Quantization of a gauge theory on a curved noncommutative space

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Abstract

We study quantization of a gauge analogon of the Grosse-Wulkenhaar model: we find divergent one-loop contributions to 1-point and 2-point Green functions. We obtain that five counterterms are necessary for renormalization and that all divergences are logarithmic.

1 Introduction

Although the renormalization of quantum field theory was the main objective of Snyder [1] and the others at the time when noncommutativity of coordinates was first introduced, its full understanding in the context of noncommutative quantum field theory and the search for renormalizable models are still open issues. This applies in particular to theories defined on the Moyal space. When one deforms a commutative field theory to a noncommutative one by replacing the ordinary by the Moyal-Weyl product, a common pattern appears: in addition to the usual divergences, the new model suffers from the so-called ultraviolet-infrared (UV/IR) mixing. In the case of scalar field the additional divergences can be handled by a modification of the propagator. Two different possibilities are known: one can add either a

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position-dependent potential term \((\bar{x} \star \phi)^2\) \cite{2,3}, or a nonlocal kinetic term \(\phi \Box^{-1} \phi\) \cite{4} to the original action. The resulting theories differ in many respects but both are renormalizable to all orders in perturbation theory.

We will follow and develop the first approach. The Grosse-Wulkenhaar (GW) model is defined by

\[
S = \int \left( \frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{\mu^2}{2} \phi \star \phi + \frac{\Omega^2}{4} (\bar{x} \phi) \star (\bar{x} \phi) + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi, \right.
\]

where \(\bar{x}_\mu = (\theta_{\mu\nu})^{-1} x^\nu\), \(\theta_{\mu\nu}\) is a constant noncommutativity tensor, \([x^\mu \star x^\nu] = i\theta^{\mu\nu}\), and the \(\star\) is the Moyal-Weyl star product \cite{3.1}. The model is fully renormalizable; in addition, a nontrivial fixed point occurs for \(\Omega = 1\) at which the \(\beta\)-function for the coupling constant vanishes. One can understand renormalizability of \((1.1)\) physically either as a consequence of an additional symmetry called the Langmann-Szabo duality \cite{5} which the model has, or as a consequence of the confinement around the origin of coordinates introduced through the quadratic potential.

Despite numerous attempts to generalize the Grosse-Wulkenhaar model to gauge theories a similarly successful gauge model has not been by now found. A difficult point roughly speaking is, how to include the oscillator or an analogous position-dependent term in a gauge invariant way. In a straightforward approach \cite{6,7}, one couples the scalar field \((1.1)\) to an external gauge field: the dynamics of the gauge field is then extracted from the divergent contributions to the one-loop effective action. The induced gauge field action however contains the explicit tadpole terms and gives rise to a non-trivial vacuum, \cite{8}. Another, simplified version of a gauge model was discussed in \cite{9,10}. The model includes only the oscillator potential for the gauge field while the tadpole terms are omitted. Hence the considered action is not gauge invariant: nonetheless the BRST invariance can be established by an appropriate choice of ghosts and auxiliary fields. Although tadpoles are not present at the tree-level in quantization they reappear as the UV-counterterms at one loop.

A different line of generalization of the GW model was proposed in \cite{11,12}. It is based on a geometric interpretation of action \((1.1)\) as a dimensionally reduced action for a scalar field on curved noncommutative space. The oscillator potential is then not external but it is rather the coupling to the curvature. This framework gives a natural prescription to define the action for gauge fields with position-dependent couplings while preserving the gauge symmetry. A specific feature of the model is that in dimensional reduction one of the gauge degrees of freedom becomes a scalar field. The model has two classical vacua one of which is trivial, \(\phi = 0\), \(A_\alpha = 0\), and suitable for quantization. Moreover, the BRST invariance of the gauge-fixed action can be easily established.

We initiate here the study of perturbative quantization of the described model \cite{12}. The outline of the paper is the following. In Section 2 we introduce our approach that is we recollect all relevant concepts and formulae. In Section 3 we derive the Feynman rules for the propagators and the vertices in momentum space. We obtain in Section 4 that the one-loop tadpole corrections do not vanish and we calculate the corresponding divergent counterterms. In Section 5 we find the one-loop divergent propagator corrections. In the concluding section we discuss our results and the work which remains to be done in the future.
2 The classical model

Our gauge model is obtained by dimensional reduction from three-dimensional noncommutative space called the truncated Heisenberg algebra. The algebra is generated by hermitian coordinates, operators $\hat{x}^1, \hat{x}^2, \hat{x}^3$ which satisfy commutation relations

\[
\begin{align*}
[\mu \hat{x}^1, \mu \hat{x}^2] &= i \epsilon (1 - \bar{\mu} \hat{x}^3), \\
[\mu \hat{x}^1, \bar{\mu} \hat{x}^3] &= i \epsilon (\mu \hat{x}^2 \bar{\mu} \hat{x}^3 + \bar{\mu} \hat{x}^3 \mu \hat{x}^2), \\
[\mu \hat{x}^2, \bar{\mu} \hat{x}^3] &= -i \epsilon (\mu \hat{x}^1 \bar{\mu} \hat{x}^3 + \bar{\mu} \hat{x}^3 \mu \hat{x}^1),
\end{align*}
\]

where $\epsilon$ is dimensionless noncommutativity parameter while $\mu$ and $\bar{\mu}$ have dimension of mass. Unlike in [12], here we will denote the noncommuting variables in a generic situation that is when representation is not specified by a hat, keeping the corresponding unhatted characters for the corresponding quantities in the Moyal-space representation. Algebra (2.1) besides the usual commutative limit $\epsilon \to 0$ has an interesting contraction $\bar{\mu} \to 0$ to the Heisenberg algebra. For $\epsilon = 1$ (2.1) has finite-dimensional $n \times n$ matrix representations which in the limit $\bar{\mu} \to 0$ (or equivalently $n \to \infty$) tend to the known unique infinite-dimensional representation of the Heisenberg algebra. We will refer to this limit as to dimensional reduction from three-dimensional truncated Heinsenberg algebra to its two-dimensional subspace $\hat{x}^3 = 0$; this interpretation, as it was shown in [12], is consistent not only at the level of algebra but also at the level of exterior algebra. In order to avoid two mass scales in the model we will also put $\bar{\mu} = \mu$ and discuss only this case.

The truncated Heisenberg algebra is a smooth noncommutative space. Its differential structure is most conveniently defined in the noncommutative frame formalism through momenta $\hat{p}_\alpha$:

\[
d\hat{f} = (e_\alpha \hat{f}) \theta^\alpha = [\hat{p}_\alpha, \hat{f}] \theta^\alpha.
\]

It is assumed that frame basis forms $\theta^\alpha$ commute with functions on the algebra, $[\hat{f}, \theta^\alpha] = 0$. We choose the momenta as

\[
\begin{align*}
\epsilon \hat{p}_1 &= i \mu^2 \hat{x}^2, & \epsilon \hat{p}_2 &= -i \mu^2 \hat{x}^1, & \epsilon \hat{p}_3 &= i \mu (\hat{x}^3 - \frac{1}{2}).
\end{align*}
\]

With this definition the differential $d$ and the derivations $e_\alpha, \alpha = 1, 2$ reduce to the differential and the derivations on the Moyal plane after the dimensional reduction.

The U(1) gauge theory on the truncated Heisenberg space was constructed in [12]. It is given by an antihermitian gauge potential $\hat{A} = \hat{A}_\alpha \theta^\alpha$; the field strength is $\hat{F} = d\hat{A} + \hat{A}^2$. The commutation relations between the momenta define almost completely the geometry; in this case they are quadratic

\[
[\hat{p}_\alpha, \hat{p}_\beta] = \frac{1}{i \epsilon} K_{\alpha \beta} + F^{\gamma}_{\alpha \beta} \hat{p}_\gamma - 2 i \epsilon Q^{\gamma \delta}_{\alpha \beta} \hat{p}_\gamma \hat{p}_\delta,
\]

with

\[
K_{12} = \frac{\mu^2}{2}, \quad F^{13}_{23} = \mu, \quad Q^{13}_{23} = \frac{1}{2}, \quad Q^{23}_{13} = \frac{1}{2}.
\]

This implies that the frame components of the field strength are

\[
\hat{F}_{\alpha \beta} = \nabla_{[\alpha} \hat{A}_{\beta]} + [\hat{A}_{\alpha}, \hat{A}_{\beta}] + 2 i \epsilon (e_{\gamma} \hat{A}_{\gamma}) Q^{\gamma \alpha \beta} + 2 i \epsilon \hat{A}_{\eta} \hat{A}_{\alpha} Q^{\gamma \alpha \beta}.
\]
The gravity-covariant derivative $\nabla_\alpha \hat{A}_\beta = e_\alpha \hat{A}_\beta - \hat{A}_\gamma \hat{\omega}^{\gamma}_{\alpha \beta}$ is here given through connection $\hat{\omega}^{\alpha}_{\beta}$, \[1\]:

\[
\begin{align*}
\hat{\omega}_{12} &= -\hat{\omega}_{21} = \mu\left(\frac{1}{2} - 2\mu \hat{x}^3\right) \theta^3, \\
\hat{\omega}_{13} &= -\hat{\omega}_{31} = \frac{\mu}{2} \theta^2 + 2\mu^2 \hat{x}^1 \theta^3, \\
\hat{\omega}_{23} &= -\hat{\omega}_{32} = -\frac{\mu}{2} \theta^1 + 2\mu^2 \hat{x}^2 \theta^3.
\end{align*}
\] (2.7)

