On the Behavior of Solutions of a System of Difference Equations

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Abstract

We establish the relation between local stability of equilibria and slopes of critical curves for a specific class of difference equations. We then use this result to give global behavior results for nonnegative solutions of the system of difference equations

\[ \begin{align*}
  x_{n+1} &= \frac{b_1 x_n}{1 + x_n + c_1 y_n} + h_1 \\
  y_{n+1} &= \frac{b_2 y_n}{1 + y_n + c_2 x_n} + h_2
\end{align*} \]

with positive parameters. In particular, we show that the system has between one and three equilibria, and that the number of equilibria determines global behavior as follows: if there is only one equilibrium, then it is globally asymptotically stable. If there are two equilibria, then one is a local attractor and the other one is nonhyperbolic. If there are three equilibria, then they are linearly ordered in the south-east ordering of the plane, and consist of a local attractor, a saddle point, and another local attractor. Finally, we give sufficient conditions for having a unique equilibrium.

Key words: difference equation, rational, global behavior, global attractivity, model, competitive system, Leslie-Gower model.

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1 Introduction

The study of dynamics of difference equations often requires that equilibria be calculated first, followed by
a local stability analysis of the equilibria. This is then complemented by other considerations (existence
of periodic points, chaotic orbits, etc.). If the analysis is applied to a class of equations dependent on one
or more real parameters, the task is complicated by the fact that a formula is not always available for
equilibria, and even if it is, determination of stability properties of parameter-dependent equilibria may be
a daunting task.

In this paper we address the study of stability character of equilibria for a parameter dependent class of
systems of difference equations. For this we develop local geometric criteria for planar systems of difference
equations, which under certain conditions determine local stability character of an equilibrium point. This
idea is used to study a class of difference equations in the plane, for which we show that the number of
equilibria determines, essentially in a unique way, the local stability character of the equilibria, as well as
the kind of global dynamics that is possible.

The class of difference equations we will focus on has its roots in biology applications. The Leslie-Gower
Model in difference equations is the two-species competition model

\[
\begin{align*}
x_{n+1} &= \frac{b_1 x_n}{1 + c_{11} x_n + c_{12} y_n} \\
y_{n+1} &= \frac{b_2 y_n}{1 + c_{21} y_n + c_{22} x_n}.
\end{align*}
\]

It is a modified version of the Beverton-Holt equation which is a well-known difference equation in Population Biology. The normalized Leslie-Gower Model

\[
\begin{align*}
x_{n+1} &= \frac{b_1 x_n}{1 + x_n + c_1 y_n} \\
y_{n+1} &= \frac{b_2 y_n}{1 + y_n + c_2 x_n},
\end{align*}
\]

was studied in detail by Liu and Elaydi [15], J. Cushing et. al [1], and Kulenović and Merino [11].

In this paper we consider the system of equations obtained by adding positive constants to the right-hand-side of the Leslie-Gower equations:

\[
\begin{align*}
x_{n+1} &= \frac{b_1 x_n}{1 + c_{11} x_n + c_{12} y_n} + H_1 \\
y_{n+1} &= \frac{b_2 y_n}{1 + c_{21} y_n + c_{22} x_n} + H_2.
\end{align*}
\]

The positive constants $H_1$ and $H_2$ in (LGI) may account for immigration. The change of variables $x_n = \frac{1}{a_{11}} \tilde{x}_n, y_n = \frac{1}{a_{22}} \tilde{y}_n$ normalizes (LGI) as follows:

\[
\begin{align*}
x_{n+1} &= \frac{b_1 x_n}{1 + x_n + c_1 y_n} + h_1 \\
y_{n+1} &= \frac{b_2 y_n}{1 + y_n + c_2 x_n} + h_2.
\end{align*}
\]

where in (LGIN) the tildes have been removed from the variables to simplify notation, and

\[
c_1 = \frac{c_{12}}{c_{22}}, \quad h_1 = c_{11} H_1, \quad c_2 = \frac{c_{21}}{c_{11}}, \quad \text{and} \quad h_2 = c_{22} H_1.
\]
The system (LGIN) is an example of a competitive system, which is defined next. Let $I$ and $J$ be intervals of real numbers and let $f : I \to I$ and $g : J \to J$ be continuous functions. Consider the system
\[
\begin{align*}
x_{n+1} &= f(x_n, y_n) \\
y_{n+1} &= g(x_n, y_n)
\end{align*}
\]
$n = 0, 1, 2, \ldots$, $(x_0, y_0) \in I \times J$ (1)
System (1) is competitive if $f(x, y)$ is non-decreasing in $x$ and non-increasing in $y$, and $g(x, y)$ is non-increasing in $y$ and non-decreasing in $x$. The map $T(x, y) := (f(x, y), g(x, y))$ associated to a competitive system (1) is said to be competitive. System (1) is strongly competitive if it is competitive, with strict coordinate-wise monotonicity of the functions $f(x, y)$ and $g(x, y)$. If $T$ is differentiable on an open set $R$, a sufficient condition for $T$ to be strongly competitive on $R$ is that the Jacobian matrix of $T$ at any $(x, y) \in R$ has nonzero entries with sign configuration
\[
\begin{pmatrix}
+ & - \\
- & +
\end{pmatrix}
\]
Competitive systems of the form (1) have been studied by many authors [2], [6], [13], [18] and others. The term competitive was introduced by Hirsch [5] (see also [4]) for systems of autonomous differential equations $x_i' = F_i(x_1, \ldots, x_n, t)$, $i = 1, \ldots, n$ satisfying $\frac{\partial F_i}{\partial x_j} \leq 0$, $i \neq j$. The main motivation for the study of these systems is the fact that many mathematical models in biological sciences may be classified as competitive (or cooperative) [16], [17], [14]. Consideration of Poincaré maps of these systems leads to the concept of competitive and cooperative in the discrete case.

