Non-Equivalence of Stochastic Optimal Control Problems with Open and Closed Loop Controls

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Abstract. For an optimal control problem of an Itô’s type stochastic differential equation, the control process could be taken as open-loop or closed-loop forms. In the standard literature, provided appropriate regularity, the value functions under these two types of controls are equal and are the unique (viscosity) solution to the corresponding (path-dependent) HJB equation. In this short note, we show that these value functions can be different in general.

Keywords. stochastic optimal control, open-loop controls, closed-loop controls

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1 Introduction

Consider the following controlled Itô’s type stochastic differential equation (SDE, for short) over a finite time horizon \([0, T]\):

\[
X_t = x_0 + \int_0^t b(s, X_s, \alpha_s) ds + \int_0^t \sigma(s, X_s, \alpha_s) dB_s, \quad t \in [0, T];
\]

with utility functional

\[
J(\alpha) := E[g(X_T)].
\]

Here \(B\) is a \(d\)-dimensional standard Brownian motion; the controlled state process \(X_t\) takes values in \(\mathbb{R}^n\); the coefficients \(b, \sigma, g\) are deterministic measurable functions with appropriate dimensions, in particular \(g\) is scalar valued; the admissible control \(\alpha \in \mathcal{A}\) takes values in a subset \(\mathcal{A}\) of some Euclidean space; and we shall leave the issue of existence and/or uniqueness of the state for (1.1) to later discussions. The optimal value, or simply the value, of the control problem is defined as:

\[
V_0 := \sup_{\alpha \in \mathcal{A}} J(\alpha),
\]

and we call \(\alpha^* \in \mathcal{A}\) an optimal control if \(J(\alpha^*) = V_0\).

The value \(V_0\) obviously relies on the choice of the admissible control set \(\mathcal{A}\). Depending on the observed information in applications, among others, the control process \(\alpha_t\) could be taken as the

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so-called open-loop or closed-loop form. An open-loop control, denoted as \( \alpha \in \mathcal{A}^o \), is such that \( \alpha \) is \( \mathbb{F}^B \)-progressively measurable, while a closed-loop control, denoted as \( \alpha \in \mathcal{A}^c \), is required to be \( \mathbb{F}^X \)-progressively measurable. Here \( \mathbb{F}^B, \mathbb{F}^X \) are the natural filtrations generated by \( B \) and \( X \), respectively. We may define the values of the control problem accordingly:

\[
V_0^o := \sup_{\alpha \in \mathcal{A}^o} J(\alpha), \quad V_0^c := \sup_{\alpha \in \mathcal{A}^c} J(\alpha).
\]

A natural question is: do we have

\[
V_0^o = V_0^c ?
\]

We remark that, typically it is more convenient to use strong formulation for open-loop controls and weak formulation for closed-loop controls, see Remark 2.2 below.

The standard literature provides a positive answer to (1.5) by using the PDE approach, see e.g. Fleming–Soner [10] and Yong–Zhou [17]. Consider the following HJB equation:

\[
\partial_t v(t, x) + H(t, x, \partial_x v(t, x), \partial^2_{xx} v(t, x)) = 0, \quad v(T, x) = g(x),
\]

where

\[
H(t, x, z, \gamma) := \sup_{a \in \mathcal{A}} \left\{ \frac{1}{2} \text{tr} \left[ \gamma \sigma \sigma^\top (t, x, a) \right] + zb(t, x, a) \right\}.
\]

Here \((t, x, z, \gamma) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{1 \times n} \times \mathbb{S}^n\), with \( \mathbb{S}^n \) being the set of all \( n \times n \) symmetric matrices. Then, provided that the coefficients \( b, \sigma, g \) have appropriate regularity and the HJB equation has a unique continuous viscosity solution \( v \), we have,

\[
V_0^o = V_0^c = v(0, x_0).
\]

Moreover, \( v(t, x) \) is the optimal value of the control problem over \([t, T]\) with initial value \( X_t = x \). The main tool for this result is the dynamic programming principle (DPP for short), from which we see that the dynamic value function \( v(t, x) \) (more precisely we should introduce \( v^o(t, x) \) and \( v^c(t, x) \)) of the control problem under each type of controls is a viscosity solution of the HJB equation and hence (1.7) follows from the uniqueness of the viscosity solution.

