Central limit theorem for Gibbsian U-statistics of facet processes

Jakub Večeřa
Charles University in Prague,
Department of Probability and Mathematical Statistics,
vecera@karlin.mff.cuni.cz

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Abstract

Special case of a Gibbsian facet process on a fixed window with a discrete orientation distribution and with increasing intensity of the underlying Poisson process is studied. All asymptotic moments for interaction U-statistics are calculated and using the method of moments the central limit theorem is derived.

1 Introduction

In the present paper we use methods developed in [6] to calculate all moments of Gibbsian U-statistics of facets in a bounded window in arbitrary Euclidean dimension. These moments are used to derive the central limit theorem for such statistics. Central limit theorems for U-statistics of Poisson processes were derived based on Malliavin calculus and the Stein method in [5].

Our calculation are based on the achievements in [1], where functionals of spatial point processes given by a density with respect to the Poisson process were investigated using the Fock space representation from [3]. This formula is applied to the product of a functional and the density and using a special class of functionals called U-statistics closed formulas for mixed moments of functionals are obtained. In processes with densities the key characteristic is the correlation function [2] of arbitrary order which is dual to kernel function of the density as a function of the Poisson process.

As in [6] we call facets some compact subsets of hyperplanes with a given shape, size and orientation. Natural geometrical characteristics of the union of the facets, based on Hausdorff measure of the intersections of pairs, triplets, etc., of facets form U-statistics. Building a parametric density from exponential family, the limitations for the space of parameters have to be given, so called submodels are investigated. In application of the moment formulas we are interested in the limit behaviour when the intensity of the reference Poisson process tends to infinity.
We restrict ourselves to the facet model with finitely many orientations corresponding to canonical vectors. In basic asymptotic properties of the studied $U$-statistics are derived. When the order of the submodel is not greater than the order of the observed $U$-statistic then asymptotically the mean value of the $U$-statistic vanishes. This leads to a degeneracy in the sense that some orientations are missing. On the other hand when the order of the submodel is greater than the order of the observed $U$-statistic then the limit of correlation function is finite and nonzero and under selected standardization $U$-statistic tends almost surely to its non-zero expectation. By changing the standardization, however, we achieve a finite non-zero asymptotic variance. In the present paper we simplify the calculation of moments so that we are able to calculate any asymptotic moment.

2 Central limit theorem

Let $Y = [0,b]^d \times \{2b\} \times \{e_1, \ldots, e_d\}$ be a space of facets (facets are $d-1$ dimensional cubes) with three parts: set of facet centres, possible sizes of facet and possible orientations of hyperplane containing facet, i.e. we consider only facet with fixed size and orientations corresponding only to elementary vectors. We denote $\eta_a$ finite Poisson process of facets with intensity function on $Y$ in form

$$\lambda(dx) = \lambda(d(z, r, \phi)) = \chi(z)dz\delta_{2b}(dr)\frac{1}{d} \otimes_{i=1}^{d} \delta_{e_i}(d\phi),$$

where we have fixed the facet size and uniform distribution of the facet orientation. We also define interaction $U$-statistics (using Hausdorff measure $H^{d-l}$ of order $d-l$)

$$G_l(x) = \sum_{(x_1, \ldots, x_l) \in \mathbf{x}_l^l} H^{d-l}(\cap_{i=1}^{l} x_i)$$

and the process $\mu_a$ with density $p(x) = c_a \exp(\nu G_d(x))$ with respect to $\eta_a$, $a \geq 1$ and $\nu$ is a real parameter $\nu < 0$. We will also use partition and diagram notation \cite{4}, $\Pi^m_l$ will be set of all partitions of $m$ rows each with $l$ elements, where each block of partition does not contain more than one element from each row and separate partitions will be denoted as $\sigma$.

Theorem 1

$$\frac{G_{d-k}(\mu^{(d)}_a) - \mathbb{E}G_{d-k}(\mu^{(d)}_a)}{a^{d-k-\frac{1}{2}}} \overset{D}{\to} Z, \ k = 1, \ldots, d - 1$$

(1)
as a tends to infinity, where $Z \sim N(0, \Sigma^2)$,

$$
\Sigma = (d-k) \frac{(d-1)^2}{d^{d-k-4}} \left( \frac{d-2}{d-k-1} \right)^2 I_k',
\tag{2}
$$

$$
I_k' = \int_{([0, b]^d \cap 2^{d-k}-1)} \mathbb{R}^{k}(\cap_{i=1}^{d-k} (s, e_i, 2b)) \mathbb{H}^{k}(\cap_{i=2}^{d-k} (s_i \cup d-k-1, e_i, 2b) \cap_{(s_1, e_1, 2b)}) \times \chi(s_1)ds_1, \ldots, \chi(s_{2(d-k)-1})ds_{2(d-k)-1}.
\tag{5}
$$

