Information about the Integer Quantum Hall Transition Extracted from the Autocorrelation Function of Spectral Determinants

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The autocorrelation function of spectral determinants (ASD) is used to probe the sensitivity of a two-dimensional disordered electron gas to the system’s size L. For weak magnetic fields the ASD is shown to depend only trivially on L which is a strong indication that all states are localized. From the nontrivial dependence of ASD on L at \( L \to \infty \) for strong magnetic fields at \( \sigma_{xy} = 1/2 \) we deduce the existence of critical wave functions. Pacs- numbers: 73.40.Hm, 72.15.Rn, 73.23.-b

In the pioneering works on localization [4, 5, 6, 7] of the Anderson model predicted a second order metal-insulator transition in dimensions larger than two. The dimension two has been found to be marginal. For noninteracting electrons in a random potential at no or weak magnetic field no extended states and thus no metal-insulator transition has been found by analytical and numerical methods [8].

However, Levine, Libby and Pruisken noticed the presence of a topological term in the field theoretical formulation of the problem (the nonlinear (NL) sigma model) and argued that this may result in the existence of extended states in a strong magnetic field [9]. They suggested that these extended states could explain the transition between plateaus in the Hall conductance, where a finite longitudinal conductance is observed experimentally [8]. There is a numerical evidence for existence of extended states in the center of the Landau band [10]. Since their wave functions decay like a power law, these states have been called critical or prelocalized.

In this paper we use the autocorrelation function of spectral determinants (ASD) to extract information about existence of critical states. The ASD is defined as follows:

\[
\bar{C}(\omega) = \frac{\langle \det(E + \frac{1}{2} \omega - H) \det(E - \frac{1}{2} \omega - H) \rangle}{C(0)},
\]

where

\[
\bar{C}(\omega) = \langle \det(E + \frac{1}{2} \omega - H) \det(E - \frac{1}{2} \omega - H) \rangle > .
\]

\( H \) is the Hamiltonian of the system under consideration, and \( E \) a central energy.

The ASD was used first to characterize the spectrum of nonintegrable quantum billiards [11, 12]. Recently, a mesoscopic disordered metal wire was studied with the ASD and was shown to contain information on the Anderson localization in such a wire [13].

Despite ASD being a non-self-averaging quantity, and not a generating function for any physical observable, one can extract from it valuable qualitative information. Namely, if there are extended states near the energy \( E \), one expects the ASD to be sensitive to the system’s size. Being more precise, in this case we expect it to depend on \( \omega^2 L^d \), where \( \eta \) is a constant, larger than the dimension \( d \). To the contrary, if all states are localized then the size dependence of the ASD can only be the trivial

\[
C = \exp(-\text{const} \tilde{V} \omega^2),
\]

where \( \tilde{V} = aL^d \) is the total volume of the sample, thus for a quasi-two-dimensional film \( a \) is its thickness, and \( d = 2 \). This can be proven using a simple model used in Refs. [14, 15]: If all states are localized with an average localization length \( L_c \), the system can be divided into \( (L/L_c)^d \) localization volumes. The levels in each localization volume obey Wigner-Dyson statistics [16, 17] and have an average level spacing of \( \Delta = 1/(\nu aL^d) \), where \( \nu \) is the average density of states. The levels in different localization volumes are almost uncorrelated. The levels of the total sample have a spacing \( \Delta = 1/(\nu aL^d) \). Taking all the \( N_c = \Delta_c/\Delta \) levels of the wire in an energy interval \( \Delta_c \), they should be uncorrelated. With a Gaussian distribution of width \( \Delta_c \), we get \( C(\omega) = \exp[-(\pi/2)\omega^2/(\Delta \Delta_c)] \) which agrees indeed with Eq. (2).

Here, we will use the ASD to address the question of localization in 2D and the integer quantum Hall transition. It allows us to answer the question if there are extended states, and where in the spectrum they are located.

