EQUIVARIANT $R$-TEST CONFIGURATIONS AND SEMISTABLE LIMITS OF $\mathbb{Q}$-FANO GROUP COMPACTIFICATIONS

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Abstract. Let $G$ be a connected, complex reductive group. In this paper, we classify $G \times G$-equivariant normal $R$-test configurations of a polarized $G$-compactification. Then for $\mathbb{Q}$-Fano $G$-compactifications, we express the $H$-invariants of its equivariant normal $R$-test configurations in terms of the combinatorial data. Based on [17], we compute the semistable limit of a K-unstable $\mathbb{Q}$-Fano $G$-compactification. As an application, we show that for the two smooth K-unstable $\mathbb{Q}$-Fano $SO_4(\mathbb{C})$-compactifications, the corresponding semistable limits are indeed the limit spaces of the normalized Kahler-Ricci flow.

1. Introduction

Let $M$ be a Fano manifold, namely, a compact Kahler manifold with positive first Chern class $c_1(M)$. Consider the following normalized Kahler-Ricci flow:

\begin{equation}
\frac{\partial}{\partial t} \omega(t) = -\text{Ric}(\omega(t)) + \omega(t), \quad \omega(0) = \omega_0,
\end{equation}

where $\omega_0$ and $\omega(t)$ denote the initial Kahler metric and the solutions of (1.1), respectively. Cao [7, Section 1] showed that (1.1) always have a global solution $\omega(t)$ for all $t \geq 0$ whenever $\omega_0 \in 2\pi c_1(M)$. A long-standing problem concerns the limiting behavior of $\omega(t)$ as $t \to \infty$. Tian-Zhu [31, 32] showed that: if $M$ admits a Kahler-Ricci soliton, then $\omega(t)$ will converge to it. However, in general $\omega(t)$ may not have a limit on $M$. The famous Hamilton-Tian conjecture (cf. [27, Section 9]) suggests that any sequence of $\{(M, \omega(t_i))\}_{i \in \mathbb{N}}$ with $t_i \to +\infty$ contains a subsequence converging to a length space $(M_\infty, \omega_\infty)$ in the Gromov-Hausdorff topology, and $(M_\infty, \omega_\infty)$ is a smooth Kahler-Ricci soliton outside a closed subset $S$ of (real) codimension at least 4. Moreover, this subsequence converges locally to the regular part of $(M_\infty, \omega_\infty)$ in the Cheeger-Gromov topology. This implies that, by taking the limit, the complex structure of $M$ may jump so that under the new complex structure there exists a Kahler-Ricci soliton.

The Gromov-Hausdorff convergency follows from Perelman [26] and Zhang [33, 36]. Tian-Zhang [29] first confirmed the whole conjecture when $\dim(M) \leq 3$. Chen-Wang [8] and Bamler [3] then solved the remaining higher dimensional cases. In fact, Bamler [3] proved a generalized version of the conjecture.

It is then natural to study the regularity of the limit space $(M_\infty, \omega_\infty)$. In fact, Tian-Zhang [29] proved that $M_\infty$ is a $\mathbb{Q}$-Fano variety whose singular set coincides

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with $S$. Concerning the further regularity of $M_\infty$, there was a folklore speculation states that $(M_\infty, \omega_\infty)$ is actually a smooth Ricci soliton, or equivalently, (1.1) always has Type-I solution. In [23], Li-Tian-Zhu disproved this folklore speculation by constructing examples of Type-II solution of (1.1). That is, a solution $\{\omega(t)\}_{t \geq 0}$ such that curvature of $\omega(t)$ is not uniformly bounded for $t \in [0, +\infty)$. More precisely, they proved

**Theorem 1.1.** [23, Theorem 1.1] Let $G$ be a connected, complex semisimple Lie group and $K$ be its maximal compact subgroup. Let $M$ be a Fano $G$-compactification which admits no Kähler-Einstein metrics. Then any solution of Kähler-Ricci flow (1.1) on $M$ with $K \times K$-invariant initial metric $\omega_0 \in 2\pi c_1(M)$ is of Type-II.

In particular, the above theorem shows that there are two smooth Fano $SO_4(\mathbb{C})$-compactification (see Section 6 below) which involves Type-II solution of (1.1). To our knowledge, these are the first examples of Type-II solution in the literature. Note that both examples are also $K$-unstable.

It is of great interest to discover the limits of the Kähler-Ricci flow (1.1) on $K$-unstable Fano group compactifications. According to [24], such a limit should be a $\mathbb{Q}$-Fano variety, admitting (weak) Kähler-Ricci solitons. Recall that a Kähler-Ricci soliton on a complex manifold $M$ is a pair $(X, \omega)$, where $X$ is a holomorphic vector field on $M$ and $\omega$ is a Kähler metric on $M$, such that

\[ \text{Ric}(\omega) - \omega = L_X(\omega), \quad (1.2) \]

where $L_X(\cdot)$ is the Lie derivative along $X$. If $X = 0$, the Kähler-Ricci soliton becomes a Kähler-Einstein metric. The uniqueness theorem in [30] states that a Kähler-Ricci soliton on a compact complex manifold, if it exists, must be unique modulo $\text{Aut}(M)$. Furthermore, $X$ lies in the center of Lie algebra of a maximal reductive subgroup $\text{Aut}_r(M) \subset \text{Aut}(M)$.

It was first showed in [24, Section 7] that the limits of (1.1) on the two smooth $K$-unstable $SO_4(\mathbb{C})$-compactifications can not be $SO_4(\mathbb{C})$-compactifications any more. In fact, this is the case for Fano compactifications of semisimple groups in general. To see this, note that on a compactification of semisimple group, the center of $\text{Aut}_r(M)$ is trivial. Hence any Kähler-Ricci soliton must be a Kähler-Einstein metric. On the other hand, it was showed in [20] that for any semisimple group $G$, the $\mathbb{Q}$-Fano $G$-compactification that admit Kähler-Einstein metrics are finite. There are of course infinitely many $\mathbb{Q}$-Fano compactifications of $G$. Hence in general the Kähler-Ricci flow (1.1) converges to a limit which is no longer a $G$-compactification.

On the other hand, Chen-Sun-Wang [9] showed the following phenomena: in general, $M$ may be degenerated to $M_\infty$ via two-step degenerations: first by a “semistable degeneration” to a normal variety $X$ with a holomorphic vector field $\Lambda$, and then a “polystable degeneration” which degenerates $(X, \Lambda)$ to $(M_\infty, \Lambda)$. Moreover, the soliton vector field on $M_\infty$ is precisely $\Lambda$ arises in the “semistable degeneration”. It is conjectured that there is an algebraic way to determine these two-step degenerations.

One approach to determining the “semistable degeneration” is minimizing the H-invariant. The H-invariant was first introduced by Tian-Zhang-Zhang-Zhu [28, Section 5] for holomorphic vector fields in the study of (1.1). Dervan-Székelyhidi [13] generalized it to special $\mathbb{R}$-test configurations. Very recently, Han-Li [17] provided a more precise expression of the H-invariant (cf. [17, Remark 2.41]) of any $\mathbb{R}$-test
configuration. Assuming the existence of a special minimizer of the H-invariant, [17] proved its uniqueness. Thus on a Fano manifold, “semistable degeneration” is the unique special R-test configuration which minimizes the H-invariant. Furthermore, the central fibre (X, Λ) is (modified) K-semistable (cf. [17] Theorem 1.2). We will call it the “semistable limit” in the following. Also, it is proved by [19] that (X, Λ) admits a unique “polystable degeneration” whose centre X₀ is modified K-polystable with respect to Λ. Hence they proved that the two degenerations in [9], and consequently the Gromov-Hausdorff limit M∞ for (1.1), depend only on the algebraic structure of M (cf. [17, Corollary 1.4]). A bit later Blum-Liu-Xu-Zhuang [3] gives an algebraic proof of the Hamilton-Tian conjecture for general log Fano pairs. By using a finite generation result, [3, Theorem 1.2] proved that the H-invariant admits a unique minimizer among a larger class of filtrations defined by valuation. Moreover, the minimizer is a special R-test configuration. In particular this proves the existence and uniqueness of “semistable degeneration”. The proof in [3] uses deep results and abstract constructions in birational geometry from the former works of its authors.

In this paper, we will apply the above algebraic approach to find the limit of (1.1) on Fano group compactifications. In particular, we will find the limit of (1.1) on the two Fano SO₄(C)-compactifications given in [23].

Let us recall the conception of group compactifications. Suppose that G is an n-dimensional connect, complex reductive group which is the complexification of a compact Lie group K. Let M be a projective normal variety. M is called a (bi-equivariant) compactification of G (or G-compactification for simplicity) if it admits a holomorphic G × G-action with an open and dense orbit isomorphic to G as a G × G-homogeneous space. (M, L) is called a polarized compactification of G if L is a G × G-linearized Q-Cartier ample line bundle on M. In particular, when K⁻¹ is an ample Q-Cartier line bundle and L = K⁻¹, we call M a Q-Fano G-compactification. For more knowledge and examples, we refer the reader to [42, 2, 11], etc.

As mentioned above, the minimizer of the H-invariant is special. Hence for our purpose, we first study the G × G-equivariant normal R-test configurations with reduced central fibre. We have the following classification result.

**Theorem 1.2.** Let (M, L) be a polarized G-compactification with moment polytope P⁺. Then the G × G-equivariant R-test configurations of (M, L) with reduced central fibre are in one-to-one correspondence with W-invariant, concave, piecewise-linear functions on

\[ \mathcal{F} = \bigcup_{w \in W} w(P⁺) \]

whose domains of linearity consist of convex rational polytopes in \( \mathfrak{M}_G \). Here W denotes the Weyl group of G with respect to a fixed maximal torus and \( \mathfrak{M} \) denotes the lattice of characters of G.

We refer to the readers Section 2.2 for precise meaning of the notations. A more precise version of Theorem 1.2 will be proved in Section 4 (see Theorem 4.4 below). With the help of Theorem 1.2, we estimate the H-invariant of G × G-equivariant normal R-test configurations via the combinatorial data. In particular we get a precise formula for the special ones. We will see that the minimizer of the

\(^1\)Our convention of the H-invariant follows [17] and differs from [13] by a sign. See Section 2.1 for detail.
H-invariant among equivariant special $\mathbb{R}$-test configurations is unique. Combining with the arguments of uniqueness in [17, Section 6] and [4] we find the “semistable degeneration”.

**Theorem 1.3.** Let $G$ be a reductive Lie group and $M$ a $\mathbb{Q}$-Fano compactification of $G$. Then there is a unique $G \times G$-equivariant special $\mathbb{R}$-test configuration $\mathcal{F}_0$ such that the $H$-invariant
\begin{equation}
H(\mathcal{F}_0) \leq H(\mathcal{F}), \quad \forall \ G \times G\text{-equivariant normal } \mathbb{R}\text{-test configuration } \mathcal{F}.
\end{equation}
Moreover, $\mathcal{F}_0$ is the “semistable degeneration” of $M$.

Theorem 1.3 will be proved in Section 5.2. Next we check the (modified) $K$-(poly)stability of the central fibre $X_0$ of $\mathcal{F}_0$. At least for the two $K$-unstable Fano $SO_4(\mathbb{C})$-compactifications, the corresponding central fibres $X_0$ are indeed (modified) $K$-polystable. This suggests that the “polystable degeneration” will be trivial and $X_0$ is actually the limit $M_\infty$ of (1.1) on these two examples. Namely,

**Theorem 1.4.** Let $M$ be a smooth $K$-unstable Fano $SO_4(\mathbb{C})$-compactification. Then the semistable limit of $M$ coincides with the limiting space of (1.1).

We will give the limits in Section 6 in terms of the combinatorial data. Moreover, we will see that the limit of (1.1) on a group compactification is always spherical, although possibly may not be a group compactification. This suggests that the class of spherical varieties forms a “closed” class when considering the moduli problem, while the class of group compactifications usually does not.

The paper is organized as follows: in Section 2 we recall $R$-test configurations, $H$-invariant as well as theory of spherical varieties. In Section 3 we overview the usual $G \times G$-equivariant normal $\mathbb{Z}$-test configurations of a $G$-compactification. In particular we study the structure of the corresponding central fibres. In Section 4 we classify the $G \times G$-equivariant normal $\mathbb{R}$-test configurations by a purely algebraic argument. In particular we classify those with reduced central fibre. In Section 5 we estimate the $H$-invariant of $G \times G$-equivariant normal $\mathbb{R}$-test configurations via the combinatorial data (see Theorem 5.1). Then we find the minimizer of $H$-invariant and prove Theorem 1.3. Finally we test the $K$-stability of the central fibre (see Proposition 5.5). In Section 6 we apply the above results to smooth $K$-unstable Fano $SO_4(\mathbb{C})$-compactifications and prove Theorem 1.4.

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## 2. Preliminaries

### 2.1. Filtrations and test configurations.
In this section we recall some basic material concerning filtrations and test configurations. We refer to the readers [13, Section 2.2] and [17, Section 2] for further knowledge.

Let $M$ be a projective variety and $L$ a very ample line bundle over $M$ so that $|L|$ gives a Kodaira embedding of $M$ into projective space. The homogenous coordinate ring (Kodaira ring) of $M$ is $R(M, L) = \oplus_{k \in \mathbb{N}} R_k$, where $R_k = H^0(M, L^k)$. 

Definition 2.1. A filtration $\mathcal{F}$ of $R$ is a family of subspaces $\{\mathcal{F}^s R_k\}_{s \in \mathbb{R}, k \in \mathbb{N}}$ of $R(M, L) = \bigoplus_{k \in \mathbb{N}} R_k$ such that

1. $\mathcal{F}$ is decreasing: $\mathcal{F}^s_1 R_k \subseteq \mathcal{F}^s_2 R_k$, $\forall s_1 \geq s_2$ and $k \in \mathbb{N}$;
2. $\mathcal{F}$ is left-continuous: $\mathcal{F}^s R_k = \bigcap_{t < s} \mathcal{F}^t R_k$, $\forall k \in \mathbb{N}$;
3. $\mathcal{F}$ is linearly bounded: There are constants $c_\pm \in \mathbb{Z}$ such that for each $k \in \mathbb{N}$, $\mathcal{F}^s R_k = 0$, $\forall s > c_+ k$ and $\mathcal{F}^s R_k = R_k$, $\forall s < c_- k$;
4. $\mathcal{F}$ is multiplicative: $\mathcal{F}^{s_1} R_{k_1} \cdot \mathcal{F}^{s_2} R_{k_2} \subseteq \mathcal{F}^{s_1 + s_2} R_{k_1 + k_2}$, for all $k_1, k_2 \in \mathbb{N}$ and $s_1, s_2 \in \mathbb{R}$.

Let $\Gamma(\mathcal{F}, k)$ be the set of values of $s$ where the filtration of $R_k$ is discontinuous. Set

\[ \Gamma_+(\mathcal{F}) := \bigcup_{k \in \mathbb{N}} (\Gamma(\mathcal{F}, k) - \min \Gamma(\mathcal{F}, k)), \]

and $\Gamma(\mathcal{F})$ the Abelian group generated by $\Gamma_+(\mathcal{F})$. We associate to each filtration $\mathcal{F}$ two graded algebras,

Definition 2.2. (1) The Rees algebra,

\[ R(\mathcal{F}) := \bigoplus_{k \in \mathbb{N}} \bigoplus_{s \in \Gamma(\mathcal{F}) + \min \Gamma(\mathcal{F}, k)} t^{-s} \mathcal{F}^s R_k, \]

and

(2) The associated graded ring of $\mathcal{F}$,

\[ \text{Gr}(\mathcal{F}) := \bigoplus_{k \in \mathbb{N}} \bigoplus_{s \in \Gamma(\mathcal{F}) + \min \Gamma(\mathcal{F}, k)} t^{-s} (\mathcal{F}^s R_k / \mathcal{F}^{s \geq} R_k). \]

There is an important class of filtrations, called the $\mathbb{R}$-test configurations, which can be considered as a generalization of the usual ($\mathbb{Z}$-)test configuration introduced in [10].

Definition 2.3. When $R(\mathcal{F})$ is finitely generated, we say that $\mathcal{F}$ is an $\mathbb{R}$-test configuration of $(M, L)$. In this case, $\text{Gr}(\mathcal{F})$ is also finitely generated. The projective scheme

\[ X_0 := \text{Proj}(\text{Gr}(\mathcal{F})) \]

is called the central fibre of $\mathcal{F}$.

When $\mathcal{F}$ is an $\mathbb{R}$-test configuration, the Abelian group $\Gamma(\mathcal{F})$ generated by (2.1) has finite rank. Denote its rank by $r_\mathcal{F}$. Then the total space $\mathcal{X}$ of $\mathcal{F}$ is

\[ \mathcal{X} = \text{Proj}_{\mathbb{C}^r\mathcal{F}}(R(\mathcal{F})). \]

Also the $\Gamma(\mathcal{F})$-grading of $\text{Gr}(\mathcal{F})$ corresponds to a (possibly real) holomorphic vector field $X$ on $X_0$, which generates a rank $r_\mathcal{F}$ torus (denote by $\mathbb{T}$) action. Note that we take the convention that the exp $(tX)$-action has weight $t^s$ on the $(\mathcal{F}^s R_k / \mathcal{F}^{s \geq} R_k)$-piece in (2.3). We call $X$ the vector field induced by $\mathcal{F}$.

Remark 2.4. By finite generation, $\mathcal{F}$ is generated by $\mathcal{F} R_{k_0}$ for some $k_0 \in \mathbb{N}_+$. By a shifting of $\mathcal{F}$, we can always normalize $\min \Gamma(\mathcal{F}, k_0) = 0$. Then $\Gamma(\mathcal{F})$ contains all points of discontinuity of $\mathcal{F}$. In the following, all $\mathbb{R}$-test configurations are assumed to be normalized in this way.

\[ \text{This differs from [17, Definition 2.12] by a sign.} \]
Remark 2.5. When \( \text{rank}(\Gamma(\mathcal{F})) = 1 \), we can embed \( \Gamma(\mathcal{F}) \) in \( \mathbb{Z} \). The above \( \mathbb{R} \)-test configuration is simply the usual \( \mathbb{Z} \)-test configuration introduced in [10, Definition 2.1].

There is an important subclass of normal \( \mathbb{R} \)-test configurations,

Definition 2.6. When \( L = K^{-1}_M \), an \( \mathbb{R} \)-test configuration \( \mathcal{F} \) is called special if the central fibre \( \mathcal{X}_0 \) is \( \mathbb{Q} \)-Fano and \( \text{Gr}(\mathcal{F}) \) is isomorphic to \( R(\mathcal{X}_0, -K_{\mathcal{X}_0}) \), the Kodaira ring of \( \mathcal{X}_0 \).

