On the summability of the spectral expansions associated with the elliptic differential operators

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Abstract. In this paper we study the summability problems of the spectral expansions associated with the elliptic partial differential operators in the spaces of distributions. In particular, the problems studied for the Schrödinger operator with the singular potential. For this operator theorems on the summability of the Fourier expansions in the generalised Holder spaces and in the Sobolev spaces of the mixed norm proved. The conditions obtained are accurate in the classes of considered distributions.

1. Introduction
The spectral theory of the differential operators is an important part of the mathematical sciences that has applications in engineering and physics. This theory started its development since the time when Fourier studied heat conduction problem in a rod and found the solution as a form of sin series. An important point is the adjustment of the obtained solution. This leads to the study of the problems of the convergence of the obtained series solutions in the various topologies. Topology depends on the understanding of the sense of the solution which mainly depends on the initial and/or boundary data.

Obtained series solutions of the problems may not be convergent in the sense that required by the study. Then the problem of summability will occur. Regularizations of the divergent series allow accurate numerically interpretations of the solutions of the problems.

The problems of the spectral theory of differential operators are described in the book [1]. Spectral expansions as generalizations of the Fourier series and integrals associated with the elliptic partial differential operators studied in the number of the papers. The problems of the convergence of the multiple trigonometric series and spectral expansions studied in [2]. The problems of the summability of the spectral expansions of the functions from the spaces of the differentiable functions are studied in [3], [4], [5] and [6]. In [7] these problems are studied in the spaces of the continuous linear differential operators.

One of the important problems is representation of the functions/distributions in the closed domain by the spectral expansions associated with the elliptic differential operators. This problem for the first time studied in [8] for the eigenfunction expansions associated with the boundary value problems for
the Laplace operator. Further these problems found its development in [9]. Later in [10], [11], [12], [13] and [14] this problem is studied in the spaces of the different functions/distributions. The problem in compact subsets of the domain is studied in [15]. In [16], [17], [18], [19] and [20] various problems of the spectral decompositions in the spaces of the Sobolev spaces are studied.

One of the problems that we study in present paper is the problem of localization of spectral expansions in the generalized Holder spaces associated with the Schrodinger operator with the singular coefficients. In the generalized Holder spaces of distributions we obtained Theorem 5 (see below). Earlier this question investigated in [24] in the classes of continuous functions and in [23] for the bi-harmonic operator only. Thus in [24] the operator considered is the same as in Theorem 5 of present paper but the space of distributions is entirely different and in [23] the operator studied is completely different than in present paper but the space of distributions considered is the same as in Theorem 5. Therefore Theorem 5 is a new result in the spaces of distributions with the generalized smoothness/singularity and includes operators with the singular coefficients.

In the present paper we also studied the problem in the Sobolev spaces with the mixed norm for the Schrodinger operator with the singular coefficients and obtained Theorem 6. Earlier the problem investigated in [14] and [25] for the Laplace operator only.

2. Preliminaries

Let \( \Omega \) a domain in \( \mathbb{R}^N \), \( N \geq 2 \) with smooth boundary \( \partial \Omega \). Let \( A(D) = \sum_{|\alpha|=2m} a_\alpha \cdot D^\alpha \) denotes a positive elliptic and symmetric differential operator with constant coefficients (see in [2]). We consider this operator as a formal differential operator with the domain \( D(A) = C_0^\infty (\Omega) \) (the space of infinitely differentiable functions). Let \( \hat{A} \) a self-adjoint positive extension of the operator \( A(D) \) in \( L_2(\Omega) \). Denote by \( \{E_\lambda\} \) corresponding family of spectral projectors

\[
\hat{A} u(x) = \int_0^\infty \lambda dE_\lambda u(x), \quad u \in L_2(\Omega).
\]

For example, if \( A(D) \) is the Laplace operator with the domain which contains those functions from \( L_2(\Omega) \) that vanishing on the boundary, then for any \( u \in L_2(\Omega) \) its spectral projection \( E_\lambda u \) is defined as partial sum of the eigenfunction expansions associated with the first boundary value problem for the Laplace operator and the operator \( \hat{A} \) defined as follows

\[
\hat{A} f = \sum_{k=1}^\infty \lambda_k f_k u_k, \quad f \in L_2(G),
\]

where \( \{u_n(x)\} \) complete orthonormal in \( L_2(G) \) system of above the mentioned eigenfunctions.

