Energy conditions in modified $f(G)$ gravity

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Abstract In this paper, we have considered flat Friedmann–Lemaître–Robertson–Walker metric in the framework of perfect fluid models and modified $f(G)$ gravity (where $G$ is the Gauss Bonnet invariant). Particularly, we have considered particular realistic $f(G)$ configurations that could be used to cure finite-time future singularities arising in the late-time cosmic accelerating epochs. We have then developed the viability bounds of these models induced by weak and null energy conditions, by using the recent estimated numerical figures of the deceleration, Hubble, snap and jerk parameters.

Keywords Relativistic fluids · Modified gravity · Stability
1 Introduction

Several interesting outcomes stem from observations of Supernovae Type Ia, cosmic microwave background radiation, etc. [1–3] have produced a revolution in the field of relativistic astrophysics and cosmology. This has created a new alluring platform for research. These ingredients have revealed that current expansion of universe is accelerating. The observational data came from, e.g., the Planck satellite [4–6], the BICEP2 experiment [7–9], and the Wilkinson Microwave anisotropy probe (WMAP) [10, 11], have illustrated that the energy fractions of the baryonic and dark matter (DM) are 5% and 27%, respectively, while that of dark energy (DE) is only 68%. The concept of modified gravity theories (MGTs) obtained by replacing the Ricci scalar in the standard Einstein–Hilbert (EH) action with some generic functions of the Ricci scalar \( f(R) \) or the combinations of the scalar and tensorial curvature invariants have been introduced by many relativistic astrophysicists. This approach has now been referred to as a standard terminology whose formulations could be considered as a viable guide to explore reason of the cosmic accelerated expansion (for further reviews on dark energy and modified gravity, see, for instance, [12–28]). The first consistent outcomes of accelerating universe from \( f(R) \) gravity was suggested by Nojiri and Odintsov [29]. There has been an interested results found on the exploration of dark source terms on the dynamical evolution of stellar systems in Einstein-\( \Lambda \) [30, 31], \( f(R) \) [32–36], \( f(R, T) \) [37–40] (\( T \) is the trace of energy momentum tensor) and \( f(R, T, R_{\mu\nu}T^{\mu\nu}) \) gravity [41–44].

Among MGTs, available in the literature, the one is Gauss Bonnet (GB) gravity which has received great attraction [45–55] and is named as \( f(G) \) gravity, where

\[
G = R - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}
\]

is a topological invariant in 4 dimensions of spacetime. The equation of motion for this gravity is required to be coupled with some scalar field or \( f(G) \) must be some arbitrary function of \( G \). This MGT could help out in the study of inflationary era, transition of acceleration from deceleration regimes, passing tests induced by solar system experiments and crossing phantom divide line for different viable \( f(G) \) models [47, 48]. It is also seen that the GB gravity is less constrained than \( f(R) \) gravity [49]. The \( f(G) \) gravity also provide an efficient platform to study various cosmic issues as an alternate to DE [50]. The \( f(G) \) gravity could also be very helpful for the study of finite time future singularities as well as the universe acceleration during late time epochs [51, 52]. Similarly, the cosmic acceleration followed by matter era could also be explained by means of some viable models in \( f(G) \) gravity [49, 50]. Different consistent \( f(G) \) models were proposed in order to pass certain solar system constraints [49, 50] which are discussed in [53] and additional bounds on \( f(G) \) models may arise from the analysis of energy conditions (ECs) [54–56]. Nojiri et al. [57] have discussed some fundamental cosmic issues, like inflation, late-time acceleration, bouncing cosmology and claimed that some modified theories of gravity, like \( f(R) \), \( f(G) \) and \( f(T) \) theories (where \( T \) is the torsion scalar) could be used as a viable mathematical tool for analyzing the clear picture of our universe.

