James’s Conjecture holds for the principal block of the Iwahori-Hecke algebra \( \mathcal{H}_{5e} \) for \( e \geq 5 \)

Low Yi Rui

September 17, 2019

Department of Mathematics, National University of Singapore,
Block S17, 10 Lower Kent Ridge Road, 119076 Singapore
E-mail: e0046869@u.nus.edu

Abstract

James’s Conjecture predicts that the decomposition numbers for blocks of the Iwahori-Hecke algebra of the symmetric group over a field of prime characteristic is equal to that over \( \mathbb{C} \) when the weight of the block is strictly less than the characteristic of the field. In this paper, we prove James’s Conjecture for the principal block of \( \mathcal{H}_{5e} \). Moreover, we also address the case when the characteristic of the field is equal to five.

1 Introduction

Suppose \( q \) is a non-zero element of a field \( \mathbb{F} \). The Iwahori-Hecke algebra \( \mathcal{H}_{F,q}(\mathfrak{S}_n) \) of the symmetric group \( \mathfrak{S}_n \), over \( \mathbb{F} \) and with parameter \( q \) is the unital associative \( \mathbb{F} \)-algebra with generators \( T_1, T_2, \ldots, T_{n-1} \) subject to the following relations:

\[
\begin{align*}
\bullet & \quad (T_i - q)(T_i + 1) = 0, \quad i \in \{1, 2, \ldots, n-1\}. \\
\bullet & \quad T_i T_j = T_j T_i, \quad |i - j| > 1. \\
\bullet & \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad i \in \{1, 2, \ldots, n-1\}. 
\end{align*}
\]
When there is no ambiguity, we denote $\mathcal{H}_{F,q}(\mathfrak{S}_n)$ by $\mathcal{H}_n$. Let $e$ be the smallest integer such that $1 + q + q^2 + \cdots + q^{e-1} = 0$, assuming throughout the paper that it exists. If $q = 1$, $\mathcal{H}_n \simeq F\mathfrak{S}_n$ and $e$ is just the characteristic of $F$. To each partition $\lambda$ of $n$, we associate a Specht Module $S^\lambda$ for $\mathcal{H}_n$. A partition is $e$-singular if it has $e$ parts of the same size. It is called $e$-regular otherwise. For an $e$-regular partition $\lambda$, $S^\lambda$ has an irreducible cosocle $D^\lambda$. The set of $D^\lambda$ as $\lambda$ ranges over all $e$-regular partitions give a complete set of distinct irreducible $\mathcal{H}_n$-modules. We denote the projective cover of $D^\lambda$ by $P^\lambda$. One of the main problems of the representation theory of $\mathcal{H}_n$ is to determine the decomposition numbers $[S^\lambda : D^\mu] = [P^\mu : S^\lambda]$. Typically, this is recorded in the decomposition matrix with rows indexed by partitions of $n$ and columns indexed by $e$-regular partitions of $n$, whose $(\lambda, \mu)$-entry is $[S^\lambda : D^\mu]$.

One of the most important outstanding problems in the modular representation theory of the symmetric groups is to determine the decomposition numbers. When the field is $\mathbb{C}$, there is an algorithm for calculating the decomposition numbers for the Iwahori-Hecke algebras. It is known that the decomposition matrix for fields of prime characteristic may be obtained from that of $\mathbb{C}$ by post-multiplying by an ‘adjustment matrix’. Therefore, we often work with adjustment matrices instead of the decomposition matrices directly when the characteristic of the field is prime. Other than for some cases with small weight, there is a great deal not known about the adjustment matrices. James’s Conjecture predicts that the adjustment matrix for a block of $\mathcal{H}_n$ is the identity matrix when the characteristic of the field is strictly less than the weight of that block. The conjecture has been proven for weights up to four by the works of Richards [14] and Fayers [6, 4]. However, Williamson found a counter-example [17] to James’s Conjecture. Nevertheless, the smallest counter-example produced in his paper occurs in the symmetric group $\mathfrak{S}_n$ where $n = 1744860$. There is considerable interest in finding smaller counter-examples.

In sections 3 and 4, we prove that the adjustment matrix for the principal block of $\mathcal{H}_{5e}$ is the identity matrix when $\text{char}(F) \geq 5$. It is hoped that some of the techniques used here could be generalised to higher weights. In section 3, the case of $\mathcal{H}_{25}$ when $\text{char}(F) = 5$ is perhaps the most interesting. In this case, the defect group of the principal block of $\mathcal{H}_{25} = F\mathfrak{S}_{25}$ is not Abelian, and experts expect it to behave differently from $\mathcal{H}_{25}$ in characteristic zero. On the other hand, Fayers’s extension of James’s Conjecture [5, Conjecture 3.1] suggests that the decomposition numbers of these two blocks are the
same. In the next section, we lay the groundwork that we need for sections 3 and 4.

2 Background and techniques

2.1 Blocks of $\mathcal{H}_n$ and abacus displays

Take an abacus with $e$ vertical runners, numbered 0, . . . , $e - 1$ from left to right, marking positions 0, 1, . . . on the runners increasing from left to right along successive 'rows'. Given a partition $\lambda$ of $n$, take an integer $r \geq \lambda'_1$, the number of parts of $\lambda$. Define $\beta_i = \lambda_i + r - i$ for $i \in \{1, \ldots, r\}$. Now, place a bead at position $\beta_i$ for each $i$. The resulting configuration is called the abacus display for $\lambda$. We remark that moving a bead up one place on its runner is akin to removing an $e$-hook from the young diagram of $\lambda$. By moving all the beads as high as possible on their runners, the resulting configuration is the abacus display for the $e$-core of $\lambda$. If its $e$-core is a partition of $n - ew$, then we call $w$ the $e$-weight of $\lambda$.

**Theorem 2.1** (Nakayama Conjecture, [11, Corollary 5.38]) Let $\lambda$ and $\mu$ be partitions of $n$. Then, $S^\lambda$ and $S^\mu$ lie in the same block of $\mathcal{H}_n$ if and only if $\lambda$ and $\mu$ have the same $e$-core.

Therefore, we may define the $e$-weight and $e$-core of a block of $\mathcal{H}_n$ simply to be the $e$-weight and $e$-core of a partition lying in that block. Let $\lambda(i)$ be the partition corresponding to the abacus display containing only a single runner, the $i^{th}$ runner. Denote the number of beads in the $i^{th}$ runner as $b_i$. Then, we may write $\lambda$ as

$$\langle 0_{\lambda(0)}, \ldots, (e - 1)_{\lambda(e-1)} | b_0, \ldots, b_{e-1} \rangle;$$

we omit $i_{\lambda(i)}$ if $\lambda(i) = \emptyset$ and omit $\lambda(i)$ if $\lambda(i) = (1)$. Additionally, we may omit $b_0, \ldots, b_{e-1}$ if it is clear which block we are dealing with. If $\lambda$ lies in the block $B$ of $\mathcal{H}_n$, we say that $B$ is the block of $e$-weight $w$ with the $\langle b_0, \ldots, b_{e-1} \rangle$ notation.

2.2 Modular Branching Rules

We use some notational conventions for modules. We write

$$M \sim M_1^{\alpha_1} + M_2^{\alpha_2} + \cdots + M_r^{\alpha_r}.$$
to indicate that $M$ has a filtration in which the factors are $M_1, \ldots, M_r$ appearing $a_1, \ldots, a_r$ times respectively. Additionally, we write $M^{\oplus a}$ to indicate the direct sum of $a$ isomorphic copies of $M$.

