Reduction of generalized Calabi-Yau structures

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(Received Nov. 16, 2006)
(Revised Feb. 11, 2007)

Abstract. A generalized Calabi-Yau structure is a geometrical structure on a manifold which generalizes both the concept of the Calabi-Yau structure and that of the symplectic one. In view of a result of Lin and Tolman in generalized complex cases, we introduce in this paper the notion of a generalized moment map for a compact Lie group action on a generalized Calabi-Yau manifold and construct a reduced generalized Calabi-Yau structure on the reduced space. As an application, we show some relationship between generalized moment maps and the Bergman kernels, and prove the Duistermaat-Heckman formula for a torus action on a generalized Calabi-Yau manifold.

1. Introduction.

Generalized Calabi-Yau structures introduced by Hitchin [7] were developed by Gualtieri [4] as a special case of generalized complex structures. It is a geometrical structure defined by a differential form, which generalizes both the concept of the Calabi-Yau structure – a non vanishing holomorphic form of the top degree – and that of the symplectic structure. In this paper, we consider a compact Lie group action on a generalized Calabi-Yau manifold.

A compact Lie group action on a generalized complex manifold was studied by Lin and Tolman in [8]. In [8], they introduced a notion of generalized moment maps for a compact Lie group action on a generalized complex manifold by generalizing the notion of moment maps for a compact Lie group action on a symplectic manifold. Using this definition, they constructed a generalized complex structure on the reduced space, which is natural up to a transformation by an exact B-field.

In the present paper, we apply the definition of a generalized moment map to a compact Lie group action on a generalized Calabi-Yau manifold, and construct a generalized Calabi-Yau structure on the reduced space. Moreover, we shall show that the reduced generalized Calabi-Yau structure is unique and has the same type as the original generalized Calabi-Yau structure (cf. Section 3).

Theorem A. Let a compact Lie group $G$ act on a generalized Calabi-Yau manifold $(M, \varphi)$ in a Hamiltonian way with a generalized moment map $\mu : M \to \mathfrak{g}^*$. If $G$ acts freely on $\mu^{-1}(0)$, then the quotient space $M_0 = \mu^{-1}(0)/G$ is a smooth manifold, and inherits a unique generalized Calabi-Yau structure $\bar{\varphi}$ which satisfies

2000 Mathematics Subject Classification. Primary 37J15; Secondary 14J32.
Key Words and Phrases. generalized Calabi-Yau structures, moment maps, Bergman kernels, the Duistermaat-Heckman formula.
where \( i_0 : \mu^{-1}(0) \to M \) is the inclusion and \( p_0 : \mu^{-1}(0) \to M_0 \) is the natural projection. Moreover, for each \( p \in \mu^{-1}(0) \),
\[
\text{type}(\varphi_p) = \text{type}(\tilde{\varphi}_p).
\]

The detailed definitions of the theorem are in Section 3. In particular, in the case that the generalized Calabi-Yau structure is induced by a symplectic structure, the reduced form is induced by the reduced symplectic form. In addition we construct an example of a Hamiltonian action on a generalized Calabi-Yau structure which is not induced by either a symplectic structure or a Calabi-Yau one. We then show some relationship between generalized moment maps and Bergman kernels (cf. Example 3.3.2 and 3.3.3 in Section 3).

We next consider that a generalized Calabi-Yau structure \( \varphi \) on a connected manifold \( M \) which has constant type \( k \). Then there exists a natural volume form \( dm = ((\sqrt{-1})^n/(2^n-k))\langle \varphi, \tilde{\varphi} \rangle \) defined by \( \varphi \), which generalizes the Liouville form on a symplectic manifold. Indeed, if \( \varphi \) is a generalized Calabi-Yau structure induced by a symplectic structure \( \omega \), then \( dm \) coincides with the Liouville form for the symplectic structure \( \omega \). Further by assuming that a compact torus \( T \) acts on \( M \) effectively. Under the assumptions, we shall show the Duistermaat-Heckman formula for the volume form \( dm \) (cf. Section 4).

**Theorem B.** Let \( (M, \varphi) \) be a 2n-dimensional connected generalized Calabi-Yau manifold which has constant type \( k \), and suppose that compact l-torus \( T \) acts on \( M \) effectively and in a Hamiltonian way. In addition, we assume that the generalized moment map \( \mu \) is proper. Then the pushforward \( \mu_*(dm) \) of the natural volume form \( dm \) under \( \mu \) is absolutely continuous with respect to the Lebesgue measure on \( \mathfrak{t} \) and the Radon-Nikodym derivative \( f \) can be written by
\[
f(a) = \int_{M_a} dm_a = \text{vol}(M_a)
\]
for each regular value \( a \in \mathfrak{t}^* \) of \( \mu \), and \( dm_a \) denotes the measure defined by the natural volume form on the reduced space \( M_a = \mu^{-1}(a)/T \).

This paper is organized as follows. In Section 2 we introduce background materials and the definition of generalized Calabi-Yau structures. In Section 3 we define the notion of generalized moment maps for a Lie group action on a generalized Calabi-Yau manifold, and construct a generalized Calabi-Yau structure on the reduced space. In addition, we discuss some relations between generalized moment maps and Bergman kernels. At last Section, we proved the Duistermaat-Heckman formula for a Hamiltonian torus action on a generalized Calabi-Yau manifold.
2. Generalized Calabi-Yau structures.

In this section we recall the definition of generalized Calabi-Yau structures. For the detail, see [4] and [7].

2.1. Clifford algebras and the spin representation.

Let \( V \) be a real vector space of dimension \( n \), and \( V^* \) be the dual space of \( V \). Then the direct sum \( V \oplus V^* \) admits a natural indefinite metric of signature \((n, n)\) defined by

\[
(X + \alpha, Y + \beta) = \frac{1}{2} (\beta(X) + \alpha(Y))
\]

for \( X + \alpha, Y + \beta \in V \oplus V^* \). Let \( T(V \oplus V^*) = \oplus_{p=0}^{\infty} (V \otimes V^*)^p \) be the tensor algebra of \( V \oplus V^* \), and define \( \mathcal{J} \) to be the two-sided ideal generated by \( \{ (X + \alpha) \otimes (X + \alpha) - (X + \alpha, X + \alpha) \mid X + \alpha \in V \oplus V^* \} \). Then we call the quotient algebra

\[
CL(V \oplus V^*) = T(V \oplus V^*)/\mathcal{J}
\]

the Clifford algebra of \( V \oplus V^* \). For each \( E, F \in CL(V \oplus V^*) \), \( E \cdot F \) denotes the multiplication induced by the tensor product.

Consider the exterior algebra \( ^\wedge V^* \) and a linear mapping \( V \oplus V^* \rightarrow \text{End}(^\wedge V^*) \) defined by

\[
(X + \alpha) \cdot \varphi = \iota_X \varphi + \alpha \wedge \varphi.
\]

Then we have

\[
(X + \alpha)^2 \cdot \varphi = \iota_X (\alpha \wedge \varphi) + \alpha \wedge \iota_X \varphi = (\iota_X \alpha) \varphi = (X + \alpha, X + \alpha) \varphi,
\]

so it can be extended to a representation of the Clifford algebra \( CL(V \oplus V^*) \rightarrow \text{End}(^\wedge V^*) \). This is called the spin representation, and a element \( \varphi \in ^\wedge V^* \) is called a spinor.

