Gibbs measures and free energies
of Ising–Vannimenus model
on the Cayley tree

Farrukh Mukhamedov, Hasan Akın and Otabek Khakimov

1 Department of Mathematical Sciences, College of Science, The United Arab Emirates University, PO Box, 15551, Al Ain, Abu Dhabi, United Arab Emirates
2 Ceyhun Atuf Caddesi, 1164. Sokak, 9/4 Çankaya, TR06460, Ankara, Turkey
3 Otabek Khakimov, Institute of Mathematics, 29, Do’rmon Yo’li str., 100125, Tashkent, Uzbekistan
E-mail: far75m@gmail.com, farrukh.m@uaeu.ac.ae, akinhasan25@gmail.com and hakimovo@mail.ru

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Abstract. In this paper, we consider the Ising–Vannimenus model on a Cayley tree for order two with competing nearest-neighbor and prolonged next-nearest neighbor interactions. We stress that the mentioned model was investigated only numerically, without rigorous (mathematical) proofs. One of the main points of this paper is to propose a measure-theoretical approach for the considered model. We find certain conditions for the existence of Gibbs measures corresponding to the model, which allowed to establish the existence of the phase transition. Moreover, the free energies and entropies, associated with translation invariant Gibbs measures, are calculated.

Keywords: exact results, renormalisation group, phase diagrams
1. Introduction

It is known [13] that the Gibbs measures are one of the central objects of equilibrium statistical mechanics. Also, one of the main problems of statistical physics is to describe all Gibbs measures corresponding to the given Hamiltonian [7]. It is well-known that such measures form a nonempty convex compact subset in the set of all probability measures. A simplest model in statistical mechanics is the Ising model which has wide theoretical interest and practical applications. There are several papers (see [9, 19]) which are devoted to the description of this set for the Ising model on a Cayley tree. However, a complete result about all Gibbs measures even for the Ising model is lacking. Later on in [21] such an Ising model was considered with next-nearest neighbor interactions on the Cayley tree for which its phase diagram was described. The considered model, in what follows, we will call as Ising–Vannimenus model. It turns out that this model has a rich enough structure to illustrate almost every conceivable nuances of statistical mechanics. Furthermore, intensive investigations were devoted to generalizations of the Ising–Vannimenus model (see [2, 5, 6, 8, 14–17, 20] for example), but most results of the existing works are numerically obtained. Therefore, in [10] it has been proposed a measure-theoretical approach (see [13]) to the mentioned model on the Cayley tree. In fact, the proposed approach, the authors used a dimer analogue of the model. Therefore, that paper cannot be considered as rigorous approach to the Ising–Vannimenus model. Hence, one of the main aims of the present paper is to develop a measure-theoretic approach (i.e. Gibbs measure formalism) to rigorously establish the phase transition for the Ising–Vannimenus model on the Cayley tree. In [1] some
attempts have been made to study the phase transition for the mentioned model within theoretical approach, but a different definition was used for the local Gibbs measures over the Cayley tree of order two.

On the other hand, recently, Gandolfo et al [12] have obtained some explicit formulae of the free energies (and entropies) according to boundary conditions (b.c.) for the Ising model on the Cayley tree. In [11], a wide class of new extreme Gibbs states for the Ising model was constructed. All these works are based on the measure-theoretic approach to study Gibbs measures for the Ising model on the Cayley trees. Therefore, our second aim is to present, as an illustration of the obtained method, to find free energies of the Ising–Vannimenus model within the developed technique.

Until now, many researchers have investigated Gibbs measures corresponding to models with nearest-neighbor interactions on Cayley trees. The aim of this paper is to propose rigorously the investigation of Gibbs measures for the Ising-Vannimenus model [21] with ternary prolonged and nearest neighbor interactions on Cayley tree.

The paper is organized as follows. In section 2, we provide necessary notations and define the Ising–Vannimenus model on the Cayley tree. In section 3, using a rigorous measure-theoretical approach, we find certain conditions for the existence of Gibbs measures corresponding to the model on the Cayley tree of an arbitrary order. To describe the Gibbs measure, we obtain a system of functional equations (which is extremely difficult to solve). Nevertheless, in section 4, we are able to succeed in obtaining explicit solutions by making reasonable assumptions, for the existence of translational invariant Gibbs measures. Furthermore, in section 5, we establish the existence of the phase transition. We note that the periodic solutions of the obtained functional equations will correspond to the periodic phases of the model. We will show that to find general solutions even in the case of translation-invariant solutions of the system is not an easy job. The case $J_p < 0$ is not easy and will require a lot of effort to explicitly find periodic solutions. Note that even for the usual Ising model (anti ferromagnetic case) up to now not all periodic solutions have been found (see [19] for the review). In section 6, as an illustration of the developed technique, one finds the free energy associated with translation-invariant Gibbs measures, which allows to calculate the corresponding entropy. Finally, section 7 contains concluding remarks and discussion of the consequences of the results.

2. Preliminaries

2.1. Ising–Vannimenus model with competing interactions on Cayley tree

A Cayley tree $\Gamma^k$ of order $k \geq 1$ is an infinite tree, i.e. a graph without cycles with exactly $k + 1$ edges issuing from each vertex. Let $\Gamma^k = (V, \Lambda)$, where $V$ is the set of vertices of $\Gamma^k$, $\Lambda$ is the set of edges. Two vertices $x$ and $y$ ($x, y \in V$) are called nearest-neighbors if there exists an edge $l \in \Lambda$ connecting them, which is denoted by $l = (x, y)$. The distance $d(x, y), x, y \in V$, on the Cayley tree $\Gamma^k$, is the number of edges in the shortest path from $x$ to $y$.

