Generalized Absorptive Polynomials and Provenance Semantics for Fixed-Point Logic

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Abstract

Semiring provenance is a successful approach, originating in database theory, to provide detailed information on the combinations of atomic facts that are responsible for the result of a query. In particular, interpretations in general provenance semirings of polynomials or formal power series give precise descriptions of the successful evaluation strategies or ‘proof trees’ for the query, and by evaluating these polynomials or power series in specific application semirings, one can extract practical information for instance about the confidence of a query or the cost of its evaluation.

While provenance analysis in databases has, for a long time, been largely confined to negation-free query languages such as conjunctive queries, positive relational algebra, Datalog, and several others, a recent approach extends this to model checking problems for logics with full negation. Algebraically this relies on new quotient semirings of dual-indeterminate polynomials or power series, and it has intimate connections with a provenance analysis of finite and infinite games. So far, this new approach has been developed mainly for first-order logic (FO) and for the positive fragment of least fixed-point logic (posLFP). What has remained open is an adequate treatment for fixed-point calculi that admit arbitrary interleavings of least and greatest fixed points such as full LFP or the modal $\mu$-calculus, but also temporal logics such as CTL.

The common approach for dealing with least fixed point inductions, as in Datalog or posLFP, is based on $\omega$-continuous semirings and Kleene's Fixed Point Theorem. It turns out that this is not sufficient for arbitrary fixed points. We show that an adequate framework for the provenance analysis of full fixed-point logics is provided by semirings that are (1) fully continuous, (2) absorptive, and (3) chain-positive. Full continuity guarantees that provenance values of least and greatest fixed-points are well-defined. Absorptive semirings provide a symmetry between least and greatest fixed-point computations and make sure that provenance values of greatest fixed points are informative. Finally, chain-positivity is not a necessary requirement in all applications, but it is responsible for having truth-preserving interpretations, which give non-zero values to all true formulae.

We further identify semirings of generalized absorptive polynomials $S^\infty[X]$ and prove universality properties that make them the most general appropriate semirings for LFP. We illustrate the power of provenance interpretations in these semirings, by relating them to provenance values of plays and strategies in the associated model-checking games. Specifically we prove that the provenance value of an LFP-formula gives precise information on the evaluation strategies in these games.

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Provenance analysis for a logical statement \( \psi \), evaluated on a finite structure \( \mathfrak{A} \), aims at providing precise information why \( \psi \) is true or false in \( \mathfrak{A} \). The approach of semiring provenance, going back to \[12\] relies on the idea of annotating the atomic facts not just by true or false, but by values from a commutative semiring \( K \), and to propagate these values through the statement \( \psi \), keeping track whether information is used alternatively (as in disjunctions or existential quantifications) or jointly (as in conjunctions or universal quantifications). Depending on the chosen semiring \( K \), the provenance value in \( K \) may then give practical information for instance concerning the confidence we may have that \( \mathfrak{A} \models \psi \), the cost of the evaluation of \( \psi \) on \( \mathfrak{A} \), the number of successful evaluation strategies for \( \psi \) on \( \mathfrak{A} \) in a game-theoretic sense, and so on. Beyond such provenance evaluations in specific application semirings, more general and more precise information is obtained by evaluations in so-called provenance semirings of polynomials or formal power series. Take, for instance, an abstract set \( X \) of provenance tokens that are used to label the atomic facts of a structure \( \mathfrak{A} \), and consider the semiring \( \mathbb{N}[X] \) of polynomials with indeterminates in \( X \) and coefficients from \( \mathbb{N} \), which is the semiring that is freely generated over \( X \). Such a labelling of the atomic facts then extends to a provenance valuation \( \pi[\psi] \in \mathbb{N}[X] \) for every negation-free first-order sentence \( \psi \in \text{FO}^+ \) or, equivalently, every Boolean query \( \psi \) from positive relational algebra RA\(^+\). This provenance valuation gives precise information about the combinations of atomic facts that imply the truth of \( \psi \) in \( \mathfrak{A} \). Indeed, we can write \( \pi[\psi] \) as a sum of monomials \( m \cdot x_1^{e_1} \ldots x_k^{e_k} \) where \( x_1, \ldots, x_k \in X \) and \( m, e_1, \ldots, e_k \geq 1 \). Each such monomial in \( \pi[\psi] \) indicates that we have precisely \( m \) evaluation strategies (or ‘proof trees’) to determine that \( \mathfrak{A} \models \psi \) that make use of the atoms labelled by \( x_1, \ldots, x_k \), and the atom labelled by \( x_i \) is used precisely \( e_i \) times by the strategy. We refer to \[12\] for precise formulations and proofs of this statement.

A similar analysis has been carried out for Datalog \[4,12\]. Due to the need of unbounded least fixed-point iterations in the evaluation of Datalog queries, the underlying semirings have to satisfy the additional property of being \( \omega \)-continuous. By Kleene’s Fixed-Point Theorem, systems of polynomial equations then have least fixed-point solutions that can be computed by induction, reaching the fixed-point after at most \( \omega \) stages. Most of the common application semirings are \( \omega \)-continuous, or can easily be extended to one that is so, but the general \( \omega \)-continuous provenance semiring over \( X \) is no longer a semiring of polynomials but the semiring of formal power series over \( X \), denoted \( \mathbb{N}^{\infty}[X] \), with coefficients in \( \mathbb{N}^{\infty} := \mathbb{N} \cup \{ \infty \} \). As above, provenance valuations \( \pi[\psi] \in \mathbb{N}^{\infty}[X] \) give precise information about the possible evaluation strategies for a Datalog query \( \psi \) on \( \mathfrak{A} \). Even though \( \mathfrak{A} \) is assumed to be finite there may be infinitely many such strategies, but each of them can use each atomic fact only a finite number of times; to put it differently, ‘proof trees’ for \( \mathfrak{A} \models \psi \) are still finite. This is closely related to the provenance analysis of reachability games on finite graphs \[4,9\].

In databases, semiring provenance has been applied to a number of other scenarios, such as nested relations, XML, SQL-aggregates, graph databases (see, e.g., the survey \[13\] as well as \[16,17\]) but for a long time, it has essentially been restricted to negation-free query languages. There have been algebraically interesting attempts to cover difference of relations \[11\] but they have not resulted in systematic tracking of negative information, and until recently there has been no convincing provenance analysis for languages with full negation. This has been a main obstacle for extending semiring provenance to other branches of logics in computer science, such as knowledge representation or verification. Indeed, while
there are many applications in databases where one can get quite far with considering only positive information, logical applications in most other areas are based on formalisms that use negation in an essential way. Fortunately, in most such formalisms, appropriate dualities between logical operators permit us to push negations through the formulae so that they are applied to atomic formulae only. Such transformations to negation normal forms are an essential ingredient of a new approach for the provenance analysis of logics with negation, such as first-order logic FO and least fixed-point logic LFP (the extension of FO with least and greatest fixed-point operators), as well as for possibly infinite games on graphs, which has recently been proposed in [8, 9]. On the algebraic side, this approach is based on new provenance semirings of polynomials and formal power series which take negation into account. They are obtained by taking quotients of traditional provenance semirings by products of positive and negative provenance tokens; they are called semirings of dual-indeterminate polynomials or dual-indeterminate power series, see [8, 9] and Sect. 2 for definitions. The semirings \( \mathbb{N}[X, X] \) of dual-indeterminate polynomials are the general provenance semirings for full first-order logic FO. Combining this with the provenance analysis of least fixed-point inductions in \( \omega \)-continuous semirings of formal power series, we obtain, by an analogous quotient construction, the semirings \( \mathbb{N}^\infty[X, X] \) of dual-indeterminate power series [9]. These are the general provenance semirings for Datalog with negated input predicates and, much more generally, also for posLFP, the fragment of LFP that consists of formulae in negation normal form such that all its fixed-point operators are least fixed-points. This is a powerful fixed-point calculus, that plays an important role in finite model theory, databases and verification. It is known that, on finite structures (but not in general), posLFP has the same expressive power as full LFP, and thus captures all polynomial-time computable properties of ordered finite structures [10].

Nevertheless, for the general objective of a provenance analysis of fixed-point calculi, the restriction to (positively used) least fixed points is not really satisfactory. The transformation from a fixed-point formula with arbitrary interleavings of least and greatest fixed points into one in posLFP is, contrary to transformations into negation normal form, not a simple syntactic translation. It goes through the Stage Comparison Theorem [15, 10] and can make a formula much longer and more complicated. Further, such transformations are not available for important fixed-point formalisms such as the modal \( \mu \)-calculus, stratified Datalog, transitive closure logics, and even simple temporal languages such as CTL. It is thus an important and interesting challenge to lay the foundations of a provenance analysis for full LFP (and infinite games with more general objectives than reachability), and to apply this approach to other fixed-point formalisms, in particular in databases and verification. In this paper we address the question, what kind of semirings are adequate for a meaningful and informative provenance analysis of unrestricted fixed-point logics, with arbitrary interleavings of least and greatest fixed points, such as full LFP or the modal \( \mu \)-calculus.

An essential requirement is that both least and greatest fixed points of our inductive constructions are well-defined. We guarantee this by requiring that the semirings are fully chain-complete which means that every chain \( C \) has not only a supremum \( \bigsqcup C \), but also an infimum \( \bigsqcap C \). In fact we work with the slightly more specific condition that the semirings are fully continuous, which additionally requires that both semiring operations are compatible

\[ \text{Of course, transformation to negation normal form is a common approach in logic. But while this is often just a matter of convenience and done for simplification, its seems indispensable for provenance semantics. Indeed, beyond Boolean semantics, negation is not a compositional logical operation: the provenance value of } \neg \varphi \text{ is not necessarily determined by the provenance value of } \varphi. \]
with suprema and infima of non-empty chains. All natural fully chain-complete semirings that we know are in fact fully continuous.

For an informative provenance semantics, there is a second important condition that is connected with the symmetry, or duality, between least and greatest fixed points. Indeed, in the Boolean setting, a greatest fixed point of a monotone operator is the complement of the least fixed of the dual operator (which is also monotone). It is this duality that permits to push negations through to the atoms and work with formulae in negation normal form. Further, a least fixed-point can be computed as the limit of an ascending chain of stages, starting at the bottom element \( \bot \) of the structure, while a greatest fixed point is obtained by a dual induction, starting at the greatest element \( \top \) and producing a descending chain of stages. To have a similar kind of symmetry in our provenance setting, we require that our semirings are absorptive. This means that \( a + ab = a \) for all \( a, b \), and we shall see that in naturally ordered semirings, this is equivalent with 1 being the greatest element (dually to \( \bot = 0 \)), and with requiring that multiplication is decreasing, i.e., \( a \cdot b \leq b \) for all \( a, b \). In particular, the powers of an element form a decreasing chain. We shall see that semirings that are absorptive and fully continuous guarantee a well-defined and informative provenance semantics for arbitrary fixed-point formulae.

Nevertheless we identify a further condition that is important for a most general provenance semantics, and in particular for guaranteeing that such a semantics is truth preserving, which means that it always gives non-zero values to true formulae. We call a semiring chain-positive if the infimum \( \bigcap \mathcal{C} \) of a chain consisting of positive elements is positive as well. Our fundamental examples of absorptive, fully continuous, and chain-positive semirings are the semirings \( \mathbb{S}^\infty[X] \) of generalized absorptive polynomials. Informally such a polynomial is a sum of monomials, with possibly infinite exponents, that are maximal with respect to absorption. For instance a monomial \( x^2y^\infty z \) occurring in a provenance value \( \pi[\psi] \) indicates an absorption-dominant evaluation strategy that uses the atom labelled by \( x \) twice, the atom labelled by \( y \) an infinite number of times, and the atom labelled by \( z \) once. This monomial absorbs all those that have larger exponents for all variables, such as for instance \( x^3y^\infty z^\infty u \), but not, say, \( x^\infty y^3 \). Absorptive polynomials thus describe shortest model-checking proofs or evaluation strategies. A precise definition and analysis of \( \mathbb{S}^\infty[X] \) will be given in Sect. 5.

We prove that the semirings \( \mathbb{S}^\infty[X] \) do indeed have universality properties that make them the most general absorptive semiring for LFP (see Theorem 23), and we shall show that this implies that, on finite structures, all fixed-point iterations for LFP-formulae in absorptive, fully continuous semirings have closure ordinals at most \( \omega \). As for the general semirings of polynomials, we can also in this case construct quotient semirings \( \mathbb{S}^\infty[X, X] \) to treat positive and negative atomic information appropriately.

In the final Sect. 6 we illustrate the power of provenance interpretations for LFP in absorptive, fully-continuous semirings, and particularly the semirings \( \mathbb{S}^\infty[X, X] \) by relating them to provenance values of plays and strategies in the associated model-checking games which in this case are parity games. Specifically we prove that, as in the case of FO and posLFP, the provenance value of an LFP-formula \( \varphi \) gives precise information on the evaluation strategies in these games.

## 2 Commutative Semirings and Provenance for First-Order Logic

**Definition 1.** A commutative semiring is an algebraic structure \( (K, +, \cdot, 0, 1) \), with \( 0 \neq 1 \), such that \( (K, +, 0) \) and \( (K, \cdot, 1) \) are commutative monoids, \( \cdot \) distributes over \( + \), and \( 0 \cdot a = a \cdot 0 = 0 \). A semiring \( K \) is naturally ordered if the relation \( a \leq b :\iff a + c = b \) for
some \( c \in K \) is a partial order. In particular, a naturally ordered semiring is \( +\)-positive, i.e. \( a + b = 0 \) implies \( a = 0 \) and \( b = 0 \). This excludes rings. All semirings considered in this paper are commutative, and naturally ordered. Further, a commutative semiring is positive if it is \( +\)-positive and has no divisors of 0 (i.e. \( a \cdot b = 0 \) implies that \( a = 0 \) and \( b = 0 \)). It is idempotent if \( a + a = a \) and absorptive if \( a + ab = a \), for all \( a, b \in K \). Obviously, every absorptive semiring is idempotent.

The standard semirings considered in provenance analysis are in fact positive, but for an appropriate treatment of negation we need semirings (of dual-indeterminate polynomials or power series) that have divisors of 0. Notice that a semiring \( K \) is positive if, and only if, the unique function \( h : K \to \{0, 1\} \) with \( h^{-1}(0) = \{0\} \) is a homomorphism from \( K \) into the Boolean semiring \( B = \{0, 1\} \).

Elements of a commutative semiring will be used as truth values for logical statements and as values for positions in games. The intuition is that \( + \) describes the alternative use of information, as in conjunctions or existential quantifications, whereas \( \cdot \) stands for the joint use of information, as in disjunctions or universal quantifications. Further, 0 is the value of false statements, whereas any element \( a \neq 0 \) of a semiring \( K \) stands for a “nuanced” interpretation of true. We briefly discuss some specific semirings that provide interesting information but about a logical statement.

- The Boolean semiring \( B = \{0, 1\} \) is the standard habitat of logical truth.
- \( N = (\mathbb{N}, +, \cdot, 0, 1) \) is used for counting evaluation strategies for a logical statement.
- \( T = (\mathbb{R}_+^\infty, \min, +, 0, 0) \) is called the tropical semiring. It is used for measuring the cost of evaluation strategies.
- The Viterbi semiring \( V = ([0, 1], \max, \cdot, 0, 1) \) is used to compute confidence scores for logical statements.
- The \( \min\text{-}\max \) semiring on a totally ordered set \((A, \leq)\) with least element \( a \) and greatest element \( b \) is the semiring \((A, \max, \min, a, b)\).

Beyond such application semirings, there are important universal provenance semirings of polynomials or formal power series. They admit to compute provenance values once and then to specialise these via homomorphisms to specific application semirings as needed.

- For any set \( X \), the semiring \( N[X] = (\mathbb{N}[X], +, \cdot, 0, 1) \) consists of the multivariate polynomials in indeterminates from \( X \) and with coefficients from \( \mathbb{N} \). This is the commutative semiring freely generated by the set \( X \).

- Admitting also infinite sums of monomials we obtain the semiring \( \mathbb{N}^\infty[X] \) of formal power series over \( X \), with coefficients in \( \mathbb{N}^\infty := \mathbb{N} \cup \{\infty\} \).
- Given two disjoint sets \( X, \overline{X} \) of “positive” and “negative” provenance tokens, together with a one-to-one correspondence \( X \leftrightarrow \overline{X} \), mapping each positive token \( x \) to its corresponding negative token \( \overline{x} \), the semiring \( N[X, \overline{X}] \) is the quotient of the semiring of polynomials \( N[X \cup \overline{X}] \) by the congruence generated by the equalities \( x \cdot \overline{x} = 0 \) for all \( x \in X \). This is the same as quotienting by the ideal generated by the polynomials \( x\overline{x} \) for all \( x \in X \). Observe that two polynomials \( g, g' \in N[X \cup \overline{X}] \) are congruent if, and only if, they become identical after deleting from each of them the monomials that contain complementary tokens. Hence, the congruence classes in \( N[X, \overline{X}] \) are in one-to-one correspondence with the polynomials in \( N[X \cup \overline{X}] \) such that none of their monomials contain complementary tokens. We call these dual-indeterminate polynomials.

- By a completely analogous quotient construction, we obtain the semiring \( \mathbb{N}^\infty[X, \overline{X}] \) of dual-indeterminate power series.

- By dropping coefficients from \( \mathbb{N}[X] \), we get the semiring \( \mathbb{B}[X] \) whose elements are just finite sets of distinct monomials. It is the free idempotent semiring over \( X \).
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- By dropping also exponents, we get the semiring \( \mathbb{W}[X] \) of finite sums of monomials that are linear in each argument. It is sometimes called the Why-semiring.
- The semiring \((\text{PosBool}(X), \lor, \land, \text{false}, \text{true})\) consists of the positive Boolean expressions over the variables \( X \), where we identify logically equivalent expressions.

Provenance for First-Order Logic. For a given finite relational vocabulary \( \tau \) and finite non-empty universe \( A \) we define the set of atoms as \( \text{Atoms}_A(\tau) = \{ R_\pi \mid R \in \tau, \, \pi \in \text{arity}(R) \} \).

The set \( \text{NegAtoms}_A(\tau) \) of negated atoms contains all negations \( \lnot R_\pi \) of atoms in \( \text{Atoms}_A(\tau) \) and we define the set of all \( \tau \)-literals on \( A \) as

\[
\text{Lit}_A(\tau) := \text{Atoms}_A(\tau) \cup \text{NegAtoms}_A(\tau) \cup \{ a = b \mid a, b \in A \} \cup \{ a \neq b \mid a, b \in A \}.
\]

Definition 2. Let \( K \) be a commutative semiring and \( \tau \) and \( A \) as above. A \( K \)-interpretation (for \( \tau \) and \( A \)) is a function \( \pi : \text{Lit}_A(\tau) \to K \) which maps all equalities and inequalities to their respective truth value 0 or 1.

We can extend the \( K \)-interpretation to any first-order formula in a natural way \[8\]. We do this by interpreting disjunctions and existential quantification as addition in the semiring, and conjunctions and universal quantification as multiplication. The only logical operation which cannot directly be interpreted by an algebraic one is negation. We deal with it syntactically, by evaluating the negation normal form \( \text{nff}(\psi) \) instead.

Definition 3. A \( K \)-interpretation \( \pi : \text{Lit}_A(\tau) \to K \) extends to a \( K \)-valuation \( \pi : \text{FO}(\tau) \to K \) by mapping a first-order sentence \( \psi(\pi) \) to a value \( \pi[\psi(\pi)] \) using the following rules

\[
\pi[\psi \lor \varphi] := \pi[\psi] + \pi[\varphi] \quad \pi[\psi \land \varphi] := \pi[\psi] \cdot \pi[\varphi] \quad \pi[\exists x \psi(x)] := \sum_{a \in A} \pi[\varphi(a)]
\]

\[
\pi[\forall x \psi(x)] := \prod_{a \in A} \pi[\varphi(a)] \quad \pi[\lnot \psi] := \pi[\text{nff}(\psi)].
\]

To ensure that the semiring valuations are somehow ‘sensible’, we want a value of 0 to still mean that the formula is false, while a non-zero value signifies some nuance of truth, with the exact meaning depending on the semiring and the application. Following \[8\], we thus say that a \( K \)-interpretation \( \pi : \text{Lit}_A(\tau) \to K \) is model-defining if for all atoms \( R_\pi \) exactly one of the two values \( \pi[R_\pi] \) and \( \pi[\lnot R_\pi] \) is zero. A model-defining \( K \)-interpretation \( \pi \) induces a unique structure \( \mathfrak{A}_\pi \) with universe \( A \) and \( \pi \in R^A \), if, and only if, \( \pi[R_\pi] \neq 0 \). Further, if \( \pi \) is model-defining and induces \( \mathfrak{A}_\pi \) then \( \pi[\varphi] \neq 0 \) implies that \( \mathfrak{A}_\pi \models \varphi \). For positive semirings, also the converse holds.

