} \). Let \( \text{eAET}'_n \) be the statement that for each sequence \( (t_i)_{i<n} \) of continuous functions, such that \( t_i \) is an...
i-ary function \( t_i : X^i \to X \), there exists a sequence \((y_i)_{i<n} \in X^n\) with \((y_i)_{i<n}\) uniformly recurrent as point in the \(n\)-fold product of \((X, T)\) and proximal to the point \((t_0, t_1(y_0), t_2(y_0, y_1), \ldots, t_{n-1}(y_0, \ldots, y_{n-2})\)). Let \(eAET'\) be the union of \(eAET'\).

To see that \(eAET\) implies \(eAET'\) let \((t_i)_{i<n}\) be given. By \(AET\) we can find a point \(y_0\) which is uniformly recurrent and proximal to \(t_0\) in the system \((X, T)\). Then applying \(eAET\) to the point \(x_0 := t_0, y_0 := y_0, x_0 := t_1(y_0)\), yields a point \(y_1\) such that \((y_0, y_1)\) is uniformly recurrent and proximal \((x_0, x_1)\) in the 2-fold product of \((X, T)\). Iterating this process yields \((y_i)_{i<n}\) as needed.

2. Minimal idempotent ultrafilters on countable algebras

A countable algebra \(A = \{A_1, A_2, \ldots\}\) is a sequence of sets closed under intersections, unions and complement. A download translation algebra is an algebra \(A\) which is additionally closed under downward translations, i.e.

\[
X \in A \Rightarrow \forall n \in \mathbb{N} (X - n \in A).
\]

A (partial) non-principal ultrafilter \(F\) for \(A\) is a subset of \(A\) that satisfies the ultrafilter axioms relativized to \(A\), i.e.

\[
\forall X, Y \in A (X \in F \land X \subseteq Y \rightarrow Y \in F) \\
\land \forall X, Y \in F ((X \cap Y) \in F) \\
\land \forall X \in F (\forall n \exists k > n k \in X) \\
\land \forall X \in A (X \in F \lor \overline{X} \in F).
\]

A partial idempotent ultrafilter \(F\), (also downward translation partial ultrafilter) is filter that satisfies the non-principal ultrafilter axioms relativized to \(A\) and the following relativized idempotency condition

\((*)\)  
\[
\forall X \in F (\{n | X - n \in F\} \in F).
\]

Note that we do not know whether \(\{n \in \mathbb{N} | X - n \in F\}\) is contained in \(A\). Therefore we cannot find a filter \(F\) which is a subset of \(A\). For this reason the filter is not given by a sequence of sets but by a predicate, i.e. \(X \in F \equiv \phi(X)\) for an arithmetical formula \(\phi\). (This definition is made relative to \(ACA_0\), since otherwise the membership property of \(F\) is not decidable.) A partial non-principal ultrafilter in the sense of the above definition is then given by \(\{X \in A | \phi(X)\}\).

In [12] Hirst considered a weaker form of idempotent ultrafilters so called almost downward translation invariant ultrafilters where (2) is replaced by the following

\[(3)\]  
\[
\forall X \in F \exists n (X - n \in F).
\]

Since \(X \cap \{n | X - n \in F\} \in F\) for a partial idempotent ultrafilter and each set in \(F\) is infinite and therefore nonempty, this condition is satisfied by any partial idempotent ultrafilter.

A (partial) minimal (idempotent) ultrafilter for \(A\) is an (idempotent) ultrafilter for \(A\) which additionally satisfies

\[(4)\]  
\[
\forall X \in F \{n | X - n \in F\} \text{ is syndetic.}
\]

**Theorem 9.** Over \(RCA_0\) the following statements are equivalent.

1. For every countable algebra \(A\) there exists a partial non-principal ultrafilter.
2. For every countable algebra \(A\) there exists a partial minimal ultrafilter.
3. \(ACA_0\).
Lemma 10. The Auslander-Ellis theorem (AET) proves that for each countable downward translation algebra $\mathcal{A}$ there exists a partial minimal idempotent ultrafilter.

