Coupling-assisted Landau-Majorana-Stuckelberg-Zener transition in two-interacting-qubit systems

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We analyse a system of two interacting spin-qubits subjected to a Landau-Majorana-Stuckelberg-Zener ramp. We show that a LMSZ transitions of the two spin-qubits are possible without an external transverse static field since its role is played by the coupling between the spin-qubits. We show how such a physical effect could be exploited to estimate the strength of the interaction between the two spin-qubits and to generate entangled states of the system by appropriately setting the slope of the ramp. Moreover, the study of effects of the coupling parameters on the time-behaviour of the Entanglement is reported. Finally, our symmetry-based approach allows us to discuss also effects stemming from the presence of a classical noise or non-Hermitian dephasing terms.

I. INTRODUCTION

The Landau-Majorana-Stuckelberg-Zener (LMSZ) scenario¹ and the Rabi one² represent two milestones among exactly solvable time-dependent semi-classical models for two-level systems. A common fundamental property of these two models is the possibility of realizing a full population inversion in a two-state quantum system. In the former case through an adiabatic passage via a level crossing, in the second case thanks to the application of a resonant π-pulse.

The LMSZ scenario, differently from the Rabi case, is an ideal model since it provides for an infinite time duration of the physical process, implying divergence of the instantaneous energy separation as time goes on. This fact creates both mathematical and physical problems in cases where amplitudes and not only probabilities are necessary, e.g. initial states presenting coherences³⁴. In such cases one can use alternatively the Allen-Eberly-Hioe⁵ model, the Demkov-Kunikke model⁶ or other models⁷⁸, where no divergence problems arise and the transition probability is rather simple.

However, despite this circumstance, it is a matter of fact that the LMSZ succeeds in providing accurate results also when applied to more realistic physical systems (finite times and different time-dependences). This relevant aspect has increased the popularity of the LMSZ model and several efforts have been done towards its generalization to the case of finite time duration⁹ and to N-level quantum systems¹⁰¹¹. Moreover, its experimental feasibility gave it a basic role in the area of quantum technology thanks also to the several sophisticated techniques developed for a precise local manipulation of the state and the dynamics of a single qubit in a chain¹²¹⁷.

In such an applicative scenario, as we know, several sources of incoherences can be present¹⁸–²¹: incoherent (mixed) states, relaxation processes (e.g., spontaneous emission) or interaction with a surrounding environment (e.g., nuclear spin bath). They generate incoherent excitation leading to departure from a perfect (ideal) population transfer. Therefore, more realistic descriptions of quantum systems subjected to LMSZ scenario comprising such effects have been proposed²²–²⁵.

In this respect, since the most relevant influence in the dynamics of a spin-qubit stems basically from the stronger coupling with its nearest neighbours, it is of fundamental importance to understand the contribution of such coupling(s) in the dynamics of the spin-qubit. Recently, indeed, the attention has been focused on double interacting spin-qubits systems subjected to LMSZ scenario²⁶–²⁸ aimed at identifying such effects and their potentiality for possible experimental techniques and protocols. Moreover, such systems, under specific conditions, behave effectively as a two-level system with relevant applicability in quantum information and computation sciences²⁹. In the references cited before, indeed, generation of entangled states²⁶ or the singlet-triplet transition¹⁵,²⁷,²⁸ in the two-qubit system under the LMSZ scenario have been studied.

With the same objective in mind, that is to characterize physical effects stemming from the coupling between two spin-qubits subjected to a LMSZ scenario, in this paper we study a two-spin-1/2 system described by a C₃-symmetry Hamiltonian model. We consider coupling terms compatible with the symmetry of the Hamiltonian, namely isotropic and anisotropic exchange interaction. The two spin-1/2’s are moreover subjected only to a LMSZ ramp with no transverse static field. We show that LMSZ transitions for the two spin-qubits are still possible thanks to the presence of the coupling, playing the role of an effective transverse field. Such an effect, we call coupling-assisted LMSZ transition, results of physical interest for two reasons. Firstly, it can be exploited to estimate the presence and the relative weight of different coupling parameters determining the symmetry of the Hamiltonian and then the dynamics of the two spins. Secondly, through such an estimation, it is possible to set the slope of the field ramp in such a way to generate asymptotic entangled states of the two qubits.

