POSITIVE SOLUTIONS AND HARMONIC MEASURE FOR SCHRÖDINGER OPERATORS IN UNIFORM DOMAINS

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Abstract. We give bilateral pointwise estimates for positive solutions of the equation
\[
\begin{cases}
-\Delta u = \omega u & \text{in } \Omega, \quad u \geq 0, \\
u = f & \text{on } \partial \Omega,
\end{cases}
\]
in a bounded uniform domain \( \Omega \subset \mathbb{R}^n \), where \( \omega \) is a locally finite Borel measure in \( \Omega \), and \( f \geq 0 \) is integrable with respect to harmonic measure \( dH^x \) on \( \partial \Omega \).

We also give sufficient and matching necessary conditions for the existence of a positive solution in terms of the exponential integrability of \( M^*(m\omega)(z) = \int_{\Omega} M(x, z)m(x)\,d\omega(x) \) on \( \partial \Omega \) with respect to \( f\,dH^{x_0} \), where \( M(x, \cdot) \) is Martin’s function with pole at \( x_0 \in \Omega \), \( m(x) = \min(1, G(x, x_0)) \), and \( G \) is Green’s function.

These results give bilateral bounds for the harmonic measure associated with the Schrödinger operator \(-\Delta - \omega\) on \( \Omega \), and in the case \( f = 1 \), a criterion for the existence of the gauge function. Applications to elliptic equations of Riccati type with quadratic growth in the gradient are given.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a nonempty, connected, open set (a domain). It is called a non-tangentially accessible (NTA) domain if it is bounded, and satisfies both the interior and exterior corkscrew conditions, and the Harnack chain condition ([JK]). For instance, any bounded Lipschitz domain is an NTA domain. The exterior corkscrew condition yields that any NTA domain is regular (in the sense of Wiener).

More generally, a uniform domain is defined as a bounded domain which satisfies the interior corkscrew condition and the Harnack chain condition. Uniform domains satisfy the local (or uniform) boundary Harnack principle ([Aik]; see [An1], [JK] for Lipschitz and NTA domains). However, they are not necessarily regular. Our main results hold for bounded uniform domains, and the regularity of $\Omega$ is not used below.

In [Ken], a slightly more general version of an NTA domain $\Omega$ is defined as a uniform domain of class $\mathcal{S}$ (Definition 1.1.20), i.e., satisfying the volume density condition, which ensures that $\Omega$ is a regular domain. Most of our results, including Theorem 1.1 and Theorem 1.2 below, hold in this setup for uniformly elliptic operators in divergence form $L = \operatorname{div}(A \nabla \cdot)$, with bounded measurable symmetric $A$, in place of the Laplacian $\triangle$, as in [JK], p. 138 and [Ken], Sec. 1.3. The same class of operators $L$ in uniform domains with Ahlfors regular boundary can be covered as well (see [Zha]).

In this paper, for simplicity, we consider mostly the case $n \geq 3$. In two dimensions, our results hold if $\Omega$ is a bounded finitely connected domain in $\mathbb{R}^2$, in particular, a bounded Lipschitz domain (see [CZ], Theorem 6.23; [Han], Remark 3.5).

Let $\omega$ be a locally finite Borel measure on $\Omega$ and let $f$ be a non-negative Borel measurable function on $\partial \Omega$. We consider the equation

$$\begin{cases} -\triangle u = \omega u & \text{in } \Omega, \quad u \geq 0, \\ u = f & \text{on } \partial \Omega. \end{cases}$$

(1.1)
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Solutions of (1.1) are understood either in the potential theoretic sense, or $d\omega$-a.e. The precise definitions are discussed in §2 below. In the case of $C^2$ domains, or bounded Lipschitz domains $\Omega$, they coincide with “very weak” solutions in the sense of Brezis (see [BCMR], [FV2], [MV], Sec. 1.2, [MR]).

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with Green’s function $G(x, y)$; then $G$ is symmetric and strictly positive on $\Omega \times \Omega$. For a Borel measure $\nu$ on $\Omega$,

(1.2) \[ G\nu(x) = \int_{\Omega} G(x, y) \, d\nu(y), \quad x \in \Omega, \]

is Green’s operator. We call $G\nu$ the Green’s potential if $G\nu \not\equiv +\infty$. For a Borel measurable function $f$ on $\partial \Omega$, define the harmonic extension $Pf$ of $f$ into $\Omega$ (the generalized solution to the Dirichlet problem) by

(1.3) \[ Pf(x) = \int_{\partial \Omega} f(z) \, dH^x(z), \quad x \in \Omega, \]

where $dH^x$ is the harmonic measure at $x$, if the integral in (1.3) exists.

A solution $u$ to (1.1) satisfies, formally, the equation

(1.4) \[ u(x) = G(u\omega)(x) + Pf(x), \quad x \in \Omega. \]

We remark that if $u \not\equiv +\infty$ satisfies (1.4), then it is a superharmonic function in $\Omega$, and $Pf$ is its greatest harmonic minorant. In particular, $u \in L^1_{\text{loc}}(\Omega, dx) \cap L^1_{\text{loc}}(\Omega, d\omega)$, and $u < +\infty$ q.e., that is, quasi-everywhere with respect to the Green capacity, see [AG], [Lan].

We also consider more general equations with an arbitrary positive harmonic function $h$ in place of $Pf$ (see §2 and §3), when irregular boundary points may come into play.

For an appropriate function $g$ on $\Omega$, we define

(1.5) \[ Tg(x) = G(g\omega)(x) = \int_{\Omega} G(x, y) \, g(y) \, d\omega(y), \]

for $x \in \Omega$, so that equation (1.4) becomes $(I - T)u = Pf$, with formal solution

(1.6) \[ u_f = \sum_{j=0}^{\infty} T^j(Pf). \]

This minimal solution $u_f$ of (1.1) satisfies

(1.7) \[ u_f(x) = G(u_f\omega)(x) + Pf(x), \quad x \in \Omega, \]

if $u_f \not\equiv +\infty$. Under conditions which guarantee the finiteness of the right side of equation (1.6) (see Theorem 1.1 and Theorem 1.2), we will see that $u_f$ defined by (1.6) gives a (generalized) solution of (1.1).
It was shown in [FV3], Lemma 2.5, that the following are equivalent: for $\beta > 0$,

(1.8) $T$ is bounded on $L^2(\Omega, \omega)$ with $\|T\| = \|T\|_{L^2(\Omega,\omega)\to L^2(\Omega,\omega)} \leq \beta^2$

and

(1.9) $\|\varphi\|_{L^2(\omega)} \leq \beta \|\nabla \varphi\|_{L^2(dx)}$, for all $\varphi \in C^\infty_0(\Omega)$.

Our results are expressed in terms of the Martin kernel $M(x, z)$. In a bounded uniform domain $\Omega \subset \mathbb{R}^n$, the Martin boundary $\triangle$ is homeomorphic to the Euclidean boundary $\partial \Omega$ ([Aik], Corollary 3; see [HW], [AG], for a bounded Lipschitz domain, and [JK], [Ken] for an NTA domain.) Martin’s kernel, defined with respect to a reference point $x_0 \in \Omega$, is given by

(1.10) $M(x, z) = \lim_{y \to z, y \in \Omega} \frac{G(x, y)}{G(x_0, y)}$, $x \in \Omega$, $z \in \partial \Omega$,

where the limit exists, and is a minimal harmonic function in $x \in \Omega$. We will see in §2 that

(1.11) $dH^x(z) = M(x, z) dH^{x_0}(z)$, $(x, z) \in \Omega \times \partial \Omega$,

for uniform domains (see [HW], p. 519; [CZ], p. 137 for Lipschitz domains; [JK], pp. 104, 115 for NTA domains). Combining (1.3) and (1.11) yields

(1.12) $P f(x) = \int_{\partial \Omega} M(x, z) f(z) dH^{x_0}(z)$, $x \in \Omega$,

for Borel measurable $f \geq 0$, whenever the integral exists. Hence, (1.6) yields

(1.13) $u_f(x) = \int_{\partial \Omega} \sum_{j=0}^{\infty} T^j M(\cdot, z)(x) f(z) dH^{x_0}(z)$, $x \in \Omega$.

We define

(1.14) $\mathcal{M}(x, z) = \sum_{j=0}^{\infty} T^j M(\cdot, z)(x)$, $(x, z) \in \Omega \times \partial \Omega$,

and

(1.15) $d\mathcal{H}^x(z) = \mathcal{M}(x, z) dH^{x_0}(z)$, $(x, z) \in \Omega \times \partial \Omega$.

Then (1.13) gives

(1.16) $u_f(x) = \int_{\partial \Omega} \mathcal{M}(x, z) f(z) dH^{x_0}(z)$

$= \int_{\partial \Omega} f(z) d\mathcal{H}^x(z)$, $x \in \Omega$. 

Comparing this last equation with equation (1.3), we see that \( d^H_x \) is harmonic measure for the Schrödinger operator \(-\triangle - \omega\).

By (1.14),

\[
M(x,z) = M(x,z) + \sum_{j=1}^{\infty} T^j M(\cdot, z)(x)
\]

\[
= M(x,z) + T M(\cdot, z)(x)
\]

\[
= M(x,z) + G(M(\cdot, z) \omega)(x).
\]

Hence \( M(x,z) \) is a superharmonic function of \( x \in \Omega \), and \( M(x,z) \) is its greatest harmonic minorant, for every \( z \in \partial \Omega \), provided \( M(\cdot, z) \not\equiv \infty \).

In fact, \( M(\cdot, z) \) is \( \omega \)-harmonic, i.e., it satisfies the Schrödinger equation

\[-\triangle u = \omega u \text{ in } \Omega.\]

Notice that \( H^x \) defined by (1.15) is not a probability measure on \( \partial \Omega \) unless \( \omega = 0 \). Letting \( f \equiv 1 \) on \( \partial \Omega \), we see by (1.16) that \( H^x \) is a finite measure on \( \partial \Omega \) if and only if \( u_1(x) < \infty \), where \( u_1 \) is the so-called gauge function defined by (1.22) below (see Corollary 1.3 for conditions under which \( u_1 < \infty \) \( d\omega \)-a.e.).

We remark that for the normalized version of \( M(x,z) \) defined by

\[
\tilde{M}(x,z) = \frac{M(x,z)}{M(x_0,z)}, \quad (x,z) \in \Omega \times \partial \Omega,
\]

where \( x_0 \in \Omega \) is to be chosen so that \( M(x_0,z) < \infty \) for every \( z \in \partial \Omega \), we have

\[
d\tilde{H}^x(z) = \tilde{M}(x,z) d\tilde{H}^{x_0}(z), \quad (x,z) \in \Omega \times \partial \Omega,
\]

which is analogous to (1.11). Obviously, \( \tilde{M}(x_0,z) = 1 \), as for the unperturbed Martin’s kernel \( M(x,z) \). Moreover, formally we have

\[
\tilde{M}(x,z) = \lim_{y \to z, y \in \Omega} \frac{G(x,y)}{G(x_0,y)}, \quad (x,z) \in \Omega \times \partial \Omega,
\]

where \( G(x,y) \) is the minimal Green’s function associated with the Schrödinger operator \(-\triangle - \omega \) (see [FNV]). Thus, \( \tilde{M}(x,z) \) serves the role of the (normalized) Martin kernel associated with the Schrödinger operator \(-\triangle - \omega \).

Nevertheless, we prefer to use the kernel \( M(x,z) \), since it does not exclude the case \( M(x_0,z) = \infty \), and is more convenient in applications. Pointwise estimates of \( \tilde{M}(x,z) \) are deduced easily from the estimates of \( M(x,z) \) discussed below.
Our bilateral estimates of $\mathcal{M}(x, z)$ (see (2.12) and (2.14) below) are stated in terms of exponentials:

\begin{equation}
M(x, z) e^{\int_{\Omega} G(x, y) \frac{M(y, z)}{M(x, z)} \, d\omega(y)} \leq \mathcal{M}(x, z) \leq M(x, z) e^{C \int_{\Omega} G(x, y) \frac{M(y, z)}{M(x, z)} \, d\omega(y)},
\end{equation}

for all $(x, z) \in \Omega \times \partial \Omega$, with an appropriate constant $C > 0$. We remark that

\begin{equation}
M(x, z) = U(x, z) M(x, z), \quad (x, z) \in \Omega \times \partial \Omega,
\end{equation}

where

\begin{equation}
U(x, z) = 1 + \frac{1}{M(x, z)} \sum_{j=1}^{\infty} T_j M(\cdot, z)(x), \quad (x, z) \in \Omega \times \partial \Omega,
\end{equation}

is the so-called conditional gauge \cite[Sec. 4.3]{CZ}.

