A position and a time for the photon

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This paper gives a constructive answer to the question whether photon states can contain or not, and to what extent, the readings of rulers and clocks. The paper first shows explicitly that, along with the momentum representation, there is room in the one photon Hilbert space for an alternative position representation. This is made possible by the existence of a self-adjoint, involutive, position operator conjugate to the momentum operator [1]. Position and momenta are shown to satisfy the Heisenberg-Weyl quantization rules in the helicity basis, which is analyzed anew from this point of view. The paper then turns to the photon’s time of arrival. By picking an appropriate photon Hamiltonian - using Maxwell equations as the photon Schrödinger equation - a conjugate time of arrival operator is built. Its interpretation, including the probability densities for the instant of arrival (at arbitrary points of 3-D space) of photon states with different helicities coming from arbitrary places, is discussed.

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Introduction

Since the advent of Special Relativity, light propagation has played a very special role among the variety of natural phenomena studied by Physics. In conjunction with rods and clocks, light signals were a main component in the construction of the -then- new theory. The universality of \( c \), the speed of light, set the new standard commensurate to the breadth of relativistic physics. While light was understood at the time as a manifestation of electromagnetic waves, the interaction of matter and radiation demanded a quantum leap:
the photon. This is the place to recall the mismatch between the prominent role played in relativity by the arrival times of light signals and the position of their group centers and wave fronts, and the lack of the corresponding operators in quantum physics [2, 3, 4]. The reason of this deficiency of the quantum theory [5] can be traced back to the fact that the photon does not have a complete set of helicity states; the state with helicity \( \lambda = 0 \) is missing. This paper aims to give a constructive answer to the long debated question whether the photon state can contain or no, and to what extent, the readings of rulers and clocks, i.e. the where’s and when’s of the photon. In view of the recent experimental advances, an appropriate formalism is necessary for a meaningful discussion of the propagation and arrival of light signals, mainly if these arrive photon by photon. The literature on this question is abundant and only a part of it is in the bibliography at the end of this paper. The reader is referred to [6] for a concise and clear introduction to this topic.

Electromagnetic waves propagate at the speed of light in vacuum. This basic tenet of relativity, has been challenged recently both from the experimental and the theoretical front. It is remarkable that the theoretical part of this question was settled long time ago by Sommerfeld [7] and Brillouin [8], who demonstrated that the electromagnetic wave fronts always move with speed \( c \). To my knowledge, the cases where the group velocity in vacuum was found larger than \( c \) where always produced by an incorrect identification of the signal center. Superluminal propagation in vacuum turned always to be a geometric artifact of the wave form and not the group speed of the wave packet. The main point to highlight here is the absence of foundational problems associated to the definitions of the group position and time of arrival of classical e.m. waves. It may be difficult to ascertain these questions in a given situation, but only from the technical point of view, not as a matter of principle.

Electromagnetic waves are composed of elementary quanta: photons. Like the rest of elementary particles, these are mathematically identified as elementary systems under the Poincare group [9, 10]. Namely, all the possible sets of values that the dynamical properties of the system can attain are connected by transformations of the group to an arbitrary, fixed, set of values, taken as standard. The different classes of elementary systems are identified by the values taken by the Casimir operators of the group. Photons are characterized by lightlike momenta \( p^2 = (p^0)^2 - p^2 = 0 \) and a helicity value that is either \( \lambda = +1 \) or \( \lambda = -1 \), there are thus two irreducible representations \((1, 0)\) and \((0, 1)\) (in the standard notation \((j, j')\)), that are combined into an irreducible entity by parity \((1, 0) \rightarrow (0, 1)\). Finally, photons
transform under the representation \((1, 0) + (0, 1)\) of the complete Poincare group.

The properties of photons are thus given by the eigenvalues of the subsets of commuting generators of the Poincare group in that representation. There are two widely used alternative subsets: a). \(\{p, \lambda\}\), that describe plane waves, suited for the description of initial and final scattering states, free flying photons etc, and b). \(\{p^0, j, j_z, parity\}\), that describe spherical photon waves that simplify the analysis of radiation, and in general, of those systems where there is a singled out space point.

The rapid pace and depth of the current advances in the manipulation of photon states are demanding a parallel progress on the theoretical side. In some cases it is necessary to describe accurately the behavior of photons that are in the near region. In others, their space time distributions in regions close to the microscopic realm, for states containing just one photon or a few of them, in states possibly entangled that spread over these regions, etc. Current experimental set ups include a variety of very, sensitive single photon devices. Even more, many of the most exciting results have been obtained experimenting with these highly nonclassical light states. On its side, the theoretical framework should be able to address simple questions like the time of arrival of a photon at the place where some device is located, the relation between the orbital angular momentum of the photon and its position at some time, etc. It is necessary at least to give a meaning to these questions, something whose very possibility has been the subject of debate [2, 11, 12, 13, 14, 15] since the early times of Quantum Mechanics. I shall avoid here repeating the well known pros and cons for the wave function of the photon [16, 17, 18, 19, 20, 21] and whether it represents the probability amplitude for finding the photon in a given region of space (technically a Borel set \(\Delta \in \mathbb{R}^3\)). Instead, I shall trace the problem back by an alternative route to non-relativistic quantum mechanics (NRQM) where the vanishing photon mass \((M = 0)\) prevents the definition of a position operator \(\hat{q}\) for the photon.

Let me first give some support to the above assertion. The dynamical properties of an elementary object, i.e. a free particle, in non-relativistic quantum theory can be traced back to its behavior under the Galileo group [22]. In particular the position operator \(\hat{q}\) is related to the Galileo boost operator \(\hat{G}\) by \(\hat{G}_i = M\hat{q}_i\), where \(M\) is the mass of the particle. For sure, it can be defined directly by \(\hat{q}_i|x\rangle = x_i|x\rangle\), so that its components are commuting, its eigenvalues real numbers, and its eigenfunctions Dirac delta functions. On the other hand, this definition has to be compatible with the fact that the momentum
operator $\hat{p}$ generates translations in configuration space, $[\hat{q}_i, \hat{p}_j] = i\hbar$. Accordingly in the momentum representation, more appropriate for a discussion of the Galileo group generators, the position operator acts as a derivative on the momenta $\hat{q}_i = i\hbar\partial/p_i$; if $\hat{p}_i$ is the basic, multiplicative operator of the representation, then $\hat{q}_i$ is derived from it. In other words, particle position emerges from the Galilean properties of the particle, specifically from its behavior under space translations.

As a consequence of the above, one may wonder that if in NRQM a massless system cannot have a properly defined position operator, then: Why expect the opposite in RQM? The contenders in the debate about the photon wave function could be aware or not about the lack of a non-relativistic $\hat{q}$ for the photon. In any case, they were undoubtedly influenced by their consequences, and induced to give up in the search of a relativistic $\hat{q}$ for $M = 0$. The fact that the photon little group is $E(2)$ instead of $SU(2)$ added the filling drop to the confusion. In some studies of the position operator for relativistic systems, photon transversality was identified as the obstacle to the existence of a photon position operator [4, 5]. Other analysis [3, 20] pointed to the lack of commutativity of its components (or better, of the Lorentz boost components) as the reason for the lack of an appropriate $\hat{q}$. Finally, M. Hawton found out [1, 23, 24] very recently a self-adjoint position operator with the required commutation rules, that operates in the photon Hilbert space as a physically acceptable position should do. This opens up the possibility of analyzing the question of photon localizability from a new and more powerful setting: A good position operator should come endowed with a probabilistic interpretation, which is the tool suited to the analysis of localizability [5, 13, 14, 15]. On the other hand, the most recent theoretical analysis [25, 26] indicate that individual photons can be localized in regions much smaller than previously thought, with entanglement (instead of wavelength) as the physical ruler [27] in the experiment [28]. This increases the value of the Hawton construct and the urge to explore its consequences.

