Quantum Otto thermal machines powered by Kerr nonlinearity

Udson C Mendes¹, José S Sales² and Norton G de Almeida¹,*

¹ Instituto de Física, Universidade Federal de Goiás, 74.001-970, Goiânia - Go, Brazil
² Campus Central, Universidade Estadual de Goiás, 75132-903, Anápolis - Go, Brazil

E-mail: norton@ufg.br

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Abstract
We study the effect of Kerr nonlinearity in quantum thermal machines having a Kerr-nonlinear oscillator as working substance and operating under the ideal quantum Otto cycle. We first investigate the efficiency of a Kerr-nonlinear heat engine and show that by varying the Kerr-nonlinear strength the efficiency surpasses in up to 2.5 times the efficiency of a quantum harmonic oscillator (QHO) Otto engine. Moreover, the Kerr-nonlinearity makes the coefficient of performance of the Kerr-nonlinear refrigerator to be as large as three times the performance of QHO Otto refrigerators. These results were obtained using realistic parameters from circuit quantum electrodynamics devices formed by superconducting circuits and operating in the microwave regime.

Keywords: quantum thermal machines, quantum Otto engine, quantum Otto refrigerator, Kerr-nonlinear oscillator

(Some figures may appear in colour only in the online journal)

1. Introduction
Quantum thermal machines (QTM) produce or consume energy by using quantum matter as working substance [1–5]. Depending on the nature of the quantum matter, bosons or fermions, and the reservoir that they are coupled, QTMs can, under certain conditions, surpass the efficiency of a classical Otto engine and even the efficiency of a classical Carnot engine. For instance, thermal engines and refrigerators in which conventional heat reservoirs are replaced by non-classical reservoirs [6–8], when operating either in the quasi-static limit and in the non-zero power limit, can surpass their classical analogues, showing efficiency larger than those obtained in the Carnot cycle [9, 10]. Recently, it was shown that it is possible to surpass Carnot efficiency, in the case of engines, and Carnot performance, in the case of refrigerators, even far from the quasi-static regime [11–14].

In absence of quantum resources, in the form of either coherence or correlations between the quantum matter and the reservoirs with which the work substance interacts [15, 16], the formalism developed by Alicki [17] provides an unambiguous definition for the quantum equivalents of heat and work. In this formalism, heat is defined as the amount of energy exchanged between the working substance and the thermal reservoir. In turn, work is defined as the change in the Hamiltonian during the cycle. This formalism was used, for example, to study an Otto engine interacting with a reservoir with effective negative temperature, which revealed the intriguing phenomenon of increased performance of an Otto quantum engine concomitant with the increasing power [18]. In addition to the use of non-classical reservoirs, techniques such as shortcut to adiabaticity have been exploited to increase the performance of thermal machines [19–21]. In this case, QTMs are engineered to obtain the same performance of the quasi-static cycle using machines operating at non-null power [22].

 Otto machines based on quantum harmonic oscillator (QHO) interacting with non-classical reservoirs and operating at finite time have been extensively investigated in the recent years. However, while implemented in real devices, quantum non-harmonic effects may not only be relevant, but

* Author to whom any correspondence should be addressed.
also lead to new phenomena. In the context of quantum optics, nonlinear interactions are responsible, for example, to generation of non-classical states of light such as squeezed states. An important nonlinear effect is the Kerr non-linearity, which appears naturally in Josephson junction-based devices, among others. Indeed, Kerr nonlinearity is at the heart of quantum information devices based on superconducting circuits [23]. For instance, Kerr nonlinearity is essential for the operation of the transmon qubit [24] and it is used to stabilize cat-states qubits [25].

In this research paper, we investigate the efficiency $\eta$ and coefficient of performance $\epsilon$ of a cyclic QTM formed by a Kerr-nonlinear oscillator (KNO) by operating an Otto cycle. We show that the efficiency and performance of a Kerr-nonlinear Otto machine outperforms the efficiency and performance of its linear QHO counterpart.

The paper is organized as follow. In section 2, we present the quantum Otto cycle and the relevant thermodynamics quantities necessary to characterize the Otto heat engine and refrigerator. In section 3, we calculate these thermodynamics quantities for the KNO Otto machine. Results are presented in section 4.1 for the KNO operating as Otto heat engine and in section 4.2 operating as an Otto refrigerator. Final remarks are presented in section 5.

2. Quantum Otto cycle

Our model consists of a KNO as the working substance and interacting with two thermal reservoirs at different temperatures to implement a quantum Otto machine (QOM). We show by varying the KNO frequency and the nonlinear coupling strength that the QOM can be controlled to either extract or perform work. The QOM is formed by two isochoric and two isentropic branches. In the first isochoric branch, the KNO is coupled to a cold-thermal reservoir, while in second branch, it is coupled to a hot-thermal reservoir. In the isentropic branches, the KNO is decoupled from the thermal reservoirs and left to evolve unitarily to complete the cycle [26]. The four strokes forming the QOM are illustrated in figure 1(b) and described in detail below.

