Can supersymmetry emerge at a quantum critical point?

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It was recently proposed that an effective space-time supersymmetry emerges at low energies when a Dirac or Weyl semimetal is tuned to a quantum critical point at which the fermion \( \psi \) and the bosonic order parameter \( \phi \) are both massless. In this paper, we examine under what circumstances such a dynamically generated low-energy SUSY could supersymmetry emerge. At the quantum critical point, a bosonic order parameter develops as a result of certain type of fermion pairing, and acquires its own dynamics after integrating out fermions. It is, however, usually dangerous to integrate out massless fermions, as such procedure might yield nonlocal contributions to the \( \phi^{2n} \) terms with integer \( n > 1 \). Such nonlocal terms are non-supersymmetric. Hence, there exists a necessary condition for supersymmetry to be generated in a quantum critical system. We apply this condition to examine some systems that were argued to display supersymmetry at low energies, and demonstrate that no supersymmetry emerges.

PACS numbers: 71.10.Hf, 74.40.Kb

Space-time supersymmetry (SUSY) is known to be the only nontrivial combination of internal and space-time symmetries. It transforms fermions into bosons, and vice versa, providing a candidate solution to the hierarchy problem \[1,2\]. Gravity can be naturally obtained once SUSY is realized by delicately tuning two or more parameters, whereas the search for SUSY in high-energy processes might yield nonlocal contributions to the \( \phi^{2n} \) terms with integer \( n > 1 \). At the microscopic level, these systems are certainly non-supersymmetric, and even non-relativistic. However, as the energy scale is lowered down to zero, there emerges an effective Lorentz symmetry and, under certain circumstances, an effective SUSY. Such a dynamically generated low-energy SUSY offers a nice platform to explore the intriguing properties of SUSY, which seems impossible in current high-energy experiments. A prominent early example is provided by the (1+1)-D Ising model, which exhibits an emergent SUSY at a multi-critical point \[3\]. In addition, a (2+1)-D Wess-Zumino supersymmetric model is constructed in optical lattices \[11\]. In these model systems, SUSY is realized by delicately tuning two or more parameters, which makes it hard to realize SUSY.

Recently, it was argued that the quantum critical point (QCP) between semimetal (SM) and superconducting (SC) phases on the surface of a 3D topological superconductor displays an emergent space-time SUSY at low energies and long distances \[10,11\]. In this system, there is only one single tuning parameter. Subsequently, SUSY was proposed to emerge in other quantum critical systems \[12,14\]. Irrespective of the differences in spatial dimension and fermion type, a common feature shared by these models is that the bosonic degree of freedom is not elementary, but arises from some sort of fermion pairing. For instance, the boson is formed by Cooper pairing in the case of SM-SC transition \[14\]. It can also come from pair density wave (PDW) order \[16\]. Let us take the SM-SC QCP as an example. The fermions \( \psi \) are gapless in the SM phase, but become fully gapped in the SC phase. The order parameter field \( \phi \) has a finite gap in the SM phase. At exactly the QCP, both fermions and bosons are gapless, described by the following effective action \[9,14,16,18\]:

\[
S = \int d\tau d^d r \left[ \bar{\psi} \gamma_0 \psi + |\partial_\mu \phi|^2 + \lambda_4 |\phi|^4 \right. \\
+ \left. g(\phi^* \psi^T i \sigma_2 \psi + h.c.) \right], \tag{1}
\]

where \( \bar{\psi} = -i \psi^\dagger \gamma_0 \) and \( \bar{\phi} = \gamma_\mu \partial_\mu \), and the \( \gamma \)-matrices are given by \( \gamma_0 = \sigma_3 \), \( \gamma_1 = \sigma_1 \), and \( \gamma_2 = \sigma_2 \) where \( \sigma_i \) are the Pauli matrices. Here, the fermion velocity \( v_F \) and the boson velocity \( c \) can be assumed to take exactly the same value, which is justified by concrete renormalization group (RG) calculations. For SUSY to be respected, the Yukawa coupling parameter \( g \) and the \( |\phi|^4 \) coupling parameter \( \lambda_4 \) need to satisfy the relation \( g^2 = \lambda_4 \). Previous RG analysis unveiled a stable infrared fixed point at which \( g^2 = \lambda_4 \) \[9,14,16,18\], which is then argued to signal the emergence of an effective SUSY.

