Mathematical modelling of the influence of yield shear stress on blood friction in a turbulent flow

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Abstract. Measurements of blood rheology indicate that human blood has a yield shear stress. Transport of oxygen in the aorta or a vein depends on blood flow rate. Therefore, it is interesting to find out how blood yield shear stress affects blood transportation if flow is turbulent. The majority of mathematical approaches deal with laminar flow of human blood, which is rather simple compared to turbulent flow modelling. This paper presents a mathematical model of fully developed turbulent flow of human blood in the aorta. The physical model assumes that blood is a non-Newtonian liquid that demonstrates yield shear stress. The main objective of the research is to examine the influence of human blood yield shear stress on turbulent properties, like friction factor, in the aorta. Available blood rheology experimental data for various concentrations of haematocrit were used in order to fit the rheological model. The rheological model together with the momentum equation and the two-equation turbulence model constitute a mathematical model of turbulent flow of human blood. Results of simulations are discussed and presented as figures and conclusions.

1. Introduction

Simulation of a blood flow is extremely difficult, as red blood cells are deformable, have a complex shape, and play a leading role in blood rheology in contrast to white cells and platelets. Concentration of haematocrit has a substantial influence on blood flow phenomena together with plasma film, which appears in close vicinity of a vein wall, containing about 90% of water and 10% of proteins. As the phenomenon of blood flow is very complex, we can find different approaches in literature concerning the development of mathematical models. Some of them treat blood as a single-phase flow of a Newtonian, or a non-Newtonian liquid, or, as a mixture of the liquid and solid phase. Apart from that, we can treat red blood cells as flexible or non-flexible solid bodies. If we consider the methods of blood flow modelling, we can recognise meso- and macroscopic approaches, as microscopic modelling refers to the scale of single atoms and molecules. If a blood flow environment is taken into account, we know that a constitutive model is effective at describing the anisotropic mechanical response of artery walls.

When analysing blood motion, one can say that the majority of mathematical approaches regard laminar flow, which is rather easy to modelling compared to a turbulent flow. Considering simulations of blood flow in micro channels at low and high concentrations of haematocrit one can mention the research of McWhirter et al. [1, 2], Peng et al. [3], Dupin et al. [4], Doddi and Bagchi [5], Fedosov et al. [6], Freund and Orescanin [7] and Krüger et al. [8]. However, their models deal with laminar blood flow. Therefore, this paper presents a mathematical model, which assumes that blood flow is turbulent, and the maximum Reynolds number does not exceed 5000. The mathematical model consists of...
averaged Navier-Stokes equations (RANS) and a turbulent stress tensor was calculated using the indirect method, which takes into account the Boussinesque hypothesis [9]. Such hypothesis utilizes turbulent viscosity, which is calculated based on the chosen two-equation turbulence model.

The main objective of the research is to examine the influence of a human blood yield shear stress on the friction factor in the human aorta. The friction factor is a crucial parameter determining the resistance of blood flow in a vein or the aorta. A higher friction factor is when higher flow resistance appears, resulting in decreased transport of oxygen. Taking into account the mathematical model of turbulent blood flow in the aorta, we examined the influence of yield stress on the friction factor.

2. Validation of rheological models
As the first step requires access to reliable physical properties of a blood, we have chosen Wells and Merrill’s experimental data [10]. Blood rheology experimental data from Wells and Merrill for several values of haematocrit were chosen in order to validate two arbitrarily chosen rheological models, described by equations (1) and (2). Both rheological models are suitable to describe the shear stress in blood flow.

– The Casson model [11]:

$$\frac{\tau}{\mu} = \frac{\tau_0}{\mu_0} + (\frac{\mu_0}{\tau_0})\gamma$$  \hspace{1cm} (1)

– The Herschel-Bulkley model [12]:

$$\tau = \tau_0 + K\gamma^n$$  \hspace{1cm} (2)

Validation of the aforementioned rheological models has been performed for human blood containing 43% of haematocrit with a density of 1060 kg/m$^3$ at a temperature of 37 °C. The following rheological parameters were obtained based on the best fitting shear stresses measured and calculated using the above rheological models:

– The Casson model, described by equation (1):

$$\tau_0=0.0144 \text{ [Pa]}; \mu_0=0.0046 \text{ [Pa s]};$$

– The Herschel-Bulkley model, described by equation (2):

$$\tau_0=0.0144 \text{ [Pa]}; K=0.020 \text{ [Pa s$^n$]}; n=0.75;$$

![Figure 1. Wells’ and Merrill’s experiments and simulation of wall shear stress in human blood containing 43% of haematocrit.](image)
Experimental data presented in figure 1 demonstrate the dependence of the shear rate on the shear stress of human blood, which contains 43% of haematocrit [9]. The Casson and Herschel-Bulkley rheological models provided almost the same results of simulated shear stress of human blood, which is presented in figure 1.

![Figure 1]({#image1})

**Figure 1.** Experimentally determined and simulated shear stress of human blood containing 43% of haematocrit.

Figure 2 presents viscosity calculated using the experimental data from Wells and Merrill [9] and simulated using the Casson and Herschel-Bulkley rheological models. It is seen that both rheological models give similar results. Concluding, one can say that both models predict well the shear stress and the viscosity of human blood. As both models gave similar predictions of shear stress and viscosity the Casson rheological model will be used in further steps as the simpler one.

3. **Physical and mathematical models**

Developing any mathematical model requires a physical model first. The physical model assumes that blood has a yield shear stress, which is in line with the aforementioned experiments. Transport of oxygen is strictly related to blood flow rate, while blood flow rate depends on the friction factor. For this reason, the research is focused on the influence of blood yield shear stress on the friction factor if flow is turbulent. Looking for simplicity of the physical model, we assumed that the blood is flowing in a rigid, smooth and horizontal aorta of constant diameter and the flow is fully developed, axially symmetrical, turbulent and homogeneous. We also assumed that the flow is stationary. Therefore, the blood is treated as a single-phase liquid with increased density and viscosity. The blood has a constant temperature equal to 37 °C.

In order to develop the mathematical model of a turbulent blood flow, the starting point are the time-averaged Navier-Stokes equations, continuity equation and boundary conditions. In order to build mathematical model, we decided to use the Random Averaged Navier-Stokes approach (RANS).

Taking into account the aforementioned assumptions, the continuity equation of blood flow can be described in the following form:

\[ \frac{\partial \rho}{\partial x} = 0 \]  (3)
while the momentum equation consists of the Random Averaged Navier-Stokes equation, the final form of which for the aforementioned assumptions in cylindrical coordinates is as follows:

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r \left( \mu \frac{\partial u}{\partial r} - \bar{p} \frac{\partial u'}{\partial r} \right) \right] = \frac{\partial p}{\partial x}$$

(4)

where the upper dash means the time averaged quantity.

The component of turbulent stress tensor, which appears in equation (4), can be designated through the indirect method using the Boussinesque hypothesis, as follows [9]:

$$-\bar{p} \frac{\partial u'}{\partial r} = \mu_t \frac{\partial u}{\partial r}$$

(5)

Several turbulence models are available in literature, which make it possible to describe the turbulent stress tensor. In this research, the Launder and Sharma turbulence model was used [13]. This particular turbulence model assumes that the turbulent viscosity, which appears in equation (5), can be designated using dimensionless analysis, as follows [13]:

$$\mu_t = f_{\mu} \frac{\bar{\rho}}{\varepsilon} k^2$$

(6)

where the kinetic energy of turbulence and its dissipation rate are derived from Navier-Stokes equations using the time-average procedure and are as follows:

- The kinetic energy of turbulence:

$$\frac{1}{r} \left[ r \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial r} \right] + \mu_t \left( \frac{\partial u}{\partial r} \right)^2 = \rho \varepsilon + 2 \mu \left( \frac{\partial k^{1/2}}{\partial r} \right)$$

(7)

- The rate of dissipation of the kinetic energy of turbulence:

$$\frac{1}{r} \left[ r \left( \mu + \frac{\mu_t}{\sigma_l} \right) \frac{\partial \varepsilon}{\partial r} \right] + C_1 \frac{\varepsilon}{k} \mu_t \left( \frac{\partial u}{\partial r} \right)^2 = C_2 [1 - 0.3 \exp(-Re_k)] \frac{\rho \varepsilon^2}{k} - 2 \frac{\mu}{\rho} \mu_t \left( \frac{\partial^2 u}{\partial r^2} \right)$$