(a). Cooling stroke: the KNO is weakly coupled to a cold thermal reservoir up to thermalization. The KNO-thermalized state is described by the Gibbs state $\rho_1 = e^{-\beta_1 H_c}/\text{Tr}(e^{-\beta_1 H_c})$, with $H_c$ being the KNO Hamiltonian and $\beta_1 = 1/k_B T_c$, with $k_B$ being the Boltzmann constant and $T_c$ the temperature of the cold reservoir.

(b). Expansion stroke. In this step, the KNO evolves unitarily from the state $\rho_1$ to $\rho_2 = U(t)\rho_1 U^\dagger(t)$, with $U(t)$ the unitary time-evolution operator. During the time evolution, the KNO Hamiltonian evolves from $H_c$ to $H_h$. As we are interested in obtaining the ultimate limit achievable by the Otto cycle, we will consider a quasi-static process. In the quasi-static limit, the specific form of the unitary time-evolution operator $U(t)$ is not important and its effect is to change the KNO frequency and nonlinearity strength.

(c). Heating stroke. In this branch, the KNO is weakly coupled to a hot-thermal reservoir at temperature $T_h$ until reaching the thermal Gibbs state $\rho_3 = e^{-\beta_3 H_h}/\text{Tr}(e^{-\beta_3 H_h})$, with $\beta_3 = 1/k_B T_h$.

(d). Compression stroke. This last step is accomplished by reversing the protocol employed to perform the expansion stroke, such that the KNO Hamiltonian transforms from $H_h$ to $H_c$ and its state evolves unitarily from $\rho_3$ to $\rho_4 = U^\dagger U(t)$.

In the complete cycle, work is either produced or consumed in the the expansion and compression strokes. Thus, the net work is defined as $W = W_{1\rightarrow 2} + W_{3\rightarrow 4}$, with $W_{1\rightarrow 2}$ the work in the expansion stroke and $W_{3\rightarrow 4}$ the work in the compression stroke. In these strokes, the QOM does not exchange heat with the thermal reservoirs. Thus, from the first law of thermodynamics, the net work is given by the variation of total energy [27]. Following reference [17], the net work is defined as

$$W = \text{Tr}(\rho_2 H_h) - \text{Tr}(\rho_1 H_c) + \text{Tr}(\rho_3 H_c) - \text{Tr}(\rho_4 H_h) .$$

In the cooling and heating strokes, the QTM does not perform or extract work. However, it exchanges heat with the thermal reservoirs. In the cooling stroke, the KNO is coupled to the cold-thermal reservoir and the heat exchanged $Q_c = Q_{4\rightarrow 1}$ takes the form

$$Q_c = \text{Tr}(\rho_1 H_c) - \text{Tr}(\rho_4 H_c) .$$

In the heating stroke, the heat exchanged $Q_h = Q_{2\rightarrow 3}$ between the KNO and the hot-thermal reservoir is

$$Q_h = \text{Tr}(\rho_3 H_h) - \text{Tr}(\rho_2 H_h) .$$

In the next section, the above formulas will be used to calculate the net work and heat for the KNO Otto cycle. Then, for a given set of parameters, we demonstrate that the KNO thermal machine can behave either as a quantum engine or as a refrigerator.
3. Work and heat in the Otto cycle of a KNO

We now calculate the net work $W$ and the heats, $Q_c$ and $Q_h$, corresponding to the Otto cycle (see figure 1(b)) described in the previous section. The working substance is formed by a quantum KNO. The quantum KNO Hamiltonian responsible for extracting work from the Otto cycle is

$$H_c = h\omega_c a^\dagger a + h\frac{K_c}{2}a^2a^\dagger a^\dagger.$$  

(4)

The first term of the Hamiltonian describes a single-mode harmonic oscillator of frequency $\omega_c$, with $a^\dagger$ and $a$ the creation and annihilation operators, respectively. The last term is the nonlinear interaction of strength $K_c$. Here, $h$ is the reduced Planck constant $h/2\pi$. Kerr Hamiltonians are easily engineered in circuit QED [23, 28] and cavity optomechanics [29] devices. These quantum devices are operated in the microwave regime. To produce work, either the frequency or the nonlinear strength of the KNO medium should vary. In circuit QED devices, the frequency and Kerr strength can be tuned by varying the capacitor charge or flux passing through a SQUID loop forming the resonator [28], thus, modifying the above Hamiltonian to

$$H_h = h\omega_h a^\dagger a + h\frac{K_h}{2}a^2a^\dagger a^\dagger.$$  

(5)