2.1.1. Equivariant \( \mathbb{R} \)-test configurations. Let \((M, L)\) be a polarized variety with a group \( \mathcal{G} \)-action. Then \( \mathcal{G} \)-acts on its Kodaira ring. Let \( \mathcal{F} \) be a filtration on \( R \). Define the action of \( \mathcal{G} \) on \( \mathcal{F} \) by

\[
(\sigma \cdot \mathcal{F})^s R_k := \sigma(\mathcal{F}^s R_k), \quad \forall s \in \mathbb{R}, k \in \mathbb{N},
\]

for any \( \sigma \in \mathcal{G} \). Clearly \( \sigma \cdot \mathcal{F} \) is also a filtration on \( R \), and it an \( \mathbb{R} \)-test configuration if and only if \( \mathcal{F} \) is. As a generalization of equivariant \( \mathbb{Z} \)-test configurations, we define

Definition 2.7. A filtration \( \mathcal{F} \) is called \( \mathcal{G} \)-equivariant if \( \mathcal{F}^s R_k \) is a \( \mathcal{G} \)-invariant space of \( R_k \) for any \( s \in \mathbb{R} \) and \( k \in \mathbb{N} \). That is

\[
\sigma(\mathcal{F}^s R_k) = \mathcal{F}^s R_k, \quad \forall s \in \mathbb{R}, k \in \mathbb{N}.
\]

As in the case of equivariant \( \mathbb{Z} \)-test configurations, it is clear that when \( \mathcal{F} \) is a \( \mathcal{G} \)-equivariant \( \mathbb{R} \)-test configuration, the projection from \( \mathcal{X} \) to the base \( \mathbb{C}^r \) is \( \mathcal{G} \)-equivariant. Note that here \( \mathcal{G} \) acts on \( \mathbb{C}^r \) trivially.

2.1.2. Filtrations and semi-valuations. Let \( \mathcal{F} \) be a filtration on \( R \). For any \( \sigma_k \in R_k, \ k \in \mathbb{N}_+ \), set

\[
\nu_{\mathcal{F}}(\sigma_k) := \max\{s | \sigma_k \in \mathcal{F}^s R_k \},
\]

and for any \( \sigma := \sum_{k \in \mathbb{N}_+} \sigma_k \) with \( (0 \neq) \sigma_k \in R_k \), set

\[
\nu_{\mathcal{F}}(\sigma) := \min\{s | \nu_{\mathcal{F}}(\sigma_k) | 0 \neq \sigma_k \in R_k \}.
\]

Then \( \nu_{\mathcal{F}}(\cdot) \) defines a semi-valuation on \( R \) (cf. [17, Section 2.2]). Conversely, given any valuation \( \nu \) with finite log-discrepancy, there is a filtration \( \mathcal{F}_{(\nu)} \) on \( R \) by (cf. [17, Example 2.2]) satisfying the above two relations,

\[
\mathcal{F}_{(\nu)} R_k := \{\sigma \in R_k | \nu(\sigma) \geq s\}.
\]

It is known that when \( \text{Gr}(\mathcal{F}) \) is an integral domain, \( \nu_{\mathcal{F}} \) is a valuation and \( \mathcal{F} = \mathcal{F}_{(\nu_{\mathcal{F}})} \), up to a shifting. In particular, this applies to special \( \mathbb{R} \)-test configurations (cf. [17, Lemma 2.11]).

2.2. The \( H \)-invariant. Given a \( \mathbb{Q} \)-Fano variety \( M \) and \( \mathcal{F} \) a special test configuration of \((M, K^{-1}_M)\). Let \( \mathcal{X}_0 \) be the central fibre and \( X \) the vector field induced by \( \mathcal{F} \). Assume that \( \mathcal{X}_0 \) has klt-singularities. Choose any smooth Kähler metric \( \omega \in 2\pi c_1(\mathcal{X}_0) \) and fix a normalized Ricci potential \( h \) of \( \omega \). Let \( \theta_X(\omega) \) be any potential of \( X \) with respect to \( \omega \). Tian-Zhang-Zhang-Zhu [28, Section 5] defined the following \( H \)-invariant of \( \mathcal{F} \),

\[
H(\mathcal{F}) = V \ln \left( \frac{1}{V} \int_{\mathcal{X}_0} e^{\theta_X(\omega) n} \right) - \int_{\mathcal{X}_0} \theta_X(\omega) e^h n,
\]
Note that $H(\mathcal{F})$ is well-defined since different choices of $\theta_X(\omega)$ only differs from each other by a constant.

Let $\mathcal{F}$ be an $\mathbb{R}$-test configuration of $(M, K_M^{-1})$. Dervan-Székelyhidi [13] generalized (2.7) to $\mathbb{R}$-test configurations via an algebraic aspect. This is recently modified by Han-Li [17]. Recall the definition in [17, 2.5]. First we associate to any $\mathbb{R}$-test configuration $\mathcal{F}$ two non-Archimedean functionals. Let $\mathcal{M}_{\mathbb{Q}}^{\text{div}}$ be the set of $\mathbb{Q}$-divisorial valuations of $M$. Denote by $\phi_F, \phi_0$ the non-Archimedean metric of $\mathcal{F}$ and the trivial test configuration (cf. [5, Section 6]), respectively. Set

$$L^{\text{NA}}(\mathcal{F}) := \inf_{v \in \mathcal{M}_{\mathbb{Q}}^{\text{div}}} \left( A_M(v) + (\phi_F - \phi_0)(v) \right),$$

where $A_M(\cdot)$ is the log-discrepancy of a divisor of $M$. Also, denote by $\Delta(\mathcal{F}(t))$ the Okounkov body of the linear series (cf. [17, Section 2.4]),

$$\mathcal{F}(t) := \{ F_{tk} \}_{k \in \mathbb{N}^+}.$$  

By Definition (3), we see that when $t \ll 0$, $\Delta(\mathcal{F}(t))$ is just the Okounkov body $\Delta$ of $(M, K_M^{-1})$ introduced in [40], and $\Delta(\mathcal{F}(t)) = \{ O \}$ when $t \gg 1$. Define a function $G_F : \Delta \to \mathbb{R}$ by

$$G_F(z) := \sup \{ t \mid z \in \Delta(\mathcal{F}(t)) \},$$

and set

$$S^{\text{NA}}(\mathcal{F}) := - \ln \left( \frac{n!}{V} \int_{\Delta} e^{-G_F(z)} dz \right).$$

The H-invariant is given by

**Definition 2.8.** Let $\mathcal{F}$ be an $\mathbb{R}$-test configuration of $(M, K_M^{-1})$. Then the H-invariant of $\mathcal{F}$ is

$$H(\mathcal{F}) := L^{\text{NA}}(\mathcal{F}) - S^{\text{NA}}(\mathcal{F}).$$

[17, Example 2.32] shows that (2.12) coincides with (2.7) on special test configurations.

2.2.1. Equivariance of the minimizer. Let $M$ be a $\mathbb{Q}$-Fano variety with a $\mathfrak{G}$-action. We want to show that the H-invariant is $\mathfrak{G}$-invariant and derive the equivariance of its minimizer. First, we have

**Proposition 2.9.** Let $M$ be a $\mathbb{Q}$-Fano variety with a $\mathfrak{G}$-action and $\mathcal{F}$ be an $\mathbb{R}$-test configuration on $K_M^{-1}$. Then

$$H(\sigma \cdot \mathcal{F}) = H(\mathcal{F}), \forall \sigma \in \mathfrak{G}. $$

**Proof.** Recall that (2.12). It suffices to prove both $L^{\text{NA}}(\sigma \cdot \cdot )$ and $S^{\text{NA}}(\cdot )$ are $\mathfrak{G}$-invariant. By (2.5), since $(\sigma \cdot F)^*R_k$ is always isomorphic to $F^*R_k$, in particular, the linear series $(\sigma \cdot F)(t) \cong F(t)$, we have

$$S^{\text{NA}}(\sigma \cdot \mathcal{F}) = S^{\text{NA}}(\mathcal{F}).$$

Recall (2.12). Note that group $\mathfrak{G}$ can act on $\mathcal{M}_{\mathbb{Q}}^{\text{div}}$ by

$$(\sigma \cdot v)(f) = v(f(\sigma^{-1})).$$

for any rational function $f$ on $M$. Thus by the definition of non-Archimedean metric (cf. [17, Definition 2.17]), the second term in (2.8) satisfies that

$$(\phi_{\sigma \cdot \mathcal{F}} - \phi_{\text{triv}})(\sigma \cdot v) = (\phi_{\mathcal{F}} - \phi_{\text{triv}})(v)$$
for any valuation \( v \).

It remains to check
\[
A_M(v) = A_M(\sigma \cdot v).
\]
Suppose that \( \mu : X \to M \) is a resolution such that \( v = \text{cord}_D(\cdot) \) for some \( c \in \mathbb{Q}_+ \) and \( D \) a prime divisor of \( X \), i.e., \( v(f) = \text{cord}_D(\mu^* \circ f) \). Then
\[
A_M(v) = c(1 + \text{ord}_D(K_X - \mu^* K_M)).
\]
Now \( (\sigma \cdot v)(f) = \text{cord}_D(\mu^* \sigma^{-1}* \circ f) \). So for morphism \( \mu_1 = \sigma^{-1} \circ \mu \), we have
\[
(\sigma \cdot v)(f) = \text{ord}_D(\mu_1^* \circ f).
\]
So
\[
A_M(\sigma \cdot v) = c(1 + \text{ord}_D(K_X - \mu_1^* K_M)).
\]
Since \( \sigma^{-1}* K_M = K_M \), we can compute
\[
A_M(\sigma \cdot v) = A_M(\sigma \cdot v) = c(1 + \text{ord}_D(K_X - \mu^* \sigma^{-1}* K_M)) = A_M(v).
\]
Now we have
\[
L^\text{NA}(\sigma \cdot F) = \inf_{v \in M^\text{vir}_{\mathbb{Q}}} (A_M(v) + (\phi_{\sigma,F} - \phi_{\text{triv}})(v))
\]
\[
= \inf_{v \in M^\text{vir}_{\mathbb{Q}}}(A_M(\sigma \cdot v) + (\phi_{\sigma,F} - \phi_{\text{triv}})(\sigma \cdot v))
\]
\[
= \inf_{v \in M^\text{vir}_{\mathbb{Q}}}(A_M(v) + (\phi_{F} - \phi_{\text{triv}})(v)) = L^\text{NA}(F)
\]
and we get the Proposition.

\[\square\]

To prove the equivariance of the minimizer of \( H(\cdot) \), we need the uniqueness theorem. The uniqueness of the minimizer has been proved in [17, Section 6] provided \( H(\cdot) \) has a special minimizer. This is the case when \( M \) is smooth. In this case, it is proved in [9, Section 3] that the semistable limit \((X, \Lambda)\) introduced on pp.2-3 already gives a special \( \mathbb{R} \)-test configuration \( F_{\text{min}} \) which is a minimizer of \( H(\cdot) \). Thus by [17, Section 6] \( F_{\text{min}} \) is the unique minimizer of \( H(\cdot) \). Later [4, Theorem 1.2] proved the uniqueness for general \( \mathbb{Q} \)-Fano varieties without the assumption of the existence of a special minimizer. It is proved by [4, Theorem 1.2] that on a \( \mathbb{Q} \)-Fano variety \( H(\cdot) \) always admits a unique minimizer \( F_{\text{min}} \) and \( F_{\text{min}} \) must be a special \( \mathbb{R} \)-test configuration.

**Corollary 2.10.** Let \( M \) be a \( \mathbb{Q} \)-Fano variety. Then the minimizer of \( H(\cdot) \) is \( \mathfrak{G} \)-equivariant.

**Proof.** Suppose that \( F_{\text{min}} \) is a minimizer of \( H(\cdot) \). By [2.13], for any \( \sigma \in \mathfrak{G} \), \( H(\sigma \cdot F_{\text{min}}) = H(F_{\text{min}}) \). By the uniqueness, \( \sigma \cdot F_{\text{min}} = F_{\text{min}} \) for any \( \sigma \in \mathfrak{G} \). Hence we get [2.6] and \( F_{\text{min}} \) is \( \mathfrak{G} \)-equivariant. \[\square\]

**2.3. Spherical varieties.** In the following we overview the theory of spherical varieties. The origin goes back to [25]. We use [18, 33] as main references.

**Definition 2.11.** Let \( \mathcal{G} \) be a connected, complex reductive group. A normal variety \( M \) equipped with a \( \mathcal{G} \)-action is called a \((\mathcal{G})\)-spherical variety if there is a Borel subgroup \( \mathcal{B} \) of \( \mathcal{G} \) acts on \( M \) with an open dense orbit.
In particular, if a subgroup \( \hat{H} \subset \hat{G} \) is called a spherical subgroup if \( \hat{G}/\hat{H} \) is a spherical variety (referred as a spherical homogenous space). For an arbitrary spherical variety \( M \), let \( x_0 \) be a point in the open dense \( \hat{B} \)-orbit. Set \( \hat{H} = \text{Stab}_{\hat{G}}(x_0) \), the stabilizer of \( x_0 \) in \( \hat{G} \). Then \( \hat{H} \) is spherical and we call \((M, x_0)\) a spherical embedding of \( \hat{G}/\hat{H} \).

2.3.1. The coloured data. Let \( \hat{H} \) be a spherical subgroup of \( \hat{G} \) with respect to the Borel subgroup \( \hat{B} \). The action of \( \hat{G} \) on the function field \( \mathbb{C}(\hat{G}/\hat{H}) \) of \( \hat{G}/\hat{H} \) is given by

\[
(g^*f)(x) := f(g^{-1} \cdot x), \ \forall g \in \hat{G}, x \in \mathbb{C}(\hat{G}/\hat{H}) \text{ and } f \in \mathbb{C}(\hat{G}/\hat{H}).
\]

A function \( f(\neq 0) \in \mathbb{C}(\hat{G}/\hat{H}) \) is called \( \hat{B} \)-semi-invariant if there is a character of \( \hat{B} \), denote by \( \chi \) so that \( b^*f = \chi(b)f \) for any \( b \in \hat{B} \). Note that there is an open dense \( \hat{B} \)-orbit in \( \hat{G}/\hat{H} \). Two \( \hat{B} \)-semi-invariant functions associated to a same character can differ from each other only by multiplying a non-zero constant.

Let \( \mathfrak{M}(\hat{G}/\hat{H}) \) be the lattice of \( \hat{B} \)-characters which admits associated \( \hat{B} \)-semi-invariant functions, and \( \mathfrak{M}(\hat{G}/\hat{H}) = \text{Hom}_{\mathbb{Z}}(\mathfrak{M}(\hat{G}/\hat{H}), \mathbb{Z}) \) its \( \mathbb{Z} \)-dual. There is a map \( \phi \) which maps a valuation \( \nu \) of \( \mathbb{C}(\hat{G}/\hat{H}) \) to an element \( \phi(\nu) \) in \( \mathfrak{M}_{\mathbb{Q}}(\hat{G}/\hat{H}) = \mathfrak{M}(\hat{G}/\hat{H}) \otimes_{\mathbb{Z}} \mathbb{Q} \) by

\[
\phi(\nu)(\chi) = \nu(f),
\]

where \( f \) is a \( \hat{B} \)-semi-invariant functions associated to \( \chi \). Again, this is well-defined since \( \hat{G}/\hat{H} \) is spherical with respect to \( \hat{B} \). It is a fundamental result that \( \phi \) is injective on \( \mathfrak{M}(\hat{G}/\hat{H}) \)-invariant valuations and the image forms a convex cone \( \nu(\hat{G}/\hat{H}) \) in \( \mathfrak{M}_{\mathbb{Q}}(\hat{G}/\hat{H}) \), called the valuation cone of \( \hat{G}/\hat{H} \) (cf. [33, Section 19]). Moreover, \( \nu(\hat{G}/\hat{H}) \) is a solid cosimplicial cone which is a (closed) fundamental chamber of a certain crystallographic reflection group, called the little Weyl group (denoted by \( \hat{W} \); cf. [33, Sections 22]). In fact, \( \hat{W} \) is the Weyl group of the spherical root system \( \Phi(\hat{G}/\hat{H}) \) of \( \hat{G}/\hat{H} \) (cf. [33, Section 30]).

Example 2.12. Let \( G \) be a connected, complex reductive group. Let \( B^+ \subset G \) be a Borel subgroup of \( G \) and \( B^- \) be its opposite group. Take \( \hat{G} = G \times G, \hat{H} = \text{diag}(G) \) and \( \hat{B} = B^+ \times B^- \). Then by the well-known Bruhat decomposition, \( \hat{H} \) is a spherical subgroup of \( \hat{G} \). Hence a \( G \)-compactification is a \( \hat{G} \)-spherical variety. By taking an involution on \( G \)

\[
\sigma(g_1, g_2) = (g_2, g_1) \ \forall g_1, g_2 \in G,
\]

we see that \( \hat{H} = \hat{G}^\sigma \). Thus \( G \cong \hat{G}/\hat{H} \) is in addition a symmetric space.

Let us fix the maximal complex torus \( T^C = B^+ \cap B^- \) of \( G \). Denote by \( \Phi \) the root system of \( (G, T^C) \) and \( \Phi_+ \subset \Phi \) the positive roots with respect to \( B^+ \). Let \( W \) be the corresponding Weyl group. By a direct computation we can identify \( \mathfrak{M}(\hat{G}/\hat{H}) \) with the lattice of weights \( \mathfrak{M}(G) \) of \( G \) via the anti-diagonal embedding,

\[
\mathfrak{M}(G) \rightarrow \mathfrak{M}(\hat{G}/\hat{H})
\]

\[
\chi \mapsto (\chi, -\chi).
\]

Further calculation shows that under this identification the spherical root system \( \Phi(\hat{G}/\hat{H}) \) is identified with \( \Phi \). Consequently, \( W \cong W \). Hence we identify \( \mathfrak{M}(\hat{G}/\hat{H}) \) with the anti-dominant (closed) Weyl chamber of \( W \) in \( \mathfrak{a} := \mathfrak{M}(G) \) (cf. [12, Sections 4-8]),

\[
-\mathfrak{a}^+ := \{ y \in \mathfrak{a} | \alpha(y) \leq 0, \ \forall \alpha \in \Phi_+ \}.
\]
A \( \hat{B} \)-stable prime divisors in \( \hat{G}/\hat{H} \) is called a colour. Denote by \( D_{\hat{B}}(\hat{G}/\hat{H}) \) the set of colours. A colour \( D \in D_{\hat{B}}(\hat{G}/\hat{H}) \) also defines a valuation on \( \hat{G}/\hat{H} \). However, the restriction of \( \varrho \) on \( D_{\hat{B}}(\hat{G}/\hat{H}) \) is usually non-injective.

**Example 2.13.** Let \( \hat{H} \supset \hat{B} \) be a parabolic subgroup of \( \hat{G} \). Then \( \varrho \) vanishes on \( D_{\hat{B}}(\hat{G}/\hat{H}) \).

It is a fundamental result that the spherical embeddings of a given \( \hat{G}/\hat{H} \) are classified by combinatorial data called the coloured fan [25].

**Definition 2.14.** Let \( \hat{H} \) be a spherical subgroup of \( \hat{G} \), \( D_{\hat{B}}(\hat{G}/\hat{H}), \mathcal{V}(\hat{G}/\hat{H}) \) be the set of colours and the valuation cone, respectively.

