Mathematical Principle Analysis of Separation Point Prediction

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Abstract. During the development of fluid mechanics, fluid separation is an important issue. So far, there is no mathematical formula to reveal and describe the essence of fluid separation. At the same time, due to the high cost and limitation of the experimental method, another method is urgently needed to predict the separation position of the fluid. After axiomatizing fluid mechanics and combining the principle of excited state of quantum mechanics, this paper reveals that fluid separation is a special form of fluid in an excited state, and deduces the state conditions of fluid separation. The research results of this paper provide new ideas for solving problems in fluid separation and engineering applications.

Keywords. Fluid separation, mathematical analysis, separation point prediction.

1. Introduction
In the past nearly a century, fluid mechanics has developed rapidly, and computational fluid dynamics has also developed rapidly. In the development of fluid mechanics, many scientific problems have appeared, such as the separation of fluids. At the beginning, Reynolds discovered the existence of fluid separation through circular pipe flow experiments\textsuperscript{[1, 2]}. At the same time, Renaut also discovered the existence of fluid transitions. Up to now, there has not been a general method to accurately define and solve the mechanism and position of the two fluid motions, which leads to many problems in theoretical derivation and engineering applications. Therefore, it has important theoretical and engineering value to accurately express and solve the fluid separation position on the mathematical level\textsuperscript{[3]}.

When describing physical phenomena, experimental methods are more accurate than theoretical derivations. Croci et al.\textsuperscript{[4]} revealed a laminar boundary layer separation in the Venturi throat. Melius et al.\textsuperscript{[5]} studied the unstable separation process of a three-dimensional airfoil. At the same time, Melius et al.\textsuperscript{[6]} also made a certain contribution to the study of the boundary layer separation. Wei et al.\textsuperscript{[7]} used the pressure distribution curve of the S809 airfoil to study the boundary layer separation of the airfoil. The above studies have proved that experimental methods can effectively describe and reveal physical phenomena, but their cost is relatively high, which makes it difficult to continue to use experimental methods to study complex problems. Therefore, it is necessary to analyze and explain some complex fluid problems at the theoretical level, such as fluid separation.

At the same time, the application of numerical simulation methods in fluid mechanics is also very extensive. Compared with the experimental method, the numerical simulation method has a lower cost.
and a wider range of applications. Therefore, when studying fluid separation [8], many researchers have used numerical simulation methods for research and analysis. Sengupta et al. [9] solved the Navier-Stokes equation to study the effect of oscillation and turbulence on fluid separation on a flat plate. Salimipour [10] studied the two-dimensional numerical simulation of the fluid around different airfoils. Ni et al. [3] used the implicit LES method to research the turbulent boundary layer separation of the backward circular slope. Jiang et al. [11] also used the DNS method to check the fluid flow separation around the cylinder.

In this article, we use mechanical methods to study fluid separation, through rigorous mathematical definitions and derivations, the essence of fluid separation and the conditions for its occurrence are given, and the fluid separation point is predicted. We first use the wave velocity and the concept of superposition state in quantum mechanics to analyze the generation of excited states mathematically. Finally, the necessary and sufficient conditions for fluid separation are studied, and the results are compared with the experimental data of fluid separation on the airfoil surface. It is found that the prediction accuracy of the separation point is within 2%. At the same time, we also compared the results with the classical experiment of flow around a cylinder.

2. Mathematical Physics Definition of Starting Point of Separation Process
The N-S equations describe the motion of flowfield. The velocity field \( u_i \) satisfies the continuity equation and the momentum equation are:

\[
\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial t} + \rho u_i u_j \delta_{ij} + f_i = 0, \quad (1)
\]

\[
\rho \frac{\partial u_i}{\partial t} + \rho u_j u_i \delta_{ij} + \sigma_{ij} + f_i = 0, \quad (2)
\]

where \( \rho \) is the density, \( f_i \) is the body force, and \( \sigma_{ij} \) is the stress state tensor, and

\[
\sigma_{ij} = -p \delta_{ij} + \nu \frac{\partial u_i}{\partial x_j} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (3)
\]

where \( p \) is the pressure, \( \delta_{ij} \) is the Kronecker symbol, and \( \mu \) is coefficient of dynamic viscosity.

Considering the fluctuation velocity in the Reynolds equation can reflect the randomness and uncertainty of turbulence. The fluctuation velocity can also help us better understand the mechanism of fluid separation from a phenomenological perspective.

**Definition 1.** The velocity field could be expressed as:

\[
u_i = u_i' + \bar{u}_i, \quad (4)
\]

where \( u_i' \) is the fluctuation velocity field and \( \bar{u}_i \) is the average velocity field, and equation (4) is shown in figure 1.

