Abstract—Motivated by recent experimental data on dichloro-tetrakis thiourea-nickel (DTN) [Soldatov et al., Phys. Rev. B 101, 104410 (2020)], a model of antiferromagnet on a tetragonal lattice with single-ion easy-plane anisotropy in the tilted external magnetic field is considered. Using the smallness of the in-plane field component, we analytically address field dependence of the energy gap in “acoustic” magnon mode and transverse uniform magnetic susceptibility in the ordered phase. It is shown that the former is non-monotonic due to quantum fluctuations, which was indeed observed experimentally. The latter is essentially dependent on the “optical” magnon rate of decay on two magnons. At magnetic fields close to the one which corresponds to the center of the ordered phase, it leads to experimentally observed dynamical diamagnetism phenomenon.

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1. INTRODUCTION

Quantum phase transitions and quantum phases of matter are among the most extensively studied subjects of contemporary condensed matter physics [1]. In this context, quantum magnets play an important role in revealing versatile physics [2–4]. Importantly, in such compounds one can add impurities in more or less controllable way [5–7], which allows studying effects of quenched disorder and so-called “dirty-boson” systems [8, 9]. Moreover, such elusive phases as Bose and Mott glasses can be observed in disordered quantum magnets [6].

Particular compounds DTN [dichloro-tetrakis thiourea-nickel, \(\text{NiCl}_2\cdot\text{SC(NH}_2\text{)}_2\)] and its bromine-doped counterpart DTNX have been studied for last several decades (see, e.g., [6, 10–15]). However, they are still able to reveal peculiar features. Their properties can be described using the model of weakly coupled antiferromagnetic (AF) chains with spin \(S = 1\) and strong single-ion easy-plane anisotropy [15] (see Fig. 1). The latter is responsible for DTN interesting magnetic properties. In the external magnetic field \(h\) along the tetragonal axis, at \(h < h_{c1}\) the system is in singlet paramagnetic phase, at \(h_{c1} < h < h_{c2}\) ordered canted AF (CAF) phase emerges, whereas at \(h > h_{c2}\) the fully polarized gapped state can be observed [10, 16]. Critical fields \(h_{c1}, h_{c2}\) and their vicinity can be discussed in terms of elementary excitations (triplons and magnons, respectively) Bose–Einstein condensation [3, 12, 16–19]. The intermediate CAF phase not very close to the critical fields can be described as a usual magnetically ordered phase and studied, e.g., by means of electron spin resonance technique [11, 20, 21]. Interestingly, such experiments reveal, for instance, nontrivial field variation of the “optical” (high-energy) magnon frequency. In our previous work [22] it was explained in the framework of conventional diagrammatic \(1/S\) expansion, where the role of strong quantum fluctuations was highlighted.

In the present paper, we continue our theoretical discussion [22]. Here we address the case of the tilted magnetic field realized experimentally [21]. Using the Kubo formalism for linear response and \(1/S\) expansion, we show that induced by the tilted field in-plane spin component inherits peculiar field-behavior of the optical magnon frequency. This leads to experimentally observed field dependence of the gap in the “acoustic” (low-energy) branch of the spectrum. Moreover, we show that the optical branch near the center of the ordered phase acquires significant damping due to two-magnon processes. So, our theory quantitatively supports qualitative discussion of the
The rest of the paper is organized as follows. In Section 2 we formulate the model, briefly recall our previous results, and employ our approach for the gap in acoustic branch derivation. Section 3 is devoted to homogeneous transverse susceptibility in the tilted field and the dynamical diamagnetism effect. We present our conclusions in Section 4. Three appendices contain cumbersome formulas and some other details of calculations.

2. GAP IN MAGNON SPECTRUM

2.1. Problem Formulation

We consider a model of an antiferromagnet with single-ion easy-plane anisotropy in the canted external magnetic field on a simple tetragonal lattice [23] (see Fig. 1). For definiteness, we take the magnetic field in the \(\mathbf{z}\) plane \((\mathbf{x}\mathbf{y}\) is the easy plane). The corresponding Hamiltonian reads

\[
\mathcal{H} = \frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + D \sum_i (\mathbf{S}_i^z)^2 - h_z \sum_i \mathbf{S}_i^z - h_x \sum_i \mathbf{S}_i^x.
\]

Here the external magnetic field is taken in the energy units \((h = g \mu_B H)\) and \(D > 0\) indicates the easy-plane anisotropy. Since our aim is to address experimentally relevant case [20, 21] of small tilt angles \(\alpha\), we will treat the part of the Hamiltonian with \(h_x \approx \alpha h\) as a small perturbation. Hence, the starting point of our analysis will be a theory from [22]. Let us briefly recall its main findings.

