Interpolatory projection technique for Riccati-based feedback stabilization of index-1 descriptor systems*

Mahtab Uddin\textsuperscript{1}[0000–0002–6526–9276], M. Monir Uddin\textsuperscript{2}[0000–0002–9817–6156], and Md. Motlubar Rahman\textsuperscript{3}[0000–0001–7040–7752]

\textsuperscript{1} Institute of Natural Sciences, United International University, Dhaka-1212, Bangladesh, mahtab@ins.uiu.ac.bd
\textsuperscript{2} Department of Mathematics and Physics, North south University, Dhaka-1229, Bangladesh, monir.uddin@northsouth.edu
\textsuperscript{3} Department of Mathematics, Jahangirnagar University, Savar, Dhaka-1342, Bangladesh, mmrmaths@gmail.com

Abstract. The aim of the work is to stabilize the unstable index-1 descriptor systems by Riccati-based feedback stabilization via a modified form interpolatory projection-based technique Iterative Rational Krylov Algorithm (IRKA). The basic IRKA is used to find Reduced Order Models (ROMs) for the stable systems conveniently but it is unsuitable for the unstable systems. In the proposed technique, we implement the initial feedback within the construction of the projectors of the IRKA approach. The Riccati solution is estimated from the ROM achieved by IRKA and hence the low-rank feedback matrix is attained. The feedback matrix for the full model is retrieved from the low-rank feedback matrix by the reverse projecting process. Finally, the applicability and efficiency of the proposed method are validated by applying to unstable index-1 descriptor systems. The simulation is done numerically using MATLAB and the achieved results are discussed in both tabular and graphical approaches.

Keywords: Interpolatory projection · Krylov subspace · Riccati equation · feedback stabilization · index-1 descriptor system.

1 Introduction

The first-order index-1 descriptor system can be written with the input-output relations by means of the sparse block-matrices as

\[
\begin{bmatrix}
E_1 & 0 \\
0 & 0 \\
E & \dot{x}(t)
\end{bmatrix}
= \begin{bmatrix}
J_1 & J_2 \\
J_3 & J_4
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix}
+ \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u(t),
\]

\[
y(t) = \begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
+ D_a u(t),
\]

* This work is funded by United International University, Dhaka, Bangladesh. It starts from October 01, 2019, and the reference is IAR/01/19/SE/18.
where \( E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{m \times n} \) and \( D \in \mathbb{R}^{m \times p} \) with very large \( n \) and \( p, m \ll n \), represent differential coefficient matrix, state matrix, control multiplier matrix, state multiplier matrix and direct transmission map respectively \[1,2\]. In some engineering applications, such as power systems models, the direct transmission remains absent and because of that \( D = 0 \). In the system (1), \( x(t) \in \mathbb{R}^n \) is the state vector and \( u(t) \in \mathbb{R}^p \) is control (input), while \( y(t) \in \mathbb{R}^m \) is the output vector and considering \( x(t_0) = x_0 \) as the initial state. Here \( x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2} \) with \( n_1 + n_2 = n \) are state vectors and other sub-matrices are sparse in appropriate dimensions and \( J_4 \) is non-singular.

Let us consider the Schur compliments attained from the index-1 descriptor system (1) as
\[
\begin{align*}
\mathcal{E} &:= E_1, \\
\mathcal{A} &:= J_1 - J_2 J_4^{-1} J_3, \\
\mathcal{B} &:= B_1 - J_2 J_4^{-1} B_2, \\
\mathcal{C} &:= C_1 - C_2 J_4^{-1} J_3, \\
\mathcal{D} &:= D - C_2 J_4^{-1} B_2.
\end{align*}
\]

Using the block-matrix properties and the Schur compliments (2), index-1 descriptor system (1) can be simplified to the generalized LTI continuous-time system as
\[
\begin{align*}
\dot{x}(t) &= \mathcal{A} x(t) + \mathcal{B} u(t), \\
y(t) &= \mathcal{C} x(t) + \mathcal{D} u(t)
\end{align*}
\]

**Lemma 1 (Equivalence of transfer functions).** Assume the transfer functions \( G(s) = C(s \mathcal{E} - \mathcal{A})^{-1} \mathcal{B} + \mathcal{D} \) and \( \mathcal{G}(s) = C(s \mathcal{E} - \mathcal{A})^{-1} B + D \) are obtained from the index-1 descriptor system (1) and the structured generalized system (3), respectively. Then, the transfer functions \( G(s) \) and \( \mathcal{G}(s) \) are identical and hence those systems are equivalent.

