Controllability of higher order fractional damped delay dynamical systems

M Sivabalan and K Sathiyanathan
Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore 641 020, Tamil Nadu, India
*Email: sivabalan8890@gmail.com

Abstract: The article is concerned the controllability results for higher order fractional damped delay dynamical systems of any different orders of Caputo derivatives. To establish a necessary and sufficient conditions for the controllability criteria of linear fractional damped delay dynamical systems by Grammian matrix. Sufficient conditions for controllability of the corresponding nonlinear damped delay dynamical systems are obtained using successive approximation technique. Examples are included to verify theoretical results.

Keywords: Controllability, Damped delay dynamical systems, Grammian matrix, Mittag-Leffler Matrix function, Iterative technique.

1. Introduction
The fractional derivative has recognized as a suitable tool in the modelling of many real-world problems in numerous fields of science and engineering. These formulations are used to represent many practical systems more accurate than integer order ones and gained significance in the fields of bioengineering, signal processing, electrochemistry, electromagnetism, thermal systems, filter design, circuit theory, nano materials, nonlinear oscillation of earthquakes and robotics [1-7].

Controllability shows a vital part in the expansion of recent mathematical control theory and it is used to influence an object's behaviour of a dynamical system to accomplish a desired goal [8,9]. The study of control problems demonstrated by fractional differential systems is significant in various problems of an applied nature. Nowadays, the controllability has applied in the field of reactor control, electric bulk power control systems, industrial and chemical process control, aerospace engineering and theory of quantum systems. Several authors studied the linear and nonlinear dynamical systems in finite dimensional spaces with the help of control theory [10-13].

Yonggang and Xiu’e [14] introduced a fractional oscillator system in which the restoring force is denoted by a term having fractional derivative and the properties of oscillation is reserved. In the fractional oscillator model, numerous specific forcing functions and their resonance were analysed by Achar et al. [15]. Tofighi [16] has described and attained the expression of the inherent damping force in the fractional oscillator system. Some authors extended the explanation to the fractional oscillator and reported that fractional oscillations have finite numbers of zeros. Solution of the fractional oscillator system is obtained by Al-rabth et al. [17] using differential transform method. Recently, Balachandran et al. [11] and Junpeng Liu et al. [18] has discussed the controllability of fractional damped dynamical systems.

To the best our knowledge, no relevant report has been described in the control problem of fractional damped delay dynamical equations of higher order. This article, we make an attempt to investigate the controllability of the higher order fractional damped delay dynamical systems. Numerical illustrations are included to clarify the theoretical results.

2. Preliminaries
This section, we recall some preliminary facts, definitions and notations [6, 7].

Definition 2.1
The Caputo fractional derivative of order \( \alpha > 0, n - 1 < \alpha \leq n \), is

\[
^{c}D_{0+}^{\alpha} h(\eta) = \frac{1}{\Gamma(n - \alpha)} \int_{0}^{\eta} (\eta - s)^{n-\alpha-1} h^{(n)}(s) ds,
\]
where \( h^{(n)}(s) = \frac{d^n h}{ds^n} \) and the function \( h(\eta) \) has absolutely continuous derivative up to order-1. For the brevity, Caputo fractional derivative \( C^{\alpha}_D z(\eta) \) is taken as \( C^{\alpha}_D \).

The Laplace transform of Caputo derivative is

\[
L\left[C^{\alpha}_D z(\eta)\right](s) = s^\alpha L[z(\eta)](s) - \sum_{k=0}^{n-1} \frac{z^k(0)s^{\alpha-k-1}}{k!}, \quad n - 1 < \alpha \leq n.
\]

**Definition 2.2**

The Mittag-Leffler function (MLF) of three parameters is described as

\[
\Xi_{\alpha, \beta}^{\gamma}(\lambda \eta^\alpha) = \sum_{k=0}^{\infty} \frac{(\gamma)_k (-\lambda)^k}{k! \Gamma(\alpha k + \beta)} \eta^{\alpha k},
\]

where \((\gamma)_n\) is a Pochhammer symbol. It is expressed as \(\gamma(\gamma+1)\cdots(\gamma+n-1)\) and \((\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}\).

