On Competitive Analysis for Polling Systems (Extended Version)

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Abstract

Polling systems have been widely studied, however most of these studies focus on polling systems with renewal processes for arrivals and random variables for service times. There is a need driven by practical applications to study polling systems with arbitrary arrivals (not restricted to time-varying or in batches) and revealed service time upon a job’s arrival. To address that need, our work considers a polling system with generic setting and for the first time provides the worst case analysis for online scheduling policies in this system. We provide conditions for the existence of constant competitive ratio for this system, and also the competitive ratios for several well-studied policies such as cyclic exhaustive, gated and $l$-limited policies, Stochastic Largest Queue policy, One Machine policy and Gittins Index policy for polling systems. We show that any policy with (1) purely static, (2) queue-length based or (3) job-processing-time based routing discipline does not have a competitive ratio smaller than $k$, where $k$ is the number of queues. Finally, a mixed strategy is provided for a practical scenario where setup time is large but bounded.

Keywords— Scheduling, Online Algorithm, Competitive Ratio, Parallel Queues with Setup Times

1 Introduction

This study has been motivated by operations in smart manufacturing systems. As an illustration, consider a 3D printing machine that uses a particular material informally called ‘ink’ to print. Jobs of the same prototype are printed using the same ink, and when a different prototype (for simplicity, say a different color) is to be printed, a different ink is required and the machine undergoes a setup that takes time to switch inks. The unprocessed jobs of the same prototype can be regarded as a ‘queue’. This problem can thus be modeled as a polling system where the server polls a queue, processes jobs, incurs a switch over time, processes another queue, and so on. In practice, besides the ink (material), other factors such as processing temperature, equipment setting and other processing requirements that require by different job prototypes will also incur setup times.

Another interesting feature of 3D printing is that it is possible to reveal the workload of each job (i.e., processing time) upon the job’s arrival. This is because before getting printed, the printing requirements such as temperature, nozzle route, printing speed, and so on are specified for the job, and using that we can easily acquire the printing time before processing. Therefore, it is unnecessary to assume that the workload of a job is stochastic at the start of processing, even though many other queueing research works do so. In this paper, we assume that the workload of jobs could be arbitrary, and could be revealed deterministically.

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upon arrival. Furthermore, the 3D printer that prints customized parts usually receives jobs with different processing requirements. Job arrivals thus could be time-varying, non-renewal, in batches, dependent, or even arrivals without a pattern. It motivates us to consider the generic polling system without imposing any stochastic assumptions on future arrivals.

The above is actually an example of the general polling system. In such a system, job arrivals are arbitrary, workload of each job is revealed deterministically upon arrival, and a setup time occurs when the server switches from one queue to another. We call such a polling system ‘general’ mainly because we do not impose any stochastic assumptions on job arrivals or service times. Polling system was firstly introduced in 1970’s and it is one of the most important system in queueing literature due to its wide application. There are practical needs for studying the polling system with general settings as in many cases, job processing requirements and arrivals are unpredictable and could be in any pattern. Besides 3D printing, many other examples of such a general polling system can be found in computer-communication systems, reconfigurable smart manufacturing systems and smart traffic systems, where arrivals could be arbitrary and service time gets revealed upon job’s arrival. Having a scheduling policy that works well in such general settings could prevent the system from performing erratically when rare events occur. Knowing the worst case performance of a policy also aids in designing reliable systems. There are also needs from theory for studying the polling system with general settings. Most of the previous works model the polling system from a stochastic standpoint by assuming certain distributions for arrival, service and switching processes. As a consequence, these works are limited in describing the average performance of certain polling systems that rely on specific stochastic assumptions. There are very few studies discussing the optimal policies or online scheduling algorithms for the polling system without stochastic assumptions due to the complexity of analysis. It is still unknown if those scheduling policies designed for specific polling systems work well in the general settings. In our paper, we study the polling system from an online optimization perspective and perform worst case analysis for the generic polling system that does not rely on any stochastic assumptions for arrival, service or switching processes. More specifically, we consider a completion time minimization problem in such a polling system and obtain the worst case performance (i.e., competitive ratios) of several widely used scheduling policies with known average performance, such as Gittins Index Policy, Stochastic Largest Queue (SLQ) as well as cyclic exhaustive and gated policies. Our work also bridges the scheduling and queueing communities by showing that some queueing policies that work well under stochastic assumptions also work well in the general scheduling settings. In this work we, for the first time, provide conditions for existence of constant competitive ratios for online policies in polling systems. Moreover, we suggest a mixed strategy in this paper that works well under practical scenarios.

1.1 Problem Statement

As mentioned earlier, the unprocessed jobs from the same prototype (or family) can be modeled as a queue. We thus consider a single server system with \( k \) parallel queues. Jobs arrive at each queue will wait until being served by the server. Figure 1.1 shows a polling system with \( k = 4 \) queues. The processing time (workload) \( p_i \) of the \( i \)th arriving job is revealed instantly upon its arrival time \( r_i \). The server can serve the jobs that are waiting in queues in any order non-preemptively. However, a setup time \( \tau \) is incurred when the server switches from one queue to another. We assume that once the setup is in progress, it can be interrupted but it is not resumable. Next time when the server switches to this queue, a setup time \( \tau \) is still needed. Information \( (r_j, p_j) \) about a future job \( j \) (for all \( j \)) remains unknown to the server until job \( j \) arrives.
in the system. The objective is to find service order for jobs and queues to minimize total completion time over all jobs, where the completion time of a job is the time period from time 0 to when the job has been served (exits the system). It is assumed that the number of jobs is an arbitrary large finite quantity. We wish to also state that previous studies of polling systems usually assume that \((r_j, p_j)\) follows certain random distributions (see [47]), but in our paper we study polling systems from an online scheduling perspective and allow polling systems to have arbitrary arrival times and processing times.

1.2 Preliminaries

In this subsection we mainly introduce some concepts and terminologies that we use in this paper. Important notations in this paper are provided in Table 1.

Machine Scheduling Problems

The polling system scheduling problem belongs to the class of one machine online scheduling problems since there is only one server (machine or scheduler) in the system. Using the notation for one machine scheduling problems [21], we write our problem as \(1 \mid r_i, \tau \mid \sum C_i\), where ‘1’ refers to the single machine in the system, \(r_i\) and \(\tau\) in the middle part of the notation refer to the release date and setup time constraints, and \(\sum C_i\) means that the objective of this scheduling problem is to minimize the total completion time. This type of notation is widely used to briefly describe the input, constraints and objective for any scheduling problem. The notation is of the type \(P_m \mid \text{constraints} \mid \text{objective}\). The first column in the notation denotes the number of machines in the system. For example, it could be 1 for one machine or \(P_m\) for \(m\) parallel machines. The middle column denotes the constraints. In the later part of this paper we would introduce other constraints such as \(\tau \leq \theta p_{\min}\) and \(p_{\max} \leq \gamma p_{\min}\), where \(p_{\max} = \sup_i p_i\) is the upper bounded workload and \(p_{\min} = \inf_i p_i\) is the lower bound workload. We say the workload variation is bounded if \(p_{\max} \leq \gamma p_{\min}\) for constant \(\gamma\). If no constraint is specified in this column, it means 1) all jobs are available at time zero, 2) no precedence constraints are imposed, 3) preemption is not allowed, and 4) no setup time exists. In this paper, we assume jobs and setup times are non-resumable and all the policies that we discuss are non-preemptive, unless we specifically point out. The last column of the notation represents the objective for the scheduling problem (default to be minimization problems). In this paper it is \(\sum C_i\), which means minimizing the total completion time. In other papers it could be \(C_{\max}\) [28], which means minimizing the maximal completion
time (makespan), or $\sum w_i C_i$ which is minimizing the weighted completion time \[22\], where $w_i$ is the weight for job $i$.

**Online and Offline Problems**

A job instance $I$ with $n(I)$ number of jobs is defined as a sequence of jobs with certain arrival times and processing times, i.e., $I = \{(r_i, p_i), 1 \leq i \leq n(I)\}$. In this work we mainly focus on an online scheduling problem, where online means $r_i$ and $p_i$ remain unknown to the server until job $i$ arrives. In contrast to the online problem, the offline problem has entire information for the job instance $I$, i.e., $(r_1, r_2, ..., r_{n(I)})$ and $(p_1, p_2, ..., p_{n(I)})$ from time 0. The offline problem is usually of great complexity. The offline version of the online problem that we want to solve, i.e., $1|r_i, \tau| \sum C_i$, is strongly NP-hard since the offline problem with $\tau = 0$ is strongly NP-hard \[27\] \[22\] \[25\]. However it is important to note that if preemption is allowed, then serving the *shortest remaining processing time* (SRPT) is the optimal policy for $1|r_i, \text{pmt}\text{n}| \sum C_i$ with $\tau = 0$ (see \[45\]). SRPT is online and polynomial-time solvable. It can be used as a benchmark for online scheduling policies to compare against.

**Scheduling Policies**

A scheduling policy $\pi$ specifies when the server should serve which job. In our paper we mainly focus on online policies. Online policies provide feasible solutions for online problems, without knowing the information of future jobs. Online policies are also called non-anticipative policies in some literatures \[56\] \[8\]. Online policies can be either deterministic or randomized. There is only one unique solution if a job instance is given to a deterministic policy. For example, SRPT is a deterministic policy. A randomized policy may toss a coin before making decision, and the decision may depend on the outcome of this coin toss \[8\]. This means for a random policy with a certain internal random variable, its solution may depend on the outcome of this internal random variable. If the same job instance is given to a randomized policy multiple times, each time the solution may be different from the others due to different outcomes by the internal random variable. Detailed discussions for randomized algorithms can be found in \[46\] \[11\]. In this paper, we mainly focus our discussion on deterministic policies.

**Competitive Ratios**

Competitive ratio is a ratio between the solution obtained by an online policy and the *benchmark*. In this paper, the optimal solution to the offline problem is the benchmark that we compare the online policy against. Thus we say the competitive ratio for an online policy is $\rho$ if $\sup_I \frac{C^\pi(I)}{C^*(I)} \leq \rho$ for any job instance $I$, where $C^\pi(I)$ is the completion time for job instance $I$ by a deterministic scheduling policy $\pi$ and $C^*(I)$ is the optimal completion time of the offline problem. We say a competitive ratio is tight if there exists an instance $I$ such that $\frac{C^\pi(I)}{C^*(I)} = \rho$.

**1.3 Related Works**

Since in this paper we analyze the polling system from a scheduling perspective, there are many research works in scheduling which are related to our work. The single machine scheduling problems that consider setup times or costs have been widely studied from various perspectives. A detailed review of the literature
Table 1: Notations

| Notations | Meaning | Notations | Meaning |
|-----------|---------|-----------|---------|
| $r_i$     | Release date, the time when job $i$ arrives in the system | $p_i$ | Workload (processing time) for job $i$ |
| $p_{max}$ | Maximum workload, $p_{max} = \max \{p_i\}$ | $p_{min}$ | Minimum workload, $p_{min} = \min \{p_i\}$ |
| $\tau$   | Setup time is $\tau$ for all queues | $I = \{r_i, p_i\}$, for $1 \leq i \leq n(I)$ | Job instance, set of jobs with information about release date and processing time for all jobs in it |
| $C_\pi^*(I)$ | Total completion time for jobs in instance $I$ under policy $\pi$ | $C^*(I)$ | Total completion time for jobs in instance $I$ under the optimal policy of the offline problem |
| $C_\pi^i$ | The completion time for job $i$ under policy $\pi$ | $n(I)$ | Number of jobs in job instance $I$ |
| $\gamma$ | Workload variation, a constant such that $p_{max} \leq \gamma p_{min}$ | $\theta$ | A constant such that $\tau \leq \theta p_{min}$ |
| $k$       | Number of queues in the system | $\kappa = \max \{ \frac{2}{\gamma}, k + 1 \}$ | Competitive ratio for cyclic-based and exhaustive-like policies, a constant |

Table 2: Competitive Ratios for Single Machine Scheduling Problem without Setup Times (i.e., $\tau = 0$)

| Problem | Deterministic | Randomized |
|---------|---------------|------------|
|         | Lower Bounds  | Upper Bounds | Lower Bounds  | Upper Bounds |
| $1|r_i, p_{\text{ptmn}}|\sum C_i$ | 1 | 1.45 | 1 | 1.45 |
| $1|r_i, p_{\text{ptmn}}|\sum w_i C_i$ | $1.073$ [15] | $1.566$ [44] | $1.038$ [15] | $\frac{7}{3}$ [42] |
| $1|r_i|\sum C_i$ | 2 [24] | 2 [24, 31, 38] | $\frac{\gamma}{\gamma - 1}$ [49] | $\frac{\gamma}{\gamma - 1}$ [11] |
| $1|r_i|\sum w_i C_i$ | 2 [24] | 2 [24] | $\frac{\gamma}{\gamma - 1}$ [49] | $\frac{\gamma}{\gamma - 1}$ [11] |
| $1|r_i, p_{\text{max}} \leq \gamma|\sum w_i C_i$ | $1 + \sqrt{\frac{4\gamma^2 + 1 - 1}{2}}$ [49] | $1 + \sqrt{\frac{4\gamma^2 + 1 - 1}{2}}$ [49] | Unknown | Unknown |
| $1|r_i, p_{\text{min}} \leq \gamma|\sum C_i$ | Numerical $[48]$ | $1 + \frac{\gamma}{1 + \sqrt{1 + \gamma (\gamma - 1)}}$ [48] | Unknown | Unknown |

can be found in [5, 3, 2, 4, 40]. Other works considering machine setup can be found in [51, 23, 37, 36, 43]. However, almost all of these works are focused on solving the offline problem where all the release times and processing times are revealed at time 0.

In this paper we are more interested in the online problem where the current state of the system is known and future is uncertain. Thus our work also falls into the class of the online single machine scheduling problem. Numerous research papers have shed light on the online single machine scheduling problem, without considering setup times. Table 2 summarizes the current state of art of competitive ratio analysis over existing online algorithms for single machine scheduling problem without setup times. Besides all the algorithms provided in Table 2, a recent work [32] provides a new method to approximate the competitive ratio for general online algorithms. However, since these works mainly focus on solving problem without setup times, the online policies provided in these works are not directly applicable to polling systems.

Only few articles so far have focused on the online single machine scheduling problem with queue setup times, i.e., the polling system. The online makespan minimization problem for the polling system is considered in [13] and an $O(1)$ algorithm is proved to exist. However the competitive ratio provided in [13] is very large.

A 3-competitive online algorithm for the polling system with minimizing the completion time is provided.
in [58], but only for the case where \( k = 2 \) and jobs are identical. To the best of our knowledge, the online algorithms for general polling systems with setup time consideration have not been well studied.