Instead in terms of potential $\hat{A}_\alpha$, the field strength can be expressed in terms of covariant coordinates $\hat{X}_\alpha = \hat{p}_\alpha + \hat{A}_\alpha$ as

\[
\hat{F}_{\alpha\beta} = 2P^{\gamma}_{\alpha\beta} \hat{X}_\gamma - F^{\gamma}_{\alpha\beta} \hat{X}_\gamma - \frac{1}{i\epsilon} K_{\alpha\beta}.
\] (2.8)

Covariant coordinate 1-form $\hat{X} = \hat{X}_\alpha \theta^\alpha$ is a difference of two connections, $\hat{A}$ and the Dirac operator $\hat{\theta} = -\hat{p}_\alpha \theta^\alpha$. It transforms therefore in the adjoint representation of $U(1)$, that is covariantly.

Performing the dimensional reduction $\hat{x}^3 = 0$, the third component of the gauge field becomes a scalar, $\hat{A}_3 = \hat{\phi}$. The components of the field strength become

\[
\begin{align*}
\hat{F}_{12} &= \hat{F}_{12} - \mu \hat{\phi} = [\hat{X}_1, \hat{X}_2] + \frac{i\mu^2}{\epsilon} + \mu \hat{\phi}, \\
\hat{F}_{13} &= D_1 \hat{\phi} - i\epsilon (\hat{p}_2 + \hat{A}_2, \hat{\phi}) = [\hat{X}_1, \hat{\phi}] - i\epsilon [\hat{X}_2, \hat{\phi}], \\
\hat{F}_{23} &= D_2 \hat{\phi} + i\epsilon (\hat{p}_1 + \hat{A}_1, \hat{\phi}) = [\hat{X}_2, \hat{\phi}] + i\epsilon [\hat{X}_1, \hat{\phi}],
\end{align*}
\]

where the gauge-covariant derivative is $D_\alpha \hat{\phi} = [\hat{p}_\alpha + \hat{A}_\alpha, \hat{\phi}]$, $\alpha = 1, 2$, and $\hat{F} = \hat{F}_{12} \theta^1 \theta^2$ is a two-dimensional field strength, $\hat{F}_{12} = \partial_1 \hat{A}_2 - \partial_2 \hat{A}_1 + [\hat{A}_1, \hat{A}_2]$.

Introducing \[2.9\] into the Yang-Mills action we obtain

\[
\mathcal{S}_{YM} = \frac{1}{2} \text{Tr} \left( (1 - e^2)(\hat{F}_{12})^2 - 2(1 - e^2)\mu \hat{F}_{12} \hat{\phi} + (5 - e^2)\mu^2 \hat{\phi}^2 + 4i\epsilon \hat{F}_{12} \hat{\phi}^2 \right) + (D_1 \hat{\phi})^2 + (D_2 \hat{\phi})^2 - e^2 (\hat{p}_1 + \hat{A}_1, \hat{\phi})^2 - e^2 (\hat{p}_2 + \hat{A}_2, \hat{\phi})^2
\] (2.10)

which defines our gauge model. It is clear that masses and couplings in \[2.10\] are fixed by the dimensional reduction procedure and parametrized by only one parameter, noncommutativity $\epsilon$. Modifications of the action \[2.10\] are possible but only at an earlier stage: for example, one can use different connection $\hat{\omega}^\alpha_{\beta}$, or define the Hodge-dual differently. Action \[2.10\] has two stationary points: $\hat{A}_\alpha = 0$, $\hat{\phi} = 0$ and $\hat{X}_\alpha = 0$. Additional minima might exist, but the corresponding equations are quite complicated and we were not able to find them in the generic case.

It is possible, in the context of the truncated Heisenberg algebra, to define the Chern-Simons action, too \[12\]. It is given by

\[
\mathcal{S}_{CS} = c\mu \text{Tr} \left( (3 - e^2)(\hat{F}_{12} - \frac{i\mu^2}{\epsilon} \hat{\phi}) + \frac{2i\epsilon}{3} ((\hat{p}_1 + \hat{A}_1)^2 + (\hat{p}_2 + \hat{A}_2)^2) (\hat{\phi} - \frac{i\mu}{2\epsilon})(\hat{\phi} - \frac{i\mu}{2\epsilon}) \right),
\] (2.11)
where \( c \) is an arbitrary constant. Adding \( S_{CS} \) to \( S_{YM} \) change of course the classical equations of motion: in principle, only the second vacuum, \( \hat{X}_\alpha = 0, \hat{X}_3 = 0 \) remains. There are however special cases, \([12]\).

### 3 Quantization

It was shown in \([12]\) that one can introduce, in a completely straightforward manner, the gauge fixing and the ghost terms to (2.10) and define a quantum action which is BRST invariant. We shall here quantize the model perturbatively. To this end, we use the Moyal-space representation. This means that fields \( \phi, A_\alpha \) are represented by functions of commuting coordinates \( x^\mu, \mu = 1, 2 \), while the algebra-multiplication is represented by the Moyal-Weyl \( \star \)-product:

\[
\chi(x) \star \phi(x) = e^{i \frac{1}{2} \theta_{\mu\nu} \partial_\mu \partial'_\nu \chi(x) \phi(x')} |_{x' \to x},
\]

where in our case \( \theta^{\mu\nu} \) is

\[
\theta^{\mu\nu} = \epsilon^{\mu} \epsilon^{\nu}.
\]

The signature is Euclidean. In the following we will often use abbreviation

\[
\tilde{x}_\mu = \epsilon_{\mu\nu} x^\nu.
\]

Also, we will redefine antihermitian fields of the previous section to hermitian ones by \( A_\alpha \rightarrow iA_\alpha, \phi \rightarrow i\phi \). The classical Yang-Mills action is then written as

\[
S_{YM} = -\frac{1}{2} \int a(F_{12})^2 - 2a\mu F_{12} \star \phi + (4 + a)\mu^2 \phi \star \phi + 4i\epsilon F_{12} \star \phi \star \phi,
\]

\[
+ [p_1 + A_1 \star \phi]^2 + [p_2 + A_2 \star \phi]^2 - \epsilon^2 (p_1 + A_1 \star \phi)^2 - \epsilon^2 (p_2 + A_2 \star \phi)^2
\]

where

\[
a = 1 - \epsilon^2.
\]

It can further be simplified using the property

\[
\{ x^\mu \star \phi \} = 2x^\mu \phi
\]

and

\[
\partial_\alpha \phi = [p_\alpha \star \phi], \quad D_\alpha \phi = [p_\alpha + A_\alpha \star \phi],
\]

which is the form which we use. The gauge is fixed by a non-covariant Lorentz gauge term:

\[
S_{gf} = \frac{a}{2} \int (\partial_\mu A^\mu)^2.
\]

Other possibilities of the gauge fixing, discussed in \([12]\), result much more complicated propagators. Finally, we add the ghost term

\[
S_{gh} = -\int \bar{c} \partial_\alpha (\partial^\alpha c + i[A_\alpha \star c]).
\]
The quantum action is the sum of all three terms,
\[ S = S_M + S_{gf} + S_{gh} = S_{kin} + S_{int}. \] (3.10)

It is BRST invariant; clearly we can add the Chern-Simons action and retain this invariance. The kinetic part of the action, after the gauge fixing, is
\[ S_{kin} = -\frac{1}{2} \int a A_\alpha \Box A^\alpha + 2 a \mu \epsilon^{\alpha \beta} (\partial_\alpha A_\beta) \phi + \phi \Box \phi - (4 + a) \mu^2 \phi^2 - 4 \mu^4 x^\alpha x_\alpha \phi^2 + 2 \bar{c} \Box c \] (3.11)

whereas the interaction reads
\[ S_{int} = -\frac{1}{2} \int 4 \epsilon_{\alpha \beta} \left( \partial^\alpha A^\beta + i A^\alpha \ast A^\beta \right) \ast \phi^2 - 2 i (\partial_\alpha \phi) [A_\alpha \ast \phi] \\
+ 2 i a \mu \epsilon_{\alpha \beta} A^\alpha \ast A^\beta \phi - 2 i a e_{\alpha \beta} \partial^\alpha A^\beta \epsilon_{\gamma \delta} A^\gamma \ast A^\delta + a (\epsilon_{\alpha \beta} A^\alpha \ast A^\beta)^2 \\
+ [A_\alpha \ast \phi] [A^\alpha \ast \phi] - \epsilon^2 \{ A_\alpha \ast \phi \} \{ A^\alpha \ast \phi \} + 2 \mu^2 \epsilon_{\alpha \beta} \{ x^\alpha \ast \phi \} \{ A^\beta \ast \phi \} \\
- i \bar{c} \partial_\alpha [A^\alpha \ast \phi]. \] (3.12)

