GRAPH COHOMOLOGY AND KONTSEVICH CYCLES

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Abstract. We use the duality between compactly supported cohomology of
the associative graph complex and the cohomology of the mapping class
to show that the duals of the Kontsevich cycles \([W_\lambda]\) correspond to polynomials
in the Miller-Morita-Mumford classes. We also compute the coefficients of the
first two terms of this polynomial. This extends the results of [Igu], giving a
more detailed answer to a question of Kontsevich [Kon92] and verifying more
of the conjectured formulas of Arbarello and Cornalba [AC96].

Introduction

This paper explains the relationship between the Kontsevich cycles in associative
graph homology and the Miller-Morita-Mumford classes in the cohomology of
the mapping class group. We use a version of the forested graph complex of Conant and
Vogtmann [CV] to go from the double dual of graph homology to the cohomology
of the mapping class group and we use our cyclic set cocycle [Igu] to evaluate the
Miller-Morita-Mumford classes on the Kontsevich cycles.

Graph homology was introduced by M. Kontsevich in [Kon93] and in [Kon94].
He constructed three graph complexes which are called the “Lie”, “associative”
and “commutative” graph complexes. The case we are studying is the associative
case. This is the cohomology of ribbon graphs (graphs with cyclic orderings of the
half-edges incident to each vertex). In [Kon93], [Kon94] Kontsevich constructed
homology and cohomology classes in the associative case. He also outlined a proof
of the theorem that the rational finitely supported cohomology of the associative
graph complex is isomorphic to the homology of the mapping class group. We use
the proof of this theorem given in [CV].

Theorem 0.1 (Kontsevich). \(H^n_c(\mathcal{G}^*; \mathbb{Q}) \cong H_n(\coprod_{g,s} BM^s_g; \mathbb{Q})\) where \(\mathcal{G}^*\) is the (rein-
dexed) associative graph complex.

In this paper we use the category of ribbon graphs \(\mathcal{F}at\) which is homotopy
equivalent to the disjoint union of classifying spaces of mapping class groups over
all \(s \geq 1\) with \(s \geq 3\) when \(g = 0\). (See [Igu02] where this is shown to follow from
Culler-Vogtmann [CV86].)

\[|\mathcal{F}at| \simeq \prod_{g,s} BM^s_g\]

Therefore, the rational homology of \(\mathcal{F}at\) is isomorphic to the rational finitely sup-
ported cohomology of the associative graph complex. We give an explicit rational
chain homotopy equivalence between the cellular chain complex of this category and

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the graph cohomology complex \( G^\ast \). Then we show that the dual Kontsevich cycles in graph cohomology are (pull-backs of) polynomials in the “adjusted” Miller-Morita-Mumford classes. (They are adjusted by subtracting certain boundary classes.)

The main results of this paper were announced in [Igu] with short proofs. This paper gives more detailed proofs, expresses them in the language of graph cohomology and also extends these results to the next case. The calculation at the end of the paper shows that for \( n \neq 1 \) we have

\[
[W_{n,1}^\ast] = 3(-2)^{n+3}(2n + 1)!!(\bar{k}_n \bar{k}_1 - \bar{k}_{n+1}) - (-2)^{n+2}(2n + 5)!!\bar{k}_{n+1}.
\]

For \( n = 1 \) we divide the right hand side by \( 2 \).

In more detail the contents of this paper are as follows. In the first section we review Kontsevich’s definition of graph homology using Conant and Vogtmann’s formula for the Kontsevich orientation of a graph. We define \( G^\ast \) to be the integral finitely supported cohomology of the associative graph complex. Thus, e.g., \( G_0^1 \) is the group of all integer valued functions \( f \) on the set of all isomorphism classes \([\Gamma] \) of oriented trivalent ribbon graphs so that \( f([\Gamma]) = 0 \) for all but a finite number of \([\Gamma] \) and \( f([-\Gamma]) = -f([\Gamma]) \) where \(-\Gamma \) is \( \Gamma \) with the opposite orientation. This complex has an augmentation map \( \epsilon : G_0^1 \to \mathbb{Q} \) given by sending each dual generator \([\Gamma]^\ast \) to \( \frac{o(\Gamma)}{|\text{Aut}(\Gamma)|} \) where \( o(\Gamma) = \pm 1 \) depending on the orientation of \( \Gamma \).

We define the integral subcomplex \( G_0^2 \) of \( G^\ast \) to be the subcomplex generated by

\[
\langle \Gamma \rangle = |\text{Aut}(\Gamma)||\Gamma|.
\]

In \( G_0^2 \) these elements have augmentation \( \pm 1 \).

For any partition \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \) of \( n = \sum \lambda_i \) we define the dual Kontsevich cycle \( W_\lambda^\ast \) to be the homomorphism

\[
W_\lambda^\ast : G_2^\ast \to \mathbb{Q}
\]

which sends each \([\Gamma]^\ast \) in the Kontsevich cycle \( W_\lambda \) to \( \frac{o(\Gamma)}{|\text{Aut}(\Gamma)|} \). Since

\[
W_\lambda^\ast(\Gamma) = o(\Gamma) = \pm 1
\]

these are integral cocycles on \( G_0^2 \). These cocycles were considered by Kontsevich in [Kon92]. They are the Poincaré duals of certain strata of the moduli space of stable curves.

In graph cohomology the dual Kontsevich cycles are linear combinations of cocycles given by partition functions associated with certain 1-dimensional \( A_\infty \) algebras. We give a detailed account of this construction, essentially repeating what Kontsevich says in [Kon94] using the Conant-Vogtmann definition of graph orientation.

For every ribbon graph \( \Gamma \) we next construct an acyclic \( \mathbb{Z} \)-augmented chain complex \( F_\ast(\Gamma) \) over \( G_\ast \) so that \( F_\ast \) gives an acyclic carrier, i.e., a functor from the category \( \mathcal{Fat} \) of all ribbon graphs to the category of augmented chain complexes over \( G_\ast \). We call it the forest carrier. This determines a chain map from the cellular chain complex \( C_\ast(\mathcal{Fat}) \) to \( G_0^2 \) which is unique up to homotopy. We show that this map is a rational homotopy equivalence by constructing a rational inverse

\[
\psi : G_0^2 \otimes \mathbb{Q} = G_\ast \otimes \mathbb{Q} \to C_\ast(\mathcal{Fat}; \mathbb{Q}).
\]

The chain map \( \psi \) is defined by dual cells \( D(\Gamma) \) modelled on the Poincaré duals of the Kontsevich cycles.
In the second section we discuss the Stasheff polyhedron. We use the Conant-Vogtmann version of Kontsevich orientation to determine the intrinsic orientation of the Stasheff polyhedron which corresponds to the sign of the simplices in the dual cell $D(\Gamma)$. One of the main purposes of this is to justify the sign convention used in [Igu]. We also use this discussion to prove that the forest carrier $F_*$ is acyclic as claimed in the previous section.

In the third section we discuss the relationship between the adjusted Miller-Morita-Mumford (MMM) classes $\tilde{\kappa}_k$ and the dual Kontsevich cycles. We review the formula for the adjusted MMM classes given by the cyclic set cocycle and we show that the dual Kontsevich cycles are polynomials in the adjusted MMM classes. This is a detailed version of a one page argument in [Igu].

The last section contains the calculation of the coefficients of $[W_{n,1}^*]$ as a polynomial in the adjusted MMM classes. We use the figures from section 2 which were drawn with this second purpose in mind.

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(1) Kontsevich cycles.
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1. Kontsevich cycles

(1) Category of ribbon graphs $\mathcal{F}at$.
(2) Associative graph cohomology $G^\ast$.
(3) Cocycles $W^\ast_\lambda$ in graph cohomology.
(4) Partition functions.
(5) The forest carrier $F_\ast$.
(6) Dual cells.

We review the basic definitions and give an explicit rational homotopy equivalence between the finitely supported cohomology of the associative graph complex and the cellular chain complex of the category of ribbon graphs.

1.1. Category of ribbon graphs $\mathcal{F}at$. By a ribbon graph (also known as fat graph) we mean a finite connected graph together with a cyclic ordering on the half-edges incident to each vertex. We will use the following set theoretic model for the objects in the category of graphs.

**Definition 1.1.** Choose a fixed infinite set $\Omega$ which is disjoint from its power set. Then by a graph we mean a finite subset of $\Omega$ (the set of half-edges) together with two partitions of the set.

1. A partition into pairs of half-edges which we call edges.
2. A partition into sets of cardinality ($=valence$) $\geq 3$ which we call vertices.

To avoid straying too far from conventional terminology we refer to the elements of a vertex as incident half-edges. Equivalently, we define incident to mean not disjoint.

If $e = \{e^-, e^+\}$ is an edge in $\Gamma$ then the vertices $v_1, v_2$ incident to $e^-, e^+$ are the endpoints of $e$. If the endpoints are equal then $e$ is a loop. If $e$ is not a loop then we can collapse $e$ to a point forming a new graph

$$\Gamma/e$$

with one fewer edge, one fewer vertex and two fewer half-edges than $\Gamma$. Set theoretically, $\Gamma/e$ is given by merging $v_1, v_2$ and deleting $e^-, e^+$.

If $\Gamma$ is a ribbon graph and $e$ is an edge in $\Gamma$ which is not a loop then $\Gamma/e$ can be given the structure of a ribbon graph in the obvious way by letting the new vertex be cyclically ordered as

$$v_\ast = (h_1, \cdots, h_n, k_1, \cdots, k_m)$$

if $v_1 = (e^-, h_1, \cdots, h_n)$ and $v_2 = (e^+, k_1, \cdots, k_m)$.

Morphisms of graphs and ribbon graphs can be given by collapsing several edges to points and by isomorphisms. In other words, certain subgraphs will be collapsed to points.

By a subgraph of a graph $\Gamma$ we mean a subset of the set of vertices together with all incident half edges and a set of edges both endpoints of which lie in the chosen set of vertices. For example, we could take all of the vertices and none of the edges. A subgraph will usually not be a graph since it usually has unpaired half-edges. The unpaired half-edges of a subgraph will be called its leaves. If the graph $\Gamma$ is connected then every subgraph is determined by its set of leaves.

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1. This occurs, e.g., if every element of $\Omega$ is a set having greater cardinality than $\Omega$.
2. A graph/subgraph is connected if it is not the disjoint union of two graphs/subgraphs.
A subgraph is a tree if it is connected and has one more vertex than edge. A forest is a disjoint union of trees. If \( F \) is a spanning\(^3 \) forest in a graph \( \Gamma \), let \( \Gamma/F \) be the graph obtained by collapsing each tree in \( F \) to a separate point. By this we mean collapse the edges of each tree to a point. Thus:

1. The edges of \( \Gamma/F \) are the edges of \( \Gamma \) which do not lie in \( F \).
2. The vertices of \( \Gamma/F \) are the sets of leaves of the component trees of \( F \).

**Definition 1.2.** A morphism of graphs \( \phi : \Gamma_0 \to \Gamma_1 \) is defined to be an isomorphism

\[
\Gamma_0/F \cong \Gamma_1
\]

for some spanning forest \( F \) in \( \Gamma_0 \). In other words, the inverse image of every edge in \( \Gamma_1 \) is an edge in \( \Gamma_0 \) and the inverse image of every vertex of \( \Gamma_1 \) is a tree in \( \Gamma_0 \).

One thing is obvious from this definition. A morphism \( \phi : \Gamma_0 \to \Gamma_1 \) is uniquely determined by the value of \( \phi^{-1}(e) \) for every half edge \( e \) in \( \Gamma_1 \). The reason is that this information specifies the forest \( F \) and also gives an isomorphism \( \Gamma_1 \cong \Gamma_0/F \).

**Proposition 1.3.** Any two morphisms \( f, g : \Gamma_0 \to \Gamma_1 \) which are equalized by a morphism \( h : \Gamma_1 \to \Gamma_2 \) are equal. I.e.,

\[
hf = hg \Rightarrow f = g.
\]

**Remark 1.4.** In category theoretic terminology, this proposition says that morphisms of graphs are monomorphisms. They are also obviously epimorphisms. I.e., they satisfy both left and right cancellation.

**Proof.** In order for \( f, g \) to be different, there must be a half-edge \( e \) in \( \Gamma_1 \) so that \( f^{-1}(e) \neq g^{-1}(e) \). But \( hf = hg \) cannot send two different half-edges of \( \Gamma_0 \) to the same half-edge in \( \Gamma_2 \). So \( h(e) \) must be a vertex \( v \). The inverse image of \( v \) is a tree \( T_0 \) in \( \Gamma_0 \) and another tree \( T_1 \) in \( \Gamma_1 \) and the leaves of both trees map bijectively onto the half-edges incident to \( v \). Consequently, \( f, g \) give the same bijection of the leaves of \( T_0 \) with the leaves of \( T_1 \). Any edge in \( T_0 \) is uniquely characterized by the partitioning of the set of leaves which would result if we cut the edge. And each interior half-edge of \( T_0 \) is determined by the corresponding subset of the set of leaves. Thus \( f^{-1}(e) = g^{-1}(e) \) which is a contradiction. \( \square \)

If \( \Gamma \) is a ribbon graph then the set of leaves of every tree in \( \Gamma \) inherits a cyclic order. Consequently, \( \Gamma/F \) has an induced structure as a ribbon graph. A graph morphism \( \phi : \Gamma_0 \to \Gamma_1 \) will be called a ribbon graph morphism if it respects these cyclic orderings, i.e., if \( \phi \) is an isomorphism of ribbon graphs.

If \( \Gamma, \Gamma' \) are ribbon graphs, let \( \text{Hom}(\Gamma, \Gamma') \) denote the set of all ribbon graph morphisms \( \Gamma \to \Gamma' \). Since this is a subset of the set of all graph morphisms, left and right cancellation hold for these morphisms as well. Thus we get the following corollary where \( \text{Aut}(\Gamma) = \text{Hom}(\Gamma, \Gamma) \) is the group of ribbon graph automorphisms of \( \Gamma \).

**Corollary 1.5.** \( \text{Aut}(\Gamma) \) acts freely on the right on \( \text{Hom}(\Gamma, \Gamma') \) and \( \text{Aut}(\Gamma') \) acts freely on the left.

\(^3\)Spanning means containing all the vertices.
Let $\mathcal{F}at$ denote the category of all ribbon graphs and ribbon graph morphisms. Let $|\mathcal{F}at|$ denote its geometric realization:

$$|\mathcal{F}at| = \bigcup_n \bigcup_{\Gamma \in \mathcal{F}at} \Delta^n / \sim$$

Then we have the following theorem which I learned from Penner [Pen87], which goes back to Strebel [Str84] but which I prove using Culler-Vogtmann [CV86]. For details, see [Igu02].

**Theorem 1.6.**

$$|\mathcal{F}at| \cong \bigcup_{g,s} BM^s_{g}$$

where $M^s_{g}$ is the mapping class group of a surface of genus $g$ with $s$ punctures.$^4$

Many ribbon graphs have an intrinsic orientation in the sense of Kontsevich graph homology.

1.2. **Associative graph cohomology $G\ast$.** Graph homology (of ribbon graphs) is rationally dual to the homology of the category of ribbon graphs. More precisely, we have an isomorphism between rational compactly supported cohomology of the associative graph complex $G\ast$ and the rational homology of the mapping class group. We will construct an integral chain map which realizes this rational equivalence.

