A NOTE ON $\delta$-STRONGLY COMPACT CARDINALS

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Abstract. In this paper we investigate more characterizations and applications of $\delta$-strongly compact cardinals. We show that, for a cardinal $\kappa$ the following are equivalent: (1) $\kappa$ is $\delta$-strongly compact, (2) For every regular $\lambda \geq \kappa$ there is a $\delta$-complete uniform ultrafilter over $\lambda$, and (3) Every product space of $\delta$-Lindelöf spaces is $\kappa$-Lindelöf. We also prove that in the Cohen forcing extension, the least $\omega_1$-strongly compact cardinal is a precise upper bound on the tightness of the products of two countably tight spaces.

1. Introduction

Bagaria and Magidor [2, 3] introduced the notion of $\delta$-strongly compact cardinals, which is a variant of strongly compact cardinals.

Definition 1.1 (Bagaria-Magidor [2, 3]). Let $\kappa$, $\delta$ be uncountable cardinals with $\delta \leq \kappa$. $\kappa$ is $\delta$-strongly compact if for every set $A$, every $\kappa$-complete filter over $A$ can be extended to a $\delta$-complete ultrafilter.

$\delta$-strongly compact cardinals, especially for the case $\delta = \omega_1$, have various characterizations and many applications, see Bagaria-Magidor [2, 3], Bagaria-da Silva [4], and Usuba [9, 10]. In this paper, we investigate more characterizations and applications of $\delta$-strongly compact cardinals.

Ketonen [7] characterized strongly compact cardinals by the existence of uniform ultrafilters, where a filter $F$ over a cardinal $\lambda$ is uniform if $|X| = \lambda$ for every $X \in F$. Ketten proved that an uncountable cardinal $\kappa$ is strongly compact cardinal if, and only if for every regular $\lambda \geq \kappa$, there exists a $\kappa$-complete uniform ultrafilter over $\lambda$. We prove a similar characterization for $\delta$-strongly compact cardinals.

Theorem 1.2. Let $\kappa$ and $\delta$ be uncountable cardinals with $\delta \leq \kappa$. Then $\kappa$ is $\delta$-strongly compact if, and only if, for every regular $\lambda \geq \kappa$, there exists a $\delta$-complete uniform ultrafilter over $\lambda$.

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In [3], Bagaria and Magidor characterized $\omega_1$-strongly compact cardinals in terms of topological spaces. Let $\mu$ be a cardinal. A topological space $X$ is $\mu$-Lindelöf if every open cover of $X$ has a subcover of size $< \mu$. An $\omega_1$-Lindelöf space is called a Lindelöf space.

Bagaria and Magidor proved that a cardinal $\kappa$ is $\omega_1$-strongly compact if and only if every product space of $\omega_1$-Lindelöf spaces is $\kappa$-Lindelöf. Using Theorem 1.2, we generalize this result as follows:

**Theorem 1.3.** Let $\delta \leq \kappa$ be uncountable cardinals. Then the following are equivalent:

1. $\kappa$ is $\delta$-strongly compact.
2. For every family $\{X_i \mid i \in I\}$ of $\delta$-Lindelöf spaces, the product space $\prod_{i \in I} X_i$ is $\kappa$-Lindelöf.

We turn to another topological property, the tightness. For a topological space $X$, the **tightness number** $t(X)$ of $X$ is the minimum infinite cardinal $\kappa$ such that whenever $A \subseteq X$ and $p \in \overline{A}$ (where $\overline{A}$ is the closure of $A$ in $X$), there is $B \subseteq A$ with $|B| \leq \kappa$ and $p \in \overline{B}$. If $t(X) = \omega$, $X$ is called a **countably tight** space.

The product of countably tight spaces need not to be countably tight: A typical example is the sequential space $S(\omega_1)$. It is a Fréchet-Urysohn space, but the square of $S(\omega_1)$ has uncountable tightness. It is also known that if $\kappa$ is regular uncountable cardinal and the set $\{\alpha < \kappa \mid \text{cf}(\alpha) = \omega\}$ has a non-reflecting stationary subset, then $t(S(\kappa)^2) = \kappa$ (see Eda-Gruenhage-Koszmider-Tamano-Todorčević [5]). In particular, under $V = L$, the tightness of the product of two Fréchet-Urysohn spaces can be arbitrary large.

We show that an $\omega_1$-strongly compact cardinal gives an upper bound on the tightness of the product of two countably tight spaces.

**Theorem 1.4.** If $\kappa$ is $\omega_1$-strongly compact, then $t(X \times Y) \leq \kappa$ for every countably tight spaces $X$ and $Y$.

We also show that an $\omega_1$-strongly compact cardinal is a **precise** upper bound in the Cohen forcing extension.

**Theorem 1.5.** Let $\mathbb{C}$ be the Cohen forcing notion, and $G$ be $(V, \mathbb{C})$-generic. Then for every cardinal $\kappa$ the following are equivalent in $V[G]$:

1. $\kappa$ is $\omega_1$-strongly compact.
2. For every countably tight spaces $X$ and $Y$ we have $t(X \times Y) \leq \kappa$.
3. For every countably tight Tychonoff spaces $X$ and $Y$ we have $t(X \times Y) \leq \kappa$.

Here we present some definitions and facts which will be used later.
**Definition 1.6.** For an uncountable cardinal \( \kappa \) and a set \( A \), let \( P_\kappa A = \{ x \subseteq A \mid |x| < \kappa \} \). A filter \( F \) over \( P_\kappa A \) is fine if for every \( a \in A \), we have \( \{ x \in P_\kappa A \mid a \in x \} \in F \).