There is a natural embedding $\mathcal{H}_{n-1} \leq \mathcal{H}_n$. If $M$ is a module for $\mathcal{H}_n$, the restriction of $M$ to $\mathcal{H}_{n-r}$ is denoted by $M \downarrow_{\mathcal{H}_{n-r}}$. Similarly, the induction of $M$ to $\mathcal{H}_{n+r}$ is denoted by $M \uparrow_{\mathcal{H}_{n+r}}$. If $B$ is a block of $\mathcal{H}_{n-r}$, we write $M \downarrow B$ to indicate the projection of $M$ to $\mathcal{H}_{n-r}$ onto $B$. Similarly, if $C$ is a block of $\mathcal{H}_{n+r}$, we write $M \uparrow B$ to indicate the projection of $M$ to $\mathcal{H}_{n+r}$ onto $C$. In this section, we describe the restriction and induction of Specht modules and simple modules.

Suppose $A$, $B$ and $C$ are blocks of $\mathcal{H}_{n-\kappa}$, $\mathcal{H}_n$ and $\mathcal{H}_{n+\kappa}$ respectively, and that there is an integer $i$ such that an abacus display for $A$ is obtained from that of $B$ by moving exactly $\kappa$ beads from runner $i$ to runner $i-1$, while an abacus display for $C$ is obtained from that of $B$ by moving exactly $\kappa$ beads from runner $i-1$ to runner $i$.

Suppose $\lambda$ is a partition in $B$, and that $\lambda^{-1}, \lambda^{-2}, \ldots, \lambda^{-r}$ are the partitions in $A$ that may be obtained from $\lambda$ by moving exactly $\kappa$ beads on runner $i$ one place to the left. Similarly, let $\lambda^{+1}, \lambda^{+2}, \ldots, \lambda^{+r}$ be the partitions in $C$ that may be obtained from $\lambda$ by moving exactly $\kappa$ beads on runner $i-1$ one place to the right. We have the following result.

**Theorem 2.2** (The Branching Rule [11, Corollary 6.2]) Suppose $A$, $B$, $C$ and $\lambda$ are as above. Then,

$$S^\lambda \downarrow_A \cong (S^{\lambda^{-1}})^{\kappa!} + (S^{\lambda^{-2}})^{\kappa!} + \cdots + (S^{\lambda^{-r}})^{\kappa!}$$

and

$$S^\lambda \uparrow_C \cong (S^{\lambda^{+1}})^{\kappa!} + (S^{\lambda^{+2}})^{\kappa!} + \cdots + (S^{\lambda^{+r}})^{\kappa!}.$$

For the discussion of the restriction and induction of simple modules, we assume that $\lambda$ is $\epsilon$-regular. The $i$-signature of $\lambda$ is the sequence of signs defined as follows. Starting from the top row of the abacus display for $\lambda$ and working downwards, write $a -$ if there is a bead on runner $i$ but no bead on runner $i-1$; write $a +$ if there is a bead on runner $i-1$ but no bead on runner $i$; write nothing for that row otherwise. Given the $i$-signature of $\lambda$, successively delete all neighbouring pairs of the form $--$ to obtain the reduced $i$-signature of $\lambda$. If there are any $-$ signs in the reduced $i$-signature of $\lambda$, we call the corresponding beads on runner $i$ normal; if there are at least $\kappa$ normal beads, then we define $\lambda^-$ to be the partition obtained by moving
the $\kappa$ highest normal beads one place to the left. If there are any $+$ signs in the reduced $i$-signature, we call the corresponding beads on runner $i - 1$ conormal; if there are at least $\kappa$ conormal beads, then we define $\lambda^+$ to be the partition obtained by moving the $\kappa$ lowest conormal beads one place to the right.

**Theorem 2.3** ([2, §2.5]) Suppose $A, B$ and $\lambda$ are as above.

- If there are fewer than $\kappa$ normal beads on runner $i$ of the abacus display for $\lambda$, then $D^\lambda \downarrow^B_A = 0$.

- If there are exactly $\kappa$ normal beads on runner $i$ of the abacus display for $\lambda$, then $D^\lambda \downarrow^B_A \cong (D^{\lambda^+})^{\oplus \kappa!}$.

- If there are fewer than $\kappa$ conormal beads on runner $i - 1$ of the abacus display for $\lambda$, then $D^\lambda \uparrow^C_B = 0$.

- If there are exactly $\kappa$ conormal beads on runner $i - 1$ of the abacus display for $\lambda$, then $D^\lambda \uparrow^C_B \cong (D^{\lambda^+})^{\oplus \kappa!}$.

**Remark** $D^\lambda \downarrow^B_A$ and $D^\lambda \uparrow^C_B$ only depends on $\lambda$, $e$, $A$, $B$ and $C$, not $F$ and $q$.

### 2.3 $v$-decomposition numbers

Let $\mathcal{P}$ be the set of all partitions. Let the quantum affine algebra, $U_v(\mathfrak{sl}_e)$ be the associative algebra over $\mathbb{C}(v)$ with generators $e_i, f_i, k_i, k_i^{-1}(0 \leq i \leq e - 1), d, d^{-1}$ subject to some relations (see [10, §4]). The Fock space representation $\mathcal{F}$ is the $U_v(\mathfrak{sl}_e)$-module with basis $\{s(\mu) : \mu \in \mathcal{P}\}$ as a $\mathbb{C}(v)$-vector space. Let $L$ be the free $\mathbb{Z}[v]$-lattice in $\mathcal{F}$ generated by $\{s(\nu) : \nu \in \mathcal{P}\}$. Moreover, let $x \mapsto \overline{x}$ be the bar involution on $\mathcal{F}$ (see [10, §6]) having the following (among other) properties:

- $\overline{b(v)x} = b(v^{-1})\overline{x}$ \quad $\forall b(v) \in \mathbb{C}(v), \forall x \in \mathcal{F}$.
- $\overline{f_i(x)} = f_i(\overline{x})$ \quad $\forall x \in \mathcal{F}$.

$\mathcal{F}$ has a distinguished basis $\{G(\mu) \mid \mu \in \mathcal{P}\}$, called the canonical basis satisfying:

- $\overline{G(\mu)} = G(\mu)$
- $G(\mu) \equiv s(\mu) \mod vL$. 

5
The \( v \)-decomposition number \( d^{(e)}_{\lambda\mu}(v) \) is the coefficient of \( s(\lambda) \) in \( G(\mu) \). Lascoux, Leclerc and Thibon have come up with the LLT algorithm \([9]\), a recursive algorithm for computing the canonical basis.

**Theorem 2.4** \([1, \text{Theorem 4.4}]\) Let \( \lambda \) and \( \mu \) be partitions of \( n \), with \( \mu \) \( e \)-regular. Then,

\[
[S^\lambda_{C,\zeta} : D^\mu_{C,\zeta}] = d^{(e)}_{\lambda\mu}(1).
\]

Consequently, the decomposition matrix for \( \mathcal{H}_{C,\zeta}(S_n) \) can be computed by the LLT algorithm.

Fix any field \( \mathbb{F} \). Let \( G^p_n(\mathbb{F}) \) be the Grothendieck group (see \([11, \text{Chapter 6, \S 1.1}]\)) of finitely generated projective \( \mathcal{H}_{F,q}(S_n) \)-modules with complex coefficients; that is the additive abelian group (with complex coefficients) generated by the symbols \([S^\rho]_p\), where \( \rho \) runs over the isomorphism classes of finitely generated projective \( \mathcal{H}_n \)-modules. These elements satisfy the relations \([S^\rho]_p = [M]_p + [N]_p\) whenever \( \rho = M \oplus N \). Therefore, the set of \([P^\lambda]_p \) as \( \lambda \) runs over all \( e \)-regular partitions of \( n \) forms a basis of \( G^p_n(\mathbb{F}) \).

