HKT MANIFOLDS: HODGE THEORY, FORMALITY AND BALANCED METRICS

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Abstract. Let $(M, I, J, K, \Omega)$ be a compact HKT manifold and denote with $\bar{\partial}$ the conjugate Dolbeault operator with respect to $I$, $\partial_J := J^{-1} \bar{J} J$, $\partial^A := [\partial, A]$ where $A$ is the adjoint of $L := \Omega \wedge -$. Under suitable assumptions, we study Hodge theory for the complexes $(\mathcal{A}^{*0}, \partial, \partial_J)$ and $(\mathcal{A}^{*0}, \partial, \partial^A)$ showing a similar behavior to Kähler manifolds. In particular, several relations among the Laplacians, the spaces of harmonic forms and the associated cohomology groups, together with Hard Lefschetz properties, are proved. Moreover, we show that for a compact HKT $\text{SL}(n, \mathbb{H})$-manifold the differential graded algebra $(\mathcal{A}^{*0}, \partial)$ is formal and this will lead to an obstruction for the existence of an HKT $\text{SL}(n, \mathbb{H})$-structure $(I, J, K, \Omega)$ on a compact complex manifold $(M, I)$. Finally, balanced HKT structures on solvmanifolds are studied.

1. Introduction

Let $(M, I, J, K)$ be a hypercomplex manifold, i.e. a smooth manifold $M$ equipped with three complex structures $I, J, K$ that anticommute with each other, and such that $IJ = K$. On a hypercomplex manifold there is a distinguished connection $\nabla$, called the Obata connection \[25\], that is torsion-free and preserves the hypercomplex structure, in the sense that

$$\nabla I = 0, \quad \nabla J = 0, \quad \nabla K = 0.$$

A Riemannian metric that is Hermitian with respect to all three complex structures $I, J, K$ is said to be hyperhermitian, and, accordingly, $(M, I, J, K, g)$ is called an hyperhermitian manifold.

On a hyperhermitian manifold the 2-form

$$\Omega := \frac{g(I \cdot, \cdot) + ig(K \cdot, \cdot)}{2}$$

is non-degenerate of type $(2, 0)$ with respect to $I$ and completely determines the hyperhermitian metric via the relation $\Omega(Z, J\bar{Z}) = g(Z, \bar{Z})$ for every $Z \in T^{1,0}M$. Furthermore there is a bijective correspondence between hyperhermitian metrics and q-positive q-real $(2, 0)$-forms, where a $(2, 0)$-form $\Omega$ is called q-real if $J\Omega = \bar{\Omega}$ and it is called q-positive if additionally satisfies $\Omega(Z, J\bar{Z}) > 0$ for every non-zero $Z \in T^{1,0}M$.

A hyperhermitian manifold is HKT, namely hyperkähler with torsion, if the associated $(2, 0)$-form satisfies

$$\partial \Omega = 0,$$

where $\partial$ is the Dolbeault operator with respect to $I$. The HKT condition is at many levels the hypercomplex analogue of the Kähler condition (see e.g. \[2\], \[5\], \[20\], \[21\], \[24\], \[33\], \[35\]). The main purpose of this work is twofold: on one hand we explore the analogies with the Kähler setting from a cohomological point of view, as done in \[20\], \[24\], \[33\], on the other hand we extend some results proved on hypercomplex nilmanifolds in \[6\] to hypercomplex solvmanifolds.

In \[34\] Verbitsky proved that a compact HKT manifold has trivial canonical bundle if and only if the holonomy group of the Obata connection is contained in $\text{SL}(n, \mathbb{H})$. Manifolds with the latter property are called $\text{SL}(n, \mathbb{H})$-manifolds. If one removes the HKT hypothesis, then it is still true that an $\text{SL}(n, \mathbb{H})$-manifold has holomorphically trivial canonical bundle. The converse has been recently disproved by Andrade-Tolcachier \[4\], however it is, for instance, true for hypercomplex nilmanifolds \[6\]. We prove here that on hypercomplex solvmanifolds the SL$(n, \mathbb{H})$ condition is equivalent to the existence

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of an invariant holomorphic trivialization of the canonical bundle (Theorem 5.4).

Another very relevant property in this context is the balanced condition. Indeed, it turns out [35] that for a compact HKT manifold \((M, I, J, K, g)\) one has that \((M, I, g)\) is balanced if and only if \(\partial \bar{\Omega}^n = 0\). In particular, if a compact HKT manifold is balanced it is necessarily \(SL(n, \mathbb{H})\). A very interesting conjecture posed by Alesker and Verbitsky predicts that this condition is also sufficient, in the sense that an HKT \(SL(n, \mathbb{H})\)-manifold admits a balanced HKT metric (which does not necessarily coincide with the initial one). Again, evidence for this conjecture is provided by nilmanifolds with invariant hypercomplex structure [6] and we shall extend this fact to solvmanifolds with invariant hypercomplex structure (Theorem 5.6).

As explained in [35], one way to approach this problem is by studying the quaternionic Calabi-Yau conjecture proposed by Alesker and Verbitsky in [3]. The solvability of the relative equation would allow to prescribe the complex volume of the HKT manifold. More precisely, for any \(q\)-positive \((2n, 0)\)-form \(\Theta\), one could find a new HKT form \(\Omega'\), compatible with the given hypercomplex structure, satisfying

\[
(\Omega')^n = \Theta.
\]

Choosing \(\Theta\) to be holomorphic would imply that \(\Omega'\) is balanced HKT. For further information on the (still unsolved in general) quaternionic Calabi-Yau conjecture see e.g. [1] [8] [13] [17] [18] [30] [38] and references therein.

Other sufficient conditions on solvmanifolds with an invariant HKT structure to have a balanced metric are given in Theorems 5.1, 5.5.

On a hypercomplex manifold \((M, I, J, K)\) endowed with a HKT structure \(\Omega\) there are three important differential operators. Denote with \(A^{p, q}(M) = A^p(I^*, M)\) the space of \((p, q)\)-forms with respect to \(I\). Then we are interested in the usual (conjugate) Dolbeault operator \(\bar{\partial}: A^{p, q}(M) \to A^{p+1, q}(M)\) and the twisted differential operator \(\partial_J: A^{p, q}(M) \to A^{p+1, q}(M)\) defined as \(\partial_J := J^{-1}\bar{\partial}J\). Because of the integrability of \(I, J, K\), these operators anticommute and both square to zero. Therefore we obtain a cochain complex \((A^{*, q}, \partial, \partial_J)\) for every fixed \(q\). This differs from the complex case, as here we obtain a single complex, while in the complex setting \(\partial\) and \(\bar{\partial}\) give rise to a double complex. Here, we restrict to study the case \(q = 0\).

Moreover, the existence of \(\Omega\) leads to the definition of the operator \(\partial^\Lambda := [\partial, \Lambda]\) where \(\Lambda\) is the adjoint of \(L := \Omega \wedge -\). Therefore we obtain a cochain complex \((A^{*, q}, \partial, \partial^\Lambda)\) for every fixed \(q\), and again we will restrict to \(q = 0\). Hence one can study cohomology groups and Hodge theory from a “complex point of view” on \((A^{*, 0}, \partial, \partial_J)\) or from a “symplectic point of view” on \((A^{*, 0}, \partial, \partial^\Lambda)\).

Concerning the latter case, one can contextualize everything in the more general setting of Lefschetz spaces and with this approach one can then prove that all the natural Laplacian operators that arise in the quaternionic Dolbeault, Bott-Chern and Aeppli cohomology groups can be defined:

\[
H^p_{\partial^\Lambda}(M) := \frac{\ker(\partial|_{A^{p, 0}(M)})}{\partial A^{p-1, 0}(M)}, \quad H^p_{\partial_J}(M) := \frac{\ker(\partial_J|_{A^{p, 0}(M)})}{\partial_J A^{p-1, 0}(M)},
\]

\[
H^p_{BC}(M) := \frac{\ker(\partial|_{A^{p, 0}(M)}) \cap \ker(\partial_J|_{A^{p, 0}(M)})}{\partial \partial_J A^{p-2, 0}(M)}, \quad H^p_{\partial^\Lambda}(M) := \frac{\ker(\partial^\Lambda|_{A^{p, 0}(M)})}{\partial A^{p-1, 0}(M) + \partial_J A^{p-1, 0}(M)},
\]

when \(M\) is compact all these groups are finite-dimensional [20], indeed, as usual, once fixed an hyper-hermitian metric, one can show that each of these cohomology group is isomorphic to the kernel of the following Laplacians acting on \((p, 0)\)-forms

\[
\Delta_{\partial} := \partial \partial^* + \partial^* \partial, \quad \Delta_{\partial_J} := \partial_J \partial_J^* + \partial_J^* \partial_J,
\]

\[
\Delta_{BC} := \partial^* \partial + \partial_J^* \partial_J + \partial \partial_J \partial_J^* \partial + \partial_J \partial \partial_J^* \partial + \partial^* \partial_J \partial_J^* \partial + \partial \partial_J^* \partial_J + \partial_J \partial \partial_J^* \partial + \partial_J^* \partial_J \partial \partial_J^* \partial + \partial_J \partial \partial_J^* \partial.
\]
For each of these we denote with a calligraphic letter the corresponding space of harmonic forms, thus, for instance, $\mathcal{H}_p^0(M) := \text{Ker}(\Delta_\partial|_{\mathcal{P}_p^0(M)})$. It is well known that on a compact Kähler manifold the spaces of Dolbeault, Bott-Chern and Aeppli-harmonic forms all coincide. We prove that the analogue result is also true for balanced HKT manifolds:

**Theorem 1.1** (Propositions 3.4, 3.8, 3.13). On a compact balanced HKT manifold $M$ the spaces of harmonic forms all coincide:

$$\mathcal{H}_\partial^p(\mathcal{M}) = \mathcal{H}_{\partial J}^p(\mathcal{M}) = \mathcal{H}_{BC}^p(\mathcal{M}) = \mathcal{H}_A^p(\mathcal{M}).$$

In particular, there are isomorphisms

$$H^p_{\partial J}(\mathcal{M}) \cong H^p_{\partial J}(\mathcal{M}) \cong H^p_{BC}(\mathcal{M}) \cong H^p_A(\mathcal{M})$$

for every $p$.

