LONG-RUN ANALYSIS OF THE STOCHASTIC REPLICATOR DYNAMICS IN THE PRESENCE OF RANDOM JUMPS

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Abstract. The effect of anomalous events on the replicator dynamics with aggregate shocks is considered. The anomalous events are described by a Poisson integral, where this stochastic forcing term is added to the fitness of each agent. Contrary to previous models, this noise is assumed to be correlated across the population. A formula to calculate a closed form solution of the long run behavior of a two strategy game will be derived. To assist with the analysis of a two strategy game, the stochastic Lyapunov method will be applied. For a population with a general number of strategies, the time averages of the dynamics will be shown to converge to the Nash equilibria of a relevant modified game. In the context of the modified game, the almost sure extinction of a dominated pure strategy will be derived. As the dynamic is quite complex, with respect to the original game a pure strict Nash equilibrium and an interior evolutionary stable strategy will be considered. Respectively, conditions for stochastically stability and the positive recurrent property will be derived. This work extends previous results on the replicator dynamics with aggregate shocks.

1. Introduction. Brownian forcing simulates everyday noise very well, however, it only assumes randomness among interactions subject to aggregate shocks, such as the “weather”. One must account for the impacts of random events that come about suddenly and make an immediate impact. There are many examples of these events, which includes earthquakes, tsunamis, volcanic explosions, floods, an increase in the level of toxicity in the environment, etc. We call these type of events anomalies. These are one-time stochastic events that have an immediate effect on the fitnesses landscape. For a bacterial population, antibiotics are considered a rare event [3]. Over fishing may also be considered as an anomaly [7]. The result of many of these anomalies is bottlenecking or gene deletion, however, not all anomalies have catastrophic impacts [8]. Well known examples of bottlenecking through catastrophic events are the northern elephant seal and the cheetah [24]. Furthermore, there are examples of a sudden increase in population, such as migration or an increase in nutrients. Runoff from farmland enters the Gulf of Mexico via the Mississippi Delta, and this sudden increase in nutrients creates an increase algae. This algal population growth is so tremendous it depletes oxygen and creates dead
zones and alters the food-chain, which fittingly may be seen as an instant negative impact to the other sea life [28].

Foster and Young [36] appear to be the first to use a stochastic differential equation to describe a random forcing of the replicator dynamics, which is represented by injecting a Brownian term directly into the system. Using Foster and Young’s idea, Fudenberg and Harris [12] derived the replicator dynamics with aggregate shocks by introducing an independent Itô integral into the fitness of each subpopulation then normalize each subpopulation. To display the validity of their model, the authors show their process converges to the replicator dynamics when the variances of the noise converge to zero. The authors characterize the long-run behavior of a two strategy game, giving stochastic analogous of the two-strategy games.

Cabrales [9] expanded upon the replicator dynamics with aggregate shocks by adding deterministic mutations from both an economic and biological perspective. The author analyzed the stable points of the dynamic after taking the mutation rate to zero, displaying the robustness of Nash equilibria. Cabrales then showed that strictly dominated strategies, under certain mutation rates, become extinct. Moreover, for the two strategy game and a different fitness than the linear fitness typical of replicator dynamics, the author showed that the mass of the invariant measure is centered at a different point than the evolutionarily stable strategy with strictly positive entries of the deterministic replicator dynamics. A strategy with strictly positive entries is called interior.

Imhof [18] analyzed Fudenberg and Harris’ model with a general finite number of strategies and determined conditions for recurrence in a neighborhood of an interior evolutionarily stable strategy, stability of pure strategies that are strict Nash equilibria, and the extinction of dominated pure strategies. Also considering the replicator dynamics with aggregate shocks, Hofbauer and Imhof [16] took their intuition from the drift of the closed form solution of the exponential growth for each strategy, and using time averages, showed similar characteristics with different conditions. The authors cast their results in terms of what they coined the “modified game”, using the payoffs of this game as a guide for their conclusions. Some of the methods established by Imhof, and Hofbauer and Imhof will be applied to the model derived in Section 3.

Considering the two strategy replicator dynamics with aggregate shocks, Vlasic [35] analyzed the dynamic where the payoffs of the game were subject to a continuous time Markov chain, modeling the impact the environment has on the game. Assuming independence of the Brownian and Markov chain noise, the author was able to derive strategy payoff inequalities similar to [12].

Mertikopoulos and Viossat [25] analyze the imitative derivation of the replicator dynamics with the assumption that a player’s payoff is subject to an exogenous stochastic noise. The authors show that this subtle assumption is equivalent to the stochastic mean dynamic of a pairwise proportional imitation process. With this evolution, the noise term that perturbs the aggregate payoffs enables situations for the noise to have no effect on the game.

Assuming the noise has the structure of the Stratonovich integral, Khasminskii and Potsepun [20] analyzed this version of the stochastic replicator dynamics. Interestingly, the authors determined for the two strategy case that this type of noise has little impact on the dynamics.

Working with $n$ strategies and a more general model than Imhof [18], Benaim, Hofbauer, and Sandholm [5] give conditions for permanence and impermanence of
the stochastic replicator dynamics; a system is permanent if the boundary of the state space is a repeller, and for the stochastic process, the dynamic is recurrent; and impermanent when there exists a subset of the state space where the system converges to the boundary with a high probability.

Sandholm and Staudigl [29] were the first to consider the evolution of a population with respect to large deviation theory to study the rate at which various types of stochastically perturbed learning dynamics converge to stable equilibrium under certain noise conditions. The authors showed in games with multiple equilibria, the convergence time may grow exponentially with the number of strategies, displaying the importance of considering the expected time for the dynamic to escape a basin of attraction.

Of the previously mentioned models, Cabrales [9] is similar but not very close to the one developed in this paper. Particularly, Cabrales adjusts for deterministic mutations and the model analyzed in this paper adjusts for stochastic impacts on fitness. Many of Cabrales’s results are given from the natural premise of letting the mutations rates go to zero, while the results in this paper assume a constant presence and displays that similarity of the integrands dictates the conditions for various stabilities. Lastly, there appears to be a natural extension of the model developed in this paper and the results of Benaïm, Hofbauer, and Sandholm [5] for the conditions of permanence and impermanence. However, this is not discussed.

There are a few authors that have modeled anomalies [4, 14], and only simple models were considered. Hanson and Tuckwell [14] use a Poisson integral to capture the impacts of the effects, and analyze the possibility of extinction. In this paper, effects with zero expectation that impact each subpopulation’s fitness are investigated. To capture this phenomenon, a compensated Poisson integral is added to the Fudenberg and Harris model, so that the expectation of this perturbation is zero and a continuous integrand models the various intensities of each impact of the anomaly. Since the process is no longer continuous, but right-continuous, many of the methods used to determine the behavior of the process are no longer applicable. The techniques applied require meticulous and tedious calculations, which are not necessary if the dynamic is almost surely continuous. This displays the complexity that needs to be considered when modeling the evolution of a populace. Similar to the replicator dynamics with aggregate shocks, the only stationary points are the vertices of the simplex.

Fudenberg and Harris [12] displayed the long-run behavior by first considering an initial condition in a subinterval of $[0, 1]$, and a solving a second-order differential equation that yields a closed form solution for the probability of the process first leaving from the right endpoint or the left endpoint. The subinterval is then extended to $[0, 1]$. Similar to this method, solving an integro-differential equation gives the probability of the process first leaving through the left or right endpoint of the subinterval. Since in general solutions to integro-differential equation are difficult to calculate, for completeness, the stochastic Lyapunov method is applied to determine the stability or instability of the pure strategies. This technique will assist in yielding analogous conditions to the standard classification of a symmetric two-strategy game. The inequalities derived are similar to the inequalities found in Imhof [18].

For a general finite number of strategies, the methods of Imhof [18] and Hofbauer and Imhof [16] will be used as a basis for proving analogous properties. Similar to the results in Imhof, conditions for the dynamic to be positive recurrent, displaying
the majority of the mass of the invariant measure is over a neighborhood of an interior evolutionarily stable strategy, and the local conditions for a pure strict Nash equilibrium to be stochastically stable will be derived. Imhof’s assumptions for the replicator dynamics with aggregate shocks are sufficient, and the assumptions for the integrands of the compensated Poisson measure are given in Section 3. As will be shown, when stochastic anomalies are taken into account, conditions for the transience and positive recurrence are more strenuous than the conditions that just account for aggregate shocks.

Taking the intuition from Hofbauer and Imhof [16], an analogous “modified game” will be derived. With respect to the modified game, the time averages of the dynamic will be shown to converge to Nash equilibria and a pure dominated strategy will be shown to become almost surely extinct. The method to show the almost sure extinction of a dominated pure strategy is an adaption of a result in Imhof [18].

The diversity of the results display the difficulties of analyzing the replicator dynamics with continuous and jump stochastic perturbations, which is a right-continuous stochastic process. In particular, both Imhof [18] and Hofbauer and Imhof [16] utilize the continuity of the replicator dynamics with aggregate shocks when applying the stochastic Lyapunov method, and rely on properties of a stochastic system with just a Gaussian noise.

2. Preliminary definitions. This section defines characteristics a stochastic process may possess. The definitions below are given for a general stochastic process and a Markov process, where the labeling of characteristics are consistent with the stochastic differential equation literature, and not that of the stochastic replicator dynamics literature.

The first characteristic defined is stochastic stability. While the notation of instability is fairly intuitive, due to the variance of a stochastic dynamic, stochastic stability is not almost sure and the definition must reflect this observation. The notion of stability will only be given for the stationary point \( x = 0 \) for a dynamic evolving on the unit interval, however, this definition has a natural extension to vertices on the simplex. Please see Khasminskii [21] for further reference.

Define \( P_{x_0} \) as the probability measure corresponding to a stochastic process \( s(t) \) when \( s(0) = x_0 \). Throughout this paper, unless stated otherwise, the initial condition for the dynamic is assumed to almost surely hold.