From (1.17) it is immediate that

\begin{equation}
e^{\int_{\Omega} G(x, y) \frac{M(y, z)}{M(x, z)} \, d\omega(y)} \leq U(x, z) \leq e^{C \int_{\Omega} G(x, y) \frac{M(y, z)}{M(x, z)} \, d\omega(y)},
\end{equation}

for all $(x, z) \in \Omega \times \partial \Omega$. We emphasize that in the exponents of (1.19) we only use the first term in the sum on the right-hand side of (1.18).

A probabilistic definition of the conditional gauge in the case $d\omega = q \, dx$ ($q \in L^1_{\text{loc}}(\Omega)$) is provided by

\begin{equation}
U(x, z) = \mathbb{E}_z^x \left[ e^{\int_0^\zeta q(X_t) \, dt} \right], \quad (x, z) \in \Omega \times \partial \Omega,
\end{equation}

where $X_t$ is a path of the Brownian motion (properly scaled to replace $\frac{1}{2} \Delta$ used in the probabilistic literature with $\Delta$) starting at $x$, $E^x$ is the conditional expectation conditioned on the event that $X_t$ exits $\Omega$ at $z \in \partial \Omega$, and $\zeta$ is the time when $X_t$ first hits $z$. Properties of the conditional gauge for potentials $q$ in Kato’s class in a bounded Lipschitz domain $\Omega$ are discussed in \cite[Ch. 7]{CZ}; in particular, $U(x, z) \approx 1$ if $U(x, z) \not\equiv +\infty$.

For general $\omega \geq 0$, we clearly have $U(x, z) \geq 1$, but $U(x, z)$ is no longer uniformly bounded from above, even if $U(x, z) \not\equiv +\infty$ and $\|T\| < 1$. Consequently, the so-called Conditional Gauge Theorem fails in this setup.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain, $\omega$ a locally finite Borel measure on $\Omega$, and $f \geq 0$ a Borel measurable function on $\partial \Omega$. 
(A) If $\|T\| < 1$ (equivalently, (1.9) holds with $\beta < 1$), then there exists a positive constant $C$ depending only on $\Omega$ and $\|T\|$ such that

$$u_f(x) \leq \int_{\partial \Omega} e^{C \int_{\Omega} G(x,y) \frac{M(y,z)}{M(x,z)} d\omega(y)} f(z) dH^x(z), \quad x \in \Omega.$$  

(B) If $u$ is a positive solution of (1.1), then $\|T\| \leq 1$ (equivalently, (1.9) holds for some $\beta \leq 1$) and

$$u(x) \geq \int_{\partial \Omega} e^{\int_{\Omega} G(x,y) \frac{M(y,z)}{M(x,z)} d\omega(y)} f(z) dH^x(z), \quad x \in \Omega.$$  

In view of (1.16), Theorem 1.1 gives estimates for the Schrödinger harmonic measure $dH^x$ in terms of the harmonic measure $dH^x$ for the Laplacian.

The solution $u_1$ of (1.1), in the case where $f$ is identically 1 on $\partial \Omega$, is called the (Feynman-Kac) gauge:

$$u_1 = 1 + \sum_{j=1}^{\infty} Tj1,$$

provided $u_1 \neq +\infty$. An equivalent probabilistic interpretation of the gauge when $d\omega = q(x) dx$ ($q \in L^1_{loc}(\Omega), q \geq 0$) is given by (see [CZ], Sec. 4.3)

$$u_1(x) = E^x \left[ e^{\int_{0}^{\tau_\Omega} q(X_t) dt} \right], \quad x \in \Omega,$$

where $X_t$ is the Brownian path (properly scaled as above) starting at $x$, $E^x$ is the expectation operator, and $\tau_\Omega$ is the exit time from $\Omega$. Notice that $u_1$ given by (1.22) is related to the conditional gauge $U(x,z)$ defined by (1.18) via the equation

$$u_1(x) = \int_{\partial \Omega} U(x,z) dH^x(z), \quad x \in \Omega.$$

In particular,

$$\inf_{z \in \partial \Omega} U(x,z) \leq u_1(x) \leq \sup_{z \in \partial \Omega} U(x,z), \quad x \in \Omega.$$

The following theorem gives sufficient and matching necessary criteria for the existence of $u_f$. For Martin’s kernel $M(x,z)$, we define the adjoint operator $M^*$ for a Borel measure $\mu$ on $\Omega$ by

$$M^* \mu(z) = \int_{\Omega} M(x,z) d\mu(x), \quad \text{for } z \in \partial \Omega.$$  

The role of $M^*$ in the following theorem is analogous to the role of the balayage operator $P^*$ in [FV2] for $C^{1,1}$ domains $\Omega$, where all integrals

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over \( \partial \Omega \) are taken with respect to surface area in place of harmonic measure.

**Theorem 1.2.** Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded uniform domain, \( \omega \) is a locally finite Borel measure on \( \Omega \), and \( f \geq 0 \) (\( f \) not a.e. 0 with respect to harmonic measure) is a Borel measurable function on \( \partial \Omega \).

Let \( x_0 \in \Omega \) be the reference point in the definition of Martin’s kernel. Let \( m(x) = \min(1, G(x, x_0)) \).

(A) There exists \( C > 0 \) (\( C \) depending only on \( \Omega \) and \( \|T\| \)) such that if \( \|T\| < 1 \) (equivalently, \((1.9)\) holds with \( \beta < 1 \)) and

\[
(1.24) \quad \int_{\partial \Omega} e^{CM^{*}(m\omega)} f \, dH_{x_0} < \infty,
\]

then \( u_f \in L^1_{\text{loc}}(\Omega, dx) \).

(B) If \( u_f \in L^1_{\text{loc}}(\Omega, dx) \), then \( \|T\| \leq 1 \) and

\[
(1.25) \quad \int_{\partial \Omega} e^{M^{*}(m\omega)} f \, dH_{x_0} < \infty.
\]

**Remark.** More general results for equation \((1.4)\) with an arbitrary positive harmonic function \( h \) in place of \( Pf \), in terms of Martin’s representation, are given in Theorem 2.5 and Theorem 3.5 below.

For \( C^{1,1} \) domains \( \Omega \) and absolutely continuous \( \omega \), Theorem 1.1 and an analogue of Theorem 1.2 were proved in the special case \( f = 1 \) in [FV2], Theorem 1.2. To see this observation, note that for a \( C^{1,1} \) domain, \( M(x, z) = P(x, z)/P(x_0, z) \), by \((1.11)\), which shows that inequalities \((1.12)\) and \((1.14)\) in [FV2] follow from Theorem 1.1 above. To see that \((1.10)\) and \((1.13)\) in [FV2] follow from Theorem 1.2, choose \( x_0 \) with \( \text{dist}(x_0, \partial \Omega) > \delta \), where \( 0 < \delta < \text{diam}(\Omega)/2 \), so that \( P(x_0, z) \) is equivalent to a constant depending only on \( \Omega \). An extension to the case of uniform domains of the criteria in [FV2] for the existence of the nontrivial gauge \( u_1 \not\equiv +\infty \) is provided by the following corollary.

**Corollary 1.3.** Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded uniform domain, and \( \omega \) is a locally finite Borel measure on \( \Omega \). Let \( x_0 \in \Omega \) be the reference point in the definition of Martin’s kernel, and \( m(x) = \min(1, G(x, x_0)) \).

(A) There exists \( C > 0 \) (\( C \) depending only on \( \Omega \) and \( \|T\| \)) such that if \( \|T\| < 1 \) and

\[
(1.26) \quad \int_{\partial \Omega} e^{CM^{*}(m\omega)} \, dH_{x_0} < \infty,
\]

then the gauge \( u_1 \) is nontrivial.
(B) If the gauge \( u_1 \) is nontrivial, then \( \| T \| \leq 1 \) and

\[
\int_{\partial \Omega} e^{M^*(m\omega)} \, dH^0 < \infty.
\]

As an application of Corollary 1.3, we consider elliptic equations of Riccati type with quadratic growth in the gradient,

\[
\begin{cases}
-\Delta v = |\nabla v|^2 + \omega & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega
\end{cases}
\]  

for locally finite Borel measures \( \omega \), in bounded uniform domains \( \Omega \subset \mathbb{R}^n \). Although (1.28) is formally related to equation (1.1) with \( f = 1 \) by the relation \( v = \log u_1 \), it is well-known that this formal relation is not sufficient to guarantee equivalence of the two equations (see §4). Nevertheless we obtain the following result.

**Theorem 1.4.** Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded uniform domain, and \( \omega \) is a locally finite Borel measure in \( \Omega \).

(A) Suppose \( \| T \| < 1 \), or equivalently (1.9) holds with \( \beta < 1 \), and (1.26) holds with a large enough constant \( C > 0 \) (depending only on \( \Omega \) and \( \| T \| \)). Then \( v = \log u_1 \in W^{1,2}_{\text{loc}}(\Omega) \) is a weak solution of (1.28).

(B) Conversely, if (1.28) has a weak solution \( v \in W^{1,2}_{\text{loc}}(\Omega) \), then \( u = e^v \) is a supersolution to (1.1) with \( f = 1 \), i.e., \( u \geq G(\omega u) + 1 \). Moreover, \( \| T \| \leq 1 \), or equivalently (1.9) holds with \( \beta = 1 \), and (1.27) holds.

**Remarks.**

1. In Theorem 1.2, \( u_f \in L^1_{\text{loc}}(\Omega, dx) \) actually yields \( u_f \in L^1(\Omega, m dx) \cap L^1(\Omega, m d\omega) \), or equivalently \( G(u_f, \omega) \neq +\infty \).

2. For bounded Lipschitz domains \( \Omega \), \( u_1 \) is a “very weak” solution in the sense of [MR]. More precisely, \( u = u_1 - 1 \) is a “very weak” solution to \( -\Delta u = \omega u + \omega \) with \( u = 0 \) on \( \partial \Omega \). Here one can use \( \phi_1 \) in place of \( m \), where \( \phi_1 \) is the first eigenfunction of the Dirichlet Laplacian in \( \Omega \) (see [AAC], Lemma 3.2). Then \( u_1 \in L^1(\Omega, \phi_1 dx) \) and \( \int_{\Omega} \phi_1 \, d\omega < +\infty \).

3. Our main results for uniform domains \( \Omega \) are based on the exponential bounds for Green’s function \( G(x, y) \) (see Theorem 2.2 below) obtained in [FNV]. Here \( G(x, y) \) is the kernel of the operator \((I - T)^{-1}\) defined by (3.15), where \( T \) is an integral operator with positive quasi-smooth kernel. The case of \( C^{1,1} \) domains \( \Omega \) and \( d\omega = q \, dx \), where \( q \in L^1_{\text{loc}}(\Omega, dx) \), was treated earlier in [FV1] for small \( \| T \| \), and in [FV2] for \( \| T \| < 1 \).

4. In the special case of Kato class potentials, or more generally, \( G \)-bounded perturbations \( \omega \) for the Schrödinger operator \( -\Delta - \omega \), it is
known that $G(x, y) \approx G(x, y)$. In this case, the gauge $u_1$ exists, and is uniformly bounded, if and only if $||T|| < 1$ (see [CZ], [Han], [Pin]).

5. For the fractional Schrödinger operator $(-\triangle)^{\alpha/2} - \omega$, criteria of the existence of the gauge $u_1$ in the case $0 < \alpha < 2$ were obtained in [FV3]. They are quite different from Corollary 1.3 and require no extra boundary restrictions on $\Omega$ like (1.26), (1.27) in the case $\alpha = 2$.

2. Pointwise estimates for $u_f$

Recall that the Martin kernel is defined by (1.10). Then $M(x, z)$ is a Hölder continuous function in $z \in \partial \Omega$ ([Aik], Theorem 3). It is worth mentioning that in uniform domains, harmonic measure may vanish on some surface balls, and so the Radon-Nykodim derivative formula $M(x, z) = \frac{dH^x}{dH^0}(z)$, which holds for NTA domains, is no longer available as a means to recover (1.10) at every point $z \in \partial \Omega$. Instead, it can be determined via (1.10), so that (1.11) still holds (see [Aik], p. 122).

In this case, the Martin representation for every nonnegative harmonic function $h$ in $\Omega$ can be expressed in the form

$$h(x) = \int_{\partial \Omega} M(x, z) \, d\mu_h(z), \quad x \in \Omega,$$

where $\mu_h$ is a finite Borel measure on $\partial \Omega$ uniquely determined by $h$.

The connection between Martin’s kernel and harmonic measure in a uniform domain is provided by the equation (see [Aik], p. 142):

$$dH^x(z) = M(x, z) \, d\mu_1(z), \quad x \in \Omega, \, z \in \partial \Omega.$$

Here $\mu_1$ is the representing measure in (2.1) for the function $h \equiv 1$.

