Remarks on zeta functions and K-theory over $\mathbb{F}_1$

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Abstract: We show that the notion of zeta functions over the field of one element $\mathbb{F}_1$, as given in special cases by Soulé, extends naturally to all $\mathbb{F}_1$-schemes as defined by the author in an earlier paper. We further give two constructions of K-theory for affine schemes or $\mathbb{F}_1$-rings, we show that these coincide in the group case, but not in general.

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Introduction

Soulé [10], inspired by Manin [12], gave a definition of zeta functions over the field of one element $\mathbb{F}_1$. We describe this definition as follows. Let $X$ be a scheme of finite type over $\mathbb{Z}$. We define the following condition: Suppose there exists a polynomial $N(x)$ with integer coefficients such that for every prime $p$ and every $n \in \mathbb{N}$, one has

$$\#X(\mathbb{F}_p^n) = N(p^n)$$

is a rational function in $p$ and $p^{-s}$. The vanishing order at $p = 1$ is $N(1)$. One may thus define

$$\zeta_X(\mathbb{Z})(s) = \prod_p Z_X(p, p^{-s})^{-1}.$$

Soulé considered in [10] the following condition: Suppose there exists a polynomial $N(x)$ with integer coefficients such that for every prime $p$ and every $n \in \mathbb{N}$, then

$$\zeta_X(\mathbb{F}_1)(s) = s^{a_0}(s-1)^{a_1} \cdots (s-n)^{a_n}.$$

In the paper [11] there is given a definition of a scheme over $\mathbb{F}_1$ as well as an ascent functor $\otimes \mathbb{Z}$ from $\mathbb{F}_1$-schemes to $\mathbb{Z}$-schemes. We say that a $\mathbb{Z}$-scheme is defined over $\mathbb{F}_1$, if it comes by ascent from a scheme over $\mathbb{F}_1$. The natural question arising is whether schemes defined over $\mathbb{F}_1$ satisfy Soulé’s condition.

Simple examples show that this is not the case. However, schemes defined over $\mathbb{F}_1$ satisfy a slightly weaker condition which serves the purpose of defining $\mathbb{F}_1$-zeta functions as well, and which we give in the following theorem.

Theorem 1 Let $X$ be a $\mathbb{Z}$-scheme defined over $\mathbb{F}_1$. Then there exists a natural number $\epsilon$ and a polynomial $N(x)$ with integer coefficients such that for every prime power $q$ one has

$$(q - 1, \epsilon) = 1 \Rightarrow \#X(\mathbb{F}_q) = N(q).$$

This condition determines the polynomial $N$ uniquely (independent of the choice of $\epsilon$). We call it the zeta-polynomial of $X$.

With this theorem, we can define the zeta function of an arbitrary $\mathbb{F}_1$-scheme $X$ as

$$\zeta_X(\mathbb{F}_1)(s) = s^{a_0}(s-1)^{a_1} \cdots (s-n)^{a_n},$$

if $N_X(x) = a_0 + a_1 x + \cdots a_n x^n$ is its zeta-polynomial.

We also define its Euler characteristic as

$$\chi(X) = N_X(1) = a_1 + \cdots + a_n.$$
This definition is due to Soulé [10]. We repeat the justification, which is based on the Weil conjectures.

Suppose that \( X/\mathbb{F}_p = X_\mathbb{Z} \times_\mathbb{Z} \mathbb{F}_p \) is a smooth projective variety over the finite field \( \mathbb{F}_p \). Then the Weil conjectures, as proven by Deligne, say that

\[
Z_{X_\mathbb{Z}}(p, T) = \prod_{k=0}^{m} (1 - p^k T)^{-a_k},
\]

satisfying \( |a_{l,j}| = p^{l/2} \), where \( b_l \) is the \( l \)-th Betti-number.

On the other hand, suppose that \# \( X(\mathbb{F}_p^n) = N(p^n) \) holds for every \( n \in \mathbb{N} \), where \( N(x) = a_0 + a_1 x + \cdots + a_n x^n \) is the zeta-polynomial, then one gets

\[
Z_{X_\mathbb{Z}}(p, T) = \prod_{k=0}^{n} (1 - p^k T)^{-a_k}.
\]

Comparing these two expressions, one gets

\[
b_l = \begin{cases} a_{l/2} & \text{if } l \text{ is even,} \\ 0 & \text{if } l \text{ is odd.} \end{cases}
\]

So \( \sum_{l=0}^{n} a_k = \sum_{l=0}^{n} (-1)^l b_l \) is the Euler characteristic.

