Magnetic fields are a very special form of elastic medium. Within astrophysical environments (magnetized stars and protogalaxies) they counteract shear and rotational distortions as well as gravitational collapse. Their vector nature allows for their extraordinary coupling with space–time curvature in the framework of general relativity. This particular coupling points out the way to study magnetic elasticity under gravitational deformation. In this context, we reveal their law of elasticity, calculate their fracture limit and subsequently argue that they ultimately lose the battle against gravitational contraction of magnetized matter. Two illustrative applications, in a neutron star and a white dwarf, accompany the results.

1. Introduction

It is well known (mainly from astrophysical studies of magnetized fluids, e.g. see [1,2], but from relativistic as well [3]) that magnetic forcelines behave like an elastic medium under their kinematic (shear or rotational) deformation. Namely, in analogy with a spring under pressure they develop tension stresses resisting their deflection. However, it is less known how to achieve a theoretical description of (elastic) magnetic distortion due to gravity [4–8] (namely space–time curvature within general relativity). In particular, the aforementioned studies have shown that the elastic behaviour in question is expressed through a magneto-curvature tension stress coming from the Ricci identities. In fact, due to their vector nature, magnetic fields present a double coupling with space–time curvature, not only via Einstein’s
equations but via the Ricci identities as well. Thus, from a relativistic point of view it has been found out that interestingly magnetic forcelines do not self-gravitate [3,7,9]. Moreover, they counteract gravitational implosion of a highly conducting fluid and potentially hold it up [5–8,10]. In reference to this problem and given the elastic behaviour of magnetic fields, one can raise the question concerning the existence of a possible elastic and a fracture magnetic limit. Moreover, if such a fracture limit exists, could one provide an estimation of it for a given magnetized (collapsing) star? Another crucial question associated with the previous ones is whether magnetic forcelines manage to disrupt gravitational collapse before reaching their fracture limit [8].

Addressing the above questions through an insightful introduction to the gravito-magnetic law of elasticity basically forms the object of the present piece of work, motivated by Mavrogiannis & Tsagas [8]. In detail, we begin with a brief presentation and mathematical description of the kinematically induced magnetic tension stresses. Then, we focus our attention on the magneto-curvature tension stress and reveal the law of magnetic elasticity under gravitational distortions. Subsequently, we move on to our principal task which consists of a theoretical calculation of the magnetic fracture limit during the gravitational collapse of magnetized matter. The aforementioned limit, illustrated by two examples, of a neutron star and a white dwarf, is ultimately used to argue that magnetic fields are not able to impede gravitational contraction before being broken. As far as we know, the results (intuitive presentation of the gravito-magnetic law of elasticity, calculation of the magnetic fracture limit and its applications) are new, appearing here for the first time in the literature.

2. Kinematically induced magnetic tension stresses

To begin with, let us consider the decomposition of the magnetic three-dimensional gradient $D_b B_a$ into its symmetric (trace-free), antisymmetric and trace part. In other words,

$$D_b B_a = D_{[b} B_{a]} + \frac{1}{3} (D^c B_c) h_{ab},$$

(2.1)

which reveals the individual tension components triggered by, and resistant to shape (i.e. $\sigma_{ab}^{(B)} = D_{[b} B_{a]}$), rotational (i.e. $\omega_{ab}^{(B)} = D_{[b} B_{a]}$) and volume distortions (i.e. $\Theta_{ab}^{(B)} = D^c B_c$) of the magnetic forcelines, respectively. Besides, at the magnetohydrodynamic limit (MHD) the tension component opposing to volume expansion/contraction (last term) vanishes (i.e. $D^c B_c = 0$ from Gauss’s law). In the above $D_a = h^b_{[a} \nabla_b$ is the projected (three-dimensional) covariant derivative operator and $h_{ab} = g_{ab} + u^a u^b$ (with $g_{ab}$ being the space–time metric and $u^a$ being a timelike four-velocity vector) an operator projecting upon the observer’s (three-dimensional) rest-space. The covariant kinematics of the magnetic tension stresses are monitored by the Ricci identities for the magnetic field

$$2 \nabla_a \nabla_b B_c = R_{abcd} B^d,$$

(2.2)

where $R_{abcd}$ is the Riemann space–time tensor. In particular, the timelike part of the above leads to propagation equations for the magnetic shear $\sigma_{ab}^{(B)}$ and vorticity $\omega_a^{(B)} = \epsilon_{abc} \omega^b c$. On the other hand, its spacelike part leads to divergence conditions (constraints) for the aforementioned quantities. The equations in question, appearing here for the first time as far as we know, could prove useful when studying the kinematics of magnetized fluids in various contexts. However, as we do not make any use of those in the present manuscript, we have chosen to place them in the brief appendix A.

3. Gravitationally induced magnetic tension stresses

In analogy with their deflection due to kinematic effects associated with the fluid’s motion, magnetic forcelines counteract their gravitational distortion. Where does the corresponding

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1 Actually the magnetic tension force vector refers to the directional derivative along the field itself. See the following analysis.
magneto-curvature tension stress come from? The answer lies in the direct coupling of magnetic fields (as vectors) with spatial curvature via the (three-dimensionally projected) Ricci identities (e.g. see [4] or [11]),

\[ 2D_{[a}B_{b]}B_{c} = -2\omega_{ab}B_{(c)} + R_{dcbg}B_{d}^g, \]  

(3.1)

where \( R_{abcd} \) represents the three-dimensional counterpart of the Riemann tensor. Note that the aforementioned coupling manifests itself at the second differentiation order.

(a) Describing the magneto-curvature tension stress

Let us consider the three-gradient of the magnetic tension force vector \( \tau_a = B^bD_bB_a \) (the non-zero tension force implies that the magnetic fieldlines are not spacelike geodesics). Employing the three-dimensional Ricci identities (3.1) we arrive at

\[ D_c\tau_a = D_cB^bD_bD_a + B^bD_bD_cB_a + 2\omega_{bc}B^bB_{(a)} + R_{dabc}B^bB^d. \]  

(3.2)

The first three terms involve kinematic effects through equation (2.1) while the last one can be envisaged as the magneto-curvature tension stress (or the gradient of the magneto-curvature tension component). If \( n^a \) is the magnetic field direction (i.e. \( B^a = Bn^a \)), the term in question can alternatively be written as

\[ s_{ac} = R_{dabc}B^bB^{d} = B^2 R_{dabc}n^b n^d = -B^2 u_{ac}, \]  

(3.3)

where \( u_{ac} \equiv -R_{dabc}n^b n^d \) can be envisaged as a kind of strain tensor, describing spatial distortions of the magnetic forcelines (for a commentary on the law of magnetic elasticity under volume gravitational distortions refer to §4). Our definition for the strain tensor is metric independent and thus essentially differs from its counterpart (4.9) in [13]. As for the stress tensor \( s_{ab} \), it includes those forces which act against (see the following discussion on the problem of gravitational collapse) spatial curvature and tend to restore the forcelines to their initial state. Overall, the meaning of (3.3) is the following. Due to spatial curvature, the magnetic fieldlines are bent and twisted. In analogy with an elastic rod under pressure, they react via the restoring stress \( s_{ab} \) which increases in proportion to the amount of deformation \( u_{ab} \) (Hooke’s law of elasticity) and the magnetic density. In fact, when appearing in the kinematic equations for a magnetized fluid, it turns out that the magneto-curvature tension stress depends on the ratio of the magnetic density over the total system’s density (i.e. matter and magnetic fields—see equation (4.1) in the following).

