**Abstract.** We study the existence and properties of birationally equivalent models for elliptically fibered varieties. In particular these have either the structure of Mori fiber spaces or, assuming some standard conjectures, minimal models with a Zariski decomposition compatible with the elliptic fibration.

1. Introduction

The geometry of elliptic surfaces is well understood by the work of Kodaira. In particular, when the Kodaira dimension of an elliptic surface is non-negative, it’s minimal model is a birationally equivalent elliptic fibration. Furthermore, it follows from Kodaira’s canonical bundle formula for relatively minimal elliptic surfaces, that the canonical bundle of the surface is related to the pullback of the canonical divisor of the base curve and a $\mathbb{Q}$-divisor $\Lambda$ supported on the loci of the image of singular curves, the discriminant locus of the fibration. The first author showed that the fibration structure on an elliptic threefold is compatible with the minimal model algorithm and in addition, that a generalization of Kodaira’s formula for the canonical divisor hold on the (relative) minimal model [7,8]. An ingredient in the proof of [7] is to show the existence of an appropriate combination of the Zariski Decomposition Theorem for surfaces with a relative version of the minimal model program. A challenge in dimension 4 (and higher) is the existence of different definition(s) of Zariski decompositions and their relation with minimal models.

This paper addresses the case of elliptic fibrations of varieties of dimension $\geq 4$ where we find a Zariski Decomposition compatible with the fibration structure and the minimal model program.

**Theorem (Theorems 23, 25, 27, Corollary 28).** Let $\pi : Y \to T$ be an elliptic fibration, $\Lambda_T$ the $\mathbb{Q}$-divisor defined in Lemma 15.

1. If $K_Y + \Lambda_T$ is not pseudo-effective
   (a) There exists a birational equivalent fibration $\overline{X} \to \overline{B}$, $\overline{X}$ with $\mathbb{Q}$-factorial terminal singularities, $(\overline{B}, \overline{\Lambda})$ with klt singularities such that $K_{\overline{X}} \equiv \overline{\pi}^*(K_{\overline{B}} + \overline{\Lambda})$.
   (b) $Y$ is birationally a Mori fiber space.

2. If $K_Y + \Lambda_T$ is pseudo-effective (or $\kappa(Y) \geq 0$) and klt flips exist and terminate in dimension $n-1$
   (a) There exists a birational equivalent fibration $\tilde{X} \to \tilde{B}$, $\tilde{X}$ minimal, $(\tilde{B}, \tilde{\Lambda})$ with $\mathbb{Q}$-factorial klt singularities such that $K_{\tilde{X}} \equiv \tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Lambda})$.
   (b) $K_Y$ birationally admits a Fujita-Zariski decomposition compatible with the elliptic fibration structure.

**Theorem (Theorem 24).** Let $\pi : Y \to T$ be an elliptic fibration, $\Lambda_T$ the $\mathbb{Q}$-divisor defined in Lemma 16. $K_Y$ birationally admits a Fujita-Zariski decomposition if and only if $K_T + \Lambda_T$ birationally admits a Fujita-Zariski decomposition.

**Corollary (Corollary 26).** Let $\pi : Y \to T$ be an elliptic fibration, $\kappa(Y) \geq 0$ and $\dim Y \leq 5$. There exists a birational equivalent fibration $\tilde{X} \to \tilde{B}$, $\tilde{X}$ minimal, $(\tilde{B}, \tilde{\Lambda})$ with $\mathbb{Q}$-factorial klt singularities such that $K_{\tilde{X}} \equiv \tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Lambda})$ and $K_Y$ birationally admits a Fujita-Zariski decomposition compatible with the elliptic fibration structure.

**Corollary (Proposition 16, Theorem 23, Corollary 33).** Let $\pi : Y \to T$ be an elliptic fibration...
(1) If dim(Y) = 4 there exists a birational equivalent fibration \( \tilde{X} \to \tilde{B} \), \( \tilde{X} \) with \( \mathbb{Q} \)-factorial terminal singularities, \((\tilde{B}, \tilde{\Lambda})\) with klt singularities such that \( K_{\tilde{X}} \equiv \tilde{\pi}^*(K_B + \tilde{\Lambda}) \) and either \( Y \) is birationally a Mori fiber space or \( \tilde{X} \) is a good minimal model.

(2) If \( s(Y) = n - 1 \), there exists a birationally equivalent fibration \( \tilde{\pi} : \tilde{X} \to \tilde{B} \) such that \( \tilde{X} \) is a good minimal model, \( K_{\tilde{X}} \equiv Q \tilde{\pi}^*(K_B + \tilde{\Lambda}) \) and \((\tilde{B}, \tilde{\Lambda})\) has klt singularities.

In Section 2, we review standard definitions and relevant results about elliptic fibrations, minimal model theory and generalized Zariski decompositions. We also highlight the different generalizations of the Zariski Decomposition, their properties and their relationship with minimal model theory and with the structure of elliptic fibrations. In Section 3, we prove properties and results of birationally equivalent fibrations over the same base (relative models). Some of these results are applied in the proofs of the Theorems in Section 4. We also prove part of the Theorems stated above. In Section 5, we study the Zariski decomposition for elliptically fibered varieties of non-negative Kodaira dimension. We end with related results on abundance and other generalize Zariski decompositions for elliptic fibrations.

Unless otherwise specified, the varieties in this paper are assumed to be complex, projective and normal.

2. Notation-Results

An elliptic fibration is a morphism, \( \pi : X \to B \), whose general fibers are genus one curves with or without a marked point. If \( \pi \) has a section, namely if the general elliptic curve has a marked point, then \( X \) is the (smooth) resolution of \( W \), the Weierstrass model of the fibration, [21].

**Theorem 1** ([5][12][20][21]). Let \( \pi : X \to B \) an elliptic fibration between smooth varieties. Assume that the ramification divisor of the fibration has simple normal crossing and that \( \pi \) is equidimensional.

(1) The \( J \)-invariants of the fibers extends to a morphism \( J : B \to \mathbb{P}^1 \).

(2) \( \pi^*(K_{X/B}) \) is a line bundle

(3) \( 12\pi^*(K_{X/S}) = O_S(\sum 12a_kD_k) \otimes J^*O_{\mathbb{P}^1}(1) \) where \( a_k \) are the rational numbers corresponding to the type of singularities over the general point of \( D_k \).

(4) \( K_{X} \equiv Q \pi^*(K_B + \pi^*(\sum m_{i,j}^{-1}Y_i)) + E - G \)

(5) \( D_i \) are irreducible components of the ramification locus \( \pi^{-1}(Y_i) \) is a multiple fiber of multiplicity \( m_{i,j} \).

(6) \( \pi^*O_X((mE)) = O_B, \forall m \) integers. Furthermore \( E|_{\pi^{-1}(C)} \) is a union of a finite numbers of proper transforms of exceptional curves, for \( C \) a general curve.

