A Framework for Systematic Study of QCD Vacuum Structure I: Kolmogorov Entropy and the Principle of Chiral Ordering

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Abstract

In this series of articles we describe a systematic approach for studying QCD vacuum structure using the methods of lattice gauge theory. Our framework incorporates four major components. (i) The recently established existence of space–time order at all scales (fundamental structure) observed directly in typical configurations of regularized path–integral ensembles. (ii) The notion of scale–dependent vacuum structure (effective structure) providing the means for representing and quantifying the influence of fluctuations at various scales on physical observables (phenomena). (iii) The unified description of gauge and fermionic aspects of the theory which facilitates a high level of space–time order in the path–integral ensembles. (iv) The strict “Bottom–Up” approach wherein the process of identifying the vacuum structure proceeds inductively, using the information from valid lattice QCD ensembles as the only input. In this work we first elaborate on the meaning of the notion of space–time order in a given configuration which is conceptually at the heart of the path–integral approach to vacuum structure. It is argued that the algorithmic complexity of binary strings associated with coarse-grained descriptions of the configuration provides a relevant quantitative measure. The corresponding ensemble averages define the ranking of different lattice theories at given cutoff by the degree of space–time order generated via their dynamics. We then introduce the set of local transformations of a configuration, chiral orderings, in which the transformed gauge connection represents an effective matrix phase acquired by chiral fermion when hopping over a given link. It is proposed that chiral orderings facilitate the evolution in the set of actions which increases the degree of space–time order while preserving the physical content of the theory, and should thus be used in the search for the fundamental QCD vacuum structure. The relation to renormalization group ideas is discussed, and the first step in general formulation of effective lattice QCD realizing the notion of scale–dependent vacuum structure is given.
1 Introduction

Understanding the vacuum structure of non-abelian gauge theories is the subject of active theoretical research. While no definitive answers have been obtained over a number of years, the subject continues to captivate the interest of many researchers since the stakes are high. Indeed, the potential fruits of such effort include, among other things, identifying the mechanism leading to confinement and spontaneous chiral symmetry breaking (SChSB) – phenomena believed to be central for the dynamics of strong interactions.

While the vacuum of quantum field theory is a state in the appropriate Hilbert space, this is not necessarily the representation of choice in the search for vacuum structure. Indeed, the discussion of related issues in QCD is frequently carried out in the Euclidean path-integral formalism. Here one does not work with states in the Hilbert space, but rather with the statistical ensemble of gauge configurations, i.e. the set of pairs $E_{QCD} \equiv \{(A, P[A] \mathcal{D}A)\}$, where $A \equiv \{A_\mu(x)\}$ collectively denotes an arbitrary space-time configuration of gauge potentials, and $P[A] \mathcal{D}A \propto \exp(-S_{QCD}[A]) \mathcal{D}A$ is its weight (“probability”) in the path integral. Since the vacuum expectation value of any operator is replaced by the corresponding ensemble average, the information stored in the quantum vacuum state is equivalent to the information encoded in the ensemble. In this sense, one can view the ensemble as a particular representation of the vacuum.

One advantage of working with classical ensembles is that the notion of “vacuum structure” acquires a natural intuitive meaning here. Indeed, the attempts to understand QCD vacuum in this language aim at finding a related ensemble $E^{STR}$ (ensemble with “structure”), which only involves a certain definite subset of configurations that share a specific kind of space-time order. The notion of “vacuum structure” is then loosely associated with the common space-time structure exhibited by these special configurations. Ideally, one should be able to describe $E^{STR}$ analytically in terms of collective degrees of freedom (objects in the gauge field) that encode this order. The (usually vaguely stated) criterion for identifying the ensemble $E^{STR}$ is that the “QCD physics is reproduced” while, at the same time, this physics should be naturally understood in terms of the associated order and/or the corresponding collective degrees of freedom. There are two highly interrelated problems with the above program which currently appear to prevent it from becoming a developed and systematic area of research.

P1: The lack of well-posed goals consistent with the fact that $E_{QCD}$ describes quantum field theory.

P2: The lack of guiding principles and the systematic framework for constructing $E^{STR}$ once such goals are given.

The aim of this series of papers is to discuss a set of ideas that have a bearing on the above issues. Most of these ideas are original, but some represent a significant expansion of the line of thought that already appeared in our previous work. In the remaining parts of this section, we will discuss in more detail why we view P1 and P2 as pressing problems, which will allow us to naturally set the stage for our approach.
1.1 Lattice QCD

One of the important reasons for using the Euclidean path integral formalism in the context of vacuum structure is that gauge theory can be conveniently regularized in this framework. Indeed, the considerations of the previous section are only meaningful when viewed as a result of the limiting procedure involving regularized versions of the theory. Gauge invariant regularization on the Euclidean space-time lattice [1] provides an appropriate framework for both the non-perturbative definition of gauge theory, and the study of the vacuum structure along the lines described above.

The power of lattice QCD in this context resides with the fact that the vacuum of a finite lattice system, namely the ensemble $E_{\text{LQCD}} = \{ (U, P(U)dU) \}$, can be fully represented by an infinite probabilistic chain of configurations $\{\ldots, U^{i-1}, U^i, U^{i+1}, \ldots\}$, such that the probability of encountering the configuration $U \equiv \{ U_{x,\mu} \in SU(3) \}$ is proportional to its weight $P(U)dU \propto \exp(-S_{\text{LQCD}}(U))dU$ in the lattice path integral (a finite-dimensional integral). At the same time, finite sections of such chains are generated explicitly in Monte Carlo simulations of lattice gauge theory on digital computers [2, 3]. In other words, a numerical simulation gives us direct access to the vacuum in the form of configurations dominating the evaluation of the regularized QCD path integral which ultimately defines the theory.

Given the above, one naturally expects lattice QCD (LQCD) to be very relevant in the quest for understanding the QCD vacuum structure. One of the points that we will emphasize throughout this series of papers is that the role of LQCD should in fact be elevated from auxiliary and secondary to primary and decisive.

1.2 The “Top–Down” Approach

For a number of years, the research in this area proceeded mainly along the line which we refer to as a “Top–Down” approach. The characteristic feature of Top–Down is that the starting point of an investigation is an immediate proposal for relevant collective degrees of freedom (e.g. instantons, magnetic monopoles, center vortices). Such a proposal is made in the continuum and is usually supplemented by a “picture” of how confinement or SchSB could be understood if the collective degrees of freedom in question would indeed turn out to be the most relevant.

The role of LQCD is rather auxiliary in this approach. The primary aspect amounts to calculations supplying details needed for an eventual analytic representation of $E_{\text{STR}}$ in terms of the collective degrees of freedom chosen, i.e. determining the densities, sizes and other geometric characteristics of the objects relevant for the description of $E_{\text{STR}}$. Such an inquiry typically proceeds via certain procedure that assigns to every configuration $U$ of the numerically-generated ensemble $E_{\text{LQCD}} = \{\ldots, U^{i-1}, U^i, U^{i+1}, \ldots\}$ an associated configuration $U_C \equiv F_C(U)$ in which the corresponding degrees of freedom $C$ are “visible”. This produces a numerical ensemble with “structure” $E_{\text{LSTR}} = \{\ldots, U^{i-1}_C, U^i_C, U^{i+1}_C, \ldots\}$ on which the needed characteristics of $E_{\text{STR}}$ can be measured. Another use of $E_{\text{LSTR}}$ could be to quantify (by measuring and comparing variety of relevant observables in $E_{\text{LQCD}}$ and $E_{\text{LSTR}}$) how “close” $E_{\text{STR}}$ is to $E_{\text{QCD}}$, but this is usually not done beyond the string tension.

There are numerous problematic issues associated with the Top–Down approach some of
which we list below.

(1) The proposal for the collective degrees of freedom is always *ad hoc* in the sense that it is motivated by certain interesting properties (mainly of topological nature) but not taking into account at all the global structure of quantum QCD ensemble. In that sense it amounts to little more than guessing (see P2). Nevertheless, it is certainly possible (but maybe not likely) to get to the root of a problem via a well-motivated guess.

(2) The procedure for selecting $U \to U_C = F_C(U)$ is not unique in existing examples, and is sometimes gauge dependent. Moreover, the aspect of non-uniqueness of $F_C$ appears to be generic. The finality of the results and arguments in favor of a given picture arrived at in a Top–Down manner thus appears to always remain in question (see P2).

(3) It is not clear how to interpret the meaning and the scope of $\mathcal{E}^{STR}$ arrived at in a Top–Down manner in the context of quantum field theory defined by $\mathcal{E}^{QCD}$. Indeed, at the face value $\mathcal{E}^{STR} = \mathcal{E}^{STR}(\mathcal{E}^{QCD}, C, F_C)$ is a function of $\mathcal{E}^{QCD}$, the collective degrees of freedom $C$, and the procedure $U \to F_C(U)$ which makes $C$ explicit in the configuration. It is hard to find a general notion in field theory which would give well-defined meaning to such generic $\mathcal{E}^{STR}$. To illustrate this further, it is customary in the Top–Down approach to interpret $\mathcal{E}^{STR}$ in terms of a phenomenon (physics) one wants to understand. For example, the ongoing attempts frequently start as an effort to understand confinement, i.e. to construct $\mathcal{E}^{STR}$ in terms of correctly chosen $C$ such that string tension ($\sigma$) would be reproduced and its origin understood. There are similar attempts to understand chiral symmetry breaking and reproducing chiral condensate. However, this approach carries ambiguities that are difficult to resolve conceptually. For example, having constructed $\mathcal{E}^{STR}_{con}$ for confinement, do we expect that it will also naturally explain chiral symmetry breaking and reproduce the condensate (and vice versa)? Such a coincidence might not be very likely since the construction of $\mathcal{E}^{STR}_{con}$ is almost certainly not unique. Indeed, it appears more reasonable to expect (and different coexisting pictures of confinement support this expectation) that there are many ways to construct $\mathcal{E}^{STR}_{con}$ reproducing just $\sigma$. Similarly, there are probably many ways to construct $\mathcal{E}^{STR}_{chi}$ which reproduce just the chiral condensate, thus supplying multitude of different “pictures” for SChSB in QCD. Moreover, the argument doesn’t stop there. One can legitimately ask how vacuum encodes (and explains) the mass of the lightest scalar glueball or, for that matter, the mass and properties of any of the other bound states. The point of these remarks is that organizing the search for vacuum structure “by phenomenon” is probably not the appropriate way to pose the problem (see P1). This simply reflects the fact that field theory is not naturally organized in terms of “phenomena”.

1.3 New Input from Lattice QCD

There were recent developments in this area that will serve as an important part of the conceptual input for our framework that we hope can overcome the issues related to P1 and P2. To fix the notation and the language, let us denote by $\mathcal{E}^{LQCD}_\phi$ the ensemble of

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1 There is no satisfactory mathematical meaning to the task of constructing the map $F_C$ for any reasonable class of $C$. In other words, the attempt to define the “$C$-content” $U_C$ of configuration $U$ appears to be ill-founded.
configurations $\phi = \tilde{\phi}(U)$ (local composite field) associated with given $\mathcal{E}^{LQCD}$.  

In terms of a probabilistic chain we have

$$\mathcal{E}^{LQCD} = \{\ldots U^{i-1}, U^i, U^{i+1}, \ldots \} \rightarrow \{\ldots \tilde{\phi}(U^{i-1}), \tilde{\phi}(U^i), \tilde{\phi}(U^{i+1}), \ldots \} \equiv \mathcal{E}_\phi^{LQCD} \equiv \{\ldots \phi^{i-1}, \phi^i, \phi^{i+1}, \ldots \} \equiv \mathcal{E}_\phi^{LQCD} \equiv$$

(1)

The probability distribution (and the associated action) governing the ensemble $\mathcal{E}_\phi^{LQCD}$ is related to the fundamental LQCD distribution $P(U)$ via

$$P_\phi(\phi) \equiv e^{-S_\phi(\phi)} / Z = \int dU \ P(U) \delta(\phi - \tilde{\phi}(U))$$

(2)

We now wish to highlight two aspects put forward in recent works [4, 5, 6, 7] dedicated to study of a particular composite field, namely the topological charge density [8, 9] associated with Ginsparg-Wilson operator [10], i.e. $q_x = \tilde{q}_x(U) = -\text{tr} \gamma_5 (1 - D_{x,x}(U)/2)$, if $D$ satisfies the canonical Ginsparg–Wilson relation.

I1: The configurations $q$ appearing in probabilistic chains representing $\mathcal{E}_\phi^{LQCD}$ exhibit an observable space-time order (structure) [5, 6].  

This order disappears after randomization of space-time coordinates of the underlying gauge field [7], and is thus a manifestation of space-time order in $\mathcal{E}^{LQCD}$.  

The conceptual content of the above result is non-trivial. Indeed, this is the first time that a definite space-time order has been observed in the physically relevant composite field evaluated directly on unmodified configurations that appear in probabilistic chains representing $\mathcal{E}^{LQCD}$. This fact gives substance to the (previously quite unfounded) expectation that QCD path integral is dominated by certain definite class of configurations exhibiting a common structure. We emphasize that this didn’t necessarily have to be the case. While the ensemble averages build up coherence over distances of order 1 fm in QCD (as measured via properly defined physical correlators), it was not a priori obvious that this coherence would manifest itself at the configuration level via explicitly detectable space-time order. Nevertheless, it now appears that this is indeed so.

I2: It was proposed [4, 6] that the patterns of structure in topological charge fluctuations described by $\tilde{q}$ associated with Ginsparg-Wilson fermions can be meaningfully studied as a function of the fermionic response scale $\Lambda_F$. The underlying proposition is that all aspects of QCD vacuum structure should be viewed in a scale-dependent manner.

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2Note that the symbol for configuration $\phi$ is distinguished from the symbol for the local map $\tilde{\phi}$ relating it to the configuration of gauge field $U$.

3The nature of the ordered structure present in these configurations is quite intriguing (see [5, 6]) but we will not discuss it here.

4Strictly speaking, the corresponding studies have so far been performed in pure-glue LQCD defined by Wilson and Iwasaki gauge actions, and using the Wilson-based overlap operator [12] to define the topological field. However, it is expected that the existence of the structure extends to generic LQCD and to generic Ginsparg-Wilson operator.
To be more explicit, the scale dependent ensemble $\mathcal{E}_{q, \Lambda^F}^{LQCD}$ for $\bar{q}$ was defined as

$$
\mathcal{E}_{q}^{LQCD} = \{\ldots q^{-1}, q^i, q^{i+1}, \ldots\} \longrightarrow \{\ldots q^{i-1, \Lambda^F}, q^{i, \Lambda^F}, q^{i+1, \Lambda^F}, \ldots\} \equiv \mathcal{E}_{q, \Lambda^F}^{LQCD}
$$

Here for arbitrary configuration $q$, the effective topological field $q^{\Lambda^F}$ is defined via $[4, 6]$

$$
q_x^{\Lambda^F} = -\sum_{\lambda \leq a\Lambda^F} (1 - \frac{\lambda}{2}) (\psi^\lambda_x)\gamma_5 \psi^\lambda_x
$$

where $\psi^\lambda$ is the eigenmode of $D$ with eigenvalue $\lambda$ and $a$ is the lattice spacing. This formalism allows to study the patterns in topological charge fluctuations as a function of scale.

### 1.4 Resolutions: “Bottom-Up” and the Fundamental Structure

Given $\mathbf{I1}$ and the host of problems with the Top–Down methodology described above, it appears highly desirable to dramatically change the point of view when approaching the problem of QCD vacuum structure in the path integral formalism. Since now there exists evidence for a directly observable order in typical representatives of $\mathcal{E}_{q, \Lambda^F}^{LQCD}$, it might no longer be necessary to rely on guessing associated with Top–Down strategy. One can instead take the attitude that a viable candidate for relevant collective degrees of freedom $C$ has to have the space-time attributes consistent with the structure observed in typical LQCD configurations. Thus, instead of making fixed $C$ a starting point of the investigation, the approach built here views determining $C$ as one of the final goals in a systematic study. We refer to this as a "Bottom–Up" approach to QCD vacuum structure. There are two crucial aspects that we wish to emphasize and to associate with this approach.

**A1:** *The sole input used in the Bottom–Up search for $\mathcal{E}_{C}^{STR}$ comes from various valid ensembles $\mathcal{E}_{q, \Lambda^F}^{LQCD}$ defining $\mathcal{E}_{C}^{QCD}$ in the continuum limit. At the technical level, this can also enter via associated ensembles $\mathcal{E}_{\bar{q}}$ of various local composite fields (see Eq. (1)).*

Indeed, the intention in the Bottom–Up approach is that its only source of information are typical configurations from $\mathcal{E}_{q, \Lambda^F}^{LQCD}$ (for various lattice formulations), and that it starts from a white page, i.e. using no assumptions at all about the nature of collective degrees of freedom $C$ in terms of which $\mathcal{E}_{C}^{STR}$ will ultimately be described. Consequently, this research will naturally proceed via intermediate phases involving the creation of suitable concepts and characteristics whose behavior in $\mathcal{E}_{q, \Lambda^F}^{LQCD}$ will provide hints about the nature of $\mathcal{E}_{C}^{STR}$. In particular, one envisions the following.

(i) **Geometry.** The Bottom–Up approach to QCD vacuum structure should start by exploring the geometric patterns occurring in the fields typical of $\mathcal{E}_{q, \Lambda^F}^{LQCD}$ $[5, 11]$. The simplest beginning is to study the structure in the basic scalar and pseudoscalar gauge invariant composite fields, i.e. action density and topological charge density. More complex studies of other composite fields, and perhaps also of fundamental gauge fields, can follow. These investigations should lead to precise geometric statements about $\mathcal{E}_{q, \Lambda^F}^{LQCD}$, which can be supported or disproved by numerical simulation.
(ii) **Picture.** The information from geometric studies will serve as a sole guide and input for an emerging picture of the relevant collective degrees of freedom. The underlying expectation is that the requirement that such picture be consistent with accumulated knowledge about geometric behavior of fields will significantly restrict the available possibilities.

(iii) **Physics.** The viability of any picture consistent with geometry will ultimately have to be assessed on physical grounds. In other words, some relevant physical implications (or predictions specific for that picture) have to be compared with physics extracted directly from $\mathcal{E}^{LQCD}$. It is worth emphasizing that stages (ii) and (iii) are naturally intertwined in that physics considerations immediately come into focus when considering any particular picture of a collective variable.

(iv) **Definition of $\mathcal{E}^{STR}$.** When the qualitative and quantitative knowledge about the relevant collective variables in QCD reaches a sufficient level of completeness via the above systematic process, it might be possible to define the degrees of freedom $C$ analytically. A precise definition of $\mathcal{E}^{STR}$ in terms of $C$ then might become possible and the understanding of QCD vacuum could be achieved.

**A2:** The vacuum structure obtained from $\mathcal{E}^{LQCD}$ in a Bottom-Up manner is fundamental[5] in the sense that it involves fluctuations of elementary fields at all scales upon taking the continuum limit, and should thus be relevant for all aspects of QCD physics.

Indeed, the fact that the structure (embodied in collective degrees of freedom $C$) is extracted from behavior in full unmodified ensembles $\mathcal{E}^{LQCD}$ implies that, in principle, no aspect of physics should be completely lost in the process of identifying the structure and the subsequent construction of $\mathcal{E}^{STR}$. We emphasize that this fact represents a non-trivial shift in the focus and the scope of the QCD vacuum research. Indeed, the line of thinking adopted in Refs. [5, 6] together with the results obtained indicate that there might exist a structure that encodes the origin of chiral symmetry breaking, confinement, the mass of the lightest pseudoscalar glueball$^5$, the mass of the 10-th excited scalar glueball and so on – all in one fundamental vacuum structure.

At the current stage, there are several missing pieces (of both conceptual and practical nature) that need to be resolved before the Bottom-Up approach to fundamental QCD vacuum structure can evolve into a developed area of research. (a) So far the practical framework only exists for a single (albeit highly relevant) composite field, namely topological charge density. While this might eventually be sufficient to uncover the nature of $C$, there should be a general framework that would allow to study an arbitrary composite field and the gauge field itself. (b) There is no accepted conceptual notion for degree of “structure” (order) in a given individual configuration of the gauge field (or a composite field for that matter). While the intuitive notion might be sufficient to proceed, the systematic framework should contain at least a concept that defines what we are attempting to do. (c) Once such notion is in place, there should exist a mechanism that guides the process of identifying $C$ via maximizing the degree of order in $\mathcal{E}^{LQCD}$ ensembles. This might require concepts that

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$^5$The masses of bound states are not always thought of as associated with vacuum structure but in very real sense they are. After all, the correlation functions encoding these masses are vacuum expectation values of appropriate operators. In fact, all the observables arise as averages over $\mathcal{E}^{QCD}$ and thus one can attempt to “understand” their origin in the vacuum structure.
do not belong to the existing framework of quantum field theory, but should nevertheless be meaningfully integrated into it and, hopefully, enrich it. Addressing these issues is the primary goal for the first two papers of this series.

1.5 Resolutions: Scale-Dependent Vacuum (Effective Structure)

We have emphasized in Sec. 1.2 that associating $E^{STR}$ with a particular phenomenon is rather problematic and unnatural in the general framework of quantum field theory. On the other hand, it has proved very useful to organize field theory calculations (and thinking) in terms of a scale. Given this and $I_2$, it is thus reasonable to expect that this will also be a productive way to deal with the QCD vacuum. In other words, we suggest the “change of basis” in which the questions about QCD vacuum are asked. In particular, instead of seeking $E^{STR}$ for various different phenomena, we suggest to seek $E^{STR}_\Lambda$, i.e. the vacuum structure relevant for physics at arbitrary typical scale $\Lambda$. Another way of saying this is that the process involving momenta up to typical value $\Lambda$ will “excite” or “see” the vacuum structure involving the corresponding momenta. Thus if we wish to understand the role of vacuum in such phenomenon, it is $E^{STR}_\Lambda$ that is relevant for such understanding.

Similarly to the case of fundamental structure, we wish to emphasize two aspects associated with the Bottom–Up program applied to uncovering the scale-dependent structure in QCD vacuum.

A3: The sole input used in the Bottom–Up search for $E^{STR}_\Lambda$ comes from various ensembles $E^{LQCD}_\Lambda$ based on $E^{LQCD}$. Alternatively, this can also enter via associated ensembles $E^{LQCD}_{\phi,\Lambda}$ of composite fields $\phi$ (see e.g. Eq. (3)).

A4: The structure obtained from $E^{LQCD}_\Lambda$ in a Bottom–Up manner is an effective structure $[4, 5]$ exhibited by elementary fields dominated by fluctuations at the momentum scale $\Lambda$. The scale-dependence introduced in this manner allows to associate a typical scale (and the corresponding effective structure) with various phenomena.

We emphasize that the notion of $E^{STR}_\Lambda$ is essential for the interpretation of the fundamental structure represented in $E^{STR}$. Indeed, the existence of such an all–encompassing structure raises a conceptual issue namely how does one “decode” the structure relevant for a particular physical phenomenon. The concept of $E^{STR}_\Lambda$ resolves this and represents an “unfolding” of the fundamental structure into an effective structure relevant for physics at scale $\Lambda$. Note that the sought for collective degrees of freedom $C$ characterizing the fundamental structure can acquire a scale-dependence $C = C(\Lambda)$ in this process.

While the discussion above summarizes our scale–dependent approach, there are clearly some crucial gaps to be filled before this can be viewed as a general theoretical/practical framework. Indeed, so far such framework (partially) exists only for the study of topological charge density. How does one generalize it to other operators and to the ensemble level? In other words, how are $E^{LQCD}_\Lambda$ and $E^{LQCD}_{\phi,\Lambda}$ defined? Can they be defined for arbitrary $E^{LQCD}$ and arbitrary $\phi$? Also, even in case of topological charge density one needs to address the question of how is the fermion–response scale $\Lambda^F$ (see Eq. (3)) related to the usual momentum scale $\Lambda$. We will start elaborating on some of these issues in the present work, and the problem will be addressed fully in the third paper of this series.
1.6 Final Introductory Remarks

We wish to make three sets of comments before we start.

(i) The ideas presented in this series of papers (and the manuscripts themselves) have been developed over the period of several years, and certain aspects of it were discussed by the author in less unified manner at various meetings. In this extended period (and before that), there appeared works that, at least to some extent and at least in some regards, align with the points of view expressed in this work. Being such, we would like to mention some of them.

The idea of using low–lying fermionic modes as an unbiased probe of vacuum structure can be viewed as a precursor to Bottom–Up. The beginnings of the “mode” approach can be traced to the set of early works [13], where they were used to assess the global properties of the underlying configuration. The proposal for using fermionic modes for systematic study of local behavior in the configuration has been made in Ref. [14]. The related early papers in this category include [15, 16, 17, 18], and later [19]. Recently, there also appeared related studies that use the eigenmodes of the covariant lattice Laplacian for similar purposes [20, 21].

As emphasized in [4], the problem with studying individual eigenmodes is that they do not provide a direct interpretation of vacuum properties in terms of gauge fields or gauge invariant composite fields. As such, they have to rely on the model assumptions (although these might be testable a posteriori). Focusing on studying the properties of gauge invariant composite fields expressible in terms of fermion eigenmodes is the natural way to proceed [4, 5, 6] with Bottom–Up. This approach has recently been taken up also in Refs. [23], which also confirmed the existence of the fundamental structure of Ref. [5]. Similar fundamental structure was observed in case of $CP^{N-1}$ models in Ref. [22]. Using a certain non–standard non–local topological density operator, vacuum properties have been very recently investigated in a largely Bottom Up manner in Ref. [24]. The recent work on using the Laplacian modes for filtering the gauge fields [21] is an interesting step in the Bottom–Up direction as well.

(ii) Throughout this series of papers we will use a rather formal mathematical language when stating certain important conclusions. We do this even though in most cases these statements are not rigorously proved at the time (referring to them as conjectures instead of theorems), and even though it is very likely that in majority of cases they might never be proved. The reason for this is not to do mathematics, but rather to benefit from borrowing its language. Indeed, the rationale for such attitude is the belief that stating intended conclusions precisely helps the process of obtaining meaningful results in case of investigating the QCD vacuum structure via path integral. The following points offer some clarification for our point of view.

• If a precise statement is eventually invalidated (numerically or otherwise), we are still left with a non–trivial positive piece of information, namely that a particular definite possibility

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6Note that even though the “tool” is in principle unbiased, it can be used in a biased way. Here we will not attempt to sort out which claims contained in all these works actually correspond to reality or can be interpreted as unbiased statements.

7We have used this approach in some of our previous works and hope to continue this practice to the largest extent possible also in our future “Bottom–Up” investigations.
is excluded.
- A precise statement that turns out to be false can frequently be shown to be approximately true in a well-defined exact sense. The information value of such proposition can thus obviously be very large. Moreover, a false statement of this kind can frequently be modified in such a way (after additional information is obtained) that it captures the truth better, or it can even evolve into an exactly valid proposition. Indeed, we are interested in a set of precise statements that get better and better constrained as more numerical data is accumulated. Such a process has a chance to converge to meaningful definitive results.
- In a lattice theory with finite degrees of freedom (where continuum results are obtained via a limiting process) everything is well defined. Moreover, as emphasized in Sec. 1.4, the crucial part of the vacuum structure research is of geometric nature. There is just no good reason not to be as specific as possible in such setting.

(iii) Since this work has many different aspects to discuss and our goal is to present it in a manner that is as coherent as possible, we decided not to include any numerical data in this series of papers. Rather, there will be several accompanying papers where certain conjectures proposed here will be supported (or possibly invalidated). Our emphasis here is on building of the framework indeed.