As we mentioned before, there are articles that study polling systems from a stochastic perspective by assuming job arrivals, service times and setup times are stochastic. Average performance of policies is considered for different types of stochastic assumptions [54]. Exact mean value analysis for cyclic routing policies with exhaustive, gated and limited service disciplines for polling system of \( M/G/1 \) type queues have been provided in [16, 17, 41, 57, 53, 17]. Service disciplines within queues (such as FCFS, SRPT and others) are discussed in [59], however the routing discipline (choose which queue to serve) and optimal service discipline between queues are not discussed in [59]. Optimal service policy for the symmetric polling system is provided by [59], where ‘symmetric’ means that jobs arrive evenly into each queue and jobs are stochastically identical. Most of the works from this stochastic perspective are restricted to average performance analysis. The structure of the optimal solution to the general polling system yet remains unknown [10, 54]. Approximating algorithms for the polling system are very few, and none of those popular policies with proven average performance have been showed the adaptivity in general settings.

In summary, so far there is no work which provides optimal or approximating scheduling policies for the polling system with a general setting and the objective of minimizing the total completion times. In addition, the conditions under which the polling system allows constant competitive ratios for online algorithms are also unknown. Besides, the competitive ratios for widely-used policies in polling systems (such as cyclic policies with exhausted and gated service discipline) remain unknown. The contribution of this paper is fourfold: (1) Our work for the first time analyzes polling systems without stochastic assumptions, evaluates policy performance by competitive ratio, provides the conditions for existence of competitive ratio in polling systems, and proves competitive ratios for some well-studied policies such as cyclic exhaustive and gated policy. (2) Our work bridges the queueing and scheduling communities by showing that some widely-used queueing policies also have decent performance in terms of worst case performance in online scheduling problems. (3) We provide a lower bound for the competitive ratio for all the possible online policies, and show that no online algorithm can have a competitive ratio smaller than this lower bound. (4) We also provide new online policies that balance future uncertainty and utilize known information, which may open up a new research direction that would benefit from revealing information and reducing variability. This paper is organized as follows: in Section 2 we consider the case of bounded workload variation and obtain competitive ratios for various policies; in Section 3 we consider the case with unbounded workload variation; in Section 4 we provide a mixed strategy to deal with a practical scenario; we make concluding remarks in Section 5.

## 2 Polling System with Bounded Workload Variation

From [15, 12] we know that in the single machine problem (no setup times) with minimizing completion times, i.e., \( 1|r_i| \sum C_i \), it is always beneficial to schedule jobs preemptively with the smallest remaining processing time (SRPT) first. The reason is that by doing this, small jobs are quickly processed thus the number of waiting jobs is reduced. This idea of scheduling small workload first can be used in polling systems as well. An efficient scheduling policy may avoid the case where a large number of jobs are waiting in the queue while a single large job is in process. It should also avoid the case when a large number of small jobs are waiting in a queue, but the server is busy serving other queues and not able to switch to this queue immediately. Thereby, the main idea of designing a scheduling policy is to avoid either of these extreme cases happening.
A good online policy should balance job priority and queue priority so that small jobs are processed in a timely manner and switching does not happen frequently. The priority of a job is usually determined by the job information, such as processing time and release date. In contrast, queue priority is usually determined by the queue information such as the number of waiting jobs, the sum of remaining workload in a queue, and so on. If the variation of processing time is large and setup times are trivial, then perhaps the server should favor the job priority more than the queue priority since a large job in process may cause a much larger delay than switching. In the case where variation of processing times is small, the server may need to favor the queue priority. We leave the discussion for unbounded variation to Section 3 and in this section we mainly discuss the case where workload variation is bounded (i.e., \( p_{\text{max}} \leq \gamma p_{\text{min}} \) for some constant \( \gamma \); recall that \( p_{\text{max}} = \sup_i p_i \) and \( p_{\text{min}} = \inf_i p_i \)). The case of bounded workload variation, in the 3D printing example that we introduced earlier in this paper, corresponds to the scenario in which jobs have similar printing requirements, i.e., jobs have similar processing times.

In this work, we suppose the decision of when to switch out from a queue and in which order the jobs get served is determined by the service discipline of a policy. The decision of which queue to serve next is determined by the routing discipline. Notice that once the variation of processing times is small, queue priority should be favored. We shall show later in this section that when workload variation is small, service discipline does not impact the policy performance greatly, as long as it follows an exhaustive-like pattern. Thereby in this section, we mainly focus our discussion on the routing discipline. There are many widely used routing disciplines and here we classify them into two types. One is called static routing discipline, which means the server always follows a fixed routing order (also called a general routing table), such as the policy provided in [7]. The other is called dynamic routing discipline, which means the server follows a dynamic routing policy which may use any available information as the processing goes on, such as Stochastic Largest Queue (SLQ) provided in [30] or selecting queues randomly such as the random routing discipline provided in [26]. For static routing disciplines, we focus our discussion on the cyclic routing policy (also called periodic), and later in this section we shall show that cyclic routing is the optimal static routing discipline in terms of the worst case performance. We will also show that random routing discipline is generally no better than static ones in terms of the worst case performance, which may be surprising and non-intuitive at beginning but it indeed reflects the underlying difference between average performance with worse case performance. Since in this section we only consider the case where workload variation is bounded, we assume the maximal processing time (across all queues) for an arbitrary job is bounded by a constant ratio of minimal possible processing time across all queues, i.e., \( p_{\text{max}} \leq \gamma p_{\text{min}} \) for some constant \( \gamma \). And we assume the setup time for all queues is fixed as \( \tau \). So our problem is denoted as \( 1 \mid r_i, \tau, p_{\text{max}} \leq \gamma p_{\text{min}} \mid \sum C_i \) where \( p_{\text{max}} \leq \gamma p_{\text{min}} \) is a constraint imposed. Unlike in [18] where \( p_{\text{min}} \) is assumed to be non-zero, here we also allow \( p_{\text{min}} = p_{\text{max}} = 0 \), for which we assume \( \gamma = 1 \).

### 2.1 Cyclic Based and Exhaustive-like Policies

In this subsection we mainly discuss policies that use cyclic routing discipline and serve each queue in an exhaustive manner. A queue is called ‘exhausted’ if there is no job left in the queue by the time the server switches out. These policies form a policy set \( \Pi_r \) which we will define later. We begin our discussion with a set of policies \( \Pi_r \) called Cyclic Exhaustive Policies with Skipping Empty Queues whose description is shown in Algorithm 1. This set contains all the policies that 1) serves queues in a cyclic way (from queue 1 to \( k \) and then again from 1 to \( k \) and so on), 2) serves each queue exhaustively, and 3) waits in the queue for a certain
Algorithm 1 Cyclic Exhaustive Policies with Skipping Empty Queues $\Pi_1$

**Require:** Instance $I$

1: server = 1; $w = 1$

2: while $I$ has not been fully processed do
3:   Process all jobs that are present in queue$_{server}$ during server's $w^{th}$ visit to queue$_{server}$ (the total number of jobs is denoted as $n_{w,server}^w$), where the length of each visit period is at most $n_{w,server}^w p_{max}$
4:   if queue$_{server} < j \leq k$ is not empty then
5:     server = server + $\min\{j\}$
6:   else if queue$_1 \leq j \leq server - 1$ is not empty then
7:     server = $\min\{j\}$; $w = w + 1$
8:   else
9:     server = $l$ if next arrival occurs at queue $l$; $w = 1$
10: end if
11: end while
12: return Total completion time $C_e(I)$

amount of time before switching out. Once the server decides to switch out, it selects the next non-empty queue to switch to. If all queues are empty, the server will idle at the last queue that it served. Note that for an arbitrary policy $\pi \in \Pi_1$, during each visit to queue $i$, the server has to serve all the available jobs in queue $i$ before switching out. After serving all the jobs in queue $i$, the server is allowed to wait in queue $i$ for some extra time to receive more arrivals. If the server processes $n_i^w$ jobs in its $w^{th}$ visit to queue $i$, it can stay in queue $i$ for at most $n_i^w p_{max}$ amount of time in this visit. If a new arrival occurs at queue $i$ during the time that the server is waiting, then $n_i^w \leftarrow n_i^w + 1$ and the server can process this job at any time before it switches out, as long as the server does not stay in the queue for time longer than the updated $n_i^w p_{max}$.

Note that if during $w^{th}$ visit to queue $i$, all the jobs have workload $p_{max}$, then the server will switch out when a queue is exhausted. Also note that we do not specify the service order of jobs for policies in $\Pi_1$. As long as a policy satisfies the description of Algorithm 1, it belongs to $\Pi_1$. The next theorem provides the competitive ratio for policies in $\Pi_1$. Since the proofs of this and some subsequent theorems are lengthy, they are provided in the appendices.

**Theorem 1.** Any policy $e \in \Pi_1$ has competitive ratio of $\kappa = \max\{\frac{3}{2} \gamma, k+1\}$ for the polling system $1 | r_i, \tau, p_{max} \leq \gamma p_{min} | \sum C_i$. When $\frac{3}{2} \gamma \leq k+1$, for arbitrary $\epsilon > 0$, there is an instance $I$ such that $\frac{C_e(I)}{C^*(I)} > \kappa - \epsilon$.

**Proof.** The main idea of the proof is to consider each ‘batch’ under the online policy, which are the jobs served by the server during each visit to a queue. The detail of the proof is in Appendix A.

Notice that if $p_i = 1$ for all jobs $i$ and $k = 2$, then $\kappa = 3$, which is the result shown in [58]. However Theorem 1 implies a stronger result since we allow $k$ to be general ($k \geq 2$) and $p_i$’s to be different. Next we show a set of policies $\Pi_2$ which keep switching even when the system is empty, called Cyclic Exhaustive Policies without Skipping Empty Queues and shown as Algorithm 2. The only difference between $\Pi_1$ and $\Pi_2$ is under any policy from $\Pi_2$, the server sets up a queue no matter if it is empty or not. Similar to policies in $\Pi_1$, policies from $\Pi_2$ allow the server to stay in the queue $i$ for no more than $n_i^w p_{max}$ amount of time for its $w^{th}$ visit, so waiting is also allowed. The policy without waiting (just exhaustively serving) belongs to $\Pi_2$ and it has
Algorithm 2 Cyclic Exhaustive policy without Skipping Empty Queues Π₂

Require: Instance I
1: server = 1; w = 1
2: while I has not been fully processed do
3:   Process all jobs that are present in queue_{server} during server’s \( w^{th} \) visit to queue_{server} (the total number of jobs is denoted as \( n_w^{server} \), where the length of each visit period is at most \( n_w^{server} p_{max} \))
4:   server = Rem(server, k) + 1
5: if server == 1 then
6:   w = w + 1
7: end if
8: end while
9: return Total completion time \( C^e(I) \)

provable average performance for \( M/G/1 \) type queues \([16, 47, 41, 57]\). Here we show the competitive ratio for the policies in \( \Pi_2 \) is also \( \kappa \).

**Theorem 2.** Any policy \( e \in \Pi_2 \) has competitive ratio of \( \kappa = \max\{\frac{3}{2} \gamma, k + 1\} \) for the polling system \( 1 | r_i, \tau, p_{max} \leq \gamma p_{min} | \sum C_i \). When \( \frac{3}{2} \gamma \leq k + 1 \), for arbitrary \( \epsilon > 0 \), there is an instance \( I \) such that \( \frac{C_e(I)}{C^*(I)} > \kappa - \epsilon \).

**Proof.** The proof of Theorem 2 is similar to the one for Theorem 1. A detailed proof is provided in Appendix C.

Notice that the policies in \( \Pi_1 \) and \( \Pi_2 \) are different only in the way that the server deals with empty queues. All the policies in \( \Pi_1 \) and \( \Pi_2 \) use an exhaustive service discipline. Besides the exhaustive discipline, the gated discipline is often discussed because its average performance for \( M/G/1 \) type queues is also provable under cyclic routing discipline without skipping empty queues \([47, 57]\). Under the gated discipline, the server only serves the jobs that are in the queue before the server sets up the queue, and jobs that arrive after that time would be left to the next cycle of service. We do not specify the service order for the gated discipline either. We let \( \Pi_3 \) be the set of policies that follow cyclic routing discipline without skipping the empty queues and serve each queue with gated discipline. We also allow server to wait after clearing a queue under \( \Pi_3 \). Once the server has set up a queue, the number of jobs that are served during this visit is determined. This is different from policies from \( \Pi_1 \) or \( \Pi_2 \). Similarly, we can have a policy set \( \Pi_4 \) in which policies are cyclic and gated, but skipping empty queues when switching. We do not provide the detailed description of \( \Pi_4 \) here since it is similar to policy set \( \Pi_3 \). Policies in \( \Pi_3 \) and \( \Pi_4 \) also have competitive ratio \( \kappa \), as shown in the following theorem.

**Theorem 3.** Any policy \( e \in \Pi_3 \cup \Pi_4 \) has competitive ratio of \( \kappa = \max\{\frac{3}{2} \gamma, k + 1\} \) for the polling system \( 1 | r_i, \tau, p_{max} \leq \gamma p_{min} | \sum C_i \). When \( \frac{3}{2} \gamma \leq k + 1 \), for arbitrary \( \epsilon > 0 \), there is an instance \( I \) such that \( \frac{C_e(I)}{C^*(I)} > \kappa - \epsilon \).

**Proof.** The proof of Theorem 3 is also similar to the one for Theorem 1. A detailed proof is provided in Appendix C.

So far we have shown the competitive ratio for policies based on exhaustive and gated disciplines, with and without skipping empty queues. Notice that all the policies in \( \Pi_i, i = 1, ..., 4 \) use the cyclic routing discipline and have the same competitive ratio. We let these policies form a policy set \( \Pi_r \), i.e., \( \Pi_r = \cup_{i=1}^{4} \Pi_i \). Again,
it is important to note that the service order of jobs during the server’s visit to a queue is not specified by \( \Pi_r \). The service order could be First Come First Serve (FCFS), Shortest Processing Time First (SPT) or any other non-preemptive processing order, but all of them result in the same competitive ratio and the competitive ratio is approximately tight when \( \frac{3}{2} \gamma \leq k + 1 \). It indicates that although revealing processing times provides additional information for job scheduling, when the workload variation is small, the revealed information does not improve the worst case performance of an online policy. It is not a coincidence that all policies in \( \Pi_r \) have the same competitive ratio. All the policies in \( \Pi_r \) follow an exhaustive-like manner, even for the gated discipline. An arbitrary gated policies from \( \Pi_3 \) would exhaust all the jobs that arrive before the queue is set up, and if a large number of jobs arrive after the queue is set up, they will anyway be exhaustively processed in the next round. So we can see the gated discipline also has some ‘exhaustive’ characteristics. In fact, Theorem 4 in the following shows that exhaustive discipline is the optimal service discipline when \( p_{max} = p_{min} \).

**Theorem 4.** For the polling system \( 1|\tau, p_{min} = p_{max}| \sum C_i \), there always exists an exhaustive policy which outperforms a non-exhaustive policy in terms of total completion time.