However, the main purpose of introducing graph homology in the present context is to fix our orientation conventions. We use the definitions given in [CV]. Since our graphs are connected this agrees with Kontsevich’s orientation convention.

**Definition 1.7** (Conant-Vogtmann). An orientation of a graph is defined to be an orientation on the vector space spanned by the set of vertices and half-edges.

This means that an orientation of $\Gamma$ can be given by ordering the vertices and orienting the edges of $\Gamma$.

**Remark 1.8** (Conant-Vogtmann). A ribbon graph has a natural orientation if all its vertices have odd valence. Since such a graph necessarily has an even number of vertices it does not matter if we put the vertex first and then the incident half-edges in cyclic order or the other way around.

**Definition 1.9** (Conant-Vogtmann). Given a graph $\Gamma$ and an edge $e$ in $\Gamma$ which is not a loop, let $\Gamma / e$ be the graph obtained from $\Gamma$ by collapsing the edge $e$. If an orientation on $\Gamma$ is given by orienting all edges and ordering the vertices so that the source of $e$ is first and its target is second, the induced orientation on $\Gamma / e$ is given by taking the coalesced vertex to be first and letting the remaining vertices and edges be ordered and oriented as before.

**Remark 1.10.** If $e$ is an oriented edge going from vertex $v_i$ to vertex $v_j$ then, when we collapse $e$, we should insert the new vertex in position $i$ and multiply the orientation by $(-1)^j$. This will give the induced orientation for $\Gamma / e$.

$^4M^s_{g}$ is the group of homotopy classes of orientation preserving self homeomorphisms of a connected Riemann surface of genus $g$ with $s$ punctures. This group maps onto the symmetric group on $s$ letters with kernel equal to the mapping class group $M_{g,s}$ of genus $g$ surfaces with $s$ marked points.
The \textit{associative graph homology complex} can now be defined. For all \( n \geq 0 \) let \( G^n \) be the free abelian group generated by all isomorphism classes \([\Gamma]\) of (connected) oriented ribbon graphs \( \Gamma \) of codimension\(^5\) \( n \) module the relation \(-[\Gamma] = [\Gamma']\) where \(-\Gamma\) is \( \Gamma \) with the opposite orientation. If \( \Gamma \) has an orientation reversing automorphism this implies that 2[\( \Gamma \)] = 0. Define the boundary operator \( \partial : G^n \to G^{n+1} \) by
\[
\partial[\Gamma] = \sum e \frac{[\Gamma/e]}{[\Gamma]}
\]
where the sum is over all edges in \( \Gamma \) which are not loops.

The compactly supported dual of this complex is the \textit{(associative) graph cohomology complex} given as follows.

**Definition 1.11.** For all \( n \geq 0 \) let \( G_n \) be the additive group of all homomorphisms \( f : G^n \to \mathbb{Z} \) so that \( f[\Gamma] \neq 0 \) for only finitely many \([\Gamma]\). (In particular, \( f[\Gamma] = 0 \) if \( \Gamma \) has an orientation reversing automorphism.) Thus \( G_n \) is generated by duals \([\Gamma]\)\(^*\) of generators of \( G^n \). The boundary map \( d : G_n \to G_{n-1} \) is given in terms of these dual generators by
\[
d[\Gamma]\ = \sum l_i [\Gamma_i]\ \in \mathbb{Z}
\]
where \( l_i \) is equal to the number of edges \( e \) in \( \Gamma_i \) so that \( \Gamma_i/e \cong \Gamma \) minus the number of edges in \( \Gamma_i \) so that \( \Gamma_i/e \cong -\Gamma \). The sum is over a basis for \( G_{n-1} \).

The coefficient \( l_i \) in (2) can be written as
\[
l_i = \frac{|\text{Hom}^+(\Gamma_i, \Gamma)| - |\text{Hom}^-(\Gamma_i, \Gamma)|}{|\text{Aut}(\Gamma)|} \in \mathbb{Z}
\]
where \( \text{Hom}^\pm(\Gamma_i, \Gamma) \) is the set of morphisms \( f : \Gamma_i \to \Gamma \) so that the orientation of \( \Gamma \) agrees/disagrees with the orientation induced from \( \Gamma_i \) by \( f \). In other words, \( l_i \) is the number of left equivalence classes of morphisms \( \Gamma_i \to \Gamma \) counted with sign.

Let \( r_i \) be the number of right equivalence classes of such maps counted with sign. Then
\[
r_i = \frac{|\text{Hom}^+(\Gamma_i, \Gamma)| - |\text{Hom}^-(\Gamma_i, \Gamma)|}{|\text{Aut}(\Gamma_i)|} \in \mathbb{Z}.
\]
So (2) can be written as
\[
d\langle \Gamma \rangle = \sum r_i \langle \Gamma_i \rangle
\]
where
\[
\langle \Gamma \rangle := |\text{Aut}(\Gamma)|[\Gamma]\*
\]

**Definition 1.12.** Let \( G^n_\mathbb{Z} \) denote the subcomplex of \( G_\ast \) generated by the elements \( \langle \Gamma \rangle \). We call \( G^\mathbb{Z}_\ast \) the \textit{integral subcomplex} of \( G_\ast \).

The boundary map in \( G^\mathbb{Z}_\ast \) can be described in terms of expanding vertices. If \( \Gamma \) is an oriented ribbon graph, each vertex of valence \( n \) can be expanded into two vertices connected by an edge in
\[
n^2 - 3n \\frac{2}{2}
\]
different ways. Each of these choices gives a ribbon graph \( \Gamma_i \) with a distinguished edge \( e \) and an isomorphism \( \Gamma_i/e \cong \Gamma \).
so that \( \Gamma_i \) is unique up to isomorphism over \( \Gamma \). We give \( \Gamma_i \) the orientation induced from \( \Gamma \) by this isomorphism. The boundary map of the integral subcomplex is then given by

\[
d(\langle \Gamma \rangle) = \sum \langle \Gamma_i \rangle.
\]

**Example 1.13.** Consider the ribbon graphs \( \Gamma_0, \Gamma'_0, \Gamma_1 \) shown in Figure 1. We take the natural orientation on the trivalent graphs \( \Gamma_0, \Gamma'_0 \) and the induced orientation on \( \Gamma_1 \) given by the isomorphism 

\[
\Gamma_1 \cong \Gamma_0/e.
\]

If we write the orientation of \( \Gamma_0 \) as 

\[
v_1e_1bcv_2e_2da \cdots
\]

then we see that the induced orientation for \( \Gamma_1 \) is \( vabcd \cdots \). The other three straight edges of \( \Gamma_0 \) are equivalent to \( e \) since \( \text{Aut}(\Gamma_0) \cong D_8 \) acts transitively on the set of 4 straight edges. So the boundary of \( [\Gamma_0] \) in the associative graph complex is given by 

\[
\partial[\Gamma_0] = 4[\Gamma_1] + 8[\Gamma'_1]
\]

where \( \Gamma'_1 \) is given by collapsing one of the curved edges of \( \Gamma_0 \).

The ribbon graph \( \Gamma_1 \) is also part of the boundary of \( [\Gamma'_0] \) since 

\[
-\Gamma_1 \cong \Gamma'_0/e'.
\]

But there are no other edges in \( \Gamma'_0 \) equivalent to \( e' \). Consequently,

\[
d[\Gamma_1]^* = 4[\Gamma_0]^* - [\Gamma'_0].
\]

Since the orders of the automorphism groups are: 2, 8, 2, respectively, we get:

\[
d(\langle \Gamma_1 \rangle) = \langle \Gamma_0 \rangle - \langle \Gamma'_0 \rangle.
\]

We will be looking at the rational cochain complex 

\[
\text{Hom}(G_*, \mathbb{Q}).
\]

This is the rational double dual of the original graph homology complex \( G^* \). Thus cocycles in this complex, such as \( W^*_a \) defined below, are “infinite cycles” in the graph homology complex.

**Remark 1.14.** Since \( G^*_n \) is a free abelian group whose generators \( \langle \Gamma \rangle \) form a \( \mathbb{Q} \)-basis for \( G_n \otimes \mathbb{Q} \), its integral dual forms a lattice

\[
\text{Hom}(G^*_n, \mathbb{Z}) \subseteq \text{Hom}(G_*, \mathbb{Q})
\]
1.3. Cocycles \( W^*_\lambda \) in graph cohomology.

**Definition 1.15.** If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) is a sequence of positive integers let \( W_\lambda \) be the set of all ribbon graphs \( \Gamma \) which are trivalent at all but \( r \) vertices \( v_1, \ldots, v_r \) which have valence \( 2\lambda_i + 3 \), resp. This set will be called the **Kontsevich cycle**. The **dual Kontsevich cycle** \( W^*_\lambda \in G^{2|\lambda|} \) (where \( |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_r \)) is given by

\[
W^*_\lambda [\Gamma]^* := \begin{cases} \frac{o(\Gamma)}{|\text{Aut}(\Gamma)|} & \text{if } \Gamma \in W_\lambda \\ 0 & \text{if } \Gamma \notin W_\lambda \end{cases}
\]

where \( o(\Gamma) = \pm 1 \) depending on whether the given orientation of \( \Gamma \) agrees with the natural orientation (Remark 1.8).

The dual Kontsevich cycle are integral cochains since they can be given by

\[
W^*_\lambda [\Gamma]^* = o(\Gamma)
\]

if \( \Gamma \in W_\lambda \). Also, note that \( W_\lambda, W^*_\lambda \) are independent of the order of the \( \lambda_i \).

In the special case when \( r = 1 \), \( W_k \) is called the **Witten cycle** and \( W^*_k \) will be called the **dual Witten cycle**. In the case \( r = 0 \), \( W_\emptyset \) is the set of all trivalent (connected) ribbon graphs and

\[
W^*_0 [\Gamma]^* = \epsilon : G_0 \to \mathbb{Q}
\]

is the map sending the dual \([\Gamma]^*\) of every trivalent graph \( \Gamma \) to \( o(\Gamma)/|\text{Aut}(\Gamma)| \). On the integral subcomplex this gives an epimorphism

\[
\epsilon : G_0^Z \to \mathbb{Z}
\]

sending each generator \( [\Gamma]^* \) to \( o(\Gamma) = \pm 1 \). We define these maps to be the **augmentation maps** for \( G_* \) and \( G^Z_* \).

We will also consider **degenerate** cases where some of the indices are zero. We interpret these 0’s as counting the number of distinct trivalent vertices:

\[
W^*_{\emptyset} [\Gamma]^* := \begin{cases} \binom{n}{k} \frac{o(\Gamma)}{|\text{Aut}(\Gamma)|} & \text{if } \Gamma \in W_\emptyset \text{ with } n \text{ vertices} \\ 0 & \text{if } \Gamma \text{ is not trivalent.} \end{cases}
\]

**Proposition 1.16.** Each dual Kontsevich cycle \( W^*_\lambda \) is an integral cocycle.

**Proof.** Let \( n = 2|\lambda| \). Then we want to show that

\[
W^*_\lambda (d [\Gamma]) = 0
\]

for all oriented ribbon graphs \( \Gamma \) of codimension \( n + 1 \). However, the only case in question occurs when \( \Gamma \) has only one even valent vertex, call it \( v_0 \). The orientation on \( \Gamma \) can be given by first taking \( v_0 \), then the incident half-edges \( e_1, \ldots, e_{2m} \), then all other vertices with their incident half-edges in cyclic order. The orientation depends on which of the half-edges at \( v_0 \) is first.

There are three cases.

- **Case 1.** The graph \( \Gamma \) has \( r - 2 \) odd valent vertices of codimension\(^6\) \( \geq 2 \). After re-indexing the \( \lambda_i \) we may assume that these codimensions are \( 2\lambda_3, 2\lambda_4, \ldots, 2\lambda_r \). The even valent vertex \( v_0 \) must have valence \( 2\lambda_1 + 2\lambda_2 + 4 \) and it needs to split into two vertices of codimension \( 2\lambda_1 \) and \( 2\lambda_2 \). There is always an even number of

\(^6\)The **codimension** of a vertex is equal to its valence minus 3.
ways to do this \((2\lambda_1 + 2\lambda_2 + 4\text{ ways if not})\) and half of them will give one sign and half the other. \((\text{The sign alternates as we rotate the half edges incident to } v_0)\) Consequently, the value of \(W_\star^\star \lambda\) on \(d[\Gamma]^\star\) will be zero.

**Case 2.** The graph \(\Gamma\) has \(r - 1\) odd valent vertices of codimension \(\geq 2\). We may assume that these codimensions are \(2\lambda_2, \cdots, 2\lambda_r\) (after re-indexing the \(\lambda_i\)). The vertex \(v_0\) must have valence \(2\lambda_1 + 4\) and it needs to split into two vertices of valence \(2\lambda_1 + 3\) and \(3\). There are \(2\lambda_1 + 4\) ways to do this and half of them will give one sign and half the other.

**Case 3.** \(v_0\) has valence \(4\). It can split into two trivalent vertices in two ways with opposite sign as we saw in Example 1.8.

Proposition 1.16 also follows from an observation of Kontsevich that \(A_\infty\) superalgebras give partition functions on ribbon graphs which are cocycles on associative graph cohomology.

1.4. **Partition functions.** In [Kon93] and [Kon94], Kontsevich explains how a finite dimensional \(A_\infty\) superalgebra \(\mathcal{A}\) gives a cocycle on the associative graph cohomology complex \(\mathcal{G}\). Kontsevich assumed that \(\mathcal{A}\) was an algebra over the real numbers. However, it is easy to see that the ground field can have any characteristic. In fact, we only need to assume that \(\mathcal{A}\) is a finitely generated free module over a commutative ring \(R\).

We will go over the definition of an \(A_\infty\) superalgebra following Getzler and Jones [GJS99]. Then we revise Kontsevich’s definition of the partition function using the Conant-Vogtmann definition of graph orientation. Finally, we examine the special case of one dimensional algebras to verify Kontsevich’s claim [Kon94] that the cocycles coming from these examples linearly span the space of polynomials in the Miller-Morita-Mumford classes. Translated into the present setting these cocycles are easily seen to be linear combinations of the dual Kontsevich cycles (which come from Kontsevich’s earlier paper [Kon92]).

**Definition 1.17.** By an \(A_\infty\) superalgebra we mean a \(\mathbb{Z}/2\)-graded algebra \(\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1\) over a commutative ring \(R\) together with a sequence of \(R\)-linear mappings

\[
m_k : \mathcal{A}^\otimes k \to \mathcal{A}, \quad k \geq 1
\]

which are homogeneous of degree \(k\) \((\text{mod } 2)\) so that for homogeneous elements \(x_1, \cdots, x_k\) we have

\[
\sum_{r+s+t=k} (-1)^u m_{r+1+t}(x_1, \cdots, x_r, m_s(x_{r+1}, \cdots, x_{r+s}), x_{r+s+1}, \cdots, x_k) = 0
\]

where \(u = r + st + s|x_1| + \cdots + s|x_r|\).

Suppose that \(\mathcal{A} \cong R^n\) is finitely generated and free as an \(R\)-module. Suppose that \(m_1 = 0\). And suppose that we have an nondegenerate even scalar product

\[
\langle , \rangle : \mathcal{A} \otimes \mathcal{A} \to R.
\]

This means the following.

