Tau functions of \((n, 1)\) curves and soliton solutions on nonzero constant backgrounds

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Abstract
We study the tau function of the KP-hierarchy associated with an \((n, 1)\) curve \(y^n = x - \alpha\). If \(\alpha = 0\) the corresponding tau function is 1. On the other hand if \(\alpha \neq 0\) the tau function becomes the exponential of a quadratic function of the time variables. By applying vertex operators to the latter we obtain soliton solutions on nonzero constant backgrounds.

Keywords KP-hierarchy · Tau function · Sato Grassmannian · Soliton · Vertex operator · Rational curve

Mathematics Subject Classification 37K40 · 35C08 · 14H70

1 Introduction
In our previous paper [1,18] the degeneration of the theta function solution (tau function) of the KP-hierarchy as a result of what we call one-step degeneration of an \((n, s)\) curve for \(n = 2, 3\) has been studied. The obtained formula expresses the degenerate tau function as a sum of combinations of exponential functions with tau functions of a lower genus curve. Repeating this process we finally come to the tau function associated with an \((n, 1)\) curve of genus zero. In this paper we derive the explicit formula for the tau function of an \((n, 1)\) curve with an arbitrary \(n \geq 2\). We use it as a seed solution in the vertex operator construction of solutions of the KP-hierarchy. In the case of \(n = 2\) the solutions obtained in this way are considered as solitons on nonzero constant backgrounds.

For \(n \geq 2\) consider the rational algebraic curve \(C_n\) defined by

\[ y^n = x - \alpha, \quad (1.1) \]
which we call an \((n, 1)\) curve. From this curve a solution of the KP-hierarchy (tau function) is constructed as follows. To this end we use the Sato Grassmannian which we denote by UGM (=universal Grassmann manifold) after Sato [21, 22]. The Sato Grassmannian is the parameter space of solutions of the KP-hierarchy and is defined as the set of certain subspaces of the vector space \(V = \mathbb{C}(z)\) of Laurent series in a variable \(z\).

Consider the point \(\infty\) of \(C_n\) and take the local coordinate \(z\) around \(\infty\) such that
\[
x = z^{-n}, \quad y = z^{-1}(1 + O(z)).
\]
This choice of the local coordinate is necessary in order to study the degeneration of solutions of the KP-hierarchy associated with an \((n, s)\) curve [18]. Define the vector space \(V_n\) by
\[
V_n = \sum_{j=0}^{\infty} \mathbb{C} y^j.
\]
By expanding \(y^j\) in the coordinate \(z\) \(V_n\) can be considered as a subspace of \(V\). Then it is known that it belongs to UGM (see [17], for example). We denote this point of UGM by \(V_n(z)\) meaning that it is obtained from \(V_n\) using the coordinate \(z\).

For each point of UGM a solution \(\tau(t)\) of the KP-hierarchy is constructed, up to constant, in the form of the Schur function expansion as
\[
\tau(t) = \sum_{\lambda} \xi_{\lambda} s_{\lambda}(t),
\]
where the summation is over all partitions, \(s_{\lambda}(t)\) is the Schur function corresponding to \(\lambda\) and \(\xi_{\lambda}\) the Plücker coordinate of the point of UGM.

In the present case the Schur function expansion of the tau function corresponding to \(V_n(z)\) begins from a nonzero constant. We define the tau function \(\tau_0(t; V_n(z))\) as that corresponding to \(V_n(z)\) normalized as
\[
\tau_0(t; V_n(z)) = 1 + \sum_{|\lambda| > 0} \xi_{\lambda} s_{\lambda}(t).
\]  
\(1.2\)

In the case \(\alpha = 0\) we easily have \(\tau_0(t; V_n(z)) = 1\). So, the actual problem here is to compute \(\tau_0(t; V_n(z))\) for \(\alpha \neq 0\). The result is
\[
\tau_0(t; V_n(z)) = e^{\frac{1}{2}(q(t) + L(t))},
\]  
\(1.3\)
where \(q(t)\) is a quadratic form in \(t = (t_1, t_2, t_3, \ldots)\) and \(L(t)\) is a linear form in \(t\) (see Theorem 3.1). It is difficult to prove such a formula directly from the Schur function expansion. The strategy here to derived the formula (1.3) is to consider a coordinate change of the corresponding wave functions [4]. The effect of a coordinate change on solutions of the KP-hierarchy may be well known (see [5], for example). It can be

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considered as a generalization of the Galilean invariance of the KdV equation which we refer to as the Galilean invariance of the KP-hierarchy. Let us give an outline of it.

Let \( u(x, t) \) be a solution of the KdV equation

\[
-4u_t + 6uu_x + u_{xxx} = 0. \quad (1.4)
\]

Then

\[
\tilde{u}(x, t) = u(x + \frac{3}{2}ct, t) + c \quad (1.5)
\]

is again a solution of (1.4) for any constant \( c \). This is the Galilean transformation of a solution of the KdV equation. It is generalized to the case of the KP-hierarchy as follows.

Let \( L(t) = \partial + \sum_{j=2}^{\infty} u_j \partial^{1-j}, \partial = d/dx, x = t_1, \) be a solution the KP-hierarchy in the Lax form:

\[
\frac{\partial L}{\partial t_j} = [B_j, L], \quad (1.6)
\]

where \( B_j = (L^j)_+ \) is the differential operator part of \( (L^j)_+ \). We denote by \( \tau(t) \) a tau function of \( L \) and by \( \Psi(t, z) \) the wave function of \( L \), respectively, that is,

\[
\Psi(t, z) = \frac{\tau(t - [z])}{\tau(t)} e^{\sum_{j=1}^{\infty} t_j z^j}, \quad [z] = (z, z^2/2, z^3/3, \ldots),
\]

\[
L \Psi = z^{-1} \Psi, \quad \frac{\partial \Psi}{\partial t_k} = B_k \Psi.
\]

Consider a change of the parameter \( z \) to \( z' \),

\[
z' = z(1 + \sum_{j=2}^{\infty} b_j z^j)^{-1},
\]

and set

\[
(z')^{-k} = z^{-k}(1 + \sum_{j=2}^{\infty} b_{k,j} z^j),
\]

\[
t'_k = t_k + \sum_{j=2}^{\infty} b_{j+k,j} t_{j+k}. \quad (t'_1, t'_2, t'_3, \ldots),
\]

Then \( \tilde{L}(t) = L(t') \) is a solution of (1.6) and \( \tilde{\Psi}(t, z') = \Psi(t', z) \) is the wave function of \( \tilde{L}(t) \). Let \( \tilde{\tau}(t) \) be a tau function of \( \tilde{L}(t) \). Then there exist a quadratic form \( \tilde{q}(t) \) and a linear form \( \tilde{L}(t) \) such that

\[
\tilde{\tau}(t) = e^{\tilde{q}(t) + \tilde{L}(t)} \tau(t'),
\]
up to constant multiples. The present case corresponds to $\tau(t) = 1$ and the special coordinate change (4.3), where $w$ and $z$ correspond to $z$ and $z'$, respectively. It explains the form of $\tau_0(t; V_n(z))$ in (1.3). In order to compute $q(t)$, $L(t)$ explicitly in our special case we need to repeat a large part of the proof of the Galilean invariance of the KP-hierarchy.

The Schur function expansion of the tau function (1.3) is also calculated explicitly by normalizing a frame of $V_n(z)$ (see Theorem 3.2). We use the Giambelli formula for Plücker coordinates [6,19] in this computation.

The second logarithmic derivative of a tau function with respect to $t_1$ (see (2.2)) gives a solution of the KP-equation (2.3). The solution corresponding to $\tau_0(t; V_n(z))$ is a constant. So, (1.3) itself does not look an interesting solution of the KP equation, although it is used to describe the degeneration of theta function solution as mentioned in the beginning. However, this is not the case. It plays a role as a seed to create various solutions of the KP-hierarchy. In fact it is well known that soliton solutions of the KP-hierarchy are constructed by applying vertex operators to the trivial solution 1 [4]. Similarly, we also get various solutions by applying vertex operators to $\tau_0(t; V_n(z))$. In particular, if $n = 2$ and $\alpha \neq 0$ solutions obtained in this way are considered as solitons with nonzero constant asymptotics (see Corollary 9.3 and Theorem 9.1). Note that for this computation to work it is indispensable to introduce the infinite number of time variables. We remark that, for the KdV and KP equations, solitons on nonzero constant asymptotics are also obtained from those with zero asymptotics by Galilean transformations.

Finally, we mention that in the paper [7] the tau function of the form (1.3) had been studied in relation with Gromov–Witten invariants and the dispersionless KP-hierarchy. In that paper all points of UGM corresponding to tau functions of the n-reduced KP-hierarchy, which are expressed in the form of the exponential of a quadratic form, are determined. The point $V_n(z)$ is a special family, depending on the parameter $\alpha$, in them. However, such explicit formula as in Theorem 3.1 of the tau function was not derived in [7] as far as the author understands.