The paper is organized as follows. In Sec. II we introduce the model and its symmetry properties on which the dynamical reduction is based. In Sec. III the application of the LMSZ scenario on both the subdynamics is performed. Moreover, physical effects stemming from the (an)isotropy of the exchange interaction are brought to light. In the subsequent Sec. IV, we emphasize the possibility of estimating the values of the coupling parameters and the generation of asymptotic
entangled states of the two spins thorough coupling-based LMSZ transitions is reported in Sec. V. Some effects of a possible interaction with a surrounding environment, providing for either a classical noisy field component or non-Hermitian terms in the Hamiltonian model, are taken into account in Sec. VI. Finally, some conclusive comments and further remarks are discussed in the last Sec. VII.

II. THE MODEL

Let us consider the following model, describing two interacting spin-qubits:

\[ H = \hbar \omega_1 \sigma_1^z + \hbar \omega_2 \sigma_2^z + \gamma_1 \sigma_1^+ \sigma_2^- + \gamma_2 \sigma_1^- \sigma_2^+ \]

where \( \sigma_i^x, \sigma_i^y \) and \( \sigma_i^z \) (\( i = 1, 2 \)) are the Pauli matrices and all the parameters may be thought as time-dependent. The matrices are represented in the following ordered two-spin basis \( \{|\pm\rangle, |\mp\rangle\} \) \((\sigma^z \pm = \pm \bar{z})\).

The \( C_2 \)-symmetry with respect to the \( z \)-direction, possessed by the Hamiltonian, causes the existence of two dynamically invariant Hilbert subspaces related to the two eigenvalues of the constant of motion \( \sigma_1^z \sigma_2^z \). Basing on such a symmetry, the time evolution operator, solution of the Schrödinger equation \( i\hbar \dot{U} = HU \), may be formally put in the following cast\(^{30}\)

\[
 U = \begin{pmatrix}
 a_+(t) & 0 & 0 & b_+(t) \\
 0 & a_-(t) & b_-(t) & 0 \\
 0 & -b_-(t) & a_-(t) & 0 \\
 -b_+(t) & 0 & 0 & a_+(t) \\
\end{pmatrix}. \tag{2}
\]

The condition \( U(0) = 1 \) is satisfied by putting \( a_\pm(0) = 1 \) and \( b_\pm(0) = 0 \). It is worth to note that \( a_\pm(t) \) and \( b_\pm(t) \) are the time-dependent parameters of the two evolution operators

\[
 U_\pm = e^{i\gamma_\pm t/\hbar} \begin{pmatrix}
 a_\pm(t) & b_\pm(t) \\
 -b_\pm(t) & a_\pm(t) \\
\end{pmatrix}, \tag{3}
\]

solutions of two independent dynamical Cauchy problems of fictitious single spin-1/2, namely \( i\hbar U_\pm = H_\pm U_\pm, U_\pm(0) = 1_\pm \), with

\[
 H_\pm = \begin{pmatrix}
 \hbar \Omega_\pm(t) & \gamma_\pm \\
 -\gamma_\pm & -\hbar \Omega_\pm(t) \\
\end{pmatrix}, \quad \gamma_\pm = (\gamma_1 \mp \gamma_2). \tag{4}
\]

Thus, it means that the solution of the dynamical problem of the two interacting spin-1/2’s is traced back to the solution of two independent problems, each one of single (fictitious) spin-1/2\(^{30}\).

The explicit expressions of \( a_\pm(t) \) and \( b_\pm(t) \) depend on the specific time-dependences of \( \omega_1(t) \) and \( \omega_2(t) \). Although, as shown before\(^{30}\), the dynamical problem of the two spins may be converted into two independent problems of single spin-1/2, we know that we are not able to find the analytical solution of the time-dependent Schrödinger equation for a spin-1/2 subjected to a generic time-dependent magnetic field (that is, in our case, for generic time-dependences of \( \Omega_\pm(t) \)). Therefore, the knowledge of specific exactly solvable time-dependent scenarios for a single spin-1/2 becomes crucial in order to get exactly solvable scenarios and hence coherence control methods for the two interacting spin system dynamics.