As Hamiltonian we take the Anderson model,

\[
H = (p - q/cA)^2/(2m) + V(x),
\]

where \( q \) is the electron charge, \( c \) the velocity of light, and \( A \) the vector potential due to an external magnetic field \( \mathbf{B} \). \( V(x) \) is a Gaussian distributed random function

\[
< V(x) > = 0, < V(x)V(x') > = \frac{\Delta \delta(x-x')}{\tau} / 2\pi aL^2,
\]

which models randomly distributed, uncorrelated impurities in the conductor.
When the time reversal symmetry is fully broken by an external magnetic field, the ASD can be represented by a functional integral over two-component Grassmann fields. This representation is invariant under $U(2)$-transformations of these Grassmann fields. Averaging over impurities, the resulting interacting theory can be decoupled by introducing a Gaussian integral over $2 \times 2$ matrices. Integrating over the Grassmann fields, doing a saddle point approximation and integrating over longitudinal (massive) modes one obtains the ASD for a quasi-two-dimensional disordered film of size $L$ and thickness $a$, for $\omega \ll 1/\tau$ as a functional integral over the transverse modes, see Refs. [19], [25]:

$$C(\omega) = \int \prod_x dQ(x) \exp(-F[Q]),$$

where

$$F[Q] = \frac{1}{8} \sigma_{xx}(E) \int d^2 x Tr \nabla Q(x)^2 + \frac{\pi}{2} \frac{\omega}{\Delta} \int \frac{d^2 x}{2} Tr \sigma_3 Q(x) - \frac{1}{16} \sigma_{xy}(E) \int d^2 x Tr \nabla x Q, \nabla y Q.$$  (7)

Here, $\sigma_3$ is the diagonal Pauli matrix and $\sigma_{xx}(E)$ and $\sigma_{xy}(E)$ are the bare longitudinal and Hall conductance, in units of $e^2/h$, of the disordered conductor on energy $E$. The transverse fluctuations are restricted by $Q^2 = 1$, which reduces the manifold of $Q$ to $U(2)/(U(1) \times U(1)) = SU(2)/U(1) = S_2$, the space of points on a sphere. This is at $\omega = 0$ the free energy of the two-dimensional O(3)-NL sigma model with a topological term.

The representation of the matrix $Q$ can be chosen as,

$$Q = UQ_c \bar{U},$$  (8)

where

$$Q_c = \begin{pmatrix} \cos \theta & i \sin \theta \\ -i \sin \theta & -\cos \theta \end{pmatrix}, U = \begin{pmatrix} \exp(i\varphi/2) & 0 \\ 0 & \exp(-i\varphi/2) \end{pmatrix}. $$  (9)

Thus, we obtain:

$$\bar{C}(\omega) = \int \prod_x (d\theta d\varphi \sin \theta) \exp(-F[\theta, \varphi]),$$  (10)

where

$$F[n] = \frac{1}{4} \sigma_{xx}(E) \int d^2 x \sum_{n=1}^2 (\partial_\mu n)^2 - \frac{i}{2} \sigma_{xy}(E) \int d^2 x n(\partial_\mu n \times \partial_\nu n) + \frac{1}{16} \sigma_{xy}(E) \int d^2 x n_3.$$  (11)

which is written in terms of the unit vector on a sphere, $n = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$.

At $\omega = 0$ the model is integrable for $\sigma_{xy} = 0$ [20] and $\sigma_{xy} = 1/2$ [21]. The exact solution of the corresponding $(1+1)$-dimensional model is known at these points. In the first case the spectrum contains a spectral gap giving a finite correlation length in the Euclidean version of the model: $\xi_c \sim \lambda \sigma_{xx} \exp(\pi \sigma_{xx})$ [20][23]. This scale differs from the localization length for unitary disorder $L_c \sim \lambda \exp(\text{const} \sigma_{xx})$. Such discrepancy may be connected with ASD being a non-self-averaging quantity [23].

Let our system be defined on a square $L_x \times L_y$ with periodic boundary conditions. We can identify the $x$-direction with the Matsubara time and treat our sigma model as an Euclidean field theory at temperature $T = 1/L_x$. The ASD can be written as a partition function

$$C(\omega) = Tr \exp(-L_x H),$$  (12)

where $H$ is the Hamiltonian on a circle of circumference $L_y$. Thus, for $L_x, L_y \to \infty$ one gets

$$C(\omega)/C(0) = \exp(-L_x L_y F(\omega/\omega_0; \xi_c/L_y))$$  (13)

where $F$ is the $\omega$-dependent part of the ground state energy density of the NL sigma model. In the limit $L_y \gg \xi_c$ the $L$-dependence disappears from $F$. In the remaining form the system’s size enters trivially:

$$C(\omega)/C(0) = \exp(-L_x L_y F(\omega/\omega_0))$$  (14)

as one would expect for a bunch of localized levels. Moreover, at $\omega << \omega_0$ the $\omega_3$ perturbation is nonsingular and one can expand the free energy in $\omega$ to get $F(\omega/\omega_0) \sim (\omega/\omega_0)^2$ (compare with (3)).