**Definition 2.3**

The Laplace transform of MLF of three parameters is described as

\[
L\left[\eta^{\alpha-1} \Xi_{\alpha, \beta}^{\gamma} (\pm \lambda \xi^\alpha)\right](s) = s^{\alpha-\gamma-\beta} (s^\alpha \mp \lambda)^{\gamma}, \quad \text{Re}(s) > 0, \text{Re}(\beta) > 0, |\lambda s - \alpha| < 1.
\]

**Lemma 2.1**

The MLF derivative of order \(q \in \mathbb{N} \) is described as

\[
\left(\frac{d}{d\eta}\right)^{(q)} \left(\eta^{\alpha-1} \Xi_{\alpha-\beta, \alpha}(A \eta^{\alpha-\beta})\right) = t^{\alpha-q-1} \Xi_{\alpha-q-1, \alpha-q+1}(A t^{\alpha-\beta}).
\]

### 3. Linear Damped Delay Systems

The linear damped delay dynamical system with fractional order takes the form

\[
C^{\alpha}_D z(\eta) - A_0 C^{\beta}_D z(\eta) = A_1 z(\eta - \tau) + A_2 v(\eta), \quad \eta \in J_P: [0, \eta_1]
\]

where \(q - 1 < \alpha < q, r - 1 < \beta < r, r \leq q - 1, z \in \mathbb{R}^n, v \in \mathbb{R}^m, A_0, A_1 \) are \(n \times n\) matrices and \(A_2\) is \(n \times m\) matrix. By utilizing Laplace, inverse Laplace and convolution of Laplace transform technique of equation (1), we get the solution of (1) which can be expressed as

\[
\begin{align*}
z(\eta) &= z(\eta; \varphi) + \int_0^{\eta} (\eta - s)^{\alpha-1} \Xi_{\alpha-\beta, \alpha}(A_0 (\eta - s)^{\alpha-\beta}) A_2 v(s) ds, \\
&\quad + A_1 \int_0^{\tau - \eta} (\eta - s - \tau)^{\alpha-1} \Xi_{\alpha-\beta, \alpha}(A_0 (\eta - s - \tau)^{\alpha-\beta}) \varphi(s) ds,
\end{align*}
\]

where

\[
z(\eta; \varphi) = \sum_{k=0}^{\alpha - 1} \eta^k \Xi_{\alpha-\beta, 1+k}(A_0 \eta^{\alpha-\beta}) - \sum_{k=0}^{\beta - 1} \eta^k \Xi_{\alpha-\beta, \alpha+k+1}(A_0 t^{\alpha-\beta})
\]

\[
+ A_1 \int_0^{\tau - \eta} (\eta - s - \tau)^{\alpha-1} \Xi_{\alpha-\beta, \alpha}(A_0 (\eta - s - \tau)^{\alpha-\beta}) \varphi(s) ds,
\]

**Definition 3.1**

System (1) is called controllable on \([0, \eta_1]\), if for every \(\varphi(\eta), \varphi'(\eta), \cdots, \varphi^{(q)}(\eta)\), and any state \(z_1 \in \mathbb{R}^n\) there exists a control input \(v \in \mathbb{R}^m\) such that the corresponding solution \(z(\eta)\) of (1) satisfies

\[
z(\eta_1) = z_1.
\]

Define the controllability Grammian matrix \(\omega\) as follows

\[
\omega = \int_0^{\eta_1} (\eta_1 - s)^{2\alpha-2} \Xi_{\alpha-\beta, \alpha}(A_0 (\eta_1 - s)^{\alpha-\beta}) A_2 A_2^T \Xi_{\alpha-\beta, \alpha}(A_0^T (\eta_1 - s)^{\alpha-\beta}) ds,
\]
where $\top$ indicates transpose of matrix.

**Theorem 3.1**

System (1) is controllable on $[0, \eta_1]$, if and only if the controllability Grammian matrix $\omega$ is nonsingular, for some $\eta_1 > 0$.

**Proof:**

Since $\omega$ is nonsingular, and so its inverse is well defined. Let $v$ be a control function and is defined by

$$
\nu(\eta) = z(\eta_1 - \eta)\alpha^{-1}A_2^T \Phi_{\alpha-\beta, \alpha}(A_0(\eta_1 - \eta)^{\alpha-\beta})\omega^{-1}(z_1 - z(\eta_1; \varphi)).
$$

(5)

Substituting $\eta = \eta_1$ in (2) and inserting (5) we get,

$$
z(\eta_1) = z(\eta_1; \varphi) + \int_{0}^{\eta_1} [(\eta_1 - s)\alpha^{-1}\Phi_{\alpha-\beta, \alpha}(A_0(\eta_1 - s)^{\alpha-\beta})A_2](\eta_1 - s)\alpha^{-1}\Phi_{\alpha-\beta, \alpha}(A_0(\eta_1 - s)^{\alpha-\beta})A_2^T \omega^{-1}(z_1 - z(\eta_1; \varphi))ds = z_1.
$$

Thus system (1) is controllable on $[0, \eta_1]$.

