On Eagleson’s theorem in the non-stationary setup

Yeor Hafouta

Abstract
A classical result due to Eagleson states (in particular) that if appropriately normalized Birkhoff sums generated by a measurable function and an ergodic probability preserving transformation converge in distribution, then they also converge in distribution with respect to any probability measure which is absolutely continuous with respect to the invariant one. In this note, we prove several quantitative and infinite-dimensional versions of Eagleson’s theorem for some classes of non-stationary stochastic processes which satisfy certain type of decay of correlations.

1. Introduction
Let \((\Omega, \mathcal{F}, \mu, T)\) be an probability preserving system (p.p.s.) and let \(f: \Omega \to \mathbb{R}^d\) be a measurable function. Then the partial sums \(S_n f = S_n f(x) = \sum_{n=0}^{n-1} f(T^n x)\) are random variables, where \(T^n = T \circ T \cdots \circ T\) and \(x\) is chosen at random according to \(\mu\) (that is, the probability that \(x\) belongs to a measurable set \(A\) is \(\mu(A)\)). Note that any discrete time vector-valued stationary process \(Y_0, Y_1, \ldots\) has the form \(Y_n = f(T^n x)\) for some p.p.s and a measurable function \(f\).

Given such a function \(f\), an important question in probability and ergodic theory is whether \((S_n f - a_n) / b_n\) converges in distribution as \(n \to \infty\), for some sequences \((a_n)\) and \((b_n)\) so that \(\lim_{n \to \infty} b_n = \infty\). A related and more general question is whether the continuous time processes \(W_n(t) = (S_{[nt]} f - a_{[nt]}) / b_n\) converge in distribution.

In certain circumstances, there is a given reference measure (for example, the Lebesgue measure) which is absolutely continuous with respect to the invariant measure \(\mu\). A classical result due to Eagleson [13] insures (for \(d = 1\)) that the weak convergence of \((S_n - a_n) / b_n\) with respect to \(\mu\) is equivalent to the weak convergence with respect to the reference measure (with the same limits), which is more natural since usually the density of \(\mu\) does not have an explicit form. Since then this result was extended to more general processes which are not necessarily real-valued, see, for instance, [27]. In particular, one can also consider vector-valued functions \(f\), as well as random continuous time processes of the form \(W_n(t) = (S_{[nt]} f - a_{[nt]}) / b_n\), which gives us a version of Eagleson’s theorem in the context of the weak invariance principle (WIP), which states that the stochastic process \(W_n(t) = n^{-1/2} S_{[nt]}(f - \mu(f))\) converge in distribution as \(n \to \infty\) in the Skorokhod space \(D([0, \infty), \mathbb{R}^d)\) toward a Gassian process (other normalizations \(b_n\) can be considered). Another application of [27] is to the, so-called, iterated WIP, which yields a certain type of smooth approximations of stochastic differential equations for suspension flows built over non-uniformly expanding or hyperbolic maps [19, Theorems 2.1, 2.2]. We also refer to [14] for additional results which also have applications to the strong invariance principle.

The results described in the latter paragraph concern partial sums \(S_n = \sum_{j=1}^{n-1} Y_n\) generated by vector-valued stationary process \(\{Y_n\}\) defined on a probability space \((\Omega, \mathcal{F}, \mu)\), and in this paper we prove certain versions of Eagleson’s theorem for non-stationary sequences of vector-valued processes \(\{Y_n\}\) satisfying certain mixing (decay of correlations) conditions which hold.

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true for many sequential dynamical systems including the ones arising as realizations of random dynamical systems, as well as for wide classes of inhomogeneous Markov chains and other mixing sequences. Note that for real-valued functions \( Y_n \), Eagleson’s results \([13]\) also apply when the tail-\( \sigma \) algebra of \( \{ Y_n \} \) is trivial, which will be the case in most of the examples we have in mind. We start with the above vector-valued case, but when \( Y_n \) are real-valued, we also prove a quantitative version, which means that we obtain explicit estimates on the convergence rate in the weak convergence with respect to a measure \( m \) which is absolutely continuous with respect to \( \mu \), in terms of the rate in the corresponding convergence with respect to the original measure \( \mu \) (for which the mixing conditions originally hold). The question of optimal convergence rate will also be addressed, as well as the problem of re-centering and re-normalizing after changing the measure. We also prove a non-stationary version of Eagleson’s theorem for continuous time stochastic processes of the form \( S_n(t) = S_{|nt|} f/b_n \) (that is, a version for the WIP). Our results, for instance, yield the WIP for the compositions of random Anosov or expanding maps considered in \([10]\) and \([11]\), with respect to the Lebesgue measure and not only with respect to the random equivariant measures (the WIP for such maps follows from the almost sure invariance principles which were obtained in \([10]\) and \([12]\)). Finally, we will also discuss a version of Eagleson’s theorem in the, so-called, iterated WIP, which we expect to have applications in smooth approximations of stochastic differential equations for non-stationary suspension flows built over random and sequential dynamical systems (that is, in a non-stationary version of \([19]\)).

### 2. Preliminaries and examples

Let \((E,F,\mu)\) be a probability space, \(X_0, X_1, X_2, \ldots\) be measurable spaces and \(X_0, X_1, X_2, \ldots\) be a sequence of measurable functions on \(E\), so that \(X_i\) takes values in \(X_i\) for each \(i\). In this paper, we are interested in sequences so that the partial sums \(S_n g = \sum_{j=0}^{n-1} g_j(X_j, X_{j+1}, \ldots)\) satisfy the central limit theorem for large classes of sequences of functions \(\{g_j\}\), namely there are sequences \((a_n)\) and \((b_n)\) which depend on \(\{g_j\}\) so that \((b_n)\) tends to \(\infty\) and \((S_n g - a_n)/b_n\) converges in distribution toward the standard normal law. In general, for the CLT to hold for a large class of sequences \(\{g_j\}\), a certain type of asymptotic independence between \(\{X_0, \ldots, X_n\}\) and \(\{X_{n+k}, X_{n+k+1}, \ldots\}\) as \(k \to \infty\) is required. As mentioned in the abstract, our standing assumption is a certain type of decay of correlations, which is a quantitative way of measuring such dependence.

**Assumption 2.1.** There exists a sequence \((\delta_n)\) which converges to 0 as \(n \to \infty\) and a set \(B\) of real integrable functions on \(E\) equipped with a ‘norm’ \(\|\cdot\|\) so that for all \(n\), a function \(s \in B\) and a bounded complex-valued function \(f = f(x_n, x_{n+1}, x_{n+2}, \ldots)\), we have

\[
\left| \int s(x) f(\overline{X}_n(x)) d\mu(x) - \int s(x) d\mu(x) \cdot \int f(\overline{X}_n(x)) d\mu(x) \right| \leq \|s\| \|f \circ \overline{X}_n\|_{\infty} \delta_n, \tag{2.1}
\]

where \(\overline{X}_n(x) = (X_n(x), X_{n+1}(x), X_{n+2}(x), \ldots)\) and \(\|f \circ \overline{X}_n\|_{\infty}\) is the essential supremum of the function \(f(\overline{X}_n(x))\) with respect to \(\mu\).

This assumption holds true in a variety of models, which will be described in Examples 2.2 and 2.3. Let us now explain why we only need \(\|f \circ \overline{X}_n\|_{\infty}\) to appear on the right-hand side of (2.1) (and not a smaller norm). Let \(g_j, j \geq 0\) be a sequence of functions on \(X_j \times X_{j+1} \times \ldots\) and set \(S_n g(x) = \sum_{j=0}^{n-1} g_j(\overline{X}_j(x))\). Then the goal in this paper is to investigate the limit (distributional) behavior of \(S_n g\) (and related infinite dimensional processes) when \(x\) is distribution according to measures \(\nu\) which are absolutely continuous with respect to \(\mu\) and \(r = d\nu/d\mu\) belongs to the \(L^p\)-closure of \(B\) for some \(p \geq 1\). The idea behind the the proofs is
that for a density \( r \in B \) and any real \( t \), integers \( 0 \leq k < n \) and a normalizing sequence \((b_n)\) so that \( \lim_{n \to \infty} b_n = \infty \), we have

\[
\int r(x)e^{itS_n g(x)/b_n}d\mu(x) = \int r(x)e^{it(S_n g(x) - S_k g(x))/b_n}d\mu(x) + O(t/b_n)A_k,
\]

where \( A_k = \int r(x)|S_k g(x)|d\mu(x) \). Now, since \( e^{it(S_n g(x) - S_k g(x))/b_n} \) is a bounded function of \( X_k \), using (2.1) we get that

\[
\int r(x)e^{itS_n g(x)/b_n}d\mu(x) = \int e^{itS_n g(x)/b_n}d\mu(x) + O(t/b_n)B_k + O(\delta_k),
\]

where \( B_k = \int (r(x) + 1)|S_k g(x)|d\mu(x) \). By choosing \( k = k_n \) appropriately so that \( \lim_{n \to \infty} k_n \to \infty \) (and other restrictions hold, depending on the result we want to prove), we see that the characteristic function of \( S_n g/b_n \) with respect to \( \nu = rd\mu \) can be controlled by the corresponding one with respect to \( \mu \) on appropriate domains.

