Wormholes, geometric flows and singularities

Oscar Lasso Andino a and Christian L. Vásconez b

a Escuela de Ciencias Físicas y Matemáticas, Universidad de Las Américas,
C/. José Queri, C.P. 170504, Quito, Ecuador
b Departamento de Física, Escuela Politécnica Nacional,
C/. Ladrón de Guevara E11-253, C.P. 170525 Quito, Ecuador

Abstract

In this paper we study the evolution of wormhole geometries under intrinsic geometric flows. We make evolve numerically the time symmetric foliations of a family of spherically symmetric asymptotically flat wormholes under the Ricci flow and under the RG-2 flow. We use some theorems adapted from the compact case for studying the evolution of different wormhole types, specially those with high curvature zones. Some metrics expand and others contract at the beginning of the flow, however, all metrics pinch-off at certain time. We present a numerical study of the evolution of wormhole singularities in three dimensions extending the theoretical estimations. Finally, we calculate numerically the Hamilton’s entropy of the surface and show that it is monotonous through the evolution.

aE-mail: oscar.lasso[at]udla.edu.ec
bE-mail: christian.vasconez[at]epn.edu.ec
1 Introduction

Wormhole spacetimes arise as quite unconventional solutions of the Einstein equations. Nowadays, they are at the center of the debate in the high energy physics community. Even though wormholes have not been measured experimentally, they seem to be very useful for studying theoretical aspects of holography [1, 2]. In [2], they use the property that an AdS wormhole appearing in AdS gravity have as a dual the thermofield double state of a quantum system\(^1\).

In the gravity side, wormholes have been studied extensively. We want to use these geometries for studying the evolution under a geometric flow of asymptotically flat spacetimes. Wormhole geometries are ideally suited for studying singularities in this kind of spacetimes. And although the evolution theorems for controlling curvature are usually stated for compact spacetimes, some results have been proved for asymptotically flat metrics [13]. In particular, we want to know how asymptotically flat metrics evolve under Ricci flow, and under its higher curvature correction: the RG-2 flow. We know that both flows will develop singularities. However, it is not known the type of singularities, and to what extent we can control the evolution of higher curvature zones. The RG-2 flow is the two loop approximation of the renormalization group of the non-linear sigma model. We show that the flow develops singularities in finite time when evolving a spherically symmetric asymptotically flat spacetime with a throat. Due to the fact that these geometries are Lorentzian, we take a time symmetric foliation in order to build the corresponding Riemannian metric\(^2\).

Geometric flows are a powerful tool for studying different problems in differential geometry. The most important result is the proof of the Thurston geometrization theorem [3, 4] by using the Ricci flow. These flows are usually very difficult to treat because of the appearance of singularities. Therefore, it is interesting to see how different geometries behave and develop singularities under a given flow.

Given a Riemannian manifold \(M^n\) whose metric \(g(\lambda)\) is parametrized by an affine parameter, we define an intrinsic geometric flow as an evolution equation of this metric. The evolution generates a family of metrics \(g(\lambda)\) through the equation

\[
\frac{\partial g_{ab}}{\partial \lambda} = \beta_{ab}(g(\lambda)); \quad g_{ab}(0) = \tilde{g}_{ab},
\]

where \(\beta_{ab}\) is a tensor built by \(g_{ab}(\lambda)\), together with its first and second derivatives. In particular, when \(\beta_{ab}(g(\lambda)) = -2R_{ab}(g(\lambda))\), we obtain the very well known Ricci flow. This flow develops singularities, and there has been a long debate about how to treat them.

There are analytical and numerical results about the evolution of spacetimes (or its Riemannian foliations) for different flows, in different contexts. In [12], the authors employed the Ricci flow as a gradient-flow in order to study the vacuum configuration of 4-dimensional

\(^1\)The thermofield double state is in the Hilbert space made of two copies of the original system. Both quantum systems can be coupled, and it is possible to make the wormhole traversable [1]. This mechanism can be used as a protocol for sending information between the two systems.

\(^2\)The evolution of the metric of a Riemannian manifold is attached to the evolution of a Lorentzian manifold built with the original Riemannian metric, where the evolution is perfectly controlled.
pure Euclidean gravity with boundary $S^1 \times S^2$, in a box of a given radius, and in a canonical ensemble of a given temperature$^3$. In the context of General Relativity, an analytic study was performed in [13], where it is shown that the area of a surface in a time-symmetric foliation of spacetime is monotonous under RG-2 flow. Moreover, it was shown that certain types of asymptotically flat metrics do not develop apparent horizons when evolved under the RG-2 flow, showing the path to studying higher flows. In [5], the authors present a numerical study of singularities in compact surfaces by making a comparison between an evolution of a family of metrics in $S^3$, with a $S^2$ neck-pinching. They found a certain type of critical behavior near the singularity formation. This critical behavior will depend of the amount of corseting. When the corseting is large, the geometry develops an $S^2$ singularity; but when it is small, the geometry converges to a sphere. An apparent similar critical behavior has been found in other types of metrics and different situations$^4$. Thus, some kind of critical behavior is expected when evolving a metric under these flows. However, remembering that the Ricci flow is the first loop approximation for the renormalization-group flow of the non-linear sigma model, it is not enough when dealing with higher curvatures. In these cases, it is necessary to consider a higher curvature flow, the RG-2 flow. We hope that criticality is going to be “enhanced”$^5$ in some way. In [10], starting with the spatial part$^6$ of a wormhole metric, the authors have found a seemingly critical behavior, which is related to the type of wormhole$^7$ considered as initial metric. Using a more powerful numerical method, we will refine the latter results.

