Eigenmodes of Dodecahedral space

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March 24, 2022

Abstract

Cosmological models where spatial sections are the Poincaré dodecahedral space $D$ have been recently invoked to give an account of the lower modes of the angular anisotropies of the cosmic microwave background. Further explorations of this possibility require the knowledge of the eigenmodes of the Laplacian of $D$. Only the first modes have been calculated numerically. Here we give an explicit form for these modes up to arbitrary order, in term of eigenvectors of a small rank matrix, very easy to calculate numerically. As an illustration we give the first modes, up to the eigenvalue $-k(k+2)$ for $k = 62$. These results are obtained by application of a more general method (presented in a previous work) which allows to express the properties of any eigenfunction of the Laplacian of the three sphere under an arbitrary rotation of SO(4).

1 Introduction

There is a long time interest for cosmological models where space is multi-rather than simply-connected (see a review in [7]). Recently, [10] claimed that a peculiar model may give an account of the two first moments in the angular power spectrum of the anisotropies of the cosmic microwave background. This model involves the Poincaré dodecahedral space $D$, whose universal covering is the three sphere $S^3$. The calculations involved the first eigenmodes of the Laplacian in $D$, which were estimated by numerical methods. However, to make predictions beyond the first moments, and to check non diagonal terms in the correlation matrix, the knowledge of a greater number of modes is required.

The eigenmodes of a spherical space $S^3/\Gamma$ are the eigenmodes of $S^3$ which are conserved by all the rotations of $\Gamma$. In general, they remain unknown. Their number, i.e., the multiplicity of the eigenspaces, was calculated by [3] as a function of the eigenvalue of the Laplacian. Recently, [8] (see also [9]) have provided analytic calculations of these modes for Lens space and Prism space. The same results were obtained later with a different method by [9]. Here we provide a new and efficient method to calculate the modes of the dodecahedral space. In [3], we constructed a special basis (B3) for the modes of $S^3$, which allowed to calculate explicitly their behaviour under any rotation of SO(4). These results are recalled in section 2. They are then applied to $D$ (section 3), which leads to the derivation of its eigenmodes.
2 Rotation properties of the eigenmodes of $S^3$

The eigenvalues of the Laplacian $\Delta$ of $S^3$ are of the form $\lambda_k = -k(k+2)$, where $k \in \mathbb{N}^+$. For a given value of $k$, they span the eigenspace $V^k$ of dimension $(k+1)^2$. The eigenmodes of the dodecahedral space $D \equiv S^3/G$ are those eigenmodes of $S^3$ which remain invariant under all holonomy transformations of $G$. To be so, it is necessary and sufficient that they are invariant under the two generators of $G$, that we call $g_\pm$.

We will use a specific basis $B_2$ of $V^k$, whose properties were detailed in [6]. This basis was originally introduced by [11]. A real version of it has been used by [9] to find the eigenmodes of lens and prism spaces. It is generated by a set of $(k+1)^2$ functions:

$$B_2 \equiv (T_{k;m_1,m_2}), \; m_1, m_2 = -k/2..k/2,$$

where $m_1$ and $m_2$ vary independently by entire increments. We know (11) that $D$ has eigenmodes only for $k$ even, that we assume from now. This ensures that $m_1$ and $m_2$ are entire. Hereafter, all sums involving $m_1$ or $m_2$ will cover the range $-k/2..k/2$.

These functions are naturally adapted to toroidal coordinates to describe $S^3$: $(\chi, \theta, \phi)$ spanning the range $0 \leq \chi \leq \pi/2$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq 2\pi$. They are conveniently defined (see [9] for a more complete description) from an isometric embedding of $S^3$ in $\mathbb{R}^4$ (as the hypersurface $x \in \mathbb{R}^4; \mid x \mid = 1$):

$$\begin{align*}
    x^0 &= r \cos \chi \cos \theta \\
    x^1 &= r \sin \chi \cos \phi \\
    x^2 &= r \sin \chi \sin \phi \\
    x^3 &= r \cos \chi \sin \theta
\end{align*}$$

where $(x^\mu)$, $\mu = 0, 1, 2, 3$ is a point of $\mathbb{R}^4$.

