The Schroedinger-Newton model as $N \to \infty$ limit of a $N$ color model.

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The generalization to $N$ colors of a recently proposed non unitary two color model for the gravitational interaction in non relativistic quantum mechanics is considered. The $N \to \infty$ limit is proven to be equivalent to the Schroedinger-Newton model, which, though sharing localization properties with the $N = 2$ model, cannot produce decoherence.

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In some recent papers [1–3] a model for gravitational interactions in nonrelativistic quantum mechanics was proposed. Starting from a gravity-free generic system, all matter degrees of freedom were duplicated and gravitational interactions were introduced between observable (green) and unobservable (red) degrees of freedom only. Once unobservable degrees of freedom are traced out, the ensuing non unitary dynamics includes both the traditional aspects of classical gravitational interactions and a form of fundamental decoherence, which may be relevant to the emergence of the classical behavior of macroscopic bodies. The model in fact treats on an equal footing mutual and self-interactions, which, by some authors, are possibly held responsible for wave function localization and/or reduction [4–7].

In particular, for bodies of ordinary density, there is a localization threshold at about $10^{11}$ proton masses and linear superpositions of two localized wave packets may decohere, either due to fast phase oscillations [1], or due to the slow spreading of the center of mass
probability density. Above the localization threshold, the latter gives rise, even starting from a single localized wave packet, to the emergence of delocalized ensembles of localized wave packets, whose entropy is slowly growing in time \[2\]. While the model was originally presented in terms of instantaneous action at a distance gravitational interactions, in Ref. \[3\] it was shown that a Hubbard-Stratonovich transformation, together with the replacement of the instantaneous interaction by a retarded one, leads to a field-theoretic interpretation. The presence of both a positive and a negative energy scalar field implies the complete cancellation among all Feynman diagrams containing divergent self-energy or vertex insertions, making the theory finite, without any mass or coupling constant renormalization \[3\].

As to the relationship between our model and Einstein gravity, the viewpoint we adhered to is that the latter may be presumably only a large scale manifestation of a fundamental theory that may well be out of reach, and whose possible non-relativistic limit is the object of our proposal \[1\]. More specifically it was hinted that the Einstein theory could arise, unlike, for instance, classical electrodynamics, not as a result of taking expectation values with respect to a pure physical state, but rather, as an effective long distance theory like hydrodynamics, from a statistical average, or equivalently by tracing out unobservable degrees of freedom from a pure metastate \[3\]. A similar point of view about gravity as "...an emergent phenomenon, in the same sense that fluid dynamics emerges from molecular physics..." is also suggested by analog models of general relativity, based on condensed matter physics and leading to consider scalar fields as possible starting points \[8\].

It was also suggested \[3\] that a suitable mean field approximation gives the Schroedinger-Newton (SN) model \[9,10\], which is a nonlinear generalization of the Schroedinger equation, where gravitational self-interactions give rise to stationary localized wave functions \[11-14\], contrary to the spreading mixed states of our model \[2\]. More generally the SN approximation cannot produce decoherence, so that, in order to estimate localization times, due to the lack of a full theory of quantum gravity, one has to invoke purely dimensional arguments \[7,14\]. On the contrary, according to the exact model, time evolution of an unlocalized state, represented as a linear superposition of many localized wave packets, can lead explicitly to
vanishing coherences and then to an ensemble of such localized states [3].

The aim of the present paper is to give a well defined procedure for passing from the original model to the SN approximation, replacing the heuristic suggestion of Ref. [3]. In doing that, the original model with just two colors, green and red, is considered as the simplest representative, for $N = 2$, of a class of $N$-color models, whereas the SN model is obtained as the $N \to \infty$ limit. While clarifying the relationship between our proposal and the SN model, this result gives in principle the possibility to develop $1/N$ expansions in analogy to what is done in ordinary condensed matter physics [15,16]. In particular, while the $N \to \infty$ limit does not involve decoherence, but only localization, $1/N$ expansions may provide approximate schemes for the evaluation of decoherence.