In this work, we evolve some wormhole geometries for larger times, (compared with those presented in [10]), and for geometries with higher curvature points. We also discuss about evolution under higher loop flows. Our method can be extended to more general flows, namely flows coupled to matter coming from the string theory. The evolution of asymptotically flat spaces, under the Ricci flow, is studied in [11]. These theorems will help us to approach the problem thinking about extending the results to higher order flows. In two$^9$ and three$^{24}$ dimensions, it is shown that the Ricci and RG-2 flows maintain asymptotic flatness during evolution.

This article is structured as follows. In Section 2 we make a brief review of the flows that we are going to use. In Section 3, we study analytically the evolution of a foliation of the Morris-Thorne wormhole under the Ricci flow, which underline the conditions for our numerical setup. In Section 4, we numerically compute the evolution of the spatial sections of the Morris-Thorne wormhole under the RG-2 flow. We continue this Section evolving different asymptotically flat geometries under the RG-2 flow. In Section 5, we discuss about the results and future work directions.

---

$^3$This numerical approach is used to find fixed points of the flow which in this particular set up, at high temperatures, corresponds to hot-flat space and two black holes: a small one and a large one

$^4$See [8] for the study of critical behavior in gravitational collapse.

$^5$The enhancement means that it will take more time, compared with the Ricci flow, to develop singularities. However, singularities will always arise at a long time.

$^6$They consider a time-symmetric foliation.

$^7$The shape function of the wormhole is parametrized by an exponent, which is associated with different types of wormholes. In particular, $\delta = 2$ is the Ellis wormhole.
2 Geometric flow equation

We will use geometric flows in order to evolve asymptotically flat wormhole metrics. We take a wormhole spacetime in the wide sense, namely a spacetime that has a minimal surface \[ 15 \]. In particular, we focus on non-compact spacetimes, which are taken as asymptotically flat. The Ricci flow is the first loop approximation of the non-linear sigma model renormalization group flow equations. The second loop approximation is the RG-2 flow \[ 6 \], which is reduced to the Ricci flow when the coupling constant is set to zero, after re-escalation. The Ricci flow had been successfully used when studying the evolution of different physical quantities \[ 7, 9, 13 \]. Even when there is no mathematical formulation of the quantization of the non-linear sigma model, it is considered that the approximation with the Ricci flow only works when curvature is small. We want to explore what happens when higher curvature flows are taken into account. Certainly, far from the throat, we expect a similar behavior as the Ricci flow. However, when curvatures go higher, near the throat, we expect to see the influence of the second loop term.

The RG-2 flow with the DeTurck term is given by

\[
\partial g_{ij}/\partial \lambda = -2R_{ij} - \frac{\alpha}{4} R^{\alpha\beta\gamma\delta} R_{j}^{\alpha\beta\gamma} + \nabla_{(i} V_{j)}. \tag{2.1}
\]

This flow is parabolic only in the zones where \( 1 + \alpha K_{ab} > 0 \), where \( K_{ab} \) is the sectional curvature of the manifold \[ 16, 17 \]. When \( \alpha = 0 \) (Ricci flow), the flow is weakly parabolic everywhere. We take a time-symmetric foliation of a spacetime\(^8\), which is in practice tantamount to drop-out the time component of the metric, getting as a result a 3-dimensional Riemannian metric. Later, we will evolve this resulting Riemannian metric. The parameter \( \lambda \) (not to be confused with physical time) is an affine parameter that labels a continuum family of Riemannian metrics, and the \( V^i \) vector generates diffeomorphisms along the flow.

3 Evolution of the Morris-Thorne wormhole

We are ready for evolving a metric. We start finding analytically the evolution of the time-symmetric foliation of the Morris-Thorne metric \[ 18 \]:

\[
ds^2 = \frac{1}{(1 - \frac{b(r, \lambda)}{r})} dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2), \tag{3.1}
\]

where \( b(r, \lambda) \) is the shape function. It satisfies that \( b(r_0, 0) = r_0 \), at the throat \( r_0 \). Replacing the latter equation in the system (2.1), we can compute the components of the RG-2 flow equation:

\[
\frac{\partial b}{\partial \lambda} = \frac{2(b - rb')}{r(r - b)} \left( 1 - \frac{\alpha b - rb'}{4r^3} \right) ; \tag{3.2}
\]

\[
0 = \left( \frac{b + rb'}{2r} \right) + \frac{\alpha}{2} \left( \frac{5b^2 - 2rb'b + r^2 b'^2}{2r^4} \right). \tag{3.3}
\]

\(^8\)A time symmetric foliation gives us a Riemannian manifold, whose extrinsic curvature is zero.
The equation (3.2) is an evolution equation for $b(r, \lambda)$. Meanwhile, equation (3.3) is a restriction over the shape function $b(r, \lambda)$.