Notice that there may be different model-defining \( K \)-interpretations that induce the same structure. Moreover, also \( K \)-interpretations that do not define a unique structure, but are compatible with a whole class of structures (over the same universe) are interesting for many applications. We refer to \[8\] for further material on this, and to the discussion of model-compatible interpretations for LFP at the end of Sect. \[5\]

3 Least Fixed-Point Logic

Least fixed-point logic, denoted LFP, extends first order logic by least and greatest fixed points of definable monotone operators on relations: If \( \psi(R, \overline{x}) \) is a formula of vocabulary \( \tau \cup \{ R \} \), in which the relational variable \( R \) occurs only positively, and if \( \overline{x} \) is a tuple of variables such that the length of \( \overline{x} \) matches the arity of \( R \), then \( [\text{lfp } R_\pi \psi(\overline{x})] \) and \( [\text{gfp } R_\pi \psi(\overline{x})] \) are also formulae (of vocabulary \( \tau \)). The semantics of these formulae is that \( \overline{x} \) is contained in the
least (respectively the greatest) fixed point of the update operator \( F_\psi : R \mapsto \{ \overline{a} : \psi(R, \overline{a}) \} \).

Due to the positivity of \( R \) in \( \psi \), any such operator \( F_\psi \) is monotone and therefore has, by the Knaster-Tarski-Theorem, a least fixed point \( \text{lf}p(F_\psi) \) and a greatest fixed point \( \text{gfp}(F_\psi) \). See e.g. \cite{[10]} for background on LFP. The duality between least and greatest fixed point implies that \( \text{gfp}(R, \psi)(\overline{a}) \equiv \neg \text{lf}p(R, \psi)[\neg \psi[R/\neg R]](\overline{a}) \) for any \( \psi \). Using this duality together with de Morgan’s laws, every LFP-formula can be brought into negation normal form, where negation applies to atoms only.

We remark that in formulae \( \text{lf}p(R, \psi)(\overline{a}) \) one could allow \( \psi \) to have other free variables besides \( \overline{a} \); these are usually called parameters of the fixed-point formula. However, at the expense of increasing the arity of the fixed-point predicates and the number of variables one can always eliminate parameters. In this paper we tacitly assume that all fixed-point formulae are parameter-free, which simplifies both provenance analysis and the construction of model-checking games.

### The fragment of positive least fixed points.

We denote by posLFP the fragment of LFP consisting of formulae in negation normal form such that all its fixed-point operators are least fixed-points. It is known that, on finite structures (but not in general), posLFP has the same expressive power as full LFP, and thus captures all polynomial-time computable least fixed-points. It is known that, on finite structures (but not in general), posLFP has the same expressive power as full LFP, and thus captures all polynomial-time computable properties of ordered finite structures \cite{[10]}.

Provenance values in \( \omega \)-continuous semirings for posLFP-sentences have been defined in \cite{[9]} in two different, but equivalent, ways. The first approach relies on the fact that the model-checking games for posLFP are reachability games and is based on a provenance analysis that gives detailed information on the strategies of players in reachability games. The second approach is a direct definition, by induction over the syntax, to extend a \( \tau \)-interpretation of \( \text{Lit} \) by \( \pi \)-interpretations of \( \text{Lit} \) and \( \text{Atoms} \text{A} \) for the atoms \( R \). (Notice that \( R \) appears only positively in \( \varphi \), so negated atoms are not needed.) The formula \( \varphi(R, \overline{a}) \) now defines, together with \( \pi \), a monotone update operator \( F_\varphi^\pi \) on functions \( g : A^m \mapsto K \). More precisely, it maps \( g \) to

\[
F_\varphi^\pi(g) : \overline{a} \mapsto \pi[R \mapsto g[\varphi(R, \overline{a})]].
\]

By Kleene’s Fixed-Point Theorem, the operator \( F_\varphi^\pi \) has a least fixed point \( \text{lf}p(F_\varphi^\pi) \) which coincides with the limit of the sequence \((g^n)_{n<\omega}\) with \( g^0 := 0 \) and \( g^{n+1} := F_\varphi^\pi(g^n) \), and which we use to define the provenance value of \( \text{lf}p(R, \varphi(R, \overline{a})[\overline{a}] \) to be \( \text{lf}p(F_\varphi^\pi)(\overline{a}) \).

### 4 Semirings for LFP

Given a naturally ordered semiring, a chain is a totally ordered subset \( C \subseteq K \). We write \( a + C \) for the set \( \{a + c \mid c \in C\} \). Provided they exist, we write \( \bigsqcup C \) and \( \bigsqcap C \) for the supremum (least upper bound) and infimum (greatest lower bound) of a set \( C \subseteq K \). We further write \( \perp \) and \( \top \) for the least and greatest elements of \( K \). A function \( f : K_1 \mapsto K_2 \) is fully continuous if it preserves suprema and infima of nonempty chains, i.e., \( f(\bigsqcup C) = \bigsqcup f(C) \) and \( f(\bigsqcap C) = \bigsqcap f(C) \) for all chains \( \emptyset \neq C \subseteq K_1 \).
Definition 4. A naturally ordered semiring $K$ is fully chain-complete if every chain $C \subseteq K$ has a supremum $\bigcup C$ and an infimum $\bigcap C$ in $K$. It is further fully continuous if its operations are fully continuous in both arguments, i.e., $a \circ \bigcup C = \bigcup (a \circ C)$ and $a \circ \bigcap C = \bigcap (a \circ C)$ for all $a \in K$, all non-empty chains $\emptyset \neq C \subseteq K$ and $\circ \in \{+,-\}$.

Examples of fully continuous semirings include the Viterbi semiring, $\mathbb{N}^\infty$, and the semirings $\mathbb{N}^\infty [X]$ of formal power series. For positive least fixed-point inductions, as in Datalog or posLFP, the common approach is to work with $\omega$-continuous semirings, where only suprema of $\omega$-chains are required, and both operations must preserve suprema. It would be tempting to work with a minimal generalization that imposes similar properties for descending $\omega$-chains, using a dual version of Kleene’s Fixed-Point Theorem. However the following example shows that applying fixed-point operators need not preserve full continuity.

Example 5. Let $K$ be a fully continuous semiring and $f : K \times K \to K$ a function that is fully continuous in both arguments. For each $x \in K$, we can consider the function $g_x : K \to K$, $g_x(y) = f(x,y)$ and, further, the function $G : K \to K$, $G(x) = \mathsf{gfp}(g_x)$. Note that $G$ is well-defined due to the continuity of $f$ and a dual version of Kleene’s Fixed-Point Theorem. Now consider $\mathsf{lfp}(G)$ and $\mathsf{gfp}(G)$. To guarantee the existence of these fixed points via Kleene’s theorem, $G$ has to be fully continuous. This is, however, not always the case. One counterexample is the Łukasiewicz semiring and $f$ due to the continuity of $g$.

We further remark that full chain-completeness is more general than the more common notion of complete lattices, used in the Knaster-Tarski fixed-point theory, as we only require suprema (and infima) of chains instead of arbitrary sets. However, based on results in [14] it follows that the two notions coincide for the absorptive, fully continuous semirings which are the primary focus of this paper.

Proposition 6. For a monotone function $f : K \to K$ on a fully chain-complete semiring, both $\mathsf{lfp}(f)$ and $\mathsf{gfp}(f)$ exist.

Proof. Consider the fixed-point iteration $(x_\beta)_{\beta \in \Omega}$ for $\mathsf{lfp}(f)$ defined above. As $K$ is a set, there must be an ordinal $\alpha \in \Omega$ with $x_\alpha = x_{\alpha + 1} = f(x_\alpha)$, so $x_\alpha$ is a fixed point of $f$. To see that $x_\alpha$ is the least fixed point, let $x'$ be any fixed point of $f$. Clearly, $\bot \leq x'$ and, by monotonicity, $f(\bot) \leq f(x') = x'$. By induction, it follows that $x_\beta \leq x'$ for all $\beta \in \Omega$. The proof for $\mathsf{gfp}(f)$ is analogous.

Proposition 7. If $K$ is an idempotent, fully chain-complete semiring, then its natural order forms a complete lattice, i.e., suprema and infima of arbitrary sets exist.

Proof. We first show that addition coincides with finite suprema, i.e. $a + b = \bigcup \{a, b\}$ for $a, b \in K$. Clearly, $a \leq a + b$ and $b \leq a + b$, so $\bigcup \{a, b\} \leq a + b$. The other direction follows from idempotence: $a + b \leq \bigcup \{a, b\} + \bigcup \{a, b\} = \bigcup \{a, b\}$.
Hence suprema of arbitrary finite sets exist (by summation). Due to an old result of Markowsky [14], chain-completeness and finite suprema imply the existence of suprema of arbitrary (possibly infinite) sets. Infima can be expressed via suprema, so $K$ forms a complete lattice under its natural order.

**Provenance semantics for LFP.** Recall that provenance semantics for posLFP is defined via the least fixed-point of the update operator $F^\phi_{\pi}$. For LFP, we analogously define $\pi[\lfp R \varphi(R, x)](\pi) := \lfp(F^\phi_{\pi})(\pi)$ and $\pi[\gfp R \varphi(R, x)](\pi) := \gfp(F^\phi_{\pi})(\pi)$. The update operators $F^\phi_{\pi}$ are always monotone and Proposition 6 thus implies the existence of the required fixed points. This can be shown by a straightforward induction on the syntax of formulae and the steps of the fixed-point iteration, see the appendix for details.

**Proposition 8.** Provenance semantics for LFP is well-defined for fully chain-complete semirings.

We further have the following fundamental property for provenance analysis. It establishes a closer connection between logic—the semantics of $\varphi$, and algebra—the semiring homomorphism $h$, and in particular, allows us to compute provenance information in a general semiring and then specialize the result to application semirings by applying homomorphisms, most prominently by working with polynomials and applying polynomial evaluation.

**Proposition 9.** Let $K_1, K_2$ be fully chain-complete semirings and let $h : K_1 \rightarrow K_2$ be a fully continuous semiring homomorphism with $h(\top) = \top$. Then for every $K_1$-interpretation $\pi$, the mapping $h \circ \pi$ is a $K_2$-interpretation and for every $\varphi \in \LFP$, we have $h(\pi[\varphi]) = (h \circ \pi)[\varphi]$.

As diagram:

\[
\begin{array}{c}
K_1 \\
\xrightarrow{\pi} \\
\xrightarrow{h} \\
\xrightarrow{\pi \circ h} \\
K_2
\end{array}
\quad \Rightarrow \quad
\begin{array}{c}
K_1 \\
\xrightarrow{\pi} \\
\xrightarrow{h} \\
\xrightarrow{\pi \circ h} \\
K_2
\end{array}
\]

**Proof sketch.** The proof is by induction on $\varphi$. For the operators $\lor, \land, \exists$ and $\forall$, the statement follows from the additivity and multiplicity of $h$.

For fixed-point formulae such as $\lfp R \varphi(R, x)$, one considers the fixed-point iterations $(x_\beta)_{\beta \in \mathbb{N}}$ for $\pi$ in $K_1$ and $(y_\beta)_{\beta \in \mathbb{N}}$ for $h \circ \pi$ in $K_2$ (notice that $x_\beta$ and $y_\beta$ are functions, e.g. $y_0 : \text{arity}(R) \rightarrow K_2, \pi \mapsto 0$ and $y_{\beta+1} = F^\varphi_{\text{fix}}(y_\beta)$). One can then show by induction that $y_\beta = h \circ x_\beta$ for all $\beta \in \mathbb{N}$. By choosing a large enough $\beta$, this covers the fixed-points of both iterations. We refer to the appendix for the full proof.

The notion of chain-completeness is based on chains of arbitrary length. We do not know whether working with ascending and descending $\omega$-chains would be sufficient in all cases, but we show in Sect. 5 that it actually suffices when working with absorptive semirings.

**Absorptive and chain-positive semirings.** Although fully continuous semirings are sufficient for the existence of fixed-points and well-defined provenance interpretations, we need further properties to guarantee that provenance values are really informative, in the sense of providing insights why a formula $\varphi$ holds. As a first such condition, we consider $K$-interpretations $\pi$ that define a fixed model and require that provenance semantics preserve the truth of $\varphi$. 
Definition 10. A semiring $K$ is truth-preserving if for every model-defining $K$-interpretation $\pi$ with induced model $\mathfrak{A}_\pi$ and every $\varphi \in \text{LFP}$, we have that $\mathfrak{A}_\pi \models \varphi \iff \pi[\varphi] \neq 0$.

Another relevant observation concerns the provenance values of greatest fixed-points. In a number of semirings, such valuations exist, but do not give any useful information. This is tied to the lack of symmetry between least and greatest fixed-point inductions in such semirings. We illustrate these points with the following examples.

Example 11. The LFP-formula $\varphi(u)$ given below expresses the existence of an infinite path from $u$, which is true on the given graph $G$:

$$\varphi(u) = [\text{gfp } R \, x, \, \exists y (E x y \land R y)](u)$$

In the Boolean semiring $\mathbb{B} = \{0, 1\}$ there is a unique $\mathbb{B}$-interpretation $\pi$ that defines $G$. The provenance semantics in $\mathbb{B}$ coincides with standard semantics and we indeed obtain $\pi[\varphi(u)] = 1$. The Viterbi semiring $\mathbb{V}$ instead allows us to assign confidence scores to the edges. If we set $\pi(E uv) = \pi(E vv) = 1$ as in the Boolean interpretation, we again obtain an overall confidence of $\pi[\varphi(u)] = 1$. However, if we instead lower the score of the self-loop to $\pi(E vv) = 1 - \varepsilon$, we obtain an overall confidence of $\pi[\varphi(u)] = 0$ due to the fixed-point iteration $1, 1 - \varepsilon, (1 - \varepsilon)^2, \ldots$. So while $\pi$ still defines the model shown above, the formula evaluates to 0 which we usually interpret as false, illustrating that the Viterbi semiring is not truth-preserving. Nevertheless, the loop occurs infinitely often in the unique infinite path from $u$ and the value 0 thus makes sense when interpreted as a confidence score. The Viterbi semiring thus provides useful information for applications.

Consider next the semiring of formal power series $\mathbb{N}^\infty [X]$. If we choose $\pi(E uv) = x$ and $\pi(E vv) = y$ (and keep the values 0 or 1 for the remaining literals), then $\pi[\varphi(u)] = 0$, as result of the iteration $T, y \cdot T, y^2 \cdot T, y^3 \cdot T, \ldots$ with infimum 0 at node $v$ (here, $T$ is the power series in which all monomials have coefficient $\infty$). Thus, $\mathbb{N}^\infty [X]$ is not truth-preserving either.

Another candidate is the semiring $\mathbb{N}^\infty$ which can be used to count proofs of formulae in FO and posLFP, However, the consideration of greatest fixed points imposes problems: Intuitively, the graph only has one infinite path that we would view as a proof of $\varphi(u)$. But setting $\pi(E uv) = \pi(E vv) = 1$ results in $\pi[\varphi(u)] = \infty$, since the iteration at node $v$ is $\infty, 1 \cdot \infty, 1 \cdot \infty, \ldots$ which stagnates immediately. Although $\mathbb{N}^\infty$ is truth-preserving, the example hints at another general issue: Multiplication with non-zero values in $\mathbb{N}^\infty$ always increases values (i.e., $a \cdot b > b$ for all $a, b > 0$). The same is true for addition, so fixed-point iterations of $\text{gfp}$-formula are likely to result in $\infty$ and do not give meaningful provenance information.

Following the fundamental property, we might expect that we can obtain the result in $\mathbb{N}^\infty$ from the computation in $\mathbb{N}^\infty [X]$ by polynomial evaluation. But the homomorphism $h : \mathbb{N}^\infty [X] \to \mathbb{N}^\infty$ induced by the evaluation $x \mapsto 1$, $y \mapsto 1$ is not fully continuous and yields $h(0) = 0 \neq \infty$. Hence evaluation of formal power series does not preserve provenance semantics in general. This is a further reason why formal power series are not the right provenance semirings for LFP.

To deal with these problems we propose to work with fully continuous semirings that are (1) absorptive, to provide useful provenance information for greatest fixed points, and (2) chain-positive, to guarantee truth-preservation.

There are several good reasons to work with absorptive semirings for fixed points. The most important one is the increase in symmetry which, for instance, helps to avoid the problem with increasing multiplication seen in $\mathbb{N}^\infty$. By requiring absorption, multiplication becomes decreasing and 1 becomes the greatest element, symmetric to addition and the least element 0. Fixed-point theory often relies on symmetry and it is thus no surprise that more symmetry leads to more useful provenance information. This can be seen in the previous example when comparing the computations of greatest fixed-points in the non-absorptive semiring $\mathbb{N}^\infty$ and the more informative Viterbi semiring.

Proposition 12. In a naturally ordered semiring $K$, the following are equivalent:
1. $K$ is absorptive, i.e., $a + ab = a$ for all $a, b \in K$,
2. \( K \) has the greatest element \( \top = 1 \), i.e., \( a \leq 1 \) for all \( a \in K \).
3. Multiplication in \( K \) is decreasing, i.e., \( a \cdot b \leq b \) for all \( a, b \in K \).

**Proof.** If \( K \) is absorptive, then \( 1 + 1 \cdot a = 1 \) and hence \( a \leq 1 \) for all \( a \in K \). Absorption further implies \( ab \leq a \) for all \( a, b \in K \).

Conversely, \( \top = 1 \) entails \( 1 \leq 1 + a \leq 1 \), so \( 1 + a = 1 \), and multiplication with \( b \) gives \( a + ab = a \). If multiplication is decreasing, then \( a \geq a \cdot (1 + b) = a + ab \). Together with \( a \leq a + ab \) (by natural order), this implies absorption.

Another motivation for considering absorptive semirings is that they give information about shortest proofs of a formula. The property \( a + ab = a \) intuitively means that a proof that contains two literals mapped to \( a \) and \( b \), thus having the value \( ab \), is absorbed by a shorter proof only using one literal, thereby having provenance value \( a \).

To see why this is useful when working with greatest fixed-points, we consider an example in the Why-semiring \( \mathcal{W}[X] \). This semiring results from polynomials \( \mathbb{N}[X] \) by dropping both coefficients and exponents, which makes it finite and thus truth-preserving, but not absorptive. This is similar to \( \mathbb{N}^{\infty} \) and although \( \mathcal{W}[X] \) provides more information about greatest fixed-points, the lack of absorptivity also here leads to undesired provenance information.

**Example 13.** Recall the formula from the previous example, now interpreted on a different graph:

\[ \varphi(u) = [\text{gfp } Rx. \exists y (Exy \land Ry)](u) \]

We consider the \( \mathcal{W}[X] \)-interpretation \( \pi \) with \( \pi(Exu) = x \) and \( \pi(Exv) = y \) that defines the above graph (with \( X = \{x, y\} \)). Here there is only one infinite path which uses the edge labelled \( x \) infinitely often. As \( \mathcal{W}[X] \) is obtained by dropping exponents, it does not allow to count the usage of \( x \), so the path simply corresponds to the monomial \( x \).

However, the iteration \( \top, x \top, x^2 \top = x \top \) at node \( u \) leads to \( \pi[\varphi(u)] = x \top = x + xy \), which additionally contains the monomial \( xy \). As there is no infinite path using both edges, \( xy \) does not correspond to an evaluation strategy of \( \varphi(u) \) on the given graph. This is another example where \( \top \neq 1 \) causes problems. And indeed, absorption would imply \( x + xy = x \) as expected. Making \( \mathcal{W}[X] \) absorptive results in the semiring \( \text{PosBool}(X) \) which provides useful provenance information, but we later see absorptive semirings that are strictly more informative.