2 The Set of Lattice Actions

As indicated by A1, our attention will be focused on the ensembles $\mathcal{E}^{LQCD}$ corresponding to various lattice theories $S^{LQCD}$ defining QCD in the continuum limit. The underlying idea is to build a conceptual framework which will guide us in the process of selecting theories judiciously in a manner that will maximize our chances of identifying the relevant collective degrees of freedom in QCD. To pursue this, we first need to specify the set of acceptable lattice actions. For most part, this discussion will be carried out in the context of gluodynamics (i.e. pure-glue QCD). However, we stress from the beginning that the full power of the approach proposed here only appears in full QCD. Indeed, certain crucial aspects will later be discussed in QCD with dynamical quarks. With that understanding, we will skip the “QCD” superscripts when denoting the actions, ensembles and related objects. Also, since we will be mostly working with lattice definitions of a continuum theory, we will mostly skip the superscript $L$ signifying the distinction. Depending on the context, we might distinguish the gluonic ($S^G$) and fermionic ($S^F$) parts of full QCD action ($S$).

Identifying proper lattice actions usually starts from classical expressions in the continuum and, to that end, we will use conventions where the gauge coupling only enters via the multiplicative factor of the gluonic part. More precisely, if $A_\mu(x) \in \text{su}(3)$ (smooth vector field of traceless anti-Hermitian matrices), then the field strength is defined via

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

while the expression for the gauge action is

$$S^G = \int d^4 x \, s^G(x) \quad s^G(x) = -\frac{1}{2g^2} \text{tr} F_{\mu\nu}(x) F_{\mu\nu}(x)$$

10
If $\psi(x)$ is a smooth Dirac field ($\psi(x) \in \mathbb{C}^{12}$ and $\bar{\psi} \equiv \psi^\dagger \gamma_0$),\(^8\) then the part of the action involving fermions reads

$$S^F = \int d^4 x \, s^F(x) \quad s^F(x) = \bar{\psi}(x)(\gamma_\mu D_\mu + m)\psi(x)$$ \hspace{1cm} (7)

where we assumed a single flavor for simplicity, and $D_\mu \psi(x) = (\partial_\mu + A_\mu(x))\psi(x)$.

The standard construction of lattice gauge theory proceeds via introducing the hypercubic lattice of points $n \in \mathbb{Z}^4$, and associating the gauge degrees of freedom $U_{n,\mu} \in \text{SU}(N)$ with links (emanating from $n$ and ending at $n+\mu$) while the fermionic degrees of freedom $\psi_n \in \mathbb{C}^{12}$ (before quantization) are associated with sites of the lattice. We then write the candidate action for latticized theory with massless quarks in the form

$$S^G = \beta \sum_n O_n(U, g) \quad S^F = \sum_{n,m} \bar{\psi}_n D_{n,m}(U, g) \psi_m$$ \hspace{1cm} (8)

where the bare gauge coupling in the conventional form $\beta = 2N/g^2$ has been introduced, and we have allowed for the possible explicit $g$-dependence of $O$ and $D$. The criteria for selecting $S^G$ and $S^F$ are the following.

(a) **Classical limit.** To establish the classical limit, the lattice structure is embedded into $\mathbb{R}^4$ via introduction of the “naive” lattice spacing $a$ and the relation $x = an$. If $A_\mu(x)$ and $\psi(x)$ are smooth fields\(^9\) defined on $\mathbb{R}^4$, then it is required that expressions (6,7) are reproduced upon transcription of (8) to the lattice via

$$U_{n,\mu} \leftarrow \mathcal{P} \exp \left( a \int_0^1 ds \, A_\mu(an + (1 - s)a\hat{\mu}) \right) \quad \psi_n \leftarrow a^{3/2} \psi(an)$$ \hspace{1cm} (9)

once the naive continuum limit $a \to 0$ is taken. Here $\mathcal{P}$ is the path ordering symbol and $\hat{\mu}$ a unit vector in direction $\mu$.\(^{10}\)

(b) **Locality.** Lattice actions are required to be local in the sense that the influence of distant fields on the contribution from site $n$ decays at least exponentially in lattice units. In other words, $\partial O_n(U, g)/\partial U_{m,\nu} \leq A e^{-\alpha|n-m|}$, with constants $A$ and $\alpha$ being independent of $U$ and $g$.\(^{11}\) Similarly, the kernel $D_{n,m}(U, g)$ should decay at least exponentially in $|n-m|$, and $|\partial D_{n,m}(U, g)/\partial U_{l,\nu}|$ exponentially in $|n-l|$ and $|m-l|$. While exponential locality might not be necessary, it is believed to be sufficient for ensuring universality.

---

\(^8\)Note that at this level we treat fermionic field as the complex-valued field and the “fermionic action” as the functional yielding the Dirac equation via the variational principle. We will thus view the transition to Grassmann variables as a part of “quantization”.

\(^9\)By “smooth” we will understand differentiable arbitrarily many times, even though a weaker condition would be sufficient. Arguments here do not depend on this detail.

\(^{10}\)With the standard definition of parallel transporter and the conventional expression for lattice covariant derivative, the link variable transports the vector from position $n + \hat{\mu}$ to $n$. Hence the path ordering prescription in Eq. (9). Simpler transcription for the gauge field, such as $U_{n,\mu} \leftarrow \exp(aA_\mu(na))$ or $U_{n,\mu} \leftarrow \exp(aA_\mu(na + a\hat{\mu}/2))$ can be used as well.

\(^{11}\)Note that partial derivatives with respect to $U_{n,\mu}$ are rather formal here since the matrix elements of $U_{n,\mu}$ are constrained. What one really means here are the partial derivatives in the canonical representation of Sec. 3.4.1, i.e. $\partial O_n(u, g)/\partial u_{n,\mu}^a$. 


(γ) Symmetries. From the point of view of universality (and convenience) it is also desirable that lattice theories preserve to the largest extent the symmetries of the action in the continuum. We thus require (i) gauge invariance, i.e. invariance under

\[ U_{n,\mu} \rightarrow G_n U_{n,\mu} G_{n+\mu}^{-1}, \quad \psi_n \rightarrow G_n \psi_n \]

(10)

with \( G_n \in SU(N) \); (ii) the invariance under transformations of the hypercubic lattice structure, i.e. lattice translations and lattice rotations; (iii) the chiral invariance of the massless fermionic action \( S^F \), i.e. invariance under infinitesimal Lüscher \[25\] transformations

\[ \psi \rightarrow \psi + i\theta \gamma_5 (1 - RD) \psi \]
\[ \bar{\psi} \rightarrow \bar{\psi} + \bar{\psi} i\theta (1 - DR) \gamma_5 \]

(11)

where \( R \) is local and \([R, \gamma_5] = 0\), which leads to Ginsparg-Wilson fermionic kernels satisfying \( \{D, \gamma_5\} = 2DR\gamma_5D\); (iv) Charge conjugation. (v) \( \gamma_5\)-hermiticity of \( D \), i.e. \( D^\dagger = \gamma_5 D \gamma_5 \).

(δ) No Doublers. The fermionic action \( S^F \) requires additional consideration. In particular, it has to be ensured that the corresponding free action \( (U_{n,\mu} \equiv 1) \) describes a single species of massless Dirac fermion. \(^{12}\)

In what follows, we will denote by \( S^G \) the set of lattice gauge actions \( S^G \) satisfying conditions \((\alpha - \gamma)\). Similarly, the set of fermionic actions satisfying \((\alpha - \delta)\) will be denoted as \( S^F \). It is worth pointing out that each element in Eq. (8) actually describes a one-parameter family of actions labeled by bare coupling \( g \). While the explicit \( g \)-dependence of lattice operators \( O \) and \( D \) can be present, the requirements \((\alpha - \gamma)\) have to be satisfied for all \( g \). For example, the correct classical limit of \( S^G \in S^G \) requires that

\[ O_0(U, g) \rightarrow -a^4 \frac{1}{4N} \text{tr} F_{\mu\nu}(0) F_{\mu\nu}(0) + O(a^5) \]

(12)

under transcription (9) independently of \( g \).

Finally, we wish to remark on the conventions that we will follow when denoting lattice entities. The distinction between lattice coordinates \( n, m, \ldots \) and the physical coordinates in \( \mathbb{R}^4 \), such as \( x = an, y = am, \ldots \) is made explicit by the change of notation. However, some confusion can arise due to the fact that while everything in Eq. (8) is dimensionless, we sometimes wish to write lattice relations in the continuum-like manner where the appropriate dimensions are in place. In that regard, we will follow the trend in the literature and not distinguish the symbols for dimensionless and continuum-like (possibly dimensionfull) lattice entities. However, to help distinguish the two cases, we will place the space-time index as a subscript if the entity is viewed as dimensionless, while we put it in parentheses if proper dimension is implied. Thus, for example, we have \( D_{n,m} \equiv D_{x,y} \equiv a D(x, y) \equiv a D(n, m) \), with the relations \( x = an, y = am \) implicitly understood. Similarly, the action \( S^F \) of Eq. (8) can be equivalently written as \( \sum_{x,y} \bar{\psi}(x) D_{x,y}(U, g) \psi(x) \) or \( a^4 \sum_{n,m} \bar{\psi}(n) D(n, m; U, g) \psi(m) \) or \( a^4 \sum_{x,y} \bar{\psi}(x) D(x, y; U, g) \psi(y) \). Note also that the link variable (being dimensionless) can take all possible forms, i.e. \( U_{n,\mu} \equiv U_{x,\mu} \equiv U(n, \mu) \equiv U(x, \mu) \).

\(^{12}\)It is worth pointing out here that the conditions we stated admit a highly singular behavior in the regions of configuration space far from classical behavior. These might become relevant for example if one claims that the corresponding classical ensemble (after quantization) can be fully represented by the probabilistic Markov chain. We will not deal with associated subtleties here.
3 Kolmogorov Entropy

In this section we start building the framework for a systematic exploration of a fundamental vacuum structure using lattice QCD in a Bottom-Up manner. Relying on I1 we will accept the proposition that even though the definition of QCD in the path integral formalism involves inherent randomness, there exists a non-trivial space-time order in configurations that dominate the evaluation of the associated regularized path integral. In an attempt to uncover the nature of this order, and thus the nature of presumably existing underlying collective degrees of freedom \( C \), we wish to study lattice definitions of QCD which tend to maximize the order and minimize the randomness. The goal of this section is to associate the amount of structure (order) in typical configurations with their Kolmogorov entropy, which we define as the Kolmogorov complexity (algorithmic complexity), of binary strings representing coarse-grained configurations. 13

The need for such conceptual step is quite clear. Indeed, while the notion of order (structure) is apparently central to the study of QCD vacuum via Euclidean path integrals (certainly so in the Bottom-Up strategy), its precise meaning is usually left unspecified. There are at least two reasons for this. First, it might be assumed that, intuitively, it is quite clear what we mean by saying that some configuration is more ordered than other. This is certainly true in extreme cases, e.g. when comparing the configuration with constant field strength to a configuration with randomly generated links, but the question becomes fuzzy in less extreme cases. Secondly, the notion is highly non-trivial conceptually, and touches upon the involved issues related to proper definition of “randomness”. We wish to emphasize in this regard, that our interest in these issues at this point is strictly theoretical. We will point out that there exists a well-defined measure of space-time order in any configuration of lattice gauge field (and various composite fields), which allows to quantify the degree of space-time order for any lattice theory. In other words, we will discuss in what sense it is meaningful to say that theory \( S_1 \) exhibits more space time order than theory \( S_2 \), but will not attempt to make these concepts into a computational scheme.

In statistical mechanics, the discussion of “order” usually goes hand in hand with the concept of entropy. Consider, for example, two classical statistical systems consisting of identical degrees of freedom, but involving different interactions among them. We are inclined to say that the system with lower entropy is more ordered than the system with higher entropy. In the same spirit, consider the pure-glue lattice gauge theory \( S(g) \in \mathcal{S}^{G} \). Upon quantization, it becomes a classical statistical system with the associated canonical ensemble \( \mathcal{E}(g) \), where the action plays the role of “energy”. We could thus use the corresponding Gibbsian entropy of \( \mathcal{E}(g) \) as a quantitative measure of “order” associated with that theory (at any fixed \( g \) and fixed lattice volume). The problem with such definition is that it obscures (and avoids) the notion of the order at the configuration level – the concept we are primarily interested in when we try to identify the collective degrees of freedom of QCD in a Bottom-Up manner. In other words, Gibbsian entropy is not assigned to a given configuration independently of an ensemble we are studying: Gibbsian entropy is strictly an ensemble concept.

13We should point out that the term “Kolmogorov entropy” is used, with different meaning, also in the theory of dynamical systems (metric entropy). However, since the measure discussed here is indeed entropy-like, we find it appropriate to use it as well.
Interestingly, the issues with similar ingredients arose in the information theory few decades ago. In particular, the Shannon’s classical theory of communication [26] treats the objects to be communicated as outcomes of a random source with associated probability distribution. The central notion here is the concept of Shannon entropy which only depends on the distribution itself, and does not explicitly refer to the “amount of information” associated with an individual object. The dual point of view was developed later by Solomonoff [27], Kolmogorov [28] and Chaitin [29], and is known as the algorithmic information theory, Kolmogorov complexity, or algorithmic complexity. Here the focus is on the object itself and it defines the information (complexity) associated with an object with the smallest number of bits required to describe it via effective computation. The standard textbook exposition to this subject is Ref. [30], and the earliest work we are aware of on utilizing this framework in physics is Ref. [31]. Bellow we discuss in what sense we can carry these concepts into the study of QCD vacuum structure. Since the notions of information theory needed are not standard in this area, we will describe them in some detail before dealing with field theories.

3.1 Shannon Entropy

Suppose we have a finite or countable set of objects (source words) \( x \in \mathcal{X} \), which are intended to be communicated to the receiver by the means of a binary string. Enumerating these objects in an arbitrary fixed manner, we associate each \( x_n \) with its order number \( n \) (for the purposes of communication) which, in turn, can be mapped surjectively to the set of finite binary strings \( \mathcal{B}^* \equiv \{0,1\}^* \), e.g. via canonical lexicographic assignment

\[
0 \leftrightarrow \epsilon, \ 1 \leftrightarrow 0, \ 2 \leftrightarrow 1, \ 3 \leftrightarrow 00, \ 4 \leftrightarrow 01, \ 5 \leftrightarrow 10, \ldots
\]

where \( \epsilon \) is an “empty word”. Given the maps, we will interchangeably denote by \( x \) the object, the integer or the string. \(^{14}\)

For a meaningful communication, the sender and the receiver have to agree both on the above maps, and on the method which reproduces \( x \) from its code \( y \). This is characterized by a definite decoding function \( D : \mathcal{B}^* \rightarrow \mathcal{B}^* \) such that \( D(y) = x \). While \( D \) does not necessarily have to have an inverse (there could be several code words representing the same source word), in this discussion we assume that the encoding substitution \( E(x) = \{ y : D(y) = x \} \) contains a unique element \( y \). If \( l(z) \) denotes the length (in bits) of arbitrary string \( z \), then the efficiency of coding a particular \( x = D(y) \) is given by \( l(y) \equiv L_D(x) \).

When communicating a stream of source words, it is advantageous to consider only codes for which recognizing the end of the current code word in the stream does not require observation of the subsequent words. Such codes are referred to as instantaneous codes or prefix codes. Formally, the subset \( A \subset \mathcal{B}^* \) is prefix–free, if \( xy \notin A \) (concatenation of \( x \) and \( y \) is not in \( A \)) for any non-empty \( x, y \in A \). The code specified by \( D \) is then called a prefix code if its domain is prefix-free.

Assume that the source words are generated by a random source specified by the prob-

\(^ {14} \) We emphasize that especially in the case of integer versus associated binary string, we will mostly make no distinction (verbal or otherwise) whatsoever, and use e.g. \( x \) in the context of an integer even if it was previously referred to as a string.
ability distribution \( P(x) \) on \( X \). \(^{15}\) In other words, we are interested in the problem of communicating the class of messages, such that the source words are distributed according to \( P(x) \). The effectiveness of the communication accomplished via particular decoding function \( D \) is then measured by the average code–word length, i.e. \( \bar{L}_D \equiv \sum_x P(x) L_D(x) \). A natural problem in this setting is to ask what is the minimal possible average code–word length, i.e. \( \bar{L} \equiv \min_D \{ L_D \} \), where the minimization goes over all one-to-one prefix codes \( D \). The result is the Shannon’s Noiseless Coding Theorem \([26, 30]\).

**Theorem 1** If \( H(P) = -\sum_x P(x) \ln P(x) \) is the Shannon entropy associated with \( P \), then

\[
H(P) \leq \bar{L} \leq H(P) + 1
\]  

The optimal average code–word length is thus (practically) equal to the Shannon entropy. The above discussion attempted to highlight two facts. (1) The concept of Shannon entropy is completely independent of the nature of the objects from \( X \) that source-words represent. In “information” terms, \( H(P) \) is the amount of information the observer (aware of \( P \)) gains when he witnesses an outcome of \( P \). It is a measure of uncertainty in \( P \). (2) We are led to Shannon entropy via the minimization of the “cost of description” of the ensemble \( \{ (x, P(x)) \} \). Indeed, the message (binary string) \( y \) such that \( D(y) = x \) can be viewed as a description of the binary string \( x \) representing some real object from \( X \). However, the optimal coding of \( x \) (its optimal description) is completely determined by \( P(x) \) and not \( x \) itself.

### 3.2 Kolmogorov Complexity

The notion of Kolmogorov complexity represents an attempt to quantify the amount of information associated with any particular binary string \( x \). If such string completely characterizes some real object from \( X \) (i.e. it is not just a string assigned to it by \textit{ad hoc} enumeration procedure, but faithfully reflects all of its attributes and no more), then it would provide an absolute amount of information stored in that object.

It turns out that a concept can indeed be defined in such a way that it will satisfy the above features for asymptotically long strings. To explain the basic idea, we consider the same setup as in the previous section in that we consider the set of decoding functions \( D: \mathbb{N}^0 \to \mathbb{N}^0 \) (or equivalently \( \mathcal{B}^* \to \mathcal{B}^* \)), and call \( y \) such that \( D(y) = x \) a description of \( x \) under \( D \). \(^{16}\) We do not need to assume that \( D \) is one-to-one (there can be several descriptions of the same object after all) and, to have maximal generality, we initially do not require the domain of \( D \) to be prefix-free. In what follows, we will refer to this general set of functions as \textit{partial} functions, where the term emphasizes that a particular function can be undefined for some arguments \( y \). Even though our goal is to define the amount of information in the \textit{individual} string \( x \) (quantified by the length of the minimal description \( y \)), it is obvious that we need to consider \( x \) in the context of all other strings from \( \mathcal{B}^* \). Indeed, in the opposite case we could simply choose any decoding function \( D \) such that \( D(1) = x \), and thus associate

---

\(^{15}\)This can clearly be thought of as a distribution on \( \mathcal{B}^* \) with the corresponding mappings assumed, and with the probability of “unused” strings occurring being zero.

\(^{16}\)We will follow the convention where the set of positive integers (natural numbers) is denoted as \( \mathbb{N} \), while the set of non-negative integers as \( \mathbb{N}^0 \). The associated finite subsets of consecutive integers will be denoted as \( \{1, 2, \ldots, m\} \equiv \mathbb{N}_m \) and \( \{0, 1, \ldots, m - 1\} \equiv \mathbb{N}_m^0 \).
a single bit of information with any particular \( x \), which is clearly not satisfactory. Thus, a natural way to proceed would appear to try to identify a function \( D_o \) such that the associated minimal description is optimal for each \( x \). More precisely,

\[
L_{D_o}(x) \leq L_D(x) \quad \forall x, \forall D \quad \text{where } L_D(x) \equiv \min_y \{ I(y) : D(y) = x \} \tag{15}
\]

where we assume that if for given \( D \) there is no \( y \) such that \( D(y) = x \), then \( L_D(x) \equiv \infty \).

However, it is easy to see that the partial function \( D_o \) with above properties doesn’t exist. Indeed, assume the opposite and choose \( x_1, x_2 \) such that \( L_{D_o}(x_1) < L_{D_o}(x_2) \), with the corresponding minimal descriptions \( x_1^*, x_2^* \) (i.e. \( D_o(x_1^*) = x_1 \) and \( D_o(x_2^*) = x_2 \)). Define the function \( D'_o(y) = D_o(y) \) for all \( y \neq x_1, x_2^* \), and \( D'_o(x_1^*) = x_2 \), \( D'_o(x_2^*) = x_1 \). This obviously leads to shortening of the minimal description for \( x_2 \) and, in particular, \( L_{D'_o}(x_2) = L_{D'_o}(x_1) < L_{D_o}(x_2) \) thus producing a contradiction with the premise that \( D_o \) is optimal.

The above argument shows that, in order to get a meaningful concept, it might be fruitful to relax the notion of an “optimal” function.\(^{17}\) A natural proposal in this direction is to consider additive optimality. In particular, the partial function \( \Phi \) is additively optimal (universal) for the set of partial functions if for every \( D \) there is a constant \( \gamma(\Phi, D) \) (independent of \( x \)), such that

\[
L_\Phi(x) \leq L_D(x) + \gamma(\Phi, D) \quad \forall x, \forall D \tag{16}
\]

Assuming that a universal function exists, there is no reason for it to be unique. However, the optimal lengths under two universal descriptions clearly differ from each other at most by an \( x \)-independent constant, i.e. \( |L_{\Phi_1}(x) - L_{\Phi_2}(x)| \leq \gamma(\Phi_1, \Phi_2) \). While this means that \( L_\Phi \) will not define a strictly absolute amount of information in a finite string, there is universality here which applies for asymptotically large ones. To see that, we first point out that any \( L_D(x) \) assigning finite value to all finite \( x \) (\( L_\Phi(x) \) obviously has to be of this kind), has an unbounded lower envelope \( L_D^{\min}(x) \equiv \min_{z \geq x} \{ L_D(z) \} \). Indeed, let \( \mathcal{B}(l) \) denote the set of all binary strings up to length \( l \). This is a finite set, and thus there exists \( \tilde{x}(x) \equiv \max\{ D(y), y \in \mathcal{B}(L_D^{\min}(x)) \} \).

By construction, for all \( z > \tilde{x}(x) \), we have \( L_D^{\min}(z) > L_D^{\min}(x) \), since all possible encodings of length \( L_D^{\min}(x) \) are simply exhausted for such \( z \). Repeating this argument for \( \tilde{x}(x) + 1 \) and recursively shows that \( L_D^{\min}(x) \) (and thus \( L_D(x) \)) is unbounded. For two universal functions \( \Phi_1, \Phi_2 \) we then have

\[
\frac{|L_{\Phi_1}(x) - L_{\Phi_2}(x)|}{L_{\Phi_1}(x)} \leq \frac{\gamma(\Phi_1, \Phi_2)}{L_{\Phi_1}(x)} \rightarrow 0 \quad \text{for} \quad l(x) \rightarrow \infty \tag{17}
\]

In this sense the additive optimality is universal for asymptotically large strings.

The above discussion suggests that one could define the amount of information in \( x \) as \( L_\Phi(x) \) using arbitrary (but fixed) \( \Phi \) from the equivalence class of universal partial functions. However, it turns out that there is still no additively optimal element \( \Phi \) on the set of partial functions. Indeed, assume the opposite and consider arbitrary increasing sequence of universal minimal encodings \( x_1^* < x_2^* < x_3^* \ldots \) (i.e. \( \Phi(x_1^*) = x_1, \Phi(x_2^*) = x_2, \ldots \)).\(^{18}\)

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\(^{17}\)Indeed, it is easy to see that definition (15) would be meaningless even if we were interested in describing \( x \) in the context of finite number of other strings, let alone in the context of \( \mathcal{B}^* \).

\(^{18}\)Note that the minimal encoding substitution \( E_D(x) = \{ y : D(y) = x, l(y) = L_D(x) \} \) can be made into a one-to-one function by fixing \( y \equiv x^* \) that appears first in the lexicographic ordering of binary strings.
Choose an arbitrary subsequence $z_i^* \equiv x_{j(i)}^*$, such that $\log_2 x_i^* < (\log_2 x_{j(i)})/2$, and define the new function $\Phi'$ via $\Phi'(y) \equiv \Phi(y)$ for $y \neq x_i^*$, and $\Phi'(x_i^*) = \Phi(x_{j(i)})$. By construction, the encodings under $\Phi'$ for all $x_{j(i)}$ will be shorter than encodings under $\Phi$ and, in particular, $L_{\Phi'}(x_{j(i)}) = l(x_i^*) \leq (l(x_{j(i)})/2 = L_\Phi(x_{j(i)})/2$. Here the inequality follows from the fact that in the canonical correspondence (13) between integers and binary strings we have $l(x) = \lceil \log_2(x + 1) \rceil$, where $\lceil \cdot \rceil$ denotes the floor function. We thus arrived at the contradiction since $L_{\Phi'}$ is not related to $L_\Phi$ via $x$-independent constant. Consequently, the set of partial functions does not have the lowest element under additive optimality.

While this appears to be a serious blow for the development of the concept we are interested in, it actually turns out to be a virtue in the end (see below). To expose the basic trick, let us consider arbitrary enumerable subset $A = \{D_1, D_2, D_3, \ldots \}$ of partial functions. We can construct the partial function (specification method) $D_o(y)$ that minorizes all elements of $A$ under additive optimality in the following way. Fix certain one-to-one correspondence $(n, y) \mapsto z \equiv \langle n, y \rangle$ (i.e. bijection $\mathbb{N}^0 \times \mathbb{N}^0 \mapsto \mathbb{N}^0$). If string (integer) $z$ represents the pair $(n, y)$, i.e. $z = \langle n, y \rangle$, then $D_o(z) \equiv D_n(y)$. By construction, the function $D_o$ can emulate all the functions $D_n$ by properly selecting its arguments $z$. Let us consider the “description length” function associated with $D_o$, i.e.

$$L_{D_o}(x) \equiv \min_z \{ l(z) : D_o(z) = x \} = \min_{n, y} \{ l(\langle n, y \rangle) : D_n(y) = x \} \quad (18)$$

and compare it to $L_{D_k}$ for arbitrary (but fixed) $k$. Denoting by $x_D^*$ the canonical minimal encoding of $x$ under $D$, there exists $x$-dependent $i$ such that

$$L_{D_o}(x) = l(\langle i, x_{D_k}^* \rangle) \leq l(\langle k, x_{D_k}^* \rangle) = l(x_{D_k}^* ) + c_k = L_{D_k}(x) + c_k \quad (19)$$

where $c_k \equiv l(\langle k, x_{D_k}^* \rangle) - l(x_{D_k}^* )$ does not depend on $x$ if the pairing bijection $(n, y) \mapsto z \equiv \langle n, y \rangle$ is chosen suitably. For example, with one of the standard prescriptions $\langle n, y \rangle = \bar{n}y$ (string $\bar{n}$ concatenated with string corresponding to $y$), where the string $\bar{n} = 1^n0n$ (meaning a string of $n$ symbols 1 followed by 0 followed by binary string corresponding to integer $n$), we have $c_k = 2l(k) + 1$. Thus, indeed, the function $D_o$ minorizes all $D_k \in A$. Assuming that things can be arranged so that $D_o$ itself is an element of $A$, then $D_o \equiv \Phi$ is a universal element for $A$ under additive optimality.