The above result remains true in the path dependent case: by abusing the notations \( b, \sigma, g \),

\[
X_t = x_0 + \int_0^t b(s, X_{[0,s]}, \alpha_s)ds + \int_0^t \sigma(s, X_{[0,s]}, \alpha_s)dB_s;
\]

\[
J(\alpha) := \mathbb{E}[g(X_{[0,T]})], \quad V_0^o := \sup_{\alpha \in \mathcal{A}^o} J(\alpha), \quad V_0^c := \sup_{\alpha \in \mathcal{A}^c} J(\alpha),
\]

where \( b, \sigma, g \) may depend on the paths of \( X \) in an adapted way, with \( X_{[0,s]} := \{X_r \mid r \in [0, s]\} \). In this case, (1.6) becomes a path dependent HJB equation, or more generally a path dependent PDE, with the same Hamiltonian \( H \):

\[
\partial_t v(t, x) + H(t, x, \partial_x v(t, x), \partial^2_{xx} v(t, x)) = 0; \quad v(T, x) = g(x).
\]

Here \( x \in C^0([0, T]; \mathbb{R}^n) \) are continuous paths, \( v(t, x) = v(t, x_{[0,t]}) \) is adapted, and \( \partial_t v, \partial_x v, \partial^2_{xx} v \) are the path derivatives of Dupire [8]. When the equation (1.10) has a unique continuous viscosity solution, then (1.7) still holds true. We refer to Zhang [18, Part III] for more details of the pathwise stochastic analysis and viscosity solutions of path dependent PDEs.
We emphasize that the above arguments require the dynamic value function \( v(t, x) \) or \( v(t, x) \) to be continuous. When \( v(t, x) \) is discontinuous, although there are some nice works on discontinuous viscosity solutions, see e.g. Barles–Perthame [2], Barron–Jensen [3], Bertsch–Dal Passo–Ughi [4], Bardi–Capuzzo–Dolcetta [1], Chen-Su [7], and Bertsch–Smarrazzo–Terracina–Tesei [5], the theory is far from complete. In particular, in this case we are not able to conclude (1.7) or (1.5) from the viscosity solution approach.

Our main purpose of this short note is to construct a counterexample which shows that (1.5) can indeed fail. This implies that, besides the practical consideration in terms of the available information, mathematically it is also crucial to choose the right type of controls, especially when the value function is discontinuous. For applications of discontinuous value functions, we refer to [1] and references cited therein. We shall remark that, for stochastic differential games, even with the desired regularity, the game values can still be very sensitive to the choice of admissible controls, see e.g. Feinstein–Rudloff–Zhang [9], Possamai–Touzi–Zhang [14], and Sun–Yong [15]. We also remark that, our analysis of the values does not depend on the existence of optimal controls.

Another important consequence of the failure of (1.5) is that an (approximately) optimal control among one type of admissible controls is not necessarily (approximately) optimal anymore among the other type.

Our counterexample is constructed based on the well-known example of Tsirelson [16], which is path dependent. In the first order case, the value function is typically discontinuous when the coefficients are discontinuous. However, for uniformly nondegenerate second order (state dependent) HJB equations, because the diffusion term has some effect of regularization, quite often the value function becomes continuous when the point is away from the terminal time, even if the coefficients (e.g. the terminal condition) are discontinuous. This is not true anymore in the path dependent case, because the regularization requires some time to take effect while the discontinuity from the coefficients could be present at any time in this case.

Our counterexample will be constructed in Section 3. In Section 2 we will formulate the problems rigorously and present some preliminary analyses on the relationship between \( V^0\varrho \) and \( V^c\varrho \).

2 The Problem Formulations and Preliminary Analyses

We now formulate the problem rigorously in the path dependent setting. Denote \( \mathbb{X}_n := C([0, T]; \mathbb{R}^n) \), equipped with the uniform norm: \( \|x\| := \sup_{0 \leq t \leq T} |x_t| \) for all \( x \in \mathbb{X}_n \). Let \( A \subset \mathbb{R}^m \) be a domain for the possible values of admissible controls. Consider the path dependent SDE (1.8) with coefficients \((b, \sigma) : [0, T] \times \mathbb{X}_n \times A \to (\mathbb{R}^n, \mathbb{R}^{n \times d}) \) and \( g : \mathbb{X}_n \to \mathbb{R} \). Throughout the paper, the following assumptions will always be in force:

- \( b, \sigma, g \) are bounded (for simplicity) and progressively measurable in all variables;
- \( b, \sigma \) are adapted in \( x \) in the sense that, for \( \varphi = b, \sigma, \varphi(t, x, a) = \varphi(t, x_{[0,t]}, a) \).

We say the system is state dependent if \( \varphi(t, x, a) = \varphi(t, x_t, a) \) for \( \varphi = b, \sigma \) and \( g(x) = g(x_T) \).

For a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) and a generic set \( E \), let \( L^0(\mathbb{F}, \mathbb{P}; E) \) denote the set of \( E \)-valued processes progressively measurable with respect to the \( \mathbb{F} \)-augmented filtration of \( \mathbb{F} \). When \( \mathbb{P} \) and/or \( E \) are clear, we may omit them and simply denote the set as \( \mathbb{L}^0(\mathbb{F}) \). Moreover, let \( \mathbb{F}^B, \mathbb{F}^X \) denote the natural filtration generated by \( B \) and \( X \), respectively.

