Gravity Dual of Spatially Modulated Phase

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We analyze an instability that may be induced by a chiral anomaly at finite densities in a strongly-coupled gauge theory. We employ the gauge/gravity duality. The effect of the chiral anomaly is represented by a Chern-Simons (CS) term in the gravity dual and it has a tendency to induce an inhomogeneous flow of the global current, namely an inhomogeneous instability, at finite densities. From the viewpoint of general relativity, this can be a new instability of charged black holes induced by the CS term. However, as far as we have analyzed based on the supergravity, the potentially unstable modes turned out to be always barely stable. In terms of the gauge theory, the stability is protected due to the coupling between the global current and the stress-energy tensor.

§1. Introduction

One of the distinctive characteristics of QCD is that the simple first-principle Lagrangian yields a great variety of the low-energy phenomena. The variety would be attributed to the strong interaction, which prevents us from direct derivation of the low-energy effective theory out of the first-principle Lagrangian. However, a systematic recipe to yield (a certain sector of) the low-energy effective theories from the first-principle Lagrangians has been proposed within the superstring theory: the gauge/gravity duality or the AdS/CFT correspondence. The gauge/gravity duality is a correspondence between some categories of strongly-interacting gauge theories and the 10-dimensional classical supergravities (or superstring theories) on curved spacetimes. This enables us to analyze the non-perturbative nature of the gauge theories in terms of the classical gravity. Although the duality is a conjecture due to lack of the complete understanding of the non-perturbative formulation of the superstring theory, it has been proposed in a very natural and logical way.

In this talk, we study an effect of a chiral anomaly of a global current at finite densities. Our final goal is to investigate QCD at finite baryon densities. However, due to technical reasons, we shall idealize the system in such a way that we can safely apply the computational technique of the gauge/gravity duality. More specifically, we consider \( SU(N_c)N = 4 \) supersymmetric Yang-Mills theory (SYM) at the large-\( N_c \) limit with the 't Hooft coupling \( \lambda \). We take \( \lambda \) to be large (\( \lambda \gg 1 \)) and we introduce a finite density of a global charge. The global symmetry we consider is the \( U(1) \) sector of the \( SU(4) \) symmetry of the \( N = 4 \) SYM. It has been known that the fermions in \( N = 4 \) SYM transform chirally under the \( \mathcal{R} \)-symmetry transformation, and there is an Adler-Bell-Jackiw (ABJ) type anomaly. We consider the \( U(1)_R \) anomaly at the finite \( U(1)_R \)-charge densities and its effects on the stability of the system.

In the gravity dual, the effect of the anomaly is already taken into account

\[ * \) This talk is based on the work with H. Ooguri and C. S. Park.\[1\]
at the classical level in terms of the Chern-Simons (CS) term\(^*\) which appears by requesting the closure of the supergravity algebra. The \(U(1)_R\)-charge corresponds to the charge of the dual black hole. Therefore, what we shall do is the stability analysis of the charged black hole in the presence of the CS interaction. From the viewpoint of general relativity, it is not quite common to take the CS interaction into Einstein’s framework. In this sense, our analysis on the black hole is new even from the viewpoint of general relativity, and we suggest a possibility of a new instability of the charged black holes induced by the CS interaction.

The organization of this report is as follows. In §2, we give a quick view on the gauge/gravity duality which may help the audience who are not familiar with the superstring theory. In §3, we outline the stability analysis. We conclude in §4.

\section*{2. Black holes as thermodynamic systems of gauge theories}

\(SU(N_c) \mathcal{N} = 4\) SYM at large-\(N_c\) with \(\lambda \gg 1\) is conjectured to be equivalent to the weakly coupled type IIB supergravity theory on \(AdS_5 \times S^5\).\(^2\) If we introduce finite temperature, the \(AdS_5\) part is modified to a 5-dimensional Schwarzschild black hole embedded in an asymptotically \(AdS_5\) geometry (AdS-BH).\(^4\) The gauge-theory side is in the “deconfinement” phase in this case.\(^*\) \(\mathcal{N} = 4\) SYM is a conformal field theory and the critical temperature of the “confinement/deconfinement” transition is zero; the system is “deconfined” as soon as the temperature is switched on.