Before formulating our main results, let us discuss two main types of examples which satisfy Assumption 2.1.

**Example 2.2 (Random and sequential dynamical systems).** Let \( X_0 = \mathcal{E} \) and \( T_0, T_1, T_2, \ldots \) be a sequence of maps so that \( T_j : X_j \to X_{j+1} \). Let \( X_0(x) = x \) and set \( X_j(x) = X_0(T_0^j x) = T_0^j x \), where \( T_m^j = T_{n+m-1} \circ \cdots \circ T_{n+1} \circ T_n \) for all \( n \) and \( m \). Then \( X_j + n = X_j \circ T_j^j \) for every \( n \) and \( j \). Therefore, \( X_n(x) \) depends only on \( X_n(x) \) and (2.1) becomes

\[
\left| \int s(x)f(T^n x)d\mu(x) - \int s(x)d\mu(x) \cdot \int f(T^n x)d\mu(x) \right| \leq \|s\| \|f \circ T^n\|_{L^1(\mu)} \delta_n.
\]

This condition (with an appropriate \( \mu \)) is satisfied for appropriate functions \( B \) and norms \( \| \cdot \| \) for many sequential and random dynamical systems, where in many of the examples, we can even replace \( \|f \circ T^n\|_{L^1(\mu)} \) with the corresponding \( L^1(\mu) \)-norm. We refer the readers to [1, 2, 5, 9, 11, 15, 17] and [20, 22] and references therein. We note that in some of these papers the case when \( T_j = T_{\theta j}^1 \) is a random stationary family of maps is considered, where \((\Omega, \mathcal{F}, P, \theta)\) is a measure preserving system, and \( T_\omega, \omega \in \Omega \) is a measurable in \( \omega \) family of maps. We note that in most of the above papers \( B \) is a normed space which is dense in \( L^p(\mu) \) for every finite \( p \geq 1 \).

Remark that in [1] the authors obtained almost sure rates of mixing for certain classes random hyperbolic maps \( T_\omega \). The authors of [1] show that these maps admit a random tower extension \((\Delta_\omega, F_\omega)\), first introduced in [3] (which generalizes [26] to the random case). The random tower inherits the random hyperbolic structure from the original maps \( T_\omega \), and after collapsing stable manifolds, the statistical properties of the original maps (with respect to the random physical measure) are reduced to the resulting “projected” random tower \((\Delta_\omega, F_\omega)\), see [1, Section 2.3]. A direct application of [1, Theorem 2.5] shows that Assumption 2.1 holds true (for \( P\text{-a.a.} \omega \)) on the projected tower with \( B = B_\omega \) being space of Hölder continuous functions on \( \Delta_\omega \) and \( \mu = \mu_\omega \), where \( \mu_\omega \) is the absolutely continuous equivariant measure (that is, \((F_\omega), \mu_\omega = \mu_{\theta_\omega})\).

**Example 2.3 (Non-stationary mixing stochastic processes).** Let \( X = \{X_j\} \) be a sequence of random variables defined on the same probability space \((\mathcal{E}, \mathcal{F}, \mu)\). For each \( n \leq m \), we denote by \( \mathcal{F}_{n,m} \) the \( \sigma \)-algebra generated by the random variables \( X_n, X_{n+1}, \ldots, X_{n+m} \). Let \( \mathcal{F}_{n,\infty} \) denote the \( \sigma \)-algebra generated by the random variables \( X_j, j \geq n \). For any two sub-\( \sigma \)-algebras \( \mathcal{G} \) and \( \mathcal{H} \) of \( \mathcal{F} \), set

\[
\alpha(\mathcal{G}, \mathcal{H}) = \sup \{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{G}, B \in \mathcal{H}\}.
\]
The coefficient $\alpha(\mathcal{G}, \mathcal{H})$ measures the dependence between $\mathcal{G}$ and $\mathcal{H}$ and it is one of the classical mixing coefficients used in the literature (often referred to as the strong mixing coefficient). For each $n \geq 1$, set

$$\alpha_n = \alpha_n(X) = \sup_{k \geq 0} \alpha(\mathcal{F}_{0,k}, \mathcal{F}_{k+n,\infty}).$$

The sequence $\{X_n\}$ is called $\alpha$-mixing if $\lim_{n \to \infty} \alpha_n = 0$. A concrete example for non-stationary $\alpha$-mixing processes are the inhomogeneous Markov chains considered in [23]. See [6–8, 24] and [25] for other examples of $\alpha$-mixing non-stationary processes.

Let $(\gamma_n)$ be a sequence which converges to 0 as $n \to \infty$ and let $B$ be the set of all functions $s$ on $\mathcal{E}$ so that for every sufficiently large $n$,

$$\beta_{1,n}(s) := ||s - E[s|X_0, \ldots, X_n]||_{L^1(\mu)} \leq \gamma_n.$$ 

It is clear that $B$ contains all the random variables of the form $s = s(X_0, \ldots, X_n)$. Let $p \geq 1$ be finite and $|| \cdot ||$ be the $L^p(\mu)$-norm. Then for all $s \in B$ so that $\int sd\mu = 1$, we have

$$\left| \int s(x)f(\overline{X}_n(x))d\mu(x) - \int s(x)d\mu(x) \cdot \int f(\overline{X}_n(x))d\mu(x) \right|$$

$$\leq \left| \int s_{[\frac{n}{2}]}(x)f(\overline{X}_n(x))d\mu(x) - \int s_{[\frac{n}{2}]}(x)d\mu(x) \cdot \int f(\overline{X}_n(x))d\mu(x) \right|$$

$$+ 2\|f \circ \overline{X}_n\|_{L^\infty} ||s - s_{[\frac{n}{2}]}||_{L^1},$$

where $s_n = E[s|X_0, \ldots, X_n]$. Since $\int sd\mu = 1$, we have

$$||s - s_{[\frac{n}{2}]}||_{L^1} \leq \gamma_{[\frac{n}{2}]} \leq ||s||_{L^p} \gamma_{[\frac{n}{2}]}.$$ 

Next, by Corollaries [18, A.1 and A.2] and since conditional expectations contract $L^p$-norms, for every $p \geq 1$, we have

$$\left| \int s_{[\frac{n}{2}]}(x)f(\overline{X}_n(x))d\mu(x) - \int s_{[\frac{n}{2}]}(x)d\mu(x) \cdot \int f(\overline{X}_n(x))d\mu(x) \right|$$

$$\leq 6 \left( \alpha \left( \mathcal{F}_{[\frac{n}{2}], \ell}, \mathcal{F}_{\ell, \infty} \right) \right)^{1-\frac{1}{p}} ||s||_{L^p} ||f(\overline{X}_n)||_{L^\infty},$$

where we use the convention $\frac{1}{\infty} := 0$. We conclude that in the above circumstances the conditions in Assumption 2.1 hold true with $||s|| = ||s||_{L^p}$ and $\delta_n = 6 \alpha \left( \mathcal{F}_{[\frac{n}{2}], \ell} \right)^{1-\frac{1}{p}} 2 \gamma_{[\frac{n}{2}]}$.

3. Vector-valued processes

Henceforth, when it is more convenient we will denote the integral of a function $f$ with respect to $\mu$ by $\mu(f)$. We will also denote by $\mu_j$ the distribution of $X_j$. For each $n$, set $Y_n = X_n \times X_{n+1} \times \ldots$. Let $d \geq 1$, $g_j : Y_j \to \mathbb{R}^d$ be a sequence of functions and $r : \mathcal{E} \to \mathbb{R}$ be a non-negative function so that $\int r(x)d\mu(x) = 1$ (that is, $r$ is a probability density with respect to $\mu$). Consider the functions $S_n : \mathcal{E} \to \mathbb{R}^d$ given by

$$S_n(x) = \sum_{j=0}^{n-1} g_j(X_j(x), X_{j+1}(x), \ldots) = \sum_{j=0}^{n-1} g_j(\overline{X}_j(x)).$$

Let $\nu$ be the probability measure on $\mathcal{E}$ defined by $d\nu = rd\mu$. We can view $S_n = S_n(x)$ as a random variable when $x$ is distributed according to either $\mu$ or $\nu$. We denote these random variables by $S_{n,\mu}$ and $S_{n,\nu}$, respectively. Our first result is the following:
THEOREM 3.1. Suppose that Assumption 2.1 holds true. Assume also that for some two conjugate exponents p and q we have that r lies in the $L^p(\mu)$-closure of $B \cap L^p(\mu)$ and $g_j \circ \mathcal{L}_j \in L^q(\mu)$ for all $j \geq 0$. Then under Assumption 2.1, for every sequence $(b_n)_n$ of positive numbers which tends to $\infty$ (as $n \to \infty$), the sequence $S_n, B_n$ converges in distribution if and only if $S_n, B_n$ converges in distribution, and in the latter case both converge toward the same limit.

