Two Dimensional QCD is a String Theory

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Abstract

The partition function of two dimensional QCD on a Riemann surface of area $A$ is expanded as a power series in $1/N$ and $A$. It is shown that the coefficients of this expansion are precisely determined by a sum over maps from a two dimensional surface onto the two dimensional target space. Thus two dimensional QCD has a simple interpretation as a closed string theory.

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1 Introduction

It has long been an outstanding problem to find an exact description of gauge theories in terms of a theory of strings. Recently one of us (DG) has explored this possibility for two dimensional QCD and has conjectured that the free energy of a gauge theory with gauge group $SU(N)$ or $U(N)$ on a 2-dimensional manifold $\mathcal{M}$ can be identified with the partition function of a closed, orientable, string theory with target space $\mathcal{M}$ \cite{DG}.

The partition function of a pure gauge theory can be easily calculated on an arbitrary orientable 2-dimensional manifold $\mathcal{M}$ of genus $G$ and area $A$, (essentially due to the fact that the action is invariant under area preserving diffeomorphisms), and is given by

$$Z_{\mathcal{M}} = \int [DA] e^{-\frac{1}{4\tilde{g}^2} \int_{\mathcal{M}} d^2x \sqrt{g} F_{\mu\nu} F^{\mu\nu}}$$

(1.1)

$$= Z(G, \lambda A, N) = \sum_R \text{dim} R \text{dim} R 2^{-2G} e^{-\frac{\lambda A}{2N} C_2(R)}$$

(1.2)

where the sum is taken over all irreducible representations of the gauge group, with dim $R$ and $C_2(R)$ being the dimension and quadratic Casimir of the representation $R$. ($\lambda$ is related to the gauge coupling $\tilde{g}$ by $\lambda = \tilde{g}^2 N$.)

The conjecture states that the free energy $W(G, \lambda A, N) \equiv \ln Z(G, \lambda A, N)$, is equal to the partition function of some string theory with target space $\mathcal{M}$, where the string coupling is identified with $1/N$, and the string tension is identified with $\lambda$.

We do not have a precise formulation of the string action or functional integral. Instead we shall relate the QCD free energy to a specific sum over maps. This can be taken as a definition of the string theory however we would still like a path integral formulation, particularly in order to draw lessons about string theory in higher dimensions. Some things are clear from the structure of the gauge theory. The string theory must have the symmetries of QCD$_2$--thus it must be invariant under area preserving diffeomorphisms. This is a feature of the Nambu (and even the Polyakov) string action. Since the free energy is an expansion in powers of $e^{-\lambda A}$ the desired string theory action must be proportional to the area of the map, like the Nambu action. However, unlike the Nambu action it would appear that folds are totally suppressed. There are two reasons for this. First, as we shall show below, we can account for the terms in the $1/N$ expansion without invoking maps with folds. In other words, the area dependence of the free energy is totally accounted for by factors of $e^{-n\lambda A}$ arising from maps that cover the target space precisely $n$ times, and by powers of the area that arise from summing over positions of the branchpoints and collapsed handles of the
maps. Second, there is no term in the $1/N$ expansion that behaves as $e^{-0\lambda A}$, which would correspond to maps with winding number zero. This is important since in string theory such maps describe the propagation of ordinary stringy particles. Even in a two dimensional string theory we would expect at least one particle—corresponding to the center of mass of the string. This is the *tachyon* that appears in non-critical two dimensional string theory with zero mass. However pure two dimensional Yang-Mills theory contains no particles. If we forbid all folds then we forbid all maps with zero winding number, since all such maps contain folds. Then the particles can be absent. Forbidding folds does not render the theory topological, since the term in the string theory action which suppresses folds need not be invariant under arbitrary diffeomorphisms of the target space.\footnote{A suggestion for modifying the usual Nambu-Goto string to suppress folds was made by Minahan \cite{Minahan}; however the term that he suggested is not invariant under area preserving diffeomorphisms.} The explicit area-dependence of the theory also keeps it from being a purely topological theory.

In \cite{polchinski} the above conjecture was explored by expanding the free energy in powers of $1/N$—the purported string coupling constant,

$$W(G, \lambda A, N) = \sum_{g=0}^{\infty} N^{2g-2} f_g^G(\lambda A).$$  \hspace{1cm} (1.3)

It was shown that the coefficients $f_g^G(\lambda A)$ have precisely the structure expected if they are given by a sum over maps of a genus $g$ manifold, $\mathcal{M}_g$, onto a genus $G$ manifold $\mathcal{M}_G$, with an action that is simply the exponential of the area,

$$f_g^G(\lambda A) = \sum_n \sum_i \omega_{g,G}^{n,i} e^{-\frac{n\lambda A}{2}} (\lambda A)^i.$$  \hspace{1cm} (1.4)

The sum over $n$ was interpreted as a sum over maps of winding number $n$, that cover the target space $n$ times, and the powers of the area in (1.4) were interpreted as coming from the contribution of branchpoints and collapsed handles of these maps. The main evidence for this interpretation was the demonstration that the QCD$_2$ expansion satisfies the bound, $g - 1 \geq n(G - 1)$, that holds for continuous maps of $\mathcal{M}_g$ onto $\mathcal{M}_G$ with winding number $n$.\footnote{A suggestion for modifying the usual Nambu-Goto string to suppress folds was made by Minahan \cite{Minahan}; however the term that he suggested is not invariant under area preserving diffeomorphisms.}

In this paper we prove that the term in (1.4) with the maximum powers of the area for given $g, G$ and $n$, namely $\omega_{g,G}^{n,i}$ with $2(g - 1) = 2n(G - 1) + i$, is equal to the number of topologically distinct continuous maps of $\mathcal{M}_g$ onto $\mathcal{M}_G$ with winding number $n$ and with $i$ branchpoints. More precisely, $i! \omega_{g,G}^{n,i}$ is given by the sum of a natural symmetry
factor over all homotopically distinct $n$-fold connected covers of $\mathcal{M}_G$. The factor of $i!$ arises because when a genus $g$ surface covers a genus $G$ surface $n$ times, there are precisely $i = 2(g - 1) - 2n(G - 1)$ branch points. Since these branch points are indistinguishable, and can be positioned anywhere on the target space, one obtains a factor of $A^i / i!$.

We will furthermore demonstrate that the remaining terms $\omega_{g,G}^{n,i}$ with $2(g - 1) > 2n(G - 1) + i$ can be interpreted in terms of branchpoints, collapsed handles, and two types of “tubes” connecting sheets of the covering space. We show that such objects contribute specific factors, which taken together form a set of “Feynman rules” for the string theory. (The rules for collapsed handles and orientation-preserving tubes were previously suggested by Minahan [3].) In the case of the torus ($G = 1$), we can interpret all of the coefficients in this fashion. For $G \neq 1$, there are some terms of lower order in $N$ for which we do not yet have a geometric interpretation.

In the next section we will discuss the $1/N$ expansion of the gauge theory defined by (1.2). We define an asymptotic expansion of (1.2), written in terms of a sum over Young tableaux. We show that when written in this fashion, the partition function factorizes into two separate and identical parts, or “chiral sectors”, coupled by a simple term. In section 3, we consider the partition function of a single chiral sector, and prove that the leading terms in the partition function are given by the sum of a symmetry factor over all branched covers of $\mathcal{M}$. In section 4, we interpret the sectors as corresponding to covers of opposite orientation, and show that for the torus all of the remaining terms in the full partition function can be interpreted in terms of further structures of the covering space which are localized at a point in $\mathcal{M}$ (collapsed handles, infinitesimal tubes, etc). In section 5, we approach the problem from a more local perspective, and rederive the results of section 3 by calculating the partition function on a single plaquette and gluing together plaquettes to form $\mathcal{M}$. The results from this section are also applicable to manifolds with boundary. In section 6, we briefly discuss how our results generalize to the case of non-orientable target spaces. Finally, in section 7, we review our results and discuss further questions. In the Appendix we derive some properties of the representations of $SU(N)$ needed for the large $N$ expansion.