We define \( Pin(V \oplus V^*) \) and \( Spin(V \oplus V^*) \), subgroups of the group consists of invertible elements of \( CL(V \oplus V^*) \) by

\[
Pin(V \oplus V^*) = \{ E_1 \cdots E_k \mid k \in \mathbb{N} \cup \{0\}, (E_i, E_i) = \pm 1 \},
\]

\[
Spin(V \oplus V^*) = \{ E_1 \cdots E_{2k} \mid k \in \mathbb{N} \cup \{0\}, (E_i, E_i) = \pm 1 \}.
\]

we call \( Pin(V \oplus V^*) \) the pin group, and \( Spin(V \oplus V^*) \) the spin group. The following proposition says a geometrical meaning of the pin and spin group.

**Proposition 2.1.1 ([1], [4]).** The pin group and the spin group have following short exact sequences.
Let $Spin_0(V \oplus V^*)$ denote the identity component of $Spin(V \oplus V^*)$. Then $\wedge^\ast V^*$ has a $Spin_0(V \oplus V^*)$-invariant bilinear form defined by

$$\langle \varphi, \psi \rangle = (\sigma(\varphi) \wedge \psi)_n,$$

where $(\cdot)_n$ indicates taking the $n$-th degree component of the form, and $\sigma : \wedge^\ast V^* \to \wedge^\ast V^*$ is an anti-homomorphism on $\wedge^\ast V^*$ defined by

$$\sigma(\varphi_1 \wedge \cdots \wedge \varphi_k) = \varphi_k \wedge \cdots \wedge \varphi_1$$

for each $\varphi_1, \ldots, \varphi_k \in \wedge^1 V^*$.

### 2.2. Pure spinors and generalized Calabi-Yau structures on a vector space.

Given a spinor $\varphi \in \wedge^\ast V^*$, we define the annihilator of $\varphi$ by

$$E_\varphi = \{ X + \alpha \in V \oplus V^* \mid (X + \alpha) \cdot \varphi = 0 \}.$$

We also define the annihilator $E_\varphi \subset (V \oplus V^*) \otimes C$ of a complex spinor $\varphi \in \wedge^\ast V^* \otimes C$ in a similar way. Since an element $X + \alpha \in E_\varphi$ satisfies

$$(X + \alpha, X + \alpha)\varphi = (X + \alpha)^2 \cdot \varphi = 0,$$

we see that if $\varphi$ is a non-zero spinor or a complex spinor, then $E_\varphi$ is isotropic with respect to the natural metric on $(V \oplus V^*) \otimes C$. In particular, we have $\dim E_\varphi \leq n$.

**Definition 2.2.1.** A spinor $\varphi \in \wedge^\ast V^*$ is called pure if $E_\varphi$ is maximally isotropic, which means that has the dimension equal to $n$. A complex spinor $\varphi \in \wedge^\ast V^* \otimes C$ with the maximal isotropic subspace $E_\varphi$ is called a complex pure spinor.

**Remark 2.2.2.** It is known that if $\varphi \in \wedge^\ast V^*$ is a pure spinor, then $\varphi \in \wedge^{ev,od} V^*$, where

$$\wedge^{ev} V^* = \wedge^0 V^* \oplus \wedge^2 V^* \oplus \cdots,$$

$$\wedge^{od} V^* = \wedge^1 V^* \oplus \wedge^3 V^* \oplus \cdots,$$

and $\varphi \in \wedge^{ev,od} V^*$ means that $\varphi$ belongs in either $\wedge^{ev} V^*$ or $\wedge^{od} V^*$.

**Example 2.2.3.** The spinor $1 \in \wedge^0 V^*$ is pure, since $E_1 = V$. 
EXAMPLE 2.2.4. A non-zero vector $\varphi \in \wedge^n V^*$ is also pure. The annihilator is $E_\varphi = V^*$.

EXAMPLE 2.2.5. If $\varphi$ is a pure spinor on $V$ and $B$ is a 2-form, then

$$\exp(B)\varphi = \left(1 + B + \frac{1}{2!} B^2 + \cdots\right) \wedge \varphi$$

is also pure. The annihilator is $E_{\exp(B)\varphi} = \{X + \alpha + \iota_X B | X + \alpha \in E_\varphi\}$, where $E_\varphi$ is the annihilator of $\varphi$.

Gualtieri shows in his thesis [4] that every pure spinor can be written by a complex 2-form and a decomposable complex form as follows.

FACT 2.2.6 ([4]). Let $\varphi$ be a complex pure spinor on $V$. Then there exists a complex 2-form $B + \sqrt{-1} \omega \in \wedge^2 V^* \otimes \mathbb{C}$ and a complex $k$-form $\Omega$ such that

$$\varphi = \exp(B + \sqrt{-1} \omega)\Omega.$$

Moreover, $\Omega$ can be written

$$\Omega = \theta^1 \wedge \cdots \wedge \theta^k$$

by some 1-forms $\theta^1, \ldots, \theta^k \in \wedge^1 V^* \otimes \mathbb{C}$, and $\omega$ is nondegenerate on a subspace $W = \{X \in V | \iota_X \Omega = 0\}$.

The degree of the form $\Omega$ is called the type of the complex pure spinor $\varphi$ and written by $\text{type}(\varphi)$.

Now we give the definition of a generalized Calabi-Yau structure on a real vector space $V$.

DEFINITION 2.2.7. Let $V$ be a real vector space of dimension $n = 2m$. A generalized Calabi-Yau structure on $V$ is a complex pure spinor $\varphi \in \wedge^{n/2} V^* \otimes \mathbb{C}$ which satisfies that $\langle \varphi, \varphi \rangle \neq 0$.

Fact 2.2.6 tells us if there exists a complex pure spinor on $V$ which satisfies $\langle \varphi, \varphi \rangle \neq 0$, then $V$ must be even dimensional. The condition $\langle \varphi, \varphi \rangle \neq 0$ has the following geometrical meaning.

FACT 2.2.8 ([1]). Let $\varphi$ and $\psi$ be pure spinors. Then they satisfy $\langle \varphi, \psi \rangle \neq 0$ if and only if their annihilators $E_\varphi$ and $E_\psi$ satisfy $E_\varphi \cap E_\psi = \{0\}$.

For the proof, see III.2.4 in [1].

EXAMPLE 2.2.9. For a symplectic form $\omega$ on $V$, we put

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\[ \varphi_\omega = \exp \sqrt{-1} \omega. \]

Then we have \( E_{\varphi_\omega} = \{ X - \sqrt{-1} \omega(X) | X \in V \otimes C \} \) and \( \dim E_{\varphi_\omega} = n \). Since \( \omega \) is non-degenerate, we have \( \langle \varphi_\omega, \varphi_\omega \rangle = \left( (-2\sqrt{-1})^m / m! \right) \omega^m \neq 0 \). Hence \( \varphi_\omega \) is a generalized Calabi-Yau structure on \( V \). The type of \( \varphi_\omega \) is equal to 0.

**Example 2.2.10.** If \( V \) has a complex structure \( J \), then for the \( \sqrt{-1} \)-eigenspace \( V^{1,0} \) of \( J^* : V^* \otimes C \rightarrow V^* \otimes C \), \( \wedge^m V^{1,0} \) is one-dimensional complex vector space. Let \( \Omega \) be a non-zero vector in \( \wedge^m V^{1,0} \). Then, we have \( E_{\Omega} = V_{0,1} \oplus V^{1,0} \) and \( \langle \Omega, \Omega \rangle = (-1)^m \Omega \wedge \Omega \neq 0 \). So \( \Omega \) is a generalized Calabi-Yau structure on \( V \). The type of \( \Omega \) is equal to \( m \).