For any non-negative number \( s \geq 0 \) and function \( u \in L_2(\Omega) \) denote the Riesz means of the spectral expansions \( E_\lambda u \) as follows:

\[
E_s^\lambda u(x) = \int_0^\frac{s}{\lambda} \left( 1 - \frac{t}{\lambda} \right)^s dE_t u(x).
\]
Note that operators $E^s_\lambda$ are integral operators with the smooth kernels $\Theta^s_\lambda(x, y, \lambda)$ from $C^\infty(\Omega \times \Omega)$ for each $\lambda > 0$ and for any distributions with the compact support $f \in \Sigma'(\Omega)$ can be defined as action with respect to the second variable

$$E^s_\lambda f(x) = \langle f, \Theta^s_\lambda(x, y, \lambda) \rangle.$$

Note, that operators $E^s_\lambda$ are continuous from $D(\Omega)$ to $\Sigma(\Omega)$, where $\Sigma(\Omega)$ denotes the space of infinitely differentiable functions with the locally convex norm defined as the maximum of the smooth function and its derivatives in compact subsets of the domain [21]. These operators can be studied in the space of distributions in the topology of the spaces of continuous linear functional. Moreover it can be studied in the classical means in the domains where distributions coincides with locally integrable functions.

3. Spectral expansions of the distributions

We study the problems in the spaces of distributions classified by the Sobolev spaces. For any real number $\alpha$ and $1 \leq p < \infty > 0$ by $W^p_\alpha(\Omega)$ denote the Sobolev spaces (see in [21]). In case $p = 2$ we have following result [15].

**Theorem 1.** Let $f \in W^{-\alpha}_2(\Omega) \cap \Sigma'(\Omega)$, $\alpha > 0$. If $s \geq \frac{N-1}{2} + \alpha$, then uniformly in each compact set $K$ from $\Omega \setminus \text{supp } f$

$$\lim_{\lambda \to \infty} E^s_\lambda f(x) = 0.$$

Moreover we have proved that a condition $s \geq \frac{N-1}{2} + \alpha$ in the theorem is precise. It is proved that if $s < \frac{N-1}{2} + l$ then for any point $x_0$ from the domain $\Omega$ there is a distribution $f \in H^{-l}(\Omega) \cap \Sigma'(\Omega)$, such that $x_0 \in \Omega \setminus \text{supp } f$ and $\lim_{\lambda \to \infty} E^s_\lambda f(x_0) = +\infty$.

Theorem 1 for the Laplace operator is proved in [7] and for the arbitrary spectral expansions associated with the elliptic operator in [15]. Case $p \neq 2$ produces some problems due to not applicability methods of the Hilbert spaces. In this case we have the following theorem [12].

**Theorem 2.** Let $f \in \Sigma \cap W^{-l}_p(\Omega)$, $l > 0$, $1 < p \leq 2$. If $s \geq \frac{N-1}{p} + l$, then uniformly on each compact $K$ from $\Omega \setminus \text{supp } f$ obtain equality

$$\lim_{\lambda \to \infty} E^s_\lambda f(x) = 0.$$

Note that later in [22] the same problem studied and obtained the necessary conditions for the summability spectral expansions of distributions with the compact support from $W^{-l}_p(\Omega)$. 

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It is natural to ask a question: If the condition \( s \geq \frac{N-1}{p} + l \) in Theorem 2 precise or not? Particular answer for this question for the special class of elliptic operators \( A_r \), \( r = 1, N-1 \) are given in [9], [18] and [20].

