HESSIAN QUARTIC SURFACES THAT ARE KUMMER SURFACES

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Abstract. In 1899, Hutchinson [Hut99] presented a way to obtain a three-parameter family of Hessians of cubic surfaces as blowups of Kummer surfaces. We show that this family consists of those Hessians containing an extra class of conic curves. Based on this, we find the invariant of a cubic surface $C$ in pentahedral form that vanishes if its Hessian is in Hutchinson’s family, and we give an explicit map between cubic surfaces in pentahedral form and blowups of Kummer surfaces.

1. Introduction

Let $C$ be a cubic surface in $\mathbb{P}^3$. Among the many interesting geometrical objects associated to $C$ is its Hessian, a quartic surface $H$ in $\mathbb{P}^3$. It was found in the nineteenth century [Seg42] that $H$ will have ten double points, and will contain ten lines through those points. Conversely, it was shown that any irreducible quartic surface containing an appropriate configuration of lines and double points would be the Hessian of a unique cubic surface.

Another class of objects of interest in classical algebraic geometry is the class of Kummer surfaces. Given an abelian surface $A$, we have an action of the group $\{\pm 1\}$ by multiplication, and we can take the quotient $K = A/\{\pm 1\}$, a surface with 16 double points. On a Kummer surface, there are 16 special curves, called tropes, each of which passing through six of the double points. It was found [Hut99] that one can choose a certain other subset $W$ of 6 of the double points, called a Weber hexad, and blow these points up, to obtain a surface $K_W$ that embeds in $\mathbb{P}^3$ as the Hessian of a cubic surface. That is, there remain 10 double points on $K_W$, and one can embed the surface such that of the sixteen tropes, ten are taken to straight lines in $\mathbb{P}^3$, and in the same configuration as referred to above. So we conclude that $K_W$ is the Hessian of some cubic surface.

Now, it is known that the moduli space of cubic surfaces is a four-dimensional normal variety, while there is only a three-parameter family of Kummer surfaces, each of which having only finitely many Weber hexads. So we can hope that the locus of cubic surfaces whose Hessians are Kummer in the above sense will be a divisor in the space of all cubic surfaces. In this paper we prove

Theorem 1.1. Let $\mathbb{P}^3$ be taken to be the hyperplane in $\mathbb{P}^4$ with $\sum_{i=0}^4 X_i = 0$. Let $C \subset \mathbb{P}^3$ be the cubic surface $\mathbb{V}(\sum_{i=0}^4 \frac{1}{\mu_i} X_i^3)$, where $\mu_0 \mu_1 \mu_2 \mu_3 \mu_4 \neq 0$, and assume $C$ is smooth. Then the Hessian of $C$,

$$\mathbb{V}(\mu_0 X_1 X_2 X_3 X_4 + \mu_1 X_0 X_2 X_3 X_4 + \mu_2 X_0 X_1 X_3 X_4 + \mu_3 X_0 X_1 X_2 X_4 + \mu_4 X_0 X_1 X_2 X_3),$$
is the blowup of a Weber hexad on a Kummer surface if and only if the coefficients \( \mu_i \) satisfy the following irreducible cubic condition:

\[
\sum_{i=0}^{4} \mu_i^3 - \sum_{i \neq j} \mu_i^2 \mu_j + 2 \sum_{i \neq j \neq k} \mu_i \mu_j \mu_k = 0.
\]

The reader familiar with the invariant theory of cubic surfaces will be curious how this locus fits in with the classical invariants. We can interpret the previous result in that language, obtaining the following.

**Corollary 1.2.** Let \( \mathbb{P}^{19} \) be the parameter space of cubic forms on \( \mathbb{P}^3 \). Let \( X \) be the locus in \( \mathbb{P}^{19} \) of cubic surfaces whose Hessians are isomorphic to blowups of Weber hexads on Kummer surfaces, embedded as in Theorem 1.1. Then \( X \) is \( SL(4) \)-invariant, and the closure of \( X \) is a divisor in \( \mathbb{P}^{19} \). If we label the classical invariants as \( I_8, I_{16}, I_{24}, I_{32}, I_{40} \), following [Hum96], then the polynomial on \( \mathbb{P}^{19} \) given by

\[
I_8 I_{24} + 8I_{32}
\]

is irreducible, is degree 32, and vanishes on \( X \).

This result, while somewhat satisfying, is only the beginning of the story. The next natural question to ask is, “If this divisor in the parameter space of \( \mu_i \)s is associated to the moduli space of Kummer surfaces with Weber hexads, then what is this correspondence?” We also answer this question. Recall that the generic abelian surface is the Jacobian of a unique genus 2 curve, say \( A = J(B) \), and that \( B \) can be specified as the double cover of \( \mathbb{P}^1 \) branched at six points \( a, b, c, d, e, f \).

It turns out that upon placing an ordering on these six points, we can specify a unique Weber hexad in \( K \). We then have the following theorem.

