The pulsing CPSD method for subcritical assemblies with pulsed sources

Daniel Ballester * and

Department of Applied Mathematics,
Polytechnic University of Valencia, 46022 Valencia, Spain

José L. Muñoz-Cobo

Department of Chemical and Nuclear Engineering,
Polytechnic University of Valencia, 46022 Valencia, Spain

Abstract

Stochastic neutron transport theory is applied to the derivation of the two-neutron-detectors cross power spectral density for subcritical assemblies when external pulsed sources are used. A general relationship between the two-detector probability generating functions of the kernel and the source is obtained considering the contribution to detectors statistics of both the pulsed source and the intrinsic neutron source. An expansion in $\alpha$-eigenvalues is derived for the final solution, which permits to take into account the effect of higher harmonics in subcritical systems. Further, expressions corresponding to the fundamental mode approximation are compared with recent results from experiments performed under the MUSE-4 European research project.

1 Introduction

In last years, researchers have shown an increasing interest on the conceptual development of accelerator-driven systems (ADS) for nuclear waste transmutation and energy production purposes. An important issue regarding its future industrial applicability is the development of a periodically subcriticality level measurement and monitoring technique, since both its operation safety and

* Corresponding author: Fax +34 963 877 669
Email addresses: dabalber@mat.upv.es (Daniel Ballester), jlcobos@iqn.upv.es (José L. Muñoz-Cobo).
its performance as a part of the nuclear fuel cycle shall be seriously affected by this variable.

Following the study of the neutron fluctuations in a multiplying medium, Courant and Wallace (1947), several static and dynamic methods have been proposed and studied for years concerning their applicability for the determination of some nuclear reactor physics parameters (see Uhrig (1970); Williams (1974); Lewins (1978); Carta and D’Angelo (1999)). Within the group of dynamic techniques, apparently the utilisation of the neutron-fission chain fluctuations for nuclear assemblies subcriticality determination was firstly suggested by Bruno Rossi (the Rossi-\(\alpha\) method). In these methods neutron detector counting rates related to individual fission-chain events must be discerned from the total counting rate, therefore these methods are applicable to subcritical systems near delayed critical conditions, Carta and D’Angelo (1999).

Further, dynamic methods based on a time-dependent external neutron source were proposed by Perez et al. (1964). Recently, the use of these methods has increased during the MUSE European experimental studies carried out at the MASURCA facility (Cadarache, France) due to their applicability in order to investigate ADS kinetic parameters. In these experiments D-D and D-T neutron sources running in pulsed mode have been used, although the utilisation of an external spallation proton source has also been thought. Anyway, measurements can be done in two different ways (Valentine et al. (2000); Degweker (2003); Ceder and Pázsit (2003)): in the first one, the neutron detector time gate is synchronized with the external neutron pulse injection, thus this method is referred to as deterministic pulsing method, whereas, in the second case, the relative delay between the neutron pulse injection and the beginning of the neutron counting time is uniformly sampled between zero and the pulsed source period, which is known as stochastic pulsing method. For the latter case, the neutron source can be assumed to have the form

\[
S(t) = k \sum_{m=-\infty}^{\infty} \delta(t - (\xi + mT)),
\]

where \(k\) is the number of protons injected per proton pulse, \(T\) is the pulsed source period, and \(\xi\) is uniformly sampled within the time interval \([0, T]\). Obviously, for the deterministic pulsing method we will put \(\xi = 0\).

On the other hand, Pál (1958) developed a general theory for the study of the stochastic neutron field. This model, complemented afterwards by Bell (1965), completely describe the stochastic neutron field in a fissile assembly. Later, Muñoz-Cobo et al. (1987) derived expressions for the variance of the number of counts in a detector and the CPSD for a Poissonian source without delayed neutrons from the general neutron stochastic transport theory. These approaches permit to go beyond classical point kinetic approximations, taking
into consideration general problems with spatial, spectral, and angular dependence. They have been extensively applied to nuclear subcriticality safety and non-destructive nuclear fuel assay problems.

Classical reactor noise methods are not correct when pulsed or correlated sources are used (Matthes et al. (1988); Behringer and Wydler (1999)). For these kind of sources, forward Kolmogorov’s approach is not valid because of the non-Markovian character of the process, while backward Green’s function description needs to go beyond Poissonian behaviour (Degweker (2003); Ballester and Muñoz-Cobo (2005)).

In our work we have derived a relationship between the source probability generating function and the kernel probability generating function when non-Poissonian spallation neutron sources are considered. We have also studied the effect of the intrinsic neutron source due to spontaneous fission occurring in major actinides forming part of the nuclear fuel of an ADS. This result can considered as a generalisation of the master equation obtained by Ballester and Muñoz-Cobo (2005) for cross statistical descriptors (problems with more than one detector).

In particular, we admit the intrinsic source spontaneous fission process to behave as a Poissonian one, albeit neutron emission multiplicity corresponding to spontaneous fission events has also been included.

In this paper we have neglected the contribution of delayed neutrons, therefore all quantities appearing here can be interpreted as prompt variables for short time-scales, in comparison with the delayed neutron precursors lifetimes.

2 General expression for the relationship between the source pgf and the kernel pgf

The derivation of the general relationship between the source pgf and the kernel pgf shall be based on two well known results (see, e.g., Lande (2003)):

- given a random variable, \( Z = X_1 + X_2 + \ldots + X_n \), \( X_i, i = 1, 2, \ldots, n \), being mutually independent discrete random variables, and \( G_{X_i}(s) = \sum_j s^j P_{X_i=j} \) being the probability generating function associated to \( X_i \), where \( P_{X_i=j} \) is the probability for the occurrence of the event \( X_i = j \), \( \sum_j P_{X_i=j} = 1 \), then the probability generating function of \( Z \) can be expressed as
  \[
  G_Z(s) = \prod_i G_{X_i}(s) ;
  \] (2)

- given a random variable \( Z = X_1 + X_2 + \ldots + X_y \), \( X \equiv X_i \ \forall i \) and \( Y \) being
mutually independent discrete random variables, then

\[ G_Z(s) = G_Y(G_X(s)). \] (3)

In our system we must consider two independent sources leading to detector counts, Ballester and Muñoz-Cobo (2005): neutrons appearing from spontaneous fission of isotopes contained within the nuclear fuel, and neutrons coming from the spallation source induced by nuclear interactions of the proton beam with the target material.

In particular, for the derivation of the cross correlation between two neutron detectors, we shall consider the two-dimensional discrete random variable \( Z = (Z_1, Z_2) \), where \( Z_i, i = 1, 2 \), is the number of neutron detections gathered by the \( i \)-th detector during the time interval \((t_{i-1}, t_i)\). And according to our previous discussion, \( Z_i = N_i + M_i \), where \( N_i, M_i \) are the number of detections registered by the \( i \)-th detector and coming from the intrinsic spontaneous fission source and the external pulsed source, respectively; equivalently, \( N = (N_1, N_2), M = (M_1, M_2) \).

In case of the intrinsic spontaneous fission source, the number of detector counts within the time interval \((t_{i-1}, t_i)\), \( N_i \), can be expressed as

\[ N_i = X_{i_1} + X_{i_2} + \ldots + X_{i_\mathcal{Y}}, \] (4)

where \( \mathcal{Y} \) is the number of spontaneous fission events occurring within the fuel material and \( X_{ij} \equiv X_i \) is the number of detector counts gathered by the \( i \)-th detector corresponding to each spontaneous fission event, both being mutually independent discrete random variables, Ballester and Muñoz-Cobo (2005); its corresponding two-dimensional discrete random variable will be denoted as \( X = (X_1, X_2) \). We can define the following probability functions:

\[
P_{n_1n_2} (d_1(t_{i_1}), d_2(t_{i_2}))\]

is the joint probability to have \( N_i = n_i, i = 1, 2 \), detector counts within the time interval \((t_{i-1}, t_i)\) upon the introduction of the neutron intrinsic source in the remote past, \( P_y(t) \) is the probability to have \( \mathcal{Y} = y \) intrinsic source spontaneous fission events within the fuel material at time \( t_f = \max\{t_{i_1}, t_{i_2}\} \) upon the introduction of this neutron source in the remote past, \( P_{x_1x_2}(r, t|d_1(t_{i_1}), d_2(t_{i_2})) \) is the joint probability to have \( X_i = x_i, i = 1, 2 \), counts in the \( i \)-th detector within the time interval \((t_{i-1}, t_i)\) after an intrinsic source disintegration event at time \( t \) and position \( r \). In addition, \( K_{n_1n_2}(\vartheta, t|d_1(t_{i_1}), d_2(t_{i_2})) \) is the joint probability of having \( N_i = n_i, i = 1, 2 \), detector counts per single neutron injected in the phase-space point \( \vartheta = (r, v, \Omega) \) at instant \( t \). Associated with these probabilities we have the source probability generating functions

