Asymptotic Regional Gradient Reduced-Order Observer

Raheam A. Al-Saphory1*, Naseif J. Al-Jawari2 and Asmaa N. Al-Janabi3
1Department of Mathematics, College of Education for Pure Science, Tikrit University, Salahaddin, IRAQ.
2,3Department of Mathematics, College of Science, Al-Mustansiriyah University, Baghdad, IRAQ.

*E-mail: saphory@tu.edu.iq

Abstract. The aim of this paper is to extend the notion of general asymptotic regional gradient observer (GARG-observer) to the reduced order case. More precisely, we study and discuss the existing of this approach in a sub-region of the considered domain. Thus, we show that the approach is enables to estimate the unknown part of the state gradient when the output function gives part of information about the region state in ω. The characterization of this notion depend on regional gradient strategic sensors (RGS-sensor) concept in order that asymptotic regional gradient reduced-order observability (ARGRO-observability) to be achieved and analyzed. An application presented to various situation cases of strategic sensors.

Key words: RGS-sensors, ARG-detectability, ARGRO-observers, Exchange system.

1. Introduction
The basic concept of an observer theory was introduced by Luenberger in [1-5]. In 1975, the observer approach is extended to a linear system of infinite dimensional case described by SCS-group operators [6]. The observation concepts given a main purpose for introducing the observer notions [7-8]. Therefore, the study of observer theory depending to the notions of sensors characterizations as in [9-11]. Thus the regional asymptotic reconstruction has been presented and explored in [12-14]. So the motivation behind this study is related to many real world problem when one cannot estimate the state of the system in whole the domain Ω, but only in a region ω of this domain as in “Figure 1” [15-16].

Figure 1. Real model (locate in-out-vents, sensors, workspace)

The objective of present research is to extend the work in [17-18] to regional gradient case for observer type reduced. The rest of paper is ordered as follows: In section2 some preliminaries and formulation of
the considered problem are stated. Section 3, recalls the definitions of ARG-stability and ARG-detectability. Section 4, introduces an approach how to build an ARGRO-observer and gives a sufficient condition of such estimator by using ARG-detectability and RGS-sensors. Numerous applications about different sensors locations has been given in the last section.

2. Some Preliminaries and Formulation of the Considered Problem

Footnotes

Let \( \mathcal{Q} = U \times ]0, \infty[ \), with smooth boundary \( \partial U \) and \( \omega \) be subregion of \( \mathcal{U} \). We denoted \( \mathcal{Q} = \mathcal{M} \times [0, \infty[ \), \( \Omega = \partial \mathcal{M} \times [0, \infty[ \). Considering systems which are designated by the following

\[
\begin{align*}
\frac{\partial x}{\partial t} (\zeta, t) & = A x(\zeta, t) + B u(t) & \mathcal{Q} \\
x(\zeta, 0) & = x(\zeta) & \mathcal{U} \\
x(\eta, t) & = 0 & \Sigma
\end{align*}
\]

Augmented by the output function

\[
\mathcal{Y}(\zeta, t) = C x(\zeta, t)
\]

where \( A \) is a second order linear differential operator, which is a generator of a \( \mathcal{S}_{\mathcal{C}_{\mathcal{S}}} \)-group on the Hilbert space \( \mathcal{X} \) [4-5]. So \( B \in \mathcal{L}(\mathbb{R}^p \times \mathbb{R}^q, \mathcal{X}) \) and \( C \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \), may be bounded or unbounded operators [8]. Thus \( \mathcal{X}, \mathcal{U} \) and \( \mathcal{Y} \) be the Sobolev spaces of type separable where \( \mathcal{X} \) is a state space, \( \mathcal{U} = \mathcal{L}^2(0, T, \mathbb{R}^p) \) is the input space and \( \mathcal{Y} = \mathcal{L}^2(0, T, \mathbb{R}^q) \) is the output space. The mathematical model (1)-(2) in Fig.2 is more general than the real model in “Figure 1”

\[\text{Figure 2. General mathematical model.}\]

The solution of the above system is specified by [6]

\[
x(\zeta, t) = S_{\mathcal{A}}(t) x_0(\zeta) + \int_0^t S_{\mathcal{A}}(t - \tau) B u(\tau) d\tau
\]

Then the related output is can be achieved by the following form in several cases [10-11]

\[
\mathcal{Y}(\zeta, t) = C x(\zeta, t)
\]

