Mechanical pressure and work cycle of confined active Brownian particles

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Abstract – We derive an analytic expression for the mechanical pressure of a generic one-dimensional model of confined active Brownian particles (ABPs) that is valid for all values of Péclet number Pe and all confining scenarios. Our model reproduces the known scaling of bulk pressure with $\text{Pe}^2$ while in strong confinement pressure scales with Pe. Our analytic results are very well reproduced by simulations of ABPs in 2D. We use the pressure formula to calculate both the work performed by an active engine and its efficiency. In particular, efficiency is maximized for work cycles with finite period and not in the limit of infinitely slow cycles as in thermodynamic engines.

Introduction. – The properties and structure formation of active systems are quite different as compared to their equilibrium counterparts [1–7]. This becomes particularly apparent in confinement [8–17]. Indeed, active particles accumulate at walls [18], interfaces [19, 20], as well as obstacles [21,22], and in denser suspensions they show motility–induced phase separation [23]. One of the key macroscopic quantities of interest is the mechanical pressure $\Pi$ that active particles exert on confining walls [10, 24–26]. Such a quantity is crucial for determining the performance of devices rectifying active motion [27–29], for work cycles that exploit active baths [30], for invasion of active particles into confining space [31], as well as evaporation [32] and wetting [33,34] in active fluids.

Predicting the value of $\Pi$ for active systems is not trivial since, due to the active nature of the particles, pressure is, in general, no longer a thermodynamic state function [35]. It is not even an intensive variable since it depends explicitly on extensive variables (like number of particles), as we show below. Several articles [10,24,25] have reported independently the characteristic scaling for the pressure, $\Pi \propto \text{Pe}^2$, where the Péclet number $\text{Pe} = \frac{v_{\text{act}} R}{D}$ depends on the active velocity $v_{\text{act}}$, the linear size $R$, and the diffusion coefficient $D$ of the particle. This relation for pressure has been derived for semi–infinite systems. However, the dynamics of active Brownian particles (ABPs) is very sensitive to the presence of boundaries [8–10] and it is not obvious that the scaling for the pressure also holds for confined ABPs. Indeed, recent numerical works [9,10] have shown that the pressure of strongly confined active particles scales as $\Pi \propto \text{Pe}$. Therefore, the scaling of $\Pi$ with Pe depends on the system size. At the moment, a comprehensive relation, valid for all confining scenarios, between the pressure and the microscopic parameters (such as active velocity and tumbling rate), which control the dynamics of ABPs, is still lacking.

In this letter we derive a closed–form expression for the pressure exerted by confined ABPs that is valid for all values of Pe and all confining scenarios. In order to do so, we consider simple ABPs that only move in one dimension either along the $x$ axis (“up” state) or against it (“down” state) and that tumble between both states. Furthermore, the ABPs experience a confining soft potential such that, in the limit of diverging potential strength, our model retrieves the case of ABPs confined in a box with hard walls. In this standard case, the calculated mechanical pressure displays multiple scalings with Pe. In particular, when particles undergo multiple tumbling...
events between subsequent collisions with the walls (diffusive regime), pressure scales as $\Pi \propto \Pe^2$ in agreement with Refs. [10, 24, 35]. In contrast, for strongly confined active colloids or for very large values of $\Pe$ (as the one attained for dry macroscopic active matter [36–41]) particles only undergo a few (if at all) tumbling events (ballistic regime) and the pressure scales as $\Pi \propto \Pe$. Our analytic predictions for the pressure are in very good agreement with results from numerical simulations of ABPs in two dimensions, with the numerical results of Refs. [9, 10] and with the expansion approach of Ref. [34] which exclusively treats the case where pressure scales as $\Pi \sim \Pe^2$. Thus, despite its simplicity our model captures the essence of the dynamics of confined ABPs for all values of $\Pe$. Hence, it can be used to predict the mechanical pressure of ABPs on both the microscopic ($\sim \mu m$) and macroscopic ($\sim cm, m$) scale in all confining scenarios.

We apply the analytical formula for the mechanical pressure to the recently introduced work cycle of active engines [35, 11] which have the capability of a bath of ABPs to perform directed work (see also a recent review on the topic Ref. [45]). We find that the work is governed by two dimensionless parameters and that the efficiency of quasi-static work cycles is optimal for a finite period in contrast to thermodynamic engines.

**Model.** — The $N$ noninteracting ABPs experience the confining potential

$$\beta U(z) = \begin{cases} 
 f(z/L-1) & z > L, \\
 0 & -L \leq z \leq L, \\
 -f(z/L+1) & z < -L,
\end{cases}$$

where $L$ is the size of the system (not including the soft walls), $f$ controls the softness of the walls, $\beta = 1/(k_BT)$, $k_B$ is the Boltzmann constant, and $T$ is the temperature. Within the overdamped regime, the time evolution of the densities for all up and down states, “up” ($\rho_{t}$) and “down” ($\rho_{l}$) states, which we express as functions of the dimensionless position $x = z/L$ and with time in units of $L^2/D$, are governed by

$$\dot{\rho}_t(x) = -\partial_x J_t - \Gamma \frac{L^2}{R^2} [\rho_t(x) - \rho_l(x)], \quad (2a)$$
$$\dot{\rho}_l(x) = -\partial_x J_l + \Gamma \frac{L^2}{R^2} [\rho_l(x) - \rho_t(x)]. \quad (2b)$$

For simplicity we do not denote explicitly the dependence on time. In Eqs. (2) we have identified the fluxes as

$$J_t(x) = -\left[\partial_x \rho_t(x) - \frac{L}{R} \Pe \rho_t(x) + \rho_t(x) \beta \partial_x U(x)\right], \quad (3a)$$
$$J_l(x) = -\left[\partial_x \rho_l(x) + \frac{L}{R} \Pe \rho_l(x) + \rho_l(x) \beta \partial_x U(x)\right], \quad (3b)$$

and we have introduced

$$\Pe = \frac{\nu_{\text{act}} R}{D}, \quad \Gamma = \gamma R^2/D, \quad (4)$$

the particle Péclet number $\Pe$ and dimensionless tumbling rate $\Gamma$ defined as tumbling rate $\gamma$ times the diffusion time scale $R^2/D$. For later use, we note that $\Pe^2/\Gamma = v_{\text{act}}^2 \gamma^{-1}/D$ is the ratio of active to passive diffusion coefficients. We note the ratio $\Pe/\Gamma = v_{\text{act}}/(\gamma R)$ can be identified with the dimensionless rotational Péclet number $\Pe_{\gamma}$.