We may find an intuitive (but not very precise) illustration why we need 5-dimensional AdS-BH. Suppose that a certain gauge theory at finite temperature is known to be dual to a gravity theory. One way to introduce the notion of temperature in gravity is to consider a black hole. It has been known that the classical dynamics of the black holes mimics the thermodynamics.\(^5\) For example, the relationship between the mass of the black hole \(M\) and the area of the horizon \(A\) is given as

\[ dM = T_H (4G_N)^{-1} dA, \]

where \(G_N\) is Newton’s constant and \(T_H\) is the Hawking temperature\(^6\) which can be identified with the temperature of the black hole. Equation (2.1) may be identified with the first law of thermodynamics if we regard \(A/(4G_N)\) as the entropy. Indeed, there are other relationships in the black-hole quantities that correspond to the zeroth, second, and the third law of thermodynamics under these identifications.\(^5\)

Let us accept the idea that the area of the horizon is proportional to the entropy of the corresponding 3+1 dimensional gauge theory. Since the entropy is an extensive quantity, it should be proportional to the volume of the space where the gauge theory lives. The entropy should be given by the “volume” rather than the “area” of the horizon, which means that the horizon need to extend into 3+1 directions. Since the horizon is the boundary between the trapped region (from which the light cannot escape) and the un-trapped region (where the light emitted outward can escape from the black hole), we need at least one more spatial direction perpendicular

\(^{*1}\) See, Witten (1998) in Refs. 2) and 3).

\(^{**}\) Here, the meaning of “deconfinement” is that the entropy is proportional to \(N_c^2\). Because of the conformal nature of the \(\mathcal{N} = 4\) SYM, the linear potential will not be realized even in this phase.
to the horizon along which we can divide the trapped and the un-trapped regions. This illustrates why we need the extra 5th direction (z-direction in our notation). Furthermore, it has been shown that a black hole embedded in flat spacetime has negative specific heat, while the specific heat can be positive if the black hole geometry is asymptotically AdS.\(^7\) This is why the 5-dimensional AdS-BH is a natural candidate for the gravity dual of a (3+1)-dimensional gauge theory at finite temperatures. Of course, the above is only an intuitive illustration and the rigorous argument to reach the gauge/gravity duality is based on the D-brane (or M-brane) picture and the superstring theory (or M-theory).\(^2\) The 5-dimensional AdS-BH is indeed conjectured to be the gravity dual of the \(N = 4\) SYM at finite temperature in the “deconfinement phase”.\(^4\)

We can also introduce the finite densities by considering a charged AdS-BH. In this case, the black-hole charge corresponds to a global charge of the corresponding gauge theory. Equation (2.1) is generalized to the first law of thermodynamics including the finite charge density. The black-hole charge couples with the 5-dimensional gauge field \(A_M\). The chemical potential is identified with \(A_0\) at the boundary (the most distant place from the horizon) under the gauge choice \(A_0(z_H) = 0\), where \(z = z_H\) is the location of the horizon. The electrically charged black hole in asymptotically AdS spacetime is known as AdS-Reissner-Nordström black hole (AdS-RN-BH). Again, the rigorous identification can only be possible by going back to the superstring theory. The \(U(1)\) charge of the AdS-RN-BH is identified with that of the \(U(1)_R\)-symmetry of the \(N = 4\) SYM.

§3. Einstein-Maxwell-Chern-Simons theory

The gravity dual of the \(N = 4\) SYM with \(U(1)_R\)-charge density is governed by the following 5-dimensional Einstein-Maxwell-Chern-Simons action\(^8\),\(^9\):

\[
S = \frac{1}{16\pi G_N} \int d^4 x dz \left[ \sqrt{-g} \left( R - 2\Lambda - \frac{1}{4} F^2 \right) + \alpha \epsilon^{IJKLM} A_I F_{JK} F_{LM} \right],
\]

(3.1)

where \(F_{IJ}\) is the field-strength of the \(U(1)\) gauge field. We choose the convention in such a way that the cosmological constant is given as \(\Lambda = -6\) and \(G_N = \pi/(2N_c^2)\). The CS coupling \(\alpha\) is fixed to be \(\alpha = 1/(2\sqrt{3})\) from the supersymmetric algebra of the supergravity. The supersymmetry is broken even at the extremal case (where \(T_H = 0\)) at finite densities.\(^9\) However, we need to require the above value of the CS coupling at the level of the Lagrangian. The AdS-RN-BH solution is given by

\[
ds^2 = \frac{1}{z^2} \left[ -f(z)(dx^0)^2 + d\vec{x}^2 + f(z)^{-1} dz^2 \right], \quad F_{z0} = F_{0z} = qz,
\]

with \(f(z) = 1 - m z^4 + \frac{q^2}{12} z^6\), \(m = z_H^{-4} + \frac{q^2 z_H^2}{12}\),