Definition 2.1. The stationary point \( x = 0 \) is said to be:

1. stable in probability if, for any \( \epsilon > 0 \),
   \[
   \lim_{x_0 \to 0} P_{x_0} \left( \sup_{t \geq 0} s(t) > \epsilon \right) = 0;
   \]

2. stochastically stable if it is stable in probability and
   \[
   \lim_{x_0 \to 0} P_{x_0} \left( \lim_{t \to \infty} s(t) = 0 \right) = 1.
   \]

For simplicity, the definitions of recurrence and transience are given for a dynamic invariant to the unit interval. The definitions have a natural extension for a process invariant to a general finite dimensional simplex. Please see [21] for the general definitions.
Definition 2.2. 1. For a Markov process \( s(t) \) and initial condition \( x_0 \), \( s(t) \) is said to be regular if for any finite time \( T > 0 \),
\[
P_{x_0} \left( \inf_{0 \leq t \leq T} s(t) = 0 \text{ or } \sup_{0 \leq t \leq T} s(t) = 1 \right) = 0
\]
2. For \( D \subseteq [0,1] \) and \( \tau_D := \inf \left\{ t \geq 0 : s(t) \in D \right\} \), a Markov process \( s(t) \) is called recurrent with respect to \( D \) if it is regular and for any finite time \( T > 0 \),
\[
P_{x_0} \left( \tau_D < \infty \right) = 1,
\]
for any initial condition \( x_0 \in D^c \). (The notation \( D^c \) stands for the complement of the set \( D \).)
3. If \( s(t) \) is not recurrent, it is called transient.
4. The process is called positive recurrent with respect to the set \( D \) if it is recurrent and \( D \subseteq (\omega_0, \omega_1) \), where \( 0 < \omega_0 < \omega_1 < 1 \).

This paper utilizes the stochastic Lyapunov method to derive the stochastic stability or instability of a vertex. Similar to the deterministic Lyapunov technique, the stochastic Lyapunov method transforms the stochastic process into a positive valued supermartingale (defined below), which implies the process is decreasing.

Definition 2.3. For \( 0 \leq s \leq t \) and a stochastic process \( X \), define \( E[X(t) | \sigma\{X(s_0) : s_0 \leq s\}] \) as the conditional expectation of the process at time \( t \) dependent on the history of the process up to time \( s \), (where \( \sigma\{X(s_0) : s_0 \leq s\} \) is the \( \sigma \)-algebra of the process up to time \( s \)). We call \( X \) a martingale if \( E[X(t) | \sigma\{X(s_0) : s_0 \leq s\}] = X(s) \), a submartingale if \( E[X(t) | \sigma\{X(s_0) : s_0 \leq s\}] \geq X(s) \), and a supermartingale if \( E[X(t) | \sigma\{X(s_0) : s_0 \leq s\}] \leq X(s) \).

3. Deriving the replicator dynamics with aggregate and instantaneous shocks. Consider a two-player symmetric game, where \( a_{ij} \) is the payoff to a player using strategy \( S_i \) against an opponent employing strategy \( S_j \), and define \( A = (a_{ij}) \) as the payoff matrix. Denote \( \Delta_n = \left\{ y \in \mathbb{R}^n : y_i > 0 \text{ for all } i \text{ and } \sum_i y_i = 1 \right\} \) as the \( n^{th} \)-dimensional simplex, and \( \bar{\Delta}_n \) as the closure. Within a population, assume that every individual is programmed to play a pure strategy \( S_j \). Let \( r_i(t) \) be the size of the subpopulation that plays strategy \( S_i \) at time \( t \), which is denoted as the \( i^{th} \) subpopulation. Furthermore, define \( r(t) := (r_1(t), \ldots, r_n(t))^T \), \( R(t) := \sum_i r_i(t) \), and \( s(t) := (s_1(t), \ldots, s_n(t))^T \) where \( s_i(t) := r_i(t) / R(t) \) (the frequency of the \( i^{th} \) strategy). When an agent in the \( i^{th} \) subpopulation is randomly matched with another agent from a well-mixed population, \( (As(t))_i \) is the average payoff for this individual, which is taken this to be the fitness of the agent. Assuming growth is proportional to fitness gives the ordinary differential equation
\[
\dot{r}_i(t) = r_i(t) (As(t))_i
\]
which yields
\[
\dot{s}_i(t) = s_i(t) \left( (As(t))_i - s(t)^T As(t) \right).
\]
This is the replicator dynamics.
Considering a biological perspective, Fudenberg and Harris [12] derived a continuous time stochastic replicator dynamics by first assuming

\[ dr_i(t) = r_i(t) \left( (As(t))_i dt + \sigma_i dW_i(t) \right), \]

for \( \sigma_i \in \mathbb{R}_+ \) and taking standard Wiener processes \( W_i(t) \) as pairwise independent. Itô's lemma then gives

\[
\begin{align*}
\frac{ds_i(t)}{s_i(t)} &= \sum_{j \neq i} s_i(t)s_j(t) \left[ ((As(t))_i - (As(t))_j) dt + \left( \sigma_i^2 s_j(t) - \sigma_j^2 s_i(t) \right) dt \right. \\
&\quad \left. + \left( \sigma_i dW_i(t) - \sigma_j dW_j(t) \right) \right].
\end{align*}
\]

(1)

For intuition, the model incorporates randomness from aggregate shocks, or population level interactions, that affects the fitness of each type. The only stationary points for this dynamic are the vertices of the simplex.

The authors then take a two strategy population, which simplifies to \( s_2(t) = 1 - s_1(t) \), and obtain the more manageable model

\[
\begin{align*}
\frac{ds_1(t)}{s_1(t)} &= s_1(t) \left[ (1 - s_1(t)) \left( a_{12} - a_{22} + \sigma_2^2 + \left( a_{11} - a_{21} - \sigma_1^2 - a_{22} - a_{12} - \sigma_2^2 \right) s_1(t) \right) dt \\
&\quad + \sigma s_1(t) \left( 1 - s_1(t) \right) dW(t) \right].
\end{align*}
\]

(2)

This dynamic is now a one-dimensional process where there are many methods and theorems to utilize in deriving the long-run behavior.

While the Fudenberg and Harris model captures the effects of everyday noise on the fitness quite well, in practice, the model would incorporate noise from anomalous events, assuming these events would occur on the tails of the distribution. For instance, the effect of superstorms on the inhabitants, the effect that high-level political elections have on the market or the (potentially global) political environment, as well as anomalous events like earthquakes and volcanic activity would be taken as part of the aggregate noise. This simplification has the potential to disregard important information that may have an impact on the long run behavior of the dynamic. Furthermore, when these events occur they may have varying intensities on their impact, which would impact the population accordingly. This intuition is taken into account below, and made technically explicit through the development of the Poisson integral.

In the event an anomaly is correlated across the population, the effect to the \( i^{th}\)-subpopulation has a quantitative representation, which we momentarily call \( h_i \), and assume the anomaly occurs via a Poisson process, say \( N(\cdot) \). In summary, the impact of the event on the \( i^{th}\)-subpopulation at time \( t \) is \( h_i \cdot N(t) \). This Poisson process is the same for each subpopulation from the assumption that the anomaly is correlated across the entire population. The more natural modeling of the impact is a function \( h_i(x) \), where \( h_i(x) \) dictates the impact of the anomaly on the \( i^{th}\)-subpopulation when the impact has “strength” \( x \in \mathbb{R} \).

To incorporate different impact sizes into the dynamic, we briefly discuss the Poisson process as a temporal and spatial random measure, and interpret an integral under this measure. For further information, please see Applebaum [2]. For a Lévy process \( X(s) \) and \( B \in \mathcal{B}(\mathbb{R}\setminus\{0\}) \) define \( \Delta X(s) := X(s) - X(s-) \), \( N(t, B) := \# \left\{ 0 \leq \right\}. \)
If $\nu([\mathbb{R}\setminus\{0\}]) = \infty$ it is possible for an anomaly to occur an infinite number of times in a finite time interval. Hence, the assumption that $\nu([0,1]) < \infty$ is quite natural.

Taking $\chi_{B}(\cdot)$ to be the identity function over the measurable set $B$, the effect on the $i^{th}$-subpopulation over the time interval $[0, t]$ is

$$
\int_{0}^{t} \int_{B} h_{i}(x) \tilde{N}(ds, dx) = \sum_{0 \leq s \leq t} h_{i}(\Delta X(s)) \chi_{B}(\Delta X(s)).
$$

Recalling that $\Delta X(s) = X(s) - X(s^{-})$, one may see the integral encompasses all jumps that have a value in set $B$ over the time period $[0, 1]$. We call $h_{i}(x)$ the \textbf{jump function} of the $i^{th}$ subpopulation. Furthermore, we assume the expected effect of the anomaly is zero. To adjust for this assumption, the deterministic integral $-\int_{0}^{t} \int_{\mathbb{R}} h_{i}(x) \nu(dx) ds$, ($ds$ is the Lebesgue measure), is added to the process so that the expectation of this impact is zero. Specifically, the measure $\tilde{N}(ds, dx) := N(ds, dx) - \nu(dx) ds$ yields $E\left[ \int_{0}^{t} \int_{\mathbb{R}} h_{i}(x) \tilde{N}(ds, dx) \right] = 0$.

Thus, the impact over a period of time $[0, t]$ is

$$
\int_{0}^{t} r_{i}(s^{-}) \int_{\mathbb{R}} h_{i}(x) \tilde{N}(ds, dx),
$$

where $r_{i}(s^{-})$ is the left limit. One may see the expectation is zero. In conclusion, a subpopulation evolves according to the process

$$
dr_{i}(t) = r_{i}(t^{-}) \left( (As(t^{-}))_{i} dt + \sigma_{i} \, dW_{i}(t) + \int_{\mathbb{R}} h_{i}(x) \tilde{N}(dt, dx) \right). \tag{3}
$$

In this dynamic, if $h_{i}(x) = -1$ then the entire subpopulation has become extinct, and if $h_{i}(x) < -1$ then the size of the subpopulation is negative. This intuition is further emphasized by the technical remark below, displaying that it is necessary to assume for every $i$, the minimum of $h_{i}(x)$ is strictly greater than $-1$. Assumption 3.1 solidifies this condition and adds extra conditions on the jump functions to ensure the model is well-defined.

\textbf{Remark 1.} If $\inf_{x \in \mathbb{R}} \{ h_{i}(x) \} > -1$ then $r_{i}(t)$ can be written explicitly as $\exp \{ Y_{i}(t) \}$ where

$$
dY_{i}(t) = \left( (As(t^{-}))_{i} dt - \frac{\sigma_{i}^{2}}{2} + \int_{\mathbb{R}} \left( \log \left[ 1 + h_{i}(x) \right] - h_{i}(x) \right) \nu(dx) \right) dt
$$

$$
+ \sigma_{i} \, dW_{i}(t) + \int_{\mathbb{R}} \log \left[ 1 + h_{i}(x) \right] \tilde{N}(dt, dx),
$$

which follows from Itô’s lemma.

To guarantee existence and uniqueness of the sample paths, the previously given assumption on the measure $\nu(\cdot)$, and the assumptions on the jump functions given below, guarantee that the stochastic differential equation generated by these perturbations is unique (see See Sato [30], Bertoin [6], or Applebaum [2] for further information).

\textbf{Assumption 3.1.} We assume that $\nu(\cdot)$ is a Borel measure and $\nu(\mathbb{R}) < \infty$. Moreover, for all $i$:

a. $h_{i}(x)$ is bounded;

b. $\inf_{x \in \mathbb{R}} \{ h_{i}(x) \} > -1$;

c. and $h_{i}(x)$ is continuously differentiable.
Hofbauer and Imhof [16] motivated their results by considering the exponential growth of the strategies and utilizing the construction of the replicator dynamics with aggregate shocks. Considering this perspective, Remark 1 generates the following definition of the modified game.

