NEWTON POLYGONS AND p-INTEGRAL BASES

LHOUSSAIN EL FADIL, JESÚS MONTES, AND ENRIC NART

Abstract. Let \( p \) be a prime number. In this paper we use an old technique of Ø. Ore, based on Newton polygons, to construct in an efficient way \( p \)-integral bases of number fields defined by a \( p \)-regular equation. To illustrate the potential applications of this construction, we show how this result yields a computation of a \( p \)-integral basis of an arbitrary quartic field in terms of a defining equation.

Introduction

In his 1923 PhD thesis and a series of subsequent papers, Øystein Ore extended the arithmetic applications of Newton polygons. Let \( f(x) \in \mathbb{Z}[x] \) be a monic irreducible polynomial, \( K \) the number field generated by a root \( \theta \) of \( f(x) \), and \( p \) a prime number. If \( f(x) \) is \( p \)-regular (Definition 1.8), Ore determined the prime ideal decomposition of \( p \) in \( K \) and the \( p \)-valuation of the index of \( \mathbb{Z}[\theta] \) inside the maximal order, in terms of combinatorial data attached to different \( \phi \)-Newton polygons, where the polynomials \( \phi(x) \) are monic lifts to \( \mathbb{Z}[x] \) of the different irreducible factors of \( f(x) \) modulo \( p \). In this paper we show that the ideas of Ore can be extended to efficiently compute a \( p \)-integral basis too. Actually, the data that is required to build a \( p \)-integral basis is a by-product of the \( \phi \)-adic developments of \( f(x) \), whose computation is necessary to build up the Newton polygons.

In section 1 we briefly recall the work of Ore, and in section 2 we prove our main theorem (Theorem 2.6). To illustrate its wide range of applicability, we use it to obtain an explicit \( p \)-integral basis of an arbitrary quartic number field, in terms of a defining equation; to this end are devoted sections 3 and 4. A few cases that cannot be solved by Theorem 2.6 are handled in section 5 after generalizing this theorem to second order Newton polygons (Theorem 5.1).

The theory of Newton polygons of higher order was developed in [Mon99] (and revised in [GMN08]) as a tool to factorize separable polynomials in \( \mathbb{Z}_p[x] \). Theorems 2.6 and 5.1 have to be considered as the first steps towards a fast algorithm to compute integral basis in number fields, based on these higher order Newton polygons [GMN09].

1. Newton polygons

1.1. \( \phi \)-Newton polygons. Let \( p \) be a prime number. Let \( \mathbb{Z}_p \) be the ring of \( p \)-adic numbers, \( \mathbb{Q}_p \), the fraction field of \( \mathbb{Z}_p \) and \( \overline{\mathbb{Q}}_p \), an algebraic closure of \( \mathbb{Q}_p \). We denote by \( v_p: \overline{\mathbb{Q}}_p^* \to \mathbb{Z} \), the \( p \)-adic valuation normalized by \( v_p(p) = 1 \); we extend \( v_p \) to a discrete valuation of \( \mathbb{Q}_p(x) \) by letting it act in the following way on polynomials:

\[
v_p(a_0 + a_1 x + \cdots + a_r x^r) := \min_{0 \leq i \leq r} \{ v_p(a_i) \},
\]
Let \( \phi(x) \in \mathbb{Z}_p[x] \) be a monic polynomial of degree \( m \) whose reduction modulo \( p \) is irreducible. We denote by \( \mathbb{F}_\phi \) the finite field \( \mathbb{Z}_p[x]/(p, \phi(x)) \), and by
\[
\text{red}: \mathbb{Z}_p[x] \twoheadrightarrow \mathbb{F}_\phi, \quad \text{ord}: \mathbb{Z}_p[x] \twoheadrightarrow \mathbb{F}_\phi
\]
the respective homomorphisms of reduction modulo \( p \) and modulo \( (p, \phi(x)) \).

Any \( f(x) \in \mathbb{Z}_p[x] \) admits a unique \( \phi \)-adic development:
\[
f(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_r(x)\phi(x)^r,
\]
with \( a_i(x) \in \mathbb{Z}_p[x], \) \( \deg a_i(x) < m \). To any coefficient \( a_i(x) \) we attach the \( p \)-adic value \( u_i = v_p(a_i(x)) \in \mathbb{Z} \cup \{\infty\} \), and the point of the plane \((i, u_i)\), if \( u_i < \infty \).

**Definition 1.1.** The \( \phi \)-Newton polygon of \( f(x) \) is the lower convex envelope of the set of points \((i, u_i)\), \( u_i < \infty \), in the Euclidean plane. We denote this open polygon by \( \mathcal{N}_{\phi}(f) \).

The length of this polygon is by definition the abscissa of the last vertex. We denote it by \( \ell(\mathcal{N}_{\phi}(f)) := r = \lfloor \deg(f)/m \rfloor \). Note that \( \deg f(x) = mr + \deg a_r(x) \).

Usually, \( f(x) \) will be a monic polynomial not divisible by \( \phi(x) \) in \( \mathbb{Z}_p[x] \) (i.e. \( a_0(x) \neq 0 \)); in this case, the typical shape of this polygon is shown in the following figure.

![Figure 1](image_url)

The \( \phi \)-Newton polygon is the union of different adjacent sides \( S_1, \ldots, S_g \) with increasing slope \( \lambda_1 < \lambda_2 < \cdots < \lambda_g \). We shall write \( \mathcal{N}_{\phi}(f) = S_1 + \cdots + S_g \). The end points of the sides are called the vertices of the polygon.

**Definition 1.2.** The polygon determined by the sides of negative slope of \( \mathcal{N}_{\phi}(f) \) is called the principal \( \phi \)-polygon of \( f(x) \) and will be denoted by \( \mathcal{N}_{\phi}^-(f) \). The length of \( \mathcal{N}_{\phi}^-(f) \) is always equal to the highest exponent \( a = \text{ord}_{\mathbb{F}_p}(f) \) such that \( \phi(x)^a \) divides \( f(x) \) in \( \mathbb{F}_p[x] \).

**Definition 1.3.** The \( \phi \)-index of \( f(x) \) is \( \deg \phi \) times the number of points with integral coordinates that lie below or on the polygon \( \mathcal{N}_{\phi}^-(f) \), strictly above the horizontal axis, and strictly beyond the vertical axis. We denote this number by \( \text{ind}_\phi(f) \).

From now on, we denote \( N = \mathcal{N}_{\phi}^-(f) \) for simplicity. The principal polygon \( N \) and the set of points \((i, u_i)\) that lie on \( N \), contain the arithmetic information we are interested in. Note that, by construction, these points \((i, u_i)\) lie all above \( N \).
We attach to any abscissa \(0 \leq i \leq \ell(N)\) the following residual coefficient \(c_i \in \mathbb{F}_\phi:\)

\[
c_i = \begin{cases} 
0, & \text{if } (i, u) \text{ lies strictly above } N \text{ or } u_i = \infty, \\
\text{red} \left( \frac{a_i(x)}{p^{u_i}} \right), & \text{if } (i, u) \text{ lies on } N.
\end{cases}
\]

Note that \(c_i\) is always nonzero in the latter case, because \(\deg a_i(x) < m\).

Let \(S\) be one of the sides of \(N\), with slope \(\lambda\). Let \(\lambda = -h/e\), with \(h, e\) positive coprime integers. We introduce the following notations:

1. the length of \(S\) is the length, \(\ell(S)\), of the projection of \(S\) to the \(x\)-axis,
2. the degree of \(S\) is \(d(S) := \ell(S)/e\),
3. the ramification index of \(S\) is \(e(S) := e = \ell(S)/d(S)\).

Note that \(S\) is divided into \(d(S)\) segments by the points of integer coordinates that lie on \(S\). These points contain important arithmetic information, encoded by a polynomial that is built with the coefficients of the \(\phi\)-adic development of \(f(x)\) to whom these points are attached.

**Definition 1.4.** Let \(s\) be the initial abscissa of \(S\), and let \(d := d(S)\). We define the residual polynomial attached to \(S\) (or to \(\lambda\)) to be the polynomial:

\[
R_\lambda(f)(y) := c_s + c_{s+e} y + \cdots + c_{s+(d-1)e} y^{d-1} + c_{s+de} y^d \in \mathbb{F}_\phi[y].
\]

**Remarks.**

1. Note that \(c_s\) and \(c_{s+de}\) are always nonzero, so that the residual polynomial has degree \(d\) and it is never divisible by \(y\).
2. The residual polynomial depends on \(\phi(x)\), but we omit the reference to \(\phi(x)\) in the notation, since this polynomial will be specified in each context.

### 1.2. Arithmetic applications of Newton polygons

In this section we recall some results of Ø. Ore on arithmetic applications of Newton polygons [Ore23, Ore28]. Modern proofs of these results can be found in [GMN08, Sec.1].

We fix from now on a monic irreducible polynomial \(f(x) \in \mathbb{Z}[x]\) of degree \(n\), and a root \(\theta \in \overline{\mathbb{Q}}\) of \(f(x)\). We denote by \(K = \mathbb{Q}(\theta)\) the number field generated by \(\theta\) and by \(\mathbb{Z}_K\) the ring of integers of \(K\). Let

\[
\text{ind}_p(f) := v_p\left((\mathbb{Z}_K : \mathbb{Z}[\theta])\right),
\]

be the \(p\)-adic value of the index of the polynomial \(f(x)\). Recall the well-known relationship, \(v_p(\text{disc}(f)) = 2\text{ind}_p(f) + v_p(\text{disc}(K))\), between \(\text{ind}_p(f)\), the discriminant of \(f(x)\) and the discriminant of \(K\).

Let \(\mathcal{P}\) be the set of prime ideals of \(K\) lying above \(p\). For any \(\mathfrak{p} \in \mathcal{P}\), we denote by \(v_\mathfrak{p}\) the discrete valuation of \(K\) associated to \(\mathfrak{p}\), and by \(e(\mathfrak{p}/p)\) the ramification index of \(\mathfrak{p}\). Endow \(K\) with the \(\mathfrak{p}\)-adic topology and fix a topological embedding \(\iota_\mathfrak{p} : K \hookrightarrow \overline{\mathbb{Q}}_\mathfrak{p}\); we have then,

\[
v_\mathfrak{p}(\alpha) = e(\mathfrak{p}/p)v_\mathfrak{p}(\iota_\mathfrak{p}(\alpha)), \quad \forall \alpha \in K.
\]

In particular, if \(\theta^\mathfrak{p} := \iota_\mathfrak{p}(\theta) \in \overline{\mathbb{Q}}_\mathfrak{p}\), then

\[
(1.2) \quad v_\mathfrak{p}(P(\theta)) = e(\mathfrak{p}/p)v_\mathfrak{p}(P(\theta^\mathfrak{p})), \quad \forall P(x) \in \mathbb{Z}[x].
\]

After Hensel’s work, we know that there is a canonical bijection between \(\mathcal{P}\) and the set of monic irreducible factors of \(f(x)\) in \(\mathbb{Z}_p[x]\). With the above notations, the irreducible factor attached to a prime ideal \(\mathfrak{p}\) is the minimal polynomial \(F_\mathfrak{p}(x) \in \mathbb{Z}_p[x]\) of \(\theta^\mathfrak{p}\) over \(\mathbb{Q}_p\).
Choose monic polynomials $\phi_1(x), \ldots, \phi_t(x) \in \mathbb{Z}[x]$ whose reduction modulo $p$ are the different irreducible factors of $f(x)$ in $\mathbb{F}_p[x]$. We have then a decomposition:

$$f(x) \equiv \phi_1(x)^{\ell_1} \cdot \cdots \cdot \phi_t(x)^{\ell_t} \pmod{p},$$

for certain positive exponents $\ell_1, \ldots, \ell_t$. We have $n = m_1 \ell_1 + \cdots + m_t \ell_t$, where $m_i = \deg \phi_i$. By Hensel's lemma, $f(x)$ decomposes in $\mathbb{Z}_p[x]$ as:

$$f(x) = F_1(x) \cdots F_t(x),$$

for certain monic factors $F_i(x) \in \mathbb{Z}_p[x]$ such that $F_i(x) \equiv \phi_i(x)^{\ell_i} \pmod{p}$.

**Definition 1.5.** For all $1 \leq i \leq t$, define $P_{\phi_i} := \{ p \in P \mid v_p(\phi_i(\overline{x})) > 0 \} = \{ p \in P \mid v_p(\phi_i(\theta^p)) > 0 \}$.

Since the polynomials $\phi_1(x), \ldots, \phi_t(x)$ are pairwise coprime modulo $p$, the set $P$ splits as the disjoint union: $P = P_{\phi_1} \cdot \cdots \cdot P_{\phi_t}$.

**Lemma 1.6.** Let $\phi(x) := \phi_i(x)$, for some $1 \leq i \leq t$. Let $P(x) \in \mathbb{Z}[x]$ be a polynomial of degree less than $\deg \phi$. Then,

$$v_p(P(\theta)) = e(p/p) v_p(P(x)), \quad \forall p \in P_{\phi}.$$

*Proof.* Take $p \in P_{\phi}$. If $P(x) = 0$ the statement is obvious; suppose $P(x) \neq 0$ and let $P(x) = p^r Q(x)$, with $\nu = v_p(P(x))$. Since $\phi(x)$ is irreducible modulo $p$, $v_p(Q(\theta)) > 0 \iff \overline{\phi(x)}$ divides $\overline{Q(x)}$ in $\mathbb{F}_p[x]$.

Since $\overline{Q(x)} \neq 0$ and $\deg \overline{Q} \leq \deg \overline{Q} = \deg \phi = v_p(\overline{\phi(x)})$, we see that $v_p(Q(\theta)) = 0$, and:

$$v_p(P(\theta)) = v_p(p^r Q(\theta))) = v_p(p^\nu) = e(p/p) v_p(P(x)).$$

\[\square\]

**Notation.** If $\mathbb{F}$ is a finite field and $\varphi(y), \psi(y) \in \mathbb{F}[y]$, we write $\varphi \sim \psi$ to indicate that the two polynomials coincide up to multiplication by a nonzero constant in $\mathbb{F}$.

