Optimal Testing of Self-Driving Cars
Jeremy Morton*, Tim A. Wheeler*, and Mykel J. Kochenderfer

Abstract—Automotive manufacturers attempting to bring autonomous vehicles to market must make the case that their product is sufficiently safe for public deployment. Much of this case will likely rely upon outcomes from real-world testing, requiring manufacturers to be strategic about how they allocate testing resources in order to maximize their chances of demonstrating system safety. This work frames the partially observable and belief-dependent problem of autonomous vehicle test scheduling as a Markov decision process, which can be solved efficiently to yield exact, optimal manufacturer testing policies. By solving for policies over a wide range of problem formulations, we are able to provide high-level guidance for manufacturers and regulators on issues relating to the real-world testing of autonomous vehicles. This guidance spans an array of topics, including circumstances under which manufacturers should continue testing despite observed incidents, when manufacturers should test aggressively, and when regulators should increase or reduce the real-world testing requirements for an autonomous vehicle.

I. INTRODUCTION

Confidence must be established in the safety of autonomous vehicles prior to their widespread release. Establishing confidence is difficult because the space of driving scenarios is vast and accidents are rare. Automotive manufacturers can build confidence by conducting test drives on public roadways and make the safety case based on the frequency of observed hazardous events like disengagements and traffic accidents. Each manufacturer must devise a testing strategy capable of providing sufficient evidence that their system is safe enough for widespread adoption. Real-world testing that is too aggressive may yield hazardous events that diminish confidence in system safety. However, a manufacturer that is reluctant to test their product may forfeit opportunities to identify and address shortcomings, and may ultimately not be able to compete in the market.

The fundamental tension between the desire to thoroughly test a product and the urgency to forego further testing in favor of bringing the product to market is not unique to the automotive industry. Prior work has studied this dilemma in the context of software engineering, where developers must determine whether to delay commercial release in order to devote more time to fixing bugs in their product [1]–[4]. For example, Chávez develops a decision-analytic rule for when to halt software testing. The rule relies on a Bayesian estimate of the bug discovery rate, and weighs costs associated with immediate or delayed bug fixing, loss of market position, and customer dissatisfaction [1].

While relevant, these analyses related to software engineering do not directly translate to the automotive industry. In software engineering, a developer is generally free to release a product whenever they like—any remaining bugs may cause consumer dissatisfaction, but will not be life-threatening. In contrast, an error-prone autonomous vehicle risks inflicting physical harm upon its riders and other traffic participants. Thus, the decision of when an autonomous driving system is safe enough for release will likely be made by policymakers not manufacturers, and manufacturers will solely decide how they should test their product in order to maximize their odds of demonstrating its safety.

The size and scope of the real-world testing burden that manufacturers will face is currently unknown. Given the rarity of the most extreme hazardous events, such as fatalities, building a statistically significant case that a system meets safety standards based on observed hazardous event rates would require an infeasible amount of real-world testing [5]–[7]. Recent research has proposed methods aimed at reducing the amount of real-world testing required from manufacturers. Such methods include testing in virtual and hardware-in-the-loop simulations, as well as limiting the scope of real-world testing to safety-critical scenarios [8]–[11]. Hence, any approach designed to derive real-world testing policies for automotive manufacturers should be flexible enough to incorporate other information sources that may exist, such as performance in simulation or safety metrics in limited trials on testing grounds.

This work investigates the autonomous vehicle test-scheduling problem, which seeks the best testing schedule to maximize the likelihood of reaching confidence in the system’s safety. The problem is defined as a belief-state Markov decision process (MDP), where high reward is obtained when one becomes certain that the vehicle is safe for release. The finite-horizon belief-MDP can be solved exactly using dynamic programming. Furthermore, this approach relies on Bayesian beliefs, which allows prior information to shape manufacturers’ decisions and alter their real-world testing burden. Several parametric studies shed insight into optimal test scheduling strategies for real-world automotive manufacturers and regulatory bodies.

II. HAZARDOUS EVENT RATES

Suppose that the safety of an autonomous vehicle (AV) is judged based on the expected frequency of hazardous driving events. Hazardous events such as collisions, emergency stops, and disengagements are measurable and their frequency of occurrence should be minimized [5]. A hazardous event frequency is the expected number of hazardous events observed when driving a given distance or a given duration of time. The ISO 26262 standard, for example, defines automotive safety integrity levels according to expected hazardous events per
Table I: Hazardous event frequencies [13], [14].
\[
\begin{array}{|c|c|}
\hline
\text{Disengagement Rate} & 0.121 \text{ per } 1000 \text{ km} \\
\text{Rate of Reported Collisions} & 12.5 \text{ per } 100 \text{ million km} \\
\text{Fatality Rate} & 0.70 \text{ per } 100 \text{ million km} \\
\hline
\end{array}
\]

This work measures hazardous events per unit distance due to the less ambiguous nature of having to drive a fixed distance and the fact that hazardous event rates tend to be reported per unit distance. Knowing the average vehicle speed allows one to convert between the two formulations.

The frequency of common hazardous events vary greatly in magnitude. As Table I illustrates, the disengagement rate differs from the fatality rate by five orders of magnitude. Waymo has covered the most test miles to date, with approximately one million test kilometers [13], but even such distances are merely sufficient for establishing confidence in the rate of likely hazardous events. Proper estimates for the fatality rate require driving a multiple of the average 70 million kilometers between collisions for human drivers [14]. Unfortunately, the rarest events are often the most critical in determining whether to release an autonomous vehicle, leading to significant uncertainty in the safety of a system even after millions of miles of testing.