III. COUPLING-BASED LMSZ TRANSITION

The possibility of controlling the quantum dynamics of a single spin-qubit while it is interacting with other neighbouring spins requires the capacity of applying local magnetic field on atomic scale. This is experimentally possible thanks to the Scanning Tunneling Microscopy (STM) consisting in applying a local magnetic field on a single spin in a chain through a STM tip\(^{12,13}\). Varying the distance between the tip spin and the one in the chain\(^{13}\) it is possible to produce a LMSZ scenario\(^{13}\) or, more in general, a time-dependent magnetic field on the spin. More in detail, it is the possibility of tuning the exchange interaction between the target spin we wish to manipulate and the spin present on the STM tip which gives rise to an effective magnetic field applied on the target spin\(^{12,13}\).

A. Collective LMSZ Dynamics

At the light of this experimental scenario, which we call STM scenario, we take into account firstly the case of a LMSZ ramp applied on the first spin such that

\[
 \hbar \omega_1(t) = \alpha t/2, \quad \hbar \omega_2(t) = 0, \quad t \in (-\infty, \infty), \tag{6}
\]

where \( \alpha \) is related to the velocity of variation of the field, \( B_t \propto \alpha \), and it is considered a positive real number without loss of generality. Let us consider, moreover, the two spins initialized in the state \( |\mp\rangle \). In this instance, the subdynamics is characterized by a LMSZ scenario where the longitudinal \( \langle z \rangle \) magnetic field produces the standard LMSZ ramp \( \hbar \Omega_\pm(t) = \hbar \omega_1(t) = \alpha t/2 \) and the transverse effective magnetic field along the \( x \)-direction is given by \( \gamma_+ \). It is well-known that the dynamical problem for such a time-dependent scenario can be exactly solved analytically and the transition probability of finding the two-spin system in the state \( |++\rangle \) coincides with the probability of finding the fictitious spin-1/2 subjected to \( H_\pm \) in its state \( |\pm\rangle \) starting from \( |\mp\rangle \), which reads\(^{1}\)

\[
 P_+ = |\langle ++|U_+(\infty)|\pm\rangle|^2 = 1 - \exp\left(-2\pi \gamma_+^2 / \hbar \alpha \right). \tag{7}
\]

If we now, instead, consider the two spins initially prepared in \( |++\rangle \), the probability for each spin-1/2 of undergoing a LMSZ transition, that is the probability of finding the two-spin system in the state \( |+-\rangle \), results

\[
 P_- = |\langle ++|U_-(\infty)|\mp\rangle|^2 = 1 - \exp\left(-2\pi \gamma_-^2 / \hbar \alpha \right). \tag{8}
\]
This time the transition probability is governed by the fictitious magnetic field given by $\gamma$. The effective longitudinal magnetic field, instead, is the same, namely $h\Omega_z(t)=\hbar\omega_l(t)=\alpha t/2$. We see that in both cases, though the absence of a constant transverse magnetic field, the LMSZ transition of both the spins is possible thanks to the presence of a coupling between them. It is important to stress that, for the cases considered before, if $\gamma_\alpha=\gamma_\beta$ (as it often happens experimentally) we cannot have transition in the first case, that is in the subdynamics involving $|++\rangle$ and $|--\rangle$. In this instance, indeed, $P_+$ happens to be 0 at any time.

**B. Isotropy Effects: Local LMSZ Transition by nonlocal Control and State Transfer**

The symmetry-based dynamical decomposition and the application of the STM LMSZ scenario in each subdynamics allow us, besides the estimation of the coupling parameters, to bring to light peculiar evolutions of physical interest. For example, if we consider $\gamma_\alpha \neq \gamma_\beta$ and the following initial condition

$$|\rangle\otimes\frac{|++\rangle+|--\rangle}{\sqrt{2}}, \quad (9)$$

the two states $|++\rangle$ and $|--\rangle$ evolve independently and applying the LMSZ ramp we have the probability $P=P_+P_-$ to find asymptotically the two-spin system in the state

$$|\rangle\otimes\frac{|++\rangle+|--\rangle}{\sqrt{2}}. \quad (10)$$

We see, that such a dynamics leaves unaffected the second spin, while it produces a LMSZ transition only on the first spin. More relevant, it is the dynamical evolution of the symmetric initial condition

