New algorithm for determining the dynamic error for the integral-square criterion

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Abstract. The paper presents a new algorithm for determining the maximum dynamic error generated by the measurement system in the case of the integral-square criterion. This algorithm is based on processing the impulse response of the considered system and can be used to analyse time-invariant linear systems. The functions for approximating the dynamic error for the integral-square criterion were determined for an assumed range of variability among the measurement system parameters and a given discretisation step. A polynomial approximation was applied, and the calculations were carried out in MATLAB.

Keywords: dynamic error, integral-square criterion, approximation function

1. Introduction

The algorithm for determining the maximum dynamic error for the case of the integral-square criterion and input signals with magnitude constraint [1] has been presented in detail in [2,3]. This algorithm concerns the necessity of solving a system of complex integral equations, the number of which depends on the number of switching times of input signal, which is not known. Thus, for a given number of integral equations included in the above system, we obtain the switching times of the signal that maximise the error of the solution. However, the problem is determining the optimal number of equations whose signal generates the highest possible error value.

This algorithm has previously been presented in [2,3] based on the assumption that in the first step, on the basis of a single equation, a signal with one switching time together with the corresponding error is determined. In the next step, based on the solution of the system of two equations, an error corresponding to a signal with two switching times is determined. The obtained error value is then compared to the value from the previous step. The number of equations is increased as long as the error obtained in the next step is greater than the error from the previous step. With this assumption, we are not sure if in the next steps we would not get a higher error value. There are also very serious difficulties regarding the possibility of solving complex integral equations, even when using the highest class of computer and dedicated software. Additionally, the error determination procedure must be performed either separately for each measurement system or in the event of a change in its dynamic properties.

The possibility of solving the equations above was assessed and the results of study of four example measurement systems were presented in paper [4]. In the summary of this paper it was assume that it is possible to solve a system with a maximum of 25 equations.

Due to the difficulties and limitations of the above-mentioned procedure, this paper presents a new algorithm for determining the maximum dynamic error for the case of the integral-square criterion, generated by a signal with magnitude constraint. It is based on iterative processing of the impulse response of the measurement system to determine the signal that maximises the error [5–7]. Additionally, for a designated set of error values from the assumed variability range among the system parameters, a procedure that allows determination the functions approximating the error to be calculated is presented. For this purpose, the polynomial approximation [8] was used to determine the errors for the voltage output accelerometer [2] as an example of the measurement system.
2. Algorithm for error determination

For linear systems, the integral-square error is represented by the convolution integral

\[ IE = \int_0^T \left[ \mathcal{K}(t-\tau) x(\tau) \right] x(t) \, dt \]  

(1)

where

\[ \mathcal{K}(t) = \int_0^T k(t-\zeta) k(\zeta-\tau) \, d\zeta \]  

(2)

is the autocorrelation function [5,6], \( x(t) \) is the input signal, \( T \) is the time for which the error is determined, \( k(t) \) is the difference between the impulse responses of the system and its standard and \( t \in (0, T) \) [7].

The maximum dynamic error is determined in the following steps:

1. Calculate the autocorrelation function \( \mathcal{K}(t) \).
2. Determine the initial signal \( x^0(t) = A \cdot \text{sign}\{\mathcal{K}(t)\} \)  

(3)

where \( A \) denotes the magnitude constraint.
3. Determine the \( j+1 \) input signal 

\[ x^{j+1}(t) = A \cdot \text{sign} \left[ \int_0^T x^j(\zeta) \mathcal{K}(\zeta-t) \, d\zeta \right] \]  

(4)

for \( j = 0, 1, \ldots, J \), where \( J \) denotes the number of iterations and is chosen in advance [5].

The integral-square error [2] is calculated by

\[ IE(x_0) = \int_0^T \left[ \int_0^T k(t-\tau) x_0(\tau) \, d\tau \right]^2 \, dt \]  

(5)

The stop condition for (4) applies if the signal \( x^{j+1}(t) \) is generated, so long as:

- the differences between the successive switching times of the signals \( x^{j+1}(t) \) and \( x^j(t) \) are no greater than the time discretisation step,
- the differences between the successive errors \( IE(x^{j+1}) \) and \( IE(x^j) \) are no greater than the assumed error accuracy.

4. Determine the approximation functions for the obtained error values in (5) from the assumed variability range of the system parameters [8].

3. Example application

The calculations were carried out for an example measurement system represented by the voltage output accelerometer, which is described by the model

\[ K_a(s) = \frac{-4\pi^2 S f_0^2}{s^2 + 4\pi f_0 \beta s + 4\pi^2 f_0^2} \]  

(6)

where: \( S \), \( \beta \) and \( f_0 \) are the voltage sensitivity, damping ratio and non-damped natural frequency, respectively [2].

Fig. 1 shows the relationship between the error \( IE(x_0) \) and time \( T \in (0, 20\,\text{ms}) \) for \( S = 4 \, \text{mV} / (\text{m}^2) \), \( f_0 = 1\,\text{kHz} \) and \( \beta = 0.05, 0.1, \ldots, 0.4 \). This relationship is nonlinear for the time interval corresponding to the nonsteady state of the impulse response. However, this nonlinearity is confined to the first six milliseconds. For all other times \( T \), this relationship is linear and can be described using the function
\[ IE(T) = a_0 + a_1 \cdot T \]  

(7)

Fig. 1. Integral-square error \( IE(x_0) \) in relation to time \( T \)

The values of coefficients \( a_0 \) and \( a_1 \) of function (7), along with the standard uncertainty \( u_A(IE) \) of approximation function, are tabulated in Table 1.

