Repdigits in $k$–generalized Pell sequence

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Abstract

Let $k \geq 2$ and let $(P_n^{(k)})_{n \geq 2-k}$ be $k$-generalized Pell sequence defined by

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \ldots + P_{n-k}^{(k)}$$

for $n \geq 2$ with initial conditions

$$P_{-2}^{(k)} = P_{-3}^{(k)} = \ldots = P_{-1}^{(k)} = P_0^{(k)} = 0, P_1^{(k)} = 1.$$

In this paper, we deal with the Diophantine equation

$$P_n^{(k)} = d \left( \frac{10^m - 1}{9} \right)$$

in positive integers $n, m, d$ with $m \geq 2$ and $1 \leq d \leq 9$. We show that repdigits in the sequence $(P_n^{(k)})_{n \geq 2-k}$, which have at least two digits, are the numbers $P_3^{(3)} = 33$ and $P_6^{(4)} = 88$.

Keywords: Repdigit, Fibonacci and Lucas numbers, Exponential Diophantine equations, Linear forms in logarithms; Baker’s method

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1 Introduction

Let $k \geq 2$ be an integer. Let the linear recurrence sequence $(G_n^{(k)})_{n \geq 2-k}$ of order $k$ define by

$$G_n^{(k)} = rG_{n-1}^{(k)} + G_{n-2}^{(k)} + \ldots + G_{n-k}^{(k)}$$

(1)
for \( n \geq 2 \) with the initial conditions \( G^{(k)}_{n(k-2)} = G^{(k)}_{n(k-3)} = \cdots = G^{(k)}_{k-1} = 0, \ G^{(k)}_0 = a, \) and \( G^{(k)}_1 = b \). For \( (a, b, r) = (0, 1, 1) \) and \( (a, b, r) = (2, 1, 1) \), the sequence \( \left( G^{(k)}_n \right)_{n \geq 2-k} \) is called \( k \)-generalized Fibonacci sequence \( \left( F^{(k)}_n \right)_{n \geq 2-k} \) and \( k \)-generalized Lucas sequence \( \left( L^{(k)}_n \right)_{n \geq 2-k} \) respectively (see [2, 3]). For \( (a, b, r) = (0, 1, 2) \) and \( (a, b, r) = (2, 2, 2) \), the sequence \( \left( G^{(k)}_n \right)_{n \geq 2-k} \) is called \( k \)-generalized Pell sequence \( \left( P^{(k)}_n \right)_{n \geq 2-k} \) and \( k \)-generalized Pell-Lucas sequence \( \left( Q^{(k)}_n \right)_{n \geq 2-k} \), respectively (see [10]). The terms of these sequences are called \( k \)-generalized Fibonacci numbers, \( k \)-generalized Lucas numbers, \( k \)-generalized Pell numbers and \( k \)-generalized Pell-Lucas numbers, respectively. When \( k = 2 \), we have Fibonacci, Lucas, Pell and Pell-Lucas sequences, \( \left( F_n \right)_{n \geq 0}, \left( L_n \right)_{n \geq 0}, \left( P_n \right)_{n \geq 0}, \) and \( \left( Q_n \right)_{n \geq 0} \) respectively.

A repdigit is a positive integer whose digits are all equal. Recently, some mathematicians have investigated the repdigits in the above sequences for \( k = 2 \) or general \( k \). In [11], Luca determined that the largest repdigits in the sequences \( \left( F_n^{(2)} \right)_{n \geq 0} \) and \( \left( L_n^{(2)} \right)_{n \geq 0} \) are \( F^{(2)}_{10} = 55 \) and \( L^{(2)}_5 = 11 \). In [9], the authors have found all repdigits in the sequences \( \left( P_n^{(2)} \right)_{n \geq 0} \) and \( \left( Q_n^{(2)} \right)_{n \geq 0} \). Here, they showed that the largest repdigits in these sequences are \( P^{(2)}_3 = 5 \) and \( Q^{(2)}_2 = 6 \). In [12], Marques proved that the largest repdigits in the sequence \( \left( F_n^{(3)} \right)_{n \geq -1} \) are \( F^{(3)}_8 = 44 \). Besides, for general case of \( k \), in [5], Bravo and Luca handled the Diophantine equation

