The Restricted Isometry Property of Block Diagonal Matrices for Group-Sparse Signal Recovery

Niklas Koep and Arash Behboodi and Rudolf Mathar
Institute for Theoretical Information Technology
RWTH Aachen University, Germany
{koep,behboodi,mathar}@ti.rwth-aachen.de

Abstract—Group-sparsity is a common low-complexity signal model with widespread application across various domains of science and engineering. The recovery of such signal ensembles from compressive measurements has been extensively studied in the literature under the assumption that measurement operators are modeled as densely populated random matrices. In this paper, we turn our attention to an acquisition model intended to ease the energy consumption of sensing devices by splitting the measurements up into distinct signal blocks. More precisely, we present uniform guarantees for group-sparse signal recovery in the scenario where a number of sensors obtain independent partial signal observations modeled by block diagonal measurement matrices. We establish a group-sparse variant of the classical restricted isometry property (RIP) for block diagonal sensing matrices acting on group-sparse vectors, and provide conditions under which subgaussian block diagonal random matrices satisfy this group-RIP with high probability. Two different scenarios are considered in particular. In the first scenario, we assume that each sensor is equipped with an independently drawn measurement matrix. We later lift this requirement by considering measurement matrices with constant block diagonal entries. In other words, every sensor is equipped with a copy of the same prototype matrix. The problem of establishing the group-RIP is cast into a form in which one needs to establish the concentration behavior of the suprema of chaos processes which involves estimating Talagrand’s $\gamma_2$ functional. As a side effect of the proof, we present an extension to Maurey’s empirical method to provide new bounds on the covering number of sets consisting of finite convex combinations of possibly infinite sets.

I. Introduction

A common problem in modern signal processing applications is that of sampling signals containing only a limited amount of information imposed by some type of low-complexity structure. The most common low-complexity structure by far manifests in the form of signal sparsity in a suitable basis or more generally in an overcomplete dictionary [1] or frame [2]. The field of compressed sensing was founded on the very idea that the number of samples required to acquire, and represent such signals should be on the order of the information-theoretic rather than the linear-algebraic dimension of the ambient signal space. This was the result of a series of landmark papers due to Candès, Tao, Romberg [3]–[6] and Donoho [7] who first showed that every $d$-dimensional vector $x$ containing at most $s$ nonzero coefficients can be perfectly reconstructed from $O(s \log(d/s))$ nonadaptive measurements of the form $y = Ax$ with $A \in \mathbb{C}^{m \times d}$, and $m \ll d$, assuming that the measurement matrix $A$ satisfies certain structural conditions. While the deterministic construction of such matrices with provably optimal scaling in terms of the information dimension of signals remains a yet unsolved problem, it is by now a well-established fact that a multitude of random ensembles in the class of subgaussian random variables are able to capture just enough information about signals of interest to allow for them to be reconstructed in polynomial time by a variety of different algorithms. More recently, it was also demonstrated that similar results can be obtained for more heavy-tailed ensembles such as measurement matrices populated by independent copies of subexponential random variables [8]. Moreover, it was established very early on that measurement matrices constructed from randomly chosen samples of basis functions in bounded orthonormal systems (BOSs) could provide similar guarantees as unstructured ensembles. Typical examples of this class of structured random matrices are partial Fourier transform matrices, partial circulant matrices generated by a subgaussian random vector or subsampled Hadamard matrices.

While unstructured random matrices are highly desirable from a theoretical perspective, practitioners are not usually free to choose measurement matrices at a whim. Instead, in most engineering applications, most structural properties of the measurement system are predetermined by the application at hand. In this work, we consider another class of structured random matrices at the intersection of purely random and highly structured measurement ensembles. In particular, we consider block diagonal measurement matrices whose blocks are either independent or identical copies of a dense subgaussian random matrix. Such measurement models appear in various applications of interest like distributed compressed sensing (DCS) [9], and the so-called multiple measurement vector (MMV) model in which one obtains multiple independent snapshots of a signal whose low-complexity structure is assumed to be stationary in time [10], [11]. Moreover, such block-wise measurement paradigms have previously been studied in image acquisition systems in order to ease both storage and energy demands of sensors, and recovery...
This acquisition model was previously addressed by Eftekhar et al. in [13] where the authors establish a lower bound on the number of measurements for subgaussian block diagonal matrices to satisfy the classical restricted isometry property (RIP), implying stability and robustness guarantees for recovery of sparse vectors. More recently the model was employed by Palzer and Maly in the context of quantized DCS with 1-bit observations. In the present work, we extend the results of [13] to more structured signal sets, namely those whose nonzero coefficients appear in groups. This type of structured sparsity frequently arises in audio [14] and image signal processing [15], e.g., in modeling the transform coefficients of the wavelet packet transform [16]. Other common applications include multiband reconstruction and spectrum sensing [17], sparse subspace clustering [19], as well as measurement of gene expression levels [20] and protein mass spectroscopy [21].

A. Notation
Throughout the paper, we denote matrices by uppercase boldface letters, vectors by lowercase boldface letters, and scalars by regular type symbols. For an integer \( n \in \mathbb{N} \), we use the common shorthand notation \([n] := \{1, \ldots, n\} \cap \mathbb{N}\). Given a norm \( \| \cdot \|_\theta \) on \( \mathbb{C}^n \) depending on some abstract parameter set \( \theta \), we write \( \mathbb{B}^\theta_n \) for the norm ball associated with \( \| \cdot \|_\theta \), i.e., \( \mathbb{B}^\theta_n \) = \( \{ x \in \mathbb{C}^n : \| x \|_\theta \leq 1 \} \). Even though we mostly work in \( \mathbb{C}^n \), we denote by \( \langle \cdot, \cdot \rangle : \mathbb{C}^n \rightarrow \mathbb{C}^n \) the bilinear form defined by \( \langle a, b \rangle = \sum_{i=1}^n a_i b_i \) for \( a, b \in \mathbb{C}^n \) rather than a sesquilinear form inducing an inner product on \( \mathbb{C}^n \). As such, the canonical \( \ell_2 \)-norm on \( \mathbb{C}^n \) is induced by \( \| a \|_2 = (a, \overline{a}) = (\overline{a}, a) \) where \( \overline{a} \) denotes the complex conjugate of \( a \in \mathbb{C}^n \). Finally, we denote the unit Euclidean sphere in \( \mathbb{C}^n \) by \( \mathbb{S}^{n-1} \). To ease notation, we will make frequent use of the following asymptotic notation: given two scalars \( a, b \in \mathbb{R} \), we write \( a \lesssim b \) if there exists a universal constant \( C > 0 \) such that \( a \leq C b \) holds. Similarly, we write \( a \gtrsim b \) to mean \( a \geq C b \).

B. Summary of Contributions
The contributions of this paper are as follows. We establish the so-called restricted isometry property for subgaussian block diagonal matrices acting on group-sparse vectors. We consider two distinct variations of block diagonal measurement matrices. First, we assume that \( \mathbf{K} \) is a block diagonal measurement matrix. In particular, we assume a vector \( \mathbf{x} \in \mathbb{C}^D \), which we decompose into \( G \) nonoverlapping groups, is observed by \( L \) sensors. For simplicity, we assume that \( D \) is an integer multiple of \( L \) such that \( D = dL \) with \( d \in \mathbb{N} \). To define the group-sparsity structure on \( \mathbf{x} \), we partition the set \([D]\) into \( G \) groups as follows.

**Definition II.1** (Group partition). A collection \( \mathcal{I} = \{ \mathcal{I}_1, \ldots, \mathcal{I}_G \} \) of subsets \( \mathcal{I}_i \subseteq [D] := \{1, \ldots, D\} \) is called a group partition of \([D]\) if \( \mathcal{I}_i \cap \mathcal{I}_j = \emptyset \ \forall i \neq j \), and \( \bigcup_{i=1}^G \mathcal{I}_i = [D] \).

Note that this definition does not assume that the elements in \( \mathcal{I}_i \) are consecutive indices, nor that the cardinality of the individual sets are identical. For simplicity of notation, we denote the size of each group by \( g_i := |\mathcal{I}_i| \) such that \( \sum_{i=1}^G g_i = D \). Moreover, we denote the cardinality of the biggest group by \( g := \max_{i \in [G]} g_i \). We emphasize that we only consider nonoverlapping group partitions in contrast with other works which often allow for efficient groups to overlap, rendering it a nontrivial task to decompose a given vector \( \mathbf{x} \) into individual groups. Some authors refer to this variant as strict group-sparsity. As we will discuss in Section III-D, the flexibility in the group structure leads to certain adversarial examples which will not allow us to correctly estimate the number for suitably chosen unitary bases. Furthermore, we show that our results reduce to previous results reported in [13]. Motivated by ideas in op. cit., we relate the problem of establishing the group-RIP to estimating certain geometric quantities associated with the suprema of chaos processes involving Talagrand’s \( \gamma_2 \)-functional. Since the methods employed in [13] do not directly apply to the group-sparse setting, we propose an alternative method to estimate the covering number at higher scales. In particular, we extend Maurey’s empirical method to sets which do not admit a polytope representation. As a side effect of the proof, we therefore provide a generalization of Maurey’s lemma to provide new bounds on the covering number of sets that consist of finite convex combinations of possibly infinite sets.

C. Organization
The paper is organized as follows. The definition of group-sparse vectors and the underlying sensing model is stated in Section II. The main results are presented in Section III where we also discuss connections to other related results in the literature. The proofs of our main theorems are given in Section IV and V. Finally, we conclude the paper in Section VI.
of measurements required to stably and robustly recover certain signals.

To properly define the concept of group-sparsity, we introduce the following notation. Denote by \( x_i \in \mathbb{C}^D \) the restriction of \( x \) to the indices in \( I_i \), i.e., \( (x_i)_j = x_j \cdot I_{\{j \in I_i\}} \) for \( j \in [D] \) where \( I_{\{j \in I_i\}} \) denotes the binary indicator function of the event \( \{j \in I_i\} \). Then a signal \( x \) is called \( s \)-group-sparse (w.r.t. the group partition \( I \)) if it is supported on at most \( s \) groups, i.e., \( x = \sum_{i \in S} x_i \) for some \( S \subseteq [G] \) with \(|S| \leq s\). We also define the following family of mixed norms.

**Definition II.2** (Group \( \ell_{p,q} \)-norms). Let \( x \in \mathbb{C}^D \). Then, for \( p \geq 1 \), the group \( \ell_{p,q} \)-norm on \( \mathbb{C}^D \) is defined as

\[
\|x\|_{\ell_{p,q}} := \left( \sum_{i=1}^{G} \|x_i\|_2^p \right)^{1/p}.
\]

As customary in the literature on sparse recovery, we extend the notation \( \|\cdot\|_{\ell_{p,q}} \) to \( p = 0 \) in which case \( \|\cdot\|_{\ell_{0}} \) corresponds to the group \( \ell_{0,0} \)-pseudonorm which counts the number of groups a vector is supported on:

\[
\|x\|_{\ell_{0,0}} := \max\{i \in [G] : x_i \neq 0\}.
\]

With this definition in place, we define the set

\[
\Sigma_{I,s} = \left\{ x \in \mathbb{C}^D : \|x\|_{\ell_{0,0}} \leq s \right\}
\]

of \( s \)-group-sparse vectors w.r.t. the group partition \( I \). In most practical real-world settings, it is unlikely that signals of interest precisely adhere to this stringent signal model. Instead, one usually assumes that real-world signals are only well-approximated by elements of \( \Sigma_{I,s} \). In particular, with the definition of the best \( s \)-term group approximation error

\[
\sigma_s(x)_{I,1} = \inf_{z \in \Sigma_{I,s}} \|x - z\|_{I,1},
\]

one commonly considers so-called compressible vectors which are characterized by the fact that \( \sigma_s(x)_{I,1} \) rapidly decays as \( s \) increases.

