LOWER SEMICONTINUITY OF QUASICONVEX BULK ENERGIES IN SBV AND INTEGRAL REPRESENTATION IN DIMENSION REDUCTION

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Abstract. A result of Larsen concerning the structure of the approximate gradient of certain sequences of functions with Bounded Variation is used to present a short proof of Ambrosio's lower semicontinuity theorem for quasiconvex bulk energies in $SBV$. It enables to generalize to the $SBV$ setting the decomposition lemma for scaled gradients in dimension reduction and also to show that, from the point of view of bulk energies, $SBV$ dimensional reduction problems can be reduced to analogue ones in the Sobolev spaces framework.

Keywords: Dimension reduction, $\Gamma$-convergence, functions of bounded variation, free discontinuity problems, quasiconvexity, equi-integrability.

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1. Introduction

Since the pioneering work [22], the modelling of thin films through dimensional reduction techniques and $\Gamma$-convergence analysis has become one of the main issues in the field of Calculus of Variations. In the membrane theory framework in nonlinear elasticity, the problem rests on the study of the (scaled) elastic energy

$$\frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W(\varepsilon)(y, \nabla v) \, dy$$

of such bodies. Here $\Omega_\varepsilon := \omega \times (-\varepsilon/2, \varepsilon/2)$, where $\omega$ is a bounded open subset of $\mathbb{R}^2$ and $\varepsilon > 0$, stands for the reference configuration of a nonlinear elastic thin film, $v : \Omega \rightarrow \mathbb{R}^3$ is the deformation field which maps the reference configuration into a deformed configuration and $W(\varepsilon) : \Omega_\varepsilon \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ is the stored energy density of the body which is a Carathéodory function satisfying uniform $p$-growth and $p$-coercivity conditions (with $1 < p < \infty$). From a mathematical point of view, the previous energy is well defined provided $v$ is a Sobolev function in $W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$.

To study the limit problem as the thickness $\varepsilon \rightarrow 0$, it will be useful to recast the energy functional over the varying set $\Omega_\varepsilon$ into a functional with a fixed domain of integration $\Omega := \omega \times (-1/2, 1/2)$. To this end, denoting by $x_\alpha := (x_1, x_2)$ the in-plane variable, we set $u(x_\alpha, x_3) := v(x_\alpha, \varepsilon x_3)$ so that, after the (now standard) change of variables

$$x_\alpha = y_\alpha, \quad x_3 = \frac{y_3}{\varepsilon},$$

we are equivalently led to study the following rescaled functional

$$\int_{\Omega} W_\varepsilon \left( x, \nabla_\alpha u \left| \frac{1}{\varepsilon} \nabla_3 u \right. \right) \, dx, \quad u \in W^{1,p}(\Omega; \mathbb{R}^3),$$

where $W_\varepsilon : \Omega \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ is the rescaled stored energy density expressed in the new variables and defined by $W_\varepsilon(x_\alpha, x_3, \xi) := W(\varepsilon)(x_\alpha, \varepsilon x_3, \xi)$. From now on, $\nabla_\alpha$ (resp. $\nabla_3$) will stand for the (approximate) gradient with respect to $x_\alpha$ (resp. $x_3$), $\xi = (\xi_\alpha | \xi_3)$ for some matrix $\xi \in \mathbb{R}^{3 \times 3}$ and...
\[ z = (z_0, z_3) \text{ for some vector } z \in \mathbb{R}^3. \] Thus in view of the \( p \)-growth of the energy, it is important to understand the structure of what we call the scaled gradient of \( u \), i.e.

\begin{equation}
(1.2) \quad \left( \nabla_\alpha u \bigg| \frac{1}{\varepsilon} \nabla_3 u \right).
\end{equation}

In particular, if \( \{ u_\varepsilon \} \subset W^{1,p}(\Omega; \mathbb{R}^3) \) is a minimizing sequence uniformly bounded in energy, up to a subsequence, there always exist \( u \in W^{1,p}(\Omega; \mathbb{R}^3) \) such that \( D_3 u = 0 \) in the sense of distributions and \( b \in L^p(\Omega; \mathbb{R}^3) \) such that \( u_\varepsilon \rightharpoonup u \) in \( W^{1,p}(\Omega; \mathbb{R}^3) \) and \((1/\varepsilon) \nabla_3 u_\varepsilon \rightharpoonup b \) in \( L^p(\Omega; \mathbb{R}^3) \). The limit function \( u \) is nothing but the deformation of the mid-plane while \( b \) is called the Cosserat vector. It seems thus natural to expect a limit model depending on the pair \((u, b)\). Unfortunately, this is still out of reach and we refer to [19] for a more detailed discussion on the subject. However, in [9] (see also [7]) a simplified model has been considered taking into account the bending moment \( \mathbf{b} \in L^p(\omega; \mathbb{R}^3) \), i.e. the average in the transverse direction \( x_3 \) of \( b \), instead of the full Cosserat vector field.

In the framework of fracture mechanics, one usually adds a surface energy term, penalizing the presence of the crack. The simplest case consists in just penalizing its area leading to the so-called Griffith’s surface energy. Moreover, the surface energy is still of Griffith’s type while the bulk energy is exactly the sum of the bulk and the surface energies. Such fracture mechanics problems belong (among others) to the class of free discontinuity problems, that is variational problems where the unknown is not only a function, but a pair set/function. Based on the idea that the deformation may be discontinuous across the crack, it is convenient to study the weak formulation, replacing the crack by the jump set of the deformation and leading to a variational problem stated in the space of (Special) Functions with Bounded Variation. Now the energy in which we are interested is

\[ \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W(\varepsilon)(y, \nabla v) \, dy + \frac{1}{\varepsilon} \mathcal{H}^2(S_v), \quad v \in SBV^p(\Omega_\varepsilon; \mathbb{R}^3), \]

where \( \nabla v \) is intended as the approximate gradient of \( v \), \( S_v \) is the jump set of \( v \) and \( \mathcal{H}^2 \) stands for the 2-dimensional Hausdorff measure. Writing as before this energy in the rescaled variables yields to

\begin{equation}
(1.3) \quad \int_{\Omega} \frac{W(\varepsilon)}{\varepsilon} \left( x, \nabla_\alpha u \bigg| \frac{1}{\varepsilon} \nabla_3 u \right) \, dx + \int_{S_u} \left| \left( \nu_u \right)_\alpha \bigg| \frac{1}{\varepsilon} \left( \nu_u \right)_3 \right| \, d\mathcal{H}^2, \quad u \in SBV^p(\Omega; \mathbb{R}^3)
\end{equation}

where \( \nu_u \) is the generalized normal to \( S_u \) and (1.2) is now referred as the approximate scaled gradient of \( u \).

The aim of this paper is to study the connections between variational problems (1.1) and (1.3), possibly taking into account the presence of the bending moment vector field. To this end, we will use as main ingredient Theorem 4.1 which extends the Decomposition Lemma for scaled gradients (see [8, Theorem 1.1] or [13, Theorem 3.1]) to the \( SBV \) setting. It states that any \( SBV \)-sequence with bounded rescaled bulk energy and whose derivative’s singular part behaves asymptotically well, can be energetically replaced, up to a set of vanishing Lebesgue measure, by a sequence of Lipschitz maps whose scaled gradient is \( p \)-equi-integrable. Thus it reduces the free discontinuity problem to a usual dimensional reduction one in the framework of Sobolev spaces. This result is nothing but a rescaled version of [21, Lemma 2.1] (see also Theorem 3.1 below). Using this structure theorem, we are able to show two integral representation theorems in \( SBV \) (Theorems 6.1 and 7.3) which say that, up to a subsequence, the functional (1.3) \( \Gamma \)-converges (in an appropriate topology) to a functional of the same kind, i.e. the sum of a bulk and a surface energy. Moreover, the surface energy is still of Griffith’s type while the bulk energy is exactly the same than that obtained in the analogue Sobolev spaces analysis. The main importance of these representation theorems relies on the fact that results on dimension reduction in Sobolev spaces can now be extended to \( SBV \) (see [7, 5, 6, 9, 22]).

Note that an integral representation result for dimensional reduction problems in \( SBV \) already exists (see [11, Theorem 2.1]). Even if this reference may seem more general from the point of view of the
hypothesis, it does not contain as special case our results because the authors made strongly use of the fact that their surface energy had to grow linearly with respect to the deformation jump. This assumption was essential in order to get compactness in $BV(\Omega;\mathbb{R}^3)$ of minimizing sequences. However, they suggested a way to remove that constraint by singular perturbation [11, Remark 2.2]. In our study we use a direct argument based on a trick introduced in [18] and which was already used in [4] in the framework of dimensional reduction. It consists in defining an artificial functional exactly as we usually do for the Γ-lim inf, except that we impose the minimizing sequences to be uniformly bounded in $L^\infty(\Omega;\mathbb{R}^3)$. Thanks to a truncation argument (see Lemma 6.2) we show that it actually coincides with the Γ-lim inf for deformations $u \in L^\infty(\Omega;\mathbb{R}^3)$ and the advantage is that now, minimizing sequences turn out to be relatively compact in $SBV(\Omega;\mathbb{R}^3)$ thanks to Ambrosio’s Compactness Theorem. We refer to [4] for a deeper insight on that subject.

To close this introduction, we wish to stress that in this paper, we are mostly interested in representation of effective bulk energies arising in 3D-2D dimensional reduction problems stated in $SBV$. For this reason we will consider a large class for such bulk energies while surface energies will be restricted to the simplified case of a Griffith’s type one. However we are convinced that the results presented here could be generalized to a larger class of surface energies.

The overall plan of the paper is as follows: after recalling some useful notations in section 2 and in order to show the technique in a more transparent way, we present in section 3 a short proof of Ambrosio’s lower semicontinuity result for quasiconvex integrands using [21, Lemma 2.1]. Then in section 4 we prove our main tool, Theorem 4.1, thanks to a slicing argument together with [21, Lemma 2.1]. To reach our goal, we need to prove a general integral representation for the Γ-limit of (1.1) in $W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3)$ as a function of the deformation and the bending moment. This is the purpose of Theorem 5.1 in section 5 which contains as particular cases [9, Theorem 3.1] (with $W_z(x,\xi) = \omega(x,\xi)$) and [7, Theorem 3.4] (with $W_z(x,\xi) = W(x,\xi)$). In section 6, we refine the analysis of section 3 adding the difficulties of dimension reduction. From the integral representation in Sobolev spaces, Theorem 5.1, we deduce an analogue result in $SBV$, Theorem 6.1, which says that the Γ-limit of (1.3) in $BV(\Omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3)$ has also an integral representation and that the bulk energy is exactly the same one that obtained in the $W^{1,p}$ analysis. This will be achieved thanks to Theorem 4.1 and a blow-up method which enables to reduce the problem to affine deformations and constant bending moments. Finally we deduce a similar result in section 7 without the presence of the bending moment.

2. Notations and preliminaries

If $\Omega \subset \mathbb{R}^N$ is an open set, we consider the Lebesgue spaces $L^p(\Omega;\mathbb{R}^d)$ and the Sobolev spaces $W^{1,p}(\Omega;\mathbb{R}^d)$ in the usual way. When needed, we will precise what topology the space $L^p(\Omega;\mathbb{R}^d)$ will be endowed. In particular we will denote by $L^p_0(\Omega;\mathbb{R}^d)$ (resp. $L^p_{weak}(\Omega;\mathbb{R}^d)$) the space $L^p(\Omega;\mathbb{R}^d)$ endowed with the strong (resp. weak) topology. Strong convergence will always be denoted by $\rightarrow$ while weak (resp. weak*) convergence will be denoted by $\rightharpoonup$ (resp. $\rightharpoonup^{\ast}$).

We denote by $\mathcal{M}(\Omega;\mathbb{R}^d)$ the space of vector valued finite Radon measures. If $\mu \in \mathcal{M}(\Omega;\mathbb{R}^d)$ and $E$ is a Borel subset of $\Omega$, we will write $\mu|_{E}$ for the restriction of $\mu$ to $E$ that is, for every Borel subset $F$ of $\Omega$, $\mu|_{E}(F) = \mu(E \cap F)$. The Lebesgue measure in $\mathbb{R}^N$ will be denoted by $\mathcal{L}^N$ while $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. We will denote by $B$ the unit ball of $\mathbb{R}^N$ and by $\omega_N := \mathcal{L}^N(B)$ its Lebesgue measure. If $x_0 \in \mathbb{R}^N$ and $\rho > 0$, $B(x_0,\rho) := x_0 + \rho B$ is the ball centered at $x_0$ with radius $\rho$. The notation $\frac{1}{|A|}$ stands for the average $\mathcal{L}^N(A)^{-1} \int_A$.

The space of Functions of Bounded Variation is denoted by $BV(\Omega;\mathbb{R}^d)$ and we refer to [3] for standard theory of $BV$ functions. We recall here few facts: if $u \in BV(\Omega;\mathbb{R}^d)$ then its distributional derivative $Du \in \mathcal{M}(\Omega;\mathbb{R}^{d \times N})$ and thanks to Lebesgue’s Decomposition Theorem, we can write $Du = D^a u + D^s u$, where $D^a u$ and $D^s u$ stand for, respectively, the absolutely continuous and singular part of $Du$ with respect to the Lebesgue measure $\mathcal{L}^N$. Let $S_u$ be the complementary of Lebesgue points of $u$. We say...
that $u$ is a Special Function of Bounded Variation, and we write $u \in SBV(\Omega; \mathbb{R}^d)$, if

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1}|_S u$$

where $\nabla u$ is the approximate gradient of $u$, $\nu_u$ is the generalized normal to $S_u$ and $u^\pm$ are the traces of $u$ on both sides of $S_u$. If $E \subset \Omega$, we say that $E$ has finite perimeter in $\Omega$ provided $\chi_E \in SBV(\Omega)$. We denote by $\partial^* E$ (resp. $\partial E$) the reduced (resp. essential) boundary of $E$. When $p > 1$, we define

$$SBV^p(\Omega; \mathbb{R}^d) := \left\{ u \in SBV(\Omega; \mathbb{R}^d) : \nabla u \in L^p(\Omega; \mathbb{R}^{d \times N}) \text{ and } \mathcal{H}^{N-1}(S_u \cap \Omega) < +\infty \right\}.$$  

We say that a sequence $\{u_n\} \subset SBV^p(\Omega; \mathbb{R}^d)$ converges weakly to some $u \in SBV^p(\Omega; \mathbb{R}^d)$, and we write $u_n \rightharpoonup u$ in $SBV^p(\Omega; \mathbb{R}^d)$, if

\[
\begin{align*}
&u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d), \\
&\nabla u_n \rightharpoonup \nabla u \text{ in } L^p(\Omega; \mathbb{R}^{d \times N}), \\
&(u_n^+ - u_n^-) \otimes \nu_{u_n} \rightharpoonup (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1}|_S u \text{ in } \mathcal{M}(\Omega; \mathbb{R}^{d \times N}).
\end{align*}
\]

If $\Omega := \omega \times I$, where $\omega$ is a bounded open subset of $\mathbb{R}^2$ and $I := (-1/2, 1/2)$, we will identify the spaces $L^p(\omega; \mathbb{R}^3)$, $W^{1,p}(\omega; \mathbb{R}^3)$ or $SBV^p(\omega; \mathbb{R}^3)$ with the space of functions $v \in L^p(\Omega; \mathbb{R}^3)$, $W^{1,p}(\Omega; \mathbb{R}^3)$ or $SBV^p(\Omega; \mathbb{R}^3)$ such that $Dv = 0$ in the sense of distributions.

