Selective pumping of a nonlinear quantum oscillator

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Abstract. Selective pumping of a quantum system under the influence of unmodulated short-range pulses of large amplitude is studied using the example of a weakly nonlinear Josephson oscillator. An exact solution is obtained in the case of small parameter anharmonicity for changes in the system energy and population of levels after impact. It is shown that due to the interference of state amplitudes, it is possible to stabilize the system and prevent undesirable excitations (energy growth). When anharmonicity is taken into account, the system stability conditions are studied numerically. The obtained results are important when analyzing leakage to high-lying levels in a transmon-qubit under the influence of short unipolar control pulses.

1. Introduction
It is known that charge qubits (e.g., transmon qubits) [1, 2] are described by the Hamiltonian of a nonlinear quantum oscillator. In this case, the two lowest levels of the oscillator play the role of a qubit, and transitions to high-lying levels are suppressed due to the non-equidistance of the spectrum, which is due to anharmonicity. It is important to note that the value of the transmon anharmonicity parameter imposes strict requirements on the duration and amplitude of microwave pulses [3, 4], which in turn limits the speed of transmon registers. If a quantum system is affected by control pulses with a wide spectrum [5-7], a similar problem arises in terms of energy transfer to the overlying states, since it is necessary to choose the optimal relationship between the distance between the levels, the pulse amplitude, and its duration.

In this paper, we have developed a method for selective pumping of a nonlinear oscillator, when a population of a small group of levels is created as a result of short unipolar pulses. The system excitation energy is chosen as the target function. The main idea of the work is to implement flexible control of the transmitted energy from a short series of unipolar pulses with suitable parameters due to interference of the amplitudes of the probabilities of state population. If the condition "the energy acquired as a result of the action of pulses is approximately equal to the distance between the lower levels" is met, the qubit levels will be selectively populated and there will be no leakage to high-energy levels. In the case when the anharmonicity of a nonlinear oscillator can be ignored, the statement is formulated as follows: the effect of a pair of pulses already allows you to transfer any required energy to the oscillator. Numerical methods are used to study the stability of state populations of a nonlinear oscillator under the influence of powerful rectangular unipolar pulses, taking into account anharmonicity.
2. Transmon-qubit. Model and basic equations

Transmon is a two-contact superconducting interferometer, each arm contains a parallel capacitance and a Josephson contact, which is characterized by electrostatic and Josephson energies [2]. In this case, the total Hamiltonian of the qubit in the presence of an external field can be reduced to the Hamiltonian of a nonlinear oscillator with external force:

\[ H = \omega a^\dagger a - \mu (a + a^\dagger)^2 + \varepsilon(t)(a + a^\dagger), \]  

where \( \omega \) is the distance between the qubit levels, \( \mu \) is the anharmonicity parameter; \( \varepsilon(t) \) is the field function (which, depending on the calibration, can be interpreted as the control magnetic field in the interferometer or the voltage at the Josephson junction).

The traditional control scheme of an oscillator with a large anharmonicity is reduced to applying a Rabi pulse to the qubit with a carrier frequency close to \( \omega \). When the external field is weak, transitions between lower levels occur with Rabi frequency (proportional to the field amplitude). To accelerate the control of a quantum system, a sequence of unipolar pulses can be used [3-7].

3. Unipolar pulses

Consider the process when a sequence of short-range pulses is applied to a transmon of the form (1), which is described by the function \( \varepsilon(t) \). Let the characteristic duration of each pulse satisfy the condition \( \tau_j \ll \omega^{-1} \), then the control function can be written as

\[ \varepsilon(t) = \sum_{j=1}^{N} \alpha_j \delta(t - t_j), \quad \alpha_j = \varepsilon_j \tau_j \]  

Let’s consider the change in the energy of the oscillator under the influence of pulses (2). It is assumed that the oscillator was in the ground state \( |\hat{0}\rangle \), which is the state with the lowest energy of the nonlinear oscillator \( H_a = \omega a^\dagger a - \mu (a + a^\dagger)^2 \), until the moment \( t_0 \) of the perturbation activation:

\[ H_{\infty} |\hat{n}\rangle = E_n |\hat{n}\rangle. \]  