**Theorem 1.7 ([2, 3]).** For uncountable cardinals \( \delta \leq \kappa \), the following are equivalent:

1. \( \kappa \) is \( \delta \)-strongly compact.
2. For every cardinal \( \lambda \geq \kappa \), there exists a \( \delta \)-complete fine ultrafilter over \( P_\kappa \lambda \).
3. For every set \( A \) with \( |A| \geq \kappa \), there exists a \( \delta \)-complete fine ultrafilter over \( P_\kappa A \).
4. For every cardinal \( \lambda \geq \kappa \), there exists an elementary embedding \( j : V \rightarrow M \) into some transitive model \( M \) such that \( \delta \leq \text{crit}(j) \leq \kappa \) and there is a set \( A \in M \) with \( |A|^M < j(\kappa) \) and \( j" \lambda \subseteq A \). Where \( \text{crit}(j) \) denotes the critical point of \( j \).

**Theorem 1.8 ([2, 3]).** If \( \kappa \) is \( \delta \)-strongly compact, then there is a measurable cardinal \( \leq \kappa \).

### 2. On uniform ultrafilters

In this section we give a proof of Theorem 1.2. It can be obtained by a series of arguments in Ketkonen [7] with some modifications.

**Lemma 2.1.** Suppose \( \kappa \) is \( \delta \)-strongly compact for some uncountable \( \delta \leq \kappa \). Then for every regular \( \lambda \geq \kappa \), there exists a \( \delta \)-complete uniform ultrafilter over \( \lambda \).

**Proof.** Fix a regular \( \lambda \geq \kappa \), and take an elementary embedding \( j : V \rightarrow M \) such that \( \delta \leq \text{crit}(j) \leq \kappa \), and there is \( A \in M \) with \( j" \lambda \subseteq A \subseteq j(\lambda) \) and \( |A|^M < j(\kappa) \). Then we have \( \sup(j" \lambda) < j(\lambda) \). Now define an ultrafilter \( U \) over \( \lambda \) by \( X \in U \iff \sup(j" \lambda) \in j(X) \). It is clear that \( U \) is a \( \delta \)-complete uniform ultrafilter over \( \lambda \). \( \square \)

For the converse direction, we need several definitions and lemmas.

Let \( U \) be an \( \omega_1 \)-complete ultrafilter over some set \( A \). Let \( \text{Ult}(V, M) \) denote the ultrapower of \( V \) by \( U \), and we identify the ultrapower with its transitive collapse. Let \( j : V \rightarrow M \approx \text{Ult}(V, U) \) be an elementary embedding induced by \( U \). Let \( \text{id}_A \) denote the identity map on \( A \), and for a function \( f \) on \( A \), let \( [f]_U \in M \) denote the equivalence class of \( f \) modulo \( U \). We know \( [f]_U = j(f)([\text{id}_A]_U) \).

**Definition 2.2.** Let \( \mu, \nu \) be cardinals with \( \mu \leq \nu \). An ultrafilter \( U \) over some set \( A \) is said to be \( (\mu, \nu) \)-regular if there is a family \( \{ X_\alpha \mid \alpha < \nu \} \) of measure one sets of \( U \) such that for every \( a \in [\nu]^{\mu} \), we have \( \bigcap_{\alpha \in a} X_\alpha = \emptyset \).
We note that if $\nu$ is regular and $U$ is $(\mu, \nu)$-regular, then $|X| \geq \nu$ for every $X \in U$.

**Lemma 2.3.** Let $\mu \leq \nu$ be cardinals where $\nu$ is regular. Let $U$ be an $\omega_1$-complete ultrafilter over some set $A$, and $j : V \rightarrow M \cong \text{Ult}(V, U)$ an elementary embedding induced by $U$. Then $U$ is $(\mu, \nu)$-regular if and only if $\text{cf}^M(\text{sup}(j^{\text{``}\nu\text{''}})) < j(\mu)$.

**Proof.** First suppose $U$ is $(\mu, \nu)$-regular, and let $\{X_\alpha \mid \alpha < \nu\}$ be a witness. Let $j(\{X_\alpha \mid \alpha < \nu\}) = \{Y_\alpha \mid \alpha < j(\nu)\}$. Let $a = \{\alpha < \text{sup}(j^{\text{``}\nu\text{''}}) \mid [\text{id}_A]_U \in Y_\alpha\} \in M$. We know $j^{\text{``}\nu\text{''}} \subseteq a$, hence $a$ is unbounded in $\text{sup}(j^{\text{``}\nu\text{''}})$, and $\text{cf}^M(\text{sup}(j^{\text{``}\nu\text{''}})) \leq |a|^M$. By the choice of $a$, we have $\bigcap_{\alpha \in a} X_\alpha \neq \emptyset$. Hence we have $|a|^M < j(\mu)$, and $\text{cf}^M(\text{sup}(j^{\text{``}\nu\text{''}})) < j(\mu)$.

