Electron–phonon interaction in a spin–orbit coupled quantum wire with a gap

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Abstract
Interaction between electron and acoustic phonon in an in-plane magnetic field induced gapped quantum wire with Rashba spin–orbit interaction is studied. We calculate acoustic phonon limited resistivity (\(\rho\)) and phonon-drag thermopower (\(S_g\)) due to two well-known mechanisms of electron–phonon interaction, namely, deformation potential (DP) and piezoelectric (PE) scattering. In the so-called Bloch–Gruneisen temperature limit, both \(\rho\) and \(S_g\) depend on temperature (\(T\)) in a power law fashion, i.e., \(\rho \propto T^\nu\) and \(S_g \propto T^\nu\). For resistivity, \(\nu\) takes the value 5 and 3 due to DP and PE scattering, respectively. On the other hand, \(\nu\) is 4 and 2 due to DP and PE scattering, respectively, for phonon-drag thermopower. Additionally, we find numerically that \(\nu\) depends on Rashba parameter (\(\alpha\)) and electron density (\(n\)). The dependence of \(\nu\) on \(\alpha\) becomes more prominent at lower density. We also study the variations of \(\rho\) and \(S_g\) with carrier density in the Bloch–Gruneisen regime. Through a numerical analysis, a similar power law dependence \(\rho \propto n^{\nu}\) and \(S_g \propto n^{\nu}\) is established in which the effective exponent \(\nu\) undergoes a smooth transition from a low density behavior to a high density behavior. At a higher density regime, \(\nu\) matches excellently with the value obtained from theoretical arguments. Approximate analytical expressions for both resistivity and phonon-drag thermopower in the Bloch–Gruneisen regime are given.

Keywords: spin–orbit coupling, electron–phonon interaction, quantum wire

(Some figures may appear in colour only in the online journal)

1. Introduction
Due to the promising applications in the area of quantum information processing [1] and device technology [2], spin dependent phenomena [3–5] in low dimensional structures have been of major interest in scientific communities for several years. The main route of spin-related phenomena is the well-known spin-orbit interaction (SOI). In semiconductor structures, SOI originates due to the inversion symmetry breaking either in the bulk or at the hetero-interface. Band bending in the heterostructure gives rise to an electric field to produce an asymmetric confining potential, which itself is responsible for generating an SOI of the Rashba type [6]. The strength of the Rashba SOI is proportional to the magnitude of the electric field generated and hence is tunable [7, 8] with the aid of an external gate voltage. Another kind of SOI, usually termed Dresselhaus SOI [9], originates due to the breaking of inversion symmetry in the bulk crystal. In general, the strength [10] of the Dresselhaus SOI is smaller than that of the Rashba SOI in the heterostructure. The case of equal strength of both SOIs is of particular importance for the future development of a non-ballistic spin field effect transistor [11].

In a quantum well, restriction of a carrier’s motion by an additional confinement in a particular direction essentially leads to the formation of a quantum wire (QW). The width of the QW is of the order of the Fermi wave length in order to allow ballistic transport [12]. Semiconductor QW with SOI is considered as a building block for a better implementation [13] of spin-field effect transistor [14]. An in-plane magnetic field along the wire direction lifts the \(k = 0\) degeneracy in a spin–orbit coupled QW, and as a result a gap is induced in the energy spectrum. Immense interest in the gapped spin–orbit coupled QW has been growing because of several proposals...
of asymmetric spin filtering [15], controlling impurities [16] by a magnetic field, topological superconducting phase [17–19], helical states [15, 18, 20, 21], etc. Recently, magnetic field induced spin–orbit gap in a one-dimensional hole gas has been realized experimentally [21].

In the present study we mainly focus on various consequences of electron-phonon interaction in a spin–orbit coupled gapped QW by calculating phonon-limited resistivity and phonon-drag thermopower in the Bloch–Gruneisen (BG) regime. In the BG regime, the resistivity differs abruptly from its equipartition behavior. An upper bound of the BG regime can be defined by the characteristic temperature $T_{BG} = 2h v_i k_F/k_B$, where $v_i$ is the sound velocity and $k_F$ is the Fermi wave vector. Below $T_{BG}$ the acoustic phonon energy is comparable with the thermal energy. Due to the smallness of the Fermi surface in semiconductor structures, acoustic phonon with wave vector $q \approx 2k_F$ cannot be excited appreciably, which in turn leads to a complicated temperature dependence of resistivity below $T_{BG}$. The existence of the BG regime in the semiconductor quantum well has been confirmed experimentally [22]. Another important quantity that can be used to probe the electron–phonon interaction in semiconductor nanostructure is the phonon-drag contribution to the thermoelectric power. A number of studies have been performed to understand the behavior of phonon-limited mobility [23–30] and phonon-drag thermopower [31–35] in quantum well and wire for several years. Recently, acoustic phonon-limited resistivity [36] and phonon-drag thermopower [37] in a Rashba spin–orbit coupled two-dimensional electron gas have been studied. It is revealed through numerical analysis that the effective exponents of the temperature dependence of both resistivity and thermopower depend significantly on the strength of the Rashba SOI.