\[
G^\text{ref}_S(s_1, s_2|d_1(t_{i_1}), d_2(t_{i_2})) = \sum_{n_1, n_2=0}^{\infty} s_1^{n_1} s_2^{n_2} P_{n_1n_2} (d_1(t_{i_1}), d_2(t_{i_2})), \] (5)
\[ G_{S,Y}(s, t_f) = \sum_{y=0}^{\infty} s^y P_y(t_f), \quad (6) \]

\[ G_{S,X}(s_1, s_2, r, t | d_1(t_{t_1}), d_2(t_{t_2})) = \sum_{x_1, x_2=0}^{\infty} s_1^{x_1} s_2^{x_2} P_{x_1x_2}(r, t | d_1(t_{t_1}), d_2(t_{t_2})), \quad (7) \]

and the kernel probability generating function

\[ G_{K}(s_1, s_2, \vartheta, t | d_1(t_{t_1}), d_2(t_{t_2})) = \sum_{n_1, n_2=0}^{\infty} s_1^{n_1} s_2^{n_2} K_{n_1n_2}(\vartheta, t | d_1(t_{t_1}), d_2(t_{t_2})). \quad (8) \]

If we apply the result given by equation (3) and consider the fact that the spontaneous fission neutron source will behave as a Poissonian source, then we find that (Bell (1965); Ballester and Muñoz-Cobo (2005))

\[ G_{S,Y}(s, t_f) = \exp \left( \int_{-\infty}^{t_f} dt \int \! dr \lambda_{sf} N(t) \rho_{sf}(r) [s - 1] \right), \quad (9) \]

\[ G_{Ssf}(s_1, s_2 | d_1(t_{t_1}), d_2(t_{t_2})) = G_{S,Y}(G_{S,X}(s_1, s_2, r, t | d_1(t_{t_1}), d_2(t_{t_2})), t_f). \quad (10) \]

where we have supposed the source, which is given by the product of its time dependent activity, \( \lambda_{sf} N(t) \), and the shape probability distribution function \( \rho_{sf}(r) \), to be introduced in the remote past. In addition, the relationship between the spontaneous disintegration source probability generating function and the kernel probability generating function is given by (Muñoz-Cobo et al. (1987); Muñoz-Cobo and Verdú (1987))

\[ G_{S,X}(s_1, s_2, r, t | d_1(t_{t_1}), d_2(t_{t_2})) = \sum_{j=0}^{I_{sf}} \varepsilon_{sf}^j \left[ \int \! dv \int \! d\Omega \frac{\chi_S(v)}{4\pi} G_{K}(s_1, s_2, \vartheta, t | d_1(t_{t_1}), d_2(t_{t_2})) \right]^j, \quad (11) \]

\( \varepsilon_{sf}^j \) being the probability of emission of \( j \) neutrons after a spontaneous fission within the fuel material, \( I_{sf} \) the maximum number of spontaneous fission neutrons emitted after a source disintegration, and \( \chi_S(v) \) their corresponding spectrum.

On the other hand, in order to obtain the relationship applicable for the proton-beam-driven spallation neutron source we can proceed in the following way: firstly, let us consider the expression for the joint probability to register \( M_1 = m_1, i = 1, 2 \), neutron counts in each neutron detector during the corresponding detector time interval \( (t_{t_i} - \tau_{ci}, t_{t_i}) \) following the injection.
of one single proton (1p) belonging to one proton source pulse injected at a random time $\xi \in (0, T)$, $T$ being the pulsed proton source period, i.e., Ballester and Muñoz-Cobo (2005),

$$P_{m_1 m_2}^{1p}(\xi | d_1 (t_f), d_2 (t_f))$$

$$= \int dr \rho_{sp}(r) \sum_{j=0}^{I_{sp}} \varepsilon_j \sum_{m_1^{(1)}, m_2^{(1)}=0}^{m_1, m_2} \cdots \sum_{m_1^{(j)}, m_2^{(j)}=0}^{m_1, m_2} \prod_{i=1}^{j} \int dv_i \int d\Omega_i$$

$$\times f_{sp}(v_i, \Omega_i) K_{m_1^{(i)} m_2^{(i)}}(r, v_i, \Omega_i, \xi | d_1 (t_f), d_2 (t_f)),$$  \hspace{1cm} (12)

restricted by the constraints $m_1^{(1)} + m_2^{(2)} + \ldots m_i^{(i)} = m_i$, $i = 1, 2$, where $\varepsilon_j$ is the probability of emission of $j$ neutrons after a spallation interaction within the target material, $I_{sp}$ the maximum number of neutrons emitted in each spallation interaction, $\rho_{sp}(r)$ is the spatial distribution function for neutrons born after a spallation interaction within the target material, and $f_{sp}(v_i, \Omega_i)$ the spectral and angular probability distribution function corresponding to these spallation neutrons. If we multiply the previous expression by $s_{m_1} s_{m_2}$ and sum up from $m_1, m_2 = 0$ to $\infty$, we get the relationship between the spallation source probability generating function and the kernel one for one proton randomly injected,

$$G_{1p}(s_1, s_2, \xi | d_1 (t_f), d_2 (t_f))$$

$$= \int dr \rho_{sp}(r) \sum_{j=0}^{I_{sp}} \varepsilon_j [T_{sp} G_{K}(s_1, s_2, \vartheta, \xi | d_1 (t_f), d_2 (t_f))]^j.$$  \hspace{1cm} (13)

where we have used the spallation operator $T_{sp} \circ = \int dv_i \int d\Omega_i f_{sp}(v_i, \Omega_i) \circ$. With this expression and taking into account the relationships expressed at the beginning of this Section, and admitting that the number of detector counts after the introduction of each proton belonging to the same source pulse can be considered as independent discrete random variables, we can derive the relationship between the spallation source and the kernel probability generating functions for the injection of $k$ protons in one proton pulse (pp) at time $\xi$,

$$G_{pp,k}(s_1, s_2, \xi | d_1 (t_f), d_2 (t_f))$$

$$= \left[ \int dr \rho_{sp}(r) \sum_{j=0}^{I_{sp}} \varepsilon_j [T_{sp} G_{K}(s_1, s_2, \vartheta, \xi | d_1 (t_f), d_2 (t_f))]^j \right]^k.$$  \hspace{1cm} (14)

But, in general, the number of protons injected per accelerator pulse might be considered to be a discrete random variable, thus, the correct expression for the relationship between the spallation source and the kernel pgfs for the injection of one proton pulse can be recast as
\[ G_{pp} (s_1, s_2, \xi |d_1 (t_{f_1}) , d_2 (t_{f_2})) = \sum_{k=0}^{I_{pp}} \varepsilon_{k}^{pp} G_{pp,k} (s_1, s_2, \xi |d_1 (t_{f_1}) , d_2 (t_{f_2})) , (15) \]

where \( \varepsilon_{k}^{pp} \) is the probability for the accelerator to inject \( k \) protons per proton pulse and \( I_{pp} \) is the maximum number of protons that can be introduced in the system per proton pulse.