**Theorem 1.7 (Theorem of Ore).** With the above notations, let $\phi(x) := \phi_i(x)$, $F(x) := F_i(x)$, for some $1 \leq i \leq t$. Suppose that

$$N_\phi^-(f) = S_1 + \cdots + S_g$$

has $g$ different sides with slopes $\lambda_1 < \cdots < \lambda_g$; then $F(x)$ admits a factorization in $\mathbb{Z}_p[x]$ into a product of $g$ monic polynomials

$$F(x) = G_1(x) \cdots G_g(x),$$

such that, for all $1 \leq j \leq g$,

1. $N_\phi(G_j)$ is one-sided, with slope $\lambda_j$,
2. $R_{\lambda_j}(G_j)(y) \sim R_{\lambda_j}(F)(y) \sim R_{\lambda_j}(f)(y)$,
3. All roots $\alpha \in \mathbb{Q}_p$ of $G_j(x)$ satisfy $v_p(\phi(\alpha)) = |\lambda_j|$. Moreover, for all $j$, let

$$R_{\lambda_j}(f)(y) = \psi_{j,1}(y)^{n_{j,1}} \cdots \psi_{j,r_j}(y)^{n_{j,r_j}},$$

be the decomposition of $R_{\lambda_j}(f)(y)$ into a product of powers of pairwise different irreducible monic polynomials in $\mathbb{F}_{\phi}[y]$; then, the polynomial $G_j(x)$ experiments a further factorization in $\mathbb{Z}_p[x]$ into a product of $r_j$ monic polynomials

$$G_j(x) = H_{j,1}(x) \cdots H_{j,r_j}(x),$$

such that, for all $1 \leq k \leq r_j$, 
(1) $N_{φ}(H_{j,k})$ is one-sided, with slope $λ_j$.

(2) $R_{λ_j}(H_{j,k})(y) ∼ ψ_{j,k}(y)^{τ_{j,k}}$.

Finally, if $n_{j,k} = 1$, the polynomial $H_{j,k}(x)$ is irreducible in $\mathbb{Z}_p[x]$ and the ramification index and residual degree of the $p$-adic field $K_{j,k}$ generated by this polynomial are given by $e(K_{j,k}/\mathbb{Q}_p) = e(S_j)$, $f(K_{j,k}/\mathbb{Q}_p) = \deg φ \cdot \deg ψ_{j,k}$. □

Definition 1.8. Let $φ(x) ∈ \mathbb{Z}[x]$ be a monic polynomial, irreducible modulo $p$. We say that $f(x)$ is $φ$-regular if for every side of $N_{φ}^-(f)$, the residual polynomial attached to the side is separable.

Choose monic polynomials $φ_1(x), . . . , φ_t(x) ∈ \mathbb{Z}[x]$ whose reduction modulo $p$ are the different irreducible factors of $f(x)$ in $\mathbb{F}_p[x]$. We say that $f(x)$ is $p$-regular with respect to this choice if $f(x)$ is $φ_i$-regular for every $1 ≤ i ≤ t$.

Although this concept depends on the choice of $φ_1(x), . . . , φ_t(x)$, in the sequel we shall just say “$f(x)$ is $p$-regular”, taking for granted that there is an implicit choice of these liftings to $Z[x]$ of the monic irreducible polynomials of $\mathbb{F}_p[x]$ that divide $f(x)$. Since the length of $N_{φ_i}^-(f)$ is equal to $\text{ord}_{φ_i}(f)$, in practice we need only to check $φ_i$-regularity for those $φ_i(x)$ with $\text{ord}_{φ_i}(f) > 1$.

If $f(x)$ is $p$-regular, Theorem [L7] provides the complete factorization of $f(x)$ into a product of irreducible polynomials in $\mathbb{Z}_p[x]$, or equivalently, the decomposition of $p$ into a product of prime ideals of $K$. Moreover, in the $p$-regular case the $p$-index of $f(x)$ is also determined by the shape of the different $φ$-Newton polygons.

Theorem 1.9 (Theorem of the index). $\text{ind}_p(f) ≥ \sum \text{ind}_φ_i(f) + \cdots + \text{ind}_φ_t(f)$, and equality holds if $f(x)$ is $p$-regular. □

2. Computation of a $p$-integral basis in the regular case

We keep the above notations for $f(x), \theta, K, \mathcal{P}$. Recall the decomposition:

$$f(x) ≡ φ_1(x)^{t_1}, . . . , φ_t(x)^{t_t} \mod p,$$

for a certain choice of monic polynomials $φ_i(x) ∈ \mathbb{Z}[x]$ of degree $m_i$, whose reduction modulo $p$ are the different irreducible factors of $f(x)$ in $\mathbb{F}_p[x]$. We fix one of these monic polynomials $φ(x) = φ_1(x)$.

Definition 2.1. The quotients attached to the $φ$-adic development (L.L) of $f(x)$ are, by definition, the different quotients $q_1(x), . . . , q_r(x)$ that are obtained along the computation of the coefficients of the development:

$$f(x) = φ(x)q_1(x) + a_0(x),$$

$$q_1(x) = φ(x)q_2(x) + a_1(x),$$

$$\ldots \quad \ldots \quad \ldots$$

$$q_{r-1}(x) = φ(x)q_r(x) + a_{r-1}(x),$$

$$q_r(x) = φ(x) \cdot 0 + a_r(x) = a_r(x).$$

Equivalently, $q_j(x)$ is the quotient of the division of $f(x)$ by $φ(x)^j$; we denote by $r_j(x)$ the residue of this division. Thus, for all $1 ≤ j ≤ r$ we have

$$f(x) = r_j(x) + a_j(x)φ(x)^j. \quad (2.1)$$

$$r_j(x) = a_0(x) + a_1(x)φ(x) + \cdots + a_{j-1}(x)φ(x)^{j-1}. \quad (2.2)$$

$$q_j(x) = a_j(x) + a_{j+1}(x)φ(x) + \cdots + a_r(x)φ(x)^{r-j}. \quad (2.3)$$
Our first aim is to compute, for some quotients \( q(x) \) attached to the \( \phi \)-adic development of \( f(x) \), the highest power \( p^a \) of \( p \) such that \( q(\theta)/p^a \) is integral.

Let \( N_\phi^-(f) = S_1 + \cdots + S_g \) be the principal \( \phi \)-Newton polygon of \( f(x) \), and denote \( \ell = \ell(N_\phi^-(f)) = \text{ord}_\phi(f) \). For any integer abscissa \( 0 \leq j \leq \ell \), let \( y_j \in \mathbb{Q} \) be the ordinate of the point of \( N_\phi^-(f) \) of abscissa \( j \). These rational numbers form an strictly decreasing sequence, and \( y_{\ell} = 0 \). Note that, by Definition 1.3.

\[
\text{ind}_\phi(f) = [y_1] + \cdots + [y_{\ell-1}].
\]

**Lemma 2.2.** For any integer abscissa \( 0 \leq j \leq \ell \), let \( 1 \leq s_j \leq g \) be the greatest index such that the projection of \( S_{s_j} \) to the \( x \)-axis contains \( j \). Then,

\[
y_j \leq y_k + (k - j)|\lambda_{s_j}|, \quad \forall 0 \leq k \leq \ell.
\]

**Proof.** This is an immediate consequence of the convexity of \( N_\phi^-(f) \). Figure 2 illustrates the two different situations that arise according to \( k \geq j \) or \( k < j \). \( \square \)

![Figure 2](image)

**Proposition 2.3.** For all \( 0 < j < \ell \), we have \( q_j(\theta)/p^{|y_j|} \in \mathbb{Z}_K \).

**Proof.** Fix \( 0 < j < \ell \). We shall show that \( v_p(q_j(\theta)) \geq e(p/p)y_j \) for all \( p \in \mathcal{P} \).

Let \( \lambda_1 < \cdots < \lambda_g \) be the slopes of the different sides of \( N_\phi^-(f) = S_1 + \cdots + S_g \).

By the Theorem of Ore, the set \( \mathcal{P}_\phi \) splits into the disjoint union of the \( g \) subsets

\[
\mathcal{P}_{s_j} = \{ p \in \mathcal{P} \mid v_p(\phi(\theta)) = e(p/p)|\lambda_{s_j}|, \quad 1 \leq s_j \leq g \}.
\]

Let \( 1 \leq s_j \leq g \) be the greatest integer such that the projection of \( S_{s_j} \) to the horizontal axis contains the abscissa \( j \).

Suppose first that \( p \in \mathcal{P}_{s_j} \) for some \( s \leq s_j \); in this case, Lemma 1.6 (1.2) and the Theorem of Ore show that, for all \( k \geq j \):

\[
v_p(a_k(\theta)\phi(\theta)^{k-j}) = e(p/p)(u_k + (k - j)|\lambda_{s_j}|) \geq e(p/p)(u_k + (k - j)|\lambda_{s_j}|) \geq e(p/p)y_j,
\]

the last inequality by Lemma 2.2. Hence, after the substitution \( x = \theta \), each summand in \( 2.3 \) has \( v_p \)-value greater than or equal to \( e(p/p)y_j \), so that \( v_p(q_j(\theta)) \geq e(p/p)y_j \) as well.

Suppose now that either \( p \nmid \phi(\theta) \) or \( p \notin \mathcal{P}_{s_j} \) for some \( s > s_j \); that is, \( v_p(\phi(\theta)) = e(p/p)\mu \) for some \( \mu < |\lambda_{s_j}| \) (\( \mu = 0 \) if \( p \nmid \phi(\theta) \) and \( \mu = |\lambda_{s_j}| \) if \( p \in \mathcal{P}_{s_j} \)). In this case, we use the identities \( 2.1 \) and \( 2.2 \), which imply, again by Lemma 1.6 and
the Theorem of Ore:
\[ v_p(q_j(\theta)) = v_p(r_j(\theta)) - j e(p/p)\mu \geq \min_{0 \leq k < j} \{ v_p(a_k(\theta)\phi(\theta)^k) \} - j e(p/p)\mu \]
\[ = v_p(a_k(\theta)\phi(\theta)^k) - j e(p/p)\mu = e(p/p)(u_{k_0} - (j - k_0)\mu) \]
\[ > e(p/p)(u_{k_0} - (j - k_0)|\lambda_{i,j}|) \geq e(p/p)(y_{k_0} - (j - k_0)|\lambda_{i,j}|) \geq e(p/p)y_j, \]
the last inequality by Lemma 2.2.

From now on we shall denote by \( q_{i,1}(x), \ldots, q_{i,\ell_i}(x) \in \mathbb{Z}[x] \) the first \( \ell_i \) quotients attached to the \( \phi_i \)-adic development of \( f(x) \). Also, we denote by \( r_{i,j}(x) \in \mathbb{Z}[x] \) the residue of the division of \( q_{i,j}(x) \) by \( \phi_i(x)^j \), as in (2.1). Finally, we denote by \( y_{i,1}, \ldots, y_{i,\ell_i} \in \mathbb{Q} \) the ordinates of the points lying on \( N_{\phi_i}(f) \) with integer abscissa.

**Lemma 2.4.** For all \( 1 \leq I \leq t \), the quotients \( q_{i,1}(x), \ldots, q_{i,\ell_i}(x) \) satisfy
\[ \text{ord}_{\phi_I}(q_{i,j}) = \begin{cases} 
\ell_i - j, & \text{if } I = i, \\
\ell_I, & \text{if } I \neq i.
\end{cases} \]

**Proof.** The first coefficient of the \( \phi_i \)-development of \( f(x) \) that is not divisible by \( p \) is \( a_{t_i}(x) \). Hence, \( \text{ord}_{\phi_I}(q_{i,j}) = \ell_i - j \) by (2.3), and \( \text{ord}_{\phi_I}(r_{i,j}) = 0 \) for all \( i \) and all \( 1 \leq j \leq \ell_i \) by (2.2). Therefore, if \( I \neq i \), then \( \ell_I = \text{ord}_{\phi_I}(f) = \text{ord}_{\phi_I}(q_{i,j}) \) by (2.1).

**Lemma 2.5.** The following elements in \( \mathbb{Z}[\theta] \) are a \( \mathbb{Z}(p) \)-basis of \( \mathbb{Z}(p)[\theta] \):
\[ \alpha_{i,j,k} := q_{i,j}(\theta)\theta^k \in \mathbb{Z}[\theta], \quad 1 \leq i \leq t, \quad 1 \leq j \leq \ell_i, \quad 0 \leq k < m_i. \]

**Proof.** Let us show first that the \( n = \ell_1 m_1 + \cdots + \ell_t m_t \) polynomials \( q_{i,j}(x)\theta^k \) are linearly independent modulo \( p \). Denote by \( Q_{i,j,k}(x) = q_{i,j}(x)\theta^k \) their reduction modulo \( p \), and suppose that for some constants \( a_{i,j,k} \in \mathbb{F}_p \) we have
\[ \sum_{i,j,k} a_{i,j,k}Q_{i,j,k}(x) = 0. \]
Consider the polynomials
\[ A_{i,j}(x) := \sum_{0 \leq k < m_i} a_{i,j,k}Q_{i,j,k}(x) = q_{i,j}(x) \sum_{0 \leq k < m_i} a_{i,j,k}x^k \in \mathbb{F}_p[x]. \]
Now, the equality (2.4) is equivalent to \( \sum_{i,j} A_{i,j}(x) = 0 \), and this implies \( A_{i,j}(x) = 0 \) for all \( i, j \). In fact, if \( A_{i,j}(x) \neq 0 \), (2.4) shows that
\[ \text{ord}_{\phi_I}(A_{i,j}) = \begin{cases} 
\ell_i - j, & \text{if } I = i, \\
\ell_I, & \text{if } I \neq i.
\end{cases} \]
Reorder \( \phi_1, \ldots, \phi_t \) by decreasing length of the principal part of the Newton polygon: \( \ell_1 \geq \cdots \geq \ell_t \), and let \( (i_0, j_0) \) be the greatest pair of indices, in the lexicographical order, with \( A_{i_0,j_0}(x) \neq 0 \); then, \( \text{ord}_{\phi_I}(\sum_{i,j} A_{i,j}(x)) = \ell_{i_0} - j_0 \), which is a contradiction. From (2.5) we deduce \( a_{i,j,k} = 0 \) for all \( i, j, k \), so that the polynomials \( Q_{i,j,k}(x) \) are \( \mathbb{F}_p \)-linearly independent.

