Remarks on Localizing Futaki-Morita Integrals At Isolated Degenerate Zeros

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Abstract

In this note we study the localization of Futaki-Morita integrals at isolated degenerate zeros by giving a streamlined exposition in the spirit of Bott [4] and implement the localization procedure for a holomorphic vector field on $\mathbb{C}P^n$ with a maximally degenerate zero, giving an essentially unique formula for the Futaki-Morita integral invariants without using a summation over multiple points. In a coming paper we will apply similar calculations to the Calabi-Futaki invariant of a Kähler blowup.

1 Introduction

Let $M$ be an $n$-dimensional compact complex manifold and $\mathfrak{h}$ the Lie algebra of holomorphic vector fields on $M$. An isolated zero $p$ of $X \in \mathfrak{h}$ is called nondegenerate if for local coordinates $(z_1, \ldots, z_n)$ centered at $p$,

$$X = \sum_{i,j} [a_{ij} z_i + O(z^2)] \frac{\partial}{\partial z_j}$$

the matrix $DX = (a_{ij})$ is invertible at $p$, i.e. $\det DX_p \neq 0$, and degenerate otherwise. Given a Hermitian metric on $M$, let $\Theta$ be the curvature of its Chern connection $\nabla$. The holomorphic localization theorem of Bott [4] (see also [13]) states:

**Theorem 1.1** (Bott [4]) Suppose $X \in \mathfrak{h}$ is such that $\text{Zero}(X)$ consists of isolated nondegenerate zeros $\{p_i\}$. For any invariant polynomial $\phi$ of degree $n$,

$$\int_M \phi \left( \frac{\sqrt{-1}}{2\pi} \Theta \right) = \sum_i \frac{\phi(DX_{p_i})}{\det DX_{p_i}}$$

(Bott [3] extended this result to vector fields with positive dimensional but still nondegenerate zero locus (nondegenerate in the sense that $DX$ is invertible in the normal direction to the zero locus).
When \( \deg(\phi) < n \) the lefthand side of (1) is of course zero for dimensional reasons. A generalization to \( \deg(\phi) > n \) was given by Futaki and Morita [12]: Let

\[
E = \mathcal{L}_X - \nabla_X
\]

where \( \mathcal{L}_X \) is the Lie derivative with respect to \( X \). It is straightforward to check \( E \) defines a smooth endomorphism \( E \in \Gamma(\text{End}(TM')) \) of the holomorphic tangent bundle \( TM' \). The Futaki-Morita integral is

\[
f_\phi(X) = \int_M \tilde{\phi}(E, \ldots, E, \frac{\sqrt{-1}}{2\pi} \Theta, \ldots, \frac{\sqrt{-1}}{2\pi} \Theta)
\]

where \( \tilde{\phi} \) is the polarization of an invariant polynomial \( \phi \) of degree \( n + k \). Futaki and Morita showed \( f_\phi : \mathfrak{h} \to \mathbb{C} \) does not depend on the choice of metric (in Bott’s theorem this follows from Chern-Weil theory) and by the same transgression argument used by Bott to prove Theorem [11] showed:

**Theorem 1.2 (Futaki-Morita [12])** Suppose that \( X \in \mathfrak{h} \) has isolated, nondegenerate zeros \( \{p_i\} \) and \( E \in \Gamma(\text{End}(TM')) \) as in (2). Then

\[
\left( \frac{n + k}{n} \right) f_\phi(X) = (-1)^k \sum_i \frac{\phi(DX_{p_i})}{\det DX_{p_i}}
\]

Futaki-Morita moreover showed that Futaki’s invariant obstructing the existence of Kähler-Einstein metrics on compact Kähler manifolds with \( c_1(M) > 0 \) can be understood within this integral invariant framework (see section 2.4).

The proof of (3) is based on exhibiting the Futaki-Morita integral as a certain Grothendieck residue via transgression, and the Bochner-Martinelli kernel provides an explicit representative for the Grothendieck residue. Using properties of the Grothendieck residue and inserting a power series expansion into the transgression argument, we will show the following extension to the case of isolated degenerate zeros:

**Theorem 1.3** If the zero locus of \( X \in \mathfrak{h} \) is a single isolated degenerate zero \( p \) such that in local coordinates centered at \( p \)

\[
z_i^\alpha_{i+1} = \sum b_{ij} X_j
\]

for some matrix \( B = (b_{ij}) \) of holomorphic functions, then

\[
\left( \frac{n + k}{n} \right) f_\phi(X) = (-1)^k \frac{1}{\prod \alpha_i!} \left. \frac{\partial^{\alpha_i}(\phi(DX) \det B)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \right|_{z=0}
\]

(4)

If Zero(\( X \)) consists of multiple isolated, possibly degenerate points then the Futaki-Morita integral is a sum over local contributions [4].
The existence of such an $\alpha$ is guaranteed by the strong Hilbert Nullstellensatz for analytic functions. In the case that $X$ has nondegenerate zeros, one may take $B = DX^{-1}$ with $\alpha_i = 0$, and (3) is immediately recovered.

