A FAMILY OF RANDOM WALKS WITH GENERALIZED DIRICHLET STEPS

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Abstract. We analyze a class of continuous time random walks in $\mathbb{R}^d$, $d \geq 2$, with uniformly distributed directions. The steps performed by these processes are distributed according to a generalized Dirichlet law. Given the number of changes of orientation, we provide the analytic form of the probability density function of the position $\{X_d(t), t > 0\}$ reached, at time $t > 0$, by the random motion. In particular, we analyze the case of random walks with two steps.

For suitable values for the basic parameters of the generalized Dirichlet probability distribution, we are able to obtain the explicit density functions of $\{X_d(t), t > 0\}$. Furthermore, by exploiting fractional Poisson processes, the unconditional probability distributions are obtained. The paper also contains some considerations on the first exit time of a slightly modified version of the above processes. This paper extends in a more general setting, the random walks with Dirichlet displacements introduced in some previous papers.

1. Introduction

Several authors over the years analyzed continuous time non-Markovian random walks, that describe motions with uniformly distributed directions on the unit sphere. These stochastic processes are called “Pearson random walks” or equivalently “random flights”. Many real phenomena can be studied by taking into account these random motions, i.e. cell motility and statistical physics problems.

Several papers (see, for instance, Stadje, 1987, and Orsingher and De Gregorio, 2007) analyze random flights in the multidimensional real spaces. In these works the main assumption concerns the underlying homogeneous Poisson process governing the changes of direction of the random walk. Poisson paced times imply that the time lapses are exponentially distributed. Unfortunately, under these assumptions, the explicit distribution of the random flight is obtained only in two spaces: $\mathbb{R}^2$ and $\mathbb{R}^4$.

In the last years, the research in this field is mainly devoted to the analysis of random motions with non-uniformly distributed lengths of the time displacements between consecutive changes of direction. Indeed, the exponentially distributed steps assign high probability mass to short intervals and for this reason they are not suitable for many important applications in physics, biology, and engineering. For example, Beghin and Orsingher (2010) introduced a random motion which changes of direction at even-valued Poisson events; this implies that the time between successive deviations is a Gamma random variable. This model can also be interpreted as the motion of particles that can hazardously collide with obstacles of different size, some of which are capable of deviating the motion. Recently, multidimensional random walks with Gamma intertimes have been also taken into account by Pogorui and Rodriguez-Dagnino (2011), (2013), Le Caër (2010) and De Gregorio and Orsingher (2012) considered the joint distribution of the time displacements as Dirichlet random variables with parameters depending on the space in which the random walker performs its motion. The Dirichlet law assigns higher probability to
time displacement with intermediate length and permits us to explicit (for suitable choices of the parameters), for each space $\mathbb{R}^d$, $d \geq 2$, the exact probability distribution of the position reached by the random motion at time $t > 0$.

De Gregorio (2012) dealt with a random flight in $\mathbb{R}^d$ with Dirichlet steps and non-uniformly distributed directions. Multidimensional random motions with a finite number of possible directions have also been studied by Di Crescenzo (2002), Lachal (2006) and Lachal et al. (2006). Ghosh et al. (2011) proved limit theorems for directionally reinforced random walks in higher dimensions.

The aim of this paper is to analyze the multidimensional random flights with generalized Dirichlet displacements. The generalized Dirichlet distribution, introduced by Connor and Mosimann (1969), has more flexible structure than classical Dirichlet distribution. This assumption leads to a family of random motions which contains, as particular cases, the processes studied in De Gregorio and Orsingher (2012) in the classical Dirichlet setting.

We describe the class of random walks studied in this paper. Let us consider a particle or a walker which starts from the origin of $\mathbb{R}^d$, $d \geq 2$, and performs its motion with a constant velocity $c > 0$. We indicate by $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ the random instants at which the random walker changes direction and denote the length of time separating these instants by $\tau_k = t_k - t_{k-1}$, $k \geq 1$. Let $N(t) = \sup\{k \geq 1 : t_k \leq t\}$ be the (random) number of times in which the random motion changes direction during the interval $[0, t]$. If, at time $t > 0$, one has that $N(t) = n$, with $n \geq 1$, the random motion has performed $n + 1$ steps. We observe that $\tau_n = (\tau_1, \ldots, \tau_n) \in S_n$, where $S_n$ represents the open simplex

$$S_n = \left\{ (\tau_1, \ldots, \tau_n) \in \mathbb{R^n} : 0 < \tau_k < t - \sum_{j=0}^{k-1} \tau_j, k = 1, 2, \ldots, n \right\},$$

with $\tau_0 = 0$ and $\tau_{n+1} = t - \sum_{j=1}^{n} \tau_j$.

The directions of the particle are represented by the points on the surface of the $d$-dimensional sphere with radius one. We denote by $\theta_{d-1} = (\theta_1, \ldots, \theta_{d-2}, \phi)$ the random vector (independent from $\tau_n$) representing the orientation of the particle; we assume that $\theta_{d-1}$ has uniform distribution on the $d$-dimensional unit sphere $\partial S^d_1$. Then, the probability density function of $\theta_{d-1}$ is equal to

$$\varphi(\theta_{d-1}) = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}} \sin^{d-2} \theta_1 \sin^{d-3} \theta_2 \cdots \sin \theta_{d-2},$$

where $\theta_k \in [0, \pi], k \in \{1, \ldots, d - 2\}$, and $\phi \in [0, 2\pi]$. Furthermore the particle chooses a new direction independently from the previous one.

Let us denote with $\{X_i(t), t > 0\}$ the process representing the position reached, at time $t > 0$, by the particle moving randomly according to the rules described above. The position $X_i(t) = (X_1(t), \ldots, X_d(t))$, is the main object of investigation. By setting $N(t) = n$, the random walk $\{X_i(t), t > 0\}$, can be described in the following manner

$$(1.2) \quad X_i(t) = c \sum_{k=1}^{n+1} Y_k \tau_k,$$

where $Y_k, k = 1, 2, \ldots, n + 1$, are independent $d$-dimensional random vectors defined as follows

$$Y_k = \begin{pmatrix} V_{1,k} \\ V_{2,k} \\ \vdots \\ V_{d-1,k} \\ V_{d,k} \end{pmatrix} = \begin{pmatrix} \cos \theta_{1,k} \\ \sin \theta_{1,k} \cos \theta_{2,k} \\ \sin \theta_{1,k} \sin \theta_{2,k} \cos \phi_k \\ \sin \theta_{1,k} \sin \theta_{2,k} \cdots \sin \theta_{d-2,k} \cos \phi_k \\ \sin \theta_{1,k} \sin \theta_{2,k} \cdots \sin \theta_{d-2,k} \sin \phi_k \end{pmatrix}.$$
and \((\theta_{1,k}, \theta_{2,k}, ..., \theta_{d-2,k}, \phi_k)\) has distribution (1.1).

A crucial role is played by the random vector \(\tau_d\). We assume that the intervals \(\tau_k\) have the following joint density function

\[
f(\tau_n; a_n, b_n, t) = C(a_n, b_n, t) \prod_{k=1}^{n} \left( \tau_k^{a_k} (t - \sum_{j=1}^{k} \tau_j)^{b_k-1} \right), \quad \tau_n \in S_n,
\]

where

\[
C(a_n, b_n, t) = \frac{1}{\Gamma(t) \prod_{k=1}^{n} \Gamma(a_k) \Gamma(b_k + \sum_{j=1}^{k} (a_j + b_j - 1))},
\]

with \(a_n = (a_1, a_2, ..., a_n), b_n = (b_1, b_2, ..., b_n)\) and \(a_1, ..., a_n, b_1, ..., b_n > 0\). The density function \(f(\tau_n; a_n, b_n, t)\) represents a rescaled generalized Dirichlet distribution (see (2.1) in Chang et al., 2010, and references therein) that we indicate by \(GD(a_n; b_n)\). By setting \(b_n = (1, 1, ..., 1, b_n)\), (1.3) becomes

\[
\Gamma\left(\sum_{k=1}^{n} a_k + b_n\right) \prod_{k=1}^{n} \Gamma(a_k) \prod_{k=1}^{n} \Gamma(b_k + \sum_{j=1}^{k} (a_j + b_j - 1)) \prod_{k=1}^{n} \tau_k^{a_k} (t - \sum_{k=1}^{n} \tau_k)^{b_n-1},
\]

which, for \(t = 1\), is the well-known Dirichlet distribution. If \(a_n = b_n = (1, ..., 1)\), \(GD(a_n; b_n)\) becomes the uniform distribution \(n!/tn\) in the simplex \(S_n\), appearing in the case of Poisson intertimes.

It is worth to point out that the stochastic process treated here can be represented by means of the triple \((\eta_{d-1}, \tau_n, \mathcal{N}(t))\) of independent random vectors where \(\eta_{d-1} = (\eta_1, ..., \eta_{d-2}, \phi)\) is the orientation of displacements (with uniform law (1.1)), \(\tau_n = (\tau_1, ..., \tau_n)\) represents the displacements (with distribution \(GD(a_n; b_n)\)) and \(\mathcal{N}(t)\) is the number of changes of direction. In the models analyzed by Orsingher and De Gregorio (2007), \(\eta_{d-1}\) has law coinciding with (1.1), \(\tau_n\) is uniformly distributed and \(\mathcal{N}(t)\) is a homogeneous Poisson process. Actually, the assumption (1.3) leads to a whole class of random walks \(\{X_d(t), t > 0\}\) (in order to avoid an heavy notation we omit the dependence on \(a_n\) and \(b_n\)). This family of processes contains, as particular case, the Dirichlet random motions analyzed in Le Caër (2010) and De Gregorio and Orsingher (2012). Furthermore, the sample paths of (1.2) appear like joined straight lines representing the randomly oriented steps with random lengths.