We remark that the equality of $\Delta_{\partial J}$ and $\Delta_\partial$ on balanced HKT manifolds is already implicitly proved in [33, Theorem 10.2]. Along the way we shall also study the Hard Lefschetz condition on these spaces (see Theorems 3.1 and 3.10).

Another interesting notion to be investigated is formality since it provides an obstruction to the $\partial\bar{\partial}$-lemma and so to the existence of HKT $SL(n, \mathbb{H})$-structures. More precisely, we prove

**Theorem 1.2** (Theorem 4.1). Let $(M, I, J, K)$ be a compact hypercomplex manifold satisfying the $\partial\bar{\partial}$-lemma, then the differential graded algebra $(A^\bullet, 0, \partial)$ is formal.

As a consequence, we obtain that triple $\partial$-Massey products vanish, which allows us to derive the following interesting obstruction for a compact complex $4n$-dimensional manifold to allow HKT $SL(n, \mathbb{H})$-structures.

**Corollary 1.3** (cf. Corollary 4.9). Let $(M, I)$ be a $4n$-dimensional compact complex manifold with holomorphically trivial canonical bundle and such that there exists a non trivial $\partial$-Massey product. Then $(M, I)$ does not admit any complex structures $J, K$ such that $(M, I, J, K)$ is hypercomplex and admits a HKT metric.

The organization of the paper is the following. In Section 2 we briefly study a class of differential graded algebras which we call “Lefschetz”. Building on the work of Tomassini and Wang [31] we define a generalization of the Hodge star operator, which allows us to take into account formal adjoints and discuss some relations between Laplacians. This algebraic picture is then applied to HKT manifolds in Section 3 leading us to the proof of Theorem 1.1. Section 4 is then devoted to investigate the notion of formality. Here we prove Theorem 1.2 and Corollary 1.3. Finally, in Section 5 we briefly study hypercomplex structures on solvmanifolds and their relation with the balanced condition.

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2. Lefschetz spaces

In this section, inspired by the algebraic treatment of Tomassini and Wang [31], we wish to push a little further their work, proving some identities between Laplacians defined in a fairly general context. We start by recalling the main definitions and results from [31] (see also [36]).

**Definition 2.1.** Let $A = \bigoplus_{p=0}^{2n} A^p$ be a direct sum of complex vector spaces. Let $L$ be a $\mathbb{C}$-linear endomorphism of $A$ such that $L(A^p) \subseteq A^{p+2}$ for $p = 0, \ldots, 2n-2$ and $L(A^{2n-1}) = L(A^{2n}) = 0$. We say that $(A, L)$ is a Lefschetz space if $L$ satisfies the Hard Lefschetz Condition (HLC), i.e.

$$L^{n-p} : A^p \to A^{2n-p}$$

is an isomorphism for all $p = 0, \ldots, n$.

If a Lefschetz space $(A, L)$ is equipped with a $\mathbb{C}$-linear endomorphism $d$ such that $d(A^p) \subseteq A^{p+1}$ for $p = 0, \ldots, 2n-1$, while $d(A^{2n}) = 0$ we call the triple $(A, L, d)$ a differential Lefschetz space.

If moreover $d^2 = 0$ then the triple $(A, L, d)$ is called a Lefschetz complex.

On a Lefschetz space we say that $\alpha \in A^p$ is a primitive form if $p \leq n$ and $L^{n-p+1} \alpha = 0$. By the HLC immediately follows the decomposition into primitive forms (see [36]), more precisely, for every $\alpha \in A^p$ there exist unique primitive $\alpha^k \in A^{p-2k}$ such that

$$\alpha = \sum_{k=0}^{\lfloor p/2 \rfloor} \frac{1}{k!} L^k \alpha^k. \tag{1}$$

As a generalization of the symplectic star operator Tomassini and Wang [31] introduced the Lefschetz star operator $*_L : A \to A$, acting on a primitive form $\beta \in A^p$ as follows:

$$*_L \frac{1}{k!} L^k \beta := (-1)^{1+2+\cdots+p} \frac{1}{(n-p-k)!} L^{n-p-k} \beta.$$ 

Clearly the definition is then extended by linearity to any $\alpha \in A^p$ via the Lefschetz decomposition (1). Notice that $*_L^2 = 1$.

The starting point of the discussion by Tomassini and Wang is the following general Demailly-Griffiths-Kähler identity [31 Theorem A].

**Theorem 2.2.** Let $(A, L, d)$ be a differential Lefschetz space and $\Lambda = *^-1_L *_L$ the dual Lefschetz operator. Define $d^\Lambda \in \text{End}(A)$ by

$$d^\Lambda |_{A^p} := (-1)^{p+1} *_L d *_L,$$

and assume that $[L, [d, L]] = 0$, then

$$[d^\Lambda, L] = d + [L, [d, L]], \quad [d, \Lambda] = d^\Lambda + [[\Lambda, d^\Lambda], L].$$

Notice that if $(A, L, d)$ is a Lefschetz complex then $d^2 = 0$ implies $(d^\Lambda)^2 = 0$. In case $[d, L] = 0$, one also obtains that

$$[d, d^\Lambda] = 0.$$ 

Therefore, on a Lefschetz complex with $[d, L] = 0$ one has that the triple $(A, d, d^\Lambda)$ is a double complex.

We summarize here the main consequences which we are interested in (cf. [31 Theorems 3.3, 3.5]).

**Theorem 2.3.** Let $(A, L, d)$ be a Lefschetz complex. Suppose $[d, L] = 0$ and denote with $\mathcal{H}^p_L$ the space of Lefschetz harmonic $p$-forms, i.e. elements $\alpha \in A^p$ such that

$$d \alpha = 0 = d^\Lambda \alpha.$$ 

Then $(\mathcal{H}^\ast_L, L)$ and $(\mathcal{H}^\ast_L, \Lambda)$ satisfy the HLC. Furthermore the following are equivalent:

- $(A^\ast, L)$ satisfies the $dd^\Lambda$-lemma, i.e.,
  $$\text{Ker } d \cap \text{Ker } d^\Lambda \cap (\text{Im } d + \text{Im } d^\Lambda) = \text{Im } dd^\Lambda$$
  - There is a Lefschetz harmonic representative in each cohomology class of $H^\ast_L$;
  - $(\mathcal{H}^\ast_L, L)$ satisfies the HLC;
  - $(\mathcal{H}^\ast_{d^\Lambda}, \Lambda)$ satisfies the HLC.
Remark 2.5. Let us consider the picture (a generalization of) the Hodge star operator, in order to do so, we need a complex structure on our Lefschetz space.

**Definition 2.4.** A Lefschetz (differential) graded algebra is a (differential) Lefschetz space 
\[ A = \bigoplus_{p=0}^{2n} A^p \] 
which is also a graded algebra that is generated by \( A^1 \) over \( \mathbb{C} \).

Let \( A \) be a Lefschetz graded algebra and assume that \( A^1 \) is equipped with an endomorphism \( J \) such that \( J^2 = -I \). We extend the action of \( J \) on \( A \) by setting on homogeneous elements
\[ J(\alpha_1 \cdots \alpha_k) = J\alpha_1 \cdots J\alpha_k, \quad \text{for every} \; \alpha_1, \cdots, \alpha_k \in A^1, \]
and then extending by \( \mathbb{C} \)-linearity.

We make the assumption that \( JL = LJ \) and consequently introduce a generalization of the Hodge star operator by setting
\[ \ast := JL \ast = *LJ \]
or equivalently
\[ \ast \frac{1}{k!} L^k \beta := (-1)^{1+2+\cdots+p} \frac{1}{(n-p-k)!} L^{n-p-k} J \beta, \]
for a primitive \( \beta \in A^p \) and then extend the definition on all \( A \) by bilinearity via the Lefschetz decomposition \([1]\). It follows that
\[ \ast^2 \big|_{A^p} = J^2 \big|_{A^p} = (-1)^p. \]

**Remark 2.5.** Let \((M,J,\omega)\) be an almost Kähler manifold, namely \( \omega \) is a symplectic structure on a smooth manifold \( M \) and \( J \) is a compatible almost complex structure. Clearly when \( J \) is integrable (and so \((M,J)\) is a complex manifold) then \((M,J,\omega)\) is a Kähler manifold. Set \( L = \omega \wedge - \) for the usual Lefschetz operator and let \( A = \bigoplus_{p=0}^{2n} A^p \) be the Lefschetz graded algebra of differential forms on \( M \). The almost complex structure \( J : TM \rightarrow TM \) naturally induces a complex structure \( J \) on \( A^1 \). Since \( \omega \) is a \((1,1)\)-form we have \( JL = LJ \) and the description above is coherent with the well-known almost Kähler case. Indeed, formula \([3]\), where \( \ast \) is the usual Hodge operator, is sometimes referred to as the Weil relation \([37]\).

Now, suppose \( A \) is equipped with a differential \( d \). Consider the dual Lefschetz operator
\[ \Lambda = \ast^{-1} L \ast = \ast^{-1} JL \ast^{-1} \ast = \ast^{-1} L \ast \]
and define as before the “Lefschetz adjoint” of \( d \), i.e. \( d^{\Lambda} \in \text{End}(A) \) given by \( d^{\Lambda} |_{A^p} := (-1)^{p+1} \ast L \ast d \ast L \).