The other side, $\omega$ is singular. Then there exists a non-zero vector $y_a$ such that

$$
y_a^T \omega y_a = y_a^T \int_{0}^{\eta_1} [(\eta_1 - s)^{2\alpha-2}\Phi_{\alpha-\beta, \alpha}(A_0(\eta_1 - s)^{\alpha-\beta})A_2A_2^T \Phi_{\alpha-\beta, \alpha}(A_0(\eta_1 - s)^{\alpha-\beta}) y_a]ds = 0,
$$

Let the initial points $\varphi(\eta) = \varphi'(\eta) = \cdots = \varphi^{(q)}(\eta) = 0$ and the final point $z_1 = y_a$. By assumption, there exists a control input $v$ on $[0, \eta_1]$ such that it steers the response from 0 to $z_1$ at $\eta = \eta_1$.

Then it follows

$$
z_1 = y_a = \int_{0}^{\eta_1} [(\eta_1 - S)^{\alpha-1}\Phi_{\alpha-\beta, \alpha}(A_0(\eta_1 - S)^{\alpha-\beta})A_2v(s)ds,
$$

then,

$$
y_a^T y_a = y_a^T \int_{0}^{\eta_1} [(\eta_1 - S)^{\alpha-1}\Phi_{\alpha-\beta, \alpha}(A_0(\eta_1 - S)^{\alpha-\beta})A_2v(s)ds = 0.
$$

This contradiction to $y_a \neq 0$. Thus $\omega$ is nonsingular.

4. **Nonlinear Damped Delay Dynamical Systems**

The nonlinear damped delay dynamical system with fractional order of the form

$$
cD^\alpha z(\eta) - A_0 cD^\beta z(\eta) = A_1z(\eta - \tau) + A_2v(\eta) + \mathcal{h}(\eta, z(\eta), z(\eta - \tau), v(\eta)), \eta \in [0, \eta_1]
$$

(6)

where the nonlinear function $\mathcal{h} : [0, \eta_1] \times R^n \times R^n \times R^m \to R^n$ being continuous.

Consider the Banach space

$$
ZB = \left\{ z(\eta) \in C([0, \eta_1] : R^n) \, : \, cD^\beta z(\eta) \in C([0, \eta_1] : R^n) \right\},
$$

with the norm $|z(\eta)|_{C} = \max_{\eta \in [0, \eta_1]} \{|z(\eta)|, \left| cD^\beta z(\eta) \right|, |v(\eta)| \}$, where $r - 1 < \beta < r$.

Further we assume the subsequent hypothesis:

(AAA1) The continuous function $\mathcal{h} : [0, \eta_1] \times R^n \times R^n \times R^m \to R^n$ and there exists positive constants $K_{c_1}$ and $L_{c_1}$ such that

$$
\|b(n, z(\eta), z(\eta - \tau), v(\eta))\| \leq K_{c_1},
$$

$$
\|b(n, z_1, z_{1+1}, v_1) - b(n, z_2, z_{2+1}, v_2)\| \leq L_{c_1} [||z_1 - z_2|| + ||z_{1+1} - z_{2+1}|| + ||v_1 - v_2||],
$$

$$
\in R^n, v_1, v_2 \in R^m.
$$

For brevity, let us define

$$
m_{c_1} = sup \left\{ \|z(\eta; \varphi)\|, \eta \in J \right\}, m_{c_2} = sup \left\{ \|\eta - s\|^{\alpha-1}\Phi_{\alpha-\beta, \alpha}(A_0(\eta - s)^{\alpha-\beta})\|, \eta, s \in J \right\},
$$

$$
m_{c_3} = sup \left\{ ||(\eta_1 - \eta)^{\alpha-1}A_2^T \Phi_{\alpha-\beta, \alpha}(A_0(\eta_1 - \eta)^{\alpha-\beta})\|, m_{c_4} = \|y\|, m_{c_5} = \|A_2\|, m_{c_6} = \|\varphi(0)\|, m_{c_7} = m_{c_1} + m_{c_2}, m_{c_8} = \|\eta - s\|^{\alpha-\beta+r-q-1}\Phi_{\alpha-\beta, \alpha-\beta-q+r}(A_0(\eta - s)^{\alpha-\beta})\|, m_{c_9} = \|\varphi(0)\|, m_{c_{10}} = m_{c_1} + m_{c_2}, m_{c_{11}} = \|\eta - s\|^{\alpha-\beta+r-q-1}\Phi_{\alpha-\beta, \alpha-\beta-q+r}(A_0(\eta - s)^{\alpha-\beta})\|
$$