As hinted at before, we consider a measurement setup in which we observe an \( s \)-group-sparse or compressible signal \( x \) by means of a block diagonal matrix \( A \) consisting of \( L \) blocks, namely

\[
A = \begin{pmatrix}
\Phi_1 & & \\
& \ddots & \\
& & \Phi_L
\end{pmatrix}.
\]

However, we assume that we only have access to the signal \( x \in \Sigma_{I,s} \) in terms of its basis expansion \( z \) in a unitary basis \( \Psi \in \mathbb{C}^{D \times L} := \{ U \in \mathbb{C}^{D \times D} : U^*U = \text{Id}_D \} \). The measurement model therefore reads

\[
y = \text{diag}\{\Phi_1\}_{i=1}^L z = \text{diag}\{\Phi_1\}_{i=1}^L \Psi x = A \Psi x. \tag{1}
\]

We will also consider an alternative measurement model in which each sensor is equipped with a copy of the same matrix \( \Phi \), i.e., \( \Phi_1 = \Phi \forall l \in [L] \). Ultimately, our goal in this paper is to provide a sufficient condition for stable and robust recovery of group-sparse signals by establishing a suitable RIP property of block diagonal matrices acting on group-sparse vectors.

**III. SIGNAL RECOVERY WITH BLOCK DIAGONAL GROUP-RIP MATRICES**

The analysis of both sensing paradigms introduced in the previous section relies on the so-called group restricted isometry property (group-RIP)—a generalization of the well-known restricted isometry property modeled on the block-sparse RIP first introduced in [22].

**Definition III.1** (Group restricted isometry property). A matrix \( A \Psi \in \mathbb{C}^{M \times D} \) with \( A \in \mathbb{R}^{M \times D} \) and \( \Psi \in \mathbb{U}(D) \) is said to satisfy the group restricted isometry property (group-RIP) of order \( s \) if, for \( \delta \in (0,1) \),

\[
(1 - \delta)\|z\|_2^2 \leq \|A \Psi x\|_2^2 \leq (1 + \delta)\|z\|_2^2 \quad \forall x \in \Sigma_{I,s}.
\]

The smallest constant \( \delta \) for which Equation (2) holds is called the group restricted isometry constant (group-RIC) of \( A \).

In combination with the above definition, the result due to Gao and Ma established in [23] which we will introduce next then implies stable and robust recovery of group-sparse signals. While the signal model employed in [23] assumes that the indices in each group \( I_i \) are linearly increasing, i.e., the signals are assumed to be block-rather than group-sparse, the proof of Theorem 1 in [23] does not explicitly rely on this structure. Furthermore, the result was originally proven in the real setting, but the proof is easily extended to the complex case. Note that such a stability and robustness result was already established in the seminal work of Eldar and Mishali [22], albeit with the necessary condition \( \delta_{2s} < \sqrt{2} - 1 \) on the group-RIP constant. These results therefore also extend to more general group partitions as defined in Definition II.1. The precise statement of this generalization is stated in the following result. For the sake of being self-contained, we provide a proof in Appendix A.

**Theorem III.1.** Let \( \hat{A} \in \mathbb{C}^{M \times D} \) be a matrix satisfying the group restricted isometry property of order \( 2s \) with constant \( \delta_{2s} < 4/\sqrt{41} \). Then for any \( \hat{x} \in \mathbb{C}^D \), and \( y = \hat{A} \hat{x} + \epsilon \) with \( \|\epsilon\|_2 \leq \epsilon \), the solution \( \hat{x}^* \) of the program

\[
\begin{align*}
\text{minimize} & \quad \|x\|_{I,1} \\
\text{s.t.} & \quad \|Ax - y\|_2 \leq \epsilon
\end{align*} \tag{P_{I,1}}
\]

satisfies

\[
\|\hat{x} - \hat{x}^*\|_2 \leq C\frac{\sigma_s(\hat{x})_{I,1}}{\sqrt{s}} + D\epsilon
\]

where the constants \( C, D > 0 \) only depend on \( \delta_{2s} \).

**Remark III.1.** (i) In the noiseless setting with \( \epsilon = 0 \), the above result immediately implies perfect recovery of all group-sparse signals as the \( s \)-term approximation error \( \sigma_s(\hat{x})_{I,1} \) vanishes as soon as \( \hat{x} \in \Sigma_{I,s} \).
The subgaussian norm of $C > D\psi$ exists a constant $C > D > 0$ which only depend on $\delta_s$. 

A. Main Results

Before stating our main result, we first recall the definition of subgaussian random variables.

**Definition III.2** (Subgaussian random variable). A zero mean random variable $X$ is called subgaussian if there exists a constant $C > 0$ such that

$$\mathbb{E}(\exp(X^2/C^2)) \leq 2.$$ 

The subgaussian norm of $X$, also known as Orlicz norm of $X$, is defined by

$$\|X\|_2 = \inf \{C > 0 : \mathbb{E}(\exp(X^2/C^2)) \leq 2\}.$$

At this point we are ready to state the main result of this paper.

**Theorem III.2.** Let $A = \text{diag}\{\Phi_l\}_{l=1}^L \in \mathbb{R}^{m \times dL}$ be a block diagonal random matrix with subgaussian blocks $\Phi_l$ whose entries are independent subgaussian zero-mean unit-variance random variables with subgaussian norm $\tau$. Let further $\Psi \in \mathcal{U}(dL)$ be a unitary matrix. Then with probability at least $1 - \eta$, the matrix $m^{-1/2} A \Psi$ satisfies the group restricted isometry property of order $s$ w.r.t. the group partition $I$, and $\delta_s \leq \delta$ if

$$m \gtrsim \delta^{-2} \left[s\mu_2^2 \log(D) \log(s)^2 (\log(G) + g \log(s\mu_2)) + \log(\eta^{-1})\right],$$

where

$$\mu_2 = \mu_2(\Psi) := \min \left\{\sqrt{d} \max_{i \in [D]} \|\psi_i\|_{I,\infty} : 1\right\},$$

and $\psi_i \in \mathbb{C}^D$ denotes the $i$-th row of $\Psi$.

In the second acquisition model in which we assume that every sensor is equipped with a copy of the same (random) measurement matrix $\Phi_l = \Phi \forall l \in [L]$, the coherence parameter $\mu_2(\Psi)$ introduced above is replaced by another parameter of the sparsity basis. To that end, we introduce the following notation. Given a sparsity basis matrix $\Phi \in \mathcal{U}(D)$, denote by $\Phi_{l} \in \mathbb{C}^{dL}$ the $l$-th partial basis expansion matrix such that $\Phi = (\Phi_1^\top, \ldots, \Phi_L^\top)^\top$. With this definition, the following result establishes the group-RIP for block diagonal subgaussian random matrices with constant block-diagonal.

**Theorem III.3.** Under the conditions of Theorem III.2, assume that $\Phi_l = \Phi$ for all $l \in [L]$ where the entries of $\Phi$ are independent subgaussian zero-mean unit-variance random variables with subgaussian norm $\tau$. Then with probability at least $1 - \eta$, the matrix $m^{-1/2} A \Psi$ satisfies the group restricted isometry property of order $s$ w.r.t. the group partition $I$, and $\delta_s \leq \delta$ if

$$m \gtrsim \delta^{-2} \left[s\omega_2^2 \log(D) \log(s)^2 (\log(G) + g \log(s\omega_2)) + \log(\eta^{-1})\right],$$

where

$$\omega_2 = \omega_2(\Psi) \quad \text{such that} \quad \mu_2(\Psi) := \min \{\sqrt{d} \max_{i \in [D]} \|\psi_i\|_{I,\infty} : 1\} = 1 = \frac{\sqrt{d} \max_{i \in [D]} \|\psi_i\|_{I,\infty}}{\sqrt{D}} \quad \text{with} \quad \mu(\Psi) = \frac{\sqrt{d} \max_{i \in [D]} \|\psi_i\|_{I,\infty}}{\sqrt{D}} \quad \text{such that} \quad \mu(\Psi) := \min \{\sqrt{d} \max_{i \in [D]} \|\psi_i\|_{I,\infty} : 1\} = 1 = \frac{\sqrt{d} \max_{i \in [D]} \|\psi_i\|_{I,\infty}}{\sqrt{D}}.$$
parameter $\omega_2(\Psi)$ w.r.t. the trivial group partition $\mathcal{I} = \{\{1\}, \ldots, \{D\}\}$ reduces to

$$\omega_2(\Psi) = \max_{e \in [D]} \|\hat{V}(e')\|_{2 \to 2}.$$  

Defining the so-called block-coherence parameter $\gamma(\Psi) := \sqrt{\omega_2(\Psi)}$ to borrow terminology from Eftekhari et al. (cf. [13, Equation (9)]), this yields the condition

$$mL \gtrsim \delta^{-2} s \mu(\Psi)^2 \log(D)^2 \log(s)^2$$

which reproduces the statement of Theorem 2 in [13].

C. Comparison to Dense Measurement Matrices

As alluded to in the introduction, it is by now a well-established fact that $O(s \log(d/s))$ nonadaptive measurements based on subgaussian random ensembles are sufficient to stably reconstruct sparse or compressible vectors from their linear projections. Moreover, this bound is fundamental in that it is known to be optimal among all encoder-decoder pairs $(A, \Delta)$ with $A \in \mathbb{C}^{m \times d}$ and decoding maps $\Delta: \mathbb{C}^m \to \mathbb{C}^d$ such that

$$\|x - \Delta(A x)\|_2 \leq C \sqrt{s} \sigma_s(x) \quad \forall x \in \mathbb{C}^d$$

for $C > 0$ [25, Chapter 10]. Such a fundamental lower bound on the required number of measurements was recently also established for the case of block-sparse vectors by Dirksen and Ullrich [26] (see also [27, Theorem 2.4]). In particular, using new results on Gelfand numbers, the authors show that stability results of the form

$$\|x - \Delta(A x)\|_2 \leq C \sqrt{s} \sigma_s(x) \quad \forall x \in \mathbb{C}^D$$

for arbitrary encoder-decoder pairs $(A, \Delta)$ require at least

$$M \geq c_1 (s \log(c G/s) + s g) \quad \text{with} \quad s > c_2$$

measurements where the constants $c_1$ and $c_2$ only depend on $C > 0$ (cf. [26, Corollary 1.2]). Perhaps most surprisingly about this result is the linear dependence on the total number of nonzero coefficients $sg$. In light of Equation (3), we also recover this scaling behavior in the total number of measurements $M$ for the block diagonal measurement setup, albeit with the additional logarithmic factor in $s$ which we conjecture to be an artifact of the proof technique employed in Section IV-D. The other polylogarithmic factors, as well as the dependence on $\mu_2(\Psi)$, on the other hand, are due to the particulars of the measurement setup compared to the situation in which we employ one densely populated measurement matrix to observe the entire signal. Whether these factors can be improved any further remains an open problem.

D. Connection to Distributed Sensing

As mentioned in the introduction, the measurement model (1) frequently appears in the context of recovering multiple versions of a vector sharing a common low-complexity structure. This model appears for instance in the context of distributed sensing where one aims to estimate the structure of a ground truth signal observed by spatially distributed sensors which each observe a slightly different version of the signal due to channel propagation effects.