As a particularly interesting example of absorptive semirings, we shall define and investigate in the next section semirings of generalized absorptive polynomials \( \mathbb{S}^{\infty}[X] \). Contrary to other fully continuous and absorptive semirings, such as the Viterbi semiring, these are truth-preserving due to the following algebraic property.

**Definition 14.** A fully chain-complete semiring \( K \) is chain-positive if for each non-empty chain \( C \subseteq K \) of positive elements, the infimum \( \bigcap C \) is positive as well.

Chain-positivity is not an indispensible requirement for a useful provenance analysis, as is shown for instance by the Viterbi semiring. However, we need this property for provenance semirings which should give insights into proofs of evaluation strategies for \( \varphi \) and thus at the very least have to preserve truth.

**Proposition 15.** A positive, fully chain-complete semiring \( K \) is chain-positive if, and only if, the unique function \( h : K \to B \) with \( h^{-1}(0) = \{0\} \) is a fully continuous semiring homomorphism. Every positive, chain-positive, fully chain-complete semiring is truth-preserving.

**Proof.** The first statement can be shown by a considering a chain and performing a simple case distinction (see the appendix for details). We prove the second part using the fundamental property. Let \( K \) be a positive, chain-positive, fully chain-complete semiring and let
5 Generalized Absorptive Polynomials

The semiring $S^\infty[X]$ of generalized absorptive polynomials is, in a well-defined sense made precise below, the most general fully continuous semiring satisfying absorption. It was introduced in [9] and generalizes the semiring of absorptive polynomials $S_{or}(X)$ from [4] by admitting exponents in $\mathbb{N}^\infty$ to guarantee chain-positivity.

**Definition 16.** Let $X$ be a finite set of provenance tokens. A *monomial* over $X$ with exponents from $\mathbb{N}^\infty$ is a function $m : X \to \mathbb{N}^\infty$. Informally, we write $m$ as $x_1^{m(x_1)} \cdots x_n^{m(x_n)}$. Monomial multiplication adds the exponents. Observe also that $x^n \cdot x^m = x^{n+m}$. For any two monomials, $m_1, m_2$ we say that $m_2$ absorbs $m_1$ if $m_2$ has smaller exponents than $m_1$. Formally, $m_1 \leq m_2$ if, and only if, $m_1(x) \geq m_2(x)$ for all $x \in X$. Since monomials are functions, this is the pointwise partial order given by the reverse order on $\mathbb{N}^\infty$.

The set of monomials inherits a lattice structure from $\mathbb{N}^\infty$ and, of course, infinite. However, it has some crucial finiteness properties [9].

**Proposition 17.** Every ascending chain and every antichain of monomials is finite.

**Definition 18.** We define $S^\infty[X]$ as the set of antichains of monomials with indeterminates from $X$ and exponents in $\mathbb{N}^\infty$. Writing an antichain as a (formal) sum of its monomials we identify it with a polynomial with coefficients 0 or 1, and call these generalized absorptive polynomials. We define polynomial addition and multiplication as usual, except that for coefficients 1+1=1, and that we keep only the maximal monomials (w.r.t. $\leq$) in the result. The empty antichain corresponds to the 0 polynomial. The 1 polynomial consists of just the monomial in which every indeterminate has exponent 0.

Since antichains of monomials are always finite, there is no difference between polynomials and power series in this case and moreover, $S^\infty[X]$ is countable. The natural order on $S^\infty[X]$ can be characterized by monomial absorption. For $P, Q \in S^\infty[X]$, we have $P \leq Q$ if, and only if, for each $m \in P$ there is $m' \in Q$ with $m \leq m'$. Hence $S^\infty[X]$ is naturally ordered. It further follows that for a set $S \subseteq S^\infty[X]$, the supremum is given by $\bigcup S = \maximals(\bigcup S)$ which is the set of $\geq$-maximal monomials in $\bigcup S$ (see below for the proof). Together with Proposition 17, it further follows that ascending chains of polynomials are always finite. Due to the exponent $\infty$ and the finiteness of $X$, there is unique smallest monomial that ensures the chain-positivity of $S^\infty[X]$.

In order to provide proofs of the algebraic properties of $S^\infty[X]$, we begin with simple observations that hold in all absorptive, fully continuous semirings. In these semirings, powers of an element $a$ always form a descending $\omega$-chain $a \geq a^2 \geq a^3 \geq \ldots$ and we denote its infimum by $a^\omega$, which we call the *infinitary power* of $a$.

**Lemma 19 (Splitting Lemma).** Let $K$ be a fully continuous semiring and let $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ be two descending $\omega$-chains. Then, $\prod_{i<\omega}(a_i \circ b_i) = (\prod_{i<\omega} a_i) \circ (\prod_{j<\omega} b_j)$, with $\circ \in \{+,-\}$. Analogous statements hold for suprema.
We only show the statement for infima, the proof for suprema is analogous. We have the following equality, where $(\ast)$ holds since $K$ is fully continuous:

$$\bigcap_{i<\omega} a_i \circ b_i = \bigcap_{i<\omega} \bigcap_{j<\omega} (a_i \circ b_j) = \bigcap_{i<\omega} (a_i \circ \bigcap_{j<\omega} b_j)$$

We prove both directions of (1). Fix $i, j$ and let $k = \max(i, j)$. Then $a_i \circ b_j \geq a_k \circ b_k \geq \prod b_k$ by monotonicity of $\circ$. As $i, j$ are arbitrary, this proves $\prod b_i \prod a_i \circ b_j$ for every $i$ by monotonicity of $\circ$. By continuity, $a_i \circ b_i \geq \prod a_i \circ b_j$ for every $i$, and thus $\prod a_i \circ b_i \geq \prod a_i \circ b_j$. \hfill $\blacksquare$

Lemma 20 (Infinitary Power). Let $K$ be an absorptive, fully continuous semiring. Then,

1. $(a+b)^\omega = a^\omega + b^\omega$ and $a^n \cdot a^\omega = a^\omega$, for $a, b \in K$ and $n \in \mathbb{N}^\omega$,
2. $(\bigcap_{i<\omega} a_i) = \prod a_i^\omega$ for any descending $\omega$-chain $(a_i)_{i<\omega}$ in $K$.

In $S^\omega[X]$, we further have an analogue of property (2) for infima:

3. $(\bigcup S)^\omega = \bigcup S^\omega$, where we write $S^\omega = \{ P^\omega \mid P \in S \}$, for any set $S \subseteq S^\omega[X]$.

Proof. For the first statement in (1), let $a, b \in K$. We clearly have $(a+b)^n \geq a^n + b^n$ (for all $n < \omega$) and hence $(a+b)^\omega \geq a^\omega + b^\omega$. For the other direction, fix $n$ and consider $(a+b)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} a^{2n-i} b^i$. Each summand is absorbed by either $a^n$ (if $i \leq n$) or by $b^n$ (if $i \geq n$), hence $a^n + b^n \geq (a+b)^{2n} \geq (a+b)^\omega$ and the claim follows. The second statement follows by continuity of multiplication: $a^n \cdot a^\omega = a^n \cdot \prod a_i^k = \prod a_i^{k+n} = a^\omega$.

For (2), we use the splitting lemma (in $(\ast)$) and the fact that we can swap infima:

$$\prod a_i^n = \prod a_i^n = \prod a_i^n = \left( \prod a_i \right)^n = \left( \prod a_i \right)^\omega$$

For the last statement, we first note that for $a, b \in K$ with $a \leq b$, we always have $a^\omega \leq b^\omega$.

That is, the infinitary power is monotone. This follows directly from the definition, as $a \leq b$ implies $a^n \leq b^n$ and thus $\prod_{n<\omega} a^n \leq \prod_{n<\omega} b^n$.

For statement (3) in $S^\omega[X]$, we compare the two sides of the equation. The direction $(\bigcup S)^\omega \geq \bigcup S^\omega$ follows from the aforementioned monotonicity. For the other direction, let $\bigcup S = m_1 + \cdots + m_k$ for a finite number of monomials $m_1, \ldots, m_k$. By statement (2), $(\bigcup S)^\omega = m_1^\omega + \cdots + m_k^\omega$. Fix one monomial $m_i$. As $\bigcup S$ is maximals $(\bigcup S)$, there is a $P \in S$ with $m_i \in P$. Hence $m_i \leq P$ and thus $m_i^\omega \leq P^\omega \leq S^\omega$ by monotonicity. As this holds for each $1 \leq i \leq k$, we can conclude $m_1^\omega + \cdots + m_k^\omega \leq S^\omega$. \hfill $\blacksquare$

Lemma 21 (Countable Chains). Let $K, K'$ be fully chain-complete semirings and $C \subseteq K$ a countable chain. Then there is a descending $\omega$-chain $(x_i)_{i<\omega}$ such that $\prod C = \prod x_i$.

Moreover, if $f : K \to K'$ is a monotone function, then additionally $\bigcap f(C) = \bigcap f(x_i)$.

Analogue statements hold for suprema.

Proof. We only show the statement involving $f$, as it implies the first, and only consider infinite $C$ (otherwise the statement is trivial). Fix a bijection $g : \omega \to C$ and recursively define $x_0 = g(0)$ and $x_{i+1} = \min(g(i+1), x_i)$. This defines an $\omega$-chain with $x_i \in C$ and thus $\prod f(x_i) \geq \prod f(C)$. Conversely, for every $c \in C$ there is an $i$ with $g(i) = c$ and thus $c \geq x_i$.

By monotonicity, $f(c) \geq f(x_i)$ and thus $\prod f(C) \geq \prod f(x_i)$. \hfill $\blacksquare$

Proposition 22. $(S^\omega[X], +, 0, 1)$ is an absorptive, fully continuous, and chain-positive semiring.
Proof. Absorption is clear from the definition. We first prove that the natural order on $S^\infty[X]$ forms a complete lattice, implying chain-completeness. For $S \subseteq S^\infty[X]$, 

$$\bigcup S = \text{maximals}(\bigcup S)$$

where $\bigcup S$ are all monomials occurring in some polynomial of $S$ and maximals ($M$) denotes the set of maximal monomials (w.r.t. $\preceq$) in the set $M$. For each $P \in S$, we have $P \subseteq \bigcup S$ and hence $P \leq \text{maximals}(\bigcup S)$, so maximals $(\bigcup S)$ is an upper bound for $S$. To see that it is the least upper bound, let $Q$ be any upper bound for $S$, so $Q \geq P$ for all $P \in S$. For each $m \in \text{maximals}(\bigcup S)$ there is a $P \in S$ with $m \in P$ and hence $m \leq P \leq Q$. It follows that maximals $(\bigcup S) \leq Q$.

For chain-positivity, consider the monomial $m_\infty$ with $m_\infty(x) = \infty$ for all $x \in X$. Then $m_\infty$ is the smallest monomial with respect to $\preceq$. Given a descending $\omega$-chain $(P_i)_{i \in \omega}$ in $S^\infty[X]$ with $P_i > 0$ for all $i$, we know that each $P_i$ must contain some monomial. These monomials must be at least as large as $m_\infty$. Hence $P_i \geq m_\infty$ for all $i$ and thus $\bigcap_{i < \omega} P_i \geq m_\infty > 0$.

What remains is to show that $S^\infty[X]$ is fully continuous. To this end, we have to prove that the two semiring operations preserve both suprema and infima of nonempty chains. In the following, let $C \subseteq S^\infty[X]$ be such a chain and let $p \in S^\infty[X]$ be a polynomial.

We first consider addition. Due to idempotency of $S^\infty[X]$, addition corresponds to the supremum and we have $p + \bigcup C = \bigcup(p, \bigcup C) = \bigcup(\biguplus(p, c) \mid c \in C) = \biguplus(p + C)$.

For infima, we show $\bigcap(p + C) \leq p + \bigcap C$. The other direction follows from monotonicity of addition. Let $m \in \bigcap(p + C)$ be a monomial. Then $m \leq \bigcap(p + C)$ and thus $m \leq p + c$ for all $c \in C$. So $m$ is absorbed by a monomial in $p + c$ which originates either from $p$ or from $c$. If $m \leq p$, then also $m \leq p + \bigcap C$ and we are done. Otherwise, we have $m \leq c$ for all $c \in C$ and hence $m \leq \bigcap C \leq p + \bigcap C$. It follows that $\bigcap(p + C) \leq p + \bigcap C$.

We now turn to the continuity of multiplication. We first show that $p \cdot \bigcup C \leq \bigcup(p \cdot C)$. The other direction holds by monotonicity of multiplication (which follows from distributivity). Ascending chains are finite, so there is a $c \in C$ with $\bigcup C = c$. Then $p \cdot \bigcup C = p \cdot c \leq \bigcup(p \cdot C)$.

It remains to show that $\bigcap(p \cdot C) \leq p \cdot \bigcap C$ (again, the other direction follows from monotonicity). We first consider the case where $p$ consists of a single monomial $m$. Let $q$ be a monomial of $\prod(m \cdot C)$. Due to absorption, we have $q \leq m \cdot c \leq m$ (for any $c \in C$). Hence $q(x) \geq m(x)$ for all $x \in X$ and we can thus write $q$ as $q = m \cdot q'$ with $q'(x) = q(x) - m(x)$ (where we set $\infty - n = \infty$ for all $n \in \mathbb{N}^\infty$). We claim that $q' \leq \bigcap C$. To see this, let $c \in C$. Then $q \leq m \cdot c$ and thus $m \cdot q' \leq m \cdot c$. By comparing the exponents, we see that $q' \leq c$ and the claim follows. Hence $q = m \cdot q' \leq m \cdot \bigcap C$. As this argument applies to all monomials of $\prod(m \cdot C)$, we have shown $\prod(m \cdot C) \leq m \cdot \bigcap C$.

For the case where $p$ consists of several monomials, so $p = m_1 + \cdots + m_k$, we exploit the continuity of addition and apply the Splitting Lemma (together with Lemma [21]):

$$\prod(p \cdot C) = \prod_{c \in C}(m_1c + \cdots + m_k c) = (\prod m_1 C) + \cdots + (\prod m_k C) \leq (m_1 \cdot \bigcap C) + \cdots + (m_k \cdot \bigcap C) = p \cdot \bigcap C.$$ 

The central property of $S^\infty[X]$ is the following universality statement which says that polynomial evaluation (into absorptive, fully continuous semirings) induces fully continuous homomorphisms. Together with the fundamental property, this makes $S^\infty[X]$ the most general fully continuous provenance semiring for LFP under the assumption of absorption. The main difficulty in the proof of this statement is the continuity requirement on infima of chains, for which we make use of König’s lemma.
Theorem 23 (Universality). Let $K$ be an absorptive, fully continuous semiring and $h : X \to K$ be a mapping of provenance tokens to values in $K$. Then $h$ extends to a uniquely defined fully continuous semiring homomorphism $h : \mathbb{S}^\infty[X] \to K$.

Proof. Due to the additivity and multiplicity requirements for homomorphisms, $h$ uniquely extends to monomials. For the exponent $\infty$, notice that continuity requires $h(x^\infty) = \prod_{n<\omega} h(x)^n$ for $x \in X$. It further follows that $h(m_1 + m_2) = h(m_1) + h(m_2)$, hence $h$ is uniquely defined on $\mathbb{S}^\infty[X]$. Care has to be taken regarding absorption. If $m_1 \leq m_2$, then $m_1 + m_2 = m_2$. Since $h$ preserves the order and $K$ is absorptive, we also have $h(m_1 + m_2) = h(m_1) + h(m_2) = h(m_2)$. It follows by induction that $h$ is well-defined.

It remains to show that $h$ is fully continuous. Ascending chains are always finite, so we only have to consider descending chains. By Lemma 21 it further suffices to consider $\omega$-chains. The only remaining observation is that

$$\bigcap_{i<\omega} h(P_i) = h\left( \bigcap_{i<\omega} P_i \right)$$

for any descending $\omega$-chain $(P_i)_{i<\omega}$ in $\mathbb{S}^\infty[X]$. The homomorphism $h$ preserves addition and is thus monotone, which entails the direction \( \geq \).

For the other direction, we first consider the case of single monomials. Let $(m_i)_{i<\omega}$ be a descending $\omega$-chain of monomials. Recall that $X$ is finite, so we can write $m_i = \prod_{x \in X} x^{m_i(x)}$. As the $m_i$ form a descending chain, the exponents $(m_i(x))_{i<\omega}$ form an ascending chain for each $x \in X$. By Lemma 19 and the definition of $h$,

$$\bigcap_{i<\omega} h(m_i) = \prod_{x \in X} \left( \bigcap_{i<\omega} h(x)^{m_i(x)} \right) \quad \text{(a)} \quad \prod_{x \in X} h(x)^\bigcup_{i<\omega} m_i(x) = h\left( \bigcap_{i<\omega} m_i \right).$$

where (a) can easily be seen by case distinction whether $\bigcup_{i<\omega} m_i(x)$ is finite or $\infty$.

For the general case of polynomials, let $P_\omega = \bigcap_{i<\omega} P_i$ be the infimum, which is of the form $P_\omega = m_1 + \cdots + m_n$. We define a second, canonical $\omega$-chain $(P_i^*)_{i<\omega}$ with the same infimum. To this end, we define the canonical monomial chain $(m_j^*)_{j<\omega}$ of a given monomial $m$ as follows (see Figure 1 for an example),

$$m_j^*(x) = \min(j, m(x)), \quad \text{for all } x \in X,$$

which satisfies the following properties needed for the proof:

1. If $m, v$ are two monomials with $m \preceq v$, then $m_j^* \preceq v_j^*$ for all $j < \omega$.
2. If $m = \bigcap_{i<\omega} m_i$ for an $\omega$-chain $(m_i)_{i<\omega}$ of monomials, then $\forall j \exists i : m_j^* \succeq m_i$.
3. In particular, $\bigcap_{j<\omega} m_j^* = m$.

The canonical polynomial chain $(P_j^*)_{j<\omega}$ is then defined by $P_j^* = (m_1)_j^* + \cdots + (m_n)_j^*$ for each $j < \omega$. We make the following observation:

Claim: $\forall j \exists i : P_j^* \geq P_i$. 

We first show that the claim implies the theorem:

\[
\prod_{i<\omega} h(P_i) \leq \prod_{j<\omega} h(P_j) = \prod_{j<\omega} \left( h((m_1)_j^*) + \cdots + h((m_n)_j^*) \right) \\
\overset{(2)}{=} \prod_{j<\omega} h((m_1)_j^*) + \cdots + \prod_{j<\omega} h((m_n)_j^*) \\
\overset{(3)}{=} h\left( \prod_{j<\omega} (m_1)_j^* \right) + \cdots + h\left( \prod_{j<\omega} (m_n)_j^* \right) \\
\overset{(4)}{=} h(m_1) + \cdots + h(m_n) = h(P_\omega),
\]

where (1) follows from the claim, (2) holds by Lemma \[\text{Lemma}\] (3) was shown above and (4) holds due to property 3 above. Hence the claim suffices to prove the theorem.

To prove the claim, assume towards a contradiction that there is a \( j \) such that \( P_j^* \not\leq P_i \) for all \( i < \omega \). Let us fix an \( i < \omega \) for the moment. Because of \( P_j^* \not\leq P_i \), there is a monomial \( m_i \in P_i \) with \( P_j^* \not\leq m_i \). Because of \( P_{i-1} \geq P_i \), there is further \( m_{i-1} \in P_{i-1} \) with \( m_{i-1} \geq m_i \). But then also \( P_j^* \not\leq m_{i-1} \) (as otherwise \( P_j^* \geq m_{i-1} \geq m_i \)). By repeating this argument, we obtain a finite chain \( m_0 \geq m_1 \geq \cdots \geq m_i \) of monomials with the property that \( m_k \in P_k \) and \( P_j^* \not\leq m_k \) for all \( 0 \leq k \leq i \).

This argument applies to all \( i < \omega \), so we obtain arbitrarily long finite chains with this property. By König’s lemma (recall that all polynomials \( P_i \) are finite), there must be an infinite monomial chain \( (m_1)_{i<\omega} \) with \( m_i \in P_i \) and \( P_j^* \not\leq m_i \) for all \( i < \omega \). Let \( m_\omega = \bigcap_{i<\omega} m_i \). Because of \( m_i \leq P_i \) for all \( i \), we have \( m_\omega \leq P_\omega \), so there is a monomial \( v \in P_\omega \) with \( m_\omega \leq v \). By considering the corresponding canonical monomial chains \( (v^*_k)_{k<\omega} \) and \( ((m_\omega)_k^*)_{k<\omega} \) at \( k = j \), we obtain a contradiction: We know from the above properties that there is an \( i \) with \( (m_\omega)_j^* \geq m_i \) and further \( v^*_j \geq (m_\omega)_j^* \). Because of \( v^*_j \in P_j^* \), we obtain \( P_j^* \geq v^*_j \geq (m_\omega)_j^* \geq m_i \), contradicting our assumption. The claim follows, closing the overall proof.