In this paper I shall analyze the properties of the photon states in Minkowski space-time. On the way, we shall devise several tools necessary to this end. The photon Hilbert space will be introduced in Sect. II, where we also present two alternative conjugate representations and show the probabilistic interpretation that they can be given. Sect. III is devoted to the explicit construction of the Hawton position operator in these representations. We revise the properties of the operator in the helicity basis and conclude that it qualifies as
a good position operator. Finally, we complete the standard picture of one photon states with additional position dependent information. Conversely, Sect. IV is devoted to the time dependent information. We first consider Maxwell equations as the photon Schrödinger equation and solve the time evolution associated to it. Then, we set up the formalism for analyzing the photon time of arrival at an arbitrary point of space. It will be necessary to split the Hilbert space into two subspaces putting the eventually detected states in one of them. In it we build the sought time operator, obtain its eigenfunctions and give the positive operator valued measure that permits a probabilistic analysis of the times of arrival of a photon in a given state at an arbitrary space position. The paper ends in Sect. V with a summary of the results, some considerations on their meaning, and indications for the use of the formalism in the analysis of some current experiments.

One photon Hilbert space

We shall work in the Coulomb gauge i.e. within the Hilbert space \( \mathcal{H} \) of one particle transverse states \( \tilde{A}_i(p) \), \( i = 1, 2, 3 \) defined on the forward light cone \( (p^0 = |p|) \) (note the symbol \( \tilde{\cdot} \), it is a label for transversality). The scalar product in \( \mathcal{H} \) is defined as

\[
(A, A') = \int d\sigma(p) \tilde{A}^*_i(p) \tilde{A}'_i(p)
\]

where \( d\sigma(p) = d^3p/2|p| \) is the measure on the light cone. An arbitrary photon state \( A \in \mathcal{H} \) can be written as

\[
\tilde{A}_i(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2|p|}} e^{ipx} \tilde{A}_i(p)
\]

Note that \( \tilde{A}_i(x) \) is solenoidal so that, in spite of being \( (A, A') = \int d^3x \tilde{A}^*_i(x) \tilde{A}'_i(x) \), \( x \) does not qualify for a good position (in poor words: \( x_i \tilde{A}(x) \) is not solenoidal, so \( x_i \) takes \( \tilde{A} \) out of the Hilbert space).

Due to Poincare invariance (that we are taking fully into account, in spite of the non covariant notation), the helicity \( W \) is a good quantum number. In fact, \( \hat{p} \) and \( \hat{W} = S(p/|p|) \) – where \( S_{ij} = -i\epsilon_{aij} \) are the spin-1 matrices – form a complete set of commuting operators on the mass shell. In the following, I shall work in momentum space where the momentum operator \( \hat{p} \) is simply represented by the vector variable \( p = (p_1, p_2, p_3) \). An arbitrary operator \( \hat{G} \) function of the momentum, that commutes with the helicity \( [\hat{G}, \hat{W}] = 0 \), shall have
eigenfunctions \( \tilde{V}_{G,\lambda}^i(p) \) satisfying

\[
(\hat{p}^2 \tilde{V}_{G,\lambda}^i(p)) = p^i \tilde{V}_{G,\lambda}^i(p), \quad (\hat{W} \tilde{V}_{G,\lambda}^i(p)) = \lambda \tilde{V}_{G,\lambda}^i(p), \quad (\hat{G} \tilde{V}_{G,\lambda}^i(p)) = G \tilde{V}_{G,\lambda}^i(p)
\]  

(3)

The second of these equations implies that \( \tilde{V}_{G,\lambda}^i(p) \propto \epsilon^i(p, \lambda) \), where \( \epsilon^i(p, \lambda) = D^j_{ij}(\Lambda^{-1} k) \epsilon_j(\lambda) \), for \( p = \Lambda k \) and \( k = (0, 0, |\Lambda^{-1} p|) \), with \( \epsilon(\lambda) \) such that \( S_3 \epsilon^i(\lambda) = \lambda \epsilon(\lambda) \). In spherical coordinates where \( p = (k, \theta, \varphi) \) and \( k = (0, 0, k) \) – i.e. taking \( \Lambda \) as a pure rotation for simplicity – the polarization vectors are given as linear combinations of the unitary vectors \( \epsilon(p, \sigma), \sigma = \theta, \varphi, k \) (or \( \sigma = 1, 2, 3 \)):

\[
e(p, k) = (\partial p/\partial k), \quad e(p, \theta) = (1/k) (\partial p/\partial \theta), \quad e(p, \varphi) = (1/k \sin \theta) (\partial p/\partial \varphi)
\]

in the standard form:

\[
\epsilon^i(p, \lambda) = -\frac{\lambda}{\sqrt{2}} \left\{ \epsilon^i(p, \theta) + i\lambda \epsilon^i(p, \varphi) \right\} \quad \lambda = \pm 1
\]

(4)

The representations of the Poincare group for the photon states are the (1,0) and (0,1), that correspond to \( \lambda = -1 \) (left polarization) and \( \lambda = 1 \) (right polarization) respectively. They are combined in the direct sum \( (1,0) \oplus (0,1) \) that becomes irreducible when parity is included in the group. No other states exist (see for instance [29] page 69) because the little group is not semi-simple. In particular, the polarizations cannot be taken as four-vectors. Even if we form the object \( \epsilon^\mu(p, \lambda) = (\epsilon^0(p, \lambda), \epsilon(p, \lambda)) \), it not only has to satisfy in every reference frame \( p_\mu \epsilon^\mu(p, \lambda) = 0 \), but also \( \epsilon^0(p, \lambda) = 0 \), or any other condition chosen to fix the gauge. In short: in spite of its four-dimensional appearance, the polarizations do not transform as four-vectors under Lorentz transformations, but as a sort of connections [30, 31]:

\[
\Lambda_\mu^\nu \epsilon^\nu(\Lambda p, \lambda) = \exp\{i\lambda \Theta(p, \Lambda)\} \epsilon^\mu(p, \lambda) + p^\mu \Omega(p, \lambda; \Lambda)
\]

(5)

where \( \Theta(p, \Lambda) \) is the Wigner rotation angle, and \( \Omega(p, \lambda; \Lambda) \) is a gauge transformation fixed by the gauge condition \( (\epsilon^0 = 0) \). Of course, the photons can be given a covariant (tensor) representation by means of the gauge invariant antisymmetric tensor \( F_{\alpha\beta}(p, \lambda) \propto i\epsilon_{\alpha\beta\mu\nu}(p^\mu \epsilon^\nu(p, \lambda) - p^\nu \epsilon^\mu(p, \lambda)) \), that gives rise to the electric and magnetic fields. With these caveats in mind, I shall continue to use the three dimensional notation to work in the
Coulomb gauge in the following, turning to [30, 31] for questions of Poincare covariance or
gauge invariance.