Let \( \mathcal{E}_n \) be the complex vector space with basis the set of symbols \([S^\nu]_p\) where \( \nu \) runs over all partitions of \( n \). (\( \mathcal{E}_n \) is the Grothendieck group of a semi-simple Iwahori-Hecke algebra.) Recall that \([S^\lambda]_p : D^\mu_{\mathcal{F}}] = [P^\mu_{\mathcal{F}} : S^\lambda_{\mathcal{F}}]. \) There is an injective homomorphism of abelian groups \( e_{\mathcal{F}} : G^p_n(\mathbb{F}) \to \mathcal{E}_n \) determined by

\[
e_{\mathcal{F}}[P^\lambda_{\mathcal{F}}] = \sum_{\nu + n} [S^\nu_{\mathcal{F}} : D^\lambda_{\mathcal{F}}][S^\nu_{\mathcal{F}}].
\]

Suppose that \( A \) and \( B \) are blocks of \( \mathcal{H}_n \) and \( \mathcal{H}_{n+1} \) respectively, and that an abacus display with \( r \) beads for \( B \) is obtained from that for \( A \) by moving a bead from runner \( k - 1 \) to runner \( k \). Let \( i \) be the residue of \( k - r \) modulo \( e \). We define \( i \)-Ind to be the group homomorphism from \( G^p_n(\mathbb{F}) \) to \( G^p_{n+1}(\mathbb{F}) \) taking \([P^\rho]_p \) to \([P \uparrow^B_A]^p\). By abusing notation, we also refer to \( i \)-Ind as the group homomorphism from \( \mathcal{E}_n \) to \( \mathcal{E}_{n+1} \) taking \([S^\nu] \) to \([S^\nu \uparrow^B_A]\).

We now describe the action of \( f_i \) on \( s(\lambda) \). Display \( \lambda \) on an abacus with \( e \) runners and \( r \) beads, where \( r \geq \lambda'_1 \). Let \( k \) be the residue class of \((i + r) \) modulo \( e \). Suppose there is a bead on runner \( k - 1 \) whose succeeding position on runner \( k \) is vacant. Let \( \mu \) be the partition whose abacus display is obtained by moving such a bead to its succeeding position. Define \( N_i(\lambda, \mu) \) to be the number of beads on runner \( k - 1 \) below the bead moved to obtain \( \mu \) minus the number of beads on runner \( k \) below the vacant position that becomes
occupied in obtaining $\mu$. Then,
\[ f_i(s(\lambda)) = \sum_{\mu} v^{N_i(\lambda, \mu)} s(\mu). \]

Note that when $v = 1$, $f_i$ acts in the same way as $i$-Ind on $\mathcal{E}_n$.

**Proposition 2.5** (Proposition 2.4])
If we write $f_i(G(\mu))$ in the form
\[ f_i(G(\mu)) = \sum_{\nu} a_\nu(v) G(\nu), \]
then $a_\nu(v) \in \mathbb{N}_0[v + v^{-1}]$ for all $\nu$.

### 2.4 Adjustment Matrices and James’s Conjecture

Denote $\mathcal{H}_{\mathfrak{c}, \lambda}(\mathfrak{S}_n)$ by $\mathcal{H}^0_n$ and $\mathcal{H}_{\mathfrak{c}, q}(\mathfrak{S}_n)$ by $\mathcal{H}_n$. 2 partitions lie in the same block of $\mathcal{H}_n$ if and only if they lie in the same block of $\mathcal{H}^0_n$ by Nakayama’s lemma. Therefore, given a block $B$ of $\mathcal{H}_n$, we may denote $B^0$ to be its corresponding block in $\mathcal{H}^0_n$.

**Theorem 2.6** ([11, Theorem 6.35]) Let $D$ and $D^0$ be the decomposition matrices for the blocks $B$ and $B^0$ respectively. Then, there is a square matrix $A$ with non-negative integer entries such that
\[ D = D^0 A. \]

We call $A$ the adjustment matrix for the block $B$. Since $D^0$ can be computed by the LLT algorithm, $D$ is often studied by considering its adjustment matrix.

**Conjecture 2.7** Let $B$ be a block of $\mathcal{H}_n$ of $e$-weight $w$. If $w < \text{char}(\mathbb{F})$, then the adjustment matrix for the block $B$ is the identity matrix.

**Theorem 2.8** (Theorem 2.5, Theorem 2.6]) Suppose $\text{char}(\mathbb{F}) \geq 5$, and that $B$ is a block of $\mathcal{H}_n$ of weight at most 4. Then, James’s Conjecture holds for $B$.

The conjecture has been proved for weights at most four. In this paper, we prove the conjecture and its extension by Fayers for the principal block of $\mathcal{H}_{5e}$ which has $e$-weight equal to 5.

**Theorem 2.9** Suppose $e \geq 5$ and $\text{char}(\mathbb{F}) \geq 5$, then the adjustment matrix for the principal block of $\mathcal{H}_{5e}$ is the identity matrix.
2.5 The Mullineux map

Let $T_1, \ldots, T_{n-1}$ be the standard generators of $\mathcal{H}_n$ defined at the beginning of this section. Let $\sharp : \mathcal{H}_n \to \mathcal{H}_n$ be the involutory automorphism sending $T_i$ to $q-1-T_i$. Given a $\mathcal{H}_n$-module $M$, define $M^\sharp$ to be the module with the same underlying vector space and with action

$$h \cdot m = h^\sharp m.$$ 

In the case of the symmetric groups when $q = 1$, $M^\sharp$ is $M \otimes \text{sgn}$, where sgn is the 1-dimensional signature representation. Let $\lambda^\circ$ be the $e$-regular partition such that $(D^\lambda)^\sharp \cong D^{\lambda^\circ}$. The map $\lambda \mapsto \lambda^\circ$ is an involutary bijection from the set of $e$-regular partitions of $n$ to itself, and is given combinatorially by Mullineux's algorithm [13] which depends only on $\lambda$ and $e$, not $F$ and $q$.

**Proposition 2.10** ([4, Lemma 4.2]) If $\lambda$ and $\mu$ are $e$-regular partitions of $n$, then $\text{adj}_{\lambda\mu} = \text{adj}_{\lambda^\circ\mu^\circ}$.

2.6 The Jantzen-Schaper formula

Let $\lambda$ be a partition and consider its abacus display, say with $k$ beads. Suppose that after moving a bead at position $a$ up its runner to a vacant position $a-ie$, we obtain the partition $\mu$. Denote $l_{\lambda\mu}$ for the number of occupied positions between $a$ and $a-ie$, and let $h_{\lambda\mu} = i$.

Further, write $\lambda \xrightarrow{\mu} \tau$ if the abacus display of $\tau$ with $k$ beads is obtained from that of $\mu$ by moving a bead at position $b-ie$ to a vacant position $b$, and $a < b$.

**Definition** Jantzen-Schaper bound

Let $p = \text{char}(F)$.

$$J_F(\lambda, \mu) = \sum_{\tau, \sigma} (-1)^{l_{\lambda\sigma} + l_{\tau\sigma} + 1} (1 + v_p(h_{\lambda\sigma})) [S_\tau^\tau : D_\tau^\mu],$$

where the sum runs through all $\tau$ and $\sigma$ such that $\lambda \xrightarrow{\sigma} \tau$, and where $v_p$ denotes the standard $p$-valuation if $p > 0$ and $v_0(x) = 0 \forall x$.

**Theorem 2.11** Jantzen-Schaper formula ([8, Theorem 4.7])

$$[S_\lambda^\lambda : D_\lambda^\mu] \leq J_F(\lambda, \mu).$$

Moreover, the left-hand side is zero if and only if the right-hand side is zero.
Corollary 2.12 If $J_{\mathcal{F}}(\lambda, \mu) \leq 1$, then

$$[S^\lambda_{\mathcal{F}} : D^\mu_{\mathcal{F}}] = J_{\mathcal{F}}(\lambda, \mu).$$

We write $\lambda \rightarrow \tau$ if there exists some $\mu$ such that $\lambda \overset{\mu}{\rightarrow} \tau$. Further, write $\lambda <_J \sigma$ if there exist partitions $\tau_0, \tau_1, \ldots, \tau_r$ such that $\tau_0 = \lambda$, $\tau_r = \sigma$ and $\tau_{i-1} \rightarrow \tau_i \forall i \in \{1, 2, \ldots, r\}$. We call $\leq_J$ the Jantzen order and it is clear that this defines a partial order on the set of all partitions, and that only partitions in the same block are comparable under this partial order. Moreover, the usual dominance order extends the Jantzen order. Combined with the fact that $\mathcal{H}_n$ is a cellular algebra, we have the following theorem.

