On Rademacher Complexity-based Generalization Bounds for Deep Learning

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Abstract
We show that the Rademacher complexity-based approach can generate non-vacuous generalisation bounds on Convolutional Neural Networks (CNNs) for classifying a small number of classes of images. The development of new Talagrand’s contraction lemmas for high-dimensional mappings between vector spaces and CNNs for general Lipschitz activation functions is a key technical contribution. Our results show that the Rademacher complexity does not explicitly depend on the network length for CNNs with some common types of activation functions such as ReLU, Leaky ReLU, Parametric Rectifier Linear Unit, Sigmoid, and Tanh.

1 Introduction
Deep models are typically heavily over-parametrized, while they still achieve good generalization performance. Despite the widespread use of neural networks in biotechnology, finance, health science, and business, just to name a selected few, the problem of understanding deep learning theoretically remains relatively under-explored. In 2002, Koltchinskii and Panchenko [1] proposed new probabilistic upper bounds on generalization error of the combination of many complex classifiers such as deep neural networks. These bounds were developed based on the general results of the theory of Gaussian, Rademacher, and empirical processes in terms of general functions of the margins, satisfying a Lipschitz condition. However, bounding Rademacher complexity for deep learning remains a challenging task. In this work, we provide some new upper bounds on Rademacher complexity in deep learning which does not explicitly depend on the length of deep neural networks. In addition, we show that Koltchinskii and Panchenko’s approach can be improved to generate non-vacuous bounds for CNNs.

1.1 Related Papers
The complexity-based generalization bounds were established by traditional learning theory aiming to provide general theoretical guarantees for deep learning. [2], [3], [4] proposed upper bounds based on the VC dimension for DNNs. [5] used Rademacher complexity to prove the bound with exponential dependence on the depth for ReLU networks. [6] and [7] uses the PAC-Bayesian analysis and the covering number to obtain bounds with polynomial dependence on the depth, respectively. [8] provided bounds with (sub-linear) square-root dependence on the depth for DNNs with positive-homogeneous activations such as ReLU.

The standard approach to develop generalization bounds on deep learning (and machine learning) was developed in seminar papers by [9], and it is based on bounding the difference between the

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generalization error and the training error. These bounds are expressed in terms of the so-called VC-dimension of the class. However, these bounds are very loose when the VC-dimension of the class can be very large, or even infinite. In 1998, several authors [10,11] suggested another class of upper bounds on generalization error that are expressed in terms of the empirical distribution of the margin of the predictor (the classifier). Later, Koltchinskii and Panchenko [11] proposed new probabilistic upper bounds on the generalization error of the combination of many complex classifiers such as deep neural networks. These bounds were developed based on the general results of the theory of Gaussian, Rademacher, and empirical processes in terms of general functions of the margins, satisfying a Lipschitz condition. They improved previously known bounds on generalization error of convex combination of classifiers. Generalization bounds for deep learning and kernel learning with Markov dataset based on Rademacher and Gaussian complexity functions have recently analysed in [12]. Analysis of machine learning algorithms for Markov and Hidden Markov datasets already appeared in research literature [13,14,15].

In the context of supervised classification, PAC-Bayesian bounds have been used to explain the generalisation capability of learning algorithms [16,17,18]. Several recent works have focused on gradient descent based PAC-Bayesian algorithms, aiming to minimise a generalisation bound for stochastic classifiers [19,20,21]. Most of these studies use a surrogate loss to avoid dealing with the zero-gradient of the misclassification loss. Several authors used other methods to estimate of the misclassification error with a non-zero gradient by proposing new training algorithms to evaluate the optimal output distribution in PAC-Bayesian bounds analytically [22,23,24]. Recently, [25] showed that uniform convergence might be unable to explain generalisation in deep learning by creating some examples where the test error is bounded by $\delta$ but the (two-sided) uniform convergence on this set of classifiers will yield only a vacuous generalisation guarantee larger than $1 - \delta$ for some $\delta \in (0,1)$. There have been some interesting works which use information-theoretic approach to find PAC-bounds on generalization errors for machine learning [26,27] and deep learning [28].

In this work, we show that the Rademacher complexity does not explicitly depend on the length of CNNs which uses some special classes of activation functions $\sigma$ such that $\sigma - \sigma(0)$ belongs to ReLU family $\mathcal{L} = \{\psi_\alpha : \psi_\alpha(x) = \text{ReLU}(x) - \alpha \text{ReLU}(-x), \ \forall x \in \mathbb{R}, \alpha \in [0,1]\}$, or odd function ones $\mathcal{O} = \{\psi : \psi(-x) = -\psi(x), \ \forall x \in \mathbb{R}\}$. Our result improves Golowich et al.'s bound [8] where the authors showed that the Rademacher complexity is square-root dependent on the depth for DNNs with ReLU activation functions.

### 1.2 Contributions

More specifically, our contributions are as follows:

- We develop new contraction lemmas for high-dimensional mappings between vector spaces which extends the Talagrand contraction lemma.
- We apply our new contraction lemmas to each layer of a CNN.
- We validate our new theoretical results experimentally on CNNs for MNIST image classifications, and our bounds are non-vacuous when the number of classes is small.

As far as we know, this is the first result which shows that the Rademacher complexity-based approach can lead to non-vacuous generalisation bounds on CNNs.

### 1.3 Other Notations

Vectors and matrices are in boldface. For any vector $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ where $\mathbb{R}$ is the field of real numbers, its induced-$L^p$ norm is defined as

$$
\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}.
$$

The $j$-th component of the vector $\mathbf{x}$ is denoted as $[\mathbf{x}]_j$ for all $j \in [n]$. 

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For $A \in \mathbb{R}^{m \times n}$ where

$$A = \begin{bmatrix}
a_{11}, & a_{12}, & \cdots, & a_{1n} \\
a_{21}, & a_{22}, & \cdots, & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}, & a_{m2}, & \cdots, & a_{mn}
\end{bmatrix}$$

we defined the induced-norm of matrix $A$ as

$$\|A\|_{p,q} = \sup_{x \neq 0} \frac{\|Ax\|_q}{\|x\|_p}.$$  \hfill (3)

For abbreviation, we also use the following notation

$$\|A\|_p := \|A\|_{p,p}. \hfill (4)$$

It is known that

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|, \hfill (5)$$

$$\|A\|_2 = \sqrt{\lambda_{\text{max}}(AA^T)}, \hfill (6)$$

$$\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|, \hfill (7)$$

where $\lambda_{\text{max}}(AA^T)$ is defined as the maximum eigenvalue of the matrix $AA^T$ (or the square of the maximum singular value of $A$).

### 2 Contraction Lemmas in High Dimensional Vector Spaces

First, we recall the Talagrand’s contraction lemma.

**Lemma 1** [29, Lemma 8, Appendix A.2.] Let $\mathcal{H}$ be a hypothesis set of functions mapping from some set $\mathcal{X}$ to $\mathbb{R}$ and $\psi$ be a $\mu$-Lipschitz function from $\mathbb{R} \to \mathbb{R}$ for some $\mu > 0$. Then, for any sample $S$ of $n$ points $x_1, x_2, \cdots, x_n \in \mathcal{X}$, the following inequality holds:

$$\frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}} \sum_{i=1}^{n} \varepsilon_i (\psi \circ h)(x_i) \right] \leq \frac{\mu}{n} \mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}} \sum_{i=1}^{n} \varepsilon_i h(x_i) \right]. \hfill (8)$$

In this section, we introduce some new versions of Talagrand’s contraction lemma for the high-dimensional mapping $\psi$ between vector spaces. The proof of the following theorem can be found in Appendix A.1.

**Theorem 2** Let $\mathcal{H}$ be a set of functions mapping from some set $\mathcal{X}$ to $\mathbb{R}^m$ for some $m \in \mathbb{Z}_+$ and

$$\mathcal{L} = \{ \psi_\alpha : \psi_\alpha(x) = \text{ReLU}(x) - \alpha \text{ReLU}(-x) \ \forall x \in \mathbb{R}, \alpha \in [0, 1] \} \hfill (9)$$

where $\text{ReLU}(x) = \max(x, 0)$.

For any $\mu > 0$, let $\psi : \mathbb{R} \to \mathbb{R}$ be a $\mu$-Lipschitz function. Define

$$\mathcal{H}_+ = \begin{cases} \mathcal{H} \cup \{-h : h \in \mathcal{H}\}, & \text{if $\psi - \psi(0)$ is odd} \\
\mathcal{H} \cup \{-h : h \in \mathcal{H}\} \cup \{|h| : h \in \mathcal{H}\}, & \text{if $\psi - \psi(0)$ others}
\end{cases} \hfill (10)$$

Then, it holds that

$$\mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right\|_\infty \right] \leq \gamma(\mu) \mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}_+} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i h(x_i) \right\|_\infty \right] + \frac{1}{\sqrt{n}} |\psi(0)|, \hfill (11)$$
where
\[
\gamma(\mu) = \begin{cases} 
\mu, & \text{if } \psi - \psi(0) \text{ is odd or belongs to } \mathcal{L} \\
2\mu, & \text{if } \psi - \psi(0) \text{ is even} \\
3\mu, & \text{if } \psi - \psi(0) \text{ others}
\end{cases}
\quad (12)
\]

Here, we define \( \psi(x) := (\psi(x_1), \psi(x_2), \ldots, \psi(x_m))^T \) for any \( x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m \).

**Remark 3** Some remarks are in order.

- Identity, ReLU, Leaky ReLU, Parametric rectified linear unit (PReLU) belong to the class of functions \( \mathcal{L} \).
- When \( \psi - \psi(0) \) is odd or belongs to \( \mathcal{L} \), the contraction constant is equal to \( \mu \), which improves Talagrand’s contraction lemma [30 Theorem 4.12] for the case \( m = 1 \) where this contraction constant is \( 2\mu \). This improvement is based on our exploitation of special properties of some function classes.
- Our results are developed based on a novel idea that we can have tighter contraction lemmas when both the class of functions \( \mathcal{H} \) and activation functions own some special properties. More specifically, in this work, we add more functions to \( \mathcal{H} \) and turn this class of functions to \( \mathcal{H}_+ \) and also limit the class of activation functions to \( \mathcal{L} \cup \{ \psi : \mathbb{R} \to \mathbb{R} : \psi(x) - \psi(0) = -(\psi(-x) - \psi(0)), \forall x \in \mathbb{R} \} \).

### 3 Rademacher Complexity Bounds for Convolutional Neural Networks (CNNs)

#### 3.1 Convolutional Neural Network Models

Let \( d_0, d_1, \ldots, d_L, d_{L+1} \) be a sequence of positive integer numbers such that \( d_0 = d \). For each \( k \in [L] \), we define a class of function \( \mathcal{F}_k \) as follows:
\[
\mathcal{F}_k := \{ f = f_k \circ f_{k-1} \circ \cdots \circ f_1 \circ f_0 : f_i \in \mathcal{G}_i \subset \{ g_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_{i+1}} \}, \quad \forall i \in \{1, 2, \ldots, k\} \},
\]
where \( f_0 : [0, 1]^d \to \mathbb{R}^{d_1} \) is a fixed function.

