BRAUER GROUP OF MODULI STACK OF STABLE PARABOLIC PGL(r)-BUNDLES OVER A CURVE

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ABSTRACT. Let $k$ be an algebraically closed field of characteristic zero. We prove that the Brauer group of moduli stack of stable parabolic PGL($r$, $k$)-bundles with full flag quasi-parabolic structures at an arbitrary parabolic divisor on a curve $X$ coincides with the Brauer group of the smooth locus of the corresponding coarse moduli space of parabolic PGL($r$, $k$)-bundles. We also compute the Brauer group of the smooth locus of this coarse moduli for more general quasi-parabolic types and weights satisfying certain conditions.

1. INTRODUCTION

Let $Y$ be a quasi-projective variety defined over an algebraically closed field $k$ of characteristic zero. The cohomological Brauer group of $Y$ is defined to be the torsion part

$$H^2(Y_{\text{et}}, \mathbb{G}_m)_{\text{torsion}} \subset H^2(Y_{\text{et}}, \mathbb{G}_m),$$

and it is denoted by $\text{Br}'(Y)$. When the variety $Y$ is smooth, it is known that the above group $H^2(Y_{\text{et}}, \mathbb{G}_m)$ is actually torsion. The Brauer group of $Y$, which is denoted by $\text{Br}(Y)$, is defined to be the Morita equivalence classes of Azumaya algebras over $Y$. Giving an Azumaya algebra over $Y$ is equivalent to giving a Brauer-Severi scheme (i.e., a projective bundle) over $Y$ in the étale topology, which is also equivalent to giving a principal PGL($r$, $k$)-bundle over $Y$. The Brauer class of a projective bundle $\mathbb{P} \rightarrow Y$ can be thought of as the obstruction for $\mathbb{P}$ to be the projectivization of a vector bundle on $Y$. For each integer $r \geq 2$, the sequence of cohomologies associated to the exact sequence

$$\{e\} \rightarrow \mu_r \rightarrow \text{SL}_r \rightarrow \text{PGL}_r \rightarrow \{e\}$$

gives a homomorphism $H^1(Y_{\text{et}}, \text{PGL}_r) \rightarrow \text{Br}'(Y)$, which induces an injective group homomorphism $i : \text{Br}(Y) \rightarrow \text{Br}'(Y)$. By a theorem of Gabber it is known that this homomorphism $i$ is surjective [dJ]. For an algebraic stack $Y$, by Brauer group of $Y$ we mean the cohomological Brauer group of $Y$.

Let $X$ be an irreducible smooth projective curve of genus $g \geq 2$ over $k$. Fix an integer $r \geq 2$ and a line bundle $\xi$ of degree $d$ on $X$. If $g = 2$, we assume that $r \geq 3$. Let $M(r, d)$ denote the moduli space of stable vector bundles on $X$ of rank $r$ and degree $d$; it is a smooth quasi-projective variety. Let $M_{\xi}(r, d) \subset M(r, d)$ denote the moduli space of stable vector bundles on $X$ with rank $r$ and determinant $\xi$. Let $\mathcal{M}(r, d)$ and $\mathcal{M}_{\xi}(r, d)$ denote the corresponding moduli stack of stable vector bundles. The Brauer group $\text{Br}(M_{\xi}(r, d))$ is cyclic of order $\text{g.c.d.}(r, d)$.
A generator for \( \text{Br}(M_\xi^a(r, d)) \) is obtained by restricting the universal projective bundle over \( X \times M_\xi(r, d) \) to \( \{x_0\} \times M_\xi(r, d) \), where \( x_0 \) is a closed point of \( X \).

Parabolic vector bundles on \( X \), denoted by \( E_\xi \), were introduced by Mehta and Seshadri in [MS]. These are vector bundles \( E \) on \( X \) with a filtration of fibers of \( E \) at a collection of finitely many points \( S \) of \( X \) and certain real numbers, called weights, attached to these filtrations. Let \( M^a(r, d) \) denote the moduli space of rank \( r \) stable parabolic vector bundles \( E_\xi \) on \( X \) whose underlying bundle \( E_\xi \) is of degree \( d \) and the parabolic structure at each parabolic point \( p_i \in S \) is of the following type:

\[
E_{p_i} =: F_{i,1} \supseteq \cdots \supseteq F_{i,a_i} \neq 0
\]

with \( \dim F_{i,j}/F_{i,j+1} := r_{i,j} \); the parabolic weight of \( F_{i,j} \) is \( a_{i,j} \in \mathbb{R} \) with

\[
0 \leq a_{i,1} < \cdots < a_{i,a_i} < 1.
\]

(see §2 for more details). It is known that \( M^a(r, d) \) is a smooth quasi–projective variety [MS]. Let \( M_\xi^a(r, d) \) denote the moduli space of stable parabolic vector bundles of rank \( r \) and above parabolic type with determinant \( \xi \), i.e., \( \mathcal{N}E = \xi \). Let \( M^a(r, d) \) and \( M_\xi^a(r, d) \) denote the corresponding moduli stack of rank \( r \) stable parabolic vector bundles with above parabolic data.

The Brauer group \( \text{Br}(M_\xi^a(r, d)) \) is isomorphic to the cyclic group \( \mathbb{Z}/m\mathbb{Z} \) [BD], where

\[
m = \text{g.c.d.}(d, r, r_{1,1}, \ldots, r_{1,a_1}, \ldots, r_{c,1}, \ldots, r_{c,a_c}).
\]

Like in the case of vector bundles, the cyclic group \( \text{Br}(M_\xi^a(r, d)) \) is generated by the Brauer class of the Brauer–Severi variety \( \mathbb{P}_{x_0} \), where \( \mathbb{P} \) is the universal projective bundle on \( X \times M_\xi^a(r, d) \) and \( x_0 \in X \).

Let \( \mathcal{N}(r) \) be the moduli stack of stable \( \text{PGL}(r, k) \) bundles and \( N(r) \) the corresponding coarse moduli space. We have the natural morphism

\[
f : \mathcal{N}(r) \longrightarrow N(r).
\]

The connected components of both \( \mathcal{N}(r) \) and \( N(r) \) are indexed by integers \( i \in [0, r-1] \); the \( i \)-th component of \( \mathcal{N}(r) \) (respectively, \( N(r) \)) is denoted by \( \mathcal{N}(r)_i \) (respectively, \( N(r)_i \)). This \( \mathcal{N}(r)_i \) (respectively, \( N(r)_i \)) is the quotient of \( \mathcal{N}(r) \) (respectively, \( N(r) \)) by the group of \( r \)-bundles \( \text{Bun}_{\mu_i} \) (respectively, by the subgroup \( \Gamma \subset \text{Pic}^0(X) \) of \( r \)-torsion points), where \( i \equiv d \mod r \) (see [GO, Remark 2.8]). The above morphism \( f \) sends \( \mathcal{N}(r) \) onto \( N(r) \).

The Brauer groups of \( \mathcal{N}(r)_i \) and \( N(r)_i \) coincide [BH10], and it fits in the exact sequence

\[
0 \longrightarrow \frac{H^2(\Gamma, k^*)}{H} \longrightarrow \text{Br}(\mathcal{N}(r)_i) \longrightarrow \mathbb{Z}/\delta\mathbb{Z} \longrightarrow 0,
\]

where \( H \subset H^2(\Gamma, k^*) \) is a subgroup of order \( \delta = \text{g.c.d}(r, d) \) [BH10, Theorem 1.2].

As in the case of vector bundles, a parabolic stable \( \text{PGL}(r, k) \)-bundle is an equivalence class of parabolic stable vector bundles \( [E_\xi] \), where two stable parabolic vector bundles \( E_\xi \) and \( E'_\xi \) are equivalent if there exists a line bundle \( L \) such that \( E'_\xi \cong E_\xi \otimes L \). The parabolic structure on \( E'_\xi \) is obtained from the parabolic structure of \( E_\xi \) by the above isomorphism.

Let \( \mathcal{P}\mathcal{N}^a(r) \) be the moduli stack of stable parabolic \( \text{PGL}(r, k) \)-bundles with full-flag parabolic structures at each parabolic points of \( S \), and let \( P\mathcal{N}^a(r) \) denote the corresponding coarse moduli
space. We have the natural morphism
\[ \tilde{g} : \mathcal{P}N^a(r) \rightarrow PN^a(r). \] (1.3)
As before, the connected components of \( \mathcal{P}N^a(r) \) and \( PN^a(r) \) are indexed by integers \( i \in [0, r-1] \); the \( i \)-th connected component of \( \mathcal{P}N^a(r) \) (respectively, \( PN^a(r) \)) is denoted by \( \mathcal{P}N^a(r)_i \) (respectively, \( PN^a(r)_i \)). The scheme \( PN^a(r)_i \) is the quotient of \( M^a_\xi(r, d) \) by the above mentioned group \( \Gamma \), while the stack \( \mathcal{P}N^a(r)_i \) is the quotient of the stack \( \mathcal{M}^a_\xi(r, d) \) by \( \text{Bun}_{\rho_i} \), where \( i \equiv d \pmod{r} \). In both cases, the action is given by parabolic tensor product with line bundles equipped with the trivial parabolic structure. The morphism \( \tilde{g} \) sends \( \mathcal{P}N^a(r)_i \) onto \( PN^a(r)_i \).