For a fixed $x^0 \in V$, called the root, we set

\[ W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \{x \in V | d(x, x^0) \leq n\} \]
and $L_n$ denotes the set of edges in $V_n$. The fixed vertex $x^0$ is called the 0th level and the vertices in $W_n$ are called the $n$th level and for $x \in W_n$ let

$$S(x) = \{y \in W_n | d(x, y) = 1\},$$

be the set of direct successors of $x \in W_{n-1}$. For the sake of simplicity we put $|x| = d(x, x^0)$, $x \in V$. Two vertices $x, y \in V$ are called the next-nearest-neighbors if $d(x, y) = 2$. Next-nearest-neighbor vertices $x$ and $y$ are called prolonged next-nearest-neighbors if they belong to the same branch, i.e. $|x| \neq |y|$, which is denoted by $\langle x, y \rangle$.

Let spin variables $\sigma(x), x \in V$, take values $\pm 1$. The Ising–Vannimenus model with competing nearest-neighbors and next-nearest-neighbors binary interactions is defined by the following Hamiltonian

$$H(\sigma) = -J_p \sum_{\langle x, y \rangle} \sigma(x) \sigma(y) - J \sum_{(x, y)} \sigma(x) \sigma(y),$$

(2.1)

where the sum in the first term ranges all prolonged next-nearest-neighbors and the sum in the second term ranges all nearest-neighbors. Here $J_p, J \in R$ are coupling constants.

Note that in [21] it is assumed that $J > 0$ and $J_p < 0$. Below we consider the model (2.1) with arbitrary sign of the coupling constants.

As usual, one can introduce the notions of Gibbs distribution of this model, limiting Gibbs distribution, pure phase (extreme Gibbs distribution), etc (see [13] and [18]).

3. Gibbs measures of the Ising–Vannimenus model

In this section we define a notion of Gibbs measure corresponding to the Ising–Vannimenus model on an arbitrary order Cayley tree. We propose a new kind of construction of Gibbs measures corresponding to the model.

Below, for the sake of simplicity, we will consider a semi-infinite Cayley tree $\Gamma^k_+$ of order $k$, i.e. an infinite graph without cycles with $k + 1$ edges issuing from each vertex except for $x^0$ which has only $k$ edges.

We consider the model where the spin takes values in the set $\Phi = \{-1, +1\}$ ($\Phi$ is called a state space) and is assigned to the vertices of the tree $\Gamma^k_+ = (V, \Lambda)$. A configuration $\sigma$ on $V$ is defined as a function $x \in V \to \sigma(x) \in \Phi$; in a similar manner one defines configurations $\sigma_n$ and $\omega$ on $V_n$ and $W_n$, respectively. The set of all configurations on $V$ (resp. $V_n$, $W_n$) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{V_n}$. Using this, for given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} 
\sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\
\omega(x), & \text{if } x \in W_n.
\end{cases}$$

It is clear that $\sigma_{n-1} \vee \omega \in \Omega_{V_n}$.

In the Ising–Vannimenus model spin takes values in $\Phi = \{-1, +1\}$ and the relevant Hamiltonian has the form

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\[ H(\sigma) = -J_p \sum_{x,y} \sigma(x)\sigma(y) - J \sum_{\langle x,y \rangle} \sigma(x)\sigma(y). \]  
(3.1)

Assume that \( h : (V \setminus \{0\}) \times (V \setminus \{0\}) \times \Phi \times \Phi \to \mathbb{R} \) is a mapping, i.e.

\[ h_{xy,uv} = (h_{xy,++}, h_{xy,+-}, h_{xy,-+}, h_{xy,--}), \]

where \( h_{xy,uv} \in \mathbb{R}, u, v \in \Phi, \) and \( x, y \in V \setminus \{0\}. \)

Now, we define the Gibbs measure with memory of length 2 on the Cayley tree as follows:

\[ \mu^{(n)}_h(\sigma) = \frac{1}{Z_n} \exp[-\beta H_n(\sigma) + \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)}]. \]
(3.2)

Here, \( \beta = \frac{1}{kT}, \sigma \in \Omega_{V_n}, \) and \( Z_n \) is the corresponding to partition function

\[ Z_n = \sum_{\sigma_n \in \Omega_{V_n}} \exp[-\beta H(\sigma_n) + \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)}]. \]
(3.3)

**Remark 3.1.** We stress that the local Gibbs measures considered in [1] were defined as follows:

\[ \tilde{\mu}^{(n)}_h(\sigma) = \frac{1}{Z_n} \exp[-\beta H_n(\sigma) + \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x)\sigma(y)\sigma(z)h_{xyz,\sigma(x)\sigma(y)\sigma(z)}] \]

which are different from (3.2).

In this paper, we are interested in a construction of an infinite volume distribution with given finite-dimensional distributions. More exactly, we would like to find a probability measure \( \mu \) on \( \Omega \) with given conditional probabilities \( \mu^{(n)}_h \), i.e.

\[ \mu(\sigma \in \Omega : \sigma|\omega = \sigma_n) = \mu^{(n)}_h(\sigma_n), \quad \text{for all } \sigma_n \in \Omega_{V_n}, n \in \mathbb{N}. \]  
(3.4)

If the measures \( \{\mu^{(n)}_h\} \) are compatible, i.e.

\[ \sum_{\omega \in \Omega_{W_n}} \mu^{(n)}_h(\sigma \cup \omega) = \mu^{(n-1)}_h(\sigma), \quad \text{for any } \sigma \in \Omega_{V_{n-1}}, \]  
(3.5)

then according to the Kolmogorov’s theorem there exists a unique measure \( \mu_h \) defined on \( \Omega \) with a required condition (3.4). Such a measure \( \mu_h \) is said to be Gibbs measure corresponding to the model. Note that a general theory of Gibbs measures has been developed in [13, 19].