\[
F^{(k)}_n = d \left( \frac{10^m - 1}{9} \right)
\]

and showed that this equation has the solutions \( (n, k, d, m) = (10, 2, 5, 2), (8, 3, 4, 2) \) in positive integers \( n, m, k, d \) with \( k \geq 2, \ 1 \leq d \leq 9 \) and \( m \geq 2 \). The same authors, in [3], considered the equation (2) for the sequence \( \left( L_n^{(k)} \right)_{n \geq 2-k} \), and they have given the solutions of this equation by \( (n, k, d, m) = (5, 2, 1, 2), (5, 4, 2, 2) \).

In this paper, we will deal with the Diophantine equation

\[
P^{(k)}_n = d \left( \frac{10^m - 1}{9} \right)
\]

in positive integers \( n, m, d \) with \( m \geq 2 \) and \( 1 \leq d \leq 9 \). We will show that the repdigits in the sequence \( \left( P^{(k)}_n \right)_{n \geq 2-k} \), which have at least two digits, are the numbers \( P^{(3)}_5 = 33 \) and \( P^{(4)}_6 = 88 \).
2 Preliminaries

It can be seen that the characteristic polynomial of the sequence \( P^{(k)}_n \) is
\[
\Psi_k(x) = x^k - 2x^{k-1} - \cdots - x - 1.
\] (4)

We know from Lemma 1 in given [17] that this polynomial has exactly one positive real root located between 2 and 3. We denote the roots of the polynomial in (4) by \( \alpha_1, \alpha_2, \ldots, \alpha_k \). Particularly, let \( \alpha = \alpha_1 \) denote positive real root of \( \Psi_k(x) \). The other roots are strictly inside the unit circle. In [4], the Binet-like formula for \( k \)-generalized Pell numbers are given by
\[
P^{(k)}_n = \sum_{j=1}^{k} \frac{(\alpha_j - 1)}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \alpha_j^n.
\] (5)

It was also showed in [4] the contribution of the roots inside the unit circle to the formula (4) is very small, namely that the approximation
\[
|P^{(k)}_n - g_k(\alpha)\alpha^n| < \frac{1}{2}
\] holds for all \( n \geq 2 - k \), where
\[
g_k(z) = \frac{z - 1}{(k + 1)z^2 - 3kz + k - 1}.
\] (7)

The proof of the following inequality is given in [14].
\[
\left| \frac{(\alpha_j - 1)}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \right| \leq 2
\] (8)
for \( k \geq 2 \), where \( \alpha_j \)'s for \( j = 1, 2, \ldots, k \) are the roots of the polynomial in (4).

Throughout this paper, \( \alpha \) denotes the positive real root of the polynomial given in (4). The following relation between \( \alpha \) and \( P^{(k)}_n \) given in [4] is valid for all \( n \geq 1 \).
\[
\alpha^{n-2} \leq P^{(k)}_n \leq \alpha^{n-1}.
\] (9)

Also, Kılıç [10] proved that
\[
P^{(k)}_n = F_{2n-1}
\] (10)
for all \( 1 \leq n \leq k + 1 \).

**Lemma 1** ([4], Lemma 3.2) Let \( k, l \geq 2 \) be integers. Then
(a) If \( k > l \), then \( \alpha(k) > \alpha(l) \), where \( \alpha(k) \) and \( \alpha(l) \) are the values of \( \alpha \) relative to \( k \) and \( l \), respectively.
(b) \( \varphi^2(1 - \varphi^{-k}) < \alpha < \varphi^2 \), where \( \varphi \) is golden ratio.
(c) \( g_k(\varphi^2) = \frac{1}{\varphi + 1} \).
(d) \( 0.276 < g_k(\alpha) < 0.5 \).
For solving the equation (3), we use linear forms in logarithms and Baker’s theory. For this, we will give some notions, theorem, and lemmas related to linear forms in logarithms and Baker’s theory.