To be specific, following Ref. [1], let $H[\psi^\dagger, \psi]$ denote the second quantized non-relativistic Hamiltonian of a finite number of particle species, like electrons, nuclei, ions, atoms and/or molecules, according to the energy scale. For notational simplicity $\psi^\dagger, \psi$ denote the whole set $\psi^\dagger_j(x), \psi_j(x)$ of creation-annihilation operators, i.e. one couple per particle species and spin component. This Hamiltonian includes the usual electromagnetic interactions accounted for in atomic and molecular physics. To incorporate gravitational interactions including self-interactions, we introduce a color quantum number $\alpha = 1, 2, ..., N$, in such a way that each couple $\psi^\dagger_j(x), \psi_j(x)$ is replaced by $N$ couples $\psi^\dagger_{j,\alpha}(x), \psi_{j,\alpha}(x)$ of creation-annihilation operators. The overall Hamiltonian, including gravitational interactions and acting on the tensor product $\bigotimes_\alpha F_\alpha$ of the Fock spaces of the $\psi_\alpha$ operators, is then given by

$$H_G = \sum_{\alpha=1}^N H[\psi_{\alpha}^\dagger, \psi_\alpha] - \frac{G}{N - 1} \sum_{j,k} m_j m_k \sum_{\alpha<\beta} \int dxdy \frac{\psi_{j,\alpha}^\dagger(x)\psi_{j,\alpha}(x)\psi_{k,\beta}^\dagger(y)\psi_{k,\beta}(y)}{|x - y|},$$

where here and henceforth Greek indices denote color indices, $\psi_\alpha \equiv (\psi_{1,\alpha}, \psi_{2,\alpha}, ..., \psi_{N,\alpha})$, and $m_i$ denotes the mass of the $i$-th particle species, while $G$ is the gravitational constant.

While the $\psi_\alpha$ operators obey the same statistics as the original operators $\psi$, we take advantage of the arbitrariness pertaining to distinct operators and, for simplicity, we choose them commuting with one another: $\alpha \neq \beta \Rightarrow [\psi_\alpha, \psi_\beta] = [\psi_\alpha, \psi_\beta]^\dagger = 0$. The metaparticle state space $S$ is identified with the subspace of $\bigotimes_\alpha F_\alpha$ including the metastates obtained
from the vacuum \(|0\rangle\rangle = \bigotimes_\alpha |0\rangle_\alpha\) by applying operators built in terms of the products \(\prod_{\alpha=1}^{N} \psi_{j_1,\alpha}^\dagger(x_\alpha)\) and symmetrical with respect to arbitrary permutations of the color indices, which, as a consequence, for each particle species, have the same number of metaparticles of each color. This is a consistent definition since the time evolution generated by the overall Hamiltonian is a group of (unitary) endomorphisms of \(S\). If we prepare a pure \(n\)-particle state, represented in the original setting - excluding gravitational interactions - by

\[ |g\rangle = \int d^n x g(x_1, x_2, ..., x_n) \psi_{j_1}^\dagger(x_1) \psi_{j_2}^\dagger(x_2) ... \psi_{j_n}^\dagger(x_n) |0\rangle, \tag{2} \]

its representative in \(S\) is given by the metastate

\[ ||g^{\otimes N}\rangle\rangle = \prod_\alpha \left[ \int d^n x g(x_1, x_2, ..., x_n) \psi_{j_1,\alpha}^\dagger(x_1) \psi_{j_2,\alpha}^\dagger(x_2) ... \psi_{j_n,\alpha}^\dagger(x_n) \right] |0\rangle\rangle. \tag{3} \]

As for the physical algebra, it is identified with the operator algebra of say the \(\alpha = 1\) metaworld. In view of this, expectation values can be evaluated by preliminarily tracing out the unobservable operators, namely with \(\alpha > 1\), and then taking the average of an operator belonging to the physical algebra. It should be made clear that we are not prescribing an ad hoc restriction of the observable algebra. Once the constraint restricting \(\bigotimes_\alpha F_\alpha\) to \(S\) is taken into account, in order to get an effective gravitational interaction between particles of one and the same color \([\|]\), the resulting state space does not contain states that can distinguish between operators of different color. The only way to accommodate a faithful representation of the physical algebra within the metastate space is to restrict the algebra.