We are now in the position to compute the net work (equation (1)) and the heats (equations (2) and (3)) in the Otto cycle described in the section 2. We start by computing $\text{Tr}(\rho_2 H_h)$, with $\rho_2 = U(t)\rho_1 U^\dagger(t)$ and $\rho_1 = e^{-\beta_h H_h}/\text{Tr}(e^{-\beta_h H_h})$:

$$\text{Tr}(\rho_2 H_h) = \text{Tr} \sum_{n=0}^{\infty} p_n^U \text{Tr}(U(t)|n\rangle\langle n|U^\dagger(t)),$$

where we defined $p_n^U = e^{-\beta_h \frac{1}{2} \left[\omega_h n + K_h (n^2 - n)/2\right]}$ [$Z_{T_h}$, the population of the $n$-Fock state, and the partition function $Z_{T_h} = \sum_{n=0}^{\infty} \text{exp} \left\{-\beta_h \frac{1}{2} \left[\omega_h n + K_h (n^2 - n)/2\right]\right\}$]. Using bosonic commutation relations and inserting the completeness relation $\sum_{m} |m\rangle\langle m| = 1$, equation (6) takes the form

$$\text{Tr}(\rho_2 H_h) = \text{Tr} \sum_{n,m=0}^{\infty} p_n^U U_{n,m}(t) \left[\omega_h m + K_h \frac{1}{2} (m^2 - m)\right],$$  

(7)

From equations (9)–(11) we observe that all three quantities depend on the population difference $\Delta p_n$. Thus, a thermal population imbalance will generate work and heat. Moreover, the Kerr nonlinearity contributes to an increase in the work available for a useful task, as well as to the heats $Q_h$ and $Q_c$ exchanged between the KNO and reservoir. Particularly, we note that work $W$, in equation (9), increases with increasing $\Delta K$, which, as we will show in the next section, is responsible for improving the engine’s efficiency. This change due to the Kerr nonlinearity in the energies exchanged between the system and its surroundings at two different temperatures, combined with a parameter adjustment to guarantee the operating conditions of a cooler or an engine can be used, as we will see, for an enhancement in both efficiency and performance of our KNO Otto machine.

4. Results

In this section, we will use the above defined work and heats to calculate the heat engine efficiency and the refrigerator performance of the KNO Otto engine. We demonstrate that KNO heat engine efficiency and refrigerator performance outperforms its counterpart formed by a simple QHO.

4.1. Heat engine powered by Kerr nonlinearity

To build a quantum Otto engine based on the Otto cycle described in section 2, we first need to assure that the engine
conditions are fulfilled. In the case of QOM, the engine exists when work is performed by the KNO while absorbing heat from the hot reservoir. Therefore, we search for a parameter regime in which $W < 0$ (equation (9)), $Q_h > 0$ (equation (11)) and $Q_c < 0$ (equation (10)) are satisfied simultaneously. In this regime, a KNO-based Otto heat engine exist and will be compared to both QHO-based Otto and Carnot machines.

Figure 2 illustrates the (a) heat $Q_h$, (b) net work $W$, and (c) heat $Q_c$ as a function of the hot-reservoir temperature for values of the Kerr-nonlinearity strength $K_c/2 = \omega_c/100$ (solid line), $K_c/2 = \omega_c/1000$ (dotted line) and $K_c = 0$ (dash dot line). The other parameters are fixed and equal to $\omega_h = 2 \pi \times 4 \text{ GHz}$, $\omega_c = 0.7 \omega_h$, $T_c = 0.1T_h$ and $K_h/2 = 0.1\omega_h$. The heat engine conditions, $W < 0$, $Q_h > 0$ and $Q_c < 0$, are satisfied for wide range of temperatures.

The best strategy to maximize the efficiency (12) is by minimizing the numerator while maximizing the denominator in equation (12). This maximization is accomplished, for example, by taking $K_c = 0$ and letting $K_h \to \infty$. However, the Kerr nonlinearities, $K_c$ and $K_h$, are constrained by experimental feasibility. For instance, in the context of circuit QED, $K_h/\omega_h$ can vary from 0.001 to 0.1 [23], and in a novel superconducting device termed *quarion* the non-linearity strength can be made as large as $1/3$ of its natural frequency [35]. Moreover, the non-linear strength $K_h$ can be externally tuned through gate voltage or flux bias [28], thus, allowing to maximize the heat engine efficiency.