Our aim here is to study the Brauer group of \( \mathcal{P}N^a(r) \) and the smooth locus of \( PN^a(r)_i \). We prove that the two Brauer groups in question coincide, and is in fact isomorphic to the kernel in (1.2).

The main results are as follows:

**Theorem 1.1.**

1. In the case of full flags (i.e. \( r_{i,j} = 1 \forall (i, j) \)), the Brauer groups of \( \mathcal{P}N^a(r)_i \) and the smooth locus of \( PN^a(r)_i \) coincide.
2. Let \( m \) as in (1.1) equals 1, and \( PN^a,sm(r) \), denote the smooth locus of \( PN^a(r)_i \). We have
\[
\text{Br}(PN^a,sm(r)_i) \cong \frac{H^2(\Gamma, k^* \mathcal{O}_U)}{H},
\]
where \( H \subset H^2(\Gamma, k^* \mathcal{O}_U) \) is as in (1.2).

Theorem 1.1(1) is proved by producing an open subscheme \( U \) of \( PN^a(r) \) whose complement is of codimension at least 3 such that the map \( \tilde{g} \) (1.3) is an isomorphism over \( U \) (cf. Corollary 4.3). For Theorem 1.1(2), we first assume the parabolic weights are sufficiently small, so that there exists a forgetful map from an open subscheme \( U_\xi^a \subset M^a_\xi(r, d) \) to \( M^a_\xi(r, d) \) by simply forgetting the parabolic structure. This map is an étale-locally trivial fibration. Next, we restrict our attention over a suitably chosen open subscheme of \( M^a_\xi(r, d) \) on which \( \Gamma \) acts freely. Quotienting by \( \Gamma \) then gives rise to a finite étale morphism, from which we obtain our result using the Hochschild-Serre spectral sequence corresponding to this finite étale map (cf. Proposition 5.1). Finally, for arbitrary generic weights, we use a result of Thaddeus [Th, § 6, 6.2] which says that the moduli spaces with different weights are actually birational.

Finally, as a remark (Remark 6.2), we discuss the case when the weights are not generic, and show that under the same conditions as Theorem 1.1, the Brauer groups can be described for moduli of parabolic stable \( PGL(r) \)-bundles for those non-generic weights which belong to a single wall.

**2. Preliminaries**

Let \( X \) be an irreducible smooth projective curve of genus \( g \geq 2 \), defined over an algebraically closed field \( k \) of characteristic zero. A vector bundle on \( X \) will always mean an algebraic vector bundle. Fix an integer \( r \geq 2 \); if \( g = 2 \), then set \( r \geq 3 \). Fix a line bundle \( \xi \) on \( X \) of degree \( d \). By a point of \( X \) we will always mean a closed point. Fix a subset
\[
S = \{ p_1, p_2, \ldots, p_s \} \subset X
\]
of $s$ distinct points; they will be referred to as parabolic points. Let $E$ be a vector bundle of rank $r$ on $X$. The fiber of $E$ over a point $z \in X$ will be denoted by $E_z$.

**Definition 2.1.** A parabolic data of rank $r$ for points of $S$ consists of the following collection: for each $p_i \in S$

- a string of positive integers $(m^i_1, m^i_2, \ldots, m^i_{l_i})$ such that $\sum_{j=1}^{l_i} m^i_j = r$, and
- an increasing sequence of real numbers $0 \leq \alpha^i_1 < \alpha^i_2 < \cdots < \alpha^i_{l_i} < 1$.

If $m^i_j = 1$ for all $1 \leq j \leq l_i$ and $1 \leq i \leq s$, we say that it is a full-flag parabolic data.

A parabolic structure on $E$ over $S$ with parabolic data of rank $r$ as above consists of the following: for each $p_i \in S$, a weighted filtration

$$E_{p_i} = E^i_1 \supset E^i_2 \supset \cdots \supset E^i_{l_i} \supset E^i_{l_i+1} = 0$$

$$0 \leq \alpha^i_1 < \alpha^i_2 < \cdots < \alpha^i_{l_i} < \alpha^i_{l_i+1} = 1$$

such that $m^i_j = \dim(E^i_j / E^i_{j+1}) \forall 1 \leq j \leq l_i, 1 \leq i \leq s$.

The above collection $\alpha := \{(\alpha^i_1 < \alpha^i_2 < \cdots < \alpha^i_{l_i})_{1 \leq i \leq s}\}$ is called the weights, and the above integer $m^i_j$ is called the multiplicity of the weight $\alpha^i_j$.

By a parabolic bundle we mean a collection of data $(E, m, \alpha)$, where $E$ is a vector bundle on $X$, while $m$ and $\alpha$ are as described above. For notational convenience, such a parabolic vector bundle will also be referred to as $E_\alpha$; the vector bundle $E$ is called the underlying bundle. A full-flag parabolic data is also called a full-flag parabolic structure.

**Definition 2.2.** Let $E_\alpha$ be a parabolic bundle on $X$ of rank $r$. The parabolic degree of $E_\alpha$ is defined as

$$\text{pardeg}(E_\alpha) := \deg(E) + \sum_{i=1}^{s} \sum_{j=1}^{l_i} m^i_j \alpha^i_j,$$

and the parabolic slope of $E_\alpha$ is defined as

$$\text{par}\mu(E_\alpha) := \frac{\text{pardeg}(E_\alpha)}{r}.$$

The parabolic bundle $E_\alpha$ is called parabolic semistable (respectively, parabolic stable) if for every proper sub-bundle $F \subset E$ we have

$$\text{par}\mu(F_\alpha) \leq \text{par}\mu(E_\alpha)$$

(respectively, $\text{par}\mu(F_\alpha) < \text{par}\mu(E_\alpha)$),

where $F_\alpha$ denotes the parabolic bundle defined by $F$ equipped with the induced parabolic structure from $E_\alpha$ (see [MS] for the details).

See [Se], [MS] for homomorphisms of parabolic bundles. The class of parabolic semistable bundles with fixed parabolic slope forms an abelian category. We refer to [Se, p. 68] for further details. In particular, it makes sense to take direct sum of parabolic semistable bundles with same parabolic slope.

**Definition 2.3.** A system of weights $\alpha$ is called generic if every semistable parabolic bundle $E_\alpha$, of given rank and degree, with weights $\alpha$ is parabolic stable. We refer to [BY] for more details.
2.1. Parabolic push-forward and pull-back. Let $X$ and $Y$ be two irreducible smooth projective curves, and let $\gamma : Y \to X$ be a finite étale Galois morphism. If $F$ is a vector bundle on $Y$ of rank $n$, then $\gamma_* F$ is a vector bundle on $X$ of rank $mn$, where $m$ is the degree of the map $\gamma$. Given a parabolic structure on $F$, there is a natural way to construct a parabolic structure on $\gamma_* F$. We refer to [BM, § 3] for details.

Let us mention a special case of parabolic push-forward which will be used here. Let $p \in X$ be a point. Let $m$ be the degree of $\gamma$. Since $\gamma$ is unramified, the inverse image $\gamma^{-1}(p)$ consists of $m$ distinct points of $Y$. Suppose we are given a full-flag parabolic data of rank $n$ with $\gamma^{-1}(p)$ as the set of parabolic points, so that the parabolic data looks like

$$q \in \gamma^{-1}(p), \quad F_q = F_1^q \supseteq F_2^q \supseteq \cdots \supseteq F_n^q \supseteq F_{n+1}^q = 0,$$

$$1 \leq \alpha_1^q < \alpha_2^q < \cdots < \alpha_n^q < \alpha_{n+1}^q = 1.$$  

Moreover, we assume that in the collection $\{\alpha_j^q \mid q \in \gamma^{-1}(p), \ 1 \leq j \leq n\}$ all numbers are distinct. Let $F_\ast^p$ be a parabolic bundle on $Y$ with this parabolic data. We shall construct a full-flag parabolic structure for $E = \gamma_* F$ at the point $p$ from this data. Note that

$$E_p = \bigoplus_{q \in \gamma^{-1}(p)} F_q.$$  

Let $\beta_1^p < \beta_2^p < \cdots < \beta_{mn}^p$ be the increasing sequence of length $mn$ obtained by ordering the numbers $\{\alpha_j^q \mid q \in \gamma^{-1}(p), \ 1 \leq j \leq n\}$. For each integer $1 \leq k \leq mn$, define a filtration of $E_p$ by the subspaces

$$E_k^p := \bigoplus_{q \in \gamma^{-1}(p)} F_{\alpha(q,k)}^q,$$

where $\alpha(q,k)$, for each point $q$, is the smallest integer $1 \leq j(q) \leq n$ satisfying the condition $\beta_k^p \leq \alpha_j^q(q)$. It is straightforward to see that $\dim(E_k^p/E_{k+1}^p) = 1 \forall \ 1 \leq k \leq mn - 1$ and $\dim E_{mn}^p = 1$. Consequently,

$$E_p = E_1^p \supseteq E_2^p \supseteq \cdots \supseteq E_{mn}^p$$

$$\beta_1^p < \beta_2^p < \cdots < \beta_{mn}^p$$

is a full-flag parabolic structure of rank $mn$ at $p$.