Considering this expression, for a periodic pulsed proton source such as that one given by equation (1), we will have

\[ G_{sp} (s_1, s_2, \xi |d_1 (t_{f_1}) , d_2 (t_{f_2})) = \prod_{m=-\infty}^{\infty} G_{pp} (s_1, s_2, \xi + mT |d_1 (t_{f_1}) , d_2 (t_{f_2}) ) \]. (16)

In general, neutron detectors will register counts coming from both, spontaneous fission and spallation, sources which can be treated as mutually independent random variables [Ballester and Muñoz-Cobo (2005)], thus the probability generating function governing both processes will be given by the product of equations (10) and (16), that is,

\[ G_{S} (s_1, s_2, \xi |d_1 (t_{f_1}) , d_2 (t_{f_2})) = G_{S}^{sf} (s_1, s_2|d_1 (t_{f_1}) , d_2 (t_{f_2})) \times G_{S}^{sp} (s_1, s_2, \xi|d_1 (t_{f_1}) , d_2 (t_{f_2})) . \] (17)

This last equation expresses the relationship between the source probability generating function and the kernel probability generating function when we consider the effect of the intrinsic spontaneous fission source and the periodic pulsed spallation neutron source, Ballester and Muñoz-Cobo (2005). In particular, for the deterministic pulsed method we just need to choose the elapsed time \( \xi \) equal to zero, whereas, in order to apply the stochastic pulsing method we will calculate the expected value of (17), \( \xi \) being uniformly sampled between zero and the proton pulse period, \( T \), i.e.,

\[ G_{S} (s_1, s_2|d_1 (t_{f_1}) , d_2 (t_{f_2})) = \langle G_{S} (s_1, s_2, \xi|d_1 (t_{f_1}) , d_2 (t_{f_2})) \rangle_{\xi} \]

\[ = \int_{0}^{T} \frac{d\xi}{T} G_{S} (s_1, s_2, \xi|d_1 (t_{f_1}) , d_2 (t_{f_2})) . \] (18)
3 The Boltzmann neutron transport equation for counting problems from the stochastic neutron transport theory

In this Section we derive the integro-differential equation governing the kernel probability generating function $G_K(s_1, s_2, \vartheta, \xi|d_1(t_1), d_2(t_2))$. First of all, we need to obtain an expression for the probability function $K_{z_{12}}(\vartheta, t|d_1(t_1), d_2(t_2))$ for both neutron detectors. It can be done using a probability balance of mutually exclusive events (Muñoz-Cobo et al. (1987, 2000)). Then we shall multiply the probability balance equation of $K_{z_{12}}(\vartheta, t|d_1(t_1), d_2(t_2))$ by the factor $s_1^z s_2^z$ and then sum up from $z_1, z_2 = 0$ to $\infty$. Next, we need to apply the known Pál’s methodology (see Pál (1958); Bell (1965); Muñoz-Cobo et al. (1987)) to obtain the non-linear transport integro-differential equation satisfied by the kernel probability generating function, Muñoz-Cobo et al. (1987):

$$H G_K(s_1, s_2, \vartheta, t|d_1(t_1), d_2(t_2)) = S(G_K(s_1, s_2, r, \Omega', t|d_1(t_1), d_2(t_2))),$$

where we have defined the general time-dependent transport operator

$$H = - \left( \frac{1}{v} \frac{\partial}{\partial t} + \Omega \cdot \nabla - \Sigma_t(r, v) \right),$$

and the non-linear kernel pgf source operator

$$S(\circ) = \Sigma_t(r, v) \left\{ C^{(0,0)}_c(r, v, t) + s_1 C^{(1,0)}_c(r, v, t) + s_2 C^{(0,1)}_c(r, v, t) \\
+ \left( C^{(0,0)}_s(r, v, t) + s_1 C^{(1,0)}_s(r, v, t) + s_2 C^{(0,1)}_s(r, v, t) \right) \right\} \times \int dv' \int d\Omega' f_s(r, v, \Omega'|v', \Omega') \circ \\
+ \sum_{j=0}^1 \left( C^{(0,0)}_j(r, v, t) + s_1 C^{(1,0)}_j(r, v, t) + s_2 C^{(0,1)}_j(r, v, t) \right) \times \left[ \int dv' \int d\Omega' \frac{\chi(r, v')}{4\pi} \right]^j,$$

where

$$f_s(r, v, \Omega'|v', \Omega') = \begin{cases} f_s(v, \Omega'|v', \Omega') & \text{for } r \notin V_{D_1}, V_{D_2}, \\
f_s^{D_1}(v, \Omega'|v', \Omega') & \text{for } r \in V_{D_1}, \end{cases}$$

represents the probability distribution function for a neutron to exit with velocity and direction within $(v', v' + dv')$ and $(\Omega', \Omega' + d\Omega')$, respectively, after
a scattering event (the superscript $D_i$ applies for the $i$-th detector volume) with an incident neutron with velocity $v$ and direction $\Omega$, whereas the spectrum of neutrons emitted following a fission event is given by

$$ \chi(r, v) = \begin{cases} 
\chi(v) & \text{for } r \notin V_{D_1}, V_{D_2}, \\
\chi_{D_i}(v) & \text{for } r \in V_{D_i},
\end{cases} $$

that is, as before, for the system volume no subscript is used, while to refer to fissions occurring within either detector volume we will add the subscript $D_i$. $I$ is the maximum number of neutrons produced after a fission event.

In addition, in expression (21) we must specify the C-probabilities:

$$ C^{(0,0)}_c (r, v, t) = 
\begin{cases} 
\frac{\Sigma_t}{\Sigma_c} & \text{if } r \notin V_{D_1}, V_{D_2}, \\
\frac{\Sigma_{D_1}}{\Sigma_c} \left(1 - \eta_c^{D_1} \Delta (d_1 (t_{f_1}))\right) & \text{if } r \in V_{D_1}, \\
\frac{\Sigma_{D_2}}{\Sigma_c} \left(1 - \eta_c^{D_2} \Delta (d_2 (t_{f_2}))\right) & \text{if } r \in V_{D_2},
\end{cases} \quad (22) $$

is the probability to have zero detector counts following a capture event within the nuclear system ($V_{SYS}$) or within one of the detectors ($V_{D_i}$, $i = 1, 2$) after a given neutron interaction at position $r$ and time $t$. $\Sigma_t$ ($\Sigma^{D_i}_t$) denotes the neutron total macroscopic cross section for the system ($i$-th detector) volume, $\Sigma_c$ ($\Sigma^{D_i}_c$) is the neutron capture macroscopic cross section, and $\eta_c^{D_i}$ accounts for the $i$-th detector capture efficiency. $\Delta (d_i (t_{f_i})) = (H (t - (t_{f_i} - \tau_{c_i})) - H (t - t_{f_i}))$ is the time window for the $i$-th detector, $H (t)$ being the characteristic or Heaviside function. Similarly, for one neutron count after a capture event in the first detector, we have

$$ C^{(1,0)}_c (r, v, t) = 
\begin{cases} 
0 & \text{if } r \notin V_{D_1}, \\
\eta_c^{D_1} \frac{\Sigma_{D_1}}{\Sigma_c} \Delta (d_1 (t_{f_1})) & \text{if } r \in V_{D_1},
\end{cases} \quad (23) $$

The expression corresponding to the case of one neutron count following a neutron capture event in the second detector can be derived in an analogous way.

Next, in equation (21) we must also specify the probability to have zero counts in both detectors after a neutron scattering event at position $r$ and time $t$, $v$
being the incident neutron velocity, i.e.,

\[
C_s^{(0,0)}(r, v, t) = \begin{cases} 
\frac{\Sigma_s}{\Sigma_t} & \text{if } r \notin V_{D_1}, V_{D_2} \\
\frac{\Sigma_s}{\Sigma_t} \left( 1 - \eta_s^{D_1} \Delta (d_1 (t_f)) \right) & \text{if } r \in V_{D_1}, \\
\frac{\Sigma_s}{\Sigma_t} \left( 1 - \eta_s^{D_2} \Delta (d_2 (t_f)) \right) & \text{if } r \in V_{D_2},
\end{cases}
\]

(24)

where \(\Sigma_s (\Sigma_s^{D_i})\) denotes the neutron scattering macroscopic cross section for the system \((i\text{-th detector})\) volume and \(\eta_s^{D_i}\) is the \(i\text{-th detector}\) scattering efficiency.

In case of one detector count, for instance, in the first detector:

\[
C_s^{(1,0)}(r, v, t) = \begin{cases} 
0 & \text{if } r \notin V_{D_1}, \\
\eta_s^{D_1} \frac{\Sigma_s}{\Sigma_t} \Delta (d_1 (t_f)) & \text{if } r \in V_{D_1}.
\end{cases}
\]

(25)

Finally, in equation (21) the probability of occurrence of a fission event with emission of \(j \geq 0\) neutrons leading to zero detector counts following a neutron interaction at position \(r\) and time \(t\) is given by

\[
C_j^{(0,0)}(r, v, t) = \begin{cases} 
\varepsilon_j \frac{\Sigma_s}{\Sigma_t} & \text{if } r \notin V_{D_1}, V_{D_2}, \\
\varepsilon_j \frac{\Sigma_s}{\Sigma_t} \left( 1 - \eta_f^{D_1} \Delta (d_1 (t_f)) \right) & \text{if } r \in V_{D_1}, \\
\varepsilon_j \frac{\Sigma_s}{\Sigma_t} \left( 1 - \eta_f^{D_2} \Delta (d_2 (t_f)) \right) & \text{if } r \in V_{D_2},
\end{cases}
\]