The event frequency of a particular hazardous event for an autonomous vehicle (AV) under test, $\lambda_{AV}$, is the expected number of hazardous events encountered when the AV is driven a reference test distance $d_{ref}$. The AV is considered safe with respect to the hazardous event when there is high confidence that the event frequency is lower than a reference threshold frequency, $\lambda_{ref}$. Such a reference frequency may be specified by regulators. Although exact values have not been established to date, we can use hazardous event rates under human driving as preliminary reference values. In this work, we assume that testing is conducted uniformly across all roadway types and driving conditions. In reality, the number of encountered hazardous events may vary according to speed limit, presence of rain or snow, or other factors, and testing that favors certain scenarios over others may draw incorrect conclusions.

Many statistical methods have been proposed for modeling the rate of hazardous events in roadway driving [15]–[18]. Among the most common are Poisson and Poisson-Gamma (negative binomial) models [19]. Wachenfeld used a Poisson distribution to model the probability of $k$ hazardous events when driving $d_{test}$ kilometers with an AV whose event frequency is $\lambda_{AV}$ as a Poisson distribution [5]:

\[
P(k; \lambda_{AV}) = \frac{1}{k!} \lambda_{AV}^k e^{-\lambda_{AV}} \quad k = 0, 1, 2, \ldots
\]

Poisson distributions capture the probability of a given number of independent, non-exhaustive events occurring in a fixed interval of time or distance given only an average rate of occurrence. The hazardous events under question tend to be extremely rare and arise from the stochastic nature of the driving environment in a manner that they can be assumed to be independent. The rarity of hazardous events also justifies the assumption that the hazardous events are non-exhaustive. Vehicles can be quickly repaired or replaced by the manufacturer in the event of a hazardous event. If many hazardous events occur, it is unlikely that the system will ever establish a high level of safety confidence.

The core objective of automotive safety validation is to establish a minimum confidence $C_{ref}$ that safety has been achieved:

\[
P(\lambda_{AV} \leq \lambda_{ref}) \geq C_{ref}.
\]

Establishing confidence thus requires establishing a belief over the AV event frequency to quantify this release criterion.

III. DECISION-THEORETIC FORMULATIONS

This section formulates the vehicle release scheduling problem using three successively simpler decision making frameworks. The first, the $\rho$POMDP, is a natural framework under which the problem can be framed. A key insight in this work is the reformulation of the vehicle release scheduling problem as a belief MDP, thereby allowing for efficient calculation of optimal policies. The third formulation is a lossless transformation of the belief MDP into an MDP, which aids in interpreting the resulting policies.

A. $\rho$POMDP

A Markov decision process (MDP) models a sequential decision making problem in a stochastic environment. It is defined by the tuple $\langle S, A, T, R, H \rangle$. At any time step $t \in [1, H]$, the system is in a state $s$ contained in the state space $S$, and the agent must select an action $a$ from the action space $A$. The transition function $T(s' | s, a)$ provides the probability of transitioning from state $s$ to state $s'$ when taking the action $a$ and the reward function $R(s, a, s')$ gives the immediate reward for the transition [20].

A policy $\pi$ for an MDP is a mapping from states to actions, where $\pi(s)$ denotes the action taken at state $s$. Decision makers seek the optimal policy that maximizes the expected accumulated reward $\sum_{t=1}^{H} r_t$, where $r_t$ is the reward for transition $t$. Optimal policies for finite-horizon MDPs can be computed using dynamic programming [21].

Many problems include uncertainty in the current state and must make inferences about the underlying state based on noisy or partial observations. Uncertainty of this form can be handled by partially observable MDPs, or POMDPs. In lieu of a state, POMDP policies operate on probability distributions over states, inferred from the past sequence of actions and observations. These beliefs, denoted $b(s)$, give the probability of being in state $s$. The space of beliefs is often continuous, which can make finding an optimal policy for a POMDP intractable [22], [23].

The reward function for a POMDP has the same form as that of an MDP, $R(s, a, s')$. In the AV release problem, however, we do not know a system’s true safety level. The reward function in such a problem must instead be dependent on the belief, $R(b, a, b')$. The term $\rho$POMDP refers to POMDPs with belief-dependent rewards that are piecewise linear and convex [24]. Solving $\rho$POMDPs is an active area of research [25].

The vehicle release scheduling problem has belief-dependent rewards, making it a $\rho$POMDP. The state is the
hazardous event rate of the vehicle, $\lambda_{AV}$, and the action is the number of drive tests $n$ of length $d_{test}$ the manufacturer conducts. The state is unobserved, and instead must be inferred from the number of hazardous events, $k$, encountered during testing. The transition function for the $\rho$POMDP in the baseline formulation does nothing; the true hazardous event rate $\lambda_{AV}$ is assumed constant. The state, action, and observation space are listed in Table II.

The reward function contains two objectives: a terminal reward for achieving confidence in the safety of the system and a penalty for every hazardous event. We use a scalar parameter $\eta \in [0,1]$ to trade off between the penalty and reward. The belief and observation-dependent reward function is written:

$$R(b,n,k,b') = \eta \text{terminmal}(b') - (1-\eta)k,$$

where the method $\text{terminmal}$ is an indicator function that returns one if the successor belief provides sufficient confidence the system is safe and zero otherwise.

### B. Belief-MDP

The $\rho$POMDP formulation is computationally difficult to solve directly. A key insight into simplifying the problem is that the belief can be represented as a Gamma distribution, providing exact and efficient belief updates. Furthermore, the reachable beliefs are all discrete offsets of the parameters of the initial Gamma distribution, thereby resulting in a discrete belief space that can be iterated over using dynamic programming. We show that while the reachable set of beliefs is infinite, optimal policies can be computed by only considering a finite number of reachable beliefs.

