(Super-)renormalizably dressed black holes

Eloy Ayón-Beato,1,2,3 Mokhtar Hassaïne,4 and Julio A. Méndez-Zavaleta1,2

1Departamento de Física, CINVESTAV–IPN, Apdo. Postal 14-740, 07000 México D.F., México
3Centro de Estudios Científicos (CECs), Casilla 1468 Valdivia, Chile
4Instituto de Matemática y Física, Universidad de Talca, Casilla 747 Talca, Chile

Black holes supported by self-interacting conformal scalar fields can be considered as renormalizably dressed since the conformal potential is nothing but the top power-counting renormalizable self-interaction in the relevant dimension. On the other hand, potentials defined by powers which are lower than the conformal one are also phenomenologically relevant since they are in fact super-renormalizable. In this work we provide a new map that allows to build black holes dressed with all the (super-)renormalizable contributions starting from known conformal seeds. We explicitly construct several new examples of these solutions in dimensions $D = 3$ and $D = 4$, including not only stationary configurations but also time-dependent ones.

I. INTRODUCTION

The nontrivial interaction between matter and geometry mathematically encoded through the equations proposed by Einstein a century ago is one of the crucial paradigms of General Relativity. As it is well-known, the nonlinearity of Einstein equations makes the task of finding physically interesting solutions very hard. In order to render this problem more easy to handle, the presence of certain symmetry can be primordial to simplify the problem or as a tool to generate nontrivial solutions from simpler ones.

As a concrete example, we mention the case of a self-gravitating scalar field conformally coupled to gravity. Indeed, in this case, the conformal symmetry of the matter source implies that the scalar curvature is zero which in turn considerably simplifies the field equations for a simple ansatz, yielding to the so-called Bekenstein black hole solution [1, 2]. This solution was in fact discovered by Bekenstein exploiting the conformal machinery enjoyed by the system by mapping the original action to a conformally invariant version [3]. This result can be proved in full generality by straightforwardly integrating the static and spherically symmetric equations of motion, which in turn implies the uniqueness of the Bekenstein black hole [4]. Nevertheless, the solution suffers from a pathological behavior due to the fact that the scalar field diverges at the horizon which makes rather obscure its physical interpretation [5]. A way of circumventing this problem is to reconsider the inclusion of the self-interaction that does not spoil the conformal invariance but this time together with a cosmological constant, whose effect is precisely to push this singularity behind the horizon. The resulting configuration is the so-called Martinez-Troncoso-Zanelli (MTZ) black hole [6], which also allows a charged generalization [7]. Due to the presence of the cosmological constant, the existence of these configurations is ensured not only for horizons with spherical topology but also with hyperbolic one. In Ref. [8], it has been shown that the inclusion of axion fields coupled to the scalar field allows the existence of black hole configurations with planar horizon topology, valid even for more general nonminimal couplings than the conformal one.

Black holes dressed by a conformal scalar field in presence of a cosmological constant were in fact first analyzed in $2 + 1$ dimensions by Martinez and Zanelli (MZ) [9]. The MZ black hole was later generalized to its conformally self-interacting version [10], and a further nonlinearly charged extension preserving the conformal invariance of the full source has been given recently in [11]. It is even known the exact gravitational collapse to the neutral self-interacting version [12].

In contrast, for higher dimensions ($D > 4$) the situation is quite different; the Bekenstein conformal mapping can be lifted, but it produces only naked singularities [13]. This result can be proved in full generality by straightforwardly integrating the static and spherically symmetric equations of motion, which in turn implies the uniqueness of the Bekenstein black hole within all the asymptotically flat black hole solutions allowed by a conformal scalar field in any dimension [14]. No similar result in such generality is known in presence of a cosmolog-
tional constant. Nevertheless, it can be shown that in the case of the simplest metric ansatz involving a single function, the only black hole solutions are those previously mentioned for $D = 3$ and $D = 4$. For completeness, we also mention that generalizing the conformal action in higher dimensions via nonminimal couplings of the scalar field with higher-curvature Lovelock terms built out of a Weyl connection (by construction conformally invariant) whose vectorial part is a pure-gauge contribution of the derivatives of the scalar field, asymptotically (A)dS black hole solutions can be obtained. Additionally, generalizing also the gravity action to the Lovelock one it is possible to find topological black hole configurations for a self-interacting scalar field with standard nonminimal coupling to gravity; some of those examples include conformally coupled scalar fields.

