Phase Space Non-commutativity and its Stability

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Abstract

We consider a generalised non-commutative space-time in which non-commutativity is extended to all phase space variables. If strong enough, non-commutativity can affect stability of the system. We perform stability analysis on a couple of simple examples and show that a system can be stabilised by introducing quartic interactions provided they satisfy phase-space copositivity. In order to conduct perturbative analysis of these systems one can use either canonical methods or phase-space path integral methods which we present in some detail.

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1. INTRODUCTION

Recently the so-called higher derivative theories have been carefully studied due to their possible role in the renormalizability of non local theories such as String Theory, Loop Quantum Gravity and non-commutative field theories.

As well known, higher derivative theories have serious constraints on their physical viability and the Ostrogradsky Instability theorem classifies all nondegenerate higher derivative theories as unstable. More precisely, the Ostrogradsky argument relies on having the highest momentum associated with the highest derivative in the theory and the energy spectrum results unbounded from below. The physical system is therefore unstable and, after quantization, negative norm states (ghosts) usually appear and the theory is not unitary. For an interesting approach see [12].

For that reason such theories have often been considered as effective field theories and, in general, one assumes that the parameters multiplying the higher derivative terms are small, justifying perturbative treatment of higher derivative operators.

Another way of tackling higher derivative theories is to introduce a Lorentz covariant cutoff operator. For certain class of such theories, the propagator poles in momentum space shift but the number of poles does not change. Barnaby and Karman claim to have proven a theorem that states that theories in which the number of poles is preserved do not contain Ostrogradsky’s instabilities. While this statement is intriguing, in the absence of complete analysis of the constraint structure of the theory à la Dirac, it is fair to say that a general proof is still lacking.

In this paper we consider a more traditional approach to non-commutativity, which introduces Lorentz breaking non-commutative parameters in phase-space. An important advantage of this class of models is the disappearance of the instabilities present in Lorentz covariant approaches. Namely, in the limit of strong non-commutativity, tachyonic modes appear, thus destabilising the theory. However, these kind of tachyonic instabilities are easy to deal with. To stabilize the theory it suffices to introduce interaction terms which have the property of copositivity.

Motivated by gravity coupled to (scalar) matter, which contains momenta in vertices, our interaction terms are bi-quadratic in both position and momentum variables. A natural framework to perform perturbative calculations in such theories is the phase space path-integral formalism (PSP), which we develop here for this
purpose. We apply PSP to define general per-
turbation expansion, and show how to apply it
in a simple one-loop case. We also check that
the PSP and canonical method give identical an-
swers.

The paper is organized as follows. In sec-
tion 2 the phase space path integral approach is
discussed. The non-commutative harmonic os-
cillator is recalled in section 3 and section 4 is de-
voted to a perturbative study of interactions in
the non-commutative harmonic oscillator. Sec-
tion 5 contains our final comments.

2. PSP TECHNIQUE

In this section we recall some preliminary de-
finitions, by the direct application to the simple
quantum harmonic oscillator with Hamiltonian

\[ H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \tag{1} \]

and then we discuss the phase space path inte-
gral formalism.

a. Preliminary definitions

The Feynman propagator defined as

\[ \langle T[x(t)x(t')] \rangle = \theta(t - t') \langle x(t)x(t') \rangle + \theta(t' - t) \langle x(t')x(t) \rangle, \tag{2} \]

obeys the simple differential equation

\[ (\frac{d^2}{dt^2} - \omega^2) \langle T[x(t)x(t')] \rangle = i\hbar \delta(t - t'), \tag{3} \]

whose solution depends on the choice of the ini-
tial state. The coefficients of the two \( \theta \) func-
tions in Eq. (2), called \textit{Wightman functions}, are com-
plex conjugate, i.e.

\[ i\Delta^+(t; t') = \langle x(t)x(t') \rangle, \]
\[ i\Delta^-(t; t') = \langle x(t')x(t) \rangle, \tag{4} \]

with

\[ i\Delta^+(t; t') = [i\Delta^-(t; t')]^*, \tag{5} \]

and satisfy the homogeneous equations

\[ (\frac{d^2}{dt^2} - \omega^2)i\Delta^+ (t; t') = 0. \tag{6} \]

If the initial state is the vacuum state one has

\[ i\Delta^+ (t; t') = \frac{\hbar}{2m\omega} e^{-i\omega(t-t')} \tag{7} \]

and

\[ i\Delta^- (t; t') = \frac{\hbar}{2m\omega} e^{i\omega(t-t')} \tag{8} \]

Let us study other T-ordered products that
we shall use later. In particular, \( \langle T[p(t)p(t')] \rangle \)
satisfies the differential equation (derived in ap-
pendix A)

\[ \left( \frac{d^2}{dt^2} - \omega^2 \right) \langle T[p(t)p(t')] \rangle = i\hbar m \omega^2 \delta(t - t'). \tag{9} \]

To evaluate the other T-ordered product

\[ \langle T[p(t)x(t')] \rangle, \tag{10} \]

one starts from the operator solution of the
equation of motion associated with the Hamil-
tonian \( \Pi \), i.e.

\[ \begin{cases} q(t) = q_0 \cos \omega t + \frac{p_0}{m\omega} \sin \omega t \\ p(t) = p_0 \cos \omega t - m\omega q_0 \sin \omega t \end{cases}, \tag{11} \]

and under the conditions

\[ \begin{cases} \langle q_0^2 \rangle = \langle \frac{\langle p_0^2 \rangle}{m^{\omega^2}} \rangle \\ \langle \{q_0, p_0\} \rangle = 0 \end{cases}, \tag{12} \]

one obtains

\[ \langle T[p(t)x(t')] \rangle = -\theta(t - t') \frac{i\hbar}{2} e^{-i\omega(t-t')} + \theta(t' - t) \frac{i\hbar}{2} e^{i\omega(t-t')} \tag{13} \]

These are the two point functions of the
ground state of the simple harmonic oscil-
lator \( \Pi \). The two point functions of more gen-
eral states such as general gaussian states are
discussed e.g. in Ref. \[18\].

b. Phase space path integral formalism

The formalism of phase space path integral
requires a detailed discussion of mixed represen-
tations. The partition function between an ini-
tial state \( |\psi_0\rangle \) at \( t_0 \) and a final state \( |\psi\rangle \) at \( t_\star \) is
defined \[^19\] as

\[
Z[J_i] = \langle \psi, t_s | \psi_0, t_0 \rangle = \left\langle \psi \right| T \left[ \exp \left( - \frac{i}{\hbar} \times \int_{t_0}^{t_s} dt \left[ H(t) - J_q q(t) - J_p p(t) \right] \right) \right| \psi_0 \rangle,
\]