In particular, since the polynomials \( q_{i,j}(x)\theta^k \) have all degree less than \( n \), the integral elements \( \alpha_{i,j,k} \) are \( \mathbb{Z} \)-linearly independent. Let \( M \) be the \( \mathbb{Z} \)-module generated by all \( \alpha_{i,j,k} \). In order to finish the proof of the lemma we need only to show that \( p \) does not divide the index \( (\mathbb{Z}[\theta]: M) \).
Take \( g(x) \in \mathbb{Z}[x] \) of degree less than \( n \) and suppose that \( pg(\theta) \in M \). We have
\[
pg(x) = \sum_{i,j,k} a_{i,j,k} q_{i,j}(x)x^k,
\]
for certain integers \( a_{i,j,k} \). Since \( Q_{i,j,k}(x) \) are \( \mathbb{F}_p \)-linearly independent, all \( a_{i,j,k} \) are divisible by \( p \) and we deduce that \( g(\theta) \in M \). \( \square \)

**Theorem 2.6.** If \( f(x) \) is \( p \)-regular, then the family of all \( \alpha_{i,j,k}/p^{[y_{i,j}]} \) is a \( p \)-integral basis of \( \mathbb{Z}_K \).

**Proof.** Let \( M := \langle \alpha_{i,j,k} \rangle_{Z} \subseteq \mathbb{Z}[\theta] \), and consider the chain of free \( \mathbb{Z}(p) \)-modules
\[
\mathbb{Z}(p)[\theta] = M \otimes_{\mathbb{Z}} \mathbb{Z}(p) \subseteq \langle \alpha_{i,j,k}/p^{[y_{i,j}]} \rangle_{Z} \otimes_{\mathbb{Z}} \mathbb{Z}(p) \subseteq \mathbb{Z}_K \otimes_{\mathbb{Z}} \mathbb{Z}(p).
\]
The \( v_p \) value of the index of \( M \otimes_{\mathbb{Z}} \mathbb{Z}(p) \) inside \( \langle \alpha_{i,j,k}/p^{[y_{i,j}]} \rangle_{Z} \otimes_{\mathbb{Z}} \mathbb{Z}(p) \) is exactly \( ind_{p}(f) + \cdots + ind_{p}(f) \), which is equal to \( ind_{p}(f) \) by the Theorem of the index. Hence, \( \langle \alpha_{i,j,k}/p^{[y_{i,j}]} \rangle_{Z} \otimes_{\mathbb{Z}} \mathbb{Z}(p) = \mathbb{Z}_K \otimes_{\mathbb{Z}} \mathbb{Z}(p) \).

This theorem covers a wide range of cases, because there is a high probability that a random monic polynomial with integer coefficients is \( p \)-regular with respect to randomly chosen lifts to \( \mathbb{Z}[x] \) of the irreducible factors modulo \( p \). To illustrate its usefulness, we apply the theorem to compute a \( p \)-integral basis of an arbitrary quartic number field. To this aim is devoted the rest of the paper.

### 3. Quartic fields: preliminaries

From now on we fix a prime number \( p \) and a monic irreducible polynomial
\[
f(x) = x^4 + ax^2 + bx + c \in \mathbb{Z}[x].
\]
We choose a root \( \theta \in \overline{\mathbb{Q}} \) of \( f(x) \) and we denote by \( K \) the quartic field generated by \( \theta \). The discriminant of \( f(x) \) is
\[
disc(f) = 16a^4c - 128a^2c^2 + 256ac^3 - 27b^4.
\]

We want to apply the methods of the last section to compute a \( p \)-integral basis of \( K \). Clearly, if \( p \) does not divide \( disc(f) \), then 1, \( \theta, \theta^2, \theta^3 \) is a \( p \)-integral basis; thus, we are interested only in the case \( p \mid disc(f) \), or equivalently, \( f(x) \) inseparable modulo \( p \). We shall discuss separately the different cases that arise according to the different possibilities for the factorization of \( f(x) \) modulo \( p \). These cases can be distinguished by the computation of \( gcd(f,f) \); however, in some concrete applications it might be useful to distinguish them directly in terms of \( a, b, c \). This is the aim of the next lemma. We denote by \( \left( -\frac{a}{p} \right) \) the Legendre symbol.

**Lemma 3.1.** Let \( f(x) = x^4 + ax^2 + bx + c \in \mathbb{Z}[x] \). Then,

(A) The polynomial \( f(x) \) factorizes modulo \( p \) as the square of a quadratic irreducible polynomial if and only if it satisfies one of the following conditions:

(A1) \( p > 2, \ p \mid b, \ p \nmid ac, \ a^2 \equiv 4c \ (mod \ p), \ \left( -\frac{aq/2}{p} \right) = -1. \)

(A2) \( p = 2, \ 2 \mid b, \ 2 \nmid ac, \)

(B) The polynomial \( f(x) \) has only one double root modulo \( p \) if and only if \( p > 2 \) and it satisfies one of the following conditions:

(B1) \( p \nmid a, \ p \mid b, \ p \mid c, \)

(B2) \( p \mid a, \ p \mid b, \ p \mid c, \ 4a^3 + 27b^2 \equiv 0 \ (mod \ p), \)
(B3) \( p \mid a, \ p \nmid b, \ p \mid c \), \( 256c^3 \equiv 27b^4 \pmod{p} \),
(B4) \( p \nmid abc, \ p \mid \text{disc}(f), \ p \nmid 2a(a^2 - 4c) + 9b^2 \).

(C) The polynomial \( f(x) \) has two different double roots modulo \( p \) if and only if it satisfies one of the following conditions:

(C1) \( p > 2, \ p \nmid a, \ p \mid b, \ p \mid c, \ a^2 \equiv 4c \pmod{p}, \ \left( -\frac{a/2}{p} \right) = 1 \).

(C2) \( p = 2, \ 2 \mid a, \ 2 \mid b, \ 2 \nmid c \).

(D) The polynomial \( f(x) \) has a triple (and not 4-tuple) root modulo \( p \) if and only if it satisfies one of the following conditions:

(D1) \( p > 3, \ p \nmid abc, \ p \mid \text{disc}(f), \ p \nmid 2a(a^2 - 4c) + 9b^2 \).

(D2) \( p = 3, \ 3 \nmid a, \ 3 \mid b, \ 3 \mid c \).

(E) The polynomial \( f(x) \) has a 4-tuple root modulo \( p \) if and only if it satisfies one of the following conditions:

(E1) \( p \mid a, \ p \mid b, \ p \mid c \),

(E2) \( p = 2, \ 2 \mid a, \ 2 \nmid b, \ 2 \nmid c \).

Proof. (A) If \( p = 2 \), the only possibility is \( f(x) \equiv (x^2 + x + 1)^2 \equiv x^4 + x^2 + 1 \pmod{2} \). If \( p > 2 \), this case is determined by the existence of an integer \( s \) such that \( x^2 + s \) is irreducible modulo \( p \) and \( f(x) \equiv (x^2 + s)^2 \pmod{p} \), or equivalently

\[ a \equiv 2s \pmod{p}, \ b \equiv 0 \pmod{p}, \ c \equiv s^2 \pmod{p}, \ \left( -\frac{s}{p} \right) = -1, \]

which is is equivalent to (A1).

(B) This case does not occur for \( p = 2 \). For \( p > 2 \), it is determined by the existence of two integers \( s, t \in \mathbb{Z} \) such that

\[ f(x) \equiv (x - s)^2(x^2 + 2sx + t) \pmod{p}, \quad t \not\equiv -3s^2 \pmod{p}, \quad t \not\equiv s^2 \pmod{p}. \]

The decomposition of \( f(x) \) modulo \( p \) implies that

\[ a \equiv t - 3s^2 \pmod{p}, \quad b \equiv 2s(s^2 - t) \pmod{p}, \quad c \equiv s^2 t \pmod{p}. \]

Suppose first that \( p \nmid c \). If \( p \mid b \), the double root of \( f(x) \) modulo \( p \) is zero, and this corresponds to (B1). If \( p \nmid b \), then \( 3.1 \) is equivalent to the fact that \( x^3 + ax + b \) has a double root modulo \( p \) and no triple root, and this is equivalent to (B2).

Suppose now that \( p \mid c \). If \( p \nmid b \), then \( 3.2 \) shows that \( p \mid (s^2 - t) \), in contradiction with \( 3.1 \). If \( p \mid b \) and \( p \mid \text{disc}(f) \), then \( f(x) \) has some multiple root \( s \) modulo \( p \) because it does not fall in case (A); hence, \( a, b, c \) satisfy \( 3.3 \) for some integer \( t \). If \( p \mid a \), the condition \( p \mid \text{disc}(f) \) is equivalent to \( 256c^3 \equiv 27b^4 \pmod{p} \) and \( f(x) \) cannot have neither triple nor 4-tuple roots modulo \( p \) because \( f'(x) \equiv 4x^3 + b \pmod{p} \) is separable modulo \( p \). Finally, if \( p \nmid a \), from \( 3.2 \) we get \( 2a(a^2 - 4c) + 9b^2 \equiv 2(t - s^2)(t + 3s^2)^2 \pmod{p} \), so that \( 3.1 \) is equivalent to \( p \nmid 2a(a^2 - 4c) + 9b^2 \).

(C) If \( p = 2 \), the only possibility is \( f(x) \equiv x^2(x+1)^2 \pmod{2} \), which is equivalent to (C2). If \( p > 2 \), then \( f(x) \) has two double roots modulo \( p \) if and only if \( f(x) \equiv (x - s)^2(x + s)^2 \pmod{p} \), for some integer \( s \) not divisible by \( p \). This implies (C1), because \( a \equiv -2s^2 \pmod{p} \) and \( \text{disc}(f) \equiv 16c(a^2 - 4c)^2 \pmod{p} \). Conversely, (C1) implies that \( f(x) \equiv 2x(2x^2 + a) \pmod{p} \) is a separable polynomial modulo \( p \), so that \( f(x) \) has neither triple nor 4-tuple roots modulo \( p \). Since \( f(x) \) falls neither in case (A) nor (B), it must have two different double roots modulo \( p \).

(D) This case is determined by the existence of an integer \( s \) such that

\[ f(x) \equiv (x - s)^3(x + 3s) \pmod{p}, \quad 4s \not\equiv 0 \pmod{p}. \]
In particular, \( p \) is necessarily odd. If \( p = 3 \), this condition is equivalent to (D2), and then \( s \equiv -b \pmod{3} \). If \( p > 3 \), condition (E3) is equivalent to

\[
(3.4) \quad \begin{align*}
    a &\equiv -6s^2 \pmod{p}, \\
    b &\equiv 8s^3 \pmod{p}, \\
    c &\equiv -3s^4 \pmod{p},
\end{align*}
\]

for some \( s \) not divisible by \( p \); clearly, this implies \( \gcd{abc, p} \mid \text{disc}(f) \). The condition \( p \mid 2a(a^2 - 4c) + 9b^2 \) is necessary too, because otherwise \( f(x) \) would satisfy (B4). Conversely, (D1) implies that \( f(x) \) has a multiple root modulo \( p \) with multiplicity three, because \( f(x) \) will be inseparable modulo \( p \) and it does not fall in neither of the cases (A), (B), (C) and (E) below.

(E) The only possibilities for \( f(x) \) modulo \( p \) are: \( f(x) \equiv (x + 1)^4 \pmod{2} \), and \( f(x) \equiv x^4 \pmod{p} \). 

A user’s guide to find the \( p \)-integral basis of \( K \) would be: check first which of the conditions (A), (B), (C), (D) or (E) of Lemma 3.1 satisfies \( f(x) \), and then look for the following equation, lemma or table:

| A1 | A2 | B | C1 | C2 | D1 | D2 | E1 | E2 |
|----|----|---|----|----|----|----|----|----|
| (4.1) | (4.2) | (4.3) | (4.5) | (4.6) | Lem. 4.2 | Lem. 4.3 | Lem. 4.4 | Table 2 |

3.1. **An iteration method.** Consider now an arbitrary monic and irreducible polynomial \( F(x) \in \mathbb{Z}[x] \) of degree four. We describe in this paragraph an iterative process that converges in some cases to an integer \( s \in \mathbb{Z} \) such that \( F(x) \) is \((x - s)\)-regular.

**Definition 3.2.** **We say that the integer \( s \) is regular if \( F(x) \) is \((x - s)\)-regular. Otherwise, we say that \( s \) is irregular.**

Let \( s \) be an integer, and denote

\[
(3.5) \quad \begin{align*}
    u_0 &:= v_p(F(s)), \quad u_1 := v_p(F'(s)), \quad u_2 := v_p(F''(s)/2), \quad u_3 := v_p(F'''(s)/6), \\
    \sigma_0 &:= F(s)/p^{u_0}, \quad \sigma_1 := F'(s)/p^{u_1}, \quad \sigma_2 := F''(s)/2p^{u_2}, \quad \sigma_3 := F'''(s)/6p^{u_3}.
\end{align*}
\]

The \((x - s)\)-adic development of \( F(x) \) is

\[
F(x) = F(s) + F'(s)(x - s) + \frac{1}{2}F''(s)(x - s)^2 + \frac{1}{6}F'''(s)(x - s)^3 + (x - s)^4,
\]

and the \((x - s)\)-Newton polygon of \( F(x) \) is the lower convex envelope of the points: \((0, u_0)\), \((1, u_1)\), \((2, u_2)\), \((3, u_3)\), and \((4, 0)\).