Another important aspect of the conformal symmetry as guiding principle is the fact that the power defining the conformal self-interaction in any dimension $D$ is just $2D/(D - 2)$. In mass units $c = 1 = h$, this implies that the conformal coupling constant is dimensionless and consequently these theories are power-counting renormalizable. From the phenomenological point of view, self-interactions with lower powers than the conformal one are also relevant, since its coupling constants have positive mass dimension and they define super-renormalizable theories. In fact, the mechanism explaining the spontaneous symmetry breaking of gauge theories is naturally modeled with these kind of contributions. Due to the recent experimental confirmation of the existence of scalar bosons in nature, exploring their self-gravitating behavior is an important task to study, even in their more extremal realization, i.e. their possible gravitational collapse to black holes. Examples of the final state of this collapse are given by the (super-)renormalizable dressed black holes found by Anabalon and Cisterna (AC) for which the scalar field is no longer conformally invariant but retains its conformal coupling. Indeed, conformal invariance is explicitly broken in their source because the potential involves additional powers of the scalar field, lower than the conformal one. One of the aim of the present paper is precisely to explain the emergence of these additional contributions to the potential by means of a generating tool as the ones emphasized at the beginning. This tool does not only explain the existence of the AC black hole, but also gives rise to many new scalar black holes supported with (super-)renormalizable self-interactions.

In this work we concretely show that the Einstein equations with cosmological constant for a self-interacting conformally invariant scalar field can be mapped to their counterpart having as source an also conformally coupled scalar field, but with the difference of being subject to a more general self-interaction that explicitly breaks conformal invariance. We prove this result by showing that, via a very precise map, the corresponding actions are proportional to each other. This mapping is realized through the composition of a shift on the original scalar field and a conformal transformation acting on the metric and the scalar field. For dimensions $D = 3, 4$ and $D = 6$ which correspond to the dimensions where the conformal power $2D/(D - 2)$ is an integer, the expression of the resulting potential involves all the integer powers of the scalar field until the conformal one yielding to a (super-)renormalizable self-interaction. Consequently with the fact that these self-interactions involve in general more coupling constants than the conformal one, this is a one-way map allowing to obtain self-gravitating solutions of a scalar field conformally coupled to gravity and self-interacting via (super-)renormalizable contributions from any self-gravitating conformal seed, but the converse is not true.

The plan of the paper is the following. In the next section, we present the new mapping from a self-gravitating conformal scalar field to another self-gravitating one which is (super-)renormalizably self-interacting and conformally coupled to gravity, both in presence of its respective cosmological constants. In dimensions $D = 4$ and $D = 3$, where black hole solutions dressed by a conformal scalar field are known, we use this mapping to explicitly exhibit new solutions of a (super-)renormalizable and conformally coupled scalar source in Secs. II and IV respectively. Finally, the achieved conclusions are given in Sec. V.

II. FROM RENORMALIZABLE TO (SUPER-)RENORMALIZABLE SOURCES

Our starting point is to consider the $D$-dimensional action of a self-gravitating conformal scalar field in presence of a cosmological constant

$$S[g, \Phi] = \int d^Dx \sqrt{-g} \left( \frac{R - \Lambda}{2\kappa} - \frac{1}{2} \partial_{\mu}\Phi \partial^{\mu}\Phi - \frac{1}{2} \xi_D R \Phi^2 - \lambda \Phi^{\frac{6}{D - 2}} \right). \quad (1)$$

Here, $R$ stands for the scalar curvature, $\Lambda$ represents the cosmological constant, $\lambda$ is the coupling constant of the conformal potential and the conformal coupling is given by

$$\xi_D = \frac{D - 2}{4(D - 1)}. \quad (2)$$

This is the precise value of the nonminimal coupling to gravity that ensures the scalar contribution to the action $1$ to be invariant, up to a boundary term, under a conformal transformation