Denote with $\leq_{se}$ the South-East partial order in the plane whose nonnegative cone is the standard fourth quadrant $\{(x, y) : x \geq 0, y \leq 0\}$, that is, $(x_1, y_1) \leq_{se} (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \geq y_2$. Competitive maps $T$ in the plane preserve the South-East ordering: $T(x) \leq_{se} T(y)$ whenever $x \leq_{se} y$. The concept of competitive (for maps) may be defined in terms of the order preserving properties of the maps. Finally we note that the terms equilibrium (of a planar system of difference equations) and fixed point (of a map) are used interchangeably in this paper.

This paper is organized as follows. In Section 2 a relation between local stability of equilibria and slopes of critical curves for a specific class of difference equations is established. In Section 3 it is shown that every solution to Eq. (LGIN) converges to an equilibrium. Section 4 contains the main result of this paper. It states that Eq. (LGIN) has between one and three equilibria, and that the number of equilibria determines global behavior as follows: if there is only one equilibrium, then it is globally asymptotically stable. If there are two equilibria, then one is a local attractor and the other one is nonhyperbolic. If there are three equilibria, then they are linearly ordered in the south-east ordering of the plane, and consist of a local attractor, a saddle point, and another local attractor. Finally, in Section 5 we give sufficient conditions for Eq. (LGIN) to have a unique equilibrium.

2 A preliminary result on critical sets

Our first result establishes a connection between local stability of the fixed point of a planar map and the slopes of certain curves at the fixed point. The result will be useful in the proof of the main result in Section 4.

Theorem 1 Let $R$ be a subset of $\mathbb{R}^2$ with nonempty interior, and let $T = (f, g) : R \to R$ be a map of class $C^p$ for some $p \geq 1$. Suppose that $T$ has a fixed point $(\bar{x}, \bar{y}) \in \text{int} R$ such that
\[
a := f_x(\bar{x}, \bar{y}), \quad b := f_y(\bar{x}, \bar{y}), \quad c := g_x(\bar{x}, \bar{y}), \quad d := g_y(\bar{x}, \bar{y})
\]
satisfy
\[ 0 < a < 1, \quad b c > 0, \quad 0 < d < 1, \quad 1 + (a + d) + a d - b c > 0 \]  \hspace{1cm} (2)

Let \( C_1, C_2 \) be the critical sets
\[ C_1 := \{ (x, y) : x = f(x, y) \} \quad \text{and} \quad C_2 := \{ (x, y) : y = g(x, y) \} \]

Then,

i. There exists neighborhood \( I \subset \mathbb{R} \) of \( \bar{x} \) and \( J \subset \mathbb{R} \) of \( \bar{y} \) such that the sets \( C_1 \cap (I \times J) \) and \( C_2 \cap (I \times J) \) are the graphs of class \( C^p \) functions \( y_1(x) \) and \( y_2(x) \) for \( x \in I \).

ii. The eigenvalues \( \lambda_1, \lambda_2 \) of the jacobian matrix of \( T \) at \((\bar{x}, \bar{y})\) are real, distinct, and the eigenvalue with larger absolute value is positive. If \( \lambda_1 \) is the larger eigenvalue, then
\[ -1 < \lambda_2 < 1 \quad \text{and} \quad \text{sign} (\lambda_1 - 1) = \begin{cases} -\text{sign} (y'_1(\bar{x}) - y'_2(\bar{x})) & \text{if} \ b < 0 \\ \text{sign} (y'_1(\bar{x}) - y'_2(\bar{x})) & \text{if} \ b > 0 \end{cases} \] (3)

Proof.

i. The existence of \( I \) and \( J \) and of smooth functions \( y_1(x) \) and \( y_2(x) \) defined in \( I \) as in the statement of the Theorem is guaranteed by the hypotheses and the Implicit Function Theorem. Moreover,
\[ y'_1(x) = \frac{1 - f_x(x, y)}{f_y(x, y)} \quad \text{and} \quad y'_2(x) = \frac{g_x(x, y)}{1 - g_y(x, y)} , \quad x \in I \] (4)