1. \(\langle a, b \rangle = 0\) if \(|a| + |b| = 1\).
2. \(\langle a, b \rangle = (-1)^{|a|} \langle b, a \rangle\). (This implies (1) if 2 is not a zero divisor in \(R\).)
3. There is a degree 0 \(R\)-linear isomorphism

\[
D : \text{Hom}_R(\mathcal{A}, R) \cong A
\]

so that \(\langle a, Df \rangle = f(a)\).
(4) For all \(x_0, \ldots, x_n \in A_0 \coprod A_1\) we have
\[
\langle m_n(x_1, \ldots, x_n), x_0 \rangle = (-1)^{n+|x_0|+n|x_0|} \langle m_n(x_0, \ldots, x_{n-1}), x_n \rangle
\]

Then we have a partition function
\[
Z_A : \mathcal{G}_*^\mathbb{Z} \to R
\]
given on any generator \(\langle \Gamma \rangle\) as follows.

First, choose an ordering for the vertices \(v_1, v_2, \ldots\) of \(\Gamma\). Next, label the half-edges incident to each \(v_i\) in reverse (clockwise) order \(e_{i1}, \ldots, e_{in_i}, e_{i0}\) (if \(v_i\) has valence \(n_i + 1\)). Let \(\epsilon_1 = \pm 1\) so that
\[
o(\Gamma) = \epsilon_1 \text{sgn}(v_1, e_{10}, e_{1n_1}, \ldots, e_{11}, v_2, e_{20}, e_{2n_2}, \ldots, e_{21}, v_3, \ldots).
\]

Choose an \(R\)-basis \(b_1, \ldots, b_n\) for \(A (b_i \in A_0 \coprod A_1)\) and a dual basis \(b_1^*, \ldots, b_n^* \in \text{Hom}_R(A, R)\) so that
\[
\langle b_i, D b_j^* \rangle = \delta_{ij}.
\]

Then the partition function is given by the state sum
\[
Z_A(\Gamma) = \epsilon_1 \sum_{\text{states}} \prod_i \langle m_n(x_{i1}, \ldots, x_{in_i}), x_{i0} \rangle \epsilon_2 \prod_j \langle D\bar{v}_j^*, Dy_j^* \rangle
\]

The sum is over all states where a state of \(\Gamma\) is given by assigning a basis element \(x_{ij}\) of \(A\) to each half edge \(e_{ij}\) of \(\Gamma\). The first product is over all vertices \(v_i\). The second product is over all edges \((h_j, \bar{h}_j)\). Here \(y_j^*\) represents the dual basis element corresponding to the basis element \(y_j\) assigned to \(h_j\) and similarly for \(\bar{v}_j\). The sign \(\epsilon_2 = \pm 1\) is the sign of the permutation of the odd half-edges (those assigned elements of \(A_1\) as basis elements) as they appear in the sequence:
\[
e_{11}, e_{12}, \ldots, e_{1n_1}, e_{10}, e_{21}, \ldots, e_{2n_2}, e_{20}, e_{31}, \ldots
\]

which places each next to its other half (placing \(\bar{h}_j\) next to and on the right of \(h_j\)).

**Theorem 1.18** (Kontsevich). \(Z_A\) is a cocycle on \(\mathcal{G}_*^\mathbb{Z}\).

**Remark 1.19.** The usual definition of the partition function has a factor of \(\frac{1}{|\text{Aut}(\Gamma)|}\):
\[
Z_A(\Gamma)^* = \frac{\epsilon_1}{|\text{Aut}(\Gamma)|} \sum_{\text{states}} \prod_i \langle m_n(x_{i1}, \ldots, x_{in_i}), x_{i0} \rangle \epsilon_2 \prod_j \langle D\bar{v}_j^*, Dy_j^* \rangle
\]

This factor disappears on the integral subcomplex \(\mathcal{G}_*^\mathbb{Z}\) since \(\langle \Gamma \rangle = |\text{Aut}(\Gamma)||\Gamma|^*\).

**Proof.** It is easy to see that the partition function is well-defined. Since each basis element \(a_{ij}\) and its dual appears exactly once, the formula for \(Z_A(\Gamma)^*\) is independent of the choice of basis. If we transpose two vertices \(v_1, v_2\) then both \(\epsilon_1\) and \(\epsilon_2\) change by a factor of \((-1)^{n_1 n_2}\). Finally, if we cyclically permute the half-edges around a vertex \(v\) of valence \(n + 1\) then the signs \(\epsilon_1, \epsilon_2\) and the value change by factors of
\[
\epsilon_1' / \epsilon_1 = (-1)^n
\]
\[
\epsilon_2' / \epsilon_2 = (-1)^{|x_0|+\sum |x_i|}
\]
\[
\text{value}' / \text{value} = (-1)^{n+|x_0|+|x_0|n}
\]

The product of these factors is 1 since the degrees of \(x_0, \ldots, x_n\) must add up to \(n\) mod 2. (Otherwise the expression \(\frac{1}{\text{Aut}(\Gamma)}\) is zero.) Thus \(Z_A\) is well-defined. It remains to show that \(Z_A\) is a cocycle.
The boundary of any generator $\langle \Gamma \rangle$ in $G^*_\mathbb{Z}$ is a sum over all vertices $v$ of $\Gamma$ of all ribbon graphs $\Gamma'$ obtained from $\Gamma$ by expanding $v$ into two vertices. For each fixed $v$ the sum of the values of $Z_A(\Gamma')$ add up to zero. To see this we label the half-edges clockwise around $v$. This means that the Conant-Vogtmann orientation starts as:

$$o(\Gamma) = \text{sgn}(v e_0 e_n \cdots e_1 v' \cdots).$$

When we expand $v$ we get $\Gamma'$ with orientation

$$o(\Gamma') = \text{sgn}(v_1 v_2 h \overline{h} e_0 e_n \cdots e_1 v' \cdots) = (-1)^n \text{sgn}(v_1 h e_{r+s} \cdots e_{r+1} v_2 e_0 e_n \cdots e_{r+s+1} \overline{h} e_{r+1} \cdots e_1 v' \cdots)$$

where $n = r + s + t$ with

$$\epsilon_1 = (-1)^u = (-1)^{st+t+n+1}$$

The corresponding terms of the partition function are:

(4) $\langle m_s(x_{r+1}, \ldots, x_{r+s}), y \rangle \langle m_{r+s+1}(x_1, \ldots, x_r, \overline{y}, x_{r+s+1}, \ldots, x_n), x_0 \rangle \langle Dy^*, Dy^* \rangle$

with associated relative sign term

$$\epsilon_2 = (-1)^s|x_1|+\cdots+s|x_r|$$

since the degrees of $x_{r+1}, \ldots, x_{r+s}, y$ must add up to $r$.

Using the identity

$$\sum_i \langle x, b_i \rangle \langle y, Db_i^* \rangle = \langle x, y \rangle$$

we see that the expression (4) contracts to

$$\langle m_{r+s+1}(x_1, \ldots, x_r, m_s(x_{r+1}, \ldots, x_{r+s}), x_{r+s+1}, \ldots, x_n), x_0 \rangle$$

when summed over all allowed values of $y, \overline{y}, y^*, \overline{y}^*$. By definition of an $A_\infty$ algebra, the product of this with $\epsilon_1 \epsilon_2$ adds up to zero if we sum over all $\Gamma'$ obtained from $\Gamma$ by expanding $v$ since $n$ is constant. We need the assumption $m_1 = 0$ since $\Gamma'$ has no bivalent vertices.

Suppose that $x = (x_0, x_1, x_2, \cdots)$ is an infinite sequence of rational numbers. Then Kontsevich points out that there is a 1-dimensional $A_\infty$ algebra $A = A_0 = \mathbb{Q}$ with scalar product in which $m_{2k}$ is multiplication by $x_{k-1}$, $m_{\text{odd}} = 0$ and $\langle a, b \rangle = ab$. Since the states of $\Gamma$ are given by assigning a basis vector to each half-edge, there is only one state and the partition function $Z_x = Z_A$, which is a sum over all states, has only one term.

**Example 1.20.** The partition function

$$Z_x : G_* \to \mathbb{Q}$$

is the cocycle defined by the equation

$$Z_x[\Gamma]^* = \frac{o(\Gamma)}{\vert \text{Aut}(\Gamma) \vert} x_0^{r_0} x_1^{r_1} \cdots$$

if $\Gamma$ is a ribbon graph with $r_i$ vertices of valence $2i + 3$ for $i = 0, 1, 2, \cdots$ and no vertices of even valence.
Since the Euler characteristic of $\Gamma$ is given by
\[ \chi(\Gamma) = -\frac{1}{2} \sum r_i (2i + 1), \]
the value of $r_0$ can be written as
\[ r_0 = -2\chi - \sum_{i \geq 1} r_i (2i + 1). \]

Thus, the partition function $Z_x$ can be given in terms of the dual Kontsevich cycles by
\[ Z_x = x_0^{-2\chi} \sum_\lambda y^\lambda W_\lambda^* \]
where $y^\lambda = \prod (x_i / x_0^{2i+1})^{r_i}$ if $\lambda = (1^{r_1}, 2^{r_2}, \cdots)$ and the sum is over all $\lambda$ so that $r_0$, as given by (5), is nonnegative. (So the right hand side of (6) is well-defined only when $\chi$ is fixed.)

Thus, if we restrict to the subcomplex of the graph cohomology complex $G_*$ generated by $[\Gamma]^*$ where $\chi(\Gamma)$ is fixed, the linear span of these partition functions is the same as the linear span of the $W_\lambda^*$ and, by Corollary 3.2 below, this is equal to the algebra generated by the adjusted Miller-Morita-Mumford classes $\tilde{\kappa}_k$ excluding $\tilde{\kappa}_0 = \chi$. (However, we lose the linear independence of the $[W_\lambda^*]$ when we restrict to this finitely generated subcomplex of $G_*$.)

Setting $x_0 = 1$ and taking partial derivatives of (6) with respect to $y^\lambda$ using finite differences, we can conclude that, for each fixed $\chi$, the dual Kontsevich cycles $W_\lambda^*$ are linear combinations of partition functions $Z_x$ for various multi-indices $x$.

Consequently, Kontsevich’s theorem (1.18) that each $Z_x$ is a cocycle implies that each $W_\lambda^*$ is a cocycle.

To pull the cocycles $W_\lambda^*$ back to the category of ribbon graphs we need to use an acyclic carrier related to the forested graph complex of $[CV]$.

1.5. The forest carrier $F_*$. Conant and Vogtmann use forested graph complexes to show that graph homology is rationally isomorphic to the cohomology of the mapping class group as claimed by Kontsevich. In our notation the forested graph complex is the total complex of an integral acyclic carrier $F_* : \mathcal{F}at \to G_*^\mathbb{Z} \subseteq G_*$ which we call the “forest carrier.”

Suppose that $\Gamma_0$ is a ribbon graph. Then we will construct a chain complex $F_*(\Gamma_0)$ generated by the isomorphism classes of all ribbon graphs $\Gamma$ which map to $\Gamma_0$. Each such object is given by a “forested ribbon graph,” i.e., a ribbon graph $\Gamma$ with a spanning forest $F$ (so that $\Gamma/F \cong \Gamma_0$). If $\Gamma_0$ has codimension $n$ then $F_*(\Gamma_0)$ will be the augmented chain complex
\[ 0 \to F_n(\Gamma_0) \to F_{n-1}(\Gamma_0) \to \cdots \to F_0(\Gamma_0) \xrightarrow{\partial} \mathbb{Z} \to 0 \]
given as follows.

Let $F_k(\Gamma_0)$ be the free abelian group generated by all isomorphism classes of codimension $k$ objects in $\mathcal{F}at$ over $\Gamma_0$ together with an orientation. In other words, generators of $F_k(\Gamma_0)$ are given by morphisms
\[ f : \Gamma \to \Gamma_0 \]
where \( \Gamma \) is a ribbon graph of codimension \( k \) together with an orientation on \( \Gamma \). Two such objects \( f_i : \Gamma_i \to \Gamma_0 \) for \( i = 1, 2 \) are isomorphic if there is an orientation preserving isomorphism \( g : \Gamma_1 \to \Gamma_2 \) so that \( f_2 \circ g = f_1 \). As usual, we equate reversal of orientation with reversal of sign. In particular, \( F_n(\Gamma_0) \) has rank 1 with two generators corresponding to the two possible orientations of \( \Gamma_0 \).

The boundary map \( d : F_k(\Gamma_0) \to F_{k-1}(\Gamma_0) \) is given by
\[
d[f : \Gamma \to \Gamma_0] = \sum [f \circ g_i : \Gamma_i \to \Gamma_0]
\]
where the sum is taken over all right equivalence classes of morphisms \( g_i : \Gamma_i \to \Gamma \) which collapse only one edge. We take the unique orientation on each \( \Gamma_i \) which induces the given orientation on \( \Gamma \).

The augmentation map \( \epsilon : F_0(\Gamma_0) \to \mathbb{Z} \) is given by
\[
\epsilon[\Gamma \to \Gamma_0] = o(\Gamma) = \pm 1.
\]

**Proposition 1.21.** Suppose that \( \Gamma_0 \) is trivalent except for \( r \) vertices \( v_1, \cdots, v_r \) which have codimensions \( n_1, \cdots, n_r \), resp. Then \( F_*(\Gamma_0) \) is based chain isomorphic to the tensor product
\[
F_*(\Gamma_0) \cong C_*(K^{n_1}) \otimes C_*(K^{n_2}) \otimes \cdots \otimes C_*(K^{n_r})
\]
where \( C_*(K^m) \) is the cellular chain complex of the \( m \) dimensional Stasheff polyhedron \( K^m \). In particular, \( F_*(\Gamma_0) \) is acyclic.

Proposition 1.21 follow from well-known properties of the Stasheff polyhedron which we will review shortly. Suppose for the moment that this is true.

For each \( \Gamma_0 \) there is a natural augmented chain map \( p : F_*(\Gamma_0) \to \mathbb{G}^Z_\ast \) given by
\[
p[f : \Gamma \to \Gamma_0] = [\Gamma].
\]
A morphism \( g : \Gamma_0 \to \Gamma_1 \) induces a chain map \( g_* : F_*(\Gamma_0) \to F_*(\Gamma_1) \) by
\[
g_*[f : \Gamma \to \Gamma_0] = [g \circ f : \Gamma \to \Gamma_1].
\]
This is a chain map over \( \mathbb{G}^Z_\ast \) in the sense that \( p \circ g_* = g_* \). Therefore, \( F_\ast \) is a functor from \( \text{Fat} \) to the category of acyclic augmented chain complexes over \( \mathbb{G}^Z_\ast \). In other words, it is an acyclic carrier. We call \( F_\ast \) the forest carrier.

The acyclic carrier \( F_\ast \) carries a unique (up to homotopy) chain map
\[
(7) \phi_\ast : C_\ast(\text{Fat}) \to \mathbb{G}^Z_\ast
\]
where \( C_\ast(\text{Fat}) \) is the cellular chain complex of the category of ribbon graphs. (So \( C_\ast(\text{Fat}) \) is the free abelian group generated by all \( n \) simplices
\[
\Gamma_\ast = (\Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_n)
\]
in the nerve of \( \text{Fat} \).)