**Definition 2.1.** (i) A weak solution of the path dependent SDE (1.8) consists of a filtered
probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) and a triplet of processes \((B, X, \alpha) \in \mathbb{L}^0(\mathbb{F}, \mathbb{P}; \mathbb{R}^d \times \mathbb{R}^n \times A)\) such that \(B\) is Brownian motion under \(\mathbb{P}\) and (1.8) holds true \(\mathbb{P}\)-a.s.

(ii) A weak solution is called a \textit{strong solution} if \(X\) and \(\alpha\) are \(\mathbb{F}^B\)-progressively measurable. \hfill \blacksquare

In this paper we do not discuss the existence and uniqueness of weak solutions, which requires further conditions on \(b, \sigma\). Instead, we shall always assume the following very mild assumption:

\(\bullet\) for any piecewise constant control \(\alpha_t\) valued in \(A\), SDE (1.8) admits a weak solution.

We now introduce the optimal values under open-loop and closed-loop controls, respectively:

\[
V_0^O := \sup \{ \mathbb{E}^\mathbb{P}[g(X)] : \text{all weak solutions of (1.8) such that } \alpha \in \mathbb{L}^0(\mathbb{F}^B) \}; \\
V_0^C := \sup \{ \mathbb{E}^\mathbb{P}[g(X)] : \text{all weak solutions of (1.8) such that } \alpha \in \mathbb{L}^0(\mathbb{F}^X) \}.
\]

\textbf{Remark 2.2.} (i) For open-loop controls, under the stronger conditions that \(b\) and \(\sigma\) are uniformly Lipschitz continuous in \(x \in \mathbb{X}_n\), one typically uses the \textit{strong formulation}. That is, we fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a Brownian motion \(B\) on it. Then for any open-loop control \(\alpha \in \mathbb{L}^0(\mathbb{F}^B)\), the SDE (1.8) admits a unique strong solution \(X \in \mathbb{L}^0(\mathbb{F}^B)\).

(ii) For closed-loop controls, it is more convenient to use \textit{weak formulation}. That is, we fix the canonical space \(\Omega := \mathbb{X}_{d+n}\), the canonical processes \((B, X)\), and set \(\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T} := \mathbb{F}^{B,X}, \mathcal{F} := \mathcal{F}_T\). Then for any closed-loop control \(\alpha \in \mathbb{L}^0(\mathbb{F}^X)\), a weak solution is mainly a probability \(\mathbb{P}\) on the canonical space \(\mathbb{X}_{d+n}\). We remark that, for given \(\alpha\), there might be multiple (or no) \(\mathbb{P}\) corresponding to \(\alpha\).

(iii) For closed-loop controls, since the utility \(\mathbb{E}^\mathbb{P}[g(X)]\) involves only the \(\mathbb{P}\)-distribution of \(X\), it is quite often that we consider instead the canonical space \(\Omega := \mathbb{X}_n\) with canonical process \(X\), especially when \(\sigma\) is non-degenerate and hence \(B\) is \(\mathbb{F}^X\)-progressively measurable under \(\mathbb{P}\). \hfill \blacksquare

\textbf{Remark 2.3.} The closed-loop control case actually includes more general situations, by increasing the dimension of the state process \(X\) when needed.

(i) For the case \(b = b(t, B, X, \alpha), \sigma = \sigma(t, B, X, \alpha), g = g(B, X)\) and/or \(\alpha_t = \alpha_t(B, X)\) depend on both \(B\) and \(X\), we can set \(\tilde{X} := (B, X)\) and consider the SDE in the form of (1.8):

\[
(2.2) \quad d\tilde{X}_t = \begin{bmatrix} 0 \\ b(t, \tilde{X}, \alpha_t) \end{bmatrix} dt + \begin{bmatrix} I_d \\ \sigma(t, \tilde{X}, \alpha_t) \end{bmatrix} dB_t.
\]

We shall remark though, in this case the coefficients \(b, \sigma, g\) are typically discontinuous in the \(B\)-component of \(\tilde{X}\), and the PDE (1.6) or PPDE (1.10) is always degenerate. Both features would contribute to the possible discontinuity of the value function.

