Deconfinement and phase diagram of bosons in a linear optical lattice with a particle reservoir

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We investigate the zero-temperature phases of bosons in a one-dimensional optical lattice with an explicit tunnel coupling to a Bose condensed particle reservoir. Renormalization group analysis of this system is shown to reveal three phases: one in which the linear system is fully phase-locked to the reservoir; one in which Josephson vortices between the one-dimensional system and the particle reservoir deconfine due to quantum fluctuations, leading to a decoupled state in which the one-dimensional system is metallic; and one in which the one-dimensional system is in a Mott insulating state.

Introduction: Ultracold atoms trapped in optical lattices is an active area of research in many-body physics [1–6]. At zero temperature and without disorder, bosons in a lattice with integer filling exhibit two distinct phases – a superfluid phase, where the phase of the wavefunctions are sharply defined on sites, and a Mott insulating phase, where the site occupation numbers are sharply defined [7–11]. This system allows novel many-body phenomena associated with unusual correlated states and quantum phase transitions [12–19].

In this work we will study a system of bosons in a one-dimensional (1D) optical lattice that is tunnel coupled to a three-dimensional (3D) Bose-condensed particle reservoir. Our goal is to understand how the reservoir impacts the states of the bosons in the optical lattice. We find that the system supports three phases: a 3D superfluid phase in which the 1D system becomes phase-locked with the reservoir; a decoupled phase (in which, under circumstances described below, the 1D system behaves like a metal); and a Mott insulating phase. These states are separated by deconfinement transitions [20], either of vortices or of tunneling events, as we describe below. The states may be distinguished both by their low-energy excitation spectra, and by their conduction properties.

Model: Our analysis begins with a Hamiltonian for lattice bosons (in number-phase representation) [7]

\[
\mathcal{H} = -t \sum_{\langle ij \rangle, \tau} \cos(\phi_{i\tau} - \phi_{j\tau}) + \frac{U}{2} \sum_{i, \tau} n_{i\tau}^2 \\
- t_R \sum_{\langle RR \rangle, \tau} \cos(\Phi_{R\tau} - \Phi_{R^\prime\tau}) + \frac{U_R}{2} \sum_{R, \tau} N_{R\tau}^2 \\
- \sum_{i, \tau} J_i \cos(\phi_{i\tau} - \Phi_{R(i)\tau}), \tag{1}
\]

where \((n_{i\tau}, \phi_{i\tau})\) and \((N_{R\tau}, \Phi_{R\tau})\) are the canonically conjugate occupation number fluctuations and phases of the bosons in the chain and in the 3D reservoir, respectively, \(t\) and \(U > 0\) \((t_R \text{ and } U_R > 0)\) are the nearest neighbor interchain hopping and on-site repulsion terms for the 1D (3D) system, and \(J_i\) is a tunneling amplitude between a site on the chain [\(i\)] and a site in the reservoir [\(R(i)\)]. The particle numbers in the real optical lattices are typically small, we nevertheless allow the fluctuations to vary from \(-\infty\) to \(+\infty\). This simplifies the calculation, and should not affect the allowed phases because the symmetries of the resulting Hamiltonian are unchanged [21].

Our geometry could be realized as a linear array of sites using red detuned light focused near the edge of a BEC cloud [12,22–24]. Note that the tight-binding form for the reservoir is adopted purely as a matter of convenience.

