EIGHTH-ORDER IMAGE MASKS FOR TERRESTRIAL PLANET FINDING

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Received 2004 November 2; accepted 2005 March 13

ABSTRACT

We describe a new series of band-limited image masks for coronagraphy that are insensitive to tip-tilt errors and other low spatial frequency optical aberrations. For a modest cost in throughput, these “eighth-order” masks would allow the Terrestrial Planet Finder Coronagraph (TPF-C) to operate with a pointing accuracy no better than that of the Hubble Space Telescope. We also provide eighth-order notch filter masks that offer the same robustness to pointing errors combined with more manageable construction tolerances: binary masks and sampled graded masks with moderate optical density requirements.

Subject headings: astrobiology — circumstellar matter — instrumentation: adaptive optics — planetary systems

1. INTRODUCTION

Coronagraphy holds great promise for imaging extrasolar planetary systems, even extrasolar terrestrial planets only \( \sim 10^{-10} \) times as bright as their host stars (e.g., Kuchner & Spergel 2003b). However, finding extrasolar terrestrial planets at contrast levels of \( \sim 10^{-10} \) using any of the present image mask designs requires either pointing accuracies at the level of a fraction of a milli-arcsecond (Kuchner & Traub 2002; Kuchner & Spergel 2003a) or apodization in the pupil plane (e.g., Aime et al. 2002; Kasdin et al. 2003), which generally carries a high penalty in throughput and inner working angle (but see also Guyon 2003; Traub & Vanderbei 2003). We offer a new series of band-limited image masks that can provide high contrast levels without pupil apodization because they are intrinsically insensitive to pointing errors and other low-order aberrations. We also provide notch filter versions of these masks that may be easier to build to the necessary tolerances.

2. EIGHTH-ORDER MASKS

2.1. Band-limited Masks and Notch Filter Masks

Here we summarize the basic definitions of band-limited masks and notch filter masks stated by Kuchner & Traub (2002) and Kuchner & Spergel (2003a). We focus on linear masks, described by functions of a single variable, \( x \). One-dimensional band-limited and notch filter masks can be combined to create a wide variety of two-dimensional masks.

An ideal linear image mask can be described by a function, \( \hat{M}(x) \), called the mask function. In our simple model of the interaction between masks and light, the mask function, also called the mask’s amplitude transmissivity, multiplies the electric field phasor of the incoming beam. The intensity transmissivity of a mask, \( |\hat{M}(x)|^2 \), multiplies the intensity of the beam. We also refer often to the mask function, the mask intensity transmissivity, and the Fourier transform of the mask function,

\[
M(u) = \int \hat{M}(x) e^{-2\pi iux} \, dx. \tag{1}
\]

Kuchner & Traub (2002) showed that if \( \hat{M}(x) \) is a notch filter function, i.e.,

\[
M(u) = 0 \quad \text{for} \quad \epsilon/2 < |u| < 1 - \epsilon/2, \tag{2}
\]

where \( \epsilon \) sets the undersizing of the Lyot stop, and if

\[
\int_{-\epsilon/2}^{\epsilon/2} M(u) \, du = 0, \tag{3}
\]

then the mask defined by \( \hat{M}(x) \) will completely remove all on-axis light in an ideal coronagraph with a uniform entrance pupil. Kuchner (2005) showed that notch filter masks are the only trivially achromatic masks that completely remove on-axis light in a one-dimensional or separable two-dimensional coronagraph. Masks that we can construct without amplifying the beam or manipulating its phase are necessarily limited to \( 0 \leq \hat{M}(x) \leq 1 \).

A band-limited mask is a notch filter mask with \( M(u) = 0 \) for \( |u| > \epsilon/2 \).

We aim to find notch filter mask functions, \( \hat{M}(x) \), that provide deep suppression of light near the optical axis, not just at the optical axis. We first derive new band-limited masks and then follow the recipes in Kuchner & Spergel (2003a) to generate useful notch filter masks based on them.

2.2. Blocking Slightly Off-Axis Light

Understanding the off-axis behavior of an ideal coronagraph with a band-limited mask is easy. A coronagraph with a band-limited image mask attenuates the intensity of an image of a point source located at an angle \( x \) by a factor of \( |\hat{M}(x)|^2 \) compared to the image the source would have if the image mask were removed while the Lyot stop remained in place (see the Appendix). In an ideal coronagraph with a band-limited mask, the point-spread function (PSF) is independent of the position of the source with respect to the optical axis; only the attenuation varies with \( x \). Hence, we can describe the way a band-limited mask attenuates sources near the optical axis, including the target star, simply by expanding \( \hat{M}(x) \) about \( x = 0 \).

If the first important term in this expansion is quadratic in \( x \), the intensity attenuation will vary as \( x^4 \). Borrowing the language...
of interferometry, we might say such a mask produces a fourth-order null. For a demonstration of why this interferometric terminology is appropriate, consider the nulling coronagraph described by Levine et al. (2003), which monochromatically synthesizes a particular band-limited mask with a fourth-order null using beam combiners.

