Effects of Modified Theories of Gravity on Neutrino Pair Annihilation Energy Deposition near Neutron Stars

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Abstract

We study the neutrino pair annihilation into electron–positron pairs ($\nu + \bar{\nu} \rightarrow e^- + e^+$) near the surface of a neutron star. The analysis is performed in the framework of extended theories of gravity. The latter induce a modification of the minimum photon sphere radius ($r_{ph\min}$) and the maximum energy deposition rate near $r_{ph\min}$, as compared to those of general relativity. These results might lead to an efficient mechanism for generating GRBs.

1. Introduction

General relativity (GR) is without a doubt the best theory of the gravitational interaction. Although its predictions have been tested to very high precision (Turyshhev 2009), there are still open questions that make GR incomplete. To address these shortcomings related to the cosmological standard model. To the high curvature regime requires curvature invariants, respectively.

The greatest success of GR. Despite these fundamental results, deviations from GR (hence from the Hilbert–Einstein action on which GR is based) are needed, and new ingredients, such as dark matter and dark energy, are required for building up self-consistent effective actions in curved spacetime (Barth & Christensen 1983; Birrell & Davies 1984; Buchbinder et al. 1992). It is worth mentioning that over the past decade some models have been proposed in which deviations from GR occur at the ultraweak-field regime by means of screening effects (Joyce et al. 2015). One introduces an additional degree of freedom (typically a scalar field) that obeys a nonlinear equation driven by the matter density, hence coupled to the environment. Screening mechanisms play a nontrivial role in what they allow to circumvent solar system and laboratory tests by suppressing, in a dynamical way, deviations from GR. In particular, the effects of the additional degrees of freedom (the scalar field) are hidden, in high-density regions, by the coupling of the field with matter while, in low-density regions, they are unsuppressed on cosmological scales. Screening mechanisms studied in the literature are the chameleon mechanism (Khoury & Weltman 2004a, 2004b), the symmetron mechanism (Hinterbichler & Khoury 2010), and Vainshtein mechanism (Vainshtein 1972). New tests of the gravitational interaction may, therefore, provide answers to these fundamental questions (Buoninfante et al. 2020a).

The aim of this paper is to investigate the effects of modified gravity on the neutrino pair annihilation efficiency.\textsuperscript{3} In particular, we focus on the process $\nu + \bar{\nu} \rightarrow e^- + e^+$, which is important for the delay shock mechanism into the Type II Supernova: indeed, at late times, from the hot proto-neutron star, the energy is deposited into the supernova envelope via neutrino pair annihilation and neutrino-lepton scattering. These processes augment the neutrino heating of the envelope generating a successful supernova explosion (Salmonson & Wilson 1999). Moreover, the process is relevant for collapsing

\textsuperscript{3} The role of gravity on the neutrino propagation has been studied both in GR (Ahluwalia & Burgard 1996; Piriz et al. 1996; Cardall & Fuller 1997; Lambiase et al. 2005; Dvornikov 2006; Visinelli 2015; Alexandre & Clough 2018; Capolupo et al. 2020; Dvornikov 2020; Swami et al. 2020) and in modified gravity (Capozziello & Lambiase 1999b; Chakraborty 2015; Buoninfante et al. 2020b).
neutron stars and for gamma-ray bursts, for which neutrino pair annihilation has been considered one of the possible sources. In Section 5 we discuss the previous results in the framework of gamma-ray bursts. Finally, in Section 6 we state our conclusions on the phenomena.

The paper is organized as follows. In Section 2 and Section 3 we introduce the formalism needed to study the energy deposition rate in a beyond GR framework. In Sections 4.1–4.6 we show our results for Charged Galileon, Einstein dilaton Gauss Bonnet, Brans Dicke, Eddington-inspired Born–Infeld, Born–Infeld generalization of Reissner–Nordstrom solution, and Higher derivative gravity, respectively. In Section 5 we discuss the results that modified theories of gravity might play a nontrivial role in the context of high gravity envelopments, such as near a neutron star.