Define the operator

\[K: x \in \mathcal{X} \rightarrow K x = C S_{\mathcal{A}}(\cdot) x \in \mathcal{Y}\]

with adjoint \( K^*: \mathcal{Y} \rightarrow \mathcal{X} \) obtained by

\[
K^* \mathcal{Y}^* = \int_0^t S_{\mathcal{A}}^*(\ell) C^* \mathcal{Y}^*(\ell) d\ell, \quad \ell \in [0, t]
\]

Contemplate the following application

\[\nabla: H^1(\mathcal{U}) \rightarrow (H^1(\mathcal{U}))^n\]
\[ \mathbf{x} \rightarrow \nabla \mathbf{x} = \left( \frac{\partial \mathbf{x}}{\partial \xi_1}, \ldots, \frac{\partial \mathbf{x}}{\partial \xi_n} \right) \]

and its adjoint denotes by \( \nabla^* \) given by

\[ \nabla^*: (H^1(\mathcal{U}))^n \rightarrow H^1(\mathcal{U}) \]

\[ x \rightarrow \nabla^* x = \mathbf{a} \]

For \( \omega \subset \mathcal{U} \) deliberate the following application

\[ \chi_\omega: (H^1(\mathcal{U}))^n \rightarrow (H^1(\omega))^n \]

\[ x \rightarrow \chi_\omega x = x|_\omega \]

with \( \chi^*_\omega \) the adjoint [4-5]. Let \( H = \chi_\omega \nabla K^* \) from \( \mathcal{O} \) into \( (H^1(\omega))^n \).

3. \textit{ARG-Detectability and Sensor}

Now we discuss and analysis the link between the concept of \textit{ARG}-Detectability and a \textit{RGS}-sensor structure in order to build an \textit{ARG}-observer for the gradient of the system state in reduced order case [13,19-20].

3.1. Definitions and characterizations

This subsection presents some definitions and characterization are needed in our research [11].

\( \diamond \) System (1)-(2) is so-called exactly regionally gradient observable \((\text{ERG-observable})\) if

\[ \text{Im } H = \text{Im } \chi_\omega \nabla K^* = (H^1(\omega))^n \]

\( \diamond \) System (1)-(2) is so-called weakly regionally gradient observable \((\text{WRG-observable})\) if

\[ \text{Im } H = \text{Im } \chi_\omega \nabla K^* = (H^1(\omega))^n \]

\( \diamond \) So system (1)-(2) may be \textit{WRG}-observable if

\[ \ker H^* = \ker K \nabla^* \chi^*_\omega = \{0\} \ [11] \]

\( \diamond \) A sensor \((D, f)\) is \textit{RGS}-sensor if the related system is \textit{WRG}-observable .

\( \diamond \) System (1) is \textit{ARG}-stable if \( \mathcal{A} \) produces a semigroup which is \textit{ARG}-stable. Further, the system (1) is \textit{ARG}-stable, if and only if there exists \( M_{\text{ARG}}, \ a_{\text{ARG}} \geq 0 \) such that,

\[ \| \chi_\omega \nabla S_{\mathcal{A}}(\cdot) \|_{(H^1(\omega))^n} \leq M_{\text{ARG}} e^{-a_{\text{ARG}} \mathcal{E}}. \forall \mathcal{E} \geq 0. \] (5)

\( \diamond \) If the semigroup \((S_{\mathcal{A}}(\mathcal{E}))_{\mathcal{E} \geq 0}\) is \textit{ARG}-stable, then \( \forall \mathbf{x}_0 \in H^1(\mathcal{U}) \), we can get

\[ \lim_{\mathcal{E} \to \infty} \| \nabla \mathbf{x}(\cdot, \mathcal{E}) \|_{(H^1(\omega))^n} = \lim_{\mathcal{E} \to \infty} \| \chi_\omega \nabla S_{\mathcal{A}}(\cdot) \mathbf{x}_0 \|_{(H^1(\omega))^n} = 0 \] (6)
\* Systems (1)-(2) is \textit{ARG}-detectable if \( H_{\text{ARG}}: R^q \to (H^1(\omega))^n \) represents an operator such that

\((A - H_{\text{ARG}} \mathcal{C})\) products a \(SCS\)-group \((S_{H_{\text{ARG}}(\epsilon)})_{\epsilon>0}\) it is \textit{ARG}-stable.