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}, \quad \Phi \mapsto \Omega^{\frac{2-D}{2}} \Phi, \quad (3)$$

where $\Omega$ is an arbitrary local function. The field equations for the conformal scalar field aris-
ing from the variation of the action \( (11) \) read

\[ G_{\mu \nu} + \Lambda g_{\mu \nu} = \kappa T_{\mu \nu}, \quad (4a) \]
\[ \Box \Phi - \xi_D R \Phi = \frac{2\Lambda D}{D-2} \Phi^{\frac{2D}{D-2}}, \quad (4b) \]

where the conformally invariant energy-momentum tensor is defined by

\[ T_{\mu \nu} = \nabla_\mu \Phi \nabla_\nu \Phi - g_{\mu \nu} \left( \frac{1}{2} \nabla_\alpha \Phi \nabla^\alpha \Phi + \lambda \Phi \right) - \xi_D (g_{\mu \nu} \Box - \nabla_\mu \nabla_\nu + G_{\mu \nu}) \Phi^2. \quad (4c) \]

Here, we show that a slightly generalization of the transformations \( (3) \) can also induce some interesting features not only for the matter source but for the full action \( (11) \). Indeed, we prove that there exists a special conformal frame, defined by taking a precise power of the conformal factor as an affine function of the scalar field, where a simple shift of the scalar field permits to transform to a new action

\[ S[\tilde{g}, \tilde{\Phi}] = (1 - a^2) S[g, \Phi], \quad (6) \]

which also describes a self-gravitating conformally coupled scalar field \( (11) \) transforms to a new action

\[ \tilde{S}[\tilde{g}, \tilde{\Phi}] = \int d^D x \sqrt{-\tilde{g}\left( \frac{\tilde{R} - 2\tilde{\Lambda}}{2\kappa} - \frac{1}{2} \tilde{\partial}_\mu \tilde{\Phi} \tilde{\partial}^\mu \tilde{\Phi} - \frac{1}{2} \xi_D \tilde{R} \tilde{\Phi}^2 - V(\tilde{\Phi}) \right) }, \quad (7a) \]

but subject to a different self-interaction potential given by

\[ V(\tilde{\Phi}) = \frac{1}{1 - a^2} \left\{ \frac{\Lambda}{\kappa} \left( 1 - a \sqrt{\kappa \xi_D} \Phi^\frac{2D}{D-2} - 1 \right) + \lambda \left[ \left( \tilde{\Phi} - \frac{a}{\sqrt{\kappa \xi_D}} \right)^\frac{2D}{D-2} - \left( - \frac{a}{\sqrt{\kappa \xi_D}} \right)^\frac{2D}{D-2} \right] \right\}, \quad (7b) \]

and in presence of a modified cosmological constant

\[ \tilde{\Lambda} = \frac{\kappa}{1 - a^2} \left\{ \frac{\Lambda}{\kappa} + \lambda \left( - \frac{a}{\sqrt{\kappa \xi_D}} \right)^\frac{2D}{D-2} \right\}. \quad (7c) \]

Notice that these precise definitions ensure that the new potential does not contain zeroth-order terms, and hence the modified cosmological constant lacks of further contributions.

We are entitled to ask the reasons for which this mapping can be relevant. Apart from generating new solutions from conformal ones, the mapping has also a nice feature in dimensions \( D = 3, 4 \) and \( 6 \) where the conformal power \( 2D/(D-2) \) is an integer. Indeed, in those cases, the resulting self-interaction \( (11) \) enhances the original conformal one with all the power-counting super-renormalizable contributions, i.e. it becomes a polynomial of degree \( 2D/(D-2) \),

\[ V(\tilde{\Phi}) = \lambda_1 \tilde{\Phi} + \lambda_2 \tilde{\Phi}^2 + \cdots + \lambda_{2D/(D-2)} \tilde{\Phi}^{2D/(D-2)}. \quad (8) \]