The notion of Kolmogorov complexity is built on two basic results from the theory of computation, namely the fact that the set of recursive (computable) functions is enumerable, and that the minorizing functions constructed according to (18) are themselves recursive, thus representing universal elements. The fact that the set of recursive functions allows for a meaningful definition of minimal description length based on additive optimality lends the feeling of inherent “rightness” for the concept attempting to quantify the absolute amount of information stored in a string. Indeed, the use of a decoding function $D$ in communicating an information on object $x$ might be viewed as pointless unless the message can be effectively decoded by computing machines. At the same time, it is generally accepted that the notion of “computability” (or effective computation) is equivalent with the notion of recursive functions, i.e. functions that can be implemented on a Turing machine. 19

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19More precisely, the partial function $D(y) = x$ (from $\mathbb{N}^0$ to $\mathbb{N}^0$) is recursive if there is a Turing machine $T$ such that for inputs $y$ for which $D(y)$ is defined, $T$ outputs $D(y)$ in a finite number of steps, while it runs forever on inputs for which $D(y)$ is not defined.
Focusing on recursive functions, the language of information theory starts to mesh with the language of computation theory. In the former case the string \( y \) such that \( D(y) = x \) is referred to as a code for \( x \) under \( D \) (or simply an argument of function \( D \)), while in the latter case one refers to the input string \( y \) placed on a tape of a Turing machine \( T \) implementing \( D \) as a *program* producing the output \( x \). The enumerability of all Turing machines (and thus of partial recursive functions) can be easily seen from the fact that one can enumerate all possible finite *lists of rules* that define the function of a valid Turing machine. In fact, each list of rules can be mapped into a unique binary string. We will assume the canonical construction of this type, which is straightforward but tedious, and can be found e.g. in Ref. [30]. Searching in the lexicographic order through all strings, let \( n(i) \) be the \( i \)-th string encoding a possible Turing machine, which we denote as \( T_i \). This enumeration of machines induces an associated enumeration of recursive functions \( D_i \) implemented by \( T_i \).\(^{20}\)

The crucial point is that the canonical enumeration is recursive in the sense that there is a Turing machine which can decide whether a given string \( n \) encodes a valid machine. From this it can be seen rather easily that applying a construction (18) to sequence \( \{D_i, i = 1, 2, \ldots \} \) yields a minorizing \( D_o \) which is itself recursive. Indeed, one can construct the machine \( T^u \) which takes the pair \( (i, y) \leftrightarrow \langle i, y \rangle \) as its input, and will simulate (on its own tape) machine \( T_i \) running program \( y \). Given \( i \), this machine will start generating binary strings in lexicographic order and check whether they represent the list of rules for a Turing machine. When encountering \( i \)-th such string \( n(i) \), it will run \( y \) using the set of rules specified by \( n(i) \) it found. By construction the machine \( T^u \) computes \( D_o \).\(^{21}\)

To summarize, what we have described above are the basic logical ingredients leading to the fact that the set of partial recursive functions contains a universal element \( \Phi \equiv D_o \) under additive optimality. This element has been identified as a function computed by the *universal* Turing machine \( T^u \), i.e. machine that can simulate the actions of any other Turing machine. Thus, the length of the shortest description under \( \Phi \) (namely \( L_\Phi(x) \)) provides an absolute measure for information stored in an asymptotically long string \( x \). It is the number of bits that have to be transmitted to the receiver who can decode it in an effective manner (using a Turing machine). In the language of computation theory it is the length of a shortest program that, without any additional input, outputs \( x \) and halts when run on \( T^u \).

We wish to make few comments at this point:

1. There are infinitely many universal Turing machines and infinitely many universal functions. Due to the universal nature of the concept discussed above, we can fix the universal machine and the universal function to be the canonical elements described above. Then the *plain Kolmogorov complexity* of string (natural number) \( x \) is defined as

\[
C(x) \equiv L_\Phi(x) \quad \Phi \text{ computed by reference machine } T^u
\]  

(20)

2. Kolmogorov complexity formalizes the intuitive notion that strings with lots of regular patterns can be described succinctly (e.g. the string 11111...1), while strings without any

\(^{20}\)It should be emphasized that while \( T_i, T_j \) always represent distinct machines for \( i \neq j \), they might be implementing the same function. The set of recursive functions is defined as the set of all distinct elements in the sequence \( \{D_i, i = 1, 2, \ldots \} \).

\(^{21}\)We note that the above description of \( T^u \)-s function is not the proof of its existence. Such proof has to demonstrate the existence of the set of rules (list of quadruples of integers) defining the machine that will function as described.
appreciable regularities are not amenable to short description no matter what method is used. Thus, apart from information-theoretic meaning of the unavoidable amount of information to be transmitted,\textsuperscript{22} it is also an absolute measure of order present in the asymptotically long string. In particular, when considering strings $x_1$ and $x_2$ of the same length, then we view $x_1$ as more ordered than $x_2$ if $C(x_1) < C(x_2)$.

(3) Contrary to the discussion of Shannon entropy, we have not restricted ourselves to the prefix (instantaneous) codes $D(y)$ so far. However, it turns out that the whole logical construction described above can be repeated if one considers only partial recursive functions with prefix-free domains. Such functions are implemented by prefix Turing machines, which only yield a halting calculation on the elements of such domain. Fixing a canonical enumeration of prefix machines \textsuperscript{23} one can construct the universal prefix machine $T^u$ according to recipe described above and use it to define the universal prefix function $\Psi$ and the corresponding prefix Kolmogorov complexity

$$K(x) \equiv L_{\Psi}(x) \quad \Psi \text{ computed by reference prefix machine } T^u \quad (21)$$

The functions $C(x)$ and $K(x)$ are asymptotically equal. However, $K(x)$ has several desirable properties which $C(x)$ does not.\textsuperscript{24} Because of this (and also to continue the comparison to Shannon’s theory) we will use the prefix version of Kolmogorov complexity in what follows.

(4) While the concept of Kolmogorov complexity is built on the notion of effective computation, the complexity functions $C(x), K(x)$ are themselves not effectively computable (recursive). Thus, the significance of the framework is mainly conceptual and theoretical. Nevertheless, many of its aspects have been successfully utilized in many varied areas. Also, the basic (and apparently deep) idea of minimal description length frequently serves as a guide for devising less general (and typically less elegant) notions that can be applied in practice in a more quantitative manner.

(5) An elementary result from the theory of Kolmogorov complexity that will be useful for us is that, if $f(x)$ is an invertible recursive function, then the complexities $C(x)$ and $C(f(x))$ differ by at most a constant independent of $x$. This is a simple consequence of the fact that to arrange for evaluation of $f(x)$ on an appropriate universal machine, one only needs to specify (input) the ordering number $i$ of the machine $T_i$ that computes $f$. Analogous statement obviously holds for $K(x)$ as well.

(6) It is sometimes instructive and useful to think about Kolmogorov complexity in terms of programs written in high-level universal languages (such as C++, LISP or Java). While the whole approach can be formalized and even used for practical purposes, we will not be concerned with this aspect here, and view it only as a tool for developing intuition.

\textsuperscript{22}We note in this context that strings for which $C(x) < l(x)$ (i.e. they allow shorter description than themselves) are referred to as compressible, while the rest of them as incompressible.

\textsuperscript{23}One way to proceed is to use a standard procedure of transforming any given Turing machine into a prefix machine. This way the canonical enumeration of Turing machines directly induces the enumeration of prefix machines.

\textsuperscript{24}One relevant difference is that, unlike plain complexity, the prefix version is subadditive, i.e. $K(xy) \leq K(x) + K(y) + O(1)$ (here $xy$ means concatenation of strings $x$ and $y$).
3.3 Kolmogorov Entropy: $\mathbb{Z}_2$ Gauge Theories

The preliminaries in the previous two subsections allow us to come back to the question of how to quantify the level of space-time order in lattice gauge theories in an interesting manner. Before dealing with SU(N) theories, it is first useful to take a simpler case. For example, consider the set of pure gauge theories $S^G$ with N=2. To every $S^G \in S^G$ assign a $\mathbb{Z}_2$ gauge theory $S^{GZ}$ by replacing $SU(2)$ variables by $\mathbb{Z}_2$ variables, and the path integral by a corresponding discrete sum. This defines the set $S^{GZ}$ of $\mathbb{Z}_2$ gauge theories. Our goal is to answer (in principle) the following generic question. Take two theories $S^{GZ}_1(\beta_1)$ and $S^{GZ}_2(\beta_2)$ ($\beta$-s fixed) both defined on a finite discretized symmetric torus, i.e. both involving $L^4$ lattice sites. Which theory possesses more space-time order?

Our aim is to approach this issue via concept that allows for even more detailed question. In particular, given two $\mathbb{Z}_2$ configurations $U_1, U_2$, which one exhibits more space-time order? We propose to answer this question using the notion of Kolmogorov complexity. Indeed, for fixed $L$, the set $\mathcal{X}$ of objects we are interested in is the set of all possible $\mathbb{Z}_2$ configurations $U$. It is clear that every $U \in \mathcal{X}$ can be completely described by a binary string $z(U)$ of length $4L^4$. Indeed, the two possible values of the field $U(n, \mu)$ associated with link $(n, \mu)$ can be represented by the two binary values $b \in \mathcal{B} \equiv \{0, 1\}$. Moreover, if we fix some one-to-one function $F(n, \mu) = k \in \{1, 2, \ldots, 4L^4\}$, then we can form a corresponding binary string via

$$z_F(U) \equiv b_1 b_2 \ldots b_{4L^4} \quad b_k = U(F^{-1}(k)) \quad (22)$$

We need to be more specific about the nature of the function $F$, which has to be defined for all $L$. In other words, $L$ has to be one of the arguments of $F$, and also (a dummy) component of the image so that $F$ is formally invertible. To do that, let us first agree that the coordinates $n \equiv (n_1, n_2, n_3, n_4)$ on the torus of size $L$ take the values $n_i \in \{0, 1, \ldots, L - 1\} = \mathbb{N}_L$, and remind ourselves that $\mu \in \{1, 2, 3, 4\} = \mathbb{N}_4$. Then we consider invertible partial functions

$$F : \mathbb{N} \times (\mathbb{N}_L)^4 \times \mathbb{N}_4 \mapsto \mathbb{N} \times \mathbb{N} \quad \text{such that} \quad (\mathbb{N}_L)^4 \times \mathbb{N}_4 \mapsto \mathbb{N}_4 \quad \text{under} \quad F(L, n, \mu) \quad (23)$$

for arbitrary fixed $L$, where we skipped the $L$-component of the domain/range in the second part of the expression (see also equation (25) below). We then define the Kolmogorov entropy of configuration $U$ relative to $F$ via

$$E^K_F(U) \equiv K(z_F(U)) \quad (24)$$

i.e. we associate it with prefix Kolmogorov complexity of $z_F(U)$.

The degree of order detected in $z^F(U)$ via its prefix complexity is a result of two factors. The first is the level of inherent space-time order present in configuration $U$, while the second is the influence of map $F$ which, although fixed for all $U$, appears rather arbitrary at this point. However, we will argue bellow that this arbitrariness is innocuous for asymptotically large configurations $U$ if we restrict ourselves to recursive functions $F$. Indeed, denoting the set of invertible recursive maps satisfying (23) as $\mathcal{F}$, let us consider $F_1, F_2 \in \mathcal{F}$, and compare the complexities $K(z_{F_1}(U)), K(z_{F_2}(U))$. Since $F_1, F_2$ are recursive, the binary strings $z_{F_1}(U)$ and $z_{F_2}(U)$ are recursively related via invertible map

$$z_{F_2}(U) = G^{F_1 F_2}(z_{F_1}(U)) \quad \forall U \quad \text{where} \quad (G^{F_1 F_2}(x))_i \equiv x_{F_1(F_2^{-1}(i))} \quad (25)$$
and we have denoted the $i$-th bit of string $x$ by $x_i$. Consequently (see comment (5) in previous section), the complexities $K(z_{F_1}(U)), K(z_{F_2}(U))$ differ at most by an $U$-independent constant $\gamma(F_1,F_2)$. We can thus conclude that Kolmogorov entropy (24) is additively optimal with respect to $F$, with arbitrary $F \in \mathcal{F}$ being a universal element.

It should be remarked that since additive optimality is inherent in the notion of Kolmogorov complexity already, it is only natural that the Kolmogorov entropy (24) has an additive arbitrariness related to the choice of $F$. While we can choose and fix any $F \in \mathcal{F}$, it is relevant here that the complexity of $F$ can affect significantly the value of Kolmogorov entropy in small configurations, and possibly mask the true level of their space-time order. Thus, contrary to the selection of universal machine, there is a natural general preference for the choice of maps $F$, namely for those with low complexity. In fact, given $T^u$, the appropriate choice $F^u$ can be constructed as follows. Let $n(i)$ be the sequence of natural numbers such that $T_{n(i)}$ is the $i$-th Turing machine computing function $F_i \in \mathcal{F}$. The set $\{K(n(i))\}$ is bounded from below and thus has a minimum $K_{\text{min}}$. We define

$$F^u \equiv F_k \quad \text{such that} \quad k = \min\{i : K(n(i)) = K_{\text{min}}\} \quad (26)$$

In other words, $F^u$ is specified by lexicographically first Turing machine computing an element of $\mathcal{F}$, whose complexity is equal to $K_{\text{min}}$. Similarly to fixing universal machine $T^u$, we now fix the map $F$ to $F^u$ and thus define the Kolmogorov entropy of configuration $U$ as

$$E^K(U) \equiv K(z(U)) \quad z(U) \text{ computed via } F^u \quad (27)$$

Given two configurations $U_1, U_2$ on the same space-time lattice, we define $U_1$ to exhibit more space-time order than $U_2$ if $E^K(U_1) < E^K(U_2)$.

We are now in the position to quantify the degree of space-time order associated with any theory $S \equiv S(\beta) \in \mathcal{S}^{GZ}$. In particular, we identify this measure with the average Kolmogorov entropy in ensemble $\mathcal{E}$ associated with $S$, namely

$$E^K[S] \equiv \langle E^K(U) \rangle_S = \sum_U E^K[U] P_S(U) \quad P_S(U) = e^{-S(U)}/Z \quad (28)$$

The meaning of $E^K[S]$ is quite clear. If the ensemble $\mathcal{E}$ is dominated by configurations with high level of space-time order, then its Kolmogorov entropy is low. Equivalently, if the individual configurations typically appearing in probabilistic chains representing $\mathcal{E}$ exhibit a high degree of space-time order, then the Kolmogorov entropy of the theory is low.

We wish to make few important remarks at this point:

1. For fixed finite $L$, Kolmogorov entropy $E^K[S,L]$ induces an order relation on the set of actions $\mathcal{S}^{GZ}$. In particular, we say that $S_1 < S_2$ if $E^K[S_1,L] < E^K[S_2,L]$. Since the elements of $\mathcal{S}^{GZ}$ are local, it is expected that any two actions can be consistently compared in the infinite volume limit, i.e. that there exists $L_{12}$ such that $E^K[S_1,L] < E^K[S_2,L]$ for

25 It is worth emphasizing that one defines the complexity of a general recursive function $f$ as $K(f) \equiv \min_i \{K(i) : T_i \text{ computes } f\}$. The reason is that, assuming a canonical order of machines, specifying $i$ is all that is needed to specify the map.

26 More precisely, it induces a linear order on the set of equivalence classes, where $S_1 \sim S_2$ if $E^K[S_1] = E^K[S_2]$. 

21
all $L > L_{12}$. As discussed extensively above, the nature of Kolmogorov entropy is such that there is an inherent arbitrariness by a constant (related to the choice of universal machine and the choice of map $F$), and the notion acquires an absolute meaning in the limit $L \to \infty$. At the same time, the above ordering relation is expected to be affected very little by this arbitrariness even at finite $L$ since $T^u$ and $F^u$ are fixed for all $U$ and all theories. Nevertheless, if one wants to ensure strict universality in the infinite–volume limit, then bounded differences of entropies need to be considered as irrelevant. In particular, if there is an $L$-independent constant $C_{12}$ such that

$$| E^K[S_1, L] - E^K[S_2, L] | < C_{12} \quad \forall L$$

then $S_1$ and $S_2$ should be viewed as possessing the same amount of space-time order. In other words, it is natural in the $L \to \infty$ limit to consider the equivalence classes of theories, with the equivalence relation defined by (29).

(2) In definition (27) we took into account that it is desirable (but not necessary) to minimize the influence of the map $F$ on the complexity $K(z(U))$. This can be done alternatively (and in some ways more elegantly) by considering the conditional Kolmogorov complexity $K(z(U)|F)$. Roughly speaking, this means that information (and necessary input into $T^u$) about $F$ doesn’t count as a contribution to the complexity of $z(U)$. We will not discuss this possibility here in detail.

(3) Since we are dealing with gauge theories, it is natural to ask how gauge freedom enters the issue of inherent space-time order. In particular, given gauge transformation $g$, the Kolmogorov entropies $E^K(U)$ and $E^K(U^g)$ can be quite different. In other words, $E^K(U)$ is not gauge invariant in the usual sense (at the configuration level). We can see three acceptable views concerning this issue. (i) Gauge invariance is not really necessary for the concept characterizing the space-time order in the configuration. Indeed, it is simply a fact that gauge freedom can introduce an unphysical space-time noise. Since this noise is integrated over in the path integral for all theories in the same manner, the ordering of theories via $E^K[S]$ preserves its intended meaning even if one is interested in gauge-invariant quantities only. (ii) We can make $E^K(U)$ gauge invariant and define the associated Kolmogorov entropy for the theory via

$$\bar{E}^K(U) \equiv \min_g K(z(U^g)) \quad \bar{E}^K[S] \equiv \langle \bar{E}^K(U) \rangle_S$$

In other words, the gauge invariant Kolmogorov entropy of $U$ is taken to be the minimal Kolmogorov complexity of $z(U^g)$ over the gauge orbit $\{U^g\}$ of $U$. Since $K(x)$ is bounded from bellow, the minimum indeed exists for arbitrary $L$. (iii) Instead of taking descriptional complexities of $z(U)$ as a measure, we could use the complexities of strings $z(\phi(U))$ describing some fixed reference gauge invariant composite field $\phi(U)$. This possibility is less general, but can certainly be exploited for comparing the theories (via $E^K_{\phi}(U)$) with respect to a given relevant composite field (i.e. topological charge density in case of QCD).

Note that considerations related to Elitzur’s theorem do not play a role here since $E^K(U)$ is a global quantity for which no sensible local definition exists.
3.3.1 Information Issues and the Information Gauge

It is interesting to elaborate on the information theory viewpoint of the definitions discussed above. Assume that the sender wishes to communicate a complete information on \( \mathbb{Z}_2 \) configuration \( U \) to a receiver via message decodable using effective computation. Then the (additively) optimal amount of information necessary to transmit is given by \( E^K(U) \).

However, if we are interested (as usually is the case) in the gauge-invariant content of the configuration, then we can further minimize the cost of transmission by communicating only \( \bar{E}^K(U) \) bits to the receiver. In effect, instead of using \( U \), we are sending a description of a particular configuration \( V = V(U) \) from the associated gauge orbit, which possesses the highest level of space-time order. Such \( V(U) \) may not be unique, but we can make it unique e.g. by selecting a configuration which appears first in the lexicographic order of associated binary strings. More precisely, if \( V(U) \) is the ordered set of such configurations, namely

\[
V(U) \equiv \{ U^{g_i} : E^K(U^{g_i}) = \bar{E}^K(U), \ z(U^{g_{i+1}}) > z(U^{g_i}) \} \quad \text{then} \quad V(U) \equiv U^{g_1}
\]

where the ordering of binary strings is defined by the canonical correspondence (13). The above prescription assigns a unique configuration to a gauge orbit of arbitrary \( U \), and thus can be viewed as a gauge–fixing prescription.

Assume now that the sender intends to communicate to the receiver information about the gauge–invariant content of the \( \mathbb{Z}_2 \) gauge theory with action \( S \), and he wishes to do that via transmitting the description of some relevant configurations. This sender would generate \( N \) independent configurations \( U_j \) from distribution \( P_S(U) \) and, instead of transmitting minimal description of \( U_j \), he would transmit the description of \( V_j \) fixed by prescription (31) in order to minimize the cost. In other words, he would prefer “fixing the gauge” accordingly. For this reason, we refer to the gauge specified above as the information gauge.

It is also important to emphasize at this point that the role of information theory in our construction is purely to assess the level of space-time order exhibited by the configuration (theory), and we are not attempting to connect it to the absolute amount of “relevant information” stored in it. The basic concepts characterizing the latter, though of utmost interest, are yet to be developed. In fact, one should readily appreciate the apparent contradictory tendencies in that regard. Indeed, the configurations with high Kolmogorov entropy require large number of bits to describe, and in that sense one can view them as storing a lot of information. However, by the very nature of our approach, we expect their relevance for physics to be rather limited. On the other hand, we hope that configurations with low Kolmogorov entropy, exhibiting a specific kind of space-time order and allowing for a short description, will contain large amount of the physically relevant information.

3.3.2 Shannon Entropy and Kolmogorov Entropy

Let us recall that we started our discussion from the fact that Gibbsian entropy, though intuitively acceptable, does not provide a measure of space-time order derived from a notion directly applicable to an individual configuration. At the same time, in the path–integral approach it is the order at the configuration level we are seeking when trying to identify the collective variables relevant in QCD. This led us to the information theory where the analog of Gibbsian entropy is represented by Shannon entropy, which explicitly relates to “ensemble” properties, and does not address the information content of individual objects from the
ensemble. However, on the information theory side there exists a notion of Kolmogorov complexity which we used to define the “Kolmogorov entropy” of a configuration, and proposed that this is the appropriate measure of its space-time order. One can then define the corresponding measure for the $\mathbb{Z}_2$ theory $S$ via the ensemble average specified by $P_S$. We thus ended up with two distinct ensemble concepts for theory $S$, namely its Kolmogorov entropy and the Shannon entropy of the associated $P_S$. What differences can one expect?

It turns out that information theory provides a partial answer to this question for theories $S$ that lead to computable probability distributions. The theorem below is a direct transcription of the relevant result for probability distributions on the set of binary strings (see e.g. Ref. [30]) to the framework developed here.

**Theorem 2** Let $S \in S^{GZ}$ (for fixed size $L$ of a symmetric torus) leads to a computable probability distribution $P_S(U)$, and $E^{Sh}[S] \equiv -\sum U P_S(U) \ln P_S(U)$. Then

$$0 \leq E^K[S] - E^{Sh}[S] \leq K(\bar{P}_S) + O(1)$$

where $\bar{P}_S(z(U)) \equiv P_S(U)$ is the induced probability distribution on $\mathcal{B}^*$. 

It needs to be emphasized here that the distribution $\bar{P}_S$ (and its computability) has to be viewed as a function of the lattice size as well, i.e. $\bar{P}_S = \bar{P}_S(x, L)$. We also recall that $K(\bar{P}_S)$ is the Kolmogorov complexity of $\bar{P}_S$ (see discussion in section 3.3 and footnote 25).

In view of the above theorem, we need to explicitly emphasize few points:

1. **Theorem 2** implies that if $\mathbb{Z}_2$ theories $S_1, S_2$ have computable probability distributions, then their order relation induced by $E^K[S]$ is the same as the one induced by $E^{Sh}[S]$ in $L \to \infty$ limit. Indeed, this follows from the equivalence relation (29) and the fact that $E^K[S]$ and $E^{Sh}[S]$ only differ by an $L$–independent constant for a theory with computable $P_S$. Thus, if it could be established that all theories in $S^{GZ}$ lead to computable distributions, one could also use Shannon entropy to distinguish different levels of space-time order in the infinite volume limit. In the specific case of $\mathbb{Z}_2$ gauge theories this would justify the use of Gibbsian entropy (see footnote 28) as a measure of space–time order in the theory. However, the corresponding analysis of $S^{GZ}$ is not available yet, and one also needs to realize that the potential extension of such result would only be valid for theories with discrete local variables.

2. In connection with the above remark we wish to emphasize that we view $E^K[S]$ as the fundamental measure of space-time order in the theory, while $E^{Sh}[S]$ is an auxiliary concept that can be used, in some cases, to estimate $E^K[S]$. For certain issues the use of $E^K[S]$ is simply insufficient. For example, consider the problem of minimizing $E^K[S]$ on some subset of $S^{GZ}$ with computable distributions. Unless $K(P_S)$ is bounded on such subset (which is not necessary the case if the subset has infinite number of elements), then minimizing $E^{Sh}[S]$ will not help us at all even in the infinite volume limit. The recurring problem that appears in similar considerations is that Kolmogorov entropy and Shannon entropy are related by a “theory–dependent” constant ($K(\bar{P}_S)$).

28 In fact, in case of $\mathbb{Z}_2$ gauge theories which we discuss, Shannon entropy is identical to Gibbs entropy. However, this is not true for probability distributions with continuous sample spaces (e.g. lattice QCD).
3.4 Kolmogorov Entropy: SU(N) Gauge Theories

The extension of the above discussion to SU(N) gauge theories is straightforward in certain regards and problematic in others. From the point of view of numerical lattice QCD it is natural to view a configuration (to a given numerical accuracy) as a binary string since that’s how it is in fact stored in the memory of a computer. While the basic idea remains to apply the Kolmogorov complexity measure to such strings and thus define the Kolmogorov entropy of a configuration/theory, the whole procedure needs to be formalized properly. Moreover, there are new conceptual problems arising due to the fact that local variables are continuous and thus involve infinitely many bits even on a finite lattice. The issues related to taking the appropriate limits are very relevant and need to be addressed.

3.4.1 Canonical Coarse-Graining of the Gauge Group

In this section we discuss a systematic way of discretizing the gauge group which is convenient for associating the SU(N) gauge configurations with binary strings. A particular way we propose is motivated by the issues of space–time order discussed here, but also by the anticipation of studying gauge theories from the point of view of information theory later. After discretizing the space-time coordinates via introduction of a lattice, it is quite natural to discretize all continuous aspects of the theory, including the continuous degrees of freedom. However, this is usually not done since it is not necessary for proper regularization of a continuum system. At the same time, continuous gauge variables allow for incorporation of exact gauge invariance in essentially continuum–like manner. In reality though, exact gauge invariance carries mainly an aspect of “elegance” in this case since it is well under control even with a discretized gauge group, and its violations boil down to a numerical precision issue (See the discussion later in this section.). Moreover, for our purposes, the role of the gauge coarse–grained approximations is to define the measure of space–time order in the original lattice theory with continuous gauge variables, and not to supply a different lattice construction of continuum QCD.