Following standard procedure [25], we construct a path-integral representation of the partition function and then use the Villain model for the three cosine terms in Eq. 1 [26]. Assuming that vortex rings in the 3D reservoir are unimportant (which will always be valid if the reservoir is sufficiently dilute), we integrate out the internal degrees of freedom of the reservoir to arrive at a partition function of the form

\[
\mathcal{Z}_{VM} = \sum_{n_{x\tau}} \exp\left\{ -\frac{1}{2} \sum_{x, \tau} \left[ \varepsilon U |n_{x\tau}|^2 + \frac{1}{\varepsilon t} |m_{x\tau}|^2 \right] \\
- \frac{1}{2L_x \beta} \sum_{q \in \omega_n} \frac{1}{\hbar(q)} \left[ |q_x m(q) + i\omega_n n(q)|^2 \right] \right\}. \tag{2}
\]

Here \(\beta = (kT)^{-1} \to \infty\), \(L_x\) is the number of sites in the 1D chain, and \(\varepsilon\) is the time slice interval. Physically, the integer variables \(m_{x\tau}\) may be understood as bond currents and \(n_{x\tau}\) as the fluctuations in the site occupation number. Eq. 2 is a form of the Bose-Hubbard model. The coefficient \(1/\hbar(q)\) contains information about the reservoir degrees of freedom, in particular the gapless collective mode it supports due to its own superfluidity,

\[
\frac{1}{\hbar(q)} = \frac{1}{\varepsilon J} + \gamma \ln \left( 1 + \frac{A^2 c_R^2}{\varepsilon t q_x^2 + \omega_n^2} \right), \tag{3}
\]
where $c_R = \sqrt{2\varepsilon t U_R}$, $\Lambda$ is the momentum cut-off, and $\gamma = (1/4\pi e^2 t g^2)[27]$.

The partition function $Z_{VM} \equiv \sum_{m,n} \exp(-\mathcal{H}_{VM})$ may be reexpressed in terms of another pair of integer fields $\phi(x,\tau)$ and $A(x,\tau)$ with $m(x,\tau) = -\partial_\tau \phi(x,\tau)$ and $n(x,\tau) = \partial_x \phi(x,\tau) + A(x,t)$, so that

$$\mathcal{H}_{VM} = \frac{1}{2K} \sum_{x,\tau} \left\{ \frac{c^2}{2} \left[ \partial_\tau \phi(x,\tau) + A(x,\tau) \right]^2 + \left[ \partial_x \phi(x,\tau) \right]^2 \right\} + \frac{1}{2L_\beta} \sum_{q_a,n_a} \frac{1}{h(q)} |\omega_n A(q)|^2.
$$

(4)

Here, $K = \varepsilon t$ and $c = \sqrt{\varepsilon^2 U t}$. Configurations for which $\nabla \phi \neq 0$ and $A = 0$ contain closed loops which may be understood as worldlines of particle-hole pairs that separate and recombine. The $A$ field, which should be regarded as residing on the time interval links, can cancel the gradient energy $(\partial_\tau \phi)^2$ on part of a closed loop configuration to form individual particle or hole trajectories; the endpoints (occurring where $\partial_\tau A(x,\tau) \neq 0$) represent tunneling events between the 3D and 1D systems. Tunneling events may be shown [20] to be dual to the vortices. Moreover, the model defined by Eq. 4 may be generalized to include a core energy $E_c$ for the vortices.

As we discuss below, this generalized Hamiltonian supports three phases. When tunneling events proliferate through the system, the 1D and 3D systems exchange particles freely (equivalently, vortices of the 1+1 dimensional system are linearly confined), and their phases become locked together to form a single superfluid. As fluctuations in $\phi$ and $A$ are decreased (by decreasing $E_c$ and/or $h$, or decreasing $K$), tunneling events bind into pairs which conserve the overall particle number in the 1D system, although closed particle-hole worldlines remain proliferated. This state may be understood as one in which Josephson vortices between the 3D and 1D system proliferate due to quantum fluctuations, effectively decoupling the two systems. For still smaller fluctuations, particle-hole worldline loops of arbitrarily large size become irrelevant, indicating that the fluctuations in the particle number on each site have been suppressed, and the system is a Mott insulator.