All the band-limited mask designs and notch filter mask designs illustrated in Kuchner & Traub (2002), Kuchner & Spergel (2003a), and Kuchner (2005) have fourth-order nulls. For example, all of the popular 1-sinc family of masks are fourth-order, 1-sinc^4 k_{1x/n} ≈ (1/6n)(k_{1x})^2. But we can design band-limited masks and notch filter masks with nulls of any order β by the methods described below, if β is a multiple of 4.

The order of the null dictates the sensitivity of the mask to optical aberrations that effectively spread the light from a target source around some region of the sky near the optical axis. Tip-tilt error (caused by pointing error, for example) is the simplest low-order aberration for us to model and a term that can easily dominate a coronagraph design’s error budget. A pointing error of Δθ will cause an intensity leak proportional to (Δθ^2)^2. A mask that is insensitive to pointing error will also defeat other low-order aberrations like defocus, coma, and astigmatism to some degree, although some low-order Zernike terms contain medium spatial frequency tails that may leak through. Medium spatial frequency errors are problematic for any coronagraph design because by definition they coincide with the search area; no mask or stop can block them without also blocking light from the planet. Shaklan & Green (2005) discuss the effects of low-order aberrations in a coronagraph with an eighth-order mask in detail.

The fractional leakage through a mispointed coronagraph with a band-limited mask is simply

$$L = \frac{\int \int I(x + \Delta \theta, y)|\hat{M}(x, y)|^2 \, dx \, dy}{\int \int I(x, y) \, dx \, dy},$$

(4)

where \(I(x, y)\) is the source intensity, i.e., the stellar disk, and Δθ is the instantaneous pointing error. For a fourth-order linear mask, the instantaneous fractional intensity leakage is

$$L = \frac{\theta_i^4 + 400 \theta_i^2 (2 \Delta \theta)^2 + 128 (4 \Delta \theta)^4}{256 \theta_i^2 \sigma_{\text{BW}}^2},$$

(5)

where \(\theta_i\) is the angular diameter of the star and \(\sigma_{\text{BW}}\) is the inner working angle of the mask, defined by |\(\hat{M}(\theta_{\text{BW}})\)| = 1/2. To derive this expression, we made the approximation that \(\hat{M}(x) = x^4\); we have corrected a numerical error in equation (17) of Kuchner & Spergel (2003a). If we assume Δθ is distributed in a Gaussian with standard deviation \(\sigma_{\Delta \theta}\), and \(\sigma_{\Delta \theta} \gg \theta_i\), then we find that the mean leakage is

$$\langle L \rangle = 1.5 (\sigma_{\Delta \theta}/\theta_{\text{BW}})^4.$$  

(6)

So if we assume that we can tolerate a leakage of \(\langle L \rangle < 3 \times 10^{-8}\), and that Δθ is much larger than the angular radius of the star, we find that we must center the star on the mask to an accuracy of

$$\sigma_{\Delta \theta} < 0.012 \theta_{\text{BW}}.$$  

(7)

Although it is easiest to interpret in terms of pointing error, this Gaussian blurring can also serve as a crude model of the effects of other low-order aberrations.

For an eighth-order mask approximated as \(\hat{M}(x) = x^8\), the instantaneous fractional intensity leakage is

$$L = \left[7 \theta_i^8 + 1120 \theta_i^6 (\Delta \theta)^2 + 17920 \theta_i^4 (4 \Delta \theta)^4 + 57344 \theta_i^2 (8 \Delta \theta)^6 + 32768 (4 \Delta \theta)^8 \right] \left(65536 \theta_i^{12} \right)^{-1},$$

(8)

and the corresponding mean fractional leakage is

$$\langle L \rangle = 52.5 (\sigma_{\Delta \theta}/\theta_{\text{BW}})^8,$$

(9)

and the pointing requirement for leakage \(\langle L \rangle < 3 \times 10^{-8}\) is

$$\sigma_{\Delta \theta} < 0.070 \theta_{\text{BW}}.$$  

(10)

A factor of \(\sim 6\) improvement over the \(\sigma_{\Delta \theta}\) tolerance for fourth-order masks. A coronagraph designed to find extrasolar terrestrial planets like the Terrestrial Planet Finder Coronagraph (TPF-C) might need \(\theta_{\text{BW}} = 60\) mas. This requirement implies a pointing tolerance of \(\sigma_{\Delta \theta} \leq 0.72\) mas using a fourth-order mask or \(\sigma_{\Delta \theta} \leq 4.2\) mas using an eighth-order mask. For comparison, the Hubble Space Telescope points to \(\sigma_{\Delta \theta} \approx 3\) mas (Burrows et al. 1991).