We considered the case of the external field along the tetragonal axis. We mainly discussed the “optical” magnon branch and its energy in the center of the Brillouin zone \(\Delta\) (which can be probed by ESR [20, 21]) employing Holstein–Primakoff representation of spin operators via bosonic ones [24] in the alternating coordinate frame suitable for the canted antiferromagnetic ordering [25]. Using quasiclassical \(1/S\)-series diagrammatic approach, we found a strong renormalization of the magnon spectrum due to the first order in \(1/S\) quantum corrections, which lead to experimentally observed [20, 21] nonmonotonous dependence \(\Delta(h)\). It is pertinent to note that our approach based on the Holstein–Primakoff transform is inapplicable at small fields \(h \ll h_{c1}\) (i.e., in the singlet phase and in the vicinity of the quantum phase transition). In this case, other techniques should be used, see, e.g., [26].

Despite being small, the transverse magnetic field \(h_x\) has an important effect on the system: it breaks the rotational symmetry of the easy \(\mathbf{x}\mathbf{y}\) plane. This results in the disappeareance of the Goldstone mode, the corresponding magnon branch becomes gapped and will be called “quasi-Goldstone” below.

Our main goal is to analytically describe the properties of the emergent gap (as a function of \(h\) and \(\alpha\)) and the magnetic susceptibility of the system. For this sake, we start from the ground state at \(\alpha = 0\) and immediately observe that \(h_x\) terms in the Hamiltonian (1) provide linear in bosonic operators terms (equivalent to the ground state instability), which in the reciprocal space read

\[
h_x \frac{\sqrt{SN}}{\sqrt{2}} (a_{k=0}^\dagger + a_{k=0}),
\]

see Appendix A for some details. We eliminate these terms using the following shift in operators:

\[
a_{k=0} = b_{k=0} + \rho, \quad a_{k=0}^\dagger = b_{k=0}^\dagger + \rho.
\]

Parameter \(\rho\) is related to the linear response of the system to \(h_x\) and can be found directly from the Hamiltonian or using the Kubo susceptibility. In any case, we found that \(\rho \propto \alpha\). Importantly, \(\rho \neq 0\) leads to new terms in the Hamiltonian, which will be used for discussing the gap in the quasi-Goldstone mode and the dynamic diamagnetism phenomenon observed in [21].
magnetic susceptibility in the tilted field. Note that the smallness of $\alpha$ allows us to omit terms $\rho^n$, $n > 2$ when recalculating the Hamiltonian for nonzero $h_x$.

### 2.2. Calculation of $\rho$

Here we derive an analytical expression for the crucial parameter $\rho$. Importantly, it is simply related to the average $S_x$ value, which is the same for all spins. Explicitly:

\[
\langle S^x \rangle = \frac{1}{N} \sum_i \langle S^x_i \rangle = \frac{\sqrt{S}}{\sqrt{2N}} \sum_i \langle a_i^+ + a_i \rangle
\]

\[
= \frac{\sqrt{S}}{\sqrt{2N}} \langle a_{k=0}^+ + a_{k=0} \rangle = \sqrt{2S} \frac{\rho}{\sqrt{SN}}.
\]

Parameter $\rho$ can be found by making linear terms in the Hamiltonian to be zero (see Eq. (2)) using the shift (3). However, in order to take into account quantum corrections, we will use a more general method of the Kubo susceptibility:

\[
\chi = \langle S^x(q, \omega) S^x(-q, \omega) \rangle_{q=0, \omega=0, h_x=0}
\]

\[
= \frac{S}{2} \langle (a_q^+ + a_{-q}) (a_{-q}^+ + a_{q}) \rangle_{q=0, \omega=0, h_x=0}
\]

\[
= -\frac{S}{2} (F_{k=0} + G_{k=0} + F_{k=0}^* + G_{k=0}).
\]

So, $\chi$ is the sum of retarded magnon Green’s functions taken at $k = (\omega = 0, k = 0)$, see Appendix A for details. The latter can be obtained at $h_x = 0$ using the conventional diagrammatic approach.