Equation (2.2) can be justified using [AC], Theorem 9.1.7 (in the special case $h \equiv 1$) for a bounded domain whose Martin boundary $\Delta$ is identified with $\partial \Omega$. It yields that, for every $f \in C(\partial \Omega)$, its harmonic extension $Pf$ via harmonic measure (1.3) can be represented in the form

$$Pf(x) = \int_{\partial \Omega} M(x, z) \, f(z) \, d\mu_1(z), \quad x \in \Omega.$$ 

By the uniqueness of the representing measure in (1.3) for all $f \in C(\partial \Omega)$, it follows that (2.2) holds.

In particular, since $M(x_0, z) = 1$ for all $z \in \partial \Omega$, letting $x = x_0$ in (2.2) yields $dH^{x_0} = d\mu_1$, and consequently (1.11) holds.

Let $\Omega$ be a bounded uniform domain in $\mathbb{R}^n$, $\omega$ a finite Borel measure in $\Omega$, and $f \geq 0$ a Borel measurable function in $\partial \Omega$ integrable with respect to harmonic measure. We consider solutions $u$ to (1.1) understood in the potential theoretic sense. Namely, a function $u$:
$\Omega \to [0, +\infty]$ is said to be a solution to (1.1) if $u$ is superharmonic in $\Omega$ ($u \not\equiv +\infty$), and

$$u(x) = G(u_\omega)(x) + Pf(x), \quad \text{for all } x \in \Omega,$$

where $Pf$ is the harmonic function defined by (1.3). Then $Pf$ is the greatest harmonic minorant of $u$, and $u \in L^1_{loc}(\Omega, \omega)$, so that $u \, d\omega$ is the associated Riesz measure of $u$, where $-\Delta u = \omega u$ in the distributional sense. In fact, if a potential theoretic solution to (2.4) exists, then $u \in L^1(\Omega, m\omega)$, where $m(x) = \min(1, G(x, x_0))$ for some $x_0 \in \Omega$; otherwise $G(u_\omega) \equiv +\infty$ (see [AG], Theorem 4.2.4).

We note that all potential theoretic solutions are by definition lower semicontinuous functions in $\Omega$. For a superharmonic function $u$, it is enough to require that equation (2.4) holds $dx$-a.e. Moreover, in a bounded uniform domain, any potential theoretic solution $u \in L^1(\Omega, mdx)$. This is not difficult to see using the estimate $G(mdx) \leq C m$ in $\Omega$, which is a consequence of the so-called 3-G inequality (see [CZ], [Han1], [Han2], [Pin]). We remark that in $C^2$ domains $Pf$ is the Poisson integral, and in fact $u \in L^1(\Omega, dx)$ (see [FV2], [MV], Theorem 1.2.). The latter is no longer true for bounded Lipschitz domains (see, e.g., [MR]).

Another useful way to define a solution of (1.1) is to require that (2.5) hold $d\omega$-a.e. More precisely, a measurable function $0 \leq u < +\infty$ $d\omega$-a.e. is said to be a solution of (1.1) with respect to $\omega$ if

$$u = G(u_\omega) + Pf \quad d\omega\text{-a.e. in } \Omega.$$

If such a solution exists, then obviously $u \in L^1_{loc}(\Omega, \omega)$, and in fact, as above, $u \in L^1(\Omega, m\omega)$.

We remark that if $f \not\equiv 0$ (with respect to $dH^x$), and (2.5) has a positive solution in this sense, then $||T|| \leq 1$ by Schur’s lemma, and consequently (1.9) holds for $\beta = 1$. It follows that $\omega(K) \leq \text{cap}(K)$ for any compact set $K \subset \Omega$. In particular, $\omega$ must be absolutely continuous with respect to the Green (or Wiener) capacity, i.e.,

$$\text{cap}(K) = 0 \implies \omega(K) = 0.$$

(See details in [FNV], [FV2].)

A connection between these two approaches is provided by the following claim used below. If $u$ is a solution of (2.5) (with respect to $\omega$), then there exists a unique superharmonic function $\hat{u} \geq 0$ in $\Omega$ such $\hat{u} = u \, d\omega$-a.e. in $\Omega$, and $\hat{u} \in L^1_{loc}(\Omega, \omega)$ is a potential theoretic solution that satisfies (2.4).
Indeed, let \( \hat{u} := G(u\omega) + Pf \) everywhere in \( \Omega \). Then \( \hat{u} = u \ d\omega \)-a.e. by (2.5), \( \hat{u} \in L^1_{\text{loc}}(\Omega, \omega) \), and consequently
\[
\hat{u}(x) = G(u\omega)(x) + Pf(x) = G(\hat{u}\omega)(x) + Pf(x) \quad \text{for all } x \in \Omega.
\]
Clearly, \( \hat{u} \) is superharmonic since \( G(u\omega) < +\infty \ d\omega \)-a.e., and hence \( G(u\omega) \) is a Green potential, and \( Pf \) is the greatest harmonic minorant of \( \hat{u} \). Thus, \( \hat{u} \) is a potential theoretic solution.

Moreover, such a superharmonic solution \( \hat{u} \) is unique: if \( \hat{v} \) is a superharmonic solution to (2.4) for which \( \hat{v} = u \ d\omega \)-a.e., it follows that
\[
\hat{v} = G(\hat{v}\omega) + Pf = G(\hat{u}\omega) + Pf = \hat{u}
\]
everywhere in \( \Omega \).

If \( \omega \) satisfies (2.6), then it is enough to require that \( u < +\infty \) and (2.4) hold q.e. Then \( u \) is a solution of (2.5) with respect to \( \omega \), and \( \hat{u} := G(u\omega) + Pf \) is a potential theoretic solution to (1.1), and \( \hat{u} \) is a quasicontinuous representative of \( u \), so that \( \hat{u} = u \) q.e.

From now on, we will not distinguish between a solution \( u \) to (2.5) understood \( d\omega \)-a.e., and its superharmonic representative \( \hat{u} = u \ d\omega \)-a.e. which satisfies (2.4) everywhere in \( \Omega \).

In particular, the solution \( u_f \) of (2.4) defined by (1.6) everywhere in \( \Omega \) is a potential theoretic (superharmonic) solution of (1.1) provided \( u_f \neq +\infty \). Indeed, for \( m \in \mathbb{N} \),
\[
\sum_{j=0}^{m} T^j(Pf)(x) = Pf(x) + T \sum_{j=0}^{m-1} T^j(Pf))(x), \quad \text{for all } x \in \Omega.
\]
Letting \( m \to \infty \), by the monotone convergence theorem we have
\[
u_f := \sum_{j=0}^{\infty} T^j(Pf) = Pf + T \sum_{j=0}^{\infty} T^j(Pf)) = Pf + G(u_f\omega)
\]
everywhere in \( \Omega \). Clearly, \( u_f \) is a superharmonic function provided \( u_f \neq +\infty \) in \( \Omega \), which occurs if and only if \( G(u_f\omega) \neq +\infty \) in \( \Omega \), or equivalently \( u_f \in L^1(\Omega, m\omega) \). Moreover, \( u_f \) is a minimal solution since, for every other solution \( u \), we obviously have, for every \( m \in \mathbb{N} \),
\[
u = G(u\omega) + Pf = G(u\omega) + \sum_{j=0}^{m} T^j(Pf) \geq \sum_{j=0}^{m} T^j(Pf).
\]
Letting \( m \to \infty \), we see that \( u \geq u_f \).
Definition 2.1. Let \((\Omega, \omega)\) be a measure space. A quasi-metric kernel \(K\) is a measurable function \(K : \Omega \times \Omega \to (0, +\infty]\) such that \(K\) is symmetric (\(K(x, y) = K(y, x)\)) and \(d = \frac{1}{K}\) satisfies
\[d(x, y) \leq \kappa (d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in \Omega,\]
for some \(\kappa > 0\), called the quasi-metric constant for \(K\).

A measurable function \(K : \Omega \times \Omega \to (0, +\infty]\) is called quasi-metrically modifiable if there exists a measurable function \(m : \Omega \to (0, +\infty]\) such that \(\tilde{K}(x, y) = \frac{K(x, y)}{m(x) m(y)}\) is a quasi-metric kernel. The function \(m\) is called a modifier for \(K\).

We will use the following result, from \([FNV]\), Corollary 3.5.

Theorem 2.2. Let \((\Omega, \omega)\) be a measure space. Suppose \(K\) is a quasi-metrically modifiable kernel on \(\Omega\) with modifier \(m\). Let \(\kappa\) be the quasi-metric constant for \(\frac{K(x, y)}{m(x) m(y)}\). For a non-negative, measurable function \(h\) on \(\Omega\), define
\[T_h(x) = \int_{\Omega} K(x, y) h(y) \, d\omega(y), \quad \text{for } x \in \Omega.\]

For \(j \in \mathbb{N}\), let \(T_j\) be the \(j^{th}\) iterate of \(T\), and let \(T_0 h = h\).

(A) If \(\|T\| < 1\), then there exists a positive constant \(C\), depending only on \(\kappa\) and \(\|T\|\), such that
\[\sum_{j=0}^{\infty} T_j m(x) \leq m(x) e^{C(T m(x))/m(x)}, \quad \text{for all } x \in \Omega.\]

(B) There exists a positive constant \(c\), depending only on \(\kappa\), such that
\[\sum_{j=0}^{\infty} T_j m(x) \geq m(x) e^{c(T m(x))/m(x)}, \quad \text{for all } x \in \Omega.\]

It is known (\([An2]\), \([Han1]\)) that in a bounded uniform domain \(\Omega\) (in particular, an NTA domain), the Green’s kernel \(G(x, y)\) is quasi-metrically modifiable, with modifier \(m(x) = \min(1, G(x, x_0))\), where \(x_0\) is any fixed point in \(\Omega\), and the quasi-metric constant of the modified kernel \(\frac{G(x, y)}{m(x) m(y)}\) is independent of \(x_0\).

In fact, in a bounded uniform domain \(\Omega \subset \mathbb{R}^n\) (\(n \geq 3\)), the following slightly stronger property (called the strong generalized triangle property) holds (\([Han1]\), p. 465):
\[|x_1 - x_2| \leq |x_1 - y| \implies \frac{G(x_1, y)}{m(x_1)} \leq \kappa \frac{G(x_2, y)}{m(x_2)},\]
for all \( x_1, x_2, y \in \Omega \), where \( \kappa \) depends only on \( \Omega \). It is known (\cite{Han1}, Corollary 2.8) that (2.9) is equivalent to the uniform boundary Harnack principle established for uniform domains in (\cite{Aik}, Theorem 1). By (2.9),

\[
\limsup_{x_1 \to z, x_1 \in \Omega} \frac{G(x_1, y)}{m(x_1)} \leq \kappa \liminf_{x_2 \to z, x_2 \in \Omega} \frac{G(x_2, y)}{m(x_2)},
\]

for all \( y \in \Omega \) and \( z \in \partial \Omega \), where \( \kappa \) depends only on \( \Omega \), because the condition \(|x_1 - x_2| \leq |x_1 - y|\) is satisfied for \( x_1 \) and \( x_2 \) sufficiently close to \( z \).

We will need the following lemma for punctured quasi-metric spaces due to Hansen and Netuka (\cite{HN}, Proposition 8.1 and Corollary 8.2); it originated in (Pinchover \cite{Pin}, Lemma A.1) for normed spaces.

**Lemma 2.3.** Suppose \( d \) is a quasi-metric on a set \( \Omega \) with quasi-metric constant \( \kappa \). Suppose \( x_1 \in \Omega \). Then

\[
\tilde{d}(x, y) = \frac{d(x, y)}{d(x, x_1) \cdot d(y, x_1)}, \quad x, y \in \Omega \setminus \{x_1\},
\]

is a quasi-metric on \( \Omega \setminus \{x_1\} \) with quasi-metric constant \( 4\kappa^2 \).

**Lemma 2.4.** Let \( \Omega \) be a bounded uniform domain with Green’s function \( G(x, y) \). Fix some \( x_0 \in \Omega \) and define Martin’s kernel \( M(x, z) \) for \( x \in \Omega \) and \( z \in \partial \Omega \) by (1.11). Then for each \( z \in \partial \Omega \), the function \( \tilde{m}(x) = M(x, z) \) is a quasi-metric modifier for \( G \), with quasi-metric constant \( \kappa \) independent of \( z \in \partial \Omega \).