In [9], we gave the explicit expression of $B_2$:

$$T_{k;m_1,m_2}(X) = C_{k;m_1,m_2} [\cos \chi e^{i\phi}]^m [\sin \chi e^{i\phi}]^m P_{d}^{(m, \ell)}[\cos(2\chi)].$$

We wrote $\ell = m_1 + m_2$, $m = m_2 - m_1$ for simplicity; $P_{d}^{(m, \ell)}$ is a Jacobi polynomial. Normalization leads to $C_{k;m_1,m_2} \equiv \sqrt{(k+1)\sqrt{(k/2+m_1)! (k/2-m_2)! \sqrt{(k/2+m_1)! (k/2-m_2)!}}}.$

Note that the basis functions $T_{k;m_1,m_2}$ have also been introduced in [2] (p. 253), with their expression in Jacobi Polynomials. They are the complex counterparts of those proposed by [9] (their equ. 19). The variation range of the indices $m_1, m_2$ here is equivalent to their condition

$$\mid \ell \mid + \mid m \mid \leq k, \; \ell + m = k \mod 2,$$

through the correspondence $\ell = m_1 + m_2$, $m = m_2 - m_1$.

2.1 Complex null vectors and roots of unity

Our calculations require the use of an other basis of $V^k$ which was introduced in [9] (and apparently ignored before in the literature):

$$B_3 \equiv (\Phi_{I,J}^I), \; I, J = 0..k.$$

This basis may only be used when $k$ even, which is assumed here. Hereafter, all sums involving $i, j, I$ or $J$ will cover the range $0, k$.
The basis functions are defined as \( \Phi_{ij} : \Phi_{ij}^k(X) \equiv (X \cdot N_{ij})^k \), from the \((k + 1)^2\) null vectors of \( \mathbb{C}^4 \),
\[
N_{ij} \equiv N(I\alpha, J\alpha) = (\cos I\alpha, i \sin J\alpha, i \cos J\alpha, \sin I\alpha).
\] (5)

Here the dot product extends the Euclidean [scalar] dot product of \( \mathbb{R}^4 \) to its complexification \( \mathbb{C}^4 \). (Complex) null vectors are defined as having zero norm. The angle \( \alpha \equiv \frac{2\pi}{k+1} \) is the argument of the \((k + 1)^{th}\) complex roots of unity. Those are the powers \( \rho' \) of
\[
\rho \equiv e^{\frac{i \pi \alpha}{k+1}} \equiv \cos \alpha + i \sin \alpha.
\] (6)

We proved in [6] that B3 form a basis of \( V^k \) (when \( k \) is even) and we gave the transformation formulae with B2. First we defined \( T_{k;1,2} \equiv P_{k;1,2} T_{k;1,2} \),
\[
P_{k;1,2} = \frac{2^{-k} k!}{(k/2 - m_1)! (k/2 + m_1)! (k + 1)^2 \ C_{k;1,2}}
\]
\[
= \frac{2^{-k} \pi k! (k + 1)^{-5/2}}{\sqrt{(k/2 + m_2)! (k/2 - m_2)!(k/2 + m_1)! (k/2 - m_1)!}}.
\]

Then the transformation reads
\[
T_{k;1,2} = \frac{1}{(k + 1)^2} \sum_{l,j=0}^{k} \rho^{l(m_1 + m_2) - j(m_2 - m_1)} \Phi_{ij}, \]
\[
\Phi_{ij} = \sum_{m_1, m_2 = -k/2}^{k/2} T_{k;1,2} \rho^{-j(m_1 + m_2) + j(m_2 - m_1)}.
\] (7)

2.2 Rotations in \( \mathbb{R}^4 \)

The main interest of the basis B3 lies in the possibility to calculate explicitly its rotation properties under an arbitrary rotation \( g \in SO(4) \). The action of a rotation \( g \in SO(4) \) on a function is noted \( R_g \colon R_g f(x) \equiv f(gx) \).

