Gaussian Polynomials and
Restricted Partition Functions with Constraints

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Abstract
We derive an explicit formula for a restricted partition function \( P_{mn}(s) \) with constraints making use of known expression for a restricted partition function \( W_m(s) \) without constraints.

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Consider the linear Diophantine equation with constraints
\[
a) \sum_{r=1}^{n} r x_r = s, \quad b) \sum_{r=1}^{n} x_r \leq m.
\] (1)

Denote by \( P_{mn}(s) \) a number of the non-negative integer solutions \( X = \{x_r\} \) of the linear system (1a,b).

The following theorem dates back to Sylvester [7] and Schur [5],

**Theorem 1.** Let \( P_{n}(s) \) be generated by the Gaussian polynomial \( G(n, m; t) \) of the finite order \( mn \)
\[
G(n, m; t) = \frac{\prod_{i=1}^{n+m}(1 - t^i) \cdot \prod_{u=1}^{n}(1 - t^u) \cdot \prod_{v=1}^{m}(1 - t^v)}{\prod_{s=1}^{mn} t^s} = \sum_{s \geq 0} P_{n}(s) \cdot t^s.
\] (2)

Then the partition function \( P_{mn}(s) \) has the following properties:
\[
P_{n}(s) = 0 \quad \text{if} \quad s > nm, \quad P_{n}(0) = P_{n}(nm) = 1,
P_{n}(s) = P_{n}(m) = P_{n}(nm - s), \quad P_{n} \left( \frac{mn}{2} - s \right) = P_{n} \left( \frac{mn}{2} + s \right), \quad P_{n}(s) - P_{n}(s - 1) \geq 0 \quad \text{for} \quad 0 < s \leq \frac{mn}{2}.
\] (3)

A most comprehensive introduction to the Gaussian polynomials \( G(n, m; t) \) and partition function \( P_{n}(s) \) with constraints (1b) is given in [1]. But nowhere an explicit expression for \( P_{mn}(s) \) was derived.

A situation is similar to the study of restricted partition function \( W_m(s) \) without constraints (see its definition in (4) and following paragraphs) up to the last decade, when its explicit expression was finally found [4].
In this paper we derive such formula for $P_n^m(s)$ making use of strong relationship of the Gaussian polynomials with the Molien generating function for restricted partition function $W_m(s)$ without constraints (1b).

Following [2, 4], recall the basic facts from the partition theory and consider the linear Diophantine equation $\sum_{r=1}^n d_r x_r = s$ without constraints (1b). Then the Molien function $M(d^m; t)$ reads,

$$M(d^m; t) = \prod_{i=1}^m \frac{1}{1 - t^{d_i}} = \sum_{s=0}^\infty W(s, d^m) t^s, \quad d^m = \{d_1, \ldots, d_m\}. \quad (4)$$

It generates a restricted partition function $W(s, d^m)$ which gives a number of partitions of $s \geq 0$ into positive integers $\{d_1, \ldots, d_m\}$, each not exceeding $s$, and vanishes, if such partition does not exist. According to Proposition 4.4.1, [6] and Schur’s theorem (see [8], Theorem 3.15.2), the function $W(s, d^m)$ is a quasi-polynomial of degree $m - 1$,

$$W(s, d^m) = \sum_{r=0}^{m-1} K_r(s, d^m) s^r, \quad K_{m-1}(s, d^m) = \frac{1}{(m - 1)! \pi_m}, \quad \pi_m = \prod_{i=1}^m d_i, \quad (5)$$

where coefficients $K_r(s, d^m)$ are periodic functions with periods dividing lcm($d_1, \ldots, d_m$). The explicit expressions for $W(s, d^m)$ were derived in [4] in a form of a finite sum over Bernoulli and Euler polynomials of higher order with periodic coefficients. Note that $W(0, d^m) = 1$.