Let \( \eta \) be an algebraic number of degree \( d \) with minimal polynomial

\[
a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \eta^{(a)}) \in \mathbb{Z}[x],
\]

where the \( a_i \)'s are integers with \( \gcd(a_0, \ldots, a_n) = 1 \) and \( a_0 > 0 \) and \( \eta^{(a)} \)'s are conjugates of \( \eta \). Then

\[
h(\eta) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log \left( \max \left\{ |\eta^{(a)}|, 1 \right\} \right) \right) \tag{11}
\]

is called the logarithmic height of \( \eta \). In particularly, if \( \eta = a/b \) is a rational number with \( \gcd(a, b) = 1 \) and \( b \geq 1 \), then \( h(\eta) = \log (\max \{ |a|, b \}) \).

We give some properties of the logarithmic height whose proofs can be found in [7]:

\[
h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{12}
\]

\[
h(\eta \gamma) \leq h(\eta) + h(\gamma), \tag{13}
\]

\[
h(\eta^m) = |m|h(\eta). \tag{14}
\]

In [14], using the above properties of the logarithmic height, the authors have proved the inequality

\[
h(g_k(\alpha)) \leq 4 \log k \text{ for } k \geq 3, \tag{15}
\]

which will be used in the main theorem, where \( g_k(\alpha) \) is as defined in [7]. Now we give a theorem deduced from Corollary 2.3 of Matveev [13] and provides a large upper bound for the subscript \( n \) in the equation (3) (also see Theorem 9.4 in [6]).

**Theorem 2** Assume that \( \gamma_1, \gamma_2, \ldots, \gamma_t \) are positive real algebraic numbers in a real algebraic number field \( \mathbb{K} \) of degree \( D \), \( b_1, b_2, \ldots, b_t \) are rational integers, and

\[
\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1
\]

is not zero. Then

\[
|\Lambda| > \exp \left( -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1A_2 \cdots A_t \right),
\]

where

\[
B \geq \max \{ |b_1|, \ldots, |b_t| \},
\]

and \( A_i \geq \max \{ Dh(\gamma_i), |\log \gamma_i|, 0.16 \} \) for all \( i = 1, \ldots, t \).
The following lemma was proved by Dujella and Pethő [8] and is a variation of a lemma of Baker and Davenport [1]. This lemma will be used to reduce the upper bound for the subscript \( n \) in the equation (3). For any real number \( x \), we let \( ||x|| = \min \{ |x - n| : n \in \mathbb{Z} \} \) be the distance from \( x \) to the nearest integer.

**Lemma 3** Let \( M \) be a positive integer, let \( p/q \) be a convergent of the continued fraction of the irrational number \( \gamma \) such that \( q > 6M \), and let \( A, B, \mu \) be some real numbers with \( A > 0 \) and \( B > 1 \). Let \( \epsilon := ||\mu q|| - M ||\gamma q|| \). If \( \epsilon > 0 \), then there exists no solution to the inequality

\[
0 < |u\gamma - v + \mu| < AB^{-w},
\]

in positive integers \( u, v, \) and \( w \) with

\[
u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.
\]

The following lemma can be found in [16].

**Lemma 4** Let \( a, x \in \mathbb{R} \). If \( 0 < a < 1 \) and \( |x| < a \), then

\[
|\log(1 + x)| < \frac{-\log(1 - a)}{a} \cdot |x|
\]

and

\[
|x| < \frac{a}{1 - e^{-\alpha}} \cdot |e^x - 1|.
\]

### 3 Main Theorem

**Theorem 5** The only solution of Diophantine equation (3) in positive integers \((n, m)\) with \(1 \leq d \leq 9\) are given by \((n, k, d, m) = (5, 3, 3, 2), (6, 4, 8, 2)\).