Applied to the basis functions, it may be developed as
\[
R_g : \Phi_{ij} \mapsto R_g \Phi_{ij} : \Phi_{ij}(x) = \Phi_{ij}(gx) = \sum_{ij=0}^k \ G_{ij}^k(g) \ \Phi_{ij}.
\] (8)

The coefficients were calculated in [5] as
\[
G_{ij}^k = \frac{(A')^k}{(k + 1)^2} \sum_{A,B=0}^k \rho^{-i(A+B-k)} \rho^{-j(A-B)} \ U \ V B,
\] (9)

where we defined \( U \equiv (\Phi), \ V \equiv (\Phi), \ A \equiv< Q_L \ a \ Q_R \cdot n_{ij} >, \ A' \equiv< Q_L \ a \ Q_R \cdot n_{ij} >, \ B \equiv< Q_L \ b \ Q_R \cdot n_{ij} >, \ D \equiv< Q_L \ d \ Q_R \cdot n_{ij} > \). Note that \( AA' = BD \).

In these formulae, the two unit quaternions \( Q_L \) and \( Q_R \) represent the rotation \( g \). The four basis quaternions are written \( j_0, j_1, j_2, j_3 \), with \( j_0 = 1 \). A quaternions is developed as \( q = q^0 j_0 \). For a complex quaternions, the \( q^i \) are complex numbers. Terms like \( Q_L \ a \ Q_R \) simply denote the quaternionic product. The brackets indicate the quaternionic scalar product, the bar the quaternionic conjugate. The null complex quaternions \( n_{ij} \) represent the null vectors \( N_{ij} \) introduced above (see details in
Also, we introduced the following peculiar null complex quaternions 
\( \alpha \equiv 1 + i j_3, \beta \equiv j_1 - i j_2 = (1 - i j_3) j_1 \) and \( \delta \equiv -j_1 - i j_2 \). They have zero norm and obey the properties
\[ < \alpha \cdot n_{IJ} > = \rho^I, < \bar{\alpha} \cdot n_{IJ} > = \rho^{-I}, < \beta \cdot n_{IJ} > = \rho^J, < \delta \cdot n_{IJ} > = \rho^{-J}. \]
The coefficients \( G_{ij}^{IJ} \) completely define the transformation properties of the basis functions under any element \( g \) of SO(4).

3 Dodecahedral space

3.1 Generators

The Poincaré Dodecahedral space has two generators \( g_\pm \), acting as \( x \mapsto g_\pm x \), for any point of \( S^3 \) represented by the unit vector \( x \) of the embedding space \( \mathbb{R}^4 \). In a certain basis of \( \mathbb{R}^4 \), they are expressed (Weeks, privred communication) as the two matrices
\[ g_\pm \equiv \begin{pmatrix} c & 0 & 1/2 & -iC \\ C & c & 0 & -1/2 \\ 1/2 & 0 & c & -C \\ 0 & 1/2 & C & c \end{pmatrix}, \]
with \( C \equiv \sqrt{5 - 1}/4, c \equiv \sqrt{5 + 1}/4 \).

It follows their (left action) complex matrix forms as \( G_\pm \equiv \begin{pmatrix} c + iC & 1/2 & 0 & -iC \\ 1/2 & c & 0 & iC \\ 0 & 1/2 & c & -C \\ 0 & 0 & -1 & c \end{pmatrix}, \)
whose action is defined by
\[ \begin{pmatrix} W \\ iZ \\ iW \end{pmatrix} \mapsto G_\pm \begin{pmatrix} W \\ iZ \\ iW \end{pmatrix}; \ W \equiv x^0 + ix^3, Z \equiv x^1 + ix^2 \in \mathbb{C}. \]

These two operators have the same eigenvalues, namely \( \lambda \equiv e^{i\pi/5} \) and \( \lambda^* \) (star means complex conjugation).