Then, by Theorem \([22]\) if \([d,L] = 0\) we have \( d^{\Lambda} = [d,\Lambda] \) and \( d = [d^{\Lambda},L] \).

We may take the “Hodge adjoints”
\[ d^{\ast} = - \ast d \ast, \quad d^{\Lambda \ast} = - \ast d^{\Lambda \ast}, \]
and obtain also \( d^{\ast} = [\Lambda, d^{\Lambda \ast}] \).

Now, we consider the following operators and we aim to study the relations between them:
\[ \Delta_d = dd^\ast + d^\ast d, \quad \Delta_{d^{\Lambda}} = d^{\Lambda} d^{\Lambda \ast} + d^{\Lambda \ast} d^{\Lambda}, \]
\[ \Delta_{d^{\ast}} = d^\ast d + d^{\ast} d^{\ast} + d^{\ast} d^{\ast} d^\ast + d^{\ast} d^{\ast} d^{\ast} d^\ast + d^\ast d^{\ast} d^{\ast} d^\ast + d^{\ast} d^{\ast} d^{\ast} d^\ast + d^{\ast} d^{\ast} d^{\ast} d^\ast. \]

We will denote with \( \mathcal{H}_d^{\ast} \) and \( \mathcal{H}_{d^{\Lambda}}^{\ast} \) the kernels of \( \Delta_d \) and \( \Delta_{d^{\Lambda}} \) respectively. All these operators where originally introduced for symplectic manifolds in \([32]\).

Since it will be useful in the following we recall that for the graded bracket \([A,B] = AB - (-1)^{\deg A \deg B} BA\), the graded Jacobi identity holds
\[ [A, [B, C]] = [[A, B], C] + (-1)^{\deg A \deg B} [B, [A, C]]. \]
Proposition 2.6. In the previous assumptions it holds
\[ \Delta_d = \Delta_{d^\Lambda} - [\Lambda, [d, d^{\Lambda^*}]] . \]
In particular, if \([d, d^{\Lambda^*}] = 0\) the kernels of \(\Delta_d\) and \(\Delta_{d^\Lambda}\) coincide, namely for every \(p\) we have
\[ \mathcal{H}_p^0 = \mathcal{H}_p^{d,\Lambda} . \]

Proof. Using \([\Lambda, d^{\Lambda^*}] = d^*\) and \([d, \Lambda] = d^\Lambda\) we obtain
\[ \Delta_d = [d, d^*] = [(d, [\Lambda, d^{\Lambda^*}]) = [[(d, \Lambda), d^{\Lambda^*}] - [\Lambda, [d, d^{\Lambda^*}]] = \Delta_{d^\Lambda} - [\Lambda, [d, d^{\Lambda^*}]] . \]

Remark 2.8. Let \((M, \omega, J, g)\) be an almost-Kähler manifold, then \([d, d^{\Lambda^*}] = 0\) and only if \([d, d^*] = 0\). In such a case \((M, \omega, J, g)\) is a Kähler manifold and we recover the usual equalities for the Laplacians in Propositions 2.6 and 2.7.

3. Hodge theory on HKT manifolds

3.1. Lefschetz spaces on HKT manifolds. We specialize the results of the previous section to the cohomology of HKT manifolds.

Let \((M, I, J, K)\) be a 4\(n\)-dimensional compact hypercomplex manifold and \(\Omega \in A^{2,0}(M)\) a non-degenerate \((2, 0)\)-form on \((M, I)\). Then, as usual we set
\[ L: A^{r,0}(M) \to A^{r+2,0}(M) , \quad L := \Omega \wedge - \]
for the Lefschetz operator.

Then \((A^{r,0}(M), L)\) is a Lefschetz space.

Moreover, if we consider as differential operator \(\partial\) (always taken with respect to \(I\)), since \(I\) is integrable, \(\partial^2 = 0\) and so \((A^{r,0}(M), L, \partial)\) defines a Lefschetz complex. If \(\partial \Omega = 0\) then we have
\[ [\partial, L] = 0 . \]

Denote with \(\mathcal{H}_{p,\partial}^0(M)\) the space of Lefschetz harmonic \((p, 0)\)-forms, i.e. forms \(\alpha \in A^{p,0}(M)\) such that
\[ \partial \alpha = 0 = \partial^\Lambda \alpha , \quad \text{where} \ \partial^\Lambda = [\partial, \Lambda] . \]

We can therefore apply the results of the previous section to infer
Theorem 3.1. Let \((M, I, J, K, \Omega)\) be a compact 4n-dimensional hypercomplex manifold and \(\Omega \in A^{2,0}(M)\) a non-degenerate \((2,0)\)-form on \((M, I)\) such that \(\partial \Omega = 0\). Then \((H^\bullet_{\partial,0}(M), L)\) and \((H^\bullet_{\partial,0}(M), \Lambda)\) satisfy the HLC. Furthermore the following are equivalent:

- \((A^\bullet,0(M), L)\) satisfies the \(\partial \partial^A\)-lemma, i.e.,
  \[
  \text{Ker} \partial \cap \text{Ker} \partial^A \cap (\text{Im} \partial + \text{Im} \partial^A) = \text{Im} \partial \partial^A;
  \]
- There is a Lefschetz harmonic representative in each Dolbeault cohomology class of \(H^\bullet_{\partial,0}(M)\);
- \((H^\bullet_{\partial,0}(M), L)\) satisfies the HLC;
- \((H^\bullet_{\partial,0}(M), \Lambda)\) satisfies the HLC.

Moreover, by the general results in the previous section we obtain

Proposition 3.2. Let \((M, I, J, K, \Omega)\) be a compact 4n-dimensional hypercomplex manifold and \(\Omega \in A^{2,0}(M)\) a non-degenerate \((2,0)\)-form on \((M, I)\) such that \(\partial \Omega = 0\). Then,

\[
\Delta_{\partial} = \Delta_{\partial^A} - [\Lambda, [\partial, \partial^A]];\]

In particular, if \([\partial, \partial^A] = 0\)

\[
\Delta_{\partial} = \Delta_{\partial^A},
\]

and for every \(p\) we have

\[
H^p_{\partial,0}(M) = H^p_{\partial,0}(M).
\]

Moreover, if \([\partial, \partial^A] = 0\)

\[
\Delta_{\partial^A} = \Delta_{\partial^A} + \partial^A \partial^A - \partial^A \partial^A = \Delta_{\partial^A} + \partial^A \partial + \partial^A \partial A.
\]

3.2. Complex Hodge theory on HKT manifolds. If we further assume that \(\Omega\) is \(q\)-positive, in the sense that \(J^4 \Omega = \Omega\) and \(\Omega(Z, JZ) > 0\) for every \(Z \in T^1_{\Omega}(M), Z \neq 0\), then it must be the HKT form corresponding to a HKT metric \(g\) on \((M, I, J, K)\). If \((M, I, J, K, g, \Omega)\) is HKT by [33] we have

\[
[\partial^*, L] = -\partial J + \theta J \wedge-
\]

where \(\theta J\) is a 1-form defined as follows. Since \(\Omega\) is non-degenerate, there exists a \((0,1)\)-form \(\bar{\theta}\) such that

\[
\bar{\partial} \Omega^n = \bar{\theta} \wedge \Omega^n
\]

then, by definition, \(\theta J := J\bar{\theta} \in A^{1,0}(M)\). Notice that \((M, I, J, K, g, \Omega)\) is balanced if and only if \(\theta J = 0\) and so for balanced HKT manifolds we have

\[
[\partial^*, L] = -\partial J.
\]

Moreover recall that a balanced HKT manifold has holonomy in \(\text{SL}(n, \mathbb{H})\). Denoting with \(A\) the adjoint of \(L\), one can easily get by duality or by applying \(J\) the following (see [33])

Proposition 3.3. Let \((M, I, J, K, g, \Omega)\) be a compact balanced HKT manifold. Then, the following identities hold

- \([\partial^*, L] = -\partial J\);
- \([\partial, A] = \partial^*,\]
- \([L, \partial^J] = -\partial\]
- \([\Lambda, \partial J] = \partial^*\).

Now, we shall show that the framework of the previous section can be used to study quaternionic cohomologies. First of all, we set \(J\alpha = J\tilde{\alpha}\) for every \(\alpha \in A^{1,0}(M)\), thus \(J\) is a complex structure on \(A^{1,0}(M)\) and naturally extends to \(A^{p,0}(M)\) by imposing compatibility with the wedge product. Since \(\Omega\) is \(q\)-real we have \(J L = L J\) and we can use (2) to define a Hodge-type operator.

We warn the reader that in this framework the operator defined by (2) slightly differs from the usual Hodge operator. To distinguish them, let us denote here \(* : H^p,0(M) \rightarrow A^{2n-p,0} (M)\) the operator defined in (2) and \(\star : H^p,q(M) \rightarrow A^{2n-p,2n-q} (M)\) the usual Hodge star operator, then one can easily show that for every \(\alpha, \beta \in A^{p,0}(M)\)

\[
\alpha \wedge * \beta = g(\alpha, \beta) \frac{\Omega^n}{n!},
\]
where $g$ here is the Hermitian product induced by the Riemannian metric on $A^{p,0}(M)$, while, by definition,

$$\alpha \wedge \hat{\beta} = g(\alpha, \beta) \frac{\Omega^n \wedge \overline{\Omega^n}}{(n!)^2}.$$  

However, we can identify the formal adjoints of $\partial$ and $\partial_J$ with respect to $\ast$ and $\hat{\ast}$ in the following way.