For explicit computations of zeta functions and Euler numbers over \( \mathbb{F}_1 \), see [6].

Next for K-theory. Based on the idea of Tits, that \( \text{GL}_n(\mathbb{F}_1) \) should be the permutation group \( \text{Per}(n) \), Soulé also suggested that

\[
K_i(\mathbb{F}_1) = \pi_i(B(\text{Per}(\infty))^+),
\]

which is known to coincide with the stable homotopy group of the spheres, \( \pi_i^s = \lim_{k \to \infty} \pi_{i+k}(S^k) \).

(The + refers to Quillen’s + construction.) More general, for a monoid \( A \), or an \( \mathbb{F}_1 \)-ring \( \mathbb{F}_A \), one has

\[
\text{GL}_n(A) = \text{GL}_n(\mathbb{F}_A) = A^n \rtimes \text{Per}(n).
\]

Setting \( \text{GL}(A) = \lim_{n \to \infty} \text{GL}_n(A) \), one lets

\[
K_i^+(A) = \pi_i(B\text{GL}(A))^+.
\]

On the other hand, one considers the category \( \mathcal{P} \) of all finitely generated projective modules over \( A \) and defines

\[
K_i^{Q}(A) = \pi_{i+1}(BQ\mathcal{P}),
\]

where \( Q \) means Quillen’s \( Q \)-construction. It turns out that \( \pi_1(BQ\mathcal{P}) \) coincides with the Grothendieck group \( K_0(\mathcal{P}) \) of \( \mathcal{P} \). If \( A \) is a group, these two definitions of K-theory agree, but not in general.

A calculation shows, that if \( A \) is an abelian group, then

\[
K_i(A) = \begin{cases} \mathbb{Z} \times A & i = 0, \\ \pi_i^s & i > 0. \end{cases}
\]

So, for general \( A \), since one has \( K^+(A) = K^+(A^\times) \), this identity completely computes \( K^+ \). Furthermore, for every \( A \) one has a canonical homomorphism \( K^+_i(A) \to K^{Q}_i(A) \).

I thank Jeff Lagarias for his remarks on an earlier version of this paper.

### 1 \( \mathbb{F}_1 \)-schemes

For basics on \( \mathbb{F}_1 \)-schemes we refer to [11].

In this paper, a ring will always be commutative with unit and a monoid will always be commutative. An ideal \( a \) of a monoid \( A \) is a subset with \( Aa \subset a \). A prime ideal is an ideal \( \mathfrak{p} \) such that \( S_\mathfrak{p} = A \setminus \mathfrak{p} \) is a submonoid of \( A \). For a prime ideal \( \mathfrak{p} \) let \( A_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1} A \) be the localization at \( \mathfrak{p} \). The spectrum of a monoid \( A \) is the set of all prime ideals with the obvious Zariski-topology (see [11]).

Similar to the theory of rings, one defines a structure sheaf \( O_X \) on \( X = \text{spec}(A) \), and one defines a scheme over \( \mathbb{F}_1 \) to be a topological space together with a sheaf of monoids, locally isomorphic to spectra of monoids.

A \( \mathbb{F}_1 \)-scheme \( X \) is of finite type, if it has a finite covering by affine schemes \( U_i = \text{spec}(A_i) \) such that each \( A_i \) is finitely generated.

For a monoid \( A \) we let \( A \otimes \mathbb{Z} \) be the monoidal ring \( \mathbb{Z}[A] \). This defines a functor from monoids to rings which is left adjoint to the forgetful functor that sends a ring \( R \) to the multiplicative monoid \( (R, \times) \). This construction is compatible with gluing, so one gets a functor \( X \mapsto X_{\mathbb{Z}} \) from \( \mathbb{F}_1 \)-schemes to \( \mathbb{Z} \)-schemes.