## 2 1/N Expansion

We will now describe the $1/N$ expansion of the partition function (1.2) when the gauge group is $SU(N)$. One might also consider other groups. Non semi-simple groups, such as $U(N)$,
involve an extra coupling and do not appear to have such a nice stringy interpretation. We shall therefore restrict our attention in this paper to $SU(N)$, although other simple groups, such as $SO(N)$, which would correspond to non-orientable strings, could be analyzed using the same techniques.

The partition function is written as a sum over representations of the gauge group $SU(N)$. We will perform an asymptotic expansion of (1.2) by first expanding the contribution from each representation separately as a series in $1/N$, and then summing over representations. This is a somewhat ambiguous procedure, since “fixing” a representation as $N \to \infty$ is not a particularly well-defined process. As we will see below, the simplest way of performing this sum only picks out half of the theory. By carefully defining the set of representations to sum over, however, we get a theory which seems to contain all the essential features of the gauge theory at fixed $N$. We believe that this is the correct asymptotic expansion of QCD$_2$, although like all asymptotic expansions it does not converge. We would expect non-perturbative corrections of order $\exp(-1/g_{\text{string}})$. These, however, are not determined by the string perturbation theory (the $1/N$ expansion). To deal with them in string theory one would need the string field theory. In our case QCD$_2$ is the string field theory! In fact in QCD$_2$ there are well defined, order $e^{-N}$, corrections to the asymptotic expansion, see [1].

Each representation $R$ is associated with a Young tableau containing some number of boxes $n$, which are distributed in rows of length $n_1 \geq n_2 \ldots \geq n_l > 0$ and columns of length $c_1, c_2 \ldots \geq c_k > 0$, where $k = n_1$, and $l = c_1$. The total number of boxes is clearly given by $n = \sum_i n_i = \sum_i c_i$ (for an example of a Young tableau, see figure [1]). The quadratic Casimir of the representation $R$ is given by

$$C_2(R) = nN + \tilde{C}(R) - \frac{n^2}{N},$$

(2.1)

where

$$\tilde{C}(R) = \sum_i n_i^2 - \sum_i c_i^2 = \sum_i n_i(n_i + 1 - 2i) = \sum_i -c_i(c_i + 1 - 2i).$$

(2.2)

The leading term in a $1/N$ expansion of the dimension of a representation $R$, whose Young tableau has $n$ boxes, is given by

$$\dim R = \frac{d_R N^n}{n!} + \mathcal{O}(N^{n-1}),$$

(2.3)

where $d_R$ is the dimension of the representation of the symmetric group $S_n$ associated with the Young tableau $R$. The $1/N$ correction terms to (2.3) can written in terms of the lengths
\[
c_2 = 6 \quad c_4 = 2 \quad c_6 = 1 \\
c_1 = 6 \quad c_3 = 4 \quad c_5 = 2 \quad c_7 = 1
\]

\[
n_1 = 7 \\
n_2 = 5 \\
n_3 = 3 \\
n_4 = 3 \\
n_5 = 2 \\
n_6 = 2
\]

\[
C_2(R) = 22N + 2 - \frac{484}{N}
\]

Figure 1: A Young tableau for a Representation of SU(N)

\( n_i \) of the Young tableau (see the Appendix.) We do not yet have an understanding of the geometrical significance of these correction terms, however, and for the most part we will not discuss them further in this paper. Note, though, that for the torus, which is the physical case of interest, the dimension term does not appear in the partition function (1.2). Thus, we will be able to completely understand the theory in the case \( G = 1 \), without having understood the significance of these correction terms.

From the equation for the quadratic Casimir (2.1), one might expect that to pick out all of the terms in the asymptotic expansion (1.4) which contain terms of the form \( e^{-\frac{n\lambda A}{2}} \), it would suffice to sum over all representations \( R \) in the set \( Y_n \) of Young tableaux with \( n \) boxes. This would lead to the partition function

\[
Z(G, \lambda A, N) = \sum_{n=0}^{\infty} \sum_{R \in Y_n} (\text{dim } R)^{2-2G} e^{-\frac{\lambda A C_2(R)}{2N}}
\]

\[
= \sum_{n} \sum_{R \in Y_n} \left( \frac{n!}{d_R} \right)^{2G-2} e^{-\frac{n\lambda A}{2}} \sum_{i=0}^{\infty} \left[ \frac{(-\lambda A \tilde{C}(R))^i}{2^i i!} N^{n(2-2G)-i} + O(N^{n(2-2G)-i-1}) \right].
\]

This expansion for the partition function, however, only contains half of the full theory. This is because there exists another set of representations whose quadratic Casimir contains a leading order term of \( nN \). We will find (2.5) to be a useful object to study nonetheless, since it contains much of the physics of the full theory.\footnote{In \cite{1} these other representations were ignored. This does not, as we shall see, modify the conclusions}

We will now discuss the other
Consider two representations $R$ and $S$, whose Young tableaux contain $n$ and $\tilde{n}$ boxes respectively, with columns of length $c_i$ and $\tilde{c}_i$. From these two representations, we can form a new representation $T$, with column lengths

$$
\begin{cases}
N - \tilde{c}_{L+1-i}, & i \leq L \\
c_{i-L}, & i > L
\end{cases},
$$

(2.6)

where $L$ is the number of boxes in the first row of the Young tableau for $S$. We will refer to this representation as $T = S\bar{R}$, or as the "composite" representation of $R$ and $S$. Note that this composite representation contains $L$ columns with $O(N)$ boxes. An example of a composite representation is shown in Figure 2. Computing the quadratic Casimir (2.1) of the composite representation $T$, we find that

$$
C_2(T) = C_2(R) + C_2(S) + \frac{2n\tilde{n}}{N}.
$$

(2.7)

Similarly, one can show that the dimension of the composite representation factorizes, up to $1/N^2$ corrections (see the Appendix),

$$
\dim T = \dim R \dim S \left[1 + O\left(\frac{1}{N^2}\right)\right],
$$

(2.8)

of that paper.
and therefore behaves, for large \( N \), as

\[
\dim T = \frac{d_R d_S N^{n+\bar{n}}}{n!\bar{n}!} \left[ 1 + O\left( \frac{1}{N^2} \right) \right].
\]  

(2.9)

From (2.7), we see that the composite representation \( SR \) has a quadratic Casimir with leading order term \((n + \bar{n})N\). Thus, in order to include all terms proportional to \( e^{-\frac{\lambda A}{2}} \) in (1.4), it is necessary to include all composite representations in the partition function sum. We have then,

\[
Z(G, \lambda A, N) = \sum_n \sum_{\bar{n}} \sum_{R \in Y_n} \sum_{S \in Y_{\bar{n}}} \left( \dim SR \right)^2 2^G e^{-\frac{\lambda A}{2} \left[ C_2(R) + C_2(S) \right]} \left[ \frac{\lambda A}{N} \right].
\]  

(2.10)

From (2.8), we see that this partition function can be factored into a product of two copies of (2.5), except for the \( 1/N^2 \) corrections from the expansion of the dimensions, and a coupling term \( e^{-\frac{\lambda A n}{N^2}} \). For the torus, the factorization is complete except for the coupling term. In the next section, we will prove that the partition function (2.5) can be rewritten as a sum over coverings of \( M \) with a fixed orientation. Since there are two possible orientations for an oriented covering, we can interpret the two factors of (2.5) in (2.10) as “chiral sectors” of a string theory which correspond to orientation-preserving and orientation-reversing maps of oriented 2-manifolds onto \( M \). The coupling term \( e^{-\frac{\lambda A n}{N^2}} \) then gives a mechanism by which these covering maps can be combined into connected covering maps of indefinite relative orientation. We will return to this interpretation of the theory in section 4.