The ECs are the basic ingredients for the deep understanding of the singularity theorem as like the theorem of black-hole thermodynamics. Hawking–Penrose singularity theorem imposed the importance of the weak energy (WE) and strong energy (SE) conditions, while the black hole second law of thermodynamics signifies null energy (NE) condition. The well-known Raychaudhuri equations could be considered
to discuss the viability of various forms of ECs [58–60]. Some of the literature review of ECs were discussed by using the classical ECs of general relativity (GR) like the phantom fields potential [61], the history of expanding universe [62–67] and the pattern movement of deceleration parameters [68,69]. The various expression for ECs are derived in $f(R)$ gravity [70] and using these formalism and techniques, some authors have pointed out some issues (cosmological) in $f(R)$ gravity [71–73]. The general formalism for ECs are derived in $f(G)$ gravity by García et al. [74]. Nojiri et al. [51] presented some specific realistic and viable $f(G)$ models by analyzing the dynamical behavior of WEC. García et al. [75] have explored some viable $f(G)$ models and checked their viability epochs by exploring ECs. Sadeghi et al. [76] have explored some $f(G)$ gravity models that could obey WEC and SEC in an era where late-time de-Sitter solution was stable. Banijamali et al. [77] analyzed the distribution of WEC for a class of consistent $f(G)$ models and claimed that power law model of the type $f(G) = \epsilon G^n$ would satisfy WEC on setting $\epsilon < 0$. Zhou et al. [78] performed a thorough phase space analysis on some $f(G)$ models to derive conditions for the cosmological viability of $f(G)$ DE models. The analysis of ECs for some specific forms of $f(R, G)$ models is performed by Atazadeh and Darabi [79].

In this paper, we have used some of the approximate values of the jerk, deceleration, Hubble as well as snap model parameters, we then apply certain limits from $f(G)$ gravity ECs on the model building variables which were suggested in paper [49]. We showed by different plots that these models in $f(G)$ gravity can satisfy the WEC and SEC in a specific region which is necessary for exploring the stability of a late time de-Sitter solutions. This work is formatted in a manner that the coming section consists of brief introduction to $f(G)$ field equations as well as modified version of ECs. In Sect. 3, we shall consider some viable $f(G)$ models in order to explore the viability epochs of ECs. The conclusions and results are summarized in the last section.

## 2 Field equation

This section is devoted to illustrate the extended version of GB gravity with its equations of motion as well as ECs. For $f(G)$ gravity, the usual EH action is modified as follows

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2} + f(G) \right] + S_M (g^{\mu\nu}, \psi),$$

(1)

where $\kappa^2 = 8\pi G \equiv 1$, $R$, $f$, $S_M (g^{\mu\nu}, \psi)$ are the Ricci scalar, arbitrary function of GB invariant and the matter action, respectively. The GB invariant quantity is

$$G = R - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\beta}R^{\mu\nu\lambda\beta},$$

(2)

where $R_{\mu\nu}$ is the Ricci tensor and $R_{\mu\nu\lambda\beta}$ is the Riemannian tensor. Upon varying the above action with respect to $g_{\mu\nu}$, we get the modified field equations for $f(G)$ gravity as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}^\text{eff},$$

(3)
where $T_{\mu\nu}^{\text{eff}}$ is dubbed as effective energy momentum tensor with its expression as follows

$$
T_{\mu\nu}^{\text{eff}} = \kappa^2 T_{\mu\nu} - 8 \left[ R_{\mu\rho\sigma\nu} + R_{\mu\nu}g_{\rho\sigma} - R_{\mu\nu}g_{\rho\sigma} + R_{\mu\sigma}g_{\nu\rho} + R_{\mu\sigma}g_{\nu\rho} \right. \\
+ \left. \frac{1}{2} (g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma}) \right] \nabla^\rho \nabla^\sigma f G + (G_f G - f) g_{\mu\nu},
$$

where subscript $G$ defines the derivation of the corresponding term with the GB term, while $T_{\mu\nu}$ is the usual stress energy momentum tensor. We model our system with the following well-known line element of Friedmann–Lemaître–Robertson–Walker (FLRW) universe

$$
ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2),
$$
in which $a(t)$ is the scale factor. We assume that this line element is filled with an ideal matter content whose energy momentum tensor is

$$
T_{\mu\nu} = \text{diag}(\rho(r), -p(r), -p(r), -p(r)).
$$

In this context, the $f(G)$ field Eq. (3) turns out to be

$$
\rho^{\text{eff}} = \frac{3}{2} H^2, \quad p^{\text{eff}} = -\left(2H' + 3H^2\right)
$$

where prime symbolizes for temporal derivations, $\rho^{\text{eff}}$ and $p^{\text{eff}}$ are effective energy density and the pressure gradient, respectively. For FLRW universe filled with perfect fluid, the expressions for effective energy density and the pressure component become

$$
\rho^{\text{eff}} = \rho + \frac{1}{2} \left[ -f(G) + 24H^2 f'(G)(H^2 + H') - 576H^3 f''(G)(4H^2 H' + 2H^2 + H H'') \right],
\quad
p^{\text{eff}} = p + \frac{1}{2} \left[ f(G) - 24H^2 f'(G)(H^2 + H') + 8\{576H^3 (4H^2 H' + 2H^2 + H \times H'')^2 f^{(3)}(G) + 24 H f''(G)(8H^4 H' + 6H^3 + 6H^3 H'' + 8H H' H'' + H^2 \times (24H^2 + H^{(3)}))\} \right]
$$