**Lemma 2** \( X \) is of finite type if and only if \( X_{\mathbb{Z}} \) is a \( \mathbb{Z} \)-scheme of finite type.

**Proof:** If \( X \) is of finite type, it is covered by finitely many affines \( \text{spec}(A_i) \), where \( A_i \) is finitely generated, hence \( \mathbb{Z}[A_i] \) is finitely generated as a \( \mathbb{Z} \)-algebra and so it follows that \( X_{\mathbb{Z}} \) is of finite type.

Now suppose that \( X_{\mathbb{Z}} \) is of finite type. Consider a covering of \( X \) by open sets of the form \( U_i = \text{spec}(A_i) \), then one gets an open covering of \( X_{\mathbb{Z}} \) by sets of the form \( \text{spec}(\mathbb{Z}[A_i]) \), with the spectrum in the ring-sense. Since \( X_{\mathbb{Z}} \) is compact, we may assume this covering finite. As \( X_{\mathbb{Z}} \) is of finite type, each \( \mathbb{Z}[A_i] \) is a finitely generated \( \mathbb{Z} \)-algebra. Let \( S \) be a generating set of \( A_i \). Then it generates \( \mathbb{Z}[A_i] \),
2 Proof of Theorem

We will show uniqueness first.

Lemma 3 For every natural number \( e \) there are infinitely many prime powers \( q \) with \((q - 1, e) = 1\).

Proof: Write \( e = 2^k m \) where \( m \) is odd. Let \( n \in \mathbb{N} \). The number \( 2^n \) is a unit modulo \( m \) and hence there are infinitely many \( n \) such that \( 2^n \equiv 1 \) modulo \( m \). Replacing \( n \) by \( n + 1 \) we see that there are infinitely many \( n \) such that \( 2^n \equiv 2 \) modulo \( m \) and hence \( 2^n - 1 \equiv 1 \) modulo \( m \). As \( 2^n - 1 \) is odd, it follows \((2^n - 1, e) = 1\) for every such \( n \).

Now for the uniqueness of \( N \). Suppose that the pairs \((e, N)\) and \((e', N')\) both satisfy the theorem. Then for every prime power \( q \) one has

\[(q - 1, ee') = 1 \Rightarrow N(q) = \# X(\mathbb{F}_q) = N'(q).\]

As there are infinitely many such prime powers \( q \), it follows that \( N(x) = N'(x) \), as claimed.

We start on the existence of \( N \). For a finite abelian group \( E \) define its exponent \( m = \exp(E) \) to be the smallest number \( m \) such that \( x^m = 1 \) for every \( x \in G \). The exponent is the least common multiple of the orders of elements of \( G \). A finitely generated abelian group \( G \) is of the form \( \mathbb{Z}^r \times E \) for a finite group \( E \). Then \( r \) is called the rank of \( G \) and the exponent of \( E \) is called the exponent of \( G \).

For a finitely generated monoid \( A \) we denote by \( \text{Quot}(A) \) its quotient group. This group comes about by inverting every element in \( A \). It has a natural morphism \( A \to \text{Quot}(A) \) and the universal property that every morphism from \( A \) to a group factors uniquely over \( A \to \text{Quot}(A) \). In the language of \( \mathbb{N} \), \( \text{Quot}(A) \) coincides with the stalk arg \( A \) at the generic point \( \eta \) of \( \text{spec}(A) \).

We define the rank and exponent to be the rank and exponent of \( \text{Quot}(A) \). Note that for a finitely generated monoid \( A \) the spectrum \( \text{spec}(A) \) is a finite set. Hence the underlying space of a scheme \( X \) over \( \mathbb{F}_1 \) of finite type is a finite set. We then define the exponent of \( X \) to be the least common multiple of the numbers \( \exp(\mathcal{O}_p) \), where \( p \) runs through the finite set \( X \).

