On the world sheet of anyon in the external electromagnetic field

D.S. Kaparulin and I.A. Retuntsev

Physics Faculty, Tomsk State University, Lenina ave. 36, Tomsk 634050, Russia

Abstract

We study the issue of description of spinning particle dynamics by means of recently proposed world sheet concept. A model of irreducible spinning particle in the 3d Minkowski space with two gauge symmetries is considered. The classical trajectories of free particle lie on a circular cylinder with a time-like axis. The direction of cylinder axis is determined by the particle momentum, and its position in space-time depends on the value of total angular momentum. The radius of cylinder is determined by the representation. The model admits inclusion of consistent interactions with a general (not necessarily uniform) electromagnetic field. The classical trajectories of the particle lie on the cylindrical hypersurface in space time, whose position is determined by the initial values of the momentum and total angular momentum. All the world paths that lie on one and the same representative in the set of hypersurfaces are connected by gauge transformations. The general construction is illustrated by the example of uniform electric field. In this case, the particle paths are shown to be general pseudotoroidal lines.

1 Introduction

The dynamics of particles with an internal angular momentum is studied for century. Frenkel has been the first to describe motion of charged particle with spin in an electromagnetic field [1]. In the case of curved background, the interest to the dynamics of spinning degrees of freedom has been pioneered by Mathisson and Papapetrou [2,3]. For a summary of studies before 1968, we cite the book [4]. The self-consistent system of equations that describes the motion of translational and internal spinning particle degrees of freedom in the general external electromagnetic and/or gravitational field
in space-time dimension $d = 4$ has been proposed in [5]. This model has been re-derived is a slightly different setting in [6]. The recent results can be found in papers [7–9] and references therein. The problem of inclusion of interaction between the charged particle and external electromagnetic and/or gravitational field exists in lower and higher dimensions [10–11]. For particular applications of the spinning particle concept, we refer articles [12–16].

The current models with an electromagnetic or gravitational coupling have one common feature: the spinning particle is considered as a point object in space-time with the internal degrees of freedom. Depending on the parametrization of internal space coordinates vector, tensor, spinor and twistor models of spinning particles are distinguished [17]. The classical dynamics of free spinning particle is characterized by its path in Minkowski space and the trajectory in internal space. In all the instances, the space-time classical trajectory is preserved by the gauge symmetries of the theory. The introduction of internal space is essential in all the models of point particle, because the space-time trajectory does not contain exhaustive information about the motion of model. For example, the space-time classical trajectory of free particle is always a straight line, while the spin vector path in the internal space is a point. The rectilinear motion describes the dynamics of translational degrees of freedom, while the internal space path makes the same job for spin. Once the space-time trajectory of the particle is of interest, the information is lost about the motion of spin.

In the article [18], a geometrical model of massive spinning particle has been proposed. The quantization of theory corresponds to the irreducible representation of the Poincare group with prescribed values of mass and spin. The classical trajectory of spinning particle is a curve on the $2d$ circular cylinder with a time-like axis. The motion of the particle is characterized by small irregular fluctuations of the path, whose value is determined by spin (in this particular model). These fluctuations has been explained by the zitterbewegung phenomenon. In the model of point particle, there is no zitterbewegung. The phenomenon is critical for description of spinning degrees of freedom in terms of space-time path: the direction of path fluctuations determines the value of spin angular momentum. The possibility of pure geometrical description of spinning particle dynamics has been unnoticed for the long time, even though the list of geometrical models has been significantly extended. The spinning particle theories with a cylindrical classical trajectories has been re-derived in [19,20]. In the theory [21], the classical trajectories are null-like helices, being isotropic paths on the circular cylinder. For geometrical models of spinning particle in space-time dimension three, we refer [10,22,23]. As for massless particles, their path are known to lie on hyperplanes [24].
A general geometric method of spinning particle dynamics description is developed in [25]. In this work, it has been observed that the positions of irreducible spinning particle lie on a cylindrical surface in Minkowski space. For massive particle the surface is a toroidal cylinder $\mathbb{R} \times \mathbb{T}^d$, where $0 \leq d \leq [(n - 1)/2]$. Here, the square brackets denote the integral part of number. The space-time submanifold that contains all the particle positions was termed a world sheet. In the space-time dimension $d = 3, 4$, the world sheets are circular cylinders with the time-like axis. The cylinder radius is the model parameter. The direction of cylinder axis is determined by the momentum. The position of the cylinder in space-time is determined by the total angular momentum. The classical paths of the particle are general cylindrical lines. In terms of particle positions, the equations of motion involve higher derivatives, and they are non-Lagrangian. Once appropriate set of auxiliary variables is introduced, the equations of motion reproduce the previously mentioned geometrical models, see [25] for example. The geometric description of spinning particle dynamics by means of the world sheet concept relies uses the conservation law of momentum and total angular momentum, so it can be applied in the case of the free particle in its current form.

In the present work, we address an issue of description charged spinning particle motion in the external electromagnetic field by means of the world sheet concept. We consider the class of models such that demonstrate the zitterbewegung already at the free level. An inconsistency is observed for the minimal electromagnetic coupling in these theories unless the external field is uniform [10,18,26]. Our main idea is that the projection of the classical trajectory onto the cylinder axis is a special point with a reduced gauge symmetry. Identifying this point (later termed the center of mass point) with the classical position of point particle from the works [5,6], we propose a variational principle for cylindrical curves that admits inclusion of consistent couplings with general (not necessarily uniform) electromagnetic field. The class of gauge equivalence of classical trajectories of the model forms a cylindrical surface whose axis is given by path of point particle. The position of this surface in space-time is determined by initial values of momentum and total angular momentum, being gauge-invariant physical observables of the model. This result means that the world sheet concept can be applied for description of spinning particle dynamics at the interacting level.

We mostly consider the simplest model in the class, a massive particle with nonzero spin that moves in three-dimensional space-time. At the free level, the world sheet is a circular hypercylinder with a time-like axis. The variational principle for cylindrical curves is given by the action functional
of paper \cite{6}, where the classical position of the particle is replaced by the center of mass coordinate. An important difference is that the dynamical variables are three-vectors, not four-vectors. The model parameters are particle’s mass, particle’s spin, and the cylinder radius. The latter quantity is an accessory parameter that distinguishes different spinning particle models with one and the same value of mass and spin. The inclusion of interactions with electromagnetic field along the lines of \cite{5,6} preserves the consistency of the model because the theories are connected by the canonical transformation of phase-space variables. This brings us to the class of consistent interactions between the electromagnetic field and spinning particle that moves a cylindrical path.

The paper is organized as follows. In Section 2, we discuss the classical dynamics of spinning particle model, whose quantization corresponds to the irreducible representation of the Poincare group. We describe geometry of the world sheet of free spinning particle, and derive the equations of motion of the theory. In Section 3, we propose a variational principle for cylindrical curves and construct the class of consistent interactions with the electromagnetic field. The model demonstrates the zitterbewegung at the free and interacting levels. In Section 4, we derive the equations of motion for the cylinder axis and propose a constructive procedure of world sheet description. The classical trajectories of interacting model are general curves on the world sheet. In Section 4, we consider the spinning particle motion in the uniform electric field. If the particle initial velocity is orthogonal to the electric vector, the world sheet is shown to be a pseudotorus, whose position is determined by the initial values of momentum and total angular momentum. The conclusion summarizes results.