The paper is organized as follows. After the introduction a brief review on the Sato Grassmannian and the KP-hierarchy is given in Sect. 2. In Sect. 3 the problems and main results are stated. Sections 4 to 7 are devoted to the proofs of main theorems. In Sect. 4 the problem of determining the tau function is reformulated in terms of the wave function. Beginning from the trivial wave function corresponding to the trivial tau function 1, the condition for the tau function which we seek for is formulated by the coordinate changed wave functions. The series expansion of the function which appears in the wave function in question is determined in Sect. 5. Based on the results of Sects. 4 and 5 Theorem 3.1 is proved in Sect. 6. In Sect. 7 Theorem 3.2 is proved by computing the Plücker coordinates of $V_n(z)$. The generating function of the coefficients of the quadratic form $q(t)$ is computed in Sect. 8. Here, the genus zero analogue of the bilinear meromorphic differential [2,3,15] of an $(n, s)$ curve with positive genus plays a crucial role. In Sect. 9 soliton solutions on nonzero constant backgrounds are computed by applying vertex operators to $\tau_0(t; V_n(z))$. In this calculation the result of Sect. 8 is crucial.
2 Sato Grassmannian

In this section we briefly review the theory of the Sato Grassmannian and the KP-hierarchy.

Let \( \tau(t) \) be a function of \( t = (t_1, t_2, \ldots) \). The KP-hierarchy is the equation for \( \tau(t) \) given by

\[
\text{Res}_{k=\infty} \tau(t-s-[k^{-1}])\tau(t+s+[k^{-1}])e^{-2\sum_{j=1}^{\infty} s_j k^j} \frac{dk}{2\pi i} = 0, \tag{2.1}
\]

where \( s = (s_1, s_2, \ldots), [k] = (k, k^2/2, k^3/3, \ldots) \). By expanding in \( s_j, j \geq 1 \), (2.1) gives an infinite number of differential equations for \( \tau(t) \) expressed in Hirota's bilinear form [4].

Set \( (x, y, t) = (t_1, t_2, t_3) \) and, for a solution \( \tau(t) \) of (2.1),

\[
u(t) = 2 \frac{\partial^2}{\partial x^2} \log \tau(t). \tag{2.2}\]

Then \( u(t) \) is a solution of the KP-equation,

\[
3u_{yy} + (-4u_t + 6uu_x + u_{xxx})_x = 0. \tag{2.3}
\]

The totality of solutions of the KP-hierarchy constitutes a certain infinite-dimensional Grassmann manifold called the Sato Grassmannian [22]. Let us recall its definition and fundamental properties.

Let \( V = \mathbb{C}((z)) \) be the vector space of Laurent series in \( z \). Define two subspaces of \( V \) by \( V_\phi = \mathbb{C}[z^{-1}], V_0 = z\mathbb{C}[[z]] \). Then \( V = V_\phi \oplus V_0 \). Let \( \pi : V \longrightarrow V_\phi \) be the projection map. Then the Sato Grassmannian which we denote by UGM (=universal Grassmann manifold) after Sato [21,22] is defined as the set of subspaces \( U \) of \( V \) such that

\[
\dim(\ker \pi|_U) = \dim(\text{coker} \, \pi|_U) < \infty.
\]

**Example 2.1** The subspace \( V_\phi \) belongs to UGM. In this case

\[
\dim(\ker \pi|_{V_\phi}) = \dim(\text{coker} \, \pi|_{V_\phi}) = 0. \tag{2.4}
\]

In general a subspace \( U \) defined by

\[
U = \sum_{j \leq 0} \mathbb{C} \xi_j, \quad \xi_j = z^j + \sum_{i=1}^{\infty} \xi_{i,j} z^i, \quad \xi_{i,j} \in \mathbb{C}, \tag{2.5}
\]

satisfies (2.4) and belongs to UGM. The totality of points of UGM corresponding to such frames is denoted by \( \text{UGM}^\phi \), which forms a cell called the big cell of UGM.
A point $U$ of UGM can be specified by its frame, that is, a basis of $U$. If we associate the infinite column vector $(a_n)_{n \in \mathbb{Z}}$ to an element $\sum a_n z^n$ of $V$, a frame of $U$ can be expressed by an $\mathbb{Z} \times \mathbb{Z}_{\leq 0}$ matrix $\xi = (\xi_{i,j})_{i \in \mathbb{Z}, j \in \mathbb{Z}_{\leq 0}}$, where $\mathbb{Z}_{\leq 0}$ denotes the set of non-positive integers.

In writing the matrix $\xi$ we follow the usual convention that the row numbers increase downward and the column numbers increase rightward. For example, the frame $\xi = (\ldots, \xi_{-1}, \xi_0)$ of $U$ given by (2.5) is represented as

$$\xi = \begin{pmatrix}
\vdots & \vdots \\
\cdots & 1 & 0 \\
\cdots & 0 & 1 \\
\cdots & (\xi_{1,-1}, \xi_{1,0}) \\
\cdots & (\xi_{2,-1}, \xi_{2,0}) \\
\vdots & \vdots 
\end{pmatrix}. \quad (2.6)$$

In general it is always possible to take a frame satisfying the following condition; there exists a negative integer $l$ such that

$$\xi_{i,j} = \begin{cases} 
1 & \text{if } j < l \text{ and } i = j \\
0 & \text{if } (j < l \text{ and } i < j) \text{ or } (j \geq l \text{ and } i < l).
\end{cases} \quad (2.7)$$

In the sequel we always take a frame which satisfies this condition, although it is not unique.

A Maya diagram $M = (m_j)_{j=0}^\infty$ is a sequence of integers such that $m_0 > m_1 > \cdots$ and $m_j = -j$ for all sufficiently large $j$. For a Maya diagram $M = (m_j)_{j=0}^\infty$ the corresponding partition is defined by $\lambda(M) = (j + m_j)_{j=0}^\infty$. By this correspondence the set of Maya diagrams and the set of partitions bijectively correspond to each other. We identify a partition with the corresponding Maya diagram.

For a frame $\xi$ and a Maya diagram $M = (m_j)_{j=0}^\infty$ define the Plücker coordinate of $\xi$ corresponding to $M$ by

$$\xi_M = \det(\xi_{m_i,j})_{i,j \leq 0}. $$

If $M$ corresponds to a partition $\lambda$, $\xi_M$ is denoted also by $\xi_\lambda$. Due to the condition (2.7) and the condition of the Maya diagram this infinite determinant can be computed by the finite determinant $\det(\xi_{m_i,j})_{k \leq -i, j \leq 0}$ for a sufficiently small $k$.

Define the Schur function $s_{(n)}(t)$ corresponding to the partition $(n)$ by

$$e^{\sum_{n=1}^\infty t_n k^n} = \sum_{n=0}^\infty s_{(n)}(t) k^n. $$

and the Schur function [12] corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ by

$$s_\lambda(t) = \det(s_{(\lambda_i-i+j)}(t))_{1 \leq i,j \leq l}. $$
We assign weight \( j \) to the variable \( t_j \). Then \( s_\lambda(t) \) becomes homogeneous of weight \( |\lambda| = \lambda_1 + \cdots + \lambda_l \).

To a point \( U \) of UGM take a frame \( \xi \) of \( U \) and define the tau function by

\[
\tau(t; \xi) = \sum_\lambda \xi_\lambda s_\lambda(t), \tag{2.8}
\]

where the sum is taken over all partitions. We call it the Schur function expansion of \( \tau(t; \xi) \).

If we change the frame \( \xi \), Plücker coordinates and consequently the tau function are multiplied by a nonzero constant due to the property of a determinant. We call \( \tau(t; \xi) \), for any frame \( \xi \) of \( U \), a tau function of \( U \). So, tau functions of a point of UGM differ by nonzero constant multiples to each other.

For a frame \( \xi \) of the form (2.5), (2.6), \( \xi(0) = 1 \) and the Schur function expansion takes the form

\[
\tau(t; \xi) = 1 + \text{p.w.t}, \tag{2.9}
\]

where p.w.t. means positive weight terms. For \( U \in \text{UGM}^\phi \) we denote \( \tau_0(t; U) \) the tau function normalized as in (2.9). We call it the normalized tau function of \( U \).

The fundamental theorem of the Sato theory is the following [22] (see also [9,13,21]).

**Theorem 2.2** [22] For a frame \( \xi \) of a point of UGM \( \tau(t; \xi) \) is a solution of the KP-hierarchy. Conversely for a formal power series solution \( \tau(t) \) of the KP-hierarchy there exists a unique point \( U \) of UGM such that \( \tau(t) \) is a tau function of \( U \).

The inverse construction from a solution \( \tau(t) \) of the KP-hierarchy to the point \( U \) of UGM is given using the wave function as follows [9,16,21,22]. The wave function \( \Psi(t; z) \) and the adjoint wave function \( \Psi^*(t; z) \) [4] corresponding to \( \tau(t) \) are defined by

\[
\Psi(t; z) = \frac{\tau(t - [z])}{\tau(t)} e^{\sum_{i=1}^\infty t_i z^{-i}} \quad \Psi^*(t; z) = \frac{\tau(t + [z])}{\tau(t)} e^{-\sum_{i=1}^\infty t_i z^{-i}}. \tag{2.10}
\]

These functions are solutions of the linear problem associated with the KP-hierarchy [4]. For the inverse construction we use the adjoint wave function. Let \( \Psi^*_i(z) \) be the Laurent series in \( z \) defined by

\[
(\tau(t)\Psi^*(t; z))|_{t=(x,0,0,0,...)} = \sum_{i=0}^\infty \Psi^*_i(z)x^i. \tag{2.11}
\]

Then

\[
U = \sum_{i=0}^\infty C\Psi^*_i(z). \tag{2.12}
\]
By this construction the following property follows. Let \( U \) be a point of UGM, \( \tau(t) \) a tau function corresponding to \( U \) and \( f(z) = e^{\sum_{i=1}^{\infty} a_i \frac{z^i}{i}} \) an invertible formal power series. Then \( f(z)U \) belongs to UGM and a tau function corresponding to it is given by

\[
e^{\sum_{i=1}^{\infty} a_i \frac{z^i}{i}} \tau(t).
\] (2.13)

3 (n,1) curve and main results

In this section the problem is formulated and main results are stated.