Solution of (6) with the initial conditions $z(\eta) = \varphi(\eta), z'(\eta) = \varphi'(\eta), \cdots, z^{(q)}(\eta) = \varphi^{(q)}(\eta)$ is given by
\[ z(\eta) = z(\eta; \varphi) + \int_0^\eta (\eta - s)^{\alpha - 1} \Phi_{\alpha - \beta, \alpha}(A_0(\eta - s)^{\alpha - \beta}) A_2 \psi(s) \, ds \\
+ \int_0^{\eta - \tau} (\eta - s)^{\alpha - 1} \Phi_{\alpha - \beta, \alpha}(A_0(\eta - s)^{\alpha - \beta}) h(s, z(s), z(s - \tau), \psi(s)) \, ds, \tag{7} \]

where \( z(\eta; \varphi) \) is defined as in (3).

**Theorem 4.1**

Let the function \( h \) satisfies the assumption (AA1). Assume that the linear damped delay system (1) is controllable. Then the nonlinear system (6) is controllable on \([0, \eta_1]\).

**Proof:**

To prove the controllability result, we use the technique of successive approximation. For that, we define

\[ z(\eta) = \varphi(\eta), \]

\[ z_{n+1}(\eta) = z(\eta; \varphi) + \int_0^\eta (\eta - s)^{\alpha - 1} \Phi_{\alpha - \beta, \alpha}(A_0(\eta - s)^{\alpha - \beta}) A_2 \psi(n(s)) \, ds \\
+ \int_0^{\eta - \tau} (\eta - s)^{\alpha - 1} \Phi_{\alpha - \beta, \alpha}(A_0(\eta - s)^{\alpha - \beta}) h(s, z_n(s), z_n(s - \tau), \psi(s)) \, ds, \tag{8} \]

where

\[ \psi_n = (\eta_1 - \eta)^{\alpha - 1} A_2^T \Phi_{\alpha - \beta, \alpha}(A_0^{\top}(\eta_1 - s)^{\alpha - \beta}) \omega^{-1} \{ z_1 - z(\eta_1; \varphi) \\
- \int_0^{\eta_1} (\eta_1 - s)^{\alpha - 1} \Phi_{\alpha - \beta, \alpha}(A_0(\eta_1 - s)^{\alpha - \beta}) h(s, z_n(s), z_n(s - \tau), \psi(s)) \, ds \}. \tag{9} \]

and \( n = 0, 1, 2 \ldots \).

Since \( \varphi(\eta) \) is a given vector and note that \( \{z_n(\eta)\} \) are the known sequence of functions. Next we have to prove that \( \{z_n(\eta)\} \) is a Cauchy sequence in \( ZB \). Then \( z_{n+1}(\eta) = \varphi(\eta) + \sum_{i=0}^n (z_i(\eta) - z_i(\eta)) \) and we need to prove that the series \( \sum_{n=0}^\infty (z_{n+1}(\eta) - z_n(\eta)) \) converges uniformly. Clearly

\[ \|\psi_n(\eta)\| \leq \|(\eta_1 - \eta)^{\alpha - 1} A_2^T \Phi_{\alpha - \beta, \alpha}(A_0^{\top}(\eta_1 - s)^{\alpha - \beta}) \omega^{-1} \| z_1 - z(\eta_1; \varphi) \| \\
+ \int_0^{\eta_1} \|(\eta_1 - s)^{\alpha - 1} \Phi_{\alpha - \beta, \alpha}(A_0(\eta_1 - s)^{\alpha - \beta}) \| h(s, z_n(s), z_n(s - \tau), \psi(s)) \| ds, \]

\[ \leq m_c_3 \{ m_c_4 + m_c_1 + m_c_2 K_c_1 \eta_1 \} = p, \]