We will now set up the specific mathematical problem which we will address in the next section. Ultimately, we wish to have a geometrical description of the coefficients in the \( 1/N \) expansion of the free energy (1.4). We will find it more convenient, however, to first work with the coefficients in the \( 1/N \) expansion of the partition function itself, which are defined through

\[
Z(G, \lambda A, N) = \sum_{g=-\infty}^{\infty} \sum_n \sum_i \eta_{g,G}^{n,i} e^{-\frac{\lambda A}{2} (\lambda A)^i N^{2-2g}}.
\]  

(2.11)

We can perform a similar expansion for the partition function of a single chiral sector,

\[
Z(G, \lambda A, N) = \sum_{g=-\infty}^{\infty} \sum_n \sum_i \zeta_{g,G}^{n,i} e^{-\frac{\lambda A}{2} (\lambda A)^i N^{2-2g}}.
\]  

(2.12)

It follows from (2.5), that when \( 2(g - 1) = 2n(G - 1) + i \), the coefficients in (2.12) are given by

\[
\zeta_{g,G}^{n,i} = \sum_R \left( \frac{n!}{d_R} \right)^{2G-2} \frac{1}{i!} \left( \frac{\hat{C}(R)}{2} \right)^i.
\]  

(2.13)
The other coefficients $\zeta_{g,G}^{n,i}$ are produced from the lower order terms in the $1/N$ expansion of the dimensions of the representations, and the term $n^2/N$ in the quadratic Casimir formula. Again, in the case of the torus, there are no corrections from dimensions, so the only other terms arise from the quadratic Casimir. For the purposes of the argument in the next section, we will temporarily ignore the term $n^2/N$. This is equivalent to changing the gauge group to $U(N)$. We will restore the extra term to the quadratic Casimir when we discuss the coupling of the two chiral sectors in section 4.

Thus, we now wish to show that the coefficients given in (2.13) can be interpreted in terms of covering maps. We are interested in covering maps which have a fixed winding number $n$, and which are only singular at points. For now, we need only consider singularities which arise from branch points of the form encountered in the 2-fold covering $z \to z^2$ of the unit disk in the complex plane. Let us consider the set $\Sigma(G,n,i)$ of $n$-fold covers of $M_G$ with $i$ branch points. If $\nu : M_g \to M_G$ is such a covering map, then $2(g-1) = 2n(G-1) + i$. We include here covering spaces which are disconnected, so that $g$ may be negative (we define the genus of a disconnected surface with Euler characteristic $\chi$ by $2 - 2g = \chi$). To every cover $\nu$, we associate a symmetry factor $|S_\nu|$, which is defined to be the number of distinct homeomorphisms $\pi$ from $M_g$ to $M_g$ such that $\nu \pi = \nu$. What we shall prove is that

$$i! \zeta_{g,G}^{n,i} = \sum_{\nu \in \Sigma(G,n,i)} \frac{1}{|S_\nu|}. \quad (2.14)$$

### 3 Counting Branched Covers

We will now proceed to prove equation (2.14) by counting the number of branched covers of $M_G$. We first consider the case with no branch points ($i = 0$).

For a fixed 2-manifold $\mathcal{M}$ of genus $G$ and of area $A$, we choose a point $p \in \mathcal{M}$. We choose a set of generators $a_1, b_1, a_2, b_2, \ldots$ for $\pi_1(\mathcal{M},p)$ whose images in $H_1(\mathcal{M})$ form a canonical homology basis; i.e., $a_i \cdot b_j = \delta_{ij}, a_i \cdot a_j = b_i \cdot b_j = 0$. With this set of generators, the single relation necessary to define $\pi_1(\mathcal{M})$ is

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_G b_G a_G^{-1} b_G^{-1} = 1. \quad (3.1)$$

Given an $n$-fold unbranched covering $\nu$ of $\mathcal{M}$, by choosing a labeling of the sheets of $\nu$ over $p$ with the integers $I = \{1, 2, \ldots, n\}$ one can construct a map from $\pi_1(\mathcal{M})$ to the permutation group $S_n$. This map is constructed by associating with each element $t \in \pi_1(\mathcal{M})$
the permutation on $I$ which arises from transporting the labels on sheets around the paths defined by lifting $t$ to the covering space.

From the definition of a covering space, this map is defined independently of which path is chosen to represent $t$, and defines a homomorphism $H_\nu : \pi_1(\mathcal{M}) \rightarrow S_n$. In fact, it is not hard to show that all such homomorphisms arise from some covering of $\mathcal{M}$ [3]. For a fixed covering $\nu$ of $\mathcal{M}$, there are $n!$ possible labelings which can be used for the sheets over $p$. Two labelings which differ by an element $\rho$ of the permutation group $S_n$ will give rise to homomorphisms $H$ and $H' = \rho H \rho^{-1}$ related through conjugation by the permutation $\rho$. Each element of the symmetry group $S_\nu$ gives rise to a permutation $\rho$ which leaves $H_\nu$ invariant. Thus, the number of distinct homomorphisms $H : \pi_1(\mathcal{M}) \rightarrow S_n$ associated with a fixed cover $\nu$ with symmetry factor $|S_\nu|$ is $n!/|S_\nu|$. It follows that to count each cover $\nu$ with a weight of $1/|S_\nu|$, it will suffice to sum over all distinct homomorphisms $H : \pi_1(\mathcal{M}) \rightarrow S_n$ with a constant weight of $1/n!$.

Since the only defining relation of $\pi_1(\mathcal{M})$ is (3.1), we can now write the weighted sum over unbranched covers as

$$
\sum_{\nu \in \Sigma(G,n,0)} \frac{1}{|S_\nu|} = \sum_{s_1,t_1,\ldots,s_G,t_G \in S_n} \left[ \frac{1}{n!} \delta \left( \prod_{i=1}^{G} s_i t_i s_i^{-1} t_i^{-1} \right) \right],
$$

where $\delta$ is a Kronecker delta function defined on $S_n$ by

$$
\delta(\rho) = \begin{cases} 
1, & \rho = \text{identity} \\
0, & \rho \neq \text{identity}. 
\end{cases}
$$

Let us illustrate this for the case of the torus $(G = 1)$. Here we must count the number of permutations, $s$ and $t$, of $n$ sheets that commute ($sts^{-1}t^{-1} = 1$). If $s$ is a permutation corresponding to a given partition of $n$ then there are precisely $n!/C_s$ permutations $t$ that commute with it, where $C_s$ is the number of distinct permutations corresponding to this partition (the order of the conjugacy class of $s$). This is because the permutation group acts transitively by conjugation on the set $P$ of permutations corresponding to a fixed partition. Thus the number of commuting elements is $n!/C_s$, and the number number of pairs $s$ and $t$ is equal to the number of partitions of $n$ (which we denote by $p(n)$) times $n!/C_s$ times $C_s$. Then we find that $\sum_{\nu \in \Sigma(1,n,0)} 1/|S_\nu| = p(n)$. Thus the leading terms in the torus partition function are given by

$$
\sum_{n} p(n) e^{-n\lambda A} = \prod_{k=1}^{\infty} (1 - e^{-k\lambda A})^{-1}. 
$$

This result coincides with the evaluation in [4].
We shall evaluate the general sum shortly, however let us now consider the case where there are \( i \neq 0 \) branch points. We count points with branching number \( j \) as consisting of \( j \) distinct branch points which have coincided, so that for instance the map from \( S^2 \) to \( S^2 \) given by \( w = z^3 \) is described as having \( i = 4 \) branch points, although it actually has two branch points, each with branching number 2 (0 and \( \infty \)). The surface \( \mathcal{M} \) can be cut along the curves \( a_1, b_1, \ldots \) to make a \( 4G \)-gon with the \( i \) branch points \( q_1, \ldots, q_i \). These branch points can be chosen in \( A_i/i! \) ways, which gives rise to the extra factor of \( i! \) in (2.14). We can construct a set of closed curves \( c_1, \ldots, c_i \) on the \( 4G \)-gon, such that \( c_j \) passes through \( p \), and is homotopic to a loop around \( q_j \), and such that the curves \( c_j \) do not intersect either one another or the curves \( a_k, b_k \) except at \( p \). (For example, the case \( G = 2, i = 4 \) is shown in figure 3.) The curves \( a_1, \ldots, a_G, b_1, \ldots, b_G, c_1, \ldots, c_i \) form a complete set of generators for \( \pi_1(\mathcal{M} \setminus \{ q_1, \ldots, q_i \}) \), which is defined by the single relation

\[
c_1 c_2 \ldots c_i a_1^{-1} b_1^{-1} a_2^{-1} b_2^{-1} \ldots a_G^{-1} b_G^{-1} = 1. \tag{3.3}
\]

