On the Jacobson element and generators of the Lie algebra $\mathfrak{grt}$ in nonzero characteristic

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Abstract

We state a conjecture (due to M. Duflo) analogous to the Kashiwara–Vergne conjecture in the case of a characteristic $p > 2$, where the role of the Campbell–Hausdorff series is played by the Jacobson element. We prove a simpler version of this conjecture using Vergne’s explicit rational solution of the Kashiwara–Vergne problem. Our result is related to the structure of the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ in characteristic $p$: we conjecture existence of a generator of $\mathfrak{grt}$ in degree $p - 1$, and we provide this generator for $p = 3$ and $p = 5$.

Let $\mathfrak{lie}_2$ be a free Lie algebra over a field $\mathbb{K}$ of characteristic zero with generators $x$ and $y$. It is a graded Lie algebra $\mathfrak{lie}_2 = \prod_{k=1}^{\infty} \mathfrak{lie}_k^2$, where $\mathfrak{lie}_2^k$ is spanned by the Lie words consisting of $k$ letters. We denote by $z = \log e^x e^y$ be the Campbell–Hausdorff series:

$$z = \sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \sum_{i,j} \frac{x^{i_1} y^{j_1} \cdots x^{i_k} y^{j_k}}{i_1! j_1! \cdots i_k! j_k!},$$

(1)

where $i = (i_1, \ldots, i_k)$, $j = (j_1, \ldots, j_k)$, $i_m, j_m \in \mathbb{Z}$ and $i_m + j_m > 0$ for all $m$, and $i_1 + \cdots + i_k + j_1 + \cdots + j_k = n$.

Let $\text{Assoc}_2$ be the associative algebra with generators $x$ and $y$, and let $\tau : \mathfrak{lie}_2 \to \text{Assoc}_2$ be the natural injection from the Lie algebra to its universal enveloping algebra. Every element $a$ in $\text{Assoc}_2$ admits a unique presentation $a = a_0 + a_1 x + a_2 y$, where $a_0 \in \mathbb{Q}$ and $a_1, a_2 \in \text{Assoc}_2$. We shall denote $a_1 = \partial_x a, a_2 = \partial_y a$.

We define the graded vector space of circular words $tr_2$ as the quotient

$$tr_2 = \text{Assoc}_2^+ / \langle (ab - ba), \ a, b \in \text{Assoc}_2 \rangle,$$

where $\text{Assoc}_2^+ = \prod_{k=1}^{\infty} \text{Assoc}^k(x, y)$ and $\langle (ab - ba), \ a, b \in \text{Assoc}_2 \rangle$ is the subspace of $\text{Assoc}_2^+$ spanned by commutators. We denote by $\tilde{\tau} : \text{Assoc}_2 \to tr_2$ the corresponding natural projection.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, and let $\rho : \mathfrak{g} \to \text{End}(V)$ be a finite dimensional representation. Then, each element $\tilde{\tau}(a)$ in $tr_2$ gives rise to a map $\rho_a : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by the formula $\rho_a(x, y) = \text{Tr}(\rho(a(x, y)))$.

The Kashiwara–Vergne conjecture [8] (now a theorem [1]) is an important problem of Lie theory which in particular implies the Duflo isomorphism [4] between the center of the universal enveloping algebra and the ring of invariant polynomials. The conjecture states that there exist elements $F(x, y)$ and $G(x, y)$ in $\mathfrak{lie}_2$ such that

$$x + y - \log e^x e^y = (e^{ad x} - 1) F(x, y) + (1 - e^{-ad y}) G(x, y)$$

(2)
and
\[ \tilde{\text{tr}}(x\partial_x F + y\partial_y G) = \frac{1}{2} \text{tr} \left( \frac{x}{e^x - 1} + \frac{y}{e^y - 1} - \frac{z}{e^z - 1} - 1 \right). \] (3)

Since the statement of the Kashiwara–Vergne conjecture uses the exponential function, it can only be defined over a field of characteristic zero. Michel Duflo [5] suggested the following question which resembles the Kashiwara–Vergne conjecture in the case of a positive characteristic. Let \( p > 2 \) be a prime, and let \( K \) be a field of characteristic \( p \).

