We obtain entropy formulas for SRB measures with finite entropy given by inducing schemes. In the first part of the work, we obtain Pesin entropy formula for the class of noninvertible systems whose SRB measures are given by Gibbs-Markov induced maps. In the second part, we obtain Pesin entropy formula for invertible maps whose SRB measures given by Young sets, taking into account a classical compression technique along the stable direction that allows a reduction of the return map associated with a Young set to a Gibbs-Markov map. In both cases, we give applications of our main results to several classes of dynamical systems with singular sets, where the classical results by Ruelle and Pesin cannot be applied. We also present examples of systems with SRB measures given by inducing schemes for which Ruelle inequality does not hold.


dification

For smooth diffeomorphisms of a Riemannian manifold, Ruelle established in \cite{52} that the entropy of an invariant probability measure is always bounded by the integral of the sum of the positive Lyapunov exponents (counted with multiplicity) with respect to that measure. Margulis first established this inequality for diffeomorphisms preserving a smooth measure. The reverse inequality was obtained by Pesin in \cite{47}, for Sinai-Ruelle-Bowen (SRB) measures, i.e. invariant probability measures whose conditional measures are absolutely continuous with respect to the Lebesgue measure along local unstable manifolds. A simpler proof of Pesin inequality was given by Mañé in \cite{43}. A characterisation of the entropy formula in terms of the SRB property was given by Ledrappier and Young in \cite{37}; see also \cite{62}. Natural versions for noninvertible smooth maps (endomorphisms) have been drawn by Liu, Qian et al in \cite{10,11,13}.

\textbf{Key words and phrases.} Piecewise smooth maps, metric entropy, entropy formula.
There is currently a vast literature addressing Ruelle inequality and Pesin entropy formula for smooth dynamical systems. Extensions of these results were obtained by Ledrappier and Strelcyn in [35] for the class of maps having points with infinite derivative introduced by Katok and Strelcyn in [31], inspired on billiard maps. Albeit, to the best of our knowledge, besides [35] and the recent work [7], not much is known on the existence of entropy formulas that can be applied to piecewise smooth maps with singular sets in general, especially in dimension greater than one. For one-dimensional dynamical systems, see e.g. [13, 26, 30, 32, 34, 39, 49]. The utility of the entropy formula can be detected, for example, in the works [5, 6, 7, 38], where it is used in an important way to prove the continuity of entropy in certain families of transformations.

Young has shown that the statistical properties – in particular, the existence of SRB measures – of some nonuniformly hyperbolic dynamical systems can be deduced by means of inducing schemes. This means that we chose a region of the phase space and define a new dynamical system in that region with some good analytical/geometric properties, considering an appropriate return transformation (not necessarily the first one) in subdomains of the considered region. An abstract framework for this strategy was developed quite successfully in [60] for systems with contractive directions and in [61] for systems without contractive directions. In recent years, the results by Young have been applied by many authors to various classes of dynamical systems, comprising, in particular, systems with singularities (including billiards). A natural question is:

To what extent the existence of an inducing scheme, by itself, guarantees an entropy formula for the SRB measure given by that inducing scheme?

In our main results, we show that the answer is affirmative whenever the entropy of the SRB measure is finite. We also characterise the finiteness of the entropy in terms of analytical properties of the inducing scheme, and give examples of SRB measures given by inducing schemes (necessarily with infinite entropy) for which Ruelle inequality does not hold.

Comparing our results with those by Katok, Ledrappier and Strelcyn in the aforementioned works [31, 35], the advantage of our approach consists essentially of two main points: i) we do not have any assumptions on the SRB measure, only analytical and geometric properties of the induced scheme, unlike (1.1) and (1.2) in [31] or conditions 2.1 and 2.2 in [35]; ii) in recent years, inducing schemes have become a widely used tool, the existence of which has been demonstrated for many classes of dynamical systems. A vast list of examples of maps with singular sets are given at Sections 7 and 15, illustrating applications of our main results. In terms of applications of the results in [31, 35] (only to billiards), the fact that the density of the SRB measure of a billiard map has a simple and well-known expression is used in an important way; see [31, Part V].

Overview. This paper is organised as follows. In Part I we obtain the entropy formula for maps without contracting directions whose SRB measures are given by Gibbs-Markov induced maps. These concepts, as well as the main results of this part, are presented in Section II. Proofs of the results are provided in the subsequent sections. In Section III we give an example of a piecewise smooth map with infinite entropy for which the integral of the Jacobian is finite. This example illustrates the failure of two classical results under slightly more general assumptions than those usually required, even for the somewhat regular SRB measures given by Gibbs-Markov induced maps:
(1) Ruelle inequality does not hold if the differentiability of the transformation is assumed only almost everywhere;
(2) the conclusion of Shannon-McMillan-Breiman Theorem is no longer valid if the generating partition is not assumed with finite entropy.

This is explained in detail in Remarks 6.2 and 6.3. In Section 7, we apply the main results of the first part of this work to some classes of piecewise smooth maps with SRB measures given by Gibbs-Markov induced maps. In Part II, we obtain the entropy formula for piecewise smooth diffeomorphisms with contracting directions whose SRB measures are given by Young sets. These concepts, as well as the main results of this part, are provided in Section 8. The proofs of the results are given in the subsequent sections. In Section 14, we give an example of system with infinite entropy for which Ruelle inequality does not hold. In Section 15, we apply our main results of the second part to some classes of piecewise smooth diffeomorphisms with SRB measures given by Young sets.

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Part I. Systems with expanding structures

Let $M$ be a Riemannian manifold and let $m$ be the Lebesgue measure on the Borel sets of $M$. Assume that $f : M \to M$ is a piecewise $C^{1+\eta}$ map, meaning that there are at most countably many pairwise disjoint open regions $M_1, M_2, \ldots$ such that $\bigcup_{i=1}^{\infty} \overline{M}_i = M$ and $f|_{\bigcup_{i=1}^{\infty} M_i}$ is a $C^{1+\eta}$ map. We will refer to $S = M \setminus \bigcup_{i=1}^{\infty} M_i$ as the singular set of $f$ and assume that $m(S) = 0$. Typically, $S$ is a set of discontinuity points or a set of points where the derivative of $f$ does not exist (possibly being infinite).

1. Gibbs-Markov induced maps

Assume that $\Delta_0 \subset M \setminus S$ is a Borel set with $m(\Delta_0) > 0$. For simplicity, the restriction of $m$ to $\Delta_0$ will still be denoted by $m$. Consider a countable $m \mod 0$ partition $P$ of $\Delta_0$ into open sets of $\Delta_0$ and a function $R : \Delta_0 \to \mathbb{N}$ constant on each element of $P$ such that,

- $f^j(\omega) \cap S = \emptyset$, for all $1 \leq j \leq R(\omega)$;
- $f^{R(\omega)}(\omega) \subset \Delta_0$.

We associate to these objects a map $F : \Delta_0 \to \Delta_0$, setting $F|_\omega = f^{R(\omega)}|_\omega$, for each $\omega \in P$.

The map $F$ will be frequently denoted by $f^R$ and called an induced map for $f$; the function $R$ will be called the recurrence time associated with the induced map. We say that $F$ is a Gibbs-Markov map if conditions (G$_1$)-(G$_5$) below are satisfied.

(G$_1$) Markov: $F$ maps each $\omega \in P$ bijectively to $\Delta_0$ ($m \mod 0$).

(G$_2$) Nonsingular: there exists a measurable function $J_F : \Delta_0 \to (0, \infty)$ such that, for every measurable set $A \subset \omega \in P$,

$$m(F(A)) = \int_A J_F(\omega) \, dm.$$ 

The function $J_F$ is called the Jacobian of $F$ (with respect to $m$). The next two properties are related to the dynamical partitions generated by $P$. Set for each $n \geq 1$

$$P_n = \bigvee_{i=0}^{n-1} F^{-i}P = \{\omega_0 \cap F^{-1}(\omega_1) \cap \cdots \cap F^{-(n-1)}(\omega_{n-1}) : \omega_0, \ldots, \omega_{n-1} \in P\}$$

and

$$P_\infty = \bigvee_{i=0}^{\infty} F^{-i}P = \{\omega_0 \cap F^{-1}(\omega_1) \cap F^{-2}(\omega_2) \cap \cdots : \omega_n \in P \text{ for all } n \geq 0\}.$$ 

(G$_3$) Generating: the $\sigma$-algebra generated by $\bigcup_{n=1}^{\infty} P_n$ coincides with $A$ ($m \mod 0$).

G$_4$) Separating: $P_\infty$ is the partition into single points of $\Delta_0$ ($m \mod 0$).
It follows from (G4) that the separation time
\[ s(x, y) = \min \{ n \geq 0 : F^n(x) \text{ and } F^n(y) \text{ lie in distinct elements of } \mathcal{P} \} \tag{2} \]
is well defined and finite for distinct points \( x, y \) in a full \( m \)-measure subset of \( \Delta_0 \). For definiteness, set the separation time equal to zero for all other points.

\((G_5)\) Gibbs: there are \( C > 0 \) and \( 0 < \beta < 1 \) such that, for all \( x, y \in \omega \in \mathcal{P} \),
\[ \log \frac{J_F(x)}{J_F(y)} \leq C \beta^s(F(x), F(y)). \]

We say that the induced map \( f^R \) has integrable recurrence times if \( R \in L^1(m) \). The next result is standard for transformations admitting Gibbs-Markov induced maps; see e.g. [3, Theorem 3.13] and [3, Corollary 3.21]. In the present context, by an SRB measure we mean an invariant probability measure that is absolutely continuous with respect to the Lebesgue measure \( m \).

**Theorem 1.1.** If \( f^R : \Delta_0 \rightarrow \Delta_0 \) is a Gibbs-Markov induced map for \( f \) with integrable recurrence times, then

1. \( f^R \) has a unique ergodic SRB measure \( \nu_0 \);
2. \( f \) has a unique ergodic SRB measure \( \mu \) with \( \mu(\Delta_0) > 0 \), which is given by
\[ \mu = \frac{1}{\sum_{j=0}^{\infty} \nu_0(\{ R > j \})} \sum_{j=0}^{\infty} f_j^*(\nu_0|\{ R > j \}). \]

In these circumstances, we say that the SRB measure \( \mu \) is given by the Gibbs-Markov induced map \( f^R \). Regarding the integral the entropy formula, in the present context we are naturally lead to consider the case where all Lyapunov exponents are positive and their sum coincides with the Jacobian \( |\det Df| \) of the map \( f \) with respect to the reference (Lebesgue) measure \( m \). As shown in [35, Proposition 2.5], this happens under very general conditions. In our first main result, we establish an entropy formula for an SRB measure \( \mu \) with finite entropy \( h_\mu(f) \) given by a Gibbs-Markov induced map; we also characterise those SRB measures with finite entropy.

**Theorem A.** Let \( f : M \rightarrow M \) be a piecewise \( C^{1+\eta} \) map with an ergodic SRB measure \( \mu \) given by a Gibbs-Markov induced map \( f^R \). Then,

1. if \( h_\mu(f) < \infty \), then
\[ h_\mu(f) = \int_M \log |\det Df| d\mu; \]
2. \( h_\mu(f) < \infty \) if, and only if,
\[ \int_{\Delta_0} R \log |\det Df^R| dm < \infty. \]

The strategy for proving the first item in Theorem A is based on building a tower extension \( (T, \nu) \) for \((f, \mu)\) and showing that
\[ h_\mu(f) = h_\nu(T) = \int \log J_T d\nu = \int \log |\det Df| d\mu. \]
The first equality is a consequence of a general result due to Buzzi for extensions with countably many fibers. The second and third equalities will be deduced in Proposition 4.3 and Lemma 5.1, respectively. The second item of Theorem A will be obtained in Lemma 4.4.

Assuming that there is some constant $C > 0$ for which $|\det Df| \leq C$, it follows from the chain rule that $|\det Df_R| \leq C^R$. We therefore have the following simple, albeit useful, consequence of Theorem A.

**Corollary B.** Let $f : M \to M$ be a piecewise $C^{1+\eta}$ map with an ergodic SRB measure $\mu$ given by a Gibbs-Markov induced map $f^R$. If $|\det Df|$ is bounded and $\int_{\Delta_0} R^2 dm < \infty$, then

$$h_\mu(f) = \int_M \log |\det Df| d\mu < \infty.$$  

In Section 6, we provide an example of a piecewise $C^\infty$ interval map $f$ with an SRB measure $\mu$ given by an induced map for which the formula in the first item of Theorem A is no longer valid in the infinite entropy case. In Section 7, we apply Theorem A and Corollary B to some classes of piecewise $C^{1+\eta}$ maps with nonempty singular sets.