In the topics of applied mathematics and engineering fields, the necessity of LTI continuous-time systems is inevitable, for instance, control theory, system automation, and mechatronics \[3,4\]. The Continuous-time Algebraic Riccati Equation (CARE) appears in many branches of engineering applications; especially in electrical fields \[5,6\]. The CARE connected to the system (3) can be defined as
\[
A^T \mathcal{E} + \mathcal{E}^T X A - \mathcal{E}^T X B B^T X \mathcal{E} + C^T C = 0.
\]

The solution \( X \) of the CARE (4) is exists and unique, if the Hamiltonian matrix corresponding to the system (3) has no pure imaginary eigenvalues \[7\]. The solution \( X \) of (4) is symmetric positive-definite and called stabilizing for the stable closed-loop matrix \( \mathcal{A} - (B B^T)^T \mathcal{E} \). Riccati-based feedback matrix plays a vital role in the stabilization approaches for unstable systems \[8\].

In Riccati-based feedback stabilization process, the optimal feedback matrix \( K^o = B^T X \mathcal{E} \) associated with the solution matrix \( X \) of the CARE (4) is essential. Using the optimal feedback matrix \( K^o \), an unstable LTI continuous-time system can be optimally stabilized by replacing \( A \) by \( \mathcal{A}_s = A - B K^o \). The stabilized
system can be written as
\[ \dot{E}x(t) = Ax(t) + Bu(t), \]
\[ y(t) = Cx(t) + Du(t). \]  

In [9] we have discussed the Riccati-based feedback stabilization technique for index-1 descriptor systems by Rational Krylov Subspace Method (RKSM) via the Linear Quadratic Regulator (LQR) approach. Also, in [10] the Low-Rank Cholesky-Factor integrated Alternative Direction Implicit (LRCF-ADI) based Kleinman-Newton method has been introduced for Riccati-based feedback stabilization of that of index-1 descriptor systems. In both of the works, we have tried to optimally stabilize power system models of the type index-1 descriptor system, derived from the Brazilian Inter-connected Power System (BIPS) models. In this work, we propose an interpolatory projection method Iterative Rational Krylov Algorithm (IRKA) for optimal feedback stabilization of those power system models.

2 IRKA for first-order systems

In this section, the IRKA approach for the first-order generalized system is discussed. Let us consider the generalized LTI continuous-time first-order system
\[ \dot{E}x(t) = Ax(t) + Bu(t), \]
\[ y(t) = Cx(t) + Du(t), \]  
where \( E \in \mathbb{R}^{n \times n} \) is non-singular, and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{m \times n} \) and \( D \in \mathbb{R}^{m \times p} \). The Reduced Order Model (ROM) of the system (6) can be written as
\[ \dot{\hat{E}}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \]
\[ \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t). \]  

We consider two sets of distinct interpolation points, \( \{\alpha_i\}_{i=1}^r \subset \mathbb{C} \) and \( \{\beta_i\}_{i=1}^r \subset \mathbb{C} \). Then the left and right transformation matrices \( W \) and \( V \), respectively, can be formed as
\[ \text{Range}(V) = \text{span}\{(\alpha_1 E - A)^{-1}Bb_1, \cdots, (\alpha_r E - A)^{-1}Bb_r\}, \]
\[ \text{Range}(W) = \text{span}\{(\beta_1 E^T - A^T)^{-1}C^Tc_1, \cdots, (\beta_r E^T - A^T)^{-1}C^Tc_r\}, \]  
where \( b_i \in \mathbb{C}^m \) and \( c_i \in \mathbb{C}^p \) are the right and left tangential directions, respectively. With these interpolation points and tangential directions the IRKA based interpolation can be achieved. For the left and right transformation matrices \( W \) and \( V \), the ROM (7) can be defined by the reduced-order matrices as follows
\[ \hat{E} := W^T E V, \quad \hat{A} := W^T A V, \]
\[ \hat{B} := W^T B, \quad \hat{C} := C V, \quad \hat{D} := D. \]
The treatment for the unstable systems is introduced and stabilization of the original system is done by the low-rank feedback matrix attained from ROM.

In the modified form of IRKA, at first the projectors \( V \) and \( W \) are determined by a set of sparse shifted linear systems. The treatment for the unstable systems is introduced and stabilization of the original system is done by the low-rank feedback matrix attained from ROM.