The corresponding (unit) quaternions \( Q_\pm \) act by left action only (as \( g_\pm x \mapsto Q_\pm q_\pm \), where \( q_\pm \) is the unit quaternion associated to the point \( x \) of \( S^3 \)). They are given by \( Q_\pm = c + C j_3 \pm j_1 / 2 \). We note \( \lambda \equiv e^{j_3 \pi/5} \) the quaternionic analog of the eigenvalue \( \lambda \).

Diagonalisation

Calculations are much easier if we adopt a different basis where one generator, say \( g_+ \), transform the complex coordinates \( W \equiv x^0 + ix^3 \) and \( Z \equiv x^1 + ix^2 \), of an arbitrary vector \( x \in \mathbb{R}^4 \), by scalar (although complex) multiplication. To find such a basis is equivalent to diagonalize the complex matrix \( G_+ \), what we have done. We express the results in quaternionic notation. We note \( u \) the quaternion which expresses the change of coordinates: by definition, \( \lambda = u Q_+ u^{-1} \). Calculations give
\[ u = 1/2 + j_2 (\sqrt{C^2 + 1/4} - C). \]
The second generator takes the form
\[ R_- \equiv u Q_- u^{-1} = \cos(\pi/5) - j_3 \sin(\pi/5) / \sqrt{5} + j_1 / 2 \sin(\pi/5) / \sqrt{5}. \] \( (10) \)

Now we continue the calculations with this basis: the two generators are \( \lambda \) (diagonal) and \( R_- \) (formula \( 10 \)).

3.2 Invariance

The diagonal character of the first generator makes the calculations easy: the basis functions \( T_{k,M_1,M_2} \) of B2 appear to be its eigenfunctions. This is analog to the case of a lens space, as seen in [5]. Thus, the functions of \( V^k \)
which are $g_4$ invariant are all combinations of the $T_{k,m_1,m_2}$ which verify
the condition $2m_2 = 0 \mod 10$. This implies $k$ even (as was derived by [11]), so that $m_1$ and $m_2$ are entire and
\[ m_2 = 0 \mod 5, \quad (11) \]
where we underline as $m_2$ a value of $m_2$ verifying this condition. It remains
to express the invariance condition with respect to the second generator.

**Second generator**
We assume $k$ even, and we adopt the basis $B_3$ of $V^k$. Simple calculations lead to estimate
\[ A_{l,j} \equiv \langle \mathcal{R}_- \alpha \cdot n_{l,j} \rangle = \mathcal{F} \rho^l + \mathcal{E} \rho^l, \quad A_{l,j}^{(i)} \equiv \langle \mathcal{R}_- \alpha \cdot n_{l,j} \rangle = \mathcal{F}^* \rho^l - \mathcal{E} \rho^l, \quad (12) \]
with $\mathcal{F} \equiv \cos(\pi/5) + i \sin(\pi/5)/\sqrt{5}, \quad \mathcal{E} \equiv 2 \sin(\pi/5)/\sqrt{5}$.

We insert the relation $\mathcal{U} \equiv \frac{\mathcal{F}}{\mathcal{E}} = \rho^l + J$ in [9]. Taking into account the
properties of the roots of unity, we obtain
\[ G_{l,j}^{(k)}(\mathcal{R}_-) = \frac{(\mathcal{A}^*)^k}{k+1} \delta^{\text{Dirac}} \frac{V^{k+1} - 1}{\rho^{l+j-2i} - 1}, \]
where $\delta^{\text{Dirac}}$ holds for $\delta^{\text{Dirac}}[[I + J - (i + j), \mod (k + 1)]$.