## 2. Tower system

In this section, we follow ideas in [60, 61] and introduce the tower system associated with an induced map and recall some useful properties of this new dynamical system. The construction of the tower will be done with more generality than needed in this section, for we be able to apply it in Section 11. Consider

- a Gibbs-Markov map $F : \Delta_0 \to \Delta_0$ on a space $\Delta_0$ with a reference measure $m$;
- the countable mod 0 partition $\mathcal{P}$ of $\Delta_0$ associated with $F$;
- a measurable function $R : \Delta_0 \to \mathbb{N}$ constant on the elements of $\mathcal{P}$.

We associate to these objects the tower

$$\Delta = \{(x, \ell) : x \in \Delta_0 \text{ and } 0 \leq \ell < R(x)\}$$

and the tower map $T : \Delta \to \Delta$, given by

$$T(x, \ell) = \begin{cases} (x, \ell + 1), & \text{if } \ell < R(x) - 1; \\ (F(x), 0), & \text{if } \ell = R(x) - 1. \end{cases}$$

For $\ell \in \mathbb{N}_0$, the $\ell^{th}$-level of the tower is

$$\Delta_\ell = \{(x, \ell) \in \Delta\}. \quad (3)$$

Observe that we use $\Delta_0$ to represent both the 0th-level (or base) of the tower and the domain of $F$ upon which the tower is built, since they are naturally identified with each other. Moreover, under this identification it is straightforward to check that $T^{R(x)}(x, 0) = F(x)$, for each $x \in \Delta_0$. We call $T^R : \Delta_0 \to \Delta_0$ the return to the base, which in this case is actually a first return map. In the same spirit, the $\ell^{th}$ level of the tower $\Delta_\ell$ is also naturally identified with the set $\{R > \ell\} \subseteq \Delta_0$. This identification allows us to extend the reference measure $m$ on $\Delta_0$ to a measure on $\Delta$, that we still denote by $m$. It happens that

$$m(\Delta) = \sum_{\ell \geq 0} m(\Delta_\ell) = \sum_{\ell \geq 0} m(\{R > \ell\}) = \int_{\Delta_0} R dm,$$
and so, the integrability of $R$ with respect to $m$ (on $\Delta_0$) is a necessary and sufficient condition for the finiteness of the measure $m$ on $\Delta$. The next result provides a unique $T$-invariant probability measure which is absolutely continuous with respect to $m$; see e.g. [3, Theorem 3.24] for a proof.

**Theorem 2.1.** Let $T : \Delta \to \Delta$ be the tower map associated with a Gibbs-Markov map $F$ and recurrence times $R \in L^1(m)$. If $\nu_0$ is the unique $F$-invariant probability measure with $\nu_0 \ll m$, then

$$\nu = \sum_{j=0}^{\infty} \nu_0\{R > j\} \sum_{j=0}^{\infty} T^j_s(\nu_0)\{R > j\}$$

is the unique $T$-invariant probability measure such that $\nu \ll m$. Moreover, $\nu$ is ergodic and the density $d\nu/dm$ is bounded from above and below by positive constants.

Set for convenience

$$\rho = \sum_{j=0}^{\infty} \nu_0\{R > j\} = \int R d\nu_0 < \infty. \quad (4)$$

The finiteness of this quantity is due to the fact we assume $R \in L^1(m)$ and $d\nu_0/dm$ is bounded from above and below by positive constants. An important feature of the tower construction is that

$$\pi : \Delta \to M \quad (x, \ell) \mapsto f^\ell(x), \quad (5)$$

is a semiconjugacy between the original system and the tower system, i.e.

$$f \circ \pi = \pi \circ T \quad \text{and} \quad \mu = \pi_* \nu,$$

where $\mu$ is the $f$-invariant probability measure given by Theorem [11]; see e.g. [3, Proposition 3.27]. This means that the tower system $(T, \nu)$ is an extension of the system $(f, \mu)$, or $(f, \mu)$ is a factor of $(T, \nu)$.

### 3. Jacobians and Natural Partitions

It is straightforward to check that the tower map $T : \Delta \to \Delta$ also has a Jacobian $J_T$ (with respect to the measure $m$ on $\Delta_0$), given by

$$J_T(x, \ell) = \begin{cases} 1, & \text{if } \ell < R(x) - 1; \\ J_F(x), & \text{if } \ell = R(x) - 1. \end{cases} \quad (6)$$

We will deduce some properties for the Jacobian $J_T$ and relate it to a natural partition associated with the tower map in a Volume Lemma. We start with some simple results for the Gibbs-Markov map $F : \Delta_0 \to \Delta_0$.

**Lemma 3.1.** There exists $C > 0$ such that, for all $\omega \in \mathcal{P}$ and $x \in \omega$,

$$\frac{1}{C} \leq \nu_0(\omega) J_F(x) \leq C.$$ 

**Proof.** Since $F$ is a Gibbs-Markov map, it follows from [(G5)] that there exists some $C_1 > 0$ such that, for all $\omega \in \mathcal{P}$ and $x, y \in \omega$,

$$\frac{1}{C_1} \leq \frac{J_F(y)}{J_F(x)} \leq C_1. \quad (7)$$
Using \([G_2]\), we obtain for all \(\omega \in \mathcal{P}\) and \(x \in \omega\),

\[
m(\Delta_0) = \int_{\omega} J_F(y) dm(y) = \int_{\omega} J_F(x) \frac{J_F(y)}{J_F(x)} dm(y),
\]

which together with (7) yields

\[
\frac{1}{C} m(\omega) J_F(x) \leq m(\Delta_0) \leq C_1 m(\omega) J_F(x).
\]

Considering that \(d\nu_0/dm\) is bounded from above and below by positive constants, we easily get the conclusion. \(\square\)

**Corollary 3.2.** There exists \(c > 0\) such that \(J_F(x) \geq c\), for \(\nu_0\) almost all \(x \in \Delta_0\).

**Proof.** It follows from Lemma 3.1 that, for all \(\omega \in \mathcal{P}\) and \(x \in \omega\),

\[
J_F(x) \geq \frac{1}{C\nu_0(\omega)} \geq \frac{1}{C}.
\]

Recall that \(\mathcal{P}\) is an \(m\) mod 0 partition of the set \(\Delta_0\) and \(\nu_0\) is a probability measure equivalent to \(m\). \(\square\)

Let \((X, \eta)\) be a measure space and \(\varphi : X \to \mathbb{R}\) a measurable function. Recall that the integral of \(\varphi\) with respect to \(\eta\) is defined whenever

\[
\int_X \varphi^+ d\eta < \infty \quad \text{or} \quad \int_X \varphi^- d\eta < \infty,
\]

where \(\varphi^\pm = \max \{0, \pm \varphi\}\). In such case,

\[
\int_X \varphi d\eta = \int_X \varphi^+ d\eta - \int_X \varphi^- d\eta.
\]

(8)

As a consequence of the next result, we have that the integral of \(\log J_F\) with respect to the probability measure \(\nu_0\) is defined.

**Corollary 3.3.** \(\int_{\Delta_0} \log J_F^- d\nu_0 < \infty\).

**Proof.** It follows from Corollary 3.2 that \(-\log J_F(x) \leq -\log \alpha\), for \(\nu_0\) almost all \(x \in \Delta_0\). This implies that \(\log J_F^-(x) \leq \max \{0, -\log \alpha\}\), for \(\nu_0\) almost every \(x \in \Delta_0\). Since \(\nu_0\) is a probability measure, we have the conclusion. \(\square\)

In the next result we relate the Jacobians of the maps \(F\) and \(T\) with respect to the reference measure. Recall that \(\rho > 0\) has been introduced in \([4]\).

**Lemma 3.4.** \(\int_{\Delta} \log J_T^\pm d\nu = \frac{1}{\rho} \int_{\Delta_0} \log J_F^\pm d\nu_0\).

**Proof.** For each \(\ell \in \mathbb{N}_0\), set

\[
\Delta_{\ell+1} = \{(x, \ell) \in \Delta : R(x) = \ell + 1\}.
\]

Using these sets and \([6]\), we obtain

\[
\log J_T^\pm(x, \ell) = \begin{cases} 
0, & \text{if } x \notin \Delta_{\ell+1}; \\
\log J_F^\pm(x), & \text{if } x \in \Delta_{\ell+1}.
\end{cases}
\]
From the expression of $\nu$ in Theorem 2.1, we deduce that
\[ \nu|_{\Delta_{\ell+1}} = \rho \nu_0|_{\{R=\ell+1\}} \] (9)

The previous considerations yield
\[ \int_{\Delta} \log J_T^\pm(x, \ell) d\nu(x, \ell) = \sum_{\ell=0}^{\infty} \int_{\Delta_\ell} \log J_T^\pm(x, \ell) d\nu(x, \ell) \]
\[ = \sum_{\ell=0}^{\infty} \int_{\Delta_{\ell+1}} \log J_F^\pm(x) d\nu_0(x) \]
\[ = \frac{1}{\rho} \int_{\Delta_0} \log J_F^\pm(x) d\nu_0(x). \]

The proof is complete. \qed

Corollary 3.5. \[ \int_{\Delta} \log J_T^- d\nu < \infty. \]

Proof. Use Corollary 3.3 and Lemma 3.4. \qed

The countable $m \mod 0$ partition $P$ of $\Delta_0$ associated with the Gibbs-Markov map $F$ naturally induces an $m \mod 0$ countable partitions on each level of the tower. Collecting all these partitions, we obtain an $m \mod 0$ partition $Q$ of the entire tower $\Delta$. A sequence of dynamically generated $m \mod 0$ partitions of $\Delta$ is then defined, for each $n \geq 1$, by
\[ Q_n = \bigvee_{i=0}^{n-1} T^{-i} Q. \] (10)

Given $(x, \ell) \in \Delta$, let $Q_n(x, \ell)$ denote the element of $Q_n$ containing the point $(x, \ell) \in \Delta$. For a proof of the following bounded distortion property for $T$ see e.g. [3, Lemma 3.30].

Lemma 3.6. There exists $C_1 > 0$ such that, for all $n \geq 0$ and $(y, k) \in Q_n(x, \ell)$,
\[ \frac{1}{C_1} \leq \frac{J_{T^n}(y, k)}{J_{T^n}(x, \ell)} \leq C_1. \]

Our goal now is to compare $m(Q_n(x, \ell))$ with the Jacobian $J_{T^n}(x, \ell)$, at least for an infinite number of times $n \in \mathbb{N}$. This is the main technical difference between inducing schemes and tower extensions we have to deal with in order to replicate the same approach, since in the former case, owing to the Markov property $G_1$, the comparison can be done for all times $n \in \mathbb{N}$. Such property does not pass down to the natural partition $Q$ of the tower but, fortunately, the strategy does not require all its strength. In the present situation, we use the approach in [7] for piecewise expanding maps, and the key feature of existence of infinitely many moments for which the respective forward image of the refined atom of a typical point has measure uniformly bounded away from zero. Here, the bound is given by the measure of the ground level. Given $(x, \ell) \in \Delta$, set
\[ \mathcal{M}(x, \ell) = \{ n \geq 1 : m(T^n(Q_n(x, \ell))) \geq m(\Delta_0) \}. \]
Lemma 3.7. The set $M(x, \ell)$ has infinitely many elements, for $m$ almost every $(x, \ell) \in \Delta$.

Proof. The identification of $TR$ and $F$ allows us to think of the partitions in (I) as partitions of the base level of the tower. We also use $P_n(x, 0)$ to denote the element of $P_n$ containing the point $(x, 0) \in \Delta_0$. Setting for each $n \geq 1$

$$R_n = \sum_{j=0}^{n-1} R \circ (T_R)^j.$$ 

it easily follows that, for $m$ almost every $(x, 0) \in \Delta_0$,

$$TR_n(Q_{R_n}(x, 0)) = T^R_n(P_n(x, 0)) = \Delta_0.$$ 

Recalling that $T$ is an upward translation between consecutive returns to the base, we have for $m$ almost all $(x, \ell) \in \Delta$

$$T^{R_n-\ell}(Q_{R_n-\ell}(x, \ell)) = T^{R_n}(Q_{R_n}(x, 0)) = \Delta_0.$$ 

This clearly gives the conclusion. \[\square\]

Note that the proof of the Lemma 3.7 provides the same conclusion for a more accurate version of $M(x, \ell)$. In fact, it gives $T^n(Q_n(x, \ell)) = \Delta_0$, for infinitely many values of $n \in \mathbb{N}$. However, a uniform lower bound for $m(T^n(Q_n(x, \ell)))$ is really what we need for the proof next lemma.