**Conjecture 1** There exist \( A(x, y) \) and \( B(x, y) \) in \( \mathfrak{lie}_2 \) over \( K \) such that
\[ [x, A(x, y)] + [y, B(x, y)] = x^p + y^p - (x + y)^p \] (4)

and
\[ \tilde{\text{tr}}(x\partial_x A + y\partial_y B) = \frac{1}{2} \text{tr}(x^{p-1} + y^{p-1} - (x + y)^{p-1}). \] (5)

Note that \( x^p + y^p - (x + y)^p \) is the Jacobson element (see e.g. [9]) in \( \mathfrak{lie}_2 \) over \( K \). We will prove a simplified version of Conjecture 1. For an arbitrary element \( a = x_{i1} \cdots x_{im} \in \text{Assoc}_2 \), we put \( a^T = (-x_{1i}) \cdots (-x_{im}) \). Consider the quotient of \( \mathfrak{t}_2 \) by the relations \( \text{tr}(a) = \text{tr}(a^T) \). We denote by \( \text{tr} \) the projection from \( \text{Assoc}_2 \) to the above quotient. Let \( \mathfrak{g} \) be a Lie algebra over \( K \) and let \( \rho \) be a finite dimensional representation of \( \mathfrak{g} \) with the property \( \rho(x)^t = -\rho(x) \) (here \( \rho(x)^t \) stands for a transposed matrix). Then, the map \( \rho_n \) only depends on \( \text{tr}(a) \). For instance, that is the case of the adjoint representation of the quadratic Lie algebra (a Lie algebra equipped with a non-degenerate invariant symmetric bilinear form). Hence, we refer to the "quadratic" case of Conjecture 1.

**Proof. (in the quadratic case)** We use the following simple facts from number theory.

**Lemma 1.** (Wilson’s Theorem) \( (p - 1)! = -1 \mod p \).

Let \( B_m \) be the \( m \)-th Bernoulli number.

**Lemma 2.** Let \( p \) be a prime and \( m \) be an even number. If \( (p - 1) \nmid m \), then \( B_m \) is a \( p \)-integer. If \( (p - 1)|m \), then \( pB_m \) is a \( p \)-integer and \( pB_m = -1 \mod p \).

See [7] for the proofs of these lemmas.

In [10] Vergne gave the following explicit solution of the Kashiwara–Vergne problem (in the quadratic case). Consider the functions
\[ \Theta(t) = \frac{1 - e^{-t}}{t}, \quad R(t) = \frac{e^t - e^{-t} - 2t}{t^2}. \]

Let \( \mathcal{R} \) be the derivation of \( \mathfrak{lie}_2 \) such that \( \mathcal{R}|_{\mathfrak{lie}_2^2} = n\text{Id}|_{\mathfrak{lie}_2^2} \). The solutions of the Kashiwara–Vergne problem are given by
\[ F(x, y) = -\Theta(-\text{ad}x)^{-1}U(x, y), \quad G(x, y) = -\Theta(-\text{ad}y)^{-1}V(x, y), \]
where \( U \) and \( V \) are defined by the equations
\[(\mathcal{R} + 1)U(x, y) = -\frac{1}{2}\Theta(\text{ad}x)\Theta(\text{ad}x)^{-1}R(\text{ad}x)(\Theta(-\text{ad}z)^{-1}x + \Theta(\text{ad}z)^{-1}y) + \frac{1}{2}\Theta(-\text{ad}x)y, \] (6)
\[(\mathcal{R} + 1)V(x, y) = -\frac{1}{2}\Theta(-\text{ad}y)\Theta(-\text{ad}z)^{-1}R(\text{ad}z)(\Theta(-\text{ad}z)^{-1}x + \Theta(\text{ad}z)^{-1}y) - \frac{1}{2}\Theta(\text{ad}y)x. \] (7)