![Figure 1. The physical phenomenon of fluid separation is to produce backflow.](image-url)
According to the physical phenomenon of fluid separation, we define fluid separation as follows:

**Definition 2.** Let \( U(x_0, \varepsilon) \) be a neighborhood of point \( x_0 \in \Omega \). When \( \exists x_k \in U(x_0, \varepsilon) \),

\[
u_i(x_k)u'_i(x_k) \leq 0, |u_i(x_k)| \leq |u'_i(x_k)|,
\]
(5)

\[u'_i(x_k) = 0,
\]
(6)

there is a separation at point \( x_k \).

And according to the definition of fluctuation velocity, the conditions of fluctuation velocity is that: let \( x_k \in \Omega \), and there is a neighborhood \( U(x_k, \varepsilon) \). For \( x_j \in \{ x_j \mid x_j \in U(x_k, \varepsilon), |x_k - x_j| < \varepsilon, \forall \varepsilon > 0 \} \), the condition of fluctuation velocity is:

\[u'_j = 0,
\]
(7)

and \( u'_j \) satisfies the following equation:

\[\frac{d u'_j}{d t} = \frac{\partial u'_j}{\partial t} + u_k u'_{j,k} \neq 0.
\]
(8)

When \( \varepsilon \to 0 \), \( x_k \) gives the coordinates of separation point.

Since the generation of fluctuation velocity is difficult to judge, it is necessary to define the unexcited state and the excited state to help us simply describe the various states as shown in figure 2.

**Figure 2.** Superposition state.

Both excited and unexcited states are flowfield states, and unexcited state will become excited state when the fluctuation velocity is generating. And the definition of velocity strain tensor is needed.

**Definition 3.** The second-order tensor \( \lambda^j_i \) satisfies:

\[u'_i = \bar{u}_j \lambda^j_i, \quad u'_i, \bar{u}_j \in \mathcal{M},
\]
(9)

and the tensor is velocity strain tensor.

We can obtain the theory of excited state. When the flowfield state is excited state, at the initial point, \( \lambda^j_i \) satisfies:

\[\lambda^j_i = 0,
\]
(10)

and the time derivative of \( \lambda^j_i \) is

\[\frac{d \lambda^j_i}{d t} = \frac{\partial \lambda^j_i}{\partial t} + u_k \lambda^j_i,k \neq 0,
\]
(11)

where \( u_k \) is the velocity field.
And we can obtain the unexcited state theory. From the above analysis, there is no fluctuation velocity in the unexcited state.

According to equations (1) and (2), the unexcited state theory could be written as:

\[
\frac{\partial \rho}{\partial t} + \rho v^i \delta^i = 0, j \in \mathcal{J}_M; 
\]

\[
\rho \frac{\partial (\bar{u} \delta^i)}{\partial t} = \rho \bar{u} \delta^i + \rho \bar{u} \delta^i = \bar{u} \delta^i, 
\]

where \( g^{(1)} = \sigma_{ik}^{(1)\ast} + f_i \) is the unexcited state external force field, \( \sigma_{ik}^{(1)} \) is the stress state tensor in unexcited state, and \( \delta^i \) is unit constant tensor.

For facilitating calculations, we definite the excited ratio tensor in the excited state:

\[
\xi^i = \delta^i + \lambda^i, 
\]

and \( \xi^i \) is the excited ratio tensor. Then the following conclusion is obtained.

\[
u_i = \bar{u} \delta^i + \bar{u} \lambda^i = \bar{u} (\delta^i + \lambda^i) = \bar{u} \xi^i.
\]

When the flowfield state is excited, \( \xi^i \) satisfies:

\[
\frac{\partial \rho}{\partial t} + \rho \bar{v}^i \xi^i = 0, j \in \mathcal{J}_M; 
\]

\[
\rho \frac{\partial (\bar{u} \xi^i)}{\partial t} + \rho \bar{u} \bar{v}^i \xi^i = \rho \bar{u} \xi^i = \bar{u} \xi^i, 
\]

where \( g^{(2)} = \sigma_{ik}^{(2)\ast} + f_i \) is the excited state external force field, \( \sigma_{ik}^{(2)} \) is the general stress state tensor, \( f_i \) is the body force, \( \epsilon_{ik}^{(1)\ast} = \xi^i \xi^j \) and \( \lambda_{ik}^{(1)\ast} = \xi^i \xi^{j\ast} \) are the adjustment tensor coefficients.