Discretization of gauge variables is made easier by the fact that SU(N) groups are compact and each element can be specified by \( N^2 - 1 \) real numbers chosen from bounded intervals. To provide a sufficient level of detail, let us focus on the group SU(3) which is most relevant for our purposes. It is convenient to start from a particular trigonometric parametrization for matrix elements of \( g \in SU(3) \) in a fundamental representation, namely [32]

\[
\begin{align*}
  g_{11} &= \cos \theta_1 \cos \theta_2 e^{i\phi_1} & g_{13} &= \cos \theta_1 \sin \theta_2 e^{i\phi_4} & g_{12} &= \sin \theta_1 e^{i\phi_3} \\
  g_{22} &= \cos \theta_1 \cos \theta_3 e^{i\phi_2} & g_{32} &= \cos \theta_1 \sin \theta_3 e^{i\phi_5} \\
  g_{21} &= \sin \theta_2 \sin \theta_3 e^{-i(\phi_4+\phi_5)} - \sin \theta_1 \cos \theta_2 \cos \theta_3 e^{i(\phi_1+\phi_2-\phi_3)} \\
  g_{33} &= \cos \theta_2 \cos \theta_3 e^{-i(\phi_1+\phi_2)} - \sin \theta_1 \sin \theta_3 e^{-i(\phi_3-\phi_4-\phi_5)} \\
  g_{23} &= -\cos \theta_2 \sin \theta_3 e^{-i(\phi_1+\phi_5)} - \sin \theta_1 \sin \theta_2 \cos \theta_3 e^{i(\phi_2-\phi_3+\phi_4)} \\
  g_{31} &= -\sin \theta_2 \cos \theta_3 e^{-i(\phi_2+\phi_4)} - \sin \theta_1 \cos \theta_2 \sin \theta_3 e^{i(\phi_1-\phi_3+\phi_5)} \\
  0 &\leq \theta_1, \theta_2, \theta_3 \leq \pi/2 & 0 &\leq \phi_1, \ldots, \phi_5 < 2\pi
\end{align*}
\]
with the invariant group measure given by

\[ dg = \frac{1}{2\pi^5} \sin \theta_1 \cos^3 \theta_1 d\theta_1 \sin \theta_2 \cos \theta_2 d\theta_2 \sin \theta_3 \cos \theta_3 d\theta_3 d\phi_1 \ldots d\phi_5 \]  

(34)

Using the above expressions we can now proceed to change the variables and define a canonical parametrization of the group where all eight parameters are chosen from a unit interval and the group measure becomes uniform (flat). In particular, we have

\[ u_1 = 1 - \cos^4 \theta_1 \quad u_2 = 1 - \cos^2 \theta_2 \quad u_3 = 1 - \cos^2 \theta_3 \quad u_{3+a} = \frac{1}{2\pi} \phi_a \]  

(35)

where

\[ u_a \in \begin{cases} [0,1], & a=1,2,3 \\ [0,1), & a=4,\ldots,8 \end{cases} \quad dg = \prod_{a=1}^{8} du_a \]  

(36)

Since group elements are uniformly distributed in the space of parameters \( u_a \), we can proceed with the discretization in the usual manner. Instead of representing a value of \( u_a \) exactly, we choose to keep only a “coarse-grained” information on it. In particular, to store \( \varepsilon \) values in the unit interval, and the “lattice spacing” assumes the existence of a bijection between the set of subintervals and the set \( B_k \) of binary strings with length \( k \). To set up such bijection let us first define a set \( \mathcal{V}_k \) of standard “lattice values” in the unit interval, and the “lattice spacing” \( \varepsilon_k \) for arbitrary \( k \), namely

\[ \mathcal{V}_k \equiv \{ i_r^{(k)} \equiv \frac{i}{2^k}, \ i = 0,1,2,\ldots,2^k-1 \} \quad \varepsilon_k \equiv \frac{1}{2^k} = r_1^{(k)} \]  

(37)

Note that there is a natural bijection between \( B_k \) and \( \mathcal{V}_k \). Indeed, every \( i \in \mathbb{N}_0^k \) has a unique binary representation of the form \( i = b_1 2^{k-1} + b_2 2^{k-2} + \ldots + b_{k-1} 2 + b_k \) and then

\[ \mathcal{V}_k \ni i_r^{(k)} = 0.b_1b_2\ldots b_k \iff b_1b_2\ldots b_k \equiv y_i^{(k)} \in B_k^* \]  

(38)

We will use the above map to define a bijection between binary strings and partitions of a unit interval by associating the values \( r_i^{(k)} \) with individual subintervals. In doing so, we wish to treat periodic parameters \( (u_4,\ldots,u_8) \) differently from non-periodic parameters \( (u_1,u_2,u_3) \) because of their different geometric meaning. In particular, we wish to end up with a discretization for which the immediate vicinity of an identity matrix corresponds to a single lattice cell rather than being split over several different ones. This requires the values of periodic parameters close to 0 and 1 being associated with the same subinterval. We thus define

\[ B_k^* \ni y_i^{(k)} \iff \begin{cases} [r_i^{(k)} + \varepsilon_k, r_i^{(k)}], & i = 0,1,\ldots,2^k-2 \\ [r_i^{(k)} + \varepsilon_k, r_i^{(k)} + \varepsilon_k], & i = 2^k-1 \end{cases} \]  

for non-periodic parameters and

\[ B_k^* \ni y_i^{(k)} \iff \begin{cases} [r_i^{(k)} - \frac{\varepsilon_k}{2}, r_i^{(k)} + \frac{\varepsilon_k}{2}], & i = 1,2,\ldots,2^k-1 \\ [r_i^{(k)} - \frac{\varepsilon_k}{2}] \cup [1 - \frac{\varepsilon_k}{2},1], & i = 0 \end{cases} \]  

(40)
for periodic ones. Let us denote the subintervals defined above as \(J_i^{(k)}\), with an implicit understanding of different definition for periodic and non-periodic cases.

The bijections (38), (39,40) imply that we can interchangeably specify a value \(r_i^{(k)}\), an interval \(y_i^{(k)}\) or a binary string \(y_i^{(k)}\) when speaking of “coarse-graining” a parameter \(u_a\) into \(k\) bits. For future reference, let us define the operation of coarse-graining explicitly. In terms of a value of the parameter we have

\[
u_a \rightarrow u_a^{(k)} \equiv r_i^{(k)} \in V_k \text{ such that } u_a \in J_i^{(k)} \tag{41}\]

At the level of binary strings, we first point out that we can associate any possible value of \(u_a\) with an infinite binary string \(z_a \equiv b_1b_2b_3\ldots\) formed by digits of its binary representation \(u_a = 0.b_1b_2b_3\ldots\). If \(u_a\) has two binary representations (e.g. \(u_a = 0.10000\ldots = 0.01111\ldots\)) we choose the option with infinitely many zeros. Note that we include \(u_a = 1\) via \(u_a = 0.11111\ldots\). With these definitions the operation of coarse-graining into \(k\) bits translates at the level of binary strings into

\[
z_a \rightarrow z_a^{(k)} \equiv y_i^{(k)} \in \mathfrak{B}_k^* \text{ such that } u_a \in J_i^{(k)} \tag{42}\]

Note that in the non-periodic case coarse-graining simply comes down to keeping first \(k\) bits of a binary representation for \(u_a\). In the periodic case it is the same after applying an appropriate periodic shift.

### 3.4.2 Coarse–Grained Configurations and Binary Strings

Consider an SU(3) configuration \(U \equiv \{U(n,\mu)\}\) on a discretized hypercubic symmetric torus of size \(L\). Emphasizing the canonical representation of the previous section, let us denote the same object as \(u \equiv \{u(n,\mu)\}\), where \(u(n,\mu) \equiv (u_1(n,\mu), u_2(n,\mu), \ldots, u_8(n,\mu))\). We associate with \(u\) (and hence with \(U\)) a sequence of \(k\)-th level coarse-grained configurations \(u^{(k)}\) defined via

\[
u \rightarrow u^{(k)} \equiv \{u^{(k)}(n,\mu)\} \equiv \{u_1^{(k)}(n,\mu), u_2^{(k)}(n,\mu), \ldots, u_8^{(k)}(n,\mu)\} \tag{43}\]

where the coarse-grained gauge parameters are defined in Eq. (41), and the “\(k\)-th level” refers to the fact that each parameter is represented by \(k\) bits. By construction \(\lim_{k \to \infty} u^{(k)} = u\) in the standard Euclidean norm.

In an analogous manner we now proceed with defining a sequence of \(k\)-th level binary strings corresponding to \(u^{(k)}\). To that effect we first define a binary string \(z^{(k)}(n,\mu)\) of length \(8k\) assigned to a given link \((n,\mu)\) via

\[
  z^{(k)}(n,\mu) \equiv z_1^{(k)}(n,\mu)z_2^{(k)}(n,\mu)\ldots z_8^{(k)}(n,\mu) \tag{44}
\]

where \(z_a^{(k)}(n,\mu)\) are defined in (42) and the operation on the right–hand side means a simple concatenation. To form a string representing the whole configuration we have to fix the recursive invertible map \(F \in \mathcal{F}\) of Eqs. (23), relating a linear order on the string to underlying space-time geometry. For arbitrary such map we then have

\[
z_F^{(k)}(U) \equiv z^{(k)}(F^{-1}(1))z^{(k)}(F^{-1}(2))\ldots z^{(k)}(F^{-1}(4L^4)) \tag{45}\]
In other words, $z^{(k)}_F$ is the concatenation of local binary strings $z^{(k)}(n, \mu)$ in the order provided by the map $F$. By construction, the length of this string is $l(z^{(k)}_F) = 32L^4k$.

Let us finally make few remarks pertaining to these issues.

(1) Note that in addition to the choice of map $F$, there is additional freedom in assembling the binary string representing the coarse–grained configuration $u^{(k)}$. For example, instead of placing different bits of $z^{(k)}_a(n, \mu)$ into a localized part of the string, we could concatenate $k$ blocks of size $32L^4$, where the $j$-th block would be formed by the $j$-th bit of $z^{(k)}_a(n, \mu)$. However, these freedoms do not affect asymptotic considerations.

(2) We emphasize that we will interchangeably use the notation $U^{(k)}$ and $u^{(k)}$ to denote the same object. In the former case one should realize that $U^{(k)}(x, \mu)$ is a matrix constructed from $u^{(k)}(x, \mu)$ via (33) rather than matrix of coarse-grained matrix elements.

(3) For future reference, we will denote the set of all configurations $U$ on the symmetric torus with $L$ lattice sites on the side as $\mathcal{U}^L$. Similarly, the finite set of $k$-th level coarse grained configurations $U^{(k)}$ will be denoted as $\mathcal{U}^{L,k} \subset \mathcal{U}^L$.

### 3.4.3 Kolmogorov Entropy

With the above preparations, we can now turn to the issue of quantifying the degree of space–time order for SU(3) (and analogously for SU($N$)) configurations/theories. Following the same path as in the $\mathbb{Z}_2$ case, the underlying idea is to apply the Kolmogorov complexity measure to binary strings describing the configurations. However, there are several problems with carrying out such a program. Among others, an immediate issue is that, even for finite $L$, the set $X$ of objects we wish to describe (set of all SU(3) configurations) is not countable. This has several undesirable consequences, the most obvious one being that we cannot form a one-to-one correspondence between binary strings from $\mathfrak{B}^* \times \mathcal{X}$ and configurations.\footnote{Note that it is not a problem to form a bijection between the sets of infinite strings and configurations at given fixed $L$. However, we need a bijection that includes all possible values of $L$.}

In order to proceed, we will abandon the notion that the binary string associated with $U$ has to encode this configuration completely.\footnote{By “complete description” we mean such that any property of a configuration can be inferred from properties of the associated string and vice-versa.} Instead, we consider a sequence of coarse–grained configurations $\{U^{(k)}, k = 1, 2, \ldots\}$, each of which approximates $U$ in an increasingly fine detail. Clearly, the ability to analyze $U^{(k)}$ for arbitrary $k$ is equivalent to analyzing $U$ fully. At the same time, as we discussed in the previous section, each $U^{(k)}$ can be completely described by a binary string $z^{(k)}_F(U)$ of length $32L^4k$. For given fixed $k$, we can thus proceed and define the $k$-th level Kolmogorov entropy of $U$ corresponding to map $F$ via

$$E^K_F(U, k) \equiv K(z^{(k)}_F(U)) \quad (46)$$

As discussed in Section 3.3, this measure is additively optimal with respect to set $\mathcal{F}$ with arbitrary $F$ being an universal element. Following the same logic as in $\mathbb{Z}_2$ case we fix $F^u$ of Eq. (26) and define

$$E^K(U, k) \equiv K(z^{(k)}(U)) \quad z^{(k)}(U) \text{ computed via } F^u \quad (47)$$
If $S \equiv S(\beta)$ is a theory from $\mathcal{S}^G$, we define its $k$-th level Kolmogorov entropy as
\[
E^K[S, k] \equiv \langle E^K(U, k) \rangle_S = \int dU E^K(U, k) P_S(U) \quad P_S(U) = e^{-S(U)}/Z
\]
i.e. as an ensemble average of Kolmogorov entropies assigned to individual configurations.

In case of $\mathbb{Z}_2$ gauge theories, we could directly associate Kolmogorov entropy with the measure indicating the degree of space–time order dynamically generated by the theory in its typical configurations. For QCD the situation is slightly more involved. Indeed, it is important to realize that binary strings $z^{(k)}(U)$ can become asymptotically large by either increasing the lattice size $L$, or by increasing the level of coarse-graining $k$. Naively, one is tempted to take $k \to \infty$ while keeping $L$ fixed in order to completely characterize the theory in a finite lattice volume. While this can certainly be done, the Kolmogorov entropy in such limit will not be a measure of space–time order at all. Rather, it will reflect how ordered are the digits of a typical local gauge parameter $u_a(n, \mu)$. Thus, it turns out that in order to obtain the construct we are interested in, we have to do just the opposite. In particular, we will say that $E^K(U, k, L)$ ($E^K[S, k, L]$) is a measure of space–time order in the configuration (theory) if and only if
\[
k \ll L^4
\]
where we have made the $L$-dependence of $E^K$ explicit.

The above considerations indicate that in dealing with issues of space–time order, we are simply required to approach any particular question at fixed coarse-graining level $k$. For example, the Kolmogorov entropy $E^K[S, k, L]$ induces a "$k$-th level" order relation on the set $\mathcal{S}^G$ of theories defined on the torus with lattice size $L$, namely
\[
S_1 <_k S_2 \quad \text{if} \quad E^K[S_1, k, L] < E^K[S_2, k, L]
\]
The underlying interpretation is that $S_1$ possesses more space–time order than $S_2$ if $S_1 <_k S_2$ (for $k \ll L^4$). Given that interactions are local, we expect that the above order relation becomes $L$-independent at sufficiently large $L$ for any two theories, thus characterizing the interactions themselves. In the $L \to \infty$ limit it is natural to consider an induced order relation on the set of equivalence classes, where theories whose Kolmogorov entropies differ at most by a constant belong to the same class (see Eq. (29)).

Finally, let us point out that in complete analogy with the $\mathbb{Z}_2$ case we can make the definition of $k$-th level Kolmogorov entropy gauge invariant by minimizing $E^K(U, k, L)$ on the gauge orbit of $U$ (see Eq. (30)). We can also gauge–fix to the information gauge via an obvious modification of Eq. (31).

### 3.4.4 The Conjecture of Hierarchic Structure

One of the main reasons behind introducing the measure of space–time order based on Kolmogorov complexity is its universality in the sense that it can capture the ordered space-time behavior in any possible form. This is important because our geometric intuition for what the domain–correlated behavior of a function can be is very limited. Even taking just a scalar function of two space–time coordinates as an example that human brain can handle relatively comfortably visually, we have absolutely no intuition on possible degrees of space–time coherence beyond extreme cases. Indeed, we tend to deal with such functions via their
graphs which we can picture mentally if they are nice smooth surfaces in 3-d space. The common intuition then is that smoother graphs (functions) are “more ordered” than rougher graphs (functions). However, even in the extreme case of very high order our intuition is incomplete since there exist very singular functions whose behavior is maximally ordered in the sense that local patterns determine their behavior everywhere.

Returning back to lattice configurations, the above–mentioned generality of the approach based on Kolmogorov complexity, while highly desirable, has certain implications that have to be dealt with. In particular, the point that needs to be emphasized at this stage is that the linear order on the set of configurations induced by $E^K(U, k, L)$ is not necessarily preserved if the “resolution” $k$ changes. More precisely, if $U_1, U_2 \in \mathcal{U}$, and $k_1 < k_2 \ll L^4$ with $U_1 <_{k_1} U_2$, then it does not necessarily follow that $U_1 <_{k_2} U_2$. Indeed, we can easily imagine a situation where first $k_1$ bits of gauge parameters in $U_1$ exhibit high space–time coherence, while being disordered in $U_2$. However, in the trailing $k_2 - k_1$ bits the situation can reverse in favor of $U_2$ if supplied with highly ordered digits. The point is that while we intuitively tend to think that leading digits are “more important” for the structure, the Kolmogorov complexity measure does not distinguish such hierarchy and the order around 1-st digit counts in principle as much as the order around the 100-th digit. We emphasize again that this is a desirable feature of Kolmogorov entropy measure since our goal is to be completely general. Thus, if there is a secret message encoded into configuration $U$ via inserting a highly space-time ordered binary string into various digits of $u$, it can still be detected by analyzing $E^K(U, k, L)$ for sufficiently large $k$.

The point we wish to convey in the above reasoning is that at the configuration level, the ranking introduced by Kolmogorov complexity has to be viewed strictly as a resolution–dependent concept in this case. In other words, it is meaningful to say that configuration $U_1$ is more ordered than configuration $U_2$ only when they are both analyzed with fixed resolution $k$. However, when we characterize the theory $S(g) \in \mathcal{S}^G$, via $E^K[S, k]$ things are expected to be much more universal in this regard. Indeed, since our theories are analytic (at least in certain corner of the configuration space), we expect that with other things being equal (such as multiplicities), the space–time order in leading digits has a more decisive influence on probability of occurrence than possibly equivalent order in higher digits. This is simply because leading digits influence the value of the action (and thus a probability of occurrence) more significantly than higher digits. In other words, it is natural to assume that a theory in $\mathcal{S}^G$ will feature typical configurations with hierarchical order. By this we mean that the relative degree of order will not increase when higher digits are included, and will in fact typically decrease. Put still differently, the higher digits in typical configurations are expected to exhibit increasingly more space–time randomness.

We wish to formalize the above observations into a definite statement concerning the measure $E^K[S, k]$ of space–time order on the set $\mathcal{S}^G$. While at present we do not have an apparatus to attempt the proof of the following statement, we propose it for future considerations because of its relevance.

**Conjecture C11.** Let $k$ be a fixed positive integer and let “$<_k$” be the linear order relation on $\mathcal{S}^G$ induced by $E^K[S, k, L]$ in the $L \to \infty$ limit. Then “$<_k$” is universal with respect to $k$, i.e. if $S_1 <_{k_1} S_2$, then $S_1 <_{k_2} S_2$, for arbitrary $k_1, k_2$ and arbitrary $S_1, S_2 \in \mathcal{S}^G$.

Few points need to be emphasized in connection with the above conjecture.
(1) The utility of Kolmogorov entropy as a measure quantifying the degree of space–time order in different theories does not rely on the validity of CI1. Indeed, in our quest for uncovering the relevant collective degrees of freedom we can use $E^K[S,k]$ at fixed resolution $k$ as a general measure distinguishing different theories. Obtaining a complete picture will require changing the resolution $k$ in either case. Nevertheless, from a theoretical standpoint it would be very appealing if CI1 could be demonstrated at least in some simplified framework. In what follows we will implicitly assume its validity.

(2) A weaker statement with similar consequences (namely existence of an absolute ranking of theories by the degree of space–time order they generate) can be obtained if one postulates the existence of $k_0[S_1,S_2]$ such that the relative ordering of $S_1$ and $S_2$ is preserved for all $k > k_0$.

(3) We remind the reader again that it is crucial in the formulation of CI1 that the infinite–volume limit is taken at fixed $k$ to define the ordering. This is to nullify the influence of various non-universal constants implicitly present in the definition of $E^K[S,k,L]$. Also, it is implicitly understood in CI1 that in $L \to \infty$ limit the linear order is defined on the equivalence classes of theories whose Kolmogorov entropies are bounded by a constant (see Eq. (29)).

3.4.5 The Associated Coarse-Grained Theory and Shannon Entropy

In section 3.3.2 we have argued that in the case of $\mathbb{Z}_2$ gauge theories it is useful to consider the Shannon entropy of the corresponding ensembles. Even though this is an auxiliary concept from our point of view, its usefulness stems from the fact that Kolmogorov entropy is a non-computable notion. Using Theorem 2, Shannon entropy gives us at least a possibility to calculate which of the arbitrary two $\mathbb{Z}_2$ gauge theories with computable probability distributions generates more space–time order in the infinite–volume limit. This raises a natural question at this point, namely how to define this auxiliary concept in case of SU(N) lattice gauge theories.

As we argued extensively above, we carry out the analysis of space–time structure at a given coarse–grained level $k$, wherein the configuration $U$ is associated with its coarse–grained counterpart $U^{(k)}$ (or $u^{(k)}$), completely represented by a binary string $z^{(k)}$. Given $S \in \mathcal{G}$ and fixed $L$, our goal is to define a probability distribution on a finite set $\mathcal{U}^{L,k}$ of coarse–grained configurations $U^{(k)}$, such that its Shannon entropy is related to Kolmogorov entropy $E^K[S,k,L]$ in the same manner as in Theorem 2 (see Eq. (32)). To do that, it is useful to think about the theory $S$ in terms of the corresponding ensemble $\mathcal{E}[S]$, which is fully represented by an appropriately chosen infinite probabilistic chain. The coarse–graining procedure corresponds to the replacement

$$\mathcal{E}[S] \equiv \{\ldots, U_{i-1}, U_i, U_{i+1}, \ldots\} \longrightarrow \mathcal{E}^{(k)}[S] \equiv \{\ldots, U_{i-1}^{(k)}, U_i^{(k)}, U_{i+1}^{(k)}, \ldots\}$$

(51)

The ensemble $\mathcal{E}^{(k)}[S]$ defines a lattice theory of discrete variables $u_n^{(k)}$, which we denote as $S^{(k)}$, so that we can write formally $\mathcal{E}^{(k)}[S] \equiv \mathcal{E}[S^{(k)}]$. The probability density of

31 Definition of action $S^{(k)}$ is more than formal though in the sense that its value for arbitrary $U^{(k)} \in \mathcal{U}^{L,k}$ can be explicitly written down (see bellow).
encountering a configuration $U \in \mathcal{U}^L$ in $\mathcal{E}[S]$ is given by $P_S(U) = \exp(-S(U))/Z$, while the probability of encountering $U^{(k)} \in \mathcal{U}^{L,k}$ in $\mathcal{E}^{(k)}[S]$ is by construction

$$P_{S^{(k)}}(U^{(k)}) \equiv \int dV \delta(U^{(k)} - V^{(k)}) P_S(V)$$

(52)

where $V^{(k)}$ is a $k$-th level coarse-grained configuration associated with $V$. Note that using our canonical representation of gauge variables, we can write the above equation in a precise (rather than formal) manner as

$$P_{S^{(k)}}(U^{(k)}) = \int \prod_{x,\mu,a} dv_a(x,\mu) \prod_{x,\mu,a} \delta(u_a^{(k)}(x,\mu) - v_a^{(k)}(x,\mu)) P_S(v)$$

(53)

Moreover, for actual evaluation of this probability one could use an explicit form without the $\delta$–functions. To do that, we define the $k$-th level displacement variables $\tau^{(k)} \equiv \{\tau^{(k)}_a(x,\mu)\}$ with values in the intervals

$$\tau^{(k)}_a(x,\mu) \in \{[0, \epsilon_k], \quad a=1,2,3 \quad \text{and} \quad [-\epsilon_k/2, \epsilon_k/2], \quad a=4,\ldots,8$$

(54)

in terms of which we can write

$$P_{S^{(k)}}(U^{(k)}) = \int d\tau^{(k)} P_S(u^{(k)} + \tau^{(k)}) \quad d\tau^{(k)} \equiv \prod_{x,\mu,a} d\tau^{(k)}_a(x,\mu)$$

(55)

where the periodicity is implicitly assumed to be taken into account for arguments of $P_S$ with $a = 4, \ldots, 8$. Equation (55) defines a theory $S^{(k)}$ of discrete variables $u^{(k)}_a$ such that its Kolmogorov entropy is equal to $k$-th level Kolmogorov entropy of $S$, namely

$$E^K[S^{(k)}] \equiv \sum_{U^{(k)}} E^K(U^{(k)}) P_{S^{(k)}}(U^{(k)}) = E^K[S, k] \equiv \int dU E^K(U, k) P_S(U)$$

(56)

where in the context of $S^{(k)}$ (left hand side; discrete variables) we have $E^K(U^{(k)}) \equiv K(z^{(k)})$, which is equal to $E^K(U, k)$ in the context of $S$ (right hand side; continuous variables).

For theory $S^{(k)}$ with computable probability distribution, there is a relation between its Kolmogorov and Shannon entropies analogous to Theorem 2 in $\mathcal{Z}_2$ case. This then induces the same relation between Shannon entropy of $S^{(k)}$ and the $k$-th level Kolmogorov entropy of $S$ which is summarized in the following theorem.

**Theorem 3** For $L, k$ fixed, let $S \in S^G$ leads to a discrete theory $S^{(k)}$ with computable probability distribution $P_{S^{(k)}}(U^{(k)})$, and $E^{Sh}[S^{(k)}] \equiv -\sum_{U^{(k)}} P_{S^{(k)}}(U^{(k)}) \ln P_{S^{(k)}}(U^{(k)})$. Then

$$0 \leq E^K[S, k] - E^{Sh}[S^{(k)}] \leq K(\tilde{P}_{S^{(k)}}) + O(1)$$

(57)

where $\tilde{P}_{S^{(k)}}(z = z^{(k)}) \equiv P_{S^{(k)}}(U^{(k)})$ is the induced probability distribution on $\mathcal{B}^*$.

To close this section, we wish to emphasize two points.

(1) The remarks given at the end of section 3.3.2 apply in the case discussed here as well.

(2) The associated coarse–grained theory $S^{(k)}$ is not exactly gauge–invariant with violations at the $k$-th bit level. This does not have any undesirable implications in this context since the role of $S^{(k)}$ is to assist us with quantifying the degree of space–time order in $S$. 

32
4 The Principle of Chiral Ordering

In the previous section we have introduced the notion of Kolmogorov entropy for SU(N) gauge theories and proposed it as a measure quantifying the degree of space–time order generated by the theory at the configuration level. Since the concept is based on Kolmogorov complexity of binary strings describing coarse–grained configurations, it is completely general in the sense that it reflects space–time order of coarse–grained configurations/theories in any form. Moreover, if we accept the hypothesis of hierarchic order (see conjecture CI1), then we have a measure providing universal ranking of theories in $S^G$ by the degree of space–time order generated in their typical configurations.

Before proceeding further, let us note that we included a detailed elaboration on Kolmogorov entropy in the previous section simply because we view the recent findings expressed in I1 (see section 1.3) as an important new aspect that needs to be properly incorporated into the framework of quantum field theory. Equipped with the notion of Kolmogorov entropy we can, at least at the conceptual level, attempt to formulate problems that are relevant for the program of uncovering the fundamental structure of QCD vacuum in the path integral formalism. In that regard, it would certainly be of crucial importance if we had means to achieve the following major goal.

Goal 1: Map out the landscape of Kolmogorov entropy in the set of actions $S^G$ and understand its behavior.

Indeed, in order to uncover the nature of relevant collective variables $C$, we would eventually prefer to deal with theories possessing low Kolmogorov entropy at finite lattice spacing. In such theories the distorting influence of randomness on $C$ is expected to be low thus giving us best chance of correctly identifying it.

To start speaking in these terms, let us clear up some terminology first. While $S^G$ is the set of individual lattice actions, we tend to speak of lattice “theories” belonging there in two different ways. Firstly, we view the quantum dynamics of finite (countable) degrees of freedom defined by any individual element $S(g)$ (g and other parameters fixed) of $S^G$ as a “theory”. Note that we talked about it in this sense when discussing the Kolmogorov entropy. For example, we can compare the space–time order of two theories differing only by values of $g$, e.g. $S(g_1) < \kappa S(g_2)$. On the other hand, it is sometimes useful to think of lattice theory as a one-parameter family of interactions $S(g)$ parametrized by bare coupling. In this case the “theory” prescribes a lattice interaction at any value of $g$ thus parametrizing the approach to the continuum limit.