ii. The characteristic polynomial of the Jacobian of \( T \),
\[ p(\lambda) = \lambda^2 - (a + d) \lambda + (a d - b c) \]
has positive discriminant, thus its roots \( \lambda_1, \lambda_2 \) are real. Since \( \lambda_1 + \lambda_2 = a + d > 0 \), then the root \( \lambda_1 \) with larger absolute value is positive, and \( |\lambda_2| < \lambda_1 \). Now by the hypothesis \( a, d \in (0, 1) \) it follows that \( \lambda_1 + \lambda_2 = a + d < 2 \), which implies \( \lambda_2 < 1 \). Since \( p(\lambda) \) has at least one positive root and since by hypothesis \( p(-1) = 1 + (a + d) + a d - b c > 0 \), we have \(-1 < \lambda_2 \). To prove the second part of (3), note that from (4), we have
\[ y'_1(\bar{x}) - y'_2(\bar{x}) = \frac{1 - a}{b} - \frac{c}{1 - d} = \frac{1 - (a + d) + a d - b c}{b (1 - d)} \]
\[ = -\frac{p(1)}{b (1 - d)} = \frac{(\lambda_1 - 1) (1 - \lambda_2)}{b (1 - d)} \] (5)

The result is a direct consequence of (5), the inequality \( \lambda_2 < 1 \) and hypotheses (2).
3 Every solution converges to an equilibrium

In this section we show that every solution to Eq. (LGIN) converges to an equilibrium. Let

\[ f(x, y) = \frac{b_1 x}{1 + x + c_1 y} + h_1, \quad \text{and} \quad g(x, y) = \frac{b_2 y}{1 + y + c_2 x} + h_2. \]

Then the map \( T(x, y) = (f(x, y), g(x, y)) \) associated with (LGIN) is

\[ T(x, y) = \left( \frac{b_1 x}{1 + x + c_1 y} + h_1, \frac{b_2 y}{1 + y + c_2 x} + h_2 \right), \quad (x, y) \in [0, \infty) \times [0, \infty) \]

For future reference we give the jacobian matrix of \( T \) at \((x, y)\):

\[
J_T(x, y) = \begin{pmatrix}
\frac{b_1 (c_1 y + 1)}{(x + c_1 y + 1)^2} & -\frac{b_1 c_1 x}{(x + c_1 y + 1)^2} \\
-\frac{b_2 c_2 y}{(c_2 x + y + 1)^2} & \frac{b_2 (c_2 x + 1)}{(c_2 x + y + 1)^2}
\end{pmatrix}
\]

By direct inspection of (7) we obtain the following result.

Lemma 1 The system of difference equations (LGIN) is strongly competitive on \([0, \infty) \times [0, \infty)\).

The following lemma has an easy proof which we skip.

Lemma 2 \( T([0, \infty) \times [0, \infty)) \subset [h_1, h_1 + b_1] \times [h_2, h_2 + b_2]\).

Theorem 2 Every solution of Eq. (LGIN) converges to an equilibrium.

Proof. It is easy matter to show that the map \( T \) is one-to-one. From (7) the determinant of \( J_T(x, y) \) is

\[
\det(Jac_T(x, y)) = \frac{b_1 b_2 (c_2 x + c_1 y + 1)}{(c_2 x + y + 1)^2 (x + c_1 y + 1)^2},
\]

which is clearly positive for \((x, y) \in [0, \infty) \times [0, \infty)\). It follows that the map \( T \) satisfies hypothesis \((H+)\) in [18]. By Lemma 4.3 of [18] and Theorem 4.2 of [18] we have that every solution of Eq. (LGIN) is eventually coordinate-wise monotone. Since by Lemma 2 every solution enters the compact set \([h_1, h_1 + b_1] \times [h_2, h_2 + b_2]\), the conclusion follows.

\[ \square \]

4 Number of equilibria and global behavior

By solving for \( y \) and \( x \), respectively, in the equations defining the critical sets \( C_1 = \{(x, y) \in \mathbb{R}^2 : x = f(x, y)\} \) and \( C_2 = \{(x, y) \in \mathbb{R}^2 : y = g(x, y)\} \) and taking derivatives we obtain the following result.

Lemma 3 All branches of the sets \[
\begin{aligned}
C_1 &= \{(x, y) \in \mathbb{R}^2 : x^2 + c_1 x y + (1 - b_1 - h_1) x - c_1 h_1 y - h_1 = 0\} \\
C_2 &= \{(x, y) \in \mathbb{R}^2 : y^2 + c_2 x y + (1 - b_2 - h_2) y - c_2 h_2 x - h_2 = 0\}
\end{aligned}
\]

are the graphs of decreasing functions of one variable.
Lemma 4 The map $T$ satisfies hypotheses (2) of Theorem 1.