\[ \|\psi_n(\eta) - \psi_{n-1}(\eta)\| \leq \|(\eta_1 - \eta)^{\alpha - 1} A_2^T \Phi_{\alpha - \beta, \alpha}(A_0^{\top}(\eta_1 - \eta)^{\alpha - \beta}) \omega^{-1} \| \\
\times \left[ \int_0^{\eta_1} \|(\eta_1 - s)^{\alpha - 1} \Phi_{\alpha - \beta, \alpha}(A_0(\eta_1 - s)^{\alpha - \beta}) \| h(s, z_n(s), z_n(s - \tau), \psi_n(s)) \| ds \right] \]

\[ \leq m_c_2 m_c_3 L_c_1 \eta_1 \{ z_{n-1}(\eta_1) - z_n(\eta_1) \} + \|z_{n-1}(\eta - \tau) - z_n(\eta - \tau)\| + \|\psi_{n-1}(\eta) - \psi_n(\eta)\|. \]

Then

\[ \|z_{n+1}(\eta) - z_n(\eta)\| \leq \int_0^{\eta} \|(\eta - s)^{\alpha - 1} \Phi_{\alpha - \beta, \alpha}(A_0(\eta - s)^{\alpha - \beta}) \| h(s, z_n(s), z_n(s - \tau), \psi_n(s)) \| ds \\
+ \int_0^{\eta} \|(\eta - s)^{\alpha - 1} \Phi_{\alpha - \beta, \alpha}(A_0(\eta - s)^{\alpha - \beta}) \| h(s, z_n(s), z_n(s - \tau), \psi_n(s)) \| ds \]

\[ \leq (m_c_3 m_c_5 L_c_1 \eta_1 \\
+ m_c_2 L_c_1) \int_0^{\eta} \{ \|z_n(s) - z_n(s)\| + \|z_n(s - \tau) - z_n(s - \tau)\| \}
+ \|\psi_n(s) - \psi_{n-1}(s)\| ds. \]
Further
\[ \|z_1(\eta) - z_0(\eta)\| \]
\[ \leq m_{c1} + m_{c6} + \int_0^\eta \| (\eta - s)^{a-1} \Phi_{\alpha-\beta,\alpha}(A_0(\eta - s)^{a-\beta}) \| \| A_2 \| \| v_0(s) \| ds \]
\[ + \int_0^\eta \| (\eta - s)^{a-1} \Phi_{\alpha-\beta,\alpha}(A_0(\eta - s)^{a-\beta}) \| \| h(s, z_0(s), z_0(s-\tau), v_0(s)) \| ds \]
\[ \leq m_{c7} + (m_{c2}m_{c5}p + m_{c2}K_{c1})\eta, \]
assuming that \( \eta_1 \geq 0 \). By induction method, to attain the estimate
\[ \|z_{n+1}(\eta) - z_n(\eta)\| \leq m_{c1}(m_{c5}m_{c3}m_{c2}L_{c1}\eta_1 + m_{c2}L_{c1}) \frac{\eta_1^{n+1}}{(n+1)!}. \]
By taking the large value of \( n \), then the right side of the above estimate can be made arbitrarily small. Therefore, the sequence \( \{z_n(\eta)\} \) is a Cauchy sequence in \( ZB \). Then \( ZB \) is complete, the \( \{z_n(\eta)\} \) converges uniformly to \( z(\eta) \) on \([0, \eta_1]\). Hence, we have
\[ z(\eta) = z(\eta; \varphi) + \int_0^\eta (\eta - s)^{a-1} \Phi_{\alpha-\beta,\alpha}(A_0(\eta - s)^{a-\beta})A_2 v(s)ds \]
\[ + \int_0^\eta (\eta - s)^{a-1} \Phi_{\alpha-\beta,\alpha}(A_0(\eta - s)^{a-\beta})h(s, z(s), z(s-\tau), v(s))ds, \]
where
\[ v(\eta) = (\eta_1 - \eta)^{a-1}A_2^T \Phi_{\alpha-\beta,\alpha}(A_0^T(\eta_1 - \eta)^{a-\beta})\omega^{-1} \left[ z_1 - z(\eta_1; \varphi) \right] \]
\[ - \int_0^{\eta_1} (\eta_1 - s)^{a-1} \Phi_{\alpha-\beta,\alpha}(A_0(\eta_1 - s)^{a-\beta})h(s, z(s), z(s-\tau), v(s))ds. \]
Taking limit as \( n \to \infty \) on both sides of \((8)\) and \((9)\). Then
\[ z_{n+1}^{(q)}(\eta) = z^{(q)}(\eta; \varphi) + \int_0^\eta (\eta - s)^{a-q-1} \Phi_{\alpha-\beta,\alpha-q}(A_0(\eta - s)^{a-\beta})A_2 v_n(s)ds \]
\[ + \int_0^\eta (\eta - s)^{a-q-1} \Phi_{\alpha-\beta,\alpha-q}(A_0(\eta - s)^{a-\beta})h(s, z_n(s), z_n(s-\tau), v_n(s))ds, \]
\[ \lim_{n \to \infty} z_{n+1}^{(q)}(\eta) = \lim_{n \to \infty} z^{(q)}(\eta; \varphi) + \int_0^\eta (\eta - s)^{a-q-1} \Phi_{\alpha-\beta,\alpha-q}(A_0(\eta - s)^{a-\beta})A_2 v_n(s)ds \]
\[ + \int_0^\eta (\eta - s)^{a-q-1} \Phi_{\alpha-\beta,\alpha-q}(A_0(\eta - s)^{a-\beta})h(s, z_n(s), z_n(s-\tau), v_n(s))ds \]
\[ = z^{(q)}(\eta). \]
Moreover, we have
\[ \left\| cD^\beta z(\eta) - cD^\beta z_{n+1}(\eta) \right\| \]
\[ = \left\| \frac{1}{\Gamma(r-\beta)} \int_0^\eta (\eta - s)^{r-\beta-1} \left[ (s - \tau)^{a-q-1} \Phi_{\alpha-\beta,\alpha-q}(A_0(s-\tau)^{a-\beta})A_0(v(\xi) - v_n(\xi)) \right. \right. \]
\[ \left. \left. + h(\xi, z(\xi), z(\xi-\tau), v(\xi)) - h(\xi, z_n(\xi), z_n(\xi-\tau), v_n(\xi)) \right] d\xi \right\| ds \]
\[ = \left\| \int_0^\eta (\eta - s)^{a-\beta+r-q-1} \Phi_{\alpha-\beta,\alpha-\beta-q+r}(A_0(\eta - s)^{a-\beta})A_2 v(s) - v_n(s) \right\| ds \]
\[ + \right\| A_0(v(\xi) - v_n(\xi)) + h(s, z(s), z(s-\tau), v(s)) - h(s, z_n(s), z_n(s-\tau), v_n(s)) \right\| ds \]
\[ \leq m_{c5} m_{c8} \eta_1 \| v(\eta) - v_n(\eta) \| + m_{c8} L_{c1} \eta_1 [\| z(\eta) - z_n(\eta) \| + \| z(\eta - \tau) - z_n(\eta - \tau) \| + \| v(\eta) - v_n(\eta) \|]. \]