By $\mathcal{A}(\omega)$ we mean the family all open subsets of $\omega$ while $\mathcal{R}(\omega)$ stands for the countable subfamily of $\mathcal{A}(\omega)$ obtained by taking all finite unions of open cubes contained in $\omega$, centered at rational points and with rational edge length.

In the sequel, we will denote by $Q' := (-1/2, 1/2)^2$ the unit cube of $\mathbb{R}^2$ and by $Q'(x_0, \rho) := x_0 + \rho Q'$ the cube centered at $x_0 \in \mathbb{R}^2$ and side length $\rho > 0$. Similarly $B' := \{x_\alpha \in \mathbb{R}^2 : |x_\alpha| < 1\}$ stands for the unit ball in $\mathbb{R}^2$ and $B'(x_0, \rho) := x_0 + \rho B'$ denotes the ball of $\mathbb{R}^2$ centered at $x_0 \in \mathbb{R}^2$ and of radius $\rho > 0$.

3. LOWER SEMICONTINUITY OF QUASICONVEX BULK ENERGIES IN $SBV$

This section is devoted to give a short proof of Ambrosio’s lower semicontinuity result for quasiconvex bulk energies in $SBV$ using the following theorem proved in [21, Lemma 2.1].

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and let $\{u_n\} \subset BV(\Omega; \mathbb{R}^d)$ be such that

\[
\begin{align*}
&\sup_{n \in \mathbb{N}} \|u_n\|_{BV(\Omega; \mathbb{R}^d)} < +\infty, \\
&\sup_{n \in \mathbb{N}} \|\nabla u_n\|_{L^p(\omega; \mathbb{R}^{d \times N})} < +\infty \quad \text{for some } p > 1, \\
&|D^s u_n|(\Omega) \to 0.
\end{align*}
\]

Then there exists a subsequence $\{n_k\} \nearrow +\infty$ and a sequence $\{w_k\} \subset W^{1,\infty}(\Omega; \mathbb{R}^d)$ such that

\[
\begin{align*}
&\sup_{k \in \mathbb{N}} \|w_k\|_{W^{1,p}(\omega; \mathbb{R}^d)} < +\infty, \\
&\{|\nabla w_k|^p\} \text{ is equi-integrable}, \\
&L^N(\{w_k \neq u_{n_k}\} \cup \{\nabla w_k \neq \nabla u_{n_k}\}) \to 0.
\end{align*}
\]

This theorem is nothing but the $BV$ counterpart of the Decomposition Lemma, [20, Lemma 1.2], in Sobolev spaces. We now use the previous result to give a short proof of Ambrosio’s lower semicontinuity result for quasi-convex bulk energies in $SBV$ (see [2, Theorem 4.3] or [3, Proposition 5.29]). This will enable us to emphasize the techniques used in this paper, occulting the difficulties of dimension reduction. The same kind of arguments will be used in section 6 to prove the lower bound of Theorem 6.1.
Theorem 3.2. Let Ω be bounded open subset of \( \mathbb{R}^N \) and \( f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty) \) be a Carathéodory function satisfying

\[
(3.1) \quad c|\xi|^p \leq f(x, s, \xi) \leq a(x) + \psi(|s|)(1 + |\xi|^p)
\]

for some \( p > 1, c > 0, a \in L^1(\Omega) \) and some increasing function \( \psi : [0, +\infty) \to [0, +\infty) \). If \( \xi \mapsto f(x, s, \xi) \) is quasiconvex for every \( s \in \mathbb{R}^d \) and a.e. \( x \in \Omega \), then

\[
\liminf_{n \to +\infty} \int_{\Omega} f(x, u_n, \nabla u_n) \, dx \geq \int_{\Omega} f(x, u, \nabla u) \, dx
\]

for any sequence \( \{u_n\} \subset SBV(\Omega; \mathbb{R}^d) \) converging in \( L^1(\Omega; \mathbb{R}^d) \) to \( u \in SBV(\Omega; \mathbb{R}^d) \) and satisfying \( \sup_n \mathcal{H}^{N-1}(S_{u_n}) < +\infty \).

Proof. The proof is divided into three steps. We first apply the blow-up method to reduce the study to an affine limit function. Then we prove that the resulting sequence can be modified, without increasing too much the energy, into another one uniformly bounded in \( L^\infty \). Finally we apply Theorem 3.1 to replace this last sequence of \( SBV \) functions by a sequence of Sobolev functions.

Step 1. Up to a subsequence, there is no loss of generality to assume the existence of nonnegative and finite Radon measures \( \lambda \) and \( \mu \in \mathcal{M}(\Omega) \) such that \( f(\cdot, u_n, \nabla u_n) \mathcal{L}^N \rightharpoonup \lambda \) and \( \mathcal{H}^{N-1} S_{u_n} \rightharpoonup \mu \) in \( \mathcal{M}(\Omega) \). To prove Theorem 3.2 it is enough to check that

\[
\lambda(\Omega) \geq \int_{\Omega} f(x, u, \nabla u) \, dx
\]

and thanks to Lebesgue’s Differentiation Theorem, it suffices to show that

\[
\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq f(x_0, u(x_0), \nabla u(x_0))
\]

for \( \mathcal{L}^N \)-a.e. \( x_0 \in \Omega \). Select \( x_0 \in \Omega \) such that

(a) \( x_0 \) is a Lebesgue point of \( u \) and \( a \) and a point of approximate differentiability of \( u \);
(b) The Radon-Nikodym derivative of \( \lambda \) with respect to \( \mathcal{L}^N \) exists and is finite;
(c) the following limit exists and

\[
(3.2) \quad \lim_{\rho \to 0^+} \frac{\mu(B(x_0, \rho))}{\omega_{N-1} \rho^{N-1}} = 0;
\]

(d) for any sequence \( \{\rho_i\} \searrow 0^+ \) there exists a subsequence \( \{\rho_{i(k)}\} \) and a \( \mathcal{L}^N \)-negligible set \( E \subset B \) such that

\[
(3.3) \quad \lim_{k \to +\infty} f(x_0 + \rho_{i(k)} y, u(x_0) + \rho_{i(k)} s, \xi) = f(x_0, u(x_0), \xi)
\]

locally uniformly in \( \mathbb{R}^d \times \mathbb{R}^{d \times N} \) for any \( y \in B \setminus E \).

Note that \( \mathcal{L}^N \)-a.e. points \( x_0 \) in \( \Omega \) satisfy these properties. Items (a) and (b) are immediate while item (d) is a consequence of [3, Lemma 5.38]. Concerning item (c), we remark that, setting

\[
\Theta(x) := \limsup_{\rho \to 0^+} \frac{\mu(B(x, \rho))}{\omega_{N-1} \rho^{N-1}},
\]

then \( \{\Theta > 0\} = \bigcup_{h=1}^{+\infty} \{\Theta \geq 1/h\} \) and using [3, Theorem 2.56], we get that \( \mathcal{H}^{N-1}(\{\Theta \geq 1/h\}) \leq h \mu(\{\Theta \geq 1/h\}) < +\infty \). Thus \( \mathcal{L}^N(\{\Theta \geq 1/h\}) = 0 \) and consequently \( \mathcal{L}^N(\{\Theta > 0\}) = 0 \).
Consider a sequence $\{\rho_k\} \searrow 0^+$ such that $0 < \rho_k < 1$, $\mu(\partial B(x_0, \rho_k)) = \lambda(\partial B(x_0, \rho_k)) = 0$ for every $k \in \mathbb{N}$ and (3.3) holds with $\rho_k$ in place of $\rho_{(k)}$. Then

$$
\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{k \to +\infty} \frac{\lambda(B(x_0, \rho_k))}{\omega_N \rho_k^N} = \lim_{k \to +\infty} \lim_{n \to +\infty} \frac{1}{\omega_N \rho_k^n} \int_{B(x_0, \rho_k)} f(x, u_n, \nabla u_n) \, dx
$$

(3.4)

$$
= \lim_{k \to +\infty} \lim_{n \to +\infty} \frac{1}{\omega_N} \int_B f(x + \rho_k y, u(x_0) + \rho_k u_n, \nabla u_n) \, dy
$$

where we set $u_{n,k}(y) = [u_n(x_0 + \rho_k y) - u(x_0)]/\rho_k$. Since $x_0$ is a point of approximate differentiability of $u$, it follows that

$$
\lim_{k \to +\infty} \lim_{n \to +\infty} \|u_{n,k} - u_0\|_{L^1(B; \mathbb{R}^d)} = 0
$$

(3.5)

where $u_0(y) := \nabla u(x_0, y)$. Moreover, by (3.2) we get that

$$
\lim_{k \to +\infty} \lim_{n \to +\infty} \mathcal{H}^{N-1}(S_{u_{n,k}} \cap B) = \lim_{k \to +\infty} \lim_{n \to +\infty} \frac{\mathcal{H}^{N-1}(S_{u_n} \cap B(x_0, \rho_k))}{\rho_k^{N-1}} = \lim_{k \to +\infty} \frac{\mu(B(x_0, \rho_k))}{\rho_k^{N-1}} = 0.
$$

(3.6)

From (3.4), (3.5) and (3.6), one can find a sequence $n(k) \nearrow +\infty$ such that, setting $v_k := u_{n(k),k}$, then $v_k \to u_0$ in $L^1(B; \mathbb{R}^d)$, $\mathcal{H}^{N-1}(S_{v_k} \cap B) \to 0$ and

$$
\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{k \to +\infty} \frac{1}{\omega_N} \int_B f(x + \rho_k y, u(x_0) + \rho_k v_k, \nabla v_k) \, dy.
$$

(3.7)

From now on, all the integrals will be restricted to the unit ball $B$.

**Step 2.** We now use the same truncation argument than in the proof of [3, Proposition 5.37]. Define $\hat{v}_k := (\sqrt{1 + |v_k - w_0|^2} - 2)^+$ so that by Theorem 3.96 and Proposition 3.64 (c) in [3], $\hat{v}_k \in SBV(B)$, $|\nabla \hat{v}_k| \leq |\nabla v_k - \nabla w_0| \mathcal{L}^N$-a.e. in $B$ and $S_{\hat{v}_k} \subset S_{v_k}$. According the Coarea Formula in BV [3, Theorem 3.40], we have that

$$
\int_0^1 \mathcal{H}^{N-1}(\partial^* \{\hat{v}_k > t\} \cap (B \setminus S_{\hat{v}_k})) \, dt \leq |D\hat{v}_k|(B \setminus S_{\hat{v}_k}) = \int_B |
abla \hat{v}_k| \, dx
$$

$$
\leq \int_{B \setminus \{|v_k - w_0| > \sqrt{3}\}} |\nabla v_k - \nabla w_0| \, dx
$$

where we have used the fact that $\nabla \hat{v}_k = 0 \mathcal{L}^N$-a.e. in $B \cap \{|v_k - w_0| \leq \sqrt{3}\}$. From (3.7) and the $p$-coercivity condition (3.1), the sequence $\{\nabla v_k\}$ is uniformly bounded in $L^p(B; \mathbb{R}^{d \times N})$ and since $p > 1$, it is equi-integrable. Using the fact that $\mathcal{L}^N(B \cap \{|v_k - w_0| > \sqrt{3}\}) \to 0$ we obtain that the right hand side of the previous relation tends to zero as $k \to +\infty$. Consequently, one can find $t_k \in (0, 1)$ such that $A_k := \{\hat{v}_k > t_k\}$ has finite perimeter in $B$ and

$$
\lim_{k \to +\infty} \mathcal{H}^{N-1}(B \cap \partial^* A_k \setminus S_{\hat{v}_k}) = 0.
$$

(3.8)

Define $\tilde{v}_k := v_k \chi_{B \setminus A_k} + w_0 \chi_{B \setminus A_k}$ so that $\tilde{v}_k \to w_0$ in $L^1(B; \mathbb{R}^d)$. As $|\tilde{v}_k| \leq t_k < 1$ in $B \setminus A_k$ it follows that $|v_k - w_0| \leq 2\sqrt{2}$ in $B \setminus A_k$ and thus

$$
\|\tilde{v}_k\|_{L^\infty(B; \mathbb{R}^d)} \leq \|v_k\|_{L^\infty(B \setminus A_k; \mathbb{R}^d)} + \|w_0\|_{L^\infty(B; \mathbb{R}^d)} \leq M
$$

(3.9)
where $M > 0$ is independent of $k$. Denoting by $v_k^-$ the exterior trace of $v_k$ on $\partial^* A_k \cap B$ oriented by the inner normal of $A_k$, Remark 3.85 in [3] implies that $|v_k^-(x)| \leq M$ for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* A_k \cap B$ and thus
\[
\int_{\partial^* A_k \cap B} |v_k^-| \, d\mathcal{H}^{N-1} \leq M \mathcal{H}^{N-1}(\partial^* A_k \cap B) < +\infty
\]
so that [3, Theorem 3.84] ensures that $\tilde{v}_k \in SBV(B; \mathbb{R}^d)$. Since $S_{\tilde{v}_k} \subset S_{v_k} \cup \partial A_k$, by (3.8) we get that
\[
\mathcal{H}^{N-1}(B \cap S_{\tilde{v}_k}) \leq \mathcal{H}^{N-1}(B \cap S_{v_k}) + \mathcal{H}^{N-1}(B \cap \partial A_k \setminus S_{v_k}) \leq \mathcal{H}^{N-1}(B \cap S_{v_k}) + \mathcal{H}^{N-1}(B \cap \partial^* A_k \setminus S_{v_k}) \to 0
\]
where we used the fact that $S_{\tilde{v}_k} \subset S_{v_k}$ and $\mathcal{H}^{N-1}(B \cap \partial A_k \setminus \partial^* A_k) = 0$. Using the locality of approximate gradients and the $p$-growth condition (3.1), we get that
\[
\int_B f(x_0 + \rho_k y, u(x_0) + \rho_k \tilde{v}_k, \nabla \tilde{v}_k) \, dy = \int_{B \setminus A_k} f(x_0 + \rho_k y, u(x_0) + \rho_k v_k, \nabla v_k) \, dy + \int_{B \cap A_k} f(x_0 + \rho_k y, u(x_0) + \rho_k w_0, \nabla u(x_0)) \, dy \leq \int_B f(x_0 + \rho_k y, u(x_0) + \rho_k v_k, \nabla v_k) \, dy + \int_{B \cap A_k} [a(x_0 + \rho_k y) + \psi(|u(x_0) + \rho_k w_0|)(1 + |\nabla u(x_0)|^p)] \, dy.
\]
By the choice of $x_0$, the sequence $\{a(x_0 + \rho_k y)\}$ is strongly converging in $L^1(B)$ to $a(x_0)$ and thus it is equi-integrable. Hence as $\mathcal{L}^N(A_k) \leq \mathcal{L}^N(\{|v_k - w_0| \geq \sqrt{3}\}) \to 0$ we deduce that the second term on the right hand side of the previous relation tends to zero as $k \to +\infty$ and thanks to (3.7) it follows that
\[
(3.10) \quad \frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq \limsup_{k \to +\infty} \frac{1}{\omega_N} \int_B f(x_0 + \rho_k y, u(x_0) + \rho_k \tilde{v}_k, \nabla \tilde{v}_k) \, dy.
\]