(14)

where \( J_q \) and \( J_p \) are different sources, coupled with position and momentum respectively. By completeness of \( q \) and \( p \) representations, the second term can be written as

\[
\int dq_* dq_0 dp_* \langle q_* | q_0 \rangle \langle p_* | p_0 \rangle \times \]

\[
F_J(p_*, t_s | q_0, t_0) \langle q_0 | \psi_0 \rangle,
\]

(15)

where \( q_* \) and \( p_* \) are the position and momentum of the system at \( t = t_s, q_0 = q(t_0) \) and

\[
F_J(p_*, t_s | q_0, t_0) = \left\langle p_* \right| T \left[ \exp \left( - \frac{i}{\hbar} \times \int_{t_0}^{t_s} dt \left[ H(t) - J_q q(t) - J_p p(t) \right] \right) \right| q_0 \rangle.
\]

(16)

Let us now consider a generic Hamiltonian \( H \) of the form

\[
H = H_0 + H_{\text{int}},
\]

(17)

where the exact solutions for \( H_0 \) are known. By dividing the time interval \( t_s - t_0 \) into \( N \) small interval and evaluating the \( N \rightarrow \infty \) limit, it turns out (see the detail in appendix B) that

\[
Z[J_i] = \int dq_* dq_0 dp_* \times \psi^*(q_*) \psi_0(q_0) \frac{e^{i \frac{q_0 - q_*}{\sqrt{2 \pi \hbar}}}}{\sqrt{2 \pi \hbar}} F_J(p_*, t_s | q_0, t_0),
\]

(18)

which can be written as

\[
Z[J_p, J_q] = \exp \left( \frac{i}{\hbar} \int_{t_0}^{t_s} dt [h J_q - \hbar \delta J_q - \hbar \delta J_p p(t)] \right) \times \]

\[
Z_0[J_p, J_q],
\]

(19)

where

\[
Z_0 \propto \int dq_* dq_0 dp_* \times \int_{t_0}^{t_s} dt \left\{ \exp \left( \frac{i}{\hbar} \int_{t_0}^{t_s} dt [p \dot{q} - H_0(p, q) + J_q q(t) + J_p p(t)] \right) \right\},
\]

and the momenta are explicitly taken into account. In the previous expressions \( Dq \) and \( Dp \) indicate the usual functional integration measure:

\[
Dq Dp = \lim_{N \rightarrow \infty} \prod_{l,m=1}^{N-1} dq_l \frac{dp_m}{\sqrt{2 \pi \hbar}},
\]

(20)

where \( q_l = q(t_l) \), \( p_m = p(t_m) \) with \( t_l \) and \( t_m \) belonging to \( [t_0, t_s] \).

The presence of the source terms, \( J_p \) and \( J_q \), permits to set up a perturbative expansion for interaction terms that are not diagonal in position operators and contain momentum dependence or for Hamiltonians with non-canonical kinetic terms.

In the following we will use \( H_0 \) in Eq. (1), with the corresponding action:

\[
S_0 = \int_{t_0}^{t_s} dt \left\{ \frac{p^2}{2m} - \frac{m \omega^2}{2} q^2 \right\}.
\]

(21)

which can be written as

\[
\frac{1}{2} \int_{t_0}^{t_s} \int_{t_0}^{t_s} dt dt' \left[ Q_1(t) D_{0}^{ij}(t; t') Q_2(t') \right]
\]

(22)

where \( Q_1(t) = q(t) \), \( Q_2(t) = p(t) \) and

\[
D_{0}^{ij}(t; t') = \left( \delta(t - t') - \frac{m \omega^2}{2} \delta(t - t') + \partial_{t'} - \frac{1}{m} \right) \delta(t - t').
\]

(23)

The next step is to evaluate the inverse two point function after integrating out \( p_* \) in Eq. (21) with \( H = H_0 \). One gets

\[
Z_0[J_p, J_q] = \int dq_* \int dq_0 \rho(q^0, q^*) \times \int_{q(t_0)=q_0}^{q(t_s)=q_*} \int_{t_0}^{t_s} dt \left\{ \frac{i}{\hbar} \int_{t_0}^{t_s} dt [p \dot{q} + H_0(p, q) + J_q q(t) + J_p p(t)] \right\},
\]

(24)

with

\[
\langle q_0 | \psi_0 \rangle \langle \psi | q_* \rangle = \rho(q^0, q^*) = \sqrt{\frac{m \omega}{2 \hbar}} \exp \left\{ - \frac{m \omega}{2 \hbar} \left[ (q^0)^2 + (q^*)^2 \right] \right\}.
\]

(25)
The direct substitution of Eq. (25) in Eq. (24) gives (b.c. = boundary conditions)

\[ Z_0[J_p, J_q] = \int dq^0 \int dq \int Dq Dp \exp \left( \frac{i}{\hbar} S_0 + \int dt \left( \frac{i m \omega}{2} (\delta(t - t^0) + \delta(t - t^*)) q^2 + J_p p + J_q q \right) \right), \]

and therefore the inverse two point function becomes

\[ \tilde{D}^{ij}_0(t; t') = \delta(t - t') \times \left( \left\{ -m \omega^2 + i m \omega (\delta(t - t^0) + \delta(t - t^*)) \right\} \delta(t - t^*) - \delta(t - t^0) + \delta_t \right) \times \frac{i}{m}. \]  

(26)

The \( H_0 \) part of the generating function can be now written as

\[ Z_0[J_i] = \int DQ_i \exp \left( \frac{i}{\hbar} \int_{t_0}^{t_*} dt \int_{t_0}^{t_*} dt' \times \left[ \frac{1}{2} Q_i(t) \tilde{D}^{ij}_0(t; t') Q_j(t') + J_i(t) \delta(t - t') Q_j(t') \right] \right), \]

(27)

where \( J^1(t) = J_q \) and \( J^2(t) = J_p \). The initial conditions on the state are now self contained, i.e.

\[ \int dq^0 \int dq \int_{q(t^0) = q_0}^{q(t^*) = q_*} Dq \cdots = \int Dq \cdots \]

where \( t_0 < t_* \).