Suppose that \( s \) satisfies one of the following initial conditions:

\[
(3.6) \quad \begin{align*}
    & (i) \quad u_2 = 0, \quad u_1 > 0, \quad u_0 > 0, \\
    & (ii) \quad u_3 = 0, \quad u_2 > 0, \quad u_1 > 0, \quad u_0 > 0, \\
    & (iii) \quad \begin{cases} 
        u_0 > 2u_2, & u_1 > \frac{3}{2}u_2, \quad u_3 \geq \frac{1}{2}u_2 > 0, \\
        \text{the side of } N_{x-s}(F) \text{ with end points } (2, u_2) \text{ and } (4, 0)
    \end{cases}
\end{align*}
\]

If \( s \) is irregular, then the unique side whose residual polynomial is inseparable has an integer slope, and the unique multiple irreducible factor of the residual polynomial has degree one. In fact, Table II displays all possible situations where an irregular \( s \) satisfies these initial conditions, and Figure 3 shows the possible shapes of \( N_{x-s}(F) \) in all these cases. When we draw two points with the same abscissa, it means that both possibilities may occur.
\textbf{Table 1.} Irregular cases for an integer $s$ satisfying one of the initial conditions (i), (ii), or (iii) of (3.6).

| $p$ | $u_0$, $\sigma_1$ | slope | residual polynomial |
|-----|-------------------|-------|---------------------|
| (i) | 2 $u_0$ even, $u_0 < 2u_1$ | $u_0/2$ | $(y + 1)^2$ |
| (i) | $u_0 > 2u_1$, $\overline{\sigma}_1 = 4\sigma_0\sigma_2$ | $u_0/2$ | $\sigma_2 \left(y + \frac{\sigma_1}{2\sigma_2}\right)^2$ |
| (ii) | 2 $u_0 > 3u_2$, $u_0 + u_2 < 2u_1$ | $(u_0 - u_2)/2$ | $(y + 1)^2$ |
| (ii) | $u_0 > 3u_2$, $u_0 + u_2 = 2u_1$ | $(u_0 - u_2)/2$ | $\sigma_2 \left(y + \frac{\sigma_1}{2\sigma_2}\right)^2$ |
| (ii) | 2 $u_0 > (3/2)u_1$, $u_1$ even, $u_1 < 2u_2$ | $u_1/2$ | $(y + 1)^2$ |
| (ii) | $u_0 > 3u_2$, $u_1 = 2u_2$ | $u_1/2$ | $\sigma_3 \left(y + \frac{\sigma_2}{3\sigma_3}\right)^2$ |
| (ii) | 2 $u_0 = 3u_2$, $u_1 = 2u_2$ | $u_0/3$ | $(y + \frac{\sigma_1}{3\sigma_2})^3$ |
| (ii) | $u_0 = 3u_2$, $u_1 = 2u_2$ | $u_0/3$ | $\sigma_3 y^3 + \sigma_1 y + \sigma_0$ |
| (ii) | $u_0 = (3/2)u_1 < 3u_2$ | $u_0/3$ | $\sigma_3 y^3 + \sigma_2 y^2 + \sigma_0$ |
| (ii) | $u_0 = 3u_2 < (3/2)u_1$ | $u_0/3$ | $\sigma_3 y^3 + \sigma_2 y^2 + \sigma_0$ |
| (iii) | 2 $u_0 + u_2$ even, $u_0 + u_2 < 2u_1$ | $(u_0 - u_2)/2$ | $(y + 1)^2$ |
| (iii) | $u_0 + u_2 = 2u_1$, $\overline{\sigma}_1 = 4\sigma_0\sigma_2$ | $(u_0 - u_2)/2$ | $\sigma_2 \left(y + \frac{\sigma_1}{2\sigma_2}\right)^2$ |

Any $s$ satisfying (3.0) will be regular with high probability. But, what can we do if we pick an irregular $s$? Let us show an efficient way to find a regular $s$.

\textbf{Lemma 3.3.} Let $s$ be an integer satisfying one of the initial conditions of (3.0), and suppose that $s$ is irregular. Let $-\delta$ be the slope of unique side whose residual polynomial is inseparable. Let $y \in \mathbb{Z}$ be any integer whose reduction modulo $p$ is the

![Figure 3](image-url)
unique multiple root of the residual polynomial, and consider the integer \( t = s + yp^\delta \).

Then, after a finite number of iterations \( s_0 := s, s_1 := t, \ldots, s_n \) of this procedure, we get a regular \( s_n \).

**Proof.** Let us deal first with the cases where \((2, u_2)\) is a vertex of \( N_{x-s}^{-}(F)\); that is, when \( s \) satisfies (i), or (iii), or the subcase \( u_0 > 3u_2, u_1 > 2u_2 \) of (ii). In these cases, if \( s \) is irregular then \( \delta = (u_0 - u_2)/2 \) is an integer and

\[
\begin{align*}
u_1 &> (u_0 + u_2)/2 \quad \text{and} \quad y \text{ is odd, if } p = 2, \\
u_1 &> (u_0 + u_2)/2 \quad \text{and} \quad 2\sigma_2y + \sigma_1 \equiv 0 \pmod{p}, \quad \text{if } p > 2.
\end{align*}
\]

Taylor expansion shows that

\[
\begin{align*}F(t) &= F(s) + F'(s) yp^\delta + \frac{1}{2} F''(s) y^2 p^{2\delta} \quad \text{(mod } p^{3\delta}), \\
F'(t) &= F'(s) + F''(s) y p^\delta \quad \text{(mod } p^{2\delta}), \\
F''(t) &= F''(s) + F'''(s) y p^\delta \quad \text{(mod } p^{\delta}).
\end{align*}
\]

We get in all cases (note that \( v_2(F''(s)) = u_2 + 1, v_2(F'''(s)) = u_3 + 1, \) if \( p = 2) \)

\[v_p(F(t)) > u_0, \quad v_p(F'(t)) > \frac{u_0 + u_2}{2}, \quad v_p(F''(t)/2) = u_2.\]

In particular, \( t \) satisfies again the initial conditions (3.1). Now, if \( v_p(F(t)) = v_p(F(s)) + 1 \) then \( F(x) \) is already \((x - t)\)-regular, because \( N_{x-t}(F) \cap ([0, 2] \times \mathbb{R}) \) has one side of length two but degree one (because \( v_p(F(t)) + v_p(F''(t)/2) \) is odd). On the other hand, if \( v_p(F(t)) \geq v_p(F(s)) + 2 \), there is at least one more point of integral coordinates lying below or on \( N_{x-t}(F) \), and

\[\text{ind}_{x-t}(F) < \text{ind}_{x-t}(F) \leq \text{ind}_{s}(F),\]

the last inequality by the Theorem of the index (Theorem 1.1). Hence, we cannot have an indefinite sequence \( s_0 := s, s_1 := t, \ldots, s_n, \ldots \) of irregular integers, because \( \text{ind}_{x-s_n}(F) \) grows strictly in each iteration and it is bounded by \( \text{ind}_{s}(F) \).

Suppose now that \( s \) satisfies (ii) and the irregularity is caused by a side with end points \((1, u_1)\) and \((3, 0)\) (rows 5 and 6 of Table 1). From Taylor expansion we deduce now

\[v_p(F(t)) > \frac{3}{2} u_1, \quad v_p(F'(t)) > u_1, \quad v_p(F''(t)/2) > \frac{1}{2} u_1, \quad v_p(F'''(t)/6) = 0.\]

In spite of the fact that \( v_p(F(t)) \) can be smaller than \( v_p(F(s)) \), we can argue as above and deduce that either \( t \) is regular, or \( \text{ind}_{x-t}(F) \) is strictly greater than \( \text{ind}_{x-s}(F) \).

Finally, suppose that \( s \) satisfies (ii) and \( N_{x-s}^{-}(F) \) has a side of length three (rows from 7 to 11 of Table 1). From Taylor expansion we deduce now

\[v_p(F(t)) > u_0, \quad v_p(F'(t)) > \frac{2}{3} u_0, \quad v_p \left( \frac{F''(t)}{2} \right) \geq \frac{1}{3} u_0, \quad v_p \left( \frac{F'''(t)}{6} \right) = 0,\]

and \( v_p(F''(t)/2) = u_0/3 \) if and only if the residual polynomial has a double (and not triple) root. We finish the argument as in the previous cases. \(\square\)

**Remark 3.4.** (1) In the cases described in rows 2, 4 and 13 of Table 1 if we choose \( y \) such that \( 2\sigma_2y + \sigma_1 \equiv 0 \pmod{p^\delta} \), we get \( v_p(F'(t)) \geq 2v_p(F'(s)) \) and the iteration process is accelerated.

(2) In the cases described in rows 10 and 11 of Table 1 all elements \( s_n \) computed along the iteration method satisfy: \( v_p(F''(s_n)/2) = u_0/3.\)
4. Explicit $p$-integral basis of quartic fields

In this section we apply Theorem 2.6 to exhibit a $p$-integral basis of the quartic number field $K$ generated by the irreducible polynomial $f(x) = x^4 + ax^2 + bx + c \in \mathbb{Z}[x]$. We shall denote throughout $\Delta := v_p(\text{disc}(f))$.

We discuss separately the different cases that arise according to the type of decomposition of $f(x)$ modulo $p$. We apply the iteration method of Lemma 3.3 only when $\Delta$ can be arbitrarily large, and even in these cases we try to give a more intrinsic description of the regular elements.

4.1. $f(x)$ is inseparable modulo $p$, but it has no multiple roots in $\mathbb{F}_p$.
Suppose $f(x)$ satisfies (A) of Lemma 3.4. If $p > 2$, we take any integer $s$ satisfying $v_p(a - 2s) > \frac{1}{2}v_p(4c - a^2)$. For instance, if $a$ is even we may take $s = a/2$. Now we take $\phi(x) = x^2 + s$ as the lift to $\mathbb{Z}[x]$ of the irreducible factor of $f(x)$ modulo $p$.
The $\phi$-adic development of $f(x)$ is

$$f(x) = \phi(x)^2 + (a - 2s)\phi(x) + bx + ((a - 2s)^2 + 4c - a^2)/4,$$

with attached quotients $q_1(x) = \phi(x) + a - 2s$, $q_2(x) = 1$. Since $2v_p(a - 2s) > v_p(4c - a^2)$, we have

$$2v_p(a - 2s) > \min\{v_p(b), v_p(4c - a^2)\} = v_p(bx + ((a - 2s)^2 + 4c - a^2)/4).$$

Hence, if we denote $\nu := \frac{1}{2}\min\{v_p(b), v_p(4c - a^2)\} \in \frac{1}{2}\mathbb{Z}$, the principal Newton polygon $N_\phi(f) = N_\nu(f)$ is one-sided with slope $-\nu$ (see Figure 4).

![Figure 4](image)

Take $\epsilon \in \mathbb{F}_p$ satisfying $\epsilon^2 + \epsilon = 0$, and denote $b_0 := b/p^{2\nu}$, $c_0 := (4c - a^2)/p^{2\nu}$. If $2\nu$ is odd, the side has degree one, whereas for $\nu \in \mathbb{Z}$, the residual polynomial attached to the side is $y^2 + b_0x + \frac{1}{4}c_0 \in \mathbb{F}_p[y]$, which is always separable. Therefore, $f(x)$ is $p$-regular and Theorem 2.6 shows that $1$, $\theta$, $q_1(\theta)/p^{\nu}$, $\theta q_1(\theta)/p^{\nu}$ is a $p$-integral basis of $K$. Since $(a - 2s)/p^{\nu}$ is an integer, the following family is a $p$-integral basis too:

$$1, \theta, \frac{\theta^2 + s}{p^{\nu}}, \frac{\theta^3 + s\theta}{p^{2\nu}}, \quad \nu = \frac{1}{2}\min\{v_p(b), v_p(4c - a^2)\}.$$

If $p = 2$ we may take $\phi(x) = x^2 + x + 1 \in \mathbb{Z}[x]$ as a lift of the irreducible factor of $f(x)$ modulo 2. The $\phi$-adic development of $f(x)$ is

$$f(x) = \phi(x)^2 - (2x - a + 1)\phi(x) + (b + 1-a)x + c - a,$$

and the two quotients attached to this development are $q_1(x) = \phi(x) - (2x - a + 1) = x^2 - x + a$, $q_2(x) = 1$. Since $v_2(2x - a + 1) = 1$ regardless of the (odd) value of $a$, the $\phi$-Newton polygon has three different possibilities according to $v_2((b + 1 - a)x + c - a) = 1, 2$ or $\geq 3$, reflected in Figure 5.
In the first and third cases, the residual polynomials attached to the sides have degree one, whereas in the second case the residual polynomial attached to the unique side of slope $-1$ is $y^2 + y + 1 \in \mathbb{F}_2[y]$, which is irreducible. Therefore, $f(x)$ is always 2-regular and Theorem 2.6 shows that the 2-integral basis of $K$ is:

$$\begin{align*}
\text{4.2) } & 1, \theta, \theta^2, \theta^3, \text{ or } 1, \theta, \frac{\theta^2 + \theta + 1}{2}, \frac{\theta^3 + \theta^2 + \theta}{2},
\end{align*}$$

according to $\min\{v_2(b + 1 - a), v_2(c - a)\} = 1$, or greater than one.