For the excited state, the \( \lambda^i \) satisfies:

\[
\rho \bar{v}^i \lambda^i + \rho \bar{v}^i \lambda^i = 0, j \in \mathcal{J}_M; 
\]

\[
\rho \frac{\partial (\bar{u} \lambda^i)}{\partial t} + \rho \bar{u} \bar{v}^i \lambda^i = \rho \bar{u} \lambda^i = \bar{u} \lambda^i, 
\]

where \( g_i = g^{(2)} - g^{(1)} = \sigma_{ik}^{(2)\ast\ast} - \sigma_{ik}^{(1)\ast\ast} \) is the excited external force field. And we obtain \( \epsilon_{ik}^{(2)} = \epsilon_{ik}^{(1)} - \delta^i \delta^j = \lambda^i \lambda^j + \lambda^j \lambda^i + \delta^i \lambda^j + \delta^j \lambda^i \), where \( \epsilon_{ik}^{(2)} \) is the adjustment tensor coefficient.

Substituting equations (12) and (13) into equations (18) and (19), the following equations can be obtained:

\[
\lambda^i = 0, 
\]

\[
\rho \bar{u} \frac{\partial \lambda^i}{\partial t} + \rho \bar{u} \lambda^i = g_i, j \in \mathcal{J}_M. 
\]

For the unexcited state, the gradient of the stress tensor could be expressed as:

\[
\sigma_{ik}^{(1)\ast\ast} = -\sigma_{ik}^{(2)\ast\ast} + \sigma_{ik}^{(1)\ast\ast} \delta^i + \mu \left( \bar{u} \bar{v}^i \delta^i + \bar{u} \bar{v}^i \delta^i \right). 
\]

And the gradient of the stress tensor could be expressed as:
The stress difference between the excited and unexcited states is:

$$
\sigma_{ik}^{J} - \sigma_{ik}^{J*} = (P^{(1)}_{ik} - P^{(1)*}_{ik}) + \mu \left( \bar{u}_{iJ}^{(1)} \lambda^{(1)}_{ik} + \bar{u}_{iJ} \lambda^{(1)*}_{ik} \right) + \mu \left( \bar{u}_{iJ}^{(2)} \lambda^{(2)}_{ik} + \bar{u}_{iJ} \lambda^{(2)*}_{ik} \right)
$$

(23)

The stress difference between the excited and unexcited states is:

$$
g_{i} = (\sigma_{ik}^{J} - \sigma_{ik}^{J*}) = (P^{(1)}_{ik} - P^{(1)*}_{ik}) + \mu \left( \bar{u}_{iJ}^{(1)} \lambda^{(1)}_{ik} + \bar{u}_{iJ} \lambda^{(1)*}_{ik} \right) + \mu \left( \bar{u}_{iJ}^{(2)} \lambda^{(2)}_{ik} + \bar{u}_{iJ} \lambda^{(2)*}_{ik} \right)
$$

(24)

The final excitation law is:

$$
\lambda^{(1)}_{ik} = 0, \quad \lambda^{(1)*}_{ik} = 0;
$$

(25)

$$
\rho u_{i} \frac{\partial \lambda^{(1)}_{ik}}{\partial t} + \rho u_{i} \lambda^{(2)}_{ik} = P + \mu \left( \bar{u}_{iJ}^{(2)} \lambda^{(2)}_{ik} + \bar{u}_{iJ} \lambda^{(2)*}_{ik} \right)
$$

(26)

where $\phi = \psi + 1$ is an auxiliary coefficient.

Because the equation cannot be solved at present, we introduce the degenerate condition [11] to solve this problem. The condition of degradation is:

$$
\frac{dx_{ik}_{j}}{dt} = \frac{\partial x_{ik}_{j}}{\partial t},
$$

(27)

When the velocity strain tensor develops from zero to $\lambda^{(1)}_{ik}$ quickly. The fluctuation velocity satisfies:

$$
\rho \frac{\partial u_{i}'}{\partial t} = \frac{P + P'_{i}}{1 + \gamma},
$$

(28)

where $\gamma = 1 - \beta$ is the dimensionless coefficient; $P_{i}$ is difference of the pressure gradient. Let $C_{ik}'$ is fluctuation velocity gradient, and $P'_{i}$ is:

$$
P'_{i} = \mu \left( \phi \frac{\partial C_{ik}'}{\partial t} + \frac{\partial C_{ik}'}{\partial t} \right) \left( u_{ik} \right)^{-1},
$$

(29)

let velocity gradient is $C_{ik}$, the viscosity coefficient $\beta = \mu C_{ik} \left( \rho u_{i} u_{j} \right)^{-1}$, where $\mu C_{ik}$ is the viscous stress; $\rho u_{i} u_{j}$ is the inertial stress.