Denote by \( \mathcal{F} := \mathcal{F}_L \). Assume that and \( f_L : \mathbb{R}^{d_L} \to \mathbb{R}^M \) for some \( M \in \mathbb{Z}_+ \).

Given a function \( f \in \mathcal{F} \), a function \( g \in \mathbb{R}^M \times [M] \) predicts a label \( y \in [M] \) for an example \( x \in \mathbb{R}^d \) if and only if
\[
g(f(x), y) > \max_{y' \neq y} g(f(x), y')
\quad (14)
\]
where \( g(f(x), y) = w_y^T f(x) \) with \( w_{y} = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \).

For a training set \( \{x_i\}_{i=1}^n \), the \( p \)-norm Rademacher complexity for the class function \( \mathcal{F}_k \) is defined as
\[
R_n(\mathcal{F}_k) := \mathbb{E}_{\epsilon} \left[ \sup_{f \in \mathcal{F}_k} \left\| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(x_i) \right\|_p \right] \quad \forall k \in [L],
\quad (15)
\]
where \( \{\epsilon_i\} \) is a sequence of i.i.d. Rademacher (taking values \( +1 \) and \( -1 \) with probability \( 1/2 \) each) random variables, independent of \( \{x_i\} \).

#### 3.2 Some Contraction Lemmas for CNNs

Based on Theorem [2] the following versions of Talagrand’s contraction lemma for different layers of CNN are derived.
Definition 4 (Convolutional Layer with Average Pooling) Let \( \mathcal{G} \) be a class of \( \mu \)-Lipschitz function \( \sigma \) from \( \mathbb{R} \to \mathbb{R} \) such that \( \sigma(0) \) is fixed. Let \( C, Q \in \mathbb{Z}^+ \), \( \{\tau_l, \tau_l^r\}_{c \in [Q]} \) be two tuples of positive integer numbers, and \( \{W_{l,c} \in \mathbb{R}^{r \times r_l}, c \in [C], l \in [Q]\} \) be a set of kernel matrices. A convolutional layer with average pooling, \( C \) input channels, and \( Q \) output channels is defined as a set of \( Q \times C \) mappings \( \Psi = \{\psi_{l,c}, l \in [Q], c \in [C]\} \) from \( \mathbb{R}^{d \times d} \) to \( \mathbb{R}^{((d-r_l+1)/\tau_l^r) \times ((d-r_l+1)/\tau_l^r)} \) such that
\[
\psi_{l,c}(x) = \sigma_{avg} \circ \sigma_{l,c}(x),
\]
where
\[
\sigma_{avg}(x) = \frac{1}{\tau_l^r} \left( \sum_{k=1}^{\tau_l^r} x_{k}, \cdots, \sum_{k=(j-1)\tau_l^r+1}^{j\tau_l^r} x_{k}, \cdots, \sum_{k=((d-r_l+1)/\tau_l^r)\tau_l^r+1}^{(d-r_l+1)/\tau_l^r} x_{k} \right),
\]
\( \forall x \in \mathbb{R}^{((d-r_l+1)/\tau_l^r)} \).

Lemma 5 (Convolutional Layer with Average Pooling) Let \( \mathcal{F} \) be a set of functions mapping from some set \( \mathcal{X} \) to \( \mathbb{R}^m \) for some \( m \in \mathbb{Z}^+ \). Consider a convolutional layer with average pooling defined in Definition 4. Then, it hold that
\[
\mathbb{E}_\varepsilon \left[ \sup_{c \in [C]} \sup_{l \in [Q]} \sup_{\psi \in \Psi} \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{l=1}^{n} \varepsilon_l \psi_{l,c} \circ f(x_l) \right\|_\infty \right] \leq \left[ \gamma(\mu) \sup_{c \in [C]} \sup_{l \in [Q]} \left( \sum_{u=0}^{r_l-1} \sum_{v=0}^{r_l^r-1} |W_{l,c}(u+1,v+1)| \right) \right] \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}_+} \left\| \frac{1}{n} \sum_{l=1}^{n} \varepsilon_l f(x_l) \right\|_\infty \right] ^+ + \frac{\| \sigma(0) \|}{\sqrt{n}},
\]
where
\[
\gamma(\mu) = \begin{cases}
\mu, & \text{if } \sigma - \sigma(0) \text{ is odd or belongs to } \mathcal{L} \\
2\mu, & \text{if } \sigma - \sigma(0) \text{ is even} \\
3\mu, & \text{if } \sigma - \sigma(0) \text{ others}
\end{cases}
\]
\( \mathcal{F}_+ = \mathcal{F} \cup \{-f : f \in \mathcal{F}\}, \) if \( \sigma - \sigma(0) \) is odd
\( \mathcal{F} \cup \{-f : f \in \mathcal{F}\} \cup \{|f| : f \in \mathcal{F}\}, \) if \( \sigma - \sigma(0) \) others.

Proof See Appendix A.7

For Dropout layer, the following holds:

Lemma 6 Let \( \psi(x) \) is the output of the \( x \) via the Dropout layer. Then, it holds that
\[
\mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi \circ f(x_i) \right\|_\infty \right] \leq \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right].
\]

Proof See Appendix A.8

The following Rademacher complexity bounds for Dense Layers.

Lemma 7 (Dense Layers) Recall the definition of \( \mathcal{L} \) in Definition 4. Let \( \mathcal{G} \) be a class of \( \mu \)-Lipschitz function, i.e.,
\[
|\sigma(x) - \sigma(y)| \leq \mu |x - y|, \quad \forall x, y \in \mathbb{R},
\]
such that $\sigma(0)$ is fixed. Let $V$ be a class of matrices $W$ on $\mathbb{R}^{d \times d'}$ such that $\sup_{W \in V} \|W\|_\infty \leq \beta$. For any vector $x = (x_1, x_2, \ldots, x_d)$, we denote by $\sigma(x) := (\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_d))^T$. Then, it holds that

$$E_x \left[ \sup_{W \in V} \sup_{f \in \mathcal{G}} \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i \sigma(Wf(x_i)) \right\|_\infty \right] \leq \gamma(\mu) \beta E_x \left[ \sup_{f \in \mathcal{G}} \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i f(x_i) \right\|_\infty \right] + \frac{\sigma(0)}{\sqrt{n}},$$

where

$$\gamma(\mu) = \begin{cases} 
\mu, & \text{if } \sigma - \sigma(0) \text{ is odd or belongs to } \mathcal{L} \\
2\mu, & \text{if } \sigma - \sigma(0) \text{ is even} \\
3\mu, & \text{if } \sigma - \sigma(0) \text{ others}
\end{cases}.$$  

Proof See Appendix A.9.

3.3 Rademacher complexity bounds for CNNs

Theorem 8 Let

$$\mathcal{L} = \{ \psi_\alpha : \psi_\alpha(x) = \text{ReLU}(x) - \alpha \text{ReLU}(-x) \ \forall x \in \mathbb{R}, \alpha \in [0, 1] \}.$$  

Consider the CNN defined in Section 3.1 where

$$[f_i(x)]_j = \sigma_i(W_i^T f_{i-1}(x)) \ \forall j \in [d_{i+1}]$$

and $\sigma_i$ is $\mu_i$-Lipschitz. In addition, $f_0(x) = [x^T, 1]^T$, $\forall x \in \mathbb{R}^d$ and $x$ is normalised such that $\|x\|_\infty \leq 1$. Let

$$\mathcal{K} = \{ i \in [L] : \text{layer } i \text{ is a convolutional layer with average pooling} \},$$

$$\mathcal{D} = \{ i \in [L] : \text{layer } i \text{ is a dropout layer} \}.$$  

We assume that there are $Q_i$ kernel matrices $W_i^{(l)}$’s of size $r_i^{(l)} \times r_i^{(l)}$ for the $i$-th convolutional layer. For all the (dense) layers that are not convolutional, we define $W_i$ as their coefficient matrices. In addition, define

$$\gamma_{cvl,i} = \gamma(\mu_i) \sup_{l \in [Q_i]} \sum_{n=1}^{r_i} \sum_{v=1}^{r_i} |W_i^{(l)}(u, v)|,$$

$$\gamma_{dl,i} = \gamma(\mu_i) \|W_i\|_\infty \text{ for } i \notin \mathcal{K},$$

where

$$\gamma(\mu_i) = \begin{cases} 
\mu_i, & \text{if } \sigma_i - \sigma_i(0) \text{ is odd or belongs to } \mathcal{L} \\
2\mu_i, & \text{if } \sigma_i - \sigma_i(0) \text{ is even} \\
3\mu_i, & \text{if } \sigma_i - \sigma_i(0) \text{ others}
\end{cases}.$$  

Then, the Rademacher complexity, $R_n(F)$, satisfies

$$R_n(F) := E_x \left[ \sup_{f \in \mathcal{F}_+} \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i f(x_i) \right\|_\infty \right] \leq F_n,$$

where $F_n$ is estimated by the following recursive expression:

$$F_i = \begin{cases} 
F_{i-1} \gamma_{cvl,i} + \frac{\sigma(0)}{\sqrt{n}}, & i \in \mathcal{K} \\
F_{i-1} \gamma_{dl,i} + \frac{\sigma(0)}{\sqrt{n}}, & i \notin (\mathcal{K} \cup \mathcal{D}) \\
F_{i-1}, & i \notin \mathcal{D}
\end{cases}$$

and $F_0 = \sqrt{\frac{2+1}{n}}$.  

6
Remark 9 For some special CNNs where all the activation functions belong to ReLU family or odd functions, Theorem 8 shows that the Rademacher complexity does not explicitly depend on the network length. This result improves Golowich et al.’s bound [8] where the authors showed that the Rademacher complexity is square-root dependent on the depth. It also improve Neyshabur et al.’s bound [5] where the authors show that the Rademacher complexity depends exponentially on the network-length.

Proof This is a direct application of Lemmas [3] [6] and [7] By the modelling of CNNs in Section 3.1 it holds that

\[ \mathcal{F}_k := \{ f = f_k \circ f_{k-1} \circ \cdots \circ f_1 \circ f_0 : f_i \in \mathcal{G}_i \subset \{ g_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_{i+1}} \}, \quad \forall i \in \{1, 2, \cdots, k\} \} \]

and \( \mathcal{F} := \mathcal{F}_L \).