## 2 Classical dynamics of irreducible spinning particle

We consider a massive spinning particle that travels in 3d Minkowski space. The particle state is determined by the values of the momentum $p$ and total angular momentum $J$, being subjected to the mass shell and spin shell conditions

\begin{equation}
(p, p) + m^2 = 0, \quad (p, J) - ms = 0.
\end{equation}

\footnote{The action principle for the irreducible spinning particle travelling cylindrical path in three-dimensional Minkowski space first proposed in \cite{10} with spinor parametrization of internal space.}
Here, \( m \) is the mass, and \( s \) is spin. We assume that \( m > 0 \). The round brackets denote the scalar product with respect to the Minkowski metric, \( (a, b) = a_\mu b^\mu \). The spin angular momentum \( \mathcal{M} \) is defined by the following relation:

\[
\mathcal{M} = J - [x, p], \quad [x, p] = \epsilon_{\mu\nu\rho} x^\mu p^\nu dx^\rho.
\]  

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The quantity \( \epsilon_{\mu\nu\rho} \) denotes the 3d Levi-Civita symbol, being antisymmetric in all indices. We use the convention \( \epsilon_{012} = 1 \). The symbol \( x \) denotes the particle coordinate. The article \([25]\) tells that \( \mathcal{M} \) is normalized in every irreducible spinning particle theory,

\[
(\mathcal{M}, \mathcal{M}) = m^2 r^2 - s^2,
\]  

with \( r \) being a real nonnegative constant with the length dimension. The value of this constant is determined by the representation. The possible models of spinning particles are characterized by the values of three independent parameters: \( m, s, \) and \( r \). The mass \( m \) and spin \( s \) determine the parameters of the spinning particle. The constant \( r \) distinguishes particle theories with one and the same value of mass and spin.

Relations \((1), (2) \) and \((3)\) imply that the possible spinning particle positions lie on a hypersurface in Minkowski space. The equation of hypersurface eventually reads

\[
(x - y(0))^2 + (n(0), x)^2 - r^2 = 0,
\]  

where \( x \) is the particle coordinate, and the notation is used,

\[
y(0) = \left[ \frac{p}{m}, \frac{J}{m} \right], \quad n(0) = \frac{p}{m} \quad \Leftrightarrow \quad p = mn(0), \quad J = m[y(0), n(0)] - sn(0). \]  

By construction, the vector \( n(0) \) is normalized and orthogonal to \( y(0) \),

\[
(n(0), n(0)) + 1 = 0, \quad (n(0), y(0)) = 0.
\]  

Equation \((4)\) determines a hypercylinder in Minkowski space with the time-like axis. The vector \( n(0) \) is tangent to the cylinder axis. The vector \( y(0) \) connects the cylinder axis and origin by the shortest

\footnote{We use a mostly positive signature of the Minkowski metric throughout the paper. All the tensor indices are raised and lowered by the metric.}
path. The quantity $r$ defines the radius of hypercylinder. The cylinder of zero radius is a straight line with the time-like tangent vector. Relation (4) is a unique condition imposed onto the particle coordinate $x$, so the hypercylinder includes all the possible spinning particle positions. This means that the circular hypercylinder with the time-like axis represents a set of massive spinning particle positions in three dimensional Minkowski space.

The conditions (1), (3) and their consequence (4) determine the classical dynamics of model. The classical trajectory represents a set of consecutive positions that are occupied by the spinning particle. Once the particle positions lie on a hypercylinder, the classical paths are cylindrical curves. The article [27] tells us that the cylindrical curves are determined by the following system of ordinary differential equations:

$$
\frac{d}{d\tau} \left( \frac{\dot{y}}{\sqrt{-\dot{y}^2}} \right) = 0, \quad (\dot{y}, \omega) = 0, \quad (\omega, \omega) - r^2 = 0.
$$

(7)

Here, $y$ denotes the coordinate of projection spinning particle current position onto the axis of hypercylinder. The difference vector $\omega$ connects the particle position and cylinder axis by the shortest path. The classical position of the particle are determined by the rule

$$
x = y + \omega.
$$

(8)

The quantity $\tau$ denotes the proper time. The dot denotes the derivative by $\tau$. The dynamical variables in the system (7) are $y$ and $\omega$. Relation (8) allows us to chose two alternative sets of independent variables: $x$ and $\omega$, $x$ and $y$. In what follows, the variable $y$ is called the coordinate of mass center. We justify this terminology by observing that the classical trajectory of free particle mass center is always a line.

The system (7) determines the class of cylindrical curves indeed. The first equation in the set (7) tells us that the curve $y(\tau)$ is a straight line with time-like tangent vector. Each line is determined by two ingredients: the direction of its tangent vector and position of initial point. If $\tau = 0$ is the initial moment, the normalized time-like vector $n(0)$ (6) is tangent to the line. The initial value $y(0)$ of the variable $y$ is the other Cauchy data. As the theory is reparametrization-invariant, there no selected initial point on the line. The second condition (5) fixes a particular choice for initial position of mass center $y(0)$. This choice has no physical meaning because it corresponds to the particular gauge fixing. Relations (5) tell us that the position cylinder axis contains all the valuable information.
about the dynamics of massive spinning particle in 3d space. Second and third conditions in the set (7) determine the relative position of the particle and cylinder axis. They tell us that the particle lies on circle of radius \( r \) with the center in point \( y \). The vector \( n(0) \) is orthogonal to the plane, which contains a circle. With the proper time being, the vector \( y \) runs over the line, and \( x \) moves along the general cylindrical path. This means that the cylindrical lines are solutions to equations (7). The classical trajectories of spinning particle described by these equations.

Let us now explain the concept of spinning particle world sheet. Equation (5) connects the parameters of hypercylinder with the momentum and total angular momentum of particle. Since this correspondence is a bijection, there is one-to-one relationship between the elements of hypercylinder set (4), (6) and particle states. This one-to-one correspondence is organized as the follows. Each particle state is determined by the values of momentum and total angular momentum, being subjected to the mass shell and spin shell conditions (1). The world sheet (4) represents the set of particle positions in space-time, the particle state is known. On the other hand, each hypercylinder of radius \( r \) determines the state of spinning particle with the mass \( m \) and spin \( s \) by the rule (5). The cylinder radius determines the norm of spin angular momentum by the rule (3). The cylindrical surface that contains all the possible positions of spinning particle was termed a world sheet in [25]. The world sheet concept reduces the problem of description of spinning particle dynamics to the task of classification of general curves on certain set of (hyper)surfaces in Minkowski space. In the case of massive spinning particles in 3d space-time, the circular hypercylinders with the time-like axis are of interest. The classical trajectories of the model are cylindrical curves, being described by equation (7). The relation (5) connects the initial data for equations (7) and values of momentum and total angular momentum of the spinning particle.

The description of spinning particle dynamics by means of the world sheet formalism has two important subtleties. First, the world sheet dimension depends on the values of model parameters. For example, the world sheet of spinning particle (4) is hypersurface (codimension 1) for \( r > 0 \). For \( r = 0 \), it is a line (codimension 2). The example of the work [28] tells us that the world sheets of maximal dimension determine spinning particle state in a unique way. For the world sheets of submaximal dimension, the information about the spin state is lost. The case \( r = 0 \) in 3d space-time is not representative because spin has no physical degrees of freedom in this model. The theories with zero and nonzero \( r \) have slightly different properties. For \( r > 0 \), the classical spinning particle trajectory can deviate from the rectilinear path, and the zittbewegung phenomenon is observed. For
$r > 0$, the particle path is always a line. This is a model without zitterbewegung at the free level. In the current article, we consider the theories with nonzero cylinder radius. As it has been mentioned above, the models with the zitterbewegung at the free level have obstructions for inclusion of consistent interactions with general electromagnetic and gravitational field. Once the world sheets of interest, the consistent couplings between the changed spinning particle and external electromagnetic field has to be constructed in the model with zitterbewegung.

Our second remark is that the spinning particle classical trajectory is always a world line, not a world sheet. The world line position in space-time determines a particle state unambiguously if the curve lies on a unique world sheet. If the world line lies on the intersection of several world sheets, multiple states can be associated with one and the same curve. In the article [25] the curves that lie on a unique world sheet has been termed typical. The paths that lie on several world sheets has been called atypical. In the case of 3d Minkowski space, the straight lines with the time-like tangent vector provide an example of atypical curves. A single line belongs to the infinite number of circular hypercylinders with one and the same direction of axis. In what follows, we ignore the rectilinear path. The other atypical curves are closed loops that lie on intersection of pairs of cylinders. To exclude loops the casuality condition $\dot{z}^0 > 0$ is imposed onto the classical paths of spinning particle. The causal classical trajectories, which are not straight lines, are spinning particle path of interest. These curves determine the state of spinning particle in a unique way.