**Step 3.** By (3.9) we have that $|D^s \tilde{v}_k|(B) \leq 2M \mathcal{H}^{N-1}(S_{\tilde{v}_k} \cap B) \to 0$ while the $p$-coercivity condition (3.1) and item (b) imply that
\[
\sup_{k \in \mathbb{N}} \|\nabla \tilde{v}_k\|_{L^p(B; \mathbb{R}^d \times N)} < +\infty.
\]
Consequently the sequence $\{\tilde{v}_k\}$ fulfills the assumptions of Theorem 3.1 so that considering a suitable (not relabeled) subsequence, there exist a Lebesgue measurable set $E_k \subset B$ and a sequence $\{w_k\} \subset W^{1,\infty}(B; \mathbb{R}^d)$ such that $\{\nabla w_k\}$ is equi-integrable, $w_k = \tilde{v}_k$ on $B \setminus E_k$ and $\mathcal{L}^N(E_k) \to 0$. From the proof of [21, Lemma 2.1], it can also be checked that $\sup_k \|w_k\|_{L^\infty(B; \mathbb{R}^d)} \leq M$. As
\[
\int_B |w_k - w_0| \, dy \leq \int_{B \setminus E_k} |\tilde{v}_k - w_0| \, dy + 2M \mathcal{L}^N(E_k) \to 0
\]
it follows that $w_k \to w_0$ in $L^1(B; \mathbb{R}^d)$ and defining the set $B_k^t := \{x \in B : |\nabla w_k(x)| \leq t\}$, relation (3.10) leads to
\[
\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq \limsup_{t \to +\infty} \limsup_{k \to +\infty} \frac{1}{\omega_N} \int_{B_k^t \setminus E_k} f(x_0 + \rho_k y, u(x_0) + \rho_k w_k, \nabla w_k) \, dy.
\]
Using now (3.3) with $\rho_t(k) = \rho_k$, we obtain that
\[
\lim_{k \to +\infty} \int_{B_k^t \setminus E_k} \left| f(x_0 + \rho_k y, u(x_0) + \rho_k w_k, \nabla w_k) - f(x_0, u(x_0), \nabla w_k) \right| \, dy = 0
\]
for each $t > 0$, implying that
\[
(3.11) \quad \frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq \limsup_{t \to +\infty} \limsup_{k \to +\infty} \frac{1}{\omega_N} \int_{B_k^t \setminus E_k} f(x_0, u(x_0), \nabla w_k) \, dy.
\]
Since $\mathcal{L}^N(E_k) \to 0$, according to the $p$-growth condition (3.1) we get that for every $t > 0$,
\begin{equation}
(3.12) \quad \int_{E_k \cap B_k^t} f(x_0, u(x_0), \nabla w_k) \, dy \leq (a(x_0) + \psi(|u(x_0)|))(1 + t^p) \mathcal{L}^N(E_k) \xrightarrow{k \to +\infty} 0.
\end{equation}

On the other hand, Chebyshev's Inequality ensures the existence of a constant $c > 0$ (independent of $k$ and $t$) such that $\mathcal{L}^N(B \setminus B_k^t) \leq c/t^p \to 0$ as $t \to +\infty$, so that the equi-integrability of $|\nabla w_k|^p$ yields to
\begin{equation}
(3.13) \quad \sup_{k \in \mathbb{N}} \int_{B \setminus B_k^t} f(x_0, u(x_0), \nabla w_k) \, dy \leq \sup_{k \in \mathbb{N}} \int_{B \setminus B_k^t} (a(x_0) + \psi(|u(x_0)|))(1 + |\nabla w_k|^p) \, dy \xrightarrow{t \to +\infty} 0.
\end{equation}

Gathering (3.11), (3.12) and (3.13), we deduce that
\[
\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq \limsup_{k \to +\infty} \frac{1}{\omega_N} \int_B f(x_0, u(x_0), \nabla w_k) \, dy
\]
and since $w_k \rightharpoonup w_0$ in $W^{1,p}(B; \mathbb{R}^d)$, we can apply [1, Theorem II-4] to conclude that
\[
\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq f(x_0, u(x_0), \nabla u(x_0)).
\]

\[\square\]

4. Structure of approximate scaled gradients

In this section we prove the following Theorem 4.1 which is a similar result than Theorem 3.1 in the context of dimension reduction. Note that it generalizes [8, Theorem 1.1] and [13, Theorem 3.1] (with obvious changes for $nD-(n-k)D$ dimensional reduction). Its proof relies on a slicing argument similar to that used in [13, Theorem 3.1]. It will be instrumental in section 6 to prove Theorem 6.1 because it will enable to replace $SBV$ minimizing sequences by Lipschitz ones without increasing the energy.

From now on, $\Omega := \omega \times I$ where $\omega$ is a bounded open subset of $\mathbb{R}^2$ and $I := (-1/2, 1/2)$.

**Theorem 4.1.** Assume that $\omega$ has a Lipschitz boundary and $p > 1$. Let $\{\varepsilon_n\} \searrow 0^+$ and $\{u_n\} \subset SBV^p(\Omega; \mathbb{R}^3)$ be such that
\begin{equation}
(4.1) \quad \sup_{n \in \mathbb{N}} \left\{ \|u_n\|_{L^\infty(\Omega; \mathbb{R}^3)} + \int_{\Omega} \left| \frac{\nabla u_n}{\varepsilon_n^2} \nabla_3 u_n \right|^p \, dx \right\} < +\infty,
\end{equation}
\begin{equation}
(4.2) \quad \int_{S_{\varepsilon_n}} \left| \frac{\nu_{u_n}}{\varepsilon_n^2} \nu_{u_n} \right| \, dH^2 \to 0
\end{equation}
and that $u_n \rightharpoonup u$ in $SBV^p(\Omega; \mathbb{R}^3)$, $(1/\varepsilon_n)\nabla_3 u_n \rightharpoonup b$ in $L^p(\Omega; \mathbb{R}^3)$ for some $u \in W^{1,p}(\omega; \mathbb{R}^3)$ and $b \in L^p(\Omega; \mathbb{R}^3)$. Then there exist a subsequence $\{\varepsilon_n\} \subset \{\varepsilon_n\}$ and a sequence $\{z_k\} \subset \mathcal{L}^3(\Omega; \mathbb{R}^3)$ such that $z_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^3)$, $(1/\varepsilon_n)\nabla_3 z_k \rightharpoonup b$ in $L^p(\Omega; \mathbb{R}^3)$, the sequence $\left\{ \left| (\nabla_3 z_k) \frac{1}{\varepsilon_n} \nabla_3 z_k \right|^p \right\}$ is equi-integrable and
\[
\mathcal{L}^3(\{z_k \neq u_{n_k}\} \cup \{\nabla z_k \neq \nabla u_{n_k}\}) \to 0.
\]

**Proof:** The proof is based on a slicing argument. We first come back to the non rescaled cylinder $\Omega_{\varepsilon_n} = \omega \times (-\varepsilon_n/2, \varepsilon_n/2)$ of thickness $\varepsilon_n$ setting $v_n(x_\alpha, x_3) := u_n(x_\alpha, x_3/\varepsilon_n)$. It follows that for each $n \in \mathbb{N}$, $v_n \in SBV^p(\Omega_{\varepsilon_n}; \mathbb{R}^3)$ and changing variable in (4.1) we get that
\begin{equation}
(4.3) \quad \sup_{n \in \mathbb{N}} \left\{ \|v_n\|_{L^\infty(\Omega_{\varepsilon_n}; \mathbb{R}^3)} + \frac{1}{\varepsilon_n} \int_{\Omega_{\varepsilon_n}} |\nabla v_n|^p \, dx \right\} < +\infty
\end{equation}
and
\begin{equation}
(4.4) \quad H^2(S_{v_n}) = \varepsilon_n \int_{S_{\varepsilon_n}} \left| \frac{\nu_{v_n}}{\varepsilon_n^2} \nu_{v_n} \right| \, dH^2.
\end{equation}
We now periodize the functions $v_n$ in the transverse direction defining
\[
\hat{v}_n(x_n, x_3) := \begin{cases} 
  v_n(x_n, -\varepsilon_n - x_3) & \text{if } -\varepsilon_n < x_3 \leq -\frac{3}{4}, \\
  v_n(x_n, x_3) & \text{if } -\frac{3}{4} < x_3 < \frac{3}{4}, \\
  v_n(x_n, \varepsilon_n - x_3) & \text{if } \frac{3}{4} \leq x_3 < \varepsilon_n.
\end{cases}
\]
Then $\hat{v}_n \in SBV^p(\omega \times (-\varepsilon_n, \varepsilon_n); \mathbb{R}^3)$ for each $n \in \mathbb{N}$ and from (4.3) and (4.4) it follows that
\[
\sup_{n \in \mathbb{N}} \left\{ \|\hat{v}_n\|_{L^\infty(\omega \times (-\varepsilon_n, \varepsilon_n); \mathbb{R}^3)} + \frac{1}{\varepsilon_n} \int_{\omega \times (-\varepsilon_n, \varepsilon_n)} |\nabla \hat{v}_n|^p \, dx \right\} < +\infty.
\]
and
\[
\mathcal{H}^2(S\hat{v}_n) = 2\varepsilon_n \int_{S\hat{v}_n} \left| (\nu_{\hat{v}_n})_2 |\nabla \hat{v}_n|_3 \right| \, d\mathcal{H}^2.
\]
We are now in a position to extend $\hat{v}_n$ by periodicity in the $x_3$ direction. Note that we do not create any additional jump set because periodicity ensures continuity at the interface of each slice. Let
\[
N_n := \begin{cases} 
  \frac{1}{4\varepsilon_n} - \frac{1}{2} & \text{if } \frac{1}{4\varepsilon_n} - \frac{1}{2} \in \mathbb{N}, \\
  \left[ \frac{1}{4\varepsilon_n} + \frac{1}{2} \right] & \text{otherwise}
\end{cases}
\]
where $[t]$ denotes the integer part of $t$. For every $i \in \{-N_n, \ldots, N_n\}$, we set $I_{n,i} := ((2i - 1)\varepsilon_n, \varepsilon_n)$ and $\Omega_{n,i} := \omega \times I_{n,i}$. Note that $N_n$ is the largest integer such that $\Omega \cap \Omega_{n,i} \neq \emptyset$ for every $i \in \{-N_n, \ldots, N_n\}$. We define the function $\tilde{v}_n$ on $\Omega(n) := \omega \times (-2N_n + 1)\varepsilon_n, (2N_n + 1)\varepsilon_n)$ by extending $\hat{v}_n$ by periodicity in the $x_3$ direction on $\Omega(n)$:
\[
\tilde{v}_n(x_n, x_3) = \hat{v}_n(x_n, x_3 - 2i\varepsilon_n) \text{ if } x_3 \in I_{n,i}.
\]
Since $\Omega \subset \Omega(n)$, $\tilde{v}_n \in SBV^p(\Omega; \mathbb{R}^3)$ and thanks to (4.5) and the definition of $N_n$, we have that
\[
\sup_{n \in \mathbb{N}} \left\{ \|\tilde{v}_n\|_{L^\infty(\Omega; \mathbb{R}^3)} + \int_{\Omega} |\nabla \tilde{v}_n|^p \, dx \right\} < +\infty
\]
while (4.6) together with (4.2) imply that
\[
\mathcal{H}^2(S\tilde{v}_n) \leq c \int_{S\tilde{v}_n} \left| (\nu_{\tilde{v}_n})_2 |\nabla \tilde{v}_n|_3 \right| \, d\mathcal{H}^2 \rightarrow 0.
\]
As a consequence of (4.7) and (4.8), the sequence $\{\tilde{v}_n\}$ fulfills the assumptions of Theorem 3.1. Hence there exist a subsequence $\{\varepsilon_{n_k}\} \subset \{\varepsilon_n\}$ and a sequence $\{w_k\} \subset W^{1,\infty}(\Omega; \mathbb{R}^3)$ such that
\[
\begin{align*}
\sup_{k \in \mathbb{N}} \|w_k\|_{W^{1,\infty}(\Omega; \mathbb{R}^3)} &< +\infty, \\
\{ |\nabla w_k|^p \} &\text{ is equi-integrable,} \\
\mathcal{L}^3(\{\nabla v_{n_k} \neq w_k\} \cup \{\nabla v_{n_k} \neq |\nabla w_k| \}) &\rightarrow 0.
\end{align*}
\]
From De La Vallée Poussin’s criterion, one can find an increasing and continuous function $\vartheta : [0, +\infty) \rightarrow [0, +\infty]$ such that $\vartheta(t)/t \rightarrow +\infty$ as $t \rightarrow +\infty$ and
\[
\sup_{k \in \mathbb{N}} \int_{\Omega} \vartheta(|\nabla w_k|^p) \, dx < +\infty.
\]
We claim that for at least half of the indexes $i \in \{-N_{n_k} + 1, \ldots, N_{n_k} - 1\}$, there holds
\[
\frac{2N_{n_k} - 1}{2} \int_{\Omega_{n_k}} [\vartheta(|\nabla w_k|^p) + |w_k|^p + |\nabla w_k|^p] \, dx \leq \int_{\Omega} [\vartheta(|\nabla w_k|^p) + |w_k|^p + |\nabla w_k|^p] \, dx.
\]
Similarly we may show that (5.1)

\[
\int_{\Omega} \left[ \frac{\varrho(|\nabla w_k|^p) + |w_k|^p + |\nabla w_k|^p}{x} \right] \, dx \geq \sum_{i \in J_k} \int_{\Omega \cap \{x \leq \frac{1}{\varepsilon}w_k \}} \left[ \frac{\varrho(|\nabla w_k|^p) + |w_k|^p + |\nabla w_k|^p}{x} \right] \, dx
\]

which is absurd. Similarly, one can show that for at least half of the indexes satisfying (4.9), we have that

\[
(4.10) \quad \frac{2N_{n_{k}}-1}{2} L^3 \Omega \setminus \{u_{n_{k}} \neq w_{k} \} \cup \{\nabla u_{n_{k}} \neq \nabla z_{k} \} \leq L^3 \{\tilde{v}_{n_{k}} \neq w_{k} \} \cup \{\nabla \tilde{v}_{n_{k}} \neq \nabla w_{k} \}.
\]

Let \( i_{k} \in \{-N_{n_{k}}+1, \ldots, N_{n_{k}}-1\} \) be such that (4.9) and (4.10) hold at the same time. Define now \( z_{k}(x_{\alpha}, x_{3}) := w_{k}(x_{\alpha}, \varepsilon_{n_{k}} x_{3} + 2\varepsilon_{n_{k}} i_{k}) \). Changing variable in (4.9) and (4.10) and using the construction of \( \tilde{v}_{n_{k}} \) from \( u_{n_{k}} \) we get that

\[
\varepsilon_{n_{k}} \frac{2N_{n_{k}}-1}{2} \int_{\Omega} \left[ \varrho \left( \left( \nabla_{\alpha} z_{k} \right) \frac{1}{\varepsilon_{n_{k}}} \nabla_{3} z_{k} \right) \right]^p + |z_{k}|^p + \left( \nabla_{\alpha} z_{k} \frac{1}{\varepsilon_{n_{k}}} \nabla_{3} z_{k} \right) \right]^p \, dx \leq \int_{\Omega} \left[ \varrho \left( |\nabla w_k|^p \right) + |w_k|^p + |\nabla w_k|^p \right] \, dx
\]

and

\[
\varepsilon_{n_{k}} \frac{2N_{n_{k}}-1}{2} L^3 \Omega \{u_{n_{k}} \neq z_{k} \} \cup \{\nabla u_{n_{k}} \neq \nabla z_{k} \} \leq L^3 \{\tilde{v}_{n_{k}} \neq w_{k} \} \cup \{\nabla \tilde{v}_{n_{k}} \neq \nabla w_{k} \}.
\]

Since \( \varepsilon_{n_{k}}(2N_{n_{k}}-1) \geq 1/4 \) for \( k \) large enough, it follows that

\[
\left\{ \sup_{k \in \mathbb{N}} \int_{\Omega} \left[ \varrho \left( \left( \nabla_{\alpha} z_{k} \frac{1}{\varepsilon_{n_{k}}} \nabla_{3} z_{k} \right) \right)^p + |z_{k}|^p + \left( \nabla_{\alpha} z_{k} \frac{1}{\varepsilon_{n_{k}}} \nabla_{3} z_{k} \right) \right]^p \, dx < +\infty, \right\}
\]

and the equi-integrability of \( \left\{ \left( \nabla_{\alpha} z_{k} \frac{1}{\varepsilon_{n_{k}}} \nabla_{3} z_{k} \right) \right\} \) follows from De La Vallée Poussin’s criterion.