The paper is organized as follows. By assuming \(\mathcal{N}(t) = n\), in Section 2 we provide the expression of the density function of the process (1.2) in integral form. For \(n = 1\), some explicit results are obtained. Section 3 gives the exact probability distributions of the random motion \(\{X_d(t), t > 0\}\), when the vectors \(a_n\) and \(b_n\) are suitably chosen. We obtain several distributions and the related random walks are compared. We will show that sometimes different choices of the basic parameters \(a_n\) and \(b_n\) lead to processes identically distributed. By resorting the fractional Poisson process (see Beghin and Orsingher, 2010) or equivalently the weighted Poisson process (as observed in Beghin and Macci, 2012), in Section 4 we obtain the unconditional density functions for the motions treated in Section 3. The last section contains some remarks on the first exit time.

2. General results

In this section we assume that the number of steps performed by the random walk is fixed and equal to \(n+1\), with \(n \geq 1\) (i.e. \(\mathcal{N}(t) = n\)). Here we consider the general case in which the random vector \(\tau_n\), representing the random displacements between consecutive deviations, follows the generalized Dirichlet distribution (1.3).
2.1. The general case. Let $\mathbf{x}_d = (x_1, \ldots, x_d)$ and $\alpha_d = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$. Let us indicate by $\|\mathbf{x}_d\| = \sqrt{\sum_{k=1}^{d} x_k^2}$ and $\langle \alpha_d, \mathbf{x}_d \rangle = \sum_{k=1}^{d} \alpha_k x_k$ the Euclidean distance and the scalar product, respectively. The conditional characteristic function of $\{\mathbf{x}_d(t), t > 0\}$ is denoted by

$$F_n(\alpha_d) = \mathbb{E} \left\{ e^{i \langle \alpha_d, \mathbf{x}_d(t) \rangle} \mid \mathcal{N}(t) = n \right\}.$$ 

We remark that at time $t > 0$, the random flight $\{\mathbf{x}_d(t), t > 0\}$ with at least two steps, is located inside the ball $B_{ct}^d = \{\mathbf{x}_d \in \mathbb{R}^d : \|\mathbf{x}_d\| < ct\}$ with center the origin of $\mathbb{R}^d$ and radius $ct$. The first result concerns the density function of $\{\mathbf{x}_d(t), t > 0\}$.

**Theorem 1.** Given $\mathcal{N}(t) = n, n \geq 1$, the density function of $\{\mathbf{x}_d(t), t > 0\}$ is equal to

$$p_n(\mathbf{x}_d, t) = \left\{ \frac{2^{\frac{d}{2}} - 1}{(2\pi)^{\frac{d}{2}} |\mathbf{x}_d|^{\frac{d}{2} - 1}} \right\}^{n+1} \int_0^{\infty} \rho^{\frac{d}{2}} J_{\frac{d}{2} - 1}(\rho |\mathbf{x}_d|) d\rho \int_{S_n} f(\mathcal{I}_d; \mathbf{a}_n, \mathbf{b}_n, t) \prod_{k=1}^{n+1} \left\{ \frac{J_{\frac{d}{2} - 1}(c_r |\alpha_d|)}{(c_r |\alpha_d|)^{\frac{d}{2} - 1}} \right\} \prod_{k=1}^{n} dr_k,$$

(2.1)

where $\mathbf{x}_d \in B_{ct}^d$ and

$$J_{\nu}(x) = \sum_{k=0}^{\infty} (-1)^k \left( \frac{x}{2} \right)^{2k+\nu} \frac{1}{k! \Gamma(k+\nu+1)}, \quad x, \nu \in \mathbb{R},$$

is the Bessel function.

**Proof.** Let us start the proof showing that under the assumption (1.3), the characteristic function of $\mathbf{x}_d(t)$ is equal to

$$F_n(\alpha_d) = \left\{ \frac{2^{\frac{d}{2}} - 1}{(2\pi)^{\frac{d}{2}} |\mathbf{x}_d|^{\frac{d}{2} - 1}} \right\}^{n+1} \int_{S_n} f(\mathcal{I}_d; \mathbf{a}_n, \mathbf{b}_n, t) \prod_{k=1}^{n+1} \left\{ \frac{J_{\frac{d}{2} - 1}(c_r |\alpha_d|)}{(c_r |\alpha_d|)^{\frac{d}{2} - 1}} \right\} \prod_{k=1}^{n} dr_k.$$

(2.2)

We can write that

$$F_n(\alpha_d) = \int_{S_n} f(\mathcal{I}_d; \mathbf{a}_n, \mathbf{b}_n, t) \mathcal{I}_n(\alpha_d; \mathcal{I}_d) \prod_{k=1}^{n} d\tau_k,$$

where

$$\mathcal{I}_n(\alpha_d; \mathcal{I}_d) = \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{d+1} \prod_{k=1}^{n} \left\{ \exp \left\{ ic \tau_k - \langle \alpha_d, \mathbf{V}_k \rangle \right\} \right\} \left\{ \frac{\Gamma(\frac{d}{2})}{(2\pi)^{\frac{d}{2}}} \sin \theta_{d-k}^2 \cdots \sin \theta_{d-2-k}^2 \right\}.$$

(2.3)

It is known that the integral $\mathcal{I}_n(\alpha_d; \mathcal{I}_d)$ (see formula (2.5) in De Gregorio and Orsingher, 2012) is equal to

$$\mathcal{I}_n(\alpha_d; \mathcal{I}_d) = \left\{ \frac{2^{\frac{d}{2}} - 1}{(2\pi)^{\frac{d}{2}} |\mathbf{x}_d|^{\frac{d}{2} - 1}} \right\}^{n+1} \prod_{k=1}^{n+1} \left\{ \frac{J_{\frac{d}{2} - 1}(c_r |\alpha_d|)}{(c_r |\alpha_d|)^{\frac{d}{2} - 1}} \right\},$$

and this leads to (2.2).

Now, by inverting the characteristic function (2.2), we are able to show that the density of the process $\{\mathbf{x}_d(t), t > 0\}$, is given by (2.1). Let us denote by

$$\Theta = \{(\theta_1, \ldots, \theta_{d-2}, \phi) \in \mathbb{R}^{d-1} : \theta_i \in [0, \pi], \phi \in [0, 2\pi], i = 1, \ldots, d - 2\}$$
and by $\psi_d$ the vector

$$\psi_d = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \cos \theta_2 \\ \vdots \\ \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{d-2} \cos \phi \\ \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{d-2} \sin \phi \end{pmatrix}.$$  

Therefore, by inverting the characteristic function (2.2) and by passing to the (hyper)spherical coordinates, we have, for $\mathbf{s}_d \in H_d^{(n)}$, that

$$p_n(\mathbf{s}_d, t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \omega, \mathbf{s}_d \rangle} f_n(\mathbf{s}_d) d\omega_1 \cdots d\omega_d$$

$$= \frac{1}{(2\pi)^d} \int_0^\infty \rho^{d-1} d\rho \int_{\Theta} e^{-i\rho \langle \omega, \mathbf{s}_d \rangle} \sin^{d-2} \theta_1 \cdots \sin \theta_{d-2} d\theta_1 \cdots d\theta_{d-2}$$

$$\times \left\{ \frac{2^{d-1} \Gamma \left( \frac{d}{2} \right) }{ \Gamma \left( \frac{d}{2} - 1 \right) } \right\}^{n+1} \int_{S_n} f(\tau_n; \mathbf{a}_n, \mathbf{b}_n, t) \prod_{k=1}^{n+1} \left\{ \frac{J_{d-1}(c\tau_k \rho)}{J_{d-1}(c\tau_k \rho) \frac{d}{2} - 1} \right\} \prod_{k=1}^{n} d\tau_k.$$

By means of formula (2.12) in De Gregorio and Orsingher (2012)

$$\int_0^\infty e^{-i\rho \langle \omega, \mathbf{s}_d \rangle} \sin^{d-2} \theta_1 \cdots \sin \theta_{d-2} d\theta_1 \cdots d\theta_{d-2} = (2\pi)^{\frac{d}{2}} \frac{J_{d-1}(\rho ||\mathbf{s}_d||)}{\rho ||\mathbf{s}_d||^{\frac{d}{2} - 1}},$$

and this concludes the proof. $\square$

**Remark 2.1.** Actually, it is possible to obtain the density function of (1.2) also if $\tau_n$ possesses an arbitrary density $g(\tau_n; t)$ on the simplex $S_n$. In this case, by means of the same arguments used in the proof of Theorem 1, we can check that the density function of (1.2) is given by the expression (2.1) where in place of $f(\tau_n; \mathbf{a}_n, \mathbf{b}_n, t)$ appears $g(\tau_n; t)$.

**Remark 2.2.** The random process $\{X_d(t), t > 0\}$ represents an isotropic random motion. The density function $p_n(\mathbf{x}_d, t)$ is rotationally invariant and it depends on the distance $||\mathbf{x}_d||$. Then we can write $p_n(\mathbf{x}_d, t) = p_n(||\mathbf{x}_d||, t)$. Furthermore, as consequence of the isotropy, the distribution of the radial process $\{R_d(t), t > 0\}$, where $R_d(t) = ||\mathbf{x}_d(t)||$, becomes

$$r^{d-1} p_n(r, t) \text{meas}(B_1^d), \quad 0 < r < ct,$$

where $\text{meas}(B_1^d) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.