The tangential interpolation projection based method IRKA have been provided in [11,12], where the authors discussed that the interpolation points and the tangential directions need to be essentially updated until attain the desired condition.

The reduced transfer function \( \hat{G}(s) \) tangentially interpolates the original transfer function \( G(s) \) at a predefined set of interpolation points and some fixed tangential directions, such that

\[
G(\alpha_i) b_i = \hat{G}(\alpha_i) b_i, \quad c_i^T G(\beta_i) = c_i^T \hat{G}(\beta_i), \quad \text{and} \\
c_i^T G(\alpha_i) b_i = c_i^T \hat{G}(\alpha_i) b_i \quad \text{when} \quad \alpha_i = \beta_i, \quad \text{for} \quad i = 1, \ldots, r. \quad (10)
\]

The following moment-matching condition needs to be satisfied

\[
c_i^T G^{(j)}(\alpha_i) b_i = c_i^T \hat{G}^{(j)}(\alpha_i) b_i, \quad j = 0, 1, \ldots, q, \quad (11)
\]

\[
c_i^T C[(\alpha_i E - A)^{-1} E]^j (\alpha_i E - A)^{-1} B b_i = c_i^T \hat{C}[(\alpha_i \hat{E} - \hat{A})^{-1} \hat{E}]^j (\alpha_i \hat{E} - \hat{A})^{-1} \hat{B} b_i,
\]

where \( C[(\alpha_i E - A)^{-1} E]^j (\alpha_i E - A)^{-1} B \) is called the \( j \)-th moment of \( G(\cdot) \), and that represents the \( j \)-th derivative of \( G(\cdot) \) evaluated at the interpolation point \( \alpha_i \). The summary of the first-order IRKA procedures are provided in Algorithm 1.

### Algorithm 1: IRKA for First-Order Systems

| Step | Description |
|------|-------------|
| 1    | Make the initial selection of the interpolation points \( \{\alpha_i\}_{i=1}^r \) and the tangential directions \( \{b_i\}_{i=1}^r \). |
| 2    | \( V = [ (\alpha_1 E - A)^{-1} B b_1, \ldots, (\alpha_r E - A)^{-1} B b_r ] \), \( W = [ (\alpha_1 E^T - A^T)^{-1} C^T c_1, \ldots, (\alpha_r E^T - A^T)^{-1} C^T c_r ] \). |
| 3    | while \( \text{not converged} \) do \|
| 4    | \( \hat{E} = W^T E V, \hat{A} = W^T A V, \hat{B} = W^T B, \hat{C} = C V \). |
| 5    | for \( i = 1, \ldots, r \) do |
| 6    | Compute \( \hat{A} z_i = \lambda_i \hat{E} z_i \) and \( y_i^T \hat{A} = \lambda_i y_i^T \hat{E} \) for \( \alpha_i \rightleftharpoons -\lambda_i \), \( b_i^* \rightleftharpoons -y_i^T \hat{B} \) and \( c_i^* \rightleftharpoons C z_i^* \). |
| 7    | end for |
| 8    | Repeat Step-2. |
| 9    | \( i = i + 1 \); |
| 10   | end while |
| 11   | Construct the reduced-order matrices by repeating Step-4. |

3 Modified IRKA for first-order index-1 descriptor systems

In the modified form of IRKA, at first the projectors \( V \) and \( W \) are determined by a set of sparse shifted linear systems. The treatment for the unstable systems is introduced and stabilization of the original system is done by the low-rank feedback matrix attained from ROM.
3.1 Sparsity preservation

For the converted generalized system \( (3) \) the transformation matrices \( W \) and \( V \) defined in \( (9) \) can be written as

\[
\text{Range}(V) = \text{span}\{(\alpha_1 \mathcal{E} - A)^{-1} B b_1, \ldots, (\alpha_r \mathcal{E} - A)^{-1} B b_r\},
\]

\[
\text{Range}(W) = \text{span}\{(\alpha_1 \mathcal{E}^T - A^T)^{-1} C^T c_1, \ldots, (\alpha_r \mathcal{E}^T - A^T)^{-1} C^T c_r\},
\]