Proof. Let $\mathcal{A} = \{A_1, A_2, \ldots\}$. We will interpret each set as a point in the Cantor-space $2^\mathbb{N}$ given by the characteristic function

$$
\chi_{A_i}(n) = \begin{cases} 
0 & \text{if } n \in A_i, \\
1 & \text{if } n \notin A_i.
\end{cases}
$$

We will use the shift $Tu(n) \mapsto u(n+1)$ as transformation. With this $(2^\mathbb{N}, T)$ becomes a compact topological dynamical system.

To treat all sets simultaneously we will use the \(N\)-fold product of this system and arrive at the system $((2^\mathbb{N})^N, T_N)$. Let

$$
(4) \quad x := \begin{pmatrix} \chi_{A_1} \\
\chi_{A_2} \\
\vdots
\end{pmatrix} \in (2^\mathbb{N})^N
$$

be the point coding all sets in the algebra. By AET there exists a point $y$ which is uniformly recurrent and proximal to $x$. By Proposition 3 there exists an increasing sequence $(n_j)_j$ such that

$$
\text{IP-lim}_{n \to FS((n_j)_j)} T^n \chi_{A_i} = (y)_i.
$$

We set

$$
\mathcal{F} := \left\{ X \subseteq \mathbb{N} \mid \text{IP-lim}_{n \to FS((n_j)_j)} (T^n \chi_X) \text{ exists and } \text{IP-lim}_{n \to FS((n_j)_j)} (T^n \chi_X)(0) = 0 \right\}
$$

or in other words

$$
\mathcal{F} = \left\{ X \subseteq \mathbb{N} \mid \exists k \text{ FS}((n_j)_j)_{j=k} \subseteq X \right\}.
$$

It is clear that $\mathcal{F}$ can be defined by an arithmetical formula and it is straightforward to check that $\mathcal{F}$ forms a filter containing only infinite sets, i.e. it is closed under finite intersection and taking supersets.

Now $\mathcal{F}$ is a partial ultrafilter for $\mathcal{A}$ since by assumption for each $X \in \mathcal{A}$ we have that $\text{IP-lim}_{n \to FS((n_j)_j)} (T^n \chi_X)$ exists. Thus, either $\text{IP-lim}_{n \to FS((n_j)_j)} (T^n \chi_X)(0)$ or $\text{IP-lim}_{n \to FS((n_j)_j)} (T^n \chi_X)(0)$ is 0 and hence either $X$ or $X^c$ is included in $\mathcal{F}$.

The filter $\mathcal{F}$ satisfies \[\text{[\text{\ref{lem:partial-ideal}]}}\]. To show \[\text{[\text{\ref{lem:arithmetic-formula}]}}\] let $A_i \in \mathcal{F}$. Since $A_i \in \mathcal{F}$ we have that $y_i(0) = 0$. Let $\varepsilon > 0$ be small enough such that such that for two points $x' = \begin{pmatrix} x'_1 \\
x'_2 \\
\vdots \end{pmatrix}$, $x'' = \begin{pmatrix} x''_1 \\
x''_2 \\
\vdots \end{pmatrix} \in (2^\mathbb{N})^N$ we have that $d(x', x'') < \varepsilon$ implies $x'_i(0) = x''_i(0)$ for that given $i$. Since $y$ is uniformly recurrent, in particular for this $\varepsilon$, we have that

$$
\{n' \in \mathbb{N} \mid (T^{n'} y_i)(0) = 0\} \text{ is syndetic.}
$$
Since \( T \) is continuous we have that
\[
\text{IP-lim}_{n \to \text{FS}((n_j)_j)} (T^n \chi_{A_i} - n') = \text{IP-lim}_{n \to \text{FS}((n_j)_j)} T^n (T^n \chi_{A_i}) = T^n' \left( \text{IP-lim}_{n \to \text{FS}((n_j)_j)} T^n \chi_{A_i} \right) = T^n' y_i.
\]
Combining this with (5) we get that
\[
\{ n' \in \mathbb{N} \mid A_i - n' \in F \}
\]
is syndetic and thus (†).
\[\square\]

Theorem 11. Over ACA\(_0\) the following statements are equivalent.