4.2. $f(x)$ has only one double root modulo $p$. Suppose $f(x)$ satisfies (B) of Lemma 3.1 and let $s_0$ be any integer such that $f(x) \equiv (x - s_0)^2g(x) \pmod{p}$, for some separable (modulo $p$) polynomial $g(x)$ such that $g(s_0)$ is not divisible by $p$. By the Theorem of Ore (Theorem 1.7), if $p$ ramifies in $\mathbb{Z}_K$ then it splits as $p\mathbb{Z}_K = p^2 q_1 q_2$ (if $g(x)$ splits modulo $p$), or $p\mathbb{Z}_K = p^3 q$ (if $g(x)$ is irreducible modulo $p$). In any case, since $p > 2$ we have $v_p(\text{disc}(K)) = 0, 1$ and $\text{ind}_p(f) = \lfloor \Delta/2 \rfloor$.

Since $s_0$ is a separable root of $f'(x)$ modulo $p$, Hensel’s lemma shows that $f'(x)$ has a $p$-adic root congruent to $s_0$ modulo $p$. Let $s \in \mathbb{Z}$ be a sufficiently good approximation of this root; more precisely: $v_p(f'(s)) > \Delta/2$. The polygon $N = N_{x-s}(f)$ has length two and

$$v_p(f'(s)) > \Delta/2 \geq \text{ind}_p(f) \geq \text{ind}_{x-s}(f) = \min\left\{\frac{1}{2}v_p(f(s)), v_p(f'(s))\right\}.$$

Hence, $v_p(f'(s)) > v_p(f(s))/2$, and this implies that $f(x)$ is $(x-s)$-regular. In particular, $\text{ind}_{x-s}(f) = \text{ind}_p(f) = \lfloor \Delta/2 \rfloor$.

The first quotient of the $(x-s)$-adic development is:

$$\begin{align*}
\text{4.3) } & q_{1,x-s}(x) = (f(x) - f(s))/(x - s) = x^3 + sx^2 + (a + s^2)x + s^3 + as + b.
\end{align*}$$

Let $q_2(x)$ be the second quotient of this $(x-s)$-development, and let $q_3(x), q_4(x)$ be the two other quotients that arise from the other factors of $f(x)$ modulo $p$. By Theorem 2.6 the following four integral elements are a $p$-integral basis:

$$q_{1,x-s}(\theta)/p', \ q_2(\theta), \ q_3(\theta), \ q_4(\theta), \ \nu = \lfloor \Delta/2 \rfloor.$$

Since $q_2(x), q_3(x), q_4(x)$ are monic polynomials with integer coefficients, the following family is a $p$-integral basis too:

$$\begin{align*}
\text{4.4) } & 1, \theta, \theta^2, \frac{(\theta^3 + s\theta^2 + (a + s^2)\theta + s^3 + as + b)/p'}{p'}, \ \nu = \lfloor \Delta/2 \rfloor.
\end{align*}$$

4.3. $f(x)$ has two double roots modulo $p$. Suppose $f(x)$ satisfies (C) of Lemma 3.1. We want to choose two regular lifts to $\mathbb{Z}$ of the two double roots modulo $p$. We show how to achieve this with different arguments for the cases $p > 2$ and $p = 2$. 

![Figure 5](image-url)
4.3.1. Two double roots, case \( p > 2 \). Recall that \( f(x) \equiv (x - s)^2(x + s)^2 \pmod{p} \), where \( s \) is a square root of \(-a/2\) modulo \( p \).

**Lemma 4.1.** Let us write \( b = Bp^m, a^2 - 4c = Ap^{m'} \), for some positive exponents \( m, m' \) and some integers \( A, B \) not divisible by \( p \). Let \( r = \min \{ m, m' \} \), \( \delta = v_p(A^2 + 8aB^2) \), and let \( s \) be a square root of \(-a/2\) modulo \( p^{r+1} \). Then,

1. If either \( m \neq m' \) or \( r \neq \delta \), then \( s \) and \(-s\) are regular lifts of the two double roots of \( f(x) \) modulo \( p \).
2. Suppose \( m = m' = \delta \) and \( s \) is irregular. Apply to \( s \) the iteration method of Lemma 3.3 to obtain a regular \( s_n \); then \(-s_n \) is regular too.
3. Suppose \( s \) and \(-s\) are regular lifts of the two double roots of \( f(x) \) modulo \( p \), and denote \( \nu_\pm = \lfloor \min \{ v_p(f(\pm s))/2, v_p(f'(\pm s)) \} \rfloor \). Then,

\[
\min(\nu_-, \nu_+) = \lfloor r/2 \rfloor, \quad \max(\nu_-, \nu_+) = \lfloor (\Delta - r)/2 \rfloor.
\]

**Proof.** Assume first \( m \neq m' \). From \( 4f(s) = (2s^2 + a)^2 + 4bs - (a^2 - 4c) \), \( f'(s) = 2s(2s^2 + a) + b \), we get \( v_p(f(\pm s)) = r, v_p(f'(\pm s)) \geq r \), so that \( s \) and \(-s\) are regular.

Assume now \( m = m' \). We have \( v_p(f'(\pm s)) = r \) in all cases. At least one of the two integers \( 4Bs - A, -4Bs - A \) is not divisible by \( p \). Suppose that \( s \) is chosen so that \( v_p(-4Bs - A) = 0 \); then \( v_p(f(-s)) = r \) and \(-s\) is regular. On the other hand, \( v_p(4Bs - A) = v_p(16B^2s^2 - A^2) \) is equal to \( \delta \), if \( \delta \leq r \), and it is greater than \( r \) otherwise. Hence, \( v_p(f(s)) = r + \delta \), if \( \delta \leq r \), and \( v_p(f(s)) > 2r \) otherwise. Thus, \( s \) might be irregular only when \( \delta = r \). This proves (1).

If \( s \) is irregular, we have \( v_p(f(s)) = 2r, v_p(f'(s)) = r \). By (3.8), all iterates \( s_n \) obtained by the method of Lemma 3.3 satisfy \( v_p(f(s_n)) > 2r, v_p(f'(s_n)) > r \). Since \( f(s_n) - f(-s_n) = 2bs_n, f'(s_n) + f'(-s_n) = 2b \), we get \( v_p(f(-s_n)) = v_p(f'(s_n)) = r \) and \(-s_n \) is regular. This proves (2).

The above arguments show that \( \nu_- = \lfloor r/2 \rfloor \) (recall that we chose \( s \) in such a way that \( \nu_- \leq \nu_+ \)). The ideal \( p\mathbb{Z}_K \) splits as the product of two ideals \( a_s, a_{-s} \), both of norm \( p^\delta \), whose factorization is determined by the respective Newton polygons \( N_{-s}(f), N_{+s}(f) \). Thus, \( v_p(\disc(K)) = 0, 1, 2 \), according to the number 0, 1, 2 of ramified ideals in the pair \( a_s, a_{-s} \). On the other hand, \( a_{-s} \) ramifies if and only if \( r \) is odd. This allows to compute \( \nu_+ \) form \( \nu_- + \nu_+ = \ind_p(f) = (\Delta - v_p(\disc(K))/2 \). This proves (3).

Lemma 4.1 shows how to find an integer \( s \) such that both \( s \) and \(-s\) are regular, and in most of the cases \( s \) is just a sufficiently good \( p \)-adic approximation to a square root of \(-a/2\). Theorem 2.5 provides then a \( p \)-integral basis:

\[
q_{1, x-s}(\theta)/p^\nu, \quad q_{2, x-s}(\theta), \quad q_{1, x+s}(\theta)/p^\nu, \quad q_{2, x+s}(\theta),
\]

where \( q_{1, x\pm s}(x) \) are given in (4.3), and \( q_{2, x\pm s}(x) \) are monic polynomials with integer coefficients. By exchanging \( s \) and \(-s\) we may assume that \( \nu_- \leq \nu_+ \), as we did in the proof of Lemma 4.1. Then,

\[
(2s\theta^2 + 2s^3 + 2as)/p^\nu = (q_{1, x-s}(\theta) - q_{1, x+s}(\theta))/p^\nu,
\]

is integral. Since \( p \nmid 2s \), the following family is a \( p \)-integral basis too.

\[
\begin{align*}
1, \quad &\theta, (\theta^2 + s^2 + a)/p^\nu, \quad (\theta^3 + s\theta^2 + (a + s^2)\theta + s^3 + as + b)/p^\nu, \\
\end{align*}
\]

with \( \nu_- = \lfloor r/2 \rfloor, \nu_+ = \lfloor (\Delta - r)/2 \rfloor \).
4.3.2. Two double roots, case \( p = 2 \). Recall that \( f(x) \equiv x^2(x + 1)^2 \mod 2 \), so that any even integer is a lift of the double root 0 modulo 2 and any odd integer is a lift of the double root 1 modulo 2.

If \( v_2(c) = 1 \), then \( N_{-}(f) \) is one-sided with slope 1/2, so that 0 is an even regular lift and \( \text{ind}_x(f) = 0 \). Similarly, if \( v_2(a + b + c + 1) = 1 \) then 1 is an odd regular lift and \( \text{ind}_{x+1}(f) = 0 \). If \( v_2(c) > 1 \) and \( v_2(a + b + c + 1) > 1 \), then:
\[
\begin{align*}
& a \equiv 1 \mod 4 \implies v_2(b) = 1 \implies v_2(f'(0)) = 1 \implies 0 \text{ is regular}, \\
& a \equiv -1 \mod 4 \implies v_2(b) > 1 \implies v_2(f'(1)) = 1 \implies 1 \text{ is regular}.
\end{align*}
\]

Hence, we apply the iteration method of Lemma 3.3 to obtain a regular lift \( s \) of an specific double root of \( f(x) \) modulo 2, and the 2-integral basis is given by

\[
\begin{align*}
\begin{array}{|c|c|c|}
\hline
v_2(c) & v_2(a + b + c + 1) & s, \text{ 2-integral basis} \\
\hline
1 & 1 & s \text{ odd} \\
1 & > 1 & s \text{ odd} \\
> 1 & 1 & s \text{ even} \\
> 1 & > 1 & a \equiv 1 \mod 4 \implies s \text{ odd} \\
& & a \equiv 3 \mod 4 \implies s \text{ even} \\
\hline
\end{array}
\end{align*}
\]

\[
(4.6)
\]

where \( \alpha = \theta^3 + s\theta^2 + (a + s^2)\theta + s^3 + as + b \), and \( \nu = \min\{v_2(f(s))/2, v_2(f'(s))\} \). In fact, if \( a \equiv 1 \mod 4 \), we take 0 as an even regular lift and we find an odd regular lift \( s \). Theorem 2.6 yields a 2-integral basis: \( q_{1,s}(\theta)/2, q_{2,s}(\theta), q_{1,s-\theta}(\theta)/2^\nu, q_{2,s-\theta}(\theta) \), where \( q_{1,s}, q_{1,s-\theta} \) are given in (4.3) and \( q_{2,s}, q_{2,s-\theta} \) are monic polynomials. In particular, \((\theta^2 + \theta)/2 \) is integral, because \( a, s \) are odd and
\[
(s\theta^2 + s^2\theta + s^3 + as)/2 = (q_{1,s-\theta} - q_{1,s}(\theta))/2
\]
is integral. Hence, \( 1, \theta, (\theta^2 + \theta)/2, q_{1,s-\theta}(\theta)/2^\nu \) is a 2-integral basis too. The case \( a \equiv -1 \mod 4 \) follows by symmetric arguments.

4.4. \( f(x) \) has a triple root modulo \( p \). Suppose \( f(x) \) satisfies (D) of Lemma 3.1

Recall that \( f(x) \equiv (x - s)^3(x + 3s) \mod p \), for some integer \( s \) not divisible by \( p \). The \( (x - s) \)-Newton polygon of \( f(x) \) has length three and the two first quotients of the \( (x - s) \)-adic development of \( f(x) \) are
\[
q_1(x) = x^3 + sx^2 + (s^2 + a)x + s^3 + as + b, \quad q_2(x) = x^3 + 2sx + 3s^2 + a.
\]

Therefore, if we choose a regular \( s \), Theorem 2.6 yields a \( p \)-integral basis:
\[
(4.7) \quad 1, \theta, \quad \frac{\theta^2 + 2s\theta + 3s^2 + a}{p^{\nu_i}}, \quad \frac{\theta^3 + s\theta^2 + (s^2 + a)\theta + s^3 + as + b}{p^{\nu_1}},
\]

where \( \nu_i = |y_i| \) and \( y_i \in \mathbb{Q} \) is the ordinate of the point on \( N_{-i}(f) \) of abscissa \( i \), for \( i = 1, 2 \). We discuss the choice of a regular \( s \) and the computation of \( \nu_1, \nu_2 \) separately for the cases \( p > 3 \) and \( p = 3 \).

4.4.1. Triple root, \( p > 3 \). The coefficients \( a, b, c \) satisfy (3.4); in particular, \(-a/6\) is a square in \( \mathbb{Z}_p \).