Table 1. Values of \( a_0 \), \( a_1 \) and \( u_A(IE) \) for \( S = 4 \ V/(m^2 \cdot s) \) and a range of \( \beta \)-values

| \( \beta \) | \( 0.05 \) | \( 0.10 \) | \( 0.15 \) | \( 0.20 \) | \( 0.25 \) | \( 0.30 \) | \( 0.35 \) | \( 0.40 \) |
|---|---|---|---|---|---|---|---|---|
| \( a_0 \) | -77.0 | -11.1 | -3.08 | -1.49 | -0.685 | -0.286 | 0.0755 | 0.0881 |
| \( a_1 \) | 19.2 | 4.96 | 2.16 | 1.24 | 0.771 | 0.508 | 0.346 | 0.233 |
| \( u_A(IE) \) | 3.75 | 0.177 | 0.127 | 0.0769 | 0.0314 | 0.0177 | 0.0584 | 0.0521 |

The standard uncertainty \( u_A(IE) \) was determined by

\[
 u_A(IE) = \sqrt{\frac{\sum_{i=1}^{n} |IE(T)_i - IE(x_0)_i|^2}{n-2}} 
\]

(8)

where \( n \) denotes the number of \( IE(x_0) \) points.

Analogously as for \( S = 4 \ V/(m^2 \cdot s) \) – Fig. 1 and Table 1, calculations were then made for \( S = (0.5, 0.5 + A_5, \ldots, 3.5) \ V/(m^2 \cdot s) \) with a step \( A_5 = 0.5 \ V/(m^2 \cdot s) \). On the basis of the obtained values of coefficients \( a_0 \) and \( a_1 \), the functions \( a_0(\beta, S) \) and \( a_1(\beta, S) \), which represent the relationships between the coefficients in (7) and the values \( \beta \) and \( S \), were determined. These functions were determined by means of a fifth-order polynomial approximation, where

\[
a_0(\beta, S) = -5.47 + 245. \beta - 3.64 \cdot S - 3911. \beta^2 + 168. \beta \cdot S - 2.41 \cdot S^2 + 2.77 \cdot 10^4 \cdot \beta^3 + 2396. \beta^2 \cdot S + 96.9 \cdot \beta \cdot S^2 - 1.75 \cdot S^3 - 8.54 \cdot 10^4 \cdot \beta^4 + 1.12 \cdot 10^4 \cdot \beta^3 \cdot S - 687. \beta^2 \cdot S^2 + 19.7 \cdot \beta \cdot S^3 - 0.136 \cdot S^4 + 9.31 \cdot 10^4 \cdot \beta^5 - 1.59 \cdot 10^4 \cdot \beta^4 \cdot S + 1221. \beta^3 \cdot S^2 - 44.9 \cdot \beta^2 \cdot S^3 + 627 \cdot 10^{-3} \cdot \beta \cdot S^4 - 502 \cdot 10^{-5} \cdot S^5 
\]

(9)

and

\[
a_1(\beta, S) = 3637.10^{-4} - 5193.10^{-4} \cdot \beta + 3759.10^{-4} \cdot S - 9562.10^{-4} \cdot S \cdot \beta + 6521.10^{-4} \cdot S^2 + 1941.10^{-4} \cdot \beta^3 - 688.10^{-3} \cdot \beta^2 \cdot S + 2194.10^{-4} \cdot \beta^2 \cdot S^2 - 1896.10^{-5} \cdot S^3 + 5544.10^4 \cdot \beta^4 - 8988.10^{-4} \cdot S \cdot \beta^3 + 8944.10^{-3} \cdot \beta^2 \cdot S^2 - 2417.10^{-4} \cdot \beta \cdot S^3 + 2785.10^{-5} \cdot S^4 - 347.10^{-3} \cdot \beta^3 + 6204.10^{-4} \cdot \beta^4 \cdot S - 5258.10^{-4} \cdot S \cdot \beta^3 \cdot S^2 + 2138.10^{-4} \cdot S^3 \cdot \beta^2 - 344.10^{-4} \cdot \beta \cdot S^4 + 2892.10^{-6} \cdot S^5 
\]

(10)
The goodness of fit for the approximation functions \( a_0(\beta, S) \) and \( a_1(\beta, S) \) are: \( SSE = 119 \), \( RMSE = 1.66 \) and \( SSE = 4.13 \), \( RMSE = 0.310 \), respectively.

The sum of squares due to error (SSE), which measures the total deviation of the response values \( a(\beta, S)_i \) from the best fit to the response values \( \tilde{a}(\beta, S)_i \), was calculated by

\[
SSE = \sum_{i=1}^{N} [a(\beta, S)_i - \tilde{a}(\beta, S)_i]^2
\]

where \( n = 64 \) is the number of data points.

The root mean square error (RMSE) was calculated by

\[
RMSE = \sqrt{\frac{SSE}{v}}
\]

where \( v = N - m = 43 \) and \( m = 21 \) is the number of fitted coefficients.

Fig. 2 shows the approximation functions \( a_0(\beta, S) \) and \( a_1(\beta, S) \) where \( \beta \) and \( S \) denote the damping ratio and sensitivity, respectively.

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4. Conclusions

The algorithm presented in the paper, which uses approximation functions, makes it possible to determine the value of the integral-square error between the function damping ratio and sensitivity. It is also possible to determine the error in the function of the third variable occurring in the analysed model of the voltage output accelerometer, i.e. the non-damped natural frequency. However, this requires extending the number of data points, and consequently the number of calculations.

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