Although the emergent SUSY is a fascinating concept, we would like to revisit this problem more carefully. Our main concern is that the dynamics of bosons may not be correctly described by the action given by Eq. (1). In the aforementioned quantum critical systems argued to display emergent SUSY, there are only fermionic degrees of freedom at the microscopic level: the bosonic field \( \phi \) originally does not exist at all. Only when the system is tuned sufficiently close to the SM-SC (or SM-PDW) QCP, could the boson \( \phi \) be induced as a result of fermion pairing. This seems to be an advantage as one only needs to tune a single parameter. However, this also leads to a severe problem: \( \phi \) is a composite bosonic field, and it

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acquires its own dynamics entirely from integrating out the fermionic degrees of freedom. For instance, \( |\phi|^4 \) term is generated by the Yukawa coupling, which means that \( \lambda_4 \) is not independent of \( g \). If \( \lambda_4 \) is a smooth function of momenta, it could be regarded as a constant at low energies. In this case, \( \lambda_4 \) and \( g \) can flow independently with varying length scale. However, if \( \lambda_4 \) receives singular quantum corrections due to the Yukawa coupling, it can no longer be regarded as a constant, and the infrared fixed point with effective SUSY may not exist. Moreover, integrating out fermions leads not only to the \( |\phi|^4 \) term, but also to an infinite number of higher order terms \( |\phi|^{2n} \), where the integer \( n > 2 \). At first glance, the terms with \( n > 2 \) are all irrelevant, and could be safely ignored [6, 14, 16]. Nevertheless, the Yukawa coupling becomes singular at the QCP, which may alter the scaling dimension of \( \phi \). Once this effect is taken into account, the \( |\phi|^6 \), \( |\phi|^8 \), and higher order terms could become important even in the low-energy region. These terms would break SUSY. Therefore, for SUSY to emerge at the SM-SC QCP, two necessary conditions should be fulfilled: first, all the \( |\phi|^{2n} \) terms with \( n > 2 \) are irrelevant operators; second, the parameter \( \lambda_4 \) does not receive nonlocal contributions from integrating out fermions. These two conditions are not properly considered in previous works seeking emergent SUSY in quantum critical systems.

In this paper, we examine whether the above necessary conditions are satisfied in the quantum critical systems that were recently argued to exhibit emergent SUSY. We will calculate the coefficients of \( |\phi|^{2n} \) terms with \( n > 1 \) by including the contributions from the Yukawa coupling, and then make a scaling analysis to verify whether these SUSY-breaking terms are relevant, marginal, or irrelevant under scaling transformations. Our finding is that no SUSY emerges due to the nonlocal contributions to the \( |\phi|^{2n} \) terms.

Thus far, SUSY is claimed to emerge in all three types of SM, including Dirac SM [3, 13, 15], Weyl SM [16], and Majorana SM [14]. The low-energy effective actions at the QCPs in these model systems have similar forms that are closely related to Eq. (1). Our following consideration will be based on Eq. (1), but our conclusion applies to all dimensions and is also independent of the specific type of fermions. As aforementioned, the dynamics of bosons comes entirely from the procedure of integrating out fermions [20, 21]. For instance, the quartic term \( |\phi|^4 \) is generated by the diagram of Fig. 1(a) and higher order terms \( |\phi|^{2n} \) is obtained by computing the diagram of Fig. 1(b). When the fermions are gapped, it is usually safe to integrate them out: the fermion gap provides an infrared cutoff that eliminates potential divergence. On the contrary, when the fermions are gapless, as what happens at a SM-SC QCP, it is dangerous to integrate them out since they are important at all energies [22, 31]. A particularly serious problem is that the \( |\phi|^{2n} \) terms may receive singular contributions from the Yukawa coupling between gapless fermions and bosons. To demonstrate this, we now compute the diagrams of Fig. 1 explicitly.