![Figure 1](An example of a polynomial ω-chain (left) and the corresponding canonical chain (right) for the proof of Theorem 143. The arrows indicate absorption between monomials of consecutive polynomials and induce a directed graph which justifies our application of König’s lemma.)
The idea to apply König’s lemma to monomial chains can also be applied to infima of chains in general and is useful for some of the later proofs.

**Proposition 24** (Characterization of Infima). Let \((P_i)_{i<\omega}\) be a descending \(\omega\)-chain in \(S^\infty[X]\). Let further \(\mathcal{M}\) be the set of descending \(\omega\)-chains \((m_i)_{i<\omega}\) of monomials with the property that \(m_i \leq P_i\) for all \(i\). Then,
\[
\bigcap_{i<\omega} P_i = \bigcup \{ \bigcap_{i<\omega} m_i \mid (m_i)_{i<\omega} \in \mathcal{M} \}.
\]

**Proof.** By definition, \(m_i \leq P_i\) and thus \(\bigcap_i m_i \leq \bigcap_i P_i\) for every chain \((m_i)_{i<\omega} \in \mathcal{M}\). Hence direction “\(\geq\)” of the proposition follows.

For the other direction, consider the infimum \(P_\omega = \bigcap P_i\). We claim that for every monomial \(m_\omega \in P_\omega\), there is a monomial chain \((m_i)_{i<\omega} \in \mathcal{M}\) with \(\bigcap_i m_i \geq m_\omega\). This is sufficient to close the proof.

To prove the claim, we use a similar argument as in the proof of the universality of \(S^\infty[X]\). Fix a monomial \(m_\omega \in P_\omega\) and, for the moment, an \(i < \omega\). We have \(P_\omega \leq P_i\), so there is a monomial \(m_i \in P_i\) with \(m_\omega \leq m_i\). As \(P_i \leq P_{i-1}\), there must be further be a monomial \(m_{i-1} \in P_{i-1}\) with \(m_i \leq m_{i-1}\). Iterating this argument yields a sequence \(m_i \leq m_{i-1} \leq m_{i-2} \leq \cdots \leq m_0\) of monomials with \(m_\omega \leq m_j\) and \(m_j \in P_i\) (for all \(j \leq i\)). This construction is possible for each \(i\), so by König’s lemma (recall that all polynomials are finite), there must be an infinite monomial chain \((m_i)_{i<\omega}\) with \(m_i \in P_i\) and \(m_i \geq m_\omega\) for each \(i\). Hence \((m_i)_{i<\omega} \in \mathcal{M}\) and the infimum is \(\bigcap_i m_i \geq m_\omega\) as claimed.

One consequence of the universality property is the existence of a most general \(S^\infty[X]\)-interpretation \(\pi^*\) by introducing variables \(X = \{x_L \mid L \in \text{Atoms}_A(\tau) \cup \text{NegAtoms}_A(\tau)\}\) for all literals and setting \(\pi^*(L) = x_L\). Any other \(K\)-interpretation \(\pi\) (where \(K\) is fully continuous and absorptive) results from \(\pi^*\) by the evaluation \(x_L \mapsto \pi(L)\) which lifts to a fully continuous homomorphism \(h\). After computing \(\pi^*[\varphi]\) once, the computation for any \(K\)-interpretation \(\pi\) is then simply a matter of applying polynomial evaluation, since \(\pi[\varphi] = h(\pi^*[\varphi])\).

**Example 25.** We recall the setting from Example 11 and first consider the model-defining \(S^\infty[X]\)-interpretation tracking the two edges labelled \(x\) and \(y\), as indicated in the left graph.

\[
\varphi(u) = [\text{gfp } R x \cdot \exists y.(E x y \land R y)](u)
\]

We obtain \(\pi[\varphi(u)] = xy^\omega\) corresponding to the infinite path \(uuuuuuu\ldots\). The confidence values from Example 11 can be obtained by polynomial evaluation: For \(h(x) = h(y) = 1\), we get \((h \circ \pi)[\varphi(u)] = 1 \cdot 1^\omega = 1\) and for \(h'(x) = 1, h'(y) = 1 - \varepsilon\) we get \((h' \circ \pi)[\varphi(u)] = 1 \cdot (1 - \varepsilon)^\omega = 0\).

Next we consider the graph on the right by setting \(\pi(E uv) = z\). There are now infinitely many infinite paths from \(u\) to \(v\). However, we obtain only two monomials due to absorption: \(\pi[\varphi(u)] = xy^\omega + z^\omega\). These correspond to the simplest infinite paths since monomials such as \(z^kxy^\omega\) (corresponding to the path \(uuuuuuu\ldots\)) are absorbed by \(xy^\omega\). In particular, we see that \(\varphi(u)\) is satisfied in all models with at least edge \(z\) or with at least edges \(x\) and \(y\).

The most general \(S^\infty[X]\)-interpretation can also be used to prove that the update operators \(F^*_x\) induced by LFP-formulae in \(S^\infty[X]\) are fully continuous. Hence Kleene’s fixed-point theorem applies and guarantees that the fixed-point iterations for \(\text{lfp}(F^*_x)\) and \(\text{gfp}(F^*_x)\) have closure ordinal at most \(\omega\). Using the universality theorem, the statement on the closure ordinal generalizes to all absorptive, fully continuous semirings – even to semirings for which Kleene’s theorem does not apply (like the Łukasiewicz semiring we considered in Example 5).
Proposition 26. Given a $S^\infty[X]$-interpretation $\pi$ and an LFP-formula $\varphi(R, F)$, the associated update operator $F^\varphi_R$ is a fully continuous function.

Proof. Let $X' = \{ x_L \mid L \in \text{Atoms}_A(\tau) \cup \text{NegAtoms}_A(\tau) \}$ and consider the most general $S^\infty[X']$-interpretation $\pi^*$ with $\pi^*(L) = x_L$. By the universal property, the mapping $x_L \mapsto \pi(L)$ extends to a fully continuous homomorphism $h_\pi : S^\infty[X'] \to S^\infty[X]$ with $\pi = h_\pi \circ \pi^*$, for any $K$-interpretation $\pi$.

In order to prove that $F^\varphi_R$ is fully continuous, we show the more general statement that for any LFP-sentence $\varphi$, the mapping $\pi \mapsto \pi[\varphi]$ is fully continuous. That is, for a chain $C$ of $S^\infty[X]$-interpretations, we have $(\bigcup C)[\varphi] = \bigcup \{ \pi[\varphi] \mid \pi \in C \}$ (and the same for infima).

The continuity of $F^\varphi_R$ follows by unraveling the definition of the update operator.

Since the set of $S^\infty[X]$-interpretations is countable, it suffices to consider $\omega$-chains $(\pi_i)_{i<\omega}$ due to Lemma 21. To simplify notation, let $\pi_\omega = \bigcup_{i<\omega} \pi_i$. Using the homomorphisms introduced above, we can reformulate the continuity statement we want to prove:

$$h_{\pi_\omega}(\pi^*[\varphi]) = \bigcup_{i<\omega} h_{\pi_i}(\pi^*[\varphi]).$$

We claim that $h_{\pi_\omega}(m) = \bigcup_{i<\omega} h_{\pi_i}(m)$ for all monomials $m$ over $X'$. Since $\pi^*[\varphi]$ consists of finitely many monomials, this implies the statement above by applying the Splitting Lemma 19. Monomials in $S^\infty[X']$ are products consisting of factors of the form $x^m_L$ for $x_L \in X'$ and $n \in N \cup \{ \infty \}$. By again resorting to Lemma 19 it suffices to show the claim for monomials of the form $x^m_L$. For such monomials,

$$h_{\pi_\omega}(x^m_L) = h_{\pi_\omega}(x_L)^m = \pi_\omega(L)^m = \bigcup_{i<\omega} \pi_i(L)^m = \bigcup_{i<\omega} h_{\pi_i}(x^m_L),$$

where $(\ast)$ can be seen by case distinction. For $n = \infty$, it follows (once again) from Lemma 19. For $n < \infty$, we can apply Lemma 20 (3).

To close the proof, we also have to consider infima, i.e. $(\bigcap C)[\varphi] = \bigcap \{ \pi[\varphi] \mid \pi \in C \}$. The argument is completely analogous, except that $(\ast)$ now requires Lemma 20 (2).

By Kleene’s fixed-point theorem (and its dualized version for greatest fixed points), the fixed-point iterations for $\text{ifp}(F^\varphi_R)$ and for $\text{gfp}(F^\varphi_R)$ both terminate at step $\omega$ (or earlier). The fundamental property then enables us to generalize this observation to other semirings. Notice that our formulation of the property (see Proposition 9) is not sufficient by itself.

What we need at this point is that fully continuous homomorphisms preserve all steps of the fixed-point iteration, which is part of the proof of the fundamental property given above.

Corollary 27. Let $K$ be an absorptive, fully continuous semiring. Given a $K$-interpretation $\pi$ and an LFP-formula $\varphi(R, F)$, the fixed-point iterations for $\text{ifp}(F^\varphi_R)$ and $\text{gfp}(F^\varphi_R)$ both have closure ordinal at most $\omega$.

Proof. The statement follows from Proposition 26 by considering the most general interpretation $\pi^*$ defined above and observing that the fully continuous homomorphism $h : S^\infty[X'] \to K$ induced by the mapping $x_L \mapsto \pi(L)$ preserves all steps of the fixed-point iteration.

As in other semirings of polynomials and power series we can also here take pairs of positive and negative indeterminates, with a correspondence $X \leftrightarrow \overline{X}$ and build the quotient with respect to the congruence generated by the equation $x \cdot \overline{x} = 0$. We thus obtain a new semiring $S^\infty[X, \overline{X}]$ which retains the properties of being absorptive, fully continuous and chain-positive and hence provides a natural framework for a provenance analysis for full
LFP and other fixed point calculi. Of course, $S^\infty[X, \overline{X}]$ is no longer positive, as $x$ and $\overline{x}$ are divisors of 0. Instead of model-defining interpretations, we consider \textit{model-compatible} interpretations $\pi$. That is, for each atom $R\overline{R}$ we either have $\pi(R\overline{R}) = x$ and $\pi(\neg R\overline{R}) = \overline{x}$ or $\{\pi(R\overline{R}), \pi(\neg R\overline{R})\} = \{0, 1\}$. Additionally, $\pi$ must not use the same indeterminate for two different atoms. We say that a model $\mathfrak{A}$ is \textit{compatible} with $\pi$ if $\mathfrak{A} \models L$ for all literals $L$ with $\pi(L) = 1$ and denote set set of compatible models by $\text{Mod}_{\pi}$.

\textbf{Proposition 28.} Let $\pi$ be a model-compatible $S^\infty[X, \overline{X}]$-interpretation and $\varphi$ an LFP-sentence. Then, $\varphi$ is $\text{Mod}_{\pi}$-satisfiable if, and only if, $\pi[\varphi] \neq 0$. Moreover, $\varphi$ is $\text{Mod}_{\pi}$-valid if, and only if, $\pi[\neg \varphi] = 0$.

\textbf{Proof.} The statement on satisfiability implies the one on validity, so we only consider the former. If $\mathfrak{A} \in \text{Mod}_{\pi}$ and $\mathfrak{A} \models \varphi$, we consider the model-defining $\mathbb{B}$-interpretation $\pi_{\mathfrak{A}}$ corresponding to the model $\mathfrak{A}$. We can obtain $\pi_{\mathfrak{A}}$ from $\pi$ by instantiating the determinates with values from $\mathbb{B}$. By the universality property, this induces a fully continuous homomorphism $h : S^\infty[X \cup \overline{X}] \rightarrow \mathbb{B}$ such that $\pi_{\mathfrak{A}} = h \circ \pi$. Since $\pi_{\mathfrak{A}}$ is model-defining, we have $h(x \cdot \overline{x}) = 0$ for all $x \in X$. Hence $h$ factors through the quotient and induces a fully continuous homomorphism $\hat{h} : S^\infty[X, \overline{X}] \rightarrow \mathbb{B}$. It follows from the fundamental property that $\hat{h}(\pi[\varphi]) = \pi_{\mathfrak{A}}[\varphi] = 1$ and, since $h(0) = 0$, we must have $\pi[\varphi] \neq 0$.

For the other direction, assume that $\pi[\varphi] \neq 0$. Then there is a monomial $m \in \pi[\varphi]$. This monomial induces an instantiation $h : X \cup \overline{X} \rightarrow \mathbb{B}$ such that $h(m) = 1$ as follows:

- If $m(x) > 0$, then $h(x) = 1$ and $h(\overline{x}) = 0$,
- if $m(\overline{x}) > 0$, then $h(x) = 0$ and $h(\overline{x}) = 1$,
- otherwise, $h(x) = 1$ and $h(\overline{x}) = 0$ (this is an arbitrary choice).

As before, $h$ lifts to a fully continuous homomorphism $\hat{h} : S^\infty[X, \overline{X}] \rightarrow \mathbb{B}$. Moreover, $\hat{h} \circ \pi$ is a model-defining $\mathbb{B}$-interpretation. It follows the induced model $\mathfrak{A}_{\hat{h} \circ \pi}$ satisfies $\varphi$, since $(\hat{h} \circ \pi)[\varphi] = h(\pi[\varphi]) \geq h(m) = 1$.

This statement shows how model-compatible interpretations can be used to reason about several models at once: Mapping certain literals to indeterminate pairs $x$ and $\overline{x}$ leaves open the truth of these literals, but still encodes the semantics of opposing literals.

\section{Game-theoretic analysis}

It has been shown in \cite{22} that the provenance analysis for FO and posLFP is intimately connected with the provenance analysis of reachability games. Evaluation strategies to establish the truth of first-order formulae are really winning strategies for reachability games on acyclic game graphs. For posLFP the situation is similar, but the associated model checking games may have cycles and thus admit infinite plays, but the winning plays for the verifying player have to reach a winning position (i.e. a true literal) in a finite number of steps. By annotating such terminal positions with semiring values and propagating these values along the edges to the remaining positions, one obtains provenance values that coincide with the syntactically defined semantics $\pi[\varphi]$.

For full LFP (or the modal $\mu$-calculus), the model checking games are parity games which are considerably more complex and do not allow for a simple propagation of values from terminal positions. Hence, we do not present here a general provenance analysis of parity games, but we show how provenance values $\pi[\varphi]$ for fixed-point formulae can be understood from a game-theoretic point of view. For first-order logic or posLFP, provenance values $\pi[\varphi]$ in $\mathbb{N}[X, \overline{X}]$ or $\mathbb{N}^\infty[X, \overline{X}]$ are sums of monomials that correspond to the evaluation strategies
for \( \varphi \) and provide information about the literals used by these strategies. We present an analogue of this statement for full fixed-point logic and the semiring \( S^\infty[X, \overline{X}] \).

**Model-checking games for LFP.** Model checking games are classically defined for a formula (assumed to be given in negation normal form) and a fixed structure \( \mathfrak{A} \) (see e.g. [2, Chap. 4]). However, the game graph of such a game depends only on the formula \( \psi \) and the universe \( A \) of the given structure \( \mathfrak{A} \). It is only the labelling of the terminal positions of the game, as winning for either the Verifier (Player 0) or the Falsifier (Player 1), that depends on which of the literals in \( \text{Lit}_A(\tau) \) are true in \( \mathfrak{A} \). Hence the definition readily generalizes to a more abstract provenance scenario where we instead label terminal positions by semiring values.

> **Definition 29.** Let \( \psi \) be an LFP-sentence in negation normal form with a relational vocabulary \( \tau \), and let \( A \) be a (finite) universe. The model checking game \( G(A, \psi) \) has positions \( \varphi(\overline{a}) \), obtained from a subformula \( \varphi(\overline{x}) \) of \( \psi \), by instantiating the free variables \( \overline{x} \) by a tuple \( \overline{a} \) of elements of \( A \). At a disjunction \( (\psi \lor \varphi) \), Player 0 (Verifier) moves to either \( \psi \) or \( \varphi \), and at a conjunction, Player 1 (Falsifier) makes an analogous move. At a position \( \exists x \varphi(\overline{a}, x) \), Verifier selects an element \( b \) and moves to \( \varphi(\overline{a}, b) \), whereas at positions \( \forall x \varphi(\overline{a}, x) \) the move to the next position \( \varphi(\overline{a}, b) \) is done by Falsifier. For every subformula of \( \psi \) of form \( \vartheta := \lfp R. \varphi(R, \overline{x})(\overline{a}) \) or \( \vartheta := \gfp R. \varphi(R, \overline{x})(\overline{a}) \) we add moves from positions \( \vartheta(\overline{a}) \) to \( \varphi(\overline{a}) \), and from positions \( R\overline{a} \) to \( \varphi(\overline{a}) \) for every tuple \( \overline{a} \). Since these moves are unique it makes no difference to which of the two players we assign the positions \( \vartheta(\overline{a}) \) and \( R\overline{a} \). The resulting game graphs \( G(A, \psi) \) may contain cycles, but the set \( T \) of terminal nodes is again a subset of \( \text{Lit}_A(\tau) \). The terminal positions of \( G(A, \psi) \) are literals in \( \text{Lit}_A(\tau) \).

These games may have cycles and thus admit infinite plays. The winning condition for infinite plays is the parity condition: We assign to each fixed-point variable a priority, which is even for greatest fixed-points and odd for least fixed points, satisfying the condition if a variable \( R \) depends on another variable \( T \) then the priority of \( R \) is smaller or equal to the priority of \( T \). An infinite play is won by Player 0 (the Verifier) if the least priority occurring infinitely often in the play is even, otherwise it is won by Player 1 (the Falsifier).

**Provenance values for plays and strategies.** Given a parity game \( G(A, \psi) \) and an absorptive, fully continuous semiring \( K \), a \( K \)-interpretation \( \pi : \text{Lit}_A(\tau) \to K \) provides a valuation of the terminal positions. Based on this, we define provenance values for plays and strategies.

> **Definition 30.** A finite play \( \rho = (\varphi_0, \ldots, \varphi_t) \) ends in a terminal position \( \varphi_t \in \text{Lit}_A(\tau) \) which we call the outcome of \( \rho \). We simply identify the provenance value of \( \rho \) with the value of its outcome, i.e. we put \( \pi[\rho] := \pi[\varphi_t] \). For an infinite play \( \rho \) we put \( \pi[\rho] := 1 \) if \( \rho \) is a winning play for the Verifier, and \( \pi[\rho] := 0 \) otherwise.

We denote by \( \text{Strat}(\varphi) \) the set of evaluation strategies for the subformula \( \varphi \) of \( \psi \), i.e. the set of all (not necessarily positional) strategies that the Verifier has from position \( \varphi \) in the parity game \( G(A, \psi) \). Every strategy \( S \in \text{Strat}(\varphi) \) induces the set \( \text{Plays}(S) \) of plays that are consistent with \( S \). Intuitively, the provenance value of a strategy is simply the product over the provenance values of all plays that it admits. However, a strategy may well admit an infinite set of plays and while it is possible to define infinite products in our setting (we refer to the appendix for details), we instead observe that the set of possible outcomes is of course finite, since there exist only finitely many literals. As a consequence, we define the provenance value for a strategy by grouping those plays with identical outcome.
Definition 31. For any strategy $S$ and any literal $L \in \text{Lit}_A(\tau)$, we write $\#_S(L) \in \mathbb{N} \cup \{\infty\}$ for the number of plays $\rho \in \text{Plays}(S)$ with outcome $L$. We then define the provenance value

$$\pi[S] := \begin{cases} \prod_{L \in \text{Lit}_A(\tau)} \pi(L)^{\#_S(L)} & \text{if all infinite } \rho \in \text{Plays}(S) \text{ are winning for Verifier} \\ 0 & \text{otherwise.} \end{cases}$$

We remark that the case for $\#_S(L) = \infty$ is well-defined, as the infinitary power $a^\infty = \prod_n a^n$ can be defined in all absorptive, fully continuous semirings. We further observe that for model-compatible interpretations in $S^\infty[X, \overline{X}]$, the value $\pi[S]$ is a single monomial.