2.2. Random walks with two steps. In general it is not possible to calculate explicetely the density function (2.1). Nevertheless some results can be obtained by setting $n = 1$. In other words we consider a random motion $\{X_d(t), t > 0\}$, which, at time $t > 0$, has performed one deviation (or two displacements). In this case the generalized Dirichlet distribution (1.3) becomes a Beta distribution with parameters $a_1$ and $b_1$. The next result represents a generalization of Theorem 2.3 in Orsingher and De Gregorio (2007).

**Theorem 2.** For $n = 1$, we have that

$$p_1(\mathbf{x}_d, t) = \frac{1}{2^d \pi^{\frac{d}{2}}} \frac{\Gamma \left( \frac{d}{2} \right)^2}{\Gamma \left( \frac{d}{2} - 1 \right)} \frac{\Gamma(a_1 + b_1)}{\Gamma(a_1) \Gamma(b_1)} \frac{1}{p_{a_1+b_1-1} ||\mathbf{x}_d||^{d-2} 2^{d-4}}$$

$$\times \int_{\frac{d}{2} - \frac{||\mathbf{x}_d||}{2}}^{\frac{d}{2} + \frac{||\mathbf{x}_d||}{2}} \tau_1^{a_1-d+1} (t - \tau_1)^{b_1-d+1} [4c^2 \tau_1 (t - \tau_1) - c^2 t^2 + ||\mathbf{x}_d||^2]^{\frac{d-2}{4}} d\tau_1$$

with $\mathbf{x}_d \in B_{ct}^d$. 

Corollary 3. If \( a_1 = b_1 = a \), we obtain that

\[
p_1(\mathbf{x}, t) = \frac{1}{2^{2a-d+1}\pi^{d/2}} \frac{\Gamma\left(\frac{d}{2}\right) \Gamma(2a) (c^2t^2 - ||\mathbf{x}||^2)^{d/2}}{\Gamma(a)^2} 2F_1\left(d - 1 - a, \frac{1}{2}, \frac{1}{2}; \frac{||\mathbf{x}||^2}{c^2t^2}\right), \quad \mathbf{x} \in \mathbb{B}_{ct}^d,
\]

where the hypergeometric function \( 2F_1(\alpha, \beta, \gamma; z) \) is defined as follows

\[
2F_1(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\gamma-1}(1-t)^{\beta-1}(1-tz)^{-\alpha} dt,
\]

for \( \Re \gamma > \Re \beta > 0, |z| < 1 \).

Proof. We can write (2.7) in the following alternative form

\[
p_1(\mathbf{x}, t) = \frac{1}{2^{2a-d+1}\pi^{d/2}} \frac{\Gamma\left(\frac{d}{2}\right) \Gamma(2a) (c^2t^2 - ||\mathbf{x}||^2)^{d/2}}{\Gamma(a)^2} 2F_1\left(d - 1 - a, \frac{1}{2}, \frac{1}{2}; \frac{||\mathbf{x}||^2}{c^2t^2}\right)
\]

In the last step above, we applied the successive substitutions \( \tau_1 = \frac{t}{2} = y \) and \( 2cy = z||\mathbf{x}|| \). From (2.8), if \( a_1 = b_1 = a \), by means of the position \( z^2 = w \), we obtain that

\[
p_1(\mathbf{x}, t) = \frac{1}{2^{2a-d+1}\pi^{d/2}} \frac{\Gamma\left(\frac{d}{2}\right) \Gamma(2a) (c^2t^2 - ||\mathbf{x}||^2)^{d/2}}{\Gamma(a)^2} 2F_1\left(d - 1 - a, \frac{1}{2}, \frac{1}{2}; \frac{||\mathbf{x}||^2}{c^2t^2}\right)
\]

with \( \mathbf{x} \in \mathbb{B}_{ct}^d \). □

Remark 2.3. From (2.9), for \( a = d = 1 \) we derive the following simplified density function

\[
p_1(\mathbf{x}, t) = \frac{1}{\pi^{d/2}} \frac{\Gamma\left(\frac{d}{2}\right) \Gamma(2(d-1)) (c^2t^2 - ||\mathbf{x}||^2)^{d/2}}{\Gamma(d-1/2)^2},
\]

while for \( a = d \) we obtain that

\[
p_1(\mathbf{x}, t) = \frac{1}{2^{d-1}\pi^{d/2}} \frac{\Gamma\left(\frac{d}{2}\right) \Gamma(2d) (c^2t^2 - ||\mathbf{x}||^2)^{d/2}}{\Gamma(d)^2} \frac{2F_1\left(d - 1, \frac{1}{2}, \frac{3}{2}; \frac{||\mathbf{x}||^2}{c^2t^2}\right)}{c^2t^2},
\]

Furthermore, since \( 2F_1\left(\frac{1}{2}, \frac{3}{2}, \frac{3}{2}; z^2\right) = \arcsin z \), (see Lebedev, 1972, pag.259, formula (9.8.5)) for \( d = 3 \) and \( a = 3/2 \), we have that

\[
p_1(\mathbf{x}, t) = \frac{2}{c^2t^2} \frac{\arcsin\left(\frac{||\mathbf{x}||}{c}\right)}{||\mathbf{x}||}.
\]

It is worth to point out that \( \mathbb{R}^3 \) represents a suitable environment for analyzing the random flight (at least for \( n = 1 \)). Indeed, for \( d = 3 \) the formula (2.7) becomes

\[
p_1(\mathbf{x}, t) = \frac{1}{2^3\pi} \frac{1}{c^2||\mathbf{x}||} \frac{\Gamma(a_1 + b_1)}{\Gamma(a_1) \Gamma(b_1)} \frac{1}{t^{a_1+b_1-1}} \int_{\frac{||\mathbf{x}||}{c}}^{\frac{||\mathbf{x}||}{c(t-\tau_1)}} r_1^{a_1-2}(t - \tau_1)^{b_1-2}d\tau_1,
\]
where $\mathbf{x}_3 \in B^3_{ct}$. In some cases, we are able to obtain the explicit form for the density function (2.10). For $a_1 \neq 1$ and $b_1 = 2$, we get

$$p_1(\mathbf{x}_3,t) = \frac{1}{2\pi(2ct)^{a_1+1}} \frac{1}{||\mathbf{x}_3||} \frac{a_1(a_1+1)}{a_1-1} [(ct+||\mathbf{x}_3||)^{a_1-1} - (ct-||\mathbf{x}_3||)^{a_1-1}]$$

and if $a_1 = 2$ the above result allows to the uniform distribution inside the three-dimensional ball with radius $ct$, that is

$$p_1(\mathbf{x}_3,t) = \frac{1}{2^2\pi (ct)^3}.$$

By setting $a_1 = 1$ and $b_1 = 2$ in (2.10), we derive that

$$p_1(\mathbf{x}_3,t) = \frac{1}{\pi(2ct)^2||\mathbf{x}_3||} \log \left( \frac{ct+||\mathbf{x}_3||}{ct-||\mathbf{x}_3||} \right),$$

which coincides with the distribution obtained in Orsingher and De Gregorio (2007) (see (2.27b) in Theorem 2.4) for uniformly distributed intertimes.

We observe that (2.10), can be also expressed as follows

$$p_1(\mathbf{x}_3,t) = \frac{1}{2\pi c^2 t^2 ||\mathbf{x}_3||} B(a_1,b_1)$$

$$\times \left[ B \left( \frac{1}{2} + \frac{||\mathbf{x}_3||}{2ct}; a_1 - 1, b_1 - 1 \right) - B \left( \frac{1}{2} - \frac{||\mathbf{x}_3||}{2ct}; a_1 - 1, b_1 - 1 \right) \right]$$

where $B(x; a, b) = \int_0^1 z^{a-1}(1-z)^{b-1}dz$ represents the incomplete Beta function, whereas $B(a, b)$ is the standard Beta function, for $a, b > 0$. It is well-known that the incomplete beta function can be expressed in terms of distribution function of a binomial random variable. In other words, the following relationship holds

$$B(x; a, b) = \sum_{k=a}^{a+b-1} \binom{a+b-1}{k} x^k (1-x)^{a+b-1-k}, \quad a, b \in \mathbb{N}.$$ 

If $a_1 - 1, b_1 - 1 \in \mathbb{N}$, we have that (2.12) turns out

$$p_1(\mathbf{x}_3,t) = \frac{1}{2\pi c^2 t^2 ||\mathbf{x}_3||} \frac{1}{(a_1 + b_1 - 1)(a_1 + b_1 - 2)} \sum_{k=a_1-1}^{a_1+b_1-3} \binom{a_1 + b_1 - 3}{k}$$

$$\times \left[ \left( \frac{1}{2} + \frac{||\mathbf{x}_3||}{2ct} \right)^{a_1+b_1-3-k} - \left( \frac{1}{2} - \frac{||\mathbf{x}_3||}{2ct} \right)^{a_1+b_1-3-k} \right]$$

$$= \frac{1}{2^{a_1+b_1} \pi (ct)^{a_1+b_1-1} ||\mathbf{x}_3||} \frac{1}{(a_1 + b_1 - 1)(a_1 + b_1 - 2)} \sum_{k=a_1-1}^{a_1+b_1-3} \binom{a_1 + b_1 - 3}{k}$$

$$\times \left[ (ct+||\mathbf{x}_3||)^{a_1+b_1-3-k} - (ct-||\mathbf{x}_3||)^{a_1+b_1-3-k} \right],$$

where $\mathbf{x}_3 \in B^3_{ct}$.