Eighth-order masks can also provide high-contrast images of extended sources, although relaxing the pointing tolerance depletes some of this power. For a fourth-order mask, equation (5) shows that the extent of a central source begins to matter when \(\theta_i > (8/3)(\Delta \theta)\), and the term corresponding to \(\theta_i^4\) cancels; i.e.,

$$\int \int \frac{\hat{M}(x) \hat{M}(y)}{M^2(x + y)} \, dx \, dy = 0.$$  

(11)

For example, we can add a term of the form \(C [1 - \cos (k_{2x})]\), otherwise known as a sin^2 mask, to any \(1 - \text{sinc}^4\) mask to create a new mask with \(\frac{d^2 \hat{M}(x)}{dx^2} \big|_{x = 0} < 0\), while still satisfying equation (3). If we start with a mask of the form \(1 - \text{sinc}^4 k_{1x} / n \approx (k_{1x})^2 / (6n)\) and add \(C (1 - \cos (k_{2x})) \approx (C/2)(k_{2x})^2\), we find that to produce an eighth-order mask, we require that \(C = (1/3n)(k_{2x})^2\). However, we do not want to add a sin^2 mask of just any random spatial frequency. We would prefer a frequency within the bandwidth of the original mask so that we do not suffer an undue throughput penalty; i.e., \(k_{2x} \approx \leq k_{1x}\). In order to minimize \(|C|\), we should pick a frequency at exactly the edge of the band; i.e., \(k_{2x} = k_{1x}\). With this constraint, we find that \(C = -1/(3n)\).

Of course, adding mask functions can violate the requirement that \(\hat{M}(x) \leq 1\). To ensure \(\hat{M}(x) \leq 1\), we can renormalize the
mask by multiplying $\hat{M}(x)$ by a constant, $N$, equal to the inverse of its maximum value.

Putting everything together and using physical units yields a series of eighth-order band-limited masks:

$$\hat{M}_{\text{BL}}(x) = N \left[ \frac{3n - 1}{3n} - \text{sinc}^n \left( \frac{\pi x e}{n \lambda_{\text{max}} f} \right) + \left( \frac{1}{3n} \right) \cos \left( \frac{\pi x e}{\lambda_{\text{max}} f} \right) \right],$$

where $f$ is the focal ratio at the mask and $\lambda_{\text{max}}$ is the longest wavelength at which the mask is to operate. Figure 1 shows $\hat{M}(x)$ for the first few linear masks in the series. The $n = 3$ design offers a good compromise between the large sidelobes of the $n = 1$ mask and the higher inner working angle–bandwidth product of the $n = 5$.

The ringing in these image masks reduces their effective throughputs. The amplitude of the additional ringing introduced by the cosine term in equation (12) falls off slowly with $n$, so simply increasing $n$ does not help much.

Fortunately, we can create another series of eighth-order masks with less ringing by combining two $1 - \text{sinc}^n$ masks instead of a $1 - \text{sinc}^n$ mask and a $\text{sinc}^2$ mask, using the same procedure that we used to construct equation (12):

$$\hat{M}_{\text{BL}}(x) = N \left[ \frac{l - m}{l} - \text{sinc}^l \left( \frac{\pi x e}{l \lambda_{\text{max}} f} \right) + \frac{m}{l} \text{sinc}^m \left( \frac{\pi x e}{m \lambda_{\text{max}} f} \right) \right].$$

This series of masks has less ringing than the series described by equation (12). It is parameterized by two integer exponents, $l$ and $m$; we assume $l > m$. Figure 2 shows $\hat{M}(x)$ for $m = 1$ and $l = 2–5$. The $m = 1$ and $l = 2–3$ masks have throughputs similar to the $n = 3$ cosine mask. Using large values of $m$ and $l$ reduces the ringing further, but it also reduces the Lyot stop throughput.

Figure 3 compares the intensity transmissivity, $|\hat{M}(x)|^2$, for the $1 - \text{sinc}^2$ fourth-order mask and the $n = 1, l = 3$ eighth-order mask. While the $1 - \text{sinc}^2$ mask has an inner working angle of $\theta_{\text{IW}} = (1.448/e)(\lambda/D)$, the $m = 1, l = 3$ eighth-order mask has an inner working angle of $\theta_{\text{IW}} = (1.788/e)(\lambda/D)$. The $m = 1, l = 3$ mask offers a good compromise between ringing and throughput and also reaches 100% transmissivity at its first maximum, a critical region for planet searching; we recommend this mask for TPF-C.

Consider a TPF-C design with $\theta_{\text{IW}} = 3\lambda/D$ using a linear fourth-order mask. This coronagraph has a bandwidth of $\epsilon = 0.4$ and a nominal Lyot stop throughput of $1 - \epsilon = 0.6$ (see Kuchner & Spergel 2003a). This fourth-order design probably requires some mild apodization of the Lyot stop to ameliorate leakage due to low-order optical aberrations, reducing the throughput to 0.35. Keeping $\theta_{\text{IW}} = 3\lambda/D$ but switching to a linear $m = 1, l = 3$ eighth-order mask would mean working at a bandwidth of $\epsilon = 0.596$ and a Lyot stop throughput of $1 - \epsilon = 0.404$. Coronagraphs with eighth-order masks should not require any Lyot stop apodization.