Just as in the unbranched case, given an \( n \)-fold cover \( \nu \) of \( \mathcal{M} \) which is branched at the points \( q_1, \ldots, q_i \), a labeling of the sheets of \( \nu \) at \( p \) gives a homomorphism from \( \pi_1(\mathcal{M} \setminus \{ q_1, \ldots, q_i \}) \) to \( S_n \). The difference, however, is that the permutations \( p_1, \ldots, p_i \) associated with the curves \( c_j \) must be in the conjugacy class \( P_n \) of permutations which switch only two elements, since we are assuming that all branch points have branching number 1. We can now generalize the expression (3.2) to include branched covers by the formula

\[
\sum_{\nu \in \Sigma(G; n, i)} \frac{1}{|S_\nu|} = \sum_{p_1, \ldots, p_i \in P_n} \sum_{s_1, t_1, \ldots, s_G, t_G \in S_n} \left[ \frac{1}{n!} \delta(p_1 \ldots p_i \prod_{j=1}^{g} s_j t_j s_j^{-1} t_j^{-1}) \right]. \tag{3.4}
\]

The sum (3.4), where the case \( i = 0 \) reduces to our previous sum (3.2), can be evaluated using standard results from the representation theory of the permutation group \( S_n \). We define the matrix associated with an element \( \rho \in S_n \) in the representation \( R \) to be \( D_R(\rho) \); the character is then given by \( \chi_R(\rho) = \text{Tr} \ D_R(\rho) \). Standard results from group theory tell us that

\[
\delta(\rho) = \frac{1}{n!} \sum_R d_R \chi_R(\rho), \tag{3.5}
\]

\[
\sum_{\rho \in S_n} \chi_R(\rho) D_R(\rho^{-1}) = \frac{n!}{d_R} I_R, \quad \text{and} \quad \sum_{\sigma \in S_n} D_R(\sigma \rho \sigma^{-1}) = \frac{n!}{d_R} \chi_R(\rho) I_R. \tag{3.6}
\]
Figure 3: Surface with Genus $G = 2$, $i = 4$ Branch Points

where $I_R$ is the identity matrix in the representation $R$. From (3.6), it follows that

$$
\sum_{\rho \in P_n} D_R(\rho) = \frac{n(n-1)}{2d_R} \chi_R(P) I_R,
$$

(3.7)

where $\chi_R(P)$ is the character of any element of $P_n$ in the representation $R$. We can now use (3.5) to rewrite (3.4) as

$$
\sum_{\nu} 1/|S_\nu| = \sum_{p_1, \ldots, p_i} \prod_{s} \frac{1}{n!} \sum_{R} d_R \chi_R(p_1 \ldots p_i \prod_{s} s_i t_i s_i^{-1} t_i^{-1}).
$$

(3.8)

From (3.6) and (3.7), we have

$$
\sum_{s, t \in S_n} D_R(sts^{-1}t^{-1}) = \sum_{s, t} D_R(sts^{-1})D_R(t^{-1}) = \sum_{t \in S_n} \frac{n!}{d_R} \chi_R(t)D_R(t^{-1})I_R = \left(\frac{n!}{d_R}\right)^2 I_R.
$$

(3.9)

We can now rewrite (3.8) as

$$
\sum_{\nu \in \Sigma(G, n, i)} 1/|S_\nu| = \sum_{R} \left(\frac{1}{n!} \frac{n!}{d_R} \frac{(n(n-1)\chi_R(P))}{2d_R} \right)^i \text{Tr} \ I_R
$$

(3.10)

$$
= \sum_{R} \left(\frac{n!}{d_R}\right)^{2G-2} \left(\frac{n(n-1)\chi_R(P)}{2d_R}\right)^i.
$$

(3.11)
In order to prove (2.14), from (2.13) it will suffice to demonstrate that
\[ \tilde{C}(R) = \frac{n(n - 1)\chi_R(P)}{d_R}. \] (3.12)

This relation can be proven as follows: clearly the operator
\[ \tilde{P} = \sum_{p \in P_n} p \] (3.13)
is diagonal in the representation \( R \), since it commutes with every element of \( P_n \) and hence with every element of \( S_n \). Assume that the representation \( R \) has been defined by first antisymmetrizing with respect to the columns (labeled by \( c_1, c_2, \ldots, c_k \)), of the Young tableau associated with \( R \), and then symmetrizing with respect to rows, (labeled by \( n_1, n_2, \ldots, n_l \)), for some specific legal labeling of the boxes with the integers \( 1, \ldots, n \) \(^\ddagger\). Choosing a normalized vector \( v \) in the representation space of \( R \), it is a straightforward exercise to calculate
\[ v^T \cdot \tilde{P} \cdot v = \sum_{j=1}^{l} \frac{n_j(n_j - 1)}{2} - Q_R \sum_{j=1}^{k} \frac{c_j(c_j - 1)}{2Q_R} = \frac{\tilde{C}(R)}{2}, \] (3.14)
where
\[ Q_R = \prod_{j=1}^{l} (n_j)!. \] (3.15)
The first term in (3.14) is the number of pairs with respect to which \( v \) is symmetrized, and whose corresponding permutations take \( v \) to itself with eigenvalue +1 (these are pairs which appear in the same row of the Young tableau ). The second term arises in a similar fashion from the pairs with respect to which \( v \) is antisymmetrized. This factor is slightly more complicated however, since the vector \( v \) is a sum of \( Q_R \) separate terms, each of which is antisymmetric with respect to a different set of \( \sum_j c_j(c_j - 1)/2 \) pairs.

Since \( \tilde{P} \) is diagonal, it follows that
\[ \text{Tr} \tilde{P} = \frac{n(n - 1)}{2} \chi(P) = \frac{\tilde{C}(R)d_R}{2}, \] (3.16)
from which (3.12) follows immediately. Thus, we have proven (2.14).

4 Tubes and Contracted Handles

We will now discuss the effects of the remaining parts of the quadratic Casimir, and the coupling term between the two sectors; we will show that combining these effects with (2.14)
allows us to completely interpret the coefficients $\omega_{g,G}^{n,i}$ from (1.4) in terms of a string theory in the case of a toroidal target space.

To begin with, recall that in the partition function for a single chiral sector, we have neglected the effects of the term $n^2/N$ in the quadratic Casimir. This term gives a multiplicative contribution of $e^{\lambda A n^2/2N^2}$ to terms in the chiral partition function corresponding to representations with $n$ boxes in their Young tableau. It has been pointed out by Minahan [3] that these extra terms have a simple geometric interpretation in terms of mappings where a handle of the covering space $\mathcal{M}_g$ is mapped to a point in $\mathcal{M}_G$, or where a pair of branch points coincide to form a “tube” connecting two sheets of the cover. To see this write the extra contribution to the exponent as $n\lambda A + n(n-1)\lambda A/2$. The first term can be associated with the contribution of a handle in $\mathcal{M}_g$ that is mapped entirely onto a single point on the target space and which lies on one of the $n$ sheets of the cover. This interpretation accounts for the factor of $n$, for the factor of $1/N^2$ (since the genus increases by one), the factor of $\lambda A$ (the position of the handle) and the factor of $1/2$ (the indistinguishability of the two ends of the handle. The second term can be associated with infinitesimally small tubes, connecting two sheets of the covering space at a single point in $\mathcal{M}_G$ (accounting for the factor $\lambda A$). Again this increases the genus by one (accounting for the $1/N^2$), and the two ends of the tube can be located on any pair of sheets of the $n$-sheeted cover of the target space (accounting for the factor $n(n-1)/2$.) Since these contributions to the maps are local they clearly exponentiate. Note that these tubes are essentially equivalent to combining two branch points at a single point. Thus, the tubes are orientation-preserving, in the sense that moving through such a tube preserves the orientation of the covering surface relative to the orientation of the target space. This is consistent with our interpretation of a single chiral sector as corresponding to covering maps with a consistent relative orientation.