As \( n \to \infty \), \( cD^\beta z_{n+1}(\eta) \to cD^\beta z(\eta) \). Clearly \( z(\eta_1) = z_1 \), the control \( v(\eta) \) transfers the system from the initial state \( \varphi(\eta) \) to \( z_1 \) in time \( \eta_1 \). Hence the equation (6) is controllable on \([0, \eta_1]\).

5. Examples

Two examples are provided to demonstrate the controllability results for proposed criteria.

Example 5.1

The problem of nonlinear fractional damped delay dynamical systems as follows

\[ cD^\frac{5}{2} z(\eta) - cD^\frac{3}{2} z(\eta) = z(\eta - 3) + v(\eta) + \frac{x(\eta) + z(\eta - 3) + v(\eta)}{x(\eta) + z(\eta - 3) + v(\eta)} \quad (10) \]

where \( \alpha = \frac{5}{2}, \beta = \frac{3}{2}, \tau = 3, A_0 = 1, A_1 = 1, A_2 = 1 \) and the nonlinear function

\[ h(\eta, z(\eta), z(\eta - \tau), v(\eta)) = \frac{x(\eta) + z(\eta - 3)}{x(\eta) + z(\eta - 3) + v(\eta)}. \]

The solution of (10) by utilizing Laplace and inverse Laplace transform technique, we have

\[ z(\eta) = \left[ \Xi_{1,2}(\eta) - \eta \Xi_{1,3}(\eta) \right] \varphi(\eta) + \left[ \eta \Xi_{1,2}(\eta) - \eta^2 \Xi_{1,3}(\eta) \right] \varphi'(\eta) + \eta^2 \Xi_{1,3}(\eta) \varphi''(\eta) + \int_0^\eta (\eta - s - 3) \Xi_{1,5}^2(\eta - s) v(s) \, ds \]

\[ + \int_0^\eta (\eta - s) \Xi_{1,5}^2(\eta - s) h(s, z(s), z(s - \tau), v(s)) \, ds. \]

Now, consider the controllability on \([0, 5]\), we have to apply Theorems 3.1 and 4.1 to prove that system (10) is controllable.

