ON ZEROS OF BILATERAL HURWITZ AND PERIODIC ZETA AND ZETA STAR FUNCTIONS

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Abstract. In this paper, we show the following:

1. Introduction and statement of main results

1.1. Real zeros of the Hurwitz and periodic zeta functions. Let \( s = \sigma + it \), where \( \sigma, t \in \mathbb{R} \). Then the Riemann zeta function is defined by

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1.
\]

It is well known that \( \zeta(s) \) is continued meromorphically and has a simple pole at \( s = 1 \) with residue 1. The Riemann zeta function \( \zeta(s) \) satisfies the functional equation

\[
\zeta(1-s) = \frac{2\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right) \zeta(s)
\]

(1.1)

(see for instance [18, (2.1.8)]). Moreover, all real zeros of \( \zeta(s) \) are simple and are located only at the negative even integers. The Riemann hypothesis is a conjecture that the Riemann zeta function \( \zeta(s) \) has its non-real zeros only on the so-called critical line \( \sigma = 1/2 \).

Next we define the Hurwitz zeta function \( \zeta(s, a) \) by

\[
\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \sigma > 1, \quad 0 < a \leq 1.
\]
The Hurwitz zeta function \( \zeta(s,a) \) is a meromorphic function with a simple pole at \( s = 1 \) whose residue is 1 (see for example [1, Chapter 12]). Note that all real zeros of \( \zeta(s,1/2) \) are simple and are located only at the non-positive even integers.

Define the periodic zeta function \( \Li_s(e^{2\pi i a}) \) by

\[
\Li_s(e^{2\pi i a}) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^s}, \quad \sigma > 1, \quad 0 < a \leq 1
\]

(see for instance [1, Exercise 12.2]). It should be mentioned that \( \Li_s(e^{2\pi i a}) \) with \( 0 < a < 1 \) is analytically continuable to the whole complex plane since the Dirichlet series of \( \Li_s(e^{2\pi i a}) \) converges uniformly in each compact subset of the half-plane \( \sigma > 0 \) when \( 0 < a < 1 \) (see for instance [9, p. 20]).

For real zeros of \( \zeta(s,a) \) and \( \Li_s(z) \) with \( |z| \leq 1 \), we can refer to [12, Section 1.1]. Recently, Matsusaka [10, Theorem 1.3] showed that for integers \( N \leq -1 \), the Hurwitz zeta function \( \zeta(s,a) \) has real zeros in \( (-N-1,-N) \) if and only if \( B_{N+1}(a)B_{N+2}(a) < 0 \), where \( B_N(a) \) is the \( N \)-th Bernoulli polynomial. Note that the cases \( N = -1 \) and \( N = 0 \) are proved by the author in [12, Theorem 1.2] and [13, Theorem 1.2], respectively.

By numerical calculation, we can see that the difference between two successive real zeros of \( \zeta(s,a) \) is not \( 2 \) in general, namely, the difference depends on \( 0 < a < 1 \). On the contrary, it is well-known that the gap between consecutive real zeros of \( \zeta(s) \) is \( 2 \) by the functional equation (1.1) and the fact that \( \zeta(s) \) does not vanish on the real line if \( \sigma \geq 0 \). Moreover, the difference between two successive real zeros of Dirichlet \( L \)-function \( L(s,\chi) \) is \( 2 \) if we assume the non-existence of any real zero of \( L(s,\chi) \) in the interval \( (0,1) \). As an unconditional result, Conrey and Soundararajan [3] proved that a positive proportion of real primitive Dirichlet characters \( \chi \), the associated Dirichlet \( L \)-function \( L(s,\chi) \) does not vanish on the positive real axis.

### 1.2. Main results.

For \( 0 < a \leq 1/2 \), let

\[
\begin{align*}
Z(s,a) &:= \zeta(s,a) + \zeta(s,1-a), \\
P(s,a) &:= \Li_s(e^{2\pi i a}) + \Li_s(e^{2\pi i (1-a)}) \\
Y(s,a) &:= \zeta(s,a) - \zeta(s,1-a), \\
O(s,a) &:= -i(\Li_s(e^{2\pi i a}) - \Li_s(e^{2\pi i (1-a)})) \\
X(s,a) &:= Y(s,a) + O(s,a) = \zeta(s,a) - \zeta(s,1-a) - i(\Li_s(e^{2\pi i a}) - \Li_s(e^{2\pi i (1-a)})).
\end{align*}
\]

We name \( Z(s,a) \) and \( Y(s,a) \) bilateral Hurwitz zeta function and bilateral Hurwitz zeta star function, respectively. In addition, we call \( P(s,a) \) and \( O(s,a) \) bilateral periodic zeta function and bilateral periodic zeta star function, respectively. Without the name of these functions, they have already appeared in an old paper of Fine [5, Equations (9), (11), (12) and (15)] and some famous text books such as Karatsuba & Voronin [7, Equations (9) and (12) in §1.4] and Koblitz [8, Problems 3 (c) and 5 (d) in §2.4]. Furthermore, it is showed in [15, Section 5] that \( Z(s,|a|) \) appears as the spectral density of some stationary self-similar Gaussian distributions (see also [17, Theorem 2.1]). Note that the functional equation and zeros on the critical line of \( Z(s,a) + P(s,a) \) have been already discussed in [14, Theorems 1.1 and 1.2].

In the present paper, we prove the following four main theorems which describe real zeros of the zeta-functions \( \Li_s(e^{2\pi i a}), Y(s,a), O(s,a), X(s,a) \) and \( P(s,a) \). It should be emphasised that the gap between successive real zeros of \( Y(s,a), O(s,a), X(s,a) \) with \( 0 < a < 1/2 \), and \( Z(s,a) \) and \( P(s,a) \) with \( 1/4 \leq a \leq 1/2 \) is always \( 2 \), namely, the gaps do not depend on \( a \) just like \( \zeta(s) \).
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Theorem 1.1. The real zeros of \( \text{Li}_s(e^{2\pi i}) \) coincide with those of \( \zeta(s) \). However, the periodic zeta function \( \text{Li}_s(e^{2\pi ia}) \) does not vanish on the real line when \( 0 < a < 1/2 \) or \( 1/2 < a < 1 \).

The theorem above asserts the non-existence of any real zero of \( \text{Li}_s(e^{2\pi ia}) \). On the contrary, the theorem below can be regarded as an analogue of the well-know fact that the all real zeros of any Dirichlet \( L \)-function \( L(s, \chi) \) with \( \Re(s) < 0 \) are located only at the negative odd integers if \( \chi \) is a primitive character with \( \chi(-1) = -1 \) (see Section 1.1).

Theorem 1.2. All real zeros of the functions \( Y(s, a) \), \( O(s, a) \) and \( X(s, a) \) with \( 0 < a < 1/2 \) are simple and are located only at the negative odd integers.

As analogues of the real zeros of \( \zeta(s) \) or \( L(s, \chi) \) with primitive characters satisfying \( \chi(-1) = 1 \), we have the following two statements.

Theorem 1.3. All real zeros of the function \( Z(s, a) \) are simple and are located only at the non-positive even integers if and only if \( 1/4 \leq a \leq 1/2 \).

Theorem 1.4. All real zeros of the function \( P(s, a) \) are simple and are located only at the negative even integers if and only if \( 1/4 \leq a \leq 1/2 \).

Next we consider the zeta function \( Z(s, a) \) and \( P(s, a) \) with \( 0 < a < 1/4 \). The asymptotic behavior of real zeros \( \beta_Z(a) \) and \( \beta_P(a) \) below are given in (3) of Propositions 4.5, 4.6 and 4.7 (see also Propositions 4.8 and 4.9).

Proposition 1.5. Assume \( 0 < a < 1/4 \). Then the function \( Z(s, a) \) have real zeros at the non-positive integers and another real zero \( \beta_Z(a) \) in \( (-\infty, 1) \) which does not appear in Theorem 1.3.

Proposition 1.6. Suppose \( 0 < a < 1/4 \). Then the function \( P(s, a) \) have precisely one simple real zero \( \beta_P(a) \) in \( (0, \infty) \). Furthermore, all real zeros in \( (-\infty, 0) \) of the function \( P(s, a) \) are simple and are located only at the negative even integers.

At the end of this subsection, we discuss the complex or non-real zeros of the zeta functions \( Y(s, a) \), \( O(s, a) \), \( X(s, a) \) \( Z(s, a) \) and \( P(s, a) \). For \( k = 3, 4, 6 \), let \( \chi_{-k} \) be the real and non-principal Dirichlet character mod \( k \). Then we define the Dirichlet \( L \)-function by \( L(s, \chi_{-k}) := \sum_{n=1}^{\infty} \chi_{-k}(n)n^{-s} \) for \( k = 3, 4, 6 \) and \( \Re(s) > 1 \). These functions can be continued to holomorphic functions on the whole complex plane by analytic continuation. For \( k = 3, 4 \), the Generalized Riemann hypothesis for \( L(s, \chi_{-k}) \) are conjectures that \( L(s, \chi_{-k}) \) have their non-real zeros only on the critical line \( \sigma = 1/2 \). Furthermore, the Generalized Riemann hypothesis for \( L(s, \chi_{-6}) \) states that all non-real zeros of \( L(s, \chi_{-6}) \) are located only on the vertical lines \( \sigma = 1/2 \) and \( \sigma = 0 \) (see Section 3.1). For \( a = 1/6, 1/4, 1/3 \) and \( 1/2 \), we have the following.

