Monodromic T-Branes And The $SO(10)_{GUT}$

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Abstract

T-branes, which are non-Abelian bound states of branes, were first introduced by Cecotti, Cordova, Heckman and Vafa [1]. They are the refined version of the monodromic branes that feature in the phenomenological F-theory models. Here, we will be interested in the T-brane corresponding to the $Z_3$ monodromy which is used to break the $E_8$ gauge group to obtain the $SO(10)_{GUT}$. This extends the results of [1] to the case of $Z_3$ monodromic T-branes used to break the $E_8$ gauge group to $SO(10) \times SU(3) \times U(1)$ and compute the Yukawa coupling with the help of the residue formula. We conclude that the Yukawa coupling, $10_H \cdot 16_M \cdot 16_M$, is non-zero for $E_7$, in complete agreement with [1], but is zero for $E_8$. Furthermore, the case of $Z_2$ monodromic T-branes used to break the $E_8$ gauge group to $E_6 \times SU(2) \times U(1)$, nothing interesting can be deduced by evaluating the Yukawa coupling $27_H \cdot 27_M \cdot 27_M$ which is dependent on whether the MSSM fermion and electroweak Higgs fields can be included in the same $27$ multiplet of a three-family $E_6$ GUT or assign the Higgs fields to a different $27_H$ multiplet where only the Higgs doublets and singlets obtain the electroweak scale energy.

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1 Introduction

To break GUT symmetries within F-theory [6–9] in order to construct phenomenologically viable models, one can either make use of Wilson lines or introduce gauge fluxes with the key ingredient being the seven-brane which wraps the four-dimensional internal subspace of the six internal directions of the compactification providing for each important element. The F-theory $E_6$ was discussed in [2] whilst the primary focus in [3], was the $E_7$ gauge group obtained in this setting.

Here, however, the focus is on T-branes, or “triangular branes,” which are novel non-Abelian bound states of branes characterized by the condition that on some loci the Higgs field is upper triangular and indeed it can be seen to be the case in [1].

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2 Refer to Appendix A for details.
where $\langle \Phi \rangle$ is upper triangular on some locus. This approach deals with the spectral equation

$$P_\Phi(z) = \det(z - \Phi) = 0$$

which when $\Phi$ belongs to the CSA is equivalent to stating that

$$\prod_i (z - \lambda_i) = 0$$

where $\lambda_i$ are the eigenvalues of $\Phi$ and they denote the directions of the intersecting branes. In the case of non-diagonalizable Higgs fields, the monodromy group is now encoded in the form of the spectral equation. Such configurations are of particular interest when systems of 7-branes are considered [10]. When considering such a background profile for $\Phi$, it has to be made sure that the following equations of motion are satisfied which read

$$\nabla_A \Phi = 0$$
$$F_{A}^{(02)5} = 0$$

for the $F$-term equations and for the $D$-term equation we have that

$$\omega \wedge F_A + \frac{i}{2} [\Phi^\dagger, \Phi] = 0.$$

In this paper, the aim is to compute the Yukawa coupling with the help of the residue formula where the required interaction terms are of the form

$$10_H \cdot 16_M \cdot 16_M.$$

The present work extends the results of [11] to the case of $Z_3$ monodromic T-branes used to break the $E_8$ gauge group to $SO(10) \times SU(3) \times U(1)$ where the case of $Z_2$ monodromic T-branes used to break the $E_7$ gauge group to $SO(10) \times SU(2) \times U(1)$ is reviewed beforehand. We will show that the Yukawa coupling, $10_H \cdot 16_M \cdot 16_M$, is non-zero for $E_7$ but for $E_8$ is zero. Moreover, the case of $Z_2$ monodromic T-branes used to break the $E_8$ gauge group to $E_6 \times SU(2) \times U(1)$, nothing interesting can be deduced by evaluating the Yukawa coupling $27_H \cdot 27_M \cdot 27_M$ which depends on whether the MSSM fermion and electroweak Higgs fields can be included in the same $27_H$ multiplet of a three-family $E_6$ GUT or assign the Higgs fields to a different $27_H$ multiplet where only the Higgs doublets and singlets obtain the electroweak scale energy. This work can also seen as an extension to [11] in pursuit of obtaining $SO(10)$ GUT symmetry especially the Flipped $SO(10)$ from various string theoretic constructions.