(1) The Auslander-Ellis theorem AET.

(2) The iterated Hindman’s theorem IHT.

(3) For every countable downward translation algebra there exists a partial minimal idempotent ultrafilter.

(4) For every countable downward translation algebra there exists a partial idempotent ultrafilter.

(5) For every countable downward translation algebra there exists an almost downward translation invariant ultrafilter (in the sense of Hirst [12]).

Proof. \(2 \Rightarrow 1\) follows from [5]. \(5 \Rightarrow 2\) follows from [12]. \(3 \Rightarrow 4 \Rightarrow 5\) is clear and \(1 \Rightarrow 3\) is Lemma 10. \[\square\]

3. Minimal idempotent ultrafilters in a higher-order setting

In this section we will work in the higher-order systems \( \text{RCA}_\omega^0 \), \( \text{ACA}_\omega^0 \) corresponding to \( \text{RCA}_0 \) and \( \text{ACA}_0 \). We refer the reader to [13] for an introduction to these systems and assume that he is familiar with the treatment of ultrafilters in these systems in [13, 14].

The statement that a minimal idempotent ultrafilter exists can be formulated in \( \text{RCA}_\omega^0 \) in the following way.

\[
(U_{\min}): \exists \mathcal{U}^2 \left( \forall X^1 \left( X \in \mathcal{U} \lor \overline{X} \in \mathcal{U} \right) \land \forall X^1, Y^1 \left( X \cap Y \in \mathcal{U} \rightarrow Y \in \mathcal{U} \right) \land \forall X^1, Y^1 \left( X, Y \in \mathcal{U} \rightarrow (X \cap Y) \in \mathcal{U} \right) \land \forall X^1 \left( X \in \mathcal{U} \rightarrow \forall n \exists k > n (k \in X) \right) \land \forall X^1 \left( X \in \mathcal{U} \rightarrow \{ n \in \mathbb{N} \mid X - n \in \mathcal{U} \} \in \mathcal{U} \right) \land \forall X^1 \left( \{ n \in \mathbb{N} \mid X - n \in \mathcal{U} \} \text{ is syndetic} \right) \land \forall X^1 \left( \{ n \in \mathbb{N} \mid X - n \in \mathcal{U} \} \right) \right)
\]

The first four lines state that \( \mathcal{U} \) is a non-principal ultrafilter. The fifth line indicates that \( \mathcal{U} \) is idempotent and the sixth that it is minimal. The last line states that \( \mathcal{U} \) respects coding of sets as characteristics functions.

In [14] we showed that the existence of idempotent ultrafilters (\( U_{\text{idem}} \)) is \( \Pi^1_2 \)-conservative over \( \text{RCA}_0^\omega + \text{IHT} \). This theorem is proved by replacing the ultrafilter occurring in a proof of a \( \Pi^1_2 \) statement by a finite sequence of partial idempotent
ultrafilters $\mathcal{F}_i i < n$ for an increasing sequence of countable algebras $(\mathcal{A}_i) i < n$ such that each $\mathcal{F}_{i+1}$ refines $\mathcal{F}_i$, in the sense that

$$\forall i \forall j < i \ (\mathcal{F}_i \cap \mathcal{A}_j = \mathcal{F}_j).$$

We indicate here how to change the construction of a downward translation partial ultrafilter $\mathcal{F}$ such that additionally for each set $X \in \mathcal{F}$ we have that

$$\{ n \in \mathbb{N} \mid X - n \in \mathcal{U} \}$$

is syndetic, and thus that they can be used to replace minimal idempotent ultrafilters.