Lemma 1 It holds

$$
\int_{Y^{[d]}} \left( \otimes_{i=1}^{l} \mathbb{R}^{d} \right) \sigma (x_1, \ldots, x_{l} \sigma) \rho_{\sigma}(x_1, \ldots, x_{l} \sigma) \mu_{\sigma} \lambda(d(x_1, \ldots, x_{l} \sigma)) = \left( \frac{d-1}{d} \right)^{|\sigma|} \int_{Y^{[d]}} \left( \otimes_{i=1}^{l} \mathbb{R}^{d} \right) \sigma (x_1, \ldots, x_{l} \sigma) \lambda(d(x_1, \ldots, x_{l} \sigma)),
\tag{6}
$$

where $Y = [0, b]^d \times \{2b\} \times \{e_1, \ldots, e_{d-1}\}$ is space of facets with an orientation missing (we can select any orientation) and $\mathbb{H}^{k}(x_1, \ldots, x_{d-k}) = \mathbb{H}^{k}(\cap_{i=1}^{d-k} x_i)$.

Proof of Lemma

The correlation function depends only on the number of selected $l$ orientations among the facets $(x_1, \ldots, x_{l} \sigma)$, then correlation function is equal to $\frac{d-l}{d}$ and thus we can write

$$
\int_{Y^{[d]}} \left( \otimes_{i=1}^{l} \mathbb{R}^{d} \right) \sigma (x_1, \ldots, x_{l} \sigma) \rho_{\sigma}(x_1, \ldots, x_{l} \sigma) \mu_{\sigma} \lambda(d(x_1, \ldots, x_{l} \sigma)) = \frac{1}{d^{|\sigma|}} \sum_{l=0}^{d} \binom{d}{l} \frac{d-l}{d} f(\sigma, l) = \frac{1}{d^{|\sigma|}} \sum_{l=0}^{d-1} \binom{d-1}{l} f(\sigma, l),
\tag{7}
$$

where $d^{|\sigma|}$ is the number of all possible orientations, $\binom{d}{l}$ is the number of possible selections of used orientations and $f(\sigma, l)$ represents the number of the possibilities, how to allocate $l$ orientations, the facets center distributions, the Hausdorff measure and most importantly it does not depend on the number of possible orientations, which we can select, because it works only with the already selected $l$ orientations.

This means that we do not need to calculate correlation function and we can omit it from the formula if we also omit one possible orientation and adjust the number of possible orientations $\binom{d-1}{l} |\sigma|$. □

Proof of Main Theorem We will explore moments of the random variable

$$
\frac{G_{d-k} \mu_{\sigma} - EG_{d-k} \mu_{\sigma}}{a^{d-k-\frac{x}{2}}},
\tag{8}
$$
which can be expressed as, of [11]

\[
\begin{align*}
E \left( G_{d-k}(\mu^{(d)}_a) - EG_{d-k}(\mu^{(d)}_a) \right)^m &= \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \left( EG_{d-k}(\mu^{(d)}_a) \right)^{m-l} EG_{d-k}'(\mu^{(d)}_a) \\
\sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \left( \int_{Y_{d-k}} H^k(\cap_{i=1}^{d-k} x_i) \rho_{d-k}(x_1, \ldots, x_{d-k}, \mu^{(d)}_a) \lambda(d(x_1, \ldots, x_{d-k})) \right)^{m-l} \\
&= \sum_{\sigma \in \Pi_{d-k}'_0} \sum_{s \in \{2, \ldots, m-1\}} a^{s+2m-2d-k}(d-1)^{2m-2d-k} \int_{\hat{Y}_{d-k}} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \left( \frac{I_k}{d^{d-k}} \right)^{m-l} (d-1)^{l-d-k} (d-1) \int_{\hat{Y}_{|\sigma|}} \left( \sum_{i=1}^l \frac{W_k}{\sigma} \right) (x_1, \ldots, x_{|\sigma|}) \lambda(d(x_1, \ldots, x_{|\sigma|})),
\end{align*}
\]

where we used first moment calculation from [9], the Lemma and

\[
I_k = \int_{[0,b]^d} H^k(\cap_{i=1}^{d-k} (s_i, 2b, e_i)) \chi(s) \ldots \chi(s_{d-k}) ds_1 \ldots ds_{d-k}. \tag{4}
\]

We are interested only in terms with higher or equal power than \(m(d-k) - \frac{m}{2}\), because terms with lower power will tend to zero with increasing \(a\), i.e. partitions fulfilling condition \(|\sigma| \geq (d-k)l - \frac{m}{2}\). Also we do not have to examine odd moments, because there is not any summand with the power of \(a\) equal to power of divisor \((d-k - \frac{1}{2})m\), thus they can be only zero or infinite, therefore if we prove that all even moments tend to some finite value, then all odd moments are equal to zero.