In the BG regime, we find analytically that both resistivity and phonon-drag thermopower in a spin–orbit coupled gapped QW maintains a power law dependence with temperature, i.e., $\rho \propto T^\nu$ and $S_g \sim T^{\nu_r}$. The effective exponent $\nu_r$ is 5 and 3 due to deformation potential (DP) and piezoelectric (PE) scattering, respectively, in the case of phonon-limited resistivity. For phonon-drag thermopower, $\nu_r$ becomes 4 and 2 due to DP and PE scattering, respectively. The exponent $\nu_T$ has also been extracted numerically and clearly undergoes a transition from the BG regime to the equipartition limit. At a relatively higher density, $\nu_r$ is in excellent agreement with the analytical results. Temperature variation of $\nu_T$ for different Rashba parameter ($\alpha$) at a fixed density has also been shown. The effect of $\alpha$ on the temperature dependence of $\nu_T$ becomes less prominent as one approaches higher density. Additionally, the dependence of both $\rho$ and $S_g$ on carrier density has been shown in which both quantities undergo a transition from a relatively low to a higher density behavior.

We organize this paper in the following way. In section 2 we present all the theoretical details. Numerical results and discussions have been reported in section 3. We summarize our work in section 4.

2. Theory

2.1. Physical system

We consider a semiconductor QW of radius $R$ in which electrons are free to move in the $z$-direction. Rashba SOI in a QW essentially breaks the spin-degeneracy [38] by shifting the non-degenerate sub-bands laterally along the wave vector. But the spectrum is still degenerate at $k = 0$. An external magnetic field $B = Bz$ along the wire direction can be used to lift this degeneracy further by inducing a gap $\Delta$ in the energy spectrum [15, 39, 40]. Now the single-particle Hamiltonian describing a gapped QW with Rashba SOI (RSOI) is written as

$$H = \left( \frac{p_z^2}{2m^*} + V(\mathbf{r}) \right) \sigma_0 + \frac{\alpha}{\hbar} \sigma_y p_z + \Delta \sigma_z,$$

where $p = \hbar k$ is the electron momentum in the $z$-direction, $m^*$ is the effective mass of electron, $\sigma_0$ is the unit matrix, $\sigma_i$'s are the Pauli spin matrices, and $\alpha$ is the strength of RSOI. Also, $\Delta = g^z \mu_B B/2$ is the Zeeman energy with $g^z$ and $\mu_B$ as the effective Lande $g$-factor and Bohr magneton, respectively. Finally, $V(\mathbf{r})$ is the confining potential in the transverse direction $r \equiv (x, y)$.

The wire is assumed to be thin enough so that only the lowest sub-band in the transverse direction is occupied by electrons. By diagonalizing equation (1), the eigen energies corresponding to the present physical system can be found in the following form

$$\epsilon_k^\pm = \frac{\hbar^2 k^2}{2m^*} + \lambda \sqrt{\alpha^2 k^2 + \Delta^2},$$

where $\lambda = \pm$ describes the branch index. Note that the energy is measured from the bottom of the lowest sub-band energy $\epsilon_{10} = \hbar^2 k_{10}^2/(2m^*)$ with $k_{10}$ as the sub-band wave vector.

The eigen functions corresponding to the $+$ and $-$ branches are, respectively, given by

$$\Psi^+(r, z) = e^{iakz}/\sqrt{2\pi} \begin{pmatrix} \cos \phi_k/2 \\ i \sin \phi_k/2 \end{pmatrix} \Phi_{10}(r),$$

and

$$\Psi^-(r, z) = e^{iakz}/\sqrt{2\pi} \begin{pmatrix} \sin \phi_k/2 \\ -i \cos \phi_k/2 \end{pmatrix} \Phi_{10}(r),$$

where $k = ak/\Delta$. The lowest sub-band wave function for $r \leq R$ is given by $\Phi_{10}(r) = J_0(k_{10}r)/\left[\sqrt{\pi k_{10}^2} J_1(k_{10}R)\right]$ with $J_\nu(x)$ as the Bessel function of order $\nu$. It is important to mention that $k_{10}R$ is the first zero of $J_0(k_{10}R)$. Outside the QW $\Phi_{10}(r)$ vanishes.
At a fixed energy, namely, Fermi energy \( \epsilon_F \), one can have the following expression for the Fermi wave vectors
\[
k_F^2 = \left( -\lambda k_\alpha + \sqrt{k_F^2 + \frac{2m^* \epsilon_F}{\hbar^2} + \frac{\Delta^2}{a^2} - \frac{\Delta^2}{a^2}} \right)^2. \tag{5}
\]
where \( k_\alpha = m^* a / \hbar^2 \). In the \( B \to 0 \) limit, equation (5) reduces to the known forms of the Fermi wave vectors of a Rashba spin–orbit coupled QW.

The velocity corresponding to the energy spectrum given in equation (2) is calculated as
\[
v_F^2 = \frac{\hbar k}{m^*} \left( 1 + \frac{\alpha^2 k}{\hbar^2 a^2} \right). \tag{6}
\]

### 2.2. Phonon-limited resistivity

In this section we shall calculate resistivity due to the electron–phonon interaction using Boltzmann transport theory. We restrict ourselves to consider only the longitudinal and transverse acoustic phonon modes. Using Drude’s formula, the resistivity is simply written as
\[
\rho = \frac{m^*}{ne^2} \left( \frac{1}{\tau} \right), \tag{7}
\]
where \( \langle 1/\tau \rangle \) is the inverse relaxation time (IRT) averaged over energy and \( n \) is the density of the electron.

The energy-averaged IRT for a specific energy branch \( \lambda \) is given by
\[
\{ \frac{1}{\tau}\epsilon_F \} = \frac{2}{k_BT} \int \text{d} \epsilon_F f(\epsilon_F) \left( 1 - f(\epsilon_F') \right) \frac{1}{\tau(\epsilon_F')}, \tag{8}
\]
where \( f(\epsilon_F') = [1 + e^{\beta(\epsilon_F^2 - \epsilon_F^2 - \epsilon_F^2)}]^{-1} \) is the Fermi–Dirac distribution function with \( \beta = (k_BT)^{-1} \). Here, the 2-factor appears to consider the \( k < 0 \) contribution since the energy spectrum is symmetric about \( k = 0 \).