**Proof.** Assume that \( P_n^{(k)} = d \left( \frac{10^m - 1}{9} \right) \) with \( n \geq 1 \), \( m, k \geq 2 \) and \( 1 \leq d \leq 9 \). If \( 1 \leq n \leq k + 1 \), then we have \( d \left( \frac{10^m - 1}{9} \right) = P_n^{(k)} = F_{2n-1} \) by [11]. In this case we get \( n = 1, 2, 3 \) by Theorem 1 given in [11]. But, these values of \( n \) yields to \( m = 1 \), a contradiction. Then we suppose that \( n \geq k + 2 \). If \( k = 2 \), then \( n \geq 4 \) and we have \( P_n = d \left( \frac{10^m - 1}{9} \right) \), which implies that \( n = 0, 1, 2, 3 \) by Theorem 1.1 given in [9]. Again, since \( m \geq 2 \), these cases are impossible. Therefore, assume that \( k \geq 3 \). Then, \( n \geq 5 \). Let \( \alpha \) be positive real root of \( \Psi_k(x) \) given in [1]. Then \( 2 < \alpha < \varphi^2 < 3 \) by Lemma [1] (b). Besides, it is seen that \( 10^{m-1} < P_n^{(k)} < 10^m \). Thus, using the inequality [9], we get

\[
(n - 2) \frac{\log 2}{\log 10} < m < (n - 1) \frac{\log 3}{\log 10} + 1,
\]

which implies that

\[
\frac{3n}{20} < m < \frac{3n}{4}
\]

(16)
for \( n \geq 5 \). Now, rearranging the equation (3) as

\[ P^{(k)}_n - g_k(\alpha)n + \frac{d}{9} = \frac{10^m}{9} - g_k(\alpha)n \]

and taking absolute value of both sides, we get

\[ \left| \frac{10^m}{9} - g_k(\alpha)n \right| < \frac{3}{2} \tag{17} \]

using the inequality (6). If we divide both sides of the inequality (17) by \( g_k(\alpha)n \), from Lemma 1, we get

\[ \left| \frac{10^m}{9} - g_k(\alpha)n \right| < \frac{3}{2} \frac{g_k(\alpha)n}{9} < \frac{1}{0.552} \cdot \frac{5.5}{\alpha^n} \tag{18} \]

In order to use the result of Matveev Theorem 2, we take

\[ (\gamma_1, b_1) := (10, m), \ (\gamma_2, b_2) := (\alpha, -n), \ (\gamma_3, b_3) := \left( \frac{9 \cdot g_k(\alpha)}{d}, -1 \right). \]

The number field containing \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) are \( \mathbb{K} = \mathbb{Q}(\sqrt{\alpha}) \), which has degree \( D = k \). We show that the number

\[ \Lambda_1 := 10^m\alpha^{-n} \frac{d(g_k(\alpha))^{-1}}{9} - 1 \]

is nonzero. Contrast to this, assume that \( \Lambda_1 = 0 \). Then

\[ d \frac{10^m}{9} = \alpha^ng_k(\alpha) = \frac{\alpha - 1}{(k+1)\alpha^2 - 3k\alpha + k - 1} \alpha^n. \]

Conjugating the above equality by some automorphism of the Galois group of the splitting field of \( \Psi_k(x) \) over \( \mathbb{Q} \) and taking absolute values, we get

\[ d \frac{10^m}{9} = \left| \frac{\alpha_i - 1}{(k+1)\alpha_i^2 - 3k\alpha_i + k - 1} \alpha_i^n \right| \]

for some \( i > 1 \), where \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_k \) are the roots of \( \Psi_k(x) \). Using (8) and that \( |\alpha_i| < 1 \), we obtain from the last equality that

\[ d \frac{10^m}{9} = \left| \frac{\alpha_i^k - \alpha_i}{\alpha_i^{k+1} - \alpha_i^{k-1} - k} \right| |\alpha_i|^n \]

\[ < 2, \]

which is impossible since \( m \geq 2 \). Therefore \( \Lambda_1 \neq 0 \). Moreover, since \( h(10) = \log 10, h(\gamma_2) = \frac{\log \alpha}{k} < \frac{\log 3}{k} \) by (11) and

\[ h(\gamma_3) = h\left( \frac{9 \cdot g_k(\alpha)}{d} \right) \leq h(9) + h(9) + h(g_k(\alpha)) \leq \log 81 + 4 \log k \leq 8 \log k \]