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2. Geodetics in a Generic External Gravitational Field

In this section we express a general way to treat the energy deposition problem. We consider a generic metric of the form

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & 0 & g_{03} \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ g_{03} & 0 & 0 & g_{33} \end{pmatrix}.$$  (1)

It is possible to define Local Lorentz tetrad as (Prasanna & Goswami 2002)

$$g_{\mu\nu} = V_{\mu}^{i} V_{\nu}^{j} \eta_{ij},$$  (2)

where

$$V_{\mu} = \begin{pmatrix} \sqrt{-g_{00} + g_{03}^{2}/g_{33}} & 0 & 0 & g_{03}/\sqrt{g_{33}} \\ 0 & \sqrt{g_{11}} & 0 & 0 \\ 0 & 0 & \sqrt{g_{33}} & 0 \\ 0 & 0 & 0 & \sqrt{g_{33}} \end{pmatrix}.$$  (3)

With the above metric, we have a Lagrangian for a close circular orbit ($\theta = \pi/2$)

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}.$$  (4)

and the generalized momenta read

$$p_{0} = \frac{\partial \mathcal{L}}{\partial \dot{t}} = g_{00} \dot{t} + \frac{1}{2} g_{03} \dot{\phi} = -E,$$  (5)

$$p_{1} = \frac{\partial \mathcal{L}}{\partial \dot{r}} = g_{11} \dot{t},$$  (6)

$$p_{3} = \frac{\partial \mathcal{L}}{\partial \phi} = g_{33} \dot{\phi} + g_{03} \dot{t} = L,$$  (7)

where $E$ and $L$ are the energy and momentum of the particle, respectively. Moreover, the Hamiltonian is defined as

$$2H = -E \dot{t} + L \dot{\phi} + g_{11} \dot{t}^{2} = \delta_{t},$$  (8)

where $\delta_{t} = 0$ for null geodetics. With the above definitions, one obtains (Prasanna & Goswami 2002):

$$U^{3} = \dot{\phi} = E \left( \frac{L}{E} + \frac{1}{2} \frac{g_{00}}{g_{03}} \left( g_{33} - \frac{1}{2} \frac{g_{03}^{2}}{g_{00}} \right)^{-1} \right);$$  (9)

$$U^{0} = \dot{t} = -\frac{E}{g_{00}} \left[ 1 + \frac{1}{2} \frac{g_{00}}{g_{03}} \left( \frac{L}{E} + \frac{1}{2} \frac{g_{03}}{g_{00}} \right) \left( g_{33} - \frac{1}{2} \frac{g_{03}^{2}}{g_{00}} \right)^{-1} \right];$$  (10)

$$r^{2} = \frac{E \dot{t} - L \dot{\phi}}{g_{11}},$$  (11)

where $L/E = b$ is the impact parameter for a massless particle.

It is possible to define the angle $\theta_{r}$, which is the angle between the trajectory and the tangent velocity in terms of the radial and longitudinal velocities (Prasanna & Goswami 2002)

$$\tan \theta_{r} = \frac{v^{l}}{v^{3}} = \frac{V_{r}^{1} v^{r} + V_{r}^{3} v^{3}}{V_{r}^{0} v^{0} + V_{r}^{3} v^{3}},$$  (12)

with $v^{0} = U^{0}/U^{1}$. These equations can be solved to find a relation between $b$ and $\theta_{r}$, and using $dr/dr$ obtained from Equation (11). In general, one would obtain an equation of the kind:

$$b = f(\cot \theta_{r}).$$  (13)

In a single orbit, $b$ is constant in each point. Thus, for a particle emitted tangentially from the surface ($\theta_{r} = \pi/2$), we can write formally that (Salmonson & Wilson 1999; Prasanna & Goswami 2002):

$$\cot \theta_{r} = f^{-1}(f(0)).$$  (14)

2.1. Photosphere

The photosphere is the last stable circular orbit for massless particles. The conditions at the photosphere are of importance when calculating the energy deposition because the neutrino emission properties sensitively depend on the photosphere temperature. In a circular orbit, defined by the condition

$$\dot{r}^{2} = f(E, L) = 0,$$  (15)

and solving with respect to $E$, one infers (Khoo & Ong 2016):

$$E^{2} = V_{\text{eff}}^{2}.$$  (16)