According to the Boltzmann transport theory, the IRT can be found in the following semi-classical form
\[
\frac{1}{\tau(\epsilon_F')} = \sum_{k,F} \left( 1 - \frac{k'}{k} \right) W_{k,F}^2 \frac{1 - f(\epsilon_F')}{1 - f(\epsilon_F')}. \tag{9}
\]

The transition probability from an initial state \( |k, \lambda \rangle \) to a final state \( |k', \lambda' \rangle \) is given by the Fermi’s Golden rule as
\[
W_{k,F}^2 = \frac{2\pi}{\hbar} \sum_Q \left| C_{Q,F} \right|^2 F(\epsilon_Q) \left| \xi_{k,F} \right|^2 \times \left\{ N_0 \delta(\epsilon_F' - \epsilon_F - \hbar\omega_Q) \delta(\epsilon_F - \epsilon_F') 
+ \left( N_Q + 1 \right) \delta(\epsilon_F' - \epsilon_F' - \hbar\omega_Q) \delta(\epsilon_F - \epsilon_F') \right\}. \tag{10}
\]

where \( Q = (q, Q) \) is the phonon wave vector with \( q_l = (q_x, q_y) \) and \( q = Q_z. N_0 \approx \left\{ e^{\beta h\omega'} - 1 \right\}^{-1} \) is the phonon distribution function, \( C_Q \) is the matrix element corresponding to the electron–phonon interaction, and \( F(\epsilon_Q) \) is the form factor arising due to the transverse confinement. The first and second terms in the braces of equation (10) correspond to the absorption and emission of acoustic phonons, respectively. Finally, the overlap integral \( \left| \xi_{k,F} \right|^2 \) coming from the spinor part of the wave function is given by
\[
\left| \xi_{k,F} \right|^2 = \frac{1 + \lambda' \cos(\phi_k - \phi_k')}{2}. \tag{11}
\]

The square of the electron–phonon matrix elements corresponding to DP and PE scatterings, are, respectively, given by
\[
\left| C_{fi} \right|^2 = \frac{D^2 hQ}{2\hbar\nu_{\text{fi}}}, \tag{12}
\]
\[
\left| C_{Q,(l)} \right|^2 = \frac{(h\lambda d_Q)^2 h A_{(l)} Q}{2\hbar\nu_{\text{sl}} q_{(l)}}, \tag{13}
\]
where \( D \) is deformation potential strength, \( h\lambda d_Q \) is the relevant PE tensor component, \( \rho_{\text{sl}} \) is the mass density, and \( \nu_{\text{sl}} \) is the longitudinal (transverse) component of sound velocity. The anisotropic factors are given by \( A_1 = 9q^2 q_{(l)}^2 / (2Q^6) \) and \( A_2 = (8q^2 q_{(l)}^2 + q_{(l)}^4) / (4Q^6) \).

Finally, the square of the form factor is defined as
\[
\left| F(\epsilon_{Q_l}) \right|^2 = \left| \int \Phi_{Q_l}^*(\mathbf{r}) e^{i\mathbf{Q} \cdot \mathbf{r}} \Phi_{Q_l} (\mathbf{r}) d\mathbf{r} \right|^2. \tag{14}
\]

In the literature, it is assumed \([41, 42]\) that the electron density is sufficiently high near the axis of the wire and vanishes everywhere, and consequently one replaces \( \Phi_{Q_l}(\mathbf{r})^2 \sim 1/(\pi R_0^2) \), where \( R_0 < R \) defines some confinement region. So in this approximation, the square of the form factor can be readily obtained in the following exact form as \( |F(\epsilon_{Q_l})|^2 = 4(\lambda q_{Q_l} R_0 / q_{Q_l} R_0) |^2 \). In the rest of the paper we will be using this expression of the form factor.

Let us now discuss the possibility of intra- and inter-branch scatterings. Generally, at low temperature, intra-branch scatterings are dominant. For inter-branch scattering to occur, large momentum transfer is needed. Since in the BG regime \( q \ll k_F \), the possibility of inter-branch scatterings is ruled out. In a spin–orbit coupled QW without the gap, the inter-branch scattering is strictly forbidden, as readily understood from equation (11), since \( \cos(\phi_k - \phi_k') \) achieves unity as \( B \to 0 \). But for \( B \neq 0 \), inter-branch scattering is possible. We have checked numerically that the inter-branch contribution is very small in comparison with the intra-branch one. So henceforth we will consider only the intra-branch scattering.