By solving Eqs. (2) in steady state using piecewise solutions in the three regions of $\beta U(x)$ (see Sec. S1 of the Suppl. Mat.), we compute the dimensionless mechanical pressure exerted on the right wall (the same results hold for the left wall),

$$\Pi = \int_{-1}^{\infty} [\rho_l(x) + \rho_l(x)] \, dx. \quad (5)$$

**Pressure.** — In order to study the case of ABPs confined within a box, we take the limit of the hard–core potential ($f \to \infty$, see Sec. S1 of the Suppl. Mat.). In this limit, $\Pi$ becomes

$$\Pi_{\infty} = \bar{\rho} \frac{R^2}{L^2} \frac{\kappa^3}{\Pe^2} \text{cosh} \kappa \gamma R \cosh \kappa \gamma R. \quad (6)$$

Here, $\bar{\rho} = NR/2L$ is the dimensionless number density and

$$\kappa = \kappa L = \frac{\sqrt{\Pe^2 + 2\Gamma \kappa}}{\gamma R}, \quad (7)$$

where $\kappa$ is the inverse of the effective length that characterizes the exponential decay of the density profile close to the wall. We remark that $\kappa$ depends solely on microscopic parameters and not on the system size. In particular, when $\kappa \gg 1$, Eq. (6) is approximated by

$$\Pi_{\infty} \simeq \bar{\rho} \frac{R^2}{L^2} \frac{\kappa^3}{\Pe^2 + 2\Gamma \kappa}. \quad (8)$$

The regime $\kappa \gg 1$ is typical for active matter as it occurs whenever either $\Pe \gg R/L$ or $\Gamma \gg R^2/L^2$. The latter means that during passive diffusion across the system, tumbling occurs frequently. Therefore, in the following we focus on the relevant case of $\kappa \gg 1$. In Sec. S2 of the Suppl. Mat. we present mathematical derivation of all limiting regimes discussed in this letter and argue that the results are valid even for moderately large values of $\kappa$.

Figures (a) and (b) show that $\Pi_{\infty}$ grows monotonically upon increasing $\Pe$, as expected. In the limit of small Péclet numbers, $\Pe < \Pe_{\text{small}} = \sqrt{2\Gamma}$, Eq. (2) gives

$$\Pi_{\text{small}} \simeq \bar{\rho} \left(1 + \frac{\Pe^2}{2\Gamma}\right), \quad \Pe < \Pe_{\text{small}}. \quad (9)$$

Thus, for vanishingly small $\Pe$, $\Pi_{\infty}$ reduces to its equilibrium value $\Pi_{\text{equ}} = \bar{\rho}$. We note that the expression in

$^1$The exponential decay and the associated decay length (Eq. (7)) are valid for all values of $\Pe$ and $L$. In particular, for small values of $\Pe$ Eq. (7) reduces to $\kappa \simeq \Gamma$, in agreement with Ref. [13].

$^2$We remark that if $L/R > 10$, $\kappa \gg 1$ provided that $\Pe > 0.1$ or $\Gamma > 0.01$. 
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Fig. 1: (a): Pressure $\Pi_\infty$ as function of Pe for diverse values of $\Gamma$ and $L = 10R$. (b): $\Pi_\infty$ as function of Pe for $\Gamma = 3/4$ and diverse values of $L/R$. The dots present the results of 2D numerical simulations of ABPs characterized by the same value of Pe, the linear size $L$ and $\Gamma$, and with no fitting parameters (see Sec. S6 of the Suppl. Mat.).

the brackets agrees with the effective temperature $T_{\text{eff}} = T(1 + Pe^2/2\Gamma)$ introduced, for example, in Refs. [46,47] (see also Sec. S3 of the Suppl. Mat.). We remark that $\Pi_\infty^0$ does not depend explicitly on system size $L$ or particle number $N$, as required for an intensive thermodynamic quantity. For $Pe \approx Pe_{\text{small}}$, activity starts to dominate the pressure. Using Eqs. [4], the condition $Pe \approx Pe_{\text{small}}$ implies $D_{act} = v_{act}/\gamma \simeq D$. Thus, the pressure starts to grow with $Pe^2$ when the active contribution to the total diffusion coefficient $D_{eff} = D + D_{act}$ becomes dominant [46,47].

For large Péclet numbers ($Pe \gg Pe_{\text{small}}$) we obtain from Eq. [5]:

$$\Pi_\infty^{\text{large}} \simeq \frac{\rho Pe^2}{2\Gamma}, \quad \text{for } Pe_{\text{large}} \gg Pe \gg Pe_{\text{small}}, \quad (10a)$$

$$\Pi_\infty^{\text{large}} \simeq \frac{\rho L}{R} Pe, \quad \text{for } Pe \gg Pe_{\text{large}}, Pe_{\text{small}}, \quad (10b)$$

where $Pe_{\text{large}} = 2FL/R$. When $Pe \gg Pe_{\text{large}}$, pressure $\Pi_\infty$ attains its asymptotic form [10b], $\Pi_\infty \simeq N Pe/2$, growing linearly in $Pe$. In this regime the pressure is no longer an intensive variable since it depends explicitly on particle number $N$.

The crossover between the different scalings of $\Pi_\infty$ with $Pe$ occurs at $Pe = Pe_{\text{large}} = 2FL/R$, i.e., when the ballistic time $L/v_{act}$ is comparable to the mean run time $1/\gamma$ between two tumbling events. Thus, in the asymptotic regime where $L/v_{act} \ll 1/\gamma$, particles only undergo a few (if at all) tumbling events between subsequent collisions with the walls (ballistic regime). As a result, they spend the large amount of time at the boundaries. Therefore, the pressure increases linearly with $Pe$ and depends explicitly only on particle number and not on system size. In the opposite case $L/v_{act} \gg 1/\gamma$ (i.e., $Pe \ll Pe_{\text{large}}$), the particles undergo multiple tumbling events between subsequent collisions with the walls (diffusive regime). In this regime, increasing $Pe$ has a twofold effect: first, it directly enhances pressure when the particles hit the wall, and second, it reduces the number of tumbling events between to subsequent collisions at the walls and thereby enhances the density of the particles at the wall. This twofold effect explains the quadratic dependence of $\Pi_\infty$ on $Pe$ in Eq. [10a], as shown in both panels of Fig. 1.

Accordingly, Fig. 1(a) shows that the scaling of the pressure with $Pe$ changes dramatically upon changing the tumbling rate $\Gamma$. At the micrometric scale, this result is crucial for confined bacterial suspensions [48,49], whose tumbling rate depends on both the biology of the bacteria as well as on external control parameters. At the macroscopic scale our result is crucial for determining the pressure of dry active matter, such as small robots [57], ants [58,59], sheep [60], and humans [60,61], just to mention a few among others.

