Efficient Algorithms for Approximate Triangle Counting

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Abstract. Counting the number of triangles in a graph has many important applications in network analysis. Several frequently computed metrics like the clustering coefficient and the transitivity ratio need to count the number of triangles in the network. Furthermore, triangles are one of the most important graph classes considered in network mining. In this paper, we present a new randomized algorithm for approximate triangle counting. The algorithm can be adopted with different sampling methods and give effective triangle counting methods. In particular, we present two sampling methods, called the \( q \)-optimal sampling and the edge sampling, which respectively give \( O(sm) \) and \( O(sn) \) time algorithms with nice error bounds (\( m \) and \( n \) are respectively the number of edges and vertices in the graph and \( s \) is the number of samples). Among others, we show, for example, that if an upper bound \( \tilde{\Delta}_e \) is known for the number of triangles incident to every edge, the proposed method provides an \( 1 \pm \epsilon \) approximation which runs in \( O(\tilde{\Delta}_e n \log \tilde{\Delta}_e \epsilon^2) \) time, where \( \tilde{\Delta}_e \) is the average number of triangles incident to an edge. Finally we show that the algorithm can be adopted with streams. Then it, for example, will perform 2 passes over the data (if the size of the graph is known, otherwise it needs 3 passes) and will use \( O(sn) \) space.

Keywords: Graphs, triangles, approximate algorithms, stream data, network analysis, complexity.

1 Introduction

Graphs are fundamental structures for modeling complex relationships between data. Examples include: Internet (where vertices are routers and edges correspond to physical links), World Wide Web (where vertices are web pages and edges correspond to hyperlinks), social networks (where vertices are humans and edges correspond to friendships), traffic data (where vertices are places or cities and edges correspond to roads) and biological networks (where vertices are proteins and edges correspond to protein interactions).

The problem of counting subgraphs of a certain class, is one of the typical problems in graph mining which in recent years has obtained considerable attentions. Triangles are one of the most important basic subgraphs. On the other hand, computation of several network indices and statistics are based on counting the number of triangles, which makes triangle counting an essential problem in network analysis. Clustering coefficient of a graph \([13]\) is defined as the normalized sum of the fraction of neighbor pairs of a
vertex of the graph that are connected. Transitivity coefficient of a graph \([11]\), is defined as the ratio between three times the number of triangles and the number of length two paths in the graph.

In terms of time complexity, the most efficient triangle counting algorithms are based on matrix multiplication. If \(A\) is the adjacency matrix of a graph \(G\), the number of triangles in \(G\) is equal to

\[
\frac{1}{6} \text{Tr}(A^3)
\]

where \(\text{Tr}(\cdot)\) denotes the trace of a matrix defined as the sum of the elements on the main diagonal of the matrix.

Time complexity of the most efficient known algorithm for matrix multiplication is \(O(n^{2.3727})\) \([6], [19]\), where \(n\) is the number of vertices of the graph (the number of rows/columns of \(A\)). The exponent of \(n\), denoted by \(\omega\), is called matrix multiplication exponent. Alon et al. give in \([1]\) a more efficient triangle counting algorithm for sparse graphs. Time complexity of their method is \(O(m^{\frac{4\omega}{\omega+1}})\), where \(m\) is the number of edges of the graph. For \(\omega = 2.3727\), this time complexity is equal to \(O(m^{1.41})\).

However, exact triangle counting methods may be inefficient when the size of the graph is large. In these cases, an approximate algorithm is preferred in the cost of losing the exact number of triangles. In recent years, many algorithms have been proposed for approximate triangle counting. A widely used technique is the sparsification technique \([17], [16], [14]\) and \([13]\). In this technique, the graph is converted into a sparse graph and the number of triangles in the sparsified graph is counted. Then, the result is scaled to the original graph. Some other methods are based on approximate computation of algebraic properties of the graph like eigenvalues of the adjacency matrix \([15]\). However, well-known methods for computing (or approximating) eigenvalues are heavily based on matrix multiplication and it is known that worst case time complexity of computing eigenvalues is the same as matrix multiplication. On the hand, such approximate triangle counting algorithms look at the algebraic methods (like approximate matrix multiplication and low rank matrix approximation) as a black-box. However, samplings in the algebraic methods are done in a way to minimize the element-wise error or the frobenius norm of the error matrix and therefore, the error of triangle counting is not minimized.