Now, the summation over \( k' \) in equation (9) can be performed with the aid of \( \delta(\epsilon_k - \epsilon_{k'}) \) given in equation (10). In the BG regime, phonon energy is comparable with the thermal energy, but much less than with the Fermi energy, i.e., \( \hbar\omega_Q \approx k_B T \ll \epsilon_F \). So one can safely make the following approximation \( f(\epsilon_k') \{ 1 - f(\epsilon_k - \hbar\omega_Q) \} \approx \hbar\omega_Q (N_0 + 1/2 \pm 1/2) \delta(\epsilon_k - \epsilon_F) \), where + and − signs correspond to the absorption and emission of acoustic
phonons, respectively. Using the previously mentioned simplification and approximation, one can find the energy-averaged IRT in the following form

\[
\left\langle \frac{1}{e^\frac{1}{Q}} \right\rangle = \frac{4\pi}{k_B T} \sum Q \left| C_{Q \alpha} \right|^2 F(q_{\perp}) \frac{2q_{Q \alpha}}{k_F} N_0(Q_0 + 1) \times \left\{ \sum_{\alpha} \left( 1 - \frac{m^2 v_x^2 g_{\perp Q \alpha}}{h k_F^2} q_{\perp} \right) \times \delta \left( q - \frac{m^2 v_x^2 g_{\perp Q \alpha}}{h k_F^2} \right) \right\}.
\]

The summation over \( Q \) in equation (15) can be transformed into an integration over \( q \) and \( q_{\perp} \) as \( \sum Q \rightarrow (1/4\pi^2) \int q_{\perp} dq_{\perp} dq_{\parallel} \). At a very low temperature (BG regime), phonon states with wave vector \( q \ll k_F \) are populated. The delta functions given in equation (15) can be approximated in the following form (see appendix for details)

\[
\delta \left( e_{k,\perp} - e_{k,\parallel} \right) \approx \frac{m^2}{h^2 k_F^2} \left\{ 1 + \frac{m^2 v_x^2 g_{\parallel Q \alpha}}{h k_F^2} q_{\perp} \right\} \times \delta \left( q - \frac{m^2 v_x^2 g_{\parallel Q \alpha}}{h k_F^2} \right),
\]

where \( \tilde{g}_{\parallel Q \alpha} = k_F(1 + 2\lambda \epsilon_a / e_k) \) with \( \epsilon_k = \sqrt{\alpha^2 k_F^2 + \Delta^2} \), \( e_a = m^2 \alpha^2 / (2h^2) \), \( g_{\parallel Q \alpha} = 1 + 2\lambda \epsilon_a / e_k \), and \( \epsilon_k = m^2 \alpha^2 / (2h^2) \). Here \( \alpha \) is defined as \( \alpha = \alpha \Delta / e_k \).

Now inserting equation (16) into equation (15) and putting the expressions for \( \tilde{g}_{\parallel Q \alpha} q_{\perp} \) we have

\[
\left\langle \frac{1}{e^\frac{1}{Q}} \right\rangle = \frac{m^2 v_x}{\pi h^2 k_B T} \int \frac{dq_{\perp} dq_{\parallel} C_{Q \alpha}^2 |F(q_{\perp})|^2 q_{\perp} Q}{k_F^2 e_k^2} \times N_0(Q_0 + 1) \delta \left( q - \frac{m^2 v_x^2 g_{\parallel Q \alpha}}{h k_F^2} \right) \times \left\{ \cos \phi_{e_{k,\perp}} - \cos \phi_{e_{k,\parallel}} \right\}
\]

\[
+ \frac{m^2 v_x^2 g_{\parallel Q \alpha}}{h k_F^2} q_{\perp} \left\{ 1 + \cos \phi_{e_{k,\perp}} + \cos \phi_{e_{k,\parallel}} \right\},
\]

where \( \phi_{e_{k,\perp}} = \phi_{e_{k,\parallel}} = \phi_{k,\perp,\parallel} \).

2.3. Phonon-drag thermopower

In the presence of a temperature gradient, diffusion motion of electrons takes place. As a result of electron-phonon interaction, the phonon gains a finite heat flux, which in turn affects the diffusion of electrons along with it from the hot to the cold end. In this way, a phonon-drag contribution to the thermoelectric power is generated. Phonon-drag thermopower is more fundamental quantity in probing electron-phonon strength experimentally.

To calculate phonon-drag thermopower, two approaches named \( Q \) and \( \Pi \) are mainly followed. In the rest, we follow only the \( Q \)-approach. We now start with the following expression [31] for the phonon-drag thermopower

\[
S_g^Q = \frac{e \tau_p}{\sigma k_B T^2} \sum_{Q, k_F, \alpha} \hbar \omega_Q f \left( e_{\perp}^Q \right) \left[ 1 - f \left( e_{\parallel}^Q \right) \right] \times W_{\parallel Q, \lambda \alpha}^Q (k, k') \left\{ \tau \left( e_{\perp}^Q \right) v_{\perp} - \tau \left( e_{\parallel}^Q \right) v_{\parallel} \right\} : v_p, \tag{18}
\]

where \( \tau_p \) is the phonon mean free time, \( L \) is the length of the sample, \( \sigma \) is the Drude conductivity, \( \tau(e_k) \) is the electron’s momentum relaxation time, \( v_{\perp} \) is the velocity of an electron as given in equation (6), and \( v_p \) is the velocity of phonon.

Finally, \( W_{\parallel Q, \lambda \alpha}^Q (k, k') \) is the transition probability by which an electron makes a transition from an initial state \( |k, \lambda \rangle \) to a final state \( |k', \lambda' \rangle \) with the absorption of an acoustic phonon.

The transition probability can be written as

\[
W_{\parallel Q, \lambda \alpha}^Q (k, k') = \frac{2 \pi}{h} \left| C_{Q \alpha} \right|^2 |F(q_{\perp})|^2 \left[ \tilde{g}_{\parallel Q \alpha}^Q q_{\perp} \right] N_0 \delta \left( e_{\perp}^Q - e_{\parallel}^Q - \hbar \omega_Q \right) \delta_{\lambda, \lambda'}.
\]

In equation (18) summation over \( k' \) is readily done using \( \delta_{\lambda, \lambda'} \) given in equation (19). Further, a slow variation of \( \tau(e_k) \) over an energy scale \( \sim \hbar \omega_Q \) is assumed. So one can use the approximation \( \tau(e_k + \hbar \omega_Q) \approx \tau(e_k) \).