For CNNs, \( f_l(x) = \sigma_l(W_l x) \) for all \( l \in [L] \) where \( W_l \in W_l \) (a set of matrices) and \( \sigma_l \in \Psi_l \) where

\[ \Psi_l = \{ \sigma_l : |\sigma_l(x) - \sigma_l(y)| \leq \mu_l |x - y|, \quad \forall x, y \in \mathbb{R} \}. \]

Then, since \( |\sigma_l|, -\sigma_l \in \Psi_l \), it is easy to see that

\[ \mathcal{F}_{l, +} \subset \Psi_l(W_l \mathcal{F}_{l-1, +}), \quad \forall l \in [L], \]

where \( \mathcal{F}_{l, +} \) is a supplement of \( \mathcal{F}_l \) defined in (32).

Therefore, by peeling layer by layer we finally have

\[ \mathbb{E}_x \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right] \leq F_L, \]

where for each \( i \in [L] \)

\[ F_i = \begin{cases} F_{i-1} \gamma_{c, i} + \frac{|\sigma_i(0)|}{\sqrt{n}}, & i \in \mathcal{K} \\ F_{i-1} \gamma_{d, i} + \frac{|\sigma_i(0)|}{\sqrt{n}}, & i \notin (\mathcal{K} \cup \mathcal{D}) \\ F_{i-1}, & i \in \mathcal{D} \end{cases} \]

and

\[ F_0 = \mathbb{E}_x \left[ \sup_{f \in \mathcal{H}_+} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right]. \]

Here, \( \mathcal{H}_+ \) is the extended set of inputs to the CNN, i.e.,

\[ \mathcal{H}_+ = \begin{cases} f_0 \cup \{-f_0\}, & \text{if } \sigma_1 - \sigma_1(0) \text{ is odd} \\ f_0 \cup \{-f_0\} \cup \{|f_0|\}, & \text{if } \sigma_1 - \sigma_1(0) \text{ others} \end{cases} . \]

Now, for the case \( \sigma_1 - \sigma_1(0) \) is odd, it is easy to see that

\[ \sup_{f \in \mathcal{H}_+} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty = \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f_0(x_i) \right\|_\infty \leq \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f_0(x_i) \right\|_2. \]

On the other hand, for the case \( \sigma_1 - \sigma_1(0) \) is general, we have

\[ \sup_{f \in \mathcal{H}_+} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \leq \max \left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f_0(x_i) \right\|_\infty, \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i |f_0(x_i)| \right\|_\infty \right\}. \]
On the other hand, we have

\[
\mathbb{E}_\varepsilon \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f_0(x_i) \right\|_2 \right] \\
\leq \frac{1}{n} \sqrt{\mathbb{E}_\varepsilon \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f_0(x_i) \right\|_2^2 \right]} \\
\leq \frac{1}{n} \sqrt{\sum_{j=1}^{d+1} \sum_{i=1}^{n} [f_0(x_i)]_j^2} \\
\leq \frac{1}{n} \sqrt{(d+1)n} \\
= \sqrt{\frac{d+1}{n}},
\]  

(45)

(46)

(47)

(48)

where (47) follows from \([f_0(x_i)]_j \leq 1\) for all \(i \in [n], j \in [d] \) when the data is normalised by using the standard method.

Similarly, we also have

\[
\mathbb{E}_\varepsilon \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f_0(x_i) \right\|_\infty \right] \leq \sqrt{\frac{d+1}{n}}.
\]  

(49)

4 Generalization Bounds for CNNs

4.1 Generalization Bounds for Deep Learning

Definition 10 Recall the CNN model in Section 3.7. The margin of a labelled example \((x, y)\) is defined as

\[ m_f(x, y) := g(f(x), y) - \max_{y' \neq y} g(f(x), y'), \]

(50)

so \(f\) misclassifies the labelled example \((x, y)\) if and only if \(m_f(x, y) \leq 0\). The generalisation error is defined as \(P(m_f(x, y) \leq 0)\). It is easy to see that \(P(m_f(x, y) \leq 0) = P(w_0^T f(x) \leq \max_{y' \in Y} w_0^T f(x))\).

Remark 11 Some remarks:

\begin{itemize}
  \item Since \(g(f(x), y) > \max_{y' \neq y} g(f(x), y')\), it holds that \(\tilde{g}(f_k(x, y)) > \max_{y' \neq y} \tilde{g}(f_k(x, y'))\) for some \(k \in [L]\) where \(\tilde{g}\) is an arbitrary function. Hence, \(P(m_f(x, y) \leq 0) \leq P(\tilde{g}(f_k(x, y)) > \max_{y' \neq y} \tilde{g}(f_k(x, y')))\), so we can bound the generalisation error by using only a part of CNN networks (from layer 0 to layer \(k\)). However, we need to know \(\tilde{g}\). If the last layers of CNN are softmax, we can easily know this function.
  
  \item When testing on CNNs, it usually happens that the generalisation error bound becomes smaller when we use almost all layers.
\end{itemize}

Now, we prove the following lemma.

Lemma 12 Let \(\mathcal{F}\) be a class of function from \(X\) to \(\mathbb{R}^m\). For CNNs for classification, it holds that

\[
\mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i m_f(x_i, y_i) \right\|_2 \right] \leq \beta(M) \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i m_f(x_i) \right\|_\infty \right],
\]

(51)

where

\[
\beta(M) = \begin{cases} 
M(2M - 1), & M > 2 \\
2M, & M = 2
\end{cases}.
\]

(52)
For $M > 2$, \[51\] is a result of [1, Proof of Theorem 11]. We improve this constant for $M = 2$.

**Proof** See Appendix A.10

Based on the above Rademacher complexity bounds and a justified application of McDiarmid’s inequality, we obtain the following generalization for deep learning with i.i.d. datasets.

**Theorem 13** Let $\gamma > 0$ and define the following function (the $\gamma$-margin cost):

$$
\zeta(x) := \begin{cases} 
0, & \gamma \leq x \\
1 - x/\gamma, & 0 \leq x \leq \gamma \\
1, & x \leq 0
\end{cases}
$$

(53)

Recall the definition of the average Rademacher complexity $R_n(F)$ in (33) and the definition of $\beta(M)$ in (52). Let $\{(x_i, y_i)\}_{i=1}^n \sim P_{xy}$ for some joint distribution $P_{xy}$ on $\mathcal{X} \times \mathcal{Y}$. Then, for any $t > 0$, the following holds:

$$
P\left\{ \exists f \in F : \mathbb{P}(m_f(x, y) \leq 0) > \inf_{\gamma \in (0, 1]} \left[ \frac{1}{n} \sum_{i=1}^n \zeta(m_f(x_i, y_i)) + \frac{2\beta(M)}{\gamma} R_n(F) + \frac{2t + \sqrt{\log \log_2(2\gamma^{-1})}}{\sqrt{n}} \right] \right\} \leq 2 \exp(-2t^2).
$$

(54)

**Proof** See Appendix A.11

**Corollary 14** (PAC-bound) Recall the definition of the average Rademacher complexity $R_n(F)$ in (33) and the definition of $\beta(M)$ in (52). Let $\{(x_i, y_i)\}_{i=1}^n \sim P_{xy}$ for some joint distribution $P_{xy}$ on $\mathcal{X} \times \mathcal{Y}$. Then, for any $\delta \in (0, 1]$, with probability at least $1 - \delta$, it holds that

$$
P\left\{ \mathbb{P}(m_f(x, y) \leq 0) > \inf_{\gamma \in (0, 1]} \left[ \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{m_f(x_i, y_i) \leq \gamma\} + \frac{2\beta(M)}{\gamma} R_n(F) + \frac{\log \log_2(2\gamma^{-1})}{n} + \frac{\log 3}{\sqrt{n}} \right] \right\} \leq 3 \exp(-2t^2) = \delta.
$$

(55)

**Proof** This result is obtain from Theorem 13 by choosing $t > 0$ such that $3 \exp(-2t^2) = \delta$.

5 **Numerical Results**

In this experiment, we use a CNN (cf. Fig. 1) for classifying MNIST images (class 0 and class 1), i.e., $M = 2$, which consists of $n = 12065$ training examples.

For this model, the sigmoid activation $\sigma$ satisfies $\sigma(x) - \sigma(0) = \frac{1}{2} \tanh \left( \frac{x}{2} \right)$ which is odd and has the Lipschitz constant $1/4$. In addition, for the dense layer, the sigmoid activation satisfies

$$
|\sigma(x) - \sigma(y)| \leq \frac{1}{4} |x - y|, \quad \forall x, y \in \mathbb{R}.
$$

(56)
Hence, by Theorem 8 and Lemma 17, it holds that $R_n(\mathcal{F}) \leq F_3$, where

\[
F_3 \leq \frac{1}{4} \|W\|_\infty F_2 + \frac{1}{2\sqrt{n}}. \tag{57}
\]

Dense layer

\[
F_2 \leq \left( \frac{1}{4} \sup_{l \in \{2, 3\}} \sum_{u=1}^3 \sum_{v=1}^3 |W^{(l)}_u(u, v)| \right) F_1 + \frac{1}{2\sqrt{n}}. \tag{58}
\]

The second convolutional layer

\[
F_1 \leq \left( \frac{1}{4} \sup_{l \in \{3\}} \sum_{u=1}^3 \sum_{v=1}^3 |W^{(l)}_u(u, v)| \right) F_0 + \frac{1}{2\sqrt{n}}. \tag{59}
\]

The first convolutional layer

\[
F_0 = \sqrt{d + 1}. \tag{60}
\]

Numerical estimation of $F_3$ gives $R_n(\mathcal{F}) \leq 0.00859$.

By Corollary 14 with probability at least $1 - \delta$, it holds that

\[
P(m_f(x, y) \leq 0) \leq \inf_{\gamma \in (0, 1]} \left[ \frac{1}{n} \sum_{i=1}^n \zeta(m_f(x_i, y_i)) \right. \right.
\]

\[
+ \frac{4M}{\gamma} R_n(\mathcal{F}) + \frac{\log \log_2(2\gamma^{-1})}{n} + \sqrt{\frac{9}{2n} \log \frac{3}{\delta}} \right] \tag{61}
\]

By setting $\delta = 5\%$, $\gamma = 0.5$, the generalisation error can be upper bounded by

\[
P(m_f(x, y) \leq 0) \leq 0.189492. \tag{62}
\]

For this model, the reported test error is 0.0028368.

Two extra experiments are given in Appendix A.12.

### 6 Comparison with Golowich et al.’s bound [8]

[8, Theorem 1,2] present bounds which depend on the square-root of network-length. In [8, Section 4], the authors present a method to convert existing bounds to depth-independence bounds. More specifically, for 1-Lipschitz, positive-homogeneous activation function such as ReLU, an upper bound on Rademacher complexity (Golowich et al.’s bound [8, Corollary 1]) is as follows:

\[
R_n(\mathcal{F}) = O \left( \prod_{j=1}^L \|W_j\|_F \max \left\{ 1, \log \left( \prod_{j=1}^L \frac{\|W_j\|_F}{\|W_j\|_2} \right) \right\} \min \left\{ \max(1, \log n)^{3/4}, \sqrt{\frac{L}{n}} \right\} \right) \tag{63}
\]
where $W_1, W_2, \cdots, W_L$ are the parameter matrices of the $L$ layers. However, this bound seems still depending on the square-root of the network-length implicitly.