The described above exposition of the world sheet concept essentially uses the conservation law of momentum and total angular momentum. This is true for free particle. In the next sections, we address two following aspects of the spinning particle dynamics. We construct the system of ordinary differential equations that describe the motion of charged spinning particle with general value of invariants $m, s, r$ (1), (2) in the electromagnetic field. We demonstrate that the class of gauge equivalence of spinning particle trajectories forms a cylindrical hypersurface in Minkowski space irrespectively to specifics of the field configuration. The quantity $r$ determines the radius of cylinder. The shape of symmetry axis of hypersurface is determined by the field configuration. The position of the world sheet is determined by the initial values of momentum and total angular momentum. The results of our study show that the concept of world sheet of spinning particle can be used for the description of dynamics even in the presence of external electromagnetic field. The classical trajectory of the particle, which lies on the world sheet, demonstrates the zitterbewegung even if the external electromagnetic is nonzero.
3 Variational principle and inclusion of interaction

In this section, we derive a variational principle for the equations (7) for cylindrical curves and propose the procedure of inclusion of interaction between the charged spinning particle and electromagnetic field. The section is organized as follows. At first, we construct the Hamiltonian variational principle for equations (7). The vector parametrization of internal space is used. The phase space-variables are subjected to constraints. We explicitly verify that the Noether charges, being associated with the space-time translations and Lorentz rotations, meet the mass shell and spin shell conditions (1). This ensures that the canonical quantization of classical theory corresponds to the irreducible representation of the Poincare group with the mass $m$ and spin $s$. The gauge symmetries of the model are identified. It shown that they connect all the paths on one and the same cylinder. Then, we propose the procedure of inclusion of interaction between the particle and electromagnetic field. The interaction preserves both gauge symmetries of the model, even though the zitterbewegung is observed at the free level.

We chose the direct product $\mathbb{R}^{2,1} \times \mathbb{R}^{2,1}$ as the configuration space of the spinning particle. The first factor is interpreted as Minkowski space, while the second one is attributed to the internal space of spinning particle. In the old-school terminology, such structure of configuration space corresponds to the vector model of spinning particle [17]. Three different choices for the dynamical variables are admissible. In the position representation, the set of generalized coordinates includes the classical particle position $x$ and difference vector $\omega$. The position representation corresponds to the most common choice for the dynamical variables in spinning particle theories, see [10] for example. In the center of mass representation, the set of generalized coordinates includes the classical position of mass center $y$ and difference vector $\omega$. If $x$ and $y$ are identified, the possibility corresponds to the spinning particle model with the world sheets of dimension 1. The classical trajectories of the point particle are straight lines. No zitterbewegung is observed at the free level. The procedure of interaction inclusion is known only in center of mass representation [5,6]. The third option for the generalized coordinates is $x$ and $y$. This corresponds to some sort of bivector model of spinning particle [20]. The relation (8) connects various representations of dynamics. We mostly use the center of mass representation is the current section because it leads to the simplest form of dynamical equations for the spinning particle classical trajectory. We return to the position representation in the process of classification of world sheets in Section 4. We do not use the bivector representation for the particle dynamics in the article.

We proceed with the introduction of canonical momenta $p, \pi$ for the generalized coordinates $y, \omega$.
in the center of mass representation. In other representations, the canonical momenta are introduced by canonical transformation that is generated by the change of generalized coordinates \( \Theta \). We do not introduce for momenta in other representations of dynamics because they are not relevant in this section. The following constraints are imposed onto the phase-space variables:

\[
\Theta_e = (p, p) + m^2 \approx 0, \quad \Theta_2 = (\omega, \omega) - r^2 \approx 0, \quad \Theta_3 = (\pi, \pi) - r^{-1} s^2 \approx 0, \\
\Theta_5 = (p, \omega) \approx 0, \quad \Theta_4 = (\omega, \pi) \approx 0, \quad \Theta_6 = (\omega, \pi) \approx 0.
\] (9)

The sign \( \approx \) means equality on the mass shell. Relations (9) have a clear origin. The first equation tells us that the linear momentum meets the mass shell condition (1). Relations \( \Theta_2, \Theta_4 \approx 0 \) imply the the difference vector \( \omega \) is normalized and orthogonal to momentum \( p \). These conditions determine relative positions of the particle and its mass center. Thee remaining relations express the vector \( \pi \) in terms of the dynamical variables \( p, \omega \). They mean that \( \pi \) is an auxiliary variable, having no independent dynamical degrees of freedom. The particular form for the constraints \( \Theta_3, \Theta_5, \Theta_6 \approx 0 \) is chosen to satisfy the spin shell condition (1). The Hamiltonian action principle for the spinning particle model reads

\[
S = \int \left( p \dot{x} + \pi \dot{\nu} - 1/2(e \Theta_e + \lambda_2 \Theta_2 + \lambda_3 \Theta_3) - \lambda_4 \Theta_4 - \lambda_5 \Theta_5 - \lambda_6 \Theta_6 \right) d\tau ,
\] (10)

The dynamical variables are the generalized coordinates \( y, \omega \), canonically conjugated momenta \( p, \pi \), and Lagrange multipliers \( e, \lambda_\alpha, \alpha = 2, \ldots, 6 \). The action functional (10) has been proposed for a massive spinning particle traveling in four-dimensional Minkowski space in the paper [6]. In that work, the generalized coordinate \( y \) has been associated with the particle position, not the coordinate of the center of mass point. In our model, the center of mass coordinate follows a rectilinear path, while the classical trajectory of the particle lies on a circular hypercylinder with the time-like axis of radius \( r \). Once the particle path is not a straight line, the particle theory (10) demonstrates zitterbewegung already at the free level.

The model (10) describes an irreducible spinning particle indeed. The momentum and total angular momentum are the conserved charges, being associated with the invariance of the theory with respect to the space-time translations and Lorentz rotations,

\[
\mathcal{P} = p, \quad \mathcal{J} = [y, p] + [\omega, \pi] .
\] (11)
The constraints (9) imply the mass shell and spin shell conditions for the quantities $P, J$,

\begin{align}
(P, P) + m^2 &= 0, \\
(P, J) - ms &= 0.
\end{align}

(12)

The spin angular momentum reads

$$\mathcal{M} = [\omega, \pi + p].$$

(13)

By construction, the vector $\mathcal{M}$ is normalized in the sense (3). Relations (12), (13) ensure that the theory (10) describes an irreducible spinning particle theory with general values of invariants $m, s, r$.

The relationship between equations (9) and (12) has an important subtlety. The constraints (9) imply that the vectors $p, [\omega, \pi]$ are normalized and collinear,

$$\frac{p}{m} = k \frac{[\omega, \pi]}{m|s|}, \quad k = \pm 1.$$

(14)

The symbol $|s|$ denotes the absolute value of $s$. The quantity $k$ can be interpreted as the chirality of the particle. For $k = 1$, the vectors $p, \omega, \pi$ form a right-handed triple. For $k = -1$, the vectors $p, \omega, \pi$ form a left-handed triple. The spin shell condition (12) follows from (9), (11) if the chirality spin have one and the same sign,

$$k = \text{sgn}(s),$$

(15)

This means that the right-handed triple of vectors $p, \omega, \pi$ describes a spinning particle with a positive value of spin. The left handed triple describes a spinning particle with a negative value of spin. Contrary to the physical intuition, the chirality is not connected to the shape of spinning particle trajectory. As we see in the next paragraph, the particle trajectories with positive or negative spin may have one and the same shape.