Proof. Set $a := f_x(x, y), b := f_y(x, y), c := g_x(x, y), d := g_y(x, y)$. Implicit differentiation of the equations defining $C_1$ and $C_2$ in (3) at $(x, y)$ gives

$$y_1'(x) = \frac{1-a}{b} \quad \text{and} \quad y_2'(x) = \frac{c}{1-d}$$

(9)

From Lemma 3, $y_1'(x) < 0$ and $y_2'(x) < 0$, and from (7), $b < 0$ and $c < 0$. It follows that $a < 1$ and $d < 1$ in (9). Furthermore, since the map $T$ is strongly competitive, $a > 0, d > 0, b < 0, c < 0$. Hence, $T$ must satisfy inequalities $0 < a < 1, 0 < d < 1$ and $bc > 0$ from hypotheses (2) of Theorem 1. Finally note that \( \det(J_T(x, y)) = ad - bc > 0 \) by (5), hence $1 + (a + d) + ad - bc > 0$. \qed

Denote with $Q_\ell(a, b), \ell = 1, 2, 3, 4$ the four regions $Q_1(a, b) := \{(x, y) \in \mathbb{R}^2 : a \leq x, b \leq y \}, Q_2(a, b) := \{(x, y) \in \mathbb{R}^2 : x \leq a, b \leq y \}, Q_3(a, b) := \{(x, y) \in \mathbb{R}^2 : x \leq a, y \leq b \}, Q_4(a, b) := \{(x, y) \in \mathbb{R}^2 : a \leq x, y \leq b \}$. 

Lemma 5 Each of the sets $Q_1(h_1, h_2)$ and $Q_3(h_1, h_2)$ contain at least one equilibrium of Eq.(LGIN).

Proof. The existence of an equilibrium of Eq.(LGIN) in the invariant and attracting set $B := [h_1, h_1 + b_1] \times [h_2, h_2 + b_2]$ from Lemma 2 is guaranteed by the Schauder Fixed Point Theorem for continuous maps on compact, convex and invariant sets. Since $B \subset Q_1(h_1, h_2)$, such equilibrium points are elements of $Q_1(h_1, h_2)$. To see that $Q_3(h_1, h_2)$ contains an equilibrium of Eq.(LGIN), note that the critical curves $C_1$ and $C_2$ can be given explicitly as functions of $x$ :

$$C_1: \quad y_1(x) = \frac{x^2 + (1-b_1-h_1)x-h_1}{c(h_1-x)}, \quad x \neq h_1$$

$$C_2:\begin{cases}
y_2+(x) = \frac{1}{2} \left( -1 + b_2 + h_2 - c_2 x + \sqrt{(-b_2 - h_2 + c_2 x + 1)^2 + 4(c_2 x h_2 + h_2)} \right) \\
y_2-(x) = \frac{1}{2} \left( -1 + b_2 + h_2 - c_2 x - \sqrt{(-b_2 - h_2 + c_2 x + 1)^2 + 4(c_2 x h_2 + h_2)} \right)
\end{cases}$$

Then,

$$\lim_{x \to -\infty} y_1(x) - y_2-(x) = \infty \quad \text{and} \quad \lim_{x \to h_1} y_1(x) - y_2-(x) = -\infty. \quad (10)$$

One can conclude from (10) and the continuity of $y_1(x), y_2-(x)$ that there exists $c < h_1$ such that $y_1(c) = y_2-(c)$. Since $x = h_1$ is a vertical asymptote of $C_1$ and $y = h_2$ is a horizontal asymptote of $C_2$, it follows from the decreasing characters of $y_1(x)$ and $y_2-(x)$ that $(c, y_1(c))$ must lie in $Q_3(h_1, h_2)$. \qed

Definition 1 Let $k$ be a positive integer. Let $(x_0, y_0) \in \mathbb{R}^2$ be an intersection point of the graphs $C_1$ and $C_2$ of two $k$-times differentiable functions $f_1, f_2$ defined in a neighborhood of $x_0$. The point $(x_0, y_0)$ is a contact point of $C_1$ and $C_2$ of order $k$ if

$$\begin{align*}
f_1^{(\ell)}(x_0) &= f_2^{(\ell)}(x_0), \quad 0 \leq \ell < k \\
f_1^{(k)}(x_0) &\neq f_2^{(k)}(x_0)
\end{align*}$$

Note that $C_1$ and $C_2$ intersect transversally at $(x_0, y_0)$ if and only if $(x_0, y_0)$ is a contact point of $C_1, C_2$ of order one.
Lemma 6 Let \( P_0 = (x_0, y_0) \) be an intersection point of two non-singular algebraic curves \( C_1 \) and \( C_2 \) in the plane that have no common component through \( P_0 \). If \( C_1 \) and \( C_2 \) have tangent lines at \( P_0 \) that are not parallel to either axis, then there exists a neighborhood \( U \) of \( P_0 \) for which both \( U \cap C_1 \) and \( U \cap C_2 \) are graphs of real analytic strictly monotonic functions \( y_1(x) \) and \( y_2(x) \) for \( x \) in some neighborhood of \( x_0 \), and the order of \( P_0 \) as a contact point of \( C_1 \) and \( C_2 \) equals the multiplicity of \( P_0 \) as an intersection point of the algebraic curves \( C_1 \) and \( C_2 \).