It remains to prove the weak convergence of \( z_{k} \) and \( (1/\varepsilon_{n_{k}}) \nabla_{3} z_{k} \). Let \( v \in L^p(\Omega; \mathbb{R}^3) \) with \( 1/p + 1/p' = 1 \), then

\[
\int_{\Omega} (z_{k} - u) \cdot v \, dx = \int_{\{z_{k} = u_{n_{k}}\}} (u_{n_{k}} - u) \cdot v \, dx + \int_{\{z_{k} \neq u_{n_{k}}\}} (z_{k} - u) \cdot v \, dx.
\]

As \( \mathcal{L}^3 \{z_{k} \neq u_{n_{k}}\} \to 0 \), it follows that \( v \chi_{\{z_{k} = u_{n_{k}}\}} \to v \) in \( L^p(\Omega; \mathbb{R}^3) \). Then, using Hölder’s Inequality, the fact that \( \{z_{k}\} \) is uniformly bounded in \( L^p(\Omega; \mathbb{R}^3) \) and that \( u_{n_{k}} \to u \) in \( L^p(\Omega; \mathbb{R}^3) \), we obtain that

\[
\lim_{k \to +\infty} \int_{\Omega} (z_{k} - u) \cdot v \, dx \leq \lim_{k \to +\infty} \int_{\Omega} (u_{n_{k}} - u) \cdot v \chi_{\{z_{k} = u_{n_{k}}\}} \, dx + \lim_{k \to +\infty} \|z_{k} - u\|_{L^p(\Omega; \mathbb{R}^3)} \|v \chi_{\{z_{k} \neq u_{n_{k}}\}}\|_{L^{p'}(\Omega; \mathbb{R}^3)} = 0.
\]

Similarly we may show that \( \nabla z_{k} \to \nabla u \) in \( L^p(\Omega; \mathbb{R}^{3 \times 3}) \) and that \( (1/\varepsilon_{n_{k}}) \nabla_{3} z_{k} \to b \) in \( L^p(\Omega; \mathbb{R}^3) \).

5. Integral representation for dimension reduction problems in Sobolev spaces involving the bending moment

Consider a Carathéodory function \( W_{\varepsilon} : \Omega \times \mathbb{R}^{3 \times 3} \to [0, +\infty) \) satisfying uniform \( p \)-growth and \( p \)-coercivity conditions: there exist \( 0 < \beta' \leq \beta < +\infty \) and \( 1 < p < +\infty \) such that

\[
(5.1) \quad \beta' |\xi|^p \leq W_{\varepsilon}(x, \xi) \leq \beta(1 + |\xi|^p)
\]
for a.e. \( x \in \Omega \) and all \( \xi \in \mathbb{R}^{3 \times 3} \). Define \( J_\varepsilon : L^p(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \times \mathcal{A}(\omega) \to [0, +\infty] \) by

\[
J_\varepsilon(u, b, A) := \left\{ \begin{array}{ll}
\int_{A \times I} W_\varepsilon \left( x, \nabla u(x) \right) \, dx & \text{if } u \in W^{1,p}(A \times I; \mathbb{R}^3), \\
0 & \text{otherwise}.
\end{array} \right.
\]

We prove the following integral representation for the \( \Gamma \)-limit.

**Theorem 5.1.** For every sequence \( \{\varepsilon_n\} \searrow 0^+ \), there exist a subsequence (not relabeled) and a Carathéodory function \( W^* : \omega \times \mathbb{R}^{3 \times 2} \times \mathbb{R}^3 \to [0, +\infty) \) (depending on the subsequence) such that for every \( A \in \mathcal{A}(\omega) \), the sequence \( J_{\varepsilon_n}(\cdot, \cdot, A) \) \( \Gamma \)-converges in \( L^p_2(A \times I; \mathbb{R}^3) \times L^p_2(A; \mathbb{R}^3) \) to \( J(\cdot, \cdot, A) \) where

\[
J(u, b, A) := \left\{ \begin{array}{ll}
\int_A W^*(x, u(x), \nabla u(x) b(\nabla u(x))) \, dx & \text{if } u \in W^{1,p}(A; \mathbb{R}^3), \\
+\infty & \text{otherwise}.
\end{array} \right.
\]

**Proof.** For every \( \{\varepsilon_n\} \searrow 0^+ \), \( u \in L^p(\Omega; \mathbb{R}^3) \), \( b \in L^p(\omega; \mathbb{R}^3) \) and \( A \in \mathcal{A}(\omega) \), let

\[
J(u, b, A) := \inf_{\{u_n, b_n\}} \left\{ \liminf_{n \to +\infty} J_{\varepsilon_n}(u_n, b_n, A) : u_n \to u \text{ in } L^p(A \times I; \mathbb{R}^3), b_n \to b \text{ in } L^p(A; \mathbb{R}^3) \right\}.
\]

Repeating word for word the (standard) proof of [9, Lemma 2.1] one can show that there exists a subsequence, still labeled \( \{\varepsilon_n\} \), such that for any \( A \in \mathcal{A}(\omega) \), \( J(\cdot, \cdot, A) \) is the \( \Gamma \)-limit in \( L^p_2(A \times I; \mathbb{R}^3) \times L^p_2(A; \mathbb{R}^3) \) of \( J_{\varepsilon_n}(\cdot, \cdot, A) \), that \( J(u, b, A) = +\infty \) if \( u \in L^p(\Omega; \mathbb{R}^3) \backslash W^{1,p}(A; \mathbb{R}^3) \) and that for every \( (u, b) \in W^{1,p}(\omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \), the set function \( J(u, b, \cdot) \) is the restriction to \( \mathcal{A}(\omega) \) of a Radon measure absolutely continuous with respect to the Lebesgue measure \( \mathcal{L}^2 \). The remaining of the proof is very close to that of [14, Theorem 1.1], thus we will only point out the main changes. Let \( \xi \in \mathbb{R}^{3 \times 2} \), \( z \in \mathbb{R}^3 \) and \( x_0 \in \omega \), define

\[
W^*(x_0, \xi, z) := \limsup_{\rho \to 0^+} \frac{\int_A W^*(x, u(x), \nabla u(x) b(x)) \, dx}{\rho^2}
\]

where we have denoted \( u_\xi(x_{0}) := \xi x_{0} \) and \( b_\xi(x_{0}) := z \). Since \( J(u_\xi, b_\xi, \cdot) \) is (the restriction of) a Radon measure absolutely continuous with respect to \( \mathcal{L}^2 \), we have for every \( A \in \mathcal{A}(\omega) \),

\[
J(u_\xi, b_\xi, A) = \int_A W^*(x_0, \xi, z) \, dx_0 = \int_A W^*(x_0, \xi, z) \, dx_0.
\]

By additivity, it is clear that

\[
J(u, b, A) = \int_A W^*(x_0, \xi, z) \, dx_0
\]

holds whenever \( u \) is piecewise affine and \( b \) is piecewise constant in \( A \) and we wish to extend (5.3) to arbitrary functions \( u \in W^{1,p}(A; \mathbb{R}^3) \) and \( b \in L^p_2(A; \mathbb{R}^3) \).

Using the lower semicontinuity of \( J \) and a suitable choice of sequence, one can show as in [14, Theorem 1.1] that \( \xi \mapsto W^*(x_0, \xi, z) \) is rank one convex. We claim that \( z \mapsto W^*(x_0, \xi, z) \) is convex. To see this let \( \theta \in [0, 1] \), \( z_1, z_2 \in \mathbb{R}^3 \) and \( \xi \in \mathbb{R}^{3 \times 2} \). Fix \( x_0 \in \omega, \rho > 0 \) and take an open set \( A \subset Q(x_0, \rho) \) such that \( \mathcal{L}^2(\partial A) = 0 \) and \( \mathcal{L}^2(A) = \theta \rho^2 \) (take e.g. \( A = Q(x_0, \sqrt{\theta} \rho) \)). Define

\[
\bar{b}_n(x_{0}) := z_1 \chi(n_{x_{0}}) + z_2(1 - \chi(n_{x_{0}}))
\]

where \( \chi \) is the characteristic function of \( A \in Q'(x_0, \rho) \) which has been extended to \( \mathbb{R}^3 \) by \( \rho \)-periodicity. Riemann-Lebesgue’s Lemma asserts that \( \bar{b}_n \to \bar{b}_{\theta z_1 + (1 - \theta) z_2} \text{ in } L^p(Q'(x_0, \rho); \mathbb{R}^3) \) and since \( J(u_\xi, \cdot, Q'(x_0, \rho)) \) is sequentially weakly lower semicontinuous in \( L^p(Q'(x_0, \rho); \mathbb{R}^3) \), it follows that

\[
J(u_\xi, b_{\theta z_1 + (1 - \theta) z_2}, Q'(x_0, \rho)) \leq \liminf_{n \to +\infty} J(u_\xi, \bar{b}_n, Q'(x_0, \rho)) = \liminf_{n \to +\infty} \left\{ J(u_\xi, \bar{b}_{z_1}, A_n) + J(u_\xi, \bar{b}_{z_2}, Q'(x_0, \rho \setminus \bar{A}_n) \right\}
\]

for a.e. \( x \in \Omega \) and all \( \xi \in \mathbb{R}^{3 \times 3} \).
where $A_n := \{ x_\alpha \in Q'(x_0, \rho) : \chi(n x_\alpha) = 1 \}$ is an open set. Note that in the last equality, we have used the fact that since $\mathcal{L}^2(\partial A_n) = 0$, then $\mathcal{J}(u_{x_\alpha}, b_{z_1}, \partial A_n) = 0$ as well and that $\mathcal{J}$ is local on open sets. Using once more the Riemann-Lebesgue Lemma together with (5.2), we get that

$$\lim_{n \to +\infty} \mathcal{J}(u_{x_\alpha}, b_{z_1}, A_n) = \lim_{n \to +\infty} \int_{Q'(x_0, \rho)} \chi(n x_\alpha) W^*(x_\alpha, \nabla \xi_{z_1}) dx_\alpha$$

and similarly for the second term of (5.4). Hence we deduce that

$$\mathcal{J}(u_{x_\alpha}, b_{z_1}, Q'(x_0, \rho)) = \theta \int_{Q'(x_0, \rho)} W^*(x_\alpha, \nabla \xi_{z_1}) dx_\alpha$$

and the convexity of $W^*(x_0, \nabla \xi_{z_1})$ arises after dividing the previous inequality by $\rho^2$ and taking the lim sup as $\rho$ tends to zero. It follows that $(\xi_{z_1}) \mapsto W^*(x_0, \nabla \xi_{z_1})$ is separately convex for a.e. $x_0 \in \omega$ and since the following $p$-growth and $p$-coercivity conditions hold

$$\beta'(|\xi|^p) \leq W^*(x_0, \nabla \xi_{z_1}) \leq \beta(1 + |\xi|^p)$$

for a.e. $x_0 \in \omega$ and all $(\xi, z) \in \mathbb{R}^3 \times \mathbb{R}^3$, we conclude that $(\xi_{z_1}) \mapsto W^*(x_0, \nabla \xi_{z_1})$ is continuous for a.e. $x_0 \in \omega$ which proves that $W^*$ is a Carathéodory function.

We now prove that (5.3) holds for any $(u, \bar{b}) \in W^{1,p}(A; \mathbb{R}^3) \times L^p(A; \mathbb{R}^3)$. By approximation and thanks to the lower semicontinuity of $\mathcal{J}(\cdot, \cdot, A)$ for the strong $W^{1,p}(A; \mathbb{R}^3) \times L^p(A; \mathbb{R}^3)$ topology, there holds

$$\mathcal{J}(u, \bar{b}, A) \leq \int_A W^*(x_\alpha, \nabla u |\bar{b}|) dx_\alpha$$

for any $(u, \bar{b}) \in W^{1,p}(A; \mathbb{R}^3) \times L^p(A; \mathbb{R}^3)$ and it remains to prove the converse inequality. This is achieved exactly as in the final step of the proof of [14, Theorem 1.1], by considering the translated functional

$$\bar{\mathcal{J}}(v, \bar{\tau}, A) := \mathcal{J}(u + v, \bar{b} + \bar{\tau}, A)$$

where $(u, \bar{b})$ are arbitrary functions in $W^{1,p}(A; \mathbb{R}^3) \times L^p(A; \mathbb{R}^3)$.

We refer to [8, 9] for more explicit formulas for the integrand $W^*$ in particular cases.

The following technical proposition states some kind of blow-up result for functionals through $\Gamma$-convergence. It will be of use in the proof of the lower bound in Theorem 6.1 because at some point, we will need to get rid of small residual terms occurring inside the integrand $W_\varepsilon$. In [5, 6, 7], this difficulty was treated thanks to a decoupling variable method which consisted in replacing the function $W_\varepsilon$ by a much more regular one thanks to Scorza-Dragoni’s Theorem and Tietze’s Extension Theorem, and the set where these two integrands did not match was controlled thanks to the equi-integrability result [8, Theorem 1.1]. This method was quite powerful in that context since the manner on which $W_\varepsilon$ was depending on $\varepsilon$ was completely known. However, in the generalized framework considered here, it does not apply anymore since we have no information on the way $W_\varepsilon$ depends on $\varepsilon$. The following blow up result, together with a diagonalization argument (see Remark 5.3 below), will enable us to overcome that problem.

**Proposition 5.2.** There exists a set $N \subset \omega$ with $\mathcal{L}^2(N) = 0$ such that for every $\{\rho_k\} \searrow 0^+$ and every $x_0 \in \omega \setminus N$, the functional $J_k : L^p(B' \times I; \mathbb{R}^3) \times L^p(B'; \mathbb{R}^3) \to [0, +\infty]$ defined by

$$J_k(u, \bar{b}) = \begin{cases} \int_{B'} W^*(x_0 + \rho_k x_\alpha, \nabla u |\bar{b}(x_\alpha)|) dx_\alpha & \text{if } u \in W^{1,p}(B'; \mathbb{R}^3) \\ +\infty & \text{otherwise,} \end{cases}$$

for any $(u, \bar{b}) \in W^{1,p}(A; \mathbb{R}^3) \times L^p(A; \mathbb{R}^3)$.
\begin{proof}
The proof relies on the Scorza-Dragoni Theorem (see e.g. [17, Chapter VIII]). For any $q \in \mathbb{N}$, there exists a compact set $K_q \subset \omega$ with $\mathcal{L}^2(\omega \setminus K_q) < 1/q$ and such that $W^*$ is continuous on $K_q \times \mathbb{R}^3 \times \mathbb{R}^3$. Let $N := \omega \setminus \bigcup_q K_q^*$ where
\begin{equation}
K_q^* := \left\{ x \in K_q : \lim_{\rho \to 0} \frac{\mathcal{L}^2(B'(x_0, \rho) \setminus K_q)}{\mathcal{L}^2(B'(x_0, \rho))} = 0 \right\}.
\end{equation}
Since $\mathcal{L}^2(K_q \setminus K_q^*) = 0$, then $\mathcal{L}^2(N) \leq \mathcal{L}^2(\omega \setminus K_q^*) = \mathcal{L}^2(\omega \setminus K_q) < 1/q \to 0$. Select a point $x_0 \in \omega \setminus N$, so that $x_0 \in K_q^*$ for some $q \in \mathbb{N}$.