**Lemma 4.2.** Let \( s_0 \) be an integer satisfying (3.4) and such that \( u_2 := v_p(a + 6s_0^2) = v_p(f''(s_0)/2) > \Delta/6 \). Denote also
\[
(4.8) \quad u_0 := v_p(f(s_0)), \quad u_1 := v_p(f'(s_0)); \quad \sigma_0 := f(s_0)/p^{u_0}, \quad \sigma_1 := f'(s_0)/p^{u_1}.
\]

(1) The integer \( s_0 \) is irregular if and only if \( 2u_0 = 3u_1 \) and \( p | \sigma_1^3 + 27\sigma_0\sigma_1^2 \).
(2) If \( s_0 \) is regular, take \( s = s_0 \); if \( s_0 \) is irregular, apply to \( s_0 \) the iteration method of Lemma 3.3 to obtain a regular \( s \). Then, the \( p \)-integral basis is given by \((4.7)\), with

\[
\begin{array}{ccc}
u_0, \nu_1 & \nu_1 & \nu_2 \\
2u_0 < 3u_1 & [2u_0/3] & [u_0/3] \\
2u_0 \geq 3u_1 & [(\Delta - u_1)/2] & [u_1/2] \\
\end{array}
\]

Proof. By Theorem 3.3 we have \( \Delta \geq 2 \text{ind}_p(f) \geq 2 \text{ind}_{x-s_0}(f) \), so that \( u_2 > \text{ind}_{x-s_0}(f)/3 \), by the hypothesis. This implies that the point \((2, u_2)\) lies strictly above \( N_{x-s_0}^-(f) \), whose shape is given in Figure 6. In fact, the conditions \( u_2 \leq u_0/3 \), \( u_2 \leq u_1/2 \) lead to a contradiction: \( \text{ind}_{x-s_0}(f) = u_2 + [(u_0 + u_2)/2] \geq 3u_2 \), or \( \text{ind}_{x-s_0}(f) = u_1 + u_2 \geq 3u_2 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Newton polygons and \( p \)-integral bases.}
\end{figure}

Suppose \( 2u_0 < 3u_1 \). The Newton polygon has only one side, with slope \(-u_0/3\). If \( 3 \nmid u_0 \), the side has degree one; if \( 3 \mid u_0 \), the side has degree three and residual polynomial \( 4s_0y^3 + \sigma_1 \), which is always separable in \( \mathbb{F}_p[y] \). Hence, \( s_0 \) is regular and \( \nu_2 = [u_0/3] \), \( \nu_1 = [2u_0/3] \).

Suppose \( 2u_0 > 3u_1 \). The Newton polygon has two sides, one of them of degree one; the other one has either degree one (if \( u_1 \) is odd) or it has degree two and residual polynomial \( 4s_0y^2 + \sigma_1 \), which is always separable in \( \mathbb{F}_p[y] \). Hence, \( s_0 \) is regular and \( \nu_2 = [u_1/2] \), \( \nu_1 = u_1 \). By the Theorem of Ore (Theorem 1.7), \( p \) ramifies in \( K \) if and only if \( u_1 \) is odd, in which case \( p\mathcal{O}_K = p^2\mathfrak{q}_1\mathfrak{q}_2 \) and \( \nu_2(\text{disc}(K)) = 1 \); hence, \( \nu_1 + \nu_2 = \text{ind}_p(f) = [\Delta/2] \), and \( \nu_1 = [(\Delta - u_1)/2] \).

Suppose \( 2u_0 = 3u_1 \). The Newton polygon has only one side, of degree three and residual polynomial \( 4s_0y^3 + \sigma_1y + \sigma_0 \). Hence, \( s_0 \) is regular if and only if \( (\sigma_1)^3 + 27\sigma_0(\sigma_0)^2 \neq 0 \). If \( s_0 \) is regular, then \( p \) is unramified in \( K \) and \( \nu_2 = u_1/2 \), \( \nu_1 = u_1 = (\Delta - u_1)/2 \). If \( s_0 \) is irregular, then all iterates of the process of Lemma 4.3 satisfy \( \nu_p(f''(s_n)/2) = u_1/2 \), \( \nu_p(f'(s_n)) > u_1 \), \( \nu_p(f(s_n)) > u_0 \) (cf. Remark 4.4); hence \( N_{x-s}^-(f) \) will have one side with end points \((2, u_1/2), (3, 0)\), so that \( \nu_2 = u_1/2 \). Also, \( \nu_p(\text{disc}(K)) = 0, 1 \), because \( p\mathcal{O}_K = \mathfrak{a}\mathfrak{q}_1\mathfrak{q}_2 \) for some ideal \( \mathfrak{a} \) of norm \( p^2 \), whose factorization depends on the final shape of \( N_{x-s}^-(f) \). Hence, \( \nu_1 = [(\Delta - u_1)/2] \). \( \square \)

4.4.2. Triple root, \( p = 3 \). Suppose \( p = 3 \) and \( 3 \mid a, 3 \nmid b, 3 \mid c \). If \( a \equiv 3 \) (mod 9), then \(-a/6\) is a square in \( \mathbb{Z}_3 \) and we can argue as in the case \( p > 3 \).

**Lemma 4.3.** Suppose \( a \equiv 3 \) (mod 9), \( 3 \nmid b, 3 \mid c \). Let \( s_0 \) be an integer such that \( u_2 := v_3(f''(s_0)/2) = v_3(a + 6s_0^2) > \Delta/6 \), and \( s_0 \equiv -b \) (mod 3). For \( i = 0, 1 \), denote also \( u_i, \sigma_i \) as in \((4.8)\).

1. The integer \( s_0 \) is irregular if and only if \( 2u_0 < 3u_1 \) and \( 3 \mid u_0 \).
2. If \( s_0 \) is regular, take \( s = s_0 \); if \( s_0 \) is irregular, take \( s_1 = s_0 + y3^{u_0/3} \), where \( y = 1 \) if \( \sigma_0 \equiv b \) (mod 3) and \( y = -1 \) otherwise. If \( s_1 \) is regular, take \( s = s_1 \);
if \( s_1 \) is irregular, apply to \( s_1 \) the iteration method of Lemma 3.3 to obtain a regular \( s \). Then, the \( p \)-integral basis is given by (4.7), with

| \( u_0, u_1 \) | \( v_3(f(s_1)) \) | \( v_3(f'(s_1)) \) | \( v_1 \) | \( v_2 \) |
|---|---|---|---|---|
| \( 2u_0 \geq 3u_1 \) | \( u_1 \) | \( u_0/3 \) | \( u_0/3 \) |
| \( 2u_0 < 3u_1, 3 \nmid u_0 \) | \( u_0 + 1 \) | \( 2u_0/3 \) | \( u_0/3 \) |
| \( 2u_0 < 3u_1, 3 \mid u_0 \) | \( u_0 + 1 \) | \( (2u_0/3) + 1 \) | \( (2u_0/3) + 1 \) |
| \( 2u_0 < 3u_1, 3 \mid u_0 \) | \( u_0 + 2 \) | \( (2u_0/3) + 1 \) | \( u_0/3 \) |
| \( 2u_0 < 3u_1, 3 \mid u_0 \) | \( u_0 + 2 \) | \( (2u_0/3) + 1 \) | \( \lfloor \Delta/2 \rfloor - (u_0/3) - 1 \) |

**Proof.** The proof runs in parallel with that of Lemma 4.2. An essential difference is that now a polynomial of the form \( y^3 + \sigma \) is always inseparable in \( \mathbb{F}_3[y] \), whereas a polynomial of the form \( y^3 + \sigma y + \tau \) is separable as long as \( \sigma \in \mathbb{F}_3 \) is nonzero.

If \( s_0 \) is irregular, we saw in (3.9) that

\[
v_3(f(s_1)) > u_0, \quad v_3(f'(s_1)) > 2u_0/3, \quad v_3(f''(s_1)) > u_0/3.
\]

If \( v_3(f(s_1)) \leq u_0 + 2 \), or \( v_3(f'(s_1)) \leq (2u_0/3) + 1 \), then \( s_1 \) is regular. If \( s_1 \) is irregular, then \( \text{ind}_{s_1}(f) \geq \text{ind}_{s_0}(f) + 3 = u_0 + 3 \), because we get at least three more points of integers coordinates below or on the \((x - s_1)\)-Newton polygon (two with abscissa 1 and one with abscissa 2). We claim that \( v_3(f''(s_1)) = (u_0/3) + 1 \) in this case. In fact, \( f''(s_1) = f''(s) + 24s_0y^{u_0/3} + 12y^2p^{2u_0/3} \); hence, \( v_3(f''(s_1)) > (u_0/3) + 1 \) would imply \( v_3(f''(s_0)) = (u_0/3) + 1 \), and we would get \( \Delta \geq 2 \text{ind}_p(f) \geq 2 \text{ind}_{s_1}(f) \geq 2u_0 + 6 = 6v_3(f''(s_0)) \), in contradiction with our hypothesis. Finally, as in the previous lemma, in this case we have \( v_3(\text{disc}(K)) = 0, 1 \), so that \( v_1 + v_2 = \text{ind}_p(f) = \lfloor \Delta/2 \rfloor \). This determines \( v_1 \) as well.

Suppose now that \( a \equiv 0, 6 \pmod{9} \). For any integer \( s \equiv -b \pmod{3} \), we have necessarily \( v_3(f''(s)) = v_3(a + 6s^2) = 1 \). The following lemma can be proved by completely analogous arguments.

**Lemma 4.4.** Suppose \( a \equiv 0, 6 \pmod{9} \), \( 3 \nmid b, 3 \mid c \). If \( v_3(f(-b)) \leq 2 \), or \( v_3(f'(-b)) = 1 \), then take \( s = -b \). If \( v_3(f(-b)) > 2 \), and \( v_3(f'(-b)) > 1 \), then take \( s \) to be any integer such that \( v_3(f(s)) > \lfloor \Delta/2 \rfloor \), and \( s \equiv -b \pmod{3} \). Then, \( s \) is regular and the 3-integral basis is given by (4.7), with

| \( v_3(f(s)) \) | \( v_3(f'(s)) \) | \( v_1 \) | \( v_2 \) |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 1 | 0 | 0 |
| \( > 2 \) | \( > 1 \) | \( \lfloor \Delta/2 \rfloor - 1 \) | 1 |

4.5. \( f(x) \) has a 4-tuple root. Suppose \( f(x) \) satisfies (E) of Lemma 3.1. 4.5.1. The three coefficients \( a, b, c \) of \( f(x) \) are divisible by \( p \). If \( v_p(a) \geq 2 \), \( v_p(b) \geq 3 \), and \( v_p(c) \geq 4 \), then the quartic polynomial \( x^4 + (a/p^2)x^2 + (b/p^3)x + (c/p^4) \) generates the same quartic field. By repeating this procedure, we can assume that either \( v_p(a) = 1 \), or \( v_p(b) \leq 2 \), or \( v_p(c) \leq 3 \). Under this assumption, a direct application of Theorem 2.6 yields a \( p \)-integral basis in most of the cases. The results are collected in Table 2. All rows of this table correspond to a situation where \( f(x) \) is \( p \)-regular, except for the third row (which reflects a regular situation only if \( v_p(a^2 - 4c) = 2 \)), and the sixth row. These irregular cases cannot be dealt with the iteration method of Lemma 3.3 because the unique side of the Newton polygon has a non-integer
Table 2. \( f(x) = x^4 + ax^2 + bx + c \) satisfies (E1) of Lemma 3.1. In rows 7 and 8, we use a regular integer \( s \) obtained by the iteration method of Lemma 3.3, and we denote \( \alpha = \theta^3 + s\theta^2 + (a + s^2)\theta + s^3 + as + b, \) \( u_0 = v_2(f(s)), \) \( u_1 = v_2(f'(s)) \).

| \( v_p(c) \) | \( v_p(b) \) | \( v_p(a) \) | \( p \) | \( \nu \) | \( p \)-integral basis |
|------------|----------------|----------------|-----|-------|------------------|
| 1          | 1              | 1              | 1   | 1, \( \theta, \theta^2, \theta^3 \) |
| > 1        | 1              | 1              | \( p \) | \( \frac{1}{2} \) \( \min \{ v_p(b), v_p(a^2 - 4c) \} \) | \( 1, \theta, \theta^2/p, \theta^3/p \) |
| 2          | \( > 1 \)      | 1              | 2   | \( 1, \theta, \theta^2 \) \( \frac{\theta^3 + (a/2)\theta}{p^{v_p(\theta)}} \) \( \frac{\theta^3 + (a/2)\theta}{p^{v_p(\theta^2)}} \) |
| > 2        | \( > 1 \)      | \( > 2 \)      | \( \theta, \theta^2/p, \theta^3/p \) |

Table 3. Expansion of the sixth row of Table 2

| \( v_2(a) \) | \( v_2(b) \) | \( v_2(2a + c - 4) \) | \( v_2(2a + b) \) | \( Q(x) \) | \( \nu \) |
|----------|-------------|---------------------|-----------------|-------|-------|
| 2        | 3           | 3                   | \( x^2 \)       | 5/4   |
| 2        | \( \geq 4 \) | \( \geq 4 \)         | \( x^2 + 2x \)  | \( \geq 4 \) |
| \( \geq 4 \) | \( \geq 4 \) | \( \geq 4 \)         | \( x^2 + 2 \)   | \( \geq 4 \) |
| \( \geq 4 \) | \( \geq 4 \) | \( \geq 4 \)         | \( x^2 + 2x \)  | \( \geq 4 \) |
| \( \geq 4 \) | \( \geq 4 \) | \( \geq 5 \)         | \( x^2 + 2x \)  | \( \geq 4 \) |

If \( v_2(a) = 2, v_2(b) \geq 4 \) and \( v_2(2a + c - 4) = 3 \), we denote \( u = v_2(b), \) \( v = v_2(c - (a^2/4)) \), \( d \) \( = (c - (a^2/4))/2^\nu \) (mod 4).

| \( u, v \) | \( d \) | \( Q(x) \) | \( \nu \) |
|----------|-------|----------|-------|
| \( u \leq v \) | \( u - 1 = v = 2w \) | \( x^2 + \frac{2}{w} \) | \( w + \frac{1}{2} \) |
| \( u \leq v \) | \( u - 1 > v = 2w \) | \( x^2 + \frac{2}{w} \) | \( w + \frac{1}{2} \) |
| \( u \leq v \) | \( u - 1 = v = 2w \) | \( x^2 + \frac{2}{w} \) | \( w + \frac{1}{2} \) |
| \( u \leq v \) | \( u - 1 > v = 2w \) | \( x^2 + \frac{2}{w} \) | \( w + \frac{1}{2} \) |
| \( u \leq v \) | \( u - 1 = v = 2w + 1 \) | \( x^2 + \frac{2}{w} \) | \( w + \frac{1}{2} \) |
| \( u \leq v \) | \( u - 1 > v = 2w + 1 \) | \( x^2 + \frac{2}{w} \) | \( w + \frac{1}{2} \) |
| \( u \leq v \) | \( u - 1 = v = 2w + 1 \) | \( x^2 + \frac{2}{w} \) | \( w + \frac{1}{2} \) |
| \( u \leq v \) | \( u - 1 > v = 2w + 1 \) | \( x^2 + \frac{2}{w} \) | \( w + \frac{1}{2} \) |

slope (equal to \(-1/2\)). They are handled in section 5 by using Newton polygons of second order.
4.5.2. The case \( p = 2, a, b \) even and \( c \) odd. We have \( f(x) \equiv (x + 1)^4 \pmod{2} \). For any odd integer \( m \), the \((x - m)\)-development of \( f(x) \) is:

\[
f(x) = (x - m)^4 + 4m(x - m)^3 + A(x - m)^2 + B(x - m) + C,
\]

with \( A = 6m^2 + a, B = f'(m), C = f(m) \). The polynomial

\[
g(x) := f(x + m) = x^4 + 4mx^3 + Ax^2 + Bx + C,
\]
defines the same quartic field \( K \), and since \( A, B, C \) are all even, the computation of a 2-integral basis is similar to the previous case. We take \( \omega := \theta - m \) as a root of \( g(x) \), and we display the results in Table 20 whose rows take into account all possible values of the triple \((v_2(A), v_2(B), v_2(C))\), except for three cases that may be discarded by a proper choice of \( m \). Lemma 4.5 whose proof is left to the reader, shows how to avoid these three bad cases.