Another classic example is that of the so-called MMV model in which a single sensor acquires various temporal snapshots of a signal whose low-complexity structure is assumed to be stationary\footnote{In particular, this model assumes the sparse support set to be constant, while amplitudes and phases of the coefficients of each vector are allowed to change between different observations.} with the intent of reducing the influence of measurement noise in a single-snapshot model. This particular model can be cast in the setting of Section V where we interpret each observation in the MMV model as an independent observation by a distinct sensor equipped with the same measurement matrix $\Psi \in \mathbb{R}^{m \times d}$.

Assuming that the ground truth signal is $s$-sparse, we can interpret both situations as trying to recover an $s$-group-sparse vector w.r.t. the group partition $\mathcal{I} = \{I_1, \ldots, I_d\}$ with

$$\mathcal{I}_i = \{i, d + i, \ldots, (L-1)d + i\}.$$  

In both situations, we assume that each signal $\tilde{x}^l = \Psi x^l \in \mathbb{C}^d$ is sparse in the same basis $\Psi \in U(d)$. We can therefore choose $\Psi^l = \text{diag}(\tilde{\Psi}^l)_{i=1}^L \in U(D)$ in Theorem III.2. This setup, however, is not able to cope with certain adversarial vectors. More precisely, due to the particular group partition structure, the knowledge about the periodicity in the support structure can not necessarily be exploited in all recovery scenarios. To see this, consider the situation in which only a single vector $x^l$ is different from 0. The vector $x = (\tilde{0}^\top, \ldots, \tilde{0}^\top, (x^l)^\top, \tilde{0}^\top, \ldots, \tilde{0}^\top)^\top$ is then by definition $s$-group-sparse (w.r.t. the group partition $\mathcal{I}$) if $x^l$ is $s$-sparse. Regardless of the sparsity basis $\Psi \in U(d)$, only the vector $y^l$ carries information about $x^l$ which implies that each matrix $\Phi_l$ should satisfy the classical restricted isometry property to recover $x$. This happens with high probability as soon as $m = O(s \log(d/s))$. In this case, instead of solving Problem (P$_{2,\mathcal{I}}$) directly, it is more favorable to solve for each $l \in [L]$ the problem

$$\begin{align*}
\text{minimize} & \quad ||x^l||_1 \\
\text{subject to} & \quad ||y^l - \Phi_l \tilde{\Psi} x^l||_2 \leq \epsilon. 
\end{align*}$$

Unfortunately, this behavior is not accurately captured by Theorem III.2 since we have by Equation (3) with $G = d$ and $g = L$ that

$$m \gtrsim c_2 \delta^{-2} \mu_2(\Psi)^2 s \log(D) \log(s)^2 (\log(d) + L \log(s)).$$

This predicts a much worse scaling behavior than what is required to solve $L$ separate problems of the form (P$_l$). The problem is ultimately rooted in the fact that independent of $\Psi \in U(d)$, only the measurements $y^l$ carry information about $x^l$.

Note that such adversarial situations had previously been discussed by van den Berg and Friedlander [28] who...
consider sufficiency conditions for noiseless joint-sparse recovery based on dual certificates. Instead of considering signals with only one $s$-sparse nonzero signal $x^l$, they consider signals $x$ in which every $x^l$ is at most $1$-sparse with $\text{supp}(x^l) \neq \text{supp}(x^{l'})$ for any $l \neq l'$. In this setting, they show that there are signals $x \in \mathbb{R}^D$ which—given the linear measurements $y = \text{diag}\{\Phi\}_{l=1}^L x$—can provably be recovered by the program

$$\begin{align*}
\text{minimize} & \quad \|x\|_1 \\
\text{s.t.} & \quad y = \text{diag}\{\Phi\}_{l=1}^L x
\end{align*}$$

but not via group $\ell_2,1$-minimization, i.e., as solutions of Problem (P2.1) with $\Psi = \text{Id}_D$, and $\epsilon = 0$.

The problem of distributed compressed sensing was also recently addressed in the context of quantized compressed sensing with binary observations by Maly and Palzer [29] who impose an additional norm constraint on each signal to avoid that $x^l = 0$. However, even with this modified signal model, the adversarial example discussed above still applies if one signal $x^l$ is exactly $s$-sparse, while any other signal $x^{l'}$ with $l' \neq l$ is $1$-sparse with the entire signal energy concentrated on the main diagonal coordinate in each vector $x^l$. The resulting signal is therefore $s$-group-sparse as in the previous example. In that case, each measurement vector $y^l$ only carries information about a single nonzero coordinate of $x^l$ which implies that each $\Phi_l$ must itself be able to recover every $(s - 1)$-sparse vector for the entire vector $x$ to be recovered as desired.

To summarize, without further restrictions on the particular signal model, it is not clear how adversarial examples as discussed above can be dealt with in order to obtain nontrivial uniform recovery guarantees. However, the conclusion of the work in [13] and our results is that sparsity or group-sparsity in a nonlocalized unitary basis such as the discrete Fourier transform (DFT) basis bears the potential to reduce the number of measurements required for stable and robust signal recovery by distributing the energy of nonzero coefficients across the entire signal support. As pointed out above, however, this requires that the sparsity basis of $x$ does not take the form of a block diagonal unitary matrix.

IV. The Group-RIP for Block Diagonal Matrices

In this section, we establish the group-RIP for general subgaussian block diagonal matrices.

A. Restricted Isometries and Suprema of Chaos Processes

We will make use of the following powerful bound on the suprema of chaos processes first established in [30, Theorem 3.1] to demonstrate that the block diagonal matrix $A\Psi \in \mathbb{C}^{M \times D}$ satisfies the group restricted isometry property with high probability on the draw of $A$. The same technique was also employed in [13] to prove the canonical restricted isometry property for block diagonal matrices consisting of subgaussian blocks. In the present work, we make use of an improved version of the bound due to Dirksen [31]. Before stating the result, we first define the following objects. Let $\mathcal{M} \subset \mathbb{C}^{m \times n}$ be a bounded set. Then the radii of $\mathcal{M}$ w.r.t.

$$\begin{align*}
\rho_0(\mathcal{M}) &= \sup_{\Gamma \in \mathcal{M}} \|\Gamma\|_F \\
\rho_{2^{-2}}(\mathcal{M}) &= \sup_{\Gamma \in \mathcal{M}} \|\Gamma\|_{2^{-2}}
\end{align*}$$

respectively. Lastly, we require the so-called $\gamma_2$-functional of $\mathcal{M}$ w.r.t.

$$\gamma_2(\mathcal{M}, \rho_{2^{-2}})$$

Definition IV.1. An admissible sequence of a metric space $(T,d)$ is a collection $\{T_r \subset T : r \geq 0\}$ where $|T_r| \leq 2^r$ for every $r \geq 1$ and $|T_0| = 1$. The $\gamma_2$ functional is defined by

$$\gamma_2(T,d) = \inf \sup_{t \in T} \sum_{r=0}^{\infty} 2^{r/2} d(t,T_r),$$

where the infimum is taken over all admissible sequences.

It is generally difficult to characterize $\gamma_2$ directly. To estimate $\gamma_2$, it is therefore customary to appeal to a classical result due to Talagrand which bounds $\gamma_2(\mathcal{M}, \rho_{2^{-2}})$ in terms of the following entropy integral of the metric space $\gamma_2(\mathcal{M}, \rho_{2^{-2}})$ [32]:

$$\gamma_2(\mathcal{M}, \rho_{2^{-2}}) \leq \int_0^\infty \sqrt{\log \#(\mathcal{M}, \rho_{2^{-2}}, \epsilon)} \, d\epsilon$$

where $\#(\mathcal{M}, \rho_{2^{-2}}, \epsilon)$ denotes the interior covering number, i.e., the cardinality of the smallest subset $\mathcal{N} \subset \mathcal{M}$ such that every point in $\mathcal{M}$ is at most $\epsilon$ apart from $\mathcal{N}$ w.r.t. the operator norm $\|\cdot\|_{2^{-2}}$. Mathematically, $\mathcal{N} \subset \mathcal{M}$ is called an $\epsilon$-net of $\mathcal{M}$ if $\forall \Gamma \in \mathcal{M} \exists \Gamma_0 \in \mathcal{N} : \|\Gamma - \Gamma_0\|_{2^{-2}} \leq \epsilon$ with $\#(\mathcal{M}, \rho_{2^{-2}}, \epsilon) = |\mathcal{N}|$ if $\mathcal{N}$ is the smallest such set. Note that the integrand of the entropy integral (5) vanishes as soon as $\epsilon \geq \rho_{2^{-2}}(\mathcal{M})$ since $\mathcal{M}$ can then be covered by a single ball $B_{2^{-2}}^{m \times n}$ centered at an (arbitrary) element of $\mathcal{M}$.

Theorem IV.1 ([31, Theorem 6.5]). Let $\mathcal{M}$ be a matrix set, and denote by $\xi$ an isotropic unit-variance subgaussian random vector with subgaussian norm $\tau$. Then, for $u \geq 1$,

$$\mathbb{P}\left( \sup_{\Gamma \in \mathcal{M}} \|\Gamma \xi\|_2^2 - \mathbb{E}\|\Gamma \xi\|_2^2 \geq c_\tau E_u \right) \leq e^{-u}$$

where $E_u = \gamma_2(\mathcal{M}, \rho_{2^{-2}}) + \rho_0(\mathcal{M}) \gamma_2(\mathcal{M}, \rho_{2^{-2}})$

$$+ \sqrt{u \rho_0(\mathcal{M}) \rho_{2^{-2}}(\mathcal{M}) + u \rho_{2^{-2}}(\mathcal{M})^2},$$

and $c_\tau$ is a constant that only depends on $\tau$.

B. Chaos Process for Block-Diagonal Group-RIP Matrices

In order to apply Theorem IV.1 to estimate the probability that $A\Psi$ as defined in Equation (1) satisfies the group restricted isometry property, first note that

$^2$The metric on $\mathcal{M}$ is the one induced by the norm $\|\cdot\|_{2^{-2}}$.