Suppose $M$ is a $\text{SL}(n, \mathbb{H})$-manifold and fix a $q$-positive $q$-real holomorphic $(2n,0)$-form $\Theta$. Define the following $L^2$-products:

$$(\alpha, \beta)_1 := \int_M g(\alpha, \beta) \frac{\Omega^n \wedge \overline{\Omega^n}}{(n!)^2} = \int_M \alpha \wedge \hat{\beta}, \quad (\alpha, \beta)_2 := \int_M g(\alpha, \beta) \frac{\Omega^n \wedge \overline{\Theta}}{n!} = \int_M \alpha \wedge \ast \beta \wedge \hat{\Theta}.$$  

Then the adjoint of $\partial$ and $\partial_J$ with respect to $(\cdot, \cdot)_1$ are $\partial^* = -\hat{\ast} \partial \ast$ and $\partial_J^* = -\hat{\ast} \partial_J \ast$, while those with respect to $(\cdot, \cdot)_2$ are $\partial^* = -\ast \partial \ast$ and $\partial_J^* = -\ast \partial_J \ast$ (cf. [24]). Since $\Theta$ is $q$-positive, there exists a real-valued function $f > 0$ such that $\Theta = f \frac{\Omega^n}{n!}$, moreover, the holomorphicity of $\Theta$ translates into the condition $\partial f + f \partial = 0$. Now, observe that $(\cdot, \cdot)_2 = (f, \cdot)_1$ thus

$$(\alpha, \partial^* \beta)_2 = (\partial \alpha, \beta)_1 = (\partial(f \alpha) - f \partial \wedge \alpha, \beta)_1 = (\alpha, \partial^* \beta)_2 + (\theta \wedge \alpha, \beta)_2$$  

and similarly, working with $\partial_J^*$ and $\partial_J^*$ one obtains

$$(\alpha, \partial_J^* \beta)_2 = (\alpha, \partial_J^* \beta)_2 - (\theta_J \wedge \alpha, \beta)_2.$$  

In particular if $M$ is balanced then $\theta = \theta_J = 0$ and $f$ is constant, so that the two $L^2$-products coincide up to a constant and $\partial^* = \partial^*$ and $\partial_J^* = \partial_J^*$. In particular the usual Laplacians obtained by means of the Riemannian Hodge star operator coincide with those Laplacians considered in Section 2 and the related results can be applied.

In particular, if $M$ is compact and if $\alpha \in A^{p,0}(M)$ one immediately obtains

$$\begin{cases}
\alpha \in \mathcal{H}^{p,0}_\partial(M) & \iff \partial \alpha = 0, \quad \partial^* \alpha = 0; \\
\alpha \in \mathcal{H}^{p,0}_J(M) & \iff \partial_J \alpha = 0, \quad \partial_J^* \alpha = 0; \\
\alpha \in \mathcal{H}^{p,0}_{BC}(M) & \iff \partial \alpha = 0, \quad \partial_J \alpha = 0, \quad \partial_J^* \partial^* \alpha = 0; \\
\alpha \in \mathcal{H}^{p,0}_A(M) & \iff \partial^* \alpha = 0, \quad \partial_J^* \alpha = 0, \quad \partial \partial_J \alpha = 0.
\end{cases}$$

Proposition 3.3 shows that $\partial^* \alpha = \partial_J^* \alpha$ and we readily obtain from Proposition 2.6

**Proposition 3.4.** Let $(M, I, J, K, \Omega)$ be a compact balanced HKT manifold, then

$$\Delta_{\partial_J} = \Delta_{\hat{\partial}}.$$  

In particular, the spaces of harmonic forms coincide, namely for every $p$ we have

$$\mathcal{H}^{p,0}_{\partial_J}(M) = \mathcal{H}^{p,0}_\partial(M).$$  

**Remark 3.5.** If we do not assume the compact HKT manifold $(M, I, J, K, \Omega)$ to be balanced we would have, in general

$$\partial^* L = -\partial_J + \theta_J \wedge -.$$  

Setting $\tau(\alpha) := \theta_J \wedge \alpha$ and $\psi(\alpha) := \theta \wedge \alpha$, then, we would get

$$\partial^* \alpha = \partial^* + \tau^*.$$  

In particular $[\partial, \partial^*] = 0$ and $\partial \tau^* = -\partial^* \tau^*$ and in such a case the Laplacians $\Delta_\partial$, $\Delta_{\partial^*}$ and $\Delta_{\partial_J}$ do not coincide, and in fact by a direct computation one gets

$$\Delta_{\partial_J} = \Delta_\partial + [\psi^*, \partial] + [\partial_J, \tau^*].$$  

Notice that $\psi^* = \iota_{\theta_J}$ and $\tau^* = \iota_{\theta_J}$.

We also observe that the condition $[\partial, \partial^*] = 0$, i.e. $\partial \theta_J = 0$ is only satisfied when the manifold is balanced, indeed $\partial \theta_J = 0$ is equivalent to $\partial_J \theta = 0$, from which we obtain

$$\partial_J \partial^* = \partial_J(\theta \wedge \Omega^n) = \partial_J \theta \wedge \overline{\Omega^n} - \theta \wedge \partial_J \overline{\Omega^n} = -\theta \wedge J^{-1} \partial \Omega^n$$

$$= -\theta \wedge \overline{J^{-1}(\bar{\theta} \wedge \Omega^n)} = \theta \wedge J \theta \wedge \overline{\Omega^n}.$$
Therefore by integrating we infer
\[ 0 = \int_M \partial_J \bar{\Omega}^n \wedge \partial \Omega^{n-1} = \int_M \partial_J \partial \bar{\Omega}^n \wedge \Omega^{n-1} = \int_M \theta \wedge J \bar{\theta} \wedge \Omega^n \wedge \Omega^{n-1} = \frac{1}{n} \int_M \|\theta\|^2 \Omega^n \wedge \bar{\Omega}^n \]
and the claim follows.

**Remark 3.6.** We can also reinterpret Proposition 3.4 as follows. We first notice that if \((M, I, J, K, \Omega)\) is a compact balanced HKT manifold. Then, \([\partial, \partial_J]\) = 0. Hence, by Propositions 2.6 and 2.7 we obtain that
\[ \Delta_{\partial} = \Delta_{\partial J} \]
and for every \(p\) we have
\[ \mathcal{H}^p(M) = \mathcal{H}^p_{\partial J}(M). \]
Moreover,
\[ \Delta_{\partial J}^\mathrm{BC} = \Delta_{\partial J} \Delta_{\partial J} + \partial^* \partial + \partial_J^* \partial_J = \Delta_{\partial J}^\mathrm{BC} + \partial^* \partial J - \partial_J^* \partial J. \]

As a consequence of Proposition 3.4, we obtain isomorphisms for the associated cohomology groups (cf. [24, Proposition 2.3] where it is noticed that an isomorphism, induced by \(J\) and conjugation with respect to \(I\), holds in general for hypercomplex manifolds).

**Corollary 3.7.** Let \((M, I, J, K, \Omega)\) be a compact HKT balanced manifold, then
\[ H^{p,0}_{\partial J}(M) \simeq H^p_{\partial J}(M). \]
In particular, we have the equalities
\[ h^{p,0}_{\partial J}(M) = h^p_{\partial J}(M). \]
Invoking Proposition 2.7 we obtain that, similarly to the Kähler case, the Laplacians \(\Delta_{\partial J}^\mathrm{BC}\) and \(\Delta_{\partial J} = \Delta_{\partial}\) are related.

**Proposition 3.8.** Let \((M, I, J, K, \Omega)\) be a compact HKT balanced manifold, then
\[ \Delta_{\partial J} = \Delta_{\partial J} \Delta_{\partial J} + \partial^* \partial + \partial_J^* \partial_J = \Delta_{\partial J}^\mathrm{BC} + \partial^* \partial J - \partial_J^* \partial J. \]
In particular, the spaces of harmonic forms coincide, namely for every \(p\) we have
\[ \mathcal{H}^p_{\partial J}(M) = \mathcal{H}^p_{\partial J}(M). \]
Consequently we obtain isomorphisms for the associated cohomology groups.

**Corollary 3.9.** Let \((M, I, J, K, \Omega)\) be a compact balanced HKT manifold, then for every \(p\),
\[ H^{p,0}_{\partial J}(M) \simeq H^p_{\partial J}(M) \simeq H^p_{\partial J}(M). \]
In particular, we have the equalities
\[ h^{p,0}_{\partial J}(M) = h^p_{\partial J}(M) = h^p_{\partial J}(M). \]
Notice that these results are the analogue of the ones proved in [27] for compact Kähler manifolds.

As a consequence of the previous results we prove that, under the same hypothesis, the Hard Lefschetz condition holds for the cohomologies \(H^i_{\partial J}(M)\), \(H^i_{\partial J}(M)\), \(H^i_{\partial J}(M)\), thus generalizing [9, Proposition 4.7].