The upper bound. Assume first that $u \in W^{1,\infty}(B'; \mathbb{R}^3)$ and $\tilde{b} \in L^\infty(B'; \mathbb{R}^3)$ and set $M := \|(\nabla_a u(\tilde{b}))\|_{L^\infty(B'; \mathbb{R}^3)}$. Then according to the $p$-growth condition (5.5)
\begin{equation}
J_k(u, \tilde{b}) = \int_{B'} W^*(x_0 + \rho_k x_a, \nabla_a u(\tilde{b})) \, dx_a
\end{equation}
As $W^*$ is uniformly continuous on $K_q \times B(0, M)$, there exists a continuous and increasing function $\eta : [0, +\infty) \to [0, +\infty)$ such that $\eta(0) = 0$ and
\begin{equation}
\int_{B' \cap \left(\frac{K_q - x_0}{\rho_k}\right)} W^*(x_0 + \rho_k x_a, \nabla_a u(\tilde{b})) = \int_{B' \cap \left(\frac{K_q - x_0}{\rho_k}\right)} W^*(x_0, \nabla_a u(\tilde{b})) \, dx_a \leq \eta(\rho_k).
\end{equation}
Gathering (5.6), (5.7) and (5.8) and passing to the limit as $k \to +\infty$ yields to
\begin{equation}
\Gamma-\limsup_{k \to +\infty} J_k(u, \tilde{b}) \leq \limsup_{k \to +\infty} J_k(u, \tilde{b}) \leq J(u, \tilde{b}).
\end{equation}
The general case follows from the density of $W^{1,\infty}(B'; \mathbb{R}^3) \times L^\infty(B'; \mathbb{R}^3)$ in $W^{1,p}(B'; \mathbb{R}^3) \times L^p(B'; \mathbb{R}^3)$, the lower continuity of the $\Gamma$-limsup and the continuity of $J$ for the strong $W^{1,p}(B'; \mathbb{R}^3) \times L^p(B'; \mathbb{R}^3)$-topology.

The lower bound. Let $(u_k, \tilde{b}_k) \in L^p(B' \times I; \mathbb{R}^3) \times L^p(B'; \mathbb{R}^3)$ and $\{(u_k, \tilde{b}_k)\} \subset L^p(B' \times I; \mathbb{R}^3) \times L^p(B'; \mathbb{R}^3)$ such that $u_k \to u$ in $L^p(B' \times I; \mathbb{R}^3)$, $\tilde{b}_k \rightharpoonup \tilde{b} \text{ in } L^p(B'; \mathbb{R}^3)$ and
\begin{equation}
\liminf_{k \to +\infty} J_k(u_k, \tilde{b}_k) < +\infty.
\end{equation}
Up to a subsequence (not relabeled) we can suppose that $u$ and $u_k \in W^{1,p}(B'; \mathbb{R}^3)$ for each $k \in \mathbb{N}$ and that $u_k \rightharpoonup u$ in $W^{1,p}(B'; \mathbb{R}^3)$. According to the Decomposition Lemma [20, Lemma 1.2] and Chacon’s Biting Lemma [3, Lemma 5.32], there is no loss of generality to assume that $\{\nabla_a u_k\}$ and $\{\tilde{b}_k\}$ are equi-integrable. Define the set $A(k) := \left\{ x_a \in B' : \|(\nabla a u_k(x_a))\|_{L^p(B')} \leq t \right\}$. From Chebyshev’s Inequality we have that $\mathcal{L}^2(B' \setminus A_k) \leq c/t^{p}$ for some constant $c > 0$ independent of $t$ and $k$ and arguing exactly as in the proof of the upper bound, one can show that for each $t > 0$,
\begin{equation}
\lim_{k \to +\infty} J_k(u_k, \tilde{b}_k) \geq \frac{1}{t^{p}} \int_{A_k(k)} W^*(x_0, \nabla a u_k(\tilde{b}_k)) \, dx_a.
\end{equation}
According to the $p$-growth condition (5.5) and (5.6),
\begin{equation}
\int_{A_k(k)} W^*(x_0, \nabla a u_k(\tilde{b}_k)) \, dx_a \leq \beta(1 + t^{p}) \mathcal{L}^2 \left( B' \setminus \left( \frac{K_q - x_0}{\rho_k} \right) \right) \to 0 \quad \text{as } k \to +\infty.
\end{equation}
while the equi-integrability of \( \{|\nabla \alpha u_k|^p\} \) and \( \{|\vec{b}_k|^p\} \) and the fact that \( L^2(B' \setminus A_k') \rightarrow 0 \) as \( t \rightarrow +\infty \) imply that
\[
\sup_{k \in \mathbb{N}} \int_{B' \setminus A_k'} W^*(x_0, \nabla \alpha u_k, \vec{b}_k) \, dx \leq \beta \sup_{k \in \mathbb{N}} \int_{B' \setminus A_k'} (1 + |\nabla \alpha u_k|^p + |\vec{b}_k|^p) \, dx \overset{t \rightarrow +\infty}{\longrightarrow} 0.
\]

Hence gathering (5.9), (5.10) and (5.11) yields to
\[
\liminf_{k \rightarrow +\infty} J_k(u_k, \vec{b}_k) \geq \liminf_{k \rightarrow +\infty} J(u_k, \vec{b}_k) \geq J(u, \vec{b})
\]
where the last inequality holds because \( J \) is sequentially weakly lower semicontinuous in \( W^{1,p}(B'; \mathbb{R}^3) \times L^p(B'; \mathbb{R}^3) \).

**Remark 5.3.** One can show that in Theorem 5.1, the value of \( \mathcal{J} \) does not change replacing \( W_{\varepsilon_n} \) by its quasiconvexification \( QW_{\varepsilon_n} \) defined by
\[
QW_{\varepsilon_n}(x, \xi) := \inf_{\varphi \in W_0^{1,\infty}((0,1)^3; \mathbb{R}^3)} \int_{(0,1)^3} W_{\varepsilon_n}(x, \xi + \nabla \varphi(y)) \, dy \quad \text{for all} \ \xi \in \mathbb{R}^{3 \times 3} \text{ and a.e. } x \in \Omega.
\]
Hence there is no loss of generality to assume in Theorem 5.1 that \( W_{\varepsilon} \) is quasiconvex. Since the weak topology on every normed bounded subsets of \( L^p(B'; \mathbb{R}^3) \) is metrizable, it follows from a diagonalization argument, Theorem 5.1, Proposition 5.2 and the fact that \( \Gamma \)-convergence of coercive and lower semicontinuous functionals on a metric space is metrizable (see \([16, \text{Theorem 10.22 (a)}]\)), that for every \( L \)
\[
\text{topology on every normed bounded subsets of } \mathbb{R}^3.
\]

We now come to the heart of this study that is dealing with a similar problem than in Theorem 5.1 but in the framework of Special functions with Bounded Variation, adding a surface energy term. Let us define
\[
\mathcal{G}_{\varepsilon}(u, \vec{b}) := \left\{ \begin{array}{ll}
\int_{\Omega} W_{\varepsilon}(x, \nabla \alpha u \frac{1}{\varepsilon} \nabla^3 u) \, dx + \int_{S_{\varepsilon}} \left| (\nu_{\varepsilon})_3 \frac{1}{\varepsilon} (\nu_{\varepsilon})_3 \right| \, dH^2 & \text{if } u \in SBV^p(\Omega; \mathbb{R}^3), \\
\int_{\Omega} W^*(x, \nabla \alpha u \vec{b}) \, dx & \text{otherwise,}
\end{array} \right.
\]
for every \( u \in \omega \setminus N, \) where \( N \subset \omega \) is the same exceptional set than in Proposition 5.2.

6. **Integral representation for dimension reduction problems in SBV involving the bending moment**

We now come to the heart of this study that is dealing with a similar problem than in Theorem 5.1 but in the framework of Special functions with Bounded Variation, adding a surface energy term. Let us define
\[
\mathcal{G}_{\varepsilon}(u, \vec{b}) := \left\{ \begin{array}{ll}
\int_{\Omega} W_{\varepsilon}(x, \nabla \alpha u \frac{1}{\varepsilon} \nabla^3 u) \, dx + \int_{S_{\varepsilon}} \left| (\nu_{\varepsilon})_3 \frac{1}{\varepsilon} (\nu_{\varepsilon})_3 \right| \, dH^2 & \text{if } u \in SBV^p(\Omega; \mathbb{R}^3), \\
\int_{\Omega} W^*(x, \nabla \alpha u \vec{b}) \, dx & \text{otherwise,}
\end{array} \right.
\]
Then, the following \( \Gamma \)-convergence result holds:

**Theorem 6.1.** For every sequence \( \{\varepsilon_n\} \searrow 0^+ \), there exists a subsequence, still labeled \( \{\varepsilon_n\} \) such that \( \mathcal{G}_{\varepsilon_n} \) \( \Gamma \)-converges in \( L^1(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \) to \( \mathcal{G} : BV(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \rightarrow [0, +\infty] \) defined by
\[
\mathcal{G}(u, \vec{b}) := \left\{ \begin{array}{ll}
\int_{\omega} W^*(x, \nabla \alpha u \vec{b}) \, dx + H^1(S_u) & \text{if } u \in SBV^p(\omega; \mathbb{R}^3), \\
+\infty & \text{otherwise,}
\end{array} \right.
\]
where \( W^* \) is given by Theorem 5.1.
The remaining of this section is devoted to prove Theorem 6.1. We will first localize the functional $G_\varepsilon$ on $A(\omega)$, and noticing that minimizing sequences are not necessarily weakly relatively compact in $BV$, we will use the same truncation argument than in [4] (see also [18]) introducing an artificial functional. Then we will show that it actually coincides with the $\Gamma$-limit whenever $\varepsilon \in BV^p(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ (see Lemma 6.2 and Remark 6.3) and it will enable us to show that for such $\varepsilon$’s the $\Gamma$-limit is a measure absolutely continuous with respect to $L^2 + H^1(\cdot, S_u)$ (see Lemma 6.6). Together with a blow up argument, this property will be useful to prove the upper bound in Lemma 6.3 while the lower bound, Lemma 6.4, will obtained thanks to Theorem 4.1 and a suitable diagonalization argument (see Remark 5.3).

6.1. Localization. We first localize our functional on $A(\omega)$ defining $G_\varepsilon : BV(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \times A(\omega) \to [0, +\infty)$ by

$$
G_\varepsilon(u, \overline{b}, A) := \begin{cases}
\int_{AxI} W_\varepsilon \left( x, \nabla u \left| -\frac{1}{\varepsilon} \nabla_3 u \right| \right) dx & \text{if } u \in SBV^p(A \times I; \mathbb{R}^3), \\
\int_{S_u \cap (AxI)} \left| (\nu_u)_\alpha \right| \cdot \frac{1}{\varepsilon} (\nu_u)_3 \right| d\mathcal{H}^2 & \text{if } u \in BV(\Omega; \mathbb{R}^3), \\
+\infty & \text{otherwise}.
\end{cases}
$$

For every sequence $\{\varepsilon_n\} \searrow 0^+$ and all $(u, \overline{b}, A) \in BV(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \times A(\omega)$, we define

$$
E(u, \overline{b}, A) := \inf \left\{ \liminf_{n \to +\infty} G_\varepsilon(u_n, \overline{b}_n, A) : u_n \to u \text{ in } L^1(A \times I; \mathbb{R}^3), \overline{b}_n \to \overline{b} \text{ in } L^p(A; \mathbb{R}^3) \right\}.
$$

Theorem 8.5 and Corollary 8.12 in [16] together with a diagonalization argument imply the existence of a subsequence, still denoted $\{\varepsilon_n\}$, such that, for any $A \in \mathcal{R}(\omega)$ (or $A = \omega$), $E(\cdot, \cdot, A)$ is the $\Gamma$-limit of $G_\varepsilon(\cdot, \cdot, A)$ in $L^1(\cdot \times I; \mathbb{R}^3) \times L^p(\cdot; \mathbb{R}^3)$. Extracting if necessary a further subsequence, one may assume that $\{\varepsilon_n\}$ is chosen so that Theorem 5.1 holds. To prove Theorem 6.1, it is enough to show that $E(u, \overline{b}, \omega) = G(u, \overline{b})$.

6.2. A truncation argument. As pointed out in [4], the main problem with the definition of $E$ in (6.1) is that minimizing sequences are not necessarily bounded in $BV(\Omega; \mathbb{R}^3)$ and thus, not necessarily weakly convergent in this space. Following [4], we define for all $(u, \overline{b}, A) \in BV(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \times A(\omega)$

$$
E_\infty(u, \overline{b}, A) := \inf \left\{ \liminf_{n \to +\infty} G_\varepsilon(u_n, \overline{b}_n, A) : u_n \to u \text{ in } L^1(A \times I; \mathbb{R}^3), \overline{b}_n \to \overline{b} \text{ in } L^p(A; \mathbb{R}^3), \sup_{n \in \mathbb{N}} \|u_n\|_{L^\infty(A \times I; \mathbb{R}^3)} < +\infty \right\}.
$$

It is immediate that $E(u, \overline{b}, A) \leq E_\infty(u, \overline{b}, A)$ while we will show that equality holds when $u$ belongs to $BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$. This will be obtained as a consequence of Lemma 6.2 below. It means that for such deformation fields $u \in BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$, strong $L^1(\Omega; \mathbb{R}^3)$-convergence and weak $BV(\Omega; \mathbb{R}^3)$-convergence are, in a sense, equivalent for the computation of the $\Gamma$-limit.