The chiral partition function (2.5) on the torus can thus be completely understood in terms of a sum over orientation-preserving covering maps. We will now choose to interpret the two copies of the chiral partition function which are coupled in (2.10) as arising from sums over orientation-preserving and orientation-reversing maps. The coupling term $-\lambda A n\tilde{n}/N^2$, then, can be interpreted as coupling an orientation-preserving cover with $n$ sheets with an orientation-reversing cover with $\tilde{n}$ sheets. Just like the extra terms in the quadratic Casimir for a single representation described above, this term is exponentiated. We can interpret it as describing infinitesimal orientation-reversing tubes which connect an arbitrary sheet of the $\tilde{n}$-sheeted cover with an arbitrary sheet of the $n$-sheeted cover. The factor of $\lambda A$ arises
as usual from the arbitrary location of this infinitesimal tube, and there is no symmetry factor because the two ends of the tube are distinguished. In addition we must introduce a multiplicative factor of $-1$ for each such tube.

In order to clarify the distinction between orientation-preserving and orientation-reversing tubes, we will now give a simple example of each. Consider the two maps from the cylinder $C = S^1 \times I = \{ z, x : |z| = 1, 0 \leq x \leq 1 \}$ to the disk $D = \{ z : |z| \leq 1 \}$ given by

\[ \nu_-(z, x) = z(1 - 2x), \]
\[ \nu_+(z, x) = \begin{cases} z(1 - 2x), & x \leq \frac{1}{2} \\ \bar{z}(2x - 1), & x > \frac{1}{2} \end{cases}. \]

These maps are both continuous 2-fold covering maps from $C$ to $D$ which are singular only at the point 0 in $D$. Both sheets of the covering $\nu_+$ have the same relative orientation with respect to a fixed orientation on $D$; we therefore see that the tube above the singularity in the covering $\nu_+$ connects two sheets of the same orientation, and we refer to this as an orientation-preserving tube. Similarly, the two sheets of $\nu_-$ have opposite relative orientation, so the tube in this covering is orientation-reversing. These covering maps are illustrated in figure 4; for clarity we have expanded the singularity of the orientation-preserving tube.

To summarize, we can now express the entire partition function for the $SU(N)$ gauge
theory in terms of a sum over the set of disconnected covering maps $\Sigma_G$,

$$Z(G, \lambda A, N) = \sum_{\nu \in \Sigma_G} \frac{(-1)^i}{|S_\nu|} e^{-\frac{\lambda A}{2} \left( [i + t + \tilde{t} + h] \right)} N^{n (2 - 2G) - 2 [t + \tilde{t} + h] - i} \left[ 1 + O \left( \frac{1}{N} \right) \right], \quad (4.3)$$

where $n$ is the winding number of $\nu$, $i$ is the number of branch points, $t$ ($\tilde{t}$) is the number of orientation-preserving (reversing) tubes, and $h$ is the number of handles which are mapped to points. Note that the $O(\frac{1}{N})$ corrections do not depend on the area. For the torus there are no correction terms and this expansion is exact.

Now that this result is proven for the partition function, the standard argument from quantum field theory can be applied to prove that logarithm of the partition function corresponds to sums over the set of connected covering maps $\tilde{\Sigma}_G$, i.e., we have

$$W(G, \lambda A, N) = \sum_{\nu \in \tilde{\Sigma}_G} \frac{(-1)^i}{|S_\nu|} e^{-\frac{\lambda A}{2} \left( [i + t + \tilde{t} + h] \right)} N^{n (2 - 2G) - 2 [t + \tilde{t} + h] - i} \left[ 1 + O \left( \frac{1}{N} \right) \right], \quad (4.4)$$

5 Plaquettes and Gluing

In this section we will rederive the results of section 2 from a more geometrical point of view. A standard approach to evaluating the gauge theory partition function (1.2) on an arbitrary manifold $M$ is to compute the partition function on a single plaquette (a disk with boundary $S^1$), and by gluing together plaquettes, form a manifold with the same topology and area as $M$. This is more or less the procedure we will follow here; we begin by showing that the partition function for a single plaquette can be described in terms of branched covers of the plaquette. We then show that gluing together plaquettes combines the separate partition functions in a way which is exactly described by gluing together the covering spaces of the plaquettes; by gluing plaquettes together to form the manifold $M$, we reproduce the results of the last section. Although this section is essentially a rederivation of the previous results, the development here may give a more geometric insight into the structure of the theory. In particular, by using a basis of traces of the holonomy of the gauge field around closed loops, the role of the two chiral sectors as describing orientation-preserving and -reversing covers is made explicit. In addition, the formalism developed here is of use in analyzing the theory on manifolds with boundary, and in understanding the effects of Wilson loops.

In this section, we will again discuss the partition function of a single chiral sector (2.3), and we will again drop the term $n^2 / N$ from the quadratic Casimir. The discussion of the previous section applies to everything done here.
Given a plaquette $\Delta$ of area $A$, if we fix the holonomy of the gauge field around the boundary of $\Delta$ to be $U$, the partition function of the gauge theory on $\Delta$ is given by

$$Z_\Delta(U) = \sum_R (\dim R) \chi_R(U) e^{-\frac{\lambda A}{N} C_2(R)}. \quad (5.1)$$

This partition function is often taken as the definition of the gauge theory on a triangulated manifold, since it is invariant under subtriangulations, and reduces to the Yang-Mills theory in the continuum limit $[2, 3]$.

The characters $\chi_R(U)$ form a natural basis for the set of class functions on the gauge group. In order to understand (5.1) in terms of covering maps, it will be useful to work with a different basis for this space. For every partition $n = n_1 + \ldots + n_k$, there is an associated class of elements of the symmetric group, consisting of those permutations with cycles of length $n_1, \ldots, n_k$. If $\sigma$ is such a permutation, we define

$$\Upsilon_\sigma(U) = \prod_{j=1}^k (\tr U^{n_j}). \quad (5.2)$$

The set of functions $\Upsilon_\sigma(U)$ also form a complete basis for the set of class functions on the gauge group. These functions are related to the characters by the relations

$$\chi_R(U) = \sum_{\sigma \in S_n} \frac{\chi_R(\sigma)}{n!} \Upsilon_\sigma(U), \quad (5.3)$$

$$\Upsilon_\sigma(U) = \sum_R \chi_R(\sigma) \chi_R(U). \quad (5.4)$$

Just as we did for the partition function (1.2) in section 3, we can expand $Z_\Delta$ as a power series in $1/N$. Once again, the partition function consists of two coupled chiral sectors. For the moment, we will consider a single chiral sector, where the partition function is given by

$$Z_\Delta(U) = \sum_n \sum_{R \in Y_n} (\dim R) \chi_R(U) e^{-\frac{\lambda A}{2N} C_2(R)}. \quad (5.5)$$

For a single sector, the leading order terms in $N$, for fixed values of $n$ and $A$, are given by

$$Z_\Delta(U) = \sum_n e^{-\frac{n \lambda A}{N}} \sum_i \frac{(\lambda A)^i}{i!} \sum_R \left[ \frac{d_R}{n!} \left( -\frac{\tilde{C}(R)}{2} \right)^i \chi_R(U) N^{n-i} + O(N^{n-i-1}) \right]. \quad (5.6)$$

This can be rewritten in terms of the class functions $\Upsilon_\sigma$, as

$$Z_\Delta(U) = \sum_n e^{-\frac{n \lambda A}{N}} \sum_i \frac{(\lambda A)^i}{i!} \sum_\sigma \left[ \frac{(-1)^i}{C_\sigma} \phi_{n,i}^{\sigma} \Upsilon_\sigma(U) N^{n-i} + O(N^{n-i-1}) \right], \quad (5.7)$$
where $C_\sigma$ is the number of permutations in the conjugacy class of $\sigma$. Plugging (5.3) into (5.6), we find

$$\phi_{n,i}^\sigma = \frac{d_RC_\sigma}{n!^2} \left( \frac{\check{C}(R)}{2} \right)^i \chi_R(\sigma).$$  \hfill (5.8)

We will now show that $\phi_{n,i}^\sigma$ is given by a sum over covers similar to that for $\zeta_{n,i}^{g,G}$. Specifically, we define $\Sigma_{\sigma}(n, i)$ to be the set of $n$-fold covers of $\Delta$ with $i$ branch points and with the additional property that the boundary of the covering space is a disjoint union of $k$ copies of $S^1$ which cover the boundary of $\Delta$ $n_1, \ldots, n_k$ times – as above, $n_j$ are the sizes of the cycles of the permutation $\sigma$. We will show that

$$\phi_{n,i}^\sigma = \sum_{\nu \in \Sigma_{\sigma}(n, i)} \frac{1}{|S_\nu|}. \hfill (5.9)$$

From an argument analogous to that leading to (3.2) and (3.4), we can write the sum over covers as

$$\sum_{\nu \in \Sigma_{\sigma}(n, i)} 1/|S_\nu| = \sum_{p_1, \ldots, p_i \in P_n} \left[ \frac{C_\sigma}{n!} \delta(p_1 \ldots p_i \sigma) \right]$$  \hfill (5.10)

$$= \sum_R \frac{d_RC_\sigma}{n!^2} \left( \frac{\check{C}(R)}{2} \right)^i \chi_R(\sigma),$$  \hfill (5.11)

so the assertion is proven.