**Theorem 3.** Let \( A(D) \in A_r \) and let \( f \in \Sigma'(R^N) \cap W_{p}^{-\alpha}(R^N), \ 1 < p \leq 2, \ \alpha > 0 \). Let \( \Omega \) a subdomain of \( R^N \setminus \text{Suppf} \). If \( s \geq \frac{N-r-1}{p} + \frac{r}{2} + \alpha \), then uniformly on each compact \( K \) from \( \Omega \)

\[
\lim_{\lambda \to x} E_{\lambda}^s f(x) = 0.
\]

This theorem proves that the condition \( s \geq \frac{N-1}{p} + l \) can be weaken for some class of operators. Moreover we prove that the condition \( s \geq \frac{N-r-1}{p} + \frac{r}{2} + \alpha \) in the theorem is precise in the class of operators \( A_r \) (see in [18]).

4. Summability on the closed domains

The problem will be more complicated if we study the summability of the spectral expansions at the points closed to the boundary, not in compact sets as in the theorems above. In case when spectral expansions associated with the eigenfunction expansions associated with the Lopatinskii boundary value problems for the elliptic self-adjoin operator \( \mathcal{A} \) in the bounded domain \( \Omega \in R^N \) with the smooth boundary \( \partial \Omega \), it was proved [10] that the necessary condition for the uniformly convergence on the closed domain \( \overline{\Omega} \) in the class of continues functions is \( s > \frac{N}{2} \). But the condition \( s > \frac{N}{2} \) is not a final in the class of all operators. Last statement follows from the fact that it can be weaken as \( s > \frac{N-1}{2} \) for the expansions in eigenfunctions of the first boundary value problem for the Laplace operator in the domain \( \Omega \) [12]-[13].

Moreover we study this problem in the Sobolev spaces of distributions in the subdomain of the closed domain \( \overline{\Omega} \). For the first time this problem is studied in [8] for the functions from the

\[
W_{p}^{\left( \frac{N}{2} + 1 \right)}(\Omega)
\]

satisfying the boundary conditions together with the repeated Laplacians. For such a functions it is proved that the Fourier expansions in eigenfunctions associated with all three boundary value problems for the Laplace operator convergence uniformly on the closed domain \( \overline{\Omega} \).

In [13] we proved that if \( p \geq 1, \ \alpha \cdot p \geq N, \ s + \alpha > \frac{N-1}{2} \), then for any continuous function \( f \) from the Nikolskii space \( \tilde{H}_{p}(G) \), the Riesz means of order \( s \), \( 0 \leq s < \frac{N-1}{2} \), of the partial
sums of its Fourier series in eigenfunction of first boundary value problem convergence uniformly to \( f \) on the closed domain \( \overline{G} \). Note that, the condition \( \alpha \cdot p \geq N \) cannot be weakening (see [3]). In the spaces of singular distributions the problem is studied in the generalized Holder classes. In the following we define these classes of distributions.

Denote \( H^\alpha_x(\Omega) \) space of the functions from \( L^2(\Omega) \) with the finite norm
\[
\| \cdot \|_{\alpha,x} = \| \cdot \|_0 + \sum_{\beta=1}^{\infty} \sup_{0<\delta<1} \frac{\omega(D^\beta,\delta)}{\delta^\alpha \chi\left(\frac{1}{\delta}\right)},
\]
where \( \alpha > 0 \), \( \alpha = \nu + \zeta \), \( \nu \) is a non-negative integer, \( 0 < \zeta \leq 1 \) and \( \chi(\delta) \) is a positive function defined in the interval \( 1 \leq \delta < \infty \). Let \( H^{-\alpha}_x(\Omega) \) is the space of the distributions on \( H^\alpha_x(\Omega) \).

**Theorem 4.** Let \( \chi(\delta) = \ln\delta \) and \( f \in \Sigma'(\Omega) \cap H^{-\alpha}_x(\Omega) \), \( \alpha > 0 \), \( s + \alpha \geq \frac{N-1}{2} \). Then the Riesz means of order \( s \), of the partial sums of the Fourier series in eigenfunction of first boundary value problem for the Laplace operator convergence uniformly to zero in any compact set \( K \) from \( \overline{\Omega} \setminus \text{Suppf} \).