The conditions of conservation of the constraints (9) imply that four of six Lagrange multipliers are expressed on the mass shell,

$$\lambda_2 = s^2 r^{-4} \lambda_3 = 0, \quad \lambda_4 = \lambda_5 = \lambda_6 = 0.$$

(16)

The quantities $e$ and $\lambda_3$ remain arbitrary functions of the proper time $\tau$. The equations of motion for the phase-space variables of the model read

$$\dot{y} = ep, \quad \dot{\omega} = \lambda_3 \pi, \quad \dot{\pi} = -\frac{s^2}{r^2} \lambda_2 \omega, \quad \dot{p} = 0.$$

(17)
In the position representation, the equations of motion for the particle coordinate $x$ and difference vector $\omega$ have the following form:

\[
\dot{x} = ep + \lambda_3 \pi, \quad \dot{\omega} = \lambda_2 \pi. \tag{18}
\]

Relations (17), (18) tell us that classical trajectory of the mass center is a line, while vectors $\omega, \pi$ rotate around the plane with normal $p$. The quantity $\lambda_3$ determines the angular velocity of rotation. The sign of $\lambda_3$ defines the direction of rotation. Since $\lambda_3$ can be positive and negative, both the directions of rotation are admissible. In the position representation, the dynamics of the vector $\omega$ determines the shape of the space-time particle trajectory. The positive direction of rotation corresponds to the right-handed cylindrical curves. The negative direction of rotation corresponds to the left-handed cylindrical curves. Since both directions of rotations are admissible in every spinning particle theory, the chirality of space-time trajectory does not related with the value of spin. The rectilinear spinning particle trajectories are selected by the condition $\lambda_3 = 0$.

Equations (17) are preserved by two gauge symmetries. The first gauge transformation generates translations along the center of mass trajectory,

\[
\delta \xi y = p \xi, \quad \delta \xi \omega = \delta \xi \pi = \delta \xi p = 0, \tag{19}
\]

where $\xi = \xi(\tau)$ is the parameter, being arbitrary function of proper time. The second gauge transformation generates rotations in $\omega, \pi$ plane,

\[
\delta \xi \omega = \pi \xi, \quad \delta \xi \pi = -s^2 r^{-4} \omega \xi, \quad \delta \xi y = \delta \xi p = 0. \tag{20}
\]

The transformation parameter is the arbitrary function of proper time $\zeta = \zeta(\tau)$. The gauge symmetries (19), (20) have clear physical meaning. The translations along the center of mass classical trajectory (19) generate the shifts of the particle path along the cylinder. The gauge symmetry (20) generates the shifts of the particle classical trajectory in the direction, which is orthogonal to the cylinder axis. These shifts can be considered as infinitesimal rotations the particle path around the axis of cylinder. As the cylinder (4) is a two-dimensional surface, we conclude that almost all classical paths on the cylinder are connected by gauge transformations. The spinning particle admits a special class of trajectories with a reduced gauge symmetry, being straight lines. We exclude the rectilinear
trajectories from our consideration because they lie on infinite number of cylinders at one and the same moment.

Let us now turn to the problem of inclusion of interactions between the spinning particle and external electromagnetic field. In the most general setting, the consistent interactions between gauge fields are associated with the deformations of action functional that preserve all the gauge symmetries and number of physical degrees of freedom. The problem of inclusion of interaction between the spinning particle and general electromagnetic field has been first solved in the model with the spinor parametrization of internal space in [5]. The same problem was solved in the vector model in [6]. In both the theories the \(d = 4\) model with zero cylinder radius is considered. The classical trajectory of the spinning particle does not demonstrate zitterbewegung at the free level. Below, we apply the procedure of the articles [5,6] for construction of consistent interactions between the charged spinning particle and electromagnetic field in the center of mass representation in the model with the general value cylinder radius \(r\). In this case, the relationship (8) between the position of center of mass representation guarantees that the classical trajectory of spinning particle demonstrate the zitterbewegung in the presence of external electromagnetic field. No assumptions about the configuration of the electromagnetic field are done, it can be general. To our knowledge, purposed model is the first spinning particle theory with zitterbewegung which admits consistent coupling with the electromagnetic field of general configuration. This result demonstrates that presence of zitterbewegung at the free level is not an obstruction for construction of consistent couplings.

At first, we explain the general idea of interaction constriction. Once the electromagnetic field is in question, the extended momentum is a relevant object. In the current article, the extended momenta is defined by the rule

\[
P_\mu = p_\mu + A_\mu(y). \tag{21}
\]

Here, the particle charge is set to \(-1\), and \(A_\mu\) denotes the vector potential of the electromagnetic field. In the formula (21), the vector potential is taken at the mass center point \(y\). In the standard definition of the extended momentum, the vector potential is taken at the current particle position \(x\). The dependence of the vector potential on the center of mass position is essential for consistency of interaction. The particle coordinate \(x\) transforms by two gauge symmetries: translations along the cylinder axis and rotations around it. Once the moment of time is fixed, the set of particle positions is given by the section of cylinder, being circle or ellipse. All the points of the section represent equivalent particle positions, which are indistinguishable from the physical viewpoint. The vector
potential describing the interaction between the charged particle and electromagnetic field must be
gauge-invariant. This is possible if it depends on the coordinates of the mass center, being the special
point with a reduced gauge symmetry. The dependence of the vector potential on the mass center
coordinate has no alternative. The analysis of paper [10] tells us that the coupling with the standard
definition of extended momentum is consistent in the anyon model if the electromagnetic field strength
meets equation of Chern-Simons type. The components of extended momentum do not commute. The
Poisson bracket read
\[ \{ P_\mu, P_\nu \} = \epsilon_{\mu\nu\rho} F^\rho(y) , \]
where \( F^\mu(y) = \epsilon_{\mu\nu\rho} \partial^\nu A^\rho(y) \) is the vector of electromagnetic field strength in the center of mass
position \( y \) (the partial derivative acts on \( y \)).

We construct the set of constraints at the interacting level as follows: the canonical momenta \( p_\mu \)
are replaced by the extended momenta \( P_\mu \), and the non-minimal coupling term is included into the
mass shell condition. By definition, we put
\[ \tilde{\Theta}_e = (P, P) - g(M, F) + m^2 \approx 0 , \quad \tilde{\Theta}_2 = (\pi, \pi) - s^2 / r^2 \approx 0 , \quad \tilde{\Theta}_3 = (\omega, \omega) - r^2 \approx 0 , \]
\[ \tilde{\Theta}_4 = (P, \omega) \approx 0 , \quad \tilde{\Theta}_5 = (P, \pi) \approx 0 , \quad \tilde{\Theta}_6 = (\pi, \omega) \approx 0 . \]
Besides the mass and spin, the interaction involves an additional parameter \( g \), being the gyromagnetic
ratio or \( g \)-factor. If the particle has no anomalous magnetic dipole moment, \( g = 2 \). The vector
\[ M = [\omega, \pi] \]
denotes the Frenkel angular momentum. Relations (1), (2) imply that the quantity \( M \) coincides with
the total angular momentum of the particle in the rest system of its mass center. The Frenkel angular
momentum is invariant with respect to the gauge transformations (19), (20). This provides another
interpretation for \( M \): (24) it is a gauge-invariant part of the total angular momentum \( J \). The Frenkel
angular momentum and spin angular momentum are equal if the cylinder radius is zero, \( r = 0 \). In
the spinor parametrization of internal space, the system of constraints has been first proposed (23)
in [5]. The vector parametrization of internal space has been used in [6]. In both mentioned above
papers, the special model with absence of zitterbewegung at free level has been considered (this model
corresponds to zero cylinder radius limit in the Lagrangian (10)). In the case of theory (23), the
cylinder radius is arbitrary positive constant. The interaction (23) is consistent because it is connected with that of the paper [5,6] by the point canonical transformation, which is induced by the coordinate change (8). In next paragraph, we demonstrate that both the gauge symmetries of free theory (10) are preserved by coupling.