Since this mapping is operated at the level of the actions and only change them by a global multiplicative factor, the solutions of the field equations \( (11) \) can be mapped to solutions of the field equations arising from the variation of the action \( (7) \) which are given by

\[ \tilde{G}_{\mu \nu} + \tilde{\Lambda} g_{\mu \nu} = \kappa \tilde{T}_{\mu \nu}, \quad (9a) \]
\[ \Box \tilde{\Phi} - \xi_D \tilde{R} \tilde{\Phi} = \frac{dV(\tilde{\Phi})}{d\tilde{\Phi}}, \quad (9b) \]

where now the new energy-momentum tensor is defined by

\[ \tilde{T}_{\mu \nu} = \tilde{\partial}_\mu \tilde{\Phi} \tilde{\partial}_\nu \tilde{\Phi} - \tilde{g}_{\mu \nu} \left( \frac{1}{2} \tilde{\partial}_\alpha \tilde{\Phi} \tilde{\partial}^\alpha \tilde{\Phi} + V(\tilde{\Phi}) \right) - \xi_D (\tilde{g}_{\mu \nu} \Box - \tilde{\nabla}_\mu \tilde{\nabla}_\nu + \tilde{G}_{\mu \nu}) \tilde{\Phi}^2, \quad (9c) \]

and involves a much more general self-interaction potential \( (7) \).

It is worth mentioning that this mapping is only effective for \( a \neq \pm 1 \), and for \( a^2 < 1 \) it preserves the unitarity of both actions. Additionally, for \( a = 0 \) it reduces to the identity. Regarding the interpretation of the parameter \( a \), notice that if the starting conformal configuration vanishes at infinity then this parameter in the scalar map \( (5) \) is just related to the constant value \( \Phi_0 \) of the bar field at infinity, i.e. \( a = \sqrt{\kappa \xi_D} \Phi_0 \). As stressed before, the metric transformation \( (5) \) is easily understood as a conformal transformation to a precise conformal frame. In contrast, the meaning of the scalar transformation \( (5) \) is more subtle, but for \( a^2 < 1 \) it may be viewed as a SL(2, \mathbb{R}) transformation.

As an interesting remark that will be relevant in what follows, one can mention that this mapping can be extended to a starting action given by \( (11) \) supplemented with an extra piece that is conformally invariant, for instance the Maxwell action in \( D = 4 \) or its nonlinear conformal extension in arbitrary dimension \( (23) \). Indeed, under the mapping \( (5) \) the additional action, being conformally invariant, would remain in principle unchanged. Hence, supplementing the conformal transformation on all the involved fields with a trivial scaling in the additional field, the full action will change by the same global factor to a bar action given by \( (7) \) together with the bar
valued version of the conformal extra piece. The procedure will become more clear in the next section since we shall include the Maxwell action in the applications of the mapping in $D = 4$.

We would like to point out that for a free conformal scalar field in the absence of cosmological constant, i.e. $\lambda = 0$ and $\Lambda = 0$, it is easy to check from both, (11) and (10), that the proposed transformation constitutes just a scaling of the self-gravitating free conformal action to itself and consequently a symmetry of the involved equations of motion. Starting in this case from the Bekenstein black hole the wormhole solution [26] is obtained, see Ref. [27] for a recent discussion on this point. In the next sections we concentrate in the cases where $\lambda \neq 0$ and $\Lambda \neq 0$ and generate new classes of solutions connected by the map to well-known conformal seeds.

### III. Generating New Solutions in $D = 4$

We now proceed to exploit the method of generating solutions for self-gravitating (super-)renormalizable scalar sources conformally coupled to gravity, which are rigged by equations (9), from known conformal solutions of equations (1). As stressed before, the map (5) can be extended in order to include an extra source that is also conformally invariant. Nevertheless, in order for the Maxwell action to be mapped with the same global symmetry denoted by $k$ and the constant $M$ related to the mass together with the electric charge $q$ are tied via the coupling constants as follows

$$q^2 = \frac{2\pi}{9} \frac{kM^2(\kappa\Lambda + 36\lambda)}{\kappa\lambda}.$$  

In the neutral limit $q = 0$ it becomes the original MTZ black hole [6] with its known fine tuning between the coupling constants, $\Lambda/\lambda = -36/\kappa$. For $\Lambda = 0 = \lambda$ and $k = 1$ the MTZ black hole reduces to the Bekenstein one [1]. The solution (13) will be our conformal seed configuration in order to generate solutions of the field equations obtained from the variation of the action at the left hand side of (12), defined by