While we are talking trivialities as to an initial metastate like in Eq. \(\|g\rangle\rangle\), that is not the case in the course of time, since the overall Hamiltonian produces entanglement between metaworlds of different color, leading, once unobservable operators are traced out, to mixed states of the physical algebra. It was shown in Ref. \([\|]\) that, for \(N = 2\), the ensuing non-unitary evolution induces both an effective interaction reproducing gravitation, and wave function localization. The proof of the former property stays conceptually unchanged for arbitrary \(N\) and is then omitted here. A peculiar feature of the model is that it cannot be obtained by quantizing its naive classical version, since the classical states corresponding to
the constraint in $\otimes \alpha F_\alpha$, selecting the metastate space $S$, have partners of all colors sitting in one and the same space point and then a divergent gravitational energy. While it is usual that, in passing from the classical to the quantum description, self-energy divergences are mitigated, in this instance we pass from a completely meaningless classical theory to a quite divergence free one. This is more transparent in a field theoretic description [3].

Let us adopt here an interaction representation, where the free Hamiltonian is identified with $\sum_{\alpha=1}^{N} H[\psi^\dagger_\alpha, \psi_\alpha]$ and the time evolution of an initially unentangled metastate $\langle \tilde{\Phi}(0) | \rangle = \otimes_{\alpha=1}^{N} | \Phi(0) \rangle_\alpha$ is represented by

$$\langle \tilde{\Phi}(t) | \rangle = T \exp \left[ \frac{iG}{(N-1)\hbar} m^2 \sum_{\alpha<\beta} \int dt \int dxdy \frac{\psi^\dagger_\alpha(x,t)\psi_\alpha(x,t)\psi^\dagger_\beta(y,t)\psi_\beta(y,t)}{|x-y|} \right] \langle \tilde{\Phi}(0) | \rangle \equiv U(t) \langle \tilde{\Phi}(0) | \rangle,$$

where for notational simplicity we are referring to just one particle species. Then, by using a Stratonovich-Hubbard transformation [17], we can rewrite $U(t)$ as

$$U(t) = \int D[\varphi] \prod_\alpha D[\varphi_\alpha] \exp \frac{ic^2}{2\hbar} \int dt dx \left[ \varphi(x,t) \nabla^2 \varphi(x,t) - \sum_\alpha \varphi_\alpha(x,t) \nabla^2 \varphi_\alpha(x,t) \right] T \exp \left[-\frac{2mc}{\hbar} \sqrt{\frac{\pi G}{N-1}} \sum_\alpha \int dt dx \left[ \varphi(x,t) + \varphi_\alpha(x,t) \right] \psi^\dagger_\alpha(x,t)\psi_\alpha(x,t) \right],$$

i.e. as a functional integral over the auxiliary real scalar fields $\varphi, \varphi_1, \varphi_2, ..., \varphi_N$, which, though less economical, for $N = 2$, than that in Ref. [3], is the simplest one for a generic $N$.