This expression allows to return to the rotation properties of the basis $B_2$ (in fact $T_{k,M_1,M_2}$ rather than $T_{k,M_1,M_2}$), that we express by the development
\[ R_{g_-} T_{k,M_1,M_2} \equiv \sum_{m_1,m_2} \Gamma_{M_1 M_2}^{m_1 m_2} (g) T_{k,m_1,m_2}, \quad (13) \]
The coefficients take the form
\[ \Gamma_{M_1 M_2}^{m_1 m_2} = \sum_{i,j} \sum_{l,j} \rho^l (M_1 + M_2 + j (M_1 - M_2) - i (m_1 + m_2) - j (m_1 - m_2)) \frac{G_{l,j}^{(k)}}{(k+1)^2} \]
\[ = \sum_{i,j} \rho^{(l-j) (M_2 + (I+j) - (m_1 + m_2 + M_1))} \frac{V^{k+1} - 1}{(k+1)^3} \sum_{j} \rho_{m_2}^{j} \mathcal{A}^k \quad (14) \]
Direct calculations allow to evaluate the last sum as
\[ (k+1) \frac{V^{l+j} (k+2-m_2)}{V^{k+1} - 1}, \]
leading finally to
\[ \Gamma_{M_1 M_2}^{m_1 m_2} = \sum_{i,j} \rho^{(l-j) (M_2 + (I+j) - (M_1 - m_1 + k/2))} \frac{V^{k+2-m_2} (\mathcal{A}^*)^k}{(k+1)^2}. \quad (15) \]

**Still a new basis**
This suggests us the introduction of still a new basis for $V^k$ (only for intermediary calculations):
\[ \tilde{T}_{\alpha M_2} \equiv \sum_{M_1} \rho^{-\alpha M_1} T_{k,M_1,M_2}, \quad \alpha = 0..k, \quad M_2 = -k/2..k/2, \quad (16) \]
with inverse formula $\tilde{T}_{k,M_1,M_2} = \frac{1}{k+1} \sum_{\alpha} \rho^\alpha M_1 \tilde{T}_{\alpha M_2}$. Its rotation properties are deduced as:
\[ R_{g_-} : \tilde{T}_{\alpha M_2} \rightarrow \frac{\rho^{(k/2-M_2)}}{k+1} \sum_{l} \rho^{2l M_2} (\mathcal{A}^*)^k Y^{l/2} \sum_{m_2} Y^{-m_2} \tilde{T}_{\alpha m_2}, \quad (17) \]
5
where the quantities $\mathcal{A}'$ and $\mathcal{V}$ have to be evaluated for the value $J = \alpha - I \mod (k + 1)$.

A first important result appears: the value of $\alpha$, for this basis, is preserved by $R_{g_{-}}$. Using the formula just above, it is straightforward to check that this implies, similarly, that $R_{g_{-}}$ preserves $m_{1}$ in the basis B2. This result will allow considerable simplification. To continue, we report in the formula above the values of $\mathcal{V} = \mathcal{A}/\mathcal{B}$ and $\mathcal{A}'$ (equ.12). This gives

$$R_{g_{-}} : \tilde{T}_{\alpha M_{2}} \mapsto \sum_{m_{2}} \gamma_{\alpha M_{2}}^{m_{2}} \tilde{T}_{\alpha m_{2}},$$

with

$$\gamma_{\alpha M_{2}}^{m_{2}} = \sum_{l} \rho^{l(1-\alpha/2)} \frac{(2M_{2}+k)}{k+1} \left[ F + E \rho^{\alpha-2l} \right]^{k/2-m_{2}} \left[ F \rho^{\alpha-2l} - E \right]^{k/2+m_{2}},$$

(18)

where $k/2 - m_{2}$ and $k/2 + m_{2}$ take their value between 0 and $k$.