\( 6 \)
by (15), we can take $A_1 := k \log 2$, $A_2 := \log 3$, and $A_3 := 8k \log k$. Also, since $m \leq 3n/4$, it follows that $B := n$. Thus, taking into account the inequality (18) and using Theorem 2, we obtain 
\[
\frac{5.5}{\alpha^n} > |A_1| > \exp(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot k^2 (1 + \log k) (1 + \log n) (k \log 10) (\log 3) (8k \log k))
\]
and so 
\[
n \log \alpha - \log(5.5) < 1.4 \cdot 30^6 \cdot 3^{4.5} k^2 (1 + \log k) (1 + \log n) (k \log 10) (\log 3) (8k \log k),
\]
where we have used the fact that $1 + \log y < 2 \log y$ for all $y \geq 3$. From the last inequality, a quick computation with Mathematica yields to
\[
n \log \alpha - \log(5.5) < \exp(1.16 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2 \cdot \log n)
\]
or
\[
n < 1.68 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2 \cdot \log n.
\] (19)
The inequality (19) can be rearranged as
\[
\frac{n}{\log n} < 1.68 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2.
\]
Using the fact that
\[
\text{if } A \geq 3 \text{ and } \frac{n}{\log n} < A, \text{ then } n < 2A \log A,
\]
we obtain
\[
n < 3.36 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2 \log (1.68 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2) \quad \text{ (20)}
\]
\[
< 3.36 \cdot 10^{12} \cdot k^4 \cdot (\log k)^2 (30.5 + 4 \log k + 2 \log(\log k))
\]
\[
< 3.36 \cdot 10^{12} \cdot k^4 \cdot (\log k)^2 (34 \log k)
\]
\[
< 1.15 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3,
\]
where we have used the fact that $30.5 + 4 \log k + 2 \log(\log k) < 34 \log k$ for all $k \geq 3$.

Let $k \in [3, 400]$. Now, let us try to reduce the upper bound on $n$ applying Lemma 3. Let
\[
z_1 := m \log 10 - n \log \alpha + \log \left[ \frac{d}{9} (g_k(\alpha))^{-1} \right].
\]
and $x := e^{z_1} - 1$. Then from (18), we get
\[
|x| = |e^{z_1} - 1| < \frac{5.5}{\alpha^n} < 0.2
\]
for $n \geq 5$. Choosing $a := 0.2$, we obtain the inequality.
by Lemma \[ \text{Lemma 4} \]. Thus, it follows that

\[ 0 < \left| m \log 10 - n \log \alpha + \log \left( \frac{d}{g} (g_k(\alpha))^{-1} \right) \right| < \frac{6.14}{\alpha^n}. \]

Dividing this inequality by \( \log \alpha \), we get

\[ 0 < |m \gamma - n + \mu| < A \cdot B^{-w}, \tag{21} \]

where

\[ \gamma := \frac{\log 10}{\log \alpha} \notin \mathbb{Q}, \quad \mu := \frac{\log \left( \frac{d}{g} (g_k(\alpha))^{-1} \right)}{\log \alpha}, \quad A := 8.86, \quad B := \alpha, \text{ and } w := n. \]

If we take

\[ M := \left[ 1.15 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3 \right], \]

which is an upper bound on \( m \) since \( m < n < 1.15 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3 \) by (20), we found that \( q_{71} \), the denominator of the 71st convergent of \( \gamma \) exceeds \( 6M \). Furthermore, a quick computation with Mathematica gives us that the value

\[ \frac{\log \left( Aq_{71}/\epsilon \right)}{\log B} \]

is less than 99.3. So, if the inequality (21) has a solution, then

\[ n < \frac{\log \left( Aq_{71}/\epsilon \right)}{\log B} \leq 99.3, \]

that is, \( n \leq 99 \). In this case, \( m < 75 \) by (16). A quick computation with Mathematica gives us that the equation \( P_n^{(k)} = d \left( \frac{10^n - 1}{9} \right) \) has the solutions for \((n, k, d, m) = (5, 3, 3, 2), (6, 4, 8, 2) \) in the intervals \( n \in [5, 99], \ m \in (2, 75) \) and \( k \in [3, 400] \). Thus, this completes the analysis in the case \( k \in [3, 400] \).