For the converse, suppose $\text{cf}^M(\text{sup}(j^{\text{``}\mu\text{''}})) < j(\mu)$. Take a function $f : A \rightarrow \nu + 1$ such that $[f]_U = j(f)([\text{id}_A]_U) = \text{sup}(j^{\text{``}\nu\text{''}})$ in $M$. Then $Z = \{x \in A \mid \text{cf}(f(x)) < \mu\} \in U$. For each $x \in Z$, take $c_x \subseteq f(x)$ such that $\text{ot}(c_x) = \text{cf}(f(x))$ and $\text{sup}(c_x) = f(x)$. Then, by induction on $i < \nu$, we can take a strictly increasing sequence $\langle \nu_i \mid i < \nu \rangle$ in $\nu$ such that $\{x \in Z \mid [\nu_i, \nu_{i+1}) \cap c_x \neq \emptyset\} \in U$ as follows. Suppose $\nu_i$ is defined for all $i < j$. If $j$ is limit, since $\nu$ is regular, we have $\sup\{\nu_i \mid i < j\} < \nu$. Then take $\nu_j < \lambda$ with $\sup\{\nu_i \mid i < j\} < \nu_j$. Suppose $j = k + 1$. Consider $c_{[\text{id}_A]_U} \subseteq j(f)([\text{id}_A]_U) = \text{sup}(j^{\text{``}\nu\text{''}})$. $c_{[\text{id}_A]_U}$ is unbounded in $\text{sup}(j^{\text{``}\nu\text{''}})$. Pick some $\xi \in c_{[\text{id}]}$ with $j(\nu_k) < \xi$, and take $\nu_j < \nu$ with $\xi < j(\nu_j)$. Then $\nu_j$ works. Finally, let $X_i = \{x \in Z \mid [\nu_i, \nu_{i+1}) \cap c_x \neq \emptyset\} \in U$. We check that $\{X_i \mid i < \nu\}$ witnesses that $U$ is $(\mu, \nu)$-regular. So take $a \in [\nu]^{\mu}$, and suppose $x \in \bigcap_{i \in a} X_i$. Then $[\nu_i, \nu_{i+1}) \cap c_x \neq \emptyset$ for every $i \in a$. Since $\langle \nu_i \mid i < \nu \rangle$ is strictly increasing, we have $|c_x| \geq \mu$, this contradicts to the choice of $c_x$. \hfill \Box

**Lemma 2.4.** Let $\kappa$ and $\delta$ be uncountable cardinals with $\delta \leq \kappa$. Then the following are equivalent:

1. $\kappa$ is $\delta$-strongly compact.
2. For every regular $\lambda \geq \kappa$, there exists a $\delta$-complete $(\kappa, \lambda)$-regular ultrafilter over some set $A$.

**Proof.** Suppose $\kappa$ is $\delta$-strongly compact. Fix a regular cardinal $\lambda \geq \kappa$, and take a $\delta$-complete fine ultrafilter $U$ over $\mathcal{P}_\kappa \lambda$. For $\alpha < \lambda$, let $X_\alpha = \{x \in \mathcal{P}_\alpha \lambda \mid \alpha \in x\} \in U$. Then the family $\{X_\alpha \mid \alpha < \lambda\}$ witnesses that $U$ is $(\kappa, \lambda)$-regular.

For the converse, pick a cardinal $\lambda \geq \kappa$. By (2), there is a $\delta$-complete $(\kappa, \lambda^+)$-regular ultrafilter $W$ over some set $A$. Take an elementary embedding $i : V \rightarrow N \cong \text{Ult}(V, W)$. We have $\text{cf}^N(\text{sup}(i^{\text{``}\lambda^+\text{''}})) < i(\kappa)$ by...
Lemma 2.3. By the elementarity of $i$, one can check that for every stationary $S \subseteq \{\alpha < \lambda^+ \mid \text{cf}(\alpha) = \omega\}$, we have that $i(S) \cap \sup(i^\alpha \lambda^+)$ is stationary in $\sup(i^\alpha \lambda^+)$ in $N$ (actually in $V$). (e.g., see [3]). Fix a stationary partition $\{S_i \mid i < \lambda \}$ of $\{\alpha < \lambda^+ \mid \text{cf}(\alpha) = \omega\}$, and let $i(\{S_i \mid i < \lambda\}) = \{S'_\alpha \mid \alpha < i(\lambda)\}$. Let $a = \{\alpha \in i(\lambda) \mid S'_\alpha \cap \sup(i^\alpha \lambda^+)$ is stationary in $\sup(i^\alpha \lambda^+)$ in $N\}$. We have $a \in N$ and $i^\alpha \lambda^+ \subseteq a$. Moreover, since $\text{cf}(\sup(i^\alpha \lambda^+)) < i(\kappa)$, we have $|a|^N < i(\kappa)$. Hence $a \in i(P_{\kappa} \lambda)$, and the filter $U$ over $P_{\kappa} \lambda$ defined by $X \in U \iff a \in i(X)$ is a $\delta$-complete fine ultrafilter over $P_{\kappa} \lambda$. □

Definition 2.5. Let $\lambda$ be an uncountable cardinal and $U$ an ultrafilter over $\lambda$. $U$ is weakly normal if for every $f : \lambda \to \lambda$ with $\{\alpha < \lambda \mid f(\alpha) < \alpha\} \in U$, there is $\gamma < \lambda$ such that $\{\alpha < \lambda \mid f(\alpha) < \gamma\} \in U$.

Lemma 2.6. Let $\lambda$ be a regular cardinal, and $\delta \leq \lambda$ an uncountable cardinal. If $\lambda$ carries a $\delta$-complete uniform ultrafilter, then $\lambda$ carries a $\delta$-complete weakly normal uniform ultrafilter as well.

Proof. Let $U$ be a $\delta$-complete uniform ultrafilter over $\lambda$, and $j : V \to M \cong \text{Ult}(V, U)$ be an elementary embedding induced by $U$. Since $U$ is uniform, we have $\sup(j^\alpha \lambda) \leq [id_\lambda]^U \in j^{\lambda}(\lambda)$. Then define $W$ by $X \in W \iff \sup(j^\alpha \lambda) \in j(X)$. It is easy to see that $W$ is a required weakly normal ultrafilter. □

The following is immediate:

Lemma 2.7. Let $\lambda$ be a regular cardinal, and $U$ an $\omega_1$-complete weakly normal ultrafilter over $\lambda$. Let $j : V \to M \cong \text{Ult}(V, U)$ be an elementary embedding induced by $U$. Then $[id_\lambda]^U = \sup(j^\alpha \lambda)$. Hence $U$ is $(\mu, \lambda)$-regular if and only if $\{\alpha < \lambda \mid \text{cf}(\alpha) < \mu\} \in U$.