### 3.3 Splitting the eigenspace

From this formula, it appears the second important result that

$$\gamma_{\alpha M_{2}}^{m_{2}} = \gamma_{M_{2}}^{m_{2}} = \sum_{l} \rho^{l(2M_{2}+k)} \frac{k+1}{k} \left[ F + E \rho^{-2l} \right]^{k/2-m_{2}} \left[ F \rho^{-2l} - E \right]^{k/2+m_{2}},$$

(19)

does not depend on $\alpha$. This allows to write the rotation formula

$$R_{g_{-}} : \tilde{T}_{\alpha M_{2}} \mapsto \sum_{m_{2}} \gamma_{M_{2}}^{m_{2}} \tilde{T}_{\alpha m_{2}}.$$

It is easily checked that this absence of dependence on $\alpha$ holds also for the $T_{k,m_{1},M_{2}}$. Finally, we are led to the transformation rule for the basis B2:

$$R_{g_{-}} : \tilde{T}_{k,M_{1},M_{2}} \mapsto \sum_{m_{2}} \gamma_{M_{2}}^{m_{2}} \tilde{T}_{k,m_{1},m_{2}},$$

(20)

with $\gamma_{M_{2}}^{m_{2}}$ given by (10).

**Summary**

We summarize the results obtained, expressed in the basis $(T_{k,m_{1},m_{2}})$:

- The $g_{+}$ invariant functions of $V^{k}$ are all combinations of the $T_{k,m_{1},m_{2}}$ which verify the condition $m_{2} = 0 \mod 5$.
- The second generator ($g_{-}$) preserves the value of $m_{1}$. This allows to consider $V^{k}$ as the direct sum of $k + 1$ sub vector-spaces $V^{k,m_{1}}$ of dimension $k + 1$, each preserved by $g_{-}$. The search for invariant functions can thus be made independently for each $V^{k,m_{1}}$.
- The rotation coefficients in $V^{k,m_{1}}$ do not depend on $m_{1}$: they are identical in each of the $V^{k,m_{1}}$. It is thus sufficient to search invariant functions in one of them (say, for $m_{1} = 0$). The other functions are given by the $k$ more copies obtained by replacing the value $m_{1} = 0$ by any of the $k$ remaining values. It results that the dimension of the space of eigen functions of the dodecahedral space is an entire multiple of $k + 1$, as was already indicated by [1].
3.4 Eigenvectors

The search of the modes of the dodecahedral space is reduced (for each even value of \( k \)) to that of the invariant eigenfunctions of \( S^3 \) in \( V^{k,m_1=0} \). Invariance with respect to \( g_+ \) implies that such a function may decomposed in the restricted basis formed by the \( T_{M_1=0,M_2} \) (we recall that underlining means \( M_2 = 0 \mod 5 \)). In this restricted basis, the rotations properties (under \( g_- \)) are expressed by the \( k_5 \ast k_5 \) matrix \( G_k \) of coefficients \( \gamma_{M_2,M_1} \) given by (19) (we define \( k_5 \equiv 1 + 2[k/10] \)). Thus, invariant functions are the eigenvector(s) of the matrix \( G_k \), corresponding to the eigenvalue 1, when they exist. We note such an eigenvector by its \( k_5 \) components \((f_{M_2})\). (Note that, when the multiplicity of the eigenvalue 1 is larger than 1, there can be several such vectors; this occurs for instance for the value \( k = 60 \), see the table.) Finally,

\[
\text{the eigenfunctions of the dodecahedral space corresponding to the eigenvalue } -k (k+2)(k \text{ even}) \text{ are all combinations of the functions }
\sum_{M_2} f_{M_2} T_{k,M_1-M_2},
\text{ where } (f_{M_2}) \text{ is an eigenvector of the matrix } G_k \text{ with eigenvalue 1, for all the entire values of } M_1 \text{ between 0 and } k.
\]

For a value of \( k \), the calculation of the \((k_5)^2\) coefficients of \( G_k \) is immediate. The search for eigenvectors also runs very easily (on MAPLE for instance). Results confirms the values of the eigenvalues and their multiplicities given by [4]. The table gives the numerical values of the first eigenvectors of the dodecahedron. MAPLE codes to calculate the modes may be obtained by request at marclr@cea.fr.
Table 1: For each value of $k$ (from 12 to 44; eigenmodes exist only for the values given), the table gives the $k_5$ components ($k_5 \equiv 1 + 2[k/10]$) of the vector $f_{M_2}$. The eigenmodes of $D$, corresponding to $\lambda_k = -k (k + 2)$, are given by all combinations $\sum_{M_2} f_{M_2} T_{k,M_1,M_2}$, where $M_2$ varies from $-k/2$ and $k/2$ and verifies [11], and $M_1$ takes all the entire values between $-k/2$ and $k/2$.