**Example 2.2.11.** Let \( \varphi \) be a generalized Calabi-Yau structure on \( V \). For each \( B \in \wedge^2 V^* \), the previous example shows that \( \exp(B) \varphi \) is pure. Moreover, the bilinear form gives \( \langle \exp(B) \varphi, \exp(B) \varphi \rangle = \langle \varphi, \varphi \rangle \neq 0 \). Hence \( \exp(B) \varphi \) is also a generalized Calabi-Yau structure on \( V \). The type of \( \exp(B) \varphi \) coincides with that of \( \varphi \).

**Example 2.2.12.** If \( \varphi_1 \) and \( \varphi_2 \) are two generalized Calabi-Yau structures on two vector spaces \( V_1 \) and \( V_2 \), and \( p_1, p_2 \) are the projections from the direct sum \( V_1 \oplus V_2 \). Then \( \varphi = p_1^* \varphi_1 \wedge p_2^* \varphi_2 \) is a generalized Calabi-Yau structure on the product. The type of \( \varphi \) is equal to the sum type(\( \varphi_1 \)) + type(\( \varphi_2 \)).

### 2.3. Generalized Calabi-Yau structures on a manifold.

Let \( M \) be a smooth manifold of dimension \( 2n \), and consider the direct sum \( TM \oplus T^* M \) of the tangent bundle and the cotangent bundle. Then there is an indefinite metric on the vector bundle \( TM \oplus T^* M \) defined by \( \langle X + \alpha, Y + \beta \rangle = \frac{1}{2} (\beta(X) + \alpha(Y)) \).

**Definition 2.3.1 ([7]).** A generalized Calabi-Yau structure on a manifold \( M \) is a closed differential form \( \varphi \in \Omega^{n\text{odd}} \otimes C \) which satisfies the following conditions.

- For each \( p \in M \), \( \varphi_p \) is a complex pure spinor on \( (T_p M \oplus T^*_p M) \otimes C \).
- At each point, \( \langle \varphi, \varphi \rangle \neq 0 \).

**Remark 2.3.2.** Generalized Calabi-Yau structures were defined by Hitchin in [7]. If a generalized Calabi-Yau structure \( \varphi \) is given, then the annihilator \( E_{\varphi} \) defines a generalized complex structure in the sense of Hitchin [7]. This shows that a generalized Calabi-Yau manifold is a special case of a generalized complex manifold. For the detail, see Proposition 1 in [7].

**Example 2.3.3.** Let \( M \) be a \( 2n \)-dimensional symplectic manifold with the symplectic form \( \omega \), and put

\[ \varphi_\omega = \exp \sqrt{-1} \omega. \]

Then we have \( E_{\varphi_\omega} = \{ X - \sqrt{-1} \omega(X) | X \in T \otimes C \} \) and \( \langle \varphi_\omega, \varphi_\omega \rangle = \left( (-2\sqrt{-1})^n / n! \right) \omega^n \neq 0 \). Since \( \omega \) is closed, \( \varphi_\omega \) is also closed. Hence \( \varphi_\omega \) is a generalized Calabi-Yau structure on \( M \).
EXAMPLE 2.3.4. Let $M$ be an $n$-dimensional complex manifold with a non-vanishing holomorphic $n$-form $\Omega$. Then $\Omega$ is pure since $E_\Omega = T_{0,1} \oplus T^{1,0}$. In addition, the bilinear form gives $(\Omega, \Omega) = (-1)^n \Omega \wedge \Omega$, which is non-vanishing. Since $\Omega$ is closed, $\Omega$ is a generalized Calabi-Yau structure on $M$.

EXAMPLE 2.3.5. If $B$ is a closed 2-form on a generalized Calabi-Yau manifold $(M, \varphi)$, then $\exp(B)\varphi$ is also a closed form. By the previous example, $\exp(B)\varphi$ is pure and $(\exp(B)\varphi, \exp(B)\varphi) \neq 0$ at each point. So $\exp(B)\varphi$ is also a generalized Calabi-Yau structure on $M$. This is called the $B$-field transform of $\varphi$.

EXAMPLE 2.3.6. If $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$ are two generalized Calabi-Yau manifolds and $p_1, p_2$ are the projections from the product manifold $M_1 \times M_2$. Then $\varphi = p_1^*\varphi_1 \wedge p_2^*\varphi_2$ is a generalized Calabi-Yau structure on the product. In particular, a product manifold of generalized Calabi-Yau manifolds is also a generalized Calabi-Yau manifold.

The local expression of a generalized Calabi-Yau structure is given by the following proposition by Gualtieri [4]. This helps us to prove the Duistermaat-Heckman formula later.

FACT 2.3.7 ([4]). An element of a generalized Calabi-Yau manifold $(M, \varphi)$ is said to be regular if it has a neighborhood where the type of $\varphi$ is constant. If $p_2^* M$ is regular, then for sufficiently small neighborhood $U_p$ of $p$, there exists a complex 2-form $B + \sqrt{-1}\omega \in \Omega^2(U_p) \otimes \mathbb{C}$ such that

$$\varphi = \exp(B + \sqrt{-1}\omega)\varphi_k$$

on $U_p$, where $k$ is the type of $\varphi_p$. Moreover, $\varphi_k$ can be written

$$\varphi_k = \theta^1 \wedge \cdots \wedge \theta^k$$

by some 1-forms $\theta^1, \ldots, \theta^k \in \Omega^1(U_p) \otimes \mathbb{C}$.

3. Reduction of generalized Calabi-Yau structures.

3.1. Generalized moment maps.

In this section we define the notion of generalized moment maps for a compact Lie group action on a generalized Calabi-Yau manifold, and construct a generalized Calabi-Yau structure on the reduced space. The definition of generalized moment maps for generalized complex cases is given by Lin and Tolman [8].

DEFINITION 3.1.1. Let a compact Lie group $G$ with its Lie algebra $\mathfrak{g}$ act on a generalized Calabi-Yau manifold $(M, \varphi)$ preserving $\varphi$. A generalized moment map is a smooth function $\mu : M \rightarrow \mathfrak{g}^*$ which satisfies

- $\mu$ is $G$-equivariant, and
- $\xi_M - \sqrt{-1}d\mu^\xi$ lies in $E_\varphi$ for all $\xi \in \mathfrak{g}$, where $\xi_M$ denotes the induced vector field on $M$ and $\mu^\xi$ is the smooth function defined by $\mu^\xi(p) = \mu(p)(\xi)$. 

A $G$-action which preserves the generalized Calabi-Yau structure $\varphi$ is called Hamiltonian if a generalized moment map exists.

Here are some examples of generalized moment maps.

**Example 3.1.2.** Let $G$ act on a symplectic manifold $(M, \omega)$ preserving $\omega$, and $\mu : M \to g^*$ be a moment map. Then $G$ also preserves the generalized Calabi-Yau structure $\varphi = \exp \sqrt{-1} \omega$, and $\mu$ is also a generalized moment map.