In this paper, we propose a new randomized algorithm for approximate triangle counting. Our method is a variation of several approximate matrix multiplication algorithms \([8], [10]\) and \([9]\). However, it does not generate any product matrix and several algorithmic aspects are different. Furthermore, the sampling methods which are crucial elements of the algorithm, are also different. The proposed algorithm can be seen as a general framework to which different sampling methods can be applied. Every sampling method gives a new triangle counting algorithm with its own error bounds and time complexity. In particular, we present two sampling methods, called the \(q\)-optimal sampling and the edge sampling, which respectively give \(O(sm)\) and \(O(sn)\) time algorithms with nice error bounds (\(m\) and \(n\) are respectively the number of edges and vertices in the graph and \(s\) is the number of samples). Among others, we show, for example, that if an upper bound \(\Delta_e^c\) is known for the number of triangles incident to every edge, the proposed method provides an \(1 \pm \epsilon\) approximation which runs in \(O(\frac{\Delta_e^c n \log n}{\Delta_e^c \epsilon^2})\).
time, where $\hat{\Delta}$ is the average number of triangles incident to an edge. As we will dis-
cuss, some existing algorithms can be seen as adoptations of the proposed algorithm
with specific sampling methods. We finally show that the algorithm can be extended
to streams. Then it, for example, will perform 2 passes over the data (if the size of the
graph is known, otherwise it needs 3 passes) and will use $O(sn)$ memory cells.

The rest of this paper is organized as follows. In Section 2, we present a new ran-
domized algorithm for approximate triangle counting. In Section 3 different sampling
methods are introduced. In Section 4 we extend the algorithm for counting triangles
in streams. An overview of related work is provided in Section 5. Finally, the paper is
concluded in Section 6.

Throughout the paper, $G$ refers to a simple (i.e. loop-free and without multiple
edges) and undirected graph. $A$ refers to the adjacency matrix of $G$. Therefore, $A$
is a square matrix consisting of 0s and 1s. $A_{ij}$ denotes the element in the $i$-th row and the
$j$-th column of $A$. $n$ and $m$ denote the number of vertices of $G$ (the number of rows and
the number of columns of $A$) and the number of edges of $G$, respectively. $\Delta$ denotes
the number of triangles in $G$. In this paper, we use an index $i$ for referring to a row (column)
in the adjacency matrix as well as for referring to the vertex of the graph corresponds
to the row (column) $i$.

## 2 The approximate triangle counting algorithm

In this section, we present a randomized algorithm for approximate triangle counting.
Suppose $A$ is an $n \times n$ matrix which is the adjacency matrix of a graph $G$. As Equation
1 shows, in order to count triangles in $G$, we can compute the trace of $A^3$. Algorithm [1]
shows the high level pseudo code of an approximate triangle counting algorithm, based
on randomized calculation of the trace of $A^3$.

In every iteration $t$ of the loop in Lines 3-10 of Algorithm [1] first the following
probabilities are computed:

$$p_1, p_2, \ldots, p_n \geq 0 \text{ such that } \sum_{i=1}^{n} p_i = 1$$

Then, an $i \in \{1, \ldots, n\}$ is selected with probability $p_i$, and the following probabil-
ities are computed:

$$q_{1i}, \ldots, q_{ni} \geq 0 \text{ such that } \sum_{j=1}^{n} q_{ji} = 1$$

Finally, an $j \in \{1, \ldots, n\}$ is selected with probability $q_{ji}$ and the number of trian-
gles is estimated by

$$\beta_t = \frac{\sum_{d=1}^{n} A_{di} A_{ij} A_{jd}}{6 p_i q_{ji}} \quad (2)$$

The final estimation, $\beta$, is the average of the estimations of different trials.

In the rest of this section, we study some important properties of Algorithm [1]
Algorithm 1: High level pseudo code of the approximate triangle counting algorithm.

TRIANGLECOUNTER

Require: A graph $G$.

