A simpler proof of zero-knowledge against quantum attacks using Grover’s amplitude amplification

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April 1, 2022

1 Introduction

Watrous\cite{Watrous2006} had presented the first proof of zero-knowledge property of a proof system against a quantum verifier. The key of the proof is the construction of a quantum simulator. In the construction, the ‘failure state’ is rotated to the ‘success’ state by a tricky operation which is initially developped for the amplification of QMA proof systems.

This manuscript presents a new and simpler construction of a simulator. In the construction, we simply amplify the success probability of a classical simulator using Grover’s amplification.

2 The Goldreich-Micali-Wigderson Graph Isomorphism Proof System

The Goldreich-Micali-Wigderson graph isomorphism protocol is a well-known example of a proof system that is perfect zero-knowledge against classical polynomial-time verifiers. In this section it is proved that this protocol is in fact zero-knowledge against polynomial-time quantum verifiers. The method can be extended to several other protocols.

2.1 The protocol

Let $x$ be a pair of graphs $(G_0, G_1)$, and $L$ be a set of pairs with $G_0 \simeq G_1$. Hereafter, $P$ denotes the prover, and $V$ the verifier.

(a) $P$ randomly chooses a permutation $\tau$ on the graph, and sends $\tau(G_0)$ to $V$.
(b) $V$ sends a random bit $a \in \{0, 1\}$ to $P$.
(c) $P$ send a permutation $\pi$, and $V$ accepts if $\tau(G_0) = \pi(G_a)$. 
To decrease the error probability, (a)-(c) are repeated for polynomially many times.

The quantum description of this classical protocol is as follows. Let $V$ and $A$ be the $V$'s workspace and a qubit which stores output of the simulator at the end the step (b), respectively. The register $V$ stores the message from $P$ to $V$ in the step (a). We also denote by $W$ for an auxiliary input $|\psi\rangle$.

The initial state is

$$|\psi\rangle |0_V\rangle |0_A\rangle |0_Y\rangle.$$ 

After the step (a),

$$|\psi\rangle \langle \psi| \otimes |0_V\rangle \langle 0_V| \otimes |0_A\rangle \langle 0_A| \otimes \frac{1}{n!} \sum_{\tau \in S_n} |\tau(G_0)\rangle \langle \tau(G_0)|.$$ 

The honest verifier will apply Hadamard transform to $|0_A\rangle$ and measure $A$ in the step (b),

$$\frac{1}{2n!} \left[ |\psi\rangle \langle \psi| \otimes |0_V\rangle \langle 0_V| \otimes \sum_{a \in \{0,1\}} |a_A\rangle \langle a_A| \otimes \sum_{\tau \in S_n} |\tau(G_0)\rangle \langle \tau(G_0)| \right].$$

In general, however, a verifier will apply an unitary transform $U_V$ on $W \otimes V \otimes A \otimes Y$, and measure $A$.

$$\frac{1}{n!} \sum_{\tau \in S_n, a \in \{0,1\}} |a_A\rangle \langle a_A| U_V (|\psi\rangle \langle \psi| \otimes |0_V\rangle \langle 0_V| \otimes |0_A\rangle \langle 0_A| \otimes |\tau(G_0)\rangle \langle \tau(G_0)|) U_V^\dagger |a_A\rangle \langle a_A|.$$ 

After this, the step (c) follows, but we omit the description of this part, for this step is easy to simulate once a simulation of the steps (a)-(b) is given.

### 2.2 A simulator

A classical simulator is constructed as follows. Assume that $G_0 \simeq G_1$. The simulator randomly chooses $b \in \{0,1\}$ and $\pi \in S_n$, and compute $\pi(G_b)$ which mimics $P$'s first message. Then it applies the operation of $V$ on the simulated message from $P$, producing an output $a \in \{0,1\}$, or the message to $P$. If $a = b$, $\pi$ chosen previously can mimic the second message from $P$ to $V$, and the simulation succeeds. If $a \neq b$, we "rewind", or abort and restart from the beginning. This successfully simulates the single round of GMW protocol with probability $\frac{1}{2}$, meaning that the simulation succeeds with high probability after some iterations.

To simulate the iterations of the single round, the simulator also has to be repeatedly run. Observe that in rewinding, the simulation only has to restart from the beginning of the present round, with the record of the final state of the previous round being copied in some registers. Otherwise, the simulation would take exponential time. In quantum case, however, this part fails because of the no-cloning principle.
Here we show how to bypass this difficulty: Grover’s amplitude amplification can increase the success probability of the simulation of each round up to 1, and thus there is no need for rewinding.