\* Then equation (6) implies \( \lim_{t \to \infty} \| \nabla x(\cdot, t) \|_{(H^1(\omega))^n} = 0. \)

\textbf{Theorem 3.1.} Assume \((D_i, f_i)_{1 \leq i \leq q}\) the measurement sensors. Then system (1)-(2) is \textit{ARG}-detectable if and only if:

(I) \( q \geq r \), \( \forall m, m = 1, ..., J \) with

\[ G_m = (G_m)_{ij} = \begin{cases} \psi_{mj}(b_i), f_i(\cdot) \geq L^2(D_i) & \text{for zone sensors} \\ \psi_{mj}(b_i) & \text{for pointwise sensors} \\ < \frac{\partial \psi_{mj}}{\partial v}, f_i(\cdot) \geq L^2(\Gamma_i) & \text{for boundary zone sensors} \end{cases} \]

where \( \sup r_m = r < \infty \) and \( j = 1, ..., r_m, \psi_{mj} \) is eigenfunctions in \((H^1(U))^n\) orthonormal in

\((H^1(\omega))^n\), and \( \lambda_m \) are the eigenvalues with multiplicity \( r_m \).

\textbf{Proof:} The proof is the same of \cite{16} with some modifications.

\textbf{Remark 3.2.} The important purpose of asymptotic \textit{ARG}-detectability that is related to the possibility for defining an \textit{ARG}-estimator of the system state from the knowledge of the output and input function.

\textbf{4. Reconstruction of \textit{ARGRO}-Observer}

In this section we extend the approach of general \textit{ARG}-observer studied in \cite{4, 18} to the reduced type. Now deliberate the following systems

\[ \begin{cases} \frac{\partial x}{\partial t}(\varsigma, t) = \mathcal{A}x(\varsigma, t) + \mathcal{B} u(t) & \mathcal{Q} \\ x(\varsigma, 0) = x_0(\varsigma) & \mathcal{U} \\ x(\eta, t) = 0 & \Sigma \end{cases} \] (7)

where the output function

\[ y(t) = \mathcal{C}x(\cdot, t) \] (8)

In this case, by using regional reduced order forms as in \cite{5}, we have
\[
\begin{align*}
\frac{dx_1}{dt}(\zeta, t) &= A_{11} x_1(\zeta, t) + A_{12} x_2(\zeta, t) + B_1 u(t) & Q \\
x_1(\zeta, 0) &= x_{01}(\zeta) & \mathcal{U} \\
x_1(\eta, t) &= 0 & \Sigma \\
\end{align*}
\] 

and

\[
\begin{align*}
\frac{dx_2}{dt}(\zeta, t) &= A_{21} x_1(\zeta, t) + A_{22} x_2(\zeta, t) + B_2 u(t) & Q \\
x_2(\zeta, 0) &= x_{02}(\zeta) & \mathcal{U} \\
x_2(\eta, t) &= 0 & \Sigma \\
\end{align*}
\]

Provided with the information function

\[
y(., t) = C x_1(\zeta, t).
\]

The aim of this section is concentrated to construct a dynamic estimator for subsystem (10) [5] which formulated via the form

\[
\begin{align*}
\frac{\partial a}{\partial t}(\zeta, t) &= A_{22} a(\zeta, t) + [B_2 u(t) + A_{21} y(\zeta, t)] & Q \\
a(\zeta, 0) &= a(\zeta) & \mathcal{U} \\
a(\eta, t) &= 0 & \Sigma \\
\end{align*}
\]

together with the output function

\[
\tilde{y}(., t) = A_{12} a(\zeta, t).
\]

So that the state \( a \) in (12) represents \( x_2 \) in (10).

In this case, the system (12)-(13) is ARG-detectable, there exists an operator \( H_{ARG} \in \mathcal{L}(\mathbb{X}|\Omega|\omega, \mathbb{X}|\omega) \) and \( (A_{22} - H_{ARG}A_{12}) \) generates an ARG-stable semi-group \( (S_{A_{22} - H_{ARG}A_{12}}(t))_{t \geq 0} \) on the space \( \mathbb{X}_2 = \mathbb{X}|\omega \). So that we can deduce that

\[
\exists M_{ARG}, \alpha_{ARG} > 0 \implies \| x_{ARG}(S_{A_{22} - H_{ARG}A_{12}}(\cdot)) \|_{(H^1(\omega))^n} \leq M_{ARG} e^{-\alpha_{ARG}t}.
\]

