Strictly proper kernel scores and characteristic kernels on compact spaces

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Abstract

Strictly proper kernel scores are well-known tool in probabilistic forecasting, while characteristic kernels have been extensively investigated in the machine learning literature. We first show that both notions coincide, so that insights from one part of the literature can be used in the other. We then show that the metric induced by a characteristic kernel cannot reliably distinguish between distributions that are far apart in the total variation norm as soon as the underlying space of measures is infinite dimensional. In addition, we provide a characterization of characteristic kernels in terms of eigenvalues and -functions and apply this characterization to the case of continuous kernels on (locally) compact spaces. In the compact case we further show that characteristic kernels exist if and only if the space is metrizable. As special cases of our general theory we investigate translation-invariant kernels on compact Abelian groups and isotropic kernels on spheres. The latter are of particular interest for forecast evaluation of probabilistic predictions on spherical domains as frequently encountered in meteorology and climatology.

1 Introduction

Probabilistic forecasts of uncertain future events are issued in a wealth of applications, see Gneiting and Katzfuss (2014) and the references therein. To assess the quality and to compare such forecasts, proper scoring rules are a well-established tool, see Gneiting and Raftery (2007), and in applications, it is usually even desirable to work with strictly proper scoring rules. A broad class of proper scoring rules are so-called kernel scores, which are constructed using a positive definite kernel. Unfortunately, however, no general conditions are available to decide whether a given kernel induces a strictly proper kernel score. As detailed below in Theorem 1.1, strict propriety of a kernel score is intimately connected to the kernel being characteristic, a notion that has been studied in the machine learning literature for a decade, see e.g. Gretton et al. (2007); Fukumizu et al. (2009); Sriperumbudur et al. (2010a, 2011) as well as the recent survey of Muandet et al. (2017) and the references therein. In this paper, we study characteristic kernels on compact spaces extending results of Micchelli et al. (2006) and Sriperumbudur et al. (2010a, 2011). As a consequence, we can characterize strictly proper kernel scores on compact Abelian groups and the practically highly relevant example of spheres.

To describe our results in more detail, let us formally introduce some of the notions mentioned above. To this end, let \((X, \mathcal{A})\) be a measurable space and let \(\mathcal{M}_1(X)\) denote the class of all probability measures on \(X\). For \(\mathcal{P} \subseteq \mathcal{M}_1(X)\), a scoring rule is a function \(S : \mathcal{P} \times X \to [-\infty, \infty]\) such that the integral \(\int S(P, x) dQ(x)\) exists for all \(P, Q \in \mathcal{P}\). The scoring rule is called proper.
with respect to \( \mathcal{P} \) if
\[
\int S(P, x) \, dP(x) \leq \int S(Q, x) \, dP(x), \quad \text{for all } P, Q \in \mathcal{P},
\]
and it is called strictly proper if equality in (1) implies \( P = Q \). Recall that if the class \( \mathcal{P} \) consists of absolutely continuous probability measures with respect to some \( \sigma \)-finite measure \( \mu \) on \( X \) then the logarithmic score \( S(P, x) := -\log p(x) \), where \( p \) is the density of \( P \), is a widely used example of a strictly proper scoring rule for density forecasts. Another well-known example is the Brier score for distributions on \( X = \{1, \ldots, m\} \) that is defined as \( S(P, i) := \sum_{j=1}^{m} p_j^2 + 1 - 2p_i \), where \( p_i = P(\{i\}) \), \( i = 1, \ldots, m \). Finally, for \( X = \mathbb{R} \), the continuous ranked probability score (CRPS) is given by
\[
S(P, x) := \int_{\mathbb{R}} |y - x| \, dP(y) - \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} |y - y'| \, dP(y) \, dP(y').
\]
It is strictly proper with respect to the class of all probability measures with finite first moment, see e.g. Gneiting and Raftery (2007, Section 4.2), and consequently it can be used to evaluate predictions of density forecasts as well as probabilistic forecasts of categorical variables. Various other examples can be found in Gneiting and Raftery (2007).

One general class of proper scoring rules are kernel scores. To this end, let \( k : X \times X \to \mathbb{R} \) be a symmetric function. We call \( k \) a kernel, if it is positive definite, that is, if
\[
\sum_{i,j=1}^{n} a_i a_j k(x_i, x_j) \geq 0
\]
for all natural numbers \( n \), all \( a_1, \ldots, a_n \in \mathbb{R} \), and all \( x_1, \ldots, x_n \in X \). It is strictly positive definite if equality in (2) implies \( a_1 = \cdots = a_n = 0 \) whenever the points \( x_1, \ldots, x_n \) are mutually distinct. Let us assume that \( k \) is measurable and define
\[
\mathcal{M}_k^1(X) := \{ P \in \mathcal{M}_1(X) \mid \int_X \sqrt{k(x, x)} \, dP(x) < \infty \}.
\]
For a bounded kernel \( k \), we have that \( \mathcal{M}_k^1(X) = \mathcal{M}_1(X) \), and the Cauchy-Schwarz inequality \( k(x, y) \leq \sqrt{k(x, x)} \sqrt{k(y, y)} \) for kernels shows that, for all \( P, Q \in \mathcal{M}_k^1(X) \), the kernel \( k \) is integrable with respect to the product measure \( P \otimes Q \).

**Definition 1.1.** The kernel score \( S_k \) associated with a measurable kernel \( k \) on \( X \) is the scoring rule \( S_k : \mathcal{M}_k^1(X) \times X \to \mathbb{R} \) defined by
\[
S_k(P, x) := -\int k(\omega, x) \, dP(\omega) + \frac{1}{2} \int \int k(\omega, \omega') \, dP(\omega) \, dP(\omega').
\]
Kernel scores are a broad generalization of the CRPS, and in fact, also the Brier score can be rewritten as a kernel score, see Gneiting and Raftery (2007, Section 5.1). However, the logarithmic score does not belong to this class. If \( X \) is a Hausdorff space and \( k \) is continuous, then Gneiting and Raftery (2007, Theorem 4) show that \( S_k \) is proper with respect to all Radon probability measures on \( X \). Their result is based on Berg et al. (1984, Theorem 2.1, p. 235), where it is fundamental that the kernel is continuous. In this respect we remark that the definition of a kernel score of Gneiting and Raftery (2007) is more general than ours as it allows for kernels being only conditionally positive definite. While this level of generality is fruitful for example in the case \( X = \mathbb{R}^d \), we believe that it is sufficient to consider only positive definite kernels for compact
spaces. Indeed, note that if $X$ is compact and separable and there is a strictly positive probability measure $\nu$ on its Borel sets, then, if we only consider kernels $k$ such that $\int_X k(x, y) \, d\nu(x)$ does not depend on $y$, we know by Bochner (1941, Theorem 2) that conditionally positive definite kernels and positive definite kernels are the same up to a constant.

In our framework, it is possible to show propriety of the kernel score without requiring continuity of the kernel using the theory of reproducing kernel Hilbert spaces (RKHS); see Theorem 1.1 below. In addition, we obtain a condition for when the kernel score is actually strictly proper.

**Theorem 1.1.** Let $k$ be a measurable kernel with RKHS $H$ with norm $\| \cdot \|_H$ and $\Phi : \mathcal{M}_1^b(X) \to H$ be the kernel embedding defined by

$$\Phi(P) := \int k(\cdot, \omega) \, dP(\omega).$$

(3)

Then the kernel score satisfies

$$\| \Phi(P) - \Phi(Q) \|^2_H = 2 \left( \int S_k(Q, x) \, dP(x) - \int S_k(P, x) \, dP(x) \right)$$

(4)

for all $P, Q \in \mathcal{M}_1^b(X)$. In particular, $S_k$ is a proper scoring rule with respect to $\mathcal{M}_1^b(X)$, and it is strictly proper if and only if $\Phi$ is injective.

In the machine learning literature a bounded measurable kernel is called characteristic if the kernel (mean) embedding $\Phi : \mathcal{M}_1(X) \to H$ defined by (3) is injective. Consequently, Theorem 1.1 shows that, for bounded measurable kernels $k$, strictly proper kernel scores $S_k$ are exactly those for which $k$ is characteristic. In particular, the wealth of examples and conditions for characteristic kernels can be directly used to find new strictly proper scoring rules and vice versa.

While Theorem 1.1 is an interesting observation for both machine learning applications and probabilistic forecasting, its proof is actually rather trivial. In the rest of the paper, we therefore focus on more involved aspects of characteristic kernels, and strictly proper kernel scores, respectively. We introduce the necessary mathematical machinery on kernels and their interaction with (signed) finite measures in Section 2. In particular we recall that a bounded measurable kernel $k$ with RKHS $H$ induces a semi-norm $\| \cdot \|_H$ on $\mathcal{M}(X)$, the space of all finite signed measures on $X$, via the kernel mean embedding, that is via the left-hand side of (4). In Section 3.1, we study this semi-norm for general $X$. In particular, Theorem 3.1 shows that for injective kernel embeddings, $\| \cdot \|_H$ fails to be equivalent to the total variation norm on $\mathcal{M}(X)$ if and only if $\dim \mathcal{M}(X) = \infty$, and Corollary 3.8 gives an even sharper result on $\mathcal{M}_1(X)$. In view of (4), these results show that the value on the right-hand side of (4) is not proportional to the (squared) total variation norm. Besides some structural results on characteristic kernels, see Lemmas 3.3 and 3.4, we further present a simple computation of the left-hand side of (4) in terms of eigenvalues and -functions of a suitable integral operator. In Section 3.2, we then exploit our general theory to obtain new results for bounded continuous kernels on locally compact Hausdorff spaces. The main question of interest is when such kernels are universal or characteristic. In Theorem 3.17 and Corollary 3.18 we give characterization results in terms of eigenfunctions of certain integral operators. We also provide insight concerning the difference of considering Borel or Radon measures on locally compact Hausdorff spaces in the study of kernel embeddings, see Theorems 3.13 and 3.14. As a result, it turns out in Theorem 3.15 that continuous characteristic kernels on compact Hausdorff spaces only exist if the spaces are metrizable. In Section 4, we apply the characterization results of Section 3.2 to translation-invariant kernels on compact Abelian groups and to isotropic kernels on spheres. All proofs can be found in Section 5.
2 Preliminaries

In this section we recall some facts about reproducing kernels and their interaction with measures. To this end, let \((X,A)\) be a measurable space. We denote the space of finite signed measures on \(X\) by \(\mathcal{M}(X)\) and write \(\mathcal{M}_+(X)\) and \(\mathcal{M}_1(X)\) for the subsets of all (non-negative) finite, respectively probability measures. Moreover, we write

\[
\mathcal{M}_0(X) := \{\mu \in \mathcal{M}(X) : \mu(X) = 0\}.
\]

As usual we equip \(\mathcal{M}(X)\) and its subsets above with the total variation norm \(\| \cdot \|_{TV}\). Recall that \(\| \cdot \|_{TV}\) is complete and hence \(\mathcal{M}(X)\) is a Banach space. Moreover, \(\mathcal{M}_0(X)\) is closed subspace of co-dimension 1, which contains, e.g. all differences of probability measures. Moreover, for every \(P \in \mathcal{M}_1(X)\) we have

\[
\mathcal{M}(X) = \mathbb{R}P \oplus \mathcal{M}_0(X) .
\]

Given a measurable function \(f : X \to \mathbb{R}\) and a measure \(\nu\) on \(X\) we write \([f]_\sim\) for the \(\nu\)-equivalence class of \(f\). Similar, we denote the space of \(p\)-times \(\nu\)-integrable functions by \(L_p(\nu)\) and the corresponding space of \(\nu\)-equivalence classes by \(L_p(\nu)\). Note that this rather pedantic notation becomes very useful when dealing with RKHSs since these spaces consist of functions that can be evaluated pointwise in a continuous fashion, whereas such an evaluation does, in general, not make sense for elements in \(L_p(\nu)\).

To formally introduce kernel mean embeddings, we need to recall the notion of Pettis integrals. To this end let, let \(H\) be a Hilbert space and \(f : X \to H\) be a function. Then \(f\) is weakly measurable, if \(\langle w, f \rangle : X \to \mathbb{R}\) is measurable for all \(w \in H\). Similarly, \(f\) is weakly integrable with respect to a measure \(\nu\) on \((X,A)\), if \(\langle w, f \rangle \in L_1(\nu)\) for all \(w \in H\). In this case, there exists a unique \(i_\nu(f) \in H\), called the Pettis integral of \(f\) with respect to \(\nu\), such that for all \(w \in H\) we have

\[
\langle w, i_\nu(f) \rangle = \int_X \langle w, f \rangle \, d\nu,
\]

see e.g. Diestel and Uhl (1977, Chapter II.3) together with the reflexivity of \(H\) and the identity \(H = H'\) between \(H\) and its dual \(H'\). Using the Hahn-Jordan decomposition, it is not hard to see that \(i_\nu(f)\) can analogously be defined for finite signed measures \(\mu\). In the following, we adopt the more intuitive notation \(\int_X f \, d\mu := i_\mu(f)\), so that the defining equation above becomes

\[
\langle w, \int_X f \, d\mu \rangle = \int_X \langle w, f \rangle \, d\mu .
\]

Furthermore, in the case of probability measures \(\mu\), we sometimes also write \(\mathbb{E}_\mu f := i_\mu(f)\). Let us now use Pettis integrals to define kernel mean embeddings. To this end, let \(H\) be an RKHS over \(X\) with kernel \(k\) and canonical feature map \(\Phi : X \to H\), that is \(\Phi(x) := k(\cdot, x)\) for all \(x \in X\). Then \(\Phi\) is weakly measurable if and only if \(\langle h, \Phi \rangle = h\) is measurable for all \(h \in H\), and therefore, we conclude that \(\Phi\) is weakly integrable with respect to some measure \(\nu\) on \(X\), if and only if \(h \in L_1(\nu)\) for all \(h \in H\). By a simple application of the closed graph theorem, the latter is equivalent to the continuity of the map \([\cdot]_\sim : H \to L_1(\nu)\). In this respect recall that \(H\) consists of measurable functions if and only if \(k\) is separately measurable, that is \(k(\cdot, x) : X \to \mathbb{R}\) is measurable for all \(x \in X\), see Steinwart and Christmann (2008, Lemma 4.24), and \([\cdot]_\sim : H \to L_1(\nu)\) is continuous if, e.g.

\[
\int \sqrt{k(x,x)} \, d\nu(x) < \infty ,
\]
Definition 2.1. \( \mu \) for all integrally positive definite with respect to \( M \).

Note that the map \( \Phi : M \rightarrow H \) defines a new semi-norm on \( X \).
Consequently, we have \( \Phi(\delta_x) = \Phi(x) \).

Note that this semi-norm is a norm, if and only if the kernel embedding \( \Phi : M \rightarrow H \) is injective and by (8) the latter is equivalent to

\[
\|\mu\|_H = \int_X k(x,x') \, d\mu(x) \, d\mu(x') > 0.
\]

for all \( \mu \in M^k \setminus \{0\} \). This leads to the following definition.

**Definition 2.1.** Let \( k \) be a measurable kernel on \( X \) and \( M \subset M^k \). Then \( k \) is called strictly integrally positive definite with respect to \( M \), if (9) holds for all \( \mu \in M \) with \( \mu \neq 0 \).

It is well-known, that using (6) the semi-norm introduced above can also be computed by

\[
\|\mu\|_H = \int_X \Phi d\mu \leq \sup_{f \in B_H} \|f \|_H \int_X \Phi d\mu = \sup_{f \in B_H} \int_X |f, \Phi| d\mu,
\]

where \( B_H \) denotes the closed unit ball of \( H \). Consequently, we have \( \|\mu\|_H \leq \|\cdot\|_H \) and if \( k \) is bounded, then \( \Phi : M(X) \rightarrow H \) is continuous with \( \|\Phi : M(X) \rightarrow H\| \leq \|k\|_{1/2} \). In particular, if \( k \) is bounded and \( \Phi : M(X) \rightarrow H \) is injective, then \( \|\cdot\|_H \) defines a new norm on \( M(X) \) that is dominated by \( \|\cdot\|_{TV} \) and that describes a Euclidean geometry with inner product (8). Unless \( \dim M(X) < \infty \), however, both norms are not equivalent, as we will see in Theorem 3.1.