Let \( X \) be a scheme over \( \mathbb{F}_1 \) of finite type. We may assume that \( X \) is connected. Let \( e \) be its exponent. Let \( q \) be a prime power and let \( D_q \) be the monoid \( \langle \mathbb{F}_q, \times \rangle \). Then \( \# X(\mathbb{F}_q) = \# X(D_q) \), where \( X(D) = \text{Hom}(D, X) \) as usual. For an integer \( k \geq 2 \) let \( C_{k-1} \) denote the cyclic group of \( k - 1 \) elements and let \( D_k \) be the monoid \( C_{k-1} \cup \{0\} \), where \( x \cdot 0 = 0 \). Note that if \( q \) is a prime power, then \( D_q \cong \langle \mathbb{F}_q, \times \rangle \), where \( \mathbb{F}_q \) is the field of \( q \) elements.

Fix a covering of \( X \) by affines \( U_i = \text{spec}A_i \). Since \( \text{spec}(D_k) \) consists of two points, the generic, which always maps to the generic point and the closed point, it follows that

\[ X(\text{spec}(D_k)) = \bigcup U_i(\text{spec}(D_k)), \]

and thus the cardinality of the right hand side may be written as an alternating sum of terms of the form

\[ \# U_{i_1} \cap \cdots \cap U_{i_k}(\text{spec}(D_k)). \]

Now \( U_{i_1} \cap \cdots \cap U_{i_k} \) is itself a union of affines and so this term again becomes an alternating sum of similar terms. This process stops as \( X \) is a finite set. Therefore, to prove the theorem, it suffices to assume that \( X \) is affine.

So we assume that \( X = \text{spec}(A) \) for a finitely generated monoid \( A \). In this case \( X(\text{spec}(D_k)) = \text{Hom}(A, D_k) \). For a given monoid morphism \( \varphi : A \to D_k \) we have that \( \varphi^{-1} \{0\} \) is a prime ideal in \( A \), call it \( p \). Then \( \varphi \) maps \( S_p = A - \{p\} \) to the group \( C_{k-1} \). So \( \text{Hom}(A, D_k) \) may be identified with the disjoint union of the sets \( \text{Hom}(S_p, C_{k-1}) \) where \( p \) ranges over \( \text{spec}(A) \). Now \( C_{k-1} \) is a group, so every homomorphism from \( S_p \) to \( C_{k-1} \) factorises over the quotient group \( \text{Quot}(S_p) \) and one gets \( \text{Hom}(S_p, C_{k-1}) = \text{Hom}(\text{Quot}(S_p), C_{k-1}) \). Note that \( \text{Quot}(S_p) \) is the group of units in the stalk \( \mathcal{O}_{\eta, p} \) of the structure sheaf, therefore does not depend on the choice of the affine neighbourhood. The group \( \text{Quot}(S_p) \) is a finitely generated abelian group. Let \( r \) be its rank and \( e \) its exponent. If \( e \) is coprime to \( k - 1 \), then there is no non-trivial homomorphism from the torsion part of \( \text{Quot}(S_p) \) to \( C_{k-1} \) and so in that case \( \# \text{Hom}(S_p, C_{k-1}) = (k - 1)^r \cdot \). This proves the existence of \( e \) and \( N \) and finishes the proof of Theorem.

Remark 1. We have indeed proven more than Theorem. For an \( \mathbb{F}_1 \)-scheme \( X \) of finite type we define \( X(\mathbb{F}_q) = \text{Hom}(\text{spec}(\mathbb{F}_q), X) \), where the Hom takes place in the category of \( \mathbb{F}_1 \)-schemes, and \( \mathbb{F}_q \) stands for the multiplicative monoid of the finite field. It follows that

\[ X(\mathbb{F}_q) \cong X_{\mathbb{Z}}(\mathbb{F}_q). \]

Further, for \( k \in \mathbb{N} \) one sets \( \mathbb{F}_k = D_k \) then this notation is consistent and we have proven above,

\[(k - 1, e) = 1 \Rightarrow \# X(\mathbb{F}_k) = N(k),\]

where \( e \) now is a well defined number, the exponent of \( X \). Further it follows from the proof, that
the degree of $N$ is at most equal to the rank of $X$, which is defined as the maximum of the ranks of the local monoids $O_p$, for $p \in X$.