3. Explicit probability density functions

From Theorem 1 emerges that in order to explicit the density function of \{$X_t(t), t > 0$\}, we should be able to calculate the following integral

$$\int_{S_n} f(\mathbf{x}_3; \mathbf{b}_n, \mathbf{d}_n, t) \prod_{k=1}^{n+1} \left( \frac{J_{2^{-1}}(ct\rho)}{(ct\rho)^{2^{-1}}} \right) \prod_{k=1}^{n} d\tau_k.$$
appearing in (2.1) or equivalently we should calculate the \( n \)-fold integral appearing in the characteristic function (2.2).

In general it is not possible to obtain the exact value of (3.1) (or (2.2)). Nevertheless, for some values of the parametric vectors \( a_n \) and \( b_n \) of the generalized Dirichlet distribution (1.3), the integral (3.1) (or (2.2)) can be worked out. Therefore in this section we study the random walks derived from the suitable choices of the parameters \( a_n \) and \( b_n \). In particular we consider two different families of “solvable random walks”. In our context with the terminology “solvable random walks”, we mean random motions, with a fixed number of steps \( n + 1 \), with isotropic density function of the following type

\[
A(c^2 t^2 - \|X_t\|^2)^b,
\]

where \( b \) is a constant depending on \( n \) and \( d \), while \( A \) is the necessary normalizing factor (depending on \( t \)). Clearly the solvable random walks represent a sub-family of the general class \( \{X_t(t), t > 0\} \).

We will use in the proof below the same approach developed in De Gregorio and Orsingher (2012). For this reason we need the following formulae

\[
\int_0^a x^\mu(a - x)^\nu J_\nu(a - x)dx = \frac{\Gamma(\mu + \frac{1}{2})\Gamma(\nu + i + \frac{1}{2})}{\sqrt{\pi}2^\nu\Gamma(\mu + \nu + i + 1)}a^{\mu+i+\frac{1}{2}}J_{\mu+i+\frac{1}{2}}(a),
\]

where \( i = 0,1 \), and \( Re \mu > -\frac{1}{2}, \ Re \nu > -\frac{i+1}{2} \) (see Gradshteyn and Ryzhik, 1980, page 743, formulae 6.581(3)-(4)), and

\[
\int_0^a \frac{J_\mu(x)J_\nu(a - x)}{x(a - x)^i}dx = \frac{\left(\frac{1}{\mu} + \frac{i}{\nu}\right)J_{\mu+i}(a)}{\frac{a^i}{i!}},
\]

where \( i = 0,1 \), and \( Re \mu > 0, Re \nu > -(i+1) \) (see Gradshteyn and Ryzhik, 1980, page 678, formulae 6.533(1)-(2)).

3.1. Solvable processes of first type. Let us denote by \( X_{d}^{h,i,j} = \{X_{d}^{h,i,j}(t), t > 0\} \) the random motion (1.2), with \( a_n \), with density function \( GD(a_n, b_n) \) with parameters \( a_n = (a_1, \ldots, a_k, \ldots, a_n) \) and \( b_n = (b_1, \ldots, b_k, \ldots, b_n) \) defined respectively as follows

\[
a_k = \begin{cases} 
    d - 1, & k \in \{1, \ldots, j\}, \\
    \frac{d}{2} - 1, & k \in \{j + 1, \ldots, n\},
\end{cases}
\]

\[
b_k = \begin{cases} 
    1, & k \in \{1, \ldots, n - 1\} \setminus \{j\}, \\
    \frac{(n - j + 1)(\frac{d}{2} - 1) + i + h + 1}{\frac{d}{2} - i}, & k = j, \\
    \frac{d}{2} - i, & k = n,
\end{cases}
\]

with \( j \in \{0, 1, \ldots, n - 1\} \), \( i, h \in \{0, 1\} \) and \( d \geq 3 \). For \( j = 0 \), we have that \( a_n = (\frac{d}{2} - 1, \ldots, \frac{d}{2} - 1) \) and \( b_n = (1, \ldots, 1, \frac{d}{2} - i) \), that is the generalized Dirichlet law reduces to the standard Dirichlet distribution. The random motions \( X_{d}^{h,i,j} \) represent a class of random walks where each combination of the indexes \( h, i, j \) defines a different process. In other words, the assumptions (3.5) and (3.6) lead to a family of processes \( X_{d}^{h,i,j} \) with \( 4n \) elements.

**Theorem 4.** Fixed \( j \in \{0, 1, \ldots, n - 1\} \) we have that the random motions \( X_{d}^{0,0,j}, X_{d}^{1,0,j}, X_{d}^{0,1,j}, X_{d}^{1,1,j} \) are identically distributed.

**Proof.** Let us start by calculating the characteristic function

\[
\mathcal{F}_{X_{d}^{h,i,j}}(\alpha) = \mathbb{E} \left\{ e^{i \alpha \cdot X_{d}^{h,i,j}} \bigg| \mathcal{N}(t) = n \right\}
\]
of $X^i_{d,j}$. Under the assumptions (3.5) and (3.6), the characteristic function (2.2) becomes

$$f_{n}^{h,j}(a_d) = \left\{ 2^{\frac{d}{2}} \Gamma \left( \frac{d}{2} \right) \right\}^{n+1} C(a_n, b_n, t) \int_{S_j} \prod_{k=1}^{j-1} \left( \tau_k^{d-2} J_{\frac{d}{2}-1}(c \tau_k ||a_d||) \right) d\tau_1 \cdots d\tau_{j-1}$$

$$\times \int_{S_j} \tau_j^{d-2}(t - \sum_{k=1}^{j-1} \tau_k)^{(n-j+1)(\frac{d}{2}-1)+h+j} J_{\frac{d}{2}-1}(c \tau_j ||a_d||) d\tau_j$$

$$\times \int_{S_n} \prod_{k=j+1}^{n} \left( \tau_k^{d-2} J_{\frac{d}{2}-1}(c \tau_k ||a_d||) \right) (t - \sum_{k=1}^{n} \tau_k)^{\frac{d}{2}-i-1} J_{\frac{d}{2}-1}(c(t - \sum_{k=1}^{n} \tau_k) ||a_d||) d\tau_{j+1} \cdots d\tau_n$$

where

$$C(a_n, b_n, t) = \frac{1}{\Gamma(2(n+1)(\frac{d}{2}-1)+h+j)(\Gamma(d-1))^n} \frac{\Gamma \left( \frac{n-j+1}{2} \right) \Gamma \left( \frac{d}{2}-i \right) \Gamma \left( 2(n-j+1)(\frac{d}{2}-1)+h+1 \right)}{\Gamma \left( \frac{n+j}{2} \right) \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d}{2}-1 \right)}$$

and

$$S_j^1 = \left\{ (\tau_1, \ldots, \tau_{j-1}) \in \mathbb{R}^{j-1} : 0 < \tau_k < t - \sum_{i=0}^{k-1} \tau_i, k = 1, \ldots, j-1 \right\},$$

$$S_j^2 = \left\{ \tau_j \in \mathbb{R} : 0 < \tau_j < t - \sum_{i=0}^{j-1} \tau_i \right\},$$

$$S_n^3 = \left\{ (\tau_{j+1}, \ldots, \tau_n) \in \mathbb{R}^{n-j} : 0 < \tau_k < t - \sum_{i=0}^{k-1} \tau_i, k = j+1, \ldots, n \right\}.$$

The first step consists in the calculation of the $(n-j)$–fold integral

$$I_1(\tau_j) = \int_{S_n^3} \prod_{k=j+1}^{n} \left( \tau_k^{d-2} J_{\frac{d}{2}-1}(c \tau_k ||a_d||) \right) \left( t - \sum_{k=1}^{n} \tau_k \right)^{\frac{d}{2}-i-1} J_{\frac{d}{2}-1}(c(t - \sum_{k=1}^{n} \tau_k) ||a_d||) d\tau_{j+1} \cdots d\tau_n.$$

We apply recursively the result (3.4). Indeed, the first integral with respect to $\tau_1$ becomes

$$\frac{1}{(c||a_d||)^{\frac{d}{2}-3-i}} \int_0^{t-\sum_{k=1}^{n} \tau_k} \frac{J_{\frac{d}{2}-1}(c \tau_k ||a_d||) J_{\frac{d}{2}-1}(c(t - \sum_{k=1}^{n} \tau_k) ||a_d||)}{c \tau_k ||a_d||} d\tau_1$$

$$= \frac{y + c \tau_1 ||a_d||}{(c||a_d||)^{\frac{d}{2}-3-i}}$$

$$= \frac{1}{(c||a_d||)^{\frac{d}{2}-3-i}} \int_0^{t-\sum_{k=1}^{n} \tau_k} \frac{J_{\frac{d}{2}-1}(y) J_{\frac{d}{2}-1}(c(t - \sum_{k=1}^{n} \tau_k) ||a_d|| - y)}{c(t - \sum_{k=1}^{n} \tau_k) ||a_d||} dy$$

$$= \frac{1}{(c||a_d||)^{\frac{d}{2}-3-i}} \Gamma \left( \frac{d}{2} - 1 \right) \left( c(t - \sum_{k=1}^{n} \tau_k) ||a_d|| - y \right)^{i}.$$
Therefore, by continuing at the same way with the successive integrations we obtain that

\[
I_1(\tau_j) = \frac{1}{(c||\alpha_d||)^{(n-j+1)(\frac{d}{2}-1)-1}} \frac{(n-j+i)!/(n-j)!}{(\frac{d}{2}-1)^{n-j}} \frac{J_{(n-j+1)(\frac{d}{2}-1)}(c(t - \sum_{k=1}^{j} \tau_k)||\alpha_d||)}{(c(t - \sum_{k=1}^{j} \tau_k)||\alpha_d||)^{\frac{d}{2}-1}}.
\]