**Proof.** The case where \( p_i = 0 \) is trivial. Now we let \( p_i = 1 \) with appropriate units. Since preemption is not allowed and all the jobs are identical, the server only needs to decide when to switch out and which queue to switch to. If there are no jobs in the queue that the server is currently serving, there are two options for the server: to continue serving the next job in this queue, or to switch to a nonempty queue and later come back to this queue again. If at time 0 the server is at queue 1 and there is an unfinished job in queue 1, then the server has to come back after it switches out. Suppose under a non-exhaustive policy \( \pi' \), the server chooses to switch to some queue(s) and come back to queue 1 at time \( T \). Say the server serves instance \( I' \) during this period. Suppose there is an adversary policy which has the same instance at time 0, and this policy chooses to serve one more job in queue 1, and then follows all decisions that policy \( \pi' \) has made (including waiting). Note that every decision policy \( \pi' \) made is available to the adversary policy because the adversary policy serves one more job before leaving queue 1. The total completion time under \( \pi' \) is \( C_{\pi'}' = 1 + T + C(I') \), and the completion time achieved by the adversary is \( C_{ad} = 1 + C(I') + n(I') \), where \( C(I') \) is the completion time of instance \( I \) served by policy \( \pi' \) during \( (0,T] \). Thus \( C_{ad} - C_{\pi'}' \leq n(I') - T \leq 0 \). Notice that the makespan of these two schemes are the same (including the final setup time of queue 1 for the adversary). The theorem is proved.

Recall that all the policies in \( \Pi_r \) use a static and cyclic routing discipline. Next we show in Theorem 5 the cyclic routing discipline results in the smallest competitive ratio among all the static routing disciplines.

**Theorem 5.** No online policy with a static or random routing discipline can guarantee a competitive ratio smaller than \( k \) for \( 1|\tau, p_{max} \leq \gamma p_{min}| \sum C_i \). Further, cyclic is the optimal static routing discipline for such a system. No online policy has a competitive ratio smaller than 2 for \( 1|\tau, p_{max} \leq \gamma p_{min}| \sum C_i \).

**Proof.** Since \( 1|\tau, p_i = 1| \sum C_i \) is a special case of \( 1|\tau, p_{max} \leq \gamma p_{min}| \sum C_i \), we only need to show the result holds for \( 1|\tau, p_i = 1| \sum C_i \). We first show the case for static routing policies. For an arbitrary policy that follows a static routing discipline, we suppose the server starts from queue 1, and queue \( k \) is the last one visited. Before the server visits queue \( k \), queue \( i \) (\( 1 \leq i \leq k - 1 \)) has been visited \( v_i \) times (suppose \( v(1) \leq v(2) \leq \ldots \leq v(k-1) \) is the ascending order for \( v_i \)'s). We construct a special job instance \( I \) by assuming that there is one job arriving at each queue \( i \) every time the server visits queue \( i \) for \( i = 1,\ldots,k-1 \). Also
we suppose there are \( n_k \) jobs at queue \( k \) at time 0 for a large \( n_k \), and say they form a batch \( b_k \). If we let
\[
g(b_k) = \frac{n_k(n_k+1)}{2},
\]
then we have
\[
C^\pi(I \cup b_k) \geq C^\pi(I) + n_k \tau \left( \sum_{i=1}^{k-1} v_i + 1 \right) + g(b_k),
\]
and
\[
C^*(I \cup b_k) \leq C^\pi(I) + n_k \tau + g(b_k) + n(I)(n_k + \tau),
\]
and \( \frac{C^\pi(I \cup b_k)}{C^*(I \cup b_k)} \geq \sum_{i=1}^{k-1} v_i + 1 \) if we let \( \tau = (n_k)^2 \) and \( n_k \to \infty \). Since for any static routing discipline we can construct a job instance like this, to achieve the smallest ratio we need \( v_i = 1 \) for \( i = 1, \ldots, k-1 \). Thus cyclic is the optimal static routing discipline. Notice a random routing policy always generates a routing table randomly. Thus the result holds for random routing policies as well.

The lower bound competitive ratio for 1\(|r_i, \tau, p_i = 1| \sum C_i\) is 2 as given in [58]. Since problem 1\(|r_i, \tau, p_i = 1| \sum C_i\) is a special case for 1\(|r_i, \tau, p_{max} \leq \gamma p_{min} \sum C_i|\) and 1\(|r_i, \tau| \sum C_i\), we know that no online algorithm can have a competitive ratio smaller than 2 for 1\(|r_i, \tau, p_{max} \leq \gamma p_{min} \sum C_i|\) or 1\(|r_i, \tau\sum C_i\). \(\square\)

So far we have shown that for problem 1\(|r_i, \tau, p_{max} \leq \gamma p_{min} \sum C_i|\), the policies in \( \Pi_r \) all have competitive ratio \( \kappa = \max\{\frac{3}{2} \gamma, k + 1\} \). From Theorem 3 we know that the smallest competitive ratio based on cyclic routing discipline is at least \( k \). The problem that whether there exists either a lower bound competitive ratio greater than \( k \) or an online policy whose competitive ratio is smaller than \( \kappa \) remains open. If we let \( \gamma \to \infty \), then \( \kappa \to \infty \). This means that the competitive ratio we provide goes to infinity in this case, but it does not mean that policies in \( \Pi_r \) all have infinite competitive ratios, since the competitive ratio is approximately tight only when \( \frac{3}{2} \gamma \leq k + 1 \) as shown in Theorems 2 and 3. However, Theorem 6 that we introduce next shows that policies in \( \Pi_r \) do not have constant competitive ratio if \( \gamma \) is infinite.

**Theorem 6.** Policies in \( \Pi_r \) do not guarantee constant competitive ratios for 1\(|r_i, \tau| \sum C_i|\).

**Proof.** We prove the theorem by giving a special job instance \( I \). We assume \( p_{min} = 0 \) and \( p_{max} = p \) so that \( \gamma = \infty \). Suppose at time 0 each of queue \( i = 2, \ldots, k \) has one job of processing time \( p \) and queue 1 has no job. At time \( \tau + \epsilon \) there are \( n \) jobs arriving at queue 1, with each of these \( n \) jobs has processing time 0. For any policy \( \pi \) in \( \Pi_r \), the server would either setup queue 1 at time 0 then switch to queue 2 at time \( \tau \), or setup queue 2 at time 0. In either of the case the server will be back to queue 1 when queue \( k \) is served in the first cycle. Then we have
\[
C^\pi(I) \geq \frac{k(k-1)}{2} p + n(k-1)p + \tau \left((n+k-1) + (n+k-2) + \ldots + n\right),
\]
and
\[
C^*(I) = \frac{k(k-1)}{2} p + \tau \left((n+k-1) + (k-1) + \ldots + 1\right) + \epsilon(n+k-1).
\]

Letting \( p = (n)^2 \) and \( n \to \infty \), we have \( \frac{C^\pi(I)}{C^*(I)} \to \infty \). \( \square \)
2.2 Other Queue-length Based Policies

Note that not all the policies with cyclic routing discipline have competitive ratio $\kappa$. Policies in $\Pi_r$ have a constant competitive ratio because they all serve as many available jobs as possible in each visit to a queue, which reduces the frequency of switching. Some other cyclic policies may not have constant competitive ratios for $1\,|\,r_i, \tau, p_{\text{max}} \leq \gamma p_{\text{min}}\,|\,\sum C_i$. We first consider a policy called $l$-limited policy. This policy is also based on the cyclic routing discipline. However, the server under $l$-limited policy only serves at most $l$ jobs during each visit to a queue, then switches to the next queue. A detailed description for this policy and its average performance for $M/G/1$ type queues can be found in [47, 17, 53]. Interestingly, as we shall show in Corollary 7, no constant competitive ratio is guaranteed by $l$-limited policy, no matter the server sets up empty queues or not.

Corollary 7. The $l$-limited policy ($l < \infty$, with or without skipping empty queues) does not have a constant competitive ratio for $1\,|\,r_i, \tau, p_{\text{max}} \leq \gamma p_{\text{min}}\,|\,\sum C_i$.

Proof. We prove this result by giving a special instance $I$. Suppose there are $(l+n)$ number of jobs ($l, n \in \mathbb{Z}_+$) at every queue at time 0, and each job has processing time $p = 1$. The server would 1) sets up the first queue, 2) serve $l$ number of jobs in the first queue, and 3) make a tour from queue 2 to queue $k$ by setting up each queue and serving $l$ jobs at each queue. After returning to queue 1, the server will again set up queue 1 and serve $l$ jobs. This process will be repeated for $n$ times before the entire instance $I$ is processed. Let $C^l(I)$ be the total completion time for the $l$-limited policy (the policies with or without skipping empty queues have the same completion time for this job instance), we have

$$\frac{C^l(I)}{C^*(I)} = \frac{knl(knl+1)}{2} + \tau (\frac{knl(knl+1)}{2} + \frac{\tau}{2} n k (k+1)).$$

If we let $\tau = (n)^2$ and $n \to \infty$, then $\frac{C^l(I)}{C^*(I)} \to \infty$. \(\square\)

Although from Theorem 6 we show policies in $\Pi_r$ do not have constant competitive ratios for $1\,|\,r_i, \tau\,|\,\sum C_i$, they do have a constant competitive ratio for $1\,|\,r_i, \tau, p_{\text{max}} \leq \gamma p_{\text{min}}\,|\,\sum C_i$. Corollary 7 shows that $l$-limited policy does not belong to $\Pi_r$, and it does not have a constant competitive ratio for either $1\,|\,r_i, \tau\,|\,\sum C_i$ or $1\,|\,r_i, \tau, p_{\text{max}} \leq \gamma p_{\text{min}}\,|\,\sum C_i$.

Next we discuss the worst case performance of a policy that selects queues based on queue length information. As provided in [30], to serve the stochastic largest queue (SLQ) is the optimal policy when the system is stochastically symmetric. Here ‘symmetric’ means that 1)if an arrival occurs, it will join one of the $k$ queues randomly but equally likely, and 2) service time for jobs in different queues are identically distributed. Note that ‘symmetric’ does not mean the inter-arrival time of jobs are fixed or job workload is revealed to be identical. It only means that the queues are equivalent from a stochastic perspective. The following proposition shows that SLQ is the optimal policy for a symmetric and stochastic polling system.

**Proposition 8.** For a symmetric polling system, the optimal policy is given by SLQ which is as follows: 1) The server serves jobs in a queue non-preemptively; 2) The server should neither idle nor switch when it is at a non-empty queue (exhaustive); 3) The server stays idling in the last queue it visits when the system is empty; 4) When a queue is finished, the server switches to the next queue with the largest number of jobs in queue.
Proof. See [30].

For a symmetric stochastic polling system, although we know the stochastic assumptions for each queue is identical, once the arrival and service times (random variables) are revealed deterministically, they may not be the same from a deterministic standpoint. The following corollary says SLQ for the symmetric system is no longer optimal when information is revealed deterministically. Further, we show that SLQ does not even have a constant competitive ratio.

**Corollary 9.** The SLQ policy does not have a constant competitive ratio for $1|r_i, \tau| \sum C_i$, thus it is not the optimal policy for $1|r_i, \tau| \sum C_i$.

**Proof.** We prove the result by giving a special instance $I$. Suppose there are only 2 queues in the system. The first queue has 1 job with processing time $p$ at time 0, while the second queue has no job at time 0 but has $n$ jobs with processing time 0 arriving at time $\tau$. SLQ will serve from queue 1 to queue 2, so that

$$C_{\text{SLQ}}(I) = \tau + 2n\tau + p + np,$$

and

$$C^*(I) = n\tau + 2\tau + p.$$

Let $p = n$ and $n \to \infty$ we have $\frac{C_{\text{SLQ}}(I)}{C^*(I)} \to \infty$.

We conclude this section by noting that all of the policies we have discussed are mainly focused on routing disciplines: policies in $\Pi_r$ use cyclic routing discipline, and SLQ are purely queue-length based. Some of these policies have constant competitive ratios when the workload variation is bounded, but when workload variation is unbounded, these policies no longer have constant competitive ratios. In the Section [3] we will introduce some job-priority based policies under the condition where workload variation is unbounded.

2.3 Simulation-based Policies

Now we consider policies that are based on simulation results. There are many simulation-based online algorithms in the literature such as the One Machine policy in [38] and the $\alpha$-scheduling provided in [11]. These policies make decisions based on the result of simulations on a virtual system. For instance, One Machine policy simulates a virtual system that has the same arrivals as the real system, however in the virtual system preemption is allowed and SRPT is the optimal policy. So One Machine policy schedules jobs in the order that they are scheduled in the virtual system. Simulation-based policies usually need to simulate a virtual online benchmark in parallel and use the simulation result. For the polling system, we may want to simulate policies on virtual instances. Here we introduce two instances $I$ and $\bar{I}$ which we call workload reduced and augmented instance respectively, such that $I$ and $\bar{I}$ are of the same arrivals as $I$ but $I$ is of processing time $p_{\text{min}}$ and $\bar{I}$ is of processing time $p_{\text{max}}$ for all jobs. We can construct policies for $I$ by simulating the policies on $I$ or $\bar{I}$ and following the simulation results. Now we show that a policy which follows the decision that $\pi \in \Pi_r$ makes on either $I$ or $\bar{I}$ has competitive ratio $\gamma(k + 1)$.
Corollary 10. A policy $\pi$ that works on $I$ but follows the decision that a policy $\bar{\pi} \in \Pi_r$ makes on either $I$ or $\bar{I}$ is of competitive ratio $\gamma(k+1)$ for $1|\tau_i,p_{\text{max}} \leq \gamma p_{\text{min}}, \tau| \sum C_i$.

Proof. Since there is only one type of jobs in either $I$ or $\bar{I}$, preemption is not needed. We suppose $\bar{\pi} \in \Pi_r$ is a policy that works on $\bar{I}$, then $C^\bar{\pi}(\bar{I}) \leq (k+1)C^\bar{\pi}(I)$ where $C^\pi(I)$ is the optimal completion time for instance $I$. Because $\bar{\pi}$ is not preemptive, when $\bar{\pi}$ starts processing a job in $\bar{I}$ at time $t$, we know the process will be done at time $t + p_{\text{min}}$. Thus we construct the policy $\pi$ by letting $\pi$ follow the same service order that $\bar{\pi}$ has. Easily we have $C^\pi(I) \leq C^\bar{\pi}(I)$ because $p_i \leq p_{\text{min}}$ for every job $i$ in instance $I$. Since $\bar{I}$ is a reduced instance, we have $C^\pi(I) \leq C^\bar{\pi}(I)$. Thus $C^\pi(I) \leq \gamma C^\bar{\pi}(I) \leq \gamma(k+1)C^\bar{\pi}(I) \leq \gamma(k+1)C^\bar{\pi}(I)$.

If $\bar{\pi}$ works on $\bar{I}$, where instance $\bar{I}$ is the augmented instance for instance $I$, with workload for all jobs in $\bar{I}$ is $p_{\text{max}}$. Then, we have $C^\pi(I) \leq C^\bar{\pi}(\bar{I}) \leq (k+1)C^\bar{\pi}(I)$. Let $C^\bar{\pi}(\bar{I})$ be the completion time for a new policy which works on $\bar{I}$ but always makes the same decisions as the optimal policy for $I$, then $C^\pi(I) \leq \gamma C^\bar{\pi}(\bar{I})$. Using $C^\bar{\pi}(\bar{I}) \leq C^\bar{\pi}(I)$ we have

$$C^\pi(I) \leq C^\bar{\pi}(\bar{I}) \leq (k+1)C^\bar{\pi}(I) \leq (k+1)C^\bar{\pi}(\bar{I}) \leq \gamma(k+1)C^\bar{\pi}(I).$$

Corollary 10 says we can simulate policies from $\Pi_r$ on both workload augmented and reduced instances and follow the decisions that simulated policies make. Interestingly we find that choosing either $I$ or $\bar{I}$ to simulate policies on results in the same competitive ratios. However, the competitive ratio for this simulation-based policy is larger than $\kappa$ defined for policies in $\Pi_r$, since $\gamma(k+1) - (k+1) \geq 0$ and $\gamma(k+1) - \frac{3}{2}\gamma > 0$.