**Proof.** Fix \( x_0 \in \Omega, z \in \partial \Omega \). As noted above, \( m(x) = \min(1, G(x, x_0)) \) is a modifier for \( G \), so that \( d(x, y) = \frac{m(x) m(y)}{G(x, y)} \) is a quasi-metric on \( \Omega \) with positive constant \( \kappa \) independent of \( x_0 \), so that

\[
\frac{m(x) m(y)}{G(x, y)} \leq \kappa \left( \frac{m(x) m(w)}{G(x, w)} + \frac{m(w) m(y)}{G(w, y)} \right),
\]

for all points \( x, y, w \in \Omega \). Suppose \( x_1 \in \Omega \) with \( x_1 \neq x_0 \). Clearly, for \( \tilde{d} \) defined by (2.11), we have

\[
\tilde{d}(x, y) = \frac{1}{m(x_1)^2} \frac{G(x, x_1) G(y, x_1)}{G(x, y)}, \quad x, y \in \Omega \setminus \{x_1\}.
\]

Then by Lemma 2.3 it follows that \( \tilde{d} \) is a quasi-metric on \( \Omega \setminus \{x_1\} \) with quasi-metric constant \( 4\kappa^2 \). Assuming that \( x, y, w \in \Omega \setminus \{x_1\} \), from the
inequality \( \tilde{d}(x, y) \leq 4\kappa^2[\tilde{d}(x, w) + \tilde{d}(y, w)] \), we deduce
\[
\frac{1}{m(x_1)^2} \frac{G(x, x_1) G(y, x_1)}{G(x, y)} \leq \frac{4\kappa^2}{m(x_1)^2} \times \left[ \frac{G(x, x_1) G(w, x_1)}{G(x, w)} + \frac{G(y, x_1) G(w, x_1)}{G(y, w)} \right].
\]

Multiplying both sides of the preceding inequality by \( m(x_1)^2 \) yields
\[
\frac{G(x, x_1) G(y, x_1)}{G(x_0, x_1) G(x, y) G(x_0, x_1)} \leq 4\kappa^2 \times \left[ \frac{G(x, x_1) G(w, x_1)}{G(x, w)} + \frac{G(y, x_1) G(w, x_1)}{G(y, w)} \right].
\]

Letting \( x_1 \to z \), with \( x_1 \in \Omega \), we have
\[
\lim_{x_1 \to z, x_1 \in \Omega} \frac{G(x, x_1)}{G(x_0, x_1)} = M(x, z) = \tilde{m}(x),
\]
by (1.10), and similarly with \( x \) replaced by \( y \) or \( w \). We obtain
\[
\tilde{m}(x) \tilde{m}(y) \leq 4\kappa^2 \left( \frac{\tilde{m}(x) \tilde{m}(w)}{G(x, w)} + \frac{\tilde{m}(w) \tilde{m}(y)}{G(w, y)} \right).
\]

\[\square\]

**Proof of Theorem 1.1.** By Lemma 2.4, \( \tilde{m}(x) = M(x, z) \) is a quasi-metric modifier for \( T \), for all \( z \in \partial \Omega \), with quasi-metric constant independent of \( z \). Hence by part (A) of Theorem 2.2 with \( \tilde{m} \) in place of \( m \), under the assumption that \( \|T\| < 1 \), (note that the estimates in Theorem 2.2 hold everywhere)
\[
M(x, z) = \sum_{j=0}^{\infty} T^j M(\cdot, z)(x) \leq M(x, z) e^{C(T M(\cdot, z))(x)/M(x, z)}
\]
(2.12)
\[
= M(x, z) e^{C \int_{\Omega} G(x, y) \frac{M(y, z)}{M(x, z)} d\omega(y)}, \quad (x, z) \in \Omega \times \partial \Omega,
\]
with \( C \) depending only on \( \Omega \) and \( \|T\| \). Substituting this estimate in (1.13) and using equation (1.11) gives (1.20). This proves part (A) of Theorem 1.1.

Suppose now that \( u \) is a solution to (1.1). Assuming without loss of generality that \( f \neq 0 \) \( dH^x \)-a.e., so that \( u \geq Pf > 0 \) is a positive solution, we see that \( Tu \leq u \), where \( 0 < u < \infty \) \( d\omega \)-a.e. Hence, \( \|T\| \leq 1 \), and consequently (1.2) holds with \( \beta = 1 \), by Schur’s lemma (see [FNV], [FV2]). In particular, (2.6) holds.
Since $Pf$ is a positive harmonic function, obviously $Pf \geq c_K > 0$ on every compact set $K \subset \Omega$, and consequently
\begin{equation}
(2.13) \quad c_K G(\chi_K \omega) \leq G(Pf \omega) \leq G(u \omega) \leq u < \infty \quad d\omega\text{-a.e.}
\end{equation}
This simple observation will be used below.

For the minimal solution $u_f$ to (1.1) given by (1.6) we have $u \geq u_f$.

Applying part (B) of Theorem 2.2 with $\tilde{m} = \tilde{M}(\cdot, z)$ in place of $m$ gives
\begin{equation}
(2.14) \quad M(x, z) = \sum_{j=0}^{\infty} T^j M(\cdot, z)(x) \geq M(x, z) e^{c(TM(\cdot, z))(x)/M(x, z)}
\end{equation}
with $c$ depending only on $\Omega$.

In fact, we can let $c = 1$ in (2.14) if instead of statement (B) of Theorem 2.2 we use a recent lower estimate of solutions obtained in [GV2], Theorem 1.2, with $q = 1$, $b = 1$, and $h = \tilde{m}$. Here $b$ is the constant in the so-called weak domination principle, which states that, for any bounded measurable function $g$ with compact support,
\begin{equation}
(2.15) \quad G(g \omega)(x) \leq h(x) \text{ in } \text{supp}(g) \implies G(g \omega)(x) \leq b \, h(x) \text{ in } \Omega,
\end{equation}
where $h$ is a given positive lower semicontinuous function on $\Omega$.

For Green’s kernel $G$, this property with $b = 1$ is a consequence of the classical Maria–Frostman domination principle (see [Hel], Theorem 5.4.8), for any positive superharmonic function $h$. We only need to verify that $G(g \omega) < \infty \; d\omega\text{-a.e.}$, which is immediate from (2.13). Hence, (2.15) holds with $b = 1$, and so (2.14) holds with $c = 1$ by [GV2], Theorem 1.2.

Consequently, by the same argument as above,
\begin{align*}
u_f(x) &= \int_{\partial\Omega} f(z) \sum_{j=0}^{\infty} T^j M(\cdot, z)(x) \, dH^x_0(z) \
&\geq \int_{\partial\Omega} e^{f_1 G(x, y)M(x, y)d\omega(y)} f(z) M(x, z) \, dH^x_0(z), \quad \text{for all } x \in \Omega,
\end{align*}
where $M(x, z) H^x_0(z) = dH^x(z)$. This yields the lower bound (1.21), The proof of part (B) of Theorem 1.1 is complete.

We complete this section with an extension of Theorem 1.1 which covers solutions of (1.4) with an arbitrary positive harmonic function $h$ in place of $Pf$. Such solutions arise naturally, because, if $u$ positive superharmonic function in $\Omega$ such that
\begin{equation}
(2.16) \quad -\triangle u = \omega u, \quad u \geq 0, \text{ in } \Omega,
\end{equation}
and if the greatest harmonic minorant of $u$ is $h > 0$, then by the Riesz decomposition theorem,

$$
(2.17) \quad u = G(u\omega) + h \quad u \geq 0, \text{ in } \Omega,
$$

where $G(u\omega) \not\equiv +\infty$, and $u\omega$ is the corresponding Riesz measure, a locally finite Borel measure in $\Omega$.

Given a positive harmonic function $h$ on $\Omega$, we will estimate the minimal solution $u_h = h + \sum_{j=1}^{\infty} T^j h$ of (2.17) and in particular give conditions for $u_h$ to exist, i.e., such that $u_h \not\equiv +\infty$. The proof is based on Martin’s representation (2.1), which takes the place of (1.3) in the proof of Theorem 1.1.

**Theorem 2.5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain, $\omega$ a locally finite Borel measure on $\Omega$, and $h$ a positive harmonic function in $\Omega$.

(A) If $\|T\| < 1$, then there exists a positive constant $C$ depending only on $\Omega$ and $\|T\|$ such that

$$
(2.18) \quad u_h(x) \leq \int_{\partial \Omega} e^{C f_0 G(x,y) \frac{M(y,z)}{M(x,z)} d\omega(y)} M(x,z) d\mu_h(z), \quad x \in \Omega.
$$

(B) If $u$ is a positive solution of (2.17), then $\|T\| \leq 1$, and

$$
(2.19) \quad u(x) \geq \int_{\partial \Omega} e^{f_0 G(x,y) \frac{M(y,z)}{M(x,z)} d\omega(y)} M(x,z) d\mu_h(z), \quad x \in \Omega.
$$

The proof of Theorem 2.5 is very similar to that of Theorem 1.1 above. We only need to integrate both sides of estimates (2.12) and (2.14) over $\partial \Omega$ against $d\mu_h(z)$ in place of $f(z) dH^{\infty_0}(z)$.

3. Existence Criteria for $u_f$

We require a few results prior to giving the proof of Theorem 1.2. The following lemma is well-known (see, for instance, [AG], Lemma 4.1.8 and Theorem 5.7.4), but we include a proof for the sake of completeness. Recall that $x_0 \in \Omega$ is a fixed reference point and $m(x) = \min(1, G(x, x_0))$.

**Lemma 3.1.** Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a domain with nontrivial Green’s function $G$. Let $K$ be a compact subset of $\Omega$ and let $\chi_K$ be the characteristic function of $K$. There exists a constant $C_K$ depending on $\Omega$, $K$, and the choice of $x_0$, such that

$$
(3.1) \quad G\chi_K(x) \leq C_K m(x), \quad x \in \Omega.
$$
Also, if $|K| > 0$, there exists a constant $c_K > 0$ depending on $\Omega$, $K$ and $x_0$ such that
\begin{equation}
G_{\chi_K}(x) \geq c_K m(x), \quad x \in \Omega.
\end{equation}

Proof. We first prove inequality (3.1). Suppose $n \geq 3$ (the case $n = 2$ is handled in a similar way with obvious modifications). We assume $|K| > 0$, else the result is trivial. We also assume that $x_0 \in K$; if not, replacing $K$ with $K \cup \{x_0\}$ does not change $G_{\chi_K}$. We first claim that there exists a constant $C_1(K)$ depending on $K$ and $x_0$ such that
\begin{equation}
G_{\chi_K}(x) \leq C_1(K),
\end{equation}
for all $x \in \Omega$. To prove this claim, we recall the standard fact that $G(x,y) \leq C|x-y|^{2-n}$ for all $x,y \in \Omega$. Let $R$ be the diameter of $K$. Then there exists $y_0 \in K$ such that $K \subseteq B(y_0, R)$. If $x \in B(y_0, 2R)$, then $K \subseteq B(x, 3R)$ and
\[
\int_K G(x,y) \, dy \leq \int_{B(x,3R)} \frac{C}{|x-y|^{n-2}} \, dy \leq C \int_0^{3R} \frac{r^{n-1}}{r^{n-2}} \, dr = CR^2.
\]
If $x \notin B(y_0, 2R)$ then $|x-y|^{2-n} \leq R^{2-n}$ for all $y \in K$, so
\[
\int_K G(x,y) \, dy \leq CR^{2-n} |K| \leq cR^2.
\]

Next we claim that there exists a constant $C_2$ depending on $\Omega$, $K$ and $x_0$ such that
\begin{equation}
G_{\chi_K}(x) \leq C_2 G(x,x_0),
\end{equation}
for all $x \in \Omega$. For this claim, let $U$ be a subdomain of $\Omega$ such that $x_0 \in U$, $K \subseteq U$ and $\overline{U} \subseteq \Omega$. If $x \in \Omega \setminus U$, then $G(x,y)$ is a positive harmonic function of $y$ in $U$, so by Harnack’s inequality (e.g., see [AG], Corollary 1.4.4), there exists a constant $C(K,U)$ such that $G(x,y) \leq C(K,U) G(x,x_0)$ for all $y \in K$. Hence
\[
\int_K G(x,y) \, dy \leq C(K,U) |K| G(x,x_0).
\]
Since a fixed domain $U$ depends only on $x_0$, $K$, and $\Omega$, we can replace $C(K,U)$ with $C(x_0,K,\Omega)$. On the other hand, suppose $x \in U$. Note that $G(z,x_0)$ is a strictly positive lower semi-continuous function of $z \in \Omega$ and hence $M = \min\{G(z,x_0) : z \in U\} > 0$, where $M$ depends on $\Omega, x_0$ and $U$, hence $K$. Hence by equation (3.3),
\[
G_{\chi_K}(x) \leq C_1(K) \leq \frac{C_1(K)}{M} G(x,x_0).
\]
Since $m(x) = \min(1,G(x,x_0))$, inequalities (3.3) and (3.4) imply inequality (3.1).
To prove inequality (3.2), let $U$ be as above. For $x \in \Omega \setminus U$, the same application of Harnack’s inequality as above gives that $G(x, y) \geq C(x_0, K, \Omega)^{-1} G(x_0, x)$ for all $x \in K$. Hence
\[
\int_K G(x, y) \, dy \geq C(K, \Omega)^{-1} |K| G(x, x_0) \geq C(K, \Omega)^{-1} |K| m(x).
\]

Now suppose $x \in \overline{U}$. Note that $G(z, y)$ is a strictly positive lower semi-continuous function of $(z, y)$ in $\Omega \times \Omega$ (see [AG], Theorem 4.1.9). Hence $C_3(\overline{U}) = \min \{ G(z, y) : (z, y) \in \overline{U} \times \overline{U} \}$ is attained at some point in the compact set $\overline{U} \times \overline{U}$. In particular, $C_3(\overline{U}) > 0$. Since $m(x) \leq 1$,
\[
\int_K G(x, y) \, dy \geq C_3(\overline{U}) |K| = C_3(x_0, K, \Omega) m(x).
\]

\[\square\]

**Lemma 3.2.** Suppose $\Omega \subset \mathbb{R}^n \ (n \geq 2)$ is a bounded uniform domain. Suppose $x_0 \in \Omega$ is a reference point for the Martin kernel. Then there exists a positive constant $c$ depending only on $x_0$ and $\Omega$ such that
\[
(3.5) \quad M(x, z) \geq c m(x), \quad \text{for all } (x, z) \in \Omega \times \partial \Omega,
\]
where $m(x) = \min(1, G(x, x_0))$.