Due to the dense form of the matrix \( A \), the above forms are usually dense and infeasible for the large-scale systems. To avoid this adversity, at each iteration shifted linear systems need to be solved as

\[
(\alpha_i \mathcal{E} - A) v_i = B b_i,
\]

\[
\text{or}, (\alpha_i E_1 - (J_1 - J_2 J_4^{-1} J_3)) v_i = (B_1 - J_2 J_4^{-1} B_2) b_i,
\]

\[
\text{or}, \begin{bmatrix} (\alpha_i E_1 - J_1) - J_2 \\ -J_3 - J_4 \end{bmatrix} v_i = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} b_i,
\]

and

\[
(\alpha_i \mathcal{E}^T - A^T) w_i = C^T c_i,
\]

\[
\text{or}, (\alpha_i E_1^T - (J_1 - J_2 J_4^{-1} J_3)^T) w_i = (C_1 - C_2 J_4^{-1} J_3)^T c_i,
\]

\[
\text{or}, \begin{bmatrix} (\alpha_i E_1 - J_1) - J_2 \end{bmatrix}^T w_i = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} c_i.
\]

Here \( \Gamma_1 \) and \( \Gamma_2 \) are the truncated terms. The explicit form of the reduced-order matrices in \( (13) \) defined in \( (9) \) are inefficient to construct feasible ROM. The sparsity preserving reduced-order matrices can be attained by following way

\[
\hat{\mathcal{E}} = W^T E_1 V, \quad \hat{A} = W^T J_1 V - (W^T J_2) J_4^{-1} (J_3 V),
\]

\[
\hat{B} = W^T B_1 - (W^T J_2) J_4^{-1} B_2, \quad \hat{C} = C_1 V - C_2 J_4^{-1} (J_3 V).
\]

3.2 Treatment for the unstable systems

If the system is unstable, a Bernoulli stabilization is required through an initial-feedback matrix \( K_0 \) to estimate \( \hat{A}_f = A - BK_0 \) and the matrix \( A \) needs to be replaced by \( \hat{A}_f \) \( (13) \). Then, the system \( (3) \) and CARE \( (4) \) need to be re-defined as

\[
\mathcal{E} \dot{x}(t) = A_f x(t) + B u(t),
\]

\[
y(t) = C x(t) + D u(t),
\]

\[
A_f^T X \mathcal{E} + \mathcal{E}^T X A_f - \mathcal{E}^T X B B^T \mathcal{E} + C^T C = 0.
\]

Then, the shifted linear systems \( (13) \) and \( (14) \) need to be redefined as

\[
(\alpha_i \mathcal{E} - A_f) v_i = B b_i,
\]

\[
\text{or}, \begin{bmatrix} (\alpha_i E_1 - (J_1 - B_1 K_0)) - J_2 \\ -J_3 - B_2 K_0 \end{bmatrix} v_i = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} b_i,
\]

and

\[
(\alpha_i \mathcal{E}^T - A_f^T) w_i = C^T c_i,
\]

\[
\text{or}, \begin{bmatrix} (\alpha_i E_1^T - (J_1 - B_1 K_0)) - J_2 \end{bmatrix}^T w_i = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} c_i.
\]
Algorithm 2: Modified IRKA for First-Order Index-1 Systems.

**Input**: \( E_1, J_1, J_2, J_3, J_4, B_1, B_2, C_1, C_2, D_a \).

**Output**: \( \hat{E}, \hat{A}, \hat{B}, \hat{C}, D_a := D_a \) and optimal feedback matrix \( K^o \).

1. Make the initial selection of the interpolation points \( \{\alpha_i\}_{i=1}^r \) and the tangential directions \( \{b_i^*\}_{i=1}^r \) & \( \{c_i\}_{i=1}^r \).

2. \( V = [v_1, v_2, \ldots, v_r], W = [w_1, w_2, \ldots, w_r] \), where \( v_i \) and \( w_i \) can be attained from (15) and (19).

3. While (not converged) do
   4. Evaluate sparsity preserving \( \hat{E}, \hat{A}, \hat{B}, \) and \( \hat{C} \) by (15).
   5. for \( i = 1, \ldots, r \) do
      6. Compute \( \hat{A}_i z_i = \lambda_i \hat{E} z_i \) and \( y_i^* \hat{A} = \lambda_i y_i^* \hat{E} \) for \( \alpha_i = -\lambda_i, b_i^* \leftarrow -y_i^* \hat{B} \) and \( c_i^* \leftarrow \hat{C} z_i^* \).
   7. end for
   8. Repeat Step-2.
   9. \( i = i + 1 \);
4. end while

10. Construct the reduced-order matrices by repeating Step-4.
11. Solve the low-rank Riccati equation (21) for \( \hat{X} \).
12. Compute the reduced-order feedback matrix \( \hat{K} = \hat{B}^T \hat{X} \).
13. Retrieve the optimal feedback matrix \( K^o = \hat{K} V^T E_1 \).