For real valued functions $g_j$, Theorem 3.1 follows from [13] when the tail $\sigma$-algebra $\mathcal{T}$ of the sequence $Y_j = g_j(X_j, X_{j+1}, \ldots)$ is trivial. Under Assumption 2.1, it is clear that $\mathcal{T}$ is trivial when the $L^1$-closure of $B$ contains all integrable $\mathcal{T}$-measurable functions. Therefore, we essentially do not consider Theorem 3.1 as a new result, but we still present a proof since later on we will adapt its arguments to obtain a quantitative version, as well as a version corresponding to the weak invariance principle.

Proof. First, for any real $t$, we set $t_n = t/b_n$. By the Levi continuity theorem, it is enough to show that for any fixed $t$ we have

$$
\lim_{n \to \infty} |\mu(t \cdot e^{it_n}S_n) - \mu(e^{it_n}S_n)| = 0.
$$

In the case when $r$ does not lie in $B$, given $\varepsilon > 0$ we can first approximate $r$ within $\varepsilon$ in $L^p(\mu)$ by a (non-negative) $s \in B$ so that $\mu(s) = 1$. Then for any real $t$, we have

$$
|\mu(t \cdot e^{it_n}S_n) - \mu(s \cdot e^{it_n}S_n)| \leq \|r - s\|_{L^p(\mu)} < \varepsilon.
$$

Therefore, it is enough to prove the theorem when $r \in B$ and the integrals $\int |S_n(x)|d\mu(x)$ and $\int r(x)|S_n(x)|d\mu(x)$ are finite for all natural $n$.

Next, since $\lim_{n \to \infty} b_n = \infty$, for any sequence $(c_k)$ there exists a (weakly increasing) sequence $(a_n)$ of natural numbers which tends to $\infty$ as $n \to \infty$ so that $c_{a_n} = o(b_n)$. It is also clear that we can assume that $a_n < n$. Consider the sequence $c_k = \int |S_k(x)|d\mu(x) + \int r(x)|S_k(x)|d\mu(x)$ and let $a_n$ be so that $c_{a_n} = o(b_n)$.

Next, by the mean value theorem,

$$
|\mu(t \cdot e^{it_n}S_n) - \mu(r \cdot e^{it_n}(S_n - S_{a_n}))| \leq |t|b_n^{-1}\mu(r \cdot |S_{a_n}|) \leq |t|c_{a_n}/b_n \to 0 \text{ as } n \to \infty.
$$

Relying on Assumption 2.1, taking into account that $\mu(r) = 1$ and that $S_n - S_{a_n}$ is a function of $\mathcal{X}_{a_n}$, we have

$$
|\mu(t \cdot e^{it_n}(S_n - S_{a_n})) - \mu(e^{it_n}(S_n - S_{a_n}))| \leq \|r\|_{\delta_{a_n}} \to 0 \text{ as } n \to \infty.
$$

Finally, by the mean value theorem,

$$
|\mu(e^{it_n}(S_n - S_{a_n})) - \mu(e^{it_n}S_n)| \leq |t|b_n^{-1}\mu(|S_{a_n}|) \leq |t|c_{a_n}/b_n \to 0 \text{ as } n \to \infty.
$$

3.1. Recentering after change of measure

In applications, it is often the case where $(S_n, B_n - E[S_n, \mu]) / b_n$ converges in distribution, and this just means that we replace $g_j$ with $g_j - \mu(g_j(\mathcal{X}_j))$ in the setup of the previous section. Applying Theorem 3.1, we infer that $(S_n, B_n - E[S_n, \mu]) / b_n$ converges in distribution, and to the same limit. The ‘centering’ term $E[S_n, \mu]$ is not natural in the latter convergence, and it is
natural to inquire whether \((S_{n,\nu} - \mathbb{E}[S_{n,\nu}])/b_n\) converges in distribution. When all the functions \(g_j\) are bounded and \(r \in B\), under Assumption 2.1 we have
\[
|\mathbb{E}[S_{n,\nu}] - \mathbb{E}[S_{n,\mu}]| \leq \sum_{j=0}^{n-1} |\mu(r g_j(\mathbf{X}_j)) - \mu(r)\mu(g_j(\mathbf{X}_j))| \leq \|r\| \sum_{j=0}^{n-1} \delta_j \|g_j\|_{\infty}.
\]
Therefore, if \(\sum_{j=0}^{n-1} \delta_j \|g_j\|_{\infty} = o(b_n)\), we obtain that the difference between \((S_{n,\nu} - \mathbb{E}[S_{n,\nu}])/b_n\) and \((S_{n,\nu} - \mathbb{E}[S_{n,\mu}])/b_n\) converges almost surely to 0, which yields the desired convergence in distribution of \((S_{n,\nu} - \mathbb{E}[S_{n,\nu}])/b_n\). Of course, the assumption that functions \(g_j\) are bounded can be weakened. For any sequence of \(M_j > 0\), we have
\[
\sum_{j=0}^{n-1} |\mu(r g_j(\mathbf{X}_j)) - \mu(r)\mu(g_j(\mathbf{X}_j))| \leq \|r\| \sum_{j=0}^{n-1} M_j \delta_j
\]
\[
+ \sum_{j=0}^{n-1} |\mu((r + 1)g_j(\mathbf{X}_j))\mathbb{E}[|g_j(\mathbf{X}_j)| > M_j]|.
\]

By the Hölder and the Markov inequalities, for any \(p_1, p_2, p_3 \geq 1\) so that \(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1\), we have
\[
|\mu((r + 1)g_j(\mathbf{X}_j))\mathbb{E}[|g_j(\mathbf{X}_j)| > M_j]| \leq \|r + 1\|_{p_1} \|g(\mathbf{X}_j)\|_{p_2} \|g(\mathbf{X}_j)\|_{p_3} M_j^{-p_2/p_3},
\]
where \(\|f\|_p := \|f\|_{L^p(\mu)}\) for any \(p\) and a vector-valued function \(f\) on \(\mathcal{E}\). This yields the following simple result:

**Proposition 3.2.** Suppose Assumption 2.1 holds, that \(r \in B\) and that there are \(p_1, p_2, p_3\) and a sequence \((M_j)\) of positive numbers so that \(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1\), \(\|r\|_{p_1} < \infty\) and
\[
\mathcal{M}_n := \sum_{j=0}^{n-1} \left( M_j \delta_j + \|g_j(\mathbf{X}_j)\|_{p_2/p_3} M_j^{-p_2/p_3} \right) = o(b_n).
\]
Then \(\mathbb{E}[S_{n,\nu}] - \mathbb{E}[S_{n,\mu}]| \leq \mathcal{M}_n = o(b_n)\) and therefore \((S_{n,\nu} - \mathbb{E}[S_{n,\mu}])/b_n\) converges in distribution if and only if \((S_{n,\nu} - \mathbb{E}[S_{n,\nu}])/b_n\) converges in distribution, and to the same limit.

For instance, when \(\sup_j \|g(\mathbf{X}_j)\|_{p_3} < \infty\) and \(p_2/p_3\) are larger than 1 and \(\delta_j \leq C j^{-2-\varepsilon}\), then we can take \(M_j = j\) and get that \(\mathcal{M}_n\) is bounded in \(n\). When \(b_n = O(n^a)\) for some \(a\), we can get rid of the power \(\varepsilon\) in the upper bound of \(\delta_j\). Of course, many other, more-explicit, moment conditions and decay rates of \(\delta_j\) can be imposed to insure that \(\mathcal{M}_n = o(b_n)\). We note that in most of the applications in Example 2.2 the space \(B\) is composed of bounded functions, and so in this case we can take \(p_1 = \infty\).

### 3.2. Renormalization after change of measure

We assume here that the functions \(g_j\) are real-valued. For random dynamical systems the CLT holds true for the corresponding centered random birkhoff sums normalized by \(b_n = \sqrt{n}\), but for sequential dynamical systems, inhomogeneous Markov chains and other non-stationary mixing sequences usually the CLT holds true (only) with the self normalizing sequence \(b_{n,\mu} = \sqrt{\text{Var}(S_{n,\mu})}\), especially because \(\text{Var}(S_{n,\mu})\) may have various asymptotic behaviors. Of course, this requires us to assume that \(\lim_{n \to \infty} b_{n,\mu} = \infty\), and we refer the readers to [15] and [6] (when \(g_j\) depends only on \((X_j, X_{j+1})\)) for characterizations of the latter convergence for several classes of sequential dynamical systems and inhomogeneous Markov chains. In Section 3.1, we showed that, under certain conditions, the weak convergence of \((S_{n,\nu} - \mathbb{E}[S_{n,\nu}])/b_{n,\mu}\) follows
from the convergence corresponding to $\mu$, and a natural question is whether the convergence of $(S_{n,\nu} - \mathbb{E}[S_{n,\nu}])/b_{n,\nu}$ can also be derived, where $b_{n,\nu} = \sqrt{\text{Var}(S_{n,\nu})}$.