**Proof.** Without loss of generality we may assume \((x_0, y_0) = (0, 0)\). Let \( y_1(t) \) and \( y_2(t) \) be such that \( U \cap C_1 \) and \( U \cap C_2 \) are parametrized by \((t, y_1(t))\) and \((t, y_2(t))\) for \( t \in I \). Then \( y_1(t) \) and \( y_2(t) \) are real analytic on \( I \). The rest of the statement follows. \( \square \)

Lemma 7 Let \( p \) and \( q \) be real analytic strictly monotone functions defined in neighborhoods of \( x_0 \) and \( y_0 \) respectively, such that \( p(x_0) = y_0 \), \( q(y_0) = x_0 \), and neither \( p \circ q \) nor \( q \circ p \) is the identity function. Let \( k \) be the order of \((x_0, y_0)\) as a contact point of \( C_1 := \{(x, y) : y = p(x)\} \) and \( C_2 := \{(x, y) : x = q(y)\} \), and let \( d_y \) and \( d_x \) be the multiplicities of \( y_0 \) and \( x_0 \) as zeros of \( y - p(q(y)) \) and \( x - q(p(x)) \) respectively. Then \( d_y = d_x = k \).

**Proof.** Consider the function \( \phi(y) := y - p(\bar{p}(y)) \), where \( \bar{p} \) is the inverse of the strictly monotone function \( p \). Clearly, \( \phi(y_0) = 0 \) in a neighborhood of \( y_0 \). It follows that \( \phi'(y_0) = 0 \) for \( \ell > 0 \). In particular, \( \phi'(y_0) = 0 \) from which we get
\[
\bar{p}'(y_0) = \frac{1}{p'(\bar{p}(y_0))} = \frac{1}{p'(x_0)}. \tag{11}
\]
Now consider the function \( \psi(y) = y - p(q(y)) \). Since \( y_0 \) is a root of \( \psi(y) \) of multiplicity \( d_y \) by hypothesis, we must have \( \psi^{(\ell)}(y_0) = 0 \) for \( 0 < \ell < d_y \) and \( \psi^{(d_y)}(y_0) \neq 0 \). In particular, \( \psi'(y_0) = 0 \) from which we have
\[
\psi'(y_0) = \frac{1}{p'(q(y_0))} = \frac{1}{p'(x_0)}. \tag{12}
\]
From (11) and (12), it follows that \( \bar{p}'(y_0) = \psi'(y_0) \). Similarly, one can show that \( \bar{p}'(y_0) = \psi'(y_0) \), \( 2 \leq \ell < d_y \). However, since \( \psi^{(d_y)}(y_0) \neq 0 \), we must have \( \bar{p}^{(d_y)}(y_0) \neq \psi^{(d_y)}(y_0) \). Hence
\[
\begin{cases}
\bar{p}^{(\ell)}(y_0) = \psi^{(\ell)}(y_0), & 0 \leq \ell < d_y \\
\bar{p}^{(d_y)}(y_0) \neq \psi^{(d_y)}(y_0)
\end{cases} \tag{13}
\]
It is a direct consequence of Definition 1 and (13) that \( d_y = k \). By a similar argument using the equation defining \( C_2 \), one can show that \( d_x = k \). \( \square \)

The main result of this paper is the following.

**Theorem 3** The following statements are true:

(i.) Eq. (LGNI) has at least one and at most three equilibria in \([0, \infty)^2\). The set of equilibrium points in \([0, \infty)^2\) is linearly ordered by \( \preceq_{se} \).

(ii.) If Eq. (LGNI) has exactly one equilibrium in \([0, \infty)^2\), then it is globally asymptotically stable.

(iii.) If Eq. (LGNI) has three distinct equilibria in \([0, \infty)^2\), say \((\overline{x}_l, \overline{y}_l), \ l = 1, \ldots, 3, \) with \((\overline{x}_1, \overline{y}_1) \preceq_{se} (\overline{x}_2, \overline{y}_2) \preceq_{se} (\overline{x}_3, \overline{y}_3), \) then \((\overline{x}_1, \overline{y}_1) \) and \((\overline{x}_3, \overline{y}_3) \) are locally asymptotically stable, while \((\overline{x}_2, \overline{y}_2) \) is a saddle point. The global stable manifold of \((\overline{x}_2, \overline{y}_2) \) is the graph of a continuous increasing function of the first variable with endpoints in the boundary of \([0, \infty)^2\), which is a separatrix of the basins of attraction of \((\overline{x}_1, \overline{y}_1) \) and \((\overline{x}_3, \overline{y}_3) \).
(iv.) If there exist exactly two equilibria in \([0, \infty)^2\), then one is locally asymptotically stable and the other is a nonhyperbolic fixed point. If \((x_1, y_1)\) and \((x_2, y_2)\) are the two equilibria with \((x_1, y_1) \preceq_{se} (x_2, y_2)\), then \(Q_2(x_1, y_1)\) is a subset of the basin of attraction of \((x_1, y_1)\), and \(Q_4(x_2, y_2)\) is a subset of the basin of attraction of \((x_2, y_2)\).