By construction,

\[(S \cdot e(p, k))_{ij} = (S_k)_{ij}, \quad \text{and} \quad (S_k)_{ij} \epsilon_j(p, \lambda) = \lambda \epsilon_i(p, \lambda), \quad \lambda = \pm 1\]  

(6)

We can now write \(\tilde{V}_{G,\lambda}(p) = g_G(p) \epsilon^i(p, \lambda)\), where \(g_G(p)\) is a \(G\) dependent function of \(p\) to be
determined in each case. By these definitions, these functions span the transverse subspace
orthogonal to \(p\), and are eigenfunctions of the helicity with eigenvalue \(\lambda\).

**Momentum and position representations**

In the trivial case of the momentum operator \(\hat{G} = p\) where \(\tilde{V}_{p,\lambda}(p') = g_p(p') \epsilon^i(p', \lambda)\), it
is straightforward to obtain \(g_p\). We want \((V_{p_1,\lambda_1}, V_{p_2,\lambda_2}) = \delta_{\lambda_1 \lambda_2} \delta^{(3)}(p_1 - p_2)\), but we only
have

\[(V_{p_1,\lambda_1}, V_{p_2,\lambda_2}) = \int d\sigma(p) \tilde{V}_{p_1,\lambda_1}^*(p) \tilde{V}_{p_2,\lambda_2}(p) = \delta_{\lambda_1 \lambda_2} \int d\sigma(p) g_{p_1}^*(p) g_{p_2}(p)\]  

(7)

where we used that \(\epsilon^*(p, \lambda) \epsilon(p, \lambda') = \delta_{\lambda \lambda'}\). This requires the value \(g_p(p') = \sqrt{2 |p'|} \delta^{(3)}(p' - p)\)
for \(g_p\), so that, apart from phases, the eigenfunctions have to be

\[\tilde{V}_{p,\lambda}(p') = \sqrt{2 |p'|} \epsilon^i(p', \lambda) \delta^{(3)}(p' - p), \quad \text{and} \quad \tilde{V}_{p,\lambda}(x) = \frac{1}{(2\pi)^{3/2}} \epsilon^i(p, \lambda) e^{ip \cdot x}\]  

(8)

Note that the second equation in (8) comes from the first one by straight application of Eq
(2).

The next step is to assume the existence of a position, namely a vector operator \(\hat{G} = \hat{q} = (\hat{q}_1, \hat{q}_2, \hat{q}_3)\), with simultaneous eigenfunctions in \(\mathcal{H}\) that can be given as \(\tilde{V}_{q,\lambda}(p) = g_q(p) \epsilon^i(p, \lambda)\). For this to be possible it is necessary that:

i. Position and helicity commute, that is \([\hat{q}^i, \hat{W}] = 0\).

ii. The three components are in involution, that is \([\hat{q}^i, \hat{q}^j] = 0\).

To these conditions we have to add that positions and momenta be canonically conjugate
operators: \([\hat{q}^i, \hat{p}^j] = i \delta^{ij}\). Finally, it will be necessary to check whether \(\hat{q}\) gives rise to a
probabilistic interpretation or not.

We shall return to the explicit construction of the position operator later on but, for
the time being, we assume its eigenfunctions exist in the transverse Hilbert space of the
photon and form an orthogonal and complete set. We first explore the implications of the orthogonality. A direct computation using \( V_{q, \lambda}(p) = g_q(p) \epsilon^i(p, \lambda) \) gives
\[
(V_{q_1, \lambda_1}, V_{q_2, \lambda_2}) = \delta_{\lambda_1 \lambda_2} \int d\sigma(p) g_{q_1}^*(p) g_{q_2}(p)
\]
but, as in the case of the momentum operator, we would like to have \((V_{q_1, \lambda_1}, V_{q_2, \lambda_2}) = \delta_{\lambda_1 \lambda_2} \delta^{(3)}(q_1 - q_2)\). This requires \( g_q(p) = (2\pi)^{-3/2} \sqrt{2|p|} e^{-i p q} \). We can now give the position eigenfunctions in both, momentum and coordinate representations:
\[
\hat{V}_{q, \lambda}^i(p) = (2\pi)^{-3/2} \sqrt{2|p|} \epsilon^i(p, \lambda) e^{-i p q} \quad \text{and} \quad \hat{V}_{q, \lambda}^i(x) = (2\pi)^{-3} \int d^3p \epsilon^i(p, \lambda) e^{i p(x-q)}
\]
It is clear from the last equation that \( x \) does not correspond to the position eigenvalue \( q \). This mismatch is due to the presence in the integrand of the \( rhs \) of momentum dependent polarization vectors that link momentum with spin. This equation summarizes much of the troublesome nature of the coordinate space wave function of the photon.

The above eigenfunctions form complete sets. By straightforward calculation using \( \sum_{\lambda} \epsilon^i(p, \lambda) \epsilon_j(p, \lambda) = \delta^{ij} - p^i p^j/|p|^2 = \delta^{ij}_\perp(p) \), we get the decompositions of the identity
\[
\sum_{\lambda} \epsilon^i(p, \lambda) \epsilon^j(p, \lambda) = \delta^{ij}_\perp(p) = \delta^{ij}(p)
\]
We have arrived at two alternative complete sets of commuting operators (momentum, helicity) and (position, helicity), whose simultaneous eigenfunctions form alternative bases of the Hilbert space \( \mathcal{H} \) of the massless \((1,0) \oplus (0,1)\) representation of the Poincare group. We recall again the difference between position and coordinate, and refer the reader back to Eq.(10) for a check. Given an arbitrary state \( \Psi \) in \( \mathcal{H} \), we could employ Dirac bracket notation to denote its components in both representations
\[
\langle p \lambda | \Psi \rangle = \int d\sigma(p') \hat{V}_{p, \lambda}^{i*}(p') \hat{\Psi}^i(p') = \frac{1}{\sqrt{2|p|}} \epsilon^{i*}(p, \lambda) \hat{\Psi}^i(p)
\]  
\[
\langle q \lambda | \Psi \rangle = \int d\sigma(p') \hat{V}_{q, \lambda}^{i*}(p') \hat{\Psi}^i(p') = (2\pi)^{-3/2} \int d^3p \epsilon^{ipq} \langle p \lambda | \Psi \rangle
\]
The relation (14) is the standard one between position and momentum representations in quantum mechanics. In fact,
\[
\langle q \lambda | p \lambda' \rangle = (2\pi)^{-3/2} \delta_{\lambda\lambda'} \exp(ipq), \quad \langle p \lambda | p' \lambda' \rangle = \delta_{\lambda\lambda'} \delta^{(3)}(p - p'), \quad \langle q \lambda | q' \lambda' \rangle = \delta_{\lambda\lambda'} \delta^{(3)}(q - q')
\]
Note also the scalar product in both representations:

\[ \langle \psi | \psi \rangle = \sum_\lambda \int d^3 p |\langle p \lambda | \Psi \rangle|^2 = \sum_\lambda \int d^3 q |\langle q \lambda | \Psi \rangle|^2 \]  

(16)

The absence of polarization vectors in the integrand in the rhs of (14) eliminates the obstructions [5, 13, 14] that prevent from considering

\[ P_\psi(\Delta, \lambda) = \int_\Delta d^3q |\langle q \lambda | \Psi \rangle|^2 \]  