Definition 2.8. Let $A$ be a non-empty set, and $U$ an ultrafilter over $A$. Let $X \in U$, and for each $x \in X$, let $W_x$ be an ultrafilter over some set $A_x$. Then the $U$-sum of $\{W_x \mid x \in X\}$ is the collection $D$ of subsets of $\{\langle x, y \rangle \mid x \in X, y \in A_x\}$ such that for every $Y$, $Y \in D \iff \{x \in X \mid \langle y \in A_x \mid \langle x, y \rangle \in Y\} \in W_x \in U$. $D$ is an ultrafilter over the set $\{\langle x, y \rangle \mid x \in X, y \in A_x\}$, and if $U$ and the $W_x$’s are $\delta$-complete, then so is $D$.

Lemma 2.9. Let $\kappa$ and $\delta$ be uncountable cardinals with $\delta \leq \kappa$. Suppose for every regular $\lambda \geq \kappa$, there exists a $\delta$-complete uniform ultrafilter over $\lambda$. Then $\kappa$ is $\delta$-strongly compact.

Proof. First suppose $\kappa$ is regular. To show that $\kappa$ is $\delta$-strongly compact cardinal, by Lemma 2.4, it is enough to see that for every regular $\lambda \geq \kappa$,
there exists a $\delta$-complete ($\kappa, \lambda$)-regular ultrafilter over $\lambda$. We prove this
by induction on $\lambda$. For the base step $\lambda = \kappa$, by Lemma 2.6, we can take
a $\delta$-complete weakly normal uniform ultrafilter $U$ over $\kappa$. Then $\{\alpha < \kappa \mid
\text{cf} (\alpha) < \kappa\} \in U$, hence $U$ is ($\kappa, \kappa$)-regular by Lemma 2.7.

Let $\lambda > \kappa$ be regular, and suppose for every regular $\mu$ with $\kappa \leq \mu < \lambda$, there exists a $\delta$-complete ($\kappa, \mu$)-regular ultrafilter $U_\mu$ over $\mu$. Fix a
$\delta$-complete weakly normal uniform ultrafilter $U$ over $\lambda$. If $\{\alpha < \lambda \mid \text{cf} (\alpha) < \kappa\} \in U$, then $U$ is ($\kappa, \lambda$)-regular by Lemmas 2.3 and 2.7, and we have done. Suppose $\{\alpha < \lambda \mid \text{cf} (\alpha) \geq \kappa\} \in U$. Let $X^* = \{\alpha < \lambda \mid \text{cf} (\alpha) \geq \kappa\}$. For $\alpha \in X^*$, let $W_\alpha = U_{\text{cf} (\alpha)}$, a $\delta$-complete ($\kappa, \text{cf} (\alpha)$)-regular ultrafilter over $\text{cf} (\alpha)$. Let $B = \{\langle \alpha, \beta \rangle \mid \alpha \in X^*, \beta < \text{cf} (\alpha)\}$. Note that $|B| = \lambda$. Let
us consider the $U$-sum of $\{W_\alpha \mid \alpha \in X^*\}$. $D$ is a $\delta$-complete ultrafilter
over $B$. We claim that $D$ is ($\kappa, \lambda$)-regular, and then we can easily take a
$\delta$-complete ($\kappa, \lambda$)-regular ultrafilter over $\lambda$.

For $\alpha \in X^*$, let $j_\alpha : V \to M_\alpha \approx \text{Ult} (V, W_\alpha)$ be an elementary embedding induced by $W_\alpha$. Let $g_\alpha : \text{cf} (\alpha) \to \alpha + 1$ be a function which represents $\sup (j_\alpha " \alpha")$. Note that, since $W_\alpha$ is ($\kappa, \text{cf} (\alpha)$)-regular, we have $\text{cf}^{M_\alpha} (\sup (j_\alpha " \alpha")) = \text{cf}^{M_\alpha} (\sup (j_\alpha " \alpha")) < j_\alpha (\kappa)$, so $\{\beta < \text{cf} (\alpha) \mid \text{cf} (g_\alpha (\beta)) < \kappa\} \in W_\alpha$.

Let $i : V \to N \approx \text{Ult} (V, D)$ be an elementary embedding induced by $D$.
Define the function $g$ on $B$ by $g (\alpha, \beta) = g_\alpha (\beta)$. We see that $\sup (i " \lambda") = [g]_D$. First, for $\gamma < \lambda$, we have $X^* \setminus \gamma \in U$, and $\{\beta < \text{cf} (\alpha) \mid g_\alpha (\beta) \geq \gamma\} \in W_\alpha$ for all $\alpha \in X^* \setminus \gamma$. This means that $\{\langle \alpha, \beta \rangle \in B \mid g (\alpha, \beta) \geq \gamma\} \in D$, and $i (\gamma) < [g]_D$. Next, take a function $h$ on $B$ with $[h]_D < [g]_D$. Then $\{\langle \alpha, \beta \rangle \in B \mid h (\alpha, \beta) < g (\alpha, \beta)\} \in D$, and $X' = \{\alpha \in X^* \mid \beta < \alpha \mid h (\alpha, \beta) < g (\alpha, \beta)\} \in W_\alpha$. For $\alpha \in X'$, we know $\{\beta < \text{cf} (\alpha) \mid h (\alpha, \beta) < g (\alpha, \beta)\} \subseteq W_\alpha$. Because $g (\alpha, \beta) = g_\alpha (\beta)$ represents $\sup (j_\alpha " \alpha")$, there is some $\gamma_\alpha < \alpha$ such that $\{\beta < \text{cf} (\alpha) \mid h (\alpha, \beta) < \gamma_\alpha\} \subseteq W_\alpha$. Now, since $U$ is weakly normal and $\gamma_\alpha < \alpha$ for $\alpha \in X'$, there is some $\gamma < \lambda$ such that $\{\alpha \in X' \mid \gamma_\alpha < \gamma\} \in U$. Then we have $[h]_D < i (\gamma) \leq \text{sup} (i " \lambda")$.