(ii) If we allow \(\alpha\) to be in \(\mathbb{L}^0(\mathbb{F})\) for the general \(\mathbb{F}\) in Definition 2.1, we may still view \(\alpha\) as a closed-loop control by considering a further enlarged state process \(\hat{X} := (\tilde{X}, \Gamma) := (B, X, \Gamma)\):

\[
(2.3) \quad d\hat{X}_t = \begin{bmatrix} 0 \\ b(t, \hat{X}, \alpha_t) \end{bmatrix} dt + \begin{bmatrix} I_d \\ \sigma(t, \hat{X}, \alpha_t) \end{bmatrix} dB_t,
\]

where \(\Gamma_t = \int_0^t \alpha_s dr\). Note that in this case \(\alpha\) is always in \(\mathbb{L}^0(\mathbb{F}^{\hat{X}})\), and hence in \(\mathbb{L}^0(\mathbb{F}^{\hat{X}})\). \hfill \blacksquare

\textbf{Remark 2.4.} In this remark we discuss some standard approaches in the literature. These approaches require appropriate regularity conditions, which we want to avoid in this paper.

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(i) For both open-loop and closed-loop controls, under appropriate regularity conditions, the dynamic value functions \( v^o(t, x) \) and \( v^c(t, x) \) would satisfy the \textit{dynamic programming principle}, which leads to the PPDE (1.10). When (1.10) has a unique continuous viscosity solution, we have \( v^o(t, x) = v^c(t, x) \) and in particular \( V_0^o = V_0^c \). Moreover, from the Hamiltonian, one can construct naturally an (approximate) optimal control which is closed-loop. In particular, even for the open-loop control problem \( V_0^o \) in (2.1), we have closed-loop (approximate) optimal controls.

(ii) Under sufficient regularity of the coefficients, any optimal open-loop control (if it exists) would satisfy the \textit{stochastic maximum principle}, a Pontryagin type maximum principle, see Peng [13] or Yong–Zhou [17]. This method is not convenient for closed-loop control though, because it involves differentiation of the closed-loop controls \( \alpha(t, x) \) with respect to \( x \). Nevertheless, the optimal open-loop control \( \alpha^o(t, B_{[0,t]} \) obtained from the stochastic maximum principle may turn out to be \( \mathbb{P}^X \)-progressively measurable, and in this case we also obtain the optimal closed-loop control \( \tilde{\alpha}^*(t, x) \) determined by: \( \tilde{\alpha}^*(t, X_{[0,t]} \) = \( \alpha^o(t, B_{[0,t]} \), \( \mathbb{P} \)-a.s.

We next provide some preliminary analyses on the relationship between \( V_0^o \) and \( V_0^c \), without invoking the viscosity solution approach. We emphasize again that in this paper the coefficients (especially \( g \)) could be discontinuous and we do not require the existence of optimal controls.

**Proposition 2.5.** Assume \( n = d \), \( \sigma \) takes values in \( \mathbb{S}^n \) and is positive definite, and \( b = b(t, x) \) does not depend on \( \alpha \). Then we have \( V_0^o \leq V_0^c \).

**Proof.** Let \( (\Omega, \mathcal{F}, \mathbb{P}, B, X, \alpha) \) be an arbitrary weak solution of (1.8) such that \( \alpha \in \mathbb{L}^0(\mathbb{R}^B) \). Note that the quadratic variation process \( \langle X \rangle \) is in \( \mathbb{L}^0(\mathbb{F}^X; \mathbb{S}^n) \), then so is \( \sigma(t, X, \alpha_t) = \left( \frac{d}{dt} \langle X \rangle_t \right)^{\frac{1}{2}} \), thanks to the assumption that \( \sigma \) is positive definite. Note that

\[
dB_t = \sigma^{-1}(t, X, \alpha_t) [dX_t - b(t, X) dt] = \left( \frac{d}{dt} \langle X \rangle_t \right)^{-\frac{1}{2}} [dX_t - b(t, X) dt].
\]

Then \( B \in \mathbb{L}^0(\mathbb{F}^X) \) and thus \( \alpha \in \mathbb{L}^0(\mathbb{F}^B) \subseteq \mathbb{L}^0(\mathbb{F}^X) \). This implies \( \mathbb{E}^\mathbb{P}[g(X)] \leq V_0^c \), hence \( V_0^o \leq V_0^c \).

Following Krylov [12], Gyongy [11], and Brunck-Shreve [6], we have the following result in the state dependent case.

**Proposition 2.6.** Assume \( b, \sigma, g \) are state dependent, and for any \( (t, x) \in [0, T] \times \mathbb{R}^n \), the set \( \{(b(t, x, a), \sigma \sigma^T(t, x, a) : a \in A) \subseteq \mathbb{R}^n \times \mathbb{R}^{n \times d} \) is convex. Then \( V_0^o \leq V_0^c \).

**Proof.** Note that we allow \( \sigma \) to be degenerate. By increasing the dimension of either \( B \) or \( X \) to \( n \lor d \), if necessary, we may assume without loss of generality that \( n = d \).