In the sequel, we need the following auxiliary fact.

**Lemma 3.1.** If \( a = \frac{N_1}{N_2}, b = \frac{N_1}{N_3} \) and \( a = \frac{N_1}{N_4}, \) then there exists \( D \in \mathbb{R} \) such that \( a = DN_1, b = DN_2, c = DN_3 \) and \( d = DN_4. \)

The next statement describes the conditions on the boundary fields \( h \) guaranteeing the compatibility of the distributions \( \{\mu^{(n)}_h\} \).

**Theorem 3.2.** The measures \( \mu^{(n)}_h, n = 1, 2, \ldots, \) in (3.2) are compatible iff for any \( x, y \in V \) the following equations hold:

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\[
\begin{cases}
    e^{h_{xy,+} + h_{xy,-}} = \prod_{z \in S(y)} \exp[h_{yz,+} + |ab|^2 + \exp[-h_{yz,-} + |b|^2]] \\
    e^{h_{xy,-} + h_{xy,+}} = \prod_{z \in S(y)} \exp[-h_{yz,-} + \exp[h_{yz,+} + |a|^2] + \exp[-h_{yz,-} + |b|^2]] \\
    e^{h_{xy,+} + h_{xy,-}} = \prod_{z \in S(y)} \exp[h_{yz,+} + |ab|^2 + \exp[-h_{yz,-} + |b|^2]] \\
\end{cases}
\]  

(3.6)

where \( a = \exp(\beta J) \) and \( b = \exp(\beta J_p) \).

**Proof.** **Necessity.** From (3.5), we have

\[
L_n \sum_{\eta \in \Omega_{W_n}} \exp[-\beta H_n(\sigma \lor \eta)] + \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)} \\
= \exp[-\beta H_n(\sigma)] + \sum_{x \in W_{n-2}} \sum_{y \in S(x)} \sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)} ,
\]  

(3.7)

where \( L_n = \frac{Z_n}{Z_n} \).

For \( \sigma \in V_{n-1} \) and \( \eta \in W_n \), we rewrite the Hamiltonian as follows:

\[
H_n(\sigma \lor \eta) = -J \sum_{(x,y) \in V_{n-1}} \sigma(x)\sigma(y) - J \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x)\eta(y) \\
- J_p \sum_{x,y \in V_{n-1}} \sigma(x)\sigma(y) - J_p \sum_{x \in W_{n-2}} \sum_{z \in S^2(x)} \sigma(x)\eta(z) \\
= H_n(\sigma_{n-1}) - J \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x)\eta(y) - J_p \sum_{x \in W_{n-2}} \sum_{z \in S^2(x)} \sigma(x)\eta(z) .
\]  

(3.8)

Therefore, the last equality with (3.7) implies

\[
L_n \sum_{\eta \in \Omega_{W_n}} \exp[-\beta H_n(\sigma_{n-1}) - \beta J \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x)\eta(y)] \\
- \beta J_p \sum_{x \in W_{n-2}} \sum_{z \in S^2(x)} \sigma(x)\eta(z) + \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)} \\
= \exp[-\beta H_n(\sigma_{n-1})] + \sum_{x \in W_{n-2}} \sum_{y \in S(x)} \sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)} .
\]  

(3.9)

Hence, one gets

\[
L_n \prod_{x \in W_{n-2}} \prod_{y \in S(x)} \prod_{z \in S(y)} \sum_{\eta \in \Omega_{W_n}} \exp[\sigma(y)\eta(z)h_{yz,\sigma(y)\eta(z)} + \beta \eta(z)(J\sigma(y) + J_p\sigma(x))] \\
= \prod_{x \in W_{n-2}} \prod_{y \in S(x)} \exp[\sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)}].
\]

Let us fix \( \langle x, y \rangle \). Then considering all values of \( \sigma(x), \sigma(y) \in \{-1, +1\} \), from (3.9), we obtain

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\[ e^{h_{xy,++}+h_{xy,--}} = \prod_{z \in S(y)} \frac{\sum_{\eta(z) \in \{\pm 1\}} \exp[\eta(z)(h_{yz,++} + \beta(J + J_p))] \sum_{\eta(z) \in \{\pm 1\}} \exp[\eta(z)(h_{yz,--} + \beta(J - J_p))] \] (3.10)

\[ e^{h_{xy,+-}+h_{xy,-+}} = \prod_{z \in S(y)} \frac{\sum_{\eta(z) \in \{\pm 1\}} \exp[-\eta(z)(h_{yz,+-} + \beta(J + J_p))] \sum_{\eta(z) \in \{\pm 1\}} \exp[-\eta(z)(h_{yz,-+} - \beta(-J + J_p))] \} (3.11)

\[ e^{h_{xy,++}+h_{xy,--}} = \prod_{z \in S(y)} \frac{\sum_{\eta(z) \in \{\pm 1\}} \exp[\eta(z)(h_{yz,++} + \beta(J + J_p))] \sum_{\eta(z) \in \{\pm 1\}} \exp[-\eta(z)(h_{yz,--} - \beta(-J + J_p))] \} . (3.12)

These equations imply the desired ones.

**SUFFICIENCY.** Now we assume that the system of equations (3.6) is valid, then from lemma 3.1 one finds

\[ e^{\sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)}} D(x, y) = \prod_{z \in S(y)} \sum_{\eta(z) \in \{\pm 1\}} \exp[\sigma(y)\eta(z)(h_{yz,\sigma(y)\eta(z)} + \beta\eta(z)(J\sigma(y) + J_p\sigma(x)))] \]

for some constant \( D(x, y) \) depending on \( x \) and \( y \).