We conclude this section by noting that all of the policies we have discussed are mainly focused on queue priority, where policies in $\Pi_r$ use cyclic routing discipline, and SLQ are purely queue-length based. These policies may have constant competitive ratios under the bounded condition for the workload. In the case when workload variation is unbounded, it is costly to process a large job and let small jobs wait. In the next section we will introduce some job-priority based policies under the condition where workload variation is unbounded.

3 Polling System with Bounded Setup Times

In this section we mainly discuss policies for the polling system with bounded setup times. Here we allow the workload variation to be unbounded. This setting in the 3D printing example corresponds to the scenario when jobs of a different color need to be printed, inks need to be switched and the switching time is bounded. When the setup times are bounded and workload variation is unbounded, an online policy may want to favor job priority instead of queue priority. For convenience of analysis, we assume switching time $\tau$ is bounded by a ratio of the minimal workload, that is $\tau \leq \theta p_{\text{min}}$. If $\tau = p_{\text{min}} = 0$, we let $\theta = 1$. Using the standard notation for scheduling problems from [21], we denote this polling problem as $1|\tau_i, \tau \leq \theta p_{\text{min}}| \sum C_i$. Notice that when setup time is small, i.e., $\theta$ is small, switching may not be the major contributor to the completion time delay. We shall show how to define a ‘small’ $\theta$ in Section 4. In this section, we mainly show that several policies which are designed for solving the problem without setup times, also work well in the polling system when setup times are small. Since job processing time is revealed upon arrival, we may want online policies to use job size information by selecting small jobs to process first.
Algorithm 3 One Machine Scheduling (OM)
1. Simulate SRPT policy on the setup time reduced instance $\tilde{I}$.
2. Schedule the jobs non-preemptively in the order of completion time of jobs by SRPT on $\tilde{I}$.

3.1 One Machine Policy and Gittins Index Policy for Polling system

In this subsection we introduce two policies that favor jobs with short processing times. We first introduce a benchmark for deriving the competitive ratio of those policies. Usually the competitive ratio $\rho$ is defined by $\sup_I \frac{C^o(I)}{C^*(I)} \leq \rho$ where $C^*(I)$ is the completion time for $I$ in the offline optimal solution. However the offline problem is strongly NP-hard. To non-rigorously show the NP-hardness, we know if no preemption is allowed, even the easier problem $|r_i| \sum C_i$ (without switching time) is strongly NP-hard. Instead of using offline optimal solution to serve as the benchmark for online policies, in this section we mainly use a lower bound of the optimal solution as the benchmark. To get a lower bound for the optimal solution in the case with unbounded processing times and bounded setup times, we introduce the idea of setup time reduced instance. Suppose instance $I$ is an arbitrary instance, the setup time reduced instance of $I$, say $\tilde{I}$, is an instance that has the same arrivals and workload as $I$ but has no setup times. The optimal scheduling policy for $I$ to minimize total completion times is SRPT. This schedule policy is also online, which is handy for online policies to emulate. The completion time of instance $I$ under SRPT is denoted as $C^o(I)$. Note that setup time does not exist in $\tilde{I}$, thus $C^o(I) \leq C^*(I)$. In our problem, we only consider policies without preemption, but using SRPT as the benchmark. When preemption is not allowed, One Machine (OM) policy is proved to be the online scheduling policy with smallest competitive ratio for $I$. When setup time is small, we can adopt OM directly into $I$, regardless of the setup times. It is important to note that OM is a simulation based policy. Under OM, SRPT is simulated in parallel and decisions for OM are based on the job sequences under SRPT. The description of OM is provided in Algorithm and the competitive ratio of OM is provided in Theorem.

**Theorem 11.** OM is a $(2+\theta)$-competitive online algorithm for the polling system $|r_i, \tau \leq \theta p_{\min}| \sum C_i$. The competitive ratio is tight when using SRPT on the reduced instance as the benchmark.

**Proof.** Let the completion time of the $j$th job under OM scheduling be $C^o_j$, and the completion time of the $j$th job completed under SRPT as $C^p_j$. Since job $j$ is also the $j$th job that completes service under SRPT, we have $\sum_{i=1}^j p_i \leq C^p_j$. Then we have $C^o_j \leq C^p_j + \sum_{i:j} C^p_{i,j} + j \tau \leq (1+\theta)C^p_j + \sum_{i=1}^j p_i \leq (2+\theta)C^p_j$. Since $C^o(I) \leq C^*(I)$, we get $\sum C^o_i \leq (2+\theta) \sum C^p_i \leq (2+\theta) \sum C^o_i$. The competitive ratio is tight when there is only one job in the instance $I$ which is available at time $0$. Suppose this job has processing time 1. Then $C^o(I) = 1$, and $C^o(I) = 1 + (1+\theta) = 2+\theta$.

OM algorithm is intuitive, easy to apply and polynomial-time solvable. Despite its simplicity, we may find it inefficient since setup times are ignored. Although each of the unnecessary switch only brings a small amount of delay, we may still want to avoid switching too often. Thus we provide another policy which is based on Gittins Index. Gittins Index policy is a well-studied method for solving problems such as the multi-armed bandit problem. Gittins Index policy is also the optimal policy for the $M/G/1$ multi-class queue scheduling problem to minimize the mean average sojourn times. Here we modify the Gittins Index policy and use it on the polling system with setup time by assigning indices to jobs and choosing the best index. We call it the Gittins Index policy for polling system (GIPP), which is shown in Algorithm. In GIPP, we first simulate SRPT, and then regard the departure time of each job under SRPT as the new ‘arrival’ time.
in GIPP. We next assign Gittins index for these newly ‘arrived’ jobs, where the Gittins index for a job with processing time $p$ is given by $\frac{1}{p}$ if this job and the server are at the same queue; if not, the Gittins index of this job is given by $\frac{1}{p+\tau}$. Among all the jobs that are waiting in the queue, we select the one with the largest Gittins index. By doing this, jobs from the queue which the server is serving may have larger indices than jobs from other queues. The server will prefer the jobs from the queue that it is currently serving, thus avoid switching frequently. Surprisingly however, GIPP does not have a competitive ratio smaller than OM. Both OM and GIPP have the same competitive ratio (see Theorems 11 and 12), when using SRPT on reduced instance as the benchmark.

**Theorem 12.** GIPP is a $(2 + \theta)$-competitive online algorithm for $1|r_i, \tau \leq \theta p_{\text{min}}|\sum C_i$. The competitive ratio is tight when using SRPT on the reduced instance as the benchmark.

**Proof.** Note both GIPP and OM simulate SRPT on $I$ and schedule job $i$ only after job $i$ has been processed in SRPT. So we can regard OM as FCFS in a job instance whose arrival times are $\{C_i^p, i = 1, 2, \ldots\}$, while GIPP serves the job with the largest Gittins index first in this instance of arrival times $\{C_i^p, i = 1, 2, \ldots\}$. We have $C_j^g \leq C_j^p + \sum_{i=1}^{j-1} \hat{p}_i$, where $\hat{p}_i$ is given by inverse of the Gittins index of job $i$. Since GIPP schedules the available jobs in the descending order of $\hat{p}_i$, we have $C_j^g \leq C_j^p + \sum_{i=1}^{j} \hat{p}_i \leq C_j^p + \sum_{i<j} C_i^p (p_i + \tau) \leq (2+\theta) C_j^p$. We give the same example as the one in Theorem 11 to show the tightness of competitive ratio: Suppose there is only one job with $p = 1$ in instance $I$, available at time 0. Then $C^g(I) = 1$, and $C^g(I) = 1 + (1+\theta) = 2+\theta$. Hence proved.

Intuitively, GIPP might perform better than OM since GIPP always schedules jobs of large indices and leaves jobs with small indices later. A job from a queue different from the server may have a small Gittins Index due to the setup time $\tau$, thus GIPP may avoid switching frequently. Surprisingly however, GIPP does not have a competitive ratio smaller than OM. Both OM and GIPP have the same competitive ratio (see Theorems 11 and 12), when using SRPT on reduced instance as the benchmark.

Next we show the lower bound competitive ratio of the problem $1|r_i, \tau = \theta p_{\text{min}}|\sum C_i$. Notice this is a special case for the problem $1|r_i, \tau \leq \theta p_{\text{min}}|\sum C_i$ if $\tau$ and $p_{\text{min}}$ are both revealed deterministically. There is no online algorithm that has a competitive ratio smaller than the lower bound for $1|r_i, \tau = \theta p_{\text{min}}|\sum C_i$.

**Theorem 13.** If $\tau = \theta p_{\text{min}}$ and $\theta \geq 0$, then there is no online algorithm whose competitive ratio is smaller than $\theta + 1$, using SRPT on the reduced instance as the benchmark.
Proof. If there is one job of processing time $p_{\min}$ in the system we have $\frac{C^\sigma(I)}{C^\sigma(\tilde{I})} \geq \frac{(1+\theta)p_{\min}}{p_{\min}} = 1 + \theta$. If $\tau = p_{\min} = 0$, then the lower bounded ratio is $\theta + 1 = 2$ as provided in [21] since we assume $\theta = 1$ in this case.

A natural question is whether this lower bound is the best lower bound that one can have. The answer remains open. There could be either an online policy whose competitive ratio is exactly equal to this lower bound, or a larger lower bound which is closer to the ratio $(2 + \theta)$.

### 3.2 Simulation-based Policies

In Section 2, we introduced the idea of simulation-based policies. A simulation-based policy usually simulates an online policy in parallel, and schedules jobs based on results of the simulated policy. In the case when setup times are bounded, we can also construct simulation-based policies, knowing that SRPT is the optimal policy for the problem without setup times. Given any instance $I$ for the polling system, there is a reduced instance $\tilde{I}$ in which arrival times are the same as $I$ but setup time is 0. There is also a setup time augmented instance $\hat{I}$, where setup time is 0 but each workload is augmented with $\tau$, i.e., $p_i \in I$ corresponds $p_i + \tau$ in $\hat{I}$. Any online algorithm can simulate policies on $\tilde{I}$ and $\hat{I}$ in parallel, and make decisions based on the simulation results. We have the following results via Corollaries 14 and 15.

**Corollary 14.** By simulating any $\rho$-competitive online algorithm that we call $\sigma$ on $\tilde{I}$ and scheduling jobs by the order of completion times under $\sigma$, we obtain an online algorithm on $I$ that is $\rho(2 + \theta)$-competitive for $1|r_i, \tau \leq \theta p_{\min}|\sum C_i$.

**Proof.** Denote the new online algorithm on $I$ as $\pi$. $C^\pi_j \leq C^\sigma_j + \sum_{i=1}^j p_i + j\tau \leq (2 + \theta)C^\sigma_j$. Thus $\sum C^\pi_j \leq \sum(2 + \theta)C^\sigma_j \leq \sum \rho(2 + \theta)C^\sigma_j$.

SRPT is the optimal policy on $\tilde{I}$ and it is online, thus it is 1-competitive for $1| r_i | \sum C_i$. By following the decision that SRPT makes we have a policy with competitive ratio $(2 + \theta)$, which is the same as the result of Theorem 11. We can also simulate online policies on augmented instance $\hat{I}$ and follow their decisions, by which we have the following corollary.

**Corollary 15.** By simulating a $\rho$-competitive online algorithm that we call $\sigma$ on $\hat{I}$ and scheduling jobs by the order of their completion times under $\sigma$, we obtain an online algorithm on $I$ that is $2\rho(1 + \theta)$-competitive for $1| r_i, \tau \leq \theta p_{\min}|\sum C_i$.

**Proof.** Denote the online algorithm on $I$ as $\pi$. We first show that $C^\rho(\hat{I}) \leq (1 + \theta)C^\rho(I)$. For an arbitrary job $\hat{p_i} \in \hat{I}$, it satisfies $\hat{p_i} = p_i + \tau \leq (1 + \theta)p_i$. Let $\hat{\sigma}$ be a policy that works on $\hat{I}$ but schedules jobs in the same order as SRPT on $I$, and serves each job the same portion as SRPT on $I$. Thus $C^\rho(\hat{I}) \leq C^\sigma(\hat{I}) \leq (1 + \theta)C^\rho(I)$. By $C^\sigma_j \leq C^\sigma_j + \sum_{i=1}^j p_i + j\tau \leq 2C^\sigma_j$ we have $C^\sigma(I) \leq 2C^\sigma(\tilde{I}) \leq 2\rho C^\rho(\tilde{I}) \leq 2\rho(1 + \theta)C^\rho(I) \leq 2\rho(1 + \theta)C^\rho(I)$.

Hence proved.

Since the optimal policy for $\hat{I}$ is SRPT, $\rho$ in Corollary 14 is at least 1. By simulating SRPT on $\hat{I}$ and following its decisions we have $C^\sigma(I) \leq (1 + \theta)C^\sigma(I)$, which is greater than ratio $(2 + \theta)$ that we get by simulating SRPT on $I$. Furthermore, if we simulate the same online policy based on either $I$ or $\hat{I}$, we have
Thus we have shown that $C$ is work-conserving, its non-service time is shorter than the non-service time of policy $I$.

Hence proved.

3.3 Other Results

So far we have discussed the cases of bounded workload variation in Section 2 and bounded setup times in this section. We next discuss the case when both setup time and workload variation are bounded. Obviously both OM and GIPP work under this scenario and have constant competitive ratios. The algorithms in $\Pi_r$ which we provide in Section 2 also have a constant competitive ratio in this case when the number of queues is fixed. However, in this special case we may have more policies with constant competitive ratios. Next we show that all the work-conserving policies (in which server never idles when the system is not empty) have a constant competitive ratio when setup time and workload variation are bounded.

**Theorem 16.** Any non-preemptive work-conserving (WC) policy on the polling system with $p_{max} \leq \gamma p_{min}$ and $\tau \leq \theta p_{min}$ is at least $(\gamma + \theta)$-competitive with respect to the optimal solution to the offline problem.