Therefore we can define an ARGRO-observer for (12)-(13) depending on [5, 9]

\[
\begin{align*}
\frac{\partial \hat{z}}{\partial t}(\zeta, t) &= A_{22} \hat{z}(\zeta, t) + [B_2 u(t) + A_{21} \tilde{y}(\zeta, t)] + H_{ARG}(\tilde{y}(., t) - A_{12} \hat{z}(\zeta, t)) & Q \\
\hat{z}(\zeta, 0) &= \hat{z}_0(\zeta) & \mathcal{U} \\
\hat{z}(\eta, t) &= 0 & \Sigma \\
\end{align*}
\]

The main result which presents the characterization of ARGRO-observer notion may be given in the next impotent result.

**Theorem 4.1.** Let (12)-(13) be an ARG-detectable system, therefore,

\[
\lim_{t \to \omega} \| \omega(\zeta, t) + H_{ARG} \tilde{y}(\zeta, t) - x_2(\zeta, t) \|_{(H^1(\omega))^n} = 0, \quad \forall \zeta \in \omega
\]
and the system of dynamic type (14) is \textit{ARGRO} -observer for the equations (12)-(13) where \(Y(\zeta, t)\) is the output given by (11) and \(\omega(\zeta, t)\) is the resolution of the following

\[
\begin{cases}
\frac{\partial \omega}{\partial t}(\zeta, t) = (A_{22} - H_{\text{ARG}} A_{12}) \omega(\zeta, t) + [A_{22} H_{\text{ARG}} - H_{\text{ARG}} A_{12}] u(t) \\
- H_{\text{ARG}} A_{11} + A_{21}] Y(\zeta, t) + [B_2 - H_{\text{ARG}} B_1]u(t) \\
\omega(\zeta, 0) = \omega_0(\zeta) \\
\end{cases}
\]

Proof. For the moment the resolution of the (14) is stated by

\[
\hat{z}(\zeta, t) = (S(A_{22} - H_{\text{ARG}} A_{12}) (t) \hat{z}_0(\zeta) + \int_0^t S(A_{22} - H_{\text{ARG}} A_{12})(t - \tau) d\tau \]

\[
[B_2 u(\tau) + A_{21} Y(\zeta, t) + H_{\text{ARG}} \hat{y}(., t)] dT
\]

From the use of equations (10) and (9), we have

\[
\hat{y}(., t) = A_{12} x(\zeta, t) = \frac{\partial \chi}{\partial t}(\zeta, t) - A_{11} \chi(\zeta, t) + B_1 u(t)
\]

Inserting (16) into (15), we obtain

\[
\begin{align*}
\hat{z}(\zeta, t) &= (S(A_{22} - H_{\text{ARG}} A_{12}) (t) \hat{z}_0(\zeta) + \int_0^t S(A_{22} - H_{\text{ARG}} A_{12})(t - \tau) d\tau \]

\[
H_{\text{ARG}} \frac{\partial \chi}{\partial t}(\zeta, \tau) d\tau +
\int_0^t S(A_{22} - H_{\text{ARG}} A_{12})(t - \tau) [B_2 u(\tau) + A_{21} Y(\zeta, t) - H_{\text{ARG}} A_{11} \chi(., \tau) -
\]

\[
- H_{\text{ARG}} B_1 u(\tau)]dT
\]

and we can get

\[
\int_0^t S(A_{22} - H_{\text{ARG}} A_{12})(t - \tau) H_{\text{ARG}} \frac{\partial \chi}{\partial t}(\zeta, \tau) d\tau = H_{\text{ARG}} x(\zeta, t) -
\]

\[
S(A_{22} - H_{\text{ARG}} A_{12})(t) H_{\text{ARG}} x(\zeta, \tau) + (A_{22} - H_{\text{ARG}} A_{21}) \int_0^t S(A_{22} - H_{\text{ARG}} A_{12})(t - \tau) H_{\text{ARG}} x(\zeta, \tau) d\tau
\]