Finally, for multiple parabolic points on $X$, we perform exactly the same construction for each parabolic point to define the parabolic push-forward.

Let $\gamma : Y \to X$ be as above. If $E_\ast^p$ is a parabolic bundle on $X$, then $\gamma^* E$ has an induced parabolic structure as follows: let $S \subset X$ be the set of parabolic points for $E$; we define $\gamma^{-1}(S)$ as the set of parabolic points for $\gamma^* E$, and for any $p \in S$ and $q \in \gamma^{-1}(p)$, give the fiber $(\gamma^* E)_q = E_p$ the same parabolic structure as $E_p$.

2.2. Moduli of parabolic PGL$(n, k)$-bundles.

Definition 2.4. A parabolic PGL$(n, k)$-bundle is an equivalence class of parabolic vector bundles, where two parabolic bundles $E_\ast$ and $E_\ast'$ are considered equivalent if there exists a line bundle $L$ such that $E_\ast' \simeq E_\ast \otimes L$ as parabolic bundles.
Let \( \mathcal{M}^a_\xi(r, d) \) denote the moduli stack of stable parabolic bundles on \( X \) of rank \( r \) and fixed determinant \( \xi \) of degree \( d \); the corresponding coarse moduli space will be denoted by \( M^a_\xi(r, d) \). The natural morphism

\[
\rho : \mathcal{M}^a_\xi(r, d) \to M^a_\xi(r, d)
\]

is a \( \mathbb{G}_m \)-gerbe; this follows from the fact that the automorphisms of a stable parabolic bundle are nonzero scalar multiplications.

Let

\[
\Gamma \subset \text{Pic}^0(X)
\]

be the group of isomorphism classes of line bundles on \( X \) of order \( r \); in other words, \( \Gamma \) is the \( r \)-torsion points of the Jacobian of \( X \). This group \( \Gamma \) acts naturally on \( PN^a_\xi(r, d) \) by parabolic tensor product. More precisely, the action of any \( L \in \Gamma \) sends any \( E_\ast \in PN^a_\xi(r, d) \) to the parabolic tensor product

\[
L \cdot (E_\ast) := E_\ast \otimes L,
\]

where \( L \) has the trivial parabolic structure.

Let \( \text{Bun}_{\mu_r} \) denote the group stack of \( \mu_r \)-bundles on \( X \). The natural map

\[
h : \text{Bun}_{\mu_r} \to \Gamma
\]

is a \( \mu_r \)-gerbe. The tensor product operation defines an action of the stack \( \text{Bun}_{\mu_r} \) on \( \mathcal{M}^a_\xi(r, d) \). This action is clearly compatible with the \( \Gamma \)-action on \( M^a_\xi(r, d) \), meaning the following diagram commutes:

\[
\begin{array}{ccc}
\text{Bun}_{\mu_r} \times \mathcal{M}^a_\xi(r, d) & \longrightarrow & \mathcal{M}^a_\xi(r, d) \\
\downarrow_{h \times \rho} & & \downarrow_{\rho} \\
\Gamma \times M^a_\xi(r, d) & \longrightarrow & M^a_\xi(r, d)
\end{array}
\]

Let \( i \equiv d \pmod{r} \) with \( i \in [0, d - 1] \). The moduli stack of parabolic stable \( \text{PGL}(r, k) \)-bundles on \( X \) of topological type \( i \), denoted by \( P\mathcal{N}^a_i(r) \), is defined to be the quotient of \( \mathcal{M}^a_\xi(r, d) \) by the action of \( \text{Bun}_{\mu_r} \).

The moduli space of parabolic stable \( \text{PGL}(r, k) \)-bundles on \( X \) of topological type \( i \) is defined similarly as the quotient of \( M^a_\xi(r, d) \) by \( \Gamma \):

\[
PN^a_i(r) := M^a_\xi(r, d)/\Gamma.
\]

We have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}^a_\xi(r, d) & \longrightarrow & M^a_\xi(r, d) \\
\downarrow & & \downarrow \\
P\mathcal{N}^a_i(r) & \longrightarrow & PN^a_i(r)
\end{array}
\]

See [GO, § 2] for more details.
3. The fixed point loci

We fix a positive integer \( r \), a subset of parabolic points of \( X \) and a line bundle \( \xi \) of degree \( d \) on \( X \). Assume that \( i \equiv d \pmod{r}, i \in [0, d-1] \) in what follows. We adopt the following notations:

- \( \mathcal{M}_{\xi}^{ss} \) := coarse moduli space of semistable vector bundles of rank \( r \) and determinant isomorphic to \( \xi \).
- \( \mathcal{M}_{\xi} \) := moduli space of stable vector bundles of rank \( r \) and determinant isomorphic to \( \xi \).
- \( \mathcal{N}(r)_1 \) := moduli space of stable PGL\((r, k)\)-bundles of rank \( r \) and topological type \( i \),
- \( \mathcal{M}_{\alpha} \) := moduli space of stable parabolic vector bundles of rank \( r \), fixed determinant \( \xi \) with full flag and weights \( \alpha \).
- \( \mathcal{P}_N_{\alpha, \xi} \) := moduli space of stable parabolic PGL\((r, k)\)-bundles of rank \( r \), topological type \( i \) with full flag and weights \( \alpha \).
- \( \Gamma \) := \{ \( L \in \text{Pic}^0(X) \mid L' \cong \mathcal{O}_X \} \cong (\mathbb{Z}/r\mathbb{Z})^2 \).

The aim in this section is to estimate the codimension of the fixed point locus of \( \mathcal{M}_{\alpha} \) under any \( r \)-torsion line bundle \( L \). This would ensure the existence of a closed subscheme of \( \mathcal{M}_{\alpha} \) (namely, the union of the fixed point locus for all \( L \in \Gamma \setminus \{ \mathcal{O}_X \} \)) of codimension at least three such that \( \Gamma \) acts freely on its complement. We remark that \( \alpha \) need not be a generic weight. We first prove a result in linear algebra which will be used.

**Lemma 3.1.** Let \( \psi \) be a diagonalizable automorphism of a \( k \)-vector space \( V \) of dimension \( r \) equipped with a filtration of subspaces

\[
V = V_r \supseteq V_{r-1} \supseteq V_{r-2} \supseteq \cdots \supseteq V_1 \supseteq 0
\]

such that \( \psi(V_i) = V_i \) for all \( 1 \leq i \leq r \). Then there exists a basis of \( V \) consisting of eigenvectors \( \{v_1, v_2, \ldots, v_r\} \) of \( \psi \) such that \( V_j = \langle v_1, \ldots, v_j \rangle \) for all \( 1 \leq j \leq r \).