To find the minimum radius $R_{\text{ph}}$, we have to impose that:

$$\frac{\partial V_{\text{eff}}}{\partial r} = 0.$$  (17)

In our work we consider neutrinos as massless particles and thus the neutrinosphere radius (the spherical surface where the stellar material is transparent to the spherical surface from which neutrinos are emitted freely) is larger or equal to the photosphere radius.
3. Neutrino Annihilation

In this section, we discuss the relativistic calculation of $\nu\bar{\nu} \rightarrow e^+ e^-$ energy deposition. Its rate per unit time and unit volume is given in general (Salmonson & Wilson 1999)

$$\dot{q} = \frac{7DG_F^2 \pi^3 \xi(5)}{2e^{\frac{3}{2}h^6}} (kT(r))^9 \Theta(r),$$

(18)

where $G_F$ is the Fermi constant, $D = 1 + 4 \sin^2 \theta_W + 8 \sin^4 \theta_W$, $\sin^2 \theta_W = 0.23$ and the plus sign is for electron neutrinos and antineutrinos while the minus sign is for muon and tau type. $T(r)$ is the temperature measured by the local observer and $\Theta(r)$ is the angular integration factor. It is possible to write

$$\Theta(r) = \int \int (1 - \Omega_x \cdot \Omega_y) d\Omega_x d\Omega_y$$

$$= 4\pi^2 \int_x^1 \int_y^1 [1 - 2 \mu_x \mu_y + \mu_x^2 \mu_y^2]$$

$$+ \frac{1}{2} (1 - \mu_x^2)(1 - \mu_y^2) \mu_x \mu_y, \quad (19)$$

where $\mu = \sin \theta_x$, $\Omega = (\mu_x, \sqrt{1 - \mu_x^2} \cos \phi, \sqrt{1 - \mu_x^2} \sin \phi)$ and $d\Omega = \cos \theta d\theta d\phi$. The result is:

$$\Theta(r) = \frac{2\pi^3}{3} (1 - x)^4 (x^2 + 4x + 5), \quad (20)$$

where $x = \sin \theta_x$ that can be obtained from Equation (14).

3.1. Redshift

The neutrino temperature varies linearly with redshift and $T(r)$ is related to the neutrino temperature at the neutrinosphere radius $R$ as (Salmonson & Wilson 1999)

$$T(r) = \frac{g_{00}(R)}{g_{00}(R)} T(R),$$

(21)

with $g_{00} = 0$. Otherwise the luminosity varies quadratically with redshift

$$L_\infty = g_{00}(R) L(R),$$

(22)

and, at the neutrinosphere, the luminosity for a single neutrino species is given by:

$$L(R) = 4\pi R^2 \frac{7}{4} \frac{a c}{4} T(R)^4,$$

(23)

where $a$ is the radiation constant. Combining these equations with Equation (18), we obtain:

$$\dot{q} = \frac{7DG_F^2 \pi^3 \xi(5)}{2e^{\frac{3}{2}h^6}} \left[ \frac{7}{4} \frac{a c}{4} \right]^{-9/4}$$

$$\times L_\infty^{9/4} \Theta(R) \left[ \frac{g_{00}(R)}{g_{00}(R)} \right]^{9/2} R^{-9/2}. \quad (24)$$

The total amount of local energy deposited by $\nu\bar{\nu} \rightarrow e^+ e^-$ for a single neutrino flavor for a unit time can be defined as:

$$\dot{Q} = \int_R^\infty 4\pi r^2 dr \sqrt{g_{11}} \dot{q}.$$  

(25)

where we integrate $\dot{q}$ from the neutrinosphere radius $R$ to infinity. It is important to state that $R$ depends on the considered situation and can vary from $R_{ph}$ to infinity: for example, for a neutron star it is possible to consider $R = R_{ph}$, while for a supernova $R = 4 - 5 M$, with $M$ being the core mass. In the case $g_{00} = g^{-1}_{11}$, one infers