Let \( n \geq 2 \) be a positive integer and \( \alpha \) a complex number. Consider the rational curve \( C_n \) defined by

\[
y^n = x - \alpha
\]  (3.1)

which we call \((n, 1)\) curve. The point of infinity of \( C_n \) corresponds to \( y = \infty \). Take a local coordinate \( z \) around \( \infty \) such that

\[
x = z^{-n}, \quad y = z^{-1}(1 - \alpha z^n)^{1/n}.
\] (3.2)

This type of the local coordinate \( z \) was used for a general \((n, s)\) curves of genus \( g \geq 1 \) in constructing the multivariate sigma functions \([2,3,15]\) and the quasi-periodic solutions of the \( n \)-reduced KP-hierarchy \([2,16]\).

Let \( V_n \) be the space of meromorphic functions on \( C_n \) which have a pole only at \( \infty \). It is nothing but the vector space generated by \( y^i, i \geq 0 \):

\[
V_n = \sum_{i=0}^{\infty} \mathbb{C} y^i.
\]

By expanding \( y^i \) in the local coordinate \( z \) we consider \( V_n \) as a subspace of \( \mathbb{C}((z)) \). We denote this subspace by \( V_n(z) \) indicating the choice of the local coordinate \( z \). Later we consider a coordinate change. Notice that

\[
y^i = z^{-i}(1 + O(z)), \quad i \geq 0.
\]

By taking linear combinations of them we have a frame of \( V_n(z) \) of the form (2.5). Therefore,

\[
V_n(z) \in \text{UGM}^\phi.
\]

Let \( \tau_0(t; V_n(z)) \) be the normalized tau function of \( V_n(z) \).

Our main theorem is

\[\textcircled{ Springe}r\]
Theorem 3.1 The following formula holds:

\[ \tau_0(t; V_n(z)) = e^{\frac{1}{2}(q(t) + L(t))}, \]

\[ q(t) = \sum_{i,j \geq 1} q_{i,j} t^i t^j, \quad L(t) = \sum_{i=1}^{\infty} L_{in} t^n. \]

Here, \( q_{i,j} = q_{j,i} \) for any \( i,j \), \( q_{i,j} = 0 \) if \( j = nm \) for some \( m \geq 1 \) or \( i + j \neq 0 \mod n \) and, for \( r \geq 1, s \geq 0, 1 \leq p \leq n - 1, i \geq 1, \)

\[ q_{nr-p,ns+p} = \alpha^{r+s} \sum_{r+s=i, r \geq 1, s \geq 0} \sum_{p=1}^{n-1} \frac{ni}{(nr-p)(ns+p)} q_{nr-p,ns+p}. \tag{3.3} \]

where, for a nonnegative integer \( n, \)

\[ \left( \begin{array}{c} x \\ n \end{array} \right) = \frac{x(x-1) \cdots (x-n+1)}{n!}, \quad n \geq 1, \quad \left( \begin{array}{c} x \\ 0 \end{array} \right) = 1. \]

The Schur function expansion of \( \tau_0(t; V_n(z)) \) can also be computed.

Theorem 3.2 The following expansion holds.

\[ \tau_0(t; V_n(z)) = 1 + \sum_{l=1}^{\infty} \sum_{r_i, s_i, p_i, q_i} A_l((r_i), (s_i), (p_i), (q_i)) S(nr_1 - p_1 - 1, ..., nr_l - p_l - 1 | ns_1 + q_1, ..., ns_l + q_l)(t), \]

\[ A_l((r_i), (s_i), (p_i), (q_i)) = (-1)^{\sum_{i=1}^{l} (ns_i + q_i + r_i)} \alpha^{\sum_{i=1}^{l} (r_i + s_i)} \det \left( \delta_{pi,qj} \left( \begin{array}{c} r_i + s_j - 1 \\ s_j \end{array} \right) \left( \begin{array}{c} s_j + \frac{p_i}{n} \\ r_i + s_j \end{array} \right) \right)_{1 \leq i, j \leq l}. \]

Here, the summation in the second term is over all \( r_i, s_i, p_i, q_i \) satisfying

\[ 1 \leq p_i, q_i \leq n - 1, \quad r_i \geq 1, \quad s_i \geq 0, \]

\[ nr_1 - p_1 > \cdots > nr_l - p_l, \]

\[ ns_1 + q_1 > \cdots > ns_l + q_l, \]

and \((m_1, \ldots, m_l | m'_1, \ldots, m'_l)\) is the Frobenius notation of a partition.

The proofs of theorems are given in subsequent sections.
4 Coordinate change and wave functions

In this section we derive the equation for \( \tau_0(t; V_n(z)) \) by considering a coordinate change and wave functions.

Let us take \( w = y^{-1} \) as another local coordinate around \( \infty \). Then

\[
x - \alpha = w^{-n}, \quad y = w^{-1}.
\]

(4.1)

By expanding elements of \( V_n \) in terms of \( w \) and identifying the ambient space \( V \) of UGM with \( \mathbb{C}((w)) \), we define the subspace \( V_n(w) \) of \( V \). Then

\[
V_n(w) = \sum_{i=0}^{\infty} \mathbb{C} w^{-i} = V_\phi \in \text{UGM}^\phi.
\]

So \( \tau_0(t; V_n(w)) = 1 \). Consider the corresponding wave and adjoint wave functions,

\[
\Psi(t; w) = e^{\sum_{i=1}^{\infty} t_i w^{-i}}, \quad \Psi^*(t; w) = e^{-\sum_{i=1}^{\infty} t_i w^{-i}}.
\]

(4.2)

Here, we change the local coordinate from \( w \) to \( z \). By (4.1) \( z \) and \( w \) are connected by

\[
z^{-n} - \alpha = w^{-n}.
\]

Therefore,

\[
w = z(1 - \alpha z^n)^{-1/n}.
\]

(4.3)

Expand

\[
(1 - \alpha z^n)^{i/n} = \sum_{m=0}^{\infty} a_{i,m} z^m,
\]

(4.4)

where

\[
a_{i,m} = \begin{cases} (-\alpha)^k \left( \frac{i}{k} \right), & m = nk \text{ for some } k \geq 0 \\ 0, & \text{otherwise}. \end{cases}
\]

(4.5)

We set \( a_{i,j} = 0 \) if \( j < 0 \) for the sake of convenience. Let us decompose \( w^{-i} \) into two parts corresponding to negative and nonnegative powers in \( z \),

\[
w^{-i} = \sum_{m=0}^{i-1} a_{i,m} z^{-(i-m)} + f_i(z), \quad f_i(z) = \sum_{m=i}^{\infty} a_{i,m} z^{m-i}.
\]

(4.6)
Then
\[
\sum_{i=1}^{\infty} t_i w^{-i} = \sum_{i=1}^{\infty} t_i \sum_{m=0}^{i-1} a_{i,m} z^{-(i-m)} + \sum_{i=1}^{\infty} t_i f_i(z).
\]

Define the new set of time variables \(\{T_i| i \geq 1\}\) by
\[
T_i = \sum_{j=i}^{\infty} a_{j,i-j} t_j. \tag{4.7}
\]

Then
\[
\sum_{i=1}^{\infty} t_i w^{-i} = \sum_{l=1}^{\infty} T_l z^{-l} + \sum_{i=1}^{\infty} t_i f_i(z). \tag{4.8}
\]

The coefficient matrix \((a_{j,i-j})_{i,j \geq 1}\) in the right-hand side of (4.7) is an upper triangular matrix whose diagonal entries are all one. So, it has the inverse which is again an upper triangular matrix with the same property. Therefore, \(t_i\) can be written as
\[
t_i = \sum_{j=1}^{\infty} b_{i,j} T_j, \tag{4.9}
\]

where \(b_{i,j} = 0\) if \(j < i\) and \(b_{i,i} = 1\) for \(i \geq 1\). Then
\[
\sum_{i=1}^{\infty} t_i f_i(z) = \sum_{j=1}^{\infty} T_j F_j(z), \tag{4.10}
\]

with
\[
F_j(z) = \sum_{m=0}^{\infty} \left( \sum_{i=1}^{j} b_{i,j} a_{i,m+i} \right) z^m. \tag{4.11}
\]

Substituting (4.8) and (4.10) into (4.2) we get
\[
\Psi(t; w) = e^{\sum_{i=1}^{\infty} T_i F_i(z)} e^{\sum_{i=1}^{\infty} T_j z^{-j}}, \quad \Psi^*(t; w) = e^{-\sum_{i=1}^{\infty} T_i F_i(z)} e^{-\sum_{i=1}^{\infty} T_j z^{-j}}.
\]

Consider \(w\) as a function of \(z\) by (4.3), \(w = w(z)\). Define new pair of functions of \(T = (T_i)\) and \(z\) by
\[
\tilde{\Psi}(T; z) = \frac{z^2}{w^2} \frac{d}{dz} e^{\sum_{i=1}^{\infty} T_i F_i^+(z)} e^{\sum_{i=1}^{\infty} T_j z^{-j}}, \quad \tilde{\Psi}^*(T; z) = e^{-\sum_{i=1}^{\infty} T_i F_i^+(z)} e^{-\sum_{i=1}^{\infty} T_j z^{-j}}. \tag{4.12}
\]
where
\[ F_i^+(z) = F_i(z) - F_i(0). \]

Then

**Lemma 4.1** The functions $\tilde{\Psi}(T; z)$ and $\tilde{\Psi}^*(T; z)$ satisfy, for any $T = (T_i)$ and $T' = (T'_i)$, the following bilinear equation,
\[
\text{Res}_{z=0} \tilde{\Psi}(T; z) \tilde{\Psi}^*(T'; z) \frac{dz}{z^2} = 0. \tag{4.14}
\]

**Proof** Obviously we have
\[
\text{Res}_{w=0} \Psi(t; w) \Psi^*(t; w) \frac{dw}{w^2} = 0.
\]

Changing the variable from $w$ to $z$ and multiplying by $e^{\sum_{i=1}^{\infty} (-T_i + T'_i) F_i(0)}$ we get (4.14).