$$\mathcal{G}_{\mu\nu} + \kappa \tilde{g}_{\mu\nu} = \kappa \left[ \bar{T}_{\mu\nu} + \frac{1}{4\pi} \left( \bar{F}_{\mu\sigma} \bar{F}^{\sigma\nu} - \frac{1}{4} \bar{g}_{\mu\nu} \bar{F}_{\alpha\beta} \bar{F}^{\alpha\beta} \right) \right],$$

$$\nabla_{\mu} \bar{F}^{\mu\nu} = 0,$$

where the potential (13a) is given in four dimensions by the (super-)renormalizable one

$$V(\Phi) = \lambda_{1} \bar{\Phi} + \lambda_{2} \bar{\Phi}^{2} + \lambda_{3} \bar{\Phi}^{3} + \lambda_{4} \bar{\Phi}^{4},$$

with couplings constants determined by

$$\lambda_{1} = -\frac{2\sqrt{6}}{27} \frac{a(\kappa\Lambda + 36a^{2}\lambda)}{(1 - a^{2})^{3}},$$

$$\lambda_{2} = \frac{a^{2}(\kappa\Lambda + 36\lambda)}{\kappa(1 - a^{2})^{3}},$$

$$\lambda_{3} = -\frac{\sqrt{6}}{9} \frac{a^{2}a^{2}\kappa\Lambda + 36\lambda}{\kappa^{1/2}(1 - a^{2})^{3}},$$

$$\lambda_{4} = \frac{1}{36} \frac{a^{4}\kappa\Lambda + 36\lambda}{(1 - a^{2})^{5}},$$

and the cosmological constant takes the following expression

$$\Lambda = \kappa\Lambda + 36a\Lambda^{4}\frac{\lambda}{\kappa(1 - a^{2})^{3}}.$$

![Equation](image_url)
The above parameterizations can be understood as follows: the transformations (16) and (15c) are just a one-parameter invertible linear map between the initial cosmological and renormalizable coupling constants \((\lambda, \lambda)\) and the final ones \((\bar{\lambda}, \lambda)\). Rewriting the rest of the parameterizations in terms of \((\bar{\lambda}, \lambda)\), via this invertible linear map, one can consider the final cosmological and renormalizable coupling constants as arbitrary and the full transformations just describe a one-parameter invertible linear map between the initial parameter subspace of the three-dimensional parameter space \((\lambda_1, \lambda_2, \lambda_3)\) characterizing the strictly super-renormalizable contributions. This is the precise subspace of the general problem with (super-)renormalizable self-interactions which is accessible via the mapping (15) from the conformal sector.

The map acting on (13) gives rise to a new charged solution of the field equations (11) that reads

\[
ds^2 = \left(\frac{r - M}{r - M}ight)^2 \left(1 - \frac{r - M}{r - M}ight)^2 dt^2 \\
+ \left[\frac{-\Lambda r^2}{3} + k \left(1 - \frac{M}{r}\right)^2 \right]^{-1} dr^2 + r^2 d\Omega^2,
\]

(17a)

\[
\bar{\Phi}(r) = \sqrt{\frac{6}{\kappa}} \frac{ar + M}{r - M} \left(1 - \frac{r - M}{r - M}\right),
\]

(17b)

\[
\bar{A}(r) = -\frac{\bar{q}}{r} dt,
\]

(17c)

where the charge \(\bar{q}\) is also tied in terms of the integration constant \(M\) via the original coupling constants as

\[
\bar{q}^2 = (1 - a^2) \frac{2\pi kM^2 (k\lambda + 36\lambda)}{9 \kappa \lambda}.
\]

(17d)

We can easily check that in the neutral limit \(\bar{q} = 0\), the relation (17d) leads to the same fine tuning between the original coupling constants characterizing the MTZ black hole,

\[
\lambda = -\frac{1}{36} \kappa\lambda,
\]

(18)

and the solution becomes the Anabalon-Cisterna solution [24] which is supported by a potential involving all the (super-)renormalizable terms except the massive one. This can easily be explained through our mapping since for this specific fine tuning (18), the coupling constant defining the mass (15c) vanishes identically. In fact, the AC solution [24] is exactly the result of applying the proposed map to the MTZ black hole [6], that’s why it must respect the MTZ fine tuning (18). This situation changes when the electric charge is introduced since the involved fine tuning is now understood as a relation between the integration constants and not as one between the coupling constants, which in turns allows the mass term to remain turned on. The AC solution represents black holes and wormholes, and this is also the case in the above charged generalization (17).