**Proof.** Since \( \psi \) is diagonalizable, for any subspace \( W \subset V \) such that \( \psi(W) = W \), the restriction of \( \psi \) to \( W \) is diagonalizable and, moreover, since diagonalizable maps are semisimple, there is a subspace \( W' \subset V \) satisfying the conditions \( \psi(W') = W' \) and \( V = W \oplus W' \). Choose any basis vector \( v_i \) for \( V_i \). Suppose \( \{v_1, v_2, \ldots, v_j\} \) has been chosen satisfying the hypothesis up to \( j \leq r - 1 \). Thus there exists a vector \( v_{j+1} \in V_{j+1} \) such that \( V_{j+1} = \langle v_{j+1} \rangle \oplus V_j \). Then \( \{v_1, \ldots, v_{j+1}\} \) satisfy the hypothesis till \( (j + 1) \). Repeating this process, the lemma follows. \( \square \)

For \( L \in \Gamma \setminus \mathcal{O}_X \), let \( \mathcal{M}_{\xi}^{L} \subset \mathcal{M}_{\xi} \) (respectively, \( \mathcal{M}_{\alpha}^{L} \subset \mathcal{M}_{\alpha} \)) be the locus of fixed points for the action of \( L \) on \( \mathcal{M}_{\xi} \) (respectively, \( \mathcal{M}_{\alpha} \)); see (2.1). If \( m = \text{ord}(L) \), choosing a nonzero section \( s_0 \in H^0(X, L^\otimes m) \), define the spectral curve

\[
Y_L := \{ v \in L \mid v^\otimes m \in s_0(X) \}.
\]
The natural projection $\gamma : Y_L \to X$ is an étale Galois covering with Galois group $\mathbb{Z}/m\mathbb{Z}$. The isomorphism class of this covering $\gamma$ does not depend on the choice of the section $s_0$. Let $N_L \subset M_{Y_L}(r/m, d)$ denote the moduli space of stable vector bundles $F$ over $Y_L$ of rank $r/m$ and degree $d$ such that $\det(\gamma_*F) \simeq \xi$. We recall the following:

**Lemma 3.2** ([BH10, Lemma 2.1]). There is a nonempty Zariski open subset $U \subset N_L$ such that $\gamma_*F \in M^L_\xi$ for all $F \in U$. Moreover, the morphism $U \to M^L_\xi$ defined by $F \mapsto \gamma_*F$ is surjective. To be more precise, $U$ is the subset of $N_L$ consisting of those bundles $F$ such that $\gamma_*F$ is also stable.

### 3.1. Estimate of codimension of fixed points.

Given a set $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of $r$ distinct elements, let $P(\alpha)$ denote the set of all possible partitions of its elements into $m$ subsets, each containing $n = r/m$ elements. Clearly, we have $|P(\alpha)| = \binom{r}{n}^m \binom{r-n}{n} \binom{r-2n}{n} \cdots \binom{n}{n}$.

Consider the spectral curve $\gamma : Y_L \to X$. Given a full-flag parabolic data of rank $r$ at the parabolic points $S$ of $X$, we would like to describe a full-flag parabolic data of rank $n$ on $\gamma^{-1}(S)$ for each element of $P(\alpha)$. For simplicity, let us start with the case of a single parabolic point $S = \{p\}$. So, we are given a full-flag parabolic data of rank $r$ at $p$. Let $\alpha$ denote its set of weights. Let

$$\text{Gal}(\gamma) = \{1, \mu, \mu^2, \ldots, \mu^{m-1}\} \subset k^*.$$ 

The action of $\text{Gal}(\gamma)$ on $\gamma^{-1}(p)$ is via multiplication. Using this, we can define a full-flag parabolic data at the points of $\gamma^{-1}(p)$ as follows: let us fix an ordering on the points of $\gamma^{-1}(p)$, say $\gamma^{-1}(p) = \{q_1, q_2, \ldots, q_m\}$, such that $\mu^i$ acts on $\gamma^{-1}(p)$ as the cyclic permutation sending $q_j$ to $q_{j+i}$, where the subscript $(j+i)$ is to be understood mod $m$. For $t \in P(\alpha)$, suppose

$$\alpha = \bigsqcup_{j=1}^m \Lambda_j$$

be the partition of the set of weights $\alpha$ according to $t$. Clearly each $\Lambda_j$ can be arranged into an increasing sequence of length $n$. We designate $\Lambda_j$ as the set of weights at $q_j$ for each $1 \leq j \leq m$. This gives a full-flag parabolic data of rank $n$ at the points of $\gamma^{-1}(p)$. Finally, for multiple parabolic points, say $S = \{p_1, \ldots, p_l\}$ and $\gamma^{-1}(p_j) = \{q_{1j}, q_{2j}, \ldots, q_{mj}\}$, then we perform the same procedure as above for each $p_j$ to get the parabolic structure upstairs. In particular, the number of possible parabolic data upstairs would be $|P(\alpha)| = \binom{r}{n}^m \binom{r-n}{n} \binom{r-2n}{n} \cdots \binom{n}{n}$.

For each $t \in P(\alpha)$, let $M^L_{Y_L}(n, d)$ denote the moduli space of stable parabolic bundles over $Y_L$ of rank $n$, degree $d$ and having full-flag parabolic structures at the points of $\gamma^{-1}(p)$ according to $t$ as described above. Let $\mathcal{N}^t_L \subset M^L_{Y_L}(n, d)$ denote the subvariety consisting of stable parabolic bundles $F_*$ such that $\det(\gamma_*F) \simeq \xi$. Define $\mathcal{N}_L := \bigsqcup_{t \in P(\alpha)} \mathcal{N}^t_L$.

**Lemma 3.3.** There is a surjective morphism $f : \mathcal{N}_L \to M^L_\xi$ given by parabolic push-forward for the étale map $\gamma$.

**Proof.** First, for simplicity let us assume that we have only one parabolic point $S = \{p\}$. The map $f$ in the statement of the send any $F_*$ to the parabolic push-forward of $F_*$ constructed in Section 2.1.
We claim that \( f(F_*) \) is parabolic semistable. To prove this, note that if \( E_* := f(F_*) \), then
\[
\gamma^*(E_*) \cong \bigoplus_{\sigma \in \text{Gal}(\gamma)} \sigma^*(F_*),
\]
where \( \sigma^*(F_*) \) has the obvious parabolic structure coming from \( F_* \). Clearly \( \text{par}_\mu(\sigma^*(F_*)) = \text{par}_\mu(F_*) \) for all \( \sigma \). Therefore \( \gamma^*(E_*) \) is a direct sum of parabolic stable bundles of same parabolic slope, which implies that \( \gamma^*(E_*) \) is parabolic semistable. Thus \( E_* \) must be parabolic semistable as well, since any sub-bundle \( E'_* \subset E_* \) with strictly larger parabolic slope would give rise to a subbundle \( \gamma^*(E'_*) \subset \gamma^*(E_*) \) of strictly larger parabolic slope, contradicting parabolic semistability of \( \gamma^*(E_*) \). Hence \( f(F_*) \) is parabolic semistable.

It can be shown that \( E_* \) is actually parabolic stable. To prove this, take any non-trivial sub-bundle \( E' \subset E \) such that \( \text{par}_\mu(E'_*) = \text{par}_\mu(E_*) \). Then
\[
\gamma^*(E'_*) \subset \gamma^*(E_*) = \bigoplus_{\sigma \in \text{Gal}(\gamma)} \sigma^*(F_*)
\]
is a sub-bundle with same parabolic slope. Now, it is clear that \( \gamma^*(E'_*) \) is also parabolic semistable. Thus it contains a parabolic stable sub-bundle \( F'_* \). We note that the sub-bundles
\[
\{ \sigma^*(F_*) \mid \sigma \in \text{Gal}(\gamma) \}
\]
are mutually non-isomorphic. Indeed, otherwise there would exist a parabolic isomorphism \( \tilde{f} \) from \( F_* \) to \( \sigma^* F_* \) for some \( \sigma \in \text{Gal}(\gamma) \setminus \{ e \} \); at a parabolic point \( y \) it should preserve the filtrations of \( F_* \) and \( F_\sigma(y) \). But since the weights of \( F_* \) at the parabolic points \( y \) and \( \sigma(y) \) are collectively distinct, and a parabolic isomorphism preserves weights, this leads to a contradiction. Therefore, it follows that all projections \( F'_* \rightarrow \sigma^* (F_*) \) except one must be zero, so that \( \gamma^*(E'_*) \) contains one of the \( \sigma^*(F_*) \). But \( \gamma^*(E'_*) \) is \( \text{Gal}(\gamma) \)-equivariant, which would force that \( \gamma^*(E'_*) = \gamma^*(E_*) \); this clearly implies that \( E'_* = E_* \). Consequently, \( E_* \) is parabolic stable.

Next, we argue that \( \text{Im}(f) \subseteq M_a^L \). Let \( E_* = f(F_*) \). There exists a tautological trivialization of \( \gamma^* L \) over \( Y_L \), which induces an isomorphism
\[
\theta : \mathcal{O}_{Y_L} \rightarrow \gamma^* L.
\]
For each \( 1 \leq i \leq m \), the induced map \( \theta_{q_i} : k \rightarrow (\gamma^* L)_{q_i} = L_p \) is given by sending \( \lambda \mapsto \lambda q_i \). This induces an isomorphism
\[
\text{Id}_F \otimes \theta : F \overset{\sim}{\rightarrow} F \otimes \gamma^* L.
\]
This, in turn, produces a canonical isomorphism \( \psi : E \cong E \otimes L \) on the underlying bundle arising from \( \gamma_*(\text{Id}_F \otimes \theta) \) followed by projection formula:
\[
\psi : E = \gamma_* F \overset{\gamma_*(\text{Id}_F \otimes \theta)}{\rightarrow} \gamma_*(F \otimes \gamma^* L) \cong E \otimes L.
\]
Now \( E_p = \bigoplus_{i=1}^m F_{q_i} \), and the map \( \psi_p : E_p \rightarrow E_p \otimes L_p \) on the fiber takes \( F_{q_i} \) to \( F_{q_i} \otimes L_p \), which clearly implies that \( \psi_p \) preserves the filtration induced on \( E_p \). Thus \( \text{Im}(f) \subseteq M_a^L \).