Finally, since $\{\beta < \text{cf} (\alpha) \mid \text{cf} (g (\alpha, \beta)) < \kappa\} \subseteq W_\alpha$ for every $\alpha \in X^*$, we have $\{\langle \alpha, \beta \rangle \in B \mid \text{cf} (g (\alpha, \beta)) < \kappa\} \subseteq D$, which means that $\text{cf}^N ([g]_D) = \text{cf}^N (\sup (i " \lambda") < i (\kappa))$, and $D$ is ($\kappa, \lambda$)-regular.

If $\kappa$ is singular, take a $\delta$-complete weakly normal uniform ultrafilter $U$ over $\kappa^+$. We have $\{\alpha < \kappa^+ \mid \text{cf} (\alpha) \leq \kappa\} \in U$, and $\{\alpha < \kappa^+ \mid \text{cf} (\alpha) < \kappa\} \in U$ since $\kappa$ is singular. Then $U$ is ($\kappa, \kappa^+$)-regular. The rest is the same to the case that $\kappa$ is regular.

This completes the proof of Theorem 1.2.

Using Theorem 1.2, we also have the following characterization of $\delta$-strongly compact cardinals.
Corollary 2.10. Let $\delta \leq \kappa$ be uncountable cardinals. Then the following are equivalent:

1. $\kappa$ is $\delta$-strongly compact.
2. For every regular $\lambda \geq \kappa$, there is an elementary embedding $j : V \rightarrow M$ into some transitive model $M$ with $\delta \leq \text{crit}(j) \leq \kappa$ and $\sup(j^{+}\lambda) < j(\lambda)$.
3. For every regular $\lambda \geq \kappa$, there is an elementary embedding $j : V \rightarrow M$ into some transitive model $M$ with $\delta \leq \text{crit}(j)$ and $\sup(j^{+}\lambda) < j(\lambda)$.

Proof. For (1) $\Rightarrow$ (2), suppose $\kappa$ is $\delta$-strongly compact. Then for every regular $\lambda \geq \kappa$, there is a $\delta$-complete fine ultrafilter over $\mathcal{P}_\kappa \lambda$. If $j : V \rightarrow M$ is the ultrapower induced by the ultrafilter, then we have that the critical point of $j$ is between $\delta$ and $\kappa$, and $\sup(j^{+}\lambda) < j(\lambda)$.

(2) $\Rightarrow$ (3) is trivial. For (3) $\Rightarrow$ (1), it is enough to see that every regular $\lambda \geq \kappa$ carries a $\delta$-complete uniform ultrafilter. Let $\lambda \geq \kappa$ be regular. Take an elementary embedding $j : V \rightarrow M$ with $\delta \leq \text{crit}(j)$ and $\sup(j^{+}\lambda) < j(\lambda)$. Define $U \subseteq \mathcal{P}(\lambda)$ by $X \subseteq U \iff \sup(j^{+}\lambda) \in j(X)$. It is easy to check that $U$ is a $\delta$-complete uniform ultrafilter over $\lambda$.

Bagaria and Magidor [3] proved that the least $\delta$-strongly compact cardinal must be a limit cardinal. We can prove the following slightly stronger result using Theorem 1.2.

For a regular cardinal $\nu$ and $f,g \in {}^{\nu}\nu$, define $f \leq^* g$ if the set $\{\alpha < \nu \mid f(\alpha) > g(\alpha)\}$ is bounded in $\nu$. A family $F \subseteq {}^{\nu}\nu$ is unbounded if there is no $g \in {}^{\nu}\nu$ such that $f \leq^* g$ for every $f \in F$. Then let $b_\nu = \min\{|F| \mid F \subseteq {}^{\nu}\nu \text{ is unbounded}\}$. Note that $b_\nu$ is regular and $\nu^+ \leq b_\nu \leq 2^{\nu}$.

Proposition 2.11. Let $\delta$ be an uncountable cardinal, and suppose $\kappa$ is the least $\delta$-strongly compact cardinal. Then for every cardinal $\mu < \kappa$, there is a regular $\nu$ with $\mu \leq \nu < b_\nu < \kappa$. As an immediate consequence, $\kappa$ is a limit cardinal.