Consequently the equation (18) becomes

\[
\int_0^t S(A_{22} - H_{\text{ARG}} A_{12})(t - \tau) H_{\text{ARG}} \frac{\partial \chi}{\partial t}(\zeta, \tau) d\tau = H_{\text{ARG}} x(\zeta, \tau) -
\]

\[
S(A_{22} - H_{\text{ARG}} A_{12})(t) H_{\text{ARG}} x(\zeta, \tau) + \int_0^t S(A_{22} - H_{\text{ARG}} A_{12})(t - \tau) (A_{22} - H_{\text{ARG}} A_{21}) H_{\text{ARG}} x(\zeta, \tau) d\tau
\]

(19) Substituting (19) into (17), we have

\[
\hat{z}(\zeta, t) = (S(A_{22} - H_{\text{ARG}} A_{12}) (t) \hat{z}_0(\zeta) - S A_{HARGA12}(t) H_{ARG} x(\zeta, \tau) + H_{ARG} x(\zeta, \tau) +
\]

\[
\int_0^t S(A_{22} - H_{\text{ARG}} A_{12})(t - \tau) [A_{22} H_{\text{ARG}} - H_{\text{ARG}} A_{21}] H_{\text{ARG}} - H_{\text{ARG}} A_{11} + A_{21}] x(\zeta, \tau) d\tau +
\]

\[
\int_0^t S(A_{22} - H_{\text{ARG}} A_{12})(t - \tau) [B_2 - H_{\text{ARG}} B_1] u(\tau) d\tau
\]

(20)
Then we have the system (14) is asymptotic

Thus, we obtain

and therefore

Now, if the sensors is a RG-strategic then from theorem 3.1 we have the following conditions holds:

(I) \( q \geq r_2 \).

(II) \( \text{rank } G_{2m} = r_{2m}, \forall m, m = 1, \ldots, J \).

So we can deduced that the system (12)-(13) is ARG-detectable, i.e., there exists an operator \( H_{\text{ARG}} \in \mathcal{L}(0, (H^1(\omega))^n) \) such that \( (\mathcal{A}_{22} - H_{\text{ARG}} \mathcal{A}_{12}) \) generates an ARG-stable semi-group \( (S_{\mathcal{A}_{22} - H_{\text{ARG}} \mathcal{A}_{12}}(t))_{t \geq 0} \) on a Hilbert space \( X_2 \).

\[ \exists M_{\text{ARG}}, \alpha_{\text{ARG}} > 0 \text{ such that } \| \chi_0 \nabla (S_{\mathcal{A}_{22} - H_{\text{ARG}} \mathcal{A}_{12}}(t)) \|_{(H^1(\omega))^n} \leq M_{\text{ARG}} e^{-\alpha_{\text{ARG}} t} \]

Thus, we obtain

\[ \|[\hat{z}(\zeta, t)]_t - x_2(\zeta, t)\|_{(H^1(\omega))^n} \leq \| \chi_0 \nabla (S_{\mathcal{A}_{22} - H_{\text{ARG}} \mathcal{A}_{12}}(t)) \|_{(H^1(\omega))^n} \|[\hat{z}(\zeta, 0)]_t - x_2(\zeta, 0)\|_{(H^1(\omega))^n} \]

\[ \leq M_{\text{ARG}} e^{-\alpha_{\text{ARG}} t} \|[\hat{z}(\zeta, 0)]_t - x_2(\zeta, 0)\|_{(H^1(\omega))^n} \rightarrow 0 \text{ as } t \rightarrow \infty \]

Then we have the system (14) is an asymptotic ARGRO-observer for the systems (12)-(13). \( \square \)
5. Some Exchange Systems Application to \textit{ARGRO-Observer}

In the subsequent we reflect two-phase systems of type exchange

\[
\frac{\partial x_1}{\partial t}(\zeta_1, \zeta_2, t) = \frac{\delta x_1}{\delta \zeta_2}(\zeta_1, \zeta_2, t) + (x_1(\zeta_1, \zeta_2, t) - x_2(\zeta_1, \zeta_2, t))
\]

\[
\frac{\partial x_2}{\partial t}(\zeta_1, \zeta_2, t) = \frac{\delta^2 x_2}{\delta \zeta_2^2}(\zeta_1, \zeta_2, t) + (x_2(\zeta_1, \zeta_2, t) - x_1(\zeta_1, \zeta_2, t))
\]

\[
x_1(\zeta_1, \zeta_2, 0) = x_{01}(\zeta_1, \zeta_2), \ x_2(\zeta_1, \zeta_2, 0) = x_{02}(\zeta_1, \zeta_2)
\]

\[
x_1(\eta_1, \eta_2, t) = 0, \ x_2(\eta_1, \eta_2, t) = 0
\]

wherever \( \mathcal{U} = \{0, 1\} \times \{0, 1\} \) through the region \( \omega = \alpha_1, \beta_1[x] \alpha_2, \beta_2 \subset \mathcal{U} \). So the measurement information \( (2) \) is known via the form

\[
\mathcal{Y}(t) = Cx(\zeta, t) = \int_D x_1(\zeta, t)f(\zeta) \, d\zeta
\]