Just as in Ref. [3], a physical interpretation of this result can be given, by considering the minimal variant of the Newton interaction in Eq. (4) aiming at avoiding instantaneous action at a distance, namely replacing $-1/|x-y|$ by the Feynman propagator $4\pi\Box^{-1} \equiv 4\pi (-\partial_t^2/c^2 + \nabla^2)^{-1}$. Then the analog of Eq. (5) holds with the d’Alembertian $\Box$ replacing the Laplacian $\nabla^2$ and the ensuing expression can be read as the mixed path integral and operator expression for the evolution operator corresponding to the field Hamiltonian

$$H_{\text{Field}} = \sum_{\alpha=1}^{N} H[\psi^\dagger_\alpha, \psi_\alpha] + \frac{1}{2} \int dx \left[ \pi^2 + c^2 |\nabla \varphi|^2 - \sum_{\alpha=1}^{N} \left( \pi^2_\alpha + c^2 |\nabla \varphi_\alpha|^2 \right) \right] + 2mc \sqrt{\frac{\pi G}{N-1}} \int dx \sum_{\alpha=1}^{N} \left\{ [\varphi + \varphi_\alpha] \psi^\dagger_\alpha \psi_\alpha \right\},$$

(6)
where $\pi = \dot{\phi}$ and $\pi_\alpha = \dot{\phi}_\alpha$ respectively denote the conjugate fields of $\phi$ and $\phi_\alpha$ and all fields are quantum operators. This can be read in analogy with nonrelativistic quantum electrodynamics, where a relativistic field is coupled with nonrelativistic matter, while the procedure to obtain the corresponding action at a distance theory by integrating out the $\phi$ fields is the analog of the Feynman’s elimination of electromagnetic field variables [18].

The resulting theory, containing the negative energy fields $\phi_\alpha$, has the attractive feature of being divergence free, at least in the non-relativistic limit, where Feynman graphs with virtual particle-antiparticle pairs can be omitted. To be specific, it does not require the infinite self-energy subtraction needed for instance in electrodynamics on evaluating the Lamb shift, or the coupling constant renormalization [19]. Here of course we refer to the covariant perturbative formalism applied to our model, where matter fields are replaced by their relativistic counterparts and the non-relativistic character of the model is reflected in the mass density being considered as a scalar coupled with the scalar fields by Yukawa-like interactions. In fact there is a complete cancellation, for a fixed color index $\alpha$, among all Feynman diagrams containing only $\psi_\alpha$ and internal $\phi$ and $\phi_\alpha$ lines, owing to the difference in sign between the $\phi$ and the $\phi_\alpha$ free propagators. This state of affairs of course is the field theoretic counterpart of the absence of direct $\psi_\alpha - \psi_\alpha$ interactions in the theory obtained by integrating out the $\phi$ operators, whose presence would otherwise require the infinite self-energy subtraction corresponding to normal ordering.

Going back to our metastate (4), the partial trace

$$M(t) \equiv Tr \left| \left| \Phi(t) \right| \right| <\{\Phi(t)\} > = \sum_{k_2, k_3, \ldots, k_N} \langle\langle k_2, k_3, \ldots, k_N | \Phi(t) \rangle \rangle \langle\langle \Phi(t) | \left| k_2, k_3, \ldots, k_N \right\rangle \rangle,$$

(7)

gives the corresponding physical state, where $\left| \left| k_2, k_3, \ldots, k_N \right\rangle \rangle \equiv \bigotimes_{\alpha=2}^N |k_\alpha\rangle_\alpha$ and $(|k\rangle_\alpha)_{k=1,2,\ldots}$ denotes an orthonormal basis in the Fock space $F_\alpha$. Then, by using Eq. (5), we can write

$$M(t) =$$

$$\int D[\phi] \prod_\alpha D[\phi_\alpha] D[\phi'] \prod_\alpha D[\phi'_\alpha] \exp \frac{i\epsilon^2}{2\hbar} \int dt dx \left[ \phi \nabla^2 \phi - \sum_\alpha \phi_\alpha \nabla^2 \phi_\alpha - \phi' \nabla^2 \phi' + \sum_\alpha \phi'_\alpha \nabla^2 \phi'_\alpha \right].$$