Let us recall that any kind of deformation can be reduced into a sum of a pure shear \( (u_{ab}) = R_{d(ab)c}n^c n^d \), a torsional one (or twisting) \( (u_{[ab]} = R_{d[ab]c}n^c n^d) \) and a hydrostatic compression \( (u^c/3)h_{ab} = (1/3)R_{c[d}n^c n^d h_{ab}) \). Hence, on splitting the magneto-curvature tension stress into its symmetric-trace-free \( (s_{(ac)}) \), antisymmetric \( (s_{[ac]} \) and trace part \( (s = s_c) \), we receive its associated component counteracting shape, rotational and volume changes, respectively, due to gravity (see appendix C). Of the aforementioned components we focus here on the last one. Hence, considering the trace of (3.3) and the double projection of the Gauss–Codacci formula (e.g. see eq. (1.3.39) in [11] and eq (92) in [8]) along the magnetic direction \( n^a \), we deduce that

\[ s = s_c = B^2 R_{bd}n^bn^d = B^2 \left( \frac{2}{3} \rho + \varepsilon + \frac{\Pi}{2} \right), \]  

(3.4)

where \( \rho \) denotes the energy density of matter; \( \Pi \equiv \Pi_{ab} n^a n^b \) and \( \varepsilon \equiv E_{ab} n^a n^b \) are the anisotropic stress \( \pi_{ab} \) and the tidal (or electric Weyl) tensor \( E_{ab} \), twice projected along the magnetic direction, respectively. In deriving the above we have employed equations (B 2) and (B 5) from the appendix. Also, assuming an ideal fluid model, the anisotropic stress terms in the above vanish. Then,

\[ \text{2Typically, within conventional elastic mechanics the strain tensor is defined to be a dimensionless symmetric quantity [12]. However, here we allow for a non-vanishing antisymmetric part taking into account any torsional deformation. Furthermore, our strain tensor has inverse square length dimensions in geometrized units.} \]

\[ \text{3For the sake of accuracy, vorticity or rotational deformations are included in the shear/shape type of distortions as well.} \]
Interestingly, we observe that closed spatial distortions (of magnetic deformation and the associated tension (presented above) opposing to it. In each case, the associated tension term turns out to have opposite sign to its triggering source. Here, being especially interested in the problem of magnetized collapse, we focus on the volume scalar propagation equations (e.g. see \([11]\)). In fact, recalling that \(\rho_{(\text{magn})} = B^2/2\), \(\pi_{ab}^{(\text{magn})} = -\delta_{ab}B_b\) and therefore \(\Pi_{ab}^{(\text{magn})} \equiv \pi_{ab}^{(\text{magn})} n^an^b = -(2/3)B^2\), it turns out that the magnetic anisotropic stress exactly cancels the magnetic energy density contribution (see also [7]).

### (b) Magneto-curvature tension and gravitational collapse

Now having in hand the expression for the magneto-curvature tension stress, how can we reveal its competitive behaviour towards the corresponding cause of magnetic deformation? An illustrative description can be achieved by making use of the shear, vorticity and volume scalar propagation equations (e.g. see [11]). In each case, the associated tension term turns out to have opposite sign to its triggering source. Here, being especially interested in the problem of magnetized collapse, we focus on the volume scalar propagation equation, known as Raychaudhuri’s equation, in combination with Euler’s equation for a magnetized fluid. More specifically, for an ideal fluid the latter reads (e.g. see [6,7])

\[
\rho + P + B^2 \hat{u}_a = D_aP - \frac{1}{2}D_aB^2 + B^bD_b\hat{u}_a + \hat{u}^bB_b\hat{u}_a, \tag{3.5}
\]

where the second and third term in the right-hand side split the magnetic Lorentz force into its pressure and tension component, respectively (\(\rho \) and \(P \) refer to the density and pressure of matter). Taking, for simplicity, the divergence of the above under the assumption of partial homogeneity \(4\) (i.e. \(D_a\rho \simeq 0 \simeq D_aP\) and \(D_aB^2 \simeq 0\) but \(D_aB_b \neq 0\)), and taking into account equation (3.1), it turns out that (e.g. refer [6,7])

\[
D^a\hat{u}_a = c_A^2R_{\alpha\beta}n^\alpha n^\beta + 2(\sigma_B^2 - \omega_B^2). \tag{3.6}
\]

Note that \(s^a \equiv c_A^2(R_{\alpha\beta}n^\alpha n^\gamma)\) (with \(c_A^2 \equiv B^2/(\rho + P + B^2)\) being the Alfvén speed) actually comes from (3.3), by contracting the indices \(a\) and \(c\), and dividing by \(\rho + P + B^2\) (it would have had the same form if we had assumed inhomogeneous contraction). In other words, it represents the magneto-curvature tension component opposing to volume (curvature induced) distortions of the magnetic force lines (or of the magnetized fluid). On the other hand, \(\sigma_B^2 \equiv D_{[a}B_{b]}D^{[b}B^{a]}/2(\rho + P + B^2)\) and \(\omega_B^2 \equiv D_{[a}B_{b]}D^{[a}B^{b]}/2(\rho + P + B^2)\) refer to the norms of magnetic tensions counteracting shape (shear) and rotational distortions, respectively.

Finally, the substitution of (3.6) into the well known Raychaudhuri formula, monitoring the magnetized fluid’s volume expansion/contraction, brings the latter into the intuitive form

\[
\dot{\Theta} + \frac{1}{2}\Theta^2 = -(R_{ab}u^au^b - c_A^2R_{ab}n^an^b) - 2(\sigma_B^2 - \omega_B^2) + \hat{u}^a\hat{u}_a. \tag{3.7}
\]

Each parenthesis in the above includes two opposite sign terms (negative sign terms in the right-hand side favour volume contraction while positive ones favour expansion [14]), the cause of magnetic deformation and the associated tension (presented above) opposing to it. Interestingly, we observe that closed spatial distortions (\(R_{ab}n^an^b > 0\), this is actually the case of stellar gravitational implosion) along the magnetic direction \(n^a\) give rise to stresses \(c_A^2R_{ab}n^an^b\) opposing to contraction while open spatial distortions (\(R_{ab}n^an^b < 0\)) generate magneto-curvature tension stresses reinforcing contraction. For the rest of this work we focus our attention on the first couple of terms (gravitational deformation of the magnetic field and its elastic reaction) in the right-hand side of equation (3.7).

\(4\) In fact, the assumption of partial homogeneity (a standard practice in singularity theorems) is not crucial regarding the core of our reasoning. In the following section, we examine the elasticity of the magnetic force lines under their volume gravitational bending. The cause of the magnetic bending (i.e. total gravitational energy) and its consequence (i.e. magnetic deformation) are explicitly independent of the partial homogeneity assumption.
4. The law of magnetic elasticity under (volume) gravitational distortions

With the present section, we move on to the essential part of our work. In detail, we discuss the law of magnetic elasticity, describe an enlightening analogy to magnetized collapse, calculate the magnetic fracture limit and apply it to the problem of gravitational collapse of compact stellar objects. The present section should be studied along with appendix B.