Shifting the \( Q_i \)'s via \( Q_i(t) \rightarrow \tilde{Q}_i(t) = Q_i(t) + \frac{i}{\hbar} \int d\bar{t} i \Delta_{ij}(t; \bar{t}) J^j(\bar{t}) \), where \( i \Delta_{ij}(t; t') \) satisfies the condition

\[ \int d\bar{t} \tilde{D}^{ij}_0(t; \bar{t}) \Delta_{jk}(\bar{t}; t') = i \hbar \delta_j^i \delta(t - t'), \]

(29)

one obtains (neglecting the \( \sim \) for the integration variables)

\[ Z_0[J_i] = \int DQ_i \times \exp \left( \frac{i}{\hbar} \int_{t_0}^{t_*} dt \int_{t_0}^{t_*} dt' \frac{1}{2} Q_i(t) \tilde{D}^{ij}_0(t; t') Q_j(t') \right) \times \exp \left( -\frac{i}{2 \hbar^2} \int_{t_0}^{t_*} dt \int_{t_0}^{t_*} dt' J^j(t) i \Delta_{ij} J^j(t') \right). \]

(30)

Integrating out the \( Q_i \)'s we finally get

\[ Z_0[J_i] = \tilde{Z}_0(t^0; t^*) \times \exp \left( -\frac{1}{2 \hbar^2} \int_{t_0}^{t_*} dt \int_{t_0}^{t_*} dt' J^j(t) i \Delta_{ij} J^j(t') \right), \]

(31)

where \( \tilde{Z}_0(t^0; t^*) \) is a \( J_i \)-independent constant.

The crucial point is to solve Eq. (29) for \( i \Delta_{ij}(t; t') \), once \( \tilde{D}^{ij}_0 \) is known, in order to obtain the generating functional of the \( H_0 \) part, \( Z_0[J_i] \). From now on we shall refer to the \( H_0 \) two point function \( \tilde{D}^{ij}_0 \) as \( \tilde{D}^{ij} \) and we focus on the problem of the inversion of Eq. (29). The solution (see appendix C) is

\[ i \Delta_{ij}(t; t') = \langle T[Q_i(t) Q_j(t')] \rangle, \]

(32)

where all the phase space two-point functions are considered. Although we worked out the computation for a specific system, this method should hold for generic Hamiltonians. Indeed, the phase space path-integral approach keeps track of the degrees of freedom of the full symplectic manifold where the evolution takes place and it is a useful starting point for perturbative calculations, in particular for systems with derivative interactions.

3. NON-COMMUTATIVE OSCILLATOR IN TWO DIMENSIONS

In this section we introduce the noncommutative oscillator in two spatial dimensions (2D). The total Hamiltonian can be conveniently split into a free part \( H_0 \) and an interacting part \( H_{\text{int}} \).

The free part of the Hamiltonian is given by,

\[ H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m \omega^2 (x'^2 + y'^2), \]

(33)

and the interaction Hamiltonian is,

\[ H_{\text{int}} = \frac{\lambda_x}{4} x^4 + \frac{\lambda_y}{4} y^4 + \frac{\lambda_{yx}}{4} x^2 y^2 + \frac{\lambda_{px}}{4} p_x^4 + \frac{\lambda_{py}}{4} p_y^4 + \frac{\lambda_{p_{yx}}}{4} p_x^2 p_y^2 \]

\[ + \frac{\lambda_{yx}}{4} x'^2 y'^2 + \frac{\lambda_{py}}{4} y'^2 p_y^2 + \frac{\lambda_{p_{py}}}{4} p_x^2 p_y^2, \]

(34)
The Hamiltonian describes a stable theory if it is copositive with respect to all directions in phase space, \((x^2, y^2, p_x^2, p_y^2)\) for large values of the coordinates \([20, 21]\), and we assume that to be the case in this work.

Canonical quantization posits that conjugate variables do not commute,

\[
[x', p'_x] = i\hbar, \quad [y', p'_y] = i\hbar, \tag{35}
\]

and all other quantities commute. A generalisation to non-commutative geometries has been studied a lot, as it is well motivated by the physics at large energies. Arguably the simplest model is the one in which spatial coordinates do not commute, \([x', y'] = i\theta xy \equiv i\theta\). Here we consider a phase space generalization of this, in which all coordinates on phase space \(q'_i\) do not commute, including the momenta,

\[
[q'_i, q'_j] = i\theta_{ij}, \tag{36}
\]

where \(q'_i \equiv \{x', y', p'_x, p'_y\}\) and \(\theta_{ij}\) is an anti-symmetric hermitean matrix, but otherwise not specified. Notable special cases are canonical quantisation, in which \(\theta_{xp} = \theta_{yp} = \hbar\) and all other elements are zero, and non-commutativity in the \(xy\) plane in which, in addition, \(\theta_{xy} \equiv \theta \neq 0\). In the general case there are 3 more non-vanishing elements of \(\theta_{ij}\): \(\theta_{xp}, \theta_{yp},\) and \(\theta_{p_x p_y}\).

3.a Non-commutativity in space

Let us now consider a simple model in which the coordinates satisfy the commutation relations

\[
[x', y'] = i\theta_{12}, \tag{37}
\]

with

\[
\theta_{12} = -\theta_{21} = \theta, \tag{38}
\]

and all the other commutators are the usual commutative ones (the indices 1 and 2 correspond to \(x\) and \(y\) directions). In terms of the usual commuting set of phase space variables, \(\{x, y, p_x, p_y\}\), the non-commuting coordinates are given by (see Ref. [22])

\[
\begin{align*}
x' &= x - \frac{\theta}{\hbar} p_y \\
p'_x &= p_x \\
y' &= y + \frac{\theta}{\hbar} p_x \\
p'_y &= p_y,
\end{align*}
\tag{39}
\]

and \(H_0\) in terms of commutative variables turns out to be

\[
H_0 = \left(\frac{1}{2m} + \frac{m\omega^2\theta^2}{8\hbar^2}\right) (p_x^2 + p_y^2) + \frac{1}{2} m\omega^2 (x^2 + y^2) - \frac{m\omega^2\theta}{2\hbar} (xp_y - yp_x), \tag{40}
\]

Following Ref. [13] we define \(M\) and \(\Omega\) as

\[
\frac{1}{2M} = \frac{1}{2m} + \frac{m\omega^2\theta^2}{8\hbar^2}, \quad M\Omega^2 = m\omega^2, \tag{41}
\]

so that \(H_0\) takes the more familiar form:

\[
H_0 = \frac{1}{2M} (p_x^2 + p_y^2) + \frac{1}{2} M\Omega^2 (x^2 + y^2) - \frac{M\Omega^2\theta}{2\hbar} (xp_y - yp_x). \tag{42}
\]

In order to gain understanding on the question of stability of this Hamiltonian, it is useful to rewrite (42) as:

\[
H_0 = \frac{1}{2M} \left( p_x + \frac{M\Omega^2\theta}{2\hbar} y \right)^2 + \frac{1}{2M} \left( p_y - \frac{M\Omega^2\theta}{2\hbar} x \right)^2 + \frac{1}{2} M\Omega^2 \left( 1 - \frac{M\Omega^2\theta^2}{4\hbar^2} \right) (x^2 + y^2), \tag{43}
\]

from where we see that the Hamiltonian is stable (it is a sum of non-negative terms) if

\[
\theta \leq \theta_c = \frac{2\hbar}{M\Omega}. \tag{44}
\]