From the last equality, we obtain

\[
\prod_{x \in W_{n-2}} \prod_{y \in S(x)} D(x, y) e^{\sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)}} = \prod_{x \in W_{n-2}} \prod_{y \in S(x)} \prod_{z \in S(y)} \sum_{\eta(z) \in \{\pm 1\}} e^{[\sigma(y)\eta(z)(h_{yz,\sigma(y)\eta(z)} + \beta\eta(z)(J\sigma(y) + J_p\sigma(x))]]. (3.13)
\]

Multiply both sides of the equation (3.13) by \( e^{-\beta H_{n-1}(\sigma)} \) and denoting

\[ U_{n-1} = \prod_{x \in W_{n-2}} \prod_{y \in S(x)} D(x, y), \]

from (3.13), one has

\[
U_{n-1} e^{-\beta H_{n-1}(\sigma)} = \prod_{x \in W_{n-2}} \prod_{y \in S(x)} \sum_{\eta(z) \in \{\pm 1\}} e^{[\sigma(y)\eta(z)(h_{yz,\sigma(y)\eta(z)} + \beta\eta(z)(J\sigma(y) + J_p\sigma(x))]} ,
\]

which yields

\[
U_{n-1} Z_{n-1}^{(n-1)}(\sigma) = \sum_{\eta} e^{-\beta H_{n}(\sigma \eta)} = \sum_{\eta} e^{[\sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)}].
\]

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This means
\[ U_{n-1}Z_{n-1}\mu^{(n-1)}_h(\sigma) = Z_n \sum_{\eta} \mu^{(n)}_h(\sigma \cup \eta). \] (3.14)

As \( \mu^{(n)}_h \) (\( n \geq 1 \)) is a probability measure, i.e.
\[ \sum_{\sigma \in \{-1,+1\}^{V_n-1}} \mu^{(n-1)}_h(\sigma) = \sum_{\sigma \in \{-1,+1\}^{V_n-1}} \sum_{\eta \in \{-1,+1\}^W} \mu^{(n)}_h(\sigma \cup \eta) = 1. \]

From these equalities and (3.14) we have \( Z_n = U_{n-1}Z_{n-1} \). This with (3.14) implies that (3.5) holds. The proof is complete. \( \square \)

According to theorem 3.2 the problem of describing the Gibbs measures is reduced to the descriptions of the solutions of the functional equations (3.6).

**Corollary 3.3.** The measures \( \mu^{(n)}_h \), \( n = 1, 2, \ldots \) satisfy the compatibility condition (3.5) if and only if for any \( n \in \mathbb{N} \) the following equation holds:
\[
\begin{cases}
  u_{xy,1} = a \prod_{z \in S(y)} \frac{b_{uyz,3}+1}{u_{yz,3}+b} \\
  u_{xy,2} = a \prod_{z \in S(y)} \frac{(b_{uyz,3}+1)u_{yz,1}}{(u_{yz,3}+b)u_{yz,1}} \\
  u_{xy,3} = a \prod_{z \in S(y)} \frac{(b_{uyz,3}+1)u_{yz,1}}{(u_{yz,3}+b)u_{yz,3}}
\end{cases}
\] (3.15)

where, as before \( a = \exp(2\beta J), b = \exp(2\beta J_1) \), and
\[
\begin{align*}
  u_{xy,1} &= a \cdot \exp(h_{xy,++} + h_{xy,--}) \\
  u_{xy,2} &= a \cdot \exp(h_{xy,--} + h_{xy,--}) \\
  u_{xy,3} &= a \cdot \exp(h_{xy,++} + h_{xy,--}).
\end{align*}
\] (3.16)

It is worth mentioning that there are infinitely many solutions of the system (3.6) corresponding to each solution of the system of equations (3.15). However, we show that each solution of the system (3.15) uniquely determines a Gibbs measure. We denote by \( \mu_u \) the Gibbs measure corresponding to the solution \( u \) of (3.15).

**Theorem 3.4.** There exists a unique Gibbs measure \( \mu_u \) associated with the function \( u = \{u_{xy}, (x,y) \in L\} \) where \( u_{xy} = (u_{xy,1}, u_{xy,2}, u_{xy,3}) \) is a solution of the system (3.15).

**Proof.** Let \( u = \{u_{xy}, (x,y) \in L\} \) be a function, where \( u_{xy} = (u_{xy,1}, u_{xy,2}, u_{xy,3}) \) is a solution of the system (3.15). Then, for any \( h_{xy,++} \in \mathbb{R} \) a function \( h = \{h_{xy}, (x,y) \in L\} \) defined by
\[
h_{xy} = \left(h_{xy,++}, \log\left(\frac{u_{xy,3}a}{u_{xy,3}}\right) - h_{xy,++}, \log\left(\frac{u_{xy,1}a}{u_{xy,1}}\right) - h_{xy,++}, \log\left(\frac{u_{xy,2}a}{u_{xy,1}}\right) + h_{xy,++}\right)
\]
is a solution of (3.6).