In other words, our analysis suggests that eighth-order masks combined with unapodized Lyot stops perform about as well as fourth-order masks combined with apodized Lyot stops in terms of throughput and robustness to pointing errors. An alternative way to provide robustness to pointing errors is to use a shaped-pupil coronagraph (Kasdin et al. 2003; Vanderbei et al. 2003a,
band-limited masks. For example, we can make a family of order notch filter masks analogous to the variety of eighth-order solutions to equation (17), we can construct a variety of eighth-order notch filter masks using the 1 fourth-order notch filter functions. We can also design low-ringing eighth-order notch filter masks using the 1 fourth-order notch filter functions. To be consistent with § 3, we refer to the various eighth-order notch filter masks by the exponents of their constituent functions (n, m, l, . . . , etc.). In the following, we provide sample calculations for making eighth-order binary and graded notch filter masks using the m = 1, l = 3 design.

4. EIGHTH-ORDER NOTCH FILTER MASKS

The functions described by equations (12) and (13) can be used in a variety of ways, e.g., to make linear masks [M(x, y) = M_{BL}(x, y)], radial masks [M(r) = M_{BL}(r)], or separable masks [M(x, y) = M_{BL}(x) M_{BL}(y)]. However, all band-limited masks are necessarily smooth graded masks. Notch filter masks offer even more design freedom and need not necessarily be smooth, making them potentially easier to manufacture than band-limited masks (Kuchner & Spergel 2003a).

Notch filter masks affect starlight and planet light the same way as band-limited masks; only their low spatial frequency parts contribute to starlight suppression. Consequently, in an eighth-order notch filter mask, only the low-frequency part needs to satisfy equation (11). In other words,

$$\frac{d^2}{dx^2}\text{[the low-frequency part of }\hat{M}(x)\text{]}_{x=0} = \int_{-\epsilon/2}^{\epsilon/2} (-2\pi i u)^2 M(u) du = 0. \quad (14)$$

Equivalently, we can say that an eighth-order notch filter mask satisfies equations (2) and (3), and also

$$\int_{-\epsilon/2}^{\epsilon/2} u^2 M(u) du = 0. \quad (15)$$

To find masks that meet these criteria, we can use a technique similar to the one employed in § 3. Since any linear combination of fourth-order notch filter mask functions will automatically satisfy equations (2) and (3), we start by writing

$$\hat{M}_{\text{notch}}(x) = N [\hat{M}_{\text{notch41}}(x) + C\hat{M}_{\text{notch40}}(x)], \quad (16)$$

where $\hat{M}_{\text{notch41}}(x)$ and $\hat{M}_{\text{notch40}}(x)$ represent different fourth-order notch filter mask functions and N ensures that $\hat{M}_{\text{notch}}(x) \leq 1$. To construct a notch filter mask that exhibits eighth-order behavior, we need to weight the linear combination so that the new notch filter function also satisfies equation (15). In other words, we find the constant C by substituting equation (16) into equation (15):

$$C = -\frac{\int_{-\epsilon/2}^{\epsilon/2} u^2 \hat{M}_{\text{notch41}}(u) du}{\int_{-\epsilon/2}^{\epsilon/2} u^2 \hat{M}_{\text{notch40}}(u) du}. \quad (17)$$

This constant should be negative; it should also satisfy |C| < 1.

By combining fourth-order notch filter functions and using the solutions to equation (17), we can construct a variety of eighth-order notch filter masks analogous to the variety of eighth-order band-limited masks. For example, we can make a family of eighth-order notch filter masks using the $1 - \cos$ and $1 - \sin^{2}$ fourth-order notch filter functions. We can also design low-ringing eighth-order notch filter masks using the $1 - \sin^{2}$ notch filter functions. To be consistent with § 3, we refer to the various eighth-order notch filter masks by the exponents of their constituent functions (n, m, l, . . . , etc.). In the following, we provide sample calculations for making eighth-order binary and graded notch filter masks using the m = 1, l = 3 design.

4.1. Eighth-Order Binary Masks

Notch filter masks can be designed to be binary: everywhere either completely opaque or completely transparent. A simple way to make such a binary mask is to assemble a mask from a collection of identical parallel stripes, for which any arbitrary notch filter mask function provides the width of each stripe. In other words, each stripe is defined by

$$\hat{M}_{\text{stripe}}(x, y) = \begin{cases} 1 & \text{where } y < \hat{M}_{\text{notch}}(x)\hat{\lambda}_{\text{min}}f, \\ 0 & \text{elsewhere}, \end{cases} \quad (18)$$

and the mask function is

$$\hat{M}_{\text{binary}}(x, y) = \sum_{j=-\infty}^{\infty} \hat{M}_{\text{stripe}}(x, y - j\hat{\lambda}_{\text{min}}f), \quad (19)$$

where $\hat{\lambda}_{\text{min}}$ is the shortest wavelength in the band of interest.

If we like, we can use the band-limited mask functions described by equation (12) or (13) in place of $\hat{M}_{\text{notch}}(x)$, resulting in a mask formed of continuous curves. However, sampled binary masks may prove to be easier to manufacture, since their features are not as small near the optical axis. We construct here an eighth-order sampled binary mask. Such a mask can be made entirely from rectangles of opaque material. Debes et al. (2004) have demonstrated the construction of sampled fourth-order masks using e-beam lithography.

