Dynamics of two planets in co-orbital motion

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Accepted 2010 April 22. Received 2010 March 31; in original form 2010 February 9

ABSTRACT

We study the stability regions and families of periodic orbits of two planets locked in a co-orbital configuration. We consider different ratios of planetary masses and orbital eccentricities; we also assume that both planets share the same orbital plane. Initially, we perform numerical simulations over a grid of osculating initial conditions to map the regions of stable/chaotic motion and identify equilibrium solutions. These results are later analysed in more detail using a semi-analytical model.

Apart from the well-known quasi-satellite orbits and the classical equilibrium Lagrangian points L₄ and L₅, we also find a new regime of asymmetric periodic solutions. For low eccentricities these are located at (Δλ, Δσ) = (±60°, ±120°), where Δλ is the difference in mean longitudes and Δσ is the difference in longitudes of pericentre. The position of these anti-Lagrangian solutions changes with the mass ratio and the orbital eccentricities and are found for eccentricities as high as ~0.7.

Finally, we also applied a slow mass variation to one of the planets and analysed its effect on an initially asymmetric periodic orbit. We found that the resonant solution is preserved as long as the mass variation is adiabatic, with practically no change in the equilibrium values of the angles.

Key words: methods: analytical – methods: numerical – celestial mechanics – planets and satellites: general – planetary systems.

1 INTRODUCTION

In the restricted three-body problem, there are different domains of stable motion associated with co-orbital motion. Each can be classified according to the centre of libration of the critical argument, σ = λ − λ′, where λ denotes the mean longitude of the minor body and λ′ the same variable for the disturbing planet. These types of motion are known as (i) tadpole orbits, corresponding to a libration of σ around L₄ or L₅; (ii) horseshoe orbits, where motion occurs around σ = 180° and encompasses both equilateral Lagrangian points; and (iii) quasi-satellite (QS) orbits, where σ oscillates around zero.

The term ‘quasi-satellites’ was originally introduced by Mikkola & Innanen (1997) and can be viewed as an extension of retrograde periodic orbits in the circular restricted three-body problem (e.g. Jackson 1913; Hénon 1969). Although not present for circular orbits, they exist for moderate to high eccentricities of the particle. In a reference frame rotating with the planet, QS orbits circle the planet like a retrograde satellite, although at distances so large that the particle is not gravitationally bounded to the planetary mass (Mikkola et al. 2006).

The first object confirmed in a QS configuration was the asteroid 2002 VE68 (Mikkola et al. 2004) with Venus as the host planet. The Earth has one temporary co-orbital object, (3753 Cruithne; Namouni 1999) and one alternating horseshoe–QS object (2002 AA29; Connors et al. 2002). The co-orbital asteroidal population in the inner Solar system was studied in Brasser et al. (2004) by numerical integrations. All QS orbits appear to be temporary, escaping in time-scales of the order of 10⁶–10⁷ yr.

Wiegert, Innanen & Mikkola (2000) numerically investigated the stability of QS orbits around the giant planets of the Solar system. Although no stable solutions were found for Jupiter and Saturn, some initial conditions around Uranus and Neptune lead to QS orbits that survive for time-scales of the order of 10⁷ yr. It thus appears that a primordial population of such objects may still exist in the Solar system. Kortenkamp (2005) used N-body simulations to model the combined effects of solar nebula gas drag and gravitational scattering of planetesimals by a protoplanet. He showed that a significant fraction of scattered planetesimals could become trapped into QS trajectories. It then seems plausible that this trapped-to-captured transition may be important not only for the origin of captured satellites but also for continued growth of protoplanets.