**Proof.**

(i.) It is a consequence of Bézout’s Theorem (Theorem 3.1, Chapter III in [19]) that the hyperbolas \(C_1\) and \(C_2\) given in (3) must intersect in at most four points. Since intersection points of \(C_1\) and \(C_2\) are precisely the equilibrium points of Eq. (LGIN), it follows that Eq. (LGIN) must have at most four equilibrium points. Lemmas 2 and 5 guarantee that at most three of these equilibria can lie in the invariant attracting box \(B := [h_1, h_2] \times [h_1 + b_1, h_2 + b_2]\) and hence in \(Q_1(h_1, h_2)\), which proves the first part of statement (i.). To see that the set of equilibrium points in \([0, \infty)^2\) is linearly ordered by \(\preceq_{se}\), note that the equilibria are precisely the intersection points of the decreasing critical curves \(T\) to be a non-attracting equilibrium. Suppose \((C_1)\) is a nonhyperbolic fixed point. If \(\text{dist}(Q_1, y, 1)\) is tangentially at \((h_1, h_2)\) and \((x, y)\) satisfies the relations \(y_1(x) = y_2(x)\) of each equilibrium in \([0, \infty)^2\). From Bezout’s Theorem, Lemma 5 and Lemma 6, each equilibrium in \([0, \infty)^2\) gives that the vertical asymptote of \((x, y)\) Stability follows from the fact that \(T^n(h_1, h_2 + b_2) \preceq_{se} T^{n+\ell}(x, y) \preceq_{se} T^n(h_1 + b_1, h_2)\) for \(\ell = 1, 2, \ldots, n = 1, 2, \ldots,\) and \(\text{dist}(T^n(h_1, h_2 + b_2), T^n(h_1 + b_1, h_2)) \to 0\).

(ii.) Let \(T\) be the map of Eq. (LGIN), and let \((x, y) \in [0, \infty)^2\). By Lemma 2 and since \(T\) is competitive, \((h_1, h_2 + b_2) \preceq_{se} T(h_1, h_2 + b_2) \preceq_{se} T(h_1 + b_1, h_2) \preceq_{se} (h_1 + b_1, h_2)\). By induction, \(T^n(h_1, h_2 + b_2) \preceq_{se} T^{n+1}(h_1, h_2 + b_2) \preceq_{se} (h_1 + b_1, h_2)\) for \(n = 1, 2, \ldots\). Then the sequences \(\{T^n(h_1, h_2 + b_2)\}\) and \(\{T^n(h_1 + b_1, h_2)\}\) are respectively monotonically increasing and decreasing, they are bounded and converge (this follows from the bounded and monotonic coordinate-wise character of the sequences). Since both must converge to a fixed point, the limit point is the unique fixed point of \(T\). Again by Lemma 2 for \((x, y) \in [0, \infty)^2\), \(T(x, y) \in B\), hence \((h_1, h_2 + b_2) \preceq_{se} T(x, y) \preceq_{se} (h_1 + b_1, h_2)\), and by induction, \(T^n(h_1, h_2 + b_2) \preceq_{se} T^{n+1}(x, y) \preceq_{se} T^n(h_1 + b_1, h_2)\) for \(n = 1, 2, \ldots\). Thus \(T^n(x, y) \to (\bar{x}, \bar{y})\) as \(n \to \infty\). This proves global attractivity of \((\bar{x}, \bar{y})\)

(iii.) Suppose Eq. (LGIN) has three distinct equilibria in \([0, \infty)^2\), say \((\bar{x}_l, \bar{y}_l)\), \(l = 1, \ldots, 3\), with \((\bar{x}_1, \bar{y}_1) \preceq_{se} (\bar{x}_2, \bar{y}_2) \preceq_{se} (\bar{x}_3, \bar{y}_3)\). Note that the equilibria are precisely the intersection points of the critical curves \(C_1\) and \(C_2\) given in (3). From Bezout’s Theorem, Lemma 5 and Lemma 6 each equilibrium in \([0, \infty)^2\) must be a contact point of \(C_1\) and \(C_2\) of order one. It follows from the remark after Definition 1 that \(C_1\) and \(C_2\) must intersect transversally at each of the three equilibria in \([0, \infty)^2\). Furthermore, solving for \(y\) and \(x\) respectively in the equations defining \(C_1\) and \(C_2\) in (3) gives that the vertical asymptote of \(C_1\) is \(x = h_1\) and the horizontal asymptote of \(C_2\) is \(y = h_2\). The asymptotes guarantee that in order to have three intersection points in \([0, \infty)^2\), the slopes of the functions \(y_1(x)\) and \(y_2(x)\) of \(C_1\) and \(C_2\) respectively, must satisfy the relations \(y_1'(\bar{x}_1) < y_2'(\bar{x}_2)\), \(y_1'(\bar{x}_2) > y_2'(\bar{x}_2)\) and \(y_1'(\bar{x}_3) < y_2'(\bar{x}_3)\). Theorem 1 then gives that \((\bar{x}_1, \bar{y}_1)\) and \((\bar{x}_3, \bar{y}_3)\) must be locally asymptotically stable, while \((\bar{x}_2, \bar{y}_2)\) must be a saddle point. The rest of the proof follows from Theorem 4 and Corollary 1 in [12].