To show the surjectivity of \( f \), let \( E_* \in M_a^L \), so there exists a parabolic isomorphism
\[
\phi_* : E_* \overset{\sim}{\rightarrow} E_* \otimes L.
\]
Suppose \((\alpha_1^p < \alpha_2^p < \cdots < \alpha_r^p)\) are the weights at \(p\). We would like to show that there exists \(F_\ast \in \mathcal{N}_L\) such that \(f(F_\ast) \cong E_\ast\). Since \(E_\ast\) is parabolic stable, it is simple, and hence any parabolic endomorphism of \(E_\ast\) is a constant scalar multiplication. As a consequence, any two parabolic isomorphisms from \(E_\ast\) to \(E_\ast \otimes L\) will differ by a constant scalar multiplication. Thus, we can re-scale \(\varphi_\ast\) by multiplying with a nonzero scalar, so that the \(m\)-fold composition

\[
\varphi_\ast \circ \cdots \circ \varphi_\ast : E_\ast \longrightarrow E_\ast \otimes L^m
\]
coincides with \(\text{Id}_{E_\ast} \otimes \tau\), where \(\tau\) is the chosen nowhere vanishing section of \(L^m\). This produces the isomorphism

\[
\varphi \circ \cdots \circ \varphi = \text{Id}_E \otimes \tau : E \longrightarrow E \otimes L^m
\]
on the underlying bundles. Then, the argument given in the proof of [BH10, Lemma 2.1] will produce a vector bundle \(F\) on \(Y_L\) with \(\gamma_\ast F \cong E\). Let us briefly recall the argument [BH10] for convenience. Consider the pull-back \(\gamma^\ast \varphi\), and compose it with the tautological trivialization of \(\gamma^\ast L\) to get a morphism

\[
\phi : \gamma^\ast E \longrightarrow \gamma^\ast E.
\]

Since \(Y_L\) is irreducible, the characteristic polynomial of \(\phi_y\) remains unchanged as \(y \in Y_L\) moves. This allows us to decompose \(\gamma^\ast E\) into generalized eigenspace sub-bundles. If \(F\) is an eigenspace sub-bundle of \(E\), then we have \(\gamma_\ast F \cong E\). Moreover, the decomposition \(\gamma^\ast(E) = \bigoplus_{\mu \in \mathbb{C}} (\mu)\) is precisely the decomposition of \(\gamma^\ast(E)\) into generalized eigenspace sub-bundles.

Our next task is to produce a parabolic structure on \(F\) so that the parabolic push-forward \(\gamma_\ast(F_\ast)\) (in the sense of Remark 2.1) coincides with \(E_\ast\). To see this, recall the description of \(\theta\) in (3.1), and notice that for any choice of \(q \in \gamma^{-1}(p)\), the map \(\phi_q\) is precisely the composition

\[
(\gamma^\ast E)_q = E_p \xrightarrow{\varphi_p} E_p \otimes L_p = (\gamma^\ast E)_q \otimes (\gamma^\ast L)_q \xrightarrow{1 \otimes (\theta_q)^{-1}} (\gamma^\ast E)_q,
\]

where \(\theta_q : k \to (\gamma^\ast L)_q = L_p\) is given by sending \(\lambda \mapsto \lambda q\). Thus, if

\[
E_p = E_1^p \supseteq E_2^p \supseteq \cdots \supseteq E_r^p \supseteq 0
\]

be the given parabolic filtration of the fiber \(E_p\), then as \(\varphi_\ast\) is a parabolic isomorphism,

\[
\forall j \in [0, r], \quad \varphi_j(E^p_j) = E^p_j \otimes L_p \Longrightarrow \phi_q(E^p_j) = E^p_j \quad \text{[from (3.2)]}.
\]

let \(\phi^s_q\) be the semisimple part of \(\phi_q\) under its Jordan-Chevalley decomposition. It is well-known that \(\phi^s_q\) can be expressed as a polynomial in \(\phi_q\) without constant coefficient. Thus

\[
\phi^s_q(E^p_j) = E^p_j \quad \forall j \in [0, r - 1].
\]

Moreover, the generalized eigenspaces of \(\phi_q\) (namely \(F_{q_j}\)'s) are the eigenspaces for \(\phi^s_q\). Thus \(\varphi^s_q : E_p \to E_p\) and the filtration \(E_p = E_1^p \supseteq \cdots \supseteq E_r^p \supseteq 0\) allow us to apply Lemma 3.1, which gives us a basis of \(E_p\), say \(\{v_1, \ldots, v_r\}\), such that each \(E_p = \langle v_k, \ldots, v_r \rangle\) and each \(v_j\) is contained in a unique \(F_{q_j}\). From this data, we can produce a full-flag parabolic structure on the fibers \(F_{q_1}, \ldots, F_{q_m}\) of \(F\) as follows:

Choose a basis \(B = \{v_1, v_2, \ldots, v_r\}\) of \(E_p\) as in Lemma 3.1, and for each \(q_j\), consider the set \(B_j := B \cap F_{q_j}\). By symmetry, each \(B_j\) consists of \(n\) elements and spans \(F_{q_j}\). Suppose that
Let \( B_i = \{ v_{i_1}, v_{i_2}, \ldots, v_{i_n} \} \), where \( i_1 < i_2 < \cdots < i_n \). Then consider the following weighted full-flag filtration at \( F_{\alpha^p} \):

\[
F_{\alpha^p} = \langle v_{i_1}, v_{i_2}, \ldots, v_{i_n} \rangle \supseteq \langle v_{i_2}, v_{i_3}, \ldots, v_{i_n} \rangle \supseteq \cdots \supseteq \langle v_{i_n} \rangle \supseteq 0,
\]

\[
\alpha^p_1 < \alpha^p_2 < \cdots < \alpha^p_n.
\]

By repeating this for all \( 1 \leq i \leq m \), a parabolic bundle \( F_* \) on \( Y_L \) is constructed. Note that \( F_* \) must be parabolic stable, because if \( F' \subset F \) is any sub-bundle such that \( \text{Par}_\mu(F'_*) \geq \text{Par}_\mu(F_*) \), then the equalities

\[
\text{pardeg}(\gamma_*(F'_*)) = \text{pardeg}(F_*^p) \quad \text{and} \quad \text{rank}(\gamma_*(F'_*)) = m \cdot \text{rank}(F')
\]

would imply that \( \gamma_*(F'_*) \subset E \) violates the condition of parabolic stability for \( E_* \). Thus \( F_* \in \mathcal{N}_L^r \) for some \( r \in P(a) \). Moreover, the parabolic push-forward of \( F_* \) under \( \gamma \) as in Remark 2.1 coincides with \( E_* \).

Finally, if the number of parabolic points on \( X \) is more than 1, then an exactly similar argument with the obvious modifications will give the result.

We note that if \( |D| = s \), then each \( \mathcal{N}_L^r \) is of dimension

\[
\dim(\mathcal{N}_L^r) = \frac{r^2}{m^2}(\text{genus}(Y_L) - 1) + 1 - g + sm \cdot \frac{\Sigma^2(m)}{2} - 1 + g
\]

\[
= (g - 1)(\frac{r^2}{m} - 1) + \frac{sr(m - 1)}{2} \quad [\because \text{genus}(Y_L) = m(g - 1) + 1].
\]

Since \( \dim(M_a) = (g - 1)(r^2 - 1) + s \cdot \frac{r(r - 1)}{2} \), from Lemma 3.3 it follows that

\[
\dim(M_a) - \dim(M_a^L) \geq \dim(M_a) - \dim(\mathcal{N}_L^r) = r^2(\frac{m - 1}{m})(g - 1 + \frac{s}{2}) \geq 3.
\]

**Corollary 3.4.** The codimension of the closed subscheme

\[
Z_a = \bigcup_{L \in \Gamma \setminus \{ \Theta \}} M^L_a \subset M_a
\]

is at least three.