In the next section, we will pursue this strategy in the three-dimensional case where there exist more conformal seed configurations.

**IV. GENERATING NEW SOLUTIONS IN \(D = 2 + 1\)**

In three dimensions, we have a priori more configurations that are interesting solutions of the Einstein equations supported by a conformal scalar field [1]. On the one hand there is the well-known Martínez-Zanelli conformal black hole [3], its self-interacting generalization [10] and a time-dependent solution describing the exact gravitational collapse to the last black hole [12]. On the other hand there also exist a different kind of time-dependent solutions named stealth configurations characterized by the peculiarity that both sides (the gravity and the matter source part) of Einstein equations vanish independently. Indeed, in three dimensions, it has been shown in [28] the existence of a time-dependent nontrivial self-interacting scalar field nonminimally coupled to gravity having an energy-momentum tensor that vanishes on the BTZ black hole [29]. Finally, there also exist conformal solutions supporting AdS-waves [30].

In this section we will provide more new examples of self-gravitating scalar field solutions with (super-)renormalizable self-interaction starting from almost all the previously mentioned conformal seeds via the mapping (15). We start by specifying the (super-)renormalizable potential (17b) resulting from the map at \(D = 3\), which will be the common support for all the generated solutions

\[
V(\bar{\Phi}) = \lambda_1 \bar{\Phi} + \lambda_2 \bar{\Phi}^2 + \lambda_3 \bar{\Phi}^3 + \lambda_4 \bar{\Phi}^4 + \lambda_5 \bar{\Phi}^5 + \lambda_6 \bar{\Phi}^6,
\]

(19a)

where the coupling constants are fixed as

\[
\lambda_1 = -\frac{3}{\sqrt{2}} \frac{a(k^2\Lambda + 512a^4\lambda)}{\kappa^{5/2}(1 - a^2)^5},
\]

(19b)

\[
\lambda_2 = \frac{15}{8} \frac{a^2(k^2\Lambda + 512a^2\lambda)}{\kappa^2(1 - a^2)^5},
\]

(19c)

\[
\lambda_3 = \frac{5}{8} \frac{a^4(k^2\Lambda + 512\lambda)}{\kappa^{7/2}(1 - a^2)^5},
\]

(19d)

\[
\lambda_4 = \frac{15}{64} \frac{a^2(k^2\Lambda + 512\lambda)}{\kappa(1 - a^2)^5},
\]

(19e)

\[
\lambda_5 = \frac{3}{128} \frac{a^4(k^2\Lambda + 512\lambda)}{\kappa^{1/2}(1 - a^2)^5},
\]

(19f)

\[
\lambda_6 = \frac{1}{512} \frac{a^6(k^2\Lambda + 512\lambda)}{(1 - a^2)^5},
\]

(19g)
along with the relationship between the new and the original cosmological constants,
\[
\bar{\Lambda} = \frac{\kappa^2 \Lambda + 512a^6\lambda}{\kappa^2(1 - a^2)^3}.
\]
(20)

As in the four-dimensional case, the above parameterizations have the following interpretation: the transformations (20) and (19a) are again a one-parameter invertible linear map between the initial cosmological and renormalizable contributions. This is the only subspace of the general problem with (super-)renormalizable self-interactions than can be probed in three dimensions from the conformal sector using the mapping (5).