The GB and the Ricci invariants for the flat FLRW spacetime are found as follows

$$
G = 24H^2 \left( H' + H^2 \right), \quad R = 6 \left( H' + 2H^2 \right).
$$

### 3 Energy conditions

In different physical scenario, the basic and fundamental tools for the study of black holes, wormholes (WHs) etc, are the ECs. The breaching of these constraints could
be fruitful to analyze the stability of WHs. The situation of exploring ECs in MGTs is quite different because the field equations differ from the Einstein equations. The ECs in GR are derived by relating \( R_{\mu\nu} \) with usual energy momentum tensor. In MGTs, such a relation is not straightforward. One must know how to relate \( R_{\mu\nu} \) with the effective forms of energy momentum tensor which will eventually give rise to the corresponding ECs. These ECs are the outcomes of Raychaudhuri’s equation for the expansion nature. In MGTs (having effective energy density and pressure), the NEC and WEC are defined as follows

\[
\text{NEC} \iff \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \\
\text{WEC} \iff \rho_{\text{eff}} \geq 0 \text{ and } \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \tag{9}
\]

while the SEC and the dominant energy condition (DEC) provide

\[
\text{SEC} \iff \rho_{\text{eff}} + 3p_{\text{eff}} \geq 0 \text{ and } \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \\
\text{DEC} \iff \rho_{\text{eff}} \geq 0 \text{ and } \rho_{\text{eff}} \pm p_{\text{eff}} \geq 0. \tag{10}
\]

We see that ECs would impose some constraints on the parameters involved in the building of \( f(G) \) models \[74\]. It has been clear that the derivative of position four vector is referred as four velocity and its double derivative is termed as four acceleration. Further, its third and fourth derivatives give jerk and snap parameters, respectively. The Hubble parameter for FLRW metric filled with an ideal matter is found as follows

\[
H = \frac{\dot{a}}{a}, \tag{11}
\]

while deceleration \( q \), jerk \( j \) and snap \( s \) parameters turn out to be

\[
q = -\frac{1}{H^2} \frac{a''}{a}, \quad j = \frac{1}{H^3} \frac{a'''}{a}, \quad s = \frac{1}{H^4} \frac{a''''}{a}. \tag{12}
\]

By means of these parameters, the derivatives of Hubble parameters become

\[
H' = -H^2(q + 1), \quad H^{(3)} = H^4(-2j - 5q + s - 3), \quad H'' = H^3(j + 3q + 2). \tag{13}
\]

Using Eqs.\((11)-(13)\), Eqs.\((7)\) and \((8)\) can be recasted as

\[
\rho_{\text{eff}} + p_{\text{eff}} = \rho + p + \frac{1}{2} \left[ 192H^8 \left( (3 - 14q - 24q^2 - 6q^3 - j(7 + 8q) + s \right) \times f''(G) + 24H^4(j + q(3 + 2q))^2 f^{(3)}(G) \right], \tag{14}
\]

\[
\rho_{\text{eff}} = \rho + \frac{1}{2} \left[ -f(G) - 24(4^4 q f'(G) + 24H^8 (j + q(3 + 2q)) f''(G) \right]. \tag{15}
\]

It is worth noticing that above equations have been expressed by taking into account arbitrary function of \( G \).
4 Specific models

In this section, we shall check the influences of some \( f(G) \) models, with vacuum (i.e. \( \rho = p = 0 \)) background, on the formulations and behavior of the ECs. In the following calculation, we would use the following specific numerical values of Hubble, deceleration, snap and jerk parameters \([80,81]\)

\[
H = 0.718, \quad q = -0.64, \quad j = 1.02, \quad s = -0.39.
\]

The following subsections would allow us to set up various configurations of FLRW models controlled by few particular \( f(G) \) models.