In a special case, when $d^m$ is a tuple of consecutive natural numbers $\{1, \ldots, m\}$, the expression for such partition function looks much more simple (see [4], formula (46)), and its straightforward calculations for $m = 1, \ldots, 12$ were presented in [2], section 6.1. For short, we denote it by $W_m(s)$. In particular, if $m$ is arbitrary large, we arrive at unrestricted partition function $W_\infty(s)$ known due to the Hardy-Ramanujan asymptotic formula and Rademacher explicit expression [1]. By definition of a restricted partition function the following equality holds,

$$W_m(s) = W_\infty(s), \quad \text{if} \quad s \leq m. \quad (6)$$

By comparison of two generating function in (2) and (4) we obtain,

$$G(n, m; t)M(m + n; t) = M(m; t)M(n; t), \quad M(m; t) = \prod_{i=1}^m \frac{1}{1 - t^{d_i}} = \sum_{s=0}^\infty W_m(s) t^s. \quad (7)$$

Substituting into (7) the polynomial representations (2) and (4) we arrive at

$$\sum_{s_1, s_2=0}^\infty P_n^m(s_1)W_{m+n}(s_2) t^{s_1+s_2} = \sum_{s_1, s_2=0}^\infty W_n(s_1)W_m(s_2) t^{s_1+s_2}. \quad (8)$$

Equating in (8) the terms with different $s_1 + s_2 = g$ we obtain for every $g$ a linear equation in $P_n^m(s)$,

$$\Delta = \sum_{s=0}^g [P_n^m(s)W_{m+n}(g - s) - W_n(s)W_m(g - s)] = 0. \quad (9)$$

For small $s$ equation (9) may be resolved easily.
Theorem 2. Let $P_n^m(g)$ be generated by the Gaussian polynomial $G(n, m; t)$, $n \leq m$, then

$$P_n^m(g) = W_\infty(g), \quad 0 \leq g \leq n,$$  \hspace{1cm} (10)

$$P_n^m(g) = W_n(g), \quad n \leq g \leq m,$$  \hspace{1cm} (11)

$$P_n^m(g) = W_n(g) + W_m(g) - W_\infty(g), \quad m \leq g \leq m + n.$$  \hspace{1cm} (12)

Proof. If $0 \leq s \leq g \leq n$, then due to (6) we have $W_n(s) = W_\infty(s)$ and $W_m(g - s) = W_{m+n}(g - s) = W_\infty(g - s)$ since $0 \leq g - s \leq n$. Substitute the above equalities into (9) and obtain (10).

If $n \leq g \leq m$, then due to (6) we have $W_m(s) = W_{m+n}(s) = W_\infty(s)$ for all $0 \leq s \leq g$. Substitute the last equalities into (9) and obtain (13).

If $m \leq g \leq m + n$ and $g = m + r$, $1 \leq r \leq n$, let us represent the l.h.s. of (9) as follows, $\Delta = \Delta_1 + \Delta_2$.

$$\Delta_1 = \left( \sum_{r=0}^{r} + \sum_{s=m}^{m+r} \right) [P_n^m(s)W_{m+n}(m + r - s) - W_n(s)W_m(m + r - s)] = \sum_{s=0}^{r} F_1(s),$$

$$F_1(s) = P_n^m(s)W_{m+n}(m + r - s) - W_n(s)W_m(m + r - s) + P_n^m(m + r - s)W_{m+n}(s) - W_n(m + r - s)W_m(s),$$

$$\Delta_2 = \sum_{s=r+1}^{m-1} [P_n^m(s)W_{m+n}(m + r - s) - W_n(s)W_m(m + r - s)], \quad \text{i.e.,}$$

$$\Delta_2 = \sum_{s=r+1}^{k} F_1(s), \quad \text{if} \quad g = 2k + 1; \quad \Delta_2 = \sum_{s=r+1}^{k-1} F_1(s) + F_2(k), \quad \text{if} \quad g = 2k,$$  \hspace{1cm} (14)

$$F_2(s) = P_n^m(s)W_{m+n}(s) - W_n(s)W_m(s).$$  \hspace{1cm} (15)

Prove that the term $\Delta_2$ vanishes and start with $F_2(k)$. Since $n \leq m$ then $(m + 1)/2 \leq k \leq m$, that implies two equalities: first, by (6) $W_{m+n}(k) = W_m(k) = W_\infty(k)$ and next, by (13) $P_n^m(k) = W_n(k)$ either $k \leq n$ or $n \leq k \leq m$. Substitute these equalities into (15) and obtain $F_2(k) = 0$.