$^3$The subgaussian property readily implies that $\xi$ is centered.
we can equivalently express the group-RIP condition in Equation (2) for \( x \in \Sigma_{L,s} \setminus \{0\} \) as
\[
\left\| \mathbf{A}\mathbf{x} \right\|_2^2 = \left\| \mathbf{A}\mathbf{x} \right\|_2^2 - 1 \leq \delta.
\]
With the definition of the set
\[
\Omega := \Sigma_{L,s} \cap \mathbb{S}^{D-1} = \left\{ x \in \mathbb{S}^{D-1} : \|x\|_{\Sigma,0} \leq s \right\}
\]
of s-group-sparse vectors on the unit Euclidean sphere, we may therefore write the group restricted isometry constant of \( \mathbf{A} \) as
\[
\delta_s = \sup_{x \in \Omega} \left\| \mathbf{A}\mathbf{x} \right\|_2^2 - 1.
\]
Next, we transform the above expression into the form required by Theorem IV.1, i.e., we rewrite the equation so that the supremum is taken over a matrix set. To that end, recall the definition of the partial basis expansion matrices \( \Psi_i \in \mathbb{C}^{L \times dL} \) with \( \Psi = (\Psi_1, \ldots, \Psi_L)^\top \). In light of Equation (1), we may now express the \( L \)-th measurement vector \( \mathbf{y}' \in \mathbb{C}^m \) of \( \mathbf{y} \in \mathbb{C}^{mL} \)
\[
\mathbf{y}' = \mathbf{F}_1 \mathbf{F}_2 \mathbf{x} = \left( \begin{array}{c}
\langle \mathbf{F}_1, \mathbf{x} \rangle \\
\vdots \\
\langle \mathbf{F}_m, \mathbf{x} \rangle 
\end{array} \right)
\]
where \( \mathbf{F}_i \in \mathbb{C}^{d} \) denotes the \( i \)-th row of the matrix \( \mathbf{F}_i \). If the blocks \( \mathbf{F}_i \) are populated by independent copies of a \( \tau \)-subgaussian random variable with unit-variance, then the vector \( \mathbf{x} = (\mathbf{x}_1^\top, \ldots, \mathbf{x}_L^\top)^\top \) is a unit-variance \( \tau \)-subgaussian random vector. Define the operator \( \mathbf{V} : \mathbb{C}^{dL} \to \mathbb{C}^{mL} \) with
\[
\mathbf{x} \mapsto \mathbf{V}(\mathbf{x}) = \text{diag} \{ \mathbf{V}(\mathbf{x}_i) \}_{i=1}^L,
\]
we therefore have \( \mathbf{A}\mathbf{x} = \mathbf{V}(\mathbf{x}) \) where \( \mathbf{A} \) denotes equality in distribution. Now note that
\[
\mathbf{E}\|\mathbf{A}\mathbf{x}\|_2^2 = \mathbf{E}\|\mathbf{V}(\mathbf{x})\|_2^2 = m\|\mathbf{x}\|_2^2
\]
which follows from the fact that the rows of the matrices \( \mathbf{F}_i \) are independent unit-variance random \( m \)-vectors with independent entries, as well as from unitarity of \( \Psi \). With Equation (6), the group restricted isometry property of the matrix \( \frac{1}{\sqrt{m}}\mathbf{A}\mathbf{x} \) can therefore be expressed as
\[
\delta_s \left( \frac{1}{\sqrt{m}}\mathbf{A}\mathbf{x} \right) = \sup_{x \in \Omega} \left| \frac{1}{\sqrt{m}}\mathbf{A}\mathbf{x} \right|_2^2 - 1
\]
where we set \( \mathcal{M} := \mathbf{V}(\Omega) = \{ \mathbf{V}(\mathbf{x}) : \mathbf{x} \in \Omega \} \). In order to apply Theorem IV.1, it remains to estimate the radius of \( \mathcal{M} \) w.r.t. the Frobenius operator norm, respectively, as well as to compute the \( \gamma_2 \)-functional of \( \mathcal{M} \) w.r.t. \( \| \cdot \|_2 \). These issues are addressed in the next two sections.

C. Radii Estimates

We begin with the estimation of \( \rho_{\mathcal{F}}(\mathcal{M}) \). To that end, first note that
\[
\|\mathbf{V}(\mathbf{x})\|_{\mathcal{F}}^2 = \|\text{diag} \{ \mathbf{V}(\mathbf{x}_i) \}_{i=1}^L\|_{\mathcal{F}}^2 = \sum_{i=1}^L \|\mathbf{V}(\mathbf{x}_i)\|_{\mathcal{F}}^2
\]
\[
= \sum_{i=1}^L \|\mathbf{V}(\mathbf{x}_i)\|_{\mathcal{F}}^2 = \sum_{i=1}^L m\|\mathbf{x}_i\|_2^2 = m\|\mathbf{x}\|_2^2.
\]
Since \( \Omega \subset \mathbb{S}^{D-1} \), this immediately implies
\[
\rho_{\mathcal{F}}(\mathcal{M}) = \sup_{\mathbf{F} \in \mathcal{M}} \left| \mathbf{F} \right|_{\mathcal{F}} = \sup_{\mathbf{x} \in \Omega} \|\mathbf{V}(\mathbf{x})\|_{\mathcal{F}} = \sqrt{m} \sup_{\mathbf{x} \in \Omega} \|\mathbf{x}\|_2 = \sqrt{m}.
\]
In order to estimate the radius \( \rho_{2 \to 2}(\mathcal{M}) \), we require a simple generalization of Hölder’s inequality to group \( l_{2,p} \)-norms on \( \mathbb{C}^D \) as defined in Definition II.1.2. We state here a specialization to the conjugate pair \( p = 1, q = \infty \).

**Lemma IV.1.** Let \( a, b \in \mathbb{C}^D \), and let \( \mathcal{I} \) be a group partition of \( \{D \} \). Then
\[
\|\langle a, b \rangle\| = \|a\|_{\mathcal{I},1} \cdot \|b\|_{\mathcal{I},\infty}
\]
where \( \langle \cdot, \cdot \rangle \) denotes the bilinear form \( \langle a, b \rangle = \sum_{i=1}^D a_i b_i \) on \( \mathbb{C}^D \).

**Proof.** By the triangle and Hölder’s inequality, we have
\[
\|\langle a, b \rangle\| \leq \sum_{i=1}^D \|a_i b_i\| \leq \sum_{i=1}^D \|a_i\|_2 \cdot \|b_i\|_2 \leq \sum_{i=1}^D \|a_i\|_2 \cdot \max_{j \in \mathcal{I}} \|b_j\|_2 = \|a\|_{\mathcal{I},1} \cdot \|b\|_{\mathcal{I},\infty}.
\]
\[\square\]
We proceed as before and compute

\[ \|V(x)\|_{2 \to 2} = \|\text{diag} \{ V_i(x) \}_{i=1}^C \|_{2 \to 2} = \max_{i \in [C]} \|V_i(x)\|_{2 \to 2} \]

where the second step follows from the fact that the operator norm of a block diagonal matrix corresponds to the maximum operator norm of the individual blocks. The last step follows because \( V_i(x) \) on its diagonal whose largest singular value is simply \( \|\Psi_i x\|_2 \). Next, we invoke the bound \( \|x\|_2 \leq \sqrt{\pi} \|x\|_\infty \) for \( x \in \mathbb{C}^n \), followed by an application of Lemma IV.1. This yields

\[ \|\Psi_i x\|_2 \leq \sqrt{d} \max_{i \in [d]} \|\Psi_i\|_{\infty} \|x\|_1 \]

where \( \Psi_i \) denotes the \( i \)-th row of \( \Psi \). Overall, we find

\[ \|V(x)\|_{2 \to 2} \leq \sqrt{d} \max_{i \in [d]} \|\Psi_i\|_{\infty} \|x\|_1 \]

\[ = \sqrt{d} \|x\|_1 \max_{i \in [d]} \|\Psi_i\|_{\infty} \]

where \( \psi_i \in \mathbb{C}^D \) denotes the \( i \)-th row of \( \Psi \). This bound is less effective for instance when \( \Psi = \text{Id}_D \) but more so when \( \Psi \) corresponds to a DFT matrix, \( \text{i.e.} \)

\[ \Psi = F_D = \frac{1}{\sqrt{D}} \left( e^{2\pi i mn/D} \right)_{0 \leq m, n \leq D-1} \]

For \( \Psi = \text{Id}_D \), we have \( \sqrt{d} \max_{i \in [D]} \|\Psi_i\|_{\infty} = \sqrt{d} \), whereas for \( \Psi = F_D \) we get \( \sqrt{d} \max_{i \in [D]} \|\Psi_i\|_{\infty} = \sqrt{g/L} \) with \( g = \max_{i \in [C]} |I_i| \) denoting the size of the largest coefficient group. To obtain an effective bound in both situations, we therefore also consider the simple bound

\[ \|V(x)\|_{2 \to 2} = \max_{i \in [d]} \|\Psi_i x\|_2 \leq \|\Psi x\|_2 \]

\[ = \|x\|_2 = \|x\|_1 \leq \|x\|_{1,1} \]

which follows from \( \|\cdot\|_p \leq \|\cdot\|_q \) for \( p \geq q \geq 1 \). Combining both estimates, we arrive at

\[ \rho_{2 \to 2}(\mathcal{M}) = \sup_{x \in \Omega} \|V(x)\|_{2 \to 2} \]

\[ \leq \sup_{x \in \Omega} \|x\|_{1,1} \min \left\{ \sqrt{d} \max_{i \in [D]} \|\Psi_i\|_{\infty,1} \right\} \]

\[ \leq \sqrt{s} \min \left\{ \sqrt{d} \max_{i \in [D]} \|\Psi_i\|_{\infty,1} \right\} \]

\[ = : \sqrt{s} \mu_2(\Psi) \]

The last inequality holds since for \( x \in \Omega = \Sigma_{I, s} \cap S^{D-1} \), we have

\[ \|x\|_{1,1} \leq \sum_{i=1}^C \|x_i\|_2 \leq \sqrt{s} \|x\|_2 \]

\[ \|x\|_{1,1} \leq \sqrt{s} \|x\|_2 \]

by the Cauchy-Schwarz inequality.

**D. Metric Entropy Bound**

Establishing a bound on the \( \gamma_2 \)-functional via Equation (5) will proceed in two steps. At small scales, we will estimate the covering number by means of a standard volume comparison argument for norm balls covered in their respective metrics. At larger scales, however, this bound will not be effective enough to yield optimal scaling behavior in \( s \). To circumvent the problem, we employ a variation of Maurey’s empirical method.

To start with, note that with \( \|x\|_V := \|V(x)\|_{2 \to 2} \), we have for \( u \geq 0 \),

\[ \mathcal{N}(\mathcal{M}, \|\cdot\|_{2 \to 2}, u) = \mathcal{N}(\Omega, \|\cdot\|_V, u) \]

Next, we may express the set \( \Omega = \sum_{I, s} \cap S^{D-1} \) of \( s \)-group-sparse signals on the unit sphere as the union of \( \mathcal{C}^D \) unit Euclidean spheres supported on \( s \) groups of a group partition \( I \). Denote for \( T \subset I \) the coordinate subspace of \( \mathcal{C}^D \) supported on the index set \( \bigcup_{T \subset I} S \subset [D] \) by \( \mathcal{C}^D_T \), \( \text{i.e.} \)

\[ \mathcal{C}^D_T = \{ x \in \mathbb{C}^D : x_S = 0 \forall S \notin T \} \]

Then we can write

\[ \Omega = \bigcup_{T \subset I, |T|=s} (\mathcal{C}^{D-1} \cap \mathcal{C}^D_T) \cup \bigcup_{T \subset I, |T|=s} (\mathcal{B}^D_T \cap \mathcal{C}^D_T). \]

The linear-algebraic dimension of the sets in this union is at most \( s g \) where again \( g \) denotes the largest group of the partition \( I \) considered in \( T \). From the volume comparison argument for norm balls covered in their associated metrics (see e.g. [33, Corollary 4.2.13]), one has that \( \mathcal{N}(\mathcal{B}^n, \|\cdot\|, t) \leq (1 + 2/t)^n \). With Equation (8), this yields for an arbitrary group index set \( T \) as above that

\[ \mathcal{N}(\Omega, \|\cdot\|_V, u) \leq \left( \frac{C}{s} \right)^n \mathcal{N}(\mathcal{B}^n, \|\cdot\|_V, u/2) \]

\[ \leq \left( \frac{C}{s} \right)^n \left( 1 + \frac{4}{u} \right)^{2s} \]

where the factor \( 1/2 \) in the covering radius of the first estimate is due to the fact that the interior covering numbers are only almost increasing by inclusion, \( \text{i.e.} \), if \( U \subset W \), then \( \mathcal{N}(U, \cdot, t) \leq \mathcal{N}(W, \cdot, t/2) \) [33, Exercise 4.2.10]. The factor 2 in the exponent of the last estimate is due to the isomorphic identification of \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \). Finally, we invoked the standard bound \( (n)_k \leq (en/k)^k \) for binomial coefficients.