## 2 The $Z_2$ Monodromy

### 2.1 Review of $SU(2)$ Field

Let us consider the spectral equation for an $SU(2)$ field along the lines of [11]:

$$P_\Phi(z) = z^2 - x$$
for which there is a $Z_2$ monodromy. In the holomorphic gauge the Higgs field is

\[
\begin{pmatrix}
0 & 1 \\
x & 0
\end{pmatrix}
\]

which is an intermediate case between a diagonal background and a nilpotent Higgs field

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

We promote this to the unitary gauge by use of a positive diagonal matrix having unit determinant

\[
\begin{pmatrix}
e^{f_1} & 0 \\
0 & e^{f_2}
\end{pmatrix}
\]

with the constraint

\[
\sum_i f_i = 0
\]

where $f_i$ are real. The $D$-term equation

\[
\omega \wedge F_A + \frac{i}{2} [\Phi^\dagger, \Phi] = 0
\]

is now replaced by the $SU(2)$ Toda equation in two complex variables

\[
\Delta f_i = C_{ij} e^{f_j}
\]

where $C_{ij}$ is the Cartan matrix of $SU(2)$.

### 2.2 The Brane Recombination

Infinitesimal perturbations to the holomorphic Higgs field are considered of the form

\[
\varphi = \text{ad}_\Phi(\xi) + h
\]

and then seen which can be gauged away to zero by the $U(2)$ transformations by way of deforming the theory via $SU(2)$ Higgs VEV

\[
\Phi = \begin{pmatrix}
0 & 1 \\
x & 0
\end{pmatrix}.
\]

After gauge fixing the most general perturbation that can be made is given by

\[
\varphi = \begin{pmatrix}
\frac{1}{2} \alpha(x, y) & 0 \\
\frac{1}{2} \alpha(x, y) & \frac{1}{2} \alpha(x, y)
\end{pmatrix}.
\]
With such a perturbation at hand, the spectral equation can be seen to be deformed by the $SU(2)$ Higgs VEV as

$$P_\Phi(z) = z^2 - x \rightarrow \left( z - \frac{1}{2} \alpha(x, y) \right)^2 - (x + \beta(x, y)).$$

Now changing coordinates yields

$$\tilde{z} - (\tilde{x} \alpha(\tilde{x}^2, \tilde{y}) + \beta(\tilde{x}^2, \tilde{y})).$$

This is interpreted as the three $D7$-branes recombining into one.

Now starting with the flat Kähler metric

$$\omega = \frac{i}{2} (dx \wedge d\bar{x} + dy \wedge d\bar{y} + dz \wedge d\bar{z})$$

changing to the new coordinates and noting that $x = \tilde{x}^2$, we have

$$\omega = \frac{i}{2} \left( (1 + 4|\tilde{x}|^2) d\tilde{x} \wedge d\bar{x} + d\tilde{y} \wedge d\bar{y} \right)$$

and therefore the recombined $D7$-brane is indeed curved.

### 2.3 $E_7 \rightarrow SO(10) \times SU(2) \times U(1)$

Here $SU(2) \times U(1)$ Higgs field is used that preserves an unbroken $SO(10)$:

$$\Phi = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \oplus (y).$$

The adjoint of $E_7$ decomposes under this breaking as

$$133 \rightarrow (1, 1)_0 \oplus (1, 3)_0 \oplus (45, 1)_0 \oplus (10, 1)_2 \oplus (10, 1)_{-2} \oplus (16, 1)_{-1} \oplus (\overline{16}, 2)_1$$

where the $U(1)$ generator is in parenthesis. We note that the required interaction terms are of the form

$$10_H \cdot 16_M \cdot 16_M.$$

We can readily identify $(10, 1)_2$ as the $10_H$ and $(16, 2)_{-1}$ as $16_M$ with

$$\varphi_{16_M} = \begin{pmatrix} \varphi_{16^+} \\ \varphi_{16^-} \end{pmatrix}$$

where the corresponding matter curve is

$$f = y^2 - x.$$
Note that the one component of the spinor doublet, namely $\phi_{16^+}$ is gauge equivalent to zero.