Proof. Fix $\mu < \kappa$. Take a regular $\nu$ as follows. If $\mu \geq \delta$, by the minimality of $\kappa$, there is a regular $\nu \geq \mu$ such that $\nu$ cannot carry a $\delta$-complete uniform ultrafilter over $\nu$. We know $\nu < \kappa$ since $\kappa$ is $\delta$-strongly compact. If $\mu < \delta$, let $\nu = \mu^+$. $\nu$ is regular with $\nu \leq \delta \leq \kappa$. We show that $b_\nu < \kappa$ in both cases. Let $\lambda = b_\nu$, and suppose to the contrary that $\lambda \geq \kappa$. By Corollary 2.10, we can find an elementary embedding $j : V \rightarrow M$ with $\delta \leq \text{crit}(j) \leq \kappa$ and $\sup(j^{+}\lambda) < j(\lambda)$. Then we have $\sup(j^{+}\nu) = j(\nu)$; Otherwise, we can take a $\delta$-complete uniform ultrafilter $U = \{X \subseteq \nu \mid \sup(j^{+}\nu) \in j(X)\}$ over $\nu$. If $\mu \geq \delta$, this contradicts to the choice of $\nu$. Suppose $\mu < \delta$. Note that $U$
is in fact \( \text{crit}(j) \)-complete. Since \( \nu \leq \delta \leq \text{crit}(j) \leq \nu \), we have \( \text{crit}(j) = \nu \). However this is impossible since \( \nu \) is successor but \( \text{crit}(j) \) is measurable.

Fix an unbounded set \( F \subseteq {}^{\nu} \nu \) with size \( \lambda \). Let \( F = \{ f_\alpha \mid \alpha < \lambda \} \). Consider \( j(F) = \{ f'_\alpha \mid \alpha < j(\lambda) \} \). Let \( \gamma = \sup(j^{\nu} \nu) < j(\lambda) \). By the elementarity of \( j \), the set \( \{ f'_\alpha \mid \alpha < \gamma \} \) is bounded in \( j^{(\nu) \nu} \) in \( M \). Thus there is \( g' \in j^{(\nu) \nu} \) such that \( f'_\alpha \leq^* g' \) for every \( \alpha < \gamma \). Take \( g \in {}^{\nu} \nu \) so that \( g'(j(\xi)) \leq j(g(\xi)) \) for every \( \xi < \nu \), this is possible since \( \sup(j^{(\nu) \nu}) = j(\nu) \). Then there is \( \alpha < \lambda \) with \( f_\alpha \not\leq^* g \). \( j(f_\alpha) = f'_{j(\alpha)} \leq^* g' \), thus there is \( \eta < \nu \) such that \( j(f_\alpha(\xi)) = f'_{j(\alpha)}(j(\xi)) \leq g'(j(\xi)) \) for every \( \xi \geq \eta \). However then \( j(f_\alpha(\xi)) \leq g'(j(\xi)) \leq j(g(\xi)) \), and \( f_\alpha(\xi) \leq g(\xi) \) for every \( \xi \geq \eta \), this is a contradiction.

\( \square \)

**Question 2.12.** For an uncountable cardinal \( \delta \), is the least \( \delta \)-strongly compact cardinal strong limit? Or a fixed point of \( \aleph \) or \( \beth \)-functions?

### 3. On Products of \( \delta \)-Lindelöf Spaces

In this section we give a proof of Theorem 1.3. The direction \((2) \Rightarrow (1)\) just follows from the proof in [3]. For the converse direction in the case \( \delta = \omega_1 \), in [3], they used an algebraic method. We give a direct proof, an idea of it come from Gorelic [6].

Now suppose \( \kappa \) is not \( \delta \)-strongly compact. By Theorem 1.2, there is a regular cardinal \( \lambda \geq \kappa \) such that \( \lambda \) cannot carry a \( \delta \)-complete uniform ultrafilter. Let \( F \) be the family of all partitions of \( \lambda \) with size \( \delta \), that is, each \( A \in F \) is a family of pairwise disjoint subsets of \( \lambda \) with \( \bigcup A = \lambda \) and \( |A| < \delta \). Let \( \{ A^\alpha \mid \alpha < 2^\lambda \} \) be an enumeration of \( F \). For \( \alpha < 2^\lambda \), let \( \delta_\alpha = |A^\alpha| < \delta \), and \( \{ A^\alpha_{\xi} \mid \xi < \delta_\alpha \} \) be an enumeration of \( A^\alpha \). We identify \( \delta_\alpha \) as a discrete space, it is trivially \( \delta \)-Lindelöf. We show that the product space \( X = \prod_{\alpha < 2^\lambda} \delta_\alpha \) is not \( \kappa \)-Lindelöf.

For \( \gamma < \lambda \), define \( f_\gamma \in X \) as follows: For \( \alpha < 2^\lambda \), since \( A^\alpha \) is a partition of \( \lambda \), there is a unique \( \xi < \delta_\alpha \) with \( \gamma \in A^\alpha_\xi \). Then let \( f_\gamma(\alpha) = \xi \).

Let \( Y = \{ f_\gamma \mid \gamma < \lambda \} \). It is clear that \( |Y| = \lambda \).

**Claim 3.1.** For every \( g \in X \), there is an open neighborhood \( O \) of \( g \) such that \( |O \cap Y| < \lambda \).

**Proof.** Suppose not. Then the family \( \{ A^\alpha_{g(\alpha)} \mid \alpha < 2^\lambda \} \) has the finite intersection property, moreover for every finitely \( \alpha_0, \ldots, \alpha_n < 2^\lambda \), the intersection \( \bigcap_{i \leq n} A^\alpha_{g(\alpha_i)} \) has cardinality \( \lambda \). Hence we can find a uniform ultrafilter \( U \) over \( \lambda \) extending \( \{ A^\alpha_{g(\alpha)} \mid \alpha < 2^\lambda \} \). By our assumption, \( U \) is not \( \delta \)-complete.