$$\dot{Q} = \frac{28DG_F^2 \pi^3 \xi(5)}{2e^{\frac{3}{2}h^6}} \left[ \frac{7}{4} \frac{a c}{4} \right]^{-9/4}$$

$$\times \int_1^\infty (x - 1)^4 (x^2 + 4x + 5) \frac{y^2 dy}{g_{00}(yR)^5}, \quad (26)$$

where $y = r/R$. It is possible to write Equation (26) as:

$$\dot{Q}_{51} = 1.09 \times 10^{-5} \frac{(M/R)}{\dot{L}_{51}^{9/4} R_6^{-3/2}}, \quad (27)$$

where $Q_{51}$ and $L_{51}$ are the total energy deposition and luminosity, respectively, in units of $10^{51}$ erg s$^{-1}$, $R_6$ is the radius in units of 10 km and

$$\dot{L}(\frac{M}{R}) = 3g_{00}(R) \frac{9}{4} \int_1^{R_{ph}} (x - 1)^4 (x^2 + 4x + 5) \frac{y^2 dy}{g_{00}(yR)^5}. \quad (28)$$

In the Newtonian case, $\mathcal{F}(0) = 1$ and, therefore, the ratio $Q_{GR}/Q_{Newt} = \mathcal{F}(M/R)$. The general form of $\mathcal{F}(M/R)$ for $g_{00} = g^{-1}_{11}$ is given by

$$\mathcal{F}(\frac{M}{R}) = 3g_{00}(R) \frac{9}{4} \int_1^{R_{ph}} (x - 1)^4 (x^2 + 4x + 5)$$

$$\times \frac{y^2 g_{11}(yR)dy}{g_{00}(yR)^2}. \quad (29)$$

4. Neutrino Deposition in Modified Gravity

In this section, we explore the neutrino pair annihilation in different models of modified gravity. The studied effect happens in a strong gravitational field, thus we will consider the black hole (BH) solutions for the chosen theories.

4.1. Charged Galileon

We investigate the neutrino pair annihilation in the charged Galileon black hole framework, a subclass of Horndeski theories. Besides nonminimal coupling between scalar and gravity, the above model also inherits an additional gauge field, which couples to the scalar sector nonminimally. The action takes the form of (Mukherjee & Chakraborty 2018)

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \beta G^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi - \eta \partial_\mu \psi \partial^\mu \psi \right]$$

$$- \frac{\gamma}{2} \left( F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) \nabla^\mu \psi \nabla_\nu \psi \right], \quad (30)$$

where $\beta \neq 0$ and $\psi$ is the gauge field. Imposing a spherical condition one could obtain an exact solution
If \( L^1 \)

\[
\text{Mukherjee & Chakraborty 2018):
\]

\[
ds^2 = -\left(1 - \frac{2M}{r} + \frac{\eta r^2}{3\beta} + \frac{\gamma(Q^2 + P^2)}{4\beta r^2}\right)dt^2
\]

\[
+ \left(1 - \frac{2M}{r} + \frac{\eta r^2}{3\beta} + \frac{\gamma(Q^2 + P^2)}{4\beta r^2}\right)^{-1}dr^2
\]

\[+ r^2d\Omega^2,
\]

where \( \eta/3\beta = -\Lambda \), with \( \Lambda \) being cosmological constant. Thus, to be consistent, \( \eta < 0 \) and \( \gamma > \beta > 0 \). It is also possible to define \( \gamma(Q^2 + P^2)/4\beta = M^2q \), finding that:

\[
g_{00}(r) = g_{11}(r)^{-1} = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \frac{M^2q}{r^2}.
\]

If \( \Lambda \neq 0 \), this metric is not flat for \( r \to 0 \) and this leads to the existence of a cosmological horizon.

We can keep \( q = 0 \): in this case, we have three real solutions for \( g_{00} = 0 \) denoting the cosmological horizon \( R_{\text{Ch}} \) along with outer and inner event horizons. The results for this metric are presented in Figure 1 (where for \( \bar{Q} \) we have integrated until \( R_{\text{Ch}} \)). The black line, corresponding to \( \Lambda = 10^{-3}M^2 \), shows an enhancement with respect to GR and it has a different behavior with respect to the other curves due to the presence of \( R_{\text{Ch}} \sim 16M \).