(17)

as the probability for finding a photon of helicity \( \lambda \) in the Borel set \( \Delta \in \mathbb{R}^3 \). In particular, \( P_\psi(\Delta) = 0 \) is possible. The reason is that, in the absence of polarization vectors in (14), the Fourier transform of \( \langle q \lambda | \Psi \rangle \) can be an entire function of \( p \). Then, according to the Plancherel-Polya theorem [32], \( P_\psi(\Delta) \) may vanish in Borel sets like \( \Delta \in \mathbb{R}^3 \). Physically this is necessary in order to cope with the absence of photons in \( \Delta \). It is the product of \( \tilde{\Psi}_i(p) \) and \( e^i(p, \lambda)/\sqrt{2|p|} \) in (13) that has to be entire. This can be so even if the presence of \( \sqrt{|p|} \) explicitly and in the \( \epsilon \)'s implies that the individual \( \tilde{\Psi}_i(p) \) can not be entire functions and, therefore, can not be given a probabilistic interpretation. In any case, it is worth to recall that the expressions in (13) are for fixed helicity. Thence photon states with fixed helicity could be localizable in spite of [13] and of theorems 1 and 2 in [14]. This is a first consequence of the until now only hypothesized existence of \( q \).

The expansion of arbitrary states in \( \mathcal{H} \) in terms of momentum and helicity eigenstates can be given inverting (13) (recall that \( \mathcal{H} \) is the space of transverse states)

\[ \tilde{\Psi}_i(p) = \sqrt{2|p|} \sum_\lambda e^i(p, \lambda) \langle p \lambda | \Psi \rangle \]  

(18)

a relation that is customarily used in conjunction with Eq. (2) to write

\[ \tilde{\Psi}_i(x) = (2\pi)^{-3/2} \sum_\lambda \int d^3p e^{ip \cdot x} e^i(p, \lambda) \langle p \lambda | \Psi \rangle \]  

(19)

This is the standard coordinate representation used to describe the one photon states and – promoting \( \langle p \lambda | \Psi \rangle \) to the ranks of annihilation operators – the photon field. The comparison of (14) and (19) indicates clearly that it is the presence of the \( p \)-dependent polarization vector in the integrands of (10) and (19) that forestalls the interpretation of \( x \) as a true position. The decompositions Eqs. (18) and (19) clearly show that the \( \tilde{\Psi}(p) \) are transverse (i.e. \( \sum_i p_i \tilde{\Psi}_i(p) = 0 \)) and the \( \tilde{\Psi}(x) \) solenoidal (i.e. \( \partial \tilde{\Psi}_i(x)/\partial x_i = 0 \)). The interested reader is referred to [20] for the use of (19) as a photon wave function and to Landau and
Peierls [12] to explore the meaning of the non-local relations between $\bar{\Psi}(x)$ and $\langle q\lambda | \Psi \rangle$. It is straightforward to show that they are

$$\bar{\psi}^i(x) = \sum_\lambda \int d^3q \bar{V}^i_{q\lambda}(x) \langle q\lambda | \psi \rangle, \quad \text{and} \quad \langle q\lambda | \psi \rangle = \int d^3x \bar{V}^*_q(x) \bar{\psi}^i(x) \quad \tag{20}$$

**Position operator**

In the previous section we have seen that there is room in the Hilbert space of one photon states for an ordinary position operator. In fact

$$\langle q'\lambda | \hat{q}_i \psi \rangle = q_i' \langle q'\lambda | \psi \rangle, \quad \text{and} \quad \langle p\lambda | \hat{q}_i \psi \rangle = i\hbar (\frac{\partial}{\partial p_i}) \langle p\lambda | \psi \rangle \quad \tag{21}$$

Here we will analyze the features of this operator as seen in the space of transverse states. Following Hawton [1], we first identify its structure by letting it to operate on the eigenfunctions $V_{q,\lambda}$. Then we shall show that it qualifies as an appropriate position operator. Concisely, our first task is to find the operator $\hat{q}^a$, $a = 1, 2, 3$ whose eigenfunctions are the $V$'s given in Eq. (10)

$$\{ \hat{q}^a \bar{V}_{q,\lambda} \}_i(p) = q_i^a \{ \bar{V}_{q,\lambda} \}_i(p) \quad \tag{22}$$

By using above the explicit form (10) of $\bar{V}_{q,\lambda}$, Hawton got after some algebra [1]

$$\{ \hat{q}^a \bar{V}_{q,\lambda} \}_i(p) = (i\delta_{ij} \nabla^a + (Q^a)_{ij}) \{ \bar{V}_{q,\lambda} \}_j(p) \Rightarrow (\hat{q}^a)_{ij} = i\delta_{ij} \nabla^a + (Q^a)_{ij} \quad \tag{23}$$

where

$$\nabla^a = \sqrt{2|p|} \frac{\partial}{\partial p^a} \frac{1}{\sqrt{2|p|}} \quad \text{and} \quad (Q^a)_{ij} = i \sum_{\sigma=\theta,\varphi,k} e_i(p,\sigma) \{ \nabla^a e_j(p,\sigma) \} \quad \tag{24}$$

As is well known, the operator ordering implied by $\nabla^a$ is necessary to make it self-adjoint with the measure $d\sigma(p)$. Notice that the attained operator (23) is independent of the helicity quantum number, being a matrix in the coordinate indices. By computing the derivatives of the basis vectors that appear in the definition of $Q^a$, one obtains the explicit expression of the operator

$$(\hat{q}^a)_{ij} = i\delta_{ij} \nabla^a + \frac{1}{|p|} [e(p,k) \wedge S_{ij}]^a - \frac{\cot \theta}{|p|} e^a(p,\varphi) W_{ij} \quad \tag{25}$$

due to Hawton [1, 23, 24], who correctly identified the first two terms as the Pryce position operator [3], and the last one as a compensating term [26] for the topological photon phase [33, 34]. She also realized that, due to the last term on the rhs of (25), $\hat{q}$ appeared as
a set of three components in involution, something that previous position operators did not meet.

A compact expression that summarizes much of the above, shows explicitly the relation between this position operator and the spinless one $i \nabla^a$, and gives a rationale for it, is

$$(\hat{q}^a)_{ij} = i \sum_{\sigma=\Theta,\varphi,k} e_i(p, \sigma) \nabla^a e_j(p, \sigma)$$

(26)

The sum in (26) includes the longitudinal polarization $e(p, k)$ but, in spite of it, $\hat{q}$ operates within the transverse subspace $\mathcal{H}$ overcoming some queries put forward by Wightman [5]. Using (26) on an arbitrary function $\Psi \in \mathcal{H}$, expanded according to (18), we obtain

$$\{\hat{q}^a \tilde{\Psi}\} = \sum_{\lambda} \int d^3p |p\lambda\rangle \frac{\partial}{\partial p^a} \langle p^i, \lambda|\Psi\rangle = \sqrt{2|p|} \sum_{\lambda} \epsilon_i(p, \lambda) e_j(p, \lambda) \langle p^i, \lambda|\Psi\rangle$$

(27)

which shows explicitly the transversality of $\hat{q} \cdot \tilde{\Psi}$. We also recall here that $[\hat{q}^a, \hat{W}] = 0$ as can be seen by explicit computation using (23) and the definition of the helicity. Putting all together, it is possible to use the familiar notations:

$$\hat{q}^a = \sum_{\lambda} \int d^3p |p\lambda\rangle \frac{\partial}{\partial p^a} \langle p^i, \lambda|\Psi\rangle, \quad \text{and} \quad \tilde{q}^a = \sum_{\lambda} \int d^3q |q\lambda\rangle q^a \langle q, \lambda|$$

(28)

for the position operator in the helicity representation $\hat{q}^a$. Finally, the wave functions can be interpreted as in the non relativistic case: given a photon in the state $\Psi$, $P_\Psi(p, \lambda) = |\langle p\lambda|\Psi\rangle|^2$ gives the probability of finding it with helicity $\lambda$ and momentum $p$, and $P_\Psi(q, \lambda) = |\langle q\lambda|\Psi\rangle|^2$ gives the probability of finding it with helicity $\lambda$ in the position $q$.