### 4. The Brauer Group of the Stack \( \mathcal{P}\mathcal{N}_{a,d} \)

We continue with the notation of Section 2. For a parabolic bundle \( E_* \), let \( \mathcal{P}(E)_a \) denote the corresponding parabolic PGL\((r, k)\)-bundle. If \( E_* \) is stable parabolic, then \( \mathcal{P}(E)_a \) is a stable parabolic PGL\((r, k)\)-bundle.

**Lemma 4.1.** Let \( U_a := M_a \setminus Z_a \) (see Corollary 3.4). Take any \( E_* \in U_a \), and let \( E'_* := E_* \otimes L \) for some \( L \in \Gamma \). Then there exists a unique isomorphism of parabolic PGL\((r, k)\)-bundles between \( \mathcal{P}(E)_a \) and \( \mathcal{P}(E'_*) \).
Proof. Since the parabolic structure of $E'_u$ is induced from $E_u$, the canonical isomorphism between the underlying PGL($r, k$)-bundles $\mathbb{P}(E)$ and $\mathbb{P}(E'_u)$ actually gives an isomorphism of parabolic PGL($r, k$)-bundles. If there are two isomorphisms between $\mathbb{P}(E)_u$ and $\mathbb{P}(E'_u)_v$, then we get an automorphism of $\mathbb{P}(E)_u$.

Let $f_u$ be a nontrivial automorphism of $\mathbb{P}(E)_u$. It evidently induces an automorphism $f$ of the underlying projective bundle $\mathbb{P}(E)$. Then there exists a line bundle $L_0$ and an isomorphism of vector bundles

$$\tilde{f} : E \rightarrow E \otimes L_0$$

which induces $f$. Taking determinant it follows that $L_0 \in \Gamma$. Moreover, it is clear that $\tilde{f}$ respects the parabolic structures on $E$ and $E \otimes L_0$, so we get an isomorphism of parabolic bundles

$$\tilde{f}_u : E_u \rightarrow E_u \otimes L_0.$$  

But $E_u \in U_u$, thus $L_0$ must be trivial. Hence $\tilde{f}_u$ is an automorphism of $E_u$. Since $E_u$ is parabolic stable, it must be a scalar multiple of the identity map. But then the induced map $f_u$ must be the identity morphism. \qed

The following proposition should be well-known to the experts, but we are providing a proof as we could not find it in the literature, and also for completeness.

**Proposition 4.2.** Let $X$ be a locally noetherian Deligne-Mumford stack, and let $Z \hookrightarrow X$ be a closed substack of codimension at least 2. Let $H^2_\text{ét}(X \setminus Z, \mathbb{G}_m) \simeq H^2_\text{ét}(X, \mathbb{G}_m)$.

**Proof.** Consider an étale cover $V \rightarrow X$. The pull-back of $Z$ produces a closed subscheme $Z'$ of $V$. We still have $\text{codim}_{V}(Z') \geq 2$ [GS, Lemma 5.1].

Let $V^{\times i} = V \times V \times \cdots \times V$ ($i + 1$ times). Now, consider the descent spectral sequence

$$E^{i,j}_1 = H^{i-j}_\text{ét}(V^{\times i}, \mathbb{G}_m) \Rightarrow H^{i+j}_\text{ét}(X, \mathbb{G}_m), \quad (i)$$

and similarly

$$E^{i,j}_1 = H^{i-j}_\text{ét}(V \setminus Z'^{\times i}, \mathbb{G}_m) \Rightarrow H^{i+j}_\text{ét}(X \setminus Z, \mathbb{G}_m). \quad (ii)$$

By purity statements for étale cohomology for smooth schemes [Ce, Theorem 1.1], we know that the open embedding $V \setminus Z' \hookrightarrow V$ induces isomorphisms on the group of units, the Picard group and the cohomological Brauer group. Thus the $E^{i,j}_1$-terms in (i) and (ii) are isomorphic for all $j \leq 2$ and all $i$. This in turn induces isomorphisms on the convergence terms for all $i + j \leq 2$, and thus we have $H^2_\text{ét}(X \setminus Z, \mathbb{G}_m) \simeq H^2_\text{ét}(X, \mathbb{G}_m).$ \qed

Next, consider

$$\mathcal{P}N_{a,i} \xrightarrow{\phi_a} \mathcal{P}N_{a,i}$$

where $\phi_a : M_a \rightarrow \mathcal{P}N_{a,i}$ denotes the quotient by $\Gamma$, and $p : \mathcal{P}N_{a,i} \rightarrow \mathcal{P}N_{a,i}$ is the coarse moduli space map. $\mathcal{P}N_{a,i}$ is a smooth Deligne-Mumford stack, as the automorphism groups of its points are finite.
Note that $\mathcal{PN}_{a,i}$ is the quotient stack $[M_a/\Gamma]$, while $\mathcal{PN}_{a,i} = M_a/\Gamma$ is the GIT quotient, and thus the map $p$ is natural map

$$p : [M_a/\Gamma] \to M_a/\Gamma.$$ 

Thus, by [St, Tag 075T, Lemma 100.33.7], $p$ is smooth. Since $\phi_a$ is a finite morphism, clearly codim$_{\mathcal{PN}_{a,i}}(\phi_a(Z_a)) \geq 2$ by Corollary 3.4. Since smooth morphisms of Artin stacks are codimension-preserving [GS, Lemma 5.1], we conclude that the closed substack

$$Z_a := p^{-1}(\phi_a(Z_a)) \hookrightarrow \mathcal{PN}_{a,i}$$

also has codimension 2.

**Corollary 4.3.** Let $\mathcal{PN}_{a,i}^{sm}$ denote the smooth locus of $\mathcal{PN}_{a,i}$. We have $\text{Br}(\mathcal{PN}_{a,i}^\circ) \simeq \text{Br}(\mathcal{PN}_{a,i}^{sm})$.

**Proof.** Use Proposition 4.2 for $Z_a \hookrightarrow \mathcal{PN}_{a,i}$, together with Lemma 4.1, which says that the map $p$ restricts to an isomorphism over $\phi_a(U_a) = \mathcal{PN}_{a,i} \setminus \phi_a(Z_a)$.

\[\Box\]

5. **Description of the Brauer Groups for Sufficiently Small Weights**

In this section, we allow ourselves to work with parabolic structures which can be consisting of partial flags as well. Assume that $a$ be a sufficiently small generic weight corresponding to a parabolic structure such that

$$\text{g.c.d.}(r, d, m_1^1, \ldots, m_i^1, \ldots, m_j^j, \ldots, m_i^j, \ldots, m_i^i) = 1.$$ 

(See Definition 2.1). By [BY, Proposition 3.2], such a generic weight exists. Since $a$ is chosen sufficiently small, there is a morphism

$$\pi : M_a \longrightarrow M^ss_\xi$$

(5.1)

that sends a parabolic bundle to the underlying vector bundle by simply forgetting the parabolic structure. $\Gamma$ acts on both $M_a$ and $M^ss_\xi$, and $\pi$ in (5.1) is equivariant under this action. Restrict $\pi$ over the stable locus $M_\xi \subseteq M^ss_\xi$. Each fiber of $\pi : \pi^{-1}(M_\xi) \to M_\xi$ is a product of flag varieties. Note that $\pi$ is flat (this is sometimes called as ‘miracle flatness theorem’ [St, Tag 00R4, Lemma 10.128.1]). In fact, $\pi$ is an étale locally trivial fibration [PP, Theorem 1.3]. The closed subset

$$Z := \bigcup_{L \in \Gamma \setminus \{\theta_x\}} M^L_\xi$$

has codimension at least three [BH10, Corollary 2.2]. Define

$$U^\circ := M_\xi \setminus Z.$$ 

Consider $\pi^{-1}(U^\circ) \subset \pi^{-1}(M_\xi)$. Clearly the action of $\Gamma$ on $M_\xi$ restricts to a free action on $U^\circ$, and hence the action of $\Gamma$ on $\pi^{-1}(U^\circ)$ is also free; indeed, if $E_\ast$ is an $L$-fixed point, then the underlying bundle $E$ is also fixed by $L$. Thus, if $\mathcal{PN}_{a,i}^{sm}$ (resp. $N(r)_i^{sm}$) be the smooth locus of $\mathcal{PN}_{a,i}$ (resp. $N(r)_i$), then we have

$$U^\circ/\Gamma \subset N(r)_i^{sm} \text{ and } \pi^{-1}(U^\circ)/\Gamma \subset \mathcal{PN}_{a,i}^{sm}.$$
A straightforward codimension estimate shows that
\[
\text{codim}_{M_a}(M_a \setminus \pi^{-1}(U)) = \text{codim}_{M_b}(M_b \setminus U) \geq 3. \tag{5.2}
\]
In view of these codimension estimates, we have
\[
\text{Br}(U/\Gamma) = \text{Br}(N(r)^{sm}) \quad \text{and} \quad \text{Br}(\pi^{-1}(U)/\Gamma) = \text{Br}(PN_{a,d}^{sm}). \tag{5.3}
\]
This enables us to work with finite étale morphisms in what follows. Consider the diagram
\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\phi'} & \pi^{-1}(U)/\Gamma \\
\downarrow \pi & & \downarrow \pi' \\
U & \xrightarrow{\phi} & U/\Gamma
\end{array}
\tag{5.4}
\]
where \(\phi\) and \(\phi'\) are both finite étale (due to the removal of all the fixed points), and \(\pi, \pi'\) are both fibrations in the étale topology. The typical fibers of \(\pi\) and \(\pi'\) are the same, with both being isomorphic to the product of flag varieties. In fact the diagram in (5.4) is Cartesian. Denote the common typical fiber of the two fiber bundles by \(F\).