The controllability Grammian matrix is defined by

\[ \omega = \int_0^5 (5 - s)^3 \left( \Xi_{1,5}^2(5 - s) \right) \left( \Xi_{1,5}^2(5 - s) \right)^T \, ds, \]

on more simplifying, we get

\[ \omega = 1761.3989 > 0, \]

is positive definite. Thus, system (10) is controllable and the nonlinear function \( h(\eta, z(\eta), z(\eta - 2), v(\eta)) \) is bounded and Lipschitz continuous, and satisfies the Lipschitz condition with the constant \( L_{c1} = 1 \) and the hypotheses of Theorem 4.1. Therefore the nonlinear equation (10) is controllable on \([0, 5]\).

Example 5.1 defines the conditions when \( A_0, A_1 \) and \( A_2 \) are constant. The next Example 5.2 demonstrate the conditions when \( A_0, A_1 \) and \( A_2 \) are matrices.

Example 5.2

The problem of nonlinear fractional damped delay dynamical system as follows

\[ cD^\frac{5}{2} z(\eta) - \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix} cD^\frac{3}{2} z(\eta) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z(\eta - 2) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} v(\eta) + h \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{z(\eta) + z(\eta - 2) + v(\eta)}{x(\eta) + z(\eta - 2) + v(\eta)} \right) \quad (11) \]

where \( \alpha = \frac{5}{2}, \beta = \frac{3}{2}, \tau = 2, A_0 = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \) and the nonlinear function

\[ h(\eta, z(\eta), z(\eta - \tau), v(\eta)) = h \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{z(\eta) + z(\eta - 2) + v(\eta)}{x(\eta) + z(\eta - 2) + v(\eta)} \right). \]

Now, we have to apply Theorems 3.1 and 4.1 to prove that system (11) is controllable on \([0, 3]\).
For \( \eta \in [0,3] \), solution (10), by utilizing Laplace and inverse Laplace transform technique we have,

\[
\begin{align*}
\mathcal{Z}(\eta) &= [\Xi_{1,1}(A_0 \eta) - \eta \Xi_{1,2}(A_0 \eta)]\varphi(\eta) + [\eta^2 \Xi_{1,3}(A_0 \eta) - \eta^2 \Xi_{1,1}(A_0 \eta)]\varphi'(\eta) + \eta^2 \Xi_{1,3}(A_0 \eta)\varphi''(\eta) \\
&+ A_1 \int_0^\eta (\eta - s - 2)^\frac{3}{2} \Xi_{1,5}^2(A_0(\eta - s - 2))\varphi(s)ds \\
&+ A_2 \int_0^\eta (\eta - s)^\frac{3}{2} \Xi_{1,5}^2(A_0(\eta - s))v(s)ds \\
&+ \int_0^\eta (\eta - s)^\frac{3}{2} \Xi_{1,5}^2(A_0(\eta - s))h(s, z(s), z(s-\tau), v(s))ds.
\end{align*}
\]

The controllability Grammian matrix is defined by

\[
\omega = \int_0^3 (3-s)^3 \Xi_{1,5}^2(A_0(3-s))A_2A_2^\top \Xi_{1,5}^2(A_0^\top(3-s))ds.
\]

The MLFs of the matrices are given by

\[
\Xi_{1,5}^2(A_0(3-s)) = \sum_{k=0}^1 \frac{(A_0(3-s))^k}{\Gamma(k + \frac{5}{2})} = \left(\begin{array}{c}
\frac{4}{3\sqrt{\pi}} \\
\frac{-8(3-s)}{15\sqrt{\pi}}
\end{array}\right),
\]

and then

\[
\Xi_{1,5}^2(A_0^\top(3-s)) = \sum_{k=0}^1 \left(\begin{array}{c}
\frac{4}{3\sqrt{\pi}} \\
\frac{-8(3-s)}{15\sqrt{\pi}}
\end{array}\right)^k = \left(\begin{array}{c}
\frac{4}{2} \\
\frac{8(3-s)}{3\sqrt{\pi}}
\end{array}\right),
\]