The pressure gradient difference is:

$$
P = p^{(1)}_{ik} - p^{(1)*}_{ik} = \Delta p_{ik},
$$

(30)

where $\Delta p_{ik}$ is the variation of the pressure gradient.

Using $\xi$ evaluates the spatial variation rate of the pressure gradient difference, and $\Delta x_{i}$ is the scale of spatial change. We obtain:

$$
\xi = \lim_{\Delta x_{i} \to 0} \frac{\Delta p_{ik}}{\Delta x_{i}} = L' p,
$$

(31)

where $L'$ is the Laplace operator. And according to equation (26), following equation could be got:

$$
\frac{\partial \theta}{\partial t} = \frac{1}{1 + \gamma} \left( \frac{P'_{ik}}{\rho} - L' p \right),
$$

(32)
where $\theta' = u'^{*,i}_{*}$ is fluctuation velocity expansion.

The divergence of characteristic pressure could be expressed, that if $u_i = 0$, then $P'^{*,i}_{*} = 0$; if $u_i \neq 0$, then:

$$P'^{*,i}_{*} = \frac{nv}{n+1}[(1+\phi) p'_i v_i - \eta p'(u_i v_i + \phi u_i v_i) C^{ii}] + \frac{\eta \theta}{(n+1)} [(1+\phi) np' - \left(f_i v_i + \phi f_i v_i\right) C^{ii}],$$

(33)

where $n$ is the spatial dimension.

Separation is the formation of a local wave velocity distribution of opposite magnitude. Therefore, $u_i' \rightarrow 0, f_i' \rightarrow 0, p' \rightarrow 0$ and $C^{ii} \rightarrow 0$, it could be considered that:

$$P'^{*,i}_{*} \rightarrow \frac{nv(1+\phi)}{n+1} \left(p'_i v_i + \frac{\eta p'}{n+1}\right).$$

(34)

According to above analysis, the point $x_j$ that separation occurs satisfies equation (35).

$$\frac{\partial \theta}{\partial t} = \frac{1}{1+\gamma} \left(\frac{L p}{\rho} - \frac{nv(1+\phi)}{n+1} \left(\frac{p'_i}{\rho} v_i + \frac{p'}{\rho} \frac{\theta}{n+1}\right)\right),$$

(35)

where $\theta'$ presents velocity divergence $u'^{*,i}_{*}$.

The streamlines cannot pass through a solid, it is shown in figure 3.

\[
\begin{align*}
\text{Figure 3. Schematic diagram of wall condition.}
\end{align*}
\]

According to equation (31), we can obtain:

If $n_i' > 0$, then $\mathcal{L}p - P'^{*,i}_{*} < 0$;

conversely, then $\mathcal{L}p - P'^{*,i}_{*} \geq 0$,

(36)

where $n_i'$ is the outer normal vector of the solid boundary.

According to the excitation law, if the flowfield is excited, there is a scalar physical variable called the flowfield excitation intensity ($\Pi$) which satisfies the following equation.

$$\Pi = \frac{1}{1+\gamma} \left(\frac{L p}{\rho} - \frac{P'^{*,i}_{*}}{\rho}\right).$$

(37)

Now we can obtain that the separation is only related to the second derivative of pressure:

$$\frac{d^2 C_{p}}{dx^2} = \chi,$$

(38)
where $C_p$ is the pressure coefficient, $x$ is the separation position, $\varepsilon \to 0$ is infinitesimal, and $n_x$ is the component of the outer normal vector. And the coefficient $\chi$ is expressed as:

$$\chi = \frac{n}{n+1}(1+\phi)\frac{C'_p}{u_{\max}}\frac{u}{u_{\infty}} \approx h\left(\frac{y}{\delta}\right),$$

(39)

where $u_{\infty} = 30m/s$ is the inflow velocity, and the mapping $h$ is expressed as the power function space. When the product of fluctuation pressure and dynamic viscosity is insignificant, $\chi \to 0$. And if we ignore the air gravity, considering $p' \approx p_{\infty}$, we can obtain:

$$\frac{C'_p}{C_p} \approx \frac{u_{\infty}^2}{u_{\max}^2}.$$  

(40)

Due to the above conditions, the separation point should satisfy:

$$n' > 0, \quad \frac{d^2C}{dx^2}(x-\varepsilon) < 0.$$  

(41)

3. Experimental Validation

We use a one-dimensional flow experiment to reveal the correctness of this theory. In the experiment, the airfoil NACA2412 with an inflow velocity of 30m/s was employed, and the condition of angle of attack 12°. The experimental device diagram is shown in figure 4, and the pressure curve is shown in figure 5.