Recently in [23] this theorem is proved for the eigenfunction expansions associated with the biharmonic operator with the Navie boundary conditions. Note, that Theorem 4 is proved in [11] and only for the Laplace operator. In present paper we prove it for the operators with singular coefficients.

5. **Expansions associated with the operators with singular coefficients**

Further we consider the operators with the singular coefficients in two dimensional bounded domains \( \Omega \subset \mathbb{R}^2 \). Let a positive function \( q(x) \) belongs to the Sobolev space \( W^{1,2}_x(\Omega) \) with singularities at a point \( x_0 \in \Omega \). Consider the Schrödinger’s operator with potential \( q \) and denote by \( u_n(x) \) sequence of eigenfunctions and by \( \lambda_n \) eigenvalues of the first boundary value problem for this operator in the domain \( \Omega \). Then uniformly by \( x \in \overline{\Omega} \) for \( \mu_0 \to \infty \) we have [24]
\[
\sum_{|\lambda_n - \mu_0| > 1} u^2_n(x) = O(\mu_0 \ln^2 \mu_0).
\] (1)

When \( q(x) = 0 \) this estimation is proved in [9].

Based on the estimate (1) we study expansions in eigenfunctions of the Schrödinger’s operator with potential \( q \) in the generalized Holder spaces we obtain the following result.

**Theorem 5.** Let \( q(x) \in W^{1,2}_x(\Omega) \) and let \( f \in \Sigma'(\Omega) \cap H^{-\alpha}_{2n\delta}(\Omega) \), \( \alpha > 0 \), \( s + \alpha \geq \frac{1}{2} \). Then the Riesz means of order \( s \), of the partial sums of the Fourier series in eigenfunction of the first
boundary value problem for the Schrödinger operator \( \Delta + q(x) \) convergence uniformly to zero on the any compact set \( K \) from \( \overline{\Omega} \setminus \text{Suppf} \).

Now we study the problem in the Sobolev spaces with the mixed norm. First define these spaces.

Let \( p = (p_1, p_2) \), where \( 1 \leq p_i \leq \infty, \ i = 1, 2 \). We say that a finite in the domain \( \Omega \) function belongs to \( L_p(\Omega) \) if it is measurable and the following mixed norm is finite

\[
\| f \|_p = \left( \int \int |f(x_1, x_2)|^{\frac{p_1}{p_2}} dx_1 \right)^\frac{p_2}{p_1},
\]

where \( f \) is supposed extending outside the domain with zero. If \( p_i = \infty \) then the integral by \( x_i \) axes in the definition above must be replaced by the essential upper bound.

Denote by \( H_{p, \alpha}^a(\Omega) \) the Banach space of functions with respect to the norm

\[
\| f \|_{H_{p, \alpha}^a(\Omega)} = \| f \|_p + \sum_{|\alpha| = \ell} \left( \sup_{x \in \Omega} |x| \right)^{\kappa} \left| \Delta^\alpha f \right|_{p}
\]

where \( \Delta^\alpha f(x) = \partial^\alpha f(x + y) - 2\partial^\alpha f(x) + \partial^\alpha f(x - y), \ \alpha = \ell + \kappa. \)

Then in the spaces \( H_{p, \alpha}^a(\Omega) \) we obtain the following result.

**Theorem 6.** Let \( q(x) \in W_2^1(\Omega) \) and let a continuous function \( f \in H_{p, \alpha}^a(\Omega) \) has a compact support in \( \Omega \). If \( \alpha > \frac{1}{2} + s \), \( \alpha = \frac{1}{p_2} + \frac{1}{p_1} \), \( 2 \leq p_1 < p_2 \), then the Riesz means of order \( s \) of the partial sums of the Fourier series in eigenfunction of the first boundary value problem for the Schrödinger operator \( \Delta + q(x) \) convergence uniformly to \( f(x) \) on the any compact set \( K \) from the closed domain \( \overline{\Omega} \).

Note, that from the statement of Theorem 1 it follows that conditions in both Theorem 5 and 6 are accurate.

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