$$\frac{|++\rangle+|--\rangle}{\sqrt{2}}\otimes|\rangle. \quad (11)$$

This time, we get the same probability $P=P_+P_-$ of finding asymptotically the two-spin system in

$$\frac{|++\rangle+|--\rangle}{\sqrt{2}}\otimes|--\rangle. \quad (12)$$

Although, as it is reasonable, we are reproducing the same dynamics but with inverted parts (the role of the two spins is interchanged), this case results more interesting since, in this instance, we generate a LMSZ transition only on the second spin by locally applying the field on the first spin. This shows how the coupling between the two spins, besides to be fundamental for the occurrence of the LMSZ transition, turns out to be a vehicle through which induce a specific dynamics on the second spin in the chain by locally manipulating the first ancilla qubit.

If we consider, instead, $\gamma_\alpha = \gamma_\beta = \gamma/2$ we know that the transition $|--\rangle \rightarrow |++\rangle$ (and *vice versa*) is suppressed. This means that if we consider as initial conditions the states in Eqs. (9) and (11), we get asymptotically, this time, the states

$$\frac{|++\rangle+|--\rangle}{\sqrt{2}}\otimes|\rangle, \quad (13a)$$

$$|\rangle\otimes\frac{|++\rangle+|--\rangle}{\sqrt{2}}, \quad (13b)$$

respectively, with probability $P = 1-\exp\{-2\pi\gamma^2/h\alpha\}$. We see that the isotropy properties of the exchange interaction consistently change the dynamics of the system. When the exchange interaction is isotropic, indeed, the asymptotic states reached by the initial conditions (9) and (11) radically change and, in these cases, the resulting physical effect is a state transfer or a state exchange between the two spin-qubits. Therefore, we may consider the different state transitions from the state (9) [(11)] to the states (10) or (13a) [(12) or (13b)] as witnesses of the isotropy or anisotropy of the exchange interaction.

**IV. COUPLING PARAMETER ESTIMATION**

It is interesting noticing that the coupling-based LMSZ transition could be used to estimate the coupling parameters. By measuring $P_+$ and $P_-$ (Eqs. (7) and (8), respectively) in a physical scenario describable by the Hamiltonian model (1), we get an estimation of $\gamma_\alpha$ and $\gamma_\beta$ and then of the two coupling parameters $\gamma_\alpha$ and $\gamma_\beta$. Supposing to know $P_+$ and $P_-$, we have indeed

$$\gamma_\alpha = \frac{1}{2} \sqrt{\frac{\hbar\gamma}{2\pi}} \left[ \log \left( \frac{1}{1-P_-} \right) - \log \left( \frac{1}{1-P_+} \right) \right], \quad (14)$$

$$\gamma_\beta = \frac{1}{2} \sqrt{\frac{\hbar\gamma}{2\pi}} \left[ \log \left( \frac{1}{1-P_+} \right) - \log \left( \frac{1}{1-P_-} \right) \right].$$

We wish to emphasize that we may estimate the coupling parameters also through the Rabi oscillations occurring in the two subspaces. Applying, indeed, a constant field $\delta_l$ on the first spin, the two probabilities $P_+$ and $P_-$ become

$$P_+ = \frac{\gamma_\alpha^2}{\hbar^2\omega_l^2 + \gamma_\alpha^2} \sin^2 \left( \sqrt{\omega_l^2 + \gamma_\alpha^2/\hbar^2} t \right),$$

$$P_- = \frac{\gamma_\beta^2}{\hbar^2\omega_l^2 + \gamma_\beta^2} \sin^2 \left( \sqrt{\omega_l^2 + \gamma_\beta^2/\hbar^2} t \right). \quad (15)$$

So, by measuring the frequency and the amplitude of the oscillations in the two cases we may get information about the the relative weights of the coupling parameters.