**Theorem 1.22.** Any chain map \( \phi_\ast \) carried by \( F_\ast \) is a rational homotopy equivalence.

**Remark 1.23.** If we consider the forest carrier \( F_\ast \) as a diagram of chain complexes over \( \mathbb{G}^Z_\ast \) we see that there is an induced chain map from the homotopy pushout of this diagram into \( \mathbb{G}^Z_\ast \). This homotopy pushout is the forested graph complex
\[
C_\ast(\text{Fat}; F_\ast) = \bigoplus_{n} \bigoplus_{\Gamma_\ast \in \mathcal{K}_n(\text{Fat})} \sigma^n \mathbb{Z}(\Gamma_\ast) \otimes F_\ast(\Gamma_0)
\]
where $\mathbb{Z}(\Gamma_\ast)$ is the free abelian group of rank one generated by $\langle \Gamma_\ast \rangle$ and $\sigma^n$ is the $n$-fold suspension operator. Since $F_\ast$ is acyclic, $C_\ast(Fat; F_\ast) \simeq C_\ast(Fat; \mathbb{Z})$. This gives the following diagram.

$$C_\ast(Fat; \mathbb{Z}) \xleftarrow{\mathbb{Z}} C_\ast(Fat; F_\ast) \xrightarrow{\mathbb{Z}} \mathbb{Z}_\ast.$$  

We are claiming that the right hand arrow, given by $p : F_\ast \to \mathbb{Z}_\ast$ for $n = 0$ and zero for $n > 0$, is a rational homotopy equivalence.

**Theorem 1.24.** The dual Kontsevich cycles pull back to well-defined integer cohomology classes

$$\phi^*[W^\lambda_\ast] \in H^{2|\lambda|}(Fat; \mathbb{Z}) \cong \prod_{g,s} H^{2|\lambda|}(M^s; \mathbb{Z}).$$

To prove Theorem 1.22 we will obtain a description of the chain map (7) and construct an explicit rational homotopy inverse using “dual cells.” The proof assumes Proposition 1.21.

### 1.6. Dual cells.

“Dual cells” are elements of $C_\ast(Fat)$ associated to every generator $\langle \Gamma \rangle \in \mathcal{G}_n^\mathbb{Z}$. Every oriented ribbon graph $\Gamma$ has many dual cells but we will see that each of them is necessarily mapped to $(-1)^{\frac{n+1}{2}} \langle \Gamma \rangle$ by any chain map carried by the forest carrier.

A rational inverse

$$\psi : \mathcal{G}_\ast^\mathbb{Z} \otimes \mathbb{Q} \to C_\ast(Fat; \mathbb{Q})$$

is given by mapping each rational generator $\langle \Gamma \rangle$ to the average dual cell in a finite model for $C_\ast(Fat; \mathbb{Q})$ given by choosing one object from every isomorphism class of ribbon graphs and taking the average over all possible dual cells which lie in this finite model.

The composition $\phi \psi$ will be the identity mapping on $\mathcal{G}_\ast^\mathbb{Z} \otimes \mathbb{Q} = \mathcal{G}_\ast \otimes \mathbb{Q}$ and the composition $\phi \psi$ will be homotopic to the identity on the finite model since it is carried by the “identity carrier.” The identity carrier is a canonical acyclic carrier which carries the identity map on the cellular chain complex of any small category. (See Lemma 1.29.)

Suppose that $\Gamma$ is an oriented ribbon graph of codimension $n$. Then a dual cell for $\Gamma$ is given by choosing one representative from every isomorphism class of ribbon graphs over $\Gamma$ (taking the identity map on $\Gamma$ as one representative). Consider all $n$-simplices

$$\Gamma_\ast = (\Gamma_0 \to \cdots \to \Gamma_n = \Gamma)$$

where $\Gamma_i$ is a representative of codimension $i$. Then the dual cell is given by the signed sum of all of these $n$-simplices:

$$D(\Gamma) = \sum o(\Gamma_\ast)(\Gamma_\ast) \in C_n(Fat)$$

where the sign $o(\Gamma_\ast) = \pm 1$ is positive iff the given orientation of $\Gamma$ agrees with the one induced from the natural orientation of the trivalent graph $\Gamma_0$. Note that, if we reverse the orientation of $\Gamma$, this sign will change. So,

$$D(-\Gamma) = -D(\Gamma).$$

It is also trivial to see that, in the case $n = 0$, we have $D(\Gamma) = \Gamma$ assuming that we take the natural orientation on $\Gamma$. 
Lemma 1.25. The boundary of the dual cell is, up to sign, a sum of dual cells:
\[ dD(\Gamma) = (-1)^n \sum \nabla D(\Gamma') \]
where the sum is taken over all chosen representatives
\[ \Gamma' \to \Gamma \]
of all isomorphism classes of ribbon graphs of codimension \( n - 1 \) over \( \Gamma \). We take the orientation on \( \Gamma' \) induced by the given map.

Proof. By Proposition 1.3, the set of isomorphism classes of objects over each \( \Gamma' \) maps monomorphically into the set of isomorphism classes of objects over \( \Gamma \). Therefore, the given choices of representatives for \( \Gamma \) gives a complete set of representatives for the objects over \( \Gamma' \) and \( D(\Gamma') \) is defined.

The boundary of \( D(\Gamma) \) is given by
\[ dD(\Gamma) = \sum_{\Gamma_i} \sum_{i=0}^{n-1} (-1)^i o(\Gamma_*)(\Gamma_0, \cdots, \hat{\Gamma}_i, \cdots, \Gamma_n = \Gamma) \]
\[ + (-1)^{n} \sum_{\Gamma_i} o(\Gamma_*)(\Gamma_0, \cdots, \Gamma_{n-1}) \]
However, the double sum is zero since, when \( \Gamma_i \) is deleted, there are exactly two ways to fill in the blank and these give opposite signs for \( o(\Gamma_*) \). The second sum is equal to the sum of \( D(\Gamma_{n-1}) \) for all possible \( \Gamma_{n-1} \). □

Lemma 1.26. Given any oriented ribbon graph \( \Gamma \) of codimension \( n \), any dual cell \( D(\Gamma) \in C_*(\text{Fat}) \) and any augmented chain map \( \phi : C_*(\text{Fat}) \to \mathbb{Z}_* \) carried by the forest carrier \( F_* \) we will have
\[ \phi(D(\Gamma)) = (-1)^{(n+1)} \langle \Gamma \rangle. \]

Proof. This will be by induction on \( n \). Suppose that \( n = 0 \). Then
\[ \phi(D(\Gamma)) = \phi(\Gamma) = \langle \Gamma \rangle \]
since the identity map \( [\Gamma \to \Gamma] \) is the unique element of \( F_0(\Gamma) \) with augmentation equal to 1.

Now suppose the statement holds for \( n - 1 \). Then by Lemma 1.26 we have
\[ \phi dD(\Gamma) = (-1)^{n} \phi \sum D(\Gamma') = (-1)^{n+\binom{n}{2}} \sum \langle \Gamma' \rangle = (-1)^{(n+1)} d \langle \Gamma \rangle. \]
However, \( F_{n+1}(\Gamma) = 0 \). So, the value of \( \phi D(\Gamma) \) in \( F_n(\Gamma) \) is uniquely determined by the value of its boundary. This forces \( \phi D(\Gamma) \) to be \((n+1) \langle \Gamma \rangle\). □

To construct a rational inverse for the chain map \( \phi \) we choose a finite model for \( \text{Fat} \). Let \( \text{Fin} \) be a full subcategory of \( \text{Fat} \) that contains exactly one object from every isomorphism class. Then \( \text{Fin} \) is a deformation retract of \( \text{Fat} \) and the cellular chain complex of \( \text{Fin} \) is a deformation retract of \( C_*(\text{Fat}) \).

If \( \Gamma \) is any oriented ribbon graph of codimension \( n \), let \( \overline{D}(\Gamma) \in C_n(\text{Fin}; \mathbb{Q}) \) be the average dual cell of \( \Gamma \) given by
\[ \overline{D}(\Gamma) = \sum_{|\text{Aut}(\Gamma_0, \cdots, \Gamma_n)|} o(\Gamma_*) (\Gamma_0 \to \cdots \to \Gamma_n \approx \Gamma) \]
where
\[ \text{Aut}(\Gamma_0, \cdots, \Gamma_n) = \text{Aut}(\Gamma_0) \times \cdots \times \text{Aut}(\Gamma_n) \]
and the sum is taken over all possible sequences of morphism in $\mathcal{F}_{\text{in}}/\Gamma$ so that $\Gamma_i$ has codimension $i$ for each $i$.

**Lemma 1.27.** If $d(\Gamma) = \sum \langle \Gamma' \rangle$ in $G^2_i$ then $dD(\Gamma) = (-1)^n \sum D(\Gamma')$.

**Proof.** The proof is analogous to the proof of Lemma 1.25. We just need to realize that when $\Gamma_i$ is deleted, there are $|\text{Aut}(\Gamma_i)|$ ways to put it back in as an isomorphic copy. Consequently,

$$dD(\Gamma) = (-1)^n \sum_{i=0}^{n-1} (-1)^i \sum_{\text{Aut}(\Gamma_0, \ldots, \Gamma_i, \ldots, \Gamma_n)} o(\Gamma_i) (\Gamma_0 \to \ldots \to \hat{\Gamma}_i \ldots \to \Gamma_n \to \Gamma)$$

Then, in the second sum, the morphism $\Gamma_{n-1} \to \Gamma$ can be uniquely factored through some $\Gamma'$ making it into a sum of terms of the form $(-1)^n D(\Gamma')$. □

This lemma says that we have a chain map

$$\psi : G^*_\mathcal{F}_{\text{in}} \otimes \mathbb{Q} \to C_* \mathcal{F}_{\text{at}}; \mathbb{Q}$$

given by $\psi(\Gamma) = (-1)^{(n+1)} D(\Gamma)$. Lemma 1.26 gives us:

**Theorem 1.28.** The composition

$$G_* \otimes \mathbb{Q} \xrightarrow{\phi} C_* \mathcal{F}_{\text{in}}; \mathbb{Q} \xrightarrow{\psi} C_* \mathcal{F}_{\text{at}}; \mathbb{Q} \xrightarrow{\phi} G_* \otimes \mathbb{Q}$$

is the identity map on $G_* \otimes \mathbb{Q} = G^*_\mathcal{F}_{\text{in}} \otimes \mathbb{Q}$ for any chain map $\phi$ carried by the forest carrier.

Finally, Theorem 1.22 follows from the following.

**Lemma 1.29.** The composition

$$C_* \mathcal{F}_{\text{in}}; \mathbb{Q} \xrightarrow{\phi} C_* \mathcal{F}_{\text{at}}; \mathbb{Q} \xrightarrow{\psi} C_* \mathcal{F}_{\text{in}}; \mathbb{Q}$$

is homotopic to the identity map.

**Proof.** To show that two chain maps are homotopic it suffices to construct an acyclic carrier that carries both of them. In this case it will be the “identity carrier.”

Let $X$ be any object of any small category $\mathcal{A}$. Then the category $\mathcal{A}/X$ of objects over $X$ is contractible since it has a terminal object $id : X \to X$ and any morphism $X \to Y$ induces a functor $A/X \to A/Y$. Consequently, the cellular chain complex $C_* (\mathcal{A}/X)$ is an acyclic carrier from $C_* (\mathcal{A})$ to itself, i.e., a functor from $\mathcal{A}$ into the the category of augmented acyclic chain complexes over $C_* (\mathcal{A})$. We call this the identity carrier since it carries the identity morphism on $C_* (\mathcal{A})$.

In order to show that $\psi \circ \phi$ is carried by the identity carrier we need to show that the chain map $\psi$ is covered by a mapping from the forest carrier to the identity carrier, i.e., we need a natural commuting diagram as follows for all objects $\Gamma$ in $\mathcal{F}_{\text{in}}$.

$$\begin{array}{ccc}
F_* (\Gamma) \otimes \mathbb{Q} & \xrightarrow{\psi} & C_* (\mathcal{F}_{\text{in}}/\Gamma; \mathbb{Q}) \\
\downarrow p & & \downarrow q \\
G_* \otimes \mathbb{Q} & \xrightarrow{\psi} & C_* (\mathcal{F}_{\text{in}}; \mathbb{Q})
\end{array}$$
Recall that the generators of \( F_\ast (\Gamma) \) are isomorphism classes \( [f : \Gamma' \to \Gamma] \) of oriented ribbon graphs over \( \Gamma \). The projection \( p \) sends this to \( \langle \Gamma' \rangle \in G_\ast \otimes \mathbb{Q} \) which then goes to the average dual cell \( \overline{D}(\Gamma') \in C_\ast (\text{Fin}/\mathbb{Q}) \). But the morphism \( f : \Gamma' \to \Gamma \) makes all the terms in the definition \( S \) of \( \overline{D}(\Gamma') \) into simplices in \( \text{Fin}/\Gamma \) and therefore defines a lifting \( \tilde{\psi} = f_\ast (\overline{D}(\Gamma')) \) of \( \overline{D}(\Gamma') \) to \( C_\ast (\text{Fin}/\Gamma; \mathbb{Q}) \) as required.

\[ \Box \]

2. Stasheff associahedra

We use several different versions of the Stasheff associahedron: the convex \( n \)-dimensional polyhedron \( K^n \), the category \( A_{n+3} = \text{simp} K^{n-3} \) of simplices in \( K^{n-3} \) and the simplicial nerve of \( A_n \) which is a triangulation of the polyhedron \( K^{n-3} \).

An outline of this section:

1. Stasheff polyhedron \( K^n \).
2. The category \( A_{n+3} \).
3. Orientation of \( K^n \).
4. Orientation of \( K^{n, \text{odd}} \).
5. Proof of Proposition 1.21.

2.1. Stasheff polyhedron \( K^n \).

**Theorem 2.1** (Stasheff [Sta63]). There is an \( n \)-dimensional convex polyhedron \( K^n \) whose elements correspond to isomorphism classes of planar metric trees with \( n+3 \) leaves of fixed length and up to \( n \) internal edges of variable length \( \leq 1 \). Two planar metric trees lie in the same open face of \( K^n \) if and only if, after collapsing all internal edges of length \( < 1 \), they become isomorphic fixing the leaves.

Let \( C_\ast (K^n) \) be the cellular chain complex of \( K^n \). Then \( C_0 (K^n) \) is freely generated by isomorphism classes of trivalent planar trees with \( n+3 \) given leaves. Recall that any such tree has a natural orientation. More generally we have the following.

**Proposition 2.2.** For all \( 0 \leq k \leq n \), \( C_k (K^n) \) is generated by isomorphism classes \( [T] \) of oriented planar trees \( T \) with \( n+3 \) fixed leaves and \( n-k \) internal edges. The boundary map is given by

\[
d[T] = \sum_{[T', e]} [T']
\]

where the sum is taken over all isomorphism classes of pairs \( (T', e) \) where \( e \) is an edge in an oriented tree \( T' \) so that \( T \cong T'/e \) with the induced orientation.