The summation over \( k \) in equation (18) can be converted into an integral over \( e_{\perp} \) by the following transformation

\[
\sum_k \rightarrow \frac{m^2 L}{2\pi h^2} \times \int \frac{1}{k_F^2} \left[ 1 - \frac{k_F}{\sqrt{k_F^2 + \frac{2m^2 v_x^2 e_k}{h^2} + \Delta^2 / \alpha^2}} \right] \left| \tilde{g}_{\parallel Q \alpha}^Q q_{\perp} \right| d e_{\perp}, \tag{20}
\]

where \( k_{\perp} \) can be obtained from equation (5).

Using equations (18)–(20) one can finally obtain the following expression for the phonon-drag thermopower for a specific branch \( \lambda \) in the BG regime as

\[
S_g^Q = \frac{m^3 v_x}{4\pi^2 \hbar^2 k_B T^2} \left| C_{Q \alpha} \right|^2 |F(q_{\perp})|^2 q_{\perp} \times Q^2 N_0(Q_0 + 1) \left[ 1 - \frac{m^2 v_x^2 g_{\parallel Q \alpha}^Q q_{\perp}}{h k_F^2} \right] \times \delta \left( q - \frac{m^2 v_x^2 g_{\parallel Q \alpha}^Q q_{\perp}}{h k_F^2} \right) \left| \tilde{g}_{\parallel Q \alpha}^Q q_{\perp} \right| \left[ \tilde{v}_{k,\perp,\lambda+\epsilon^Q} - \tilde{v}_{k,\perp,\lambda} \right] \cdot v_p, \tag{21}
\]

where \( N_0^Q = \left[ 1 - \lambda k_a / \sqrt{k_a^2 + \frac{2m^2 v_x^2 e_k}{h^2} + \Delta^2 / \alpha^2} \right] \). Note that in deriving equation (21) we have used the approximation
\[ f(\epsilon_k)\{1 - f(\epsilon_k + h\omega_Q)\} \approx h\omega_Q(N_Q + 1)\delta(\epsilon_k - \epsilon_F) \] as earlier.

2.4. Approximate analytical results in BG regime

We shall now derive some approximate analytical expressions for phonon-limited resistivity and phonon-drag thermopower in the BG regime.

In the BG regime, we have \( q \ll k_F \). In this limit one can use the following approximation, \( \phi_{\epsilon_F}^\perp \approx 1 \). Again, phonon energy is comparable with the thermal energy in the BG regime, i.e., \( \nu_q q \sim k_B T \). So we have \( q_k R_0 = k_B T R_0/\nu_q \), which is much lower than unity, and consequently one can approximate [34] the form factor as \( |F(q)|^2 \approx 1 \).

Under the previously mentioned approximations, equation (17) takes the following form

\[
\left| \frac{1}{\tau}\right| \approx \frac{D}{\pi \hbar k_F^4}\left( \frac{m^* v_d}{\hbar k_F}\right)^3 \left( k_B T \right)^3.
\]

(23)

Now inserting \( |C_q|^2 \) as given in equations (12) and (13) and using the standard integral \( \int_0^{\infty} e^{\epsilon^2}/(\epsilon^2 - 1)^2 = \zeta(2) \), in equation (22) one can derive the following expressions for the energy-averaged IRT corresponding to DP, longitudinal, and transverse PE scatterings, respectively, as

\[
\left| \frac{1}{\tau}\right|_{DP} \approx \frac{D^3 g_0^2 \zeta(5)}{\pi \hbar^3 k_F^4}\left( m^* v_d/\hbar k_F \right)^3 \left( k_B T \right)^5.
\]

(24)

\[
\left| \frac{1}{\tau}\right|_{PE,l} \approx \frac{9 (\epsilon h\lambda)^2 g_0^2 \zeta(3)}{2 \pi \hbar^3 v_d^2 k_F^4}(m^* v_d/\hbar k_F)^3 \left( k_B T \right)^3.
\]

and

\[
\left| \frac{1}{\tau}\right|_{PE,t} \approx \frac{(\epsilon h\lambda)^2 g_0^2 \zeta(3)}{4 \pi \hbar^3 v_d^2 k_F^4}(m^* v_d/\hbar k_F)^3 \left( k_B T \right)^3
\]

\[
\times \left\{ 1 + 8 \left( \frac{m^* v_d}{\hbar k_F} \right)^4 \right\}.
\]

(25)

In deriving approximate analytical results for phonon-drag thermopower, we expand velocity in equation (6) and retain terms up to \( q \) since \( q \ll k_F \) in the BG regime. We find the approximate expression for the following quantity as

\[
v_{k_F+q} - v_{k_F} = \left( \frac{h}{m^*} + \lambda \frac{\tilde{\alpha}^3}{\hbar a \Delta} \right) q.
\]

(26)

Inserting equation (26) in equation (21) and doing the integration over \( q \) we arrive at the following approximate result for the phonon-drag thermopower

\[
S_{\delta}^{\perp} = -\frac{m^* v_d N_F}{4\pi \hbar^2 k_F^2} \left( \frac{m^* v_d}{\hbar^2 k_F} \right)^3 \left( \frac{h}{m^*} + \lambda \frac{\tilde{\alpha}^3}{\hbar a \Delta} \right)
\]

\[
\times \int dq \left[ C_q \left( \frac{1}{N_Q} \right) \right] \left( N_Q + 1 \right).
\]

(27)