According to Eq. (1), the free fermion propagator is

\[
G_0(k) = \frac{1}{k^2} = \frac{1}{\lambda^2},
\]

(2)

The vertex correction given by Fig. 1(a) is defined as

\[
\lambda_4(q_1, q_2, q_3) = -g^4 \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left[ G_0(p) \sigma_2 G_0^T(-p - q_1) \times \sigma_2 G_0(p + q_1 + q_2) \sigma_2 \times G_0^T(-p - q_1 - q_2 - q_3) \sigma_2 \right].
\]

(3)

where \( D = d + 1 \) and we have utilized the fact \( q_4 = -q_1 - q_2 - q_3 \). Apparently, the vertex \( \lambda_4 \) depends on three independent external momenta \( q_1, q_2, q_3 \), which makes it extremely difficult to get an analytic expression. Fortunately, the possible singularity comes mainly from very small external momenta [28, 29]. We therefore assume that \( q_1 = q_2 = q_3 = q \), and then compute \( \lambda_4(q, q, q) \). After tedious calculations, with details given in the Appendix, we find that

\[
\lambda_4(q, q, q) \propto g^4 \left| q \right|^{-2D + 1}
\]

(4)

in \( D = 2 + 1 \), and

\[
\lambda_4(q, q, q) \propto g^4 \left| q \right|^{-3D + 1}
\]

(5)

in \( D = 3 + 1 \). It turns out that \( \lambda_4(q, q, q) \) is as singular as \( 1/|q| \) for \( D = 2 + 1 \), but is a constant for \( D \geq 4 \). The loop diagram Fig. 1(a) was computed in previous works [3, 14, 16, 18], which did not find any singular contribution. The reason is that in these works all the external momenta are supposed to vanish. This treatment is certainly applicable if the fermions are gapped, but might not be appropriate at the QCP where the fermions are strictly gapless and play an important role even at zero energy. In case the bosonic field \( \phi \) is a SC or a PDW, the vertex correction given by Fig. 1(a) is dominated by the internal fermion propagators whose momenta are smaller than those of external boson propagators [28, 29].

![FIG. 1: The Feynman diagram for the \( |\phi|^4 \) term in the effective bosonic action (left) and the schematic diagram for \( |\phi|^{2n} \) (right). Here, the solid line represents the free fermion propagator and the dash line the boson propagator.](image-url)
Nevertheless, if all the external momenta are forced to vanish, these important contributions are artificially discarded. Indeed, this assumption amounts to require that the fermion momenta cannot be smaller than the boson momenta, which is not appropriate. In order not to miss such contributions, the external momenta should be kept finite. As showed in our calculations, for $D = 2 + 1$ the $|\phi|^4$ term becomes nonlocal after integrating out gapless fermions. Such singular contributions need to be incorporated in the scaling analysis.

Notice here that the absence of singularity in $\lambda_4$ at $(3 + 1)$-D does not mean that integrating out gapless fermions is justified, since there are still an infinite number of higher order terms. To address this issue, we now compute the diagram of Fig. 1(b) which is given by

$$\lambda_{2n} = -g^{2n} \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left[ \sigma_2 G_0(p) \sigma_2 G_0^T(-p - q_1)\sigma_2 \ldots \sigma_2 G_0(p) \right],$$

where for notational simplicity the explicit dependence of $\lambda_{2n}$ on external momenta is omitted. We adopt the same simplification as what we have done in the computation of $\lambda_4$, and set all the external momenta to be identical. Under this approximation, we obtain

$$\lambda_{2n} \sim g^{2n} \left( \frac{1}{q^2} \right)^{n - \frac{D}{2}}.$$  \hspace{1cm} (7)

The detailed derivation is given in the Appendix. This result implies that the singularity induced by integrating out gapless fermions becomes more severe as $n$ grows. Owing to this property, the space-time SUSY emergent in $(3 + 1)$-D is also not well justified, since the $|\phi|^{2n}$ terms may not be irrelevant.

After obtaining $\lambda_4(q, q)$ and $\lambda_{2n}(q, ..., q)$, we are now ready to make a scaling analysis by taking into account the nonlocal corrections. For this purpose, we re-write the effective action as follows

$$S_f^0 = \int \frac{d^D p}{(2\pi)^D} \psi(p) \left( \gamma_0 p_0 + c_f \gamma \cdot p \right) \psi(p),$$

$$S_b^0 = \int \frac{d^D k}{(2\pi)^D} \phi\left( k_0 + v_f |k^2| \right) \phi(k),$$

$$S_f = \int \frac{d^D p}{(2\pi)^D} \frac{d^D k}{(2\pi)^D} G \left[ \psi^T(p_0, p) i\sigma_2 \psi(k_0, k) \right] \times \phi \left( k - p, k_0 - p_0 + h.c. \right),$$

$$S_b = \sum_{n=1}^{\infty} u_{2n} \left( \int \frac{d^D k}{(2\pi)^D} \right)^{2n-1} \left( \frac{1}{k^2} \right)^{n - \frac{D}{2}} |\phi(k)|^{2n}.$$  \hspace{1cm} (10)

where $u_{2n} = g^{2n}/v_f^d$. Using the scaling dimension in momentum space, we have

$$[p_0] = z, \quad [p_1] = 1, \quad [v_f] = [v_b] = z - 1,$$

$$[p] = \left[ \sqrt{v_f^2 (v_b^2) p^2 + v_b^2 p_0^2} \right] = z.$$  \hspace{1cm} (12)