**Remark 2.3.** This is more or less a tautology since we choose the geometric orientation of the faces to make this algebraic statement true. We also note that trees with fixed leaves have no nontrivial automorphisms. Therefore,

\[
(T) = |\text{Aut}(T)||[T] = [T].
\]

**Proof.** Let \( T_0 \) be a trivalent planar tree with \( n+3 \) fixed leaves \( h_0, \ldots, h_{n+2} \) (in cyclic order) and \( n \) internal edges \( e_1, \ldots, e_n \). Since \( T_0 \) is trivalent, it has an intrinsic orientation. If we collapse the edges \( e_1, \ldots, e_k \) in that order we get a tree \( T_k \) of codimension \( k \) with the induced orientation. The trees \( T_k \) for various \( k \) are related by

\[
T_k = T_{k-1}/e_k
\]
The $k$ dimensional face of $K^n$ corresponding to the tree $T_k$ consists of isomorphism class of trees $T$ having edges $e_1, \ldots, e_k$ of variable length $< 1$ and the other edges of length equal to 1. Let $x_1, \ldots, x_k$ be the lengths of $e_1, \ldots, e_k$ then we choose the geometric orientation of this face of $K^n$ (in a neighborhood of the vertex $T_0$) by taking these coordinates in opposite order: $$ (x_k, \ldots, x_1). $$

When $x_k$ reaches 1 we get to the face corresponding to $T_{k-1}$. The orientation of the $k-1$ face is therefore related to that of the $k$-face by the “first vector points outward” rule which is standard. □

2.2. The category $A_n$. Let $A_{n+3}$ be the category of faces of $K^n$ with inclusion maps as morphisms. Then the geometric realization of the nerve of $A_{n+3}$ is homeomorphic to $K^n$. A homeomorphism

$$ \phi : |A_{n+3}| \to K^n $$

is given by sending each object (=vertex in the nerve) to some point in the interior of the corresponding face of $K_n$ and extending linearly over the simplices.

Take an $n$-simplex

(9) $$ T_* = (T_0 \to T_1 \to \cdots \to T_n) $$

in $A_{n+3}$ which is nondegenerate in the sense that $T_0$ is trivalent and each $T_i$ is obtained from $T_{i-1}$ by collapsing one edge. Then, by induction, each tree $T_i$ obtains an induced orientation from the intrinsic orientation of $T_0$. The orientation of the tree $T_n$ gives a geometric orientation of the polyhedron $K^n$ as explained in the proof of Proposition 2.2.

Suppose that an orientation for the tree $T_n$ representing the top cell of $K^n$ is given. (For example, when $n$ is even, we can take the intrinsic orientation of $T_n$.) Then the algebraic orientation $o(T_*)$ of the $n$-simplex $T_*$ in (9) is defined to be $\pm 1$ depending on whether the given orientation of $T_n$ is equal to the one induced from $T_0$. The dual cell $D(T_n)$ is defined by

$$ D(T_n) = \sum o(T_*)T_* \in C_n(A_{n+3}) $$

where the sum is over all nondegenerate $n$-simplices $T_*$. This is an $n$-chain in the cellular chain complex of $A_{n+3}$. It represents a triangulation of the top cell of $K^n$ with algebraic orientation which does not agree with the geometric orientation.

2.3. Orientation of $K^n$.

**Lemma 2.4.** The embedding

$$ \Delta^n \to K^n $$

given by the $n$ simplex (9) has degree $(-1)^{\frac{n+1}{2}}$ with respect to the orientation of $K^n$ corresponding to the orientation of $T_n$ induced from that of $T_0$.

**Remark 2.5.** This implies that, given any orientation of $T_n$, the corresponding geometric orientation of $K^n$ agrees with

$$ \psi[T_n] := (-1)^{\frac{n+1}{2}} D(T_n). $$
Proof. For any $0 \leq k \leq n$ we claim that the map 

$$\sigma^k : \Delta^k \to F_k$$

given by $T_0 \to \cdots \to T_k$ has degree $(-1)^{\binom{k+1}{2}}$ where $F_k$ is the face of $K^n$ corresponding to $T_k$ with the induced orientation.

This statement holds for $k = 0$. Suppose it holds for $k - 1$. Since the front $k - 1$ face $\Delta^{k-1}$ is opposite the $k$-th vertex $T_k$, its orientation is equal to $(-1)^k$ times the induced orientation from $\Delta^k$. Whereas, we are orienting the faces to make Proposition 2.2 true, i.e., the orientation of $F_{k-1}$ is the one induced from $F_k$. Consequently the degree of $\sigma^k$ is

$$(-1)^{\binom{k}{2}}(-1)^k = (-1)^{\binom{k+1}{2}}.$$ 

Putting $k = n$ we get the lemma. \(\square\)

Now suppose that $n = 2k$. Then $T_{2k}$ has an intrinsic orientation which determines an intrinsic orientation for the polyhedron $K^{2k}$.

**Theorem 2.6.** The sign of the embedding

$$\sigma^{2k} : \Delta^{2k} \to K^{2k}$$

given by the $2k$ simplex $(T_0 \to T_1 \to \cdots \to T_{2k})$ is as given in [Igu].

In [Igu] the sign of $\sigma^{2k}$ is defined to be the sign of the permutation of $2k + 3$ letters:

$$\text{sgn}(\sigma^n) = \text{sgn}(a_1, a_2, a_3, b_1, b_2, \cdots, b_{2k})$$

where $a_i, b_j$ are the regions (components of the complement of the tree in the disk in which it can be embedded with leaves on the boundary) given as follows. The regions which bound the edge $e_1$ (so that $T_1 = T_0/e_1$) are $a_1, a_3$ (in either order). Let the two other regions which touch $e_1$ be $a_2, b_1$ so that they are $a_1, a_2, a_3, b_1$ in cyclic order. For $i \geq 2$ let $v_i$ be the vertex of $e_i$ furthest away from $e_1$ and let $b_i$ be the region which touches $e_i$ at the point $v_i$. Then $\text{sgn}(\sigma^n)$ is equal to the sign of the permutation of $a_1, a_2, a_3, b_1, \cdots, b_{2k}$ which puts these regions into the correct cyclic order.

Proof. We have three definitions of the orientation of $K^{2k}$:

(a) The intrinsic orientation of $K^{2k}$ induced from the intrinsic orientation of $T_{2k}$.

(b) The orientation of $K^{2k}$ induced from that of $T_0$ by the sequence of maps $T_0 \to \cdots \to T_{2k}$.

(c) The orientation of $K^{2k}$ induced from $\Delta^{2k}$ by the map $\sigma^{2k}$.

The statement we are trying to prove is that (a) and (c) differ by the sign convention given above. We know, by Lemma 2.4, that (b) and (c) differ by the sign

$$(-1)^\binom{2k+1}{2} = (-1)^k.$$ 

Therefore, it suffices to show that the sign difference between (a) and (b), which is equal to $o(T_s)$ by definition, is given by

$$o(T_s) = (-1)^k \text{sgn}(\sigma^{2k}) = \text{sgn}(a_1, a_2, a_3, b_{2k}, \cdots, b_1).$$

But this is a special case of the following lemma. \(\square\)
Lemma 2.7. Suppose that $T_0$ is a planar tree with internal edges $e_1, \cdots, e_{2k}$ and $2k+1$ vertices all trivalent except for $v_0$ which has valence $2n+1$. Let $v_i$ be the vertex on $e_i$ furthest away from $v_0$. Let $T_1, \cdots, T_{2k}$ be given by $T_i = T_{i-1}/e_i$. Then the difference $o(T_*)$ between the intrinsic orientation of $T_{2k}$ and the one induced from $T_0$ is given by

$$o(T_*) = (-1)^k \text{sgn}(a_1, \cdots, a_{2n+3}, b_1, \cdots, b_{2k})$$

where $a_i$ are the regions around $v_0$ and $b_i$ is the region which touched $e_i$ only at $v_i$.

Proof. First we claim that the truth value of this statement remains unchanged if we permute the edges $e_1, \cdots, e_{2k}$. To see this suppose we change the order of $e_i, e_{i+1}$. Then $T_i$ will become a different tree and the orientations of $T_{i+1}, \cdots, T_{2k}$ will be reversed. This comes from the proof that $\partial^2 = 0$ in graph homology. Switching $e_i, e_{i+1}$ will also transpose the labels $b_i, b_{i+1}$. So the sign (11) will also change so the relative sign remains unchanged.

By permuting the order of the edges we may assume that each tree $T_i$ has only one vertex $v_0$ which is not trivalent and $e_{i+1}$ becomes an edge in $T_i$ connecting $v_0$ to $v_{i+1}$. Consequently, $T_{2k-2}$ has an intrinsic orientation and, by induction on $k$, this orientation differs from the one induced from $T_0$ by

$$o(T_0, \cdots, T_{2k-2}) = (-1)^{k-1} \text{sgn}(a_1, \cdots, a_{2n+3}, b_1, \cdots, b_{2k-2})$$

To prove that (12) implies (11) we look at the difference between the two signs and the difference between the intrinsic orientation of $T_{2k}$ and the one induced from the intrinsic orientation of $T_{2k-2}$. The statement that we need is exactly the statement of the lemma in the case $k = 1$.

Now we assume that $k = 1$. As before, we may assume that $e_1$ connects $v_0$ to $v_1$ but there are two possibilities for $e_2$.

Case 1. $e_2$ connects $v_0$ and $v_2$.

If we let $a_1, \cdots, a_m$ be the regions at $v_0$ from $e_1$ to $e_2$ as shown in Figure 2 then the intrinsic orientation of $T_0$ and induced orientations of $T_1, T_2$ are

$$(T_0) \quad v_0 e_1^+ h_1 h_2 \cdots h_{m-1} e_2^+ h_{m+2} \cdots h_{2n+3} v_1 e_1^+ h_{2n+4} h_{2n+5} v_2 e_2^+ h_m h_{m+1}$$

![Figure 2](image-url)
The induced orientation of $T$ are the halves of $e$ of $a$ of $h$ so the permutation sign is $\sgn(a_1, \ldots, a_{2n+3}, b_1, b_2) = (-1)^m$ multiplying by $(-1)^k = -1$ we get (11).

Case 2. $e_2$ connects $v_1$ and $v_2$.
Here there are two subcases as shown in Figure 3.

2a $b_1$ is clockwise from $b_2$.
2b $b_2$ is clockwise from $b_1$.

In subcase 2a we have the following orientations on $T_0, T_1, T_2$ induced from the intrinsic orientation of $T_0$.

$$(T_0) \quad v_0 e_1 h_2 \cdots h_{2n+2} v_1 e^{+}_1 h_{2n+3} e^-_2 v_2 e_{2}^+ h_{2n+4}$$

$$(T_1) \quad -v_0 h_2 \cdots h_{2n+3} e^-_2 v_2 e_{2}^+ h_{2n+4} = v_0 e_2 v_2 e_{2}^+ h_1 \cdots h_{2n+5}$$

$$(T_2) \quad v_0 h_2 \cdots h_{2n+5}$$

The induced orientation of $T_2$ is negative the natural orientation. But

$$\sgn(a_1, \ldots, a_{2n+3}, b_1, b_2) = +1$$

since the regions are in the correct cyclic order. Thus the lemma holds in this case.

In subcase 2b the induced orientations on $T_0, T_1, T_2$ are:

$$(T_0) \quad v_0 e^-_1 h_2 \cdots h_{2n+2} v_1 e^{+}_1 h_{2n+3} e^-_2 v_2 e_{2}^+ h_{2n+4}$$

$$(T_1) \quad -v_0 h_2 \cdots h_{2n+3} e^-_2 v_2 e_{2}^+ h_{2n+4} = -v_0 e_2 v_2 e_{2}^+ h_1 \cdots h_{2n+5}$$

$$(T_2) \quad v_0 h_2 \cdots h_{2n+5}$$

The induced orientation on $T_2$ is equal to the natural orientation. But

$$\sgn(a_1, \ldots, a_{2n+3}, b_1, b_2) = -1$$

so the lemma holds in this final case.  \hfill \square
2.4. Orientation of $K^{odd}$. Suppose that $n$ is odd. Then the planar trees in $K^n$ have an even number $(n + 3)$ of leaves. An orientation of these trees is given by taking a fixed cyclic ordering of these leaves, starting at one point and going counterclockwise. This gives an orientation of a tree with one vertex and thus of the top cell of $K^n$. For any other tree $T \in K^n$ an orientation is given by choosing an ordering of the internal edges of $T$. The orientation given by this ordering is the one which induces the chosen orientation on the one vertex tree by collapsing the edges in order.

We can now prove Proposition 2.21.

2.5. Proof of Proposition 2.21. Recall that we have a ribbon graph $\Gamma_0$ all of whose vertices are trivalent except for $v_1, \ldots, v_r$ which have valence $n_1 + 3, \ldots, n_r + 3$. For each $i$ we choose a cyclic ordering of the half-edges at $v_i$ and take the orientation on $\Gamma_0$ given by $v_1$ followed by its half-edges, $v_2$ followed by its half-edges, etc.

Not suppose that for each $i$, $T_i$ is a planar tree with $n_i + 3$ leaves representing an $m_i$-face of $K^{n_i}$. Thus $T_i$ has $n_i - m_i$ internal edges. Choose an ordering of these edges. Let $\Gamma_0(T_1, \ldots, T_r)$ denote the graph obtained from $\Gamma_0$ by replacing $v_i$ by $T_i$. Then the $T_i$ will form a forest in $\Gamma_0$. Let $e_{i1}, e_{i2}, \cdots$ be the internal edges of $T_i$. Take the orientation on $\Gamma_0(T_1, \cdots, T_r)$ so that, if the edges $e_{ij}$ are collapsed in lexicographic order, we get the chosen orientation on $\Gamma_0$.

Let

$$
\phi : C_{m_1}(K^{n_1}) \otimes \cdots \otimes C_{m_r}(K^{n_r}) \to F_m(\Gamma_0)
$$

where $m = m_1 + \cdots + m_r$ be given by

$$
\phi([T_1] \otimes \cdots \otimes [T_r]) = [\Gamma_0(T_1, \cdots, T_r) \to \Gamma_0].
$$

We claim that this gives a chain isomorphism

$$
\phi : C_*(K^{n_1}) \otimes \cdots \otimes C_*(K^{n_r}) \to F_*(\Gamma_0).
$$

Since $\phi$ sends basis elements to basis elements, it suffices to show that $\phi$ is a chain map. But this is straightforward.

The boundary of $[T_1] \otimes \cdots \otimes [T_r]$ is, by Proposition 2.22, equal to

$$
\sum_{i=1}^r (-1)^{m_1 + \cdots + m_{i-1}} \sum [T_1] \otimes \cdots \otimes [T_{i-1}] \otimes [T_i'] \otimes [T_{i+1}] \otimes \cdots \otimes [T_r]
$$

where the second sum is over all pairs $(T_i', e_{i0})$ so that $T_i' / e_{i0} \cong T_i$. But $(-1)^{m_1 + \cdots + m_{i-1}}$ is also the sign of the permutation which brings the edge $e_{i0}$ to the beginning in the ordering of all edges of $\Gamma_0(T_1, \cdots, T_i', \cdots, T_r)$. So

$$
d\phi([T_1] \otimes \cdots \otimes [T_r]) = \sum (-1)^{m_1 + \cdots + m_{i-1}} \phi([T_1] \cdots [T_i'] \cdots [T_r])
$$

as required.