Theorem 1.3 follows from a simple power series expansion in Bott’s transgression argument and application of well-known properties of Grothendieck residues. Surprisingly it does not seem to have received use in the literature although it has certainly been pointed out in related contexts [2] [15] [5] [7]. We give a complete presentation, hopefully contributing to the available exposition on Bott-style localization. The calculations in the last section serve to illustrate localization at a degenerate zero, even if the results are standard. We remark that Proposition 4.1 is essentially unique in that any vector field with a maximally degenerate zero on $\mathbb{C}P^n$ is equivalent to the one used, and thus any formula for Futaki-Morita invariants on $\mathbb{C}P^n$ not involving a summation over fixed points will be of the form arrived at.

One application of localization at degenerate zeros is to calculations on blow-ups: If $X$ is a holomorphic vector field with nondegenerate zero at $p$, the blowup $\text{Bl}_p(M)$ at $p$ admits a holomorphic lift $\tilde{X}$ of $X$. Zeros of $\tilde{X}$ in the exceptional divisor may very well be degenerate, depending on the linearization of $X$ at $p$. We will study this in a forthcoming paper, in particular extending results of Li and Shi concerning the Futaki invariant of Kähler surface blow-ups [14]. The calculations used will be extensions of that in Proposition 4.1.

The paper is organized as follows: In section 2 we recall background material on invariant polynomials, Grothendieck residues, the Bochner-Martinelli kernel, and the Futaki invariant for clarity. In section 3 we give a complete proof of the main theorem, which may in particular be read as a self-contained proof of the results of Bott and Futaki-Morita. In section 4 we give our main calculation.

## 2 Background

### 2.1 Invariant Polynomials

Let $\mathfrak{gl}(n, \mathbb{C})$ denote the spaces of $n \times n$ matrices over $\mathbb{C}$. An invariant polynomial $\phi : \mathfrak{gl}(n, \mathbb{C}) \rightarrow \mathbb{C}$ is a homogeneous polynomial in the entries of $\mathfrak{gl}(n, \mathbb{C})$ such that $\phi(A) = \phi(gAg^{-1})$ for all $g \in GL(n, \mathbb{C})$.

We will consider two sources of input for an invariant polynomial $\phi$:

1. Let $X \in \mathfrak{h}$ be a holomorphic vector field vanishing at $p$ and consider $A = DX$. As coordinate change about $p$ has the effect of conjugating $DX$, $\phi(DX)$ is locally a well-defined holomorphic function.

2. Let $E \in \Omega^k(\text{End}(T'M'))$. Locally $E$ is a $k$-form valued matrix that transforms according to $E_\alpha = g_{\alpha\beta}E_\beta g_{\alpha\beta}^{-1}$, where $g_{\alpha\beta}$ are the usual transition functions for
By the invariance hypothesis, $\phi(E) \in \Omega^*(M, \mathbb{C})$ given by point-wise evaluation in local coordinates is well-defined.

2.2 Grothendieck Residues

Let $U$ be an open ball about the origin in $\mathbb{C}^n$ and consider holomorphic functions $f_1, \ldots, f_n \in \mathcal{O}(U)$ such that the origin is an isolated zero of $f = (f_1, \ldots, f_n)$. The Grothendieck residue of

$$\omega = \frac{h(z) dz^1 \wedge \cdots \wedge dz^n}{f_1(z) \cdots f_n(z)} \quad h \in \mathcal{O}(U)$$

at 0 is defined to be

$$\operatorname{Res}_0(\omega) = \left(\frac{1}{2\pi \sqrt{-1}}\right)^n \int_\Gamma \omega. \quad (5)$$

where $\Gamma$ is the real $n$-cycle $\Gamma = \{ z \mid |f_i(z)| = \varepsilon_i \}$, oriented by

$$d(\arg(f_1)) \wedge \cdots \wedge d(\arg(f_n)) > 0.$$

Linearity of the residue is immediate, as is the fact that $\operatorname{Res}_0 \omega$ depends only on the homology class $\Gamma \in H_n(U - D, \mathbb{Z})$ and cohomology class $[\omega] \in H^n_{DR}(U - D)$ where $D_i = f_i^{-1}(0)$ and $D = \bigcup D_i$.