Theorem 2.13 Suppose $\lambda$ and $\mu$ are partitions of $n$, with $\mu$ e-regular. Then,

- $[S^\mu : D^\mu] = 1$;
- $[S^\lambda : D^\mu] > 0 \Rightarrow \mu \geq_J \lambda$.

Corollary 2.14 Suppose $\lambda$ and $\mu$ are e-regular partitions lying in a block $B$ of $\mathcal{H}_n$. Then,

- $\text{adj}_{\mu\mu} = 1$;
- $\text{adj}_{\lambda\mu} > 0 \Rightarrow \mu \geq_J \lambda$.

It is difficult to check that $\mu \not\geq_J \lambda$ by inspection. To this end, we introduce the product order on partitions. Let $\lambda$ be a partition, displayed on an abacus with $e$ runners and $N$ beads. Suppose that the beads having positive $e$-weights are at positions $a_1, a_2, \ldots, a_r$ with weights $w_1, w_2, \ldots, w_r$ respectively. The induced $e$-sequence of $\lambda$, denoted $s(\lambda)_N$, is defined as

$$\bigcup_{i=1}^r(a_i, a_i - e, \ldots, a_i - (w_i - 1)e),$$

where $(b_1, b_2, \ldots, b_s) \sqcup (c_1, c_2, \ldots, c_t)$ denotes the weakly decreasing sequence obtained by rearranging terms in the sequence $(b_1, \ldots, b_s, c_1, \ldots, c_t)$. Note that $s(\lambda)_N \in \mathbb{N}_0^w$, where $w$ is the $e$-weight of $\lambda$.

We define a partial order $\geq_P$ on the set of partitions by: $\mu \geq_P \lambda$ if and only if $\mu$ and $\lambda$ have the same $e$-core and $e$-weight, and $s(\mu)_N \geq s(\lambda)_N$ (for sufficiently large $N$) in the standard product order of $\mathbb{N}_0^w$. 
Lemma 2.15 (\cite{16}, Lemma 2.9) 

\[ \lambda \leq_J \mu \Rightarrow \lambda \leq_P \mu. \]

Therefore, \( \mu \nRightarrow_P \lambda \Rightarrow \text{adj}_{\lambda \mu} = 0. \)

Theorem 2.16 (\cite{17}, Theorem 1) Suppose \( \lambda \) and \( \mu \) are partitions of \( n \), with \( \mu \) e-regular, and let \( d_{\lambda \nu}^{(e)}(v) \) denote the derivative of the \( v \)-decomposition number \( d_{\lambda \nu}^{(e)}(v) \) with respect to \( v \). Then

\[ J_C(\lambda, \mu) = d_{\lambda \mu}^{(e)}(1). \]

Corollary 2.17 Suppose \( \lambda \) and \( \mu \) are partitions lying in the principal block \( B \) of \( H_{5e} \), \( e \geq 5 \) and \( p = \text{char}(F) \geq 5 \). Moreover, suppose that \( \lambda \) is not of the form \( \langle i_5 \rangle \). Additionally, suppose that \( \text{adj}_{\nu \mu} = 0 \) for all e-regular partitions \( \nu \) such that \( \lambda <_J \nu <_J \mu \), and that \( d_{\lambda \mu}^{(e)}(v) \in \{0, v\} \). Then, \( \text{adj}_{\lambda \mu} = 0. \)

Proof Suppose \( \lambda <_J \nu <_J \mu \). Then,

\[ [S_\nu^v : D_\mu^v] = \sum_{\nu \leq \sigma \leq \mu} \text{adj}_{\sigma \mu}[S_\sigma^v : D_\sigma^v] = [S_\nu^v : D_\nu^v], \]

where the first equality is due to the definition of adjustment matrices, Theorem 2.13 and Corollary 2.14, and the second equality is due to our assumptions in the statement. Since \( \lambda \) is not of the form \( \langle i_5 \rangle \), \( v_p(h_{\lambda \sigma}) = 0 \) for all \( \sigma \) and \( \tau \) such that \( \lambda \nRightarrow \tau \). Hence,

\[ J_F(\lambda, \mu) = J_C(\lambda, \mu). \]

By the previous theorem, \( J_C(\lambda, \mu) = 0 \) or 1 when \( d_{\lambda \mu}^{(e)}(v) = 0 \) or \( v \) respectively. Therefore, \( J_F(\lambda, \mu) = J_C(\lambda, \mu) \leq 1 \) and we have

\[ [S_\lambda^\lambda : D_\mu^\mu] = J_C(\lambda, \mu) = J_F(\lambda, \mu) = [S_\lambda^\lambda : D_\mu^\mu]. \]

By the definition of adjustment matrices, Theorem 2.13 and Corollary 2.14,

\[ [S_\lambda^\lambda : D_\mu^\mu] = \sum_{\lambda \leq \sigma \leq \mu} \text{adj}_{\sigma \mu}[S_\sigma^\lambda : D_\sigma^\mu] = [S_\lambda^\lambda : D_\mu^\mu] + \text{adj}_{\lambda \mu}. \]

So, \( \text{adj}_{\lambda \mu} = 0 \) as required. \( \blacksquare \)

10
2.7 The row removal theorem

**Theorem 2.18** ([7, Theorem 6.18]) Suppose \( \lambda \) and \( \mu \) are partitions of \( n \), with \( \mu \) \( e \)-regular, and that \( \lambda_1 = \mu_1 \). Define
\[
\lambda^2 = (\lambda_2, \lambda_3, \ldots), \quad \mu^2 = (\mu_2, \mu_3, \ldots).
\]
Then
\[
[S^\lambda : D^\mu] = [S^{\lambda^2} : D^{\mu^2}].
\]

**Corollary 2.19** Suppose \( \lambda \) and \( \mu \) are distinct partitions of \( n \), with \( \mu \) \( e \)-regular, and that \( \lambda_1 = \mu_1 \). Moreover, suppose that James’s Conjecture holds for the block containing \( \lambda^2 \) and \( \mu^2 \). Then \( \text{adj}_{\lambda \mu} = 0 \).

**Proof** Since James’s Conjecture holds for the block containing \( \lambda^2 \) and \( \mu^2 \), we have
\[
[S_F^{\lambda^2} : D_F^{\mu^2}] = [S_C^{\lambda^2} : D_C^{\mu^2}],
\]
hence
\[
[S_F^\lambda : D_F^\mu] = [S_C^\lambda : D_C^\mu]
\]
by the previous theorem. However,
\[
[S_F^\lambda : D_F^\mu] = \sum_{\lambda \leq j \sigma \leq j \mu} \text{adj}_{j \sigma \mu}[S_C^\lambda : D_C^\sigma]
\]
\[
= [S_C^\lambda : D_C^\mu] + \text{adj}_{\lambda \mu} + \sum_{\lambda < j \sigma < j \mu} \text{adj}_{j \sigma \mu}[S_C^\lambda : D_C^\sigma].
\]
The terms in the final sum are all non-negative, so \( \text{adj}_{\lambda \mu} = 0 \) as required.

2.8 Lowerable partitions

**Proposition 2.20** Suppose that \( \text{char}(\mathbb{F}) \geq 5 \), \( B \) is a block of \( \mathcal{H}_n \) of weight 5, and \( C \) is a block of \( \mathcal{H}_{n-1} \) of weight less than 5. Let \( \lambda \) and \( \mu \) be distinct \( e \)-regular partitions lying in \( B \) such that \( D^\mu \downarrow_C \neq 0 \), while \( D^\lambda \downarrow_C \) is either zero or simple. Then \( \text{adj}_{\lambda \mu} = 0 \).