**Example 3.1.3.** Let $G$ act on a connected Calabi-Yau $n$-fold $(M, \Omega)$, where $\Omega$ is a non-vanishing holomorphic $n$-form. If the $G$-action is Hamiltonian, $\xi_M$ must be anti-holomorphic for all $\xi \in \mathfrak{g}$. However induced vector fields must be real, so we have $\xi_M = 0$. In particular, the $G$-action is trivial and the generalized moment map is regarded as a linear functional on the Lie algebra $\mathfrak{g}$.

**Example 3.1.4.** If $G$ acts on two generalized Calabi-Yau manifolds $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$, preserving both $\varphi_1$ and $\varphi_2$. Let $\mu_1$ and $\mu_2$ are generalized moment maps for these actions. Then the diagonal action of $G$ on the product manifold $M_1 \times M_2$ preserves the generalized Calabi-Yau structure $\varphi = p_1^* \varphi_1 \wedge p_2^* \varphi_2$, where $p_1$ and $p_2$ are the projections from the product $M_1 \times M_2$. Moreover $\mu = \mu_1 \circ p_1 + \mu_2 \circ p_2$ is a generalized moment map for this action.

### 3.2. Generalized Calabi-Yau structure on the reduced space.

Let a compact Lie group $G$ act on a generalized Calabi-Yau manifold $(M, \varphi)$ in a Hamiltonian way with a generalized moment map $\mu : M \to g^*$. Suppose that $G$ acts freely on $\mu^{-1}(0)$. Then $0$ is a regular value and the quotient space

$$M_0 = \mu^{-1}(0)/G$$

is a manifold. The purpose of 3.2 is to prove Theorem A in Introduction. By restricting to an appropriate neighborhood of $\mu^{-1}(0)$, we may assume that $G$ acts freely on $M$. The following lemmas are required for the proof of the theorem.

**Lemma 3.2.1.** Under the assumptions above, let $\mathfrak{g}_M$ be the subbundle of $TM$ generated by the fundamental vector fields $\xi_M$ for $\xi \in \mathfrak{g}$, and $d\mu$ be the subbundle of $T^*M$ generated by the differential $d\mu^\eta$ for $\eta \in \mathfrak{g}$. Then we have

1. $T_p \mu^{-1}(0) = (d\mu)^0_p$,
2. $\ker(p_{\mathfrak{g}_M})_p = (\mathfrak{g}_M)_p$, and
3. $T_p \mu^{-1}(0)/(\mathfrak{g}_M)_p = (d\mu)^0_p/(\mathfrak{g}_M)_p$,

where $p \in \mu^{-1}(0)$ and $(d\mu)^0_p = \{X \in T_p M | (d\mu^\xi)_p(X) = 0 \ (\xi \in \mathfrak{g})\}$ is the annihilator of $(d\mu)_p$.

**Proof.** For each $\xi \in \mathfrak{g}$, the smooth function $\mu^\xi$ vanishes on $\mu^{-1}(0)$. So $(d\mu^\xi)_p(X) = 0$ for all $X \in T_p \mu^{-1}(0)$. This implies that $T_p \mu^{-1}(0) \subset (d\mu)^0_p$. In addition, because $\dim T_p \mu^{-1}(0) = \dim (d\mu)^0_p$, the first claim holds. Since $(\mathfrak{g}_M)_p \subset \ker(p_{\mathfrak{g}_M})_p$ and
$p_0 : \mu^{-1}(0) \rightarrow M_0$ is a submersion, the second claim holds. Now it is easy to see the last claim. \hfill \Box

The following lemma will help us to prove that the reduced form does not vanish anywhere.

**Lemma 3.2.2.** Under the assumptions above, let $\pi : (TM \oplus T^*M) \otimes C \rightarrow TM \otimes C$ be the natural projection. Then we have

$$\dim_C(T\mu^{-1}(0) \otimes C) \cap \pi(E_\varphi)_p = \dim_C \pi(E_\varphi)_p - \dim G$$

for each $p \in \mu^{-1}(0)$.

**Proof.** For a subspace $W \subset (TM \oplus T^*M)_p \otimes C$, we denote by $W^\perp$ the annihilator of $W$ with respect to the natural metric on $(TM \oplus T^*M)_p \otimes C$. Then, since $E_\varphi$ is maximal isotropic, we have

$$E_\varphi = E_\varphi^\perp,$$

and $W^\perp \cap (E_\varphi)_p = (W + (E_\varphi)_p)^\perp$.

If $X \in (T\mu^{-1}(0) \otimes C) \cap \pi(E_\varphi)_p$, then it satisfies that

$$X \in \pi(E_\varphi)_p,$$

and $d\mu^\xi(X) = 0$ for each $\xi \in \mathfrak{g}$. Thus we have $X \in \pi((\mathfrak{g}_M \otimes C)^\perp \cap E_\varphi)_p$. Conversely, if $X \in \pi((\mathfrak{g}_M \otimes C)^\perp \cap E_\varphi)_p$, then we also have $d\mu^\xi(X) = 0$ for each $\xi \in \mathfrak{g}$. So we have

$$X \in (T\mu^{-1}(0) \otimes C) \cap \pi(E_\varphi)_p.$$

This shows that

$$(T\mu^{-1}(0) \otimes C) \cap \pi(E_\varphi)_p = \pi((\mathfrak{g}_M \otimes C)^\perp \cap E_\varphi)_p.$$ 

Since the kernel of $\pi : (TM \oplus T^*M)_p \otimes C \rightarrow T_pM \otimes C$ is equal to $T_p^*M \otimes C$, we have

$$(\mathfrak{g}_M \otimes C)_p^\perp \cap (E_\varphi)_p \cap T_p^*M \otimes C = ((\mathfrak{g}_M \otimes C) + E_\varphi)_p \cap T_p^*M \otimes C = \pi((\mathfrak{g}_M \otimes C) + E_\varphi)_p^0 = \pi(E_\varphi)_p^0,$$

and thus

$$\dim_C \pi((\mathfrak{g}_M \otimes C)^\perp \cap E_\varphi)_p = \dim_C ((\mathfrak{g}_M \otimes C)^\perp \cap (E_\varphi)_p) - \dim_C \pi(E_\varphi)_p^0.$$ 

In addition, by $(\mathfrak{g}_M \otimes C)_p \cap (E_\varphi)_p = \{0\}$, we obtain the dimension

$$\dim_C ((\mathfrak{g}_M \otimes C) + E_\varphi)_p^1 = \dim C \pi((\mathfrak{g}_M \otimes C) + E_\varphi)_p^1 = \dim M - \dim G.$$
Hence we have
\[
\dim_{C}(T_{p}\mu^{-1}(0) \otimes C) \cap \pi(E_{\varphi})_{p} = \dim_{C} \pi((\mathfrak{g}_{M} \otimes C)^{1} \cap (E_{\varphi}))_{p} \\
= \dim_{C}(\mathfrak{g}_{M} \otimes C)^{1} \cap (E_{\varphi})_{p} - \dim_{C} \pi(E_{\varphi})_{p} \\
= \dim_{C} \pi(E_{\varphi})_{p} - \dim G,
\]
this completes the proof.