3. Miller-Morita-Mumford classes

We review the definition of the adjusted Miller-Morita-Mumford classes. We pull them back to the graph cohomology complex $G_*$ and we describe what happens when we evaluate them on the dual Kontsevich cycles.

(1) Cyclic set cocycle.
(2) Adjusted MMM classes in $H^{2k}(G_*; \mathbb{Q})$.
(3) Cup products of adjusted MMM classes.
(4) Computing the numbers $b^*_{n^*}$.
(5) Kontsevich cycles in terms of MMM classes.
(6) Computing $a^\mu_{\lambda}$.

Suppose that

$$\Sigma_g^s \to E \overset{p}{\to} B$$

is a compact manifold bundle where $\Sigma_g^s$ is an oriented connected surface of genus $g$ and $s$ unordered distinguished points. Let

$$\pi : \tilde{B} \to B$$

be the $s$-fold covering space given by the $s$ distinguished point in each fiber of $E$. Then the vertical tangent bundle of $E$ is an oriented 2-plane bundle and therefore has an Euler class $e(E) \in H^2(E; \mathbb{Z})$. The push-down of the $k + 1$-st power of this class is the Miller-Morita-Mumford class

$$\kappa_k (E) = p_* (e(E)^{k+1}) \in H^{2k}(B; \mathbb{Z}).$$

The restriction of the Euler class $e(E)$ to $\tilde{B} \subseteq E$ gives another Euler class $e(\tilde{B}) \in H^2(\tilde{B}; \mathbb{Z})$. The push-down of the $k$-th power of this second class is the boundary class

$$\gamma_k (E) = \pi_* (e(\tilde{B})^k) \in H^{2k}(B; \mathbb{Z})$$

The adjusted or punctured Miller-Morita-Mumford classes $\tilde{\kappa}_k (E)$ are given by

$$\tilde{\kappa}_k (E) = \kappa_k (E) - \gamma_k (E).$$

The surface bundle $\Sigma_g^s \to E \to B$ is classified by a map $f : B \to BM_g^s$ and all three cohomology classes defined above are pull-backs of universal classes

$$\kappa_k, \gamma_k, \tilde{\kappa}_k \in H^{2k}(M_g^s; \mathbb{Z}) \cong H^{2k}(Fat_g^s; \mathbb{Z}).$$

By the fundamental results of Morita [Mor87] and Miller [Mil86] the Miller-Morita-Mumford classes $\kappa_k$ and the first $s$ boundary classes $\gamma_1, \cdots, \gamma_s$ are algebraically independent over $\mathbb{Q}$ in the stable range (given by Harer stability [Har85]). The classes $\gamma_r$ for $r > s$ are polynomials in $\gamma_1, \cdots, \gamma_s$. For example,

$$\gamma_3 = 3\gamma_1\gamma_2 - \frac{1}{2}\gamma_1^3$$

if $s = 2$. Since the category $Fat$ is the disjoint union of $Fat_g^s$ for all $g, s$, this also extends to $\tilde{\kappa}_0 = 1 - 2g - s$ and we have the following.

**Theorem 3.1.** The universal adjusted Miller-Morita-Mumford classes

$$\tilde{\kappa}_k \in H^* (Fat; \mathbb{Q})$$

for $k \geq 0$ are algebraically independent.

In [Igu02], a combinatorial rational cocycle is constructed for the punctured class $\tilde{\kappa}_k$. It is called the “cyclic set cocycle.”
3.1. Cyclic set cocycle. Let $Z$ be the category of cyclically ordered sets and cyclic order preserving monomorphisms. Then it is well known that 

$$|Z| \simeq BU(1).$$

The $k$-th power of the first Chern class of the canonical complex line bundle over $Z$ is given by the unadjusted cyclic set cocycle $c^k_Z$ whose value on a $2k$-simplex 

$$C_* = (C_0 \to C_1 \to \cdots \to C_{2k})$$

is given by

$$c^k_Z(C_*) = \frac{\sum sgn(a_0, a_1, \cdots, a_{2k})}{(-2)^k(2k - 1)!!|C_0| \cdots |C_{2k}|}$$

where the sum is taken over all choices of elements $a_i \in C_i - C_{i-1}$. (In our notation we pretend that the maps in $C_*$ are inclusion maps. Strictly speaking, $a_i$ should be an element of $C_{2k}$ which lies in the image of $C_i$ but not in the image of $C_{i-1}$.)

The unadjusted cyclic set cocycle has the following obvious property.

**Proposition 3.2.** In order for the unadjusted cyclic set cocycle $c^k_Z$ to be nonzero on $C_*$ it is necessary (but not sufficient) for each $C_i$ to be larger than $C_{i-1}$, i.e., 

$$|C_0| < |C_1| < \cdots < |C_{2k}|.$$

In [Igu02] it is shown that the adjusted Miller-Morita-Mumford class $\widetilde{\kappa}_k$ is given on the category of ribbon graphs by evaluating the unadjusted cyclic set cocycle (13) at each vertex and dividing by $-2$. Since there is already a factor of $(-2)^k$ in the denominator it seems reasonable to define the (adjusted) cyclic set cocycle $\widetilde{c}^k_Z$ by 

$$\widetilde{c}^k_Z(C_*) = \frac{1}{-2} c^k_Z(C_*) = \frac{\sum sgn(a_0, a_1, \cdots, a_{2k})}{(-2)^{k+1}(2k - 1)!!|C_0| \cdots |C_{2k}|}.$$

**Theorem 3.3 ([Igu02]).** The adjusted rational Miller-Morita-Mumford class $\widetilde{\kappa}_k \in H^{2k}(\text{Fat}; \mathbb{Q})$ is given on a $2k$-simplex 

$$\Gamma_* = (\Gamma_0 \to \cdots \to \Gamma_{2k})$$

in $\text{Fat}$ by evaluating the cyclic set cocycle $\widetilde{c}^k_Z$ on every vertex of $\Gamma_0$ counted with multiplicity.\textsuperscript{7}

We use the notation $\widetilde{c}^k_{\text{Fat}}$ to denote this rational cocycle on the cellular chain complex of $\text{Fat}$ and we refer to it by the same name, i.e., $\widetilde{c}^k_{\text{Fat}}$ is the cyclic set cocycle on the category of ribbon graphs.

By definition, $\widetilde{c}^k_{\text{Fat}}$ satisfies the following important condition.

**Proposition 3.4.** The value of $\widetilde{c}^k_{\text{Fat}}$ on a $2k$ can be nonzero only if there is at least one vertex $v_0$ of $\Gamma_0$ whose image in each $\Gamma_i$ has greater valence than its image in $\Gamma_{i-1}$.

Using dual cells we can pull back the adjusted Miller-Morita-Mumford classes to the rational cohomology of the graph cohomology complex $G_*$.\textsuperscript{7}

\textsuperscript{7}The multiplicity of a vertex is defined to be the valence minus 2.
3.2. **Adjusted MMM classes in** $H^{2k}(G_\ast; \mathbb{Q})$. Recall that there is a chain homotopy equivalence

$$\psi : G_\ast \otimes \mathbb{Q} \to C_\ast(Fat; \mathbb{Q})$$

given up to sign by sending each rational generator $\langle \Gamma \rangle \in G_n$ to the average dual cell of $\Gamma$ inside a finite model $\text{Fin}$. 

$$\psi(\langle \Gamma \rangle) = (-1)^{\frac{n+1}{2}} D(\Gamma).$$

Since the cyclic set cocycle $\bar{\kappa}_{\text{Fat}}$ has the same value on isomorphic simplices, its value on any dual cell of $\Gamma$ is the same. Therefore, its value of the average dual cell $D(\Gamma)$ is equal to its value on any particular dual cell $D(\Gamma)$.

**Definition 3.5.** The cyclic set cocycle

$$\bar{\kappa}_G : G_{2k} \to \mathbb{Q}$$
on the graph cohomology complex $G_\ast$ is given by

$$\bar{\kappa}_G(\Gamma) = (-1)^k \bar{\kappa}_\text{Fat} (D(\Gamma))$$

for any choice of dual cell $D(\Gamma)$.

**Theorem 3.6.** The cyclic set cocycle $\bar{\kappa}_G$ represents the adjusted Miller-Morita-Mumford class:

$$[\bar{\kappa}_G] = \psi^*([\bar{\kappa}_n]) \in H^{2k}(G_\ast; \mathbb{Q}).$$

It is clear from the definition of the cyclic set cocycle that it can only be nonzero on the Witten cycle $W_k$ and it has the same value on $\langle \Gamma \rangle$ for every element $\Gamma$ of $W_k$ with natural orientation. Therefore, $\bar{\kappa}_G$ is proportional to the dual Witten cycle $W^*_k$. The proportionality constant was computed in [Igu].

**Theorem 3.7.** We have the following equation of rational cocycles on the associative graph cohomology complex $G_\ast$.

$$W^*_k = (-2)^{k+1}(2k + 1)! \bar{\kappa}_G^k$$

**Proof.** As we showed in Theorem 2.8 and Remark 2.5, the sign convention used in [Igu] for every simplex in the dual cell of any $\Gamma \in W_k$ is given by $(-1)^k o(\Gamma_\ast)$ which agrees with the coefficient of the same term in

$$\psi(\Gamma) = (-1)^k \sum o(\Gamma_\ast)(\Gamma_0, \cdots, \Gamma_{2k}).$$

Putting these together we get the following which can be interpreted as statement about the relationship between rational cohomology classes in graph cohomology, the category of ribbon graphs or the mapping class group.

**Corollary 3.8 (Igu).** The dual Witten cycle $[W^*_n]$ is a multiple of the adjusted Miller-Morita-Mumford class:

$$[W^*_n] = (-2)^{n+1}(2n + 1)! \bar{\kappa}_n.$$
For example, we have:

\[ W^*_0 = -2\tilde{\kappa}_0 \]
\[ W^*_1 = 12\tilde{\kappa}_1 \]
\[ W^*_2 = -120\tilde{\kappa}_2 \]
\[ W^*_3 = 1680\tilde{\kappa}_3 \]
\[ W^*_4 = -30240\tilde{\kappa}_4 \]

3.3. Cup products of adjusted MMM classes. We now consider cup products of the \( \tilde{\kappa}_k \)'s for \( k \geq 1 \). If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \) is a partition of \( n \) in the sense that \( \lambda_i > 0 \) and \( n = \sum \lambda_i \) then let \( \tilde{\kappa}_\lambda \) denote the cup product

\[ \tilde{\kappa}_{\lambda_1} \cup \cdots \cup \tilde{\kappa}_{\lambda_s} \in H^{2n}(\mathcal{F}at; \mathbb{Z}). \]

Since the \( \tilde{\kappa}_{\lambda_i} \) are even degree classes, this cup product does not depend on the order of the \( \lambda_i \)’s. However, at the chain level, the order does matter. (The order matters in \( \mathcal{F}at \) but not in the graph cohomology complex by Corollary 3.22 below.) Let \( \tilde{c}_\lambda^{\mathcal{G}_{2n}} \) denote the cup product

\[ \tilde{c}_\lambda^{\mathcal{G}_{2n}} = \tilde{c}_{\lambda_1}^{\mathcal{G}_{2n}} \cup \cdots \cup \tilde{c}_{\lambda_s}^{\mathcal{G}_{2n}} \in C^{2n}(\mathcal{F}at; \mathbb{Q}) \]

and let

\[ \tilde{c}_\lambda^{\mathcal{G}_s} = \psi^* \tilde{c}_\lambda^{\mathcal{G}_{2n}} \in \operatorname{Hom}(\mathcal{G}_{2n}, \mathbb{Q}) \]

denote the pull-back of \( \tilde{c}_\lambda^{\mathcal{G}_{2n}} \) to \( \mathcal{G}_s \). Then, by Theorem 3.3, we have

\[ [\tilde{c}_\lambda^{\mathcal{G}_s}] = \psi^* \tilde{\kappa}_\lambda. \]

Lemma 3.9. If \( \tilde{c}_\lambda^{\mathcal{G}_s} \) is nonzero on \( (\Gamma) \in \mathcal{G}_{2n} \) then \( \Gamma \) must lie in some Kontsevich cycle \( W_\mu \) where \( \lambda \) is a refinement of \( \mu \).

Definition 3.10. \( \lambda = (\lambda_1, \cdots, \lambda_s) \) is a refinement of \( \mu = (\mu_1, \cdots, \mu_r) \) and we write

\[ \lambda \leq \mu \]

if each \( \mu_i \) is a sum of \( \lambda_j \)'s so that, if we let \( \pi(i) \) be the set of these indices \( j \), then \( \pi \) is a partition of the set \( \{1, 2, \cdots, s\} \) into \( r \) parts and

\[ \lambda_{\pi(i)} := \sum_{j \in \pi(i)} \lambda_j = \mu_i. \]

We say that \( \pi \) represents \( \lambda \) as a refinement of \( \mu \).

Proof. In order for the cup product \( (16) \) to be nonzero on a dual cell \( D(\Gamma) \) where \( \Gamma \in \mathcal{G}_{2n} \) there must be a nondegenerate \( 2n \)-simplex (i.e., where none of the morphisms are isomorphisms)

\[ \Gamma_0 \rightarrow \Gamma_1 \rightarrow \cdots \rightarrow \Gamma_{2n} \equiv \Gamma \]

so that

\[ \tilde{c}_{\lambda_1}^{\mathcal{G}_{2n}}(\Gamma_0 \rightarrow \cdots \rightarrow \Gamma_{2\lambda_1}) \neq 0, \]
\[ \tilde{c}_{\lambda_2}^{\mathcal{G}_{2n}}(\Gamma_{2\lambda_1} \rightarrow \cdots \rightarrow \Gamma_{2\lambda_1+2\lambda_2}) \neq 0, \]

etc.

By Proposition 3.4 \( \Gamma_{2\lambda_1} \) must have a vertex of multiplicity \( 2\lambda_1 \) and it must have a vertex which increases in multiplicity by \( 2\lambda_2 \) by the time it gets to \( \Gamma_{2\lambda_1+2\lambda_2} \). This implies that \( \Gamma_{2\lambda_1+2\lambda_2} \) must have either a vertex of multiplicity \( 2\lambda_1+2\lambda_2 \) or two vertices of multiplicity \( 2\lambda_1, 2\lambda_2 \), resp.
By induction, $\Gamma = \Gamma_{2\lambda_1 + \cdots + 2\lambda_{s-1}} = \Gamma_{2n-2\lambda_s}$ must lie in a Kontsevich cycle $W_\nu$ so that $(\lambda_1, \cdots, \lambda_{s-1})$ is a refinement of $\nu$ as a partition of $n - \lambda_s$. In order for $c_{\mathcal{F}_{rat}}^{n-\lambda_s}$ to be nonzero on the back $\lambda_s$ face of $\Gamma_*$, the graph $\Gamma$ must have a vertex which increases in multiplicity by $2\lambda_s$ by the time it gets to $\Gamma_{2n}$. Thus $\Gamma_{2n}$ must lie in $W_\mu$ where either $\mu = (\nu, \lambda_s)$ or $\mu$ is equal to $\nu$ with one of the $\nu_i$ increased by $\lambda_s$. In either cases,

$$\lambda \leq (\nu, \lambda_s) \leq \mu$$

as claimed. 

\[\square\]

**Lemma 3.11.** Suppose that $\Gamma \in W_\mu$ where $\mu = (\mu_1, \cdots, \mu_r)$ is a partition of $n$ and $\lambda$ is a refinement of $\mu$. Then the value of $\tilde{c}_G^{\lambda}$ on $\langle \Gamma \rangle$ depends only on the ordered partition $\lambda$ and the unordered partition $\mu$ (and is independent of the choice of $\Gamma \in W_\mu$).