(8)

The turbulence damping function ($f_{\mu}$) in equation (6) and the turbulent Reynolds number in equation (8) were defined by the Launder and Sharma turbulence model [13], as follows:

$$f_{\mu} = 0.09 \exp \left[ \frac{-3.4}{\left( 1 + Re_k \right)^{1/2}} \right]$$

(9)

$$Re_t = \frac{\bar{p} k^2}{\varepsilon \mu}$$

(10)

Taking into account the apparent viscosity concept, one can determine the shear stress for a Newtonian liquid, as follows:

$$\tau = \mu_{app} \gamma$$

(11)

The apparent viscosity concept means that for shear thinning blood, the apparent viscosity decreases as the shear rate increases [12]. Taking into account equation (11), one can develop the equation for apparent viscosity using Casson’s model, as following [15]:
\[ \mu_{\text{app}} = \frac{\mu_{\infty}}{1 - \left(\frac{\tau_w}{\mu_{\infty}}\right)^{1/2}} \left[ \left(\frac{\tau_w}{\mu_{\infty}}\right)^{1/2} + \mu_{\infty}^{1/2} \right]^2 \] (12)

Dynamic viscosity \( \mu \), which appears in equations (4), (7), (8) and (10), should be replaced by apparent viscosity expressed by equation (12). Finally, the mathematical model of turbulent human blood flow in the aorta consists of equations (4), (7) and (8), together with complimentary relations (5), (6), (9), (10) and (12). The model assumes non slip velocity at the aorta wall, i.e. \( U=0 \) and \( k=0, \varepsilon=0 \). As flow is axially symmetrical, it is assumed that \( \partial U/\partial r=0, \partial k/\partial r=0, \partial \varepsilon/\partial r=0 \). Constants in the Launder and Sharma turbulence model are as follows: \( C_1=1.44; C_2=1.92; \sigma_k=1.0; \sigma_\varepsilon=1.3 \) [13]. The mathematical model has been solved for 80 nodal points not uniformly distributed on the aorta radius \( R=0.004 \) [m]. Most of the nodal points were localized in close vicinity of the aorta wall. The number of nodal points was set experimentally to provide nodally independent simulations. Computations were made using own computer code [15]. The set of partial differential equations (4), (7), (8) was solved by taking into account TDMA approach, with iteration procedure, using control volume method [16]. Iteration cycles are repeated until criterion of convergence, defined by equation (13), is achieved.

\[ \sum_j \left| \frac{\phi_j^n - \phi_j^{n-1}}{\phi_j^n} \right| \leq 0.0005 \] (13)

4. Results of simulations

The yield shear stress of a human blood describes a critical stress below which no flow takes place. Several researchers confirmed the importance of yield shear stress in a flow. Some of them explained its nature. Michaels and Bolger provided a comprehensive explanation of phenomenon of yield shear stress [17, 18]. They reasoned that a yield shear stress has two components: a true network strength, which must be overcome for motion to occur at all, and a creep energy dissipation effect accompanying the collisions between flocs. They considered the flocs to be the basic unit of the suspension and that aggregates of flocs formed at low shear rates. The flocs were smaller than the aggregates and shear tends to produce more dense flocs. If the blood flow rate in the aorta starts from zero and is increasing, we go through regimes of laminar, transient and turbulent flow.

Non-Newtonian behaviour of a blood flow indicates that changes of wall shear stresses resulting in changes of apparent viscosity, which is expressed by equation (12). Figure 3 presents calculated apparent viscosity for different wall shear stresses of human blood flow containing 43% of haematocrit. In the range of a wall shear stress from 5 to 50 [Pa] there exists laminar, transient and turbulent flow in the aorta with a radius of \( R=0.004 \) [m]. For example, if the wall shear stress equals \( \tau_w=30 \) [Pa], the Reynolds number, defined by equation (14), is \( Re=4000 \), which means that blood flow is fully turbulent.