Now fix \( n \geq 1 \). Since \( |W_n-1| = k^{n-1} \) and \( |S(x)| = k \) we get \( |L_n \setminus L_{n-1}| = k^n \). Let \( \sigma \) be any configuration on \( \Omega_{V_n} \). Denote
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\[ \mathcal{N}_{1,n}(\sigma) = \{ (x, y) \in L_n \setminus L_{n-1} : \sigma(x) = 1, \sigma(y) = 1, x \in W_{n-1}, y \in S(x) \} \]

\[ \mathcal{N}_{2,n}(\sigma) = \{ (x, y) \in L_n \setminus L_{n-1} : \sigma(x) = 1, \sigma(y) = -1, x \in W_{n-1}, y \in S(x) \} \]

\[ \mathcal{N}_{3,n}(\sigma) = \{ (x, y) \in L_n \setminus L_{n-1} : \sigma(x) = -1, \sigma(y) = 1, x \in W_{n-1}, y \in S(x) \} \]

\[ \mathcal{N}_{4,n}(\sigma) = \{ (x, y) \in L_n \setminus L_{n-1} : \sigma(x) = -1, \sigma(y) = -1, x \in W_{n-1}, y \in S(x) \} . \]

We have

\[
\prod_{x \in W_{n-1}, y \in S(x)} \exp \left\{ h_{xy,\sigma(x)\sigma(y)}(x)\sigma(y) \right\} = \prod_{(x,y)\in\mathcal{N}_{1,n}(\sigma)} \exp \left\{ h_{xy,++} \right\} \prod_{(x,y)\in\mathcal{N}_{2,n}(\sigma)} a \cdot \exp \left\{ h_{xy,++} \right\} u_{xy,3} \\
\times \prod_{(x,y)\in\mathcal{N}_{3,n}(\sigma)} u_{xy,1} \prod_{(x,y)\in\mathcal{N}_{4,n}(\sigma)} u_{xy,1} \exp \left\{ h_{xy,++} \right\} u_{xy,1} \exp \left\{ h_{xy,++} \right\} \prod_{(x,y)\in\mathcal{N}_{3,n}(\sigma)} u_{xy,1} \prod_{(x,y)\in\mathcal{N}_{4,n}(\sigma)} u_{xy,1} .
\]

By means of the last equality, from (3.2) and (3.3) we find

\[ \mu_{\mathbf{u}}^{(n)}(\sigma) = \sum_{\omega \in \Omega_{V_n}} \exp \left\{ -\beta H_{\beta}^{(n)}(\sigma) \right\} \prod_{x \in W_{n-1}, y \in S(x)} \exp \left\{ h_{xy,\sigma(x)\sigma(y)}(x)\sigma(y) \right\} \]

\[ = \sum_{\omega \in \Omega_{V_n}} \exp \left\{ -\beta H_{\beta}^{(n)}(\sigma) \right\} \prod_{(x,y)\in\mathcal{N}_{2,n}(\sigma)} u_{xy,3} \prod_{(x,y)\in\mathcal{N}_{3,n}(\sigma)} u_{xy,1} \prod_{(x,y)\in\mathcal{N}_{4,n}(\sigma)} u_{xy,1} \exp \left\{ h_{xy,++} \right\} \prod_{(x,y)\in\mathcal{N}_{3,n}(\sigma)} u_{xy,1} \prod_{(x,y)\in\mathcal{N}_{4,n}(\sigma)} u_{xy,1} .
\]

(3.17)

One can see the right hand side of (3.17) does not depend to \( h_{xy,++} \). So, we can say that each solution \( \mathbf{u} \) of the system (3.15) uniquely determines only one Gibbs measure \( \mu_{\mathbf{u}} \). \( \square \)

**Remark 3.2.** Hence, due to theorem 3.4 there exists a phase transition for the model (3.8) with \( J_0 = 0 \) if and only if the equation (3.15) has at least two solutions.

**Remark 3.3.** We point out that in the original work [21] modulated phases were found in the frustrated regime, when the next-nearest-neighbor interaction \( J_p \) is negative. To obtain these kinds of phases, one needs to find periodic solutions\(^5\) of the equation (3.15). We will show that to find general solutions even in the case of translation-invariant ones of the system is not an easy job. Our main aim in this paper is first rigorously establish the existence of the phase transition by finding translation-invariant solutions of the system. The case \( J_p < 0 \) is not easy and will require a lot of effort to explicitly find periodic solutions. Note that even for the usual

\(^5\) Periodicity of the solution \( \mathbf{u} = \{ u_{xy}, (x, y) \in L \} \) can be defined via representing the tree as a free group. We refer the reader to [19] for detail information.

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Ising model (anti ferromagnetic case), up to now, not all periodic solutions have been found (see [19] for the review).

4. The existence of Gibbs Measures

In this section we are going to establish the existence of Gibbs measures by analyzing the equation (3.15).

Recall that \( u = \{u_{xy}\}_{(x,y) \in L} \) is a translation-invariant function, if one has \( u_{xy} = u_{zw} \) for all \( (x, y), (z, w) \in L \). A measure \( \mu_u \), corresponding to a translation-invariant function \( u \), is called a translation-invariant Gibbs measure.

Solving the equation (3.15), in general, is rather very complex. Therefore, let us first restrict ourselves to the description of its translation-invariant solutions. Hence, (3.15) reduces to the following one

\[
\begin{align*}
  u_1 &= a \left( \frac{bu_2 + 1}{u_3 + b} \right)^k \\
  u_2 &= a \left( \frac{(bu_2 + 1)u_3}{(u_3 + b)u_1} \right)^k \\
  u_3 &= a \left( \frac{bu_2 + 1}{u_2 + b}u_1 \right)^k.
\end{align*}
\]

4.1. Solution of the system (4.1)

In this subsection, we are aiming to study the set of all solutions of the system (4.1).