Let \( p \) be a prime. It is easy to show that the lowest homogeneous degree in \( x \) and \( y \) of a term of \( F \) with non-\( p \)-integer coefficient is \( p - 1 \). We expand the \((p - 1)\)-st homogeneous component \( F_{p-1} \) of \( F \) in powers of \( p \): \( F_{p-1} = \sum_{n=\infty}^{\infty} f_n p^n \). It is easy to see that the lowest power of \( p \) in this...
expansion is \(-2\): the \(\frac{1}{p}\) coming from \(R(\text{ad}z) = 2(\text{ad}z)^2 + (\text{ad}z)^3 + (\text{ad}z)^5 + \ldots\) is multiplied by the \(\frac{1}{p}\) coming from the inverse of \(R + 1\). However, the following computation shows that the coefficient \(f_{-2}\) is actually equal to zero. By the definition of the Campbell–Hausdorff series (1) we see that \(z_1 = x + y\). We have
\[
\frac{f_{-2}}{p^2} = -\frac{1}{2} \cdot \frac{1}{p} \cdot \frac{(\text{ad}z_1)^{p-2}}{p!}(x + y),
\]
and so
\[
f_{-2} = -\frac{1}{2} \frac{(\text{ad}z_1)^{p-2}}{(p-1)!}(x + y) = -\frac{1}{2} \frac{(\text{ad}(x + y))^{p-2}}{(p-1)!}(x + y) = 0.
\]
Thus, the \(p\)-adic expansion of \(F_{p-1}\) has the form \(F_{p-1} = \frac{f_{-1}}{p} + \sum_{n=0}^{\infty} f_n p^n\). The same calculation for \(G\) gives \(G_{p-1} = \frac{g_{-1}}{p} + \sum_{n=0}^{\infty} g_n p^n\). Consider the \(p\)-th homogeneous part of equation (2). The left-hand side yields
\[
\frac{\text{ad}x}{1!} F_{p-1} + \frac{\text{ad}x^2}{2!} F_{p-2} + \ldots + \frac{\text{ad}(x^{p-1})}{(p-1)!} F_1 + \frac{-\text{ady}}{1!} G_{p-1} + \frac{(-\text{ady})^2}{2!} G_{p-2} + \ldots + \frac{(-\text{ady})^{p-1}}{(p-1)!} G_1,
\]
and the right-hand side is of the form
\[
-\sum_{k=1}^{p} \frac{(-1)^{k-1} k}{k} \sum_{i,j} x^{i_1} y^{j_1} \ldots x^{i_k} y^{j_k},
\]
where \(i = (i_1, \ldots, i_k), j = (j_1, \ldots, j_k), i_m, j_m \in \mathbb{Z}\) and \(i_m + j_m > 0\) for all \(m\), and \(i_1 + \ldots + i_k + j_1 + \ldots + j_k = p\). Expanding the above expressions in powers of \(p\) and comparing the coefficients at \(\frac{1}{p}\), we have
\[
\text{ad}x \cdot f_{-1} + \text{ady} \cdot g_{-1} = -\frac{x^p + y^p}{(p-1)!} - (x + y)^p \mod p.
\]
Using Lemma (1), we obtain
\[
\text{ad}x \cdot f_{-1} + \text{ady} \cdot g_{-1} = x^p + y^p - (x + y)^p \mod p.
\]
Next, consider equation (3). It is easy to see that the lowest homogeneous degree in \(x\) and \(y\) with non-\(p\)-integer coefficient is \(p - 1\). Consider the \((p - 1)\)-st homogeneous part of the equation and expand it in powers of \(p\). Comparing the coefficients at \(\frac{1}{p}\), we obtain
\[
\text{tr}(x \partial_x f_{-1} + y \partial_y g_{-1}) = \frac{p B_{p-1}}{2(p-1)!} (x^{p-1} + y^{p-1} - (x + y)^{p-1}) \mod p.
\]
Using Lemma (2) we obtain
\[
\text{tr}(x \partial_x f_{-1} + y \partial_y g_{-1}) = \frac{1}{2} (x^{p-1} + y^{p-1} - (x + y)^{p-1}) \mod p.
\]
We put \(A(x, y) = f_{-1}(x, y)\) and \(B(x, y) = g_{-1}(x, y)\).