Let us analyze the case when $\alpha = 0$. Here, equations (3.2) and (3.3) are reduced,

\[
\frac{\partial b}{\partial \lambda} = \frac{2(b - rb')}{r^2}; \tag{3.4}
\]

\[
0 = \frac{b + rb'}{r}. \tag{3.5}
\]

Figure 1: $C_r(\lambda)$ as a function of $r$, for $\lambda = 1, .., 10$. The initial metric has a throat at $r_o = 1$. The red-horizontal line is the size of the throat at $\lambda = 0$.

The restriction equation (3.5) can be solved, leading to $b(r, \lambda) = C_r(\lambda)/r$. It shows that the shape function will change when it is evolved under the flow. In particular, replacing $b(r, \lambda)$ in equation (3.4), we obtain\(^\text{10}\):

\[
C_r(\lambda) = -r^2 W \left[ \frac{b_o^2}{r^2} \exp \left( \frac{4\lambda - b_o^2 r}{r^3} \right) \right], \tag{3.6}
\]

where $W$ is the Lambert function\(^\text{11}\). In Fig. 1, we plot the evolution of $C_r(\lambda)$, for different values of $\lambda$. The red line is the constant value $b_o = 1$ (size of the throat at $\lambda = 0$). Then, after the evolution under the Ricci flow, the shape function starts changing, and therefore the throat will change its size. Moreover, the shape function far from the throat will not change showing that asymptotic flatness is maintained during evolution. It is very difficult to see how this change happens in these coordinates. This is the simplest solution of the system, if we want to go further, numerical methods are necessary. Moreover, when we consider the RG-2 flow, it is really difficult to find an analytic solution of the system (3.4)-(3.5). In order to study what is really happening at the throat, we have to change to a more useful set of coordinates. We need a coordinate system that somehow helps us to see the evolution of the size of the throat directly.

\(^9\)In this Section, we consider the RG-2 flow without the DeTurck term
\(^\text{10}\)We have assumed that $\frac{dr}{d\lambda} = 0$
\(^\text{11}\)The Lambert function is defined as the inverse function of $f(W) = We^W$. 

5
3.1 Initial wormhole metric

The metric (in its original form) does not let us see the evolution of the throat size. Thus following [10], we take as the initial metric the spherically symmetric ansatz

\[ ds^2 = e^{2\Psi(\lambda, \rho)} \left( d\rho^2 + R^2(\lambda, \rho)(d\phi^2 + \sin^2(\theta)d\phi^2) \right), \]  

(3.7)

with a DeTurck vector defined -because of spherical symmetry- as:

\[ V^i = V(\lambda, \rho) \partial_\rho. \]  

(3.8)

We will take \( \Psi(t, \rho) = 0 \). Therefore, the metric (3.7) is reduced to:

\[ ds^2 = d\rho^2 + R^2(\lambda, \rho)(d\theta^2 + \sin^2(\theta)d\phi^2). \]  

(3.9)

Using the latter metric and equation (3.8), we obtain the RG-2 flow equations (2.1):

\[ \frac{\partial R}{\partial \lambda} = \partial^2_\rho R + \frac{(\partial_\rho R)^2}{R} - \frac{1}{R} - \frac{\alpha}{8} \left[ \frac{(\partial_\rho R)^2}{R} \right] + V \partial_\rho R; \]  

(3.10)

\[ \partial_\rho V = -2 \frac{\partial^2_\rho R}{R} - \frac{\alpha}{2} \frac{(\partial^2_\rho R)^2}{R^2}. \]  

(3.11)

We note that when \( \alpha = 0 \) the Ricci flow equation is recovered, with its correspondent restriction. In the next Section, we will solve numerically the equations system (3.10)-(3.11). For the moment, we start solving the Ricci flow system. A first approach was made in [10], where the authors suggested that for a given family of wormholes there is a criticality (when \( \delta = 1.259 \)). This means that they proposed a metric that remains the same when it is evolved by the Ricci flow. Our results confirm this statement, but in a limited given zone. However, if the flow evolve further, the throat will always develop a singularity. It is worth to note that in [10], combinations of pure finite-difference schemes are employed, while we will use a pseudo-spectral method (appropriate for studying viscous fluids) to comparing the evolution of different initial metrics, under the Ricci and the RG-2 flows, respectively. In order to do so, we need to set initial and boundary conditions.

The asymptotic flatness in both directions is translated to the Dirichlet conditions

\[ R(\lambda, \rho_{\text{max}}) = R_{\text{max}}; \]  

(3.12)

\[ V(\lambda, \rho_{\text{max}}) = 0. \]  

(3.13)

The condition about the existence of the throat is imposed as a Newman condition

\[ \partial_\rho R|_{\rho=0} = 0. \]  

(3.14)

The asymptotic flatness imposes that \( R \sim |\rho| + \mathcal{O}(l_1 \ln (|\rho|/l_2)) \). Then, using equation (3.11), we are able to show that \( V \sim \mathcal{O}(1/\rho^5) \). It is our immediate goal to find an initial metric, and a \( V \) vector, that satisfies all the latter requirements.
The time symmetric foliation of the well known Morris-Thorne wormhole [18] is

\[
\left.\begin{array}{l}
ds^2 = \frac{1}{g(r)}dr^2 + r^2(d\theta^2 + \sin(\theta)^2d\phi^2); \\
g(r) = 1 - \frac{b(r)}{r},
\end{array}\right\} (3.15)
\]