An alternate description of the Grothendieck residue that employs a degree $2n - 1$ de Rham class is as follows: Let $U_i = U - D_i$ and consider the open cover $\{U_i\}$ of $U^* = U - \{0\}$. The meromorphic form $\omega$ can be thought of as a Čech $(n-1)$-coycle for the sheaf of holomorphic forms on $U^*$, which is trivially closed as there are only $n$ open sets in the cover. We denote by $\eta_\omega$ the image of $\left(\frac{1}{2\pi \sqrt{-1}}\right)^n \omega$ under the Dolbeault isomorphism $\dot{\bar{\mathbb{H}}}^{n-1}(U^*, \Omega^*) \cong H^{n,n-1}(U^*)$, and since $d = \partial$ on forms of type $(n, n-1)$ we may think of $\eta_\omega$ as an element of $H^{2n-1}_{DR}(U^*)$ $\cong \mathbb{C}$. Here we are using that $U^*$ has the homotopy type of the $2n - 1$ sphere.

It then turns out the Grothendieck residue is precisely the image of the following sequence of maps:

$$\operatorname{Res}_0 : \left(\frac{1}{2\pi \sqrt{-1}}\right)^n \omega \mapsto \eta_\omega \mapsto \int_{S^{2n-1}} \eta_\omega \in \mathbb{C} \quad (6)$$

We refer to Griffiths and Harris [13] for the calculation.

Lemma 2.1 (Transformation Rule) Suppose that $g = (g_1, \ldots, g_n)$ satisfies the same hypotheses as $f$ above and moreover that

$$g_i(z) = \sum_j b_{ij}(z)f_j(z),$$

where $b_{ij}(z)$ are holomorphic functions on $U$. Then

$$\operatorname{Res}_0(h(z) dz^1 \wedge \cdots \wedge dz^n) = \sum_i b_{i0}(z)f_i(z) \operatorname{Res}_0(\omega).$$
for some matrix $B(z) = (b_{ij}(z))$ of holomorphic functions. Then for any $h(z) \in \mathcal{O}(U)$,
\[
\text{Res}_0 \frac{h(z)dz^1 \wedge \cdots \wedge dz^n}{f_1(z) \cdots f_n(z)} = \text{Res}_0 \frac{h(z) \det B(z)dz^1 \wedge \cdots \wedge dz^n}{g_1(z) \cdots g_n(z)}
\]

We refer to p. 657-659 of [13] for a full proof. The key idea is the notion of a good deformation of $f$, namely a family $f_t = (f_{1,t}, \ldots, f_{n,t})$ of holomorphic functions on $U$ satisfying the same hypotheses as $f_i$, continuous in $t$, with $f_0 = f$, and such that for $t > 0$ the Jacobian of $f_t$ is invertible. A Sard’s Theorem argument proves the existence of such a good deformation. The lemma follows by establishing the transformation law in the case of an invertible Jacobian and taking an appropriate limit as $t \to 0$.

### 2.3 Bochner-Martinelli Formula

The Bochner-Martinelli kernel is defined on $\mathbb{C}^n \times \mathbb{C}^n$ by
\[
\beta(w, z) = C_n \sum_{i=1}^{n} \frac{(w_i - z_i) \Phi_i(w - z) \wedge \Phi(w)}{\|w - z\|^{2n}}
\]

where
\[
C_n = (-1)^{n(n-1)/2} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \\
\Phi_i(w) = (-1)^{i-1} dw^1 \wedge \cdots \wedge \hat{dw}^i \wedge \cdots \wedge dw^n \\
\Phi(w) = dw^1 \wedge \cdots \wedge dw^n
\]

Key properties are:

1. $\overline{\partial} \beta(w, z) = 0$ as a function of $w$ away from the diagonal $w = z$.

2. The constant $C_n$ is such that $\int_{\partial B(0)} \beta(w, 0) = 1$, where $B(0)$ is any ball in $\mathbb{C}^n$ centered at 0 and integration is with respect to $w$.