4.1 Model 1

First, we assume a model containing combinations of power law and logarithmic \( f(G) \) corrections \([82]\)

\[
f(G) = \alpha G^n + \beta G \log[G],
\]

where \( \alpha, n \) and \( \beta \) are constants. The dynamics presented by this model is found to be in agreement with the data presented by the same cosmographic parameters \([83]\). This upon substituting in Eq. (15), we obtain the effective energy density as

\[
\rho^{\text{eff}} = \frac{1}{q^2} (24\beta H^4 q (j + q (3 + q)) - 24^n (-1 + n) \alpha (-H^4 q)^n (nj + q (3n + (-1 + 2n)q))) \geq 0,
\]

while the sum of effective pressure energy density can be obtained, after using Eqs.(14) and (16), as follows

\[
p^{\text{eff}} + \rho^{\text{eff}} = \frac{1}{3q^3} (3q (24\beta H^4 q - 24^n (-1 + n) \alpha (-H^4 q)^n) (j + q (3 + 2q))
- \left(24\beta H^4 q + 24^n n (2 - 3n + n^2) \alpha (-H^4 q)^n\right) (j + q (3 + 2q))^2
+ q \left(24\beta H^4 q - 24^n (-1 + n) \alpha (-H^4 q)^n\right) (-3 + 5q + 18q^2 + 6q^3 + j (4 + 8q - s))) \geq 0.
\]

To get an exact solution from the above two inequalities (17) and (18), for the parameters \( \alpha, \beta \) and \( n \), is a quite hard task. In order to achieve this goal, we would consider specific value of \( \alpha = 0.3 \) and plot \( \rho^{\text{eff}} \) and \( \rho^{\text{eff}} + p^{\text{eff}} \) as a function of \( \beta \) and \( n \) as shown in Fig. 1. One can see the validity of WECs from the Fig. 1.

Now we will discuss the constraints which is required for the validity of WEC, i.e., for \( \rho^{\text{eff}} \geq 0 \). The validity of WEC is guaranteed if
Fig. 1 WEC plots for $f(G)$ model given in Eq. (16). Here, left and right plots are showing the behaviors of $\rho^{\text{eff}}$ and $\rho^{\text{eff}} + p^{\text{eff}}$ with respect to $\beta$ and $n$ for $\alpha = 0.3$, respectively.
Next, we consider another realistic formulation of Model 2

\[ f(G) = \alpha G^n (\beta G^m + 1), \]  

(19)

where \( \alpha, \beta \) and \( m \) are the arbitrary constants and \( n \) is a positive constant. This model could be useful to understand the finite time future singularities \[84\]. The outcomes of this model are found to be in agreement with the local test as well as the cosmological bounds \[85\]. By making use of Eq. (19), the effective energy density has been found to be

\[
\rho_{\text{eff}} = 24^n \alpha \left( -H^4 q \right)^n \left( -1 + n - 24^m \beta \left( -H^4 q \right)^m + 24^m m \beta \left( -H^4 q \right)^m + 24^m n \beta \right) \times \left( -H^4 q \right)^m - \frac{1}{q^2} \left( j + q (3 + 2q) \right) \left( 24^m (-1 + m) m \beta \left( -H^4 q \right)^m + n^2 \{1 + \right.ight.

\[
\left. \times 24^m \beta \left( -H^4 q \right)^m \right) + n \left( -1 + 24^m (-1 + 2m) \beta \left( -H^4 q \right)^m \right) \right) \geq 0, \quad (20)
\]

while the combination of effective pressure and energy density becomes

\[
\rho_{\text{eff}} + p_{\text{eff}} = \frac{1}{q^2} 3^{1+n} 8^n \alpha \left( -H^4 q \right)^n \left( -3q (j + q (3 + 2q)) \right) \left( 24^m (-1 + m) m \beta \left( -H^4 q \right)^m \right) + n^2 \left( 1 + 24^m \beta \left( -H^4 q \right)^m \right) + n^2 \left( 1 + 24^m \beta \left( -H^4 q \right)^m \right) + 3n^2 \left( -1 + 24^m (-1 + m) \beta \left( -H^4 q \right)^m \right) + n \left( 2 + 24^m \beta \left( -H^4 q \right)^m \right) \left( -1 + 24^m (-1 + m) \beta \left( -H^4 q \right)^m \right) + n^2 \left( 1 + 24^m \beta \left( -H^4 q \right)^m \right) + n \left( -1 + 24^m \right. \right.

\[
\left. \times \left( -1 + 2m \beta \left( -H^4 q \right)^m \right) \right) \left( -3 + 5q + 18q^2 + 6q^3 + j (4 + 8q) - s \right) \geq 0. \quad (21)
\]