6
\[
\begin{align*}
&\left[\bigotimes_{\alpha=2}^{N} \alpha \langle \Phi(0) \rangle \right] T^{-1} \exp \left[ \frac{2mc}{\hbar} \sqrt{\frac{\pi G}{N-1}} \sum_{\alpha=2}^{N} \int dt dx \left[ \varphi'(x,t) + \varphi'_\alpha(x,t) \right] \psi'_\alpha(x,t) \psi_\alpha(x,t) \right] \\
&T \exp \left[ -\frac{2mc}{\hbar} \sqrt{\frac{\pi G}{N-1}} \sum_{\alpha=2}^{N} \int dt dx \left[ \varphi(x,t) + \varphi_\alpha(x,t) \right] \psi_\alpha(x,t) \psi'_\alpha(x,t) \right] \left[ \bigotimes_{\alpha=2}^{N} \alpha \langle \Phi(0) \rangle \right] \\
&1 \langle \Phi(0) \rangle T^{-1} \exp \left[ \frac{2mc}{\hbar} \sqrt{\frac{\pi G}{N-1}} \sum_{\alpha=2}^{N} \int dt dx \left[ \varphi'(x,t) + \varphi'_1(x,t) \right] \psi'_1(x,t) \psi_1(x,t) \right].
\end{align*}
\]

To study the \( N \to \infty \) limit of this expression, replace the products \( \varphi \psi'_1 \psi_1 \) and \( \varphi' \psi'_1 \psi_1 \) respectively with \( \tilde{\varphi} \psi'_1 \psi_1 \) and \( \tilde{\varphi}' \psi'_1 \psi_1 \), inserting simultaneously the \( \delta \)-functionals \( \delta[\varphi - \tilde{\varphi}] = \int D[g] \exp(i\frac{2mc}{\hbar} \sqrt{\frac{\pi G}{N-1}} \int dt dx g[\tilde{\varphi} - \varphi]), \delta[\varphi' - \tilde{\varphi}'] = \int D[g'] \exp(i\frac{2mc}{\hbar} \sqrt{\frac{\pi G}{N-1}} \int dt dx g'[\tilde{\varphi}' - \varphi']). \)

Then we can perform functional integration on \( \varphi, \varphi' \) and \( \varphi_\alpha, \varphi'_\alpha \) for \( \alpha = 2, 3, \ldots, N \) and get

\[
M(t) =
\]
space integrals have vanishing mean square deviations (as can be checked at any perturbative order), though their commutators with the other terms are $O(N^0)$, ($\hat{\varphi}, \hat{\varphi}'$ integrations imply $g, g' \sim \psi^\dagger \psi$). The latter property forces us, when defining the expression explicitly as the limit as $dt \to 0$ of time ordered products of time evolution operators during time intervals $dt$, to keep the factors depending on $g$ and $g'$ in the right place, while the former one allows for the replacement of the exponential of these terms with the exponential of their average in the metastate at that time instant. As a result, if we remember that

$$\langle \psi^\dagger_\alpha(x) \psi_\alpha(x) \rangle = \langle \psi^\dagger_1(x) \psi_1(x) \rangle,$$

we have

$$M(t) \overset{N \to \infty}{\sim}$$

$$\int D[\varphi] D[\varphi'] D[g] D[g'] \int D[\varphi_1] D[\varphi'_1] \exp \frac{i c^2}{2 \hbar} \int dt dx \left[ -\varphi_1 \nabla^2 \varphi_1 + \varphi'_1 \nabla^2 \varphi'_1 \right]$$

$$T \exp \frac{-2mc \pi G}{N} \int \exp \left[ \left[ \varphi(x) + \varphi_1(x) \right] \psi^\dagger_1(x) \psi_1(x) - g(x) \left[ \varphi(x) + \frac{m}{2c} \sqrt{\frac{G N}{\pi}} \int dy \frac{\langle \psi^\dagger_1(y) \psi_1(y) \rangle}{|x-y|} \right] \right] |\Phi(0)\rangle_1$$