At variance with these results, in the case of the general (non-restricted) three-body problem, although equilateral solutions and horseshoe orbits are well known, QS configurations have only been studied very recently. Hadjidemetriou, Psychoyos & Voyatzis (2009) performed a detailed study of periodic orbits in the 1/1 mean-motion resonance (MMR) for fictitious planetary systems.
with different mass ratios. They found that stable QS solutions occur for \( \sigma = \Delta \lambda = \lambda_2 - \lambda_1 = 0^\circ \) and \( \Delta \sigma = \sigma_2 - \sigma_1 = 180^\circ \), where the subscripts identify each planet. Unstable trajectories were found at \( \sigma = 180^\circ \), \( \Delta \sigma = 0^\circ \). Although at present there are no confirmed cases of exoplanets in QS configurations, Goździewski & Konacki (2006) found that the radial velocity curves of the HD 82943 and HD 128311 planets could correspond to co-orbital motion in highly inclined orbits. Numerical simulations of both systems show QS trajectories, instead of Trojan orbits as initially believed.

In this work we aim to revisit the 1/1 MMR in the planar planetary three-body problem, trying to identify possible domains of stable solutions and their location in the phase space. Section 2 presents several dynamical maps constructed from numerical simulations for different initial conditions. These maps allow us to identify stable fixed points and periodic orbits as well as the domains of regular motions. In Section 3 we develop a semi-analytical model for co-orbital planets, which is then applied in Section 4 to calculate the families of stable periodic orbits. In the same section, we also present a brief study of the effects of an adiabatically slow mass variation in one of the planetary bodies. Finally, conclusions close the paper in Section 5.

2 DYNAMICAL MAPS WITH EQUAL-MASS PLANETS

Consider two planets with masses \( m_1 \) and \( m_2 \) in coplanar orbits around a star with mass \( m_0 = M_{\odot} \). We will begin considering the case \( m_2 = m_1 \); other mass ratios will also be discussed in later sections. Let \( a_1 \) denote the semimajor axes, \( e_1 \) the eccentricities, \( \lambda_1 \) the mean longitudes and \( \sigma_1 \) the longitudes of pericentre. All orbital elements considered in this paper are assumed astrocentric and osculating. Throughout this work, \( m_1 \) will be our ‘reference’ planet: its mass will be fixed at one Jovian mass \( (m_1 = M_{\text{Jup}}) \) and the system scaled to initial condition \( a_1 = 1 \) au. The angular variables for co-orbital motion will then be defined as \( \sigma = \lambda_2 - \lambda_1 \) and \( \Delta \sigma = \sigma_2 - \sigma_1 \).

As pointed out by Hadjidemetriou et al. (2009), for equal-mass planets the periodic orbits are such that are located at \( a_1 = a_2 \) and \( e_1 = e_2 \). Accordingly, we fixed the semimajor axes and eccentricities and constructed a 100 × 100 grid of initial conditions varying both \( \sigma \) and \( \Delta \sigma \) between 0° and 360°. Each point in the grid was then numerically integrated over 3000 orbital periods using a Bulirsch–Stoer-based N-body code, and we calculated the averaged MEGNO chaos indicator \( \langle Y \rangle \) (Cincotta & Simó 2000) to identify regions of regular or chaotic motion. Results are shown in Fig. 1 for six values of the initial eccentricities \( e_1 \); dashed regions correspond to unstable orbits while white was used to identify stable solutions. An analysis of these plots shows the following characteristics.

(i) For low initial eccentricities \( (e_1 = 0.05) \) the maps show two disconnected strips of regular motion, corresponding to motion around \( \sigma = \pm 60^\circ \) and any value of \( \Delta \sigma \).

(ii) For moderate low to intermediate initial eccentricities \( (e_1 = 0.15 \text{ and } e_1 = 0.30) \), the vertical strips of regular motion become thinner and slightly distorted. A new stable domain is now present, associated with QS orbits and located around \( \sigma = 0^\circ \).

(iii) For high initial eccentricities \( (e_1 \geq 0.40) \), the domain of QS orbits increases and covers a significant portion of the plane of initial conditions. Conversely, the distorted vertical strips shrink and each seems to break into two islands of stable motion. The smaller islands encompass equilateral solutions, although they almost disappear for \( e_1 = 0.70 \). The larger islands correspond to a different type of asymmetric solution, and their locations tend towards the centre of the plots as the eccentricities increase.