In this parameter range, the model does not show significant differences with respect to GR.

If we keep \( q \neq 0 \), we have the shape for the energy deposition described in Figure 2. It is possible to notice the enhancement of the maximum amount of energy deposition, up to a factor of 2, with respect to the GR case \( (q = 0) \) and the shift in the minimum photosphere radius \( R_{\text{ph}} \).

For values of \( R/M \), where energy deposition is not defined in GR, we have, however, extended the definition \( \bar{Q}/Q_{\text{Newt}} = \mathcal{F}(M/R) \).

**Figure 1.** Ratio of energy deposition \( \bar{Q} \) for the metric in Equation (32) to the total Newtonian energy deposition \( Q_{\text{Newt}} \) for different values of \( \Lambda \). The green curve shows the GR energy deposition for comparison.

(Mukherjee & Chakraborty 2018):

\[
ds^2 = -\int d^4 x \sqrt{-g} \left(\frac{R}{2} - \frac{1}{2} \psi_{\mu\nu} \psi_{\mu\nu} + \alpha \psi L_{\text{GB}}\right).
\]

where \( \psi \) is a scalar field and \( L_{\text{GB}} \) is the Gauss–Bonnet invariant: \( L_{\text{GB}} = R^2 - 4R_{\alpha\beta}R_{\alpha\beta} + R_{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \). The solution considered is the Sotiriou–Zhou solution, valid for small \( \bar{\alpha} = \alpha/4M^2 \) (solution in perturbation theory) (Sotiriou & Zhou 2014):

\[
ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2d\Omega^2,
\]

with:

\[
f(r) = \left(1 - \frac{2m}{r}\right)\left(1 + \sum A_n \bar{\alpha}^n\right);
\]

\[
h(r) = \left(1 - \frac{2m}{r}\right)^{-1}\left(1 + \sum B_n \bar{\alpha}^n\right).
\]

where, to the second order:

\[
A_1 = B_1 = 0;
\]

\[
A_2 = \frac{49}{40m^3r^3} - \frac{49}{20m^2r^2} - \frac{137}{30mr^3};
\]

\[
B_2 = \frac{49}{40m^3r^3} + \frac{29}{20m^2r^2} + \frac{19}{10mr^3} - \frac{203}{15r^4} - \frac{436m}{15r^5} - \frac{184m^2}{3r^6}.
\]

With the above metric we obtain the shape for energy deposition shown in Figure 3. The maximum value taken for \( \bar{\alpha} \), considering the perturbative regime of the solution, showed an increase of 50% for the maximum amount of energy deposition with respect to GR.

**Figure 2.** Ratio of energy deposition \( \bar{Q} \) for the metric in Equation (32) to the total Newtonian energy deposition \( Q_{\text{Newt}} \) for \( \Lambda = 0 \) (a similar shape is obtain for a different value of \( \Lambda \)). For different values of \( q \) we have different values for the photosphere radius. The green curve shows the GR energy deposition for comparison.

4.2. Einstein Dilaton Gauss Bonnet Gravity

In this section we discuss the solution in spherical symmetry as a subclass of the Horndesky theory and corresponding to Einstein dilaton Gauss Bonnet gravity. The action is
4.3. Brans Dicke Theory

In this section we discuss $\nu\psi$ annihilation in the Brans Dicke theory. This represents a generalization of GR, where gravitational effects are in part due to geometry, and in part due to a scalar field. The action is (Brans & Dicke 1961)

$$S = \int d^4x \sqrt{-g} \left[ \psi R + \frac{16\pi}{c^2} L - \omega(\psi) \right].$$

(35)

where $L$ is the Lagrangian density of all the matter, including the whole nongravitational field, $\psi$ is a scalar field, and $\omega$ is its Lagrangian density.