**Proof.** Let $\hat{I}$ be the workload and setup time augmented instance that all jobs are of workload $\hat{p} = p_{max} + \tau \leq (\gamma + \theta)p_{min}$, setup time is 0 in $\hat{I}$ and arrivals are the same as $I$. Note that any non-preemptive work-conserving policy on $\hat{I}$ is optimal since processing times for jobs in $\hat{I}$ are identical. Let $\hat{\sigma}$ be a non-preemptive work-conserving policy on $\hat{I}$, and $\hat{\sigma}$ is a policy that works on $\hat{I}$ and serves jobs in the same order as $\sigma$ does in $I$. Then $\hat{\sigma}$ is work-conserving since $\sigma$ never idles when there are unfinished jobs in system, and therefore $C^\hat{\sigma}(\hat{I}) = C^\hat{\sigma}(\hat{I})$. Now we show $C^\hat{\sigma}(\hat{I}) \leq (\gamma + \theta)C^\sigma(I)$. Let $\hat{S}_i^*$ be the starting time of job $i$ in $\hat{I}$ under the optimal solution and $S_i^*$ be the starting time of job $i$ in $I$ under the optimal solution. Let $\delta$ be a policy that works on $\hat{I}$ and finishes each job $i$ at $(\gamma + \theta)C_i^\hat{\sigma}$. Notice that $(\gamma + \theta)C_i^\hat{\sigma} = (\gamma + \theta)(S_i^* + p_i) \geq (\gamma + \theta)S_i^* + \hat{p}$. Let $W_i^*$ be the non-service times of the optimal policy on $I$ from time 0 to $S_i^*$, i.e., sum of setup times and idling times till $S_i^*$. Then $(\gamma + \theta)S_i^* = (\gamma + \theta)(W_i^* + \sum_{j=1}^{i-1} p_j) \geq (\gamma + \theta)W_i^* + (i-1)\hat{p} \geq \hat{S}_i^*$. The last inequality holds because $\hat{S}_i^* - (i-1)\hat{p}$ is the non-service time of the optimal policy on $\hat{I}$. Since the the optimal policy on $\hat{I}$ is work-conserving, its non-service time is shorter than the non-service time of policy $\delta$ which is $(\gamma + \theta)W_i^*$. Thus we have shown that $\delta$ is a feasible schedule on $\hat{I}$. In summary, we have

$$C^\delta(I) \leq C^\hat{\sigma}(\hat{I}) = C^\hat{\sigma}(\hat{I}) \leq C^\sigma(I) = (\gamma + \theta)C^\sigma(I).$$

Hence proved.

So far we have shown the existence of constant competitive ratios under different assumptions for polling systems. Table 3 summarizes the competitive ratios that we prove in this paper. We find from Table 3 that when either $p_{max} \leq \gamma p_{min}$ or $\tau \leq \theta p_{min}$ holds we can have constant competitive ratio for polling systems. In fact in many practical scenarios either the workload (processing time) is bounded or the setup time is bounded or both, where constant competitive ratio algorithms exist. In the following theorem we show that certain type of policies cannot achieve competitive ratios smaller than $k$.

**Theorem 17.** There is no online policy with competitive ratio smaller than $k$ for $1 \mid r_i, \tau \mid \sum C_i$ if its routing discipline is static or random, or is purely queue-length based, or purely job-priority based.
Proof. Theorem 5 shows that no policy with static or random routing discipline has competitive ratio smaller than $k$. To show that it holds for any queue-length based routing policy $\pi_1$, we give an instance $I$ where there is one job at each queue at time 0, with the job in queue 1 having workload $p$ and jobs in other queues having workload 0. Since $\pi_1$ is purely queue-length based, it treats all queues equally since the queue lengths are equal. Suppose the server serves from queue 1 to queue $k$, we then have

$$C_{\pi_1}(I) \geq p + (k-1)p + \tau \frac{k(k+1)}{2},$$

and

$$C^*(I) = p + \tau \frac{k(k+1)}{2}.$$  

Letting $p \to \infty$, we have $\frac{C_{\pi_1}(I)}{C^*(I)} \geq k$.

To show that the result holds for any purely job-priority based policy $\pi_2$, we give an instance $I'$ where there are $n$ jobs at queue 1 and one job at queue $i = 2, \ldots, k$ at time 0, with all jobs having workload 0. Since $\pi_2$ is purely job priority based, it does not consider any queue information. Notice both OM and GIPP are purely job priority based. Suppose $\pi_2$ serves $I'$ from queue $k$ to queue 1, thus

$$C_{\pi_2}(I') \geq \tau ((n+k-1) + (n+k-2) + \ldots + n),$$

and

$$C^*(I') = \tau ((n+k-1) + (k-1) + \ldots + 1).$$

Letting $n \to \infty$, we have $\frac{C_{\pi_2}(I')}{C^*(I')} \geq k$. Hence proved. \qed
strategy that balance both queue and job priorities. We use this idea and introduce a mixed strategy in Section 4 to show that a better performance can be achieved when we balance queue and job priorities.

3.4 Clearing Problem

In Subsection 3.1 we show both OM and GIPP are \((2 + \theta)\) competitive, however the lower bound competitive ratio is \((1 + \theta)\) as shown in Theorem 13. This means either there exists a better algorithm with competitive ratio smaller than \((2 + \theta)\) or the actual lower bound is greater than \((1 + \theta)\). However, in this subsection we show the lower bound \((1 + \theta)\) is not trivial for the clearing problem. Note that the clearing problem is a special case of the online problem, with all arrivals happen at time 0. We show in this subsection that for the clearing problem, both of OM and GIPP have a tight competitive ratio \((1 + \theta)\). We denote the clearing problem as \(1|r_i = 0, \tau \leq \theta p_{\text{min}}|\sum C_i\). Moreover, since there is no arrival in the future, both OM and GIPP do not need to simulate SRPT in parallel. We thus modify OM and GIPP, by truncating the simulation part, then OM schedules jobs by smallest workload first and GIPP schedules jobs by the largest Gittins index first.

Theorem 18. OM and GIPP both have competitive ratio \(\theta + 1\) for \(1|r_i = 0, \tau \leq \theta p_{\text{min}}|\sum C_i\).

Proof. There is no idling time in schedule \(C^o\). Suppose jobs are indexed in ascending workload order so that \(p_i^r \leq p_2^r \leq \ldots \leq p_n^r\), then \(C_j^o \leq \sum_{i=1}^{j} p_i^r + \tau j\). Since \(C_j^o = \sum_{i=1}^{j} p_i^r\), we have \(C_j^o \leq (1 + \theta)C_j^p\), thus \(C^o(I) \leq (1 + \theta)C^p(I)\).

We now follow jobs scheduled by GIPP. Let \(p_i^q\) be the inverse of Gittins index for \(i\)th job scheduled by GIPP. If \(p_i^q\) is available for GIPP to schedule, then we have \(p_i^q \leq p_i^r + \tau\). Now we consider the case in which \(p_i^q\) is not available to GIPP as GIPP serves \(p_i^r\) before \(p_i^q\). Let \(S(i)\) be the set of jobs served by SRPT but not by GIPP after the \(i\)th schedule. At time \(C_{i-1}^q\), if \(S(i - 1) = \emptyset\), then \(p_i^q\) is available for GIPP to schedule. Thus we assume \(S(i - 1)\) is non-empty. Choose an arbitrary job \(p^*\) from \(S(i - 1)\) we have \(p_i^q \leq p^* + \tau \leq p_i^r + \tau\), so that \(p_i^q \leq p_i^r + \tau\). Therefore \(C_j^q = \sum_{i=1}^{j} p_i^q \leq \sum_{i=1}^{j} p_i^r + j\tau\) and \(C^o(I) \leq (1 + \theta)C^p(I)\).

We know the clearing problem with fixed number of queues is polynomial time solvable [35, 18]. The problem with dynamic number of queues is solvable using a 2-approximating algorithm provided in [12]. However here we say that if \(p_{\text{min}} > 0\) and \(\tau\) is bounded, simple algorithms exist for the clearing problem and their competitive ratios are proved to be tight and optimal. The results look worse than competitive ratio 2, however here we use SRPT as the benchmark, which means the actual competitive ratio could be smaller than \(\theta + 1\). The work [12] does not show the tightness of algorithms. Before presents a mixed strategy in Section 4 we next present algorithms for the offline algorithm.

3.5 On Approximating Algorithms for the Generalized Offline Problem

In this subsection we discuss the case in which job information is available to the server at time 0 but arrival times in the future are known, unlike the clearing problem where all jobs are available at time 0. Because the focus of this paper is mainly on solving the online scheduling problem, we do not wish to discuss the offline problem in detail. The main purpose of this subsection is to model the offline problem and provide an approximation method as well. To make the formulation more comprehensive and general, we add another constraint known as precedence, through the use of priorities. If job \(j\) has higher priority than job \(i\), we note
Proof. We let $C$ polynomial time via the ellipsoid algorithm [39, 22], we obtain the optimal solution to this linear problem as $S$.

**Theorem 19.** The linear problem which minimizes $\sum w_i C_i$ under constraints (3.1), (3.2) and (3.5) is solvable in polynomial time via the ellipsoid algorithm [39, 22], we obtain the optimal solution to this linear problem as $\bar{C}_1, ..., \bar{C}_n$, and then schedule the jobs in order of non-decreasing $\bar{C}_i$. We denote this schedule as $\bar{C}_i$.

Notice the constraint (3.3) is non-linear, we relax it by using the method provided in [22]. We thus have

$$\sum_{i=1}^n p_i C_i \geq \sum_{j=1}^n p_j \left( \sum_{i=1}^j p_i \right) = \frac{1}{2} \left( p(I)^2 + p^2(I) \right), \quad (3.4)$$

where $p(I) = \sum_{i=1}^n p_i$ and $p^2(I) = \sum_{i=1}^n p_i^2$.

We extend Inequality (3.4) to every subset of instance $I$, so that for arbitrary $S \subseteq I$ we have

$$\sum_{i \in S} p_i C_i \geq \frac{1}{2} \left( p(S)^2 + p^2(S) \right). \quad (3.5)$$

Notice the linear problem which minimizes $\sum w_i C_i$ with only constraints (3.1), (3.2) and (3.5) is solvable in polynomial time via the ellipsoid algorithm [39, 22], we obtain the optimal solution to this linear problem as $\bar{C}_1, ..., \bar{C}_n$, and then schedule the jobs in order of non-decreasing $\bar{C}_i$. We denote this schedule as $\bar{C}_i$.

**Theorem 19.** The Schedule by $\bar{C}_i$ is a $(3 + 2\theta)$-competitive algorithm for $1|\tau, \tau \leq \theta p_{min}, prec| \sum w_i C_i$.

**Proof.** We let $\bar{C}_i$ be the actual completion time for job $i$ by following schedule by $\bar{C}_i$. We assume $\bar{C}_1 \leq ... \leq \bar{C}_n$ are completion times on solving $\min \sum w_i C_i$ under constraints (3.1), (3.2) and (3.5), then for any $S = \{1, 2, ..., j\}$ we have

$$\bar{C}_j \sum_{i=1}^j p_i \geq \sum_{i=1}^j \bar{C}_i p_i \geq \frac{1}{2} \left( p(S)^2 + p^2(S) \right).$$

Since $\bar{C}_j \leq \max_{i=1}^j r_i + \sum_{i=1}^j p_i + j \tau$ and $\sum_{i=1}^j r_i \leq \bar{C}_j$, therefore $\bar{C}_j \leq (3 + 2\theta)\bar{C}_j$. Suppose $C^*_i$ is the optimal completion time for job $i$ in the original problem with constraints (3.1) and (3.3), because constraint (3.5) is a relaxation of (3.3), then $\sum w_i C_i \leq \sum w_i C^*_i$. Thus $\sum w_i \bar{C}_i \leq (3 + 2\theta) \sum w_i C^*_i$. \qed

This competitive ratio looks worse than the ratio $2 + \theta$ achieved by online policies, however we should notice in this offline problem we have precedence constraints. This also points to the problem with precedence...
Algorithm 5 Deterministic Mixed Strategy $\pi^m$

1: while The system is not empty do
2: Use $\pi^e$: When a service is done, serve the next job in the queue with shortest processing time first; switch to the next non-empty queue when the current queue has been exhausted
3: if There is an arrival whose workload is greater than $\eta p_{\text{min}}$ then
4: if The server is serving then
5: Finish the current job
6: else if The server is setting up then
7: Halt setting up and the server stays in the current queue
8: end if
9: end if
10: end if
11: end while
12: return Total completion time $C^\pi_m(I)$

is harder than the one without. The discussion for the offline precedence and release time constrained scheduling problem can be found in [14, 38, 22], but none of them consider a setup time constraint.

4 A Mixed Strategy

In Section 2 we provided a policy set $\Pi_r$. Policies in $\Pi_r$ are based on cyclic routing discipline, and they have competitive ratio $\kappa$ when workload variation is bounded. When workload variation is unbounded, we may want to use GIPP, with competitive ratio $\theta + 2$. In reality, it is common to see cases where workload variation is unbounded, and setup times are large but bounded. For example in the reconfigurable manufacturing, robust components are used to process customized jobs and also designed to reduce setup times to make the manufacturing system more efficient. It is very rare to see unbounded setup times [33]. In the 3D printing example, jobs are usually highly customized and heterogeneous, and the workload variation could be large in this case. Besides, when jobs from a very different prototype is received, many setup steps need to be performed so that setting up the 3D printer would take a large amount of time (usually bounded). Thus in this section we discuss the problem where setup time is large but bounded, and workload variation is unbounded. From Section 3 we know that this online problem can be solved by OM or GIPP, however, if a large workload is rare, we may want to adopt a policy $\pi^e$ from $\Pi_r$ most of time when it gives a competitive ratio smaller than $\theta + 2$. This motivates us to construct a mixed strategy such that if there is no workload greater than a threshold, say $\eta p_{\text{min}}$, then a policy from $\Pi_r$ is applied, resulting in a competitive ratio $\max\{\frac{3}{2} \eta, k+1\}$; if there is a new arrival with workload greater than $\eta p_{\text{min}}$, then GIPP is applied so competitive ratio is $\theta + 2$. If $\kappa(\eta) = \max\{\frac{3}{2} \eta, k+1\} \leq \theta + 2$, this mixed strategy has a better expected performance than simply using GIPP for a finite job instance, as we will see later. Specifically, we assume that the exhaustive policy $\pi^e$ serves continuously without waiting and skips empty queues when switching. Within a queue we let $\pi^e$ serve following Shortest Processing Time First (SPT). It helps reduce the queue length within each queue, though this does not reflect on the competitive ratio. A formal statement of this mixed strategy $\pi^m$ is provided in Algorithm 5.

Theorem 20. If $p_{\text{min}} > 0$, $\eta \leq \theta$, $\kappa(\eta) \leq \theta + 2$ and workload is drawn from a known distribution, then

$$E\left[\frac{C^\pi_{\text{mix}}(I)}{C^*(I)}\right] \leq \nu(\eta),$$
where \( \nu(\eta) = \kappa(\eta)\mu(\eta)^{n(I)} + (\theta + 2)(1 - \mu(\eta)^{n(I)}) \) and \( \mu(\eta) = P(p_i \leq \eta p_{\min}) \), prior to revealing.

**Proof.** Let \( p_{\min} = 1 \). If all the jobs are of workload smaller than \( \eta \), then throughout the busy period, \( \pi^m = \pi^c \) and \( \frac{C^\pi^m(I)}{C^\pi^c(I)} \leq \kappa(\eta) \). Now we show that if there is a workload in the busy period with workload greater than \( \eta \), then \( \frac{C^\pi^m(I)}{C^\pi^c(I)} \leq \theta + 2 \). Suppose \( I = I_1 \cup I_2 \), where \( I_1 \) is the job instance that is served by \( \pi^c \) in \( I \), and \( I_2 \) is the rest of \( I \) which is served by GIPP. Let \( S_2 \) be the time when GIPP is triggered. Then \( C^\pi^m(I) = C^\pi^c(I_1) + S_2 n(I_2) + C^g(I_2) \) where \( C^g(I_2) \) is the completion time for GIPP triggered at time \( S_2 \).