**Proposition 3.3.** Under Assumption 2.1, set $G_k = g_k \circ \overline{X}_k$, and assume that $\mathbb{E}_{\mu}[G_k] = 0$ for all $k$. Suppose also that $\delta_k = C_1 b^k$ for some $C_1 > 0$ and $b \in (0,1)$, and that

$$\max(||G_k||, ||r \cdot G_k||) \leq b \delta^{-ak} \text{ for some } a, b > 0 \text{ and all } k \geq 0.$$  

(3.1)

Moreover, assume that there are $c_0, C_0 > 0$, $\beta \in (0,1)$ and $p > 3$ so that for any $k \geq 0$

$$||G_k||_p \leq C_0 e^{c_0 k^\beta}$$  

(3.2)

and $||r||_p < \infty$. Then there is $C > 0$ so that for every $n \geq 1$,

$$|\mathbb{E}[S^2_{n,\mu}] - \mathbb{E}[S^2_{n,\nu}]| \leq C.$$

Therefore,

$$|\text{Var}(S_{n,\mu}) - \text{Var}(S_{n,\nu})| \leq C + M_{n}^2,$$

where $M_n$ comes from Proposition 3.2. Hence, when $M_n = o(b_{n,\mu})$ and $b_{n,\mu}$ tends to $\infty$ as $n \to \infty$, then

$$\lim_{n \to \infty} \frac{b_{n,\mu}}{b_{n,\nu}} = 1$$

and therefore $(S_{n,\nu} - \mathbb{E}[S_{n,\nu}])/b_{n,\nu}$ converges in distribution if and only $(S_{n,\nu} - \mathbb{E}[S_{n,\nu}])/b_{n,\nu}$ converges in distribution, and to the same limit.

In the circumstances of Example 2.2, condition (3.1) holds true when $|| \cdot ||$ is an Hölder norm (as in [17, 22] or [15]) or some total variation norm (as in [9]) and $g_j(\overline{X}_j(x)) = h_j(T_{0j} x)$, where $||h_j||$ are uniformly bounded in $j$. In the circumstances of Example 2.3, the norm $|| \cdot ||$ is some $L^p$-norm and so (3.1) will be satisfied if the functions $G_k$ are bounded $L^q$ for $q > p$ and $||r||_{q'} < \infty$, where $1/p = 1/q + 1/q'$.

**Proof of Proposition 3.3.** First, we have

$$|\mathbb{E}[S^2_{n,\mu}] - \mathbb{E}[S^2_{n,\nu}]| \leq 2 \sum_{0 \leq k \leq j < n} |\text{Cov}_{\mu}(r, G_k G_j)|.$$  

(3.3)

Let $0 \leq k \leq j < n$ be so that $j \geq (a + 1)k$, where $a$ comes from (3.1). Moreover let $\beta < \alpha < 1$, where $\beta$ comes from (3.2) and set

$$\tilde{G}_j = G_j \mathbb{I}(||G_j|| \leq e^{j^n}).$$

Let $q_2$ be the conjugate exponent of $p_2 = p/2$ and $q_3$ be the conjugate exponent of $p_3 = p/3$. Let us write

$$\text{Cov}_{\mu}(r, G_k G_j) = \text{Cov}_{\mu}(r, G_k \tilde{G}_j) + D_{k,j}.$$  

Using (3.2) and the Hölder and the Markov inequalities, we have

$$|D_{k,j}| \leq \left(\|r G_k G_j\|_{p_3}^{1+q_3/q_3} + \|G_k G_j\|_{p_2}^{1+q_2/q_2}\right) e^{-c_j \alpha} \leq C_2 e^{-c_2 j^n}$$

where $c = \min(p_2/q_2, p_3/q_3)$ and $c_2$ and $C_2$ are some positive constants. The contribution to the right-hand side of (3.3) coming from $D_{j,k}$, with $j \geq (a + 1)k$ is therefore controlled by

$$\sum_{j=0}^{n-1} \sum_{k=0}^j e^{-c_2 j^n} \leq \sum_{j=1}^\infty j e^{-c_2 j^n} < \infty.$$
Now we will control the contribution coming from \( \text{Cov}_\mu(r, G_k \tilde{G}_j) \), when \( j \geq (a + 1)k \). First, using (3.2) and that \( ||r||_3 < \infty \), we have
\[
|\mu(rG_k)| \leq ||r||_2 ||G_k||_2 \leq C_0 ||r||_2 e^{c_0k^\alpha} \leq C_0 ||r||_2 e^{c_0j^\alpha}.
\]
Next, using (2.1) with \( n = j \) and \( s = rG_k \), (3.1) we get that
\[
\left| \mathbb{E}_\mu[rG_k \tilde{G}_j] \right| \leq C_0 ||r||_2 e^{c_0j^\alpha} ||\mathbb{E}_\mu[\tilde{G}_j]|| + C_1 be^{\alpha_0} \delta^{j^\alpha-k^\alpha}.
\]
Moreover,
\[
\left| \mathbb{E}_\mu[G_k \tilde{G}_j] \right| \leq be^{\alpha_0} \delta^{j^\alpha-k^\alpha},
\]
where we have also used that \( G_k \) is centered. Next, using the Markov inequality, we have that
\[
\left| \mathbb{E}_\mu[G_j] \right| = ||\mathbb{E}[G_j 1(\{|G_j| > e^{j\alpha}\})]|| \leq ||G_j||_p^{1+q/p} e^{-(n/p)j^\alpha},
\]
where \( q \) is the conjugate exponent of \( p \) (and \( p \) comes from (3.2)). We conclude that, in absolute value, the contribution to the right-hand side of (3.3) coming from the pairs \( j \) and \( k \) so that \( j \geq (a + 1)k \) does not exceed a constant times
\[
\sum_{j=0}^{n-1} e^{j\alpha} \sum_{k=0}^{j/(a+1)} \delta^{j^\alpha-k^\alpha} + \sum_{j=0}^{n-1} (j+1)e^{-c_2j^\alpha} \leq c \sum_{j=0}^{\infty} e^{j\alpha} \delta^{j/(a+1)} + \sum_{j=0}^{\infty} (j+1)e^{-c_2j^\alpha} < \infty,
\]
where \( c \) is some constant. Now we estimate the contribution coming from pairs \( (k, j) \) such that \( k \leq j \leq (a + 1)k \). First, by the Markov and the Hölder inequalities and (3.2), we have
\[
|\text{Cov}_\mu(r, G_k G_j)| \leq |\text{Cov}_\mu(r, G_k G_j 1(|G_k G_j| \leq e^{j\alpha}))| + C_4 e^{-c_4j^\alpha},
\]
where \( C_4 \) and \( c_4 \) are some positive constants. Using now (2.1) with \( s = r \) and \( f \) so that \( f \circ X_k = G_j G_k 1(|G_j G_k| \leq e^{j\alpha}) \), we get that
\[
|\text{Cov}_\mu(r, G_k G_j)| \leq ||r||_\alpha \delta^k e^{j\alpha} + C_4 e^{-c_4j^\alpha}.
\]
Therefore, there are constants \( C_5, C_6 > 0 \) so that
\[
\sum_{k=0}^{n-1} \sum_{j=k}^{(a+1)k} |\text{Cov}_\mu(r, G_k G_j)| \leq C_5 \sum_{k=0}^{\infty} (k+1)e^{(a+1)^\alpha k^\alpha} \delta^k + C_6 \sum_{k=0}^{\infty} (k+1)e^{-c_4k^\alpha} < \infty.
\]

**Remark 3.4.** The arguments in the proof of Proposition 3.3 show that the conclusion of the proposition holds true if we assume that \( ||G_k||_p \leq \delta^{-u}k \) for some sufficiently small \( u \) and all \( k \geq 0 \).