(iv) Due to symmetry present in the dynamical system, the results are invariant to transformations of the type \( (\sigma, \Delta \sigma) \rightarrow (-\sigma, -\Delta \sigma) \). In fact, since \( m_1 = m_2 \), both equilateral solutions are actually the same solution, since we can pass from one to the other just by redefining the reference planet. However, since later sections will discuss the case \( m_2 \neq m_1 \), we prefer to treat both equilateral solutions separately.

Although MEGNO is a very efficient tool to identify chaotic motion, it is not suited to distinguish between different types of regular orbits (e.g. fixed points, periodic orbits, etc.). Sometimes this task is performed with a Fourier transform of the numerical data (e.g. Michtchenko, Beaugé & Ferraz-Mello 2008a,b); however, here we have chosen a different route. Starting from the output of each numerical simulation, we calculated the amplitudes of oscillation in each angular variable. Initial conditions with zero amplitude in \( \sigma \) correspond to \( \sigma \)-family periodic orbits of the co-orbital system, while solutions with zero amplitude in \( \Delta \sigma \) will correspond to periodic orbits of the so-called \( \Delta \sigma \)-family (see Michtchenko et al. 2008a,b). Finally, stationary solutions of the averaged problem, identified as intersections of both families, may be thought

![Figure 1. Results of numerical integrations of initial conditions in a grid in the \((\sigma, \Delta \sigma)\) plane. Planetary masses were taken equal to \( m_1 = m_2 = M_{\text{Jup}} \) and initial semimajor axes equal to \( a_1 = a_2 = 1 \) au. Regions of regular motion are shown in white, while the dashed regions correspond to chaotic and unstable trajectories.](https://academic.oup.com/mnras/article-abstract/407/1/390/985383)
as analogous to the apsidal corotation resonances (ACR) found in other MMR (e.g. Beauç, Ferraz-Mello & Michtchenko 2003). The equilateral Lagrangian solutions will appear as ACR in these plots.

The grey-scale graphs in Fig. 2 show values of the amplitudes in \( \sigma \) (left) and \( \Delta \sigma \) (right) for four of the plots shown in Fig. 1. White regions represent initial conditions with semi-amplitudes smaller than 2° and thus indicate the families of periodic orbits in each angle. Darker regions correspond to increasing amplitudes (up to 45°) and denote initial conditions with quasi-periodic motion. The dashed areas are unstable solutions. Finally, it is worthwhile mentioning that symmetric configurations may correspond to either an alignment (\( \Delta \sigma = 0° \)) or an anti-alignment of the apses (\( \Delta \sigma = \pm 180° \)) while asymmetric configurations have stationary values of \( \Delta \sigma \) different from the above.

For low eccentricities (\( e_1 = 0.05 \)), we observe four asymmetric ACR solutions. Two are the well-known Lagrangian equilateral solutions located at \( (\sigma, \Delta \sigma) = (\pm 60°, \pm 60°) \). By analogy with the restricted problem, we will denote them \( L_3 \) and \( L_4 \). As far as we know, the remaining two ACR have not been previously reported and are located at approximately \( (\sigma, \Delta \sigma) = (\pm 60°, \mp 120°) \). We have called them anti-Lagrangian solutions and they are connected to the classical equilateral Lagrangian solutions by the \( \sigma \)-family of periodic orbits. By analogy, we have denoted the new solutions as

\[
AL_4: \sigma \in [0, 180°] \quad \Delta \sigma \in [180°, 360°] \\
AL_5: \sigma \in [180°, 360°] \quad \Delta \sigma \in [0°, 180°] .
\]

As with all previous stationary solutions, these asymmetric points are found at \( a_1 = a_2 \).