3.3 Optimal feedback matrix from the reduced-order models

This section includes the main contribution of the article. Using the reduced-order matrices defined in (15), we can find the reduced-order form of the system (3) as

\[
\hat{E} \dot{x}(t) = \hat{A} x(t) + \hat{B} u(t),
\]

\[
y(t) = \hat{C} x(t) + \hat{D} u(t),
\]

Now, we employ the ROM (20) to determine the approximate optimal feedback matrix for the original model. If the ROM (20) can be attained once, we can solve the corresponding low-rank CARE defined as

\[
\hat{A}^T \hat{X} \hat{E} + \hat{E} \hat{X} \hat{A} - \hat{E} \hat{X} \hat{B} \hat{B}^T \hat{X} \hat{E} - \hat{C}^T \hat{C} = 0.
\]

The low-rank CARE (21) can be solved for symmetric positive-definite matrix \( \hat{X} \) by applying the M ATLAB care command. Then, the stabilizing feedback matrix for the ROM (20) is \( \hat{K} = \hat{B}^T \hat{X} \hat{E} \), and the ROM based approximation for the optimal feedback matrix \( K^o \) can be retrieved for stabilizing the full model defined as

\[
K^o = \hat{B}^T \hat{X} V^T E_1 = \hat{K} V^T E_1,
\]

where \( V \) is the right transformation matrix. The modified form of the IRKA for the first-order index-1 systems is provided in Algorithm 2.

At the end, using the optimal feedback matrix \( K^o \), the system (3) can be optimally stabilized as (5).
4 Numerical results

To justify the accuracy and effectiveness of the proposed algorithm, it has been applied to the data generated in some large-scale real-world models. The experiments are carried out with MATLAB® R2015a (8.5.0.197613) on a board with Intel® Core™ i5 6200U CPU with a 2.30 GHz clock speed and 16 GB RAM. For the numerical experiments, the following model examples are used.

4.1 Brazilian Inter-connected Power System (BIPS) models

The Brazilian Inter-connected Power System (BIPS) models are the most convenient examples of the index-1 descriptor systems [14]. The following Table-1 provides the details about the unstable power system models [mod - 606, mod - 1998, and mod - 2476], where the names of the models are considered according to their number of states.

Table 1. Structure of the unstable power system models

| Dimensions | 7135 | 15066 | 16861 |
|------------|------|-------|-------|
| States     | 606  | 1998  | 2476  |
| Algebraic variables | 6529 | 13068 | 14385 |
| Inputs     | 4    | 4     | 4     |
| Outputs    | 4    | 4     | 4     |

4.2 Reduced-order models of BIPS models

To find the stabilizing feedback matrix for unstable BIPS model, at first the ROMs are need to be computed. The ROMs of the models mod - 606, mod - 1998, and mod - 2476 are computed with the dimensions 30, 70, and 100, respectively. In the numerical computations, for every models we have taken the truncation tolerance $10^{-3}$ for the relative error and maximum number of iterations $i_{max} = 150$. For the convenience of time and compactness of this work, the analysis of the results found for the model mod - 2476 will be narrated.

In Figure 1, the comparison of full model and 100 dimensional ROM of mod - 2476 is provided. From Figure 1a it is observed that the transfer functions of the full model and the ROM are identical. Figure 1b and Figure 1c illustrate the absolute error and relative error of the ROM, respectively. From those figures it is evident that the absolute and relative error are significantly small.

4.3 Analysis of the eigenvalues

The Figure 2 depicts the magnified eigenvalues of the original system and the stabilized system. From the figure it can be said that using the proposed technique the unstable eigenvalues of the original system can be sufficiently stabilized.

1 https://sites.google.com/site/rommes/software
Figure 1. Comparison of full model and reduced order model for $mod = 2476$. 

(a) Sigma plot

(b) Absolute error

(c) Relative error
Interpolatory projection technique for Riccati-based feedback stabilization

4.4 Stabilization of step-responses

The Figure 3 represents the stabilization of the dominant step-responses of the target model. To reduce the size of the paper, only two step-responses are accounted. In Figure 3a, the step-responses of the original and stabilized systems are shown for second input/first output, whereas in Figure 3b it has shown for fourth input/third output. From the figurative comparison, it is revealed that the modified form of IRKA can be efficiently applied for the feedback stabilization of the unstable index-1 descriptor systems.