**RG analysis:** A method for constructing a momentum shell RG for Hamiltonians such as Eq. 4 was developed in Ref. [20]. We replace the integer fields $\phi$ and $A$ with the continuous fields $\varphi(x,\tau)$ and $a(x,\tau)$ in Eq. 4 and add terms of the form $-y \int dx d\tau \cos[2\pi \varphi(x,\tau)]$ and $-\frac{1}{2} \int dx d\tau \cos[2\pi a(x,\tau)]$ so that the resulting effective Hamiltonian has the same symmetries as the original one [21]. In this replacement, $y = \exp(-E_c)$ is the usual vortex fugacity, and choosing $y_a \sim \int d^2q/(h(q))$ approximately reproduces the action associated with a worldline endpoint (i.e., a tunneling event). We then integrate out short wavelength degrees of freedom $|A/b < |q_a|, |w_n|/c < A$ with $b = \exp(l)$ to lowest order in $y$ and $y_a$, and rescale lengths, times, and fields according to $x = bx', \tau = br', \varphi'(x',\tau') = \varphi(x,\tau)$, and $a(x,\tau) = a'(x',\tau')/b$. This choice preserves the terms that are lowest order in gradients in the quadratic part of the Hamiltonian. Because the $a$ field shrinks upon rescaling, it is natural to expand $\cos[2\pi a(x,\tau)]$ in its argument, producing a quadratic term of the form $\frac{1}{2} \rho |a(x,\tau)|^2$ that contributes to the fixed point. The higher order vertices generated by this expansion contribute to the renormalization of $\rho$ but are themselves irrelevant. Note the initial value of $\rho$ is $4\pi^2 y_a$.

The fixed points that emerge from this procedure have the form

$$\mathcal{H}_* = \frac{1}{2K} \int dx d\tau \left[ c^2 \left| \partial_\tau \varphi(x,\tau) + a(x,\tau) \right|^2 + \left| \partial_x \varphi(x,\tau) \right|^2 + \rho K |a(x,\tau)|^2 \right].
$$

(5)

The last term in $\mathcal{H}_*$ is very important: since tunneling events are specified by $\partial_\tau a \neq 0$, when $\rho \neq 0$ they are bound into equal and opposite pairs. Unbinding occurs if $\rho(\ell)$ scales to zero. To lowest order in $y_a$, its scaling relation is

$$\frac{d \ln \rho}{dt} = -\left( \frac{2\pi\Lambda^2 K}{\sqrt{\rho K}} \right) e^{-2t}.
$$

(6)

For small $\rho$ one can easily show that the term $-y \int dx d\tau \cos[2\pi \varphi(x,\tau)]$ is strongly irrelevant. The resulting scaling flows are shown in Fig. 1. As may be seen, in general $\rho$ scales to a point along a fixed line; if its initial value is small enough (as occurs for small $\gamma$ and large $J$), that point is at $\rho = 0$. A remarkable feature of Eq. 6 for such flows is that $\rho(\ell) = 0$ at a *finite* value of $\ell = \ell^*$, because of the singularity as $\rho \to 0$. If the initial value of $\rho$ is increased, $\ell^*$ eventually diverges, defining a transition point above which $\rho$ scales to a non-zero value. *This represents a confinement transition for tunneling events* [20,28].

The dual representation of this system is obtained by performing a Poisson resummation on $A$ and $\phi$ in Eq. 4. The resulting Hamiltonian is very similar in form to Eq. 4 [29], and may be understood as a model with vortex degrees of freedom rather than tunneling events. Because of this, there is a second transition, in which vortices go from a bound to an unbound state, in the same continuous fashion as seen above for tunneling events. In terms of $\varphi$ and $a$ degrees of freedom, this corresponds to a phase in which arbitrarily large particle-hole loops are negligible in the partition function. Pairs of tunneling events are then linearly confined [20].