Fourth-order sampled masks are defined by the following prescription (Kuchner & Spergel 2003a):

$$\hat{M}_{\text{notch}}(x) = \hat{M}_{\text{samp4}}(x) - \hat{M}_{0}, \quad (20)$$

where

$$\hat{M}_{\text{samp4}}(x) = \hat{P}(x) \left\{ \hat{M}_{\text{BL4}}(x) \Delta x \sum_{k} \delta(x - (k + \zeta)\Delta x) \right\}, \quad (21a)$$

$$M_{\text{samp4}}(u) = P(u) \left[ M_{\text{BL4}}(u) * \sum_{k} \delta \left( u - \frac{k}{\Delta x} \right) e^{-2\pi i u \zeta} \right], \quad (21b)$$

and

$$\hat{M}_{0} = \int_{-\epsilon/2}^{\epsilon/2} \hat{M}_{\text{samp4}}(u) du = \int_{-\infty}^{\infty} M_{\text{BL4}}(u) P(u) du \quad \left|_{x=0} \right., \quad (22)$$

Here $M_{\text{BL4}}$ represents any fourth-order band-limited mask function, k ranges over all integers, and asterisks indicate convolution. The sampling points are offset from the mask center by a fraction $\zeta$ of $\Delta x$. The kernel, $\hat{P}(x)$, can represent the “beam” of a nanofabrication tool. It should be normalized so that $\int_{-\infty}^{\infty} \hat{P}(x) dx = 1$, and $P(x)$ must be everywhere $\leq 1/(\Delta x)$, so $M_{\text{samp4}}(x)$ remains $\leq 1$. The constant $M_{0}$ ensures that the mask satisfies equation (3). Although the sampled mask is derived from $\hat{M}_{\text{BL4}}(x)$, the function being sampled is $M_{\text{BL4}}(x) - \hat{M}_{0}$. 
Combining equation (16) and equation (20), we have

\[ \tilde{M}_{\text{notch}}(x) = N \left[ \tilde{M}_{\text{notch,4}}(x) + C \tilde{M}_{\text{notch,4}}(x) \right] \]

\[ = N \left\{ \left[ \tilde{M}_{\text{samp,4}}(x) - \tilde{M}_0 \right] + C \left[ \tilde{M}_{\text{samp,4}}(x) - \tilde{M}_0 \right] \right\}, \]

where \( \tilde{M}_{\text{samp,4}}(x) \) and \( \tilde{M}_{\text{samp,4}}(x) \) are sampled versions of the fourth-order band-limited functions \( M_{BL4}(x) \) and \( M_{BL4}(x) \)
described by equation (21a). The constants \( \tilde{M}_0 \), \( \tilde{M}_0 \), and \( C \) ensure that \( \tilde{M}_{\text{notch,4}}(x) \) and \( \tilde{M}_{\text{notch,4}}(x) \) satisfy both equation (3) and equation (15). The constant \( C \) is given by

\[ C = -\frac{\int_{-\infty}^{t/2} u^2 P(u)M_{BL4}(u) \, du}{\int_{-\infty}^{t/2} u^2 P(u) \, du} . \] (25)

To make an eighth-order sampled notch filter mask, the function that we sample is \( N \left( \left[ \tilde{M}_{\text{notch,4}}(x) - \tilde{M}_0 \right] + C \tilde{M}_{\text{notch,4}}(x) - \tilde{M}_0 \right) \).

Figure 4 shows a plot of this function to illustrate how \( \zeta \) may be chosen. This example uses the \( m = 1, l = 3 \) sampled eighth-order mask, meaning \( M_{BL4}(x) = 1 - \sin(x) \) and \( M_{BL4}(x) = 1 - \sin^2(x) \). To guarantee that \( \tilde{M}_{\text{notch}}(x) \geq 0 \), the parameter \( \zeta \) must be in the range \( |\zeta| \leq \zeta_0 \), where \( \zeta_0 \) is defined by the condition \( \tilde{M}_{\text{BL4}}(\zeta_0 / \lambda_{\min}) + C \tilde{M}_{\text{BL4}}(\zeta_0 / \lambda_{\min}) = \tilde{M}_0 + C \tilde{M}_0 \). For our binary mask, we choose \( \zeta = \zeta_0 \), to make the central rectangles contiguous.

The bandwidth of a mask should be chosen conservatively; e.g., \( \lambda_{\max} \) should be somewhat larger than the longest wavelength at which the detector is sensitive so a filter with a finite slope can remove all the extraneous light. Band-limited masks and notch filter masks leak light at wavelengths longer than \( \lambda_{\max} \); notch filter masks also leak light at wavelengths shorter than \( \lambda_{\min} \). At a fixed inner working angle, increasing \( \lambda_{\max} \) necessitates increasing \( \epsilon \) and thereby decreasing the throughput. Decreasing \( \lambda_{\min} \) means spacing the stripes and samples in a notch filter mask closer together.