The variational principle for the interacting model reads

$$S = \int (p\dot{y} + \pi \dot{\omega} - 1/2(e\tilde{\Theta}_e + \lambda_2\tilde{\Theta}_2 + \lambda_3\tilde{\Theta}_3) - \lambda_4\tilde{\Theta}_4 - \lambda_5\tilde{\Theta}_5 - \lambda_6\tilde{\Theta}_6) d\tau.$$  \hspace{1cm} (25)

The dynamical variables are generalized coordinates $y, \omega$, momenta $p, \pi$, and Lagrange multipliers $e, \lambda_\alpha, \alpha = 2, \ldots, 6$. The condition of conservation of constraints (23) allows us to determine four Lagrange multipliers of six,

$$\lambda_6 = 0, \quad \lambda_2 = \frac{s^2}{r^4}\lambda_3, \quad \lambda_4 = \frac{1}{2} \frac{(g - 2)(\pi, P, F) - g(\pi, \partial)(M, F)}{m^2 - (g + 1)(M, F)} ,$$

$$\lambda_5 = -\frac{1}{2} \frac{(g - 2)(\omega, P, F) - g(\omega, \partial)(M, F)}{m^2 - (g + 1)(M, F)} .$$  \hspace{1cm} (26)

The quantities $e$ and $\lambda_3$ remain arbitrary functions of proper time. The following Lagrange equations follows from the least action principle for the functional (25):

$$\frac{dy}{d\tau} = e\left\{ P + \frac{1}{2} \frac{(g - 2)(\pi, P, F) - g(\pi, \partial)(M, F)}{m^2 - (g + 1)(M, F)} \omega - (\omega \leftrightarrow \pi) \right\} ,$$  \hspace{1cm} (27)

$$\frac{dP}{d\tau} = e\left\{ \frac{g}{2} \partial(M, F) + \frac{1}{2} \frac{(g - 2)(\pi, P, F) - g(\pi, \partial)(M, F)}{m^2 - (g + 1)(M, F)} \omega - (\omega \leftrightarrow \pi) , F \right\} ,$$  \hspace{1cm} (28)

$$\frac{d\pi}{d\tau} = -\frac{s^2}{r^4}\lambda_3\omega - e\frac{1}{2} \frac{(g - 2)(\pi, P, F) - g(\pi, \partial)(M, F)}{m^2 - (g + 1)(M, F)} P ,$$  \hspace{1cm} (29)

$$\frac{d\omega}{d\tau} = \lambda_3\pi - e\frac{1}{2} \frac{(g - 2)(\omega, P, F) - g(\omega, \partial)(M, F)}{m^2 - (g + 1)(M, F)} P .$$  \hspace{1cm} (30)

The first equation in this system expresses the spinning particle extended momentum in terms of center of mass coordinate and its velocity. If the electromagnetic field strength is nonzero, the velocity $\dot{y}$ and extended momentum $P$ are not collinear. Equation (28) impose the second order differential equation for spinning particle classical trajectory, $y = y(\tau)$. Relations (29), (30) determine the time evolution of internal space variables $\omega, \pi$. This evolution is not casual because $\omega$ has one independent
component modulo constraints (23). The time derivative of this component is determined by the Lagrange multiplier $\lambda_3$, being arbitrary function of proper time.

The system of equations (27), (28), (29), (30) is preserved by two gauge symmetries that generalize free infinitesimal transformations (19), (20).

\[
\delta_\xi y = \left\{ P + \frac{1}{2} \frac{(g - 2)(\pi, P, F) - g(\pi, \partial)(M, F)}{m^2 - (g + 1)(M, F)} \omega - (\omega \leftrightarrow \pi) \right\} \xi, \tag{31}
\]

\[
\delta_\xi P = \left\{ \frac{g}{2} \partial(M, F) + \frac{1}{2} \frac{(g - 2)(\pi, P, F) - g(\pi, \partial)(M, F)}{m^2 - (g + 1)(M, F)} \omega - (\omega \leftrightarrow \pi), P \right\} \xi, \tag{32}
\]

\[
\delta_\xi \pi = -\frac{1}{2} \frac{(g - 2)(\pi, P, F) - g(\pi, \partial)(M, F)}{m^2 - (g + 1)(M, F)} P \xi, \tag{33}
\]

\[
\delta_\xi \omega = -\frac{1}{2} \frac{(g - 2)(\omega, P, F) - g(\omega, \partial)(M, F)}{m^2 - (g + 1)(M, F)} P \xi, \tag{34}
\]

and

\[
\delta_\xi y = \delta_\xi P = 0, \quad \delta_\xi \omega = \pi \zeta, \quad \delta_\xi \pi = -r^{-4} s^2 \zeta. \tag{35}
\]

The gauge transformation parameters are arbitrary functions of proper time $\xi, \zeta$. The geometric meaning of the transformations is the same as in the free case. The first gauge symmetry generates shifts of particle position along the center of mass trajectory. The second gauge generates rotations around the current center of mass position. The extended momenta $P$ determines the normal to the plane of rotation.

### 4 Motion of mass center and world sheet

In this section, we consider the general properties of spinning particle dynamics in external electromagnetic field. At first, we observe that the spin has no physical degrees of freedom in 3d space-time. This allows us to construct a closed system of equations that describes the time evolution of center of mass coordinate, extended momentum, and Frenkel angular momentum. Then, the Frenkel angular momentum is expressed from this system, and we get equations of motion for the mass center trajectory in terms of variables $y$ and $P$. The exclusion of the extended momentum from this system (which we do not make here) leads the the second-order differential equation for center of mass path of the spinning particle. The existence of this system is a consequence of small space-time dimension.
In the space-time dimension $d = 4$, where spin has two physical polarizations, the center of mass trajectory does not determine the particle dynamics. Then, we consider the spinning particle dynamics in the position representation. We show that the spinning particle trajectories lie on an cylindrical hyper surface in Minkowski space and provide a constructive procedure of world sheet construction.

We begin our consideration from the construction of a closed system of evolutionary equations for three variables: $y$, $P$, and $M$. The double vector product formula
\[ [a, [b, c]] = c(a, b) - b(a, c), \] (36)
brings equations (27), (28) to the desired form
\[
\frac{1}{e} \frac{dy}{d\tau} = P + \frac{1}{2} \left[ M, \frac{(g - 2)[P, F] - g\partial(M, F)}{m^2 - (g + 1)(M, F)} \right], \quad \frac{1}{e} \frac{dP}{d\tau} = \frac{g}{2} \partial(M, F) + \left[ \frac{1}{e} \frac{dy}{d\tau}, F \right]. \] (37)

Relations (29), (30) determine the motion of the Frenkel angular momentum,
\[
\frac{1}{e} \frac{dM}{d\tau} = \left[ P, \frac{1}{e} \frac{dP}{d\tau} - \frac{g}{2} \partial(M, F) \right]. \] (38)

The dynamical equations (37), (38) are supplemented by the constraints
\[
(P, P) - g(M, F) + m^2 \approx 0, \quad (M, M) + s^2 \approx 0. \] (39)

\[ [P, M] = 0. \] (40)

The conditions (39), (40) are consequences of constraints (23) and definition of the variable $M$ (24). Equations (39) express the mass shell and spin shell conditions for $P, M$. Relation (40) implies that vectors of extended momentum and Frenkel angular momentum are collinear. Relations (37), (38) represent a three-dimensional analog of the Frenkel equations, which has been derived in [6]. The spacial feature of three-dimensional case is that the Frenkel angular momentum $M$ can be expressed algebraically in terms of particle momentum and the center of mass coordinate $y$. This leads to the self-consistent system for the variables $y, P$.