Ensure: The approximate number of triangles in $G$.

1: Let $A$ be the $n \times n$ adjacency matrix of $G$
2: $\beta \leftarrow 0$
3: for $t = 1$ to $s$
4: Compute $p_1, \ldots, p_n$
5: Select $i \in \{1, \ldots, n\}$ with the probability $p_i$
6: Compute $q_{ij}, \ldots, q_{in}$
7: Select $j \in \{1, \ldots, n\}$ with the probability $q_{ji}$
8: $\beta_t \leftarrow \sum_{d=1}^{n} A_{di} A_{ij} A_{jd} p_i q_{ji}$
9: $\beta \leftarrow \beta + \beta_t$
10: end for
11: return $\beta$

Lemma 1. In Algorithm[7] for every $t \in \{1, \ldots, s\}$ we have:

$$E(\beta_t) = \frac{\text{Tr}(A^3)}{6}$$ (3)

Proof. For every $t \in \{1, \ldots, s\}$, we have:

$$E(\beta_t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sum_{d=1}^{n} A_{di} A_{ij} A_{jd} p_i q_{ji} \right)$$
$$= \frac{1}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{d=1}^{n} A_{di} A_{ij} A_{jd}$$
$$= \frac{1}{6} \text{Tr}(A^3)$$

On the other hand, $\beta_t$’s are independent random variables and $\beta$ is the sum of $s$ independent random variables $\beta_1 \ldots \beta_s$ divided by $s$. Therefore

$$E(\beta) = \frac{E(\sum_{t=1}^{s} \beta_t)}{s} = \frac{sE(\beta_t)}{s} = E(\beta_t) = \frac{\text{Tr}(A^3)}{6}$$

Lemma 2. In Algorithm[7] variance of every random variable $\beta_t$ is

$$\text{Var}(\beta_t) = \frac{1}{36} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sum_{d=1}^{n} A_{di} A_{ij} A_{jd} \right)^2 \right) - \left( \frac{\text{Tr}(A^3)}{6} \right)^2$$ (4)

Proof. We have:

$$\text{Var}(\beta_t) = E(\beta_t^2) - (E(\beta_t))^2$$
Then
\[
E(\beta_t^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\sum_{d=1}^{n} \sum_{j=1}^{n} A_{di} A_{ij} A_{jd}}{6 p_i p_j q_{ji}} \right)^2 p_i p_j q_{ji} = \frac{1}{36} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\sum_{d=1}^{n} A_{di} A_{ij} A_{jd}}{p_i q_{ji}} \right)^2
\]
(5)

and
\[
(E(\beta_t))^2 = \frac{1}{36} (\text{Tr}(A^3))^2
\]
(6)

Therefore
\[
\text{Var}(\beta_t) = \frac{1}{36} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\sum_{d=1}^{n} A_{di} A_{ij} A_{jd}}{p_i q_{ji}} \right)^2 - (\text{Tr}(A^3))^2 \right)
\]
(7)

Since \( \beta \) is the average of \( s \) independent copies of \( \beta_t \), then
\[
\text{Var}(\beta) = \frac{\text{Var}(\beta_t)}{s} = \frac{1}{36 s} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\sum_{d=1}^{n} A_{di} A_{ij} A_{jd}}{p_i q_{ji}} \right)^2 - (\text{Tr}(A^3))^2 \right)
\]
(8)

For a vertex \( i \) of the graph, local triangles of \( i \) are triangles which are incident to \( i \). The number of local triangles of \( i \), denoted by \( \Delta_i \), equals to
\[
\Delta_i = \frac{1}{2} \sum_{j=1}^{n} \sum_{d=1}^{n} A_{di} A_{ij} A_{jd}
\]

Let \( \{i, j\} \) be an edge of the graph. Local triangles of \( \{i, j\} \) are triangles for which \( \{i, j\} \) is an edge. The number of local triangles of \( \{i, j\} \), denoted by \( \Delta_{\{i,j\}} \), is equal to
\[
\Delta_{\{i,j\}} = \sum_{d=1}^{n} A_{di} A_{ij} A_{jd}
\]