**Proof of Theorem A.** For each \( p \in \mu^{-1}(0) \), we denote by \((\varphi_{s})_{p}\) the \( s \)-th degree component of \( \varphi_{p} \in \wedge^{ev/od}T_{p}M \otimes C \). Then, by the definition of the generalized moment map, we have
\[
i_{\xi_{M}}\varphi_{s} = \sqrt{-1}d\mu^{s} \wedge \varphi_{s-2} = 0
\]
for each \( \xi \in \mathfrak{g} \). Moreover, the identity \( T_{p}\mu^{-1}(0) = (d\mu)_{p}^{0} \) in Lemma 3.2.1 tells us that the \((s-1)\)-form \( i_{\xi_{M}}(\varphi_{s})_{p} \) vanishes on \( T_{p}\mu^{-1}(0) \). So by identifying the tangent space \( T_{p}M_{0} \) with \( T_{p}\mu^{-1}(0)/\langle \mathfrak{g}_{M} \rangle_{p} \) (see Lemma 3.2.1, (3)), we obtain a well-defined complex \( s \)-form \( (\tilde{\varphi}_{s})_{[p]} \) on \( T_{p}M_{0} \) by
\[
(\tilde{\varphi}_{s})_{[p]}([X_{1}, \cdots, [X_{s}]] = (i_{0}^{p}\varphi_{s})_{p}(X_{1}, \cdots, X_{s}),
\]
where \( X_{1}, \cdots, X_{s} \in T_{p}\mu^{-1}(0) \). Thus we have a complex form \( (\tilde{\varphi})_{[p]} \in \wedge^{ev/od}T_{p}M_{0} \otimes C \) defined by
\[
(\tilde{\varphi})_{[p]} = (\tilde{\varphi}_{k})_{[p]} + (\tilde{\varphi}_{k+2})_{[p]} + \cdots,
\]
where \( k \) is the type of \( \varphi_{p} \). \( G \)-invariance of the form \( \varphi \) tells us that the definition of \( (\tilde{\varphi})_{[p]} \) does not depend on a representative \( p \in \mu^{-1}(0) \). So we get the reduced form \( \tilde{\varphi} \in \Omega^{ev/od} \otimes C \). It is clear that \( \tilde{\varphi} \) satisfies that \( p^{0}_{0}\tilde{\varphi} = i_{0}^{p}\varphi \) and \( d\tilde{\varphi} = 0 \).

Next we shall show that \((\tilde{\varphi})_{[p]} \neq 0 \). It is sufficient to show that \((i_{0}^{p}\varphi_{k})_{p} \neq 0 \). Suppose that \( \dim M = 2n \) and \( \dim G = l \). Then Lemma 3.2.2 tells us
\[
\dim_{C}(T_{p}\mu^{-1}(0) \otimes C) \cap \pi(E_{\varphi})_{p} = 2n - k - l.
\]
So we can take a basis
\[
\{e_{1}, \cdots, e_{2n-k-l}, u_{1}, \cdots, u_{k}, v_{1}, \cdots, v_{l}\}
\]
of \( T_{p}M \otimes C \), where \( \{e_{1}, \cdots, e_{2n-k-l}, u_{1}, \cdots, u_{k}\} \) is a basis of \( T_{p}\mu^{-1}(0) \otimes C \), and \( \{e_{1}, \cdots, e_{2n-k-l}, v_{1}, \cdots, v_{l}\} \) is a basis of \( \pi(E_{\varphi}) \). Since \((\varphi_{k})_{p} \neq 0 \), so we have
\[
(\varphi_{k})_{p}(u_{1}, \cdots, u_{k}) \neq 0.
\]
This shows that \((i_0^*\varphi_k)_p \neq 0\).

Now we say that an element \(X + \tilde{\alpha} \in \langle TM_0 \oplus T^*M_0 \rangle_p \otimes C\) satisfies the compatibility condition if there exists \(X \in T_p\mu^{-1}(0) \otimes C\) and \(\alpha \in T^n_p M \otimes C\) such that \((p_0)_p X = \tilde{X}\), \(p_0^*\tilde{\alpha} = i_0^*\alpha\), and that \((i_0)_p(X) + \alpha \in (E_\varphi)_p\). We denote by \(E_0\) the set of elements \(X + \tilde{\alpha} \in \langle TM_0 \oplus T^*M_0 \rangle_p \otimes C\) which satisfy the compatibility condition.

Then, for each \(\tilde{X} + \tilde{\alpha} \in E_0\), we have

\[
p_0^*(t\chi \tilde{\varphi} + \tilde{\alpha} \wedge \tilde{\varphi}) = i_0^*_0(t(i_{(\alpha)})_X \varphi + \alpha \wedge \varphi) = 0.
\]

So we can see \(E_0 \subset E_\varphi\) because \(p_0\) is a submersion. Moreover, since \(E_\varphi\) is isotropic, we have \(\dim C E_0 \leq \dim C E_\varphi \leq 2(n - l)\). Let us show the equality \(\dim C E_0 = 2(n - l)\).

Since \(\dim C(T_p\mu^{-1}(0) \otimes C) \cap \pi(E_\varphi)_p = 2n - k - l\), we can take

\[
X_1 + \alpha_1, \ldots, X_{2n-l-k} + \alpha_{2n-l-k} \in E_\varphi,
\]

which are linearly independent and \(X_i \in T_p\mu^{-1}(0) \cap \pi(E_\varphi)_p\) for \(i = 1, \ldots, 2n - l - k\). Since

\[
t_{\xi_i} \alpha_i = (\alpha_i, \xi_M) = (X_i + \alpha_i, \xi_M - d\mu^i) = 0
\]

for each \(\xi \in g\), \(\alpha_i\) descends to a form \(\tilde{\alpha}_i \in \wedge^{ev/od} T^*_{p_0} M_0 \otimes C\). If we take

\[
\tilde{X}_i = (p_0)_p X_i
\]

then we have \(\tilde{X}_i + \tilde{\alpha}_i \in E_0\). Furthermore, since \(\ker (p_0)_p = (g_M)_p\) has dimension \(l\), and it is contained in \(T_p\mu^{-1}(0) \cap \pi(E_\varphi)_p\), so we may assume that

\[
\tilde{X}_1 + \tilde{\alpha}_1, \ldots, \tilde{X}_{2(n-l)-k} + \tilde{\alpha}_{2(n-l)-k}
\]

are linearly independent.

On the other hand, by Fact 2.2.6, we can take \(\theta^1, \cdots, \theta^k \in T^n_p M \otimes C\) which satisfy

\[
(\varphi_k)_p = \theta^1 \wedge \cdots \wedge \theta^k.
\]

Then, since \((\varphi_k)_p\) satisfies \(i_{\xi} (\varphi_k)_p = 0\) for each \(\xi \in g\), so does \(\theta^i\) for \(i = 1, \cdots, k\). Hence \(\theta^i\) descends to a 1-form \(\tilde{\theta}^i \in \wedge^{ev/od} T^*_{p_0} M_0 \otimes C\). Then \(\tilde{\theta}^i \in E_0\), and

\[
p_0^*(\tilde{\theta}^1 \wedge \cdots \wedge \tilde{\theta}^k) = i_0^*_0(\theta^1 \wedge \cdots \wedge \theta^k)
\]

\[
= i_0^*_0((\varphi_k)_p)
\]

\[
\neq 0.
\]
This shows that $\hat{\theta}^1, \ldots, \hat{\theta}^k$ are linearly independent. Thus we have

$$\dim C E_0 = 2(n - l), \text{ and } E_0 = E_{\tilde{\varphi}},$$

in particular $E_{\tilde{\varphi}}$ is maximal isotropic.

Furthermore, since $E_x$ does not have a real vector except for 0, neither does $E_0$. So we also have

$$(E_{\tilde{\varphi}})|_p \cap (E_{\tilde{\varphi}})|_{p'} = \{0\}.$$ 

This shows that $\tilde{\varphi}$ is a generalized Calabi-Yau structure on $M_0$.