If the shape function \(b(r)\) is written as

\[
b(r) = \frac{b_0}{r^{\delta-1}}.
\] (3.16)

where \(b_0\) has units of length and \(\delta > -1/2\); the metric (3.15) becomes

\[
ds^2 = \frac{dr^2}{1 - (\frac{\rho}{r_0})^\delta} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2).
\] (3.17)

In order to write this wormhole metric in the form (3.9), we set

\[
\left.\begin{array}{l}
r = R(0, \rho); \\
\partial_\rho R(0, \rho) = \sqrt{g(R(0, \rho))}.
\end{array}\right\} (3.18, 3.19)
\]

Then, it is direct to obtain

\[
\left.\frac{\partial R(0, \rho)}{\partial \rho} = \sqrt{1 - \left(\frac{r_0}{R(0, \rho)}\right)^\delta}\right. (3.20)
\]

The solution to the previous differential equation is given in terms of the hypergeometric function \(\text{$_2F_1(a, b, c, z)$}\) as

\[
\text{$_2F_1\left(\frac{1}{2}, -\frac{1}{\delta}, \frac{1 - \delta}{\delta}, \left(\frac{r_0}{R(0, \rho)}\right)^\delta\right)$} R(0, \rho) = |\rho| + \text{$_2F_1\left(\frac{1}{2}, -\frac{1}{\delta}, \frac{1 - \delta}{\delta}, (r_0)^\delta\right)$},
\] (3.21)

where we have used the condition \(R(0, 0) = 1\) for determining the integration constant.

From the previous equation, we see that \(R(0, \rho)\) is dependent on lambda. When \(\delta = 2\) we obtain the Ellis wormhole, and equation (3.21) would be reduced to \(R^2(0, \rho) = \rho^2 + r_0^2\). In Figure 3 we have plotted the initial condition \(R(\rho, 0)\) with \(\delta = 1.3\). In the right panel, we can see a revolution plot of (3.21). We can see that there is a zone of saddle points (the throat). In compact 3-dimensional manifolds it is expected that the singularity appears at the throat. As we focus in asymptotically-flat spaces, we expect that asymptotic flatness forces to slow down the appearance of singularities, although at the end they will arise.

### 4 Wormhole evolution

Contrary to what happens with the Ricci flow, the evolution of any Riemannian metric under RG-2 flow has a restriction over the value of the scalar curvature. In other words,
the system (3.10)-(3.11) is going to be a parabolic system if and only if $1 + \alpha R_s/4 > 0$, where $R_s$ is the scalar curvature of the metric. When solving the system numerically, we have to ensure that the restriction is satisfied. Moreover, we will normalize using the length scale $b_o$. Therefore, at the end, the equations will be independent of $b_o$. Thus,

$$
\hat{R} = R/b_o; \quad \hat{\rho} = \rho/b_o; \quad \hat{t} = t/b_o^2.
$$

(4.1)

In what follows, we will omit the hat-notation. The RG2-flow system has been solved in a periodic spatial domain $D = [-L, L]$, which is discretized with $N_\rho$ grid points in $D$. For the numerical evolution, we implement a pseudo-spectral method. The advances in $\lambda$ will be done with finite differences. However, as we want to study initial conditions involving high gradients in $D$, spectral methods are employed to computing first- and second-order derivatives respect to $\rho$. Computational cost of this method is similar to that of the pure finite-differences approach (not shown here), due to the domains size. The step $\Delta \lambda$ has been chosen in such a way that the Courant-Friedrichs-Lewy condition is always satisfied. For all of our cases of study, the dimensionless function $V$ has to satisfy the conditions explained in Section 3.1. Thus, we use

$$
V = \left( \frac{\rho - L}{10L} \right)^2 \exp \left[ -\frac{(\rho - \mu)^2}{2L} \right],
$$

(4.2)

with $\mu = -3\pi + \sqrt{8 + 9\pi^2}/2$. In the system (3.10)-(3.11), the value $\alpha$ has to be taken in such a way that the influence of the higher curvature term becomes important. Remembering that the RG-2 flow comes from a perturbative expansion, we will work with $\alpha$’s as small as possible. Instead, we will look for initial metrics with zones with big curvatures. The family of metrics (3.21) is parametrized by $\delta$. When $\delta = 1$, we obtain the spatial sections of the Schwarzschild wormhole. When $\delta = 2$, we get the Morris-Throne wormhole. We will explore what happens during the evolution of metrics defined by different values of $\delta$. 

Figure 2: Initial condition $R(\rho, 0)$, for $\delta = 1.3$. 