Any arbitrary operator $\hat{O}$ defined in the Hilbert space $\mathcal{H}$ can be represented in the two different bases introduced above. Using the scalar product (1) and the relation (18) we get:

$$\langle \Phi, \hat{O} \Psi \rangle = (2\pi)^{-3/2} \int d\sigma(p) \sum_{ij} \bar{\Phi}_i(p) \hat{O}_{ij} \tilde{\Psi}_j(p)$$

(29)

$$= (2\pi)^{-3/2} \int d^3p \sum_{\lambda\lambda'} \langle \Phi|p\lambda\rangle \hat{O}_{\lambda\lambda'} \langle p, \lambda'|\Psi\rangle$$

(30)

so that there is a well defined relation between the operator’s expressions in both bases:

$$\hat{O}_{\lambda\lambda'}(p, \partial/\partial p) = \frac{1}{\sqrt{2|p|}} \sum_{ij} \epsilon_i(p, \lambda) \hat{O}_{ij}(p, \partial/\partial p) \epsilon_j(p, \lambda') \sqrt{2|p|}$$

(31)

Applying this to the canonically conjugate momenta $\tilde{p}^a_{ij} = \delta_{ij} p^a$ and position (26) operators, we get them in the helicity basis as

$$\hat{p}^a_{\lambda\lambda'} = \delta_{\lambda\lambda'} p^a; \quad \text{and} \quad \tilde{q}^a_{\lambda\lambda'} = \delta_{\lambda\lambda'} i \frac{\partial}{\partial p^a}$$

(32)
We have thus recovered the old Heisenberg-Weyl quantization rules as anticipated in (21). They are valid in the helicity basis and only in it because, due to helicity conservation, additional terms appear in other frames to compensate for the effects of $\partial/\partial p$ on the momentum dependent polarizations. As said above, the meaning of the derivative in the helicity basis is that of a covariant derivative. In fact, it is related to the standard covariant derivative introduced in ref [26] to account for the topological phase [33, 34] of the photon.

$$(i\nabla^a)_{\lambda\lambda'} = \delta_{\lambda\lambda'} \left[ i \frac{\partial}{\partial p^a} + \lambda D^a(p) \right]$$

(33)

where $D^a(p) = \cot \theta e^a(p, \varphi)/|p|$. Notice, by the way, that the helicity operator $\hat{W}_{ij}$ transforms to the helicity quantum number $\lambda$ in the transformation from the spin basis onto the helicity basis. The spin matrix also undergoes a similar transformation

$$\hat{S}^a_{\lambda\lambda'} = \delta_{\lambda\lambda'} \lambda e^a(p, k)$$

(34)

The transformation to the helicity basis analyzed above $\hat{O}_{ij} \rightarrow \hat{O}_{\lambda\lambda'}$ should not be mistaken with the similarity transformation to the photon frame, namely:

$$\hat{O}^a_{\sigma\sigma'} = \sum_{ij} e_i(p, \sigma) \hat{O}^a_{ij} e_j(p, \sigma'), \text{ with } \{\sigma, \sigma'\} = \{k, \theta, \varphi\}$$

(35)

This is simply the rotation from the fixed cartesian axes to the axes lying along the unit vectors $e(p, \sigma)$. A most striking case occurs for the spin matrix that, by (35), becomes

$$\hat{S}^a_{\sigma\sigma'} = i \sum_{\sigma''} \epsilon_{\sigma\sigma'\sigma''} e^a(p, \sigma'')$$

Notice the difference with (34):

$$[\hat{S}^a, \hat{S}^b]_{\sigma\sigma'} = i \sum_c \epsilon_{abc} \hat{S}^c_{\sigma\sigma'}, \text{ while } [\hat{S}^a, \hat{S}^b]_{\lambda\lambda'} = 0$$

(36)

The vanishing of the second commutator is of no surprise as there are no remains of the spin matrices in the helicity representation. The dimension of the spin space is 3, while the helicity basis is a sum of two one-dimensional representations. Even after putting together the two parity related representations (1,0) and (0,1) to have both helicities, the would-be helicity 0 eigenstate is outside the representation space. The relation of these facts with gauge invariance was discussed [30, 31] a long time ago. From the geometric point of view, the transformation $\hat{O}_{ij} \rightarrow \hat{O}_{\lambda\lambda'}$ is a projection from $\mathbb{R}^3$ onto the transverse subspace spanned by $e(p, \theta)$ and $e(p, \varphi)$, along with the appropriate label rearrangements. As a result, the operator products must be handled with caution: in general their transform shall not coincide with the product of the operators' transforms. We saw this in Eq. (36) for the
spin matrices. The same occurs for the angular momentum and the boost operators and, in
general, for all the operators that pull the states out of the transverse space. This sometimes
led to define the helicity representation through a non singular transformation applied to the
spin representation:

\[ \hat{O}_{ij}'(p, \partial/\partial p) = \frac{1}{\sqrt{2|p|}} \sum_{rs} R_{is}^{-1}(p) \hat{O}_{rs}(p, \partial/\partial p) R_{sj}(p) \sqrt{2|p|} \]  

(37)

where \( R(p) \) can be any of the rotations from the standard momentum \((0, 0, |p|)\) to \( p \). Denoting by \( e(\sigma), \sigma = 1, 2, 3 \) three unitary vectors along the fixed axis

\[ e_i(p, \sigma) = R_{ij}(p) e_j(\sigma) \]

(38)

Notice that the intrinsic arbitrariness in \( R \), due to the invariance of the standard momentum
under rotations around it, is removed by the choice of fixed vectors \( e_i(\sigma) \) in (38). Note also
that the helicity representation operators are obtained by projecting (37) on the fixed helicity
basis:

\[ \hat{O}_{\lambda\lambda'} = \sum_{ij} \epsilon_i^{(\lambda)}(\lambda') \hat{O}_{ij}' \epsilon_j^{(\lambda')} \]

(39)

where, according to (4) and (38), \( \epsilon_i(\lambda) = \frac{\lambda}{\sqrt{2}} \{ \epsilon_i(1) + i\lambda\epsilon_i(2) \} \). Needless to say that (39) is
invertible only within the subspace orthogonal to the standard momentum.

The application of the above results to the specific case of the electromagnetic field
completes the standard picture of the one photon state with additional, position dependent,
information. A photon in a state \( A \) can be given as

\[ A^i(x) = (2\pi)^{-3/2} \sum_{\lambda} \int \frac{d^3p}{\sqrt{2|p|}} e^{ipx} \epsilon^i(p, \lambda) \langle p\lambda| A \rangle \]

(40)
or in the alternative form

\[ A^i(x) = \sum_{\lambda} \int dq V_{q,\lambda}^i(x) \langle q \lambda| A \rangle \]

(41)
this reinforces the interpretation of \( \hat{q} \) as a position operator and of \( V_{q,\lambda}(x) \) (given explicitly
in (10) as the configuration space amplitude of the photon state localized at \( q \) with helicity
\( \lambda \).