1) **Conjugate Prior:** We extend Wachenfeld’s work by representing the belief over the expected hazardous event rate $\lambda$ using the Gamma distribution with density:

$$p(\lambda_{AV} ; \alpha, \beta) = \frac{\beta^\alpha \lambda_{AV}^{\alpha-1} e^{-\beta \lambda_{AV}}}{\Gamma(\alpha)},$$

where $\lambda_{AV} \in (0, \infty)$ and the belief is parameterized by the shape $\alpha > 0$ and the rate $\beta > 0$. The mean and variance under a Gamma distribution are $\alpha/\beta$ and $\alpha/\beta^2$, respectively.

We write a belief over the hazardous event rate as

$$\lambda_{AV} \sim \text{Gamma}(\alpha, \beta).$$

The Gamma distribution was chosen because it is a conjugate prior to the Poisson distribution. That is, the posterior belief formed by updating a Gamma distribution with an observation produces a new belief that is also a Gamma distribution. If $k$ hazardous events are observed in the course of performing $n$ drive tests, each with reference test distance $d_{test}$, the Bayesian posterior belief is updated according to:

$$\lambda_{AV} \sim \text{Gamma}(\alpha + k, \beta + n).$$

Such a Gamma belief over the mean of a Poisson-distributed random variable is known as a Poisson-Gamma model. These models are generally preferred to Poisson models in cases where data exhibits over-dispersion, as is often the case with automotive accident data [19].

The confidence in the safety of the system can be quantified according to:

$$P(\lambda_{AV} \leq \lambda_{ref}; \alpha, \beta) = \int_0^{\lambda_{ref}} p(\lambda; \alpha, \beta) \, d\lambda.$$  

When $\alpha$ and $\beta$ are integer values, the quantity above can be expressed as

$$P(\lambda_{AV} \leq \lambda_{ref}; \alpha, \beta) = 1 - e^{-\beta \lambda_{ref}} \sum_{m=0}^{\alpha-1} \frac{(\beta \lambda_{ref})^m}{m!}.$$  

The mean and variance under a Poisson-Gamma model are $\alpha/\beta$ and $\alpha/\beta^2$, respectively.

It can be shown that if the hazardous event rate is distributed according to a Poisson-Gamma model, then the number of hazardous events encountered during $n$ trials is governed by the negative binomial distribution [26]:

$$P(k; n, \alpha, \beta) = \binom{k + n \alpha - 1}{k} \left( \frac{1}{1 + \beta} \right)^k \left( \frac{\beta}{1 + \beta} \right)^{n \alpha}.$$  

For brevity, we denote the distribution above as $\text{NB}(n \alpha, \beta)$. The expected number of hazardous events encountered in $n$ trials is $n \alpha/\beta$.

2) **Formulation:** The vehicle release problem is formulated as a belief MDP. Each state parameterizes a Gamma distribution over the expected hazardous event rate. The state space is thus the Gamma distribution parameters $(\alpha, \beta)$, with shape $\alpha > 0$ and rate $\beta > 0$. The actions are still the number of drive tests to conduct. The transition function is the belief update Eq. (6), and the reward function is:

$$R(\alpha, \beta, n, k) = \eta \text{terminmal}(\alpha + k, \beta + n) - (1-\eta)k.$$  

where $\alpha$ and $\beta$ parameterize the initial belief and the method $\text{terminmal}$ is run on the posterior belief. The belief-MDP representation is summarized in Table II. If $\alpha$ and $\beta$ in the prior belief are integers, then they will still be integers in the posterior. The set of reachable beliefs is thus countably infinite.

### C. MDP

Intuition is boosted by making a lossless transformation to an MDP whose states consist of the number of driving tests conducted and the number of hazardous events observed. Optimal policies obtained for the resulting MDP have equivalent counterparts in the belief MDP. Let the state space $S$ be the set of all tuples $(K, N)$, corresponding to the cumulative number of hazardous events $K$ encountered in $N$ total drive tests. The action space $A$ is the set of all nonnegative integers corresponding to the number of drive tests to perform in the next
quarter. We first present the baseline MDP formulation, where the MDP state is equivalent to the belief over vehicle safety \((K, N) \Leftrightarrow (\alpha, \beta)\). We then explain how this formulation can be extended to account for manufacturer innovation and prior beliefs over vehicle safety.

The reward function for the MDP is still belief-dependent, but is expressed in terms of the current state, number of test drives, and number of observed adverse events. The reward function for the baseline problem formulation is given by:

\[
R(K, N, n, k) = \eta \mathbb{I}_{\text{terminal}}(K + k, N + n) - (1 - \eta)k. \tag{11}
\]

In the test-scheduling problem, we seek an optimal policy \(\pi^* : \mathcal{S} \to \mathcal{A}\) that prescribes the optimal number of drive tests to perform in each state.

1) **Maximizing Immediate Reward:** We first demonstrate how to find a policy that maximizes the immediate reward obtained by a manufacturer. We know the optimal test-scheduling policy will maximize the expected reward:

\[
\pi^*(K, N) = \arg \max_k \mathbb{E} \left[ R(K, N, n, k) \right], \tag{12}
\]

where \(k \sim \text{NB}(nK, N)\).

We know that \(\mathbb{E}[(1 - \eta)k] = (1 - \eta)nK\) \(\frac{K}{N}\), but computing \(\mathbb{E}_k[\eta \mathbb{I}_{\text{terminal}}(K + k, N + n)]\) requires taking an expectation over a countably infinite set of values for \(k\). However, close inspection of Eq. (8) reveals that if \(P(\lambda_N \leq \lambda_{\text{ref}}; K + k, N + n) \leq C_{\text{ref}}\) for some \(\tilde{k}\), then \(P(\lambda_N \leq \lambda_{\text{ref}}; K + k, N + n) \leq C_{\text{ref}}\) for all \(k \geq \tilde{k}\). Thus, it is possible to calculate \(\mathbb{E}_k[\eta \mathbb{I}_{\text{terminal}}(K + k, N + n)]\) efficiently by iterating until a value of \(k\) is encountered such that it is not possible to reach a terminal state.