With the help of the new semi-norm \( \|\cdot\|_H \) on \( M^k \) we can now define a semi-metric on \( M^k_+ \) by setting

\[
\gamma_k(P,Q) := \|P - Q\|_H = \sup_{f \in B_H} \int f \, dP - \int f \, dQ.
\]
for \( P, Q \in \mathcal{M}^f_k(X) \). Here, we note that the second equality, which follows from our considerations above, has already been shown by Sriperumbudur et al. (2010a, Theorem 1). Similarly, the following definition is taken from Fukumizu et al. (2008); Sriperumbudur et al. (2010a).

**Definition 2.2.** A bounded measurable kernel \( k \) on \( X \) is called characteristic, if the kernel mean embedding \( \Phi_k : \mathcal{M}_1(X) \to H \) is injective.

Clearly, \( k \) is characteristic, if and only if \( \gamma_k \) is a metric, and a literal repetition of Sriperumbudur et al. (2010a, Lemma 8) shows:

**Proposition 2.3.** Let \((X, \mathcal{A})\) be a measurable space and \( k \) be a bounded measurable kernel on \( X \). Then the following statements are equivalent:

i) \( k \) is strictly integrally positive definite with respect to \( \mathcal{M}_0(X) \).

ii) \( k \) is characteristic.

Now, let \( \nu \) be a measure on \( X \), \( k \) be a measurable kernel on \( X \), and \( H \) be its RKHS. We further assume that the map \( I_{k,\nu} : H \to L_2(\nu) \) given by \( f \mapsto [f]_\sim \) is well-defined and compact. For an example of such a situation recall Steinwart and Scovel (2012, Lemma 2.3), which shows that \( I_{k,\nu} \) is Hilbert-Schmidt if

\[
\int_X k(x,x) \, d\nu(x) < \infty. \tag{11}
\]

Obviously, the latter holds, if e.g. \( k \) is bounded and \( \nu \) is a finite measure. Now assume that \( I_{k,\nu} \) is well-defined and compact. Then, the associated integral operator \( T_{k,\nu} : L_2(\nu) \to L_2(\nu) \), defined by

\[
t_{k,\nu}f = \left[ \int_X k(x,\cdot)f(x) \, d\nu(x) \right]_\sim, \quad f \in L_2(\nu),
\]

satisfies \( T_{k,\nu} = I_{k,\nu} \circ I_{k,\nu}^* \), see e.g. Steinwart and Scovel (2012, Lemma 2.2), where \( I_{k,\nu}^* \) denotes the adjoint of \( I_{k,\nu} \). In particular, \( T_{k,\nu} \) is compact, positive, and self-adjoint, and if (11) is satisfied, \( T_{k,\nu} \) is even nuclear. Moreover, the spectral theorem in the form of Steinwart and Scovel (2012, Lemma 2.12) gives us an at most countable, ordered family \((\lambda_i)_{i \in I} \subset (0, \infty)\) converging to 0 and a family \((e_i)_{i \in I} \subset H\) such that:

- \((\lambda_i)_{i \in I}\) are the non-zero eigenvalues of \( T_{k,\nu} \) including multiplicities,
- \([e_i]_\sim\) is an \( L_2(\nu) \)-ONS of the corresponding eigenfunctions with

\[
\text{span}\{[e_i]_\sim : i \in I\}^{L_2(\nu)} = [H]^{L_2(\nu)}, \tag{12}
\]

- \((\sqrt{\lambda_i} e_i)_{i \in I}\) is an ONS in \( H \).

Here, we say that an at most countable family \((\alpha_i)_{i \in I} \subset [0, \infty)\) converges to 0, if either \( I = \{1, \ldots, n\} \) or \( I = \mathbb{N} \) and \( \lim_{i \to \infty} \alpha_i = 0 \).

Note that for nuclear \( T_{k,\nu} \), we additionally have \( \sum_{i \in I} \lambda_i < \infty \), and if \( k \) is bounded, \( \|e_i\|_\infty < \infty \) holds for all \( i \in I \). Finally, Steinwart and Scovel (2012, Theorem 3.1) show that the injectivity of \( I_{k,\nu} : H \to L_2(\nu) \) is equivalent to either of the following statements:

i) \((\sqrt{\lambda_i} e_i)_{i \in I}\) is an ONB of \( H \).
ii) For all $x, x' \in X$ we have

$$k(x, x') = \sum_{i \in I} \lambda_i e_i(x)e_i(x').$$  \hspace{1cm} (13)$$

Obviously, if one of these conditions is true, then $H$ is separable, and Steinwart and Scovel (2012, Corollary 2.10) show that the convergence in (13) is absolute and we even have $k(\cdot, x) = \sum_{i \in I} \lambda_i e_i(x)e_i$ with unconditional convergence in $H$.

Let us now recall some notions related to measures on topological spaces, see e.g. Bauer (2001, Chapter IV) for details. To this end, let $(X, \tau)$ be a Hausdorff (topological) space and $\nu$ be a Borel measure on its Borel-$\sigma$-algebra $B(X)$. Then $\nu$ is a Borel measure, if $\nu(K) < \infty$ for all compact $K \subset X$ and $\nu$ is called strictly positive, if $\nu(O) > 0$ for all non-empty $O \in \tau$. Moreover, a finite measure $\nu$ on $B(X)$ is a (finite) Radon measure if it is regular, i.e. if for all $B \in B(X)$ we have

$$\nu(B) = \sup\{\nu(K) : K \text{ compact and } K \subset B\} = \inf\{\nu(O) : O \text{ open and } B \subset O\}.$$ 

A finite signed Radon measure is simply the difference of two finite Radon measures. In the following, we denote the space of all finite signed Radon measures by $\mathcal{M}^*(X)$ and the cone of (non-negative) finite Radon measures by $\mathcal{M}^+_*(X)$. As usual, $\mathcal{M}^*(X)$ is equipped with the norm of total variation. Obviously, every finite Radon measure is a finite Borel measure, and by Ulam’s theorem, see e.g. Bauer (2001, Lemma 26.2), the converse implication is true if $X$ is a Polish space. In this respect recall that compact, metrizable spaces are Polish.

Now let $X$ be a locally compact Hausdorff space and $C_0(X)$ be the space of continuous functions vanishing at infinity. As usual, we equip $C_0(X)$ with the supremum norm. Then, Riesz’s representation theorem for locally compact spaces, see e.g. Hewitt and Stromberg (1965, Theorem 20.48 together with Definition 20.41, Theorem 12.40, Definition 12.39, and a simple translation into the real-valued case using Theorem 12.36) shows that

$$\mathcal{M}^*(X) \to C_0(X)’$$

$$\mu \mapsto (f \mapsto \langle f, \mu \rangle := \int_X f \, d\mu)$$  \hspace{1cm} (14)$$

is an isometric isomorphism. In the compact case, in which $C_0(X)$ coincides with the space of continuous functions $C(X)$, this can also be found in e.g. Dunford and Schwartz (1958, p. 265, Theorem IV.6.3).

Given a locally compact Hausdorff space $X$, a continuous kernel $k$ on $X$ with RKHS $H$ is called universal if $H \subset C_0(X)$ and $H$ is dense in $C_0(X)$ with respect to $\| \cdot \|_\infty$. Note that for compact $X$ the inclusion $H \subset C_0(X)$ is automatically satisfied. Examples of universal kernels as well as various necessary and sufficient conditions for universality can be found in e.g. Steinwart (2001); Micchelli et al. (2006); Sriperumbudur et al. (2011); Chen et al. (2016) and the references mentioned therein.

3 New Characterizations

In this section we first compare the norms $\| \cdot \|_H$ and $\| \cdot \|_{TV}$ and show that in infinite dimensional they are never equivalent. By establishing some structural result for characteristic kernels, we then demonstrate that characteristic kernels cannot reliably distinguish between distributions that are far away with respect to $\| \cdot \|_\infty$. Note that for compact $X$ the inclusion $H \subset C_0(X)$ is automatically satisfied. Examples of universal kernels as well as various necessary and sufficient conditions for universality can be found in e.g. Steinwart (2001); Micchelli et al. (2006); Sriperumbudur et al. (2011); Chen et al. (2016) and the references mentioned therein.
3.1 General results

In this subsection we investigate the semi-norm \( \| \cdot \|_H \) on \( \mathcal{M}(X) \) for bounded kernels and general \( X \). We begin with a result that compares \( \| \cdot \|_H \) with \( \| \cdot \|_{TV} \).

**Theorem 3.1.** Let \((X, \mathcal{A})\) be a measurable space and \( H \) be the RKHS of a bounded and measurable kernel \( k \) on \( X \) such that the kernel embedding \( \Phi : \mathcal{M}(X) \to H \) is injective. Then, the following statements are equivalent:

i) The space \( \mathcal{M}(X) \) is finite dimensional.

ii) The norms \( \| \cdot \|_H \) and \( \| \cdot \|_{TV} \) on \( \mathcal{M}(X) \) are equivalent.

iii) The norm \( \| \cdot \|_H \) on \( \mathcal{M}(X) \) is complete.

iv) The kernel embedding \( \Phi : \mathcal{M}(X) \to H \) is surjective.

Theorem 3.1 shows that for most cases of interest \( (\mathcal{M}(X), \| \cdot \|_H) \) is not a Hilbert space. To illustrate the fourth statement of Theorem 3.1, recall that the space \( H_{\text{pre}} := \{ \Phi(\mu) : \mu \in \text{span}\{\delta_x : x \in X\} \} \) is dense in \( H \), see e.g. Steinwart and Christmann (2008, Theorem 4.21). Moreover, the space \( \text{span}\{\delta_x : x \in X\} \) is, in a weak sense, dense in \( \mathcal{M}(X) \), and therefore it is natural to ask whether every \( f \in H \) is of the form \( f = \Phi(\mu) \) for some \( \mu \in \mathcal{M}(X) \). Theorem 3.1 tells us that the answer is no, unless \( \mathcal{M}(X) \) is finite dimensional. In this respect recall that it has been recently mentioned by Simon-Gabriel and Schölkopf (2016) that the kernel embedding \( \Phi \) is, in general, not surjective. However, the authors do not provide any example, or conditions, for non-surjective \( \Phi \).

Two examples of non-surjective kernel embeddings \( \Phi \) are provided by Pillai et al. (2007, Section 3), while our Theorem 3.1 shows that actually all injective \( \Phi \) fail to be surjective whenever we have \( \dim \mathcal{M}(X) = \infty \).

Our next goal is to show that for characteristic kernels on infinite dimensional spaces \( \mathcal{M}(X) \) there always exists probability measures that have maximal \( \| \cdot \|_{TV} \)-distance but arbitrarily small \( \| \cdot \|_H \)-distance. To this end, we need a couple of preparatory results. We begin with the following lemma that investigates the effect of \( 1_X \in H \).

**Lemma 3.2.** Let \((X, \mathcal{A})\) be a measurable space and \( H \) be the RKHS of a bounded and measurable kernel \( k \) on \( X \). If \( 1_X \in H \), then \( \mathcal{M}_0(X) \) is \( \| \cdot \|_H \)-closed in \( \mathcal{M}(X) \), and if \( k \) is, in addition, characteristic, then the kernel embedding \( \Phi : \mathcal{M}(X) \to H \) is injective.

The next simple lemma computes the \( \| \cdot \|_H \)-norm of measures if \( H \) is an RKHS of the sum of two kernels.

**Lemma 3.3.** Let \((X, \mathcal{A})\) be a measurable space, and \( k_1, k_2 \) be bounded measurable kernels on \( X \) with RKHSs \( H_1 \) and \( H_2 \). Let \( H \) be the RKHS of the kernel \( k = k_1 + k_2 \). Then for all \( \mu \in \mathcal{M}(X) \) we have

\[
\| \mu \|^2_H = \| \mu \|^2_{H_1} + \| \mu \|^2_{H_2}.
\]

In particular, if \( k_1 \) is characteristic or has an injective kernel embedding, then the same is true for \( k \).

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In Sriperumbudur et al. (2010a, Corollary 11) it has already be show that the sum of two bounded, continuous translation-invariant kernels on $\mathbb{R}^d$ is characteristic, if at least one summand is characteristic. Lemma 3.3 shows that this kind of inheritance holds in the general case.

Our next lemma considers products of kernels. In particular it shows that such products can only be characteristic if the involved factors are strictly integrally positive definite.

**Lemma 3.4.** Let $(X_1, A_1)$ and $(X_2, A_2)$ be measurable spaces and $k_1, k_2$ be a bounded, measurable kernels on $X_1$ and $X_2$, respectively. We denote the RKHSs of $k_1$ and $k_2$ by $H_1$ and $H_2$. Moreover, let $H$ be the RKHS of the kernel $k := k_1 \cdot k_2$ on $X_1 \times X_2$. Then, for all $\mu_1 \in \mathcal{M}(X_1)$ and $\mu_2 \in \mathcal{M}(X_2)$ we have

$$\|\mu_1 \otimes \mu_2\|_H = \|\mu_1\|_{H_1} \cdot \|\mu_2\|_{H_2}.$$  

In particular, if $\dim \mathcal{M}(X_1) \geq 2$ and $\dim \mathcal{M}(X_2) \geq 2$, and $k$ is characteristic, then $k_1$ and $k_2$ are strictly integrally positive definite with respect to $\mathcal{M}(X_1)$ and $\mathcal{M}(X_2)$, respectively.

At first glance it seems that Lemma 3.4 contradicts Sriperumbudur et al. (2010a, Corollary 11), which shows that the product of two bounded, continuous, translation-invariant kernels on $\mathbb{R}^d$ is characteristic on $\mathbb{R}^d$ as soon as at least one factor is characteristic. However, a closer look reveals that their result considers the restriction of the product to the diagonal, whereas we treat the unrestricted kernel. Later in Corollary 3.16, we will see that, on compact spaces, the product of two strictly integrally positive definite kernels gives a strictly integrally positive definite kernel on the product.

The following lemma compares strictly integrally positive definite kernels with respect $\mathcal{M}(X)$ and $\mathcal{M}_0(X)$. In an implicit form, it has already been used, and a similar statement is Simon-Gabriel and Schölkopf (2016, Theorem 32).

**Lemma 3.5.** Let $(X, A)$ be a measurable space, and $k$ be a bounded measurable kernel on $X$. Moreover, let $\mathcal{M} \subset \mathcal{M}(X)$ be a subspace with $\mathcal{M} \cap \mathcal{M}_1(X) \neq \emptyset$ and $\mathcal{M}_0 := \mathcal{M} \cap \mathcal{M}_0(X)$. Then the following statements are equivalent:

i) $k$ is strictly integrally positive definite with respect to $\mathcal{M}_0$.

ii) $k + 1$ is strictly integrally positive definite with respect to $\mathcal{M}$.

iii) $k + 1$ is strictly integrally positive definite with respect to $\mathcal{M}_0$.

We have already seen in Theorem 3.1 that in the infinite-dimensional case the norms $\| \cdot \|_H$ and $\| \cdot \|_{TV}$ are not equivalent on $\mathcal{M}(X)$. Intuitively, this carries over to the subspace $\mathcal{M}_0(X)$. The following result confirms this intuition as long as $\mathcal{M}_0(X)$ is a $\| \cdot \|_H$-closed subspace of $\mathcal{M}(X)$.

**Theorem 3.6.** Let $(X, A)$ be a measurable space such that $\dim \mathcal{M}(X) = \infty$ and $H$ be the RKHS of a bounded and measurable kernel $k$ on $X$ such that the kernel embedding $\Phi : \mathcal{M}(X) \to H$ is injective. If $\mathcal{M}_0(X)$ is a $\| \cdot \|_H$-closed subspace of $\mathcal{M}(X)$, then $\| \cdot \|_H$ and $\| \cdot \|_{TV}$ are not equivalent on $\mathcal{M}_0(X)$.