Recall the following characterization of the KP-hierarchy in terms of wave functions.

**Theorem 4.2** [4,8] Suppose we have formal series of the form
\[
\Phi(t; z) = (1 + \sum_{j=1}^{\infty} \Phi_j(t) z^j) e^{\sum_{i=1}^{\infty} \tau_i z^{-i}}, \tag{4.15}
\]
\[
\Phi^*(t; z) = (1 + \sum_{j=1}^{\infty} \Phi^*_j(t) z^j) e^{-\sum_{i=1}^{\infty} \tau_i z^{-i}}, \tag{4.16}
\]

which satisfy
\[
\text{Res}_{z=0} \Phi(t; z) \Phi^*(t'; z) \frac{dz}{z^2} = 0,
\]
for any $t$ and $t'$. Then there exists a solution $\tau(t)$ of the KP-hierarchy, unique up to constant multiples, such that $\Phi(t; z)$ and $\Phi^*(t; z)$ are the corresponding wave and adjoint wave functions.

By (4.3), (4.12), (4.13) it is obvious that $\tilde{\Psi}(T; z)$ and $\tilde{\Psi}^*(T; z)$ have the expansions of the form (4.15) and (4.16), respectively. Therefore,

**Corollary 4.3** There exists a unique, up to constant multiples, a solution $\tau(T)$ of the KP-hierarchy with the time variables $T = (T_i)$ such that
\[
\tilde{\Psi}(T; z) = \frac{\tau(T - [z])}{\tau(T)} e^{\sum_{i=1}^{\infty} T_i z^{-i}}, \tag{4.17}
\]
\[
\tilde{\Psi}^*(T; z) = \frac{\tau(T + [z])}{\tau(T)} e^{-\sum_{i=1}^{\infty} T_i z^{-i}}. \tag{4.18}
\]
Next we prove

**Proposition 4.4** The point of UGM corresponding to $\tau(T)$ in Corollary 4.3 is $V_n(z)$.

**Proof** We prove the proposition by calculating the expansion of $\tilde{\Psi}^*(T_1, 0, 0, \ldots; z)$ in $T_1$. Let

$$g(z) = w^{-1} = z^{-1} + \sum_{m=0}^{\infty} a_{1,m+1} z^m.$$  

Then, by the definition,

$$V_n(z) = \sum_{i=0}^{\infty} C g(z)^i.$$  

On the other hand

$$\tilde{\Psi}^*(T_1, 0, 0, \ldots; z) = e^{-T_1(z^{-1} + F^+(z))}.$$  

We have

$$F_1(z) = \sum_{m=0}^{\infty} a_{1,m+1} z^m,$$

since $b_{1,1} = 1$. Therefore,

$$\tilde{\Psi}^*(T_1, 0, 0, \ldots; z) = e^{-T_1(g(z) - a_{1,1})}.$$  

It follows that

$$\text{Span}_C \left\{ \partial_{T_1}^i \tilde{\Psi}(T_1, 0, 0, \ldots; z) |_{T_1=0} | i \geq 0 \right\} = \sum_{i=0}^{\infty} C g(z)^i = V_n(z),$$

which completes the proof of the proposition.$\square$

**Corollary 4.5** The tau function $\tau(T)$ in Corollary 4.3 is a nonzero constant multiple of $\tau_0(T; V_n(z))$.

**5 Expansion coefficients of $F^+_i(z)$**

In this section we compute the expansion coefficients of $F^+_i(z)$ which appears in the wave and the adjoint wave functions (4.12), (4.13).
Set

\[-F_i^+(z) = \sum_{j=1}^{\infty} c_{i,j} \frac{z^j}{j}. \tag{5.1}\]

**Proposition 5.1** The following properties hold.

(i) \(c_{i,j} = c_{j,i}\)

(ii) \(c_{i,j} = 0\) if \(i + j \neq 0 \mod. n\).

(iii) For \((i, j) = (ns + p, nr - p)\) with \(r \geq 1, s \geq 0, 1 \leq p \leq n\),

\[c_{i,j} = \alpha^{r+s} \frac{pr}{r+s} \left( \frac{r - \frac{p}{n}}{r} \right) \left( \frac{s + \frac{p}{n}}{s} \right). \tag{5.2}\]

(iv) If \(j = 0 \mod. n\), \(c_{i,j} = 0\).

By the definition (4.11) of \(F_i(z)\)

\[c_{i,j} = -j \sum_{l=1}^{i} b_{l,i} a_{l,j+l}. \tag{5.3}\]

So we need to compute \((b_{i,j}) = (a_{j,j-i})^{-1}\).

**Lemma 5.2** The following properties are valid.

(i) \(b_{i,j} = 0\) if \(i \neq j \mod. n\).

(ii) For \(r, s \geq 0, 1 \leq p \leq n\),

\[b_{nr+p,ns+p} = \alpha^{s-r} \left( \frac{s + \frac{p}{n}}{s - r} \right). \tag{5.4}\]

**Proof** Since \(a_{i,j} = 0\) for \(j \neq 0 \mod. n\), the equation (4.7) can be written as

\[T_{nr+p} = \sum_{s=r}^{\infty} a_{ns+p,n(s-r)} t_{ns+p}, \tag{5.5}\]

with \(r, s \geq 0, 1 \leq p \leq n\). It follows that \(b_{i,j} = 0\) unless \(i = j \mod. n\). Then (4.9) reduces to

\[t_{nr+p} = \sum_{s=0}^{\infty} b_{nr+p,ns+p} T_{ns+p}. \tag{5.6}\]

So, it is sufficient to prove that \(\{b_{i,j}\}\) given by (5.4) satisfy, for any \(r, r' \geq 0\),

\[\sum_{s=r}^{r'} a_{ns+p,n(s-r)} b_{ns+p,nr'+p} = \delta_{r,r'}. \tag{5.7}\]
By the definition (4.5) of $a_{i,j}$

$$a_{ns+p,n(s-r)} = (-\alpha)^{s-r} \left( \frac{s + \frac{p}{n}}{s - r} \right). \quad (5.8)$$

Substituting (5.4) and (5.8) into (5.7) we have

$$\alpha^{r-r} \sum_{s=r}^{r'} (-1)^{s-r} \left( \frac{s + \frac{p}{n}}{s - r} \right) \left( \frac{r' + \frac{p}{n}}{r' - s} \right) = \delta_{r,r'}. \quad (5.9)$$

Let us prove this equation. By computation we have

$$\left( \frac{s + \frac{p}{n}}{s - r} \right) \left( \frac{r' + \frac{p}{n}}{r' - s} \right) = \left( \frac{r' + \frac{p}{n}}{r' - r} \right) \left( \frac{r' - r}{r' - s} \right).$$

Then

LHS of (5.9) = $\alpha^{r-r} \left( \frac{r' + \frac{p}{n}}{r' - r} \right) \sum_{s=r}^{r'} (-1)^{s-r} \left( \frac{r' - r}{r' - s} \right)$

$$= \alpha^{r-r} \left( \frac{r' + \frac{p}{n}}{r' - r} \right) \sum_{s=0}^{r'-r} (-1)^{s} \left( \frac{r' - r}{s} \right)$$

$$= \delta_{r,r'}.$$

$\square$

**Proof of Proposition 5.1** Notice that, by (5.3), if $c_{i,j} \neq 0$ then there exists $l$ such that $l = i, j + l = 0 \mod n$. Then $i = l = -j \mod n$. So $c_{i,j} = 0$ if $-i \neq j \mod n$. This proves (ii).