- A coloured cone is a pair \((C, R)\), where \( R \subset D_{\hat{B}}(\hat{G}/\hat{H}), O \notin \varrho(R) \) and \( C \subset \mathcal{R}_0(\hat{G}/\hat{H}) \) is a strictly convex cone generated by \( \varrho(R) \) and finitely many elements of \( \mathcal{V}(\hat{G}/\hat{H}) \) such that the intersection of the relative interior of \( C \) with \( \mathcal{V}(\hat{G}/\hat{H}) \) is non-empty;
- Given two coloured cones \((C, R)\) and \((C', R')\), We say that a coloured cone \((C', R')\) is a face of another coloured cone \((C, R)\) if \( C' \) is a face of \( C \) and \( R' = R \cap \varrho^{-1}(C') \);
- A coloured fan is a collection \( F \) of finitely many coloured cones such that the face of any its coloured cone is still in it, and any \( v \in \mathcal{V}(\hat{G}/\hat{H}) \) is contained in the relative interior of at most one of its cones.

Now we state the classification theorem of spherical embeddings (cf. [18, Theorem 3.3]),

**Theorem 2.15.** There is a bijection \((M, x_0) \to F_M\) between spherical embeddings of \( \hat{G}/\hat{H} \) up to \( \hat{G} \)-equivariant isomorphism and coloured fans. There is a bijection \( \mathcal{Y} \to (C_{\mathcal{Y}}, R_{\mathcal{Y}}) \) between the \( \hat{G} \)-orbits in \( M \) and the coloured cones in \( F_M \). An orbit \( \mathcal{Y} \) is in the closure of another orbit \( \mathcal{Z} \) in \( M \) if and only if the coloured cone \((C_{\mathcal{Z}}, R_{\mathcal{Z}})\) is a face of \((C_{\mathcal{Y}}, R_{\mathcal{Y}})\).

2.3.2. **Line bundles and polytopes.** Let \( M \) be a complete spherical variety, which is a spherical embedding of some \( \hat{G}/\hat{H} \). Let \( L \) be a \( \hat{G} \)-linearized line bundle on \( M \). In the following we will associated to \((M, L)\) several polytopes, which encode the geometric structure of \( M \).

**Moment polytope of a line bundle.** Let \((M, L)\) be a polarized spherical variety. Then for any \( k \in \mathbb{N} \) we can decompose \( H^0(M, L^k) \) as direct sum of irreducible \( \hat{G} \)-representations,

\[
H^0(M, L^k) = \bigoplus_{\lambda \in P_{+, k}} \hat{V}_\lambda,
\]

where \( P_{+, k} \) is a finite set of \( \hat{B} \)-weights and \( \hat{V}_\lambda \) the irreducible representation of \( \hat{G} \) with highest weight \( \lambda \). Set

\[
P_+ := \bigcup_{k \in \mathbb{N}} (\frac{1}{k} P_{+, k}).
\]

Then \( P_+ \) is indeed a polytope in \( \mathfrak{M}_0(\hat{G}/\hat{H}) \). We call it the **moment polytope** of \((M, L)\). Clearly, the moment polytope of \((M, L^k)\) is \( k \)-times the moment polytope of \((M, L)\) for any \( k \in \mathbb{N}_+ \).
Polytope of a divisor. Denote by \( I_\hat{G}(M) = \{ D_A | A = 1, \ldots, d_0 \} \) the set of \( \hat{G} \)-invariant prime divisors in \( M \). Then any \( D_A \in I_\hat{G}(M) \) corresponds to a 1-dimensional cone \((C_A, \emptyset) \in F_M\). Denote by \( u_A \) the prime generator of \( C_A \). Recall that \( D_B(\hat{G}/\hat{H}) \) is the set of colours, which are \( B \)-stable but not \( \hat{G} \)-stable in \( M \). Any \( B \)-stable \( \mathbb{Q} \)-Weil divisor can be written as

\[
\mathfrak{d} = \sum_{A=1}^{d_0} \lambda_A D_A + \sum_{D \in D_B(\hat{G}/\hat{H})} \lambda_D D
\]

for some \( \lambda_A, \lambda_D \in \mathbb{Q} \). Set

\[
D_M := \emptyset \cup \{ R \subset D_B(\hat{G}/\hat{H}) | \exists (C, R) \in F_M \}
\]

By [6] Proposition 3.1, \( \mathfrak{d} \) is further a \( \mathbb{Q} \)-Cartier divisor if and only if there is a rational piecewise linear function \( l_\emptyset(\cdot) \) on \( F_M \) such that

\[
\lambda_A = l_\emptyset(u_A), \quad A = 1, \ldots, d_0 \quad \text{and} \quad \lambda_D = l_\emptyset(\emptyset(D)), \quad \forall D \in D_M.
\]

It is further proved in [6] Section 3 that when \( \mathfrak{d} \) is ample \( l_\emptyset(-x) : \mathcal{M}_\emptyset(\hat{G}/\hat{H}) \rightarrow \mathbb{R} \) is the support function of some convex polytope \( \Delta_\emptyset \). We call the \( \Delta_\emptyset \) the polytope of \( \mathfrak{d} \).

Suppose that \( s \) is a \( \hat{B} \)-semi-invariant section of \( L \) with respect to a character \( \chi \). Let \( \mathfrak{d} \) be the divisor of \( s \). We have

**Proposition 2.16.** ([6] Proposition 3.3) The two polytopes \( P_+ \) and \( \Delta_\emptyset \) are related by

\[
P_+ = \chi + \Delta_\emptyset.
\]

When \( L = K_M^{-1} \), there is a divisor \( \mathfrak{d} \) of \( L \) in form of \((2.14)\),

\[
\mathfrak{d} = \sum_{A=1}^{d_0} D_A + \sum_{D \in D_B(\hat{G}/\hat{H})} n_D D,
\]

where \( n_D \) are explicitly obtained in [15]. In particular, \( n_D \equiv 2 \) for group compactifications. We associated to \( \mathfrak{d} \) one more polytope as the convex hull

\[
\Delta^*_\emptyset = \text{Conv}\{ u_A | A = 1, \ldots, d_0 \} \cup \{ \frac{\emptyset(D)}{n_D} | D \in D_B(\hat{G}/\hat{H}) \}.
\]

In particular, the coloured fan \( F_M \) can be recovered from \( \Delta^*_\emptyset \) by taking all coloured cones \( (\text{Cone}(F), \emptyset^{-1}(F)) \) for all faces \( F \) of \( \Delta^*_\emptyset \) such that \( \text{RelInt}(\text{Cone}(F)) \cap \mathcal{V}(\hat{G}/\hat{H}) \neq \emptyset \). We will use this construction in Section 3.

2.3.3. The Okounkov body of a spherical variety. For each dominant weight \( \lambda \) of \( \hat{G} \), there is a Gel’fand-Tsetlin polytope \( \Delta(\lambda) \) which has the same dimension with the maximal unipotent subgroup \( \hat{N}_u \) of \( \hat{G} \) (cf. [39]). It is known that

\[
\dim(V_\lambda) = \text{number of integral points in } \Delta(\lambda).
\]

Let \( M \) be a \( \hat{G} \)-spherical variety. It is proved in [40] that the Okounkov body \( \Delta \) of \( M \) is given by the convex hull

\[
\Delta := \text{Conv}\left( \bigcup_{k \in \mathbb{N}_+} \bigcup_{\lambda \in P_+} \mathfrak{m}(\hat{G}/\hat{H}) \left( \lambda, \frac{1}{k} \Delta(k\lambda) \right) \right) \subset \mathcal{M}_\emptyset(\hat{G}/\hat{H}) \oplus \mathbb{R}^{\dim(N_u)}.
\]
Note that the Gel’fand-Tsetlin polytope $\Delta(\hat{\lambda})$ is linear in $\hat{\lambda}$. Thus $\Delta$ is a convex polytope in $\mathfrak{M}_{R}(\hat{G}/\hat{H}) \oplus \mathbb{R}^{\dim(X_{0})}$. It is a fibration over $\mathcal{T}_{+}$ so that the fibre at each $\hat{\lambda} \in \mathcal{T}_{+} \cup \frac{1}{k} \mathfrak{M}(\hat{G}/\hat{H})$ is $\frac{1}{k} \Delta(k\hat{\lambda})$.

**Notations.** Now we fix the notations in the following sections. We denote by

- $K$: a connected, compact Lie group;
- $G = K^{C}$: the complexification of $K$, which is a complex, connected reductive Lie group;
- $J$: the complex structure of $G$;
- $T$: a fixed maximal torus of $K$ and $T^{C}$ its complexification;
- $B^{+}$: a chosen positive Borel group of $G$ containing $T^{C}$ and $B^{-}$ the opposite one;
- $\mathfrak{a} := Jt$: the non-compact part of $t^{C}$ and $\mathfrak{a}^{*}$ the dual of $\mathfrak{a}$;
- $\Phi$: the root system with respect to $G$ and $T^{C}$;
- $W$: the Weyl group with respect to $G$ and $T^{C}$;
- $\Phi_{+}$: a chosen system of positive roots in $\Phi$ determined by $B^{+}$ and $\Phi_{+,s} \subset \Phi_{+}$ the simple roots;
- $\mathfrak{a}_{+}$ and $\mathfrak{a}_{+}^{*}$: the dominant Weyl chamber with respective to $\Phi_{+}$ in $\mathfrak{a}$ and $\mathfrak{a}^{*}$, respectively;
- $(\cdot, \cdot)$: a fixed $W$-invariant inner product on $\mathfrak{a}$;
- For any dominant weight $\lambda$ of $G$, denote by $V_{\lambda}$ the irreducible representation of $G$ with highest weight $\lambda$ and $\nu_{\lambda}$ the highest weight vector. Also denote by $V_{\lambda}^{C}$ the dual representation of $V_{\lambda}$. Then $V_{\lambda}^{C}$ has a vector $\nu_{\lambda}^{C}$ of lowest weight $-\lambda$;
- $\text{Ad}_{a}(\cdot) := \sigma(\cdot)\sigma^{-1}$: the conjugate of some subgroup or Lie algebra by some element $\sigma$.
- $\hat{G} = G \times G$, $\hat{T} = T \times T$ and $\hat{B}^{+} = B^{-} \times B^{+}$;
- $\hat{U}^{+} \subset \hat{B}^{+}$: the maximal unipotent subgroup in $\hat{B}^{+}$;
- $\hat{\Phi}$, $\hat{\Phi}_{+}$: the roots and positive roots with respect to $\hat{G}$ and $\hat{B}^{+}$, respectively;
- $\mathfrak{V}(\cdot)$: the valuation cone of some spherical homogeneous space;
- $\mathfrak{M}(\cdot)$: certain lattice of weights and $\mathfrak{M}(\cdot) = \text{Hom}_{\mathbb{Z}}(\mathfrak{M}(\cdot), \mathbb{Z})$;
- $\pi_{\nu}$: the projection from $\mathfrak{M}(T)$ to $\mathfrak{M}(\hat{G}/\hat{H})$.

### 3. Equivariant normal $\mathbb{Z}$-test configurations

In this section we overview useful results on the equivariant normal $\mathbb{Z}$-test configurations of a group compactification. Then we compute some combinatorial data of the central fibre.

#### 3.1. The classification results

The equivariant $\mathbb{Z}$-test configurations of general spherical varieties are studied in [11]. The following Proposition is a special case of [11] Theorem 3.30 for group compactifications.

**Proposition 3.1.** Let $M$ be a $\mathbb{Q}$-Fano $G$-compactification. Then for any $\Lambda \in \mathfrak{M}(T) \cap \pi_{\nu}^{-1}(\mathfrak{a}_{+})$ and $m \in \mathbb{N}_{+}$, there is a $\hat{G}$-equivariant normal test configuration $(\hat{X}, \hat{\mathcal{L}})$ of $(M, K^{M})$ with irreducible central fibre $X_{0}$. Moreover, the central fibre $X_{0}$ of $\hat{X}$ is a $\hat{G}$-equivariant embedding of $\hat{G}/H_{0}$ for some spherical subgroup $H_{0} \subset \hat{G}$ and the $\mathbb{C}^{*}$-action on $X_{0}$ is given by

$$e^{\mathcal{A}} \cdot gH_{0} = \hat{g} \Lambda(e^{\mathcal{A}})H_{0}, \ \forall e^{\mathcal{A}} \in \mathbb{C}^{*}.$$
In addition, two vectors \((\Lambda, m)\) and \((\Lambda', m)\) generate the same test configuration if \(\pi_{\nu}(\Lambda) = \pi_{\nu}(\Lambda')\).

Indeed, up to multiplying \((\Lambda, m)\) by a sufficiently divisible integer, we can do the above construction for any \(\Lambda \in \mathfrak{N}(T) \cap \pi_{\nu}^{-1}(\overline{\mathcal{F}})\) and \(m \in \mathbb{Q}_+\).

We briefly recall the construction of \([11]\). Let \(d\) be a \(B\)-invariant divisor of \(L\) and \(\Delta_0^\ast\) the corresponding polytope given by \((2.10)\). The coloured cone \(F_X\) of \(X\) consists of all cones of the following three types, which has non-empty intersection with the relative interior of \(Y(\hat{G}/\text{diag}(G))\),

\[
\begin{aligned}
(Cone(F), \varrho^{-1}(Cone(F)));
(Cone(F \cup \{(0, 1)\}), \varrho^{-1}(Cone(F \cup \{(0, 1)\})));
(Cone(F \cup \{(-\Lambda, -m)\}), \varrho^{-1}(Cone(F \cup \{(-\Lambda, -m)\})).
\end{aligned}
\]

where \(F\) runs over all faces of \(\Delta_0^\ast\). Then \(X\) is a complete spherical embedding of \((\hat{G} \times \mathbb{C}^\ast)/\text{diag}(G) \times \{e\})\) and there is a \(\hat{G} \times \mathbb{C}^\ast\)-equivariant surjective map \(\pi_X\) of \(X\) to \(\mathbb{C}P^1\). Clearly \(X_0 \coloneqq \pi_X^{-1}(0)\) corresponds to the one-dimensional coloured cone \((-\Lambda, m, 0)\). The line bundle \(L\) is defined by the \(B\)-invariant divisor

\[
\delta = m_0\left(\sum_{A=1}^{d_0} D_A \times \mathbb{C}^\ast + \sum_{D \in \mathfrak{D}(\hat{G}/\mathfrak{H})} n_D D \times \mathbb{C}^\ast + (C_0 - 2\Lambda_0(\rho))X_0,\text{red},\right)
\]

where \(m_0, C_0 \in \mathbb{N}_+\) are sufficiently divisible constants so that \(\delta\) is an ample Cartier divisor. The number \(m_0\) is called the exponent of \((X, L)\). It is also proved by \([11, 3.24]\) that \(H_0\) is a spherical subgroup of \(\hat{G}\).

We will also use the following inverse of Proposition \(3.1\) later.

**Proposition 3.2.** For any \(\hat{G}\)-equivariant normal test configuration \(X\) of \(M\) with irreducible central fibre \(X_0\), there is an integral vector \((\Lambda, 0, m)\) \(\in (\Phi_+ + \mathfrak{N}(T)) \oplus \mathfrak{N}(T) \oplus \mathbb{Z}\) such that \(X\) is constructed from \((\Lambda, 0, m)\) by using Proposition \(3.1\).

**Proof.** By gluing \(X\) with a trivial family

\[
M \times \mathbb{C} \to M
\]

along \(\mathbb{C}^\ast \subset \mathbb{C}\), we get an \(\hat{G}\)-equivariant family \(\hat{X}\)

\[
\hat{\pi} : \hat{X} \to \mathbb{C}P^1
\]

over \(\mathbb{C}P^1\) such that \(\hat{\pi}^{-1}(0) = X_0\) and \(\hat{\pi}^{-1}(t) = X\) for \(t = \infty\) and any \(t \neq 0\) in \(\mathbb{C}\).

Note that the total space \(\hat{X}\) is a \((\hat{G} \times \mathbb{C}^\ast)\)-compactification. Consider the coloured fan \(F_X\) of \(\hat{X}\). Since the central fibre \(X_0\) is irreducible, it is a single \(\hat{G} \times \mathbb{C}^\ast\)-invariant divisor, which is associated to a 1-dimensional cone in \(F_{\hat{X}}\). Let \((-\Lambda_1, -\Lambda_2, -m) \in \mathfrak{N}(T) \times \mathbb{Z}\) be the generator of this cone. Then \((\Lambda_1, \Lambda_2) \in \mathfrak{N}(T) \cap \pi_{\nu}^{-1}(\overline{\mathcal{F}})\).

Take \(\Lambda = \Lambda_1 - \Lambda_2\), it is direct to check that \(X\) can be constructed from \((\Lambda, 0, m)\) by using Proposition \(3.1\).

\(\square\)

3.2. Combinatorial data of the central fibre. In the following we will determine some combinatorial data of \(X_0\). Let \(\Phi_+, s = \{\alpha_1, \ldots, \alpha_r\}\) be the simple roots in \(\Phi_+\). Then each \(\alpha_i \in \Phi_+, s\) defines a Weyl wall \(W_{\alpha_i}\) of the dominant Weyl chamber...
a_+ of a. As \( \Lambda \in \pi_\nu^{-1}(\mathfrak{a}_+^\vee) \), we can assume that \( \Lambda \in \pi_\nu^{-1}(W_{\alpha_i}) \) for \( i = 1, \ldots, i_0 \) but away from other Weyl walls, or equivalently,

\[
\alpha_i(\Lambda_1 - \Lambda_2) = 0, \quad i = 1, \ldots, i_0,
\]

for simple roots \( \alpha_1, \ldots, \alpha_{i_0} \in \Phi_{+,s} \) and

\[
\alpha_i(\Lambda_1 - \Lambda_2) > 0, \quad i = i_0 + 1, \ldots, r,
\]

for the remaining simple roots \( \alpha_{i_0+1}, \ldots, \alpha_r \). Also, let \( \alpha_{r+1}, \ldots, \alpha_{s_1} \) be positive roots in \( \Phi_+ \setminus \Phi_{+,s} \) which can be written as linear combination of \( \alpha_1, \ldots, \alpha_{i_0} \). Denote by \( \alpha_{s_1+1}, \ldots, \alpha_{s_2} \in \Phi_{+,s} \) the remaining positive roots in \( \Phi_+ \setminus \Phi_{+,s} \).

As mentioned before, \( X_0 \) is an equivariant embedding of some spherical homogeneous space \( \hat{G}/H_0 \). For our latter use, we compute the data of \( H_0 \).

**Proposition 3.3.** Suppose that \( \Lambda = (\Lambda_1, \Lambda_2) \in \mathcal{H}(T) \oplus \mathcal{H}(T) \cong \mathcal{H}(\tilde{T}) \) satisfies (3.3)-(3.4). Then the central fibre \( X_0 \) is a \( G \)-equivariant compactification of \( \hat{G}/H_0 \), where \( H_0 \) is a subgroup of \( G \) with Lie algebra

\[
h_0 = \text{diag}((\Lambda_2 - \Lambda_1)^\perp) \oplus \mathbb{C}(\Lambda_1, \Lambda_2)
\]

(3.5) \(
\oplus \oplus_{i=1, \ldots, i_0; r+1, \ldots, s_1} (\mathbb{C}(X_{\alpha_i}, X_{\alpha_i}) \oplus \mathbb{C}(X_{-\alpha_i}, X_{-\alpha_i}))
\)

(3.6) \(
\oplus \oplus_{j=i_0+1, \ldots, r; s_1+1, \ldots, s_2} (\mathbb{C}(0, X_{\alpha_j}) \oplus \mathbb{C}(X_{-\alpha_j}, 0)).
\)

Here \( (\Lambda_2 - \Lambda_1)^\perp \) is the orthogonal complement of \( \mathbb{C}(\Lambda_2 - \Lambda_1) \) in \( a \).