3.1 Propagators
The first property of (3.11) which one observes is that, although the gauge fixing removes the mixed quadratic terms \((\partial_\alpha A_\beta)(\partial_\beta A_\alpha)\), the mixing between \(\phi\) and \(A_\alpha\) remains. This means that \(A_\alpha\) and \(\phi\) do not propagate independently. Apparently, only the scalar \(\phi\) is coupled to the oscillator potential \(x^\alpha x_\alpha\). The value \(a = 0\) of the parameter \(a = 1 - \epsilon^2\) is special: for \(a = 0\) the gauge field does not propagate, that is the corresponding kinetic term vanishes and the model becomes degenerate. On the other hand \(a = 0\) would seem to be a preferred choice because for this value representations of the truncated Heisenberg algebra are finite matrices. We shall see that though some of the expressions which we will calculate much simplify for \(a = 0\), the structure of divergences is in fact similar for all values of \(a\).

Since scalar and gauge field are mixed we consider them as a multiplet of fields, \((A^\mu \phi)\). The kinetic term can be rewritten as
\[ S_{kin} = -\frac{1}{2} \int \left( A^\mu \phi \right) \left( \begin{array}{cc} a \Box \delta_{\mu \nu} & -a \mu \epsilon_{\mu \nu} \partial^K \\ a \mu \epsilon_{\nu \eta} \partial^K & K^{-1} - a \mu^2 \end{array} \right) \left( \begin{array}{c} A^\nu \phi \end{array} \right) + 2 \bar{c} \Box c, \] (3.13)

where \(K^{-1}\) denotes the operator
\[ K^{-1} = \Box - 4 \mu^4 x_\alpha x^\alpha - 4 \mu^2. \] (3.14)

The corresponding kinetic matrix
\[ G^{-1} = \left( \begin{array}{cc} a \Box \delta_{\mu \nu} & -a \mu \epsilon_{\mu \nu} \partial^K \\ a \mu \epsilon_{\nu \eta} \partial^K & K^{-1} - a \mu^2 \end{array} \right) \] (3.15)
cannot be diagonalized easily because of the mixing of position and momentum variables introduced through $K^{-1}$. It is however possible to find its inverse, the propagator $G$:

$$G = \begin{pmatrix} \frac{1}{a} \Box^{-1} \delta_{\mu\nu} - \mu^2 \Box^{-1} \epsilon_{\mu\nu} \partial^\delta K \epsilon_{\nu\eta} \partial^\eta \Box^{-1} K & -\mu \Box^{-1} \epsilon_{\mu\nu} \partial^\delta K \\ \mu K \epsilon_{\nu\eta} \partial^\eta \Box^{-1} K & K \end{pmatrix}. \quad (3.16)$$

The propagation of ghosts is decoupled and the ghost propagator $G_{gh}$ cannot be diagonalized easily because of the mixing of position and momentum variables introduced through $K^{-1}$. It is however possible to find its inverse, the propagator $G$:

$$G^{-1} = \frac{1}{a} \Box^{-1} \delta_{\mu\nu} - \mu^2 \Box^{-1} \epsilon_{\mu\nu} \partial^\delta K \epsilon_{\nu\eta} \partial^\eta \Box^{-1} K.$$  

We will use mostly the propagator kernels in momentum representation. In accordance with the conventions for the Fourier transformation given in Appendix I we have

$$\tilde{G}_{gh}^{-1}(p, q) = -(2\pi)^2 p^2 \delta(p + q), \quad (3.17)$$

$$\tilde{G}^{-1}(p, q) = \begin{pmatrix} -ap^2 \delta_{\mu\nu} (2\pi)^2 \delta(p + q) & -ia \mu \epsilon_{\mu\nu} p^2 (2\pi)^2 \delta(p + q) \\ -ia \mu \epsilon_{\nu\eta} q \eta (2\pi)^2 \delta(p + q) & K_{gh}^{-1}(p, q) - ap^2 (2\pi)^2 \delta(p + q) \end{pmatrix}. \quad (3.18)$$

Note that in our conventions all momenta are incoming. This will, apart from a somewhat unusual factor $\delta(p + q)$ in the propagators, also reflect later when we define ‘long’ and ‘short’ variables.

$K^{-1}$ is the kinetic operator for the scalar field. Its kernel in coordinate space is given by

$$K^{-1}(x, y) = (\Box - 4\mu^4 x_\mu x^\mu - 4\mu^2) \delta^2(x - y); \quad (3.19)$$

the inverse is the so-called Mehler kernel. In two dimensions the Mehler kernel is given by

$$K(x, y) = -\frac{1}{8\pi} \int_0^\infty \frac{\omega d\tau}{\sinh \omega \tau} e^{-\frac{\tau^2}{4}(x-y)^2 \coth \frac{\omega \tau}{2} + (x+y)^2 \tanh \frac{\omega \tau}{2}} \omega^{-\tau}, \quad (3.20)$$

or in momentum space,

$$\tilde{K}(p, q) = -\frac{\pi}{4\mu^4} \int_0^\infty \frac{\omega d\tau}{\sinh \omega \tau} e^{-\frac{1}{4\mu^2}(p+q)^2 \coth \frac{\omega \tau}{2} + (p-q)^2 \tanh \frac{\omega \tau}{2}} \omega^{-\tau}. \quad (3.21)$$

Parameter $\omega$ in the last formula is by dimension a frequency; it is in fact the frequency of the quantum-mechanical harmonic oscillator of mass $m$ of which $(3.21)$ is the Green function. Here, $m\omega = 2\mu^2$. The mass of the scalar field on the other hand is $\mu_0$; it enters the Mehler kernel through exponential term $e^{-\frac{\omega^2}{2m^2} \tau} = e^{-\frac{\omega^2}{4\mu^2} \omega \tau}$. In our case this mass is fixed $\mu_0 = 2\mu$ and therefore we have in $(3.21)$ factor $e^{-\omega \tau}$.

One usually introduces dimensionless parameter $\alpha = \omega \tau$ so the Mehler kernel becomes

$$\tilde{K}(p, q) = -\frac{\pi}{4\mu^4} \int_0^\infty \frac{d\alpha}{\sinh \alpha} e^{-\frac{1}{4\mu^2}(p+q)^2 \coth \frac{\alpha}{2} + (p-q)^2 \tanh \frac{\alpha}{2}} \alpha^{-\alpha}, \quad (3.22)$$

or $\xi = \coth \frac{\alpha}{2}$ so we have

$$\tilde{K}(p, q) = -\frac{\pi}{4\mu^4} \int_1^\infty \frac{d\xi}{\xi} \frac{\xi - 1}{\xi + 1} e^{-\frac{1}{4\mu^2}(p+q)^2 \xi + (p-q)^2 \frac{\xi}{2}}. \quad (3.23)$$
The importance of the form (3.21) of the Mehler kernel is that using it one can easily perform
the free-field limit $\omega \to 0$. This limit, in particular, determines the prefactors in kernels (3.20)
and (3.21): using the Schwinger parametrization and
\[
\lim_{\sigma \to 0} \frac{1}{2\pi \sigma^2} e^{-\frac{(p+q)^2}{2\sigma^2}} = \delta^2(p + q)
\] (3.24)
we find the limiting value,
\[
\tilde{K}(p, q)|_{\omega \to 0} = -\frac{(2\pi)^2}{p^2 + \mu_0^2} \delta^2(p + q).
\] (3.25)

The Mehler kernel $K(x, y)$ is the contraction of two scalar fields in coordinate space:
\[
K(x, y) = \phi(x)\phi(y) = \frac{1}{(2\pi)^4} \int dk dl \tilde{\phi}(k)\tilde{\phi}(l)e^{-ikx - ily};
\] (3.26)
in momentum space we write
\[
\tilde{\phi}(k)\tilde{\phi}(l) = \tilde{K}(k, l).
\] (3.27)
In the following, in order to alleviate the notation we will omit the tilde sign in the Fourier
transformation, so we will distinguish for example $\phi(x)$ from $\tilde{\phi}(p)$ by the value of the argument
only. We thus write
\[
\tilde{\phi}(k)\tilde{\phi}(l) \equiv \phi(k)\phi(l) = K(k, l),
\] (3.28)
and in analogy
\[
A_{\sigma}(k)\phi(l) = -i\mu \frac{\epsilon_{\sigma\beta}k^\beta}{k^2} K(k, l),
\] (3.29)
\[
\phi(k)A_{\sigma}(l) = -i\mu K(k, l) \frac{\epsilon_{\sigma\rho}l^\rho}{l^2},
\] (3.30)
\[
A_{\rho}(k)A_{\sigma}(l) = -\frac{(2\pi)^2}{a} \delta_{\rho\sigma} \delta(k + l) + (-i\mu)^2 \frac{\epsilon_{\rho\sigma}k^\nu}{k^2} K(k, l) \frac{\epsilon_{\sigma\tau}l^\tau}{l^2},
\] (3.31)
\[
\bar{c}(k)c(l) = -\frac{(2\pi)^2}{k^2} \delta(k + l).
\] (3.32)
### 3.2 Vertices