Lemma 3.8 (Volume Lemma). There exists $C_0 > 0$ such that, for $m$ almost all $(x, \ell) \in \Delta$ and all $n \in M(x, \ell)$,

$$\frac{1}{C_0} \leq m(Q_n(x, \ell))J_{T^n}(x, \ell) \leq C_0.$$ 

Proof. For all $n \geq 1$ and $m$ almost all $(x, \ell) \in \Delta$, we may write

$$m(T^n(Q_n(x, \ell))) = \int_{Q_n(x, \ell)} J_{T^n}(y, k) \, dm(y, k)$$

$$= \int_{Q_n(x, \ell)} \frac{J_{T^n}(y, k)}{J_{T^n}(x, \ell)} J_{T^n}(x, \ell) \, dm(y, k). \tag{11}$$

It follows from (11) and Lemma 3.6 that

$$m(\Delta) \geq m(T^n(Q_n(x, \ell))) \geq C_1^{-1} J_{T^n}(x, \ell) m(Q_n(x, \ell)),$$

and consequently,

$$J_{T^n}(x, \ell)m(Q_n(x, \ell)) \leq C_1 m(\Delta).$$

In addition, it follows from (11), Lemma 3.6 and the definition of the set $M(x, \ell)$ that, for all $n \in M(x, \ell)$,

$$m(\Delta_0) \leq m(T^n(Q_n(x, \ell))) \leq C_1 J_{T^n}(x, \ell) m(Q_n(x, \ell)),$$

and therefore,

$$C_1^{-1} m(\Delta_0) \leq J_{T^n}(x, \ell)m(Q_n(x, \ell)).$$

Taking $C_0 = \max \{m(\Delta)C_1, (C_1^{-1} m(\Delta_0))^{-1}\}$, we are done. \[\square\]
4. Entropy of the tower system

The goal of this section is to deduce the entropy formula for the tower system. We could appeal to Rohlin’s formula, which gives the entropy transformations with a generating partition as the integral of the Jacobian of the transformation with respect to the invariant measure; see e.g. [58, Theorem 9.7.3]. However, to avoid introducing new concepts, we provide in Proposition 4.3 a self-contained proof of the entropy formula for the tower system. This uses a general version of Birkhoff Ergodic Theorem for functions whose integral is defined; recall (8). The need of this is due to the fact that nothing guarantees the integrability of $\log J_T^\pm$ with respect to $\nu$. The integrability of $\log J_T^-$ given by Corollary 3.5 will be sufficient for our purpose.

Lemma 4.1 (Generalised Birkhoff Theorem). Let $\Phi : X \to X$ preserve an ergodic probability measure $\eta$ and $\varphi : X \to \mathbb{R}$ be such that $\int_X \varphi \, d\eta$ is defined. For $\eta$ almost every $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\Phi^i(x)) = \int_X \varphi \, d\eta.$$  

Proof. If $\varphi \in L^1(\eta)$, the conclusion is just the classical case of Birkhoff Ergodic Theorem. Otherwise, one of the functions $\varphi^\pm$ belongs in $L^1(\eta)$ and the integral of the other one is equal to $+\infty$. Suppose for definiteness that $\int \varphi^+ \, d\eta = +\infty$. In such case, we have $\int \varphi \, d\eta = +\infty$. Set for each $n \geq 1$

$$S_n \varphi = \sum_{j=0}^{n-1} \varphi \circ \Phi^j.$$  

Note that both $S_n \varphi$ and $\int \varphi \, d\eta$ are linear in $\varphi$. Since $\varphi = \varphi^+ - \varphi^-$, it is enough to show that, for $\eta$ almost every $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} S_n \varphi^+(x) = +\infty. \quad (12)$$  

Set for each $N \geq 1$,

$$\varphi_N = \min\{N, \varphi^+\}.$$  

By the Monotone Convergence Theorem,

$$\lim_{N \to \infty} \int \varphi_N \, d\eta = \int \varphi^+ \, d\eta = +\infty. \quad (13)$$  

Also, for all $x \in X$ and $n, N \geq 1$,

$$\frac{1}{n} S_n \varphi^+(x) \geq \frac{1}{n} S_n \varphi_N(x). \quad (14)$$  

Since $\eta$ is a finite measure and $\varphi_N$ is bounded, we have $\varphi_N \in L^1(\eta)$, for all $N \geq 1$. It follows from Birkhoff Ergodic Theorem that, for $\eta$ almost every $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} S_n \varphi_N(x) = \int \varphi_N \, d\eta,$$  

which together with (13) and (14) yields (12). \qed
In Lemma 4.2 below, we show that the entropy of the natural partition $Q$ introduced in Section 3 gives the entropy of the tower system. Recall that

$$h_\nu(T, Q) = \inf_{n \geq 1} \frac{1}{n} H_\nu(Q_n)$$

(15)

and

$$H_\nu(Q_n) = - \sum_{\omega \in Q_n} \nu(\omega) \log \nu(\omega).$$

**Lemma 4.2.** $h_\nu(T) = h_\nu(T, Q)$.

**Proof.** By the separation property (G\textsubscript{4}) for the Gibbs-Markov map $T^R : \Delta_0 \to \Delta_0$, we have that $P_\infty$ is the partition into single points of $\Delta_0$. This implies that $Q_\infty$ is the partition into single points of $\Delta$. Since $Q_1 \leq Q_2 \leq \cdots$, it follows from [46, Corollary 5.12] that

$$h_\nu(T) = \lim_{n \to \infty} h_\nu(T, Q_n).$$

On the other hand, it is a general fact that $h_\nu(T, Q) = h_\nu(T, Q_n)$, for all $n \geq 1$. This completes the proof. $\square$

In the next result, we establish the entropy formula for tower systems for which the natural partition of the tower has finite entropy. It follows from Lemma 4.2 and (15) that the finiteness of $H_\nu(Q)$ is actually equivalent to the finiteness of the entropy of the tower system. The example in Section 6 shows that the conclusion is no longer valid if the entropy is not finite.

**Proposition 4.3.** If $H_\nu(Q) < \infty$, then

$$h_\nu(T) = \int_{\Delta} \log J_T d\nu.$$ 

**Proof.** First of all, note that, as we assume $H_\nu(Q) < \infty$, Shannon-McMillan-Breiman Theorem can be used to obtain $h_\nu(T, Q)$. Taking $(x, \ell) \in \Delta$ a generic point with respect to $\nu$ (or $m$), the proof is produced with the following ingredients:

1. $Q$ is a generating partition (Lemma 4.2);
2. Shannon-McMillan-Breiman Theorem;
3. $d\nu/dm$ is bounded from above and below by positive constants (Theorem 2.1);
4. $M(x, \ell)$ has infinitely many elements (Lemma 3.7);
5. Volume Lemma (Lemma 3.8);
6. Chain Rule for the Jacobian;
7. Generalised Birkhoff Theorem for $\log J_T$ (Corollary 3.5 & Lemma 4.1);
and prepared as follows:

\[
H_\nu(T) = h_\nu(T, Q) = \lim_{n \to \infty} -\frac{1}{n} \log \nu(Q_n(x, \ell)) = \lim_{n \to \infty} -\frac{1}{n} \log m(Q_n(x, \ell)) \\
= \lim_{n \to \infty} \frac{1}{n} \log J_T(x) \\
= \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log J_T(T^i(x, \ell)) \\
= \int_\Delta \log J_T^c d\nu.
\]

This gives the conclusion. \(\square\)

We finish this section with a result that will be used to obtain the equivalence in the second item of Theorem A.

**Lemma 4.4.** \(H_\nu(Q) < \infty\) if, and only if, \([\Delta_0 \log J_F dm < \infty].\)

**Proof.** Set for each \(n \geq 1\)

\[
R_n = \{\omega \in \mathcal{P} : R(\omega) = n\}.
\]

By definition of the tower, for each \(\omega \in R_n\), there are exactly \(n\) elements \(\tilde{\omega} \in Q\) above \(\omega\) and, by Theorem 2.1,

\[
\nu(\tilde{\omega}) = \frac{1}{\rho} \nu_0(\omega),
\]

with \(\rho\) as in \((4)\). Hence,

\[
H_\nu(Q) = -\sum_{\tilde{\omega} \in Q} \nu(\tilde{\omega}) \log \nu(\tilde{\omega}) \\
= \frac{1}{\rho} \sum_{n \geq 1} \sum_{\omega \in R_n} n\nu_0(\omega) \log \left(\frac{\rho}{\nu_0(\omega)}\right) \\
= \frac{1}{\rho} \int_{\Delta_0} R(x) \log \left(\frac{\rho}{\nu_0(\mathcal{P}(x))}\right) d\nu_0(x). \tag{16}
\]

Using Lemma 3.1 we get

\[
\frac{\rho}{C} J_F(x) \leq \frac{\rho}{\nu_0(\mathcal{P}(x))} \leq C \rho J_F(x),
\]

and so

\[
\log \frac{\rho}{C} + \log J_F(x) \leq \log \left(\frac{\rho}{\nu_0(\mathcal{P}(x))}\right) \leq \log(C\rho) + \log J_F(x). \tag{17}
\]
It follows from (16) and (17) that
\[
\log \frac{\rho}{C} \int_{\Delta_0} R d\nu_0 + \int_{\Delta_0} R \log J_{F}d\nu_0 \leq \rho H_{\nu}(Q) \leq \log(C\rho) \int_{\Delta_0} R d\nu_0 + \int_{\Delta_0} R \log J_{F}d\nu_0.
\]
Since \(d\nu_0/dm\) is bounded from above and below by positive constants and \(\int_{\Delta_0} R d\nu_0 < \infty\), we get the conclusion. \(\square\)

5. Entropy of the original system

Here, we complete the proof of Theorem A. Since the tower system \((T, \nu)\) is an extension of \((f, \mu)\) with countably many fibers, we may use [16, Proposition 2.8] to get
\[
h_{\mu}(f) = h_{\nu}(T). \tag{18}
\]
By Lemma 4.2 and (15), we have
\[
h_{\nu}(T) = h_{\nu}(T, Q) = \inf_{n \geq 1} \frac{1}{n} H_{\nu}(Q_n). \tag{19}
\]
Since \(H_{\nu}(Q) \leq H_{\nu}(Q_n)\), for all \(n \geq 1\), it follows from (19) and Lemma 4.4 that
\[
h_{\nu}(T) < \infty \iff H_{\nu}(Q) < \infty \iff \int_{\Delta_0} R \log J_{F}dm < \infty, \tag{20}
\]
which together with (18) yields
\[
h_{\mu}(f) < \infty \iff \int_{\Delta_0} R \log J_{F}dm < \infty.
\]
This gives the second item of Theorem A. The first item is a consequence of (18), (20), Proposition 4.3 and the next result.