Thinking now in terms of this second interpretation, consider the set of pure-glue lattice gauge theories contained in $S^G$. Upon quantization, each theory $S(g) \in S^G$ is associated with ensembles $\mathcal{E}(g)$ which determine the lattice quantum averages for various observables. To obtain values in physical units, one usually fixes the appropriate quantity to its physical value, independently of $g$, which introduces the lattice spacing $a$. From the physical point of view it makes clearly more sense to compare space–time order of two theories at the same lattice spacing rather than the same bare coupling, i.e. to view the theory $S$ as being parametrized by lattice spacing rather than bare coupling. Since this is what we will do, let

\[32\text{In fact, we naturally expect that typical configurations will become increasingly spatially ordered as } g \text{ decreases. In other words, we expect that } S(g_1) < \kappa S(g_2) \text{ for any } g_1 < g_2.\]
In other words, all lattice theories setting the statement that all actions for all theories \( S(g) \), in each of which its dimensionless value is \( m^S(g) \). For given \( S(g) \) this defines the lattice spacing \( a^S(g) \) via

\[
a^S(g) \bar{m} \equiv m^S(g)
\]

For simplicity, assume that the set of other interesting physical observables consists of masses \( \bar{m}_i \), of all the other stable particles in the theory, which we want to predict. In this restricted setting the statement that all actions \( S(g) \in \mathcal{S}^G \) define the same continuum theory (universality) means that

\[
\lim_{g \to 0} \frac{m^S_i(g)}{a^S(g)} = \bar{m} \lim_{g \to 0} \frac{m^S_i(g)}{m^S(g)} = \bar{m}_i \quad \forall i, \text{ independent of } S
\]

In other words, all lattice theories \( S \in \mathcal{S}^G \) will predict the same ratios of masses of stable particles in the continuum \( (g \to 0) \) limit. Note that in the above considerations we have implicitly assumed that all theories in \( \mathcal{S}^G \) have a second order phase transition at \( g = 0 \). If we further assume that the lattice correlation lengths \( \xi^S(g) \equiv 1/m^S_i(g) \) are monotonically increasing as \( g \to 0^+ \) (at least for \( g < \hat{g} \), where \( \hat{g} \) is independent of \( S \)), then we can invert the functions \( a^S(g) \) and talk about lattice actions parametrized by lattice spacing \( a \). In other words, we set \( S'(a) \equiv S(g = g^S(a)) \), but in what follows we will keep the same symbol \( (S) \) for both dependences to simplify the notation.

With all the above pieces in place, it is tempting to conclude that, at least at the conceptual level, it is now clear how to proceed with the program of identifying the fundamental structure of QCD vacuum (collective variables \( C \)). Being restricted to some finite range of non-zero lattice spacings \( a \) by practical considerations, we could minimize the Kolmogorov entropy \( E^K[S(a), k] \) in the set of actions at fixed \( a \) to identify the action \( S^o(a) \) which maximizes the space–time order. We would then study the corresponding ensemble \( \mathcal{E}^o(a) \) in a Bottom-Up manner to extract as much information as possible about the nature of space–time order present. The lattice spacing could then be decreased and the procedure repeated in an attempt to gain the information about the sought-for collective variables \( C \) in the continuum limit.

As simple and appealing this might sound, it is most likely a wrong way to pose the relevant task (even at the level of principle, leaving out the practical considerations). Indeed, apart from the issues of existence and uniqueness of \( S^o(a) \) (which might not be serious in this case \[33\]), there is a fatal flaw in the program as formulated above. Indeed, assume that we start the minimization to determine \( S^o(a) \) from the action \( S^{ini}(a) \) such that theory \( S^{ini} \) is well in the scaling region for lattice spacing \( a \) considered. Thus, for \( S^{ini} \) we have that \( m^S_{ini}(a)/a \approx \bar{m}_i \) for physical masses \( \bar{m}_i \). Since the minimization of \( E^K \) makes no reference to correlation lengths other than \( \xi^S(a) \equiv 1/m^S(a) = 1/a\bar{m} \) (recall that lattice spacing is

\[33\]We note that such considerations were actually part of the reason why we chose to parametrize theories in terms of \( a \) rather than \( g \). Indeed, in the latter case, the set \( \mathcal{S}^G \) restricted to actions at fixed \( g \) most likely does not have a bottom element with respect to \( E^K \). The other reason obviously is that when we compare Kolmogorov entropies for two actions with the same number of lattice degrees of freedom and fixed \( a \), we are comparing the degrees of space–time order in the same physical volume.
kept fixed), there is no reason to expect that the scaling behavior will be preserved during the minimization. In other words, the masses of stable particles (other than the one fixing the lattice spacing) would most likely drift away from their physical values. Thus, even though the Kolmogorov entropy might be low in $S^0(a)$, we would certainly not want to use it to make conclusions about the nature of space–time order in the continuum limit, no matter how small $a$ might be. These considerations suggest that in an attempt to develop a systematic program for studying space–time structure in QCD vacuum, we would ideally (in principle) like to be able to do the following.

Goal 2: Starting from some action $S^{\text{ini}}(a)$, we would like to move in the set $S^G$ toward actions with lower Kolmogorov entropy such that if $S^{\text{ini}}(a)$ is in the scaling region, then the theories reached will remain in the scaling region, and if $S^{\text{ini}}(a)$ is not in the scaling region, then the theories reached will not diverge further away from it.

As we emphasized along the way, the above goal specifies the “in principle” strategy for identifying a fundamental structure in QCD vacuum. Indeed, it appears unlikely that at the practical level we could proceed via direct constrained minimization of Kolmogorov entropy in the set of actions. Instead, we will view Goal 2 as a conceptual guide against which our attempts should measure their potential merit. The second part of this article deals with a proposal in that direction.

To start describing our approach, it is instructive to first highlight the crucial obstacles to practical realization of Goal 2 which need to be addressed. (α) There is no standard explicit way of parametrizing the elements of $S^G$ and thus no standard practical way of “moving in the set of actions”. (β) There is no clear underlying principle that would guide us to move in $S^G$ while “preserving physics” as demanded by Goal 2. One could view the renormalization group (RG) blocking as a procedure of such kind, except that the decimation of the lattice poses a problem. At the heart of our method is the proposal for such principle. (γ) Since Kolmogorov entropy is not computable, we cannot directly minimize it or at least check aposteriori by explicit calculation that it became lower during the procedure. One thus needs to discuss why is the chosen evolution in the set of actions believed to lower the Kolmogorov entropy. In the sections that follow we will focus on the above issues in turn.

4.1 Configuration-Based Deformation of the Action

In this section we describe a generic way of “moving” in the set of actions based on the local deformation of individual configurations. We will use the fact that it is not important for us to know analytically (or otherwise) the exact form of the action we work with. Indeed, we only need to be able to inspect the ensemble associated with the action and to make sure that it represents a valid element of $S^G$. Thus, consider a map

$$\mathcal{M} : U^L \rightarrow U^L \quad U_{n,\mu} \rightarrow \mathcal{M}_{n,\mu}(U)$$

Let $S \in S^G$ and $\mathcal{E}_S$ be the associated ensemble. The map $\mathcal{M}$ then induces the new ensemble

$$\mathcal{E}_S \equiv \{\ldots U^{i-1}, U^i, U^{i+1}, \ldots\} \rightarrow \{\ldots \mathcal{M}(U^{i-1}), \mathcal{M}(U^i), \mathcal{M}(U^{i+1}), \ldots\} \equiv \mathcal{E}_{S^M}$$

where the old probability density $P_S(U) = e^{-S(U)}/Z_S$ is modified to

$$P_{S^M}(U) = \int dV P_S(V) \delta\left(U - \mathcal{M}(V)\right)$$
and is associated with the “action” $S^M$ given by

$$e^{-S^M(U)} \equiv P_{SM}(U) \ Z_{SM} \quad Z_{SM} = Z_S \quad (63)$$

We now impose the following requirements on the map $M$:

(i) **Symmetries.** We require that the symmetries of the original action $S(U) \in \mathcal{S}^G$ are also present in new action $S^M(U)$. One can easily inspect that this is guaranteed if $M_{n,\mu}$ transforms in the same way as $U_{n,\mu}$ under respective symmetry transformations. Thus, for example

$$M_{n,\mu}(U) \rightarrow G_n \ M_{n,\mu}(U) \ G_{n+\mu}^{-1} \quad \text{under} \quad U_{n,\mu} \rightarrow G_n \ U_{n,\mu} \ G_{n+\mu}^{-1} \quad (64)$$

and similarly for the symmetries of the hypercubic lattice structure.

(ii) **Locality.** Considering the configuration in the canonical representation (see Sec. 3.4.1), i.e. $U = \{U_{n,\mu}\} \equiv \{u_{n,\mu}\} = u$ we require the existence of constants $C, \gamma$ such that

$$\frac{\partial M^n_{n,\mu}(u)}{\partial u^b_{m,\nu}} \leq C e^{-\gamma|n-m|} \quad (65)$$

independently of $u$.

(iii) **Classical Limit.** If $A_\mu(x)$ is a classical (smooth) field (see footnote 9), and $U_{n,\mu}(a)$ is its discretization (9) with classical lattice spacing $a$, then we require that $M_{n,\mu}(U, a)$ is also its valid discretization. In other words, that

$$U_{n,\mu}(a) - M_{n,\mu}(U, a) = O(a^2) \quad (66)$$

Note that the nature of the above conditions (especially (ii) and (iii)) is such that it is implicitly assumed that $M$ is defined for arbitrary $L$. In the following, we will denote the set of maps satisfying the conditions (i)–(iii) as $\mathcal{F}$.

We propose that the action $S^M$ defined via $M \in \mathcal{F}$ satisfying (i)–(iii) will generically be a valid element of $\mathcal{S}^G$. It needs to be emphasized that this conclusion is not the result of a rigorous proof at this stage. Indeed, while the symmetries are directly guaranteed by (i), it is not straightforward to precisely formulate and prove the relevant statement regarding locality of $S^M$ based on (65), or the statement on the classical limit based on (66). On the other hand, from the physical point of view it is quite clear that the transition from $S$ to $S^M$ will not cause any major pathologies if (i)–(iii) are satisfied. Indeed, consider arbitrary set of local operators $\{O_\alpha\}$ and the related set $\{O^M_\alpha\}$, where

$$O^M_\alpha(U) \equiv O_\alpha(M(U)) \quad (67)$$

The conditions (i)–(iii) guarantee that $O^M_\alpha$ will inherit the transformation properties, locality and classical limit of $O_\alpha$, while at the same time

$$\langle O_\alpha(U) \rangle_{SM} = \langle O^M_\alpha(U) \rangle_S \quad (68)$$

Thus the expectation values of arbitrary local operators (and their correlation functions) in $S^M$ correspond to values in original theory $S$ (element of $\mathcal{S}^G$) for manifestly “valid” discretizations of the same continuum operators. Roughly speaking, our overall level of rigor here is analogous to that of Wilson’s renormalization group (RG) approach [33], wherein one moves in the set of actions via RG transformation.

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34 This is not to say that it is impossible to formulate and prove such statements (with appropriate restrictive conditions stipulated). However, we will not deal with these rather involved issues rigorously here.
4.2 The Principle of Chiral Ordering

Consider a configuration-based deformation $S^M$ of action $S$, given by an ultralocal map $M \in \mathcal{F}$. By ultralocality we mean that $M_{n,\mu}(U)$ does not depend on $U_{m,\nu}$ for $|n - m| > R$ larger than some configuration–independent positive constant $\kappa$. Using the same arguments as those invoked in the case of RG transformations, we can conclude that the behavior of $S^M$ at large lattice distances is the same as that of $S$. In other words, the bound-state masses $\bar{m}^S$ in lattice units are expected to be the same (on an infinite lattice) as $\bar{m}^{S^M}$. However, there are several setbacks associated with such generic approach to moving in the set of actions. 

(a) Transition from $S$ to $S^M$ can strongly affect observables for which the gauge field fluctuations at the scale of $\kappa$ lattice spacings are crucial (i.e. observables that are not determined entirely by long–distance behavior of correlators), even on an infinite lattice.

(b) On a finite lattice, even the estimates of masses in $S^M$ can change from the estimates in $S$. Moreover, after applying the deformation many times, all observables in $S^M$ on a finite lattice can change uncontrollably (both in lattice and physical units) relative to $S$. There is nothing to protect them from changing.

(c) For the generic choice of $M$, (which typically can be interpreted as a “smoothing” operation), the repeated application of the map on given configuration ($M^k(U)$) is not expected to converge to a non-trivial limit. Indeed, one rather expects the convergence to a trivial configuration with gauge potentials vanishing ($\bar{U}_{n,\mu} = I$ (identity matrix)), or a diverging behavior. Thus, even if Kolmogorov entropy of $M(U)$ decreases relative to $U$ for all relevant configurations, (which also means that $E^k[S^M, L] < E^k[S, L]$), we cannot expect a unique/interesting result of minimizing procedure based on the repeated application of $M$. Taken together, the above points serve to illustrate that to approach Goal 2 via configuration–based deformation of the action with generic $M$ is not expected to succeed. We also note that these issues become slightly more complicated in the non-ultralocal case since lattice correlation lengths in $S^M$ can change (by at most the range of non-ultralocality) relative to $S$, and thus the masses (and the lattice spacing) can change accordingly even on an infinite lattice.

In this section we wish to propose a class of maps $M$ which, we believe, can potentially alleviate most of the problems encountered in generic case. It is quite obvious from the above discussion that such maps must be a very special subset of all possibilities and, in an attempt to identify them, we will proceed quite heuristically. Indeed, our only initial guide will be the striking success of linking the topological density operator to chiral symmetry, which resulted in 11 of Sec. 1.3, i.e. the observation of space–time order at the fundamental level [5]. It will become clear soon in the process however, that the approach potentially carries much deeper connotations. Indeed, we will use chiral symmetry here in different and more general manner. Consider a fermionic action $S^F$ specified by lattice Dirac kernel $D$. Apart from all conditions imposed on $D(U)$ by the fact that $S^F$ is an element of $S^F$ (which includes requirement of chiral symmetry), we impose an additional and highly non-trivial requirement. In particular,

$$D_{n,n+\mu}(U) = D^f_{\mu} \times \bar{U}_{n,\mu} \quad \forall n, \mu, \forall U \quad \bar{U}_{n,\mu} \in SU(N)$$

where $D^f_{\mu}$ is a free fermion matrix (no color indices) representing hopping by vector $m$, namely

$$(D^f_{\mu})_{\alpha,\beta} = D_{n,n+m}(I)_{\alpha,\beta} \quad \text{color index } a \text{ fixed.}$$
In other words we require that the color-spin matrix $D_{n,n+\mu}(U)$ is a direct product of a free fermion matrix $D_{\mu}^f$ and an SU(N) “matrix phase” $\tilde{U}_{n,\mu}$. Consequently, the fermionic action $S^F$ specified by $D$ satisfying (69) defines a map $M^D$ which associates with every configuration $U$ a new configuration $\tilde{U}$ via

$$U \rightarrow M^D(U) \quad M^D_{n,\mu}(U) \equiv \tilde{U}_{n,\mu} \quad (71)$$

We will call $\tilde{U} = M^D(U)$ a chirally ordered version of $U$ according to $D$. One can easily check that the map $M^D$ satisfies the conditions (i)–(iii) of Sec. 4.1. Consequently, $M^D$ is expected to induce a valid deformation of the gauge action $S \rightarrow S^{M^D} \in S^G$. Action $S^{M^D}$ will be referred to as a chirally ordered version of $S$ according to $D$.

Few crucial points pertaining to the above definition of chiral ordering need to be made clear at this point.

(1) While it is not straightforward to prove that the elements of $S^F$ satisfying (69) exist, their existence does not appear to contradict any requirements imposed on valid fermionic actions. Moreover, later in this paper, we will claim that there exist converging sequences of elements in $S^F$ (with the limit that could be outside the set) that will satisfy (69) to arbitrary precision. Accepting conditionally the possibility, let us denote the subset of such special fermionic actions as $S^{F,C}$. Note also that from now on we will frequently identify the action $S^F$ with its defining kernel $D$ and so we will interchangeably write e.g. $S^F \in S^F$ and $D \in S^F$.

(2) According to Eq. (69) the operation of chiral ordering replaces the link $U_{n,\mu}$ with the interaction matrix phase $\tilde{U}_{n,\mu}$ involved in “hopping” of chiral fermion (quark) from site $n + \mu$ to site $n$. In other words, it is a measure of how much is the elementary hopping of chiral fermion affected by the presence of non-zero gauge field relative to the dynamics of free fermion. Thus, as far as chiral fermion is concerned, the physics content of configuration $U$ is very much embodied in configuration $\tilde{U} = M^D(U)$. This is our rationale for the expectation that the relevant physics encoded in gauge action $S$ is very close to physics in $S^{M^D}$. We take this as a basis for the proposition that moving in the set of actions via chiral ordering can potentially solve the problem $(\beta)$ stipulated in the introduction to Sec. 4.

(3) We emphasize that the condition (69) can be easily satisfied by lattice fermionic actions that are not chirally symmetric. For example, the Wilson–Dirac operator $D_W$ (which is not an element of $S^F$) satisfies it with $\tilde{U}_{n,\mu} = U_{n,\mu}$, i.e. with $M^{D_W}$ being an identity map. It is thus the chiral symmetry of the probing fermion that turns the chiral ordering condition into a highly non-trivial requirement. This is natural since we view the chirality of lattice fermion as a necessary requirement for properly regularizing the continuum physics. At the technical level, this comes as a result of non-ultralocality [34] of $D \in S^F$ in gauge variables. Indeed, as a consequence of Ginsparg–Wilson relation, the matrix $D_{n,n+\mu}(U)$ is expected to receive contributions from all possible gauge paths starting at $n$ and ending at $n + \mu$.

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35 Indeed, the locality and the symmetries follow from the locality and the symmetries of $S^F$. The classical limit is also preserved as one can see from the expansion of $D_{n,n+\mu}(U)$ in terms of gauge–covariant paths.

36 The non-trivial part in constructing such elements obviously comes from reconciling the requirement of chiral symmetry (Ginsparg–Wilson relation) with equation (69). See also point (3).

37 See also points (a)-(c) in the beginning of this section.
Condition (69) demands that these contributions add up to an effective SU(N) phase. This is clearly a highly complex requirement, and the associated $\mathcal{M}^D$ is a highly complex map. Viewed as a requirement for the fermionic action, Eq. (69) represents a condition that is independent from the usual ones imposed on elements of $S^F$. In particular, it is not implied by standard symmetries. This is why we sometimes use the term “principle” when referring to it. However, our present aim in imposing Eq. (69) is not to define a better lattice fermionic theory, but to provide a guide for evolution in the set of gauge actions that suits our goal of studying the QCD vacuum structure. In that regard, we view the principle of chiral ordering as the following generic statement.

The Principle of Chiral Ordering: The configuration–based deformation of the gauge action defined by map $\mathcal{M}$ representing the effective SU(N) interaction phase associated with elementary hopping of lattice chiral fermion has a very mild effect on lattice physics. This applies to both long and short–distance behavior (in lattice units).

Thus, what we mean by the principle of chiral ordering is the broad statement according to which using the maps $\mathcal{M}_{n,\mu}$ with tendency to capture the effective interaction phase of chiral fermion are particularly suitable for exploring the nature of space–time order in QCD. We emphasize that the potential usefulness of this observation does not crucially depend on the fact that the set $S^{F,C} \subset S^F$ is not empty. Indeed, we view the condition (69) as a clearest expression for the underlying idea, which we can use as a guide for extracting the approximate matrix phase corresponding to arbitrary chiral lattice fermion. This will be discussed more fully in Sec. 5.

4.3 The “Ordering” Conjecture

In the previous two subsections we have addressed the points $(\alpha)$, $(\beta)$ put forward in the opening part of Sec. 4. In particular, we have suggested how to move in the set of actions (via configuration–based deformation) while minimally changing the lattice physics (via principle of chiral ordering). In these considerations we have already implicitly assumed, as the name “chiral ordering” suggests, that this will also lead to lowering of the associated Kolmogorov entropy, i.e. that the point $(\gamma)$ is automatically satisfied. To prove the relevant rigorous statement within the framework of Kolmogorov complexity is a difficult task that we will not attempt here. However, we need to make it clear that our considerations assume the validity of the following conjecture.

Conjecture CI2. Let $k$ be an arbitrary but fixed positive integer. Then $E^K[S^{\mathcal{M}^D}, k, L] < E^K[S, k, L]$ for sufficiently large $L$.

In other words, we assert that chiral ordering leads to a gauge theory with lower Kolmogorov entropy. Note that the only restriction on lattice size $L$ in CI2 is related to applicability of Kolmogorov complexity concepts. Also, we do not restrict the validity of CI2 to chiral orderings defined by $D \in S^{F,C}$, i.e. we propose that it is valid also for the approximate orderings performed with arbitrary $D \in S^F$, which we will discuss later in this paper. It should also be emphasized that we do not claim that the conjecture is strictly valid at the

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38 This does not necessarily mean that the condition cannot be phrased in the language of symmetry. However, if it does, then the associated symmetry is different from the standard ones.
configuration level, i.e. it is not necessary true that $E^K(\mathcal{M}^D(U),k) < E^K(U,k)$ for arbitrary $U \in U^L$, even though it is reasonable to expect that it is true for most configurations.

The heuristic argument for validity of the “ordering” conjecture is tied to the principle of chiral ordering and is as follows. The underlying idea behind the principle of chiral ordering is that the physical content of a configuration mostly resides in the $SU(N)$ interaction phases associated with elementary hopping of a lattice chiral fermion. At the same time, the path integral approach to QCD vacuum implicitly relies on the assumption that the fluctuations associated with “physics” exhibit high degree of space–time order (and will lead to identification of collective variables $C$), while the physically irrelevant fluctuations are mostly disordered. Thus, by identifying and keeping the physically relevant part of the fluctuations, we should increase the order on average, and thus to lower the Kolmogorov entropy. Needless to say, it would be very illuminating if the statement analogous to CI2 could be proved rigorously at least in some model situation.

4.4 Repeated Chiral Ordering and Perfectly Chirally Ordered Configurations

It is interesting (and in the case of approximate chiral ordering very important) to think about repeating the chiral ordering procedure iteratively. To start speaking in these terms, let us denote the configuration obtained from $U$ in $j$-th step as $U^{(j)}$. More precisely

$$U^{(0)} \equiv U \quad U^{(j)} \equiv (\mathcal{M}^D)^j(U), \quad j = 1, 2, \ldots$$

At the same time, the original gauge action $S \equiv S^D_U(0)$ evolves into $S^D_U(j)$ after $j$ steps. The configuration $U^{(1)}(= \bar{U})$ represents the interaction–phase content of $U^{(0)}(= U)$. Is the interaction–phase content of $U^{(1)}$ equal to itself, i.e. is $U^{(1)}$ equal to $U^{(2)}$? Even though this would seem natural, it is not necessarily so. On the other hand, from the point of view of the principle of chiral ordering, the configurations stable under map $\mathcal{M}^D$ are clearly special. In fact, we expect them to be the configurations that reflect the space–time order present in the vacuum in the clearest possible manner. Let us thus define the corresponding set precisely.

**Definition 1.** Let $D(U) \in \mathcal{S}^F$ be an arbitrary chiral lattice Dirac operator defined on a latticized torus of size $L$ (i.e. $U \in U^L$). The configuration $\hat{U}$ satisfying the condition

$$D_{n,n+\mu}(\hat{U}) = D^f_{\mu} \times \hat{U}_{n,\mu} \quad \forall n, \mu$$

is said to be perfectly chirally ordered. The subset of all such configurations will be denoted as $U^L,D$.

We emphasize that in the above definition we do not restrict ourselves to Dirac operators that are elements of $\mathcal{S}^F,C$. Indeed, the set of perfectly chirally ordered configurations is defined for arbitrary chirally symmetric action, and will play an important role in the procedure of approximate chiral ordering discussed later in this paper.

As is obvious from its definition, the set $U^L,D$ is not empty for arbitrary $D$, since it contains a trivial configuration with $U_{n,\mu} \equiv I^c$ (“free–field” configuration; $I^c$ is an identity in color space). However, the notion of perfectly chirally ordered configuration (and the whole program of chiral ordering) would not be of much interest if this was the only possibility.
Thus, the following conjecture is to be viewed as one of the cornerstones for the approach to vacuum structure proposed in this paper.

**Conjecture C1.** The set \( U^{L,D} \) contains non-trivial configurations for arbitrary \( D \in S^F \) for sufficiently large \( L \). More precisely, it contains configurations with non-vanishing gauge invariant composite fields.

An important feature of the Conjecture C1 is that one can attempt its numerical verification for overlap fermions [12].

Returning now to the repeated exact chiral ordering, let us consider the sequences \( \{ U^{(j)}, j = 0, 1, 2, \ldots \} \) and \( \{ S^{MD}_{(j)}, j = 0, 1, 2, \ldots \} \) defined above for \( D \in S^{F,C} \). According to the principle of chiral ordering and Conjecture C12 we expect that, with increasing \( j \), we increasingly reduce the randomness present in the configuration, while changing lattice physics very mildly. Thus, in some sense, our approach expresses the view that random fluctuations are mostly responsible for violating the condition (69), while the relevant fluctuations ("physics") tend to satisfy it. For the elements of \( U^{L,D} \) this process ends immediately – they are invariant configurations under \( M^D \). For \( U \notin U^{L,D} \) it is expected that the amount of removed noise decreases fast with \( j \) and, asymptotically, it is not possible to remove any more randomness. In other words, the elements of \( \{ U^{(j)}, j = 0, 1, 2, \ldots \} \) behave asymptotically as elements of \( U^{L,D} \). This expectation is expressed in the following conjecture.

**Conjecture C2.** Let \( U \in U^L \) and \( D \in S^{F,C} \). (a) The limit \( \lim_{j \to \infty} U^{(j)} \) exists generically and is an element of \( U^{L,D} \). (b) There exist configurations \( U \) for which this limit is non-trivial at sufficiently large \( L \).

Note that by “generically” we mean up to some possible small set of measure zero which we postulate not to be relevant in the continuum limit. We wish to emphasize the following points.

(1) The non-trivial part of conjecture C2 is the generic existence of the limit and the fact that there exist configurations for which the limit is non-trivial. Indeed, if \( \lim_{j \to \infty} U^{(j)} = \hat{U} \) exists, then it is easy to see that it must be an element of \( U^{L,D} \) since

\[
\lim_{j \to \infty} || U^{(j)} - M_{n,\mu}^D(U^{(j)}) || = \lim_{j \to \infty} || U^{(j)} - U^{(j+1)} || = 0
\]

and so \( M_{n,\mu}^D(\hat{U}) = \hat{U}_{n,\mu} \). In the above equations ||.|| denotes an arbitrary matrix norm. The existence of a non-trivial limit (at the configuration level) for the procedure of chiral ordering is one of the main features that, we believe, distinguishes this approach from a deformation based on a generic map \( M \) (See point (c) in Sec. 4.2.). In a way we are suggesting that the deformation that preserves the interaction phases (and hence the physics) avoids converging to trivial configurations where physics is clearly changed. Instead, the tendency to preserve physics is strong enough so that a convergence to non-trivial invariant configurations arises.

(2) As a result of conjecture C2 the sequence \( \{ S^{MD}_{(j)}, j = 0, 1, 2, \ldots \} \) converges to a non-trivial action \( \hat{S}(S, M^D) \). By “non-trivial” we mean that the corresponding action assigns a non-zero probability density to at least some non-trivial configurations. Note that while all individual actions \( S^{MD}_{(j)} \) are expected to be valid elements of \( S^G \), we do not necessarily claim that the limiting action \( \hat{S} \) is also such an element. Indeed, while the classical limit and
the symmetries are surely preserved, the action could in principle become asymptotically non-local. However, we emphasize that even if this happened in some cases, the relevant message is that $\hat{S}$ can be approximated to arbitrary precision by an element of $S^G$.