If \(\theta = \theta_c\) it is marginally stable and if \(\theta > \theta_c\) both \(x\) and \(y\) directions in \(H_0\) are unstable (tachyonic) in the sense that the oscillator frequencies squared are negative. This instability can be cured by adding to the free Hamiltonian quartic
interactions as in Eq. (34) with $\lambda_x, \lambda_y > 0$ and with other couplings in (33) zero or positive. \(^1\)

The classical ground state of the sub-critical system with $\theta < \theta_c$ is then simply, $p_x = p_y = x = y = 0$, such that the ground state energy vanishes, $E_{0}^{\text{cl}} = 0$. In the critical case when $\theta = \theta_c$ the ground state of $H_0$ is given by, $p_x = -M\Omega(\theta/\theta_c)y_0$, $p_y = M\Omega(\theta/\theta_c)x_0$, with $x_0$ and $y_0$ arbitrary (in this case both $x$ and $y$ are flat directions such that it costs no energy to shift along $x$ and $y$ directions). Also in this case the ground state energy is zero in classical theory, $E_{0}^{\text{cl}} = 0$. Finally, when $\theta > \theta_c$, both $x$ and $p$ condense via spontaneous symmetry breaking and the ground state is given by,

$$x_0^2 = \frac{M\Omega^2}{\lambda_x} \left( \frac{\theta^2}{\theta_c^2} - 1 \right), \quad y_0^2 = \frac{M\Omega^2}{\lambda_y} \left( \frac{\theta^2}{\theta_c^2} - 1 \right),$$

$$p_{x,0} = -M\frac{\theta}{\theta_c}y_0, \quad p_{y,0} = M\frac{\theta}{\theta_c}x_0 \quad (45)$$

and the corresponding ground state energy is negative,

$$E_{0}^{\text{cl}} = -\frac{1}{4}M^2\Omega^4 \left( \frac{1}{\lambda_x} + \frac{1}{\lambda_y} \right). \quad (46)$$

A similar result was obtained in the context of a 3D supersymmetric harmonic oscillator with spin non-commutativity, see Ref. [24, 25]. We now pause to summarize what we have found so far. If non-commutativity in space is strong enough, it can induce condensation of coordinates that strongly resembles spontaneous symmetry breaking, which is one of the principal results of this work. Consequently, the ground state energy is negative and and the particle is moving in circles with a constant velocity $||\vec{v}||$, which is in its ground state given by

$$v_{x,0} = \frac{p_{x,0}}{M} = -\Omega^2 \frac{\sqrt{M/\lambda_x}}{\sqrt{1 - \theta_c^2/\theta^2}},$$

$$v_{y,0} = \frac{p_{y,0}}{M} = \Omega^2 \frac{\sqrt{M/\lambda_y}}{\sqrt{1 - \theta_c^2/\theta^2}}. \quad (47)$$

i.e. as a consequence of the space non-commutativity, we will perceive the particle moving without any ‘apparent’ reason. Furthermore, recalling the structure of the Hamiltonian in an external Abelian gauge field $A_{\mu} \equiv (A_0, A_x, A_y, A_z)$, whose kinetic part reads,

$$H_{0,\text{gauge}} = \frac{\vec{p} - (e/c)\vec{A}}{2M}^2 \quad (48)$$

we see that, as already shown in [20] and many other works, the noncommutativity in space can be modeled by a fictitious gauge field,

$$A_x = -\frac{c M^2 \Omega^2 \theta}{e} y, \quad A_y = \frac{c M^2 \Omega^2 \theta}{e} x, \quad (49)$$

which yields constant magnetic field in the $z$ direction,

$$B_z = -\frac{c M^2 \Omega^2 \theta}{e} \frac{1}{\hbar}. \quad (50)$$

The corresponding equivalent Lorentz force $F_L = (e/c)\vec{v} \times \vec{B}$ is then,

$$F_x = -\frac{p_y}{M} \frac{M^2 \Omega^2 \theta}{\hbar} = -2\Omega \frac{\theta}{\theta_c} p_y$$

$$F_y = \frac{p_x}{M} \frac{M^2 \Omega^2 \theta}{\hbar} = 2\Omega \frac{\theta}{\theta_c} p_x, \quad (51)$$

which will cause particles to rotate in the $xy$ plane with a cyclotron angular frequency, $\omega_c = (eB)/(Mc)$, given by,

$$\omega_c = 2\Omega \frac{\theta}{\theta_c}. \quad (52)$$

In the quantum mechanical case when $\theta = \theta_c$ and interactions are switched off, the problem can be reduced to the two oscillator problem in a constant magnetic field pointing orthogonally to the $xy$ plane. The energy of the ground and excited states are then famously given by the Landau levels, whose energy is quantized as,

$$E_n = \hbar \omega_c \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots \quad (53)$$

When $\theta < \theta_c$ the oscillators will in their ground state rest on average at zero, and one can show that the ground state energy is equal to, $E_0 = \hbar \omega_c/2$. However, in their excited state they can exhibit cyclotron rotation in the $xy$ plane. For example, if both $x$ and $y$ oscillators are in a coherent state, then the center of a Gaussian wave function will exhibit cyclotron oscillations in the $xy$ plane.

\(^1\) As already mentioned above, a weaker criterion of positiveness suffices to make the theory stable.
3.b Non-commutativity in phase space

We can extend the non-commutative geometry to the phase space via the relations

\[ [p'_x, p'_y] = i\theta_{p_x p_y}, \]  

\[ \theta_{p_x p_y} = -\theta_{p_y p_x} = \eta, \]  

analogous as in eq. (30).

All the other commutation relations are the same as in the case discussed previously. Here we are introducing phase space non-commutativity in a simple but unnatural way, note that the same result can be obtained via purely physical arguments [23].

The homeomorphism between regular and non-commutative geometry can be expressed via the coordinate change

\[
\begin{align*}
    x' &= C \left( x - \frac{\eta}{2\hbar} p_y \right), \\
    p'_x &= C \left( p_x + \frac{\eta}{2\hbar} y \right), \\
    y' &= C \left( y + \frac{\eta}{2\hbar} p_x \right), \\
    p'_y &= C \left( p_y - \frac{\eta}{2\hbar} x \right),
\end{align*}
\]

with

\[
C = \frac{1}{\sqrt{1 + \frac{\eta}{4\hbar^2}}}, \quad \tilde{\theta} = \frac{\theta}{1 + \frac{\eta}{4\hbar^2}}, \quad \tilde{\eta} = 1 + \frac{\eta}{2\hbar^2}.
\]

If one substitutes eq. (56) in eq. (33) the free part of the Hamiltonian can be recast in the same form as in eq. (32) but with redefined parameters and variables, namely

\[ H_0 = \frac{1}{2M} \left( p_x^2 + p_y^2 \right) + \frac{1}{2} \tilde{M} \tilde{\Omega}^2 (x^2 + y^2) + \frac{\tilde{M} \tilde{\Omega}^2 \tilde{\theta}}{2\hbar} (xp_y - yp_x), \]

where

\[
\tilde{M} = m \left( 1 + \frac{\eta}{2\hbar^2} \right), \quad \tilde{M} \tilde{\Omega}^2 = \frac{m \omega^2}{1 + \frac{\eta}{4\hbar^2}} \left( 1 + \frac{1}{(m\omega)^2} \frac{\tilde{\eta}^2}{4\hbar^2} \right), \quad \tilde{\theta} = \frac{\tilde{\eta}^2}{1 + \frac{1}{(m\omega)^2} \tilde{\eta}^2}.
\]

The limit \( \eta \to 0 \) is well behaved and produces the results (31)–(32) obtained in the case of space non-commutativity.