**Theorem 1.3.** Let \( a, b, c = 0, d = 1, e, f = \infty \) be six distinct points in \( \mathbb{P}^1 \). Let \( B \) be the double cover of \( \mathbb{P}^1 \) branched at these six points, and let \( A \) be the abelian surface \( J(B) \). Let \( W \subset A \) be the Weber hexad

\[
\{0, b + c - 2a, c + d - 2a, d + e - 2a, e + f - 2a, f + b - 2a\}.
\]

Then the surface \( K_W \) obtained by blowing up \( K = A/\{\pm 1\} \) at \( W \) can be embedded in \( \mathbb{P}^3 \) as the Hessian of the surface

\[
V\left( \sum_{i=0}^{4} \frac{1}{\mu_i} X_i^3 \right),
\]

where the coefficients \( \mu_i \) are given by

\[
\mu_0 = a(1-b), \quad \mu_1 = e(1-a), \quad \mu_2 = b(e-a), \quad \mu_3 = (e-b), \quad \mu_4 = (a-b)(1-e).
\]

Conversely, if \( \mu_0 \mu_1 \mu_2 \mu_3 \mu_4 \neq 0 \) and

\[
\sum_{i=0}^{4} \mu_i^3 - \sum_{i \neq j} \mu_i^2 \mu_j + 2 \sum_{i \neq j \neq k} \mu_i \mu_j \mu_k = 0,
\]
then let
\[ a = \frac{\mu_0 + \mu_3 + \mu_4 - \mu_1 - \mu_2}{2\mu_3}, \]
\[ b = \frac{\mu_1 + \mu_2 + \mu_3 - \mu_0 - \mu_4}{2\mu_2}, \]
\[ e = \frac{\mu_0 + \mu_3 - \mu_4 - \mu_1 - \mu_2}{\mu_0 + \mu_3 + \mu_4 - \mu_1 - \mu_2} \]
be points in \( \mathbb{P}^1 \). If these points are all distinct, and none of these points are 0, 1, or \( \infty \), then let \( B \) be the double cover of \( \mathbb{P}^1 \) branched at \( a, b, c = 0, d = 1, e, f = \infty \), and let \( K_W \) be the blown up Kummer specified above. Then the Hessian surface
\[ H = \mathcal{W}(\mu_0 X_0 X_1 X_2 X_3 X_4 + \mu_1 X_0 X_2 X_3 X_4 + \mu_2 X_0 X_1 X_3 X_4 + \mu_3 X_0 X_1 X_2 X_4 + \mu_4 X_0 X_1 X_2 X_3) \]
is isomorphic to \( K_W \).

This still is not the whole story as it should be. If one recalls that any smooth cubic surface can be obtained as the blow up of \( \mathbb{P}^2 \) at six points, the question arises, given a genus 2 curve \( B \) and a Weber hexad \( W \subset J(B) \), what six points we should blow up in \( \mathbb{P}^2 \) to obtain a cubic surface \( C \) whose Hessian \( H \) is isomorphic to \( K_W \). As far as this author is aware, this question has not been answered. Failing that, we present proofs of the above two theorems, in the hope that our techniques can be extended to answer the remaining questions about Kummer Hessian surfaces.

2. The geometry of Hessian quartic surfaces

We will begin our exploration by collecting some results about the geometries of Hessian surfaces and Kummer surfaces, and then apply them to the theorems at hand. First, let us describe some of the geometry of the Hessian of a generic cubic surface. Let \( \mathbb{P}^3 \) be the hyperplane
\[ \mathcal{W} (\sum_{i=0}^{4} X_i) \subset \mathbb{P}^4, \]
as above, and for \( 0 \leq i < j \leq 4 \), let
\[ \ell_{ij} = \mathcal{W}(X_i, X_j) \subset \mathbb{P}^3. \]
Let \( L = \bigcup_{i<j} \ell_{ij} \). Then we have the following lemma.

**Lemma 2.1.** Let \( H \) be a quartic form on \( \mathbb{P}^4 \) that vanishes on \( L \). Then \( H \) is in the linear span of
\[ (X_0 X_1 X_2 X_3, X_0 X_1 X_2 X_4, X_0 X_1 X_3 X_4, X_0 X_2 X_3 X_4, X_1 X_2 X_3 X_4). \]
As a result, \( H \) is double at the 10 points \( p_{ijk} = \mathcal{W}(X_i, X_j, X_k) \).

**Proof.** For the first statement, we observe that \( \mathcal{W}(H, X_0) \) by assumption contains the four lines \( \ell_{01}, \ell_{02}, \ell_{03}, \ell_{04} \). So \( H \) lies in the ideal \( (X_0, X_1 X_2 X_3 X_4) \). Applying this symmetrically yields the result. The second statement is immediate. \( \square \)

Observe that if \( \mu_0 \mu_1 \mu_2 \mu_3 \mu_4 \neq 0 \), then the cubic surface \( \mathcal{W}(\sum_{i=0}^{4} \frac{1}{\mu_i} X_i^3) \) has Hessian equal to
\[ H = \mu_0 X_1 X_2 X_3 X_4 + \mu_1 X_0 X_2 X_3 X_4 + \mu_2 X_0 X_1 X_3 X_4 + \mu_3 X_0 X_1 X_2 X_4 + \mu_4 X_0 X_1 X_2 X_3. \]
Clebsch showed \cite{Sal82} that the generic cubic surface is isomorphic to an essentially unique cubic surface of this (pentahedral) form, but this result is largely irrelevant to our work here, so simply recall the result, and do not pursue it further.