For the \( m = 1, l = 3 \) mask with \( \theta_0 = 3 \lambda_{\max} / D \), spacing \( \Delta x = \lambda_{\min} / f \), and bandpass 0.5–0.8 \( \mu \)m, we find that \( \tilde{M}_0 = 0.0630889 \), \( \tilde{M}_0 = 0.0188218 \), \( C = -0.33935486 \), and \( \zeta_0 = 0.25941279 \). Table 1 lists normalization constants and sampled mask parameters for eighth-order notch filter masks of various inner working angles using a top-hat kernel, \( \tilde{P}(x) = (D / \lambda_{\min}) \Pi(xD / \lambda_{\min}) \), and a 0.5–0.8 \( \mu \)m bandpass.

### Table 1: Sampled Eighth-Order Mask Parameters

| Mask \(^a\) | \( N \)^b | \( \theta_0(\lambda_{\max}/D) \) | \( \epsilon \) | \( \zeta_0 \) | \( \tilde{M}_0 \) | \( \tilde{M}_0 \) | \( C \) | \( \zeta_0 \) | OD\(_{\max}\)^c |
|---|---|---|---|---|---|---|---|---|---|
| \( n = 1 \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) | \( l = 2 \) | \( l = 3 \) | \( l = 4 \) |
| 0.966115405054 | 0.960497945651 | 0.959860814806 | 0.999927046667 | 0.999967637078 | 0.999991487843 | 0.994355716928 | 0.992296789001 | 0.99920502046 | 0.999959092620 | 1.000006649135 | 1.856758172445 | 1.856208735161 | 1.434216871605 | 1.429552473250 | 1.427349701514 | 1.312506672966 | 1.308947970039 | 1.306220598720 |
| 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 |
| 0.453 | 0.340 | 0.272 | 0.487 | 0.366 | 0.292 | 0.533 | 0.400 | 0.320 | 0.578 | 0.434 | 0.347 | 0.557 | 0.412 | 0.334 | 0.596 | 0.447 | 0.357 | 0.637 | 0.478 | 0.382 |
| 0.01092315 | 0.0016692719 | 0.00395309 | 0.00632078 | 0.00357716 | 0.00227903 | 0.00505061 | 0.00284944 | 0.00182510 | 0.00445606 | 0.00251629 | 0.00169075 | 0.00825681 | 0.00459272 | 0.00289028 | 0.00630889 | 0.00035664 | 0.00106373 | 0.00227076 | 0.00540801 | 0.00305104 |
| 0.01239960 | 0.00923329 | 0.00592117 | 0.0183265 | 0.01068997 | 0.00682024 | 0.0229541279 | 0.01275232 | 0.00818418 | 0.02664059 | 0.01499184 | 0.00961517 | 0.01646433 | 0.00904466 | 0.00955415 | 0.01828618 | 0.00636948 | 0.00673939 | 0.001738457 | 0.01499184 | 0.01067322 |
| 0.25945500 | 0.25945657 | 0.25958912 | 0.25954428 | 0.25953913 | 0.25958482 | 0.25941279 | 0.25952294 | 0.25957404 | 0.25937869 | 0.25950368 | 0.25956186 | 0.25942680 | 0.25953456 | 0.25957920 | 0.25941279 | 0.25948279 | 0.25937869 | 0.25950368 | 0.25956186 |
| 8.804 | 9.012 | 7.971 | 7.969 | 8.969 | 9.757 | 8.760 | 8.868 | 9.649 | 7.744 | 8.751 | 9.534 | 7.923 | 8.977 | 9.710 | 7.882 | 8.890 | 9.674 | 8.909 | 9.603 |

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\(^a\) Values of \( n \) for masks based on eq. (12). Values of \( l \) are for masks based on eq. (13) with \( m = 1 \).

\(^b\) Normalization constant for \( \zeta = \zeta_0 \) and \( \lambda_{\min} \) sampling.

\(^c\) For a graded mask with \( \zeta = \zeta_0 \).

\(^d\) Suggested for TPF-C.
construction errors probably still limit the broadband performance of these masks. We suggest that sampled graded masks may be easier to construct than smooth graded masks. Graded masks produce large phase errors, but it may be possible to correct the phase of these sampled masks using transparent strips of varying thickness. In addition, as Kuchner & Spergel (2003a) pointed out, sampled masks can be designed so that unlike smooth masks, they do not require their darkest regions to be perfectly opaque. This flexibility limits the demands on the lithography tool used to make the masks. The $1 - \text{sinc}^2$ mask with $\theta_{\text{FW}} = 2.9 \lambda_{\text{min}} / D$ and $\epsilon = 0.4$ can be built with a maximum optical density of 4. The $\text{sinc}^2$ mask with $\epsilon = 0.4$ can be built with a maximum optical density of 3.