For $\sigma_{xy} = 1/2$ the spectrum is gapless and at the critical point the model is equivalent to the level $k = 1$ SU(2) Wess-Zumino-Novikov-Witten model (WZNW1) perturbed by the marginal current-current interaction [21]. This model is in the same universality class as the spin-1/2 isotropic Heisenberg antiferromagnetic chain. Since the topological term does not change the classical equations of motion and therefore does not contribute to the perturbation theory expansion in powers of $\sigma_{xx}^{-1}$, the above result is non-perturbative. The WZNW1- model can be represented as a Gaussian model with a particular set of scaling dimensions:

$$S = \int d^2 x [\frac{1}{2} (\nabla \Phi)^2 + g J^a \tilde{J}^a]$$  (15)

where $J^a, \tilde{J}^a$ are currents satisfying the level $k = 1$ Kac-Moody algebra. The quantity $\xi_c$ serves as the ultraviolet cut-off for theory (15) so that this model is valid only for distances greater than $\xi_c$. The coupling $g$ is proportional to $\sigma_{xx} - \sigma^*$. At the critical point the operator $Tr \sigma_3 Q = 2 \cos \theta$ becomes $F \cos \sqrt{2} \pi \Phi$, where $F$ is a prefactor whose exact value is currently unknown. Hence the equivalent representation of the model (11) at distances greater than $\xi_c$ and $\omega/\omega_0 << 1$ is
\[
S = \frac{1}{2} \int d^2x [(\nabla \Phi)^2 + \pi i \frac{\omega}{\omega_0} F \cos \sqrt{2} \pi \Phi + g J^a \bar{J}^a]
\] (16)

At \( g > 0 \) the last term is marginally irrelevant and can be neglected. After that we get the sine-Gordon model which is exactly solvable though most of the results have been obtained for real \( \omega \). The reality of \( \omega \) gives rise to interesting new features (see [24]). The quantity \( \omega_0 \) is proportional to the inverse correlation length of the original sigma model squared \( \omega_0 \sim D/\xi^2 \). Since the new theory is defined on distances greater than \( \xi \), the correlation functions of the exponentials are normalized as

\[
< \exp(i\beta\Phi(x)) \exp(-i\beta\Phi(x)) > \sim \left( \frac{\xi}{|x|} \right)^{\beta^2/2}/\pi
\]

For the \( n_3 \)-operator at the critical point \( \beta = \sqrt{2\pi} \). From the poor man’s scaling one gets then:

\[
C(\omega)/C(0) = \exp[-f(\omega^2 L^3)]
\]

where \( f(x) \sim x \) at \( x << x_0 \), \( x_0 = \omega_0^2 c^3 \) (here we put \( L_x = L_y = L \)). This is different from the trivial size dependence Eq. (1) one obtains when all states are localized. Thus this ASD remains sensitive to the system’s size at any \( \omega << \omega_0 \), which implies the existence of extended states.

The renormalized coupling constant of the O(3) sigma model discussed in this work is not directly related to the physical, renormalized conductivity \( \sigma_{xx} \). Nevertheless one can use the phase diagram of this model to draw qualitative conclusions about physical conductivities. First of all, we have the striking fact that the beta function of the WZNW1- model is parabolic at the critical point:

\[
\beta(g) \sim g^2
\]

This means that there is some critical amount of disorder, beyond which the function \( C(\omega) \) becomes insensitive to the boundaries in large samples even at \( \sigma_{xy} = 1/2 \). It is the most likely that this corresponds to localization.

Then, the beta-function of the supersymmetric sigma model, which generates the physical observables [23], should have the same quadratic behaviour at the critical point. Such conjecture was first made by Polyakov [24]. Although we can not deduce the exact value of the critical conductivity \( \sigma^*(E) \) as a function of the scattering rate \( 1/\tau \), we do conclude the existence of a Kosterlitz-Thouless transition as a function of the disorder strength.