So a Hessian in this family contains ten lines and ten double points, such that each line passes through three of the double points and each double point lies on three of the lines. We now give one result each about the double points and the lines of a Hessian quartic $H$.

**Proposition 2.2.** Projection away from the node $p_{012} = \mathbb{V}(X_0, X_1, X_2)$ gives a rational map from $\mathbb{V}(H)$ to $\mathbb{P}^2$, which is generically 2-to-1. The branch locus of this map is the union of two cubic curves in $\mathbb{P}^2$, with equations

\[ (sX_0X_1X_2 + (\mu_0X_1X_2 + \mu_1X_0X_2 + \mu_2X_0X_1)(-X_0 - X_1 - X_2)) \]
\[ (\bar{s}X_0X_1X_2 + (\mu_0X_1X_2 + \mu_1X_0X_2 + \mu_2X_0X_1)(-X_0 - X_1 - X_2)) \]

where $s$ and $\bar{s}$ are the roots of

\[ s^2 - 2(\mu_3 + \mu_4)s + (\mu_3 - \mu_4)^2 = 0. \]

**Proof.** The first statement follows because a generic line through $p_{012}$ meets $\mathbb{V}(H)$ twice there, and at two other points. The second statement follows by computing the discriminant of $H$, viewed as a quadric in $X_3$. \hfill \square

Observe that the two cubics given are tangent to the quadric $\mu_0X_1X_2+\mu_1X_0X_2+\mu_2X_0X_1 = 0$ at the three points $\mathbb{V}(X_i, X_j)_{0 \leq i < j \leq 2}$, and so meet each other to order two there. These points are the images of the lines in $H$ through $p_{012}$. The cubic curves have their remaining three intersections transverse, all along the line $X_0 + X_1 + X_2 = 0$. This line is the image of the line $\ell_{34}$. We next prove a result about this line.

**Proposition 2.3.** The plane $\mu_4X_3 + \mu_3X_4 = 0$ is tangent to $H$ at every point of the line $\ell_{34}$.

**Proof.** This is immediate from the equation of the surface. \hfill \square

Observe that the intersection of this plane with the surface then consists of the line $\ell_{34}$, counted twice, and a conic with equation $\mu_0X_1X_2+\mu_1X_0X_2+\mu_2X_0X_1 = 0$, the same conic referred to above.

Finally, observe that if a cubic surface $C$ has a node at a point $p$, then its Hessian is also nodal at $p$, with the same tangent cone. Conversely, if a Hessian in our four-parameter family acquires a node other than the ten coordinate points, the corresponding cubic surface also acquires a node. So, if we restrict our attention to Hessians $H$ of smooth cubic surfaces, we may assume that $H$ contains only ten nodes, and that the discriminant sextics described above are smooth away from the six images of the nodes.

3. **The geometry of Kummer surfaces**

For this section, we will largely follow the development in \cite{GH94}, with one exception. Since we have already made use of subscripted numbers for our cubic surface in pentahedral form, we will begin with six distinct points labelled $a, b, c, d, e, f \in \mathbb{P}^1$. So begin with these points, and let $B \to \mathbb{P}^1$ be the genus 2 curve that is the double cover branched over these six points. We will also label
the ramification points in $B$ by the letters $a \ldots f$. Then the Jacobian of $B$ is an abelian surface $A$, with 16 two-torsion points, and these correspond to the divisors

\[ 0, \quad \{ b - a, c - a, \ldots, f - a \}, \quad \{ b - 2a, \ldots, e + f - 2a \}. \]

Recall that a theta divisor on $A$ is an image of the curve $B$ under a map $p \mapsto p - D$, where $D$ is some divisor of degree 1 on $B$. If $D$ is any “two-torsion” point, i.e., if $2D \sim 2a$, then the theta-divisor given will pass through 6 of the two-torsion points of $A$. We will refer to these 16 divisors on $A$ as tropes, and will label the trope corresponding to the divisor $D$ by the symbol $\Theta_D$. Note that they give 16 distinguished subsets of 6 two-torsion points. We will now define a different sort of set of six two-torsion points, called a Weber hexad. A Weber hexad is a set of 6 points of the two-torsion of $A$ such that 10 of the tropes each contain 3 of the points of the hexad, and the other 6 tropes each contain exactly one of the points of the hexad. For example, the six points

\[ 0, b + c - 2a, c + d - 2a, d + e - 2a, e + f - 2a, f + b - 2a \]

have this property: only 0 lies on $\Theta_a$, etc. On a given abelian surface, there are exactly 192 Weber hexads, which can be obtained from the one above (each one 60 times) by acting by translation by the two-torsion of $A$, and by acting on $a \ldots f$ with the group $S_6$.

