Dynamic Hurst Parameter Estimation of Multi-fractional Processes in Impulse Noise Environment

Shuyuan Zhao¹,², Hu Sheng³*, Tianshuang Qiu¹

¹ School of Electronic and Information Engineering, Dalian University of Technology, Dalian, China
² School of Electrical Engineering, Dalian University of Science and Technology, Dalian, China
³ School of Electrical and Information Engineering, Dalian Jiaotong University, Dalian, China

*Corresponding author e-mail: hu.sheng@djtu.edu.cn

Abstract. Hurst parameter estimation is the key issue for long range dependent system modeling and data prediction. Dynamic Hurst parameter estimation of data with impulse noise is difficult. In this research, the Dynamic Hurst parameter of multi-fractional signal is estimated using sliding windowed R/S method and generalized Hurst exponent. Simulation results show that sliding windowed R/S method and generalized Hurst exponent can effectively analyze the Hurst parameter of impulse noise contaminated signals.

Keywords: Hurst parameter estimation, generalized Hurst method, dynamic Hurst parameter

1. Introduction

Self similarity analysis has been wildly investigated in many fields, such as biomedicine, economics, hydrology[1-3], etc. Self similarity, which can be measured by Hurst parameter, reflects the correlation between local and global signals. Traditionally, Hurst parameter \( H \) is a constant between 0 and 1. However, many research results show that the constant Hurst parameter cannot accurately describe some complex physical phenomena and complex non-stationary signals. The main reason is that constant Hurst parameter can not accurately describe the multi-scale or local memory characteristics of these stochastic processes.

Self similarity analysis is seriously affected by impulse noise, so it is difficult to accurately estimate the Hurst parameter of this kind of signals. In this research, dynamic Hurst parameter of simulated multi-scaling data is estimated. There are two kinds of simulated signals, which are multi-fractional Gaussian signal and impulse noise contaminated fractional Gaussian signal[4-6].

The paper is organized as follows. In Section 2, the theory of R/S method and generalized Hurst method are introduced briefly. Section 3 provides the evaluation results of dynamic Hurst parameter for simulated multi-fractional signals. The conclusion of this study is presented in Section 4.
2. Hurst parameter estimation methods

Hurst parameter estimation is very important for long-range dependent signal modeling and prediction. Some estimation methods were proposed in research results and adopted in financial data, biomedical data, hydrological data, network traffic data[1-3], etc. The traditional R/S estimation method provides a standardized time series statistical method to reveal the long-term correlation in random process. R/S estimation method is one of the most commonly used Hurst index estimation methods[7]. The basic idea is to study the changes of time series under different time scales, which can be divided into uncorrelated time series and correlated time series. Statistical characteristics of the objective existence of self similarity between the whole and the local are calculated. Because of the short-term dependence of the traditional R/S estimation method, the accuracy of the algorithm is not high, and it is affected by the noise pulse. Therefore, generalized Hurst method is used in this research to estimate the Hurst parameter.

Generalized Hurst method estimate the Hurst parameter by computing the q-order moments of the data increment [8-10]. The q-order moments of a stochastic variable \( S(t) \) is defined as

\[
K_q(\tau) = \frac{\langle [S(t+\tau) - S(t)]^q \rangle}{\langle S(t)^q \rangle}.
\]  

(1)

There \( \tau \) can vary between 1 and \( \tau_{max} \). The generalized Hurst parameter \( H_q(q) \) is defined as

\[
K_q(\tau) \propto \tau^{\frac{q}{\alpha(q)}}.
\]  

(2)

Taking logarithm of both sides for equation (2), we can get the following linear relationship

\[
\ln(K_q^{\alpha(q)}(t, \tau)) = qH^{\alpha(q)}(q)\ln(\tau) + C,
\]  

(3)

and \( C \) is a constant. Hurst parameter \( H_q(q) \) can be evaluate by the slope of a curve, which is the logarithmic chart of \( \ln \tau \) and \( \ln(K_q^{\alpha(q)}(t, \tau)) \).