Lemma 5.1. \(\int_{\Delta} \log J_{T} d\nu = \int_{M} \log |\det Df| d\mu.\)

Proof. Consider the measurable partition \(Q_{R} = \{\Delta_{n}^{\ell}\}\) of \(\Delta\), defined for all \(\ell \in \mathbb{N}_0\) and \(n > \ell\) by
\[
\Delta_{n}^{\ell} = \{(x, \ell) \in \Delta: R(x) = n\}.
\]
Clearly, the natural partition \(Q\) is a refinement of \(Q_{R}\). Recalling (6), we have
\[
J_{T}|_{\Delta_{n}^{\ell}}(x, \ell) = \begin{cases} 1, & \text{if } n > \ell + 1; \\ J_{fR}(x), & \text{if } n = \ell + 1. \end{cases}
\]
Writing for simplicity \(J_{f} = |\det Df|\) and using the chain rule, we have for each \(x \in \Delta_0\) with \(R(x) = n\)
\[
J_{fR}(x) = J_{f}(f^{n-1}(x)) \cdots J_{f}(f(x)) \cdot J_{f}(x).
\]
The previous considerations yield
\[
\int_{\Delta} \log J_{T}(x, \ell) d\nu(x, \ell) = \sum_{\ell=0}^{\infty} \int_{\Delta_{\ell}} \log J_{T}(x, \ell) d\nu(x, \ell)
= \sum_{\ell=0}^{\infty} \int_{\Delta_{\ell+1}} \log J_{fR}(x) d\nu(x, \ell)
\]
\[\begin{align*}
&= \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} \int_{\Delta_{\ell+1}} \log J_f(f^i(x)) \, d\nu(x, \ell) \\
&= \sum_{\ell=0}^{\infty} \sum_{n>\ell} \int_{\Delta_n} \log J_f(f^\ell(x)) \, d\nu(x, \ell) \\
&= \sum_{\ell=0}^{\infty} \int_{\Delta_{\ell}} \log J_f(f^\ell(x)) \, d\nu(x, \ell) \\
&= \sum_{\ell=0}^{\infty} \int_{\Delta_{\ell}} \log J_f(\pi(x, \ell)) \, d\nu(x, \ell) \\
&= \int_{\Delta} \log J_f(\pi(x, \ell)) \, d\nu(x, \ell).
\end{align*}\]

Finally, observing that \(\pi_*\nu = \mu\), we get
\[\int_{\Delta} \log J_f \circ \pi \, d\nu = \int_M \log J_f \, d\pi_*\nu = \int_M \log J_f \, d\mu.\]

The proof is complete. \(\square\)

6. INFINITE ENTRPY

In this section, we give an example of a piecewise smooth interval map having an SRB measure given by a Gibbs-Markov induced map and not satisfying the entropy formula. To build this example, we are actually going to give

(i1) an interval \(\Delta_0\) with finite Lebesgue measure \(m\);

(i2) a piecewise \(C^\infty\) Gibbs-Markov map \(F: \Delta_0 \to \Delta_0\) with associated partition \(P\);

(i3) a measurable function \(R: \Delta_0 \to \mathbb{N}\) constant on each element of \(P\), with \(R \in L^1(m)\).

The tower map \(T: \Delta \to \Delta\) associated with these objects can easily be thought of as a piecewise \(C^\infty\) map of an interval with countably many smoothness domains and
\[J_T = |T'|\]
on each smoothness domain. In this case, the return to the base of the tower \(T^R: \Delta_0 \to \Delta_0\), which is naturally identified with the map \(F\), is a Gibbs-Markov induced map for \(T\). Therefore, we need to present objects as in \(\{i1\} \{i3\}\) such that for the corresponding tower map \(T: \Delta \to \Delta\) we have
\[h_\nu(T) = \infty \quad \text{and} \quad \int_\Delta \log J_T \, d\nu < \infty, \quad (21)\]

where \(\nu\) is the unique \(T\)-invariant probability measure such that \(\nu \ll m\), given by Theorem 2.1. We are going to use the continuous map \(\phi: [0, 1/e] \to [0, 1/e]\), given by \(\phi(0) = 0\) and
\[\phi(x) = -x \log x,\]
for each \(x > 0\). Note that, for all \(0 < x < 1/e\),
\[\phi'(x) = -\log x - 1 > 0 \quad \text{and} \quad \phi''(x) = -\frac{1}{x} < 0. \quad (22)\]
Hence, $\phi$ is an increasing concave function. Since $\phi$ is continuous at 0 and $\phi(1/e) = 1/e$, then $\phi$ is a bijection. For each $n \geq 2$, set

$$a_n = \phi^{-1}\left(\frac{1}{n^2 \log n}\right).$$

and

$$b_n = \sum_{i=2}^{n} a_i.$$

Set also $b_1 = 0$. In Lemma 6.1 below we show that

$$\sum_{n=2}^{\infty} a_n < +\infty.$$

Consider $\Delta_0 = [0, b]$ and the $m \mod 0$ partition $\mathcal{P} = \{\omega_n\}_{n \geq 2}$ of $\Delta_0$, where

$$\omega_n = (b_{n-1}, b_n),$$

for each $n \geq 2$. We define $F : \Delta_0 \to \Delta_0$, mapping each $\omega_n$ linearly onto $\Delta_0$. In this way, we have for each $n \geq 2$

$$J_F|_{\omega_n} = |F'|_{\omega_n} = \frac{b}{a_n} \geq \frac{b}{a_2} > 1.$$

It is not difficult to check that $F$ is a Gibbs-Markov map; see e.g. [3, Lemma 3.3]. We also define a map $R : \Delta_0 \to \mathbb{N}$, setting for all $n \geq 2$

$$R|_{\omega_n} = n.$$

Finally, consider the tower map $T : \Delta \to \Delta$ as in Section 2 associated with these objects. Observe that, for each $n \geq 2$,

$$m(\{R = n\}) = m(\omega_n) = a_n.$$

It follows from the construction above that

$$\int_{\Delta_0} R \, dm = \sum_{n \geq 2} na_n,$$

$$\int_{\Delta_0} \log J_F \, dm = \sum_{n \geq 2} a_n (\log b - \log a_n) = \log b \sum_{n \geq 2} a_n + \sum_{n \geq 2} \phi(a_n),$$

$$\int_{\Delta_0} R \log J_F \, dm = \sum_{n \geq 2} na_n (\log b - \log a_n) = \log b \sum_{n \geq 2} na_n + \sum_{n \geq 2} n\phi(a_n).$$

In Lemma 6.1 below, we show that $\sum_{n \geq 2} na_n < +\infty$, which then gives $R \in L^1(m)$. By Lemma 3.3,

$$\int_{\Delta} \log J_T \, d\nu = \frac{1}{\rho} \int_{\Delta_0} \log J_F \, d\nu_0,$$

with $\rho = \int R \, d\nu_0 < \infty$ as in [4]. Recalling the second item of Theorem A and (24)-(25) above, we obtain (21) from the next result.

**Lemma 6.1.**

1. $\sum_{n \geq 2} a_n < +\infty$;
2. $\sum_{n \geq 2} na_n < +\infty.$
(3) $\sum_{n\geq 2} \phi(a_n) < \infty$;
(4) $\sum_{n\geq 2} n\phi(a_n) = \infty$.

**Proof.** Note that $\phi(a_n) = 1/(n^2 \log n)$, for each $n \geq 2$. Therefore, (3) is obvious and (4) can be easily obtained by the integral test. Note also that (2) implies (1). It remains to verify the second item. For each $n \geq 2$,

$$a_n = \phi^{-1}\left(\frac{1}{n^2 \log n}\right) = \phi^{-1}\left(\frac{1}{n^2 \log n}\right) - \phi^{-1}(0) = \left(\phi^{-1}\right)'(c_n) \frac{1}{n^2 \log n},$$

for some $c_n$ between 0 and $1/(n^2 \log n)$. It follows from (22) that, for each $n \geq 2$,

$$\left(\phi^{-1}\right)'(c_n) = \frac{1}{-\log \phi^{-1}(c_n) - 1}.$$

Using that $\phi^{-1}$ is an increasing concave function, we have for all $n \geq 3$

$$0 \leq \phi^{-1}(c_n) \leq \phi^{-1}\left(\frac{1}{n^2 \log n}\right) \leq \frac{1}{n^2 \log n} \leq \frac{1}{n^2}.$$

It follows that

$$\left(\phi^{-1}\right)'(c_n) \leq \frac{1}{2 \log n - 1} \leq \frac{1}{\log n},$$

for all $n \geq 3$, and so

$$na_n \leq \frac{1}{n \log^2 n}. \quad (26)$$

Now, by the Cauchy condensation test,

$$\sum_{n \geq 2} \frac{1}{n \log^2 n} \leq \sum_{n \geq 2} 2^n \frac{2^n}{2^n \log^2 2^n} = \sum_{n \geq 2} \frac{1}{n^2 \log^2 2} < \infty.$$

Since this last series converges and (26) holds for all $n \geq 3$, we get the second item. \qed

**Remark 6.2.** The system $(T, \nu)$ is a counterexample to Ruelle inequality in the setting of maps which are differentiable only almost everywhere. Indeed, we have for $\nu$ almost every $(x, \ell) \in \Delta$,

$$\lim_{n \to \infty} \frac{1}{n} |(T^n)(x(x, \ell))| = \int_{\Delta} \log |T'| d\nu < \infty = h_{\nu}(T).$$

This is the reverse of Ruelle inequality.

Regarding the previous remark, it will be interesting to note that Ruelle inequality is obtained in [14, Proposition 1] for Lipschitz maps. Actually, the proof uses only that the map is uniformly continuous and differentiable almost everywhere, the latter being assured by Rademacher Theorem. The system $(T, \nu)$ also illustrates how important is the uniform continuity for the conclusion of [14, Proposition 1].

**Remark 6.3.** The system $(T, \nu)$ illustrates that the conclusion of Shannon-McMillan-Breiman Theorem does not hold in case we do not assume that the generating partition has finite entropy. Indeed, in the previous sections, condition $H_{\nu}(Q) < \infty$ (which is equivalent to $h_{\nu}(T) < \infty$) is used only in step (2) of the proof of Proposition 4.3. Therefore, equalities (3)-(7) in that proof remain valid for $(T, \nu)$, yielding for every generic point $(x, \ell) \in \Delta$

$$\lim_{n \to \infty} \frac{1}{n} \log \nu(Q_n(x, \ell)) = \int_{\Delta} \log |T'| d\nu < \infty = h_{\nu}(T).$$
7. Applications

In the subsections below, we apply Theorem A and Corollary B to some classes of piecewise smooth maps having SRB measures given by Gibbs-Markov induced maps. We admit the possibility that in some of the cases the entropy formula has already be obtained by other methods. Anyway, we include them all here as a way of illustrating how vast and general the applications of our main results in this first part can be. We present some classes of concrete examples, but we could have placed more abstract classes such as those dealt with in [27] or [28].

7.1. Lorenz-like maps. Here, we consider a class of piecewise expanding one-dimensional maps with a singular point, introduced by Guckenheimer and Williams in [29] Section 2. They appear as the quotient of a Poincaré return map for a geometric model of the Lorenz flow; see [9, Section 3.3.2.1] for details. Such quotient maps are modeled by a family of maps with a singular point, introduced by Guckenheimer and Williams in [29, Section 2].

Moreover, for all derivatives are both equal to zero. It was proved in [44] that for all \( \alpha \) such that the one-dimensional quotient map has chaotic behaviour, no longer robustly, but contracting Lorenz attractor and managed to show that the one-dimensional quotient map associated with the Poincaré return map has chaotic behaviour, no longer robustly, but persistently in a measure theoretical sense: it is given by a family of maps having SRB measures given by Gibbs-Markov induced maps. We

7.2. Rovella maps. Considering a geometric construction similar to that in [29], reversing only a relation in the eigenvalues of the singularity, Rovella introduced in [51] the so-called contracting Lorenz attractor and managed to show that the one-dimensional quotient map associated with the Poincaré return map has chaotic behaviour, no longer robustly, but persistently in a measure theoretical sense: it is given by a family of maps \( f_a : I \to I \) of the interval \( I = [-1/2, 1/2] \) satisfying the following properties:

(1) \( f \) is a \( C^1 \) local diffeomorphism in \( I \setminus \{0\} \) with \( f(0^+) = -1/2 \) and \( f(0^-) = 1/2 \);
(2) there is \( \lambda > 1 \) such that \( f'(x) \geq \lambda \), for all \( x \in I \setminus \{0\} \);
(3) there is \( 0 < \alpha < 1 \) such that \( f'(x) = |x|^{-\alpha} \), for all \( x \in I \setminus \{0\} \).

It is well-known that such a map \( f \) admits a unique ergodic SRB measure \( \mu \); see e.g. [57, Corollary 3.4]. Moreover, it was proved in [27, Theorem 1] that \( \mu \) is given by a Gibbs-Markov induced map. It is easily verified that \( \mathcal{P} = \{(-1/2, 0), (0, 1/2)\} \) is a \( \mu \) mod 0 generating partition. Since \( \mathcal{P} \) is finite, we have \( H_\mu(\mathcal{P}) < \infty \), and so \( h_\mu(f, \mathcal{P}) < \infty \). It follows that \( h_\mu(f) < \infty \), and Theorem A then yields

\[
    h_\mu(f) = \int_I \log |f'| d\mu.
\]

Moreover, for all \( a \in [0, 1] \)

(5) there is \( 0 < \alpha < 1 \) such that \( (f'_a(x) \approx |x|^{\alpha} \), for all \( x \in I \setminus \{0\} \);
(6) \( f_a \) is \( C^0 \) in \( I \setminus \{0\} \) with derivatives depending continuously on \( a \);
(7) the functions \( a \mapsto f_a(-1) \) and \( a \mapsto f_a(1) \) have derivative 1 at \( a = 0 \).