The change in energy is determined by the expression:

\[ E(t) = \langle \hat{0} | U^{\dagger}(t, t_0) \left( \omega a^\dagger a - \mu (a + a^\dagger)^2 \right)^n U(t, t_0) |\hat{0}\rangle, \]  

where the propagator \( U(t, t_0) \) is from the equation:

\[ i \frac{\partial}{\partial t} U(t, t_0) = \left( \omega a^\dagger a - \mu (a + a^\dagger)^2 + \varepsilon(t)(a + a^\dagger) \right) U(t, t_0), \quad U(t_0, t_0) = I. \]  

Solution of Eq. (5) for a control function of the form Eq. (2) is written as:

\[ U(t_N, 0) = e^{-i\omega t_N a^\dagger a} e^{-i\mu t_N (a + a^\dagger)^2} \cdots e^{-i\omega t_1 a^\dagger a} e^{-i\mu t_1 (a + a^\dagger)^2} e^{-i\omega t_0 a^\dagger a} e^{-i\mu t_0 (a + a^\dagger)^2}. \]  

Expression (6) will be used below for numerical analysis.

Transitions occur in a two-dimensional Hilbert subspace of qubit states \( |\hat{0}\rangle \) and \( |\hat{1}\rangle \) under strong anharmonicity, so the transmon Hamiltonian can be written as

\[ H_2 = \begin{pmatrix} \omega & 0 \\ 0 & -\mu \end{pmatrix} + \varepsilon(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]  

Taking into account Eq. (7), it is easy to rewrite the propagator Eq. (6) in the truncated subspace \( |\hat{0}\rangle \) and \( |\hat{1}\rangle \), which will correspond to the standard dynamics of a qubit controlled by the function \( \varepsilon(t) \). It should be noted that the anharmonicity parameter \( \mu \) in the case of a typical transmon is usually \( \sim 200 \text{ MHz} \), at \( \omega \approx 5.4 \text{ GHz} \) [4], so when the qubit is strongly excited, it is necessary to expand the subspace of States to include the state of the next level: \( |\hat{2}\rangle \), to which a leakage is possible. In this approximation, the transmon Hamiltonian is written as

\[ H_3 = D + \varepsilon(t)V, \]  

where matrices are defined as
\[
D = \begin{pmatrix}
-3\mu & 0 & 0 \\
0 & \omega - 9\mu & 0 \\
0 & 2\omega - 6\mu & 0
\end{pmatrix}, \quad V = \begin{pmatrix}
0 & \sqrt{2}\left(1 - \frac{3}{2}\mu\right) & 0 \\
\sqrt{2}\left(1 - \frac{3}{2}\mu\right) & 0 & -\frac{9}{2}\mu \\
0 & -\frac{9}{2}\mu & 0
\end{pmatrix}.
\]

For this case, the propagator Eq. (6) can be represented as:
\[
U(t_N, 0) = e^{-i\omega a V} e^{-iD(t_{N-1} - t_N)} e^{-i\omega a V} \cdots e^{-iD(t_1 - t_2)} e^{-i\omega a V} e^{-iD(t - t_1)} e^{-i\omega a V}.
\]  
(9)

Substituting the propagator Eq. (9) in the expression (4), it is easy to calculate numerically the change in the transmon energy for any sequence of pulses.

4. **Selective pumping of a linear quantum oscillator**

Let's get an expression for changing the energy of a linear oscillator under the influence of an arbitrary control function \(\varepsilon(t)\). The basic states of a linear oscillator, \(H = \omega a^\dagger a\) are the Fock States: \(a^\dagger a|n\rangle = n|n\rangle\), which we denote \(|n\rangle\). The Hamiltonian of the excited oscillator has the form:

\[
H = \omega a^\dagger a + \varepsilon(t) (a + a^\dagger). \tag{10}
\]