If \( i \) is not connected to \( j \), the number of local triangles of \( \{i, j\} \) is equal to 0. The following holds between \( \Delta_i \) and \( \Delta_{\{i,j\}} \):
\[
\Delta_i = \frac{1}{2} \sum_{j=1}^{n} \Delta_{\{i,j\}}.
\]

Let \( \Delta \) refer to the number of triangles in the graph. We have:
\[
\Delta = \frac{1}{2} \sum_{i=1}^{n} \Delta_i.
\]

Using the notion of local triangles of edges, Equation 8 can be re-written as:
\[
\text{Var}(\beta) = \frac{1}{36 s} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\Delta_{\{i,j\}}^2}{p_i q_{ji}} - \frac{\Delta^2}{s}
\]
(9)

**Lemma 3.** If in Algorithm 1 probabilities \( p_1, p_2, \ldots, p_n \) and \( q_{1|i}, q_{2|i}, \ldots, q_{n|i} \) are accessible in constant time, its time complexity will be \( O(sn) \).

**Proof.** The loop in Lines 3-10 is performed for \( s \) times. Inside the loop, the most time consuming step is Line 8 which takes \( O(n) \) time. Therefore, time complexity of the algorithm is \( O(sn) \). ■
3 Sampling methods

In this section, we present a number of sampling methods. First in Section 3.1 the optimal sampling is investigated. Then, since the optimal sampling might be computationally expensive, other near-optimal samplings are introduced.

3.1 Optimal sampling

Lemma 4 introduces the probabilities and error bound of the optimal sampling.

Lemma 4. If for \(1 \leq i \leq n\), \(p_i\)'s are equal to

\[
p_i = \frac{\Delta_i}{3\Delta}
\]

and then after selecting \(i\), for \(1 \leq j \leq n\), \(q_{j|i}\)'s are

\[
q_{j|i} = \frac{\Delta_{\{i,j\}}}{2\Delta_i}
\]

the variance of \(\beta\) presented in Equation 8 is minimized. The minimized variance is 0.

Since the error bound of the optimal sampling is 0, it gives an exact triangle counting algorithm. On the other hand, time complexity of computation of \(p_i\)'s in Equation 10 is the same as time complexity of exact triangle counting. Therefore, using Algorithm 1 with the optimal sampling is the same (in both accuracy and complexity) as using an exact triangle counting algorithm.

3.2 \(q\)-optimal sampling

In the \(q\)-optimal sampling, every vertex \(i\), \(1 \leq i \leq n\), is selected by some strategy (which can be performed in \(O(1)\) time). For example, they are selected uniformly at random (therefore \(p_i = \frac{1}{n}\)), or they are selected proportional to their degrees (therefore, \(p_i = \frac{\deg(i)}{2m}\), where \(\deg(i)\) refers to the degree of \(i\)). Then, every vertex \(j\) is selected in a way to minimize \(\text{Var}(\beta)\).

Lemma 5. In the \(q\)-optimal sampling, after choosing a vertex \(i\), if every vertex \(j\) is selected with probability

\[
q_{j|i} = \frac{\Delta_{\{i,j\}}}{2\Delta_i}
\]

the variance of \(\beta\) is minimized.

Proof. In order to minimize \(\text{Var}(\beta)\), we need to minimize \(\sum_{j=1}^{n} \frac{\Delta_{\{i,j\}}^2}{q_{j|i}}\), because other parts of \(\text{Var}(\beta)\) are independent of \(j\). We define

\[
f(q_1|i, q_2|i, \ldots, q_n|i) = \sum_{j=1}^{n} \frac{\Delta_{\{i,j\}}^2}{q_{j|i}}
\]
and substitute $q_{n|i}$ by $1 - \sum_{j'=1}^{n-1} q_{j'|i}$ and form equations $\frac{\partial f}{\partial q_{j|i}} = 0$, for $1 \leq j \leq n - 1$. We get

$$\frac{\Delta_{(i,j)}}{q_{j|i}} = \frac{\Delta_{(i,n)}}{(1 - \sum_{j'=1}^{n-1} q_{j'|i})^2}$$

$$\Rightarrow q_{j|i} = \frac{1 - \sum_{j'=1}^{n-1} q_{j'|i}}{\Delta_{(i,n)}} \Delta_{\{i,j\}}$$ (13)