These two inequalities (20) and (21) are much complicated to find the exact analytical expression for the parameters. So, we shall fix some parameters by putting them equal
Fig. 2  WEC plots for $f(G)$ model mentioned in Eq. (16). Here, the left and right plots are representing the regions where $\rho_{\text{eff}} > 0$ and $\rho_{\text{eff}} + p_{\text{eff}} > 0$ with respect to $\alpha$, $\beta$ and $n$, respectively. We see that the WEC is satisfied for the considered range of parameters.
to specific values. For simplicity, we let $\alpha = 1$, $\beta = 1$ and plot $\rho_{\text{eff}}$ and $\rho_{\text{eff}} + p_{\text{eff}}$ which are the function of $m$ and $n$ only, as shown in Fig. 3. It can be observed from this figure that WEC is also valid for the model (19).

Now, we will check the constraints on parameters for the validity of WEC. For this purpose, let $\beta = 1$ and we found those regions under which WEC is valid. For $\rho_{\text{eff}} \geq 0$, we require

1. $\alpha > 0$ with any value of $m$.
2. $m < 0$ with a very small $n$.

Similarly for $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$, we require

1. $\alpha > 0$ with any value of $m$.
2. $-1 < m < 0$ with a very small $n$.

The region plots for WEC are shown in Fig. 4, in which the left plot is for the effective energy density while the right plot is for the summation of effective energy density and pressure.

4.3 Model 3

It would be interesting to analyze another realistic model in $f(G)$ gravity [51]

$$f(G) = \frac{a_1 G^n + b_1}{a_2 G^n + b_2}, \quad (22)$$

where $a_1$, $b_1$, $a_2$, $b_2$ and $n$ are the arbitrary constants, with $n > 0$. This model could be helpful in the study of finite time future singularities as well as the late time cosmic acceleration. The effective energy density for this model becomes

$$\rho_{\text{eff}} = \frac{-1}{(a_2 24^m (-H^4q)^m + b_2)^3} \left( \frac{24^m m}{q^2} (j + q (2q + 3)) (a_2 b_1 - a_1 b_2) (-H(t)^4 q)^m (a_2 24^m (m + 1) (-H^4 q)^m - b_2 m + b_2) + 24^m m (a_2 b_1 - a_1 b_2) (-H^4 q)^m (a_2 24^m (-H^4 q)^m + b_2) + (a_1 24^m (-H^4 q)^m + b_1) (a_2 24^m (-H^4 q)^m + b_2)^2 \right) \geq 0, \quad (23)$$

and the combination of the effective energy density and pressure becomes

$$\rho_{\text{eff}} + p_{\text{eff}} = \frac{1}{q^3 (a_2 24^m (-H^4 q)^m + b_2)^4} 3^{m-1} 8^m m (a_2 b_1 - a_1 b_2) (-H^4 q)^m \times ((j + q (2q + 3))^2 (a_2^2 576^n (m^2 + 3m + 2)) (-H^4 q)^2 + a_2 b_2 \times (-2^{3m+2}) 3^m (m^2 - 1) (-H^4 q)^m + b_2^2 (m^2 - 3m + 2))$$
Fig. 3. Plots of WEC for $f(G)$ model given in Eq. (19). In this figure, the left and right plots show the distributions of $\rho^{\text{eff}}$ and $\rho^{\text{eff}} + p^{\text{eff}}$ with respect to $m$ and $n$ with $\alpha = 1$, $\beta = 1$, respectively.
Fig. 4. WEC validity regions for $f(G)$ model mentioned in Eq. (19). Here the left plot indicates $\rho_{\text{eff}} > 0$ while the right plot indicates $\rho_{\text{eff}} + p_{\text{eff}} > 0$ with respect to $\alpha, m$ and $n$ with $\beta = 1$. 
\[-q(j(8q+4)+6q^3+18q^2+5q-s-3)(a_224^m(-H^4q)^m
\]
\[+b_2)(a_224^m(m+1)(-H^4q)^m-b_2m+b_2)-3q(j+q(2q+3))
\]
\[\times(a_224^m(-H^4q)^m+b_2)(a_224^m(m+1)(-H^4q)^m-b_2m+b_2))\geq0.
\]

(24)

As this model contains five parameters, i.e., \(a_1, a_2, b_1, b_2, m\), so we will fix some of these parameters by assigning some specific values. For simplicity, we let \(b_1 = -1, b_2 = 1\). Now, the constraints on other parameters are (for \(\rho_{eff} \geq 0\) with \(m > 0\))

1. \(a_1 < 0\) with \(a_2 > -1\).
2. \(a_1 > 1\) with \(a_2 < 0\).