### 3.3. Quantitative versions for scalar-valued functions

Let \( X \) and \( Y \) be random variables. Recall that the Kolmogorov (uniform) metric \( d_K(X,Y) \) between the laws of \( X \) and \( Y \) is given by
\[
d_K(X,Y) = \sup_{t \in \mathbb{R}} |P(X \leq t) - P(Y \leq t)|.
\]
We have the following (well known) version of the, so-called, Berry–Esseen inequality:

**Lemma 3.5.** Let \( X \) and \( Y \) be two real-valued random variables, and let \( \varphi_X \) and \( \varphi_Y \) be their characteristic functions, respectively. Let \( Z \) be another random variable which has a bounded density function \( f_Z \). Then for every \( T > 0 \), we have
\[
d_K(X,Y) \leq 4c d_K(Y,Z) + \int_{-T}^{T} \left| \frac{\varphi_X(t) - \varphi_Y(t)}{t} \right| dt + \frac{2\|f_Z\|_\infty^2 e^2}{T}.
\]
where \( \|f_Z\|_\infty = \sup f_Z \) and \( c > 0 \) is some absolute constant which can be taken to be the root of the equation
\[
\int_0^{c/2} \frac{\sin^2 x}{x^2} = \frac{\pi}{4} + \frac{1}{8}.
\]
In particular,
\[
d_K(X, Z) \leq (4c + 1)d_K(Y, Z) + \int_{-T}^{T} \left| \frac{\varphi_X(t) - \varphi_Y(t)}{t} \right| dt + \frac{2\|f_Z\|_\infty c^2}{T}.
\]

**Proof.** Taking \( b = 1 \) at the beginning of [21, Section 4.1], we get that
\[
d_K(X, Y) \leq \int_{-T}^{T} \left| \frac{\varphi_X(t) - \varphi_Y(t)}{t} \right| dt + 2T \sup_{x \in \mathbb{R}} \int_{c/T}^{c/T} |P(Y \leq x + y) - P(Y \leq x)| dy.
\]

Next, it clear that for every \( x \in \mathbb{R} \),
\[
\int_{c/T}^{c/T} |P(Y \leq x + y) - P(Y \leq x)| dy \leq \int_{-c/T}^{-c/T} |P(Z \leq x + y) - P(Z \leq x)| dy + 2cd_K(Y, Z)/T.
\]

Lemma 3.5 makes it possible to estimate \( d_K(S_{\nu,n}/b_n, Z) \) by means of \( d_K(S_{\mu,n}/b_n, Z) \), namely we can estimate the error in the weak convergence of \( S_{\nu,n}/b_n \) by means of the error term in the weak convergence of \( S_{\mu,n}/b_n \).

**Theorem 3.6.** Let Assumption 2.1 hold, and suppose that \( r \in B \) and that the functions \( g_j \) are real-valued. Then for every positive integer \( \rho < n, T \geq 1 \) and a random variable \( Z \) with a bounded density function \( f_Z \), we have
\[
d_K(S_{\nu,n}/b_n, Z) \leq (4c + 1)d_K(S_{\mu,n}/b_n, Z)
\]
\[
+ 2T \mu(|S_{\nu,n}((1 + r))| b_n) + 4\delta_p |r| \ln T + \frac{2\|f_Z\|_\infty c^2}{T} + \frac{2\mu((r + 1)|S_{\nu,n}|)}{b_n T},
\]
where \( c \) comes from Lemma 3.5.

**Proof.** Let \( t \neq 0 \) and set \( t_n = t/b_n \). As in the proof of Theorem 3.1 for every \( \rho < n \), we have
\[
|\mu(re^{it_n S_n}) - \mu(e^{it_n S_n})| \leq |t|b_n^{-1} I_1(\rho) + |\mu(re^{it_n (S_{\nu,n} - S_{\rho})}) - \mu(e^{it_n (S_{\nu,n} - S_{\rho})})|,
\]
where
\[
I_1(\rho) = \int |S_\rho(x)|(1 + r(x))d\mu(x).
\]
When \(|t| \leq 1/T\) we will not use the above, and instead we will use the estimate
\[
|\mu(re^{it_n S_n}) - \mu(e^{it_n S_n})| \leq |(\mu(re^{it_n S_n}) - 1) - (\mu(e^{it_n S_n}) - 1)| \leq \mu((r + 1)|S_n|)|t_n|.
\]
Therefore, with \( Y = S_{\nu,n}/b_n \) and \( X = S_{\nu,n}/b_n \), for every \( \rho, T \geq 1 \),
\[
\int_{-T}^{T} \left| \frac{\varphi_X(t) - \varphi_Y(t)}{t} \right| dt \leq \int_{-1/T}^{1/T} \left| \frac{\varphi_X(t) - \varphi_Y(t)}{t} \right| dt + 2TI_1(\rho)/b_n
\]
\[
+ \int_{1/T \leq |t| \leq T} \left| \frac{\mu(re^{it_n (S_{\nu,n} - S_{\rho})}) - \mu(e^{it_n (S_{\nu,n} - S_{\rho})})}{t} \right| dt \leq 2\mu((r + 1)|S_{\nu,n}|)(b_n T)^{-1}
\]
\[
+ 2TI_1(\rho)/b_n + 4\delta_p |r| \ln T,
\]
where in the last inequality we have used Assumption 2.1. The theorem follows now from Lemma 3.5 and the above estimate.

We remark that
\[ \mu((r + 1)|S_n|) \leq \|r + 1\|_{L^2} \]
and so when \( S_n \) has zero \( \mu \)-mean and \( \|r\|_{L^2} < \infty \), we get that the above expression is of order \( \sigma_n = \sqrt{\text{Var}(S_n)} \). Hence, the contribution of the last expression in the right-hand side of (3.4) is of order \( 1/T \) when \( b_n \approx \sigma_n \), which is the case in most applications we have in mind. When, in addition, \( r \) is bounded, \( g_j \) are uniformly bounded and \( \delta_j \leq c_1 e^{-c_2 j} \) for some positive \( c_1 \) and \( c_2 \) then by taking \( T \leq A n \) (for some \( A > 0 \)) and \( \rho = c \ln b_n \) for a sufficiently large \( c \), we get
\[ d_K(S_{n,\nu}/b_n, Z) \leq (4c + 1)d_K(S_{\mu,n}/b_n, Z) + C(b_n^{-1} T \ln b_n + 1/T). \]
Taking \( T = b_n^{1/4} \), we get that
\[ d_K(S_{n,\nu}/b_n, Z) \leq (4c + 1)d_K(S_{\mu,n}/b_n, Z) + Cb_n^{-1/2} \ln b_n. \]
Since \( T \) and its reciprocal appear in the right-hand side of (3.4), we do not expect to get better rates than the above only under Assumption 2.1 (of course, certain rates can be obtained when \( \delta_j \) diverges polynomially fast to 0 and when \( r \) and \( g_j \) only satisfy certain moment conditions).

**Remark 3.7.** When \( S_{n,\nu} \) is not centered (but \( S_{n,\mu} \) is), then it is desirable to get estimates on \( d_K(b_n^{-1} S_{n,\nu}, Z) \), where \( Y = Y - \mathbb{E}[Y] \) for every random variable \( Y \). Applying [16, Lemma 3.3] with \( a = \infty \) yields that
\[ d_K(b_n^{-1} S_{n,\nu}, Z) \leq 3d_K(b_n^{-1} S_{n,\mu}, Z) + (1 + 4\|f_Z\|_{\infty})\mathbb{E}[S_{n,\nu}] - \mathbb{E}[S_{n,\nu}]/b_n. \]
The first expression on the right-hand side was estimated in Theorem 3.6, while the second expression was estimated in Section 3.1. Using the above Lemma 3.3 together with Proposition 3.3 we can also get rates in the CLT for \( (S_{n,\nu} - \mathbb{E}[S_{n,\nu}])/b_n, \nu \) from given rates in the corresponding CLT for \( (S_{n,\mu} - \mathbb{E}[S_{n,\mu}])/b_n, \mu \).