**Proof** This is essentially the same as [6, Proposition 2.17]. Let \( B^0 \) and \( C^0 \) be the blocks of \( \mathcal{H}_n^0 \) and \( \mathcal{H}_{n-1}^0 \) respectively corresponding to \( B \) and \( C \).
The modular branching rules which are characteristic-free imply that there is an $e$-regular partition $\hat{\mu}$ in $C$ such that $D_{\hat{\mu}} F \downarrow C$ is an indecomposable module with simple socle $D_{\hat{\mu}}$, while $D_{\hat{\mu}} C \downarrow C_0$ is an indecomposable module with simple socle $D_{\hat{\mu}}$. Moreover, we have $[D_{\hat{\mu}} F \downarrow C: D_{\hat{\mu}}] = 0$; because $D_{\hat{\mu}} C \downarrow C_0$ is either simple or zero, and if the former occurs, the modular branching rules show that it will be different from $D_{\hat{\mu}} C$.

Let $T$ be the ‘simple branching matrix’ from $B$ to $C$, with rows indexed by $e$-regular partitions in $B$ and columns by $e$-regular partitions in $C$, and with the $(\nu, \sigma)$-entry being the composition multiplicity $[D_{\nu} F \downarrow C: D_{\sigma}]$. Let $T^0$ be the simple branching matrix from $B^0$ to $C^0$ defined analogously. Using the fact that restriction is an exact functor, we have $T^0 Z = AT$, where $Z$ and $A$ are the adjustment matrices for $C$ and $B$ respectively. James’s Conjecture holds for $C$ [6, Theorem 2.6], therefore

$$T^0 = AT.$$ 

Comparing the $(\lambda, \mu)$-entries of both sides yields

$$0 = [D_{\lambda} C \downarrow C_0: D_{\mu}] = \sum_{\nu} \text{adj}_{\lambda\nu}[D_{\nu} F \downarrow C: D_{\mu}] = \text{adj}_{\lambda\mu}[D_{\mu} F \downarrow C: D_{\mu}] + \sum_{\nu \neq \mu} \text{adj}_{\lambda\nu}[D_{\nu} F \downarrow C: D_{\mu}].$$

Since every term of the sum is non-negative and $[D_{\mu} F \downarrow C: D_{\mu}] > 0$, we conclude that $\text{adj}_{\lambda\mu} = 0$. □

**Definition** If $\lambda$ and $\mu$ satisfy the conditions of the proposition above, we say that $(\lambda, \mu)$ is lowerable.

### 3 The principal block of $\mathcal{H}_{25}$

Let $B$ be the principal block of $\mathcal{H}_{5e}$, $e \geq 5$.

**Lemma 3.1** If $\mu$ is an $e$-regular partition in $B$, then there is some block $C$ of $\mathcal{H}_{5e-1}$ of weight less than 5 such that $D^\mu \downarrow_C \neq 0$.

**Proof** Since $\mathcal{H}_{5e-1}$ is a unital subalgebra of $\mathcal{H}_{5e}$, we have $D^\mu \downarrow_{\mathcal{H}_{5e-1}} \neq 0$; in particular, $D^\mu \downarrow_C \neq 0$ for some block $C$ of $\mathcal{H}_{5e-1}$. Clearly, every block of $\mathcal{H}_{5e-1}$ has weight less than 5, so the result follows. □
Corollary 3.2  Suppose \( \lambda \) and \( \mu \) are \( e \)-regular partitions in \( B \). If there is no block \( C \) of \( \mathcal{H}_{5e-1} \) such that \( D^\lambda \downarrow_C \) is reducible, then \( \text{adj}_{\lambda\mu} = 0 \).

**Proof**  Suppose \( \lambda \) and \( \mu \) are as in the statement. By the previous lemma, there is a block \( C \) of \( \mathcal{H}_{5e-1} \) such that \( D^\mu \downarrow_C \neq 0 \). By assumption, \( D^\lambda \downarrow_C \) is zero or simple. Therefore \((\lambda, \mu)\) is lowerable and Proposition 2.20 implies that \( \text{adj}_{\lambda\mu} = 0 \).

The only \( e \)-regular partitions \( \lambda \) in \( B \) such that \( D^\lambda \) is reducible after restricting to some block of \( \mathcal{H}_{5e-1} \) are:

- \( \langle i_3, 2 \rangle \), \( i \in \{1, 2, \ldots, e-1\} \).
- \( \langle i_{2^2}, j \rangle \), \( 1 \leq i \leq j-1 \).
- \( \langle j, i_{2^2} \rangle \), \( j \leq i-2 \).

In light of the corollary, we need only consider these rows of the adjustment matrix in order to prove James’s Conjecture for the block \( B \).

The weight 4 block \( C \) with the \( (5^{i-1}, 6, 4, 5^{e-i-1}) \) notation is the only block of \( \mathcal{H}_{5e-1} \) such that \( D^\lambda \downarrow_C \) is reducible. Hence, if \( D^\mu \downarrow_D \neq 0 \) for some block \( D \neq C \), then \((\lambda, \mu)\) would be lowerable and by Proposition 2.20, \( \text{adj}_{\lambda\mu} = 0 \). Therefore, we may assume that \( D^\mu \downarrow_D = 0 \) for every block \( D \) of \( \mathcal{H}_{5e-1} \) other than \( C \).

For the rest of this section, we assume that \( e = 5 \).

3.1  \( \lambda = \langle i_{3,2} \rangle \)

By Corollary 2.19, \( \text{adj}_{\lambda\mu} \neq 0 \) \( \Rightarrow \mu_1 > \lambda_1 \). Those \( \mu \) satisfying \( \mu_1 > \lambda_1 \) are:

- \( \langle i_5 \rangle \)
- \( \langle i_4, i+1 \rangle \) where \( i \leq 3 \)
- \( \langle 0, i_4 \rangle \)

\( \text{adj}_{\lambda\mu} = \text{adj}_{\lambda^5\mu^5} \) by Proposition 2.10. Therefore, we may also assume that \( \mu_i^5 > \lambda_i^5 \). We calculate \( \lambda^5 \) and \( \mu^5 \) for all of the pairs above (see the tables at the end of the paper) and find that none of them satisfy this condition. Therefore, \( \text{adj}_{\lambda\mu} = 0 \) for every \( e \)-regular \( \mu \) in the block \( B \).
3.2 \( \lambda = \langle i_{22}, j \rangle \) or \( \langle j, i_{22} \rangle \)

For the same reasons as in the former case, we only consider e-regular \( \mu \) such that \( \mu_1 > \lambda_1 \). At this stage, the possibilities for \( \mu \) are:

- \( \langle i_5 \rangle \)
- \( \langle i_4, i + 1 \rangle \) where \( i \leq 3 \)
- \( \langle 0, i_4 \rangle \)
- \( \langle i_{2,3} \rangle \)
- \( \langle i_3, (i + 1)_{12} \rangle \)
- \( \langle i_3, (i + 1)_2 \rangle \) where \( i \leq 3 \)
- \( \langle 0_2, 4_3 \rangle \) where \( i = 4 \)
- \( \langle 0_3, 4_2 \rangle \) where \( i = 4 \)
- \( \langle i_3, i + 1, i + 2 \rangle \) where \( i \leq 2 \)
- \( \langle 0, i_3, i + 1 \rangle \) where \( i \leq 3 \)
- \( \langle 0, 1, i_3 \rangle \)
- \( \langle i_2, (i + 1)_2, i + 2 \rangle \) where \( i \leq 2 \)
- \( \langle 0, i_2, (i + 1)_2 \rangle \) where \( i \leq 3 \)