(3.2)

where \(m\) and \(q\) parametrize the mass and the charge of the black hole, respectively. The Hawking temperature is given by \(T_H = (\pi z_H)^{-1} (1 - q^2 z_H^6/24)\). The location of

\(^*\) The signature of the metric is diag(−, +, +, +, +).
the boundary is \( z = 0 \). The CS term does not affect the AdS-RN-BH solutions; the above is exactly the same as the solution to the Einstein-Maxwell theory without the CS term. However, the stability analysis will be affected by the CS term.

The geometry is stable if any fluctuation around the solution decays along the time evolution, while it is unstable if some fluctuation glows along the time evolution. The fluctuation we shall consider in this talk is that of the 5-dimensional \( U(1) \) gauge field which carries finite momentum along the \( x^3 \) direction:

\[
\delta A_i = f_i(z)e^{-i\omega t + ikx^3},
\]

where \( i = 1, 2 \). We find that these modes may be unstable due to the CS interaction.\(^*)\)

In order to demonstrate this, let us consider the fluctuation in the 5-dimensional Maxwell theory on the Minkowski spacetime as an exercise:

\[
S_{\text{Maxwell}} = \int d^4x dz \left[ -\frac{1}{4} F^2 + \frac{\alpha}{3!} \epsilon^{IJKLM} A_I F_{JK} F_{LM} \right].
\]

(3.4)

We assume the presence of a constant electric flux \( F_{z0} = E \). The equations of motion at the linear order of the fluctuations of (3.3) are given by

\[
\partial^2 f_1 - 4\alpha E(ik) f_2 = 0, \quad \partial^2 f_2 + 4\alpha E(ik) f_1 = 0,
\]

(3.5)

where we have employed the Lorenz gauge \( \partial^M A^M = 0 \). Due to the CS interaction, the two modes are mixed. The equations are diagonalized by considering the circularly polarized modes \( f_\pm \equiv f_1 \pm if_2 \):

\[
(\omega^2 - k^2) f_\pm \mp 4\alpha E k f_\pm = 0,
\]

(3.6)

from which we can read the dispersion relation as

\[
\omega^2 - (k \pm 2\alpha E)^2 = -(2\alpha E)^2.
\]

(3.7)

There are two important features here. The minimum of \( \omega^2 \) is realized at \( k = \pm 2\alpha E \) but not at \( k = 0 \), and \( \omega \) is imaginary in the region \( 0 < |k| < 4\alpha E \). The circularly polarized modes are unstable in this region and the instability occurs spatially inhomogeneously due to the non-zero momentum.

Let us repeat the similar analysis on the black-hole geometry. We need to take care of two more facts in the gravity theory: 1) the presence of the Breitenlohner-Freedman (BF) bound,\(^{11}\) and 2) the fluctuation of the \( U(1) \) Maxwell field couples with that of the gravitons. The presence of the negative mass-squared means the instability on the flat spacetime. However, the fluctuation with negative mass-squared is not necessarily unstable on the AdS spacetime.\(^{11}\) The BF bound gives the lower bound of the mass-squared for stability. The statement 2) means that we need to consider the combination of the fluctuations of the gravitons and the Maxwell field.

Let us start the analysis at the parameter region where we expect the system is most unstable. We expect that the system is most unstable at zero temperature. The reason is that the instability with finite momentum induces spontaneous

\(^{11}\) The CS induced instability has been found in the context of a phenomenological holographic QCD in Ref. 10).
breakdown of the translational/rotational symmetry. It is natural to expect that the spontaneous symmetry breaking occurs in the lower-temperature regime. Another point is that the effect of the CS interaction is proportional to the electric field; we should look at the region where we have the largest electric field in the dual geometry. We can easily see from Eq. (3.2) that the electric field in AdS-RN-BH takes the maximum at the horizon. Therefore, the best starting point for us is to study the zero-temperature limit (with finite density) and near the horizon. The AdS-RN-BH at zero temperature is known as an extremal black hole whose near-horizon geometry is $AdS_2 \times \mathbb{R}^3$. Let us consider the stability on this geometry.