As the eccentricities grow (e.g. \( e_1 = 0.40 \)), the QS region at \( (\sigma, \Delta \sigma) = (0°, 180°) \) causes a distortion and compression of the stable asymmetric domain. The anti-Lagrangian zone seems less affected and surrounded by a larger island of stable motion. This effect is even more pronounced for \( e_1 = 0.60 \) and \( e_1 = 0.70 \) where the stable domain around \( L_4 \) and \( L_5 \) almost disappears. The regions around \( AL_4 \) and \( AL_5 \) are still visible, although they also decrease in size and their location approaches the unstable symmetric periodic orbit located at \( \sigma = \Delta \sigma = 180° \).

The decrease in the size of the stable regions around the asymmetric ACR solutions is accompanied by a significant increase in the stable domains around QS orbits, which, for high eccentricities, seem to cover a large proportion of the plane. Inside this region, we also note two families of periodic orbits: the \( \sigma \)-family which is restricted to a small region around \( \Delta \sigma = 180° \) and a smaller \( \sigma \)-family close to the zero value of the resonant angle.

Table 1 summarizes the detected stable stationary solutions in the planar planetary three-body problem, as well as their location in the plane of angular variables for low eccentricities.

### Table 1. Approximate location for the stable ACR solutions in the \((\sigma, \Delta \sigma)\) plane, calculated from the dynamical maps with \( e_1 = e_2 = 0.15 \). For equal-mass planets, all stationary solutions occur for \( a_1 = a_2 \).

| \( \sigma \) (deg) | \( \Delta \sigma \) (deg) |
|-------------------|-------------------|
| QS                | 0                 | 180               |
| \( L_4 \)         | 60                | 60                |
| \( L_5 \)         | 300               | 300               |
| \( AL_4 \)        | 60                | 240               |
| \( AL_5 \)        | 300               | 120               |

2.1 Motion around the stationary solutions

In order to visualize the dynamics of stable orbits outside the ACR, we integrated several orbits with initial elements \( a_1 = a_2 = 1 \) au, \( e_1 = e_2 = 0.4, \sigma = 0 \) and different values of \( \Delta \sigma \). Each initial condition was chosen along line A drawn in Fig. 1 for \( e_1 = 0.40 \). Results are shown in Fig. 3. The left-hand frame shows the orbital evolution in the \((e_2, \Delta \sigma)\) plane, while the right-hand frame presents the variation of \((e_2, \sigma)\). In both cases, the numerical output was
and AL will show a small-amplitude circulation around the corre-
roughly at $/Delta_1 \varpi$ without reaching the fixed points. Fi-
tation of each solution in $(\sigma_4, \Delta \sigma)$ plane (left-hand frame) shows two centres of os-
45 $^\circ$. As before, we see a smooth transition in the dynamical be-
and AL, the minimum distance coincides with the initial condition.

same as in the previous plots. The initial values of $\Delta \sigma$ were varied from 0$^\circ$ to 360$^\circ$, and in each case $\sigma$ was chosen along line B in Fig. 1 for $e_i = 0.4$ ($\sigma$-family).

The $(\sigma_2, \Delta \sigma)$ plane (left-hand frame) shows two centres of osci-
correspond to the stable ACR solution discussed previously (QS, $L_4$, and $L_5$). Each of the other plots shows the orbital representation of each solution in $(x, y)$ astrocentric Cartesian coordinates.
3 SEMI-ANALYTICAL MODEL

One drawback in the previous numerical approach is the excessive CPU time required for the construction of each dynamical map. In order to extend these results to other values of the parameter space (e.g. planetary masses, eccentricities), it is useful to construct a semi-analytical model for the co-orbital motion.

Such a model can be developed along similar lines to other MMR (e.g. Michtchenko, Beaugé & Ferraz-Mello 2006; Michtchenko et al. 2008a,b). It requires two main steps: first, a transformation to adequate resonant variables and, secondly, a numerical averaging of the Hamiltonian with respect to short-period terms. Both tasks are detailed below.