For active colloids, such as Janus particles, $\Gamma$ is controlled by the rotational diffusion coefficient $D_{rot}$, which depends on their size and shape. For spherical particles $D_{rot} = \frac{\gamma}{2\Gamma}$, which in 2D equals $\gamma$ so that $\Gamma = 3/4$. For this case, Fig. 1(b) presents $\Pi_\infty$ versus $Pe$ for diverse system sizes. Interestingly, for typical values of the Péclet number ($Pe \approx 1 - 100$) and system sizes ($L \gtrsim 100R$) that have been investigated experimentally [50] or numerically [51], our model predicts $\Pi_\infty \propto Pe^2$, in agreement with Ref. [10,24,25,35,51,52]. However, for smaller system sizes $L \approx 10R$ or for very large values of the Péclet number, $Pe \gg Pe_{\text{large}}$, the asymptotic behavior $\Pi_\infty \propto Pe$ is retrieved.

In order to check the validity of our expression against more realistic models, we performed 2D simulations of spherical ABPs characterized by $\Gamma = 3/4$, where the particle orientations diffused on the unit circle (see Sec. S6 of the Suppl. Mat. for more details on the simulations). The results are included in Fig. 1(b). Interestingly, without using any fitting parameters, the agreement between the theoretical predictions and the results of the numerical simulations is very good for all values of $Pe$ and $L$ that we tested. Hence, our simple two–state model captures the essence of the dynamics of confined ABPs [55].

Compressibility. – Having an explicit expression for the pressure [cf. Eq. (4)], we can calculate explicitly the dimensionless compressibility $H_\infty = -\langle \partial \Pi_\infty / \partial L \rangle^{-1}/L$ (see [53]).

Typically, $\Gamma \gtrsim 100$. Even upon genetically switching off tumbling, the lower bound is $\Gamma \simeq 1$ due to rotational diffusion. In addition, in chemical gradients $\Gamma$ can vary by a factor of five [10].
less work per particle along a cycle reads:

$$W = \int_{L_1}^{L_2} \Pi_{\infty}(L, Pe) \frac{2dL}{R} \int_{L_1}^{L_2} \Pi_{\infty}(L, Pe) \frac{2dL}{R}.$$  \hfill (12)

Figure 3(b) shows that for small system sizes the work performed by the system grows linearly with $L_1$ (for fixed ratio $L_2/L_1$). In this regime the work is insensitive to $\Gamma$ [blue and cyan curves as well as red and orange curves lie on top of each other in Fig. 3(b)], while the overall amount of work depends on $Pe$. Upon increasing $L_1$ further, the work $W$ reaches a plateau. Here, $W$ increases upon decreasing $\Gamma$ for both values of $Pe$, since tumbling reduces the pressure of the expanding system. The dependence of $W$ on $\Gamma$ shows that the work performed by the active system explicitly depends on the tumbling rate, i.e., on some microscopic time scale. Such a dependence does not occur in passive systems and therefore is a signature of the active nature of the system under study. In contrast, the dependence of $W$ on $Pe$ is clear since the active motion of the ABPs generates the force with which they push against the wall.

All these observations can be rationalized by considering the limit $\kappa_c \gg 1$ in Eq. (12) which gives

$$W \approx W_0 \ln \left[ \frac{1 + L_2/L}{1 + L_1/L} \right] - \ln \left[ \frac{L_2}{L_1} \right].$$  \hfill (13)

Here, we have identified the effective length $\Lambda$ and strength $W_0$ of the work cycle,

$$\Lambda = \frac{Pe^2 R}{2\Gamma \sqrt{Pe^2 + 2\Gamma}}, \quad W_0 = 1 + \frac{Pe^2}{2\Gamma}.$$  \hfill (14)

After rescaling work by $W_0$ and system size by $\Lambda$, for $W_0 \gg 1$ all curves from Fig. 3(b) collapse onto one master curve, as demonstrated in Fig. 3(c). In the regime $Pe \gg Pe_{\text{small}} = \sqrt{2\Gamma}$ the parameters become

$$\Lambda \approx \frac{v_{\text{act}}}{\gamma}, \quad W_0 \approx \frac{v^2}{D\gamma} = \frac{D_{\text{act}}}{D},$$  \hfill (15)

i.e., the threshold length $\Lambda$ is the typical distance traveled by the particle between two tumbling events, whereas the work strength $W_0$ is proportional to the ratio of active to passive diffusion coefficients. In particular, for large systems sizes, $L_1, L_2 \gg \Lambda$, we have

$$W \approx (W_0 - 1) \ln(L_2/L_1) \approx W_0 \ln(L_2/L_1).$$  \hfill (16)

Thus, the work per particle over one cycle solely depends on $W_0$ and its dependence on system size is reminiscent of the work done by a passive ideal gas.

**Efficiency.** – Finally, we define the efficiency of the work cycle as the ratio of the total work performed by the system composed of $N$ particles

$$W^{\text{tot}} = NW$$  \hfill (17)
In the case under study, we estimate the velocity via the Péclet number 
\[ \frac{\Delta L}{\tau} \]
where \( \Gamma \) accounts for the energy spent in dissipative processes. In the following we assume that the change in the value of Péclet number occurs with no additional friction forces can be expressed as
\[ W_{\text{irr}} = \frac{\mathcal{P}}{\Gamma} \]
Accordingly, the dissipated power is
\[ W = W_{\text{sys}} + W_{\text{irr}} = \mathcal{P} \Gamma + 4W \Delta L^2 / \tau \]
where \( \mathcal{P} \) and \( W \) are phenomenological parameters encoding, respectively, for the power dissipated by the active particle, which depends on the specific propulsion mechanism, and for the effective friction of the container. Maximizing the efficiency with respect to \( \tau \) amounts to minimizing \( W_{\text{irr}} \) which gives an optimal time of the cycle
\[ \tau_{\text{opt}} = \left( \frac{4W \Delta L^2}{\mathcal{P}} \right)^{1/2} \]
Accordingly, even for quasi-static expansions, the efficiency \( \eta \) is maximized for a finite cycle time \( \tau \) in stark contrast to the quasi-static limit \( \tau \to \infty \) of thermodynamic engines for which \( W_{\text{irr}} \) is replaced by the heat \( Q \) that is independent of \( \tau \). This is in agreement with the numerical results of Ref. [59]. Interestingly, \( \tau_{\text{opt}} \) depends on the ratio between the dissipation in the system, \( 4W \Delta L^2 \) and that due to the active bath \( \mathcal{P} \). Hence, active engines exploiting many particles should be run at short cycling times, whereas the opposite holds for smaller particle numbers.