Lemma 6.2. Let $A \in A(\omega)$, $u \in BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ and $\overline{b} \in L^p(\omega; \mathbb{R}^3)$. If $\{u_n\} \subset SBV^p(A \times I; \mathbb{R}^3)$ is such that $u_n \to u$ in $L^1(A \times I; \mathbb{R}^3)$, $\frac{1}{\varepsilon_n} \int_I \nabla_3 u_n(\cdot, x_3) dx_3 \to \overline{b}$ in $L^p(A; \mathbb{R}^3)$ and the following limit

$$
L := \lim_{n \to +\infty} G_{\varepsilon_n} \left( u_n, \frac{1}{\varepsilon_n} \int_I \nabla_3 u_n(\cdot, x_3) dx_3, A \right)
$$

exists and is finite. Then, for any $n \geq 0$ one can find $C > 0$ and $\{w_n\} \subset SBV^p(A \times I; \mathbb{R}^3)$ such that $w_n \to u$ in $L^1(A \times I; \mathbb{R}^3)$, $\frac{1}{\varepsilon_n} \int_I \nabla_3 w_n(\cdot, x_3) dx_3 \to \overline{b}$ in $L^p(A; \mathbb{R}^3)$, $\sup_n \|w_n\|_{L^\infty(A \times I; \mathbb{R}^3)} \leq C$ and

$$
L \geq \liminf_{n \to +\infty} G_{\varepsilon_n} \left( w_n, \frac{1}{\varepsilon_n} \int_I \nabla_3 w_n(\cdot, x_3) dx_3, A \right) - \eta.
$$
Proof. Let us define a smooth truncation function \( \varphi_i \in C_c^1(\mathbb{R}^3; \mathbb{R}^3) \) satisfying
\[
(6.2) \quad \varphi_i(s) = \begin{cases} 
 s & \text{if } |s| < e^i, \\
 0 & \text{if } |s| \geq e^{i+1}
\end{cases} \quad \text{and} \quad |\nabla \varphi_i(s)| \leq 2.
\]

Let \( w_{n,i} := \varphi_i(u_n) \), thanks to the Chain Rule formula [3, Theorem 3.96], \( w_{n,i} \in SBV^p(A \times I; \mathbb{R}^3) \) and
\[
(6.3) \quad \begin{cases} 
 \|w_{n,i}\|_{L^\infty(A \times I; \mathbb{R}^3)} \leq e^{i+1}, \\
 S_{w_{n,i}} \subset S_{u_n}, \\
 \nabla w_{n,i} = \nabla \varphi_i(u_n) \nabla u_n \quad \mathcal{L}^3\text{-a.e. in } A \times I.
\end{cases}
\]

Since \( u \in L^\infty(\Omega; \mathbb{R}^3) \), we can choose \( i \) large enough (\( i \geq m := \lfloor \ln(\|u\|_{L^\infty(\Omega; \mathbb{R}^3)}) \rfloor + 1 \)) so that \( u = \varphi_i(u) \) and thus according to (6.2)
\[
(6.4) \quad \|w_{n,i} - u\|_{L^p(A \times I; \mathbb{R}^3)} = \|\varphi_i(u_n) - \varphi_i(u)\|_{L^1(A \times I; \mathbb{R}^3)} \leq 2\|u_n - u\|_{L^1(A \times I; \mathbb{R}^3)}.
\]

Since (a subsequence of) \( u_n \rightarrow u \) a.e. in \( A \times I \) and \( \nabla \varphi_i \) is continuous, it follows that \( \nabla \varphi_i(u_n) \rightarrow \nabla \varphi_i(u) = \text{Id a.e. in } A \times I \) as \( n \rightarrow +\infty \). Take \( v \in L^p(A; \mathbb{R}^3) \) where \( 1/p + 1/p' = 1 \), as \( |\nabla \varphi_i(u_n)^T v| \leq 2|v| \in L^p(A) \), the Dominated Convergence Theorem implies that \( \nabla \varphi_i(u_n)^T v \rightharpoonup v \) in \( L^p(A \times I; \mathbb{R}^3) \) and thus
\[
\lim_{n \rightarrow +\infty} \int_A \left( \frac{1}{\varepsilon_n} \int_I \nabla_3 w_{n,i}(x_\alpha, x_3) \, dx_3 \right) \cdot v(x_\alpha) \, dx_\alpha = \lim_{n \rightarrow +\infty} \int_{A \times I} \frac{1}{\varepsilon_n} \nabla_3 u_n \cdot (\nabla \varphi_i(u_n)^T v) \, dx
\]
\[
= \int_A \overline{b} \cdot v \, dx_\alpha,
\]
where we used the fact that \( (1/\varepsilon_n) \nabla_3 u_n \rightharpoonup \overline{b} \) in \( L^p(A \times I; \mathbb{R}^3) \) and \( \overline{b} = \int_I b(\cdot, x_3) \, dx_3 \). Hence
\[
(6.5) \quad \frac{1}{\varepsilon_n} \int_I \nabla_3 w_{n,i}(\cdot, x_3) \, dx_3 \rightharpoonup \overline{b} \text{ in } L^p(A; \mathbb{R}^3), \text{ for all } i \geq m.
\]

The growth condition (5.1), (6.2) and (6.3) imply that
\[
\int_{A \times I} W_{\varepsilon_n} \left( x, \nabla_\alpha w_{n,i}, \frac{1}{\varepsilon_n} \nabla_3 w_{n,i} \right) \, dx
\]
\[
\leq \int_{\{|u_n| < e^i\}} W_{\varepsilon_n} \left( x, \nabla_\alpha u_n, \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \, dx + \beta \mathcal{L}^3(\{|u_n| \geq e^{i+1}\})
\]
\[
+ \int_{\{e^i \leq |u_n| < e^{i+1}\}} W_{\varepsilon_n} \left( x, \nabla \varphi_i(u_n) \nabla_\alpha u_n, \frac{1}{\varepsilon_n} \nabla \varphi_i(u_n) \nabla_3 u_n \right) \, dx
\]
\[
\leq \int_{A \times I} W_{\varepsilon_n} \left( x, \nabla_\alpha u_n, \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \, dx + \beta e^{-i} \|u_n\|_{L^1(A \times I; \mathbb{R}^3)}
\]
\[
+ 2^p \beta \int_{\{e^i \leq |u_n| < e^{i+1}\}} \left( \left| \nabla_\alpha u_n \frac{1}{\varepsilon_n} \nabla_3 u_n \right| \right)^p \, dx,
\]
where we have used Chebyshev’s Inequality. Since \( \nu_{w_{n,i}}(x) = \pm \nu_{u_n}(x) \) for \( \mathcal{H}^2\)-a.e. \( x \in S_{w_{n,i}} \), (6.3) yields to
\[
(6.7) \quad \int_{S_{w_{n,i}} \cap (A \times I)} \left| \left( \nu_{w_{n,i}} \right)_\alpha \frac{1}{\varepsilon_n} \left( \nu_{w_{n,i}} \right)_3 \right| \, d\mathcal{H}^2 \leq \int_{S_{u_n} \cap (A \times I)} \left| \left( \nu_{u_n} \right)_\alpha \frac{1}{\varepsilon_n} \left( \nu_{u_n} \right)_3 \right| \, d\mathcal{H}^2.
\]
Let $M \in \mathbb{N}$, from (6.6) and (6.7), a summation for $i = m$ to $M$ implies that
\[
\frac{1}{M - m + 1} \sum_{i=m}^{M} \int_{A \times I} W_{\varepsilon_n} \left( x, \nabla_\alpha w_{n,i} \mid \frac{1}{\varepsilon_n} \nabla_3 w_{n,i} \right) \, dx + \int_{S_{w_n, \cap (A \times I)}} \left( \left( \nu_{w_n} \right)_\alpha \mid \frac{1}{\varepsilon_n} \left( \nu_{w_n} \right)_3 \right) \, d\mathcal{H}^2 \\
\leq \int_{A \times I} W_{\varepsilon_n} \left( x, \nabla_\alpha w_{n} \mid \frac{1}{\varepsilon_n} \nabla_3 w_{n} \right) \, dx + \int_{S_{w_n, \cap (A \times I)}} \left( \left( \nu_{w_n} \right)_\alpha \mid \frac{1}{\varepsilon_n} \left( \nu_{w_n} \right)_3 \right) \, d\mathcal{H}^2 + \frac{c}{M - m + 1},
\]
where
\[
c = \beta \sup_{n \in \mathbb{N}} \| u_n \|_{L^1(A \times I; \mathbb{R}^3)} \sum_{i \geq 1} e^{-i} + 2^p \beta \sup_{n \in \mathbb{N}} \left\| \left( \nabla_\alpha u_n \mid \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right\|_p^{L^p(A \times I; \mathbb{R}^{3 \times 3})} < +\infty.
\]

We may find some $i_n \in \{m, \ldots, M\}$ such that, setting $w_n := w_{n,i_n}$, then
\[
\int_{A \times I} W_{\varepsilon_n} \left( x, \nabla_\alpha w_{n} \mid \frac{1}{\varepsilon_n} \nabla_3 w_{n} \right) \, dx + \int_{S_{w_n, \cap (A \times I)}} \left( \left( \nu_{w_n} \right)_\alpha \mid \frac{1}{\varepsilon_n} \left( \nu_{w_n} \right)_3 \right) \, d\mathcal{H}^2 \\
\leq \int_{A \times I} W_{\varepsilon_n} \left( x, \nabla_\alpha w_{n} \mid \frac{1}{\varepsilon_n} \nabla_3 w_{n} \right) \, dx + \int_{S_{w_n, \cap (A \times I)}} \left( \left( \nu_{w_n} \right)_\alpha \mid \frac{1}{\varepsilon_n} \left( \nu_{w_n} \right)_3 \right) \, d\mathcal{H}^2 + \frac{c}{M - m + 1}.
\]
Moreover, in view of (6.4) and (6.5), $w_n \to u$ in $L^1(A \times I; \mathbb{R}^3)$, $\frac{1}{\varepsilon_n} \int_I \nabla_3 w_n(\cdot, x_3) \, dx_3 \to \bar{b}$ in $L^p(A; \mathbb{R}^3)$ and (6.3) implies that $\| w_n \|_{L^\infty(A \times I; \mathbb{R}^3)} \leq e^{\eta n + 1} \leq e^{M + 1}$. The proof is achieved passing to the limit as $n$ tends to $+\infty$ in (6.8) and choosing $M$ large enough so that $c/(M - m + 1) \leq \eta$.

**Remark 6.3.** As a consequence of Lemma 6.2, we get that for any $A \in \mathcal{R}(\omega)$ (or $A = \omega$), every $u \in BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ and every $\bar{b} \in L^p(\omega; \mathbb{R}^3)$, then $\mathcal{E}(u, \bar{b}, A) = \mathcal{E}_\infty(u, \bar{b}, A)$.

**Remark 6.4.** A similar statement of Lemma 6.2 can be proved in the framework of Sobolev spaces, replacing $\mathcal{G}_{\varepsilon_n}$ by $\mathcal{F}_{\varepsilon_n}$.

**Remark 6.5.** Using a relaxation argument in $SBV^p$ as in the proof of [4, Lemma 3.4] and Lemma 6.2, one can show that if $u \in SBV^p(\omega; \mathbb{R}^3) \cap L^\infty(\omega; \mathbb{R}^3)$ and if $\bar{b} \in L^p(\omega; \mathbb{R}^3)$, the value of $\mathcal{E}_\infty$ does not change replacing $W_{\varepsilon_n}$ by is quasiconvexification $QW_{\varepsilon_n}$ defined in (5.12). The main point is that the diagonalization argument can still be used despite the weak $L^p(\omega; \mathbb{R}^3)$-convergence of the bending moment since the dual of $L^p(\omega; \mathbb{R}^3)$ is separable. Hence we may assume without loss of generality that $W_{\varepsilon}$ is quasiconvex. In particular (see [15, Lemma 2.2, Chapter 4]), the following $p$-Lipschitz condition holds,
\[
| W_{\varepsilon}(x, \xi_1) - W_{\varepsilon}(x, \xi_2) | \leq c(1 + | \xi_1 |^{p-1} + | \xi_2 |^{p-1}) | \xi_1 - \xi_2 |, \text{ for all } \xi_1, \xi_2 \in \mathbb{R}^{3 \times 3} \text{ and a.e. } x \in \Omega.
\]

Lemma 6.2 and Remark 6.3 are essential for the proof of the following result because they allow us to replace strong $L^1(\Omega; \mathbb{R}^3)$-convergence of any minimizing sequence by strong $L^p(\omega; \mathbb{R}^3)$-convergence.

**Lemma 6.6.** For all $u \in SBV^p(\omega; \mathbb{R}^3) \cap L^\infty(\omega; \mathbb{R}^3)$ and all $\bar{b} \in L^p(\omega; \mathbb{R}^3)$, $\mathcal{E}_\infty(u, \bar{b}, \cdot)$ is the restriction to $A(\omega)$ of a Radon measure absolutely continuous with respect to $\mathcal{L}^2 + \mathcal{H}^1 \ll S_u$.

**Proof.** Let $u \in SBV^p(\omega; \mathbb{R}^3) \cap L^\infty(\omega; \mathbb{R}^3)$, $A \in \mathcal{A}(\omega)$ and assume first that $\bar{b}$ is smooth. Then taking $u_n(x_\alpha, x_3) := u(x_\alpha) + \varepsilon_n x_3 \bar{b}(x_\alpha)$ and $\bar{b}_n(x_\alpha) := \bar{b}(x_\alpha)$ as test functions for $\mathcal{E}_\infty(u, \bar{b}, A)$ and using the $p$-growth condition (5.1), we get that
\[
\mathcal{E}_\infty(u, \bar{b}, A) \leq \beta \int_A \left( 1 + | \nabla u |^p + | \bar{b} |^p \right) \, dx_\alpha + \mathcal{H}^1(S_u \cap A).
\]
The same inequality holds for arbitrary functions $\bar{b} \in L^p(\omega; \mathbb{R}^3)$ thanks to the density of smooth maps into $L^p(\omega; \mathbb{R}^3)$ and the sequential weak lower semicontinuity of $\mathcal{E}_\infty(u, \cdot, A)$ in $L^p(A; \mathbb{R}^3)$. The remaining
of the proof is very classical and is essentially the same than that of [4, Lemma 3.6]. As usual, the most delicate point is to prove the subadditivity of $\mathcal{E}_\infty(u, \overline{b}, \cdot)$ and this is done by gluing together suitable minimizing sequences by means of a cut-off function. The argument still works with the presence of the bending moment since the cut-off function is chosen independently of $x_3$. One should once more be careful when applying a diagonalization argument because of the weak convergence in $L^p$. As already mentioned in Remark 6.5, it is still allowed in the case where we include the bending moment since dual of $L^p$ is separable.

As a consequence of Lemma 6.6 and Lebesgue’s Decomposition Theorem, there exists a $L^2$-measurable function $f$ and a $\mathcal{H}^1 \ll S_u$-measurable function $g$ such that for every $A \in A(\omega)$,

$$\mathcal{E}_\infty(u, \overline{b}, A) = \int_A f \, d\mathcal{L}^2 + \int_{A \cap S_u} g \, d\mathcal{H}^1. \tag{6.11}$$

Since the measures $\mathcal{L}^2$ and $\mathcal{H}^1 \ll S_u$ are mutually singular, $f$ is the Radon-Nikodým derivative of $\mathcal{E}_\infty(u, \overline{b}, \cdot)$ with respect to $\mathcal{L}^2$,

$$f(x_0) = \lim_{\rho \to 0} \frac{\mathcal{E}_\infty(u, \overline{b}, B'(x_0, \rho))}{\mathcal{L}^2(B'(x_0, \rho))}, \quad \text{for } \mathcal{L}^2\text{-a.e. } x_0 \in \omega$$

and $g$ is the Radon-Nikodým derivative of $\mathcal{E}_\infty(u, \overline{b}, \cdot)$ with respect to $\mathcal{H}^1 \ll S_u$,

$$g(x_0) = \lim_{\rho \to 0} \frac{\mathcal{E}_\infty(u, \overline{b}, B'(x_0, \rho))}{\mathcal{H}^1(S_u \cap B'(x_0, \rho))}, \quad \text{for } \mathcal{H}^1\text{-a.e. } x_0 \in S_u.$$

### 6.3. The upper bound

We first show the upper bound. To this end, we will use the locality property of the $\Gamma$-limit proved in the previous subsection when $u \in BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ and the analogue $\Gamma$-convergence result in Sobolev spaces (Theorem 5.1).

**Lemma 6.7.** For all $u \in BV(\Omega; \mathbb{R}^3)$ and all $\overline{b} \in L^p(\omega; \mathbb{R}^3)$, $\mathcal{E}(u, \overline{b}, \omega) \leq \mathcal{G}(u, \overline{b})$.

**Proof.** It is enough to consider the case where $\mathcal{G}(u, \overline{b}) < +\infty$ and thus $u \in SBVP(\omega; \mathbb{R}^3)$. In fact, we will first restrict to the case where $u \in L^\infty(\omega; \mathbb{R}^3) \cap SBVP(\omega; \mathbb{R}^3)$ because thanks to Remark 6.3, it allows us to replace $\mathcal{E}$ by $\mathcal{E}_\infty$. According to (6.11) and the definition of $\mathcal{G}$, we must show that $g(x_0) \leq 1$ for $\mathcal{H}^1\text{-a.e. } x_0 \in S_u$ and $f(x_0) \leq W^*(x_0, \nabla u(x_0), \overline{b}(x_0))$ for $\mathcal{L}^2\text{-a.e. } x_0 \in \omega$.