Then we can take a partition \( A \) of \( \lambda \) with size \( < \delta \) such that \( A \not\in U \) for every \( A \in \mathcal{A} \). We can take \( \alpha < 2^\lambda \) with \( \mathcal{A} = A^\alpha \). However then \( A^\alpha_{g(\alpha)} \in U \), this is a contradiction. \( \square \)
For each \( g \in X \), take an open neighborhood \( O_g \) of \( g \) with \( |O_g \cap Y| < \lambda \). Let \( \mathcal{U} = \{O_g \mid g \in X\} \). \( \mathcal{U} \) is an open cover of \( X \), but has no subcover of size \( < \lambda \) because \( |Y| = \lambda \). Hence \( \mathcal{U} \) witnesses that \( X \) is not \( \lambda \)-Lindelöf, and not \( \kappa \)-Lindelöf. This completes our proof.

By the same proof, we have:

**Corollary 3.2.** Let \( \kappa \) be an uncountable cardinal, and \( \delta < \kappa \) a cardinal. Then the following are equivalent:

1. \( \kappa \) is \( \delta^+ \)-strongly compact.
2. Identifying \( \delta \) as a discrete space, for every cardinal \( \lambda \), the product space \( \delta^\lambda \) is \( \kappa \)-Lindelöf.

4. On Products of Countably Tight Spaces

We prove Theorems 1.4 and 1.5 in this section. For a topological space \( X \) and \( Y \subseteq X \), let \( \overline{Y} \) denote the closure of \( Y \) in \( X \).

**Lemma 4.1.** Let \( S \) be an uncountable set and \( U \) a \( \sigma \)-complete ultrafilter over the set \( S \). Let \( X \) be a countably tight space, and \( \{O_s \mid s \in S\} \) a family of open sets in \( X \). Define the set \( O \subseteq X \) by \( x \in O \iff \{s \in S \mid x \in O_s\} \in U \) for \( x \in X \). Then \( O \) is open in \( X \).

**Proof.** It is enough to show that \( \overline{X \ \setminus \ O} \subseteq X \ \setminus \ O \). Take \( x \in \overline{X \ \setminus \ O} \), and suppose to the contrary that \( x \notin X \ \setminus \ O \). We have \( \{s \in S \mid x \in O_s\} \in U \).

Since \( X \) is countably tight, there is a countable \( A \subseteq X \ \setminus \ O \) with \( x \in \overline{A} \). For each \( y \in A \), we have \( \{s \in S \mid y \notin O_s\} \in U \). Since \( A \) is countable and \( U \) is \( \sigma \)-complete, there is \( s \in S \) such that \( y \notin O_s \) for every \( y \in A \) but \( x \in O_s \). Then \( A \subseteq X \ \setminus \ O_s \). Since \( O_s \) is open, we have \( \overline{X \ \setminus \ O_s} \subseteq X \ \setminus \ O_s \). Hence \( x \in \overline{A} \subseteq \overline{X \ \setminus \ O_s} \subseteq X \ \setminus \ O_s \), and \( x \notin O_s \). This is a contradiction. \( \square \)

The following proposition immediately yields Theorem 1.4.

**Proposition 4.2.** Suppose \( \kappa \) is \( \omega_1 \)-strongly compact, and \( \mu \leq \kappa \) the least measurable cardinal. Let \( I \) be a set with \( |I| < \mu \), and \( \{X_i \mid i \in I\} \) a family of countably tight spaces. Then \( t(\prod_{i \in I} X_i) \leq \kappa \). More precisely, for every \( A \subseteq \prod_{i \in I} X_i \) and \( f \in \overline{A} \), there is \( B \subseteq A \) such that \( |B| < \kappa \) and \( f \in \overline{B} \).

**Proof.** Take \( A \subseteq \prod_{i \in I} X_i \) and \( f \in \overline{A} \). We will find \( B \subseteq A \) with \( |B| < \kappa \) and \( f \in \overline{B} \).

Since \( \kappa \) is \( \omega_1 \)-strongly compact, we can find a \( \sigma \)-complete fine ultrafilter \( U \) over \( \mathcal{P}_\kappa(\prod_{i \in I} X_i) \). Note that \( U \) is in fact \( \mu \)-complete. We show that \( \{s \in \mathcal{P}_\kappa(\prod_{i \in I} X_i) \mid f \in \overline{A \cap s}\} \in U \). Suppose not and let \( E = \{s \in \mathcal{P}_\kappa(\prod_{i \in I} X_i) \mid f \notin \overline{A \cap s}\} \in U \). For each \( s \in E \), since \( f \notin \overline{A \cap s} \), we can
choose finitely many \(i_0^n, \ldots, i_n^n \in I\) and open sets \(O_{i_k}^n \subseteq X_{i_k}\) respectively such that \(f(i_k^n) \in O_{i_k}^n\) for every \(k \leq n\) but \(\{g \in A \cap s \mid \forall k \leq n (g(i_k^n) \in O_{i_k}^n)\}\) = \(\emptyset\). Since \(U\) is \(\mu\)-complete and \(|I| < \mu\), we can find \(i_0, \ldots, i_n \in I\) such that \(E' = \{s \in E' \mid \forall k \leq n (i_k = i_k^n)\} \in U\).