In reference to equation (3.7) we particularly observe that the magneto-curvature tension stress,
\[ s^* = -c^2_A u, \]  
(4.1)
counteracts the magnetized fluid’s gravity,
\[ R_{ab} u^a u^b = (1/2)(\rho + 3P + B^2) > 0, \]  
namely the cause of magnetic (volume) distortion in the form of \( u \equiv -R_{ab} u^a u^b \) (so that \( u < 0 \) is associated with closed spatial sections and compression). We plausibly require that \( R_{ab} u^a u^b > 0 \) (\( u < 0 \)) at all times during gravitational contraction. In complete analogy with (3.7), the symmetric-trace-free and the antisymmetric counterparts of (4.1) can be obtained via the shear and the vorticity propagation formulae, respectively, along with Euler’s equation of motion. However, here we examine the magnetic elasticity against gravitationally induced (volume) distortions.

(a) Insight into the law of magnetic elasticity

The meaning of (4.1)\(^5\) is that the tension stress \( s^* \), tending to restore the magnetic field into its initial (undeformed) state, is proportional to the distortion of the magnetic forcelines (refer to the appendices B and C)\(^6\)
\[ u \equiv -R_{ab} u^a u^b = -\left(\frac{2}{3}\right) \rho - E. \]  
(4.2)
The minus sign in the right-hand side of (4.1) implies that \( s^* \) acts against the increasing magnetic distortion \( u \). Note that in deriving the above we have taken into account eq. (1.3.41) from Tsagas et al.\(^7\) [11], as well as relations (B 2), (B 5) from the appendix B. It is straightforward to see that condition \( E = -(2/3)\rho \) corresponds to the natural (undeformed) ‘volume’ state of the magnetic field, where \( s^* = 0 \). The proportionality factor \( 0 < c^2_A < 1 \) (note the difference to (3.3)) is always positive and its definition implies that the greater the magnetic density contribution to the total fluid’s density, the more rigid the magnetic fieldlines are (or the more they resist to their deformation). In other words, equation (4.1) is a relativistic expression of Hooke’s law of elasticity for a gravitationally distorted magnetic field, frozen into a highly conducting fluid. Nevertheless, in contrast to an elastic spring, the proportionality factor \( c^2_A \) is not a constant but a variable quantity (a function of the ratio \( \rho/B^2 \)). Moreover, although Hooke’s law is an approximate relation valid for sufficiently small deformations, equation (4.1) seems to be valid for any deformation, given that the Ricci identities (3.1) hold. Therefore, from our point of view, magnetic fields appear to keep their elastic behaviour as well as to satisfy Hooke’s law of elasticity no matter how big their deformation is.

Even if magnetic forcelines do not present an elastic limit\(^7\) under their gravitational bending, one expects that they can support a finite amount of distortion. Thus, we expect that there must be at least a fracture limit of the magnetic fieldlines, predicted by the exact formula (4.1).\(^8\) The significance of such a limit becomes clear on considering for instance the astrophysical/cosmological phenomenon of magnetized gravitational collapse. In particular, magnetic fields are known not to self-gravitate as well as to have the potential to impede

\(^5\)The expression in question has appeared several times in past works (e.g. see [5–8]) but it was not recognized or envisaged as an expression of Hooke’s law of elasticity and therefore was not given its full interpretation presented here.

\(^6\)Written here for an ideal (magnetized) fluid.

\(^7\)The elastic limit refers to that value of distortion beyond which the elastic medium is unable to return to its initial state. Mathematically speaking, on setting the external forces equal to zero, the deformation becomes zero as well. Of course we do not know any such example of material in nature.

\(^8\)By contrast, Hooke’s law for elasticity, being a linear approximation—valid for small values of deformation—does not and could not contain such information.
gravitational implosion from reaching a space–time singularity. Before proceeding to a definition and theoretical calculation of the magnetic fracture limit under gravitational distortions, we present our approach to magnetized contraction through an analogy.

(b) Our approach to magnetized gravitational collapse—an illuminating analogy

We study gravitational collapse of a magnetized fluid. In other words, we study the contraction of a medium which behaves elastically along a specific direction \( n^a \parallel B^a \). The above statement points out the crucial difference between ‘ordinary’ and magnetized gravitational collapse, which we illustrate via the following analogue.9

Consider an elastic water balloon filled with water.10 In this case, water stands for the fluid while the elastic medium (balloon containing the water) stands for the magnetic field. Both the balloon and the magnetic field, force, in a sense, the water/fluid to behave elastically. Now imagine that we moderately tighten the water balloon with our hand (the tightening stands for gravitational contraction). For generally anisotropic tightening we will observe that the water balloon decreases in length along one or two specific directions while the remaining (two or one, respectively) directions experience expansion. This happens because the water balloon’s volume remains practically unchanged (it contains always the same water volume) and the individual dimensions, \( 1 + 2 \), are those which actually change. Ultimately, if the tightening becomes too big (the external pressure much greater than the elastic restoration force), the balloon breaks releasing the (no longer elastic) water. We expect that the collapsing magnetized fluid should present an analogue behaviour, with the exception of two essential differences. First, the magnetized fluid behaves elastically only along the magnetic direction. Second, its total volume changes. In fact, we approach the above described elasticity along the magnetic direction \( n^a \) by adopting the assumption \( u'^a = n^bD_bu_a = 0 \). (i.e. homogeneous velocity along the magnetic fieldlines–streamlines in this case). In particular, \( u'^a = 0 \) (i.e. homogeneous velocity along the magnetic fieldlines) implies that the magnetic force lines are envisaged as streamlines of the fluid. Crucially, the aforementioned assumption leads to \( \Sigma \equiv \sigma_{ab}n^an^b = -\Theta/3 \) (\( \Sigma \) appears in deriving equation (3.3) through the Gauss–Codacci formula—see (1.3.39) in [11]), and therefore to \( \Theta_{ab}n^an^b = 0 \), where \( \Theta_{ab} \equiv \sigma_{ab} + (\Theta/3)h_{ab} \) represents the volume expansion/contraction tensor. In parallel, note that motion along the magnetic direction \( n^a \) (i.e. \( B^a = Bn^a \)) is still possible due to non-vanishing acceleration and vorticity. Therefore, what does our picture of magnetized collapse, given by \( \Theta_{ab}n^an^b = 0 \) (recall that \( B^a \equiv Bn^a \)), look like? In accordance with the spirit of the above described analogy, one or two of the spatial directions orthogonal to \( n^a \) contract, while the fluid along \( n^a \) moves with acceleration determined by the magnetic stresses11. Finally, could the magnetic direction length increase or decrease unceasingly? Within the context of our analogy, the answer should be negative, and is given in the following subsections.

Overall, it seems that \( u'^a = n^bD_bu_a = 0 \) is an assumption imposed by the problem itself, namely by the elastic behaviour of the magnetized fluid along direction \( n^a \parallel B^a \).

(c) Magnetic fracture limit and gravitational contraction

In the first place, we claim that the fracture limit must correspond to a maximum of the magnetic deformation with respect to proper time. However, given that as the deformation of an elastic medium increases, so do the internal tension stresses acting against it; it often happens that the maximum deformation coincides with the maximum resisting tension stress (see the following §4d). In reference to our case, because the law of magnetic elasticity, equation (4.1), is valid

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9 The present section should be studied along with appendix B.