Now, the issue is how to calculate \( x_2(\zeta, t) \). For this step, reflect the following formula

\[
\frac{\partial x}{\partial t} = \left[ \begin{array}{cc} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{array} \right] \left[ \begin{array}{cc} x_1 \\ x_2 \end{array} \right]
\]

somewhere

\[
\mathcal{A}_{11} = \frac{\partial^2 x_1}{\partial \zeta_2^2}(\zeta_1, \zeta_2, t) + 1, \ \mathcal{A}_{22} = \frac{\partial^2 x_2}{\partial \zeta_2^2}(\zeta_1, \zeta_2, t) + 1 \quad \text{and} \quad \mathcal{A}_{12} = \mathcal{A}_{21} = -1.
\]

By the use of the previous theorem \( 4.1 \), \textit{ARGRO-}estimator can build for \( (22) \) if the measuring sensor \( (D, f) \) remains \( RG \)-strategic.

\[
\frac{\partial^2 x_1}{\partial \zeta_2^2}(\zeta_1, \zeta_2, t) = \frac{\delta^2 x_1}{\delta \zeta_2^2}(\zeta_1, \zeta_2, t) + (x_1(\zeta_1, \zeta_2, t)
\]

\[
-x_2(\zeta_1, \zeta_2, t))
\]

\[
x_1(\zeta_1, \zeta_2, 0) = x_{01}(\zeta_1, \zeta_2)
\]

\[
x_2(\eta_1, \eta_2, t) = 0
\]

The measuring information is achieved by

\[
\mathcal{Y}(t) = \int_D x_1(\zeta, t) f(\zeta) \, d\zeta 
eq 0
\]

under the detectability property, we possess the important formula

\[
\lim_{n \to \infty} \| (\mathcal{W}(\cdot, t) + H_{ARG} x(\cdot, t)) - x(\cdot, t) \|_{(H^1(\omega))^n} = 0
\]

where

\[
\frac{\partial \mathcal{W}}{\partial t}(\zeta_1, \zeta_2, t) = \frac{\delta^2 \mathcal{W}}{\delta \zeta_2^2}(\zeta_1, \zeta_2, t) + ((1 + H_{ARG}) \mathcal{W}(\zeta_1, \zeta, t)
\]

\[
+ (1 - H_{ARG}) (1 + H_{ARG}) - 1)(\zeta_1, \zeta_2, t)
\]

\[
\mathcal{W}(\zeta_1, \zeta_2, 0) = \mathcal{W}_0(\zeta_1, \zeta_2)
\]

\[
\mathcal{W}(\eta_1, \eta_2, t) = 0
\]

Reflect the process system well-defined by

\[
\frac{\partial x_1}{\partial t}(\zeta_1, \zeta_2, t) = \frac{\delta^2 x_1}{\delta \zeta_2^2}(\zeta_1, \zeta_2, t) + x_2(\zeta_1, \zeta_2, t)
\]

\[
-x_1(\zeta_1, \zeta_2, t))
\]

\[
x_2(\zeta_1, \zeta_2, 0) = x_{02}(\zeta_1, \zeta_2)
\]

\[
x_2(\eta_1, \eta_2, t) = 0
\]

with \( \mathcal{U} = (0, 1) \times (0, 1) \) and the output function
\( \Psi(t) = C x_t(\cdot, t) \) (27)