In the above calculations we have assumed that the particle density follows adiabatically the change in volume \( \Delta L \). We recall that the relaxation time of the system can be estimated as
\[ \tau_{\text{relax}} \approx \min \left( \frac{\Delta L^2}{D}, \frac{\Delta L}{\nu_{\text{act}}} \right) \]
Hence, the adiabatic assumption we made in deriving Eq. (12) is fulfilled when \( \tau_{\text{relax}} \ll \tau_{\text{opt}} \). For \( \tau_{\text{relax}} = \)
\footnote{For bacteria, \( \mathcal{P} \) amounts to the metabolic cost of keeping the bacteria alive and swimming. For diffusiophoretic colloids, \( \mathcal{P} \) amounts to the power dissipated in order to keep the imbalance in the bulk chemical potentials of the reactants and the reaction products. For light driven phoretic colloids, \( \mathcal{P} \) amounts to the power dissipated (per particle) by the light source. For ABPs, under the assumption that all the energy “consumed” by the internal mechanism responsible for active displacement is transformed into motion, we have \( \mathcal{P}_{\text{ABP}} = F \cdot v_0 = \frac{\Delta L^2}{\tau} \nu_{\text{act}} \), where \( F \) is the effective driving force.}

Fig. 3: a) Scheme of the work cycle of the active engine in the Pe–L plane. b) Work \( W \) performed by one cycle of the active engine as function of \( L_1 \) with \( L_2 = 2L_1 \) and \( Pe_1 = 0 \) for diverse values of \( Pe_2 = Pe \) and \( \Gamma \). c) Rescaled data of panel (b)

Particular, it does not depend on \( N \). Assuming instantaneous changes of \( Pe \), the velocity is \( 2(L_2 - L_1)/\tau = 2\Delta L/\tau \) and \( W_{\text{sys}} = 4W \Delta L^2 / \tau \) [57,58], where \( W \) plays the role of an effective friction coefficient. All in all we obtain
\[ W_{\text{irr}} = W_{\text{irr}}^{\text{pec}} + W_{\text{irr}}^{\text{sys}} = \mathcal{P} \Gamma + 4W \Delta L^2 / \tau \]
and hence the dissipated work during the cycle reads
\[ W_{\text{sys}} = \dot{W} \tau = 4W \frac{\Delta L^2}{\tau} \]
We remark that, in leading order, \( W_{\text{sys}} \) does not depend on the parameters characterizing the active particles, in particular, it does not depend on \( N \). Assuming instantaneous changes of \( Pe \), the velocity is \( 2(L_2 - L_1)/\tau = 2\Delta L/\tau \) and \( W_{\text{sys}} = 4W \Delta L^2 / \tau \) [57,58], where \( W \) plays the role of an effective friction coefficient. All in all we obtain
\[ W_{\text{irr}} = W_{\text{irr}}^{\text{pec}} + W_{\text{irr}}^{\text{sys}} = \mathcal{P} \Gamma + 4W \Delta L^2 / \tau \]
where \( \mathcal{P} \) and \( W \) are phenomenological parameters encoding, respectively, for the power dissipated by the active particle, which depends on the specific propulsion mechanism, and for the effective friction of the container. Maximizing the efficiency with respect to \( \tau \) amounts to minimizing \( W_{\text{irr}} \) which gives an optimal time of the cycle
\[ \tau_{\text{opt}} = \left( \frac{4W \Delta L^2}{\mathcal{P}} \right)^{1/2} \]

Accordingly, even for quasi-static expansions, the efficiency \( \eta \) is maximized for a finite cycle time \( \tau \) in stark contrast to the quasi-static limit \( \tau \to \infty \) of thermodynamic engines for which \( W_{\text{irr}} \) is replaced by the heat \( Q \) that is independent of \( \tau \). This is in agreement with the numerical results of Ref. [59]. Interestingly, \( \tau_{\text{opt}} \) depends on the ratio between the dissipation in the system, \( 4W \Delta L^2 \) and that due to the active bath \( \mathcal{P} \). Hence, active engines exploiting many particles should be run at short cycling times, whereas the opposite holds for smaller particle numbers.

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$\Delta L/v_{\text{act}}$ that leads to

$$N \ll \frac{4W_0 v_{\text{act}}^2}{P}. \quad (26)$$

Interestingly, Eq. (26) shows that the maximum efficiency can be attained by small systems, whereas larger systems containing a larger number of particles $N$, will be suboptimal.

**Conclusions.** – Based on a one-dimensional model for run-and-tumble particles, we have derived an analytic expression for the mechanical pressure ABPs exert on bounding walls. In the limit of large systems we reproduce the well-known scaling of the bulk pressure with $P_e^2$. In contrast, for either strongly confined micrometric ABPs or macroscopic ABPs with very large $P_e$, the pressure scales with $P_e$ and is no longer an intensive variable. We clearly rationalize the regimes where the different scalings are observed. Furthermore, two-dimensional Brownian dynamics simulations of ABPs quantitatively agree with our analytic expression and thereby show its generality.

Our analytic formula for pressure allows to systematically explore basic features of confined active systems between bulk- and surface-driven behavior. For the recently introduced active engines we have calculated the work performed during one cycle in the quasi-static limit. It explicitly depends on the characteristic time scale $\Gamma^{-1}$, a feature that is absent in conventional thermodynamic engines. Furthermore, the efficiency is maximized at a finite cycle rate due to the inherent dissipation, in clear contrast to thermodynamic engines where infinitely small rates avoid dissipation. Surprisingly, such an optimal cycling time is typical of “small engines” i.e., those engines exploiting a small number of active particles.

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SUPPLEMENTAL MATERIAL

S1. Model

In this section we analyze the dynamics of \( N \) noninteracting active Brownian particles (ABPs) confined in 1D and suspended in an equilibrium thermal bath. The ABPs are characterized by an “internal state” that determines the preferred direction of motion. When the particles are in the “up” state, the active contribution \( v_{\text{act}} \) to the overall velocity points to the right (parallel to \( x \) axis), while in “down” state it points to the left (antiparallel to \( x \) axis). The particles can randomly hop between the two states with the rate \( \gamma \). Additionally, we assume that the particles experience a confining potential

\[
\beta U(z) = \begin{cases} 
  f \frac{z - L}{L} & \text{for } z > L, \\
  0 & \text{for } -L \leq z \leq L, \\
  -f \frac{z + L}{L} & \text{for } z < -L,
\end{cases}
\]  

(S1)
such that when the center of mass of a particle is not in the region \(-L \leq z \leq L\) there is a constant force pushing the particle back into the region. The choice of a piecewise linear potential allows us for further analytical insight.