The non-equivalence of $\| \cdot \|_H$ and $\| \cdot \|_{TV}$ on $\mathcal{M}_0(X)$ has already been observed in some particular situations. For example, Sriperumbudur et al. (2010a, Theorem 23) show that for universal kernels on compact metric spaces, $\gamma_k$ metrizes the weak topology (in probabilist’s terminology) on $\mathcal{M}_1(X)$, and since for $\dim \mathcal{M}(X) = \infty$ this weak topology is strictly coarser than the $\| \cdot \|_{TV}$-topology, we see that $\| \cdot \|_H$ and $\| \cdot \|_{TV}$ cannot be equivalent for such kernels. In addition, the non-equivalence can also be obtained from Sriperumbudur et al. (2010a, Theorems 21 and 24) for
other continuous kernels on certain metric spaces. Finally recall that \( \mathcal{M}_0(X) \) is a \( \| \cdot \|_H \)-closed subspace of \( \mathcal{M}(X) \) if \( 1_X \in H \) by Lemma 3.2.

With the help of Theorem 3.6 the next result shows that characteristic kernels cannot reliably distinguish between distributions that are far away in total variation norm.

**Theorem 3.7.** Let \( (X,A) \) be a measurable space such that \( \dim \mathcal{M}(X) = \infty \) and \( H \) be the RKHS of a characteristic kernel \( k \) on \( X \). Then for all \( \varepsilon > 0 \) there exist distributions \( Q_1, Q_2 \in \mathcal{M}_1(X) \) such that \( \| Q_1 - Q_2 \|_{TV} = 2 \) and \( \| Q_1 - Q_2 \|_H \leq \varepsilon \).

Theorem 3.7 only shows that there are some distributions that cannot be reliably distinguished. The following corollary shows that such distributions actually occur everywhere.

**Corollary 3.8.** Let \( (X,A) \) be a measurable space such that \( \dim \mathcal{M}(X) = \infty \) and \( H \) be the RKHS of a characteristic kernel \( k \) on \( X \). Then for all \( P \in \mathcal{M}_1(X) \), \( \delta \in (0,2] \), and \( \varepsilon \in (0,\delta) \) there exist \( Q_1, Q_2 \in \mathcal{M}_1(X) \) such that \( \| P - Q_i \|_{TV} \leq \delta \) for \( i = 1,2 \), \( \| Q_1 - Q_2 \|_{TV} = \delta \), and \( \| Q_1 - Q_2 \|_H \leq \varepsilon \).

The next goal of this subsection is to investigate the \( \| \cdot \|_H \)-norm with the help of the eigenvalues and \(-\)functions of the integral operator \( T_{k,\nu} \). We begin with the following lemma that computes the inner product (8) by these eigenvalues and \(-\)functions.

**Lemma 3.9.** Let \( (X,A,\nu) \) be \( \sigma \)-finite measure space and \( k \) be a bounded, measurable kernel with RKHS \( H \) for which \( I_{k,\nu} : H \to L_2(\nu) \) is compact and injective. Then, for all \( \mu_1, \mu_2 \in \mathcal{M}(X) \), we have

\[
\int_X \int_X k(x,x') \, d\mu_1(x) \, d\mu_2(x') = \sum_{i \in I} \lambda_i \cdot \left( \int_X e_i \, d\mu_1 \right) \cdot \left( \int_X e_i \, d\mu_2 \right),
\]

where \( (\lambda_i)_{i \in I} \subset (0,\infty) \) and \( (e_i)_{i \in I} \subset H \) are as at (13).

For an interpretation of this lemma, we write, for bounded measurable \( f : X \to \mathbb{R} \) and \( \mu \in \mathcal{M}(X) \),

\[
\langle f, \mu \rangle := \int_X f \, d\mu.
\]

Combining Lemma 3.9 with (8) we then have

\[
\langle \Phi(\mu_1), \Phi(\mu_2) \rangle_H = \sum_{i \in I} \lambda_i \langle e_i, \mu_1 \rangle \langle e_i, \mu_2 \rangle.
\]

In other words, all calculations regarding inner products and norms of the kernel embedding \( \mu \mapsto \Phi(\mu) \) can be carried over to a weighted \( \ell_2 \)-space.

To formulate the following theorem, we denote, for a \( \sigma \)-finite measure \( \nu \) on \( (X,A) \), the set of all \( \nu \)-probability densities contained in \( L_2(\nu) \) by \( \Delta(\nu) \), that is

\[
\Delta(\nu) := \left\{ [h]_\sim \in L_2(\nu) \cap L_1(\nu) : [h]_\sim \geq 0 \text{ and } \int_X h \, d\nu = 1 \right\}.
\]

Moreover, we write \( \mathcal{P}_2(\nu) := \{ h \, d\nu : [h]_\sim \in \Delta(\nu) \} \) for the corresponding set of probability measures.

With the help of these preparations we can now formulate the following theorem that characterizes non-characteristic kernels on \( \mathcal{P}_2(\nu) \) and that also establishes a result similar to Theorem 3.7 for non-characteristic kernels.
Theorem 3.10. Let \((X, \mathcal{A}, \nu)\) be \(\sigma\)-finite measure space and \(k\) be a bounded, measurable kernel with RKHS \(H\) for which \(I_{k, \nu} : H \to L_2(\nu)\) is compact and injective. Then for all \([h], [g] \in \Delta(\nu)\) and \(P := h \, d\nu, Q := g \, d\nu\) the kernel mean distance can be computed by
\[
\gamma_k^2(P, Q) = \sum_{i \in I} \lambda_i \langle [h - g], [\varepsilon_i] \rangle_{L_2(\nu)}^2.
\]

Moreover, the following statements are equivalent:
1) There exist \(Q_1, Q_2 \in \mathcal{P}_2(\nu)\) with \(Q_1 \neq Q_2\) and \(\gamma_k^2(Q_1, Q_2) = 0\).
2) There exists an \([f] \in L_1(\nu) \cap [H]_z^c\) with \([f] \neq 0\) and \(\int_X f \, d\nu = 0\).
3) There exist \([h_1], [h_2] \in \Delta(\nu)\) with \([h_1] \neq [h_2]\) such that for all \(i \in I\) we have
\[
\langle [h_1], [\varepsilon_i] \rangle_{L_2(\nu)} = \langle [h_2], [\varepsilon_i] \rangle_{L_2(\nu)}.
\]

Moreover, if one, and thus all, statements are true we actually find for all \(P \in \mathcal{P}_2(\nu)\) and \(\varepsilon \in (0, 2)\) some \(Q_1, Q_2 \in \mathcal{P}_2(\nu)\) with \(\|P - Q_1\|_{TV} \leq \varepsilon, \|Q_1 - Q_2\|_{TV} = \varepsilon, \) and \(\gamma_k^2(Q_1, Q_2) = 0\).

Equation (15) can also be used to show that under certain circumstances \(\|\cdot\|_H\) cannot reliably identify, for example, the uniform distribution. The following result, which is particularly interesting in view of Section 4, illustrates this.

Corollary 3.11. Let \((X, \mathcal{A}, \nu)\) be a probability space and \(k\) be a bounded, measurable kernel with RKHS \(H\) for which \(I_{k, \nu} : H \to L_2(\nu)\) is compact and injective. Assume that there is one eigenfunction \(e_{i_0}\) with \(e_{i_0} = 1_X\). In addition assume that there are constants \(c_1 > 0\) and \(c_\infty < \infty\) with \(\|e_i\|_{L_1(\nu)} \geq c_1\) and \(\|e_i\|_{\infty} \leq c_\infty\) for all \(i \in I\). For \(\alpha := c_\infty^{-1}\) and \(j \neq i_0\) consider the signed measure \(Q_j := (1_X + \alpha e_j) \, d\nu\). Then \(Q_j\) is actually a probability measure and for \(P := \nu\) we have
\[
\|P - Q_j\|_{TV} \geq c_1 c_\infty^{-1},
\]
\[
\|P - Q_j\|_H = \lambda_j c_\infty^{-2}.
\]

The last result of this subsection provides some necessary conditions for characteristic kernels.

Corollary 3.12. Let \((X, \mathcal{A}, \nu)\) be finite measure space and \(k\) be a bounded, measurable kernel with RKHS \(H\) for which \(I_{k, \nu} : H \to L_2(\nu)\) is compact and injective. Then the following statements are true:
1) If \(\text{codim}[H] \geq 2\) in \(L_2(\nu)\), then \(k\) is not characteristic.
2) If \(\text{codim}[H] \geq 1\) in \(L_2(\nu)\) and \(1_X \in H\), then \(k\) is not characteristic.

3.2 Continuous Kernels on Locally Compact Subsets

In this subsection, we apply the general theory developed so far to bounded continuous kernels on locally compact Hausdorff spaces \((X, \tau)\).

Let us begin with some preparatory remarks. To this end, let \(k\) be a bounded and continuous kernel on \(X\) whose RKHS \(H\) satisfies \(H \subset C_0(X)\). In the following we call such a \(k\) a \(C_0(X)\)-kernel. Our goal in this section is to investigate when \(C_0(X)\)-kernels are universal or characteristic. We begin with the following result that provides a necessary condition for the existence of strictly integrally positive definite kernels.
Theorem 3.13. Let \((X, \tau)\) be a locally compact Hausdorff space with \(\mathcal{M}^*(X) \neq \mathcal{M}(X)\). Then no \(C_0(X)\)-kernel is strictly integrally positive definite with respect to \(\mathcal{M}(X)\).

Note that Sriperumbudur et al. (2011) restrict their considerations to characteristic kernels on locally compact Polish spaces, for which we automatically have \(\mathcal{M}^*(X) = \mathcal{M}(X)\) by Ulam’s theorem, see e.g. Bauer (2001, Lemma 26.2). Some other papers, however, do not carefully distinguish between Borel and Radon measures, which, at last consequence, means that their results only hold if we additionally assume \(\mathcal{M}^*(X) = \mathcal{M}(X)\). Theorem 3.13 shows that this restriction is natural, and actually no restriction at all. Furthermore note that for compact spaces \(X\) one can use Theorem 3.13 to show that \(\mathcal{M}^*(X) = \mathcal{M}(X)\) is necessary for the existence of characteristic kernels. We skip such a result since later in Theorem 3.15, we will be able to show an even stronger result.

Before we formulate our next result we need a bit more preparation. To this end, let \(k\) be a \(C_0(X)\)-kernel on a locally compact space \((X, \tau)\). Then we have \(\mathcal{H} \subset C_0(X)\) and a quick closed-graph argument shows that the corresponding inclusion operator \(I : \mathcal{H} \to C_0(X)\) is bounded. By the identification \(C_0(X)' = \mathcal{M}^*(X)\) in (14) and the simple calculation

\[
\langle Ih, \mu \rangle_{C_0(X), \mathcal{M}^*(X)} = \int_X h \, d\mu = \int_X \langle h, k(x, \cdot) \rangle_{\mathcal{H}} \, d\mu(x) = \langle h, \Phi(\mu) \rangle_{\mathcal{H}},
\]

which holds for all \(h \in H, \mu \in \mathcal{M}^*(X)\) we further find that the adjoint \(I'\) of \(I\) is given by \(I' = \Phi\). This simple observation leads to the following characterization, which has already been shown for compact spaces \(X\) by Micchelli et al. (2006, Proposition 1) and for locally compact Polish spaces by Sriperumbudur et al. (2011, Proposition 4). Although the proof of the latter paper also works on general locally compact Hausdorff spaces, we decided to add the few lines for the sake of completeness.

Theorem 3.14. Let \((X, \tau)\) be a locally compact Hausdorff space and \(k\) be a \(C_0(X)\)-kernel. Then the following two statements are equivalent:

i) \(k\) is strictly integrally positive definite with respect to \(\mathcal{M}^*(X)\).

ii) \(k\) is universal.

With the help of Theorem 3.14 we can now show that for characteristic kernels on compact spaces \(X\) it suffices to consider metrizable \(X\). A similar result for universal kernels, which is included in the following theorem, has already been derived by Steinwart et al. (2006).

Theorem 3.15. For a compact topological Hausdorff space \((X, \tau)\) the following statements are equivalent:

i) There exists a universal kernel \(k\) on \(X\).

ii) There exists a continuous characteristic kernel \(k\) on \(X\).

iii) \(X\) is metrizable, i.e. there exists a metric generating the topology \(\tau\).

If one and thus all statements are true, \((X, \tau)\) is a compact Polish space and \(\mathcal{M}^*(X) = \mathcal{M}(X)\).

Theorem 3.15 shows that on compact spaces we may only expect universal or characteristic kernels, if the topology is metrizable. Since in this case we have \(\mathcal{M}^*(X) = \mathcal{M}(X)\), Theorem 3.14 and Proposition 2.3 show the well-known result that every universal kernel is characteristic. In general, the converse implication is not true, but adding some structural requirements, both notions may coincide. The following corollary illustrates this by showing that for product kernels universal and characteristic kernels coincide.
Corollary 3.16. Let \((X_1, \tau_1)\) and \((X_2, \tau_2)\) be non-trivial compact metrizable spaces and \(k_1, k_2\) be continuous kernels on \(X_1\) and \(X_2\), respectively. For the kernel \(k := k_1 \cdot k_2\) on \(X_1 \times X_2\) the following statements are then equivalent:

i) \(k\) is universal.

ii) \(k\) is characteristic.

iii) \(k_1\) and \(k_2\) are universal.

Our next theorem, which provides a characterization of universal kernels with the help of the eigenfunctions of the integral operator \(T_{k, \nu}\), is an extension of Micchelli et al. (2006, Corollary 5) from compact to arbitrary locally compact Hausdorff spaces. Before we present it, let us first make some preparatory remarks. To this end, let \(\nu\) be a strictly positive and \(\sigma\)-finite Borel measure on \(X\). For the matter of concreteness note that if \(X\) contains a dense, countable subset \((x_i)_{i \geq 1}\) then \(\nu := \sum_{i \geq 1} \delta_{x_i}\) satisfies these assumptions and therefore we always find such measures on e.g. compact metric spaces. Now, let \(k\) be a bounded and continuous kernel on \(X\) satisfying (11). Then \(H\) consists of continuous functions and Steinwart and Scovel (2012, Corollary 3.5) show that (13) holds for all \(x, x' \in X\), and consequently, the assumptions of Lemma 3.9, Theorem 3.10, and Corollary 3.12 are satisfied.

With these preparations we can now formulate the following characterization of universal kernels, where we note that the equivalence between i) and ii) has essentially been shown in (Sriperumbudur et al., 2010b, Proposition 12).

Theorem 3.17. Let \((X, \tau)\) be a locally compact Hausdorff space, \(\nu\) be a strictly positive, \(\sigma\)-finite Borel measure on \(X\), and \(k\) be a \(C_0(X)\)-kernel satisfying (11). In addition, let \((e_i)_{i \in I}\) be the eigenfunctions of \(T_{k, \nu}\) in (13). Then the following statements are equivalent:

i) \(k\) is universal.

ii) For all \(\mu \in \mathcal{M}^*(X)\) satisfying \(\int_X e_i \, d\mu = 0\) for all \(i \in I\) we have \(\mu = 0\).

iii) The space \(\text{span}\{e_i : i \in I\}\) is dense in \(C_0(X)\).

If one, and thus all, statements are true and \(\nu \in \mathcal{M}^*(X)\), then \([(e_i)_{i \in I}]_{\sim}\) is an ONB of \(L_2(\nu)\).

Our next result characterizes universal and characteristic kernels on compact spaces with the help of the eigenfunctions and -values of a suitable \(T_{k, \nu}\). In view of Theorem 3.15 it suffices to consider compact spaces that are Polish.