When we design eighth-order graded notch filter masks, we can reduce the required maximum optical density by beginning the sampling at $\zeta = 0$, so long as the spacing between the samples is large enough to straddle the valleys shown in Figure 4. Choosing $\Delta x = \lambda_{\text{min}} f$ satisfies this condition for all the masks listed in Table 1. Figure 6 shows a graded version of the $m = 1$, $l = 3$ eighth-order mask described in § 4.1. The mask is defined by $\hat{M}(x, y) = \hat{M}_{\text{notch}}(x)$; its optical density is $-2 \log_{10} |\hat{M}_{\text{notch}}(x)|^2$. To make the darkest stripe of the mask as transparent as possible, we chose $\zeta = 0$. With this choice, the darkest stripe of the mask has optical density $-2 \log_{10} |\hat{M}_{\text{notch}}(0)| \approx 7.882$. Table 1 lists the maximum optical densities ($\text{OD}_{\text{max}}$) of sampled graded masks with $\zeta = 0$.

5. SUMMARY

We offer a series of eighth-order masks that are relatively insensitive to tip-tilt errors and other low spatial frequency aberrations; in a coronagraph using one of these masks, the rms pointing error only needs to be managed to a few milliarcseconds, no better than the pointing accuracy of the Hubble Space Telescope. Eighth-order notch filter masks retain most benefits of using fourth-order masks—broadband capabilities, reasonably high throughput, and small inner working angle—permitting extremely high dynamic range coronagraphy suitable for terrestrial planet finding using a popular optical layout.

In particular, we suggest a binary mask designed for TPF-C at 0.5–0.8 $\mu$m composed of opaque strips whose shapes are
described by equation (24) with \( m = 1, l = 3, \epsilon = 0.596, N = 1.434216871605, M_0 = 0.00630889, M_0^* = 0.01882618, C = -0.33935486, \) and \( \zeta_0 = 0.25941279. \) This mask provides 40% Lyot stop throughput and requires an f/115 or slower beam, assuming the mask can be manufactured with an rms accuracy of 20 nm. The rms pointing required for achieving starlight suppression of \( 10^{-10} \) with this mask in the search area is \( \sigma_{\Delta \alpha} \approx 4.2 \) mas for stars of diameter up to \( \sim 2.4 \) mas. If the mask is used on a telescope with better pointing accuracy, it can achieve contrast levels of \( 10^{-10} \) on targets with even larger diameters.

We also provide a graded version of this design, whose amplitude transmissivity is described by equation (24) using the above parameters but with \( \zeta = 0. \) This mask offers the same performance as the above binary version, but it allows easier e-beam fabrication because it only requires optical densities \( \leq 7.882. \) Other eighth-order masks can provide less ringing at the cost of inner working angle or Lyot stop throughput.

We thank Stuart Shaklan and Joseph Green for helpful conversations and for delaying the publication of their paper on low-order aberrations in coronagraphs with eighth-order masks until this paper was ready. M. J. K. acknowledges the support of the Hubble Fellowship Program of the Space Telescope Science Institute. J. C. and J. G. acknowledge support by NASA with grants NAG5-12115 and NAG5-11427, NSF with grants AST 01-38235 and AST 02-43090, the UCF-UF Space Research Initiative program, and the JPL TPF program.

APPENDIX

We prove, for a monochromatic coronagraph with a notch filter mask, a binary entrance aperture of finite size, and a Lyot stop that is perfectly opaque everywhere the entrance aperture is opaque, that (1) the PSF shape is the absolute square of the Fourier transform of the Lyot stop amplitude transmissivity independent of the position of the source on the sky, and (2) the PSF is attenuated by the intensity transmissivity of the band-limited part of the mask evaluated at the source position. Kuchner & Traub (2002) demonstrated this principle for a sin\(^2\) mask; this more general proof applies to any two-dimensional notch filter mask.

As usual, we examine a coronagraph comprising an entrance aperture, \( A, \) an image mask, \( M, \) and a Lyot stop, \( L, \) each of which is represented by a complex-valued function. We use the notational conventions of Kuchner & Traub (2002) and Kuchner & Spergel (2003a): letters with hats represent image-plane quantities. The image-plane coordinates are \( x = (x, y), \) and the pupil-plane coordinates are \( \mathbf{u} = (u, v). \)

Monochromatic light propagates through the coronagraph as follows:

1. An incoming wave incident on the entrance aperture creates a field with amplitude \( E(u). \) When an incoming wave interacts with a stop or mask, the function representing the mask multiplies the wave’s complex amplitude. So after the wave interacts with the entrance aperture, the amplitude becomes \( A(u) \times E(u). \)
2. After the entrance aperture, the beam propagates to an image plane, where the new field amplitude is the Fourier transform of the pupil plane field amplitude, \( A(x) = \mathcal{F}(x); \) denotes convolution. In this plane, the beam interacts with the image mask, and the field amplitude becomes \( M(x) = [A(x) \times E(x)]. \)
3. Next the beam propagates to a second pupil plane, where the field amplitude is \( M(u) = [A(u) \times E(u)]. \) In this second pupil plane, the wave interacts with a Lyot stop, changing the field amplitude to \( F(u) = L(u) \times (M(u) = [A(u) \times E(u)]). \)
4. At last, the beam propagates to the final image plane, where the final image field is \( \hat{F}(x) \), the Fourier transform of \( F(u). \) For a point source, the intensity of the final image is proportional to the absolute value of this quantity squared.