$$\langle \Phi(0) | T^{-1} \exp \left[ \frac{2mc}{\hbar} \pi G \right] \int \exp \left[ \left[ \varphi'(x) + \varphi'_1(x) \right] \psi^\dagger_1(x) \psi_1(x) - g'(x) \left[ \varphi'(x) + \frac{m}{2c} \sqrt{\frac{G N}{\pi}} \int dy \frac{\langle \psi^\dagger_1(y) \psi_1(y) \rangle}{|x-y|} \right] \right] \rangle,$$

where $\langle \psi^\dagger_1(y) \psi_1(y) \rangle \equiv \langle \tilde{\Phi}(t) | \psi^\dagger_1(y, t) \psi_1(y, t) | \tilde{\Phi}(t) \rangle$. By integrating over $g, g', \varphi, \varphi'$, we get

$$M(t) \overset{N \to \infty}{\sim}$$

$$\int D[\varphi_1] D[\varphi'_1] \exp \frac{i c^2}{2 \hbar} \int dt dx \left[ -\varphi_1 \nabla^2 \varphi_1 + \varphi'_1 \nabla^2 \varphi'_1 \right]$$

$$T \exp \left[ \frac{i G}{\hbar} m^2 \int dt \int dx dy \frac{\psi^\dagger(x) \psi(x) \langle \psi^\dagger(y) \psi(y) \rangle}{|x-y|} - \frac{i 2mc}{\hbar} \sqrt{\frac{\pi G}{N}} \int dt dx \varphi_1(x) \psi^\dagger(x) \psi(x) \right] |\Phi(0)\rangle$$

$$\langle \Phi(0) | T^{-1} \exp \left[ \frac{-i G}{\hbar} m^2 \int dt \int dx dy \frac{\psi^\dagger(x) \psi(x) \langle \psi^\dagger(y) \psi(y) \rangle}{|x-y|} + \frac{i 2mc}{\hbar} \sqrt{\frac{\pi G}{N}} \int dt dx \varphi'_1(x) \psi^\dagger(x) \psi(x) \right]$$

omitting the by now irrelevant index in $\psi_1$, and finally, after integrating out $\varphi_1, \varphi'_1$. 
lim_{N \to \infty} M(t) \equiv |\Phi(t)\rangle \langle \Phi(t)| = \nabla \exp \left[ \frac{iG}{\hbar} m^2 \sum_{\alpha < \beta} \int dt \int dxdy \frac{\psi^\dagger(x, t) \psi(x, t) \langle \Phi(t) | \psi^\dagger(y, t) \psi(y, t) | \Phi(t) \rangle}{|x - y|} \right] |\Phi(0)\rangle

\langle \Phi(0)| T^{-1} \exp \left[ \frac{-iG}{\hbar} m^2 \sum_{\alpha < \beta} \int dt \int dxdy \frac{\psi^\dagger(x, t) \psi(x, t) \langle \Phi(t) | \psi^\dagger(y, t) \psi(y, t) | \Phi(t) \rangle}{|x - y|} \right] \rangle, \quad (11)

where the normalization is automatically correct, as the resulting dynamics, though non-linear, is unitary. It should be remarked that, to make the derivation more rigorous, the Newton potential should be replaced with a regularized potential like $1/(|x - y| + \lambda)$ with $\lambda > 0$, and the Laplacian with the corresponding inverse. One should first take the limit $N \to \infty$, then take $dt \to 0$ and remove the regularization, $\lambda \to 0$.

Eq. (11) coincides with the time evolution of the SN model in the interaction representation, which is then the $N \to \infty$ limit of our model. Just as in condensed matter physics, this limit suppresses fluctuations (quantum fluctuations here) and preserves mean field features only. In the present case the $N \to \infty$ limit keeps the presence of localized states, but wipes out the non unitary evolution and then the ability of generating decoherence.

In conclusion the present construction is the first derivation of the SN model from a well defined quantum model producing both classical gravitational interactions, localization and decoherence. In fact the model was usually presented, up to now, as some sort of mean field approximation of a not yet well specified theory incorporating self-interactions.

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