(7) \( G \) is an effective \( \mathbb{Q} \)-divisor and codim \( \pi(G) \geq 2 \)

(8) \( \pi^* \left( \frac{m_{i,j}^{-1}}{m_{i,j}} \right) + E - G \) is effective.

**Definition 2.** \( \Delta \overset{def}{=} \pi^*(K_{X/S}) \equiv O_S(\sum a_kD_k) \otimes \sum \frac{1}{12} J^*O_{\mathbb{P}^1}(1) \)

\( \Lambda \overset{def}{=} \pi^*(K_{X/S}) + \sum \frac{m_{i,j}^{-1}}{m_{i,j}} Y_i \equiv O_S(\sum a_kD_k) \otimes \sum \frac{1}{12} J^*O_{\mathbb{P}^1}(1) \otimes O_S(\sum \frac{m_{i,j}^{-1}}{m_{i,j}} Y_i). \)

The pairs \((B, \Delta)\) and \((B, \Lambda)\) are klt. More generally, in the following let \( B \) a normal (projective) variety, \( D_i \) prime divisors with \( D \overset{def}{=} \sum a_iD_i \) and \( B + D \) \( \mathbb{Q} \)-Cartier. We say that \((B, D)\) is a log pair.

**Definition 3.**
(i) A log resolution of \((B, D)\) is a resolution \( f : \tilde{B} \to B \), such that the union of \( \sum a_i f_i^{-1}(D_i) \), the strict transform of \( D \), and the exceptional locus of \( f \) are supported on divisors with simple normal crossings.

(ii) We then write \( K_{\tilde{B}} + \sum a_i f_i^{-1}(D_i) = f^*(K_B + D) + \sum a(E_j, X, D)E_j \), where \( a(E_j, X, D) \) are the discrepancies.

(iii) The pair \((B, D)\) is terminal if for any (equivalently for every) log resolution \( f \), \( a(E_j, X, D) > 0, \forall j \).

klt: if for any (equivalently for every) log resolution \( f \), \( a(E_j, X, D) > -1, \forall j \).

lc: (log canonical) if for any (equivalently for every) log resolution \( f \), \( a(E_j, X, D) \geq -1, \forall j \).

dlt: (divisorially log terminal) if there is a log resolution \( f \) such that \( a(E_j, X, D) > -1 \) for every exceptional divisors \( E_j \).

plt: (purely log terminal) if for any log resolution \( f \), \( a(E_j, X, D) > -1 \), for every coefficient of an exceptional divisor \( E_j \).
In the following \((X, D)\) is always a lc pair.

**Definition 4.** Minimal models, log minimal models etc.

Min. Model.: \(\hat{X}\) is a minimal model if \((\hat{X}, 0)\) has terminal singularities, \(K_{\hat{X}}\) is nef and \(K_{\hat{X}}\) is \(\mathbb{Q}\)-factorial.

Neg. Contr.: \(\psi: B \rightarrow \hat{B}\) is a \((K_B + D)\)-negative contraction if 
\[\psi^{-1}\] does not contract any divisor and there exists a resolution \(\hat{B}\)

\[
\begin{array}{c}
\hat{B}
\end{array}
\begin{array}{c}
\text{g}
\end{array}
\begin{array}{c}
(B, D)
\end{array}
\begin{array}{c}
\text{h}
\end{array}
\begin{array}{c}
\psi
\end{array}
\begin{array}{c}
(B, \psi_*(D))
\end{array}
\]

such that \(g^*(K_{\hat{B}} + D) - h^*(K_{\hat{B}} + \hat{D}) = \sum a_j E_j\), \(a_j > 0\) and \(E_j\) exceptional for \(h\).

LMM-A: \((B, \psi_*(D))\) is a log minimal model for \((B, D)\) if 
\(\psi\) is a \((K_B + D)\)-negative contraction and \((K_{\hat{B}} + \psi_*(D))\) is nef.

LBM: \((\hat{B}, \hat{D})\) is a log birational model of \((B, D)\) if 
\(\psi: B \rightarrow \hat{B}\) is birational and \(\hat{D} \overset{\text{def}}{=} \psi_*(D) + E\), where \(E\) is the reduced exceptional divisor of \(\psi^{-1}\).

LMM-B [2]: A log birational model \((\hat{B}, \hat{D})\) is a log minimal model for \((B, D)\) if 
\((\hat{B}, \hat{D})\) is \(\mathbb{Q}\)-factorial dlt, \((K_{\hat{B}} + \hat{D})\) is nef and \(a(E_j, B, D) < a(E_j, \hat{B}, \hat{D})\), for \(E_j\) divisor in \(B\), exceptional for \(\psi\).

GOOD A log minimal model \((\hat{B}, \psi_*(D))\) is good if \(K_{\hat{B}} + \psi_*(D)\) is semi-ample.

**Remark 5.** The definition LMM-B allows for \(\psi^{-1}\) exceptional divisors. Furthermore, the Negativity Lemma implies that a log minimal model according to A is a log minimal model in the sense of B; the two definitions are equivalent for plt pairs [2].

**Definition 6 (Zariski Decompositions, [2] - [5] - [13,19] - [22]).** Let \(X\) be a normal, projective variety with a proper map \(\pi: X \rightarrow Z\) and \(D\) a \(\mathbb{R}\)-divisor on \(X\). We have that \(D = P + N\) is called:

W: A Weak Zariski decomposition over \(Z\), if \(P\) is \(\pi\)-nef and \(N\) is effective.

FZ-A: A Fujita-Zariski decomposition over \(Z\), if it is a Weak Zariski decomposition and we have that for every projective birational morphism \(f: W \rightarrow X\), where \(W\) is normal, and \(f^* D = P' + N'\) with \(P'\) nef over \(Z\), then we have \(P' \leq f^* P\).

CKM: A CKM-Zariski decomposition over \(Z\), if it is a Weak Zariski decomposition and we have that \(\pi_* \mathcal{O}_X(mP) \cong \pi_* \mathcal{O}_X(mD)\) is an isomorphism for all \(m \in \mathbb{N}\).

If we have that \(Z = \text{Spec}(\mathbb{C})\) then we will refer to \(D = P + N\) as simply the (Weak, Fujita, CKM) Zariski decomposition. Additionally, for the case where \(Z = \text{Spec}(\mathbb{C})\) and \(X\) smooth, we have the following original definition of the Fujita-Zariski decomposition.

Num. Fixed: Let \(E\) be an effective \(\mathbb{Q}\)-divisor and \(L\) be a \(\mathbb{Q}\)-divisor on \(X\). \(E\) clutches \(L\) if, for any effective \(\mathbb{Q}\)-divisor \(F\) where \(L - F\) is nef, we have that \(E - F\) is effective. \(E\) is numerically fixed by \(L\) if for any birational morphism \(\pi: W \rightarrow X\) we have that \(\pi^* E\) clutches \(\pi^* L\).