Now, as discussed in the introduction, if we identify points on $A$ with their negatives, we obtain a surface $K$ with sixteen double points, the image of the two-torsion. We can desingularize these nodes by blowing up, or equivalently by blowing up the two-torsion points on $A$ before taking the quotient. Observe that if we blow up the six points of a Weber hexad, we will be left with a surface with 10 nodes, just like our Hessians above. In fact, one makes the following claim, first noticed by Hutchinson [Hut99].

**Proposition 3.1.** Let $a \ldots f, B, A, K$ be as above. Let

\[ W = \{ 0, b + c - 2a, c + d - 2a, d + e - 2a, e + f - 2a, f + b - 2a \} \subset A. \]

If one maps $A$ to projective space using the linear series $|4\Theta_a - 2W|$, one gets a map to a quartic surface $K_W \subset \mathbb{P}^3$, with 10 nodes, with the following properties. There exist 5 planes in $\mathbb{P}^3$ such that $K_W$ is nodal at the intersection of any three of these planes, and contains the line that is the intersection of any two of these planes. Further, the image of each of the point of $W$ is a conic in $K_W$.

**Proof.** One may check using homological criteria that the linear series $|4\Theta_a - 2W|$ has rank 4, and that all of its sections are even functions, so identify points with their negatives. So, the map is $\rho : A \rightarrow \mathbb{P}^3$, and factors through $K$. The points of $W$ are in the base locus, so get blown up. Now, the self-intersection $(4\Theta_a - 2W)^2 = 8$, which divided by two gives 4, so $K_W$ is a quartic surface, as stated. Also, if $E_0$ is the exceptional divisor over 0 $\in A$, then $E_0$ is part of the ramification locus of the map. So if $C_0$ is the image of $E_0$ in $\mathbb{P}^3$, and $\omega$ the hyperplane class in $\mathbb{P}^3$, then $C_0 \cdot \omega = E_0 \cdot (4\Theta_a - 2W) = 2$, and $C_0$ is a conic. Likewise the remaining points of $W$ also map to conics.
Now, for clarity we will introduce the notation $p_\lambda$ for the two-torsion point we have been calling $\lambda \in A$. Observe that
\[
(\Theta_b - p_0 - p_{b+c-2a} - p_{c+b-2a}) + \\
(\Theta_d - p_0 - p_{d+e-2a} - p_{c+d-2a}) + \\
(\Theta_{b+c-a} - p_{b+c-2a} - p_{d+e-2a} - p_{c+d-2a}) + \\
(\Theta_{c+d-a} - p_{c+d-2a} - p_{c+a-2a} - p_{d+e-2a}) = 4\Theta_a - 2W.
\]
So these four tropes have coplanar image, and are all lines. Letting the group \(\mathbb{Z}/5\mathbb{Z}\) act by \((bcdef)\), we get five such planes, and the result.

Hutchinson largely ignores the conics coming from \(W\), because his purpose is to study low-degree curves that arise on every Hessian quartic, not solely on the Kummer surfaces. In the next section, we will take the opposite approach, and study those Hessian surfaces that do contain conics like these, and find that this extra class of curves is enough to make a surface Kummer.

3.1. Labelling. Before we proceed to our main results, we pause here for a discussion of labelling. To conform with our names for the 5 planes in the discussion of the Hessian, we will let
\[
P_0 = \mathbb{V}(X_0) = \text{span}(\rho(p_b), \rho(p_c), \rho(p_d)), \\
P_1 = \mathbb{V}(X_1) = \text{span}(\rho(p_c), \rho(p_d), \rho(p_e)), \\
P_2 = \mathbb{V}(X_2) = \text{span}(\rho(p_d), \rho(p_e), \rho(p_f)), \\
P_3 = \mathbb{V}(X_3) = \text{span}(\rho(p_e), \rho(p_f), \rho(p_b)), \\
P_4 = \mathbb{V}(X_4) = \text{span}(\rho(p_f), \rho(p_b), \rho(p_c)).
\]
Then we may label the lines on \(K_W\) as \(\ell_{ij}\) as before, and obtain another set of names for the ten nodes. However, the real interest lies in what we can say about the conics. Observe that the conic \(C_0\) meets the tropes
\[
\Theta_d, \Theta_f, \Theta_c, \Theta_e, \Theta_b,
\]
that is, the lines
\[
\ell_{02}, \ell_{24}, \ell_{41}, \ell_{13}, \ell_{30}.
\]
So, we would do well to associate to \(C_0\) the cyclic ordering \((02413)\). Similarly, one finds that the other exceptional divisors should be assigned the labels
\[
C_{b+c-a} \sim (03214), \\
C_{c+d-a} \sim (01432), \\
C_{d+c-a} \sim (04312), \\
C_{e+f-a} \sim (01324), \\
C_{f+b-a} \sim (03421).
\]
This accounts for 6 of the 12 cyclic orders on five letters. The astute reader will ask to what do the other cyclic orders correspond. The astute reader will also have noticed that a conic is a plane curve, and since a plane meets a quartic surface in a degree 4 curve, each conic on \(K_W\) must be coplanar with a residual conic on \(K_W\). The answer, of course, is that we should label these residual conics with the complementary orderings.
Now take one of these conics, say the one labelled \((ijklm)\), and consider how it meets the lines on \(K_W\) through a node \(p_{rst}\). If \(rst\) are consecutive letters in the cyclic order \((ijklm)\), then the conic will meet the lines \(\ell_{rs}\) and \(\ell_{st}\), but not \(\ell_{rt}\). If the letters \(rst\) are not consecutive in \((ijklm)\), for example, \(ijl\), then the conic will only meet the one line \(\ell_{ij}\), while its residual will meet the other two lines.