Denote \( \sqrt{u_1} = x_1, \sqrt{u_2} = x_2, \sqrt{u_3} = x_3, \sqrt{a} = a \). Then from (4.1) we obtain

\[
\begin{align*}
  x_1 &= \tilde{a} \frac{bx_2 + 1}{x_3 + b} \\
  x_2 &= \tilde{a} \frac{(bx_2 + 1)x_3}{(x_3 + b)x_1} \\
  x_3 &= \tilde{a} \frac{(bx_2 + 1)x_1}{(x_1 + b)x_3}.
\end{align*}
\]

Define the following sets

\[
\begin{align*}
  \mathcal{A}_1 &= \{ x \in \mathbb{R}^3_+ : x_1 = x_2 \}, \\
  \mathcal{A}_2 &= \{ x \in \mathbb{R}^3_+ : x_1 = x_3 \}, \\
  \mathcal{A}_3 &= \{ x \in \mathbb{R}^3_+ : x_2 = x_3 \}, \\
  \mathcal{A} &= \{ x \in \mathbb{R}^3_+ : x_1 = x_2 = x_3 \}.
\end{align*}
\]

**Proposition 4.1.** Let \( x \) be a solution of (4.2). Then \( x \in \mathcal{A} \) if and only if \( x \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \).

**Proof.** Assume that \( x_1 = x_2 \), then multiplying the second and the third equalities of (4.2) and dividing the obtained one by the first equality, one finds

\[
\frac{x_2x_3}{x_1} = \tilde{a} \frac{bx_2 + 1}{x_2^k + b}.
\]

The assumption yields \( x_3 = \tilde{a} \frac{bx_2 + 1}{x_2^k + b} \). Now inserting the last equality into the third one of (4.2), we obtain \( x_1 = x_3 \).
Let us suppose \( x_1 = x_3 \), then from the first and the third equations of (4.2), one gets \( x_1^k + b = x_2^k + b \), which implies \( x_1 = x_2 \).

Now assume that \( x_2 = x_3 \). In this case dividing the second by the third equations of (4.2) we obtain \( \left( \frac{x_2}{x_1} \right)^k = 1 \), which yields \( x_1 = x_3 \). This completes the proof. \( \square \)

A natural question arises: Does there exist a solution on \( \mathbb{R}^3_+ \setminus A \)? We will try to answer it in the next section 4.2.

Now we consider the case \( u := u_1 = u_2 = u_3 \). Then denoting \( z = bu \) from (4.1) we obtain

\[
\left( \frac{z + 1}{z + b^2} \right)^k = \frac{b^{k-1}}{a} z. \tag{4.3}
\]

To solve the last equation we apply the following well-known fact.

**Proposition 4.2** ([18]). The equation

\[
\left( \frac{1 + x}{b + x} \right)^{m-1} = ax
\]

(with \( x \geq 0, m \geq 2, a > 0, b > 0 \)) has one solution if either \( m = 2 \) or \( b \leq \left( \frac{m}{m-2} \right)^2 \). If \( m > 2 \) and \( b > \left( \frac{m}{m-2} \right)^2 \) then there exist \( \eta_1(b, m), \eta_2(b, m) \) with \( 0 < \eta_1(b, m) < \eta_2(b, m) \) such that the equation has three solutions if \( \eta_1(b, m) < a < \eta_2(b, m) \) and has two solution if either \( a = \eta_1(b, m) \) or \( a = \eta_2(b, m) \). In fact

\[
\eta_i(b, m) = \frac{1}{x_i} \left( \frac{1 + x_i}{b + x_i} \right)^{m-1},
\]

where \( x_1, x_2 \) are solutions of

\[
x^2 + [2 - (b - 1)(m - 2)]x + b = 0.
\]

Hence, according to proposition 4.2 the equation (4.3) can be solved under certain conditions which provide sufficient conditions for the existence of Gibbs measures.

### 4.2. Solution on \( \mathbb{R}^3_+ \setminus A \)

In this section, for the sake of simplicity, we assume that the order of the tree is two, i.e. \( k = 2 \).

Suppose that \( x \in \mathbb{R}^3_+ \setminus A \) is a solution of (4.2). Due to proposition 4.1 we assume that \( x_1 = x, x_2 = mx, x_3 = tx \), where \( x, m, t > 0, m \neq 1, t \neq 1 \) and \( m \neq t \). It then follows from (4.2) that

\[
\begin{align*}
  x &= \frac{\alpha b^2 x^2 + 1}{t x^2 + b} \\
  mx &= \frac{\alpha t^2 b m^2 x^2 + 1}{t^2 x^2 + b} \\
  tx &= \frac{\alpha b^2 x^2 + 1}{m x^2 + b}
\end{align*} \tag{4.4}
\]

Note that \( mt = 1 \) if and only if \( m = t = 1 \). Indeed, multiplying the second and the third equations of (4.4), and dividing the result by the first one, we get
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\[ mtx = \frac{b m^2 x^2 + 1}{m^2 x^2 + b}. \]

The last equality with the first equation of (4.2) implies that \( m = t = 1 \) if \( mt = 1 \). So, in the current setting, we may assume that \( mt \neq 1 \). From (4.4) one finds

\[
\begin{align*}
    b(m^2 - m)t^2 x^2 &= m - t^2 \\
    (m^2 t - 1)t^2 x^2 &= b(1 - t^3).
\end{align*}
\]

(4.5)

Since \( x > 0 \), we get \( 1 < t < m^{-2} \) or \( m^{-2} < t < 1 \). Plugging (4.5) into the first equation of (4.4), one finds

\[ x = \tilde{a}b^{-1} \frac{m^2 t - 1}{t(m^2 - m)}. \]

(4.6)

Substituting (4.6) into (4.5) yields

\[ \tilde{a}^2 (m^2 t - 1)^2 = b(m^2 - m)(m - t^2). \]

The last equality with (4.5) implies that \( (x, mx, tx) \in \mathbb{R}_+^3 \setminus A \) is a solution of (4.4) if and only if the parameters \( m \) and \( t \) satisfy the following equations

\[
\begin{align*}
    b^2(m^2 - m)(1 - t^3) &= (m - t^2)(m^2 t - 1) \\
    \tilde{a}^2(m^2 t - 1)^2 &= b(m^2 - m)(m - t^2) \\
    1 < t < m^{-2} \text{ or } m^{-2} < t < 1
\end{align*}
\]

where \( x \) is defined by (4.6).