(3) At the operational level, the special significance of the elements of $U^{L,D}$ (invariant configurations) comes from the fact that their probability of occurrence for actions reached via repeated chiral ordering cannot decrease. In fact, the total probability of these configurations and their small deformations will grow and, according to the conjecture C2, they will eventually dominate the path integral. Thus, to the extent to which the principle of chiral ordering reflects reality (i.e. the extent to which the effective interaction phases affecting chiral fermion encapsulate the physics), we suggest that it is the elements of $U^{L,D}$ that should be examined in order to determine the nature of collective degrees of freedom relevant in QCD vacuum.

(4) Note that the actions reached via the procedure of chiral ordering, while describing pure–glue theory, become functions of lattice Dirac kernel $D$ describing the associated chiral fermions. This “appearance” of structure associated with fermions in gauge part of the action will be a recurring theme in this series of papers and, in fact, it is very natural in the context of full QCD. Indeed, we will elaborate on the usefulness of tying the gauge and the fermion structure later in this paper and extensively in the in the second paper of this series.

(5) We emphasize that everything discussed here for the case of exact chiral ordering (Conjecture C2 in particular), also applies for various cases of approximate chiral ordering that will be defined in Sec. 5. This is important since it gives the possibility to test the ideas proposed here, namely by using the overlap Dirac operator as a starting point. We also remind the reader that the set of perfectly chirally ordered configurations $U^{L,D}$ is already defined for all chirally symmetric operators $D \in S^F$.

4.5 More on the Rationale for the Principle of Chiral Ordering

It is sometimes emphasized that the physical content of the gauge field is associated with its influence on a charged particle. This is usually expressed by saying that the wave function of a particle (charged under the gauge group) acquires a non-integrable (path-dependent) matrix phase when traveling over a specific path in space–time [35, 36]. Let us impose this condition in the lattice theory for the fermion in the fundamental representation and an “infinitesimal” path so that we can avoid the discussion about restricting the particle to a particular macroscopic path and/or about the interference of contributions from different paths. The elementary displacement of the particle in the lattice theory at finite cutoff is not really “infinitesimal” of course. However, we will impose it as if it was in order to mimic the behavior in the continuum theory. The quantum dynamics of the particle is described by its propagator, and for Dirac particle of mass $m$ we obtain the condition

$$ (D + m)^{-1}_{n,n+\mu}(U) = (D^f + m)^{-1}_{\mu} \times \tilde{U}_{n,\mu} \quad \forall n, \mu, \forall U \quad \tilde{U}_{n,\mu} \in SU(N) \quad (74) $$

It is worth noting that the above motivating discussion in the continuum uses the Hamiltonian language for the probing particle. This is actually problematic in the lattice theory with Ginsparg–Wilson fermions due to non-ultralocality [34] (see also [37]) and thus the discussion cannot be carried out entirely within the lattice context. Thus, Eq. (74) should be viewed as a discretized version of the relation in the continuum where the connection of the propagator formalism to Hamiltonian is formally made.
where \((D_f + m)^{-1}\) is a free (translation invariant) propagator (no color indices). The above equation is to be compared with Eq. (69).

Assuming that the fermionic actions satisfying the above condition exist, or using its approximate version with arbitrary \(D \in S^F\), the map \(\mathcal{M} : U \rightarrow \tilde{U}\) represents a valid deformation of the gauge action for non-zero positive \(m\) (constant in lattice units). Indeed, the only concern is locality which is however expected to be present for finite lattice mass. One could thus view Eq. (74) as a condition defining the chiral ordering along the similar lines as we did for standard chiral ordering Eq. (69). However, the above line of reasoning would not have much meaning if the configuration \(\tilde{U}\) depended strongly on the mass \(m\) chosen in the map. In other words, it would not be natural if the intended definition of physical content of \(U\) depended strongly on the mass of the probing particle. The consequences associated with demanding the independence of map \(\mathcal{M}\) on \(m\) can be analyzed at large \(m\). Indeed, the spectrum of \(D \in S^F\) is bounded, and we can choose the value of \(m\) larger than its upper bound, i.e. \(m > 2\) for the standard overlap operator. We then have

\[
(D + m)^{-1} = \frac{1}{m} \left( \mathbb{1} - \frac{D}{m} + \frac{D^2}{m^2} - \frac{D^3}{m^3} + \ldots \right)
\]

which is a converging Taylor-like expansion in \(t \equiv 1/m\). The requirement that the configuration \(\tilde{U}\) is independent of \(m\) then directly leads to the set of conditions

\[
D_{n,n+\mu}(U) = D_{\mu}^f \times \tilde{U}_{n,\mu}, \quad (D^2)_{n,n+\mu}(U) = (D_f^f)_{\mu}^2 \times \tilde{U}_{n,\mu}, \quad \ldots
\]

for arbitrary positive powers of \(D\). Thus the chiral ordering condition Eq. (69) represents a leading term (in \(t\)) which facilitates the independence of the “propagating phase” on the mass of the (heavy) probing fermion. This further supports our proposition that the effective interaction phase associated with elementary hopping of chiral lattice fermion tends to preserve the lattice physics.

We finally add two relevant remarks.

1. From our discussion in this section it is tempting to conclude that the chiral ordering prescription given by Eq. (74) is more fundamental than that of Eq. (69), and that the principle of chiral ordering formulated in Sec. 4.2 should be taken with respect to this relation. This may in fact turn out to be true. However, from Eq. (75) we see that the two are quite closely related. Moreover, the perfectly ordered configurations with respect to (74) form a subset of those that are perfect under (69). We will thus not lose any physical information by proceeding with the original principle of chiral ordering which can be numerically investigated with computational power currently available.

2. Classical configurations (see footnote 9) belong to the set of perfectly chirally ordered configurations in the classical continuum limit, i.e. on an asymptotically large lattice. This is true for both versions based on Eqs. (69) and (74). To make it more precise, consider a classical field \(A_\mu(x)\) on a symmetric torus of physical size \(L_p\). Introducing a “classical” lattice spacing via \(L_p = L a\) we construct the corresponding configuration \(U^c(A_\mu, a)\) on an \(L^4\) lattice via Eq. (9) (see also footnote 10). We then have in case of Eqs. (69) that

\[
D_{n,n+\mu}(U^c(A_\mu, a)) - D_{\mu}^f \times U^c(A_\mu, a) = \mathcal{O}(a^2)
\]
i.e. the linear term in $a$ vanishes. In this classical case it is a simple consequence of gauge invariance. We emphasize this fact to reassure ourselves that the principle of chiral ordering is compatible with the requirement of classical correspondence. It obviously doesn’t imply that classical configurations necessarily dominate the path integral for actions $S_{(j)}^{MD}$ at fixed large $j$ in the quantum continuum limit.

5 Approximate Chiral Ordering

Our discussion of the principle of chiral ordering in the previous section was mostly based on the assumption that there exist valid fermionic actions (elements of $S^F$) for which the condition (69) is strictly satisfied (elements of $S^{F,C}$). This allowed us to proceed with our reasoning in a straightforward manner. However, we also made it clear that the basic idea is meaningful (and can be pursued) without this assumption. In this section we define various forms of approximate chiral ordering procedure (map $M^D$) which can be realized e.g. with the standard overlap operator [12]. It will turn out that our discussion will also shed some light on how should one view the possible existence of (non-empty) set $S^{F,C}$.

5.1 Various Forms of Approximate Chiral Ordering

The proposition that the physical content of the gauge field is largely associated with the interaction phase of the probing chiral fermion can be implemented in many different ways. In an ideal case, these would all lead to equivalent behavior. However, it is also possible that some are superior to others in the sense that the degree to which the procedure tends to preserve the physical content of the configuration/theory can vary. We discuss some of the possibilities bellow. If not stated explicitly, the maps defined bellow are understood to be elements of $F$, i.e. they satisfy conditions (i)–(iii) of Sec. 4.1.

(I) The simplest way to proceed is to consider the formula (69) specifying the chirally ordered link $\bar{U}_{n,\mu}$ in the exact case, and use it as a basis for the ordering map also in the inexact case. The resulting map $M^D$ is then a composition of three maps $M^D = M^{(3)} \circ M^{(2)} \circ M^{(1)}$ specified by the following steps.

(a) Projection to color space, i.e.

$$M^{(1)} : U_{n,\mu} \rightarrow M \equiv \frac{1}{4} \text{tr}^s \left[ (D_{n,n+\mu}(\mathbb{I}))^{-1} D_{n,n+\mu}(U) \right]$$

$$= \frac{1}{4} \text{tr}^s \left[ (D_{\mu}^f \times \mathbb{I}^c)^{-1} D_{n,n+\mu}(U) \right]$$

$$\equiv \frac{1}{4} \text{tr}^s \left[ (D_{\mu}^f)^{-1} D_{n,n+\mu}(U) \right] \quad (77)$$

where $\text{tr}^s$ denotes a spinorial trace, $^{40} \mathbb{I}^c$ is an identity in the color space, and $\mathbb{I}$ is a configuration of such identity matrices. $^{41}$ The last form is just a shorthand for the first two in this

$^{40}$If $O$ is a matrix in color–spin space, then $\text{tr}^s O$ is the matrix in the color space with matrix elements $(\text{tr}^s O)_{a,b} = \sum_{\alpha} O_{\alpha a,\alpha b}$, where $\alpha$ denotes the spinorial and $a, b$ the color indices.

$^{41}$Note that if $D_{\mu}^f$ happens to be non-invertible (which can occur e.g. in the case of the overlap operator for special values of negative mass and the Wilson parameter), the map $M^{(1)}$ is still well defined by Eq. (69).
definition. One can easily see that in the case of exact chiral order- ing (69) we would have $\bar{U}_{n,\mu} = M$ and thus $\mathcal{M}^D = \mathcal{M}^{(1)}$.

(b) Unitary projection. This is provided by the polar decomposition which offers a natural and unique definition of the “matrix phase” for a non-singular square matrix in direct analogy to the phase of a non-zero complex number. According to the polar decomposition theorem, if matrix $M$ is non-singular then it can be uniquely decomposed as

$$M = M^u M^h \quad \text{such that} \quad (M^u)\dagger M^u = \mathbb{I}^c, \quad (M^h)\dagger = M^h$$

and $M^h$ is positive semidefinite. This associates a “matrix phase” (unitary matrix $M^u$) with matrix $M$. \(^{42}\) Thus the map $\mathcal{M}^{(2)}$ is defined by \(^{43}\)

$$\mathcal{M}^{(2)} : M \mapsto M^u \equiv M \frac{1}{\sqrt{M^\dagger M}}$$

where $\sqrt{\cdot}$ denotes a positive semidefinite branch of the square–root. \(^{44}\)

(c) $SU(N)$ projection. Unitary matrix $M^u$ can be mapped on the element of $SU(N)$ via rescaling by the $N$–th root of its determinant. Selecting the $N$–th root that leads to minimal modification of $M^u$ (in arbitrary matrix norm) specifies a unique prescription, namely

$$\mathcal{M}^{(3)} : M^u \mapsto \bar{U}_{n,\mu} \equiv e^{-i\varphi/N} M^u \quad \det M^u \equiv e^{i\varphi} \quad \varphi \in (-\pi, \pi]$$

The full map $\mathcal{M}^D = \mathcal{M}^{(3)} \circ \mathcal{M}^{(2)} \circ \mathcal{M}^{(1)}$ can be expressed in a single explicit formula if so desired, and is an element of $\mathcal{F}$.

(II) Using the same logic as in method (I), we can choose to perform the approximate chiral ordering via more “continuum–like” formula than that of Eq. (77). Indeed, consider a Clifford decomposition of $D^f_{\mu}$, namely

$$D^f_{\mu} = \sum_{a=1}^{16} B^a_{\mu} \Gamma^a, \quad a = 1, 2, \ldots 16$$

where $B^a_{\mu}$ are complex numbers and $\Gamma^a$ are the elements of an orthogonal Clifford basis normalized such that $\text{tr} \Gamma^a \Gamma^b = 4 \delta_{a,b}$. Let us denote by $B_{\mu\nu}$ the Clifford component of $D^f_{\mu}$ corresponding to basis element $\gamma^a_{\nu}$, i.e.

$$B_{\mu\nu} \equiv B^a_{\mu} \quad \text{such that} \quad \Gamma^a = \gamma^a_{\nu}$$

We then define the projection to color space (map $\mathcal{M}^{(1)}$) via

$$\mathcal{M}^{(1)} : U_{n,\mu} \mapsto M \equiv \frac{1}{4B_{\mu\mu}} \text{tr}^s \left[ \left( \gamma_{\mu} \times \mathbb{I}^c \right) D_{n,n+\mu}(U) \right] \equiv \frac{1}{4B_{\mu\mu}} \text{tr}^s \gamma_{\mu} D_{n,n+\mu}(U)$$

\(^{42}\)Note that the decomposition also holds for singular matrices but in that case it is not unique. Also, the matrix phase $M^u$ defined by (78) should be called the left matrix phase since one can also define the right phase by switching the order of matrices. Left and right phases are equal for normal matrices $M$.

\(^{43}\)Projection to the gauge group based on the polar decomposition (steps (b) and (c) here) was used by several authors in different contexts. Reference [38] is the earliest we are aware of.

\(^{44}\)We implicitly assume here that the cases when $M$ is strictly singular will essentially never be encountered in practice. One can also take more strict attitude and use the polar decomposition with an additional condition to fix $M^u$ uniquely also in singular cases. We will not elaborate on this here.
where the second form in the above equation is an abbreviation for the first form. One can easily check that if \( D \in S^{F,C} \) (exact chiral ordering) we would have \( \tilde{U}_{n,\mu} = M \) and thus \( \mathcal{M}^D = \mathcal{M}^{(1)} \). In the approximate case we define \( \mathcal{M}^D = \mathcal{M}^{(3)} \circ \mathcal{M}^{(2)} \circ \mathcal{M}^{(1)} \), where \( \mathcal{M}^{(2)} \) and \( \mathcal{M}^{(3)} \) are specified in method (I) above.

**III** The non–trivial aspect of the chiral ordering condition (69) resides in the fact that it forces a factorization of local spin–color matrix in the respective degrees of freedom. This gives meaning to the statement that the fermion acquires a matrix phase when it hops from site \( n \) to site \( n + \mu \). In other words, it ensures the fact that all fermionic degrees of freedom at site \( n \) get rotated by the gauge field in an identical manner. It is thus sensible to define the approximate chiral ordering prescription based on the “degree of factorizability”. For example, we can construct the map \( \mathcal{M} \) in the same way as in case I but replace \( \mathcal{M}^{(1)} \) with

\[
\mathcal{M}^{(1)} : U_{n,\mu} \rightarrow M \quad \text{such that} \quad \min_{\tilde{M} \in \mathbb{C}^{N \times N}} F(\tilde{M}) = F(M) \tag{84}
\]

where \( \mathbb{C}^{N \times N} \) is the set of \( N \times N \) complex matrices and

\[
F(\tilde{M}) \equiv || D_{n,n+\mu}(U) - D^f_\mu \times \tilde{M} || \tag{85}
\]

with \( ||.|| \) denoting a matrix norm. One can easily check that for non–zero \( D^f_\mu \) the solution of the above minimization problem in Frobenius norm (i.e. \( ||A||^2 \equiv \text{tr} A^\dagger A \) for matrix \( A \)), is unique and given by

\[
M = \frac{1}{\text{tr}[(D^f_\mu)^\dagger D^f_\mu]} \text{tr}^s \left( (D^f_\mu)^\dagger D_{n,n+\mu}(U) \right) \tag{86}
\]

Comparing the above expression with Eq. (77) we see that the two projections to color space are identical if

\[
(D^f_\mu)^\dagger D^f_\mu = \frac{\text{tr}[(D^f_\mu)^\dagger D^f_\mu]}{4} \mathbb{1}^s \tag{87}
\]

i.e. when \( D^f_\mu \) is a unitary matrix rescaled by a real number. This is true e.g. for the overlap Dirac operator but not necessarily in general. Note that if \( D \in S^{F,C} \) (case of exact chiral ordering) then it has to satisfy both Eqs. (77,86) and, consequently, the unitarity condition (87) also has to be satisfied.

Let us also point out that when replacing \( D^f_\mu \) in the minimization problem (84,85) with arbitrary spinorial matrix \( S \), then the solution is given by (86) with \( D^f_\mu \) replaced by \( S \). Consequently, one can easily check that the color–space projection \( \mathcal{M}^{(1)} \) of case II is equivalent to minimization of \( F(M) \) for \( S = B_{\mu\gamma\mu} \) (\( \mu \) not summed).

If one wishes to focus the definition of color projection on the spin-color factorizability property, then one can obtain a prescription that is different (but approximately equal in generic case) from those in cases I and II by relaxing the condition that the spinorial factor is fixed. More precisely, one can associate the projection to the color space with the following minimization problem.

\[
\mathcal{M}^{(1)} : U_{n,\mu} \rightarrow M \quad \text{such that} \quad \min_{\tilde{M} \in \mathbb{C}^{N \times N}, \tilde{S} \in \mathbb{C}^{4 \times 4}} F_1(\tilde{M}, \tilde{S}) = F_1(M, S) \tag{88}
\]

where

\[
F_1(\tilde{M}, \tilde{S}) \equiv || D_{n,n+\mu}(U) - \tilde{S} \times \tilde{M} || \tag{89}
\]
Note that the above problem is invariant under simultaneous rescaling $S \rightarrow \alpha S$, $M \rightarrow \alpha^{-1}M$, and we fix the resulting degeneracy of solutions by requiring that the $\gamma_\mu$-component in Clifford expansion of $S$ coincides with that of $D^f_\mu$, i.e.

$$\text{tr} \gamma_\mu S = 4B_{\mu\mu}$$  \hspace{1cm} (90)

At the numerical level the solution can be obtained e.g. iteratively by alternating the solutions of type (86) with spinorial/color matrix factors fixed. The resulting map $\mathcal{M}^{(1)}$ is then composed with $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(3)}$ of $I$ to obtain the full map $\mathcal{M}^D$.

(IV) In the remaining three cases we list prescriptions that are based on the spin–color factorizability property of exact chiral ordering, but avoid steps (b), (c) (i.e. unitarity projection and SU(N) projection). In other words, they are based on a direct minimization in SU(N) (rather than $\mathbb{C}^{N \times N}$). As a result, they will not be expressed in the explicit form, but rather defined only as solutions to the corresponding minimization problems. The uniqueness is hard to demonstrate in these cases, and will be assumed. The actual numerical work will reveal if this is reasonable or if it will be necessary to impose additional conditions to make them unique. We thus define the map $\mathcal{M}^D$ for the current case via the solution to the following minimization problem

$$\mathcal{M}^D: U_{n,\mu} \rightarrow \bar{U}_{n,\mu} \text{ such that } \min_{V \in \text{SU}(N)} F(V) = F(\bar{U}_{n,\mu})$$ \hspace{1cm} (91)

where

$$F(V) \equiv ||D_{n,n+\mu}(U) - D^f_\mu \times V||$$ \hspace{1cm} (92)

The above map represents perhaps the most natural generalization of exact chiral ordering prescription (69) to arbitrary $D \in \mathcal{S}^F$.

(V) The prescription that formally looks more “continuum–like” and represents the analog of case II is given by map (91) with $F(V)$ given by

$$F(V) \equiv ||D_{n,n+\mu}(U) - B_{\mu\mu} \gamma_\mu \times V||$$ \hspace{1cm} (93)

(VI) Finally, the analog of case III (Eqs. (88–90)) is specified by the minimization problem

$$\mathcal{M}^D: U_{n,\mu} \rightarrow \bar{U}_{n,\mu} \text{ such that } \min_{V \in \text{SU}(N), \bar{S} \in \mathbb{C}^{4 \times 4}} F_1(V, \bar{S}) = F_1(\bar{U}_{n,\mu}, S)$$ \hspace{1cm} (94)

where

$$F_1(V, \bar{S}) \equiv ||D_{n,n+\mu}(U) - \bar{S} \times V||$$ \hspace{1cm} (95)

Note that since $V$ is constrained to be unitary, it is not necessary to impose the analog of condition (90) in this case for SU(3) group.

5.2 Repeated Approximate Chiral Ordering

Various versions of approximate chiral ordering defined in the previous section represent some of the natural choices but certainly do not exhaust all possibilities. In fact, given arbitrary
\( D \in S^F \), we have not explicitly defined the set of acceptable maps \( M^D \) that we would consider approximate chiral orderings. Let us thus put forward the following definition.

**Definition 2.** Let \( D(U) \in S^F \) be an arbitrary chiral lattice Dirac operator. We call the map \( M^D \) an approximate chiral ordering defined by \( D \) if (a) \( M^D \in F \); (b) the elements of \( U^{L,D} \) (perfectly ordered configurations) are its fixed points, namely

\[
M^D(\hat{U}) = \hat{U}, \quad \hat{U} \in U^{L,D}
\]  

(c) it approximately satisfies the chiral ordering relation for generic configurations, i.e.

\[
\epsilon_{n,\mu} \equiv \frac{\|D_{n,n+\mu}(U) - D_{\mu} \times M^D_{n,\mu}(U)\|}{\|D_{n,n+\mu}(U)\|} \ll 1 \quad \forall n, \mu
\]  

As usual, by “generic” we mean all up to a possible set of measure zero which we assume not to be relevant.

The reason for introducing the approximate chiral ordering is that basically all the features discussed for the exact case are expected to carry over generically to the approximate case. In other words, the principle of chiral ordering (see Sec. 4.2) is viewed as a robust statement about the nature of the gauge field. In that regard, we wish to explicitly mention the following points.

(1) As emphasized in Sec. 4.3, we assume that there exist approximate chiral orderings for which the ordering conjecture C12 is valid.

(2) Fixing arbitrary \( D \in S^F \) and a particular approximate chiral ordering \( M^D \), we can consider the sequences \( \{U^{(j)}, j = 0, 1, 2, \ldots\} \) and \( \{S_{(j)}^{M^D}, j = 0, 1, 2, \ldots\} \) defined by repeated application of \( M^D \) as in the exact case. As \( j \) increases, it is expected that the configurations (and the ensembles representing the theory) will become less affected by randomness with lattice physics changing very mildly. Very important expectation is that the evolved configurations will asymptotically approach the elements of \( U^{L,D} \) and the ensembles will approach a non-trivial lattice theory. To express this, we modify the conjecture C2 of exact chiral ordering as follows.

**Conjecture C2a.** Let \( U \in U^L \) and \( D \in S^F \). There exist approximate chiral orderings \( M^D \) such that (a) \( \lim_{j \to \infty} U^{(j)} \) exists generically and is an element of \( U^{L,D} \); (b) there exist configurations \( U \) for which this limit is non-trivial at sufficiently large \( L \).

Needless to say, performing numerical tests of the principle of chiral ordering (as well as conjectures C1 and C2a) in the context of overlap Dirac operator will be a crucial step towards making the construction proposed here viable.

### 5.3 Dual View of the Chiral Ordering Evolution

Let us recall that the evolution of arbitrary configuration \( U \) under chiral ordering (exact or approximate) corresponding to \( D \in S^F \) is characterized by the sequence \( \{U^{(j)}, j = 0, 1, 2, \ldots\} \), which in turn induces the evolution of gauge theory \( S \) in the set of actions, namely \( \{S_{(j)}^{M^D}, j = 0, 1, 2, \ldots\} \) where the actions are defined via the associated configuration–based
deformation. However, it is also sometimes useful to think of chiral ordering as an evolution of
the underlying chiral fermionic action or equivalently a Dirac operator. Let us thus consider
a sequence of operators \( \{ D^{(j)}, j = 0, 1, 2, \ldots \} \) defined via
\[
D^{(j)}(U) \equiv D(U^{(j)}) = D\left( (M^D)^j(U) \right) \quad j = 1, 2, \ldots \quad D^{(0)}(U) \equiv D(U)
\]  
(98)

Since \( M^D \) is an element of \( \mathcal{F} \), it is easy to check that \( D^{(j)} \in \mathcal{S}^F \) for arbitrary \( j \). Moreover,
the free Dirac operators associated with \( D^{(j)} \) coincide with that of \( D \), i.e.
\[
D^{(j)}(\mathbb{I}) \equiv D(\mathbb{I}) \quad j = 1, 2, \ldots
\]
(99)

The basic property of \( D^{(j)} \) built in by construction is that
\[
U^{(j)} = (M^D)^j(U) = M^{D^{(j-1)}}(U)
\]
(100)

In other words, many iterations of chiral ordering using \( D \) can be thought of as a single
iteration with another element of \( \mathcal{S}^F \). Assuming the validity of conjecture \( \text{C2a} \) (C2) we can
conclude that for generic configurations \( U \) there exist limits
\[
\lim_{j \to \infty} U^{(j)} \equiv \hat{U} \quad \lim_{j \to \infty} D^{(j)}(U) \equiv \hat{D}(U)
\]
(101)
satisfying the relations
\[
\hat{D}_{n,\mu}(U) = D_{n,\mu}(\hat{U}) = \hat{D}_{n,\mu}(\hat{U}) = D^f_{\mu} \times \hat{U}_{n,\mu}
\]
(102)

We can then conclude the following. (i) The operator \( \hat{D} \) satisfies the exact chiral ordering
property (69) regardless of whether the initial operator \( D \) satisfies it or not. In fact, it
is an operator which yields a fixed-point configuration (perfectly ordered configuration;
element of \( \mathcal{U}^{L,D} \)) in one step. (ii) Operators \( D \) and \( \hat{D} \) share some of their perfectly ordered
configurations.

The above considerations suggest that the validity of conjecture \( \text{C2a} \) is sufficient to
conclude that the set \( \mathcal{S}^{F,C} \subset \mathcal{S}^F \) is non-empty. Indeed, every \( D^{(j)} \) of chirally ordered
sequence is an element of \( \mathcal{S}^F \), and one thus naturally expects that the limit \( \hat{D} \) belongs there
as well, in which case \( \hat{D} \in \mathcal{S}^{F,C} \). However, we caution that the limiting action could become
asymptotically non-local in some cases. Nevertheless, even if that happens, the important
result of these considerations is that the elements of \( \mathcal{S}^F \) can satisfy the chiral ordering
condition (69) to arbitrary precision.

6 Partial Chiral Ordering

In our treatment of the principle of chiral ordering we required that the ordering condition
(69) be satisfied for all lattice links. In other words, well-defined interaction phases were
assumed to exist for all possible elementary space–time “hops” of chiral fermion, and for all
(generic) configurations. However, one could also take a view (and some results of [5] could
be interpreted that way) that there is a sizable fraction of elementary space–time filaments
(links) which are not used by the propagating fermion. For such links the interaction phase

49
would not have to be well-defined. Turning the argument around, one could also say that the regions of space–time where condition (69) is satisfied to high accuracy are highly susceptible to fermion propagation, while the ones where (69) is satisfied poorly form sort of an “excluded” space–time.

To incorporate the above possibility into our reasoning (and to be able to put it eventually to the test), requires few changes in the formalism and we will describe them in this section. However, our overall line of logic will not change at all. In fact, we will essentially trace the same steps as we did in case of full chiral ordering, and will mostly concentrate on what needs modification. Note also that we will not use different notation for analogous concepts in the two cases which will always be distinguished by the context (either “full” or “partial” chiral ordering) in the future.