Proposition 1.7. Assume \( a = 1/6, 1/4, 1/3 \) or \( 1/2 \). Then the Riemann hypothesis holds if and only if
\[
\bullet \text{ all non-real zeros of } Z(s, a) \text{ are on the vertical lines } \sigma = 1/2 \text{ and } \sigma = 0, \text{ or } \\
\bullet \text{ all non-real zeros of } P(s, a) \text{ are on the vertical lines } \sigma = 1/2 \text{ and } \sigma = 1.
\]

Proposition 1.8. We have the following.
\[
\bullet \text{ For each } k = 3, 4, \text{ all non-real zeros of } Y(s, 1/k) \text{ and } O(s, 1/k) \text{ are on the vertical line } \sigma = 1/2 \text{ if } \\
\text{ and only if the Generalized Riemann hypothesis for } L(s, \chi_{-k}) \text{ is true.}
\]
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1. All non-real zeros of \( Y(s, 1/6) \) and \( O(1 - s, 1/6) \) are on the vertical lines \( \sigma = 1/2 \) and \( \sigma = 0 \) if and only if the Generalized Riemann hypothesis for \( L(s, \chi_{-a}) \) is true.

2. For each \( k = 3, 4, 6 \), all non-real zeros of \( X(s, 1/k) \) are on the vertical line \( \sigma = 1/2 \) if and only if the Generalized Riemann hypothesis for \( L(s, \chi_{-k}) \) is true.

On the contrary, when \( a \neq 1/2, 1/3, 1/4, 1/6 \), we have the following for non-periodic complex zeros of the functions \( \Li_k(e^{2\pi ia}) \), \( O(s, a) \), \( Y(s, a) \) and \( X(s, a) \), \( Z(s, a) \) and \( P(s, a) \).

**Proposition 1.9.** Let \( a \in \mathbb{Q} \cap (0, 1/2) \backslash \{1/6, 1/4, 1/3\} \). Then for any \( \delta > 0 \), there exist positive constants \( C_a^0(\delta) \) and \( C_a^1(\delta) \) such that the function \( P(s, a) \) has more than \( C_a^0(\delta)T \) and less than \( C_a^0(\delta)T \) complex zeros in the rectangle \( 1 < \sigma < 1 + \delta \) and \( 0 < t < T \) if \( T \) is sufficiently large. Moreover, for any \( 1/2 < \sigma_1 < \sigma_2 < 1 \), there are positive numbers \( C_a^1(\sigma_1, \sigma_2) \) and \( C_a^2(\sigma_1, \sigma_2) \) such that the function \( P(s, a) \) has more than \( C_a^1(\sigma_1, \sigma_2)T \) and less than \( C_a^2(\sigma_1, \sigma_2)T \) non-trivial zeros in the rectangle \( \sigma_1 < \sigma < \sigma_2 \) and \( 0 < t < T \) when \( T \) is sufficiently large.

Furthermore, there exist positive numbers \( C_a^0(\delta) \) and \( C_a^1(\delta) \) such that the function \( \Li_k(e^{2\pi ia}) \) with \( a \in \mathbb{Q} \cap (0, 1/2) \backslash \{1/6, 1/4, 1/3\} \), and the functions \( Z(s, a) \) and \( Y(s, a) \) with \( a \in (0, 1/2) \backslash \mathbb{Q} \) or \( a \in \mathbb{Q} \cap (0, 1/2) \backslash \{1/6, 1/4, 1/3\} \) have the property mentioned above.

This paper is organized as follows. In Section 2, we give some remarks on linear relations between Dirichlet \( L \)-functions and zeta functions composed by \( \zeta(s, a) \) and \( \Li_k(e^{2\pi ia}) \) and their real zeros. We give proofs of Proposition 1.8, Theorems 1.1 and 1.2 in Section 3. We prove Theorems 1.3 and 1.4, Propositions 1.5, 1.6, 1.7 and 1.9 in Section 4.

2. Linear relations and remarks on real zeros

2.1. Linear relations. First, we show that each of the functions \( Z(s, r/q) \), \( P(s, r/q) \), \( Y(s, r/q) \) and \( O(s, r/q) \) essentially can be expressed as a linear combination of Dirichlet \( L \)-functions if \( 0 < r < q \) are relatively prime integers.

**Proposition 2.1.** Let \( 0 < r < q \) be relatively prime integers, \( \varphi \) be the Euler totient function. Put \( \overline{\chi}(r) := (1 + \chi(1))\overline{\chi}(r) \) and \( G_r(\chi) := (1 + \chi(1))\sum_{n=1}^q e^{2\pi irn/q}\chi(n) \). Then one has

\[
Z(s, r/q) = \frac{q^s}{\varphi(q)} \sum_{\chi \mod q} \overline{\chi}(r)L(s, \chi), \quad P(s, r/q) = \frac{1}{\varphi(q)} \sum_{\chi \mod q} G_r^+(\overline{\chi})L(s, \chi).
\]

Furthermore, for \( 0 < 2r < q \), it holds that

\[
Y(s, r/q) = \frac{q^s}{\varphi(q)} \sum_{\chi \mod q} \overline{\chi}(r)L(s, \chi), \quad O(s, r/q) = \frac{-i}{\varphi(q)} \sum_{\chi \mod q} G_r^-(\overline{\chi})L(s, \chi).
\]

**Proof.** When \( 0 < r < q \) are relatively prime integers, we have

\[
\zeta(s, r/q) = \sum_{n=0}^\infty \frac{1}{(n+r/q)^s} = \sum_{n=0}^\infty \frac{q^s}{(r+qn)^s} = \frac{q^s}{\varphi(q)} \sum_{\chi \mod q} \overline{\chi}(r)L(s, \chi),
\]

\[
\Li_k(e^{2\pi ir/q}) = q^{-s} \sum_{n=1}^q e^{2\pi irn/q} \xi(s, n/q) = \frac{1}{\varphi(q)} \sum_{\chi \mod q} G(r, \overline{\chi})L(s, \chi).
\]
Note that $\varphi(q) \leq 2$ if and only if $q = 1, 2, 3, 4, 6$, which plays important role in the proof of Proposition 1.9. By (2.3) and the definitions of the functions $Z(s, r/q), P(s, r/q), Y(s, r/q)$ and $O(s, r/q)$, we have (2.1) and (2.2).

Next we show that the Dirichlet $L$-function essentially can be written by a linear combination of the functions $Z(s, r/q), P(s, r/q), Y(s, r/q)$ or $O(s, r/q)$.

**Proposition 2.2.** Let $\chi$ be a primitive Dirichlet character, $G(\chi)$ be the Gauss sum associated to $\chi$, and $0 < r < q$ be relatively prime integers. If $\chi(-1) = 1$, we have

\[(2.4)\]

$$L(s, \chi) = \frac{1}{2q^s} \sum_{r=1}^{q} \chi(r)Z(s, r/q) = \frac{1}{2G(\chi)} \sum_{r=1}^{q} \chi(r)P(s, r/q).$$

When $\chi(-1) = -1$, one has

\[(2.5)\]

$$L(s, \chi) = \frac{1}{2q^s} \sum_{r=1}^{q} \chi(r)Y(s, r/q) = \frac{i}{2G(\chi)} \sum_{r=1}^{q} \chi(r)O(s, r/q).$$

**Proof.** When $0 < r < q$ are relatively prime integers, it holds that

\[(2.6)\]

$$L(s, \chi) = \frac{1}{G(\chi)} \sum_{r=1}^{q} \sum_{n=1}^{\infty} \frac{\chi(r+nq)\chi(r)}{(r+nq)^s} = \frac{1}{G(\chi)} \sum_{r=1}^{q} \chi(r)_{r/q},$$

$$L(s, \chi) = \frac{1}{G(\chi)} \sum_{r=1}^{q} \frac{\chi(r)e^{2\pi ir/q}}{n^s} = \frac{1}{G(\chi)} \sum_{r=1}^{q} \chi(r)_{r/q}.$$ 

Suppose that $\chi$ is even, namely, $\chi(-1) = 1$. Then we have

\[2q^s L(s, \chi) = \sum_{r=1}^{q} \chi(r)_{r/q} + \sum_{r=1}^{q} \chi(r)_{r/q} = \sum_{r=1}^{q} \chi(r)_{r/q},\]

from (2.6) and $\chi(q-r) = \chi(-r) = \chi(r)$. Similarly, one has

\[2G(\chi)L(s, \chi) = \sum_{r=1}^{q} \chi(r)_{r/q} + \sum_{r=1}^{q} \chi(r)_{r/q} = \sum_{r=1}^{q} \chi(r)_{r/q}.\]

Similar arguments apply to the case $\chi(-1) = -1$. Thus, we have (2.4) and (2.5). \qed

**2.2. Remarks on real zeros.** We have the following remarks by theorems in Section 1.2, the equations (2.1), (2.2), (2.4) and (2.5).