FZ-B: \(D = P + N\) is a Fujita-Zariski decomposition if \(N\) is numerically fixed by \(D\).

Additionally, if we also assume that \(D\) is pseudo-effective we have the sectional decomposition (sometimes called the Nakayama-Zariski decomposition).

NZ: Let \(A\) be a fixed ample divisor on \(X\). Given a prime divisor \(\Gamma\) on \(X\), define
\[\sigma_\Gamma(D) = \min\{\text{mul}_\Gamma(D')| D' \geq 0, D' \sim_\mathbb{Q} D + cA \text{ for some } c > 0\}\]

This definition is independent of the choice of \(A\). It was also shown in [22] that for only finitely many \(\Gamma\) is \(\sigma_\Gamma(D) > 0\). This allows us to define the following decomposition.

Let \(N_\sigma(D) = \sum \sigma_\Gamma(D) \Gamma\) and \(P_\sigma(D) = D - N_\sigma(D)\), then we call \(D = P_\sigma(D) + N_\sigma(D)\) the sectional decomposition. If we have also that \(P_\sigma(D)\) is nef then we refer to this as the Nakayama-Zariski decomposition of \(D\).
$D$ birationally admits a (Weak, Fujita, CKM, Nakayama) Zariski decomposition over $Z$ if there exists some resolution $f : Y \rightarrow X$ such that $f^*(D)$ has a (Weak, Fujita, CKM, Nakayama) Zariski decomposition over $Z$.

Remark 7. There is a nesting of the above generalized Zariski decompositions as listed:

1. A Nakayama-Zariski decomposition (a sectional decomposition with nef positive part) is a Fujita-Zariski decomposition.
2. A Fujita-Zariski decomposition is a CKM-Zariski decomposition.
3. These are all Weak Zariski decompositions.
4. There are CKM-Zariski decompositions that are not Fujita-Zariski decompositions.
5. It is not known if there are Fujita-Zariski decompositions that are not Nakayama-Zariski decompositions.

Below we list some technical properties, relations and similarities of the different versions of the generalized Zariski decompositions.

**Proposition 8** ([5 Cor. 1.9; Lemma 1.22]). Let $X$ be a smooth projective variety with $E$ an effective $\mathbb{Q}$-divisor that is numerically fixed by a Cartier divisor $L$.

1. Let $F$ be the smallest Cartier divisor such that $F - E$ is effective, then we have the following isomorphism of graded rings:

$$
\bigoplus_{t \geq 0} H^0(X, tL) = \bigoplus_{t \geq 0} H^0(X, tL - tF)
$$

2. $L - E$ admits a Fujita-Zariski decomposition if and only if $L$ admits a Fujita-Zariski decomposition. Additionally, the nef parts of the decompositions are the same.

**Proposition 9** ([5 Prop. 1.10] - [6 Lemma 2.16]). Let $f : M \rightarrow S$ be a surjective morphism of manifolds with connected fibers. Let $X$ be a divisor on $M$ such that $\dim f(X) < \dim S$. Suppose that for every irreducible component $Z$ of $f(X)$ with $\dim Z = \dim S - 1$, there is a prime divisor $D$ on $M$ such that $f(D) = Z$ and $D \not\subset \text{Supp}(X)$.

1. $X$ is numerically fixed by $X + f^*L$ for any $\mathbb{Q}$-Cartier divisor on $S$.
2. For any pseudoeffective $\mathbb{R}$-divisor $L$ on $S$, $D \leq N_\sigma(f^*L + D)$ and $P_\sigma(f^*L + D) = P_\sigma(f^*L)$.

**Proposition 10** ([5 Prop. 1.24]). Let $f : M \rightarrow S$ be a surjective morphism of manifolds with $L$, a $\mathbb{Q}$-Cartier divisor, on $S$ and $R$ an effective $\mathbb{Q}$-divisor on $M$ such that $\dim f(R) \leq \dim S - 2$. Then $f^*L + R$ birationally admits a Fujita-Zariski decomposition if and only if $L$ birationally admits a Fujita-Zariski decomposition.

**Proposition 11** ([22 Prop. V.1.14]). Let $D$ be a pseudo effective $\mathbb{R}$-divisor, then

- $N_\sigma(D) = 0$ if and only if $D$ is movable.
- If $D - E$ is movable for an effective divisor $E$, then $N_\sigma(D) \leq E$.

**Proposition 12.** When $Z = \text{Spec}(\mathbb{C})$ and $X$ is smooth, the two definitions of the Fujita-Zariski decomposition are equivalent.

Proof. Let $D = P + N$ be a Fujita-Zariski decomposition in the sense of $FZB$. We will show that this implies the properties of $FZ-A$. Let $f : X' \rightarrow X$ be a birational morphism with $f^*(D) = P' + N'$ where $P'$ is nef and $N'$ is an effective $\mathbb{Q}$-Cartier divisor. We have that $N$ is numerically fixed by $D$ and so $f^*(N)$ clutches $f^*(D)$. As $f^*(D) - N' = P'$ is nef, we have that $N' - f^*(N)$ is effective. But we know that $N' = f^*(D) - P'$ and $f^*(N) = f^*(D) - f^*(P)$. So by replacing and simplifying we have that $f^*(P) - P'$ is effective.

Let $D = P + N$ be a Fujita-Zariski decomposition in the sense of $FZ-A$ and we will show that $N$ is numerically fixed by $D$, so given a birational morphism $f : X' \rightarrow X$ we will show that $f^*(N)$ clutches $f^*(D)$. Thus given an effective $\mathbb{Q}$-divisor $N'$ such that $P' := f^*(D) - N'$ is nef, we want to show that $N' - f^*(N)$ is effective. We can assume that $X'$ is normal, otherwise we can resolve singularities to get $\pi : W \rightarrow X'$, where we have that showing $\pi^*(N' - f^*(N))$ is effective is sufficient to show that $N' - f^*(N)$ is effective on $X'$. Thus without loss of generalities, we can assume $X'$ is normal. Since $P + N$ is a Fujita-Zariski decomposition in the sense of $FZ-A$, we have that $f^*(P) - P'$ is effective. Replacing with $f^*(P) = f^*(D) - f^*(N)$ and $P' = f^*(D) - N'$, we get that $N' - f^*(N)$ is effective. This completes the argument that shows the two definitions are equivalent.
Remark 13.  

(1) The original definition of the Fujita-Zariski decomposition in [5] is equivalent to our definition of a divisor birationally admitting a Fujita-Zariski decomposition, which will be accounted for in later arguments.