For $i = ns + p$, $j = nr - p$, $s \geq 0$, $r \geq 1$, $1 \leq p \leq n$, we have, using Lemma 5.2,

$$c_{ns+p,nr-p} = -(nr - p) \sum_{l=1}^{ns+p} b_{l,ns+p} a_{l,nr-p+l}$$

$$= -(nr - p) \sum_{k=0}^{s} b_{nk+p,ns+p} a_{nk+p,n(r+k)}$$

$$= -(nr - p) \alpha^{r+s} \sum_{k=0}^{s} (-1)^{r+k} \left( \frac{s + \frac{p}{n}}{s - k} \right) \left( \frac{k + \frac{p}{n}}{r + k} \right). \quad (5.10)$$

By computation we have

$$(nr - p) \left( \frac{s + \frac{p}{n}}{s - k} \right) \left( \frac{k + \frac{p}{n}}{r + k} \right) = (-1)^{r-1} p \left( \frac{s + \frac{p}{n}}{s} \right) \left( \frac{r - \frac{p}{n}}{r} \right) \left( \frac{r + s}{s} \right) \left( \frac{s - k}{s - k} \right).$$
Then
\[ c_{ns+p,nr-p} = \alpha^{r+s} \left( \frac{s+\frac{p}{n}}{s} \right) \left( \frac{r-\frac{p}{n}}{r} \right) \left( \frac{r+s}{s} \right)^{-1} \sum_{k=0}^{s} (-1)^k \left( \frac{r+s}{s-k} \right). \]
\[ = (-1)^s \alpha^{r+s} \left( \frac{s+\frac{p}{n}}{s} \right) \left( \frac{r-\frac{p}{n}}{r} \right) \left( \frac{r+s}{s} \right)^{-1} \sum_{k=0}^{s} (-1)^k \left( \frac{r+s}{k} \right). \]

(5.11)

**Lemma 5.3** The following formula holds,
\[ \sum_{k=0}^{s} (-1)^k \left( \frac{r+s}{k} \right) = (-1)^s \left( \frac{r+s-1}{s} \right). \]

**Proof** We have
\[ \sum_{k=0}^{s} (-1)^k \left( \frac{r+s}{k} \right) = \sum_{k=0}^{s} (-1)^k \left( \left( \frac{r+s-1}{k} \right) + \left( \frac{r+s-1}{k-1} \right) \right) \]
\[ = \sum_{k=0}^{s-1} (-1)^k \left( \frac{r+s-1}{k} \right) + (-1)^s \left( \frac{r+s-1}{s} \right) \]
\[ + \sum_{k=1}^{s} (-1)^k \left( \frac{r+s-1}{k-1} \right) \]
\[ = (-1)^s \left( \frac{r+s-1}{s} \right). \]

Applying Lemma 5.3 to (5.11) we get (iii) of the proposition.
If \( p = n \) the right-hand side of (5.2) is zero, since
\[ \left( \frac{r-\frac{p}{n}}{r} \right) = 0. \]
So \( c_{i,j} = 0 \) if \( j = 0 \) mod. \( n \). Thus, (iv) of the proposition is proved.
Let us prove (i) of the proposition. It is sufficient to prove \( c_{ns+p,nr-p} = c_{nr-p,ns+p}, \)
\( r \geq 1, s \geq 0, 1 \leq p \leq n - 1 \). Using (5.2) we have
\[ c_{nr-p,ns+p} = c_{n(r-1)+n-p,n(s+1)-(n-p)} \]
\[ = \alpha^{r+s} \left( \frac{(s+1)(n-p)}{r+s} \right) \left( \frac{r-1+\frac{n-p}{n}}{r-1} \right) \left( \frac{s+1+\frac{n-p}{n}}{s+1} \right) \]
\[ = \alpha^{r+s} \left( \frac{(s+1)(n-p)}{r+s} \right) \left( \frac{r-\frac{p}{n}}{r-1} \right) \left( \frac{s+\frac{p}{n}}{s+1} \right). \]
Using
\[
(n - p) \left( \begin{array}{c}
    r - \frac{p}{n} \\
    r - 1
\end{array} \right) = rn \left( \begin{array}{c}
    r - \frac{p}{n} \\
    r
\end{array} \right),
\]
\[
\left( \begin{array}{c}
    s + \frac{p}{n} \\
    s + 1
\end{array} \right) = \frac{p}{n(s + 1)} \left( \begin{array}{c}
    s + \frac{p}{n} \\
    s
\end{array} \right),
\]
we get
\[
c_{nr-p,ns+p} = \alpha^{r+s} \frac{pr}{r+s} \left( \begin{array}{c}
    r - \frac{p}{n} \\
    r
\end{array} \right) \left( \begin{array}{c}
    s + \frac{p}{n} \\
    s
\end{array} \right).
\]

\[
= c_{ns+p,nr-p}.
\]

\[\square\]

6 Proof of Theorem 3.1

We shall find the function \( \tau(T) \) which satisfies (4.17) and (4.18) in the form
\[
\tau(T) = e^{\frac{1}{2} (q(T|T) + L(T))},
\]  
(6.1)
\[
q(T|S) = \sum_{i,j=1}^{\infty} q_{i,j} T_i S_j, \quad L(T) = \sum_{i=1}^{\infty} L_i T_i,
\]  
(6.2)

where \( q_{i,j} = q_{j,i} \) for any \( i, j, S = (S_1, S_2, \ldots) \). The equation (4.18) is equivalent to
\[
\frac{\tau(T + [z])}{\tau(T)} = e^{q(T|[z]) + \frac{1}{2} (q([z][z]) + L([z]))}
\]
which, by (6.1), (6.2), is rewritten as
\[
q(T|[z]) + \frac{1}{2} (q([z][z]) + L([z])) = \sum_{i,j=1}^{\infty} c_{i,j} T_i \frac{z^j}{j}.
\]  
(6.3)

This equation is satisfied if we set
\[
q(T|[z]) = \sum_{i,j=1}^{\infty} c_{i,j} T_i \frac{z^j}{j},
\]  
(6.4)
\[
q([z][z]) + L([z]) = 0.
\]  
(6.5)
These equations are solved if we take

\[ q_{i,j} = c_{i,j}, \quad (6.6) \]

\[ L_k = - \sum_{i+j=k} \frac{k}{ij} c_{i,j}. \quad (6.7) \]

Now Theorem 3.1 follows from the following lemma which is used in the proof of Theorem 4.2 in [4].

**Lemma 6.1** [4] Let \( \tau_i(t), i = 1, 2, \) be two functions of \( t = (t_1, t_2, \ldots) \) such that

\[ \frac{\tau_1(t + [z])}{\tau_1(t)} = \frac{\tau_2(t + [z])}{\tau_2(t)} \]

Then \( \tau_2(t) = c\tau_1(t) \) for some constant \( c. \)

**Proof** Let \( f(t) = \tau_2(t)/\tau_1(t). \) Then the condition is

\[ f(t + [z]) = f(t). \quad (6.8) \]

Using

\[ f(t + [z]) = e^{\sum_{i=1}^{\infty} \frac{z^i}{i!} \partial_i f(t)} = \sum_{i=0}^{\infty} z^i p_i(\tilde{\partial}) f(t), \quad (6.9) \]

where \( \partial_i = \partial/\partial t_i \) and \( \tilde{\partial} = (\partial_1, \partial_2/2, \partial_3/3, \ldots), \) we have

\[ \sum_{i=1}^{\infty} z^i p_i(\tilde{\partial}) f(t) = 0. \quad (6.10) \]

Therefore,

\[ p_i(\tilde{\partial}) f(t) = 0, \quad i \geq 1. \quad (6.11) \]

Since

\[ p_i(\tilde{\partial}) = \frac{\partial_i}{i} + \text{(terms containing only } \partial_j, j < i), \]

we get \( \partial_i f(t) = 0 \) for any \( i \geq 1. \)

By Corollary 4.5 both \( \tau(T) \) constructed above and \( \tau_0(T; V_n(z)) \) satisfy the equation (4.18). Therefore, \( \tau(T) = c\tau_0(T; V_n(z)) \) for some constant \( c \) by Lemma 6.1. By setting \( T_j = 0 \) for all \( j \) we have \( c = 1. \) Then we have (3.3), (3.4) by (6.6), (6.7), Proposition 5.1. Thus, Theorem 3.1 is proved.
7 Proof of Theorem 3.2

By the definition of $V_n(z)$ the following set of functions give a basis of $V_n(z)$,

$$
h_{ni}(z) := z^{-ni}, \quad i \geq 0,
$$

$$
h_i(z) := y^i = z^{-i(1 - \alpha z^{-n})}, \quad i \geq 1, \quad i \neq 0 \text{ mod. } n.
$$

Then it is obvious that there exists the unique basis $\tilde{h}_i, i \geq 1$ with the property

$$
\tilde{h}_i = z^{-i} + O(z)
$$

where $O(z)$ denotes an element in $z\mathbb{C}[[z]]$.

It is given explicitly as

**Proposition 7.1** The basis $\{\tilde{h}_i(z)\}$ is given by

$$
\tilde{h}_{nr}(z) = z^{-nr},
$$

$$
\tilde{h}_{nr+p}(z) = z^{-nr-p} + \sum_{k=1}^{\infty} (-1)^k \alpha^{k+r} \binom{k+r-1}{r} \binom{r+p}{k+r} z^{nk-p}, \quad (7.1)
$$

where $r \geq 0, 1 \leq p \leq n - 1$.

**Proof** It is sufficient to prove that $\tilde{h}_{nr+p}(z)$ is a linear combination of $h_{nr'+p}(z), r' \geq 0$ for each $p$. We prove it by induction on $r$.