Transforming the interaction terms to momentum space we obtain the following 3-vertices:

1. $\frac{2i}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \cos \frac{k \wedge q}{2} \epsilon_{\rho\sigma} p^\rho A^\sigma(p) \phi(q) \phi(k)$
2. $\frac{2i}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \sin \frac{q \wedge k}{2} p^\rho \phi(p) A_\rho(k) \phi(q)$
3. $\frac{-4i\mu^2 \epsilon}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \cos \frac{k \wedge q}{2} \epsilon_{\rho\sigma} \partial_\rho \phi(p) A^\sigma(k) \phi(q)$
4. $\frac{\alpha}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \sin \frac{q \wedge k}{2} p_\rho A^\rho(p) A_\rho(q) A^\tau(k)$
5. $\frac{i\alpha}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \sin \frac{q \wedge p}{2} \epsilon_{\rho\sigma} A^\rho(p) A^\sigma(q) \phi(k)$
6. $\frac{2i}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \sin \frac{q \wedge k}{2} p_\alpha \bar{c}(p) c(q) A^\alpha(k),$ 

where we denoted

\[ \epsilon_{\alpha\beta} = \frac{\epsilon}{\mu^2} p_\alpha q_\beta = \frac{\epsilon}{\mu^2} p \cdot \bar{q}. \quad (3.33) \]

Clearly only the vertex (3), containing the derivative of $\delta$-function, comes from the position-dependent terms in the interaction and breaks the translation invariance. There are no such terms in 4-vertices as we have:

7. $\frac{2}{(2\pi)^5} \int dp \, dq \, dk \, dl \, \delta(p + q + k + l) \sin \frac{k \wedge p}{2} \sin \frac{l \wedge q}{2} \delta_\rho^\sigma A_\rho(p) A_\sigma(q) \phi(k) \phi(l)$
8. $\frac{2\epsilon^2}{(2\pi)^5} \int dp \, dq \, dk \, dl \, \delta(p + q + k + l) \cos \frac{k \wedge p}{2} \cos \frac{l \wedge q}{2} \delta_\rho^\sigma A_\rho(p) A_\sigma(q) \phi(k) \phi(l)$
9. $\frac{2\epsilon}{(2\pi)^5} \int dp \, dq \, dk \, dl \, \delta(p + q + k + l) \sin \frac{q \wedge p}{2} \cos \frac{l \wedge k}{2} \epsilon_{\rho\sigma} A_\rho(p) A_\sigma(q) \phi(k) \phi(l)$
10. $\frac{\alpha}{2(2\pi)^6} \int dp \, dq \, dk \, dl \, \delta(p + q + k + l) \sin \frac{q \wedge p}{2} \sin \frac{l \wedge k}{2} \epsilon_{\rho\sigma} A_\rho(p) A_\sigma(q) \epsilon^{\lambda\tau} A_\lambda(k) A_\tau(l).$

This completes the list of the Feynman rules of our theory.

### 4 Tadpoles: one-loop divergences

We start the calculation of the quantum corrections from the simplest, tadpole diagram. Since we have vertices with three external lines such diagrams a priori exist. Moreover, as the
Translation invariance is broken and the momentum is not conserved along the propagator, they do not vanish. The scalar field tadpole is the expectation value

\[ T_\phi \equiv T(r) = -\langle \phi(r) S \rangle. \] (4.1)

Nonvanishing contributions can be graphically represented as in the picture. It is worth stressing that propagators are drawn either by a simple flat line or by a mixed line, corresponding to respectively \( \phi \phi \) and \( \phi A_\mu \).

We need contributions from each of the 3-vertices. From vertex (1) we obtain

\[ T_{\phi,1} = \frac{2i\epsilon}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \cos \frac{k \wedge q}{2} \epsilon_{\rho\sigma} p^\rho \times \left( \phi(r) A^\sigma(p) \phi(q) \phi(k) + \phi(r) \phi(q) A^\sigma(p) \phi(k) + \phi(r) \phi(k) A^\sigma(p) \phi(q) \right) \]

\[ = \frac{2\mu \epsilon}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \cos \frac{k \wedge q}{2} K(r, p, q, k), \]

where we introduced the cyclic product of two Mehler kernels

\[ K(r, p, q, k) = K(r, p) K(q, k) + K(r, q) K(p, k) + K(r, k) K(p, q). \] (4.3)

Obviously, \( K \) is invariant under permutations of its factors which is similar to the property

\[ K(p, q) = K(q, p) = K(-p, -q) \] (4.4)

of the Mehler kernel. Analogously the other 3-vertices give

\[ T_{\phi,2} = -\frac{2\mu}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \sin \frac{k \wedge q}{2} \frac{p \cdot \tilde{k}}{k^2} K(r, p, q, k) \] (4.5)

\[ T_{\phi,3} = \frac{4\mu^3 \epsilon}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \cos \frac{k \wedge q}{2} \frac{k}{k^2} \frac{\partial}{\partial p} K(r, p, q, k) \]

\[ T_{\phi,4} = \frac{\alpha \mu^3}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \sin \frac{q \wedge p}{2} \frac{p \cdot \tilde{q}}{p^2 q^2} K(r, p, q, k) \]

\[ T_{\phi,5} = -\frac{\alpha \mu^3}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \sin \frac{q \wedge p}{2} \frac{p \cdot \tilde{q}}{p^2 q^2} K(r, p, q, k). \]

\[ T_{\phi,6} = 0. \]
The ghost contribution is zero, \( T_{\phi,4} + T_{\phi,5} = 0 \) and 4-vertices do not contribute. Therefore for the scalar-field tadpole we obtain:

\[
T(r) = \frac{2\mu}{(2\pi)^4} \int dp \, dq \, dk \, \delta(p + q + k) \left( \epsilon \cos \frac{p \wedge q}{2} \left( 1 + 2\mu^2 \frac{p_\sigma}{p^2} \frac{\partial}{\partial q_\sigma} \right) + \sin \frac{q \wedge p}{2} \frac{p \cdot \tilde{q}}{p^2 q^2} \right) K(r, p, q, k).
\]  

(4.6)

In similar manner we can calculate the gauge-field tadpole,

\[
T_\mu \equiv T_\mu(r) = -\langle A_\mu(r) S_{\text{int}} \rangle = \sum_{j=1}^6 T_{A_\mu,j};
\]

(4.7)

By noticing that there is a simple relation between the two expressions, (4.6) and (4.8):

\[
T_\nu(r) = -i\mu \frac{\tilde{r}_\nu}{r^2} T(r) + B_\nu(r)
\]

(4.9)

where

\[
B_\mu(r) = \frac{i}{(2\pi)^2 a} \int dp \, dq \, \delta(p + q - r)
\]

(4.10)

\[
\times \left( 2\epsilon \cos \frac{p \wedge q}{2} \frac{\epsilon_{\mu\alpha}}{r^2} (p^\alpha - 2\mu^2 \frac{\partial}{\partial p^\alpha}) + \sin \frac{q \wedge p}{2} \left( \frac{p_\mu}{r^2} + a\mu^2 \frac{\tilde{r}_\mu p \cdot \tilde{q}}{r^2 q^2} \right) \right) K(p, q).
\]

One notices that there is a simple relation between the two expressions, (4.6) and (4.8):

\[
T_\nu(r) = -i\mu \frac{\tilde{r}_\nu}{r^2} T(r) + B_\nu(r)
\]

(4.9)

where

\[
B_\mu(r) = \frac{i}{(2\pi)^2 a} \int dp \, dq \, \delta(p + q - r)
\]

(4.10)

\[
\times \left( 2\epsilon \cos \frac{p \wedge q}{2} \frac{\epsilon_{\mu\alpha}}{r^2} (p^\alpha - 2\mu^2 \frac{\partial}{\partial p^\alpha}) + \sin \frac{q \wedge p}{2} \left( \frac{p_\mu}{r^2} + a\mu^2 \frac{\tilde{r}_\mu p \cdot \tilde{q}}{r^2 q^2} \right) \right) K(p, q).
\]

In fact, as we are using the multiplet of fields \((A_\mu, \phi)\), it is more practical to change diagrammatic representation of the propagators and use one, ‘doubled’ line for both fields in the multiplet as given below. Then the tadpole is also a doublet: the corresponding diagram is

\[
\begin{array}{c}
\text{Diagram} \\
\end{array}
\]