\textbf{Remark.} In order to use a similar strategy for proving Conjecture 1 in the general case we need a control of the \(1/p\) behavior of coefficients of a rational solution of the Kashiwara–Vergne conjecture. The solution of [1] uses Kontsevich integrals over configuration spaces, and \textit{a priori} it is defined over \(\mathbb{R}\). The existence of rational solution follows by linearity, but there is no control over coefficients.

In [2], the Kashiwara–Vergne problem was related to the theory of Drinfeld’s associators. By analogy, this relation suggests a link between Conjecture 1 and the structure of the Grothendieck–Teichmüller Lie algebra over a field of characteristic \(p\).
Definition 3. The algebra $t_n$ is the quotient of the free Lie algebra with $n(n - 1)/2$ generators $t^i, j = t^j, i$ by the following relations

$$[t^i, j, k] = 0$$

if all indices $i, j, k$ are distinct, and

$$[t^i, j + t^j, k, t^i, k] = 0$$

for all triples of distinct indices $i, j,$ and $k$.

Below we will use the following statement (see [3]).

Lemma 4. $t_4 \cong \mathbb{K}t^{1,2} \oplus \mathfrak{lie}(t^{1,2}, t^{2,3}) \oplus \mathfrak{lie}(t^{1,4}, t^{2,4}, t^{3,4})$, where $\mathfrak{lie}(t^{1,4}, t^{2,4}, t^{3,4})$ is an ideal acted by $\mathbb{K}t^{1,2} \oplus \mathfrak{lie}(t^{1,2}, t^{2,3})$, and $\mathfrak{lie}(t^{1,3}, t^{2,3})$ is an ideal in $\mathbb{K}t^{1,2} \oplus \mathfrak{lie}(t^{1,2}, t^{2,3})$ acted by $\mathbb{K}t^{1,2}$.

Definition 5. The Grothendieck–Teichmüller Lie algebra is the Lie algebra spanned by the elements $\psi \in \mathfrak{lie}_2$ satisfying the following relations:

$$\psi(x, y) = -\psi(y, x),$$

$$\psi(x, y) + \psi(y, z) + \psi(z, x) = 0,$$

where $z = -x - y$,

$$\psi(t^{1,2}, t^{2,3}) + \psi(t^{1,3}, t^{2,3}) = \psi(t^{2,3}, t^{3,4}) + \psi(t^{1,2}, t^{2,3}),$$

where the latter takes place in $t_4$ and $t^{i,j,k} = t^i, j + t^j, k$.

The Deligne–Drinfeld conjecture [3] states that, over a field of characteristic zero, $\mathfrak{grt}$ is a graded free Lie algebra with generators $\sigma_{2n-1}, n = 1, 2, \ldots$ of degree $\deg(\sigma_{2n-1}) = 2n - 1$. This conjecture is numerically verified up to degree 16. Consider the algebra $\mathfrak{grt}$ over a field of characteristic $p > 2$. Conjecture 2 (see below) suggests existence of a generator of $\mathfrak{grt}$ in the degree $p - 1$.

Consider the function $\psi(x, y)$ such that $\psi(-x - y, x) = A(x, y)$ and $\psi(-x - y, y) = B(x, y)$, where $A$ and $B$ are solutions of (15). Such a function exists because the solutions of (15) can always be chosen symmetric: $A(x, y) = B(y, x)$.

We define another grading on $\mathfrak{lie}_2$. The depth of a Lie monomial is defined as the number of $y$’s entering this monomial. The depth of a Lie polynomial is the smallest depth of its monomials.

Lemma 6. The polynomial $\psi(x, y)$ is of depth one.