By (4.4), (4.5)

$$
h_{nr+p}(z) = z^{-nr-p} + \sum_{k=1}^{\infty} a_{nr+p,nk} z^{n(k-r)-p}.
$$

For $r = 0$ we have

$$
h_p(z) = z^{-p} + \sum_{k=1}^{\infty} (-\alpha)^k \binom{p}{k} z^{nk-p} = \tilde{h}_p(z).
$$

Suppose that the assertion is valid until $r$. Dividing $h_{n(r+1)+p}(z)$ to negative and nonnegative power parts in $z$ we have

$$
h_{n(r+1)+p}(z) = z^{-n(r+1)-p} + \sum_{l=1}^{r+1} a_{n(r+1)+p,nl} z^{-n(r+1-l)-p}
$$

$$
+ \sum_{k=1}^{\infty} a_{n(r+1)+p,n(k+r+1)} z^{nk-p}.
$$

(7.2)
We erase the middle term by subtracting linear combination of $\tilde{h}_{nk+p}(z)$, $0 \leq k \leq r$. Set

$$h'_{n(r+1)+p}(z) = h_{n(r+1)+p}(z) - \sum_{l=1}^{r+1} a_{n(r+1)+p,n_l} \tilde{h}_{n(r+1-l)+p}(z).$$

We shall show $h'_{n(r+1)+p}(z) = \tilde{h}_{n(r+1)+p}(z)$.

Using the assumption of induction we have

$$h'_{n(r+1)+p}(z) = z^{-n(r+1)-p} + \sum_{k=1}^{\infty} a_{n(r+1)+p,n(k+r+1)} - \sum_{l=1}^{r+1} (-1)^k a_{n(r+1)+p,n_l} \alpha^{k+r+1-l}$$

\[
\times \left( \frac{k-r-l}{r+1-l} \left( k + r + 1 - l \right) \right) z^{nk-p}. \tag{7.3}
\]

By (4.5) the coefficient of $z^{nk-p}$ of the right-hand side is

$$(-\alpha)^{k+r+1} \left( \binom{r+1 + \frac{p}{n}}{r+1+k} \right)$$

\[
- \sum_{l=1}^{r+1} (-1)^l r^{l+r+1} \left( \binom{r+1 + \frac{p}{n}}{l} \left( r + 1 - l + \frac{p}{n} \right) \left( k + r + 1 - l \right) \right) \left( k + r - l \right). \tag{7.4}
\]

Notice the equation

$$\left( \binom{r+1 + \frac{p}{n}}{l} \right) \left( \frac{r+1 - l + \frac{p}{n}}{k + r + 1 - l} \right) = \left( \binom{r+1 + \frac{p}{n}}{k + r + 1} \right) \left( \frac{k + r + 1}{l} \right).$$

Then the right-hand side of (7.4)

$$= (-\alpha)^{k+r+1} \left( r + 1 + \frac{p}{n} \right) \left( k + r + 1 \right) \left( k - 1 + l \right) \left( k - 1 + l \right) \left( r + 1 + \frac{p}{n} \right) \left( k + r + 1 - l \right).$$

\[\square\]

Lemma 7.2 The following equation holds:

$$1 - \sum_{l=0}^{r} (-1)^l \left( \binom{k + r + 1}{r + 1 - l} \right) \left( \frac{k - 1 + l}{l} \right) = (-1)^{r+1} \left( \frac{k + r}{r + 1} \right).$$
Proof It is sufficient to prove the following identity for polynomials in \(x\):

\[
1 - \sum_{i=0}^{r} (-1)^i \left( \frac{x + r + 1}{r + 1 - i} \right) \left( \frac{x - 1 + l}{l} \right) = (-1)^{r+1} \left( \frac{x + r}{r + 1} \right).
\]

It is easily proved by examining the zeros and the coefficients of \(x^{r+1}\) of both sides. We leave the details to the reader. \(\square\)

By Lemma 7.2 we finally have

\[
h_{n(r+1)+p}(z) = z^{-n(r+1)-p} + \sum_{k=1}^{\infty} (-1)^k \alpha^{k+r+1} \left( \frac{r + 1 + \frac{p}{n}}{r + 1 + k} \right) z^{nk-p}
\]

which is equal to \(\tilde{h}_{n(r+1)+p}(z)\). This completes the proof of Proposition 7.1. \(\square\)

By Proposition 7.1

\[
\xi = [\ldots, \hat{h}_1(z), \hat{h}_0(z)]
\]

is a frame of \(V_n(z)\) of the form (2.6). To determine the Schur function expansion of \(\tau_0(t; V_n(z))\) we have to compute the Plücker coordinates of \(\xi\). The Plücker coordinates corresponding to hook diagrams is easily computed as

**Lemma 7.3** Let \(\xi' = [\ldots, \xi_{-2}', \xi_{-1}', \xi_0']\) be a frame of a point of UGM of the form (2.6), that is,

\[
\xi'_j = z^j + \sum_{i=1}^{\infty} x_{i,j} z^i, \quad j \leq 0.
\]

Then, for \(i, j \geq 0\),

\[
\xi'_{(i,j)} = (-1)^j x_{i+1,-j}.
\]

By this lemma and Proposition 7.1 we get

**Corollary 7.4** (i) \(\xi_{(i,j)} = 0\) if \(j = 0 \mod n\) or \(i + j + 1 \neq 0 \mod n\).

(ii) For \(r \geq 1, s \geq 0, 1 \leq p, q \leq n - 1\),

\[
\xi_{(nr-p-1|ns+q)} = \delta_{p,q} (-1)^{ns+q+r} \left( s + 1 \right) \left( r + s \right) \alpha^{s+r}.
\]

Finally, let us recall the Giambelli formula for Plücker coordinates.

**Theorem 7.5** [6,19] Let \(\xi'\) be the frame of a point of UGM which is the same form as in Lemma 7.3. Then for \(l \geq 1, a_1 > \cdots > a_l \geq 0, b_1 > \cdots > b_l \geq 0\),

\[
\xi'_{(a_1, \ldots, a_l|b_1, \ldots, b_l)} = \det \left( \xi'_{(a_i|b_j)} \right)_{1 \leq i, j \leq l}.
\]
Now Theorem 3.2 follows from Theorem 7.5 and Corollary 7.4.

8 Generating function of $q_{i,j}$

In this section we calculate the generating function of $\{q_{i,j}\}$ which is used to compute the action of vertex operators on $\tau_0(t; V_n(z))$.

Let
\[
G(z) = (1 - \alpha z^n)^{\frac{1}{n}}, \quad H(z_1, z_2) = z_2 G(z_1) - z_1 G(z_2). \tag{8.1}
\]

Since $H(z_1, z_1) = 0$, $H(z_1, z_2)/(z_2 - z_1)$ is holomorphic near $(0, 0)$. It has the expansion of the form
\[
\frac{H(z_1, z_2)}{z_2 - z_1} = 1 + \text{h.o.t},
\]
where h.o.t means terms containing $z_1^i z_2^j$ with $i + j \geq 1$. Thus, $\log(H(z_1, z_2)/(z_2 - z_1))$ can be expanded to a power series in $z_1, z_2$.

Proposition 8.1 The following expansion holds,
\[
\partial_{z_1} \partial_{z_2} \log \left( \frac{H(z_1, z_2)}{z_2 - z_1} \right) = \sum_{i,j=1}^{\infty} q_{i,j} z_1^{i-1} z_2^{j-1}, \quad \partial_{z_i} = \partial / \partial z_i. \tag{8.2}
\]

Remark 8.2 For $(p_1, p_2) \in C_n \times C_n$, $p_i = (x_i, y_i)$, the bilinear differential
\[
dz_1 dz_2 \log H(z_1, z_2) = dp_1 dp_2 \log (y_1 - y_2) = dp_2 \frac{\sum_{j=0}^{n-1} y_1^j y_2^{n-j}}{(x_1 - x_2) y_1^{n-1}} dx_1
\]
is the genus zero analogue of that of an $(n, s)$ curve with positive genus which is used in the construction of the multi-variate sigma function $[2,3,15]$.

Proof of Proposition 8.1 We have
\[
\partial_{z_1} \partial_{z_2} \log \left( \frac{H(z_1, z_2)}{z_2 - z_1} \right) = \partial_{z_1} \partial_{z_2} \log \left( 1 - \frac{z_1 G(z_2)}{z_2 G(z_1)} \right) - \frac{1}{(z_2 - z_1)^2}. \tag{8.3}
\]

Let us expand the right-hand side near $(z_1, z_2) = (0, 0)$ in $C^2$ assuming
\[
\left| \frac{z_1}{z_2} \right| < 1, \quad \left| \frac{z_1 G(z_2)}{z_2 G(z_1)} \right| < 1, \quad z_1 z_2 \neq 0.
\]
For example,

\[
\frac{1}{(z_2 - z_1)^2} = \sum_{m=1}^{\infty} m^{m-1} z_2^{-m-1}.
\]

Since the left-hand side of (8.3) is holomorphic at \((0, 0)\), only nonnegative powers of \(z_1, z_2\) should remain in the expansion of the right-hand side under the assumption above. Define the symbol \([\quad]_+\) by

\[
\left[ \sum_{i, j=0}^{\infty} e_{i, j} z_1^i z_2^j \right]_+ = \sum_{i, j=0}^{\infty} e_{i, j} z_1^i z_2^j.
\]

Then

\[
\partial_{z_1} \partial_{z_2} \log \left( \frac{H(z_1, z_2)}{z_2 - z_1} \right) = \left[ \partial_{z_1} \partial_{z_2} \log \left( 1 - \frac{z_1 G(z_2)}{z_2 G(z_1)} \right) \right]_+.
\]

Write

\[
\left[ \partial_{z_1} \partial_{z_2} \log \left( 1 - \frac{z_1 G(z_2)}{z_2 G(z_1)} \right) \right]_+ = \sum_{i, j=1}^{\infty} \tilde{q}_{i, j} z_1^{i-1} z_2^{j-1}.
\]

We shall show that \(\tilde{q}_{i, j} = q_{i, j}\).