In the following, instead of the number density of particles \( \tilde{\rho}(z) \), we use the dimensionless quantities which we get by rescaling the distance \( x = z/L \) and by multiplying \( \tilde{\rho} \) by the particle size \( R \)

\[
\rho(x) = R \tilde{\rho}(z = xL).
\]  

(S2)
The potential in rescaled variables is

\[
\beta U(x) = \begin{cases} 
  f(x - 1) & \text{for } x > 1, \\
  0 & \text{for } -1 \leq x \leq 1, \\
  -f(x + 1) & \text{for } x < -1,
\end{cases}
\]  

(S3)
and the normalization condition is

\[
\int_{-\infty}^{\infty} \tilde{\rho}(z) \, dz = N \quad \Rightarrow \quad \int_{-\infty}^{\infty} \rho(x) \, dx = N \frac{R}{L}.
\]  

(S4)
The time evolution of the dimensionless probability distributions of particles in the “up state” \( \rho_\uparrow(x) \) and “down state” \( \rho_\downarrow(x) \) are governed by

\[
\dot{\rho}_\uparrow(x) = -\partial_x J_\uparrow - \Gamma \frac{L^2}{R^2} \left[ \rho_\uparrow(x) - \rho_\downarrow(x) \right],
\]  

(S5a)
\[
\dot{\rho}_\downarrow(x) = -\partial_x J_\downarrow + \Gamma \frac{L^2}{R^2} \left[ \rho_\uparrow(x) - \rho_\downarrow(x) \right],
\]  

(S5b)
where we have used the dimensionless time, measured in \( L^2/D \) units. We have identified the fluxes as

\[
J_\uparrow(x) = -\left[ \partial_x \rho_\uparrow(x) - \frac{L}{R} \rho_\uparrow(x) + \rho_\uparrow(x) \beta \nabla U(x) \right],
\]  

(S6a)
\[
J_\downarrow(x) = -\left[ \partial_x \rho_\downarrow(x) + \frac{L}{R} \rho_\downarrow(x) + \rho_\downarrow(x) \beta \nabla U(x) \right],
\]  

(S6b)
with

\[
\text{Pe} = \frac{v_{\text{act}} R}{D}, \quad \Gamma = \frac{\gamma R^2}{D}.
\]  

(S7)
By introducing the total (dimensionless) density \( \rho(x) \) and the density difference \( \delta \rho(x) \)

\[
\rho(x) = \rho_\uparrow(x) + \rho_\downarrow(x), \quad \delta \rho(x) = \rho_\uparrow(x) - \rho_\downarrow(x),
\]  

(S8)
we can rewrite Eqs. (S5) as

\[
\dot{\rho}(x) = -\partial_x J_\rho(x),
\]  

(S9a)
\[
\dot{\delta \rho}(x) = -\partial_x J_{\delta \rho}(x) - 2\Gamma \frac{L^2}{R^2} \delta \rho(x),
\]  

(S9b)
with
\[
J_\rho(x) = -\partial_x \rho(x) + \frac{L}{R} \text{Pe} \delta \rho(x) - \rho(x) \beta \partial_x U(x),
\]
\[
J_{\delta \rho}(x) = -\partial_x \delta \rho(x) + \frac{L}{R} \text{Pe} \rho(x) - \delta \rho(x) \beta \partial_x U(x).
\]
We assume that there are no particles away from the box
\[
\rho(x = \pm \infty) = 0, \quad \delta \rho(x = \pm \infty) = 0,
\]
which additionally implies that for \(x = \pm \infty\) there are no fluxes. In the steady state \((\dot{\rho}(x) = \delta \rho(x) = 0)\) Eqs. \ref{S9a} and \ref{S10a} give
\[
\delta \rho(x) = \frac{1}{\text{Pe} L} \partial_x \rho(x) + \rho(x) \beta \partial_x U(x).
\]
Substituting the above expression in Eq. \ref{S9b}, using Eq. \ref{S10b}, after some algebra we get the general equation
\[
\partial_x^2 \rho + 2 (\partial_x^2 \rho) \beta \partial_x U + (\partial_x \rho) \left(3\beta \partial_x^2 U - \frac{L^2}{R^2} \text{Pe}^2 + (\beta \partial_x U)^2 - 2\Gamma \frac{L^2}{R^2}\right)
+ \rho \left(\beta \partial_x^2 U + 2(\beta \partial_x U) \beta \partial_x^2 U - 2\Gamma \frac{L^2}{R^2} \beta \partial_x U\right) = 0.
\]
For \(-1 < x < 1\), where \(U(x) = 0\), Eq. \ref{S13} simplifies to
\[
\partial_x^3 \rho_c - (\partial_x \rho_c) \left(\text{Pe}^2 + 2\Gamma\right) \frac{L^2}{R^2} = 0,
\]
Since we require \(\rho(x)\) to have the symmetry \(x \mapsto -x\), the solution of the above equation is
\[
\rho_c(x) = A_c \cosh(\kappa_c x) + B_c, \quad \kappa_c^2 = (\text{Pe}^2 + 2\Gamma) \frac{L^2}{R^2},
\]
where \(A_c\) and \(B_c\) are, yet to be determined, constants.
For \(x > 1\), where \(\beta \partial_x U = f\), Eq. \ref{S13} takes the form
\[
\partial_x^3 \rho_t + 2f(\partial_x^2 \rho_t) + \left(f^2 - \frac{L^2}{R^2} \text{Pe}^2 - 2\Gamma \frac{L^2}{R^2}\right) \partial_x \rho_t - 2\Gamma \frac{L^2}{R^2} f \rho_t = 0,
\]
and the solution is
\[
\rho_t(x) = A_1 e^{\kappa_1 x} + A_2 e^{\kappa_2 x} + A_3 e^{\kappa_3 x},
\]
where \(A_1, A_2\) and \(A_3\) are constants, and \(\kappa_1 \geq \kappa_2 \geq \kappa_3\) are the roots of the polynomial
\[
Q(\kappa) = \kappa^3 + 2f\kappa^2 + \kappa \left(f^2 - \frac{L^2}{R^2} \text{Pe}^2 - 2\Gamma \frac{L^2}{R^2}\right) - 2\Gamma \frac{L^2}{R^2} f.
\]
Since \(Q(0) = -2\Gamma \frac{L^2}{R^2} f < 0\) and \(Q(-f) = f \frac{L^2}{R^2} \text{Pe}^2 > 0\), the above polynomial has three different, real roots; \(\kappa_1\) is positive and \(\kappa_2, \kappa_3 < 0\). Using the Cardano’s formula, after some algebra, we get
\[
\kappa_1 = -\frac{2f}{3} + \frac{2}{3} \sqrt{f^2 + \frac{L^2}{R^2} \left(2\Gamma + \text{Pe}^2\right) \cos \frac{\theta}{3}},
\]
\[
\kappa_2 = -\frac{2f}{3} + \frac{1}{3} \sqrt{f^2 + \frac{L^2}{R^2} \left(2\Gamma + \text{Pe}^2\right) \left(\sqrt{3} \sin \frac{\theta}{3} - \cos \frac{\theta}{3}\right)},
\]
\[
\kappa_3 = -\frac{2f}{3} + \frac{1}{3} \sqrt{f^2 + \frac{L^2}{R^2} \left(2\Gamma + \text{Pe}^2\right) \left(\sqrt{3} \sin \frac{\theta}{3} + \cos \frac{\theta}{3}\right)},
\]
where the angle \(\theta\) is given by
\[
\sin \theta = \left[\left(f^2 + 3 \frac{L^2}{R^2} \left(2\Gamma + \text{Pe}^2\right)\right)^3 - \left(f^3 + 9 \frac{fL^2}{R^2} \left(\Gamma - \text{Pe}^2\right)\right)^2\right]^{1/2}, \quad \cos \theta = \frac{f^3 + 9 \frac{fL^2}{R^2} \left(\Gamma - \text{Pe}^2\right)}{\left[f^2 + 3 \frac{L^2}{R^2} \left(\text{Pe}^2 + 2\Gamma\right)\right]^{3/2}}.
\]
Due to the symmetry $x \mapsto -x$ there is no need to separately consider the case of $x < -1$.