If the oscillator was in the ground state at the moment \(t_0\) of switching on \(\varepsilon(t)\), then

\[
E(t) = \omega \langle 0| a^\dagger a(t)|0\rangle, \tag{11}
\]

where the Heisenberg representation is used for \(a(t) = U^\dagger(t, t_0) a U(t, t_0)\) and \(a^\dagger(t) = U^\dagger(t, t_0) a^\dagger U(t, t_0)\). The operator \(a(t)\) obeys the equation:

\[
\dot{a}(t) = i\omega a(t) + \varepsilon(t).
\]  
(12)

The solution of Eq. (12) has the form

\[
a(t) = ae^{-i\omega(t-t_0)} + \int_{t_0}^{t} \varepsilon(t')e^{-i\omega(t-t')} dt'. \tag{13}
\]

Then from Eqs. (12) and (13) it follows for a complete change of energy (at \(t_0 \to -\infty, t \to \infty\)):

\[
E = \omega \left| \int_{-\infty}^{\infty} \varepsilon(t)e^{i\omega t} dt \right|^2.
\]  
(14)

Thus, for a sequence of \(N\) – short-range pulses, the energy change is written as:

\[
E = \omega \left( \sum_{j=1}^{N} (\alpha_j)^2 + \sum_{j=1}^{N} \alpha_j \alpha_k e^{i(t_j - t_k)} \right). \tag{15}
\]

As you can see, the second term reflects the interference nature of the impact on the system, which allows us to get a predetermined change in energy. In particular, for two pulses we have

\[
E = \omega \left( \alpha_1^2 + \alpha_2^2 + 2\alpha_1 \alpha_2 \cos(\omega T) \right), \quad \text{where} \quad T = t_2 - t_1. \tag{16}
\]

It can be seen that for the same signal amplitudes \(\alpha = \alpha_1 = \alpha_2\), a quantum of energy will be transmitted to the oscillator if the ratio \(\cos(\omega T) = \frac{1}{2\alpha - 1}\) is met.

5. **Numerical simulation of the transmon dynamics**

The three-level approximation, which is described by expressions (4), together with Eqs. (8) and (9), is very important for the qubit realization in practice. For this case, the propagator for two pulses is defined as: \(U(T, 0) = e^{-i\omega v} e^{-iD T} e^{-i\omega v}\), where \(T = t_2 - t_1\). We will briefly describe the energy evolution. The first pulse excites an oscillator in the ground state. Then there is a process of free evolution, expressed by a diagonal matrix \(D\), which shifts the phases of wave functions in three channels. After that, the states are mixed again due to the action of the second pulse. Since Eq. (4) contains the
operator $U^\dagger(T,0) = e^{i\omega T} e^{i\alpha T}$, describing the inverse process in time, or interference of waves in the forward and reverse directions in time will lead to an oscillatory dependence of energy on time $T$ like the expression (16).

We calculate numerically the change in energy under the influence of two short pulses depending on the time interval between them $T$ (measured in time units $\omega^{-1}$). Let the amplitudes be defined as $\alpha = \alpha_1 = \alpha_2$ ($\alpha / \omega = 1$), and the nonlinearity parameter is $\mu / \omega = 1/25$. With this anharmonicity, the transition levels are as follows: $E_0 / \omega = 0$, $E_1 / \omega = 0.78$ and $E_2 / \omega = 1.91$ (see the figure 1). The energy oscillations occur for any number of levels of the nonlinear oscillator.

Figure 1. Dependence of the transmon energy on the interval between two pulses with the same amplitudes. Energy and time are measured in dimensionless units.

6. Conclusion
We have shown that under the influence of short-range and wide-band pulses of large amplitude, selective excitation of a quantum system is possible, using the example of a weakly nonlinear Josephson oscillator (transmon). In the case where the transmon anharmonicity can be ignored, an exact solution is presented for changing the energy and population levels after exposure to an arbitrary sequence of unipolar pulses. It is shown that due to the interference of states, it is possible to stabilize the system and prevent undesirable energy growth. Taking into account anharmonicity, the conditions for selective excitation of a transmon are studied numerically. The studied behavior of the nonlinear system makes it possible to analyze the leakage of a qubit to high-lying levels when it is exposed to unipolar control pulses. Practical implementation of the required sequences of acting pulses is possible on the basis of digital Josephson circuits.

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