Indeed, studying the stability of the system, one obtains eq. (43) in terms of these redefined parameters, and the critical value is now realized as

\[ \tilde{\theta} = \tilde{\theta}_c = \frac{2\hbar}{M \tilde{\Omega}}, \]

granting exactly the same considerations as before, but with the suitably rescaled non-commutative parameter \( \theta \to \tilde{\theta} \).

3.c PSP Feynman rules for non-commutative oscillator

In what follows we construct two-point functions for a simple Gaussian initial state in the presence of the free Hamiltonian (33). These correlators will be used for studying the interactions (34) in perturbation theory. Since our interactions are higher order in both position and momentum variables, the phase space formalism developed in section 2 is particularly suitable. For simplicity we consider here only the sub-critical case (\( \theta < \theta_c \)) and leave the interesting super-critical case for future work.

The Hamilton operator equations can be conveniently written in a matrix form as,

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} 0 & \frac{\Omega \Omega}{\hbar} & 0 & 1 \\ -\frac{\Omega \Omega}{\hbar} & 0 & 0 & 1 \\ 0 & -M \Omega^2 & 0 & 0 \\ 0 & 0 & -\frac{\Omega \Omega}{\hbar} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}.
\]

Making an Ansatz, \( q_i \propto \exp(\lambda t) \), one obtains four eigenvalues for \( \lambda \),

\[
\lambda^2_{\pm} = -\Omega^2 \left( 1 \pm \frac{\theta^2}{\theta_c^2} \right).
\]

This means that – when \( \theta < \theta_c \) – the four fundamental solutions are oscillatory harmonic func-
tions, \(^2\)

\[
\cos(\Omega_+ t), \cos(\Omega_- t), \sin(\Omega_+ t), \sin(\Omega_- t),
\]

where

\[
\Omega_\pm = \Omega \sqrt{1 \pm \frac{\theta^2}{\theta_c^2}}.
\]

The general solution for \(x(t)\) is

\[
x(t) = \frac{x_0}{2} (\cos \Omega_+ t + \cos \Omega_- t) + \frac{y_0}{2} (\sin \Omega_+ t - \sin \Omega_- t) + \frac{p_{x0}}{2M\Omega} (\sin \Omega_+ t + \sin \Omega_- t) + \frac{p_{y0}}{2M\Omega} (\cos \Omega_- t - \cos \Omega_+ t)
\]

where \(x_0 = x(0), y_0 = y(0), p_{x0} = p_x(0),\) and \(p_{y0} = p_y(0)\). The solutions for the others dynamical variables can be found in Appendix D.

The Wightman functions evaluate to,

\[
\left\{\begin{array}{l}
i\Delta_{ij}^+(t; t') = \langle Q_i(t)Q_j(t') \rangle \\
i\Delta_{ij}^-(t; t') = \langle Q_j(t')Q_i(t) \rangle
\end{array}\right.
\]

(66)

indeed, the diagonalization of \(H_0\) \(^3\) allows to calculate the expectation values we need to determine \(i\Delta_{ij}(t; t')\). The only non-zero expectation values turn out to be

\[
\langle x^2 \rangle = \langle y^2 \rangle = \frac{\hbar}{2M\Omega}, \\
\langle p_{x0}^2 \rangle = \langle p_{y0}^2 \rangle = \frac{\hbar M\Omega}{2}, \\
\langle x_0 p_{x0} \rangle = \langle y_0 p_{y0} \rangle = \frac{i\hbar}{2}.
\]

(67)

Some comments on the two point functions are now in order. In the limit \(\theta \to 0\), the two propagators

\[
i\Delta_{xx}^+(t; t') = \frac{\hbar}{4M\Omega} \left( e^{-i\Omega_+ t} + e^{-i\Omega_- t} \right),
\]

\[
i\Delta_{pp_{xx}}^+(t; t') = \frac{\hbar M\Omega}{4} \left( e^{-i\Omega_+ t} + e^{-i\Omega_- t} \right).
\]

(68)

\(^2\) When \(\theta > \theta_c\), the fundamental solutions either exponentially grow or decay in time with the rate given by \(\kappa_\pm = \sqrt{-\lambda_\pm} = \Omega \sqrt{(\theta/\theta_c)^2 + 1}\). The exponentially growing behavior comes from the tachyonic mode mentioned above.

The solutions for the others dyagonal interaction potentials and the PSP are now in order. In the limit \(\theta \to 0\), the fundamental solutions either exponentially grow or decay in time with the rate given by \(\kappa_\pm = \sqrt{-\lambda_\pm} = \Omega \sqrt{(\theta/\theta_c)^2 + 1}\). The exponentially growing behavior comes from the tachyonic mode mentioned above.

reproduce the well known Wightman functions of the commutative case. Moreover the non-commutative parameter generates two distinct frequencies \(\Omega_\pm\) and unusual correlations, as for example

\[
\langle p_x(t) y(t') \rangle = \frac{\hbar}{4} \left( e^{-i\Omega_+ (t-t')} - e^{-i\Omega_- (t-t')} \right),
\]

(69)

that disappear in the continous limit \(\theta \to 0\).