For each \(i_k\), let \(O_{i_k} \subseteq X_{i_k}\) be a set defined by \(x \in O_{i_k} \iff \{s \in E' \mid x \in O_{i_k}^n\} \in U\). By lemma 4.1, each \(O_{i_k}\) is open in \(X_{i_k}\) with \(f(i_k) \in O_{i_k}\). Since \(f \in \mathcal{A}\), there is \(h \in A\) such that \(h(i_k) \in O_{i_k}\) for every \(k \leq n\). Because \(U\) is fine, we can take \(s \in E'\) with \(h \in A \cap s\) and \(h(i_k) \in O_{i_k}\) for every \(k \leq n\). Then \(h \in \{g \in A \cap s \mid \forall k \leq n (g(i_k^n) \in O_{i_k}^n)\}\), this is a contradiction. \(\square\)

**Note 4.3.**

1. The restriction \(|I| < \mu\) in Proposition 4.2 cannot be eliminated. If \(I\) is an infinite set and \(\{X_i \mid i \in I\}\) is a family of \(T_1\) spaces with \(|X_i| \geq 2\), then \(t(\prod_{i \in I} X_i) \geq |I|\); For each \(i \in X\) take distinct points \(x_i, y_i \in X\). For each finite subset \(a \subseteq I\), define \(f_a \in \prod_{i \in I} X_i\) by \(f_a(i) = x_i\) if \(\gamma \in a\), and \(f_a(i) = y_i\) otherwise. Let \(X = \{f_a \mid a \in [I]^{<\omega}\}\), and \(g\) the function with \(g(i) = x_i\) for \(i \in I\).
   
   Then \(g \in \mathcal{Y}\) but for every \(Y \subseteq X\) with \(|Y| < |I|\) we have \(g \notin \mathcal{Y}\).

2. On the other hand, we do not know if Proposition 4.2 can be improved as follows: If \(\kappa\) is the least \(\omega_1\)-strongly compact and \(I\) is a set with size \(< \kappa\), then the product of countably tight spaces indexed by \(I\) has tightness \(\leq \kappa\).

Recall that the Cohen forcing notion \(\mathbb{C}\) is the poset \(2^{<\omega}\) with the reverse inclusion order.

**Lemma 4.4.** Let \(\kappa\) be a cardinal which is not \(\omega_1\)-strongly compact. Let \(\mathbb{C}\) be the Cohen forcing notion, and \(G\) be \((V, \mathbb{C})\)-generic. Then in \(V[G]\), there are regular \(T_1\) Lindelöf spaces \(X_0^n\) and \(X_1^n\) such that \(X_0^n\) and \(X_1^n\) are Lindelöf for every \(n < \omega\), but the product space \(X_0 \times X_1\) has an open cover which has no subcover of size \(< \kappa\).

**Proof.** Let \(X_0^n\) and \(X_1^n\) be spaces constructed in the proof of Proposition 3.1 in [9]. We know that \(X_0 \times X_1^n\) has an open cover which has no subcover of size \(< \kappa\). In addition, we can check that \(X_0^n\) and \(X_1^n\) are Lindelöf for every \(n < \omega\) (see the proof of Proposition 3.9 in [9]). \(\square\)

For a Tychonoff space \(X\), let \(C_p(X)\) be the space of all continuous functions from \(X\) to the real line \(\mathbb{R}\) with the pointwise convergent topology. For a topological space \(X\), the Lindelöf degree \(L(X)\) is the minimum infinite cardinal \(\kappa\) such that every open cover of \(X\) has a subcover of size \(\leq \kappa\). Hence \(X\) is Lindelöf if and only if \(L(X) = \omega\).

**Theorem 4.5** (Arhangel’skii-Pytkeev [1, 8]). Let \(X\) be a Tychonoff space, and \(\nu\) a cardinal. Then \(L(X^n) \leq \nu\) for every \(n < \omega\) if and only if \(t(C_p(X)) \leq \nu\).

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\( \nu \). In particular, each finite power of \( X \) is Lindelöf if and only if \( C_p(X) \) is countably tight.

**Proposition 4.6.** Let \( \kappa \) be a cardinal which is not \( \omega_1 \)-strongly compact. Let \( \mathbb{C} \) be the Cohen forcing notion, and \( G \) be \( (V, \mathbb{C}) \)-generic. Then in \( V[G] \), there are regular \( T_1 \) Lindelöf spaces \( X_0 \) and \( X_1 \) such that \( C_p(X_0) \) and \( C_p(X_1) \) are countably tight and \( t(C_p(X_0) \times C_p(X_1)) \geq \kappa \).

**Proof.** Let \( X_0 \) and \( X_1 \) be spaces in Lemma 4.4. By Theorem 4.5, \( C_p(X_0) \) and \( C_p(X_1) \) are countably tight. It is clear that \( C_p(X_0) \times C_p(X_1) \) is homeomorphic to \( C_p(X_0 \oplus X_1) \), where \( X_0 \oplus X_1 \) is the topological sum of \( X_0 \) and \( X_1 \). We have \( L((X_0 \oplus X_1)^2) \geq L(X_0 \times X_1) \geq \kappa \), hence \( t(C_p(X_0) \times C_p(X_1)) \geq \kappa \) by theorem 4.5 again. \( \square \)

Combining these results we have Theorem 1.5:

**Corollary 4.7.** Let \( \mathbb{C} \) be the Cohen forcing notion, and \( G \) be \( (V, \mathbb{C}) \)-generic. Then for every cardinal \( \kappa \) the following are equivalent in \( V[G] \):

1. \( \kappa \) is \( \omega_1 \)-strongly compact.
2. For every countably tight spaces \( X \) and \( Y \) we have \( t(X \times Y) \leq \kappa \).
3. For every countably tight Tychonoff spaces \( X \) and \( Y \) we have \( t(X \times Y) \leq \kappa \).
4. For every regular \( T_1 \) Lindelöf spaces \( X \) and \( Y \), if \( C_p(X) \) and \( C_p(Y) \) are countably tight then \( t(C_p(X) \times C_p(Y)) \leq \kappa \).

Theorem 1.5 is a consistency result, and the following natural question arises:

**Question 4.8.** In ZFC, is the least \( \omega_1 \)-strongly compact cardinal a precise upper bound on the tightness of the products of two countably tight spaces? How about Fréchet-Urysohn spaces?

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