4.1 The Ricci flow

It has been suggested that, under Ricci-flow evolution, the wormhole pinches off (or expand) at the throat forever, depending on the values of initial data parameters \([10]\). Here, we present a numerical study of the evolution of the initial condition \(R(0, \rho)\) (given in (3.21)) under the Ricci flow. In Table 1, we present the setup for three different runs. These parameters were chosen in order to test the evolution at different characteristic scales.

| RUN  | \(L\)  | \(N_{\rho}\) |
|------|--------|-----------|
| RUN 1| \(2\pi\) | 64        |
| RUN 2| \(4\pi\) | 256       |
| RUN 3| \(10\pi\) | 512       |

Table 1: Simulations setup.

Figure 3 shows the initial conditions \(R(\rho, 0)\) used for RUN 1 (left panel), RUN 2 (middle panel), and RUN 3 (right panel). Each initial condition is a numerical solution of equation (3.21), for different values of \(\delta\). In particular, we take

\[
\delta = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.1, 1.2, 1.253, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9\}.
\]

In this Figure, the scale of colors departs from black, for \(\delta = 0.1\) (lower line), and arrives to light orange, for \(\delta = 1.9\) (higher line). We will keep the same scale of colors throughout this Section. We note that for all of the initial conditions, \(R(0, 0) = 1\).

The set of initial conditions, for RUN 1, RUN 2, and RUN 3 are evolved with the Ricci flow, i.e., setting \(\alpha = 0\) in the system (3.10)-(3.11). In particular, we follow the minimum point of the throat \(R_{th} \equiv R(0, \lambda)\), as a function of \(\lambda\). The three panels of Figure 4 report the evolution of \(R_{th}\) for RUN 1 (left panel), RUN 2 (middle panel), and RUN 3 (right panel), and for the different values of \(\delta\), previously chosen. We note that for early stages of the simulation, \(\lambda \lesssim 0.5\), no difference can be seen among the three runs, for the corresponding values of \(\delta\). After this initial stage, and for values of \(\delta > 0.3\), the characteristic length
(dimension of the simulation box) affects the evolution of $R_{th}$, taking this value quicker to zero when $D$ decreases. In fact, for this range of values of $\delta$, $R_{th}$ remains positive for longer periods of $\lambda$ depending on $D$. We could name $\delta \approx 0.7$ as a “critical” value. However, it is clear, when comparing the three runs, that this behavior is apparent. As it is known from the mathematical literature, these kind of intrinsic flows smooth-out the geometries when diffusing curvature from high curvature points to the lower ones. In our case, the asymptotic flatness of the metric plays an important role. Even though the asymptotic flatness tend to slow down the convergence to a singularity, at the end the flow wins and the singularity appears. Moreover, we note that the increase (or reduction) of the throat is not unlimited. In the cases of initial conditions with $\delta < 0.7$, the throat decreases from the beginning of each one of our simulations. However, in the cases of initial conditions with $\delta \geq 0.7$, the increment of $R_{th}$ is temporal. Even for RUN 3, where no decrease exist in the chosen interval of $\lambda$ (when $\delta \geq 0.7$), we could presume that $R_{th}$ will eventually decrease if a longer interval is chosen.

Figure 4: Evolution of $R_{th}$ in function of $\lambda$, using different values of $\delta$ (see the text) in RUN 1 (left panel), RUN 2 (middle panel), and RUN 3 (right panel). The scale of colors departs from black ($\delta = 0.1$), and arrives to light orange ($\delta = 1.9$).

4.2 From Ricci flow to RG-2 flow

Now, we study the evolution of the wormhole metric (3.9) under the RG-2 flow, i.e., we take $\alpha \neq 0$ when solving the system (3.10)-(3.11). Once we consider higher curvatures, the Ricci flow is not a good approximation anymore, we must consider higher curvature flows. The RG-2 flow will control better the evolution of curvature. We have solved numerically the RG-2 flow equations for different values of $\alpha$, however, as expected, there is few differences respect to the evolution under the Ricci flow. Since the parameter $\alpha$ comes from a perturbative expansion\footnote{The parameter $\alpha$ is related with the length of a string in the non-linear sigma model.}, it has to be small. Therefore, in order to see the effect of the term $R_{\alpha \beta \gamma} R^\alpha_{\beta \gamma}$ we must built an initial wormhole metric such that the Riemmann curvature tensor is high at some points. We focus in wormholes with strong curvatures. We take as initial metric the wormhole function (3.21), where we have have dropped the throat section, and embedded a half of an ellipse with minor axis $a = 2$ and major axis $b = 6$. This initial condition $R(\rho, 0)$ is presented in Fig. 5.
Figure 5: Initial condition $R(\rho, 0)$, for $\delta = 1.3$, with an embedded ellipse with minor axis $a = 2$ and major axis $b = 6$.

We follow the evolution of this initial condition, with different values of $\alpha$. We see in Figure 6 three evolution snapshots—at $\lambda = 5$ (left panel), $\lambda = 50$ (middle panel), and $\lambda = 80$ (right panel)—of the initial condition depicted in Figure 5 ($\delta = 3$), with $\alpha = 0.15$. Moreover, in Figure 7, we show the evolution of $R_{th}$ with $\alpha = 0$ (red line), compared with $\alpha = 0.001$ (black line), $\alpha = 0.01$ (blue line), and $\alpha = 0.15$ (purple line). We note that for $\alpha = 0.15$, and $\lambda \gtrsim 0.1$, $R_{th}$ decreases slowly respect to the other cases. As it was previously noted in Subsection 3.1, we expect that this behavior becomes more notorious in regions where the curvature is higher. For example, the panels of Figure 8 show the evolution of $R^*(\lambda) \equiv R(\rho = 0.8, \lambda)$, for $\alpha = 0$ (black-solid line), and $\alpha = 0.15$ (red-dashed line). It is clear that for early stages of the simulation $\lambda \lesssim 0.1$ (left panel), the value of $\alpha$ is quite important differentiating the future of $R^*$. After this early stage (right panel), the evolution of $R^*$ is similar for both values of $\alpha$.