The Bochner-Martinelli kernel may be used to construct an explicit representative of the class $\eta_\omega$ in (6): Given $f, \omega, \eta_\omega$ as before, let $F : U \to \mathbb{C}^n \times \mathbb{C}^n$ be $F(z) = (z + f(z), z)$. It follows that
\[
\eta_\omega = h(z) F^* \beta(w, z)
\]
is a distinguished representative of the class $\left[ \left( \frac{1}{2\pi\sqrt{-1}} \right)^n \omega \right]$. In other words,
\[
\text{Res}_0(\omega) = C_n \int_{\partial B(0)} h(z) \sum_{i=1}^{n} \frac{(-1)^{i-1} \bar{f}_i df \wedge \cdots \wedge \hat{df}_i \wedge \cdots \wedge \bar{d}f_n \wedge dz_1 \wedge \cdots \wedge dz_n}{\|f\|^{2n}}
\]

(7)
2.4 Futaki Invariant

Let $M$ be an $n$-dimensional compact Kähler manifold. Establishing the existence of various canonical metrics on $M$ is one of the central problems in Kähler geometry. See [16] for a survey. In the search for Kähler-Einstein metrics, the first Chern class $c_1(M)$ is necessarily definite or zero according to the sign of the Ricci curvature, imposing a strong topological restriction. The celebrated works of Yau [18] and Aubin, Yau [1], [18] settled existence and uniqueness of Kähler-Einstein metrics in the cases of $c_1(M) = 0$ and $c_1(M) < 0$, respectively. When $M$ has positive first Chern class there are well-known obstructions and the problem has only recently been settled in the work of Chen-Donaldson-Sun [6]; see also Tian [17].

We recall Futaki’s obstruction to Kähler-Einstein metrics when $c_1(M) > 0$. Choose a Kähler metric $\omega \in 2\pi c_1(M)$. Since $\text{Ric}(\omega) \in 2\pi c_1(M)$ as well, by the $\partial \bar{\partial}$-lemma

$$\text{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} F_\omega$$

for some real-valued function $F_\omega$ (defined up to addition of a constant). The metric $\omega$ is called Kähler-Einstein if $F_\omega$ is constant.

Futaki [9], [10] defined what is now called the Futaki invariant

$$\text{Fut}(X, \omega) = \int_M X(F_\omega) \omega^n$$

and showed the definition does not depend on the choice of $\omega$ within its Kähler class. The vanishing of $\text{Fut}(X, \omega)$ is thus necessary for the existence of a Kähler-Einstein metric.

Futaki and Morita [11], [12] showed that the Futaki invariant may be understood within the Futaki-Morita integral invariant framework. Specifically, they proved

$$\text{Fut}(X, \omega) = f_\phi(X)$$

where $\phi$ is the invariant polynomial $\phi(A) = \text{Tr}(A^{n+1})$. By (3), when $X$ has isolated nondegenerate zeros $\{p_i\}$,

$$\text{Fut}(X, \omega) = -\frac{1}{n+1} \sum_i \frac{\text{Tr}(DX_{p_i})^{n+1}}{\det DX_{p_i}}$$

3 Proof of Theorem 1.3

We now turn to the proof of Theorem 1.3, which will use:

**Lemma 3.1** Suppose $f = (f_1, \ldots, f_n)$ is holomorphic and has an isolated zero at $z = 0$, and let $B = (B_{ij})$ be a matrix of holomorphic functions such that
\[ z_i^{\alpha_i+1} = \sum_{i,j=1}^n B_{ij} f_j \]

for some \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n \). Then

\[
\text{Res}_0 \left[ \frac{h(z) dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n} \right] = \frac{1}{\prod \alpha_i!} \left. \frac{\partial^{|\alpha|} \left( h(z) \det B \right)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \right|_{z=0}
\]

where \( |\alpha| = \sum \alpha_i \).

**Proof** Since Lemma 2.1 holds for possibly singular \( B \),

\[
\text{Res}_0 \left[ \frac{h(z) dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n} \right] = \text{Res}_0 \left[ \frac{h(z) \det B \, dz_1 \wedge \cdots \wedge dz_n}{z_1^{\alpha_1+1} \cdots z_n^{\alpha_n+1}} \right].
\]

Expand the holomorphic function \( h(z) \det B \) in a neighborhood of \( z = 0 \):

\[
h(z) \det B = \sum_{\gamma \geq 0} \frac{1}{\gamma!} \left. \frac{\partial^{\gamma} \left( h(z) \det B \right)}{\partial z_1^{\gamma_1} \cdots \partial z_n^{\gamma_n}} \right|_{z=0} z_1^{\gamma_1} \cdots z_n^{\gamma_n}
\]

By linearity of the Grothendieck residue

\[
\text{Res}_0 \left[ \frac{h(z) dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n} \right] = \sum_{\gamma \geq 0} \frac{1}{\gamma!} \left. \frac{\partial^{\gamma} \left( h(z) \det B \right)}{\partial z_1^{\gamma_1} \cdots \partial z_n^{\gamma_n}} \right|_{z=0} \text{Res}_0 \left[ \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{\alpha_1-\gamma_1+1} \cdots z_n^{\alpha_n-\gamma_n+1}} \right]
\]

The lemma then follows from the definition of Grothendieck residue and the multidimensional Cauchy integral formula, which shows all terms with \( \gamma_i \neq \alpha_i \) vanish while terms with \( \gamma_i = \alpha_i \) produce a residue of 1.