Indeed, let $A$ be the term inside the bracket in (63), and define

$$\beta = \min_j \frac{\|W_j\|_F}{\|W_j\|_2} \geq 1. \quad (64)$$

Then, from (63) we have

$$A \geq \prod_{j=1}^{L} \|W_j\|_F \min \left\{ \frac{\max\{1, \log n\}^{3/4} \sqrt{\max\{1, L \log \beta\}}}{n^{1/4}}, \sqrt{\frac{L}{n}} \right\}. \quad (65)$$

For the general case, it holds that $\beta > 1$. Hence, we have

$$A = \Omega(\sqrt{L}), \quad (66)$$

which depends on the square-root of the network-length.

Our Rademacher complexity bound in Theorem 8 does not depend on the network-length explicitly for CNNs with some special classes of activation functions, including ReLU family and classes of old activation functions.

7 Limitations

The result that our Rademacher complexity bound does not explicitly depend on network length is only proved for CNNs with some classes of activation functions.
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A Appendix

A.1 Proof of Theorem 2

The proof of Theorem 2 is a combination of the following contraction lemmas.

Lemma 15 (Contraction for Linear Mapping) Let \( \mathcal{G} \) be a class of functions from \( \mathbb{R}^p \rightarrow \mathbb{R}^p \). Let \( V_{p,q} \) be a class of matrices \( W \) on \( \mathbb{R}^{p \times q} \) such that \( \sup_{W \in V_{p,q}} \|W\|_{p,q} \leq \nu \). Then, it holds that

\[
\mathbb{E}_x \left[ \sup_{W \in V} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i W f(x_i) \right] \leq \nu \mathbb{E}_x \left[ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right].
\]

(67)

Especially, we have

\[
\mathbb{E}_x \left[ \sup_{W \in V} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i W f(x_i) \right] \leq \sup_{W \in V_{\infty,\infty}} \|W\|_{\infty,\infty} \mathbb{E}_x \left[ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right].
\]

(68)

Lemma 16 Let \( \mathcal{H} \) be a set of functions mapping \( X \) to \( \mathbb{R}^m \) and \( \mathcal{H}_+ = \mathcal{H} \cup \{h : h \in \mathcal{H}\} \) and \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \psi(x) = \text{ReLU}(x) - \alpha \text{ReLU}(-x) \forall x \) for some \( \alpha \in [0,1] \). Then, for any \( p \geq 1 \) it holds that

\[
\mathbb{E}_x \left[ \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right\|_p \right] \leq \mu \mathbb{E}_x \left[ \sup_{h \in \mathcal{H}_+} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right\|_p \right].
\]

(69)

Identity, ReLU, Leaky ReLU, Parametric rectified linear unit (PReLU) belong to the class of functions \( \mathcal{L} := \{\psi : \psi(x) = \text{ReLU}(x) - \alpha \text{ReLU}(-x) \forall x, \text{ for some } \alpha \in \mathbb{R}\} \).

Lemma 17 Let \( \mathcal{H} \) be a set of functions mapping \( X \) to \( \mathbb{R}^m \). Define

\[
\mathcal{H}_+ = \mathcal{H} \cup \{-h : h \in \mathcal{H}\}.
\]

(70)

For any \( \mu > 0 \), let \( \psi : \mathbb{R}^m \rightarrow \mathbb{R}^m \) such that \( |\psi_j(x) - \psi_j(x')| \leq \mu |x_j - x'_j|, \forall (x,x') \in \mathbb{R}^m \times \mathbb{R}^m, \forall j \in [m] \) and \( \psi - \psi(0) \) is odd. In addition, \( \psi_j(0) \) does not depend on \( j \). Then, it holds that

\[
\mathbb{E}_x \left[ \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right\|_\infty \right] 
\leq \mu \mathbb{E}_x \left[ \sup_{h \in \mathcal{H}_+} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i h(x_i) \right\|_\infty \right] + \frac{1}{\sqrt{n}} \sup_{j \in [m]} |\psi_j(0)|.
\]

(71)

Here, we define \( \psi(x) := (\psi(x_1), \psi(x_2), \cdots, \psi(x_m))^T \) for any \( x = (x_1, x_2, \cdots, x_m)^T \in \mathbb{R}^m \).

Then, the following is a direct result of Lemma 17 by setting \( \psi_j(x) = \psi(x_j) \) for all \( j \in [m], x \in \mathbb{R}^m \) for some \( \mu \)-Lipschitz function \( \psi : \mathbb{R} \rightarrow \mathbb{R} \).

Corollary 18 Let \( \mathcal{H} \) be a set of functions mapping \( X \) to \( \mathbb{R}^m \). Define

\[
\mathcal{H}_+ = \mathcal{H} \cup \{-h : h \in \mathcal{H}\}.
\]

(72)

For any \( \mu > 0 \), let \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) such that \( |\psi(x) - \psi(x')| \leq \mu |x - x'|, \forall (x,x') \in \mathbb{R} \times \mathbb{R} \) and \( \psi - \psi(0) \) is odd. Then, it holds that

\[
\mathbb{E}_x \left[ \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right\|_\infty \right] 
\leq \mu \mathbb{E}_x \left[ \sup_{h \in \mathcal{H}_+} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right\|_\infty \right] + \frac{1}{\sqrt{n}} |\psi(0)|.
\]

(73)

Here, we define \( \psi(x) := (\psi(x_1), \psi(x_2), \cdots, \psi(x_m))^T \) for any \( x = (x_1, x_2, \cdots, x_m)^T \in \mathbb{R}^m \).
Lemma 19 Let $\mathcal{H}$ be a set of functions mapping $\mathcal{X}$ to $\mathbb{R}^m$. Define
$$\mathcal{H}_+ = \mathcal{H} \cup \{-h : h \in \mathcal{H}\} \cup \{h : h \in \mathcal{H}\}. \quad (74)$$

For any $\mu > 0$, let $\psi : \mathbb{R} \to \mathbb{R}$ such that $|\psi(x) - \psi(x')| \leq \mu|x - x'|$, $\forall (x, x') \in \mathbb{R} \times \mathbb{R}$ and $\psi - \psi(0)$ is even. Then, it holds that
$$\mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right\|_\infty \right] \leq 2\mu \mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}_+} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right\|_\infty \right] + \frac{1}{\sqrt{n}} |\psi(0)|. \quad (75)$$

Here, we define $\psi(x) := (\psi(x_1), \psi(x_2), \cdots, \psi(x_m))^T$ for any $x = (x_1, x_2, \cdots, x_m)^T \in \mathbb{R}^m$.

Lemma 20 Let $\mathcal{H}$ be a set of functions mapping $\mathcal{X}$ to $\mathbb{R}^m$. Define
$$\mathcal{H}_+ = \mathcal{H} \cup \{-h : h \in \mathcal{H}\} \cup \{h : h \in \mathcal{H}\}. \quad (76)$$

For any $\mu > 0$, let $\psi : \mathbb{R} \to \mathbb{R}$ such that $|\psi(x) - \psi(x')| \leq \mu|x - x'|$, $\forall (x, x') \in \mathbb{R} \times \mathbb{R}$. Then, it holds that
$$\mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right\|_\infty \right] \leq 3\mu \mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}_+} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right\|_\infty \right] + \frac{1}{\sqrt{n}} |\psi(0)|. \quad (77)$$

Here, we define $\psi(x) := (\psi(x_1), \psi(x_2), \cdots, \psi(x_m))^T$ for any $x = (x_1, x_2, \cdots, x_m)^T \in \mathbb{R}^m$.

These lemmas are proved in the next appendices.

A.2 Proof of Lemma 15

For any $W \in \mathcal{V}$, observe that
$$\left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i W f(x_i) \right\|_q = \left\| W \left( \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right) \right\|_q \leq \left\| W \right\|_{p,q} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_p \leq \nu \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_p. \quad (80)$$

Hence, (67) is a direct application of this fact.

This concludes our proof of Lemma 15.

A.3 Proof of Lemma 16

Observe that
$$\psi(x) = ReLU(x) - \alpha ReLU(-x)$$
$$= \frac{x + |x|}{2} - \alpha \frac{-x + |x|}{2}$$
$$= \frac{1 + \alpha}{2} x + \frac{(1 - \alpha)}{2} |x|. \quad (83)$$
Then, for any $p \geq 1$ we have

$$
\frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{h \in H} \left\| \sum_{i=1}^n \varepsilon_i \psi(h(x_i)) \right\|_p \right]
\leq \left( \frac{1 + \alpha}{2} \right) \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{h \in H} \left\| \sum_{i=1}^n \varepsilon_i h(x_i) \right\|_p \right]
\quad \leq \left( \frac{1 + \alpha}{2} \right) \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{h \in H} \left\| \sum_{i=1}^n \varepsilon_i h(x_i) \right\|_p \right]
\quad \leq \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{h \in H} \left\| \sum_{i=1}^n \varepsilon_i h(x_i) \right\|_p \right],
$$

where (85) follows from Minkowski’s inequality [31], and (86) follows from the fact that $|h| \in H_+$ if $h \in H$.

A.4 Proof of Lemma 17

First, we have

$$
\mathbb{E}_\varepsilon \left[ \sup_{h \in H} \left( \frac{1}{n} \sum_{i=1}^n \varepsilon_i \psi(h(x_i)) \right) \right]
\leq \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{h \in H} \left( \sum_{i=1}^n \varepsilon_i \left( \psi(h(x_i)) - \psi(0) \right) \right) \right] + \mathbb{E}_\varepsilon \left[ \sup_{h \in H} \left( \sum_{i=1}^n \varepsilon_i \psi(0) \right) \right]
\leq \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{h \in H} \left( \sum_{i=1}^n \varepsilon_i \psi(h(x_i)) \right) \right] + \mathbb{E}_\varepsilon \left[ \left( \sum_{i=1}^n \varepsilon_i \psi(0) \right)^2 \right]
\leq \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{h \in H} \left( \sum_{i=1}^n \varepsilon_i \psi(h(x_i)) \right) \right] + \sup_{j \in [m]} \left| \psi_j(0) \right| \frac{1}{\sqrt{n}}.
$$

where (87) follows from the triangular property of the $\infty$-norm [31], and (89) follows from Cauchy-Schwarz inequality and the assumption that $\psi_j(0)$ does not depend on $j$.