(i) To build the concept of chiral ordering we started with the definition of the special class of fermionic actions $S_{F,C}$, which in the case of full chiral ordering was defined as the subset of elements from $S_F$ satisfying condition (69). To modify this for the partial case let $\Omega_1$ be the set of all link coordinates (i.e. all $(n,\mu)$ of a given lattice) and $\mathcal{L} \subset \Omega_1$ its arbitrary subset. We say that $D \in S_{F,C}$ if $D \in S_F$ and if for generic configuration $U$ there exists a non-empty $\mathcal{L}(U)$ such that

$$D_{n,n+\mu}(U) = D_{\mu}^I \times \bar{U}_{n,\mu} \quad \forall (n, \mu) \in \mathcal{L} \quad \bar{U}_{n,\mu} \in SU(N) \quad (103)$$

The associated partial chiral ordering map $\mathcal{M}^D$ is defined via

$$U \rightarrow \mathcal{M}^D(U) \quad \mathcal{M}^D_{n,\mu}(U) \equiv \begin{cases} \bar{U}_{n,\mu} , \quad (n, \mu) \in \mathcal{L} \\ U_{n,\mu} , \quad (n, \mu) \notin \mathcal{L} \end{cases} \quad (104)$$

Thus, the gauge links for which the corresponding interaction phase is not well defined will not get changed under the transformation. One can easily see that $\mathcal{M}^D \in \mathcal{F}$ thus inducing a valid deformation of the gauge action. It is expected that the chiral ordering conjecture CI2 applies and the Kolmogorov entropy of $S^D$ will not grow relative to $S$.

(ii) The analysis of the repeated exact partial chiral ordering has the same ingredients as in the exact case. For arbitrary $D \in S_F$ we can define the set of perfectly ordered configurations $\mathcal{U}^{L,D}$ in complete analogy with Definition 1 with Eq. (73) required to be valid only on some non-empty set $\mathcal{L}(U)$ of links. Direct analogues of conjectures C1 and C2 are expected to hold.

(iii) The implementation of approximate partial chiral ordering requires a non-trivial modification relative to the full case. Indeed, it is not possible just to take e.g. the cases (I)–(VI) of Sec. 5.1 and restrict them to a subset of links. The point is that for generic $D \in S_F$ the equation (103) will not be satisfied exactly for any links, and thus the selection of $\mathcal{L}$ is part of the constructed map. The set $\mathcal{L}$ is configuration–dependent and the rule specifying it must be chosen such that the resulting $\mathcal{M}^D$ is local. Below we discuss few examples that conform to our requirements.

(Ip) Consider the map $\mathcal{M}^D$ for case (IV) of full approximate chiral ordering with arbitrary $D \in S_F$. For given $U \in \mathcal{U}^L$ let $\epsilon_{n,\mu}(U)$ be the relative residues of Eq. (97), and let $\delta > 0$. 

50
Then we define the map $\mathcal{M}_D^\delta$ by

$$U \rightarrow \mathcal{M}_D^\delta(U)$$

where

$$(\mathcal{M}_D^\delta)_{n,\mu}(U) \equiv \begin{cases} \mathcal{M}^D_{n,\mu}(U), & \epsilon_{n,\mu}(U) < \delta \\ U_{n,\mu}, & \epsilon_{n,\mu}(U) \geq \delta \end{cases}$$ (105)$$

We emphasize that $\mathcal{M}_D^\delta$ is local and thus the element of $\mathcal{F}$. Depending on the value of $\delta$ it will chirally order all the links or some subset $\mathcal{L}$. We can obviously base the construction of $\mathcal{M}_D^\delta$ on other full approximate chiral orderings in an analogous manner.

(IIp) Qualitatively different kind of prescription can be obtained as follows. Consider the matrix element $D_{n,n+\mu}(U)$ and the configuration $\tilde{U}(V)$ obtained from $U$ by replacing $U_{n,\mu}$ with $V \in \text{SU}(N)$ while keeping all the other link variables the same i.e.

$$\tilde{U}_{m,\nu}(V) \equiv \begin{cases} V, & (m,\nu) = (n,\mu) \\ U_{m,\nu}, & (m,\nu) \neq (n,\mu) \end{cases}$$ (106)$$

Furthermore, let $\mathfrak{U}_{n,\mu}(U)$ denote the set of all $V \in \text{SU}(N)$ such that $D_{n,n+\mu}(\tilde{U}(V))$ leads to an exact spin–color factorization of chiral ordering type, i.e.

$$\mathfrak{U}_{n,\mu}(U) \equiv \{ V \in \text{SU}(N) : D_{n,n+\mu}(\tilde{U}(V)) = D^f_{\mu} \times \tilde{U}_{n,\mu}(V), \tilde{U}_{n,\mu}(V) \in \text{SU}(N) \}$$ (107)$$

and let $V_0$ be the element of $\mathfrak{U}_{n,\mu}$ closest to $U_{n,\mu}$, i.e. such that

$$\min_{\mathfrak{U}_{n,\mu}} || U_{n,\mu} - V || = || U_{n,\mu} - V_0 ||$$ (108)$$

In the last equation we have implicitly assumed that if $\mathfrak{U}_{n,\mu}$ is non-empty then $V_0$ is unique generically. We then define

$$\mathcal{M}^D_{n,\mu}(U) \equiv \begin{cases} U_{n,\mu}(V_0), & \text{if } \mathfrak{U}_{n,\mu} \neq \emptyset \\ U_{n,\mu}, & \text{if } \mathfrak{U}_{n,\mu} = \emptyset \end{cases}$$ (109)$$

The above equation defines the approximate chiral ordering $\mathcal{M}^D$ (element of $\mathcal{F}$) with the property that it only modifies links for which the corresponding spin–color factorization is achievable in a prescribed manner. It is thus a partial chiral ordering prescription.

(IIIp) The disadvantage of the partial chiral ordering defined above is that it is not clear at this point how to carry it out in practice. In other words, there are no straightforward numerical ways to determine $V_0$, and they will have to be developed. We thus also advocate another possibility. Recall that one of the main purposes of approximate chiral ordering is to determine the structure of perfectly chirally ordered configurations obtained by repeated application of map $\mathcal{M}^D$. A natural way to proceed in this regard is to enforce “local perfectness” in the map $\mathcal{M}^D$. By this we mean the following. Using the same notation as in case (IIp) let $\mathfrak{V}^p_{n,\mu}(U)$ denotes the set of all possible “perfect links” $\hat{U}_{n,\mu}$ associated with fixed $(n,\mu)$, i.e.

$$\mathfrak{V}^p_{n,\mu}(U) \equiv \{ \hat{U}_{n,\mu} \in \text{SU}(N) : D_{n,n+\mu}(\hat{U}(\hat{U}_{n,\mu})) = D^f_{\mu} \times \hat{U}_{n,\mu} \}$$ (110)$$

Note that $\delta$ is not to be viewed as a free parameter that one can change at will when changing the configuration or the lattice spacing. The map $\mathcal{M}_D^\delta$ has to be fixed when investigating the evolution in the set of actions.
We now select the element \( \hat{U}_{n,\mu}^0 \) of \( \mathcal{V}_{n,\mu}^p(U) \) (if any) closest to \( U_{n,\mu} \), i.e. such that
\[
\min_{\mathcal{V}_{n,\mu}^p} ||U_{n,\mu} - \hat{U}_{n,\mu}|| = ||U_{n,\mu} - \hat{U}_{0,\mu}||
\]  
and define the map \( \mathcal{M}^D \) via
\[
\mathcal{M}^D_{n,\mu}(U) \equiv \begin{cases} 
\hat{U}_{0,\mu}^0, & \text{if } \mathcal{V}_{n,\mu}^p \neq \emptyset \\
U_{n,\mu}, & \text{if } \mathcal{V}_{n,\mu}^p = \emptyset 
\end{cases}
\]
Note that the important difference between the sets \( \mathcal{V}_{n,\mu}(U) \) and \( \mathcal{V}_{n,\mu}^p(U) \) is that in the former case we have twice as many variables to solve for defining condition than in the latter case (16 versus 8 in SU(3) case), and thus the set \( \mathcal{V}_{n,\mu}^p(U) \subset \mathcal{V}_{n,\mu}(U) \) is naively expected to be quite small generically. In fact, one expects at most a finite set of solutions in the perfect case. However, the comparison to case (IIp) can only be made by studying these solutions on specific backgrounds typical for the starting theory \( S \). Practically relevant feature of the perfect case is that one can attempt to find the elements of \( \mathcal{V}_{n,\mu}^p(U) \) in a straightforward manner iteratively.

(iv) The content of conjecture C2a carries over naturally to the partial chiral ordering case and the analogous statement is expected to be valid. One can also take a dual view and think in terms of evolving Dirac operator as in the full case.

Let us finally remark that the usefulness of the idea of partial chiral ordering can only be decided by studying the above options (full and partial) explicitly. Indeed, it is an open question whether QCD dynamics leads to the phenomenon of excluded space–time, i.e. whether the natural splitting of space–time into regions of well-defined and ill-defined chiral phase actually occurs in typical configurations dominating the QCD path integral.

7 Chiral Ordering and the Renormalization Group

As discussed extensively in the previous parts of this paper, the search for the fundamental QCD vacuum structure in the path integral formalism naturally leads to exploring the behavior of Kolmogorov entropy in the set of lattice QCD actions. Moreover, we proposed that a suitable tool to guide us in this search might be the iterated chiral ordering transformations defining certain trajectories in this set. Thus, at least superficially, there exist some similarities here with the Renormalization Group (RG) concepts and techniques. The aim of this section is to suggest, and make it plausible, that the connection might be more than superficial.

In case of RG the important structure in the set of actions is represented by renormalized trajectories associated with different RG transformations. The lattice actions on renormalized trajectory are expected to yield exact predictions (no cutoff effects) for all spectral quantities (or any physical quantities if the operators are correspondingly improved) [39]. While not clear a priori at all, it is tempting to think that renormalized trajectories will also represent a relevant feature from the point of view of Kolmogorov entropy. Such expectation is tied to the fact that, in certain situations, it appears valid to intuitively associate the physical (continuum) content of the lattice theory with order in typical configurations (clearly
defined collective variable \( C \), while to view scaling violations as being caused by an excess of “disorder” superimposed on top of it. Indeed, consider the theory \( S^G(a(\beta)) \in S^G \) and follow the trajectory of actions as \( a \) is being changed towards the continuum limit. As the lattice spacing is decreased, the scaling violations become smaller while the Kolmogorov entropy is expected to decrease since the associated distribution of configurations is becoming more sharply peaked. In this sense the transition from lattice spacing \( a_1 \) to lattice spacing \( a_2 < a_1 \) can be intuitively viewed as a “noise reduction” in typical configurations of \( S^G(a_1) \). Looking now in the orthogonal direction in the set of actions (lattice spacing fixed), the actions on renormalized trajectories represent the solutions to the problem of maximizing the information content of the lattice theory relative to the continuum. In other words, among the actions at the particular fixed cutoff, they carry the most information about the continuum limit. As such, the underlying order shaping this physics is expected to be closely related to the one associated with the physics of the continuum limit. Thus, in analogy to approaching the continuum limit, it is natural to expect that departures from this specific order (moving away from renormalized trajectory) are caused by extra randomness present in the dynamics defined by the corresponding actions. In other words, one tends to conclude that actions on renormalized trajectories correspond to local minima of Kolmogorov entropy at fixed lattice spacing.

While the above considerations appear quite plausible, they are strictly heuristic. At the same time, the conclusion on renormalized trajectories is rather non-trivial. \(^{46}\) In this section we extend the concepts needed to introduce the principle of chiral ordering in a manner that brings the relation to the renormalization group ideas more to the forefront. We will then argue that, in this extended framework, the above conclusion on renormalized trajectories suggests itself naturally. In fact, we will propose that the iterated chiral ordering can drive gauge actions close to renormalized trajectories of particular RG transformations.

### 7.1 RG Transformations Based on Chiral Ordering

Let \( D \in S^F \) and \( M^D \) be an arbitrary full approximate chiral ordering map. \(^{47}\) The action of \( M^D \) on \( U_{n,\mu} \) is determined by \( D_{n,n+\mu}(U) \) and it represents the interaction matrix phase associated with “elementary hopping” of chiral fermion from \( n+\mu \) to \( n \). In the same manner, we can think of an effective interaction matrix phase associated with fermion moving from \( n+2\mu \) to \( n \). Such map will be based on \( D_{n,n+2\mu} \) but otherwise be identical (in form) to \( M^D \).

To illustrate this explicitly, let us consider the approximate chiral ordering \( M^D \) of case (IV) (see Eqs. (91,92)). The associated map \( M^{D,2} \) will be given by

\[
M^{D,2} : U_{n,\mu} \longrightarrow M^{D,2}_{n,\mu}(U) \equiv \tilde{U}_{n,\mu} \quad \text{such that} \quad \min_{V \in SU(N)} \quad F(V) = F(\tilde{U}_{n,\mu}) \quad (113)
\]

\(^{46}\)We point out that in case of approaching the continuum limit in given lattice theory, one can support the conclusion of minimal Kolmogorov entropy with the fact that all correlation lengths increase sufficiently close to the limit. This is not the case when approaching the action on renormalized trajectory in which case the correlation lengths are merely adjusted to reproduce the continuum masses. This is one of the reasons why the conclusion of minimal Kolmogorov entropy is non-trivial in this case.

\(^{47}\)Note that if it happens that \( D \in S^{F,C} \) then all approximate chiral ordering prescriptions define the same map, namely the associated exact chiral ordering.

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53
where
\[ F(V) \equiv || D_{n,n+2\mu}(U) - D^f_{2\mu} \times V || \] (114)

In general, we associate with arbitrary approximate chiral ordering map \( M^D \) the family of maps \( M^{D,s} \) defined by the formal replacement

\[ M^D \rightarrow M^{D,s} \quad \text{under} \quad \begin{cases} D_{n,n+\mu}(U) \rightarrow D_{n,n+s\mu}(U) \\ D^f_\mu \rightarrow D^f_{s \mu} \end{cases} \quad s = 1, 2, \ldots \] (115)

where obviously \( M^{D,1} = M^D \).

It is now straightforward to define the associated family of RG transformations with scaling factor \( s \). For simplicity, we will work formally on the infinite lattice. The discussion can be repeated for the finite lattice in a straightforward manner if one lets the lattice sizes \( L \) to be the appropriate factors of powers of \( s \). The RG transformation \( R^{D,s} \) is defined by

\[ U \rightarrow R^{D,s}(U) : U_{n,\mu} \rightarrow R^{D,s}_{n,\mu}(U) \equiv M^{D,s}_{n,\mu}(U) \] (116)

where reverting to the original lattice is already performed. Note that \( R^{D,1} \equiv M^D \). Repeated application of \( R^{D,s} \) on \( U \) induces the sequence of configurations \( U^{s,j} \equiv (R^{D,s})^j(U) \), \( j = 0, 1, 2, \ldots \) with \( U^{s,0} \equiv U \). Starting from the gauge action \( S \equiv S^{(R^{D,s})}_{(0)} \in S^G \) represented by the corresponding ensemble, this repeated map defines the sequence of actions \( S^{(R^{D,s})}_{(j)} \), \( j = 0, 1, 2, \ldots \) in \( S^G \), namely the RG trajectory. In fact, because of the freedom in choosing \( s \), we have defined a family of RG trajectories with different scaling factors \( s = 2, 3, \ldots \). Chiral ordering procedure \((s = 1)\) represents the limiting case in this family and we will refer to the corresponding trajectory as the \textit{chiral ordering trajectory}.

To close this section, we need to discuss a particular aspect of the above transformations that one needs to keep in mind. In a generic case of RG transformation with scaling factor \( s \), one argues on intuitive physical grounds that the long-distance behavior in the induced theory of blocked variables is exactly the same as in the original theory. This is expressed in the fact that the correlation lengths (before reverting to the original lattice) remain exactly the same. As a result we find (after reverting to the original lattice) that \( \xi_i \rightarrow \xi_i/s \) in the transformed theory for all correlation lengths \( \xi_i \). Consequently, the lattice spacing (being based on one of the correlation lengths) gets magnified as \( a \rightarrow sa \). The conclusion that interactions on renormalized trajectory specify perfect actions (no cutoff effects) relies heavily on the fact that the above scaling of correlation lengths is \textit{exact}. This is difficult to prove rigorously for generic non-linear RG transformations. Moreover, even in case of linear transformations (e.g. for scalar fields) the argument assumes that the transformations are strictly ultralocal, i.e. that the new variable does not depend at all on old variables outside the considered block. At the same time, it is not difficult to see that if the dependence on variables outside the block is non-ultralocal but local, i.e. that the influence of variables outside the block is bounded by the exponential decay with characteristic lattice length \( \alpha \), then the correlation lengths of blocked variables can in principle change at most by \( \alpha \). If such change occurs, it will certainly make little difference close to the critical surface \((g = a = 0)\) when the correlation lengths are very large. However, in principle it can make a difference when transforming a system with lattice correlation lengths of order \( \alpha \).
The RG transformations proposed here are based on Ginsparg–Wilson fermionic kernels that are necessarily non-ultralocal in fermionic variables [34] and expected to be non-ultralocal also in gauge variables which is of relevance here. Consequently, we will not claim that actions on the corresponding renormalized trajectories are perfect at arbitrary lattice spacing (correlation length). Rather, we will implicitly understand that they are close to perfect down to correlation lengths comparable to $\alpha$. Since there are Ginsparg–Wilson actions with $\alpha \approx 1$ (see e.g. Ref. [40]), this is not an essential restriction. Moreover, as we emphasized all along, the nature of the RG transformation based on chiral ordering is expected to be such that the physics of the gauge field remains largely preserved even at small lattice distances. We thus expect that the correlation lengths will only change by the amount smaller than $\alpha$. The main purpose of the above remarks is to convey two messages. (1) Even though the RG transformations proposed here are non-ultralocal, we expect that “perfectness” on the renormalized trajectory is still a valid concept. (2) Because of non-ultralocality we admit the possibility that the correlation lengths of blocked variables can in principle change by very small amounts.

7.2 Renormalized Trajectory and the Line of Perfect Chiral Order

While the chiral ordering transformation $M^D$ and the RG transformations of Eq. (116) have been put on the same footing via the family of maps $R^{D,s}$, $s = 1, 2, \ldots$, we would conventionally tend to think of them in rather different terms. Starting with RG transformations $R^{D,s}$, $s = 2, 3, \ldots$, we accept the standard picture of the RG flow for Yang–Mills theory in the set of actions. In other words, for given $s$ we assume that there exists a unique fixed point (FP) of $R^{D,s}$ on the critical surface ($g = a = 0$), and that there is a single weakly relevant coupling associated with this FP, namely the gauge coupling. Starting with arbitrary action $S(a(g)) \in S^G$ close to critical surface ($g$ very small), the RG trajectory will first run towards the position of the FP and then away towards larger couplings (larger lattice spacings). The limiting procedure where the bare gauge coupling of the starting action $S(a)$ approaches zero (critical surface) defines a trajectory running from FP towards larger couplings, namely the renormalized trajectory corresponding to $R^{D,s}$. The generic flow of actions (starting from $S$ not necessarily very close to critical surface) in relation to renormalized trajectory is shown schematically in Fig. 1 (left).

Turning now to the chiral ordering map $M^D$, one could naively view it as nothing more than a very sophisticated way of smoothing the gauge field. Generic procedures of this type are sometimes distrusted if used inappropriately. However, as we argued extensively, our suggestion here is that chiral ordering is considerably more fine-tuned than generic

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48We emphasize that we do not claim that non-ultralocality necessarily forces the small change of correlation length in the transformed system. We only say that this is possible with non-ultralocal transformations. Also, if one wants to be more precise, our underlying assumption here is that performing the blocking operation arbitrarily many times (without reverting to original lattice) leads asymptotically only to a finite total change of correlation length. Then the concept of perfect action is meaningful only up to correlation length comparable to this total change which is expected to be of order $\alpha$.

49It is worth mentioning here that standard RG transformations in momentum space correspond to non-ultralocal (and sometimes even non-local) transformations in position space. Nevertheless, they are widely used albeit for analyzing different questions than those relevant here.
Figure 1: The expected flow under transformations $R^{D,s}$ for $s > 1$ (left) and $s = 1$ (right). $K_i$ collectively denotes the set of couplings other than the gauge coupling. The line of perfect chiral order plays the role analogous to that of the renormalized trajectory.

smoothing since it tends to preserve the defining feature of the gauge field, namely its influence on the charged particle. This point of view is expressed by the principle of chiral ordering and, in more definite terms, by conjectures $C1$, $C2$ and $C2a$. These imply that the evolution in the set of actions under $\mathcal{M}^D$ could actually be very constrained, and the tendency to preserve physics leads to the fact that the actions $S_{(j)}^{\mathcal{M}^D} \equiv S_{(j)}^{R^{D,1}}, j = 0, 1, 2, \ldots$ converge to the limiting action $\hat{S}(S, \mathcal{M}^D)$ (fixed point) under repeated application of $\mathcal{M}^D$. Fixing the starting lattice theory $S(a)$ parametrized by its lattice spacing, we obtain a one-parameter line of fixed points $\hat{S}(a, \mathcal{M}^D)$. We will refer to this line as the line of perfect chiral order. Note that the lattice spacing of $\hat{S}(a, \mathcal{M}^D)$ is not expected to be exactly $a$, but rather $\hat{a}(a)$ since correlation lengths can change by small amounts upon the repeated chiral ordering. However, because of tendency to preserve physics, we expect that $\hat{a} \approx a$ with increased accuracy as $a \to 0$. We should also mention that by writing $\hat{S}(a, \mathcal{M}^D)$ we implicitly assumed that the line of perfect chiral order only depends on transformation $\mathcal{M}^D$ and not on the starting theory chosen (analogue of the uniqueness of the fixed point in the RG case). We thus propose a flow pattern for $\mathcal{M}^D \equiv R^{D,1}$ as shown in Fig. 1 (right).

Given two arbitrary and unrelated RG transformations, it is not expected that the positions of their corresponding fixed points and RG trajectories are highly correlated. However, it is clear that various transformations in the family $R^{D,s}$ are strongly related by the fact that they are all defined from a single and highly “fine-tuned” object – the chirally symmetric lattice Dirac operator. Moreover, all these transformations are based upon the same underlying principle: they express the effective matrix phase associated with the propagating chiral fermion. Thus, for example, one would expect that the evolution of the action under a single $s = 4$ transformation $R^{D,4}$ is quite similar to repeating two $s = 2$ transformations $R^{D,2}$. Consequently, one would also expect that the corresponding fixed points and RG trajectories for different scaling factors are closely positioned in the set of actions. At the same time, as made clear by the discussion in this section, RG trajectories for $s = 2, 3, \ldots$ are the direct analogs of the line of perfect chiral order in the $s = 1$ case. Consequently, it is

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50By “generic” smoothing we mean any map that tends to average the gauge fields locally. Examples of this could be the APE smearing [41] or the “stout link” operation [42].
natural to conclude that the line of perfect chiral order runs in the vicinity of RG trajectories corresponding to transformations $\mathcal{R}^{D,s}$.

Accepting the above scenario entails two noteworthy implications:

(i) Repeated chiral ordering brings lattice gauge theory close to particular RG trajectories and thus, due to the conjecture $\text{CI2}$, the regions around such RT–s are special in that their Kolmogorov complexity is very low. Thus, the framework of chiral ordering is compatible with the intuitive reasoning put forward at the beginning of this section.

(ii) To the extent that the arguments put forward here will turn out to be valid, one can also expect that chiral ordering might be capable of not only “preserving the physical content” as stipulated by $\text{Goal 2}$, but also to improve the scaling behavior. Indeed, since theories in the immediate vicinity of RT are expected to have small cutoff effects, it can happen that the various correlation lengths will readjust themselves in order to mimic more closely the continuum behavior.

Finally, let us mention that we carried out this discussion in the context of full chiral ordering. Adaptation to the case of partial chiral ordering is possible but depends to a large extent on which particular ordering is used. We thus postpone the detailed discussion for this case to dedicated publications.

8 Effective Lattice QCD I.

The final topic we will deal with in this first paper of the series is the construction of the framework for studying the effective QCD vacuum structure and its scale dependence (see Sec. 1.5 and points A3 and A4). More precisely, here we will only discuss a specific issue (in fact a particular version of it) relevant for this problem. The topic will be discussed further in the second paper, and in its full generality in the third paper of this series.

We will examine the problem of defining the theory $S_{\Lambda_F}$ associated with $S \in S^G$, for arbitrary fermionic response scale $\Lambda_F$. The particular approach proposed here is based on the approximate chiral ordering, and will not allow us to do it for arbitrary $S \in S^G$. However, with every $S \in S^G$ we will associate another valid lattice action(s) with the same physics content (in the spirit of the principle of chiral ordering), for which it can be done. Viewed alternatively, the construction applies for arbitrary $S \in S^G$ but with restricted set of observables, i.e. we will not be able to define the mean values of all lattice operators $O(U)$ in $S_{\Lambda_F}$, but only a subset of it. However, this subset is sufficiently rich so that this is in fact a very mild restriction.

Let $S \in S^G$, $D \in S^F$ and let $\mathcal{M}^D \in \mathcal{F}$ be an arbitrary approximate chiral ordering based on $D$. As discussed in Secs. 4.1,4.2, the map $\mathcal{M}^D$ can be used to associate with $S$ another gauge theory $S^{M^D}$ via the transformation of the ensemble

$$\mathcal{E}_S = \{\ldots U^{i-1}, U^i, U^{i+1}, \ldots\} \longrightarrow \{\ldots \mathcal{M}^D(U^{i-1}), \mathcal{M}^D(U^i), \mathcal{M}^D(U^{i+1}), \ldots\} \equiv \mathcal{E}_{SM^D}$$

To define the action $S^{M^D}_{\Lambda_F}$ of effective lattice QCD at fermionic response scale $\Lambda_F$ via its ensemble, we first define the chiral ordering map $\mathcal{M}^{D,\Lambda_F}$ at the corresponding scale in lattice units ($\lambda_F = \Lambda_F a$). To do that, we will proceed in the same way as in definition of the effective
topological charge density in Refs. [4, 6], i.e. we will perform the eigenmode expansion of the corresponding matrix element of $D$. Indeed, the map $\mathcal{M}^D$ depends on the collection of spin–color matrices $D_{n,n+\mu}$ and we can write

$$D_{n,n+\mu} = \sum_j \psi_n^j \lambda_j (\psi_{n+\mu}^j)^\dagger \quad D_{n,n+\mu}^{\lambda F} \equiv \sum_{j:|\lambda_j| \leq |\lambda_F|} \psi_n^j \lambda_j (\psi_{n+\mu}^j)^\dagger$$

(118)

where $\psi^j$ is an eigenmode of $D$ with eigenvalue $\lambda_j$ and, in what follows, we will always assume that the labeling of eigenmodes respects the ordering of eigenvalues by magnitude, i.e. if $j < k$ then $|\lambda_j| \leq |\lambda_k|$. The map $\mathcal{M}^{D,\lambda_F}$ is then defined via a replacement

$$\mathcal{M}^{D}_{n,\mu} \rightarrow \mathcal{M}^{D,\lambda_F}_{n,\mu} \quad \text{under} \quad D_{n,n+\mu} \rightarrow D^{\lambda F}_{n,n+\mu} \quad D^f_\mu \rightarrow D^{f,\lambda F}_\mu$$

(119)

Note that in the above equations $\lambda_F \in [0,|\lambda|_{max}]$, where $|\lambda|_{max}$ is the maximal possible magnitude of an eigenvalue. The effective lattice QCD $S_{\lambda_F}^{O_D}$ associated with $S^{O_D}$ is then specified by its ensemble

$$\mathcal{E}_S = \{ \ldots U^{i-1}, U^i, U^{i+1}, \ldots \} \rightarrow \{ \ldots \mathcal{M}^{D,\bar{a}\Lambda_F}(U^{i-1}), \mathcal{M}^{D,\bar{a}\Lambda_F}(U^i), \mathcal{M}^{D,\bar{a}\Lambda_F}(U^{i+1}), \ldots \} \equiv \mathcal{E}_{S_{\lambda_F}^{O_D}}$$

(120)

with the new probability distribution (action $S_{\lambda_F}^{O_D}(U)$) given by equations (62, 63). Note that $\bar{a} = \bar{a}(a)$ is the lattice spacing of $S^{O_D}$. According to the principle of chiral ordering we expect that $\bar{a}(a) \approx a$.