Let us define forms

\[
\phi_r = \binom{n+k}{r} \tilde{\phi}(E, \ldots, E, \Theta, \ldots, \Theta) \in \Omega^{r,r}(M, \mathbb{C})
\]

where \( \tilde{\phi} \) is the polarization of invariant polynomial \( \phi \) of degree \( n+k \) and \( E \) as in (2). It is \( \phi_n \) in which we are ultimately interested for dimensional reasons.

Also let \( \hat{M} = M - \bigcup B_r(p_i) \) where \( B_r(p_i) \) denotes small disjoint balls about the \( p_i \in \text{Zero}(X) \). Upon choice of a Hermitian metric \( g \) on \( M \), define

\[
\eta(\cdot) = \frac{g(\cdot, \bar{X})}{\|X\|^2} \in \Omega^{1,0}(\hat{M})
\]

\[
\Phi_i = \eta \wedge \phi_i \wedge (\bar{\partial} \eta)^{n-i-1} \in \Omega^{n,n-1}(\hat{M})
\]
\[
\Phi = \sum_{i=0}^{n-1} \Phi_i = \eta \wedge \sum_{i=0}^{n-1} \phi_i \wedge (\bar{\partial}\eta)^{n-i-1} \in \Omega^{n,n-1}(\tilde{M})
\]

**Lemma 3.2** With the above definitions,

1. \( \bar{\partial}\phi_i = i_X\phi_{i+1} \) for \( i = 0, \ldots, n - 1 \)

2. \( i_X\bar{\partial}\eta = 0 \)

3. \( i_X\bar{\partial}\Phi_i = i_X\phi_i \wedge (\bar{\partial}\eta)^{n-i} - i_X\phi_{i+1} \wedge (\bar{\partial}\eta)^{n-i-1} \) for \( i = 0, \ldots, n - 1 \)

As a result, on \( \tilde{M} \):

\[
\bar{\partial}\Phi + \phi_n = 0 \quad \text{(9)}
\]

**Proof** We first show

\[
\bar{\partial}E = i_X\Theta \quad \text{(10)}
\]

As \( L_X \) preserves the type of a form when \( X \) is holomorphic, and \( L_X = i_Xd + di_X \)

by Cartan’s formula, it follows from the decomposition \( d = \partial + \bar{\partial} \) that

\[
i_X\bar{\partial} + \bar{\partial}i_X = 0 \quad \text{(11)}
\]

Equation (10) follows by computing \( \bar{\partial}E \) applied to a local holomorphic section \( \sigma \) of \( TM' \):

\[
\bar{\partial}(E\sigma) = \bar{\partial}(L_X\sigma - i_X\nabla\sigma)
\]

\[
= 0 + i_X\bar{\partial}\nabla\sigma \quad \text{(using (11))}
\]

\[
= i_X\Theta\sigma
\]

1.) Using the symmetry of \( \tilde{\phi} \), equation (10), and that \( \bar{\partial}\Theta = 0 \),

\[
\bar{\partial}\phi_i = \binom{n}{i} \bar{\partial}\tilde{\phi}(E, \ldots, E, \Theta, \ldots, \Theta)
\]

\[
= \binom{n}{i} (n-i)\tilde{\phi}(E, \ldots, E, \bar{\partial}E, \Theta, \ldots, \Theta)
\]

\[
= \binom{n}{i} (n-i)\tilde{\phi}(E, \ldots, E, i_X\Theta, \Theta, \ldots, \Theta)
\]

\[
= \binom{n}{i} \frac{(n-i)}{(i+1)} i_X\tilde{\phi}(E, \ldots, E, \Theta, \ldots, \Theta)
\]

\[
= i_X\phi_{i+1}
\]
2.) Since $i_X \eta = 1$,

$$0 = \bar{\partial} (i_X \eta) = -i_X (\bar{\partial} \eta).$$

We are again using $i_X \bar{\partial} = -\bar{\partial} i_X$ as in (11).