For the optimal solution, we have \( C^*(I) = C^*(I_1 \cup I_2) \geq C^*(I_1) + R_2 n(I_2) + C^p(I_2) \), where \( R_2 \) is the time when a job with workload greater than \( \eta \) arrives and \( C^p(I_2) \) is the completion time for \( I_2 \) under SRPT. Since \( R_2 \leq S_2 \leq R_2 + \max\{\theta, \eta\} = R_2 + \theta \) by \( \eta \leq \theta \), we have,

\[
\frac{C^\pi^m(I)}{C^*(I)} \leq \frac{C^\pi^c(I_1) + S_2 n(I_2) + C^g(I_2)}{C^*(I_1) + R_2 n(I_2) + C^p(I_2)} \leq \frac{C^\pi^c(I_1) + (R_2 + \theta) n(I_2) + C^g(I_2)}{C^*(I_1) + R_2 n(I_2) + C^p(I_2)}.
\]

If \( R_2 \geq \theta \), then we have \( \frac{C^\pi^m(I)}{C^*(I)} \leq \max\{\kappa(\eta), 2, \theta + 2\} = \theta + 2 \). If \( R_2 < \theta \), then the server is setting up when the new arrival occurs. Setup is aborted immediately and GIPP is started, thus \( S_2 = R_2 \) and

\[
\frac{C^\pi^m(I)}{C^*(I)} \leq \frac{C^g(I_2)}{C^p(I_2)} \leq \theta + 2.
\]

Therefore

\[
E[\frac{C^\pi^m(I)}{C^*(I)}] = E[\frac{C^\pi^m(I)}{C^*(I)}|I_2 = \emptyset]P(I_2 = \emptyset) + E[\frac{C^\pi^m(I)}{C^*(I)}|I_2 \neq \emptyset]P(I_2 \neq \emptyset)
\]

\[
\leq \kappa(\eta)\mu(\eta)^{n(I)} + (\theta + 2)(1 - \mu(\eta)^{n(I)}).
\]

Hence proved. \( \Box \)

If only a policy from \( \Pi_r \) is used when \( I_2 \neq \emptyset \), then the expected performance is smaller than \( \kappa = \max\{\frac{2}{3} \gamma, k + 1\} \), which may be greater than \( \theta + 2 \) when \( \gamma \) is large. Also we know that if only GIPP is applied, the expected performance is bounded by \( \theta + 2 \). Since \( \nu(\eta) = \kappa(\eta)\mu(\eta)^{n(I)} + (\theta + 2)(1 - \mu(\eta)^{n(I)}) \), the Deterministic Mixed Strategy has an expected performance bound smaller than \( \kappa(\eta) \) or \( (\theta + 2) \). Note that \( \kappa(\eta) = \max\{\frac{2}{3} \eta, k + 1\} \).

If \( \frac{2}{3} \eta \leq k + 1 \), then \( \kappa(\eta) = k + 1 \). The optimal \( \eta^* \) for minimizing \( \nu(\eta) \) in this case is given by \( \frac{2}{3}(k + 1) \) since \( \mu(\eta) \) is an increasing function of \( \eta \). If \( \frac{2}{3} \eta \geq k + 1 \), then \( \nu(\eta) = (\theta + 2) + \mu(\eta)^{n(I)}(\frac{2}{3} \eta - (\theta + 2)) \), and the optimal value \( \eta^* \) can be obtained by solving

\[
\min \quad (\theta + 2) + \mu(\eta)^{n(I)}(\frac{2}{3} \eta - (\theta + 2))
\]

s.t. \( \frac{2}{3}(k + 1) \leq \eta \leq \min\{\frac{2}{3}(\theta + 2), \theta\} \) \hspace{1cm} (4.1)

Notice \( \eta = \frac{2}{3}(k + 1) \) is a feasible solution to System (4.1), thus by solving System (4.1) we can obtain the optimal solution to \( \nu(\eta) \), which is \( \eta^* \). Note that \( \nu(\eta^*) \) is smaller than \( \kappa(\eta^*) \) and \( \theta + 2 \).

In the online problem where the information of future jobs is unknown to the server, the server cannot actually optimize the System (4.1). A practical way is letting \( \eta = \frac{2}{3}(k + 1) \) for the Deterministic Mixed Strategy, which
eventually results in an expected performance bound smaller than $\theta + 2$ for a finite $n(I)$. It is important to note that Deterministic Mix Strategy has expected performance bound smaller than $\max\{\kappa, \theta + 2\}$, however this only happens when $p_i$ values are drawn from a single distribution and the number of jobs $n(I)$ is finite. It is also important to point out that $\nu(\eta)$ is an expected value. The real competitive ratio of this strategy is $\theta + 2$. However, this strategy shows that by balancing queue and job priorities, one could design policies with better performance. This strategy also gives us insights of revealing future information, and shows that if one can reveal or estimate the number of jobs in a busy period as well as the workload distribution, then the System (4.1) is solvable and a smaller expected competitive ratio may be obtained. In a future research study we will discuss how to estimate future information and use it for better system performance.

5 Concluding Remarks and Future Works

In this paper we consider scheduling policies in the polling system without stochastic assumptions. Our analysis provides a novel way to classify scheduling policies for polling systems by considering their worst case performance, i.e., competitive ratio. It allows one to describe the performance of some policies even when their average performance is intractable. Conditions for the existence of constant competitive ratio are discussed and the worst case performance for several well-studied polling system scheduling policies are provided. We show that both cyclic exhaustive policy and cyclic gated policy have a constant competitive ratio $\kappa$ for problem $1 | r_i, \tau, p_{max} \leq \gamma p_{min} | \sum C_i$, but they do not have a constant competitive ratio for the problem $1 | r_i, \tau | \sum C_i$. Interestingly, we find cyclic is the optimal static routing discipline, and when cyclic routing discipline is adopted, revealing the processing times for jobs is not helpful in reducing the competitive ratio if $\frac{3}{2} \gamma \leq k + 1$. We also find some policies with provable average performance do not have constant competitive ratios for $1 | r_i, \tau, p_{max} \leq \gamma p_{min} | \sum C_i$ such as $l$-limited policy and SLQ policy. We provide online policies that balance well the future uncertainty and current information availability, such as GIPP. Besides, we show that if the routing discipline for an online policy relies only on a routing table (static or random), queue-length, or job processing times, then the competitive ratio of this policy cannot be smaller than $k$. We then provide a mixed strategy which performs better that $\Pi_r$ and GIPP when number of jobs in finite. Our analysis suggests a policy with competitive ratio smaller than $k$ may need to incorporate more information besides job processing times or number of jobs in queue. However, the question that whether there exists an online policy with constant competitive ratio for the problem $1 | r_i, \tau | \sum C_i$ without any bound conditions for workload and setup times remains open. Also, it is unclear if there exists a better lower bound competitive ratio for all the online policies. A future problem to consider will be searching for online policies with smaller competitive ratios and deriving a better competitive ratio lower bound for all the online policies.

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A Proof for Theorem 1 in the Main Paper

In this section we mainly provide the proof for Theorem 1 of our original paper. We first introduce a fact that will be useful later.

**Fact 21.** For positive numbers \( \{a_i, b_i\}_{i=1}^{n} \), we have \( \sum_{i=1}^{n} \frac{a_i}{b_i} \leq \max_{i=1}^{n} \{ \frac{a_i}{b_i} \} \).

**Proof.** Without loss of generality, assume \( \frac{a_n}{b_n} = \max_{i=1}^{n} \{ \frac{a_i}{b_i} \} \), then for any \( i \) we have \( \frac{a_i}{b_i} \geq \frac{a_n}{b_n} \), thus \( a_i b_n \geq a_n b_i \) holds for all \( i = 1, ..., n \). Since \( \frac{a_n}{b_n} = \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} = \frac{\sum_{i=1}^{n} b_i - b_n \sum_{i=1}^{n-1} a_i}{\sum_{i=1}^{n} b_i} = \frac{\sum_{i=1}^{n} (a_i b_n - a_n b_i)}{b_n \sum_{i=1}^{n} b_i} \geq 0 \), we have \( \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \leq \frac{a_n}{b_n} = \max_{i=1}^{n} \{ \frac{a_i}{b_i} \} \). \( \square \)

In Theorem 1 we want to show \( \sup_I \frac{C_{\text{opt}}(I)}{C_{*}(I)} \leq \rho \) for some constant \( \rho \). However, by Fact 21 we only need to show that this inequality holds for the instance processed in each busy period. Here we first introduce the concept of busy periods. If there is at least one job in the system, we say the system is busy, otherwise it is empty. When the system is empty, the server is not serving under the online policy. The status of the system under the online policy can be described as a busy period following an empty period, and then following by another busy period, and so on. There are two types of busy periods that we are interested in. Type I busy period (denoted as I-B) is the busy period in which the server under online policy resumes work without setting up. This is because the server was idling at the last queue it served (say queue \( i \)) after the previous busy period, and the first arrival in the new busy period also occurs at queue \( i \). Type II busy period (denote as II-B) is the busy period in which the server resumes work with a setting up, which is because the new arrival occurs at a queue different from the queue that the server was idling at. We will consider these two types of busy
period separately in the proof. To show the online policy has competitive ratio \( \rho \), from Fact \( \text{21} \) we only need to show that \( \sup_I \frac{C^e_I}{C^*(I)} \leq \rho \) holds for any instance \( I \) processed in a busy period.

In the following we only focus our discussion in a single busy period. Since the server in the online policy serves queues in a cyclic way, without loss of generality, we assume that the server serves from queue 1 to queue \( k \), and then switches back to queue 1, and so on. A cycle (round) starts when the server visits queue 1 and ends when it visits queue 1 the next time. If at some time point a queue, say queue \( k \), is empty and skipped by the server, we still say that queue \( i \) has been visited in this cycle, with setup time 0. We define the job instance served by the server in its \( w \)th visit to queue as \( b^w_i \) (for \( i = 1, \ldots, k \)), and we call each \( b^w_i \) a batch. Batch \( b^w_i \) is a subset of job instance \( I \). In each cycle, there are \( k \) batches served by the online policy, and some of them may be empty but not all of them (if all of them are empty then the system is empty and the server would idle at queue 1). We let \( I^w = \bigcup_{i=1}^k b^w_i \). For each batch \( b^w_i \) with the number of jobs \( n(b^w_i) = n_i^w \), \( S^w_i \) is the earliest time when a job from \( b^w_i \) starts being processed under the online policy, and \( R^w_i \) is the earliest release date (arrival time) over all jobs from batch \( b^w_i \). Notice that \( R^w_i \leq S^w_i \). Each batch \( b^w_i \) may be processed by the optimal offline policy in a different way from the online policy. Suppose \( E^w_i \) is the earliest time when a job in batch \( b^w_i \) starts service under the optimal offline policy. Note \( E^w_i \) may differ from \( S^w_i \). Before time \( S^w_i \), we know all the batches \( (\bigcup_{j=1}^{w-1} I^j) \cup (\bigcup_{j=1}^{l-1} b^w_j) \) have been served by the online policy. However in the optimal offline policy, only some jobs from these batches have been served. We suppose \( q^w_i \) number of jobs in \( (\bigcup_{j=1}^{w-1} I^j) \cup (\bigcup_{j=1}^{l-1} b^w_j) \) have been served by the optimal offline policy before time \( E^w_i \). Note that \( q^w_i \) is just a number instead of a job instance. We let \( C^e(I) \) be the cumulative completion times of all jobs in job instance \( I \) under online policy \( e \in \Pi_1 \), and \( C^*(I) \) be the cumulative completion times of all jobs in \( I \) under the offline optimal policy. For convenience, we let \( g(I) = \frac{n(I)(n(I)+1)}{2} \) for job instance \( I \), which is the sum of arithmetic sequence from 1 to \( n(I) \). Also, \( g(I) \) can be regarded as the completion time of \( I \) when 1) all the jobs are available at time 0, 2) each of them are of processing time 1, and 3) no setup time is considered. All the notations are summarized in Table 4 of this document. Before going to the proof of Theorem \( \text{1} \) we first provide an example to show how the total completion time is characterized.

**Example 22.** Suppose there is a job instance \( I \) with \( n(I) = n_1 + n_2 \) jobs which arrives at time \( R \). Each of the job has processing time 1. Under policy \( \pi \) the server starts to process the first \( n_1 \) jobs, and idles for time \( W \), and then processes the rest of \( n_2 \) jobs without idling. The total completion time for \( I \) is given by

\[
C^\pi(I) = (R + 1) + (R + 2) + \ldots + (R + n_1) + (R + n_1 + W + 1) + (R + n_1 + W + 2) + \ldots + (R + n_1 + W + n_2)
\]

\[
= (n_1 + n_2)R + n_2W + g(I).
\]

Notice the completion time of \( I \) is made up of three components. The first component \((n_1 + n_2)R\) is because all the jobs in \( I \) arrive at time \( R \). The second term \( n_2W \) is because the rest \( n_2 \) jobs wait for another \( W \) amount of time. The third term \( g(I) \) is the pure completion time if we process jobs one by one without idling.

Having showed the idea of calculating total completion times in Example \( \text{22} \) now we move on to show the proof of Theorem \( \text{1} \). We next introduce the idea of truncated optimal schedule. Notice that the optimal policy may not always be work-conserving (i.e., never idles when there are jobs in the system). The optimal policy may wait at some queue in order to receive more jobs which will arrive in the future. The truncated optimal solution is defined by the completion time for the optimal offline problem with subtracting the completion
time caused by idling, which is shown in Figure A.1. There is a waiting (idling) period $W$ between $b_1^1$ and $b_1^2$ in Figure A.1. The truncated optimal solution is given by $C^*(b_1^1 \cup b_2^1 \cup b_3^1) - W(n_1^2 + n_2^1 + n_3^1)$. We use $C^i(I)$ to denote the total completion time for instance $I$ under the truncated optimal schedule. Note that the truncated optimal solution is always a lower bound for the real optimal solution.

**Lemma 23.** Suppose $I$ is a job instance, $b$ is a batch and $p_{\min} = 1$, then $C^i(I \cup b) \geq C^i(I) + g(b) + En(b) + n(b)(n(I) - q)$, where $E$ is the time when the server starts serving batch $b$ in the truncated optimal solution, and $q$ is the number of jobs in $I$ that are served before time $E$.