**Theorem 3.10.** Let \((M, I, J, K, \Omega)\) be a compact 4n-dimensional balanced HKT manifold, then for every \(i\),
\[ L^{n-i} : \mathcal{H}^{i,0}_{\partial J}(M) \to \mathcal{H}^{2n-i,0}_{\partial J}(M), \]
\[ L^{n-i} : \mathcal{H}^{i,0}_{\partial J}(M) \to \mathcal{H}^{2n-i,0}_{\partial J}(M), \]
\[ L^{n-i} : \mathcal{H}^{i,0}_{\partial J}(M) \to \mathcal{H}^{2n-i,0}_{\partial J}(M) \]
are isomorphisms.
In particular,
\[ h^{i,0}_{\partial J} = h^{i,0}_{\partial J} = h^{i,0}_{\partial J} = h^{2n-i,0}_{\partial J} = h^{2n-i,0}_{\partial J}. \]
Proof. In view of Propositions 3.4, 3.8 it is sufficient to prove that
\[ L^{n-i} : H^i_\partial(M) \to H^{2n-i,0}_\partial(M) \]
are isomorphisms. Notice that by hypothesis \( \partial \Omega = \partial J \Omega = 0 \), hence
\[ [\partial, L] = 0, \quad [\partial J, L] = 0. \]
Let \( \alpha \in H^i_\partial(M) = H^i_\partial(M) \). Then,
\[ \partial \alpha = 0, \quad \partial J \alpha = 0, \quad \partial^* \alpha = 0, \quad \partial^* J \alpha = 0. \]
As a consequence
\[ \partial(L^{n-i} \alpha) = L^{n-i} \partial \alpha = 0, \]
and, using \([\partial^*, L] = -\partial J\),
\[ \partial^*(L^{n-i} \alpha) = L^{n-i} \partial^* \alpha - (n-i)L^{n-i-1} \partial J \alpha = 0. \]
Hence, \( L^{n-i} \alpha \in H^{2n-i,0}_\partial(M) \). The result follows from \( \Omega \) being non-degenerate. \( \square \)

Notice that combining this result with Theorem 3.1 we have

Corollary 3.11. Let \((M, I, J, K, \Omega)\) be a compact 4\( n \)-dimensional balanced HKT manifold, then it satisfies the \( \partial \partial^* \)-lemma and there exists a Lefschetz harmonic representative in each Dolbeault cohomology class of \( H^*_\partial(M) \).

Remark 3.12. Notice that, in general, on a compact \( 4n \)-dimensional HKT manifold \((M, I, J, K, \Omega)\) we cannot expect the HLC for \( H^i_\partial\). Indeed, there are examples of HKT manifolds with \( K_{(M, I)} \) holomorphically non-trivial and \( H^2_\partial = H^{2n,0}_\partial \simeq H^0_\partial \simeq H^2_\partial = \{0\} \) (e.g. quaternionic Hopf surfaces). But HLC would imply that
\[ L^n : H^0_\partial(M) \to H^0_\partial(M), \]
is an isomorphism, which is absurd.

Proposition 3.13. Let \((M, I, J, K, \Omega)\) be a compact 4\( n \)-dimensional balanced HKT manifold, then for every \( p \) we have
\[ H^p_{BC}(M) = H^p_A(M). \]
Proof. We first show the inclusion \( H^p_{BC}(M) \subseteq H^p_A(M) \). Let \( \alpha \in H^p_{BC}(M) \). By Propositions 3.4, 3.8 \( \alpha \in H^p_\partial(M) = H^p_\partial(M) \), namely
\[ \partial \alpha = 0, \quad \partial J \alpha = 0, \quad \partial^* \alpha = 0, \quad \partial^* J \alpha = 0. \]
Hence, \( \alpha \in H^p_A(M) \). The opposite inclusion \( H^p_A(M) \subseteq H^p_{BC}(M) \) follows from Theorem 3.10 and [20, Remark 21], indeed for every \( p \),
\[ h^p_{BC}(M) = \overline{h}^{2n-p,0}(M) = h^p_A(M). \]
As a corollary we have

Corollary 3.14. Let \((M, I, J, K, \Omega)\) be a compact balanced HKT manifold, then for every \( p \),
\[ H^p_{BC}(M) \simeq H^p_A(M). \]
4. Formality of HKT manifolds

It is well known that formality in the sense of Sullivan is an obstruction to Kählerianity, more precisely compact complex manifolds satisfying the $\partial\bar{\partial}$-lemma are formal (see [12]). However, notice that the HKT condition does not imply formality, indeed there are examples of non tori nilmanifolds that are HKT but it is well known that non tori nilmanifolds are not formal in the sense of Sullivan [22].

In this section we study formality for compact hypercomplex manifolds. We first recall some definitions. Let $(\mathcal{A}, d_A)$ and $(\mathcal{B}, d_B)$ be two differential graded algebras (DGA for short) over a field $\mathbb{K}$. A DGA-homomorphism between $\mathcal{A}$ and $\mathcal{B}$ is a $\mathbb{K}$-linear map $f : \mathcal{A} \rightarrow \mathcal{B}$ such that

\begin{align*}
\text{i)} & \quad f(A^i) \subset B^i; \\
\text{ii)} & \quad f(\alpha \cdot \beta) = f(\alpha) \cdot f(\beta); \\
\text{iii)} & \quad d_B \circ f = f \circ d_A.
\end{align*}

Any DGA-homomorphism $f : (\mathcal{A}, d_A) \rightarrow (\mathcal{B}, d_B)$ induces a DGA-homomorphism in cohomology

$$H(f) : (H^\bullet(\mathcal{A}, d_A), 0) \rightarrow (H^\bullet(\mathcal{B}, d_B), 0).$$

A DGA-homomorphism $f : (\mathcal{A}, d_A) \rightarrow (\mathcal{B}, d_B)$ is called quasi-isomorphism if $H(f)$ is an isomorphism.

Two DGA $(\mathcal{A}, d_A)$ and $(\mathcal{B}, d_B)$ are said to be equivalent if there exists a sequence of quasi-isomorphisms of the following form:

$$(\mathcal{A}, d_A) \overset{\sim}{\longrightarrow} (\mathcal{B}, d_B).$$

A DGA $(\mathcal{A}, d_A)$ is called formal if $(\mathcal{A}, d_A)$ is equivalent to a DGA $(\mathcal{B}, d_B = 0)$.

We show now that for a compact hypercomplex manifold $M$ of the following form:

$$(C_1, d_{C_1}) \overset{\sim}{\longrightarrow} (C_2, d_{C_2}) \overset{\sim}{\longrightarrow} \cdots \overset{\sim}{\longrightarrow} (C_n, d_{C_n}) \overset{\sim}{\longrightarrow} (B, d_B).$$

A compact hypercomplex manifold $(M, I, J, K)$ is formal if $(\mathcal{A}, d_A) = (M, I, J, K)$ is formal by the following

**Theorem 4.1.** Let $(M, I, J, K)$ be a compact hypercomplex manifold satisfying the $\partial\bar{\partial}$-lemma, then the DGA $(\mathcal{A}, d_A)$ is formal.

In order to prove this Theorem, we will need three lemmas.

**Lemma 4.2.** Let $(M, I, J, K)$ be a compact hypercomplex manifold satisfying the $\partial\bar{\partial}$-lemma, then the natural inclusion

$$i : (A^{\bullet, 0}(M) \cap \operatorname{Ker} \partial_J, \partial) \rightarrow (A^{\bullet, 0}(M), \partial)$$

is a DGA quasi-isomorphism.

**Proof.** Notice that $(A^{\bullet, 0}(M) \cap \operatorname{Ker} \partial_J, \partial)$ is a DGA and the inclusion

$$i : (A^{\bullet, 0}(M) \cap \operatorname{Ker} \partial_J, \partial) \rightarrow (A^{\bullet, 0}(M), \partial)$$

is a morphism of DGAs. We are left to prove that the map induced in cohomology

$$H\partial(i) : H\partial(A^{\bullet, 0}(M) \cap \operatorname{Ker} \partial_J, \partial) \rightarrow H\partial(A^{\bullet, 0}(M), \partial)$$

is an isomorphism.

We first prove that $H\partial(i)$ is injective. Fix $k$, and let $[\alpha] \in H\partial(A^{k, 0}(M) \cap \operatorname{Ker} \partial_J, \partial)$ such that $H\partial(i)([\alpha]) = [\alpha]_{\partial J} = 0$, hence

$$\alpha \in \operatorname{Ker} \partial_J \cap \operatorname{Im} \partial = \operatorname{Im} \partial J$$

i.e., $\alpha = \partial (\partial_J \beta)$ for some form $\beta \in A^{k-2, 0}(M)$ and clearly $\partial_J \beta \in A^{k-1, 0}(M) \cap \operatorname{Ker} \partial_J$, hence

$$[\alpha] = 0 \in H\partial(A^{k, 0}(M) \cap \operatorname{Ker} \partial_J, \partial)$$

and so $H\partial(i)$ is injective.

We now prove that $H\partial(i)$ is surjective. Let $a \in H\partial(A^{k, 0}(M), \partial)$, $a = [\alpha]$ with $\partial \alpha = 0$. Consider,

$$\partial_J a \in \operatorname{Im} \partial_J \cap \operatorname{Ker} \partial = \operatorname{Im} \partial J$$

and so $a \in H\partial(A^{k, 0}(M) \cap \operatorname{Ker} \partial_J, \partial)$, hence $H\partial(i)$ is surjective.
hence $\partial_j\alpha = \partial_j\partial\beta$ for some $\beta$. Therefore, $\partial_j(\alpha - \partial\beta) = 0$ and $\partial(\alpha - \partial\beta) = \partial\alpha = 0$. This means that $\alpha - \partial\beta$ defines a class in $H_{\partial_j}^{k,0}(A^{\bullet,0}M \cap \text{Ker} \partial_j, \partial)$ and

$$H_{\partial}(i)([\alpha - \partial\beta]) = [\alpha - \partial\beta]_{\partial} = [\alpha] = a,$$

concluding the proof. \hfill \square

**Lemma 4.3.** Let $(M, I, J, K)$ be a compact hypercomplex manifold satisfying the $\partial \partial_j$-lemma, then the natural projection

$$p : (A^{\bullet,0}(M) \cap \text{Ker} \partial_j, \partial) \rightarrow \left(H_{\partial_j}^{\bullet,0}(M), \partial\right)$$

is a DGA quasi-isomorphism.