Let us define
\[ X = V \otimes A \otimes Y \otimes B \otimes Z, \]
where \( Z \) and \( B \) stores random bits specifying a permutation \( \pi \) on the graph and a random bit \( b \), respectively.

Let us denote by \( A \) a unitary operation corresponding to the classical simulator other than rewinding part,
\[
\begin{align*}
A |\psi\rangle |0_X\rangle &= \frac{1}{\sqrt{2^n}} \sum_{b \in \{0,1\}, \pi \in S_n} \langle U_V |\psi\rangle |0_V\rangle |0\rangle |\pi(G_b)) \rangle |b\rangle |\pi\rangle.
\end{align*}
\]

We apply amplitude amplification to this operation. Define a unitary transform \( S_0^\varphi, S_1^\varphi \) in \( W \otimes X \) by
\[
\begin{align*}
S_0^\varphi : &= (\varphi - 1) I_{W \otimes V} \otimes |0_X\rangle \langle 0_X| + 1, \\
S_1^\varphi : &= (\varphi - 1) \Pi A + 1.
\end{align*}
\]

where \( \Pi \) is the projection onto success event,
\[
\Pi : = \sum_{b \in \{0,1\}} I_{W \otimes V} \otimes |b\rangle \langle b| \otimes I_{Y} \otimes |b\rangle \otimes I_{Z}.
\]

These phase factors are chosen according to lemma 3 in [1].

Observe that \( a = b \) occurs with probability \( \frac{1}{2^2} \), for all the state \( |\psi\rangle \) because \( b \in \{0,1\} \) is uniformly random, and does not affect the input of \( U_V \). This assures us the identity
\[
\langle 0_X| A^{-1} \Pi A|0_X\rangle = \frac{1}{2} I_{W}.
\]

More rigorously, this is true for the following equalities holds for any \( |\psi\rangle \):
\[
\begin{align*}
\| \Pi A |\psi\rangle |0_X\rangle \|^2 &= \frac{1}{2n^3} \left\| \sum_{a,b \in \{0,1\}, \pi \in S_n} I_{W \otimes V} \otimes |a\rangle \otimes I_{Y} \otimes |a\rangle \otimes I_{Z} (U_V |\psi\rangle |0_V\rangle |0\rangle |\pi(G_b)) \rangle |b\rangle |\pi\rangle \right\|^2 \\
&= \frac{1}{2n^3} \left\| \sum_{b \in \{0,1\}, \pi \in S_n} I_{W \otimes V} \otimes |b\rangle \otimes I_{Y} \otimes I_{Z} (U_V |\psi\rangle |0_V\rangle |0\rangle |\pi(G_b)) \rangle |b\rangle |\pi\rangle \right\|^2 \\
&= \frac{1}{2n^3} \left\| \sum_{b \in \{0,1\}, \pi \in S_n} I_{W \otimes V} \otimes |b\rangle \otimes I_{Y} (U_V |\psi\rangle |0_V\rangle |0\rangle |\pi(G_b)) \rangle |b\rangle \langle \pi\rangle \right\|^2 \\
&= \frac{1}{2n^3} \left\| \sum_{b \in \{0,1\}, \pi \in S_n} I_{W \otimes V} \otimes |b\rangle \otimes I_{Y} (U_V |\psi\rangle |0_V\rangle |0\rangle |\pi(G_b)) \rangle |b\rangle \langle \pi\rangle \right\|^2
\end{align*}
\]
\[
\frac{1}{2^n n!} \sum_{\pi \in S_n} \sum_{\tau \in \{0, 1\}^n} \| \mathbf{I}_{W \otimes V} \otimes \langle b | \otimes \mathbf{I}_v (U_V |\psi\rangle |0\rangle |\pi(G_0)\rangle \| \|^2
\]

\[
= \frac{1}{2^n n!} \sum_{\pi \in S_n} 1 = \frac{1}{2},
\]

where in the third line, \( \tau(G_0) = G_1 \). Using the equation (1), as shortly described, we can explicitly check the following identity

\[
AS_0^\dagger A^{-1} S_0^\dagger A | \psi \rangle |0_x\rangle = (i-1) \Pi A |\psi\rangle |0_x\rangle.
\]

(2)

Measure \( B \) and \( Z \), and compute \( \pi(G_b) \), and store its result some register, say \( Z' \). Trace out the register. Then, the final state is

\[
\frac{1}{n!} \sum_{\pi \in S_n} \langle a_A | a_A | U_V \langle \psi | \otimes |0_V\rangle \langle 0_V | \otimes |a_A\rangle \langle \pi(G_0) | \pi(G_0) \rangle U_V^\dagger |a_A\rangle \langle a_A | |
\]

\[
\otimes | \tau(G_a) \rangle \rangle | \tau(G_a) \rangle .
\]

This shows that \( \pi(G_b), W \otimes V \otimes A \otimes Y \), and \( Z \) mimics the message from \( P \) to \( V \) in the step (a), the \( V \)'s final state in the step (b) and the message from \( V \) to \( P \), and the message from \( P \) to \( V \) in the step (c), respectively.