4. PERTURBATIVE EXPANSION

The usual perturbative methods to study the typical anharmonic correction in commutative configuration space, i.e. \(\lambda x^4\), to order \(O(\lambda)\) is based on the evaluation of (see Eq. (19))

\[
Z[J_x] = \tilde{Z}_0(t_0, t_*), \frac{1}{e^{\frac{\hbar}{2\Delta} J_x \Delta J_x}}
\]

\[
\simeq \tilde{Z}_0(t_0, t_*) \left[ 1 - \frac{i\lambda}{4} \int_{t_0}^{t_*} dt \left( \frac{\hbar \delta}{i\delta J_x(t)} \right)^4 \right]
\]

\[
+ O(\lambda^2) e^{-\frac{i}{2\Delta} J_x \Delta J_x} \bigg|_{J_x = 0}.
\]

(70)

In particular the \(O(\lambda)\) term turns out to be

\[
-3i\frac{\lambda}{4} \int_{t_0}^{t_*} \langle i\Delta(\tau; \tau) \rangle^2 d\tau.
\]

(71)

On the other hand, one could be interested in non-diagonal interaction potentials and the PSP is precisely a method to set up a perturbative expansion for such potentials by introducing the sources \(J_i(t)\). In these cases the general fourth order functional derivative of the partition function associated with \(H_0\) is given by

\[
\frac{\delta^4 Z_0[J_i]}{\delta J_a(t) \delta J_b(t') \delta J_c(t'') \delta J_d(t'''')} \bigg|_{J_i = 0} = \frac{\hbar^4 \delta^4 Z_0[J_i]}{\delta J_a(t) \delta J_b(t') \delta J_c(t'') \delta J_d(t''')} + i\Delta_{ab}(t; t'') i\Delta_{ac}(t; t'') + i\Delta_{bd}(t'; t'') i\Delta_{bc}(t'; t'')
\]

(72)

As a simple example to outline the perturbative approach in phase space, let us consider the
interaction term for the non-commutative anharmonic oscillator in 2D
\[ H_{\text{int}} = \frac{\lambda}{4} (x'^2 + y'^2)^2, \quad (73) \]
where for simplicity we suppose
\[ \lambda_x = \lambda_y = \frac{\lambda_{xy}}{2} \equiv \lambda \quad (74) \]
in eq. (33). By the substitutions in Eq. (39), the interaction part \( H_{\text{int}} \) becomes a 4th-order polynomial function in \( x, y, p_x, \) and \( p_y \). Following the same procedure to get Eq. (70), the \( \mathcal{O}(\lambda) \) correction can be written as
\[ - i \lambda \sum_{n=0}^{4} \int_{t_0}^{t_*} d\tau \left( \frac{\theta}{2\hbar} \right)^n \times \]
\[ f_n \left( \frac{\delta}{\delta J_1(\tau)}, \frac{\delta}{\delta J_2(\tau)} \right), \quad (75) \]
where \( f_n \) are 4th-order homogeneous polynomials in the functional derivatives and all odd \( n \) terms give no contribution due to the odd powers of the variables in the functional integral. Substituting the explicit expressions for \( f_n \) in (75), the \( \mathcal{O}(\lambda) \) correction is given by
\[ Z[0] = Z[t_0, t_*] \left[ 1 - i \lambda \left( \frac{\hbar^2}{4(M\Omega)^2} + \theta^2 \right) \]
\[ + (M\Omega)^2 \frac{\theta^4}{48\hbar^2} (t_* - t_0) + \mathcal{O}(\lambda^2) \right], \quad (76) \]
consistent with the result obtained by standard perturbative techniques \( [13] \).

Another interesting case is that of a potential dependent on \( p'_x \) and \( p'_y \). If we suppose \( H_{\text{int}} \) is non-commutative. In the measure of the functional integral \( [28] \), whose determinant is,
\[ \left( \frac{1 - \frac{\delta^2}{4\hbar^2}}{1 + \frac{\delta^2}{4\hbar^2}} \right)^2, \quad (78) \]
whose inverse square root contributes to the partition function \( Z \).

In this case the \( \mathcal{O}(\lambda) \) contribution is
\[ \frac{Z[0]}{Z[t_0, t_*]} = 1 - \frac{3i}{4} \frac{(t_* - t_0)}{1 - \left( \frac{\delta^2}{4\hbar^2} \right)^2} \]
\[ \left\{ \frac{3}{4} \left( \frac{\hbar}{M\Omega} \right)^2 \left[ 3 \left( \frac{h}{2\hbar} \right)^4 \left( \lambda_x + \lambda_y + (\lambda_{px} + \lambda_{py}) \right) \right] \]
\[ + \frac{3}{8} \delta^2 (\lambda_x + \lambda_y) + \frac{3}{8} \delta^2 (\lambda_{px} + \lambda_{py}) \]
\[ + \frac{3}{4} \left( \frac{\hbar}{2\hbar} \right)^4 \left( \lambda_{px} + \lambda_{py} + (\lambda_x + \lambda_y) \right) \left( \frac{\theta}{2\hbar} \right)^4 \right\} + \mathcal{O}(\lambda^2). \quad (79) \]

5. CONCLUSIONS AND OUTLOOK

One of the central results of this work has to be the realisation that strong enough non-commutative parameters can induce condensation of coordinates that resembles spontaneous symmetry breaking, leading to a constant fictitious magnetic field in the \( z \)-direction
\[ B_z = -\frac{c}{e} \frac{M^2\Omega^2\theta}{2\hbar}. \quad (80) \]

Moreover, in this paper a new phase space path integral method has been formulated and investigated for interacting theories. The proposed perturbative expansion has the advantage of treating a very broad class of interacting Hamiltonians by a straightforward generalization of the usual techniques in configuration space.

The PSP approach has been discussed for an exactly solvable system and the perturbative expansion has been explicitly applied to the non-commutative anharmonic oscillator which, in terms of commutative dynamical variables,
produces an interaction potential which is a 4-th order polynomial in \( x, y, p_x \) and \( p_y \).

Of course, the perturbative approach in phase space can be applied to physical systems with general momentum and position dependent interactions.

Moreover, the technique studied in this paper is the starting point for developing similar perturbative expansions in quantum field theories. The generalization requires subtle modifications but, in principle could be a powerful tool to study perturbative expansion on non-uniform backgrounds, typical of higher order derivative theories \[27,28\].

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**APPENDIX A: NON STANDARD TIME ORDERED OPERATORS**

We are interested in determining the equation for the time evolution of \( \langle T[p(t)p(t')] \rangle \). Let us define the operator

\[
O_F(t) = T[p(t)p(t')]=\theta(t-t')p(t)p(t')+\theta(t'-t)p(t')p(t),
\]

and take the derivative w.r.t. \( t \) to obtain

\[
\frac{d}{dt}O_F(t) = \delta(t-t')\left[ \frac{dp(t)}{dt},p(t') \right]_{t=t'} + \theta(t-t')\frac{dp(t)}{dt}p(t') + \theta(t'-t)p(t')\frac{dp(t)}{dt}.
\]

The commutator of course vanishes and the second derivative gives

\[
\frac{d^2}{dt^2}O_F(t) = \delta(t-t')\left[ \frac{dp(t)}{dt},p(t') \right]_{t=t'} + \theta(t-t')\frac{d^2p(t)}{dt^2}p(t') + \theta(t'-t)p(t')\frac{d^2p(t)}{dt^2}.
\]