### 3.3.1. Optimal convergence rates

Consider the case when \( b_n = n^{-\frac{1}{2}} \) (or \( b_n \approx n^{\frac{1}{2}} \)) and
\[ d_K(S_{n,\mu}/b_n, Z) = O(n^{-1/2}), \]
where \( Z \) is a standard normal random variable. The rate \( n^{-1/2} \) is optimal, while Theorem 3.6 is not likely to yield optimal rates for \( d_K(n^{-\frac{1}{2}} S_{n,\nu}, Z) \) even when \( \delta_j \) decays exponentially fast to 0 as \( j \to \infty \) and \( r \) and \( \sup_j |g_j| \) are bounded (in this case we have managed to obtain the rate \( n^{-1/4} \ln n \)). In many situations (see [9, 11, 15] and [17]), there exist normed spaces \( B_n \) of functions on some measurable spaces \( \mathcal{E}_n \), a family of operators \( \mathcal{L}_{z}^{(n)} : B \to B_n, z \in \mathbb{C} \) and a family of probability measures \( \mu_n \) on \( \mathcal{E}_n \) so that for every \( s \in B, n \geq 1 \) and \( z \in \mathbb{C} \),
\[ \mu(s \cdot e^{z S_n}) = \mu_n(\mathcal{L}_{z}^{(n)} s). \]
Moreover, the norm on \( B_n \) is larger than the \( L^1(\mu_n) \)-norm and there exists \( \epsilon > 0, C > 0 \) and a function \( R : [0, \infty) \to \mathbb{R} \) so that \( R(t^2) \) is integrable and for every \( t \in [-\epsilon, \epsilon], n \geq 1 \) and a function \( s \) with \( \int s d\mu = 0 \),
\[ \|\mathcal{L}_{i t}^{(n)} s\| \leq C\|s\|t|R(t^2). \]
In applications, such estimates follow from analyticity (in \( z \)) assumptions on the operators \( \mathcal{L}_{z}^{(n)} \) together with a complex sequential Ruelle–Perron–Frobenius theorem (see, for example, [15, Theorem 3.3]). When \( r - 1 \in B \), then with \( t_n = tn^{-1/2} \),
\[ |\mu(re^{i t_n S_n}) - \mu(e^{i t_n S_n})| = |\mu_n(\mathcal{L}_{i t_n}^{(n)}(r - 1))| \leq C_1\|t_n^{-\frac{1}{2}} R(t^2), \]
where we have used that $\mu(r-1) = 0$. Taking $T \approx \delta \sqrt{n}$ in Lemma 3.5, we get that
\[
d_K(S_{\nu,n}/b_n, Z) \leq C(4c + 1)d_K(S_{\mu,n}/b_n, Z) + C\left(\frac{2\|f\|_{\infty}c^2}{\delta} + C_1 \int R(t^2)dt\right)n^{-1/2}
\]
and so the optimal rate of convergence is preserved in the above circumstances. When $\nu$ lies in the $(\epsilon, \delta)$-neighborhood of $\nu_0$, we claim that $S_{\nu,n}$ is also a tight family. Indeed, for any $\epsilon > 0$ there exists a compact set $K_{\epsilon}$ (of paths in the Skorokhod space) so that
\[
\sup_n \mu\{S_{\nu,n}/b_n \notin K_{\epsilon}\} < \epsilon.
\]
We claim that $S_{\nu,n}/b_n$ is also a tight family. Indeed, for any $C > 0$, set
\[
\eta_C = \mu(|r| I(|r| > C)).
\]
Since $r \in L^1(\mu)$, we have $\lim_{C \to \infty} \eta_C = 0$. For every positive $\epsilon$ and $C$, we have
\[
\nu\{S_{\nu,n} \notin K_{\epsilon}\} = \mu(rI(S_{\nu,n} \notin K_{\epsilon})) \leq C\mu(I(S_{\nu,n} \notin K_{\epsilon})) + \mu(rI(|r| > C) \leq \eta_C + C\epsilon.
\]
Given $\epsilon' > 0$, we first take $C$ large enough so that $\eta_C < \frac{1}{2}\epsilon'$, and then, after fixing this $C$, we take $\epsilon$ so that $C\epsilon < \frac{1}{2}\epsilon'$. Then set $K_{\epsilon}$ satisfies
\[
\sup_n \nu\{S_{\nu,n} \notin K_{\epsilon}\} < \epsilon'.
\]
Tightness and the convergence of all the finite-dimensional distributions, expect from the ones which involve members of a certain set of functions $t$ which depend only on the target limiting

4. Infinite-dimensional results

4.1. The weak invariance principle

Let $g_j : X_j \to \mathbb{R}^d$, $j \geq 0$ be vector-valued functions and for every $t \geq 0$ and $n \geq 1$, consider the function
\[
S_n(t) = \sum_{n=0}^{[nt]-1} g_j \circ X_j = \sum_{n=0}^{[nt]-1} g_j(X_j(x)).
\]
For every fixed $n$, we can view $S_n(t)$ as a continuous time process by considering $x$ as a random variable whose distribution is either $\mu$ or $\nu = rd\mu$, where $r$ is a density function. Let $S_{n,\mu}(t)$ and $S_{n,\nu}(t)$ be the resulting continuous time processes.

**Theorem 4.1.** Suppose that for some two conjugate exponents $p$ and $q$, we have that $r$ lies in the $L^p(\mu)$-closure of $B \cap L^p(\mu)$ and $g_j \circ X_j \in L^q(\mu)$ for all $j \geq 0$. Let $(b_n)$ be a sequence of positive numbers so that $\lim_{n \to \infty} b_n = \infty$. Then, under Assumption 2.1, if the sequences of processes process $S_{n,\mu}(\cdot)/b_n$ converges in distribution in the Skorokhod space $D([0, \infty), \mathbb{R}^d)$ as $n \to \infty$, then the processes $S_{n,\nu}/b_n$ converges in distribution to the same limit. If $r$ is positive ($\mu$-almost surely), then the convergence with respect to $\mu$ can be derived from the convergence with respect to $\nu$.

**Proof.** Suppose that $S_{n,\mu}(\cdot)/b_n$ converges in distribution. Then $S_{n,\mu}(\cdot)/b_n$ is a tight family, namely for every $\epsilon > 0$ there exists a compact set $K_{\epsilon}$ (of paths in the Skorokhod space) so that
\[
\sup_n \mu\{S_{n,\mu}/b_n \notin K_{\epsilon}\} < \epsilon.
\]
We claim that $S_{n,\nu}/b_n$ is also a tight family. Indeed, for any $C > 0$, set
\[
\eta_C = \mu(|r| I(|r| > C)).
\]
Since $r \in L^1(\mu)$, we have $\lim_{C \to \infty} \eta_C = 0$. For every positive $\epsilon$ and $C$, we have
\[
\nu\{S_{\nu}(\cdot) \notin K_{\epsilon}\} = \mu(rI(S_{\nu}(\cdot) \notin K_{\epsilon})) \leq C\mu(I(S_{\nu}(\cdot) \notin K_{\epsilon})) + \mu(rI(|r| > C) \leq \eta_C + C\epsilon.
\]
Given $\epsilon' > 0$, we first take $C$ large enough so that $\eta_C < \frac{1}{2}\epsilon'$, and then, after fixing this $C$, we take $\epsilon$ so that $C\epsilon < \frac{1}{2}\epsilon'$. Then set $K_{\epsilon}$ satisfies
\[
\sup_n \nu\{S_{\nu}(\cdot) \notin K_{\epsilon}\} < \epsilon'.
\]
distribution, is equivalent to weak convergence in the Skorokhod space (see [4, Theorem 15.1]).

Therefore, what is left to prove in order to get the convergence in distribution of \( S_{n,\nu} \) is that one can derive the convergence of a given finite-dimensional distribution of \( S_{n,\nu} \) from the convergence of the corresponding finite-dimensional distribution of \( S_{n,\mu} \) (in fact, we will show that these two convergences are equivalent). Consider the sequence

\[
c_k = \int |S_k(x)| d\mu(x) + \int r(x)|S_k(x)| d\mu(x)
\]

and let \( (a_n) \) be a sequence of positive integers so that \( c_{a_n} = o(b_n) \) and \( \lim_{n \to \infty} a_n = \infty \). It is clear that we can assume that \( a_n = o(n) \). Let \( s_1, \ldots, s_m \in (0, \infty) \) and \( t \in \mathbb{R}^{dm} \). For every \( n \geq 1 \), we write \( t_n = t/b_n \). Set

\[
V_n(x) = (S_{[ns_1]}(x), \ldots, S_{[ns_m]}(x))
\]

and

\[
U_n(x) = (S_{a_n}(x), \ldots, S_{a_n}(x)).
\]

We first assume that \( r \in B \). By the mean value theorem, for all sufficiently large \( n \), we have

\[
|\mu(r e^{it_n V_n}) - \mu(r e^{it_n (V_n - U_n)})| \leq C_m |t| b_n^{-1} \mu(|r \cdot |S_{a_n}||) \leq C_m |t| c_{a_n}/b_n \to 0 \quad \text{as} \quad n \to \infty,
\]

where \( C_m \) is some constant which depend only on \( m \) (the number of functions \( s_i \)). Relying on Assumption 2.1, taking into account that \( \mu(r) = 1 \) and that \( V_n - U_{a_n} \) is a function of \( \overline{X}_{a_n} \), we have

\[
|\mu(r e^{it_n (V_n - U_n)}) - \mu(e^{it_n (V_n - U_n)})| \leq \|r\| \delta_{a_n} \to 0 \quad \text{as} \quad n \to \infty.
\]

Finally, by the mean value theorem, we have

\[
|\mu(e^{it_n (V_n - U_n)}) - \mu(e^{it_n V_n})| \leq C_m |t| b_n^{-1} \mu(|S_{a_n}|) \leq C_m |t| c_{a_n}/b_n \to 0 \quad \text{as} \quad n \to \infty.
\]

We conclude that for every \( t \in \mathbb{R}^{md} \),

\[
\lim_{n \to \infty} |\mu(r e^{it_n V_n}) - \mu(e^{it_n V_n})| = 0
\]

and the claim about the equivalence between the convergence of the finite-dimensional distributions follows from the Levi continuity theorem. The reduction to the case when \( r \) is only in the \( L^p(\mu) \)-closure of \( B \) relies on the inequalities

\[
|\mu(r e^{it_n V_n}) - \mu(s e^{it_n V_n})| \leq \|r - s\|_{L^p}
\]

and

\[
\int s(x)|S_n(x)| d\mu(x) \leq \int r(x)|S_n(x)| d\mu(x) + \|s - r\|_{L^p} \|S_n\|_{L^q} < \infty,
\]

where \( p \) and \( q \) come from the assumptions of the theorem.

\[ \square \]

**Remark 4.2.** When \( |\mathbb{E}[S_{n,\nu}] - \mathbb{E}[S_{n,\mu}]| = o(b_n) \) (see Section 3.1 for conditions insuring that), we also obtain the convergence of \( (S_{[nt],\nu} - \mathbb{E}[S_{[nt],\nu}])/b_n \) from the convergence of \( (S_{[nt],\mu} - \mathbb{E}[S_{[nt],\mu}])/b_n \).