\[ Re_{\text{ap}} = \frac{\rho_b U_b 2R}{\mu_{\text{app}}} \] (14)
After analysing equation (12) and figure 3, one can say that influence of the yield shear stress on the apparent viscosity of human blood increases when wall shear stress decreases. It is known that in laminar flow, the wall shear stress is low compared to a turbulent one. In order to demonstrate this phenomenon, we made a simulation of the influence of the yield shear stress on the apparent viscosity of human blood for different values of wall shear stresses, which is presented in figure 4. As an example, let us assume that the yield shear stress equals 0.03 [Pa]. In such a case, the apparent viscosity of human blood is $\mu_{app}=0.0047$ [Pa s] or $\mu_{app}=0.00418$ [Pa s] depending on the wall shear stress, which is respectively $\tau_w=5$ [Pa] and $\tau_w=60$ [Pa] – see figure 4. In such a case, decrease of relative apparent viscosity is about 11%. If the wall shear stress equals to $\tau_w=5$ [Pa] the flow becomes laminar, while for $\tau_w=60$ [Pa], it is turbulent. This clearly means that the importance of the yield shear stress in a turbulent blood flow is lower compared to its importance in a laminar flow.

After analysing figure 4, it can be seen that under the laminar flow regime, which exists for the wall shear stress $\tau_w=5$ [Pa], the apparent viscosity substantially increases with the yield stress increase. In such a case, for yield stress $\tau_0=0$ [Pa], the apparent viscosity equals $\mu_{app}=0.004$ [Pa s], while for yield shear stress $\tau_0=0.05$ [Pa], the relative increase of apparent viscosity is about 23.5%. However, if the wall
shear stress increases, which is due to transient and turbulent flow regimes, the rate of increase of apparent viscosity drops, which is seen in figure 4 for \( \tau_0 = 15; 30; 60 \) [Pa]. To clarify this, let us consider a blood flow at \( \tau_0 = 30 \) Pa, which corresponds to a Reynolds number of \( \text{Re} = 4000 \). For such a case, the relative increase of apparent viscosity equals to about 9% at \( \tau_0 = 0.05 \) [Pa], comparing to its value at \( \tau_0 = 0 \) [Pa]. This phenomenon is even more pronounced if the wall shear stress equals 60 [Pa]. Concluding, one can say that the influence of the yield shear stress on the apparent viscosity of human blood is significant if the flow becomes laminar, and is less important for a turbulent flow.

Lee et al. examined two rheological models, namely the Casson and the Herschel-Bulkley models, looking for best fit for the experiments on human blood [19]. They concluded that the yield shear stress value is \( \tau_0 = 14.4 \) [mPa] for the Casson model and \( \tau_0 = 32.5 \) [mPa] for the Herschel-Bulkley model. Their study showed that the Casson model is more suitable than the Herschel-Bulkley model for representing the non-Newtonian characteristics of blood viscosity. Taking into account the achievements of Lee at al. [19], the numerical simulation of friction factor \( \lambda \) was performed. Simulations were performed for turbulent blood flow in the aorta with a radius \( R = 0.004 \) [m]. Two values of yield shear stresses proposed by Lee at al. [19] were chosen: \( \tau_0 = 0.0144 \) [Pa] and \( \tau_0 = 0.0325 \) [Pa]. Simulations of friction factor \( \lambda \) for turbulent blood flow for two different yield shear stresses are presented in figure 5. Simulations were made for Reynolds numbers from 2900 to 5000. Results clearly demonstrate there are no differences of friction factor for the two different values of yield shear stress, as both predictions lie on the same curve. The results confirmed that the influence of the yield shear stress on the friction factor in turbulent human blood flow can be neglected.

![Figure 5](image_url)

**Figure 5.** Simulation of the influence of Reynolds number on friction factor for human blood containing 43% of haematocrit, for two different yield stresses.