(2) If $K_B + \Delta$ has a log minimal model, then it birationally has a Fujita (also CKM and Weak) Zariski decomposition; Birkar shows this explicitly as part of the argument of [2, Thm 1.5].

(3) A Fujita-Zariski decomposition and a Nakayama-Zariski decomposition of a divisor is unique. A CKM-Zariski and Weak Zariski decomposition of a divisor need not be unique.

(4) Each of the above generalized Zariski decomposition for the canonical divisor has a different role in birational geometry and their relations to minimal models. The Nakayama-Zariski decomposition is more attune to work with abundance and good minimal models as seen in [6]. The Fujita-Zariski decomposition aligns with minimal models as seen below, and the CKM-Zariski decomposition is more focused on the canonical ring and as a result on the canonical model.

(5) Recent work in [9, 11] and [2], shows that the Weak Zariski decomposition is sufficient to ensure the existence of minimal models.

Theorem 14 ( [10,23,24] - [11,16] - [3] - [1]). We have the following results in the theory of minimal models.

(1) Flips for klt pairs exists in all dimension.

(2) Any sequence of klt flips terminate in dimension 3.

(3) A klt pair in dimension up to 4 either it admits minimal model or it is birational to a Mori Fiber space.

(4) The abundance conjecture holds for klt pairs of dimension $\leq 3$. Thus klt pairs of dimension up to 3 admit a good minimal model or are birational to a Mori Fiber space.

(5) General type klt pairs admit a good minimal model.

3. Relative minimal models, the canonical bundle formula

We recall the following application of Hironaka’s flattening theorem:

Lemma 15. Let $\Pi : Y \to T$ be an elliptic fibration between varieties. Then there exist birational equivalent fibrations

$$
\begin{array}{cccc}
Y & \leftarrow & X_0 & \leftarrow & X_1 & \leftarrow & X \\
\downarrow \Pi & & \downarrow \pi_0 & & \downarrow \pi_1 & & \downarrow \pi \\
T & \leftarrow & B_0 & \leftarrow & B_1 & \leftarrow & B
\end{array}
$$

where $X_0$, $X$, $B_0$, and $B_1$ are smooth, $\pi_1$ is flat and $\pi : X \to B$ satisfy the hypothesis of Theorem 7. Without loss of generality we can also assume $B_1 = B$.

If $\psi_0 : B_0 \to B_0$ and $\psi : B \to T$ denotes the composition of the birational morphism in the bottom row, set:

$\Lambda_{B_0} = \psi_0^*(\Lambda)$ and $\Lambda_T = \psi^*(\Lambda)$.

Proposition 16. Let $X_0 \to B_0$ be an elliptic fibration between smooth varieties. Let $X_1$, $X$ and $B$ be as in the above Lemma 15. Then $\kappa(X_0) = \kappa(K_{B_0} + \Lambda_0)$.

Proof. The statement holds for $\dim X_0 = 2$. When $\dim X_0 = 3$, from the proof of [7] Theorem 3.2] we can deduce that $\kappa(X) = \kappa(X, K_X + G) = \kappa(X, K_B + \Lambda) + E = \kappa(X, K_B + \Lambda)$, as in [7, Proposition 1.3]. These arguments can be extended also to $\dim X_0 \geq 4$; we leave the details to the reader.

3.1. Relative minimal models.

Example 3.1.

(1) Let $\pi : X \to B$ be an elliptic fibration, $X$ with terminal singularities and let $D$ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ on $X$ which is non $\pi$-nef. Then the log relative minimal model Theorems of [10] and [18] for the klt pair $(X, \varepsilon D)$ give a birational map $\mu : X \dashrightarrow X_r$, which contracts $D$, but $X_r$ has klt terminal singularities, not necessarily terminal singularities.

This is precisely what happens in Example 1.1 of [8].
(2) In addition, Corollary \[19\] shows that in \([8\) Example 1.1] no birationally equivalent elliptic fibration \(\pi_r : X_r \to B, X\) with terminal singularities, such that \(K_{X_r} \equiv_{\mathbb{Q}} \pi_r^*(K_B + \Lambda)\) can exist because \(K_X \equiv \pi^*(K_B + \Lambda) - D\), with \(D\) effective.

In particular the relative log minimal model results of \([10\) and \([18\) applied to the klt pair \((X, \epsilon F)\) do not give the result of the following Theorem \[17\]. See also \([19\).

**Theorem 17.** Let \(\pi : X \to B\) be an elliptic fibration between \(\mathbb{Q}\)-factorial varieties. Assume that \(\text{codim} \text{Sing}(X) \geq 3\) and that \(K_X \equiv_{\mathbb{Q}} \pi^*(L) + F\), where \(F\) is an effective \(\mathbb{Q}\)-divisor such that no irreducible component \(F_j\) of \(F\) is \(\pi^\ast \equiv\) nef.

(i) \(K_X\) is \(\pi^\ast \equiv\) nef.

(ii) If in addition \(X\) has terminal singularities there is a relative good minimal model for \(X\) over \(B\), that is, there exists a birational equivalent elliptic fibration \(\pi_r : X_r \to B\), such that \(X_r\) has \(\mathbb{Q}\)-factorial terminal singularities, \(K_{X_r} \equiv \pi_r^\ast \equiv_{\mathbb{Q}} \pi_r^\ast (\Gamma_j)\), for some \(\Gamma_j\). Then

- (i) \(K_X\) is \(\pi^\ast \equiv\) nef.
- (ii) \(K_X\) is \(\pi^\ast \equiv\) nef.
- (iii) There exists a \(\mathbb{Q}\)-divisor \(\Lambda\) such that \(L = K_B + \Lambda_r, \Lambda_r\) defined as in Lemma \[17\] and \((B, \Lambda_r)\) has klt singularities.

**Proof.** Note that the locus of terminal singularities has codimension \(\geq 3\).

(i) Let \(F_j\) be an irreducible component of \(F\). If \(\text{codim} \pi(F_j) \geq 2\), then there is an effective curve \(\gamma\) such that \(F_j \cdot \gamma < 0\). If \(\text{codim} \pi(F_j) = 1\), let \(C\) be a general curve in \(B, p\) a point in \(C\) and in the support of \(\pi(F_j)\). Consider the elliptic surface \(S \overset{\text{def}}{=} \pi^\ast \equiv C, S \to C\) and conclude that there is an effective (exceptional) curve \(\gamma\) in the fiber over \(p\) such that \(K_X \cdot \gamma = F_j \cdot \gamma < 0\).