Like in the proof of Lemma 10 and in [14] the partial idempotent ultrafilters will be of the following form.

$$\mathcal{F}((n_i)_i) := \{ X \mid \exists m \mathbb{F}_s((n_i)_{i=m}^\infty) \subseteq X \}. $$

Note that the filter here is not given by a formula anymore but by a higher-order object. To construct this object from the sequence $(n_i)_i$ one in general needs $\mu$.

The construction in the proof of Lemma 10 can be summed up in the following way. Let $\mathcal{A}$ be an countable algebra and $x \in 2^{\mathbb{N}}$ the point in the system \((2^{\mathbb{N}})^\mathbb{N}, T^\mathbb{N}\) corresponding to $\mathcal{A}$ via (4). Then any uniformly recurrent point $y$ proximal to $x$ gives rise of partial minimal idempotent ultrafilter of the form $\mathcal{F}((n_i)_i)$.

The following lemma gives a reversal to the construction.

We will write $\mathcal{F}$-lim for the limit along a filter $\mathcal{F}$, i.e. $\mathcal{F}$-lim$_n x_n = x$ if for all $\varepsilon$ the set $\{ n \mid d(x_n, x) < \varepsilon \}$ is contained in $\mathcal{F}$.

**Lemma 12.** Let $\mathcal{A} = \{ A_1, A_2, \ldots \}$ be an countable algebra and let $\mathcal{F}$ be a partial minimal idempotent ultrafilter.

For $x$ as in (4) we have that

$$\mathcal{F} \text{-lim}_n (T^\mathbb{N})^n(x) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} =: y$$

and that $y$ is uniformly recurrent and proximal to $x$ and for all $i$ we have that $A_i \in \mathcal{F}$ iff $y_i(0) = 0$.

**Proof.** For each set $X$ we have that

$$\mathcal{F} \text{-lim}_n (T^\mathbb{N})^n(\chi_X)$$

exists and equals 0 iff $X \in \mathcal{F}$.

Thus $\mathcal{F} \text{-lim}_n (T^\mathbb{N})^n(x) = y$ exists and $A_i \in \mathcal{F}$ iff $y_i(0) = 0$.

Since the sets $O_{i,k,b} := \{ x \in (2^{\mathbb{N}})^\mathbb{N} \mid x_i(k) = b \}$ form a subbase of the topology of $(2^{\mathbb{N}})^\mathbb{N}$ it suffices to show that

$$\{ n \mid T^n y_i(k) = y_i(k) \}$$

is syndetic for each $i, k$,

to obtain that $y$ is uniformly recurrent.

This follows from

$$T^n y_i(k) = (T^k + n y_i)(0) = \left( \mathcal{F} \text{-lim}_{n'} T^{k+n+n'} (\chi A_i) \right)(0)$$

$$= \begin{cases} 0 & \text{if } (A_i - k) - n \in \mathcal{F}, \\ 1 & \text{if } (A_i - k) - n \in \mathcal{F}. \end{cases}$$

In [14] the filters are called downward translation partial ultrafilter.
For simplicity we assume that \((A_i - k) \in F\) then by the minimality of \(F\) we have \(T^n y_i(k) = 0 = y_i(k)\) for a syndetic sets of \(n\). The case of \((A_i - k) \notin F\) is similar.

Now we show that \(x, y\) are proximal. Let \(X_\varepsilon := \{ n \mid d(T^n(x), y) < \varepsilon \}\). By assumption \(X_\varepsilon \in F\) for each \(\varepsilon\). Fix \(\varepsilon > 0\). By the properties of \(F\) we have
\[
X' := X_\varepsilon \cap \{ n \mid X_\varepsilon - n \notin F \} \in F.
\]
Let \(m\) be an arbitrary element of \(X'\) and choose \(\delta > 0\) such that \(d(z, y) < \delta\) implies \(d(T^m z, T^m y) < \varepsilon\). Now let \(n \in X_\varepsilon \cap X' \in F\) be such that \(n > m\). Then \(d(T^n x, y) < \delta\); thus \(d(T^{m+n} x, T^m y) < \varepsilon\) and \(m + n \in X_\varepsilon\). We get
\[
d(T^m x, T^m y) \leq d(T^n x, y) + d(y, T^{m+n} x) + d(T^{m+n} x, T^m y) < 3\varepsilon.
\]

In particular, the above construction shows that there is a one-to-one correspondence between points \(y\), that are uniformly recurrent and proximal to \(x\) as in (4), and partial minimal idempotent ultrafilters. Moreover, this construction shows that each partial idempotent ultrafilter \(F\) on a algebra \(A\) is equal to an partial idempotent ultrafilter of the form \(F((n_i))\) in the sense that \(F \cap A = F((n_i)) \cap A\).