10 We assume that there is no water or air flow in or out of the balloon.

11 In general, the fluid rotates as well.
at all times during the collapse (for any deformation), it is necessary\textsuperscript{12} that the fracture limit corresponds to a double maximum, of the magnetic deformation and the magneto-curvature tension as well (i.e. $u_{fr} = 0 = \delta_{fr}$).\textsuperscript{13} Besides, a maximum of spatial deformation along a direction $n^a$, i.e. $u = -\mathcal{R}_{ab}n^an^b$, does not make physical sense by itself within the problem of magnetized gravitational collapse. The quantity $u$ should increase monotonically without ever reaching a maximum value. It is only through the magnetic field presence (with its elastic behaviour encoded by (4.1)), appearing in $s^a$ but not directly in $u$, that a maximum of the latter acquires physical beingness together with a parallel maximum of the former. Beyond that maximum value the fieldlines of the magnetized fluid are expected to be broken, so that eventually and discontinuously $s^a$ becomes zero. Within the present section, we consider the implications of the condition $u_{fr} = 0$. In the following section, we proceed to the full determination of the magnetic fracture limit.

Thus, in mathematical terms, we proceed to the differentiation of $u \equiv -\mathcal{R}_{ab}n^an^b$ with respect to proper time (associated with the fluid’s motion), which leads to

$$\dot{u} = -\frac{2}{3} \dot{\rho} - \dot{\mathcal{E}} = \frac{1}{2} \Theta (\rho + P + B^2 + 3E),$$

(4.3)

assuming an ideal fluid model and homogeneity of implosion. In particular, on deriving the above we have taken into account the continuity equation as well as the propagation equation for the tidal tensor, projected twice along the magnetic direction, namely (see appendix D)

$$\dot{\rho} = -\Theta (\rho + P) \quad \text{and} \quad \dot{\mathcal{E}} = \frac{3}{2} \Theta E + \frac{1}{6} \Theta (\rho + P - 3B^2).$$

(4.4)

Note that within the framework of magnetized gravitational contraction we require the satisfaction of the following conditions (i.e. positive curvature magnetic deformation and increasing magnetic distortions)

$$u < 0 \Leftrightarrow \mathcal{E} > -\frac{2}{3} \rho \quad \text{and} \quad \dot{u} > 0 \Leftrightarrow \mathcal{E} < -\frac{1}{3} (\rho + P + B^2).$$

(4.5)

Overall, $\mathcal{E}$ is required to satisfy the condition: $-(2/3)\rho < \mathcal{E} < -(1/3)(\rho + P + B^2)$, with $\rho > P + B^2$. Subsequently, setting equation (4.3) equal to zero, we find out that the function’s critical point corresponds to

$$\mathcal{E}_{fr} = -\frac{1}{3} (\rho + P + B^2)_{fr} < 0 \quad \text{and} \quad u_{fr} = -\frac{1}{3} (\rho - P - B^2)_{fr},$$

(4.6)

where the negative sign requirement of $u_{fr}$ implies the condition $\rho_{fr} > P_{fr} + B_{fr}^2$, in accordance with (4.5). Essentially, the critical point in question is determined by the values of the matter and magnetic density (a relation between $P$ and $\rho$ can always be assumed through an equation of state, see §4d). Now in reference to the problem of magnetized gravitational collapse we face the following question. \textit{Do magnetic fieldlines affect the fate of gravitational collapse? In particular, will they manage to impede contraction towards a singularity or will they be inevitably broken beforehand?} Our experience shows that under increasing external tension elastic media are ultimately broken. Let us examine the case of magnetic elasticity under increasing gravity. Envisaged under a cause-consequence (causal) perspective, the magneto-curvature tension $c_A^2 \mathcal{R}_{ab}n^an^b$ is the exclusive result of the system’s gravity $\mathcal{R}_{ab}u^au^b$ (recall equation (3.7)).\textsuperscript{14} Hence, our argument is the following: If $s^a$ turns out to be smaller than the fluid’s gravity at the fracture limit, then it should have also been smaller earlier during the collapse. In such a case magnetic fieldlines are not able

\textsuperscript{12}The Alfvén speed, $c_A^2 = 1/(\beta + 1)$ with $\beta \equiv (\rho + P)/B^2$, is also a monotonically increasing function of proper time during gravitational contraction. This happens because the magnetic density, $B^2 \propto a^{-\beta}$ (refer to Mavrogiannis & Tsagas [8]), with $a$ being the fluid’s scale factor, increases faster than the matter density and pressure under a polytropic or barotropic equation of state (see the following section and appendix D).

\textsuperscript{13}The monotonic increase of $u$ and $s^a$ during contraction is given by the problem’s nature.

\textsuperscript{14}Besides, during the principally relativistic phenomenon of collapse, gravity is plausibly pointed out as the dominant term, cause of magnetic deformation.
to impede contraction towards a singularity (refer also to Mavrogiannis & Tsagas [8]). In other words, our collapse criterion reads

\[ s^* = -(c_A^2 u)_{fr} < (R_{ab} u^a u^b)_{fr} \]  \hspace{1cm} (4.7)

Although we have not yet fully determined the magnetic fracture limit, by comparing the aforementioned relativistic terms at the critical point (4.6) we explicitly arrive at (recall that \( 0 < c_A^2 < 1 \))

\[ \frac{c_A^2(\rho - P - B^2)_{fr}}{3} < \frac{1}{2} (\rho + 3P + B^2)_{fr}. \]  \hspace{1cm} (4.8)

We observe that the criterion (4.7) is clearly satisfied at the limit and it was consequently satisfied earlier. Therefore, the magnetic force lines are expected to reach their fracture limit before managing to impede the contraction. It is worth noting that we have reached the aforementioned conclusion without yet assuming a specific equation of state. Our result is in fact based on the assumptions of an ideal MHD fluid model and of its homogeneous contraction. In the following section, it becomes evident that condition (4.8) is satisfied at the fully determined fracture limit.

(d) Complete specification of the magnetic fracture limit—applications to neutron stars and white dwarfs

In the present section, we fully determine the magnetic fracture limit as a double maximum, of the magnetic deformation \( u \) and the magneto-curvature tension stress \( s^* \) as well. In practice, taking the dot derivative of (4.1) under the condition \( \dot{u}_{fr} = 0 \) we get

\[ (s^*)_{fr} = (c_A^2 u)_{fr} = -\frac{1}{B^2(1 + \beta)^2} \left[ \dot{\beta} - \beta \frac{2}{B} \dot{B}(\rho + P) \right]_{fr}, \]  \hspace{1cm} (4.9)

where \( \beta \equiv (\rho + P)/B^2 \) (see appendix E for details). Assuming a polytropic equation of state (i.e. \( P = k \rho^\gamma \), where \( k \) and \( \gamma \) are constant parameters), and setting (4.9) equal to zero we arrive at the condition

\[ \gamma P^2 + (\gamma - 1)\rho P - \rho^2 = 0. \]  \hspace{1cm} (4.10)

Note that in deriving the above we have taken into account the propagation equation for the magnetic field at the MHD, namely \( \dot{B} = -\epsilon B \) (see [8]). Moreover, it is simply verified that expression (4.9) is positive for \( P < \rho/\gamma \), and negative in the opposite case. The real solutions of the above quadratic are\(^{15}\)

\[ P_{fr} = k\rho_{fr}^\gamma = \frac{\rho_{fr}}{\gamma} \Leftrightarrow \rho_{fr} = \left( \frac{1}{\gamma k} \right)^{1/(\gamma - 1)} \frac{1}{\gamma - 1} \]  \hspace{1cm} and  \hspace{1cm} \[ P_{fr} = -\rho_{fr}, \]  \hspace{1cm} (4.11)

of which we accept the former and reject the latter, under the consideration of ordinary collapsing matter. Note that equation (4.11a) provides the values (in geometrized units) of matter density and pressure at the magnetic fracture limit. Overall, we have associated the magnetic fracture limit with the unique (double) maximum of \( u \) and \( s^* \) with respect to proper time. The aforementioned point (a theoretical approach to the magnetic fracture limit) is certainly expected to slightly differ, in practice, from the actual instant in which the magnetic field lines are broken (i.e. \( s^* = 0 \)).