Expand the logarithmic function as

\[
\log \left( 1 - \frac{z_1 G(z_2)}{z_2 G(z_1)} \right) = -\sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{G(z_2)}{G(z_1)} \right)^m \left( \frac{z_1}{z_2} \right)^m.
\]

We further expand

\[
\left( \frac{G(z_2)}{G(z_1)} \right)^m = (1 - \alpha z_1^n)^{-m/n} (1 - \alpha z_2^n)^m = \sum_{k, l=0}^{\infty} (-\alpha)^{k+l} \binom{-m/n}{k} \binom{m/n}{l} z_1^{nk} z_2^{nl}.
\]

Substituting this expression into (8.6) and writing \(m = ni + p, i \geq 0, 1 \leq p \leq n\), we get

RHS of (8.6)

\[
= -\sum_{p=1}^{n} \sum_{i=0}^{\infty} \frac{1}{ni + p} \sum_{k, l=0}^{\infty} (-\alpha)^{k+l} \binom{ni + p}{k} \binom{p}{l} z_1^{n(k+i)+p} z_2^{n(-i) - p}
\]

\[
= -\sum_{p=1}^{n} \sum_{i=0}^{\infty} \frac{1}{ni + p} \sum_{s \geq i, r \geq -i} (-\alpha)^{r+s} \binom{ni + p}{s} \binom{p}{r} z_1^{ns+p} z_2^{nr-p}.
\]
Then
\[
\left[ \frac{\partial z_1}{\partial z_2} \log \left( 1 - \frac{z_1}{z_2} \frac{G(z_2)}{G(z_1)} \right) \right]_+ = \sum_{p=1}^{n} \sum_{r \geq 1, s \geq 0} \tilde{q}_{ns+p,nr-p} z_1^{ns+p-1} z_2^{nr-p-1},
\]
where
\[
\tilde{q}_{ns+p,nr-p} = (-1)^{r+s+1} \alpha^{r+s} \sum_{i=0}^{s} \frac{(ns + p)(nr - p)}{ni + p} \left( -i - \frac{p}{n} \right) \frac{(s + \frac{p}{n})}{s - i} \left( r + i \right). 
\]

By this expression we see that \( \tilde{q}_{i,j} = 0 \) if \( i + j \neq 0 \mod n \).

A computation shows
\[
(ns + p)(nr - p) \left( -i - \frac{p}{n} \right) \frac{(s + \frac{p}{n})}{s - i} \left( r + i \right) = (-1)^{r+s+i-1} p(ni + p) \frac{r!s!}{(r + i)!(s - i)!} \left( s + \frac{p}{n} \right)(r - \frac{p}{n}).
\]

Then
\[
\tilde{q}_{ns+p,nr-p} = pr!s!\alpha^{r+s} \left( s + \frac{p}{n} \right) \left( r - \frac{p}{n} \right) \sum_{i=0}^{s} \frac{(-1)^i}{(r + i)!(s - i)!}. \tag{8.8}
\]

Using Lemma 5.3 we easily have
\[
\sum_{i=0}^{s} \frac{(-1)^i}{(r + i)!(s - i)!} = \frac{1}{(r + s)(r - 1)!s!}.
\]

Substituting it to (8.8) we get
\[
\tilde{q}_{ns+p,nr-p} = \alpha^{r+s} \frac{pr}{r+s} \left( s + \frac{p}{n} \right) \left( r - \frac{p}{n} \right) = q_{ns+p,nr-p}.
\]

Thus, the proposition is proved. \( \square \)

**Corollary 8.3** (i)
\[
eq (1 - \alpha z^n)^{\frac{1-n}{n}}. \tag{8.10}
\]

(8.9)
Proof (i) Integrating (8.2) twice and using
\[ \frac{H(z_1, 0)}{-z_1} = \frac{H(0, z_2)}{z_2} = 1, \]
we get the result.
(ii) is obtained by taking the limit \( z_2 \to z_1 \) in (i).
\[ \square \]

9 Soliton solution on the nonzero constant background

Applying vertex operators to \( \tau_0(t; V_2(z)) \) it is possible to obtain soliton solutions on nonzero backgrounds. In this section we compute them explicitly.

Let us consider the vertex operator [4],
\[ X(p, q) = e^{\sum_{m=1}^{\infty} t_m (p^m - q^m)} e^{\sum_{m=1}^{\infty} (-p^{-m} + q^{-m}) \frac{\partial m}{m}}, \partial_m = \frac{\partial}{\partial t_m}. \] (9.1)

Then
\[ X(p_1, q_1)X(p_2, q_2) = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)} : X(p_1, q_1)X(p_2, q_2) :. \] (9.2)

where the normal ordering symbol : \( : \) signifies to move differential operators \( \partial_n \) to the right. It then follows the following properties,
\[ X(p_1, q_1)X(p_2, q_2) = X(p_2, q_2)X(p_1, q_1) \quad \text{for any } p_i, q_j, \] (9.3)
\[ X(p_1, q_1)X(p_2, q_2) = 0 \quad \text{if } p_1 = p_2 \text{ or } q_1 = q_2. \] (9.4)

Let \( M, N \) be two positive integers, \( p_i, q_j, a_{i,j}, 1 \leq i \leq M, 1 \leq j \leq N \) complex parameters such that \( \{p_i, q_j\} \) are all nonzero and mutually distinct. Consider the operator
\[ g = e^{\sum_{i=1}^{M} \sum_{j=1}^{N} a_{i,j} X(p_i, q_j)}. \] (9.5)

It is well known [4] that, for a solution \( \tau(t) \) of the KP-hierarchy, \( g \tau(t) \) is a solution of the KP-hierarchy if it is well defined as a formal power series in \( t \). In particular, soliton solutions are obtained by taking \( \tau(t) = 1 \) [4]. Here, we compute \( g \tau_0(t; V_n(z)) \).

Let us set \( [M] = \{1, \ldots, M\} \). For a nonnegative integer \( l \) we denote \( \binom{[M]}{l} \) the set of all \( l \)-element subsets of \( [M] \). If \( I = \{i_1, \ldots, i_l\} \in \binom{[M]}{l} \), we assume \( i_1 < \cdots < i_l \) unless otherwise stated.

Set, in general,
\[ \Delta(x_1, \ldots, x_N) = \prod_{i<j} (x_i - x_j), \quad \eta(t; z) = \sum_{i=1}^{\infty} t_i z^i. \]
For \( I \in \binom{[M]}{I} \) we define
\[
\eta_I(p) = \sum_{i \in I} \eta(t; p_i),
\]
and similarly for \( \eta_J(q), J \in \binom{[N]}{I} \). We denote by \( J^c \in \binom{[N]}{N-I} \) the complement of \( J \) in \([N]\) for \( J \in \binom{[N]}{I} \).

Set
\[
\tilde{a}_{i,j} = a_{i,j} \prod_{m=1, m \neq j}^{N} \frac{q_j - q_m}{p_i - q_m}.
\] (9.6)

Define the \( N \times (N + M) \) matrix \( B = (b_{i,j}) \) by
\[
B = \begin{pmatrix}
1 & \ldots & 0 & \tilde{a}_{1,1} & \ldots & \tilde{a}_{M,1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & \tilde{a}_{N,1} & \ldots & \tilde{a}_{M,N}
\end{pmatrix}.
\]

For \( K = (k_1, \ldots, k_N) \in \binom{[M+N]}{N} \) \( B_K \) denotes the minor determinant corresponding to the columns specified by \( K \):
\[
B_K = \det(b_{i,k_j})_{1 \leq i, j \leq N}.
\]

Set
\[
(k_1, \ldots, k_{M+N}) = (q_1, \ldots, q_N, p_1, \ldots, p_M).
\]

For \( K \in \binom{[M+N]}{N} \) define
\[
\eta_K(\kappa) = \sum_{k \in K} \eta(t; \kappa_k), \quad \eta_K(\kappa^{-1}) = \sum_{k \in K} \eta(t; \kappa_k^{-1}).
\]

\[
\Delta_K(\kappa) = \prod_{i < j, i,j \in K} (\kappa_i - \kappa_j).
\]

To make the result simple we introduce the function
\[
\tau^{(n)}(g|t) = \Delta(q_1, \ldots, q_N)e^{\sum_{j=1}^{N} \eta(t; q_j) g} \tau_0(t - \sum_{j=1}^{N} [q_j^{-1}]; V_n(z)),
\] (9.7)
which is obviously a solution of the KP-hierarchy.