It is believed that deviations of \( \sigma_{xy} \) from \( 1/2 \) are relevant, leading to a finite correlation length \( \xi \sim (\sigma_{xy} - 1/2)^{-\nu} \) with \( \nu \approx 7/3 \) (see [3] for a review). In addition there is numerical [27] and experimental evidence [28], and it was conjectured on phenomenological grounds [29] that there is a semi-elliptic separatrix in the flow diagram which separates the quantum Hall state from the insulating state. According to [27], [28] the Hall resistivity is quantized to \( \rho_{xy} = 1 \) along this separatrix, yielding a semi circle defined by \( \sigma^2_{xx} + (\sigma_{xy} - 1/2)^2 = 1/4 \). Additional evidence for such a flow diagram comes from bifurcation theory: If the beta function would cross zero, there would be a repulsive node and a saddle point on the \( \sigma_{xy} = 1/2 \) line. Demanding that the beta function only touches zero, which is the rigorous result obtained from the O(3) NL sigma model, these two points merge and one has a fold- Hopf bifurcation with one separatrix. Therefore we conjecture the following set of RG equations for the real conductances in the vicinity of the critical point:

\[
\frac{\partial \epsilon_n}{\partial \ln L} = -\epsilon_n^2.
\]

where \( \epsilon_n \) is the amplitude of deviation of the vector \( (\sigma_{xx}, \sigma_{xy}) \) normal to the separatrix defined by \( \rho_{xy} = 1 \), or \( \sigma^2_{xx} + (\sigma_{xy} - 1/2)^2 = 1/4 \). And

\[
\frac{\partial \epsilon_i}{\partial \ln L} = \frac{1}{\nu} \epsilon_i
\]

where \( \epsilon_i \) denotes the change of that vector along the separatrix. This system of equations generates the scaling portrait shown in Fig. 1.

![FIG. 1. The two-parameter flow diagram in the integer quantum hall regime. The full lines are based on the exact result. The broken lines are conjectured](image)

Recently, it has been shown that the SUSY (supersymmetric) NL sigma model can be mapped at \( \sigma_{xy} = 1/2 \mod (1) \) on the Hamiltonian of superspin chains [31].

It was argued that the SUSY model with the topological term does not have a classical critical point in two dimensions and therefore the critical point may appear only when it is driven under renormalization to another theory. The results given above do suggest strongly that the WZNW1- model is an important part of this theory.

We note that the flow diagram, as conjectured by Khmelnitskii [30] seems inconsistent at small bare longitudinal conductivities with the one obtained above. A derivation of the former was given by Pruisken based on a dilute instanton approximation in the calculation of the beta- function of the N- replica NL sigma model [3].
The validity of that derivation is limited to large conductivities, however, as noticed before. In fact, for large bare conductivities both flow diagrams do agree with each other.

The approach presented here can also be used for understanding the influence of electron-electron interactions on the localization. In that case one studies the replica function, which is the autocorrelation function of the partition function of the interacting electrons. Considering only spinless fermions, the electron-electron interaction breaks the $U(2)$- invariance of the Grassmann fields. Accordingly, as first derived by A. Finkel'stein [22], decoupling the three interaction terms (one produced by averaging over impurities which couples both partition functions and two real interaction terms between the fermions in each partition function), and integrating out perturbatively the Hubbard-Stratonovich fields originating from the real fermion interactions, leads to the $O(3)$ NL sigma model action where the symmetry is broken by the presence of the interactions. This results in an additional term of the form,

$$\frac{U}{\Delta} \int \frac{dx^2}{L^2} Tr[Q(x) \sigma_3] \sigma_3^2.$$  

(20)

where $U$ is a spatially averaged measure of the electron-electron interaction. We note that the addition of this asymmetry can lead to the existence of transitions in two dimensions: The $O(2)$ NL sigma model has a Kosterlitz-Thouless transition, and the Ising model has a phase transition in 2D. Thus, we can understand that the presence of interaction may indeed result in the existence of a metal-insulator transition in two dimensions.

In summary we have reconsidered the Anderson localization in two dimensions and the integer quantum Hall transition from the perspective of the level statistics.

For weak magnetic fields our results indicate that all states are localized in the infinite volume limit.

In strong magnetic fields we have found that when the bare Hall conductance is half integer $\sigma_{xy} = 1/2, 3/2, \ldots$, there do exist critical states at the corresponding energy. The resulting flow diagram shows a Kosterlitz-Thouless transition at a critical disorder strength beyond which there is no extended state.

We note that Quantum Hall-Insulator- transitions have been recently observed experimentally [33].

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