In particular, if $\omega$ is a locally finite Borel measure in $\Omega$ such that $M^*(m \omega) \neq +\infty$, then $m \in L^2(\Omega, \omega)$.

**Proof.** Fix $z \in \partial \Omega$. Let $B(x_0, r) \subset \Omega$, where $0 < r \leq \frac{1}{3} \text{dist}(x_0, \partial \Omega)$. Since $M(\cdot, z)$ is a positive harmonic function in $\Omega$, by Harnack’s inequality in $B(x_0, 2r)$, there exists a constant $c > 0$ depending only on $x_0$ and $r$ such that $M(x, z) \geq c M(x_0, z)$, for all $x \in B(x_0, r)$ where $M(x_0, z) = 1$. Hence,
\[
(3.6) \quad M(x, z) \geq c > 0, \quad \text{for all } x \in B(x_0, r).
\]

For $x \in \Omega \setminus B(x_0, r)$, we argue that by the 3-G inequality in a bounded uniform domain ($n \geq 3$),
\[
\frac{G(x, x_0) G(x_0, y)}{G(x, y)} \leq C \left( |x - x_0|^{2-n} + |y - x_0|^{2-n} \right),
\]
for all $y \in \Omega$, where $C$ depends only on $\Omega$, see [Han1]. Hence, for $x, y \in \Omega \setminus B(x_0, r)$,
\[
\frac{G(x, y)}{G(x_0, y)} \geq C^{-1} \frac{G(x, x_0)}{|x - x_0|^{2-n} + |y - x_0|^{2-n}} \geq C^{-1} 2r^{n-2} G(x, x_0).
\]

(For $n = 2$, an analogue of the 3-G inequality holds in any bounded domain [Han2].) Letting $y \to z$, where without loss of generality we
may assume that \( y \in \Omega \setminus B(x_0, r) \), we deduce
\[
(3.7) \quad M(x, z) \geq C^{-1}2r^{n-2}G(x, x_0), \quad \text{for all } x \in \Omega \setminus B(x_0, r).
\]
Combining estimates (3.6) and (3.7) yields (3.5).

If \( \omega \) is a locally finite Borel measure in \( \Omega \) such that \( \star M(m \omega) \neq +\infty \), then for some \( z \in \partial \Omega \), by (3.5)
\[
\int_{\Omega} m^2 d\omega \leq c \star M(m \omega)(z) < +\infty, \quad \text{i.e.,} \quad m \in L^2(\Omega, \omega).
\]

**Lemma 3.3.** Suppose \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) is a bounded uniform domain. Suppose \( \mu \) is a finite Borel measure with compact support in \( \Omega \). Let \( z \in \partial \Omega \). Then
\[
(3.8) \quad \lim_{x \to z, x \in \Omega} \frac{G(\mu)(x)}{G(x, x_0)} = \int_{\Omega} M(y, z) d\mu(y) = \star M \mu(z).
\]
In addition, if \( z \) is a regular point of \( \partial \Omega \), then
\[
(3.9) \quad \lim_{x \to z, x \in \Omega} \frac{G(\mu)(x)}{m(x)} = \int_{\Omega} M(y, z) d\mu(y) = \star M \mu(z).
\]

**Proof.** By (1.10), if \( y \in \Omega \) and \( x_j \to z \) (\( x_j \in \Omega \)), then
\[
\lim_{j \to \infty} G(y, x_j)/G(x_j, x_0) = M(y, z).
\]
As in the proof of Lemma 3.1, we denote by \( U \) a relatively compact domain in \( \Omega \) that contains both \( x_0 \) and \( K \). Since \( x_j \to z \), \( z \in \partial \Omega \), we have that \( x_j \notin \overline{U} \) for \( j \geq j_0 \). Then \( G(y, x_j) \) is a harmonic function of \( y \in U \), and for each \( j \geq j_0 \), by Harnack’s inequality,
\[
G(y, x_j) \leq C(K, U) G(x_0, x_j), \quad \text{for all } y \in K.
\]
Since \( \mu \) is a finite measure, we obtain (3.8) by the dominated convergence theorem.

If \( z \) is a regular point of \( \partial \Omega \), then \( G(x_j, x_0) \to 0 \) as \( j \to \infty \), and consequently \( m(x_j) = G(x_j, x_0) \) for \( j \) large enough. Hence, (3.9) follows from (3.8).

In Lemma 3.3, \( \mu \) is a finite Borel measure with compact support in \( \Omega \). We remark that more generally, for \( \mu \) only locally finite,
\[
(3.10) \quad \liminf_{x \to z, x \in \Omega} \frac{G(\mu)(x)}{G(x_0, x)} \geq \int_{\Omega} M(x, z) d\mu(x),
\]
for \( z \in \Delta \) (a Martin boundary point), by Fatou’s Lemma. In fact, by [AG], Theorem 9.2.7, for any Green’s potential \( G \mu \) and \( z \in \Delta_1 \) (a
Martin boundary point where $\Omega$ is not minimally thin),

$$(3.11) \quad \liminf_{x \to z, x \in \Omega} \frac{G_{\mu}(x)}{G(x_0, x)} = \int_{\Omega} M(x, z) \, d\mu(x).$$

For uniform domains, $\Delta = \Delta_1 = \partial \Omega$, so that (3.11) holds for all $z \in \partial \Omega$. We could use this fact in our proof below, but we prefer the more elementary approach in Lemma 3.3. The compact support restriction can be removed later in the proof by exhausting $\Omega$ with a sequence of nested domains $\Omega_j$, and using the monotone convergence theorem.

**Proof of Theorem 1.2**

(A) Suppose $\|T\| < 1$ and (1.24) holds. Define

$$(3.12) \quad Gf(x) = \int_{\Omega} G(x, y)f(y) \, dy, \quad x \in \Omega.$$ 

Let $G_1 = G$, and let $G_j(x, y)$ be the kernel of the $j^{th}$ iterate $T^j$ of $T$ defined by (1.5), so that

$$(3.13) \quad T^j h(x) = \int_{\Omega} G_j(x, y) h(y) \, d\omega(y).$$

Then $G_j$ in (3.13) is determined inductively for $j \geq 2$ by

$$(3.14) \quad G_j(x, y) = \int_{\Omega} G_{j-1}(x, w) G(w, y) \, d\omega(w).$$

We define the minimal Green’s function associated with the Schrödinger operator $-\Delta - \omega$ to be

$$(3.15) \quad G(x, y) = \sum_{j=1}^{\infty} G_j(x, y), \quad \text{for all } x, y \in \Omega.$$ 

The corresponding Green’s operator is

$$(3.16) \quad Gf(x) = \int_{\Omega} G(x, y)f(y) \, dy, \quad x \in \Omega.$$ 

Let $K$ be a compact set in $\Omega$. Denote by $u_K$ a solution to the equation

$$\begin{cases} \quad -\Delta u = \omega u + \chi_K \quad \text{in } \Omega, & u \geq 0, \\
\quad u = 0 \quad \text{on } \partial \Omega. \end{cases}$$

In other words,

$$(3.17) \quad u_K = G(u_K \omega) + G\chi_K.$$ 

By Lemma 3.1, $G\chi_K(x) \approx m(x)$ in $\Omega$ if $m(x) = \min(1, G(x, x_0))$. Without loss of generality we may assume that $m \in L^2(\Omega, \omega)$; otherwise
$M^*(m\omega) \equiv +\infty$ by Lemma 3.2 and condition (1.24) is not valid. It follows that $G_{\chi_K} \in L^2(\Omega, \omega)$. But $||T|| < 1$, so that $u_K = (I - T)^{-1}G_{\chi_K} \in L^2(\Omega, \omega)$, and the series in (3.18) converges in $L^2(\Omega, \omega)$ (and hence $d\omega$-a.e.). In particular, $G(u_K \omega) \not\equiv \infty$.

From this fact it is immediate that the minimal superharmonic solution to (3.17) is given by

$$u_K(x) := G(u_K \omega) + G_{\chi_K} = (I - T)^{-1}G_{\chi_K}(x)$$

(3.18)

$$= \sum_{j=0}^{\infty} T^j(G_{\chi_K})(x) = \int_K G(x, y) \, dy,$$

for all $x \in \Omega$.

By equation (1.6),

$$u_f(x) = Pf(x) + \sum_{j=1}^{\infty} T^j(Pf)(x)$$

$$= Pf(x) + \int_{\Omega} G(x, y) \, Pf(y) \, d\omega(y),$$

for all $x \in \Omega$. Integrating both sides of this equation over $K$ with respect to $dx$,

$$\int_K u_f(x) \, dx = \int_K Pf(x) \, dx + \int_K \int_{\Omega} G(x, y) \, Pf(y) \, d\omega(y) \, dx$$

(3.19)

$$= \int_K Pf(x) \, dx + \int_{\Omega} \int_K G(x, y) \, dx \, Pf(y) \, d\omega(y)$$

$$= \int_K Pf(x) \, dx + \int_{\Omega} u_K(y) \, Pf(y) \, d\omega(y),$$

by Fubini’s theorem, equation (3.18) and the symmetry of $G$.

The term $\int_K Pf(x) \, dx$ is finite because (1.24) guarantees that $f$ is integrable with respect to harmonic measure, so $Pf$ is not identically infinite, and so is harmonic. Thus to prove that $u_f \in L^1(K, dx)$, it suffices to show that $u_K Pf \in L^1(\Omega, \omega)$.

By (1.12) and Fubini’s theorem,

$$\int_{\Omega} u_K(y) \, Pf(y) \, d\omega(y) = \int_{\partial\Omega} \int_{\Omega} M(y, z) u_K(y) \, d\omega(y) \, f(z) \, dH^0(z).$$

We claim that

$$\int_{\Omega} M(y, z) u_K(y) \, d\omega(y) \leq C_K e^{CM^*(m\omega)(z)},$$

(3.20)

if $z$ is a regular point of $\partial\Omega$. Assuming (3.20) for the moment, the set of irregular boundary points $E \subset \partial\Omega$ is known to be Borel and polar,
i.e., \( \text{cap}(E) = 0 \) ([AG], Theorem 6.6.8), and consequently negligible, i.e., of harmonic measure zero ([AG], Theorem 6.5.5). Therefore (3.20) yields

\[
(3.21) \quad \int_{\Omega} u_K(y) P f(y) \, d\omega(y) \leq C_K \int_{\partial \Omega} e^{CM^*(m\omega)(z)} f(z) \, dH^{x_0}(z).
\]

Hence our assumption (1.24) guarantees that \( u_K P f \in L^1(\Omega, \omega) \).

To prove (3.20), let us assume first that \( \omega \) is compactly supported. Then as mentioned above after (3.17), \( u_K \in L^2(\Omega, \omega) \). Hence, by Cauchy’s inequality, \( d\mu = u_K d\omega \) is a finite compactly supported measure. By equation (3.18), Lemma 3.1, and Theorem 2.2,

\[
(3.22) \quad u_K(x) \leq C_K \sum_{j=0}^{\infty} T^j m(x) \leq C_K m(x) e^{CG(m\omega)(x)/m(x)},
\]

since \( Tm = G(m\omega) \). Using the trivial estimate \( m(\cdot) \leq G(x_0, \cdot) \), followed by (3.17) and then (3.22),

\[
(3.23) \quad \frac{G(u_K \omega)(x)}{G(x, x_0)} \leq \frac{G(u_K \omega)(x)}{m(x)} \leq \frac{u_K(x)}{m(x)} \leq C_K e^{CG(m\omega)(x)/m(x)},
\]

for \( x \in \Omega \). Applying (3.8) with \( d\mu = u_K d\omega \) and then (3.9) with \( d\mu = m d\omega \),

\[
\int_{\Omega} M(y, z) u_K(y) \, d\omega(y) = \lim_{x \to z, x \in \Omega} \frac{G(u_K \omega)(x)}{G(x, x_0)} \leq \lim_{x \to z, x \in \Omega} C_K e^{CG(m\omega)(x)/m(x)} = C_K e^{M^*(m\omega)(z)},
\]

where the regularity of \( z \in \partial \Omega \) is used only at the last step. Hence (3.20) is established for compactly supported measures \( \omega \).