To estimate \( \mathcal{N}(\Omega, \|\cdot\|_V, u) \) at higher scales, we develop a variation on Maurey’s empirical method, also known as Maurey’s lemma [34]. In general, Maurey’s lemma is concerned with the following question. Given a vector \( x \) in the convex hull of a finite set \( U \subset \mathbb{R}^n \), how many elements of \( U \) are needed to approximate \( x \) to within a desired level of accuracy? Maurey’s empirical method answers this question by constructing a sequence of random vectors, and estimating the number of elements required for the expected average to fall below a predefined distance to \( x \).
To frame the problem in the appropriate context, first note that we have by the Cauchy-Schwarz inequality that

\[ \frac{\Omega}{\sqrt{s}} < \mathbb{B}^D_{L^1} \]
as argued before. We can therefore bound the entropy integral over an interval \( I \subset [0, \sqrt{\mu_x}(\Psi)] \) by

\[
\int_I \sqrt{\log \mathcal{N}(\Omega, \| \cdot \|_V, \varepsilon)} \, d\varepsilon
\leq 2\sqrt{s} \int_I \sqrt{\log \mathcal{N}(\mathbb{B}^D_{L^1}, \| \cdot \|_V, \varepsilon/\sqrt{s})} \, d\varepsilon
\]

where in the last step we adjusted the covering radius by a factor of 1/2 due to the inclusion bound for covering numbers, before performing a change of variable. Unless the number of groups in the partition \( T \) is identical to the ambient dimension \( D \), the group \( \ell^2_x \) unit ball cannot be expressed as the convex hull of a finite set as it does not admit a polytope representation.\(^4\) We will circumvent this problem by an additional covering argument.

Let \( x \in \mathbb{B}^D_{L^1} \) such that \( \sum_{j} \| x_{T_j} \|_2 \leq 1 \), and denote by \( S \subseteq [G] \) the index set of nonzero groups of \( x \). Then we can express \( x \) as

\[
x = \sum_{j \in S} x_{T_j} = \sum_{j \in S} \frac{x_{T_j}}{\left\| x_{T_j} \right\|_2} \left\| x_{T_j} \right\|_2
\]

where \( S^D_{T_j} \) denotes the subset of the unit sphere supported on the index set \( T_j \). Since Maurey’s lemma is concerned with the estimation of the covering number of the convex hull of a finite point cloud w.r.t. an arbitrary seminorm, the argument does not immediately extend to the current setting. This is due to fact for every \( x \in \mathbb{B}^D_{L^1} \), the dictionary

\[
U_x := \left\{ g_j \in S^D_{T_j} : g_j = \frac{x_{T_j}}{\left\| x_{T_j} \right\|_2} \forall j \in S \right\}
\]
such that \( x \in \text{conv}(U_x) \) depends on the particular choice of \( x \). In other words, since \( \mathbb{B}^D_{L^1} \) does not generally admit a polytope representation, there exists no finite set \( U \subset C^D \) such that \( \mathbb{B}^D_{L^1} = \text{conv}(U) \). To circumvent this issue, we equip each unit sphere \( S^D_{T_j} \) in a coordinate subspace supported on an index set \( T_j \) with its own net. To that end, denote by \( N_j \subset S^D_{T_j} \) an \( (\varepsilon/2) \)-net of \( S^D_{T_j} \) in terms of the norm \( \| \cdot \|_V \). Moreover, denote by

\[
\pi_j : C^D \rightarrow N_j : x \mapsto \arg\min_{z \in N_j} \| x - z \|_V
\]

the canonical projection on the net \( N_j \), and define

\[
\pi(x) = \arg\min_{\pi_j(x), j \in [G]} \| x - \pi_j(x) \|_V.
\]

Next, observe that since we have that \( \| x_{T_j} \|_2 > 0 \) for all \( j \in S \) and \( \sum_{j \in S} \| x_{T_j} \|_2 \leq 1 \), the weights \( \| x_{T_j} \|_2 \) define a probability distribution on \([B]\). We can therefore construct a random vector \( z \)

\[
P\left( z = \frac{x_{T_j}}{\| x_{T_j} \|_2} \right) = \| x_{T_j} \|_2 \quad \forall j \in S.
\]
such that \( Ez = x \). Next, consider a sequence of \( K \)

independent copies \( z_1, \ldots, z_K \) of \( z \). Then we have by the triangle inequality that

\[
\| x - \frac{1}{K} \sum_{i=1}^{K} \pi(z_i) \|_V
\leq \| x - \frac{1}{K} \sum_{i=1}^{K} z_i \|_V + \frac{1}{K} \sum_{i=1}^{K} \| \pi(z_i) - z_i \|_V
\leq \| x - \frac{1}{K} \sum_{i=1}^{K} (z_i - Ez_i) \|_V + \frac{1}{K} \sum_{i=1}^{K} \| z_i - \pi(z_i) \|_V
\]

where we used the fact that \( Ez = x \). Taking expectation on both sides of the inequality now yields

\[
E \left\| x - \frac{1}{K} \sum_{i=1}^{K} \pi(z_i) \right\|_V
\leq \frac{1}{K} \sum_{i=1}^{K} E \| z_i - \pi(z_i) \|_V.
\]

We focus on the second term on the right-hand side first, and note that

\[
\frac{1}{K} \sum_{i=1}^{K} E \| z_i - \pi(z_i) \|_V
\leq \frac{1}{K} \sum_{i=1}^{K} \sum_{j \in S} \| x_{T_j} \|_2 \cdot \left\| \frac{x_{T_j}}{\| x_{T_j} \|_2} - \pi \left( \frac{x_{T_j}}{\| x_{T_j} \|_2} \right) \right\|_V
\leq \frac{1}{K} \sum_{i=1}^{K} \sum_{j \in S} \| x_{T_j} \|_2 \cdot \varepsilon \quad \frac{\varepsilon}{2} \leq \varepsilon
\]

which follows from the fact that the operator \( \pi \) takes the subvector of \( x \) supported on the \( j \)-th normalized group to the closest point in the \( (\varepsilon/2) \)-net of the respective unit sphere supported on \( T_j \). To deal with the first term in Equation (11), we invoke the Giné-Zinn symmetrization principle\(^{[35]}\) which yields

\[
\frac{1}{K} E \left\| \sum_{i=1}^{K} (z_i - Ez_i) \right\|_V
\leq \frac{2}{K} E \left\| \sum_{i=1}^{K} \xi_i z_i \right\|_V
\leq \frac{2}{K} E \left\| \sum_{i=1}^{K} \xi_i V(z_i) \right\|_2 \to 2
\]

where \( \xi_i \) are independent Rademacher variables. The last step follows from the definition \( \| \cdot \|_V = \| V(\cdot) \|_{2 \to 2} \), as well as from linearity of the operator \( V \).

To control the Rademacher sum on the right-hand side,
we invoke a noncommutative Khintchine inequality for operator norms established in [13, Lemma 9]. To that end, we fix randomness in the sequence \((z_i)\), for the moment (i.e., we condition on \((z_i)\)), and find

\[
E \left\| \sum_{i=1}^{K} \xi_i V(z_i) \right\|_{2 \to 2} \leq \sqrt{\log(mL)} \left( \sum_{i=1}^{K} \|V(z_i)\|_{2 \to 2}^2 \right)^{1/2}
\]

\[
\leq \sqrt{\log(M)} \left( \sum_{i=1}^{K} E \|V(z_i)\|_{2 \to 2}^2 \right)^{1/2}
= \sqrt{\log(M)} \left( \sum_{j \in S} \left\| \sum_{i=1}^{K} x_{I_j} \cdot \left( \frac{x_{I_j}}{\|x_{I_j}\|_2} \right) . \left( \frac{x_{I_j}}{\|x_{I_j}\|_2} \right) \right\|_{2 \to 2}^2 \right)^{1/2}
= \sqrt{\log(M)} \left( \sum_{j \in S} \left\| x_{I_j} \cdot \left( \frac{x_{I_j}}{\|x_{I_j}\|_2} \right) \right\|_{2 \to 2}^2 \right)^{1/2}
\]

where the second step follows from Jensen’s inequality and concavity of \((-1)^{1/2}\). Note that we have by an earlier calculation that

\[
\left\| V \left( \frac{x_{I_j}}{\|x_{I_j}\|_2} \right) \right\|_{2 \to 2} \leq \mu_{\lambda}(\Psi) \left\| \frac{x_{I_j}}{\|x_{I_j}\|_2} \right\|_{L^2,1} = \mu_{\lambda}(\Psi).
\]

Invoking this bound, and summing over the \(\|x_{I_j}\|_2\), we therefore arrive at

\[
\frac{1}{K} \left\| x - \frac{1}{K} \sum_{i=1}^{K} z_i \right\|_{V} \leq 2 \mu_{\lambda}(\Psi) \sqrt{\frac{\log(M)}{K}}.
\]

Overall, we have shown that

\[
E \left\| x - \frac{1}{K} \sum_{i=1}^{K} \pi(z_i) \right\|_{V} \leq 2 \mu_{\lambda}(\Psi) \sqrt{\frac{\log(M)}{K}} + \frac{\varepsilon}{2} \leq \varepsilon.
\]

This implies that for the choice

\[
K \geq \frac{16 \mu_{\lambda}(\Psi)^2 \log(M)}{\varepsilon^2},
\]

there exists at least one realization of the random vector

\[
\hat{z} := \frac{1}{K} \sum_{i=1}^{K} \pi(z_i)
\]

which is at most \(\varepsilon\) apart from \(x\) w.r.t. the norm \(\|\cdot\|_{V}\). Denote now by \(\nu = \max_{\gamma \in [G]} |N_{\gamma}|\) the cardinality of the biggest net \(N_{\gamma}\). Since each random vector \(z_i\) belongs to one of the \(G\) unit spheres \(S_{G}^{D-1}\), there are at most \(G\nu\) possible realizations of the random vector \(\pi(z_i)\). This in turn implies that there are at most \((G\nu)^K\) choices for the sum \(\hat{z}\). In order to cover the unit spheres \(S_{G}^{D-1}\), we return to the volume comparison argument, and find for \(g_{j} := |I_j|\)

\[
|N_{j}| = \Omega(S_{G}^{D-1}, \|\cdot\|_{V}, \varepsilon/2) \leq \Omega(B_{G}^{D}, \|\cdot\|_{2}, \varepsilon/4) \leq \left(1 + \frac{2}{\varepsilon/4}\right)^{g_j}
\]

and therefore

\[
\nu \leq \left(1 + \frac{8}{\varepsilon}\right)^{g_j}.
\]

Combining this estimate with the choice of \(K\) according to Equation (12), we finally find

\[
\sqrt{\log \Omega_{\lambda}(\Omega, \|\cdot\|_{V}, \varepsilon)} \lesssim \sqrt{K \log(G\nu)}
\]

\[
\lesssim \sqrt{K \left( \sqrt{\log(G)} + \sqrt{\log(\nu)} \right)}
\]

\[
\lesssim \mu_{\lambda}(\Psi) \sqrt{\log(D)} / \varepsilon \left( \sqrt{\log(G)} + g \log \left(1 + \frac{8}{\varepsilon}\right) \right). (13)
\]