(4.9)

Though we have the results for the tadpole, it is not easy to understand the structure of divergences in the obtained expressions. This is difficult first of all because we are dealing with the Mehler kernel in which results are in the form of a parameter integral. In addition, (4.6-4.8) contain \( K(r, p, q, k) \), that is the products of two kernels. In order to find divergences and the form of counterterms we apply the method developed in [10, 13]: we amputate the leg of the tadpole, multiply it with the corresponding external field then and integrate. The amputated tadpole graph is obtained by multiplication with the inverse propagator:
\[
\left( \frac{\tau_\mu(s)}{\tau(s)} \right) = \frac{1}{(2\pi)^2} \int dr \, G^{-1}(s, -r) \left( T_\nu(r) T(r) \right);
\]

from (4.6-4.8) we get
\[
\tau_\mu(s) = -as^2 \beta_\mu(s), \quad (4.11)
\]
\[
\tau(s) = ia s^\nu \beta_\nu(s) + \frac{1}{(2\pi)^2} \int dr \, K^{-1}(s, -r) T(r). \quad (4.12)
\]

### 4.1 Counterterms

It is clear that our formulae simplify considerably for \( a = 0 \), so let us calculate first this part of the tadpole divergences. This will help us to understand better the framework we are working in; the \( a \)-linear terms we will calculate in the sequel. After momentum integrations we have

\[
\tau_\mu(s) \mid_{a=0} = \frac{i\epsilon}{4\mu^2} \bar{s}_\mu \int_1^\infty d\xi \left( \xi - 1 \right) e^{-\frac{s^2}{4\mu^2}\xi} \quad (4.13)
\]
\[
\tau(s) \mid_{a=0} = -\frac{\epsilon}{2\mu} \int_1^\infty d\xi \left( \frac{\xi - 1}{\xi + 1} \right) e^{-\frac{s^2}{4\mu^2}\xi}. \quad (4.14)
\]

Both integrals are finite in variable \( \xi \) but the result is divergent in external momentum \( s \) in the infrared region, \( s = 0 \):

\[
\tau_\mu(s) = 4i\mu^2 \bar{s}_\mu \frac{\bar{s}^\nu}{s^4} e^{-\frac{s^2}{4\mu^2}} \quad (4.15)
\]
\[
\tau(s) = -\frac{1}{\mu} e^{-\frac{s^2}{4\mu^2}} \left( E_0 \left( \frac{s^2}{2\mu^2} \right) - E_1 \left( \frac{s^2}{2\mu^2} \right) \right). \quad (4.16)
\]

where \( E_0 \) and \( E_1 \) are the exponential integrals reviewed shortly in Appendix I. As mentioned before, the corresponding counterterms can be found by multiplying by external field and integrating. As the tadpoles are divergent only at \( s = 0 \) we can Taylor-expand the external field around this value and integrate term by term. For the gauge field we obtain

\[
\frac{1}{(2\pi)^2} \int d^2s \, \tilde{A}^\mu(s) \tau_\mu(-s) = -\frac{4i\mu^2}{(2\pi)^2} \int d^2s \, \tilde{s}_\mu \frac{s^2}{s^4} e^{-\frac{s^2}{4\mu^2}} \tilde{A}^\mu(s) \quad (4.17)
\]

\[
= -\frac{4i\mu^2}{(2\pi)^2} \int d^2s \, \tilde{s}_\mu e^{-\frac{s^2}{4\mu^2}} \left( \tilde{A}_\mu(0) + \frac{\partial \tilde{A}_\mu}{\partial s_\rho}(0) s_\rho + \frac{1}{2!} \frac{\partial^2 \tilde{A}_\mu}{\partial s_\rho \partial s_\sigma}(0) s_\rho s_\sigma + \ldots \right).
\]

Here and in the following, as we shall see, only the initial terms are divergent. In (4.17) the first integral is obviously zero while the second one can be calculated in polar coordinates, \( s^1 = s \cos \varphi, s^2 = s \sin \varphi, d^2s = s \, ds \, d\varphi \). Using

\[
\int_0^{2\pi} \frac{s_\alpha s_\beta}{s^2} d\varphi = \pi \delta_{\alpha\beta}, \quad (4.18)
\]

* In this subsection we reintroduce the tilde to distinguish between a field and its Fourier transform, that is to clarify the form of counterterms both in momentum and in position space.
we obtain
\[ \frac{1}{(2\pi)^2} \int d^2s \tilde{A}^\mu(s)\tau_\mu(-s) = -\frac{4i\mu^2}{(2\pi)^2} \epsilon_{\mu\alpha} \frac{\partial \hat{A}_\mu(0)}{\partial s_\rho} \pi \delta_{\alpha\rho} \int_0^\infty \frac{ds}{s} e^{-\frac{2}{\mu^2}s} \]
\[ = -\frac{i\mu^2}{2\pi} \epsilon_{\mu\rho} \frac{\partial \hat{A}_\mu(0)}{\partial s_\rho} \Gamma(0) = -\frac{\mu^2}{2\pi} \Gamma(0) \int d^2x \tilde{x}_\mu \tilde{A}^\mu(x). \quad (4.19) \]

The integral has a logarithmic divergence. Higher than linear terms in the expansion contain higher orders of \( s \) and therefore they converge at the lower bound; convergence at \( s = \infty \) is guaranteed by the exponentially decreasing factor \( e^{-\frac{s^2}{\mu^2}} \).

In similar way we can calculate the divergence in \( \tau(s) \). As before, the infinite contribution comes from the lower bound, \( s = 0 \); we will focus therefore on the behavior of the integral only at this point. Expanding \( \tilde{\phi}(s) \) and the exponential integrals \( E_0(\frac{s^2}{2\mu^2}) \) and \( E_1(\frac{s^2}{2\mu^2}) \) around zero, we obtain
\[ \frac{1}{(2\pi)^2} \int d^2s \tilde{\phi}(s)\tau(-s) = \quad (4.20) \]
\[ = -\frac{1}{\mu(2\pi)^2} \int d^2s e^{-\frac{s^2}{\mu^2}} \left( \frac{2\mu^2}{s^2} e^{-\frac{s^2}{\mu^2}} + \gamma + \log \frac{s^2}{2\mu^2} + \ldots \right) \left( \tilde{\phi}(0) + \frac{\partial \tilde{\phi}}{\partial s_\alpha}(0) s_\alpha + \ldots \right). \]
The first, divergent, term gives
\[ -\frac{\mu}{\pi} \int_0^\infty \frac{ds}{s^2} e^{-\frac{s^2}{2\mu^2}} \tilde{\phi}(0) = -\frac{\mu}{2\pi} \Gamma(0) \tilde{\phi}(0) = -\frac{\mu}{2\pi} \Gamma(0) \int d^2x \phi(x), \quad (4.21) \]
and the divergence is again logarithmic. Other terms in \( (4.20) \) give finite contributions including the log term, as the integral of the logarithm vanishes at the lower bound, \( \int \log s ds = s \log s - s \).

We obtained for \( a = 0 \) only two counterterms which regularize the tadpole diagrams at one loop:
\[ \int d^2x \phi, \quad \int d^2x \tilde{x}^\mu * A_\mu. \quad (4.22) \]
Are there more counterterms in linear order in \( a \)? In fact the answer is negative: though the integrals which one calculates become more complicated, they do not bring additional divergences. To see this we start again with \( \tau_\mu \) denoting
\[ \tau_\mu(s) = \tau_\mu(s) | + \Delta \tau_\mu(s); \quad (4.23) \]
the difference is
\[ \Delta \tau_\mu(s) = -a \frac{i\mu^2}{(2\pi)^2} \hat{s}_\mu \int dp dk \delta(p + k - s) \sin \frac{p \wedge k}{2} \frac{p \cdot \hat{k}}{p^2 \hat{k}^2} K(p, k). \quad (4.24) \]

\[ ^1 \text{As in the previous calculation we use} \]
\[ \int d^2x \phi(x) = \frac{1}{(2\pi)^2} \int d^2p d^2x \tilde{\phi}(p)e^{-ipx} = \frac{1}{(2\pi)^2} \int d^2p \tilde{\phi}(p)(2\pi)^2 \delta^2(p) = \delta(0). \]
The additional integral can be expressed in parameter form using Schwinger parametrization. After momentum integrations we obtain

\[
\Delta \tau_{\mu} = \frac{ie\alpha}{8s_{\mu}} \int_1^\infty d\xi \frac{\xi - 1}{\xi + 1} \int_0^\infty d\beta \frac{1}{(1 + 4\mu^2\beta\xi)^2 - \epsilon^2\xi^2} e^{-\frac{\mu^2}{s_{\mu}}\left((8\mu^2\beta + \frac{1}{4\pi^2}\frac{1}{(1 + 2\mu^2\beta\xi)^2} + \epsilon^2\xi^2\right)}
\]