As an example, consider the term

$$\phi_{n,1}^P = \frac{1}{2(n-2)!},$$  \hfill (5.12)

where $P \in P_n$ is a permutation consisting of a single pair exchange. For any value of $n$, the only covering space of $\Delta$ with a single branch point has a boundary which consists of $n - 2$ single covers of the boundary of $\Delta$, and one double cover. Thus, $P$ is in the only conjugacy class with $\phi_{n,1}^\sigma \neq 0$. The symmetry factor of the unique cover is exactly $2(n - 2)!$, which accounts for the denominator in (5.12).

To summarize what we have shown so far in this section, we can write the highest order terms in the chiral partition function (5.5) in terms of a sum over all coverings $\nu$ of $\Delta$,

$$Z_\Delta(U) = \sum_{\nu} \left[ \frac{(-1)^i}{|S_\nu|} e^{-\frac{2\pi \lambda A}{l} \frac{1}{i!} \prod_j (\text{Tr} \hat{U}_j) N^{n-i} + O(N^{n-i-1})} \right],$$  \hfill (5.13)

where $n$ is the winding number of $\nu$, $i$ is the number of branch points, and $\hat{U}_j$ are the holonomies of the pullback of the gauge field to the covering space. We will now show that
this formula continues to hold when we glue plaquettes together, so that for an arbitrary
orientable manifold \( M \) of genus \( G \), area \( A \), and with \( l \) boundary components, the highest
order terms in the chiral partition function are given by a sum over all coverings of \( M \),
\[
Z_\Delta(U_1, \ldots, U_l) = \sum_\nu \left[ \frac{(-1)^i}{|S_\nu|} e^{-\frac{\lambda A\lambda}{i!}} \prod_j (\text{Tr} \, \hat{U}_j) N^{(2-2G-l)_i} + \mathcal{O}(N^{(2-2G-l)_i+1}) \right],
\]
where again \( \hat{U}_j \) are the holonomies of the pullback of the gauge field around the boundaries
of the covering space. Note that this formula could also have been derived directly by
arguments similar to those above; it is perhaps more instructive, however, to derive this
result through the following argument.

We will prove (5.14) by induction on the number of internal edges in a triangulation of
\( M \). (By a triangulation, we mean a decomposition into plaquettes which are joined along
edges – the plaquettes need not be triangular; an internal edge is an edge which is not on the
boundary). The result (5.13) is the case where there is a single plaquette, with 0 internal
edges. What we now need to show is that when we add a new internal edge, either by gluing
together two manifolds \( M \) and \( N \) along a common edge, or by gluing together two edges
of a single manifold \( M \), the relation (5.14) continues to hold. The approach we use to glue
together plaquettes is similar to methods which have been widely used in the context of
\( U(N) \) lattice gauge theories in the large \( N \) limit[7].

In order to glue together the partition function along an edge, we use the integral
\[
\int dUU_{i_1 j_1} \ldots U_{i_n j_n} U^{ \ell_{1_1} \ldots \ell_{n_1}} U^{ \ell_{1_n} \ldots \ell_{n_n}} = \frac{1}{N^n} \left( \sum_{\sigma \in S_n} \delta_{i_1 \ell_{1_1}} \delta_{i_2 \ell_{1_2}} \ldots \delta_{i_n \ell_{1_n}} \delta_{j_1 \ell_{2_1}} \ldots \delta_{j_n \ell_{2_n}} \right) + \mathcal{O}(N^{-n-1}),
\]
where \( \sigma = (\sigma_1, \ldots, \sigma_n) \) ranges over all permutations of the integers 1, \ldots, \( n \) (we have normal-
ized \( \int dU = 1 \)). We glue along an edge by integrating over the integral of the gauge field
along the edge. Formula (5.13) is essentially the statement that to highest order in \( N \),
the products of traces of \( U \) combine according to the Wick expansion familiar from matrix
model theories. This formula has a simple geometric interpretation, which is that when two
plaquettes are glued together along a common boundary, their covers are glued together
in all possible ways consistent with their boundary structure. The induction step follows
immediately from this integral, as can be seen from the following arguments:

If we take two manifolds, \( M \) and \( N \), the set of \( n \)-fold coverings of the glued manifold
\( M \cup N \) is given by taking all \( n \)-fold covers of \( M \) and gluing them to all \( n \)-fold covers of \( N \)
in all possible ways along the common edge. The winding number is fixed, and the genus
and the number of branch points are additive. The holonomies around the boundaries of the
new covering space are formed by contracting the holonomies around the old boundaries in
exactly the way specified by (5.15). The number of edges on the glued manifold is one less
than the sum of the numbers of edges on $\mathcal{M}$ and $\mathcal{N}$, so the factor of $N^{-n}$ in (5.13) fixes the
exponent of $N$ in (5.14) correctly. Finally, a short combinatorial argument shows that the
symmetry factors combine in the correct fashion to give the symmetry factor of the glued
cover.

In a similar way, one can go through the details of gluing a single manifold $\mathcal{M}$ to itself
along an internal edge. There are two possible forms that such a gluing might take. If the
two copies of the internal edge which are being glued together are in the same boundary
component, the gluing has the net effect of adding one to the number of edges. The factor
of $N^{-n}$ from (5.15) correctly fixes the exponent of $N$ in (5.14). In the other case, the two
edges are in different boundary components. In this case, the genus is increased by one, and
the number of edges decreases by one. Again, the power of $N$ is fixed correctly. Finally, one
can contract an edge to a point. This decreases the number of edges by one and contributes
a factor of $N^n$ to the partition function.

As an example, let us consider the gluing together of two plaquettes $\Delta$ and $\Delta'$ along a
common edge to form a new plaquette $\Lambda$. We will take the areas of the plaquettes $\Delta$ and $\Delta'$
to be $A$ and $A'$. The partition function on $\Lambda$ can be written as

$$Z_{\Lambda}(VW) = \int dU Z_{\Delta}(VU) Z_{\Delta'}(U^{-1}W), \quad (5.16)$$

where $VW, VU$, and $U^{-1}W$ are the holonomies of the gauge field around the boundaries of
$\Lambda, \Delta$, and $\Delta'$ respectively. From (5.7) and (5.8), we can write the first few terms in the
partition functions for $\Delta$ and $\Delta'$:

$$Z_{\Delta}(VU) = e^{-\frac{\lambda A}{2} \left( \text{Tr} VU \right) N + \ldots}$$
$$+ e^{-\gamma A} \left[ \frac{1}{2} (\text{Tr} VU)^2 N^2 - \frac{\lambda A}{2} (\text{Tr} VUVU) N + \ldots \right]$$
$$+ e^{-\frac{3\gamma A}{2}} \left[ \frac{1}{6} (\text{Tr} VU)^3 N^3 + \ldots \right] + \ldots \quad (5.17)$$

$$Z_{\Delta'}(U^{-1}W) = e^{-\frac{\lambda A}{2} \left( \text{Tr} U^{-1}W \right) N + \ldots}$$
$$+ e^{-\gamma A} \left[ \frac{1}{2} (\text{Tr} U^{-1}W)^2 N^2 - \frac{\lambda A}{2} (\text{Tr} U^{-1}WU^{-1}W) N + \ldots \right]$$

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\[ +e^{-\frac{3\lambda A}{2}} \left[ \frac{1}{6} (\text{Tr} \ U^{-1} W)^3 N^3 + \ldots \right] + \ldots \]  