**Proof.** By [11] Proposition 3.23, we can find an \( x_0 \) in \( X_0 \) whose \( \hat{G} \times \mathbb{C}^* \)-orbit is open dense in \( X_0 \) and is isomorphic to \( (\hat{G} \times \mathbb{C}^*) / \hat{H}_0 \) for some spherical \( \hat{H}_0 \subset (\hat{G} \times \mathbb{C}^*) \). Also, \( x_0 \) can be realized as

\[
x_0 = \lim_{\mathbb{C}^* \ni t \to 0} (\Lambda(t), t^m) \tilde{x}_0
\]

(3.6) for some \( \tilde{x}_0 \) in the open dense \( \hat{G} \times \mathbb{C}^* \)-orbit of \( X \).

We first compute \( \hat{H}_0 \). Consider the base point \( \tilde{x}_0 \) of the open dense orbit of \( X \) in (3.6). Its stabilizer in \( \hat{G} \times \mathbb{C}^* \) is

\[
\hat{H} = \text{diag}(G) \times \{ e \},
\]

whose Lie algebra is spanned by

\[
(X, X, 0), \quad X \in t;
\]

\[
(X_\alpha, X_\alpha, 0), \quad \alpha \in \Phi_+;
\]

\[
(X_{-\alpha}, X_{-\alpha}, 0), \quad \alpha \in \Phi_+.
\]

Recall that for each \( t \in \mathbb{C} \), \( (\Lambda(t), e^{mt}) \tilde{x}_0 \) has stabilizer \( \text{Ad}_{(\Lambda(t), e^{mt})} \hat{H} \), whose Lie algebra is spanned by

\[
\text{Ad}_{(\Lambda(t), e^{mt})}(X, X, 0) = (X, X, 0), \quad X \in t;
\]

\[
\text{Ad}_{(\Lambda(t), e^{mt})}(X_\alpha, X_\alpha, 0) = (e^{\alpha(t)}X_\alpha, e^{\alpha(t)}X_\alpha, 0), \quad \alpha \in \Phi_+;
\]

(3.7) \[
\text{Ad}_{(\Lambda(t), e^{mt})}(X_{-\alpha}, X_{-\alpha}, 0) = (e^{-\alpha(t)}X_{-\alpha}, e^{-\alpha(t)}X_{-\alpha}, 0), \quad \alpha \in \Phi_+.
\]

By (3.3) and the above relations, we have

\[
\text{Ad}_{(\Lambda(t), e^{mt})}(X_\alpha, X_\alpha, 0) = (X_\alpha, X_\alpha, 0)
\]

(3.8)
(3.10) \[ \text{Ad}(\Lambda(e^t), e^{mt}) (X_{-\alpha}, X_{-\alpha}, 0) = (X_{-\alpha}, X_{-\alpha}, 0) \]
for all \( \alpha \in \{\alpha_1, \ldots, \alpha_{i_0}, \alpha_{r+1}, \ldots, \alpha_{s_1}\} \).

On the other hand, by (3.12), as \( e^t \to 0 \) we have
\[ e^{-\alpha(L_2) t} \text{Ad}(\Lambda(e^t), e^{mt}) (X_{-\alpha}, X_{-\alpha}, 0) = (e^{-\alpha(L_2) t} X_{-\alpha}, X_{-\alpha}, 0) \to (0, X_{-\alpha}, 0) \]
and
\[ e^{\alpha(L_1) t} \text{Ad}(\Lambda(e^t), e^{mt}) (X_{-\alpha}, X_{-\alpha}, 0) = (X_{-\alpha}, e^{\alpha(L_1) t} X_{-\alpha}, 0) \to (X_{-\alpha}, X_{-\alpha}, 0) \]
for all \( \alpha \in \{\alpha_{i_0+1}, \ldots, \alpha_r, \alpha_{s_1+1}, \ldots, \alpha_{n-\tau}\} \).

It is direct to see that \( (\Lambda, m) \in \hat{h}_0 \). Hence the Lie algebra \( \hat{h}_0 \) of \( \hat{H}_0 \) is
\[ \hat{h}_0 = (\text{diag}(t) \times \{0\}) \oplus C(\Lambda_1, \Lambda_2, m) \]
\[ \oplus (\oplus_{i=1, \ldots, i_0; r+1, \ldots, s_1} C(X_{-\alpha_i}, X_{-\alpha_i}, 0) \oplus C(X_{-\alpha_i}, X_{-\alpha_i}, 0)) \]
\[ \oplus (\oplus_{i=i_0+1, \ldots, r; s_1+1, \ldots, n-\tau} C(0, X_{-\alpha_i}, 0) \oplus C(X_{-\alpha_i}, 0, 0)) \]
which is understood as a Lie sub-algebra in \( g \oplus g \oplus C \). Corollary 3.4 then follows directly from the above relation. \( \square \)

By the above Proposition we can show that \( H_0 \) is even horosymmetric in the sense of [12, Definition 2.1].

**Corollary 3.4.** Under the assumption of Proposition 3.3, the homogeneous space \( \hat{G}/H_0 \) is horosymmetric. Its anticanonical line bundle has isotropic character
\[ (3.11) \quad \chi = \sum_{j=i_0+1, \ldots, r; s_1+1, \ldots, n-\tau} (\alpha_j, -\alpha_j). \]

**Proof.** To see that \( H_0 \) is horosymmetry, consider the following parabolic subgroup \( \hat{P} \subset \hat{G} \) with Lie algebra
\[ \hat{p} = (t \oplus t) \oplus (\oplus_{i=1, \ldots, i_0; r+1, \ldots, s_1} C(X_{\pm \alpha_i}, 0) \oplus C(0, X_{\pm \alpha_i})) \]
\[ \oplus (\oplus_{j=i_0+1, \ldots, r; s_1+1, \ldots, n-\tau} C(X_{-\alpha_j}, 0) \oplus C(0, X_{-\alpha_j})), \]
and Levi group \( \hat{L} = L \times L \) in \( \hat{P} \), where \( L \) has Lie algebra
\[ L = t \oplus (\oplus_{i=1, \ldots, i_0; r+1, \ldots, s_1} C(X_{\pm \alpha_i}, C X_{-\alpha_i}). \]

By Proposition 3.3 \( H_0 \subset P \) and the unipotent radical
\[ (3.12) \quad \hat{P}^u \subset H_0. \]

Define an involution \( \Theta \) on \( \hat{I} \) whose eigenspace of +1 is
\[ \text{diag}((\Lambda_2 - \Lambda_1)^\pm) \oplus C(\Lambda_1, \Lambda_2) \oplus \oplus_{i} C(\pm \alpha_i, X_{\pm \alpha_i}), \]
and eigenspace of -1 is
\[ \text{antidiag}((\Lambda_2 - \Lambda_1)^\pm) \oplus C(\Lambda_2, -\Lambda_1) \oplus \oplus_{i} C(X_{\pm \alpha_i}, -X_{\pm \alpha_i}), \]
where \( \text{antidiag}(V) \) denotes the anti-diagonal embedding of \( V \) in \( V \times V \).

Since all \( \alpha_i \)'s are in \( (\Lambda_2 - \Lambda_1)^\pm \), it is not hard to check that \( \Theta \) is a morphism of the Lie algebra \( \hat{I} \). Hence it defines a complex involution \( \Theta \) on \( \hat{L} \). It is direct to check that the neutral component of the fixed points
\[ (\hat{L})^\Theta = \hat{L} \cap H_0 \subset H_0. \]
Combing with (3.12), we see that $H_0$ is horosymmetry. Relation (3.11) then follows from [12] Example 3.1.

3.2.1. The equivariant automorphism. Now we compute $\text{Aut}_{\hat{G}}(X_0)$, the group of $\hat{G}$-equivariant automorphisms of $X_0$. Fix a $W$-invariant inner product $\langle \cdot , \cdot \rangle$ on $a$ which extends the Killing form on $a \cap [a, a]$. Take $a_1 = a \cap (\cap_{i=1}^{l_0} \ker(\alpha_i))$ and $a_2$ the orthogonal complement of $a_1$ in $a$. Let $\hat{A}_1, \hat{A}_2$ be two toruses of $\hat{G}$ defined by

$$\hat{A}_1 = \exp(a_1 \oplus C a_1), \quad \hat{A}_2 = \exp(a_2 \oplus C a_2).$$

We conclude from (3.7)-(3.10) that the centralizer

$$C_G(H_0) \cap N_G(H_0) = \hat{A}_1.$$

By [11] Proposition 3.21 and 3.24, for the adapted Levi group $\hat{B}$,

$$N_G(H_0) = H_0(C_B(H_0) \cap N_G(H_0)) = H_0(\hat{A}_1) = H_0(\exp(a_1) \times \{ e \}).$$

By [33] Proposition 1.8, the group of $\hat{G}$-equivariant automorphisms of $X_0$,\n
$$\text{Aut}_{\hat{G}}(X_0) \cong \text{Aut}_{\hat{G}}(G/H_0) \cong N_G(H_0)/H_0.$$

Thus, we have

**Lemma 3.5.** Let $A_1 = \exp(a_1) \subset T$. Then

$$\text{Aut}_{\hat{G}}(X_0) \cong A_1.$$

The Lemma will be used for computing the valuation cone of $\hat{G}/H_0$.

3.2.2. The valuation cone of $\hat{G}/H_0$. In this section we compute the valuation cone $\mathcal{V}(\hat{G}/H_0)$. We will adopt the formal curve method in [33] Section 24. Suppose that it holds (3.3) - (3.4). Set

$$\hat{U}_1 = \exp(\oplus_{i=1, ..., i_0, r+1, ..., s_1}(\mathbb{C}(0, X_{\alpha_i}) \oplus \mathbb{C}(X_{-\alpha_i}, 0))$$

and

$$\hat{U}_2 = \exp(\oplus_{j=i_0+1, ..., r+s_1+1, ..., s_2}(\mathbb{C}(0, X_{\alpha_i}) \oplus \mathbb{C}(X_{-\alpha_i}, 0)).$$

Then

$$\hat{U}^+ = \hat{U}_1 \cdot \hat{U}_2.$$

Also, define

$$l = t^C \oplus (\oplus_{i=1, ..., i_0, r+1, ..., s_1}(\mathbb{C}X_{\alpha_i} \oplus \mathbb{C}X_{-\alpha_i}))$$

$$L = \exp(l).$$

Then $L$ is a reductive subgroup of $G$, $T$ is a maximal compact torus of $L$ and $\Phi_L = \{ \pm \alpha | i = 1, ..., i_0, r+1, ..., s_1 \}$ is its root system. Moreover, $\Phi_{L, +} = \Phi_L \cap \Phi_+$ and $\Phi_{L, +, s} = \Phi_L \cap \Phi_{+, s}$ are the positive and the simple roots, respectively. Set

$$\hat{L} = L \times L.$$

To apply the formal curve method, we need the following

**Lemma 3.6.** A formal curve $\hat{G}(t)$ in $\hat{G} \times \mathbb{C}^*$ can be decomposed as

$$\hat{G}(t) = \hat{G}[[t]] \cdot \hat{A}(t) \cdot \hat{L}[[t]] \cdot \hat{U}_2(t).$$

Consequently,

$$\hat{G}(t)H_0 = \hat{G}[[t]] \cdot \hat{A}(t) \cdot \hat{L}[[t]]H_0.$$
Thus \( \gamma \) By applying an action of (3.19), we see that for a generic choice of (\( \gamma \) \( \gamma \) \( \gamma \)),

\[
G(t) = G[t] \cdot \hat{A}_1(t) \cdot \hat{U}_1(t) \cdot \hat{U}_2(t)
\]

where in the last line we use the fact that \( \hat{A}_1 \) commutes with \( \hat{U}_1 \). Combining with the fact that \( \hat{U}_2 \subset H_0 \), we have

\[
G(t)H_0 = G[t] \cdot \hat{A}_2(t) \cdot \hat{U}_1(t) \cdot \hat{A}_1(t)H_0.
\]

Set \( L_{ss} := [\hat{L}, \hat{L}] \). By the last line of (3.17), we can rewrite

\[
G(t) = G[t] \cdot \hat{L}_{ss}(t) \cdot \hat{A}_1(t) \cdot \hat{U}_2(t)
\]

Here we used the Cartan decomposition in \([33, \text{Section 24}]\) for \( \hat{L}_{ss}(t) \) in the last line. Since \( \hat{A}_1 \) commutes with \( \hat{L}_{ss} \), we get (3.15). Note that by Proposition 3.3, \( \hat{U}_2 \subset H_0 \). We get (3.16).

Now we can compute the valuation cone

**Proposition 3.7.** Under the assumption of Proposition 3.3, the valuation cone \( V(G/H_0) \) of \( G/H_0 \) can be identified with the anti-dominant Weyl chamber with respective to \( \Phi_{L,+} \),

\[
\{ x \in a | \alpha(x) \leq 0, \alpha \in \Phi_{L,+} \} = \bigcap_{i=1}^{i_0} \{ x \in a | \alpha_i(x) \leq 0 \}.
\]

**Proof.** We use the arguments of \([33, \text{Section 24}]\). By Lemma 3.6, every \( v \in \mathfrak{a} \) is proportional to a punctured curve in \( G/H_0 \). It suffices to compute the order

\[
\nu(f((g_1, g_2) \gamma(t)H_0)), \text{ as } t \to 0,
\]

for a rational function \( f \) in \( G/H_0 \) and a generic \( (g_1, g_2) \in \hat{G} \). Decompose \( v = (v_1, v_2) \) such that

\[
v_1 \in \hat{a}_1, \ v_2 \in \hat{a}_2.
\]

Thus \( \gamma(t) = e^{v_1(t)} \cdot e^{v_2(t)} \). By Lemma 3.3, we have

\[
\nu(f((g_1, g_2) \gamma(t)))) = \text{ord}_{t=0} f(e^{v_1(t)}(g_1, g_2)e^{v_2(t)}H_0).
\]

By applying an action of \( N_{\hat{L}}(\hat{T}) \), we may further assume that

\[
\alpha_i(v_2) \leq 0 \text{ for } i = 1, ..., i_0.
\]

On the other hand, the \( \hat{B}^+ \)-eigenfunctions are of form

\[
f_\lambda(g_1, g_2, w) = \langle v_\lambda, w^{-1}(g_2g_1^{-1})v_\lambda \rangle,
\]

where \( \lambda \) is any weight in \( \mathfrak{M}(\hat{G}/H_0) \) such that \( \langle \lambda, \alpha_i \rangle \geq 0, \ i = 1, ..., i_0 \). By (3.19)-(3.20), we see that for a generic choice of \( (g_1, g_2), \)

\[
\text{ord}_{t=0} f_\lambda(e^{v_1(t)}(g_1, g_2)e^{v_2(t)}H_0) = \lambda(v_1) + \lambda(v_2) = \lambda(v).
\]
Also, by Proposition 3.3, \( \text{diag}(\exp((\Lambda_1 - \Lambda_2)^\perp)) \cdot e^{(\Lambda_1, \Lambda_2)^t} \) acts trivially on \((\hat{G} \times \mathbb{C}^*)/H_0\), so
\[
\mathcal{V}(\hat{G}/H_0) = \{-(y_1, y_2) \in \mathfrak{a}|\alpha_i(y_2 - y_1) \geq 0, i = 1, \ldots, i_0\}/(\text{diag}((\Lambda_2 - \Lambda_1)^\perp) + \mathbb{C}(\Lambda_1, \Lambda_2))
\]
\[
\cong \{y \in \mathfrak{a}|\alpha_i(y) \leq 0, i = 1, \ldots, i_0\}.
\]
We conclude the Proposition since \( \{\alpha_i| i = 1, \ldots, i_0\} = \Phi_{L,+} \) are precisely the simple roots in \( \Phi_{L,+} \).

Proposition 3.7 will be used to test the (modified) K-stability of \( X_0 \) in Section 5.

### 3.2.3. Moment polytope of \( X_0 \)

By Proposition 3.1 when \( X \) is a special \( \mathbb{Z} \)-test configuration, \( X_0 \) is a spherical embedding of \( \hat{G}/H_0 \). Recall (3.2), \((X, \mathcal{L})\) is a complete spherical embedding of \((\hat{G} \times \mathbb{C}^*)/(\text{diag}(G) \times \{e\})\) with moment polytope
\[
\mathcal{P} := \{m_0(y, y') \in a_+^* \oplus \mathbb{R}|0 \leq y' \leq C_0 - (\Lambda_1 - \Lambda_2)(y) - m, \ y \in P_+\}.
\]
Thus, if \( X \) is special, the central fibre \( X_0 \) corresponds to the facet
\[
\{m_0(y, y') \in a_+^* \oplus \mathbb{R}|y' = C_0 - (\Lambda_1 - \Lambda_2)(y) - m, \ y \in P_+\} \subset \partial \mathcal{P}.
\]
Hence we have

**Proposition 3.8.** Suppose that \( X \) is a \( \hat{G} \)-equivariant special \( \mathbb{Z} \)-test configuration. Then there is a constant \( C_0 > 0 \) such that for each \( k \in \mathbb{N}_+ \) we can decompose \( H^0(X_0, K_{X_0}^{-k}) \) as direct sum of irreducible \( \hat{G} \times \mathbb{C}^* \)-representations
\[
H^0(X_0, K_{X_0}^{-k}) = \bigoplus_{\lambda \in \mathfrak{t}^* \cap 2\mathbb{N}} V_\lambda \otimes V_\lambda^* \otimes E_{\mathfrak{a}^*}(kC_0 - (\Lambda_1 - \Lambda_2)(\lambda)),
\]
where \( V_\lambda \) is the irreducible \( G \)-representation of highest weight \( \lambda \), \( E_\lambda \) is the 1-dimensional representation of \( \mathbb{C}^* \) of weight \( q \). Consequently, the moment polytope of \((X_0, K_{X_0}^{-1})\) is \( P_+ \).

### 4. Filtrations and equivariant \( \mathbb{R} \)-test configurations

In this section, we classify the \( \hat{G} \)-equivariant normal \( \mathbb{R} \)-test configurations of a polarized \( G \)-compactionification. For simplicity, we write \( \mathfrak{M} \) in short of \( \mathfrak{M}(G) \). Recall that for a polarized \( G \)-compacification \((M, L)\) with moment polytope \( P_+ \), we can decompose \( H^0(M, L^k) \) into direct sum of irreducible \( \hat{G} \)-representations [2] Section 2.1,
\[
R_k = H^0(M, L^k) = \bigoplus_{\lambda \in \mathfrak{t}^* \cap 2\mathbb{N}} \text{End}(V_\lambda),
\]
where \( V_\lambda \) is the irreducible \( G \)-representation of highest weight \( \lambda \) and \( \text{End}(V_\lambda) \cong V_\lambda \otimes V_\lambda^* \). The Kodaira ring of \( M \) is given by
\[
R(M, L) = \bigoplus_{k \in \mathbb{N}} R_k.
\]
Suppose that \( \mathcal{F} \) is a \( \hat{G} \)-equivariant filtration on \( R \). Then by Definition 2.4 (1)-(2), we have
\[
\mathcal{F}^s R_k = \bigoplus_{\lambda \geq s} \text{End}(V_\lambda),
\]
where \( s \) is an integer.
where we associated to each \( \text{End}(V_\lambda) \) a number \( s_\lambda^{(k)} \). Recall the Abelian group \( \Gamma(\mathcal{F}) \) defined after (2.1). Under the normalization of Remark 2.4, we see that the corresponding Rees algebra (2.2) reduces to

\[
(4.4) \\
R(\mathcal{F}) = \bigoplus_{k \in \mathbb{N}} \bigoplus_{s \in \Gamma(\mathcal{F})} \bigoplus_{\lambda \in \mathbb{N}^+ \cap \mathfrak{M}} s_\lambda^{(k)} t^{-s} \text{End}(V_\lambda).
\]

\( \mathcal{F} \) is an \( \mathbb{R} \)-test configuration if and only if (4.4) is finitely generated.