(26)

where \(\varepsilon_j (\varepsilon_j^{D_1})\) accounts for the probability to emit \(j\) neutrons after a fission event within the system \((i\text{-th detector})\) volume, \(\Sigma_f (\Sigma_f^{D_1})\) denotes the neutron fission macroscopic cross section for the system \((i\text{-th detector})\) volume and \(\eta_f^{D_i}\) is the \(i\text{-th detector}\) fission efficiency. Whereas, if we consider, e.g., one detector count registered by the first detector,

\[
C_j^{(1,0)}(r, v, t) = \begin{cases} 
0 & \text{if } r \notin V_{D_1}, \\
\eta_f^{D_1} \varepsilon_j \frac{\Sigma_s}{\Sigma_t} \Delta (d_1 (t_f)) & \text{if } r \in V_{D_1}.
\end{cases}
\]

(27)

Expression (19) must fulfill the final condition \(G_K (s_1, s_2, \vartheta, t|d_1 (t_f), d_2 (t_f)) = 1\) for \(t > t_f = \max \{ t_{f_1}, t_{f_2} \}\), due to the causality principle, and the boundary condition \(G_K (s_1, s_2, r_B, v, \Omega, t|d_1 (t_f), d_2 (t_f)) = 1\) for \(\mathbf{n} \cdot \Omega > 0\), i.e., for neutrons injected outwardly at a convex boundary (Bell (1965); Muñoz-Cobo et al. (1987)).
Further, according to Bartlett’s procedure, we can derive the first factorial moment of the number of detector counts per single neutron injected in the system at the phase-space point $\vartheta$ and at time $t$ as

$$\bar{z}_i (\vartheta, t|d(t_f)) = \frac{\partial}{\partial s_i} G_K (s_1, s_2, \vartheta, t|d_i (t_f)) \bigg|_{s_1,s_2=1},$$

(28)

whereas the cross second factorial moment of the number of detector counts per single neutron injected can be defined as

$$\bar{z}_1 z_2 (\vartheta, t|d(t_f)) = \frac{\partial^2}{\partial s_1 \partial s_2} G_K (s_1, s_2, \vartheta, t|d_1 (t_f), d_2 (t_f)) \bigg|_{s_1,s_2=1}.$$  

(29)

Thus, applying the operator $\partial/\partial s_i|_{s_1,s_2=1}$ to the expression corresponding to the transport integro-differential equation satisfied by $G_K (s_1, s_2, \vartheta, t|d_1 (t_f), d_2 (t_f))$, equation (19), we get

$$\left( -\frac{1}{v} \frac{\partial}{\partial t} - L^+ \right) \bar{z}_i = S^+_{D_i},$$

(30)

where the time-independent adjoint transport operator $L^+$, Muñoz-Cobo et al. (1987), and

$$S^+_{D_i} (\vartheta, t) = S^+_{D_i} (\vartheta) \times S^+_{D_i} (t) = \left[ \eta_c^{D_i} \Sigma_c^{D_i} + \eta_s^{D_i} \Sigma_s^{D_i} + \eta_f^{D_i} \Sigma_f^{D_i} \right] \times \left[ H (t - (t_i - \tau_{c_i})) - H (t - t_i) \right].$$

(31)

This magnitude will be non-zero only for $r \in V_{D_i}$ and $t \in (t_i - \tau_{c_i}, t_i]$. The solution $\bar{z}_i$ of equation (30) must fulfil the boundary condition $\bar{z}_i = 0$ for $n \cdot \Omega > 0$, on a convex boundary, and the time-reversed causality condition, i.e., it must vanish at the end of the measurement period $\tau_{c_i}$. Indeed, equation (30) reveals the nature of $\bar{z}_i$ as an adjoint generalised Green’s function driven by the adjoint importance source $S^+_{D_i}$, Muñoz-Cobo et al. (1987). Consequently, for the forward transport problem we shall write

$$\left( \frac{1}{v} \frac{\partial}{\partial t} - L \right) \phi = S_1,$$

(32)

where $L$ is the time-independent direct transport operator, Bell and Glasstone (1973), and

$$S_1 (\vartheta, t) = S^f_1 (\vartheta, t) + S^{sp}_1 (\vartheta, t)$$
\[
S_{\text{sf}}^{1}(\vartheta) \times S_{\text{sp}}^{1}(t) + S_{\text{sf}}^{1}(\vartheta) \times S_{\text{sp}}^{1}(t),
\]
that is, the total neutron source can be expressed as the sum of the intrinsic and the spallation neutron sources,

\[
S_{\text{sf}}^{1}(\vartheta) = \bar{\nu}_{\text{sf}} N_{0} \rho_{\text{sf}}(\mathbf{r}) \frac{\lambda_{S}(v)}{4\pi},
\]

\[
S_{\text{sf}}^{1}(t) = \lambda_{\text{sf}} N(t) / N_{0} \equiv \lambda_{\text{sf}},
\]

where we assume the initial number of nuclei corresponding to the spontaneous disintegration neutron source to be constant in our time-scale,

\[
S_{\text{sp}}^{1}(\vartheta) = \bar{\nu}_{\text{pp}} \bar{\nu}_{\text{sp}} \rho_{\text{sp}}(\mathbf{r}) f_{\text{sp}}(v, \Omega),
\]

\[
S_{\text{sp}}^{1}(t) = \sum_{m=\infty}^{\infty} \delta(t - (\xi + mT)),
\]

with \( \bar{\nu}_{w} = \sum_{j} l_{j} \bar{\varepsilon}_{j}^{w} \), \( w = \text{sf}, \text{sp}, \text{pp} \).

Now, the forward neutron flux satisfies the initial condition \( \phi(t = -\infty) = 0 \), and the boundary condition \( \phi = 0 \) for \( \mathbf{n} \cdot \Omega < 0 \), on a convex boundary. A proper choice of the corresponding boundary and final conditions for the adjoint function makes the associated bilinear concomittance to vanish, and, hence, due to the commutation relation

\[
\langle z_{i}|S_{1}\rangle = \langle S_{D_{i}}^{+}||\phi\rangle,
\]

where Dirac’s notation for the inner product is used, and where both terms account for the average number of detector counts during its counting interval: at the left hand side we have the inner product of the neutron source strength, \( S_{1} \), (neutrons emitted at a given phase-space point and time) and the number of counts gathered at detector \( i \) per single neutron introduced at a given phase-space point and time, \( z_{i} \); equivalently, the right hand side term expresses the inner product of the effective macroscopic neutron detection cross section for the same detector, \( S_{D_{i}}^{+} \), and the neutron flux, \( \phi \), \( \text{[Múñoz-Cobo et al. (2000)]} \).

Similarly, we can make use of (29) to find an expression for the cross second factorial moment of the number of detector counts per single neutron introduced:

\[
\left(-\frac{1}{v} \frac{\partial}{\partial t} - L^{+}\right) \overline{z_{1}z_{2}} = S_{D_{1}D_{2}}^{+},
\]
where the importance source for the cross second factorial moment can be recast as

\[
\frac{S_{D_1D_2}^+ (\vartheta, t)}{\Sigma_t (\mathbf{r}, v)} = C_s^{(1,0)} (\mathbf{r}, v, t) \int d\mathbf{v}' \int d\Omega' f_s (\mathbf{r}, v, \Omega|v', \Omega') \bar{z}_1 (\mathbf{r}, v', \Omega, t|d_1 (t_f)) \\
+ C_s^{(0,1)} (\mathbf{r}, v, t) \int d\mathbf{v}' \int d\Omega' f_s (\mathbf{r}, v, \Omega|v', \Omega') \bar{z}_2 (\mathbf{r}, v', \Omega, t|d_2 (t_f)) \\
+ \sum_{j=1}^l j C_j^{(1,0)} (\mathbf{r}, v, t) \int d\mathbf{v}' \int d\Omega' \chi (\mathbf{r}, v') /4\pi \bar{z}_1 (\mathbf{r}, v', \Omega, t|d_1 (t_f)) \\
+ \sum_{j=1}^l j C_j^{(0,1)} (\mathbf{r}, v, t) \int d\mathbf{v}' \int d\Omega' \chi (\mathbf{r}, v') /4\pi \bar{z}_2 (\mathbf{r}, v', \Omega, t|d_2 (t_f)) \\
+ \sum_{j=2}^l j (j - 1) \sum_{r_1, r_2=0, r_1 + r_2 = 1}^1 C_j^{(r_1, r_2)} (\mathbf{r}, v, t) \int d\mathbf{v}' \int d\Omega' \chi (\mathbf{r}, v') /4\pi \bar{z}_1 (\mathbf{r}, v', \Omega, t|d_1 (t_f)) \bar{z}_2 (\mathbf{r}, v', \Omega, t|d_2 (t_f)).
\]