Here, $z$ is the dynamical exponent. Within the standard RG framework, one chooses to regard the free boson action as the free fixed point, which then leads to the following scaling behaviors:

$$[\psi(p)] = -\frac{2z + d}{2}, \quad [\phi(k)] = -\frac{3z + d}{2},$$

$$[g] = \frac{3z - d}{2}, \quad [u_{2n}] = (3z - d)n + d(1 - z).$$  \hspace{1cm} (13)

According to Eq. (7), for $n = 1$ we see that the one-loop polarization function behaves as

$$\Pi_2(q) \sim g^2 \left( q^2 \right)^{\frac{d}{2} - 1},$$

which means that $z = 1$ when the polarization is added to the free boson action. As pointed out in Refs. [33, 34], $z$ is not altered by higher order corrections because this value is guaranteed by the $U(1)$ gauge invariance of action Eq. (11). In the case of $z = 1$, it is clear that $[u_{2n}] = 0$ for $d = 3$, thus all the $|\phi|^{2n}$ terms are marginal. For $d < 3$, $[u_{2n}] > 0$ and there are infinite number of relevant self-coupling terms. This clearly indicates that, at the QCP of $d < 3$ system, the one-loop polarization $\Pi_2(q)$ is more important than the free boson action given by Eq. (11) in the infrared region. As a consequence, it is no longer appropriate to regard the term $S_b^0$ as the free fixed point. To make a legitimate scaling analysis, we follow the scheme proposed in Refs. [33, 34]: discard the free action of boson, and determine the scaling dimension of boson $\phi$ by taking Yukawa coupling as the starting fixed point. Within this scheme, one can show that

$$[\psi(p)] = -\frac{2z + d}{2}, \quad [\phi(k)] = -d,$$

$$[g] = 0, \quad [u_{2n}] = d(1 - z).$$  \hspace{1cm} (15)

By using these scaling behaviors, it is easy to verify that the free boson action is irrelevant. For $z = 1$, all the $|\phi|^{2n}$ vertices are marginal operators, which is independent of the dimension. One can infer from this fact that all the bosonic self-coupling terms are equally important at low energies and should be considered simultaneously.

We now discuss the impact of nonlocal $|\phi|^4$ and $|\phi|^{2n}$ terms on the possible emergent SUSY. In the absence of such nonlocal terms, the action given in Eq. (10) is a well-defined local quantum field theory. The bosons are obtained by integrating out high-energy fermionic modes, one can readily show that $\lambda_{2n}$ is just a constant. In this case, all the self-coupling boson terms but $|\phi|^4$ are irrelevant, and it can be verified by RG calculations [3, 14, 15, 16] that there is a stable infrared fixed point at which an effective SUSY emerges. Indeed, for a local effective theory of SM-SC or SM-PDW QCP, there
is a relation $\lambda_4 \sim g^4$ and the system contains only one free parameter. As long as this parameter has a stable infrared fixed point, SUSY emerges naturally at low energies \cite{12, 14, 15, 16}, and the action of Eq. (1) is unchanged under the SUSY transformations:

\begin{equation}
\delta_\eta \psi = i2i\phi^* \eta - \frac{1}{2} \phi^2 \sigma_2 \eta^T, \quad \delta_\eta \phi = i\eta \delta \phi - \frac{1}{2} \phi^* \eta^T i\sigma_2,
\end{equation}

\begin{equation}
\delta_\eta \phi = -\eta \phi, \quad \delta_\eta \phi^* = \eta \psi. \quad (16)
\end{equation}

However, when the theory becomes nonlocal due to the singular contributions from Yukawa coupling, we have shown that all the nonlocal terms of boson are marginal and $u_{2n} \propto g^{2n}$. Once $g$ flows to a stable, non-Gaussian infrared fixed point, the relation $u_{2n} \propto (g^*)^{2n}$ is always satisfied. Since there is always a singularity in each term of Eq. (11), the whole action does not respect any SUSY. In the special case that $g$ flows to a Gaussian fixed point with $g^* = 0$, the SUSY transformations are respected. However, this is a trivial case because at such a fixed point the fermions and bosons are non-interacting.