Let us find a closed system of equations for the dynamical variables $y, P$. Condition (40) is solved

\[ \text{We note that the signs in the right hand side of relation are sensitive to the signature of the metric.} \]
by the following ansatz (initial form) for $M$:

$$M = -\frac{s}{m}\gamma P, \quad (41)$$

where $\gamma$ is an unknown dimensionless function. The function $\gamma$ must be positive because the Frenkel angular momentum and extended momentum are collinear for $s < 0$ and anti-collinear for $s > 0$, see the comments to the formula (14). In the free limit $A, F = 0, \gamma = 1$. The constraints (39) determine the current value of $\gamma$ in the presence of interaction. On substituting the ansatz (41) into the conditions (39), we obtain the cubic equation for $\gamma$,

$$z\gamma^3 + \gamma^2 - 1 = 0, \quad z = gs(P, F)/m^3. \quad (42)$$

The positive root that tends to 1 in the limit $g \to 0$ is relevant. The trigonometric formula gives the following exact solution for $\gamma$:

$$\frac{1}{\gamma} = \cos \left( \frac{1}{3} \arcsin \sqrt{\frac{27z}{2}} \right) + \frac{1}{\sqrt{3}} \sin \left( \frac{1}{3} \arcsin \sqrt{\frac{27z}{2}} \right). \quad (43)$$

If the field $F$ is weak, the solution involves a small parameter $z$. The Maclaren expansions for the quantities $\gamma$ and $1/\gamma$ read

$$\frac{1}{\gamma} = 1 + \frac{1}{2}z - \frac{3}{8}z^2 + O(z^3), \quad \gamma = 1 - \frac{1}{z} - \frac{5}{8}z^2 + O(z^3). \quad (44)$$

Relation (41), (43) allow us to eliminate the variable $M$ in terms of the mass center position $y$ and extended momentum $P$. The final form of the equations of motion for $y$ and $P$ reads

$$\frac{1}{e} \frac{dy}{d\tau} = \left( 1 - \frac{1}{2} \frac{(g-2)z\gamma}{g+(g+1)z} \right) P - \frac{s}{2m} \frac{g(g-2)\gamma(1+z\gamma)}{g+(g+1)z\gamma} F + \frac{s}{2m} \frac{g\gamma[P, \partial]z}{g+(g+1)z\gamma}, \quad (45)$$

$$\frac{1}{e} \frac{dP}{d\tau} = -\frac{1}{2}m^2\gamma \partial z + \left[ \frac{1}{e} \frac{dy}{d\tau}, F \right].$$

Here, the functions $z$ and $\gamma$ are defined in (42), (43) (see (44) for Maclaren series expansion for $\gamma$). The evolutionary equations (45) are complimented by the mass shell constraint

$$(P, P) + m^2(1 + z\gamma(z)) \approx 0. \quad (46)$$
By construction, relations (45), (46) determine a self-consistent system of equations describing the classical trajectory of the mass center. The system (47), (48) has no analog in the higher space-time dimensions, where the particle spin has physical degrees of freedom.

The system (47), (48) involves the model parameters, mass and spin, in a complicated way. As the spin of all real particles is proportional to the Plank constant \( \hbar \) (we put \( s = \sqrt{3/4} \hbar \) for electron), the quantity \( z \) can be considered as small parameter in practical applications, at least if the electromagnetic field is not too strong. In this setting, it is interesting to consider the spin as small parameter and derive approximate equations of motion, which include contributions up certain order in \( s \) (or in \( z \)). Formula (44) determines the decomposition of \( \gamma \) up to the second order in \( z \). With the same precision, we get the following representations for equations (45):

\[
\frac{1}{e} \frac{dy}{d\tau} = \left( 1 - \frac{1}{2} \frac{g - 2}{g} z + \frac{(g - 2)(3g + 2)}{4g^2} z^2 \right) P - \frac{(g - 2)s}{2m} \left( 1 - \frac{1}{2} \frac{g + 2}{2g} z \right) F - \frac{s}{2m} [P, \partial] z + O(s^3),
\]

(47)

\[
\frac{1}{e} \frac{dP}{d\tau} = - \frac{1}{2} m^2 \left( 1 - \frac{1}{2} z \right) \partial z + \left[ \frac{1}{e} \frac{dy}{d\tau}, F \right] + O(s^3).
\]

The constraint (46) takes the following form:

\[
(P, P) + m^2 \left( 1 + z - \frac{1}{2} z^2 + O(s^3) \right) \approx 0.
\]

(48)

Equations (47), (48) account all the possible corrections for the spinning particle classical trajectory that are at least quadratic in \( s \). The contributions, that involve the electromagnetic field strength, account for the deviation of spinning particle classical trajectory caused by the presence of spin. The gradient contributions, which involve \( \partial z \), account the interaction between the particle magnetic moment and external field. The case \( s = 0, z = 0 \) corresponds to the option of scalar particle. If \( z = 0, s \neq 0 \) the particle has spin, but its magnetic moment vanishes. Equations (47), (48) tell us that the particle trajectories of spinning and spinless particle are different even if the gyromagnetic ratio is zero. This means that the spin has the mechanism of influence on the particle path, which is not caused by the coupling of interaction of magnetic moment and electromagnetic field.

The classical trajectory of the mass center in the phase-space is a curve. The classical trajectory of the particle with the initial vales of the center of mass coordinate \( y(0) \) and extended momenta \( P(0) \)
is denoted as follows:

\[ y^\mu = Y^\mu_{\cdot}(\tau; y(0), P(0)) , \quad P = P^\mu_{\cdot}(\tau; y(0), P(0)) , \quad \mu = 0, 1, 2. \]  \hspace{1cm} (49)

Here, the proper time \( \tau \) is the parameter on the curve; the initial moment of time is \( \tau = 0 \). There are four independent initial data among the quantities \( y(0), P(0) \). At first, \( y(0) \) and \( P(0) \) meet the mass shell constraint condition (46). The constraint reduced the number of independent initial data by one. The triviality of other initial data is connected to the gauge invariance of equations (47). The particular choice of the parameter on the curve has no sense in the reparametrization-invariant models. The fixation of initial point of the particle trajectory corresponds to a particular gauge fixing, which is a natural ambiguity. Hereinafter, we assume that

\[ (y(0), P(0)) = 0 . \]  \hspace{1cm} (50)

In the absence of electromagnetic field, this condition implies that the vector \( y(0) \) connects the origin and mass center trajectory by the shortest path (see relations (5)). This interpretation is not valid in the interacting theory because the vectors \( \dot{y} \) and \( P \) have different direction, in general. Relations (24), (40), (42), (43) express the vectors \( y(0), P(0) \) in terms of initial values of the momentum and total angular momentum \( p(0), J(0) \) by the rule

\[ p(0) = P(0) - A(y(0)) , \quad J(0) = \left[ y(0), P(0) - A(y(0)) \right] - \frac{s}{m} \gamma(z(0)) P(0) . \]  \hspace{1cm} (51)

The notation \( z(0) \) is used for initial value of the quantity \( z \) (43). The solution of equations (51) with respect to \( y(0) \) and \( P(0) \) expresses the initial data in terms of the values of momentum and total angular momentum. It is not possible to express \( y(0) \) and \( P(0) \) explicitly. An approximate representation is obtained by the fixed-point iteration method for the system of equations

\[ P(0) = p(0) + A(y(0)) , \quad y(0) = \frac{1}{1 + z(0)\gamma(z(0))} \left[ \frac{P(0)}{m}, \frac{J(0)}{m} \right] . \]  \hspace{1cm} (52)

The free solution (5) determines the zero approximation for unknowns \( y(0), P(0) \). As we see, the position of the classical trajectory of spinning particle mass center contains all the valuable information about the dynamics of the model.