The following central result justifies our game-theoretic analysis and precisely characterizes provenance semantics $\pi[\psi]$ in terms of strategies in the associated model checking game.

Theorem 32. Let $\psi \in \text{LFP}$, and let $\pi : \text{Lit}_A(\tau) \to K$ be a $K$-interpretation into an absorptive, fully continuous semiring $K$.

$$\pi[\psi] = \bigsqcup \{\pi[S] \mid S \in \text{Strat}(\psi)\}.$$ 

Consider now specifically the semiring $S^\infty[X, \overline{X}]$ and model-compatible interpretations. By the above theorem, the provenance value of a sentence $\psi$ is then a sum of monomials $x_1^{e_1} \cdots x_k^{e_k}$, each of which corresponds to a strategy $S$ for Verifier that uses precisely the literals labelled by $x_1, \ldots, x_k$, and each literal $x_i$ is used precisely $e_i$ many times, that is, there are $e_i$, plays consistent with $S$ that have outcome $x_i$. By using dual indeterminates $X$ and $\overline{X}$, we further see whether a literal or its negation are used and we disregard inconsistent strategies that use both an atom and its negation. We may thus view the strategy $S$ as a witness for the truth of $\psi$. Conversely, each strategy $S$ for Verifier is represented in $\pi[\psi]$ in the sense that some monomial $x_1^{e_1} \cdots x_k^{e_k}$ in $\pi[\psi]$ absorbs the value $\pi[S]$. That is, $x_1^{e_1} \cdots x_k^{e_k}$ corresponds to a strategy $S'$ that uses the same or fewer literals compared to $S$.

In this sense, provenance semantics in absorptive, fully continuous semirings, and most prominently in $S^\infty[X, \overline{X}]$, provide detailed information about evaluation strategies of sentences $\psi$. Because of absorption, we do not obtain information about all evaluation strategies, as in first-order logic and $\mathbb{N}[X, \overline{X}]$, but instead only about the absorption-dominant strategies (corresponding to absorption-maximal monomials). These are strategies that allow the fewest possible different outcomes and are thus the simplest or canonical evaluation strategies.

6.1 Examples

Before we prove Theorem 32, let us illustrate provenance values for strategies with two examples. Since model checking games become large even for simple formulae and small universes, we only consider a small graph with two nodes. The formula, on the other hand, features alternating least and greatest fixed points which is arguably the most difficult case to analyse and leads to more complicated parity games that need several different priorities.

Example 33. Consider the formula $\varphi(u)$ below which expresses that there is a path from $u$ on which $P$ holds infinitely often. We evaluate $\varphi(u)$ using the model-compatible $S^\infty[X, \overline{X}]$-interpretation $\pi$ over $A = \{u, v\}$ indicated on the right, with $\pi(Pu) = 0$ and $\pi(Pv) = 1$.

$$\varphi(u) = \left[ \text{gfp} X x. \left[ \text{lfp} Y x. \exists y (Exy \land ((Xy \land Py) \lor Yy)) \right] (x) \right] (u)$$

\[ u \quad \xymatrix{ x_2 \ar@{~>}[d] \ar@{~>}[r] & v \ar@{~>}[d] \ar@{~>}[l]^x \ar@{~>}[r] & P \ar@{~>}[l]_{y_1} \ar@{~>}[r]_{y_2} & \bullet \ar@{~>}[l]^y } \]
Intuitively, witnesses for \( \varphi(u) \) are simply infinite paths that infinitely often visit \( v \). There are infinitely many such paths, but the simplest ones (in terms of the different edges they use) are the paths \( uvvuv \ldots \) and \( uvuwv \ldots \) which correspond to the monomials \( x_2y_1^\infty \) and \( x_2^2y_2^\infty \). And indeed, \( \pi[\varphi(u)] = x_2y_1^\infty + x_2^2y_2^\infty \). Notice that the edge \( x_1 \) does not appear in the result and we can conclude that its existence does not affect the truth of \( \varphi(u) \).

Let us now consider the evaluation strategies for \( \varphi(u) \) from the game-theoretic perspective. The complete model checking game (with abbreviated node labels) is shown below, where rounded nodes belong to Verifier, rectangular nodes to Falsifier and the small numbers indicate the priorities assigned to fixed-point relations. Terminal positions are highlighted by dashed borders and include the value assigned by \( \pi \). There are four positions for which Verifier can make a decision: The two nodes labeled \( \exists y(\ldots) \) and the two disjunctions in the center of the figure. Hence there are 16 positional strategies in total. One of these strategies is highlighted in gray and has the provenance value \( x_2y_1^\infty \), as there is one play ending in \( Euv \) and there are arbitrarily long plays ending either in \( Euv \) or in \( Pu \) depending on the choices of Falsifier. Most of the other 15 strategies allow infinite paths with least priority 1 and thus have provenance value 0 (for instance by choosing the cycle \( \exists y(\ldots) \to Euv\ldots \to (Xu \land Pu) \lor Y u \to Y u \)). The only remaining strategy has the provenance value \( x_2^2y_2^\infty \). One can further observe that non-positional strategies only lead to monomials with additional variables which are then absorbed, so we indeed obtain \( \pi[\varphi(u)] = x_2y_1^\infty + x_2^2y_2^\infty \).

If we are just interested in the question which literals are needed to satisfy \( \varphi(u) \) or, equivalently, in which models \( \varphi(u) \) holds, we can drop all exponents and obtain \( \pi[\varphi(u)] = x_2y_1 + x_2^2y_2 \). This is the same result we would obtain in the semiring \( \text{PosBool}(X, X) \) (the dual-indeterminate version of \( \text{PosBool}(X) \)). We see that \( \varphi(u) \) holds in all models that satisfy \( Pu \), \( \neg Pu \) and additionally contain at least the edges \( x_2 \) and \( y_1 \), or at least \( x_2 \) and \( y_2 \).

**Example 34.** In the previous example, the interpretation of the evaluation strategies in terms of infinite paths was straightforward. If we instead consider the negated formula \( \neg \varphi(u) \), which states that there are only finitely many occurrences of \( P \) on all paths from \( u \), witnesses for the truth of \( \varphi(u) \) are more complex and are best understood through the model checking game. We first bring \( \neg \varphi(u) \) into negation normal form using the duality laws of LFP:

\[
\neg \varphi(u) \equiv \lfp x. \lfp y. \forall y (\neg Exy \lor ((Xy \lor \neg Py) \land Yy)) (u)
\]
We analyse the game as in the previous example (the game graph is shown above). Again, the provenance values of non-positional strategies are absorbed by the values of positional ones. The players have basically switched roles, but Verifier can still make relevant decisions for only four nodes. One possible strategy is highlighted in gray above and has the provenance value $\prod_2$.

Notice the exponent 2, as the position $\forall u(\neg Ev_u \lor \ldots)$ tells us that $\forall u \neg Ev_u$ holds in the semiring $\prod_2$. It is possible, albeit tedious and not straightforward, to verify this result by manually determining the infimum of the nested fixed-point iteration. The interpretation of the result beyond the provenance values of non-positional strategies are absorbed by the values of positional ones. The only finite plays has the provenance value $\prod_1$ and two with value $\prod_2$. The other positional strategy with only finite plays has the provenance value $\prod_1 \prod_2$. Most of the positional strategies with infinite plays admit an infinite play with priority 1 and thus have value 0, except for the two strategies with values $\prod_2$, and $\prod_1 \prod_2$, respectively. We thus obtain $\pi[\neg \varphi(u)] = \prod_1 + \prod_1 \prod_2 + \prod_2 + \prod_1 \prod_2$.

It is possible, albeit tedious and not straightforward, to verify this result by manually determining the infimum of the nested fixed-point iteration. The interpretation of the result beyond the model checking game is not as clear as in the previous example. We can, however, again omit the exponents and, due to absorption, obtain the value $\pi[\neg \varphi(u)] = \prod_1 + \prod_2$ in PosBool$(X, \overline{X})$. This tells us that $\neg \varphi(u)$ is satisfied in all models that lack at least edge $x_2$ or have no outgoing edges from $v$ – exactly complementary to the models we determined for $\varphi(u)$.

### 6.2 Proof of Theorem 32

The proof of Theorem 32 is more involved than the proofs presented so far. We first show that it holds in the semiring $S^\infty[X]$ and afterwards make use of the universality property to generalize it to all absorptive, fully continuous semirings. We begin with some notation for model checking games and strategies. The inductive proof is then based on the notion of truncations, which are essentially prefixes of strategy trees. We extend the definition of the value $\pi[S]$ of a strategy to these prefixes and show by induction that truncations of increasing size correspond to the steps of the fixed-point iteration for fixed-point formulae.

For the proof in $S^\infty[X]$, we always fix a universe $A$, a signature $\tau$ (which we usually omit), a $S^\infty[X]$-interpretation $\pi$ and consider model checking games $G(A, \varphi)$ which we abbreviate by just $G(\varphi)$. To make the positions of the game precise, let $G(\varphi) = (V, V_0, V_1, E, \Omega)$, where $(V, E)$ is a directed graph and $V = V_0 \cup V_1$ are the positions owned by Verifier and Falsifier, respectively. The priorities are given by the node labeling $\Omega : V \to \mathbb{N}$. To avoid the special case of infinite plays that are losing and thus lead to $\pi[S] = 0$, we only consider strategies for Verifier in which all infinite plays are winning for Verifier and denote their set by $W(\varphi) \subseteq \text{Strat}(\varphi)$. For $v \in V$, we denote the set of successor positions by $Strat(v)$.
\( vE = \{w \mid (v, w) \in E\} \). Positions \( v \) with \( vE = \emptyset \) are called terminal positions. A play from position \( v_0 \) is a (finite or infinite) sequence \( \rho = v_0v_1v_2\ldots \) such that \((v_i, v_{i+1}) \in E\) for all \( i \). If \( \rho \) is finite, it must end in a terminal position (which we call the outcome of \( \rho \)).

The tree unraveling of a game \( G(\varphi) \), as defined in [3], is the tree \( T(G(\varphi), v_0) = (V^#, V_0^#, V_1^#, E^#) \) where \( V^# \) is the set of all finite paths \( \tau \) from \( v_0 \), \( V_0^# \subseteq V \) is the set of those finite paths ending in a node of player \( \sigma \) and \( E^# = \{(\tau v, \tau v') \mid \tau v \in V^# \text{ and } (v, v') \in E\} \). For \( \tau, \tau' \in V^# \), we write \( \tau \sqsubseteq \tau' \) if \( \tau \) is a prefix of \( \tau' \). For a node \( \tau v \), we call \( \tau(v) = v \) the position of \( \tau v \).

It is often convenient to identify a node \( \tau v \) in the tree unraveling with its position in the original game. For \( \tau = v_0 \ldots v_k \), we write \(|\tau| = k + 1\) for the length of \( \tau \). Following [3], we view strategies as subtrees of the tree unraveling.

**Definition 35.** A strategy \( S \) of player \( \sigma \in \{0, 1\} \) from \( v_0 \) in \( G \) is a subtree of \( T(G, v_0) \) of the form \( S = (W, F) \) with \( W \subseteq V^# \) and \( F \subseteq (W \times W) \cap E^# \) that satisfies the following conditions. Let \( \hat{V}_\sigma^# \) be the set \( V_\sigma^# \) without terminal nodes (i.e., leaves).

1. \( W \) is closed under predecessors: if \( \tau v \in W \), then also \( \tau \in W \);  
2. player \( \sigma \) makes unique choices: if \( \tau \in W \cap \hat{V}_\sigma^# \), then \(|\tau F| = 1\);  
3. all choices of the opponent are considered: if \( \tau \in W \cap \hat{V}_{\lnot \sigma}^# \), then \( \tau F = \tau E^# \).

A play \( \rho \) is consistent with \( S \) if the corresponding path in \( T(G, v_0) \) is contained in \( S \). The strategy \( S \) is winning if all plays consistent with \( S \) are winning (for player \( \sigma \)).

**Strategy Truncations**

**Definition 36.** Let \( S = (W, F) \) be a strategy in \( G(\varphi) = (V, V_0, V_1, E, \Omega) \) and let \( R \) be a relation symbol of arity \( r \). Nodes \( v \in V \) with \( V(v) = \overline{R} \) (for some \( \overline{a} \in A^r \)) are called \( R \)-nodes. For \( \tau v = v_0v_1\ldots v_k \in W \), we define

\[ |\tau|_R = |\{i \mid V(v_i) = \overline{R} \text{ for some } \overline{a} \in A^r \}| \]

as the number of \( R \)-nodes occurring along the path. The \((R, n)\)-truncation of \( S \) is the tree \((W', F')\) defined as follows. Its nodes are finite sequences over \( V \cup \{\infty\} \), where \( \infty \) is a special symbol which marks the nodes at which we cut off subtrees of \( S \). For \( n \geq 1 \), we define

\[ W' = \{\tau \in W \mid |\tau|_R < n\} \cup \{\tau \infty \mid \tau \overline{a} \in W, |\tau|_R = n - 1\} \]

\[ F' = F \cap (W' \times W') \cup \{\{\tau, \tau \infty\} \mid \tau \infty \in W\} \]

For \( n = 0 \), we instead set \( W' = \{\infty\} \) and \( F' = \emptyset \). If \( R \) is clear from the context, we write \( S|_n \) for the \((R, n)\)-truncation of \( S \).

We lift the definition of the provenance value \( \pi[S] \) to truncations \( \pi[S|_n] \) by treating \( \infty \) as an additional literal. That is, given some value \( \pi(\infty) \),

\[ \pi[S|_n] = \pi(\infty)^{\#_a(\infty)} \cdot \prod_{L \in \text{Lit}_a} \pi(L)^{\#_a(L)} \]

**Lemma 37.** Let \( \pi \) be an \( S^{\infty}[X] \)-interpretation.

1. Let \( \varphi = [\text{lfp } R \overline{a}, \overline{a}](\overline{y}) \) and let \( S \in W(\varphi(\overline{x})) \). If we extend \( \pi \) by \( \pi(\infty) = 0 \), then

\[ \bigcup_{n < \omega} \pi[S|_n] = \pi[S] \]
2. Let $\varphi = [\mu R \varphi \vartheta](\overline{y})$ and let $S \in W(\varphi(\overline{y}))$. If we extend $\pi$ by $\pi(\infty) = 1$, then

$$\bigcap_{n<\omega} \pi[S|_n] = \pi[S]$$

**Proof.** For (1), note that $\pi[S|_n] = 0$ whenever $S$ has a path with at least $n$ $R$-nodes (so $S|_n$ contains an $\infty$-node). We show that there is a $k$ such that all paths of $S$ have less than $k$ $R$-nodes. Assume towards a contradiction that this is not the case. Then consider the subgraph of $S$ consisting of all nodes from which a path to an $R$-node exists. This subgraph must then have paths of arbitrary length and by König's lemma (note that $S$ is finitely branching), it must have an infinite path. This is a contradiction, as this infinite path would contain an infinite number of $R$-nodes and would thus be losing. Hence $\pi[S|_n] = \pi[S]$ for all $n \geq k$ and the claim follows.

For (2), we first note that the truncations $\pi[S|_n]$ indeed form a chain. The reason is that $\pi(\infty) = 1$ is the greatest element, so replacing subtrees of $S$ by $\infty$ leads to a larger provenance value. Using the splitting lemma, the infimum can be written as follows (we can ignore the value $\pi(\overline{y})$ appearing in the provenance value, as $1$ is also the neutral element).

$$\bigcap_{n<\omega} \pi[S|_n] = \prod_{L \in \text{Lit}_A} \bigcap_{n<\omega} \pi(L)^{\#S|_n}(L) = \prod_{L \in \text{Lit}_A} \pi(L)^{c_L}, \text{ where } c_L = \bigcup_{n<\omega} \#S|_n(L)$$

The main observation is that each node of $S$ is eventually contained in $S|_n$ (for sufficiently large $n$). Consider a literal $L$. If $\#S(L)$ is finite, then for sufficiently large $n$, we have $\#S|_n(L) = \#S(L)$ and thus $c_L = \#S(L)$. If $\#S(L) = \infty$, then for each $k$ there is a sufficiently large $n$ such that $\#S|_n(L) \geq k$ and thus $c_L = \infty$, which closes the proof. ▮

**The Puzzle Lemma**

**Lemma 38 (Puzzle Lemma).** Let $\varphi = [\mu R \varphi \vartheta](\overline{y})$, let $r$ be the arity of $R$ and let $\overline{y} \in A^r$. Let $\pi$ be an $S^\infty[\overline{X}]$-interpretation extended by $\pi(\infty) = 1$. Let further $(S_i)_{i<\omega}$ be a family of strategies in $W(\varphi(\overline{y}))$ such that $(\pi[S_i|_n])_{i<\omega}$ is a descending chain. Then there is a winning strategy $S \in W(\varphi(\overline{y}))$ with $\pi[S] \geq \bigcap_{i<\omega} \pi[S_i|_n]$.

For an intuition why this result is not obvious, we consider the following example. The key problem is that the strategies $S_i$ can all be different. In particular, it can happen that
for every $i$, the provenance value of the truncation $S_i\!\mid_i$ is larger than the value of the full strategy $S_i$. The insight of the lemma is that we can always use one of the truncations $S_i\!\mid_i$ (for sufficiently large $i$) to construct a strategy $S$ with the desired property. This construction has to be done carefully to ensure that the resulting strategy $S$ is winning.

**Example 39.** Consider the following setting:

$\varphi_{\text{infpath}}(u) = [\text{gfp } R x. \exists y (E x y \land R y)](u)$

Let $S_i$ be the strategy corresponding to the infinite path that cycles $i-1$ times via $x$, then uses edge $y$ and finally cycles via $z$. The $i$-truncation then cuts off $S_i$ after taking the edge $y$ and we obtain the provenance values

$\pi[S_i] = x'yz^\infty$ and $\prod_{i<\omega} \pi[S_i\!\mid_i] = \prod_{i<\omega} x'y = x^\infty y$.

We see that the infimum only contains the variables $x$ and $y$, although there is no winning strategy with this provenance value. Instead, we obtain $S$ by repeating the cycling part of any truncation $S_i\!\mid_i$ (without the problematic literal $y$). This results in the strategy $S$ with value $\pi[S] = x^\infty$ that corresponds to the path always cycling via $x$. This path is not consistent with any of the strategies $S_i$. In general, we have to make sure that the additional plays in $S$ (which result from the repetition of $S_i\!\mid_i$) are always winning.

As a first step, we can apply the splitting lemma to the infimum and obtain:

$\prod_{i<\omega} \pi[S_i\!\mid_i] = \prod_{L \in \text{Lit}_A} \pi(L)^{n_L}$ with $n_L = \bigcup_{i<\omega} \# S_i\!\mid_i(L)$

Literals with $n_L = \infty$ (such as the edge $x$ in the example) can appear arbitrarily often in $S$, so they do not impose any restrictions. If $n_L < \infty$, then we must have $\# S(L) \leq n_L$ to guarantee that the provenance value of $S$ is larger than the infimum. We therefore call literals $L$ with $n_L < \infty$ (such as the edge $y$) problematic. The outline of the proof is as follows:

- We decompose the trees $S_i\!\mid_i$ into layers based on the appearance of $R$-nodes.
- We choose a sufficiently large $i$ such that there is one such layer in $S_i\!\mid_i$ which does not contain any problematic literals at all.
- We construct $S$ by first following $S_i\!\mid_i$ and then repeating this layer ad infinitum. For the construction, we collect several subtrees (which we call puzzle pieces) from this layer which we can then join together to form the repetition.
- The form of the puzzle pieces ensures that $S$ is winning. In particular, we only join the pieces at $R$-nodes. Paths through infinitely many pieces are thus guaranteed to satisfy the parity condition.

**Decomposition into layers.** Fix an $i$ and let $S_i\!\mid_i = (W, F)$. We call each node $\tau \in W$ with $V(\tau) = R \overline{\tau}$ (for any $\overline{\tau} \in A^*$) an $R$-node. If an $R$-node happens to be a leaf, we speak of an $R$-leaf.