We now ask the question: When can SUSY exist at a quantum critical system? There are generically three possibilities.

1) If some model system respects SUSY at the microscopic level, the non-renormalization theorem \cite{1, 2} guarantees that the mass and coupling constants are not renormalized by the quantum corrections. There exist certain diagrams that have the same topology as Fig. 1 but contain internal boson propagators. As long as the SUSY is respected, these two types of diagrams cancel each other because they are equal in absolute value but opposite in sign \cite{1, 2}. In this case, the SUSY is intrinsic, not emergent, and could be realized only if one can carefully tune a number of free parameters.

2) If the boson appearing in Eq. (1) is a composite particle and formed by certain type of fermion pairing, either Cooper pairing \cite{14, 15} or PDW pairing \cite{16}, then its dynamics, embodied in the self-coupling terms $|\phi|^{2n}$, is generated by integrating out fermions. While it is usually valid to integrate out high-energy fermions, nonlocal contributions could be induced in the process of integrating out the gapless fermions. It turns out that no SUSY emerges in such systems. The existence and impact of the nonlocal contributions should be seriously considered when one is trying to realize emergent SUSY at certain QCP.

3) Now suppose that there is a quantum critical system described by the action given by Eq. (1), and that the boson $\phi$ is entirely independent of the fermion $\psi$. In this case, the boson has its own dynamics. However, the self-coupling terms $|\phi|^{2n}$ still receive quantum corrections from the Yukawa coupling and then become nonlocal. Therefore, there is still no emergent SUSY even if boson $\phi$ is not formed by fermion pairing.

In summary, SUSY could emerge at a QCP if the Yukawa coupling between gapless fermions and bosons does not generate any nonlocal correction to the effective action of bosons. Otherwise, SUSY cannot emerge. It seems most possible to realize emergent SUSY in fully gapped systems, such as the models considered in Ref. \cite{11}. Although one needs to adjust two or even more parameters to realize SUSY in these systems, at least there are not severe infrared divergence and nonlocal coupling terms.

The authors thank Jing-Rong Wang and Jing Wang for helpful discussion, and acknowledge the financial support by the National Natural Science Foundation of China under Grant 11574285.

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1. Nonlocal contributions to $|\phi|^4$ and $|\phi|^{2n}$

After including all the possible higher order self-coupling terms, the boson action can be written in the momentum space as

$$S_b = \int \frac{d^Dk}{(2\pi)^D} \left[ \phi^\dagger(k) D^{-1}(k) \phi(k) + \sum_{n=2}^{\infty} \left( \int \frac{d^Dk}{(2\pi)^D} \right)^{2n-1} \lambda_{2n} |\phi(k)|^{2n} \right],$$

where $D = d + 1$ and $D^{-1}(k) = D_0^{-1}(k) - \Pi(k)$ with $D_0^{-1}(k) = k^2$. The diagram of Fig. 1(a) is calculated as follows:

$$\lambda_4(q_1, q_2, q_3) = -g^4 \int \frac{d^Dp}{(2\pi)^D} \text{Tr} \left[ G_0(p)\sigma_2 G_0^T(-p - q_1)\sigma_2 G_0(p + q_1 + q_2)\sigma_2 G_0^T(-p - q_1 - q_2 - q_3) \right]$$

$$= -g^4 \int \frac{d^Dp}{(2\pi)^D} \text{Tr} \left[ G_0(p)G_0(p + q_1)G_0(p + q_1 + q_2)G_0(p + q_1 + q_2 + q_3) \right]$$

$$= -g^4 \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2(p + q_1)^2(p + q_1 + q_2)^2(p + q_1 + q_2 + q_3)^2} \cdot \left\{ \left[ p \cdot (p + q_1) \right] \left[ (p + q_1 + q_2) \cdot (p + q_1 + q_2 + q_3) \right] - \left[ p \cdot (p + q_1 + q_2) \right] \left[ (p + q_1) \cdot (p + q_1 + q_2 + q_3) \right] + \left[ p \cdot (p + q_1 + q_2 + q_3) \right] \left[ (p + q_1) \cdot (p + q_1 + q_2) \right] \right\}. \quad (19)$$

In the above derivation, we have used the relation $\sigma^y G_0^T(p)\sigma^y = G_0(-p)$. In general, it is hard to evaluate the above integration in the presence of three freely varying outline momenta. To simplify analytical calculation, we choose to assume that $q_1 = q_2 = q_3$. After tedious but straightforward computations, we find that

$$\lambda_4(q, q, q) = \begin{cases} \frac{-(20\sqrt{33}+423)g^4}{12288}, & D=3 \\ Cg^4, & D=4 \end{cases} \quad (20)$$

where $C$ is a negative constant.