Corollary 3.18. Let \((X, \tau)\) be a compact metrizable space and \(k\) be a continuous kernel with RKHS \(H\). Moreover, let \((\lambda_i)_{i \in I} \subset [0, \infty)\) be a family converging to 0 and \((e_i)_{i \in I} \subset C(X)\) be a family such that \(\text{span}\{e_i : i \in I\}\) is dense in \(C(X)\) and

\[
  k(x, x') = \sum_{i \in I} \lambda_i e_i(x) e_i(x')
\]

holds for all \(x, x' \in X\). If there is a strictly positive, finite and regular Borel measure \(\nu\) on \(X\) such that \([(e_i)_{i \in I}]_{\sim}\) is an ON in \(L_2(\nu)\), then:

i) \(k\) is universal if and only if \(\lambda_i > 0\) for all \(i \in I\).

ii) If \(e_{i_0} = 1_X\) for some \(i_0 \in I\), then \(k\) is characteristic if and only if \(\lambda_i > 0\) for all \(i \neq i_0\).

iii) If \(1_X \in H\) and \(e_i \neq 1_X\) for all \(i \in I\), then \(k\) is characteristic if and only if \(\lambda_i > 0\) for all \(i \in I\).
4 Characteristic kernels on spaces with additional structure

In this section, we apply the developed theory to translation-invariant or isotropic kernels on compact Abelian groups or spheres, respectively.

4.1 Compact Abelian Groups

In this subsection we apply the theory developed so far to translation-invariant kernels on compact Abelian groups. Here the main difficulty lies in the fact that one traditionally considers kernels on groups that are \( \mathbb{C} \)-valued, while we are only interested in \( \mathbb{R} \)-valued kernels. Although at first glance, one may not expect any problem arising from this discrepancy, it turns out that it actually does make a difference when constructing an ONB of \( L_2(\nu) \) with the help of characters as soon as we have more than one self-inverse character.

Our first goal is to make the introducing remarks precise. To this end, let \( (G, +) \) be a compact Abelian group, and \( \nu \) be its normalized Haar measure. As usual, we write \( L_2(G) := L_2(\nu) \), and for later use recall that \( \nu \) is strictly positive and regular, see e.g. Hewitt and Ross (1963, p. 193/4). Moreover, let \( (\hat{G}, \cdot) \) be the dual group of \( G \), which consists of characters \( e : G \rightarrow \mathbb{T} \), where \( \mathbb{T} \) denotes, as usual, the unit circle in \( \mathbb{C} \), see e.g. Hewitt and Ross (1963, Chapter Six) and Folland (1995, Chapter 4.1). For notational purposes, we assume that we have another group \( (I, +) \) that is isomorphic to \( (\hat{G}, \cdot) \) by some mapping \( i \mapsto e_i \). This gives \( e_{i+j} = e_i e_j \), \( e_0 = 1_G \), and since we further have \( ee = 1_G \) for all \( e \in \hat{G} \), our notation also yields \( e_{-i} = \bar{e}_i \). In particular, we have \( \Re e_{-i} = \Re e_i \) and \( \Im e_{-i} = -\Im e_i \) for all \( i \in I \), and for all \( i \in I \) with \( i = -i \) the latter equality immediately yields \( \Im e_i = 0 \). Finally, for \( i \in I \) and \( x, y \in G \) we have

\[
e_i(-y + x) = \frac{e_i(x)}{e_i(y)} = e_i(y)e_i(x) = e_{-i}(y)e_i(x),
\]

and from this it is easy to derive both, \( \Re e_i(-x) = \Re e_i(x) \) and \( \Im e_i(-x) = -\Im e_i(x) \), as well as

\[
\Re e_i(-y + x) = \Re e_i(x) \Re e_i(y) + \Im e_i(x) \Im e_i(y).
\]

Note that for \( i \in I \) with \( i = -i \), the latter formula can be simplified using \( \Im e_i = 0 \).

Now, it is well-known that \( (\{e_i\}_{i \in I}) \) is an ONB of \( L_2(G, \mathbb{C}) \), see e.g. Folland (1995, Corollary 4.26) and using this fact a quick application of the Stone-Weierstrass theorem shows that \( (\nu e_i)_{i \in I} \) is dense in \( \mathcal{C}(G, \mathbb{C}) \). To construct a corresponding ONB in \( L_2(G) \). To this end, we write \( I_0 := \{i \in I : i = -i\} \) for the set of all self-inverse elements of \( I \). Moreover, we fix a partition \( I_+ \cup I_- = I \setminus I_0 \) such that \( i \in I_+ \) implies \( -i \in I_- \) for all \( i \in I \setminus I_0 \). Obviously, the sets \( I_0, I_+, I_- \) form a partition of \( I \). Let us now define the family \( (e^*_i)_{i \in I} \) by

\[
e^*_i := \begin{cases} 
\Re e_i & \text{if } i \in I_0 \\
\sqrt{2} \Re e_i & \text{if } i \in I_+ \\
\sqrt{2} \Im e_i & \text{if } i \in I_- .
\end{cases}
\]

The next result shows that \( (e^*_i)_{i \in I} \) is the desired family.

**Lemma 4.1.** Let \( (G, +) \) be a compact Abelian metric group. Then each family \( (e^*_i)_{i \in I} \) given by (17) is an ONB of \( L_2(G) \) and \( \operatorname{span}\{e^*_i : i \in I\} \) is dense in \( \mathcal{C}(G) \). Finally, we have \( \|e^*_i\|_{\infty} \leq \sqrt{2} \) for all \( i \in I \).
In the following, we call a kernel $k$ on an Abelian group $(G,+)$ translation invariant, if there exists a function $\kappa : G \to \mathbb{R}$ such that $k(x,x') = \kappa(-x + x')$ for all $x,x' \in G$. Clearly, $k$ is continuous, if and only if $\kappa$ is. The following lemma provides a representation of translation invariant kernels.

**Lemma 4.2.** Let $(G,+)$ be a compact Abelian group, $(e_i^*)_{i \in I}$ be a family of the form (17), and $k : G \times G \to \mathbb{R}$ be a bounded, measurable function. Then the following statements are equivalent:

i) $k$ is a bounded, measurable, and translation invariant kernel on $G$.

ii) There exists a summable family $(\lambda_i)_{i \in I} \subset [0,\infty)$ such that

$$k(x,x') = \sum_{\lambda_i > 0} \lambda_i e_i(x)e_i^*(x') = \sum_{\lambda_i > 0} \lambda_i \text{Re} e_i(x-x'),$$

where the series converge absolutely for all $x,x' \in G$ as well as uniformly in $(x,x')$.

If one, and thus both, statements are true, then $k$ is continuous, $(\lambda_i)_{i \in I}$ are all, possibly vanishing, eigenvalues of $T_{k,\nu}$, and $(e_i^*)_{i \in I}$ is an ONB of the corresponding eigenfunctions.

For an interpretation of the representation (18) recall that $\kappa(-x+x') = k(x,x')$ for all $x,x' \in G$ and hence (18) gives

$$\kappa(x) = \sum_{\lambda_i > 0} \lambda_i \text{Re} e_i(-x) = \sum_{\lambda_i > 0} \lambda_i \text{Re} e_i(x)$$

for all $x \in G$. Consequently, the second equality in (18) is Bochner’s theorem, see e.g. Hewitt and Ross (1970, Theorem 30.3), in the case of $\mathbb{R}$-valued kernels on compact Abelian groups. Unlike this classical theorem, however, the second equality in (18) also describes how the representing measure of $\kappa$ on $\hat{G}$ is given by the eigenvalues of $T_k$ or $T_k^C$. In the following, we do not need this information, in fact, we only mentioned the second equality to provide a link to the existing theory. Instead, the first equality in (18), which replaces the characters of $G$ by the eigenfunctions of $T_k$, is more important for us, since this equality actually is the Mercer representation of $k$ in the sense of (13) and therefore the theory developed in the previous sections becomes applicable.

The next result is an extension of Theorem 3.15 to translation-invariant kernels on compact Abelian groups.

**Theorem 4.3.** Let $(G,+)$ be a compact Abelian group. Then the following statements are equivalent:

i) $G$ is metrizable.

ii) $\hat{G}$ is at most countable.

iii) There exists a translation-invariant universal kernel on $G$.

iv) There exists a universal kernel on $G$.

v) There exists a translation-invariant characteristic kernel on $G$.

vi) There exists a continuous characteristic kernel on $G$.

Note that the equivalence between i) and ii) can also be shown without using translation-invariant kernels, see e.g. Morris (1979, Proposition 3) or Hewitt and Ross (1963, Theorem 24.15). Our proof, to the contrary, is solely RKHS-based.

Our next result characterizes universal and characteristic translation-invariant kernels on compact Abelian groups. In view of Theorem 4.3, it suffices to consider the metrizable case.
Corollary 4.4. Let \((G, +)\) be a compact metrizable Abelian group and \(k\) be a translation-invariant kernel on \(G\) with representation (18). Then we have:

\begin{enumerate}
  \item \(k\) is universal if and only if \(\lambda_i > 0\) for all \(i \in I\).
  \item \(k\) is characteristic if and only if \(\lambda_i > 0\) for all \(i \neq 0\).
\end{enumerate}

Corollary 4.4 generalizes Sriperumbudur et al. (2010a, Theorem 14 and Corollary 15) from \(\mathbb{T}^d\) to arbitrary compact metrizable Abelian groups. Moreover, recall that these authors also provide a couple of translation-invariant characteristic kernels on \(\mathbb{T}\) that enjoy a closed form.

As mentioned in the beginning of this section, the major difficulty in deriving a Mercer representation (18) for translation-invariant kernels is the handling of self-inverse characters other than the neutral element. The simplest example of a group \(G\) whose dual \(\hat{G}\) contains more than one self-inverse is the quotient group \((\mathbb{Z}_2, +)\) of \((\mathbb{Z}, +)\) with its subgroup \(2\mathbb{Z}\). Indeed, besides the neutral element \(e_0\), \(\hat{\mathbb{Z}_2}\) only contains the character \(\hat{e}_1\) given by \(\hat{e}_1(0) := 1\) and \(\hat{e}_1(1) := -1\). Clearly, this gives \(e_1^2 = e_0\) and thus \(\hat{e}_1\) is self-inverse. Now note that a function \(k : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{R}\) can be uniquely described by a 2-by-2 matrix \(K = (k(x, x'))_{x, x' \in \mathbb{Z}_2}\) and a simple calculation shows that \(k\) is a kernel if and only if \(k(0, 1) = k(1, 0), k(0, 0) \geq 0, k(1, 1) \geq 0,\) and \(k(0, 0)k(1, 1) \geq k^2(0, 1)\). Moreover, \(k\) is translation-invariant as soon as it is constant on its diagonal, and in this case the previous conditions reduce to

\[
k(0, 1) = k(1, 0) \quad \text{and} \quad k(0, 0) = k(1, 1) \geq |k(0, 1)|.
\]

Now, let \(k\) be a translation-invariant kernel on \(\mathbb{Z}_2\) and \(\lambda_0, \lambda_1 \geq 0\) be the coefficients in (18). Then a simple calculation shows that the describing matrix \(K\) is given by

\[
K = \begin{pmatrix}
\lambda_0 + \lambda_1 & \lambda_0 - \lambda_1 \\
\lambda_0 - \lambda_1 & \lambda_0 + \lambda_1
\end{pmatrix},
\]

and therefore it is not hard to see by Corollary 4.4 that \(k\) is characteristic, if and only if \(k(0, 0) \neq k(0, 1)\). Similarly, \(k\) is universal, if and only if \(k(0, 0) \neq \pm k(0, 1)\).

While this example seems to be rather trivial, it already has some important applications. For example, assume that our input space \(X\) is a product space for which some components belong to a compact metrizable Abelian group, while the remaining components are only allowed to attain the values 0 and 1. In other words, \(X\) is of the form \(X = G \times \mathbb{Z}^d_2\), where \(G\) is a compact metrizable Abelian group and \(d \geq 1\). Now, an intuitive way to construct a (translation-invariant) characteristic kernel \(k\) on \(X\) is to take a product \(k := k_C \cdot k_D\), where \(k_C\) and \(k_D\) are kernels on \(G\) and \(\mathbb{Z}^d_2\), respectively. By Corollary 3.16 we then know that \(k\) is characteristic (or universal) if and only if both \(k_C\) and \(k_D\) are universal. Clearly, if \(k_D\) is itself a product of kernels \(k_1, \ldots, k_d\) then \(k_D\) is almost automatically translation-invariant and universal. Indeed, if all \(k_i\) satisfy (19) with \(k_i(0, 0) \neq \pm k_i(0, 1)\), then each \(k_i\) is translation-invariant and universal, and thus so is \(k_D\). It seems fair to say that most “natural” choices of \(k_i\) will satisfy these assumptions. On the other hand, translation-invariant universal kernels \(k_C\) on \(G\) are completely characterized by Corollary 4.4, and thus it is straightforward to characterize all translation-invariant characteristic kernels \(k\) on \(G \times \mathbb{Z}^d_2\) of product type \(k := k_C \cdot k_D\). However, their representation (18) is a bit more cumbersome. Indeed, any element \((e, \omega) \in \hat{G} \times \mathbb{Z}^d_2 = G \times \mathbb{Z}^d_2\) with self-inverse \(e \in \hat{G}\) and arbitrary \(\omega \in \mathbb{Z}^d_2\) is self-inverse, where the equality in the sense of group isomorphisms can e.g. be found in Folland (1995, Proposition 4.6). Consequently, the set \(I_0\), which intuitively may be viewed as a small set of unusual characters, may actually be rather large.
Note that the set \( \mathbb{Z}_2 \) appears in data analysis settings whenever we have categorical variables with two possible values, which quite frequently is indeed the case. Now we have seen around (19) that most natural choices of kernels on \( \mathbb{Z}_2 \) are actually translation-invariant, and for these kernels the results of this subsection applies. Similarly, if we have a categorical variable with an even number \( m \) of possible values that have a cyclic nature, for example hours of a day or months of a year, then \( \mathbb{Z}_m \) can be used to describe these values, and kernels that respect the cyclic nature are translation-invariant. Clearly \( m/2 \) is self-inverse in \( \mathbb{Z}_m \), and therefore our theory again applies.

For more on structural properties of compact Abelian groups as well as for further examples we refer to Hewitt and Ross (1963, §25) and Hofmann and Morris (2013, Chapter 8).

4.2 Spheres

In this subsection we consider isotropic kernels \( k \) on \( S^d \) for \( d \geq 1 \), that is kernels of the form \( k(x, y) = \psi(\theta(x, y)) \), where \( \theta(\cdot, \cdot) = \arccos(\cdot, \cdot) \) is the geodesic distance on \( S^d \) and \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product on \( \mathbb{R}^{d+1} \). Following Gneiting (2013), let \( \Psi_d \) be the class of all continuous functions \( \psi \) on \([0, \pi] \) such that \( k(x, y) = \psi(\theta(x, y)) \) is a kernel on \( S^d \), and define \( \Psi_\infty := \cap_d \Psi_d \). We write \( \Psi^+_d \subset \Psi_d \) for the class of functions that induce a strictly positive definite kernel on \( S^d \), and set \( \Psi^+_\infty := \cap_d \Psi^+_d \). It holds that \( \Psi_{d+1} \subset \Psi_d \) and \( \Psi^+_{d+1} \subset \Psi^+_d \), see Gneiting (2013, Corollary 1).

The following two theorems are our main results on characteristic kernels on \( S^d \).

**Theorem 4.5.** Let \( k \) be an isotropic kernel on \( S^d \) induced by some \( \psi \in \Psi_{d+2} \) or by some \( \psi \in \Psi^+_d \). Then the following statements are equivalent:

i) \( k \) is characteristic.

ii) \( k \) is universal.

iii) \( k \) is strictly positive definite on \( S^d \), that is, \( \psi \in \Psi^+_d \).

Theorem 4.5 shows in particular that any \( \psi \in \Psi^+_{d+1} \) induces a characteristic kernel on \( S^d \). For the practically most relevant case of \( S^2 \), all the parametric families of isotropic positive definite functions in Gneiting (2013, Table 1) are in \( \Psi^+_3 \) and thus all of them are characteristic and yield strictly proper kernel scores on \( S^2 \) by Theorem 1.1 and Theorem 4.5.