With this Lagrangian, expressing the line element in the isotropic form, we obtain the solution (Brans & Dicke 1961)

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2d\Omega^2,$$

(36)

where

$$\lambda = \sqrt{(C + 1)^2 - C(1 - C\omega/2)}$$

$$e^{2\alpha} = e^{2\alpha_0} \left[ 1 - \frac{B}{r} \right]^{\frac{C\lambda}{C + 1}},$$

$$e^{2\beta} = e^{2\beta_0} \left[ 1 + \frac{B}{r} \right]^{\frac{C\lambda}{C + 1}},$$

$$\psi = \psi_0 \left[ 1 - \frac{B}{r} \right]^{\frac{-\lambda}{C + 1}},$$

with $\omega$ being positive constant and

$$\alpha_0 = \beta_0 = 0,$$

$$\psi_0 = \frac{4 + 2\omega}{3 + 2\omega},$$

$$C \sim \frac{1}{2 + \omega},$$

$$B \sim \frac{M}{2\sqrt{\psi_0}}.$$

Using this metric, we obtain the shape for energy deposition in Figure 4. Even with this model we have an enhancement of about 50% respect to the maximum value of $Q/Q_{\text{Newt}}$ 30 in GR.

4.4. Eddington-inspired Born–Infeld Black Hole Solution

In this section we turn our attention on the spherically symmetric solution that Baados and Ferreira considered for the line element in the Born–Infeld model coupled with an electric field (Beltran Jimenez et al. 2018). The action can be written as:

$$S = \frac{1}{\epsilon k^2} \int d^4x \sqrt{-\det(g_{\mu\nu} + \epsilon R_{\mu\nu}(\Gamma))}$$

$$- \lambda \sqrt{-\det g_{\mu\nu}} + S_m(g_{\mu\nu}, \psi_m),$$

(37)

where $\psi_m$ is the matter field, $\Gamma$ is the connection, and $\epsilon$ and $\lambda$ are parameters. The line element reads as:

$$ds^2 = -\psi(r)^2f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2,$$

(38)

where

$$\psi(r) = \frac{r^2}{\sqrt{r^4 + (\epsilon/\lambda)Q^2}};$$

(39)

$$f(r) = \frac{r\sqrt{\epsilon Q + \lambda r^4}}{\lambda r^4 - \epsilon Q^2}$$

$$\times \left[ (3r^2 - Q^2 - (\lambda - 1)r^4/\epsilon)\sqrt{\epsilon Q^2 + \lambda r^4} + 4/r^3 \sqrt{\epsilon Q^2 + \lambda r^4} ight]$$

$$\times \left[ iQ \sqrt{\epsilon Q^2 + \lambda r^4} \Gamma^2 \left( \frac{i}{4} \right) + \frac{4}{3} \sqrt{\epsilon Q^2 + \lambda r^4} \right]$$

$$\times F \left[ \text{arcsinh} \left( \frac{i}{Q} \sqrt{\frac{\lambda}{r}} - 1 \right) - 2\sqrt{\lambda M} \right],$$

(40)

with

$$F(\beta, \alpha) = \int_0^\beta (1 - \alpha^2 \sin^2 \theta)^{-1/2} d\theta.$$
Using this metric we obtain the shape for the energy deposition in Figure 5. Even in this case we have a maximum increase of about 50%.

4.5. Born–Infeld Generalization of Reissner–Nordstrom Solution

In the Born–Infeld model, it is also possible to write the generalization of the Reissner–Nordstrom solution as (Breton 2002):

$$ds^2 = -\psi dt^2 + \psi^{-1}dr^2 + r^2d\Omega^2,$$

(42)

where

$$\psi = 1 - \frac{2M}{r} + \frac{2}{3}b^2r^2 \left( 1 - \sqrt{1 + \frac{Q^2}{b^2r^4}} \right) + \frac{2Q^2}{3r} \sqrt{\frac{b}{Q}} \left( \arccos \left( \frac{br^2}{Q} - 1 \right) \sqrt{\frac{1}{2}} \right).$$

(43)

The Reissner–Nordstrom solution characterizes the final state of a charged star, having as its uncharged limit the Schwarzschild black hole. It is interesting to investigate the neutrino annihilation energy deposition in its nonlinear electromagnetic generalization.