(ii) We only need to prove that \(K_X \equiv_{\mathbb{Q}} \pi^\ast (L)\). In fact, the first part of the statement follows, from the proof of Theorem 2.12 in \([10\), which generalizes \([18\). In fact the hypothesis (2) of that Theorem is in fact satisfied by \([9\). As in the case of threefolds, we can assume without loss of generality that there is a birational morphism \(\mu : X \to X_r\), with \(K_X \equiv \mu^\ast (K_{X_r}) + \sum a_i E_i, E_i \mu\)-exceptional and \(a_i > 0\). We then have \(\mu^\ast (K_{X_r} - \pi_r^\ast (L)) = F - \sum a_i E_i\). If \(F\) is not \(\mu\)-exceptional, we can take \(\gamma\) as in part (i) and conclude by contradiction, since \(0 \leq \mu^\ast (K_{X_r} - \pi_r^\ast (K_B + L)) \cdot \gamma = (F - \sum a_i E_i) \cdot \gamma < 0\).

(iii) It follows from Lemma \[15\] and \([21\ Theorem 0.4\].

**Lemma 18** (Lemma 1.5 \([8\)). Let \(\pi_i : X_i \to B, i = 1, 2\) be birationally equivalent elliptic fibrations, \(X_i\) with terminal singularities and

\[K_{X_i} \equiv \pi_i^\ast (K_B + \Lambda) + F_i, F_i \mathbb{Q}\text{-Cartier} \mathbb{Q}\text{-divisor}.\]

Then \(F_1\) is \(\mathbb{Q}\)-effective if and only if \(F_2\) is.

**Corollary 19** (Corollary 1.4 \([8\)). Let \(\pi : X \to B\) be an elliptic fibrations, \(X\) with terminal singularities and \(K_X \equiv \pi^\ast (K_B + \Lambda) + F, F\) for some \(\mathbb{Q}\text{-Cartier} \mathbb{Q}\text{-divisor} F\). Then \(F\) is \(\mathbb{Q}\)-effective if and only if there exists a birational equivalent elliptic fibration \(\pi_r : X_r \to B, X\) with terminal singularities, such that \(K_{X_r} \equiv_{\mathbb{Q}} \pi_r^\ast (K_B + \Lambda)\).

**Corollary 20.** Let \(\pi : X_0 \to B_0\) be an elliptic fibrations between manifolds such that the ramification locus has simple normal crossing as in Theorem \[13\]. Assume that either \(\pi\) is equidimensional or there are no multiple fibers. Then there exists a good minimal model \(X_r\) of \(X\) over \(B\), that is a birational map \(\mu : X \to X_r\) and a morphism \(\pi_r : X_r \to B\) such that diagram commutes, \(K_{X_r}\) is \(\pi_r\)-nef, \(K_X \equiv_{\mathbb{Q}} \pi_r^\ast (K_B + \Lambda)\) and \(X_r\) has terminal singularities.

**Proof.** In fact if there are no multiple fiber \(F = E - G\) is effective; if \(\pi\) is equidimensional then \(F = E\) is effective.

3.2. Birational equivalent elliptic fibrations, minimality, Mori fiber spaces and the canonical bundle formula.
Lemma 21. Let $\pi: X \to B$ be an elliptic fibration between manifolds. Assume that the ramification divisor of the fibration has simple normal crossing as in Theorem 7. Then there is a birationally equivalent elliptic fibration $\pi_r: X_r \to B_r$ such that $X_r$ has terminal singularities, $(B_r, \Lambda_r)$ has klt singularities and $K_{X_r} \equiv \pi_r^*(K_{B_r} + \Lambda_r)$.

Proof. There is a relative good minimal model $\pi_r: X_r \to B_r$ such that $X_r$ has terminal singularities, $(B_r, \Lambda_r)$ has klt singularities and $K_{X_r} \equiv \pi_r^*(K_{B_r} + \Lambda_r)$. By the proof of Theorem 2.12 in [10] and [9], [10] generalizes [18]. In particular there exist a birational morphism $\phi: B_r \to B$, a birationally equivalent fibration $\pi_r: X_r \to B_r$ and $L_r$ such that $K_{X_r} \equiv \pi_r^*(L)$ and $\pi_r = \phi \cdot \pi_r$. $K_{X_r} = \pi^*(K_{B_r} + \Lambda)$ and $(B_r, \Lambda)$ is klt.

Corollary 22. Let $\pi: X \to B$ be an elliptic fibration between manifolds such that the ramification locus has simple normal crossing as in Theorem 7.

1. If $K_B + \Lambda$ is not pseudo effective, there exists a birational equivalent fibration $\tilde{X} \to \tilde{B}$, $\tilde{X}$ with terminal singularities, $(\tilde{B}, \tilde{\Lambda})$ with klt singularities such that $K_{\tilde{X}} \equiv \tilde{\pi}^*(K_B + \tilde{\Lambda})$. In addition $\tilde{X}$ is birationally a Mori fiber space.

2. If $K_B + \Lambda$ is pseudo effective and klt flips exist and terminate in dimension $n - 1$, then there exists a birational equivalent fibration $\tilde{X} \to \tilde{B}$, $\tilde{X}$ minimal, with terminal singularities, $(\tilde{B}, \tilde{\Lambda})$ with klt singularities such that $K_{\tilde{X}} \equiv \tilde{\pi}^*(K_B + \tilde{\Lambda})$.

Proof. Let $\pi_r: X_r \to B_r$, a birationally equivalent elliptic fibration as in Lemma 21. If $K_B + \Lambda$ is not pseudo effective, then by [9] Corollary 1.3.2 $\tilde{B}$ is birationally a Mori fiber space, that is there exist a is a $(K_B + \Lambda)$-negative birational contraction $\psi: B \dasharrow \tilde{B}$ and a morphism $f: \tilde{B} \to Z$ with connected fibers such that $\dim(Z) < \dim(B)$ and $(\tilde{K}_{\tilde{B}} + \psi_{\ast} \Lambda_f)$ is anti-ample for a general fiber $F$ of $f$. We then conclude by applying Corollary 2.13 in [10] to every birational contraction and flip in $\psi$. This shows (1). Part (2) follows similarly.

Theorem 23. Let $\pi: Y \to T$ be an elliptic fibration

1. If $\kappa(Y) \geq 0$ and klt flips exist and terminate in dimension $n - 1$, then there exists a birational equivalent fibration $\tilde{X} \to \tilde{B}$, $\tilde{X}$ minimal, with terminal singularities, $(\tilde{B}, \tilde{\Lambda})$ with klt singularities such that $K_{\tilde{X}} \equiv \tilde{\pi}^*(K_B + \tilde{\Lambda})$.

2. Let be $\Lambda_T$ the $\mathbb{Q}$-divisor defined in Lemma 17. If $K_Y + \Lambda_T$ is not pseudo-effective there exists a birational equivalent fibration $\tilde{X} \to \tilde{B}$, $\tilde{X}$ with terminal singularities, $(\tilde{B}, \tilde{\Lambda})$ with klt singularities such that $K_{\tilde{X}} \equiv \tilde{\pi}^*(K_B + \tilde{\Lambda})$. In addition $\tilde{X}$ is birationally a Mori fiber space.