**Proposition 5.1.** Let \(\alpha\) be a sufficiently small weight, and \(PN_{a,d}^{sm}\) (resp. \(N(r)^{sm}\)) be the smooth locus of \(PN_{a,i}\) (resp. \(N(r)_i\)), where \(i \equiv d(\bmod r)\). Let \(U \subset M_\delta\) be as above. Then \(\text{Br}(PN_{a,d}^{sm})\) can be identified with the kernel of the natural surjective map
\[
\text{Br}(N(r)^{sm}) = \text{Br}(U/\Gamma) \longrightarrow \text{Br}(U) = \text{Br}(M_\delta) \cong \mathbb{Z}/\delta\mathbb{Z},
\]
where \(\delta = \text{g.c.d.}(r, d)\).

**Proof.** The Leray spectral sequence for the map \(\pi : \pi^{-1}(U) \to U\) gives us the exact sequence
\[
0 \to H^1(U, \pi_*\mathbb{G}_m) \xrightarrow{\pi^*} \text{Pic}(\pi^{-1}(U)) \to H^0(U, R^1\pi_*\mathbb{G}_m) \tag{5.5}
\]
We have \(\pi_*\mathcal{O}_{\pi^{-1}(U)} = \mathcal{O}_U\) since \(\pi\) is flat and global sections of the fibers are trivial; this implies \(\pi_*\mathbb{G}_m = \mathbb{G}_m\). Also, since \(U\) is simply connected, \(R^1\pi_*\mathbb{G}_m\) is the constant sheaf with stalk \(\text{Pic}(F)\), where \(F\) is a typical fiber of \(\pi\) (for details, see [BD, Lemma 3.1].)

Similarly, for \(\pi' : \pi^{-1}(U)/\Gamma \to U/\Gamma\) we get:
\[
0 \to H^1(U/\Gamma, \pi'_*\mathbb{G}_m) \xrightarrow{\pi'^*} \text{Pic}(\pi^{-1}(U)/\Gamma) \to H^0(U/\Gamma, R^1\pi'_*\mathbb{G}_m) \tag{5.6}
\]
Again for similar reasons as above, \(\pi'_*\mathbb{G}_m = \mathbb{G}_m\). We claim that \(R^1\pi'_*\mathbb{G}_m\) is the constant sheaf with stalk \(\text{Pic}(F)\). This is because, as remarked above, \(R^1\pi'_*\mathbb{G}_m\) becomes the constant sheaf \(\text{Pic}(F)_{U}\) after pull-back to the étale cover \(U \longrightarrow U/\Gamma\), and the action of \(\Gamma\) on \(\text{Pic}(F)\) is trivial.

Thus the exact sequences of (5.5) and (5.6) become the following, respectively:
\[
0 \to \text{Pic}(U) \xrightarrow{\pi^*} \text{Pic}(\pi^{-1}(U)) \to \text{Pic}(F) \tag{5.7}
\]
\[
0 \to \text{Pic}(U/\Gamma) \xrightarrow{\pi'^*} \text{Pic}(\pi^{-1}(U)/\Gamma) \to \text{Pic}(F) \tag{5.8}
\]
Consequently, we get the following diagram:

\[
0 \longrightarrow \text{Pic}(U'/\Gamma) \xrightarrow{\phi'^*} \text{Pic}(\pi^{-1}(U)/\Gamma) \xrightarrow{j'^*} \text{Pic}(F) \quad (5.9)
\]

where \(j, j'\) denote the isomorphism of \(F\) with some fiber, and \(g'\) is the induced map.

**Claim 5.1.** The homomorphism \(g'\) in (5.9) is the identity map.

**Proof of claim.** First, note that for a point \(y \in U'/\Gamma\), identifying \(F\) with \(\pi'^{-1}(y)\),

\[
\phi'^{-1}(F) = \phi'^{-1}(\pi'^{-1}(y)) = \pi^{-1}(\phi^{-1}(y)) \simeq \pi^{-1}([\{y\} \times \Gamma])
\]

\[
= \pi^{-1}(y) \times \Gamma
\]

\[
\simeq F \times \Gamma,
\]

and hence \(\text{Pic}(\phi'^{-1}(F)) = \bigoplus_{i=1}^{[\Gamma]} \text{Pic}(F)\). Therefore, \(g'\) can be expressed as the composition

\[
\text{Pic}(F) = \text{Pic}(\pi'^{-1}(y)) \xrightarrow{\phi'^*} \bigoplus_{i=1}^{[\Gamma]} \text{Pic}(F) \longrightarrow \text{Pic}(F),
\]

where the last arrow is the projection to any one of the components. It is easy to see that this is the identity map, which proves the claim. \(\square\)

Note that \(\Gamma\) acts trivially on both \(\text{Pic}(U')\) and \(\text{Pic}(\pi^{-1}(U'))\) and both are torsion-free, which implies that

\[ H^1(\Gamma, \text{Pic}(U')) = H^0(\Gamma, \text{Pic}(U')) = 0 \quad \text{and} \quad H^1(\Gamma, \text{Pic}(\pi^{-1}(U'))) = H^0(\Gamma, \text{Pic}(\pi^{-1}(U'))) = 0, \]

since \(\Gamma\) is finite. The five-term exact sequence corresponding to the Hochschild-Serre spectral sequence associated to \(\phi\) and \(\phi'\) [Mi, III Theorem 2.20] then produces following two exact sequences:

\[
0 \longrightarrow \chi(\Gamma) \xrightarrow{f} \text{Pic}(U'/\Gamma) \xrightarrow{\phi'^*} \text{Pic}(U') \longrightarrow H^2(\Gamma, k^*) \longrightarrow \text{Br}(U'/\Gamma) \longrightarrow \text{Br}(U') \simeq \mathbb{Z}/\delta \mathbb{Z} \longrightarrow 0 \quad (5.10)
\]

\[
0 \longrightarrow \chi(\Gamma) \xrightarrow{f'} \text{Pic}(\pi^{-1}(U')/\Gamma) \xrightarrow{\phi'^*} \text{Pic}(\pi^{-1}(U')) \longrightarrow H^2(\Gamma, k^*) \longrightarrow \text{Br}(\pi^{-1}(U')/\Gamma) \longrightarrow \text{Br}(\pi^{-1}(U')) \longrightarrow 0 \quad (5.11)
\]

whose rows and columns are exact. Note that due to the codimension estimates in (5.2),

\[
\text{Br}(U') \cong \text{Br}(M_1) \cong \mathbb{Z}/\delta \mathbb{Z}, \quad \text{and} \quad \text{Br}(\pi^{-1}(U')) \cong \text{Br}(M_a) = 0 [BD, \text{Theorem 1.1}].
\]

Let \(M := \text{coker}(f)\) and \(M' := \text{coker}(f')\). Using the snake lemma in the diagram

\[
0 \longrightarrow \chi(\Gamma) \xrightarrow{f} \text{Pic}(U'/\Gamma) \xrightarrow{\phi'^*} M \longrightarrow 0
\]

\[
0 \longrightarrow \chi(\Gamma) \xrightarrow{f'} \text{Pic}(\pi^{-1}(U')/\Gamma) \xrightarrow{\phi'^*} M' \longrightarrow 0
\]
we conclude that

\[ \text{Pic}(F) \cong \frac{M'}{M}. \tag{5.12} \]

Set \( H := \text{coker}(\phi^*) \) and \( H' := \text{coker}(\phi'^*) \) (see the diagram in (5.4)). The group \( H \) is cyclic of order \( \delta \) [BH10, Theorem 1.2]. Using the snake lemma in the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & M & \rightarrow & \text{Pic}(U^*) & \rightarrow & H & \rightarrow & 0 \\
| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M' & \rightarrow & \text{Pic}(\pi^{-1}(U^*)) & \rightarrow & H' & \rightarrow & 0
\end{array}
\]

and from (5.9), it follows that

\[ 0 \rightarrow \frac{M'}{M} \rightarrow \text{Pic}(F) \rightarrow \frac{H'}{H} \rightarrow 0. \tag{5.13} \]