Summing $q_{j|i}$’s, for $1 \leq j \leq n - 1$, and doing simplifications, we get

$$\sum_{j=1}^{n-1} q_{j|i} = \frac{\sum_{j'=1}^{n-1} \Delta_{(i,j)}}{\Delta_{(i,n)}} \Delta_{\{i,j\}}$$

Putting the value of $\sum_{j=1}^{n-1} q_{j|i}$ into Equation (13) and doing simplifications, we get the value of $q_{j|i}$ for the $q$-optimal sampling:

$$q_{j|i} = \frac{\Delta_{(i,j)}}{\sum_{j'=1}^{n} \Delta_{\{i,j'\}}} = \frac{\Delta_{(i,j)}}{2\Delta_i}$$

If vertices $i$ are selected proportional to their degrees, the variance of $\beta$ in the $q$-optimal sampling will be

$$\text{Var}(\beta) = \frac{2m}{9s} \sum_{i=1}^{n} \frac{\Delta_i^2}{\text{deg}(i)} - \frac{\Delta^2}{s}$$ (14)

and if they are selected selected uniformly at random, $\text{Var}(\beta)$ will be

$$\text{Var}(\beta) = \frac{n}{9s} \sum_{i=1}^{n} \Delta_i^2 - \frac{\Delta^2}{s}$$ (15)

The motivation for selecting vertices $i$ proportional to their degrees is that, as studied in [15], in most of real-world networks, vertices of higher degrees have higher number of local triangles. Then, for every two vertices $i$ and $i'$, if it holds that $\text{deg}(i) \geq \text{deg}(i')$ implies $\Delta_i \geq \Delta_{i'}$, it can be shown that selecting vertices $i$ proportional to their degrees gives a better sampling than choosing them uniformly at random.

If the $q$-optimal sampling is used, in every iteration of the loop in Lines 3-10 of Algorithm 1, probabilities $p_i$ and $q_{j|i}$ can be computed in $O(m)$ time. On the other hand, it takes $O(n)$ time to compute $\beta_t$. Therefore, time complexity of Algorithm 1 with the $q$-optimal sampling will be $O(sm^2)$.

The $q$-optimal sampling can provide efficient $1 \pm \epsilon$ approximations, specifically if some information on local triangles of vertices is available. For example, consider the version of the $q$-optimal sampling where vertices $i$ are selected uniformly at random.
(p_i = \frac{1}{n_i}). Suppose that there exists an already known value \( \overline{\Delta}^v \) such that for every vertex \( i \), \( \Delta_i \leq \overline{\Delta}^v \). Then, in every iteration of the loop in Lines 3-10 of Algorithm 1 a random variable \( X_t \) can be defined as \( X_t = \frac{\hat{\Delta}_i}{n_i \Delta_i} = \frac{\hat{\Delta}_i}{\overline{\Delta}^v} \), where \( \Delta_i(t) \) is the number of local triangles of the vertex selected in iteration \( t \). We have: \( E(X_t) = \frac{\hat{\Delta}_i}{n_i \Delta_i} = \frac{\overline{\Delta}^v}{\overline{\Delta}^v} \), where \( \overline{\Delta}^v \) is the average number of local triangles of vertices. By Chernoff bound we obtain:

\[
\Pr \left[ \frac{1}{s} \sum_{t=1}^{s} X_t - \frac{\Delta_i}{\Delta_i} > \epsilon \frac{\overline{\Delta}^v}{\overline{\Delta}^v} \right] \leq \exp \left( -\frac{\epsilon^2 s \overline{\Delta}^v}{2 \Delta_i} \right) \tag{16}
\]

If \( s = \Omega(\frac{\overline{\Delta}^v \log n}{\Delta_i \epsilon^2}) \), then \( \frac{\overline{\Delta}^v}{s} \sum_{t=1}^{s} X_t \) approximates \( \Delta \) within a factor of \( \epsilon \) with probability at least \( 1 - n^{-c} \) for any constant \( c \). This gives an \( O(\frac{\Delta_i \log n}{\Delta_i \epsilon^2}) \) time algorithm which approximates \( \Delta \) within a factor of \( \epsilon \). Specifically, if \( \overline{\Delta}^v \) is greater than \( \overline{\Delta}^v \) only by a factor of a constant, time complexity of the algorithm will be \( O(\frac{\log n}{\epsilon^2}) \).