**V. ENTANGLEMENT**

A precise estimation of the coupling parameters is useful also to generate entangled states of the two spins. By the knowledge of them, indeed, we may set the parameter $\alpha$ in
order to get asymptotically \( P_\pm = 1/2 \), generating so an entangled state. Indeed, if the two spins start from state \(|-| \> \) or \(|+| \> \), being the dynamics unitary, they reach asymptotically the pure state \((|+| + e^{i\theta}|-|)/\sqrt{2} \) in the first case and \((|+| + e^{i\theta}|-|)/\sqrt{2} \) in the second case, which are a maximally entangled states. The asymptotic curves of the Concurrence\(^8\), in fact, when the two-spin system is initialized in \(|-| \> \) or \(|+| \> \), read respectively

\[
C = 2|c_{++} c_{--}| = 2 \sqrt{P_+ (1 - P_+)} = 2 \sqrt{(1 - e^{-2\pi \beta_+}) e^{-2\pi \beta_+}},
\]

and

\[
C = 2|c_{+-} c_{-+}| = 2 \sqrt{P_- (1 - P_-)} = 2 \sqrt{(1 - e^{-2\pi \beta_-}) e^{-2\pi \beta_-}},
\]

and they exhibit a maximum for \( \beta_+ = \beta_- = \log(2)/2\pi \approx 0.11 \). In the previous expressions we put \( \beta_+ = \gamma_+^2/\hbar \alpha \) and \( \beta_- = \gamma_-^2/\hbar \alpha \), while \( c_{++} \) and \( c_{--} \) (\( c_{+-} \) and \( c_{-+} \)) are the asymptotic amplitudes of the states \(|+| \> \) and \(|-| \> \) \((|+| \> \) and \(|-| \> \)), respectively. Therefore, \( \log(2)/2\pi \) is exactly the value the LMSZ parameters \( \beta_+ \). \( \beta_- \) must have to realize the generation of the entanglement states \(|+| + e^{i\theta}|-|)/\sqrt{2} \) when the two spins start from \(|-| \> \) or \(|+| \> \), respectively. Figure 1a reports the two curves for \( \beta_-/2 = \beta_+ = \beta \).

We may verify this fact by investigating the behaviour of the Concurrence in time. To this end, the exact solutions of the two time-dependent parameters determining the time evolution operators \( U_+ \) and \( U_- \) in Eq. (3), related to each sub-dynamics, are necessary and the reads, namely\(^9\)

\[
a_\pm = \frac{\Gamma_\pm (1 - i\beta_\pm)}{\sqrt{2\pi}} \times \begin{pmatrix} D_{\pm \beta_\pm} \left( \sqrt{2} e^{-i\pi/4} \tau \right) & D_{-1 + i\beta_\pm} \left( \sqrt{2} e^{3\pi/4} \tau \right) \\ D_{1 + i\beta_\pm} \left( \sqrt{2} e^{i\pi/4} \tau \right) & D_{-1 + i\beta_\pm} \left( \sqrt{2} e^{-3\pi/4} \tau \right) \end{pmatrix},
\]

\[
b_\pm = \frac{\Gamma_\pm (1 - i\beta_\pm)}{\sqrt{2\pi \beta}} e^{i\pi/4} \times \begin{pmatrix} -D_{\pm \beta_\pm} \left( \sqrt{2} e^{-i\pi/4} \tau \right) & D_{-1 + i\beta_\pm} \left( \sqrt{2} e^{3\pi/4} \tau \right) \\ D_{1 + i\beta_\pm} \left( \sqrt{2} e^{i\pi/4} \tau \right) & D_{-1 + i\beta_\pm} \left( \sqrt{2} e^{-3\pi/4} \tau \right) \end{pmatrix},
\]

and

\[
\tau = \sqrt{\alpha/\hbar} t \text{ is a time dimensionless parameter; } \tau \text{ identify the initial time instant. If the system starts, e.g., from the state } |\pm| \text{ the amplitudes result}
\]

\[
c_{++} = b_+, \quad c_{--} = a^*_+, \quad c_{+-} = c_{-+} = 0,
\]

and the related time-behaviour of the Concurrence \( C = |b_+||a_+| \) for \( \beta_+ = 0.1 \) is reported in Fig. 1b. We see, as expected, that such a choice of the LMSZ parameter generate a maximally entangled state of the two spin-qubits. It is important to point out that, on the basis of Eqs. (17), the parameter \( \beta_- \) determines not only the asymptotic value of the Concurrence but also its time behaviour. This fact is confirmed and can be appreciated by Figs. 2a and Fig. 2b reporting the Concurrence against the dimensionless parameter \( \tau \) for \( \beta_+ = 1/2 \) and \( \beta_+ = 2 \), respectively. The physical meaning of the asymptotic vanishing of \( C \) in Fig. 2b is that for the specific value of \( \beta_+ \) the system evolves quite adiabatically towards the factorized states \(|-| \> \). On the contrary, in Figs. 2a the slope of the ramp induces a non adiabatic evolution towards a coherent not factorizable superposition of \(|+| \) and \(|-| \).