For each $n \geq 0$, we define the sets

$W_{\leq n} = \{ \tau \in W \mid |\tau|_R \leq n \}, \quad W_{\leq n}^+ = W_{\leq n} \cup \{ \tau v \in W \mid \tau \in W_{\leq n}, v \in V \}$

We sort the nodes $\tau \in W$ into layers based on the number of $R$-nodes on the path to $\tau$. For now, think of a layer as a forest in which all roots and most of the leaves are $R$-nodes.
A visualization of a strategy. The gray nodes form the first layer (for $k = 1$). The two trees in this layer are puzzle pieces, the left one has an infinite winning path.

The $R$-leaves of one layer are the root nodes of the next layer, apart from this layers do not overlap. The constant $k$ controls the thickness of the layer (the maximal number of $R$-nodes that can occur on paths through the layer).

For any $j \geq 1$, the $j$-th layer is the subgraph of $S_i\mid_i$ induced by the node set

$$W_j = W_{<j \cdot k} \setminus W_{\leq (j-1) \cdot k} \quad \text{where} \quad k = |A| + 2.$$  

Note that each tree in a layer is a strategy (i.e., satisfies conditions (1)-(3) of Definition 35) except for its leaves. See Figure 3 for a visualization.

### Avoiding problematic literals.

Let $n = \sum \{ n_L \mid L \in \text{Lit}_A, n_L < \infty \}$ be the sum of the problematic $n_L$, which is an upper bound on the number of problematic literals appearing in any truncation $S_i\mid_i$. Note that $n$ is always finite. We now choose any $i$ such that:

$$i \geq (n + 1) \cdot k = (n + 1) \cdot (|A| + 2)$$

From now on, we only work with $S_i\mid_i = (W, F)$. Consider the layers $W_1, \ldots, W_{n+1}$ of $S_i\mid_i$. First assume that there is a $j$ such that $W_j = \emptyset$. By definition of the layers, we thus have $|\tau|_R \leq (j - 1) \cdot k < i$ for all $\tau \in S_i\mid_i$. But this means that each path in $S_i\mid_i$ has less than $i$ $R$-nodes. By definition of the truncation, this means that $S_i\mid_i = S_i$. In this case we can simply set $S = S_i$ and are done.

Otherwise, there are $n + 1$ nonempty layers and at most $n$ occurrences of problematic literals. Hence there must be a layer $j$ such that $W_j$ does not contain any problematic literals. In the following, we concentrate only on this layer $W_j$.

### Collecting puzzle pieces.

We want to build the strategy $S$ from the prefix of $S_i\mid_i$ up to layer $W_j$ and then continue by always repeating the layer $W_j$. Because $W_j$ does not contain any problematic literals, this eventually yields $\pi[S] \geq \pi[S_i\mid_i]$ as required.

Let $T$ be one of the components in $W_j$, so $T$ is a tree. We call a path in $T$ winning if it is infinite or ends in a terminal position, so it corresponds to a play consistent with $S_i$.
Paths ending in $R$-leaves of $T$ (which could be continued by leaving the layer $W_j$) are not considered to be winning.

**Definition 40.** A puzzle piece $P = (W', F')$ is a subtree of $W_j$ such that

(a) The root of $P$ is an $R$-node,
(b) For each inner node $\tau \in P$, we have $\tau F' = \tau F$ ($P$ contains all successors),
(c) Each maximal path through $P$ is either winning or ends in an $R$-node.

A puzzle piece $P$ with root $\tau$ matches a node $\tau' \in W_j$ if $V(\tau) = V(\tau')$. A complete puzzle is a set of puzzle pieces such that for each piece in the set and all $R$-leaves $\tau$ of this piece, the set contains a puzzle piece that matches $\tau$.

First observe that $T$ itself is a puzzle piece: Each maximal path through $T$ which does not end in an $R$-node must visit less than $k$ $R$-nodes. If we append this path to the unique path from the root of $S_j$ to the root of $T$, then the resulting path contains less than $i$ $R$-nodes. Hence the path is not truncated in $S_i$, so it is also a maximal path of $S_i$ and thus winning. However, a single piece does not make a complete puzzle. Instead, we collect smaller pieces from $T$ by the following process:

1. Initialize $L = \{\hat{\tau}\}$ where $\hat{\tau}$ is the root of $T$ (which is an $R$-node).
2. Pick a node $\tau \in L$ and remove it from $L$ (if $L$ is empty, terminate).
3. If we have already found a puzzle piece matching $\tau$, go to step (2).
4. Let $P$ be the subgraph of $T$ induced by the following set of nodes. Then $P$ is a puzzle piece matching $\tau$ and we add it to our set of pieces.

$$W' := \{\tau' \in T \mid \tau \subseteq \tau' \text{ and there is no } R\text{-node } \tau'' \text{ with } \tau \subseteq \tau'' \subseteq \tau'\}$$

5. For each $\pi \in A'$: If $P$ has a leaf $\tau'$ with $V(\tau') = R\pi$, add one such leaf $\tau'$ to $L$.
6. Go back to step (2).

If the definition of $P$ in step (4) is correct, then this process clearly terminates after finding at most $|A'|$ puzzle pieces and the resulting set of pieces is a complete puzzle. For step (4), recall the definition of $W_j$. For the root of $T$, we have $|\hat{\tau}|_R = (j - 1)k + 1$ and $W_j$ contains in particular the nodes $\tau$ with $(j - 1)k < |\tau|_R \leq jk$.

Assume that in (2), we picked a node $\tau$ with $|\tau|_R = n$ (for some $n$). By definition of $W'$, the piece $P$ only contains nodes $\tau'$ with $|\tau'|_R \leq n + 1$. In particular, the leaves that we add to $L$ in step (5) all satisfy $|\tau'|_R \leq n + 1$. We start with $|\hat{\tau}|_R = (j - 1)k + 1$ and perform at most $k - 2$ iterations, hence we always have $|\tau|_R < jk$ for all $\tau \in L$.

This guarantees that $P$ is always a puzzle piece in step (4): Inner nodes of $P$ cannot be $R$-nodes and hence $P$ always contains all successors of inner nodes, so (b) is satisfied. For (c), assume towards a contradiction that there is a maximal path through $P$ which does not end in an $R$-node and is not winning. This path is also a path in $S_j$ and because all infinite paths of $S_j$ are winning, the path must be finite. Because all terminal positions in $S_j$ are winning, the path must end in a leaf of $T$ which is not a leaf of $S_j$. But such leaves of $T$ must be $R$-nodes by definition of the layers, which is a contradiction. Hence (c) holds as well and $P$ is a puzzle piece.

We proceed in the same way for all other components of $W_j$ and obtain a complete puzzle for each component. The overall result is the union of all these puzzles, which is again a complete puzzle. An illustration of such a puzzle (as individual pieces and in assembled form) is shown in Figure 4; the next step is to perform the assembly.
Completing the puzzle. We now have a complete puzzle with a matching piece for all root nodes of $W_j$ (these are precisely the $R$-leaves of the preceding layer $W_{j-1}$). All that remains is to join the pieces together to form the strategy $S$. Note that puzzle pieces can contain infinite paths or even infinitely many $R$-leaves. We therefore construct $S$ recursively layer by layer.

- $S_0$ is the subgraph induced by $W^+_{j \leq (j-1)k}$, i.e., the prefix of $S_i$ up to layer $W_j$. By definition of the layers, all leaves of $S_0$ are either leaves of $S_i$ or $R$-leaves of $W_{j-1}$.
- Given $S_n$, we construct $S_{n+1}$ as follows. Recall that for $\tau = v_0 \ldots v_l \in S_n$, we write $|\tau| = l$ for its length (which equals the depth of $\tau$ in $S_n$). Consider the set

$$X = \{ x \in S_n \mid x \text{ is an } R\text{-leaf of } S_n \text{ with } |x| = n \}$$

Because $S_n$ is finitely branching (as we construct it from subtrees of $S_i$), this set is finite. Moreover, each $x \in X$ is either the $R$-leaf of a puzzle piece or, initially, the root of one component of $W_j$. In both cases, the complete puzzle contains a piece matching $\tau$. The tree $S_{n+1}$ results from $S_n$ by replacing all leaves $\tau \in X$ with the unique puzzle piece matching $\tau$. Then $S_{n+1}$ has no more $R$-leaves at depth $n$ (note that the puzzle pieces we collected always consist of at least two nodes).

When we replace $\tau \in X$ by a piece $P$, we rename the nodes of $P$ accordingly, to match our definition of strategies (if $\hat{\tau}$ is the root of $P$, we rename each node $\hat{\tau}' \in P$ to $\tau'$ when adding it to $S_{n+1}$).

- We define $S = \bigcup_{n < \omega} S_n$, so $S$ contains no more $R$-leaves.

Then $S$ is a strategy: Each node $\tau \in S$ corresponds to a node $\tau' \in S_i$ (either $\tau'$ is an inner node of a puzzle piece, or $\tau' \in S_0$) and $\tau$ has the same successors as $\tau'$. Moreover, the provenance value is $\pi[S] \geq \pi[S_i]$ as desired, because the repetition of puzzle pieces does not contain any problematic literals. Lastly, $S$ is a winning strategy: Consider a play consistent with $S$ and the corresponding maximal path through $S$. If the path is finite, it
ends in a leaf of $S$ which corresponds to a leaf of $S_i$ and is therefore winning. If the path visits infinitely many puzzle pieces (whose root nodes are $R$-nodes), then it visits infinitely many $R$-nodes and is thus winning by the parity condition (note that $R$ belongs to the outermost fixed-point formula in $\varphi$). If the path is infinite and stays in $S_0$, then it corresponds to an infinite path of $S_{|i}$, and is thus winning. Otherwise, the path is infinite, leaves $S_0$ at some point and visits only finitely many puzzle pieces. This means that it must from some point on stay in one piece, so it is winning by definition of puzzle pieces. We have therefore completed the puzzle (lemma).

\section*{The Main Proof}

We are now ready to prove the central result for $S^\infty[X]$: 

$$\pi[\varphi(\pi)] = \bigsqcup\{\pi[S] \mid S \in \mathcal{W}(\varphi(\pi))\}$$

The interesting part is the proof for fixed-point formulae such as $\varphi(\pi) = \lfloor R \varphi, \vartheta(\pi) \rfloor(\pi)$ or $[\mathsf{gfp} R \varphi, \vartheta(\pi)](\pi)$. A strategy $S \in \mathcal{W}(\varphi(\pi))$ may then look as in the picture below. We write $L/\ldots$ to denote either a literal or an infinite path (without occurrences of $R$-nodes).

The strategy $S$ must first move to $\vartheta(\pi)$ and thus contains a winning strategy from $\vartheta(\pi)$ in $G(\varphi)$. If we only consider the strategy from $\vartheta(\pi)$ up to the first occurrence of an $R$-node, as indicated above, we obtain a winning strategy for the game $G(\vartheta(\pi))$. Note that in $G(\vartheta(\pi))$, $R$-nodes are terminals and are thus winning.

In $G(\varphi(\pi))$, these $R$-nodes are not terminals. Hence $S$ must further contain substrategies for these $R$-nodes ($R\varphi$ and $R\pi$ above). Because the positions $\varphi(\pi)$ and $R\varphi$ must both be followed by $\vartheta(\pi)$, we can view the substrategy from $R\varphi$ as a winning strategy $S_{\varphi}$ for $G(\varphi(\pi))$ (as highlighted above).

We thus see that each winning strategy $S$ in $G(\varphi(\pi))$ can be decomposed into a prefix $S_{\vartheta}$ which we can identify with a winning strategy for $G(\vartheta(\pi))$ and, for all $R$-leaves of $S_{\vartheta}$, substrategies which are winning strategies in $G(\varphi)$. Conversely, every winning strategy $S_{\vartheta}$ for $G(\vartheta(\pi))$ can be combined with winning strategies from $\varphi(\pi), \varphi(\pi), \ldots$ for all the $R$-leaves of $S_{\vartheta}$ to form a winning strategy in $G(\varphi(\pi))$.

If we build the strategy by starting with $S_{\vartheta}$ but then appending the truncations $S_{\vartheta}|_{n+1}$ and $S_{\varphi}|_{n}$ instead of $S_{\vartheta}, S_{\varphi}$ (as indicated by the dashed lines in the picture), then the result is the $n+1$-truncation $S|_{n+1}$ of a winning strategy $S \in \mathcal{W}(\varphi(\pi))$, because $S_{\vartheta}$ contains at most one $R$-node on each path. We exploit this observation in an inductive proof that relates the $n$-truncations of winning strategies with the $n$-th step of the fixed-point iteration.

\begin{proof}
 Induction on the negation normal form of $\varphi(\pi)$:

\end{proof}
\( \varphi(\overline{a}) = L \in \text{Lit} \): Then \( \mathcal{G}(\varphi(\overline{a})) \) consists only of a terminal position (which is winning). There is only one (trivial) strategy with \( \pi[S] = \pi(L) = \pi(\varphi(\overline{a})) \).

\( \varphi(\overline{a}) = \varphi_1(\overline{a}) \lor \varphi_2(\overline{a}) \). The game \( \mathcal{G}(\varphi(\overline{a})) \) is shown on the right. Each strategy \( S \) for \( \mathcal{G}(\varphi(\overline{a})) \) makes a unique choice at \( \varphi(\overline{a}) \) and thus either consists of a strategy \( S_1 \) for \( \mathcal{G}(\varphi_1(\overline{a})) \) or a strategy \( S_2 \) for \( \mathcal{G}(\varphi_2(\overline{a})) \), but not both. Conversely, a strategy \( S_i \) for \( \mathcal{G}(\varphi_i(\overline{a})) \) lifts to a strategy \( S \) from \( \varphi(\overline{a}) \) (for \( i \in \{0, 1\} \)). We thus have:

\[
\bigcup \{ \pi[S] \mid S \in \mathcal{W}(\varphi(\overline{a})) \} = \bigcup \{ \pi[S_i] \mid S_i \in \mathcal{W}(\varphi_i(\overline{a})) \}, \quad i \in \{0, 1\}
\]

\[= \bigcup \{ \pi[S_1] \mid S_1 \in \mathcal{W}(\varphi_1(\overline{a})) \} \cup \bigcup \{ \pi[S_2] \mid S_2 \in \mathcal{W}(\varphi_2(\overline{a})) \}\]

\[= \pi[\varphi_1(\overline{a})] \cup \pi[\varphi_2(\overline{a})] = \pi[\varphi_1(\overline{a})] + \pi[\varphi_2(\overline{a})] = \pi[\varphi(\overline{a})] \]

\( \varphi(\overline{a}) = \varphi_1(\overline{a}) \land \varphi_2(\overline{a}) \). The reasoning is similar: Each strategy \( S \) for \( \mathcal{G}(\varphi(\overline{a})) \) consists of both a strategy \( S_1 \) for \( \mathcal{G}(\varphi_1(\overline{a})) \) and a strategy \( S_2 \) for \( \mathcal{G}(\varphi_2(\overline{a})) \). The converse direction (all \( S_1 \) and \( S_2 \) together induce a strategy \( S \)) holds as well.

If \( S \) consists of the two strategies \( S_1 \) and \( S_2 \), then we further have \( \pi[S] = \pi[S_1] \cdot \pi[S_2] \) by definition of the provenance value and Lemma 20. The claim follows by induction and continuity of \( S^\infty[A] \):

\[
\pi[\varphi_1(\overline{a})] \cdot \pi[\varphi_1(\overline{a})] = \bigcup \{ \pi[S_i] \mid S_i \in \mathcal{W}(\varphi_1(\overline{a})) \} \cdot \bigcup \{ \pi[S_2] \mid S_2 \in \mathcal{W}(\varphi_2(\overline{a})) \}
\]

\[= \bigcup \{ \pi[S_i] \cdot \pi[S_2] \mid S_i \in \mathcal{W}(\varphi_1(\overline{a})) \} \quad \text{for } i \in \{1, 2\}
\]

\[= \bigcup \{ \pi[S] \mid S \in \mathcal{W}(\varphi(\overline{a})) \} \]

The cases for \( \exists \) and \( \forall \) follow by the same arguments (with \( |A| \) instead of 2 child nodes).

For fixed-point formulae, we use the decomposition into \( S_0 \) and \( S_F, S^* \) as motivated above. Consider the fixed-point iteration \((f_\beta)_{\beta \in \text{On}}\) for \( \varphi = [\text{lfp } R^\overline{a}, \vartheta]\overline{a} \) or \( \varphi = [\text{gfp } R^\overline{a}, \vartheta]\overline{a} \), where \( R \) has arity \( r \).

We first show by induction that the following holds for all \( n < \omega \) and all \( \overline{a} \in A^r \), where we set \( \pi(\infty) = 0 \) for \( \varphi = [\text{lfp } R^\overline{a}, \vartheta]\overline{a} \) and \( \pi(\infty) = 1 \) for \( \varphi = [\text{gfp } R^\overline{a}, \vartheta]\overline{a} \):

\[
f_n(\overline{a}) = \bigcup \{ \pi[S_\beta] \mid S \in \mathcal{W}(\varphi(\overline{a})) \}
\]

For \( n = 0 \), we trivially have \( f_0 = 0 \) and \( \pi[S_0] = 0 \) for least fixed points and \( f_0 = 1 \) and \( \pi[S_0] = 1 \) for greatest fixed points.

For the induction step \( n \rightarrow n + 1 \), we first show \( f_{n+1}(\overline{a}) \subseteq \bigcup \{ \pi[S_{\beta+1}] \mid S \in \mathcal{W}(\varphi(\overline{a})) \} \).

To simplify notation, we set \( \text{Lit}_A = \text{Lit}_A \setminus \{ \overline{b} \mid \overline{b} \notin A^r \} \). By the induction hypothesis on \( \vartheta \) and on \( f_n \), we can write \( f_{n+1} \) as follows:

\[
f_{n+1}(\overline{a}) = [R/f_n]_{\vartheta}(\overline{a}) = \bigcup \{ \pi[R/f_n][S_\beta] \mid S_\beta \in \mathcal{W}(\varphi(\overline{a})) \}
\]

\[= \bigcup \{ \prod_{L \in \text{Lit}_{\overline{a}}} \pi(L)^{\nu_L} \cdot \prod_{\overline{b} \in A^r} \left( \bigcup \{ \pi[S^*_\overline{a}] \mid S^*_\overline{a} \in \mathcal{W}(\varphi(\overline{b})) \} \right)^{\nu_{\overline{b}}} \mid S_\beta \in \mathcal{W}(\varphi(\overline{a})) \}
\]

Notice that here we consider suprema of arbitrary sets. It follows from the considerations in [11] that if multiplication preserves suprema of finite sets and of chains, it also preserves suprema of arbitrary sets.

In idempotent semirings, finite suprema are simply finite sums which are preserved by distributivity.
where we set \( n_L = \#S_\alpha(L) \) and \( b_L = \#S_\alpha(R\vec{b}) \), depending on \( S_\alpha \). Let us first fix a strategy \( S_\alpha \) and \( \vec{b}, b \in A' \) and consider the term \((*)\). Recall that absorptive polynomials are finite and that for a set \( S \), \( \bigcup S = \) maximals \((\bigcup S) \). We can thus write
\[
(*) = (\pi[S_\alpha^1 | n_1] + \cdots + \pi[S_\alpha^n | n]) b_L
\]
for some \( S_\alpha^1, \ldots, S_\alpha^n \in W(\bar{b}(\vec{b})) \). If \( b_L = \infty \), then by Lemma 20
\[
(*) = \pi[[S_\alpha^1 | n_1]^\infty + \cdots + \pi[S_\alpha^n | n]^\infty
\]
Otherwise, \( b_L = l < \infty \). Then each monomial of \((*)\) is of the form
\[
\pi[[S_\alpha^i | n_i] \cdots \pi[S_\alpha^l | n]]
\]
where \( i_1, \ldots, i_l \in \{1, \ldots, n\} \).

To prove that \( f_{n+1}(\vec{b}) \leq \bigcup \{ \pi[S_{|n+1}] | S \in W(\bar{b}(\vec{b})) \} \), we show that each monomial of \( f_{n+1}(\vec{b}) \) is absorbed by the right-hand side. Given the above considerations, we know that these monomials are of the form
\[
m = \prod_{L \in \text{Lit}_\alpha} \pi(L)^{n_L}, \quad n_L = n_L + \sum_{b \in A'} \#S_{|n+1}(L) \cdot \infty + \sum_{b \in A'} \#S_{|n+1}^i(L) - \#S_{|n+1}^i(L)
\]
for some \( S_{\alpha} \in W(\bar{b}(\vec{b})) \) (which defines \( n_L \)) and some \( S_{\alpha}^1, S_{\alpha}^2 \in W(\bar{b}(\vec{b})) \) (for all \( b, i \)).