Note that we could add zero components to the polarization vectors to form a fourvector-
like object \( A_\mu(x) \). However, as shown in (5) this object does not transform as a fourvector,
but inhomogeneously as a connection:

\[ U[\Lambda]A_\mu(x, \lambda)U[\Lambda^{-1}] = \Lambda_\nu^\mu (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2p}} e^{-ipAx} \{ \epsilon_\nu(p, \lambda) - p_\nu \Omega(p, \lambda; \Lambda) \} \langle p\lambda| A \rangle \]

(42)
A tensor-like covariant object for the photon can be defined as

$$F_{\mu\nu}(x) = (2\pi)^{-3/2} \sum_\lambda \int \frac{d^3p}{\sqrt{2|p|}} e^{ipx} \{ p_\mu \epsilon_\nu(p, \lambda) - p_\nu \epsilon_\mu(p, \lambda) \} \langle p\lambda | A \rangle$$ (43)

This contains the same information that $A_\mu$, but transforms as an antisymmetric tensor; its components constitute the electric and magnetic fields. In the next section we shall use (43) to construct and solve the photon Maxwell equations in vacuum.

**Time operator**

The Galilean boost operator in the NR representations for elementary systems has only three components $\hat{G}_i, i = 1, 2, 3$. Each of them is associated to a component of the position operator. There is no room in the representation space for an additional boost associated to the time. In fact, time is invariant under Galilean boosts. Most likely, this is hidden among the reasons [2, 35, 36, 37, 38, 39, 40, 41, 42, 43] for the difficulty of finding a time operator that behave just like the other position operators, something tempting in the study of individual particles. As explained in [43], this is a misleading approach to the role of time in quantum mechanics and its use led to several dead ends in the past. A question to be taken into account is that time translations are given in terms the Hamiltonian $H$ which is not independent, but is fixed as a function of the other operators in the representation by the mass shell condition. The same is to be expected for a time operator, instead of the addition of a new independent component to the position, a given function of the basic operators, whose explicit form depends on the very properties of the representation.

Leaving aside the far-reaching but still out of reach question of the status of Newtonian time in quantum mechanics, and the search of its corresponding operator – if any – it is possible to identify [44, 45] time-like properties of quantum systems. The simplest of these appears for free elementary particles in the form of the time of arrival at a fixed position $X$, a deceptively simple property whose analysis in quantum mechanics is very delicate and full of traps. Classically it is a derived quantity, a function of the initial position and momentum whose values are either “never” (when $X$ is out of the particle trajectory), or a real number that solves the equations of motion for $t$ as a function of $X$. Notice that some kind of integrability is implicit for this classical notion to work properly [46]. Quantization requires the splitting of the Hilbert space into two orthogonal subspaces, one that contains
the never detected states, and other that contains the eventually detected ones. The time of
decay acts within this last subspace \([44, 47]\), where it is represented by an operator which
is not self-adjoint but only maximally symmetric. Its eigenstates are not orthogonal, so that
instead a projector valued measure, the statistics of the times of arrival at a detector can
only be analyzed in terms of positive operators \([48]\). Below, I shall show how this works for
the photon.

First, we shall study the time dependence hidden in the bracket \(\langle p\lambda | A(t) \rangle\). Taking into
account that \(p^2 = 0\), and that \(p \epsilon(p, \lambda) = 0\), we get from (43) the equations \(\partial^\mu F_{\mu\nu} = 0\).
These cover the full set of Maxwell equations due to the duality relation among null fields
in vacuum \([31]\):

\[
F^{\mu\nu}(x, \lambda) = i \frac{\lambda}{2} \epsilon^{\mu
u\rho\sigma} F_{\rho\sigma}(x, \lambda) \Rightarrow \begin{cases}
F_{ij} = \epsilon_{ijk}(E_k + i\lambda B_k) \\
F_{0k} = -i\lambda(E_k + i\lambda B_k)
\end{cases} \quad (44)
\]

We now have two equations \(\partial^i F_{i0} = 0\) and \(\partial^0 F_{0j} + \partial^i F_{ij} = 0\). Calling \(F(x, \lambda) = E(x, \lambda) +
i\lambda B(x, \lambda)\), the first equation reduces to the divergenceless condition \(\nabla F(x, \lambda) = 0\) and the
second to the Schrödinger equation \([3, 16, 18, 20, 21]\):

\[
i \frac{\partial F(x, \lambda)}{\partial t} = \lambda c \nabla \wedge F(x, \lambda) \quad (45)
\]

Recalling the expression for the spin matrix \((S^i)_{jk} = i\epsilon_{ijk}\), this equation can be given a more
transparent form:

\[
i \frac{\partial F_i(x, \lambda)}{\partial t} = \lambda (Sp)_{ij} F_j(x, \lambda) \quad (46)
\]

where \(p = -i\nabla_x\). Using now (43) in the definition of \(F\) we obtain

\[
F(x, \lambda) = (2\pi)^{-3/2} \int d^3p \sqrt{2|p|} i e^{ipx} \epsilon(p, \lambda) \langle p\lambda | A(t) \rangle \quad (47)
\]

therefore, \((Sp)_{ij} F_j = \lambda |p| F_j\) and hence,

\[
i \frac{\partial F(x, \lambda)}{\partial t} = c |p| F(x, \lambda) \quad (48)
\]

Therefore, the photon Hamiltonian is \(H = c |p| \delta_{\lambda\lambda'}\) in the helicity basis.

Due to the simple expression of the photon Hamiltonian, the position operator in the in
the helicity basis evolves quite simply:

\[
\frac{d\hat{q}^a}{dt} = i[\hat{q}^a, c\sqrt{\hat{p}^2}] = c \frac{\hat{p}^a}{|\hat{p}|}, \quad \frac{d\hat{p}^a}{dt} = 0 \quad \Rightarrow \quad \hat{q}^a(t) = \hat{q}^a + c \frac{\hat{p}^a}{|\hat{p}|} t, \quad \hat{p}^a(t) = \hat{p}^a \quad (49)
\]
In the first of these Eqs. we displayed explicitly the hamiltonian obtained from the mass shell condition $H(p) = c\sqrt{p^2}$, but then we continue to denote it by $|p|$ as before. $\hat{q}^a$ and $\hat{p}^a$ are the position and momentum operators of the photon at $t = 0$, while $\hat{q}^a(t)$ is the position operator at a time $t$ (we are working in the Heisenberg picture). All this is trivial and prompts to the seemingly innocent question: When does the photon arrives at a position $\mathbf{x}$? This is the alternative to the standard quantum mechanical question: Where is the photon at time $t$? Much in the same way that the question “where?” brings a position operator whose probability distribution is used to ascertain at what place?, the question “when?” demands the introduction of a time operator, with the meaning of time of arrival (i.e. at what instant?), to close the logical loop. Several obstructions prevent the existence of such an operator in the standard formulation of quantum mechanics. The earliest one, which turned out to be the strongest, was formulated in the early days of quantum theory by Pauli [2]: The simultaneous existence of unitary representations for both, energy and time conjugate operators, is incompatible with a bounded hamiltonian, opening up at least a sort of infrared catastrophe (see however [43]). This led to renounce to the self-adjointness of the time operator, keeping its maximal symmetry only (i.e. $T^* = T$). In these conditions, time eigenstates are no longer orthogonal (unless some steering regularization is used [44]) and, instead of the traditional projector valued measure, it is necessary to turn to a positive operator valued measure [48] to interpret the formalism. The interested reader is referred to the review [49] to get an account of these and related issues, and references to the original literature.