It is easy to check that \( \left( \sqrt{b}, 3\sqrt{b}, \frac{\sqrt{b}}{2} \right) \) is a solution of (4.4), if \( \tilde{a} = \frac{6}{7}\sqrt{b} \) and \( b = \sqrt{\frac{11}{6}} \).

5. The existence of phase transition

In this section, we restrict ourselves to the case \( k = 2 \) and

\[ h_{xy,++} = h_{xy,--} = h_1 \text{ and } h_{xy,+-} = h_{xy,-+} = h_2. \]

(5.1)

The analysis of the solution of the equations (3.6) is rather tricky. In this section, we will study the translation-invariant solutions.

Denoting \( \ln u_1 = h_{xy,++} = h_{xy,--} \) and \( \ln u_2 = h_{xy,+-} = h_{xy,-+} \) for any \( x, y \in V \), from (3.6) one can produce

\[ u_1^2 = \left( \frac{a^2 b^2 u_1 u_2 + 1}{a^2 u_1 u_2 + b^2} \right)^2 = u_2^2, \]

(5.2)

where \( a = e^{\beta J} \) and \( b = e^{\beta Jp} \). This means \( u_1 = u_2 \), therefore letting \( u := u_1 = u_2 \), we have

\[ u = \frac{(ab)^2 u^2 + 1}{a^2 u^2 + b^2}. \]

Now putting \( c := a^2 \) and \( d := b^2 \), then the last equation reduces to

\[ u = g(u), \]

(5.3)
where
\[ g(u) = \frac{cdu^2 + 1}{cu^2 + d}. \] (5.4)

Note that if there is more than one positive solutions of (5.3), then we have more than one translation-invariant Gibbs measures corresponding to the solution of (5.3).

**Proposition 5.1.** The equation \( u = \frac{cdu^2 + 1}{cu^2 + d} \) has one solution if either \( c \leq 1 \) or \( d < 3 \). If \( d \geq 3 \) then there exist \( \eta_1(d), \eta_2(d) \) with \( 0 < \eta_1(d) < \eta_2(d) \) such that equation (5.3) has three solutions if \( \eta_1(d) < c < \eta_2(d) \) and has two solutions if either \( \eta_1(d) = c \) or \( \eta_2(d) = c \).

The proof of proposition 5.1 can be done by using the similar method as in [10].

From the proposition we infer that if one has \( \beta J_p > (\ln 3)/2 \) then there are three translation-invariant Gibbs measures for the model, which yields the existence of the phase transition.

Let us give an illustrative example. Figure 1 shows that there are 3 positive fixed points of the function (5.4), if we take \( J = -1.85, J_p = 4.5, T = 2.6 \). In figure 1, we can find three fixed points of the function \( g \) as \( u_1 = 0.0316222, u_2 = 4.86623, u_3 = 26.9681 \) corresponding to the parameters \( J = -1.85, J_p = 4.5, T = 2.6 \). We have 3 TIGMs associated with the fixed points \( u_1 = 0.0316222, u_2 = 4.86623, u_3 = 26.9681 \). Therefore, the phase transition for the model (2.1) occurs.

### 6. Free energy

In this section, we study the free energy of depending on the boundary conditions for the Ising–Vannimenus model on the Cayley tree. From the previous section we know that for any boundary condition satisfying the equations (3.6) there exist Gibbs measures corresponding to the model. Recall that the partition function of the model is

\[ Z_n = Z_n(\beta, h) = \sum_{\sigma \in \Omega_n} \exp \left\{ -\beta H_n(\sigma) + \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \sigma(x) \sigma(y) h_{xy, \sigma(x)\sigma(y)} \right\}. \] (6.1)
Then the free energy is defined as follows:

\[ F(\beta, h) = \lim_{n \to \infty} \frac{1}{|V_n|} \ln Z_n(\beta, h). \] (6.2)

In this section, we discuss the behavior of the free energy of the model, as function of boundary conditions. As before, we consider the boundary conditions (5.1), i.e.

\[ h_{xy,++} = h_{xy,--} = \tilde{h}_1 \quad \text{and} \quad h_{xy,-+} = h_{xy,--} = \tilde{h}_2, \quad \forall (x, y) \in L. \] (6.3)

**Proposition 6.1.** The free energies corresponding to the translation-invariant (TI) boundary conditions with (6.3) exist and are given by

\[ F_{\text{TI}}(\beta, h_i) = -\frac{1}{\beta} \log \left[ 2 \cosh(h_i + \beta(J + J_p)) \cosh(h_i + \beta(J - J_p)) \right], \] (6.4)

where \( h_i, (i = 1, 2, 3) \), is the variety such that \( u_i = e^{h_i} \) which is a solution of (5.3).