One can check above mentioned constraints through the left plot of Fig. 5. The validity regions for \(\rho_{eff} + \rho_{eff} \geq 0\) would impose some constraints on the parameters \(a_1\) and \(a_2\). It is seen that \(a_1\) depends on the choice of \(a_2\) but if \(a_1 < 0\) and \(a_2 < 0\) with \(m > 0\) then these give \(\rho_{eff} + \rho_{eff} \geq 0\) as shown in the right plot of Fig. 5.

5 Summary

In the present paper, we have explored the influence of modified GB gravity models on the existence of realistic configurations of cosmological perfect fluid models. The investigation of ECs are closely associated with the realistic picture of the traversable WH solutions. To avoid the use of exotic matter content at the WH throat, the exploration of viable and well-consistent models is an alluring objective. We have considered the behavior of FLRW metric filled with an ideal fluid. The \(f(G)\) field equations turn out to be highly non-linear that could not be solved without taking certain physically consistent assumptions. From \(f(G)\) field equations, we have evaluated general energy inequalities relation. We have considered three different modified GB gravity models, i.e., \(f(G) = \alpha G^n + \beta G \log[G]\), \(f(G) = \alpha G^n (\beta G^m + 1)\) and \(f(G) = \frac{a_1G^n+b_1}{a_2G^n+b_2}\).

We have checked the behavior of ECs by taking into account all of the above mentioned modified GB models and perfect fluid. Then, the recent calculated values of the parameter Hubble, deceleration, jerk and snap are used with the different specific viable \(f(G)\) models. We plotted the regions where NEC and WEC hold against various parameters of \(f(G)\) gravity. The graphical features show some results given as follows:

(i) By considering higher curvature corrections induce from \(f(G) = \alpha G^n + \beta G \log[G]\) model, the effectiveness of WEC could be attained by setting positive values of \(\alpha\) and \(n > 1\) with any \(\beta \in (-5, 5)\) or by taking \(\alpha \) and \(\beta\) to be positive with \(n\) less than \(-1\). Further, we claimed that the breaching of WEC could be nullified by considering positive values of \((n, \alpha, \beta)\) tetrad. The validity of NEC could be achieved by taking \(n > 1\) along with the positive values of \(\alpha\) for any \(\beta\) or by setting \(\beta\) to be positive and very little value of \(n\) with for all \(\alpha\).

(ii) In the realm of \(f(G) = \alpha G^n (\beta G^m + 1)\), the WEC would be valid under two possibilities. One with positive \(\alpha\) with any \(m\) and other with negative \(m\) and very
Fig. 5 WEC validity epochs for $f(G)$ model given in Eq. (22). Here, left plot indicates $\rho_{\text{eff}} > 0$, while the right plot describes $\rho_{\text{eff}} + p_{\text{eff}} > 0$. 
little value of $n$. The validity of NEC could be attained by setting $m \in (-1, 0)$ with small $n$ or by considering positive numeric value of $\alpha$ with any $m$. It is worthy to mention that in this analysis, we have assumed $\beta$ to be unity.

(iii) Our next considered model is quiet complicated, as it comprises of five parameters, i.e., $a_1$, $a_2$, $b_1$, $b_2$, $m$. In order to handle such situation, we have fixed some of these parameters to get estimated validity epochs of WEC and NECs. Thus, in $f(G) = \frac{\alpha_1 G^n + \alpha_2 G^m}{\alpha_2 G^n + \alpha_2}$, the WEC will be valid, if one takes $b_1 = -1$, $b_2 = 1$ along with negative value of $a_1$, positive value of $m$ and $a_2$ to be greater than -1. Further, by setting positive $m$, negative $a_2$ and $a_1$ to be greater than unity (for details see Fig. 5). The NEC violation could be avoided by taking negative $a_1$ and $a_2$ with $m > 0$ as shown in Fig. 5.

Finally, it is remarked that the exploration of viable $f(G)$ models performed in this paper could easily be extended for the case that there exists convenient usual complicated matter content within FLRW metric. The corresponding analysis may lead to some significant qualitative outcomes in comparison with the discussion of pure gravity. It will be executed elsewhere.

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