The final image field, \( \hat{F}(x), \) and its Fourier transform are linear functions of \( A(u), L(u), \) and also \( M(u). \) This last property allows us to study masks by decomposing them into Fourier components, computing \( F(u) \) or \( \hat{F}(x) \) for each one, and then summing the final field amplitudes back together.

Consider a point source providing a field \( \hat{E}(x) = \delta(x - x_1) \) in the plane of the sky and a harmonic mask function \( M(u) = \delta(u - u_1). \) The field after the entrance pupil is \( A(u) \times \exp(-2\pi i u \cdot x), \) and the field in the first image plane is \( A(x - x_1). \) The field after the image mask is \( \exp(2\pi i u_1 \cdot x)A(x - x_1). \) The field in the second pupil plane is \( A(u - u_1) \times \exp[-2\pi i (u - u_1) \cdot x]. \) The field after the Lyot stop is

\[
F(u) = L(u)A(u - u_1)e^{-2\pi i (u - u_1) \cdot x_1} \quad \text{for a harmonic mask.} \tag{A1}
\]

Let \( A \) be binary (everywhere equal to 1 or 0), and let \( A \) represent the support of \( A \) and \( \mathcal{L} \) represent the support of \( L. \) If \( \mathcal{L} \subseteq \mathcal{A}, \) then there is some set \( \mathcal{P} \subseteq \mathbb{R}^2 \) for which \( L(u)A(u - u_1) = L(u) \) for \( u_1 \in \mathcal{P}. \) If \( \mathcal{A} \) is finite in extent, then there is also some set \( \mathcal{Q} \subseteq \mathbb{R}^2 \) for which \( L(u)A(u - u_1) = 0 \) for \( u_1 \in \mathcal{Q}. \)

Under these circumstances, there are three kinds of harmonic image masks:

- \( u_1 \in \mathcal{P}. \) —For these harmonic masks, the field after the Lyot stop is uniform in amplitude with a phase gradient \( x_1. \)
- \( u_1 \in \mathcal{Q}. \) —For these masks, the field inside the Lyot stop is zero.
- \( u_1 \notin (\mathcal{P} \cup \mathcal{Q}). \) —For these masks, the field after the Lyot stop does not have uniform amplitude.

These three kinds of harmonic masks correspond to the three kinds of virtual pupils illustrated in Figure 6 of Kuchner & Traub (2002).

A band-limited mask is defined to be a continuous sum of harmonic masks of the first variety:

\[
\hat{M}(x) = \int_{\mathcal{P}} M(u_1) e^{2\pi i u_1 \cdot x} du_1, \tag{A2}
\]
A notch filter mask is defined to be a continuous sum of harmonic masks of the first and second varieties:

\[ \hat{M}(x) = \int_{u_1 \in (\mathbb{R}^2 \cap Q)} M(u_1)e^{2\pi i u_1 \cdot x} \, du_1. \]  

Combining this expansion and equation (A1) using the linear property of \( F(u) \) described above, we find that in a coronagraph with a notch filter mask, the field amplitude after the Lyot stop is

\[
F(u) = \int_{u_1 \in (\mathbb{R}^2 \cap Q)} M(u_1)L(u)A(u - u_1)e^{-2\pi i (u - u_1) \cdot x_1} \, du_1
\]

\[
= \int_{u_1 \in \mathbb{P}} M(u_1)L(u)e^{-2\pi i (u - u_1) \cdot x_1} \, du_1
\]

\[
= L(u)e^{-2\pi i u \cdot x_1} \int_{u_1 \in \mathbb{P}} M(u_1)e^{2\pi i u_1 \cdot x_1} \, du_1.
\]

To interpret this equation, let us define the band-limited part of \( \hat{M}(x) \) as

\[
\hat{M}_{BL}(x) = \int_{u_1 \in \mathbb{P}} M(u_1)e^{2\pi i u_1 \cdot x} \, du_1.
\]

Now we can write

\[
F(u) = \hat{M}_{BL}(x_1)L(u)e^{-2\pi i u \cdot x_1} \, du_1 \quad \text{for a notch filter mask.}
\]

The final image field is the Fourier transform of this quantity, \( \hat{F}(x) = \hat{M}_{BL}(x_1)\hat{L}(x - x_1) \), and the final image intensity is the absolute square of the Fourier transform of this quantity,

\[
|\hat{F}(x)|^2 = |\hat{M}_{BL}(x_1)|^2 |\hat{L}(x - x_1)|^2 \quad \text{for a notch filter mask.}
\]

In other words, for a notch filter mask, the PSF shape is \(|\hat{L}(x)|^2\), independent of \(x_1\), the position of the source on the sky. The PSF is attenuated by a factor \(|\hat{M}_{BL}(x_1)|^2\), the amplitude transmissivity of the band-limited part of the mask evaluated at the source position.

The band-limited part of a notch filter mask can generally be found by applying a low-pass filter to the mask function.

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