**Definition 3.2.** For a symmetric payoff matrix $A$, define the matrix $\tilde{A}$ where $\tilde{a}_{ij} = a_{ij} - \frac{1}{2} \sigma_i^2 + \int_\mathbb{R} \left( \log \left[ 1 + h_i(x) \right] - h_i(x) \right) \nu(dx)$. The matrix $A$ will be labeled as the payoff matrix of the *original game*, and $\tilde{A}$ as the payoff matrix of the *modified game*.

One may see that $\log \left[ 1 + h_i(x) \right] - h_i(x) \leq 0$ for all $x$. While this negative term may appear to be counterintuitive, especially considering the potential that large positive jumps would have on the dynamic of the population, consider the phenomena of a positive jump closely followed by the negative value of this jump. From the exponential growth, the combined value of these jumps is not zero, but (approximately) $(1 + h(x))(1 - h(x)) = 1 - h^2(x)$.

To assist in developing intuition about the evolution of the population, the rest of this section will explore the two strategy model. Section 4 will derive a method to calculate the closed form solution for the two strategy game. However, since this solution is difficult to calculate, the stochastic Lyapunov method will be applied to yield an evolution of the population analogous to the classical characterization of a two strategy game. In the latter sections, a general finite number of strategies will be examined. For the modified game, the time averages of the dynamic will be shown to converge to a Nash equilibrium, and dominated pure strategies in this modified game will be shown to almost surely become extinct. With respect to the original game, conditions for stochastic stability of a pure strict Nash equilibrium and positive recurrence property generated from the interior evolutionarily stable strategy will be derived. The lack of homogeneity of the results display the hindrance the complexity of the dynamic adds to the analysis.

Applying Itô’s lemma to $s_1(t)$ yields the *replicator dynamics with aggregate and instantaneous shocks*

$$
\begin{align*}
    ds_1(t) &= \left[ s_1(t-)s_2(t-) \left( (As(t-))_1 - (As(t-))_2 + s_2(t-)\sigma_2^2 - s_1(t-)\sigma_1^2 \right) \\
    &\quad + \int_\mathbb{R} \left( \frac{s_1(t-)}{s_1(t-)+s_2(t-)}h_1(x) - \frac{s_1(t-)}{s_1(t-)+s_2(t-)}h_2(x) \right) \nu(dx) \right] dt \\
    &\quad + s_1(t-)s_2(t-) \left( \sigma_1 dW_1(t) - \sigma_2 dW_2(t) \right) \\
    &\quad + \int_\mathbb{R} \left( \frac{s_1(t-)}{s_1(t-)+s_2(t-)}h_1(x) - \frac{s_1(t-)}{s_1(t-)+s_2(t-)}h_2(x) \right) \tilde{N}(dt, dx).
\end{align*}
$$

(4)
Solving for $ds_2(t)$ gives a similar equality. This particular version of Itô’s lemma can be found in Applebaum (Theorem 4.4.7 [2]) or Gihman and Skorohod (Part II Chapter 2 §6 [13]). Similar to the replicator dynamics with aggregate shocks, the only stationary points for this dynamic are the vertices of the simplex, which is the case when the population consists of only one strategy. The following proposition ensures that the model is well-defined. This proposition may be readily seen by considering Remark 1, which tells us that $r_i(t)$ is almost surely positive for all time and all $i$ and implies $s(t)$ is in the simplex for all finite time $t$.

**Proposition 1.** For arbitrary $y \in \Delta_n$ and $n \in \mathbb{N}$, we have $P_y(s(t) \in \Delta_n$, for all $t \geq 0) = 1$.

By Proposition 2.1 we have the equality $s_2(t) = 1 - s_1(t)$. From this observation, we may focus on the dynamic of

$$ds_1(t) = s_1(t) \left(1 - s_1(t)\right) \left[a_{12} - a_{22} + \sigma_2^2ight] + \int_\mathbb{R} \left(\frac{h_1(x) - h_2(x)}{s_1(t)[h_1(x) - h_2(x)] + 1 + h_2(x)} + h_2(x) - h_1(x)\right) \nu(dx)$$

$$+ \left(a_{11} - a_{21} - \sigma_1^2 + a_{22} - a_{12} - \sigma_2^2\right)s_1(t)\left(1 - s_1(t)\right) + \sigma s_1(t)\left(1 - s_1(t)\right) dW(t)$$

$$+ \int_\mathbb{R} \frac{s_1(t)(1 - s_1(t)) \left|h_1(x) - h_2(x)\right|}{s_1(t)[h_1(x) - h_2(x)] + 1 + h_2(x)} \tilde{N}(dt, dx),$$

where $\sigma \coloneqq \sqrt{\sigma_1^2 + \sigma_2^2}$ and $W(t) \coloneqq \frac{\sigma_1 W_1(t) - \sigma_2 W_2(t)}{\sigma}$.

From Equation (5), if $h_1(x) = h_2(x)$ for every $x \in \mathbb{R}$ then the jump terms disappear, leaving the replicator dynamics with aggregate shocks. While this technical note may seem counterintuitive, it is well-known for $\hat{A} := \begin{pmatrix} a_{11} + c & a_{12} \\ a_{21} + c & a_{22} + c \end{pmatrix}$, or

$\hat{A} := \begin{pmatrix} a_{11} & a_{12} + c \\ a_{21} & a_{22} + c \end{pmatrix}$, the equality $(\hat{A}p)_i - p \cdot \hat{A}p = (Ap)_i - p \cdot Ap$ holds for $i = 1, 2$ [17]. Furthermore, in application if an anomaly were to affect the environment in such a way that the subpopulations are equally impacted, an equal change in each respective fitness is to be expected, and there would be no change in the frequencies. If these two functions are not equal then different jump sizes affect the dynamic differently. This observation will be made more explicit in Section 4.

**4. The long run behavior for a two strategy game.** In this section we derive a way to calculate the closed form solution of the long run behavior of a two strategy game. For the general model this calculation is a daunting task, and for completeness, the stochastic Lyapunov method will be applied to extract characteristics about the long run behavior of the dynamic.

Using the equality $s_2(t) = 1 - s_1(t)$, the analysis is simplified by only considering the dynamic of $s_1(t)$ with respect to Equation (5). To assist with the analysis, define

$$\alpha(y) := y(1 - y) \left[a_{12} - a_{22} + \sigma_2^2 + \int_\mathbb{R} (h_2(x) - h_1(x)) \nu(dx)\right]$$
\begin{align*}
\beta(y) & := \frac{1}{2} \sigma^2 y^2 (1 - y)^2, \\
\gamma(y, x) & := y + \frac{y(1 - y) [h_1(x) - h_2(x)]}{y[h_1(x) - h_2(x)] + 1 + h_2(x)} = \frac{y[1 + h_1(x)]}{y[h_1(x) - h_2(x)] + 1 + h_2(x)}.
\end{align*}

Taking \(L\) as the infinitesimal generator of \(s_1(t)\) and an appropriate function \(f\), we have
\[Lf(\cdot) = \alpha(\cdot)f'(\cdot) + \beta(\cdot)f''(\cdot) + \int_{\mathbb{R}} \left( f(\gamma(\cdot, x)) - f(\cdot) \right) \nu(dx),\]
(Theorem 2 Part II Chapter 2 §9 [13]). Using this infinitesimal generator, one may derive the long-run behavior and display how the noise would affect the payoffs of the original game. Take \(0 < y_1 < y_2 < 1\), define \(\tau_{y_1y_2}(y_0) = \inf_{t \geq 0} \{ s_1(t) \notin (y_1, y_2) \} \) \(s_1(0) = y_0\), \(\pi_{y_2y_1}(y_0) = P(s_1(\tau_{y_1y_2}(y_0)) = y_2)\), and \(\pi_{y_1y_2}(y_0) = P(s_1(\tau_{y_1y_2}(y_0)) \leq y_1)\). Considering an integro-differential equation of the form
\[Lu(y) = \alpha(y)u'(y) + \beta(y)u''(y) + \int_{\mathbb{R}} \left[ u(\gamma(y, x)) - u(y) \right] \nu(dx) = 0 \text{ for } y \in (y_1, y_2), \tag{6}\]
with the conditions \(u(y) = 0\) for \(y \in [0, y_1]\), and \(u(y) = 1\) for \(y \in [y_2, 1]\). The papers of Henry Tuckwell [34] and Mario Abundo [1] tell us that solving this integro-differential equation gives \(\pi_{y_2y_1}(y_0)\), (interchanging the initial conditions will give \(\pi_{y_1y_2}(y_0)\)). To apply these theorems, all moments of \(\tau_{y_1y_2}(y_0)\) have to exist, i.e., \(E[\tau_{y_1y_2}^{k}(y_0)] < \infty\) for every \(k \in \mathbb{Z}_+\). This condition is satisfied by Theorem 5.1 and Remark 2. This differential equation is similar to the one Fudenberg and Harris solved to derive their results. Given the result in Theorem 5.2, one would expect inequalities in the form of the modified game in Definition 3.2.

Solving this integro-differential equation is a very difficult task, and to display the long-run behavior of this process, we will derive local conditions for the dynamic to be either stochastically stable or unstable in probability near a pure strategy. To accomplish this goal, we follow the method derived in [35] (and the references there in) where the author applied Lyapunov functions to extract characteristics for the dynamic to be stochastically stable or unstable in probability at the stationary points \(x = 0\) and \(x = 1\).

Define the stochastic Lyapunov functions \(V^0_0(y) = y\), and \(V^\infty_0(y) = y^{-1}\), \(V^0_1(y) = 1 - y\), and \(V^\infty_1(y) = (1 - y)^{-1}\). Applying the operator to each of the Lyapunov functions and simplifying yields the equalities:

I. \(LV^0_0(y) = \left[ \alpha(y) + \int_{\mathbb{R}} \frac{h_1(x) - h_2(x)}{y[h_1(x) - h_2(x)] + 1 + h_2(x)} \nu(dx) \right] (1 - y) \cdot y; \)

II. \(LV^\infty_0(y) = \left[ -\alpha(y) + \sigma^2 (1 - y) + \int_{\mathbb{R}} \frac{h_2(x) - h_1(x)}{1 + h_2(x)} \nu(dx) \right] (1 - y) \cdot y^{-1}; \)

III. \(LV^0_1(y) = \left[ -\alpha(y) + \int_{\mathbb{R}} \frac{h_2(x) - h_1(x)}{y[h_1(x) - h_2(x)] + 1 + h_2(x)} \nu(dx) \right] y \cdot (1 - y); \)
IV. and $LV_i^\infty(y) = \left[ \alpha(y) + \sigma^2 y + \int_\mathbb{R} \left( \frac{h_1(x) - h_2(x)}{1 + h_1(x)} \right) \nu(dx) \right] y \cdot (1 - y)^{-1}$.