Finding an optimal policy for each state requires finding a value of \(n\) that maximizes the expected reward. However, we cannot just loop over all values of \(n\) to find the optimal value, as the number of possibilities are countably infinite. Hence, we desire an upper bound on \(n^*\), the optimal amount of drive tests to perform. Note that the expected reward has a lower bound of zero, as it is always an option to not perform any drive tests and receive no reward. Furthermore, it is clear that \(\mathbb{I}_{\text{terminal}}(\cdot)\) has an upper bound of one. Using these properties, we find an upper bound on \(n^*\):

\[
0 \leq \mathbb{E}_k[\eta \mathbb{I}_{\text{terminal}}(K + k, N + n^*)] - (1 - \eta)k \tag{14}
\]

\[
\leq \eta - (1 - \eta)n^* \frac{K}{N} \tag{15}
\]

\[
\Rightarrow n^* \leq \frac{\eta}{1 - \eta} K \frac{N}{k} \tag{16}
\]

2) **Baseline Test-Scheduling Problem:** We now formulate a decision problem under which vehicle testing decisions can be made over the course of multiple quarters. In this sequential decision problem, we must consider the possible rewards that can be received far into the future, not just immediate rewards. If we wish to derive an optimal testing schedule over the course of \(T\) quarters, then we must solve for optimal policies that are functions of both the state \((K, N) \in \mathcal{S}\), as well as the current time step \(t \in \{1, 2, \ldots, T\}\).

For each state and time step, the aim is to determine the optimal utility, which captures the expected reward that will be received at all future time steps through following the optimal policy. According to the Bellman equation, the optimal utility \(U^*(K, N, t)\) will satisfy the recurrence [21]:

\[
U^*(K, N, t) = \max_n \mathbb{E}_k \left[ R(K, N, n, k) + \gamma U^*(K + k, N + n, t + 1) \right], \tag{17}
\]

where \(\gamma \in [0, 1]\) is the discount factor, a parameter that discounts rewards received in future time steps. We can limit the number of values of \(n\) that we must consider by using the upper bound derived in Eq. (14). Furthermore, for any terminal state \((K, N)\) it is assumed that \(U^*(K, N, t) = 0\) for any time step \(t\).

The optimal policy for the terminal time step \(T\) will be equivalent to the optimal policy under the immediate reward model, as it is not possible to attain any subsequent rewards. Thus, the optimal policy and associated utility can be found for each state \((K, N) \in \mathcal{S}\) at the terminal time step, and then a single backward sweep through time can be performed using Eq. (17) to calculate optimal utilities and policies at all preceding time steps.

In order to calculate the expected utility in future time steps \(\mathbb{E}_k[U^*(K + k, N + n, t + 1)]\), we must again take an expectation over a countably infinite number of hazardous events. However, it can be shown that if \(U^*(K + \tilde{k}, N + n, t + 1) = 0\) for some non-terminal state \((K + \tilde{k}, N + n)\), then \(U^*(K + k, N + n, t + 1) = 0\) for all \(k \geq \tilde{k}\) (see appendix). Thus, we only consider the utility of the future states up to the point that a state is encountered that is both non-terminal and has zero utility.

3) **Innovation Bonus:** The baseline test-scheduling problem described above does not allow for the safety of an autonomous driving system to improve over time through research and development. Hence, we propose a modification to the baseline problem formulation that allows the safety of a system to evolve stochastically over time through an innovation bonus. To accommodate this change, the state space is augmented with an additional integer-valued innovation term \(\beta_t\) that alters the expected number of hazardous events that will be encountered in future testing.

Between subsequent quarters, the change in innovation level \(\Delta \beta_t\) is drawn according to a categorical distribution over integer values and scaled linearly according to the number of drive tests performed. The scaling of innovation with drive tests models the ability of manufacturers to identify and remedy problems with their system in the course of real-world testing. For a given state \((K, N, \beta_t) \in \mathcal{S}\), the number of hazardous events encountered in \(n\) drive tests is distributed according to a negative binomial distribution that accounts for the current innovation level:

\[
k \sim \text{NB}(nK, N + \beta_t). \tag{18}
\]

Thus, while the innovation bonus cannot alter the number of hazardous events that have already occurred, it can make further hazardous events more or less likely depending on the value of \(\beta_t\). Positive innovation bonuses decrease the expected
number of hazardous events and negative innovation bonuses increase the expected number. In the absence of an innovation bonus, the distribution over future hazardous events is the same as the distribution in the baseline test-scheduling problem. Hence, the exact form of the innovation distribution will guide manufacturer decisions about when they should perform drive tests, as it can influence their beliefs about whether the safety of the system will change relative to its observed performance.

4) Prior Belief: The test-scheduling formulations presented above have assumed the absence of any relevant information that might influence a manufacturer’s belief about the safety of an autonomous driving system. However, in testing a real-world system, there may exist several relevant sources of information that would guide expectations about how a system will perform in future drive tests. For example, Table I informs us that the hazardous event frequencies for different types of events can vary by several orders of magnitude. Thus, in the course of driving the distance required to observe one fatality, we may develop a high-confidence estimate for a vehicle’s disengagement rate. If we assume that the disengagement rate is correlated with the fatality rate for an autonomous driving system, then we can use this relevant experience to bias our belief over the number of fatalities we will observe in future drive tests. Furthermore, if an autonomous driving system has undergone extensive testing in a high-fidelity simulation environment, its performance in simulation could likewise affect our belief.