**Remark 2.** As the proof of Theorem 1 shows, the zeta-polynomial $N_X$ of $X$, does actually not depend on the structure sheaf $O_X$, but on the subsheaf of units $O_X^\times$, where for every open set $U$ in $X$ the set $O_X^\times(U)$ is defined to be the set of sections $s \in O_X(U)$ such that $s(p)$ lies in $O_X^\times_p$ for every $p \in U$. We therefore call $O_X^\times$ the *zeta sheaf* of $X$.

## 3 K-theory

In this section we give two definitions of K-theory over $\mathbb{F}_1$ and we show that they do coincide for groups, but not in general. This approach follows Quillen [9].

### 3.1 The $+$-construction

Let $A$ be a monoid. Recall from [1] that $\text{GL}_n(A)$ is the group of all $n \times n$ matrices with exactly one non-zero entry in each row and each column, and this entry being an element of the unit group $A^\times$. We also write $A^\times$ as the stalk $A_\{\}$ at the closed point $c$ of $\text{spec}(A)$. In other words, we have

$$\text{GL}_n(A) \cong A^n \rtimes \text{Per}(n),$$

where $\text{Per}(n)$ is the permutation group in $n$ letters, acting on $A^n$ by permuting the co-ordinates.

There is a natural embedding $\text{GL}_n(A) \hookrightarrow \text{GL}_{n+1}(A)$ by setting the last co-ordinate equal to 1. We define the group

$$\text{GL}(A) \equiv \lim_{\longrightarrow} \text{GL}_n(A).$$

Similar to the K-theory of rings [9] for $j \geq 0$ we define

$$K_j^+(A) \equiv \pi_j(\text{BGL}(A)^+),$$

where $\text{BGL}(A)$ is the classifying space of $\text{GL}(A)$, the $+$ signifies the $+$-construction, and $\pi_j$ is the $j$-th homotopy group. For instance, $K_1^+(\mathbb{F}_1)$ is the $j$-th stable homotopy group of the spheres [8].

### 3.2 The $\Sigma$-construction

A category is called *balanced*, if every morphism which is epi and mono, already has an inverse, i.e., is an isomorphism.

Let $C$ be a category. An object $I \in C$ is called *injective* if for every monomorphism $M \rightarrow N$ the induced map $\text{Mor}(N, I) \rightarrow \text{Mor}(M, I)$ is surjective. Conversely, an object $P \in C$ is called *projective* if for every epimorphism $M \rightarrow N$ the induced map $\text{Mor}(P, M) \rightarrow \text{Mor}(P, N)$ is surjective. We say that $C$ has enough injectives if for every $A \in C$ there exists a monomorphism $A \hookrightarrow I$, where $I$ is an injective object. Likewise, we say that $C$ has enough projectives if for every $A \in C$ there is an epimorphism $P \twoheadrightarrow A$ with $P$ projective.

A category $C$ is pointed if it has an object 0 such that for every object $X$ the sets $\text{Mor}(X, 0)$ and $\text{Mor}(0, X)$ have exactly one element each. The zero object is uniquely determined up to unique isomorphism. In every set $\text{Mor}(X, Y)$ there exists a unique morphism which factorises over the zero object, this is called the zero morphism. In a pointed category it makes sense to speak of kernels and cokernels. Kernels are always mono and cokernels are always epimorphisms. A sequence

$$0 \rightarrow X \xrightarrow{i} Y \xrightarrow{j} Z \rightarrow 0$$

is called *strong exact*, if $i$ is the kernel of $j$ and $j$ is the cokernel of $i$. We say that the sequence *splits*, if it is isomorphic to the natural sequence

$$0 \rightarrow X \rightarrow X \oplus Z \rightarrow Z \rightarrow 0.$$ 

Assume that kernels and cokernels always exist. Then every kernel is the kernel of its cokernel and every cokernel is the cokernel of its kernel. For a morphism $f$ let $\text{im}(f) = \ker(\text{coker}(f))$ and $\text{coim}(f) = \text{coker}(\ker(f))$. If $C$ has enough projectives, then the canonical map $\text{im}(f) \rightarrow \text{coim}(f)$ has zero kernel and if $C$ has enough injectives, then this map has zero cokernel.

Let $C$ be a pointed category and $\mathcal{E}$ a class of strong exact sequences. The class $\mathcal{E}$ is called *closed under isomorphism*, or simply *closed* if every sequence isomorphic to one in $\mathcal{E}$, lies in $\mathcal{E}$. Every morphism occurring in a sequence in $\mathcal{E}$ is called an $\mathcal{E}$-morphism.