In order to determine the integration constants we first impose the normalization condition (see Eq. $S21$)

$$\tilde{\rho} = \frac{NR}{2L} = \int_0^\infty \rho(x) \, dx = \frac{A_\kappa}{\kappa_c} \sinh \kappa_c + B_c + \frac{A_1}{\kappa_1} e^{\kappa_1 x} \bigg|_1^\infty + \frac{A_2}{\kappa_2} e^{\kappa_2 x} \bigg|_1^\infty + \frac{A_3}{\kappa_3} e^{\kappa_3 x} \bigg|_1^\infty.$$

(S21)

Since $\kappa_1 > 0$, the only way to satisfy the above condition is to require $A_1 = 0$. The resulting relation is

$$\tilde{\rho} = \frac{A_\kappa}{\kappa_c} \sinh \kappa_c + B_c - \frac{A_2}{\kappa_2} e^{\kappa_2 x} = \frac{A_3}{\kappa_3} e^{\kappa_3 x}.$$

(S22a)

Three more relations come from the requirement that $\rho$, $\delta \rho$, $J_\rho$, and $J_{\delta \rho}$ are continuous at $x = 1$ (discontinuity of probability would lead to an infinite flux). The requirement of continuity of $\rho$ gives

$$A_c \cosh \kappa_c + B_c = A_2 e^{\kappa_2 x} + A_3 e^{\kappa_3 x},$$

(S22b)

the continuity of $\delta \rho$ gives

$$A_c \kappa_c \sinh \kappa_c = A_2 \kappa_2 e^{\kappa_2 x} + A_3 \kappa_3 e^{\kappa_3 x} + f \left(2 A_2 e^{\kappa_2 x} + A_3 e^{\kappa_3 x}\right),$$

(S22c)

and the continuity of $J_\rho$ gives (we have used Eq. $S22$ to simplify the formula)

$$A_c \kappa_c^2 \cosh \kappa_c = A_2 \kappa_2^2 e^{\kappa_2 x} + A_3 \kappa_3^2 e^{\kappa_3 x} + 2f \left(2 A_2 \kappa_2 e^{\kappa_2 x} + A_3 \kappa_3 e^{\kappa_3 x}\right) + f^2 \left(2 A_2 e^{\kappa_2 x} + A_3 e^{\kappa_3 x}\right).$$

(S22d)

Since $J_\rho(x) = 0$, there is no equation coming from the requirement that $J_\rho$ is continuous at $x = 1$. The solution of the linear Eqs. $S22$ is

$$A_c = -\tilde{\rho} \frac{\kappa_2 \kappa_3 \kappa_c (f + \kappa_2) (f + \kappa_3)}{G_2 \cosh \kappa_c - G_1 \sinh \kappa_c},$$

(S23a)

$$B_c = \tilde{\rho} \frac{\kappa_2 \kappa_3 \kappa_c \left[\left(|f + \kappa_2| (f + \kappa_3 + \kappa_2^2) \cosh \kappa_c - (2 f + \kappa_2 + \kappa_3) \kappa_c \sinh \kappa_c\right)\right]}{(G_2 \cosh \kappa_c - G_1 \sinh \kappa_c)},$$

(S23b)

$$A_2 = \tilde{\rho} \frac{e^{-\kappa_2 \kappa_2 \kappa_3 \kappa_c^2 (f + \kappa_3)} \left|(f + \kappa_3) \sinh \kappa_c - \kappa_c \cosh \kappa_c\right|}{(G_2 \cosh \kappa_c - G_1 \sinh \kappa_c)},$$

(S23c)

$$A_3 = \tilde{\rho} \frac{e^{-\kappa_2 \kappa_3 \kappa_3 \kappa_c^2 (f + \kappa_2)} \left|(f + \kappa_2) \sinh \kappa_c - \kappa_c \cosh \kappa_c\right|}{(G_2 \cosh \kappa_c - G_1 \sinh \kappa_c)}.$$

(S23d)

with

$$G_1 = \kappa_2 \kappa_3 (f + \kappa_2) (f + \kappa_3) - \kappa_c^2 \left[(f + \kappa_2)^2 - (f + \kappa_3)^2\right] \sinh \kappa_c - \kappa_c \cosh \kappa_c \left[(f + \kappa_2) - (f + \kappa_3)\right],$$

(S23e)

$$G_2 = \kappa_2 \kappa_3 \kappa_c (f + \kappa_2) (f + \kappa_3) - \kappa_c^2 \left[(f + \kappa_2 + \kappa_3) - (f + \kappa_3 + \kappa_2)\right].$$

(S23f)

Accordingly, the dimensionless mechanical pressure (measured in $k_B T/R$ units) of $N$ noninteracting particles is

$$\Pi = \int_1^\infty \beta \partial_x U(x) \rho(x) \, dx = \int_1^\infty f \rho(x) \, dx$$

$$= \tilde{\rho} f \kappa_c^2 \kappa_c^2 \left[(f + \kappa_2)^2 - (f + \kappa_3)^2\right] \sinh \kappa_c - \kappa_c \cosh \kappa_c \left[(f + \kappa_2) - (f + \kappa_3)\right] \cosh \kappa_c.$$

(S24)