**Remark.** Put $\mathfrak{D} := \bigcup_{\sigma \neq \chi Z(n, n + 1)}$. It is widely known that Dirichlet $L$-functions $L(\sigma, \chi)$ with $\chi(-1) = -1$ do not vanish in $\mathfrak{D}$ from the functional equations and the Euler products. By Theorem 1.2, the functions $Y(\sigma, a)$ and $O(\sigma, a)$ do not vanish in the open set $\mathfrak{D} \cup (0, 1)$. Furthermore, the equation (2.5) holds for both $\sigma \in \mathfrak{D}$ and $\sigma \in (0, 1)$. Hence it may be natural to expect that $L(\sigma, \chi)$ with $\chi(-1) = -1$ do not vanish for not only $\sigma \in \mathfrak{D}$ but also $\sigma \in (0, 1)$.

**Remark.** On the contrary, the functions $Z(s, a)$ and $P(s, a)$ have real zeros in the open set $\mathfrak{D} \cup (0, 1)$ if and only if $0 < a < 1/6$ or $1/6 < a < 1/4$ from Theorems 1.3, 1.4 and

\[(2.7)\]

$$Z(0, 1/6) = P(1, 1/6) = 0.$$
which is proved by (4.4) and (4.8). Recall that by (2.4), the Dirichlet $L$-function $L(s, \chi)$ with $\chi(-1) = 1$ can be expressed as a linear combination of the functions $Z(s, r/q)$ and $P(s, r/q)$, where $0 < r < q$ are relatively prime integers. In addition, the functions $Z(s, r/q)$ and $P(s, r/q)$ with $0 < r/q < 1/6$ have a real zero in the interval $(0, 1)$ from (2) of Proposition 4.5. Despite of these facts, it is expected that any Dirichlet $L$-function $L(s, \chi)$ with $\chi(-1) = 1$ does not vanish in the interval $(0, 1)$.

**Remark.** There is a possibility that the two remarks above indicate that proving the non-existence of real zeros in $(0, 1)$ for Dirichlet $L$-functions with even characters is more difficult than that with odd characters.

3. Proofs of the main results of $\text{Li}_x(e^{2\pi i a}), Y(s, a)$ and $O(s, a)$

3.1. **Proof of Proposition 1.8.** For each $k = 3, 4, 6$, we show that $Y(s, 1/k), O(s, 1/k)$ and $X(s, 1/k)$ can be essentially expressed as $L(s, \chi_{-k})$.

**Proof of Proposition 1.8.** Let $a = 1/3$ and $\Re(s) > 1$. Then one has

$$Y(s, 1/3) = 3^s \sum_{n=0}^{\infty} \frac{1}{(3n+1)^s} - 3^s \sum_{n=0}^{\infty} \frac{1}{(3n+2)^s} = 3^s L(s, \chi_{-3}).$$

In addition, it holds that

$$O(s, 1/3) = \sum_{n=1}^{\infty} \frac{e^{2\pi in/3}}{s^n} - \sum_{n=1}^{\infty} \frac{e^{-2\pi in/3}}{s^n} = \sum_{n=0}^{\infty} \frac{\sqrt{3}}{(3n+1)^s} - \sum_{n=0}^{\infty} \frac{\sqrt{3}}{(3n+2)^s} = \sqrt{3} L(s, \chi_{-3}).$$

Therefore, we have

$$X(s, 1/3) = Y(s, 1/3) + O(s, 1/3) = (3^s + \sqrt{3}) L(s, \chi_{-3}).$$

Similarly, we can show that

$$Y(s, 1/4) = 4^s L(s, \chi_{-4}), \quad O(s, 1/4) = 2 L(s, \chi_{-4}), \quad X(s, 1/4) = (4^s + 2) L(s, \chi_{-4}).$$

Finally, consider the case $a = 1/6$. For $\Re(s) > 1$, we have

$$Y(s, 1/6) = \sum_{n=0}^{\infty} \frac{1}{(n+1/6)^s} - \sum_{n=0}^{\infty} \frac{1}{(n+5/6)^s} = 6^s \sum_{n=0}^{\infty} \frac{1}{(6n+1)^s} - 6^s \sum_{n=0}^{\infty} \frac{1}{(6n+5)^s}$$

$$= 6^s L(s, \chi_{-6}) = 6^s (1 + 2^{-s}) L(s, \chi_{-3}) = (6^s + 3^s) L(s, \chi_{-3})$$

(see for instance [2, Lemma 10.2.1]). On the other hand, one has

$$O(s, 1/6) = \sum_{n=0}^{\infty} \frac{\sqrt{3}}{(6n+1)^s} + \sum_{n=0}^{\infty} \frac{\sqrt{3}}{(6n+2)^s} - \sum_{n=0}^{\infty} \frac{\sqrt{3}}{(6n+4)^s} - \sum_{n=0}^{\infty} \frac{\sqrt{3}}{(6n+5)^s}$$

$$= \sqrt{3} L(s, \chi_{-6}) + 2^{-s} \sqrt{3} L(s, \chi_{-3}) = \sqrt{3} (1 + 2^{-s}) L(s, \chi_{-3}).$$

Therefore, one has

$$X(s, 1/6) = Y(s, 1/6) + O(s, 1/6) = (3^s (1 + 2^s) + \sqrt{3} (1 + 2^{-s})) L(s, \chi_{-3}).$$
Now we show the factor $3^s(1 + 2^s) + \sqrt{3}(1 + 2^{1-s})$ has zeros only on $\sigma = 1/2$. Put

$$f(s) := \frac{3^s}{\sqrt{3}}, \quad g(s) := \frac{1 + 2^{1-s}}{1 + 2^s}.$$  

Obviously, one has $|f(s)| = 1$ for $\sigma = 1/2$, $|f(s)| > 1$ if $\sigma > 1/2$ and $|f(s)| < 1$ when $\sigma < 1/2$, and $|g(s)| = 1$ for $\sigma = 1/2$. Now we show that $|g(s)| < 1$ when $\sigma > 1/2$ and $|g(s)| > 1$ when $\sigma < 1/2$. Suppose $\sigma > 1/2$ and $\cos(t \log 2) \geq 0$. Then, by

$$g(\sigma + it) = \frac{1 + 2^{1-\sigma} \cos(t \log 2) - i2^{1-\sigma} \sin(t \log 2)}{1 + 2^{\sigma} \cos(t \log 2) + i2^{\sigma} \sin(t \log 2)},$$

we have $|g(s)| < 1$. Next assume $\sigma > 1/2$ and $-1 \leq \cos(t \log 2) < 0$. In this case, we only have to show the inequality

$$1 + 2^{2-\sigma} \cos(t \log 2) + 2^{2-2\sigma} < 1 + 2^{1+\sigma} \cos(t \log 2) + 2^{2\sigma}$$

which is equivalent to $(2^{2-\sigma} - 1 + \sigma) \cos(t \log 2) < 2^{2\sigma} - 2^{2-2\sigma}$. From the factorization

$$2^{2\sigma} - 2^{2-2\sigma} - (2^{1+\sigma} - 2^{2-\sigma}) = 2^{-2\sigma}(2^{2\sigma} - 2)((2^{\sigma} - 1)^2 + 1),$$

for $\sigma > 1/2$ and $-1 \leq \cos(t \log 2) < 0$, we get

$$(2^{2-\sigma} - 2^{1+\sigma}) \cos(t \log 2) \leq 2^{1+\sigma} - 2^{2-\sigma} < 2^{2\sigma} - 2^{2-2\sigma}.$$  

Therefore, we have $|g(s)| < 1$ when $\sigma > 1/2$. By using $g(1-s) = 1/g(s)$, we immediately obtain $|g(s)| > 1$ when $\sigma < 1/2$. $\square$

### 3.2. Proof of Theorem 1.1

Let $0 < a < 1$. For $\sigma > 0$, it is known that

$$(3.1) \quad \text{Li}_s(e^{2\pi ia}) = \sum_{n=1}^{\infty} \frac{e^{2\pi ina}}{n^s} = \frac{e^{2\pi ia}}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - e^{2\pi ia}} \, dx$$

(see for example [12, (2.9)]). It should be noted that the series $\sum_{n=1}^{\infty} n^{-s} e^{2\pi ina}$ with $0 < a < 1$ converges uniformly on compact subsets in the half-plane $\sigma > 0$ by Abel’s summation formula (see for instance [9, p. 20]). From the following functional equation

$$(3.2) \quad \text{Li}_s(e^{2\pi ia}) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left( e^{\pi i(1-s)/2} \zeta(1-s, a) + e^{-\pi i(1-s)/2} \zeta(1-s, 1-a) \right),$$

we can extend the definition of $\text{Li}_s(e^{2\pi ia})$ over the entire $s$-plane when $0 < a < 1$ (see for instance [1, Exercises 12.2 and 12.3]). By an easy computation, we have

$$(3.3) \quad \Im\left( \frac{e^{2\pi ia}}{e^x - e^{2\pi ia}} \right) = \frac{e^x \sin 2\pi a}{(e^x - \cos 2\pi a)^2 + \sin^2 2\pi a}.$$  