Generally, the search of a time of arrival operator has been undertaken in the case of one space dimension. The present author studied the case of the free particle in three space dimensions [47] concluding that the detected states are confined within a subspace of the whole Hilbert space. Outside it is the realm of no detection, that is, of those states for which the time of arrival is “never.” In classical terms, they miss the detector, whose efficiency – a different question – is assumed to be 1. On physical grounds, detection requires a constraint: that the particle momentum is parallel to the line joining $\mathbf{q}$ with the arrival (detector) position $\mathbf{z}$. This vector constraint and the free particle Hamiltonian form a first class system. The use of this formulation made possible the obtention in [47] of a very simple solution for the time of arrival in three dimensional space, basically an extension of the 1-D results.
A time operator for photons requires of a 3-D setting. Not only because the transverse-vector character of the electromagnetic field stripes the 1-D approach of credibility. Also from the practical side: we shall analyze later on the arrival of photons through inhomogeneous media, where direction changes will in general take place, whose description needs of more than the mere distance covered. Therefore, we shall examine the first class constrained system that evolves in the detected subspace as the first step towards the time of arrival of photons.

The task can be formulated in very simple terms: If \( z \) is the detector position, at what time \( t \) is \( \hat{q}^a(t) = z^a \)? In other words: How to invert the equation (49), namely \( z^a = \hat{q}^a + c (\hat{p}^a / \hat{p}) t \), to get \( t \)? Several comments are in order here. First, we are promoting \( t \) to the category of a q-number, while demoting \( \hat{q}(t) \) to a given, external, parameter. This is the very task to accomplish to define a time operator. Second, the evolution equations that we have to invert to obtain the time of arrival is the set (49) of three equations, one per component, depending on a unique parameter \( t \). To be compatible, they have to satisfy the constraint:

\[
L_a(z) = \epsilon_{abc} (\hat{q}_b - z_b) \hat{p}_c = 0
\]

(50)

To quantize this constrained system we borrow from the method of Dirac [50]. Classically, the constraint guarantees that the orbital angular momentum of the particle is \( z \wedge \mathbf{p} \), so that \( z \) is a point of its trajectory. The total Hamiltonian formed by adding the constraints to the original one is:

\[
H^\uparrow(z) = c\sqrt{p^2} + \mu_a L_a(z)
\]

(51)

where \( q^a \) and \( p^a \) are the dynamical variables to become operators after quantization, the \( \mu \)'s are Lagrange multipliers, and \( z \) is an external vector parameter corresponding to the detection position. The system is first class:

\[
\{ L_a(z), L_b(z) \} = \epsilon_{abc} L_c(z), \quad \{ L_a(z), H^\uparrow(z) \} = \epsilon_{abc} \mu_b L_c(z)
\]

(52)

where \( \{ , \} \) indicate Poisson brackets as this is still classical dynamics. The evolution of the constraints does not produce additional (secondary) constraints. Hence, the Hamiltonian (51) is enough to account for the evolution of the constrained system.

When quantizing the system, the vanishing of the constraint translates into a kind of subsidiary condition:

\[
\left( \hat{L}_a(z) \hat{\Phi} \right)_i(p) = 0
\]

(53)
The set of vectors of the Hilbert space $\tilde{\Phi}_i(p)$ that satisfy the above equation form the subspace $H_z$ of the states that could, eventually, be detected at $z$. On the other hand, using (27), Eq. (53) can be written as:

$$\epsilon_{abc} \sqrt{2|p|} \sum_{\lambda} \epsilon_i(p, \lambda) \left\{ i \frac{\partial}{\partial p_b} - z_b \right\} p_c \langle p\lambda|\Phi \rangle = 0 \quad (54)$$

whose solution is

$$\langle p\lambda |\Phi_z \rangle = e^{-ipz} \Phi(|p|, \lambda, z) \quad (55)$$

Note that the functions $\Phi$ may depend on the modulus of the momentum (but not on its direction), on the helicity and, possibly, on the detection point $z$. However, this last dependence has to be switched off to maintain translational invariance.

We now proceed to invert Eq. (49). First of all we write down the action of the position operator on the detected subspace:

$$\hat{q}^a \langle p\lambda |\Phi_z \rangle = e^{-ipz} \left\{ z^a + i \frac{p^a}{|p|} \frac{\partial}{\partial |p|} \right\} \Phi(|p|, \lambda) \quad (56)$$

Hence, with $z^a$ a parameter and $\hat{t}(z)$ an operator, Eq. (49) reads:

$$z^a = \hat{q}^a + c \frac{p^a}{|p|} \hat{t}(z), \Rightarrow \quad e^{-ipz} \left\{ z^a + i \frac{p^a}{|p|} \frac{\partial}{\partial |p|} \right\} \Phi(|p|, \lambda) + c \frac{p^a}{|p|} \hat{t}(z) e^{-ipz} \Phi(|p|, \lambda) = 0 \quad (57)$$

This equation has to be valid whatever the function $\Phi$ chosen. This serves to define $\hat{t}()$: It is precisely the operator that transforms (57) into an identity in $H_z$, that is:

$$\hat{t}(z) \approx -i e^{-ipz} \frac{\partial}{\partial |p|} e^{ipz}, \hat{t}(z) = -i e^{-ipz} \frac{1}{|p|} \frac{\partial}{\partial |p|} |p| e^{ipz} \quad (58)$$

The symbol $\approx$ at the left indicates equal up to operator ordering, something that we fix by the condition that $\hat{t}$ be maximally symmetric in the integration by parts with the measure $d^3p$ of the $|p\lambda\rangle$ basis. This produces the operator defined at the right hand side, that we shall use as the time of arrival operator in what follows. It depends parametrically on $z$ in as much the same way as the operators depend parametrically on $t$. Incidentally, this makes us recall that we are working in the Heisenberg picture at $t = 0$. It is straightforward to show that, when some time $t_0$ has elapsed, the time operator shifts to $\hat{t}(z, t_0) = \hat{t}(z) - t_0$, and that the arrival occurs at a time $t_z$ such that $\hat{t}(z, t_z) = 0$.