**Proof.** Notice that the projection

$$p : (A^{\bullet,0}(M) \cap \text{Ker} \partial_j, \partial) \rightarrow \left(H_{\partial_j}^{\bullet,0}(M), \partial\right)$$

is a morphism of DGAs. We are left to prove that the map induced in cohomology

$$H_{\partial}(p) : H_{\partial_j}^{\bullet,0}(A^{\bullet,0}(M) \cap \text{Ker} \partial_j, \partial) \rightarrow H_{\partial_j}^{\bullet,0}(H_{\partial_j}^{\bullet,0}(M), \partial)$$

is an isomorphism. We first prove that $H_{\partial}(p)$ is injective. Fix $k$, and let $[\alpha] \in H_{\partial_j}^{k,0}(A^{\bullet,0}(M) \cap \text{Ker} \partial_j, \partial)$ such that $H_{\partial}(p)([\alpha]) = 0$. Hence,

$$\alpha \in \text{Im} \partial \cap \text{Ker} \partial_j = \text{Im} \partial \partial_j,$$

i.e., $\alpha = \partial(\partial_j\beta)$ for some form $\beta \in A^{k-2,0}(M)$ and clearly $\partial_j\beta \in A^{k-1,0}(M) \cap \text{Ker} \partial_j$, hence

$$[\alpha] = 0 \in H_{\partial_j}^{\bullet,0}(H_{\partial_j}^{\bullet,0}(M), \partial)$$

and so $H_{\partial}(p)$ is injective. The surjectivity of $H_{\partial}(p)$ is immediate. \hfill \square

**Lemma 4.4.** Let $(M, I, J, K)$ be a compact hypercomplex manifold satisfying the $\partial \partial_j$-lemma, then $\partial$ is the trivial operator on $H_{\partial_j}^{\bullet,0}(M)$.

**Proof.** Fix $k$ and let $a = [\alpha]_{\partial_j} \in H_{\partial_j}^{k,0}(M)$, namely $\partial_j\alpha = 0$. Now

$$\partial a = [\partial\alpha]_{\partial_j}$$

and

$$\partial\alpha \in \text{Im} \partial \cap \text{Ker} \partial_j = \text{Im} \partial \partial_j,$$

so $\partial\alpha = \partial_j\partial\beta$ for some $\beta$, giving $\partial a = [\partial_j\partial\beta]_{\partial_j} = 0 \in H_{\partial_j}^{k+1,0}(M)$, concluding the proof. \hfill \square

Now we are able to prove Theorem 4.1

**Proof.** Under the assumptions and as a consequence of Lemmas 4.2, 4.3, 4.4, we have the following diagram of quasi-isomorphisms of DGAs,

$$
\begin{array}{ccc}
(A^{\bullet,0}(M) \cap \text{Ker} \partial_j, \partial) & \xrightarrow{i_{\text{qis}}} & (A^{\bullet,0}(M), \partial) \\
& & \xleftarrow{\text{qis}} \downarrow p \\
& & (H_{\partial_j}^{\bullet,0}(M), 0)
\end{array}
$$

hence, by definition, $(A^{\bullet,0}(M), \partial)$ is a formal DGA. \hfill \square

As a consequence of [20, Theorem 6] and Theorem 4.1, we obtain

**Corollary 4.5.** Let $(M, I, J, K, \Omega)$ be a compact HKT $\text{SL}(n, \mathbb{H})$-manifold, then the DGA $(A^{\bullet,0}(M), \partial)$ is formal.

We recall now the definition of triple Massey products of a DGA in our setting.
Definition 4.6. Let $a = [\alpha] \in H^{p,0}_\partial(M)$, $b = [\beta] \in H^{r,0}_\partial(M)$ and $c = [\gamma] \in H^{s,0}_\partial(M)$ such that $a \cup b = 0 \in H^{p+q,0}_\partial(M)$ and $b \cup c = 0 \in H^{q+r,0}_\partial(M)$; more precisely suppose that $\alpha \wedge \beta = \partial \lambda$ and $\beta \wedge \gamma = \partial \mu$ for some $\lambda \in A^{p+q-1,0}$, $\mu \in A^{q+r-1,0}$. The triple $\partial$-Massey product of $a, b, c$ is defined as

$$\langle a, b, c \rangle := [\lambda \wedge \gamma - (-1)^p \alpha \wedge \mu] \in \frac{H^{p+q+r-1,0}_\partial(M)}{H^{p+q,0}_\partial + H^{q+r,0}_\partial + H^{p+r-1,0}_\partial + H^{p+q+r-1,0}_\partial}.$$ 

Then, since for a formal DGA the associated Massey products vanish we have the following

Corollary 4.7. Let $(M, I, J, K)$ be a compact hypercomplex manifold satisfying the $\partial\bar{\partial}$-lemma, then the triple $\partial$-Massey products vanish.

Hence, we have

Theorem 4.8. Let $(M, I, J, K, \Omega)$ be a compact HKT $SL(n, \mathbb{H})$-manifold, then the triple $\partial$-Massey products vanish.

In particular, triple $\partial$-Massey products are an obstruction to the existence of a HKT $SL(n, \mathbb{H})$-structure on a compact hypercomplex manifold. More precisely,

Corollary 4.9. Let $(M, I)$ be a $4n$-dimensional compact complex manifold such that there exists a non trivial $\partial$-Massey product, then $(M, I)$ does not admit any complex structures $J, K$ such that $(M, I, J, K)$ is hypercomplex and admits a HKT $SL(n, \mathbb{H})$-structure.

Notice that, in fact, by [3] if a nilmanifold $N$ admits an invariant HKT structure $(I, J, K, \Omega)$ then the complex structures $I, J, K$ are abelian and in such a case the triple $\partial$-Massey products are trivial. Indeed, we prove in general the following

Theorem 4.10. Let $N = \Gamma \backslash G$ be a $2n$-dimensional nilmanifold and let $I$ be an invariant abelian complex structure on $N$. Then, the triple $\partial$-Massey products are all zero.

Proof. Since $I$ is an invariant abelian complex structure on $N$, there exists a co-frame of invariant $(1,0)$-forms $\{\varphi^i\}_{i=1,...,n}$ on $(N, I)$ such that

$$\partial \varphi^i = 0, \quad \text{for } i = 1, \ldots, n.$$ 

Since $I$ is abelian, by [11] the Dolbeault cohomology of $N$ can be computed using only invariant forms, hence

$$H^{p,0}_\partial(N) \cong H^{r,0}_\partial(\mathfrak{g}^C) = \langle \varphi^{i_1} \wedge \ldots \wedge \varphi^{i_r} \rangle_{1 \leq i_1 \ldots i_r \leq n}$$

for $r = 1, \ldots, n$, where, denoting with $\mathfrak{g} = \text{Lie}(G)$, $H^{\bullet,\bullet}_\partial(\mathfrak{g}^C)$ denotes the cohomology of the differential bigraded algebra $\Lambda^{\bullet,\bullet}(\mathfrak{g}^C)$ with respect to the operator $\partial$. In order to construct a triple $\partial$-Massey product let $a = [\alpha] \in H^{p,0}_\partial(N)$, $b = [\beta] \in H^{r,0}_\partial(N)$ such that $a \cup b = 0 \in H^{p+q,0}_\partial(N)$, hence $a \cup b = 0 \in H^{p+q,0}_\partial(\mathfrak{g}^C)$, namely there exists an invariant $(p + q - 1,0)$-form $\lambda$ such that

$$\alpha \wedge \beta = \partial \lambda.$$ 

But, on invariant $(r,0)$-forms the operator $\partial$ vanishes and so we can take the primitive $\lambda = 0$ itself. A similar conclusion is obtained taking the third class in the definition of $\partial$-Massey products. This means that we cannot construct non trivial $\partial$-Massey products since both $\lambda$ and $\mu$ in the definition of $\partial$-Massey products would be zero.

An immediate consequence of this result combined with [6] Theorem 4.6 is the following

Theorem 4.11. Let $N = \Gamma \backslash G$ be a $4n$-dimensional nilmanifold and let $(I, J, K, \Omega)$ be an invariant HKT structure on $N$. Then, the triple $\partial$-Massey products are all zero.

Therefore, a relevant application of Corollary 4.9 can be given on solvmanifolds.
Example 4.12. Consider the 8-dimensional almost abelian Lie algebra $\mathfrak{g}$ with structure equations

$$[e_8, e_2] = e_4, \quad [e_8, e_3] = e_5.$$ 

Let $G$ be the associated solvable simply connected Lie group. Then, by [10] $G$ admits a lattice $\Gamma$ such that $S := \Gamma \backslash G$ is a solvmanifold. Define the complex structure setting as global co-frame of $(1,0)$-forms

$$\varphi^1 = e^1 + ie^8, \quad \varphi^2 = e^2 + ie^3, \quad \varphi^3 = e^4 + ie^5, \quad \varphi^4 = e^6 + ie^7.$$ 

The complex structure equations become

$$d\varphi^1 = d\varphi^2 = d\varphi^3 = 0, \quad d\varphi^3 = \frac{i}{2} \varphi^{12} + \frac{i}{2} \varphi^{21}.$$ 

Note that the form $\varphi_{1234}$ is closed, so $S$ has holomorphically trivial canonical bundle. We now construct a non trivial triple $\partial$-Massey product. Take $[\varphi^1] \in H^1_{\partial}(S)$, $[\varphi^2] \in H^1_{\partial}(S)$ and $[\varphi^3] \in H^1_{\partial}(S)$. Notice that $\varphi^1 \wedge \varphi^2 = \partial(-2i \varphi^3)$ and $\varphi^2 \wedge \varphi^3 = 0$. Hence, the $\partial$-Massey product is given by

$$[-2i \varphi^3 \wedge \varphi^2] \in \frac{H^7_{\partial}(S)}{[\varphi^1] \cup H^1_{\partial}(S) + H^3_{\partial}(S) \cup [\varphi^3]},$$

and this class is clearly non trivial. Therefore, by Corollary [24] the complex manifold $(S, I)$ does not admit any complex structures $J, K$ such that the solvmanifold $(S, I, J, K)$ is hypercomplex and admits a SL$(2, \mathbb{H})$ HKT structure.