**Lemma 4.5.** Let \( m, m' \) be odd integers and let \( f(x + m) = x^4 + 4mx^3 + Ax^2 + Bx + C \), \( f(x + m') = x^4 + 4mx'^3 + A'x^2 + B'x + C' \).

If \( v_2(A) > 2, v_2(B) > 3 \) and \( v_2(C) = 4 \), then for \( m' = m + 2 \), we get \( v_2(A') > 2, v_2(B') > 3 \) and \( v_2(C') > 4 \).

If \( v_2(A) > 4, v_2(B) = 6 \) and \( v_2(C) = 8 \), then for \( m' = m + 4 \), we get \( v_2(A') = 4, v_2(B') > 6, v_2(C') > 8 \).

If \( v_2(A) = 4, v_2(B) = 6 \) and \( v_2(C) > 8 \), then for \( m' = m + 4 \), we get \( v_2(A') > 4, v_2(B') > 6 \) and \( v_2(C') > 8 \).

Most of the rows of Table 4 correspond to a situation where \( g(x) \) is 2-regular, with some exceptions that we discuss now.

In rows 5, 18 and 24, regularity is achieved by applying the iteration method of Lemma 3.3. The cases of rows 4, 16 and 17 are handled in section 5 by using Newton polygons of second order.

In rows 10 and 11, we consider the polynomial \( h(x) = g(2x)/16 = x^4 + 2x^3 + A'x^2 + B'x + C' \), where \( A' = A/4, B' = B/8, C' = C/16 \). This polynomial is another defining equation of our quartic field \( K \).

If \( v_2(A) = 2, v_2(B) > 3, v_2(C) = 4 \) (row 10), then \( h(x) \equiv (x^2 + x + 1)^2 \pmod{2} \), and the discussion is similar to that of section 4.3. In the table below, we display a choice of \( \phi(x) \in \mathbb{Z}[x] \) such that \( \phi(x) \equiv x^2 + x + 1 \pmod{2} \) and \( h(x) \) is \( \phi \)-regular.

We denote \( u = v_2(B' + 1 - A') \), \( v = v_2(C' - (A' - 1)^2/4) \) in this table.

| \( A' \pmod{4} \) | \( u, v \) | \( \phi(x) \) |
|-----------------|-----------------|-----------------|
| 1               | \(-1\)           | \((x^2 + x - 1)\) |
| \(-1\)          | \(-1\) \( u = v = 2w \) | \((x^2 + (1 + 2^w)x + \frac{A - 1}{2})\) |
| \(-1\)          | \(-1\) \( u > v = 2w \) | \((x^2 + 1 + 2^w + \frac{A - 1}{2^w})\) |
| \(-1\)          | \(-1\) \( u > v = 2w \) | \((x^2 + x + \frac{A - 1}{2} + 2^w)\) |

We display the 2-integral basis provided by Theorem 2.6 in Table 6.

If \( v_2(A) = 2, v_2(B) > 3, v_2(C) > 4 \) (row 11), then \( h(x) \equiv x^2(x + 1)^2 \pmod{2} \), and the discussion is similar to that of section 4.3. We display the results in Table 7. For any integer \( s \) the first quotient of the \((x - s)\)-development of \( h(x) \), evaluated at the root \( \tau := \omega/2 \) of \( h(x) \) is:

\[
\alpha_s := \tau^3 + (s + 2)\tau^2 + (s^2 + 2s + A')\tau + s^3 + 2s^2 + A's + B'.
\]
If \( s \) is regular, the ordinate of the point on \( N_{x-s}(h) \) of abscissa one is:

\[
\nu_s := \min \left\{ \frac{1}{2} v_2(h(s)), v_2(h'(s)) \right\}
\]

Finally, in row 25, we consider the polynomial \( h(x) = g(4x)/64 = x^4 + mx^3 + A'x^2 + B'x + C' \), where \( A' = A/16, B' = B/64, C' = C/256 \). This polynomial defines the same quartic field \( K \), and it satisfies \( h(x) \equiv x^3(x + 1) (\text{mod } 2) \); we take \( \tau := \omega/4 \) as a root of \( h(x) \) in \( K \). A 2-integral basis is obtained by the computation of a regular lift \( s \) of the triple root modulo 2, by the iteration method of Lemma 3.3. Theorem 2.6 yields:

\[
1, \tau, \tau^2 + (2s + m)\tau + 2s^2 + 2ms + A', \\
\tau^3 + (s + m)\tau^2 + (s^2 + ms + A')\tau + s^3 + ms^2 + A's + B'
\]

where \( \nu_2 = \min \left\{ \frac{1}{2} v_2(h(s)), \frac{1}{2} v_2(h'(s)), v_2(h''(s)) \right\} \) is the ordinate of the point on \( N_{x-s}(h) \) of abscissa two, and \( \nu_1 = \min \left\{ \frac{1}{2} (v_2(h(s)) + v_2), v_2(h'(s)) \right\} \) that of the point of abscissa one.

5. Newton polygons of second order

The theory of Newton polygons of higher order was developed in [Mon99] (and revised in [GMN08]) as a tool to factorize separable polynomials in \( \mathbb{Z}_p[x] \). It conjecturally yields a fast algorithm to compute integral basis in number fields [GMN09]. In this section we shall use second order polygons directly addressed to cover the few cases of section 4 where a quartic polynomial is not \( p \)-regular and the iteration method of Lemma 3.3 cannot be applied because the irregular side of the Newton polygon has non-integer slope. In these cases we can find a \( p \)-integral basis by generalizing Theorem 2.6 to certain Newton polygons of second order.

5.1. Second order \( p \)-integral bases. Let \( F(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x] \) be an irreducible polynomial; let \( \theta \in \mathbb{Q} \) be a root of \( F(x) \), and \( K \) the quartic field generated by \( \theta \). Let \( p \) be a prime number and denote

\[
u_i = v_p(a_i), \quad \sigma_i = a_i/p^{\nu_i}, \quad i = 0, 1, 2, 3.
\]

We assume throughout this section that \( u_0 = 2, u_1 > 1, u_2 \geq 1, u_3 \geq 1 \), and \( R(y) := y^2 + \sigma_2y + \sigma_0 \in \mathbb{F}_p[y] \) is inseparable. Hence, \( N_x(F) \) has only one side, with length four, degree two and slope \(-1/2\) (see Figure 7).

![Figure 7](image-url)
TABLE 4. $f(x)$ satisfies (E2) of Lemma 5.1 and $g(x) = f(x + m) = x^4 + 4mx^3 + Ax^2 + Bx + C$. In the first twelve rows we take $m = 1$; in the rest of the rows we consider other odd values of $m$ to avoid some bad cases (see Lemma 5.3). In rows 5, 18, 24, we use a regular integer $s$ obtained by the iteration method of Lemma 5.3. We denote $\alpha = \omega^3 + (s + 4m)\omega^2 + (A + 4ms + s^2)\omega + s^3 + 4ms^2 + As + B$, and $\nu = \min\{v_2(g(s)) + v_2(A)/2, v_2(g'(s))\}$.

\[
\begin{array}{|c|c|c|c|}
\hline
v_2(C) & v_2(B) & v_2(A) & 2\text{-integral basis} \\
\hline
1 & 1 & 1 & 1, \omega, \omega^2, \omega^3 \\
> 1 & > 1 & 1, \omega, \omega^2, \omega^3/2 \\
2 & 1, 1 & 1, \omega, \omega^2/2, \omega^3/2 \\
2 & > 1 & 1, \omega, \omega^2/2, \omega^3/2 \\
3 & > 2 & 1, \omega, \omega^2/2, \omega^3/2 \\
4 & 2 & 1, \omega, \omega^2/2, \omega^3/4 \\
4 & > 2 & 1, \omega, \omega^2/2, \omega^3/4 \\
4 & > 3 & 1, \omega, \omega^2/2, \omega^3/4 \\
5 & 3 & 1, \omega, \omega^2/2, \omega^3/8 \\
5 & > 4 & 1, \omega, \omega^2/2, \omega^3/8 \\
6 & 4 & 1, \omega, \omega^2/2, \omega^3/16 \\
6 & > 5 & 1, \omega, \omega^2/2, \omega^3/16 \\
6 & > 6 & 1, \omega, \omega^2/2, \omega^3/16 \\
7 & 5 & 1, \omega, \omega^2/2, \omega^3/32 \\
7 & > 5 & 1, \omega, \omega^2/2, \omega^3/32 \\
7 & > 6 & 1, \omega, \omega^2/2, \omega^3/32 \\
8 & > 6 & 1, \omega, \omega^2/2, \omega^3/32 \\
8 & > 7 & 1, \omega, \omega^2/2, \omega^3/32 \\
8 & > 8 & 1, \omega, \omega^2/2, \omega^3/32 \\
8 & > 9 & 1, \omega, \omega^2/2, \omega^3/32 \\
\hline
\end{array}
\]

Choose a $p$-local integer $y \in \mathbb{Z}_p(p)$ whose reduction modulo $p$ is the double root of $R(y)$; for instance, take $y = 1$ if $p = 2$. Choose an arbitrary $z \in \mathbb{Z}_p(p)$ and consider the monic polynomial $\phi(x) = x^2 + zpx - yp \in \mathbb{Z}_p(p)[x]$. Let $v_p^{(2)}$ be the $p$-adic valuation of second order of $\mathbb{Q}_p(x)$ defined in [GMN08, Sec.2.2]. We recall the following properties of $v_p^{(2)}$:

1. $v_p^{(2)}(m) = 2v_p(m)$, for all $m \in \mathbb{Z}_p$,
2. $v_p^{(2)}(x) = 1, v_p^{(2)}(\phi(x)) = 2$,
3. $v_p^{(2)}(mx + n) = \min\{v_p^{(2)}(mx), v_p^{(2)}(n)\}$, for all $m, n \in \mathbb{Z}_p$.

Let $F(x) = \phi(x)^2 + a_1(x)\phi(x) + a_0(x)$ be the $\phi$-adic development of $F(x)$. The $\phi$-Newton polygon of the second order, $N^{(2)}_\phi(F)$, is by definition the lower convex envelope of the set of points $(i, v_p^{(2)}(a_i(x)\phi(x)^i))$, $i = 0, 1, 2$, of the Euclidean plane. The possible shapes of this polygon are displayed in Figure 8.
TABLE 5. Expansion of the fifth row of Table 4. The 2-integral basis is 1, \(\omega\), \(Q(\omega)/2^{\nu}\), \(Q(\omega)/2^{\nu+(1/2)}\).

| \(v_2(A)\) | \(v_2(B + 8)\) | \(v_2(2A + C + 4)\) | \(Q(x)\) | \(\nu\) |
|---------|----------------|-----------------|--------|--------|
| 2       | 3              | \(x^2\)         | 5/4    |
| 2       | 4              | \(x^2 + 2\)     | 7/4    |
| 2       | \(\geq 4\)    | 3               | 7/4    |
| 2       | \(\geq 5\)    | 4               | 9/4    |
| \(\geq 3\) | \(\geq 4\)   | 3               | 5/2    |
| \(\geq 3\) | \(\geq 4\)   | \(x^2 + 2\)     | 7/4    |
| \(\geq 3\) | \(\geq 4\)   | \(x^2 + 2\)     | 2      |

If \(v_2(A) \geq 3\), \(v_2(B + 8) \geq 4\), and \(v_2(2A + C + 4) = 3\), we denote \(u = v_2(B + 8 - 2A)\), \(v = v_2(C - ((A - 4)^3/4))\), \(d = (C - ((A - 4)^3/4))/2^\nu\) (mod 4), \(e = (B + 8 - 2A)/2^\nu\) (mod 4).

| \(u, v\) | \(d\) | \(e\) | \(Q(x)\) | \(\nu\) |
|----------|-------|-------|----------|--------|
| \(u < v\), \(u = v = 2w\) | \(1 + \frac{3}{2}\) | 1 | \(x^2 + 2x - 2 + \frac{3}{2}\) | \(w + \frac{1}{2}\) |
| \(u = v = 2w + 1\) | \(1 + \frac{1}{2}\) | \(-1\) | \(x^2 + (2 + 2^\nu) x - 2 + \frac{3}{2} + 2^w\) | \(w + \frac{3}{2}\) |
| \(u = v = 2w + 1\) | \(-1 + \frac{1}{2}\) | \(-1\) | \(x^2 + (2 + 2^\nu) x - 2 + \frac{3}{2}\) | \(w + \frac{3}{2}\) |
| \(u = v = 2w + 1\) | \(-1 + \frac{1}{2}\) | \(-1\) | \(x^2 + (2 + 2^\nu) x - 2 + \frac{3}{2}\) | \(w + \frac{3}{2}\) |
| \(u = v = 2w + 1\) | \(-1 + \frac{1}{2}\) | \(-1\) | \(x^2 + (2 + 2^\nu) x - 2 + \frac{3}{2}\) | \(w + \frac{3}{2}\) |
| \(u = v = 2w + 1\) | \(-1 + \frac{1}{2}\) | \(-1\) | \(x^2 + (2 + 2^\nu) x - 2 + \frac{3}{2}\) | \(w + \frac{3}{2}\) |
| \(u > v = 2w + 1\) | \(-1\) | \(-1\) | \(x^2 + (2 + 2^\nu) x - 2 + \frac{3}{2}\) | \(w + 1\) |