With this metric we obtain the shape for the energy deposition in Figure 6. It is possible to see a relevant enhancement (or suppression) of the annihilation energy released up to 200% in the case of $Q = M$ and $b = 0.3/M$.

4.6. Higher Derivative Gravity Analytical Solution

Finally, in this section we study the analytic solution in higher-order gravity. The action can be expressed as

$$S = \int d^4x \sqrt{-g} \left( \gamma R - \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta R^2 \right),$$

(44)

where $\alpha$, $\beta$, and $\gamma$ are constants and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. The non-Schwarzschild BH solution is of the form

$$(\text{Kokkotas et al. 2017}):$$

$$ds^2 = -\left( 1 - \frac{r_0}{r} \right) A(r, p) dt^2 + \frac{B(r, p)^2 dr^2}{\left( 1 - \frac{r_0}{r} \right) A(r, p)} + r^2 d\Omega^2.$$  

(45)

The expressions of the metric components $A(r, p)$ and $B(r, p)$ are quite involved and are not reported here (see Kokkotas et al. 2017), while $p$ is a parameter of the model defined as

$$p = \frac{r_0}{2\alpha},$$

(46)

where $r_0$ is the horizon radius. The parameter $p$ goes from 0.876, corresponding to the merger of the Schwarzschild and non-Schwarzschild solutions, to 1.14 where the non-Schwarzschild solution almost vanishes. The behavior of the energy deposition, for high values of the parameter $p$, is represented in Figure 7. In such a case, we obtain a reduction of the energy released by the neutrino pair annihilation, up to a reduction of a factor of 6 for $p = 1.14$.

5. Gamma-Ray Burst Analysis in Extended Theories of Gravity

Results of the previous section show that modified gravity provides, in almost all cases, a nontrivial deviation from GR behavior. Such a deviation is extremely relevant for neutron star (NS) and black hole evolution, as well as for the gamma-ray burst (GRB) phenomena. In the latter case, with which we are interested in this section, we shall highlight that neutrino pair annihilation might partially contribute.

GRBs and their possible connection with neutrino production in compact stars is currently a field of high interest. They represent the biggest mystery in high energy astronomy. There are two kinds of GRBs (Kouveliotou et al. 1993):

4 More precisely, the Schwarzschild metric is also an exact solution of the Einstein–Weyl theory for all $p$. However, at some minimal nonzero value of $p$, $p_{\text{min}}$, there appears (in addition to the Schwarzschild solution) a non-Schwarzschild branch, that is, a solution that describes the asymptotically flat black hole, characterized by a mass that decreases as $p$ grows, and vanishes at some $p_{\text{max}}$. The range of values of $p$ are those discussed in the text, i.e., $p \in [0, 0.84, 1.14]$. 

\[ \text{Figure 5. Ratio of total energy deposition } Q \text{ for the metric in Equation (38) to the total Newtonian energy deposition } Q_{\text{Newt}} \text{ for } \epsilon = Q = \lambda = 1 \text{ and various values of mass. The green curve shows the GR energy deposition for comparison.} \]

\[ \text{Figure 6. Ratio of total energy deposition } Q \text{ for the metric in Equation (43) to the total Newtonian energy deposition } Q_{\text{Newt}} \text{ for three values of the parameter } b \text{ and } Q = M. \text{ The green curve shows the GR energy deposition for comparison.} \]
First evidence of long GRBs associated with SNe were derived by studying GRB 980425 (Galama et al. 2000) and GRB 030329 (Hjorth et al. 2003; Matheson et al. 2003; Stanek et al. 2003). Regarding short GRBs, there is a scarcity of information. Particularly relevant in these frameworks is the reaction \( \nu + \bar{\nu} \rightarrow e^- + e^+ \) since the \( e^- \) pairs may further give rise to gamma rays, which could be a possible explanation for the observed GRBs. Previous analysis of the reaction \( \nu + \bar{\nu} \rightarrow e^- + e^+ \) near a neutron star based on Newtonian gravity (i.e., in the regime \( r \gg R_s \), where \( R_s \) is the Schwarzschild radius) has been performed in Goodman et al. (1987) and Cooperstein et al. (1986). The inclusion of gravitational effects for static stars was developed in Salmond & Wilson (1999, 2003). The inclusion of the rotation of stars was studied in Mallick et al. (2013) and Kovacs et al. (2010). These effects are nonnegligible in strong gravitational regimes, as well as in theories of gravity that extended GR, as we shall discuss. Indeed, it is possible to make a simple evaluation of the energy emitted from neutrino pair annihilation considering Equation (27), with a total luminosity from NS for neutrinos of the order \( \sim O(10^{33} \text{ erg s}^{-1}) \) and radius \( R \) of the order \( \sim O(20 \text{ km}) \) (Perego et al. 2017):