5. Discussion and conclusions

When the flow of human blood in the aorta is considered, it is usually assumed that such flow is laminar. However, it is known that under some circumstances, like physical activity, the flow of human blood in the aorta can be turbulent. Additionally, it is worth to remind that pulsating flow of human blood demonstrates increase of turbulence during deceleration phase. Therefore, assuming that the blood flow in the aorta is turbulent, it is interesting to know if blood yield shear stress plays an important role in transporting oxygen. For this reason, the mathematical model of fully developed and stationary blood flow in the aorta was developed. Of course, the mathematical model is simplified and does not take into account aorta flexibility and the complex nature of a blood, especially that red blood cells are deformable. Nevertheless, numerical simulations confirmed that under the turbulent blood flow regime, the influence of the yield shear stress on the blood friction factor is not important compared to laminar blood flow.
We know that viscosity affects shearing stress, which increases blood friction. Higher blood friction results in lower blood flow rate and, as a consequence, lower transportation of oxygen. However, if blood flow becomes turbulent, the importance of viscosity decreases, as turbulence is a major player affecting blood flow properties. There are two main reasons affecting such behaviour of blood. Firstly, blood yield shear stress is relatively low. Secondly, the importance of apparent viscosity in turbulent flow is low, as turbulence plays a crucial role in blood transportation. If turbulence is taken into consideration, the turbulent viscosity, described by equation (6), plays a dominant role. Taking into account figure 3 and figure 4 it is clear that as the wall shear stress increases, the blood apparent viscosity decreases. However, if the yield stress increases, the apparent viscosity increases as well (assuming that the wall shear stress is constant). In conclusion, one can say that the wall shear stress and the yield shear stress affect blood apparent viscosity oppositely. Figure 5 explicitly shows that as the wall shear stress increases, the blood apparent viscosity decreases. However, if the yield stress increases, the apparent viscosity increases as well (assuming that the wall shear stress is constant). In conclusion, one can say that the wall shear stress and the yield shear stress affect blood apparent viscosity oppositely. Figure 5 explicitly shows that the blood friction factor lies on the same line for two different yield shear stresses equal to 14.4 [mPa] and 32.5 [mPa]. The presented results confirmed that if turbulent human blood flow is taken into consideration, the importance of the yield shear stress is marginal.

The research was carried out for human blood containing 43% of haematocrit. We can anticipate that for lower concentrations of haematocrit, the influence of the yield shear stresses on the human blood friction factor can be neglected. However, for concentrations of haematocrit higher than 43%, it is difficult to anticipate if the influence of the yield shear stress on the friction factor is still marginal. Such simulations are important if the influence of medications on blood flow transportation, for known concentration of haematocrit, is considered.

Concluding, on the base of numerical simulations, it is possible to withdraw following conclusions:

1. If human blood is considered the Casson rheological model is adequate to predict dependence of the shear rate on the shear stress.
2. Influence of the yield shear stress on the apparent viscosity of human blood is significant if the flow becomes laminar, and is marginal for a turbulent flow.
3. Influence of the yield shear stress on the friction factor in turbulent human blood flow can be neglected.

**Nomenclature**

- $C_i$ – constants in the Launder and Sharma turbulence model, $i=1, 2$
- $f_\mu$ – turbulence damping function
- $j$ – number of nodal points
- $k$ – kinetic energy of turbulence [m$^2$/s$^2$]
- $K$ – coefficient in the Herschel-Bulkley rheological model [Pa s$^n$]
- $n$ – power exponent in the Herschel-Bulkley rheological model / number of iteration cycle
- $p$ – static pressure [Pa]
- $r, x$ – distance from symmetry axis / axial coordinate [m]
- $R$ – inner aorta radius [m]
- $Re$ – Reynolds number
- $u', v'$ – fluctuating components of blood velocity [m/s]
- $U$ – blood velocity [m/s]

**Greek symbols**

- $\gamma$ – shear rate (shear deformation rate) [1/s]
- $\Phi$ – general dependent variable $\Phi=U, k, \varepsilon$
- $\varepsilon$ – rate of dissipation of kinetic energy of turbulence [m$^2$/s$^3$]
- $\lambda$ – friction factor
- $\mu$ – blood viscosity [Pa s]
- $\mu_\infty$ – coefficient in Casson rheological model
- $\rho$ – blood density [kg/m$^3$]
\( \sigma_e \) – effective Prandtl-Schmidt number for \( \varepsilon \)

\( \tau, \tau_0 \) – shear stress / yield shear stress [Pa]

**Subscripts:**

app – apparent viscosity

b – bulk (cross sectional averaged value)

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