The phases described above have physically different characteristics. This can be seen most directly in the collective mode spectra, which in principle may be measured in light scattering experiments [30,31]. We find these by examining the density-density correlation function $\langle n(-q_x, -\omega) n(q_x, \omega) \rangle$ near the fixed point Hamiltonians representing the various phases. Generally, this contains a broad response for $\omega \gg \epsilon_H$, due to modes in the reservoir. Beyond this, in the superfluid phase [Fig. 2(a)],...
we find [30] a sharp resonance (collective mode) at \( \omega_1 = c_R q_x - \delta \omega \), where \( \delta \omega = c_R \Lambda \exp\left(-\left(c^2/\gamma K\right)/(q_x^2(c_R^2 - c^2))\right) \).

This rapidly approaches the reservoir continuum as \( q_x \to 0 \), indicating that number fluctuations in the 1D system will strongly mix with those of the reservoir in the long wavelength limit. This is consistent with our interpretation of this phase as a single 3D superfluid. (Note that although the term containing the logarithmic singularity is irrelevant in the RG sense, it nevertheless has the physical effect of “pulling” the collective mode very close to the edge of the reservoir modes at small enough wavevectors.) In the intermediate phase we find two collective modes, the superfluid mode of the reservoir at \( \omega_1 \) and another linear mode at \( \omega_2 = c q_x \sqrt{\rho K/(\rho K + c^2)} \) [Fig. 2(b)]. For an appropriate geometry we will show this leads to metallic behavior. In the Mott insulator phase, the reservoir becomes decoupled from the 1D chain in the long-wavelength limit, and supports a gapped mode at \( \omega_3 = \sqrt{c^2 q_x^2 + 2\pi^2 K/E_c} \) [Fig. 2(c)].

Conductance: The different characters of the phases may also be seen in the conductance [32] of the system. To measure this, one has to attach a particle source and lead as shown in Fig. 3. The conductance quantifies the current injected by this source with chemical potential \( \mu_s > 0 \), draining into the 3D reservoir which is held at zero chemical potential. (Note that a link has been removed to ensure that the current flows in the 1D chain before tunneling into the reservoir.) This system could be fabricated in a Josephson-junction array [7].

The source and lead introduce two extra degrees of freedom \([\phi_L(\tau), N_L(\tau)]\) to our model. We treat the particle source as an ideal reservoir with Hamiltonian \( H_L = \mu_s N_L; \) current conservation at the site \((x = 0)\) where current is injected specifies \( N_L \) in terms of the other variables through a constraint in the partition function, \( \Pi_\tau \{ \partial_x m(x = 0, \tau) + \partial_\tau n(x = 0, \tau) - dN_L(\tau)/d\tau = 0 \}. \) The conductance is given by \( G(\omega) = \omega(N_L(-\omega) N_L(\omega)) \).

The analysis proceeds in a fashion similar to what is described above for the uniform chain. However, because of the lead there is an additional term in our effective Hamiltonian of the form \(-\delta y \int \, d\tau \cos[2\pi a(x = 0, \tau)]\), reflecting the fact that the rate of tunneling events at the \( x = 0 \) site is different than at other sites. The scaling relation for \( \delta y(\ell) \) takes the form [30]

\[
\frac{d\delta y(\ell)}{d\ell} \approx \delta y(\ell) \left[ 1 - \frac{\pi^2 K A^2 |\alpha(\ell)|^2}{\sqrt{\rho K}} \right].
\]

for small \( y, y_n, \) and \( \delta y, \) with \( \alpha(\ell) = \exp(-\ell) \). It is important to recognize that in the 3D superfluid state, \( \rho(\ell) \to 0 \) for finite \( \ell \equiv \ell^*, \) so that \( \delta y(\ell) \) will be driven to zero at \( \ell = \ell^* \). Thus the non-uniformity of tunneling events along the chain is irrelevant in this state. In this situation, we can compute the conductance using the fixed point Hamiltonian (Eq. 5) for \( \rho = 0 \) to find a true superconducting response, \( G(\omega) \propto -i/(\omega + i\delta) \).