Then
Theorem 9.1 Let \( p_i, q_j, a_{i,j} \) are complex parameters such that \{\( p_i, q_j \)\} are all nonzero and mutually distinct. Then

\[
\tau^{(n)}(g|t) = \tilde{\tau}^{(n)}(g|t)\tau_0(t; V_n(z)).
\] (9.8)

Here,

\[
\tilde{\tau}^{(n)}(g|t) = \sum_{I \in \left(\begin{array}{c} M+N \\hline N \end{array}\right)} B_I C_I(\kappa)e^{\eta_I(\kappa)-\sum_{i \in I} q(t[k_i^{-1}])},
\]

\[
C_I(\kappa) = \prod_{i<j, i,j \in I} \left( \kappa_i \kappa_j H(\kappa_j^{-1}, \kappa_i^{-1}) \right) \prod_{i \in I} (1 - \alpha \kappa_i^{-n})^{\frac{1-n}{n}}.
\] (9.9)

Proof Using (9.2)–(9.4) we have

\[
g = \prod_{j=1}^{N} \left( 1 + \sum_{i=1}^{M} a_{i,j} X(p_i, q_j) \right)
\]

\[
= \sum_{l=0}^{N} \sum_{1 \leq j_1 < \cdots < j_l \leq N} \sum_{i_1, \ldots, i_l=1}^{M} a_{i_1,j_1} \cdots a_{i_l,j_l} \prod_{r<s} (p_{ir} - p_{is})(q_{jr} - q_{js})
\]

\[\times : X(p_{i_1}, q_{j_1}) \cdots X(p_{i_l}, q_{j_l}) :.
\]

Rewriting \( a_{i,j} \) in terms of \( \tilde{a}_{i,j} \) using (9.6) and noting that the normal ordering parts are symmetric in \( \{p_{i_r}\} \) and \( \{q_{j_s}\} \), respectively, we get

\[
g = \sum_{l=0}^{N} \sum_{J = \{j_1, \ldots, j_l\} \subset [N]} \sum_{I \subset [M]} B_{(j^c, I)} \begin{pmatrix} \Delta(q_{j_1'}, \ldots, q_{j_{N-l}'}, p_i, \ldots, p_i) \\ \Delta(q_1, \ldots, q_N) \end{pmatrix}
\]

\[\times : X(p_{i_1}, q_{j_1'}) \cdots X(p_{i_l}, q_{j_l'}) :,
\]

where \( J^c = \{j_1', \ldots, j'_{N-l}\} \subset [N] \) and \( (J^c, I) = (j_1', \ldots, j'_{N-l}, i_1, \ldots, i_l) \). On the other hand

\[
e^{\sum_{j=1}^{N} \eta_j(t, q_j) - q_{j-1}^{-1}} \times X(p_{i_1}, q_{j_1'}) \cdots X(p_{i_l}, q_{j_l'}) : \tau_0(t - \sum_{j=1}^{N} [q_{j-1}^{-1}]; V_n(z))
\]

\[= e^{\eta_I(p) + \eta_{J^c}(q)} \tau_0(t - \sum_{i \in I} [p_i^{-1}] - \sum_{j \in J^c} [q_j^{-1}]; V_n(z)).
\]
Then
\[
\tilde{\tau}^{(n)}(g|t) = \sum_{I=0}^{N} \sum_{J=(j_1, \ldots, j_l) \in \binom{[N]}{l}} \sum_{I=(i_1, \ldots, i_l) \in \binom{[M]}{l}} B_{J^c, I} \Delta_q \eta_J \eta_I \tau_0(t - \sum_{i \in I} \gamma_{i-1}^I - \sum_{j \in J} \gamma_{j-1}^J; V_n(z)),
\]
where \( q_{J^c} = (q_{j_1}, \ldots, q_{j_{N-l}}) \), \( p_I = (p_{i_1}, \ldots, p_{i_l}) \). By considering the summation in \( l, I, J \) as that over the indices \( (J^c, I) \) we have
\[
\tilde{\tau}^{(n)}(g|t) = \sum_{K \in \binom{[M+N]}{N}} B_K \Delta_K(k) \eta_K(k) \tau_0(t - \sum_{k \in K} \gamma_{k-1}^K; V_n(z)),
\]
Now the theorem follows from

**Lemma 9.2** For parameters \( z_i, 1 \leq i \leq N \), the following equation is valid:
\[
\tau_0(t - \sum_{j=1}^N \gamma_{j}; V_n(z)) = e^{-\sum_{j=1}^N q(t[j])} \prod_{j=1}^N (1 - \alpha z_j) \prod_{i < j}^N H(z_i, z_j) \tau_0(t; V_n(z)).
\]

**Proof** Using the bilinearity of \( q(t|S) \) and the relation (6.5) we have
\[
\tau_0(t - \sum_{j=1}^N \gamma_{j}; V_n(z)) = e^{-\sum_{j=1}^N q(t[j]) + \sum_{i < j} q(t[j])} \tau_0(t; V_n(z))
\]
Then the assertion follows from Corollary 8.3.

Let
\[
u(t) = 2 \frac{\partial^2}{\partial x^2} \log \tau^{(n)}(g|t) = 2q_{1,1} + 2 \frac{\partial^2}{\partial x^2} \log \tilde{\tau}^{(n)}(g|t)
\]
be a solution of (2.3). By (3.3) in Theorem 3.1 we have
\[
q_{1,1} = \begin{cases} \frac{n}{2} & n = 2 \\ 0 & n > 2. \end{cases}
\]
Thus, we have

**Corollary 9.3** Let \( \tilde{\tau}^{(2)}(g|t) \) be given by (9.9) with \( n = 2 \). Then
\[
u(t) = \alpha + 2 \frac{\partial^2}{\partial x^2} \log \tilde{\tau}^{(2)}(g|t),
\]
\[\text{ Springer}\]
is a solution of the KP equation (2.3).

**Remark 9.4** If $\alpha = 0$ then $u(t)$ given by (9.12) becomes a well-known soliton solution [4,10,11,14,20]. If parameters are chosen such that $\tilde\tau^2(g|t)$ is positive for real variables $t$, the second term of the right-hand side of (9.12) decays exponentially as $(t_1, t_2)$ goes to infinity except finite number of directions. Thus, $u(t)$ can be considered as a soliton solution on a nonzero constant background if $\alpha \neq 0$.

**Remark 9.5** For real $\kappa_j$, $1 \leq j \leq M + N$, $\alpha$, $t$, the function $\tilde\tau^{(n)}(g|t)$ is nonnegative up to overall constant if, for example, $\kappa_1 > \cdots > \kappa_{M+N} > \alpha^{1/n} \geq 0$ and $B_I \geq 0$ for all $I \in \{M+N\}$.

Here are some examples of $\tilde\tau^2(g|t)$.

**Example 9.6** In the case $N = M = 1$ and $B = (1, b)$ we have

\[
\tilde\tau^2(g|t) = (1 - \alpha \kappa_1^{-2} - \frac{1}{2} \eta(t; \kappa_1) - q(t|\kappa_1^{-1})) = b(1 - \alpha \kappa_2^{-2} - \frac{1}{2} \eta(t; \kappa_2) - q(t|\kappa_2^{-1})).
\]  
\[\text{(9.13)}\]

**Example 9.7** In the case $N = 1$, $M \geq 1$, $B = (b_1, \ldots, b_{M+1})$, $b_1 = 1$, we have

\[
\tilde\tau^2(g|t) = \sum_{j=1}^{M+1} b_j (1 - \alpha \kappa_j^{-2} - \frac{1}{2} \eta(t; \kappa_j) - q(t|\kappa_j^{-1})).
\]  
\[\text{(9.14)}\]

**Example 9.8** In the case $N = M = 2$ and

\[
B = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix},
\]

we have

\[
\tilde\tau^2(g|t) = C_{12} (\kappa) e^{\eta(t; \kappa_1) + \eta(t; \kappa_2) - q(t|\kappa_1^{-1}) - q(t|\kappa_2^{-1})}
\]

\[
+ c C_{13} (\kappa) e^{\eta(t; \kappa_1) + \eta(t; \kappa_3) - q(t|\kappa_1^{-1}) - q(t|\kappa_3^{-1})}
\]

\[
+ d C_{14} (\kappa) e^{\eta(t; \kappa_1) + \eta(t; \kappa_4) - q(t|\kappa_1^{-1}) - q(t|\kappa_4^{-1})}
\]

\[
- a C_{23} (\kappa) e^{\eta(t; \kappa_2) + \eta(t; \kappa_3) - q(t|\kappa_2^{-1}) - q(t|\kappa_3^{-1})}
\]

\[
- b C_{24} (\kappa) e^{\eta(t; \kappa_2) + \eta(t; \kappa_4) - q(t|\kappa_2^{-1}) - q(t|\kappa_4^{-1})}
\]

\[
+ (ad - bc) C_{34} (\kappa) e^{\eta(t; \kappa_3) + \eta(t; \kappa_4) - q(t|\kappa_3^{-1}) - q(t|\kappa_4^{-1})},
\]

where

\[
C_{ij} (\kappa) = \kappa_i \kappa_j H(\kappa_i^{-1}, \kappa_j^{-1})(1 - \alpha \kappa_i^{-2})^{-\frac{1}{2}} (1 - \alpha \kappa_j^{-2})^{-\frac{1}{2}}.
\]
To study the solution as a function of \((x, y, t) := (t_1, t_2, t_3)\) the following lemma, which is calculated using Corollary 8.3, is useful.

**Lemma 9.9** For \(n = 2\) the following equations hold.

(i) \[\sum_{j=1}^{\infty} q_1,j \frac{z^j}{j} = z^{-1} \left( 1 - (1 - \alpha z^2)\frac{1}{2} \right).\]

(ii) \[\sum_{j=1}^{\infty} q_2,j \frac{z^j}{j} = 0.\]

(iii) \[\sum_{j=1}^{\infty} q_3,j \frac{z^j}{j} = z^{-3} \left( 1 - (1 - \alpha z^2)\frac{1}{2} \left(1 + \frac{1}{2} \alpha z^4\right)\right).\]

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