Now consider the strategy which starts with \( S_\alpha \), and for each \( b \in A' \), we replace all \( R\vec{b} \)-leaves of \( S_\alpha \) by either \( S_\alpha^b \) (if there are infinitely many such leaves) or by the strategies \( S_\alpha^b, \ldots, S_\alpha^n \) if there are \( l < \infty \) such leaves (it does not matter in which order these strategies are assigned to the leaves). The result is a strategy \( S \in W(\bar{b}(\vec{b})) \).

We further see that \( S_{|n+1} \) results from \( S_\alpha \) in the same way if we replace leaves by the truncations \( S_{|n}^b \) instead of \( S_\alpha^b \) (and \( S_{|n}^i \) instead of \( S_{|n}^i \)), because \( S_\alpha \) contains \( R \)-nodes only as leaves. By this construction, we see that \( \#S_\alpha(L) = n_L \) for each \( L \in \text{Lit}_\alpha \) and hence \( m = \pi[S_{|n+1}] \) and thus \( m \leq \bigcup \{ \pi[S_{|n+1}] | S \in W(\bar{b}(\vec{b})) \} \) as claimed.

For the other direction, we fix a strategy \( S \in W(\bar{b}(\vec{b})) \) and show that \( \pi[S_{|n+1}] \leq f_{n+1}(\vec{b}) \). In the case that \( \pi(\infty) = 0 \) and \( \infty \) appears in \( S_{|n+1} \), we have \( \pi[S_{|n+1}] = 0 \) and there is nothing to show. If \( \pi(\infty) = 1 \), the appearance of \( \infty \) does not affect the provenance value \( \pi[S_{|n+1}] \). We again decompose \( S \) into a prefix \( S_\alpha \) corresponding to a winning strategy in \( G(\bar{b}(\vec{b})) \) and substrategies from all \( R \)-leaves of \( S_\alpha \).

We first consider any \( \vec{b} \in A' \) for which \( \#S_\alpha(R\vec{b}) = \infty \) such that we have infinitely many such substrategies from \( R\vec{b} \)-leaves. We make the following claim: There is a strategy \( S_\alpha^b \in W(\bar{b}(\vec{b})) \) such that the strategy \( S' \) which is like \( S \) but uses \( S_\alpha^b \) for all of the infinitely many \( R\vec{b} \)-leaves of \( S_\alpha \) satisfies \( \pi[S_{|n+1}] \leq \pi[S'_{|n+1}] \).

Proof of the claim: Let \( (S_\alpha^b)_{i < \omega} \) be the family of all substrategies \( S \) uses from \( R\vec{b} \)-leaves of \( S_\alpha \). As in the proof of the Puzzle Lemma, we call a literal \( L \) problematic if \( \#S_\alpha(L) < \infty \). Let \( L \) be a problematic literal. Then there can only be finitely many \( i \) with \( \#S_\alpha(L) > 0 \) (otherwise \( L \) would occur infinitely often in \( S_{|n+1} \)). As \( \omega \) is infinite while the number of literals is finite, there is an \( i < \omega \) such that \( S_\alpha^i \) contains no problematic literals at all. We then set \( S' = S_\alpha^i \) and the claim follows.

Due to this claim and because \( A' \) is finite, we obtain a strategy \( S' \) with \( \pi[S'_{|n+1}] \leq \pi[S''_{|n+1}] \) such that for all \( \vec{b} \in A' \) with \( \#S_\alpha(R\vec{b}) = \infty \), \( S' \) uses the same strategy from all \( R\vec{b} \)-leaves of \( S_\alpha \). From the other direction, we know that
\[
f_{n+1}(\vec{b}) \geq \prod_{L \in \text{Lit}_\alpha} \pi(L)^{n_L} \cdot \prod_{b \in A'} \left( \bigcup \{ \pi[S_\alpha^1 | n_1] | S_\alpha^1 \in W(\bar{b}(\vec{b})) \} \right)^{b_L}
\]
We have already shown that the fixed-point iteration has closure ordinal $\omega$. Consider the strategies $S'$ uses from the $R$-leaves of $\{S\}$: For $\vec{b} \in A'$ with $\#S_{\vec{b}}(R\vec{b}) = \infty$, let $\{S\}_{\vec{b}}$ be the strategy that $S'$ uses from all $R\vec{b}$-leaves. For $\vec{b}$ with $\#S_{\vec{b}}(R\vec{b}) = l < \infty$, let $\{S\}_{\vec{b}}^1, \ldots, \{S\}_{\vec{b}}^l$ be the strategies $S'$ uses from the $R\vec{b}$-leaves of $\{S\}$. Let further
\[
n'_L = n_L + \sum_{\vec{r} \in A'} \#\{S\}_{\vec{r}^n}(L) \cdot \infty + \sum_{\vec{r} \in A'} \#\{S\}_{\vec{r}^1}(L) \cdots \#\{S\}_{\vec{r}^n}(L)
\]
We can apply commutativity and the lemma on infinitary powers to conclude
\[
\pi[S]_{n+1} = \prod_{L \in \text{Lit}_{\alpha}} \pi(L)^{n_L} \cdot \prod_{\vec{r} \in A'} \pi[\{S\}_{\vec{r}^n}] \cdot \prod_{\vec{r} \in A'} \pi[\{S\}_{\vec{r}^1}] \cdots \prod_{\vec{r} \in A'} \pi[\{S\}_{\vec{r}^n}] \leq \prod_{\vec{r} \in A'} \pi(L)^{n_L} \cdot \prod_{\vec{r} \in A'} \pi[\{S\}_{\vec{r}^1}] \cdots \prod_{\vec{r} \in A'} \pi[\{S\}_{\vec{r}^n}]
\]
This proves the inductive claim. We now show that for least and greatest fixed points,
\[
f_\omega(\vec{y}) = \bigsqcup_{n<\omega} \{\pi[S] \mid S \in \mathcal{W}(\varphi(\vec{y}))\}
\]
For $\varphi = [\mathbf{fp} R \vec{x}, \vartheta](\vec{y})$, this follows via Lemma 37 by swapping suprema:
\[
f_\omega(\vec{y}) = \bigsqcup_{n<\omega} f_n(\vec{y}) = \bigsqcup_{n<\omega} \bigsqcup_{\pi[S] \subseteq \mathcal{W}(\varphi(\vec{y}))} \{\pi[S] \mid S \in \mathcal{W}(\varphi(\vec{y}))\} = \bigsqcup_{n<\omega} \bigsqcup_{\pi[S] \subseteq \mathcal{W}(\varphi(\vec{y}))} \{\pi[S] \mid S \in \mathcal{W}(\varphi(\vec{y}))\}
\]
For $\varphi = [\mathbf{gfp} R \vec{x}, \vartheta](\vec{y})$, the proof is more difficult and requires the Puzzle Lemma. We first note that one direction is trivial (using Lemma 37 in the last step):
\[
\bigcap_{n<\omega} \{\pi[S] \mid S \in \mathcal{W}(\varphi(\vec{y}))\} \subseteq \bigcap_{n<\omega} \{\pi[S] \mid S \in \mathcal{W}(\varphi(\vec{y}))\} = \{\pi[S] \mid S \in \mathcal{W}(\varphi(\vec{y}))\}
\]
For the other direction, we use the characterization of infima of Proposition 23:
\[
f_\omega(\vec{y}) = \bigcap_{n<\omega} \bigcup_{\pi[S] \subseteq \mathcal{W}(\varphi(\vec{y}))} \{\pi[S] \mid S \in \mathcal{W}(\varphi(\vec{y}))\} = \bigcup_{n<\omega} \{m_n \mid (m_n)_{n<\omega} \in \mathcal{M}\}
\]
where $\mathcal{M}$ is defined as in the characterization, based on the chain $(P_n)_{n<\omega}$. By definition of $P_n$, we have $m_n = \pi[S_n] \subseteq \mathcal{M}$ for some strategy $S_n \in \mathcal{W}(\varphi(\vec{y}))$ (for each $n < \omega$). Consider one monomial chain $m = (m_n)_{n<\omega} \in \mathcal{M}$. By the Puzzle Lemma, there is a strategy $S_m \in \mathcal{W}(\varphi(\vec{y}))$ such that $\pi[S_m] \subseteq \bigcap_{n} m_n$. This implies the nontrivial direction:
\[
f_\omega(\vec{y}) = \bigcup_{n<\omega} \{m_n \mid m = (m_n)_{n<\omega} \in \mathcal{M}\} \subseteq \bigcup_{n<\omega} \{\pi[S_m] \mid m \in \mathcal{M}\}
\]
We have already shown that the fixed-point iteration has closure ordinal $\omega$, so $\pi[\varphi(\vec{y})] = f_\omega(\vec{y})$, finally closing the proof. \qed
Generalization

We generalize the result in $S^\infty[X]$ to all absorptive, fully continuous semirings. To this end, we first need two lemmas on the properties of fully continuous homomorphisms.

Lemma 41. Let $h : S^\infty[X] \to K$ be a semiring homomorphism (not necessarily fully continuous) into an absorptive, fully continuous semiring $K$. Then $\bigsqcup h(S) = h(\bigsqcup S)$ holds for arbitrary sets $S \subseteq S^\infty[X]$.

Proof. We first remark that the natural order on $K$ forms a complete lattice (by Proposition 7), so the supremum $\bigsqcup h(S)$ is well defined. Since $h$ preserves addition and thus the natural order, the direction $h(\bigsqcup S) \geq \bigsqcup h(S)$ is trivial. Let $\bigsqcup S = m_1 + \cdots + m_k$ for some monomials $m_1, \ldots, m_k$ and consider one monomial $m_i$. Since $\bigsqcup S$ is maximals ($\bigsqcup S$), there is a $P \in S$ with $m_i \in P$. Then $h(\bigsqcup S) \geq h(P) \geq h(m_i)$. This holds for each $1 \leq i \leq k$ and implies $h(\bigsqcup S) = h(m_1 + \cdots + m_k) = h(\bigsqcup S)$.

Lemma 42. Let $K_1$ and $K_2$ be absorptive, fully continuous semirings and let $h : K_1 \to K_2$ be a fully continuous semiring homomorphism. Let further $\pi$ be a $K_1$-interpretation, $\varphi \in LFP$ and $S$ a strategy in $G(\varphi)$. Then $h(\pi[S]) = (h \circ \pi)[S]$.

Proof. Clearly, $h$ preserves finite products. It thus suffices to prove that $h$ preserves infinitary powers. Let $a \in K_1$. Then $h(a^\infty) = h(\bigsqcap_i a^i) = \bigsqcap_i h(a^i) = \bigsqcap_i h(a)^i = h(a)^\infty$.

We can now finally prove the main result in its general formulation by considering the most general $S^\infty[X]$-interpretation together with the above lemmas.

Proof of Theorem 32. Consider the most general $S^\infty[X]$-interpretation $\pi^*$ with $\pi^*(L) = x_L$ and $X = \{x_L \mid L \in \text{Atoms}_A(\tau) \cup \text{NegAtoms}_A(\tau)\}$. By the universality theorem, there is a fully continuous homomorphism $h : S^\infty[X] \to K$ with $\pi = h \circ \pi^*$ (induced by the mapping $x_L \mapsto \pi(L)$). Then,

\[
\begin{align*}
\pi[\varphi(\overline{x})] &= h(\pi^*[\varphi(\overline{x})]) = h(\bigsqcup \{\pi^*[S] \mid S \in \mathcal{W}(\varphi(\overline{x}))\}) \\
&= \bigsqcup \{h(\pi^*[S]) \mid S \in \mathcal{W}(\varphi(\overline{x}))\} = \bigsqcup \{\pi[S] \mid S \in \mathcal{W}(\varphi(\overline{x}))\}.
\end{align*}
\]

7 Outlook

We have laid foundations for the semiring provenance analysis of full fixed-point logics, with arbitrary interleavings of least and greatest fixed points, as part of the general project of developing provenance semantics of logical languages used in various branches of computer science. Next steps will include the specific analysis of important logics such as temporal logics like CTL and CTL*, dynamic logics such as PDL, the modal $\mu$-calculus, description logics (see initial work in [3]) etc., and this will in particular have to take into account algorithmic and complexity theoretic issues.

A further interesting problem is the provenance analysis of parity games. In this paper, we have seen that provenance values of fixed-point formulae can be related to and understood as provenance values for parity games, but we have not analysed how provenance values of infinite plays and strategies are actually computed. This requires solving much more complicated equation systems that those for reachability games described in [9].
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A Omitted Proofs

For the sake of completeness, we present below few rather technical proofs we omitted from or only sketched in the main text.

**Proposition 8.** Provenance semantics for LFP is well-defined for fully chain-complete semirings.

It suffices to prove that update operators \( F_\pi^x \) are monotone, which guarantees the existence of their least and greatest fixed points by Proposition 8. Notice that \( F_\pi^x \) does not operate on the semiring \( K \), but on functions \( A^{\pi} \rightarrow K \). These functions form a semiring under pointwise operations that inherits most of the properties from \( K \), so Proposition 8 still applies. This also means that we compare two \( K \)-interpretations \( \pi_1 \) and \( \pi_2 \) pointwise, so \( \pi_1 \leq \pi_2 \) if, and only if, \( \pi_1(L) \leq \pi_2(L) \) for all literals \( L \); functions \( g : A^K \rightarrow K \) are compared in the same way. We split the monotonicity proof into two parts, which together imply Proposition 8.

**Lemma A1.** Let \( K \) be a fully chain-complete semiring and \( \vartheta(R, \pi) \) an LFP-formula. If \( \pi[\vartheta] \) is monotone in \( \pi \), then the update operator \( F_\pi^x \) is monotone.

**Proof.** Let \( k \) be the arity of \( R \) and let \( g_1, g_2 : A^k \rightarrow K \) with \( g_1 \leq g_2 \). To simplify notation, let \( g_1' = F_\pi^x(g_1) \) and \( g_2' = F_\pi^x(g_2) \). Due to \( g_1 \leq g_2 \), we also have \( \pi[R/g_1] \leq \pi[R/g_2] \). Then \( g_1' \leq g_2' \), as for all \( \pi \in A^k \): \( g_1'(\pi) = \pi[R/g_1][\vartheta(\pi)] \leq \pi[R/g_2][\vartheta(\pi)] = g_2'(\pi) \), due to the monotonicity assumption on \( \pi[\vartheta] \).

**Proposition A2.** Let \( K \) be a fully chain-complete semiring. Then \( \pi[\varphi] \) is monotone with respect to \( \pi \), so given two \( K \)-interpretations \( \pi_1 \) and \( \pi_2 \), the following implication holds for all LFP-sentences \( \varphi(\pi) \): \( \pi_1 \leq \pi_2 \implies \pi_1[\varphi(\pi)] \leq \pi_2[\varphi(\pi)] \)

**Proof.** Fix \( K \)-interpretations \( \pi_1 \leq \pi_2 \). The proof is an induction on the negation normal form of \( \varphi \).

- For literals, \( \pi_1[R(\pi)] = \pi_1(R(\pi)) \leq \pi_2(R(\pi)) = \pi_2[R(\pi)] \). The same holds for negative literals \( \varphi = \neg R(\pi) \) (and for equality atoms).
- If \( \varphi = \varphi_1 \lor \varphi_2 \), then \( \pi_1[\varphi] = \pi_1[\varphi_1] + \pi_1[\varphi_2] \) for \( i \in \{1, 2\} \) and the claim follows by induction and monotonicity of \( + \). The interpretation of \( \land \) via \( \cdot \) is analogous.
- If \( \varphi = \exists x. \vartheta(x) \), then \( \pi_1[\varphi] = \sum_{a \in A} \pi_1[\vartheta(a)] \). Recall that the universe \( A \) is finite, so the claim again follows by induction and monotonicity of \( + \). The case for \( \forall \) is analogous.
- If \( \varphi = \lfloor R. \vartheta(\pi) \rfloor \) where \( R \) has arity \( k \), we proceed by induction on the fixed-point iterations \( \lfloor g_{\beta} \rfloor \) for \( \pi_1 \) and \( \lfloor f_{\beta} \rfloor \) for \( \pi_2 \). By the induction hypothesis and the previous lemma, \( F_\pi^x \) and \( F_\pi^{\varphi} \) are monotone and hence the fixed-point iterations are well-defined. We prove by induction on \( \beta \) that \( g_{\beta} \leq f_{\beta} \) for all \( \beta \). We now give the proof for \( g_\beta \leq f_\beta \).

- \( x_0 = 0 = y_0 \).
- For \( \beta + 1 \), we have \( \pi_1[R/g_{\beta}] \leq \pi_2[R/f_{\beta}] \) by the induction hypothesis for \( \beta \). Applying the outer induction hypothesis for \( \vartheta \) then yields:

\[
g_{\beta+1}(\pi) = F_\pi^{\varphi}(g_{\beta})(\pi) = \pi_1[R/g_{\beta}][\vartheta(\pi)] \leq \pi_2[R/f_{\beta}][\vartheta(\pi)] = F_\pi^{\varphi}(f_{\beta}) = f_{\beta+1}
\]

- For limit ordinals \( \lambda \), we have \( g_{\lambda} = \bigsqcup \{ g_{\beta} \mid \beta < \lambda \} \leq \bigsqcup \{ f_{\beta} \mid \beta < \lambda \} = f_{\lambda} \) since we know that \( g_{\beta} \leq f_{\beta} \) for all \( \beta < \lambda \).

This ends the inner induction. For a sufficiently large ordinal \( \alpha \), we have

\[
\pi_1[\varphi(\pi)] = \lfloor F_\pi^{\varphi}(\pi) \rfloor = g_{\alpha}(\pi) \leq f_{\alpha}(\pi) = \lfloor F_\pi^{\varphi}(\pi) \rfloor = \pi_2[\varphi(\pi)]
\]

The proof for gfp-formulae is analogous.
Proposition 9. Let $K_1$, $K_2$ be fully chain-complete semirings and let $h : K_1 \rightarrow K_2$ be a fully continuous semiring homomorphism with $h(\top) = \top$. Then for every $K_1$-interpretation $\pi$, the mapping $h \circ \pi$ is a $K_2$-interpretation and for every $\varphi \in \text{LFP}$, we have $h(\pi[\varphi]) = (h \circ \pi)[\varphi]$.

As diagram:

\[
\begin{array}{ccc}
\text{LFP} & \xrightarrow{h} & K_2 \\
\downarrow h & \circ \pi & \downarrow h \\
K_1 & \xrightarrow{\pi} & \text{Lit}_A(\tau)
\end{array}
\]

Proof. We prove that for all LFP-formulae $\varphi(\overline{\tau})$ in negation normal form, $h(\pi[\varphi(\overline{\tau})]) = (h \circ \varphi)[\varphi(\overline{\tau})]$ holds for all $K$-interpretations $\pi$ and all tuples $\overline{\tau}$ from the universe $A$. The proof is by induction on the syntax of $\varphi$.