By contrast, if \( \rho \) remains finite as \( \ell \to \infty \), \( \delta y \) will necessarily grow, and we need to look for a new fixed point. To do this, we integrate the RG flows to a scale \( \ell_0 \) for which the irrelevant operators may be ignored. We then integrate out \( \varphi \) and \( a(\omega \neq 0, \tau) \) and recollect some irrelevant terms to restore the \( \sqrt{\rho a(x = 0, \tau)} \) term to its cosine form \( \cos[2\pi a(\ell_0) a(x = 0, \tau)] \), and arrive at an effective Hamiltonian for the lead site in the chain,

\[
H_{eff}^{L} = \frac{c}{2K} \sqrt{\frac{\rho K}{\rho K + c^2}} \sum_{\omega_n} |w_n||a(x = 0, \omega_n)|^2 + y_0(\ell_0) \sum_{\tau} \cos[2\pi a(\ell_0) a(x = 0, \tau)].
\]

We then modify the RG so that \( a'(x = 0, \tau') = a(x = 0, \tau) \) to preserve the form of the quadratic term in \( H_{eff}^{L} \); the scaling relation obeyed by \( y_0 \) is then

\[
\frac{dy_0(\ell)}{d\ell} \approx y_0(\ell) \left[ 1 - \frac{2\pi^2 K}{\rho K + c^2} \left| a(\ell_0) \right|^2 \right].
\]

It is important to recognize that \( a(\ell_0) \) does not shrink as it did in Eq. 7. Thus since \( \rho \) is small, \( y_0 \) is irrelevant. Our fixed point is then \( H_{eff}^{L} \) with \( y_0 = 0 \). In this case the conductance is finite as \( \omega \to 0 \), with \( G(\omega) \propto 1/\sqrt{\rho} \). This metallic behavior is surprising in light of the linear mode supported by this phase. It is a result of the very limited phase space available for fluctuations in one dimension [32]. For this reason the transport properties of this intermediate phase is distinct from that of the 3D superfluid phase.

The conductance in the deconfined vortex state can be computed straightforwardly using the dual representation of the model. The result unsurprisingly is an insulating response \( G(\omega) \sim \omega \). (Details will be presented elsewhere [30].)

In summary, we have shown that bosons in a one-dimensional optical lattice that exchanges particles with a bulk superfluid supports three distinct states, with different collective mode spectra and conductances.

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[1] D. Jaksch et al., Phys. Rev. Lett. 81, 3108 (1998).
[2] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).
[3] P. Pedri et al., Phys. Rev. Lett. 87, 220401 (2001).
[4] C. J. Pethick and H. Smith, Bose-Einstein Condensation in Dilute Gases, Cambridge University Press, New York, 2002.
[5] H. Moritz, T. Stöferle, M. Köhl, and T. Esslinger, Phys. Rev. Lett. 91, 250402 (2003).
It is interesting to speculate that, for low boson (fracion-al) densities and weak coupling to the reservoir, the decoupled phase will be in the Tonks-Girardeau regime, for which the bosons display behavior similar to fermions. See B. Paredes et al., Nature 429, 277 (2004). Our model is specific to integer filling and cannot describe this regime.

FIG. 1. Schematic diagram of RG flows for the scaling relations of $\rho$ and $y$ from initial microscopic values to a fixed line. $\rho = 0$ represents a deconfined phase for tunneling events. For $\rho > 0$ they are bound in pairs such that the net particle flow from the reservoir vanishes. Heavy line separates flows for the two kinds of states.

FIG. 2. Collective mode spectra obtained from fixed point Hamiltonians. In the 3D superfluid phase there is a single sharp mode (a). The 1D metallic phase supports two gapless modes (a) and (b). In the Mott insulator phase, there is only a single gapped mode (c). Note all three phases also contain a continuum of modes due to the reservoir (shaded region).

FIG. 3. Josephson junction realization of the system with source and drain to measure conductances of the phases.