**Figure 1:** (Color online) a) The two curves of the Concurrence in Eq. (16a) (full blue line) and Eq. (16b) (red dashed line) for \( \beta_-/2 = \beta_+ = \beta \); b) Time behaviour of Concurrence for the initial condition \(|-| \> \) and \( \beta_+ = 0.1 \).

**Figure 2:** (Color online) Time behaviour of the Concurrence against the dimensionless parameter \( \tau = \sqrt{\alpha/\hbar} t \) during a LMSZ process when the system starts from the state \(|-| \> \) for a) \( \beta_+ = 1/2 \) and b) \( \beta_+ = 2 \).

Analogous results would be got by studying the LMSZ process when the two spin-qubits start from the state \(|+| \> \). In this case, only the states \(|+| \> \) and \(|-| \> \) would be involved and the LMSZ parameter determining the different Concurrence regimes would be \( \beta_- \). For such initial conditions, then, the ratio \( \beta_+ / \beta_- \), imposing precise relationships between the coupling parameters \( \gamma_+ \) and \( \gamma_- \), does not matter.

Such a ratio, conversely, result determinant for other initial conditions, e.g., the one considered in Eq. (10). In this case the amplitudes read

\[
c_{++} = a_+, \quad c_{--} = -b^*_+, \quad c_{+-} = a_-, \quad c_{-+} = -b^*_-. \]

In Figs. 3a-3f we may appreciate the influence of both the ratio \( \beta_-/\beta_+ \) and the free parameter \( \beta_+ \); the former influences only qualitatively the behaviour of the Concurrence, while the latter determines both a qualitative and quantitative such a behaviour. Also this time, we note that for high values of \( \beta_+ \) the Concurrence comes back to zero witnessing an asymptotic factorized state, while, for small values of \( \beta_+ \), the level of Entanglement remains non-vanishing also for large time.
meaning that an entangled superposition of the four standard basis states is reached.

We emphasize that the same calculations of the Concurrency may be analogously proposed in the case of presence of the DM interaction. In such an instance the roles of $\gamma_+^2$ and $\gamma_-^2$ in the expressions of $\beta_+$ and $\beta_-$, are played by $\gamma_+^2 + \Gamma_+^2$ and $\gamma_-^2 + \Gamma_-^2$, respectively.

VI. EFFECTS OF CLASSICAL NOISE

In experimental physical contexts involving atoms, ions and molecules investigated and manipulated by application of lasers and fields, the presence of noise in the system stemming from the coupling with a surrounding environment is unavoidable. Though a lot of technological progresses and experimental experiences have been developed, it is necessary to introduce such decoherence effects in the theoretical models for a better understanding and closer description of the experimental scenarios. There exist different approaches to treat the influence of a thermal bath; one is to consider the presence of classical noisy fields, e.g., from the presence and the influence of a surrounding nuclear spin bath.

In the last reference the authors study a noisy LMSZ scenario for a $N$-level system. They take into account a time-dependent magnetic field $\eta(t)$ only in the $z$-direction and characterized by the following time correlation function $\langle \eta(t)\eta(t') \rangle = 2G\delta(t-t')$. The authors show that for a spin-1/2 and for large values of $G$ the LMSZ transition probability changes as

$$P_+ = \frac{1 - \exp(-2\pi g^2/\hbar)}{2},$$

where $g$ is the energy contribution due to the coupling of the spin-1/2 with the constant transverse magnetic field and $\alpha$ is the ramp of the longitudinal magnetic field. We see that the value of $G$, provided that it is large, does not influence the transition probability. The unique effect of the noisy component is the loss of coherence, since it cannot generate transitions between the two diabatic states, being only in the same direction of the quantization axis. In this way the transition probability, as reasonable, results hindered by the presence of the noise, since, for $g^2/\alpha \gg 1$, at most the system reaches the maximally mixed state.