The next two examples show that the converse of Corollary [7] (and hence Theorem 4.8) does not hold in general. The first one is a compact HKT manifold which is not SL$(n, \mathbb{H})$, while the second one is SL$(n, \mathbb{H})$ but does not admit any HKT metric.

Example 4.13. Consider SU$(3)$ equipped with a homogeneous hypercomplex structure $(I, J, K)$ as constructed in [23, 29]. There is an HKT metric on SU$(3)$ compatible with this hypercomplex structure [21, 26]. By [28] the holonomy of the Obata connection on SU$(3)$ is GL$(2, \mathbb{H})$ and, in fact, we claim that the $\partial \partial_I$-lemma cannot hold on SU$(3)$.

To see this, we observe that from [18], there exists a unitary co-frame $\{Z^1, \ldots, Z^4\}$ of $(1,0)$-forms (with respect to $I$) on the Lie algebra of SU$(3)$ such that the HKT form is

$$\Omega = Z^{12} + Z^{34} = \frac{1}{2} \partial Z^2.$$ 

Now, if the $\partial \partial_I$-lemma hold we would have that $\Omega = \partial \partial_I f$ for some function $f$, but since the HKT form is $q$-positive, by E. Hopf’s maximum principle $f$ would be constant and thus $\Omega = 0$ which is a contradiction.

On the other hand the triple $\partial$-Massey products are all zero because the same coframe satisfies

$$\partial Z^1 = 0, \quad \partial Z^2 = 2Z^{12} + 2Z^{34}, \quad \partial Z^3 = (1 + 3i)Z^{13}, \quad \partial Z^4 = (1 - 3i)Z^{14},$$

which shows that $H^1_{\partial}(M) \simeq (Z^1)$ and $H^3_{\partial}(M) = 0$ for $i > 1$.

Example 4.14. Consider the nilmanifold $M = \Gamma \backslash G$ whose structure equations of the Lie algebra $\mathfrak{g}$ of $G$ are given by (see [21] Example 1)

$$de^1 = de^2 = de^3 = de^4 = de^5 = 0, \quad de^6 = e^{12} + e^{34}, \quad de^7 = e^{13} - e^{24}, \quad de^8 = e^{14} + e^{23},$$

where we use the standard notation $e^{ij} = e^i \wedge e^j$. Define the following hypercomplex structure

$$Ie^1 = e^2, \quad Ie^3 = e^4, \quad Ie^5 = e^6, \quad Ie^7 = e^8,$$

$$Je^1 = e^3, \quad Je^2 = -e^4, \quad Je^5 = e^7, \quad Je^6 = -e^8.$$ 

Then a co-frame for invariant $(1,0)$-forms with respect to $I$ on $M$ is given by

$$\varphi^1 = e^1 - ie^2, \quad \varphi^2 = e^3 - ie^4, \quad \varphi^3 = e^5 - ie^6, \quad \varphi^4 = e^7 - ie^8$$

and the complex structure equations become

$$d\varphi^1 = d\varphi^2 = 0, \quad d\varphi^3 = -\frac{1}{2}(\varphi^1 \varphi^2 + \varphi^2 \varphi^1), \quad d\varphi^4 = \varphi^1.$$
Since the hypercomplex structure is not abelian it does not admit any compatible HKT metric (see also [24]). The conjugate Dolbeault cohomology in bidegree $(p,0)$ is given by

\[ H^{1,0}_\partial(M) \cong \langle \varphi, \varphi^2, \varphi^3 \rangle, \quad H^{2,0}_\partial(M) \cong \langle \varphi^{13}, \varphi^{23}, \varphi^{14}, \varphi^{24} \rangle, \]

We now construct a non trivial triple $\partial$-Massey product. Take $[\varphi^1] \in H^{1,0}_\partial(M)$, $[\varphi^2] \in H^{1,0}_\partial(M)$ and $[\varphi^3] \in H^{1,0}_\partial(M)$. Notice that $\varphi^1 \wedge \varphi^2 = \partial \varphi^1$ and $\varphi^2 \wedge \varphi^3 = 0$. Hence, the $\partial$-Massey product is given by

\[ [\varphi^4 \wedge \varphi^2] \in \frac{H^{2,0}_\partial(M)}{[\varphi^1] \cup H^{1,0}_\partial(M) + H^{1,0}_\partial(M) \cup [\varphi^2]} , \]

and this class is clearly non trivial.

5. Balanced HKT solvmanifolds

The following result can be seen as a generalization of [6] Proposition 4.11] where it is proven that for a hyperhermitian nilmanifold with abelian hypercomplex structure the metric is balanced. Notice that in the nilpotent case the HKT assumption is automatic due to [14].

**Theorem 5.1.** Let $(\Gamma\backslash G, I, J, K)$ be a $4n$-dimensional solvmanifold with an invariant abelian hypercomplex structure. Then, every invariant hyperhermitian metric $g$ is balanced.

**Proof.** Let $g$ be an invariant hyperhermitian metric on $\Gamma\backslash G$, then by [14] $g$ is HKT. We will denote with $(I, J, K, \Omega, g)$ the induced structure on $G$. Since $\Omega$ is HKT the Bismut connections associated to $I, J, K$ coincide and we will denote them uniquely with $\nabla^B$. Since $\Gamma\backslash G$ is a solvmanifold then $G$ is unimodular. Hence, by [6] Lemma 2.4] the Lee form $\tau_J$ associated to $(J, g)$ is given by

\[ \tau_J(X) = \text{tr} \left( \frac{1}{2} J \nabla^B_{jX} \right) \]

for any $X \in \mathfrak{g}$. Now we argue as in the proof of [6] Proposition 4.11] to show that $\tau_J = 0$ and so $g$ is balanced with respect to $J$. The argument is similar for $I$ and $K$.

Let $X_1, JX_1, KX_1, \cdots, X_n, JX_n, KX_n$ be an orthonormal basis of $\mathfrak{g}$. Now using that $\nabla^B$ preserves $I, J, K$ and that $g$ is quaternionic Hermitian we have

\[
\begin{align*}
\text{tr} \left( J \nabla^B_{jX} \right) &= \sum_{j=1}^{n} g(J \nabla^B_{jX} X_j, X_j) + \sum_{j=1}^{n} g(J \nabla^B_{jX} I X_j, I X_j) \\
&\quad + \sum_{j=1}^{n} g(J \nabla^B_{jX} J X_j, J X_j) + \sum_{j=1}^{n} g(J \nabla^B_{jX} K X_j, K X_j) \\
&= \sum_{j=1}^{n} g(J \nabla^B_{jX} X_j, X_j) + \sum_{j=1}^{n} g(J I \nabla^B_{jX} X_j, I X_j) \\
&\quad + \sum_{j=1}^{n} g(J K \nabla^B_{jX} X_j, K X_j) \\
&= \sum_{j=1}^{n} g(I J \nabla^B_{jX} X_j, X_j) - \sum_{j=1}^{n} g(J I \nabla^B_{jX} X_j, I X_j) \\
&\quad + \sum_{j=1}^{n} g(I K \nabla^B_{jX} X_j, K X_j) \\
&= \sum_{j=1}^{n} g(J \nabla^B_{jX} X_j, X_j) - \sum_{j=1}^{n} g(J \nabla^B_{jX} J X_j, J X_j) \\
&\quad + \sum_{j=1}^{n} g(I \nabla^B_{jX} J X_j, X_j) - \sum_{j=1}^{n} g(I \nabla^B_{jX} K X_j, K X_j) \\
&= \sum_{j=1}^{n} g(J \nabla^B_{jX} X_j, X_j) - \sum_{j=1}^{n} g(J \nabla^B_{jX} J X_j, X_j) \\
&\quad + \sum_{j=1}^{n} g(I \nabla^B_{jX} J X_j, X_j) - \sum_{j=1}^{n} g(I \nabla^B_{jX} K X_j, X_j) = 0.
\end{align*}
\]
Corollary 5.2. Let $(\Gamma \backslash G, I, J, K)$ be a 4n-dimensional solvmanifold with an invariant abelian hypercom-
plex structure. Suppose that there exists an HKT structure $\Omega$ on $(\Gamma \backslash G, I, J, K)$. Then, there exists a
balanced abelian HKT structure on $\Gamma \backslash G$.

Proof. By [15] there exists an invariant HKT structure $\hat{\Omega}$ on $(\Gamma \backslash G, I, J, K)$. Now, by Theorem 5.1 we have
that $\hat{\Omega}$ is balanced. □

Remark 5.3. Notice that, differently from the nilpotent case (cf. [6]), the converse of Theorem 5.1 is not true. Indeed, in [7] it is provided an example of a balanced HKT solvmanifold with an hypercomplex structure that is not abelian.

In [34] Verbitsky showed that an $SL(n, \mathbb{H})$-manifold has holomorphically trivial canonical bundle. Andrada and Tolcachier [4] found a counterexample to the converse exhibiting an hypercomplex solvmanifold that is not $SL(n, \mathbb{H})$ but admits a non-invariant holomorphic section of the canonical bundle. We indeed now show that the $SL(n, \mathbb{H})$ condition is equivalent to the existence of an invariant holomorphic section of the canonical bundle.

Theorem 5.4. Let $(M := \Gamma \backslash G, I, J, K, g)$ be a hypercomplex solvmanifold, then the holonomy of the
Obata connection $\nabla$ is contained in $SL(n, \mathbb{H})$ if and only if the canonical bundle admits an invariant holomorphic section.