Consider the term $\Delta_2$ with $g = 2k$ in (14), where $1 \leq r < s < k \leq m$ and $n \leq m + r - s \leq m$, and write four equalities:

$$W_{m+n}(m + r - s) = W_m(m + r - s), \quad W_{m+n}(s) = W_m(s),$$

$$P_n^m(m + r - s) = W_n(m + r - s), \quad P_n^m(s) = W_n(s).$$  \hspace{1cm} (16)

which implies $\Delta_2 = 0$. The term $\Delta_2$ with $g = 2k + 1$ in (14) vanishes also by the same reasons (16).

Thus, instead of (9), we arrive at equation $\sum_{s=0}^{r} F_1(s) = 0$ where $0 \leq s \leq r \leq n$. However, according to (10) we have $W_{m+n}(s) = W_m(s) = W_n(s) = W_\infty(s)$, so we obtain

$$\sum_{s=0}^{r} W_\infty(s) [W_{m+n}(m + r - s) - W_m(m + r - s) + P_n^m(m + r - s) - W_n(m + r - s)] = 0,$$

where $m < m + r - s \leq m + n$. Denote $g = m + r - s$ and arrive at (14) that proves Theorem. \qed
Corollary 1.
\[ \lim_{m \to \infty} P^m_n(s) = W_n(s), \quad \lim_{n \to \infty} P^m_n(s) = W_m(s). \]

Further extension of Theorem 2 on higher \( s \geq m + n \) loses its generality, that indicates a necessity to develop another approach. Equations (9) with different \( g \) represent the linear convolution equations with a triangular Toeplitz matrix. They can be solved using the inversion of the Toeplitz matrix (see [3], Chapt. 3). We will give another representation of a general solution of (9) which can be found due to triangularity of the Toeplitz matrix.

**Theorem 3.** Let \( P^m_n(s) \) be generated by the Gaussian polynomial \( G(n, m; t) \), then
\[ P^m_n(g) = \sum_{r=0}^{g-1} \left( \sum_{s=0}^{g-r} W_n(s) W_m(g - r - s) - W_{m+n}(g - r) \right) \Phi_r(m + n), \quad (17) \]
where coefficients \( \Phi_r(m + n) \) are related to \( W_{m+n}(s) \) and defined as follows,
\[ \Phi_r(m + n) = \sum_{q_1=1}^{r} \sum_{q_r=1}^{q_1} \frac{q!}{q_1! \cdots q_r!} \prod_{k=1}^{r} (-1)^{q_k} W_{m+n}(k), \quad \sum_{k=1}^{r} kq_k = r, \quad \sum_{k=1}^{r} q_k = q. \quad (18) \]

**Proof.** Consider linear convolution equations with a triangular Toeplitz matrix
\[ P(g) = T(g) + \sum_{s=0}^{g-1} P(s) U(g - s), \quad P(0) = T(0) = 1, \quad (19) \]
where two known functions \( T(g), \ U(g) \) and unknown function \( P(g) \) are considered only on the non-negative integers. The successive recursion of (19) gives
\[ P(g) = T(g) + T(g - 1) U(1) + \sum_{s=0}^{g-2} P(s) \cdot \left[ U(g - s) + U(1) U(g - 1 - s) \right] \]
\[ = T(g) + T(g - 1) U(1) + T(g - 2) \left[ U(2) + U^2(1) \right] + \]
\[ \sum_{s=0}^{g-3} P(s) \left[ U(g - s) + U(1) U(g - 1 - s) + \left[ U(2) + U^2(1) \right] U(g - 2 - s) \right]. \]

By induction we can arrive at
\[ P(g) = \sum_{r=0}^{k-1} T(g - r) \Phi_r(U) + \sum_{s=0}^{g-k} P(s) \sum_{r=0}^{k-1} U(g - r - s) \Phi_r(U), \quad 1 \leq k \leq g, \quad (20) \]
where polynomials \( \Phi_r(U) \) are related to the restricted partition number \( W_g(r) \) of positive integer \( r \) into non-negative parts, none of which exceeds \( g, \)
\[ \Phi_r(U) = \sum_{q_1=1}^{r} \sum_{q_r=1}^{q_1} \frac{q!}{q_1! \cdots q_r!} \prod_{k=1}^{r} U^{q_k}(k), \quad \sum_{k=1}^{r} kq_k = r, \quad \sum_{k=1}^{r} q_k = q. \quad (21) \]
A sum \( \sum_{q_1, \ldots, q_r} \) in (21) is taken over all distinct solutions \( \{q_1, \ldots, q_r\} \) of the two Diophantine equations (21) with fixed \( q \). Below we present expressions for the four first polynomials \( \Phi_r(U) \),