By Proposition 2.10 as well as Lemma 2.15 and Corollary 2.19, we may further assume that \( \mu^c >_P \lambda^c \) and \( \mu_1^c > \lambda_1^c \). We calculate \( \lambda^c \) and \( \mu^c \) using the Mullineux map for all of the pairs above (see the table at the end of the section) and list the pairs \((\lambda, \mu)\) of partitions satisfying these two conditions:

1. \( (\langle 2, 4_{22} \rangle, \langle 0_3, 4_2 \rangle) \)

1°. \( (\langle 3, 1_{22} \rangle, \langle 1_2, 2_2, 3 \rangle) \)

2. \( (\langle 1, 4_{22} \rangle, \langle 0_3, 4_2 \rangle) \)

2°. \( (\langle 4, 1_{22} \rangle, \langle 1_2, 2_2, 3 \rangle) \)
3. \((\langle 1, 4_2 \rangle, \langle 0, 1, 4_3 \rangle)\)
3°. \((\langle 4, 1_2 \rangle, \langle 0, 1_3, 2 \rangle)\)
4. \((\langle 4, 3_2 \rangle, \langle 3_3, 4_2 \rangle)\)
4°. \((\langle 3, 2_2 \rangle, \langle 2_2, 3_2, 4 \rangle)\)
5. \((\langle 1, 3_2 \rangle, \langle 3_3, 4_2 \rangle)\)
5°. \((\langle 4, 2_2 \rangle, \langle 2_2, 3_2, 4 \rangle)\)

Note that the list has been numbered so that if case \(k\) is \((\lambda, \mu)\), then case \(k°\) is \((\lambda°, \mu°)\). Therefore, cases \(k\) and \(k°\) are essentially the same since \(\text{adj}_{\lambda\mu} = \text{adj}_{\lambda°\mu°}\) by Proposition 2.10.

We apply the LLT algorithm and find that in cases 2° and 4, \(d_{5\mu}^{(5)}(v) = 0\) and in cases 1° and 5°, \(d_{5\mu}^{(5)}(v) = v\). Since \(\text{adj}_{\nu\mu} = 0\) whenever \(\lambda <_P \nu <_P \mu\) in cases 2°, 4, 5°, Corollary 2.17 rules them out directly.

We now know more entries of the adjustment matrix, so we are able to deduce that \(\text{adj}_{\nu\mu} = 0\) whenever \(\lambda <_P \nu <_P \mu\) for case 1° as well, thereby ruling it out by Corollary 2.17 again.

In the next section, we will have an argument which establishes that \(\text{adj}_{\lambda\mu} = 0\) for case 3. Therefore, \(\text{adj}_{\lambda\mu} = 0\) for all cases and we may conclude that the adjustment matrix for this block is the identity matrix.

4 The principal block of \(H_{5e}, e \geq 6\)

Let \(B\) be the principal block of \(H_{5e}, e \geq 6\). As mentioned in the beginning of the last section, a pair of \(e\)-regular partitions \((\lambda, \mu)\) in \(B\) are lowerable unless \(\lambda\) is of the form:

- \(\langle i_3, 2 \rangle\) where \(1 \leq i \leq e - 1\)
- \(\langle i_2, j \rangle\) where \(1 \leq i \leq j - 1\)
- \(\langle j, i_2 \rangle\) where \(j \leq i - 2\)

and moreover, \(D_{\mu}^\lambda \downarrow D = 0\) for every block \(D\) of \(H_{5e-1}\) other than \(C\), where \(C\) is the weight 4 block with the \(\langle 5^{i-1}, 6, 4, 5^{e-i-1} \rangle\) notation.
4.1 $\lambda = \langle i_{3,2} \rangle$

By Corollary 2.19, $\text{adj}_{\lambda\mu} \neq 0 \Rightarrow \mu_1 > \lambda_1$. Those $\mu$ satisfying $\mu_1 > \lambda_1$ are:

- $\langle i_5 \rangle$
- $\langle i_4, i + 1 \rangle$
- $\langle 0, i_4 \rangle$

Proposition 2.10 gives $\text{adj}_{\lambda\mu} = \text{adj}_{\lambda^e\mu^e}$. Therefore, we may also assume that $\mu_1^e > \lambda_1^e$. We calculate $\lambda^e$ and $\mu^e$ for all of the pairs above and find that none of them satisfy this condition. Therefore, $\text{adj}_{\lambda\mu} = 0$ for every $e$-regular $\mu$ in the block $B$.

4.2 $\lambda = \langle i_{2,2}, j \rangle$ or $\langle j, i_{2,2} \rangle$

For the same reasons as in the former case, we only consider $e$-regular $\mu$ such that $\mu_1 > \lambda_1$. At this stage, the possibilities for $\mu$ are:

- $\langle i_5 \rangle$
- $\langle i_4, i + 1 \rangle$
- $\langle 0, i_4 \rangle$
- $\langle i_{3,2} \rangle$
- $\langle i_3, (i + 1)_1 \rangle$
- $\langle i_3, (i + 1)_2 \rangle$
- $\langle 0_2, (e - 1)_3 \rangle$ where $i = e - 1$
- $\langle 0_3, (e - 1)_2 \rangle$ where $i = e - 1$
- $\langle i_3, i + 1, i + 2 \rangle$
- $\langle 0, i_3, i + 1 \rangle$
- $\langle 0, 1, i_3 \rangle$
- $\langle i_2, (i + 1)_2, i + 2 \rangle$
• \(\langle 0, i_2, (i + 1)_2 \rangle\)

By proposition 2.10, we may assume that \(\mu_i^e > \lambda_i^e\). Therefore, the only possibilities for \((\lambda, \mu)\) are the following:

1. \(\langle i_2^e, j \rangle, \langle 0, 1, i_3 \rangle\) where \(j \geq i + 2 \geq 4\) and \(e \geq 6\)

\(1^\circ.\) \(\langle e - j, (e - i)_2^e \rangle, \langle 0, 1, (e - i)_3 \rangle\)

2. \(\langle i_2^e, i + 1 \rangle, \langle 0, 1, i_3 \rangle\) where \(e \geq 6\)

\(2^\circ.\) \(\langle (e - i)_2^e, e - i + 1 \rangle, \langle 0, 1, (e - i)_3 \rangle\)

3. \(\langle 1_2^e, j \rangle, \langle 0, 1_3, 2 \rangle\) where \(j \geq i + 2 = 3\) and \(e \geq 6\)

\(3^\circ.\) \(\langle e - j, (e - 1)_2^e \rangle, \langle 0, 1, (e - 1)_3 \rangle\)

4. \(\langle 3_2^e, 5 \rangle, \langle 3_2, 4_2, 5 \rangle\) where \(e = 6\)

\(4^\circ.\) \(\langle 1, 3_2^e \rangle, \langle 3_2, 4_2, 5 \rangle\) where \(e = 6\)

Note that the list has been numbered so that if case \(k\) is \((\lambda, \mu)\), then case \(k^\circ\) is \((\lambda^\circ, \mu^\circ)\).

We use the product order to check that \(\mu \npreceq_P \lambda\) in cases 1 and \(2^\circ\). In case \(3^\circ\), \(\mu \npreceq_P \lambda\) when \(e - j \geq 2\). In case 4, we use the LLT algorithm to calculate \(d_{2^6}^{(6)}(v) = 0\), hence \(adj_{\lambda \mu} = 0\) by Corollary 2.17. By Lemma 2.15, we are left with the single case

\[(\lambda, \mu) = \langle (1, e - 1)_2^e, \langle 0, 1, (e - 1)_3 \rangle \rangle.\]

From now on, we shall assume that \(e \geq 5\) to include case 4 of section 3. Let \(D\) be the weight 4 block with the \(\langle 5^{e-2}, 6, 4 \rangle\) notation.