Now we work out the following integral

\[
I_2(\tau_{j-1}) = \int_{S^2_j} \tau_j^{d-2}(t - \sum_{k=1}^{j} \tau_k)(n-j+1)(\frac{d}{2}-1)+h J_{(n-j+1)(\frac{d}{2}-1)}(c(t - \sum_{k=1}^{j} \tau_k)||\alpha_d|| - y)dy
\]

by applying (3.3). Therefore, we get that

\[
I_2(\tau_{j-1}) = \frac{1}{(c||\alpha_d||)^{(2(n-j+1)+2)(\frac{d}{2}-1)+h+1}} \frac{(n-j+i)!/(n-j)!}{(\frac{d}{2}-1)^{n-j}} \int_{0}^{t-\sum_{k=1}^{j-1} \tau_k} dy \frac{d-1}{2} J_{\frac{d}{2}-1}(y)
\]

\[
\times (c(t - \sum_{k=1}^{j-1} \tau_k)||\alpha_d|| - y)^{n-j+1}(\frac{d}{2}-1)+h J_{(n-j+1)(\frac{d}{2}-1)}(c(t - \sum_{k=1}^{j-1} \tau_k)||\alpha_d|| - y)dy
\]

\[
\times \int_{S^2_j} \prod_{k=1}^{j-1} \left( \tau_k^{d-2} J_{\frac{d}{2}-1}(c\tau_k)||\alpha_d||) \right) \frac{\tau_j^{d-1} J_{\frac{d}{2}-1}(c\tau_j)||\alpha_d||}{} \frac{\tau_j^{d-1} J_{\frac{d}{2}-1}(c\tau_j)||\alpha_d||}{(d-1)/(d-1)+h+1} \Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})
\]

\[
\times \left( \sum_{k=1}^{j-1} \tau_k \right)^{d-1} \frac{\Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})}{\sqrt{2\pi} \Gamma((n-j+2)(\frac{d}{2}-1)+h+1)}
\]

The last step consists in the calculation of

\[
(3.10) \quad \int_{S^2_j} \prod_{k=1}^{j-1} \left( \tau_k^{d-2} J_{\frac{d}{2}-1}(c\tau_k)||\alpha_d||) \right) \frac{\tau_j^{d-1} J_{\frac{d}{2}-1}(c\tau_j)||\alpha_d||}{} \frac{\tau_j^{d-1} J_{\frac{d}{2}-1}(c\tau_j)||\alpha_d||}{(d-1)/(d-1)+h+1} \Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})
\]

The first integral in (3.10) becomes

\[
\frac{(n-j+i)!/(n-j)!}{(c||\alpha_d||)^{(2(n-j+1)+4)(\frac{d}{2}-1)+h+1}} \frac{\Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})}{\sqrt{2\pi} \Gamma((n-j+2)(\frac{d}{2}-1)+h+1)}
\]

\[
\times \int_{0}^{t-\sum_{k=1}^{j-1} \tau_k} dy \frac{d-1}{2} J_{\frac{d}{2}-1}(y)
\]

\[
\times (c(t - \sum_{k=1}^{j-1} \tau_k)||\alpha_d||)^{n-j+1}(\frac{d}{2}-1)+h+\frac{1}{2} J_{(n-j+1)(\frac{d}{2}-1)}(c(t - \sum_{k=1}^{j-1} \tau_k)||\alpha_d|| - y)dy
\]

\[
\times \int_{S^2_j} \prod_{k=1}^{j-1} \left( \tau_k^{d-2} J_{\frac{d}{2}-1}(c\tau_k)||\alpha_d||) \right) \frac{\tau_j^{d-1} J_{\frac{d}{2}-1}(c\tau_j)||\alpha_d||}{} \frac{\tau_j^{d-1} J_{\frac{d}{2}-1}(c\tau_j)||\alpha_d||}{(d-1)/(d-1)+h+1} \Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})
\]

\[
\times \left( \sum_{k=1}^{j-1} \tau_k \right)^{d-1} \frac{\Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})}{\sqrt{2\pi} \Gamma((n-j+2)(\frac{d}{2}-1)+h+1)}
\]

\[
\times \left( \sum_{k=1}^{j-1} \tau_k \right)^{d-1} \frac{\Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})}{\sqrt{2\pi} \Gamma((n-j+2)(\frac{d}{2}-1)+h+1)}
\]

\[
\times \left( \sum_{k=1}^{j-1} \tau_k \right)^{d-1} \frac{\Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})}{\sqrt{2\pi} \Gamma((n-j+2)(\frac{d}{2}-1)+h+1)}
\]

\[
\times \left( \sum_{k=1}^{j-1} \tau_k \right)^{d-1} \frac{\Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})}{\sqrt{2\pi} \Gamma((n-j+2)(\frac{d}{2}-1)+h+1)}
\]

\[
\times \left( \sum_{k=1}^{j-1} \tau_k \right)^{d-1} \frac{\Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})}{\sqrt{2\pi} \Gamma((n-j+2)(\frac{d}{2}-1)+h+1)}
\]

\[
\times \left( \sum_{k=1}^{j-1} \tau_k \right)^{d-1} \frac{\Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})}{\sqrt{2\pi} \Gamma((n-j+2)(\frac{d}{2}-1)+h+1)}
\]

\[
\times \left( \sum_{k=1}^{j-1} \tau_k \right)^{d-1} \frac{\Gamma((n-j+1)(\frac{d}{2}-1)+h+\frac{1}{2})}{\sqrt{2\pi} \Gamma((n-j+2)(\frac{d}{2}-1)+h+1)}
\]
Theorem 4

Finally, the characteristic function of $X_d^{h,i,j}$ is equal to

$$
\mathcal{F}_n^{h,i,j}(\alpha_d) = \left\{ \begin{array}{ll}
\frac{\alpha_d^{1-n-j}/(n-j)!}{(\alpha_d + 1)^{n-j}} \Gamma((n-j+1)(\frac{d}{2} - 1) + h + \frac{1}{2}) \\
\times \frac{\Gamma(2(n+1)(\frac{d}{2} - 1) + h + 1)}{(\Gamma(d-1))^2(\Gamma(\frac{d}{2} - 1))^2} \Gamma((n-j+1)(\frac{d}{2} - 1) + 1 - i)
\end{array} \right.
$$

For $h = 1$, by applying the duplication formula for Gamma functions, we have that

$$
\Gamma \left( \frac{d}{2} \right) = \sqrt{\pi} 2^{\frac{d}{2} - 1} \Gamma \left( \frac{d-1}{2} \right)
$$

and after careful calculations we obtain that

$$
\mathcal{F}_n^{h,i,j}(\alpha_d) = 2^{(n+1)(\frac{d}{2} - 1) + \frac{1}{2}} \Gamma \left( (n+1) \left( \frac{d}{2} - 1 \right) + \frac{j + 1}{2} \right)
$$

which does not depend on $i$. For $h = 0$ we take into account the following relationships

$$
\Gamma \left( (n-j+1) \left( \frac{d}{2} - 1 \right) + \frac{1}{2} \right) = \sqrt{\pi} 2^{-2(n-j+1)(\frac{d}{2} - 1)} \Gamma \left( (n-j+1) \left( \frac{d}{2} - 1 \right) + \frac{1}{2} \right)
$$

$$
\sqrt{\pi} \Gamma \left( (n+1) \left( \frac{d}{2} - 1 \right) + j + 1 \right) = 2^{(n+1)(\frac{d}{2} - 1) + \frac{j + 1}{2}} \Gamma \left( (n+1) \left( \frac{d}{2} - 1 \right) + \frac{j + 1}{2} \right)
$$

which leads to (3.11).

Fixed $j \in \{0, 1, ..., n-1\}$, the processes $X_d^{0,0,j}$, $X_d^{1,0,j}$, $X_d^{0,1,j}$, $X_d^{1,1-j}$ have the same characteristic function (3.11) and then the same probability distribution.
Theorem 5. The random process \( \{X_{t}^{h,i,j}(t), t > 0\} \) has density function given by

\[
p_{n}(X_{t}, t) = \frac{\Gamma((n+1)(\frac{d}{2}-1)+\frac{j}{2}+1)\left(c^{2}t^{2} - ||X_{t}||^{2}\right)n(\frac{d}{2}-1)+\frac{j}{2}-1}{\pi^{d/2}(ct)^{2(n+1)(\frac{d}{2}-1)+j}}, \quad X_{t} \in B_{ct}^{d},
\]

with \( j \in \{0, 1, \ldots, n-1\} \) and \( d \geq 3 \).