The last claim is clear because $\text{type}(\varphi_p) = \text{type}(\tilde{\varphi}|_p) = k$. \hfill $\square$

**Remark 3.2.3.** The reduction for other levels can be done by taking the coadjoint orbit. The detailed statement is as follows. Let a compact Lie group $G$ act on a generalized Calabi-Yau manifold $(M, \varphi)$ Hamiltonian way with a generalized moment map $\mu : M \rightarrow \mathfrak{g}^*$. For each $a \in \mathfrak{g}^*$, $\mathcal{O}_a$ denotes the coadjoint orbit of $a$. Suppose that $G$ acts on $\mu^{-1}(\mathcal{O}_a)$ freely. Then the quotient space $M_a = \mu^{-1}(\mathcal{O}_a)/G$ is a manifold and has unique generalized Calabi-Yau structure $\tilde{\varphi}$ which satisfies that

$$p^*_a \tilde{\varphi} = i^*_a \varphi$$

and

$$\text{type}(\varphi_p) = \text{type}(\tilde{\varphi}|_p),$$

for all $p \in \mu^{-1}(\mathcal{O}_a)$, where $i_a : \mu^{-1}(\mathcal{O}_a) \rightarrow M$ is the inclusion and $p_a : \mu^{-1}(\mathcal{O}_a) \rightarrow M_a$ is the natural projection. In addition, we have $\dim M_a = \dim M + \dim \mathcal{O}_a - 2 \dim G$.

**Example 3.2.4.** Let $G$ act on a symplectic manifold $(M, \omega)$ preserving $\omega$, and let $\mu : M \rightarrow \mathfrak{g}^*$ be a moment map. Then $G$ also acts on $(M, \varphi_{\omega})$ Hamiltonian way and $\mu$ is a generalized moment map. Moreover if we assume that $G$ acts freely on $\mu^{-1}(0)$, then we get the reduced symplectic structure $\tilde{\omega}$ and the reduced generalized Calabi-Yau structure $\tilde{\varphi}_{\omega}$ on the reduced space $M_0$. Then $\tilde{\varphi}_{\omega}$ coincides with the generalized Calabi-Yau structure $\varphi_{\omega}$ induced by the reduced symplectic structure $\tilde{\omega}$.

**Example 3.2.5.** Let $G$ act on a Calabi-Yau manifold $(M, \Omega)$. If the $G$-action is Hamiltonian, then the action is trivial and the generalized moment map $\mu$ is regarded as a linear functional on the Lie algebra $\mathfrak{g}$. So the reduced space $M_0$ coincides with either $M$ or the empty set.

**Remark 3.2.6.** Lin and Tolman showed the existence of a generalized complex structure on the reduced space in [8]. The generalized complex structure induced by the reduced generalized Calabi-Yau structure coincides with the reduced generalized complex structure from the generalized complex structure induced by the original generalized Calabi-Yau structure.
3.3. Relationship to Bergman kernels.

We introduce a Hamiltonian action on a generalized Calabi-Yau structure which is not induced from either a symplectic structure or a Calabi-Yau one here. Let $D \subset C^m+n$ be a Reinhardt bounded domain, that is, a bounded domain which the standard action of $(m+n)$-dimensional torus $T^{m+n}$ on $C^{m+n}$ leaves $D$ invariant. For each $w = (w_1, \cdots, w_m) \in C^m$, $D_w$ denotes the slice of $D$ at $w$,

$$D_w = \{(z_1, \cdots, z_{m+n}) \in D \mid z_j = w_j \quad (j = 1, \cdots, m)\}.$$ 

If the slice $D_w$ is not empty, we can regard $D_w$ as a Reinhardt bounded domain in $C^n$ naturally. Let $K_w(z) = K_w(z, z)$ be the Bergman kernel function of $D_w$, and $\Omega_w = ((\sqrt{-1})/2)\partial\bar{\partial}\log K_w$ be the Kähler form of the Bergman metric on $D_w$. Then the natural action of $S^1$ on $D_w$ preserves $\Omega_w$, and

$$\mu_w = -\frac{1}{2} \sum_{j=1}^n \frac{\partial}{\partial z_j} \left( \log K_w \right)$$

is a moment map for this action. Note that the function $\mu_w$ is real and $S^1$-invariant since the real function $\log K_w$ is $S^1$-invariant and the fundamental vector field $\xi$ induced by the $S^1$-action is given by

$$\xi = \sqrt{-1} \sum_{j=1}^n \left\{ z_j \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_j} \right\}.$$ 

Now we assume that the Bergman kernel $K_w$ depends smoothly on $w$. Then we can define a smooth function $K$ on $D$ by

$$K(w, z) = K_w(z) : D \rightarrow R,$$

and a complex form $\varphi$ on $D$ by

$$\varphi = dw_1 \wedge \cdots \wedge dw_m \wedge \exp \sqrt{-1} \Omega,$$

where $\Omega = (\sqrt{-1}/2)\partial\bar{\partial}\log K$. It is easy to see that the complex form $\varphi$ is a generalized Calabi-Yau structure on $D$, and the $S^1$-action on $D$ defined by

$$e^{\sqrt{-1} \varphi} (w_1, \cdots, w_m, z_1, \cdots, z_n) = (w_1, \cdots, w_m, e^{\sqrt{-1} \varphi} z_1, \cdots, e^{\sqrt{-1} \varphi} z_n),$$

preserves $\varphi$. 

Reduction of generalized Calabi-Yau structures

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THEOREM 3.3.1. Let $\mu$ be a smooth function on $D$ defined by

$$
\mu(w_1, \cdots, w_m, z_1, \cdots, z_n) = \mu_w(z_1, \cdots, z_n) = -\frac{1}{2} \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} \left( \log K \right).
$$

Then the function $\mu$ is a generalized moment map for the $S^1$ action on $D$ defined above.

PROOF. Let $\xi$ be the fundamental vector field for this action. Then $S^1$-invariance of the function $\log K$ implies that $\mu$ is a $S^1$-invariant real-valued function. By simple calculation, we have

$$
\iota_{\xi} \Omega \left( \frac{\partial}{\partial z_i} \right) = \Omega \left( \sqrt{-1} \sum_{j=1}^{n} \left( z_j \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_j} \log K \right) \right)
$$

$$
= \sqrt{-1} \sum_{j=1}^{n} z_j \left( \frac{\sqrt{-1}}{2} \frac{\partial^2}{\partial z_i \partial z_j} (\log K) \right)
$$

$$
= \frac{\partial}{\partial z_i} \left( -\frac{1}{2} \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} (\log K) \right)
$$

$$
= \frac{\partial}{\partial z_i} \left( -\frac{1}{2} \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} (\log K) \right)
$$

$$
= \frac{\partial \mu}{\partial z_i},
$$

and $\iota_{\xi} \Omega \left( \frac{\partial}{\partial w_j} \right) = \frac{\partial \mu}{\partial w_j}$ similarly. Hence we have $d\mu = \iota_{\xi} \Omega$, and we can check easily that $\mu$ is a generalized moment map for this action. \qed

EXAMPLE 3.3.2. Let $D$ be an $(m + n)$-dimensional polydisc,

$$
D = (D^1)^{m+n} = \{ (z_1, \cdots, z_{m+n}) \mid |z_j| < 1 \ (j = 1, \cdots, m+n) \}.
$$

For each $w \in (D^1)^m = \{ (w_1, \cdots, w_m) \in \mathbb{C}^m \mid |w_j| < 1 \ (j = 1, \cdots, m) \}$, $D_w$ denote the slice of $D$ at $w$,

$$
D_w = \{ (z_1, \cdots, z_n) \in \mathbb{C}^n \mid |z_j| < 1 \ (j = 1, \cdots, n) \}.
$$

Then $D_w$ is a polydisc on $\mathbb{C}^n$, and

$$
K_w = \frac{1}{\pi^n} \frac{1}{\prod_{j=1}^{n} (1 - |z_j|^2)^2}
$$

is the Bergman kernel function of $D_w$. Since the Bergman kernel $K_w$ does not depend on $w$. 