Figure 3 shows the KNO heat engine efficiency $\eta$ as a function of the hot-reservoir temperature for the set of parameters used in figure 2. We observe that the efficiency increases with increasing temperature $T_h$, reaching an efficiency of approximately 75% for high temperatures. Furthermore, the efficiency $\eta/\eta_{\text{Carnot}}$ of our quantum Otto KNO is always larger than one, thus, indicating the the KNO efficiency surpasses the QHO Otto efficiency $\eta_{\text{Otto}} = 1 - \omega_c/\omega_h$. However, the KNO efficiency $\eta/\eta_{\text{Otto}}$ does not surpass the Carnot efficiency $\eta_{\text{Carnot}}/\eta_{\text{Otto}} = [1 - T_c/T_h]/[1 - \omega_c/\omega_h] = 3$ (dashed line). We also note that $\eta$ depends weakly on the cold-reservoir Kerr non-linearity strength. Indeed, for $K_c < K_h$, the efficiency is always larger than $\eta_{\text{Otto}}$. For $K_c > K_h$, the heat engine condition $W < 0$ and $Q_h > 0$ is only satisfied at low temperatures, but the heat engine efficiency is smaller than $\eta_{\text{Otto}}$ (not shown in figure 3).

We observe in figure 3 that KNO Otto heat engine efficiency can reach approximately 2.5 times the QHO heat engine’s efficiency. Thus, these results show that the Kerr non-linear
interaction enhances the efficiency of heat engines in comparison to QHO heat engines. It is important to emphasize that, in the context of circuit QED, both the frequency and Kerr non-linearity strength can be tuned simultaneously [28].

4.2. Refrigerator powered by Kerr nonlinearity

In this section, we extend the formalism presented in the previous sections to investigate the role of Kerr nonlinearity in the efficiency and coefficient of performance. Indeed, for the Kerr nonlinearity strength $\epsilon$ must be made as large as possible, while taking the limit of $\Delta K/\Delta \omega$ going to zero.

Figure 4 shows that the coefficient of performance $\eta_{\text{QHO}} = \omega_c/\Delta \omega$ is recovered for $K_c/\Delta K = \omega_c/\Delta \omega$ and, as expected, in the absence of Kerr non-linearities $K_c = K_h = 0$. In order to maximize the, the ratio $K_c/\omega_c$ must be fixed. At finite temperature, $\epsilon$ is always larger than the coefficient of performance of QHO refrigerator $\epsilon_{\text{QHO}} = \omega_c/\Delta \omega = 1/3$ and smaller than coefficient of performance of the Carnot’s refrigerator $\epsilon_{\text{Carnot}} = T_c/(T_h - T_c) = 7/3$.

The QHO Otto coefficient of performance $\epsilon_{\text{QHO}} = \omega_c/\Delta \omega$ is retrieved for $K_c/\Delta K = \omega_c/\Delta \omega$ and, as expected, in the absence of Kerr non-linearities $K_c = K_h = 0$. In order to maximize the $\epsilon$, the ratio $K_c/\omega_c$ must be made as large as possible, while taking the limit of $\Delta K/\Delta \omega$ going to zero.

Figure 5 shows that, for certain parameters, the improvement in performance can exceed 3 times the ideal performance $\eta_{\text{Otto}}$ of the QHO refrigerator. Thus, the above results demonstrate that the KNO Otto refrigerator can be implemented in a circuit QED device and its performance easily surpasses the performance of the QHO refrigerator in a Otto cycle.

5. Final remarks

We investigated the role played by Kerr nonlinearity powering QTM operating in Otto cycle. By using realistic parameters taken from circuit QED [23, 28, 35], we demonstrated that both the engine efficiency $\eta$ and the refrigerator coefficient of performance $\epsilon$ are enhanced, surpassing their QHO counterparts $\eta_{\text{Otto}}$ and $\epsilon_{\text{Otto}}$ and limited by the Carnot’s efficiency and coefficient of performance. Indeed, for the
parameters used, taking into account the experimental feasibility in the context of the circuit QED. Kerr nonlinearity enables to achieve gains of up to 2.5 times for heat engines and above 3 times for refrigerators in comparison with the QHO Otto machines. Differently from the QHO Otto machines, KNO 3 times for refrigerators in comparison with the QHO Otto to achieve gains of up to 2.5 times for heat engines and above. In the context of the circuit QED, Kerr nonlinearity enables to achieve gains of up to 2.5 times for heat engines and above.

Also, in future studies, we will consider finite-time effects and the use of shortcut to adiabaticity techniques to improve efficiency and performance of Kerr nonlinear quantum Otto machines [32, 33, 37].

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

ORCID iDs

Udson C Mendes https://orcid.org/0000-0002-1230-8366
Norton G de Almeida https://orcid.org/0000-0001-8517-6774

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