We can see that, depending on the value of $\alpha$, we need to make evolve a metric with a very high curvature in order to see a difference between both flows. The initial condition depicted in Fig. 5 has two throats (at each side of the ellipse) and can still be called a wormhole. But, since it was constructed for satisfying curvature requirements, it might...
not be a solution of the Einstein equations. The evolution after a long time under de RG-2 flow is very similar to the Ricci flow case. We note that when the Ricci flow develops a singularity the RG-2 flow is able to continue a little bit further.

Figure 7: Evolution of $R_{th}$ in the system (3.10)-(3.11), with $\alpha = 0$ (red line), $\alpha = 0.001$ (black line), $\alpha = 0.01$ (blue line), and $\alpha = 0.15$ (purple line). Only the last curve is not superimposed to the rest of values of $\alpha$.

Figure 8: $\lambda$-evolution of $R^\ast(\lambda)$ (see text). The right panel shows the initial instants of the evolution of $R^\ast$ with the system (3.10)-(3.11), and with $\alpha = 0$ (black-solid line), and $\alpha = 0.15$ (red-dashed line).

4.3 Entropy and energy

When a manifold evolves under a intrinsic flow, we can compute the evolution of certain quantities, which somehow let us characterize what is happening with whole geometry. The Hamilton’s entropy is inspired in the classical Boltzmann definition, but it uses curvature. In [24], it was shown that the formula for Hamilton’s entropy can also be used for asymptotically flat spaces, and for the flow without normalization\textsuperscript{13}. Therefore, we can compute

\textsuperscript{13}For the normalized flow case see [23].
numerically the entropy of the wormhole surface, and see how this entropy behaves under the RG-2 flow.

By definition the Hamilton’s entropy for the RG-2 flow is

$$S_H = \int_{\mathcal{M}} \left( R_s \log(R_s) + \frac{\alpha}{4} R_s^2 \right) dv,$$

(4.3)

where $dv$ is the volume element, and $R_s$ is the scalar curvature. This entropy is monotonous under the RG-2 flow\(^{14}\) and we will show that for our wormholes evolution it is indeed the case. See appendix A. At the same time that the entropy of the surface of revolution grows with $\lambda$, the enclosed volume shrinks maintaining the asymptotic flatness. If the surface would be compact and we would had enforce the constancy of the volume, the surface will develop singularities with a similar geometry of those of the Ricci flow\(^{15}\).

Complementary, we take advantage of the fluid nature of the system (3.10)-(3.11) for defining and computing an “energy density” $\epsilon \equiv R^2/2$, for each instant $\lambda$. This quantity would behave as its counterpart—the kinetic energy density in Navier-Stokes models. In Figure 9, we have plotted the energy and Hamilton’s entropy values for every $\lambda$ and for both flows. It is clear that both quantities are monotonous under their evolution. At first sight the energy evolution of the Ricci flow is very similar to the RG-2 flow energy, but there are some subtleties. In order to see these differences, in Figure 10, we have plotted the energy difference of both flows $\Delta \epsilon$. The form of $\Delta \epsilon$ shows that the RG-2 flow is dealing better with higher curvature. Moreover, it approaches differently the singularities. At the beginning of the evolution, where the asymptotic flatness of the spacetime forces the

---

\(^{14}\)Depending on the chosen sign, this entropy will be increasing or decreasing.

\(^{15}\)In the Ricci flow with surgery (used in the proof of the Thurston geometrization theorem), Perelman was able to remove the singularities and let the flow continues towards one of the 8 Thurston geometries. In fact, the requirement was the constancy of the volume form, multiplied by an exponential factor.
wormhole to reduce the area. Since the $\epsilon$ is, in some way, a measure of the curvature, we can see that each flow acts differently when smoothing curvatures, but at the end the result is the same, both flows tend to reduce the curvature and finally develop singularities.

![Energy difference between both flows](image)

Figure 10: Energy difference between both flows. There is a maximum before the difference starts going to zero.

## 5 Discussion and Conclusions

We have evolved different wormhole geometries under two different flows: the Ricci flow and the RG-2 flow. When curvatures are small there is practically no difference between both flows. However, when curvatures grow we can clearly see differences. Although both flows develop singularities, the RG-2 flow is able to manage high curvatures in a better way. We have used a general definition of wormhole, although we have studied only the asymptotically flat case. Moreover, we have studied the evolution of wormhole-like metrics that are not solutions of Einstein Equations. We have found that the wormhole pinches-off no matter what kind of initial condition we start with, sooner or later we will have a singularity.