$k=12$: .70352647068144845281, -.10050378152592120757 i, .70352647068144845294

$k= 20$: .70702906968084266661, .010397486318835921579 i, -.0018904520579701675214, .010397486318835921658 i, .70702906968084267112

$k=24$: .70697192267503596680, -.012403016187281332779 i, .010397486318835921658 i, .70702906968084267112

$k=30$: .70709634963126361072+.28263881117495907229 e-2 i, .1035288865625736179 e-4+.2590096518795837646 e-2 i, .235436319189053680400 e-3+.9411715328415742893 e-6 i, 0, .235436319189053680400 e-3+.9411715328413699545 e-6 i, .10352888656225582837 e-4+.2590096518795837623 e-2 i, .70709634963126361072+.28263881117495713938 e-2 i

$k=32$: .70708591611133287319, .54182828820791792693 e-2 i, .35336627491820729542 e-3, .221290156349361997 e-3 i, .35336627491820731478 e-3, .54182828820791791292 e-2 i, .70708591611133284832

$k=36$: .70708473425088585344, -.55056653389946616956 e-2 i, -.78652361985638024036 e-3, .70373165987149811020 e-3 i, -.78652361985638024036 e-3, -.55056653389946616956 e-2 i, .70708473425088584255

$k=40$: .707106610000066274387, .49002537075583660713 e-3 i, -.44188504840885072726 e-4, .39518175060953083014 e-5 i, .71851227383553307597 e-6, .39518175060953083356 e-5 i, .44188504840885377434 e-4, .49002537075583232202 e-3 i, .707106610000067363741

$k=42$: .70710034184580288923, -.30174979026251895955 e-2 i, .20583312770835158828e-4, .1972220851380132964 e-4 i, 0, .1972220851380132964 e-4 i, -.29583312770835191330 e-4, .30174979026251928239 e-2 i, -.70710034184580303513

$k= 44$: .70709762382971759036, .39558850229318122280 e-2 i, .13964602030803151449 e-3, -.15649985034520771577 e-4 i, .2187857439065900270 e-4 i, -.15649985034520777432 e-4 i, .13964602030803151858 e-3, .39558850229318113024 e-2 i, .7070976238297176255
Table 1, continued: idem, for values of $k$ from 48 to 56

| $k$     | Real Part | Imaginary Part |
|---------|-----------|----------------|
| 48      | .70709753814957595208 | -.36096867863497570254 e-2 i |
|         | -2.0595810492863306199 e-3 | .6189233738632857047 e-4 i |
|         | .56297546508021977011 e-4 | .6189233738632854996 e-4 i |
| 50      | .70710677432308054872 | -.98118423819072486562 e-4 i |
|         | .88473375505798110701 e-5 | .79771076275668425524 e-6 i |
|         | -.72519160250664405750 e-7 | .13686767861056995799 e-19 i |
|         | .72519160250661739763 e-7 | -.79771076275733196796 e-6 i |
| 52      | .70710632382180878363 | .80371462823607746218 e-3 i |
|         | -.29191620936716760517e-4 | .12714219501894409823 e-5 i |
|         | -.45691726334935131233e-6 | -.1475757620755658245e-6 i |
|         | -.45691726334935055995e-6 | .12714219501895343589 e-5 i |
| 54      | .70709995013538603217 | -.31081316489467430301 e-2 i |
|         | -.47140520459254381578 e-5 | -.33908093663673236269 e-5 i |
|         | -.15713506819751032060 e-5 | 0 |
|         | .15713506819751014572 e-5 | .33908093663673236906 e-5 i |
|         | .47140520459254356244 e-5 | .31081316489467431476 e-2 i |
| 56      | .707101658218236325 | .26907309019741289751 e-2 i |
|         | .69176592337984333471 e-4 | -.59344090988012596170 e-5 i |
|         | .63998529496876216108 e-6 | .20222732830614883787 e-5 i |
|         | .63998529496876549920 e-6 | -.59344090988012761721 e-5 i |
|         | .6917659233798705779 e-4 | .26907309019741259726 e-2 i |