Moreover, \( s \in \Gamma(\mathcal{F}) \) satisfies (4.6). As in [5, Proposition 2.15], for sufficiently large \( r_0 \in \mathbb{N}_+ \) we may assume that the Rees algebra \( R(\mathcal{F}) \) in (4.4) is

Moreover, \( s \in \Gamma(\mathcal{F}) \) satisfies (4.6). As in [5, Proposition 2.15], for sufficiently large \( r_0 \in \mathbb{N}_+ \) we may assume that the Rees algebra \( R(\mathcal{F}) \) in (4.4) is

4.1. Classification of \( \tilde{G} \)-equivariant normal \( \mathbb{R} \)-test configurations. Let \( (M, L) \) be a polarized \( G \)-compactification. Recall [1, 2]. The closure \( Z \) of a maximal torus \( T^\mathbb{C} \) in \( M \), together with \( L|_Z \) is a polarized toric variety. The polytope \( P \) associated to \( (Z, L|_Z) \) is a \( W \)-invariant, rational convex polytope in \( \mathfrak{M}_Q \). In fact, it holds \( P_+ = P \cap \mathfrak{a}_+^\mathbb{Q} \). Under the normalization of Remark 2.4, we have

**Theorem 4.1.** Let \( (M, L) \) be a polarized \( G \)-compactification with moment polytope \( P_+ \). Then for any \( G \)-equivariant normal \( \mathbb{R} \)-test configuration \( \mathcal{F} \) of \( (M, L) \), there is a \( W \)-invariant, concave, piecewise-linear function \( f \) on \( \overline{P} \) whose domains of linearity consist of rational polytopes in \( \mathfrak{M}_Q \) such that \( \min f = 0 \) and

\[
(4.6) \\
s_\lambda^{(k)} = \max \{ s \in \Gamma(\mathcal{F}) | s \leq k f(\frac{\lambda}{k}) \}, \forall \lambda \in \overline{P}_+ \cap \mathfrak{M} \text{ and } k \in \mathbb{N}.
\]

Moreover, \( s_\lambda^{(k)} = k f(\frac{\lambda}{k}) \) if \( \frac{1}{k} \lambda \) is a vertex of the domains of linearity of \( f \).

Conversely, given any such \( f \) and \( r_0 \in \mathbb{N}_+ \) so that the domains of linearity of \( r_0 f(\frac{1}{r_0} \lambda) \) in \( r_0 P_+ \) consist of integral polytopes in \( \mathfrak{M} \). Denote by \( \Gamma_{r_0}(\text{Vert}(f)) \) the Abelian group generated by

\[
\{ r_0 f(\frac{1}{r_0} \lambda) | \lambda \text{ is a vertex of a domain of linearity of } f \}.
\]

Then for any finitely generated Abelian group \( \Gamma \) containing \( \Gamma_{r_0}(\text{Vert}(f)) \), the collection of points of discontinuity

\[
(4.7) \\
s_\lambda^{(k)} = \sup \{ s \in \Gamma | s \leq k f(\frac{\lambda}{k}) \}, \forall \lambda \in \overline{P}_+ \cap \mathfrak{M} \text{ and } k \in \mathbb{N}.
\]

together with (4.6) defines a \( \tilde{G} \)-equivariant normal \( \mathbb{R} \)-test configuration \( \mathcal{F} \) of \( (M, L) \) satisfying (4.6).

**Proof.** The proof will be divided into two parts according to the two directions.

**Part-1: From equivariant normal \( \mathbb{R} \)-test configurations to (4.6).**

Suppose that \( \mathcal{F} \) is a \( \tilde{G} \)-equivariant normal \( \mathbb{R} \)-test configuration of \( (M, L) \). By (2.4), up to replacing \( L \) by some \( L^w \) with \( r_0 \in \mathbb{N}_+ \), we may assume that the Rees algebra \( R(\mathcal{F}) \) in (4.4) is a normal ring.

We are going to construct a function \( f \) satisfying (4.6). As in [5, Proposition 2.15], for sufficiently large \( r_0 \in \mathbb{N}_+ \) we may assume that the Rees algebra (4.4) is
generated by the piece \( k = 1 \). We can choose \( r_0 \) sufficiently divisible so that even each vertex of \( r_0 \) is also integral. Also, without loss of generality, we can assume that \( s^{(1)}_\lambda \geq 0 \) for all \( \lambda \in \overline{P_+} \cap \mathfrak{M} \).

Let \( \lambda \in \overline{P_+} \cap \mathfrak{M} \) and \( \mu_1, \ldots, \mu_k \in \lambda + \overline{P_+} \cap \mathfrak{M} \) so that

\[
\text{End}(V_\lambda) \subset \text{End}(V_{\mu_1}) \cdot \cdots \cdot \text{End}(V_{\mu_k}).
\]

By (4.5) we have

\[
s^{(k)}_\lambda \geq k \sum_{j=1}^{k} s^{(1)}_{\mu_j}.
\]

Since (4.4) is generated by the piece \( k = 1 \), we get for any \( k \in \mathbb{N}_+ \) and \( k \overline{P_+} \cap \mathfrak{M} \),

\[
s^{(k)}_\lambda = \max \{ k \sum_{i=1}^{r} c_i \alpha_i : \mu_i \in \overline{P_+} \cap \mathfrak{M}, \text{End}(V_\lambda) \subset \text{End}(V_{\mu_1}) \cdot \cdots \cdot \text{End}(V_{\mu_k}) \} \geq 0.
\]

(4.8)

Let \( M_z = M \cap z(\mathfrak{g}) \) and \( M_{sc} = \text{Span}_Z \Phi_{+,s} \oplus M_z \). Also, let \( \{ \varpi_1, \ldots, \varpi_r \} \) be the fundamental weights with respect to \( \Phi_{+,s} = \{ \alpha_1, \ldots, \alpha_r \} \). It follows

\[
M \subset M_{sc} = \text{Span}_Z \{ \varpi_1, \ldots, \varpi_r \} \oplus M_z.
\]

Then there is an \( n_G \in \mathbb{N} \) so that

\[
n_G \cdot M \subset n_G \cdot M_{sc} \subset M_{ad} \subset M \subset M_{sc}.
\]

(4.9)

**Step-1.1:** Comparison of points of discontinuity. We first show that for any \( \lambda, \mu \in \overline{P_+} \cap \mathfrak{M} \) satisfying

\[
\lambda = \mu - \sum_{i=1}^{r} c_i \alpha_i, \quad 0 \leq c_i \in \mathbb{Q} \quad \text{for all } i = 1, \ldots, r,
\]

it holds

\[
s^{(1)}_\lambda \geq s^{(1)}_\mu.
\]

(4.11)

Otherwise, if \( s^{(1)}_\lambda < s^{(1)}_\mu \), by (4.3) we see that

\[
t^{-s^{(1)}_\mu} \text{End}(V_\lambda) \not\subset R(F).
\]

Let \( e_\lambda \) be the highest weight vector in \( \text{End}(V_\lambda) \). We see that for any \( k \in \mathbb{N}_+ \),

\[
(t^{-s^{(1)}_\mu} e_\lambda)^k \in t^{-ks^{(1)}_\mu} \text{End}(V_{k\lambda}).
\]

(4.13)

Note that by (4.10),

\[
\lambda \in \text{Conv}\{w(\mu)|w \in W\}.
\]

Hence by (4.2) Lemma 1], there is a \( k_0 \in \mathbb{N}_+ \) such that

\[
V_{k_0 n_G \lambda} \subset V_{\mu}^{\otimes k_0 n_G}.
\]

On the other hand, by (4.9),

\[
n_G (\mu - \lambda) \in \text{Span}_{\mathbb{Z}_k} \Phi_{+,s}.
\]

Combining with (4.2) Proposition 4], we have

\[
\text{End}(V_{k_0 n_G \lambda}) \subset \text{End}(V_{\mu})^{k_0 n_G}.
\]

We get from (4.13) that

\[
(t^{-s^{(1)}_\mu} e_\lambda)^{k_0 n_G} \in (t^{-s^{(1)}_\mu} \text{End}(V_\mu))^{k_0 n_G} \subset R(F).
\]
Thus $t^{-\lambda_{(1)}}e_{\lambda}$ is integral over $R(\mathcal{F})$. Since $R(\mathcal{F})$ is normal,

$$t^{-\lambda_{(1)}}e_{\lambda} \in R(\mathcal{F}),$$

which contradicts to (4.12) and we conclude (4.11).

Step 1.2: Construction of $f$. In view of (4.10), for simplicity we will write “$\mu \geq \lambda$” whenever $\lambda$ and $\mu$ satisfy (4.10). We claim that for each $k \in \mathbb{N}_+, \mu, \lambda_1, ..., \lambda_l \in kP_+ \cap \mathfrak{M}$ and constants $0 \leq c_1, ..., c_l \leq 1$ satisfying

$$\mu = \sum_{i=1}^{l} c_i \lambda_i,$$

and

$$\sum_{i=1}^{l} c_i = 1,$$

it always holds

$$(4.16) \quad s_{\mu}^{(k)} \geq \sup\{ s \in \Gamma(\mathcal{F}) | s \leq \sum_{i=1}^{l} c_i s_{\lambda_i}^{(k)} \}. $$

Otherwise, suppose that there are $\mu, \lambda_1, ..., \lambda_l \in kP_+ \cap \mathfrak{M}$ and $0 \leq c_1, ..., c_l \leq 1$ satisfying (4.14)-(4.15) but (4.16) fails. Then

$$s_{\mu}^{(k)} < \sum_{i=1}^{l} c_i s_{\lambda_i}^{(k)} =: \hat{s}_{\mu}^{(k)} \in \Gamma(\mathcal{F}),$$

where we can choose $\hat{s}_{\mu}^{(k)} \in \Gamma(\mathcal{F}) \cap s_{\mu}^{(k)} \leq s_{\lambda_i}^{(k)}$. Let $e_{\mu} \in \text{End}(V_{\mu})$ be the highest weight vector, as in (4.12), we have

$$t^{-\hat{s}_{\mu}^{(k)}}e_{\mu} \not\in R(\mathcal{F}).$$

On the other hand, we can choose $n_0, m_0 \in \mathbb{N}_+$ so that $n_0 c_j \in \mathbb{N}$ and $m_0 \hat{s}_{\lambda_i}^{(k)} \in \Gamma(\mathcal{F})$ for $j = 1, ..., l$. Then

$$(t^{-\hat{s}_{\mu}^{(k)}}e_{\mu})^{n_0m_0} = t^{-\sum_{j=1}^{l} n_0m_0 c_j \hat{s}_{\lambda_j}^{(k)}}(e_{\mu})^{n_0m_0} = t^{-\sum_{j=1}^{l} n_0m_0 c_j \hat{s}_{\lambda_j}^{(k)}} \text{End}(V_{n_0m_0\mu}).$$

However, by (4.14) and (4.13),

$$t^{-\sum_{j=1}^{l} n_0m_0 c_j \hat{s}_{\lambda_j}^{(k)}} \text{End}(V_{n_0m_0\mu}) \subset (t^{-m_0 \hat{s}_{\lambda_1}^{(k)}} \text{End}(V_{m_0\lambda_1}))^{n_0c_1} \cdot ... \cdot (t^{-m_0 \hat{s}_{\lambda_l}^{(k)}} \text{End}(V_{m_0\lambda_l}))^{n_0c_l} \subset R(\mathcal{F}).$$

Hence $(t^{-\hat{s}_{\mu}^{(k)}}e_{\mu})^{n_0}$ is integral in $R(\mathcal{F})$. Since $s_{\lambda_i}^{(k)} m_0 \leq s_{\lambda_i}^{(m_0k)}$, we conclude that $t^{-\hat{s}_{\mu}^{(k)}}e_{\mu} \in R(\mathcal{F})$ since $R(\mathcal{F})$ is normal, which contradicts with (4.17). Hence (4.16) is true.

Note that $s_{\mu}^{(1)} \in \Gamma(\mathcal{F})$ for each $\mu \in \mathcal{P}_+ \cap \mathfrak{M}$. Take $k = 1$ in (4.8) and (4.19), we can define a piece-wise linear concave function $f : \mathcal{P}_+ \to \mathbb{R}$ so that $f(0) = 0$ and

$$(4.18) \quad s_{\mu}^{(1)} = \max\{ s \in \Gamma(\mathcal{F}) | s \leq f(\mu) \}, \forall \mu \in \mathcal{P}_+ \cap \mathfrak{M}.$$

Indeed, consider the convex hull of

$$(4.19) \quad \mathcal{P}'_+ := \{ (\lambda, s_{\lambda}^{(1)}) | \lambda \in \mathcal{P}_+ \cap \mathfrak{M} \}.$$
It is a convex polytope in $\mathbf{a}^+ \oplus \mathbb{R}$ such that

$$
\text{Conv}(P'_+) = \{(x, y) \in \mathbf{a}^+ \oplus \mathbb{R} | x \in P_+, 0 \leq y \leq f(x)\}
$$

for some concave function $f$. Obviously, $s^{(1)}\mu \leq f(\mu)$ and the equality holds if $\mu$ is a vertex of a domain of linearity. Combining with (4.10), we get (4.18). Also, by the normalization condition $\min \Gamma(F, 1) = 0$ and (4.18), there must be a vertex $\mu$ of $P \cap P_+$ so that $s^{(1)}\mu = 0$. Hence $\min f = f(\mu) = 0$.

It is easy to see that each domain of linearity of $f$ is a convex polytope with vertices in $P_+ \cap \mathcal{M}$. In fact, any domains of linearity of $f$ is the projection of a facet of Conv($P'_+$) on the roof $\{(x, f(x)) | x \in P_+\}$. But the vertices of such a facet must lie in $P'_+$. Hence it projects to a point in $P_+ \cap \mathcal{M}$.

By (4.11), the gradient of $f$ (in the sense of subdifferential),

$$
\nabla f \subset (-\mathbf{a}^+).
$$

Hence by using the $W$-action we can extend $f$ to a $W$-invariant piecewise-linear function, which is globally concave on $P'_+$.

**Step-1.3: Proof of (4.6).** It remains to prove that (4.6) holds for any $k \in \mathbb{N}^+$. Fix any $\mu \in kP_+ \cap \mathcal{M}$. By (4.8), we can assume that there are $\lambda_1, ..., \lambda_k \in P_+ \cap \mathcal{M}$ so that

$$
\sum_{i=1}^{k} s^{(1)}\lambda_i = \text{End}(V\mu) \subset \text{End}(V\lambda_1) \cdot \cdot \cdot \text{End}(V\lambda_k).
$$

In particular,

$$
\sum_{i=1}^{k} \lambda_i \geq \mu.
$$

We have

$$
\frac{1}{k} s^{(k)}\mu = \frac{1}{k} \sum_{i=1}^{k} s^{(1)}\lambda_i \leq \frac{1}{k} \sum_{i=1}^{k} f(\lambda_i) \leq f(\frac{\sum_{i=1}^{k} \lambda_i}{k}).
$$

By (4.21) and (4.20) we get

$$
\frac{1}{k} s^{(k)}\mu \leq f(\frac{\sum_{i=1}^{k} \lambda_i}{k}) \leq f(\frac{\mu}{k}).
$$

Recall that the vertices of each domain of linearity of $f$ is in $\mathcal{M}$. Suppose that $\Omega$ is the domain of linearity which contains $\frac{\mu}{k}$ whose vertices are $\lambda_1, ..., \lambda_l \in \mathcal{M}$, then there are non-negative constants $c_1, ..., c_l$ such that

$$
\mu = \sum_{i=1}^{l} c_i \lambda_i, \quad \sum_{i=1}^{l} c_i = 1.
$$

Since $f$ is linear on $\Omega$, we get

$$
\sum_{j=1}^{l} c_j s^{(1)}\lambda_j = kf(\frac{\mu}{k}).
$$
On the other hand, by (4.16), we have

\[ s^{(k)}_{\mu} \geq \sup \{ s \in \Gamma(\mathcal{F}) \mid s \leq \sum_{j=1}^{l} c_j s^{(k)}_{\lambda_j} \} \]

\[ \geq \sup \{ s \in \Gamma(\mathcal{F}) \mid s \leq k \sum_{j=1}^{l} c_j s^{(1)}_{\lambda_j} \} = \sup \{ s \in \Gamma(\mathcal{F}) \mid s \leq kf(\frac{\lambda}{k}) \} , \]

where we used (4.8) and (4.23) in the last inequality.

Combining with (4.22), up to replacing \( f(\cdot) \) by \( \frac{1}{r_0} f(r_0 \cdot) \) we get

\[ s^{(k)}_{\lambda} = \sup \{ s \in \Gamma(\mathcal{F}) \mid s \leq kf(\frac{\lambda}{k}) \} , \quad \forall \lambda \in kP_+ \cap \mathfrak{M} \text{ and } k \in \mathbb{N}. \]

The relation (4.6) then follows from the above equality and the fact that \( s^{(k)}_{\lambda} \in \Gamma(\mathcal{F}) \). Also, from Step-1.2 we see that after this scaling, \( f \) is still a \( \mathcal{W} \)-invariant, concave piecewise linear function on \( P \). But its domains of linearity consists of convex polytopes with vertices in \( \mathfrak{M}_Q \).

**Part-2: The inverse direction.**

**Step-2.1: Construction of the \( \mathbb{R} \)-test configuration.** Given an \( f \) satisfying the assumption of Theorem 4.1, we can fix an \( r_0 \in \mathbb{N} \) so that the domains of linearity of the function

\[ f(x) = r_0 f(\frac{1}{r_0} x) : r_0 P_+ \rightarrow \mathbb{R} \]

consist of polytopes with vertices in \( \mathfrak{M} \). Replacing \( L \) by \( L^{r_0} \), we may assume that \( r_0 = 1 \). By concavity and (4.20), it holds

\[ (k_1 + k_2) f(\frac{\mu}{k_1 + k_2}) \geq (k_1 + k_2) f(\frac{\lambda_1 + \lambda_2}{k_1 + k_2}) \geq k_1 f(\frac{\lambda_1}{k_1}) + k_2 f(\frac{\lambda_2}{k_2}) \]

for any two \( \lambda_i \in k_i P_+ \cap \mathfrak{M} \), \( i = 1, 2 \) and \( \mu \in (k_1 + k_2) P_+ \cap \mathfrak{M} \) so that \( \lambda_1 + \lambda_2 \geq \mu \). Combining with [12, Proposition 4], it is direct to check that (4.3) and (4.7) define a \( \mathcal{G} \)-equivariant filtration \( \mathcal{F} \) of \( (M, L) \). Then we prove that \( \mathcal{F} \) is an \( \mathbb{R} \)-test configuration. We have two case:

**Case-2.1.1.** \( \Gamma \) is a discrete subgroup in \( \mathbb{R} \). In this case (4.7) is reduced to

\[ s^{(k)}_{\lambda} = \max \{ s \in \Gamma \mid s \leq kf(\frac{\lambda}{k}) \} , \quad \forall \lambda \in kP_+ \cap \mathfrak{M} \text{ and } k \in \mathbb{N}. \]

Clearly \( \Gamma(\mathcal{F}) \subset \Gamma \) is finitely generated and \( \mathcal{F} \) is an \( \mathbb{R} \)-test configuration.