After doing the integration over \( q \), we finally obtain the following expressions for the phonon-drag thermopower due to DP, longitudinal, and transverse PE scatterings, respectively

\[
S_{\delta}^{\perp} |_{DP} \approx \frac{k_B D^2}{e \hbar^2 v_d^2} m^* v_d k_F^4 \left( k_B T \right)^2.
\]

(28)

and

\[
S_{\delta}^{\perp} |_{PE,l} \approx \frac{k_B}{e} \frac{9 v_d}{4} \left( \frac{m^* v_d}{\hbar k_F} \right)^3 \left( k_B T \right)^2.
\]

(29)

Here \( P_{\delta}^{(0)} \) is defined as

\[
P_{\delta}^{(0)} = \frac{m^* v_d^2 N_F}{8\pi^2 \nu_d \hbar v_d^2 k_F^4} \left( \frac{h}{m^*} + \lambda \frac{\tilde{\alpha}^3}{\hbar a \Delta} \right)
\]

(31)

Let us now provide here a systematic comparison between the results obtained for a QW (present case) and two-dimensional electron system (2DES) with the RSOI in the BG regime. In this context, we calculate the following quantity \( S_{\phi} \rho^{-1} \) for both quasi-2DES and QW. In the case of a quasi-2DES [36, 37], using approximate analytical expressions for \( p \) and \( S_{\phi} \) in the BG regime, one can obtain the following result:

\[
S_{\phi} \rho^{-1} \approx -G_{\phi} k_{F0} k_{2d}/T
\]

for DP and longitudinal PE scattering. Here, \( G_{\phi} = e\nu_{d,l} \nu_{d,t} /4\pi \), \( k_{2d} \approx 1 - 2k_F^2/k_{F0} \), and finally, \( k_{F0} \) is the Fermi wave vector obtained via \( k_{F0} = \sqrt{2 m_{2d}} \) with \( n_{2d} \) as the carrier concentration in 2D. The result corresponding to the transverse PE scattering is easily obtained by multiplying the previous result by a factor of \( \nu_{d,t}^2/\nu_{d,l}^2 \). Note that \( \rho(S_{\phi}) \) represents the total resistivity (phonon-drag thermopower), which is obtained by summing up the contributions coming from individual energy branches. In the present case, with \( B \neq 0 \), the total resistivity or phonon-drag thermopower cannot be obtained because of the complicated structure of \( k_F^2 \) and other quantities as evident from equations (23)–(25) and (28)–(30). However, in the \( B \to 0 \) limit, total quantities can be obtained easily. In this limit, we obtain \( S_{\phi} \rho^{-1} |_{1d} \approx -G_{\phi} k_{F0} k_{1d}/(2T) \) due to DP and longitudinal PE scattering, where \( k_{1d} \approx 1 - (k_d/k_F)^2 \) and \( k_d^2 = n\pi/2 \) with \( n \) as the electron density in 1D. Multiplying \( S_{\phi} \rho^{-1} |_{1d} \) by \( \nu_{d,t}^2/\nu_{d,l}^2 \), one can obtain the corresponding result.
for the transverse PE case. However, the functional forms of \(\rho - S_{g11}\) and \(\rho - S_{g12}\) are different, but in both cases we essentially obtain \(\rho \sim -S_T\), which confirms Herring’s law [43].

3. Results and discussions

From equations (23)–(25) and (28)–(30) it is revealed that phonon-limited resistivity and phonon-drag thermopower in the BG regime depend on temperature in a power law fashion, i.e., we have \(\rho \sim \nu S g T\). The effective exponent \(\nu T\) becomes 5 and 3 due to DP and PE scattering, respectively, in the case of resistivity. On the other hand, for phonon-drag thermopower, \(\nu T\) is 4 and 2 corresponding to DP and PE scattering, respectively. However, the integrals over \(\perp q\) in equations (17) and (21) have been evaluated numerically for both DP and PE scattering mechanisms to show the explicit temperature dependence of \(\rho\) and \(S_g\).

For the numerical calculation, the following material parameters, appropriate for an InAs QW, have been considered: \(m^* = 0.036m_e\), with free electron mass \(m_e\), \(g^* = -8\), \(\rho_n = 5.68 \times 10^3 \text{ Kg m}^{-3}\), \(v_d = 4.41 \times 10^3 \text{ m s}^{-1}\), \(v_{sd} = 2.35 \times 10^3 \text{ m s}^{-1}\), \(D = 5.08 \text{ eV}\), \(h_{14} = 3.5 \times 10^8 \text{ V m}^{-1}\), \(n_0 = 10^7 \text{ m}^{-3}\), \(R_0 = 10 \text{ nm}\), and \(\alpha_0 = 10^{-11} \text{ eVm}\). The value of the external magnetic field is taken to be \(B = 0.3 \text{ T}\).

Figure 1 shows the temperature variation of phonon-limited resistivity for different densities, namely \(n = 3n_0, 5n_0, 7n_0\), and \(9n_0\). The value of the Rashba parameter is considered to be \(\alpha = 3\alpha_0\). Figure 1 clearly demonstrates a crossover from the low temperature BG regime to a high temperature equipartition regime (in which \(\rho \sim T\)). For both DP and PE scattering mechanisms, \(\rho\) decreases as \(n\) increases. The resistivity due to PE scattering is higher in magnitude than with DP scattering. The exponent \(\nu T\) of the temperature dependence of \(\rho\) can be defined as \(\nu T = d \log \rho / d \log T\), which is extracted numerically, and its variation with temperature has been shown in the insets of figure 1. It is clear that the temperature variation of \(\nu T\) depends on electron density. At lower density, namely \(n = 3n_0\), the exponent \(\nu T\) shows a clear deviation from the limiting case (i.e., \(\nu T = 5\) due to DP and \(\nu T = 3\) for PE scattering). As density increases, the BG temperature regime becomes more stable. This numerically obtained BG regime is in excellent agreement with the approximated analytical results. As temperature increases, \(\nu T\) approaches its equipartition value, i.e., \(\nu T = 1\).