**Theorem 4.6.** Let \( k \) be an isotropic kernel on \( S^d \) induced by \( \psi \in \Psi_\infty \). Then the following statements are equivalent:

i) \( k \) is characteristic.

ii) \( k \) is universal.

iii) \( k \) is strictly positive definite, that is, \( \psi \in \Psi^+_\infty \).

Theorem 4.6 is analogous to the result of Sriperumbudur et al. (2011, Proposition 5) for radial kernels on \( \mathbb{R}^d \).

For the proofs of Theorems 4.5 and 4.6, we need to introduce some preliminaries. By Schoenberg (1942) the functions in \( \Psi_\infty \) have a representation of the form

\[
\psi(\theta) = \sum_{n=0}^{\infty} b_n(\cos(\theta))^n, \quad \theta \in [0, \pi],
\]  

(20)
where \((b_n)_n\) is a summable sequence of non-negative coefficients termed the \(\infty\)-Schoenberg sequence of \(\psi\). Schoenberg (1942) also showed that the functions in \(\Psi_d\) have a representation as

\[
\psi(\theta) = \sum_{n=0}^{\infty} b_{n,d} \frac{C_n^{(d-1)/2}(\cos \theta)}{C_n^{(d-1)/2}(1)}, \quad \theta \in [0, \pi],
\]

where \((b_{n,d})_n\) is a summable sequence of non-negative coefficients termed the \(d\)-Schoenberg sequence of \(\psi\), and \(C_n^\lambda, \lambda > 0, n \in \mathbb{N}_0\) are the Gegenbauer polynomials; see the Digital Library of Mathematical Functions (DLMF) (2011, 18.3.1). For \(\lambda = 0\), we set \(C_n^0(\cos \theta) := \cos(n\theta)\).

**Definition 4.7.** A sequence of non-negative real numbers \((b_n)_{n \in \mathbb{N}_0}\) fulfills condition b, if \(b_n > 0\) for \(\infty\)-many even and \(\infty\)-many odd integers.

**Remark 4.8.** For \(\psi \in \Psi_\infty\) or \(\psi \in \Psi_d\), \(d \geq 2\), the induced isotropic kernel is strictly positive definite if and only if its Schoenberg sequence fulfills condition b, see Menegatto (1992, 1994) and Chen et al. (2003). For \(d = 1\), condition b remains a necessary condition for \(\psi \in \Psi_1^+\) as shown by Menegatto (1995). However, it is not sufficient any more. A simple sufficient condition for \(\psi \in \Psi_1^+\) that is useful for our purposes but which is not necessary is that there is \(n_0\) such that \(b_{n,1} > 0\) for all \(n \geq n_0\). See Menegatto et al. (2006) for a necessary and sufficient condition in the case \(d = 1\).

**Lemma 4.9.** If \(\psi \in \Psi_{d+2}\), then it is strictly positive definite on \(S^d\) if and only if \(b_{n,d} > 0\) for all \(n \in \mathbb{N}_0\).

The set \(\Psi_1^+ \setminus \Psi_{d+2} \subset \Psi_d\) is not empty and also contains elements with \(b_{n,d} > 0\) for all \(n \geq 0\). To construct an explicit example, take any summable sequence \((b_n)_{n \in \mathbb{N}_0}\) of positive real numbers such that \(b_2 > (d(d+3)/2)b_0\). Let \(\psi\) be the function with \(d\)-Schoenberg sequence \((b_n)_{n \in \mathbb{N}_0}\). Then \(\psi \in \Psi_d^+\), fulfills \(b_{n,d} > 0\) for all \(n \geq 0\), and, by Gneiting (2013, Corollary 4), it is not a member of \(\Psi_{d+2}\).

**Lemma 4.10.** If \(\psi \in \Psi_{d+1}^+\), then \(b_{n,d} > 0\) for all \(n \geq 0\).

After these preliminary considerations concerning Schoenberg sequences, we will now show Theorem 4.5 by applying Corollary 3.18. To this end, let \((e_{n,j})_{n \in \mathbb{N}_0, j=1,\ldots,N(d,n)}\) denote an orthonormal basis of spherical harmonics on \(S^d\). The polynomial \(e_{n,j}\) has order \(n\) and \(N(d,n) = \binom{n+d}{n} - \binom{n+d-2}{n-2}\); see for example Groemer (1996, Theorem 3.1.4), where we note that he works on \(S^{d-1}\) while we work on \(S^d\). In particular, \(e_{0,0} = 1\). By Groemer (1996, Theorem 3.3.3),

\[
C_n^{(d-1)/2}(\langle x, y \rangle) = \frac{1}{N(d,n)} \sum_{j=1}^{N(d,n)} e_{n,j}(x)e_{n,j}(y), \quad (21)
\]

hence any isotropic kernel on \(S^d\) induced by \(\psi \in \Psi_d\) has a Mercer representation of the form (13) with \(\lambda_{n,j} = b_{n,d}/N(d,n)\). Moreover, Groemer (1996, Corollary 3.2.7) shows that the space \(\text{span}\{e_{n,j} : n \in \mathbb{N}_0, j = 1,\ldots,N(d,n)\}\) is dense in \(C(S^d)\).

Similar to Sriperumbudur et al. (2010a, Theorem 9) for translation invariant kernels on \(\mathbb{R}^d\), Corollary 3.18 yields the following theorem.

**Theorem 4.11.** The kernel induced by \(\psi \in \Psi_d\) is

i) universal if and only if \(b_{n,d} > 0\) for all \(n \geq 0\).
ii) characteristic if and only if $b_{n,d} > 0$ for all $n \geq 1$.

For the proof of the converse of Theorem 4.6, we need the following proposition.

**Proposition 4.12.** Let $k$ be a kernel on $S^d$ induced by $\psi \in \Psi_\infty$. If $k$ is characteristic, then $\psi$ is strictly positive definite, that is $\psi \in \Psi_\infty^+$.

## 5 Proofs

### 5.1 Proofs related to Section 1

For the proof of Theorem 1.1 we assume that the general results on kernel mean embeddings recalled in Section 2 up to Definition 2.2 of characteristic kernels are available. Note that these results do not involve kernel scores at all, so that there is no danger of circular reasoning.

**Proof of Theorem 1.1:** As explained around (7), the condition $P \in M^2(X)$ ensures that $\Phi(P)$ defined by (3) is indeed an element of $H$. Now (4) follows from (8), namely

$$
\|\Phi(P) - \Phi(Q)\|_H^2 = \iint k(x,y) \, dP(x) \, dP(y) + \iint k(x,y) \, dQ(x) \, dQ(y) - 2 \iint k(x,y) \, dP(x) \, dQ(y) \\
= 2 \left( \int S_k(P,x) \, dQ(x) - \int S_k(Q,x) \, dQ(x) \right).
$$

The remaining assertions directly follow from (4).

### 5.2 Proofs related to Section 3.1

**Proof of Theorem 3.1:** $iv) \Rightarrow iii)$. By assumption, $\Phi : M(X) \to H$ is bijective and since $H$ is complete, so is $\| \cdot \|_H$ on $M(X)$.

$iii) \Rightarrow ii)$. Consider the identity map $id : (M(X), \| \cdot \|_{TV}) \to (M(X), \| \cdot \|_H)$. Since we have already seen in Section 2 that

$$
\|\mu\|_H \leq \|\cdot\|_{L_1} : H \to \|\cdot\|_{TV} \leq \|k\|_{\infty} \cdot \|\mu\|_{TV}
$$

holds for all $\mu \in M(X)$, the identity map is continuous. In addition, it is, of course, bijective, and since both $(M(X), \| \cdot \|_{TV})$ and $(M(X), \| \cdot \|_H)$ are complete, the open mapping theorem, see e.g. Megginson (1998, Corollary 1.6.8) shows that both norms are equivalent.

$ii) \Rightarrow i)$. By the arguments of Diestel et al. (1995, page 63f) the space $(M(X), \| \cdot \|_{TV})$ is a so-called $L_{1,\lambda}$-space for all $\lambda > 1$, see Diestel et al. (1995, page 60) for a definition, while the Euclidean structure of $(M(X), \| \cdot \|_H)$, which is inherited from $(H, \| \cdot \|_H)$, shows that $(M(X), \| \cdot \|_H)$ is an $L_{2,\lambda}$-space for all $\lambda > 1$. Let us now assume that $\dim M(X) = \infty$. Then, Diestel et al. (1995, Corollary 11.7) shows that $(M(X), \| \cdot \|_{TV})$ has only trivial type 1, while $(M(X), \| \cdot \|_H)$ has optimal type 2. However, the definition of the type of a space, see Diestel et al. (1995, page 217), immediately shows that equivalent norms always share their type, and hence $\| \cdot \|_H$ and $\| \cdot \|_{TV}$ are not equivalent on $M(X)$.

$i) \Rightarrow iv)$. If $M(X)$ is finite dimensional, the $\sigma$-algebra $\mathcal{A}$ is finite. Since $k$ is assumed to be measurable, we then see that $k$ can only attain finitely many different values, and consequently, the canonical feature map $\Phi : X \to H$ also attains only finitely many different values, say $f_1, \ldots, f_m \in H$. Using Steinwart and Christmann (2008, Theorem 4.21), we conclude that $H = \text{span}\{f_1, \ldots, f_m\}$. We now define $A_i := \Phi^{-1}(\{f_i\})$ for $i = 1, \ldots, m$. By construction $A_1, \ldots, A_m$
form a partition of $X$ with $A_i \in \mathcal{A}$ and $A_i \neq \emptyset$ for all $i = 1, \ldots, m$. Let us fix some $x_i \in A_i$. For the corresponding Dirac measures we then have $\Phi(\delta_{x_i}) = \Phi(x_i) = f_i$ and since $\Phi : \mathcal{M}(X) \to H$ is linear we find $\text{span}\{f_1, \ldots, f_m\} \subset \Phi(\mathcal{M}(X))$. This shows the surjectivity of the kernel embedding.

**Proof of Lemma 3.2:** Let $(\mu_n) \subset \mathcal{M}_0(X)$ be a sequence that converges to some $\mu \in \mathcal{M}(X)$ in $\| \cdot \|_H$. Then we have $\mu_n(X) = 0$ for all $n \geq 1$ and we need to show $\mu(X) = 0$. The latter, however, follows from (10), namely

$$|\mu(X)| = \left| \int 1_X \, d(\mu_n - \mu) \right| \leq \|1_X\|_H \|\mu_n - \mu\|_H \to 0.$$ 

Let us now assume that $k$ is characteristic. Then Proposition 2.3 shows that $k$ is strictly integrally positive definite with respect to $\mathcal{M}_0(X)$, and hence $\|\mu\|_H > 0$ for all $\mu \in \mathcal{M}_0(X) \setminus \{0\}$. Now let $\mu \in \mathcal{M}(X) \setminus \mathcal{M}_0(X)$ and $P$ be some probability measure on $X$. By (5) we then find an $\alpha \in \mathbb{R}$ and some $\mu_0 \in \mathcal{M}_0(X)$ such that $\alpha P + \mu_0 = \mu$, and since $\mu \notin \mathcal{M}_0(X)$ we actually have $\alpha \neq 0$. Using this decomposition, and (10) we now find

$$\|\mu\|_H = \|\alpha P + \mu_0\|_H \geq \frac{1}{\|1_X\|_H} \left| \int 1_X \, d(\alpha P + \mu_0) \right| = \frac{|\alpha|}{\|1_X\|_H} > 0$$

and hence $\Phi : \mathcal{M}(X) \to H$ is injective, see (9).

**Proof of Lemma 3.3:** By the definition of the $\| \cdot \|_H$-norm and (8) we have

$$\|\mu\|_H^2 = \int_X \int_X k_1(x, x') + k_2(x, x') \, d\mu(x) \, d\mu(x') = \|\mu\|_{H_1}^2 + \|\mu\|_{H_2}^2.$$ 

Moreover, if $k_1$ is characteristic, then $\|\mu\|_{H_1}^2 > 0$ for all $\mu \in \mathcal{M}_0(X) \setminus \{0\}$ by Proposition 2.3 and the just established formula then yields $\|\mu\|_H^2 > 0$ for these $\mu$. Consequently, $k$ is characteristic. Repeating the argument for $\mathcal{M}(X)$ yields the last assertion.

**Proof of Lemma 3.4:** By the definition of the $\| \cdot \|_H$-norm and (8) we have

$$\|\mu_1 \otimes \mu_2\|_H^2 = \int_{X_1 \times X_2} \int_{X_1 \times X_2} k_1(x_1, x_1') \cdot k_2(x_2, x_2') \, d\mu_1(x_1) \, d\mu_2(x_2) \, d\mu_1(x_1') \, d\mu_2(x_2')$$

$$= \int_{X_1 \times X_1} \int_{X_2 \times X_2} k_1(x_1, x_1') \cdot k_2(x_2, x_2') \, d\mu_1(x_1) \, d\mu_2(x_2) \, d\mu_1(x_1') \, d\mu_2(x_2')$$

$$= \|\mu_1\|_{H_1}^2 \cdot \|\mu_2\|_{H_2}^2,$$

where we note that the application of Fubini’s theorem is possible, since the kernels are bounded and the measures are finite. Now assume that $k$ is characteristic, but, say, $k_1$ is not strictly integrally positive definite with respect to $\mathcal{M}(X_1)$. Then there exists a $\mu_1 \in \mathcal{M}(X_1)$ with $\mu_1 \neq 0$ but $\|\mu_1\|_{H_1} = 0$. Moreover, since $\dim \mathcal{M}(X_2) \geq 2$, the decomposition (5) gives a $\mu_2 \in \mathcal{M}_0(X_2)$ with $\mu_2 \neq 0$. Let us define $\mu := \mu_1 \otimes \mu_2$. Our construction then yields $\mu \neq 0$ and $\mu_1 \cdot \mu_2 = \mu_1(X_1) \cdot \mu_2(X_2) = 0$, that is $\mu \notin \mathcal{M}_0(X_1 \times X_2)$, while the already established product formula shows $\|\mu\|_H = \|\mu_1\|_{H_1} \cdot \|\mu_2\|_{H_2} = 0$. By Proposition 2.3 we conclude that $k$ is not characteristic.

**Proof of Lemma 3.5:** Let $\Phi_1$ denote the canonical feature map of the kernel $k_1 = 1_{X_1 \times X}$ and $H_1$ its RKHS. Moreover, we write $H + 1$ for the RKHS of $k + 1$. For later use, we note that (8) implies $\langle \Phi_1(\mu), \Phi_1(\mu_0) \rangle_{H_1} = 0$ for all $\mu \in \mathcal{M}(X)$ and $\mu_0 \in \mathcal{M}_0(X)$.

(iii) $\Rightarrow$ (ii). Let us fix a $P \in \mathcal{M} \cap \mathcal{M}_1(X)$. Similar to (5) we then have $\mathcal{M} = \mathbb{R}P \oplus \mathcal{M}_0$. Indeed, “$\supset$” and $\mathbb{R}P \cap \mathcal{M}_0 = \{0\}$ are trivial and for $\mu \in \mathcal{M}$ it is easy to see that $\mu - \mu(X)P \in \mathcal{M}_0$.  

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Now, we need to prove $\|\mu\|_{H+1} > 0$ for all $\mu = \alpha P + \mu_0 \in \mathbb{R}P \oplus \mathcal{M}_0$ with $\mu \neq 0$. By (iii) we already know this in the case $\alpha = 0$, and thus we further assume $\alpha \neq 0$. Then Lemma 3.3 and (8) together with our initial remark yield
\[
\|\mu\|_{H+1}^2 = \|\mu\|_H^2 + \langle \Phi_1(\mu), \Phi_1(\mu) \rangle_{H_1} \\
= \|\mu\|_H^2 + \alpha^2 \|P\|_{H_1}^2 + 2\alpha \langle \Phi_1(P), \Phi_1(\mu_0) \rangle_{H_1} + \|\mu_0\|_{H_1}^2 \\
= \|\mu\|_H^2 + \alpha^2
\]
and since $\alpha \neq 0$, we conclude $\|\mu\|_{H+1}^2 > 0$.