By the modular branching rules, \(D^{\tilde{\mu}}\) is the only simple module in block \(D\)
that upon induction to block $B$ has $D^\mu$ as a direct summand. Similarly, the only Specht modules in $D$ that upon induction to $B$ have a filtration with a factor of $S^\lambda$ are $S^{\lambda_0}$ and $S^{\lambda_1}$. By Proposition 2.5, we may write $f_{e-1}(G(\tilde{\mu}))$ in the form

$$f_{e-1}(G(\tilde{\mu})) = \sum_{\nu} a_\nu(v)G(\nu), \quad (1)$$

where $a_\nu(v) \in \mathbb{N}_0[v + v^{-1}]$ for all $\nu$.

If we manage to show that $a_\lambda(v) = 0$, then we have that

$$P^\tilde{\mu}_{\tilde{C}} \uparrow_B^D \cong \bigoplus_{\nu \neq \lambda} a_\nu(1)P^\nu_C$$

(2)

since $f_{e-1}$ acts like $(e-1)$-Ind when $v = 1$. Since James’s Conjecture holds for blocks of weight four,

$$\mathbf{e}_C([P^\tilde{\mu}_C \uparrow_B^D]^p) = \mathbf{e}_F([P^\tilde{\mu}_F \uparrow_B^D]^p).$$

By equation (2), the left-hand side is $\sum_{\nu \neq \lambda} a_\nu(1)\mathbf{e}_C([P^\nu_C]^p)$. On the other hand, the right-hand side contains the term $\mathbf{e}_F([P^\mu_F]^p) = \mathbf{e}_C([P^\mu_C]^p) + \text{adj}_{\lambda\mu}\mathbf{e}_C([P^\lambda_C]^p)$. Since $\mathbf{e}_C$ is injective and the set of $[P^\nu_C]^p$ as $\nu$ runs over all partitions of $n$ is a linearly independent set of $G_p^n(C)$, $\text{adj}_{\lambda\mu}$ must be zero.

**Proposition 4.1** $a_\lambda(v) = 0$

**Proof** By the definition of $v$-decomposition numbers, $G(\tilde{\mu}) = \sum_{\tilde{\nu} \in D} d^{(e)}_{\tilde{\nu} \tilde{\mu}}(v)s(\tilde{\nu})$.

So,

$$f_{e-1}(G(\tilde{\mu})) = \sum_{\tilde{\nu} \in D} d^{(e)}_{\tilde{\nu} \tilde{\mu}}(v)f_{e-1}(s(\tilde{\nu})). \quad (3)$$

In this sum, only the terms $\tilde{\nu} = \tilde{\lambda}_0$ and $\tilde{\nu} = \tilde{\lambda}_1$ may contribute to the coefficients of $s(\lambda)$. Let $D^{(i)}$ be the weight 4 block with the $\langle 5^i, 6, 5e-i-2, 4 \rangle$ notation and $E$ be the weight 2 block with the $\langle 4, 5e-2, 6 \rangle$ notation. Modular branching rules tell us that (see figures 1 and 2)

\[
S^{\tilde{\lambda}_0} \downarrow D^{(e-3)} \downarrow D^{(e-4)} \cdots \downarrow D^{(2)} \downarrow D^{(1)} \downarrow D^{(0)} \downarrow E \cong (S^{\tilde{\lambda}_0})^2,
\]

\[
D^{\tilde{\mu}} \downarrow D^{(e-3)} \downarrow D^{(e-4)} \cdots \downarrow D^{(2)} \downarrow D^{(1)} \downarrow D^{(0)} \downarrow E \cong (D^{\tilde{\mu}})^{\oplus 2}.
\]
Using the product order, we see that $\hat{\mu} \leq_P \hat{\lambda}_0$. Therefore,

$$[S^{\hat{\lambda}_0} : D^{\hat{\mu}}] = 0 \Rightarrow [S^{\hat{\lambda}_0} : D^{\tilde{\mu}}] = 0.$$ 

Hence, $d^{(e)}_{\tilde{\lambda}_0 \tilde{\mu}}(v) = 0$ and we can conclude that the term $\tilde{\nu} = \tilde{\lambda}_0$ in (3) has no contribution. Let us now focus our attention on the term $\tilde{\nu} = \tilde{\lambda}_1$. Since $f_{e-1}(s(\tilde{\lambda}_1)) = s(\lambda)$, the coefficient of $s(\lambda)$ in (3) must be $d^{(e)}_{\tilde{\lambda}_1 \tilde{\mu}}(v) \in v \mathbb{N}_0[v]$. On the other hand, the coefficient of $s(\lambda)$ in (1) is $a_{\lambda}(v) + \sum_{\lambda \leq \lambda \nu} a_{\nu}(v)d^{(e)}_{\lambda \nu}(v)$.

If $a_{\lambda}(v) \neq 0$, then the coefficient of $s(\lambda)$ in (1) would have either constant terms or negative powers of $v$ since $a_{\nu}(v) \in \mathbb{N}_0[v + v^{-1}]$ for all $\nu$. This is a contradiction.

Therefore, $a_{\lambda}(v) = 0$ and $\text{adj}_{\lambda \mu} = 0$. 