\( K(w, z) = K_w(z) : D \rightarrow \mathbb{R} \)

is a smooth function on \( D \), and thus we get a generalized Calabi-Yau structure on \( D \),

\[ \varphi = dw_1 \wedge \cdots \wedge dw_m \wedge \exp \sqrt{-1} \Omega, \]

where \( \Omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log K \). The natural \( S^1 \)-action defined above preserves \( \varphi \), and we have a generalized moment map \( \mu \) for this action,

\[ \mu = - \sum_{j=1}^{n} \frac{|z_j|^2}{1 - |z_j|^2}. \]

On the other hand, since the total space \( D \) and the parameter space \((D^1)^m\) are also Reinhardt bounded domains, they have Kähler forms induced by their Bergman kernels. So they have also generalized Calabi-Yau structures induced by their Kähler forms, and they are preserved by the natural \( S^1 \)-actions on them. By simple calculations, we get moment maps for their actions,

\[ \mu_D = - \left( \sum_{i=1}^{m} \frac{|w_i|^2}{1 - |w_i|^2} + \sum_{j=1}^{n} \frac{|z_j|^2}{1 - |z_j|^2} \right) \]

on \( D \), and

\[ \mu_{D^m} = - \sum_{i=1}^{m} \frac{|w_i|^2}{1 - |w_i|^2} \]

on \( D^m \). Then they satisfy the following additive relation;

\[ \mu_D = \mu_{D^m} + \mu. \]

**Example 3.3.3.** Let \( D \) be an \((m + n)\)-dimensional complex ball

\[ D = D^{m+n} = \left\{ (w_1, \cdots, w_m, z_1, \cdots, z_n) \in \mathbb{C}^{m+n} \left| \sum_{j=1}^{m} |w_j|^2 + \sum_{j=1}^{n} |z_j|^2 < 1 \right. \right\}. \]

For each \( w \in D = \left\{ (w_1, \cdots, w_m) \in \mathbb{C}^m \left| \sum_{j=1}^{m} |w_j|^2 < 1 \right. \right\} \), \( D_w \) denote the slice of \( D \) at \( w \),

\[ D_w = \left\{ (z_1, \cdots, z_n) \in \mathbb{C}^n \left| \sum_{j=1}^{n} |z_j|^2 < 1 - \sum_{j=1}^{m} |w_j|^2 \right. \right\}. \]
Then $D_w$ is also a complex ball on $\mathbb{C}^n$, and

$$K_w = \frac{n!}{\pi^n} \frac{1 - \sum_{j=1}^n |w_j|^2}{1 - (\sum_{j=1}^n |w_j|^2)^{n+1}}$$

is the Bergman kernel function of $D_w$. Since the Bergman kernel $K_w$ depends smoothly on $w$,

$$K(w, z) = K_w(z) : D \longrightarrow \mathbb{R},$$

is a smooth function on $D$, and thus we get a generalized Calabi-Yau structure on $D$,

$$\varphi = dw_1 \wedge \cdots \wedge dw_n \wedge \exp \sqrt{-1} \Omega,$$

where $\Omega = \frac{1}{2} \partial \bar{\partial} \log K$. The natural $S^1$-action on $D$ preserves $\varphi$, and we have a generalized moment map $\mu$ for this action,

$$\mu = - \frac{n+1}{2} \frac{1 - \sum_{j=1}^n |w_j|^2}{1 - (\sum_{j=1}^n |w_j|^2 + \sum_{j=1}^n |z_j|^2)}.$$ 

As in the case of the previous example, we have moment maps for the natural actions of $S^1$ on $D$ and $D^m$ which are derived from their Bergman kernels,

$$\mu_D = - \frac{m+n+1}{2} \frac{1}{1 - (\sum_{j=1}^m |w_j|^2 + \sum_{j=1}^n |z_j|^2)}$$

on $D$, and

$$\mu_{D^m} = - \frac{m+1}{2} \frac{1}{1 - \sum_{j=1}^m |w_j|^2}$$

on $D^m$. They have the following multiplicative relation;

$$\mu_D = - \frac{2(m+n+1)}{(m+1)(n+1)} \mu_{D^m} \cdot \mu.$$

4. The Duistermaat-Heckman formula.

4.1. The Duistermaat-Heckman measures and the reduced volumes.

Let $(M, \varphi)$ be a $2n$-dimensional connected generalized Calabi-Yau manifold which has constant type $k$, and suppose that compact $l$-torus $T$ acts on $M$ effectively and in a Hamiltonian way. In addition, we assume that the generalized moment map $\mu$ is proper. Then we have a natural volume form.
\[ dm = \frac{(-1)^n}{2^{n-k}} \langle \varphi, \tilde{\varphi} \rangle. \]

The volume form \( dm \) defines a measure on \( M \). Our second purpose is to prove the Duistermaat-Heckman formula in this case.

Let \( t \) denote the Lie algebra of \( T \), and \( t^r_{\text{reg}} \) denote the subset of \( t^r \) consisting of the regular values of \( \mu \). If \( a \in t^r \) is a regular value of \( \mu \) and \( p \in \mu^{-1}(a) \), then the stabilizer group

\[ T_p = \{ g \in T | g \cdot p = p \} \]

is finite. So if \( T \)-action on \( \mu^{-1}(a) \) is not free, the quotient space \( M_d = \mu^{-1}(a)/T \) is an orbifold. In this case, there exists a complex differential form on \( M_d \) which, in each local representation is a generalized Calabi-Yau structure on \( \mathbb{R}^{2(n-k)} \), and satisfies

\[ p^*_a \tilde{\varphi} = i_a^* \varphi, \]

where \( i_a : \mu^{-1}(a) \to M \) is the inclusion and \( p_a : \mu^{-1}(a) \to M_a \) is the natural projection. We call it a generalized Calabi-Yau structure on an orbifold \( M_a \).

Since \( \mu \) is proper, \( t^r_{\text{reg}} \) is a dense open subset, and \( t^r \setminus t^r_{\text{reg}} \) has measure 0 because of Sard’s theorem. The following lemma is due to Appendix B in [5].

**Lemma 4.1.1.** Suppose that \( M \) is connected and \( T \) acts on \( M \) effectively. Then the set \( M_{\text{free}} \) on which \( T \) acts freely is equal to the complement of a locally finite union of submanifolds of codimension \( \geq 2 \). In particular \( M_{\text{free}} \) is open, connected, dense, and \( M \setminus M_{\text{free}} \) has measure 0. Also \( (\mu_*)_p \) is surjective for all \( p \in M_{\text{free}} \).