Note that \( x = 0 \) is a discontinuity point for each \( f_a \) and also a critical point: the side derivatives are both equal to zero. It was proved in [44] that for all \( a \in E \), the map \( f_a \) has some ergodic SRB measure, and it was shown in [8, Corollary B] that the SRB
measure is unique. Moreover, it follows from [8] Theorem A and [2] Theorem 4.1 that the SRB measure \( \mu_a \) is given by an induced Gibbs-Markov map for which \( m(\{ R > n \}) \) decays exponentially fast to zero with \( n \). This in particular implies that \( \int R^2 dm < \infty \). Since \( f' \) is bounded, it follows from Corollary [3] that, for each \( a \in E \),
\[
h_{\mu_a}(f_a) = \int_I \log |f'_a| d\mu_a.
\]

7.3. Intermittent maps. In this subsection, we consider a class of maps which contain as a particular case the model studied by Liverani, Saussol and Vaienti in [12]. Given \( \alpha > 0 \), consider the interval \( I = [0, 1] \), a point \( z_0 \in (0, 1) \) and a map \( f : I \to I \), defined by
\[
f_\alpha(x) = \begin{cases} 
g_0(x), & \text{if } 0 \leq x \leq z_0; 
g_1(x), & \text{if } z_0 < x \leq 1. 
\end{cases}
\]
(27)

Assume \( g_1 \) is a \( C^2 \) map with derivative strictly greater than one mapping \((z_0, 1]\) diffeomorphically to \((0, 1]\), and \( g_0 \) is such that
1. \( f_\alpha(0) = 0 \) and \( f_\alpha(z_0) = 1 \);
2. \( f'_\alpha(0) = 1 \) and \( f'_\alpha(x) > 1 \) for \( x \in (0, z_0] \);
3. \( f_\alpha \) is \( C^2 \) on \((0, z_0]\) and \( f''_\alpha(x) \approx x^{\alpha-1} \).

Note that \( f_\alpha \) is \( C^{1+\alpha} \) on \( I \setminus \{ z_0 \} \), for each \( 0 < \alpha < 1 \), with a discontinuity at \( z_0 \). The model in [12] is given by the particular choices of
\[
z_0 = \frac{1}{2}, \quad g_0(x) = x + 2^\alpha x^{\alpha+1} \quad \text{and} \quad g_1(x) = 2x - 1.
\]
(28)

For \( \alpha \geq 1 \), the Dirac measure at zero is a physical measure for \( f_\alpha \) and its basin covers \( m \) almost all of \( I \); see e.g. [3] Theorem 3.59. Therefore, \( f_\alpha \) has no SRB measure for \( \alpha \geq 1 \).

For \( 0 < \alpha < 1 \), the map \( f_\alpha \) has a unique SRB measure \( \mu_\alpha \) given by a Gibbs-Markov induced map; see e.g. [3] Theorem 3.59. It is easily verified that \( P = \{(0, z_0), (z_0, 1)\} \) is a \( \mu_\alpha \) mod 0 generating partition. Since \( P \) is finite, we have \( H_{\mu_\alpha}(P) < \infty \), and so \( h_{\mu_\alpha}(f_\alpha, P) < \infty \). It follows that \( h_{\mu_\alpha}(f_\alpha) < \infty \). Theorem [4] then yields
\[
h_{\mu_\alpha}(f_\alpha) = \int_I \log |f'_\alpha| d\mu_\alpha.
\]

7.4. Singular intermittent maps. Here, we consider a family of maps \( f : S^1 \to S^1 \) of the circle \( S^1 = [-1, 1]/\sim \), with \(-1 \sim 1 \), introduced by Cristadoro, Haydn, Marie and Vaienti in [24], combining the infinite derivative in Section 7.1 with the intermittency phenomenon in Section 7.3. The image \( f(x) \) depends on a parameter \( \gamma > 1 \) and is defined implicitly for \( x > 0 \) by
\[
x = \begin{cases} 
\frac{1}{2\gamma}(1 + f(x)^\gamma), & \text{if } 0 \leq x \leq \frac{1}{2\gamma}; 
f(x) + \frac{1}{2\gamma}(1 - f(x)^\gamma), & \text{if } \frac{1}{2\gamma} < x \leq 1.
\end{cases}
\]
and for \( x < 0 \) by \( f(x) = -f(-x) \). The map \( f \) is \( C^1 \) on \( S^1 \setminus \{ 0 \} \) and \( C^2 \) on \( S^1 \setminus \{ 0, 1 \} \), with
\[
\lim_{x \to \pm 0} f'(x) = +\infty.
\]
For the limit case \( \gamma = 1 \), this is the well-known doubling map, and for the particular value \( \gamma = 2 \), this is the map considered in [4] as an example of a system with positive frequency of
hyperbolic times but non-integrability of the first hyperbolic time. As observed in [24], the Lebesgue measure \( m \) is \( f \)-invariant for all \( \gamma > 1 \). Moreover, it is given by an induced Gibbs-Markov map; see [24] Section 3. It is easily verified that \( P = \{ (-1,0), (0,1) \} \) is an \( m \) mod 0 generating partition. Since \( P \) is finite, we have \( H_m(P) < \infty \), and so \( h_m(f,P) < \infty \). This implies that \( h_m(f) < \infty \). Theorem \([A]\) then yields

\[
h_m(f) = \int_{S^1} \log |f'| \, dm.
\]

### 7.5. Skew-product intermittent maps.

In this subsection, we consider a family of two-dimensional piecewise smooth maps introduced by Bahsoun, Bose and Duan in [10], based on a previous skew-product random map in [11]. Consider the interval \( I = [0,1] \) and, for each \( \alpha > 0 \), let \( f_\alpha \colon I \to I \) be defined as in (27) with the particular choices of \( z_0, g_0, g_1 \) as in (28). Take real numbers \( \alpha_0, \alpha_1, p_0, p_1 > 0 \) such that \( 0 < \alpha_0 < \alpha_1 \leq 1 \) and \( p_0 + p_1 = 1 \). Consider the \( C^{1+\alpha_0} \) skew-product transformation \( f : I \times I \to I \times I \) by

\[
f(x, y) = (f_\alpha(y)(x), \varphi(y)),
\]

where

\[
\alpha(y) = \begin{cases} 
\alpha_0, & \text{if } y \in [0, p_0); \\
\alpha_1, & \text{if } y \in [p_0, 1];
\end{cases}
\]

and

\[
\varphi(y) = \begin{cases} 
y, & \text{if } y \in [0, p_0); \\
p_1, & \text{if } y \in [p_0, 1].
\end{cases}
\]

Note that the discontinuity points of \( f \) are given by the lines \( x = 1/2 \) and \( y = p_0 \). Therefore, \( f \) has a partition into four smoothness domains. It follows from the results in [10] Section 3 that \( f \) has a unique SRB measure \( \mu \) given by a Gibbs-Markov induced map. It is easily verified that \( P = \{ (0,1/2) \times (0, p_0), (0,1/2) \times (p_0, 1), (1/2, 1) \times (0, p_0), (1/2, 1) \times (p_0, 1) \} \) is a \( \mu \) mod 0 generating partition. Since \( P \) is finite, we have \( H_\mu(P) < \infty \), and so \( h_\mu(f, P) < \infty \). It follows that \( h_\mu(f) < \infty \). Theorem \([A]\) then yields

\[
h_\mu(f) = \int_{I \times I} \log |f'| \, d\mu.
\]

### Part II. Systems with hyperbolic structures

Let \( M \) be a finite dimensional compact Riemannian manifold \( M \). Let \( \text{dist} \) be the distance on \( M \) and \( m \) be the Lebesgue (volume) measure on the Borel sets of \( M \) induced by the Riemannian metric. Given a submanifold \( \gamma \subset M \), we use \( \text{dist}_\gamma \) to denote the distance on \( \gamma \) and \( m_\gamma \) to denote the Lebesgue measure on \( \gamma \) induced by the restriction of the Riemannian metric to \( \gamma \). Let \( f : M \to M \) be a \( C^{1+\eta} \) piecewise diffeomorphism, meaning that there is a countable number of pairwise disjoint open regions \( M_1, M_2, \ldots \) such that \( \bigcup_{k \geq 1} M_k = M \) and \( f|_{\bigcup_{k \geq 1} M_k} \) is a \( C^{1+\eta} \) diffeomorphism onto its image. We refer to

\[
S = M \setminus \bigcup_{k \geq 1} M_k
\]

as the singular set of \( f \). Typically, \( S \) can be a set of critical points or points where the derivative of \( f \) does not exist (possibly, discontinuity points).
8. Young sets

We say that \( \Gamma \) is a continuous family of \( C^1 \) disks in \( M \), if there are a compact metric space \( K \), a unit disk \( D \) in some \( \mathbb{R}^k \) and an injective continuous function \( \Phi: K \times D \to M \) such that

- \( \Gamma = \{ \Phi(\{x\} \times D) : x \in K \} \);
- \( \Phi \) maps \( K \times D \) homeomorphically onto its image;
- \( x \mapsto \Phi(\{x\} \times D) \) defines a continuous map from \( K \) into \( \operatorname{Emb}^1(D, M) \), where \( \operatorname{Emb}^1(D, M) \) denotes the space of \( C^1 \) embeddings of \( D \) into \( M \).

Note that the disks in \( \Gamma \) have all the same dimension (of the disk \( D \)) denoted by \( \dim \Gamma \).

We say that a compact set \( \Lambda \subset M \) has a product structure, if there are continuous families of \( C^1 \) Pesin stable disks \( \Gamma^s \) and unstable disks \( \Gamma^u \) such that

- \( \Lambda = (\bigcup_{\gamma \in \Gamma^s} \gamma) \cap (\bigcup_{\gamma \in \Gamma^u} \gamma) \);
- \( \dim \Gamma^s + \dim \Gamma^u = \dim M \);
- each \( \gamma \in \Gamma^s \) meets each \( \gamma \in \Gamma^u \) in exactly one point.

We say that \( \Lambda_0 \subset \Lambda \) is an s-subset, if \( \Lambda_0 \) has a product structure with respect to families \( \Gamma_0^s \) and \( \Gamma_0^u \) such that \( \Gamma_0^s \subset \Gamma^s \) and \( \Gamma_0^u = \Gamma^u \); u-subsets are defined similarly. Let \( \gamma^*(x) \) denote the disk in \( \Gamma^s \) containing the point \( x \in \Lambda \), for \( * = s, u \). Consider the holonomy map \( \Theta_{\gamma,\gamma'}: \gamma \cap \Lambda \to \gamma' \cap \Lambda \), defined for each \( x \in \gamma \cap \Lambda \) by

\[
\Theta_{\gamma,\gamma'}(x) = \gamma^s(x) \cap \gamma'.
\]  

We say that a compact set \( \Lambda \) is a Young set if \( \Lambda \) has a product structure given by continuous families of \( C^1 \) disks \( \Gamma^s \) and \( \Gamma^u \) such that conditions (Y1) below are satisfied.

(Y1) Markov: there is a sequence \((\Lambda_i)_{i \geq 1}\) of pairwise disjoint s-subsets of \( \Lambda \) such that

- \( m_\gamma(\Lambda \cap \gamma) > 0 \) and \( m_\gamma(\Lambda \setminus \bigcup_{i \geq 1} \Lambda_i) \cap \gamma = 0 \), for all \( \gamma \in \Gamma^u \);
- for all \( i \geq 1 \), there is \( R_i \in \mathbb{N} \) such that \( f^{R_i}(\Lambda_i) \) is a u-subset and, for all \( x \in \Lambda_i \),

\[
f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)) ;
\]
- for all \( i \geq 1, 0 \leq j \leq R_i \) and \( x \in \Lambda_i \),

\[
f^j(\gamma^s(x)) \cap S = \emptyset \quad \text{and} \quad f^j(\gamma^u(x)) \cap S = \emptyset.
\]

This allows us to introduce the recurrence time \( R: \Lambda \to \mathbb{N} \) and the return map \( f^R: \Lambda \to \Lambda \) of the Young set \( \Lambda \), setting for each \( i \geq 1 \)

\[
R|_{\Lambda_i} = R_i \quad \text{and} \quad f^{R_i}|_{\Lambda_i} = f^{R_i}|_{\Lambda_i}.
\]  

Note that \( R \) and \( f^R \) are defined on a full \( m_\gamma \) measure subset of \( \Lambda \cap \gamma \), for each \( \gamma \in \Gamma^u \).