3. Time variable Hurst parameter estimation of multi-fractional stable processes

In this section, the time variable Hurst parameters of simulated signals are estimated using sliding windowed R/S method[11] and generalized Hurst method. Two signals with local self similarity are generated, and the analysis results are provided.

3.1 Multi-fractional processes generation

It has been researched that the multi-fractional signals can be generated using variable-order fractional calculus. The multi-fractional stable signal can be simulated by variable-order integration of alpha stable process[12].

The variable order fractional Gaussian signal can be expressed as

\[
Y_{H(t)}(t) = \mathcal{D}_t^{-\alpha(t)} \omega(t) = \frac{1}{\Gamma(\alpha(t))} \int_0^t (t - \tau)^{\alpha(t) - 1} \omega(\tau) d\tau, 0 < \alpha(t) < 1/2,
\]  

(4)

where \( H(t) = 1/2 + \alpha(t) \), \( \omega(t) \) is a white Gaussian noise, and \( \mathcal{D}_t^{-\alpha(t)} \) is the variable order fractional integral[13].
Based on this theory, the multi-fractional Gaussian signal is generated as Figure 1. The variable Hurst parameter $H(t)$ of signal in Figure 1 is $H(t) = at + b$, where $a = 4 \times 10^{-4}$, $b = 0.5$.

3.2 Dynamic Hurst parameter estimation of multi-fractional Gaussian signal

Sliding windowed R/S method and generalized Hurst method are used to estimate the multi-fractional Gaussian signal. Figure 2 shows the estimation results of multi-fractional Gaussian signal based on R/S method. The estimation results of generalized Hurst method is provided in Figure 3, where $q = 2$. The blue dashed line in figures is true Hurst parameter value, and red solid line is estimated parameter.

**Figure 1.** Multi-fractional Gaussian signal

**Figure 2.** Time variable Hurst estimation results of multi-fractional Gaussian signal based on sliding windowed R/S method

**Figure 3.** Time variable Hurst estimation results of multi-fractional Gaussian signal based on generalized Hurst method
It can be seen from Figure 2 that the estimation result sliding windowed R/S method is inaccurate. The estimated variable Hurst parameter is obviously lower than true value when $300 < t < 500$ and $720 < t < 820$. Figure 3 shows that the estimation result of generalized Hurst method is much better than sliding windowed R/S method. The estimated variable Hurst parameter is close to true value in all ranges of $t$.

3.3 Dynamic Hurst parameter estimation of multi-fractional Gaussian signal with impulse noise

In order to investigate the estimation results of the impulsive data, 20dB alpha stable noise with $\alpha = 1.5$ is added into multi-fractional Gaussian signal in Figure 1. Sliding windowed R/S method and generalized Hurst method are used to estimate the impulsive data. Figure 4 shows the estimation results of multi-fractional Gaussian signal with 20dB alpha stable noise based on R/S method. The estimation results of generalized Hurst method is provided in Figure 5, where $q = 2$. The blue dashed line in figures is true Hurst parameter value, and red solid line is estimated parameter.

![Figure 4. Time variable Hurst estimation results of multi-fractional Gaussian signal with 20dB alpha stable noise based on sliding windowed R/S method](image)

![Figure 5. Time variable Hurst estimation results of multi-fractional Gaussian signal with 20dB alpha stable noise based on generalized Hurst method](image)

It can be seen from Figure 4 that the estimation result sliding windowed R/S method is inaccurate as well. However, compare with the Figure2, the impulse noise almost has no effect on the estimation results. Figure 5 shows that the estimation result of generalized Hurst method is better than sliding windowed R/S method, but the impulse noise has some effect on the estimation results. The estimation results are lower than that in multi-fractional Gaussian signal.
3.4 Error analysis of generalized Hurst method

In order to evaluate the Hurst parameter estimation method, the mean square error of R/S method and generalized Hurst method are calculated. The mean square errors of two kinds of data, which are multi-fractional Gaussian signal, and multi-fractional Gaussian signal with 20dB alpha stable noise are calculated. Table 1 shows the mean square errors results, where the first row of data is the error analysis of multi-fractional signal, the second row multi-fractional signal with 20dB alpha stable noise. It can be seen that the mean square errors of generalized Hurst method in three signals are much less than sliding windowed R/S method.