**Proof of Theorem 1.1.** The first statement is easily shown by

$$\text{Li}_s(e^{\pi i}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = -\zeta(s) + \sum_{n=1}^{\infty} \frac{2}{(2n)^s} = (2^{1-s} - 1) \zeta(s).$$

Suppose $0 < a < 1/2$. From (3.1) and (3.3), it holds that

$$(3.4) \quad \Im(\text{Li}_s(e^{2\pi ia})) > 0, \quad \sigma > 0, \quad 0 < a < 1/2.$$
We have \( \text{Li}_0(e^{2\pi ia}) \neq 0 \) according to
\[
(3.5) \quad \text{Li}_0(e^{2\pi ia}) = \frac{e^{2\pi ia}}{1 - e^{2\pi ia}}, \quad a \neq 1
\]
(see for example [19, Section 2]). It should be noted that we can prove \( \Re(\text{Li}_0(e^{2\pi ia})) \neq 0 \) by (4.12).
For \( \sigma > 1 \), it holds that
\[
(3.6) \quad \frac{\partial}{\partial a} \zeta(s, a) = \sum_{n=0}^{\infty} \frac{\partial}{\partial a} (n+a)^{-s} = -s \sum_{n=0}^{\infty} (n+a)^{-s-1} = -s \zeta(s+1, a).
\]
Moreover, we have \( \zeta(\sigma, a) > 0 \) if \( \sigma > 1 \) by the series expression of \( \zeta(s, a) \). Hence one has
\[
(3.7) \quad \zeta(\sigma, a) > \zeta(\sigma, 1-a) > 0, \quad \sigma > 1, \quad 0 < a < 1/2.
\]
Hence, the equation
\[
e^{\pi i(1-\sigma)/2} \zeta(1-\sigma, a) + e^{-\pi i(1-\sigma)/2} \zeta(1-\sigma, 1-a) = 0
\]
contradicts to the facts that \( |e^{\pi i(1-\sigma)}| = 1 \) and
\[
|\zeta(1-\sigma, a)| > |\zeta(1-\sigma, 1-a)|, \quad \sigma < 0, \quad 0 < a < 1/2
\]
which is proved by (3.7). Thus we have \( \text{Li}_\sigma(e^{2\pi ia}) \neq 0 \) when \( \sigma < 0 \) and \( 0 < a < 1/2 \) by the functional equation (3.2) and the fact \( \Gamma(1-\sigma) > 0 \) if \( \sigma < 0 \).
We can prove \( \text{Li}_\sigma(e^{2\pi ia}) \neq 0 \) when \( 1/2 < a < 1 \) similarly since one has
\[
\Im(\text{Li}_\sigma(e^{2\pi ia})) < 0, \quad \sigma > 0, \quad 1/2 < a < 1,
\]
\[
\zeta(\sigma, 1-a) > \zeta(\sigma, a) > 0, \quad \sigma > 1, \quad 1/2 < a < 1
\]
from (3.4) and (3.7), respectively. \( \square \)

3.3. Proof of Theorem 1.2. It is widely known that we have
\[
(3.8) \quad \zeta(s, a) = \frac{a^{1-s}}{s-1} + f(s, a),
\]
where \( f(s, a) \) is an analytic function for all \( s \in \mathbb{C} \) (see for instance [1, Theorem 12.21]). Hence the function \( Y(s, a) \) is entire when \( 0 < a < 1/2 \) by using (3.8). Furthermore, the function \( O(s, a) \) is entire if \( 0 < a < 1/2 \) since \( \text{Li}_s(e^{2\pi ia}) \) with \( 0 < a < 1/2 \) is an entire function (see Section 3.2).
For simplicity, we put
\[
\Gamma(\pi) := \frac{\Gamma(s)}{(2\pi)^s}, \quad \Gamma_{\cos}(s) := 2\Gamma(s) \cos\left(\frac{\pi s}{2}\right), \quad \Gamma_{\sin}(s) := 2\Gamma(s) \sin\left(\frac{\pi s}{2}\right).
\]
The following functional equation is well-known (see for instance [9, Theorem 2.3.1])
\[
(3.9) \quad \zeta(1-s, a) = \Gamma_{\cos}(s) \sum_{n=1}^{\infty} \frac{\cos 2\pi na}{n^s} + \Gamma_{\sin}(s) \sum_{n=1}^{\infty} \frac{\sin 2\pi na}{n^s}, \quad \sigma > 1.
\]
Note that the equation above holds for \( \sigma > 0 \) when \( 0 < a < 1 \). From (3.9), we have
\[
(3.10) \quad Y(1-s, a) = 2\Gamma_{\sin}(s) \sum_{n=1}^{\infty} \frac{\sin 2\pi na}{n^s} = \Gamma_{\sin}(s) O(s, a).
\]
On the other hand, by the functional equation

\[ \text{Li}_{1-s}(e^{2\pi ia}) = \Gamma_1(s)(e^{\pi i/2}\zeta(s,a) + e^{-\pi i/2}\zeta(s,1-a)), \quad 0 < a < 1, \]

(see for example [1, Exercises 12.2]), it holds that

\[ O(1-s,a) = \Gamma_{\sin}(s)(\zeta(s,a) - \zeta(s,1-a)) = \Gamma_{\sin}(s)Y(s,a). \]

**Proof of Theorem 1.2 for** \( Y(s,a) \). For \( \sigma > 1 \), we have

\[ Y(s,a) = a^{-s} - (1-a)^{-s} + \sum_{n=1}^{\infty} n^{-s}\left((1 + \frac{a}{n})^{-s} - \left(1 + \frac{1-a}{n}\right)^{-s}\right). \]

The last series converges absolutely when \( \sigma > 0 \) by

\[ \left(1 + \frac{a}{n}\right)^{-s} - \left(1 + \frac{1-a}{n}\right)^{-s} = \frac{s}{n} + \frac{s(s+1)}{2} \frac{a^2 - (1-a)^2}{n^2} + \cdots. \]

Therefore, we obtain

\[ Y(\sigma,a) > 0, \quad \sigma > 0, \quad 0 < a < 1/2 \]

from \((n+a)^{-\sigma} > (n+1-a)^{-\sigma}\) if \( 0 < a < 1/2 \). On the other hand, one has \( \Im(\text{Li}_1(e^{2\pi i(1-a)})) < 0 \) by (3.4). Thus we obtain

\[ O(\sigma,a) > 0, \quad \sigma > 0, \quad 0 < a < 1/2. \]

Thus, by (3.10), all real zeros of \( Y(1-s,a) \) with \( \sigma > 0 \) and \( 0 < a < 1/2 \) is caused by \( \sin(\pi s/2) = 0 \) with \( \sigma > 0 \). Therefore, all real zeros of the function \( Y(s,a) \) with \( \sigma \leq 0 \) are simple and are located only at \( s = 1 - 2n \), where \( n \in \mathbb{N} \).

**Proof of Theorem 1.2 for** \( O(s,a) \). By (3.14), we only have to show the case \( \sigma \leq 0 \). From (3.12) and (3.13), all real zeros of \( O(1-s,a) \) with \( \sigma > 0 \) and \( 0 < a < 1/2 \) is deduced by \( \sin(\pi s/2) = 0 \). Hence, the function \( O(\sigma,a) \) with \( \sigma \leq 0 \) vanishes only at \( s = 1 - 2n \), where \( n \in \mathbb{N} \).

**Proof of Theorem 1.2 for** \( X(s,a) \). From the functional equations (3.10) and (3.12), one has

\[ X(1-s,a) = \Gamma_{\sin}(s)X(s,a). \]

On the other hand, the inequalities (3.13) and (3.14) imply

\[ X(s,a) := Y(s,a) + O(s,a) > 0, \quad \sigma > 0, \quad 0 < a < 1/2. \]

Therefore, all real zeros of the function \( X(s,a) \) are simple and are located only at \( s = 1 - 2n \), where \( n \in \mathbb{N} \) by the functional equation (3.15).\[ \square \]
4. Proofs of the main results of $Z(s,a)$ and $P(s,a)$

4.1. Proof of Proposition 1.7. We prove that $Z(s,a)$ and $P(s,a)$ can be essentially expressed as $\zeta(s)$ for each $a = 1/2, 1/3, 1/4, 1/6$.

Proof of Proposition 1.7. For any positive integer $m$, one has

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \sum_{l=1}^{m} \sum_{n=0}^{\infty} \frac{1}{(mn+l)^\sigma} = \frac{1}{m^\sigma} \sum_{l=1}^{m} \zeta(s,l/m)$$

when $\sigma > 1$. By using the equation above, we have

(4.1) \hspace{1cm} Z(s,1/2) = 2\zeta(s,1/2) = 2(2^s-1)\zeta(s),

(4.2) \hspace{1cm} Z(s,1/3) = 3^s\zeta(s) - \zeta(s,1) = (3^s-1)\zeta(s),

(4.3) \hspace{1cm} Z(s,1/4) = 4^s\zeta(s) - \zeta(s,1/2) - \zeta(s,1) = 2^s(2^s-1)\zeta(s),

(4.4) \hspace{1cm} Z(s,1/6) = 6^s\zeta(s) - \zeta(s,1/3) - \zeta(s,1/2) - \zeta(s,2/3) - \zeta(s) = (2^s-1)(3^s-1)\zeta(s).