We are interested in deriving an expression for the pressure $\Pi$ in the limit $f \to \infty$, i.e., in the case in which the particle is confined in a box $(x \in [-1 : 1])$. From Eq. $S18$ for large $f$ we have

$$\kappa_1 = 2f \frac{L^2}{R^2} \frac{1}{f^2} + O\left(\frac{1}{f^2}\right),$$

(S25a)

$$\kappa_2 = -f + \frac{L}{R} \frac{1}{f} + O\left(\frac{1}{f^2}\right),$$

(S25b)

$$\kappa_3 = -f - \frac{L}{R} \frac{1}{f} + O\left(\frac{1}{f^2}\right).$$

(S25c)
Using the above asymptotic expansion, after some algebra we get

\[ G_1 = -\Pe^2 \frac{L^2}{R^2} f^2 + O (f), \]
\[ A_c = \frac{\kappa_c \Pe^2}{\Pe^2 \sinh \kappa_c + 2 \Gamma \kappa_c \cosh \kappa_c} + O \left( \frac{1}{f} \right), \]
\[ G_2 = 2 \Gamma \frac{L^2}{R^2} \kappa_c f^2 + O (f), \]
\[ B_c = \frac{2 \Gamma \kappa_c \cosh \kappa_c}{\Pe^2 \sinh \kappa_c + 2 \Gamma \kappa_c \cosh \kappa_c} + O \left( \frac{1}{f} \right). \] 

(S26a) (S26b)

Finally, from Eq. (S24) we get

\[ \Pi = \frac{\kappa_c^3 \cosh \kappa_c}{\Pe^2 \sinh \kappa_c + 2 \Gamma \kappa_c \cosh \kappa_c} - \frac{1}{\beta} \frac{\kappa_c^3}{\Pe^2 \sinh \kappa_c + 2 \Gamma \kappa_c \cosh \kappa_c} + O \left( \frac{1}{f^2} \right), \] 

therefore

\[ \Pi_{\infty} = \lim_{f \to \infty} \Pi = \frac{\kappa_c^3 \cosh \kappa_c}{\Pe^2 \sinh \kappa_c + 2 \Gamma \kappa_c \cosh \kappa_c}. \] 

(S27)

From Eqs. (S15) and (S23) for \(-1 \leq x \leq 1\)

\[ \rho_{\infty} (x) = \lim_{f \to \infty} \rho (x) = \rho_c \, \frac{\Pe^2 \cosh (\kappa_c x) + 2 \Gamma \kappa_c \cosh \kappa_c}{\pe^2 \sinh \kappa_c + 2 \Gamma \kappa_c \cosh \kappa_c}. \] 

(S29a)

\[ \delta \rho_{\infty} (x) = \lim_{f \to \infty} \delta \rho (x) = \rho_c \, \frac{\beta}{\Pe} \frac{R}{L} \frac{\kappa_c^2 \Pe \sinh (\kappa_c x)}{\Pe^2 \sinh \kappa_c + 2 \Gamma \kappa_c \cosh \kappa_c}. \] 

(S29b)

For \(|x| > 1\) we get \(\rho_{\infty} (x) = \delta \rho_{\infty} (x) = 0\). 

S2. Pressure in different limiting cases

In order to study the properties of the pressure in different limits, we expand the general result (S28) (Eq. (6) of the main text) in the limit of large \(\kappa_c\)

\[ \Pi_{\infty} = \frac{\kappa_c^3 \cosh \kappa_c}{\Pe^2 \sinh \kappa_c + 2 \Gamma \kappa_c \cosh \kappa_c} = \frac{\kappa_c^3}{\Pe^2 + 2 \Gamma \kappa_c} + O \left( e^{-2 \kappa_c} \right), \] 

which proves Eq. (S5) of the main text. We note that the corrections to the above equation decay exponentially fast upon increasing \(\kappa_c\), therefore this approximation is reliable even for moderate values of \(\kappa_c\). For the discussion of physical arguments justifying this limit, see the main text. Further expansion of (S30) for large \(\kappa_c\) gives

\[ \Pi_{\infty} \simeq \frac{\kappa_c^3}{\Pe^2 + 2 \Gamma \kappa_c} = \rho \left( 1 + \frac{\Pe^2}{2 \Gamma} \right) = \rho \left( 1 + \frac{\Pe^2}{2 \Gamma} \right) + O \left( \frac{\Pe^2}{2 \Gamma \kappa_c} \right). \] 

(S31)

In the limiting case \(\Pe \ll \Pe_{\text{small}} = \sqrt{2} \Gamma\), the correction term in the above equation is very small. Therefore for \(\Pe \ll \Pe_{\text{small}}\) the approximation (S31) is again reliable even for moderate values of \(\kappa_c\). This proves Eq. (9) of the main text.

Finally, we study the pressure in the limit \(\Pe \gg \Pe_{\text{small}}\). In this limit

\[ \kappa_c = \sqrt{\Pe^2 + 2 \Gamma \frac{L}{R}} \approx \Pe \frac{L}{R}. \] 

(S32)

Using Eq. (S32), Eq. (S31) can be simplified to

\[ \Pi_{\infty} \simeq \frac{\kappa_c^3}{\Pe^2 + 2 \Gamma \kappa_c} = \frac{\kappa_c^3}{\Pe^2 + 2 \Gamma \kappa_c} = \rho \frac{L}{R} \frac{\Pe^2}{\Pe^2 + 2 \Gamma R}. \] 

(S33)

where \(\Pe_{\text{large}} = 2 \Gamma L / R\). This proves that

\[ \Pi_{\infty} \simeq \frac{\kappa_c^3}{\Pe^2 + 2 \Gamma \kappa_c} = \rho \frac{\Pe^2}{2 \Gamma} \] 

for \(\Pe \ll \Pe_{\text{large}}\) and \(\Pe \gg \Pe_{\text{small}}\),

(S34a)

\[ \Pi_{\infty} \simeq \frac{\kappa_c^3}{\Pe^2 + 2 \Gamma \kappa_c} = \rho \frac{\Pe}{\Pe_{\text{large}}} \] 

for \(\Pe \gg \Pe_{\text{large}}\) and \(\Pe \gg \Pe_{\text{small}}\),

(S34b)
which is equivalent to Eq. (11) of the main text.

In the above derivation we have not used the fact that $\kappa_c$ is large. Therefore, like Eq. (S30), the result is reliable even for moderate values of $\kappa_c$.

We note that, there is a range of parameters of the system for which $P_{\text{small}}$ and $P_{\text{large}}$ are of the same order, or even $P_{\text{small}} > P_{\text{large}}$. In that case, the regime where $\Pi_\infty \propto P_e^2$ (S34a) is not present and, upon increasing $P_e$, the behavior of the pressure changes directly from independent of $P_e$ (Eq. (S31)) to linear in $P_e$ (Eq. (S34b)).