**Figure 4.** Experimental device.  
**Figure 5.** Pressure curve (1 is the upper surface and 2 is the lower surface).
The derivative curves are shown in figure 5 and figure 6. Observe curve 1 in figure 6. From figure 7, when the normal vector component in is greater than 0, the separation occurs at \( x = 0.762 \), and the second derivative of the pressure on the upper surface is less than 0.

The experimental results are shown in figure 8. The separation point is \( x = 0.755 \) and region E is the separation bubble \( x \in (0.5, 2.5) \cup (2.5, 6) \). Because of the viscosity of the fixed wall, region D is an error region.

The prediction accuracy of this method is:

\[
\Delta = \frac{0.755 - 0.762}{0.755} = 0.927\%
\] (42)

Due to the existence of measurement errors in the experiment, it is conservatively estimated that the theoretical prediction accuracy should be within 2%.
At the same time, in order to verify the correctness of the theory, we consider the pressure distribution formula of the ideal fluid around a cylinder:

\[ C_p = 1 - 4 \sin^2 \alpha. \]  

(43)

The second derivative of the pressure distribution is:

\[ \frac{d^2 C_p}{d\alpha^2} = 8 \left( \cos^2 \alpha - \sin^2 \alpha \right) = 8 \cos 2\alpha. \]  

(44)

According to the separation theorem in the paper, when the divergence of the characteristic pressure is 0, there is:

\[ \cos 2\alpha = 0. \]  

(45)

Solving the above formula can get

\[ \alpha = \frac{\pi}{4} + k \frac{\pi}{2}, k \in N. \]  

(46)

In theory, the ideal fluid does not have a separation effect, but if the viscosity is extremely small and the pressure basically meets the above form, separation will occur near 135° and 225°. If the viscosity increases further, we observe the pressure distribution curve. According to the conclusion in the paper, separation will occur when the curve changes from concave to convex at the inflection point. For subcritical flow (curve 3 in figure 9), the separation will occur at about 75°; for supercritical flow (curve 2 in figure 9) separation will occur at around 121°.

![Figure 9. Experimental results of flow around a cylinder.](image)

4. Conclusion

The determination of the separation point is of great significance to the design and testing of the aircraft. At the same time, the determination of the separation position also greatly promotes the development of flow control technology. In order to determine the separation position, after axiomatizing fluid mechanics, combined with the excited state principle of quantum mechanics, it is concluded that fluid separation is the special form of fluid in the excited state, and the state conditions for fluid separation are deduced. The conservative estimation accuracy of the calculation results of this method is less than 2% that compared with the experimental results. And this paper rigorously proves the relationship between the occurrence of separation and the pressure distribution curve mathematically.
References

[1] Hall M G 1981 Computational fluid dynamics. A revolutionary force in aerodynamics 5th Computational Fluid Dynamics Conference (Palo Alto: American Institute of Aeronautics and Astronautics)

[2] Vivian H, Lerome C and Morice P 1987 Computational fluid dynamics in France 8th Computational Fluid Dynamics Conference (Honolulu: American Institute of Aeronautics and Astronautics)

[3] Moulinec C and Emerson D 2019 Flow separation control over a rounded ramp with spanwise alternating wall actuation Phys. Fluids 31(1) 015101

[4] Croci K, Ravelet F, Danlos A, Robinet J C and Barast L 2019 Attached cavitation in laminar separations within a transition to unsteadiness Phys. Fluids 31(6) 063065

[5] Melius M S, Mulleners K and Cal R B 2018 The role of surface vorticity during unsteady separation Phys. Fluids 30(4) 045108

[6] Melius M S, Mulleners K and Cal R B 2016 Numerical investigation of the role of free-stream turbulence in boundary-layer separation J. Fluid Mech. 851 289-321

[7] Wei B, Gao Y, Wang L and Li D 2019 Analysis of flow transition and separation on oscillating airfoil by pressure signature J. Mech. Sci. and Technol. 33 279-288

[8] Istvan M S, Kurelek J W and Yarusevych S 2018 Turbulence intensity effects on laminar separation bubbles formed over an airfoil AIAA J. 56 1335

[9] Sengupta A and Tucker P 2020 Effects of forced frequency oscillations and free stream turbulence on the separation induced transition in pressure gradient dominated flows Phys. Fluids 32(10) 104105

[10] Salimpour E 2019 A modification of the $k – k_i - \omega$ turbulence model for simulation of short and long separation bubbles Comput. Fluids 181 67

[11] Jiang H Y 2020 Separation angle for flow past a circular cylinder in the subcritical regime Phys. Fluids 32(1) 014106