We begin introducing the usual mass-weighted Poincaré canonical variables (e.g Laskar 1990) for each planet $m_i$: 

\[ \lambda_i = m_i \sqrt{m_i \sigma_i} \]

\[ \sigma_i = G_i - L_i = -L_i \left(1 - \sqrt{1 - \sigma_i^2}\right), \]

where $\mu_i = \kappa^2(m_0 + m_i)$, $\kappa$ denotes the gravitational constant and $m_i$ is the reduced mass of each body, given by

\[ m_i = \frac{m_i m_0}{m_0 + m_i}. \]

The Hamiltonian function $F$ can be expressed as $F = F_0 + F_1$, where $F_0$ corresponds to the two-body contribution and has the form

\[ F_0 = -\sum_{i=1}^{3} \frac{\mu_i^2 m_i^3}{2L_i^2}. \]

The second term, $F_1$, is the disturbing function which can be written as

\[ F_1 = -\kappa^2 m_2 m_3 \frac{1}{\Delta} + T_1, \]

where $\Delta$ is the instantaneous distance between the two planets and $T_1$ is the indirect part of the potential energy of the gravitational interaction (see Laskar 1990; Laskar & Robutel 1995 for more details).

For initial conditions in the vicinity of co-orbital motion, we define the following set of planar resonant canonical variables $(I_1, I_2, \kappa, \Delta \sigma, \sigma, \sigma_2, Q, q)$, where

\[ \sigma = \lambda_2 - \lambda_1; \]

\[ I_1 = \frac{1}{2} (L_2 - L_1); \]

\[ \Delta \sigma = \sigma_2 - \sigma_1; \]

\[ I_2 = \frac{1}{2} (G_2 - G_1 - L_2 + L_1); \]

\[ q = \sigma_2 + \sigma_1; \]

\[ J_1 = \frac{1}{2} (G_1 + G_2); \]

\[ Q = \lambda_1 + \lambda_2 - q; \]

\[ J_2 = \frac{1}{2} (L_1 + L_2). \]

where $J_1 = \frac{1}{2} \mathcal{A} \mathcal{M}$ and $J_2 = \frac{1}{2} \kappa$. A generic action $\varphi$ of the disturbing function can be written as

\[ \varphi = j_1 \lambda_1 + j_2 \lambda_2 + j_3 \sigma_1 + j_4 \sigma_2, \]

where $j_k$ are integers. In terms of the new angles, the same argument may be written as

\[ \varphi = \frac{1}{2} \left((j_2 - j_1) \sigma + (j_4 - j_3) \Delta \sigma + (j_1 + j_2) Q\right). \]

Since $\sigma$ is a cyclic angle, the associated action $\mathcal{A} \mathcal{M}$ is a constant of motion (total angular momentum) of the system.

The next step is an averaging of the Hamiltonian over the fast angle $Q$. This procedure can be performed numerically, allowing us to evaluate the averaged Hamiltonian $\bar{F}$ as

\[ \bar{F}(I_1, I_2, \sigma, \Delta \sigma; K, \mathcal{A} \mathcal{M}) = \frac{1}{2\pi} \int_0^{2\pi} F dQ. \]

In the averaged system, $K$ is a new integral of motion which, in analogy to other MMR (e.g. Michtchenko et al. 2008a), we call the scaling parameter.

$\bar{F}$ then constitutes a system with 2 degrees of freedom in the canonical variables $(I_1, I_2, \sigma, \Delta \sigma)$, parametrized by the values of both $K$ and $\mathcal{A} \mathcal{M}$. Since the numerical integration depicted in equation (9) is equivalent to a first-order averaging of the Hamiltonian function (e.g. Ferraz-Mello 2007), only those periodic terms (7) with $j_1 + j_2 = 0$ remain in $\bar{F}$. In consequence, we can rewrite the generic resonant argument of the averaged system as

\[ \varphi = j_2 \sigma + j_3 \Delta \sigma, \]

where the index $j_2, j_3$ are integers that may take any value in the interval $(-\infty, \infty)$.