From now on, we can assume that \( k > 400 \). Then we can see from (20) that the inequality

\[ n < 1.15 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3 < \varphi^{k/2} \tag{22} \]

holds for \( k > 400 \).

By Lemma 7 given in (15), we have

\[ g_k(\alpha)\alpha^n = \frac{\varphi^{2n}}{\varphi + 2} + \frac{\delta}{\varphi + 2} + \eta \varphi^{2n} + \eta \delta, \tag{23} \]

where

\[ |\delta| < \frac{\varphi^{2n}}{\varphi^{k/2}} \text{ and } |\eta| < \frac{3k/2}{\varphi^{k}}. \tag{24} \]
So, using (17), (23) and (24), we obtain
\[
\left| \frac{d_{10} m_n}{9} - \varphi^{2n} \right| = \left| \left( \frac{d_{10} m_n}{9} - g_k(\alpha) \alpha^n \right) + \frac{\delta}{\varphi + 2} + \eta \varphi^{2n} + \eta \delta \right| \quad (25)
\]
\[
\leq \left| \frac{d_{10} m_n}{9} - g_k(\alpha) \alpha^n \right| + \frac{|\delta|}{\varphi + 2} + |\eta| \varphi^{2n} + |\eta| |\delta|
\]
\[
< \frac{3}{2} \frac{\varphi^{2n}}{\varphi^{k/2} (\varphi + 2)} + \frac{3k \varphi^{2n}}{2 \varphi^k} + \frac{3k \varphi^{2n}}{2 \varphi^{3k/2}}.
\]

Dividing both sides of the above inequality by $\frac{\varphi^{2n}}{\varphi + 2}$, we get
\[
\left| 10^n \varphi^{2n} \frac{d_{9}}{9} (\varphi + 2) - 1 \right| < \frac{3 (\varphi + 2)}{2 \varphi^{2n}} + \frac{1}{\varphi^{k/2}} + \frac{3k (\varphi + 2)}{2 \varphi^k} + \frac{3k (\varphi + 2)}{2 \varphi^{3k/2}} \quad (26)
\]
\[
< \frac{0.15}{\varphi^{k/2}} + \frac{1}{\varphi^{k/2}} + \frac{0.005}{\varphi^{k/2}} + \frac{0.005}{\varphi^{k/2}} = \frac{1.16}{\varphi^{k/2}},
\]
where we have used the facts that
\[
\frac{3k (\varphi + 2)}{2 \varphi^k} < \frac{0.005}{\varphi^{k/2}} \quad \text{and} \quad \frac{3k (\varphi + 2)}{2 \varphi^{3k/2}} < \frac{0.005}{\varphi^{k/2}} \quad \text{for} \quad k > 400.
\]

In order to use the result of Matveev Theorem 2, we take
\[
(\gamma_1, b_1) := (10, m), \quad (\gamma_2, b_2) := (\varphi, -2n), \quad (\gamma_3, b_3) := \left( \frac{d (\varphi + 2)}{9}, 1 \right).
\]

The number field containing $\gamma_1$, $\gamma_2$, and $\gamma_3$ are $K = \mathbb{Q}(\sqrt{5})$, which has degree $D = 2$. We show that the number
\[
\Lambda_1 := 10^n \varphi^{-2n} \frac{d_{9}}{9} (\varphi + 2) - 1
\]
is nonzero. Contrast to this, assume that $\Lambda_1 = 0$. Then $10^n \varphi^{2n} (\varphi + 2) = \varphi^{2n}$ and conjugating this relation in $\mathbb{Q}(\sqrt{5})$, we get $10^n \varphi^{2n} (\beta + 2) = \beta^{2n}$, where $\beta = \frac{1 - \sqrt{5}}{2} = \varphi$. So, we have
\[
\frac{\varphi^{2n}}{\varphi + 2} = \frac{\beta^{2n}}{\beta + 2},
\]
which implies that
\[
\frac{\varphi^{4n}}{\varphi + 2} = \frac{1}{\beta + 2} < 1.
\]
The last inequality is impossible for $n \geq 5$. Therefore $\Lambda_1 \neq 0$. Moreover, since
\[
h(\gamma_1) = h(10) = \log 10, \ h(\gamma_2) = h(\varphi) \leq \frac{\log \varphi}{2}
\]
and
\[
h(\gamma_3) \leq h(9) + h(d) + h(\varphi) + h(2) + \log 2 \leq \log 324 + \frac{\log \varphi}{2},
\]