To establish our final bound on the \(\gamma_2\)-functional of \(M\), we split the entropy integral into two parts. We then control the first part via the volume comparison estimate (9), and bound the second integral via Equation (10) followed by an application of Equation (13). For the first integral, this yields

\[
\int_{0}^{\lambda} \sqrt{\log \Omega(\Omega, \|\cdot\|_{V}, \varepsilon)} \, d\varepsilon
\]

\[
\lesssim \int_{0}^{\lambda} \sqrt{s} \log(eG/s) + 2s \log(1 + 4/s) \, d\varepsilon
\]

\[
\lesssim \lambda \sqrt{s} \log(eG/s) + \lambda \sqrt{2s} \log(5e/\lambda)
\]

where the last estimate follows from [25, Lemma C.9]. For the second integral, we find

\[
\int_{\lambda}^{\lambda} \sqrt{\log \Omega(\Omega, \|\cdot\|_{V}, \varepsilon)} \, d\varepsilon
\]

\[
\lesssim 2 \sqrt{s} \mu_{\lambda}(\Psi) / \lambda \left[ \int_{ \lambda/(2\sqrt{s}) }^{ \lambda/(2\sqrt{s}) } \sqrt{\log(G)} / \varepsilon \, d\varepsilon \right] + \int_{\lambda/(2\sqrt{s})}^{\lambda} \sqrt{g \log(1 + 8/s)} / \varepsilon \, d\varepsilon.
\]

For the last integral, note that \(\sqrt{\log(1 + 1/t)}\) is monotonically decreasing in \(t\). Hence, we have that

\[
\int_{a}^{b} \sqrt{\log \left(1 + \frac{1}{t}\right)} \, dt \leq \sqrt{\log \left(1 + a^{-1}\right)} \log(b/a).
\]

This yields

\[
\int_{\lambda}^{\lambda} \sqrt{\log \Omega(\Omega, \|\cdot\|_{V}, \varepsilon)} \, d\varepsilon
\]

\[
\lesssim 2 \sqrt{s} \mu_{\lambda}(\Psi) \sqrt{\log(D)} \log(\sqrt{s} \mu_{\lambda}(\Psi) / \lambda) \left( \sqrt{\log(G)} + \sqrt{g \log(1 + 16\sqrt{s}/\lambda)} \right).
\]

Simplifying both expressions by absorbing numerical constants into the implicit constant in the notation, and
collecting both estimates finally yields
\[
\gamma_2(\mathcal{M}, \| \cdot \|_{2 \to 2}) \
\leq \lambda \sqrt{s} \log(G/s) + \lambda g \log(\log(1/\lambda)) \
+ \sqrt{s} \mu_2(\Psi) \sqrt{\log(D) \log(s \mu_2(\Psi))} \sqrt{\log(G)} \
+ g \log(s \mu_2(\Psi))
\]
which, for the choice \( \lambda = \mu_2(\Psi)/\sqrt{s} \), ultimately results in
\[
\gamma_2(\mathcal{M}, \| \cdot \|_{2 \to 2}) \
\leq \mu_2(\Psi) \sqrt{\log(G/s)} + \mu_2(\Psi) \sqrt{g \log(s \mu_2(\Psi))} \
+ \mu_2(\Psi) \sqrt{\log(D) \log(s \mu_2(\Psi))} \sqrt{\log(G)} \
+ \mu_2(\Psi) \sqrt{g \log(s \mu_2(\Psi))}.
\]
To establish our main result, it remains to invoke Theorem IV.1 after collecting our estimates for \( \rho_F(\mathcal{M}), \rho_2(\mathcal{M}) \) and \( \gamma_2(\mathcal{M}, \| \cdot \|_{2 \to 2}) \). This concludes the proof of Theorem III.2.

V. THE GROUP-RIP FOR BLOCK DIAGONAL MATRICES WITH CONSTANT BLOCK-DIAGONAL

Let us now turn turn to the scenario in which each sensor is equipped with a copy of the same measurement matrix \( \Phi \), i.e., we observe
\[
y = A\Psi x = \begin{pmatrix} \Phi & \cdots & \Phi \Psi_1 x \\ & \ddots & \vdots \\ & & \Phi \Psi_L x \end{pmatrix} = \begin{pmatrix} \Phi \Psi_1 x \\ \vdots \\ \Phi \Psi_L x \end{pmatrix}.
\]
While we could use the same transformations \( V_i \) as in the case of unique per-sensor matrices, and set
\[
A\Psi x \overset{d}{=} \begin{pmatrix} V_1(x) \\ \vdots \\ V_L(x) \end{pmatrix} \xi := V'(x) \xi
\]
with \( \xi \in \mathbb{R}^{nm} \) a unit-variance \( \tau \)-subgaussian random vector, the lack of a (block) diagonal structure in \( V' \) complicates the calculation of both \( \rho_2 \) and \( \gamma_2 \) as we cannot concisely express the operator norm in terms of a mixed \((\ell_\infty, \ell_2)\) vector norm as in Equation (7). However, since we only require \( \| A\Psi x \|_2^2 \) and \( \| V'(x) \xi \|_2^2 \) to be identical in distribution to apply Theorem IV.1, we are free to reorder the rows of \( V'(x) \). To that end, we define the operator
\[
\hat{V}(x) = \begin{pmatrix} (\Psi_1 x)^T \\ \vdots \\ (\Psi_L x)^T \end{pmatrix} \in \mathbb{C}^{L \times d}.
\]
Then we have with the block diagonal matrix
\[
\hat{V}(x) := \begin{pmatrix} \hat{V}(x) \\ \vdots \\ \hat{V}(x) \end{pmatrix} \in \mathbb{C}^{mL \times md}
\]
with \( m \) copies of \( \hat{V}(x) \) on its diagonal that \( \| A\Psi x \|_2^2 \overset{d}{=} \| \hat{V}(x) \xi \|_2^2 \). As before, we define the set \( \hat{M} = \hat{V}(\Omega) \) so that
\[
\mathbb{P} \left( \sup_{x \in \Omega} \left\| \frac{1}{\sqrt{m}} A\Psi x \right\|_2^2 - 1 \geq \delta \right) = \left( \frac{1}{m} \sup_{\mathbf{r} \in \hat{M}} \| \mathbf{r} \xi \|_2^2 - \mathbb{E} \| \mathbf{r} \xi \|_2^2 \right) \geq \delta.
\]
It remains to estimate the radii of \( \hat{M} \), as well as its metric entropy integral. Unsurprisingly, we mostly proceed in the same way as before. For convenience of notation, we associate with \( \hat{V} \) the norm \( \| \cdot \|_\hat{V} \) on \( \mathbb{C}^D \) induced by \( \| \cdot \|_{\hat{V}} = \| \hat{V}(\cdot) \|_{2 \to 2} \).

First, note that
\[
\| \hat{V}(x) \|_F^2 = \sum_{i=1}^m \| \hat{V}(x) \|_{F} = m \sum_{i=1}^L \text{tr}(\hat{V}(x)\hat{V}(x)^*) 
\]
and therefore
\[
\rho_F(\hat{M}) = \sup_{\mathbf{r} \in \hat{M}} \| \mathbf{r} \|_F = \sup_{x \in \Omega} \| \hat{V}(x) \|_F = \sup_{x \in \Omega} \sqrt{m} \| x \|_2 = \sqrt{m}.
\]
Next, denote as before by \( S \subset [G] \) the index set of nonzero groups of \( x \in \mathbb{C}^D \) w.r.t. \( T \). Then we have due to linearity of \( \hat{V} \), and consequently linearity of \( \hat{V} \) that
\[
\| \hat{V}(x) \|_{2 \to 2} = \| \hat{V}(x) \|_F = \left[ \sum_{i=1}^G \| \hat{V}(x_{\mathcal{T}_i}) \|_{2 \to 2} \right]^{1/2} 
\]
\[
\leq \sum_{i \in S} \| x_{\mathcal{T}_i} \|_2 \cdot \| \hat{V} \left( \frac{x_{\mathcal{T}_i}}{\| x_{\mathcal{T}_i} \|_2} \right) \|_{2 \to 2} 
\]
\[
\leq \| x \|_{\mathcal{T}_1} \max_{i \in S} \sup_{x \in S} \left\| \frac{x_{\mathcal{T}_i}}{\| x_{\mathcal{T}_i} \|_2} \right\|_{2 \to 2} 
\]
\[
\leq \| x \|_{\mathcal{T}_1} \max_{i \in [G]} \sup_{u \in \mathbb{C}^{D_i-1}} \left\| \frac{\hat{V}(u)}{\| x \|_2} \right\|_{2 \to 2}.
\]
In the edge case where the number of groups \( G \) coincides with the ambient dimension \( D \) (i.e., in case of regular sparsity rather than group-sparsity), the supremum in Equation (14) can be easily computed as each sphere \( S^{D_i-1} \) reduces w.l.o.g. to a two-element5 set \( \{ \pm e^i \} \) where \( e^i \in \mathbb{R}^D \) denotes the \( i \)-th canonical unit vector. However, the same does not hold for \( G < D \) which does not allow us to compute (14) numerically. To circumvent the computability issue, we estimate the supremum as follows.