5 Conclusions

In this work, we have proposed and discussed a modified form of interpolatory projection-based technique Iterative Rational Krylov Algorithm (IRKA) for the Riccati-based feedback stabilization of unstable index-i descriptor systems. We have implemented the proposed method to unstable power system models derived from Brazilian Interconnected Power System (BIPS). To pursue the desired goal, the Reduced-Order Model (ROM) based low-rank CARE has been solved by the MATLAB library command care and corresponding low-rank feedback matrix has estimated. The optimal feedback matrix for stabilizing the full model has retrieved. We have shown the sparse form of the construction of the transformation matrices and introduced an approach to imply the initial feedback matrix for the treatment of the unstable systems. For the validation of the proposed method, the numerical results gained by the method has justified by tabular and graphical methods. It has been observed that the ROMs attained by the proposed method properly represent the full model. From the stabilization of the eigenvalues and step-responses, it is evident that the proposed method is efficient for the stabilization of unstable descriptor systems.
Figure 3. Stabilization of step-responses $mod - 2476$. 
References

1. M. S. Hossain and M. M. Uddin, “Iterative methods for solving large sparse lyapunov equations and application to model reduction of index 1 differential-algebraic-equations,” Numerical Algebra, Control & Optimization, vol. 9, no. 2, pp. 173–186, 2019.
2. P. Benner, J. Saak, and M. M. Uddin, “Reduced-order modeling of index-1 vibrational systems using interpolatory projections,” in 2016 19th International Conference on Computer and Information Technology (ICCIT). IEEE, 2016, pp. 134–138.
3. M. M. Uddin, “Computational methods for model reduction of large-scale sparse structured descriptor systems,” Ph.D. dissertation, Otto-von-Guericke Universität Magdeburg, 2015.
4. P. Benner, J. Saak, and M. M. Uddin, “Structure preserving model order reduction of large sparse second-order index-1 systems and application to a mechatronics model,” Mathematical and Computer Modelling of Dynamical Systems, vol. 22, no. 6, pp. 509–523, 2016.
5. E. K. W. Chu, “Solving large-scale algebraic riccati equations by doubling,” in Talk presented at the Seventeenth Conference of the International Linear Algebra Society, Braunschweig, Germany, vol. 22, 2011.
6. W. Chen and L. Qiu, “Linear quadratic optimal control of continuous-time lti systems with random input gains,” IEEE Transactions on Automatic Control, vol. 61, no. 7, pp. 2008–2013, 2016.
7. H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, Matrix Riccati equations in control and systems theory. Springer, 2012.
8. E. Bänsch, P. Benner, J. Saak, and H. Weichelt, “Riccati-based boundary feedback stabilization of incompressible Navier-Stokes flow,” DFG-SPP1253, Preprint SPP1253-154, 2013.
9. M. Uddin, M. H. Khan, and M. M. Uddin, “Riccati based optimal control for linear quadratic regulator problems,” in 2019 5th International Conference on Advances in Electrical Engineering (ICAEE). IEEE, 2019, pp. 290–295.
10. M. Uddin, M. H. Khan, and M. M. Uddin, “Efficient computation of riccati-based optimal control for power system models,” in 2019 22nd International Conference of Computer and Information Technology (ICCIT). IEEE, 2019, pp. 260–265.
11. Y. Xu and T. Zeng, “Optimal $H_2$ model reduction for large scale MIMO systems via tangential interpolation,” International Journal of Numerical Analysis and Modeling, vol. 8, no. 1, pp. 174–188, 2011.
12. P. Benner and J. Saak, “Efficient balancing-based MOR for large-scale second-order systems,” Math. Comput. Model. Dyn. Syst., vol. 17, no. 2, pp. 123–143, 2011.
13. P. Benner and T. Stykel, “Model order reduction for differential-algebraic equations: A survey,” Surveys in Differential-Algebraic Equations IV, p. 107, 2017.
14. F. Freitas and A. S. Costa, “Computationally efficient optimal control methods applied to power systems,” IEEE transactions on power systems, vol. 14, no. 3, pp. 1036–1045, 1999.
15. R. Leandro, A. S. e Silva, I. Decker, and M. Agostini, “Identification of the oscillation modes of a large power system using ambient data,” Journal of Control, Automation and Electrical Systems, vol. 26, no. 4, pp. 441–453, 2015.