4 FAMILIES OF PERIODIC ORBITS

In the averaged system defined by $\bar{F}$ exact zero-amplitude ACR solutions are given by the stationary conditions

\[ \frac{\partial \bar{F}}{\partial \sigma} = \frac{\partial \bar{F}}{\partial \Delta \sigma} = \frac{\partial \bar{F}}{\partial I_1} = \frac{\partial \bar{F}}{\partial I_2} = 0 \]

and can therefore be identified as extrema of the averaged Hamiltonian function. In this section, we will use this approach to estimate the families of different ACR as a function of the planetary masses and eccentricities and compare the results with numerical integrations of the exact equations of motion.

4.1 Families of symmetric ACR. QS

We begin calculating the exact stationary solutions, corresponding to QS configurations, as a function of the eccentricities and for different values of the planetary masses. As mentioned in Hadjidemetriou et al. (2009), the locations and stability of the ACR do not appear to be dependent on the individual values of the masses, but only on their ratio $m_2/m_1$.

In all cases, the stationary values of the canonical momenta $L_i$ are such that $n_1 = n_2$, where $n_i$ are the mean motions of the planets. For equal-mass planets, this reduces to the condition $a_1 = a_2$. Finally, the angles of the exact ACR always remain locked at $(\sigma, \Delta \sigma) = (0^\circ, 180^\circ)$. Hadjidemetriou et al. (2009) presented similar plots for the same mass ratios.

Fig. 6 shows the families of stable zero-amplitude QS orbits for selected mass ratios: $m_2/m_1 = 1/3$. All solutions occur for $m_2/m_1 = 1$ and $m_2/m_1 > 1$. For $m_2/m_1$, all solutions occur for and $e_1 = e_2$. Due to the intrinsic symmetry of the dynamical system, the family of stationary solutions for $m_2/m_1 = 1/3$ is a mirror image of the solution for $m_2/m_1 = 3$, since it may be obtained by simply interchanging $e_1$ with $e_2$. In the case of $m_2/m_1 = 3$, we note that $e_2 < e_1$ for $e_2 < 0.565$, while $e_2 > e_1$ for more elliptic orbits.
Fig. 6 also shows the solutions for $m_2/m_1 = 2, 5, 20$ mass ratios. For mass ratios smaller than unity, the solutions are mirror images with respect to the family $m_2/m_1 = 1$. Note that the families of stable solutions approach $e_1 = e_2$ as $m_2 \rightarrow m_1$. However, as the mass ratio tends towards the restricted three-body problem, the eccentricity of the smaller mass approaches unity. Finally, the solution $e_1 = 0.565$ is common to all the QS families and corresponds to a global extrema of the Hamiltonian in this plane. A similar structure was already noted by Michtchenko et al. (2006) for other MMR.

4.2 Families of asymmetric ACR solutions. $L_4$ and $AL_4$

The same procedure can also be applied to the Lagrangian $L_4$ and anti-Lagrangian $AL_4$ configurations. Recall that the dynamical maps (Fig. 2) showed a symmetry with respect to the transformation $(\sigma, \Delta \sigma) \rightarrow (-\sigma, -\Delta \sigma)$, so the results discussed here can also be applied to the $L_4$ and $AL_4$ solutions, by applying the same operation on the variables.

The ACR solution associated with the Lagrangian solution $L_4$ shows no variation in the angles, maintaining constant both angles at $60^\circ$. The solutions remain stable for initial conditions up to eccentricities $e_1 = 0.7$. However, $AL_4$ shows significant changes as a function of the eccentricities. Fig. 7 shows the equilibrium values of both angles for the family of $AL_4$, as a function of the eccentricity of the smallest planet, for several values of the mass ratio $m_2/m_1$. The resonant angle $\sigma$ increases monotonically from $60^\circ$, at quasi-circular orbits, towards $\sim 180^\circ$ for near-parabolic trajectories. As the mass ratio increases, the maximum value of the resonant angle decreases, reaching $\sigma = 150^\circ$ for a mass ratio of $m_2/m_1 = 10$.