In the general case, consider an exhaustion \( \Omega = \bigcup_{k=1}^{\infty} \Omega_k \), where \( \{\Omega_k\} \) is a family of nested, relatively compact subdomains of \( \Omega \). Without loss of generality we may assume that \( x_0 \in \Omega_k \), for all \( k \in \mathbb{N} \).

In \( \Omega \times \Omega \), define the iterated Green’s kernels \( G_j^{(k)}(x, y) \) for \( j \in \mathbb{N} \), and \( G^{(k)}(x, y) = \sum_{j=1}^{\infty} G_j^{(k)}(x, y) \), as in (3.14), (3.15), except with \( \omega \) replaced by \( \omega_k \), \( k \in \mathbb{N} \). Let \( u_K^{(k)} = G^{(k)} \chi_K \). By repeated use of the monotone convergence theorem, we see that \( G_j^{(k)}(x, y) \) increases monotonically as \( k \to \infty \) to \( G_j(x, y) \) for each \( j \), \( G^{(k)}(x, y) \) increases monotonically to \( G(x, y) \), and \( u_K^{(k)} \) increases monotonically to \( u_K \). Applying
the compact support case gives
\[ \int_{\Omega} M(y, z) u_K^{(k)}(y) \chi_{\Omega_k}(y) \, d\omega(y) \leq C_K e^{CM^*(m\omega_k)}(z) \]
\[ \leq C_K e^{CM^*(m\omega)}(z). \]

Then, as \( k \to \infty \), the monotone convergence theorem yields (3.20).

(B) Suppose \( u_f \in L^1_{\text{loc}}(\Omega, dx) \), where \( f \neq 0 \) a.e. relative to harmonic measure, and
\[
u = T u_f + Pf \quad \text{on } \Omega. \]
So \( T u_f \leq u_f \), where \( 0 < u_f < \infty \) \( \omega \)-a.e. It follows by Schur’s lemma that
\[ \|T\|_{L^2(\omega) \to L^2(\omega)} \leq 1. \]
It remains to show that (1.25) holds. We remark that this condition follows immediately from (1.21) with \( x = x_0 \) provided \( u_f(x_0) < \infty \).
Since this is not necessarily the case, we proceed as follows.

Choose any compact set \( K \subseteq \Omega \) with \( |K| > 0 \). By Lemma 3.1 and Theorem 2.2,
\[ u_K(x) = \sum_{j=0}^{\infty} T_j G \chi_K(x) \geq c_K \sum_{j=0}^{\infty} T^j m(x) \geq c_K m(x) e^{c(T^j m(x))/(m(x))}, \]
for all \( x \in \Omega \). In fact, we can let \( c = 1 \) in the preceding estimate, exactly as in the proof of (2.14) above, by using [GV2], Theorem 1.2 with \( q = 1 \), \( h = m \), and \( b = 1 \). Notice that \( m \) is a superharmonic function in \( \Omega \), and so the Maria-Frostman domination principle yields (2.15) with \( b = 1 \) and \( h = m \).

By inequality (3.24), equation (3.17) and inequality (3.11),
\[ e^{T^j m(x)/(m(x))} \leq c_K^{-1} \frac{u_K(x)}{m(x)} \leq c_K^{-1} \left( \frac{G(u_K \omega)(x)}{m(x)} + G \chi_K(x) \right) \]
\[ \leq c_K^{-1} \frac{G(u_K \omega)(x)}{m(x)} + C_K^{-1}. \]

Let \( z \in \partial \Omega \) be a regular point. Applying Lemma 3.3 with \( d\mu = md\omega \) on the left side of (3.25) (recalling that \( Tm = G(m\omega) \)), and with \( d\mu = u_K \omega \) on the right side, we obtain
\[ e^{M^*(m\omega)}(z) \leq c_K^{-1} \int_{\Omega} M(y, z) u_K(y) \, d\omega(y) + C_K^{-1}, \]
if \( \omega \) has compact support in \( \Omega \). By the same exhaustion process that was used in the opposite direction, (3.26) holds for \( \omega \) locally finite in \( \Omega \).
Since the set of irregular points in $\partial \Omega$ has harmonic measure 0, as noted above, we can integrate (3.26) over $\partial \Omega$ with respect to $f \, dH^x_0$ and apply Fubini’s theorem to obtain
\[
\int_{\partial \Omega} e^{M^* (m \omega)(z)} f(z) \, dH^x_0(z) \leq C \frac{C_1}{C} \left( \int_{\Omega} \int_{\partial \Omega} M(y, z) f(z) \, dH^x_0(z) u_K(y) \, d\omega(y) \right) + C_K \frac{c}{c} \left( \int_{\partial \Omega} f(z) \, dH^x_0(z) \right),
\]
using equation (1.12). Since $u_K P f \in L^1(\Omega, \omega)$ by (3.19), we have condition (1.25).

**Remark.** For part (A) of Theorem 1.2 and Corollary 1.3, if $\Omega$ is a bounded $C^{1,1}$ domain, or a bounded Lipschitz domain with sufficiently small Lipschitz constant, then $G \chi_\Omega \approx m$ (see, for instance, [AAC], Theorem 1.1 and Remark 1.2(i)). Hence, \[ \int_{\Omega} M(x, z) \, dx \leq C, \] where $C$ does not depend on $z \in \partial \Omega$. Then one can replace $\chi_K$ above with $\chi_\Omega$ and obtain that $u_f \in L^1(\Omega, dx)$ with
\[
\int_{\Omega} u_f(x) \, dx \leq C \left( \int_{\partial \Omega} f(z) \, dH^x_0(z) + C_K \left( \int_{\partial \Omega} e^{C \cdot M^* (m \omega)(z)} f(z) \, dH^x_0(z) \right) \right).
\]
In the same way that Theorem 2.5 generalizes Theorem 1.1, there is a complete analogue of Theorem 1.2 for solutions of equation (2.17), with an arbitrary positive harmonic function $h$ in place of $P f$. It gives sufficient and matching necessary conditions for the existence of solutions whose pointwise estimates are provided in Theorem 2.5. The primary difference in this case is that $\mu_h$ is not necessarily zero on the set of irregular points of $\partial \Omega$. Hence we need to consider
\[
\varphi(z) = \liminf_{x \to z, x \in \Omega} \max(1, G(x, x_0)), \\
\psi(z) = \limsup_{x \to z, x \in \Omega} \max(1, G(x, x_0)),
\]
for $z \in \partial \Omega$. Note that $\varphi = \psi = 1$ at regular boundary points. The following result is a generalization of Lemma 3.3 which allows us to control the behavior of $\varphi$ and $\psi$ at irregular points in a uniform domain.

**Lemma 3.4.** Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded uniform domain, for $n \geq 2$. Suppose $\mu$ is a finite Borel measure with compact support in $\Omega$. Let $z \in \partial \Omega$. Then
\[
1 \leq \varphi(z) \leq \psi(z) \leq \kappa \varphi(z) \leq \kappa C_1, \quad z \in \Omega,
\]
for constants \( \kappa \) and \( C_1 \), where \( \kappa \) depends only on \( \Omega \) and \( C_1 \) depends only on \( \text{dist}(x_0, \partial \Omega) \). Moreover, for all \( z \in \partial \Omega \),

\[
\limsup_{x \to z, x \in \Omega} \frac{G \mu(x)}{m(x)} = \psi(z) M^* \mu(z) \leq \kappa \varphi(z) M^* \mu(z)
\]

\[
= \kappa \liminf_{x \to z, x \in \Omega} \frac{G \mu(x)}{m(x)}.
\]

**Proof.** The inequalities \( 1 \leq \varphi(z) \leq \psi(z) \) are trivial. The inequality \( \psi(z) \leq \kappa \varphi(z) \) follows from inequality (2.10) with \( y = x_0 \) and the observation that \( \max(1, G(x, x_0)) = G(x, x_0)/m(x) \). Since \( x \to z \), we may assume that \( |x - x_0| \geq c_1 \) for any \( c_1 < \text{dist}(x_0, \partial \Omega) \), for \( x \) close enough to \( z \). Then

\[
G(x, x_0) \leq c(n) |x - x_0|^{2-n} \leq c(n) c_1^{2-n},
\]

where we suppose again that \( n \geq 3 \) (the case \( n = 2 \) is treated in a similar way). Hence,

\[
\psi(z) \leq C_1 = \max \left( 1, c(n) [\text{dist}(x_0, \partial \Omega)]^{2-n} \right), \quad \text{for all } z \in \partial \Omega,
\]

and consequently (3.28) holds.

To prove (3.29), note that by (3.8),

\[
\limsup_{x \to z, x \in \Omega} \frac{G \mu(x)}{m(x)} = \limsup_{x \to z, x \in \Omega} \frac{G(x, x_0)}{m(x)} \frac{G \mu(x)}{G(x, x_0)} = \psi(z) M^* \mu(z)
\]

and

\[
\liminf_{x \to z, x \in \Omega} \frac{G \mu(x)}{m(x)} = \liminf_{x \to z, x \in \Omega} \frac{G(x, x_0)}{m(x)} \frac{G \mu(x)}{G(x, x_0)} = \varphi(z) M^* \mu(z).
\]

Hence, (3.29) is immediate from (3.28). \( \square \)

**Theorem 3.5.** Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded uniform domain, \( \omega \) is a locally finite Borel measure on \( \Omega \), and \( h \) is a positive harmonic function in \( \Omega \). Let \( x_0 \in \Omega \) be the reference point in the definition of Martin’s kernel. Let \( m(x) = \min(1, G(x, x_0)) \), and let \( \mu_h \) be the Martin’s representing measure for \( h \).

(A) There exists \( C > 0 \) (\( C \) depending only on \( \Omega \) and \( \|T\| \)) such that if \( \|T\| < 1 \) (equivalently, \( (1.3) \) holds with \( \beta < 1 \)) and

\[
(3.30) \int_{\partial \Omega} e^{C \varphi(z) M^*(\omega)(z)} \, d\mu_h(z) < \infty,
\]

then \( u_h = \sum_{j=0}^{\infty} T^j h \in L^1_{\text{loc}}(\Omega, dx) \) is a positive solution to (2.17).
(B) If \( u \in L^1_{\text{loc}}(\Omega, dx) \) is a positive solution of (2.17), then \( \|T\| \leq 1 \) and
\[
\int_{\partial \Omega} e^{\psi(z)M^*(m\omega)(z)} d\mu_h(z) < \infty.
\]

Proof. The proof follows the lines of the proof of Theorem 1.2, so we only sketch the differences. Let \( K \subseteq \Omega \) be compact with \( |K| > 0 \). Replacing \( Pf \) with \( h \), we obtain
\[
\int_K u_h(x) \, dx = \int_K h(x) \, dx + \int u_K(y) h(y) \, d\omega(y)
\]
instead of (3.19). Using Martin’s representation (2.1) instead of (1.12),
\[
\int_{\Omega} u_K(y) h(y) \, d\omega(y) = \int_{\partial \Omega} \int M(y,z) u_K(y) \, d\omega(y) \, d\mu_h(z).
\]

For part (A), it suffices to show that \( u_K h \in L^1(\Omega, d\omega) \). We claim that
\[
\int_{\Omega} M(y,z) u_K(y) \, d\omega(y) \leq C_K e^{CG(m\omega)(x_0)/m(x_j)} \int_{\Omega} M(y,z) u_K(y) \, d\omega(y) + C_K e^{CG(m\omega)(x_0)/m(x_j)},
\]
for all \( z \in \partial \Omega \), where \( C_1 \geq 1 \) is the constant in (3.28), which depends only on \( x_0 \) and \( \Omega \). Assuming this claim, then (3.33) implies (3.31) since
To prove (3.35), let $x_j$ be a sequence of points such that
\[
\lim_{j \to \infty} G(m\omega(x_j)) = \lim_{w \to z} \sup_{w \in \Omega} G(m\omega(w)) = \frac{\psi(z) M^*(u_K\omega)(x_j)}{m(x_j)}.
\]
By (3.29) with $\mu = u_K\omega$, and recalling that $G(m\omega) = Tm$,
\[
e^{\psi(z) M^*(m\omega)(z)} \leq \lim_{j \to \infty} Tm(x_j)/m(x_j)
\]
\[
\leq \lim_{j \to \infty} \sup_{w \in \Omega} \frac{G(u_K\omega)(x_j)}{m(x_j)} + C_K c^{-1} K G(u_K\omega)(x_j) m(x_j),
\]
which establishes (3.35). \qed

4. Nonlinear elliptic equations of Riccati type

In this section we treat equation (1.28). The definition of solutions of (1.28) is consistent with our approach in the previous sections.