Define $\tilde{\psi}(x) := \psi(x) - \psi(0)$ for all $x \in \mathbb{R}^m$. Then, we have $\tilde{\psi}(0) = 0$, and $\tilde{\psi}$ satisfies $|\tilde{\psi}_j(x) - \tilde{\psi}_j(x')| \leq \mu |x_j - x'_j|$ for all $x, x' \in \mathbb{R}^m, j \in [m]$. In addition, by our assumption, $\tilde{\psi}$ is odd.

Let

$$
\Psi = \{ \tilde{\psi} : \mathbb{R}^m \to \mathbb{R}^m, \text{ st. } \tilde{\psi}(-x) = -\tilde{\psi}(x), |\tilde{\psi}_j(x) - \tilde{\psi}_j(y)| \leq \mu |x_j - y_j| \forall x, y \in \mathbb{R}^m, j \in [m] \}.
$$

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It follows that

\[
\mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \tilde{\psi}_j(h(x_i)) \right\|_\infty \right] = \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{j \in [m]} \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \tilde{\psi}_j(h(x_i)) \right\| \right] 
\]

\[
\leq \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{s \in \{-1,+1\}^m} \sup_{j \in [m]} \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^{n} \varepsilon_i \tilde{\psi}_j(h(x_i)) \right) \right] \]

\[
= \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{s \in \{-1,+1\}^m} \sup_{j \in [m]} \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^{n} \varepsilon_i \tilde{\psi}_j(h(x_i)) \right) \right] \]

\[
= \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{s \in \{-1,+1\}^m} \sup_{j \in [m]} \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^{n} \varepsilon_i \tilde{\psi}_j(h(x_i)) \right) \right] \]

\[
\leq \frac{1}{n} \mathbb{E}_\varepsilon \left[ \sup_{s \in \{-1,+1\}^m} \sup_{j \in [m]} \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^{n} \varepsilon_i \tilde{\psi}_j(h(x_i)) \right) \right] \]

where \((96)\) follows by defining \(\tilde{\psi}^{(s)} = (s_1 \tilde{\psi}_1, s_2 \tilde{\psi}_2, \ldots, s_m \tilde{\psi}_m)\) for any \(s \in \{-1,+1\}^m\), \((97)\) follows from the fact that \(\tilde{\psi}^{(s)} \in \Psi\) for any fixed \(s\), and \((98)\) follows from the definition of \(\mathcal{H}_+\).

Now, we have

\[
\mathbb{E}_\varepsilon \left[ \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} \sum_{i=1}^{n} \varepsilon_i \tilde{\psi}_j(h(x_i)) \right] 
\]

\[
= \mathbb{E}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n} \left[ \mathbb{E}_{\varepsilon_n} \left[ \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) + \varepsilon_n \tilde{\psi}_j(h(x_n)) \right] \right].
\]

where

\[
u = \frac{n-1}{n} \varepsilon_i \tilde{\psi}_j(h(x_i)).
\]

Since \(\varepsilon_n\) is uniformly distributed over \([-1, 1]\), we have

\[
\mathbb{E}_{\varepsilon_n} \left[ \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) + \varepsilon_n \tilde{\psi}_j(h(x_n)) \right] = \frac{1}{2} \mathbb{E}_{\varepsilon_n} \left[ \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) + \tilde{\psi}_j(h(x_n)) \right] 
\]

\[
+ \frac{1}{2} \left( \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) - \tilde{\psi}_j(h(x_n)) \right).
\]
Hence, we have

$$
\mathbb{E}_\varepsilon \left[ \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} \sum_{i=1}^n \varepsilon_i \tilde{\psi}_j(h(x_i)) \right] \\
= \frac{1}{2} \mathbb{E}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}} \left[ \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) + \tilde{\psi}_j(h(x_n)) \right] \\
+ \frac{1}{2} \mathbb{E}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}} \left[ \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) - \tilde{\psi}_j(h(x_n)) \right]
$$

(102)

$$
\mathbb{E}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}} \left[ \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) + \tilde{\psi}_j(h(x_n)) \right] \\
+ \frac{1}{2} \mathbb{E}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}} \left[ \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} -u_{n-1}(h, j) - \tilde{\psi}_j(h(x_n)) \right]
$$

(103)

$$
\mathbb{E}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}} \left[ \frac{1}{2} \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) + \tilde{\psi}_j(h(x_n)) \right] \\
+ \frac{1}{2} \left( \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} -u_{n-1}(h, j) - \tilde{\psi}_j(h(x_n)) \right) 
$$

(104)

where (103) follows from the fact that \((-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_{n-1})\) is a tuple of independent Rademacher random variables which has the same distribution as \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})\).

Now, given any \(j \in [m]\) and \(\tilde{\psi} \in \Psi\) we have

$$
\sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) + \tilde{\psi}_j(h(x_n))
$$

(105)

$$
= \sup_{h \in \mathcal{H}_+} u_{n-1}(-h, j) + \tilde{\psi}_j(-h(x_n))
$$

(106)

where (105) follows from the assumption that \(h \in \mathcal{H}_+\) if and only if \(-h \in \mathcal{H}_+,\) and (106) follows from the assumption that \(\tilde{\psi}\) is odd for any \(\tilde{\psi} \in \Psi.\)

Hence, for any arbitrarily small \(\delta > 0\) there exists \(j_0 \in [m], \tilde{\psi}_0 \in \Psi\) and \(h_1, h_2 \in \mathcal{H}\) such that

$$
\sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) + \tilde{\psi}_j(h(x_n)) \leq u_{n-1}(h_1, j_0) + \tilde{\psi}_0(j_0) + \psi_0(j_0) + \delta, 
$$

(107)

and

$$
\sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} -u_{n-1}(h, j) - \tilde{\psi}_j(h(x_n)) \leq -u_{n-1}(h_2, j_0) - \tilde{\psi}_0(j_0) + \psi_0(j_0) + \delta. 
$$

(108)
It follows that

\[
\frac{1}{2} \left( \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}^+} u_{n-1}(h, j) + \tilde{\psi}_j(h(x_n)) \right) \\
+ \frac{1}{2} \left( \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}^+} -u_{n-1}(h, j) - \tilde{\psi}_j(h(x_n)) \right) \\
\leq \frac{1}{2} \left( u_{n-1}(h_1, j_0) + \tilde{\psi}_{0,j_0}(h_1(x_n)) \right) \\
+ \frac{1}{2} \left( -u_{n-1}(h_2, j_0) - \tilde{\psi}_{0,j_0}(h_2(x_n)) \right) + \delta \\
= \frac{1}{2} \left( u_{n-1}(h_1, j_0) - u_{n-1}(h_2, j_0) \right) \\
+ \frac{1}{2} \left( \tilde{\psi}_{0,j_0}(h_1(x_n)) - \tilde{\psi}_{0,j_0}(h_2(x_n)) \right) + \delta \\
\leq \frac{1}{2} \left( u_{n-1}(h_1, j_0) - u_{n-1}(h_2, j_0) \right) + \frac{\mu}{2} \left| [h_1(x_n)]_{j_0} - [h_2(x_n)]_{j_0} \right| \quad (111) \\
= \frac{1}{2} \left( u_{n-1}(h_1, j_0) - u_{n-1}(h_2, j_0) \right) + \frac{\mu}{2} s_{12,n} \left( [h_1(x_n)]_{j_0} - [h_2(x_n)]_{j_0} \right) \quad (112) \\
= \frac{1}{2} \left( u_{n-1}(h_1, j_0) + \mu s_{12,n} [h_1(x_n)]_{j_0} \right) + \frac{1}{2} \left( -u_{n-1}(h_2, j_0) - \mu s_{12,n} [h_2(x_n)]_{j_0} \right) \quad (113) \\
\leq \sup_{s_{12} \in (-1,+1)} \frac{1}{2} \left( u_{n-1}(h_1, j_0) + \mu s_{12} [h_1(x_n)]_{j_0} \right) + \frac{1}{2} \left( -u_{n-1}(h_2, j_0) - \mu s_{12} [h_2(x_n)]_{j_0} \right) \quad (114) \\
\leq \sup_{s_{12} \in (-1,+1)} \frac{1}{2} \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}^+} u_{n-1}(h, j) + \mu s_{12} [h(x_n)]_{j} \\
+ \frac{1}{2} \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}^+} -u_{n-1}(h, j) - \mu s_{12} [h(x_n)]_{j} \quad (115) \\
\leq \sup_{s_{12} \in (-1,+1)} \frac{1}{2} \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}^+} u_{n-1}(h, j) + \mu s_{12} [h(x_n)]_{j} \\
+ \frac{1}{2} \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}^+} u_{n-1}(h, j) - \mu s_{12} [h(x_n)]_{j}, \quad (116)
\]

where \( s_{12,n} := \text{sgn} \left( [h_1(x_n)]_{j_0} - [h_2(x_n)]_{j_0} \right) \) in (112), and (116) follows from the fact that \( -\tilde{\psi} \in \Psi \) if \( \tilde{\psi} \in \Psi \).

From (116) we obtain

\[
\frac{1}{2} \left( \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}^+} u_{n-1}(h, j) + \tilde{\psi}_j(h(x_n)) \right) \\
+ \frac{1}{2} \left( \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}^+} -u_{n-1}(h, j) - \tilde{\psi}_j(h(x_n)) \right) \quad (117) \\
\leq \sup_{s_{12} \in (-1,+1)} \mathbb{E}_{\tilde{\varepsilon}_n} \left[ \sup_{\tilde{\psi} \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}^+} u_{n-1}(h, j) + \mu \tilde{\varepsilon}_n s_{12} [h(x_n)]_{j} \right] \quad (118)
\]

for some Rademacher random variable \( \tilde{\varepsilon}_n \) which is independent of \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}) \).
Since \( \tilde{\varepsilon}_n s_{12} \sim \tilde{\varepsilon}_n \) for any fixed \( s_{12} \in \{-1, +1\} \), from (118) we have

\[
\frac{1}{2} \left( \sup_{\psi \in \Psi} \sup_{j \in [m]} \sup_{h \in \mathcal{H}_+} u_{n-1}(h, j) + \tilde{\psi}_j(h(x_n)) \right) + \frac{1}{2} \left( \sup_{\psi \in \Psi} \sup_{j \in [m]} h \in \mathcal{H}_+ \right) u_{n-1}(h, j) - \tilde{\psi}_j(h(x_n)) \right) 
\leq \mathbb{E}_{\tilde{\varepsilon}_n} \left[ \sup_{\psi \in \Psi} \sup_{j \in [m]} h \in \mathcal{H}_+ \right] u_{n-1}(h, j) + \mu \tilde{\varepsilon}_n \left[ h(x_n) \right] \right].
\]

(119)

(120)