Let us now consider the particle motion in the position representation. The constraints (23) imply
that the difference vector $\omega$ \(^{(8)}\) is orthogonal the extended momentum and normalized in each moment of proper time (see the relations $\Theta_2, \Theta_4 \approx 0$). Condition \(^{(8)}\) expresses the difference vector in terms of the particle coordinate center of mass position. Relation \(^{(49)}\) determines the classical trajectory of mass center in the parametrical form. Combining \(^{(8)}, \>(^{49)}\) with the constraints $\Theta_2, \Theta_4 \approx 0 \,(^{23})$, we get the following relations for particle position $x$ and proper time $\tau$:

$$
(x - Y^{\mu}_{cl}, P^{\mu}_{cl}) = 0, \quad (x - Y^{\mu}_{cl}, x - Y^{\mu}_{cl}) - r^2 = 0.
$$

These equations determine the world sheet in implicit form. For each particular value of $\tau$, relations \(^{(53)}\) tell us that the particle position $x$ lies on a circle of radius $r$, with the extended momentum $P$ being the normal to the plane. The vector $Y_{cl}$ determines the position of the circle center. With the proper time being the classical position of the mass center moves along its trajectory \(^{(49)}\). The set of possible particle positions moves along a cylindrical hypersurface that surrounds the mass center classical trajectory. This hypersurface coincides with a spinning particle world sheet. In the explicit form, the world sheet equation for the particle coordinates follows from the system \(^{(53)}\), if the proper time variable $\tau$ is expressed in terms of current particle position $x$. In the case of free particle,

$$
Y^{\mu}_{cl}(\tau; y(0), p(0)) = y^{\mu}(0) + p^{\mu}(0)\tau, \quad P^{\mu}_{cl}(\tau; y(0), p(0)) = p^{\mu}(0),
$$

and the solution reads

$$
\tau = -\frac{(x, p(0))}{m^2}.
$$

The world sheet of spinning particle, being determined by the relations \(^{(53)}\), is the hypercylinder \(^{(4)}\). In all the instances, the hypersurface \(^{(53)}\) represents a set of all possible spinning particle positions with the fixed values of momentum and total angular momentum. The classical trajectories of the spinning particle are general lines on the world sheet. This result demonstrates that concept of the world sheet of spinning particle is viable at the interacting level. Another important observation is that the spinning particle trajectory can deviate from the smooth curve, being the center of mass path. This means that the zitterbewegung phenomenon is observed at the interacting level.
5 Motion in the uniform electric field

In this section, we describe the particle motion in the uniform electric field. The vector potential and field strength are chosen in the following form:

\[ A(x) = (Ex^1, 0, 0), \quad F = (0, 0, E), \tag{56} \]

where \( x^1 \) is the space-time coordinate and \( E \) is the field strength. The field is termed "electric" because only time component of the vector potential is nonzero. In the first part of this section, we find the mass center classical trajectory. The motion in the direction of field strength vector is uniform. Its projection of the trajectory onto the orthogonal to the field strength vector plane is shown to be hyperbola. The shape of the classical path depends on spin, so the trajectories of spinning and scalar particles are different. In the second part of the section, we derive the spinning particle world sheet. If the vector of initial velocity is orthogonal to the fields strength, the world sheet is pseudotorus with the inner radius \( r \). The outer radius of pseudotorus is determined by the particle mass, particle spin, electric field strength.

Let us first determine the center of mass classical trajectory. In the case of uniform electric field \( \tag{56} \), the equations of motion \( \tag{45} \) read

\[
\frac{1}{e} \frac{dy^0}{d\tau} = \left( 1 - \frac{1}{2} \frac{(g-2)z\gamma(z)}{g + (g+1)z\gamma(z)} \right) P^0, \quad \frac{1}{e} \frac{dy^1}{d\tau} = \left( 1 - \frac{1}{2} \frac{(g-2)z\gamma(z)}{g + (g+1)z\gamma(z)} \right) P^1, \tag{57}
\]

\[
\frac{1}{e} \frac{dy^2}{d\tau} = \left( 1 - \frac{1}{2} \frac{(g-2)z\gamma(z)}{g + (g+1)z\gamma(z)} \right) P^2 - \frac{1}{2m} \frac{g(g-2)(1+z\gamma(z))\gamma(z)}{g + (g+1)z\gamma(z)} E, \tag{58}
\]

\[
\frac{1}{e} \frac{dP^0}{d\tau} = -\left( 1 - \frac{1}{2} \frac{(g-2)z\gamma(z)}{g + (g+1)z\gamma(z)} \right) EP^1, \quad \frac{1}{e} \frac{dP^1}{d\tau} = -\left( 1 - \frac{1}{2} \frac{(g-2)z\gamma(z)}{g + (g+1)z\gamma(z)} \right) EP^0, \tag{59}
\]

\[
\frac{1}{e} \frac{dP^2}{d\tau} = 0.
\]

Here, the variable \( z \) denotes a dimensionless combination \( \tag{42} \) of the extended momentum \( P \), field strength \( E \), gyromagnetic ratio \( g \), mass \( s \) and spin \( s \). The univariate function \( \gamma(z) \) is defined in \( \tag{43} \). Equation \( \tag{44} \) determines two leading orders of decomposition of \( \gamma \) in the small parameter \( z \).

The dynamical variables are the center of mass position coordinates \( y^\mu \) and components extended momenta \( P^\mu \). The evolutionary equations are supplemented by the constraint \( \tag{46} \). The arbitrary function \( e = e(\tau) \) denotes einbein, and \( \tau \) is the proper time. We are interested in the solution \( y(\tau), \)
$P(\tau)$ to the equations (57), (58) with the initial condition $y(\tau)|_{\tau} = y(0), P(\tau)|_{\tau=0} = P(0)$.

Equations (57), (58) have two obvious integrals of motion $(P, P) = \text{const}$ and $(P, E) = \text{const}$. This implies conservation of quantities $z$ and $\gamma_4(42), (43)$. With account of this fact, equations (57), (58) reduce to a linear system with respect to the dynamical variables $y^\mu, P^\mu$. The motion is uniform in the direction of field. The projection of the center of mass velocity is determined by the value of momentum component of momentum $P^2(0)$, being connected to the auxiliary quantity $z(0)$ (42). If the spinning particle center of mass classical trajectory lies in the plane orthogonal to the field, the equation $\dot{y}^2 = 0$ determines the selected value of momentum projection $P^2(0)$ of constant $z(0)$,

$$P^2(0) = \frac{(g - 2)s^2}{2m}E + O(s^2), \quad z(0) = \frac{(g - 2)s^2}{2gm^6}E^2 + O(s^3). \quad (59)$$

It is interesting to mention that the first-order correction for $P^2(0)$ vanishes for the particle without anomalous magnetic moment. In the gauge

$$\frac{1}{e} = \left(1 - \frac{1}{2g + (g + 1)z(z)}\right), \quad (60)$$

the solution to the system (57), (58) with the initial condition reads

$$y^0 = y^0(0) \cosh \frac{\tau}{E} + y^1(0) \sinh \frac{\tau}{E} + \frac{P^1(0)}{E}, \quad y^1 = y^0(0) \sinh \frac{\tau}{E} + y^1(0) \cosh \frac{\tau}{E} + \frac{P^0(0)}{E}, \quad (61)$$

$$y^2 = y^2(0) + \left(P^2(0) - \frac{m^2z(0)(g - 2)(1 + z(0)\gamma(z(0))\gamma(z(0)))}{P^2(0)}\right)\tau. \quad (62)$$

For the extended momentum $P^\mu$, the classical trajectory reads

$$P^0 = P^0(0) - Ey^0(0) \sinh \frac{\tau}{E} + Ey^1(0)(\cosh \frac{\tau}{E} - 1), \quad (63)$$

$$P^1 = P^1(0) - Ey^0(0)(\cosh \frac{\tau}{E} - 1) + Ey^1(0) \sinh \frac{\tau}{E}, \quad P^2 - P^2(0) = 0. \quad (64)$$