\[
\dot{Q} = 1.2116 \times 1.09 \times 10^{-5}(10^2)^{9/42} 3/2 \text{ erg s}^{-1} F(R) \quad (47)
\]

\[
= 1.48 \times 10^{50} \text{ erg s}^{-1} F(R), \quad (48)
\]

where we use the smallest value possible for \( D \) and the function \( F(R) \) depends on the theory under consideration (see Equation (27)). For GR, \( F \sim 30 \) considering that in a neutron star neutrinos are emitted from the photosphere. Therefore, we obtain \( \dot{Q} \sim 4.41 \times 10^{31} \text{ erg s}^{-1} \), which is inferior to the GRB emitted energy rate \( \sim 5 \times 10^{32} \text{ erg s}^{-1} \) (Mallick et al. 2013). As shown in our work, the maximum of \( F \) in theories beyond GR can be different and, in some cases, we have a variation up to a factor of 3, obtaining

\[
\dot{Q}_{\text{max}} = 1.3 \times 10^{32} \text{ erg s}^{-1}, \quad (49)
\]

which is almost of the same order of magnitude as the GRB emission energy rate.

As these results suggest, it is possible that theories of gravity beyond GR may efficiently contribute, as a consequence of the reduction of the photosphere radius (hence the gravitational effects turn out to be enhanced), to the GRB emission through the neutrino pair annihilation mechanism. This could be a testbed for probing nonstandard gravity described by GR.

6. Conclusions

In this paper, we have provided a systematic way to treat the neutrino pair annihilation \( \nu \bar{\nu} \rightarrow e^+ e^- \) and we have analyzed the process in the framework of various models of gravity beyond GR. We extended the general relativistic calculations of neutrino heating rates, showing an increase of a factor \( \sim 27 \) with respect to Newtonian calculations, to Charged Galileon, Einstein dilaton Gauss–Bonnet, Brans Dicke, Eddington-inspired Born–Infeld, Born–Infeld generalization of Reissner–Nordström solution, and Higher derivative gravity models, respectively, finding some relevant differences in the minimum photon sphere radius \( R_{\text{ph}} \) and in the maximum of the energy deposition rate near \( R_{\text{ph}} \).

We have shown that in the opportune range of parameters, one can obtain relevant enhancement (up to a factor of 3) or suppression (up to a factor of 1/3) of the maximum rate and a shift in the minimum photosphere radius.

These results can be extremely important in the context of neutron star merging (for which one has to consider the value of the rate at \( R_{\text{ph}} \)) or in the supernovae, where it is relevant to consider \( R = 4-5M \). In particular, we would like to state that the enhancement shown in some model of gravity beyond GR could be relevant for GRBs, for which neutrino pair annihilation has been proposed as a possible source. Indeed, although the source of gamma-ray bursts is still undetermined, neutrino pair annihilation is not the best candidate due to the fact that some extra energy is needed. More precisely, in the case of short GRBs, a factor of 2 or 3 up to 10 (Perego et al. 2017) is needed to generate all the gamma-ray bursts observed. As we have highlighted in this paper, in some modified theories of GR it is possible to partially obtain the extra energy needed. Moreover, it is important to remark that in our analysis we have not considered any trapping for the neutrinos, but they may be trapped producing an increase in the temperature (Ghosh et al. 1996). Thus other changes in the energy deposition rate could be obtained.

We finally conclude, pointing out that deviations from GR could significantly increase all the energy deposition processes in the NS and supernova envelope, suggesting that further investigations have to be done in these frameworks.

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