Several important points regarding the definition and meaning of effective lattice QCD now need to be discussed.

(i) It needs to be stressed that, at this point, we have not yet given the precise definition of $D^{f,\lambda_F}_\mu$ (i.e. of the effective free matrix element) in Eq. (119). At the same time, the definition of $\mathcal{M}^{D,\lambda_F}_{n,\mu}$ requires both $D^{\lambda F}_{n,n+\mu}$ and $D^{f,\lambda_F}_\mu$ for all prescriptions of approximate chiral ordering we discussed except for the case (VI). We will rectify this in the following subsection devoted to this point.

(ii) We wish to reiterate that the map $\mathcal{M}^{D,\lambda_F}$ associates with arbitrary $U \in \mathcal{U}^L$ an effective chirally ordered configuration $\bar{U}^{\lambda_F} \in \mathcal{U}^L$ since

$$\lim_{\lambda_F \rightarrow |\lambda|_{max}} \bar{U}^{\lambda_F} = \lim_{\lambda_F \rightarrow |\lambda|_{max}} \mathcal{M}^{D,\lambda_F}(U) = \mathcal{M}^D(U) = \bar{U}$$

(121)

i.e. $\bar{U}^{\lambda_F}$ is an expansion of $\bar{U}$ (not $U$). This is, of course, why $S_{\lambda_F}^{O_D}$ given by (120) defines an effective description of $S^{O_D}$ rather than that of $S$. However, at the same time, following the reasoning in Sec. 4.1, we can view the map $\mathcal{M}^D$ as the means of associating with arbitrary local operator $O_{\alpha}(U)$, that we wish to evaluate in $S$, a new local operator $O_{\alpha}^{\lambda_F}(U) \equiv O_{\alpha}(\mathcal{M}^D(U))$ with the same classical limit. The map $\mathcal{M}^{D,\lambda_F}$ then defines an effective operator $O_{\alpha}^{\lambda_F}$ (to be valued in $S$) via

$$O_{\alpha}^{\lambda_F}(U) \equiv O_{\alpha}(\mathcal{M}^{D,\lambda_F}(U))$$

(122)

51Note that the expansion in the form (118) applies when $D$ is a normal matrix, e.g. for the overlap Dirac operator. If $D$ is not normal then we can still use the expansion in the bi–orthogonal left–right eigensystem.
Thus, from the point of view of action $S$, we can define the effective description at fermionic response scale $\Lambda_F$ for all operators that can be obtained as $O^\lambda_D(U)$ for some $O_\alpha$.

(iii) It is tempting to think that for the definition of $D^{\lambda_F}_{n,n+\mu}$ (and thus of $\mathcal{M}^{D,\lambda_F}$) it would be advantageous to utilize the trick used in [4, 6] to define the effective topological density i.e. to eigenmode–expand $(D - |\lambda|_{\text{max}})_{n,n+\mu} = D_{n,n+\mu}$ rather than $D_{n,n+\mu}$ itself. This would seemingly give the low–lying modes the largest weight. However, contrary to the case of topological density, it turns out that $\gamma_5$–Hermiticity forces the two expansions be identical.

(iv) We emphasize that the purpose and the meaning of effective lattice QCD is quite different from that of the common notion of effective field theory. In the latter, one constructs new degrees of freedom and couples them via effective interaction designed so that it takes into account the influence of high-frequency fluctuations that have been eliminated from the theory. However, in case of effective lattice QCD the nature of field variables is preserved (they are still the same gauge fields), and the interaction is changed only to filter out the high-frequency fluctuations in a meaningful manner. This effective interaction is non–local with the corresponding range being controlled by the scale $\Lambda_F$. The continuum limit of the theory $S^{\mathcal{M}^{D,\Lambda_F}}(\bar{a}(a))$ is achieved via bringing the driving theory $S(a)$ to the continuum limit $(a \to 0)$ while keeping $\Lambda_F$ fixed. The values of measurable quantities in the continuum theory $S^{\mathcal{M}^{D,\Lambda_F}}(\bar{a}(a))$ will differ from those of the full continuum theory $S_{\Lambda_F}^{\mathcal{M}^{D,\Lambda_F}}$, and this difference must then be directly ascribed to the influence of high–frequency fluctuations in the gauge field. In the language of the typical vacuum configurations, the space–time vacuum structure acquires a scale–dependence via $\Lambda_F$, and the relation of this structure to the aforementioned differences will teach us how is vacuum structure at different scales related to various physical phenomena. In other words, the purpose of effective lattice QCD is to provide us with the scale–dependent picture of QCD vacuum.

(v) While we have carried out the discussion of effective lattice QCD in the context of pure–glue QCD, it can be clearly repeated without any change for full QCD. In fact, for given full lattice QCD defined by $S^{\mathcal{G}} \in S^{\mathcal{G}}$ and $S^{\mathcal{F}} \in S^{\mathcal{F}}$, there is an obvious natural choice of Dirac operator on which we base our chiral ordering transformation $\mathcal{M}^{\mathcal{D}}$, namely the $D$ that specifies $S^{\mathcal{F}}$.

(vi) At the technical level, our way of defining effective QCD is similar to the recently discussed procedure of Laplacian filtering [21], wherein the authors define a “filtered link” via performing the eigenmode expansion of relevant matrix element of the covariant Laplacian.

8.1 Expansion of the Free Matrix Element

In this section we come back to the issue of specifying $D^{\lambda_F}_{\mu}$, which is necessary for definition of $\mathcal{M}^{D,\lambda_F}$ with approximate chiral orderings we discussed (except case (VI)). Our goal is to preserve the physical meaning of chiral ordering procedure in the effective case. In other words, we want that the map $\mathcal{M}^{D,\lambda_F}$ can still be interpreted as extracting the matrix phase acquired by fermion when hopping from $n + \mu$ to $n$ relative to the free case. The issue is what exactly do we mean by “free case” in effective theory. There are several ways to proceed and we will describe them in turn.
(α) The most straightforward possibility is to expand the free operator $D^f$ (no color indices in this case) \(^{52}\) in its own eigenmodes $\chi^j$. The relevant matrix element is

$$D^f_{\mu,\lambda_F} \equiv D^f_{n,n+\mu,\lambda_F} \equiv \sum_{j:|\omega_j| \leq \lambda_F} \chi^j_n \omega_j (\chi^j_{n+\mu})^\dagger \quad D^f \chi^j = \omega_j \chi^j \quad (123)$$

While this is a sensible definition satisfying our requirement, it might not be suitable if we are interested in (or restricted to) studying the low-energy behavior in finite physical volume $V_p = (La)^4$ (fixed as continuum limit is approached). Indeed, with antiperiodic boundary conditions in time direction, there will be a non-zero lower bound of the free spectrum $|\omega|_{\min}(L)$ such that $|\omega|_{\min}(L)/a \propto 1/(La)$ converges to a finite physical value $\Lambda_F^{\min}$ as $a \to 0$, when $V_p$ is held fixed. Thus, in this case we will not be able to define the map $M^{D,a}\Lambda_F$ for $\Lambda_F < \Lambda_F^{\min}$. In other words, the effective lattice theory $S^{LM}_{\Lambda_F}$ will not be defined for $\Lambda_F < \Lambda_F^{\min}$. \(^{53}\)

(β) The possible issue with definition (α) is that for $\Lambda_F$ close to $\Lambda_F^{\min}$ there will be an increasingly larger mismatch between the number of eigenmodes contributing to $D^f_{\mu,\lambda_F}$ and $D^f_{n,n+\mu,\lambda_F}$ as the continuum limit is approached. This comes about as a result of the fact that we force the scale (lattice spacings $a$) determined in interacting theory also on the free dynamics. This is not natural. Indeed, one should rather match the free theory to the interacting one by the “number of degrees of freedom”, which are measured in this case by the number of eigenmodes included in the expansions. In other words, it is meaningful to ask what is an effective matrix phase acquired by interacting fermion described by $N$ eigenmodes relative to the dynamics described by $N$ free eigenmodes. To implement this properly, let us recall that, due to $\gamma_5$–Hermiticity, the eigenmodes of $D$ with complex eigenvalues are naturally paired, i.e. for each such eigenmode $\psi^j$ of $D$ there is a mode $\psi^k = \gamma_5^j \psi^j$ with complex–conjugated eigenvalue ($\lambda_k = \lambda_j^+$). The unpaired eigenmodes have real eigenvalues and reflect the topology of the underlying gauge field. The eigenmodes of $D^f$ are all paired via the $\gamma_5$ operation. For given fixed $U$, let $N_0$ be the number of zeromodes of $D(U)$, and let $N_p(\alpha)$ is the number of paired modes such that $|A_j| \leq \alpha > 0$. By definition, $N_p(\alpha)$ is an even number.

We then have

$$D^f_{\mu,\lambda_F} \equiv D^f_{n,n+\mu,\lambda_F} \equiv \sum_{j \leq N_p(\lambda_F)} \chi^j_n \omega_j (\chi^j_{n+\mu})^\dagger \quad (124)$$

Two points need to be emphasized with regard to the above definition.

(i) Since there can be degeneracies in the free spectrum, we implicitly assume that the ordering of the eigenmodes within the degenerate subspace is fixed by some arbitrary rule, and that the $\gamma_5$–conjugated modes follow one another when the sequence $\{(|\omega_j, \chi^j|, j = 1, 2, \ldots \}$ is formed.

(ii) Note that there is still seemingly a small mismatch between the number of modes contributing to $D^f_{n,n+\mu}$ (equal to $N_0 + N_p(\lambda_F)$) and the number of modes contributing to $D^f_{\mu,\lambda_F}$ (equal to $N_p(\lambda_F)$). However, one can easily convince himself that the real modes in fact do not contribute to $D^f_{n,n+\mu}$.

\(^{52}\)Note that since $D^f$ is translation invariant we frequently used $D^f_m$ (single index) as a shorthand for $D^f_{n,n+m}$, \(\forall n\). The meaning of single and double–indexing is thus clear.

\(^{53}\)Note that this is not necessarily a bad thing since $1/(La)$ is a natural infrared scale for this system.
Using the above construction, the map $M^{D^2,\lambda F}_\mu$ (and thus action $S^{M^D}_\mu$) is defined by equations (118,119) and (124). To see more explicitly how is this done for some basic forms of approximate chiral ordering, consider the case (I). We have $M^D = M^{(3)} \circ M^{(2)} \circ M^{(1)}$, where only the map $M^{(1)}$ (projection to color space) will need modification when transiting from $M^D$ to $M^{D^2,\lambda F}_\mu$. In particular, in place of Eq. (77) we will have

$$
(U_{n,\mu})_{a,b} \longrightarrow M^F_{a,b} = \frac{1}{4} \text{tr}^s\left[\left( D^f_{\mu} \times I^c \right)^{-1} D_{n,n+\mu}^{\lambda F}(U) \right]_{a,b} \\
= \frac{1}{4} \sum_{j:|\lambda_j| \leq \lambda_F} \lambda_j \left( \psi^j_{n+\mu} \right)_b \left( D^f_{\mu} \right)^{-1} \left( \psi^j_a \right)_a
$$

where $(\psi^j_a)_a$ is the 4-component object (color index fixed to $a$) and $D^F_{\mu} \times I^c$ is given by (124).

For approximate chiral ordering of case (II) we will have in a similar manner

$$
(U_{n,\mu})_{a,b} \longrightarrow M^F_{a,b} = \frac{1}{4B^a_{\mu}} \text{tr}^s\left[ \gamma_\mu D^F_{n,n+\mu} \left( U \right) \right]_{a,b} \\
= \frac{1}{4B^a_{\mu}} \sum_{j:|\lambda_j| \leq \lambda_F} \lambda_j \left( \psi^j_{n+\mu} \right)_b \gamma_\mu \left( \psi^j_a \right)_a
$$

where $B^F_{\mu}$ is the $\gamma_\mu$ component in the Clifford decomposition of $D^F_{\mu} \times I^c$. We proceed in a straightforward manner also in the remaining cases of the approximate chiral ordering.

(The final possibility that we wish to discuss treats $D_{n,n+\mu}(U)$ and $D_{n,n+\mu}(I) = D^f_{\mu} \times I^c$ in a symmetric manner with respect to the eigenmode expansions performed. In particular, we will expand both operators in interacting eigenmodes $\psi^j$. To do this, let $\hat{\chi}^j$ denotes the eigenmode of $D(I) = D^f \times I^c$ with eigenvalue $\omega_j$, i.e. $\hat{\chi}^j$ has both the spin and color indices and is related to eigenmodes of $D^f$ in an obvious manner. We can write $D(I)$ as

$$
D(I) = \sum_j \hat{\chi}^j \omega_j \left( \hat{\chi}^j \right)^\dagger = \sum_{k_1,k_2} c_{k_1,k_2} \psi^{k_1} \left( \psi^{k_2} \right)^\dagger \text{ where } c_{k_1,k_2} \equiv \left( \psi^{k_1} \right)^\dagger D(I) \psi^{k_2} \quad (127)
$$

which can be truncated to define

$$
D^{\lambda F}_{n,n+\mu}(I) \equiv \sum_{k_1,k_2} c_{k_1,k_2} \psi^{k_1}_n \left( \psi^{k_2} \right)^\dagger_{n+\mu} \quad (128)
$$

The aspect that needs to be discussed with regard to the above definition is that, contrary to cases (a), (\beta) where spin and color variables are a priori separated, the matrix $D^{\lambda F}_{n,n+\mu}(I)$ is not expected to be exactly expressible as a direct product in the corresponding subspaces. If this was possible and we could write it as $D^f_{\mu} \times I^c$; then we would use this correspondence as a definition of $D^f_{\mu} \times I^c$ in this case. However, as it stands, we need to take an additional step. While there are other ways to proceed, \footnote{One immediate possibility would be to define $D^f_{\mu} \times I^c$ by minimizing the norm $||D^{\lambda F}_{n,n+\mu}(I) - D^f_{\mu} \times I^c||$ with respect to $D^f_{\mu} \times I^c$.} we wish to explicitly discuss the approach where, instead of defining $D^f_{\mu} \times I^c$ explicitly and proceeding as in cases (a), (\beta), we simply replace the factors $D^f_{\mu} \times I^c$ in various definitions of approximate chiral orderings with $D^{\lambda F}_{n,n+\mu}(I)$.}
To see this explicitly, let us consider some interesting cases of approximate chiral ordering. In case (I) we can replace the projection to color space $M^{(1)}$ (see Eq. (77)) directly with

$$U_{n,\mu} \rightarrow M^{\lambda_F} \equiv \frac{1}{4} \text{tr}^s \left[ (D_{n,n+\mu}^{\lambda_F}(\mathbb{I}))^{-1} D_{n,n+\mu}^{\lambda_F}(U) \right]$$

(129)

thus defining the map $M^{D,\lambda_F}$ by composing it with $M^{(2)}$ and $M^{(3)}$. For case (IV) we define the map $M^{D,\lambda_F}$ via the following minimization problem.

$$M^{D,\lambda_F} : U_{n,\mu} \rightarrow \bar{U}_{n,\mu}^{\lambda_F} \quad \text{such that} \quad \min_{V \in SU(N)} F(V) = F(\bar{U}_{n,\mu}^{\lambda_F})$$

(130)

where

$$F(V) \equiv || D_{n,n+\mu}^{\lambda_F}(U) - D_{n,n+\mu}^{\lambda_F}(\mathbb{I}) (\mathbb{I}^s \times V) ||$$

(131)

where $\mathbb{I}^s$ is the identity in spinor space.

Finally, we emphasize that practical utility of cases $(\alpha) - (\gamma)$ is to be decided numerically by studying which expansion leads to the most accurately satisfied chiral ordering condition.

9 Summary

The purpose of this series of articles is to discuss in detail the set of ideas that, we think, can form a basis of a consistent framework for systematic study of QCD vacuum structure in the path integral formalism. While the history of using the lattice definition of QCD to pursue these issues is rather long, it has not matured into a well-defined subject with accepted scope, goals, and a standard collection of tools. Thus, while in the areas such as hadron spectroscopy one walks on the firm ground of well-defined quantitative notions, the realm of QCD vacuum structure is typically associated with constant stumbling over (at best) intuitive concepts such as “typical configuration”, “object in the vacuum”, “space–time structure”, etc.. More importantly, the ultimate goal of the effort (the analog of obtaining the mass) is the part of the research question here. In other words, solving the problem of QCD vacuum means different things to different researchers, even though (vaguely) stated goals such as “understanding confinement” and “understanding chiral symmetry breaking” are of universal interest.

Nevertheless, the common aspect of various efforts can be summed up rather succinctly as follows. We are trying to replace the QCD path integral ensemble $E^{QCD}$ (defined via some regularized limiting procedure, e.g. using lattice gauge theory) with the ensemble $E^{STR}$ which is defined in terms of some collective variables $C$ that arise naturally as a result of strong dynamics. The analysis of strong interactions in terms of $C$ is expected to be very transparent, and the space–time nature of $C$ will represent the space–time structure of the QCD vacuum in the path integral formalism. Stating the problem in this way, there are two kinds of issues that need to be addressed for there to be a chance that a systematic framework emerges. 

(i) First, it needs to be clarified what is the relation of $E^{QCD}$ (full theory of strong interactions) to $E^{STR}$ that we seek to find (see P1). 

(ii) Secondly, once the underlying goal of the effort is specified, it is necessary to have a set of guidelines/tools that will make the inquiries into specific questions possible and streamlined (see P2). Our suggestions with regard to (i) and (ii) can be schematically described as follows.
(i) Given the existence of the fundamental topological structure observable in regularized QCD ensembles [5], we propose that the ensemble $\mathcal{E}^{STR}[C]$ can be regarded as fully equivalent to $\mathcal{E}^{QCD}$ for all questions of physical interest. In other words, we can basically view the transition from $\mathcal{E}^{QCD}[A]$ to $\mathcal{E}^{STR}[C]$ as a complicated change of variables in the QCD path integral. Analogously to the case of topological vacuum, we refer to the structure entailed by variables $C$ as a fundamental QCD vacuum structure. 55 Constructing $\mathcal{E}^{STR}$ will not provide us with direct understanding of various phenomena in strong interactions because of scale dependence inherent in field–theory description. To achieve such understanding, we propose that it is also necessary to study the effective vacuum structure which captures the dependence of vacuum properties on the scale of fluctuating elementary fields [4, 6]. The effective structure is specified by the ensemble $\mathcal{E}_\Lambda^{STR} \equiv \mathcal{E}^{STR}[C(\Lambda), \Lambda]$ describing the effective QCD at scale $\Lambda$. 56 Thus, in summary, the ultimate goal in our approach is to construct the ensembles $\mathcal{E}^{STR}$ and $\mathcal{E}_\Lambda^{STR}$ for all $\Lambda$.

(ii) The above goal can be systematically pursued using the lattice definition of QCD and, in particular, by studying the space–time structure in numerically–generated ensembles of finite systems in a Bottom–Up manner (see section 1.4). This means that we will attempt to arrive at the definition of $\mathcal{E}^{STR}$ and $\mathcal{E}_\Lambda^{STR}$ inductively, using the observed structure in the regularized ensembles as the only input (see A1–A4). In other words, the criterion for selecting the appropriate collective variables $C$ (or $C(\Lambda)$) is identified in this approach with actual observability of space–time attributes associated with $C$ in the regularized path–integral ensembles.

Having the basic overall scheme described above in mind, the content of this series of papers can be described as an attempt to elevate it to the state where its various elements are properly defined to the largest extent, and where it is possible to practically pursue its goals using the currently available methods of lattice gauge theory. The present work addresses three topics related to these issues.

(I) When searching for fundamental vacuum structure in a Bottom–Up manner, we can take advantage of a freedom that is inherent in the process of regularization. In particular, the continuum theory can be defined in many ways using different lattice regularizations (theories in the set of valid lattice actions). Since the underlying goal is to identify the space–time patterns in configurations dominating the regularized path integral, we naturally prefer lattice theories for which these configurations exhibit a high degree of space–time order. However, such reasoning assumes that there exists a well–defined notion for “degree of space–time order” in an arbitrary configuration. We argued that Kolmogorov entropy (Kolmogorov complexity) of binary strings describing coarse–grained configurations provides the appropriate quantitative measure. The ensemble average of Kolmogorov entropy associated

55Note that discovering the nature of variables $C$ does not necessarily imply the analytic solution of QCD. However, if QCD can be solved analytically, then specifying $C$ would clearly mean an important step in that direction.

56For more precise meaning of “effective” in this case see comment (iv) of Sec. 8. More detailed discussion will be given in the third paper of this series.

63
with any theory then defines the ranking of regularizations (at a particular fixed cutoff) by the degree of space–time order the corresponding lattice interaction generates. We wish to make a few comments regarding these issues.

\((\alpha)\) The Kolmogorov entropy measure is highly universal, capturing the space–time order in essentially any form. There is a price that we have to pay for this generality. Indeed, if one wishes to consider Kolmogorov entropy as a practical tool (rather than a conceptual construct), then one needs to emphasize that the entropy \(E^K(U, k)\), while well defined, is not computable in general. Also, the use of pseudo–random (deterministic) numbers in actual simulations will most likely lead to a systematic error in determining the average value \(E^K[S, k]\) for theory \(S\). Thus, if pseudo–random numbers are used for generating actual ensembles, then we implicitly assume that our conclusions are not affected by this error.

\((\beta)\) We included a rather detailed discussion of the information theory aspects in Sec. 3. This is not just to introduce the Kolmogorov entropy, but also because we believe that the language and methods of information theory are both appropriate and fruitful for this area of research. Indeed, there are many relevant questions regarding the vacuum structure that can be both properly formulated and potentially solved by using the framework of information theory. For instance, regarding the issue at hand, one can attempt to formulate less general but computable measures (based on the quantity of information) that would mimic the role of Kolmogorov entropy.

\((\gamma)\) It is also worth mentioning that Kolmogorov entropy and the information viewpoint are quite useful for providing intuitive understanding in situations that are otherwise difficult to grasp. For example, Kolmogorov entropy of the Wilson gauge action at currently studied couplings is probably rather high since no observable structure could be identified in its typical configurations when standard ultralocal operators for composite fields were used. From the information theory viewpoint it is clear that in order to reach the theory with significantly lower Kolmogorov entropy, it is necessary that it is related to Wilson theory by a highly complex map. In other words, this theory must be defined by an action that is difficult to work with computationally. Thus, the actions possessing low Kolmogorov entropy are expected to be very complicated. It is for related reasons that we are able to observe a well-defined structure in the configurations of overlap–based topological density (which is difficult to compute) but not in the ultralocal definitions which are comparably “simple”.

\((\text{II})\) In order to identify the space–time nature of collective variables \(C\) in regularized configurations most easily, one would prefer to work with lattice theories that maximize the degree of space–time order. However, it is not sufficient (even as a conceptual goal) to simply minimize the Kolmogorov entropy in the set of actions (see \textit{Goal 1} and \textit{Goal 2} of Sec. 4). Indeed, one should rather perform a minimization constrained by the requirement that the physics content of the theory remains largely preserved in the process. As a tool for achieving this in practice, we propose the configuration–based deformation of the action defined via \textit{chiral ordering transformation}. Chiral ordering transformation associates with an arbitrary gauge configuration \(U\) a new configuration \(\bar{U}\) such that \(\bar{U}_{n,\mu}\) represents an effective SU(N) matrix phase associated with hopping of chiral fermion from \(n + \mu\) to \(n\). Since the transformation is constructed to represent the local physical meaning of the gauge field (local influence on a charged particle), it is expected that both short and long distance (in lattice units) proper-
ties of the theory will change very little upon the deformation (principle of chiral ordering). The repeated application of chiral ordering transformation on a given ensemble induces the evolution in the set of actions. We propose that there exist nontrivial configurations on a finite lattice that are stable under the chiral ordering transformation (perfectly chirally ordered configurations), and that the evolution in the set of actions can lead to theories where such configurations play a dominant role. We wish to mention the following points related to these issues.

(a) Two basic forms of chiral ordering transformations were discussed. The full ordering where all links of the lattice are forced to acquire their effective value dictated by chiral fermion, and the partial ordering where only a subset of all links is changed. The second option was introduced in order to incorporate the possibility that the QCD path integral is dominated by configurations containing regions that are not accessible to fermion propagation. Indeed, if that was the case then the effective interaction phase would not have to be well defined for links over which the propagation is prohibited. Detailed numerical studies will be necessary to determine which option is closer to that being realized by QCD dynamics.

(β) Connections to RG ideas suggest themselves rather naturally in our framework, and they were discussed in Sec. 7. We defined the family of RG transformations with scale factor $s$ via chiral orderings extended over straight paths containing $s$ links. The original chiral ordering can then be viewed as a limiting case of these transformations with $s = 1$. We proposed that actions on renormalized trajectories corresponding to generic RG transformations represent important features in the landscape of Kolmogorov entropy over the set of actions. The particular suggestion is that there are deep local minima of Kolmogorov entropy in the vicinity of renormalized trajectories. Assuming that chiral ordering evolution indeed leads to fixed points belonging to the set of actions, we argued that this conclusion can be made plausible at least for renormalized trajectories corresponding to RG transformations based on chiral ordering.

(III) In the Bottom-Up approach, the construction of $E_{STR}^{S}$ (effective structure) is to be guided by the space–time behavior in effective QCD ($E_{QCD}^{S}$), which can be defined by its ensemble $E_{QCD}^{S}$. The ensemble $E_{QCD}^{S}$ is in turn to be defined as a continuum limit of lattice-regularized ensembles $E_{LQCD}^{S}$ with $\Lambda$ fixed. However, such regularizations have not been defined yet. Here we suggested a particular form of the first step toward the general definition of this kind. Guided by the fruitfulness of bringing in the chiral fermion structure into definition of lattice topological field, we proposed lattice theories defined at a given fermionic response scale $\Lambda_F$. This definition is based on the chiral ordering transformation discussed here and, in this regard, we wish to briefly mention the following two points.

(α) It needs to be emphasized that in our framework the search for the fundamental and effective structure is completely unified and streamlined under the umbrella of the principle of chiral ordering. Indeed, in a nutshell we are contemplating the following chain of logic. Consider an arbitrary lattice action $S \in S^G$. If the goal is to examine its ensemble to obtain an information on the fundamental structure, we suggest that it is beneficial to make a transition to the action $S_M^{D}$ representing a configuration–based deformation of $S$ constructed via chiral ordering map $M^{D}$, since this is expected to lower the Kolmogorov
entropy. Next, we propose to consider the effective theories $S^M_D$ of $S^M_D$, whose definition is made possible by the fact that theory became a function of Dirac kernel $D$. This will provide information on the effective structure in a manner fully consistent with fundamental structure since $\lim_{\Lambda_F \to \Lambda_{\max}} S^M_D = S^M_D$.

(β) As is quite clear from our discussion, the framework proposed here is particularly natural in the context of full QCD. Indeed, in this case the choice of $D \in S^F$ in chiral ordering transformations is fixed by the fact that $D$ is part of the definition of the theory itself.

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