(40)

The cross second factorial moment \(\bar{z}_1 \bar{z}_2\) must satisfy the same time-reversed and boundary conditions as \(\bar{z}_i\), but for \(t_f = \max \{t_1, t_2\}\). Hence, it can be viewed as a generalised adjoint function, now driven by the adjoint source, \(S_{D_1D_2}^+\), that is, the product of the detector cross sections and the spectral and angular weighted neutron importances. Again, the commutation relation leads to the identity

\[
\langle \bar{z}_1 \bar{z}_2 | S_1 \rangle = \langle S_{D_1D_2}^+ | \varphi \rangle.
\]

(41)

In order to express the adjoint problem in terms of instantaneous detector counting rates, we shall divide adjoint transport equations (30) and (39) by \(\tau_{c_i}\) and \(\tau_{c_1} \tau_{c_2}\) and then calculate the limits \(\lim_{\tau_{c_i} \downarrow 0}\) and \(\lim_{\tau_{c_1} \tau_{c_2} \downarrow 0}\), respectively. As a consequence, we shall write

\[
\left( -\frac{1}{v} \frac{\partial}{\partial t} - L^+ \right) \hat{z}_i = \dot{S}_{D_i}^+,
\]

(42)

\[
\left( -\frac{1}{v} \frac{\partial}{\partial t} - L^+ \right) \hat{z}_1 \hat{z}_2 = \ddot{S}_{D_1D_2}^+,
\]

(43)

where, by definition,

\[
\hat{z}_i (\vartheta, t - t_f) = \lim_{\tau_{c_i} \downarrow 0} \bar{z}_i (\vartheta, t|d_i (t_f)),
\]

(44)
\[ \dot{S}_{Di}^+ (\vartheta, t - t_f) = \lim_{\tau_c, \tau_i \downarrow 0} \frac{S_{Di}^+ (\vartheta, t|d_i (t_f))}{\tau_c}, \quad \text{(45)} \]

since, taking into account equations (42) and (45), \( \dot{z}_i (\vartheta, t - t_f) \) can be regarded as a displacement kernel, \textit{Muñoz-Cobo et al.} (1987).

\[ \ddot{z}_1 \ddot{z}_2 (\vartheta, t - t_{f1}, t - t_{f2}) = \lim_{\tau_{c1}, \tau_{c2} \downarrow 0} \frac{z_1 z_2 (\vartheta, t|d_1 (t_{f1}), d_2 (t_{f2}))}{\tau_{c1} \tau_{c2}}, \quad \text{(46)} \]

\[ \ddot{S}_{D1D2}^+ (\vartheta, t - t_{f1}, t - t_{f2}) = \lim_{\tau_{c1}, \tau_{c2} \downarrow 0} \frac{S_{D1D2}^+ (\vartheta, t|d_1 (t_{f1}), d_2 (t_{f2}))}{\tau_{c1} \tau_{c2}}. \quad \text{(47)} \]

Next, we look for a solution to the Boltzmann neutron transport equation (32) for the direct flux, \( \phi (\vartheta, t) \), satisfying the \( \alpha \)-modes expansion:

\[ \phi (\vartheta, t) = \phi_{sf} (\vartheta, t) + \phi_{sp} (\vartheta, t) = \sum_j \varphi_j (\vartheta) \zeta_j (t), \quad \text{(48)} \]

where we assume the eigenfunctions \( \varphi_j (\vartheta) \) to form a complete basis in the corresponding Hilbert space (Bell and Glasstone (1979); Carta and D’Angelo (1999)). These must obey the \( \alpha \)-eigenvalue equation

\[ L \varphi_j (\vartheta) = \frac{\alpha_j}{v} \varphi_j (\vartheta). \quad \text{(49)} \]

Similarly, for the adjoint flux instantaneous rate we will have

\[ \dot{z}_i (\vartheta, t - t_f) = \sum_j \varphi_{D1j}^+ (\vartheta) \zeta_{Di}^+ (t - t_f), \quad \text{(50)} \]

and the \( \alpha \)-eigenvalue equation

\[ L^+ \varphi_{D1j}^+ (\vartheta) = \frac{\alpha_j}{v} \varphi_{D1j}^+ (\vartheta). \quad \text{(51)} \]

Beneath this ansatz, it is obvious that we must put \( \varphi_{D1j}^+ (\vartheta) = \varphi_{D2j}^+ (\vartheta) \equiv \varphi_j^+ (\vartheta) \). The adjoint and forward eigenfunctions satisfy the biorthogonal relation, \textit{Bell and Glasstone} (1979), i.e.,

\[ \left( \frac{1}{v} \varphi_n^+, \varphi_m \right) = \delta_{nm} \left( \frac{1}{v} \varphi_n^+, \varphi_n \right), \quad \text{(52)} \]

where the phase-space inner product is defined by \( (a, b) = \int \! \! dr \! \! dv \! \! d\Omega \, a (\vartheta) \, b (\vartheta) \).

If we introduce the ansatz (48) in equation (32) and apply the Fourier transform operator to both sides of it, we shall obtain the Fourier transform of
the $j$-th flux instantaneous rate time-dependent term, which, on account of identities (49), (51), and (52), reads as

$$
ζ_j(ω) = \frac{1}{iω - α_j} \left[ 2πλsf \left( \frac{S_{sf}^f, ϕ_j^+}{\bar{v}_j ϕ_j^+} \right) δ(ω) + \left( \frac{S_{sp}^p, ϕ_j^+}{\bar{v}_j ϕ_j^+} \right) \sum_{m=-∞}^{∞} e^{-iω(ξ+mT)} \right]
$$

with

$$
(S_{sf}^f, ϕ_j^+) = \int \int \int dΩ \bar{ν}_{sf} N_0 ρ_{sf}(r) \frac{χ_s(v)}{4π} ϕ_j^+(r, v, Ω),
$$

$$
(S_{sp}^p, ϕ_j^+) = \int \int \int dΩ \bar{ν}_{pp} \bar{ν}_{sp} N_0 ρ_{sp}(r) f_{sp}(v, Ω) ϕ_j^+(r, v, Ω),
$$

In the same way, from (42) we can deduce the expression corresponding to the Fourier transform of the $j$-th adjoint flux instantaneous rate time-dependent term for the $i$-th neutron detector:

$$
ζ_j^+(ω) = -\frac{1}{iω + α_j} \left( S_{Di}^i, ϕ_j \right).
$$

where $S_{Di}^i = S_{Di}^i(θ)$ is given by (31).

4 Analytical expression for the cross power spectral density with pulsed sources

4.1 The deterministic pulsing method

In order to obtain the analytical expressions corresponding to the factorial moments of the number of counts of both neutron detectors we can apply again Bartlett’s procedure (Bartlett (1955)) to the expression corresponding to the source probability generating function. As we have outlined previously, we can do it taking into account two different situations: in the first case, we can calculate factorial moments corresponding to the deterministic pulsing method.

We are interested in the well known cross covariance function, Papoulis (1991), defined as

$$
Ξ(d_1(t_{f_1}), d_2(t_{f_2})) = \left. \frac{∂^2 G_S}{∂s_1 ∂s_2} \right|_{s_1, s_2 = 1} - \left. \frac{∂G_S}{∂s_1} \right|_{s_1, s_2 = 1} × \left. \frac{∂G_S}{∂s_2} \right|_{s_1, s_2 = 1},
$$

15
where \( G_S \equiv G_S (s_1, s_2 | d_1 (t_{f_1}), d_2 (t_{f_2})) \) is given by (17) with \( \xi = 0 \), i.e., it is the difference between the cross second factorial moment of the number of detector counts gathered by both detectors and the product of their first factorial moments. It can be recast as

\[
\Xi (d_1 (t_{f_1}), d_2 (t_{f_2})) = \langle S^+_{D_1D_2} | \phi \rangle (d_1 (t_{f_1}), d_2 (t_{f_2})) + \Delta \Xi (d_1 (t_{f_1}), d_2 (t_{f_2})), \tag{58}
\]

where the first term takes into account the contribution coming from multiplicative processes within the system and the detector volumes due to fission events and detections. The latter can be de facto neglected if we admit that the volume occupied by detectors is small in comparison with the system volume. In addition, the second term in (58) stems from the non-Poissonian behaviour of both neutron sources.