Using this one-to-one correspondence one obtains the following lemma.

**Lemma 13.** RCA\(\omega_1^1 + (\mu) + eAET\) proves the following. Given a countable algebra \(A = \{ A_1, A_2, \ldots \}\) and a partial minimal idempotent ultrafilter \(F\). Then for each countable algebra \(A'\) extending \(A\) there exists a sequence \((n_i)_i\) such that \(F((n_i)_i)\) is a partial minimal idempotent ultrafilter and \(F((n_i)_i) \cap A = F\).

**Proof.** By Lemma 12 there is a point \(y_1 \in (2^N)^N\) that is uniformly recurrent and proximal to the point \(x_1\) as in (4) (for the algebra \(A\)) and with \((y_1)_i(0) = 0 \iff A_i \in F\). Now let \(x_2\) be as in (4) now for the algebra \(A'\) then by the eAET there exists an \(y_2\) such that \((y_1, y_2)\) is uniformly recurrent and proximal to \((x_1, x_2)\). By the construction in Lemma 10 there exists an partial minimal idempotent ultrafilter \(F = F((n_i)_i)\) for the algebra \(A'\). Since the membership of \(X \in A\) only depends on \(y_1\) we have that \(F((n_i)_i) \cap A = F\). □

Replacing Theorem 15 in 14 with the previous lemma one can now show the following variant of Theorem 9 of 14.

**Theorem 14.** The system ACA\(\omega_1^1 + (\mu) + \Pi_1\) is \(\Pi_2\)-conservative over ACA\(\omega_1^1 + eAET\).

**Proof.** One first notes that the Section 4 of 14 goes through unchanged since \((U_{\min})\) differs from \((U_{dem})\) only by
\[
\{ n \in \mathbb{N} \mid X - n \notin U \}\text{ is syndetic}
\]
which is arithmetic and can be made quantifier free using \((\mu)\).

Now one just replaces Theorem 15 in 14 in the construction of the approximation of the ultrafilter with Lemma 13. □

Since the sequence of algebras is directly given as terms in 14 the principle eAET suffices to carry out the above construction and one obtains.

**Corollary 15.** The system ACA\(\omega_1^1 + (\mu) + \Pi_1\) is \(\Pi_2\)-conservative over ACA\(\omega_1^1 + eAET\).

**Theorem 16.** ACA\(\omega_1^1 + (\mu) + (U_{\min}) \vdash eAET\).
Proof. Let \((t_i)_{i<k}\) be as in the definition of \(e\text{AET}'\) and \(U\) be a minimal idempotent ultrafilter. By Theorem 19.26 of [11], \(U\)-\(\lim n T^n(t_0) =: y_0\) is uniformly recurrent and proximal to \(t_0\). Since \(U\) is a minimal idempotent ultrafilter also
\[
U\text{-}\lim n T^n \left( \begin{array}{c} t_0 \\ t_1(y_0) \end{array} \right) = \left( \begin{array}{c} y_0 \\ y_1 \end{array} \right)
\]
even exists and is again uniformly recurrent (now in \((X \times X, T \times T)\)) and proximal to \(t_0, t_1(y_0)\). Iterating this construction yields a solution to \(e\text{AET}'\). □

Corollary 17 (to Theorem 14). Theorem 14 remains true if one replaces the system by the following.

\[
\text{ACA}_0^\omega + (\mu) + \text{IHT} \vdash \forall f \left( \exists U \; [U \text{ is a minimal idempotent ultrafilter extending } t_F(f)] \rightarrow \exists g \; A(f, g) \right)
\]

where \(t_F\) is a closed term such that \(t_F(f)\) codes a partial minimal idempotent ultrafilter. (Cf. Remark 20 of [14].)