In figure 2 we show the temperature dependence of phonon-drag thermopower due to DP and PE scattering. \(S_g\) decreases as density increases. In this case we also extract the exponent \(\nu T = d \log S_g / d \log T\) of the temperature dependence of \(S_g\). Similar to the resistivity case, the temperature dependence of \(\nu T\) also depends on the density as depicted in the insets. At higher density, the BG regime is obtained in which \(S_g \sim T^4\) due to DP and \(S_g \sim T^2\) due to PE scattering. The magnitude of \(S_g\) due to PE scattering is higher than that due to DP scattering.
Figure 2. Temperature dependence of phonon-drag thermopower due to DP and PE scatterings is shown. Different values of density, namely $n = 3n_0$, $5n_0$, $7n_0$, and $9n_0$, are considered. The strength of the RSOI is fixed at $\alpha = 3\alpha_0$. The temperature dependencies of the effective exponent of the phonon-drag thermopower are shown in the insets of both panels.

Figure 3. The temperature variation of the effective exponent of resistivity, i.e., $\nu_T = d\log\rho/d\log T$ for various values of $\alpha$, namely, $\alpha = 0$, $3\alpha_0$, and $5\alpha_0$, is shown. The left panel shows DP scattering in which the upper and lower panels correspond to $n = 2n_0$ and $6n_0$. Similarly, the right panel describes PE scattering for $n = 2n_0$ and $6n_0$. 
Let us now discuss the following important point. The boundary of the BG regime is defined by the characteristic temperature $T_{BG} = 2\hbar k_F^2 / k_B$. For a typical value of electron density, say, $n = 5n_0$, we have $T_{BG} \sim 6$ K. But it is obtained numerically such that the BG regime exists for a small range of temperature below 1 K.

The temperature dependence of $\nu_T$ not only depends on the density, but also on the Rashba parameter $\alpha$. These facts are depicted in figures 3 and 4 in which the temperature dependences of $\nu_T$ corresponding to $\rho$ and $S_g$ for different $\alpha$ are shown. When density is low, the temperature variation depends significantly on $\alpha$. At a relatively higher density, the effect of $\alpha$ on this temperature dependence is not so prominent for both $\rho$ and $S_g$. A similar effect of $\alpha$ on the temperature dependence of $\rho$ or $S_g$ in a Rashba spin–orbit coupled two-dimensional electron gas in the BG regime has been addressed recently [36, 37].

In figures (5) and (6) we have shown how $\rho$ and $S_g$ depend on the electron density at a fixed temperature in the BG regime. In equations (23)–(25) and (28)–(30) one can notice that both IRT (and consequently $\rho$) and phonon-drag thermopower show a power law dependence with electron density through the Fermi wave vectors at a fixed temperature. So in general we can write $\rho$ or $S_g \sim n^{\nu_n}$, where the exponent $\nu_n$ corresponding to $\rho$ and $S_g$ can be obtained by taking negative logarithmic differentiation of $\rho$ or $S_g$ with respect to $n$, i.e., $\nu_n = -d \log \rho / d \log n$. Let us now estimate $\nu_n$ from equations (23)–(25) and (28)–(30). It is well known that the Fermi wave vector scales with density as $k_F^0 \sim n$ in one dimension. In our case $k_F^0$ depends on density in a complicated way, as seen from equation (5). Nevertheless, we can find $k_F^0 \sim n$ since $k_F \Delta \alpha \ll \hbar k_F^2$. From equations (23)–(25) one finds $(\tau^{-1}) \sim n^{-3}$ and as a result we have $\rho \sim n^{-6}$. The phonon-drag thermopower depends on density as $S_g \sim n^{-5}$ as seen from equations (28)–(30). However, solving equations (17) and (21) numerically, we find that $\nu_n$ undergoes a crossover from a relatively lower density behavior to a higher density behavior for both $\rho$ and $S_g$. As density increases, $\nu_n$ approaches the values obtained from asymptotic expressions, i.e., $\nu_n = 6$ for $\rho$ and $\nu_n = 5$ for $S_g$. Note that at a higher density the same values of $\nu_n$ are obtained due to DP and PE scattering for both cases of $\rho$ and $S_g$. But at lower densities, $\nu_n$ differs significantly due to DP and PE scattering.

Although a gap $\Delta$ is considered in the energy spectrum, its magnitude is much smaller than that corresponding to the Rashba spin-splitting, i.e., $\Delta \ll \hbar k_F$. The main purpose for considering $\Delta$ is to see whether inter-branch transitions are happening or not. But in the BG regime, the possibility of inter-branch scattering has been ruled out. So the qualitative results do not change significantly due to the presence of $\Delta$ in the energy spectrum.
4. Summary