The secular angle $\Delta \sigma$ shows a slightly more complex behaviour. Initially it increases from $\sim 240^\circ$ until it reaches a maximum value close to $\sim 260^\circ$, after which it once again decreases towards $\Delta \sigma \sim 180^\circ$. The planetary eccentricity corresponding to the maximum in the secular angle increases with the mass ratio, approaching the parabolic limit for $m_2/m_1 \sim 10$.

As shown in Figs 1 and 2, the size of the stable region around each asymmetric solution decreases with the increase of $e_i$ and practically disappears as the angles approach $180^\circ$. For quasi-parabolic orbits, only the region around $AL_4$ is discernible. Thus, for high eccentricity planets in co-orbital motion, it appears that the $AL_4$ and $AL_5$ asymmetric solutions are more regular than the classical equilibrium Lagrangian solutions $L_4$ and $L_5$.

The values of the planetary eccentricities at $AL_4$ for different mass ratios are presented in Fig. 8. Contrary to the QS trajectories, there appears to be a purely linear dependence between $e_2$ and $e_1$ as a function of the mass ratio. In fact, a simple numerical analysis of the results appears to indicate that

$$e_1 \simeq \left( \frac{m_2}{m_1} \right) e_2.$$  \hspace{1cm} (12)
Thus, for mass ratios approaching the restricted three-body problem (with $m_2 \rightarrow 0$) it should be expected that the eccentricity of the massive planet $m_1$ at the $AL_4$ solution would tend towards zero.

Finally, the equilibrium values of the semimajor axes also change as a function of the mass ratio. Here, however, it is easy to see from the stationary conditions (11) that a zero-amplitude $AL_4$ trajectory is characterized by the relation $n_1 = n_2$. For equal-mass planets, this reduces to $a_1 = a_2$.

The families of stationary solutions presented in this section were calculated using our semi-analytical model. In order to compare them with actual numerical simulations of the exact equations, we choose four solutions from Fig. 7 with $e_1 = 0.2$, but corresponding to different mass ratios. Each was then numerically integrated for several orbital periods, assuming zero initial values for the cyclic angular variables $q$ and $Q$. Results are shown in Fig. 9, where the top frame presents the trajectories in the plane ($e_1 \cos \sigma, e_1 \sin \sigma$) and the bottom frame in the plane ($e_2 \cos \Delta \sigma, e_2 \sin \Delta \sigma$). Each initial condition shows a small amplitude oscillation around the stationary value, which presents a very good agreement with the family of $AL_4$ solutions calculated with our model (black curve).

4.3 Adiabatic mass variation in $AL_4$

As a final analysis, in this section we study the orbital evolution of a system initially near $AL_4$, when the mass of one of the planets is decreased adiabatically. This question is raised for three reasons. First, as shown by Lee (2004), for two planets in a 2/1 MMR, a sufficiently slow change in one of the masses will preserve the resonant configuration and allow us to calculate the variation of the ACR as a function of $m_2/m_1$. In other words, this approach provides a different numerical test of our semi-analytical model and an alternative way to calculate the stationary orbits. Secondly, the results will also allow us to test the robustness of the new asymmetric co-orbital solutions $AL_4$ and see how they respond to changes in the parameters of the system. Finally, we wish to analyse the behaviour of these new solutions in the limit of the restricted three-body problem, corresponding to $m_2 = 0$.

Fig. 10 shows a typical example. Initial conditions correspond to an $AL_4$ solution for $m_2/m_1 = 1$ and $e_1 = 0.2$. While $m_1$ was maintained fixed, $m_2$ was varied linearly down to $m_2 = 0$ in a time-scale of $10^6$ orbital periods. We checked using other timescales, finding no significant variations. This guarantees that we are effectively in the adiabatic regime.