Let us first treat the surface term. By virtue of (6.10) with $A = B'(x_0, \rho)$, we have that for $\mathcal{H}^1\text{-a.e. } x_0 \in S_u$,

$$g(x_0) = \lim_{\rho \to 0} \frac{\mathcal{E}_\infty(u, \overline{b}, B'(x_0, \rho))}{\mathcal{H}^1(S_u \cap B'(x_0, \rho))} \leq \lim_{\rho \to 0} \frac{1}{\mathcal{H}^1(S_u \cap B'(x_0, \rho))} \left\{ \beta \int_{B'(x_0, \rho)} (1 + |\nabla u|^p + |\overline{b}|^p) \, dx + \mathcal{H}^1(S_u \cap B'(x_0, \rho)) \right\}$$

$$= \lim_{\rho \to 0} \frac{\mu(B'(x_0, \rho))}{\mathcal{H}^1(S_u \cap B'(x_0, \rho))} + 1,$$

where we set $\mu := \beta(1 + |\nabla u|^p + |\overline{b}|^p) \mathcal{L}^2$. But since $\mu$ and $\mathcal{H}^1 \ll S_u$ are mutually singular, we have for $\mathcal{H}^1\text{-a.e. } x_0 \in S_u$

$$\lim_{\rho \to 0} \frac{\mu(B'(x_0, \rho))}{\mathcal{H}^1(S_u \cap B'(x_0, \rho))} = 0,$$

which shows that $g(x_0) \leq 1$ for $\mathcal{H}^1\text{-a.e. } x_0 \in S_u$. 
Concerning the bulk term, choose \( x_0 \in \omega \) to be a Lebesgue point of \( u, \nabla_\alpha u, \overline{b} \) and \( W^*(\cdot, \nabla_\alpha u(\cdot)\overline{b}(\cdot)) \) and such that

\[
\lim_{\rho \to 0} \frac{\mathcal{H}^1(S_u \cap B'(x_0, \rho))}{\mathcal{L}^2(B'(x_0, \rho))} = 0.
\]

Remark that \( L^3 \) almost every points \( x_0 \) in \( \omega \) satisfy these properties and set \( u_0(x_\alpha) := \nabla_\alpha u(x_\alpha) \) and \( \overline{b}_0(x_\alpha) := \overline{b}(x_\alpha) \). For every \( \rho > 0 \), Theorem 5.1 implies the existence of a sequence \( \{v^\rho_n\} \subset W^{1,p}(B'(x_0, \rho) \times I; \mathbb{R}^3) \) such that \( v^\rho_n \rightharpoonup u_0 \) in \( L^p(B'(x_0, \rho) \times I; \mathbb{R}^3) \) (thus a fortiori in \( L^1(B'(x_0, \rho) \times I; \mathbb{R}^3) \)),

\[
\lim_{n \to +\infty} \int_{B'(x_0, \rho) \times I} W_\varepsilon_n \left( x, \nabla_\alpha v^\rho_n \left| \frac{1}{\varepsilon_n} \nabla_\alpha v^\rho_n \right| \right) dx = \int_{B'(x_0, \rho)} W^*(x_\alpha, \nabla_\alpha u(x_\alpha)\overline{b}(x_\alpha)) dx_\alpha.
\]

Since \( u_0 \in L^\infty(\omega; \mathbb{R}^3) \), by Lemma 6.2 and Remark 6.4, for any \( \eta > 0 \) we can find a sequence \( \{w^\rho_n\} \subset W^{1,p}(B'(x_0, \rho) \times I; \mathbb{R}^3) \) and \( C_\rho > 0 \) such that \( \sup_n \|w^\rho_n\|_{L^\infty(B'(x_0, \rho) \times I; \mathbb{R}^3)} \leq C_\rho \), \( w^\rho_n \to u_0 \) in \( L^p(B'(x_0, \rho) \times I; \mathbb{R}^3) \),

\[
\limsup_{n \to +\infty} \int_{B'(x_0, \rho) \times I} W_\varepsilon_n \left( x, \nabla_\alpha w^\rho_n \left| \frac{1}{\varepsilon_n} \nabla_\alpha w^\rho_n \right| \right) dx \leq \int_{B'(x_0, \rho)} W^*(x_\alpha, \nabla_\alpha u(x_\alpha)\overline{b}(x_\alpha)) dx_\alpha + C_\rho \mathcal{L}^2(B'(x_0, \rho)) \eta.
\]

Thanks to (5.5) and the separately convex character of \( W^*(x_\alpha, \cdot) \) (see the proof of Theorem 5.1), it follows that \( W^*(x_\alpha, \cdot) \) is \( p \)-Lipschitz. Thus our choice of \( x_0 \) implies that

\[
\lim_{\rho \to 0} \limsup_{n \to +\infty} \int_{B'(x_0, \rho) \times I} W_\varepsilon_n \left( x, \nabla_\alpha u^\rho_n \left| \frac{1}{\varepsilon_n} \nabla_\alpha u^\rho_n \right| \right) dx \leq W^*(x_\alpha, \nabla_\alpha u(x_\alpha)\overline{b}(x_\alpha)) + \eta
\]

and from the coercivity condition (5.1), we get

\[
\sup_{\rho > 0, n \in \mathbb{N}} \int_{B'(x_0, \rho) \times I} \left\| \left( \nabla_\alpha u^\rho_n \left| \frac{1}{\varepsilon_n} \nabla_\alpha u^\rho_n \right| \right) \right\|^p dx < +\infty.
\]

Let \( \overline{b}_k \in C_c^\infty(\omega; \mathbb{R}^3) \) be such that \( \overline{b}_k \to \overline{b} \) in \( L^p(\omega; \mathbb{R}^3) \) and define

\[
u^\rho_{n,k}(x) := u(x_\alpha) + \varepsilon_n x_3 (\overline{b}_k(x_\alpha) - \overline{b}(x_\alpha)) + u^\rho_n(x_\alpha, x_3) - \nabla_\alpha u(x_\alpha) x_\alpha.
\]

Then, \( u^\rho_{n,k} \to u \) in \( L^1(B'(x_0, \rho) \times I; \mathbb{R}^3) \),

\[
\lim_{n \to +\infty} \int_{B'(x_0, \rho) \times I} W_{\varepsilon_n} \left( x, \nabla_\alpha u^\rho_{n,k} \left| \frac{1}{\varepsilon_n} \nabla_\alpha u^\rho_{n,k} \right| \right) dx \leq W^*(x_\alpha, \nabla_\alpha u(x_\alpha)\overline{b}(x_\alpha)) + \eta
\]

and from the coercivity condition (5.1), we get

\[
\sup_{\rho > 0, n \in \mathbb{N}} \int_{B'(x_0, \rho) \times I} \left\| \left( \nabla_\alpha u^\rho_{n,k} \left| \frac{1}{\varepsilon_n} \nabla_\alpha u^\rho_{n,k} \right| \right) \right\|^p dx < +\infty.
\]

Let \( \overline{b}_k \in C_c^\infty(\omega; \mathbb{R}^3) \) be such that \( \overline{b}_k \to \overline{b} \) in \( L^p(\omega; \mathbb{R}^3) \) and define

\[
u^\rho_{n,k}(x) := u(x_\alpha) + \varepsilon_n x_3 (\overline{b}_k(x_\alpha) - \overline{b}(x_\alpha)) + u^\rho_n(x_\alpha, x_3) - \nabla_\alpha u(x_\alpha) x_\alpha.
\]

Then, \( u^\rho_{n,k} \to u \) in \( L^1(B'(x_0, \rho) \times I; \mathbb{R}^3) \),

\[
\lim_{n \to +\infty} \int_{B'(x_0, \rho) \times I} W_{\varepsilon_n} \left( x, \nabla_\alpha u^\rho_{n,k} \left| \frac{1}{\varepsilon_n} \nabla_\alpha u^\rho_{n,k} \right| \right) dx \leq W^*(x_\alpha, \nabla_\alpha u(x_\alpha)\overline{b}(x_\alpha)) + \eta
\]

and from the coercivity condition (5.1), we get

\[
\lim_{\rho \to 0} \limsup_{n \to +\infty} \int_{B'(x_0, \rho) \times I} W_\varepsilon_n \left( x, \nabla_\alpha u^\rho_{n,k} \left| \frac{1}{\varepsilon_n} \nabla_\alpha u^\rho_{n,k} \right| \right) dx \leq W^*(x_\alpha, \nabla_\alpha u(x_\alpha)\overline{b}(x_\alpha)) + \eta
\]

and from the coercivity condition (5.1), we get

\[
\sup_{\rho > 0, n \in \mathbb{N}} \int_{B'(x_0, \rho) \times I} \left\| \left( \nabla_\alpha u^\rho_{n,k} \left| \frac{1}{\varepsilon_n} \nabla_\alpha u^\rho_{n,k} \right| \right) \right\|^p dx < +\infty.
\]
Thus from (6.12), we obtain
\[
f(x_0) \leq \liminf_{\rho \to 0} \liminf_{k \to +\infty} \liminf_{n \to +\infty} \int_{B'(x_0, \rho) \times I} W_{\varepsilon_n} \left( x, \nabla_{\alpha} u(x_0) - \nabla_{\alpha} u(x_0) + \nabla_{\alpha} w^p_n(x) \right) + \varepsilon_n x_3 \nabla_{\alpha} b_k(x_0) \left( \frac{1}{\varepsilon_n} \nabla_{\alpha} w^p_n(x) + b_k(x_0) - b(x_0) \right) \, dx.
\]

Relations (6.9), (6.13), (6.14) and Hölder’s inequality yield
\[
f(x_0) \leq \liminf_{\rho \to 0} \liminf_{k \to +\infty} \liminf_{n \to +\infty} \left\{ \int_{B'(x_0, \rho) \times I} W_{\varepsilon_n} \left( x, \nabla_{\alpha} w^p_n \left( \frac{1}{\varepsilon_n} \nabla_3 w^p_n(x) \right) \right) dx 
+ c \int_{B'(x_0, \rho) \times I} \left( 1 + |\nabla_{\alpha} u(x_0) - \nabla_{\alpha} u(x_0)|^{p-1} + |b_k(x_0) - b(x_0)|^{p-1} \right) \left( |\nabla_{\alpha} u(x_0) - \nabla_{\alpha} u(x_0)| + \varepsilon_n |\nabla_{\alpha} b_k(x_0) + b_k(x_0) - b(x_0)| \right) dx \right\}
\leq W^*(x_0, \nabla_{\alpha} u(x_0), b(x_0)) + \eta
\]
\[
+ c \limsup_{\rho \to 0} \left\{ \int_{B'(x_0, \rho)} \left( 1 + |\nabla_{\alpha} u(x_0) - \nabla_{\alpha} u(x_0)|^{p} + |b(x_0) - b(x_0)|^{p} \right) \, dx \right\}^{(p-1)/p}
\times \left\{ \int_{B'(x_0, \rho)} \left( |\nabla_{\alpha} u(x_0) - \nabla_{\alpha} u(x_0)|^{p} + |b(x_0) - b(x_0)|^{p} \right) \, dx \right\}^{1/p}.
\]

Thanks to our choice of \( x_0 \) and letting \( \eta \to 0 \), we conclude that \( f(x_0) \leq W^*(x_0, \nabla_{\alpha} u(x_0), b(x_0)) \) for \( L^2 \)-a.e. \( x_0 \in \omega \) which completes the proof in the case where \( u \in L^\infty(\omega; \mathbb{R}^3) \cap SBV^p(\omega; \mathbb{R}^3) \). The general case can in turn be treated by approximation exactly as in the proof of [4, Lemma 3.8]. \( \square \)

6.4. The lower bound. Let us now prove the lower bound. The proof is essentially based on Theorem 4.1 and a blow up argument.

**Lemma 6.8.** For all \( u \in BV(\Omega; \mathbb{R}^3) \) and all \( \bar{b} \in L^p(\omega; \mathbb{R}^3) \), \( \mathcal{E}(u, \bar{b}, \omega) \geq \mathcal{G}(u, \bar{b}) \).

**Proof.** It is not restrictive to assume that \( \mathcal{E}(u, \bar{b}, \omega) < +\infty \). By \( \Gamma \)-convergence, there exists a sequence \( \{u_n\} \subset SBV^p(\Omega; \mathbb{R}^3) \) such that \( u_n \rightharpoonup u \) in \( L^1(\Omega; \mathbb{R}^3) \), \( \frac{1}{\varepsilon_n} \int_\Omega \nabla_3 u_n(x, x_3) \, dx \rightharpoonup \bar{b} \) in \( L^p(\omega; \mathbb{R}^3) \) and
\[
\lim_{n \to +\infty} \int_\Omega \left( \int_{\varepsilon_n} W_{\varepsilon_n} \left( x, \nabla_{\alpha} u_n \left( \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right) \, dx \right) + \int_{\varepsilon_n} \left| \left( \nu_{u_n} \right)_{\alpha} \left( \frac{1}{\varepsilon_n} \nu_{u_n} \right) \right| \, dH^2 \right) = \mathcal{E}(u, \bar{b}, \omega).
\]

Arguing exactly as in the proof of [4, Lemma 3.9], we can actually show that \( u \in SBV^p(\omega; \mathbb{R}^3) \) and that \( u_n \rightharpoonup u \) in \( SBV^p(\Omega; \mathbb{R}^3) \). Now for every Borel set \( E \subset \omega \), define the following sequences of Radon measures:
\[
\lambda_n(E) := W_{\varepsilon_n} \left( \cdot, \nabla_{\alpha} u_n \left( \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right) L^3 \mathcal{L}(E \times I) + \left| \left( \nu_{u_n} \right)_{\alpha} \left( \frac{1}{\varepsilon_n} \nu_{u_n} \right) \right| \mathcal{H}^2 \mathcal{L}(S_{u_n} \cap (E \times I))
\]
and
\[
\mu_n(E) := \left| \left( \nu_{u_n} \right)_{\alpha} \left( \frac{1}{\varepsilon_n} \nu_{u_n} \right) \right| \mathcal{H}^2 \mathcal{L}(S_{u_n} \cap (E \times I)).
\]

Then for a subsequence (not relabeled), there exist nonnegative and finite Radon measures \( \lambda \) and \( \mu \in \mathcal{M}(\omega) \) such that \( \lambda_n \rightharpoonup \lambda \) and \( \mu_n \rightharpoonup \mu \) in \( \mathcal{M}(\omega) \). By the Besicovitch Differentiation Theorem ([3,
Theorem 2.22], one can find three mutually disjoint nonnegative Radon measures $\lambda^a$, $\lambda^j$ and $\lambda^c$ such that $\lambda = \lambda^a + \lambda^j + \lambda^c$ where $\lambda^a \ll \mathcal{L}^2$ and $\lambda^j \ll \mathcal{H}^1 \mathcal{L}^2 S_u$. It is enough to check that

$$\frac{d\lambda^j}{d\mathcal{H}^1 \mathcal{L}^2 S_u}(x_0) \geq 1,$$  \hspace{1cm} \text{for } \mathcal{H}^1 \text{-a.e. } x_0 \in S_u

and

$$\frac{d\lambda^a}{d\mathcal{L}^2}(x_0) \geq W^*(x_0, \nabla \alpha u(x_0) \mid b(x_0)), \quad \text{for } \mathcal{L}^2 \text{-a.e. } x_0 \in \omega.$$  \hspace{1cm} (6.17)

Indeed, if (6.16) and (6.17) hold, we obtain from (6.15) that

$$E(u, b, \omega) \geq \lambda(\omega) = \lambda^a(\omega) + \lambda^j(\omega) + \lambda^c(\omega) \geq \int_{\omega} W^*(x_0, \nabla \alpha u(x_0) \mid b(x_0)) dx + \mathcal{H}^1(S_u) = G(u, b).$$

We first prove (6.16). Fix a point $x_0 \in S_u$ such that $d\lambda^j/d\mathcal{H}^1 \mathcal{L}^2 S_u(x_0)$ exists and is finite and remark that $\mathcal{H}^1 \text{-a.e. points in } S_u$ satisfy this property. Let $\{\rho_k\} \downarrow 0$ be such that $\lambda(\partial B(x_0, \rho_k)) = 0$ for each $k \in \mathbb{N}$. Then,

$$\frac{d\lambda}{d\mathcal{H}^1 \mathcal{L}^2 S_u}(x_0) = \frac{d\lambda}{d\mathcal{H}^1 \mathcal{L}^2 S_u}(x_0) = \lim_{k \to +\infty} \frac{\lambda(B(x_0, \rho_k))}{\mathcal{H}^1(S_u \cap B'(x_0, \rho_k))} = \lim_{k \to +\infty} \lim_{n \to +\infty} \frac{\lambda_n(B(x_0, \rho_k))}{\mathcal{H}^1(S_u \cap B'(x_0, \rho_k))} \geq \lim_{k \to +\infty} \lim_{n \to +\infty} \frac{\mathcal{H}^2(S_u \cap (B'(x_0, \rho_k) \times I))}{\mathcal{H}^1(S_u \cap B'(x_0, \rho_k))}.$$

By [3, Theorem 4.36], we have that

$$\lim_{n \to +\infty} \mathcal{H}^2(S_u \cap (B'(x_0, \rho_k) \times I)) \geq \mathcal{H}^1(S_u \cap B'(x_0, \rho_k))$$

hence we obtain (6.16).