### 3.3 Edge sampling

In the edge sampling, first a vertex \( i \) is selected by some strategy (which can be done in \( O(1) \) time). Then, a neighbor \( j \) of \( i \) is selected by some (probably different) strategy which also can be done in \( O(1) \) time. Since in this sampling computation of probabilities is done in \( O(1) \) time, time complexity of the algorithm is \( O(sn) \).

For example, \( i \) can be selected uniformly at random (therefore, \( p_i = \frac{1}{n_i} \)), then, for every \( j \), if \( j \) is a neighbor of \( i \), \( q_{ji} \) is equal to \( \frac{1}{\text{deg}(i)} \); otherwise it is 0. This case is similar to the methods which uniformly sample an edge and count the number of triangles incident to it and scale the result. The first algorithm proposed in [12] and partially the algorithm of [5] are examples of such methods. In this case, variance of \( \beta \) will be

\[
\text{Var}(\beta) = \frac{n}{36s} \sum_{i=1}^{n} \left( \deg(i) \sum_{j=1}^{n} \Delta_{(i,j)} \right)^2 - \frac{\Delta^2}{s} \tag{17}
\]

In the second case of the edge sampling, \( i \) is chosen with probability \( p_i = \frac{\deg(i)}{\deg(1)} \). Then, similar to the first case, for every vertex \( j \), if \( j \) is a neighbor of \( i \), \( q_{ji} \) will be \( \frac{1}{\deg(1)} \). Otherwise, it will be 0. Variance of \( \beta \) in this case is:

\[
\text{Var}(\beta) = \frac{m}{18s} \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_{(i,j)}^2 - \frac{\Delta^2}{s} \tag{18}
\]

Similar to the \( q \)-optimal sampling, the edge sampling can provide efficient \( 1 \pm \epsilon \) approximations, specifically if some information on local triangles of edges is available. For example, consider the second case and suppose that there exists a known value \( \overline{\Delta}^v \) such that for every edge \( \{i,j\} \), \( \Delta_{(i,j)} \leq \overline{\Delta}^v \). Then, in every iteration of the loop in Lines 3-10 of Algorithm 1 a random variable \( X_t \) can be defined as \( X_t = \frac{\hat{\Delta}_i}{m \overline{\Delta}^v} = \)
with probability



if our focus is the storage capabilities. Arrives one item at a time, and the algorithm has access to limited computation and the use of the data stream model. A data stream is an ordered sequence in which data

In many applications like World Wide Web and social networks, the dataset is too large to load it into the main memory. A widely used approach to address this problem is proximate triangle counting are gathered. Then, for every

\[
\Pr \left[ \frac{1}{s} \sum_{t=1}^{s} X_t - \frac{\hat{\Delta}^e}{\Delta^e} > \epsilon \frac{\hat{\Delta}^e}{\Delta^e} \right] \leq \exp \left( -\frac{\epsilon^2 s \hat{\Delta}^e}{2 \Delta^e} \right)
\]

and if \( s = \Omega(\frac{\hat{\Delta}^e \log n}{\Delta^e \epsilon^2}) \), then \( \frac{m \hat{\Delta}^e}{s} \sum_{t=1}^{s} X_t \) approximates \( \Delta \) within a factor of \( \epsilon \) with probability at least \( 1 - n^{-c} \) for any constant \( c \). This gives an \( O(\frac{\hat{\Delta}^e \log n}{\Delta^e \epsilon^2}) \) time algorithm which approximates \( \Delta \) within a factor of \( \epsilon \). Specifically, if \( \hat{\Delta}^e \) is greater than \( \hat{\Delta}^e \) only by a factor of a constant, time complexity of the algorithm will be \( O(\frac{n \log n}{\epsilon^2}) \).