3.) By the first two parts of the lemma and $i_X \eta = 1$,

$$i_X \bar{\partial} \Phi_i = i_X \left[ \phi_i \wedge (\bar{\partial} \eta)^n - \eta \wedge \bar{\partial} \phi_i \wedge (\bar{\partial} \eta)^{n-i} \right]$$

$$= i_X \left[ \phi_i \wedge (\bar{\partial} \eta)^n - \eta \wedge i_X \phi_{i+1} \wedge (\bar{\partial} \eta)^{n-i} \right]$$

$$= i_X \phi_i \wedge (\bar{\partial} \eta)^n - i_X \eta \wedge i_X \phi_{i+1} \wedge (\bar{\partial} \eta)^{n-i}$$

$$= i_X \Phi_i \wedge (\bar{\partial} \eta)^n - i_X \Phi_{i+1} \wedge (\bar{\partial} \eta)^{n-i}$$

We now prove (9):

$$i_X \bar{\partial} \Phi = i_X \bar{\partial} \sum_{i=0}^{n-1} \Phi_i$$

$$= \sum_{i=0}^{n-1} i_X \phi_i \wedge (\bar{\partial} \eta)^n - i_X \phi_{i+1} \wedge (\bar{\partial} \eta)^{n-i-1}$$

$$= i_X \phi_0 \wedge (\bar{\partial} \eta)^n - i_X \phi_n$$

$$= -i_X \phi_0$$

where we have used that $i_X \phi_0$ is trivially 0. Thus $i_X (\bar{\partial} \Phi + \phi_n(\Theta)) = 0$ on $\hat{M}$ and so

$$\bar{\partial} \Phi + \phi_n = 0$$

since $i_X$ is injective on top degree forms away from Zero($X$).

With these preliminaries out of the way, we are ready to prove Theorem 1.3. The transgression formula (9) reduces calculation to a neighborhood of Zero($X$):

$$\int_M \phi_n = \lim_{\epsilon \to 0} \int_{M_{\epsilon}} \phi_n$$

$$= -\lim_{\epsilon \to 0} \int_{M_{\epsilon}} \bar{\partial} \Phi \quad \text{(by (9))}$$

$$= -\lim_{\epsilon \to 0} \int_{M_{\epsilon}} d\Phi \quad \text{(since $\Phi$ is type $(n, n-1)$)}$$

$$= -\lim_{\epsilon \to 0} \sum_i \int_{\partial B_{\epsilon}(p_i)} \Phi \quad \text{(by Stokes' Theorem)} \quad (12)$$
These local contributions will be computed using a Hermitian metric $g$ that is Euclidean on a neighborhood of each $p_i$ (although the form $\Phi$ depends on the choice of $g$, by Futaki and Morita’s work $f_\phi(X)$ does not). To be precise, consider the open cover of $M$ by disjoint $U_i = B_\epsilon(p_i)$ and $U_0 = M - \bigcup B_{\epsilon/2}(p_i)$. Let $\{\rho_i\}$ be a partition of unity subordinate to this cover and $g_i$ be the Euclidean metric on $U_i$ for $i \neq 0$, and let $g_0$ be any Hermitian metric on $U_0$. Then $g = \sum \rho_i g_i$ is the Hermitian metric on $M$ we work with.

In the Euclidean metric, $\eta = \sum \frac{\bar{X}^i dz^i}{\|X\|^2}$ so that

$$\bar{\partial} \eta = \sum \frac{\bar{X}^i \wedge dz^i}{\|X\|^2} - \sum \frac{\bar{X}^i X^j \bar{X}^j \wedge dz^i}{\|X\|^4}$$

Notice that the second term of $\bar{\partial} \eta$ wedged with itself is zero by symmetry, as it is when wedged with $\eta$. We therefore find by direct computation

$$\eta \wedge (\bar{\partial} \eta)^{n-1} = -(-1)^{n(n-1)/2} (n-1)! \sum_i (-1)^{i-1} \bar{X}^i d\bar{X}^i \wedge \cdots \wedge \bar{X}^i \wedge \cdots \wedge d\bar{X}^n \wedge dz^1 \wedge \cdots \wedge dz^n$$

In terms of the Grothendieck residue ([7]), for any holomorphic $h$ we have

$$\left(\frac{-1}{2\pi i}\right)^n \int_{\partial B_{\epsilon/2}(p)} h(z) \eta \wedge (\bar{\partial} \eta)^{n-1} = (-1)^{n+1} \text{Res}_p \left[ \frac{h(z) dz^1 \wedge \cdots \wedge dz^n}{X^1 \cdots X^n} \right]$$

(13)

Since $g$ is Euclidean near $p \in \text{Zero}(X)$, $\Gamma^k_{ij} = 0$ and so

$$E|_{B_{\epsilon/2}(p)} = -\frac{\partial X^j}{\partial z^k} \frac{\partial}{\partial z^j} \otimes dz^k$$