We suppose that the initial position $y(0)$ and momentum satisfy constraint (46) and condition (50). Formula (51) expresses the initial data $y(0), P(0)$ in terms of the initial values of momentum and total
momentum \( p(0) \), \( J(0) \). The exact solution for the variables \( y^\mu(0) \), \( \mu = 0, 1, 2 \) reads

\[
y^0(0) = -\frac{1}{m^2} \frac{\epsilon^{ij} p^j(0) J^i(0)}{1 + z(0) \gamma(z(0))},
\]

\[
y^1(0) = \frac{1}{m^2} \frac{p^0(0) J^2(0) - p^2(0) J^0(0)}{1 + z(0) \gamma(z(0))} \left( 1 - \frac{1}{m^2} \frac{E J^2(0)}{1 + z(0) \gamma(z(0))} \right)^{-1},
\]

\[
y^2(0) = \frac{1}{m^2} \frac{p^1(0) J^0(0) - p^0(0) J^1(0)}{1 + z(0) \gamma(z(0))} - \frac{1}{m^4} \frac{E J^1(0)}{1 + z(0) \gamma(z(0))} \frac{p^0(0) J^2(0) - p^2(0) J^0(0)}{1 + z(0) \gamma(z(0))} \times \left( 1 - \frac{1}{m^2} \frac{E J^2(0)}{1 + z(0) \gamma(z(0))} \right)^{-1}.
\]

The initial value of the extended momentum \( P^\mu(0) \), \( \mu = 0, 1, 2 \) is expressed as follows:

\[
P^0(0) = p^0(0) + \frac{E}{m^2} \frac{p^0(0) J^2(0) - p^2(0) J^0(0)}{1 + z(0) \gamma(z(0))} \left( 1 - \frac{1}{m^2} \frac{E J^2(0)}{1 + z(0) \gamma(z(0))} \right)^{-1},
\]

\[
P^i(0) = p^i(0), \quad i = 1, 2.
\]

Relations (61)-(66) determine the classical trajectory of the mass center of the particle with the prescribed initial values of momentum \( p(0) \) and total angular momentum \( J(0) \).

Let us explain the geometric meaning of equations (61)-(64). Relations (61), (62) determine the classical trajectory of mass center of the particle in space-time. The classical trajectory (63), (64) of extended momentum is consequence of definition of \( P \) in terms of particle velocity. Due to this fact, the space-time trajectory of mass center contains all the valuable information about the dynamics of the model. The initial values of momentum and total angular momentum determine the positions of mass center classical path (61), (62) in space-time by the rule (65), (66). Equation (61) tells us that the projection of the mass center path onto the orthogonal to the fields strength vector plane is a hyperbola. The center of symmetry has the coordinates

\[
\hat{y}^0(0) = y^0(0) + \frac{P^1(0)}{E}, \quad \hat{y}^1(0) = y^1(0) + \frac{P^0(0)}{E}.
\]

The focal distance \( \Phi \) of hyperbola is the function of momentum projection onto the direction of field,

\[
\Phi(z(0)) = m E \left( 1 + z(0) \gamma(z(0)) + \frac{m^4 E^2}{g^2 s^2} z^2(0) \right)^{1/2}.
\]
The focal distance depends on the value of spin, so the shape of paths of scalar and spinning particle is different. Equation (62) determines the deviation of the mass center trajectory in the direction of field. The path (61), (62) is planar curve if the condition \( \dot{y}^2 = 0 \). In this case, the quantities \( \hat{y}_0, \hat{y}^1 \) and \( y_2(0) \) serve as the coordinates of hyperbola center. The focal distance of hyperbola reads

\[
\Phi(z) = \frac{m}{E} \left( 1 + \frac{s^2 (g - 2)(3g + 2)}{8m^4 E^2} + O(s^3) \right).
\]  

Here, we note that the corrections to the focal distance are at least quadratic in spin. In the first order in spin, the classical paths of spinning and scalar particle, with the initial velocity vector being orthogonal to the field strength, have one and the same shape. In the second-order approximation in \( s \), the classical paths of spinning and scalar particles are different.

The world sheet of spinning particle in the uniform electric field is determined by the conditions (53), where equations (61), (62) describe the classical trajectory of mass center. As in the general case, we obtain the system of two equations for the unknown vector \( x \) and parameter on the curve \( \tau \). A single equation for the particle coordinate follows from this system if the variable \( \tau \) is expressed from system in terms of the positions \( x \), and initial data \( y(0), P(0) \) (or \( p(0), J(0) \)). It is not possible to express \( \tau \) from the system (53), (61), (62) for general initial data explicitly, so the world sheet is determined in the implicit form. The general result of the previous paragraph tells us that it a cylindrical surface of radius \( r \) surrounding the mass center classical trajectory. The problem of world sheet construction simplifies in the case \( \dot{y} = 0 \) of motion in the orthogonal to the field direction plane. The system (61), (62) becomes algebraic with respect to the unknowns \( x, \tau \). The quantity \( \tau \) is expressed from equations by the application of the techniques of resultants. The world sheet equation eventually reads

\[
-(x^0 - \hat{y}^0)^2 + (x^1 - \hat{y}^1)^2 + (x^2 - \hat{y}^2)^2 + \Phi^2(z) - r^2 + 4\Phi^2(z)(- (x^0 - \hat{y}^0)^2 + (x^1 - \hat{y}^1)^2) = 0,
\]  

where \( \Phi(z) \) is determined in (59), (68). The hypersurface, being defined by equation (70), is a pseudotorus of external radius \( \Phi(z) \) and internal radius \( r \). The constant vector \( \hat{y}(0) = (\hat{y}^0(0), \hat{y}^1(0), y^2(0)) \) denotes the position of symmetry center of pseudotorus. The vector of electric field strength (56) determines the direction of symmetry axis of pseudotorus. Equations (59), (67), (65), (66) express the components \( y^\mu(0), \mu = 0, 1, 2 \), in terms of initial values of the momentum and total angular momen-
tum $p(0), J(0)$. This final result demonstrates that spinning particles trajectories can be considered as curves on the set of hypersurfaces even in the presence of external electromagnetic field.

6 Conclusion

In this article, we have studied an issue of description of dynamics of spinning particle by means of recently proposed world sheet concept. We have considered the class of massive spinning particle models in three-dimensional Minkowski space whose quantization corresponds to the irreducible representation of the Poincare group. The classical path of the free particle lie on a hypercylinder with the time like axis. The vector of momentum determines the direction of cylinder axis. The position of the hypercylinder in space-time is determined by the value of total angular momentum. The radius of cylinder is determined by the representation. All the paths that lie on one and the same cylinder are connected by the gauge transformations.

Using the ideas of the papers [5, 6], we have proposed a constructive procedure of inclusions of consistent interactions between the particle and general electromagnetic field. The coupling is not minimal. The electromagnetic field potential and its strength are taken not at the particle position, but in a special point with reduced gauge symmetry, termed a particle mass center. As we see, such reinterpretation of the dynamical variables preserves gauge invariance and ensures that the gauge equivalence of the particle paths forms a cylindrical hypersurface in Minkowski space. The axis of cylinder is associated with the path of point massive particle, whose dynamics is described by the equations of the papers [5, 6].

The specifics of three-dimensional model is that the spin has no physical degrees of freedom. In this case, the center of mass classical trajectory contains all the valuable information about the dynamics of the particle. We derive a system of ordinary differential equations that describe the center of mass path. The dynamical variables of this system are classical position of the mass center and conjugated momentum, while all internal space variables are excluded. This system has no analogs in higher space-time dimensions. Assuming that the mass center trajectory is known, we propose the procedure of construction of a hypersurface that includes all the particle paths.

The general constructions are illustrated in the case of uniform electric field. The particle’s mass center equations of motion are explicitly solved for general mass and spin. If the center of mass velocity vector is orthogonal to the field strength vector, the center of mass path is a hyperbola.
The focal distance of hyperbola depends on mass, spin, and electric field strength. The world sheet of the spinning particle is a pseudotorus whose outer radius is determined by the particle mass and electric field strength. The inner radius of pseudotorus equals to the cylinder, which is the free particle world sheet. The relationship between the pseudotorus parameters and initial values of the particle momentum and total angular momentum is explicitly established.

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E-mail address: dsc@phys.tsu.ru, retuntsev.i@phys.tsu.ru.