Thus, there exists a set \( \Lambda' \subset \Lambda \) intersecting each \( \gamma \in \Gamma^u \) in a full \( m_\gamma \) measure subset, such that \( (f^R)^n(x) \) belongs in some \( \Lambda_i \), for all \( n \geq 0 \) and \( x \in \Lambda' \). Given \( x, y \in \Lambda' \), we define the separation time

\[
s(x, y) = \min \{ n \geq 0 : (f^R)^n(x) \text{ and } (f^R)^n(y) \text{ lie in distinct } \Lambda_i \},
\]

with the convention that \( \min(\emptyset) = \infty \). For definiteness, we set the separation time equal to zero for all other points. For the remaining conditions, we consider constants \( C > 0 \) and \( 0 < \beta < 1 \), only depending on \( f \) and \( \Lambda \).

(Y2) Contraction on stable disks: for all \( \gamma \in \Gamma^s \) and \( x, y \in \gamma \),

- \( \text{dist}(f^n(y), f^n(x)) \leq C\beta^n \), for all \( n \geq 0 \).
(Y$_3$) Expansion on unstable disks: for all $i \geq 1$, $\gamma \in \Gamma^u$ and $x, y \in \gamma \cap \Lambda_i$,
- $\operatorname{dist}((f^R)^n(y), (f^R)^n(x)) \leq C\beta^{s(x,y)-n}$, for all $n \geq 0$;
- $\operatorname{dist}(f^j(y), f^j(x)) \leq C \operatorname{dist}(f^R(x), f^R(y))$, for all $1 \leq j \leq R_i$.

(Y$_4$) Gibbs: for all $i \geq 1$, $\gamma \in \Gamma^u$ and $x, y \in \gamma \cap \Lambda_i$,
- $\log \frac{|Df^R|_{T_x \gamma}}{|Df^R|_{T_y \gamma}} \leq C \beta^{s(x,y)}$.

(Y$_5$) Regularity of the stable holonomy: for all $\gamma, \gamma' \in \Gamma^u$, the measure $(\Theta_{\gamma, \gamma'})_* m_\gamma$ is absolutely continuous with respect to $m_{\gamma'}$ and its density $\rho_{\gamma, \gamma'}$ satisfies
- $\frac{1}{C} \leq \rho_{\gamma, \gamma'} \leq C$;
- $\log \frac{\rho_{\gamma, \gamma'}(x)}{\rho_{\gamma, \gamma'}(y)} \leq C \beta^{s(x,y)}$, for all $x, y \in \gamma' \cap \Lambda$.

We say that a Young set has integrable recurrence times if $R$ is integrable with respect to the measure $m_\gamma$, for some $\gamma \in \Gamma^u$. And so, for all $\gamma \in \Gamma^u$, by (Y$_5$). In the next two results, we see the utility of Young sets (with integrable recurrence times) for obtaining SRB measures; see [60, Section 2.2] or [3, Theorem 4.7 & Theorem 4.9] for a proof. In the present context, by an SRB measure we mean an invariant probability measure whose conditionals on local unstable leaves are absolutely continuous with respect to the conditional Lebesgue measure on those leaves.

**Theorem 8.1.** Assume that $f^R : \Lambda \to \Lambda$ is the return map of a Young set with integrable recurrence times. Then,

1. $f^R$ has a unique ergodic SRB measure $\mu_0$;
2. $f$ has a unique ergodic SRB measure $\mu$ with $\mu(\Lambda) > 0$, which is given by

$$
\mu = \frac{1}{\sum_{j=0}^{\infty} \mu_0(\{R > j\})} \sum_{j=0}^{\infty} f^j_\Lambda(\mu_0(\{R > j\})).
$$

We will refer to the measure $\mu$ as the SRB measure given by the Young set $\Lambda$. It is not difficult to see that both $\mu_0$ and $\mu$ have $\dim \Gamma^s$ negative Lyapunov exponents and $\dim \Gamma^u$ positive Lyapunov exponents. Bearing in mind the classical Oseledets Theorem, we define for $\mu$-almost (thus $\mu_0$-almost) every point $x \in M$

$$
Df^u(x) = Df|_{E^u_x} \quad \text{and} \quad Df^R_u(x) = Df^R|_{E^u_x},
$$

where $E^u_x$ is the direct sum of the subspaces in the Oseledets decomposition of $T_x M$ associated with the positive Lyapunov exponents. Our main result in this part establishes in particular the classical Pesin entropy formula for SRB measures given by Young sets; recall [35, Proposition 2.5].

**Theorem C.** Let $f : M \to M$ be a piecewise $C^{1+\eta}$ diffeomorphism with an ergodic SRB measure $\mu$ given by a Young set $\Lambda$ with recurrence time $R$. Then,

1. if $h_\mu(f) < \infty$, then

$$
h_\mu(f) = \int_M \log |Df^u| d\mu;
$$
(2) \( h_\mu(f) < \infty \) if, and only if, for some \( \gamma_0 \in \Gamma^u \),
\[
\int_{\gamma_0 \cap \Lambda} R \log |\det D f^R|_{\gamma_0} \, dm_{\gamma_0} < \infty.
\]

The strategy for proving the first item of Theorem C is slightly more involving than that for Theorem A. We consider a tower extension \((\hat{T}, \hat{\nu})\) of \((f, \mu)\), but also a Gibbs-Markov quotient map \(F\) of \(f\) with an invariant measure \(\nu_0\) and the tower system \((T, \nu)\) associated with \((F, \nu_0)\), as is Section 2. We also consider the natural extensions \((\hat{T}^#, \hat{\nu}^#)\) and \((T^#, \nu^#)\) of \((\hat{T}, \hat{\nu})\) and \((T, \nu)\), respectively. Finally, we prove that, taking \(\rho > 0\) as in (4), we have
\[
h_\mu(f) = h_{\hat{\nu}^#}(\hat{T}^#) = h_{\nu^#}(T^#) = h_{\nu}(T) = \int_M \log |\det Df_u| \, d\mu < \infty.
\]

The first equality is a consequence of a general result due to Buzzi for extensions with countably many fibers. The remaining equalities in the first line are a consequence of a result due to Demers, Wright and Young establishing that the measure preserving systems \((\hat{T}^#, \hat{\nu}^#)\) and \((T^#, \nu^#)\) are isomorphic, and a result due to Rohlin which establishes that the entropy of the natural extension is equal to the entropy of the original system. The four equalities in the second line will be obtained in Proposition 4.3, Lemma 3.4, Lemma 13.1 and Lemma 13.2, respectively.

Assuming that there is \(C > 0\) for which \(|\det Df_u| \leq C\), it follows from the chain rule that \(|\det Df^R|_{\gamma_0}| \leq C^R\). We therefore have the following simple, albeit useful, consequence of Theorem C.

**Corollary D.** Let \(f : M \to M\) be a piecewise \(C^{1+\eta}\) diffeomorphism with an ergodic SRB measure \(\mu\) given by a Young set \(\Lambda\) with recurrence time \(R\). If \(|\det Df_u|\) is bounded and \(\int_{\gamma_0 \cap \Lambda} R^2 dm_{\gamma_0} < \infty\) for some \(\gamma_0 \in \Gamma^u\), then
\[
h_\mu(f) = \int_M \log |\det Df_u| \, d\mu < \infty.
\]

In Section 14, we provide an example of a piecewise \(C^\infty\) diffeomorphism \(f\) with an SRB measure \(\mu\) with infinite entropy given by a Young set for which the formula in the first item of Theorem C is no longer valid. In Section 15, we apply Theorem C and Corollary D to some classes of piecewise smooth diffeomorphisms with nonempty singular sets.

### 9. Quotient Return Map

In this section, we introduce a quotient map associated with the return map of a set with a Young structure, by collapsing stable leaves. Let \(\Lambda\) be a Young set with return map \(f^R : \Lambda \to \Lambda\). Fixing some \(\gamma_0 \in \Gamma^u\), consider \(\Theta_{\gamma_0} : \Lambda \to \gamma_0 \cap \Lambda\), setting for each \(x \in \Lambda\)
\[
\Theta_{\gamma_0}(x) = \gamma^u(x) \cap \gamma_0.
\]
We define the quotient map of \(f^R\) as
\[
F : \gamma_0 \cap \Lambda \to \gamma_0 \cap \Lambda \quad x \mapsto \Theta_{\gamma_0} \circ f^R(x).
\]
It is easily verified that
\[ F \circ \Theta_{\gamma_0} = \Theta_{\gamma_0} \circ f^R. \] (34)
To simplify notation, the restriction of \( m_{\gamma_0} \) to \( \gamma_0 \cap \Lambda \) will still be denoted by \( m_{\gamma_0} \). The proof of the next result is given in [3, Proposition 4.2].

**Proposition 9.1.** The quotient map \( F : \gamma_0 \cap \Lambda \to \gamma_0 \cap \Lambda \) is Gibbs-Markov with respect to the \( m_{\gamma_0} \mod 0 \) partition \( \mathcal{P} = \{ \gamma_0 \cap \Lambda_1, \gamma_0 \cap \Lambda_2, \ldots \} \) of \( \gamma_0 \cap \Lambda \).

The proof of [3, Proposition 4.2] gives that, for all \( \omega \in \mathcal{P} \),
\[ J_F|_{\omega} = (\rho_{\gamma_1, \gamma_0} \circ f^R) \cdot | \det Df^R|_{\gamma_0} |, \] (35)
with \( \gamma_1 \in \Gamma^u \) such that \( f^R(\omega) \subset \gamma_1 \) and \( \rho_{\gamma_1, \gamma_0} \) as in (Y5). By Theorem 1.1, the quotient map \( F \) has a unique ergodic invariant probability measure absolutely continuous with respect to \( m_{\gamma_0} \).

**Proposition 9.2.** Let \( \Lambda \) be a Young set and \( F : \gamma_0 \cap \Lambda \to \gamma_0 \cap \Lambda \) be a quotient of the return map \( f^R : \Lambda \to \Lambda \). If \( \mu_0 \) is the unique SRB measure for \( f^R \), then \( \nu_0 = (\Theta_{\gamma_0})_* \mu_0 \) is the unique \( F \)-invariant probability measure such that \( \nu_0 \ll m_{\gamma_0} \).

This last result, together with (34), shows that \( (f^R, \mu_0) \) is an extension of \( (F, \nu_0) \).

10. **Tower extension**

As in Section 2, consider the tower associated with \( f^R : \Delta \to \Delta \),
\[ \Delta = \{ (x, \ell) : x \in \Lambda \text{ and } 0 \leq \ell < R(x) \}, \]
and the tower map \( \hat{T} : \hat{\Delta} \to \hat{\Delta} \), given by
\[ \hat{T}(x, \ell) = \begin{cases} (x, \ell + 1), & \text{if } \ell < R(x) - 1; \\ (f^R(x), 0), & \text{if } \ell = R(x) - 1. \end{cases} \]
As before, the base \( \hat{\Delta}_0 \) of the tower \( \hat{\Delta} \) is naturally identified with the set \( \Lambda \), and each level \( \hat{\Delta}_\ell \) with the set \( \{ R > \ell \} \subset \Lambda \). This allows us to refer to stable and unstable disks through points in the tower, naturally considering the corresponding disks of their representatives in the base. Also, the set \( \Lambda \) is identified with the base level \( \hat{\Delta}_0 \) and the return map \( f^R : \Lambda \to \Lambda \) is identified with the return to the base \( T^R : \hat{\Delta}_0 \to \hat{\Delta}_0 \).

We have seen in Theorem [8.1] that the return map \( f^R \) has a unique SRB measure. In Theorem [10.1] below, we show that this SRB measure gives rise to a unique ergodic SRB measure for \( \hat{T} \). Since each level of the tower is identified with a subset of the set with a product structure, it still makes sense to talk about SRB measures for the tower map \( \hat{T} \).