We can make use of the information provided by (4.11a), together with an initial setting, in order to predict how much the volume of a given magnetized (collapsing) star has changed until reaching its magnetic fracture limit. In particular, with the aid of (D 2) (see appendix D), we deduce that the relation between matter density/scale factor at the fracture limit (i.e. \( \rho_{fr} \) and

\(^{15}\)It is easy to check that \( c_A^2 \) is an increasing function of proper time (\( \epsilon \theta < 0 \) is always assumed) for \( P < \rho/\gamma \) while decreasing for \( P > \rho/\gamma \).
The above formula points out the sensible conclusion that the more dense a star is, the earlier it reaches its fracture limit. Furthermore, substituting (4.11) into (4.8) we deduce once again (formally this time) that the magnetic force lines are broken before managing to impede contraction. In the following, we consider, as an application, two illustrative examples of a neutron star and a white dwarf.

(i) Neutron star of mass $M = 1.5M_\odot$, radius $R = 10 \text{ km}$ and average magnetic field $B \sim 10^{12} \text{ G}$

A neutron star with the above mentioned characteristics has matter and magnetic densities $\rho \sim 10^{-14} \text{ cm}^{-2}$ and $B^2 \sim 10^{-27} \text{ cm}^{-2}$, in geometrized units (i.e. $c = 1 = G$). For conversion between cgs and geometrized units see e.g. the appendix of Hartle [15]). Assuming that the matter of the star consists in average of non-relativistic neutrons, the parameters of its polytropic equation of state are $\gamma = 5/3$ and $k = 5.3802 \times 10^{9} \text{ cgs} \sim 10^{6.7} \text{ cm}^{4/3}$ (e.g. refer to Shapiro & Teukolsky [16] or Bielich [17] for details regarding the parameters’ values). Finally, from (D2) written in the initial conditions we determine the value of the integration constant, $C = (\rho_0^{2/3} + k)/a_0^{2} \sim 10^{9.3} a_0^{-2}$.

Taking into account that $B^2 \propto a^{-6}$ (refer to Mavrogiannis & Tsagas [8]), and substituting all the aforementioned values in (4.12) we find out that

$$a_{fr} \sim 10^{-0.7} a_0 \sim 10^{-1} a_0, \quad \rho_{fr} \sim 10^2 \rho_0 \sim 10^{-12} \text{ cm}^{-2} \quad \text{and} \quad B^2_{fr} \sim 10^6 B^2_0 \sim 10^{-21} \text{ cm}^{-2}. \quad (4.13)$$

Therefore, we expect that the magnetic force lines will be broken when the neutron star’s radius becomes about ten times smaller (that is 1 km) than its initial (equilibrium) value. In parallel, the magnetic/matter density ratio will have grown by four orders of magnitude, i.e. $(B^2/\rho)_{fr} \sim 10^4 (B^2/\rho)_0 \sim 10^{-9}$.

(ii) White dwarf of mass $M = 0.6M_\odot$, radius $R = 1.4 \times 10^{-2} R_\odot$, and average magnetic field $B \sim 10^{6} \text{ G}$

The white dwarf in question has matter and magnetic densities $\rho \sim 10^{-23} \text{ cm}^{-2}$ and $B^2 \sim 10^{-39} \text{ cm}^{-2}$, in geometrized units. Assuming that the stellar fluid mainly consists of ultra-relativistic electrons, the parameters of its polytropic equation of state read $\gamma = 4/3$ and $k \sim 10^{15} \text{ cgs} \sim 10^{4.3} \text{ cm}^{2/3}$. Under the above mentioned initial conditions the integration constant of (D2) gives $C \sim 10^{7.6} a_0^{-2}$. Overall, recalling once again that $B^2 \propto a^{-6}$ we find out that

$$a_{fr} \sim 10^{-4.8} a_0 \sim 10^{-5} a_0, \quad \rho_{fr} \sim 10^{28} \rho_0 \sim 10^5 \text{ cm}^{-2} \quad \text{and} \quad B^2_{fr} \sim 10^{28.8} B^2_0 \sim 10^{-10} \text{ cm}^{-2}. \quad (4.14)$$

We deduce that the stellar fluid reaches its magnetic fracture limit when its radius shrinks to approximately one hundred thousand times its initial value (that is the fracture radius is some hundreds of metres). As for the magnetic/matter density ratio, it increases by about an order of magnitude, i.e. $(B^2/\rho)_{fr} \sim 10^{0.8} (B^2/\rho)_0 \sim 10^{-15}$.

5. Discussion

Overall, the essence of our reasoning lies in that gravitational deformation of magnetic force lines is governed by Hooke’s law of elasticity, originating from the Ricci identities. However, there are two basic features distinguishing gravitational distortions of magnetic force lines from mechanical distortions of elastic materials. Firstly, unlike mechanical distortions of elastic materials, Hooke’s law in the form of (4.1) is not an approximate expression only valid for small magnetic deformations (thus magnetic force lines do not seem to have an elastic limit). In contrast, as long as Ricci identities are an appropriate definition of spatial curvature for large values of the latter (advanced stages of gravitational collapse), the law in question consists of an exact expression,
valid for any size of distortion. Secondly, the proportionality factor in the elasticity law (4.1) is a variable instead of a constant quantity.

Based on the aforementioned law we have calculated the magnetic fracture limit under gravitational volume distortions, and subsequently applied it in two explicit cases of a neutron star and a white dwarf. Considering our results as new, we raise the fundamental problem regarding the observational–experimental (and further theoretical) verification of the gravito-magnetic elasticity law. Although magnetic elasticity under great gravitational distortions is practically not a subject offered for study in earthly laboratories, progress towards the experimental path could alternatively and in the first place be achieved by examining magnetic distortions under progressively increasing rotations. Finally, focusing on the magnetic reaction against gravity, the present work peaks with the argument that magnetic force lines reach their fracture limit before managing to impede gravitational contraction. Interestingly, knowing how the magnetic field lines explicitly behave during gravitational implosion could hopefully shed light on the various evolution phases of astrophysical objects. The motivation for the above sentence essentially comes from considering that many stars or protogalactic clouds are associated with (even small) magnetic fields which are rapidly increasing during their collapse.

Wherever they appear in our universe, either in astrophysical or in cosmological environments, magnetic fields are impelled by gravity to manifest their extraordinary elastic features. The phenomena arising from those properties await our exploration.