Proof. We get the density function of \( X_{t}^{h,i,j} \) by inverting the characteristic function (3.11). Hence, we can write that

\[
p_{n}(X_{t}, t) = \frac{1}{(2\pi)^{d}} \int_{B_{ct}^{d}} e^{-i\langle \omega, X_{t}\rangle} \mathcal{F}_{n}^{h,i,j}(\omega) d\omega_{1} \cdots d\omega_{d}
\]

\[
= \frac{1}{(2\pi)^{d}} \int_{0}^{\infty} \rho^{d-1} d\rho \int_{\Theta} e^{-i\rho \cdot \omega} \sin^{d-2} \theta_{1} \cdots \sin \theta_{d-2} d\theta_{d-2}
\]

\[
\times \frac{2^{(n+1)(\frac{d}{2}-1)+\frac{j}{2}}}{(2\pi)^{d/2}} \frac{\Gamma\left((n+1)\left(\frac{d}{2}-1\right)+\frac{j}{2}+1\right)}{\Gamma\left((n+1)(\frac{d}{2}-1)+\frac{j}{2}+1\right)} \frac{J_{n+1}\left(\frac{d}{2}-1\right)+\frac{j}{2}+1}{(ct\rho)^{2(n+1)(\frac{d}{2}-1)+j}}
\]

\[
= \frac{2^{(n+1)(\frac{d}{2}-1)+\frac{j}{2}}}{(2\pi)^{d/2}} \frac{\Gamma\left((n+1)\left(\frac{d}{2}-1\right)+\frac{j}{2}+1\right)}{\Gamma\left((n+1)(\frac{d}{2}-1)+\frac{j}{2}+1\right)} \int_{0}^{\infty} \rho^{d-1} J_{n+1}\left(\frac{d}{2}-1\right)+\frac{j}{2}+1
\]

\[
\times \frac{J_{n+1}\left(\frac{d}{2}-1\right)+\frac{j}{2}+1}{(ct\rho)^{2(n+1)(\frac{d}{2}-1)+j}} \frac{\sin^{d-2} \theta_{1} \cdots \sin \theta_{d-2} d\theta_{d-2} \cdots d\theta_{d-2}}{\sin^{d-2} \theta_{1} \cdots \sin \theta_{d-2} d\theta_{d-2}}
\]

where in the last step we have used the formula (see Gradshteyn-Ryzhik, 1980, pag. 692, formula 6.575.1) with a correction in the bounds of \( \mu \) and \( \nu \)

\[
\int_{0}^{\infty} J_{n+1}\left(\alpha x\right) J_{n+1}\left(\beta x\right) x^{\mu-\nu} dx = \frac{\Gamma\left((n+1)\left(\frac{d}{2}-1\right)+\frac{j}{2}+1\right)}{\Gamma\left((n+1)(\frac{d}{2}-1)+\frac{j}{2}+1\right)} \frac{\sin^{d-2} \theta_{1} \cdots \sin \theta_{d-2} d\theta_{d-2} \cdots d\theta_{d-2}}{\sin^{d-2} \theta_{1} \cdots \sin \theta_{d-2} d\theta_{d-2}}
\]

Remark 3.1. As the reader can check (3.12) coincides with (3.2) with

\[
A = \frac{1}{(ct)^{d/2}(2\pi)^{d/2}(n+1)(\frac{d}{2}-1)+j}, \quad b = n \left(\frac{d}{2}-1\right)+\frac{j}{2}-1.
\]

Furthermore, for \( j = 0 \), the density (3.12) reduces, as expected, to (2.11) in De Gregorio and Orsingher (2012).

Remark 3.2. From (3.12), we observe that the process \( \{X_{t}^{h,i,j}(t), t > 0\} \) has uniform density function in \( B_{ct}^{d} \), that is

\[
p_{n}(X_{t}, t) = \frac{1}{(ct)^{d/2} \text{meas}(B_{ct}^{d})} = \frac{\Gamma\left(\frac{d}{2}\right)}{(ct)^{d/2}2\pi^{d/2}}
\]

if \( d = 3, j = 1, n = 1 \) and \( d = 4, j = 0, n = 1 \).
Let us denote by $P_n(\cdot \in A) = P(\cdot \in A|N(t) = n)$, where $A$ is a Borel subset of $\mathbb{R}^d$. Bearing in mind the considerations done in Remark 2.2 about the isotropy of the random process, we can write that

$$P_n(\{X_d^{h,i,j}(t) \in B_d^z\}) = P_n(||X_d^{h,i,j}(t)|| < z)$$

$$= \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^z r^{d-1} p_n(r,t) dr$$

$$= \frac{2}{(ct)^{2(n+1)(\frac{d}{2}-1)+\frac{d}{2}}} \frac{\Gamma((n+1)(\frac{d}{2}-1)+\frac{d}{2})}{\Gamma(n(\frac{d}{2}-1)+\frac{d}{2})} \int_0^z r^{d-1}(c^2r^2 - r^2)^{n(\frac{d}{2}-1)+\frac{d}{2}-1} dr$$

$$= \frac{\Gamma((n+1)(\frac{d}{2}-1)+\frac{d}{2})}{\Gamma(n(\frac{d}{2}-1)+\frac{d}{2})} \int_0^z r^{d-1}(1-r)^{n(\frac{d}{2}-1)+\frac{d}{2}-1} dr$$

$$= \frac{B(\frac{d}{2}, \frac{d}{2}, n(\frac{d}{2}-1)+\frac{d}{2})}{B(\frac{d}{2}, n(\frac{d}{2}-1)+\frac{d}{2})},$$

with $z < ct$. Then, if $\frac{d}{2}$ and $n(\frac{d}{2}-1)+\frac{d}{2}$ assume integer values (happening when both $\frac{d}{2}$ and $\frac{d}{2}$ are even), by means of the relationship (2.12), we get that

$$P_n(\{X_d^{h,i,j}(t) \in B_d^z\}) = \sum_{k=\frac{d}{2}}^{(n+1)(\frac{d}{2}-1)+\frac{d}{2}} \left( (n+1)(\frac{d}{2}-1)+\frac{d}{2} \right) \left( \frac{z^2}{c^2r^2} \right)^k \left( 1 - \frac{z^2}{c^2r^2} \right)^{(n+1)(\frac{d}{2}-1)+\frac{d}{2}-k},$$

with $z < ct$.

3.2. Solvable processes of second type. Let us indicate by $\{Y_d^{h,i}(t), t > 0\}$ the random walk (1.2) with vector $\tau_n$ having probability distribution (1.3) with parameters

$$a_n = \left( \frac{d}{h} - 1, ..., \frac{d}{h} - 1 \right),$$

$$b_n = \left( 1, ..., 1, \frac{d}{h} - i \right),$$

with $h \in \{1, 2\}$, $i \in \{0, 1\}$ and $d \geq 2$. These random flights coincide with the random walks with standard Dirichlet steps studied (independently) by De Gregorio and Orsingher (2012) and Le Caër (2010) in the case $i = 1$. The case $i = 0$ has been treated by Le Caër (2010) (actually the vector of parameters has values in a different order).

Furthermore, we consider the random motions $\{Z_d^{h,i}(t), t > 0\}$ defined by (1.2) and with lengths of the interval between consecutive changes of direction possessing generalized Dirichlet distribution (1.3). In this case the vector $a_n$ is defined as in (3.14) while $b_n$ has entries given by

$$b_k = \begin{cases} 1, & k \in \{1, ..., n-1\} \setminus \{j\}, \\ 2, & k = j, \\ \frac{d}{h} - 1, & k = n, \end{cases}$$

with $j \in \{1, ..., n-1\}$, $h \in \{1, 2\}$ and $d \geq 2$.

**Theorem 6.** Fixed $j \in \{1, ..., n-1\}$ and $h \in \{1, 2\}$, the random processes $\{Y_d^{h,i}(t), i \in \{0, 1\}$, and $\{Z_d^{h,i}(t), t > 0\}$, are identically distributed.
Proof. As observed by Le Caër (2010) (see Section 5 of the cited reference) the random motions \( \mathbf{Y}_{d,0} \) and \( \mathbf{Y}_{d,1} \) have the same probability distribution and then possess the same characteristic function. The characteristic function of \( \mathbf{Y}_{d,1} \) has been obtained in De Gregorio and Orsingher (2012), formula (2.1), and yields

\[
\hat{F}_{n,1}^{d,1}(\omega_{d}) = \frac{2^{n+1}}{\Gamma((d-1)\frac{1}{2})} \frac{\Gamma((n+1)(d-1) + \frac{1}{2})}{\Gamma((n+1)\frac{1}{2})} f_{n,1}^{d,1}(\omega_{d}),
\]

where \( d \geq 2 \), while the characteristic function of \( \mathbf{Y}_{d,2} \) has been obtained in De Gregorio and Orsingher (2012), formula (2.2), and yields

\[
\hat{F}_{n,2}^{d,1}(\omega_{d}) = \frac{2^{n+1} \Gamma((n+1)(d-1) + 1)}{\Gamma((n+1)(d-1)\frac{1}{2})} f_{n,2}^{d,1}(\omega_{d}),
\]

where \( d \geq 3 \). In order to complete the proof we show that the characteristic function of the random flight \( Z_{d,j} \) coincides with (3.17) for \( h = 1 \) and (3.18) for \( h = 2 \). We use similar arguments to those adopted in the proof of Theorem 4 and then we omit some details.