Proof. It follows from Corollary 22 and Proposition 10.

4. Non negative Kodaira dimension, minimal models, Zariski decomposition and the canonical bundle formula

We show how standard conjectures in the theory of minimal models imply a birational Fujita-Zariski decomposition for the canonical divisor for elliptic fibrations with non-negative Kodaira dimension. We combine properties of two definitions of the Fujita-Zariski decomposition. From [2], we use the relationship between Fujita-Zariski decomposition and minimal model theory. From the definition as in Fujita’s paper, [5], we use the relationship between Fujita-Zariski decomposition and the properties of numerically fixed divisors. We show a relationship between the total space and base space of an elliptic fibration through a birational Fujita-Zariski decomposition via the canonical bundle formula in Theorem 1.

4.1. Generalized Zariski Decompositions for Elliptic Fibrations.

Theorem 24. Given an elliptic fibration $X_0 \to B_0$, there exist a birationally equivalent fibration $X \to B$ and a $\mathbb{Q}$-divisor $\Lambda$ on $B$ such that $K_X$ birationally admits a Fujita-Zariski decomposition if and only if $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition where $X$ and $(B, \Lambda)$ are as in Lemma 17.
Proof. Without loss of generality we can assume that $X_0$ and $B_0$ are smooth, with ramification divisor $\Lambda_0$ having simple normal crossing. As in Lemma 15 we have:

$$
\begin{array}{c}
\Xa{X_0} \xleftarrow{\nu} \Xa{X_1} \xleftarrow{\pi_0} X \\
\Xa{B_0} \xleftarrow{F} \Xa{B_1} \xleftarrow{\nu} B
\end{array}
$$

where all the horizontal maps are birational morphisms, $X_1$ is the resolution of the flattening of $\pi_0$ and and $\pi : X \to B$ is as in Theorem 17. We have $K_X = \pi^*(K_B + \Lambda) + E - G$ where $(B, \Lambda)$ is a klt pair of dimension $n - 1$.

Assume that $K_X$ birationally admits a Fujita-Zariski decomposition. Without loss of generality, we assume that $K_X$ admits a Fujita-Zariski decomposition, in the sense of FZ-A, equivalently, FZ-B, as in Definition 6 and Remark 13. Then we have

$$
P + N = K_X = \pi^*(K_B + \Lambda) + E - G$$

with $P, N$ as in Definition 5. We will show that $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition.

We have $G$ is a $\nu$-exceptional effective divisor, since $X_1 \to B_1$ is equidimensional, as it is a flat morphism over a smooth base, and codim$(\pi(G)) \geq 2$. Furthermore $K_X = \nu^*(K_{X_0}) + F$, with $F$ an effective $\nu$-exceptional divisor, since $X$ and $X_0$ are smooth. Then $F$ is numerically fixed by $K_X$ and $F + G$ is numerically fixed by $\nu^*(K_{X_0}) + F + G = K_X + G$, [5, Prop. 1.10]. Since $F$ is numerically fixed by $K_X$ and $K_X = P + N$ is a Fujita-Zariski decomposition then $K_X = \nu^*(K_{X_0})$ has a Fujita-Zariski decomposition by Lemma 8. Similarly, since $F + G$ is numerically fixed by $\nu^*(K_{X_0}) + F + G = K_X + G$, thus $K_X + G$ admits a Fujita-Zariski decomposition. In both cases $P$ is the nef part of the decomposition. It follows that

$$
\pi^*(K_B + \Lambda) + E = P + N + G
$$

is a Fujita-Zariski decomposition. Since $E$ is also numerically fixed by $\pi^*(K_B + \Lambda) + E$ (Theorem 1 and [5, Prop 1.10]); then $\pi^*(K_B + \Lambda)$ also admits a Fujita-Zariski decomposition (Lemma 8). Then $K_B + \Lambda$ birationally also admits a Fujita-Zariski decomposition by Proposition 10.

Assume now that $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition. Without loss of generality we assume that $K_B + \Lambda = P_\Lambda + N_\Lambda$ is a Fujita-Zariski decomposition. We have $\pi^*(K_B + \Lambda) = \pi^*(P_\Lambda) + h^*(N_\Lambda)$ is then a Fujita-Zariski decomposition (Proposition 11), with $\pi^*(P_\Lambda)$ the nef portion of the decomposition. The canonical bundle formula $K_X = \pi^*(K_B + \Lambda) + E - G$ and [5, Prop 1.10] imply that $E$ is numerically fixed by $\pi^*(K_B + \Lambda) + E$. We have then a Fujita-Zariski decomposition for $K_X + G = \pi^*(K_B + \Lambda) + E = \pi^*(P_\Lambda) + \pi^*(N_\Lambda) + E$ with nef part $\pi^*(P_\Lambda)$. Similarly, with Lemma 8 applied to $G$, we deduce that $K_X$ admits a Fujita-Zariski decomposition of the form

$$
K_X = \pi^*(P_\Lambda) + \pi^*(N_\Lambda) + E - G.
$$

Here $\pi^*(N) + E - G$ is effective and $\pi^*(P_\Lambda)$ is nef.

\[\text{\begin{center} \hline \end{center}}\]

**Theorem 25.** Let $\pi_0 : X_0 \to B_0$ be an elliptic fibration, $\dim X_0 = n$ and $\kappa(X_0) \geq 0$. Assume the existence of minimal models for klt pairs of non negative Kodaira dimension in dimension $n - 1$. There exist birationally equivalent fibrations and birational morphisms $\phi_B$ and $\phi_B$

$$
\begin{array}{c}
\Xa{X_0} \xleftarrow{\pi_0} X \\
\Xa{(B, \Lambda)} \xleftarrow{(\nu, \nu)} (B, \Lambda)
\end{array}
$$

such that $K_X = \nu^*(K_B + \Lambda) + \nu^*\Gamma + E - G$ is a Fujita-Zariski decomposition of $K_X = \pi^*(K_B + \Lambda) + E - G$.
where
- \((\bar{B}, \bar{\Lambda})\) is a log minimal model of the klt pair \((B, \Lambda)\)
- \(\Gamma\) is an \(\phi_{\bar{B}}\)-exceptional effective \(\mathbb{Q}\)-divisor.
- \(P = \epsilon^*(K_{\bar{B}} + \bar{\Lambda})\) is the nef part and \(N = \tilde{\pi}^*\Gamma + E - G\) the effective part of the Fujita-Zariski decomposition.