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Appendix A. Propagation equations and constraints for the kinematically induced magnetic tension stresses

In order to arrive at the propagation equation for $\sigma^{(B)}_{ab}$, we make the following steps. First, project equation (2.2) along the timelike four-velocity $u^\alpha$; second, project orthogonal to $u^\alpha$ with the aid of $h^\alpha_{ab}$ and with respect to both indices (removing thus timelike terms); third, take the symmetric and trace-free part of the resulting relation. The equation in question finally reads

$$
\dot{\sigma}^{(B)}_{ab} = -\Theta\dot{u}_{(a}B_{b)} + 2\dot{u}_{(a}\omega_{b)}cB_c + D_{(a}\dot{B}_{b)} - \left(\sigma_{c(a} + \omega_{c(a} + \frac{1}{3}\Theta h_{c(a}\sigma_{b)}ight)
$$

$$
- \left(\sigma_{c(a} + \omega_{c(a} \omega^{(B)}_{b)} - \frac{1}{2}B_{(a}q_{b)}\right),
$$

(A1)

where $\dot{u}_a = u^b\nabla_b u_a$ is the fluid’s acceleration and $q_a$ its flux vector. On deriving the above we have taken into account eq. (1.3.1) of [11], as well as that

$$
\nabla_a B_b = D_a B_b - u_a \dot{B}_b + u_b (\nabla_a u^d) B_d + u_a u_b \dot{u}^c B_c \quad \text{and} \quad R_{a(b|c)d} u^d B^b = \frac{1}{2} B_{(b} q_{c)},
$$

(A2)

where eqs (1.2.6), (1.2.8) and (1.2.11) of [11] have been used on finding the latter of the above. Following a similar procedure but taking the antisymmetric part of (2.2) (via contraction with the three-dimensional Levi-Civita pseudotensor $\epsilon_{abc}$) this time, we arrive at the propagation equation

16 An index with bar denotes that the associated component has been projected orthogonal to $u^\alpha$. 

for the magnetic tension induced by twisting effects

\[
\omega^{(B)}_{\alpha} = -3\epsilon_{\alpha bc}u^b \sigma^c \cdot d \rho^d - \epsilon_{\alpha bc}D^b \frac{\sigma^c}{d \rho^d} - \epsilon_{\alpha bc}D^b B^c - \epsilon_{\alpha bc}\left(\sigma^{bd} + \omega^{dh} + \frac{1}{3}H^{bd}\right)D_d B^c - (i\mu B_b)\omega_a + (i\mu \omega_b)B_a + H_{ab}B^b - \frac{1}{2}\epsilon_{abc}B^b q^c, \tag{A 3}
\]

where \(H_{ab}\) is the magnetic Weyl component and we have taken into account that

\[
\dot{B}_a = -\frac{2}{3}\Theta B_a + (\sigma_{ab} + \epsilon_{abc}\omega^c)B^b \quad \text{and} \quad \epsilon_{abc}\epsilon^{bcd}u_c B_d = H_{ab}B^b - \frac{1}{2}\epsilon_{abc}B^b q^c. \tag{A 4}
\]

Note that equation (A 4\(a\)) is an expression of Faraday’s law at the MHD limit. On the other hand, the spacelike part of (2.2) leads to the divergence conditions for the aforementioned quantities. In detail, we start from the three-dimensional Ricci identities (3.1). Subsequently, we take either its trace or its contraction with \(\epsilon_{abc}\). The former case leads to

\[
D^b \omega^{(B)}_{ab} = \text{curl} \omega^{(B)}_{ab} + 2\omega_{ab}\left(-\frac{2}{3}\Theta B^a + \sigma^b B^c\right) + \mu\omega_a - 2\omega^2 B_a + R_{ab}B^b, \tag{A 5}
\]

while the latter leads to

\[
D^a \omega^{(B)}_{a} = \frac{1}{6}\Theta B - 2\sigma_{ab}\omega^a B^b. \tag{A 6}
\]

In (A 5) \(R_{ab}\) represents the three-dimensional Ricci tensor. On deriving the above we have made use of (A 4\(a\)) as well as of

\[
\omega^a B_a = \frac{\mu}{2} \quad \text{and} \quad \epsilon^{abc}R_{dcba}B^d = -\frac{2}{3}\Theta B - 4\sigma_{ab}\omega^a B^b, \tag{A 7}
\]

where \(\mu\) is the charge density and (A 7\(a\)) is an expression of Gauss’s law at the MHD limit. It is worth noting that for zero rotational distortions (i.e. \(\omega_{ab} = 0\)) of the magnetic field equations (A 1) and (A 5) significantly simplify to

\[
\dot{\sigma}^{(B)}_{ab} = -\Theta u_{(a}B_{b)} + D_{(a}\dot{B}_{b)} - \left(\sigma_{c(a} + \frac{1}{3}H_{c(a}\right)\sigma^{c(b)}_{b)} \quad \text{and} \quad \nabla^b \sigma^{(B)}_{ab} = R_{ba}B^b. \tag{A 8}
\]

Overall, equations (A 1), (A 3), (A 5) and (A 6) determine the kinematics of the magnetic tension stresses triggered by shear and vorticity effects.

**Appendix B. Our approach to magnetized contracting flow in technical terms**

\(18\) The present appendix unit provides a technical supplement to §4\(b\), within the main text. Throughout the present manuscript we encounter several times products of tensors with the spacelike (unit) vector field \(n^a\) (such that \(n_a u^a = 0\) and \(n_a n^a = 1\)), which is taken parallel to the magnetic field lines (i.e. \(B^a = En^a\)). In particular, our calculations often involve the quantities \(\Sigma = \sigma_{ab}n^a n^b\) and \(\Sigma_a = \dot{h}_a B^b \sigma_{bc}n^c\) (with \(\dot{h}_{ab} = g_{ab} + u_a u_b - n_a n_b = h_{ab} - n_a n_b\), satisfying \(\dot{h}_{ab} n^b = 0\)). Concerning the former, it can be written as

\[
\Sigma \equiv \sigma_{ab}n^a n^b \equiv D_{(b} u_{a)} n^a n^b = D_{(b} u_{a)} n^a n^b - \frac{1}{3}\Theta h_{ab} n^a n^b = u^b_{(a} n^{b)} - \frac{1}{3}\Theta, \tag{B1}
\]

where \(u^\prime_{a} = n^b D_b u_{a}\). Note that the term \(u^\prime_{a} n^a\) is not generally zero because the prime (′) denotes an actual spatial derivative operator. This means that ′ does not generally satisfy the product (Leibniz) rule between (the timelike) \(n^a\) and (the spacelike) \(n^a\). Our approach consists thus of setting \(u^\prime_{a} n^a\) equal to zero. The quantity in question vanishes either when prime differentiation

\(17\) An ideal fluid (i.e. \(q_a = 0\)) has been assumed in the first equation.

\(18\) The present appendix section is envisaged as a correction of Appendix A in [8]. In particular, we point out here that eqs (102) and (106) of the aforementioned work are valid, not generally, but within a specific framework.
obeys the product rule\(^{19}\) or when \(u'_a = n^b D_b u_a = 0\). Hence, we have

\[
\Sigma = -\frac{\theta}{3} \quad \text{for} \quad u'_a = 0. \tag{B2}
\]

The last condition implies that the fluid velocity is homogeneous along the magnetic forcelines. In other words, our approximation is equivalent with considering the magnetic forcelines as streamlines of the fluid.