This result is of particular interest in our case since the addition of the noisy component $\eta(t)$ leaves completely unaffected the symmetry-based Hamiltonian transformation and the validity of the dynamics-decoupling procedure. Thus, also in this case, the dynamical problem of the two-qubit system may be converted into two independent spin-1/2 problems affected by a random fluctuating $z$-field. Thus, we may write easily the transition probabilities when the two spins are subjected to a unique homogeneous field influenced by the noisy component considered before. We have precisely

$$P_+ = \frac{1 - \exp(-2\pi g^2/\hbar)}{2}, \quad \omega_1(t) = \omega_2(t) = [\alpha t + \eta(t)]/4,$$

Finally, we underline that in the authors considered a system of two spin-1/2’s interacting only through the term $\hat{S}_1^z \hat{S}_2^z$ and subjected to the same magnetic field consisting in a Gaussian pulse uniformly rotating in the $x-y$ plane and a LMSZ ramp in the $z$-direction. They showed that the coupling between the two spins enhances significantly the probability to drive adiabatically the two-spin system from the separate state $|\rangle$ to the entangled state $(|++\rangle + |+-\rangle)/\sqrt{2}$. In this case the procedure to generate an entangled state is different from the scenario considered here because of the different symmetries of the Hamiltonians ruling the two-spin dynamics. Indeed, in the Hamiltonian commutes with $\hat{S}_z$ and consequently two dynamically invariant Hilbert subspaces exist: one of dimension three and the other of dimension one. The three-dimensional subspace is spanned by the states $|++\rangle$, $|+--\rangle$, $|--+\rangle$ and $|--\rangle$, making possible the preparation of the entangled state of the two spin-1/2’s by an adiabatic passage when they start from the separate state $|--\rangle$. In our case, instead, $\hat{S}_z$ is not constant while the integral of motion is $\hat{S}_1^z \hat{S}_2^z$. The symmetries of the Hamiltonian, thus, generate two bi-dimensional dynamically invariant Hilbert subspaces: one spanned by $|++\rangle$ and $|+-\rangle$ and the other by $|+--\rangle$ and $|--\rangle$. Then, in our case, the transition between the states considered in the other work is impossible since such states belong to different invariant subspaces.

![Figure 3: (Color online) Time behaviour of the Concurrence against the dimensionless parameter $\tau = \sqrt{\alpha}/\hbar$ during a LMSZ process when the system starts from the state $|++\rangle + |+-\rangle)/\sqrt{2}$ for $\beta_+ = 1/2$ and a) $\beta_+ = 1/2$, b) $\beta_+ = 2$; c) $\beta_+ = 1/2$, d) $\beta_+ = 2$; e) $\beta_+ = 1/10$, f) $\beta_+ = 10$.](image-url)
We underline that the transition probability $P_-$ vanishes in case of an unique homogeneous magnetic field. In this case, indeed, the effective field ruling the two-spin dynamics is zero, namely $\Omega_-(t) = 0$. Moreover, for $\gamma = \gamma$, we would have no physical effects, since, in such a case, also $P_+$ would result zero.

Another way to face with the problem of open quantum systems is to use non-Hermitian Hamiltonians effectively incorporating the information of the fact that the system they describe is interacting with a surrounding environment\cite{34-39}. We may suppose, for example, that the spontaneous emission from the up-state to the down-one is negligible and that some mechanism makes the up-state $|+\rangle$ irreversibly decaying out of the system with rate $\xi$ and $\xi'$ for the first and second spin-1/2, respectively. It is well known that we can phenomenologically describe such a scenario by introducing the non-Hermitian terms $i\xi \hat{\sigma}_z^1/2$ and $i\xi' \hat{\sigma}_z^2/2$ in our Hamiltonian model. Analogously to the case of a noisy field component, also the introduction of these terms does not alter the symmetry of the Hamiltonian model getting a simple redefinition of the parameters in front of the operators $\hat{\sigma}_z^1$ and $\hat{\sigma}_z^2$. The symmetry-based transformation, thus, leads us to two independent non-Hermitian two-level models. In the same way we may exploit the results got for a single qubit with a decaying state subjected to the LMSZ scenario\cite{22,23,25} and reread them in terms of the two-spin-1/2 language. We know that the decaying rate affects only the time-history of the transition probability but not, surprisingly, its asymptotic value\cite{22}. However, this result is valid for the ideal LMSZ scenario; considering the more realistic case of a limited time window, indeed, it has been demonstrated that a decaying rate-dependence for the population of the up-state arises\cite{53}.