**Remark 4.3.** Theorem 4.1 shows that the weak invariance principles which follow from the results in [10] and [12] hold true also when starting from the Lebesgue measure on the underlying manifold, and not only from the equivariant random measure \( \mu_\omega \).
4.2. The iterated weak invariance principle

In this section, we will discuss a version of Eagleson’s theorem for the iterated weak invariance principle. Since the latter is less known than the usual WIP, we will first describe the context in which it naturally arises.

Let \( d, e \in \mathbb{N}, a : \mathbb{R}^d \to \mathbb{R}^d \) be a function of class \( C^{1+} \) and \( b : \mathbb{R}^d \to \mathbb{R}^{d+e} \) be a function of class \( C^{2+} \). Consider the stochastic differential equation (SDE)

\[
dX = \left( a(X) + \frac{1}{2} \sum_{\alpha, \beta, \gamma} D^{\beta \gamma} \phi^\alpha b^\beta(X) b^{\alpha \gamma}(X) \right) dt + b(X) \circ dW, \quad X(0) = \xi, \tag{4.1}
\]

where \( W \) is a standard \( d \)-dimensional Brownian motion and \( \xi \) is a fixed vector in \( \mathbb{R}^d \). Here we sum over \( 1 \leq \alpha \leq d, 1 \leq \beta, \gamma \leq e \), and \( b^{\alpha \gamma} \) and \( b^\beta \) denote the \((\alpha, \gamma)\)-th entry and \( \beta \)-th column, respectively, of \( b \). In [19, Theorem 2.2], for suspension flows \( \{\phi_s : s \geq 0\} \) built over non-uniformly expanding or hyperbolic dynamical systems \((X, T, \mu)\), it was shown that for any centered function \( v : X \to \mathbb{R}^d \), the solution \( X_\alpha \) of the SDE

\[
dX_\alpha = a(X_\alpha) dt + b(X_\alpha) dW_\alpha, \quad X_\alpha(0) = \xi
\]

weakly converges in \( C([0, \infty), \mathbb{R}^d) \) toward the solution of (4.1) as \( n \to \infty \), where

\[
W_n(t) = \frac{1}{\sqrt{n}} \int_0^t v \circ \phi_s ds \quad \text{and} \quad dW_n = \frac{dW}{\sqrt{n}} dt.
\]

Next, let \( W_n(t) \) be the \( e \times e \)-dimensional process whose entries are given by

\[
W_n^{\beta \gamma}(t) = \int_0^t W_n^{\beta} dW_n^{\gamma}, \quad 1 \leq \beta, \gamma \leq e.
\]

A key ingredient in the proof of [19, Theorem 2.2] is the iterated WIP [19, Theorem 2.1] which states that \((W_n, W_n)\) weakly converges as \( n \to \infty \) toward a process with a certain structure. The proof of the latter was based on a discretization argument, where Kelly and Melbourne showed that it is enough to prove the convergence of \((S_n, S_n)\) toward the latter process, where

\[
S_n(t) = n^{-1/2} \sum_{j=0}^{[nt]-1} v \circ T^j
\]

and

\[
S_n^{\beta \gamma}(t) = \int_0^t S_n^{\beta} dS_n^{\gamma} = n^{-1} \sum_{0 \leq i < j \leq [nt]-1} v^\beta \circ T^i \cdot v^\gamma \circ T^j, \quad 1 \leq \beta, \gamma \leq e.
\]

In fact, the iterated WIP needed is with respect to the measure \( \nu = rd\mu \), where \( r \) is the underlying roof function defining the suspension flow, and not with respect to \( \mu \). In order to settle this, the authors of [19] used a version of Eagleson’s theorem from [27] which applies, in particular, to the iterated WIP.

A natural question arising here is whether the smooth approximation still holds for random suspension flows (see, for instance [20] for the definition), namely if we replace \( T \) with a random dynamical system (as describe in Example 2.2) and the roof function \( r \) with a random roof function. It seems to us that the main obstacle in such a generalization is proving a version of Eagleson’s theorem for the iterated WIP in the random dynamics setup. In what follows, we will provide such results in the more general non-stationary setup of this paper.

Let \((\mathcal{E}, \mathcal{F}, \mu)\) and \( X_0, X_1, \ldots \) be as specified in Section 2. Let \( d, m \in \mathbb{N} \). For each \( 1 \leq i \leq d \), let \( g_j^{(i)} : \mathcal{Y}_j \to \mathbb{R}^{d_i}, j \geq 0 \) be a sequence of vector-valued functions, where \( d_i \geq 1 \). Set

\[
S_n g^{(i)} = \sum_{j=0}^{n-1} g_j^{(i)} \circ X_j.
\]
Moreover, for each \( d + 1 \leq k \leq d + m \) let \( f_j^{k,u} \) and \( f_j^{k,v} \), \( j \geq 0 \) be sequences of real-valued functions on \( \mathcal{Y}_j \), where \( 1 \leq u \leq q_k \) and \( 1 \leq v \leq p_k \) and \( p_k, q_k \in \mathbb{N} \). Consider the matrix \( A_n f_k \) given by

\[
(A_n f_k)_{u,v} = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} f_i^{k,u} \circ X_i \cdot f_j^{k,v} \circ X_j = \sum_{i=0}^{n-1} f_i^{k,u} \circ X_i \cdot (S_n f_i^{k,v} - S_{i+1} f_i^{k,v}),
\]

We consider \( A_n f_k \) as an \( \mathbb{R}^{p_k \cdot q_k} \)-valued function. For each \( t \in \mathbb{R} \), consider the vector-valued function \( V_n(t) \) on \( \mathcal{E} \) given by \( V_n(t) = (L_n(t), R_n(t)) \), where

\[
L_n(t) = n^{-\frac{1}{2}} \{ S_{[nt]} g^{(1)}, \ldots, S_{[nt]} g^{(d)} \}
\]

and

\[
R_n(t) = n^{-1} (A_{[nt]} f_{d+1}, \ldots, A_{[nt]} f_{d+m}).
\]

Let \( r \) be a function in the \( L^{s_1} (\mu) \)-closure of \( B \cap L^{s_1} (\mu) \), for some \( s_1 \geq 1 \), where \( B \) comes from Assumption 2.1. Assume also that \( r \geq 0 \) and that \( \int r \, d\mu = 1 \). Let us introduce an additional (moment) assumption.

**Assumption 4.4.** There are \( p_1 \geq 1 \) so that \( s_1 \geq p_1^* = p_1/(p_1 - 1) \) and \( \varepsilon \in (0, 1) \) such that for every \( d + 1 \leq k \leq d + m \) and \( 1 \leq v \leq q_k \), we have

\[
\| S_n f_i^{k,v} \|_{L^{p_1^*}(\mu)} \leq C n^{1-\varepsilon},
\]

(4.2)

where \( C \) is some constant. Moreover,

\[
f_i^{k,u} \circ X_i \in L^{s_3} (\mu)
\]

for every \( i \geq 0 \), \( d + 1 \leq k \leq d + m \) and \( 1 \leq u \leq q_k \), where \( s_3 \) is some real number (where \( 1^* := \infty \)) and

\[
\frac{1}{s_3} = 1 - \frac{1}{s_1} - \frac{1}{p_1}.
\]

Here, we use the conventions \( \frac{1}{\infty} := 0 \) and \( \frac{1}{0} := \infty \).

The condition (4.2) holds true with \( \varepsilon = 1/2 \) in the random dynamics setup for appropriate classes of random non-uniformly expanding or hyperbolic maps (or for Markov chains in random dynamical environments [17, Chapter 6]).

Let \( \nu \) be the probability measure on \( \mathcal{E} \) given by \( d\nu = r \, d\mu \). Our main result here is the following theorem.

**Theorem 4.5.** Under the Assumptions 2.1 and 4.4, if the continuous time process \( V_n(\cdot) \) converges in distribution in the Skorokhod space with respect to the measure \( \mu \), then it also converges in distribution in the Skorokhod space with respect to the measure \( \nu \) (and to the same limit). Finally, if \( r > 0 \) (\( \mu \)-a.s.), then the convergence with respect to \( \mu \) can be derived from the convergence with respect to \( \nu \).