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Leftrightarrow$ i). For $\mu_0 \in \mathcal{M}_0 \subset \mathcal{M}_0(X)$, Lemma 3.3 together with our initial remark shows
\[
\|\mu_0\|_{H+1}^2 = \|\mu_0\|_H^2 + \|\mu_0\|_{H_1}^2 = \|\mu_0\|_H^2.
\]
From this equality the equivalence immediately follows. \(\square\)

**Proof of Theorem 3.6:** For some fixed $P \in \mathcal{M}_1(X)$ we consider the map
\[
\pi : \mathbb{R}P \oplus \mathcal{M}_0(X) \to \mathcal{M}(X) \\
\alpha P + \mu_0 \mapsto \mu_0.
\]
By (5), $\pi$ is a linear map $\pi : \mathcal{M}(X) \to \mathcal{M}(X)$ with $\|\cdot\|_\mathcal{M}$ and $\pi = \mathcal{M}_0(X)$, and hence a projection onto $\mathcal{M}_0(X)$. Moreover, $\mathcal{M}_0(X)$ is $\|\cdot\|_\mathcal{M}$-closed by assumption and therefore $\pi$ is $\|\cdot\|_\mathcal{M}$-continuous, see Megginson (1998, Theorem 3.2.14). Let us now assume that $\|\cdot\|_H$ and $\|\cdot\|_{TV}$ were equivalent on $\mathcal{M}_0(X)$. Our goal is to show that this assumption implies the equivalence of $\|\cdot\|_H$ and $\|\cdot\|_{TV}$ on $\mathcal{M}(X)$. To this end we fix some sequence $(\mu_n) \in \mathcal{M}(X)$ and some $\mu \in \mathcal{M}(X)$ with $\|\mu_n - \mu\|_H \to 0$. By (5) we then find $\alpha_n, \alpha \in R$ with $\mu_n = \alpha_n P + \pi(\mu_n)$ and $\mu = \alpha P + \pi(\mu)$.

Using the $\|\cdot\|_H$-continuity of $\pi$, we then find $\|\pi(\mu_n) - \pi(\mu)\|_{TV} \to 0$, and since we assumed that $\|\cdot\|_H$ and $\|\cdot\|_{TV}$ are equivalent on $\mathcal{M}_0(X)$, we conclude that $\|\pi(\mu_n) - \pi(\mu)\|_{TV} \to 0$. In addition, we have $\|P\|_H > 0$ by the injectivity of the kernel embedding, and therefore
\[
|\alpha_n - \alpha| \cdot \|P\|_H \leq \|\mu_n - \mu\|_H + \|\pi(\mu_n) - \pi(\mu)\|_H \to 0
\]
shows that $\alpha_n \to \alpha$. Combining these considerations, we find
\[
\|\mu_n - \mu\|_{TV} \leq |\alpha_n - \alpha| \cdot \|P\|_{TV} + \|\pi(\mu_n) - \pi(\mu)\|_{TV} \to 0.
\]
Summing up, we have seen that $\|\mu_n - \mu\|_H \to 0$ implies $\|\mu_n - \mu\|_{TV} \to 0$ for every sequence $(\mu_n) \in \mathcal{M}(X)$, and consequently, the identity map $\text{id} : (\mathcal{M}(X), \|\cdot\|_H) \to (\mathcal{M}(X), \|\cdot\|_{TV})$ is continuous. This yields $\|\cdot\|_{TV} \leq \|\cdot\|_H$ on $\mathcal{M}(X)$, and since we also have $\|\cdot\|_H \leq \|k\|_{\infty} \|\cdot\|_{TV}$ we see that $\|\cdot\|_H$ and $\|\cdot\|_{TV}$ are indeed equivalent on $\mathcal{M}(X)$. This, however, contradicts Theorem 3.1, and hence our assumption that $\|\cdot\|_H$ and $\|\cdot\|_{TV}$ were equivalent on $\mathcal{M}_0(X)$ is false. \(\square\)

**Proof of Theorem 3.7:** Let us first assume that $1_X \in H$. By Lemma 3.2 we then see that the kernel embedding $\Phi : \mathcal{M}(X) \to H$ is injective, and Theorem 3.6 thus shows that $\|\cdot\|_H$ and $\|\cdot\|_{TV}$ are not equivalent on $\mathcal{M}_0(X)$. Since $\|\cdot\|_H$ is dominated by $\|\cdot\|_{TV}$, we consequently find a sequence $(\mu_n) \in \mathcal{M}_0(X)$ and some $\delta > 0$ such that $\|\mu_n\|_H \to 0$ and $\inf_{n \geq 1} \|\mu_n\|_{TV} \geq \delta$.

Let us consider the Hahn-Jordan decomposition $\mu_n = \mu_n^+ - \mu_n^-$ with $\mu_n^+, \mu_n^- \in \mathcal{M}_+(X)$. Since $\mu_n \in \mathcal{M}_0(X)$, we have $\mu_n^+(X) = \mu_n^-(X)$. We define
\[
Q^{(1)}_n := \frac{\mu_n^+}{\mu_n(X)} \quad \text{and} \quad Q^{(2)}_n := \frac{\mu_n^-}{\mu_n(X)}.
\]
Clearly, this yields $Q_n^{(1)}, Q_n^{(2)} \in M_1(X)$ and

$$\|Q_n^{(1)} - Q_n^{(2)}\|_{TV} = \frac{\mu_n^+(X) + \mu_n^-(X)}{\mu_n^+(X)} = 2.$$ \hfill \hbox{	extup{1}}

Moreover, we have

$$\|Q_n^{(1)} - Q_n^{(2)}\|_H = \frac{\|\mu_n\|_H}{\mu_n^+(X)} = \frac{\|\mu_n\|_H}{\frac{1}{2}\|\mu_n\|_{TV}} \leq 2\delta^{-1}\|\mu_n\|_H,$$

and by choosing $n$ sufficiently large we can therefore guarantee $\|Q_n^{(1)} - Q_n^{(2)}\|_H \leq \varepsilon$.

Let us now consider the case $1_X \notin H$. By Lemma 3.3 we then see that the kernel $\tilde{k} := k + 1$ is characteristic. Let us write $H_1 := \mathbb{R}1_X$ for the RKHS of the constant kernel $k_1 := 1_{X \times X}$. Then $x \mapsto (k(x, \cdot), k_1(x, \cdot)) \in H \times H_1$ is a feature map of the kernel $k + 1$ in the sense of (Steinwart and Christmann, 2008, Definition 4.1) if $H \times H_1$ is equipped with the usual Hilbert space norm $\|(w, w_1)\| := \sqrt{\|w\|_H^2 + \|w_1\|_{H_1}^2}$ and consequently an application of (Steinwart and Christmann, 2008, Theorem 4.21) shows that the RKHS $\tilde{H}$ of $k + 1$ is given by $\tilde{H} = H + H_1$. The latter yields $1_X \in \tilde{H}$. The already considered case thus shows that for all $\varepsilon > 0$ there exist distributions $Q_1, Q_2 \in M_1(X)$ such that $\|Q_1 - Q_2\|_{TV} = 2$ and $\|Q_1 - Q_2\|_H \leq \varepsilon$. Moreover, for $\mu := Q_1 - Q_2 \in M_0(X)$ we see by Lemma 3.3 and (8) that

$$\varepsilon^2 \geq \|Q_1 - Q_2\|_H^2 = \|\mu\|_H^2 = \|\mu\|_H^2 + \int \int 1_{X \times X} d\mu d\mu = \|\mu\|_H^2 + \mu(X)\mu(X) = \|\mu\|_H^2 = \|Q_1 - Q_2\|_H^2$$

we have shown the assertion in the second case, too. \hfill \hbox{\textup{2}}

\textbf{Proof of Corollary 3.8:} Let us fix some $P \in M_1(X)$, $\delta \in (0, 1]$ and $\varepsilon \in (0, 2\delta)$. By Theorem 3.7 there then exist distributions $\tilde{Q}_1, \tilde{Q}_2 \in M_1(X)$ with $\|\tilde{Q}_1 - \tilde{Q}_2\|_{TV} = 2$ and $\|\tilde{Q}_1 - \tilde{Q}_2\|_H \leq \varepsilon$.

We define $Q_i := (1 - \delta)P + \delta\tilde{Q}_i \in M_1(X)$ for $i = 1, 2$. A simple calculation then shows that

$$\|Q_1 - Q_2\|_{TV} = \delta\|\tilde{Q}_1 - \tilde{Q}_2\|_{TV} = 2\delta,$$

and analogously, we find $\|Q_1 - Q_2\|_H \leq \delta\varepsilon \leq \varepsilon$. Finally, we have $\|P - Q_i\|_{TV} = \delta\|P - \tilde{Q}_i\|_{TV} \leq 2\delta$, and hence we obtain the assertion for $\delta := 2\delta$. \hfill \hbox{\textup{3}}

\textbf{Proof of Lemma 3.9:} Let us first assume that $\mu_1, \mu_2 \in M_+(X)$. For $x, x' \in X$ we define

$$\tilde{k}(x, x') := \sum_{i \in I} \lambda_i |e_i(x)||e_i(x')|.$$ \hfill \hbox{\textup{4}}

Then the Cauchy-Schwarz inequality together with (13) gives

$$|\tilde{k}(x, x')| = \sum_{i \in I} \lambda_i |e_i(x)||e_i(x')| \leq \sum_{i \in I} \lambda_i e_i^2(x) \cdot \sum_{i \in I} \lambda_i e_i^2(x') = k(x, x) \cdot k(x', x')$$

for all $x, x' \in X$, and since $k$ is bounded, we conclude that $\tilde{k}$ is also bounded, and hence $\tilde{k} \in L_2(\mu_1 \otimes \mu_2)$. Moreover, for finite $J \subset I$ and $x, x' \in X$ we have

$$\left| \sum_{i \in J} \lambda_i e_i(x) e_i(x') \right| \leq \tilde{k}(x, x').$$
Using Fubini’s theorem, (13), Lebesgue’s dominated convergence theorem, and yet another time Fubini’s theorem, we thus find
\[
\int_X \int_X k(x, x') \, d\mu_1(x) \, d\mu_2(x') = \int_{X \times X} \sum_{i \in I} \lambda_i e_i(x) e_i(x') \, d(\mu_1 \otimes \mu_2)(x, x')
\]
\[
= \sum_{i \in I} \int_X \sum_{x' \in X} \lambda_i e_i(x) e_i(x') \, d(\mu_1 \otimes \mu_2)(x, x')
\]
\[
= \sum_{i \in I} \int_X \int_X \lambda_i e_i(x) e_i(x') \, d\mu_1(x) \, d\mu_2(x') .
\]
From the latter the assertion immediately follows.

Finally, let us assume that \(\mu_1, \mu_2 \in \mathcal{M}(X)\). Using the Hahn decomposition \(\mu_1 = \mu_1^+ - \mu_1^-\) and \(\mu_2 = \mu_2^+ - \mu_2^-\) as well as the fact that the expressions on the left and right of the desired equation are linear in the involved measures, we then obtain the assertion by the already established case.

**Proof of Theorem 3.10:** Clearly, the signed measure \(\mu := P - Q\) has the \(\nu\)-density \(h - g\) and we have \([h - g]_\sim \in L_2(\nu) \cap L_1(\nu)\). Now we find (15) by (8) and Lemma 3.9, namely
\[
\gamma_k^2(P, Q) = \|\mu\|_H^2 = \int_X \int_X k(x, x') \, d\mu(x) \, d\mu(x') = \sum_{i \in I} \lambda_i \cdot \left( \int_X e_i \, d\mu \right)^2
\]
\[
= \sum_{i \in I} \lambda_i \cdot \left( \int_X e_i \cdot (h - g) \, d\nu \right)^2 .
\]

\(ii) \Rightarrow i)\). We split \(f\) into \(f = f^+ - f^-\) with \(f^+ \geq 0\) and \(f^- \geq 0\). By our assumption we then know that
\[
c := \int_X f^+ \, d\nu = \int_X f^- \, d\nu > 0
\]
For \(\delta > 0\), \(h \in \Delta(\nu)\) and \(P := h \, d\nu \in \mathcal{P}_2(\nu)\) we define \(h_1 := (1 + \delta c)^{-1} (h + \delta f^+)\) and \(h_2 := (1 + \delta c)^{-1} (h + \delta f^-)\) and consider the corresponding measures \(Q_1 := h_1 \, d\nu\) and \(Q_2 := h_2 \, d\nu\). The construction immediately ensures \(Q_1, Q_2 \in \mathcal{P}_2(\nu)\). Moreover, we have
\[
\|P - Q_1\|_{TV} = \int_X |h - h_1| \, d\nu = \int_X \left| \frac{h + \delta ch}{1 + \delta c} - \frac{h + \delta f^+}{1 + \delta c} \right| \, d\nu
\]
\[
= \int_X \left| \frac{\delta ch - \delta f^+}{1 + \delta c} \right| \, d\nu
\]
\[
\leq \frac{\delta}{1 + \delta c} \int_X |ch| + |f^+| \, d\nu
\]
\[
= \frac{2\delta c}{1 + \delta c} ,
\]
and analogously we find \(\|P - Q_2\|_{TV} \leq 2\delta c/(1 + \delta c)\). In addition, we have
\[
\|Q_1 - Q_2\|_{TV} = \frac{\delta}{1 + \delta c} \int_X |f^+ - \delta f^-| \, d\nu = \frac{\delta}{1 + \delta c} \int_X |f| \, d\nu = \frac{2\delta c}{1 + \delta c}
\]
and by using that \(\{2\delta c/(1 + \delta c) : \delta > 0\} = (0, 2)\) we obtain the norm conditions for \(Q_1\) and \(Q_2\). Finally, (15) yields
\[
\gamma_k^2(Q_1, Q_2) = \sum_{i \in I} \lambda_i ([h_1 - h_2], [e_i]_\sim, 2_{L_2(\nu)}) = \frac{\delta}{1 + \delta c} \sum_{i \in I} \lambda_i ([f], [e_i]_\sim, 2_{L_2(\nu)}) = 0 ,
\]
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which shows $i$) as well as the final assertion.

$i) \Rightarrow iii)$. Let $Q_1$ and $Q_2$ be according to $i)$ and $[h_1]_\sim, [h_2]_\sim \in \Delta(\nu)$ be their $\nu$-densities. Then $Q_1 \neq Q_2$ implies $[h_1]_\sim \neq [h_2]_\sim$, and (15) yields

$$0 = \gamma^2_{\nu}(P, Q) = \sum_{i \in I} \lambda_i \langle [h_1 - h_2]_\sim, [e_1]_\sim \rangle_{L^2(\nu)}^2.$$ 

Since $\lambda_i > 0$ for all $i \in I$, we then conclude that $\langle [h_1 - h_2]_\sim, [e_1]_\sim \rangle_{L^2(\nu)}^2 = 0$ for all $i \in I$, which in turn implies $iii)$.

$iii) \Rightarrow ii)$. We define $f := h_1 - h_2$. Clearly, we have $[f]_\sim \in L_2(\nu) \cap L_1(\nu)$ and $f \neq 0$. Moreover, $h_1, h_2 \in \Delta(\nu)$ gives

$$\int_X f \, d\nu = \int_X h_1 \, d\nu - \int_X h_2 \, d\nu = 1 - 1 = 0.$$ 

Finally, the equality $\langle [h_1]_\sim, [e_1]_\sim \rangle_{L^2(\nu)} = \langle [h_2]_\sim, [e_1]_\sim \rangle_{L^2(\nu)}$, which holds for all $i \in I$, implies $\langle [f]_\sim, [e_1]_\sim \rangle_{L^2(\nu)} = 0$ for all $i \in I$, and hence $[f]_\sim \in [H]_\sim^\perp$.