S3. Effective Temperature

Eq. (9) of the main text (and Eq. (S31)) suggests to introduce the effective temperature

$$T_{\text{eff}} = T \left(1 + \frac{P_e^2}{2\Gamma}\right),$$

(S35)

as it is done in Refs. [1,2] (Refs. [46] and [47] of the main text). However, we remark that in our case this is only possible in the regime $P_e \ll P_{\text{small}}$, for which Eq. (9) of the main text holds. As shown in Eq. (10) of the main text (see Eq. (S34)), in the regime $P_e \gg P_{\text{large}}$ we cannot introduce $T_{\text{eff}}$ anymore since we are not in the limit of large tumbling rate $\Gamma$, where $\Gamma$ can be eliminated adiabatically.

S4. Comparison with Ref. [3]

In order to compare our results with Ref. [3] (Ref. [35] of the main text) we calculate

$$\frac{\Pi_\infty}{\rho_\infty(x = 0)} = \frac{(P_e^2 + 2\Gamma) \cosh(\kappa_c)}{P_e^2 + 2\Gamma \cosh(\kappa_c)},$$

(S36)

which for $\kappa_c \gg 1$ reduces to

$$\frac{\Pi_\infty}{\rho_\infty(x = 0)} = \frac{P_e^2}{2\Gamma} + 1,$$

(S37)

the result reported in Ref. [3]. Note that $\kappa_c = \frac{L}{R} \sqrt{P_e^2 + 2\Gamma} \gg 1$ can be attained for systems whose size is

$$L \gg \frac{R}{\sqrt{P_e^2 + 2\Gamma}}.$$  

(S38)

The discrepancy between our formula (Eq. (9) of the main text and Eq. (S28)) and Ref. [3] is relevant for weakly active systems, i.e., for $P_e \lesssim 1$ and $\Gamma \lesssim 1$. We note that, thanks to our approach, we can compute the full value of $\Pi_\infty$ on the top of its “deviation” from the ideal gas law, $\Pi_\infty/\rho_\infty(x = 0)$.

S5. Compressibility

Here we derive the expression for the compressibility

$$H_\infty = -\frac{1}{L} \left( \frac{\partial \Pi_\infty}{\partial L} \right)^{-1} = \frac{1}{\bar{\rho}} \left( \frac{L}{R} \right)^2 \frac{(P_e^2 \sinh \kappa_c + 2\Gamma \kappa_c \cosh \kappa_c)^2}{\kappa_c^4 (2\Gamma \cosh \kappa_c + P_e^2)} = \frac{1}{\Pi_\infty} \frac{(2\Gamma \kappa_c \cosh \kappa_c + P_e^2 \sinh \kappa_c) \cosh \kappa_c}{\kappa_c (P_e^2 + 2\Gamma \cosh^2 \kappa_c)}. 

(S39)$$

Notice that in the limit $\kappa_c \gg 1$ the last expression reduces to

$$H_\infty = \frac{1}{\Pi_\infty} \frac{2\Gamma \kappa_c + P_e^2}{2\Gamma \kappa_c}. 

(S40)$$
S6. Numerical simulations

In this section we describe the numerical simulations of Brownian dynamics of ABPs we have performed to support the results reported in this manuscript. The comparison with exact formulae is presented in Fig. 1(b) of the main text.

We consider a two-dimensional system of \( N \) particles confined in a square box of size \( 2L \times 2L \). Each particle is described by a position vector \( x_i(t) \) and an angle \( \theta_i(t) \) defining the orientation, where \( i = 1, 2, \ldots, N \) labels the particles and \( t \) is the time. The equations of motion are

\[
\begin{align*}
\dot{x}_i(t + dt) &= x_i(t) + w_i(t) \, dt = x_i(t) + \left[ v_i(t) + \sqrt{2D} \, \mathcal{R}_i(t) \right] \, dt, \\
\dot{\theta}(t + dt) &= \theta(t) + \sqrt{2D/\Gamma} \, \chi(t) \, dt,
\end{align*}
\]  

(S41a)

(S41b)

where \( dt \) is the time step, \( w_i \) is the average velocity of the particle over the time \( dt \), \( v_i = [v_{act \cos} \theta_i(t), v_{act \sin} \theta_i(t)] \) is the active velocity, and \( \mathcal{R}_i \) and \( \chi \) denote independent random variables with normal distribution with zero mean value and unit variance that model the random noise. We note that in Eq. (S41b) we have assumed the rotational diffusion constant \( D_r = D/\Gamma = \gamma \) to be equal to the tumbling rate of the particles in 1D model discussed in the main text. This simple assumption is enough to observe the agreement between 1D model and numerical simulations in 2D shown in Fig. 1(b) of the main text.

In the simulation we assume periodic boundary conditions in vertical direction. If the equations of motion Eq. (S41) move the \( i \)-th particle beyond the left or right wall, we assume that \( x_i(t + dt) = x_i(t) \) (resetting the position of the particle) and over the time \( dt \) the average force exerted on the wall by the particle is

\[
F_i = \frac{k_BT}{D} w_i \cdot n,
\]  

(S42)

where \( D/(k_BT) \) is the mobility given by the Einstein relation, and \( n \) is the unit normal vector of the wall (\( n = [1,0] \) for the right wall and \( n = [-1,0] \) for the left wall). During the simulation, we have been calculating sums of all the forces exerted by the particles in a given time step, we average this quantity over all time steps, and divide it by the length of the wall to obtain the pressure.

Since the pressure presented in Fig. 1(b) of the main text is normalized by the pressure of ideal gas, the exact value of \( T \) (and the height of the system which is assumed to be 2L) is not relevant. Moreover, changing \( D \) and \( R \) (with \( L/R \) fixed) is equivalent to rescaling of \( dt \). Therefore, for the simulation we have taken \( k_BT = D = R = 1 \). For several values of the parameters we have checked that when \( dt \lesssim 0.01 \), the pressure is de facto independent on the exact value of \( dt \); therefore we have assumed \( dt = 0.01 \).

For each value of \( Pe, \Gamma \) and \( L/R \) we have prepared the initial configuration by placing \( N = 200 \) particles in the box randomly and with random orientation. Then, the simulation was run for \( 10^6 \) time steps in order to relax the initial condition. Finally, for \( 10^7 \) time steps the pressure was measured. The simulation for each set of parameters has been repeated 100 times and the average pressure and its standard deviation calculated. In each case the standard deviation of the pressure is much smaller that the size of points in the plot.

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