(iv.) Suppose Eq. (LGIN) has two distinct equilibria in \([0, \infty)^2\), say \((\bar{x}_1, \bar{y}_1)\) and \((\bar{x}_2, \bar{y}_2)\). Note that \((\bar{x}_1, \bar{y}_1)\) and \((\bar{x}_2, \bar{y}_2)\) cannot both be contact points of \(C_1\) and \(C_2\) of order one. Indeed if they were, then as a result of the remark after Definition 1 \(C_1\) and \(C_2\) would have to intersect transversally at \((\bar{x}_1, \bar{y}_1)\) and \((\bar{x}_2, \bar{y}_2)\). Thus by Theorem 1 both equilibria would have to be locally asymptotically stable. But this would lead to a contradiction since by Theorem 4 of [3], at least one of the equilibria has to be a non-attracting equilibrium. Suppose \((\bar{x}_2, \bar{y}_2)\) is contact with \((\bar{x}_1, \bar{y}_1)\) and \((\bar{x}_2, \bar{y}_2)\). Hence by Theorem 1 \((\bar{x}_1, \bar{y}_1)\) must be a non-hyperbolic fixed point and \((\bar{x}_2, \bar{y}_2)\) must be locally asymptotically stable. The statement on basins of attraction is a consequence of \(T\) being competitive and the fact that \(Q_2(x_1, y_1)\) and \(Q_4(x_2, y_2)\)
are invariant sets for $T$. Indeed, if $(x,y) \in Q_2(x_1,y_1)$ then $T^n(x,y) \in Q_2(x_1,y_1)$. Since $\{T^n(x,y)\}$ converges to a fixed point by Theorem 2, it must converge to the only fixed point in $Q_2(x_1,y_1)$, namely, $(x_1,y_1)$. A similar argument applies if $(x,y) \in Q_4(x_2,y_2)$.

\[ \square \]

5 A global attractivity result

A version of the following theorem was proved first in [9]. It is given here for easy reference.

**Theorem 4** Let $[a,b]$ and $[c,d]$ be intervals of real numbers and let $f : [a,b] \to [a,b]$ and $g : [c,d] \to [c,d]$ be continuous functions that satisfy the following properties (a) and (b):

(a) $f(x,y)$ is non-decreasing in $x$ and non-increasing in $y$, and $g(x,y)$ is non-increasing in $y$ and non-decreasing in $x$.

(b) For every $(m,M), (\overline{m}, \overline{M}) \in [a,b] \times [c,d]$,

\[
\begin{align*}
    m &= f(m,M), \quad \overline{m} = f(\overline{m}, M), \\
    M &= f(M,m), \quad \overline{M} = g(m, \overline{M}) \quad \implies \quad m = \overline{m}, \quad M = \overline{M}
\end{align*}
\]

Then the system of difference equations

\[
\begin{align*}
    x_{n+1} &= f(x_n, y_n) \\
    y_{n+1} &= g(x_n, y_n)
\end{align*}
\]

has a unique equilibrium $(\overline{x}, \overline{y})$ in $[a,b] \times [c,d]$, and the unique equilibrium is a global attractor.

Next we give a sufficient condition for the existence of a unique equilibrium.

**Theorem 5** If at least one of the following conditions is satisfied

\[
\begin{align*}
    (a) \quad &1 - b_1 + h_1 + c_1 h_2 \geq 0 \quad \text{and} \quad 1 - b_2 + h_2 + c_2 h_1 \geq 0 \\
    (b) \quad &c_1 c_2 \leq 1
\end{align*}
\]

then Eq. (LGIN) has a unique equilibrium in $[0, \infty)^2$.

**Proof.**

(a) Suppose

\[
1 - b_1 + h_1 + c_1 h_2 \geq 0 \quad \text{and} \quad 1 - b_2 + h_2 + c_2 h_1 \geq 0.
\]

From direct inspection of (17) one can see that the functions $f(x,y)$ and $g(x,y)$ in $T(x,y) = (f(x,y), g(x,y))$ satisfy hypothesis (a) of Theorem 4. To verify hypothesis (b) of Theorem 4, note that for $T$ as in (6), the system of equations in (14) is given by

\[
\begin{align*}
    m &= \frac{b_1 m}{1 + m + c_1 M} + h_1 \\
    M &= \frac{b_1 M}{1 + M + c_1 m} + h_1
\end{align*}
\] and

\[
\begin{align*}
    \overline{m} &= \frac{b_2 \overline{m}}{1 + \overline{m} + c_2 \overline{M}} + h_2 \\
    \overline{M} &= \frac{b_2 \overline{M}}{1 + \overline{M} + c_2 \overline{m}} + h_2
\end{align*}
\]

\(9\)
Algebraic manipulation of the equations in (18) yields the equation

\[
c_2(M - m)[(m - h_1) + (M - h_1) + (1 - b_1 + h_1 + c_1 h_2)] \\
+ c_1(M - m m)[(m - h_2) + (M - h_2) + (1 - b_2 + h_2 + c_2 h_1)] = 0
\] (19)

Using hypothesis (17) and the facts \(m, M \geq h_1\) and \(m, M \geq h_2\), we obtain that in equation (19), \(m = M\) and \(m = M\) which shows that hypothesis (b) holds. Then Theorem 3 implies that there is a unique equilibrium.