\[
= \frac{ie\alpha}{8s_{\mu}} \int_1^\infty d\xi d\eta \frac{\xi - 1}{\xi + 1} \frac{1}{(2\eta - 1 - \epsilon\xi)(2\eta - 1 + \epsilon\xi)} e^{-\frac{\mu^2}{s_{\mu}}\left((\xi + \frac{1}{4\pi^2}(\epsilon^2\xi - \frac{\epsilon^2}{\xi})\right)},
\]

where we introduced a new variable \( \eta = 1 + 2\mu^2\xi\beta \). Similarly for the scalar-field tadpole \( \Delta \tau(s) = \tau(s) - \tau(s)|_{a=0} \) we have

\[
\Delta \tau(s) = \frac{ae}{8s_{\mu}} \int_1^\infty d\xi d\eta \frac{\xi - 1}{\xi + 1} \frac{1}{(2\eta - 1 - \epsilon\xi)(2\eta - 1 + \epsilon\xi)} e^{-\frac{\mu^2}{s_{\mu}}\left((\xi + \frac{1}{4\pi^2}(\epsilon^2\xi - \frac{\epsilon^2}{\xi})\right)}
\]

\[
+ \frac{2ae}{\mu} e^{-\frac{\mu^2}{4s_{\mu}}(1+\epsilon^2)} \int_1^\infty d\zeta \left( \frac{2}{\zeta} - \frac{1}{\zeta^2} \right) e^{-\frac{\mu^2}{4s_{\mu}}(1+\epsilon^2)\zeta}.
\]

The second line of (4.26) can be integrated in terms of the exponential integrals and it is convergent for all \( s \). The double integral is the same for both corrections (4.26) and (4.25); it is regular, too. This we can verify by analyzing the integral in the potentially divergent region \( s = 0 \), in which the exponential can be replaced by 1. The integral is then

\[
\int_1^\infty d\xi d\eta \frac{\xi - 1}{\xi + 1} \frac{1}{(2\eta - 1 - \epsilon\xi)(2\eta - 1 + \epsilon\xi)} = \frac{1}{2} \int_1^\infty d\xi \left( \frac{\xi - 1}{\xi + 1} \frac{1}{\xi^2 - \epsilon^2\xi^2} \right)
\]

\[
= \frac{1}{4\epsilon} \int_1^\infty \frac{d\xi}{\xi} \left( \frac{\xi - 1}{\xi + 1} \log \left| \frac{\xi - \epsilon}{\xi + \epsilon} \right| \right),
\]

and finite for all values of \( \epsilon \). For \( \epsilon = 1 \) for example the value of the integral is \( -\frac{\pi^2}{8} \) while for general \( \epsilon \) it obtains additional terms proportional to the PolyLog functions. We thus arrive at a very nice conclusion, that there are no new divergences in the tadpole diagrams for \( a \neq 0 \) apart from those given in (4.22).

### 5 Propagators: one-loop divergences

The one-loop propagator corrections can be calculated along similar lines except that calculations are longer and more complicated. We denote

\[
P_{\phi(r)\phi(s)} \equiv P(r, s) = -\langle \phi(r)\phi(s)\rangle S_{\text{int}},
\]

\[
P_{\phi(r)A_\mu(s)} \equiv P_{\mu}(r, s) = P_{\mu}(\mu, s, r) = -\langle \phi(r)A_\mu(s)\rangle S_{\text{int}},
\]

\[
P_{A_\nu(r)A_\mu(s)} \equiv P_{\nu}(\nu r, \mu s) = -\langle A_\nu(r)A_\mu(s)\rangle S_{\text{int}}.
\]

The one-loop corrections contain now three field contractions so we introduce auxiliary functions

\[
N(r, s; p, q, k, l) = K(r, s)K(k, l)K(p, q) + \text{all pairings of arguments}
\]

\[
\equiv N(r, s; p, q, k, l) + K(r, s)K(p, q, k, l).
\]
The product of three Mehler kernels $N(r, s, p, q, k, l)$ contains 15 terms that is, all permutations of the arguments. The $N(r, s; p, q, k, l)$ on the other hand does not change under permutations of the first two and of the last four. To simplify expressions for the 2-point Green functions we introduce further:

$$N(r, s) = \int dp dq dk dl \, \delta(p + q + k + l) N(r, s; p, q, k, l)$$

$$\times \left( 2 \frac{p \cdot q}{p^2 q^2} \left( \sin \frac{q \wedge p}{2} \sin \frac{l \wedge k}{2} + \cos \frac{q \wedge p}{2} \cos \frac{l \wedge k}{2} \right) + \frac{1}{2} \frac{p \cdot q}{p^2 q^2} \sin \frac{q \wedge p}{2} \cos \frac{l \wedge k}{2} \right)$$

$$- a \left( 2 \frac{p \cdot q}{p^2 q^2} \cos \frac{k \wedge p}{2} \cos \frac{l \wedge k}{2} + \frac{\mu^2}{2} \frac{p \cdot q}{p^2 q^2} \frac{k \cdot \tilde{l}}{k^2 l^2} \sin \frac{q \wedge p}{2} \sin \frac{l \wedge k}{2} \right) \right) \right) (5.3)$$

$$K_{\nu}(s, r) = \int dp dk dl \, \delta(p + k + l - r) K(s, p, k, l)$$

$$\times \left( 4 \frac{\tilde{\nu}}{p^2} \left( \cos \frac{r \wedge p}{2} \cos \frac{l \wedge k}{2} - \sin \frac{r \wedge p}{2} \sin \frac{l \wedge k}{2} \right) - \frac{\nu}{p^2} \sin \frac{r \wedge p}{2} \cos \frac{l \wedge k}{2} \right)$$

$$- a \left( 4 \frac{\tilde{\nu}}{p^2} \cos \frac{k \wedge p}{2} \cos \frac{l \wedge k}{2} - 2\mu^2 \frac{\nu}{p^2} \frac{k \cdot \tilde{l}}{k^2 l^2} \sin \frac{r \wedge p}{2} \sin \frac{l \wedge k}{2} \right) \right) \right) (5.4)$$

$$K_{\nu} K(r, s) = \int dp dq \frac{2}{q^2} K(r, -p) K(s, p) \left( 2 - a \left( \cos^2 \frac{q \wedge p}{2} - \frac{\mu^2}{p^2} \sin^2 \frac{q \wedge p}{2} \right) \right) (5.5)$$

$$Q(r) = \frac{1}{r^2} \int dq \frac{2}{q^2} \sin^2 \frac{q \wedge r}{2} \right) \right) (5.6)$$

$$J(s) = \int dk dl \, \delta(k + l - s) \cos \frac{k \wedge l}{2} K(k, l) \right) \right) (5.7)$$
The propagator corrections are then given by

\[ P_{\phi(r)\phi(s)} = \frac{\mu^2}{(2\pi)^6} N(r, s) + \frac{2}{(2\pi)^4 a} KK(r, s) \] (5.8)

\[ P_{\phi(r)A_\mu(s)} = -i\mu \frac{\bar{s}_\mu}{s^2} P_{\phi(r)\phi(s)} - \frac{i\mu}{(2\pi)^4 a} \frac{1}{s^2} K_{\mu}(r, s) + \frac{4i\mu}{(2\pi)^2 a} \frac{s_\mu}{s^2} K(r, s) Q(s), \] (5.9)

\[ P_{A_\nu(r)A_\mu(s)} = \mu^2 \frac{\bar{s}_\mu}{s^2} \bar{r}_\nu P_{\phi(r)\phi(s)} - i\mu \frac{\bar{s}_\mu}{s^2} P_{\phi(s)A_\nu(r)} - i\mu \frac{\bar{r}_\nu}{r^2} P_{\phi(r)A_\mu(s)} + \frac{4}{a^2} \frac{\delta_{\mu\nu}(r + s)}{s^2} Q(r) \] (5.10)

\[ + \frac{1}{(2\pi)^2 a^2} \frac{1}{r^2 s^2} \int dk dl \delta(k + l - r - s) K(k, l) \]

\[ \times \left( -4\delta_{\mu\nu} \left( \sin\frac{s \wedge r}{2} - \sin\frac{l \wedge k}{2} + \cos\frac{s \wedge r}{2} - \cos\frac{l \wedge k}{2} \right) - \epsilon_{\nu\mu} \sin\frac{s \wedge r}{2} \cos\frac{l \wedge k}{2} \right. \]

\[ + \left. a \left( 4\delta_{\mu\nu} \cos\frac{k \wedge r}{2} \cos\frac{l \wedge s}{2} + 4\mu^2 \frac{k_\nu l_\mu}{k^2 l^2} \sin\frac{k \wedge r}{2} \cos\frac{l \wedge s}{2} + 2\mu^2 \epsilon_{\nu\mu} \frac{k \cdot \tilde{l}}{k^2 l^2} \sin\frac{s \wedge r}{2} \sin\frac{l \wedge k}{2} \right) \right). \]