The trace in the adjoint of $\mathfrak{e}_7$ produces the following invariant tensors

$$\text{Tr}([t_{10,i}, t_{16,\alpha}^M]t_{16,\beta}^N) \propto (C\Gamma_i)_{\alpha\beta}\epsilon^{MN}$$

which is generated group theoretically by contraction with a $\Gamma$ matrix of the $SO(10)$ Clifford algebra, where $\alpha, \beta$ are spinor indices, $i$ is a vector index, and $C$ denotes the standard charge conjugation matrix.

The Yukawa coupling can now be evaluated simply as

$$W_{10\cdot16\cdot16} = \text{Res}_{(0,0)} \left[ \frac{\text{Tr}([\eta_{10}, \eta_{16}]\phi_{16})}{(y)(y^2 - x)} \right]$$

$$= \text{Res}_{(0,0)} \left[ \frac{(C\Gamma_i)_{\alpha\beta}\phi_{16}^\alpha\phi_{16}^\beta - \phi_{10}^i}{(x)(y)} \right]$$

where

$$\eta_{10_H} = \frac{1}{2}\phi_{10_H}$$

and

$$\eta_{16} = -\left(\frac{\phi_{16^-}}{y\phi_{16^-}}\right).$$

This coupling requires a single field, $\phi_{16^-}$, to participate twice in the trilinear Yukawa coupling and is known to give mass to exactly one generation of SM matter $16$'s.

## 3 The $Z_3$ Monodromy

### 3.1 Review of $SU(3)$

We follow [14] to outline some of the key ideas to paint the picture. Let us begin by considering the spectral equation for an $SU(3)$ field:

$$P_3(z) = z^3 - x$$

for which there is a $Z_3$ monodromy.

In the holomorphic gauge the Higgs field is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & 0 & 0 \end{pmatrix}$$

which is an intermediate case between a diagonal background and a nilpotent Higgs field

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$
We promote this to the unitary gauge by use of a positive diagonal matrix having unit determinant

\[
\begin{pmatrix}
e^{f_1} & 0 & 0 \\
0 & e^{f_2} & 0 \\
0 & 0 & e^{f_3}
\end{pmatrix}
\]

with the constraint

\[
\sum_i f_i = 0
\]

where \(f_i\) are real. The \(D\)-term equation

\[
\omega \wedge F_A + \frac{i}{2} [\Phi^\dagger, \Phi] = 0
\]

is now replaced by the \(SU(3)\) Toda equation in two complex variables

\[
\Delta f_i = C_{ij} e^{f_j}
\]

where \(C_{ij}\) is the Cartan matrix of \(SU(3)\) which is given by

\[
\begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}
\]

The components for the unitary transformation for the nilpotent Higgs field \(\Phi\) satisfy

\[
\begin{align*}
\partial_1 \partial_1 f_1 &= 2e^{f_1} - e^{f_2}, \\
\partial_1 \partial_1 f_2 &= -e^{f_1} + 2e^{f_2}.
\end{align*}
\]

### 3.2 The Brane Recombination

Infinitesimal perturbations to the holomorphic Higgs field are considered of the form

\[
\varphi = \text{ad}_\Phi (\xi) + h
\]

and then seen which can be gauged away to zero by the \(U(3)\) transformations by way of deforming the theory via \(SU(3)\) Higgs VEV

\[
\Phi = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
x & 0 & 0
\end{pmatrix}
\]

After gauge fixing, as was shown in [4], the most general perturbation that can be made is given by

\[
\varphi = \begin{pmatrix}
\frac{1}{3} \alpha(x, y) & 0 & 0 \\
0 & \frac{1}{3} \alpha(x, y) & 0 \\
\gamma(x, y) & \beta(x, y) & \frac{1}{3} \alpha(x, y)
\end{pmatrix}
\]
With such a perturbation at hand, the spectral equation can be seen to be deformed by the $SU(3)$ Higgs VEV as

$$P_{\Phi}(z) = z^3 - x \to \left( z - \frac{1}{3} \alpha(x,y) \right) \left( \left( z - \frac{1}{3} \alpha(x,y) \right)^2 - \beta(x,y) \right) - (x + \gamma(x,y))$$

which to first order reads

$$z^3 - z^2 \alpha(x,y) - z \beta(x,y) - x - \gamma(x,y).$$

Now changing coordinates to

$$(\tilde{x}, \tilde{y}, \tilde{z}) = (z, y, P_{\Phi}(z))$$

yields

$$\tilde{z} - (\tilde{x}^2 \alpha(\tilde{x}^3, \tilde{y}) + \tilde{x} \beta(\tilde{x}^3, \tilde{y}) + \gamma(\tilde{x}^3, \tilde{y})).$$

This is interpreted as the three $D7$-branes recombining into one.