The near-horizon geometry of the extremal AdS-RN-BH is given by

$$ds^2 = \frac{1}{12\xi^2}(-dx_0^2 + d\xi^2) + d\vec{X}^2, \quad F_{\xi 0} = 2\sqrt{6}\frac{1}{12\xi^2}, \quad (3.8)$$

where $\vec{X} \equiv \vec{x}/z_H$, $\xi \equiv [12(1 - z/z_H)]^{-1}$. It is convenient to perform the Kaluza-Klein (KK) reduction to the $AdS_2$ so that the vector fields along $\vec{X}$ become scalars in the 2-dimensional point of view. If we ignore the coupling with the gravitons, the fluctuations (3.3) have the eigenvalues of the mass-squared whose minimum is $-8$. The BF bound for the scalars on the $AdS_2$ geometry is given by $m^2 > -3$ in the present notation, which suggests a possible instability at this stage. What happens if we take the coupling to the gravity into account? A straightforward but lengthy computation leads us to the minimum eigenvalue of the mass-squared of the linear combinations of the graviton and the Maxwell field as

$$m_{\min}^2 = \frac{E^2}{2(4\alpha^2 + 1)^2}(-64\alpha^6 - 24\alpha^4 + 6\alpha^2 - (16\alpha^4 + 4\alpha^2 + 1)^{3/2} + 1) = -2.96804 \cdots > -3, \quad (3.9)$$

where $E = 2\sqrt{6}$. Therefore, the gravitons prevent the system from the instability. However, the fluctuation we have considered is almost marginal in the sense that the absolute value of the mass-squared is just about 1% smaller than that for the BF bound. We emphasize that the supersymmetry is broken in the present case and we have no reason to expect the system is stable a priori.

What we have studied so far is the stability of the AdS-RN-BH geometry (3.2) only at the vicinity of the horizon at zero temperature. We can also analyze the stability in the full geometry at arbitrary temperature if we employ the numerical method. We have performed the numerical analysis in Ref. 1), but we could not find any example of unstable geometry if we employ the CS coupling coming from the supergravity as far as we have analyzed. Interestingly, the geometries are always barely stable in such a way that the CS coupling $\alpha$ is smaller than the critical value for the onset of the instability within the margin of 0.4%, as far as we have studied.

§4. Discussion

We have analyzed the stability of AdS-RN-BH in the presence of the CS term. The CS term under the presence of the electric field has a tendency to make the
circularly-polarized modes unstable. However, as far as we have analyzed, the charged black holes that are dual to the gauge theory are always barely stable against the linear perturbation.\(^1\) It is interesting to study whether there is any systematic reason why the geometry is always barely stable. Once we find an unstable example, on the other hand, the unstable mode carries finite momentum and the system would exhibit spatially inhomogeneous transition which reminds us of the phase transition modeled by Brazovskii.\(^{13}\) Even without the instability, the CS term induces the following effects: 1) it mixes the 5-dimensional Maxwell fields of different polarizations and 2) it deforms the dispersion relation in such a way that the minimum of the spectrum is located at finite momentum. In terms of the gauge theory, the normalizable mode of the 5-dimensional Maxwell field is dual to the expectation value of the corresponding global current. The instability means that the current evolves even without any external force. The normalizable mode of the graviton corresponds to the stress-energy tensor of the gauge theory; the coupling between the global current and the stress-energy tensor is important to prevent the system from the instability. The CS term represents the effect of the chiral anomaly of the global symmetry and all the peculiar phenomena we have discussed are attributed to the anomaly and the finite density. It is certainly worthwhile studying the effects of the anomaly at finite densities in more detail.

### Acknowledgements

The author thanks H. Ooguri and C. S. Park for collaboration. Discussions during the YIPQS international workshop on “New Frontiers in QCD 2010” held at Yukawa Institute for Theoretical Physics, were useful.

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\(^1\) Similar analysis on the CS induced instability for larger class of geometry has been reported recently in Ref. 12), where the authors could not find any unstable example within their analysis.