Note that the sign of each of the functions $LV_i$ is dictated by the term in the square brackets. Since each term in the square brackets is continuous, plugging in the values $y = 0$ or $y = 1$ appropriately and determining conditions for the terms at these points to be strictly negative will give an interval where the function $LV_i$ is strictly negative. This logic leads us to the following assumptions.

**Assumption 4.1.** For process defined by Equation (5) and the points $x = 0$ and $x = 1$, respectively, we take either:

(i.) $\sigma_2^2 + \int_\mathbb{R} \left( h_2(x) - h_1(x) + \frac{h_1(x) - h_2(x)}{1 + h_2(x)} \right) \nu(dx) + a_{12} < a_{22}$

(ii.) or $a_{22} < a_{12} - \sigma_2^2 + \int_\mathbb{R} \left( h_2(x) - h_1(x) + \frac{h_1(x) - h_2(x)}{1 + h_2(x)} \right) \nu(dx)$;

(iii.) and $\sigma_1^2 + \int_\mathbb{R} \left( h_1(x) - h_2(x) + \frac{h_2(x) - h_1(x)}{1 + h_1(x)} \right) \nu(dx) + a_{21} < a_{11}$

(iv.) or $a_{11} < a_{12} - \sigma_2^2 + \int_\mathbb{R} \left( h_1(x) - h_2(x) + \frac{h_2(x) - h_1(x)}{1 + h_1(x)} \right) \nu(dx)$.

As will be seen below, the combination of these inequalities will give conditions for the long-run behavior of the dynamic. While the inequalities given are similar to that of Imhof [18], interestingly, the integral term stays the same for affecting the stability or instability of the pure strategies. The integrand does not purely consist of the difference of the jump functions, but also the negative difference of the jump functions weighted by one plus the size of the jump. This displays the importance of the size of the impact on a subpopulation, as well as the correlation of the perturbation across the population.

Before the combinations of the inequalities can be utilized, we must verify that each inequality gives us a characteristic for stability or instability of a pure strategy. The theorem below verifies these characteristics, and the argument is very similar to the proofs in Theorems 5.3.1, 5.4.1, and 5.4.2 in [21]. An adjustment is needed for the stopping times since the replicator dynamics with aggregate and instantaneous shocks is only right-continuous. For brevity, the proof is omitted.

**Theorem 4.2.** For the dynamic defined by Equation (5):

(i) if Assumption 4.1(i) holds, then $x = 0$ is stochastically stable;

(ii) if Assumption 4.1(ii) holds, then $x = 0$ is unstable in probability;

(iii) if Assumption 4.1(iii) holds, then $x = 1$ is stochastically stable;

(iv) if Assumption 4.1(iv) holds, then $x = 1$ is unstable in probability.

The following propositions will be proved by utilizing the characteristic of the dynamic near each stationary point to infer the global long-run behavior, displaying intuition about the game. Each proposition is a stochastic analog of the classic two-strategy games.

**Proposition 2.** For the dynamic defined by Equation (5), if Assumptions 4.1(i) and 4.1(iv) hold, then for any initial condition, $s_1(t)$ converges to 0 almost surely.

**Proof.** By Theorem 4.2(iv), there exists a $\delta > 0$ such that, for $x \in (1 - \delta, 1)$ and an $\alpha > 0$, $LV_i^\infty(y) \leq -\alpha$. This tells us that $E_x[\tau_{[0, 1-\delta]}] < \infty$. Also, by Theorem 4.2(i), there exists an $\epsilon > 0$ such that for $x \in (0, \epsilon)$ we have $P_x \left( \lim_{t \to \infty} s_1(t) = 0 \right) > 0$. 

Note that the sign of each of the functions $LV_i$ is dictated by the term in the square brackets. Since each term in the square brackets is continuous, plugging in the values $y = 0$ or $y = 1$ appropriately and determining conditions for the terms at these points to be strictly negative will give an interval where the function $LV_i$ is strictly negative. This logic leads us to the following assumptions.
We now show that the process is transient, which implies that \( s_1(t) \) converges to 0 almost surely. To accomplish this, Theorem 3 in Myen and Tweedie [27] is applied. Take a constant \( \epsilon_0 \), where \( 0 < \epsilon_0 < \epsilon \), and for the Lebesgue measure \( \lambda \) and \( A \in \mathfrak{B}(0,1) \), define \( \Phi(A) = \lambda(A \cap (\epsilon_0,1 - \delta)) \). One might see that the measure \( \Phi \) is an irreducible measure (see [10, 26, 27] for further information). For \( x \in (\epsilon_0,\epsilon) \), notice that \( P_x(\tau_{\epsilon,1} = \infty) > 0 \). Therefore, the assumptions of Theorem 3 hold, and hence the dynamic is transient. \( \square \)

The proof of the proposition applied the characteristics of instability at \( x = 1 \) and stochastic stability at \( x = 0 \) to show almost sure convergence to the point \( x = 0 \). For the case when \( x = 0 \) is unstable and \( x = 1 \) is stochastically stable, the argument to show that the dynamics converges to \( x = 1 \) is identical. This logic gives the following corollary.

**Corollary 1.** If Assumptions 4.1(ii) and 4.1(iii) hold, then for any initial condition, \( s_1(t) \) converges to 1 almost surely.

The next proposition shows when both stationary points are stochastically stable, the process is transient and will converge to either endpoint almost surely, displaying the property of the coordination game. Due to the complexity of the dynamic the method is unable to derive the exact probability for the process to converge to \( x = 0 \) or \( x = 1 \).

**Proposition 3.** For the dynamic defined by Equation (5), if Assumptions 4.1(i) and 4.1(iii) hold, then \( s_1(t) \) is transient and converges to 0 or 1 almost surely.

**Proof.** Take an arbitrarily small \( \epsilon > 0 \). By Theorem 4.2(i) and 4.2(iii), there exists a \( \delta > 0 \) such that \( P_x\left( \lim_{t \to \infty} s_1(t) = 0 \right) \geq 1 - \epsilon \) for \( x < \delta \), or \( P_x\left( \lim_{t \to \infty} s_1(t) = 1 \right) \geq 1 - \epsilon \) for \( x > 1 - \delta \). Define \( F = \left\{ \lim_{t \to \infty} s_1(t) = 1 \text{ or } 0 \right\} \), and define \( \hat{\tau}_\delta \) as the first time the process leaves the interval \([\delta,1-\delta]\).

Theorem 5.1 tells us that \( \hat{\tau}_\delta \) has finite mean. Therefore, the strong Markov property yields \( P_x(F) = E_x\left[ E_{s_1(\hat{\tau}_\delta)}[I_F] \right] \geq 1 - \epsilon \). Since \( \epsilon \) was arbitrary, the statement follows. \( \square \)

Finally, when both stationary points are unstable, the process will be positive recurrent in the simplex, displaying the stochastic property of an interior ESS. For an interior ESS, a small amount of noise would consistently favor one strategy over the other, but only for a small period of time. This dynamic would drive the positive recurrent property. For a general number of strategies, Theorem 7.1 displays how an interior ESS is impacted by the noise. The proposition below gives general conditions for this property to occur.

**Proposition 4.** For the dynamic defined by Equation (5), if Assumptions 4.1(ii) and 4.1(iv) hold, then \( s_1(t) \) is positive recurrent.

**Proof.** By Theorem 4.2(iv), there exists a \( \delta > 0 \) and \( \alpha_1 > 0 \) such that for any \( x \in (1 - \delta,1) \) we have \( LV(x,i) \leq -\alpha_1 \). Thus \( E_x[\tau_{(0,1-\delta)}] < \infty \). Moreover, Theorem 4.2(ii) says there exists an \( \epsilon > 0 \) and \( \alpha_0 > 0 \) such that for \( x \in (0,\epsilon) \) we have \( LV(x) \leq -\alpha_0 \). Thus \( E_x[\tau_{(\epsilon,1)}] < \infty \). Therefore, the process hits the set \((\epsilon,1-\delta)\) in finite time. The strong Markov property tells us that the process is positive recurrent in \((\epsilon,1-\delta)\). Theorem 5.1 yields that the process is positive recurrent for any strict subinterval of \((0,1)\). \( \square \)
5. Time averages and local analysis of Nash equilibria in the modified and original game. For a symmetric two player game with a general number of strategies, Imhof [18], and Hofbauer and Imhof [16], exhibited many useful techniques to derive characteristics about the long-run behavior of the replicator dynamics with aggregate shocks. The rest of the paper is devoted to answering questions posed by Imhof, and some questions asked by Hofbauer and Imhof. Although some of Imhof’s methods are applicable, since the replicator dynamics with aggregate and instantaneous shocks is only right-continuous, a nontrivial extension of these techniques, as well as new methods, are necessary to analyze the dynamics.

Following the work of Hofbauer and Imhof [16], Theorems 5.2 and 6.1 show that when the time averages of the dynamics converges, it converges to a Nash equilibria of the modified game, and a dominated pure strategy of the modified game eventually becomes extinct. Theorems 5.1 and 7.1 are considered with equilibria of the original game and the effect of the stochastic perturbations.

To define the n-dimensional model, take \( \{ e_1, e_2, \ldots, e_n \} \) as the standard orthonormal basis of \( \mathbb{R}^n \). For the Euclidean norm denoted by \( | \cdot | \), define \( U_\delta(y') := \{ y \in \Delta_n : |y' - y| < \delta \} \), and for a Borel set \( G \), recall \( \tau_G := \inf \{ t > 0 : s(t) \in G \} \).

For the general n strategy model, \( s_i(t) \) has the form

\[
\begin{align*}
    ds_i(t) &= s_i(t-) \left[ (As(t-))_i - \sum_j s_j(t-)(As(t-))_j + \sum_j s_j(t-)\sigma_j^2 - s_i(t-)\sigma_i^2 \\
    &+ \int_\mathbb{R} \left( \frac{1 + h_i(x)}{1 + \sum_j s_j(t-)h_j(x)} - 1 + \sum_j s_j(t-)h_j(x) - h_i(x) \right) \nu(dx) \right] dt \\
    &+ s_i(t-) \left( \sigma_i dW_i(t) - \sum_j s_j(t-)\sigma_j dW_j(t) \right) \\
    &+ s_i(t-) \int_\mathbb{R} \left( \frac{1 + h_i(x)}{1 + \sum_j s_j(t-)h_j(x)} - 1 \right) \tilde{N}(dt, dx).
\end{align*}
\]

Define \( h(x) = (h_1(x), h_2(x), \ldots, h_n(x))^T \) and \( W(t) = (W_1(t), \ldots, W_n(t))^T \).