As before, to obtain divergent parts we amputate the external propagator legs by multiplying from the left and from the right by the inverse propagator; the amputated 2-point function, a $2 \times 2$ matrix $\Pi(r, s)$, is

\[ \Pi(r, s) = \frac{1}{(2\pi)^4} \int dp dq G^{-1}(r, -p)P(p, q) G^{-1}(-q, s). \] (5.11)

Again we first calculate the $1/a$- and $a$-constant divergent parts of (5.11) which we denote by $\Pi|$. We obtain

\[ \Pi_{\rho\sigma}(\rho r, \sigma s) = 4\delta_{\rho\sigma}(r + s)s^2 Q(s) - \frac{1}{(2\pi)^2} (4\delta_{\rho\sigma} \cos\frac{r \wedge s}{2} - \epsilon_{\rho\sigma} \sin\frac{r \wedge s}{2}) \bar{J}(r + s) \] (5.12)

\[ \Pi_{\rho}(\rho r, s) = \frac{2i\mu}{(2\pi)^2} \int dk dl \delta(k + l - r - s) \cos\frac{s \wedge l}{2} K(k, l) \left( \frac{4k_\rho}{k^2} \cos\frac{r \wedge k}{2} - \frac{k_\nu}{k^2} \sin\frac{r \wedge k}{2} \right) \] (5.13)

\[ \Pi(r, s) = \frac{8}{a} \delta(r + s) \int \frac{dl}{l^2} - 4 \delta(r + s) \int \frac{dl}{l^2} \cos\frac{s \wedge l}{2} \]

\[ \left( \frac{\mu^2}{(2\pi)^2} \cos\frac{r \wedge s}{2} \int dk dl \delta(k + l - r - s) \left( 4\frac{k \cdot \tilde{l}}{k^2 l^2} \cos\frac{k \wedge l}{2} - \frac{k \cdot \tilde{l}}{k^2 l^2} \sin\frac{k \wedge l}{2} \right) K(k, l) \right). \] (5.14)

These integrals can be naturally rewritten using the ‘short variable’ $u = r + s$, the difference between the incoming and the outgoing momentum, and the ‘long variable’ $v = r - s$ which is done in Appendix II.
The $a$-linear part $\Delta \Pi$ is given by

$$\Delta \Pi = \frac{a}{(2\pi)^2} \int \frac{dk dl}{2} \delta(k + l - r - s) K(k, l) \left( \frac{k \cdot l}{k^2 l^2} \right)$$

(5.15)

$$\times \left( 4 \delta_{\rho \sigma} \cos \frac{k \wedge r}{2} \cos \frac{l \wedge s}{2} + 4 \mu^2 \frac{k_{\rho} l_{\sigma}}{k^2 l^2} \sin \frac{k \wedge r}{2} \sin \frac{l \wedge s}{2} + 2 \mu^2 \epsilon_{\rho \sigma} \frac{k \cdot \tilde{l}}{k^2 l^2} \sin \frac{s \wedge r}{2} \sin \frac{l \wedge k}{2} \right)$$

(5.16)

$$\Delta \Pi_{\rho \sigma} = -4 i a \mu (2\pi)^2 \int \frac{dk dl}{2} \delta(k + l - r - s) K(k, l) \left( \frac{k_{\rho}}{k^2} \cos \frac{k \wedge r}{2} \cos \frac{l \wedge s}{2} \right)$$

$$-4 i a \mu (2\pi)^2 \int \frac{dk dl}{2} \delta(k + l - r - s) K(k, l) \left( \frac{k_{\rho}}{k^2} \cos \frac{k \wedge r}{2} \cos \frac{l \wedge s}{2} \right)$$

(5.17)

The one-loop contributions to the propagators are now rather long integrals. Therefore the calculation and analysis of the corresponding divergences is placed in Appendix II. It is interesting to mention however that many terms cancel in the course of calculation, and in very encouraging way. At the end we obtain only three divergent contributions to the 2-point functions at one loop; the corresponding counterterms are

$$\int \frac{d^2 x}{A_{\mu} \star A^\mu}, \quad \int \frac{d^2 x}{x \phi \star \phi} \quad \text{and} \quad \int \frac{d^2 x}{\tilde{x}^\mu \star A_{\mu} \star \phi}.$$  

(5.18)

Again the forefactors are proportional to integrals which diverge logarithmically.

6 Conclusions and outlook

As we said at the beginning, our paper is devoted to an analysis of renormalizability properties of the BGM model [12], which is a gauge analogon of the GW model.

The classical model was constucted in [12] and some of its properties were explored there: the equations of motion, the vacuum solutions, the BRST symmetry. Here we study perturbative quantization of the model. The model is apparently quite complicated: it contains a scalar field and a gauge field mixed already at the level of propagators; the interaction is described by ten vertices. Therefore the renormalizability analysis has not been completed yet, though a considerable amount of work has been done here.

Let us resume it shortly. First, the model was represented on the two-dimensional Moyal space and then the Feynman rules of the theory were derived. As it was impossible to diagonalize the kinetic term, we treated the fields as a multiplet, as in supersymmetry. The propagator became a $2 \times 2$ matrix containing the Mehler kernels in all matrix elements, which means that the background curvature influences propagation of all fields. We then calculated the one-loop quantum corrections to the tadpoles and to the propagators, leaving the vertex corrections for the subsequent work. The quantum corrections which we obtained
are divergent: all logarithmically. Notably the tadpole terms do not vanish. This property can be related to the non-conservation of the momentum; it appears in similar models. Expressed as counterterms, the 1-point function divergences are \( \int \phi \) and \( \int \tilde{x}_\mu \star A^\mu \). Obviously these terms are not present in the initial action \( (3.4) \). The 2-point functions also have divergent corrections, \( \int \phi \star \phi \), \( \int A_\mu \star A^\mu \) and \( \int \{ \tilde{x}_\mu \star A^\mu \} \star \phi \).

There are two ways to understand counterterm \((4.22)\) and \((5.18)\) in our model. One possibility is to interpret these terms as indication that the trivial vacuum \( \phi = 0, A_\mu = 0 \) we started with is unstable under quantization, and that the quantum vacuum is of the form

\[
\phi = \alpha, \quad A_\mu = \beta \tilde{x}_\mu, \tag{6.1}
\]

which the second classical vacuum of our theory has. Expansion around \((6.1)\) obviously gives all terms which we obtained as divergences, and some additional ones. The second possibility is that all counterterms add up to Chern-Simons action \((2.11)\): this would mean that \( S_{CS} \) should be included in the classical action.

To complete our analysis and come to conclusive results we have to perform a couple more steps. First, we need to calculate corrections to the vertices: this will help us to decide whether the origin of divergences is a shift of the vacuum or the Chern-Simons term (or both). Also, to obtain and compare the coefficients in the counterterms one should find a systematic way to quantify divergences in the parameter integrals. At present, we were able only to analyze the type of divergences, as when there are two parameter integrals the expressions are quite complicated and as a rule, impossible to solve exactly in terms of special functions. One should also do the power counting and estimate higher-order contributions. And finally, it is to be expected that for particular values of parameters \( a, \mu \) and \( c \) our model has specific renormalizability properties. All these points we plan to address in our future work.

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### 7 Appendix I

**i) Fourier transformation conventions**

\[
f(x) = \frac{1}{(2\pi)^2} \int \hat{f}(p) e^{-ipx} \]
\[
f(x, y) = \frac{1}{(2\pi)^4} \int d^2p \, d^2q \, \hat{f}(p, q) e^{-ipx - iqy}.
\]
With this convention in the $n$-point functions all momenta are incoming. Thus for example the Fourier transformation of the identity operator is

$$I(x, y) = \delta(x - y) = \frac{1}{(2\pi)^2} \int d^2p \, d^2q \, \tilde{I}(p, q) e^{-ipx - iqy}$$

$$\tilde{I}(p, q) = (2\pi)^2 \delta(p + q)$$

$$\overline{FG}(p, q) = \frac{1}{(2\pi)^2} \int d^2r \, \overline{F}(p, r) \overline{G}(-r, q)$$

A useful formula is

$$\int d^2p \, K^{-1}(r, -p) K(p, q, k, l) = (2\pi)^4 (\delta(r + q) K(k, l) + \delta(r + k) K(q, l) + \delta(r + l) K(q, k)).$$

**ii) Gaussian integrals**

$$\frac{1}{p^2} = \int_0^\infty d\beta \, e^{-\beta p^2}$$

$$\int e^{-ap^2 + bp} \, d^2p = \frac{\pi}{a} e^{b^2/4a}$$

$$\int \rho \alpha e^{-ap^2 + bp} \, d^2p = \frac{\pi}{2a^2} \frac{b_\alpha}{b} e^{b^2/4a}$$

$$\int \rho_\alpha \frac{p}{p^2} e^{-ap^2 + bp} \, d^2p = \frac{2\pi}{b^2} \left( \frac{\delta_{\alpha\beta} b^2 - 2b_\alpha b_\beta}{b^2} + \frac{b_\alpha b_\beta}{2a} \right) e^{b^2/4a}$$