Let us prove that (6.17) holds at every point $x_0 \in \omega \setminus N$ (where $N \subset \omega$ is the exceptional set introduced in Proposition 5.2) which is a Lebesgue point of both $\nabla \alpha u$ and $\mathcal{L}^2$, a point of approximate differentiability of $u$ such that

$$\frac{d\lambda^a}{d\mathcal{L}^2}(x_0) = \frac{d\lambda}{d\mathcal{L}^2}(x_0)$$

exists and is finite and satisfying

$$\lim_{\rho \to 0} \mu(B'(x_0, \rho))/2\rho = 0.$$  \hspace{1cm} (6.18)

It turns out that $\mathcal{L}^2 \text{-a.e. points } x_0 \in \omega \text{ satisfy these property. Indeed, the verification of (6.18) is similar to the one of (3.2) used in the proof of Theorem 3.2. As before, let } \{\rho_k\} \downarrow 0^+ \text{ be such that}
\(\lambda(\partial B'(x_0, \rho_k)) = 0\) for every \(k \in \mathbb{N}\), then
\[
\frac{d\lambda}{d\mathcal{L}^2}(x_0) = \lim_{k \to +\infty} \frac{\lambda(B'(x_0, \rho_k))}{\mathcal{L}^2(B'(x_0, \rho_k))} = \lim_{k \to +\infty} \lim_{n \to +\infty} \frac{\lambda_n(B'(x_0, \rho_k))}{\mathcal{L}^2(B'(x_0, \rho_k))} \geq \limsup_{k \to +\infty} \limsup_{n \to +\infty} \frac{1}{\mathcal{L}^2(B'(x_0, \rho_k))} \int_{B'(x_0, \rho_k) \times I} W_{\varepsilon_n} \left( x, \nabla u_{n,k} \right) dx,
\]
(6.19)
changing variables in the surface term and thanks to (6.18), it yields to
\[
\lambda \left( \partial B'(x_0, \rho_k) \right) = 0
\]
where \(u_{n,k}(x_0, x_3) = \left[ u_n(x_0 + \rho_k x_3) - u(x_0) \right] / \rho_k\). Since \(x_0\) is a point of approximate differentiability of \(u\), we have that
\[
\lim_{k \to +\infty} \lim_{n \to +\infty} \int_{B'(x_0, x_3)} \left| u_{n,k}(x) - \nabla u(x_0) x_3 \right| dx = 0
\]
and using the fact that \(x_0\) is a Lebesgue point of \(\overline{b}\), for every \(v \in L^p(B'; \mathbb{R}^3)\) we get that
\[
\lim_{k \to +\infty} \lim_{n \to +\infty} \int_{B'} \left( \frac{p_k}{\varepsilon_n} \right) \int_{I} \nabla u_{n,k}(x_0, x_3) dx_3 \cdot v(x_0) dx_3 = \int_{B'} \overline{b}(x_0) \cdot v dx_0.
\]
Changing variables in the surface term and thanks to (6.18), it yields to
\[
\limsup_{k \to +\infty} \limsup_{n \to +\infty} \int_{S_{\epsilon_n, k} \cap (B'(x_0, x_3))} \left| \left( \nu_{u_{n,k}} \right)_\alpha \right| \frac{1}{\varepsilon_n} \left( \nu_{u_{n,k}} \right)_3 \right| d\mathcal{H}^2
\]
\[
= \limsup_{k \to +\infty} \limsup_{n \to +\infty} \frac{1}{p_k} \int_{S_{\epsilon_n, k} \cap (B'(x_0, x_3))} \left| \left( \nu_{u_{n,k}} \right)_\alpha \right| \frac{1}{\varepsilon_n} \left( \nu_{u_{n,k}} \right)_3 \right| d\mathcal{H}^2
\]
\[
\leq \limsup_{k \to +\infty} \limsup_{n \to +\infty} \frac{\mu_n(B'(x_0, x_3))}{p_k}
\]
(6.22)
because \(\mu(\partial B'(x_0, \rho_k)) \leq \lambda(\partial B'(x_0, \rho_k)) = 0\). Set
\[
M := \max \left\{ \left( \frac{L^2(B')}{B'} \left( \left| \frac{d\lambda}{d\mathcal{L}^2}(x_0) \right| + 1 \right) \right)^{1/p}, \overline{b}(x_0) \right\} < +\infty.
\]
From (6.19)-(6.22), using a diagonalization argument, the fact that \(L^p(B'; \mathbb{R}^3)\) is separable and Remark 5.3, we can find a sequence \(n(k) \nearrow +\infty\) such that, setting \(\delta_k := \varepsilon_n(k) / p_k\), \(v_k := u_{n(k), k}, u_0(x_0) := \nabla u(x_0) x_3\) and \(\overline{b}_0(x_0) := \overline{b}(x_0)\), then \(\delta_k \to 0, v_k \to u_0\) in \(L^1(B' \times I; \mathbb{R}^3)\), \(\frac{1}{\delta_k} \int_{B'} \nabla v_k(x, x_3) dx_3 \to \overline{b}_0\) in \(L^p(B'; \mathbb{R}^3)\),
\[
\lim_{k \to +\infty} \int_{S_{\epsilon_k}} \left| \left( \nu_{v_k} \right)_\alpha \right| \frac{1}{\delta_k} \left( \nu_{v_k} \right)_3 \right| d\mathcal{H}^2 = 0,
\]
(6.24)
\[
\frac{d\lambda}{d\mathcal{L}^2}(x_0) \geq \limsup_{k \to +\infty} \frac{1}{\mathcal{L}^2(B')} \int_{B'(x_0, x_3)} W_{\varepsilon_n(k)} \left( x_0 + \rho_k x_3, x_3, \nabla v_k \left| \frac{1}{\delta_k} \nabla v_k \right| \right) dx,
\]
(6.25)
and for every $(u, b) \in L^p(B' \times I; \mathbb{R}^3) \times L^p(B'; \mathbb{R}^3)$ with $\|b\|_{L^p(B'; \mathbb{R}^3)} \leq M$, the $\Gamma$-limit in $L^p_\ast(B' \times I; \mathbb{R}^3) \times L^p(B'; \mathbb{R}^3)$ of
\[
\left\{ \int_{B' \times I} W_{\varepsilon_n(k)} \left( x_0 + \rho_k x_\alpha, x_3, \nabla_\alpha u \right) \frac{1}{\delta_k} \nabla_3 u \right\} dx
\]
coincides with
\[
\left\{ \int_{B'} W^\ast(x_0, \nabla_\alpha u(b)) dx_\alpha \right\} \text{ if } u \in W^{1,p}(B' \times I; \mathbb{R}^3),
\]
\[
+ \infty \text{ otherwise,}
\]
from (6.24), (6.25) and (a slight variant of) Lemma 6.2, for any $0 < \eta < 1$, there exist a constant $C > 0$ and \{w_k\} \subset SBV^p(B' \times I; \mathbb{R}^3) such that $w_k \rightharpoonup u_0$ in $L^1(B' \times I; \mathbb{R}^3)$, \[\frac{1}{\delta_k} \int_I \nabla_3 w_k(\cdot, x_3) dx_3 \rightarrow \overline{b}_0\] in $L^p(B'; \mathbb{R}^3)$, sup_k \|w_k\|_{L^\infty(B' \times I; \mathbb{R}^3)} \leq C.

Thus by our choice of the subsequence $\eta$ and using the $p$-coercivity condition (5.1) and [3, Theorem 4.36], the sequence \{w_k\} converges weakly to $u$ in $SBV^p(\Omega; \mathbb{R}^3)$ and it fulfills the assumptions of Theorem 4.1. Thus, for a not relabeled subsequence, one can find another sequence \{z_k\} \subset W^{1,\infty}(B' \times I; \mathbb{R}^3) such that $z_k \rightharpoonup u_0$ in $W^{1,p}(B' \times I; \mathbb{R}^3)$, \[\frac{1}{\delta_k} \int_I \nabla_3 z_k(\cdot, x_3) dx_3 \rightarrow \overline{b}_0\] in $L^p(B'; \mathbb{R}^3)$, \[\left\{ \left| \nabla_\alpha z_k \right| \frac{1}{\delta_k} \nabla_3 z_k \right\}^p \] is equi-integrable and $L^3(\{z_k \neq w_k\} \cup \{\nabla z_k \neq \nabla w_k\}) \rightarrow 0$. Hence
\[
\frac{d\lambda}{d\mathcal{L}^2}(x_0) \geq \limsup_{k \rightarrow +\infty} \frac{1}{\mathcal{L}^2(B')} \int_{\{w_k = z_k\}} W_{\varepsilon_n(k)} \left( x_0 + \rho_k x_\alpha, x_3, \nabla_\alpha w_k \left| \frac{1}{\delta_k} \nabla_3 z_k \right| \right) dx - \eta
\]
and using the $p$-growth condition (5.1), the fact that \[\left\{ \left| \nabla_\alpha z_k \right| \frac{1}{\delta_k} \nabla_3 z_k \right\}^p \] is equi-integrable and that $L^3(\{z_k \neq w_k\}) \rightarrow 0$ we get,

\[
\limsup_{k \rightarrow +\infty} \int_{\{w_k \neq z_k\}} W_{\varepsilon_n(k)} \left( x_0 + \rho_k x_\alpha, x_3, \nabla_\alpha z_k \left| \frac{1}{\delta_k} \nabla_3 z_k \right| \right) dx = 0.
\]

As a consequence
\[
\frac{d\lambda}{d\mathcal{L}^2}(x_0) \geq \limsup_{k \rightarrow +\infty} \frac{1}{\mathcal{L}^2(B')} \int_{B' \times I} W_{\varepsilon_n(k)} \left( x_0 + \rho_k x_\alpha, x_3, \nabla_\alpha z_k \left| \frac{1}{\delta_k} \nabla_3 z_k \right| \right) dx - \eta
\]
and by the $p$-coercivity condition (5.1) and (6.23),
\[
\left\| \frac{1}{\delta_k} \int_I \nabla_3 z_k(\cdot, x_3) dx_3 \right\|_{L^p(B'; \mathbb{R}^3)} \leq M, \quad \|\overline{b}_0\|_{L^p(B'; \mathbb{R}^3)} \leq M.
\]
Thus by our choice of the subsequence $n(k)$ and Remark 5.3, we get that
\[
\frac{d\lambda}{d\mathcal{L}^2}(x_0) \geq W^\ast(x_0, \nabla_\alpha u(x_0)) - \eta.
\]
Letting $\eta$ tend to zero completes the proof of (6.17). \hfill \Box

**Remark 6.9.** Note that it seems difficult to think of applying the decoupling variable method introduced in [7] and further developed in [5, 6]. Indeed, this generalized framework has the drawback that we have no information on the way that $W_\varepsilon$ depends on $\varepsilon$, and it requires application of such abstract results as metrizability of $\Gamma$-convergence. Remark also that the same kind of blow-up argument considered here
could have been used in [5, 6, 7] in place of the decoupling variable method, in order to treat the presence
of the spatial variable.

7. CASE WITHOUT BENDING MOMENT

In this last section, we deduce from Theorem 6.1 a similar result without the presence of the bending
moment. Define $I_\varepsilon : L^p(\Omega; \mathbb{R}^3) \to [0, +\infty]$ by

$$I_\varepsilon(u) := \begin{cases} \intW(x, \nabla\alpha u\frac{1}{\varepsilon}\nabla_3 u) \; dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

In [12, theorem 2.5], it has been proved the following integral representation result:

**Theorem 7.1.** For every sequence $\{\varepsilon_n\} \searrow 0^+$, there exist a subsequence (not relabeled) and a Carathéodory
function $\hat{W} : \omega \times \mathbb{R}^{3 \times 2} \to [0, +\infty)$ (depending on the subsequence) such that the sequence $I_{\varepsilon_n}$ $\Gamma$-converges
in $L^p_{\omega}(\Omega; \mathbb{R}^3)$ to $I$ where

$$I(u) = \begin{cases} \int_{\omega} \hat{W}(x_\alpha, \nabla\alpha u) \; dx_\alpha & \text{if } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

We refer to [22, 12, 7, 5, 6] for more explicit formulas in particular cases.

**Remark 7.2.** As it has been pointed out in [7] in the case where $W_\varepsilon$ was independent of $\varepsilon$ (see also [9]),
it can still be seen here that

$$\hat{W}(x_0, \xi) = \min_{z \in \mathbb{R}^3} W^*(x_0, \xi | z)$$

for all $\xi \in \mathbb{R}^{3 \times 2}$ and a.e. $x_0 \in \omega$.

Define now $F_\varepsilon : BV(\Omega; \mathbb{R}^3) \to [0, +\infty]$ by

$$F_\varepsilon(u) := \begin{cases} \intW(x, \nabla\alpha u\frac{1}{\varepsilon}\nabla_3 u) \; dx + \int_{S_u} \left| (\nu_\alpha)_{\frac{1}{\varepsilon}\nabla_3 \nu} \right| \; d\mathcal{H}^2 & \text{if } u \in SBV^p(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

As a consequence of Theorem 6.1, Theorem 7.1, Remark 7.2 and a standard measurability selection
criterion (see e.g. [17, Theorem 1.2, Chapter VIII]) we get the following integral representation result for
dimension reduction problems in $SBV$ without bending moment:

**Theorem 7.3.** For every sequence $\{\varepsilon_n\} \searrow 0^+$, there exists a subsequence, still labeled $\{\varepsilon_n\}$ such that
$F_{\varepsilon_n}$ $\Gamma$-converges in $L^1(\Omega; \mathbb{R}^3)$ to $F : BV(\Omega; \mathbb{R}^3) \to [0, +\infty]$ defined by

$$F(u) := \begin{cases} \int_{\omega} \hat{W}(x_\alpha, \nabla\alpha u) \; dx_\alpha + \mathcal{H}^1(S_u) & \text{if } u \in SBV^p(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\hat{W}$ is given by Theorem 7.1.

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