**Remark 3.12.** We will denote this number by

\[(17)\quad b_\mu^\lambda := \tilde{c}_G^{\lambda}(\Gamma) = \psi^* c_{\mathcal{F}_{rat}}^{\lambda}(\Gamma) = (-1)^n c_{\mathcal{F}_{rat}}^{\lambda} D(\Gamma) \in \mathbb{Q}.
\]

It is obvious that the order of the $\mu_i$’s is not important. We will later show (Corollary 3.22) that the order of the $\lambda_j$’s is also not important.

**Proof.** If $\Gamma$ lies in $W_\mu$ then $\Gamma$ has vertices $v_1, \cdots, v_r$ of codimension $2\mu_1, \cdots, 2\mu_r$. In any dual cell for $\Gamma$, each of these vertices is expanded to a tree in all possible ways (up to isomorphism). The rest of the graph is left fixed. However, the cyclic set cocycle is only evaluated on the vertices of these trees. Since the orientation of the simplices in the dual cell depend only on these trees and the value of the cocycle depends only on the trees, which in turn depend only on the numbers $\mu_1, \cdots, \mu_r$ the value of $c_{\mathcal{F}_{rat}}^{\lambda}$ on $D(\Gamma)$ depends only on $\mu$. \[\square\]

Putting these two lemmas together we get the following.

**Theorem 3.13.** Any cup product of cyclic set cochains

$$\tilde{c}_G^{\lambda} = \psi^* \left( c_{\mathcal{F}_{rat}}^{\lambda} \cup \cdots \cup c_{\mathcal{F}_{rat}}^{\lambda} \right)$$

can be expressed as a rational linear combinations of dual Kontsevich cycles by

$$\tilde{c}_G^{\lambda} = \sum b_\mu^\lambda W_\mu^*$$

where the sum is over all partitions $\mu$ of $n = \sum \lambda_j$ so that $\lambda$ is a refinement of $\mu$ and $b_\mu^\lambda$ is given by (17) above.

**Remark 3.14.** At the level of cohomology this theorem implies that

$$\tilde{\kappa}_\lambda = \sum b_\mu^\lambda [W_\mu^*].$$

Once we show that the cohomology classes $[W_\mu^*]$ are linearly independent (Corollary 3.21), then this equation can be used to define $b_\mu^\lambda$. We can then conclude that $b_\mu^\lambda$ and $\tilde{c}_G^{\lambda}$ are independent of the order of $\lambda$ (Corollary 3.22).
3.4. Computing the numbers $b^\mu_\lambda$. We first show that the computation of the numbers $b^\mu_\lambda$ can be reduced to the case when $\mu = n$ is the trivial partition of $n$.

**Lemma 3.15.** If $\mu = (\mu_1, \cdots, \mu_r)$ and $\lambda = (\lambda_1, \cdots, \lambda_s)$ are partitions of $n$ then

$$b^\mu_\lambda = \sum_{\pi} \prod_{i=1}^{r} b^{\mu_i}_{\lambda_{\pi(i)}}$$

where we sum over all partitions $\pi$ of the set $\{1, \cdots, s\}$ into $r$ parts $\pi(i)$ representing $\lambda$ as a refinement of $\mu$ in the sense that $\mu_i = \sum_{j \in \pi(i)} \lambda_j$.

**Example 3.16.** Take the partition $\mu = (5, 3)$ of $n = 8$ and the refinement $\lambda = (3, 2, 1, 1, 1)$. Then

$$b^5_3 = b^5_{3,2,1,1,1} = b^5_{3,2}b^3_{1,1,1} + 3b^5_{3,1,1}b^2_{2,1} + b^5_{2,1,1,1}b^3_3.$$ 

These terms come from the 5 ways in which $(5, 3)$ can be refined to $(3, 2, 1, 1, 1)$. They are

$$(\{3, 2\}, \{1, 1, 1\}), 3 \times (\{3, 1, 1\}, \{2, 1\}), (\{2, 1, 1, 1\}, \{3\}).$$

There are three ways to do the second refinement depending on which 1 goes to the right (to 3 in $(5, 3)$).

**Proof.** The number $b^\mu_\lambda$ from \[17\] times $(-1)^n$ is given by evaluating the cup product $\tilde{c}^\lambda_{\Gamma \at}$ of the cocycles $\tilde{c}^\lambda_{\Gamma \at}$ on every term of the dual cell $D(\Gamma)$ of any ribbon graph $\Gamma$ in $W_\mu$.

The dual cell is given by choosing graphs over $\Gamma$ and taking nondegenerate 2n-simplices

$$\Gamma_s = (\Gamma_0 \rightarrow \cdots \rightarrow \Gamma_{2n} \cong \Gamma)$$

(times $o(\Gamma_s)$) where each $\Gamma_i$ is a chosen representative.

Look at all the terms in the dual cell $D(\Gamma)$ which begin with a fixed $\Gamma_0$ (fixed as an object over $\Gamma$). We will see that the sum of the values of the cup product $\tilde{c}^\lambda_{\Gamma \at}$ on these terms is a sum of products corresponding to the sum of products on the right hand side of \[18\].

Let $v_1, \cdots, v_r$ be the non-trivalent vertices of $\Gamma$ and let $T^1, \cdots, T^r$ be the trees in $\Gamma_0$ which collapse to these vertices. Thus $T^i$ has $2\mu_i$ edges which are naturally ordered up to even permutation.

If the cup product $\tilde{c}^\lambda_{\Gamma \at} = \cup \tilde{c}^\lambda_{\Gamma \at}$ is nontrivial on a 2n-simplex $\Gamma_s$ beginning with $\Gamma_0$ then to each index $p$ the cocycle $\tilde{c}^\lambda_{\Gamma \at}$ must be nontrivial on

$$\Gamma_{2n_p} \rightarrow \Gamma_{2n_p+1} \rightarrow \cdots \rightarrow \Gamma_{2n_p+2\mu_p}$$

where $n_p = \lambda_1 + \lambda_2 + \cdots + \lambda_{p-1}$. This means the edges which collapse in this sequence must all lie in the same tree $T^i$. Let $i = f(p)$. Then $f$ is an epimorphism

$$f : \{1, \cdots, s\} \rightarrow \{1, \cdots, r\}$$

so that $\mu_i$ is equal to the sum of the $\lambda_p$ for all $p \in f^{-1}(i)$. In other words, $\pi = f^{-1}$ represents $\lambda$ as a refinement of $\mu$.

The value of the cocycle $\tilde{c}^\lambda_{\Gamma \at}$ on the middle 2$\lambda_p$-simplex \[19\] depends only on the sequence of edges in $T^i$ which are collapsing. Thus, if we fix the epimorphism \[20\], then, for each index $i$, the order in which the edges of $T^i$ collapse in \[19\] varies.
independently of the edges of \( T^j \) for \( j \neq i \). Consequently, the sum of products becomes a product of sums \((\pi = f^{-1} \text{ still being fixed})\).

\[
\sum_{\Gamma} \prod_{i=1}^{r} \prod_{p \in \pi(i)} c_{\mathcal{F}_{at}}(\Gamma_{2n_p}, \cdots, \Gamma_{2n_p+2\lambda_p}) = \prod_{i=1}^{r} \sum_{T^i} \prod_{p \in \pi(i)} c_{\mathcal{F}_{at}}(T^i_{2n_p}, \cdots, T^i_{2n_p+2\lambda_p})
\]

where \( T^i_k \) is the inverse image in \( \Gamma_k \) of the vertex \( v_i \) of \( \Gamma \).

But a different example with \( r = 1 \) gives the same sum:

\[
(-1)^{\mu_i} b_{\lambda_p(i)}^\mu = \sum_{\pi} \prod_{i=1}^{r} (-1)^{\mu_i} b_{\lambda_p(i)}^\mu
\]

Taking the product over all \( i \) and the sum over all \( \pi \) we get

\[
(-1)^n b_n^\mu = \sum_{\pi} \prod_{i=1}^{r} (-1)^{\mu_i} b_{\lambda_p(i)}^\mu
\]

which is the same as (18) since \( n = \sum \mu_i \). \( \square \)

We can now compute some of the numbers \( b_{\lambda}^\mu \). We start with the following case which follows from Theorem 3.7.

**Lemma 3.17.** In the case of the trivial partitions \( \lambda = \mu = n \) we have:

\[
b_n^\mu = \frac{1}{(-2)^{n+1}(2n+1)!!}
\]

By Lemma 3.15 this give the following.

**Proposition 3.18.** If the partition \( \lambda \) of \( n \) has \( m_i \) terms equal to \( \lambda_i \) for \( i = 1, \cdots, r \) (so that \( \sum m_i \lambda_i = n \)) then

\[
b_{\lambda}^\mu = \prod_{i=1}^{r} m_i! \left( b_{\lambda_i}^\mu \right)^{m_i} = \prod_{i=1}^{r} \frac{m_i!}{((-2)^{\lambda_i+1}(2\lambda_i+1)!!)^{m_i}}.
\]

**Proof.** The factor of \( \prod m_i! \) is equal to the number of ways that the partition \( \lambda \) refines itself. \( \square \)

### 3.5. Kontsevich cycles in terms of MMM classes

We are now ready to show that the dual Kontsevich cycles \( W^*_\mu \) represent polynomials in the adjusted Miller-Morita-Mumford classes \( \tilde{\kappa}_k \) (as we claimed in [Igu]). We will then conclude that their cohomology classes \([W^*_\mu]\) are linearly independent as promised in Remark 3.14.

To avoid circular reasoning, we must assume at this point that the numbers \( b_{\lambda}^\mu \) may depend on the order of the parts of \( \lambda \). Therefore, we take the ordering of both \( \lambda \) and \( \mu \) to be nonincreasing: \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \). If \( \lambda \) is a refinement of \( \mu \) then \( \lambda \leq \mu \) in lexicographic order. Consequently, the matrix

\[
B_n = (b_{\lambda}^\mu)
\]

is upper triangular. (A priori this uses only some of the numbers \( b_{\lambda}^\mu \).) The diagonal entries \( b_{\lambda}^\lambda \) are nonzero by Proposition 3.18 so \( B_n \) is invertible. Let \( A_n \) be the inverse matrix

\[
A_n = B_n^{-1} = (a_{\lambda}^\mu).
\]

The entries of this matrix are rational numbers uniquely determined by the equation

\[
\sum_{\nu} a_{\nu}^\mu b_{\nu}^\lambda = \delta_{\lambda}^\mu.
\]
The main theorem is the following.

**Theorem 3.19.** The cohomology classes of the dual Kontsevich cycles are polynomials in the adjusted Miller-Morita-Mumford classes:

\[ [W^*_\mu] = \sum_{\lambda} \alpha_{\mu}^\lambda \tilde{\kappa}_{\lambda}. \]

**Remark 3.20.** This formula holds in the rational cohomology of \( G^* \), and in the integral cohomology modulo torsion of the category of ribbon graphs and the mapping class group \( M^*_g \) for all \( g, s \geq 1 \) (and for \( g = 0, s \geq 3 \)).

**Proof.** Since the matrix \( B_n \) is invertible, the system of linear equations given in Theorem 3.13 has the unique solution

\[ W^*_\mu = \sum_{\lambda} \alpha_{\mu}^\lambda \tilde{\kappa}_G^\lambda \]

provided that \( \lambda \) is in nonincreasing order. \( \square \)

Since the coefficients \( \alpha_{\mu}^\lambda \) form an invertible matrix and the monomials \( \tilde{\kappa}_{\lambda} \) are linearly independent by Theorem 3.1 we have the following.

**Corollary 3.21.** The cohomology classes \( [W^*_\mu] \) are linearly independent over \( \mathbb{Q} \).

This, in turn, implies the following as explained in Remark 3.14.

**Corollary 3.22.** The cocycles \( \tilde{c}_G^\lambda \) and the numbers \( b^\mu_{\lambda} \) are independent of the ordering of the partitions \( \lambda, \mu \).

Combining these with Theorem 3.13 we get the following.

**Corollary 3.23.** The cohomology classes \( [W^*_\mu] \) form a \( \mathbb{Q} \)-basis for the polynomial algebra generated by the adjusted Miller-Morita-Mumford classes \( \tilde{\kappa}_k \) for \( k \geq 1 \).

3.6. **Computing \( a_{\mu}^\lambda \).** We will use the defining equation (21) to determine the numbers \( a_{\mu}^\lambda \) in some simple cases.

**Proposition 3.24.** When \( \lambda = \mu = n \) we have

\[ a_n^n = \frac{1}{b_n^n} = (-2)^{n+1}(2n+1)!!. \]

To simplify the notation we write \( a_n \) and \( b_n \) instead of \( a_n^n \) and \( b_n^n \).

**Proposition 3.25.** When \( \lambda = \mu = (\lambda_1^{m_1}, \ldots, \lambda_r^{m_r}) \) with \( \sum m_i \lambda_i = n \) then

\[ a_{\lambda}^\lambda = \frac{1}{b_{\lambda}^\lambda} = \prod_{i=1}^{r} \frac{((-2)^{k_i+1}(2k_i+1)!!)^{m_i}}{m_i!}. \]

For example, \( a_{2,2}^2 = 120^2/2 = 7200 \) and \( a_{1,1,1}^{1,1,1} = 12^3/3! = 288 \). These numbers are not always integers. For example,

\[ a_{1,5}^5 = \frac{12^5}{5!} = \frac{10368}{5}. \]

**Corollary 3.26.** The dual Kontsevich cycle \( [W^*_\lambda] \) is a polynomial of degree \( \sum m_i \) in the adjusted Miller-Morita-Mumford classes with leading term

\[ \prod_{i=1}^{r} \frac{((-2)^{\lambda_i+1}(2\lambda_i+1)!!)^{m_i}}{m_i!}. \]
In order to compute the remaining terms we need to compute \( a_{n,k}^{\lambda} \) for \( \lambda \neq \mu \). The first case is \( a_{n,k}^{n+k} \) which occurs in the equation

\[
(W_{n,k}^\ast) = a_{n,k}^{n+k} \kappa_n \kappa_k + a_{n,k}^{n+k} \tilde{\kappa}_{n+k}.
\]

The defining equation for \( a_{n,k}^{n+k} \) is

\[
a_{n,k}^{n+k} b_{n+k} + a_{n,k}^{n+k} \tilde{\kappa}_{n+k} = 0.
\]

From this and Proposition 3.24 we get:

\[
a_{n,k}^{n+k} = (-2)^{2n+2k+2}(2n+1)!!(2k+1)!!
\]

if \( n \neq k \) and

\[
a_{n,n}^{n,n} = -(2)^{4n+3}((2n+1)!!)^2
\]

\[
a_{n,n}^{2n} = -(2)^{4n+2}(4n+1)!!((2n+1)!!)^2 \tilde{\kappa}_{n,n}^{-2n}
\]

In the next section we will compute \( b_{n,k}^{n+k} \) in the special case \( k = 1 \).