The equations above imply Proposition 1.7 for $Z(s,1/6)$ with $a = 1/2, 1/3, 1/4, 1/6$.

Obviously, for any positive integer $m$, we have

$$\sum_{n=1}^{\infty} \frac{m}{(mn)^\sigma} = \sum_{l=1}^{m} \sum_{n=1}^{\infty} \frac{e^{2\pi iln/m}}{n^\sigma} = \sum_{l=1}^{m} \text{Li}_s(e^{2\pi il/m}), \quad \sigma > 1.$$

From this equality, one has

(4.5) \hspace{1cm} P(s,1/2) = 2\text{Li}_s(e^{\pi i}) = 2(2^{1-s}-1)\zeta(s),

(4.6) \hspace{1cm} P(s,1/3) = 3^{1-s}\zeta(s) - \zeta(s) = (3^{1-s}-1)\zeta(s),

(4.7) \hspace{1cm} P(s,1/4) = 4^{1-s}\zeta(s) - \text{Li}_s(e^{\pi i}) - \zeta(s) = 2^{1-s}(2^{1-s}-1)\zeta(s),

(4.8) \hspace{1cm} P(s,1/6) = 6^{1-s}\zeta(s) - \text{Li}_s(e^{2\pi i/3}) - \text{Li}_s(e^{\pi i}) - \text{Li}_s(e^{4\pi i/3}) - \zeta(s) = (2^{1-s}-1)(3^{1-s}-1)\zeta(s).

Hence we obtain Proposition 1.7 for $P(s,a)$ with $a = 1/2, 1/3, 1/4, 1/6$. \qed

4.2. Lemmas. First, we show the following functional equations (see also [14, the proof of Theorem 1.1]).

Lemma 1.1. Assume $0 < a \leq 1/2$. The function $Z(s,a)$ has the functional equation

(4.9) \hspace{1cm} Z(1-s,a) = \Gamma_{\cos}(s)P(s,a).

Furthermore, the function $P(s,a)$ satisfies the function equation

(4.10) \hspace{1cm} P(1-s,a) = \Gamma_{\cos}(s)Z(s,a).
Proof. From (3.9), it holds that
\[ Z(1-s,a) = 2\Gamma_\cos(s) \sum_{n=1}^{\infty} \frac{\cos 2\pi na}{n^s} = \Gamma_\cos(s)P(s,a) \]
which proves the functional equation (4.9). We remark that the equation above holds for not only \( \sigma > 1 \) but also \( \sigma > 0 \) if \( 0 < a < 1 \) by applying Abel's summation formula to the Dirichlet series of \( P(s,a) \). On the other hand, by (3.11) we have
\[ P(1-s,a) = \Gamma_\cos(s)\zeta(s,a) + \Gamma_\cos(s)\zeta(s,1-a) = \Gamma_\cos(s)Z(s,a) \]
which shows the functional equation (4.10).

Next consider the values of \( \lim_{\sigma \to 1} Z(s,a) \), \( Z(0,a) \), \( P(1,a) \) and \( P(0,a) \).

Lemma 4.2. Let \( 0 < a \leq 1/2 \). Then we have
\[ \lim_{\sigma \to 1+0} Z(\sigma,a) = - \lim_{\sigma \to 1-0} Z(\sigma,a) = \infty, \quad Z(0,a) = 0, \]
\[ P(1,a) = -2\log(2\sin \pi a), \quad P(0,a) = -1. \]

Proof. From (3.8), it holds that
\[ \lim_{\sigma \to 1+0} Z(\sigma,a) = \infty, \quad \lim_{\sigma \to 1-0} Z(s,a) = -\infty. \]
Furthermore, by (3.8) and (4.10), one has
\[ P(0,a) = \lim_{s \to 1} \frac{2\Gamma(s)a^{-s} + (1-a)^{1-s}}{(2\pi)^s} \cos \left( \frac{s\pi}{2} \right) = \frac{4 - \pi}{2\pi} = -1. \]
Note that we can prove the equation above by (3.5). When \( 0 < \theta < 2\pi \), it is well-known that
\[ \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n} = -\log \left( 2 \sin \frac{\theta}{2} \right) + i\left( \frac{\pi}{2} - \frac{\theta}{2} \right). \]
Hence we have \( P(1,a) = -2\log(2\sin \pi a) \). Moreover, one has
\[ Z(0,a) = \lim_{s \to 1} \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{s\pi}{2} \right) P(s,a) = 0 \]
from the functional equation (4.9).

We prove the following by modifying the proof of [4, Lemma 2.3].

Lemma 4.3. Let \( 0 < a < 1/4 \). Then the function
\[ ( -\log(\cos 2\pi a) )^{-\sigma} \Gamma(s)P(\sigma,a) \]
is strictly increasing for \( \sigma > 0 \).
\textbf{Proof.} For $0 < a < 1$ and $\sigma > 0$, one has

\begin{equation}
(4.13) \quad \sum_{n=1}^{\infty} \frac{\cos 2\pi n a}{n^{\sigma}} = \Re \left( \frac{e^{2\pi i a}}{\Gamma(\sigma)} \int_{0}^{\infty} \frac{x^{\sigma-1}}{e^x - e^{2\pi i a}} \, dx \right)
\end{equation}

from (3.1). We can easily see that \begin{equation}
(4.14) \quad \Re \left( \frac{e^{2\pi i a}}{e^x - e^{2\pi i a}} \right) = \frac{e^x \cos 2\pi a - 1}{(e^x - \cos 2\pi a)^2 + \sin^2 2\pi a}.
\end{equation}

Define $G(a, x)$ by the right hand side of the equation above. Then we can find \begin{align*}
e^x \cos 2\pi a - 1 &< 0 \quad \text{when } 0 < x < -\log(\cos 2\pi a), \\
e^x \cos 2\pi a - 1 &\geq 0 \quad \text{when } x \geq -\log(\cos 2\pi a),
\end{align*}
if $0 < a < 1/4$. On the other hand, the function given in this lemma can be rewritten by
\begin{align*}
\frac{1}{2} \alpha^{-\sigma} \Gamma(\sigma) P(\sigma, a) &= \int_{0}^{\alpha} G(a, x) \left( \frac{x}{\alpha} \right)^{\sigma} \frac{dx}{x} + \int_{\alpha}^{\infty} G(a, x) \left( \frac{x}{\alpha} \right)^{\sigma} \frac{dx}{x},
\end{align*}
where $\alpha := -\log(\cos 2\pi a) > 0$. The first integral is strictly increasing in $\sigma > 0$ since one has $G(a, x) < 0$ and $(x/\alpha)^{\sigma}$ is strictly decreasing in $\sigma$ when $0 < x < \alpha$. Similarly, the second integral is also strictly increasing in $\sigma > 0$ since one has $G(a, x) > 0$ and $(x/\alpha)^{\sigma}$ is increasing in $\sigma$ when $x \geq \alpha$. Therefore, the function $\alpha^{-\sigma} \Gamma(\sigma) P(\sigma, a)$ is strictly increasing for $\sigma > 0$ when $0 < a < 1/4$. \hfill \Box

Finally, we show the functions $Z(\sigma, a)$ and $P(\sigma, a)$ are strictly decreasing with respect to $0 < a \leq 1/2$ for fixed $0 < \sigma \neq 1$.

\textbf{Lemma 4.4.} Let $0 < a < 1/2$ and $\sigma > 0$. Then one has
\begin{align*}
\frac{\partial}{\partial a} Z(\sigma, a) &< 0 \quad (\sigma \neq 1), \\
\frac{\partial}{\partial a} P(\sigma, a) &< 0.
\end{align*}

\textbf{Proof.} From (3.6) it holds that
\begin{equation}
\frac{\partial}{\partial a} Z(s, a) = s \sum_{n=0}^{\infty} \left( (n+1-a)^{-s-1} - (n+a)^{-s-1} \right) = -sY(s+1, a)
\end{equation}
for $s \neq 1$ and $\sigma > 0$. The equation above and (3.13) imply Lemma 4.4 for $Z(\sigma, a)$.

Suppose $\sigma > 2$. Then it holds that
\begin{equation}
\frac{\partial}{\partial a} P(s, a) = 2 \sum_{n=1}^{\infty} \frac{\partial}{\partial a} \frac{\cos 2\pi n a}{n^{s}} = -4\pi \sum_{n=1}^{\infty} \frac{\sin 2\pi n a}{n^{s-1}} = -2\pi O(s-1, a).
\end{equation}

The formula above can be continued to the whole complex plane $\mathbb{C}$. On the other hand, by (3.13), (3.14), Theorem 1.2 and the continuity of the functions $Y(\sigma, a)$ and $O(\sigma, a)$ with respect to $\sigma \in \mathbb{R}$, we have $Y(\sigma, a) > 0$ and $O(\sigma, a) > 0$ when $\sigma > -1$. Hence, we have $(\partial/\partial a) P(\sigma, a) < 0$ for $\sigma > 0$. \hfill \Box
4.3. When $\sigma \in (0, 1)$. The second and third statement of the proposition below are analogues of (1) and (2) in [4, Theorem 1.2], respectively.