It follows $\phi(E)|_{B_{\epsilon/2}(p)} = (-1)^{n+k} \phi(DX)$. And as $\Theta = 0$ near $p$ as well,

$$\Phi|_{B_{\epsilon/2}(p)} = \eta \wedge \phi_0 \wedge (\bar{\partial} \eta)^{n-1} = (-1)^{n+k} \phi(DX) \eta \wedge (\bar{\partial} \eta)^{n-1}$$

(14)

We finish the proof by continuing the above calculation with these observations,
\[
\left(\frac{n+k}{n}\right) f_\phi(X) = \int_M \left(\frac{\sqrt{-1}}{2\pi}\right)^n \phi_n
\]
\[= -\lim_{\epsilon \to 0} \sum_i \int_{\partial B_{\epsilon/2}(p_i)} \left(\frac{\sqrt{-1}}{2\pi}\right)^n \Phi \quad \text{(by (12))}
\]
\[= -\lim_{\epsilon \to 0} \sum_i \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{\partial B_{\epsilon/2}(p_i)} (-1)^{n+k} \phi(DX) \eta \wedge (\bar{\eta})^{n-1} \quad \text{(by (14))}
\]
\[= -(-1)^{n+k} \sum_i (-1)^{n+1} \text{Res}_{p_i} \left[ \frac{\phi(DX) dz^1 \wedge \cdots \wedge dz^n}{X^1 \cdots X^n} \right] \quad \text{(by (13))}
\]
\[= (-1)^k \sum_i \frac{1}{\prod \alpha_i!} \left. \frac{\partial^{\alpha_i}| (\phi(DX) \det B) |}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \right|_{z=0} \quad \text{(by Lemma 2.1)}
\]

4 Localization at a maximally degenerate zero on \(\mathbb{C}P^n\)

In this section we illustrate Theorem 1.3 by computing Futaki-Morita invariants for a holomorphic vector field on \(\mathbb{C}P^n\) with a maximally degenerate zero. Proposition 4.1 in particular gives a localization formula for Chern numbers of \(\mathbb{C}P^n\) without a summation over multiple points. As the maximally degenerate vector field we use is unique up to coordinate change, such a formula is essentially unique.

Let \(A \in \mathfrak{sl}(n + 1, \mathbb{C})\) be zero everywhere except for a diagonal of 1’s above the main diagonal. \(A\) induces a holomorphic vector field \(X = \sum A_{ij}Z_j \frac{\partial}{\partial Z_i}\) in homogeneous coordinates (we let the indices for \(A\) begin at 0 here). This vector field has a single zero at \(p = [1, 0, \ldots, 0]\), which is isolated and of maximal degeneracy. Changing to nonhomogeneous coordinates \(z_i = Z_i/Z_0\) for \(i = 1, \ldots, n\) on \(U_0 = \{Z_0 \neq 0\}\),

\[
X = \sum_{j=1}^{n-1} (z_{j+1} - z_1 z_j) \frac{\partial}{\partial z_j} + (-z_1 z_n) \frac{\partial}{\partial z_n}
\]

so that

\[
DX|_{U_0} = \begin{bmatrix}
-2z_1 & 1 & 0 & \cdots & 0 \\
-z_2 & -z_1 & \ddots & \ddots & \vdots \\
\vdots & 0 & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 1 \\
-z_n & 0 & \cdots & 0 & -z_1
\end{bmatrix}
\]

In order to implement Theorem 1.3 we need to find \(B\) such that \(z_i^{\alpha_i+1} = \sum B_{ij} X_j\).
To do this systematically choose $k \in \mathbb{Z}$ such that $2^k < n + 1 \leq 2^{k+1}$. One may observe from (15)

$$z_1^{n+1} = (-z_1)^{n-1}X_1 + (-z_1^{n-2})X_2 + \cdots + (-z_1)X_{n-1} + (-1)X_n$$

$$z_2^n = z_n X_{n-1} + (-z_{n-1})X_n$$

while by completely factoring differences of squares in $z_j^{2^k} - (z_1 z_j)^{2^k}$, we have for $j = 1, \ldots, n-2$

$$z_{j+1}^{2^k} = (z_j^{2^k}) z_1^{2^k} + X_j \prod_{i=0}^{k-1} (z_{j+1}^{2^i} + z_1^{2^i} z_j^{2^i})$$