Denote by \( u \) an arbitrary unit-normalized 1-group-

5In light of the linearity of \( \hat{V} \), this in turn implies the supremum in (14) is taken over a singleton set.
sparse vector w.r.t. the group partition \( \mathcal{I} \). Then
\[
\| \tilde{V}(u) \|_{2 \to 2} = \sup_{x \in \mathbb{R}^2} \| \tilde{V}(u) x \|_2 \leq \| T \| \sup_{x \in \mathbb{R}^2} \| \tilde{V}(u) x \|_\infty
\]
\[
= \sqrt{L} \sup_{x \in \mathbb{R}^2} \| \tilde{V}(u, x) \|_2 = \sqrt{L} \sup_{x \in \mathbb{R}^2} \max_{i \in [L]} \langle u, \tilde{\Psi}_i z \rangle
\]
\[
\leq \sqrt{L} \sup_{x \in \mathbb{R}^2} \max_{i \in [L]} \| \tilde{\Psi}_i z \|_{\mathcal{I}, \infty}
\]
\[
= \sqrt{L} \sup_{x \in \mathbb{R}^2} \max_{i \in [L]} \| \tilde{\Psi}_i z \|_{\mathcal{I}, \infty}
\]
where we used the fact that \( \| u \|_{\mathcal{I}, 1} = 1 \) since \( u \) is a unit-norm vector supported on a single group in \( \mathcal{I} \). Expanding the supremum, we find
\[
\sup_{x \in \mathbb{R}^2} \| \tilde{\Psi}_i z \|_{\mathcal{I}, \infty}
\]
\[
= \sup_{x \in \mathbb{R}^2} \max_{i \in [G]} \| \tilde{\Psi}_i z \|_{\mathcal{I}, \infty}
\]
\[
= \max_{i \in [G]} \| \tilde{\Psi}_i z \|_{\mathcal{I}, \infty}
\]
where \( \tilde{\Psi}_i z \in \mathbb{C}^{d \times |\mathcal{I}_i|} \) denotes the submatrix of \( \tilde{\Psi}_i \) restricted to the columns indexed by \( \mathcal{I}_i \). The two estimates therefore yield
\[
\| \tilde{V}(x) \|_{2 \to 2} \leq \| x \|_{\mathcal{I}, 1} \sqrt{L} \max_{i \in [G]} \| (\tilde{\Psi}_i z) \|_{2 \to 2}, \quad (15)
\]
Unfortunately, this bound is too loose in the previously discussed case where \( G = D \) as it does not reduce to
\[
\| \tilde{V}(x) \|_{2 \to 2} \leq \| x \|_{\mathcal{I}, 1} \max_{i \in [D]} \| \tilde{V}(e_i^i) \|_{2 \to 2}
\]
which immediately follows from Equation (14). In other words, the bound does not reduce to the natural bound we obtain in the sparse setting. To remedy the situation, we also consider the following bound. Note that for \( i \in S = \{ i \in [G] : x_{ij} \neq 0 \} \), we have
\[
\| \tilde{V} \left( x \right) \|_{2 \to 2} \leq \sup_{u \in \mathbb{S}^{p-1}} \| \tilde{V}(u) x \|_2 = \sup_{u \in \mathbb{S}^{p-1}} \| \sum_{j \in \mathcal{I}_i} u_j \tilde{V}(e_i^i) \|_{2 \to 2}
\]
\[
\leq \sqrt{L} \max_{j \in \mathcal{I}_i} \| \tilde{V}(e_i^i) \|_{2 \to 2}
\]
Combining both estimates in the parameter
\[
\omega_\mathcal{I}(\Psi) = \min \left\{ \sqrt{\rho} \max_{i \in [D]} \| \tilde{V}(e_i^i) \|_{2 \to 2}, \sqrt{L} \max_{i \in [G]} \| (\tilde{\Psi}_i z) \|_{2 \to 2} \right\}
\]
we find
\[
\| \tilde{V}(x) \|_{2 \to 2} \leq \| x \|_{\mathcal{I}, 1} \omega_\mathcal{I}(\Psi)
\]
which finally yields
\[
\rho_{2 \to 2}(\hat{M}) = \sup_{x \in \Omega} \| \tilde{V}(x) \|_{2 \to 2} \leq \sqrt{\omega_\mathcal{I}(\Psi)}
\]
Now note that we have
\[
\| \tilde{V}(x) \|_{2 \to 2} = \| \tilde{V}(\hat{x}) \|_{2 \to 2} \leq \| \tilde{V}(\hat{x}) \|_{2 \to 2} \leq \| \tilde{V}(\hat{x}) \|_{F} = \| (\tilde{\Psi}_1 \xi, \ldots, \tilde{\Psi}_L \xi) \|_{2 \to 2} = \| \tilde{\Psi} \xi \|_{2 \to 2}^2 = \| x \|_{2 \to 2}^2.
\]
Estimating the \( \gamma_2 \)-functional of \( \hat{M} \) by means of the metric entropy integral
\[
\gamma_2(\hat{M}, \| \cdot \|_{2 \to 2}) \lesssim \int_0^{\rho_{2 \to 2}(\hat{M})} \sqrt{\log(\mathcal{N}(\hat{M}, \| \cdot \|_{2 \to 2}, \varepsilon))} d\varepsilon = \int_0^{\sqrt{\omega_\mathcal{I}(\Psi)}} \sqrt{\log(\mathcal{N}(\Omega, \| \cdot \|_{2 \to 2}, \varepsilon))} d\varepsilon
\]
therefore proceeds identically to the derivation in Section IV-D, and thus
\[
\gamma_2(\hat{M}, \| \cdot \|_{2 \to 2}) \lesssim \sqrt{\omega_\mathcal{I}(\Psi)} \sqrt{\log(D) \log(s) \left( \sqrt{\log(G)} \sqrt{g \log(\omega_\mathcal{I}(\Psi))} \right)}. \quad (16)
\]
In particular, as in the case of Theorem III.2, Theorem III.3 immediately follows by invoking Theorem IV.1 with the respective estimates of \( \rho_{1} \), \( \rho_{2 \to 2} \), and \( \gamma_2 \) of \( \hat{M} \), \( \| \cdot \|_{2 \to 2} \).

VI. Conclusion

In this paper, we established conditions on the number of measurements required to stably and robustly estimate group-sparse vectors by means of block diagonal measurement matrices whose blocks either consist of independent or identical copies of subgaussian matrices. Appealing to a powerful concentration bound on the suprema of chaotic processes, we derived conditions on the number of measurements required for subgaussian block diagonal random matrices to satisfy the so-called group restricted isometry property. This generalizes an earlier result due to Eftekhari et al. who first established a similar result for the canonical sparsity model. Although certain adversarial group partitions including the distributed sensing model may lead to suboptimal scaling of the number of measurements, such cases are generally avoided if signals
are group-sparse in nonlocalized sparsity bases whose basis matrices are not block diagonal. In this case, our results predict almost optimal scaling behavior up to logarithmic factors.

ACKNOWLEDGMENT

The authors would like to thank Holger Rauhut for many fruitful discussions on the topics addressed in the present paper.

APPENDIX A
PROOF OF THEOREM III.1

In general, necessary and sufficient conditions for sparse recovery depend on the so-called null space property (NSP) which ensures that the null space of the measurement matrix does not contain any sparse vectors of a certain order besides the zero vector. In this section, we provide a similar sufficient condition for group-sparse recovery. The group-sparse NSP is a natural generalization of the block-sparse NSP, which was originally introduced in [23]. Similar to the proofs in the block-sparse case of op. cit., the structure of our proof follows the example of the respective proof in the canonical sparsity setting (cf. [25, Chapter 4 and 6]).

A. Robust Group-NSP

Definition A.1 (ℓ2-robust group-NSP). Given q ≥ 1, a matrix A ∈ C^{M×D} is said to satisfy the ℓ2-robust group null space property (group-NSP) of order s with respect to ∥·∥ and constants ρ ∈ (0, 1) and τ > 0 if for all v ∈ C^D and for all S ⊂ {1, 2, ..., q} with |S| = s,

∥v_S∥ ≤ ρ ∥v_{S^c}∥ + τ ∥Av∥.

Theorem A.1. Suppose that the matrix A ∈ C^{M×D} satisfies the ℓ2-robust group null space property of order s with respect to ∥·∥ and constants ρ ∈ (0, 1) and τ > 0. Then for any x, z ∈ C^D,

∥z - x∥ ≤ C ∥v_S∥ + D ∥Av∥,

where C = \frac{(1 + ρ)^2}{1 - ρ} and D = \frac{(3 + ρ)^2}{1 - ρ}.

Proof. We introduce the following notation used throughout the rest of the proof. Given a group partition I = \{I_1, ..., I_G\} and a group index set S ⊂ [G], we denote by I_S the subpartition \{I_i : i ∈ S\}. Moreover, we denote by I_{S^c} the subpartition consisting of the groups indexed by S = [G] \ S. Finally, with slight abuse of notation, we write x_{I_S} for the vector x ∈ C^D restricted to the index set \bigcup_{i ∈ S} I_i.

The ℓ2-robust group-NSP directly implies that for any x, z ∈ C^D and v = z - x, we have

∥v∥ ≤ ∥v_S∥ + ∥v_{S^c}∥,

where

∥v∥ ≤ \frac{ρ}{\sqrt{s}} ∥v_{S^c}∥ + τ ∥Av∥ + ∥v_{S^c}∥.

We first provide a bound for ∥v∥ in terms of ∥v∥_{I,z}. Denote by \{T_1, ..., T_G\} the nonincreasing group rearrangement of I such that

∥v_{T_1}∥ ≥ ∥v_{T_2}∥ ≥ · · · ≥ ∥v_{T_G}∥.

We choose S as the index set of the best s-term group approximation of v which implies that

∥v_S∥^2 = \sum_{j=s+1}^G ∥v_{T_j}∥^2

≤ \left( \frac{1}{s} \sum_{j=1}^s ∥v_{T_j}∥^2 \right) \left( \sum_{j=s+1}^G ∥v_{T_j}∥^2 \right)

≤ \frac{1}{s} ∥v∥_{I,z}^2.

Applying this inequality to (16) therefore yields

∥v∥_2 ≤ \frac{1 + ρ}{\sqrt{s}} ∥v∥_{I,z} + τ ∥Av∥.

Next we bound ∥v∥_{I,z}. First note that if the ℓ2-robust group-NSP holds, the Cauchy-Schwarz inequality implies the following bound on the group ℓ1-norm:

∥v_S∥_{I,z} ≤ ρ ∥v_{T_{z^*}}∥_{I,z} + τ ∥Av∥.

Invoking Equation (18), we have

∥v∥_{I,z} = ∥v_S∥_{I,z} + ∥v_{T_{z^*}}∥_{I,z}

≤ (1 + ρ) ∥v_{T_{z^*}}∥_{I,z} + τ ∥Av∥.

Here S can be chosen differently as before. We apply Equation (18) once again in combination with the following result which is easily adopted to the group-sparse setting from [25, Lemma 4.15].

Lemma A.1. Consider group-sparse signals with G groups and partition I. For S ⊂ [G], vectors x, z ∈ C^D and v = z - x, we have

∥v_{T_S}∥_{I,z} ≤ ∥v∥_{I,z} - ∥x∥_{I,z} + ∥v_S∥_{I,z} + 2∥x_{T_{z^*}}∥_{I,z}.

We apply Equation (18) to the above inequality to obtain

∥v_{T_S}∥_{I,z} ≤ ∥v∥_{I,z} - ∥x∥_{I,z} + ∥v_S∥_{I,z} + 2∥x_{T_{z^*}}∥_{I,z}

≤ ∥v∥_{I,z} - ∥x∥_{I,z} + ρ ∥v_{T_{z^*}}∥_{I,z} + τ ∥Av∥ + 2∥x_{T_{z^*}}∥_{I,z},

which implies that

∥v∥_{I,z} ≤ \frac{1 - ρ}{1 - ρ} ∥v∥_{I,z} + τ ∥Av∥.
and consequently from (19) that
\[ \|\mathbf{v}\|_{I,1} \leq \frac{1 + \rho}{1 - \rho} (\|z\|_{I,1} - \|x\|_{I,1} + 2 \|x^*\|_{I,1}) + \frac{2\sqrt{\tau}}{1 - \rho} \|A\mathbf{v}\|. \]

The choice of \( S \) to minimize the right hand side is the support for the best \( s \)-term group approximation of \( x \). Combined with (17), the theorem follows.

Since the result above holds for any \( x \) and \( z \), choosing \( x = \hat{x} \) and \( z = x^* \) with \( x^* \) denoting a minimizer of Problem (P_{I,1}) immediately implies the following theorem.

**Theorem A.2.** Suppose that \( A \in \mathbb{C}^{M \times D} \) satisfies the \( \ell_2 \)-robust group-NSP of order \( s \) with constants \( 0 < \rho < 1 \) and \( \tau > 0 \). Then for all \( \mathbf{x} \in \mathbb{C}^D \), and \( \mathbf{y} = \hat{\mathbf{x}} + \mathbf{e} \) with \( \|e\|_2 \leq \epsilon \), any solution \( x^* \) of the program Problem (P_{I,1}) approximates \( x \) with error
\[ \|\mathbf{x} - x^*\|_2 \leq \frac{C}{\sqrt{s}} \left( \sigma_1(\mathbf{v}) + \tilde{D}\epsilon \right) \]
with \( C, D > 0 \).

The \( \ell_2 \)-robust group-NSP provides a necessary and sufficient condition for recovery of group-sparse vectors. In the next section, we establish that the group-NSP implies the robust group-NSP and therefore yields a sufficient condition for stable and robust recovery of group-sparse vectors.