In summary, the plane containing \(C_{(ijklm)}\) and \(C_{(ikmjl)}\) corresponds to one of the 6 subgroups of order 5 in \(S_5\). The elements of order 5 are all conjugate under the action of \(S_5\), but break into two orbits under the action of \(A_5\), with each element conjugate to its inverse. In one of these \(A_5\)-orbits we get the 6 cyclic orders we assigned to the exceptional divisors in \(K_W\), while in the other \(A_5\)-orbit we get the labels of the residual conics.

4. Conics on Hessian quartic surfaces

Now, we return to the question of when a Hessian quartic surface contains “extra” conics. We observed in section 2 that every Hessian quartic surface contains 10 conics, each lying in the tangent plane to one of the ten lines. We saw that the conic in the plane of \(\ell_{34}\) met all three of the lines through the node \(p_{012}\). In the last section, we saw that in a Kummer Hessian surface, there existed twelve other conics, not coplanar with any node, and that for each node, each of these conics met exactly one or two of the lines through that node. So, we assume that we have a surface \(H\) in the four-parameter family we have been studying, that \(H\) contains a conic \(C\) in its smooth locus such that \(C\) meets the lines \(\ell_{01}\) and \(\ell_{02}\), but not \(\ell_{12}\).

We will find that this condition is necessary and sufficient for the Hessian to be Kummer, and so prove Theorem 1.1. We begin with the following lemma.

**Lemma 4.1.** Let \(H\) be the quartic surface

\[
\mathbb{V}(\sum_{i=0}^{4} \mu_i \prod_{j \neq i} X_j),
\]

and assume \(\prod_{i=0}^{4} \mu_i \neq 0\), so \(H\) is a Hessian, and that \(H\) has only the ten nodes it should, i.e., the cubic surface is smooth. Assume there exists a conic \(C\) contained in the smooth locus of \(H\) such that \(C\) meets the lines \(\ell_{01}\) and \(\ell_{02}\), but not \(\ell_{12}\). Then the \(\mu_i\) satisfy the following irreducible cubic form:

\[
\sum_{i=0}^{4} \mu_i^3 - \sum_{i \neq j} \mu_i^2 \mu_j + 2 \sum_{i \neq j \neq k} \mu_i \mu_j \mu_k = 0.
\]

**Proof.** To begin with, since \(C\) meets \(\ell_{01}\) at a smooth point, it must be tangent to the plane \(\mu_1 X_0 + \mu_0 X_1 = 0\), and likewise tangent to the plane \(\mu_2 X_0 + \mu_0 X_2 = 0\). Now, let \(\pi : K_W \to \mathbb{P}^2\) be the projection away from the point \(p_{012}\), and let \(Q = \pi(C)\), a conic. Our observations have placed four linear conditions on \(Q\), so \(Q\) must lie in the pencil

\[
(\mu_0 X_1 X_2 + \mu_1 X_0 X_2 + \mu_2 X_0 X_1) + \alpha X_0^2.
\]

Now our goal is to find which \(Q\) in this pencil can have a conic in its preimage. Since \(Q\) passes through the images of the lines \(\ell_{01}\) and \(\ell_{02}\), and with the right tangent direction, the pullback \(\pi^{-1}(Q)\) will always contain these lines, each counted twice. The rest of the preimage will then be a quartic curve, dominating \(Q\) and mapping 2-to-1 to it. This quartic will be branched over \(Q\) at four points, the remaining
intersections of $Q$ with the discriminant locus away from the lines through $p_{012}$. To have the preimage decompose, these four intersections must coincide in pairs.