Table 1. Mean square errors results

| Numble | Signal                                      | R/S method | Generalized Hurst |
|--------|---------------------------------------------|------------|-------------------|
| 1      | Multi-fractional Gaussian signal            | 0.0314     | 0.0102            |
| 2      | Multi-fractional Gaussian with 20dB noise   | 0.0294     | 0.0124            |

4. Conclusion

This research investigates the estimation of dynamic Hurst parameter. Generalized Hurst method and sliding windowed R/S method are used in two kinds of signal, which are multi-fractional Gaussian signal, and 20dB contaminated multi-fractional Gaussian signal. The dynamic Hurst parameter of the two signals is \( H(t) = at + b \). The estimation results and the mean square error analysis show that the sliding windowed R/S method has good robustness against impulse noise, but it has low accuracy. Generalized Hurst method can accurately estimate the dynamic Hurst parameter, but it is affected by impulse noise.

Acknowledgments

The study was supported by “The Doctoral Scientific Research Foundation of Liaoning Province, China (Grant No. 20170520215)”, “Natural science research project of Liaoning Provincial Department of Education, China (Grant No. JD2019014)”

References

[1] N. R. Danganan, R. Tamangan, N. Guba-Natan. The Identification of Long Memory Process in the Asean-4 Stock Markets by Fractional and Multifractional Brownian Motion. Social Science Electronic Publishing, 2006, 65:1-2.

[2] S. V. Muniandy, R. Murugan. Inhomogeneous scaling behaviors in Malaysian foreign currency exchange rates. Physica A, 2001, 301(1-4):407-428.

[3] H. Tyralis, P. Dimitriadis, D. Koutsoyiannis. On the long-range dependence properties of annual precipitation using a global network of instrumental measurements. Advances in Water Resources, 2018, 111:301-318.

[4] B. B. Mandelbrot, J. W. Van Ness M J W V . Fractional Brownian Motions, Fractional Noises and Applications[J]. SIAM Review, 1968, 10(4):422-437.

[5] H. P. Corporation. Fractal Time Series—A Tutorial Review. Mathematical Problems in Engineering,2010,(2009-11-09), 2009, 2010(1024-123X):242-256.

[6] A. Dabiri, E. A. Butcher. Stable fractional Chebyshev differentiation matrix for the numerical solution of multi-order fractional differential equations Nonlinear Dynamics, 2017, 90(1):185-201.

[7] B. B. Mandelbrot. Computer experiments with fractional Gaussian noises. Water Resources Research, 1969, 5(1):242-259.

[8] T. D. Matteo, T. Aste, M. M. Dacorogna. Long-term memories of developed and emerging markets: Using the scaling analysis to characterize their stage of development. Journal of Banking & Finance, 2005, 29(4):0-851.

[9] T. D. Matteo, T. Aste, M. M. Dacorogna. Scaling behaviors in differently developed markets.
Physica A, 2003, 324(1-2):183-188.

[10] T. D. Matteo. Multi-scaling in finance[J]. Quantitative Finance, 2007, 7(1):21-36.

[11] H. Sheng, H. G. Sun, Y. Q. Chen, T. S. Qiu. Tracking performance and robustness analysis of Hurst estimators for multifractional processes. IET Signal Processing, 2012, 6(3):213-226.

[12] H. Sheng, Y. Q. Chen, et al. Synthesis of multifractional Gaussian noises based on variable-order fractional operators. Signal Processing, 2011, 91(7):1645-1650.

[13] C. F. Lorenzo, T. T. Hartley. Variable Order and Distributed Order Fractional Operators. Nonlinear Dynamics, 2002, 29(1-4):57-98.