With a little work, one may see that

\[
ds(t) = D^1(s(t-))dt + D^2(s(t-))dW(t) + \int_\mathbb{R} D^3(s(t-))\tilde{N}(dt, dx),
\]

where

\[
D^1(y) := \left[ \text{diag}(y_1, \ldots, y_n) - yy^T \right] \left[ A - \text{diag}(\sigma_1^2, \ldots, \sigma_n^2) \right] y \\
+ \int_\mathbb{R} \left(y h(x)^T - \text{diag}(h_1(x), \ldots, h_n(x)) \right) y \nu(dx),
\]

\[
D^2(y) := \left[ \text{diag}(y_1, \ldots, y_n) - yy^T \right] \text{diag}(\sigma_1, \ldots, \sigma_n),
\]

and

\[
D^3(y) := \left( \frac{1}{1 + y^T h(x)} \text{diag}(1 + h_1(x), \ldots, 1 + h_n(x)) - \text{diag}(1, \ldots, 1) \right) y.
\]
Denote $A_J$ as the infinitesimal generator for the dynamics defined by Equation (8). By Theorem 2 (Part II Chapter 2 §6) in Gihman and Skorohod [13], the infinitesimal generator has the form

$$A_J f(y) = \sum_j \tilde{D}_j^1(y) \frac{\partial f}{\partial y_j}(y) + \frac{1}{2} \sum_{j,k} \gamma_{jk}(y) \frac{\partial^2 f}{\partial y_j \partial y_k}(y) + \int_{\mathbb{R}} \left( f(D^3(y) + y) - f(y) \right) \nu(dx),$$

where $D_j$ is the $j$th coordinate of the function $D$, $\gamma_{jk}(y) := \sum_l c_{jl}(y)c_{kl}(y)$ for $c_{jl}(y) := \begin{cases} y_j(1 - y_j)\sigma_j, & j = l \\ -y_jy_l\sigma_l, & j \neq l \end{cases}$, and

$$\tilde{D}_j^1(y) := y_j(e_i - y)y^T \left[ A - \text{diag}(\sigma_1^2, \ldots, \sigma_n^2) \right] y + y_i \int_{\mathbb{R}} \left( \sum_k y_kh_k(x) - h_i(x) \right) \nu(dx).$$

Imhof [18] showed that the replicator dynamics with aggregate shocks will almost surely evolve arbitrarily close to a vertex in the simplex, i.e., a pure strategy $e_k$ for some $k \in \{1, \ldots, n\}$. Since the only stationary points of the dynamics are the vertices of the simplex, this is how one would expect the dynamics to behave. However, the process may only stay near a pure strategy for a small period of time. For instance, given a non-Nash pure strategy, the noise may favor this strategy, causing this strategy to be temporarily dominant. The dynamics will evolve close to the neighborhood of this pure non-Nash strategy, where the process will only stay in this neighborhood for a short period of time. If the strategy is a strict Nash equilibrium, under appropriate noise conditions, there is a high probability that the process will evolve to this stationary point. This behavior is shown below, where the proof is crucial in displaying that all moments of the hitting time are finite, an essential characteristic for analyzing stochastic processes. This property is shown in Remark 2.

The proof of this behavior is an extension of the derivation given by Imhof [18], where the dynamic is composed with a locally positive bounded function to assist in displaying this property. Since the jump functions are bounded, the result is fairly natural.

**Theorem 5.1.** Take $s(t)$ to be an $n$-dimensional replicator dynamics with aggregate shocks defined by Equation (8), an arbitrary payoff matrix $A$, and for $\epsilon > 0$, define $\tau_\epsilon := \inf \left\{ t > 0 : s_k(t) \geq 1 - \epsilon \text{ for some } k \in \{1, \ldots, n\} \right\}$. Then for $y \in \Delta_n$,

$$E_y[\tau_\epsilon] < \infty, \text{ and } P_y \left( \sup_{t > 0} \max_{1 \leq i \leq n} s_i(t) = 1 \right) = 1.$$

**Proof.** To show $E_y[\tau_\epsilon] < \infty$, we apply the infinitesimal generator to a positive function and explicitly derive a negative-constant as an upper bound, then apply Dynkin’s formula. This result is then used to finish the proof.

For $\alpha > 0$ and $y \in \Delta_n$ define the positive function $g(y) = ne^{\alpha} - \sum_k e^{\alpha y_k}$, define the modified payoff matrix $\tilde{A} := A - \text{diag}(\sigma_1^2, \ldots, \sigma_n^2)$, and recall $A_J$ as the infinitesimal generator of the dynamics. Then
\[ A_{jg}(y) = -\alpha \sum_k y_k (e_k - y)^T \hat{A} y e^{\alpha y_k} - \frac{\alpha^2}{2} \sum_k y_k^2 \left( \sigma_k^2 (1 - y_k)^2 + \sum_{j \neq k} \sigma_j^2 y_j \right) e^{\alpha y_k} \]

\[ - \alpha \int_{\mathbb{R}} \sum_k y_k \left( \sum_j y_j h_j(x) - h_k(x) \right) e^{\alpha y_k} \nu(dx) \]

\[ + \int_{\mathbb{R}} \left[ \sum_k \exp(\alpha y_k) - \sum_k \exp \left( \frac{\alpha y_k (1 + h_k(x))}{1 + \sum_j y_j h_j(x)} \right) \right] \nu(dx) \]

\[ := I + II + III, \]

defined respectively. We now derive upper bounds for the terms \( I, II, \) and \( III. \)

For \( \sigma_{\min} := \min\{\sigma_1, \ldots, \sigma_n\} \) and a constant \( \beta > 0 \) such that \( \left| (e_k - y)^T \hat{A} y \right| \leq \beta \) for all \( y \in \Delta_n \), one may see that

\[ I = -\alpha \sum_k y_k (e_k - y)^T \hat{A} y e^{\alpha y_k} - \frac{\alpha^2}{2} \sum_k y_k^2 \left( \sigma_k^2 (1 - y_k)^2 + \sum_{j \neq k} \sigma_j^2 y_j \right) e^{\alpha y_k} \leq \alpha \sum_k y_k e^{\alpha y_k} \beta - \frac{\alpha \sigma_{\min}^2}{2} y_k (1 - y_k)^2. \] \hspace{1cm} (9)

Considering term \( II, \) for \( \kappa_{\max} := \sup_{x \in \mathbb{R}} \max\{h_1(x), \ldots, h_n(x)\}, \) \( \kappa_{\min} := \inf_{x \in \mathbb{R}} \min\{h_1(x), \ldots, h_n(x)\}, \) and \( M := \int_{\mathbb{R}} \left( \kappa_{\max} - \kappa_{\min} \right) \nu(dx), \) we have the inequality

\[ II = \alpha \int_{\mathbb{R}} \sum_k y_k \left( - \sum_j y_j h_j(x) + h_k(x) \right) e^{\alpha y_k} \nu(dx) \leq \alpha \sum_k y_k e^{\alpha y_k} M. \] \hspace{1cm} (10)

Recalling the inequality \(-e^x \leq -1 - x \) for \( x > 0, \) we determine that

\[ III = \int_{\mathbb{R}} \left[ \sum_k \exp(\alpha y_k) - \sum_k \exp \left( \frac{\alpha y_k (1 + h_k(x))}{1 + \sum_j y_j h_j(x)} \right) \right] \nu(dx) \]

\[ \leq \sum_k \int_{\mathbb{R}} \left[ \exp(\alpha y_k) - 1 - \frac{\alpha y_k (1 + h_k(x))}{1 + \sum_j y_j h_j(x)} \right] \nu(dx) \]

\[ \leq \sum_k \alpha y_k \int_{\mathbb{R}} \left[ \sum_{n=1}^{\infty} \frac{(\alpha y_k)^{n-1}}{n!} \right] \nu(dx) \leq \alpha \sum_k y_k \exp(\alpha y_k) \nu(\mathbb{R}). \] \hspace{1cm} (11)

Collecting Equations (9), (10), and (11) yields

\[ A_{jg}(y) \leq \alpha \sum_k y_k e^{\alpha y_k} \left[ \left( \beta + M + \nu(\mathbb{R}) \right) - \frac{\alpha \sigma_{\min}^2}{2} y_k (1 - y_k)^2 \right] \]

To finish deriving an upper bound on \( A_{jg}(y), \) take an arbitrarily small \( \epsilon > 0 \) and \( y \in \Delta_n \) such that \( y_i \leq 1 - \epsilon \) for all \( i. \) For the vector \( y, \) there is at least one \( y_k \) such that \( y_k \geq \frac{1}{n}. \) Now, choose \( \alpha > 0 \) large enough that \( \frac{\alpha \sigma_{\min}^2}{2} y (1 - y)^2 \geq \left( \beta + M + \nu(\mathbb{R}) \right) y_k + 1. \) Plugging the vector \( y \) into \( A_{jg}(\cdot) \) yields

\[ A_{jg}(y) \leq \alpha \left( \beta + M + \nu(\mathbb{R}) \right) \sum_{k: y_k < 1/n} y_k e^{\alpha y_k} \]
Letting $T_n$ denote logic in Theorem 5.1, we conclude that many of the results in this paper require all of the moments of Remark 2. (the corresponding filtration), which shows that $\tilde{T}$ we will construct a stopping time that is almost surely greater than the stopping time property. To show that all moments exist, for a general moment $k$ exists, i.e., for all $n$ we have $A<T_n$ we have

$$\sum_{y \geq 1/n}\alpha \left(-\left(\beta + M + \nu(R)\right) - 1\right)$$

$$\leq \alpha \left(\beta + M + \nu(R)\right)(n - 1)\frac{e^{\alpha/n}}{n} + \alpha \frac{e^{\alpha/n}}{n} \left(-\left(\beta + M + \nu(R)\right) - 1\right)$$

$$= -\alpha \frac{e^{\alpha/n}}{n}.$$

Now by Dynkin’s formula, for every finite $T > 0$,

$$0 \leq \mathbb{E}_y \left[ g(s(\tau \wedge T)) \right] = g(y) + \mathbb{E}_y \left[ \int_0^{\tau \wedge T} A_J g(s(t)) dt \right]$$

$$\leq ne^\alpha - \alpha \frac{e^{\alpha/n}}{n} \mathbb{E}_y \left[ \tau \wedge T \right].$$

Letting $T \to \infty$, the monotone convergence theorem gives the inequality $\mathbb{E}_y[\tau_e] \leq n^2 \frac{e^\alpha}{\alpha}$.