(5.18)

Plugging these expressions into (5.16), we can compute the integral term by term;

\[ \int dU e^{-\frac{\lambda A}{2}} e^{-\frac{\lambda A'}{2}} (\text{Tr} \ VU)(\text{Tr} \ U^{-1} W) N^2 = e^{-\frac{\lambda (A+A')}{2}} (\text{Tr} \ VW) N, \]

(5.19)

\[ \int dU e^{-\lambda A} e^{-\lambda A'} \frac{1}{4} (\text{Tr} \ VU)^2 (\text{Tr} \ U^{-1} W)^2 N^4 = e^{-\lambda (A+A')} \frac{1}{2} (\text{Tr} \ VW)^2 N^2 + O(N), \]

(5.20)

\[ \int dU e^{-\lambda A} e^{-\lambda A'} \frac{1}{2} N^3 \left[ \frac{\lambda A}{2} (\text{Tr} \ VUVU)(\text{Tr} \ U^{-1} W)^2 + \frac{\lambda A'}{2} (\text{Tr} \ VU)^2 (\text{Tr} \ U^{-1} W U^{-1} W) \right] \]

(5.21)

\[ = e^{-\lambda (A+A')} \frac{\lambda (A + A')}{2} (\text{Tr} \ VWVVW) N + O(1). \]

(5.22)

Combining these results, we get

\[ Z_\Delta(VW) = e^{-\frac{\lambda (A+A')}{2}} [(\text{Tr} \ VW) N + \ldots] \]

\[ + e^{-\lambda (A+A')} \left[ \frac{1}{2} (\text{Tr} \ VW)^2 N^2 + \frac{\lambda (A + A')}{2} (\text{Tr} \ VWVV) N + \ldots \right] \]

\[ + e^{-\frac{3\lambda (A+A')}{2}} \left[ \frac{1}{6} (\text{Tr} \ VW)^3 N^3 + \ldots \right] + \ldots, \]

(5.23)

which is exactly what we expect, since the partition function is invariant under subtriangulation. As a particular example, we will now describe the interpretation of (5.22) from the geometric point of view as a gluing of covering spaces. This term as it appears in the partition function (5.23) is associated with covers of \( \Lambda \) with winding number 2 and 1 branch point. The two terms in (5.21) correspond to the two ways such a cover can be constructed by gluing covers of \( \Delta \) and \( \Delta' \); let us take the first term, which arises from gluing together a cover of \( \Delta \) with one branch point with a cover of \( \Delta' \) with no branch points. This gluing is illustrated in figure 5. There is a single 2-fold cover of \( \Delta \) with a single fixed branch point. The branch point can be anywhere on \( \Delta \), which accounts for the factor of \( \lambda A \). This cover has a symmetry factor of 2. There is also a single cover of \( \Delta' \) with no branch points, also with a symmetry factor of 2. These two covers can be joined in exactly two possible ways along the common edge. Both of these joinings, however, give topologically equivalent covers of \( \Lambda \). The factor of 2 fixes the correct symmetry factor of this single cover, which is 2. Thus, the first term in (5.22) corresponds to a cover of \( \Lambda \) with a single branch point, which lies in \( \Delta \). The other term in (5.21) corresponds similarly to covers of \( \Lambda \) with a branch point lying in \( \Delta' \).
To review the results of this section: we have shown by induction that (5.14) describes the highest order terms in the chiral partition function for an arbitrary manifold with boundary as a sum over all coverings of the manifold. In particular, we have reproduced the result of section 3 by a more geometric argument. Note that the fact that (5.14) is exact for the torus is not explained by this argument, in which we have only retained highest order terms in $N$. It is interesting to note that the only other target space for which (5.14) has no lower order corrections is the annulus (the genus 0 surface with two boundary components.)

We can now discuss what happens when we return the extra terms to the quadratic Casimir and couple the two chiral sectors. The effects of the term $n^2/N$ and the coupling term $\lambda \tilde{a} n \tilde{b} / N^2$ are the same as was discussed in the previous section. In the case of a single plaquette, or of any manifold with boundary, however, there is now an extra coupling, due to the fact that

$$\chi_{\bar{S}R}(U) \neq \chi_S(U)\chi_R(U).$$  \hspace{1cm} (5.24)

In fact, $\chi_{\bar{S}R}(U)$ is the character of the only irreducible representation occurring in the tensor product representation $\bar{S} \otimes R$ which has a quadratic Casimir with leading term $(n + \tilde{n})N$. The effect of this coupling is precisely to cancel any folds which might occur by contracting a factor of $U$ with a factor of $U^\dagger$ from boundaries of covering sheets with opposite orientations. This can be seen as follows:

We have shown that the representations $R$ with $n$ boxes can be associated with linear combinations of $n$th order invariant polynomials in $U$, which are then associated with contours around a covering of a plaquette with a fixed orientation. If one had a contour around the boundary in the opposite direction, one would have a factor of $U^\dagger$. In fact, the representations $\bar{S}$ are exactly the complex conjugates of the representations $S$, and therefore their characters are linear combinations of polynomials of order $\tilde{n}$ in $U^\dagger$. This gives a natural
understanding of why one chiral sector involves covers with one orientation, and the other
chiral sector involved covers with the opposite orientation. To return to the question of folds,
the effect of the extra coupling implicit in the appearance of $\chi_{SR}$ in the partition function is
to subtract all terms where any factors of $U$ and $U^\dagger$ from the same plaquette are contracted.
From the point of view developed in this section, this is exactly the suppression of folds
which is responsible for allowing the theory to be described in terms of maps without folds.

Finally, let us note that the results achieved here for manifolds with boundary are also of
use in discussing the partition function when Wilson loops are inserted on the target space
manifold. This subject will be described in a later paper.

6 Non-Orientable Target Spaces

So far we have restricted our attention to orientable target spaces. However it is easy
to extend the discussion of QCD and of the string theory to the non-orientable case. The
partition function for QCD on a general non-orientable manifold has been derived by Witten
[6]. The main difference is that only self-conjugate representations ($R = \bar{R}$) contribute. Thus,
for example, the partition function for a manifold consisting of a genus $G$ surface to which
$K$ copies of the Klein bottle are attached is,

$$Z_{G,K} = \sum_{R = \bar{R}} (\dim R)^{2-2G-2K} e^{-\frac{\lambda A}{2N} C_2(R)}. \quad (6.1)$$

The representations that survive in the large $N$ limit are now the composite representations
discussed in section 2, except that $S = R$ and $n = \bar{n}$. Because of this we see that only
even winding numbers occur. The Casimir operator of these self-conjugate representations
simplifies,

$$C_2(\bar{R}R) = 2(nN + \bar{C}(R)). \quad (6.2)$$

Therefore, in the $1/N$ expansion the terms that we associated with collapsed handles and
tubes cancel completely, due to the $(-1)$ associated with orientation reversing tubes. Thus,
for example, the partition function of the Kein bottle, ($G = 0, K = 1$), is given by

$$Z_{0,1} = \sum_{R = \bar{R}} e^{-n\lambda A} \frac{\lambda A\bar{C}(R)}{N}, \quad (6.3)$$

which can be interpreted as precisely a sum over even branched covers of the Klein bottle.
From the point of view of the orientable string theory we have developed here, the fact that \( n = \tilde{n} \) for non-orientable surfaces has a simple geometric interpretation in terms of covering maps. Any cover, connected or disconnected, of a non-orientable surface by an orientable surface, must always have an equal number of sheets with each of the two possible relative orientations. This is because if we consider the permutation on sheets associated with a curve around which the orientation of the target space is reversed, this permutation must take sheets with one relative orientation to sheets with the other relative orientation. Thus, each cycle for such a permutation must contain an equal number of sheets with each orientation. It is therefore clear, that the partition function of a single chiral sector over a non-orientable surface vanishes, and that only terms with \( n = \tilde{n} \) contribute to the complete theory.