From (104) and (120) we obtain

\[
\mathbb{E}_\varepsilon \left[ \sup_{\psi \in \Psi} \sup_{j \in [m]} h \in \mathcal{H}_+ \right] \sup_{i=1}^n \varepsilon_i \tilde{\psi}_j(h(x_i)) \]

\[
\leq \mathbb{E}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}} \left[ \mathbb{E}_{\tilde{\varepsilon}_n} \left[ \sup_{\psi \in \Psi} \sup_{j \in [m]} h \in \mathcal{H}_+ \right] u_{n-1}(h, j) + \mu \tilde{\varepsilon}_n \left[ h(x_n) \right] \right] \]

(121)

\[
= \mathbb{E}_{\tilde{\varepsilon}_n} \left[ \mathbb{E}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}} \left[ \sup_{\psi \in \Psi} \sup_{j \in [m]} h \in \mathcal{H}_+ \right] u_{n-1}(h, j) + \mu \tilde{\varepsilon}_n \left[ h(x_n) \right] \right].
\]

(122)

By continuing this process (peeling) for \( n - 1 \) more times, we have

\[
\mathbb{E}_\varepsilon \left[ \sup_{\psi \in \Psi} \sup_{j \in [m]} h \in \mathcal{H}_+ \right] u_{n-1}(h, j) + \tilde{\varepsilon}_n \mu \left[ h(x_n) \right] \]

\[
\leq \mu \mathbb{E}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n} \left[ \sup_{j \in [m]} h \in \mathcal{H}_+ \sup_{i=1}^n \varepsilon_i \left[ h(x_n) \right] \right] \]

(123)

\[
= \mu \mathbb{E}_\varepsilon \left[ \sup_{j \in [m]} h \in \mathcal{H}_+ \sup_{i=1}^n \varepsilon_i \left[ h(x_n) \right] \right] \]

(124)

\[
\leq \mu \mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}_+} \left\| \sum_{i=1}^n \varepsilon_i h(x_n) \right\| \right] \]

(125)

\[
= \mu \mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}_+} \left\| \sum_{i=1}^n \varepsilon_i h(x_n) \right\| \right].
\]

(126)

From (90) and (126), we obtain

\[
\mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i \psi(h(x_i)) \right\| \right] \]

\[
\leq \mu \mathbb{E}_\varepsilon \left[ \sup_{h \in \mathcal{H}_+} \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i h(x_n) \right\| \right] + \sup_{j \in [m]} \psi_j(0) \left( \frac{1}{\sqrt{n}} \right).
\]

(127)

This concludes our proof of Lemma 17.

A.5 Proof of Lemma 19

Since \( \psi(x) \) is even, it holds that

\[
\mathbb{E} \left[ \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i \psi(h(x_i)) \right\| \right] = \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i \psi(h(x_i)) \right\| \right],
\]

(128)

Define

\[
\tilde{\psi}(x) := \psi(x1\{x > 0\}) - \psi(-x1\{x < 0\}) \quad \forall x \in \mathbb{R}.
\]

(129)
Then, it is easy to see that \( \tilde{\psi} \) is an odd function.

On the other hand, we also have
\[
\tilde{\psi}(|x|) = \psi(|x|), \quad \forall x \in \mathbb{R},
\]
so
\[
\mathbb{E} \left[ \sup_{h \in \mathcal{H}} \frac{1}{n} \left| \sum_{i=1}^{n} \varepsilon_i \psi(|h(x_i)|) \right| \right] = \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \frac{1}{n} \left| \sum_{i=1}^{n} \varepsilon_i \tilde{\psi}(|h(x_i)|) \right| \right].
\]

Furthermore, for all \( x, y \in \mathbb{R} \) we have
\[
|\psi(x) - \tilde{\psi}(y)| \leq |\psi(x \mathbf{1}\{x > 0\}) - \psi(y \mathbf{1}\{y > 0\})| + |\psi(x \mathbf{1}\{x < 0\}) - \psi(y \mathbf{1}\{y < 0\})|,
\]
\[
\leq \mu |x \mathbf{1}\{x > 0\} - y \mathbf{1}\{y > 0\}| + \mu |x \mathbf{1}\{x < 0\} - y \mathbf{1}\{y < 0\}|
\]
\leq \mu |x - y|.
\]

Now, observe that
\[
|x \mathbf{1}\{x > 0\} - y \mathbf{1}\{y > 0\}| = \frac{|x + |x| - y + |y|}{2}
\]
\leq \frac{1}{2} |x - y| + \frac{1}{2} \sum_{i=1}^{L} ||x| - |y||
\]
\leq |x - y|
\]
\leq |x - y|.
\]

Similarly, we also have
\[
|x \mathbf{1}\{x < 0\} - y \mathbf{1}\{y < 0\}| \leq |x - y|.
\]

From \(\text{(133)}, \text{(136)}, \text{and (137)}\) we obtain
\[
|\tilde{\psi}(x) - \tilde{\psi}(y)| \leq 2\mu |x - y|, \quad \forall x, y \in \mathbb{R}.
\]

Hence, by Lemma \(\text{[18]}\) we have
\[
\mathbb{E} \left[ \sup_{h \in \mathcal{H}} \frac{1}{n} \left| \sum_{i=1}^{n} \varepsilon_i \tilde{\psi}(|h(x_i)|) \right| \right] \leq 2\mu \mathbb{E} \left[ \sup_{h \in \mathcal{H}_+} \frac{1}{n} \left| \sum_{i=1}^{n} \varepsilon_i h(x_i) \right| \right],
\]
\[
\leq 2\mu \mathbb{E} \left[ \sup_{h \in \mathcal{H}_+} \frac{1}{n} \left| \sum_{i=1}^{n} \varepsilon_i h(x_i) \right| \right],
\]
\[
\mathbb{E} \left[ \sup_{h \in \mathcal{H}} \frac{1}{n} \left| \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right| \right] \leq 2\mu \mathbb{E} \left[ \sup_{h \in \mathcal{H}_+} \frac{1}{n} \left| \sum_{i=1}^{n} \varepsilon_i h(x_i) \right| \right].
\]

A.6 Proof of Lemma \(\text{[20]}\)

For any general function \(\psi\), we can represent as
\[
\psi(x) = \frac{\psi(x) + \psi(-x)}{2} + \frac{\psi(x) - \psi(-x)}{2}, \quad \forall x \in \mathbb{R}.
\]

It is easy to see that \(\frac{\psi(x) + \psi(-x)}{2}\) is an even function with \(\mu\)-Lipschitz. Besides, \(\frac{\psi(x) - \psi(-x)}{2}\) is an odd function with \(\mu\)-Lipschitz. Hence, by using triangle inequality, Lemma \(\text{[17]}\) and Lemma \(\text{[19]}\) we have
\[
\mathbb{E} \left[ \sup_{h \in \mathcal{H}} \frac{1}{n} \left| \sum_{i=1}^{n} \varepsilon_i \psi(h(x_i)) \right| \right] \leq (2\mu + \mu) \mathbb{E} \left[ \sup_{h \in \mathcal{H}_+} \frac{1}{n} \left| \sum_{i=1}^{n} \varepsilon_i h(x_i) \right| \right].
\]
A.7 Proof of Lemma \[5\]

Let
\[
\mathbf{1}_{r^2} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix},
\]
\[
\mathbf{0}_{r^2} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix},
\]
and
\[
\mathbf{A} = \frac{1}{r^2} \begin{bmatrix}
\mathbf{1}_{r^2} & \mathbf{0}_{r^2} & \mathbf{0}_{r^2} & \cdots & \mathbf{0}_{r^2} & \mathbf{0}_{r^2} \\
\mathbf{0}_{r^2} & \mathbf{1}_{r^2} & \mathbf{0}_{r^2} & \cdots & \mathbf{0}_{r^2} & \mathbf{0}_{r^2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{0}_{r^2} & \mathbf{0}_{r^2} & \mathbf{0}_{r^2} & \cdots & \mathbf{1}_{r^2}
\end{bmatrix} \in \mathbb{R}^{[(d-r_1+1)^2/r^2] \times [(d-r_1+1)^2/r^2]}.
\]

Then, for all \(\mathbf{x} \in \mathbb{R}^{d \times d \times c}\) and \(l \in [Q], c \in [C]\), we have
\[
\psi_{l,c}(\mathbf{x}) = \sigma_{\text{avg}} \circ \sigma_{l,c}(\mathbf{x}),
\]
where
\[
\sigma_{\text{avg}}(\mathbf{x}) = \mathbf{A} \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^{[(d-r_1+1)^2/r^2]}.
\]

Now, for all \(\mathbf{x}, \mathbf{y} \in \mathbb{R}^{[(d-r_1+1)^2/r^2]}\) we have
\[
\left\| \sigma_{\text{avg}}(\mathbf{x}) - \sigma_{\text{avg}}(\mathbf{y}) \right\|_{\infty}
\leq \frac{1}{r^2} \max_{j \in \llbracket[(d-r_1+1)^2/r^2]\rrbracket} \sum_{k=(j-1)r_2^2+1}^{jr_2^2} |x_k - y_k|,
\]
\[
\leq \left\| \mathbf{x} - \mathbf{y} \right\|_{\infty}.
\]

Hence, we have
\[
\left\| \mathbf{A} \right\|_{\infty} \leq 1.
\]

Hence, by Lemma \[\overline{15}\] we have
\[
\mathbb{E} \left[ \sup_{c \in [C]} \sup_{l \in [Q]} \sup_{\nu, u \in \Psi} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \psi_{l,c} \circ f(\mathbf{x}_i) \right\|_{\infty} \right]
\leq \mathbb{E} \left[ \sup_{c \in [C]} \sup_{l \in [Q]} \sup_{\nu, u \in \Psi} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sigma_{\text{avg}} \circ \sigma_{l,c} \circ f(\mathbf{x}_i) \right\|_{\infty} \right]
\leq \mathbb{E} \left[ \sup_{c \in [C]} \sup_{l \in [Q]} \sup_{\nu, u \in \Psi} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sigma_{l,c} \circ f(\mathbf{x}_i) \right\|_{\infty} \right].
\]

In addition, for all \(\mathbf{x} \in \mathbb{R}^{d \times d \times c}\),
\[
\sigma_{l,c}(\mathbf{x}) = \left\{ \hat{x}_c(a, b) \right\}_{a,b=1}^{d-r_1+1},
\]
\[
\hat{x}_c(a, b) = \sigma \left( \sum_{u=0}^{r_1-1} \sum_{v=0}^{r_1-1} x(a+u, b+v, c) W_{l,c}(u+1, v+1) \right).
\]