Proposition 4.5. We have the following:

1. Let $1/6 < a < 1/4$ and $0 < \sigma < 1$. Then we have $Z(\sigma, a) < 0$ and $P(\sigma, a) < 0$.

2. When $0 < a < 1/6$, the functions $Z(\sigma, a)$ and $P(\sigma, a)$ have precisely one simple zero in the open interval $(0, 1)$.

3. For $0 < a < 1/6$, let $\beta_Z(a)$ and $\beta_P(a)$ denote the unique zero of $Z(\sigma, a)$ and $P(\sigma, a)$ in $(0, 1)$, respectively. Then the function $\beta_Z(a) : (0, 1/6) \to (0, 1)$ is a strictly decreasing $C^\infty$-diffeomorphism and $\beta_P(a) : (0, 1/6) \to (0, 1)$ is a strictly increasing $C^\infty$-diffeomorphism. Furthermore, as $a \to +0$, it holds that
   \[
   \beta_Z(a) = 1 - 2a + 2a^2 \log a + O(a^2), \quad \beta_P(a) = 2a - 2a^2 \log a + O(a^2).
   \]

Proof. First we show (1) for $P(\sigma, a)$. Let $1/6 < a < 1/4$. Then we have
   \[
   \lim_{\sigma \to +0} \left( -\log(\cos 2\pi a) \right)^{-\sigma} \Gamma(\sigma) P(\sigma, a) = -\infty,
   \]
   \[
   \lim_{\sigma \to 1-0} \left( -\log(\cos 2\pi a) \right)^{-\sigma} \Gamma(\sigma) P(\sigma, a) = \frac{2 \log(2 \sin \pi a)}{\log(\cos 2\pi a)} < 0
   \]
   from (4.12). Hence we have $P(\sigma, a) < 0$ for all $0 < \sigma < 1$ by Lemma 4.3. Recall that
   \[
   \Gamma(\sigma) > 0 \quad \text{and} \quad \cos(\sigma/2) > 0, \quad 0 < \sigma < 1.
   \]
   Hence we have (1) for $Z(\sigma, a)$ by the functional equation (4.9).

Next assume $0 < a < 1/6$. Then, from (4.12), we have
   \[
   \lim_{\sigma \to +0} \alpha^{-\sigma} \Gamma(\sigma) P(\sigma, a) = -\infty, \quad \lim_{\sigma \to 1-0} \alpha^{-\sigma} \Gamma(\sigma) P(\sigma, a) > 0,
   \]
   where $\alpha := -\log(\cos 2\pi a) > 0$. Thus the function $P(\sigma, a)$ has precisely one simple zero $0 < \beta_P(a) < 1$ by Lemma 4.3. From the functional equation (4.9), we have
   \[
   Z(1 - \beta_P(a), a) = \frac{2 \Gamma(\beta_P(a))}{(2\pi)^{\beta_P(a)}} \cos\left( \frac{\pi \beta_P(a)}{2} \right) P(\beta_P(a), a) = 0.
   \]
   Therefore, by (4.16), we have $\beta_Z(a) = 1 - \beta_P(a)$ which implies (2) for $Z(\sigma, a)$.

Recall that $Z(\sigma, a)$ with $0 < \sigma < 1$ is strictly decreasing with respect to $0 < a \leq 1/2$ from Lemma 4.4. Assume $0 < a_1 < a_2 < 1/6$. Then one has
   \[
   0 = Z(\beta_Z(a_1), a_1) > Z(\beta_Z(a_1), a_2).
   \]
   From the uniqueness of the zero of $Z(\sigma, a)$, it holds that
   \[
   Z(\sigma, a_2) < 0 \quad \text{if and only if} \quad 0 < \beta_Z(a_2) < \sigma < 1.
   \]
   Thus we have $\beta_Z(a_1) > \beta_Z(a_2)$. Therefore, we have the monotonicity of $\beta_Z$ which implies that $\beta_Z$ is injective. Fix $\sigma \in (0, 1)$. Then we have
   \[
   \lim_{a \to +0} Z(\sigma, a) = \lim_{a \to +0} \left( a^{-\sigma} + \zeta(\sigma, 1+a) + \zeta(\sigma, 1-a) \right) = \infty.
   \]
On the other hand, from \((4.4)\) and \([18, \text{(2.12.4)}]\), we have

\[ Z(\sigma, 1/6) = (2^\sigma - 1)(3^\sigma - 1)\zeta(\sigma) < 0. \]

Hence, there exists \(0 < a < 1/6\) such that \(Z(\sigma, a) = 0\). Therefore, \(\beta_Z\) is surjective. By \((3.6)\) and the holomorphy of \(\zeta(s, a)\), two variable function

\[ Z(\cdot, \cdot) : (0,1) \times (0,1/6) \to \mathbb{R} \]

is \(C^\infty\). Hence the function \(\beta_Z(a)\) is \(C^\infty\) from \((\partial / \partial a)Z(\sigma, a) < 0\), which is proved in Lemma \(4.4\), the implicit function theorem and \((2)\) of this proposition. Similarly, we can see that the inverse of \(\beta_Z(a)\) is also \(C^\infty\) from the inverse function theorem. By using \(P(a) = 1 - \beta_Z(a), (\partial / \partial a)P(\sigma, a) < 0\) and modifying the argument above, we can show that \(\beta_P(a)\) is strictly increasing \(C^\infty\)-diffeomorphism.

When \(\sigma > 1\) and \(0 < a \leq 1/2\), it holds that

\[ \zeta(s, 1+a) - \zeta(s) = a^s \sum_n \frac{(1+n)^{-s}}{n} = a^s \sum_n \left( -s \frac{a}{n} + \frac{s(s+1)a^2}{2} \frac{1}{n^2} - \ldots \right). \]

The series above converges absolutely if \(\sigma > 0\). Hence, for \(\sigma \geq 1/2\), one has

\[ \zeta(\sigma, 1+a) - \zeta(\sigma) = O(1), \quad \zeta(\sigma, 1-a) - \zeta(\sigma) = O(1). \]

Therefore, by \((3.8)\), it holds that

\[
Z(\sigma, a) = \zeta(\sigma, a) + \zeta(\sigma, 1-a) = a^{-\sigma} + \zeta(\sigma, 1+a) + \zeta(\sigma, 1-a)
\]

\[
= a^{-\sigma} + 2\zeta(\sigma) + O(1) = a^{-\sigma} + \frac{2}{\sigma-1} + O(1)
\]

when \(1 - \sigma > 0\) and \(a > 0\) are sufficiently small. Take \(\sigma = \beta := \beta_Z(a)\). Then we obtain

\[(4.17) \quad \beta - 1 = -2a^\beta + O((1 - \beta) a^\beta).\]

One has \(\beta - 1 \ll a^\beta \ll a^{1/2} \) from the assumption \(1/2 \leq \sigma < 1\). Hence we have

\[ a^\beta = a \exp((\beta - 1) \log a) = a + (\beta - 1) \log a + O((\beta - 1)^2 |a \log a|^2). \]

In particular, one has \(a^\beta \ll a\). By substituting the estimates above into \((4.17)\), we obtain

\[(4.18) \quad \beta - 1 = -2a + 2(1 - \beta) a \log a + O((1 - \beta) a).\]

Especially, we have \(\beta - 1 \ll a\) and \(\beta - 1 = -2a + O(a^2 |\log a|)\). Substituting these estimates into \((4.18)\), we get

\[
\beta - 1 = -2a + (2a + O(a^2 |\log a|)) a \log a + O(a^2) = -2a + 2a^2 \log a + O(a^2).
\]

Therefore, we obtain the asymptotic formulas \((4.15)\) according to \(\beta_P(a) = 1 - \beta_Z(a)\). \(\square\)
4.4. When \( \sigma > 1 \) or \( \sigma < 0 \). In this subsection, we consider the case \( \sigma \notin [0,1] \).