Let us first consider the conformal seed configuration which corresponds to the self-interacting version of the MZ conformal black hole [9] found in [10], which is a solution of the field equations (4),
\[
ds^2 = -\left(\frac{r^2}{l^2} - \frac{2\alpha B^3}{r} - 3\alpha B^2\right)dt^2 + \left(\frac{r^2}{l^2} - \frac{2\alpha B^3}{r} - 3\alpha B^2\right)^{-1}d\varphi^2 + r^2d\varphi^2,
\]
(21a)
\[
\Phi(r) = \sqrt{\frac{8B}{\kappa(r + B)}},
\]
(21b)
\[
\alpha = \frac{\kappa^2 - 512\lambda l^2}{\kappa^2 l^2},
\]
(21c)
where the cosmological constant is chosen to be negative $\Lambda = -1/l^2$ and $B$ is an integration constant. Hence, as in the four-dimensional case, a solution of the field equations (4) with the (super-)renormalizable potential given by (19) can be constructed from the conformal seed (21), and the resulting configuration is given by
\[
ds^2 = \left(\frac{a\sqrt{B} + \sqrt{r + B}}{r + B}\right)^4 \left[\left(\frac{r^2}{l^2} - \frac{2\alpha B^3}{r} - 3\alpha B^2\right)dt^2 + \left(\frac{r^2}{l^2} - \frac{2\alpha B^3}{r} - 3\alpha B^2\right)^{-1}dr^2 + r^2d\varphi^2\right],
\]
(22a)
\[
\tilde{\Phi}(r) = \frac{8\sqrt{B} + a\sqrt{r + B}}{\kappa a\sqrt{B} + \sqrt{r + B}}.
\]
(22b)
We remark that, as in the four-dimensional case, one could have started from the charged version of the solution (21) given in [11], which is supported by a conformally invariant nonlinear electrodynamics [25]. Hence, in this case the map will yield to a nonlinearity charged version of the configuration (22) based on the same conformal electrodynamics, but where the scalar field self-interact with the non-conformal (super-)renormalizable potential (19).

As a second conformal seed, we now consider the exact gravitational collapse of the self-interacting version of the MZ conformal black hole (21) derived in [12]. This is a time-dependent solution which in the limit when time goes to infinity reduces to the configuration (21). We stress that it is one of the few examples of an exact gravitational collapse in the literature. Due to time dependence it is more convenient to write the metric solution in the Eddington-Finkelstein coordinates as
\[
ds^2 = -f(u)^{-2/3}\left(\frac{r^2}{l^2} - \frac{2\alpha B^3}{r}f(u)\right)du^2 + 2f(u)^{-1/3}du dr + r^2d\varphi^2,
\]
(23a)
\[
\Phi(u) = \sqrt{\frac{8B}{\kappa rf(u)^{-4/3} + B}},
\]
(23b)
\[
f(u) = \tanh\left(\frac{3aBu}{8}\right),
\]
(23c)
where $B$ is an integration constant and $\alpha$ is defined in terms of the coupling constants in (21c). Notice that in the limit of $u \to \infty$ we have $f(u) = 1$, i.e. the black hole solution (21) is just the final state of the evolutive solution (23). As before, starting from this conformal seed solution, one can generate a new time-dependent solution of the field equations (4) with a (super-)renormalizable potential fixed by (19)
\[
ds^2 = \left(\frac{a\sqrt{B} + \sqrt{r + B}}{r + B}\right)^4 \left[\left(\frac{r^2}{l^2} - \frac{2\alpha B^3}{r}f(u)\right)^{-2/3}du^2 + 2f(u)^{-1/3}du dr + r^2d\varphi^2\right],
\]
(24a)
\[
\tilde{\Phi}(u) = \sqrt{\frac{8\sqrt{B} + a\sqrt{r(u)^{-4/3} + B}}{\kappa a\sqrt{B} + \sqrt{r(u)^{-4/3} + B}}},
\]
(24b)
Consequently, the final state of this evolution $u \to \infty$ is just the previously generated stationary solution (22).

As the last conformal seed, we consider again a time-dependent configuration but of a different sort. It represents a conformal stealth [28] overflying the BTZ black hole [29]. The stealths are particular nontrivial solutions of Einstein equations where both of its sides vanish inde-
pendently, that is
\[ G_{\mu\nu} - \frac{1}{l^2} g_{\mu\nu} = 0 = \kappa T_{\mu\nu}. \] (25)

The left hand side produces the BTZ black hole \[29\] assuming only rotational symmetry \[30\], while the right hand side has nontrivial solutions only when this black hole is static \[28\]. The resulting conformal stealth is given in the Eddington-Finkelstein coordinates by
\[
\begin{align*}
 ds^2 &= - \left( \frac{r^2}{l^2} - M \right) du^2 - 2 dudr + r^2 d\varphi^2, \quad (26a) \\
 \Phi(u,r) &= \sqrt{\frac{8}{\kappa}} \frac{1}{\sqrt{\sigma(u,r)}}, \quad (26b) \\
 \sigma(u,r) &= \sqrt{\frac{8}{2\lambda - h^2}} \left[ r \cosh\left( \frac{\sqrt{M} u}{l} \right) \right. \\
 &\quad + \sqrt{M} \sinh\left( \frac{\sqrt{M} u}{l} \right) \bigg] + h. \quad (26c)
\end{align*}
\]