Composing the first map with (5.12) gives

\[ \text{Pic}(F) \cong \frac{M'}{M} \rightarrow \text{Pic}(F) \]

which is the identity map in Claim 5.1, and hence

\[ \frac{H'}{H} = 0 \implies H \cong H'. \]

From the exact sequences in (5.10) and (5.11) we get the following two exact sequences:

(i) \[ 0 \rightarrow H \rightarrow H^2(\Gamma, k^*) \rightarrow \text{Br}(U^*/\Gamma) \rightarrow \text{Br}(U^*) \cong \mathbb{Z}/\delta\mathbb{Z} \rightarrow 0, \]

(ii) \[ 0 \rightarrow H' \rightarrow H^2(\Gamma, k^*) \rightarrow \text{Br}(\pi^{-1}(U^*)/\Gamma) \rightarrow 0. \]

Since both \( PN_{a,i} \) and \( N(r) \) are normal projective varieties and the open subsets \( U^*/\Gamma \) and \( \pi^{-1}(U^*)/\Gamma \) both have complement of codimension 2, it follows that \( \pi^* \) induces isomorphism

\[ k^* = H^0(U^*/\Gamma, \mathbb{G}_m) \rightarrow H^0(\pi^{-1}(U^*)/\Gamma, \mathbb{G}_m) = k^*, \]

and thus it induces isomorphism \( H^2(\Gamma, k^*) \rightarrow H^2(\Gamma, k^*) \), which takes \( H \) to \( H' \). Thus, from the exact sequences (i) and (ii) we finally conclude that

\[ \text{Br}(U^*/\Gamma) \cong \text{Br}(\pi^{-1}(U^*)/\Gamma). \]

Applying (5.3) proves our claim. \( \square \)

6. Brauer Groups for Arbitrary Generic Weights

We once again work with full-flag parabolic structures. For any such generic weight \( \alpha \), let us denote as usual the smooth locus of \( PN_{a,i} \) by \( PN_{a,i}^{\text{sm}} \). Let \( \alpha \) and \( \beta \) be two generic weights separated by a single wall \( W \) (cf. [BY, § 2]). In [Th, § 5, page 11] it is shown that there exists a closed subscheme \( Y_a \hookrightarrow M_a \) whose image is precisely the locus of those parabolic bundles that are \( \alpha \)-stable but not \( \beta \)-stable, and thus \( M_a \setminus Y_a \) is the open subset of \( M_a \) consisting precisely of those parabolic bundles which are both \( \alpha \)-stable as well as \( \beta \)-stable. Similarly, there exists a closed
subscheme \( Y_\beta \subset M_\beta \) with exactly similar properties as \( Y_\alpha \) when \( \alpha \) and \( \beta \) are interchanged. Thus there exists a natural isomorphism

\[
M_\alpha \setminus Y_\alpha \cong M_\beta \setminus Y_\beta. \tag{6.1}
\]

Moreover, both \( Y_\alpha \) and \( Y_\beta \) have codimension at least 2 in \( M_\alpha \) and \( M_\beta \) respectively.

From their descriptions it is clear that both \( U_\alpha = M_\alpha \setminus Y_\alpha \) and \( U_\beta = M_\beta \setminus Y_\beta \) are \( \Gamma \)-invariant, and their identification in (6.1) is \( \Gamma \)-equivariant. Thus, if we consider the closed subschemes \( Z_\alpha \) and \( Z_\beta \) of \( M_\alpha \) and \( M_\beta \) respectively as in Corollary 3.4, then \( V_\alpha := U_\alpha \setminus Z_\alpha \) and \( V_\beta := U_\beta \setminus Z_\beta \) are \( \Gamma \)-invariant open subsets of \( M_\alpha \) and \( M_\beta \) respectively; but now \( \Gamma \) acts freely on these sets. It follows that the isomorphism in (6.1) descends to an isomorphism

\[
V_\alpha / \Gamma \cong V_\beta / \Gamma, \tag{6.2}
\]

and moreover, under the quotients by the \( \Gamma \)-action \( \phi_\alpha : M_\alpha \to PN_{a,i} \) and \( \phi_\beta : M_\beta \to PN_{b,i} \), we have \( V_\alpha / \Gamma \subset PN_{a,i}^{sm} \) and \( V_\beta / \Gamma \subset PN_{b,i}^{sm} \). Of course, since \( \phi_\alpha \) is a finite morphism, the complement of \( V_\alpha / \Gamma \) in \( PN_{a,i}^{sm} \) will still be of codimension at least 2; the same is true if we replace \( \alpha \) by \( \beta \). Hence we conclude that when \( \alpha \) and \( \beta \) are in adjacent chambers separated by a single wall,

\[
Br(PN_{a,i}^{sm}) \cong Br(PN_{b,i}^{sm}). \tag{6.3}
\]

Now, since there are only finitely many walls, we can arrange the collection of chambers in a sequence, say \( C_1, C_2, \ldots, C_N \), such that \( C_1 \) contains a sufficiently small weight, and for each \( 1 \leq j < N \), the chambers \( C_j \) and \( C_{j+1} \) are separated by a single wall. Choose generic weights \( \alpha_j \in C_j \) for each \( j \), such that \( \alpha_1 \) is a sufficiently small weight. By (6.3), we have

\[
Br(PN_{a,i}^{sm}) \cong Br(PN_{a,i+1}^{sm}), \quad \forall \ 1 \leq j \leq N.
\]

Thus we obtain the following:

**Corollary 6.1.** For any two generic weights \( \alpha \) and \( \beta \) corresponding to full-flag type,

\[
Br(\ PN_{a,i}^{sm} ) \cong Br(\ PN_{b,i}^{sm} ).
\]

Thus, Proposition 5.1 remains valid for arbitrary generic weights of full-flag type.

**Remark 6.2.** In fact, under the condition \( g.c.d.(r, d, m_1, \ldots, m_i, \ldots, m^r, \ldots, m^s) = 1 \), we can also describe the Brauer groups for the non-generic weights which lie on a single wall (that is, those weights which do not lie on the intersection of two or more hyperplanes). Let \( \gamma \) be such a weight lying on a single wall \( W \). Under the hypothesis, we know a generic weight exists [BY, Proposition 3.2]. It follows that in a small enough neighborhood of \( \gamma \), we can choose two generic weights \( \alpha \) and \( \beta \) which are separated by \( W \) only, and \( \gamma \) is the point of intersection between \( W \) and the line joining \( \alpha \) and \( \beta \). By [BH95, Lemma 3.6], any \( \alpha \)-parabolic stable bundle is \( \gamma \)-parabolic semistable, which induces a \( \Gamma \)-equivariant morphism \( \varphi_\alpha : M_\alpha \to M_\gamma \) by simply replacing the weights. Note that \( M_\gamma \) is no longer smooth. Let \( U_\gamma \subset M_\gamma \) be the open subset consisting of \( \gamma \)-parabolic stable bundles. From [BH95, Theorem 3.1] it follows that \( \varphi_\gamma \) is an isomorphism between the open subsets \( U_\alpha := \varphi_\alpha^{-1}(U_\gamma) \) and \( U_\gamma \), and moreover, both of them
have complements of codimension at least 2. Since $U_\gamma$ is $\Gamma$-invariant, this isomorphism is $\Gamma$-equivariant. The rest of the argument is similar to above. Namely, if we consider the closed subschemes $Z_a$ and $Z_\gamma$ of $M_a$ and $M_\gamma$ respectively as in Corollary 3.4, then $V_a := U_a \setminus Z_a$ and $V_\gamma := U_\gamma \setminus Z_\gamma$ are also $\Gamma$-equivariantly isomorphic under $\phi_a$, and thus descends to an isomorphism

$$V_a/\Gamma \simeq V_\gamma/\Gamma.$$ 

Since $\Gamma$ acts freely on both $V_a$ and $V_\gamma$, we have $V_a/\Gamma \subset PN_{a,d}^{sm}$ and its complement will still have codimension 2. Of course, the same is true for $V_\gamma/\Gamma \subset PN_{\gamma,d}^{sm}$ as well. From this we conclude that

$$\text{Br}(PN_{a,d}^{sm}) \simeq \text{Br}(PN_{\gamma,d}^{sm}).$$

Note that $\text{Br}(PN_{a,d}^{sm})$ is already known from Proposition 5.1. Of course, the same remains true if $\alpha$ is replaced by $\beta$.

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