The eigenfunctions $\langle p\lambda|t_z \rangle$ of $\hat{t}(z)$ obtained by solving the eigenvalue equation $\hat{t}(z) \langle p\lambda|t_z \rangle = t \langle p\lambda|t_z \rangle$ are proportional to $\exp i(Ht - pz)/|p|$. Due to the fact that
\[
\lim_{|p| \to 0} (|p\rangle \langle p| t \lambda) \neq 0, \text{ being } |p| = 0 \text{ the lower bound of the Hamiltonian, the operator (58) can not be self-adjoint. This is similar to the case of the radial momentum } p_r = i r^{-1} (\partial / \partial r) r \text{ in three space dimensions (but notice that this does not prevent us from using } H = p^2_r / 2m + l(l+1)/2mr^2 \text{ as Hamiltonian; what is done in this case is to restrict the behavior of the wave function at the boundary } r = 0). \text{ The time operator commutes with the helicity so, taking into account that } \langle p^\lambda | q^{\lambda'} \rangle = \delta^{\lambda \lambda'} (2\pi)^{-3/2} \exp(-i pq), \text{ we could write}
\]
\[
\langle p^\lambda' | t^\lambda; z \rangle = \langle p^\lambda' | H e^{i H t} | z^\lambda \rangle \Rightarrow |t^\lambda; z\rangle = \frac{1}{H} e^{i H t} |z^\lambda\rangle
\]

This notation is highly symbolic mainly due to the fact that \( z \) is just an external parameter belonging to the experimental set-up, the observer’s will, etc, so that there is nothing like \( z \) among the properties of the particle. However, the above expression is correct if one considers that \( |z^\lambda\rangle \) is the eigenket of the particle’s position operator with eigenvalue \( z \).

Writing the detected states (55) in the same form may through some light on the meaning of the notation:
\[
\langle p^\lambda | \Phi_z \rangle = \langle p^\lambda | \Phi(H, \lambda) \rangle |z^\lambda\rangle
\]

This is the effect of the subsidiary condition (53): it projects on the detector position \( z \), keeping only that part of the state that is in s-wave relative to \( z \) (hence the \( \Phi \) dependence on \( |p| \) alone). The lack of completeness this produces on the time operator, and the associated interpretation, was discussed with some detail in ref. [47] for the relativistic massive spinless particle. I shall repeat here the two main results tailored to the massless, helicity \( \pm 1 \), photon case:

1. The time eigenstates are not orthogonal:
\[
\sum_{\lambda'} \int d^3p \quad \langle t^\lambda; z | p^\lambda' \rangle \langle p^\lambda' | t^\lambda''; z \rangle = \delta_{\lambda \lambda'} \frac{1}{2 \pi^2} \int_0^\infty d|p| e^{i|p|(t'-t)} = \delta_{\lambda \lambda'} \frac{1}{2 \pi^2} \frac{i}{t' - t + i \epsilon}
\]

2. The basis is complete only within \( \mathcal{H}_z \). In other words, the projection over states orthogonal to the detector position is excluded from the decomposition of the identity in terms of time eigenstates:
\[
\sum_{\lambda'} \int_{-\infty}^{+\infty} dt \quad \langle p^\lambda | t^\lambda'; z \rangle \langle t^\lambda'; z | p''^\lambda'' \rangle = \delta_{\lambda \lambda'} \frac{2\pi}{|p|^2} \delta(|p| - |p''|) \langle p^\lambda | z^\lambda \rangle \langle z^\lambda | p''^\lambda'' \rangle \Rightarrow \sum_{\lambda'} \int_{-\infty}^{+\infty} dt \quad \langle p^\lambda | t^\lambda'; z \rangle \langle t^\lambda'; z | \Phi_z \rangle = \langle p^\lambda | \Phi_z \rangle \quad \forall \Phi_z \in \mathcal{H}_z
\]
We recall that due to the non-orthogonality of the states, the spectral decomposition of the operator

\[ \hat{t}(z) = \sum_{\lambda} \int dt \ t \ | t \ \lambda; \ z \rangle \langle t \ \lambda; \ z | \]  

(63)
defines a positive operator valued measure only (not a projector valued one) that can be used to give the probability that the arrival of the state \( \psi \) at the detector occurs in the time \( t \) as

\[ P_\psi(t; z) = \sum_{\lambda} |\langle t \ \lambda; \ z | \psi \rangle|^2. \]

This really is a probability density, something not reflected on the notation for the sake of simplicity. Note also that, due to the lack of completeness, the probability of eventually arriving at \( z \) (in any time) \( P_\psi(z) = \int dt \ P_\psi(t; z) \leq 1 \), and may be zero for states \( \psi \notin \mathcal{H}_z \). Finally, the mean value of the time of arrival at \( z \) of a particle in the state \( \psi \) is \( t_\psi(z) = \langle \psi | \hat{t}(z) | \psi \rangle / P_\psi(z) \), this excludes counterfactuals (the case \( P_\psi(z) = 0 \)) as it should.

Conclusions

Photons are eventually detected through their interaction with matter. Gauge invariance singles out the minimal coupling of the potentials to the currents \( J_\mu(x, t) \) of additively conserved quantum numbers (the electric charge in this case). The coupling to Pauli like currents (e. g. \( \partial^\nu \bar{\psi} \sigma_{\mu\nu} \psi \)) would never produce the finite amplitudes for absorbing or emitting soft photons observed experimentally. These facts and their implications have been thoroughly analyzed by S. Weinberg [31] and other authors. Of course, by means of a canonical transformation [51, 52] the minimal coupling interaction can be cast into a multipolar form. We will take into account this structure of the interactions when discussing the detection of photons. Assume for simplicity the case of broad-band photodetectors for which the counting rates are given by the energy density of the field \( \langle \psi | \hat{E}^\dagger(x, t) \cdot \hat{E}(x, t) | \psi \rangle \). Then, the probability that the state \( \psi \) be localized (in the plain sense of being detected by a detector) in a neighborhood \( \Delta \) around \( (x, t) \) is given in terms of the fraction of the total energy that is within \( \Delta \).

Absolute localization of quantum mechanical systems [5] -that is, the condition that the probability of finding the system out of some finite volume vanish- is such a strong condition that it violates causality [53, 54]. In other words, any free particle initially confined in a finite volume, continues in it forever, or immediately spreads to infinity. This result applies to free relativistic and non relativistic particles, to complex systems, in the presence of interactions,
etc. Surprisingly the only requirement for this is that the system Hamiltonian be bounded from below. This prompts the question of what is the maximal degree of localization to be expected for a particle. An important clue [55] is that – even if the localization outside the finite volume is not absolute, but exponentially bounded tails are permitted – the probability spreads out to infinity faster than with any finite propagation speed.

The limits to photon localization have been strengthened recently [26] based on simple physical requirements to be satisfied by one photon states. Basically, these are: a) That \( \langle p \lambda | \psi \rangle \) can be given a probabilistic interpretation (something that we discussed below (16)) and b) That the Hamiltonian be bounded from below. Then, the Paley-Wiener Theorem VIII [56] says that the fall-off of the photon wavefunction \( \langle q \lambda | \psi \rangle \) as \( |q| \to \infty \) is slower than \( \exp(-a|q|^r) \), where \( a, r \) are positive constants and \( r < 1 \). As the physical requirements noted above apply to all types of particles, the same occurs to the limit of almost exponential localization: it applies to all kind of particles. This puts photons at the same level than the other particles in what refers to localization. A recent analysis of spontaneous emission from excited atoms [27] has shown the possibility of producing entangled atom-photon states where the photon wave packets have Gaussian tails. This explicit breaking of the barrier of exponential localization is a product of the entangled final state.

These results shall likely find their application in the field of quantum information, and shall promote new developments in quantum optics. Some necessary tools like good position [1] and time of arrival operators and their associated probabilistic interpretations are provided in this paper. We are completing a detailed analysis of the application of these tools to the tunneling through photonic band gaps, to HOM [57] interferometry and entanglement [27, 58], and to the superluminal propagation detected in several experiments [59, 60, 61].

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