Finally, take $\epsilon = 1/m$ for $m \in \mathbb{N}$. Then $P_y \left( \sup_{t > 0} \max\{s_1(t), \ldots, s_n(t)\} \geq 1 - 1/m \right) = 1$, and therefore

$$1 = P_y \left( \bigcap_{m=1}^{\infty} \left\{ \sup_{t > 0} \max\{s_1(t), \ldots, s_n(t)\} \geq 1 - 1/m \right\} \right)$$

$$= P_y \left( \sup_{t > 0} \max\{s_1(t), \ldots, s_n(t)\} = 1 \right).$$

**Remark 2.** Many of the results in this paper require all of the moments of $\tau_e$ to exists, i.e., for all $k \in \mathbb{N}$, $\mathbb{E}_y[\tau_e^k] < \infty$. Inspecting the proof of Theorem 5.1 displays that the only characteristic of $\tau_e$ that was utilized to show $\mathbb{E}_y[\tau_e] < \infty$ was the stopping time property. To show that all moments exist, for a general moment we will construct a stopping time that is almost surely greater than $\tau_e^k$, and hence $\mathbb{E}_y[\tau_e^k] < \infty$. Define $\tilde{\tau}_e = \max\{1, \tau_e\}$, which one may see is a stopping time. For $k \in \mathbb{N}$ and for $t < 1$ we have $\{\tilde{\tau}_e^k \leq t\} = \emptyset$, and for $t \geq 1$, $\{\tilde{\tau}_e^k \leq t\} = \{\tilde{\tau}_e \leq t + 1\} \in \mathcal{F}_t$ (the corresponding filtration), which shows that $\tilde{\tau}_e^k$ is a stopping time. Using the logic in Theorem 5.1, we conclude that $\mathbb{E}_y[\tilde{\tau}_e^k] < \infty$. Therefore, $\mathbb{E}_y[\tau_e^k] \leq \mathbb{E}_y[\tilde{\tau}_e^k] < \infty$.

Recall a strategy $p \in \Delta_n$ is called a Nash equilibrium if for $q \in \Delta_n$ where $q \neq p$ then $q^T Ap \leq p^T Ap$, and a strict Nash equilibrium if $q^T Ap < p^T Ap$. Hofbauer and Imhof [16] give incredible insight into how the noise will affect the underlying game by first considering the time averages of the dynamics. The result is the derivation of, what the authors called, the “modified game”. For the two strategy game it should be noted that this modified game is exactly the closed form solution derived by Fudenberg and Harris [12]. The authors then use the modified game to direct for their results.

Applying their method to the replicator dynamics with aggregate and instantaneous shocks, the theorem below shows the dynamics will almost surely average to a Nash equilibria of the modified game given in Definition 3.2.
Theorem 5.2. \( (i) \) For almost every \( \omega \in \Omega \) for which \( \lim_{T \to \infty} \frac{1}{T} \int_0^T s(t, \omega)dt \) exist, the limit converges to a Nash equilibrium of the game \( \tilde{A} \).

\( (ii) \) Suppose there exists a sequence of times \( T_k \), where \( T_k \to \infty \) and

\[
\frac{\log(s_i(T_k))}{T_k} \to 0
\]

for all \( i = 1, \ldots, n \). Then for almost every \( \omega \) where

\[
y(\omega) := \lim_{k \to \infty} \frac{1}{T_k(\omega)} \int_0^{T_k(\omega)} s(t, \omega)dt
\]

exists, \( y \) is a Nash equilibrium for the game \( \tilde{A} \).

Proof. The proof will follow the logic given in the analogous theorem in Hofbauer and Imhof [16]. Statement \( (i) \) will be proved, and since the proof of statement \( (ii) \) is similar, the proof will be omitted.

From Remark 1, one may see almost surely that

\[
d\log(r_i(t)) = \left( (As(t-))_i - \frac{\sigma_i^2}{2} + \int_{\mathbb{R}} \log \left[ 1 + h_i(x) \right] \nu(dx) \right) dt
\]

\[+ \sigma_i dW_i(t) + \int_{\mathbb{R}} \log \left[ 1 + h_i(x) \right] \tilde{N}(dt, dx)
\]

(12)

Equation (12) then yields

\[
d\left( \log(s_i(t)) - \log(s_j(t)) \right) = d\left( \log(r_i(t)) - \log(r_j(t)) \right) = d\log(r_i(t)) - d\log(r_j(t))
\]

\[= \left[ (\tilde{A}s(t-))_i - (\tilde{A}s(t-))_j \right] dt + \sigma_i dW_i(t) - \sigma_j dW_j(t) + \int_{\mathbb{R}} \log \left[ \frac{1 + h_i(x)}{1 + h_j(x)} \right] \tilde{N}(dt, dx).
\]

(Although the Poisson terms are correlated, since the jump functions are restricted to their respective subpopulations, the separation of terms hold.) By independence of the Brownian and Poisson noise, and Equation (16) in the proof of Theorem 6.1, the event

\[
\Omega_0 = \left\{ \lim_{t \to \infty} \frac{W_i(t)}{t} = \lim_{t \to \infty} \frac{\int_{\mathbb{R}} \log \left[ 1 + h_i(x) \right] \tilde{N}(dt, dx)}{t} = 0 \text{ for } i = 1, \ldots, n \right\}
\]

has probability 1.

To account for the possibility that the dynamics may converge to more than one Nash equilibrium, the rest of the analysis will consider sample paths of the process. For \( \omega \in \Omega_0 \) such that \( \lim_{T \to \infty} \frac{1}{T} \int_0^T s(t, \omega)dt := y(\omega) \) exists, if \( y_i > 0 \) then there exists an increasing sequence of stopping times, \( \{T_k(\omega)\} \), where \( \lim_{k \to \infty} T_k(\omega) = \infty \) and \( s_i(T_k(\omega), \omega) > y_i/2 \). We may then conclude that \( \lim_{k \to \infty} \log \left( r_i(T_k(\omega), \omega) \right) / T_k(\omega) = 0 \).

Therefore,

\[
\left( \tilde{A}y(\omega) \right)_i - \left( \tilde{A}y(\omega) \right)_j = \lim_{k \to \infty} \frac{1}{T_k(\omega)} \int_0^{T_k(\omega)} \left[ \left( \tilde{A}s(t, \omega) \right)_i - \left( \tilde{A}s(t, \omega) \right)_j \right] dt
\]

\[= \lim_{k \to \infty} \frac{\log \left( s_i(T_k(\omega), \omega) \right) - \log \left( s_j(T_k(\omega), \omega) \right)}{T_k} \geq 0,
\]
Theorem 5.2 gives great insight into the global evolution of the dynamics, but says very little about the local evolution. Particularly, given the process is almost surely in a neighborhood of a pure strict Nash equilibrium, what is the probability that the dynamics will converge to this strategy. If there is more than one pure strict Nash equilibrium, the probability will not be 1. To understand why this convergence is not almost sure, consider when there are at least two pure strict Nash equilibria. There exists a nonzero event where the noise will temporarily weaken this strategy, pushing the dynamics out of this current neighborhood, creating a potential for the dynamics to evolve towards another strict Nash equilibrium.

For this analysis, the impact that compensated random jumps and Gaussian perturbations have on the stability of the replicator dynamics to a pure strict Nash equilibrium of the original game will be examined. The focus on the original game comes from the complexity that the dynamics bring to the characterization of the process. The method follows the technique derived in the analogous theorem in Imhof [18].

For the rest of this section, take the pure strategy \( S_k \) as a strict Nash Equilibrium, i.e., \( a_{kk} > a_{jk} \) for all \( j \neq k \). For the matrix \( \hat{A} := A - \text{diag}(\sigma^2_1, \ldots, \sigma^2_n) \), define \( \beta := \max \{ |a_{ji}| : 1 \leq j, i \leq n \} \). Since each jump term is able to impact the stochastic stability by temporarily strengthening or weakening a strategy, we define the functions \( \psi^k_{\min}(x) := \min_{j \neq k} h_j(x) \) and \( \psi^k_{\max}(x) := \max_{j \neq k} h_j(x) \), and define the integral

\[
I_k = \int_{\mathbb{R}} \left( h_k(x) - \psi^k_{\min}(x) + \frac{\psi^k_{\max}(x) - h_k(x)}{1 + h_k(x)} \right) \nu(dx).
\]

For the two strategy game, note that \( I_1 \) and \( I_2 \) are the equivalent integrals in Assumption 4.1. To derive this property, we compose the dynamics with the function \( g(y) = 1 - y_k \), which enables one to focus on the evolution of this strategy.

**Theorem 5.3.** Take the payoff matrix \( A \) and the process \( s(t) \) defined in Equation (8). Assume that for the pure strategy \( S_k \) and the corresponding variance \( \sigma^2_k \), we have the inequality \( a_{kk} > a_{jk} + \sigma^2_k \) for all \( j \neq k \). Furthermore, for \( \alpha > 0 \), where \( \alpha + a_{jk} < a_{kk} - \sigma^2_k \), assume \( \alpha - I^k > 0 \). Then for any \( \epsilon > 0 \), there exists a neighborhood of \( e_k \), say \( U \subset \Delta_n \), such that for any \( y \in U \),

\[
P_y \left( \lim_{t \to \infty} s(t) = e_k \right) \geq 1 - \epsilon.
\]

**Proof.** Take \( \hat{A} \) defined above, the infinitesimal generator \( A_{J} \), and a stochastic Lyapunov function \( g(y) = 1 - y_k \). The bulk of the proof consists of showing there exists a neighborhood of \( e_k \), say \( U \), such that \( A_{J}g(y) \leq -cy(y) \) for a constant \( c > 0 \) and all \( y \in U \). After this neighborhood is established, invoking results in Gihman and Skorohod [13] finishes the proof.

Applying \( A_{J} \) to \( g(y) \) yields

\[
A_{J}g(y) = -y_k(e_k-y)^T \hat{A} y - y_k \int_{\mathbb{R}} \left( h_k(x) - \frac{\sum_j y_j h_j(x)}{1 + \sum_j y_j h_j(x)} + \sum_j y_j h_j(x) - h_k(x) \right) \nu(dx).
\]

From these two terms, we will factor out the function \( 1 - y_k \) through a series of inequalities.
Thus for \( \tilde{c} \) \( \sum_{i \neq k} \alpha_i q_i U_i \) dominated by \( \tilde{c} \) \( \sum_{i \neq k} \alpha_i y_i + y_k^2 \left( -(1 - y_k) \alpha_{kk} + \sum_{i \neq k} \alpha_{ij} y_j \right) \) finishes the proof. 

6. Dominated strategies of the modified game. A strategy \( q \) is called dominated by \( p \) if for any strategy, the payoff is better when employing \( p \) rather than \( q \). This is written as \( q' A' p' < p' A' p' \) for all \( p' \in \Delta_n \) and a general payoff matrix \( A' \). From an evolutionary perspective, one would expect this strategy to die out
over time. Although Imhof [18] did not state his result with respect to the modified game derived in [16], the theorem may be written in terms of a dominated pure strategy in this modified game.