The top graph of Fig. 10 shows the evolution of the orbital eccentricities as a function of the mass ratio. As soon as $m_2/m_1$ departs from unity, the value of $e_2$ increases while $e_1$ decreases. The broken black curve that can be seen over the continuous curve shows the predicted value of $e_1$ applying relation (12) to each value of $e_2$. The agreement is excellent, giving an additional corroboration to this empirical relationship between the eccentricities. It must be noted that neither the total angular momentum $\mathcal{AM}$ nor the scaling parameter $K$ is preserved during the mass change. The bottom plot of Fig. 10 shows the behaviour of the angular values during the mass variation. The equilibrium values of both $\sigma$ and $\Delta \sigma$ remain practically unchanged.
5 CONCLUSIONS

We studied the stability regions and families of periodic orbits of two-planet systems in the vicinity of a 1/1 MMR (i.e. co-orbital configuration). We considered different ratios of planetary masses and orbital eccentricities; we also assumed that both planets share the same orbital plane (coplanar motion).

As a result we identified two separate regions of stability as follows, each with two distinct modes of motion.

(i) **QS region.** Originally identified by Hadjidemetriou et al. (2009) for the planetary problem, QS orbits correspond to oscillations around an ACR located at \((\sigma, \Delta \sigma) = (0^\circ, 180^\circ)\). Although not present for quasi-circular trajectories, they fill a considerable portion of the phase space in the case of moderate to high eccentricities.

   We also found a new regime, associated with stable orbits displaying oscillations around \((\sigma, \Delta \sigma) = (0^\circ, 0^\circ)\), even though this point is unstable and corresponds to a collision between the two planets.

(ii) **Lagrangian region.** Apart from the previous symmetric solutions, we also found two distinct types of asymmetric ACR orbits in which both \(\sigma\) and \(\Delta \sigma\) oscillate around values different from \(0^\circ\) or \(180^\circ\). The first is the classical equilateral Lagrangian solution associated with local maxima of the averaged Hamiltonian function. Independently of the mass ratio \(m_2/m_1\) and their eccentricities, these solutions are always located at \((\sigma, \Delta \sigma) = (\pm 60^\circ, \pm 60^\circ)\). However, the size of the stable domain decreases rapidly for increasing eccentricities, being practically undetectable for \(e_i > 0.7\).

The second type of asymmetric ACR corresponds to local minima of the averaged Hamiltonian function. We dubbed them anti-Lagrangian solutions \((AL_4\) and \(AL_5)\). For low eccentricities, they are located at \((\sigma, \Delta \sigma) = (\pm 60^\circ, \mp 120^\circ)\). Each is connected to the classical \(L_4\) and \(L_5\) solutions through the \(\sigma\)-family of periodic orbits in the averaged system. Contrary to the classical equilateral Lagrangian solution, their location in the plane \((\sigma, \Delta \sigma)\) varies with the planetary mass ratio and eccentricities. Although their stability domain also shrinks for increasing values of \(e_i\), they do so at a slower rate than the classical Lagrangian solutions and are still appreciable for eccentricities as high as \(\sim 0.7\).

Finally, we also applied an ad hoc adiabatically slow mass variation to one of the planetary bodies and analysed its effect on the \(AL_4\) configuration. We found that the resonant co-orbital solution was preserved, with practically no change in the equilibrium values of the angles. The eccentricities, however, varied with the larger planet approaching a quasi-circular orbit as the smaller planet had its eccentricity increased. These solutions still exist in the limit of the restricted three-body problem (i.e. \(m_2 \to 0\)), although both types of asymmetric solutions \((L_4\) and \(AL_4)\) have different geometries. While the first are true stationary solutions in the unaveraged system, the latter are periodic orbits around the classical equilateral Lagrangian points.

6 ACKNOWLEDGMENTS

This work has been supported by the Argentinean Research Council (CONICET), the Brazilian National Research Council (CNPq) and the São Paulo State Science Foundation (FAPESP). The authors also gratefully acknowledge the CAPES/Secyt programme for scientific collaboration between Argentina and Brazil.
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