**iii) Exponential integrals**

$$E_\nu(z) = \int_1^\infty \frac{e^{-zt}}{t^\nu}, \quad \text{Re} z > 0$$

$$E_\nu(z) \sim \frac{e^{-z}}{z} (1 + O\left(\frac{1}{z}\right)), \quad |z| \to \infty$$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!} (\psi(n) - \log z) - \sum_{k=0}^\infty \frac{(-z)^k}{(k-n+1)k!}$$

### 8 Appendix II

Let us discuss some of the integrals which appear in the one-loop propagator divergences and obtain the corresponding counterterms. The $\Pi_{\rho\sigma}$ is the simplest as it contains divergent integral which appeared before,

$$\mathcal{I}(u) = \int dk \, dl \, \delta(k + l - u) \cos \frac{k \wedge l}{2} K(k, l) = -\frac{\pi^2}{2\mu^2} \int_1^\infty d\xi \, \frac{\xi - 1}{\xi + 1} e^{-\frac{\mu^2}{4\pi^2} \xi}.$$
The second component, $\Pi\mid_{\rho}$, or more precisely its part containing the Mehler kernel is given by

$$\frac{\mu^2}{(2\pi)^2} \cos \frac{r \wedge s}{2} \int dk dl \delta(k + l - r - s) \left( \frac{k \cdot l}{k^2 l^2} \cos \frac{k \wedge l}{2} - \frac{k \cdot \tilde{l}}{k^2 l^2} \sin \frac{k \wedge l}{2} \right) K(k, l)$$

$$= -\frac{1}{8\mu^2} \cos \frac{r \wedge s}{2} \int \int_{1}^{\infty} d\xi d\eta \left( \frac{\xi - 1}{\xi + 1} \frac{4\eta - 2 + 2\xi^2 - \xi \eta}{\eta(2\eta - 1)^2 - \xi^2} e^{-\frac{\mu^2}{8\eta \xi} \left( \xi + \frac{1}{\eta} (\xi - \frac{1}{\eta}) \right)} \right),$$

and finally, in the $\Pi_{\rho}\mid$ propagator part the integration gives using $2s \wedge r = -u \wedge v$,

$$\frac{2i\mu}{(2\pi)^2} \int dk dl \delta(k + l - r - s) \cos \frac{s \wedge l}{2} K(k, l) \left( \frac{4k_{\rho}}{k^2} \cos \frac{r \wedge k}{2} - \frac{k_{\rho}}{k^2} \sin \frac{r \wedge k}{2} \right)$$

$$= \frac{i(2\pi)^4}{4\mu} \frac{1}{u^2} \int_{1}^{\infty} \frac{d\xi}{(\xi + 1)^2} e^{-\frac{\pi^2}{8u^2 \xi} \left( (4 - \xi) \tilde{u}_\rho \cos \frac{u \wedge v}{4} + (4\xi - 1) u_\rho \sin \frac{u \wedge v}{4} \right)}$$

$$- \frac{i(2\pi)^4}{4\mu} \int_{1}^{\infty} d\xi \frac{\xi - 1}{\xi + 1} e^{-\frac{\mu^2}{8u^2 (u^2 + v^2) \xi} \left( \frac{1}{u - \xi v^2} (u - \xi v^2) \right)}$$

$$\times \left( \cos \frac{u \wedge v}{2} (4\tilde{u}_\rho - \xi \tilde{v}_\rho)(u^2 - \xi^2 v^2) + 2(u_\rho - 4\xi v_\rho) u \cdot \xi \tilde{v} \right)$$

$$+ \sin \frac{u \wedge v}{2} ((u_\rho - 4\xi v_\rho)(u^2 - \xi^2 v^2) - 2(4\tilde{u}_\rho - \xi \tilde{v}_\rho) u \cdot \xi \tilde{v}) \right).$$

Let us analyze the corresponding counterterms. We start with the simplest, $\Pi_{\rho\sigma}\mid$: as before, we multiply it by the external fields and integrate. The first summand of the $\Pi_{\rho\sigma}\mid$ has a $\delta$-function; of course there is no overall translation invariance, and the momentum conservation is broken by the second part of $\Pi_{\rho\sigma}\mid$, in which the Mehler kernel plays a role of a smeared $\delta(r + s)$. Denoting

$$\int dr \; ds \; A^\rho(r) \Pi_{\rho\sigma}\mid_{(r, -s)} A^\sigma(s) = \int dr \; du \; A^\rho(r) A^\sigma(-r + u) \Pi_{\rho\sigma}\mid_{(-r, r - u)} \equiv (1) + (2),$$

where $u = r + s$, we have

$$(1) = 2 \int d^2r \; d^2u \; A^\rho(r) A^\rho(-r + u) \delta(u) \int \frac{d^2q}{q^2} (1 - \cos(q \wedge r))$$

$$= 4\pi \int d^2r \; A^\rho(r) A^\rho(-r) \int_{0}^{\infty} \frac{dq}{q} - 2 \int d^2r \; A^\rho(r) A^\rho(-r) \int d^2q \int_{0}^{\infty} d\beta e^{-\beta q^2} \cos(q \wedge r)$$

$$= 2\pi \left( 2 \int_{0}^{\infty} \frac{dq}{q} - \Gamma(0) \right) \int d^2r \; A^\rho(r) A^\rho(-r).$$
where

\[ \text{The other part is} \]

\[ \text{The divergence is logarithmic.} \]

\[ \text{counterterm proportional to} \]

\[ \text{terms are all convergent. Therefore we conclude that the divergent part of } \]

\[ \text{The leading divergence in the last expression is } 2\pi \int_0^\infty \frac{dq}{q} \int d^2 r A^\rho(r) A_\rho(-r), \text{ the remaining terms are all convergent. Therefore we conclude that the divergent part of } \]

\[ \text{The leading potentially divergent term after the change of variables } 2\eta - 1 = \zeta \text{ becomes} \]

\[ \text{The divergence is logarithmic.} \]

\[ \text{The second counterterm can also be divided into two:} \]

\[ \text{where} \]

\[ \text{The other part is} \]

\[ \text{The leading potentially divergent term after the change of variables } 2\eta - 1 = \zeta \text{ becomes} \]

\[ \text{21} \]
and it is finite. Therefore the divergence comes only from (3) and it is proportional to
\[
\frac{1}{(2\pi)^2} \int d^2 r \, \phi(r) \phi(-r) = \int d^2 x \, \phi(x) \phi(x).
\] (8.2)

Again it is logarithmic.

By similar reasoning we find that there are no divergences in \( \Pi_\rho | \) terms; however, in the \( a \)-linear part of the 2-point function a divergent term appears. We denote
\[
\int \int dr ds A^\rho(r) \Pi_\rho(-r, -s) \phi(s) = (5) + (6).
\]

Part (5) is finite, while for part (6) we obtain
\[
(6) = -\frac{4i\mu a}{(2\pi)^2} \int \int dr ds A^\rho(r) \phi(s) \int dk dl \, \delta(k + l - r - s) K(k, l) \frac{k_\mu}{k^2} \cos \frac{r \wedge l}{2} \cos \frac{r \wedge s}{2}
\]
\[
= \frac{ia}{16\mu} \int \int dr ds A^\rho(r) \phi(s) \cos \frac{r \wedge s}{2} u^\alpha \int_1^\infty d\xi (\xi - 1) e^{-\frac{u^2}{8\mu^2} (1 + \epsilon^2)}
\]
\[
\times \left( \frac{1}{(1 - \epsilon^2)(1 + \xi)} + \frac{2(\epsilon - 1)(1 + \epsilon \xi)}{2(\epsilon + 1)(1 - \epsilon \xi)} \right).
\]

After the parameter integrations the only divergent contribution is of the form
\[
(6) = -\frac{i\mu a \epsilon^2}{2} \int \int dr du A^\rho(r) \phi(-r + u) \cos \frac{r \wedge u}{2} u^\alpha \epsilon^{-\frac{u^2}{8\mu^2} u^2}.
\]

As usual, we expand around \( u = 0 \) and obtain
\[
(6) = -\frac{i\mu a \epsilon^2}{2} \int \int dr ds A^\rho(r) \left( \phi(-r) + u^\rho \partial_\rho \phi(-r) + \ldots \right) \frac{\tilde{u}_\rho}{u^4}.
\]

Using the polar coordinates we see that the first term gives vanishing contribution while the second term is
\[
-\frac{i}{(2\pi)^2} \int d^2 r \, e^{\rho\alpha} A_\rho(r) \partial_\alpha \phi(-r) = \int d^2 x \, \tilde{x}^\rho A_\rho \phi.
\] (8.3)

The divergence is again logarithmic.

Infinities appear also in \( \Delta \Pi_{\rho\sigma} \) and \( \Delta \Pi \) but they are of the same forms (8.1) and (8.2).

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