So for each of the cubic curves making up the branch locus, we ask which elements of the pencil meet it a third time non-reducedly. So let

$$E_s = \mathbb{V}(sX_0X_1X_2 + (\mu_0X_1X_2 + \mu_1X_0X_2 + \mu_2X_0X_1)(-X_0 - X_1 - X_2)).$$

Since $H$ is assumed to have only the ten nodes, $E_s$ will be smooth, so an elliptic curve, and there will be exactly four elements of the pencil where the $g_2^3$ given by residual intersection with the conic branches. Two of these are uninteresting: we already know the preimage of the conic

$$\mu_0X_1X_2 + \mu_1X_0X_2 + \mu_2X_0X_1,$$

and the double line $X_0^2$ pulls back to the plane $P_0$. So we are left with a quadratic equation in $\alpha$ indicating which conics meet $E_s$ interestingly. This quadratic is

$$T(s, \alpha) = 4\mu_0\alpha^2 + [(s - \mu_0)^2 + (\mu_2 - \mu_1)^2 - 2(s + \mu_0)(\mu_2 + \mu_1)]\alpha + 4s\mu_1\mu_2.$$

We want for there to be a conic that meets both $E_s$ and $E_{\bar{s}}$ interestingly, so we take the resultant of $T(s, \alpha)$ and $T(\bar{s}, \alpha)$, to find for what values of $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4$ these quadratics have a common solution. We find that the resultant is

$$512\mu_0\mu_1\mu_2\mu_3\mu_4 \left[ \sum_{i=0}^{4} \mu_i^3 - \sum_{i \neq j} \mu_i^2\mu_j + 2 \sum_{i \neq j \neq k} \mu_i\mu_j\mu_k \right],$$

and since we are not interested in the cases where some $\mu_i = 0$, we keep only the last factor.

Observe that this form is symmetric in the five variables. Now, since this cuts out an irreducible threefold in the space of $\mu$s, we have almost proven Theorem 1.1 We deal with the remaining issues below. But while we have the quadratics $T(s, \alpha)$ and $T(\bar{s}, \alpha)$ in hand, we observe that their difference is linear in $\alpha$, and solving, we may write

$$\alpha = \frac{2\mu_1\mu_2}{\mu_0 + \mu_1 + \mu_2 - \mu_3 - \mu_4}.$$

We now return to the theorem, which we restate as follows:

**Theorem 4.2.** Let $H$ be the quartic surface

$$\mathbb{V}(\sum_{i=0}^{4} \mu_i \prod_{j \neq i} X_j),$$

and assume $\prod_{i=0}^{4} \mu_i \neq 0$, so $H$ is a Hessian, and that $H$ has only the ten nodes it should, i.e., the cubic surface is smooth. Assume the $\mu_i$ satisfy the cubic form

$$\sum_{i=0}^{4} \mu_i^3 - \sum_{i \neq j} \mu_i^2\mu_j + 2 \sum_{i \neq j \neq k} \mu_i\mu_j\mu_k = 0.$$

Then there exists a Kummer surface $K$ and a Weber hexad $W$ such that $H \cong K_W$.

**Proof.** We begin by showing that for any $H$ satisfying this hypothesis, there are only finitely many conics satisfying the hypothesis of the lemma. But this is easy, since at most two choices of $\alpha$ can give acceptable image conics $Q$, and each of these will have at most two preimages in $H$. In fact, we observe that since $s \neq \bar{s}$, only one choice of $\alpha$ will work.
Next, we show that from the existence of such a conic $C$, we can deduce the existence of eleven others meeting different subsets of the lines, as in section 3.1. We know that $C$ meets the lines $\ell_{01}$ and $\ell_{02}$. If we look at its intersection with the plane $P_2$, we see that it must hit another line, which we will assume is $\ell_{23}$. We then intersect it with the plane $P_4$, and conclude that it must hit $\ell_{14}$ and $\ell_{34}$. So we label our conic $C_{(01432)}$. As in the proof of the lemma, we may project away from $p_{012}$ and pull back, residuating $C_{(01432)}$ to a conic, which we will call $C_{(01342)}$, and we can verify that it meets the appropriate lines and so deserves that name. Similarly, we can project the Hessian away from $p_{014}$, $p_{134}$, $p_{234}$, and $p_{023}$, and obtain four other conics. Residuating each of these six conics in the plane containing them gives us our total of twelve. Using the observation that only two conics will meet, $\ell_{01}$ and $\ell_{02}$, but not $\ell_{12}$, and the image under $S_5$ of this fact, we can conclude that these are the only twelve interesting conics on $H$.

Now to prove the theorem, we observe that the conics in one $A_5$-orbit, say $C_{(02413)}$, $C_{(03214)}$, $C_{(01432)}$, $C_{(04312)}$, $C_{(01324)}$, $C_{(03421)}$, are all disjoint, and each had self-intersection $-2$. So we may blow these down to obtain a 16-nodal K3 surface, which is known to be Kummer.

Given this theorem, we may now restate, and prove, Corollary 1.2.

**Corollary 4.3.** Let $\mathbb{P}^{19}$ be the parameter space of cubic forms on $\mathbb{P}^3$. Let $X$ be the locus in $\mathbb{P}^{19}$ of cubic surfaces whose Hessians are isomorphic to blowups of Weber hexads on Kummer surfaces, embedded as in Theorem 1.1. Then $X$ is $SL(4)$-invariant, and the closure of $X$ is a divisor in $\mathbb{P}^{19}$. If we label the classical invariants as $I_8, I_{16}, I_{24}, I_{32}, I_{40}$, following [Hun96], then the polynomial on $\mathbb{P}^{19}$ given by

$$I_8 I_{24} + 8I_{32}$$

is irreducible, is degree 32, and vanishes on $X$.