**Proof.** We suppose the optimal solution is given, and now we consider the total completion time of $I \cup b$ under truncated optimal solution. If all the jobs from $b$ are served after $I$ in the optimal solution, then we have $C^i(I \cup b) \geq C^i(I) + g(b) + En(b)$. If not, we let $\delta(b) = C^i(I \cup b) - C^i(I)$ be the additional completion time incurred by inserting $b$ into $I$. Notice that $\delta(b)$ is minimized when all jobs in $b$ has $p_{\min} = 1$. If we can show that $\delta(b) \geq g(b) + En(b) + n(b)(n(I) - q)$ with every job in $b$ having $p_{\min} = 1$, we can then prove the lemma. So we assume here that every job in $b$ has $p_{\min} = 1$. Since the earliest time to process batch $b$ in the truncated optimal solution is $E$, if we combine all jobs in $b$ altogether and serve them in one batch from time $E$ to time $E + n(b)$, then $\delta(b)$ is again minimized since all the jobs in $b$ have the smallest processing time.

So in the following we show that by inserting batch $b$ at time $E$, the additional completion time incurred is at least $g(b) + En(b) + n(b)(n(I) - q)$. By inserting batch $b$ into $I$ from time $E$ to $E + n(b)$, some jobs from $I$ served after $E$ in the original truncated optimal solution (with total number $(n(I) - q)$) are moved after batch $b$, resulting an increase of delay $n(b)(n(I) - q)$ for these jobs. Besides, inserting a batch $b$ at time $E$ increases the total completion time by $g(b) + En(b)$. So inserting a batch $b$ can increase at least $g(b) + En(b) + n(b)(n(I) - q)$ amount of completion time. We thus prove the lemma. \(\square\)

To make the proof of Lemma 23 easier to understand, an example is given in Figure A.2. The first schedule in Figure A.2 is the truncated optimal for the batch $b_1^1 \cup b_2^1 \cup b_3^1$. The second schedule is the truncated optimal schedule for $b_1^1 \cup b_2^1 \cup b_3^1 \cup b$, where $b$ is separated into two parts. If all jobs in $b$ are of workload $p_{\min} = 1$, it is
always beneficial to schedule all jobs of \( b \) in the same batch, which is shown as the third schedule in Figure A.2. Notice that in Figure A.2, \( q = n(b_1^1 + b_1^2) = n_1^1 + n_2^1 \).

**Lemma 24.** Suppose \( I \) is a job instance, \( b \) is a batch and \( p_{\min} = 1 \), then \( C^*(I \cup b) \geq C^*(I) + g(b) + E^*n(b) \), where \( E^* \) is the time when the server starts serving batch \( b \) in the optimal solution.

**Proof.** Since the earliest service time in the optimal solution for \( b \) is at \( E^* \), we have the minimal total completion time for \( b \) is \( g(b) + E^*n(b) \). Hence proved.

We now introduce a benchmark for the online policy by combining the optimal solution and the truncated optimal solution. We let \( C_{m}(I) = \alpha C^*(I) + (1 - \alpha) C^t(I) \) be the benchmark, where \( \alpha = \frac{1}{1+k} \). We notice that \( C^*(I) \geq C_m(I) \geq C^t(I) \) from the fact that \( C^*(I) \geq C^t(I) \). If we have \( C_e(I) C_m(I) \leq \rho \), then we can show that \( C_e(I) C^*(I) \leq \kappa - \epsilon \).

Next we restate Theorem 1 in the main paper and describe the proof.

**Theorem 25. (Theorem 1 in the main paper)** Any policy in \( \Pi_1 \) has competitive ratio of \( \kappa = \max \{ \frac{3}{2} \gamma, k + 1 \} \) for the polling system \( 1 | r_i, \tau, p_{\max} \leq \gamma p_{\min} | \sum C_i \). When \( \frac{3}{2} \gamma \leq k + 1 \), for arbitrary \( \epsilon > 0 \), there is an instance \( I \) such that \( \frac{C_e(I)}{C^*(I)} > \kappa - \epsilon \).

**Proof.** We prove the theorem by induction. In this proof we only consider an instance \( I \) that the cyclic policy \( e \in \Pi_1 \) serves in a busy period. By induction we can finally conclude that \( \frac{C^*(I)}{C^*(I)} \leq \kappa \) for instance \( I \), which also implies that \( \frac{C^*(I)}{C^*(I)} \leq \kappa \). We first show that the batches served by the online policy in the first cycle, i.e., \( I^1 \in I \), satisfies \( \frac{C^*(I^1)}{C^*(I)} \leq \kappa \). We next prove that if the result holds true for \( \cup_{j=1}^{w-1} I^j \), then it also holds for \( (\cup_{j=1}^{w-1} I^j) \cup b_w^1 \). We next show the result holds true for \( (\cup_{j=1}^{w-1} I^j) \cup (\cup_{i=1}^{l+1} b_i^w) \) if the result holds for \( (\cup_{j=1}^{w-1} I^j) \cup (\cup_{i=1}^{l+1} b_i^w) \).

To start, we first assume \( p_{\min} = 1 \) and \( p_{\max} = \gamma \) (the case where \( p_{\min} = 0 \) is similar). Notice that \( I^1 = \cup_{i=1}^{l+1} b_i^1 \) is the union of batches served in the first cycle under the online policy. Without loss of
generality, we assume the server serves from queue 1 to queue \( k \) in each cycle. Knowing that \( I^1 = \bigcup_{i=1}^{k} b_i^1 \), we let \( n_{(k)}^1 \geq n_{(k-1)}^1 \geq \ldots \geq n_{(1)}^1 \) be the descending permutation of \( (n^1_1, \ldots, n^1_k) \), and \( E^1 = \min_{i=1}^{k} E^1_i, S^1 = \min_{i=1}^{k} S^1_i \). Then we have (with explanation given later)

\[
C^t(I^1) \geq g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(k-j+1)}^1 + E^1 \sum_{i=1}^{k} n_i^1, \tag{A.1}
\]

and

\[
C^c(I^1) \leq \gamma g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(j)}^1 + S^1 \sum_{i=1}^{k} n_i^1. \tag{A.2}
\]

The RHS of Inequality \( \text{(A.1)} \) is the minimal completion time of a list which has the same number of jobs in each queue as \( I^1 \) and all of these jobs arrive at time \( E^1 \) with each job having processing time 1. The first term \( g(I^1) \) is the pure completion time. The second term is because \( n_{(k)}^1 \geq n_{(k-1)}^1 \geq \ldots \geq n_{(1)}^1 \), if all batches are available at time \( E^1 \) and there is no further arrivals, the best order of serving the batches is to serve from the longest one to the shortest one. Note that the optimal policy may start without setting up since it may be the same queue that the server was idling at and resumed. In the case where there is no setup for the first queue, the completion time incurred by setup is \( \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(k-j+1)}^1 \). The third term in RHS of Inequality \( \text{(A.1)} \) is because the entire service process for the optimal solution starts from \( E^1 \). Therefore, the RHS of Inequality \( \text{(A.1)} \) is a lower bound for \( C^t(I^1) \). The RHS of inequality \( \text{(A.2)} \) is the upper bound for the online policy, which says that the online policy may serve batches from the shortest to the longest, starting from time point \( S^1 \), and all jobs are of the maximal workload \( \gamma \). Since we consider I-B in this case, the server in the online policy does not set up for the first batch as the server was idling in the same queue as the new arrival. So the completion time resulted by setup is upper bounded by \( \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(j)}^1 \).

We let \( Z(k) = \sum_{i=1}^{k} \sum_{j=i}^{k} n_{(j)}^1 \) and \( Z'(k) = \sum_{i=1}^{k} \sum_{j=i}^{k} n_{(k-j+1)}^1 \) then

\[
\frac{Z(k)}{Z'(k)} = \frac{kn_{(k)}^1 + (k-1)n_{(k-1)}^1 + \ldots + n_{(1)}^1}{kn_{(1)}^1 + (k-1)n_{(2)}^1 + \ldots + n_{(k)}^1} \leq \frac{kn_{(k)}^1 + kn_{(k-1)}^1 + \ldots + kn_{(1)}^1}{n_{(1)}^1 + n_{(2)}^1 + \ldots + n_{(k)}^1} \leq k.
\]

From Fact \( 21 \) we have

\[
\frac{C^c(I^1)}{C^t(I^1)} \leq \frac{\gamma g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(i)}^1 + S^1 \sum_{i=1}^{k} n_i^1}{g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(k-j+1)}^1 + E^1 \sum_{i=1}^{k} n_i^1} \leq \frac{\gamma g(I^1) + \tau (Z(k) - \sum_{i=1}^{k} n_i^1) + E^1 \sum_{i=1}^{k} n_i^1}{g(I^1) + \tau (Z'(k) - \sum_{i=1}^{k} n_i^1) + E^1 \sum_{i=1}^{k} n_i^1} \leq \max\{\gamma, k, 1\} < \kappa. \tag{A.3}
\]

The Inequality \( \text{(A.3)} \) follows from the fact that \( \tau \leq \min_{i=1}^{k} S_i^1 = \min_{i=1}^{k} \{R_i^1\} \leq \min_{i=1}^{k} E_i^1 \) because there must be a busy period happening before an I-B. So far we have shown \( C^c(I^1) \leq C^t(I^1) \leq C^m(I^1) \). Now we prove that \( C^c((\cup_{j=1}^{w-1} I_j^1) \cup b_i^w) \leq \kappa C^m((\cup_{j=1}^{w-1} I_j^1) \cup b_i^w) \). Clearly if \( n_i^w = 0 \) then the conclusion holds. Now we let \( \bar{n} = \sum_{j=1}^{w-1} \sum_{i=1}^{k} n_i^j + n_i^w \) and suppose \( n_i^w \neq 0 \), we then have
\[
C^c((\cup_{j=1}^{w-1} I^j) \cup b_1^w) \leq C^c((\cup_{j=1}^{w-1} I^j) + \gamma g(b_1^w) + n_1^w \left( \gamma \sum_{i=1}^{k} n_i^{w-1} + k \tau + S_1^{w-1} \right), \quad (A.4)
\]

and

\[
C^m((\cup_{j=1}^{w-1} I^j) \cup b_1^w) = \alpha C^c((\cup_{j=1}^{w-1} I^j) \cup b_1^w) + \gamma g(b_1^w)
\geq \alpha C^c((\cup_{j=1}^{w-1} I^j) + \gamma E_1^w n_1^w + \gamma g(b_1^w)
\geq (1 - \alpha)C^c((\cup_{j=1}^{w-1} I^j) + (1 - \alpha)g(b_1^w)
\geq (1 - \alpha)E_1^w n_1^w + (1 - \alpha)n_1^w (\bar{n} - q_1^w), \quad (A.5)
\]

where

\[
E_1^w \geq S^1 + q_1^w, \quad (A.6)
\]

and

\[
E_1^{w*} \geq R_1^{w} \geq S_{1}^{w-1} + n_{1}^{w-1}. \quad (A.7)
\]

The RHS of Inequality (A.4) is because the completion time of batch \(b_1^w\) is bounded by \(\gamma\) times its pure completion time \(g(b_1^w)\), which is the maximal pure completion time for \(b_1^w\) (if all the workload in \(b_1^w\) is \(p_{\text{max}} = \gamma\)), plus \(n_1^w\) times the maximal starting time \(\gamma \sum_{i=1}^{k} n_i^{w-1} + k \tau + S_1^{w-1}\). Inequality (A.5) follows from Lemmas 23 and 24 directly. Inequality (A.6) holds because before \(E_1^w\), the server has served \(q_1^w\) number of jobs. Inequality (A.7) is because the earliest time to serve \(b_1^w\) is no earlier than \(R_1^w\). From Inequalities (A.4, A.5, A.6 and A.7) we have

\[
\frac{C^c((\cup_{j=1}^{w-1} I^j) \cup b_1^w)}{C^m((\cup_{j=1}^{w-1} I^j) \cup b_1^w)} \leq \frac{C^c((\cup_{j=1}^{w-1} I^j) + \gamma g(b_1^w) + n_1^w \left( \gamma \sum_{i=1}^{k} n_i^{w-1} + k \tau + S_1^{w-1} \right)}{\alpha C^c((\cup_{j=1}^{w-1} I^j) + \gamma g(b_1^w) + n_1^w \left( \gamma \sum_{i=1}^{k} n_i^{w-1} + k \tau + S_1^{w-1} \right)}
\leq \max\left\{ \frac{C^c((\cup_{j=1}^{w-1} I^j) \cup b_1^w)}{C^c((\cup_{j=1}^{w-1} I^j) \cup b_1^w)}, \frac{\gamma g(b_1^w) + n_1^w \left( \gamma \sum_{i=1}^{k} n_i^{w-1} + k \tau + S_1^{w-1} \right)}{\gamma S_1^{w} - n_1^{w-1} + (1 - \alpha)\gamma \bar{n}} \right\}
\leq \max\{\kappa, \gamma, k + 1, \frac{\gamma}{1 - \alpha}\}.
\]

Notice that \(\max\{\kappa, \gamma, k + 1, \frac{\gamma}{1 - \alpha}\} = \max\{\kappa, \gamma, k + 1, \frac{\gamma k + 1}{k}\} \leq \max\{\kappa, \gamma, k + 1, \frac{3}{2}\gamma\} = \kappa\) from the fact that \(k \geq 2\) and \(\alpha = \frac{1}{k+1}\).