**Proof of Corollary 3.11:** Let $h_j := (1_X + \alpha e_j)$ be the $\nu$-density of $Q_j$. Using $\|e_j\|_\infty \leq c_\infty$ and $\alpha = c_\infty^{-1}$, we then find $h_j \geq 0$, and since $[e_{i_0}]_\sim \perp [e_j]_\sim$, we further find

$$\int_X h_j \, d\nu = 1 + \alpha \int_X e_j e_{i_0} \, d\nu = 1.$$ 

Consequently, $Q_j$ is a probability measure. Moreover, we find

$$\|P - Q_j\|_{TV} = \int_X |1_X - h_j| \, d\nu = \int_X |\alpha e_j| \, d\nu \geq c_1 c_\infty^{-1}$$ 

and (15) implies

$$\|P - Q_j\|_{H}^2 = \sum_{i \in I} \lambda_i \langle [1_X - h_j]_\sim, [e_1]_\sim \rangle_{L^2(\nu)}^2 = \sum_{i \in I} \lambda_i \langle [-\alpha e_j]_\sim, [e_1]_\sim \rangle_{L^2(\nu)}^2 = \lambda_2 c_\infty^{-2}.$$ 

This shows the assertion. \hfill \Box

**Proof of Corollary 3.12:** In both cases it suffices to show $ii)$ of Theorem 3.10.

$i)$. Since $\text{codim}[H]_\sim \geq 2$ there exist linearly independent $[f_1]_\sim, [f_2]_\sim \in [H]_\sim^\perp$. If $\int_X f_2 \, d\nu = 0$ then there is nothing left to prove, and if $\int_X f_2 \, d\nu \neq 0$ then a quick calculation shows that

$$f := f_1 - \frac{\int_X f_1 \, d\nu}{\int_X f_2 \, d\nu} \cdot f_2$$

is the desired function.

$ii)$. Since $\text{codim}[H]_\sim \geq 1$, there exists an $[f]_\sim \in [H]_\sim^\perp \setminus \{0\}$ and from $1_X \in H$ we conclude that $\int_X f \, d\nu = \langle [f]_\sim, [1_X]_\sim \rangle_{L^2(\nu)} = 0$.

\hfill \Box

### 5.3 Proofs related to Section 3.2

**Proof of Theorem 3.13:** We fix a $C_0(X)$-kernel $k$ and a finite signed measure $\mu \in \mathcal{M}(X) \setminus \mathcal{M}^*(X)$. Then $f \mapsto \int_X f \, d\mu$ defines a bounded linear operator $C_0(X) \to \mathbb{R}$, and by (14) there thus exists a $\mu^* \in \mathcal{M}^*(X)$ such that

$$\int_X f \, d\mu = \int_X f \, d\mu^*$$

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for all \( f \in C_0(X) \). By \( H \subset C_0(X) \) and (10) we conclude that \( \|\mu - \mu^*\|_H = 0 \) while our construction ensures \( \mu \neq \mu^* \).

**Proof of Theorem 3.14:** Using the already observed identity \( I' = \Phi \) and Megginson (1998, Theorem 3.1.17) we see that \( I \) has a dense image if and only if \( \Phi : M^*(X) \to H \) is injective. By (9) we then conclude that \( k \) is universal if and only if \( k \) is strictly integrally positive definite with respect to \( M^*(X) \).

**Proof of Theorem 3.15:** \( i) \iff iii) \). This has been shown in Steinwart et al. (2006, Theorem 2). Moreover, it is well-known that compact metrizable Hausdorff spaces are Polish. The equality \( M^*(X) = M(X) \) then follows from Ulam’s theorem, see e.g. Bauer (2001, Lemma 26.2).

\( i) \Rightarrow ii) \). If there exists a universal kernel \( k \), then we have already shown that \( M^*(X) = M(X) \). Consequently, \( k \) is strictly integrally positive definite with respect to \( M(X) \) by Theorem 3.14, and thus characteristic by Proposition 2.3.

\( ii) \Rightarrow i) \). Assume that there exists a characteristic kernel \( k \). By Proposition 2.3 we know that \( k \) is strictly integrally positive definite with respect to \( M_0(X) \). Then \( k + 1 \) is a bounded and continuous kernel, which is strictly integrally positive definite with respect to \( M(X) \) by Lemma 3.5. Using \( M^*(X) \subset M(X) \) and Theorem 3.14 we conclude that \( k \) is universal.

**Proof of Corollary 3.16:** \( i) \Rightarrow ii) \). Since \( (X_1 \times X_2, \tau_1 \otimes \tau_2) \) is a compact metrizable space, we have \( M^*(X_1 \times X_2) = M(X_1 \times X_2) \), and hence the implication follows from Proposition 2.3 and Theorem 3.14.

\( ii) \Rightarrow iii) \). Since \( (X_1, \tau_1) \) and \( (X_2, \tau_2) \) are assumed to be non-trivial, we find \( \dim M(X_1) \geq 2 \) and \( \dim M(X_2) \geq 2 \). Now \( iii) \) follows from Lemma 3.4 and Theorem 3.14.

\( iii) \Rightarrow i) \). This can be shown by the theorem of Stone-Weierstraß, see e.g. Steinwart et al. (2016, Lemma A.5) for details.

**Proof of Theorem 3.17:** Before we begin, we write \( E := \text{span}\{e_i : i \in I\} \) and denote the RKHS of \( k \) by \( H \).

\( ii) \iff iii) \). Via the isomorphism (14) between \( C_0(X)' \) and \( M^*(X) \) we easily see that \( ii) \) is equivalent to the statement \( \varphi|_E = 0 \iff \varphi' = 0 \) for all \( \varphi' \in C_0(X)' \) and by Hahn-Banach’s theorem, see e.g. Dunford and Schwartz (1958, p. 64, Corollary II.3.13), the latter is equivalent to \( iii) \).

\( iii) \Rightarrow i) \). The chain of inclusions \( E \subset H \subset C_0(X) \) immediately gives the desired implication.

\( i) \Rightarrow iii) \). Clearly, we have \( E = \text{span}\{\sqrt{\lambda}e_i : i \in I\} \) and since \( (\sqrt{\lambda}e_i)_{i \in I} \) is an ONB of \( H \), we conclude that \( E^H = H \). Let us now fix an \( \varepsilon > 0 \) and an \( f \in C_0(X) \). Since \( k \) is universal, there then exists an \( h \in H \) with \( \|f - h\|_\infty \leq \varepsilon \) and for this \( h \) our initial consideration yields an \( e \in E \) with \( \|e - h\|_H \leq \varepsilon \). Combining both estimates we find

\[
\|f - e\|_\infty \leq \|f - h\|_\infty + \|h - e\|_\infty \leq \varepsilon + \|k\|_\infty \|h - e\|_H \leq (1 + \|k\|_\infty) \varepsilon,
\]

and hence \( E \) is dense in \( C_0(X) \).

To check the last statement, let us assume that \( E \) is dense in \( C_0(X) \). Using that \( \nu \) is finite we then see that \( E \) is also dense in \( C_0(X) \) with respect to \( \|\cdot\|_{L_2(\nu)} \), and since \( C_0(X) \) is dense in \( L_2(\nu) \) by the regularity of \( \nu \), see e.g. Bauer (2001, Theorem 29.14), we conclude that \( E \) is dense in \( L_2(\nu) \). This shows that \( [E]^\sim = \text{span}\{e_i^\sim : i \in I\} \) is dense in \( L_2(\nu) \), and therefore \( ([e_i^\sim])_{i \in I} \) is indeed an ONB of \( L_2(\nu) \).

**Proof of Corollary 3.18:** \( i) \). Let us first assume that \( \lambda_i > 0 \) for all \( i \in I \). By Steinwart and Scovel (2012, Lemma 2.6 and Theorem 2.11) we then see that \( k \) is of the form considered in Theorem 3.17, and since \( \text{span}\{e_i : i \in I\} \) is dense in \( C_0(X) \) by our assumption, we conclude that

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$k$ is universal. To show the converse implication we set $I^* := \{ i \in I : \lambda_i > 0 \}$ and assume that $k$ is universal but $I^* \neq I$. By the definition of $I^*$ we have

$$k(x, x') = \sum_{i \in I^*} \lambda_i e_i(x)e_i(x')$$

for all $x, x' \in X$, and by Steinwart and Scovel (2012, Lemma 2.6 and Theorem 2.11) we thus conclude that $(e_i)_{i \in I^*}$ is the family considered in Theorem 3.17. Consequently, this sub-family $\{ (e_i)_{i \in I^*} : i \in I^* \}$ is actually universal by the ONB in $L_2(\nu)$, which, however, is impossible for $I^* \neq I$.

(ii). Let us first assume that $k$ is characteristic and that there exists an $j \neq i_0$ with $\lambda_j = 0$. We have $[e_j]_{\sim} \in L_1(\nu)$ because $\nu(X) < \infty$. Moreover, $[e_j]_{\sim} \perp [e_i]_{\sim}$ for all $i \neq j$ implies both $\int_X e_j \, d\nu = 0$ and $[e_j]_{\sim} \in [H]_{\sim}$, where in the last step we used (12). Consequently, $k$ cannot be characteristic by Theorem 3.10.

Conversely, assume that $\lambda_i > 0$ for all $i \neq i_0$. If $\lambda_{i_0} > 0$ then $k$ is actually universal by the already established part \(i\), and thus characteristic. If $\lambda_{i_0} = 0$, then the kernel

$$k(x, x') + 1 = \sum_{i \neq i_0} \lambda_i e_i(x)e_i(x') + e_{i_0}(x)e_{i_0}(x')$$

is universal by part \(i\) and thus $k$ is characteristic by Theorem 3.14, Lemma 3.5, Proposition 2.3, and $\mathcal{M}^+(X) = \mathcal{M}(X)$.

(iii). If $\lambda_i > 0$ for all $i \in I$, then \(i\) shows that $k$ is universal, and by Theorem 3.14, Proposition 2.3, and $\mathcal{M}^+(X) = \mathcal{M}(X)$ we conclude that $k$ is characteristic. To show the converse implication, we assume that $k$ is characteristic and there is an $i_0 \in I$ with $\lambda_{i_0} = 0$. Since $\{ (e_i)_{i \in I} \}$ is an ONS in $L_2(\nu)$ and $[e_{i_0}]_{\sim} \notin [H]_{\sim} = \text{span}\{ [e_i]_{\sim} : \lambda_i > 0 \}$, see (12), we conclude that $[e_{i_0}]_{\sim} \in [H]_{\sim}$. On the other hand, $1_X \in H$ gives $[1_X]_{\sim} \in [H]_{\sim}$, and thus we find

$$0 = \langle [e_{i_0}]_{\sim}, [1_X]_{\sim} \rangle_{L_2(\nu)} = \int_X e_{i_0} \, d\nu.$$ 

Finally, $[e_{i_0}]_{\sim} \neq 0$ is obvious and $[e_{i_0}]_{\sim} \in L_1(\nu)$ follows from $\nu(X) < \infty$, so that Theorem 3.10 shows that $k$ is not characteristic. \(\square\)

5.4 Proofs related to Section 4.1

**Proof of Lemma 4.1:** To check that $(e_i^+)_{i \in I}$ is an ONS, we first observe that the equivalences $i = j \iff -i = -j$ and $i = -j \iff -i = j$ imply for $a_i, a_j \in \{-1, 1\}$ that

$$\langle [e_i + a_i \bar{e}_i]_{\sim}, [e_j + a_j \bar{e}_j]_{\sim} \rangle_{L_2(G, \mathbb{C})} = \langle [e_i]_{\sim} + a_i [e_{-i}]_{\sim}, [e_j]_{\sim} + a_j [e_{-j}]_{\sim} \rangle_{L_2(G, \mathbb{C})}$$

$$= \langle [e_i]_{\sim}, [e_j]_{\sim} \rangle_{L_2(G, \mathbb{C})} + a_j \langle [e_i]_{\sim}, [e_{-j}]_{\sim} \rangle_{L_2(G, \mathbb{C})} + a_i \langle [e_{-i}]_{\sim}, [e_j]_{\sim} \rangle_{L_2(G, \mathbb{C})} + a_i a_j \langle [e_{-i}]_{\sim}, [e_{-j}]_{\sim} \rangle_{L_2(G, \mathbb{C})}$$

$$= (1 + a_i a_j) \delta_{i,j} + (a_i + a_j) \delta_{i,-j}.$$ 

Let us first consider $i, j \in I_0$. Then we have $i = j$ if and only if $i = -j$, that is $\delta_{i,j} = \delta_{i,-j}$, and thus we find

$$\langle [e^+_i]_{\sim}, [e^+_j]_{\sim} \rangle_{L_2(G, \mathbb{C})} = \frac{1}{4} \langle [e_i + \bar{e}_i]_{\sim}, [e_j + \bar{e}_j]_{\sim} \rangle_{L_2(G, \mathbb{C})} = \delta_{i,j}.$$ 

Similarly, for $i, j \in I_+$, we cannot have $\delta_{i,j} = 1$, since this would imply $i \in I_-$, and hence we obtain

$$\langle [e^+_i]_{\sim}, [e^+_j]_{\sim} \rangle_{L_2(G, \mathbb{C})} = \frac{1}{2} \langle [e_i + \bar{e}_i]_{\sim}, [e_j + \bar{e}_j]_{\sim} \rangle_{L_2(G, \mathbb{C})} = \delta_{i,j},$$

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and for $i, j \in I_-$ we find by an analogous reasoning that
\[
\langle [e_i^*]_\sim, [e_j^*]_\sim \rangle_{L_2(G)} = \frac{1}{2} \langle [e_i - \bar{e}_i]_\sim, [e_j - \bar{e}_j]_\sim \rangle_{L_2(G, \mathbb{C})} = \delta_{i,j} .
\]
For the mixed cases, in which $i$ and $j$ belong to different partition elements $I_0$, $I_+$, or $I_-$ we clearly have $i \neq j$, and hence the above calculation reduces to
\[
\langle [e_i + a_i \bar{e}_i]_\sim, [e_j + a_j \bar{e}_j]_\sim \rangle_{L_2(G, \mathbb{C})} = (a_i + a_j) \delta_{i,j} .
\]
Now, if $i \in I_0$ and $j \notin I_0$, then we cannot have $i = -j$ and hence we obtain $\langle [e_i^*]_\sim, [e_j^*]_\sim \rangle_{L_2(G)} = 0$.

For the same reason we find for $i \in I_+$ and $j \in I_-$ with $i \neq -j$ that $\langle [e_i^*]_\sim, [e_j^*]_\sim \rangle_{L_2(G)} = 0$.

Finally, for $i \in I_+$ and $j \in I_-$ with $i = -j$ we have
\[
\langle [e_i^*]_\sim, [e_j^*]_\sim \rangle_{L_2(G)} = \frac{1}{2} \langle [e_i + \bar{e}_i]_\sim, i : [e_j - \bar{e}_j]_\sim \rangle_{L_2(G, \mathbb{C})} = -i \cdot (1 - 1) \cdot \delta_{i,-j} = 0 ,
\]
and therefore we conclude that $\{ [e_i^*]_\sim \}_{i \in I}$ is an ONS in $L_2(G)$.

Our next goal is to show that span$\{ e_i^* : i \in I \}$ is dense in $C(G)$. To this end, we fix an $f \in C(G)$ and an $\varepsilon > 0$. Since $f \in C(G, \mathbb{C})$ and $(e_i)_{i \in I}$, is dense in $C(G, \mathbb{C})$, there then exists a finite set $J \subset I$ and $(a_j)_{j \in J} \subset \mathbb{C}$ such that
\[
\sup_{x \in G} \left| \sum_{j \in J} a_j e_j(x) - f(x) \right| < \varepsilon ,
\]
and from this conclude that
\[
\sup_{x \in G} \left| \sum_{j \in J} \Re a_j \Re e_j(x) - \sum_{j \in J} \Im a_j \Im e_j(x) - f(x) \right| = \sup_{x \in G} \left| \Re \left( \sum_{j \in J} a_j e_j(x) \right) - \Re f(x) \right|
\]
\[
\leq \sup_{x \in G} \left| \sum_{j \in J} a_j e_j(x) - f(x) \right| < \varepsilon .
\]
In other words, the span of $(\Re e_i)_{i \in I} \cup (\Im e_i)_{i \in I}$ is dense in $C(G)$. However, our initial considerations showed that $\Re e_i = \Re e_{-i}$ and $\Im e_i = -\Im e_{-i}$ for all $i \in I$ as well as $\Im e_i = 0$ for all $i \in I_0$, and therefore span$\{ e_i^* : i \in I \}$ is dense in $C(G)$, too.