| $k$     | Real Part | Imaginary Part |
|---------|-----------|----------------|
| 56      | .707101658218236325 | .26907309019741289751 e-2 i |
|         | .69176592337984333471 e-4 | -.59344090988012596170 e-5 i |
|         | .63998529496876216108 e-6 | .20222732830614883787 e-5 i |
|         | .63998529496876549920 e-6 | -.59344090988012761721 e-5 i |
|         | .6917659233798705779 e-4 | .26907309019741259726 e-2 i |
|         | .707101658218236410534 | 9 |
Table 1, continued: idem, for values of $k$ from 60 to 62

| $k$  | Values                                                                 |
|------|------------------------------------------------------------------------|
| 60   | 6.9394960301828269967-.5086496302764001912 $i$                         |
|      | , .29538129642425900098-.64161677487791105693 $i$                   |
|      | , -.25742682470776540173e-2-.1168376322439293660 $e-2$ $i$          |
|      | , -.3634253284660713864e-4+.77853455569667120805 $e-4$ $i$          |
|      | , .1105720865150194216e-4+.499659178915788800096 $e-5$ $i$          |
|      | , .2149925071811890501e-5-.46857540636509975228 $e-5$ $i$          |
|      | , -.39964493552999000974e-5-.182680033433521902658 $e-5$ $i$       |
|      | , .21499250718136508514e-5-.685754063652609287 $e-5$ $i$          |
|      | , .1105720861498572522e-4+.499659178915788800006 $e-5$ $i$          |
|      | , -.3634253284620033877e-4+.778534555696838907558 $e-4$ $i$        |
|      | , -.2574268247073301674e-2-.116837632246853776 $e-2$ $i$          |
|      | , .29358129642426822804-.64161677487790595997 $i$                 |
|      | , .69349460301835349645-.5086496298031253450 $i$              |
|      | , .80848705655829822065e-2+.70706052583865137173 $i$             |
|      | , .26862969516112067927e-3+.6528267307451706578 $e-4$ $i$        |
|      | , .23918191153754173532e-6-.99923360845873892481 $e-6$ $i$        |
|      | , .18133448495383542660e-6+.98554877105180951336 $e-8$ $i$        |
|      | , -.9467201255100653876e-9+.110228784146687929246 $e-7$ $i$       |
|      | , -.26567291467002942075e-8+.90083380949540372946 $e-9$ $i$       |
|      | , .4022454688128971313e-9+.94397507086428917031 $e-9$ $i$        |
|      | , -.2656729146691520685e-8+.90083380949346250026 $e-9$ $i$        |
|      | , -.9467201255285644106e-9+.1102287144643159845 $e-7$ $i$        |
|      | , .181334484954122990052e-6-.98554877083976930526 $e-8$ $i$       |
|      | , .23918191155535933834e-6-.99923368050776437049 $e-6$ $i$       |
|      | , -.26862969516570937280e-3+.6528267306944956583 $e-4$ $i$       |
|      | , .808487055988914170656e-2+.70706052584512272298 $i$            |
| 62   | .70710674764980394726-.2175110731334935062 $e-3$ $i$               |
|      | , .10807153948145142931 $e-4$.18045880549024964468 $e-6$ $i$       |
|      | , .55301892005028276440 $e-7$.11642503580014562160 $e-7$ $i$       |
|      | , -.11642503580010046955 $e-7$ $i$.5530189200504956638 $e-7$ $i$ |
|      | , -.18045880549866284252 $e-6$ $i$.10807153948148682328 $e-4$ $i$|
|      | , .21751107313369991548 $e-3$ $i$.70710674765011395485$]           |

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