In the linear spin-wave theory we have

\[
\chi_0 = \frac{1}{2SJ_0},
\]

which yields

\[
\langle S^x \rangle = \frac{h_x}{2SJ_0},
\]

\[
\rho = \frac{h_x}{\sqrt{NS}}.
\]

One sees that the linear spin-wave theory predicts constant susceptibility.

When $1/S$ corrections are taken into account, the susceptibility behavior becomes much more interesting. Using Eq. (5) with renormalized by magnon interaction Green’s functions, we obtain

\[
\chi_{1/S}(h) = \frac{P(h)}{2\Delta(h)},
\]

where

\[
P(h) = 2E_k - 2B_k + \Sigma_k + \Sigma_{-k} - 2\Pi_{k|k|=0},
\]

\[
\Delta(h) = \varepsilon_k + (\Sigma_k - \Sigma_{-k})/2
\]

\[
+\{E_k(\Sigma_k + \Sigma_{-k}) - 2B_k\Pi_{k|k|=0}/2\varepsilon_k|_{k=0} = 0.
\]

2.3. Gap at $h_x \neq 0$

After the shift in Bose-operators (3), multiple terms $\rho^n$, $n \geq 1$ emerge in the Hamiltonian in addition to those discussed in Appendix A. Some details are presented in Appendix B.

Linear in Bose-operators terms are canceled by proper choice of $\rho$ (see previous Subsection). We also neglect “decay” terms $\delta \mathcal{H}_0$ (and the higher order in Bose-operators ones) because they provide small in $1/S$ corrections. Moreover, the subject of the calculations below is magnon with $k = k_0$ and of minimal energy, which cannot acquire a finite lifetime due to such processes.

So, we focus on corrections to the bilinear part of the Hamiltonian. Importantly, they include normal and umklapp parts, which read
respectively. Explicit formulas for the coefficients are presented in Appendix B.

Next, one can see that the normal corrections can be directly plugged into the equation for the magnon spectrum of the linear theory (A.9), whereas the umklapps require some additional calculations. They contribute to both normal and anomalous self-energy parts; corresponding diagrams are shown in Fig. 4.

However, at \( k = k_0 \) only the normal terms contribute to the spectrum since the umklapp corrections cancel each other (see Appendix B). So, the equation for the gap in the spectrum reads:

\[
\Delta_{h_1} = \sqrt{2E_{k_0}(E_{k_0} + \bar{E}_{k_0})},
\]

Using the expressions for \( \bar{E}_{k_0} \) (B.1) and \( \bar{E}_{k_0} \) (B.2) and taking the value of parameter \( \rho \) calculated within the Kubo formalism, we get

\[
\Delta_{h_1} = \sqrt{2E_{k_0}A_h S\rho} \sqrt{NS},
\]

where

\[
\rho = \frac{h_x}{S\rho}. 
\]

The first term in Eq. (16) is much smaller than the others and it is significant only near the critical fields \( h_{c1} \) and \( h_{c2} \). In the intermediate field region (on which we mainly focus) it can be safely neglected. Finally, we obtain the following result for the gap value:

\[
\Delta_{h_1} = \sqrt{E_{k_0}A_h(s^2)}. 
\]

Results of our calculations are presented in Fig. 5 for a particular parameter set. Note that the gap is pro-

\[
\text{Field dependencies of frequencies of “quasi-Goldstone” (} k = k_0; \text{ black curve) and “optical” (} k = 0; \text{ blue curve) magnons. Parameters } \alpha = \pi/60, D = 8.0 \text{ K, } J_c = 2 \text{ K, and } J_a = 0.2 \text{ K were used.}
\]
portional to the average $S_x$ value (cf. Fig. 3). Moreover, its behavior is similar to the one, observed experimentally [20, 21].

3. MAGNETIC SUSCEPTIBILITY

Here we discuss the uniform transverse magnetic susceptibility of the system in the tilted external field. A crucial point here is the significant damping of the optical magnon with $k = 0$, which is addressed below.