A balanced pointed category $C$, together with a closed class $\mathcal{E}$ of strong exact sequences is called a *quasi-exact category* if

- for any two objects $X, Y$ the natural sequence
  $$0 \rightarrow X \rightarrow X \oplus Y \rightarrow Y \rightarrow 0$$
  belongs to $\mathcal{E}$,

- the class of $\mathcal{E}$-kernels is closed under composition and base-change by $\mathcal{E}$-cokernels, likewise, the class of $\mathcal{E}$-cokernels is closed under composition and base change by $\mathcal{E}$-kernels.

Let $(C, \mathcal{E})$ be a quasi-exact category. We define the category $QC$ to have the same objects as $C$, but a morphism $f : X \rightarrow Y$ is a morphism in $QC$ if and only if $f$ is in $\mathcal{E}$. The composition in $QC$ is the same as in $C$.
morphism from $X$ to $Y$ in $QC$ is an isomorphism class of diagrams of the form

$$
\begin{array}{ccc}
S & \rightarrow & Y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
X & \rightarrow & Y,
\end{array}
$$

where the horizontal map is a $E$-kernel in $C$ and the vertical map is a $E$-cokernel. The composition of two $Q$-morphisms

$$
\begin{array}{ccc}
S & \rightarrow & Y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
X & \rightarrow & Y,
\end{array}
\quad
\begin{array}{ccc}
T & \rightarrow & Z \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
Y & \rightarrow & Z,
\end{array}
$$

is given by the base change $S \times_Y T$ as follows,

$$
\begin{array}{ccc}
S \times_Y T & \rightarrow & T \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
S & \rightarrow & Y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
X & \rightarrow & Y.
\end{array}
$$

Every $E$-kernel $i : X \rightarrow Y$ gives rise to a morphism $i_1$ in $QC$, and every $E$-cokernel $p : Z \rightarrow Z$ gives rise to a morphism $p' : X \rightarrow Z$ in $QC$. By definition, every morphism in $QC$ factorizes as $i p'$ uniquely up to isomorphism.

Let $(C, E)$ be a small quasi-exact category. Then the classifying space $BQC$ is defined. Note that for every object $X$ in $QC$ there is a morphism from $0$ to $X$, so that $BQC$ is path-connected. We consider the fundamental group $\pi_1(BQC)$ as based at a zero $0$ of $C$.

**Theorem 4** The fundamental group $\pi_1(BQC)$ is canonically isomorphic to the Grothendieck group $K_0(C) = K_0(C, E)$.

**Proof:** This proof is taken from [2], where it is done for exact categories, we repeat it for the convenience of the reader. The Grothendieck group $K_0(C, E)$ is the abelian group with one generator $[X]$ for each object $X$ of $C$ and a relation $[X] = [Y][Z]$ for every strong exact sequence

$$
\begin{array}{ccc}
0 & \rightarrow & Y \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
X & \rightarrow & Z \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
0 & \rightarrow & 0
\end{array}
$$

in $E$. According to Proposition 1 of [2], it suffices to show that for a morphism-inverting functor $F : QC \rightarrow Sets$ the group $K_0(C)$ acts naturally on $F(0)$ and that the resulting functor from the category $F$ of all such $F$ to $K_0(C)$-sets is an equivalence of categories.

For $X \in C$ let $i_X$ denote the zero kernel $0 \rightarrow X$, and let $j_X$ be the zero cokernel $X \rightarrow 0$. Let $F'$ be the full subcategory of $F$ consisting of all $F$ such that $F(X) = F(0)$ and $F(i_X) = \text{id}_{F(0)}$ for every $X$. Any $F \in F$ is isomorphic to an object of $F'$, so it suffices to show that $F'$ is equivalent to $K_0(C)$-sets. So let $F \in F'$, for a kernel $i : X \rightarrow Y$ we have $i i_X = i_Y$, so that $F(i) = \text{id}_{F(0)}$. Given a strong exact sequence