Besides, it is worth noting that the double projection of the volume expansion/contraction tensor, \(\Theta_{ab} = \sigma_{ab} + (\Theta/3) h_{ab}\), along a spatial direction \(n^a\) gives

\[
\Theta_{ab} n^a n^b = u'_a n^a. \tag{B3}
\]

Interestingly, taking into account (B2), the above expression becomes \(\Theta_{ab} n^a n^b = 0\). Crucially, in our problem of magnetized gravitational collapse (i.e. \(B^2 = B n^2\)), \(\Theta_{ab} n^a n^b = 0\) translates into the following physical assumption. During collapse, one or two of the spatial directions orthogonal to \(n^a\) contract, while the fluid along \(n^a\) moves with acceleration determined by the magnetic stresses. **Could the magnetic direction length increase or decrease incessantly? Our answer to the problem is definitely no.** Instead, we claim that there is a magnetic fracture limit (in the sense of elastic dynamics), which we calculate within the main text.

In reference to the other shear component of interest, namely \(\Sigma_a\), we have

\[
\Sigma_a \equiv \tilde{h}_a^b \sigma_b c n^c \equiv \tilde{h}_a^b n^c D_{(c} u_{b)} = \tilde{h}_a^b n^c D_{(c} u_{b)} = \frac{1}{2} \Sigma_a + \frac{1}{2} \epsilon_{ab} \Omega^b + \frac{1}{2} \tilde{h}_a^b u'_b, \tag{B4}
\]

which subsequently leads to

\[
\Sigma_a = \epsilon_{ab} \Omega^b + \tilde{h}_a^b u'_b \quad \text{or} \quad \Sigma_a = \epsilon_{ab} \Omega^b \quad \text{for} \quad u'_a = 0. \tag{B5}
\]

Moreover, we take into account that the projection of Faraday’s law (within ideal MHD—see eq. (41) in [8]) normal to the magnetic direction \(n^a\), gives

\[
\Sigma_a = \epsilon_{ab} \Omega^b + \alpha_a, \tag{B6}
\]

where \(\alpha_a \equiv \tilde{h}_a^b n_b\). Comparing the above with (B5a), we find out that

\[
\alpha_a = \tilde{h}_a^b u'_b \quad \text{and} \quad \alpha_a = 0 \quad \text{for} \quad u'_a = 0. \tag{B7}
\]

The last expression will prove useful in appendix D. We make use of equations (B2) and (B5) in various calculations throughout this manuscript.

### Appendix C. Magneto-curvature tension stresses

Following discussion in §3a the magneto-curvature tension stresses associated with shear, rotational and volume curvature distortions are\(^{20}\)

\[
s_{(ac)} = B^2 R_{d(ac)b} n^b n^d
\]

\[
= B^2 \left[ \epsilon_{(ab)(c)} \epsilon_{d} \epsilon_{e} \epsilon_{f} \epsilon_{g} \epsilon_{h} \epsilon_{i} \epsilon_{j} \epsilon_{k} \epsilon_{l} \epsilon_{m} \epsilon_{n} \epsilon_{o} \epsilon_{p} \epsilon_{q} \epsilon_{r} \epsilon_{s} \epsilon_{t} \epsilon_{u} \epsilon_{v} \epsilon_{w} \epsilon_{x} \epsilon_{y} \epsilon_{z} \right] \left( \Sigma_n (a n_c) + 4 \Sigma_n (a n_c) \right), \tag{C1}
\]

\(^{19}\)Let us consider firstly the decomposition \(D_b n_a = h_b^e h^a_e \nabla_a n_e = \nabla_a n_e + \tilde{h}_a^b n_e - u_a (D_b u_b) n^e\). Projecting the above along \(n^b\) and recalling that \(D_b u_a = \sigma_{ab} + \omega_{ab} + (\Theta/3) h_{ab}\), we receive

\[
n^b D_b n_a = n^b \nabla_b n_a - u_a (n^b \nabla_b u_b) n^a = n^b \nabla_b n_a - \left( \Sigma + \frac{\Theta}{3} \right) u_a. \]

We observe that under the condition \(\Sigma = -\Theta/3\), the above equation recasts into \(n^b \nabla_b u_a = n^b D_b u_a\) (e1). Secondly, recalling \(\nabla_b n_a = D_b n_a - u_a h_{ab}\), we observe that \(n^b \nabla_b u_a = n^b D_b u_a\) (e2) always. Overall, it is clear that via expressions (e1), (e2), the prime derivative obeys the product rule between \(u^a\) and \(n^a\). In other words, we deduce that (see also our initial equation (B1)) \(u'_a n^a = (u_a n^a) - u_a n^a = 0\) if and only if \(\Sigma = -\Theta/3\).

Note that the last conclusion is not directly implied by (B1). In fact, the latter, under condition (B2), leads to \(u'_a n^a = 0\) but not necessarily to Leibniz’s rule for prime differentiation.

\(^{20}\)On deriving equations (C1)–(3.4) we make use of the so-called Gauss–Codacci formula (e.g. see eq. (1.3.39) in [11]).
\[ s_{[ac]} = B^2 R_{d[a]b} n^b n^d = B^2 \left( \Pi_{[u]n_c] - \frac{4\Theta}{3} \Sigma_{[u]n_c} \right) \]  
(C2)

and

\[ s = s^c_c = B^2 R_{bd} n^b n^d = B^2 \left( \frac{2}{3} \rho + \mathcal{E} + \frac{\Pi}{2} \right), \]  
(C3)

where \( \pi_{ab} \) and \( E_{ab} \) are the anisotropic stress and the tidal (or electric Weyl) tensors, respectively. Moreover, we have \( \Pi \equiv \pi_{ab} n^a n^b \), \( \Pi_a \equiv \tilde{h}_a b n^c \pi_{bc} \), \( \xi \equiv E_{ab} n^a n^b \), \( \xi_{ab} \equiv (\tilde{h}_a c \tilde{h}_b d - (1/2) \tilde{h}_{ab} \tilde{h}^{cd}) E_{cd} \), \( \Sigma \equiv \sigma_{ab} n^a n^b = -\Theta/3 \) and \( \Sigma_a \equiv \tilde{h}_a b n^c \pi_{bc} = \epsilon_{ab} \Omega^b \) (see appendix B for the last two expressions), with \( \epsilon_{ab} \equiv \epsilon_{abc} n^c \) being the two-dimensional counterpart of the Levi-Civita pseudotensor, \( \Omega_a \equiv \tilde{h}_a b \epsilon_{ob} \) and \( \tilde{h}_{ab} \equiv \tilde{h}_{ab} - n_a n_b \) an operator projecting orthogonal to the magnetic field direction \( n^a \). We observe that tidal effects (electric Weyl components) are associated with shape and volume magnetic distortions only. Assuming an ideal fluid model, the anisotropic stress terms in the above vanish. Then, of particular interest is that the deformation due to gravitational compression/expansion in (C3) is determined by the density of matter and the tidal tensor projected along the magnetic fieldlines.