Now starting with the flat Kähler metric

$$\omega = \frac{i}{2} (dx \wedge d\overline{x} + dy \wedge d\overline{y} + dz \wedge d\overline{z})$$

changing to the new coordinates and noting that $x = \tilde{x}^3$, we have

$$\omega = \frac{i}{2} ((1 + 9|\tilde{x}|^4) d\tilde{x} \wedge d\overline{\tilde{x}} + d\tilde{y} \wedge d\overline{\tilde{y}}$$

and therefore the recombined $D7$-brane is indeed curved.

**3.3 $E_8 \to SO(10) \times SU(3) \times U(1)$**

Here $SU(3) \times U(1)$ Higgs field is used that preserves an unbroken $SO(10)$:

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & 0 & 0 \end{pmatrix} \oplus (y).$$

The adjoint of $E_8$, following [5], decomposes as

$$248 \to (1, 1)_0 \oplus (1, 8)_0 \oplus (45, 1)_0 \oplus (1, 3)_{-4} \oplus (1, \overline{3})_{4} \oplus (10, 3)_2 \oplus (10, \overline{3})_{-2} \oplus (16, 1)_3 \oplus (\overline{16}, 1)_{-3} \oplus (16, 3)_{-1} \oplus (\overline{16}, \overline{3})_{1}$$

where the $U(1)$ generator is in parenthesis. We note that the required interaction terms are of the form

$$10_H \cdot 16_M \cdot 16_M.$$
We can readily identify \((10,3)_2\) as the \(10_H\) and \((16,3)_{-1}\) as \(16_M\). The \(16_M\) is in the fundamental of the \(SU(3)\)

\[
\varphi_{16_M} = \begin{pmatrix}
\varphi^1_{16_M} \\
\varphi^2_{16_M} \\
\varphi^3_{16_M}
\end{pmatrix}.
\]

The torsion equation can be solved using the adjugate matrix

\[
\eta_{16_M} = \begin{pmatrix}
4y^2 & x & -2xy \\
-2y & 4y^2 & x \\
1 & -2y & 4y^2
\end{pmatrix} \begin{pmatrix}
\varphi^1_{16_M} \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
4y^2\varphi^1_{16_M} \\
-2y\varphi^1_{16_M} \\
\varphi^1_{16_M}
\end{pmatrix}.
\]

The \(10_H\) transforms in the fundamental of \(SU(3)\) as well and as a result replacing \(y\) with \(-2y\) the solution of the torsion equation is found to be

\[
\eta_{10_H} = \begin{pmatrix}
16y^2 & x & 4xy \\
4y & 16y^2 & x \\
1 & 4y & 16y^2
\end{pmatrix} \begin{pmatrix}
\varphi^1_{10_H} \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
16y^2\varphi^1_{10_H} \\
4y\varphi^1_{10_H} \\
\varphi^1_{10_H}
\end{pmatrix}.
\]

Using that the matter curve corresponding to \(A\) and \(B\) are

\[f_A = 8y^3 + x, \quad f_B = 64y^3 - x\]

and again using the fact that the trace in the adjoint of \(\mathfrak{e}_8\) is

\[
\text{Tr}([t^{A}_{10,i}, t^{B}_{16,a}, t^{C}_{16,\beta}]) \propto (C\Gamma_i)_{\alpha\beta}\epsilon^{ABC}
\]

allowing the Yukawa coupling to be evaluated simply as

\[W_{10\cdot16\cdot16} = 0\]

as was expected since it was noted in [1] that for \(n \geq 2\) and arbitrary \(a\) and \(b\), matter curves of the form

\[f = ay^n + bx\]

will always yield zero for the computation of the trilinear Yukawa coupling.