Definition 4.1. A nonnegative function $v \in W^{1,2}_{\text{loc}}(\Omega)$ is a solution of (1.28) if $v$ is a weak solution in $\Omega$, i.e.,
\[(4.1)\quad \int_{\Omega} \nabla v \cdot \nabla h \, dx = \int_{\Omega} |\nabla v|^2 h \, dx + \int_{\Omega} h \, d\omega, \quad \text{for all } h \in C^\infty_0(\Omega),\]
and $v$ has a superharmonic representative (denoted also by $v$) in $\Omega$ whose greatest harmonic minorant is the zero function.

Since $v \in W^{1,2}_{\text{loc}}(\Omega)$, it is easy to see that (4.1) is equivalent to
\[(4.2)\quad -\Delta v = |\nabla v|^2 + \omega \quad \text{in } D'(\Omega),\]
i.e., $v$ is a distributional solution in $\Omega$. In other words, by the Riesz decomposition theorem (AG, Sec. 4.4), $|\nabla v|^2 + \omega$ is the Riesz measure associated with $-\Delta v$, and $v$ satisfies the integral equation
\[(4.3)\quad v = G(|\nabla v|^2 + \omega) \quad \text{in } \Omega.\]
In bounded Lipschitz domains, (4.3) is equivalent to $v$ being a very weak solution of (1.28) in the sense of [MR].

Via the relation $v = \log u$, solutions $v$ of (1.28) correspond formally to solutions $u$ of (1.1) with $f = 1$, i.e.,
\[(4.4)\quad \begin{cases}
-\Delta u = \omega u, & u > 0 \quad \text{in } \Omega, \\
u = 1 & \text{on } \partial \Omega.
\end{cases}\]
The minimal solution \(u_1\) to (4.4) (the gauge) is given by (1.22).

Earlier results on (1.28) were obtained in [HMV], where the problem was posed of finding precise conditions on the boundary behavior of \(\omega\) that ensure the existence of solutions.

The precise relation between solutions to (4.4) and (1.28) is complicated, as discovered by Ferone and Murat (see [FM1], [FM2], or Remark 4.2 in [FV2]). In the special case of smooth domains and absolutely continuous \(\omega\), the problem was studied by the authors in [FV2], where the condition of the exponential integrability of the balayage of \(m\omega\) appeared for the first time. In that setup, it was shown that if \(u_1\) is the minimal solution of (4.4), then \(v = \log u_1\) is a solution of (1.28). However, if \(v\) is a solution to (1.28) then \(u = e^v\) is in general only a supersolution to (4.4).

In Theorem 1.4, we treat general measures \(\omega\) and uniform domains \(\Omega\) based on the results of the previous sections. We take this opportunity to give further details on some points in the arguments presented in [FV2], Sec. 4. We also improve the constant in the exponent of the necessary condition (exponential integrability of the balayage).

**Proof of Theorem 1.4.** First suppose that \(\|T\| < 1\) and (1.26) holds with sufficiently large \(C > 0\). By Corollary 1.3, the Schrödinger equation (4.4) has a positive solution \(u = 1 + G\omega\). (This solution was called \(u_1\) in the statement of Corollary 1.3.) Then \(u \in L^1_{loc}(\Omega, d\omega)\) and \(u\) satisfies the integral equation \(u = 1 + G(\omega u)\). Therefore \(u : \Omega \to [1, +\infty]\) is defined everywhere as a positive superharmonic function in \(\Omega\) and hence is quasi-continuous by the known properties of superharmonic functions.

In particular, the infinity set \(E = \{x \in \Omega : u(x) = +\infty\}\) has zero capacity, \(\text{cap}(E) = 0\), and \(u \in W^{1,p}_{loc}(\Omega)\) when \(p < \frac{n}{n-1}\). In fact, \(u \in W^{1,2}_{loc}(\Omega)\) as shown in [JMV], Theorem 6.2, but the proof of this stronger property is more involved, and it will not be used below.

Define \(d\mu = -\Delta u = \omega u\), where a solution \(u \in L^1_{loc}(\Omega, \omega)\) to (4.4) is understood as in §2 above. Notice that \(u = \frac{d\mu}{d\omega}\) is the Radon–Nikodym derivative defined \(d\omega\)-a.e. Let \(v = \log u\). Then \(0 \leq v < +\infty\) \(d\omega\)-a.e., \(v\) is superharmonic in \(\Omega\) by Jensen’s inequality, and \(v \in W^{1,2}_{loc}(\Omega)\) (see [HKM], Theorem 7.48; [MZ], Sec. 2.2).

We claim that (4.1) holds. We will apply the integration by parts formula

\[
\int_\Omega g \, d\rho = -\langle g, \Delta r \rangle = \int_\Omega \nabla g \cdot \nabla r \, dx,
\]
where \( g \in W^{1,2}(\Omega) \) is compactly supported and quasi-continuous in \( \Omega \), and \( \rho = -\Delta r \) where \( r \in W^{1,2}_{\text{loc}}(\Omega) \) is superharmonic (see, e.g., [MZ], Theorem 2.39 and Lemma 2.33). This proof would simplify if we could apply (4.5) with \( g = \frac{h}{u}, \rho = \mu, \) and \( r = u \), for \( h \in C^\infty_0(\Omega) \). However, we do not use the property \( u \in W^{1,2}_{\text{loc}}(\Omega) \), so we need an approximation argument. For \( k \in \mathbb{N} \), let

\[
u_k = \min(u, e^k), \quad \mu_k = -\Delta u_k.
\]

Clearly \( u_k \) and \( v_k \) are superharmonic, hence \( \mu_k \) is a positive measure. Moreover, \( u_k \) and \( v_k \) belong to \( W^{1,2}_{\text{loc}}(\Omega) \cap L^\infty(\Omega) \) (see [HKM], Corollary 7.20).

Let \( h \in C^\infty_0(\Omega) \). We invoke (4.5) with \( g = \frac{h}{u_k}, \rho = \mu_k, \) and \( r = u_k \). Note that \( u_k \geq 1 \), \( g \) is compactly supported since \( h \) is, and \( g \in W^{1,2}(\Omega) \) since \( u_k \in W^{1,2}_{\text{loc}}(\Omega) \) and \( h \in W^{1,\infty}(\Omega) \) is compactly supported. Then by (4.5), we have

\[
\int_{\Omega} \frac{h}{u_k} \, d\mu_k = \int_{\Omega} \nabla \left( \frac{h}{u_k} \right) \cdot \nabla u_k \, dx
= \int_{\Omega} \frac{\nabla h}{u_k} \cdot \nabla u_k \, dx - \int_{\Omega} \frac{\nabla u_k^2}{u_k^2} h \, dx
= \int_{\Omega} \nabla h \cdot \nabla v_k \, dx - \int_{\Omega} |\nabla v_k|^2 h \, dx.
\]

As mentioned above, \( v \in W^{1,2}_{\text{loc}}(\Omega) \), and consequently \( \nabla v_k = \nabla v \) a.e. on \( \{v < k\} \), and \( \nabla v_k = 0 \) a.e. on \( \{v \geq k\} \) (see [MZ], Corollary 1.43). Hence,

\[
\lim_{k \to \infty} \int_{\Omega} \nabla h \cdot \nabla v_k \, dx = \int_{\Omega} \nabla h \cdot \nabla v \, dx,
\]
\[
\lim_{k \to \infty} \int_{\Omega} |\nabla v_k|^2 h \, dx = \int_{\Omega} |\nabla v|^2 h \, dx
\]

by the dominated convergence theorem.

Since \( u \) is superharmonic, \( u \) is lower semi-continuous, so the set \( \{x \in \Omega : u(x) > e^k\} \equiv \{u > e^k\} \) is open, and the measure \( \mu_k = -\Delta u_k \) is supported on the closed set \( \{u \leq e^k\} \) where \( u = u_k \). Hence \( u = u_k \) \( d\mu_k \)-a.e., and

\[
\int_{\Omega} \frac{h}{u_k} \, d\mu_k = \int_{\Omega} \frac{h}{u} \, d\mu_k.
\]

We next show that, for any continuous function \( h \) with compact support in \( \Omega \),

\[
\lim_{k \to \infty} \int_{\Omega} \frac{h}{u} \, d\mu_k = \int_{\Omega} \frac{h}{u} \, d\mu.
\]
Without loss of generality we assume here that $h \geq 0$. Otherwise we apply the argument below to $h_+$ and $h_-$ separately.

Notice that $u_k \uparrow u$, and consequently $\mu_k \rightharpoonup \mu$ weakly in $\Omega$, by the weak continuity property (see, for instance, [TW] in a rather more general setting), i.e.,

$$\lim_{k \to \infty} \int_\Omega \phi \, d\mu_k = \int_\Omega \phi \, d\mu$$

for all continuous functions $\phi$ with compact support in $\Omega$. It follows (see [Lan], Lemma 0.1) that

$$\lim \inf_{k \to \infty} \int_\Omega \phi \, d\mu_k \geq \int_\Omega \phi \, d\mu$$

for all lower semicontinuous functions $\phi$ with compact support in $\Omega$. The function $\frac{k}{u}$ is obviously upper semicontinuous with compact support, so by (4.8) applied to $-\frac{k}{u}$, we deduce

$$\lim \sup_{k \to \infty} \int_\Omega \frac{k}{u} \, d\mu_k \leq \int_\Omega \frac{k}{u} \, d\mu.$$

To prove an estimate in the opposite direction, we claim that $\mu_k \geq \mu$ on the closed set $F_k = \{x \in \Omega : u(x) \leq e^k\}$. It is enough to prove that

$$\mu_k(K) \geq \mu(K), \quad \text{for every compact set } K \subset F_k.$$  (4.10)

We verify (4.10) by using another approximation argument based on a version of Lusin’s theorem for certain Green potentials (the so-called semibounded potentials, see [Fug], Sec. 2.6). Notice that $u = G\mu + 1$, where $d\mu = u \, d\omega$, and $u < \infty$ $d\omega$-a.e., as discussed in §2. Moreover, $u < \infty$ on $\Omega \setminus E$, i.e., outside the infinity set $E$, which is obviously a Borel set such that $\mu(E) = 0$ since $\omega(E) = 0$.

This is also a consequence of the fact that $E$ is a set of zero capacity, and $\omega(E) \leq \text{cap}(E)$, which follows immediately from (1.9). In fact, the condition $\mu(E) = 0$ is equivalent to absolute continuity of $\mu$ with respect to capacity, i.e., $\text{cap}(K) = 0 \implies \mu(K) = 0$ for all compact sets $K \subset \Omega$.

Consequently (see [Fug], Theorem 2.6; [Hel], Theorem 4.6.3), there exists an increasing sequence of compactly supported measures $\mu^j$ such that $u^j = G\mu^j + 1 \in C(\Omega)$, so that $\mu^j(K) \uparrow \mu(K)$, for every compact set $K \subset \Omega$, and $G\mu^j \uparrow G\mu$ on $\Omega$, as $j \to \infty$. It follows that $u^j \uparrow u$, and so $\min(u^j, e^k) \uparrow \min(u, e^k) = u_k$ as $j \to \infty$, which yields that the corresponding Riesz measures $\mu^j_k$ associated with the superharmonic functions $\min(u^j, e^k)$ have the property $\mu^j_k \rightharpoonup \mu_k$ weakly in $\Omega$ as $j \to \infty$. 
Without loss of generality we may assume that actually \( u^j(x) < u(x) \) for all \( x \in \Omega \). Otherwise we replace \( u^j \) with \( \epsilon_j u^j \), where \( \epsilon_j \uparrow 1 \) is a strictly increasing sequence of positive numbers. Then all the properties of \( u^j \) remain true.

Obviously, \( F_k \subset G^j_k \) where \( G^j_k = \{ x \in \Omega : u^j(x) < e^k \} \) is an open set for every \( j, k \in \mathbb{N} \), since \( u^j \in C(\Omega) \). Clearly, \( u^j = \min(u^j, e^k) \) on \( G^j_k \), and so \( \mu^j \) coincides with \( \mu^j_k \) on \( G^j_k \). In particular, \( \mu^j_k(K) = \mu^j(K) \) for every compact set \( K \subseteq F_k \subset G^j_k \).