Proof. By definition, we have $\psi(x, y) = A(y, -x - y)$, so we must prove that $A(y, -x - y)$ is of depth one, i.e., that $A(x, y) = c \text{ad}_y^{p-2} x + \ldots$, where $c \neq 0$. In equation (14), we consider the homogeneous part of degree $p - 2$ in $y$:

$$[x, A_{xy}^{p-2}] + [y, B_{zxy}^{p-1}] = (x^p + y^p - (x + y)^p)_{x^2 y^{p-2}},$$

where the indices $x^i y^{p-i}$ denote the corresponding homogeneous degree parts of the expressions. By the definition of the Campbell–Hausdorff series, we have $A_{xy}^{p-2} = c \text{ad}_y^{p-2} x$ and $B_{zxy}^{p-1} = \sum b_{i,j} \text{ad}_y^i \text{ad}_x^j y$, where the sum is taken over all $l_1$ and $l_2$ such that $l_1 \geq 0$, $l_2 \geq 0$, and $l_1 + l_2 = p - 2$.

Suppose $c = 0$. Then $(x^p + y^p - (x + y)^p)_{x^2 y^{p-2}} = [y, B_{zxy}^{p-1}]$. Consider the injection $\tau$ to the universal enveloping algebra. The image under $\tau$ of the right-hand side of the above equation is a sum of monomials either beginning or ending with $y$, so it does not contain the monomial $x y^{p-2} x$, whereas the left-hand side of this equation does contain such a monomial. Thus, $c \neq 0$, and so $\psi$ is of depth one.
Conjecture 2  The function \( \psi(x, y) \) belongs to \( \grt \).

We verify this conjecture for \( p = 3 \) and \( p = 5 \).

The case \( p = 3 \).

The solution of (4,5) is given by \( A(x, y) = [y, x] \) and \( B(x, y) = [x, y] \). Then \( \psi(x, y) = -[x, y] \).

We verify conditions (10-12).

Condition (10). We have \(-\psi(y, x) = [y, x] = -[x, y] = \psi(x, y)\).

Condition (11). We have
\[
\psi(x, y) + \psi(y, -x - y) + \psi(-x - y, x) = -[x, y] - [y, -x - y] - [-x - y, x] = -[x, y] + [y, x] + [y, x] = -3[x, y] = 0 \mod 3.
\]

Condition (12). We write \((ij)\) for \( t_{i,j} \). We have
\[
\psi((12) + (23), (24)) + \psi((13) + (23), (34)) - \psi((12), (23) + (24)) - \psi((13), (23) + (34)) + \psi((12), (23)) = 0.
\]

The case \( p = 5 \).

The solution of (4,5) is given by \( A(x, y) = [x, [x, y]] + [y, [x, y]] + 2[y, [y, x]] \) and \( B(x, y) = [y, [y, [y, x]] + [x, [y, [y, x]]] + 2[x, [x, [y, x]]]] \). Then \( \psi(x, y) = 2[x, [x, [y, x]]] + 2[y, [y, [y, x]]] + 3[y, [x, [y, x]]] \).

Let us verify the conditions (10,12).

Condition (10). We have \(-\psi(y, x) = -2[y, [y, [y, x]]] - 2[x, [x, [y, x]]] - 3[x, [y, [y, x]]] = 2[y, [y, [y, x]]] + 2[x, [x, [y, x]]] + 3[x, [y, [y, x]]] = \psi(x, y)\).

By direct calculation, we obtain that \( \psi(x, y) + \psi(y, -x - y) + \psi(-x - y, x) = 0 \mod 5 \), which gives (11).

A lengthy calculation using Lemma (4) gives equation (12) modulo 5. \( \square \)

The kernel of the projection \( \pi : \grt \rightarrow \grt/[\grt, \grt] \) contains only the elements of depth greater or equal to 2 (see [6] for details). Thus, Conjecture 2 together with Lemma 6 would give a generator of \( \grt \) in degree \( p - 1 \).

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