4 Triangle counting in streams

In many applications like World Wide Web and social networks, the dataset is too large to load it into the main memory. A widely used approach to address this problem is the use of the data stream model. A data stream is an ordered sequence in which data arrives one item at a time, and the algorithm has access to limited computation and storage capabilities.

In this section, we extend the algorithm presented in Section 2 to streams. While our focus is the \( q \)-optimal sampling where vertices \( i \) are selected uniformly at random, the other sampling methods can be extended to streams in a similar way. Due to lack of space, we here omit details. If the number of vertices of the graph, \( n \), is already known, our algorithm will need 2 passes over the data. Otherwise, an extra pass will be needed to find it. First, \( s \) integers (vertices) \( i \) are selected independently at random with uniform probability \( \frac{1}{n} \). Then:

- During the first pass, for every \( i \), the neighborhood vector of \( i \), denoted by \( T^i \), is formed. The neighborhood vector of \( i \), shows which vertices of the graph are a neighbor of \( i \) and which ones are not. For every vertex \( j \), if \( \{i, j\} \) is an edge of the graph, \( T^i_j \) is set to 1, otherwise, it is set to 0.

- During the second pass, for every vertex \( i \) and for all vertices \( j \), the number of local triangles of the edge between \( i \) and \( j \) is calculated. To do so, for every \( i \) a vector \( P^i \) of size \( n \) is used, where \( P^i_j \) stores the number of local triangles of \( \{i, j\} \). When an edge \( A_{i_d} \) is visited, if there exists an edge between \( i \) and \( j \) (i.e. \( T^i_j = 1 \)) and an edge between \( i \) and \( d \) (i.e. \( T^i_d = 1 \)), a triangle consisting of the vertices \( i, j \) and \( d \) is found. Therefore, \( P^i_j \) and \( P^i_d \) and \( z^i \) are increased by 1. \( z^i \) is used to store the number of local triangles of \( i \).

During these two passes, the information required for \( q \)-optimal sampling and approximate triangle counting are gathered. Then, for every \( i \), a \( j \in \{1, \ldots, n\} \) is selected with probability \( q_{ji} = \frac{\Delta_{\{i, j\}}}{2\Delta^e} = \frac{P^i_j}{2z^i} \) and the approximate number of triangles is calculated.

Every pass needs \( O(sn) \) space from the main memory. After twice passing over the stream and calculating \( T^i \)'s, \( P^i \)'s and \( z^i \)'s, the approximate number of triangles can be
determined in \( O(1) \) time. We can store random variables \( X_t \), if \( \Delta_t \) is known. In this case, we will need \( O(\frac{\Delta_t \log n}{\Delta_t \epsilon^2}) \) space to provide a \( 1 \pm \epsilon \) approximation. Therefore, we can present the following theorem:

**Theorem 1.** There is a 2-pass algorithm (if \( n \) is already known, otherwise it will need 3 passes) to count the number of triangles in a stream of edges which needs \( O(n) \) memory space and constant update time and its error guarantee obeys Equation 15. Specifically, with space usage \( O(\frac{\Delta_t \log n}{\Delta_t \epsilon^2}) \), it gives a \( 1 \pm \epsilon \) approximation.

Similar results can be presented for other samplings.

5 Related work

Buriol et al. [5] presented one of the first approximate triangle counting algorithms. In their method, an edge and a vertex are selected by random and it is checked whether they form a triangle or not. Then, the fraction of tests which form a triangle is scaled and returned as an estimation of the number of triangles. In this method, if \( s = \log \frac{1}{\delta} \epsilon^2 \left( \frac{T_0 + T_1 + T_2 + T_3}{T_3} \right) \) independent trials are done, where \( T_i \) is the number of triples of vertices with \( i \) edges, with probability at least \( 1 - \delta \), the estimated number of triangles is between \( (1 - \epsilon)T_3 \) and \( (1 + \epsilon)T_3 \). However, \( T_0 + T_1 + T_2 \) can be very large compared to \( T_3 \).