4 Discussion And Conclusion

The low-energy string-derived model of [11] was constructed in the free fermionic formulation [12] of the four-dimensional heterotic string in which the space-time vector
bosons are obtained solely from the untwisted sector and generate the observable and hidden gauge symmetries:

\[
\text{observable} : \quad SO(6) \times SO(4) \times \sum_{i=1}^{3} U(1)_i \\
\text{hidden} : \quad SO(4)^2 \times SO(8) .
\]

where the $E_6$ combination being

\[
U(1)_\zeta = \sum_{i=1}^{3} U(1)_i ,
\]

which is anomaly free whereas the orthogonal combinations of $U(1)_{1,2,3}$ are anomalous.

Motivated by such string-derived low-energy effective models the Flipped $SO(10)$ was derived from the F-theory $E_6$ as was discussed in [2] and investigated further in [3], where the gauge group was $E_7$. Another possibility, that was explored in [3], was that of nonperturbative heterotic vacua arising from the Horava-Witten theory. The space of solutions of type A contains exactly one vacua over the Hirzebruch surfaces for any allowed value of $r$ for $6 \leq s \leq 24$ and the corresponding appropriate choice for $\lambda$ with

\[
s \text{ even}, \quad e - r \text{ even}, \quad \lambda = \pm 1, \pm 3.
\]

| $s$ | $e(r; \lambda)$ |
|-----|-----------------|
| 6   | $3r + 6 + \frac{1}{\lambda} \in \mathbb{Z}$ |
|     | $\lambda = \pm 1$ |

whereas the space of solutions of Type B

\[
r \text{ even}, \quad \lambda = \pm \frac{1}{2}, \pm \frac{3}{2},
\]

are given by

\[
s = 6, \quad e \left( r; \lambda = \pm \frac{1}{2} \right) = 3r + 6 + \frac{1}{\lambda} = 3r + 6 \pm 2.
\]

\[\text{3See Appendix B for the rules which allow the construction of realistic, viable vacua with } E_6 \text{ GUT symmetry where the base manifold is taken to be the Hirzebruch surfaces.}\]
In this paper, the aim was to compute the Yukawa coupling with the help of the residue formula where the required interaction terms are of the form

\[ 10_H \cdot 16_M \cdot 16_M \]

serving as an extension to the results of [1] to the case of \( Z_3 \) monodromic T-branes used to break the \( E_8 \) gauge group to \( SO(10) \times SU(3) \times U(1) \). We conclude that the Yukawa coupling, \( 10_H \cdot 16_M \cdot 16_M \), is non-zero for \( E_7 \), in complete agreement with [1], but for \( E_8 \) is zero. Furthermore, the case of \( Z_2 \) monodromic T-branes used to break the \( E_8 \) gauge group to \( E_6 \times SU(2) \times U(1) \), nothing interesting can be deduced by evaluating the Yukawa coupling \( 27_H \cdot 27_M \cdot 27_M \) which depends on whether the MSSM fermion and electroweak Higgs fields can be included in the same \( 27 \) multiplet of a three-family \( E_6 \) or assign the Higgs fields to a different \( 27_H \) multiplet where only the Higgs doublets and singlets obtain the electroweak scale energy. The work presented here can also be viewed as an extension to [3] in pursuit of obtaining \( SO(10) \) GUT symmetry especially the Flipped \( SO(10) \) from various string theoretic constructions.

A The F-Theory Construct

A.1 \( E_6 \)

\[ E_8 \supset E_6 \times SU(3)_\perp \]

with

\[ 248 \rightarrow (78, 1) + (1, 8) + (27, 3) + (\bar{27}, \bar{3}) \]

where the inhomogeneous Tate form for \( E_6 \) is given by

\[ x^3 - y^2 + b_1 xyz + b_2 x^2 z^2 + b_3 y z^2 + b_4 x z^3 + b_5 z^5 = 0. \]

In the spectral cover approach the \( E_6 \) representations are distinguished by the weights \( t_{1,2,3} \) of the \( SU(3)_\perp \) Cartan subalgebra subject to the traceless condition

\[ \sum_{i=1}^{3} t_i = 0 \]

while the \( SU(3)_\perp \) adjoint decomposes into singlets.