Let us start with the case \( h = 1 \). We have that the characteristic function of \( Z_{d,j} \) becomes

\[
\hat{F}_{n,1}^{d,1}(\omega_{d}) = \left\{ \begin{array}{c}
2^{d-1} \Gamma\left(\frac{d}{2}\right) \\
\prod_{k=1}^{j-1} \left\{ c_{d-2} \frac{J_{d-1}^{1}(c\tau^{k}_{d,1}|\omega_{d}|)}{(c\tau^{k}_{d,1}|\omega_{d}|)^{\frac{d}{2}-1}} \right\} \right.
\times \int_{S_{j-1}^{d}} \prod_{k=1}^{j-1} \left\{ c_{d-2} \frac{J_{d-1}^{1}(c\tau^{k}_{d,1}|\omega_{d}|)}{(c\tau^{k}_{d,1}|\omega_{d}|)^{\frac{d}{2}-1}} \right\} \, dt_{1} \cdots dt_{j-1}
\times \int_{S_{j}^{d}} \prod_{k=1}^{j} \left\{ c_{d-2} \frac{J_{d-1}^{1}(c\tau^{k}_{d,1}|\omega_{d}|)}{(c\tau^{k}_{d,1}|\omega_{d}|)^{\frac{d}{2}-1}} \right\} \, dt_{1} \cdots dt_{n},
\end{array} \right.
\]

where \( S_{j-1}^{d}, S_{j}^{d} \) and \( S_{n-j}^{d} \) are defined by (3.7), (3.8) and (3.9) respectively. By applying the result (3.3) for \( i = 0 \) we obtain that the \( (n-j) \)-fold integral appearing in (3.19) becomes

\[
\hat{F}_{n,1}^{d,1}(\omega_{d}) = \int_{S_{n-j}^{d}} \prod_{k=j+1}^{n-j} \left\{ c_{d-2} \frac{J_{d-1}^{1}(c\tau^{k}_{d,1}|\omega_{d}|)}{(c\tau^{k}_{d,1}|\omega_{d}|)^{\frac{d}{2}-1}} \right\} \, dt_{j+1} \cdots dt_{n}
\]

and then, by taking into account the result (3.3) for \( i = 1 \), we get that

\[
\hat{F}_{n,2}^{d,1}(\omega_{d}) = \int_{0}^{\gamma} \prod_{k=1}^{j} \left\{ c_{d-2} \frac{J_{d-1}^{1}(c\tau^{k}_{d,1}|\omega_{d}|)}{(c\tau^{k}_{d,1}|\omega_{d}|)^{\frac{d}{2}-1}} \right\} \, dt_{j+1} \cdots dt_{n}
\]
Finally, by using again (3.3) for $i = 1$ recursively, the last $(j - 1)$-fold integral is equal to

$$
\int_{S_{j-1}^d} \prod_{k=1}^{j-1} \left\{ \tau_k^{-1} J_{\frac{n+1}{2} - 1} \left( c \tau_k ||\alpha_d|| \right) \right\} J_2(\tau_{j-1}) d\tau_1 \cdots d\tau_{j-1}
$$

and then by plugging (3.20) into (3.19), we obtain that $\tilde{F}_{n,j}^1(\alpha_d)$ coincides with (3.17).

For $h = 2$, we can use steps similar to those adopted above together with the result (3.4). Hence, we conclude that the characteristic function $\tilde{F}_{n,j}^2(\alpha_d)$ of $Z_{2,j}^d$ is equal to (3.18).

**Remark 3.3.** The solvable random walks $Y_{1,i}^d$ and $Z_{1,j}^d$, with $i \in \{1, 2\}$, $j \in \{1, ..., n-1\}$ and $d \geq 2$, have the same probability density function

$$
\Gamma\left(\frac{n+1}{2} - 1\right) \frac{\left(c^2r^2 - ||\alpha_d||^2\right) \frac{n}{2} - 1}{\pi^{d/2}(c^2r^2)^{n/2}}
$$

which is obtained by inverting (3.17) (see (2.10) in De Gregorio and Orsingher, 2012).

At the same way we have that $Y_{2,i}^d$ and $Z_{2,j}^d$, with $i \in \{1, 2\}$, $j \in \{1, ..., n-1\}$ and $d \geq 3$, have the same probability density function

$$
\Gamma\left(\frac{n+1}{2} - 1\right) \frac{\left(c^2r^2 - ||\alpha_d||^2\right) \frac{n}{2} - 1}{\pi^{d/2}(c^2r^2)^{n/2}}
$$

which is obtained by inverting (3.18) (see (2.11) in De Gregorio and Orsingher, 2012).

Similar considerations to those leading to (3.13), hold for the probabilities $P_n(Y_{h,i}^d(t) \in B_z^d)$ and $P_n(Z_{h,j}^d(t) \in B_z^d)$ ($0 < z < ct$).

4. **Unconditional probability distributions**

**4.1. General considerations.** So far, we have analyzed the random process (1.2), with a fixed number of deviations. Formally, we can write the unconditional density function of (1.2) by means of the probability distribution $P(N(t) = n)$ of $\mathcal{N}(t)$, with $n \geq 0$. We recall that $\mathcal{N}(t)$ is independent from $Y_{d-1}$ and $\mathcal{R}_n$.

Let $d\mathcal{X}_d = (dx_1, ..., dx_d)$, the absolutely component of the probability distribution of $\{\mathcal{X}_d(t), t > 0\}$ is given by

$$
p(\mathcal{X}_d, t) = \frac{P(\mathcal{X}_d(t) \in d\mathcal{X}_d)}{\prod_{k=1}^{d} dx_k} = \sum_{n=1}^{\infty} p_n(\mathcal{X}_d, t) P(N(t) = n),
$$

where $p_n(\mathcal{X}_d, t)$ is equal to (2.1) and $\mathcal{X}_d \in B^d_{ct}$. Furthermore the distribution of the process $\{\mathcal{X}_d(t), t > 0\}$ has a discrete component given by the probability that the random motion at time $t > 0$ hits the edge $\partial B^d_{ct} = \{\mathcal{X}_d \in \mathbb{R}^d : ||\mathcal{X}_d|| = ct\}$. This probability emerges when the random walk does not change the initial direction, i.e. $\mathcal{N}(t) = 0$. Hence, we can write

$$
P(\mathcal{X}_d(t) \in \partial B^d_{ct}) = P(\mathcal{N}(t) = 0).
$$

Therefore, the random process $\{\mathcal{X}_d(t), t > 0\}$ has support in the closed ball $\overline{B}_{ct}^d = B^d_{ct} \cup \partial B^d_{ct}$ and the complete density function (in sense of generalized functions) reads

$$
p(\mathcal{X}_d, t)1_{\overline{B}_{ct}^d}(\mathcal{X}_d) + P(\mathcal{N}(t) = 0)\delta(\mathcal{X}_d - \mathcal{X}_d) + \sum_{n=1}^{\infty} p_n(\mathcal{X}_d, t) P(N(t) = n),
$$
where \( \delta(\cdot) \) is the Dirac’s delta function.

4.2. Unconditional probability distributions for solvable random walks. We focus our attention on the random process \( \mathbf{X}^{h,i,j}_d \) introduced in Section 3.1. Under a suitable assumption on the law of the random number of changes of direction \( N(t) \), we are able to explicit the probability distribution (or equivalently (4.2)) for \( \mathbf{X}^{h,i,j}_d \). For \( \mathbf{Y}^{h,i,j}_d \) and \( \mathbf{Z}^{h,i,j}_d \) this problem has been tackled in De Gregorio and Orsingher (2012).

In order to obtain the unconditional probability distributions of \( \mathbf{X}^{h,i,j}_d \), we introduce the fractional Poisson process in the spirit of the paper by Beghin and Orsingher (2009). Let \( \{N^j_d(t), t>0\} \) be the fractional Poisson process with distribution

\[
(4.3) \quad P(N^j_d(t)=n) = \frac{1}{E \left[ \lambda t, d+j \right]} \frac{(\lambda t)^n}{\Gamma((n+1)\left(\frac{d}{2}-1\right)+\frac{j}{2}+1)}, \quad d \geq 3, n \geq 0, j \geq 0,
\]

where \( E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)} \), \( x \in \mathbb{R}, \alpha, \beta > 0 \), is the generalized Mittag-Leffler function. For \( \alpha = \beta = 1 \) the Mittag-Leffler function \( E_{1,1}(x) \) becomes the exponential function and then in this case \( \{N^j_d(t), t>0\} \) reduces to the standard Poisson process.

As observed in Beghin and Macci (2012), the fractional Poisson process \( \{N^j_d(t), t>0\} \) belongs to the class of weighted Poisson processes. Indeed, (4.3) can be written as follows

\[
P(N^j_d(t)=n) = \frac{w_n P(N(t)=n)}{\sum_{k=0}^{\infty} w_k P(N(t)=k)}, \quad d \geq 3, n \geq 0,
\]

where \( w_k = \frac{k!}{\Gamma((k+1)\left(\frac{d}{2}-1\right)+\frac{j}{2}+1)} \) represents the weights of \( \{N^j_d(t), t>0\} \) and \( P(N(t)=k) = e^{-\lambda t}(\lambda t)^k/k! \) is the probability distribution of a standard homogeneous Poisson process.

Remark 4.1. Let us analyze some properties of the aforementioned fractional Poisson process. The probability generating function of \( \{N^j_d(t), t>0\} \) becomes

\[
G_{N^j_d}(u,t) = E_{\frac{d}{2}-1,\frac{d+j}{2}}(\lambda t u), \quad |u| \leq 1,
\]

and since

\[
\frac{d}{dx} E_{\alpha,\beta}(ax) = \frac{\alpha}{\nu} [E_{\alpha,\beta+1}(ax) + (1-\beta)E_{\alpha,\beta}(ax)]
\]

we obtain that

\[
\mathbb{E} \left\{ N^j_d(t) \right\} = \left. \frac{d}{du} G_{N^j_d}(u,t) \right|_{u=1}
\]

\[
= \frac{2\lambda t}{(d-2)E_{\frac{d}{2}-1,\frac{d+j}{2}}(\lambda t)} \left[ E_{\frac{d}{2}-1,\frac{d+j}{2}}(\lambda t) + \left( 1 - \frac{d+j}{2} \right) E_{\frac{d}{2}-1,\frac{d+j}{2}-1}(\lambda t) \right].
\]

The following theorem concerns the main result of this section.