An interesting result is the fact that the apparent criticality described in [10] appears only in certain zone. If we let the flow evolve further, we can see that all the wormholes pinch-off for every $\lambda$. We were able to reproduce the results for the RG-2 flow, and therefore we were able to deal with geometries with really high curvatures. Due to the fact that higher loops flows will add more higher curvature terms, and the behavior for low curvatures is the same for both flows. We can safely argue that for the type of wormhole solutions studied here, the evolution is the same at all loops.

Another interesting fact is that in our evolution the growing of the entropy, enforces the reduction of the enclosed volume. If the wormhole has a throat, which introduces the curvature in the wormhole, it starts expanding but the asymptotic flatness forces the throat to retreat. Therefore, in order to maintain asymptotic flatness, the surfaces shrinks and the enclosed volume will reduce.
It is known that compact surfaces of constant curvature shrink to a point when evolved with Ricci flow [22]. A similar behavior was expected with RG-2 flow evolution. However, in our set up, we enforced asymptotic flatness and therefore the flow should either go to flat space or develops a singularity.

An interesting problem could be the evolution of asymptotically AdS wormholes [2]. A family of these wormholes is interpreted as connector of entangled particles. Therefore, in this context, the development of singularities would mean disentanglement. We could check what happens with dual fields in the conformal theory when its bulk counterpart are evolved under a geometric flow.

The extended Hamilton’s entropy computed for the surface is a good indicator of what is going on through the evolution. The entropy is going to be monotonous, and according to our estimation, it will be an increasing function of $\lambda$. When the volume is reduced, $S_H$ grows and $\epsilon$ decreases. We note that the geometric entropy has been used for finding bounds on curvature evolution, but a physical interpretation is still lacking. Recently, the interest for understanding this (another kind of entropies) have increased. In [26] the authors study the entropy functionals of a gradient flow (the Ricci flow). They show that these entropies provide a good definition for the distance, in the context of the infinite distance conjecture in the swampland approach. The behavior of these entropies has not been studied extensively, and since there are many candidates for a geometric entropy, we believe that it deserves more study [24]. Until now, we have shown that the entropy and the energy are monotonous but we should study what happens with different geometries before arriving to a physical interpretation.

**Acknowledgments**

C.L.V was partially supported by EPN internal projects PII-DFIS-2019-01 and PII-DFIS-2019-04.

**A Hamilton’s Entropy**

Here we make a brief review of the Hamilton’s entropy and its properties. In order to find some bounds to the curvature evolution under Ricci flow, in [25] Hamilton defined a geometric entropy, in analogy to the thermal entropy [16]. The Hamilton’s entropy is defined for closed surfaces of positive curvature and is given by

$$S_H = \int_M (R_s \log(R_s)) \, dv,$$  \hspace{1cm} (A.1)

---

\[16\] It is known that Boltzman have used as a definition of entropy $S_B = \rho \ln\rho$ where $\rho$ was a density phase space. Later on, Gibbs generalized the Boltzman entropy to the form $S_G = \sum p_i \ln p_i$, where $p_i$ are probabilities.
where $R_s$ is the scalar curvature of the manifold $\mathcal{M}$ and $dv$ is the volume form in two dimensions. $S_H$ is monotonous under normalized Ricci flow evolution (see Proposition 5.39 in [22]). When the scalar curvature is negative we have to use another, although very similar, formula for the entropy:

$$S_{Hw} = \int_M (R_s - w) \log(R_s - w) dv, \quad (A.3)$$

where

$$w(r) = \frac{r}{1 - \left(1 - \frac{r}{w_0}\right) e^{rt}}. \quad (A.4)$$

In this case of a negative curvature, $S_{Hw}$ is not necessarily monotonous, but is bounded from above (see Proposition 5.44 in [22]). These results can be extended for the RG-2 flow. In [23] there is a generalization of the Hamilton’s entropy for the RG-2 flow, this new entropy is given by

$$S_H = \int_M \left(R_s \log(R_s) + \frac{\alpha}{4} R_s\right) dv, \quad (A.5)$$

where, as before, $R_s$ is the scalar curvature of the metric, and $dv$ is the volume form in two dimensions. The definition has been extended to asymptotically flat spaces, in [24]. For our wormhole metric (3.9) the scalar curvature is

$$R_s = -\frac{2(1 - (R')^2 + 2RR'')}{R^2}. \quad (A.6)$$

Therefore we can calculate the surface entropy at every time of the evolution.

References

[1] P. Gao, D. L. Jafferis and A. C. Wall, “Traversable Wormholes via a Double Trace Deformation,” JHEP 12 (2017), 151, DOI: 10.1007/JHEP12(2017)151 arXiv:1608.05687 [hep-th].

[2] J. Maldacena, D. Stanford and Z. Yang, “Diving into traversable wormholes,” Fortsch. Phys. 65 (2017) no.5, 1700034 DOI: 10.1002/prop.201700034, arXiv:1704.05333[hep-th].

[3] G. Perelman, “The Entropy formula for the Ricci flow and its geometric applications,” arXiv: math/0211159[math-dg].

\[17\] The normalized Ricci flow on surfaces is defined as

$$\frac{\partial g_{ij}}{\partial \lambda} = -2 \left(R_s - \frac{\int_M R_s dv}{\int_M dv}\right) \quad (A.2)$$
[4] G. Perelman, “Ricci flow with surgery on three-manifolds,” arXiv:math/0303109[math-dg].