Below, we use the block representation in which \( |\psi\rangle |0_x\rangle \) writes

\[
|\psi\rangle |0_x\rangle = \begin{bmatrix} |\psi\rangle & 0 \end{bmatrix}.
\]

In that representation,

\[
A^{-1} \Pi A = \begin{bmatrix} \frac{1}{2} \mathbf{I}_W \Pi_{A,12} & \Pi_{A,12} \\ \Pi_{A,12} & * \end{bmatrix},
\]

\[
S_0^a = \begin{bmatrix} \mu \mathbf{I}_W & 0 \\ 0 & \mathbf{I}_X \end{bmatrix}.
\]

Therefore,

\[
AS_1^\dagger A^{-1} S_0^a A |\psi\rangle |0_x\rangle
\]

\[
= AS_1^\dagger \left( (i-1) A^{-1} \Pi A + \mathbf{I} \right) |\psi\rangle |0_x\rangle
\]

\[
= AS_1^\dagger \left[ \left( \frac{i-1}{2} + 1 \right) |\psi\rangle \right]
\]

\[
= A \left[ \frac{i (i-1)}{2} |\psi\rangle \right]
\]

\[
= (i-1) A \left[ \frac{i}{i} |\psi\rangle \right]
\]

\[
= (i-1) A -1 \Pi A |\psi\rangle |0_x\rangle
\]

\[
= (i-1) \Pi A |\psi\rangle |0_x\rangle.
\]
This is our assertion (2).

2.3 Watrous’s simulator revisited

Instead of doing Grover’s amplitude amplification, we can perform the measurement \( \Pi \) to the state \( A|\psi\rangle|0_X\rangle \). If the success event is observed, we are done. This occurs with probability \( \frac{1}{2} \). Otherwise, the state of the system collapses to \( \sqrt{2}(I - \Pi)A|\psi\rangle|0_X\rangle \), and \( AS_0^{-1}A^{-1} \), or reflection about \( A|\psi\rangle|0_X\rangle \) maps this state to \( \sqrt{2}\Pi A|\psi\rangle|0_X\rangle \), which corresponds to success.

This simulation is the same as the one presented in [2], although the presentation is different.

3 When success probability is not \( \frac{1}{2} \)

3.1 Amplification operations

The construction in the previous section seemingly depends on the fact that the success probability equals \( \frac{1}{2} \). In the section, we show that if we have

\[
A^{-1}\Pi A = \begin{bmatrix}
\lambda I_W & \Pi_{A,12}^1 \\
\Pi_{A,12} & \Pi_{A,22}
\end{bmatrix}
\]

our method works for any success probability \( \lambda \), if proper phase shifts are introduced. Especially, we have to check that repetition of the amplification works in the same as the case where the auxiliary input \( |\psi\rangle \) is absent.

Then, the identity

\[
(A^{-1}\Pi A)^2
\]

\[
= \begin{bmatrix}
\lambda^2 I_W + \Pi_{A,12}^1\Pi_{A,12} & \lambda \Pi_{A,12} + \Pi_{A,12}^1\Pi_{A,22} \\
\lambda \Pi_{A,12} + \Pi_{A,22}\Pi_{A,12} & \Pi_{A,12}\Pi_{A,12} + \Pi_{A,22}^2
\end{bmatrix}
\]

implies

\[
(\lambda^2 - \lambda) I_W + \Pi_{A,12}^1\Pi_{A,12} = 0
\]

\[
(\lambda - 1) \Pi_{A,12} + \Pi_{A,22}\Pi_{A,12} = 0
\]

\[
\Pi_{A,12}\Pi_{A,12} + \Pi_{A,22}^2 = \Pi_{A,22}.
\]