**B. Group-NSP and Robust Group-NSP**

In light of the previous section, it suffices to prove that the group-NSP of order \( 2s \) with constant \( \delta \) implies the robust group-NSP in order to prove Theorem III.1. Inspired by \cite[Chapter 6]{Maurey}, consider the sets \( S_0, S_1, \ldots, S_t \) such that \( S_t \) is defined as the index of \( s \) largest groups in \( \bigcup_{j < t} S_j \). If the group-NSP is established for \( S_0 \), which yields the largest possible \( \|\mathbf{v}_{I,0}\|_2 \), then it holds also for all \( S_j \). Assuming the group-NSP holds, we have \( \|A\mathbf{v}_{I,0}\|_2 = (1 + \delta)\|\mathbf{v}_{I,0}\|_2 \) with \( |t| < \delta \) and therefore, we can bound \( \|A\mathbf{v}_{I,0}\|_2 \) by
\[ \|A\mathbf{v}_{I,0}\|_2^2 = \left\langle A\mathbf{v}_{I,0}, A(v - \sum_{k \geq 1} \mathbf{v}_{I,k}) \right\rangle \]
\[ = \left\langle A\mathbf{v}_{I,0}, A\mathbf{v} \right\rangle - \sum_{k \geq 1} \left\langle A\mathbf{v}_{I,0}, A\mathbf{v}_{I,k} \right\rangle \]
\[ \leq \|A\mathbf{v}_{I,0}\|_2^2 \|A\mathbf{v}\|_2 - C_t \sum_{k \geq 1} \|\mathbf{v}_{I,k}\|_2 \|\mathbf{v}_{I,k}\|_2 \]
where the last inequality follows from Lemma A.2 given at the end of this section with \( C_t = \sqrt{\delta^2 - \tau^2} \). Using \( \|A\mathbf{v}_{I,0}\|_2 = \sqrt{1 + \delta^2} \|\mathbf{v}_{I,0}\|_2 \), we arrive at an expression similar to the group-NSP, namely:
\[ (1 + t)\|\mathbf{v}_{I,0}\|_2 \leq C_t \sum_{k \geq 1} \|\mathbf{v}_{I,k}\|_2 + \sqrt{1 + t}\|A\mathbf{v}\|_2. \] (20)

Although \( \sum_{k \geq 1} \|\mathbf{v}_{I,k}\|_2 \leq \|\mathbf{v}_{I,0}\|_{I,1} \), we need an additional \( 1/\sqrt{s} \) term to get the group-NSP. Invoking \cite[Lemma 6.14]{Maurey}, we immediately obtain
\[ \sum_{k \geq 1} \|\mathbf{v}_{I,k}\|_2 \leq \frac{1}{\sqrt{s}} \|\mathbf{v}_{I,0}\|_{I,1} + \frac{1}{4}\|\mathbf{v}_{I,0}\|_2. \]

Next, we apply the above inequality to (20) which—after standard manipulations—yields
\[ \|\mathbf{v}_{I,0}\|_2 \leq \frac{\delta}{\sqrt{1 - \delta^2 - \tau^2}} \frac{1}{\sqrt{s}} \|\mathbf{v}_{I,0}\|_{I,1} \]
\[ + \frac{\sqrt{1 + \delta}}{\sqrt{1 - \delta^2 - \tau^2}} \|A\mathbf{v}\|_2. \]

Therefore the group-NSP holds with \( \rho \) and \( \tau \) given by
\[ \rho = \frac{\delta}{\sqrt{1 - \delta^2 - \tau^2}} \quad \text{and} \quad \tau = \frac{\sqrt{1 + \delta}}{\sqrt{1 - \delta^2 - \tau^2}}. \]

This holds provided that \( \rho < 1 \) which is equivalent to \( \delta < 4/\sqrt{4t} \). The constants \( C \) and \( D \) follow accordingly. The claim follows.

It remains to establish the following result.

**Lemma A.2.** Suppose that the matrix \( A \in \mathbb{C}^{M \times D} \) satisfies the group-NSP of order \( 2s \). For two disjoint sets \( S_0, S_1 \subset [G] \) with cardinality \( s \),
\[ \|\langle A\mathbf{v}_{I,S_0}, A\mathbf{v}_{I,S_1} \rangle\|_2 \leq \sqrt{\delta^2 - \tau^2}\|\mathbf{v}_{I,S_0}\|_2\|\mathbf{v}_{I,S_1}\|_2. \]

**Proof.** To start with, we normalize the two vectors to have unit \( \ell_2 \)-norm by defining the auxiliary vectors \( \mathbf{u} := \mathbf{v}_{I,S_0}/\|\mathbf{v}_{I,S_0}\|_2 \) and \( \mathbf{w} := \mathbf{v}_{I,S_1}/\|\mathbf{v}_{I,S_1}\|_2 \). Fix \( \alpha, \beta > 0 \). Then
\[ 2|m(A\mathbf{u}, \mathbf{w})| = \frac{1}{\alpha + \beta} (\|A(\alpha\mathbf{u} + \mathbf{w})\|_2 - \|A(\beta\mathbf{u} - \mathbf{w})\|_2^2) \]
\[ - (\alpha^2 - \beta^2)\|A\mathbf{u}\|_2^2 \leq \frac{1}{\alpha + \beta} (1 + \delta)\|\alpha\mathbf{u} + \mathbf{w}\|_2^2 \]
\[ - (1 - \delta)\|\beta\mathbf{u} - \mathbf{w}\|_2^2 - (\alpha^2 - \beta^2)(1 + t)\|\mathbf{u}\|_2^2 \]
\[ \leq \frac{1}{\alpha + \beta} [(1 + \delta)(\alpha^2 + 1)^2 \]
\[ - (1 - \delta)(\beta^2 + 1)^2 - (\alpha^2 - \beta^2)(1 + t)] \]
\[ \leq \frac{1}{\alpha + \beta} [\alpha^2(\delta - 1) + \beta^2(\delta + 1) + 2\delta]. \]

Choosing \( \alpha = (\delta + t)/\sqrt{\delta^2 - \tau^2} \) and \( \beta = (\delta - t)/\sqrt{\delta^2 - \tau^2} \) completes the proof.

**APPENDIX B**

**EXTENSION OF MAUREY’S LEMMA**

In this appendix, we provide a result of independent interest which establishes an extended version of Maurey’s empirical method.

**Theorem B.1** (Maurey’s Extended Empirical Lemma). Let \( X \) be a normed space, and let \( U_1, \ldots, U_B \) be \( B \) compact subsets of \( X \). Assume that for every \( L \in \mathbb{N} \) and
(u_1, \ldots, u_L) \in (\bigcup_{j=1}^B U_j)^L$ the following holds:
\[ E \left\| \sum_{j=1}^L \epsilon_j u_j \right\|_X \leq A \sqrt{L} \]
where $(\epsilon_j)_j$ is a Rademacher vector and $A > 0$. Then for every $u > 0$,
\[ \log \mathcal{N}(\text{conv}(U_1, \ldots, U_B), \|\cdot\|_X, u) \leq (A/u)^2 \log \sum_{i=1}^B \mathcal{N}(U_i, \|\cdot\|_X, u/2) \]
where
\[ \text{conv}(U_1, \ldots, U_B) = \left\{ \sum_{i=1}^B \alpha_i u_i : \sum_{i=1}^B \alpha_i = 1, \alpha_i \geq 0, u_i \in U_i \ \forall i \in [B] \right\}. \]

Proof. The proof consists in first covering the sets $U_i$ individually using independent nets, followed by using the convex combinations of elements of the individual nets to cover the convex hull of $\bigcup_i U_i$.

Consider a vector $x$ in $\text{conv}(U_1, \ldots, U_B)$. Then
\[ x = \alpha_1 u_1 + \cdots + \alpha_B u_B \]
with $u_i \in U_i$ and $\alpha_i \in [0, 1]$ for $i \in [B]$ with $\sum_i \alpha_i = 1$. We first find an upper bound for the covering number of the convex hull of $u_1, \ldots, u_B$. Define a random variable $Z$ such that for all $i \in [B]$
\[ \mathbb{P}(Z = u_i) = \alpha_i. \]
We have $E(Z) = x$. Consider $Z_1, \ldots, Z_L$ as $L$ independent copies of $Z$ and set
\[ Y = \frac{1}{L} \sum_{i=1}^L Z_i. \]
Using the Rademacher symmetrization argument we have:
\[ E \left\| x - Y \right\|_X = \frac{1}{L} E \left\| \sum_{i=1}^L (Z_i - x) \right\|_X \leq \frac{2}{L} E \left\| \sum_{i=1}^L \epsilon_i Z_i \right\|_X \leq 2A/\sqrt{L}. \]
where the last part is due to the assumption in the theorem and the fact that $(\epsilon_i)_i$ is an independent symmetric Rademacher sequence. As a result, if $L$ is chosen equal to $16(A/u)^2$, we have
\[ E \left\| x - Y \right\|_X \leq u/2. \]
Therefore, one can find at least one realization of the random vector $Y = \frac{1}{L} \sum_{i=1}^L Z_i$ such that
\[ \left\| x - Y \right\|_X \leq u/2. \]
Consider now for each set $U_i$ a cover $\mathcal{N}_i$ with covering radius $u/2$ with $|\mathcal{N}_i| = \mathcal{N}(U_i, \|\cdot\|_X, u/2)$. Given a vector $z \in U_i$, let $\pi_i(z)$ be its best approximation by the elements of the net $\mathcal{N}_i$:
\[ \pi_i : X \to \mathcal{N}_i, \quad \pi_i(z) = \arg \min_{w \in \mathcal{N}_i} \|z - x\|_X. \]
Furthermore, for a vector $z \in X$, define
\[ \pi(z) = \arg \min_{\pi_i(z), i \in [B]} \|z - \pi_i(z)\|_X \]
for $z \neq 0$, and $\pi(z) = 0$ otherwise. Therefore, $\|z - \pi(z)\|_X \leq u/2$. By the triangle inequality and the definition above, we have
\[ \left\| x - \frac{1}{L} \sum_{i=1}^L \pi_i(z) \right\|_X \geq \left\| x - \frac{1}{L} \sum_{i=1}^L \pi_i(z) \right\|_X - \frac{1}{L} \sum_{i=1}^L \|z_i - \pi(z_i)\|_X \geq \left\| x - \frac{1}{L} \sum_{i=1}^L \pi_i(z) \right\|_X - u/2. \]
Hence,
\[ \left\| x - \frac{1}{L} \sum_{i=1}^L \pi_i(z) \right\|_X \leq u. \]
Since there are at most $\left| \bigcup_{i=1}^B \mathcal{N}_i \right|^L$ different sums of the form $\sum_{i=1}^L \pi_i(z)$, one can cover the convex hull $\text{conv}(U_1, \ldots, U_B)$ with $(\sum_{i=1}^B \mathcal{N}(U_i, \|\cdot\|_X, u/2))^L$ balls of radius $u$.

References

[1] H. Rauhut, K. Schnass, and P. Vandergheynst, “Compressed sensing and redundant dictionaries”, IEEE Transactions on Information Theory, vol. 54, no. 5, pp. 2210–2219, 2008.
[2] P. G. Casazza and G. Kutyniok, Finite frames: Theory and applications, Springer, 2012.
[3] E. J. Candés and T. Tao, “Decoding by linear programming”, IEEE Trans. Inf. Theor., vol. 51, no. 12, pp. 4203–4215, Dec. 2005, issn: 0018-9448. doi: 10.1109/TIT.2005.858979.
[4] E. J. Candès, J. K. Romberg, and T. Tao, “Stable signal reconstruction from incomplete and inaccurate measurements”, Communications on Pure and Applied Mathematics, vol. 59, no. 8, pp. 1207–1223, 2006, issn: 1097-0312, doi: 10.1002/cpa.20124. [Online]. Available: http://dx.doi.org/10.1002/cpa.20124.
[5] ——, “Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information”, IEEE Trans. Information Theory, vol. 52, pp. 489–509, 2006.
[6] E. J. Candès and T. Tao, “Near-optimal signal recovery from random projections: Universal encoding strategies?”, IEEE Transactions on Information Theory, vol. 52, no. 12, pp. 5406–5425, Dec. 2006, issn: 0018-9448. doi: 10.1109/TIT.2006.885507.
12, no. 4, pp. 929–989, Nov. 1984. DOI: 10.1214/aop/1176993138. [Online]. Available: https://doi.org/10.1214/aop/1176993138.