**Proof of Theorem 4.5.** First if \( V_n(\cdot) \) converges in distribution with respect to \( \mu \), then \( \{ V_n \} \) is a tight family. Arguing exactly as in the proof of Theorem 4.1, we obtain that it is also a tight family with respect to the measure \( \nu \). Therefore, it remains to show that the finite-dimensional distributions converge.

Let \( t_1, \ldots, t_p \) be positive real numbers and set

\[
Q_n = (V_n(t_1), V_n(t_2), \ldots, V_n(t_p)).
\]

We first need the following elementary result.
Lemma 4.6. For any two sequences \((c_n)\) and \((q_n)\) of real numbers such that \(\lim_{n \to \infty} q_n = \infty\), there exists a (weakly increasing) sequence \((b_n)\) of natural numbers which tends to \(\infty\) as \(n \to \infty\) so that for any other sequence \((a_n)\) of natural numbers which tends to \(\infty\) and satisfies \(a_n \leq b_n\), we have \(c_{a_n} = o(q_n)\).

Remark 4.7. We now observe that for two pairs of sequences \((c_n)\) and \((q_n)\) as in Lemma 4.6, we can choose a sequence \((b_n)\) compatible with both of those pairs. More precisely, take two pairs of sequences \((c_n), (q_n)\) and \((c'_n), (q'_n)\) such that \(\lim_{n \to \infty} q_n = \lim_{n \to \infty} q'_n = \infty\). Let \((b_n)\) be a sequence given by Lemma 4.6 for the pair \((c_n), (q_n)\). Furthermore, let \((b'_n)\) be a sequence given by Lemma 4.6 for the pair \((c'_n), (q'_n)\). Set \(b''_n := \min\{b_n, b'_n\}, n \in \mathbb{N}\). Then, \((b''_n)\) is weakly increasing and for any sequence of natural numbers \((a_n)\) such that \(a_n \leq b''_n\) and \(\lim_{n \to \infty} a_n = \infty\), we have that \(c_{a_n} = o(q_n)\) and \(c'_{a_n} = o(q'_n)\).

Corollary 4.8. Under Assumptions 2.1 and 4.4, there exists a sequence \((b_n)\) which tends to \(\infty\) as \(n \to \infty\) such that for any other sequence \((a_n)\) of natural numbers such that \(a_n \leq b_n\) for all \(n\) and \(\lim_{n \to \infty} a_n = \infty\), for any relevant \(i, k, u, v\), we have that

\[
\lim_{n \to \infty} \delta_{1,i}(n) = \lim_{n \to \infty} \delta_{2,i}(n) = \lim_{n \to \infty} \delta_{1,k,u,v}(n) = \lim_{n \to \infty} \delta_{2,k,u,v}(n) = 0,
\]

where

\[
\delta_{1,i}(n) = n^{-1/2} \mu(|S_{a_n}g^{(i)}|), \quad \delta_{2,i}(n) = n^{-1/2} \mu(|r \cdot S_{a_n}g^{(i)}|),
\]

\[
\delta_{1,k,u,v}(n) = n^{-1} \mu\left(\sum_{i=0}^{a_n-1} f_{i}^{k,u} \circ X_{i} \cdot (S_{n} f_{i}^{k,v} - S_{i+1} f_{i+1}^{k,v})\right),
\]

and

\[
\delta_{2,k,u,v}(n) = n^{-1} \mu\left(r \cdot \sum_{i=0}^{a_n-1} f_{i}^{k,u} \circ X_{i} \cdot (S_{n} f_{i}^{k,v} - S_{i+1} f_{i+1}^{k,v})\right).
\]

Proof. First, by (4.2) for all \(0 \leq i < n\), we have that

\[
\|S_{n} f_{i}^{k,v} - S_{i} f_{i}^{k,v}\|_{L^{p_1}(\mu)} \leq 2C n^{1-\varepsilon}.
\]

Standard applications of the Hölder inequality yield that for \(l = 1, 2\),

\[
\delta_{l,k,u,v}(n) \leq n^{-1} \sum_{i=0}^{a_n-1} \|f_{i}^{k,u} \circ X_{i}\|_{L^{s_2}(\mu)} 2C_{l} C n^{1-\varepsilon} = 2C_{l} C n^{-\varepsilon} c_{k,u,a_n}
\]

where \(C_{1} = 1, C_{2} = \|r\|_{L^{s_1}(\mu)}\) and

\[
c_{k,u,a_n} = \sum_{i=0}^{a_n-1} \|f_{i}^{k,u} \circ X_{i}\|_{L^{s_3}(\mu)}.
\]

Here \(s_1\) and \(s_3\) come from Assumption 4.4. By applying Lemma 4.6 with

\[
c_{n} = \max_{i} \left(\mu(|S_{n}g^{(i)}|) + \mu(|r S_{n}g^{(i)}|)\right) \quad \text{and} \quad q_{n} = n^{1/2}
\]

and then with

\[
c_{n} = \max_{k,u} c_{k,u,n}, \quad \text{and} \quad q_{n} = n^{\varepsilon},
\]

we complete the proof of the corollary, taking into account Remark 4.7.

\[\square\]

In order to complete the proof of Theorem 4.5, we need the following result.
PROPOSITION 4.9. Suppose that Assumptions 2.1 and 4.4 hold true. Then the sequence of random vector-valued variables $Q_n$ converges in distribution with respect to $\mu$ if and only if it converges in distribution with respect to $\nu$ (and in the latter case the limiting distributions are equal).

Proof. Let $D$ denote the dimension of the range of the (random) functions $Q_n$. By the Levi continuity theorem, in order to prove the proposition it is enough to show that for all $s \in \mathbb{R}^D$ we have

$$\lim_{n \to \infty} |\mu(re^{isQ_n}) - \mu(e^{isQ_n})| = 0.$$  \hfill (4.3)

Since we can approximate $r$ in $L^{\infty}(\mu)$ by non-negative functions $s \in B \cap L^{\infty}(\mu)$ satisfying $\mu(s) = 1$ and since

$$|\mu(re^{isQ_n}) - \mu(se^{isQ_n})| \leq \|r - s\|_{L^{\infty}(\mu)},$$

we conclude that it is enough to prove (4.3) in the case when $r \in B \cap L^{\infty}(\mu)$. Note that Assumption 4.4 is left unchanged after such a reduction.

Let $(a_n)$ be a sequence such that the conclusion of Corollary 4.8 holds. It is clear that we can assume without loss of generality that $a_n < n$. For $1 \leq i \leq d$ and $1 \leq \rho < n$, set

$$G_{n,\rho}g^{(i)} = S_n g^{(i)} - S_{\rho} g^{(i)}$$

which is a function of the variable $\tilde{X}_\rho$. Then for all $1 \leq \ell \leq p$, we have

$$S_{[nt]\ell} g^{(i)} = S_{a_{[nt]\ell}} g^{(i)} + G_{[nt]\ell, a_{[nt]\ell}} g^{(i)}.$$

Next, for every $d + 1 \leq k \leq m$ and $1 \leq \rho < n$, set

$$(G_{n,\rho} f_k)_{u,v} = \sum_{i=0}^{n-1} f_{i}^{k,u} \circ \tilde{X}_i \cdot (S_n f^{k,v} - S_{i+1} f^{k,v}).$$

Then $G_{n,\rho} f_k$ is a function of $\tilde{X}_\rho$ and for every $1 \leq \ell \leq p$, we have

$$A_{[nt]\ell} f_k = \sum_{i=0}^{a_{[nt]\ell} - 1} f_{i}^{k,u} \circ \tilde{X}_i \cdot (S_{[nt]\ell} f^{k,v} - S_{i+1} f^{k,v}) + G_{[nt]\ell, a_{[nt]\ell}} f_k.$$
as \( n \to \infty \). Finally, using Assumption 2.1, taking into account that \( \mu(r) = 1 \) and that the function \( H = e^{i \tilde{Q}_n} \) is a bounded function of the variable \( \mathcal{X}_{\alpha_n} \), where \( p_n = p_n(t) = \min\{n t_1, \ldots, n t_p\} \), we have

\[ |\mu(e^{i \tilde{Q}_n}) - \mu(e^{i \tilde{Q}_n})| = O(\alpha_n), \]

and thus, as \( \lim_{n \to \infty} a_n = \infty \),

\[ |\mu(e^{i \tilde{Q}_n} r) - \mu(e^{i \tilde{Q}_n})| \to 0, \]

when \( n \to \infty \). It remains to observe that (4.3) follows from the last three assertions. \( \square \)

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Yeor Hafouta
Department of Mathematics
The Ohio State University 100 Math Tower
231 W 18th Ave
Columbus, OH 43210
USA
yedor.hafouta@mail.huji.ac.il
hafuta.1@osu.edu