7 Conclusions

We will now briefly review the status of the general program of interpreting 2-dimensional gauge theories as string theories. We have shown here that the coefficients \( \omega_{g,G}^{n,i} \) in the asymptotic expansion of the partition function for the \( SU(N) \) gauge theory on a genus \( G \) surface have a simple interpretation as a sum over maps from a genus \( g \) surface to a genus \( G \) surface, when \( 2(g - 1) = 2n(G - 1) + i \). When the target space is a torus (\( G = 1 \)), we can understand all coefficients \( \omega_{g,G}^{n,i} \) in terms of such maps, so we have a complete understanding of the geometric structure of this partition function. Since this is the physical case of interest, namely a flat target manifold, we can claim with confidence that QCD2 is equivalent to a theory of maps of a two dimensional internal space onto the target space—i.e. a string theory.

When the genus of \( \mathcal{M} \) is not 1, extra terms arise in the partition function from the lower order terms in the \( 1/N \) expansion of the dimensions, corrections to Equation (2.3). These terms give rise to extra contributions to the coefficients \( \omega_{g,G}^{n,i} \) when

\[
2(g - 1) > 2n(G - 1) + i.
\]

We do not yet have a geometric understanding of these terms which can be related to a string theory picture. Since they do not occur on the torus they must be related to the global properties of the target space.

For those terms which we do understand, in particular for the case of the torus, the natural next step is to attempt to write down a string theory whose partition function is
equal to the free energy of the gauge theory. As discussed above, the natural conclusion of this paper is that the string action is something like the Nambu action plus a term that totally suppresses folds but otherwise does not contribute, say by contributing infinite (zero) action when the map has (does not have) folds. What is this term? One possibility is to simply introduce a constraint into the functional integral that eliminates folds. This can be done by adding to the action a term,

$$S_{\text{fold}} = \int d^2\xi \sqrt{g} \lambda (n - 1),$$

where $\lambda$ is a Lagrange multiplier field and $n$ is the normal to the embedded surface, $n = \text{sign} \left[ \det \left( \frac{\partial x^\mu}{\partial \xi^\alpha} \right) \right]$. In two dimensions the normal is a scalar, taking values $\pm 1$, and the discontinuities occur at the folds. However this is not a very elegant term and it does not generalize to higher dimensions. Another possibility is that there exist extra fermionic fields on the surface, which have zero modes for the folded maps and therefore eliminate them from the sum. We would have to add in addition corresponding bosonic fields that would cancel the contribution of the fermions for allowed maps. These fields might give extra corrections for target spaces with $G \neq 1$, thus accounting for the extra terms in the $1/N$ expansion of the dimensions. They might also explain the factor of $(-1)$ that occurs for orientation-reversing tubes.

The next step in understanding the string picture of QCD$^2$ is to incorporate fermions. First, one should calculate the correlation functions of Wilson loops. These can be calculated for arbitrary loops. However, for complicated loops, on arbitrary manifolds, the calculation is very involved. One needs to expand, in powers of $1/N$, not just the dimensions and Casimirs of general representations of $SU(N)$, but also their 6-j symbols. Also, these correlation functions depend not just on the total area and genus of the manifold but also on the areas inclosed by all non self-intersecting portions of the loop. Real quarks are even more complicated. In principle they can be treated once one knows how to handle arbitrary Wilson loops, with any number of self intersections.

Finally one should try to construct a string theory for QCD in higher dimensions, including four. This will no longer be so simple to evaluate. Two dimensions is clearly a very degenerate case both for gauge theories, since there are no glueball states – there being no transverse dimensions, and for string theory, due to the simple nature of maps and the apparent suppression of folds. In higher dimensions there will be a full spectrum of glueballs. Correspondingly we expect that the string theory will be more complex. Presumably this is
because the term in the string action that suppresses extrinsic curvature will now contribute to the weight of the maps. It is only for two dimensional maps that the extrinsic curvature is either zero or infinity. However we are confident that a string representation exists. In QCD we expect the physics to be continuous as we vary the dimension between two and four. As for string theory normally one expects trouble for \( d > 2 \) due to the tachyonic nature of the center of mass degree of freedom. However, for the QCD string this mode does not appear for \( D = 2 \), as it does for the Liouville string, and therefore we have no reason to expect a tachyon to appear as we go above two dimensions. The fold-preventing term in the string action keeps the string rigid, preventing the proliferation of thin tubes which presumably correspond to the tachyonic mode of the string, and thus eliminates this instability! Also, in this two dimensional string theory there is, as in QCD\( _2 \), no sign of the dilaton field, whose spatial dependence is responsible for the breaking of Lorenz invariance in the Liouville two-dimensional string theory. We might hope that its generalization to higher dimensions will yield a Lorenz invariant theory, with no gravitons or gauge mesons.

A Dimensions and Casimirs for Large \( N \)

In this appendix we will analyse the dimensions and Casimirs of the representations of \( SU(N) \) that survive in the large \( N \) limit.

As discussed in section 2 a representation \( R \), associated with a Young tableau containing rows of length \( n_1, \ldots, n_l \) has a quadratic Casimir given by

\[
C_2(R) = N \sum n_i + \sum_i n_i (n_i - (2i - 1)) - \frac{n^2}{N},
\]

(A.1)

The only representations which can survive as \( N \to \infty \) are those which either have a finite number of total boxes, or which have a finite number of columns of length of order \( N \).

These are precisely the composite representations, \( T = \bar{S}R \), considered in section 2. It is straightforward to show that their quadratic Casimir is given by (2.7).

The dimension of a representation of \( SU(N) \) whose Young tableaux has rows of length \((n_1, n_2, \ldots, n_k)\) is given by Weyl’s formula,

\[
\dim R = \frac{\prod_{1 \leq i < j \leq N} (h_i - h_j)}{\prod_{1 \leq i < j \leq N} (i - j)}; \quad \text{where} \quad h_i = N + n_i - 1.
\]

(A.2)

As shown in [1] one can separate out of this formula the dimension \( d_R \) of the representation of the symmetric group of \( n \equiv \sum_{i=1}^{k} n_i \) objects, corresponding to the partition
\[ n_1 \geq n_2 \geq \ldots \geq n_k, \]
\[
d_R = d_{[n_1, n_2, \ldots, n_k]} = n! \prod_{1 \leq i < j \leq k} \frac{(h_i - h_j)}{(i - j)},
\]
(A.3)
to derive
\[
dim R = \frac{d_R}{n!} \prod_{i=1}^{k} \frac{(N + n_i - i)!}{(N - i)!}
\]
(A.4)
This formula is well suited for a \( \frac{1}{N} \) expansion when the total number of boxes, \( n \), is fixed as \( N \to \infty \). Note that \( \frac{(N+n_i-i)!}{(N-i)!} = N^{n_i} \prod_{j=1}^{n_i} \left( 1 + \frac{j-i}{N} \right) \), where the index \( j \) (\( i \)) in this formula runs over the columns (rows) of the tableau. Thus we can write the dimension as,
\[
dim R = \frac{d_R N^n}{n!} \prod_{v} \left( 1 + \frac{\Delta_v}{N} \right),
\]
where the product runs over all the cells of the tableau and \( \Delta_v \) is defined for each cell to be the column index minus the row index.

Now consider the dimensions of the composite representations \( T = \bar{S}R \). It is a straightforward algebraic exercise to separate the expression (A.2) into a product over the rows of \( R \) and of \( S \) respectively times a cross-term. The result is that
\[
\dim T = \dim R \dim S Q[R, S] = \prod_{i,j} \frac{(N+1-i-j)(N+1-i-j+n_i+\tilde{n}_j)}{(N+1-i-j+n_i)(N+1-i-j+\tilde{n}_j)},
\]
(A.7)
where the product is over the rows of \( R \) (\( S \)) of length \( n_i(\tilde{n}_j) \). Note that whenever the representation \( R \) or \( S \) is trivial, then \( Q = 1 \). Furthermore, since both \( n = \sum_i n_i \) and \( \tilde{n} = \sum_j \tilde{n}_j \) are finite, \( Q \to 1 \) as \( N \to \infty \). In fact
\[
Q[R, S] \sim 1 - \frac{n\tilde{n}}{N^2} + \frac{n\tilde{C}(S) + \tilde{n}\tilde{C}(R)}{N^3} + O \left( \frac{1}{N^4} \right).
\]
(A.8)

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