Let \( (a_1, b_1) \times (a_2, b_2) \) be studied region provided with

\[
\varphi_{ij}(\xi, \zeta) = \frac{2}{\sqrt{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}} \sin i\pi \left( \frac{\xi - \alpha_1}{\beta_1 - \alpha_1} \right) \sin j\pi \left( \frac{\zeta - \alpha_2}{\beta_2 - \alpha_2} \right)
\]

and

\[
\lambda_{ij} = -\left( \frac{i^2}{(\beta_1 - \alpha_1)^2} + \frac{j^2}{(\beta_2 - \alpha_2)^2} \right) \pi^2, \quad i, j \geq 1
\] (28)

\[
\varphi_{ij}(\xi, \zeta) = \frac{2}{\sqrt{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}} \sin i\pi \left( \frac{\xi - \alpha_1}{\beta_1 - \alpha_1} \right) \sin j\pi \left( \frac{\zeta - \alpha_2}{\beta_2 - \alpha_2} \right)
\]

\[
\lambda_{ij} = -\left( \frac{i^2}{(\beta_1 - \alpha_1)^2} + \frac{j^2}{(\beta_2 - \alpha_2)^2} \right) \pi^2, \quad i, j \geq 1
\] (29)

5.1. Case 1

Reflect case 1 in rectangular sensor supports which is illustrated in “Figure 3” and characterize by equations (26)-(27). So the measuring output is specified by

\[
\Psi(t) = \int_0^D x_2(\xi, \zeta, t) f(\xi, \zeta) d\xi d\zeta,
\] (30)

wherever \( D \subset U \), is the position of the sensor in zone type.

![Figure 3. Case 1 of the domain, region and sensor.](image)

So that the sensor \((D, f)\) may be enough for deriving an ARGO-observer, and \( \exists H_{\text{ARG}} \in (A_{22} - H_{\text{ARG}}A_{12}) \) creates ARG-stable. Thus we have

\[
\lim_{\epsilon \to 0} \| (\nu(\xi, \zeta, t) + H_{\text{ARG}} x_2(\xi, \zeta, t)) - x(\xi, \zeta, t) \|_{(H^1(\omega))^n} = 0,
\]

where

\[
\begin{align*}
\frac{\partial \nu}{\partial t}(\xi, \zeta, t) &= \frac{\partial^2 \nu}{\partial \xi^2}(\xi, \zeta, t) + ((1 + H_{\text{ARG}}) \nu(\xi, \zeta, t) \\
&+ (1 - H_{\text{ARG}}) \frac{\partial x_2}{\partial \xi}(\xi, \zeta, t) + (H_{\text{ARG}} H_{\text{ARG}} - 1)(\xi, \zeta, t)) U \\
\nu(\xi_1, \xi_2, 0) &= \nu_0(\xi_1, \xi_2) \quad U \\
\nu(\xi_1, \xi_2, t) &= 0 \quad \Theta
\end{align*}
\]

(31)

**Proposition 5.1.** Supposing \( D = [\xi_{01} - l_1, \xi_{01} + l_1] \times [\xi_{02} - l_2, \xi_{02} + l_2] \subset U \) as in Fig.3. Thus the system (31) is not ARGO-observer for the systems (26)-(30), if for any \( i_0, i_0 (\xi_{01} - \alpha_1)/(\beta_1 - \alpha_1) \) and \( i_0 (\xi_{02} - \alpha_2)/(\beta_2 - \alpha_2) \) is rational number and \( f \) stays symmetric around the \( x_{01} = \xi_{01} \).

**Proof.** Supposing \( i_0 = 1, (\xi_{01} - \alpha_1)/(\beta_1 - \alpha_1) \) and \( i_0 (\xi_{02} - \alpha_2)/(\beta_2 - \alpha_2) \in Q \), then there exists \( j_0 \geq 1 \) such that \( \text{Sin}(j_0 \pi c_1/\beta_1 \alpha_1) = 0 \). But
\( y(t) = \langle f_1, \varphi_{io/0} \rangle = \left( \frac{4}{(\beta_1-\alpha_2)(\beta_2-\alpha_2)} \right)^{1/2} \int_{a_2-\alpha_2}^{a_2+\alpha_2} \int_{a_1-\alpha_1}^{a_1+\alpha_1} f_1(\zeta_1, \zeta_2) \right) \)

\[
\sin \left[ \frac{\int_{0}^{\pi} \zeta_1}{(\beta_1-\alpha_2)} \right] \sin \left[ \frac{\int_{0}^{\pi} \zeta_2}{(\beta_2-\alpha_2)} \right] d\zeta_1 d\zeta_2
\]

If \( f \) satisfy symmetry property for \( x_{01} = \zeta \), this implies

\[
y(t) = \langle f_1, \varphi_{io/0} \rangle = 0.
\]

Therefore, the dynamic system (31) is not ARGRO-observer for the systems (26)-(30). □

5.2. Case 2

Deliberate case 2 in circular supports which is shown in Fig.4., and described by equations (26)-(32). So the measuring output is stated by

\[
y(t) = \int_{D} x_1(r, \theta, t) f(r, \theta) d\theta,
\]

where \( D = (r, \theta) \subset U \), is the position of the using sensor.” Figure 4”.