\[
\begin{align*}
\Phi_0(U) &= 1, \\
\Phi_1(U) &= U(1), \\
\Phi_2(U) &= U(2) + U^2(1), \\
\Phi_3(U) &= U(3) + 2U(2)U(1) + U^3(1),
\end{align*}
\]

and in (27) for the other two. The total number of algebraically independent terms, contributing to the polynomial \( \Phi_r(U) \), is equal \( W_r(r) \), while the sum of coefficients at these terms is equal \( 2^{r-1} \), e.g.,\( W_3(3) = 3, 1 + 2 + 1 = 2^2 \). The terms comprising \( \Phi_r(U) \) may be calculated with Mathematica Software using \textit{IntegerPartitions}[r].

Put \( k = g \) into (20) and obtain finally,

\[
P(g) = \sum_{r=0}^{g-1} [T(g - r) + U(g - r)] \cdot \Phi_r(U).
\]

Comparing (19) and (8) we conclude,

\[
T(g) = \sum_{s=0}^{g} W_n(s) W_m(g - s), \quad U(s) = -W_{m+n}(s).
\]

Substituting (24) into (23) we immediately arrive at (13, 14).

Illustrate the usage of formulas (17,18) and apply them to calculate the partition function \( P_{m,n}(s) \) with small \( m, n \), e.g., \( m = 2, n = 3 \).

\[
P_3^2(g) = \sum_{r=0}^{g-1} \sum_{s=0}^{g-r} [W_3(s)W_2(g - r - s) - W_5(g - r)] \Phi_r(5).
\]

For this aim we need expressions for \( W_2(s), W_3(s) \) and \( W_5(s) \), found in [2], section 6.1,

\[
\begin{align*}
W_2(s) &= \frac{s}{2} + \frac{3}{4} + \frac{1}{4} \cos \pi s, \\
W_3(s) &= \frac{s^2}{12} + \frac{s}{2} + \frac{47}{72} + \frac{2}{9} \cos \frac{2\pi s}{3} + \frac{1}{8} \cos \pi s, \\
W_5(s) &= \frac{s^4}{2880} + \frac{s^3}{96} + \frac{31s^2}{288} + \frac{85s}{192} + \frac{s}{64} \cos \pi s + \frac{50651}{86400} + \frac{1}{16} \left( \cos \frac{\pi s}{2} + \sin \frac{\pi s}{2} \right) + \\
&\quad \frac{2}{27} \cos \frac{2\pi s}{3} + \frac{2}{25} \cos \frac{4\pi s}{5} + \frac{15}{128} \cos \pi s,
\end{align*}
\]

and polynomials \( \Phi_r(U), 0 \leq r \leq 5 \), presented in (22) and also given below,

\[
\begin{align*}
\Phi_4(U) &= U(4) + 2U(3)U(1) + U^2(2) + 3U(2)U^2(1) + U^4(1), \\
\Phi_5(U) &= U(5) + 2U(4)U(1) + 2U(3)U(2) + 3U(3)U^2(1) + 3U^2(2)U(1) + \\
&\quad 4U(2)U^3(1) + U^5(1).
\end{align*}
\]

Making use of (26), calculate \( W_2(r), W_3(r), W_5(r), 0 \leq r \leq 5 \), and find

\[
\Phi_0(5) = \Phi_5(5) = 1, \quad \Phi_1(5) = \Phi_2(5) = -1, \quad \Phi_3(5) = \Phi_4(5) = 0.
\]
Substitute the values for $W_k(r), k = 2, 3, 5,$ and $\Phi(r), 0 \leq r \leq 5,$ into (25) and obtain

$$
P_3^2(0) = P_3^2(1) = P_3^2(5) = P_3^2(6) = 1, \quad P_3^2(2) = P_3^2(3) = P_3^2(4) = 2,
$$

that satisfies the straightforward calculation of the Gaussian polynomial $G(3, 2; t),$

$$
G(3, 2; t) = 1 + t + 2t^2 + 2t^3 + 2t^4 + t^5 + t^6.
$$

It is easy to verify that the values in (28) satisfy also Theorem 2.

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