Suppose we have relevant experience that shows a hazardous event frequency \( \mu \), and we would like to use it to construct a prior for our belief over \( \lambda_{AV} \), the hazardous event frequency of interest. As a prior belief over \( \lambda_{AV} \), we can assume it is drawn from a Gamma distribution with mean \( \mu \) and user-specified variance \( \sigma^2 \). A small value of \( \sigma^2 \) would indicate high confidence that the true value of \( \lambda_{AV} \) is likely to be close to \( \mu \), while a large value of \( \sigma^2 \) would indicate that \( \mu \) may not be very informative about the true value of \( \lambda_{AV} \).

We introduce the values \( \alpha_0 \) and \( \beta_0 \), which will parameterize the prior distribution over \( \lambda_{AV} \):

\[
\lambda_{AV} \sim \text{Gamma}(\alpha_0, \beta_0).
\]  

Using the mean and variance of the Poisson-Gamma model, we find the following relations [27]:

\[
\mu = \frac{\alpha_0}{\beta_0}, \quad \sigma^2 = \frac{\alpha_0}{\beta_0^2}.
\]  

(20)

Rearranging and solving, we find that \( \beta_0 = \mu/\sigma^2 \) and \( \alpha_0 = \mu \beta_0 \). By performing driving tests, we can incrementally update our belief over the hazardous event frequency of interest through the posterior distribution:

\[
\lambda_{AV} \sim \text{Gamma}(\alpha_0 + K, \beta_0 + N).
\]  

(21)

By incorporating this prior belief, the baseline test-scheduling problem is modified in two ways. First, the reward function changes, as the inclusion of prior knowledge alters the amount of experience required to reach a terminal state:

\[
\text{terminal}(K, N) = 1 \{ P(\lambda_{AV} \leq \lambda_{ref}, K + \alpha_0, N + \beta_0) \geq C_{ref} \}.
\]  

(22)

Furthermore, prior belief can alter the distribution over the number of hazardous events that may be encountered during driving tests. The distribution over hazardous events is given by:

\[
k \sim \text{NB}(\eta(K + \alpha_0), N + \beta_0 + \beta_I),
\]  

(23)

where \( \beta_I \) is the innovation bonus described previously.

An example five-quarter sequential policy is given in Fig. 1. The innovation distribution is described in the figure caption, and the remaining problem parameters can be found in Table III. In each policy plot, the terminal states can be found in the dark blue region of the upper left corner and the states where no testing occurs can be found in the lighter blue region in the bottom right corner. Note that testing tends to occur in states adjacent to the terminal states. Furthermore, fewer drive tests tend to occur in each state at early time steps, which can be attributed to the influence of the discount factor.

| \( C_{ref} \) | \( \lambda_{ref} \) | \( \eta \) | \( \gamma \) | \( \mu \) | \( \sigma^2 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.95            | 1               | 0.95            | 1               | 0.5             | 0.1             |

Table III: Problem parameters for policy in Fig. 1
Table IV: Minimum reward ratio required for any testing when \( \frac{K}{N} > \lambda_{\text{ref}} \).

| \( C_{\text{ref}} \) | 0.90 | 0.95 | 0.99 |
|---------------------|------|------|------|
| \( \left( \frac{\eta}{1-\eta} \right)_{\text{min}} \) | \( 3.50 \times 10^5 \) | \( 1.80 \times 10^5 \) | \( 2.52 \times 10^4 \) |

IV. PARAMETRIC STUDIES

In the following sections, we present a series of takeaways that result from studying optimal policies obtained through the procedures outlined above. These takeaways are meant to provide insight to manufacturers as they grapple with high-level decisions about vehicle testing. While the exact circumstances for each automated vehicle will vary and cannot be modeled exactly, the following analysis can provide guidance to manufacturers and regulators about how the testing and certification of autonomous vehicles can be accelerated in a safe manner.

A. If only concerned about immediate reward, a manufacturer should avoid drive tests when the observed hazardous event rate exceeds \( \lambda_{\text{ref}} \).

Of particular interest in the test scheduling problem are scenarios where an autonomous driving system has not demonstrated the requisite safety level in prior testing. For example, consider a vehicle that has undergone one drive test and experienced two hazardous events. If we assume that \( \lambda_{\text{ref}} = 1 \), then the observed hazardous event frequency for this vehicle exceeds the reference value. Such an outcome is troubling; while outside factors may have contributed to the observed hazardous events, a hypothesis that the system is sufficiently safe now requires heavy skepticism. Hence, it is worth pondering the circumstances under which it may be optimal to continue testing an autonomous driving system with a subpar safety record.

If a manufacturer is only concerned about immediate reward then they completely discount reward that can be received in subsequent quarters. Thus, they must weigh the reward for achieving sufficient confidence that safety requirements are met against the penalty associated with hazardous events that occur during testing. By modifying the structure of the reward function, we can incentivize or discourage testing in different regions of the state space. In Table IV, we present the minimum reward ratio \( \left( \frac{\eta}{1-\eta} \right) \) required in order for any drive tests to be administered when the observed hazardous event rate exceeds the reference value \( \lambda_{\text{ref}} = 1 \) (i.e. when \( K/N > 1 \)). Results are presented for confidence levels of 90%, 95%, and 99%.