Hence, we have
\[
\left\| \sigma_{l,c}(\mathbf{x}) - \sigma_{l,c}(\mathbf{y}) \right\|_{\infty}
\leq \mu \max_{a \in [d-r_1+1]} \max_{b \in [d-r_1+1]} \sum_{u=0}^{r_1-1} \sum_{v=0}^{r_1-1} \left| W_{l,c}(u+1, v+1) x(a+u, b+v, c) - W_{l,c}(u+1, v+1) y(a+u, b+v, c) \right|
\leq \mu \sum_{u=0}^{r_1-1} \sum_{v=0}^{r_1-1} \left| W_{l,c}(u+1, v+1) \right| \left\| \mathbf{x} - \mathbf{y} \right\|_{\infty}.
\]
Since the convolution is linear, it is also easy to see that $\sigma_{l,c}$ is the composition of a linear map and a point-wise activation map. Hence, by Lemma 15 and Theorem 2 we have

$$
E \left[ \sup_{c \in [C]} \sup_{l \in [Q]} \sup_{\sigma_{l,c} \in \Psi} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} \sigma_{l,c} \circ f(x_{i}) \right\|_{\infty} \right] 
\leq \left[ \gamma(\mu) \sup_{c \in [C]} \sup_{l \in [Q]} \left( \sum_{u=0}^{r_{l}-1} \sum_{v=0}^{r_{l}-1} |W_{l,c}(u + 1, v + 1)| \right) \right] E \left[ \sup_{f \in \mathcal{F}^{+}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} f(x_{i}) \right\|_{\infty} \right] + \frac{|\sigma(0)|}{\sqrt{n}}. 
$$

Finally, from (153) and (158) we obtain

$$
E \left[ \sup_{c \in [C]} \sup_{l \in [Q]} \sup_{\psi_{l,c} \in \Psi} \sup_{f \in \mathcal{F}^{+}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} \psi_{l,c} \circ f(x_{i}) \right\|_{\infty} \right] 
\leq \left[ \gamma(\mu) \sup_{c \in [C]} \sup_{l \in [Q]} \left( \sum_{u=0}^{r_{l}-1} \sum_{v=0}^{r_{l}-1} |W_{l,c}(u + 1, v + 1)| \right) \right] E \left[ \sup_{f \in \mathcal{F}^{+}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} f(x_{i}) \right\|_{\infty} \right] + \frac{|\sigma(0)|}{\sqrt{n}}.
$$

(A.8) Proof of Lemma 6

This is a direct result of Lemma 17, where $\tilde{\psi}_{j}(x) = x_{j}$ or 0 at each fixed $j$. Hence, we have

$$
|\tilde{\psi}_{j}(x) - \tilde{\psi}_{j}(y)| \leq |x_{j} - y_{j}|
$$

for all vectors $x$ and $y$.

(A.9) Proof of Lemma 7

This is a direct result of Theorem 2 and Lemma 15.

(A.10) Proof of Lemma 12

For $M > 2$, (51) is a result of [1, Proof of Theorem 11]. Now, we prove (51) for $M = 2$. Observe that

$$
E_{\varepsilon} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} m_{f}(x_{i}, y_{i}) \right\| \right] 
= E_{\varepsilon} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} \left( [f(x_{i})]_{y_{i}} - \sup_{y' \neq y_{i}} [f(x_{i})]_{y'} \right) \right\| \right] 
\leq E_{\varepsilon} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} [f(x_{i})]_{y_{i}} \right\| \right] + E_{\varepsilon} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} \sup_{y' \neq y_{i}} [f(x_{i})]_{y'} \right\| \right].
$$

(161)
Now, we have

\[
\mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i [f(x_i)]_{y_i} \right| \right]
\]

\[
= \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i [f(x_i)]_{y_i} \sum_{y=1}^{M} \mathbf{1}_{\{y_i=y\}} \right| \right] 
\]

\[
= \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i [f(x_i)]_{y_i} \mathbf{1}_{\{y_i=y\}} \right| \right] 
\]

\[
\leq \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i [f(x_i)]_{y_i} \mathbf{1}_{\{y_i=y\}} \right| \right] 
\]

\[
\leq \frac{1}{2} \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i [f(x_i)]_{y_i} (2 \mathbf{1}_{\{y_i=y\}} - 1) \right| \right] 
\]

\[
+ \frac{1}{2} \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i [f(x_i)]_{y_i} \right| \right] 
\]

\[
= \frac{1}{2} \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i [f(x_i)]_{y_i} \right| \right] 
\]

\[
+ \frac{1}{2} \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i [f(x_i)]_{y_i} \right| \right] 
\]

\[
= \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i [f(x_i)]_{y_i} \right| \right] 
\]

\[
\leq \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right] 
\]

\[
= M \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right], 
\]

where (167) follows from the fact that \((2 \mathbf{1}_{\{y_1=y\}} - 1)\varepsilon_1, (2 \mathbf{1}_{\{y_2=y\}} - 1)\varepsilon_2, \cdots, (2 \mathbf{1}_{\{y_n=y\}} - 1)\varepsilon_n\) has the same distribution as \((\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)\).
On the other hand, we also have

\[
\mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq y_i} \left[ |f(x_i)|_{y'} \right] \right\} \right]
\]

\[
= \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq y_i} \left[ f(x_i) \right] \sum_{y=1}^{M} \mathbb{1}_{\{y_i = y\}} \right\} \right]
\]

\[
= \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq y_i} \left[ f(x_i) \right] \mathbb{1}_{\{y_i = y\}} \right\} \right]
\]

\[
\leq \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq y_i} \left[ f(x_i) \right] \mathbb{1}_{\{y_i = y\}} \right\} \right]
\]

\[
\leq \frac{1}{2} \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq y_i} \left[ f(x_i) \right] (2 \mathbb{1}_{\{y_i = y\}} - 1) \right\} \right]
\]

\[
+ \frac{1}{2} \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq y_i} \left[ f(x_i) \right] \right\} \right]
\]

\[
= \frac{1}{2} \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq y_i} \left[ f(x_i) \right] \right\} \right]
\]

\[
+ \frac{1}{2} \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq y_i} \left[ f(x_i) \right] \right\} \right]
\]

\[
= \sum_{y=1}^{M} \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq y_i} \left[ f(x_i) \right] \right\} \right],
\]

where (175) follows from the fact that \((2 \mathbb{1}_{\{y_i = y\}} - 1)\varepsilon_1, (2 \mathbb{1}_{\{y_2 = y\}} - 1)\varepsilon_2, \ldots, (2 \mathbb{1}_{\{y_n = y\}} - 1)\varepsilon_n\) has the same distribution as \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\).

Now, for each fixed \(y \in \{1, \ldots, M\}\) and \(M = 2\), let \(\hat{y} = \{1, \ldots, M\} \setminus \{y\}\) we have

\[
\mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq \hat{y}} \left[ f(x_i) \right] \right\} \right]
\]

\[
= \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \left[ f(x_i) \right] \right\} \right]
\]

\[
\leq \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\} \right].
\]

It follows from (176) and (178) that

\[
\mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sup_{y' \neq y_i} \left[ f(x_i) \right] \right\} \right]
\]

\[
\leq M \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\} \right].
\]

From (162), (170), and (179), for \(M = 2\) we have

\[
\mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i m_f(x_i, y_i) \right\} \right] \leq 2M \mathbb{E}_\varepsilon \left[ \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\} \right].
\]
A.11 Proof of Theorem 13

Let \((x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_n, y'_n)\) is an i.i.d. sequence with distribution \(P_{XY}\) which is independent of \(X^nY^n\). Define

\[
E(f) := \mathbb{E}_{X'Y'} \left[ \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x'_i, y'_i)) \right].
\] (181)

Now, let \(D = \{(x_i, y_i) : i \in [n]\}\), and let \(\tilde{D} = \{(x_i, y_i) : i \in [n]\}\) be a set with only one sample different from \(D\), i.e. the \(k\)-th sample is replaced by \((\tilde{x}_k, \tilde{y}_k)\). Define

\[
\hat{E}_D(f) := \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i))
\] (182)

and

\[
\Phi(D) := \sup_{f \in F} E(f) - \hat{E}_D(f),
\] (183)

which is a function of \(n\) independent random vectors \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) where \((x_i, y_i) \sim P_{XY}\) for all \(i \in [n]\). Since \(0 \leq \zeta(x) \leq 1\) for all \(x \in \mathbb{R}\), from (181) and (182) we have

\[
|\Phi(\tilde{D}) - \Phi(D)| \leq \sup_{f \in F} \left| \frac{\zeta(m_f(x_k, y_k)) - \zeta(m_f(\tilde{x}_k, \tilde{y}_k))}{n} \right|
\] (184)

\[
\leq \frac{1}{n}.
\] (185)

By McDiarmid’s inequality \([32]\), with probability at least \(1 - \exp(-2t^2)\) we have

\[
\sup_{f \in F} \left( \frac{1}{n} \mathbb{E}_{X'Y'} \left[ \sum_{i=1}^{n} \zeta(m_f(x'_i, y'_i)) \right] - \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)) \right)
\leq \mathbb{E}_{XY} \left[ \sup_{f \in F} \left( \mathbb{E}_{X'Y'} \left[ \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x'_i, y'_i)) \right] - \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)) \right) \right] + \frac{t}{\sqrt{n}}.
\] (186)
Now, let \( \zeta(x) := \zeta(x) - \zeta(0) \), which is a \( 1/\gamma \)-Lipschitz function with \( \zeta(0) = 0 \). Then, we have

\[
\mathbb{E}_{XY} \left[ \sup_{f \in \mathcal{F}} \left( \mathbb{E}_{X|Y} \left[ \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i', y_i')) \right] - \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)) \right] \right] 
\leq \mathbb{E}_{XY} \left[ \sup_{f \in \mathcal{F}} \left( \mathbb{E}_{X|Y} \left[ \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i', y_i')) \right] - \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)) \right] \right] 
\leq \mathbb{E}_{XY} \left[ \sup_{f \in \mathcal{F}} \left( \mathbb{E}_{X|Y} \left[ \frac{1}{n} \sum_{i=1}^{n} (\zeta(m_f(x_i', y_i')) - \zeta(m_f(x_i, y_i))) \right] \right] \right] 
\leq \frac{1}{\gamma} \mathbb{E}_{XY} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} (m_f(x_i', y_i') - m_f(x_i, y_i)) \right] 
\leq \frac{1}{\gamma} \mathbb{E}_{xy} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (m_f(x_i', y_i') - m_f(x_i, y_i)) \right]
\leq \frac{1}{\gamma} \mathbb{E}_{xy} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i m_f(x_i, y_i) \right] 
+ \frac{1}{\gamma} \mathbb{E}_{XY} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i m_f(x_i, y_i) \right] 
= \frac{1}{\gamma} \mathbb{E}_{XY} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i m_f(x_i, y_i) \right] 
\leq \frac{2\beta(M)}{\gamma} \mathbb{E}_{XY} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right]
\]

where (192) follows from [33, Lemma 25], and (196) follows from Lemma 12. From (196), with probability at least \( 1 - \exp(-2t^2) \) we have

\[
\sup_{f \in \mathcal{F}} \left( \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i', y_i')) - \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)) \right) 
\leq \frac{2\beta(M)}{\gamma} \mathbb{E}_{XY} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right] + \frac{t}{\sqrt{n}}.
\]