\[
\omega = \int_0^3 (3-s)^3 \left(\begin{array}{c}
\frac{4}{3\sqrt{\pi}} \\
\frac{-8(3-s)}{15\sqrt{\pi}}
\end{array}\right) \left(\begin{array}{c}
\frac{4}{2} \\
\frac{8(3-s)}{3\sqrt{\pi}}
\end{array}\right) ds,
\]

and evaluating, we get

\[
\omega = \begin{pmatrix}
12.8343 & -54.0872 \\
-54.0872 & 275.4781
\end{pmatrix}
\]

so that \( \det(\omega) = 610.1293 > 0 \). Thus, linear equation is controllable. The nonlinear function \( h(\eta, z(\eta), z(\eta-\tau), v(\eta)) \) is bounded and Lipschitz continuous, and satisfies the Lipschitz condition with the constant \( L_{L_1} = 1 \) and the hypotheses of Theorem 4.1. Therefore the nonlinear equation (11) is controllable on \([0,3]\).

6. Conclusion
The present paper has discussed about the controllability of linear and nonlinear fractional damped delay dynamical systems. Sufficient conditions for controllability criteria are established by constructing the Grammian matrix and the technique of successive approximation. In addition to that, examples are given to verify the theorem. Likewise, our forthcoming work is the direction to examine the controllability of higher order fractional damped delay dynamical systems with various kinds of delays in control variables such as distributed delay and multiple time varying delays in control.

References
[1] Bagley R L and Torvik A 1983 A Theoretical basis for the application of fractional calculus to viscoelasticity J. Rheol. 27 pp 201-210
[2] Chow T S 2005 Fractional dynamics of interfaces between soft-nanoparticles and rough substrates Phy. Lett. A 342 pp 148-155
[3] He J H 1998 Nonlinear oscillation with fractional derivative and its applications *Int. Conf. Vibrating Eng.*. 98 pp288-291

[4] Hilfer R 2000 *Applications of Fractional Calculus in Physics* World Scientific Publisher Singapore

[5] Ichise M, Nagayanagi Y and Kojima T 1971 Analog simulation of non-integer order transfer functions for analysis of electrode processes *J. Electroanalytical Chem.*. 33 pp 253-265

[6] Kilbas A A, Srivastava H M and Trujillo J J 2006 *Theory and Applications of Fractional Differential Equations* Elsevier Amsterdam

[7] Miller K S and Ross B 1993 *An Introduction to the Fractional Calculus and Fractional Differential Equations* Wiley and Sons New York

[8] Matignon D and d’Andrea-Novel B 1996 Some results on controllability and observability of finite dimensional fractional deferential systems *Proc. IAMCS IEEE Conf. Systems Man Cybernetics* Lille France 9-12 pp 952-956

[9] Shamardan A B and Moubarak M R A 1999 Controllability and observability for fractional control systems *J. Fractional Calculus* 15 pp 25-34

[10] Bettayeb M and Djennoune S 2008 New results on the controllability and observability of fractional dynamical systems *J. Vibration Cont.* 14 pp1531-1541

[11] Balachandran K, Govindaraj V, Rivero M and Trujillo J J 2015 Controllability of fractional damped dynamical systems *Appl. Math. Comput.* 257 pp 66-73

[12] Joice Nirmala R, Balachandran K, Rodriguez-Germa L and Trujillo J J 2016 Controllability of non-linear fractional delay dynamical systems *Reports Math. Phy.* 77 pp 87-104

[13] Sivabalan M and Sathiyanathan K 2017 controllability results for nonlinear higher order fractional delay dynamical systems with distributed delays in control *Global J. Pure Appl. Math.* 13 pp 7969-7989

[14] Yongyong K and Xiue Z 2010 Some comparison of two fractional oscillators *Physica B Condens. Matter* 405 pp 369-373

[15] Achar B N N, Hanneken J W and Clarke T 2002 Response characteristics of a fractional oscillator *Physica A Stat. Mech. Appl.* 309 pp 275-288

[16] Tofighi A 2003 The intrinsic damping of the fractional oscillator *Physica A* 329 pp 29-34

[17] Al-rabtah A, Erturk V S and Momani S 2010 Solutions of a fractional oscillator by using differential transform method *Comp. Math. Appl.* 59 pp 1356-1362

[18] Junpeng Liu, Suli Liu and Huilai Li 2017 Controllability result of nonlinear higher order fractional damped dynamical system *J. Nonlinear Sci. Appl.* 10 pp 325-337