**Case-2.1.2.** \( \Gamma \) is a not discrete. Then \( \Gamma \) is everywhere dense in \( \mathbb{R} \). In this case (4.7) is reduced to

\[ s^{(k)}_{\lambda} = kf(\frac{\lambda}{k}) , \quad \forall \lambda \in kP_+ \cap \mathfrak{M} \text{ and } k \in \mathbb{N}. \]

Consequently \( \Gamma(\mathcal{F}) \) is generated by a finite set \( \{ f(\lambda) \mid \lambda \in P_+ \cap \mathfrak{M} \} \). Again \( \mathcal{F} \) is an \( \mathbb{R} \)-test configuration.

**Step-2.2: Normality of the total space.** It remains to prove that the corresponding Rees algebra

\[ \mathbb{R}(\mathcal{F}) = \oplus_{k \in \mathbb{N}} \oplus_{s \in \Gamma(\mathcal{F}) \cap \mathfrak{M}} \oplus_{\lambda \in kP_+ \cap \mathfrak{M}} t^{-s} \text{End}(V_\lambda). \]

is normal. Since

\[ (\mathbb{R}(\mathcal{F}) \subset \mathbb{R}' = \oplus_{k \in \mathbb{N}} \oplus_{s \in \Gamma(\mathcal{F}) \cap \mathfrak{M}} \oplus_{\lambda \in kP_+ \cap \mathfrak{M}} t^{-s} \text{End}(V_\lambda) \]
is a normal ring, it suffices to show that \( R(\mathcal{F}) \) is integrally closed in \( R' \). This is equivalent to show that any \( \bar{s} \in R' \setminus R(\mathcal{F}) \) is not integral in \( R(\mathcal{F}) \).

Assume there is some \( \bar{s} \in R' \setminus R(\mathcal{F}) \) integral over \( R(\mathcal{F}) \). It suffices to deal with \( \bar{s} \) of the following form:

\[
\bar{s} = \sum_{i=1}^{d} t^{-s_i} \sigma_{\tau_i},
\]

where each \( \tau_j \in k(j)P_+ \cap \mathfrak{m} \) for some \( k(j) \in \mathbb{N} \),

\[
(0 \neq) \sigma_{\tau_j} \in \text{End}(V_{\tau_j}) \) and \( s_j > k(j)f\left(\frac{1}{k(j)}\tau_j\right), \ j = 1, \ldots , d.
\]

Since \( \bar{s} \) is integral over \( R(\mathcal{F}) \), there is some \( q \in \mathbb{N}_+ \) such that for any integer \( l \in \mathbb{N}_+ \),

\[
\bar{s}^l \in R(\mathcal{F}) + \left( R(\mathcal{F}) \bar{s} \right)^l + \cdots + R(\mathcal{F}) \bar{s}^q.
\]

Consider the convex hull of \( \{w(\tau_i) | w \in W, i = 1, \ldots , d\} \) which is a vertex of this convex hull. Then for any \( p \in \mathbb{N}_+ \), \( p\tau \) can not be dominated by \( \sum_{i=1}^{p} \tau_i \), whenever there is a \( \tau_i \in \{\tau_1, \ldots , \tau_d\} \setminus \{\tau\} \). Take the component \( t^{-s} \sigma_{\tau} \) of \( \bar{s} \) in its decomposition (4.24). Suppose that the degree of \( t^{-s} \sigma_{\tau} \) is \( m \) (that is, \( \sigma_{\tau} \in R_m \)). Then

\[
m(f(\frac{1}{m}\tau)) < s.
\]

For any \( l \in \mathbb{N}_+ \), we can decompose \( \bar{s}^l \) as (4.24). By Lemma 4.2 below, there is a nonzero component of \( \bar{s}^l \) in \( t^{-ls} \text{End}(V_{\tau}) \) with degree \( lm \), we denote it by \( t^{-ls} \sigma_{\tau} \).

Decompose \( \bar{s}^i, i = 0, \ldots , q \) as (4.24). Then all their components have the form \( t^{-w} \sigma_{\gamma} \) for some degree \( k \). All such triples \( (w, \gamma, k) \) form a finite set \( S \). Thus

\[
\bar{s}^l = \sum_{(w, \gamma, k) \in S} R(\mathcal{F}) t^{-w} \sigma_{\gamma}.
\]

Consider the component \( t^{-ls} \sigma_{\tau} \) of \( \bar{s}^l \). Since \( S \) is finite, up to passing to a subsequence, there is some \( (w, \gamma, k) \in S \) such that

\[
t^{-ls} \sigma_{\tau} \in t^{-w} \text{End}(V_{\gamma}) R(\mathcal{F})_{lm-k}, \ l \in \mathbb{N}_+.
\]

Thus, we have

\[
t^{-ls} \sigma_{\tau} \in t^{-w} \text{End}(V_{\gamma}) t^{-r} \text{End}(V_{\mu})
\]

for some \( t^{-r} \text{End}(V_{\mu}) \subseteq R(\mathcal{F})_{lm-k} \), where \( ls = w + r \),

\[
\gamma + \mu \geq l\tau,
\]

and

\[
r \leq (lm-k)f(\frac{\mu}{lm-k}).
\]

Here in (4.27) we used [42, Proposition 4]. Hence

\[
s = \frac{w}{l} + \frac{r}{l} \leq \frac{w}{l} + (m - \frac{k}{l})f(\frac{1}{lm-k}\mu)
\]

\[
\leq \frac{w}{l} + (m - \frac{k}{l})f(\frac{\tau - \frac{1}{l}\gamma}{m - \frac{k}{l}}), \text{ for sufficiently large } l \in \mathbb{N}_+.
\]
where in the last line we used (4.27) and (4.20). Sending $l \to +\infty$ we see that
\[ s \leq mf\left(\frac{1}{m}\right), \]
which contradicts to (4.26). Hence (4.25) is not true and we conclude that $R(F)$ is normal.

□

To complete the arguments in Step-2.2, we need the following

**Lemma 4.2.** Suppose that $\sigma \in \text{End}(V_{\mu}) \subset R_k$ for some $k \in \mathbb{N}$. Then for any $q \in \mathbb{N}_+$, $\sigma^q$ has non-zero component in $\text{End}(V_{q\mu})$.

**Proof.** Since $\text{End}(V_{\mu})$ is an irreducible $\hat{G}$-representation, there is a $\hat{g}_0 \in \hat{G}$ so that $\hat{g}_0(\sigma)$ has non-zero component on the subspace generated by the highest weight vector. Thus for any $q \in \mathbb{N}_+$, $\hat{g}_0(\sigma^q)$ has non-zero component in $\text{End}(V_{q\mu})$. We conclude the Lemma since $\sigma^q = \hat{g}_0^{-1}(\hat{g}_0(\sigma^q))$ and the fact that $\text{End}(V_{q\mu})$ is $\hat{G}$-invariant. □

**Remark 4.3.** When $f$ is rational, we can take $k$ to be the smallest positive integer so that the set
\[ \left\{ k(\lambda, s) | 0 \leq s \leq f(\lambda), \ \lambda \in P_+ \right\} \subset M_\mathbb{R} \oplus \mathbb{R} \]
is an integral polytope and $\Gamma = \mathbb{Z}$. Theorem 4.1 then reduces to the classification theorem of $\hat{G}$-equivariant normal $\mathbb{Z}$-test configurations [2, Section 2.4] (based on [1, Section 4.2]).

We can further classify $\hat{G}$-equivariant normal $\mathbb{R}$-test configurations with reduced central fibre by using Theorem 4.1

**Theorem 4.4.** Let $(M, L)$ be a polarized $G$-compactification with moment polytope $P_+$. Then for any $\hat{G}$-equivariant normal $\mathbb{R}$-test configuration $F$ of $(M, L)$ with reduced central fibre, there is a $W$-invariant, concave, piecewise-linear function $f \geq \min f = 0$ on $P_+$ whose domains of linearity consist of rational polytopes in $M_\mathbb{Q}$, such that
\[ s^{(k)}(\lambda) = kf(\frac{\lambda}{k}), \ \forall \lambda \in kP_+ \cap M \text{ and } k \in \mathbb{N}, \]
and vice versa.

Theorem 1.2 follows directly from Theorem 4.4. In the following we will call $f$ in (4.28) the function associated to $F$ and denote $F = F_f$.

**Proof of Theorem 4.4.** We divide the proof in two parts.

**Part-1: Necessity of (4.28).** Suppose that $F$ is given so that $\text{Gr}(F)$ defined by (2.3) contains no nilpotent element. Let $f$ be the function defined in Theorem 4.1. We will show (4.28) holds. Otherwise, by (4.9) there is a $\lambda_0 \in l_0P_+ \cap M$ for some $l_0 \in \mathbb{N}_+$ so that
\[ (\Gamma(F) \ni)s^{(l_0)}(\lambda_0) \leq l_0f(\frac{\lambda_0}{l_0}). \]
Let $\sigma_0 \in \text{End}(V_{\lambda_0})$ be a highest weight vector. Then $\sigma_0^{\otimes k} \in \text{End}(V_{k\lambda_0})$ has (real) weight $t^{-ks^{(l_0)}(\lambda_0)}$ for any $k \in \mathbb{N}_+$. 


On the other hand, choose a set of positive generators \( \{ s_1, ..., s_{r_F} \} \) of \( F \) and \( s_M := \max \{ s_1, ..., s_{r_F} \} \). Then there is a \( k_0 \in \mathbb{N}_+ \) so that
\[
k_0(l_0 f(\frac{\lambda_0}{l_0}) - s_{(\lambda_0)}) \geq s_M.
\]
Hence there must be an \( s' \in \Gamma(\mathcal{F}) \) so that \( k_0 s_{(\lambda_0)} < s < k_0 l_0 f(\frac{\lambda_0}{l_0}) \) and consequently
\[
s_{k_0 \lambda_0} > k_0 s_{(\lambda_0)}.
\]
Hence \( \text{End}(V_{k_0 \lambda_0}) \subset \mathcal{F}^{k_0 s_{(\lambda_0)}} R_{k_0 l_0} \) and in \((2.3)\) the piece
\[
t^{-k_0 s_{(\lambda_0)}} \mathcal{F}^{k_0 s_{(\lambda_0)}} R_{k_0 l_0} / \mathcal{F}^{k_0 s_{(\lambda_0)}} R_{k_0 l_0}
\]
contains no \( \text{End}(V_{k_0 \lambda_0}) \)-factor. Hence \( \sigma_0^s k \) descends to 0 in \( \text{Gr}(\mathcal{F}) \). In other words, \( \sigma_0^k = 0 \) in \( \text{Gr}(\mathcal{F}) \) and \( \sigma_0 \) is nilpotent, a contradiction.

**Part-2: Sufficiency of** \((4.28)\). Suppose that \((4.28)\) holds. Define
\[
\mathcal{F}^s R_k := \bigoplus_{\lambda \in \mathcal{P}_+, k f(\lambda/k) \geq s} \text{End}(V_\lambda), \quad \forall k \in \mathbb{N}_+.
\]
We will show \((2.3)\) contains no nilpotent element. Otherwise, there are a \( \sigma \in \text{End}(V_{\lambda_0}) \) for some \( \lambda_0 \in l_0 \mathcal{P}_+ \cap \mathfrak{M} \) and some \( k_0 \in \mathbb{N}_+ \) so that \( \sigma_0^k = 0 \) in \( \text{Gr}(\mathcal{F}) \).

By Lemma \((4.2)\) we can assume that \( \sigma \) has non-zero component on the direction of highest weight vector. Hence \( \sigma_0^k \) has non-zero component in \( \sigma \in \text{End}(V_{k_0 \lambda_0}) \). It must hold
\[
t^{-k_0 s_{(\lambda_0)}} \mathcal{F}^{k_0 s_{(\lambda_0)}} R_{k_0 l_0} / \mathcal{F}^{k_0 s_{(\lambda_0)}} R_{k_0 l_0}
\]
contains no \( \text{End}(V_{k_0 \lambda_0}) \)-factor. This implies
\[
s_{k_0 \lambda_0} > k_0 s_{(\lambda_0)},
\]
a contradiction to \((4.28)\). \(\square\)

**Remark 4.5.** Given any \( f \) satisfying the assumption of Theorem \((4.4)\), we can construct \( \mathcal{F}_f \) by choosing the following data in Theorem \((4.4)\):
- \( k \) is the smallest positive integer so that the domains of linearity of \( f \) in \( \mathcal{P}_+ \) consists of integral polytopes in \( \mathfrak{M} \).
- \( \Gamma \) is the group generated by \( \{ s_{\lambda}^{(k)} | s_{\lambda}^{(k)} \text{ is given by } (4.28) \} \).

It is direct to check that \( \Gamma(\mathcal{F}_f) = \Gamma \).

**Corollary 4.6.** Let \((M, L)\) be a polarized \( \hat{G} \)-compactification. Then for any \( \Lambda \in \overline{\mathcal{M}}_\tau \) there is a \( \hat{G} \)-equivariant special \( \mathbb{R} \)-test configuration \( \mathcal{F}_\Lambda \) of \((M, L)\), so that the central fibre \( X_0 \) is a \( \hat{G} \)-spherical variety and admits an action of the torus \( \exp(t\Lambda) \subset \hat{G} \). Conversely, for any \( \hat{G} \)-equivariant special \( \mathbb{R} \)-test configuration \( \mathcal{F} \) of \((M, L)\), there is a \( \Lambda \in \overline{\mathcal{M}}_\tau \) such that \( \mathcal{F} = \mathcal{F}_\Lambda \).

**Proof.** It suffices to show that a \( \hat{G} \)-equivariant normal \( \mathbb{R} \)-test configuration \( \mathcal{F} \) is special if and only if the associated function \( f \) given by Theorem \((4.4)\) is affine on \( \overline{\mathcal{P}}_+ \). Given any \( \Lambda \in \overline{\mathcal{M}}_\tau \), define
\[
f_\Lambda = \max_{\mu \in \overline{\mathcal{P}}_+} \Lambda(\mu) - \Lambda(\lambda).
\]
Let \( \mathcal{F}_\Lambda \) be the \( \hat{G} \)-equivariant normal \( \mathbb{R} \)-test configuration associated to \( f_\Lambda \) defined by Theorem \((4.4)\) Then its centre \( X_0 \) is reduced.
As in [17, Section 2.2] (see [17, end of p.9 - beginning of p.10]), we can perturb $\Lambda$ to some $\Lambda' \in \mathcal{N}_0$ so that $\mathcal{F}_\Lambda$ and $\mathcal{F}_{\Lambda'}$ have the same central fibre. In fact, $\Lambda'$ can be any rational vector in the Lie algebra of $\exp(\Lambda)$ sufficiently close to $\Lambda$. Hence by Proposition 3.1, $\mathcal{N}_0$ is irreducible. Note that when (4.28) holds, $\mathcal{N}_0$ is also reduced. By [33, Theorem 15.20] $\mathcal{N}_0$ is normal, in particular it is a $\tilde{G}$-spherical variety. We see that $\mathcal{F}_\Lambda$ is special.

When (4.29) holds, by (2.3) and (4.28) we see that the vector field induced by $\mathcal{F}_\Lambda$ is $\Lambda$. Hence we get the first part of the Corollary.

Conversely, suppose that $\mathcal{F}$ is a $\tilde{G}$-equivariant special $\mathbb{R}$-test configuration. Then $\mathcal{N}_0$ is reduced and irreducible. Since the Rees algebra $R(\mathcal{F})$ is finitely generated, we can assume that it can be generated by

$$\bigoplus_{0 \leq k \leq r_0, k \in \mathbb{Z}_+} \bigoplus_{s \in \Gamma(\mathcal{F}, k)} \bigoplus_{\lambda \in k\mathcal{F}_+ \cap \mathbb{N}} s^{(k)}_{\lambda} t^{-s} \text{End}(V_\lambda).$$

Hence we can perturb each $s^{(k)}_{\lambda}$ for $k \in [0, \ldots, r_0]$ and $\lambda \in k\mathcal{F}_+ \cap \mathbb{N}$ to a rational number $s^{(k)}_{\lambda'}$ sufficiently close to it so that it defines a $\mathbb{Z}$-test configuration $\mathcal{F}'$ with the same central fibre. By Proposition 3.2, the function $f'$ associated to $\mathcal{F}'$ is affine on $\mathcal{F}_+^3$. As for small perturbations, the number of domains of linearity of $f'$ can not be smaller than that of $f$, we get the Corollary. $\square$

There is also an algebraic proof of Corollary 4.6 in the Appendix. See Proposition 7.1 below.

4.2. Approximation of an equivariant $\mathbb{R}$-test configuration. In the following we approximate a $\tilde{G}$-equivariant normal $\mathbb{R}$-test configuration by a sequence of $\mathbb{Z}$-test configurations as [17, Definition-Proposition 2.15]. More precisely, given such an $\mathbb{R}$-test configuration $\mathcal{F}$, we can construct a sequence of $G$-equivariant normal $\mathbb{Z}$-test configurations $\{\mathcal{F}_p\}_{p \in \mathbb{N}_+}$ so that the filtration of $\mathcal{F}_p$ on $\bigoplus_{k \in \mathbb{N}} R_p k$ is induced by $\mathcal{F}_\mathbb{Z}$ on the $R_p$-piece (cf. [17, Section 2.2]),

$$(\mathcal{F}_\mathbb{Z})^s R_p = \mathcal{F}^{[s]} R_p.$$  

Indeed, let $f$ be the functions associated to $\mathcal{F}$ given by Theorem 4.1. Define

$$f_p(\mu) = \min(\varphi(\mu) | \varphi(\mu) \text{ is concave and } \varphi(1, \lambda) \geq \frac{s^{(p)}_{\lambda}}{p}, \lambda \in \mathbb{P} \mathcal{F}_+ \cap \mathbb{N}).$$

Then $f_p$ is the $\mathbb{Z}$-test configuration defined by $f_p$ (cf. Remark 3.3).