Next we can divide equation (58) by \( \tau_{c_1} \tau_{c_2} \) and then apply the limits \( \lim_{\tau_{c_1}, \tau_{c_2} \rightarrow 0} \) in order to derive the expression corresponding to the second order instantaneous rate of the cross covariance function:

\[
\tilde{\Xi} (t - t_{f_1}, t - t_{f_2}) = \langle \tilde{S}^+_{D_1D_2} | \phi \rangle (t - t_{f_1}, t - t_{f_2}) + \Delta \tilde{\Xi} (t - t_{f_1}, t - t_{f_2}), \tag{59}
\]

where, by definition,

\[
\tilde{\Xi} (t - t_{f_1}, t - t_{f_2}) = \lim_{\tau_{c_1}, \tau_{c_2} \rightarrow 0} \frac{\Xi (d_1 (t_{f_1}), d_2 (t_{f_2}))}{\tau_{c_1} \tau_{c_2}}, \tag{60}
\]

\[
\Delta \tilde{\Xi} (t - t_{f_1}, t - t_{f_2}) = \lim_{\tau_{c_1}, \tau_{c_2} \rightarrow 0} \frac{\Delta \Xi (d_1 (t_{f_1}), d_2 (t_{f_2}))}{\tau_{c_1} \tau_{c_2}}, \tag{61}
\]

with

\[
\langle \tilde{S}^+_{D_1D_2} | \phi \rangle (t - t_{f_1}, t - t_{f_2}) = \langle \tilde{S}^+_{D_1D_2} | \phi^S \rangle (t - t_{f_1}, t - t_{f_2}) + \langle \tilde{S}^+_{D_1D_2} | \phi^P \rangle (t - t_{f_1}, t - t_{f_2})
\]

\[
= \sum_{j,k,l} \tilde{\nu}^2 D \left( \Sigma_{s} \varphi_j, \varphi_k^+, \varphi_l^+ \right) \int_{-\infty}^{t_{f_1}} dt \zeta_j (t) \zeta^+_{D_1k} (t - t_{f_1}) \zeta^+_{D_2l} (t - t_{f_2}) \tag{62}
\]

\( D = \frac{\nu (\nu - 1)}{\bar{\nu}^2} \) being system Diven’s factor, \( \bar{\nu} = \sum_j \bar{\epsilon}_j, \quad \nu (\nu - 1) = \sum_j j (j - 1) \bar{\epsilon}_j \), and where we have defined the phase-space inner product

\[
\left( \Sigma_{s} \varphi_j, \varphi_k^+, \varphi_l^+ \right) = \int_{V_{S}V_{S} + V_{B}} dr \int dv \int d\Omega \Sigma_{s} (r, v) \varphi_j (r, v, \Omega)
\]

\[
\times \int dv' \int d\Omega' \frac{\chi (r, v', \Omega') \varphi_k^+ (r, v', \Omega')}{4\pi} \tag{63}
\]
\[
\times \int \text{d}v'' \int d\Omega'' \frac{\chi(r, v'')}{4\pi} \varphi_i^+(r, v'', \Omega''),
\]
(63)

and

\[
\Delta \ddot{\Xi}(t - t_{f_1}, t - t_{f_2}) = \Delta \ddot{\Xi}_{sf}(t - t_{f_1}, t - t_{f_2}) + \Delta \ddot{\Xi}_{sp}(t - t_{f_1}, t - t_{f_2}),
\]
(64)

which arises from the non-Poissonian nature of the intrinsic and the external spallation sources, respectively,

\[
\Delta \ddot{\Xi}_{sf}(t - t_{f_1}, t - t_{f_2}) = \sum_{j,k} \bar{\nu}_{sf} D_{sf} \left( S^1_{sf}, \varphi_j^+, \varphi_k^+ \right) \int_{-\infty}^{t_f} dt \zeta_{D_{D_1j}}(t - t_{f_1}) \zeta_{D_{2k}}(t - t_{f_2}),
\]
(65)

\[
\Delta \ddot{\Xi}_{sp}(t - t_{f_1}, t - t_{f_2}) = \sum_{j,k} \left[ \bar{\nu}_{sp} D_{sp} \left( S^1_{sp}, \varphi_j^+, \varphi_k^+ \right) \right. \left. + (D_{pp} - 1) \left( S^1_{sp}, \varphi_j^+ \right) \left( S^1_{sp}, \varphi_k^+ \right) \right] \times \int_{-\infty}^{t_f} dt \sum_{m=-\infty}^{\infty} \delta(t - mT) \zeta_{D_{D_1j}}(t - t_{f_1}) \zeta_{D_{2k}}(t - t_{f_2}),
\]
(66)

where \( D_w = \nu_w (\nu_w - 1)/\nu_w^2 \), \( w = \text{sf, sp, pp} \), is Diven’s factor for the spontaneous fission (intrinsic) source, the spallation neutron production source, and the pulsed proton source, respectively, with \( \nu_w (\nu_w - 1) = \sum_j j(j - 1) \zeta_j^w \). Furthermore, in the last two expressions we have introduced the following inner products:

\[
\left( S^1_{sf}, \varphi_j^+, \varphi_k^+ \right) = \bar{\nu}_{sf} N_0 \int \text{d}r \rho_{sf}(r)
\times \int \text{d}v' \int d\Omega' \frac{\chi_{sf}(v')}{4\pi} \varphi_j^+(r, v', \Omega')
\times \int \text{d}v'' \int d\Omega'' \frac{\chi_{sf}(v'')}{4\pi} \varphi_k^+(r, v'', \Omega''),
\]
(67)

\[
\left( S^1_{sp}, \varphi_j^+, \varphi_k^+ \right) = \bar{\nu}_{sp} \bar{\nu}_{sp} \int \text{d}r \rho_{sp}(r)
\times \int \text{d}v' \int d\Omega' \chi_{sp}(v', \Omega') \varphi_j^+(r, v', \Omega')
\times \int \text{d}v'' \int d\Omega'' \chi_{sp}(v'', \Omega'') \varphi_k^+(r, v'', \Omega'').
\]
(68)
Without loss of generality, we can assume that the upper integral limit in (57) can be selected in such a way that \( t_f > \max \{ t_1, t_2 \} \), and then admit that \( t_f \to \infty \). Next, we can define the time delay between the final instant of both detector intervals as \( \tau = t_{t_2} - t_{t_1} \), and then apply the operator \( f \, d\tau \, e^{-i\omega \tau} \) to equation (59) to derive the Fourier transform of \( \tilde{\Xi}(t - t_1, t - t_2) \), i.e., the cross power spectral density:

\[
\text{CPSD} = \mathcal{F} \left[ \left\langle \tilde{S}^+_{D_1D_2}\phi \right\rangle \right] + \mathcal{F} \left[ \Delta \tilde{\Xi} \right],
\]

where, on account of equations (53) and (56) for \( \xi = 0 \),

\[
\mathcal{F} \left[ \left\langle \tilde{S}^+_{D_1D_2}\phi \right\rangle \right] = \mathcal{F} \left[ \left\langle \tilde{S}^+_{D_1D_2}\phi^{sf} \right\rangle \right] + \mathcal{F} \left[ \left\langle \tilde{S}^+_{D_1D_2}\phi^{sp} \right\rangle \right],
\]

\[
\mathcal{F} \left[ \left\langle \tilde{S}^+_{D_1D_2}\phi^{sf} \right\rangle \right] = \sum_{j,k,l} \bar{v}^2 D \left( \Sigma_l \varphi_j, \varphi_k^+, \varphi_l^+ \right) \left( \frac{S_{D_1}^{sf}, \varphi_j}{\frac{1}{v} \varphi_j, \varphi_j} \right)
\times \left( \frac{S_{D_1}^{sf}, \varphi_k}{\frac{1}{v} \varphi_k, \varphi_k} \right) \left( \frac{S_{D_2}^{sf}, \varphi_l}{\frac{1}{v} \varphi_l, \varphi_l} \right) \lambda_{sf} \left( \omega - i\alpha_k \right) \left( \omega + i\alpha_l \right),
\]

\[
\mathcal{F} \left[ \left\langle \tilde{S}^+_{D_1D_2}\phi^{sp} \right\rangle \right] = \sum_{j,k,l} \bar{v}^2 D \left( \Sigma_l \varphi_j, \varphi_k^+, \varphi_l^+ \right) \left( \frac{S_{D_1}^{sp}, \varphi_j}{\frac{1}{v} \varphi_j, \varphi_j} \right) \left( \frac{S_{D_2}^{sp}, \varphi_k}{\frac{1}{v} \varphi_k, \varphi_k} \right) \left( \frac{S_{D_2}^{sp}, \varphi_l}{\frac{1}{v} \varphi_l, \varphi_l} \right)
\times \frac{\left( \exp(-\alpha_j T) - 1 \right)^{-1} - \left( \exp(-\left(\alpha_k + i\omega \right) T) - 1 \right)^{-1}}{\left( \omega + i(\alpha_j - \alpha_k) \right) \left( \omega + i\alpha_l \right)},
\]