In summary, we have studied various features of acoustic phonon-limited resistivity and phonon-drag thermopower in a Rashba spin–orbit coupled semiconductor QW with an in-plane magnetic field induced gap. Two mechanisms of electron–phonon interaction, namely, DP and PE scatterings, are taken into consideration. In the BG regime a power law dependence of both resistivity and phonon-drag thermopower with temperature have been obtained analytically. We find the exponent ($\nu_T$) of the temperature dependence, which takes the values 5 and 3, corresponding to the DP and PE scattering, respectively, in the case of resistivity. $\nu_T$ becomes 4 and 2 in the case of phonon-drag thermopower due to DP and PE scattering, respectively. Through a numerical calculation, we have shown a transition in resistivity from the BG to the equipartition regime. Numerically, it is also found that $\nu_T$ depends on both density and the Rashba parameter. At a higher density, $\nu_T$ matches well with that obtained from the analytical calculation for both $\rho$ and $S_g$, or in other words, a BG regime is established at higher density. The effect of SOI on $\nu_T$ is found to be more prominent in a low density regime. Finally, the dependence of $\rho$ and $S_g$ on the carrier density is also discussed. An approximate analytical calculation shows $\rho \sim n^{-6}$ and $S_g \sim n^{-5}$ in the BG regime. This dependence on $n$ has been confirmed through a numerical analysis at higher densities. The results obtained in the present case have also been compared with the corresponding results for a spin–orbit coupled two-dimensional electron system and we obtain in both cases $S_g \rho^{-1} \sim T^{-1}$, which affirms Herring’s law.

Appendix

In this appendix we shall perform an explicit derivation of the term $\delta(\epsilon_{k_F+q} - \epsilon_{k_F}) \mp \hbar \omega_F$ as given in equation (16). From equation (2) one can write

$$\epsilon_{k_F+q} = \frac{\hbar^2 (k_F + q)^2}{2m^*} + \lambda \sqrt{\alpha^2 (k_F + q)^2 + \Delta^2}. \quad (A.1)$$

Now defining $\epsilon_{k_F} = \sqrt{(\alpha k_F^2 + \Delta^2)}$ and assuming $q \ll k_F^2$, the second term in equation (A.1) can be expanded up to $q^2$ as

$$\epsilon_{k_F+q} = \epsilon_{k_F} + \frac{\alpha^2 k_F q}{\epsilon_{k_F}} + \frac{\alpha^2 q^2}{2\epsilon_{k_F}} \left(1 - \frac{\alpha^2 k_F^2}{\epsilon_{k_F}}\right). \quad (A.2)$$

We then have

$$\epsilon_{k_F+q} - \epsilon_{k_F} = \frac{\hbar^2}{2m^*} \tilde{g}_\nu^\lambda \left(q^2 + 2q \tilde{k}_F^\lambda \tilde{g}_\nu^\lambda\right), \quad (A.3)$$

where $\tilde{g}_\nu^\lambda$ and $\tilde{k}_F^\lambda$ are defined earlier.

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**Figure 5.** The dependence of phonon-limited resistivity on carrier density for different temperatures, namely, $T = 0.5$ K, 1 K, and 2 K, is shown. We consider $a = 2\alpha$. Panels (a) and (b) are due to DP and PE scattering. In panels (c) and (d) we show the variation of the quantity $\nu_n = -d \log \rho / d \log n$ with $n$ due to DP and PE scattering, respectively.
Since we are dealing with the BG regime in which $q \ll k^\perp$, the term $q^2$ in equation (A.3) can be neglected. So from the energy conservation $\epsilon_k^\alpha - \epsilon_{k^\perp} + \hbar \omega_Q = 0$, one can obtain $q = (m^\alpha v_k / \hbar k_F^\perp)Q$ with $Q = \sqrt{q^2 + q^2 \pm \lambda}$. Since the coefficient $m^\alpha v_k / (\hbar k_F^\perp) \ll 1$, consequently, we have $q \ll Q$, which in turn forces us to write the following expression

$$q = \frac{m^\alpha v_k}{\hbar k_F^\perp}q_L.$$  

(A.4)

We now calculate the delta function corresponding to the absorption case, which can be obtained in the following form

$$\delta(\epsilon_{k_{\perp}+q} - \epsilon_{k_{\perp}} - \hbar \omega_Q) = \frac{2m^\alpha}{\hbar^2 \tilde{g}_a^\perp} \times \delta \left( q^2 + 2q k_F^\perp \frac{\tilde{g}_a^\perp}{\tilde{g}_a^\perp} - \frac{2m^\alpha v_k q_L}{\hbar k_F^\perp} \right)$$

$$= \frac{2m^\alpha}{\hbar^2 \tilde{g}_a^\perp} \frac{1}{q_+ - q_-} \times \left\{ \delta(q - q_+) + \delta(q - q_-) \right\}.$$  

(A.5)

where $q_\pm = \left( -k_F^\perp + \sqrt{k_F^\perp + C^\perp q_L} \right)/\tilde{g}_a^\perp$ with $C^\perp = 2m^\alpha v_k / \hbar$. With the approximation $q \ll k^\perp_F$ one can find $q_+ = m^\alpha v_k q_L/(\hbar k_F^\perp)$ and $q_- = -2k_F^\perp / \tilde{g}_a^\perp - m^\alpha v_k q_L/(\hbar k_F^\perp)$. Since we are considering the BG regime, one may ignore the term $\delta(q - q_-)$ in equation (A.5). A similar analysis can be done for the emission case. Now it is straightforward to obtain equation (16) from equation (A.5).

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Figure 6. The dependence of phonon-drag thermopower on carrier density for different temperatures, namely, $T = 0.5 \text{K}$, $1 \text{K}$, and $2 \text{K}$, is shown. We consider $\alpha = 2\alpha_0$. Panels (a) and (b) are due to DP and PE scattering. In panels (c) and (d) we show the variation of the quantity $\nu = -d \log S_x / d \log n$ with $n$ due to DP and PE scattering, respectively.
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