Finally, $\nu$ is regular, and therefore $C(G)$ is dense in $L_2(G)$, see e.g. Bauer (2001, Theorem 29.14). Consequently, span$\{ e_i^* : i \in I \}$ is dense in $L_2(G)$, and therefore $(e_i^*)_{i \in I}$ is an ONB of $L_2(G)$. The estimate $\|e_i^*\|_\infty \leq \sqrt{2}$ follows from $|e_i(x)| = 1$ for all $x \in G$.

**Proof of Lemma 4.2:** Let $\nu$ be the Haar measure of $G$.

\( i) \Rightarrow ii) \). For a character $e_i \in \hat{G}$ and $x \in G$ a simple calculation shows
\[
\int_G k(x, x') e_i(x) \, d\nu(x') = \int_G k(-x + x') e_i(x') \, d\nu(x') = \int_G k(x') e_i(x + x') \, d\nu(x') = \lambda_i e_i(x) ,
\]
where $\lambda_i := \int_G k(x') e_i(x') \, d\nu(x')$. Now, since $k$ is $\mathbb{R}$-valued, the integral operator $T_k^C : L_2(G, \mathbb{C}) \to L_2(G, \mathbb{C})$ is self-adjoint and the previous calculation shows that each character $e_i$ gives an eigenvector $[e_i]_\sim \in L_2(G, \mathbb{C})$ of $T_k^C$ with eigenvalue $\lambda_i$. Using that eigenvalues of self-adjoint operators are real numbers, we then find
\[
T_k[\Re e_i]_\sim = T_k^C[\Re e_i]_\sim = \lambda_i[\Re e_i]_\sim \\
T_k[\Im e_i]_\sim = T_k^C[\Im e_i]_\sim = \lambda_i[\Im e_i]_\sim
\]
for all $i \in I$. Now recall that for $i \in I_{0}$ we have $\text{Im} \, e_{i} = 0$ and therefore these functions $\text{Im} \, e_{i}$ are not eigenvectors of $T_{k} : L_{2}(G) \to L_{2}(G)$. By Lemma 4.1 we thus conclude that $(|e_{i}|)_{i \in I}$ is an ONB of eigenvectors of $T_{k}$ with corresponding eigenvalues $(\lambda_{i})_{i \in I}$, and in particular, there are no further eigenvalues than these. Moreover, $(\lambda_{i})_{i \in I}$ is summable since (11) holds. Moreover, for $i \in I$ we have $\lambda_{i} = \lambda_{-i}$, where we note that for $i \notin I_{0}$ the corresponding eigenvalues have thus a geometric multiplicity of at least two. Since $\nu$ is finite and strictly positive, we thus see that $k$ enjoys a Mercer representation (13) for the index set $I^{*} := \{i \in I : \lambda_{i} > 0\}$ and the sub-family $(e_{i})_{i \in I^{*}}$. Moreover, for $x, x' \in G$ this representation yields

$$
k(x, x') = \sum_{\lambda_{i} > 0} \frac{\lambda_{i} e_{i}(x)e_{i}(x')}{\sum_{\lambda_{i} > 0} \lambda_{i}}
= \sum_{i \in I : \lambda_{i} > 0} \lambda_{i} \text{Re} \, e_{i}(x) \text{Re} \, e_{i}(x') + 2 \sum_{i \in I : \lambda_{i} > 0} \lambda_{i} (\text{Re} \, e_{i}(x) \text{Re} \, e_{i}(x') + \text{Im} \, e_{i}(x) \text{Im} \, e_{i}(x'))
= \sum_{i \in I : \lambda_{i} > 0} \lambda_{i} \text{Re} \, e_{i}(-x' + x) + 2 \sum_{i \in I : \lambda_{i} > 0} \lambda_{i} \text{Re} \, e_{i}(-x' + x)
= \sum_{\lambda_{i} > 0} \lambda_{i} \text{Re} \, e_{i}(-x' + x),
$$

where in the second to last step we used (16) and the last step rests on $\text{Re} \, e_{i} = \text{Re} \, e_{-i}$. In addition, $\sup_{i \in I} \|e_{i}^{*}\|_{\infty} \leq \sqrt{2}$ together with the summability of $(\lambda_{i})_{i \in I}$ quickly shows that the series converge both absolutely and uniformly. Finally, the continuity of $k$ follows from the uniform convergence in (18) and $e_{i}^{*} \in C(G)$ for all $i \in I$.

(ii) $\Rightarrow$ (i). We first observe that Lemma 4.1 together with Steinwart and Scovel (2012, Lemma 2.6) shows that (18) does indeed define a kernel $k$, and its translation invariance is built into the construction. Clearly, $k$ is measurable and $\sup_{i \in I} \|e_{i}^{*}\|_{\infty} \leq \sqrt{2}$ together with the summability of $(\lambda_{i})_{i \in I}$ shows that $k$ is bounded.

**Proof of Theorem 4.3:** The equivalences (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) have already been shown in Theorem 3.15, and (iii) $\Rightarrow$ (iv) is trivial. In addition, if one of the conditions (iii) - (vi) is satisfied, Theorem 3.15 shows that $\mathcal{M}^{*}(G) = \mathcal{M}(G)$, where in the case of (v) we additionally need Lemma 4.2. Therefore Lemma 3.5 together with Proposition 2.3 and Theorem 3.14 shows that a continuous kernel $k$ is characteristic if and only if $k + 1$ is universal. This yields the equivalences (iv) $\Leftrightarrow$ (vi), and by the last statement in Lemma 4.2, also (iii) $\Leftrightarrow$ (v). It thus suffices to show that (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). If $\hat{G}$ is at most countable, so is $I$, and hence there exists a family $(\lambda_{i})_{i \in I}$ with $\lambda_{i} > 0$ for all $i \in I$ and $\sum_{i \in I} \lambda_{i} < \infty$. The kernel $k$, which is constructed by (18), is then translation-invariant and continuous, and by Corollary 3.18 it is also universal.

(i) $\Rightarrow$ (ii). By Schurle (1979, Theorem 3.2.11 and Corollary 3.3.2) we know that $G$ is completely regular and hence Conway (1990, Theorem V.6.6.) shows that $G$ is metrizable if and only if $C(G)$ is separable. We therefore see that $C(G)$ is separable. In addition, since $\nu$ is regular, Bauer (2001, Theorem 29.14) shows that $C(G)$ is dense in $L_{2}(G)$. Consequently, $L_{2}(G)$ is separable, and since $(e_{i})_{i \in I}$ is an ONB of $L_{2}(G)$, we conclude that $I$, and thus $\hat{G}$, is at most countable.

**Proof of Corollary 4.4:** Since the Haar measure $\nu$ on $G$ is a finite, regular, and strictly positive Borel measure, we see by Lemma 4.1 that all assumptions of Corollary 3.18 are satisfied. Moreover, we have $e_{*}^{*} = 1_{G}$. Now the assertions follow from Corollary 3.18.
5.5 Proofs related to Section 4.2

Proof of Proof of Theorem 4.5: We first consider the case $\psi \in \Psi_{d+2}$. If the kernel $k$ on $\mathbb{S}^d$ induced by $\psi$ is strictly positive definite then Lemma 4.9 implies that $b_{n,d} > 0$ for all $n \geq 0$. By Theorem 4.11, $k$ is then universal and characteristic. Conversely, if $k$ is universal or characteristic then $b_{n,d} > 0$ for all $n \geq 1$ by Theorem 4.11, thus it is strictly positive definite by Remark 4.8.

In the case $\psi \in \Psi_{d+1}^+$, we have $\psi \in \Psi_d^+$, and hence it suffices to show that $k$ is universal and characteristic. This, however, follows from Lemma 4.10 and Theorem 4.11. \qed

Proof of Theorem 4.6: Suppose that the kernel $k$ is induced by $\psi \in \Psi_\infty^+$. Then $b_{n,d} > 0$ for all $n \in \mathbb{N}_0$ by Lemma 4.10 and hence the kernel is universal and characteristic by Theorem 4.11. Suppose now that $k$ is characteristic. By Proposition 4.12 we obtain that $\psi \in \Psi_\infty^+$. \qed

Proof of Lemma 4.9: It is clear from the results of (Chen et al., 2003) that $b_{n,d} > 0$ for all $n \in \mathbb{N}_0$ is a sufficient condition for $\psi$ being strictly positive definite. Suppose now that $\psi \in \Psi_{d+2} \cap \Psi_d^+$. Gneiting (2013, Corollary 4) implies that if $b_{2k+2,d} > 0$ ($b_{2k+1,d} > 0$) for some $k$, then $b_{2k'+2,d} > 0$ ($b_{2k'+1,d} > 0$) for all $k' \leq k$. This yields the claim by Remark 4.8. \qed

Proof of Lemma 4.10: This is shown in Gneiting (2013, Proof of Corollary 1(b)). \qed

Proof of Proposition 4.12: Assume that $\psi$ is not strictly positive definite, or, by Remark 4.8, does not satisfy condition $b$. We will show that it cannot be characteristic.

First, we construct a special class of probability densities $p$ on $\mathbb{S}^d$ such that we explicitly know the integrals
\[
c_{k,j}(p) := \int_{\mathbb{S}^d} e_{k,j}(y)p(y) \, d\sigma(y)
\] (22)
with respect to the basis of spherical harmonics. Here, $\sigma$ is the surface area measure on $\mathbb{S}^d$ normalized such that $\int_{\mathbb{S}^d} d\sigma = 1$. Fix $v_0 \in \mathbb{S}^d$ and $a \in [-1,1] \setminus \{0\}$. We have
\[
|C_n^{(d-1)/2}(x)| \leq C_n^{(d-1)/2}(1), \quad x \in [-1,1],
\]
see (DLMF, 18.14.4 for $d \geq 2$), and therefore, for all $n \geq 1$, $x \in \mathbb{S}^d$, we have
\[
p_{n,a}(x) := 1 + \frac{a}{2} \frac{C_n^{(d-1)/2}(v_0,x)}{C_n^{(d-1)/2}(1)} \geq 0,
\] (23)
and $\int_{\mathbb{S}^d} p_{n,a}(x) \, d\sigma(x) = 1$, where for the last equality we used that $C_n^{(d-1)/2}(v_0,\cdot)$ is a spherical harmonics of degree $n$ and thus orthogonal to $e_{0,0} = 1_{\mathbb{S}^d}$. Consequently, $p_{n,a}$ is a probability density function on $\mathbb{S}^d$ with respect to the surface area measure $\sigma$. Note that $p_{n,a}$ and $p_{n',a}$ induce different probability measures on $\mathbb{S}^d$ for $n \neq n'$. We obtain
\[
c_{k,j}(p_{n,a}) = \delta_{k,0} + \delta_{k,n} \frac{a}{N(d,k)} e_{k,j}(v_0)
\] (24)
using (21), and the Funk-Hecke Theorem (Groemer, 1996, Theorem 3.4.1) yields that
\[
\int_{\mathbb{S}^d} (x,y)^n e_{k,j}(y) \, d\sigma(y) = \lambda_k^n e_{k,j}(x), \quad x \in \mathbb{S}^d, j = 1, \ldots, N(d,k),
\]
where
\[
\lambda_k^n = \frac{\Gamma((d+1)/2)}{\sqrt{\pi} \Gamma(d/2)} C_k^{(d-1)/2}(1)^{-1} \int_{-1}^{1} t^n C_k^{(d-1)/2}(t)(1 - t^2)^{(d-2)/2} \, dt.
\]
Since the family \((e_{n,j})_{n\in\mathbb{N}_0, j=1,...,N(d,k)}\) is an ONB of \(L_2(\mathbb{S}^d)\), we obtain the following Mercer representation of the bounded and continuous kernel \((x,y)\mapsto (x,y)^n\) on \(\mathbb{S}^d\)

\[
\langle x, y \rangle^n = \sum_{k=0}^{\infty} \sum_{j=1}^{N(d,k)} \lambda^n_{k,j} e_{k,j}(x)e_{k,j}(y),
\]

where for each \(x\) the convergence is uniform in \(y\) by Steinwart and Scovel (2012, Corollary 3.5).

Using that \(C_k^{(d-1)/2}\) is even for \(k\) even and odd for \(k\) odd, one obtains that \(\lambda^n_k = 0\) if \(k - n\) is odd. The \(C_k^{(d-1)/2}\) are orthogonal with respect to the weight function \((1 - t^2)^{(d-2)/2}\) DLMF, 18.3.1, therefore \(\lambda^n_k = 0\) for \(k > n\). Finally, the formula DLMF, 18.17.37 for the Mellin transform yields that

\[
\lambda^n_k = \frac{\pi 2^{d-n-1} \Gamma(k + d -1) \Gamma(n+1)}{k! \Gamma(\frac{d-1}{2}) \Gamma(\frac{k + d + n}{2}) \Gamma(\frac{n-k+2}{2})} > 0 \quad k \leq n, \quad k - n \text{ even}.
\]

For a probability density \(p\) on \(\mathbb{S}^d\), we have by (20) and (25) for \(x \in \mathbb{S}^d\),

\[
\int_{\mathbb{S}^d} k(x,y)p(y) \, d\sigma(y) = \sum_{n=0}^{\infty} \sum_{k=0}^{N(d,k)} \sum_{j=1}^{b_n \lambda^n_k} e_{k,j}(x)c_{k,j}(p)
\]

\[
= \sum_{k=0}^{\infty} \sum_{j=1}^{N(d,k)} z_k c_{k,j}(p)e_{k,j}(x),
\]

where \(z_k = \sum_{n=0}^{\infty} b_n \lambda^n_k\) and \(c_{k,j}(p)\) is defined at (22). If \(b_n = 0\) for all even \(n \geq n_0\) then \(z_k = 0\) for all even \(k \geq n_0\). If \(b_n = 0\) for all odd \(n \geq n_0\) then \(z_k = 0\) for all odd \(k \geq n_0\).

Let us start with the case that \(b_n = 0\) for all even \(n \geq n_0\), i.e. \(z_k = 0\) for all even \(k \geq n_0\). For all \(m \in \mathbb{N}_0\), we have \(c_{k,j}(p_{2m,a}) = 0\) for \(k\) odd by (24), where \(p_{n,a}\) is defined at (23). Hence, for \(2m \geq n_0\) and \(x \in \mathbb{S}^d\), we obtain

\[
\int_{\mathbb{S}^d} k(x,y)p_{2m,a}(y) \, d\sigma(y)
\]

\[
= \sum_{k=0}^{n_0} \sum_{j=1}^{N(d,k)} z_k \left(\delta_{k,0} + \delta_{k,2m} \frac{a}{N(d,k)} e_{k,j}(v_0)\right) e_{k,j}(x) = z_0,
\]

which shows that the kernel mean embedding maps all these densities to the constant function with value \(z_0\). Consequently, \(k\) is not characteristic.

Suppose now that \(b_n = 0\) for all odd \(n \geq n_0\), i.e. \(z_k = 0\) for all odd \(k \geq n_0\). For all \(m \in \mathbb{N}_0\), we have \(c_{k,j}(p_{2m+1,a}) = \delta_{k,0}\) for \(k\) even by (24). Hence, for \(2m+1 \geq n_0\) and \(x \in \mathbb{S}^d\), we obtain

\[
\int_{\mathbb{S}^d} k(x,y)p_{2m+1,a}(y) \, d\sigma(y)
\]

\[
= z_0 + \sum_{k=1,k \text{ odd}}^{n_0} \sum_{j=1}^{N(d,k)} z_k \left(\delta_{k,2m+1} \frac{a}{N(d,k)} e_{k,j}(v_0)\right) e_{k,j}(x) = z_0,
\]

which again shows that the kernel mean embedding maps all these densities to the constant function with value \(z_0\). Consequently, \(k\) is not characteristic.
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