Let \( (\Omega, \mathcal{F}, \mathbb{P}, B, X, \alpha) \) be an arbitrary weak solution of the state dependent SDE (1.1) such that \( \alpha \in \mathbb{L}^0(\mathbb{F}^B) \). By setting \( Y := X - x_0 \) and \( Z := X \) in [6, Theorem 3.6], we have

(i) there exists a measurable function \( (\hat{b}, \hat{\sigma}): [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{S}^n \) such that

\[
\hat{b}(t, X_t) = \mathbb{E}^\mathbb{P}[b(t, X_t, \alpha_t)|X_t], \quad \hat{\sigma}^2(t, X_t) = \mathbb{E}^\mathbb{P}[\sigma \sigma^T(t, X_t, \alpha_t)|X_t];
\]

(ii) there exists a probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \), a Brownian motion \( \hat{B} \), and a process \( \hat{X} \) such that

\[
\hat{X}_t = x_0 + \int_0^t \hat{b}(s, \hat{X}_s) ds + \int_0^t \hat{\sigma}(s, \hat{X}_s) d\hat{B}_s, \quad \hat{\mathbb{P}}\text{-a.s.}
\]
(iii) for any $t$, the $\hat{P}$-distribution of $\hat{X}_t$ is equal to the $P$-distribution of $X_t$.

Since $\{(b(t,x,a),\sigma_\alpha^\top(t,x,a) : a \in A\}$ is convex, by (2.4) there exists a measurable mapping $\hat{\alpha} : [0,T] \times \mathbb{R}^n \to A$ such that

\begin{equation}
\hat{b}(t,X_t) = b(t,X_t,\hat{\alpha}(t,X_t)), \quad \hat{\sigma}^\top(t,X_t) = \sigma_\alpha^\top(t,X_t,\hat{\alpha}(t,X_t)).
\end{equation}

Moreover, there exists a mapping $Q : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ such that $Q(t,x)$ is an orthogonal matrix and $\hat{\sigma}(t,x) = \sigma(t,x,\hat{\alpha}(t,x))Q(t,x)$. Denote $\hat{B}_t := \int_0^t Q(s,\hat{X}_s)dB_s$, which is still an $\hat{P}$-Brownian motion. Then (2.5) and (2.6) imply

$$
\hat{X}_t = x_0 + \int_0^t b(s,\hat{X}_s,\hat{\alpha}(s,\hat{X}_s))ds + \int_0^t \sigma(s,\hat{X}_s,\hat{\alpha}(s,\hat{X}_s))d\hat{B}_s, \quad \hat{P}\text{-a.s.}
$$

This means that $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}^{\hat{B},\hat{X}}, \hat{P})$ and $(\hat{B}, \hat{X}, \hat{\alpha}(\hat{X}))$ is a weak solution to (1.1) and $\hat{\alpha}(\hat{X}) \in \mathbb{L}^0(\mathbb{F}^{\hat{X}}, \hat{P})$, and thus $\mathbb{E}^\hat{P}[g(\hat{X}_T)] \leq V_0^\hat{\alpha}$. Finally, by (iii) we have $\mathbb{E}^P[g(X_T)] = \mathbb{E}^\hat{P}[g(\hat{X}_T)] \leq V_0^\alpha$.

We remark that, in (iii) above, only the marginal distributions are equal. In general the $\hat{P}$ (joint) distribution of the process $\hat{X}_{[0,T]}$ does not coincide with the $P$ distribution of $X_{[0,T]}$, so we are not able to extend these arguments to the path dependent case.

**Proposition 2.7.** Under the following two conditions we have $V_0^\alpha \leq V_0^\beta$:

(i) $\sigma = \sigma(t,x)$ does not depend on $\alpha$ and is uniformly Lipschitz continuous in $x$;

(ii) $b = \sigma \lambda$ where the function $\lambda : [0,T] \times \mathbb{R}^n \to \mathbb{R}^d$ is bounded, continuous in $a$, and uniformly Lipschitz continuous in $x$.

**Proof.** Let $(\Omega, \mathcal{F}, \mathbb{P}, B, X, \alpha)$ be an arbitrary weak solution of (1.8) such that $\alpha \in \mathbb{L}^0(\mathbb{F}^X)$. It suffices to show that $\mathbb{E}^\mathbb{P}[g(X_T)] \leq V_0^\alpha$. For this purpose, we denote

$$
B^\alpha_t := B_t + \int_0^t \lambda(s,X_s,\alpha_s)ds, \quad d\mathbb{P}^\alpha := \frac{d\mathbb{P}^\alpha}{d\mathbb{P}} := e^{-\int_0^T \lambda(s,X_s,\alpha_s)dB_s - \frac{1}{2} \int_0^T |\lambda(s,X_s,\alpha_s)|^2ds}.
$$