- For positive literals, we have $h(\pi[R \overline{\tau}]) = h(\pi[R \overline{\tau}]) = (h \circ \pi)(R \overline{\tau}) = (h \circ \pi)[R \overline{\tau}]$. The same holds for negative literals.
- For $\varphi = \varphi_1 \land \varphi_2$ (and, analogously, for $\lor$, $\exists$, $A$) we use that $h$ is a semiring homomorphism: $h(\pi[\varphi_1]) = h(\pi[\varphi_1]) \cdot h(\pi[\varphi_2]) = (h \circ \pi)[\varphi_1] \cdot (h \circ \pi)[\varphi_2] = (h \circ \pi)[\varphi]$.
- For $\varphi = [\text{gfp} \ R \overline{\tau}, \vartheta](\overline{\tau})$ with $R$ of arity $k$, we consider the fixed-point iteration $(g_\beta)_{\beta \in \text{On}}$ for $\pi$ in $K_1$ and the iteration $(f_\beta)_{\beta \in \text{On}}$ for $h \circ \pi$ in $K_2$. We show by induction that $h \circ g_\beta = f_\beta$ for all $\beta \in \text{On}$, so $h$ preserves all steps of the fixed-point iteration.
- For $\beta = 0$, we have $g_0 = f_0 : A^k \rightarrow K_1 \rightarrow \top$. Then $h \circ g_0 = f_0$, as $h(\top) = \top$.
- For successor ordinals, we can apply the induction hypothesis. Notice that

\[
\begin{align*}
g_{\beta+1}(\overline{\tau}) &= F^g_{\beta+1}(g_\beta)(\overline{\tau}) = \pi|R/g_\beta]\vartheta(\overline{\tau}) \\
f_{\beta+1}(\overline{\tau}) &= F^g_{h \circ \pi}(f_\beta)(\overline{\tau}) = (h \circ \pi)(R/g_\beta]\vartheta(\overline{\tau})
\end{align*}
\]

In $(\ast)$, we use the induction hypothesis $h \circ g_\beta = f_\beta$. Using the outer induction hypothesis on $\vartheta$, we obtain

\[
(h \circ g_{\beta+1})(\overline{\tau}) = h(\pi[R/g_\beta]\vartheta(\overline{\tau})) = (h \circ \pi[R/g_\beta]\vartheta(\overline{\tau})) = f_{\beta+1}(\overline{\tau})
\]

- For limit ordinals, we exploit that $h$ is fully continuous:

\[
h(\pi[\varphi(\overline{\tau})]) = h(\bigwedge \{g_\beta(\overline{\tau}) \mid \beta < \lambda\}) = \bigwedge \{h(g_\beta(\overline{\tau})) \mid \beta < \lambda\} = \bigwedge \{f_\beta(\overline{\tau}) \mid \beta < \lambda\} = f_\lambda(\overline{\tau})
\]

This closes the proof for $\text{gfp}$-formulae, as for sufficiently large $\beta$, we have

\[
h(\pi[\varphi(\overline{\tau})]) = h(g_\beta(\overline{\tau})) = f_\beta(\overline{\tau}) = (h \circ \pi)[\varphi(\overline{\tau})]
\]

The proof for $\text{lfp}$-formulae is analogous.

Proposition 15. A positive, fully chain-complete semiring $K$ is chain-positive if, and only if, the unique function $h : K \rightarrow \mathbb{B}$ with $h^{-1}(0) = \{0\}$ is a fully continuous semiring homomorphism. Every positive, chain-positive, fully chain-complete semiring is truth-preserving.

Proof. The second statement was already shown in the main text of the paper, so we only prove the first statement here. Let $h$ be defined as in the proposition. If $h$ is fully continuous and $C \subseteq K$ a chain of positive elements, then $h(\bigcap C) = \bigcap h(C) = \bigcap \{1\} = 1$. As $h(0) = 0$, it follows that $\bigcap |C| > 0$.

For the other direction, assume that $K$ is chain-positive. First note that $h$ is a semiring homomorphism, since $h(0) = 0$, $h(1) = 1$ and $h$ preserves addition and multiplication as
K is positive. To show that h is fully continuous, let C ⊆ K be a chain. If C = \{0\}, then h(\bigcup C) = h(0) = 0 = \bigcap \{0\} = \bigcup h(C). Otherwise, there is a c ∈ C with c > 0 and hence \bigcup h(C) ≥ h(c) = 1 and h(\bigcup C) ≥ h(c) = 1, so \bigcup h(C) = h(\bigcup C) = 1. For infima, first assume that 0 ∈ C. Then h(\bigcap C) = h(0) = 0 = \bigcap h(C). Otherwise, we have \bigcap h(C) = \bigcap \{1\} = 1 and h(\bigcap C) = 1 due to the chain-positivity of K.

\section{On Infinite Products of Plays}

In Sect. 4, we have introduced the provenance value π[\mathcal{S}] of a strategy \mathcal{S} informally as the (possibly infinite) product over the values π[\rho] of all \rho ∈ \text{Plays}(\mathcal{S}). The formal definition of π[\mathcal{S}^\iota], on the other hand, avoids infinite products by grouping together those plays with the same outcome. Here we discuss how infinite products can be formulated in our setting of absorptive, fully continuous semirings, in order to justify our definition of \pi[\mathcal{S}^\iota]. We restrict our interest to products over countable domains which are relatively easy to analyse (in fact, finite domains would suffice as there are only finitely many different outcomes of plays).

For this section, we always assume the following setting. Let K be an absorptive, fully continuous semiring and let A ⊆ K be a countable set. Let I be an arbitrary index set and \((x_i)_{i \in I}\) a family over A, that is, with \(x_i \in A\) for all \(i \in I\). Notice that a single element \(a \in A\) can occur several times, or even infinitely often, in the family \((x_i)_{i \in I}\) and we allow index sets I of arbitrary cardinality. Since A is countable, we fix an enumeration \(A = \{a_0, a_1, a_2, \ldots\}\) (not to be confused with the elements \(x_i\) of the family). We are interested in the (possibly infinite) product of the family \((x_i)_{i \in I}\) which we denote by \(\prod_{i \in I} x_i\).

\subsection{Definition via Finite Subproducts}

In [5], infinite sums in complete monoids are defined using the supremum over all finite subsums. Here we propose an analogous definition for infinite products.

\[\prod_{i \in I} x_i := \bigcap \{ \prod_{i \in F} x_i \mid F \subseteq I, F \text{ finite} \}\]

We write \(\hat{\prod}\) to clearly distinguish this product from the usual finite product \(\prod\). To simplify notation, we write \(F \subseteq_{\text{fin}} I\) to express that \(F\) is a finite subset of \(I\) and use the abbreviation \(x_F := \prod_{i \in F} x_i\) for finite subproducts. Then, \(\prod_{i \in I} x_i = \bigcap\{x_F \mid F \subseteq_{\text{fin}} I\}\).

Some properties of infinite products are immediate, based on this definition. Further properties and an alternative definition are discussed in the remainder of this section.

\begin{proposition}

The infinite product (in the setting described above)

\begin{enumerate}
\item coincides with finite multiplication, i.e., \(\prod_{i \in I} x_i = \prod_{i \in I} x_i\) for finite \(I\),
\item is commutative, i.e., \(\prod_{i \in I} x_i = \prod_{i \in I} x_{f(i)}\), where \(f : I \to I\) is a bijection,
\item satisfies \(\prod_{i \in \emptyset} x_i = 1\), and \(\prod_{i \in I} x_i = 0\) if \(x_i = 0\) for some \(i \in I\).
\end{enumerate}

\end{proposition}

\begin{proof}

For (1), assume that \(I\) is finite and \(F \subseteq I\). Then \(x_F \geq x_I\) because of absorption, hence \(\prod_{i \in I} x_i = \bigcap\{x_F \mid F \subseteq_{\text{fin}} I\} = x_I = \prod_{i \in I} x_i\). For (2), consider some \(F \subseteq_{\text{fin}} I\) and let \(F' = f^{-1}(F)\). Then \(F' \subseteq_{\text{fin}} I\) and \(\prod_{i \in F} x_i = \prod_{i \in F'} x_{f(i)}\). Writing \(x_{F'}\) for \(\prod_{i \in F} x_{f(i)}\), it follows that \(\prod_{i \in I} x_i = \bigcap\{x_{F'} \mid F' \subseteq_{\text{fin}} I, F' = f^{-1}(F)\} = \bigcap\{x_F \mid F \subseteq_{\text{fin}} I\} = \prod_{i \in I} x_{f(i)}\).

For (3), \(\prod_{i \in \emptyset} x_i = x_{\emptyset} = 1\). If \(x_i = 0\) then \(x_{(i)} = 0\) and hence \(\prod\{x_F \mid F \subseteq_{\text{fin}} I\} \leq 0\).
\end{proof}
B-II Products as Chains

In order to justify our definition of $\pi[S]$, we need a form of associativity for infinite products. Consider a partition $I = I_1 \cup I_2$. We then need $\prod_{i \in I} x_i = (\prod_{i \in I_1} x_i) \cdot (\prod_{i \in I_2} x_i)$ to group together plays with identical outcome.

Since the product is defined as an infimum, this is related to the full continuity of multiplication. However, recall that continuity applies only to infima of chains. We therefore reformulate the product $\prod_{i \in I} x_i$ as the infimum of an $\omega$-chain. Here we rely on $A$ being countable, so $A = \{a_0, a_1, \ldots\}$. We need some further notation. For $a \in A$ and $F \subseteq I$ we write $\#_F(a)$ for the number of occurrences of $a$ in $(x_i)_{i \in F}$, that is, $\#_F(a) = \{|i \in F \mid x_i = a\}$ which we understand as a number in $\mathbb{N} \cup \{\infty\}$. In case of $F = I$, we simply write $\#(a)$ for $\#_I(a)$. Using this notation, we define the $\omega$-chain $(\tilde{x}_n)_{n<\omega}$ as follows:

$$\tilde{x}_n := \prod_{0 \leq k < \min(n, |A|)} (a_k)^{\#_k}$$

Notice that $(\tilde{x}_n)_{n<\omega}$ is indeed a descending chain because of absorption. Moreover, each finite subset of $A$ is eventually contained in the elements $a_0, \ldots, a_{n-1}$ considered in $\tilde{x}_n$, which leads to the following observation.

**Proposition B4.** It holds that $\prod_{i \in I} x_i = \prod_{n<\omega} \tilde{x}_n$.

**Proof.** Recall that $\prod_{i \in I} x_i = \bigcap\{x_F \mid F \subseteq \text{fin} \, I\}$. We show both directions of the equality. We first fix any $F \subseteq \text{fin} \, I$. Since $F$ is finite, there is $n < \omega$ such that $\{x_i \mid i \in F\} \subseteq \{a_0, \ldots, a_{n-1}\}$. Then $x_F \geq \tilde{x}_n$, by comparing the exponents of each $a_k$ (with $k < n$) in $x_F$ and $\tilde{x}_n$. It follows that $\prod_{i \in I} x_i \geq \prod_{n<\omega} \tilde{x}_n$.

For the other direction, fix $n$ and consider $\tilde{x}_n$ as defined above. Intuitively, we want to choose an appropriate set $F \subseteq \text{fin} \, I$ that contains sufficiently many occurrences of the values $a_0, \ldots, a_{n-1}$. However, in case of $\#a_k = \infty$ this is not possible. Instead, we define a family $(F_j)_{j<\omega}$ of increasingly large finite subsets of $I$. For each $j$, choose $F_j \subseteq \text{fin} \, I$ in such a way that for all $k < n$, we have $\#_{F_j}(a_k) = \{|i \in F_j \mid x_i = a_k\} = \min(j, \#a_k)$. Then, the finite products $(x_{F_j})_{j<\omega}$ form a chain and we can apply the Splitting Lemma to see that $\bigcap_{j<\omega} x_{F_j} = \tilde{x}_n$. It follows that $\prod_{i \in I} x_i = \bigcap\{x_F \mid F \subseteq \text{fin} \, I\} \leq \bigcap\{x_{F_j} \mid j < \omega\} = \tilde{x}_n$. This holds for all $n < \omega$, hence $\prod_{i \in I} x_i \leq \prod_{n<\omega} \tilde{x}_n$.

We have reformulated the infinite product as infimum of an $\omega$-chain and can now exploit that multiplicity is fully continuous to prove further properties of infinite products.

**Proposition B5 (Associativity).** For each partition $I = I_1 \cup I_2$, it holds that

$$\prod_{i \in I} x_i = (\prod_{i \in I_1} x_i) \cdot (\prod_{i \in I_2} x_i).$$

In particular, given $c \in A$ it holds that $c \cdot \prod_{i \in I} x_i = \prod_{i \in I \cup \{c\}} x_i$ where $x_* = c$.

**Proof.** The definition of the chain $(\tilde{x}_n)_{n<\omega}$ for the product $\prod_{i \in I} x_i$ on the left-hand side depends on the values $\#_a = \#_I(a)$ for $a \in A$. For the two products on the right-hand side, let $(\tilde{x}_{n}^{(1)})_{n<\omega}$ and $(\tilde{x}_{n}^{(2)})_{n<\omega}$ be the corresponding chains that are defined analogously using $\#_{I_1}(a)$ and $\#_{I_2}(a)$, respectively.

First observe that $\#_a = \#_{I_1}(a) + \#_{I_2}(a)$ for each $a \in A$. Since we fixed an enumeration of $A$, it follows from the definitions of the three chains that $\tilde{x}_n = \tilde{x}_n^{(1)} \cdot \tilde{x}_n^{(2)}$ for each $n < \omega$.
Proposition B6. We revisit the proof of Proposition B5. If \( \delta \) and we define the (infinite) product as

\[
\hat{\prod}_{i \in I} x_i = \left( \prod_{n < \omega} \hat{x}_n^{(1)} \right) \cdot \left( \prod_{n < \omega} \hat{x}_n^{(2)} \right) = \prod_{n < \omega} \hat{x}_n = \hat{\prod}_{i \in I} x_i
\]

The statement on multiplication with \( \vdash \) follows by considering the partition \( I' = \{ \star \} \cup I \). \( \blacktriangleleft \)

The last property we consider is compatibility with semiring homomorphisms \( h : K \to K' \). Finite products are preserved by \( h \) and we can generalize this to infinite products if we require \( h \) to be fully continuous. This applies in particular to the homomorphisms induced by polynomial evaluation in \( S^\infty[\mathbb{X}] \). Recall that \( K \), and now also \( K' \), are absorptive and fully continuous.

\textbf{Proposition B7.} Let \( h : K \to K' \) be a fully continuous homomorphism. Then,

\[
h\left( \prod_{i \in I} x_i \right) = \hat{\prod}_{i \in I} h(x_i).
\]

\textbf{Proof.} We revisit the proof of \( \prod_{i \in I} x_i = \prod_{n < \omega} \hat{x}_n \) in Proposition B5. For each \( F \subseteq \mathbb{N} \), we have shown that there is an \( n \) with \( x_F \geq \hat{x}_n \). Hence \( h(x_F) \geq h(x_n) \) by monotonicity of semiring homomorphisms.

Conversely, for each \( n \) we have constructed a family \( (F_j)_{j \omega} \) such that \( \prod_{j < \omega} x_{F_j} = \hat{x}_n \).

Using the continuity of \( h \), it follows that \( h(\hat{x}_n) = h(\prod_{j < \omega} x_{F_j}) = \prod_{j < \omega} h(x_{F_j}) \geq \prod \{ h(x_F) \mid F \subseteq \mathbb{N} \} \).

Combining both directions, we get \( \prod \{ h(x_F) \mid F \subseteq \mathbb{N} \} = \prod_{n < \omega} h(\hat{x}_n) \).

The proof of the proposition now simply follows from the continuity assumption:

\[
h\left( \prod_{i \in I} x_i \right) = h\left( \prod_{n < \omega} \hat{x}_n \right) = \prod_{n < \omega} h(x_n) = \prod \{ h(x_F) \mid F \subseteq \mathbb{N} \}
\]

Clearly, \( h(x_F) = \prod_{i \in F} h(x_i) \) and it follows that \( \prod \{ h(x_F) \mid F \subseteq \mathbb{N} \} = \prod_{i \in I} h(x_i). \) \( \blacktriangleleft \)

\textbf{B-III Alternative Definition}

Infinite products can be defined in different ways and while the definition in terms of finite subproducts is natural, one could also define the product recursively. We show that the notion of infinite products is robust by observing that the recursive definition is equivalent. Let \( (x_\beta)_{\beta < \alpha} \) be a family over \( A \) indexed by an ordinal \( \alpha \in \text{On} \). The (possibly infinite) product of this family can be defined recursively by

\[
\pi_0 := 1,
\]

\[
\pi_{\beta + 1} := x_\beta \cdot \pi_\beta,
\quad \text{for ordinals } \beta \in \text{On},
\]

\[
\pi_\lambda := \prod \{ \pi_\beta \mid \beta < \lambda \},
\quad \text{for limit ordinals } \lambda \in \text{On},
\]

and we define the (infinite) product as \( \prod_{\beta < \alpha} x_\beta := \pi_\alpha \).

\textbf{Proposition B7.} In the above setting, \( \prod_{\beta < \alpha} x_\beta = \prod_{\beta \in \alpha} x_\beta \).

\textbf{Proof.} We prove by induction that \( \pi_\delta = \prod_{\beta \in \delta} x_\beta \) holds for all \( \delta \leq \alpha \). For \( \delta = 0 \), we have shown \( \prod_{\beta \in \delta} x_\beta = 1 \) above. For \( \delta + 1 \), we have \( x_\delta \cdot \pi_\delta = x_\delta \cdot \prod_{\beta \in \delta} x_\beta = \prod_{\beta \in \delta + 1} x_\beta \) by Proposition B5. If \( \delta \) is a limit ordinal, then \( \prod \{ \pi_\beta \mid \beta < \delta \} = \prod \{ \prod_{\gamma \in \beta} x_\gamma \mid \beta < \delta \} \).

By applying the definition of the infinite product in terms of the infimum over all finite subproducts, this is equal to \( \prod \{ x_F \mid F \subseteq \mathbb{N}, \beta < \delta \} = \prod \{ x_F \mid F \subseteq \mathbb{N}, \delta \} = \prod_{\beta \in \delta} x_\beta. \) \( \blacktriangleleft \)
B-IV Back to Games: Products of Plays

The infinite product we defined and analysed above can be used to properly define the provenance values of strategies as products over all plays. Recall that the value \( \pi[\rho] \) of a play \( \rho \) with outcome \( L \) is simply the semiring value \( \pi(L) \), and the value of infinite plays is either 0 or 1. The value of a strategy \( S \) was defined as

\[
\pi[S] := \begin{cases} 
\prod_{L \in \text{Lit}(\tau)} \pi(L)^{\#_S(L)} & \text{if all infinite } \rho \in \text{Plays}(S) \text{ are winning for Verifier} \\
0 & \text{otherwise.}
\end{cases}
\]

We first check that our setting applies: In Sect. 6, we assumed an absorptive, fully continuous semiring \( K \). We further always assume the universe to be finite. In particular, the number of literals and hence also the number of different values \( \pi[\rho] \) is finite and thus countable. The number of plays, on the other hand, can well be infinite and even uncountable. We can thus model the product over all values \( \pi[\rho] \) as an infinite product which finally justifies our definition of \( \pi[S] \) in all absorptive, fully continuous semirings.

**Proposition B8.** In the setting of Sect. 6 it holds that

\[
\pi[S] = \prod_{\rho \in \text{Plays}(S)} \pi[\rho]
\]

**Proof.** First observe that whenever there is an infinite play that is losing for Verifier, such that \( \pi[S] = 0 \), this play has the value \( \pi[\rho] = 0 \) and thus also \( \prod_{\rho \in \text{Plays}(S)} \pi[\rho] = 0 \). If no such play exists, we group plays with identical outcome, making use of the associativity of infinite products (see Proposition B5). For each literal \( L \), let \( \text{Plays}_L(S) \) be the set of plays with outcome \( L \). We further write \( \text{Plays}_1(S) \) for the set of infinite plays that are winning for Verifier. If we denote the finite set of literals by \( \{L_1, \ldots, L_n\} \), we obtain the partition \( \text{Plays}(S) = \text{Plays}_1(S) \cup \text{Plays}_{L_1}(S) \cup \cdots \cup \text{Plays}_{L_n}(S) \) and can apply Proposition B5:

\[
\prod_{\rho \in \text{Plays}(S)} \pi[\rho] = \prod_{\rho \in \text{Plays}_1(S)} \pi[\rho] \cdot \prod_{L \in \text{Lit}(\tau)} \left( \prod_{\rho \in \text{Plays}_L(S)} \pi[\rho] \right)
\]

\[
= \prod_{\rho \in \text{Plays}_1(S)} 1 \cdot \prod_{L \in \text{Lit}(\tau)} \left( \prod_{\rho \in \text{Plays}_L(S)} \pi(L) \right)
\]

Infinite products of a single value, that is, \( \prod_{i \in I} a_i \) with \( a_i = a \) for all \( i \in I \), have the value \( \prod\{x_F \mid F \subseteq \text{fin} \} = \prod\{a[F] \mid F \subseteq \text{fin} \} = a^\infty \). We can thus conclude

\[
\prod_{\rho \in \text{Plays}(S)} \pi[\rho] = 1 \cdot \prod_{L \in \text{Lit}(\tau)} \left( \pi(L)^{\#_S(L)} \right) = \pi[S].
\]

Consider the rather artificial formula \( \varphi(u) = \text{gfp } Rx. Rx \land Rx(u) \) which allows Falsifier to repeatedly make a binary choice in the corresponding game.