3.1. Magnon Damping

In [22], we were particularly interested in the energy of the optical magnon with $k = 0$. For our current purposes, its damping also becomes important. Theoretically, the finite magnon lifetime is connected with the possibility for it to decay; in our case on two other magnons.

Explicitly, the damping comes from the imaginary part of magnon Green’s function self-energies (see Eq. (A.15)). Moreover, here we can neglect small $h_x$-induced terms in the Hamiltonian. Separating the imaginary part of the corresponding denominator from the real one, we get it in the first order in $1/S$:

$$\Gamma(k) = \text{Im} \left[ \frac{\Sigma_k - \Sigma_{-k}}{2} + \frac{E_k(\Sigma_k + \Sigma_{-k}) - 2B_k\Pi_k}{2E_k} \right].$$  \hspace{1cm} (18)

Corresponding imaginary parts of normal and anomalous self-energy parts are due to $\mathcal{H}_s$ terms (see Eq. (A10)) where a magnon with momentum $k$ can decay on magnons with momenta $k - p + k_0$ and $p$. Below, we only discuss the damping of magnon with $k = 0$ and use the on-shell approximation. The corresponding damping is $h$-dependent, we denote it as $\Gamma(h)$.

Our calculations show that $\Gamma(h)$ becomes zero at the critical fields $h_{c2}$ and $h = 0$ (here it is pertinent to note that in our approach the field $h_{c3}$ does not emerge) and has a maximum with significant value near the ordered phase center. This result is illustrated in Fig. 6 for particular parameter set.

3.2. Susceptibility

The generalization of Eq. (5) for the static transverse spin susceptibility is the sum of retarded Green’s functions:

$$\chi^x(\omega, k = 0) = -\text{Re} \left\{ \frac{P(h) + 2E_{k=0} - 2E_{k=0}}{\omega^2 - (\Delta - i\Gamma)^2} \right\}. \hspace{1cm} (20)$$

Importantly, this formula has two types of poles, corresponding to “stable” acoustic magnon ($\delta \to +0$) and to damped optical magnon. So, an interplay between them determines uniform susceptibility as a function of frequency $\chi^x(\omega)$. Moreover, the magnitude of optical magnon damping $\Gamma$ plays a crucial role here (see Fig. 7).

In the absence of the damping ($\Gamma = 0$), the real part of the susceptibility at $\omega > 0$ has two hyperbolic features associated with acoustic and optical magnon energies. However, when the damping grows up (what happens near the ordered phase center, see Fig. 6) the qualitative behavior of $\text{Re} \chi^x(\omega)$ changes. Region of $\text{Re} \chi^x(\omega) > 0$, pronounced near $\omega = \Delta$ at small $\Gamma$, fades away. Thus, in our analytical consideration, we observe the change of the susceptibility sign in that frequency domain, which was called “dynamic diamagnetism” in [21]. Moreover, our interpretation agrees with the one from this paper on a qualitative level, since two-magnon processes are the reason for large $\Gamma$ in our calculations.

4. DISCUSSION AND CONCLUSIONS

We study antiferromagnet with large single-ion easy-plane anisotropy in the tilted external field at small tilt angles in the ordered canted phase. In this case, the susceptibility is proportional to the average $S_x$ value (cf. Fig. 3). Moreover, its behavior is similar to the one, observed experimentally [20, 21].
case, the in-plane field component is small and we use linear response theory (Kubo formalism) to calculate the corresponding corrections to the ground state. This allows us to obtain corrections to the Hamiltonian, which appear due to the in-plane field component. Also, the latter breaks the rotational symmetry with respect to the tetragonal axis. Importantly, this leads to the gap in magnon spectrum emergence, which is non-monotonic under the external field variation. Theoretically, this feature is inherited from also nonmonotonic behavior of the energy of the optical magnon in the center of the Brillouin zone. It was studied in [22] and connected with strong quantum fluctuations-induced renormalization of this energy. Moreover, we show that our result for the gap field-dependence is in good qualitative agreement with the experimental findings of [20], where DTN was studied.