A.2 \( E_7 \)

\[ E_8 \supset E_7 \times SU(2)_\perp \]

with

\[ 248 \rightarrow (133, 1) \oplus (1, 3) \oplus (56, 2) \]

where the inhomogeneous Tate form for \( E_7 \) is given by

\[ x^3 - y^2 + b_1 xyz + b_2 x^2 z^2 + b_3 y z^2 + b_4 x z^3 + b_5 z^5 = 0. \]
A.3 The Gauge Enhancements

A.3.1 \( SO(10) \)

\[
\Delta = -16b_2^3b_3^2z^7 + \left( -27b_4^4 - 8b_1^2b_2^2b_3^2 + 72b_2b_4b_3^3 + 4b_1b_2(9b_3^2 + 4b_2b_4)b_3 + 16b_2^2(b_4^2 - 4b_2b_6) \right)z^8 + \mathcal{O}(z^9)
\]

\[
= z^7 \left[ -16b_2^3b_3^2 + \left( -27b_4^4 - 8b_1^2b_2^2b_3^2 + 72b_2b_4b_3^2 + 4b_1b_2(9b_3^2 + 4b_2b_4)b_3 + 16b_2^2(b_4^2 - 4b_2b_6) \right)z + \mathcal{O}(z^9) \right]
\]

| deg(\( \Delta \)) | Type | Gauge Group | Object Equation |
|-----------------|------|-------------|----------------|
| GUT            | 7    | \( D_5 \)  | \( SO(10) \)  |
| Matter Curve   | 8    | \( D_6 \)  | \( SO(12) \)  |
|                |      |             | \( S : z = 0 \) |
| Matter Curve   | 8    | \( E_6 \)  | \( E_6 \)      |
|                |      |             | \( P_{10} : b_3 = 0 \) |
| Yukawa Points  | 9    | \( E_7 \)  | \( E_7 \)      |
|                |      |             | \( b_2 = b_3 = 0 \) |
|                |      |             | \( b_3 = b_4^2 - 4b_2b_6 = 0 \) |

A.3.2 \( E_7 \)

\[
\Delta = z^9 \left[ -1024b_4^4 + ((b_1^2 + 4b_2)^2 - 96b_1b_3)b_4^2 + 72(b_1^2 + 4b_2)b_4b_6 - 432b_6^2 \right]z + \mathcal{O}(z^9)
\]

| deg(\( \Delta \)) | Type | Gauge Group | Object Equation |
|-----------------|------|-------------|----------------|
| GUT            | 9    | \( E_7 \)  | \( E_7 \)      |
|                |      |             | \( S : z = 0 \) |
| Matter Curve   | 10   | \( E_8 \)  | \( E_8 \)      |
|                |      |             | \( b_4 = 0 \)  |

B The Hirzebruch Surfaces \( F_r \)

The rules for constructing realistic, viable vacua with \( E_6 \) GUT symmetry where the base manifold \( B \) are taken to be the Hirzebruch surfaces, \( F_r \) are given. We arrive at the following conditions modified for the \( E_6 \) observable gauge group:
B.1 The Semistability Condition

The semistability condition offers a choice: either

\[ \lambda \in \mathbb{Z} \]

and

\[ s \text{ even, } e - r \text{ even} \]

or

\[ \lambda = \frac{2m - 1}{2}, \ m \text{ even, } r \text{ even}. \]

B.2 The Involution Conditions

The involution conditions are

\[ \sum \kappa_i = \eta \cdot c_1(B = F_r) = 2e + 2s - rs. \]

B.3 The Effectiveness Condition

The effectiveness condition boils down to

\[ s \leq 24, \ \text{and} \ 12r + 24 \geq e \]

with

\[ \sum \kappa_i^2 \leq 100 + \frac{9}{4\lambda} - 9\lambda \]

and

\[ \sum \kappa_i^2 \leq 4 + \frac{9}{4\lambda} - 9\lambda + \sum \kappa_i. \]

B.4 The Commutant Condition

The commutant condition for \( E_6 \) becomes

\[ \eta \geq 3c_1 \]

which implies that

\[ s \geq 6, \ \text{and} \ e \geq 3r + 6. \]
B.5 The Three Family Condition

The three family condition reads

\[-rs^2 + 3rs + 2es - 6e - 6s = \frac{6}{\lambda}.\]

Solving the three family condition for \(e\) assuming that the value of \(s\) is known leads to

\[e(r; \lambda) = \frac{1}{2s - 6}\left(rs^2 - 3rs + 6s + \frac{6}{\lambda}\right).\]

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