Proof. As in Theorem 24 we have the birationally equivalent fibrations:

\[
\begin{array}{c}
\ X_0 \ 
\ \ 
\pi_0 \ 
\downarrow \ 
\ X_1 \ 
\downarrow \ 
\ X \ 
\ 
\pi \ 
\ 
\bar{B}_0 \ 
\ 
\bar{B}_1 \ 
\pi_0 \ 
\ 
\bar{B} \ 
\end{array}
\]

and \(K_X = \pi^*(K_B + \Lambda) + E - G\), where \((B, \Lambda)\) is a klt pair of dimension \(n - 1\). By the hypotheses \(0 \leq \kappa(X) = \kappa(B,K_B + \Lambda)\) (Proposition 19) and existence of minimal models for klt pairs of dimension \(n - 1\), \((B, \Lambda)\) has log minimal model \((\bar{B}, \bar{\Lambda})\). Let \(\tilde{\pi}\) be a common log resolution of \((B, \Lambda)\) and \((\bar{B}, \bar{\Lambda})\) and \(\tilde{X}\) be a resolution of \(\tilde{\pi} \times_B \bar{B}\). As in Theorem 24 we can assume without loss of generalities \(\tilde{X} = X\). We have the following commutative diagram:

\[
\begin{array}{c}
\ X_0 \ 
\ \ 
\pi_0 \ 
\downarrow \ 
\ X_1 \ 
\downarrow \ 
\ X \ 
\ 
\tilde{\pi} \ 
\ 
\bar{B}_0 \ 
\ 
\bar{B}_1 \ 
\pi_0 \ 
\ 
\bar{B} \ 
\end{array}
\]

By the Negativity Lemma, [17, Lemma 3.39], we have \(g^*(K_B + \Lambda) = h^*(K_{\bar{B}} + \bar{\Lambda}) + \Gamma\) with \(\Gamma\) effective and \(h\)-exceptional. From the arguments of [2, Thm. 1.5], \(g^*(K_B + \Lambda) = h^*(K_{\bar{B}} + \bar{\Lambda}) + \Gamma\) is a Fujita-Zariski decomposition of \(g^*(K_B + \Lambda)\) with \(h^*(K_{\bar{B}} + \bar{\Lambda}) = P_{\Lambda}\) the nef part and \(\Gamma = N_A\). Then \(K_B + \Lambda\) birationally admits a Fujita-Zariski decomposition and so by the arguments of Theorem 24 we have that:

\[
K_X = \pi^*h^*(K_{\bar{B}} + \bar{\Lambda}) + \pi^*(\Gamma) + E - G
= \epsilon^*(K_{\bar{B}} + \bar{\Lambda}) + \pi^*(\Gamma) + E - G.
\]

\[\blacksquare\]

Corollary 26. Under the assumption of the hypothesis and notation of Theorem 24, the canonical model of \(X\) is isomorphic to the log canonical model of \((B, \Lambda)\).

Proof. A Fujita-Zariski decomposition is a CKM-Zariski decomposition [7]. In Theorems 24 and 25 we showed that \(P = \pi^*(P_{\Lambda})\). Then, up to a change in grading the canonical rings of \(X\) and \((B, \Lambda)\) are isomorphic and the canonical models are isomorphic. 

4.2. Minimal Models and Elliptic Fibrations. We now use our results of Zariski decomposition and the relative elliptic fibrations (Theorems 25 and 17) to give a different proof of part (2) in Theorem 23. Note that the statement is stronger. In particular \(\bar{B}\) is \(\mathbb{Q}\)-factorial.

Theorem 27. Let \(\pi_0 : X_0 \to B_0\) be elliptic fibration, with \(\dim(X) = n\) and \(\kappa(X) \geq 0\).

Assume one of the following:

1. Assume the existence of log minimal models for klt pairs of non negative Kodaira dimension in dimension \(n - 1\).
2. Any sequence of flips for generalized klt pairs of dimension at most \(n - 2\) terminates and \(K_B + \Lambda\) admits a weak Zariski decomposition.

There exists a birationally equivalent fibration \(\tilde{\pi} : \tilde{X} \to \tilde{B}\) such that

- \(\tilde{B}\) is normal and \(\mathbb{Q}\)-factorial.
There exists an effective divisor $\bar{\Lambda}$ on $\bar{B}$ such that $(\bar{B}, \bar{\Lambda})$ is a klt pair. 

$\bar{X}$ has at worst terminal singularities. 

$K_{\bar{X}} \equiv_{q} \bar{\pi}^{*}(K_{\bar{B}} + \bar{\Lambda})$ 

$K_{\bar{X}}$ is nef

Proof. Assumption (1) or (2) ensures the existence of a minimal model for $(B, \Lambda)$. The argument for existence of minimal model from assumption (2) follows from [9, Thm. 1]. Thus, let $(\bar{B}, \bar{\Lambda})$ be a minimal model as in Theorem 25.

We have the following diagram:

```
\begin{tikzcd}
X_0 \arrow[r, swap] & X \arrow[l] \arrow[dr, phantom, cross out] & \bar{B} \arrow[dl, phantom, cross out] \\
& \arrow[ul, swap, \pi] & \arrow[dl, \bar{\pi}] \\
B_0 \arrow[u, \kappa] & (B, \Lambda) \arrow[rr, dashed] & & (B, \bar{\Lambda}) \arrow[u, \bar{\pi}, swap] \arrow[ll, \phi]
```

and the following Fujita-Zariski decomposition of the canonical divisor of $K_{X}$:

$K_{X} = \epsilon^{*}(K_{\bar{B}} + \bar{\Lambda}) + \bar{\pi}^{*}\Gamma + E - G$, $\epsilon^{*}(K_{\bar{B}} + \bar{\Lambda})$ nef and $\bar{\pi}^{*}\Gamma + E - G$ effective.

We apply Theorem [17] to the relative MMP with respect to $\epsilon: X \to \bar{B}$:

```
\begin{tikzcd}
X \arrow[r] & \bar{X} \arrow[l] \\
& \bar{B} \arrow[dl, \bar{\pi}] \arrow[ul, \pi] \\
(B, \Lambda) \arrow[rr, dashed] & & (B, \bar{\Lambda}) \arrow[u, \bar{\pi}, swap] \arrow[ll, \phi]
```

To apply Theorem [17] we want to show that no component of the effective divisor $\bar{\pi}^{*}\Gamma + E - G$ is a pullback of some $Q$-divisor on $\bar{B}$.

It is sufficient to show that no component of $\bar{\pi}^{*}\Gamma$ and $E$ contains the pullback of a divisor on $\bar{B}$, since they contain all the components of $\bar{\pi}^{*}\Gamma + E - G$. $\Gamma$ is contracted by $\phi_{\bar{B}}$ thus $\bar{\pi}^{*}\Gamma$ cannot contain the pullback of a divisor on $\bar{B}$. The components of $E$ can map down to a space of codimension 1 or to a space of codimension $\geq 2$ on $\bar{B}$.

We then need to show that when $\epsilon_{*}(D)$ has codimension one in $\bar{B}$, then $D$ does not contain the fiber over the points in its image on $\bar{B}$.