Choose a set of nonnegative generators $\{e_j\}_{j=1}^{r_\mathcal{F}}$ of $\Gamma(\mathcal{F})$ and denote $\delta(1) = \max_j \{e_j\}$. Then

$$0 \leq f(\lambda) - \frac{1 + \delta}{p} \leq \frac{s^{(p)}_{\lambda}}{p} \leq f(\lambda),$$

we have

$$0 \leq f(\lambda) - f_p(\lambda) \leq \frac{1 + \delta}{p}, \lambda \in \mathcal{F}_+.$$  

Hence we get the uniform convergency,

$$f_p \Rightarrow f, \text{ on } \mathcal{F}_+ \text{ as } p \to +\infty.$$  

Precisely, to apply Proposition 3.2 one needs to rescale $\Gamma(\mathcal{F}')$ so that it coincides with the standard lattice $\mathbb{Z}$. Hence $(\Lambda, 0, m)$ in Proposition 6.2 should be taken as $(k' \nabla f', 0, 1)$ for some sufficiently large $k'$.
5. H-INVARIANT AND SEMISTABLE LIMIT

In this section, we estimate the H-invariant of a general $\hat{G}$-equivariant normal $\mathbb{R}$-test configuration of a $\mathbb{Q}$-Fano $\hat{G}$-compactification. In particular we compute its precise value for $\hat{G}$-equivariant special $\mathbb{R}$-test configurations. Then we find its minimizer and prove Theorem 1.3.

5.1. Reduction of the H-invariant. In this section we express the H-invariant of an equivariant normal $\mathbb{R}$-test configuration in terms of its associated function.

**Theorem 5.1.** Let $(M, L)$ be a $\mathbb{Q}$-Fano $\hat{G}$-compactification with moment polytope $P_+$. Let $\mathcal{F}$ a $\hat{G}$-equivariant normal $\mathbb{R}$-test configuration of $(M, K_M^{-1})$ and $f$ the function defined in Theorem 4.1. Then up to adding a uniform constant,

$$H(\mathcal{F}) \geq \ln \left( \frac{1}{V} \int_{P_+} e^{-f(y)} \pi(y) dy \right),$$

and the equality holds if $\mathcal{F}$ is special.

Note that a special $\mathbb{R}$-test configuration can be defined by a valuation on $G$ and has reduced central fibre. The proof of the Theorem is a combination of (2.12) and the following Lemmas 5.2-5.3.

**Lemma 5.2.** Under the assumption of Theorem 5.1, up to adding a uniform constant,

$$S^{\text{NA}}(\mathcal{F}) \leq -\ln \left( \frac{1}{V} \int_{P_+} e^{-f(y)} \pi(y) dy \right),$$

and the equality holds if $\mathcal{F} = \mathcal{F}_f$, the $\hat{G}$-equivariant normal $\mathbb{R}$-test configuration with reduced central fibre defined in Theorem 4.4.

**Proof.** Recall (2.11). We need to find the Okounkov bodies. By (4.1), (4.3) and (4.6), we have

$$\mathcal{F}^t R_k = \bigoplus_{\lambda \in F_{+}^{\mathbb{C}}} \mathfrak{S}_\lambda^{(k)} \text{End}(V_\lambda), \forall k \in \mathbb{N} \text{ and } t \in \mathbb{R}.$$ 

On the other hand, by (4.5), $\mathcal{S}_\lambda^{(k)} \leq k f(\frac{1}{k} \lambda)$. Thus the Okounkov bodies

$$\Delta(\mathcal{F}^t R_k) \subset \text{Conv} \left( \bigcup_{\lambda \in F^+_{\mathbb{C}}} \mathfrak{S}_\lambda^{(k)} f(\lambda/k) \geq t (\lambda, \Delta(\lambda)) \right),$$

and the equality holds if (4.28) is true. Combining with (2.10) and (5.2), the Okounkov body of $\mathcal{F}_\lambda^{(t)} := \{\mathcal{F}^t R_k\}_{k \in \mathbb{N}^+}$ is

$$\Delta(\mathcal{F}^{(t)}) = \text{Conv} \left( \bigcup_{k=1}^{+\infty} \frac{1}{k} \Delta(\mathcal{F}^t R_k) \right) \subset \Delta \cap \{f(\lambda) \geq t\} =: \Delta_{f \geq t},$$

and equality holds if (4.28) is true.

Recall (2.18). Each $z \in \Delta$ can be decomposed as $z = (\lambda, z')$, where $\lambda \in F^\mathbb{C}_+$ and $z' \in \mathbb{R}^{\dim(N_u)}$. Set $\Delta_\lambda = \{z' | (\lambda, z') \in \Delta\}$. By (2.10) and (5.2),

$$G_\mathcal{F}(z) \leq \sup \{z | z \in \Delta_{f \geq t}\} = f(\lambda), \text{ for } z = (\lambda, z') \in \Delta_\lambda \subset \Delta,$$

with the equality holds for all $z \in \Delta$ if (4.28) holds.
We want to decompose the measure $dz$ on $\Delta$. By [17, Theorem 2.5], the Dirac type measure

$$\nu_k := \frac{n!}{k^n} \sum_{\mathbf{z} \in \Delta \text{ is an integral point}} \delta_{\mathbf{z}}$$

converges weakly to $dz$ on $\Delta$,

$$dz = \lim_{k \to +\infty} \nu_k.$$ 

We may rewrite (5.4) as

$$\nu_k = \frac{n!}{k^n} \sum_{\lambda \in \mathbb{P}_{\mathbb{R}^+} \cap \frac{1}{k} \mathbb{M}} \left( \sum_{\mathbf{z'} \in \Delta(k\lambda) \text{ is an integral point}} \delta_{(\lambda, \mathbf{z'})} \right).$$

Recall the Weyl character formula [38, Section 3.4.4],

$$\dim(V_\lambda \otimes V_\lambda^*) = \frac{\prod_{\alpha \in \Phi_+} \langle \alpha, \rho \rangle + k\lambda \rangle^2}{\prod_{\alpha \in \Phi_+} \langle \alpha, \rho \rangle^2}, \quad \forall \lambda \in \mathbb{a}_+ \cap \mathfrak{m}.$$ 

By (2.18), for any continuous function $\phi$ on $\Delta$, which only depends on $\lambda \in \mathbb{P}_{\mathbb{R}^+}$, we have

$$\int_{\Delta} \phi \nu_k = \frac{n!}{k^n} \sum_{\lambda \in \mathbb{P}_{\mathbb{R}^+} \cap \frac{1}{k} \mathbb{M}} \left( \sum_{\mathbf{z'} \in \Delta(k\lambda) \text{ is an integral point}} \phi(\mathbf{z'}) \right) \phi(\lambda) \frac{\prod_{\alpha \in \Phi_+} \langle \alpha, \rho + k\lambda \rangle^2}{\prod_{\alpha \in \Phi_+} \langle \alpha, \rho \rangle^2}.$$ 

Sending $k \to +\infty$, by (5.5) and [41, Section 1.4],

$$\int_{\Delta} \phi dz = \int_{\mathbb{P}_{\mathbb{R}^+}} \phi(\lambda) \frac{\pi(\lambda)}{\prod_{\alpha \in \Phi_+} \langle \alpha, \rho \rangle^2} d\lambda.$$ 

By (2.11), (5.3) and (5.6), we have

$$S^{NA}(F) = -\ln \left( \frac{1}{V} \int_{\Delta} e^{-G_F(z)} dz \right)$$

$$\leq -\ln \left( \frac{1}{V} \int_{\mathbb{P}_{\mathbb{R}^+}} e^{-f(\lambda)} \pi(\lambda) d\lambda \right) + \ln \prod_{\alpha \in \Phi_+} \langle \alpha, \rho \rangle^2,$$

and the equality holds provided (4.28). Since $\ln \prod_{\alpha \in \Phi_+} \langle \alpha, \rho \rangle^2$ is a constant depending only on $G$, we get the Lemma. \hfill \Box

**Lemma 5.3.** Under the assumption of Theorem 5.1, we have

$$L^{NA}(F) \geq f(2\rho),$$

and the equality holds if $F$ is defined by a valuation on $G$.

**Proof.** To compute $L^{NA}(F)$, we first deal with the case when $f$ is rational. In this case, $F$ is a $\mathbb{Z}$-test configuration of some exponent $m_0 \in \mathbb{N}_+$. The $L^{NA}$-functional in (2.8) is computed by the log-canonical threshold (cf. [17, Example 2.31]),

$$L^{NA}(F_\lambda) = \text{lct}_{(X_0 - (\frac{m_0}{\lambda} \mathcal{L} + \mathcal{K}_X - \frac{1}{\lambda} \mathcal{K}_{\mathbb{C}^{\mathbb{R}}}) - \mathcal{X}_0)}(\mathcal{X}_0) - 1.$$ 

We have
Since \( f \) is rational, we may assume that \( f \) has domains of linearity \( \Omega_1, \ldots, \Omega_{N_f} \) so that
\[
f(y) = C_a - \Lambda_a(y), \quad \forall y \in \Omega_a, \ a = 1, \ldots, N_f,
\]
where each \( \Lambda_a \in \overline{\mathbb{R}}^+ \cap \mathbb{Q} \) and \( C_a \in \mathbb{Q} \). Consequently,
\[
f(y) = \min_a\{C_a - \Lambda_a(y)\}, \quad \forall y \in \overline{\mathcal{P}}_e.
\]
Without loss of generality we may assume that \( f \geq 0 \). Set
\[
\mathcal{P}_+ = \{(y, t) | y \in \mathcal{P}_+, 0 \leq t \leq f(y)\},
\]
Then \( m_0\mathcal{P}_+ \) is an integral polytope which is the moment polytope of \((\mathcal{X}, \mathcal{L})\).

Let \( \{F_A\}_{A=1, \ldots, d_+} \) be the outer facets of \( \mathcal{P}_+ \). Then each facet
\[
\hat{F}_A = \{(y, t) \in m_0\mathcal{P}_+ | y \in m_0F_A\}, \ A \in \{1, \ldots, d_+\},
\]
of \( m_0\mathcal{P}_+ \) corresponds to a \( \hat{G} \times \mathbb{C}^* \)-invariant divisors \( \hat{Y}_A \) of \( \mathcal{X} \). There are also prime boundary divisors \( \hat{Y}_\infty \) that corresponds to \( m_0\mathcal{P}_+ \cap \{0\} \) and \( \hat{Y}_{0,a}, a = 1, \ldots, N_f \) that corresponds to the piece
\[
\{(y, t) \in m_0\mathcal{P}_+ | t = m_0C_a - \Lambda_a(y), y \in \overline{\Omega_a}\}.
\]
Recall that \((\mathcal{X}, \mathcal{L})\) is a polarized \( \hat{G} \times \mathbb{C}^* \)-compactification. The colours are exactly given by the closures
\[
\hat{D}_a = \overline{D_a \times \mathbb{C}^*}, \quad D_a \text{ is a colour of } \mathcal{M}.
\]
Let \( Y_A \) be the \( \hat{G} \)-invariant divisor of \( \mathcal{M} \) that corresponds to \( F_A \). Since the divisor
\[
-K_M = \sum_A Y_A + 2 \sum _\alpha D_\alpha,
\]
we get
\[
-K_{\mathcal{X}} = \sum_A \hat{Y}_A + \sum _a \hat{Y}_{0,a} + \hat{Y}_\infty + 2 \sum _\alpha \hat{D}_\alpha.
\]
For \( a = 1, \ldots, N_f \), denote by \( m_a \) the smallest positive integer so that \( m_a\Lambda_a \in \mathfrak{N} \). As in [34] Proof of Theorem 14], the pull-back of \(-K_{\mathbb{C}P^1}\) by the projection \( \pi_{\mathcal{X}} : \mathcal{X} \to \mathbb{C}P^1 \) is
\[
(5.10) \quad -\pi_{\mathcal{X}}^* K_{\mathbb{C}P^1} = \hat{Y}_\infty + \sum _a m_a \hat{Y}_{0,a},
\]
and
\[
(5.11) \quad \mathcal{X}_0 = \sum _a m_a \hat{Y}_{0,a}.
\]
On the other hand, note that the Cartier line bundle \( K_M^{-m_0} \) has a \( \hat{B} \)-semi-invariant section of weight \( 2m_0\rho \) [11 Section 3.2.4]. As in Section 3.1, \( \mathcal{L} \) has a \( \hat{B} \times \mathbb{C}^* \)-semi-invariant section of weight \( 2m_0\rho \) whose divisor is
\[
\mathcal{L} = m_0 \left( \sum_A \hat{Y}_A + \sum _a m_a (C_a - 2\Lambda_a(\rho)) \hat{Y}_{0,a} + 2 \sum _\alpha \hat{D}_\alpha \right).
\]
Combing this with \(5.9\)-\(5.11\), we get
\[
D_c = -(\frac{1}{m_0}L + K_X - \pi_X^*K_{\mathbb{P}^1}) + cX_0 \\
= \sum_a (1 + cm_a - (C_a - 2\Lambda_a(\rho) + 1)m_a)\hat{Y}_a.
\]
Recall that \((X, -K_X)\) is always a log canonical pair [11, Section 5]. We get
\[
lct(X, D_0) := \sup\{c|(1 + cm_a - (C_a - 2\Lambda_a(\rho) + 1)m_a) \leq 1, \ a = 1, ..., N_f\}
\]
\[
= \min_a (C_a - 2\Lambda_a(\rho) + 1) = 1 + f(2\rho) = f(2\rho) + 1.
\]
Hence
\[
L^{N_A}(F) = f(2\rho)
\]
for any \(\hat{G}\)-equivariant normal \(\mathbb{Z}\)-test configuration \(F\).

For a general \(G \times G\)-equivariant normal \(\mathbb{R}\)-test configuration \(F\) with associated function \(f\), we can choose a sequence of approximating \(G \times G\)-equivariant normal \(\mathbb{Z}\)-test configurations \(F_p\) with associated function \(f_p\) constructed in Section 4.2. By the above computation,
\[
L^{N_A}(F_p) = f_p(2\rho), \ p \in \mathbb{N}_+.
\]
On the other hand, by [17] Remark 2.29, 3.32,
\[
\lim_{p \to +\infty} L^{N_A}(F_p) \leq L^{N_A}(F),
\]
with the equality holds if \(F\) is defined by a valuation on \(G\). Combining with (4.31) and (5.12) we get the Lemma. 

5.2. Minimizer of the H-invariant. In this section we study the minimizer of the H-invariant and finish the proof of Theorem 1.3.

Proof of Theorem 1.3. Let \(F\) be any \(\hat{G}\)-equivariant normal \(\mathbb{R}\)-test configuration and \(f\) the function defined in Theorem 4.1. Let \(\Omega_0\) be any of its domain of linearity that contains \(2\rho\). Then
\[
f|_{\Omega}(y) = (f(2\rho) + \Lambda(2\rho)) - \Lambda(y),
\]
for some \(\Lambda \in \mathbb{R}_+\). On the other hand, since \(f\) is concave,
\[
f(y) \leq (f(2\rho) + \Lambda(2\rho)) - \Lambda(y), \ \forall y \in \mathbb{R}_+.
\]
By Theorem 5.1 we get
\[
H(F) \geq \ln \left( \frac{1}{V} \int_{P_+} e^{-f(y)+f(2\rho)} \pi(y)dy \right)
\]
\[
\geq \ln \left( \frac{1}{V} \int_{P_+} e^{N(y-2\rho)} \pi(y)dy \right) =: \mathcal{H}(\Lambda).
\]
It is direct to check that \(\mathcal{H}(\cdot)\) is strictly convex and proper on \(\mathbb{R}_+\). It admits a unique minimizer \(\Lambda_0 \in \mathbb{R}_+\). By Corollary 4.6 there is a \(\hat{G}\)-equivariant special \(\mathbb{R}\)-test configuration \(F_{\Lambda_0}\) with H-invariant
\[
H(F_{\Lambda_0}) = \mathcal{H}(\Lambda_0) = \min_{\Lambda \in \mathbb{R}_+} \mathcal{H}(\Lambda).
Here the first equality follows from Theorem 5.1 and fact that $\mathcal{F}_{\Lambda_0}$ is special. Hence we get (1.3). On the other hand, by Corollary 2.10 $\mathcal{F}_{\Lambda_0}$ is also the minimizer of $H(\cdot)$ among all filtrations. We conclude that $\mathcal{F}_{\Lambda_0}$ is the semistable degeneration.

We want to test the K-polystability of the central fibre. The following barycenter condition will be used.

**Lemma 5.4.** Let $\mathcal{F}_{\Lambda_0}$ be the minimizer in Theorem 1.3. Suppose that $(\Lambda_0, 0) \in \mathfrak{t}$ satisfies (3.3)-(3.4). Set $\Xi_0 = \text{Span}_{\mathbb{R}_+} \alpha_1, ..., \alpha_i$. Then

$$ b(\Lambda_0) := \frac{\int_{P_+} y_i e^{\Lambda_0(y)} \pi dy}{\int_{P_+} e^{\Lambda_0(y)} \pi dy} \in 2\rho + \Xi_0. \quad (5.13) $$

**Proof.** Let $\{\varpi_i\}_{i=1, ..., r}$ be the fundamental weights with respect to $\Phi_{+, s}$. That is

$$ \varpi_i(\alpha_j) = \frac{1}{2} |\alpha_j|^2 \delta_{ij}, \quad 1 \leq i, j \leq r. $$

Hence the Weyl wall orthogonal to $\alpha_i$ is

$$ W_{\alpha_i} = \text{Span}_{\mathbb{R}} \{ \pm \alpha_1, ..., \pm \alpha_i-1, \alpha_i, \pm \alpha_i+1, ..., \pm \alpha_r \}. $$

By (3.4) we can write

$$ \text{RelInt}(\cap_{i=1, ..., i_0} W_i) \ni \Lambda_0 = \sum_{j=i_0+1}^r c_j \varpi_j, \quad c_j > 0. $$

Hence, $\Lambda_0$ is also an interior minima of $H|_{\cap_{i=1, ..., i_0} W_i(\cdot)}$. We have

$$ 0 = \frac{\partial H}{\partial \varpi_j}(\Lambda_0) = \varpi_j(b(\Lambda_0) - 2\rho) \int_{P_+} e^{\Lambda_0(y)} \pi dy, \quad j = i_0 + 1, ..., r. \quad (5.14) $$

On the other hand, $\Lambda_0$ is a boundary minima in the half space $\{ y | \varpi_i(y) \geq 0 \}$ for $i = 1, ..., i_0$. Thus

$$ 0 \leq \frac{\partial H}{\partial \varpi_i}(\Lambda_0) = \varpi_i(b(\Lambda_0) - 2\rho) \int_{P_+} e^{\Lambda_0(y)} \pi dy, \quad i = 1, ..., i_0. \quad (5.15) $$

Note that for any $i \in \{1, ..., r\}$,

$$ \{ y | \varpi_i(y) \geq 0 \} = \text{Span}_{\mathbb{R}_{\geq 0}} \{ \pm \alpha_1, \pm \alpha_1-1, \alpha_i, \pm \alpha_i+1, ..., \pm \alpha_r \}. $$

Combining the above relation with (5.14)-(5.15), we get (5.13).

Combining with Proposition 3.7 we have

**Proposition 5.5.** Suppose that $\mathcal{F}_{\Lambda_0}$ is the minimizer in Theorem 1.3 so that $\Lambda_0$ satisfies (3.3)-(3.4). Then the central fibre $X_0$ of $\mathcal{F}_{\Lambda_0}$ is $\mathcal{G}$-equivariantly modified K-semistable with respect to the vector field $\Lambda_0$. In addition, if (5.13) is strict, i.e.

$$ b(\Lambda_0) \in 2\rho + \Xi_0, \quad (5.16) $$

then $X_0$ is modified K-polystable and the Kähler-Ricci flow (1.1) on $M$ converges to $(X_0, \Lambda_0)$. 