TABLE 6. Expansion of the tenth row of Table 4. We take \(h(x) = g(2x)/16 = x^4 + 2x^3 + A'x^2 + B'x + C'\), with \(A', C'\) odd, \(B'\) even. We denote \(u = v_2(B' + 1 - A')\), \(v = v_2(C' - ((A' - 1)^2/4))\), \(r = \min(u, v)\), \(e = (B' + 1 - A')/2^\nu\) (mod 4), \(d = (C' - ((A' - 1)^2/4))/2^\nu\) (mod 4), \(k = A' + 2^u\) (mod 8). The 2-integral basis is 1, \(\tau\), \(Q(\tau)/2^\nu\), \(\tau Q(\tau)/2^\nu\), where \(\tau = \omega/2\) is a root of \(h(x)\).

| \(A' \equiv 1\) (mod 4) | \(Q(x)\) | \(\nu\) |
|--------------------------|--------|--------|
| \(C' \equiv 1\) (mod 4) or \(v_2(B') = 1\) | \(x^2\) | 0     |
| otherwise                | \(x^2 + x + 1\) | 1     |

If \(A' \equiv 3\) (mod 4)

| \(r\) odd | \(Q(x)\) | \(\nu\) |
|------------|--------|--------|
| \(u > v = 2w\) | \(u = 2w + 1\) or \(d = 1\) | \(x^2 + x + A' - 1\) | \(w\) |
| \(u > v = 2w\) | otherwise | \(x^2 + x + A' - 1\) + 2^w | \(w + 1\) |
| \(u = v = 2w\) | \(d = \frac{1}{2} + \frac{e}{2}\), or \((w = 1, e = 1)\), or \((w > 1, e = -1)\) | \(x^2 + x + A' - 1\) | \(w\) |
| \(u = v = 2w\) | otherwise | \(x^2 + (1 + 2^w)x + \frac{A' - 1}{2}\) | \(w + 1\) |
| \(u = 2w < v\) | \(e = 1 + 2^w\), or \((v = 2w + 1, k = 3)\), or \((v > 2w + 1, k = 7)\) | \(x^2 + x + A' - 1\) | \(w\) |
| \(u = 2w < v\) | otherwise | \(x^2 + (1 + 2^w)x + \frac{A' - 1}{2} + 2^w\) | \(w + 1\) |
Theorem 5.1. With the above notations, let the first quotient of the \( \phi \) computed in terms of the higher order indices \([GMN08, Thm. 4.18]\). In our situation are separable. In the regular case, the polynomial, and this is the only property we need in what follows. We define points lying on the side determine a nonzero coefficient of this second order residual whose degree coincides with the degree of the side \([GMN08, Sec. 2.5]\). Only the order, then the following elements are a

\[
p_1 \; \theta, \; Q_1, \; \theta Q_1, \; p^{1/2}, \; \nu := \frac{Y}{2} - 1.
\]

Proof. Arguing as in Proposition \([2.3]\) one checks easily that \( v_p(Q(\theta)) \geq e(p/p)\nu \), for all prime ideals \( p \) of \( K \) lying above \( p \). This is proved for Newton polygons of arbitrary order in \([GMN09, Prop. 3.6]\). On the other hand, along the proof of Proposition \([2.3]\) we saw that \( v_p(\theta) \geq e(p/p)/2 \), for all prime ideals \( p \) of \( K \) lying
Thus, polynomial is a separable polynomial of the form \( \sigma \), with

\[ v_p \left( (M: \mathbb{Z}_p[\theta]) \right) = \left\lceil \frac{Y}{2} \right\rceil + \left\lfloor \frac{Y+1}{2} \right\rfloor - 2 = |Y| - 2 = \text{ind}_p(f). \]

In the rest of the section we apply Theorem 5.1 to compute a \( p \)-integral basis in four subcases of Table 5. Actually, in the discussion of some of these cases we just show how to choose a polynomial \( \phi(x) \) such that \( F(x) \) is \( \phi \)-regular in second order.

5.2. Case \( p > 2 \), \( F(x) = f(x) = x^4 + ax^2 + bx + c \), \( v_p(a) = 1 \), \( v_p(b) > 1 \), \( v_p(c) = 2 \). This case corresponds to row 3 of Table 2. The polynomial \( f(x) \) is \( x \)-regular if and only if \( v_p(a^2 - 4c) = 2 \), in which case, \( 1, \theta, \theta^2/p, \theta^3/p \) is a \( p \)-integral basis. In the irregular case, \( v_p(a^2 - 4c) > 3 \), we choose \( \phi(x) = x^2 + (a/2) \), leading to a \( \phi \)-adic development: \( f(x) = \phi(x)^2 + bx + c - (a^2/4) \). Denote \( u = v_p^2(bx + c - (a^2/4)) = \min\{2v_p(b) + 1, 2v_p(c - (a^2/4))\} \). The \( \phi \)-Newton polygon of second order has only one side, with end points \((0, u)\), \((2, 4)\). This side has degree one if \( v_p(b) < v_p(a^2 - 4c) \) (because then \( u \) is odd), and degree two otherwise: in the latter case the residual polynomial is a separable polynomial of the form \( \sigma y^2 + \tau \) for some nonzero \( \sigma, \tau \in \mathbb{F}_p \). Thus, \( f(x) \) is always \( \phi \)-regular in second order. We have \( Y = (u+4)/2 \), and \( \nu = u/4 \); however, in Table 2 we displayed \( \nu = \frac{1}{2} \min\{v_p(b), v_p(a^2 - 4c)\} \) because it yields too the right values of \( |\nu|, |\nu + (1/2)| \).

5.3. Case \( p = 2 \), \( F(x) = f(x) = x^4 + ax^2 + bx + c \), \( v_2(a) > 1 \), \( v_2(b) > 1 \), \( v_2(c) = 2 \). This case corresponds to row 6 of Table 2. For the different choices of \( \phi(x) \) specified in Table 2 the polynomial \( f(x) \) is \( \phi \)-regular in second order. We discuss in some detail only the cases covered by \( \phi(x) = x^2 - 2 \). The \( \phi \)-adic development of \( f(x) \), and the first quotient of this development are:

\[ f(x) = \phi(x)^2 + (a + 4)\phi(x) + bx + c + 2a + 4, \quad Q(x) = \phi(x) + a + 4 = x^2 + a + 2. \]

If \( v_2(a) = 2 \), \( v_2(b) \geq 4 \) and \( v_2(c + 2a + 4) \geq 4 \), the \( \phi \)-Newton polygon of second order of \( f(x) \) can have different shapes; in Figure 9 we display \( N[^{(2)}]_\phi(f) \) in all other cases.

In the first three cases of Figure 9, \( f(x) \) is \( (x^2 - 2) \)-regular and we obtain the 2-integral basis \( (5.1) \), with the following values of \( Y \) and \( \nu \). We also indicate a polynomial \( Q(x) \), simpler than \( x^2 + a + 2 \), and for which \( (5.1) \) is still a 2-integral basis.
In the fourth case of Figure 9, the second order residual polynomial is \((y + 1)^2\) and \(f(x)\) is \((x^2 - 2)\)-irregular; hence, we must look for other choices for \(\phi(x)\).

5.4. **Case** \(p = 2\), \(F(x) = g(x) = x^4 + 4x^3 + Ax^2 + Bx + C, v_2(A) > 1, v_2(B) > 1, v_2(C) = 2\). This case corresponds to row 4 of Table 4. For the different choices of \(\phi(x)\) specified in Table 3 the polynomial \(g(x)\) is \(\phi\)-regular in second order. The discussion is completely analogous to that of the previous case.

5.5. **Case** \(p = 2\), \(F(x) = g(2x)/16 = x^4 + 2mx^3 + A'x^2 + B'x + C', v_2(A') > 1, v_2(B') > 1, v_2(C') = 2\). This case corresponds to rows 16, 17 of Table 4 with \(A' = A/4, B' = B/8, C' = C/16\). The polynomial \(F(x)\) is always \(\phi\)-regular in second order for \(\phi(x) = x^2 - 2\). The \(\phi\)-development is:

\[
F(x) = \phi(x)^2 + (2mx + 4 + A')\phi(x) + (4m + B')x + 2A' + C' + 4,
\]

and the first quotient is \(Q(x) = x^2 + 2mx + 2 + A'\). We have \(v_2(\phi(x)) = 5\) in all cases, and \(v_2((2mx + 4 + A')\phi(x)) = 5\) or \(\geq 6\) according to \(v_2(B') \geq 3\) or \(v_2(B') = 2\). Therefore, if \(\tau\) is a root of \(F(x)\), Theorem 5.1 yields a 2-integral basis

\[
1, \tau, Q(\tau)/2^{[\nu]}, Q(\tau)/2^{[\nu + (1/2)]},
\]

with the values of \(Y\) and \(\nu\) indicated in the table below. Actually, we also indicate a polynomial \(Q(x)\), simpler than \(x^2 + 2mx + 2 + A'\), and for which (5.2) is still a 2-integral basis.

| \(v_2(B')\) | \(Y\) | \(\nu\) | \(Q(x)\) |
|--------------|-----|-----|----------|
| \(\geq 3\)   | 9/2 | 5/4 | \(x^2\)  |
| 2            | 5   | 3/2 | \(x^2 + 2\) |

In Table 4 we express the basis (5.2) in terms of the root \(\omega = 2\tau\) of the polynomial \(g(x)\).

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### Table 8. Choice of $\phi(x)$ for which $f(x) = x^4 + ax^3 + bx + c$ is $\phi$-regular in second order. Here $v_2(a) > 1$, $v_2(b) > 1$, $v_2(c) = 2$, and $u = v_2(b)$, $v = v_2(c - (a^2/4))$, $d = (c - (a^2/4))/2^u$ (mod 4).

| $v_2(a)$ | $v_2(b)$ | $v_2(2a + c - 4)$ | $u,v$ | $d$ | $\phi(x)$ |
|---------|---------|-------------------|-------|-----|-----------|
| 2       |         |                   |       |     | $x^2 - 2$ |
| 3       | 3       |                   |       |     | $x^2 - 2$ |
| 3       | $\geq 4$ |                   |       |     | $x^2 - 2x - 2$ |
| 2       | $\geq 4$ | $u < v$           |       |     | $x^2 + \frac{d}{2}$ |
| 2       | $\geq 4$ | $u = v = 2w$      |       |     | $x^2 + \frac{d}{2} + 2^u$ |
| 2       | $\geq 4$ | $u = v = 2w + 1$ |       |     | $x^2 + 2^u x + \frac{d}{2}$ |
| 2       | $\geq 4$ | $u > v = 2w$      |       | 1   | $x^2 + 2^u x + \frac{d}{2} + 2^w$ |
| 2       | $\geq 4$ | $u > v = 2w + 1$ |       |     | $x^2 + 2^u x + \frac{d}{2} + 2^w + 1$ |
| $\geq 3$ | $\geq 4$ | 3                 |       |     | $x^2 - 2$ |
| $\geq 3$ | $\geq 4$ | 4                 |       |     | $x^2 - 2x - 2$ |
| $\geq 3$ | $\geq 4$ | $\geq 5$          |       |     | $x^2 - 2x + 2$ |

### Table 9. Choice of $\phi(x)$ for which $g(x) = x^4 + 4x^3 + Ax^2 + Bx + C$ is $\phi$-regular in second order. Here $v_2(A) > 1$, $v_2(B) > 1$, $v_2(C) = 2$, and $u = v_2(B + 8 - 2A)$, $v = v_2(C - ((A - 4)/2)^2)$, $d = (C - ((A - 4)/2)^2)/2^u$ (mod 4).

| $v_2(A)$ | $v_2(B + 8)$ | $v_2(2A + C - 4)$ | $u,v$ | $d$ | $\phi(x)$ |
|---------|-------------|-------------------|-------|-----|-----------|
| 2       |             |                   |       |     | $x^2 - 2$ |
| 3       | $\geq 4$   |                   |       |     | $x^2 - 2$ |
| 2       | $\geq 4$   | $u < v$           |       |     | $x^2 + 2x - 2 + \frac{d}{2}$ |
| 2       | $\geq 4$   | $u = v = 2w$      |       |     | $x^2 + 2x - 2 + \frac{d}{2} + 2^u$ |
| $\geq 3$ | $\geq 4$   | $u = v = 2w + 1$ |       |     | $x^2 + (2 + 2^u)x - 2 + \frac{d}{2} + 2^w + 1$ |
| $\geq 3$ | $\geq 4$   | $u = v = 2w + 1$ |       |     | $x^2 + (2 + 2^u)x - 2 + \frac{d}{2} + 2^w + 2^u$ |
| $\geq 3$ | $\geq 4$   | $u > v = 2w$     |       |     | $x^2 + (2 + 2^u)x - 2 + \frac{d}{2} + 2^w$ |
| $\geq 3$ | $\geq 4$   | $u > v = 2w + 1$ |       |     | $x^2 + (2 + 2^u)x - 2 + \frac{d}{2}$ |
| $\geq 3$ | $\geq 4$   | $\geq 4$          |       |     | $x^2 - 2$ |