This ends the proof of Theorem 2.9.
| $\mu$          | Conditions                                                                 | $\mu^\circ$                  |
|---------------|-----------------------------------------------------------------------------|-------------------------------|
| $\langle i_{3,2} \rangle$ | $i = e - 1$  
$1 \leq i \leq e - 2$ | $\langle 1_{22}, 2 \rangle$  
$\langle 0, (e - i)_{22} \rangle$ |
| $\langle j, i_{22} \rangle$ | $j = 0, i \geq 2$  
$1 \leq j \leq i - 2$ | $\langle (e - i)_{3,2} \rangle$  
$\langle (e - i)_{22}, e - j \rangle$ |
| $\langle i_{22}, j \rangle$ | $j \geq i + 2 \geq 3$  
$j = i + 1, i = 1$  
$j = i + 1, i \geq 2$ | $\langle e - j, (e - i)_{22} \rangle$  
$\langle (e - 1)_{3,2} \rangle$  
$\langle (e - i)_{22}, e - i + 1 \rangle$ |
| $\langle i_{5} \rangle$ | $i = 4, e = 5$  
$i = 3, e = 5$  
$i = 2, e = 5$  
$i = 1, e = 5$  
$i = e - 1, e \geq 6$  
$i = e - 2, e \geq 6$  
$i = e - 3, e \geq 6$  
$i = e - 4, e \geq 6$  
$1 \leq i \leq e - 5, e \geq 6$ | $\langle 1_{2}, 2, 3, 4 \rangle$  
$\langle 0_{2}, 2, 3, 4 \rangle$  
$\langle 0_{2}, 1, 3, 4 \rangle$  
$\langle 0_{2}, 1, 2, 4 \rangle$  
$\langle 0_{2}, 1, 2, 4 \rangle$  
$\langle 0_{1}, 3, 4, 5 \rangle$  
$\langle 0_{1}, 3, 4, 5 \rangle$  
$\langle 0_{1}, 2, 4, 5 \rangle$  
$\langle 0_{1}, 2, 3, 5 \rangle$  
$\langle 0_{1}, 2, 3, 5 \rangle$  
$\langle 0_{1}, 2, 3, 5 \rangle$  
$\langle 0_{1}, 2, 3, 5 \rangle$  
$\langle 0_{1}, 2, 3, 5 \rangle$ |
| $\langle i_{4}, i + 1 \rangle$ | $i = e - 2$  
$2 \leq i \leq e - 3$  
$i = 1$ | $\langle 0_{2}, 1, 3, 4 \rangle$  
$\langle 0_{1}, 1, (e - i)_{12}, e - i + 1 \rangle$  
$\langle 0_{2}, 1, (e - 1)_{12} \rangle$ |
| $\langle 0, i_{4} \rangle$ | $i = e - 1$  
$i = e - 2$  
$i = e - 3$  
$2 \leq i \leq e - 4, e \geq 6$  
$i = 1$ | $\langle 0_{1}, 2, 3 \rangle$  
$\langle 0_{1}, 2, 3 \rangle$  
$\langle 0_{1}, 2, 3 \rangle$  
$\langle 0_{1}, 2, 3 \rangle$  
$\langle 0_{1}, 2, 3 \rangle$ |
| $\langle i_{3}, (i + 1)_{12} \rangle$ | $1 \leq i \leq e - 2$ | $\langle 0, (e - i - 1)_{12}, (e - i)_{2} \rangle$ |
| $\langle i_{3}, (i + 1)_{2} \rangle$ | $i = e - 2, e \geq 7$  
$i = e - 3, e \geq 7$  
$i = e - 4, e \geq 7$  
$i \leq e - 5, e \geq 7$  
$i = 4, e = 6$  
$i = 3, e = 6$  
$i = 2, e = 6$  
$i = 1, e = 6$  
$i = 3, e = 5$  
$i = 2, e = 5$  
$i = 1, e = 5$ | $\langle 2, 3, 4, 5, 6 \rangle$  
$\langle 0, 3, 4, 5, 6 \rangle$  
$\langle 0, 1, 4, 5, 6 \rangle$  
$\langle 0, 1, 2, 4, 5, 6 \rangle$  
$\langle 0, 2, 3, 4, 5 \rangle$  
$\langle 0, 2, 3, 4, 5 \rangle$  
$\langle 0, 2, 3, 4, 5 \rangle$  
$\langle 0, 2, 3, 4, 5 \rangle$  
$\langle 0, 2, 3, 4, 5 \rangle$  
$\langle 0, 2, 3, 4, 5 \rangle$  
$\langle 0, 2, 3, 4, 5 \rangle$  
$\langle 0, 2, 3, 4, 5 \rangle$  
$\langle 0, 2, 3, 4, 5 \rangle$
| $\mu$          | Conditions                  | $\mu^c$                  |
|---------------|-----------------------------|--------------------------|
| $\langle 0_2, (e - 1)_3 \rangle$ | $i = e - 1$                 | $\langle 0, 1_{1,2}, 2 \rangle$ |
| $\langle 0_3, (e - 1)_2 \rangle$ | $i = e - 1, e \geq 6$       | $\langle 1_2, 2, 3, 4 \rangle$ |
|               | $i = 4, e = 5$              | $\langle 1_2, 2_2, 3 \rangle$ |
| $\langle i_3, i + 1, i + 2 \rangle$ | $i = e - 3, e \geq 6$       | $\langle 0, 3_{12}, 4, 5 \rangle$ |
|               | $3 \leq i \leq e - 4, e \geq 7$ | $\langle 0, (e - i)_{12}, e - i + 1, e - i + 2 \rangle$ |
|               | $i = 2$                     | $\langle 0_2, (e - e)_{12}, e - 1 \rangle$ |
|               | $i = 1$                     | $\langle 0_2, (e - 1)_{12} \rangle$ |
| $\langle 0, i_3, i + 1 \rangle$ | $i = 1$                     | $\langle 0, 1_{1,2}, (e - 1)_3 \rangle$ |
|               | $2 \leq i \leq e - 2$       | $\langle 0, (e - i)_{2,1}, e - i + 1 \rangle$ |
| $\langle 0, 1, i_3 \rangle$ | $i = e - 1$                 | $\langle 0, 1_3, 2 \rangle$ |
|               | $2 \leq i \leq e - 2$       | $\langle 0, 1_{1,2}, (e - i)_3 \rangle$ |
| $\langle i_2, (i+1)_2, i + 2 \rangle$ | $i = 4, e = 7$              | $\langle 3_2, 4, 5, 6 \rangle$ |
|               | $i = e - 3, e \geq 8$       | $\langle 3, 4, 5, 6, 7 \rangle$ |
|               | $i = 3, e = 7$              | $\langle 0_2, 4, 5, 6 \rangle$ |
|               | $i = e - 4, e \geq 8$       | $\langle 0, 4, 5, 6, 7 \rangle$ |
|               | $3 \leq i \leq e - 5, e \geq 8$ | $\langle 0, 1, e - i, e - i + 1, e - i + 2 \rangle$ |
|               | $i = 2, e \geq 7$           | $\langle 0_2, 1, e - 2, e - 1 \rangle$ |
|               | $i = 2, e = 6$              | $\langle 0_2, 4_2, 5 \rangle$ |
|               | $i = 3, e = 6$              | $\langle 3_2, 4_2, 5 \rangle$, self-dual |
|               | $i = 2, e = 5$              | $\langle 3_3, 4_2 \rangle$ |
|               | $i = 1, e = 5$              | $\langle 0_3, 4_2 \rangle$ |
|               | $i = 1, e \geq 6$           | $\langle 0_2, 1_2, e - 1 \rangle$ |
| $\langle 0, i_2, (i + 1)_2 \rangle$ | $i = e - 2$                 | $\langle 0, 2_2, 3, 4 \rangle$ |
|               | $2 \leq i \leq e - 3$       | $\langle 0, 1_{1,2}, (e - i)_2, e - i + 1 \rangle$ |
|               | $i = 1$                     | $\langle 0_2, 1_{1,2}, (e - 1)_2 \rangle$ |
References

[1] S. Ariki, On the decomposition numbers of the Hecke algebra of G(m,1,n), *J. Math. Kyoto Univ.* **36** (1996), 789-808.

[2] J. Brundan, A. Kleshchev, Representation theory of the symmetric groups and their double covers, in: *Groups, Combinatorics & Geometry Durham 2001* (pp. 3153), World Sci. Publishing, River Edge, NJ, 2003.

[3] J. Chuang, H. Miyachi, K.M. Tan, Kleshchev’s decomposition numbers and branching coefficients in the fock space, *Trans. Amer. Math. Soc.* **360** (2008), 1179-1191.

[4] M. Fayers, Decomposition numbers for weight three blocks of symmetric groups and IwahoriHecke algebras, *Trans. Amer. Math. Soc.* **360(3)** (2008), 1341-1376.

[5] M. Fayers, An extension of James’s Conjecture, *Int. Math. Res. Notices* (2007), no. 10 Art. ID rnm032.

[6] M. Fayers, James’s Conjecture holds for weight four blocks of IwahoriHecke algebras, *Journal of Algebra* **317** (2007), 593633.

[7] G. James, The decomposition matrices of GL_n(q) for n 10, *Proc. London Math. Soc.* **60(3)** (1990), 225265.

[8] G. James and A. Mathas, A q-analogue of the Jantzen-Schaper theorem, *Proc. London Math. Soc.* **74(3)** (1997), 241-274.

[9] A. Lascoux, B. Leclerc and J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, *Comm. Math. Phys.* **181** (1996), 205263.

[10] B. Leclerc, Symmetric functions and the Fock space, *Symmetric Functions 2001: Surveys of Developments and Perspectives*.

[11] A. Mathas, IwahoriHecke Algebras and Schur Algebras of the Symmetric Group, *University Lecture Series* **15**, American Mathematical Society, 1999.
[12] A. O. Morris and J. B. Olsson, On p-quotients for spin characters, *Journal of Algebra* **119** (1988), 5182.

[13] G. Mullineux, Bijectons on p-regular partitions and p-modular irreducibles of the symmetric groups, *J. London Math. Soc.* **20**(2) (1979), 6066.

[14] M. Richards, Some decomposition numbers for Hecke algebras of general linear groups, *Math. Proc. Cambridge Philos. Soc.* **119** (1996), 383-402.

[15] S. Ryom-Hansen, The Schaper formula and the Lascoux, Leclerc and Thibon-algorithm, *Letters in Mathematical Physics* **64** (2003), 213-219.

[16] K.M. Tan, Beyond Rouquier partitions, *Journal of Algebra* **321** (2009), 248-263.

[17] G. Williamson, Schubert calculus and torsion explosion, *J. Amer. Math. Soc.* **30** (2017).