□

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**Proposition 5.5.** Suppose that $\mathcal{F}_{\Lambda_0}$ is the minimizer in Theorem 1.3 so that $\Lambda_0$ satisfies (3.3)-(3.4). Then the central fibre $X_0$ of $\mathcal{F}_{\Lambda_0}$ is $\mathcal{G}$-equivariantly modified K-semistable with respect to the vector field $\Lambda_0$. In addition, if (5.13) is strict, i.e.

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then $X_0$ is modified K-polystable and the Kähler-Ricci flow (1.1) on $M$ converges to $(X_0, \Lambda_0)$. 

□
Proof. By Theorem 1.3 \( X_0 \) is normal. Thus it is a \( \mathbb{Q} \)-Fano spherical variety. To compute the combinatorial data of \( X_0 \) it is more convenient to realize it via a \( \mathbb{Z} \)-test configuration. For this purpose, we can slightly perturb \( \Lambda_0 \) to a rational \( \Lambda'_0 \) so that \( \mathcal{F}_{\Lambda'_0} \) is a \( \mathbb{Z} \)-configuration whose central fibre is also \( X_0 \). Note that by [1\textsuperscript{7} Section 2.2] (see [1\textsuperscript{7} end of p.9 - beginning of p.10]), \( \Lambda'_0 \) should be chosen in the Lie algebra of \( \exp(t\Lambda_0) \). Hence \( (\Lambda'_0, 0) \) also satisfies (3.3) - (3.4). By Proposition 3.7 the valuation cone of \( X_0 \) is given by (3.18). On the other hand, by Proposition 3.8 the moment polytope of \( (\Lambda'_0, K^{-1}) \) is also \( P_+ \). Recall the group \( L \) given by (3.14) and denote by \( Z(\hat{L}) \) its centre. By Lemma 3.3 and [30], \( \Lambda_0 \in \text{Aut}_G(X_0) = Z(\hat{L}) \). Hence by Lemma 5.4 and [11 Theorem 5.3], \( (\Lambda_0, \Lambda_0) \) is \( G \)-equivariantly modified K-semistable and \( \Lambda_0 \) is the soliton vector field on \( X_0 \). Note that by Corollary 4.6 \( \mathcal{F}_{\Lambda_0} \) induces the \( \exp(t\Lambda_0) \)-action on \( X_0 \).

Now we prove the second part. If (5.16) holds, we can show that \( (X_0, \Lambda_0) \) is (modified) \( \hat{G} \)-uniformly Ding-polystable in the sense of [16 Definition 5.17]. Once this is proved, [16 Theorem 6.3] will imply that the modified Ding functional (with respect to \( \Lambda_0 \)) on \( X_0 \) is \( \hat{G} \)-coercive. Note that a \( \mathbb{Q} \)-Fano spherical variety always has klt singularities (cf. [11 Section 5]). By [16 Theorem 3.5] \( X_0 \) admits a (singular) K"{a}hler-Ricci soliton (with soliton vector field \( \Lambda_0 \)). Hence \( (X_0, \Lambda_0) \) is modified K-polystable (cf. [16 Theorem 6.3]). By the uniqueness theorem [17 Theorem 1.3], the “polystable degeneration” is trivial. Hence \( X_0 \) is the limiting space of \( (X_k, \Lambda_k) \).

It remains to show the (modified) \( \hat{G} \)-uniform Ding-polystability provided (5.16) holds. Denote by \( W_L \) the Weyl group of \( \Phi_L \) (see Section 3.2.2 above). It is well-known that any \( \hat{G} \)-equivariant normal \( \mathbb{Z} \)-test configuration \( \mathcal{F} \) corresponds to a rational, concave, \( W_L \)-invariant piecewise linear function \( f \) on the \( W_L \)-invariant convex polytope \( P_0 := \cup_{w \in \Phi_+, L} w(P_+) \). As in the first part of the proof of Lemma 6.3 one gets

\[
L^{\Lambda_0}(\mathcal{F}) = f(2\rho).
\]

On the other hand, for the \( \mathbb{Z} \)-test configuration \( \mathcal{F} \), it holds

\[
\mathcal{F}^k R_k = \oplus_{\lambda \in \mathfrak{T}_{G^{\mathbb{Z}}}} \mathbb{C}[k \mathbb{R}(\lambda/(k))]_{\geq k} \text{End}(V_\lambda), \quad \forall k \in \mathbb{N}.
\]

Using the Riemann-Roch formula of [11 Section 1.4],

\[
\text{vol}_{e^{\Lambda_0}}(\mathcal{F}) = \lim_{n \to +\infty} \frac{n!}{k^n} \sum_{\lambda \in \mathfrak{T}_{G^{\mathbb{Z}}} \cap \mathfrak{m}, |k \mathbb{R}(\lambda/(k))| \geq k} e^{\Lambda_0(\lambda/k)} \dim(\text{End}(V_\lambda))
\]

\[
= \int_{\mathcal{P}_+ \cap \{f \geq s\}} e^{\Lambda_0(y)} \pi dy.
\]

Taking \( g = e^{\Lambda_0(y)} \) in the formula [16 Eq. (5.46)] and combining with the above relation, one concludes

\[
E^{\Lambda_0}(\mathcal{F}) = \frac{\int_{\mathcal{P}_+} s \text{vol}_{e^{\Lambda_0}}(\mathcal{F})}{\int_{\mathcal{P}_+} e^{\Lambda_0(y)} \pi dy} = \frac{\int_{\mathcal{P}_+} f e^{\Lambda_0(y)} \pi dy}{\int_{\mathcal{P}_+} e^{\Lambda_0(y)} \pi dy}.
\]

Thus the modified non-Archimedean Ding functional

\[
D^{\Lambda_0}(\mathcal{F}) := -E^{\Lambda_0}(\mathcal{F}) + L^{\Lambda_0}(\mathcal{F}) = \frac{\int_{\mathcal{P}_+} f e^{\Lambda_0(y)} \pi dy}{\int_{\mathcal{P}_+} e^{\Lambda_0(y)} \pi dy} + f(2\rho).
\]
Then we compute the modified non-Archimedean J-functional. Let $\mathcal{X}$ be the total space of $\mathcal{L}$. Then it is a polarized compactification of $\tilde{G}/H \times \{e\}$ with moment polytope $\mathcal{P}_\mathcal{L} = \{(y, t) | 0 \leq t \leq f(y), \ y \in P_+\}$. The modified non-Archimedean J-functional

$$J_{\Lambda_0}^{NA}(\mathcal{F}) = \frac{1}{L} \cdot \mathcal{L} \cdot L_\mathcal{CP}^n \cdot E_{\Lambda_0}^{NA}(\mathcal{F}).$$

By (5.17) it suffices to compute the first term on the right-hand-side. To compute the intersection number, we use the method of [33, Section 18]. Note that for any $\epsilon > 0$, the Newton polytope of the ample line bundle $L_\epsilon := \epsilon \mathcal{L} + L_\mathcal{CP}^1$ is $\mathcal{P}_\epsilon := \mathcal{P}_\mathcal{L} + (P_+ \times \{0\}) \subset \mathfrak{M}_R \oplus \mathbb{R}$ (see Figure-1 below). By [33, Corollary 18.28],

$$\frac{1}{(n+1)!} (\mathcal{L}_\epsilon)^{(n+1)} = \int_{\mathcal{P}_\epsilon} \pi dy \wedge dt = \epsilon \max f \cdot \int_{P_+} \pi dy + O(\epsilon^2), \quad \epsilon \to 0^+.$$

Hence

$$\mathcal{L} \cdot L_\mathcal{CP}^n = n! \frac{d}{d\epsilon}_{\epsilon=0} (\mathcal{L}_\epsilon)^{(n+1)} = n! \max f \cdot \int_{P_+} \pi dy = L^n \cdot \max f.$$

Thus we get

$$J_{\Lambda_0}^{NA}(\mathcal{F}) = \frac{\int_{P_+} (\max f - f)e^{\Lambda_0(y)}\pi dy}{\int_{P_+} e^{\Lambda_0(y)}\pi dy}.$$

Recall Lemma [33, Lemma 3.5]. The twist $\mathcal{F}(\xi)$ of $\mathcal{F}$ by an element $\xi$ in $\mathfrak{z}(\text{aut}(\mathcal{X}_0)) = \mathfrak{z}(\hat{l})$ is associated to the function $f_\xi(y) = f(y) + \xi(y)$. Hence

$$J_{\Lambda_0}^{NA}(\mathcal{F}(\xi)) = \frac{\int_{P_+} (\max f_\xi - f_\xi)e^{\Lambda_0(y)}\pi dy}{\int_{P_+} e^{\Lambda_0(y)}\pi dy}.$$

Note that $\xi \in \mathfrak{z}(\hat{l})$ is $W_L$-invariant.

On the other hand, using the argument of [22, Proposition 4.5], one can prove there is a constant $\epsilon_0 > 0$ so that

$$D_{\Lambda_0}^{NA}(\mathcal{F}) \geq \epsilon_0 J_{\Lambda_0}^{NA}(\mathcal{F})$$

holds for any concave $W_L$-invariant function $f$ satisfying the normalized condition

$$\max f = f(2\rho - 2\rho_L) = 0.$$

In fact, we apply the argument of [22, Proposition 4.5] to the convex function $u = -f$ and weight $e^{\Lambda_0(y)}\pi(y)$. 
Here $2\rho - 2\rho_L = \sum_{\alpha \in \Phi_+ \setminus \Phi_{+L}} \alpha \in z^+(l)$. Note that $f$ can always be normalized by subtracting a $W_L$-invariant affine function. We get

$$D_{\Lambda_0}^{\text{NA}}(\mathcal{F}) \geq \epsilon_0 \inf_{\xi \in \hat{z}(l)} J_{\Lambda_0}^{\text{NA}}(\mathcal{F}(\xi)).$$

Hence $(\lambda_0, \Lambda_0)$ is modified $\hat{G}$-uniformly Ding-polystable.

**Remark 5.6.** Combining (5.13) (or (5.16), respectively) with [11, Theorem 5.3], one directly concludes that $(\lambda_0, \Lambda_0)$ is $\hat{G}$-equivariantly modified K-semistable (or K-polystable, respectively). Apply the arguments of [37, Theorem 1.1] to the weighted $\delta$-invariant defined in [4, Section 4.1] instead of the usual one, one concludes the corresponding modified K-stability regardless the $\hat{G}$-action.

**Remark 5.7.** When $\Lambda_0 \in \hat{z}(g)$, we see that $\hat{H}_0 = \text{diag}(G) \times \mathbb{C}^*$ and the corresponding $\mathcal{F}_{\Lambda_0}$ is indeed a product test configuration. In this case $\lambda_0 = M$. If in addition (5.16) holds, then $M$ admits a Kähler-Ricci soliton. See also [21, Section 5].

### 6. Application to $SO_4(\mathbb{C})$-compactifications

In [11, Example 5.12], Delcroix showed two K-unstable smooth Fano compactifications of $SO_4(\mathbb{C})$ by giving their moment polytopes. In this section we will determine the limits of (1.1) on these $SO_4(\mathbb{C})$-compactifications by using Theorem 1.3 and Proposition 5.5. In this way we show Theorem 1.4.

To describe the polytopes in detail, choose a coordinate on $a^*$ such that the basis are the generator of $\mathcal{M}$. Then the positive roots are $\alpha_1 = (1, -1)$, $\alpha_2 = (1, 1)$. Thus, $2\rho = (2, 0)$,

$$a^*_+ = \{x > y > -x\}, \quad 2\rho + \Xi = \{-2 + x > y > 2 - x\},$$

and $\pi(x, y) = (x - y)^2(x + y)^2$.

For both of $P_+$, the barycenter of $P_+$, the barycenter $b(0) \notin \overline{2\rho + \Xi}$. Hence the corresponding $SO_4(\mathbb{C})$-compactifications admit no Kähler-Einstein metrics. Moreover, the Futaki invariant vanishes since the center of automorphisms group is finite. Hence there are also no other Kähler-Ricci solitons on those compactifications. It is proved in [23] that the Kähler-Ricci flow on them develops Type-II solutions.

![Figure-2](image-url)
Case-(1). The polytope is
\[ P_+ = \{ y > -x, x > y, 2 - x > 0, 2 + y > 0, 3 - x + y > 0 \}. \]
By using a Wolframe Mathematica 8 program, we get the critical point of \( \mathcal{H}(\cdot) \),
\[ \Lambda_0 = s(1, -1), \text{ where } s \in (0.15210775, 0.15210800). \]
We see that \( \Lambda_0 \in \ker(\alpha_2) \). We can write \( \mathcal{X}_0 \) as a \( \hat{G}/H_0 \)-compactification where \( \hat{G} = SO_4(\mathbb{C}) \times SO_4(\mathbb{C}) \), and \( H_0 \subset \hat{G} \) whose Lie algebra
\[ \mathfrak{h}_0 = \mathbb{C}(\alpha_2, \alpha_2) \oplus \mathbb{C}(\alpha_1, 0) \oplus (\mathbb{C}(X_{\alpha_2}, X_{\alpha_2}) \oplus \mathbb{C}(X_{-\alpha_2}, X_{-\alpha_2})) \oplus (\mathbb{C}(0, X_{\alpha_1}) \oplus \mathbb{C}(X_{-\alpha_1}, 0)). \]
Thus the valuation cone
\[ \mathcal{V}(\hat{G}/H_0) = \{(x, y)|\alpha_2(x, y) = x + y \geq 0 \}. \]
The polytope of \( \mathcal{X}_0 \) remains the same as (1).
It is direct to check that
\[ \alpha_2(b(\Lambda_0) - 2\rho) > 0. \]
By Proposition 5.5, we see that the limit \( \mathcal{X}_0 \) is indeed modified K-polystable with respect to \( \Lambda_0 \). Thus \( (\mathcal{X}_0, \Lambda_0) \) is the desired limit.
Case-(2). The polytope is
\[ P_+ = \{ y > -x, x > y, 2 - x > 0, 2 + y > 0, 3 - x + y > 0, 5 - 2x + y > 0 \}. \]
Again, by using a Wolframe Mathematica 8 program, we can check that \( \mathcal{H}(\cdot) \) has no critical point in \( 2\rho + \partial\Xi \). Hence
\[ \Lambda_0 \in 2\rho + \text{RelInt}(\Xi), \]
which lies in neither \( \ker(\alpha_1) \) nor \( \ker(\alpha_2) \). We see that the central fibre \( \mathcal{X}_0 \) is a \( \hat{G}/H_0 \)-compactification with \( \hat{G} = SO_4(\mathbb{C}) \times SO_4(\mathbb{C}) \), and \( H_0 \subset \hat{G} \) whose Lie algebra
\[ \mathfrak{h}_0 = \mathbb{C}(\Lambda_0, 0) \oplus \mathbb{C}(\Lambda_0^\perp, \Lambda_0^\perp) \oplus (\oplus_{i=1,2}(\mathbb{C}(0, X_{\alpha_i}) \oplus \mathbb{C}(X_{-\alpha_i}, 0))). \]
Here \( \Lambda_0^\perp \) is any vector in a orthogonal to \( \Lambda_0 \). Since \( \Lambda_0 \) does not perpendicular to any simple root, by Proposition 5.3 the central fibre \( \mathcal{X}_0 \) is a horospherical variety. Hence it always admits a Kähler-Ricci soliton with soliton field \( \Lambda_0 \).

7. Appendix: An algebraic proof of Corollary 4.6

In this Appendix, we give an algebraic proof of Corollary 4.6. Recall that a \( \hat{G} \)-equivariant normal \( \mathbb{R} \)-test configuration \( \mathcal{F} \) with reduced central fibre is special if and only if \( \text{Gr}(\mathcal{F}) \) is an integral ring. It suffices to show

**Proposition 7.1.** Suppose that (1.28) holds. Then the algebra \( \text{Gr}(\mathcal{F}) \) defined by (2.3) is integral if and only if \( f \) is affine on \( P_+ \).

**Proof.** Assume that \( \text{Gr}(\mathcal{F}) \) is integral. We show that \( f \) is affine. Otherwise, we can take two domains of linearity \( Q_1, Q_2 \subset P_+ \) so that they intersect along a common facet. Take \( \lambda_i \in Q_i \cap \mathfrak{M}_Q \) so that the line segment \( \lambda_1 \lambda_2 \subset Q_1 \cup Q_2 \). Up to replacing \( P_+ \) by some \( k_0P_+ \), we can assume \( \lambda_i \in Q_i \cap \mathfrak{M} \) for \( i = 1, 2 \).
Let $\sigma_1 \in \text{End}(V_{\lambda_1})$ be a highest weight vector. Then $\sigma_1$ has real weight $t^{\lambda_1^{(1)}}$. Consequently, $\sigma_1 \otimes \sigma_2 \in \text{End}(V_{\lambda_1+\lambda_2})$ has real weight $t^{\lambda_1^{(1)}-\lambda_2^{(1)}}$. On the other hand, for $\lambda_1 + \lambda_2 \in 2P_+ \cap \mathfrak{m}$, by [128],

$$s_{\lambda_1+\lambda_2}^{(2)} = 2f(\frac{1}{2}(\lambda_1 + \lambda_2)) > s_{\lambda_1}^{(1)} + s_{\lambda_2}^{(2)},$$

where the last inequality follows from the concavity of $f$ and the fact that $\lambda_i$'s lie in different domains of linearity. Hence $\sigma_1 \cdot \sigma_2 = 0$ in $\text{Gr}(\mathcal{F})$. A contradiction to the assumption that $\text{Gr}(\mathcal{F})$ is integral.

Conversely, assume that $f$ is affine. We will show that $\text{Gr}(\mathcal{F})$ is integral. Otherwise, there are $(0 \neq) \sigma_i \in \text{End}(V_{\lambda_i}), i = 1, 2$ so that $\sigma_1 \cdot \sigma_2 = 0$ in $\text{Gr}(\mathcal{F})$. By Lemma 4.2, we can assume that each $\sigma_i$ is a highest weight vector. Assume that $\lambda_i \in k_iP_+$. Then by [128], $\sigma_1$ has real weight $t^{-k_1f(\lambda_i/k_1)}$. Consequently, $\sigma_1 \otimes \sigma_2 \in \text{End}(V_{\lambda_1+\lambda_2})$ has real weight $t^{-k_1f(\lambda_1/k_1)-k_2f(\lambda_2/k_2)}$ with

$$k_1f(\frac{\lambda_1}{k_1}) + k_2f(\frac{\lambda_2}{k_2}) = (k_1 + k_2)f(\frac{\lambda_1 + \lambda_2}{k_2 + k_2}).$$

Here we used the fact that $f$ is affine. Note that the right-hand side is just the real weight of the $\text{End}(V_{\lambda_1+\lambda_2})$-piece in $\text{Gr}(\mathcal{F})$. Hence $\sigma_1 \cdot \sigma_2 \neq 0$, a contradiction. The Proposition is proved. \hfill \Box

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