Choose any partition \(\sigma_s \in \Pi_{d-k}'_0\), \(s \in \{2, \ldots, m-1\}\), \(|\sigma| \geq (d-k)l - \frac{m}{2}\), which has no pure singleton rows, i.e. each row is connected to another by some block of this partition. There are partitions \(\sigma_u \in \Pi_{d-k}'_0\), \(u > s\), which have only additional singleton rows compared to \(\sigma_s\) and it holds

\[
\begin{align*}
&\left( \frac{d-1}{d} \right)^{|\sigma_s|} \int_{\hat{Y}_{|\sigma_s|}} \left( \sum_{i=1}^l \frac{W_k}{\sigma_s} \right) (x_1, \ldots, x_{|\sigma_s|}) \lambda(d(x_1, \ldots, x_{|\sigma_s|})) = \\
&\left( \frac{d-1}{d} \right)^{|\sigma_s|} \int_{\hat{Y}_{|\sigma_s|}} \left( \sum_{i=1}^l \frac{W_k}{\sigma_s} \right) (x_1, \ldots, x_{|\sigma_s|}) \lambda(d(x_1, \ldots, x_{|\sigma_s|})) \times \\
&\left( \frac{d-1}{d} \right)^{(d-k)(u-s)} \left( \int_{\hat{Y}_{d-k}} H^k(\cap_{i=1}^{d-k} x_i) \lambda(d(x_1, \ldots, x_{d-k})) \right)^{u-s} = \\
&\left( \frac{d-1}{d} \right)^{|\sigma_s|} \int_{\hat{Y}_{|\sigma_s|}} \left( \sum_{i=1}^l \frac{W_k}{\sigma_s} \right) (x_1, \ldots, x_{|\sigma_s|}) \lambda(d(x_1, \ldots, x_{|\sigma_s|})) \times \\
&\left( \frac{I_k}{d^{d-k}} \right)^{u-s}.
\end{align*}
\]
because the singleton rows can be separated into another integral which can be calculated in the same way as the expectation. We can see that all partitions which have additional singleton rows compared to \( \sigma_s \) contain common term \( s \).

When we sum over all such partitions and omit the common term we get

\[
\sum_{l=s}^{m} \binom{l}{s} \binom{m}{l} (-1)^{m-l} \left( \frac{I_k}{d^{d-k}} \left( \frac{d-1}{d-k} \right) \right)^{m-l+s-l} = \\
\sum_{l=0}^{m-s} \binom{m-s}{l+s} \binom{m}{l+s} (-1)^{m-s-l} \left( \frac{I_k}{d^{d-k}} \left( \frac{d-1}{d-k} \right) \right)^{m-s} = \\
\binom{m}{s} \left( \frac{I_k}{d^{d-k}} \left( \frac{d-1}{d-k} \right) \right)^{m-s} \sum_{l=0}^{(m-s)l} (-1)^{m-s-l} = \\
\binom{m}{s} \left( \frac{I_k}{d^{d-k}} \left( \frac{d-1}{d-k} \right) \right)^{m-s} (1-1)^{m-s} = 0,
\]

where \( \binom{l}{s} \) is number of partitions are same as \( \sigma_s \) but with additional \( l-s \) singleton rows. In this way we can cancel out all relevant partitions with at least one pure singleton row or which does not belong to partition \( \Pi_{m-k} \). So there remain only partitions without singleton rows, belonging to \( \Pi_{m-k} \) and with number of blocks higher than \( (d-k-\frac{1}{2})m \), which leaves us only with partitions, which have each row connected exactly to one another row by one block of two elements.

There are \( (m-1)!!(d-k)^m \) such partitions, where \( (m-1)!! \) is number of possible divisions of rows into pairs and \( (d-k)^m \) is number of possibilities how to select common elements among the rows and for such partition \( \sigma \) it holds

\[
\left( \frac{d-1}{d} \right)^{(d-k-\frac{1}{2})m} \int_{\mathcal{C}(d-k-\frac{1}{2})m} \left( \otimes_{i=1}^{d-k} \mathcal{X}^{k}_{\sigma} \right) (x_1, \ldots, x_{d-k-\frac{1}{2}}) \lambda(d(x_1, \ldots, x_{d-k-\frac{1}{2}})) = \\
\left( \frac{d-1}{d} \right)^{(d-k-\frac{1}{2})m} \left( \frac{(d-1)^{1/2}}{(d-1)^{d-k-\frac{1}{2}}} \right) \left( \frac{d-2}{d-k-1} \right) I_k^m,
\]

where \( (d-1)^{\frac{1}{2}} \) is the number of possible selections of orientations of common elements which are among the pair of rows, \( (d-k)^m \) is the number of possible selection of the orientation of singleton elements, \( (d-1)^{(d-k-\frac{1}{2})m} \) is the number of all possible orientations.

This gives us

\[
E \left( \frac{G_{d-k}(\mu_{a}^{(d)}) - E G_{d-k}(\mu_{a}^{(d)})}{a^{d-k-\frac{1}{2}}} \right)^m = 0, \text{ odd } m \\
= (m-1)!! \Sigma^m, \text{ even } m,
\]

where \( \Sigma = (d-k)^{(d-1)^{\frac{1}{2}}} (d-k-1)^{d-k-\frac{1}{2}} I_k^m \) and thus we have asymptotically normal distribution.

\( \square \)
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