$$
\begin{array}{ccc}
0 & \rightarrow & X \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
Y & \rightarrow & Z \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
0 & \rightarrow & 0,
\end{array}
$$

we have $j' i_{Z'}^! = i_{j_X}^!$, hence $F(j') = F(j_X^!)$ in $\text{Aut}(F(0))$. Also,

$$
F(j_X^!) = F(j_X^!j_Z^!) = F(j_X^!)F(j_Z^!).
$$

So by the universal property of $K_0(C)$, there is a unique homomorphism from $K_0(C)$ to $\text{Aut}(F(0))$ such that $[X] \mapsto F(j_X^!)$. So we have a natural action of $K_0(C)$ on $F(0)$, hence a functor from $F'$ to $K_0(C)$-sets given by $F \mapsto F(0)$.

The other way round let $S$ be a $K_0(C)$-set, and let $F_S : QC \rightarrow Sets$ be the functor defined by $F_S(X) = S$, $F_S(i_{j(Y)})$ is multiplication by $[\ker j]$ on $S$. To see that this is indeed a functor, it suffices to show that $F_S(j' i_{Z'}) = F_S(j')$. It holds $j_i = i_{j_S} j_1$, where $i_1$ and $j_1$ are given by the cartesian diagram

$$
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
Z & \rightarrow & Y.
\end{array}
$$

It follows $F_S(j' i_{Z'}) = F_S(i_{j_S} j_1) = [\ker j_1]$. Using the cartesian diagram one sees that $\ker j_1$ is isomorphic to $\ker j$. It is easy to verify that the two functors given are inverse to each other up to isomorphism, whence the theorem.

□

This theorem motivates the following definition,

$$
K_i(C, E) \overset{\text{def}}{=} \pi_{i+1}(BQC).
$$

For a monoid $A$ we let $P$ be the category of finitely generated pointed projective $A$-modules, or rather a small category equivalent to it, and we set

$$
K^0_i(A) \overset{\text{def}}{=} K_i(P, E),
$$

where $E$ is the class of sequences in $P$ which are strong exact in the category of all modules. These categories are considered with the cartesian closed structure.
sequences all split, which establishes the axioms for a quasi-exact category.

The two $K$-theories we have defined, do not coincide. For instance for the monoid of one generator $A = \{1, a\}$ with $a^2 = a$ one has

$$K^+_{i}(A) = \mathbb{Z}, \quad K_{i}^{Q}(A) = \mathbb{Z} \times \mathbb{Z}.$$  

The reason for this discrepancy is that $K^+_{i}(A)$ only depends on the group of units $A^\times$, but $K_{i}^{Q}(A)$ is sensible to the whole structure of $A$. So these two $K$-theories are unlikely to coincide except when $A$ is a group, in which case they do, as the last theorem of this paper shows,

**Theorem 5** If $A$ is an abelian group, then $K^+_{i}(A) = K_{i}^{Q}(A)$ for every $i \geq 0$.

**Proof:** For a group each projective module is free, hence the proof of Grayson [3] of the corresponding fact for rings goes through. \[\square\]

So, if $A$ is a group, this defines $K_i(A)$ unambiguously. In particular, computations of Priddy [8] show that $K_i(\mathbb{F}_1) = \pi_i^*$ is the $i$-th stable homotopy group of the spheres. Based on this, one can use the $\mathbb{Q}$-construction to show that if $A$ is an abelian group, then

$$K_i(A) = \begin{cases} \mathbb{Z} \times A & i = 0, \\ \pi_i^* & i > 0. \end{cases}$$

For an arbitrary monoid $A$ we conclude that $K^+_{i}(A) = K^+_{i}(A^\times) = K_{i}(A^\times)$, which we now can express in terms of the stable homotopy groups $\pi_i^*$.

Further, for every $A$ one has a canonical homomorphism $K^+_{i}(A) \to K_{i}^{Q}(A)$ given by the map $K^{Q}(A^\times) \to K^{Q}(A)$. The latter comes about by the fact that every projective $A^\times$-module is free. Note that general functoriality under monoid homomorphism is granted for $K^+$, but not for $K^{Q}$. This contrasts the situation of rings, and has its reason in the fact that not every projective is a direct summand of a free module.

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