where, owing to the application of the deterministic pulsing method, we just need to assume that \( t_{t_1} = I_p T \), with \( I_p \to \infty \), whereas

\[
\mathcal{F} \left[ \Delta \tilde{\Xi} \right] = \mathcal{F} \left[ \Delta \tilde{\Xi}^{sf} \right] + \mathcal{F} \left[ \Delta \tilde{\Xi}^{sp} \right],
\]

\[
\mathcal{F} \left[ \Delta \tilde{\Xi}^{sf} \right] = \sum_{j,k} \bar{v}_{sf}^2 D_{sf} \left( S_{sf}^{sf}, \varphi_j^+, \varphi_k^+ \right) \left( \frac{S_{D_1}^{sf}, \varphi_j}{\frac{1}{v} \varphi_j^+, \varphi_j} \right) \left( \frac{S_{D_2}^{sf}, \varphi_k}{\frac{1}{v} \varphi_k^+, \varphi_k} \right)
\times \lambda_{sf} \left( \omega - i\alpha_j \right) \left( \omega + i\alpha_k \right),
\]

\[
\mathcal{F} \left[ \Delta \tilde{\Xi}^{sp} \right] = \sum_{j,k} \bar{v}_{sp} D_{sp} \left( S_{sp}^{sp}, \varphi_j^+, \varphi_k^+ \right) + (D_{pp} - 1) \left( S_{sp}^{sp}, \varphi_j^+ \right) \left( S_{sp}^{sp}, \varphi_k^+ \right)
\]

18
\[
\frac{(S_1^{+}, \varphi_j) (S_2^{+}, \varphi_k)}{(\frac{1}{v} \varphi^+_{j}, \varphi_j)} \left(\frac{1}{v} \varphi^+_{k}, \varphi_k\right) \frac{(\exp(-(\alpha_j + i\omega)T) - 1)^{-1}}{(i\omega - \alpha_k)}.
\]

(75)

4.2 The stochastic pulsing method

Similarly, expressions for the factorial moments of the detector number of counts can be derived for the stochastic pulsing method, just applying Bartlett’s procedure to expression (18). Now the cross covariance function will be defined as

\[
\langle \Xi (d_1(t_1), d_2(t_2)) \rangle_{\xi} = \left\langle \frac{\partial^2 G_S}{\partial s_1 \partial s_2} \bigg|_{s_1,s_2=1} \right\rangle_{\xi} - \left\langle \frac{\partial G_S}{\partial s_1} \bigg|_{s_1,s_2=1} \right\rangle_{\xi} \times \left\langle \frac{\partial G_S}{\partial s_2} \bigg|_{s_1,s_2=1} \right\rangle_{\xi},
\]

(76)

where \( G_S \equiv G_S(s_1, s_2, \xi | d_1(t_1), d_2(t_2)) \) is given by (17), which can be recast as

\[
\langle \Xi (d_1(t_1), d_2(t_2)) \rangle_{\xi} = \langle \langle \hat{z}_1 \hat{S}_1^{sp} \rangle \langle \hat{z}_2 \hat{S}_1^{sp} \rangle \rangle_{\xi} - \langle \langle \hat{z}_1 \hat{S}_1^{sp} \rangle \langle \hat{z}_2 \hat{S}_1^{sp} \rangle \rangle_{\xi}
\]

\[
+ \langle \hat{z}_1 \hat{z}_2 \hat{S}_1 \rangle_{\xi} + \langle \Delta \Xi \rangle_{\xi},
\]

(77)

where \( S_1^{sp} \equiv S_1^{sp}(\vartheta, t) \) is given by the second term of expression (33). The first and second term of (77) stem from the time correlation introduced by the stochastic pulsing method (see Ceder and Pázsit (2003); Ballester and Muñoz-Cobo (2005)) and, obviously it only involves the pulsed neutron source.

The cross power spectral density shall be derived following the same steps as before, i.e., dividing equation (77) by \( \tau_{c_1} \tau_{c_2} \), then applying the limits \( \lim_{\tau_{c_1} \tau_{c_2} \rightarrow 0} \) to the expression obtained, and, finally, calculating its Fourier transform. Thus, we shall write, on account of the commutation relations (38), (41),

\[
\langle \text{CPSD} \rangle_{\xi} = \mathcal{F} \left[ \langle \langle \hat{S}_1^{+} \hat{S}_2^{+} \rangle \langle \hat{S}_1^{+} \hat{S}_2^{+} \rangle \rangle_{\xi} \right] - \mathcal{F} \left[ \langle \langle \hat{S}_1^{+} \hat{S}_2^{+} \rangle \langle \hat{S}_1^{+} \hat{S}_2^{+} \rangle \rangle_{\xi} \right]
\]

\[
+ \mathcal{F} \left[ \langle \hat{S}_1^{+} \hat{S}_2^{+} \rangle_{\xi} \right] + \mathcal{F} \left[ \langle \Delta \Xi \rangle_{\xi} \right],
\]

(78)

where, as before,

\[
\mathcal{F} \left[ \langle \hat{S}_1^{+} \hat{S}_2^{+} \rangle_{\xi} \right] = \mathcal{F} \left[ \langle \hat{S}_1^{+} \hat{S}_2^{+} \rangle \hat{s}^{sf} \right] + \mathcal{F} \left[ \langle \hat{S}_1^{+} \hat{S}_2^{+} \rangle \hat{s}^{sp} \right].
\]

(79)
F \left[ \langle \Delta \tilde{z}_{\xi} \rangle \right] = F \left[ \Delta \tilde{z}_{sf} \right] + F \left[ \langle \Delta \tilde{z}_{sp} \rangle_{\xi} \right]. \tag{80}

The external pulsed source contribution to the term arising from the system cross covariance, equation (79), under the stochastic pulsing method becomes

\begin{align*}
F \left[ \langle \langle \hat{S}^+_{D_1 D_2} | \phi_{sp} \rangle \rangle_{\xi} \right] &= \sum_{j,k,l} \bar{v}^2 D \left( \Sigma_t \varphi_j, \varphi_k^+, \varphi_l^+ \right) \left( S^+_{11}, \varphi_j^+ \right) \left( 1 / \varphi_j^+ \right) \\
&\times \left( S^+_{D_1}, \varphi_k \right) \left( S^+_{D_2}, \varphi_l \right) \frac{T^{-1}}{\left( 1 / \varphi_k^+ \right) \left( 1 / \varphi_l^+ \right)} \left( -\alpha_j \right) \left( \omega - i\alpha_k \right) \left( \omega + i\alpha_l \right). \tag{81}
\end{align*}

On the other hand, the time correlation term stemming from the stochastic pulsing method in (77), after dividing by \( \tau_{c_1} \tau_{c_2} \) and calculating the limits \( \lim_{\tau_{c_1}, \tau_{c_2} \downarrow 0} \), is given by, Ballester and Muñoz-Cobo (2005),

\begin{align*}
\langle \langle \hat{S}^+_{D_1} | \phi_{sp} \rangle \langle \hat{S}^+_{D_2} | \phi_{sp} \rangle \rangle_{\xi} - \langle \langle \hat{S}^+_{D_1} | \phi_{sp} \rangle \rangle_{\xi} \langle \langle \hat{S}^+_{D_2} | \phi_{sp} \rangle \rangle_{\xi} \\
&= \sum_{j,k} \left( S^+_{sp}, \varphi_j^+ \right) \left( S^+_{sp}, \varphi_k^+ \right) \frac{1}{T^2} \int_{-\infty}^{t_i} dt' \int_{-\infty}^{t_i} dt'' \zeta^+_{D_1j} \left( t' - t_{i_1} \right) \\
&\times \zeta^+_{D_2k} \left( t'' - t_{i_1} - \tau \right) \sum_{m=-\infty, m\neq 0}^{\infty} \exp \left( \frac{2\pi m}{T} (t' - t'') \right), \tag{82}
\end{align*}

and applying now the operator \( \int d\tau \exp (-i\omega \tau) \), we find that

\begin{align*}
F \left[ \langle \langle \hat{S}^+_{D_1} | \phi_{sp} \rangle \langle \hat{S}^+_{D_2} | \phi_{sp} \rangle \rangle_{\xi} \right] - F \left[ \langle \langle \hat{S}^+_{D_1} | \phi_{sp} \rangle \rangle_{\xi} \langle \langle \hat{S}^+_{D_2} | \phi_{sp} \rangle \rangle_{\xi} \right] \\
&= \sum_{j,k} \left( S^+_{sp}, \varphi_j^+ \right) \left( S^+_{sp}, \varphi_k^+ \right) \frac{2\pi}{T^2} \left( S^+_{D_1}, \varphi_j \right) \left( S^+_{D_2}, \varphi_k \right) \\
&\times \sum_{m=-\infty, m\neq 0}^{\infty} \frac{\delta \left( \omega - 2\pi m / T \right)}{\left( \omega - i\alpha_j \right) \left( \omega + i\alpha_k \right)}. \tag{83}
\end{align*}

We can similarly proceed to calculate the second term appearing in equation (80):

\begin{align*}
F \left[ \langle \Delta \tilde{z}_{sp} \rangle_{\xi} \right] &= \sum_{j,k} \bar{\nu}_{sp} D_{sp} \left( S^+_{11}, \varphi_j^+, \varphi_k^+ \right) + \left( D_{pp} - 1 \right) \left( S^+_{sp}, \varphi_j^+ \right) \left( S^+_{sp}, \varphi_k^+ \right)
\end{align*}
This term can effectively become negative for certain experimental conditions, Ballester et al. (2005).