**Appendix D. (a) Temporal evolution of the matter density under a polytropic equation of state (b) Propagation equation for tidal stresses along the magnetic force lines**

Considering an ideal, polytropic (i.e. \( P = k \rho^\gamma \), with \( k \) and \( \gamma \) constants) fluid at the MHD limit, the continuity equation, \( \dot{\rho} = -\Theta (\rho + P) \) \( (\Theta = 3a/a, a \) denoting the scale factor of the fluid’s volume), reads the following explicit Bernoulli form

\[ \frac{d\rho}{d\eta} + \frac{3}{a} \rho - \frac{3k}{a} \rho^\gamma = 0. \]  
(D1)

The equation in question accepts the general solution

\[ \rho = [Ca^{-3(1-\gamma)} - k]^{1/(1-\gamma)}, \]  
(D2)

with \( C \) (note that \( C > 0 \) for \( k > 0 \)) being the integration constant. In the cases of non-relativistic neutrons (\( \gamma = 5/3 \)) and ultra-relativistic electrons (\( \gamma = 4/3 \)) the above equation recasts into

\[ \rho = (Ca^2 - k)^{-3/2} \]  
and \[ \rho = (Ca - k)^{-3} \]  
(D3)

respectively. Obviously, the pressure of matter, \( P = k \rho^\gamma \), increases faster than its density for \( \gamma > 1 \) (i.e. the cases we consider). We employ the above equations in determining the magnetic fracture limit of a neutron star and a white dwarf in the main text (see §4d).

In the following second part of the present appendix unit we present some details regarding the propagation equation for \( \mathcal{E} \) used in §4c.

In particular, for homogeneous collapse of a magnetized ideal fluid the propagation equation for the tidal tensor reads

\[ \dot{E}_{(ab)} = -\Theta E_{ab} - \frac{1}{2} (\rho + P) \sigma_{ab} - \frac{1}{2} \pi_{ab} - \frac{1}{6} \Theta \pi_{ab} + 3 \sigma_{[a} c \left( E_{b]c} - \frac{1}{6} \pi_{b]c} \right) \]

\[ + \epsilon_{cd(a} \left[ 2 \epsilon^{e} H_{b) d} - \omega^e \left( E_{b]d} + \frac{1}{2} \pi_{b]d} \right) \right], \]  
(D4)

where \( H_{ab} \) is the so-called magnetic Weyl component and \( \pi_{ab} \) is exclusively sourced from the magnetic field. Projecting the above twice along the magnetic direction \( n^a \) we ultimately arrive at

\[ \dot{\mathcal{E}} = -\frac{3}{2} \Theta \mathcal{E} + \frac{1}{6} \Theta (\rho + P) - \frac{1}{2} \Theta B^2. \]  
(D5)

It is worth noting that in deriving the above we have taken into account that \( \Sigma = -\Theta/3 \) and \( \Sigma_a = \epsilon_{ab} \Omega^b \) (where for the inner product of two arbitrary vectors \( k^a \) and \( l^a \) with the two-dimensional
Levi-Civita tensor we have $\epsilon_{ab} k^a b^b = 0$. Moreover, we have made use of $E_{ab} n^a n^b = \mathcal{E}_a \alpha^a = 0$, where $\alpha_a \equiv \hat{n}_a \cdot \hat{h}_b$ (see equation (B.7)).

### Appendix E. Some auxiliary calculations

In reference to equation (4.9) (i.e. temporal derivative of the Alfvén speed) in §4d, we employ the continuity equation and the equation of state ($P = k \rho^\gamma$) to calculate the dot derivative of matter density and pressure, i.e. $\dot{\rho} = -\Theta (\rho + P)$ and $\dot{\rho} = \gamma (P/\rho) \dot{\rho}$. Furthermore, we make use of the law of magnetic contraction, $\vec{B} = -\Theta \vec{B}$ (under the ideal MHD approximation of a magnetized fluid, and condition (B.2)), introduced and described in [8]. Therefore, equation (4.9) recasts into

$$ (\dot{c}_A^2)_{tr} = - \frac{1}{B^2 (1 + \beta)^2} \left[ \dot{\rho} + \dot{P} - \frac{2}{B} \dot{\mathcal{B}} (\rho + P) \right] = - \frac{\Theta (1 + \beta)^2}{B^2} \left[ \rho + (1 - \gamma) P - \frac{\rho^2}{\rho} \right] = 0, \quad (E.1) $$

which clearly leads to (4.10). Alternatively, assuming a barotropic equation of state (i.e. $P = w \rho$ with $w =$ constant), equation (4.9) transforms into

$$ (\dot{c}_A^2)_{tr} = - \frac{1}{B^2 (1 + \beta)^2} \left[ \dot{\rho} + \dot{P} - \frac{2}{B} \dot{\mathcal{B}} (\rho + P) \right] = - \frac{\Theta \rho}{B^2 (1 + \beta)^2} (1 - w^2) = 0, \quad (E.2) $$

from which we deduce that $w = \pm 1$ or $P = \pm \rho$. The negative solution is directly rejected (ordinary collapsing matter is assumed) while the negative one leads to $u_{tr} = -(1/3)(\rho - P - B^2)_{tr} = B^2 / 3 > 0$ (see equation (4.6)), also not accepted because $u$ must be negative at all times during the collapse. Hence, it seems that a barotropic equation of state is not appropriate for describing magnetized gravitational collapse, of neutrons stars or white dwarfs.

Finally, we mention here for reference the values of the second temporal derivative of $u$ and the first temporal derivative of $\mathcal{E}$ at the fracture limit. In detail, the dot differentiation of (D.5) under the condition (4.6), as well as $\dot{\rho} = -\Theta (\rho + P)$ and $\dot{\mathcal{B}} = -\Theta \mathcal{B}$, leads to

$$ \dot{u}_{tr} = \frac{1}{2} \Theta \left( 3 \mathcal{E} - \frac{\dot{\rho}}{3} + 2 B \mathcal{B} \right)_{tr} = \frac{2}{3} \Theta (\rho + P)_{tr} < 0, \quad (E.3) $$

where $\Theta < 0$ (for contraction), and $\dot{\mathcal{E}}_{tr} = -(2/3) \dot{\mathcal{B}}_{tr}$ is verified by definition of $u$ (i.e. $u \equiv -(2/3) \rho - \mathcal{E}$). Similarly, the improper-time differentiation of (4.3) under the condition (4.6), as well as the aforementioned propagation formulae, $P = k \rho^\gamma$ and equation (4.11), leads to

$$ \ddot{u}_{tr} = \frac{1}{2} \Theta (3 \dot{\mathcal{E}} + \dot{\rho} + \dot{\mathcal{B}} + 2 B \dot{\mathcal{B}})_{tr} = -(\Theta^2 B^2)_{tr} < 0. \quad (E.4) $$

Therefore, magnetic deformation $u$ presents a maximum at the fracture point. Moreover, we observe that multiplying the condition $u > u_{tr}$ ($u_{tr}$ is more negative than $u$) by $-c_A^2 c_A^2 (E.1)_{tr} < 0$, leads to the verification of the associated maximum for $s$ (note that $c_A^2 < c_A^2 (E.1)_{tr}$)

$$ u > u_{tr} \rightarrow c_A^2 (E.1)_{tr} s < c_A^2 s_{tr} \rightarrow s < s_{tr}. \quad (E.5) $$

Similarly, it is straightforward to confirm that $u < u_{tr}$ implies $s > s_{tr}$.

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Note 21: Recall that $c_A^2 \equiv (1 + \beta)^{-1}$, with $\beta \equiv (\rho + P)/B^2$, increases during contraction because $B^2$ increases faster than $\rho + P$ for a polytropic or barotropic fluid (see the previous appendix section).
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