By Girsanov Theorem, we know that $\mathbb{P}^\alpha \sim \mathbb{P}$ (meaning that they are equivalent) and $B^\alpha$ is a $\mathbb{P}^\alpha$-Brownian motion. Note that

\begin{equation}
X_t = x_0 + \int_0^t \sigma(s,X_s)dB^\alpha_s.
\end{equation}

Fix a probability space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ and a Brownian motion $B^0$ on it. Under (i) the SDE

\begin{equation}
X^0_t = x_0 + \int_0^t \sigma(s,X^0_s)dB^0_s, \quad \mathbb{P}^0\text{-a.s.}
\end{equation}

has a unique strong solution $X^0$. Now compare (2.7) and (2.8) we see that the $\mathbb{P}^\alpha$-distribution of $(B^\alpha, X)$ is equal to the $\mathbb{P}^0$-distribution of $(B^0, X^0)$. Since $\alpha \in \mathbb{L}^0(\mathbb{F}^X, \mathbb{P})$, we may write it as $\alpha(t,X)$. Then

\begin{equation}
\mathbb{E}^\mathbb{P}[g(X_T)] = \mathbb{E}^\mathbb{P}^\alpha [(M^\alpha_T)^{-1}g(X_T)] = \mathbb{E}^\mathbb{P}^\alpha [N^\alpha_T g(X_T)],
\end{equation}

where $N^\alpha_T := \exp \left( \int_0^T \lambda(s,X^0_s,\alpha(s,X^0_s))dB^0_s - \frac{1}{2} \int_0^T |\lambda(s,X^0_s,\alpha(s,X^0_s))|^2ds \right)$.
For fixed $\mathbb{P}^0$, there exist piecewise constant processes $\alpha^n(t, X^0) = \sum_{i=0}^{n-1} \alpha^n(t_i, X^0)1_{(t_i, t_{i+1})}(t)$ such that $\lim_{n \to \infty} \mathbb{E}^{\mathbb{P}^0}\left[ \int_0^T |\alpha^n(t, X^0) - \alpha(t, X^0)|^2 dt \right] = 0$. Then by (ii) one can easily show that

$$
\lim_{n \to \infty} \mathbb{E}^{\mathbb{P}^0}[N^n_T g(X^0)] = 0,
$$

and hence

$$
\lim_{n \to \infty} \mathbb{E}^{\mathbb{P}^0}[N^n_T g(X^0)] = \mathbb{E}^{\mathbb{P}^0}[N^0_T g(X^0)].
$$

For each $n$, by the Girsanov theorem we have $\mathbb{E}^{\mathbb{P}^0}[N^n_T g(X^0)] = \mathbb{E}^{\mathbb{P}^n}[g(X^0)]$, where $\mathbb{P}^n \sim \mathbb{P}^0$ is a probability measure, $B^n_t := B^0_t - \int_0^t \lambda(s, X^0, \alpha^n(s, X^0)) ds$ is an $\mathbb{P}^n$-Brownian motion, and

$$
X^0_t = x_0 + \int_0^t b(s, X^0, \alpha^n(s, X^0)) ds + \int_0^t \sigma(s, X^0) dB^n_s.
$$

Since $\alpha^n$ is piecewise constant, and $b, \sigma$ are uniformly Lipschitz continuous in $x$, by induction on $i$ one can easily show that $F^X \subset F^B$, where the augmentation is under $\mathbb{P}^0$ and equivalently under $\mathbb{P}^n$. Then $\alpha^n(t, X^0) \in L^{1,0}(\mathbb{P}^B, \mathbb{P}^0)$, namely is an open-loop control. This implies that $\mathbb{E}^{\mathbb{P}^0}[N^n_T g(X^0)] = \mathbb{E}^{\mathbb{P}^n}[g(X^0)] \leq V^0$. Then by (2.9) and (2.10) we have $\mathbb{E}^\mathbb{P}[g(X)] = \mathbb{E}^{\mathbb{P}^0}[N^n_T g(X^0)] \leq V^0$, and therefore $V^c \leq V^0$.

Note that in the above proof, continuity of $g$ is not needed. Therefore, we allow $g$ to be discontinuous. Combine Propositions 2.6 and 2.7, we immediately have the following.

**Corollary 2.8.** Assume $b, \sigma, g$ are state dependent, and

(i) $\sigma = \sigma(t, x)$ does not depend on $\alpha$ and is uniformly Lipschitz continuous in $x$;

(ii) $b = \sigma \lambda$ where the function $\lambda : [0, T] \times \mathbb{R}^n \times A \to \mathbb{R}^d$ is bounded, continuous in $a$, uniformly Lipschitz continuous in $x$, and the set $\{ \lambda(t, x, a) : a \in A \} \subset \mathbb{R}^d$ is convex.

Then $V^0 = V^c$.