Now we consider the normalized Haar measure \( dt \) on \( T \). Then the measure \( dt \) induces the Lebesgue measure \( dX \) on its Lie algebra \( t \), and we obtain the dual Lebesgue measure \( d\zeta \) on \( t^r \). The assumption that \( \mu \) is proper implies that the pushforward \( \mu_* (dm) \) of \( dm \) under \( \mu \) induces a measure in \( t^r \). We call it the Duistermaat-Heckman measure. In view of Lemma 4.1.1, we obtain \( M \setminus M_{\text{free}} \) has measure 0 and \( \mu|_{M_{\text{free}}} : M_{\text{free}} \to t^r \) is a submersion. This shows that \( \mu_*(dm) \) is absolutely continuous with respect to the Lebesgue measure \( d\zeta \). So there exists a Borel measurable function \( f \) on \( t^r \) which satisfies

\[ \mu_*(dm) = f d\zeta. \]

The corresponding Duistermaat-Heckman formula is stated in Theorem B in Introduction. For the proof, we need the following lemma.

**Lemma 4.1.2.** For each regular point \( p \in M \) of the generalized moment map \( \mu \), there exists a neighborhood \( U_p \) of \( p \) and a complex 2-form \( B + \sqrt{-1} \omega \in \Omega^2(U_p) \otimes \mathbb{C} \) such that \( \varphi = \exp(B + \sqrt{-1} \omega) \varphi_k \) on \( U_p \), and \( \iota_{\xi} \omega = d\varphi \) for all \( \xi \in t \).
PROOF. By Fact 2.3.7, there exists a neighborhood $U_p$ and a complex 2-form $\tilde{B} + \sqrt{-1}\omega \in \Omega^2(U_p) \otimes \mathcal{C}$ such that $\varphi = \exp(\tilde{B} + \sqrt{-1}\omega)\varphi_k$ on $U_p$. Moreover, there exists a local frame $\theta^1, \cdots, \theta^{2n}$ of $\wedge^1 T^*M$ such that $\varphi_k = \theta^1 \wedge \cdots \wedge \theta^k$ on $U_p$. So we may assume that $\tilde{B} + \sqrt{-1}\omega$ can be written

$$\tilde{B} + \sqrt{-1}\omega = \sum_{i,j} c_{ij} \theta^i \wedge \theta^j,$$

where $c_{ij}$ is a smooth complex function on $U_p$. In addition, since $p$ is a regular point, so $(t_M)_p$ has dimension $l$. Hence we may assume that $t_M$ has dimension $l$ on $U_p$.

Now consider the dual basis $\{X_1, \cdots, X_n\}$ of $\{\theta^1, \cdots, \theta^n\}$, and take an arbitrary Riemannian metric on $M$. Then we can define a complex 1-forms $\eta^1, \cdots, \eta^n$ on $U_p$ defined by

$$\eta^i(\xi_M) = \sqrt{-1}d\mu^i(X_i)$$

for $\xi_M \in t_M$, and vanishes on the orthogonal complement of $t_M$. Then we define a complex 2-form $B + \sqrt{-1}\omega$ on $U_p$ by

$$B + \sqrt{-1}\omega = \tilde{B} + \sqrt{-1}\omega + \sum_{s=1}^k \eta^s \wedge \theta^s.$$ 

It is clear that $\varphi = \exp(B + \sqrt{-1}\omega)\varphi_k$ on $U_p$ and

$$t_{\xi_M}(B + \sqrt{-1}\omega)(X_i) = \left( \sum_{i,j}^k c_{ij} \theta^i \wedge \theta^j \right)(\xi_M, X_i) + \left( \sum_{s=1}^k \eta^s \wedge \theta^s \right)(\xi_M, X_i)$$

$$= \left( \sum_{s=1}^k \eta^s \wedge \theta^s \right)(\xi_M, X_i)$$

$$= \sum_{s=1}^k \eta^s(\xi_M)\theta^s(X_i)$$

$$= \eta^s(\xi_M)$$

$$= \sqrt{-1}d\mu^s(X_i),$$

for each $\xi \in t$ and $i = 1, \cdots, k$. On the other hand, since $\xi_M - \sqrt{-1}d\mu^k \in E_{\mathcal{C}}$ for each $\xi \in t$, so we have

$$(t_{\xi_M}(B + \sqrt{-1}\omega) - \sqrt{-1}d\mu^k) \wedge \varphi_k = 0.$$

Thus for $i = k + 1, \cdots, 2n$, we obtain
convex neighborhood of a straight lines through a where takes the value $/C6$ and in particular we have $\mathrm{pa}^*(\complement M) = \{0\}$, this shows that $\mathrm{pa}^*(\complement M)$ is contained in a finite union of submanifolds (or suborbifolds) of $M$. Note that the complement of $\mathrm{pa}^*(\complement M)$ is free $\mathrm{pa}^*(\complement M)$ is locally given by the $\mu^{-1}(a)$-orbits such that for each $p \in \mu^{-1}(U)$, draw the horizontal curves lying over the straight lines through $a$ and $b = \mu(p)$. This defines a $T$-equivariant projection $\Phi : \mu^{-1}(U) \longrightarrow \mu^{-1}(a)$ such that for each $b \in U$ the restriction $\Phi|_{\mu^{-1}(b)} : \mu^{-1}(b) \longrightarrow \mu^{-1}(a)$ is a $T$-equivariant diffeomorphism and

$$\mu \times \Phi : \mu^{-1}(U) \longrightarrow U \times \mu^{-1}(a)$$

is a trivialization. Using this trivialization and Fubini theorem, we have that $f(a)$ is equal to the volume of $\mu^{-1}(a)$ with respect to the quotient of $dn$ by $\mu^*d\zeta$. In addition, by Lemma 4.1.2 $dm/\mu^*d\zeta$ is locally given by the $(2n-l)$-form

$$i^*_a(\varphi_k \wedge \varphi_k) \wedge \frac{1}{(n-k-l)!} (i^*_a \omega)^{n-k-l} \wedge \eta,$$

where $\omega$ is a 2-form given by the lemma above and $\eta$ is an $l$-form which on the $T$-orbits takes the value $\pm 1$ on an $l$-tuple $(X_1, \ldots, X_l)$ such that $dX(X_1, \ldots, X_l) = 1$.

Note that the complement of $p_a(M_{\text{free}} \cap \mu^{-1}(a)) = (M_{\text{free}})_a$ has measure 0 for the projection $p_a : \mu^{-1}(a) \longrightarrow M_a$ because the complement of $(M_{\text{free}})_a$ is equal to the image of a finite union of submanifolds (or suborbifolds) of $\mu^{-1}(a)$ of codimension $\geq 2$. Since $p_a : M_{\text{free}} \cap \mu^{-1}(a) \longrightarrow (M_{\text{free}})_a$ is a principle $T$-fibration and $\text{vol}(T) = 1$, we get that the volume of $M_{\text{free}} \cap \mu^{-1}(a)$ is equal to the volume of $(M_{\text{free}})_a$ with respect to the measure $dm_a$ induced by the reduced generalized Calabi-Yau structure on $M_a$. Because the complement of $(M_{\text{free}})_a$ has measure 0, we have proved the formula. □

Remark 4.1.3. For the density function $f$, one can show that $f$ is a piecewise polynomial of degree at most $n - l - k$. Moreover, in the case that $M$ is compact, the localization formula holds by applying the Atiyah-Bott-Berline-Vergne localization theorem. Detailed statements and proofs can be seen in [9].
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