Then $\bar{\pi}_{*}(D)$ is also effective divisor. But $\bar{\pi}_{*}(D)$ cannot be contracted by $\phi_{\bar{B}}: \bar{B} \to B$, in fact, if so $D$ would map to a space of codimension $\geq 2$ on $B$ and since $(B, \Lambda)$ is a log minimal model of $(B, \Lambda)$, $\bar{\pi}_{*}(D)$ would also be contracted by $\phi_{\bar{B}}$. Since $\bar{\pi}_{*}(D)$ is not contracted by $\phi$, then $D$ is exceptional, in the sense of Theorem [1] in particular $D$ does not contain preimage of general points on its image in $\bar{B}$ and $D$ is not a pullback of a divisor on $\bar{B}$ and a fortiori of $\bar{B}$ also.

By Theorem [17] we will have $K_{\bar{X}} \equiv_{q} \bar{\pi}^{*}(K_{\bar{B}} + \Lambda)$ and $K_{\bar{X}}$ is nef since it is numerically the pullback of a log canonical divisor of a log minimal model. $\bar{X}$ has at worst terminal singularities since it is obtained from running a relative MMP on a smooth variety.

Corollary 28. Assume the existence of minimal models for klt pairs in dimension $n-1$ with non-negative Kodaira dimension. Given an elliptic $n$-fold, $\pi: X \to B$, then we have that $K_{X}$ birationally admits a Fujita-Zariski decomposition.

Corollary 29. Let $\pi: Y \to T$ be an elliptic fibration with $\dim(Y) = n$ and $\kappa(Y) \geq 0$. If generalized klt flips terminate in dimension up to $n-1$, then any minimal model program for $Y$ terminates.

Proof. Theorem [25] establishes a weak Zariski decomposition for $K_{X}$ and the results follow from [9, Thm. 1].
Since minimal model exist for klt pairs of non-negative Kodaira dimension of dimension up to 4 we have the following:

Corollary 30. An elliptically fibered variety of dimension $n \leq 5$ with non-negative Kodaira dimension has a birationally equivalent fibration $\tilde{\pi} : \tilde{X} \to \tilde{B}$ where $\tilde{X}$ is a minimal model and $K_{\tilde{X}} \equiv_0 \tilde{\pi}^*(K_B + \Lambda)$.

Theorem 31. Assume termination of flips for dlt pairs in dimension $n - 2$. Let $X \to B$ and $(B, \Lambda)$ as in Theorem 13 $X$ has a minimal model if and only if $(B, \Lambda)$ has a log minimal model.

Proof. $K_X$ birationally admits a Fujita-Zariski decomposition if and only if $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition (Theorem 13). If $(B, \Lambda)$ has a log minimal model then following the argument in the proof of Theorem 27 we can construct a minimal model of $X$.

If $X$ has a minimal model, the arguments of [2 Thm. 1.5] show that $K_X$ birationally admits a Fujita-Zariski decomposition. Then $K_B + \Lambda$ birationally admits a Fujita-Zariski decomposition (Theorem 13). Now since $\dim(B) = n - 1$, $(B, \Lambda)$ has a log minimal model [2 Thm. 1.5]. □

4.3. Abundance and Elliptic Fibrations.

Corollary 32. Let $\pi_0 : X_0 \to B_0$ be an elliptical fibration, $\dim X_0 = n$ and $\kappa(X_0) \geq 0$. Assume the existence of good minimal models for klt pairs of non negative Kodaira dimension in dimension $n - 1$. Then the Fujita-Zariski decomposition in Theorem 26 is also a Nakayama-Zariski decomposition.

Proof. Using the notation and set up as in Theorem 25 the Fujita-Zariski decomposition of $K_X$ is given by:

$$K_X = \epsilon^*(K_B + \Lambda) + \tilde{\pi}^*(\Gamma) + E - G$$

By assumption $K_X$ is pseudoeffective and so it also has a Nakayama-Zariski decomposition:

$$K_X = P_{\sigma}(K_X) + N_{\sigma}(K_X);$$

we will show that $\epsilon^*(K_B + \Lambda) = P_{\sigma}(K_X)$ and $\tilde{\pi}^*(\Gamma) + E - G = N_{\sigma}(K_X)$.

As $(\tilde{B}, \Lambda)$ is a good minimal model, $K_{\tilde{B}} + \Lambda$ is semiample. From the arguments in Theorem 25 we have that $\tilde{\pi}^*(\Gamma) + E - G$ is $\epsilon$-degenerate thus by [6] Lemma 2.16 we have:

$$\tilde{\pi}^*(\Gamma) + E - G \leq N_{\sigma}(K_X)$$

$$P_{\sigma}(\epsilon^*(K_B + \Lambda)) = P_{\sigma}(K_X)$$

From [6] Lemma 2.9, we have that for any pseudoeffective divisor $D$, we have $N_{\sigma}(D)$ is contained in $B_-(D)$ where:

$$B_-(D) = \bigcup_{\epsilon > 0} Bs(D + \epsilon A)$$

and $Bs(F)$ denotes the base locus of $F$ and $A$ is any ample divisor. The definition is independent of the choice of $A$. Now as we have that $\epsilon^*(K_B + \Lambda)$ is semiample, we must have that $B_-(\epsilon^*(K_B + \Lambda)) = \emptyset$, so that $N_{\sigma}(\epsilon^*(K_B + \Lambda)) = 0$. This implies that:

$$P_{\sigma}(K_X) = \epsilon^*(K_B + \Lambda)$$

and

$$N_{\sigma}(K_X) = \tilde{\pi}^*(\Gamma) + E - G.$$ □

Corollary 33. [Proposition 16, Theorem 23] Let $\pi : Y \to T$ be an elliptic fibration

(1) If $\dim(Y) = 4$ there exists a birational equivalent fibration $\bar{X} \to \bar{B}$, $\bar{X}$ with $\mathbb{Q}$-factorial terminal singularities, $(\bar{B}, \bar{\Lambda})$ with klt singularities such that $K_{\bar{X}} \equiv \bar{\pi}^*(K_B + \bar{\Lambda})$ and either $Y$ is birationally a Mori fiber space or $\bar{X}$ is a good minimal model.

(2) If $\kappa(Y) = n - 1$, there exists a birationally equivalent fibration $\tilde{\pi} : \tilde{X} \to \tilde{B}$ such that $\tilde{X}$ is a good minimal model, $K_{\tilde{X}} \equiv_0 \tilde{\pi}^*(K_B + \Lambda)$ and $(\bar{B}, \bar{\Lambda})$ has klt singularities.

Proof. It follows from Proposition 16, Theorem 14 and Theorem 23 See also [18] Thm. 4.4, [6] Cor. 4.5. □
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