After establishing the ground state in the tilted field, our formalism also allows for studying transverse spin susceptibility. It is shown that there are two crucial points in this derivation. The first one is the significant damping of the optical magnons, which is connected with decay on two magnons. The second one is accurate accounting for the umklapp terms in the Hamiltonian.

As a result, using realistic DTN parameters, we theoretically explain the effect of dynamic diamagnetism, which was experimentally observed in [21]. Moreover, two-magnon processes plays important role in our derivation, which lies in agreement with the interpretation proposed in that paper.

It is pertinent to mention here, that the best fit of the experimental data was not our goal, and we use some realistic parameters of DTN just for illustration purposes. Furthermore, due to the quasi-1D nature of DTN, the dependence of the results on the parameter values is strong and the question of higher order corrections in the 1/S series can be also raised. Nevertheless, we find that our qualitative conclusions are true in a wide range of parameters. Moreover, we can report that the used parameters result in a semi-quantitative agreement with the experimental data.

Our theory and its results should be compared with the approach developed in [27]. The advantages of the latter are its elegance and simple analytical formulas as the yield. However, it is valid only for magnetic fields close to the center of the ordered phase and relies on J/D smallness. In our approach, the calculations are much more cumbersome, but the methods are more general. The latter allows us to address the transverse susceptibility in the same framework as the gap in the magnon spectrum.

**APPENDIX A**

**HAMILTONIAN AT h_x = 0**

Here we briefly remind the basic technique, which was used in our previous study [22] (see also [12, 25]). Hamiltonian (1) can be transformed into bosonic one using conventional Holstein-Primakoff spin-operators representation [24] in the approximate form (which allows taking into account first 1/S corrections):

\[
S_{\alpha}^{\sigma} + i S_{\beta}^{\sigma} = \sqrt{2S}a_{\alpha}^{+}\left(1 - \frac{a_{\alpha}^{2}a_{\beta}^{2}}{2S}\right),
\]

\[
S_{\alpha}^{\sigma} - i S_{\beta}^{\sigma} = \sqrt{2S}\left(1 - \frac{a_{\alpha}^{2}a_{\beta}^{2}}{2S}\right)a_{\alpha},
\]

\[
S_{\gamma}^{\sigma} = -S + a_{\alpha}^{+}a_{\gamma}.
\]
in a local coordinate frame, suitable for canted antiferromagnetic order description [\( k_0 = (\pi, \pi, \pi) \) is the AF vector]:

\[
S_i^x = S_i^\ast^x, \\
S_i^y = S_i^\ast^y \cos \theta + S_i^z \exp(ik_0 \cdot \mathbf{r}_0) \sin \theta, \quad (A.2) \\
S_i^z = S_i^\ast^z \cos \theta - S_i^y \exp(ik_0 \cdot \mathbf{r}_0) \sin \theta. 
\]

As a result we have \( \mathcal{H} = \sum_{n=0}^d \mathcal{H}_n \); each term contains \( n \) Bose-operators. Explicitly:

\[
\mathcal{H}_1 = iS^{3/2}\sqrt{N}(L\mathbf{a}_k - L\mathbf{a}_k^\dagger), \quad (A.3) \\
L = \frac{1}{\sqrt{2}}\left[(2J_0 + 3\mathcal{D}) \cos \theta - \frac{\hbar}{S}\right] \sin \theta. \quad (A.4)
\]

This term can be eliminated by taking proper canting angle \( \theta \), which depends on the external field. Next, the bilinear part is the following:

\[
\mathcal{H}_2 = \sum_k a_k^\dagger a_k^\ast E_k + \frac{S}{2} \sum_k (a_k^\dagger a_{-k}^\ast + a_k^\ast a_{-k}^\dagger) B_k, \quad (A.5) \\
E_k = S(J_0 + J_k) \cos^2 \theta + S(J_0 + D) \sin^2 \theta, \quad (A.6) \\
B_k = S(J_k - D) \sin^2 \theta. \quad (A.7)
\]

Here the Fourier transform of the exchange interaction reads

\[
J_k = 2J_\perp \cos k_z + J_\parallel (\cos k_x + \cos k_y). \quad (A.8)
\]

So, the magnon spectrum in the linear theory is given by:

\[
\varepsilon_k = \sqrt{E_k^2 - B_k^2}. \quad (A.9)
\]

Since the exchange interaction Fourier transform \( J_k \) obeys the property \( J_0 = -J_{-k} \), this spectrum is gapless with \( \varepsilon_{k_0} = 0 \) in agreement with the Goldstone’s theorem.