The other method which is efficient only for triangle-dense graphs is [2]. Their method is based on reducing triangle counting to estimating the zero-th, first and second frequency moments. They showed that there is a streaming algorithm that for any adjacency stream of a graph, computes an \( (\epsilon - \delta) \) approximation of the number of triangles using space:

\[
O \left( \frac{1}{\epsilon^3} \times \log \frac{1}{\delta} \times \left( \frac{T_1 + T_2 + T_3}{T_3} \right)^3 \times \log n \right)
\]

Tsourakakis [15] proposed a triangle counting algorithm based on computing the largest eigenvalues of the adjacency matrix of an undirected graph to approximate both the total as well as the local number of triangles in the graph. Tsourakis’s method exploits the following property: the total number of triangles in an undirected graph is \( \frac{1}{6} \sum_{j=1}^{n} \lambda_j^3 \), where \( \lambda_j \) is the \( j \)-th eigenvalue of \( A \). Using the Lanczos method [7], the eigenvalues are generated from the biggest one to the smallest one. An approximation is done when the smallest generated eigenvalue contributes very little to the total number of triangles. However, it is known that time complexity of finding eigenvalues is the same as time complexity of matrix multiplication. For approximating the number of local triangles of a vertex \( i \), he exploited the following property: \( \Delta_i = \frac{\sum_{j=1}^{n} \lambda_j^3 u_{i,j}^2}{2} \), where \( \lambda_j \) is the \( j \)-th eigenvalue and \( u_{i,j} \) is the \( j \)-th entry of the \( i \)-th eigenvector of \( A \).

In [16], the authors proposed the DOULION algorithm for approximate triangle counting. In this method, every edge \( e \) of the graph is removed with a sparsification
probability $1 - p$. If it survives, the weight $\frac{1}{p}$ is assigned to it. Then, the number of triangles in the original graph is approximated as the number of triangles in the sparsified graph times $\frac{1}{p^3}$. The following error bound was provided for this method:

$$\text{Var}(\Delta) = \Delta(p^3 - p^6) + 2k(p^5 - p^6)$$

where $k$ is the number of pairs of triangles which are edge-disjoint.

The randomized algorithm of [14] colors the vertices of the graph with $N = \frac{1}{p}$ colors uniformly at random, counts triangles whose vertices have the same color, and scales that count appropriately. The authors showed that for enough large values of $p$, their estimation of the number of triangles is concentrated around its expectation. In [13], the authors combined the sparsification techniques and the idea of vertex partitioning (into high degrees and low degrees) presented in [1]. They developed an $O(m + \frac{m^2 \log n}{\Delta^*})$ time $(1 \pm \epsilon)$-approximation algorithm.

In [3], the authors studied the problem of approximate local triangle counting in large graphs. Their approximation algorithms are based on the idea of min-wise independent permutations [4]. Their algorithms operate in a semi-streaming fashion, using $O(n)$ space in main memory and performing $O(\log n)$ passes over the edges of the graph.

6 Conclusions

In this paper, we proposed a new randomized algorithm for approximate triangle counting. The algorithm provides a general framework which can be adopted with different sampling techniques and give methods with different time complexities and error bounds. For example, it can be adopted with the $q$-optimal sampling and the edge sampling, presented in the paper, and give linear time algorithms for approximate triangle counting. We showed that if an upper bound $\tilde{\Delta}^e$ is known for the number of triangles incident to every edge, the proposed method provides an $1 \pm \epsilon$ approximation which runs in $O(\frac{\tilde{\Delta}^e n \log n}{\Delta^* \epsilon^2})$ time, where $\tilde{\Delta}^e$ is the average number of triangles incident to an edge.

Also, if an upper bound $\tilde{\Delta}^v$ is known for the number of triangles incident to every vertex, the proposed method provides an $1 \pm \epsilon$ approximation which runs in $O(\frac{\tilde{\Delta}^v m \log n}{\tilde{\Delta}^v \epsilon^2})$ time, where $\tilde{\Delta}^v$ is the average number of triangles incident to a vertex.

Finally we showed the algorithm can be extended to streams. Then it, for example, will perform 2 passes over the data (if the size of the graph is known, otherwise it needs 3 passes) and will use $O(sn)$ space.

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