Proof. Assume $M$ is a $SL(n, \mathbb{H})$-manifold, then there exists a q-positive holomorphic section $\eta$ of the
canonical bundle. We claim that $\eta$ must be invariant. To prove this we argue along the lines of [16, Proposition 2.1]. Let $\Theta$ be an invariant q-positive section of the canonical bundle (a priori not holomorphic). Since both $\eta$ and $\Theta$ are q-real and q-positive there exists a positive real-valued function $f$ such that $\eta = f\Theta$. Therefore $0 = \partial\eta = \partial f \wedge \Theta + f \partial \Theta$, i.e. $\partial \Theta = -\bar{\partial}(\log f) \wedge \Theta$ because $f$ is positive. Since $\Theta$ is invariant so is $\partial \Theta$ and thus there exists an invariant $(0,1)$-form $\alpha$ such that $\bar{\partial}(\log f) = \alpha$. We now apply the well-known Belgun’s symmetrization process [9] that for any $k$-form $\beta$ on $M$ returns an invariant $k$-form $\mu(\beta)$. Since the hypercomplex structure is invariant $\mu$ preserves $\bar{\partial}$ and we obtain
$$\bar{\partial}(\log f) = \alpha = \mu(\alpha) = \mu(\bar{\partial}(\log f)) = \bar{\partial}\mu(\log f) = 0$$

because the symmetrization of a function is constant. In particular $f$ is constant, showing that $\eta = f\Theta$ is invariant.

Conversely, let $\eta$ be an invariant nowhere vanishing holomorphic section of the canonical bundle. The fact that $\text{Hol}(\nabla) \subseteq SL(n, \mathbb{H})$, follows from the fact that $\eta$ is parallel with respect to $\nabla$, which can be proved along the lines of [34, Theorem 3.2]. □

Now we prove the following

Theorem 5.5. Let $(M := \Gamma \backslash G, I, J, K, g)$ be a solvmanifold with an invariant holomorphic section of
the canonical bundle and invariant HKT structure. Then $g$ is balanced.

Proof. Let $\bar{\eta}$ be an invariant non-vanishing $\partial$-closed section of $A^{0,2n}(M)$, hence
$$\bar{\Omega}^n = c\bar{\eta}$$

with $c$ constant. Since $d\bar{\eta} = 0$, then $d\bar{\Omega}^n = 0$ and so $\bar{\partial}\bar{\Omega}^n = 0$ proving that $g$ is balanced. □

As a consequence we confirm the conjecture by Alesker and Verbitsky on solvmanifolds with invariant hypercomplex structure.

Theorem 5.6. Let $(M := \Gamma \backslash G, I, J, K)$ be a $SL(n, \mathbb{H})$-solvmanifold with invariant hypercomplex structure. Suppose that there exists an HKT metric on $M$. Then there exists a balanced HKT structure on $M$.

Proof. Since $(M := \Gamma \backslash G, I, J, K, g)$ is an HKT solvmanifold with a $SL(n, \mathbb{H})$ structure, then the canonical bundle of $M$ has an invariant section and, by [15], there exists an invariant HKT structure on $M$. Hence, by the previous result the associated Hermitian metric is balanced. □
References

[1] S. Alesker, E. Shelukhin, A uniform estimate for general quaternionic Calabi problem (with appendix by Daniel Barlet), Adv. Math. 316 (2017), 1–52.
[2] S. Alesker, M. Verbitsky, Plurisubharmonic functions on hypercomplex manifolds and HKT-geometry, J. Glob. Anal. 16 (2006), 375–399.
[3] S. Alesker, M. Verbitsky, Quaternionic Monge-Ampère equations and Calabi problem for HKT-manifolds, Israel J. Math. 176 (2010), 109–138.
[4] A. Andrada, A. Toliccacher, On the canonical bundle of complex solvmanifolds and applications to hypercomplex geometry, e-print arXiv:2307.16673.
[5] B. Banos, A. Swann, Potentials for Hyper-Kähler Metrics with Torsion, Classical and Quantum Gravity 21(13) (2004), 3127–3135.
[6] M. L. Barberis, I. Dotti, M. Verbitsky, Canonical bundles of complex nilmanifolds, with applications to hypercomplex geometry, Math. Res. Lett. 16 (2009), no. 2, 331–347.
[7] M. L. Barberis, A. Fino, New HKT manifolds arising from quaternionic representations, Math. Z. 267 (2011), 717–735.
[8] L. Bedulli, G. Gentili, L. Vezzoni, A parabolic approach to the Calabi-Yau problem in HKT geometry, Math. Z., 302 (2022), 917–933.
[9] F. A. Belgun, On the metric structure of non-Kähler complex surfaces, Math. Ann., 317 (2000), no. 1, 1–40.
[10] C. Bock, On low-dimensional solvmanifolds, Asian J. Math. 20 (2016), no. 2, 199–262.
[11] S. Console, A. Fino, Dolbeault cohomology of compact nilmanifolds, Transform. Groups 6 (2001), no. 2, 111–124.
[12] P. Deligne, Ph. A. Griffiths, J. Morgan, D. P. Sullivan, Real homotopy theory of Kähler manifolds, Invent. Math. 20 (1975), no. 3, 245–274.
[13] S. Dinew, M. Sroka, On the Alesker-Verbitsky conjecture on hyperKähler manifolds. Geom. Funct. Anal. 33 (2023), no.4, 875–911.
[14] I. Dotti, A. Fino, Hyperkähler torsion structures invariant by nilpotent Lie groups, Class. Quantum Gravity 19 (2002), 551–562.
[15] A. Fino, G. Grantcharov, Properties of manifolds with skew-symmetric torsion and special holonomy, Adv. in Math. 189(2) (2004), 429–450.
[16] A. Fino, A. Otal, L. Ugarte, Six-dimensional solvmanifolds with holomorphically trivial canonical bundle. Int. Math. Res. Not. IMRN 2015, no. 24, 13757–13799.
[17] G. Gentili, L. Vezzoni, The quaternionic Calabi conjecture on abelian hypercomplex nilmanifolds viewed as tori fibrations, Int. Math. Res. Not. IMRN 2022, no. 12, 9499–9528.
[18] G. Gentili, L. Vezzoni, A remark on the quaternionic Monge-Ampère equation on foliated manifolds, Proc. Amer. Math. Soc., 151 (2023), 1263–1275.
[19] G. Gentili, J. Zhang, Fully non-linear elliptic equations on compact manifolds with a flat hyperkähler metric, J. Geom. Anal. 32 (2022), no. 9, Paper No. 229, 38 pp.
[20] G. Grantcharov, M. Lejmi, M. Verbitsky, Existence of HKT metrics on hypercomplex manifolds of real dimension 8, Adv. Math. 320 (2017), 1135–1157.
[21] G. Grantcharov, Y. S. Poon, Geometry of hyperKähler connections with torsion, Comm. Math. Phys. 213(1) (2000), 19–37.
[22] K. Hasegawa, Minimal models of nilmanifolds, Proc. Amer. Math. Soc., 106 (1989), 65–71.
[23] D. Joyce, Compact hypercomplex and quaternionic manifolds, J. Differential Geom. 35 (1992), no. 3, 743–761.
[24] M. Lejmi, P. Weiler, Quaternionic Bott-Chern cohomology and existence of HKT metrics, q. J. Math. 68 (2017), no. 3, 705–728.
[25] M. Obata, Affine connections on manifolds with almost complex, quaternionic or Hermitian structures, Japan. J. Math. 26 (1956), 43–79.
[26] A. Opfermann, G. Papadopoulos, Homogeneous HKT and QKT manifolds, e-print arXiv:math-ph/9907026.
[27] M. Schweitzer, Autour de la cohomologie de Bott-Chern, arXiv:0709.3528v1 [math. AG].
[28] A. Soldatenkov, Holonomy of the Obata connection on SU(3), Int. Math. Res. Not. IMRN 2012, no. 15, 3483–3497.
[29] Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen, Extended supersymmetric σ-models on group manifolds. I. The complex structures. Nuclear Phys. B 308 (1988), no. 2-3, 662–698.
[30] M. Sroka, The C0 estimate for the quaternionic Calabi conjecture, Adv. Math. 370 (2020), 107237.
[31] A. Tomassini, X. Wang, Some results on the hard Lefschetz condition, Internat. J. Math. 29 (2018), no. 13, 1850095.
[32] L.-S. Tseng, S.-T. Yau, Cohomology and Hodge Theory on Symplectic manifolds: I, J. Differ. Geom. 91 (2012), no. 3, 383–416.
[33] M. Verbitsky, HyperKähler manifolds with torsion, supersymmetry and Hodge theory, Asian J. Math. 6 (2002), no. 4, 679–712.
[34] M. Verbitsky, Hypercomplex manifolds with trivial canonical bundle and their holonomy. Moscow Seminar on Mathematical Physics. II, 203–211, Amer. Math. Soc. Transl. Ser. 2, 221, Adv. Math. Sci., 60, Amer. Math. Soc., Providence, RI, 2007.
[35] M. Verbitsky, Balanced HKT metrics and strong HKT metrics on hypercomplex manifolds, *Math. Res. Lett.* **16** (2009), no. 4, 735–752.

[36] X. Wang, Notes on variation of Lefschetz star operator and T-Hodge theory, (2017), e-print arXiv:1708.07332.

[37] A. Weil, Introduction à l’Étude des Variété Kählériennes, *Publications de l’Institut de Mathématique de l’Université de Nancago* VI, Hermann, Paris, 1958.

[38] J. Zhang, Parabolic quaternionic Monge-Ampère equation on compact manifolds with a flat hyperKähler metric, *J. Korean Math. Soc.* **59** (2022), no.1, 13–33.

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