Now suppose the result holds for \(\tilde{b} = (\cup_{j=1}^{w-1} I^j) \cup b_1^w\) where \(w \geq 2\), and we want to show it also holds for \(\tilde{b} \cup b_{l+1}^w = (\cup_{j=1}^{w-1} I^j) \cup (\cup_{i=1}^{l} b_i^w)\) for \(l < k\) by induction, where \(n_{l+1}^w \neq 0\). We abuse the notion by letting \(\bar{n} = \sum_{j=1}^{w} \sum_{i=1}^{k} n_i^j + \sum_{i=1}^{l} n_i^j\) be the number of jobs served before \(b_{l+1}^w\), and \(\bar{n}_{l+1}^w = \sum_{j=l+1}^{k} n_j^w + \sum_{j=1}^{l} n_j^w\) be the number of jobs served between \(S_{l+1}^{w-1}\) and \(S_{l+1}^w\). Because the server stays in queue \(i\) at the \(kth\) visit for no more than time \(\gamma n_i^k\) and serves \(n_i^k\) jobs, we have
\[ C^c(\bar{b} \cup b^{w}_{t+1}) \leq C^c(\bar{b}) + \gamma g(b^{w}_{t+1}) + n^{w}_{t+1} \left( \gamma \bar{n}^{w}_{t+1} + k\tau + S^{w-1}_{t+1} \right), \]

and

\[ C^m(\bar{b} \cup b^{w}_{t+1}) = \alpha C^* (\bar{b} \cup b^{w}_{t+1}) + (1 - \alpha)C^t(\bar{b} \cup b^{w}_{t+1}) \]
\[ \geq \alpha C^*(\bar{b}) + \alpha E^{w}_{t+1} n^{w}_{t+1} + \gamma g(b^{w}_{t+1}) \]
\[ + (1 - \alpha)C^*(\bar{b}) + (1 - \alpha)g(b^{w}_{t+1}) \]
\[ + (1 - \alpha)E^{w}_{t+1} n^{w}_{t+1} + (1 - \alpha)n^{w}_{t+1} (\bar{n} - q^{w}_{t+1}), \]

where

\[ E^{w}_{t+1} \geq \tau + q^{w}_{t+1}, \]

and

\[ E^{w*}_{t+1} \geq R^{w}_{t+1} > S^{w-1}_{t+1} + n^{w-1}_{t+1}. \]

Similar to our discussion above, we have

\[ \frac{C^c(\bar{b} \cup b^{w}_{t+1})}{C^m(\bar{b} \cup b^{w}_{t+1})} \leq \frac{C^c(\bar{b}) + \gamma g(b^{w}_{t+1}) + n^{w}_{t+1} \left( \gamma \bar{n}^{w}_{t+1} + k\tau + S^{w-1}_{t+1} \right)}{\alpha C^*(\bar{b}) + (1 - \alpha)C^t(\bar{b}) + \alpha S^{w-1}_{t+1} n^{w}_{t+1} + g(b^{w}_{t+1}) + (1 - \alpha)n^{w}_{t+1} n^{w}_{t+1} + (1 - \alpha)\bar{n}^{w}_{t+1}} \]
\[ \leq \max \left\{ \frac{C^c(\bar{b})}{C^m(\bar{b})}, \gamma \frac{g(b^{w}_{t+1})}{S^{w-1}_{t+1} + (1 - \alpha)\bar{n}} \right\} \]
\[ \leq \max\{ \kappa, \gamma, k + 1, \frac{\gamma}{1 - \alpha} \} = \kappa. \]

Now we show that results hold for the second type of busy period, i.e., II-B. For II-B, the first arrival occurs in a different queue from where the server was idling, thus the server starts this period with a setup. Note that the very first busy period is also a II-B. If \( I^1 \) belongs to the first busy period, then

\[ C^t(I^1) \geq g(I^1) + \tau \sum_{i=1}^{k} \sum_{j=i}^{k} n^{1}_{(k-j+1)}, \]

\[ C^c(I^1) \leq \gamma g(I^1) + \tau \sum_{i=1}^{k} \sum_{j=i}^{k} n^{1}_{(j)} + \tau \sum_{i=1}^{k} n^{1}_{i}. \]
Thus
\[
\frac{C^c(I^1)}{C^o(I^1)} \leq \frac{\gamma g(I^1) + \tau Z(k) + \tau \sum_{i=1}^{k} n_i^1}{g(I^1) + \tau Z^o(k)} \leq \max\{\gamma, k + 1\} \leq \kappa.
\]

If \(I^1\) does not belong to the first busy period, then
\[
C^o(I^1) \geq g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(i-j+1)}^1 + E^1 \sum_{i=1}^{k} n_i^1,
\]
and
\[
C^c(I^1) \leq \gamma g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(i-j+1)}^1 + S^1 \sum_{i=1}^{k} n_i^1.
\]

Thus if \(R^1 \geq 2\tau\), then \(\frac{R^1}{R^1 - \tau} \leq 2\), then

\[
\frac{C^c(I^1)}{C^o(I^1)} \leq \frac{\gamma g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(i-j+1)}^1 + S^1 \sum_{i=1}^{k} n_i^1}{g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(i-j+1)}^1 + E^1 \sum_{i=1}^{k} n_i^1}
\]
\[
\leq \frac{\gamma g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(i-j+1)}^1 + (R^1 + \tau) \sum_{i=1}^{k} n_i^1}{g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(i-j+1)}^1 + R^1 \sum_{i=1}^{k} n_i^1}
\]
\[
\leq \max\{\gamma, k, 2\}
\]
\[
< \kappa.
\]

Inequality (A.8) follows from \(E^1 \geq R^1 = \min_{i=1}^{k} R_i^1 \geq \tau\) and \(S^1 = R^1 + \tau\) because the server would immediately set up the queue where a new arrival occurs after an idling period. If \(R^1 \leq \tau\) then \(I^1\) belongs to the very first busy period, which we have discussed. If \(\tau < R^1 < 2\tau\), then the online policy has only scheduled at most one batch before \(R^1\). Since the new busy period belongs to II-B, both online and optimal policy in this busy period start from processing a queue different from the queue processed in the previous busy period. We then have \(E^1 \geq 2\tau\) and

\[
\frac{C^c(I^1)}{C^o(I^1)} \leq \frac{\gamma g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(i-j+1)}^1 + (R^1 + \tau) \sum_{i=1}^{k} n_i^1}{g(I^1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(i-j+1)}^1 + 2\tau \sum_{i=1}^{k} n_i^1}
\]
\[
\leq \frac{\gamma g(I^1) + \tau Z(k) + R^1 \sum_{i=1}^{k} n_i^1}{g(I^1) + \tau Z^o(k) + \tau \sum_{i=1}^{k} n_i^1}
\]
\[
\leq \max\{\gamma, k, 2\} \leq \kappa.
\]

Discussion for \((\cup_{j=1}^{w-1} I_j) \cup b_{l+1}^w\) for \(l = 0, ..., k - 1\) is similar to our discussion for I-B.

Now we prove the approximate tightness argument of this theorem by constructing a special instance \(I\).
When $\frac{3}{2}\gamma \leq (k + 1)$, we have $\kappa = k + 1$. Suppose at time 0 there is one job with workload $\gamma$ at queue 1 and one job with $p = 1$ at the other queues. At time $\gamma + \tau + \epsilon_1$ (with small $\epsilon_1 > 0$) a batch $b_1^2$ arrives at queue 1 and each job in $b_1^2$ has workload $p = 1$. We thus have

$$C^e(I) \geq g(I) + n_1^2(k + 1)\tau + \frac{k(k + 1)}{2}\tau,$$

and

$$C^*(I) \leq \gamma g(I) + n_1^2(\tau + \epsilon_1) + \frac{k(k + 1)}{2}\tau + (k - 1)\epsilon_1.$$

Thus when $\tau = (n_1^2)^2$ there is an $n_1^2$ such that for arbitrary $\epsilon$,

$$\frac{C^e(I)}{C^*(I)} > 1 + k - \epsilon.$$

The theorem also holds for $p_{\text{max}} = 0$, for simplicity we do not show the proof here.

\[\Box\]

B Proof for Theorem 2 in the Main Paper

**Theorem 26. (Theorem 2 in the main paper)** Any policy in $\Pi_2$ has competitive ratio of $\kappa = \max\{\frac{3}{2}\gamma, k + 1\}$ for the polling system $1 | r_i, \tau, p_{\text{max}} \leq \gamma p_{\text{min}} | \sum C_i$. When $\frac{3}{2}\gamma \leq k + 1$, for arbitrary $\epsilon > 0$, there is an instance $I$ such that $\frac{C^e(I)}{C^*(I)} > \kappa - \epsilon$.

**Proof.** The proof is similar to the one for Cyclic with Skipping the Empty Queue, however this time we only need to show $C^{ew}(I) \leq \kappa C^m(I)$ for any busy period $I$ where $C^{ew}(I)$ is the completion time by a policy from $\Pi_2$. Notice we no longer need to consider different cases for I-B and II-B because even when the system is idling, the cyclic policy still keeps setting up queues in cycle. When a new arrival occurs after the system being empty for some time, we simply regard this time $R^1$ as the beginning of a busy period. Without loss of generality, we assume the server begins serving with queue 1 in this busy period. Let $n_{(k)}^1 \geq n_{(k-1)}^1 \geq ... \geq n_{(1)}^1$ be the descending permutation of $(n_1^1, ..., n_k^1)$, $R^1 = \min_{i=1}^k R_i^1$, $E^1 = \min_{i=1}^k E_i^1$ and $S^1 = \min_{i=1}^k S_i^1$. Notice some of $n_{(i)}^1$ may be zero (not all of them) but the server still sets up the queue even if the queue is empty. We have

$$C^m(I_1) \geq g(I_1) + \tau \sum_{i=2}^k \sum_{j=i}^k n_{(k-j+1)}^1 + E^1 \sum_{i=1}^k n_i^1,$$

and

$$C^{ew}(I_1) \leq \gamma g(I_1) + \tau k \sum_{i=1}^k n_{(i)}^1 + R^1 \sum_{i=1}^k n_i^1.$$
To show $\frac{C^w(I)}{C^m(I)} \leq \kappa$, by abusing the notation a little, we first let $Z(k) = k \sum_{i=1}^{k} n_i^{1_i}$ and $Z'(k) = \sum_{i=1}^{k} \sum_{j=i}^{k} n_{i-j+1}^{1_i}$, so

$$Z(k) = \frac{kn_i^{1_i} + kn_{i-k+1}^{1_i} + \ldots + kn_{(k-1)}^{1_i}}{kn_1^{1_i} + (k-1)n_2^{1_i} + \ldots + n_k^{1_i}} \leq \frac{kn_i^{1_i} + kn_{i-k+1}^{1_i} + \ldots + kn_{(k-1)}^{1_i}}{n_1^{1_i} + n_2^{1_i} + \ldots + n_k^{1_i}} \leq k.$$ 

Thus

$$\frac{C^w(I)}{C^m(I)} \leq \frac{\gamma g(I_1) + \tau k \sum_{i=1}^{k} n_i^{1_i} + R^1 \sum_{i=1}^{k} n_i^{1_i}}{g(I_1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{i-j+1}^{1_i} + E^1 \sum_{i=1}^{k} n_i^{1_i}} \leq \frac{\gamma g(I_1) + \tau k \sum_{i=1}^{k} n_i^{1_i} + \tau \sum_{i=1}^{k} n_i^{1_i}}{g(I_1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{i-j+1}^{1_i} + \tau \sum_{i=1}^{k} n_i^{1_i}} \leq \max\{\gamma, k+1\}$$

Inequality [B.1] holds because if $R^1 \leq \tau$, then

$$\frac{\gamma g(I_1) + \tau k \sum_{i=1}^{k} n_i^{1_i} + R^1 \sum_{i=1}^{k} n_i^{1_i}}{g(I_1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{i-j+1}^{1_i} + E^1 \sum_{i=1}^{k} n_i^{1_i}} \leq \frac{\gamma g(I_1) + \tau k \sum_{i=1}^{k} n_i^{1_i} + \tau \sum_{i=1}^{k} n_i^{1_i}}{g(I_1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{i-j+1}^{1_i} + \tau \sum_{i=1}^{k} n_i^{1_i}} \leq \max\{\gamma, k+1\}.$$ 

And if $R^1 > \tau$, then

$$\frac{\gamma g(I_1) + \tau k \sum_{i=1}^{k} n_i^{1_i} + R^1 \sum_{i=1}^{k} n_i^{1_i}}{g(I_1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{i-j+1}^{1_i} + E^1 \sum_{i=1}^{k} n_i^{1_i}} \leq \frac{\gamma g(I_1) + \tau (k+1) \sum_{i=1}^{k} n_i^{1_i} + (R^1 - \tau) \sum_{i=1}^{k} n_i^{1_i}}{g(I_1) + \tau \sum_{i=1}^{k} \sum_{j=i}^{k} n_{i-j+1}^{1_i} + (R^1 - \tau) \sum_{i=1}^{k} n_i^{1_i}} \leq \max\{\gamma, k+1\}.$$

The discussion for $(\cup_{j=1}^{y} I_j) \cup b_{k+1}$ is similar to proof of Theorem 1. \qed

\section{Proof for Theorem 3 in the Main Paper}

\textbf{Theorem 27. (Theorem 3 in the Main Paper)} Any policy in $\Pi_3$ has competitive ratio of $\kappa = \max\{\frac{3}{2}\gamma, k+1\}$ for the polling system $1 \mid r, \tau, p_{\max} \leq \gamma p_{\min} \mid \sum C_i$. When $\frac{3}{2}\gamma \leq k+1$, for arbitrary $\epsilon > 0$, there is an instance $I$ such that $\frac{C^w(I)}{C^m(I)} > \kappa - \epsilon$.

\textit{Proof.} We only show the proof for the policy without skipping the empty queue. The proof is similar to the proof for Theorem \ref{thm:main}. We prove the theorem by induction. Again for simplicity, we assume $p_{\min} = 1$ so that $p_{\max} = \gamma$. We want to show $C^w(I) \leq \kappa C^m(I)$ holds for any instance $I$, where $C^w(I)$ is the completion time for any policy $g \in \Pi_3$. Again we assume the server routes from queue 1 to queue $k$ in each cycle. Let $n_{(k)}^{1_i} \geq n_{(k-1)}^{1_i} \geq \ldots \geq n_{(1)}^{1_i}$ is the descending permutation of $(n_1^1, \ldots, n_k^1)$ and $E^1 = \min_{i=1}^{k} E_i^1$, $S^1 = \min_{i=1}^{k} S_i^1$, we have
\[ C^t(I_1) \geq g(I_1) + \tau \sum_{i=2}^{k} \sum_{j=i}^{k} n_{(k-j+1)}^i + E^1 \sum_{i=1}^{k} n_i^1, \]

and

\[ C^g(I_1) \leq g(I_1) + \tau k \sum_{i=1}^{k} n_i^1 + S^1 \sum_{i=1}^{k} n_i^1. \]

The rest of discussions are similar to the proof of Theorem 1, except now we have \( E_{t+1}^w > R_{t+1}^w \) because the policy is gated.

| Notation | Meaning | Notation | Meaning |
|----------|---------|----------|---------|
| \( b^w_i \), \( i = 1, \ldots, k \) | The job instance that are served by the cyclic online policy during the \( w \)th visit (\( w \)th cycle) to queue \( i \), within a busy period | \( I^w \), \( w = 1, 2, \ldots \) | \( I^w = \bigcup_{i=1}^{k} b^w_i \); the union of instances that are served by the online policy during the \( w \)th cycle within a busy period |
| \( n^w_i = n(b^w_i) \), \( i = 1, \ldots, k \) | The number of jobs in batch \( b^w_i \) | \( \alpha = \frac{1}{k+1} \) | A constant |
| \( S^w_i \), \( i = 1, \ldots, k \) | The time when the online policy starts to serve batch \( b^w_i \) | \( S^1 = \min_{i=1}^{k} \{ S^1_i \} \) | The earliest staring time for \( I^1 \) by the online policy |
| \( R^w_i \), \( i = 1, \ldots, k \) | The earliest release time (arrival time) of batch \( b^w_i \) | \( R^1 = \min_{i=1}^{k} \{ R^1_i \} \) | The earliest release time of \( I^1 \) |
| \( E^w_i \) | The time when the truncated optimal offline policy starts to serve batch \( b^w_i \) | \( E^1 = \min_{i=1}^{k} \{ E^1_i \} \) | The earliest time when the truncated optimal offline policy starts to serve \( I^1 \) |
| \( E^{w*}_i \) | The time when the optimal offline policy starts to serve batch \( b^w_i \) | \( q^w_i, i = 1 \ldots k \) | The jobs in \((\bigcup_{j=1}^{w-1} I^j) \cup (\bigcup_{i=1}^{k} b^w_i)\) that have been served by the optimal policy before \( E^w_i \) |
| \( g(I) = \frac{n(I)(n(I)+1)}{2} \) | Pure completion time for instance \( I \) | Busy period | The time period between two consecutive empty periods |
| I-B | Type I busy period. The server start the new busy period without setting up | II-B | Type II busy period. The server start the new busy period by setting up |

Table 4: List of Notations for Appendix