It is interesting to note that the mapped solution given by
\[
\begin{align*}
 ds^2 &= \left( \frac{a}{\sqrt{\sigma(u,r)}} + 1 \right)^4 \left[ - \left( \frac{r^2}{l^2} - M \right) du^2 \\
 &\quad - 2 dudr + r^2 d\varphi^2 \right], \\
 \Phi(u,r) &= \sqrt{\frac{8}{\kappa}} \frac{1 + a\sqrt{\sigma(u,r)}}{a + \sqrt{\sigma(u,r)}}, \quad (27b)
\end{align*}
\]
is as usual a new solution of the field equations \[29\] exhibiting also time dependence, but now it is not longer a stealth configuration \[25\] for the bar fields.

V. CONCLUSIONS

Here, we provide new examples of self-gravitating solutions in presence of a cosmological constant \( \Lambda \) for scalar fields conformally coupled to gravity, allowed to self-interact with themselves via a potential where all the super-renormalizable contributions, defined by the positive mass-dimension coupling constants \( \lambda_1, \ldots, \lambda_{(D+2)/(D-2)} \), and the renormalizable one described by the dimensionless coupling constant \( \lambda_{2D/(D-2)} \) are turned on. This was possible due to the introduction of a new one-parameter mapping connecting any self-gravitating conformal scalar solution with the previous configurations. The map consists in a conformal transformation for the metric and a \( \text{SL}(2,\mathbb{R}) \) transformation for the scalar field. All the studied cases allow the following interpretation of the mapping: the sector of the general problem with (super-)renormalizable self-interactions that can be probed with a conformal counterpart, is the one where the cosmological and renormalizable coupling constants, \( \Lambda \) and \( \lambda_{2D/(D-2)} \), are arbitrary but from the \( (D+2)/(D-2) \)-dimensional parameter space \( (\lambda_1, \ldots, \lambda_{(D+2)/(D-2)}) \) of the super-renormalizable contributions only a one-parameter subspace described by the mapping is accessible.

We systematically use well-known conformal solutions in the literature as seed configurations in this map to generate the new (super-)renormalized dressed solutions. Concretely, we exhibit a charged version of the AC solution at \( D = 4 \) \[24\], starting from the charged version \[17\] of the MTZ black hole \[6\]. We explain why the AC configurations have no mass term in the potential and how this mass term appears now due to the presence of the electric charge. Many other similar examples are generated in \( D = 3 \). The first of them is generated from the generalization of the MZ black hole \[8\] including a conformal self-interaction \[10\]. A second one is generated from the exact gravitational collapse of the previous conformal black hole \[12\], giving this time a (super-)renormalizable dressed time-dependent configuration. As a last example, the conformal stealth overlying the static BTZ black hole is used as seed \[28\]. The resulting configuration is again time dependent but, since the map mix the gravity and matter contributions, the resulting solution is not longer a stealth configuration.

All the generated (super-)renormalized dressed configurations have the same asymptotic as their conformal seeds since the involved conformal factors take unit value at infinity, i.e. they are all asymptotically (A)dS spacetimes. On the contrary, the (super-)renormalized self-interacting scalar fields no longer vanish at infinity as its conformal counterparts. In fact, the parameter of the introduced mapping defines the constant value of the final fields at infinity. However, it cannot be interpreted as a new integration constant since as previously discussed this parameter appears in the action probing the strictly super-renormalizable sector.

Finally, at dimension \( D = 6 \) our generating technique can also provide (super-)renormalized dressed solutions, this time for potentials built from cubic polynomials. However, the only known conformal seeds in this dimension to our knowledge are those consisting of stealth configurations and AdS-waves \[32\]. The relevant stealth examples live on flat spacetime \[33\] and (A)dS \( 6 \) spacetime \[31\]. We leave the exploration of the resulting six-dimensional configurations for future work.

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