Define also

\[
|\text{succ}\rangle : = \frac{1}{\sqrt{\lambda}}\Pi A|\psi\rangle|0_X\rangle = \frac{1}{\sqrt{\lambda}} A \begin{bmatrix}
\lambda |\psi\rangle \\
\Pi_{A,12} |\psi\rangle
\end{bmatrix},
\]

\[
|\text{fail}\rangle : = \frac{1}{\sqrt{1-\lambda}}(I - \Pi)A|\psi\rangle|0_X\rangle = \frac{1}{\sqrt{1-\lambda}} A \begin{bmatrix}
(1 - \lambda) |\psi\rangle \\
-\Pi_{A,12} |\psi\rangle
\end{bmatrix}.
\]
Then we have

\[ A S_0^\phi A^{-1} S_1^\phi |\text{succ}\rangle = A S_0^\phi A^{-1} S_1^\phi A \cdot A^{-1} |\text{succ}\rangle \]

\[
\begin{align*}
&= \frac{A}{\sqrt{1 - \lambda}} \left[ \phi \{ \lambda (\varphi - 1) + 1 \} (\varphi - 1) P_{A,12} \right. \\
&\left. + \phi(\varphi - 1) I_{A,12} \right] \left[ (\varphi - 1) P_{A,22} + I \right] \left[ \frac{\lambda}{\varphi} |\psi\rangle \right] \\
&= \frac{A}{\sqrt{1 - \lambda}} \left[ (\varphi - 1) \phi \{ \lambda (\varphi - 1) + 1 \} - \phi(\varphi - 1) I_{A,12} P_{A,22} \right] |\psi\rangle \\
&= \frac{A}{\sqrt{1 - \lambda}} \left[ (\varphi - 1) \phi \{ \lambda (\varphi - 1) + 1 \} + (\lambda^2 - \lambda) \phi(\varphi - 1) \right] |\psi\rangle \\
&= \frac{A}{\sqrt{1 - \lambda}} \left[ \frac{1}{\lambda \varphi} I_{A,12} |\psi\rangle \right] \\
&= -\sqrt{\lambda} |\text{fail}\rangle.
\]

Therefore, the linear space spanned by \{ |\text{succ}\rangle, |\text{fail}\rangle \} is invariant by the action of \( A S_0^\phi A^{-1} S_1^\phi \). Especially, in the \( \phi = \varphi = -1 \) case,

\[ -A S_0^\phi A^{-1} S_1^\phi |\text{succ}\rangle = (1 - 2\lambda) \lambda |\text{succ}\rangle - 2\sqrt{\lambda (1 - \lambda)} |\text{fail}\rangle \]

\[ -A S_0^\phi A^{-1} S_1^\phi |\text{fail}\rangle = 2\sqrt{\lambda (1 - \lambda)} |\text{succ}\rangle + (1 - 2\lambda) |\text{fail}\rangle \]

and \(-A S_0^\phi A^{-1} S_1^\phi\) corresponds to one step of Grover’s search. Therefore, trivially, the repetition of the our amplification works in the same manner as the case where the auxiliary input is absent. Also, by choosing the phase factors property, we can control the speed of the amplification as in [1].

### 3.2 Computational zero-knowledge proof systems for NP

As is mentioned in subsection 4.2 in [2], a zero-knowledge proof system for Graph 3-Coloring (G3C) yields a zero-knowledge proof for any problem in NP. [2] presents a simulator for a classical proof system which is secure against attack
by any quantum verifier. In this subsection, we present a new construction of
simulator for this proof system.

In the construction of [2], the essential part is the amplification of the success
probability of a simulator $A$ which succeeds with probability $\frac{1}{m}$ with $m$ being a
polynomially-bounded function of the input length $n$.

We can construct such an amplification using Grover’s amplitude amplification
as is studied in the previous subsection.

On the other hand, the amplification used in [2] can be described in the
language of Grover’s amplitude amplification as follows. First, apply $A$ to the
initial state $|\psi\rangle|0_x\rangle$, and apply the measurement $\Pi$. If the success event is
observed, the simulation will be successful, and this success event occurs with
the probability $\frac{1}{m}$. Otherwise, the state collapses to $|fail\rangle$, at which point the
reflection operator $AS_0^{-1}A^{-1}$ is applied. This changes the state to

$$\sqrt{\frac{2}{m}}|\text{succ}\rangle + \sqrt{1 - \frac{2}{m}}|\text{fail}\rangle,$$

and the measurement $\Pi$ is applied to this, producing $|\text{succ}\rangle$ with the probability
$\frac{2}{m}$. The process continues in this way, with each iteration yielding a successful
simulation with probability at least $\frac{1}{m}$.

References

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[2] J. Watrous, ”Zero-knowledge against quantum attacks”, quant-ph/0511020
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