We also have the interaction part of the Hamiltonian. It consists of two terms, the first one is

\[
\mathcal{H}_3 = iS^{\ast} \sin \theta \cos \theta \frac{4\sqrt{2}N}{4} \times \sum_{123=0}^N (a_i^\dagger a_j^\ast a_k^\dagger a_l^\dagger + V_i + V_j) \frac{V_k + V_l}{2}, \quad (A.10) \\
V_k = \left[2J_{k_0} - 8J_k + 10D - \frac{\hbar}{S} \cos \theta\right],
\]

where we denote \( k_1, k_2, k_3 \) as 1, 2, 3, and the momentum conservation law reads

\[
k_1 + k_2 + k_3 = k_0. \quad (A.11)
\]

The second one is the following:

\[
\mathcal{H}_4 = \frac{1}{\sum_{1234=0}^N} U_{1234} a_i^\dagger a_j^\ast a_k^\ast a_l^\dagger \\
+ \frac{1}{\sum_{1234=0}^N} V_{123} (a_i^\dagger a_j^\ast a_k^\dagger a_l^\ast + a_i^\ast a_j a_k a_l)
\]

with the conventional momentum conservation law. Here

\[
U_{1234} = \left[1 - 2\sin^2 \theta J_{4,i} - \cos^2 \theta (J_1 + J_4) \right] \\
+ D \left[1 - \frac{3}{2}\sin^2 \theta \right], \quad (A.13) \\
V_{123} = \left[D \sin^2 \theta - \frac{(J_1 + J_2 + J_3) \sin^2 \theta}{12}\right]. \quad (A.14)
\]

For nonlinear corrections to magnon spectrum analysis, it is convenient to use normal \( (G_k = \langle a_k, a_k^\dagger \rangle) \) and anomalous \( (F_k = \langle a_k^\ast, a_k^\dagger \rangle) \) Green’s functions. The solution of the system of Dyson’s equation for these functions reads

\[
G_k = \frac{\omega + E_k + \Sigma_k}{D_{a_k}}, \quad (A.15) \\
F_k = -\frac{B_k + \Pi_k}{D_{a_k}}, \quad (A.16) \\
D_k = \omega^2 - \varepsilon_k^2 - E_k (\Sigma_k + \Sigma_{-k}) + 2B_k \Pi_k + \omega (\Sigma_k - \Sigma_{-k}), \quad (A.17)
\]

where \( \Sigma_k, \Pi_k \) are normal and anomalous self-energy parts, respectively. They can be calculated perturbatively.

**APPENDIX B**

**CORRECTIONS TO THE HAMILTONIAN AT SMALL \( \hbar \)**

When the external magnetic field is tilted, nonzero \( \hbar \) appears. It leads to uniform \( \langle S^z \rangle \) component. In our theory, it is related to the parameter \( \rho \). Nonzero \( \rho \) provides various additional terms in the Hamiltonian, which can be altogether denoted as \( \delta \mathcal{H} \).

Linear term \( \delta \mathcal{H}_1 \) vanishes when \( \rho \) is properly chosen.

The bilinear part of \( \delta \mathcal{H}_3 \) can be divided onto normal and umklapp contributions (see Eqs. (12) and (13)). Explicit expressions for the corresponding coefficients in \( \delta \mathcal{H}_3 \) are the following:

\[
E_k = -\frac{\hbar \rho}{2S\sqrt{2N}} + 8U_{0k0} \frac{\rho^2}{NS} + 12V_{0k0} \frac{\rho^2}{NS}, \quad (B.1) \\
\overline{B}_k = -\frac{\hbar \rho}{2S\sqrt{2N}} + 8U_{kk00} \frac{\rho^2}{NS} + 12V_{0k0} \frac{\rho^2}{NS}. \quad (B.2)
\]
These terms are analogous to the contributions in Eq. (A5). For $\delta^E\hat{c}^N$, the corresponding equations read

$$X_k = \sin \theta \cos \theta (V_0 + V_k) \frac{\rho}{4 \sqrt{2} NS}, \quad (B.3)$$

$$Y_k = \sin \theta \cos \theta (V_{k-k_0} + V_k) \frac{\rho}{4 \sqrt{2} NS}. \quad (B.4)$$