(b) Suppose \(c_1 c_2 \leq 1\). Applying the transformation \(x = X + h_1, y = Y + h_2\) to the equations defining the sets \(C_1, C_2\) in Lemma 3 gives rise to the equations

\[
\begin{align*}
(a) & \quad (1 - c_1 c_2)X^4 + B X^3 + C X^2 + D X + b_1^2 h_1^2 = 0 \\
(b) & \quad (1 - c_1 c_2)Y^4 + B Y^3 + C Y^2 + D Y + b_2^2 h_2^2 = 0
\end{align*}
\] (20)

where \(B, C, D, B, C, D\) and \(\overline{D}\) are functions of \(b_1, b_2, c_1, c_2, d_1, d_2\). We consider two cases.

Case 1: Suppose \(c_1 c_2 < 1\). Since the leading coefficient of (20a) is \(-c_1 c_2\) and the constant coefficient is \(b_1^2 h_1^2\), the product of the roots of (20a) is \(\frac{b_1^2 h_1^2}{1 - c_1 c_2} > 0\). Similarly, the product of the roots of (20b) is \(\frac{b_2^2 h_2^2}{1 - c_1 c_2} > 0\). From Lemma 5 each equation in (20) must have a positive root \(\alpha\) and a negative root \(\beta\). Furthermore, none of the two equations can have complex roots. Indeed if, say, (20a) had a pair of complex conjugate roots \(z\) and \(\overline{z}\), then the product of the roots of (20a) given by \(\alpha \beta |z|^2\) would be negative, which is impossible. It follows that the set of roots of each equation in (20) must have the sign structure \(\{+, +, -, -\}\). As a result, the set of equilibria of Eq.(LGIN) must have one of the sign structures (i.) \(\{(+, +), (-, -), (+, +), (-, -)\}\) or (ii.) \(\{(+, +), (-, -), (+, +), (-, +)\}\).

The sign structure given by (i.) is not possible since otherwise, there must exist two equilibria of Eq.(LGIN) in \(\mathbb{R}_2^+\) and two equilibria outside \(\mathbb{R}_2^+\). By the proof of Theorem 3 part iv., one of the equilibria in \(\mathbb{R}_2^+\) must have multiplicity two while the other equilibrium in \(\mathbb{R}_2^+\) must have multiplicity one. Thus the sum total of the multiplicities of all the equilibrium points must be at least five, contradicting Bézout’s Theorem. Hence when \(c_1 c_2 < 1\), the only possible sign structure for the set of equilibria of Eq.(LGIN) is (ii.), which implies that Eq.(20a) has a unique equilibrium in \(Q_1(0,0)\), and hence Eq.(LGIN) has a unique equilibrium in \(Q_1(h_1, h_2)\).

Case 2: Suppose \(c_1 c_2 = 1\): In this case, the equations in (20) reduce to

\[
(1 - b_1 - c_1 + b_2 c_1)X^3 + C_1 X^2 + D_1 X + b_1^2 h_1^2 = 0
\] (21)

where \(C_1, D_1, C_1, D_1\) and \(D_1\) are functions of \(b_1, c_1, h_1, b_2, c_2, h_2\). We consider three cases: \(-b_1 - c_1 + b_2 c_1\) is negative, zero, and positive. If \(-b_1 - c_1 + b_2 c_1 \neq 0\), the product of the roots of the first equation in (21) is \(-\frac{b_1^2 h_1^2}{1 - b_1 - c_1 + b_2 c_1}\) and the product of the roots of the second equation in (21) is \(\frac{b_2^2 h_2^2 c_1}{1 - b_1 - c_1 + b_2 c_1}\). Note that from Lemma 5, we have that each equation in (21) must have a positive root and a negative root. Thus the equations in (21) cannot have complex roots due to their cubic nature, since complex roots of real polynomials always occur in conjugate pairs.
If \( 1 - b_1 - c_1 + b_2 c_1 < 0 \), the sets of roots of the equations in (21) must have sign structures \( \{+, -, -\} \) and \( \{+, +, -\} \), respectively, by an argument similar to the one already used in case 1. Hence the set of equilibria of Eq.(LGIN) must have the sign structure \( \{(+, +), (-, -), (-, +)\} \). If \( 1 - b_1 - c_1 + b_2 c_1 > 0 \), the sets of roots of the equations in (21) must have sign structures \( \{+, +\} \) and \( \{+, +, -\} \), respectively. Hence the set of equilibria of Eq.(LGIN) must have the sign structure \( \{(+, +), (-, -), (+, -)\} \). If \( 1 - b_1 - c_1 + b_2 c_1 = 0 \), the equations in (21) reduce to a pair of quadratic equations. It follows from Lemma 5 that each equation must have a positive root and a negative root. So the set of roots of each equation must have the sign structure \( \{+, -\} \). Hence the set of equilibria of Eq.(LGIN) must have the sign structure \( \{(+, +), (-, -), (+, -)\} \), which implies that Eq.(20) has a unique equilibrium in \( Q_1(0,0) \), and hence Eq.(LGIN) has a unique equilibrium in \( Q_1(h_1, h_2) \).

\[ \square \]

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