[5] D. Garfinkle and J. Isenberg, “Numerical studies of the behavior of Ricci flow”, in Geometric Evolution Equations, Vol. 367, 2005, DOI:10.1090/conm/367/06750, arXiv:math/0306129[math-dg].

[6] K. Gimre, C. Guenther and J. Isenberg , “A geometric introduction to the 2-loop renormalization group flow”. J. Fixed Point Theory Appl. 14, 3–20 (2013), DOI:10.1007/s11784-014-0162-7, arXiv:1312.6049v1[math.DG].

[7] E. Woolgar, “Some Applications of Ricci Flow in Physics,” Can. J. Phys. 86 (2008) 645, DOI:10.1139/P07-146, arXiv:0708.2144[hep-th].

[8] C. Gundlach and J. M. Martin-Garcia, “Critical phenomena in gravitational collapse,” Living Rev. Rel. 10 (2007), 5, DOI:10.12942/lrr-2007-5, arXiv:0711.4620[gr-qc].

[9] S. N. Solodukhin, “Entanglement entropy and the Ricci flow,” Phys. Lett. B 646 (2007) 268, DOI:10.1016/j.physletb.2007.01.031, arXiv: hep-th/0609045[hep-th].

[10] V. Husain and S. S. Seahra, “Ricci flows, wormholes and critical phenomena,” Class. Quant. Grav. 25 (2008) 222002 DOI:10.1088/0264-9381/25/22/222002, arXiv:0808.0880[gr-qc].

[11] T. Oliynyk and E. Woolgar, “Asymptotically Flat Ricci Flows,” Communications in Analysis and Geometry 15 (2007), 535-568 DOI:10.4310/CAG.2007.v15.n3.a4, arXiv:math/0607438[math.DG].

[12] M. Headrick and T. Wiseman, “Ricci flow and black holes,” Class. Quant. Grav. 23 (2006) 6683, doi:DOI:10.1088/0264-9381/23/36/006, arXiv:hep-th/0606086.

[13] Ó. Lasso Andino, “RG-2 flow, mass and entropy,” Class. Quant. Grav. 36 (2019) 065011 doi:DOI:10.1088/1361-6382/ab05f6, arXiv: 1806.10031[gr-qc].

[14] O. Lasso Andino, “RG-2 flow and black hole entanglement entropy,” arXiv:1905.00102[hep-th].

[15] M. Visser and D. Hochberg, “Generic wormhole throats,” Annals Israel Phys. Soc. 13 (1997), 249, arXiv:gr-qc/9710001[gr-qc].

[16] K. Gimre, C. Guenther and J. Isenberg, “Short-time existence for the second order renormalization group flow in general dimensions,” Proc. Am. Math. Soc. 143 (2015) no.10, 4397, arXiv: 1401.1454[math.DG].

[17] T. A. Oliynyk, “The 2nd order renormalization group flow for non-linear sigma models in 2 dimensions,” Class. Quant. Grav. 26 (2009) 105020, DOI:10.1088/0264-9381/26/10/105020, arXiv:0904.1241[hep-th].
[18] M. S. Morris and K. S. Thorne, “Wormholes in space-time and their use for interstellar travel: A tool for teaching general relativity,” Am. J. Phys. 56 (1988) 395. DOI: 10.1119/1.15620.

[19] T. Muller, “Exact geometric optics in a Morris-Thorne wormhole spacetime,” Phys. Rev. D 77 (2008) 044043. DOI: 10.1103/PhysRevD.77.044043.

[20] G. J. Olmo, D. Rubiera-Garcia and A. Sanchez-Puente, “Geodesic completeness in a wormhole spacetime with horizons,” Phys. Rev. D 92 (2015) no.4, 044047 DOI: 10.1103/PhysRevD.92.044047, arXiv:1508.03272.

[21] S. W. Kim and H. Lee, “Exact solutions of a charged wormhole,” Phys. Rev. D 63 (2001) 064014, DOI: 10.1103/PhysRevD.63.064014, arXiv:gr-qc/0102077.

[22] B. Chow, D. Knopf. “The Ricci flow: An introduction. American Mathematical Society. Mathematical Surveys and Monographs, Vol.110 (2004).

[23] V. Branding, ”The normalized second order renormalization group flow on closed surfaces”, Advances in Theoretical and Mathematical Physics, Volume 20 (2016), Number 5, DOI: 10.4310/ATMP.2016.v20.n5.a7, arXiv:1503.07462.

[24] O. Lasso Andino, C. Vásconez Christian. “Work in progress”

[25] R. Hamilton, “The Ricci flow on surfaces”, in: Mathematics and General Relativity, Santa Cruz, CA, 1986, in: Contemporary Mathematics, vol. 71,American Mathematical Society, Providence, RI, 1988, pp. 237–262

[26] A. Kehagias, D. Lüst and S. Lüst, “Swampland, Gradient Flow and Infinite Distance,” JHEP 04 (2020), 170. DOI: 10.1007/JHEP04(2020)170, arXiv:1910.00453[hep-th].