Considering, as in Section 2 the map
\[ \pi : \hat{\Delta} \to M, \quad (x, \ell) \mapsto f^\ell(x), \] (36)
we also have
\[ f \circ \pi = \pi \circ \hat{T}. \] (37)
The next result gives in particular that the push-forward under \( \pi \) of the unique ergodic SRB measure for the tower map coincides with the ergodic SRB measure given by Theorem [8.1]. A proof of this result can be found in [3, Theorem 4.11].
Theorem 10.1. Let $\hat{T}$ the tower map associated with the return map $f^R$ of a Young set $\Lambda$ with integrable recurrence times. If $\mu_0$ is the unique SRB measure for $f^R$, then

$$\hat{\nu} = \frac{1}{\sum_{j=0}^\infty \mu_0(\{R > j\})} \sum_{j=0}^\infty \hat{T}^j(\mu_0|\{R > j\})$$

is the unique SRB measure for $\hat{T}$. Moreover, $\hat{\nu}$ is ergodic and $\mu = \pi_* \hat{\nu}$ is the unique SRB measure for $f$ with $\mu(\Lambda) > 0$.

Together with (37), this means that the tower system $(\hat{T}, \hat{\nu})$ is an extension of $(f, \mu)$.

11. Quotient tower

Here, we analyse the relationship between the tower associated with the return map and the tower associated with the quotient of the return map. Fix some $\gamma_0 \in \Gamma^u$ and consider the quotient map $F: \gamma_0 \cap \Lambda \to \gamma_0 \cap \Lambda$, as in (33). By Proposition 9.1, $F$ is a Gibbs-Markov map with respect to the $m_{\gamma_0}$ mod 0 partition $\mathcal{P} = \{\gamma_0 \cap \Lambda_1, \gamma_0 \cap \Lambda_2, \ldots\}$ of $\gamma_0 \cap \Lambda$. Notice that

$$R|_{\gamma_0 \cap \Lambda_i} = R|_{\Lambda_i} = R_i,$$

(38)

Since $R$ is constant in the elements of $\mathcal{P}$, we can consider the tower map $T: \Delta \to \Delta$ of the Gibbs-Markov map $F$ with recurrence time $R$. As before, we still denote the reference measure on $\Delta$ by $m_{\gamma_0}$. Since $\gamma_0 \cap \Lambda \subset \Lambda$, it follows that, for all $\ell \geq 0$,

$$\Delta_\ell \subset \hat{\Delta}_\ell \quad \text{and} \quad T|_{\Delta_\ell} = \hat{T}|_{\Delta_\ell},$$

(39)

with $\hat{\Delta}_\ell$ and $\hat{T}$ as in the beginning of this Section [10]. Hence, it makes sense to consider the map

$$\Theta : \hat{\Delta} \to \Delta,$$

$$\Theta(x, \ell) = (\Theta_{\gamma_0}(x), \ell).$$

(40)

It is straightforward to check that

$$\hat{T} \circ \Theta = \Theta \circ T,$$

(41)

thus $\Theta$ being a semiconjugacy between the tower maps $\hat{T}$ and $T$. Observe that $\Theta$ is not countable-to-one, and so we cannot invoke [16, Proposition 2.8] to obtain $h_\hat{\nu}(\hat{T}) = h_\nu(T)$. This will be deduced later, by mean of natural extensions. The next result shows that $(\hat{T}, \hat{\nu})$ is in fact an extension of $(T, \nu)$; see [3, Proposition 4.13] for a proof.

Proposition 11.1. If $\hat{\nu}$ is the ergodic SRB measure for $\hat{T}$, then $\Theta_* \hat{\nu}$ is the unique ergodic $T$-invariant probability measure absolutely continuous with respect to $m_{\gamma_0}$.
Hence, the measure preserving systems \((f, \mu)\) and \((T, \nu)\) are both factors of \((\hat{T}, \hat{\nu})\), and we have the commuting diagram

\[
\begin{array}{ccc}
(M, \mu) & \xleftarrow{\pi} & (\hat{\Delta}, \hat{\nu}) & \xrightarrow{\Theta} & (\Delta, \nu) \\
\downarrow f & & \downarrow \hat{T} & & \downarrow T \\
(M, \mu) & \xleftarrow{\pi} & (\hat{\Delta}, \hat{\nu}) & \xrightarrow{\Theta} & (\Delta, \nu)
\end{array}
\]

12. Natural extensions

Now, we turn our attention to the tower systems \((T, \nu)\) and \((\hat{T}, \hat{\nu})\). Heuristically, it is natural to expect these two systems have the same entropy, since we are in a certain sense just ignoring the stable direction, where no dynamical information is produced. The formal way in which we will deduce this fact is via natural extensions. Let us briefly recall this concept, in a general setting. Let \(\phi : X \to X\) be a measure preserving transformation of a probability measure space \((X, \mathcal{A}, \eta)\). Set

\[
X^\# = \left\{ (x_1, x_2, \ldots) \in \prod_{i=1}^{\infty} X : \phi(x_{i+1}) = x_i \right\}
\]

and the map \(\phi^\# : X^\# \to X^\#\), given by

\[
\phi^\#(x_1, x_2, \ldots) = (\phi(x_1), x_1, x_2, \ldots).
\]

Consider the \(\sigma\)-algebra \(\mathcal{A}^\#\) in \(X^\#\) generated by cylinders of the form

\[
[A_1, \ldots, A_k] = \{(x_1, x_2, \ldots) \in X^\# : x_i \in A_i, \text{ for all } 1 \leq i \leq k \},
\]

where \(A_i \in \hat{\mathcal{A}}\), for all \(1 \leq i \leq k\). It is easily verified that \(\phi^\#\) preserves the probability measure \(\eta^\#\) defined in the cylinders by

\[
\eta^\#([A_1, \ldots, A_k]) = \eta (A_k \cap \phi^{-1}(A_{k-1}) \cap \cdots \cap \phi^{-k+1}(A_0)).
\]

Moreover, the map \(\pi^\# : X^\# \to X\), given by \(\pi^\#(x_1, x_2, \ldots) = x_1\), is a semiconjugacy between \(\phi^\#\) and \(\phi\) and \(\pi^\# \eta^\# = \eta\). The measure preserving system \((\phi^\#, \eta^\#)\) is called the natural extension of \((\phi, \eta)\). A classical result due to Rohlin gives that the entropies of these two measure preserving systems coincide, i.e.

\[
h_{\eta^\#}(\phi^\#) = h_{\eta}(\phi).
\]

see [49, Section 3.3] or [50, Section 9.9]. For the natural extensions of the tower systems \((\hat{T}, \hat{\nu})\) and \((T, \nu)\), it is proved in [25, Appendix B] that the transformation \(\Theta^\# : \hat{\Delta}^\# \to \Delta^\#\), given by

\[
\Theta^\#((x_1, \ell_1), (x_2, \ell_2), \ldots) = (\Theta(x_1, \ell_1), \Theta(x_2, \ell_2), \ldots)
\]

is an isomorphism of the measure preserving systems \((\hat{T}^\#, \hat{\nu}^\#)\) and \((T^\#, \nu^\#)\). This in particular implies that

\[
h_{\hat{\nu}^\#}(\hat{T}^\#) = h_{\nu^\#}(T^\#).
\]
13. Entropy of the original system

In this section, we complete the proof of Theorem C. Since \((\hat{T}, \hat{\nu})\) is an extension of \((f, \mu)\) with countably many fibers, it follows from [16, Proposition 2.8] that
\[
h_\mu(f) = h_\hat{\nu}(\hat{T}). \tag{44}
\]
Moreover, by (12) and (13),
\[
h_\hat{\nu}((\hat{T})^#) = h_{\hat{\nu}^#}(T^#) = h_\nu(T). \tag{45}
\]
Using (20), (44) and (45) we get
\[
h_\mu(f) < \infty \iff \int_{\Delta_0} R \log J_FM_{\gamma_0} < \infty. \tag{46}
\]
On the other hand, using (35) and (Y_5), we obtain
\[
\int_{\Delta_0} R \log J_FM_{\gamma_0} < \infty \iff \int_{\Delta_0} R \log |\det Df| d\mu_0 < \infty. \tag{47}
\]
The second item of Theorem C is then a consequence of (46) and (47). Now, we prove the first item of Theorem C. Assuming \(h_\mu(f) < \infty\), it follows from (20), (44), (45) and Proposition 4.3 that
\[
h_\mu(f) = \int_{\Delta} \log J_T d\nu,
\]
and from Lemma 3.4
\[
\int_{\Delta} \log J_T d\nu = \frac{1}{\rho} \int_{\gamma_0 \cap \Lambda} \log J_F d\nu_0,
\]
with \(\rho\) as in (4). By the last two displayed formulas, the proof of Theorem C will be complete as soon as we get
\[
\int_{\gamma_0 \cap \Lambda} \log J_F d\nu_0 = \rho \int_M \log |\det Df| d\mu.
\]
This will be obtained in the next two lemmas. Recalling (1), (38) and Proposition 9.2, we have in this case
\[
\rho = \sum_{j \geq 0} (\Theta_{\gamma_0})_\mu_0(\{R|_{\gamma_0 \cap \Lambda} > j\}) = \sum_{j \geq 0} \mu_0(\{R > j\}). \tag{49}
\]

Lemma 13.1. \(\int_{\gamma_0 \cap \Lambda} \log J_F d\nu_0 = \int_{\Lambda} \log |\det Df| d\mu_0.\)

Proof. Using (34), we easily see that, for every \(x \in \Lambda,\)
\[
F(\Theta_{\gamma(x),\gamma_0}(x)) = \Theta_{\gamma_0(f(x)),\gamma_0}(f(x)).
\]
It follows from (Y_5) that, for all \(\gamma, \gamma' \in \Gamma^u,\) the transformation \(\Theta_{\gamma,\gamma'}\) has a Jacobian with respect to the measures \(m_\gamma\) and \(m_{\gamma'}\) on \(\gamma\) and \(\gamma',\) respectively. Using the Chain Rule and applying logarithms, we have for each \(x \in \Lambda\)
\[
\log J_F(\Theta_{\gamma(x),\gamma_0}(x)) + \log J_{\Theta_{\gamma(x),\gamma_0}}(x) = \log J_{\Theta_{\gamma_0(f(x)),\gamma_0}}(f(x)) + \log |\det Df(R(x))|. \tag{50}
\]

Now, observing that \( \Theta_{\gamma u}(x), \gamma 0(x) = \Theta_{\gamma 0}(x) \) and \((\Theta_{\gamma 0})_* \mu_0 = \nu_0 \) (recall Proposition 9.2), we have
\[
\int_{\Lambda} \log J_F(\Theta_{\gamma u}(x), \gamma 0(x)) d\mu_0(x) = \int_{\Lambda} \log J_F(\Theta_{\gamma 0}(x)) d\mu_0(x) = \int_{\gamma 0 \cap \Lambda} \log J_F(x) d\nu_0(x). \tag{51}
\]
Since the measure \( \mu_0 \) is \( f_R \)-invariant, we also have
\[
\int_{\Lambda} \log J_F(\Theta_{\gamma u}(f_R(x), \gamma 0(f_R(x)))) d\mu_0(x) = \int_{\Lambda} \log J_F(\Theta_{\gamma 0}(f(x))) d\mu_0(x) = \int_{\gamma 0 \cap \Lambda} \log J_F(x) d\nu_0(x). \tag{52}
\]
From (50), (51) and (52) we get the conclusion. □

**Lemma 13.2.** \( \int_{\Lambda} \log |\det Df_R^u| d\mu_0 = \rho \int_{M} \log |\det Df_u| d\mu. \)

**Proof.** Set for each \( n \geq 1 \)
\[
P_n = \{ x \in \Lambda : R(x) = n \}.
\]
By the chain rule, we have for each \( x \in P_n \),
\[
det Df_R^u(x) = \det Df_u(f^{n-1}(x)) \cdots \det Df_u(f(x)) \cdot det Df_u(x).
\]
It follows that
\[
\int_{\Lambda} \log |\det Df_R^u| d\mu_0 = \sum_{n=1}^{\infty} \int_{P_n} \log |\det Df_R^u| d\mu_0
\]
\[
= \sum_{n=1}^{\infty} \int_{M} \log |\det Df_R^u| d(\mu_0|P_n)
\]
\[
= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \int_{M} \log |\det Df_u| \circ f^j d(\mu_0|P_n)
\]
\[
= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \int_{M} \log |\det Df_u| d(f^j_*(\mu_0|P_n))
\]
\[
= \sum_{n=1}^{\infty} \int_{M} \log |\det Df_u| d \left( \sum_{n=0}^{\infty} f^n_*(\mu_0|\{R > n\}) \right)
\]
\[
= \int_{M} \log |\det Df_u| d \left( \sum_{n=0}^{\infty} f^n_*(\mu_0|\{R > n\}) \right).
\]
By Theorem 8.1
\[
\int_{M} \log |\det Df_u| d \left( \sum_{n=0}^{\infty} f^n_*(\mu_0|\{R > n\}) \right) = \sum_{j=0}^{\infty} \mu_0(\{R > j\}) \int_{M} \log |\det Df_u| d\mu.
\]
Using (49), we finish the proof. □
Here, we adapt the example in Section 6 to build a piecewise $C^\infty$ diffeomorphism with an SRB measure given by a Young set for which the entropy formula does not hold. Consider the following objects, as described in Section 6:

1. the interval $\Delta_0$ with finite Lebesgue measure $m$;
2. the $m \mod 0$ partition $\mathcal{P} = \{\omega_n\}_{n \geq 2}$ of $\Delta_0$;
3. the piecewise $C^\infty$ Gibbs-Markov map $F : \Delta_0 \to \Delta_0$ with associated partition $\mathcal{P}$;
4. the function $R : \Delta_0 \to \mathbb{N}$ such that $R|_{\omega_n} = n$, for all $n \geq 2$.