![Figure 4](image)

Figure 4. Case 2 of the domain, region and sensor.

So that \((D, f)\) may be enough to ensure ARGRO-observer [8], and by employing the same way in case 1 implies the following

\[
\lim_{t \to 0} \| (w(r, \theta, t) + H_{ARG} x_2(r, \theta, t)) - x(r, \theta, t) \|_{(H^1(\omega))^n} = 0,
\]

where

\[
\begin{align*}
\frac{\partial w}{\partial t} (r, \theta, t) &= \frac{\partial^2 w}{\partial \zeta^2} (r, \theta, t) + ((1 + H_{ARG}) w (r, \theta, t) \\
&\quad + (1 - H_{ARG}) \frac{\partial x_2}{\partial \zeta} (r, \theta, t) + (H_{ARG}^2 - 1) (r, \theta, t)) \quad Q
\end{align*}
\]

\[
\begin{align*}
w (r, \theta, 0) &= w_0 (r, \theta) \quad U \\
w (r, \theta, 0) &= 0 \quad \Theta
\end{align*}
\]

Then, we have the following result:

**Proposition 5.2.** Supposing \( D = D(c, r) \subset U \), \( c = (c_1, c_2) \). So the process (33) is not ARGRO-observer for (26)-(32), if \( \forall i_0, i_0(c_{01} - \alpha_1)/(\beta_1 - \alpha_2) \) and \( i_0(c_{02} - \alpha_2)/(\beta_2 - \alpha_2) \) is rational number in order \( f \) satisfy symmetry property around the \( x_{01} = c_{01} \).

**Proof.** Assuming \( i_0 = 1 \), then and \( \exists j_0 \geq 1 \). \( \exists \cos(j_0 \pi c_1/\beta_1 \alpha_2) = 0 \). Thus the measuring information (32) with some modification variables can be characterized by \( x_2 = c_1 + \hat{r} \cos \theta \), and \( x_2 = c_2 + \hat{r} \sin \theta \). Thus we have
\[ Y(t) = \langle f_1, \varphi_{i_o, t_0} \rangle = \left( \frac{4}{(\beta_1-\alpha_1)(\beta_2-\sigma_2)} \right)^{1/2} \int_0^{2\pi} \int_0^r f_1(c_1 + \hat{r} \cos \theta, c_2 + \hat{r} \sin \theta) \sin \left[ \frac{\lambda_0 \pi (c_1 + \hat{r} \sin \theta)}{(\beta_1-\alpha_1)} \right] \sin \left[ \frac{\lambda_0 \pi (c_1 + \hat{r} \sin \theta)}{(\beta_2-\sigma_2)} \right] \hat{r} \, d\hat{r} \, d\theta \]

Since \( f_1 \) is symmetric about \( x_2 = c_1 \), the function

\[(\hat{r}, \theta) \rightarrow f_1(c_1 + \hat{r} \cos \theta, c_2 + \hat{r} \sin \theta) \cos \left[ \frac{\lambda_0 \pi (c_1 + \hat{r} \sin \theta)}{(\beta_2-\sigma_2)} \right] \]

is symmetric on \([0, \pi]\) about \( \theta = \pi/2 \) for all \( \hat{r} \). But the function

\[(\hat{r}, \theta) \rightarrow \sin \left[ \frac{\lambda_0 \pi + \gamma \sin \theta}{(\beta_2-\sigma_2)} \right] \]

verify anti-symmetry property on \([0, \pi]\) around \( \pi/2 \). Therefore,

\[ Y(t) = \langle f_1, \varphi_{i_o, t_0} \rangle = 0 \]

and hence, the process \((33)\) is not \textit{ARGRO}-observer for \((26)-(32)\).□

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\section*{6. Conclusion}

The existing of an \textit{ARGRO}-observer for a class of DPS have been introduced and characterized. Precisely, we have specified an approach for construct an \textit{ARGRO}-estimator which rebuild a gradient of the state in the reflected region \( \omega \). For the future work, may be study the possibility to extend these results to the case of semi-linear in distributed parameter systems.

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