By the choice of $k$ and the above expression for $z_1^{n+1}$, these expressions recursively give $z_{j+1}^{2^k}$ as a linear combination of the $X_j$ and thus contain the information necessary to form the desired matrix $B = (B_{ij})$ with $\alpha_1 = n, \alpha_n = 1$, and $\alpha_i = 2^k - 1$ for $i = 2, \ldots, n-1$. It then follows from Theorem 1.3 that

$$\binom{n + k}{n} f_\phi(X) = (-1)^k \frac{1}{n!(2^k - 1)!} \frac{\partial (\phi(DX) \det B)}{\partial z_1^n (\partial z_1)(\partial z_2)(\partial z_{n-1})} \bigg|_{z = 0} (17)$$

By using standard determinant properties, we find

$$\det B = (-1)^n (z_n + z_1 z_{n-1}) \prod_{j=1}^{n-2} \prod_{i=0}^{k-1} (z_{j+1}^{2^i} + z_1^{2^i} z_j^{2^i})$$

Nearly all $z_2, \ldots, z_{n-1}$ derivatives of det $B$ evaluated at $z_2 = \cdots = z_{n-1} = 0$ yield zero or a term with $z_1^m$ where $m > n$, which may be ignored. The exception is when all $(2^k - 1)$ derivatives for each of $z_2, \ldots, z_{n-1}$ in (17) are applied to det $B$, yielding

$$\frac{\partial \det B}{(\partial z_2)(\partial z_{n-1})} \bigg|_{z_2 = \cdots = z_{n-1} = 0} = (-1)^n [(2^k - 1)!]^{n-2} (z_1^n + z_n)$$

or when only one of these $(2^k - 1)$ derivatives is not applied, giving

$$\frac{\partial \det B}{(\partial z_2)^{2^k-1} (\partial z_{n-1})^{2^k-1}} \bigg|_{z_2 = \cdots = z_{n-1} = 0} = (-1)^n \frac{[(2^k - 1)!]^{n-2}}{2^k - 1} z_1^{2^k} z_{n-1}$$

With these observations, (17) is evaluated to give

**Proposition 4.1** Let $X$ be the maximally degenerate vector field on $\mathbb{C}P^n$ given in (15). For any invariant polynomial $\phi$ of degree $n + k$, the Futaki-Morita integral is

$$\left. \binom{n + k}{n} f_\phi(X) = (-1)^{n+k} \frac{1}{n!} \left( \frac{\partial^n \phi(DX)}{\partial z_1^n} + \sum_{j=2}^{n} \frac{\partial}{\partial z_1^n \partial z_j} (\phi(DX) \cdot z_j^1) \right) \right|_{z = 0}$$
where $DX$ is as in (16).

A few simple cases of note:

(i) Let $\phi(A) = \det(A)$, so $k = 0$ and $f_{\phi}(X)$ calculates the Euler characteristic $\chi(CP^n)$. From (16),

$$\det(DX) = (-1)^n \left[2z_1^n + \sum_{j=2}^{n} z_j z_1^{n-j} \right]$$

Inserting this into Proposition 4.1 yields

$$\chi(CP^n) = \frac{(-1)^n}{n!} \left( \frac{\partial^n \det(DX)}{\partial z_1^n} + \sum_{j=2}^{n} \frac{\partial}{\partial z_1} \left( (\det(DX) \cdot z_j) \right) \right) \bigg|_{z=0}$$

$$= \frac{1}{n!} \left( 2n! + \sum_{j=2}^{n} \frac{\partial}{\partial z_1} (z_j z_1^{n-j}) \right)$$

$$= n + 1$$

(ii) Take $\phi(A) = [\text{Tr}(A)]^n$. From (16),

$$\text{Tr}(DX) = -(n + 1)z_1$$

By Proposition 4.1

$$\int c_1^n = \frac{(-1)^n}{n!} \frac{\partial}{\partial z_1} \left( (-n + 1)z_1 \right) \bigg|_{z=0}$$

$$= (n + 1)^n$$

One could at this point compute the entire cohomology ring of $CP^n$ as in [13], but without the complicated summations.

(iii) Take $\phi(A) = [\text{Tr}(A)]^{n+1}$, so that $f_{\phi}(X)$ calculates the Futaki invariant as in [8]. By (16) again,

$$\phi(DX) = [-(n + 1)z_1]^{n+1}.$$ 

It is immediate from Proposition 4.1 that Fut$(CP^n, X) = 0$ as there are no derivatives of appropriate order. Of course this is necessary; the Fubini-Study metric on $CP^n$ is well-known to be Kähler-Einstein.

(iv) Similarly, one could check that $f_{\phi}(X)$ vanishes for $\phi(A) = \text{Tr}(A) \det A$ as there are again no derivatives of appropriate order. This vanishing was observed by Futaki to always be the case [11].
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