**Proof.** Inside $\mathbb{P}^{19}$, consider the 4-plane $P$ of cubic forms

$$\sum_{i=0}^{4} \lambda_i X_i^3,$$

where as usual $X_0 + X_1 + X_2 + X_3 + X_4 = 0$. We have been concentrating our attention on the locus in this family where no $\lambda_i$ equals zero, and have been writing $\mu_i = \frac{1}{\lambda_i}$. If we pull back our cubic condition

$$\sum_{i=0}^{4} \mu_i^3 - \sum_{i \neq j} \mu_i^2 \mu_j + 2 \sum_{i \neq j \neq k} \mu_i \mu_j \mu_k = 0$$

in the $\mu_i$ variables to the 4-plane $P$, we see that we have described an open subset of a degree 12 threefold $T \subset P \subset \mathbb{P}^{19}$. The locus $X$ is the $SL(4)$-orbit of $T$, and since $T$ has finite stabilizer in $SL(4)$, the closure of $X$ is a divisor in $\mathbb{P}^{19}$.

We next look for those $SL(4)$-invariants on $\mathbb{P}^{19}$ which vanish on $X$. These must vanish on $T$, a degree 12 threefold inside a $\mathbb{P}^4$ in $\mathbb{P}^{19}$. The ring of invariants of cubic forms has no element of degree 12, but if we refer to [Hun96] or [Sal82] to recall how the invariants restrict to $T$, we will find that the invariant

$$I = I_8 I_{24} + 8I_{32}$$
vanishes on $T$. Indeed, this irreducible degree 32 invariant cuts out the closure of $X$ in $\mathbb{P}^{19}$, and $T$ restricts on $P$ to an irreducible degree 12 polynomial, multiplied by $(\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4)^4$.

5. Finding the correspondence

So given a Hessian quartic surface satisfying the hypothesis of Theorem 1.1, we know it to be $K_W$ for some choice of Kummer surface and Weber hexad, and the next question is to which genus 2 curve it corresponds and to which Weber hexad. We answer this by observing that the lines $\ell_{ij}$ correspond to tropes on the Kummer, and that the three nodes on a line and the right choice of three places where conics meet the line give six points on $\mathbb{P}^{1}$ which specify the genus 2 curve $B$. So our task is to explicitly find the six planes containing the conics.

To do this, we return to the pullback of the conic $Q$ of the last section. So for any Hessian $H$, not necessarily Kummer, let

$$\alpha = \frac{2\mu_1\mu_2}{\mu_0 + \mu_1 + \mu_2 - \mu_3 - \mu_4},$$

and let

$$Q = (\mu_0X_1X_2 + \mu_1X_0X_2 + \mu_2X_0X_1) + \alpha X_0^2,$$

a singular quadric surface. Then as above, the intersection of $Q$ with $H$ consists of two lines, each counted twice, and a quartic elliptic curve $F$. This quartic elliptic curve is the base locus of a pencil of quadrics,

$$\langle Q, \alpha\mu_0X_3X_4 + (\mu_1X_2 + \mu_2X_1 + \alpha X_0)(\mu_3X_4 + \mu_4X_3) \rangle.$$

Since we are looking, in the Kummer case, for an element of this pencil that decomposes into two planes, we begin by looking at the singular elements of the pencil. We find, inter alia, the following result.

**Proposition 5.1.** Let $\mu_0\mu_1\mu_2\mu_3\mu_4 \neq 0$, and assume

$$\alpha = \frac{2\mu_1\mu_2}{\mu_0 + \mu_1 + \mu_2 - \mu_3 - \mu_4}, \quad \beta = \frac{2\mu_3\mu_4}{\mu_0 + \mu_3 + \mu_4 - \mu_1 - \mu_2}$$

are finite. Then let

$$R = (\mu_1X_2 + \mu_2X_1)(\mu_3X_4 + \mu_4X_3) + \alpha(\mu_0X_3X_4 + \mu_3X_0X_4 + \mu_4X_0X_3) + \beta(\mu_0X_1X_2 + \mu_1X_0X_2 + \mu_2X_0X_1) + \alpha\beta X_0^2.$$

Then $R$ is always singular, i.e., of rank $\leq 3$, with singular point

$$[\mu_1 + \mu_2 - \mu_3 - \mu_4 : -\mu_1 : -\mu_2 : \mu_3 : \mu_4].$$

Further, $R$ has rank $\leq 2$, i.e., decomposes, exactly if

$$\sum_{i=0}^{4} \mu_i^3 - \sum_{i \neq j} \mu_i^2\mu_j + 2 \sum_{i \neq j \neq k} \mu_i\mu_j\mu_k = 0.$$