From the table, we see that the reward associated with reaching a terminal state must be at least hundreds to thousands of times larger than the cost associated with a hazardous event in order for any drive testing to be performed in cases where an autonomous driving system has not met the reference safety standard in previous testing. It is unrealistic to assume that any manufacturer would care so little about hazardous events, in particular fatalities, that they would have such a lopsided reward function. Therefore, we conclude that manufacturers should not make an all-at-once attempt to prove the safety of an autonomous vehicle if previous tests have shown the system to have an inadequate safety record.

B. Expected innovation and prior beliefs can justify more testing.

In contrast with manufacturers who are myopic in pursuing immediate rewards, the sequential test-scheduling problem allows a manufacturer to reason about how their vehicle’s safety record may improve in future testing. The two primary mechanisms that can justify such optimism are (1) expectations of improved system performance through innovation and (2) prior beliefs that suggest the vehicle is likely to operate safely in the future. In practice, innovation or prior beliefs may make it optimal to perform drive tests in certain states \( \langle K, N \rangle \in S \) where testing would not occur in the baseline sequential test-scheduling problem.

This more aggressive testing strategy is shown in Fig. 2. The baseline region of the plot illustrates the states where it would be optimal to perform any drive tests during the first of five quarters in the baseline problem associated with Table III. Furthermore, the plot shows how the testing region varies with either the inclusion of an innovation bonus described in the caption of Fig. 1 or the incorporation of a prior belief with \( \mu = 0.5 \) and \( \sigma^2 = 0.1 \). Both sources of optimism cause the region of the state space where testing occurs to extend downward, which signifies that testing is occurring in states associated with higher observed hazardous event rates. An additional effect of incorporating prior experience can be noticed as well, as the upper bound of the testing region is decreased. In this case, the prior experience inspires greater confidence in the safety of the system, which reduces the testing burden required to prove safety and increases the number of terminal states.

Figure 2 provides illustrative examples of scenarios where manufacturers should be willing to tolerate higher hazardous rates.
event rates and continue testing. With the innovation bonus, manufacturers expect their system to become safer over time, and hence assume that the hazardous event rates encountered in future testing will be lower. In contrast, relevant experience can provide confidence that the system is in fact already safer than what has been observed so far, and that higher hazardous event rates may be anomalous and likely to decrease in future drive tests.

C. A patient manufacturer tests incrementally, but may tolerate higher hazardous event rates.

Manufacturers are motivated to be the first to market. They must balance the risk of aggressive testing against the relative safety of letting the research and development team improve vehicle safety over a longer time horizon. In the sequential test-scheduling problem, the balance between this desire for immediate over long-term rewards is controlled by a discount factor $\gamma$. In the limit as $\gamma \to 0$, all future rewards are neglected, and the policy in each quarter should be identical to the policy for a manufacturer that only cares about immediate rewards. However, as the value of the discount factor approaches one, future rewards are valued as much as immediate rewards, and the optimal testing strategy for a manufacturer can change considerably.

The change in testing strategy as $\gamma \to 1$ manifests itself in two ways, both of which are most apparent in early time steps. First, in states where testing does occur, the number of drive tests tend to decrease. Second, there is an increase in the number of states where testing occurs. Table V shows the average number of drive tests performed in each state and the fraction of nonterminal states where testing occurs as the discount factor is varied between 0.5 and 1 for the first quarter of a test-scheduling problem that otherwise is defined by the parameters given in Table III. While these results are for a single problem instantiation, they are illustrative of a trend that is observed in sequential test-scheduling problems with a variety of reward structures, innovation distributions, and forms of relevant experience.

The first change in testing strategy, where fewer drive tests are performed in each state, comes from tests being delayed until the end of the finite time horizon. Reward is not discounted, so there is no penalty in delaying a large portion of testing until the final quarter.

The second change in testing strategy, that a higher discount factor makes it optimal to perform tests in more states, is somewhat counterintuitive. The states where testing becomes optimal as $\gamma \to 1$ are associated with higher hazardous event rates. As such, it is undesirable to perform too many drive tests in these states, which means it may take several quarters of testing before a manufacturer can achieve sufficient confidence that their system is safe. Because this terminal reward signal can be quite delayed for states associated with a higher hazardous event rate, testing is only worthwhile if a manufacturer is patient and does not heavily discount future rewards relative to immediate rewards.

There is extensive interplay between the effect of manufacturer patience and the optimism engendered by the innovation bonus and prior beliefs. In states associated with higher hazardous event rates, expected innovation or outside information can spur a manufacturer to believe that hazardous event rates could be lower in future drive tests. However, depending on their level of confidence in this belief, manufacturers may only find it worthwhile to perform a limited number of tests. Hence, such restrained testing is most justified in cases where a delayed terminal reward for making it to market still outweighs the costs of hazardous events that occur during testing.

D. In the sequential decision problem, it can be optimal to perform drive tests even though the observed hazardous event rate exceeds $\lambda_{\text{ref}}$.

Manufacturers that only value immediate rewards do not perform incremental testing or consider how the performance of their system may improve over time. Instead, they only use the observed performance of their system in order to predict how it will fare in future testing. In previous sections, we have shown how expected improvement, relevant experience, and the discount factor all interact to guide manufacturer decisions when vehicle testing is performed over the course of several quarters instead of all at once. Furthermore, we have shown how these components of the sequential test-scheduling problem can inspire manufacturers to be more aggressive about testing in states associated with higher hazardous event rates.