Taking this observation into consideration, an extension of Imhof’s conditions is given where a dominated pure strategy of the modified game defined in Definition 3.2 will be analyzed. The theorem displays that accounting for anomalies increases the likelihood that a pure strategy in the original game becomes a dominated strategy in the modified game, and hence, overtime becomes extinct.

Since the method employed in Theorem 3.1 [18] is quite natural, an adjustment of this derivation is applied, showing that the frequency of the agents employing the dominated strategy almost surely decreases at an exponential rate.

**Theorem 6.1.** With respect to the modified game $\hat{A}$, take the pure strategy $S_k$ to be dominated by the mixed strategy $p \in \Xi_n$. Then for any $y \in \Delta_n$,

$$P_y \left( \lim_{t \to \infty} s_k(t) = 0 \right) = 1.$$  

**Proof.** For a dominating mixed strategy $p$, define $G(t) = \log \left( s_k(t) \right) - \sum_j p_j \log \left( s_j(t) \right)$. Itô’s lemma yields

$$G(t) = G(0) + \int_0^t e^T_k \tilde{A}s(u)du - t\frac{\sigma_k^2}{2} + t \int_{\mathbb{R}} \left( \log \left( 1 + h_k(x) \right) - h_k(x) \right) \nu(dx)$$

$$- \int_0^t p^T \tilde{A}s(u)du + t \sum_j p_j \frac{\sigma_j^2}{2} - t \sum_j p_j \int_{\mathbb{R}} \left( \log \left( 1 + h_j(x) \right) - h_j(x) \right) \nu(dx)$$

$$+ \sigma_k W_k(t) - \sum_j p_j \sigma_j W_j(t) + \int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{\prod_j \left( 1 + h_j(x) \right)^{p_j}} \right) \tilde{N}(dx, du)$$

$$= G(0) + \int_0^t \left( e^T_k \tilde{A}s(u) - p^T \tilde{A}s(u) \right) du$$

$$+ \sigma_k W_k(t) - \sum_j p_j \sigma_j W_j(t) + \int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{\prod_j \left( 1 + h_j(x) \right)^{p_j}} \right) \tilde{N}(dx, du).$$

(15)

To understand the behavior of the dynamic, the drift, diffusion, and Poisson terms will be examined. For $\hat{\sigma} := \left[ (1 - p_k)^2 \sigma_k^2 + \sum_{j \neq k} p_j^2 \sigma_j^2 \right]^{1/2}$, one may see that $\hat{W}(t) := \left[ \sigma_k W_k(t) - \sum_j p_j \sigma_j W_j(t) \right] / \hat{\sigma}$ is a standard Wiener process.

For the integral $\int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{\prod_j \left( 1 + h_j(x) \right)^{p_j}} \right) \tilde{N}(dx, du)$, notice that the integrand is not dependent on the time variable. Hence

$$\int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{\prod_j \left( 1 + h_j(x) \right)^{p_j}} \right) N(dx, du)$$
is a compound Poisson process. Since \( \nu(\mathbb{R}) < \infty \), Theorem 36.5 in Sato [30] tells us that almost surely

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \int_\mathbb{R} \log \left( \frac{1 + h_k(x)}{\prod_j \left(1 + h_j(x)\right)^p_j} \right) \nu(dx, du) = \int_\mathbb{R} \log \left( \frac{1 + h_k(x)}{\prod_j \left(1 + h_j(x)\right)^p_j} \right) \nu(dx)
\]

Therefore

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \int_\mathbb{R} \log \left( \frac{1 + h_k(x)}{\prod_j \left(1 + h_j(x)\right)^p_j} \right) \tilde{N}(dx, du) = 0 \quad \text{a.s.} \tag{16}
\]

Finally, define the constant \( K = \min_{q \in \Delta_n} \{ p^T \tilde{A}q - e_k^T \tilde{A}q \} \), where the dominated assumption tells us that \( K > 0 \).

Getting a better bound on \( G(t) \), notice \( P_Y \) almost surely

\[
G(t) \leq G(0) - tK + \tilde{W}(t) + \int_0^t \int_\mathbb{R} \log \left( \frac{1 + h_k(x)}{\prod_j \left(1 + h_j(x)\right)^p_j} \right) \tilde{N}(dx, du). \tag{17}
\]

The proof will conclude by showing \( s_k(t) \) almost surely goes to zero at an exponential rate. For \( 0 < \delta < K \), consider the limit

\[
\limsup_{t \to \infty} s_k(t) \exp [t\delta] \leq \limsup_{t \to \infty} \left[ G(t) + t\delta \right]
\]

\[
\leq \limsup_{t \to \infty} \left[ G(0) + t(\delta - K) + \delta \tilde{W}(t) + \int_0^t \int_\mathbb{R} \log \left( \frac{1 + h_k(x)}{\prod_j \left(1 + h_j(x)\right)^p_j} \right) \tilde{N}(dx, du) \right]. \tag{18}
\]

The Law of the Iterated Logarithm tell us that

\[
\limsup_{t \to \infty} \left[ G(0) + \delta \tilde{W}(t) - 3\sigma_{\max} \sqrt{t \log \log t} \right] = 0
\]

almost surely. But, \( 3\sigma_{\max} \sqrt{t \log \log t} < t(K - \delta) \), for large enough \( t \). Therefore, including Equation (16) yields almost surely that

\[
\limsup_{t \to \infty} \left[ G(0) + t(\delta - K) + \delta \tilde{W}(t) + \int_0^t \int_\mathbb{R} \log \left( \frac{1 + h_k(x)}{\prod_j \left(1 + h_j(x)\right)^p_j} \right) \tilde{N}(dx, du) \right] = 0.
\]

Hence, \( P_Y \left( s_k(t) = o(e^{-\delta t}) \right) = 1 \). ☐

7. Conditions for positive recurrence for an interior evolutionarily stable strategy for the original game. A strategy \( p \in \Delta_n \) is called an evolutionarily stable strategy (ESS) if \( q^T Ap \leq p^T Ap \) for all \( q \in \Delta_n \), and for \( q \in \Delta_n \) where \( q \neq p \) and \( q^T Ap = p^T Ap \) then \( q \cdot Aq < p \cdot Aq \). The result in this section gives conditions for the replicator dynamics with aggregate and instantaneous shocks to be positive recurrent, which is a strong recurrence property. Imhof [18] considered an interior evolutionarily stable strategy where the payoff matrix is conditional negative definite, which is defined below. From these assumptions, the author was then able to derive conditions for the process to be positive recurrent, and approximated the
mass of the invariant measure in a neighborhood of the evolutionarily stable strategy in the original game. The mass in this neighborhood dissipates as the intensity of the noise increases. The assumptions for the theorem are similar to Imhof’s, but are more stringent. The complexity that the compensated Poisson term adds to the replicator dynamics with aggregate shocks may considerably decrease the mass of the invariant measure over the neighborhood of the evolutionarily stable strategy. Since the process is right-continuous, a different and more tedious method is needed to show the dynamics are positive recurrent. Imhof’s results are used to help display that this property holds.

To derive how far away the process is from the ESS, Shannon information theory provides the Kullback-Leibler distance, denoted as

$$d(y, p) = \sum_j p_j \log \left( \frac{p_j}{y_j} \right),$$

where \( \log \left( \frac{p_j}{y_j} \right) = 0 \) if \( p_j = 0 \) or \( y_j = 0 \). Quite naturally, the Kullback-Leibler distance function will be used as a Lyapunov function and yields a means to determine how much noise this strategy may survive before being overtaken by the stochastic forcing terms. As previously noted, Theorem 5.2 gives intuition about the intensity of the stochastic terms.

For the assumption of the theorem, we call a matrix \( A \) conditionally negative definite if for \( y \in \mathbb{R}^n \setminus \{0\} \) where \( 1^T y = 0 \), we have \( y^T A y < 0 \). Hence, restricted to the tangent space of the deterministic replicator dynamics. Finally, recall the definitions of \( U_\delta(y) \) and \( \tau_\mathcal{G} \), given in Section 5.

**Lemma (Imhof [18]).** Suppose that \( A \) is an \( n \times n \) (\( n \geq 2 \)) conditionally negative definite matrix, define \( \overline{A} = \frac{1}{2} (A + A^T) \) and let \( \lambda_2 \) be the second largest eigenvalue of

$$D := \overline{A} - \frac{1}{n} A11^T - \frac{1}{n} 11^T \overline{A} + \frac{1}{n} A11^T.$$

Then

$$\max_{x^T 1 = 0, x \neq 0} \frac{x^T D x}{x^T x} = \max_{x^T 1 = 0, x \neq 0} \frac{x^T A x}{x^T x} = \lambda_2 < 0.$$

As Imhof noted, one may consider \( |\lambda_2| \) as the intensity of the attraction to an ESS, and the theorem below yields conditions for the noise not to override this attraction. However, as previously observed, the intensity of the noise may severely weaken this attraction without overriding the recurrence.

**Theorem 7.1.** Take \( s(t) \) defined in Equation (8), \( p \in \Delta_n \) an ESS for our payoff matrix \( A \), \( \lambda_2 \) as the second largest eigenvalue of \( D \), for \( D \) defined in the previous lemma. Finally, define

$$\kappa_J^2 = \frac{1}{2} \sum_j p_j \sigma_j^2 - \frac{1}{2} \sum_j \sigma_j^2 + \int_{\mathbb{R}} \max_k h_k(x) \nu(dx)$$

$$+ \sum_j p_j \int_{\mathbb{R}} \log \left( \frac{1 + \max_k h_k(x)}{1 + h_j(x)} \right) \nu(dx) - \sum_j p_j \int_{\mathbb{R}} h_j(x) \nu(dx).$$