**Proof.** This can all be checked by computing the necessary determinants. 

Observe that $R$ is symmetric with respect to the notational involution

$$*_1 \leftrightarrow *_3,$$

$$*_2 \leftrightarrow *_4,$$

$$\alpha \leftrightarrow \beta,$$
and in the case that $H$ is Kummer, we know that $R$ breaks into the planes containing $C_{(03214)}$ and $C_{(01432)}$. Also, the proposition provides for us a point on the intersection of these planes. If let $S_5$ act on this proposition, so to speak, by relabelling the variables, we obtain points on each of the intersections of the six planes we are interested in. For example, the plane $P_{(03214)}$ contains the following five points:

\[\begin{align*}
&\left[\mu_1 + \mu_2 - \mu_3 - \mu_4 : -\mu_1 : -\mu_2 : \mu_3 : \mu_4\right] \\
&\left[-\mu_0 : \mu_0 + \mu_3 - \mu_2 - \mu_4 : \mu_2 : -\mu_3 : \mu_4\right] \\
&\left[\mu_0 : -\mu_1 : \mu_2 + \mu_4 - \mu_0 - \mu_2 : -\mu_4\right] \\
&\left[\mu_0 : -\mu_2 : -\mu_3 : \mu_2 + \mu_3 - \mu_0 - \mu_1\right] \\
&\left[-\mu_0 : \mu_1 : \mu_0 + \mu_4 - \mu_1 - \mu_3 : \mu_3 : -\mu_4\right]
\end{align*}\]

Taking minors of this matrix, we obtain an equation for the plane $P_{(03214)}$ with coefficients cubic in the $\mu_i$s, and likewise for the other five planes. More interestingly, we can find the intersections of the conics $C_{(01324)}$, $C_{(03421)}$, and $C_{(01432)}$ with the line $\ell_{01}$, that is to say, the locations of the points $p_{e+f-2a}$, $p_{f+b-2a}$, $p_{c+d-2a}$ on the trope $\Theta_{x+d-a}$. This gives the following theorem.

**Theorem 5.2.** If $\mu_0\mu_1\mu_2\mu_3\mu_4 \neq 0$ and

\[
\sum_{i=0}^{4} \mu_i^3 - \sum_{i \neq j} \mu_i^2 \mu_j + 2 \sum_{i \neq j \neq k} \mu_i \mu_j \mu_k = 0,
\]

and if the Hessian quartic surface $H$ given by

\[\mathbb{V}(\sum_{i=0}^{4} \mu_i \prod_{j \neq i} X_j)\]

has only ten nodes, then $H$ is Kummer. Specifically, let $B$ be the branched cover of $\mathbb{P}^1$ over

\begin{align*}
a &= \frac{\mu_1 + \mu_4 - \mu_0 - \mu_2 - \mu_3}{2\mu_3}, \\
b &= \frac{2\mu_2}{\mu_0 + \mu_4 - \mu_1 - \mu_2 - \mu_3}, \\
c &= 0, \\
d &= -1, \\
e &= \frac{\mu_0 + \mu_3 - \mu_1 - \mu_2 - \mu_4}{\mu_1 + \mu_2 - \mu_0 - \mu_3 - \mu_4}, \\
f &= \infty.
\end{align*}

Then these six points will be distinct, so $B$ will be a smooth genus 2 curve. Let $K$ be its Kummer surface, and let $W$ be the Weber hexad

\[\{0, b + c - 2a, c + d - 2a, d + e - 2a, e + f - 2a, f + b - 2a\} \subset K.\]

Then $H \cong K_W$. 
Conversely, if \( \{a, b, c = 0, d = -1, e, f = \infty\} \) are six distinct points on \( \mathbb{P}^1 \), and \( B \) is the genus 2 curve branched over those six points, and \( K \) and \( W \) are as usual, the surface \( K_W \) can be embedded as a Hessian, with equation

\[
H = \mathcal{V}(\mu_0X_1X_2X_3X_4+\mu_1X_0X_2X_3X_4+\mu_2X_0X_1X_3X_4+\mu_3X_0X_1X_2X_4+\mu_4X_0X_1X_2X_3),
\]

where the coefficients \( \mu_i \) are given by

\[
\begin{align*}
\mu_0 &= a(b + 1), \\
\mu_1 &= e(a + 1), \\
\mu_2 &= b(a - e), \\
\mu_3 &= e - b, \\
\mu_4 &= (a - b)(e + 1).
\end{align*}
\]

Proof. Given the previous theorem and the proposition, this reduces to a computation.

6. Suggestions for Further Research

As stated in the introduction, this exploration is by no means done. To begin with, there is the problem of finding 6 points in \( \mathbb{P}^2 \) to blow up to obtain the cubic surfaces associated to these Hessians. Igor Dolgachev has presented a candidate sextuple, but this author cannot see a good technique to answer his question.

Another intriguing line of research is broached by observing that among all cubic surfaces, there is a codimension one subfamily of singular cubic surfaces. On the moduli space, this divisor meets the Kummer divisor studied in this paper, and the two divisors are everywhere tangent along their intersection. However, the condition of smoothness of the cubic surface has only barely made its presence known in the results of this paper. A related question is brought up by asking what happens if we allow our genus 2 curve to degenerate.

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