Under certain circumstances, it is optimal to perform driving tests even though previous tests have yielded an observed hazardous rate that exceeds $\lambda_{\text{ref}}$. The gray curves in Fig. 3 show the maximum observed hazardous event rate, $\langle K/N \rangle_{\text{max}}$, under which it would be optimal to perform any testing for the testing policies shown in Fig. 1. These curves illustrate multiple instances where it is optimal to perform testing even though the hazardous event rate exceeds $\lambda_{\text{ref}} = 1$. The values of $\langle K/N \rangle_{\text{max}}$ are plotted against the total number of drive tests performed, meaning that hazardous event rates associated with a smaller number of drive tests exhibit a higher level of variance. Hence, it is clear that manufacturers can choose to be more aggressive with testing in cases where there is more uncertainty in a system’s estimated safety level.

While we have shown that it can be optimal to perform drive tests in cases where the observed hazardous event rate is higher than the reference standard, it is worth stressing that such circumstances are rare. The black curves in Fig. 3 illustrate optimal release decisions for the baseline problem associated with Table III (i.e. the same problem without the innovation bonus or relevant experience). These curves show that under such circumstances it is never optimal to perform testing when the hazardous event rate exceeds $\lambda_{\text{ref}}$. Thus, in many cases it is not optimal to continue testing when previous performance has shown a system to be insufficiently safe. However, the following factors would make it more
likely that it is worthwhile to continue testing an autonomous driving system that has already experienced a high number of hazardous events:

1) High uncertainty in the hazardous event rate (few drive tests have been performed).
2) Many more quarters remaining for testing.
3) High reward for reaching a terminal state relative to penalty for hazardous events.
4) High likelihood of improvement through innovation.
5) Prior belief provides confidence that the system should be safer than previous drive tests suggest.
6) High discount factor ($\gamma \approx 1$).

E. Incorporating prior beliefs can change the amount of required testing.

Aside from influencing testing decisions for manufacturers, prior beliefs can also provide insights about how regulators may be able to alleviate the immense real-world testing burden necessary to prove the safety of an automated vehicle. As noted previously, the amount of real-world testing required to establish sufficient confidence that the fatality rate for an autonomous driving system is lower than that of human drivers is extensive and likely impractical. However, rather than gaining confidence entirely through drive tests, it may be possible to augment driving experience with other sources of information, such as performance in simulation, to reach the same level of confidence.

The stronger the belief that the relevant experience is predictive of the safety level of the automated vehicle, the stronger it will bias the estimate of the hazardous event rate for that system. This confidence in the predictive value of the experience is encoded in its estimated variance, $\sigma^2$, and will need to be determined on a case-by-case basis. In Fig. 4, we see the effect that different values of $\mu$ and $\sigma^2$ have on the real-world testing burden required to prove system safety. The top three plots show the change in the number of terminal states in a 50 × 50 state space ($K \in \{1, \ldots, 50\}, N \in \{1, \ldots, 50\}$) for three values of $\mu$ and a range of variance values when $C_{\text{ref}} = 0.95$ and $\lambda_{\text{ref}} = 1$. Reaching a terminal state means no additional testing is required to prove system safety. Thus, an increase in the number of terminal states means that the testing burden is reduced, while a decrease in the number of terminal states signifies that even more testing should be performed to prove system safety.

Three distinct behaviors can be observed for different values of $\mu$. In the top left plot, when $\mu = 0.5$, we see what is deemed Type 1 behavior, where at low variance values there is an increase in the number of terminal states, with the change in terminal states decreasing as the variance becomes large and hence non-informative. In contrast, in the top right plot, when $\mu = 1$ we see Type -1 behavior, where the number of terminal states decreases for all variance values and the change tends toward zero for large variances. In the top middle plot, when $\mu = 0.9$, we see Type 0 behavior, which like Type 1 behavior exhibits an increase in the number of terminal states at low variance values and little net effect at high variances. Interestingly, however, at intermediate variance values it shows a decrease in the number of terminal states (and hence an increased testing burden). This behavior likely arises because $\mu$ is relatively close to $\lambda_{\text{ref}}$, and at moderate variance levels the uncertainty is high enough that we must take seriously the possibility that the hazardous event rate for the vehicle may actually exceed the reference standard.

The bottom three plots of Fig. 4 show that the testing burden can change as $\mu$ and $C_{\text{ref}}$ vary. For $C_{\text{ref}} \geq 0.95$, we see that for $\mu < \lambda_{\text{ref}}$ there is either Type 1 or Type 0 behavior, while for $\mu > \lambda_{\text{ref}}$ there is always Type -1 behavior. This provides two practical takeaways for regulators. First, if a manufacturer can demonstrate through relevant experience that their hazardous event rate may be lower than the reference standard, then it is possible that their testing burden can be reduced, although this depends on the assumed predictive power of such experience. Second, if relevant experience suggests that an autonomous driving system may not be sufficiently safe (e.g. the collision rate in simulation exceeds the reference standard), then there should always be an enhanced real-world testing burden on the manufacturer.

F. When should the most testing be performed?

We have studied the conditions necessary to make it worthwhile for a manufacturer to perform any drive tests, but we have not yet addressed a related question: under which conditions should a manufacturer perform the most testing? In particular, it is worth considering which observed hazardous
event rates should make a manufacturer most motivated to aggressively test their automated vehicles. To provide a clearer answer to this question, Fig. 5 plots the optimal number of drive tests to perform in a terminal time step against the observed hazardous event rate with $\eta = 0.99$ and $C_{\text{ref}} = 0.95$. The high terminal reward and lack of opportunity for future drive tests in this problem provides a strong incentive for performing a large amount of immediate testing. With a large amount of testing performed, it is easier to see a trend in the results.

The points in Fig. 5 are colored according to the number of drive tests already performed, and hence the color can be seen to represent the variance in the hazardous event rate estimate. The lighter points therefore exhibit highest variance, while the darker points have the lowest variance. This figure shows that the largest number of drive tests tend to occur in states with low hazardous event rates and high variance. Hence, manufacturers should be most aggressive about testing in circumstances when their observed hazardous event rate is low, but they have not yet performed sufficient testing to achieve confidence that their system meets the reference safety standard.