By using Hamilton’s equations we finally have

\[
\left( -\frac{d^2}{dt^2} - \omega^2 \right) O_F(t) = \hbar m \omega^2 \delta(t-t').
\]

**APPENDIX B: EVALUATION OF THE TRANSITION AMPLITUDE**

In this appendix we find a suitable expression for Eq. \((81)\) by a discretization procedure. For the sake of simplicity we shall suppress the source terms that will be rewritten at the end of the appendix. Starting with 2 sub-intervals in \([t_0,t_s]\),

\[
\langle p_s, t_s | q_0, t_0 \rangle = \int dq_1 dq_2 \langle p_s | q_2 \rangle \times \langle q_2 | U(t_s,t_1) | q_1 \rangle \langle q_1 | U(t_1,t_0) | q_0 \rangle,
\]

where

\[
U(t,t') = T \left[ \exp \left( -\frac{i}{\hbar} \int_t^{t'} H(\tau)d\tau \right) \right].
\]

In Eq. \((85)\) we have used two times the identity

\[
1 = \int dq|q\rangle \langle q|
\]

and \( t_1 \) is the central point of the time interval.

Recalling that

\[
\langle q | p \rangle = \frac{e^{ipq/\hbar}}{\sqrt{2\pi\hbar}}
\]

we get

\[
\langle p_s, t_s | q_0, t_0 \rangle = \int dq_1 dq_2 \frac{e^{-ip_s q_2/\hbar}}{\sqrt{2\pi\hbar}} \times \langle q_2 | U(t_s,t_1) | q_1 \rangle \langle q_1 | U(t_1,t_0) | q_0 \rangle.
\]
Let’s now consider the matrix element
\[
\langle q_2 | U(t_2, t_1) | q_1 \rangle = \langle q_2 | e^{-\frac{i}{\hbar} \delta t \frac{p^2}{2m}} e^{-\frac{i}{\hbar} \delta t V(q)} | q_1 \rangle
\]
\[
= \int dp_1 \langle q_2 | e^{-\frac{i}{\hbar} \delta t \frac{p^2}{2m}} | p_1 \rangle \langle p_1 | e^{-\frac{i}{\hbar} \delta t V(q)} | q_1 \rangle
\]
\[
= \frac{i}{\hbar} \delta t \exp \left( -\frac{i}{\hbar} \delta t H(p_1, q_1) + \frac{i}{\hbar} p_1 (q_2 - q_1) \right),
\]
where \( \delta t = (t_2 - t_1) \), or, in general \((t_i - t_{i-1})\). Therefore for just two intervals, we have
\[
\langle p_*, t_* | q_0, t_0 \rangle = \int dq_1 dq_2 \frac{e^{-\frac{i}{\hbar} \delta q_1 q_2}}{\sqrt{2\pi \hbar}} \int dp_0 dp_1 \frac{e^{-\frac{i}{\hbar} \delta p_1}}{(2\pi \hbar)^2} \times \exp \left( \frac{i}{\hbar} \delta t \left( p_1 (q_2 - q_1) - H(p_1, q_1) + p_0 (q_1 - q_0) - H(p_0, q_0) \right) \right),
\]
and if we consider \( N \) intervals we get
\[
\langle p_*, t_* | q_0, t_0 \rangle = \prod_{l=1}^{N} \int dq_l \frac{e^{-\frac{i}{\hbar} \delta q_l}}{\sqrt{2\pi \hbar}} \times \exp \left( \frac{i}{\hbar} \delta t \left( p_l \frac{q_l - q_{l-1}}{\delta t} - H(q_{l-1}, q_l) \right) \right)
\]
After substituting in Eq. (85) one has
\[
\langle p_*, t_* | q_0, t_0 \rangle = \prod_{l=1}^{N} \int dq_l \frac{e^{-\frac{i}{\hbar} \delta q_l}}{\sqrt{2\pi \hbar}} \prod_{l=0}^{N-1} \int dp_l \frac{e^{-\frac{i}{\hbar} \delta p_l}}{\sqrt{2\pi \hbar}} \times \exp \left( \frac{i}{\hbar} \delta t \sum_{l=0}^{N-1} \left( p_l \frac{q_{l+1} - q_l}{\delta t} - H(q_l, p_l) \right) \right)
\]
\[
= \int d\bar{q} N \frac{e^{-\frac{i}{\hbar} \delta \bar{q} N}}{\sqrt{2\pi \hbar}} \prod_{l=1}^{N} \int dq_l \prod_{l=0}^{N-1} \int dp_l \frac{e^{-\frac{i}{\hbar} \delta p_l}}{\sqrt{2\pi \hbar}} \times \exp \left( \frac{i}{\hbar} \delta t \sum_{l=0}^{N-1} \left( p_l \frac{q_{l+1} - q_l}{\delta t} - H(q_l, p_l) \right) \right)
\]
In the limit \( N \to \infty \) one obtains
\[
F_J(p_*, t_* | q_0, t_0) = \int d\bar{q} N \frac{e^{-\frac{i}{\hbar} \delta \bar{q} N}}{\sqrt{2\pi \hbar}} \int q(t_*) = q_N \int \bar{q}(t_0) = q_0 \int d\bar{q} Dp \exp \left( \frac{i}{\hbar} \int_{t_0}^{t_*} dt \left( \bar{p} \dot{q} - H(q, \bar{p}) + J_q \dot{q}(t) + J_p p(t) \right) \right)
\]
where the source terms have been again included.