Proposition 4.6. Assume \( \sigma > 1 \). We have the following:

1. Let \( 0 < a \leq 1/2 \). Then we have \( Z(\sigma,a) > 0 \) for all \( \sigma > 1 \).
2. Let \( 0 < a < 1/6 \). Then we have \( P(\sigma,a) > 0 \) for all \( \sigma > 1 \).
3. When \( 1/6 < a < 1/4 \), the functions \( P(\sigma,a) \) have precisely one simple zero in \( (1,\infty) \). Furthermore, \( \beta_p(a) : (1/6,1/4) \to (1,\infty) \) is a strictly increasing \( C^\infty \)-diffeomorphism and

\[
\beta_p(a) = -\frac{\log(\cos 2\pi a)}{\log 2} + O(\cos 2\pi a), \quad a \to 1/4 - 0.
\]

Proof. The statement (1) is immediately proved by

\[
Z(\sigma,a) = \zeta(\sigma,a) + \zeta(\sigma,1-a) = \sum_{n=0}^{\infty} \left((n+a)^{-\sigma} + (n+1-a)^{-\sigma}\right) > 0.
\]

Suppose \( 0 < a < 1/6 \). From (4.12) and the series expression of \( P(s,a) \), we have

\[
P(1,a) = -2\log(2\sin \pi a) > 0, \quad \lim_{\sigma \to \infty} P(\sigma,a) = 2\cos 2\pi a > 0.
\]

Hence it holds that

\[
\alpha^{-1}\Gamma(1)P(1,a) > 0, \quad \lim_{\sigma \to \infty} \alpha^{-\sigma}\Gamma(\sigma)P(\sigma,a) = \infty,
\]

where \( \alpha := -\log(\cos 2\pi a) > 0 \). Recall that the function \( \alpha^{-\sigma}\Gamma(\sigma)P(\sigma,a) \) is strictly increasing for \( \sigma > 0 \) from Lemma 4.3. Therefore, one has \( P(\sigma,a) > 0 \) for all \( \sigma > 1 \) if \( 0 < a < 1/6 \) since \( \Gamma(\sigma) > 0 \) when \( \sigma > 1 \).

Suppose \( 1/6 < a < 1/4 \). Then we have

\[
P(1,a) = -2\log(2\sin \pi a) < 0, \quad \lim_{\sigma \to \infty} P(\sigma,a) = 2\cos 2\pi a > 0.
\]

Thus, from Stirling’s formula for the gamma function, one has

\[
\alpha^{-1}\Gamma(1)P(1,a) < 0, \quad \lim_{\sigma \to \infty} \alpha^{-\sigma}\Gamma(\sigma)P(\sigma,a) = \infty,
\]

Hence, the function \( P(\sigma,a) \) has precisely one simple zero in \( (1,\infty) \) by the monotonicity of the function \( \alpha^{-\sigma}\Gamma(\sigma)P(\sigma,a) \) proved in Lemma 4.3. By using \( (\partial/\partial a)P(\sigma,a) < 0 \) proved in Lemma 4.4 and modifying the proof of (3) of Proposition 4.5, we have that \( \beta_p(a) \) is strictly increasing with respect to \( 1/6 < a < 1/4 \). The monotonicity of \( \beta_p(a) \) implies that \( \beta_p \) is injective. Moreover, we obtain that \( \beta_p(a) \) is a \( C^\infty \)-function, especially, continuous function, from Lemma 4.4 and the argument appeared in the proof of (3) of Proposition 4.5. On the other hand, it holds that

\[
\lim_{a \to 1/6} \lim_{\sigma \to 1} P(\sigma,a) = \lim_{\sigma \to 1} P(\sigma,a) = P(1,1/6) = 0,
\]

\[
\lim_{a \to 1/4} \lim_{\sigma \to \infty} P(\sigma,a) = \lim_{\sigma \to \infty} P(\sigma,a) = 0.
\]

Therefore, \( \beta_p(a) \) is surjective by the equations above, the intermediate value theorem, the continuity and monotonicity of \( \beta_p(a) \). Hence \( \beta_p(a) \) is a \( C^\infty \)-diffeomorphism. The inverse of \( \beta_p(a) \) is also a \( C^\infty \)-function by Lemma 4.4 and the inverse function theorem.
Now let \( a - 1/4 > 0 \) be sufficiently small and \( \sigma > 1 \) be sufficiently large. For any \( m \in \mathbb{N} \), one has

\[
(4.20) \quad \left| \sum_{n=m}^{\infty} \frac{\cos 2\pi na}{n^\sigma} \right| \leq \sum_{n=m}^{\infty} \frac{1}{n^\sigma} \leq m^{-\sigma} + \int_{m}^{\infty} \frac{dx}{x^\sigma} = m^{-\sigma} + \frac{m^{1-\sigma}}{\sigma - 1}.
\]

Put \( \beta = \beta_P(a) \). Then we have

\[
0 = P(\beta, a) = \cos 2\pi a + \frac{\cos 4\pi a}{2^\beta} + \frac{\cos 6\pi a}{3^\beta} + \frac{\cos 8\pi a}{4^\beta} + \frac{\cos 10\pi a}{5^\beta} + \cdots.
\]

By applying (4.20) to terms of the right-hand side of the formula above except for the first and second terms, we have

\[
0 = \cos 2\pi a + 2^{-\beta} \cos 4\pi a + O(3^{-\beta}).
\]

Hence, from \( 3^{-\beta} \ll 2^{-\beta} \), one has

\[
(4.21) \quad 2^{-\beta} = O(\cos 2\pi a).
\]

According to the triple-angle formula, we obtain

\[
\frac{2\beta}{3^\beta} \cos 6\pi a = \frac{2\beta}{3^\beta} (4\cos^3 2\pi a - 3\cos 2\pi a) = O(\cos 2\pi a).
\]

By using (4.20) and (4.21), we get

\[
2^\beta \left| \sum_{n=4}^{\infty} \frac{\cos 2\pi na}{n^\beta} \right| \leq 2^{-\beta} + 4 \frac{2^{-\beta}}{\beta - 1} = O(\cos 2\pi a).
\]

Therefore, it holds that

\[
0 = 2^\beta \cos 2\pi a + \cos 4\pi a + 2^\beta \sum_{n=3}^{\infty} \frac{\cos 2\pi na}{n^\beta} = 2^\beta \cos 2\pi a + \cos 4\pi a + O(\cos 2\pi a).
\]

Hence, by the equation above, one has

\[
\beta \log 2 = \log \left( \frac{-\cos 4\pi a + O(\cos 2\pi a)}{\cos 2\pi a} \right) = -\log(\cos 2\pi a) + \log \left( -\cos 4\pi a + O(\cos 2\pi a) \right)
\]

\[
= -\log(\cos 2\pi a) + \log(1 - 2\cos^2 2\pi a + O(\cos 2\pi a)) = -\log(\cos 2\pi a) + O(\cos 2\pi a).
\]

The formula above implies the asymptotic formula (4.19). \(\square\)

**Proposition 4.7.** Suppose \( \sigma < 0 \). We have the following:

1. Let \( 0 < a < 1/4 \). Then all real zeros of \( \mathcal{P}(s,a) \) are simple and are located only at the negative even integers.

2. Let \( 0 < a < 1/6 \). Then all real zeros of \( \mathcal{Z}(s,a) \) are simple and are located only at the negative even integers.

3. If \( 1/6 < a < 1/4 \), we have \( \mathcal{Z}(-2n,a) = 0 \), \( n \in \mathbb{N} \) and \( \mathcal{Z}(1 - \beta_P(a),a) = 0 \), where \( \beta_P(a) : (1/6,1/4) \rightarrow (1,\infty) \) is appeared in Proposition 4.6. In addition, for any fixed \( l \in \mathbb{N} \), there exists precisely one \( 1/6 < a_l < 1/4 \) such that the zero of \( \mathcal{Z}(s,a_l) \) at \( s = -2l \) is double. Moreover, we have

\[
(4.22) \quad \beta_Z(a) := 1 - \beta_P(a) = \frac{\log(\cos 2\pi a)}{\log 2} + 1 + O(\cos 2\pi a), \quad a \rightarrow 1/4 - 0.
\]
\textbf{Proof}. The statements above are easily proved by Proposition 4.6, the functional equations (4.9) and (4.10). The bijectivity of $\beta_P(a) : (1/6, 1/4) \to (1, \infty)$ implies the uniqueness of $1/6 < a_l < 1/4$ such that $Z(-2l, a_l) = 0$. In this case, the all real zeros of $Z(s, a)$ are located only at the negative even integers. However, the real zero at $s = -2l$ is not simple but double. \hfill $\square$

4.5. \textbf{Proofs of Theorems 1.3 and 1.4}. By using (4.4) and (4.8), we have the following.

\textbf{Proposition 4.8}. When $0 < a < 1/4$, the function $Z(\sigma, a)$ have zeros at the non-positive integers and real zero $\beta_Z(a)$ in $(\infty, 1)$.

\textbf{Proof}. By (4.4), the function $Z(s, a)$ has a double real zero at $\sigma = 0$ when $a = 1/6$. Hence we have this proposition from (3) of Propositions 4.5 and 4.7. \hfill $\square$

\textbf{Proposition 4.9}. When $0 < a < 1/4$, the function $P(\sigma, a)$ have precisely one simple zero $\beta_P(a)$ in $(0, \infty)$. Furthermore, the function $\beta_P(a) : (0, 1/4) \to (0, \infty)$ is a strictly increasing $C^\infty$-diffeomorphism.

\textbf{Proof}. From (4.8), the function $P(s, a)$ has a simple real zero at $\sigma = 1$ when $a = 1/6$. Hence, $\beta_P(a) : (0, 1/4) \to (0, \infty)$ is a bijection from (3) of Propositions 4.5 and 4.6. We can see that $\beta_P(a)$ is a strictly increasing $C^\infty$-function by $\partial/\partial a)P(\sigma, a) < 0$ proved in Lemma 4.4 and the method used in the proof of (3) of Proposition 4.5. We can show that the inverse of $\beta_P(a)$ is also a $C^\infty$-function likewise. \hfill $\square$

In order to prove Theorems 1.3 and 1.4, we show the following.