V. CONCLUSIONS

This work introduced a method for deriving optimal testing strategies for autonomous vehicles. The problem is formulated as a finite-horizon Markov decision process, where policies map a vehicle’s observed safety record to a recommendation for how many drive tests should be carried out in the next quarter. By performing parametric studies on a set of optimal
testing policies, we arrive at a series of insights that can guide automotive manufacturers as they determine how to approach testing their products in the real world. We discussed the range of assumptions that manufacturers must make in order to continue testing their product in circumstances where previous testing has shown it to be insufficiently safe. Additionally, we showed how regulators can use information obtained from sources other than road testing to reduce the testing burden for an autonomous vehicle. Finally, we demonstrated that manufacturers should perform the most drive testing in cases where their observed hazardous event rate is low, but there is high uncertainty in the estimate.

The goal of this project is to provide manufacturers with high-level takeaways about relevant considerations relating to vehicle testing, not to derive an optimal testing policy for any particular vehicle. As such, manufacturers may wish to modify the problem formulation in areas that might lack the appropriate level of fidelity. For example, our analysis does not account for the cost of performing drive tests because reasoning about the exact form of this cost would complicate analysis without providing much additional insight. Instead, it is left to manufacturers to tailor this analysis in ways that enhance its relevance to their product. To facilitate this analysis, the code for this project has been released to the public, and can be found at https://github.com/sisl/av_testing.

**APPENDIX A**

**PROOF THAT SUBSEQUENT STATES HAVE ZERO UTILITY**

Here, we prove that if $U^*(K, N, t) = 0$ for some state $(K, N)$, then $U^*(K + \ell, N, t) = 0$ for all positive integers $\ell$. We limit the proof in this section to $\ell = 1$ and the terminal time step $T$, but it follows inductively that the same analysis holds for any greater $\ell$ and for all time steps $t$. We know that the optimal utility at the terminal time step satisfies:

$$U^*(K, N, T) = \max_n \mathbb{E}_k [\eta \text{ isterminal}(K + k, N + n)] - (1 - \eta)k,$$

where $k \sim \text{NB}(nK, N)$. If we encounter a state $(K, N)$ where $U^*(K, N, T) = 0$, then we know that

$$\mathbb{E}_k [\eta \text{ isterminal}(K + k, N + n)] \leq (1 - \eta)\frac{K}{N}$$

for all $n$.

Now consider a state $(K + 1, N)$. From Eq. (8), we know that

$$\mathbb{E}_{k \sim \text{NB}(n(K + 1), N)} [\text{ isterminal}(K + k + 1, N + n)]$$

$$\leq \mathbb{E}_{k \sim \text{NB}(n(K + 1), N)} [\text{ isterminal}(K + k, N + n)].$$

We would like to show that

$$\mathbb{E}_{k \sim \text{NB}(nK, N)} [\text{ isterminal}(K + k, N + n)]$$

$$\leq \mathbb{E}_{k \sim \text{NB}(nK, N)} [\text{ isterminal}(K + k, N + n)].$$

For each $n$, we can say that

$$\mathbb{E}_{k \sim \text{NB}(nK, N)} [\text{ isterminal}(K + k, N + n)]$$

$$= \sum_{k=0}^{k(n)} P(k; n, K, N),$$

where $k(n)$ is the largest value of $k$ such that $\text{ isterminal}(K + k, N + n)$ takes on a value of one. Hence, the above quantity represents the cumulative distribution function over $k$.

For the negative binomial distribution, the cumulative distribution function is given in terms of the regularized beta function $\mathcal{I}_p(\cdot)$ [28]. We have that

$$\mathbb{E}_{k \sim \text{NB}(n(K + 1), N)} [\text{ isterminal}(K + k, N + n)]$$

$$= \mathcal{I}_p(nK + n, k(n) + 1)$$

and

$$\mathbb{E}_{k \sim \text{NB}(nK, N)} [\text{ isterminal}(K + k, N + n)]$$

$$= \mathcal{I}_p(nK, k(n) + 1),$$

where $p = N/(1 + N)$. The regularized beta function has the property that $\mathcal{I}_p(a + 1, b) \leq \mathcal{I}_p(a, b)$. Thus, we find that the inequality in Eq. (27) holds.

We denote the expected utility of performing $n$ drive tests as $U^{(n)}(\cdot)$. For each $n$, it follows that

$$U^{(n)}(K + 1, N, T) =$$

$$\mathbb{E}_k [\eta \text{ isterminal}(K + k + 1, N + n)] - (1 - \eta)k],$$

where $k \sim \text{NB}(n(K + 1), N)$. Using the inequalities in Eq. (26) and Eq. (27), we find that

$$U^{(n)}(K + 1, N, T) \leq \mathbb{E}_k [\eta \text{ isterminal}(K + k, N + n)]$$

$$- (1 - \eta)n\frac{K + 1}{N}$$

where $k \sim \text{NB}(nK, N)$. Finally, we use the result from Eq. (25) to find that:

$$U^{(n)}(K + 1, N, T) \leq (1 - \eta)nK\frac{1}{N} - (1 - \eta)n\frac{K + 1}{N}$$

$$= -(1 - \eta)\frac{n}{N} \leq 0.$$ (34)

Thus, we see that performing any positive number of drive tests has a negative expected utility, and hence $U^*(K + 1, N, T) = 0$.

**ACKNOWLEDGMENT**

This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-114747 and upon work supported by Robert Bosch LLC. We gratefully acknowledge the helpful comments received from anonymous reviewers.