We then compute the diagram of Fig. 1(b), which is given by

$$\lambda_{2n} = -g^{2n} \int \frac{d^Dp}{(2\pi)^D} \text{Tr} \left[ G_0(p)\sigma_2 G_0^T(-p - q_1)\sigma_2 \ldots G_0(p + \sum_{i=1}^{2n-2} q_i) \sigma_2 G_0^T(-p - \sum_{i=1}^{2n-1} q_i) \sigma_2 \right]$$

$$= -g^{2n} \int \frac{d^Dp}{(2\pi)^D} \text{Tr} \left[ G_0(p)G_0(p + q_1)\ldots G_0(p + \sum_{i=1}^{2n-1} q_i) \right]$$

$$= -g^{2n} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{\prod_{i=0}^{2n-1} (p + a_i)^2} \cdot \left[ \sum_{i_1=1}^{2n-1} (-1)^{r(a_{i_1}, a_{i_2}, \ldots a_{i_2n-1})} p \cdot (p + a_i) \right] \times (p + a_{i_2}) \cdot (p + a_{i_3}) \times \ldots \times (p + a_{i_{2n-2}}) \cdot (p + a_{i_{2n-1}}), \quad (21)$$
where \( a_i = \sum_{i=1}^{j} q_i \), \((a_{i_1}, a_{i_2}, ..., a_{i_{2n-1}})\) is an arbitrarily array for \((a_1, a_2, ..., a_{2n-1})\) and \( r(a_{i_1}, a_{i_2}, ..., a_{i_{2n-1}})\) is the total replacement number of the array \((a_{i_1}, a_{i_2}, ..., a_{i_{2n-1}})\) with a condition that \( i_j < i_k \) is satisfied in every inner product term \((p + a_{i_j}) \cdot (p + a_{i_k})\). By setting \( q_1 = q_2 = ... = q_{2n-1} = q \) and introducing \((2n-1)\) parameters, \( x_1, x_2, ..., x_{2n-1} \), to carry out Feynman parametrization, we eventually get

\[
\lambda_{2n}(q) = \frac{-g^{2n}}{(4\pi)^{D/2}} \int dF_{2n-1} \left( \frac{1}{f(x_1, x_2, ..., x_{2n-1})} \right)^{n-\frac{D}{2}} \left[ \frac{\Gamma(n - \frac{D}{2}) \Gamma(n + \frac{D}{2})}{\Gamma(\frac{D}{2}) \Gamma(2n)} \right] \Gamma(2n) \sum_{j=1}^{n-1} Y(D, j) \frac{\Gamma(2n - j - \frac{D}{2}) \Gamma(j + \frac{D}{2})}{\Gamma(2n - \frac{D}{2}) \Gamma(2n)} \left( \frac{1}{q^2} \right)^{n-\frac{D}{2}}. \tag{22}
\]

In this expression, \( \int dF_{2n-1} \) is an integration measure defined as \( \int dF_{2n-1} = \int_0^1 \int_0^{1-x_1} ... \int_0^{1-x_1-...-x_{2n-2}} dx_1 dx_2 ... dx_{2n-1} \), and \( f(x_1, x_2, ..., x_{2n-1}) = \sum_{j=1}^{2n-1} j^2 x_j - \left( \sum_{j=1}^{2n-1} x_j \right)^2 \). \( Y(D, j) \) is a constant that is determined by the space-time dimension \( D \) and the value of \( j \). Generically, this integral cannot be zero for all values of \( n \). At least, it is easy to verify that for small values of \( n \), such as \( n = 2, 3, 4 \), this integral is nonzero. Summarizing the above results, we have

\[
\lambda_{2n}(q) \propto g^{2n} \left( \frac{1}{q^2} \right)^{n-\frac{D}{2}}.
\]