**Remark 2.9.** Under the conditions in Corollary 2.8, obviously the dynamic value functions for the control problem on $[t, T]$ with initial value $x$ are also equal: $v^0(t, x) = v^c(t, x) =: v(t, x)$. However, we emphasize here that $g$ can be discontinuous and $\sigma$ can be degenerate, then $v$ can also be discontinuous. One trivial example is: $b = 0, \sigma = 0$, then $v(t, x) = g(x)$ for all $t$, which will be discontinuous if $g$ is so.

### 3 A Counterexample

In this section we construct a counterexample that $V^0$ and $V^c$ are indeed not equal. We first recall the following well-known result of Tsirelson [16].

**Lemma 3.1.** Let $t_0 := T$ and, for $k = -1, -2, \cdots, t_k \downarrow 0$ as $k \to -\infty$. Define

$$
\theta(x) := x - [x] \quad \forall x \in \mathbb{R}; \quad [x] \text{ is the greatest integer no more than } x,
$$

$$
\mu(t, x) := \theta\left(\frac{x_{t_k} - x_{t_{k-1}}}{t_k - t_{k-1}}\right), \quad x \in \mathbb{R}, \quad t \in [t_k, t_{k+1}), k \leq -1.
$$

Then the following SDE has no strong solution:

$$
X_t = \int_0^t \mu(s, X_s) ds + B_t.
$$
We note that there is a typo in the statement of [16, Theorem]. In the definition of the coefficient $A$ (our $\mu$ here), the domain $t \in [t_k, t_{k-1})$ should be $t \in [t_k, t_{k+1})$ as in (3.1).

We shall construct the counterexample in the setting of Remark 2.3 (i). Set $n = d = 1$, so the $X$ in (2.2) is two dimensional. We will use the notation $\tilde{x} = (\omega, x) \in \mathcal{X}_2$, where $\omega$ and $x$ refer to the paths of $B$ and $X$, respectively.

**Example 3.2.** Let $A := [0, 1]$, $x_0 = 0, b(t, \tilde{x}, a) := a, \sigma(t, \tilde{x}, a) := 1$, for $(t, \tilde{x}, a) \in [0, T] \times \mathcal{X}_2 \times A$, namely SDE (1.8) (or say, the second equation of (2.2)) becomes:

\begin{equation}
X_t = \int_0^t \alpha_s ds + B_t.
\end{equation}

Moreover, $g(\tilde{x}) := 1_D(\tilde{x})$, where, for $\tilde{x} = (\omega, x) \in \mathcal{X}_2$,

\begin{equation}
\alpha^*(t, \tilde{x}) := 0 \lor \left(\limsup_{h \to 0} \frac{(x - \omega)_t - (x - \omega)(t-h)^+}{h}\right) \land 1,
\end{equation}

\begin{equation}
D := \left\{ \tilde{x} \in \mathcal{X}_2 : \int_0^T |\alpha^*(t, \tilde{x}) - \mu(t, x)|dt = 0 \right\}.
\end{equation}

Then $V_0^* = 0 < 1 = V_0^\circ$.

**Proof.** We first prove $V_0^\circ = 1$. Since $g \leq 1$, it is clear that $V_0^\circ \leq 1$. Next, by Girsanov theorem, the SDE (3.2) has a unique (in law) weak solution $(\Omega, \mathcal{F}, \mathbb{P}, B, X)$. Denote $\tilde{X} := (B, X)$ as usual and consider the closed-loop control $\alpha_k := \mu(t, X)$, which is obviously $\mathbb{F}^X$-progressively measurable. Note that $\mu$ takes values in $[0, 1]$. Then $(\Omega, \mathcal{F}, \mathbb{F}^\tilde{X}, \mathbb{P}, B, X, \alpha)$ is a weak solution to SDE (3.3) in the sense of Definition 2.1 and $\alpha \in L^0(\mathbb{F}^X)$. Therefore, $V_0^\circ \geq \mathbb{E}[g(\tilde{X})] = \mathbb{P}(\tilde{X} \in D)$. By (3.2), it is clear that

\begin{equation}
\limsup_{h \to 0} \frac{(X - B)_t - (X - B)(t-h)^+}{h} = \limsup_{h \to 0} \frac{1}{h} \int_0^t \mu(s, X)ds = \mu(t, X), \quad dt \times d\mathbb{P}\text{-a.s.}
\end{equation}

Then

\begin{equation}
\alpha^*(t, \tilde{X}_t) = \mu(t, X_t), \quad dt \times d\mathbb{P}\text{-a.s.}, \quad \text{and thus } \tilde{X} \in D, \quad \mathbb{P}\text{-a.s.}
\end{equation}

This implies $V_0^\circ \geq \mathbb{P}(\tilde{X} \in D) = 1$, and therefore, $V_0^\circ = 1$.