\section*{VII. CONCLUSIVE REMARKS}

In this work we considered a physical system of two interacting spin-1/2’s whose coupling comprises the terms stemming from the anisotropic exchange interaction. Moreover, each of them is subjected to a local field linearly varying over time. The $C_2$-symmetry (with respect to the quantization axis $\hat{z}$) possessed by the Hamiltonian allowed us to identify two independent single spin-1/2 sub-problems nested in the quantum dynamics of the two spin-qubits. This fact gave us the possibility of decomposing the dynamical problem of the two spin-1/2’s into two independent problems of single spin-1/2. In this way, our two-spin-qubit system may be regarded as a four-level system presenting an avoided crossing for each pair of instantaneous eigenenergies related to the two dynamically invariant subspaces. This aspect turned out to be the key to solve exactly and easily the dynamical problem, bringing to light several physically relevant aspects.

In case of time-dependent Hamiltonian models, such a symmetry-based approach and the reduction to independent problems of single spin-1/2 has been used also in other cases\cite{40,42}. This fact permits a deep understanding of the quantum dynamics of the spin systems with consequent potential applications in quantum information and computation.

We underline, in addition, that the dynamical reduction exposed in Sec. II is independent of the time-dependence of the fields. Thus, we may consider also different exactly solvable time-dependent scenarios\cite{43-49} for the two subdynamics, resulting, of course, in different two-spin dynamics and physical effects.

In this paper, we showed that, although the absence of a transverse chirp\cite{36} or constant field, MLSZ transitions are still possible, precisely from $|--\rangle$ to $|++\rangle$ and from $|+\rangle$ to $|--\rangle$ (the two couples of states spanning the two dynamically invariant Hilbert spaces related to the symmetry Hamiltonian). Such transitions occur thanks to the presence of the coupling between the spins which plays as effective static transverse field in each subdynamics.

It is worth noticing that, in our model, the two MLSZ sub-dynamics are ruled either by different combinations of the externally applied fields (when the local fields are different) or by the same field (under the STM scenario, that is when one local field is applied on just one spin). In the latter case we showed the possibility of 1) a non-local control, that is to manipulate the dynamics of one spin by applying the field on the other one and 2) a state exchange/transfer between the two spins. We brought to light how such effects are two different replies of the system depending on the isotropy properties of the exchange interaction.

Concerning the interaction terms, each subdynamics is characterized by different combinations of the coupling parameters. This aspect has relevant physical consequences since, as showed, by studying the transition probability in the two subspaces, it is possible both to evaluate the presence of different interaction terms and to estimate their weights in ruling the dynamics of the two-spin system. We brought to light how the estimation of the coupling parameters could be of relevant interest since, through this knowledge, we may set the slope of variation of the LMSZ ramp as to generate asymptotically entangled states of the two spin-1/2’s. Moreover, we reported the exact time-behaviour of the Entanglement for different initial conditions and we analysed how the coupling parameters can determine different Entanglement regimes and asymptotic values.

Finally, we emphasized how our symmetry-based analysis results useful also to get exact results when a classical random field component or non-Hermitian terms are considered to take into account the presence of a surrounding environment interacting with the system. In this case, the dynamics decomposition is unaffected by the presence of the noise or the dephasing terms and then we may apply the results previously reported for a two-level system\cite{22,23,24} and reread them in terms of the two spin-1/2’s.

Two challenging problems naturally extending the investigation here reported are 1) that considering the interaction of two qutrits in place of two qubits and 2) that taking into account the coupling of the two spins with a quantum bath\cite{50} in place of the interaction with a classical random field.
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