Theorem 7. If we assume that the number of deviations is a fractional Poisson process with distribution (4.3), the absolutely continuous component of the unconditional probability distributions of \( \mathbf{X}^{h,i,j}_d \), \( j \geq 0, d \geq 3 \), is equal to:

\[
p^j(\mathbf{x}_d,t) = \frac{P(\mathbf{X}^{h,i,j}_d \in d\mathbf{x}_d)}{\prod_{k=1}^{d} d\mathbf{x}_k} = \frac{\lambda t e^{t^2 - ||\mathbf{x}_d||^2} E_{\frac{d}{2}-1,\frac{d+j}{2}}(\frac{\lambda t e^{t^2 - ||\mathbf{x}_d||^2}}{\text{ct}^{d/2}})^{\frac{j}{2}+1}}{E_{\frac{d}{2}-1,\frac{d+j}{2}}(\lambda t)},
\]

with \( \mathbf{x}_d \in \mathbb{B}^d_{ct} \).
Proof. For $x_d \in B_{ct}^d$, we can write that

$$p^j(x_d, t) = \sum_{n=1}^{\infty} P(N_d^j(t) = n)p_n^j(x_d, t)$$

$$= \frac{1}{\pi^{d/2}E_{\frac{d}{2}-1, \frac{d}{2}+1}(\lambda t)} \sum_{n=1}^{\infty} (\lambda t)^{n} (c^2 t^2 - ||x_d||^2)^n (\frac{d}{2} - 1)^{1/n}$$

$$= \frac{1}{\pi^{d/2}E_{\frac{d}{2}-1, \frac{d}{2}+1}(\lambda t)} \lambda t(c^2 t^2 - ||x_d||^2)^{\frac{d}{2} - 1} \sum_{n=0}^{\infty} (\lambda t)^{n} (c^2 t^2 - ||x_d||^2)^n (\frac{d}{2} - 1)^{1/n}$$

Some suitable adjustments conclude the proof. □

Remark 4.2. Since $E_{1,1}(x) = e^x$ and $E_{1,2}(x) = \frac{x^2}{2}$, from (4.6) for $d = 4$ and $j = 0$, we get that

$$p^0(x_4, t) = \frac{x^2}{\pi^2 c^4 t^4} \frac{\exp\left\{\frac{1}{2c^2 t^2} (c^2 t^2 - ||x_4||^2)\right\}}{\exp\{\lambda t\} - 1},$$

(see Remark 3.2 in De Gregorio and Orsingher, 2012), while for $d = 4$ and $j = 1$, one has that

$$p^1(x_4, t) = \frac{1}{\pi^2 c^4 t^4} \frac{\exp\left\{\frac{1}{2c^2 t^2} (c^2 t^2 - ||x_4||^2)\right\} - 1}{E_{1,3}(\lambda t)},$$

with $x_4 \in B_{ct}^4$.

For the considerations done in the previous section we have that the probability distribution of $X_{d, h, i, j}^d$ admits the following discrete component

$$P(X_{d, h, i, j}^d \in \partial B_{ct}^d) = \frac{1}{E_{\frac{d}{2}-1, h}^{\frac{d}{2}+1}(\lambda t)} \frac{1}{\Gamma\left(\frac{d+j}{2}\right)}.$$

From (4.2), it is immediate to observe that the distance processes $R_{d, h, i, j}^d = \{R_{d, h, i, j}^d(t), t > 0\}$, where $R_{d, h, i, j}^d(t) = ||X_{d, h, i, j}^d(t)||$, have probability density function (in the sense of generalized functions) given by

$$r^{d-1} \text{meas}(B_1^d)p^j(r, t)1_{(0, ct)}(r) + \frac{1}{E_{\frac{d}{2}-1, \frac{d}{2}+1}(\lambda t)} \frac{1}{\Gamma\left(\frac{d+j}{2}\right)} \delta(ct - r).$$

Theorem 8. For $p \geq 1$, the $p$-th moment of $R_{d, h, i, j}^d$, $d \geq 3$, becomes

$$E(R_{d, h, i, j}^d)^p = \frac{(ct)^p}{E_{\frac{d}{2}-1, \frac{d}{2}+1}(\lambda t)} \left\{ \frac{\Gamma\left(\frac{p+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \lambda t E_{\frac{d}{2}-1, d+\frac{d+j}{2}-1}(\lambda t) + \frac{1}{\Gamma\left(\frac{d+j}{2}\right)} \right\}.$$

Proof. We have to calculate

$$E(R_{d, h, i, j}^d)^p = \int_0^{ct} r^{p+d-1} \text{meas}(B_1^d)p^j(r, t)dr + \frac{(ct)^p}{E_{\frac{d}{2}-1, \frac{d}{2}+1}(\lambda t)} \frac{1}{\Gamma\left(\frac{d+j}{2}\right)}.$$

Now, we focus our attention on the integral appearing in the above equality. One has that

$$\int_0^{ct} r^{p+d-1} \text{meas}(B_1^d)p^j(r, t)dr$$

$$= \frac{2\lambda t}{\Gamma\left(\frac{d}{2}\right)(ct)^{d(\frac{d}{2} - 1)}E_{\frac{d}{2}-1, \frac{d}{2}+1}(\lambda t)} \int_0^{ct} r^{p+d-1} (c^2 t^2 - r^2)^{\frac{d+j}{2}-2} E_{\frac{d}{2}-1, \frac{d+j}{2}} \left(\frac{\lambda t(c^2 t^2 - r^2)^{\frac{d}{2} - 1}}{(ct)^{d-2}}\right) dr.$$
\[ (5.1) \]
\[ \frac{\Gamma(d/2)}{\pi^{d/2}} \sin^{d-2} \theta_1 \sin^{d-3} \theta_2 \cdots \sin \theta_{d-2}, \]
where \( \theta_k \in [0, \pi], k \in \{1, \ldots, d-2\} \), and \( \phi \in [0, \pi] \). Therefore, at time \( t > 0 \) the random walk is located inside \( K_{ct}^d = K_{ct}^d \cup \partial K_{ct}^d \), where \( K_{ct}^d = \{ x_{d-1} \in \mathbb{R}^{d-1}, x_d > 0 : ||x_d|| < ct \} \). We indicate by \( \{ \mathbf{x}_d(t), t > 0 \} \) the position of this random motion within \( K_{ct}^d \). Furthermore, it is easy to check that the probability distributions of this new class of random walks coincide with those obtained in the previous sections (up to the normalizing constant 2).

Let
\[ (5.2) \]
\[ T_a = \inf(t > 0 : \mathbf{x}_d(t) \notin K_{ct}^d) \]
be the first exit time from the set \( K_{ct}^d, a > 0 \) (or equivalently the first passage time in the edge \( \partial K_{ct}^d \)). Since the sample paths of \( \{ \mathbf{x}_d(t), t > 0 \} \) under the assumption (5.1) move toward the edge \( \partial K_{ct}^d \), we have that

\[ (5.3) \]
\[ P(T_a > t) = P(\mathbf{x}_d(t) \in K_{ct}^d) = \int_{K_{ct}^d} p(\mathbf{x}_d, t)dx_1 \cdots dx_d, \quad t > a/c, \]
or equivalently
\[ P(T_a < t) = P(\mathbf{x}_d(t) \in K_{ct}^d \setminus K_{ct}^d) = \int_{K_{ct}^d \setminus K_{ct}^d} p(\mathbf{x}_d, t)dx_1 \cdots dx_d, \quad t > a/c, \]
where \( p(\mathbf{x}_d, t) \) is given by (4.1).

Clearly for \( t < a/c \) the above probabilities are equal to one and zero respectively. Furthermore, the probability distribution of the hitting time \( T_a \) admits a discrete component
\[ P(T_a = a/c) = P(\mathbf{x}_d(a/c) \in \partial K_{ct}^d) = P(N(a/c) = 0), \]
appearing if, at time \( t = a/c \), the random walk has not still changed the initial direction.

\textbf{Remark 5.1.} The planar process \( \mathbf{y}_2^{1,1} \) introduced in Section 3.2 (modified with the assumption (5.1)) coincides with the random flight with exponential displacements (that is \( \mathbf{z}_2 \) is uniformly
distributed) and we have that (see Stadje, 1987)

\[
p(x_2, t) = \frac{\lambda}{\pi c} \exp \left\{ -\lambda t + \frac{\lambda}{c} \sqrt{c^2 t^2 - ||x_2||^2} \right\}, \quad x_2 \in S^2_c.
\]

This result implies that

\[
P(T_{a} > t) = 1 - \exp \left\{ -\lambda t + \frac{\lambda}{c} \sqrt{c^2 t^2 - a^2} \right\}, \quad t > a/c.
\]

For the random motions of \(X_{d,0}^{h,1,0}\) and \(X_{d,1,1}^{h,1,1}\), modified with the assumption (5.1), in the results (4.7) and (4.8) appear 2 in the normalizing constant. In these cases we obtain respectively that

\[
P(T_{a} > t) = \frac{\exp\{\lambda t\} - \exp\left\{ \frac{\lambda}{c^2}(c^2 t^2 - a^2) \right\} \left( 1 + \frac{a^2}{c^2} \right)^{4 h,i}}{\exp\{\lambda t\} - 1}, \quad t > a/c,
\]

and

\[
P(T_{a} > t) = \frac{\frac{1}{\lambda^2} \left[ \exp\{\lambda t\} - \exp\left\{ \frac{\lambda}{c^2}(c^2 t^2 - a^2) \right\} \left( 1 + \frac{a^2}{c^2} \right)^{4 h,i} \right] - \frac{2 a}{(c^2 t^2 - a^2)^{h,i}}}{E_{1,3}(\lambda t)}, \quad t > a/c.
\]

Remark 5.2. By means of the random time \(T_{a}\), we can define the killed random process \(\{X_{d,0}^{h}(t), t > 0\}\) as follows

\[
X_{d,0}^{h}(t) = \begin{cases} X_{d}(t), & t < T_{a}, \\ \beta_d, & t \geq T_{a}, \end{cases}
\]

where \(\beta_d\) is a cemetery point. Therefore when \(X_{d,0}^{h}(t)\) leaves \(S^2_c\), it lies in the point \(\beta_d\).

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