Assume that \( 0 < \kappa_J < \frac{n}{n-1} \sqrt{|\lambda_2|} \min_{1 \leq j \leq n} p_j \), \( A \) is conditionally negative definite, and that \( \int_{\mathbb{R}} \left( \sum_j (p_j - y_j) h_j(x) - 1 \right) \nu(dx) < 0 \) holds for all \( y \in \Delta_n \). Then for \( \delta > 0 \)
such that $\delta^2 > \kappa^2_j/|\lambda_2|$, $y \in \Delta_n$, and $t > 0$, we have the inequalities
\[
\mathbb{E}_y [\tau_{\mathcal{U}_\delta(p)}] \leq \frac{d(y, p)}{|\lambda_2|\delta^2 - \kappa^2_j}.
\]
(19)
and
\[
\mathbb{E}_y \left[ \frac{1}{t} \int_0^t |s(u) - p|^2 du \right] \leq \frac{1}{|\lambda_2|} \left( \frac{d(y, p)}{t} + \kappa^2_j \right).
\]
(20)
Lastly, an invariant measure for the replicator dynamics with aggregate and instantaneous shocks, which we call $\pi(\cdot)$, exists, is unique, and satisfies the inequality
\[
\pi\left( U_\delta(p) \right) \geq 1 - \frac{\kappa^2_j}{|\lambda_2|\delta^2}.
\]
(21)
Proof: For $p \in \Delta_n$ an ESS for the original game $A$, define the function $v(y) = \sum_j p_j \log (p_j/y_j)$. Applying the infinitesimal generator $A_j$ to $v$, we see that
\[
A_j v(y) = (y - p)^T A y - \frac{1}{2} \sum_j y_j^2 \sigma_j^2 + \frac{1}{2} \sum_j p_j \sigma_j^2 \sum_j p_j \int_R \left( h_j(x) - \sum_k y_k h_k(x) \right) v(dx) + \frac{1}{2} \sum_j p_j \int_R \log \left( \frac{1 + \sum_k y_k h_k(x)}{1 + h_j(x)} \right) v(dx)
\]
\[
\leq (y - p)^T A y - \frac{1}{2} \sum_j y_j^2 \sigma_j^2 + \frac{1}{2} \sum_j p_j \sigma_j^2 + \int_R \max_k h_k(x) v(dx)
\]
\[
+ \sum_j p_j \int_R \log \left( \frac{1 + \max_k h_k(x)}{1 + h_j(x)} \right) v(dx) - \sum_j p_j \int_R h_j(x) v(dx).
\]
The conditionally negative definite assumption yields the inequality
\[
(y - p)^T A y \leq (y - p)^T A (y - p) \leq \lambda_2 |y - p|^2.
\]
Moreover, Cauchy-Schwarz yields $1 \leq \left( \sum_j y_j^2 \sigma_j^2 \right) \sum_j \sigma_j^{-2}$, which gives
\[
-\frac{1}{2} \sum_j y_j^2 \sigma_j^2 \leq -\frac{1}{2} \sum_j \sigma_j^{-2}.
\]
Thus, for $y \in \Delta_n$,
\[
A_j v(y) \leq \lambda_2 |y - p|^2 + \kappa^2_j.
\]
To show the dynamics will eventually evolve close to the ESS, we use the inequality above and assume the dynamic is outside of the specified neighborhood of $p$, $U_\delta(p)$. The assumption $\delta^2 > \kappa^2_j/|\lambda_2|$ tells us for $y \in \Delta_n \setminus U_\delta(p)$, the inequality $A_j v(y) \leq \lambda_2 \delta^2 + \kappa^2_j$ holds. By Itô’s lemma, the process $v(s(t)) - (\lambda_2 \delta^2 + \kappa^2_j)t$ is a local supermartingale on the interval $[0, \tau_{\mathcal{U}_\delta(p)}]$. Therefore $v(y) \geq (|\lambda_2| \delta^2 - \kappa^2_j) \mathbb{E}_y [\tau_{\mathcal{U}_\delta(p)}]$, which shows the inequality in Equation (19). The strong Markov property tells us that $s(t)$ is recurrent in the set $U_\delta(p)$. Furthermore, by choosing a $\delta_0 > 0$ where $\kappa_j/\sqrt{|\lambda_2|} < \delta_0 < \frac{n}{n - 1} \min p_j$ one can see that $\Delta_n \setminus \left\{ \Delta_n \cap U_\delta(p) \right\} = \emptyset$. Thus, $s(t)$ never hits the boundary, and for the remainder of the proof, we may choose any $\delta > 0$ for which the inequality holds.
To derive Equation (20), define \( \tau_k = \inf\{t > 0 : v(s(t)) \geq k\} \), where \( k > v(y) \). Applying Dynkin’s formula we see that

\[
0 \leq E_y \left[ v(s(t \wedge \tau_k)) \right] = v(y) + E_y \left[ \int_0^{t \wedge \tau_k} A_f v(s(u)) du \right] \leq v(y) \\
+ \lambda_2 E_y \left[ \int_0^{t \wedge \tau_k} |s(u) - p|^2 du \right] + \kappa_2^2 E_y [t \wedge \tau_k]
\]

Since \( t \wedge \tau_k \to t \) as \( k \to \infty \), the bounded convergence theorem yields the inequality.

The recurrence property shown is weak, while one would expect a stronger recurrence property, i.e., an invariant measure. To accomplish this task we need to show the recurrence properties converge in total variation to an invariant measure, which will be done by applying Theorem 5.2 in Down et al [10]. In order to satisfy the hypotheses of the theorem, we need to show the dynamic is \( \psi \)-irreducible (page 1674 [10]) and aperiodic (page 1675 [10]).

To show the \( \psi \)-irreducible condition, define the Borel measure \( \psi(O) = M \left( O \cap U_\delta(p) \right) \), where \( M \) is the Lebesgue measure, and \( \eta_O := \int_0^\infty 1_{\{s(t) \in O\}} dt \), which is the occupancy time in the set \( O \). Since the process is recurrent in \( U_\delta(p) \), if \( \psi(O) > 0 \) then \( E_y[\eta_O] > 0 \).

To show the aperiodic condition, we need to find a small Borel set \( B \) and a time \( T \) such that the transition probability \( P_x(t, B) > 0 \) for all \( t \geq T \) and all \( y \in B \). A clear candidate for \( B \) is the set \( U_\delta(p) \). Before we show this conditions holds, we note that since the Poisson measure is generated by a Lévy process, (and so the initial condition for a Lévy process is the Dirac measure \( \delta_0 \)), and independent of all the Wiener processes, the jumps are only dependent on time.

To show this condition holds, we follow the proof of Claim 1 given in [23]. Since \( \nu(\mathbb{R}) < \infty \), we may rewrite \( s(t) \) as \( s(t) = y + \int_t^t \hat{D}^1(s(t-)) dt + \int_0^t D^1(s(t-)) dW(t) + \int_0^t \int_R D^1(s(t-)) N(dt, dx) \), where

\[
\hat{D}^1(y) = \left[ \text{diag}(y_1, \ldots, y_n) - yy^T \right] \left[ A - \text{diag}(\sigma_1^2, \ldots, \sigma_n^2) \right] y \\
+ \int_R \left( yh(x)^T - \text{diag}(h_1(x), \ldots, h_n(x)) \right) \nu(dx).
\]

For the finite interval \([0, t']\), there is a positive probability \( P_y \) that a jump does not occur. On this event, \( s(t) \) agrees with the process \( l(t) = y + \int_0^t \hat{D}^1(l(t)) dt + \int_0^t D^2(l(t)) dW(t) \). Thus, considering Theorem 2.1 in Imhof [18], the condition holds.

Lastly, we need to show there exists a function \( V \), where \( V \geq 1 \), and constants \( c,b > 0 \) such that \( A_f V(\cdot) \leq -cV(\cdot) + b1_{U_\delta(p)}(\cdot) \). Define \( V(y) = K + \prod_i y_i^{-p_i} \), where \( K \) is a positive constant which will later be determined. After applying the infinitesimal generator to \( V(y) \), since simplifying is a strenuous process, only key points in the inequality are given. Hence,
By the assumptions, $C$ where helpful discussions, tremendous guidance, insightful comments. and therefore Equation (21) follows.

To finish the inequality, we note that

$$A_j V(y) = (y - p)^T \lambda y \cdot \prod_i y_i^{-p_i} + \int_R \sum_j (p_i - y_j) h_j(x) \nu(dx) \cdot \prod_i y_i^{-p_i}$$

$$+ \sum_j p_i (1 - y_j) \sigma_j^2 \prod_i y_i^{-p_i} - \frac{1}{2} \sum_i \sum_{k \neq i} p_i p_k \left( (1 - y_i) \sigma_i^2 + y_k \sigma_k^2 \right) \prod_i y_i^{-p_i}$$

$$+ \int_R \left( V(D^3(y) + y) - V(y) \right) \nu(dx)$$

$$\leq \left( \lambda_2 |p - y|^2 + \sum_j p_i (1 - y_j) \sigma_j^2 + \int_R \left( \sum_j (p_j - y_j) h_j(x) - 1 \right) \nu(dx) \right)$$

$$- \frac{1}{2} \sum_i \sum_{k \neq i} p_i p_k \left( (1 - y_i) \sigma_i^2 + y_k \sigma_k^2 \right) \prod_i y_i^{-p_i} + \int_R \frac{1 + \max_j h_j(x)}{1 + \min_j h_j(x)} \nu(dx)$$

$$:= C(y) \prod_i y_i^{-p_i} + \vartheta,$$

where $C(y) := \lambda_2 |p - y|^2 + \sum_j p_i (1 - y_j) \sigma_j^2 + \int_R \left( \sum_j (p_j - y_j) h_j(x) - 1 \right) \nu(dx) - \frac{1}{2} \sum_i \sum_{k \neq i} p_i p_k \left( (1 - y_i) \sigma_i^2 + y_k \sigma_k^2 \right)$ and $\vartheta := \int_R \frac{1 + \max_j h_j(x)}{1 + \min_j h_j(x)} \nu(dx)$.

To finish the inequality, we note that

$$C(y) \prod_i y_i^{-p_i} + \vartheta = \left( \frac{C(y) \prod_i y_i^{-p_i}}{V(y)} + \frac{\vartheta}{V(y)} \right) V(y)$$

$$= \left( \frac{C(y) \prod_i y_i^{-p_i}}{K + \prod_i y_i^{-p_i}} + \frac{\vartheta}{K + \prod_i y_i^{-p_i}} \right) V(y)$$

$$\leq \left( C(y) + \frac{\vartheta}{K} \right) V(y).$$

By the assumptions, $C(y) < 0$ for $y \in \Delta_n \setminus \cup_b(p)$. Taking $K$ large enough so that $C(y) + \frac{\vartheta}{K} < 0$ for all $y \in \Delta_n \setminus \cup_b(p)$ and $V \geq 1$, we are able to find constants $c, b > 0$ such that $A_j V(y) \leq -c V(y) + b \mathbf{1}_{\cup_b(p)}(y)$ holds for all $y \in \Delta_n$.

Defining $O^C := \Delta_n \setminus O$ and $\pi(\cdot)$ as the invariant measure, we have

$$\pi \left( \mathcal{U}_b(p)^C \right) = \lim_{t \to \infty} \mathbb{E}_y \left[ \frac{1}{t} \int_0^t \mathbb{E}_U(t) \rho(s(u)) \, du \right]$$

$$\leq \lim_{t \to \infty} \mathbb{E}_y \left[ \frac{1}{t} \int_0^t \left| s(u) - p \right|^2 \, du \right] \leq \frac{\kappa^2}{\lambda_2 |\delta|^2},$$

and therefore Equation (21) follows.

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