It remains to show that $V_0^\circ = 0$. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B, X, \alpha)$ be an arbitrary weak solution to SDE (3.3) with open-loop control $\alpha \in L^0(\mathbb{F}^B)$. Note that in this case $X \in L^0(\mathbb{F}^B)$ is a strong solution, then $\tilde{X} \in L^0(\mathbb{F}^B; \mathbb{R}^2)$. For the $t_k$ in Lemma 3.1, introduce:

\begin{equation}
E_k := \left\{ \int_0^{t_k} |\alpha_t - \mu(t, X)|dt = 0 \right\}, \quad k \leq -1, \quad E_\infty := \lim_{k \to -\infty} E_k.
\end{equation}

Note that $E_k \uparrow E_\infty$ as $k \to -\infty$. Clearly $E_k \in \mathcal{F}_{t_k}^B$, then by the Blumenthal 0-1 law we have $\mathbb{P}(E_\infty) = 0$ or 1. If $\mathbb{P}(E_\infty) = 0$, since $\{\tilde{X} \in D\} = E_0 \subset E_\infty$, then $\mathbb{E}[g(\tilde{X})] = \mathbb{P}(\tilde{X} \in D) = 0$, which is the desired equality we want. So from now on we assume by contradiction that $\mathbb{P}(E_\infty) = 1$.

For each $k \leq -1$, introduce $\alpha^k, X^k \in L^0(\mathbb{F}^B)$ as follows:

\begin{equation}
\alpha^k_t := \alpha_t, \quad t \in [0, t_k); \quad \alpha^k_t := \theta \left( \frac{B_{ti} - B_{t_{i-1}} + \int_{t_{i-1}}^{t_i} \alpha^k_s ds}{t_i - t_{i-1}} \right), \quad t \in [t_i, t_{i+1}), \quad i = k, \ldots, -1;
\end{equation}

\begin{equation}
X^k := \int_0^{t_k} \alpha^k_s ds + B_s.
\end{equation}
Note that, for $i < k$,

$$
\alpha_t = \mu(t, X) = \theta \left( \frac{B_{t_i} - B_{t_{i-1}} + \int_{t_{i-1}}^{t_i} \alpha_s ds}{t_i - t_{i-1}} \right), \quad dt \times d\mathbb{P}\text{-a.s. on } [t_i, t_{i+1}) \times E_k.
$$

Now for $n < k$, since $E_k$ is increasing as $k \to -\infty$, clearly $(\alpha^n_t, X^n_t) = (\alpha^k_t, X^k_t) = (\alpha_t, X_t)$ for $t \leq t_n$, and for $i = n, \cdots, k-1$, by applying (3.6) for $n$ and (3.7) for $k$ we see that

$$
(\alpha^n_t, X^n_t) = (\alpha^k_t, X^k_t), \quad \alpha^k_t = \mu(t, X^k_t),
$$
on $[t_i, t_{i+1}) \times E_k$. Then by applying (3.6) for both $n$ and $k$ and recalling (3.1) we see that (3.8) holds on $[t_i, t_{i+1}) \times E_k$ for $i = k, \cdots, -1$ as well. That is, (3.8) holds $dt \times d\mathbb{P}$-a.s. on $[0, T] \times E_k$ for all $n \leq k$. Denote

$$
\hat{\alpha} := \limsup_{k \to -\infty} \alpha^k \in L^0(\mathbb{F}^B), \quad \hat{X} := \limsup_{k \to -\infty} X^k \in L^0(\mathbb{F}^B).
$$

Then, by (3.6) and (3.8) we have

$$
(\hat{\alpha}_t, \hat{X}_t) = \mu(t, \hat{X}), \quad \hat{X}_t = \int_0^t \hat{\alpha}_s ds + B_t, \quad dt \times d\mathbb{P}\text{-a.s. on } [0, T] \times E_k \text{ for each } k, \text{ and thus } dt \times d\mathbb{P}\text{-a.s. on } [0, T] \times E_{\infty}. \quad \text{By the assumption } \mathbb{P}(E_{\infty}) = 1, \text{ we see that (3.10) holds } dt \times d\mathbb{P}\text{-a.s. on } [0, T] \times \Omega. \text{ This implies that } \hat{X} \text{ is a strong solution of SDE (3.2), which is a desired contradiction. So } \mathbb{P}(E_{\infty}) = 0 \text{ for all weak solutions with open loop controls, and therefore } V_0^o = 0.
$$

**Remark 3.3.** In this remark we present some related interesting questions we would like to explore in the future research.

(i) The above counterexample relies heavily on the path dependence of the terminal condition $g$, and control only enters in the drift. Is it possible to construct a counterexample such that all the coefficients are state dependent, and/or the control appears in the diffusion as well?

(ii) Regardless whether the open-loop and closed-loop dynamic value functions are equal or not, they might be discontinuous in general. Is it possible to establish the connection between these values functions and the so called discontinuous viscosity solutions of the HJB equation?

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