Appendix C: Solution of the inversion problem

In this appendix we address the solution of Eq. (29) for \( i\Delta_{ij} \) assuming \( D^{ij} \) from Eq. (27). For the sake of simplicity we usually omit the time dependence on the functions. The system defined by the matrix (29), which we can write as
\[
\Delta_{ij} = \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & \Delta_4 \end{pmatrix}
\]
has two types of equations with homogeneous and Dirac-\( \delta \) sources. The equations involve either indices 1 and 3 or 2 and 4. Let’s write the equations for the case(1,3):
\[
\begin{aligned}
-\omega^2 \Delta_1(t; t') - \partial_t \Delta_3(t; t') &= \hbar \delta(t - t') \\
\partial_t \Delta_1(t; t') - \frac{1}{m} \Delta_3(t; t') &= 0 \\
\text{b.c. on } \Delta_1(t; t').
\end{aligned}
\]
From the second equation one has
\[
\partial_t \Delta_1(t; t') = \frac{1}{m} \Delta_3(t; t'),
\]
and by taking the derivative w.r.t. \( t \) and substituting \( \partial_t \Delta_3(t; t') \) into the system in Eq. (95) we get
\[
\begin{aligned}
-\omega^2 \Delta_1(t; t') - \partial_t \Delta_3(t; t') &= \hbar \delta(t - t') \\
\Delta_3(t; t') &= m \partial_t \Delta_1(t; t') \\
\text{b.c. on } \Delta_1(t; t').
\end{aligned}
\]
From Eq. (3) and the previous expression we
know that \( \Delta_1 \) is a Feynman type propagator:
\[
i\Delta_1(t; t') = \theta(t - t') i\Delta_1^+(t; t') + \theta(t' - t) i\Delta_1^-(t; t')
\] (98)
where \( \Delta_1^\pm \) have to satisfy the homogenous equation
\[
m(-\partial_t^2 - \omega^2) i\Delta_1^\pm(t; t') = 0,
\] (99)
i.e.
\[
i\Delta_1^+(t; t') = A_1^+(t') e^{-i\omega t} + B_1^+(t') e^{i\omega t}. \]
(100)
One gets analogous results for the system (2,4),
which can be reduced to the system of eqs.
\[
\begin{cases}
(-\partial_t^2 - \omega^2) \Delta_4 = m\omega^2 \hbar \delta(t - t') \\
\partial_t \Delta_4(t; t') = -\frac{m\omega}{\hbar} \Delta_4(t; t') \\
b.c. on \ \Delta_2(t; t').
\end{cases}
\] (101)
Again \( \Delta_4 \) is a Feynman type propagator.

We have now to impose the boundary conditions. For \( \Delta_1(t; t') \) the boundary conditions are
\[
\begin{cases}
m\delta(t - t_0)(i\omega - 1) \Delta_1(t; t') = 0 \\
m\delta(t - t^*) (i\omega + 1) \Delta_1(t; t') = 0.
\end{cases}
\] (102)
By calling \( \Delta t = t - t' \), for \( t = t_0 \) we have for the step function \( \theta(\Delta t) = 0 \) since \( t' > t_0 \) while in the case \( t = t^* \), \( \theta(-\Delta t) = 0 \) and we get the conditions for \( \Delta_1^+ \) and \( \Delta_1^- \), namely
\[
\delta(t - t_0)(i\omega - 1) \Delta_1^-(t; t') = 0,
\delta(t - t^*)(i\omega + 1) \Delta_1^+(t; t') = 0.
\] (103)
Following the previous analysis in the configuration space in Ref. [18], we interpret \( i\omega \) as a \( \partial_t \) applied to \( \Delta_1^- \) and as \( -\partial_t \) if applied to \( \Delta_1^+ \) and we obtain
\[
i\Delta_1^+(t; t') = A_1^+(t') e^{-i\omega t},
i\Delta_1^-(t; t') = B_1^-(t') e^{i\omega t}. \] (104)
Finally by deriving Eq. (98) w.r.t. \( t \) one gets
\[
i\partial_t \Delta_1(t; t') = \delta(\Delta t)[i\Delta_1^+ - i\Delta_1^-] + 
\theta(\Delta t) i\partial_t \Delta_1^+ + \theta(-\Delta t) i\partial_t \Delta_1^-.
\] (105)
Eq. (3) and Eq. (105) implies
\[
\delta(t - t') [i\Delta_1^+ - i\Delta_1^-] = 0,
\] (106)
and a second derivative gives
\[
\partial_t^2 i\Delta_1(t; t') = 
\delta(t - t') [-i\partial_t \Delta_1^+ (t; t') - i\partial_t \Delta_1^- (t; t')] + 
\theta(t - t') i\partial_t^2 \Delta_1^+ (t; t') + \theta(t' - t) i\partial_t^2 \Delta_1^- (t; t').
\] (107)
By adding and subtracting in the RHS of Eq. (107) the term \( \omega^2 i\Delta_1 \), and using Eq. (99) one can handle the terms not proportional to the \( \delta(t - t') \). After substituting in Eq. (104) a direct comparison with Eq. (97) gives
\[
\begin{cases}
i\Delta_1^+(t; t') = \frac{\hbar}{2m\omega} e^{-i\omega(t-t')} \\
i\Delta_1^-(t; t') = \frac{\hbar}{2m\omega} e^{i\omega(t-t')}.
\end{cases}
\] (108)
Finally it is possible to obtain \( \Delta_3 \) by using Eq. (96):
\[
i\Delta_3(t; t') = 
- \theta(-\Delta t) \frac{\hbar}{2} e^{-i\omega(t-t')} + \theta(-\Delta t) \frac{\hbar}{2} e^{i\omega(t-t')}.
\] (109)
Analogously we can find \( \Delta_4 \):
\[
i\Delta_4(t; t') = \theta(t - t') \frac{\hbar m\omega}{2} e^{-i\omega(t-t')} + 
\theta(t - t') \frac{\hbar m\omega}{2} e^{i\omega(t-t')}.
\] (110)
\( \Delta_2 \) can be evaluated from the system in Eq. (101),
\[
i\Delta_2(t; t') = \theta(-\Delta t) \frac{\hbar}{2} e^{-i\omega(t-t')} + 
- \theta(-\Delta t) \frac{\hbar}{2} e^{i\omega(t-t')}.
\] (111)
APPENDIX D. DYNAMICAL SOLUTION

The quantum solutions of system in Eq. (61) are

\[
\begin{align*}
x(t) &= \frac{x_0}{2} (\cos \Omega t + \cos \Omega t) + \\
&+ \frac{y_0}{2} (\sin \Omega t - \sin \Omega t) + \\
&+ \frac{p_{x0}}{2M\Omega} (\cos \Omega t - \cos \Omega t) + \\
&+ \frac{p_{y0}}{2M\Omega} (\sin \Omega t + \sin \Omega t); \\
y(t) &= \frac{y_0}{2} (\cos \lambda t + \cos \lambda t) + \\
&+ \frac{x_0}{2} (\sin \Omega t - \sin \Omega t) + \\
&+ \frac{p_{x0}}{2M\Omega} (\cos \Omega t - \cos \Omega t) + \\
&+ \frac{p_{y0}}{2M\Omega} (\sin \Omega t + \sin \Omega t); \\
p_x(t) &= \frac{p_{x0}}{2} (\cos \Omega t + \cos \Omega t) + \\
&+ \frac{x_0 M\Omega}{2} (\sin \Omega t + \sin \Omega t) + \\
&+ \frac{y_0 M\Omega}{2} (\cos \Omega t - \cos \Omega t); \\
p_y(t) &= \frac{p_{y0}}{2} (\cos \Omega t + \cos \Omega t) + \\
&+ \frac{p_{x0}}{2} (\sin \Omega t - \sin \Omega t) + \\
&+ \frac{x_0 M\Omega}{2} (\sin \Omega t + \sin \Omega t) + \\
&+ \frac{y_0 M\Omega}{2} (\cos \Omega t - \cos \Omega t).
\end{align*}
\]

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