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by (13), we can take $A_1 := 2 \log 10$, $A_2 := \log \varphi$, and $A_3 := \log \left(324^2 \varphi\right)$. Also, since $m < 3n/4$, we can take $B := 2n$. Thus, taking into account the inequality (26) and using Theorem 2, we obtain

$$\frac{k}{2} \log \varphi - \log(1.16) < 2.59 \cdot 10^{13} \cdot (1 + \log 2n)$$

or

$$k < 2.16 \cdot 10^{14} \cdot \log 2n,$$

where we have used the fact that $(1 + \log 2n) < 2 \log 2n$ for $n \geq 5$. On the other hand, from (22), we get

$$\log 2n < \log \left(2.3 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3\right)$$

$$< 35.4 + 4 \log k + 3 \log(\log k)$$

$$< 40 \log k.$$ 

So, from (27), we obtain

$$k < 2.16 \cdot 10^{14} \cdot 40 \log k,$$

which implies that

$$k < 3.5 \cdot 10^{17}. \quad (28)$$

To reduce this bound on $k$, we use Lemma 3. Substituting this bound of $k$ into (22), we get $n < 1.14 \cdot 10^{90}$, which implies that $m < 8.55 \cdot 10^{89}$ by (16).

Now, let

$$z_2 := m \log 10 - 2n \log \varphi + \log \left(\frac{d}{9} (\varphi + 2)\right).$$

and $x := 1 - e^{z_2}$. Then

$$|x| = |1 - e^{z_2}| < \frac{1.16}{\varphi^{k/2}} < 0.6$$

by (26). Choosing $a := 0.6$, obtain the inequality

$$|z_2| = |\log(x + 1)| < \frac{\log(5/2)}{0.6} \cdot \frac{1.16}{\varphi^{k/2}} < \frac{1.78}{\varphi^{k/2}}$$

by Lemma 4. That is,

$$0 < \left| m \log 10 - 2n \log \varphi + \log \left(\frac{d}{9} (\varphi + 2)\right)\right| < \frac{1.78}{\varphi^{k/2}}.$$ 

Dividing both sides of the above inequality by $\log \varphi$, we get

$$0 < |m \gamma - 2n + \mu| < A \cdot B^{-w}. \quad (29)$$
where
\[ \gamma := \frac{\log 10}{\log \varphi} \notin \mathbb{Q}, \quad \mu := \frac{\log \left( \frac{3}{4} (\varphi + 2) \right)}{\log \varphi}, \quad A := 3.7, \quad B := \varphi, \text{ and } w := k/2. \]

If we take \( M := 8.55 \cdot 10^{89} \), which is an upper bound on \( m \), we found that \( q_{180} \), the denominator of the 180 th convergent of \( \gamma \) exceeds 6M. Furthermore, a quick computation with Mathematica gives us that the value

\[ \frac{\log (Aq_{180}/\epsilon)}{\log B} \]

is less than 444.7. So, if the inequality (29) has a solution, then

\[ \frac{k}{2} < \frac{\log (Aq_{180}/\epsilon)}{\log B} \leq 444.7, \]

that is, \( k \leq 889 \). Hence, from (22), we get \( n < 2.25 \cdot 10^{29} \), which implies that \( m < 1.69 \cdot 10^{29} \) since \( m < 3n/4 \) by (10). If we apply Lemma 5 again with \( M := 1.69 \cdot 10^{29} \), we found that \( q_{60} \), the denominator of the 60 th convergent of \( \gamma \) exceeds 6M. After doing this, then a quick computation with Mathematica show that the inequality (29) has solution only for \( k < 313 \). This contradicts the fact that \( k > 400 \). This completes the proof. \( \Box \)

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