It follows that, with probability at least \( 1 - \exp(-2t^2) \),

\[
\mathbb{E}_{X|Y} \left[ \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i', y_i')) \right] \leq \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)) 
+ \frac{2\beta(M)}{\gamma} \mathbb{E}_{XY} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right] + \frac{t}{\sqrt{n}} \quad \forall f \in \mathcal{F},
\]

or

\[
\mathbb{E}[\zeta(m_f(x, y))] \leq \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)) 
+ \frac{2\beta(M)}{\gamma} \mathbb{E}_{XY} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right] + \frac{t}{\sqrt{n}} \quad \forall f \in \mathcal{F}.
\]
Now, observe that
\[
\mathbb{E}[\zeta(m_f(x, y))] = \mathbb{P}[m_f(x, y) \leq 0] + \mathbb{E}[\zeta(m_f(x, y))]0 \leq m_f(x, y) \leq \gamma] \mathbb{P}[0 \leq m_f(x, y) \leq \gamma] \quad (201)
\]
\[
\geq \mathbb{P}(m_f(x, y) \leq 0). \quad (202)
\]
From (200) and (202), with probability at least \(1 - \exp(-2t^2)\),
\[
\mathbb{P}[m_f(x, y) \leq 0] \leq \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i))
\]
\[
+ \frac{2\beta(M)}{\gamma} \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right] + \frac{t}{\sqrt{n}}, \quad \forall f \in \mathcal{F}. \quad (203)
\]
Now, let \(\gamma_k = 2^{-k}\) for all \(k \in \mathbb{N}\). For any \(\gamma \in (0, 1]\), there exists a \(k \in \mathbb{N}\) such that \(\gamma \in (\gamma_k, \gamma_{k-1}]\). Then, by applying (203) with \(t\) being replaced by \(t + \sqrt{\log k}\) and \(\zeta(\cdot) = \zeta_k(\cdot)\) where
\[
\zeta_k(x) := \begin{cases} 0, & \gamma_k \leq x \\ 1 - \frac{x}{\gamma_k}, & 0 \leq x \leq \gamma_k \\ 1, & x \leq 0 \end{cases}
\]
with probability at least \(1 - \exp(-2(t + \sqrt{\log k})^2)\), we have
\[
\mathbb{P}[m_f(x, y) \leq 0] \leq \frac{1}{n} \sum_{i=1}^{n} \zeta_k(m_f(x_i, y_i))
\]
\[
+ \frac{2\beta(M)}{\gamma} \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right] + \frac{t + \sqrt{\log k}}{\sqrt{n}}, \quad \forall f \in \mathcal{F}. \quad (205)
\]
By using the union bound, from (205), with probability at least \(1 - \sum_{k \geq 1} \exp(-2(t + \sqrt{\log k})^2)\), it holds that
\[
\mathbb{P}[m_f(x, y) \leq 0] \leq \inf_{k \geq 1} \left\{ \frac{1}{n} \sum_{i=1}^{n} \zeta_k(m_f(x_i, y_i))
\right. 
\]
\[
+ \frac{2\beta(M)}{\gamma} \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right] + \frac{t + \sqrt{\log k}}{\sqrt{n}}, \quad \forall f \in \mathcal{F}. \quad (206)
\]
On the other hand, it is easy to see that
\[
\frac{1}{\gamma_k} \leq \frac{2}{\gamma}, \quad (207)
\]
\[
\frac{1}{n} \sum_{i=1}^{n} \zeta_k(m_f(x_i, y_i)) \leq \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)), \quad (208)
\]
\[
\sqrt{\log k} \leq \sqrt{\log \log_2 \frac{1}{\gamma_k}} \leq \sqrt{\log \log_2 \frac{2}{\gamma}}, \quad (209)
\]
\[
\sum_{k \geq 1} \exp(-2(t + \sqrt{\log k})^2) \leq \sum_{k \geq 1} k^2 e^{-2t^2} = \frac{\pi^2}{6} e^{-2t^2} \leq 2e^{-2t^2}. \quad (210)
\]
Hence, by combining (207)–(210), and (206), with probability at least \(1 - 2 \exp(-2t^2)\), it holds that
\[
\mathbb{P}[m_f(x, y) \leq 0] \leq \inf_{\gamma \in (0, 1]} \left\{ \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i))
\right. 
\]
\[
+ \frac{2\beta(M)}{\gamma} \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i) \right\|_\infty \right] + \frac{t + \sqrt{\log_2(2\gamma^{-1})}}{\sqrt{n}}, \forall f \in \mathcal{F}. \quad (211)
\]
From (211) we have

\[
P[m_f(x, y) \leq 0] \leq \inf_{\gamma \in (0, 1]} \left[ \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)) \right]
+ \frac{2\beta(M)}{\gamma} E \left[ \sup_{f \in F} \left\| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(x_i) \right\|_{\infty} \right] + t + \frac{\sqrt{\log \log (2\gamma^{-1})}}{\sqrt{n}}, \quad \forall f \in F. \tag{212}
\]

This concludes our proof of Theorem 13.

A.12 Extra Numerical Results

A.12.1 Experiment 2

```python
class DenseLayer(nn.Module):
    def __init__(self, input_shape, output_dim):
        super(DenseLayer, self).__init__()
        self.fc = nn.Linear(input_shape, output_dim)

    def forward(self, x):
        return self.fc(x)

def build_cnn(input_shape, output_dim):
    model = nn.Sequential(
        DenseLayer(input_shape, 32),
        nn.ReLU(),
        DenseLayer(32, 64),
        nn.ReLU(),
        DenseLayer(64, output_dim),
        nn.ReLU()
    )
    return model
```

Figure 2: CNN model with ReLU activations

In this experiment, we use a CNN (cf. Fig. 2) for classifying MNIST images (class 0 and class 1), i.e., \( M = 2 \), which consists of \( n = 12665 \) training examples.

For this model, we use ReLU for the first two convolutional layers, and the sigmoid \( \sigma \) for the dense layer which satisfies \( \sigma(x) - \sigma(0) = \frac{1}{2} \tanh \left( \frac{x}{2} \right) \) (an odd function with Lipschitz constant 1/4).

Hence, by Theorem 8 and Lemma 17, it holds that \( R_n(F) \leq F_3 \), where

\[
F_3 \leq \frac{1}{4} \| W \|_{\infty} F_2 + \frac{1}{2\sqrt{n}}, \tag{213}
\]

Dense layer

\[
F_2 \leq \left( \sup_{l \in [64]} \sum_{u=1}^{3} \sum_{v=1}^{3} \left| W^{(l)}_2(u, v) \right| \right) F_1 + \frac{1}{2\sqrt{n}}. \tag{214}
\]

The second convolutional layer

\[
F_1 \leq \left( \sup_{l \in [32]} \sum_{u=1}^{3} \sum_{v=1}^{3} \left| W^{(l)}_1(u, v) \right| \right) F_0 + \frac{1}{2\sqrt{n}}. \tag{215}
\]

The first convolutional layer

\[
F_0 = \sqrt{\frac{d+1}{n}}. \tag{216}
\]

Numerical estimation of \( F_3 \) gives \( R_n(F) \leq 0.0476 \).
By Corollary 14 with probability at least $1 - \delta$, it holds that
\[
P(m_f(x, y) \leq 0) \leq \inf_{\gamma \in (0, 1)} \left[ \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)) \right] + \frac{4M}{\gamma} \mathcal{R}_n(F) + \sqrt{\frac{\log \log(2\gamma^{-1})}{n}} + \sqrt{\frac{2}{n} \log \frac{3}{\delta}} \quad (217)
\]
By setting $\delta = 5\%, \gamma = 1$, the generalisation error can be upper bounded by
\[
P(m_f(x, y) \leq 0) \leq 0.412806. \quad (218)
\]
For this model, the reported test error is 0.0009456.

A.12.2 Experiment 3

```python
model = keras.Sequential(
    [layers.Input(shape=input_shape),
     layers.Conv2D(32, kernel_size=(3, 3), activation="sigmoid"),
     layers.AveragePooling2D(pool_size=(2, 2)),
     layers.Conv2D(64, kernel_size=(3, 3), activation="sigmoid"),
     layers.AveragePooling2D(pool_size=(2, 2)),
     layers.Flatten(),
     layers.Dropout(0.5),
     layers.Dense(2, activation="softmax"),
    ]
)
model.summary()
```

**Figure 3:** CNN model with sigmoid activations

In this experiment, we use a CNN (cf. Fig. 3) for classifying MNIST images (class 0 and class 1), i.e., $M = 2$, which consists of $n = 12665$ training examples.

For this model, the sigmoid activation $\sigma$ satisfies $\sigma(x) - \sigma(0) = \frac{1}{2} \tanh \left( \frac{x}{2} \right)$ which is odd and has the Lipschitz constant $1/4$. In addition, for the dense layer, the sigmoid activation satisfies
\[
|\sigma(x) - \sigma(y)| \leq \frac{1}{4} |x - y|, \quad \forall x, y \in \mathbb{R}. \quad (219)
\]
For this example, we assume that we compare the outputs at the layer right before the softmax layer to bound the generalisation error. Then, by Theorem 8 and Lemma 17 it holds that $\mathcal{R}_n(F) \leq F_2$, where
\[
F_2 \leq \left( \frac{1}{4} \sup_{l \in [64]} \sum_{u=1}^{3} \sum_{v=1}^{3} |W^{(l)}_2(u, v)| \right) F_1 + \frac{1}{2\sqrt{n}}, \quad (220)
\]
The second convolutional layer
\[
F_1 \leq \left( \frac{1}{4} \sup_{l \in [32]} \sum_{u=1}^{3} \sum_{v=1}^{3} |W^{(l)}_1(u, v)| \right) F_0 + \frac{1}{2\sqrt{n}}, \quad (221)
\]
The first convolutional layer
\[
F_0 = \sqrt{\frac{d + 1}{n}}. \quad (222)
\]
Numerical estimation of $F_2$ gives $\mathcal{R}_n(F) \leq 0.03074$. 


By Corollary 14 with probability at least $1 - \delta$, it holds that

$$\mathbb{P}(m_f(x, y) \leq 0) \leq \inf_{\gamma \in (0, 1]} \left[ \frac{1}{n} \sum_{i=1}^{n} \zeta(m_f(x_i, y_i)) \right]$$

$$+ \frac{4M}{\gamma} \mathcal{R}_n(\mathcal{F}) + \sqrt{\frac{\log \log_2(2\gamma^{-1})}{n}} + \sqrt{\frac{2}{n} \log \frac{3}{\delta}}$$

(223)

By setting $\delta = 5\%$, $\gamma = 1$, the generalisation error can be upper bounded by

$$\mathbb{P}(m_f(x, y) \leq 0) \leq 0.2775.$$  

(224)

For this model, the reported test error is 0.001418.