5 Fundamental mode approximation and discussion

Following the derivation obtained in the previous Section, we can also apply the fundamental mode approach to the expression corresponding to the CPSD when the stochastic pulsing method is used:

\[
\langle \text{CPSD} \rangle_\xi = \left( \varrho_1 + \varrho_2 \sum_{m=\pm \infty, m \neq 0} \delta \left( \frac{\omega - \frac{2\pi m}{T}}{\omega^2 + \alpha_0^2} \right) \right) \frac{1}{\omega^2 + \alpha_0^2},
\]

(85)

\[
\varrho_1 = \left\{ \bar{v}^2 D \left( \Sigma_t \varphi_0, \varphi_0^+, \varphi_0^+ \right) \left( \frac{S_{1}^{sf+sp}, \varphi_0^+}{\frac{1}{\bar{v}} \varphi_0^+} \right) \left( \frac{1}{\bar{v}} \varphi_0^+ \right) \frac{1}{\omega^2 + \alpha_0^2} + \bar{v}_s^2 D_{sf} \left( \frac{S_{1}^{sf}, \varphi_0^+, \varphi_0^+}{\frac{1}{\bar{v}} \varphi_0^+} \right) \left( \frac{1}{\bar{v}} \varphi_0^+ \right) \frac{1}{\omega^2 + \alpha_0^2} \right\}
\]

(86)

\[
\varrho_2 = \left( \frac{S_{1}^{sp}, \varphi_0^+}{\frac{1}{\bar{v}} \varphi_0^+} \right) \left( \frac{S_{1}^{sp}, \varphi_0^+}{\frac{1}{\bar{v}} \varphi_0^+} \right) \frac{2\pi}{T^2} \left( \frac{S_{1}^{sf+sp}, \varphi_0^+}{\frac{1}{\bar{v}} \varphi_0^+} \right) \left( \frac{S_{D_1}^{sf}, \varphi_0^+}{\frac{1}{\bar{v}} \varphi_0^+} \right) \left( \frac{1}{\bar{v}} \varphi_0^+ \right).
\]

(87)

where we have defined the total neutron source strength \( S_{1}^{sf+sp} \equiv S_{1}^{sf+sp} (\vartheta, t) = \lambda_{sf} S_{1}^{sf} (\vartheta, t) + T^{-1} S_{1}^{sp} (\vartheta, t). \)

Equation (85) apparently seems to be similar to that obtained by Rugama et al. (2004) albeit a deeper examination permits to understand an important difference: in the latter case, the expression for the CPSD for the stochastic pulsing method was derived assuming a quasi-Poissonian behaviour of the pulsed external source, i.e., with no delay time averaging. That means that the expression obtained in this reference should be exactly equal to our expression for the CPSD in Section 4 under the deterministic pulsing method, only without the contribution of (75), which stems from the non-Poissonian behaviour of the periodic pulsed source. But it is clear that, in any case, equation (69) is a completely bounded function for any non-zero value of the frequency, \( \omega \), which cannot produce the response obtained at MUSE-4 experiments, also
Table 1

Values of \((-\alpha_0)\) obtained for the configuration SC0 (with pilot rod inserted) of the MASURCA subcritical assembly used during MUSE-4 experiments, Soule et al. (2004). The result of the eigenvalue fitting by means of the Feynman-\(\alpha\) method is reported by Ballester et al. (2005).

| Method                              | \((-\alpha_0)\) (rad \cdot s^{-1}) |
|-------------------------------------|----------------------------------|
| Stochastic Pulsing CPSD method      | 13258\(\pm\)273                  |
| Stochastic Pulsing Feynman-\(\alpha\) method | 13646\(\pm\)515            |

reported by Rugama et al. (2004), for the stochastic pulsing CPSD method. Unlikely, we have shown that those spectral lines appearing at frequencies which are multiples of the accelerator frequency are indeed not produced by the system cross covariance contribution, but they are a prima facie of the time self-correlation introduced by the stochastic pulsing method.

In addition, it has been recently shown that the utilisation of deterministic external pulsed sources, such as those used in MUSE-4 experiments, can make the nuclear process to behave as a sub-Poissonian one, Ballester et al. (2005). It can occur when the contribution of non-Poissonian term of the pulsed source becomes negative. That means that under those conditions the time-independent term \(\rho_1\) can be effectively negative for very deterministic pulsed sources. But, anyway, it is of common practice to use Bode’s diagram of the magnitude in order to represent graphically a system response function such as (85), therefore, it is not necessary to consider the sign of \(\rho_1\) provided that, from a practical viewpoint, we only need to fit its magnitude for those points of the graph where \(\omega\) is not a multiple of the accelerator frequency, together with the eigenvalue \(\alpha_0\). Notice that the value of \(\rho_2\) cannot be determined from this technique (we will need a further integral condition), but in any case it is completely useless for a practical purpose. That means that when the stochastic pulsing CPSD method is applied to determine the value of the subcriticality level of a nuclear system, only two fitting parameters must be considered, in contrast with the stochastic pulsing Feynman-\(\alpha\) technique, where three parameters must be fitted, Ballester et al. (2005).

In Figure 1 we show some experimental points reported in that graph corresponding to the stochastic pulsing CPSD method of Rugama et al. (2004) (Figure 6), together with equation (85) conveniently fitted. This particular experiment, corresponds to the configuration SC0 of the MASURCA subcritical assembly used during MUSE-4 studies, Soule et al. (2004), with a D-D pulsed source. The value of the prompt neutron time constant obtained by Rugama et al. (2004), which is shown in Table 1, is effectively very similar to that reported using other noise techniques for the same conditions.
Fig. 1. Stochastic CPSD pulsing method obtained during MUSE-4 experiments for the SC0 configuration of the MASURCA subcritical assembly. Points of the experimental curve (points) from Rugama et al. (2004) are compared with equation (85) fitted with \((-\alpha_0) = 13258 \pm 273 \text{ rad} \cdot \text{s}^{-1}\) (solid). The accelerator period is equal to \(1 \text{ ms}\).

6 Conclusions

In the present work we have dealt with the applicability of stochastic-neutron-field-based methods for the study of the neutron counting statistics in a nuclear system. We have derived the generalised two-detectors relationship between the probability generating functions of the kernel and the source for sub-critical assemblies when pulsed neutron sources are used together with the intrinsic neutron source coming from spontaneous fission events within the fuel material, Ballester and Muñoz-Cobo (2003). It has been done within the stochastic neutron transport theory framework, which permits to understand how the general transport problem is influenced by its spatial, spectral and angular dependence.

Further, we have followed Pál-Bell’s methodology for the derivation of the integro-differential Boltzmann transport equation, and applied the formalism described by Muñoz-Cobo et al. (1987) in order to calculate the chosen statistical descriptor.

In Section 4 an expansion in \(\alpha\)-eigenvalues for the cross covariance and the CPSD of two-detectors stochastic counting rates have been obtained. The contribution of higher harmonics in subcritical monitoring problems shall play an important role in ADS assemblies. In this case, the excitement of higher modes could be relevant in situations of normal operation, and it will increase as the reactor departs from the criticality condition.

In Section 5 we have compared the expression obtained for the stochastic pulsing CPSD method with experimental data obtained during the MUSE-4 European project. The value of the prompt neutron time constant fitted is
comparable with others methods. In addition, the reduced number of fitting parameters makes this method suitable as a subcriticality monitoring technique.

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