4. Some computations

(1) The degenerate case \( n = 0 \)
(2) Computation of \( b_{n,1}^{n+1} \)
(3) Conjectures

We will compute the numbers \( b_{n,1}^{n+1} \) and obtain \( a_{n,1}^{n+1} \) and the coefficients of \( [W_{n,1}^\ast] \) when expressed as a polynomial in the adjusted Miller-Morita-Mumford classes.

4.1. The degenerate case \( n = 0 \). First we consider the degenerate case \( n = 0 \). In this case it is easy to compute \( W_{k,0}^\ast \) and work backwards since this cocycle counts the number of pairs of vertices, one of codimension \( 2k \) and the other of codimension \( 0 \) (i.e., trivalent). The number of trivalent vertices of a graph with one vertex of multiplicity \( 2k+1 \) is \( t \) where

\[
t + 2k + 1 = -2\kappa - 2\kappa_0.
\]

So, \( t = -2\kappa_0 - (2k + 1) \) and

\[
(W_{k,0}^\ast) = t(W_k^\ast) = (-2\kappa_0 - (2k + 1))(-2)^{k+1}(2k+1)!!\kappa_k
\]

(23) \[
(W_{k,1}^\ast) = (-2)^{k+2}(2k+1)!!\kappa_k \kappa_0 - (2k + 1)(-2)^{k+1}(2k+1)!!\kappa_k.
\]

The right hand side should be divided by 2 when \( k = 0 \).

For the case \( k = 1 \) this gives

\[
(W_{1,0}^\ast) = [W_{0,1}^\ast] = -24\kappa_0 \kappa_1 - 36\kappa_1.
\]

Consequently, \( a_{0,1}^1 = -36 \) and

\[
b_{0,1}^1 = \frac{1}{2\cdot 9} a_{0,1}^1 = -\frac{1}{8}.
\]
4.2. Computation of $b_{n,1}^{n+1}$. To obtain $b_{n,1}^{n+1}$ in general we use the formula from Lemma 3.11:

$$b_{n,1}^{n+1} = (-1)^{n+1} (c_{F_{at}}^{n} \cup e_{F_{at}}^{1}) D(\Gamma)$$

where $\Gamma$ is any graph in the Witten cycle $W_{n+1}$, i.e., a ribbon graph which is trivalent except at one vertex of multiplicity $2n + 2$.

The dual cell $D(\Gamma)$ is a sum

$$D(\Gamma) = \sum_{\Gamma_{s}} o(\Gamma_{s})(\Gamma_{0}, \ldots, \Gamma_{2n+2} = \Gamma).$$

When we evaluate the cup product $(-1)^{n+1} (c_{F_{at}}^{n} \cup e_{F_{at}}^{1})$ we get

$$b_{n,1}^{n+1} = (-1)^{n+1} \sum_{\Gamma_{s}} o(\Gamma_{s})c_{F_{at}}^{n}(\Gamma_{0}, \ldots, \Gamma_{2n}) \frac{1}{(\Gamma_{2n}, \Gamma_{2n+1}, \Gamma)}.$$

There are three possible configurations for $\Gamma_{2n}$. They are Case 1, given in Figure 2 and Case 2a,2b, given in Figure 3 from section 2. In all cases, $\Gamma_{2n}$ is odd-valent so it has a natural orientation. The orientation of the sequence $\Gamma_{s}$ can then be expressed as a product of two orientations:

$$o(\Gamma_{0}, \ldots, \Gamma_{2n}, \Gamma', \Gamma) = o(\Gamma_{0}, \ldots, \Gamma_{2n})o(\Gamma_{2n}, \Gamma', \Gamma)$$

where we write $\Gamma' = \Gamma_{2n+1}$. Consequently, for any fixed $\Gamma_{2n}$, we get

$$(-1)^{n} \sum_{\Gamma_{s}} o(\Gamma_{0}, \ldots, \Gamma_{2n})c_{F_{at}}^{n}(\Gamma_{0}, \ldots, \Gamma_{2n}) \left( - \sum_{\Gamma'} o(\Gamma_{2n}, \Gamma', \Gamma)c_{F_{at}}^{1}(\Gamma_{2n}, \Gamma', \Gamma) \right).$$

The first factor is $b_{n}$ regardless of $\Gamma_{2n}$ so

$$\frac{b_{n}^{n+1}}{b_{n}} = - \sum_{\Gamma_{2n}, \Gamma'} o(\Gamma_{2n}, \Gamma', \Gamma)c_{F_{at}}^{1}(\Gamma_{2n}, \Gamma', \Gamma).$$

If we denote the sequence $\Gamma_{2n} \rightarrow \Gamma' \rightarrow \Gamma$ by $\Gamma_{s}$ and substitute $1/b_{n} = a_{n}$ we get

(24)

$$a_{n}b_{n}^{n+1} = - \sum_{\Gamma_{s}} o(\Gamma_{s})c_{F_{at}}^{1}(\Gamma_{s}).$$

Recall that

$$c_{F_{at}}^{1}(\Gamma_{s}) = \sum_{v_{0}} \frac{\mu(v_{0})}{4} \sum_{C_{1}, C_{2}} \text{sgn}(a, b, c) |C_{1}| |C_{2}|$$

with the first sum being over all vertices $v_{0}$ of $\Gamma_{2n}$ where $\mu(v_{0})$ is the multiplicity of $v_{0}$ and $|C_{1}|$ is the valence of the image of $v_{i}$ in $\Gamma_{2n+1}$. The sign sum

$$\sum \text{sgn}(a, b, c) = \text{sgn}(x_{1} \cdots x_{2n+5})$$

is the number of times that the letters $a, b, c$ occur in the correct cyclic order in the word $w = x_{1} \cdots x_{2n+5}$ minus the number of times it occurs in the other cyclic order in $w$ where the $j$th letter $x_{j}$ of $w$ is equal to $b_{j}$ ($b_{0} = a, b_{1} = b, b_{2} = c$) if the $j$th region in the complement of the graph reaches $v_{0}$ at the $j$th step ($\Gamma_{2n+i}$).

The sum (24) breaks up into three parts depending on the graph $\Gamma_{2n}$.

Case 1. Suppose the graph $\Gamma_{2n}$ is given by Figure 2 and $T' = T_{2n}/C_{1}$. Then

$$o(\Gamma_{s}) = (-1)^{m}$$

The number of times this same configuration (with fixed $1 \leq m \leq 2n + 2$) occurs is $2n + 5$.

The cyclic set cocycle $c_{F_{at}}^{1}(\Gamma_{s})$ has two terms
(1) The center vertex has multiplicity $2n + 1$ and average sign
\[
\frac{\text{sgn} \left( a^m ca^{2n-m+3} b \right)}{(2n+3)(2n+4)(2n+5)} = \frac{2n-2m+3}{(2n+3)(2n+4)(2n+5)}
\]
for a contribution of
\[
\frac{(2n+1)(2n-2m+3)}{4(2n+3)(2n+4)(2n+5)}
\]

(2) The vertex $v_1$ has multiplicity 1 and average sign
\[
\frac{\text{sgn} \left( a^3 b^{m-1} c b^{2n-m+2} \right)}{3(2n+4)(2n+5)} = -\frac{2n-2m+3}{(2n+4)(2n+5)}
\]
The total value of the cocycle is
\[
\tilde{c}^{1}_{\text{rat}}(\Gamma_{*}) = \frac{(2n-2m+3)(2n+1-(2n+3))}{4(2n+3)(2n+4)(2n+5)} = \frac{-(2n-2m+3)}{2(2n+3)(2n+4)(2n+5)}.
\]
Multiply this by $(-1)^m(2n+5)$ and sum over all $1 \leq m \leq 2n+2$ to get
\[
\frac{-1}{2(2n+3)(2n+4)} \sum_{m=1}^{2n+2} (-1)^m(2n-2m+3) = \frac{n+1}{(2n+3)(2n+4)}.
\]

Case 2a. Suppose $\Gamma_{2n}$ is given by Figure 3a and $\Gamma' = \Gamma_{2n}/\epsilon_1$. Then
\[
o(\Gamma_{*}) = \text{sgn}(a_1, \ldots, a_{2n+3}, b_1, b_2) = 1.
\]
This configuration occurs $2n + 5$ times and the cyclic set cocycle has two terms.

(1) The center vertex has multiplicity $2n + 1$ and average sign
\[
\frac{\text{sgn} \left( a^{2n+3} b c \right)}{(2n+3)(2n+4)(2n+5)} = \frac{1}{(2n+4)(2n+5)}
\]
for a contribution of
\[
\frac{2n+1}{4(2n+4)(2n+5)}
\]

(2) The vertex $v_1$ has multiplicity 1 and average sign
\[
\frac{\text{sgn} \left( a^2 cab^{2n+1} \right)}{3(2n+4)(2n+5)} = -\frac{-(2n+1)}{3(2n+4)(2n+5)}
\]
So
\[
\tilde{c}^{1}_{\text{rat}}(\Gamma_{*}) = \frac{2n+1}{6(2n+4)(2n+5)}.
\]
Multiply this by $2n + 5$ for a total of
\[
\frac{2n+1}{6(2n+4)}.
\]

Case 2a'. Suppose that $T_{2n}$ is the same (Figure 3a) but $\Gamma' = \Gamma_{2n}/\epsilon_2$. Then $o(\Gamma_{*}) = -1$. The configuration still occurs $2n + 5$ times and the cyclic set cocycle again has two terms.

(1) At $v_1$ we have
\[
\frac{1}{4} \frac{\text{sgn} \left( a^2 b a c^{2n+1} \right)}{3 \cdot 4(2n+5)} = \frac{2n+1}{48(2n+5)}.
\]
(2) At \( v_2 \) we get
\[
\frac{1}{4} \frac{\text{sgn}(ba^3 c^{2n+1})}{3 \cdot 4(2n + 5)} = -\frac{3(2n + 1)}{48(2n + 5)}.
\]
So,
\[
\tilde{c}^{1}_{\text{Fat}}(\Gamma_*) = -\frac{(2n + 1)}{24(2n + 5)}
\]
making the total in this case
\[
\frac{2n + 1}{24}.
\]
Case 2b is the same as Case 2a. (The sign changes twice.) So
\[
a_{n} b_{n+1}^{n+1} = \frac{n + 1}{(2n + 3)(2n + 4)} + \frac{2n + 1}{3(2n + 4)} + \frac{2n + 1}{12}.
\]
Simplifying this expression we get:
\[
a_{n} b_{n+1}^{n+1} = \frac{2n + 5}{12} - \frac{1}{2(2n + 3)}.
\]
Multiplying by \( a_{1} a_{n} b_{n+1}^{n+1} = 12(-2)^{n+2}(2n + 3)!! \) we get
\[
a_{n+1} a_{n} b_{n+1}^{n+1} = (-2)^{n+2}(2n + 5)!! + 3(-2)^{n+3}(2n + 1)!!
\]

**Theorem 4.1.** For \( n \neq 1 \) we have
\[
[W_{n,1}^{*}] = 3(-2)^{n+3}(2n + 1)!!(\tilde{\kappa}_{n} \tilde{\kappa}_{1} - \tilde{\kappa}_{n+1}) - (-2)^{n+2}(2n + 5)!! \tilde{\kappa}_{n+1}.
\]
For \( n = 1 \) we divide the right hand side by 2.

For example, we have
\[
[W_{0,1}^{*}] = -24\tilde{\kappa}_{0} \tilde{\kappa}_{1} - 36\tilde{\kappa}_{1}
\]
\[
[W_{1,1}^{*}] = 72\tilde{\kappa}_{1}^2 + 348\tilde{\kappa}_{2}
\]
\[
[W_{2,1}^{*}] = -1440\tilde{\kappa}_{2} \tilde{\kappa}_{1} - 13680\tilde{\kappa}_{3}
\]
\[
[W_{3,1}^{*}] = 20160\tilde{\kappa}_{3} \tilde{\kappa}_{1} + 312480\tilde{\kappa}_{4}
\]
\[
[W_{4,1}^{*}] = -362880\tilde{\kappa}_{4} \tilde{\kappa}_{1} - 8285760\tilde{\kappa}_{5}
\]

This agrees with the calculation when \( n = 0, k = 1 \) and also agrees with the calculation of Arbarello and Cornalba. The sign difference comes from the fact that they use the opposite sign for all \( \kappa_{\text{even}} \). (What we call \( \tilde{\kappa}_{k} \) is what they would call \( (-1)^{k+1} \kappa_{k} \) restricted to the open moduli space of curves.) The calculations of give cohomology classes which they showed act as Poincaré duals of the Kontsevich cycles with respect to products of boundary cycles. Therefore, Theorem 3.1 and Theorem 3.19 already suggests that they must be correct.

### 4.3. Conjectures.

The formula in Theorem 4.1 has an apparent symmetry which also appears in the formula when it is rephrased as follows.
\[
(25) \quad [W_{n,k}^{*}] = (-2)^{k+2}(2k + 1)!!(\tilde{\kappa}_{n} \tilde{\kappa}_{0} - \tilde{\kappa}_{k}) - (-2)^{k+1}(2k + 3)!! \tilde{\kappa}_{k}.
\]
This leads to the following conjecture.

**Conjecture 4.2.**
\[
[W_{n,k}^{*}] = (-2)^{n+k+2}(2n+1)!!(2k+1)!!(\tilde{\kappa}_{n} \tilde{\kappa}_{k} - \tilde{\kappa}_{n+k}) - (-2)^{n+k+1}(2n+2k+3)!! \tilde{\kappa}_{n+k}.
\]
The right hand side should be divided by 2 if \( n = k \).
Remark 4.3. First, note that this conjectured formula can be simplified using the numbers $a_n = (-2)^{n+1}(2n+1)!!$:

$$[W^*_{n,k}] = a_n a_k (\tilde{\kappa}_n \tilde{\kappa}_k - \tilde{\kappa}_{n+k}) + \frac{1}{2} a_{n+k+1} \tilde{\kappa}_{n+k}.$$ 

Next, we also note that it is symmetrical in $n,k$. And finally, in the first new case when $n = k = 2$ it gives

$$[W^*_{2,2}] = \frac{120^2}{2} (\tilde{\kappa}_2^2 - \tilde{\kappa}_4) + \frac{665280}{4} \tilde{\kappa}_4 = 7200 \tilde{\kappa}_2^2 + 159120 \tilde{\kappa}_4,$$

which agrees with Arbarello and Cornalba \[AC96\].

Remark 4.4. Michael Kleber and I have made further progress on calculation and interpretation of the coefficients $b^n_\lambda$. So far we verified Conjecture 4.2 for all $k \leq 7$. In particular this proves \[(26)\] . We also found that

$$[W^*_{1,1,1}] = 288 \tilde{\kappa}_1^3 + 4176 \tilde{\kappa}_1 \tilde{\kappa}_2 + 20736 \tilde{\kappa}_3.$$ 

This, together with Theorem \[131\] and Corollary \[3.8\] verifies all calculations of the coefficients $a^\mu_\lambda$ given by Arbarello and Cornalba. Details will be given in a subsequent joint paper.

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