\textbf{Proposition 4.10}. Let $1/4 \leq a \leq 1/2$. Then all real zeros of $Z(s, a)$ are simple and are located only at the non-positive even integers.

\textbf{Proof}. One has $Z(\sigma, a) > 0$ when $\sigma > 1$ from (1) of Proposition 4.6. When $1/4 \leq a \leq 1/2$, one has $-1 \leq \cos 2\pi a \leq 0$. Hence for all $x > 0$, we have

$$\Re \left( \frac{e^{2\pi ia}}{e^x - e^{2\pi ia}} \right) = \frac{e^x \cos 2\pi a - 1}{(e^x - \cos 2\pi a)^2 + \sin^2 2\pi a} < 0$$

by (4.14). The inequality above and (4.13) imply

$$P(\sigma, a) = 2 \sum_{n=1}^{\infty} \frac{\cos 2\pi na}{n^\sigma} = \Re \left( \frac{e^{2\pi ia}}{\Gamma(\sigma)} \int_0^\infty \frac{2x^{\sigma-1}}{e^x - e^{2\pi ia}} \, dx \right) < 0, \quad \sigma > 0.$$  

It is widely known that $\Gamma(s)$ is analytic when $\sigma > 0$ and

$$\Gamma(\sigma) > 0, \quad \sigma > 0.$$  

Therefore, all real zeros of $Z(1-s, a)$ with $1/4 \leq a \leq 1/2$ and $\sigma > 0$ come from $\cos(\pi s/2) = 0$ with $\sigma > 0$ by (4.9), (4.23) and (4.24). Hence, every real zero of $Z(s, a)$ with $1/4 \leq a \leq 1/2$ and $\sigma < 1$ is caused by $\cos(\pi(1-s)/2) = 0$ with $\sigma < 1$ which is equivalent to that $s$ is a non-positive even integer. Furthermore, we can easily see that all real zeros of $Z(s, a)$ are simple by the equation above. \hfill $\square$

\textbf{Proposition 4.11}. Assume $1/4 \leq a \leq 1/2$. Then all real zeros of the function $P(s, a)$ are simple and are located only at the negative even integers.
Proof. We have \( P(\sigma, a) < 0 \) when \( \sigma > 0 \) and \( 1/4 \leq a \leq 1/2 \) by (4.23). Moreover, we have \( P(0, a) = -1 \) from (4.12). Therefore, all real zeros of \( P(1 - s, a) \) with \( 1/4 \leq a \leq 1/2 \) and \( \sigma > 1 \) come from \( \cos(\pi s/2) = 0 \) with \( \sigma > 1 \) by (4.10), (4.24) and (1) of Proposition 4.6. Obviously, all real zeros of the function \( P(s, a) \) are simple by the equation above or the functional equation (4.10). □

Now we are in position to prove Theorems 1.3 and 1.4

Proof of Theorem 1.3. When \( 0 < a < 1/6 \), \( Z(s, a) \) has precisely one simple real zero in the open interval \((0, 1)\) from (3) of Proposition 4.5. If \( a = 1/6 \), \( Z(s, a) \) has a double real zero at \( \sigma = 0 \) by (4.4).

Let \( 1/6 < a < 1/4 \). Then, from (3) of Proposition 4.7, \( Z(s, a) \) has a double real zero at \( \sigma = -2l, l \in \mathbb{N} \) or a simple real zero for \( \sigma < 0 \) and \( \sigma \neq -2l \). Proposition 4.10 implies the case \( 1/4 \leq a \leq 1/2 \). □

Proof of Theorem 1.4. From Proposition 4.9, the functions \( P(\sigma, a) \) have precisely one simple zero in \((0, \infty)\) when \( 0 < a < 1/4 \). Proposition 4.11 implies the case \( 1/4 \leq a \leq 1/2 \). □

4.6. Proofs of Propositions 1.5, 1.6 and 1.9. We prove the remaining propositions.

Proofs of Propositions 1.5 and 1.6. These are easily proved by Propositions 4.5, 4.6 and 4.7, 4.8 and 4.9.

We prove Proposition 1.9 only for the function \( Z(s, a) \). We can show this proposition for other zeta functions similarly.

Proof of Proposition 1.9. The upper bound for the number of zeros of \( Z(s, a) \) is proved by the Bohr-Landau method (see [18, Theorem 9.15 (A)]), the mean square of \( \zeta(s, a) \) (see [9, Theorem 4.2.1]) and the inequality

\[
\int_{2}^{T} |Z(\sigma + it, a)|^2 dt \leq 2 \int_{2}^{T} |\zeta(\sigma + it, a)|^2 dt + 2 \int_{2}^{T} |\zeta(\sigma + it, 1 - a)|^2 dt, \quad \sigma > 1/2.
\]

First assume that \( a \neq 1/2, 1/3, 1/4, 1/6 \) is rational. Then the lower bounds for the number of zeros of \( Z(s, a) \) with \( \sigma > 1 \) and \( 1/2 < \sigma < 1 \) are prove by (2.1), [16, Corollary], [6, Theorem 2], the definition of the zeta function \( Z(s, a) \) and the fact that \( \varphi(q) \leq 2 \) if and only if \( q = 1, 2, 3, 4, 6 \).

Next suppose \( a \) is transcendental and \( 1/2 < \sigma < 1 \). Then \( \zeta(s, a) \) and \( \zeta(s, 1 - a) \) have the joint universality by [11, Theorem 5]. Hence we can prove Proposition 1.6 in this case by modifying the proof of [6, Theorem 2] (see also [6, Theorem 3]).

Finally, consider the case \( a \) is transcendental and \( \sigma > 1 \). We can easily see that the set

\[
\{ \log(n + a) : n \in \mathbb{N} \cup \{0\} \} \cup \{ \log(n + 1 - a) : n \in \mathbb{N} \cup \{0\} \}
\]

is linearly independent over \( \mathbb{Q} \). Hence, by using the Davenport and Heilbronn method (see for example [9, p. 162]), the function \( Z(s, a) \) has more than \( C_n^4(T) \) in the rectangle \( 1 < \sigma < 1 + \delta \) and \( 0 < t < T \) when \( T \) is sufficiently large. □

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References

[1] T. M. Apostol, Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics, Springer, New York, 1976.

[2] H. Cohen, Number theory. Vol. II. Analytic and modern tools (Graduate Texts in Mathematics, 240. Springer, New York, 2007).

[3] J. B. Conrey and K. Soundararajan, Real zeros of quadratic Dirichlet L-functions. Invent. Math. 150 (2002), no. 1, 1–44.

[4] K. Endo and Y. Suzuki, Real zeros of Hurwitz zeta-functions and their asymptotic behavior in the interval (0, 1), J. Math. Anal. Appl. 473 (2019), no. 2, 624–635.

[5] N. J. Fine, Note on the Hurwitz zeta-function, Proceedings of the American Mathematical Society. 2, (1951), 361–364.

[6] J. Kaczorowski and M. Kulas, On the non-trivial zeros off line for L-functions from extended Selberg class, Monatshefte Math. 150 (2007), no. 3, 217–232.

[7] A. A. Karatsuba and S. M. Voronin, The Riemann zeta-function. Translated from the Russian by Neal Koblitz. De Gruyter Expositions in Mathematics, 5. Walter de Gruyter & Co., Berlin, 1992.

[8] N. Koblitz, Introduction to elliptic curves and modular forms. Second edition. Graduate Texts in Mathematics, 97. Springer-Verlag, New York, 1993.

[9] A. Laurinčikas and R. Garunkštis, The Lerch zeta-function. Kluwer Academic Publishers, Dordrecht, 2002.

[10] T. Matsusaka, Real zeros of the Hurwitz zeta function, Acta Arithmetica, 183 (2011), no. 12, 2352–2367.

[11] H. Mishou, The joint universality theorem for a pair of Hurwitz zeta functions. J. Number Theory 131 (2011), no. 12, 2352–2367.

[12] T. Nakamura, Real zeros of Hurwitz-Lerch zeta and Hurwitz-Lerch type of Euler-Zagier double zeta functions. Math. Proc. Cambridge Philos. Soc. 160 (2016), no. 1, 39–50.

[13] T. Nakamura, Real zeros of Hurwitz-Lerch zeta functions in the interval (−1, 0). J. Math. Anal. Appl. 438 (2016), no. 1, 42–52.

[14] T. Nakamura, Functional equation and zeros on the critical line of the quadrilateral zeta function, J. Number Theory 233 (2022), 432–455.

[15] T. Nakamura, The values of zeta functions composed by the Hurwitz and periodic zeta functions at integers, arXiv:2006.03300

[16] E. Saias and A. Weingartner, Zeros of Dirichlet series with periodic coefficients, Acta Arith. 140 (2009), no. 4, 335–344.

[17] Y. G. Sinai, Self-similar probability distributions. Theory Probab. Appl. 21 (1976), 64–80.

[18] E. C. Titchmarsh, The theory of the Riemann zeta-function, Second edition. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986.

[19] Wikipedia, Polylogarithm. https://en.wikipedia.org/wiki/Polylogarithm

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