We are going to introduce an extra contracting direction and slightly modify the standard tower construction associated with these objects in order to obtain a piecewise $C^\infty$ diffeomorphism $f : M \to M$ of a rectangle $M$ in $\mathbb{R}^2$. In addition, we will show that $f$ has an ergodic SRB measure $\mu$ given by a Young set such that $h_\mu(f) = \infty$ and $\int_M \log |\det Df_u| d\mu < \infty$. (53)

Set $\Delta = \{(x, \ell) : x \in \Delta_0$ and $0 \leq \ell < R(x)\}$
and
$M = \Delta \times [0,1]$. As observed in Section 6, the set $\Delta$ can be identified with an interval in $\mathbb{R}$, and so $M$ can be identified with a two-dimensional rectangle. Taking some $\lambda \in (0, 1/2]$, consider $f : M \to M$ given by

$$f(x, \ell, y) = \begin{cases} 
(x, \ell + 1, \lambda y), & \text{if } \ell < R(x) - 1; \\
(F(x), 0, \lambda + \lambda^2 + \cdots + \lambda^{R(x)-1} + \lambda^{R(x)} y), & \text{if } \ell = R(x) - 1.
\end{cases}$$

For each $n \geq 2$ and $\ell \geq 0$, consider the open set

$$M_{n, \ell} = \omega_n \times \{\ell\} \times (0,1).$$

It is easily verified that these sets are pairwise disjoint and $\bigcup_{n, \ell} M_{n, \ell} = M$. Let us now show that $f|_{\bigcup_{n, \ell} M_{n, \ell}}$ is a $C^\infty$ diffeomorphism onto its image. Recalling that $F$ is affine on each $\omega_n = \{x \in \Delta_0 : R(x) = n\}$, it is enough to prove that $f$ is injective. For this, we just need to show that, given $x_n \in \omega_n$, $x_k \in \omega_k$ with $n \neq k$ and $y, z \in (0,1)$, we have

$$f(x_n, n-1, y) \neq f(x_k, k-1, z).$$

Assume for definiteness $n > k$. Since

$$f(x_n, n-1, y) = (F(x_n), 0, \lambda + \lambda^2 + \cdots + \lambda^{n-1} + \lambda^n y)$$
and

$$f(x_k, k-1, z) = (F(x_k), 0, \lambda + \lambda^2 + \cdots + \lambda^{k-1} + \lambda^k z)$$

we have that the difference of the third coordinates of these two images is

$$\lambda^k + \cdots + \lambda^{n-1} + \lambda^n y - \lambda^k z > \lambda^n y > 0.$$ 

This implies that $f$ is injective. Now, taking

$$\Lambda = \Delta_0 \times \{0\} \times [0,1] \subset M$$
we have that \( \Lambda \) has a product structure given by the continuous families of disks
\[
\Gamma^s = \{ x \times \{ 0 \} \times [0,1] : x \in \Delta_0 \} \quad \text{and} \quad \Gamma^u = \{ \Delta_0 \times \{ y \} : y \in [0,1] \}.
\]
Set for all \( i \geq 2 \)
\[
\Lambda_i = \omega_i \times \{ 0 \} \times [0,1] \quad \text{and} \quad R_i = i.
\]
It is easily verified that \( (\Lambda_i)_{i \geq 2} \) is a family of pairwise disjoint \( s \)-subsets of \( \Lambda \) and the return map \( f^R \) associated with these objects as in (30) satisfies conditions \( (Y_1)-(Y_5) \) in Section 8, thus showing that \( \Lambda \) is a Young set with integrable recurrence times. Let \( \mu \) be the SRB measure for \( f \) given by Theorem 8.1. Note that the quotient map associated with \( f^R \) on the disk \( \gamma_0 = \Delta_0 \times \{ 0 \} \times \{ 0 \} \) is the map \( F \), under the natural identification of \( \gamma_0 \) and \( \Delta_0 \).

Moreover, \( \rho_{\gamma_1, \gamma_0} = 1 \), for all \( \gamma_1 \in \Gamma^u \). It follows from (25), (35) and Lemma 6.1 that
\[
\int_{\gamma_0 \cap \Lambda} R \log |\det Df^R_{\gamma_0}| \, d\gamma_0 = \int_{\Delta_0} R \log J_F \, d\gamma_0 = \infty.
\]
Using Theorem C, we obtain \( h_\mu(f) = \infty \). On the other hand, it follows from (24), (48) and Lemma 6.1 that
\[
\int_M \log |\det Df_u| \, d\mu = \frac{1}{\rho} \int_{\Delta_0} \log J_F \, dm < \infty.
\]
This concludes the proof of (53).

15. Applications

In the next subsections we apply Theorem C and Corollary D to some classes of piecewise smooth diffeomorphisms with SRB measures given by Young sets. The case of billiard maps has already been considered in [35] through the approach in [31].

15.1. Piecewise hyperbolic maps. Here, we apply our results to a class of piecewise hyperbolic diffeomorphisms studied by Young in [60], in dimension two, and by Chernov in [19], in any finite dimension. Concrete examples of such systems include the family of Lozi-like mappings in [59]; see also [23, 45, 53]. Let \( M \) be a compact \( d \)-dimensional Riemannian manifold, for some \( d \geq 2 \), possibly with boundary. Assume that \( f : M \to M \) satisfies the following conditions:

1. \( f \) is a piecewise \( C^2 \) diffeomorphism from \( M \) into itself: there is a finite number of pairwise disjoint open regions \( (M_i)_i \), with \( M = \bigcup_i M_i \), whose boundaries are \( d-1 \) submanifolds such that
   (a) \( f\big|_{(\bigcup_i M_i)} \) is injective;
   (b) \( f\big|_{M_i} \) can be extended to a \( C^2 \) diffeomorphism of \( \overline{M_i} \) onto its image, for all \( i \).
2. \( f \) is uniformly hyperbolic: there exist \( Df \)-invariant cone families \( C^u \) and \( C^s \) on \( M \) and \( \lambda > 1 \) such that, for all \( i \) and \( x \in \overline{M_i} \)
   (a) \( |Df_x(v)| \geq \lambda |v| \), for all \( v \in C^u_x \);
   (b) \( |Df_x^{-1}(v)| \geq \lambda |v| \), for all \( v \in C^s_x \).

We shall refer to \( S = M \setminus \bigcup_i M_i \) as the singularity set. Note that we allow \( \bigcup_i f(\overline{M_i}) \subset M \), so that \( M \) can be a trapping region for an attractor. For \( n \geq 1 \), denote by
\[
S_n = S \cup f^{-1}(S) \cup \cdots \cup f^{-n+1}(S)
\]
the singularity set for \( f^n \).
(3) The angle between $S$ and $C_u$ is bounded away from 0.

(4) There is $n \geq 1$ such that the multiplicity of any point in $S^n$ is smaller than $\lambda^n - 1$.

The results in [19, 60] show that $f$ has an ergodic SRB measure $\mu$ given by a Young set with the tail of recurrence times decaying exponentially fast. In particular, the square of the recurrence time function is integrable with respect to Lebesgue measure on any unstable leaf in the family that defines the Young set. Since $|\det Df_u|$ is bounded, it follows from Corollary D that

$$h_\mu(f) = \int_M \log |\det Df_u| \, d\mu < \infty.$$ 

15.2. Billiard maps. Here, we apply the main results of this part to some classes of billiard maps. Maps of this type have been introduced by Sinai in [55], and can be described as follows. Let $\Gamma_1, \dots, \Gamma_d$ be pairwise disjoint simply connected $C^3$ curves in the torus $\mathbb{T}^2$, which can be interpreted as the boundaries of scatters. Consider the billiard flow on the domain $\mathbb{T}^2 \setminus \bigcup_i \text{int}(\Gamma_i)$, where each $\text{int}(\Gamma_i)$ stands for the interior of the curve $\Gamma_i$, generated by the motion of point particles traveling at unit speed and having elastic reflections at the boundary $\bigcup_i \Gamma_i$. This flow has the Poincaré section

$$M = \bigcup_i \Gamma_i \times [-\pi/2, \pi/2],$$

the first coordinate giving the collision point in $\bigcup_i \Gamma_i$ and the second one the angle of the trajectory with the normal to $\bigcup_i \Gamma_i$ at the collision point. We are interested in the first return map $f : M \to M$, known as the billiard map. This is essentially a piecewise hyperbolic diffeomorphism as in Subsection 15 with the difference that $Df$ is not bounded, due to the tangential reflections corresponding to angles $\pm \pi/2$. Considering $\bigcup_i \Gamma_i$ parametrised by arc length $x$ and $\theta \in [-\pi/2, \pi/2]$, it is known that $f$ preserves an ergodic measure $\mu$ given by

$$d\mu = k \cos \theta dx d\theta,$$

where $k > 0$ is a normalizing constant; see e.g. [20, Section 2.12]. In the next two subsections, we use Theorem C to deduce the entropy formula for $\mu$ in two special cases of billiards. Due to the unboundedness of $Df$, we cannot apply Corollary D in this context.

15.2.1. Dispersing billiards. Assume that the curves $\Gamma_1, \dots, \Gamma_d$ have strictly positive curvature. In this case, Young proved in [60] that the measure $\mu$ is given by a Young set with exponential tail of recurrence times if the billiard has finite horizon, i.e. when the time between collisions is uniformly bounded. This conclusion was extended by Chernov to billiards with infinite horizon in [18]. In order to apply Theorem C we need to ensure that the entropy $h_\mu(f)$ is finite. This follows easily from [12, Lemma 3.6] in the finite horizon case. In general, this may be deduced from [17, Corollary 2.4 & Example 3.1]. Therefore, using Theorem C we get

$$h_\mu(f) = \int_M \log |\det Df_u| \, d\mu < \infty.$$ 

15.2.2. Semi-dispersing billiards. Chernov and Zhang considered a class of billiards for which the curvature of the curves $\Gamma_1, \dots, \Gamma_d$ vanishes at some points. As observed in [22], if there is no periodic trajectory that hits the boundary at flat points only, then a certain power of the collision map is uniformly hyperbolic. To avoid this situation, and for
simplicity, they assume that there is one such periodic trajectory of period two that runs between two flat points. Moreover, the boundary near these flat points is given by
\[ y = \pm (1 + |x|^\alpha), \quad \alpha > 2, \]
in some rectangular coordinate system \((x, y) \in \mathbb{R}^2\). As a byproduct of the results by Chernov and Zhang, we have that the measure \(\mu\) is given by a Young set with tail of recurrence times decaying as \(O((\log n)^{\beta+1}/n^{\beta+1})\), for \(\beta = (\alpha + 2)/(\alpha - 2)\); see conditions (F1)-(F2) in [22, Section 3] and the proof of [21, Theorem 4]. This implies that the recurrence times are integrable for all \(\alpha > 2\). In order to apply Theorem C, we need to ensure that \(h_\mu(f) < \infty\). This follows from [17, Corollary 2.4] and [50, Corollary 1.3]. Therefore, using Theorem C we get
\[ h_\mu(f) = \int_M \log |\det Df_u|d\mu < \infty. \]

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