**Proof.** From (5.1) one finds

\[ D(x, y)e^{h_{xy,++}} = \prod_{z \in S(y)} \left[ e^{h_{yz,++} + \beta(J + J_p)} + e^{-h_{yz,++] - \beta(J + J_p)} \right] \]

\[ = \prod_{z \in S(y)} 2e^{\frac{h_{yz,++} - h_{yz,--}}{2}} \cosh \left[ \frac{h_{yz,++} - h_{yz,--}}{2} + \beta(J + J_p) \right]. \] (6.5)

\[ D(x, y)e^{-h_{xy,-+}} = \prod_{z \in S(y)} \left[ e^{h_{yz,++] + \beta(J - J_p)}} + e^{-h_{yz,++] - \beta(J - J_p)} \right] \]

\[ = \prod_{z \in S(y)} 2e^{\frac{h_{yz,++} - h_{yz,--}}{2}} \cosh \left[ \frac{h_{yz,++} - h_{yz,--}}{2} + \beta(J - J_p) \right]. \] (6.6)

Multiply the equations (6.5) and (6.6), we then obtain

\[ D(x, y) = 4 \prod_{z \in S(y)} b(y, z), \] (6.7)

where

\[ b(y, z) = e^{\frac{h_{yz,++} - h_{yz,--}}{2}} \left( \cosh \left[ \frac{h_{yz,++} - h_{yz,--}}{2} + \beta(J + J_p) \right] \cosh \left[ \frac{h_{yz,++} - h_{yz,--}}{2} + \beta(J - J_p) \right] \right)^{\frac{1}{2}}. \]

Hence, one finds

\[ U_{n-1} = \prod_{x \in W_{n-2}} \prod_{y \in S(x)} D(x, y) \]

\[ = 4^{|W_{n-1}|} \prod_{y \in W_{n-1}} \prod_{z \in S(y)} b(x, y) \]

\[ = 4^{|W_{n-1}|} e^{\sum_{y \in W_{n-1}} \sum_{z \in S(y)} \ln b(x, y)}. \]
Denoting \( a(x, y) = \ln b(x, y) \), from the last equality we get

\[
Z_n = U_{n-1}Z_{n-1} = 4|V_{n-1}| \sum_{y \in V_{n-1}} \sum_{z \in S(y)} a(y, z) \sum_{y_1 \in V_{n-2}} \sum_{z_1 \in S(y_1)} a(y_1, z_1) \ldots \sum_{y_0 \in V_0} a(y_0, z_0).
\]

Therefore, from (6.3) we have

\[
F_{TI_1}(\beta, h) = -\lim_{n \to \infty} \frac{|V_{n-1}|}{\beta|V_n|} \ln D(x, y)
= -\frac{1}{\beta} \ln \left[ 2e^{\tilde{h}_1 - \tilde{h}_2} \cosh \left( \frac{\tilde{h}_1 + \tilde{h}_2}{2} + \beta(J + J_p) \right) \cosh \left( \frac{\tilde{h}_1 + \tilde{h}_2}{2} + \beta(J - J_p) \right) \right].
\]

(6.9)

Due to (5.2) we may assume that \( e^{\tilde{h}_1} = e^{\tilde{h}_2} \). So, due to proposition 5.1, under certain conditions, there exist three solutions \( u_i (i = 1, 2, 3) \) of (5.3). Denoting \( h_i = \ln u_i \) from (6.9), one finds

\[
F_{TI_1}(\beta, h_i) = -\frac{1}{\beta} \ln \left[ 2 \cosh(h_i + \beta(J + J_p)) \cosh(h_i + \beta(J - J_p)) \right].
\]

(6.10)

This completes the proof.

In order to draw the free energy \( F_{TI_1}(\beta, h) \) as a function of \( \beta \), we consider the fixed points of the function (5.4). Some particular plots are shown in figure 2, where \( u_1 = 0.260261 \), \( u_2 = 1.18483 \) and \( u_3 = 3.52491 \) are the fixed points of the function \( g \) corresponding to the parameters \( J_p = 4.5, J = -1.85 \) and \( T = 2.6 \).

Let us compute the entropy

Figure 2. The free energies \( F_{TI_1}(\beta, h) \) (blue color line, for \( u_1 = 0.260261 \), (red color, for \( u_2 = 1.18483 \)) and (black color line, for \( u_3 = 3.52491 \)). Here \( J_p = 4.5, J = -1.85 \).
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\[
S(\beta, h_i) = -\frac{dF(\beta, h_i)}{dT} = \frac{1}{d\beta} \frac{dF(\beta, h_i)}{\beta^2} \\
= -\ln \left[ 2 \cosh(h_i + \beta(J - J_p)) \cosh(h_i + \beta(J + J_p)) \right] \\
= \frac{\beta(J - J_p) \tanh(h_i + \beta(J - J_p)) + \beta(J + J_p) \tanh(h_i + \beta(J + J_p))}{\beta^4}. \tag{6.11}
\]

7. Conclusions

In the present paper, we have considered the Ising–Vannimenus model on an arbitrary order Cayley tree with competing nearest-neighbor, prolonged next-nearest neighbor interactions. Recently, the mentioned model was investigated only numerically, without rigorous (mathematical) proofs [10]. We have proposed a measure-theoretical approach in order to study the translation invariant Gibbs measures associated with the model. Under certain conditions the existence of Gibbs measures of the Ising–Vannimenus model is obtained. Then we have established the existence of the phase transition. Moreover, an explicit formulae of the free energies corresponding to the translation invariant Gibbs measures is found. Also, we have calculated the entropies corresponding to the mentioned free energies.

We point out that for the Ising–Vannimenus model, the free energies and entropies associated with various known boundary conditions such as ART [3], Bleher–Ganikhodjaev [4], Zachary [22], have not been investigated, yet. Explicit formulaes of the free energies and entropies for the mentioned boundary conditions will be calculated in the future publications.

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