Since \( \mu^j_k \to \mu_k \) weakly, it follows by \((4.8)\) applied to the lower semicontinuous function \(-\chi_K\) that
\[
\limsup_{j \to \infty} \mu^j_k(K) \leq \mu_k(K).
\]
Hence,
\[
\mu(K) = \lim_{j \to \infty} \mu^j(K) = \limsup_{j \to \infty} \mu^j_k(K) \leq \mu_k(K),
\]
which proves \((4.10)\). Consequently,
\[
\liminf_{k \to \infty} \int_{\Omega} \frac{h}{u} \, d\mu_k \geq \liminf_{k \to \infty} \int_{F_k} \frac{h}{u} \, d\mu_k \geq \int_{\Omega \setminus E} \frac{h}{u} \, d\mu,
\]
where \( E \) is the infinity set of \( u \). As mentioned above, \( \mu(E) = 0 \), so \((4.11)\) actually yields
\[
\liminf_{k \to \infty} \int_{\Omega} \frac{h}{u} \, d\mu_k \geq \int_{\Omega} \frac{h}{u} \, d\mu.
\]
Combining the preceding inequality with \((4.9)\) proves \((4.7)\).

In fact, \( \mu_k \) coincides with \( \mu \) on the set \( G_k = \{ x \in \Omega : u(x) < e^k \} \), i.e.,
\[
\mu_k(K) = \mu(K), \quad \text{for every compact set } K \subseteq G_k.
\]
To prove \((4.12)\), notice that the set \( G_k \) is finely open (see \([AG]\), Sec. 7.1). Let \( U_k = \{ x \in \Omega : u(x) > e^k \} \), and \( \lambda = \chi_{U_k} \mu \). Then clearly \( G\lambda \leq G\mu = u \) in \( \Omega \), and so \( G\lambda < e^k \) on \( G_k \). Moreover, \( \lambda(G_k) = 0 \) since \( U_k \) and \( G_k \) are disjoint. Hence by \([Fug]\), Theorem 8.10, \( G\lambda \) is finely harmonic on \( G_k \).

On the other hand, let
\[
\bar{\mu} = \mu_k - \mu|_{F_k},
\]
where \( \mu_k \) is supported on the closed set \( F_k = \Omega \setminus U_k \). By \((4.10)\), \( \bar{\mu} \) is a nonnegative measure on \( \Omega \). Clearly, \( G\bar{\mu} \leq G\mu_k = u_k \leq e^k \) in \( \Omega \). Since
is finely harmonic on $G_k$. Hence applying [Fug], Theorem 8.10 in the opposite direction, we deduce that $\tilde{\mu}(G_k) = 0$, so $\tilde{\mu}(K) = \mu_k(K) - \mu(K) = 0$ for every compact set $K \subset G_k$. The proof of (4.12) is complete.

As noted above, $u = d\mu/d\omega$ is the Radon–Nikodym derivative defined $d\omega$-a.e., and $\mu(E) = \omega(E) = 0$, where $E = \{x \in \Omega : u(x) = \infty\}$, hence

$$
\int_{\Omega} h \, d\omega = \int_{\Omega} \frac{h}{u} \, d\mu = \lim_{k \to \infty} \int_{\Omega} \frac{h}{u_k} \, d\mu_k.
$$

Passing to the limit as $k \to \infty$ in (4.6), we obtain

$$
\int_{\Omega} h \, d\omega = \int_{\Omega} \nabla h \cdot \nabla v \, dx - \int_{\Omega} |\nabla v|^2 \, h \, dx,
$$

for all $h \in C_0^\infty(\Omega)$, which justifies equation (4.1).

By the Riesz decomposition theorem,

$$
v = G(-\Delta v) + g = G(|\nabla v|^2 + \omega) + g,
$$

where $g$ is the greatest harmonic minorant of $v$. Since $v \geq 0$, a harmonic minorant of $v$ is 0, so $g \geq 0$. It follows from (4.13) and the equation $u = G(u\omega) + 1$ that

$$
g \leq v = \log u = \log (G(u\omega) + 1) \leq G(u\omega).
$$

Since $G(u\omega)$ is a Green potential, the greatest harmonic minorant of $G(u\omega)$ is 0, therefore $g = 0$. Hence $v$ is a solution of (1.28). This completes the proof of Theorem 1.4 (A).

Conversely, suppose $v \in W^{1,2}_{loc}(\Omega)$ is a solution of equation (1.28), that is, $v = G(|\nabla v|^2 + \omega)$. Then $v \geq 0$ is superharmonic, $d\nu = |\nabla v|^2 \, dx + d\omega$ is the corresponding Riesz measure, and (4.2) holds. Let $v_k = \min(v, k)$ and $\nu_k = -\Delta v_k$, for $k = 1, 2, \ldots$. Clearly, $v_k \in W^{1,2}_{loc}(\Omega) \cap L^\infty(\Omega)$ is superharmonic.

Next, as in the proof of (4.10) above, we observe that $v_k \geq v$ on the set $F_k = \{x \in \Omega : v(x) \leq k\}$. To verify this claim, it is enough to check that

$$
\nu_k(K) \geq \nu(K), \text{ for every compact set } K \subseteq F_k.
$$

The preceding inequality is deduced again using the approximation argument based on [Hel], Theorem 4.6.3. It requires the existence of a Borel set $E \subset \Omega$ such that $G\nu < \infty$ on $\Omega \setminus E$, and $\nu(E) = 0$. Let $E = \{x \in \Omega : v(x) = \infty\}$. Then $E$ is a Borel set and $\text{cap}(E) = 0$. We need to show that $\nu(E) = 0$. 

It is known (see [HMV], Lemma 2.1) that since \( v \in W^{1,2}_{\text{loc}}(\Omega) \) is a solution to (4.2), then
\[
\int_{\Omega} h^2 d\nu = \int_{\Omega} |v|^2 h^2 dx + \int_{\Omega} h^2 d\omega \leq 4 \int_{\Omega} |\nabla h|^2 dx,
\]
for all \( h \in C_0^{\infty}(\Omega) \). It follows immediately that \( \nu(F) \leq 4 \text{cap}(F) \) for all compact (and hence Borel) sets \( F \). Since \( \text{cap}(E) = 0 \), we see that \( \nu(E) = 0 \), which completes the proof of (4.14).

We remark that actually \( \nu_k = \nu \) on \( G_k \), where \( G_k = \{ x \in \Omega : v(x) < k \} \), exactly as was shown above for \( \mu_k = \mu \) on \( G_k \) (with \( e^k \) in place of \( k \)). However, we do not need this fact in the remaining part of the proof.

Since \( \nabla v = \nabla v_k \) dx-a.e. on \( F_k \), and \( \nabla v_k = 0 \) dx-a.e. outside \( F_k \), it follows from (4.14) that
\[
- \Delta v_k = \nu_k \geq \chi_{F_k} \nu = |\nabla v_k|^2 + \chi_{F_k} \omega,
\]
as measures. In other words,
\[
- \Delta v_k = \nu_k = |\nabla v_k|^2 + \chi_{F_k} \omega + \lambda_k,
\]
where \( \lambda_k \) is a nonnegative measure in \( \Omega \) supported on \( F_k \). In fact, as discussed above, \( \lambda_k = 0 \) outside the set \( \{ x \in \Omega : u(x) = k \} \).

Let \( u = e^v \geq 1 \), \( u_k = e^{v_k} \) and \( \mu_k = -\Delta u_k \). Clearly, \( \nabla u_k = \nabla v_k e^{v_k} \), so \( u_k \in W^{1,2}_{\text{loc}}(\Omega) \cap L^\infty(\Omega) \). We claim that
\[
(4.17) \quad \mu_k = -\Delta u_k = -\Delta v_k e^{v_k} - |\nabla v_k|^2 e^{v_k} \geq 0.
\]
To prove (4.17), we use integration by parts (4.5) with \( g = he^{v_k} \), where \( h \in C_0^{\infty}(\Omega) \), and \( v_k \) in place of \( r \):
\[
\int_{\Omega} h e^{v_k} dv_k = \int_{\Omega} \nabla(h e^{v_k}) \cdot \nabla v_k \, dx
\]
\[
= \int_{\Omega} e^{v_k} \nabla h \cdot \nabla v_k \, dx + \int_{\Omega} h |\nabla v_k|^2 e^{v_k} \, dx
\]
\[
= \int_{\Omega} \nabla h \cdot \nabla u_k \, dx + \int_{\Omega} h |\nabla v_k|^2 e^{v_k} \, dx
\]
\[
= \int_{\Omega} h \, d\mu_k + \int_{\Omega} h |\nabla v_k|^2 e^{v_k} \, dx.
\]
Hence, first applying (4.17) and then (4.16), we obtain
\[
\langle h, \mu_k \rangle = \int_\Omega h \, d\mu_k = \int_\Omega h e^{v_k} \, d\nu_k - \int_\Omega h |\nabla v_k|^2 e^{v_k} \, dx = \int_\Omega h e^{v_k} \chi_{F_k} \, d\omega + \int_\Omega h e^{v_k} \, d\lambda_k = \int_\Omega h e^v \chi_{F_k} \, d\omega + \int_\Omega h e^v \, d\lambda_k.
\]
From the preceding equation it follows that, for all \( h \in C_0^\infty(\Omega), h \geq 0, \)
\begin{equation}
(4.18) \quad \langle h, \mu_k \rangle \geq \int_\Omega h u \chi_{F_k} \, d\omega \geq 0.
\end{equation}
Since \( v_k, \) and hence \( u_k, \) is lower semicontinuous, it follows that \( u_k \) is superharmonic in \( \Omega. \)

Clearly, \( u = \lim_{k \to +\infty} u_k \) is a superharmonic function in \( \Omega \) as the limit of the increasing sequence of superharmonic functions \( u_k, \) since \( u = e^v \not\equiv \infty. \) Moreover, as mentioned above, the infinity set \( E \) on which \( u = e^v = \infty \) has zero capacity, and \( \omega(E) \leq \nu(E) \leq 4 \text{cap}(E), \) so \( \omega(E) = 0. \)

Since \( -\Delta u_k = \mu_k \to \mu \) weakly in \( \Omega, \) where \( \mu = -\Delta u, \) passing to the limit as \( k \to \infty \) in (4.18) and using the monotone convergence theorem on the right-hand side yields
\[
\langle h, \mu \rangle \geq \int_{\Omega \setminus E} h u \, d\omega = \int_\Omega h u \, d\omega \geq 0.
\]
Hence \( u \) is superharmonic, and
\begin{equation}
(4.19) \quad -\Delta u \geq \omega u \quad \text{in} \quad \Omega
\end{equation}
in the sense of measures.

It follows from (4.19) that \( \tilde{\omega} = -\Delta u - \omega u \) is a non-negative measure in \( \Omega, \) so by the Riesz decomposition theorem
\[
u = G(-\Delta u) + g = G(\omega u) + G\tilde{\omega} + g \geq G(\omega u) + g,
\]
where \( g \) is the greatest harmonic minorant of \( u. \) Since \( u \geq 1, \) i.e., 1 is a harmonic minorant of \( u, \) it follows that \( g \geq 1, \) and consequently,
\begin{equation}
(4.20) \quad u \geq G(\omega u) + 1 = Tu + 1,
\end{equation}
for \( T \) defined by (1.5). Since \( u \geq Tu, \) it follows by Schur’s test that \( \|T\|_{L^2(\Omega,\omega) \to L^2(\Omega,\omega)} \leq 1, \) and hence (1.9) holds with \( \beta = 1. \)
Iterating (4.20) and taking the limit, we see that
\[
\phi \equiv 1 + G\omega = 1 + \sum_{j=1}^{\infty} G_j\omega = 1 + \sum_{j=1}^{\infty} T^j1 \leq u < +\infty \text{ a.e.},
\]
and
\[
\phi = G(\omega\phi) + 1.
\]
Hence \(\phi\) is a positive solution of (4.4). Thus (1.26) holds by Corollary 1.3 (B). This completes the proof of Theorem 1.4 (B). □

Remarks. 1. As in [FV2] for smooth domains and \(\omega \in L^1_{\text{loc}}(\Omega)\), our sufficiency results hold in uniform domains for signed measures \(\omega\), if \(\omega\) is replaced with \(|\omega|\) both in the spectral conditions (1.8), (1.9), and conditions (1.26), (1.27).

2. The lower pointwise estimates of solutions in Theorem 1.1(B) are still true for signed measures \(\omega\), under some additional assumptions (see [GV1]). However, the upper pointwise estimates Theorem 1.1(A) are no longer true in general, unless we replace \(\omega\) with \(|\omega|\).

3. It is still unclear under which (precise) additional assumptions on the quadratic form of \(\omega\) the main existence results and upper estimates of solutions remain valid. Some results of this type are discussed in [JMV], but without the prescribed boundary conditions.

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