Proof. Use Lemma 12 to obtain the first approximation of the ultrafilter. Then continue using Lemma 13 as in the proof of Theorem 14. □

Theorem 18. Over \(\text{ACA}_0^\omega + (\mu)\) the statement
\[
\forall f \left( \exists U \; [U \text{ is a minimal idempotent ultrafilter extending } t_F(f)] \right)
\]
for a suitable term \(t_F\) proves \(e\text{AET}\).

Proof. Let \(x_1, y_1, x_2\) be given as in the definition of \(e\text{AET}\) then by Proposition 3 and the proof of Lemma 10 there exists an increasing sequence \((n_i)\) such that
\[
\text{IP-}\lim_{n \to \text{FS}(n_i)_{\infty}} T^n x_1 = y_1
\]
and
\[
F = \{ X \subseteq \mathbb{N} | \exists k \; \text{FS}(n_i)_{i=k} \subseteq X \}
\]
is a partial minimal idempotent ultrafilter for the algebra generated by \(\{ n \in \mathbb{N} | d(T^n x_1, y_1) < 2^{-k} \}\). It is easy to see that \(F\) is definable from \(x_1, y_1\) using \(\mu\). For a minimal idempotent ultrafilter \(U\) extending \(F\) we have then
\[
U\text{-}\lim n T^n \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right)
\]
and by Theorem 19.26 of [11] the point \(\left( \begin{array}{c} y_1 \\ y_2 \end{array} \right)\) is uniformly recurrent and proximal to \(\left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)\). □

Remark 19. Like in [14] Remark 19] all the previous results on minimal idempotent ultrafilter over \(\mathbb{N}\) also apply to minimal idempotent ultrafilters over any other countable semigroup \(G\). The proofs are formulated such that neither commutativity nor any other specify property of \(\mathbb{N}\) has been used. However, for the construction of the partial minimal idempotent ultrafilter one then need the Auslander-Ellis theorem for this particular semigroup.

For \(\text{AET}\) the analysis Blass, Hirst, Simpson shows that \(\text{AET}\) for a semigroup \(G\) is equivalent to \(\text{ACA}_0\) plus \(\text{IHT}\) for this group which is, if \(G\) is not trivial, equivalent to \(\text{IHT}\) for \(\mathbb{N}\). For \(e\text{AET}, e\text{AET}'\) we do not where their variant for different groups is
4. Discussion and Questions

**Question 1.** What is the strength of \( \text{eAET} \) and \( \text{eAET}' \)?

A set \( X \subseteq \mathbb{N} \) is called **central** if one of the following equivalent conditions holds.

1. There exists a compact topological dynamical system \( (X, T) \) and points \( x, y \in X \) with \( x, y \) are proximal and \( y \) is uniformly recurrent (in other words they satisfy the conclusion of \( \text{AET} \)) such that
   \[
   X = \{ n \mid d(T^n x, y) < \varepsilon \}
   \]
   for an \( \varepsilon \).
2. \( X \) is an element of a minimal idempotent ultrafilter.
3. \( X \) is syndetic and an IP-set (i.e. contains a set of the form \( \text{FS}(Y) \) for an infinite set \( Y \)).

see [9, Chap. 8 §3] for the original definition and [11] for this equivalences.

It is easy to see that \( \text{AET} \) proves that each finite partition of \( \mathbb{N} \) contains a central set, see [9, Theorem 8.8]. In the same way \( \text{eAET} \) proves that each finite partition of a central set contains a central set (in other words being central is partition stable), see Remark at the end of Chap. 8 §3 of [9]. It is open whether \( \text{AET} \) is sufficient for this.

**Question 2.** What is the strength of the statement that each finite partition of a central set contains a central set?
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