Importantly, the umklapps provide corrections to the normal and anomalous self-energy parts $\rho^2$ (they are combined in pairs, see Fig. 4):

$$\Sigma_k^U = X_k^2 G_{ak-k_0} + Y_k^2 G_{ak-k_0-k}$$

$$\Pi_k^U = X_k^2 F_{ak-k_0} + Y_k^2 F_{ak-k_0-k} - X_k Y_k G_{ak-k_0} - X_k Y_k G_{ak-k_0-k}. \quad (B.5)$$

However, one can see at the momentum $k = k_0$ condition $X_k = Y_k$ is satisfied, so $\Sigma_k^U = -\Pi_k^U$. This means that there is no correction to the energy of magnon with $k = k_0$ from the umklapp processes and only the normal terms are important in this context.

**APPENDIX C**

**DERIVATION OF TRANSVERSE SUSCEPTIBILITY IN TILTED FIELD**

Here, for calculations, it is also convenient to use normal and anomalous Green’s functions:

$$G_k = \langle b_k F_k^\dagger \rangle, \quad F_k^\dagger = \langle b^\dagger_{k-k_0} \rangle. \quad (C.1)$$

where $k = (\omega, \mathbf{k})$. These functions obey the following system of Dyson’s equations

$$G_k = G_k^0 + G_k^0 \Sigma G_k + G_k^0 (B_k + \overline{B}_k + \Pi_k) F_k, \quad (C.2)$$

$$F_k = G_k^0 (B_k + \overline{B}_k + \Pi_k) G_k + G_k^0 \Sigma F_k,$$

where the bare Green’s function reads

$$G^0 = \frac{1}{\omega - E_k - \overline{E}_k + i\delta}. \quad (C.3)$$

After solving the system of equations (C.2) we obtain the following expression for normal and anomalous Green’s functions

$$G_k = \frac{\omega + E_k + \overline{E}_k + \Sigma_k}{D_{ak}}, \quad (C.4)$$

$$F_k = -\frac{B_k + \overline{B}_k + \Pi_k}{D_{ak}}. \quad (C.5)$$

Here the denominator reads

$$D_{ak} = \omega^2 - \epsilon_k^2 - E_k (\Sigma_k + \Sigma_{-k} + 2 \overline{E}_k) + 2 B_k (\Pi_k + \overline{B}_k) + \omega (\Sigma_k - \Sigma_{-k}). \quad (C.6)$$

To calculate the susceptibility (19), it is necessary to use the retarded Green’s functions $G^R$ and $F^R$, which are related to the causal ones (C.4) by the following relation:

$$G^R_{ak} = \theta (\omega) G^0_{ak} + \theta (-\omega) G^*_{ak} \quad (C.7)$$

and the same for $F$. The real parts of retarded functions are equal to the corresponding causal ones. So, we get

$$\text{Re}[G^R_{ak}] = -\text{Re} \left\{ 2E_k + \Sigma_k + \Sigma_{-k} - 2B_k - 2\Pi_k \overline{E}_k \frac{(2E_k + \Sigma_k + \Sigma_{-k} - 2B_k - 2\Pi_k)}{\omega^2 - (\Delta - i\Gamma)^2} \right\}. \quad (C.8)$$

Using Eqs. (B.5) and (B.6) at $k = 0$ and Eq. (10), the real part of the uniform transverse spin susceptibility can be finally expressed as

$$\text{Re}[\chi^R_{ak=0}] = -\text{Re} \left\{ \rho (\omega) + 2E_k + 2E^U_{k=0} - 2\overline{B}_k - 2\overline{B}^U_{k=0} \overline{E}_k \frac{(2E_k + \Sigma_k + \Sigma_{-k} - 2B_k - 2\Pi_k)}{\omega^2 - (\Delta - i\Gamma)^2} \right\}, \quad (C.9)$$

where

$$K = 2(X_k + Y_k) (E_k + \overline{E}_k). \quad (C.10)$$

**CONFLICT OF INTEREST**

The authors declare that they have no conflicts of interest.

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