Gaugino and meson condensates in $\mathcal{N} = 1$ SQCD from
Seiberg-Witten curves

Michael Chesterman∗

Department of physics, Karlstad University, S-651 88 Karlstad, Sweden and
Nordita, Blegdamsvej 17, Copenhagen, DK-2100, Denmark.

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Abstract

We calculate gaugino and meson condensates in $\mathcal{N} = 1$ SQCD theory with $SU(N_c)$ gauge group and $N_f < 2N_c$ matter flavours, by deforming the pure $\mathcal{N} = 2$ Super-Yang-Mills plus fundamental matter action with a mass term for the adjoint scalar superfield. This follows similar recent work by Konishi and Ricco for the case without fundamental matter.

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∗Electronic address: Michael.Chesterman@kau.se
I. INTRODUCTION

The gaugino and meson vacuum condensates of SQCD are fundamental features of the theory, which are associated with the presence of a mass gap, confinement and instantons. Over the years, various methods have emerged for calculating the values of these condensates.

An indirect but simple way is to calculate the minima of one of two related non-perturbative quantum superpotentials: the Veneziano-Yankielovicz-Taylor (VYT) superpotential \[^{1}\] , which was conjectured on the basis of reproducing the correct \(U(1)\) anomalies of the theory, or the Affleck-Dine-Seiberg (ADS) superpotential \[^{2}\] , which was deduced using holomorphy arguments, and the \(U(1)_{A}\) selection rule. For the ADS superpotential, which only yields the meson condensate, one can obtain the gaugino condensate by using a suitable Konishi anomaly equation. Note that both of these superpotentials only exist when the number of flavours is less than the number of colours \(N_f < N_c\). For the region \(N_f \geq N_c\), alternative techniques for calculating the condensates are needed.

There are also direct methods, as described in \[^{3}\] . The so-called strong coupling instanton (SCI) and weak coupling instanton (WCI) one-loop calculations \[^{4}\] , though developed for calculating the gaugino condensate in pure \(\mathcal{N} = 1\) SYM, can also be applied to SQCD by taking into account the differing one-loop beta function. The meson condensate can then be obtained from a Konishi anomaly equation. The results of the two methods disagree by a factor of 4/5, though it is widely believed that the SCI calculation is not reliable \[^{4}\] and that we should believe the WCI calculation, which also agrees with the results from the VYT and ADS superpotentials. The main problem with the SCI approach seems to be that in the strong coupling confining phase of the theory, semi-classical instanton calculations are not valid. In another interesting technique \[^{5}\] which could be generalized to the case of fundamental matter, space-time is compactified on \(R^3 \times S^1\), in order that the semi-classical instanton calculation is valid.

In our approach, we use the intimate relationship between \(\mathcal{N} = 1\) and \(\mathcal{N} = 2\) SYM, extending Konishi and Ricco’s calculation \[^{3}\] to the case of fundamental matter with \(SU(N_c)\) gauge group and \(N_f < 2N_c\) flavours. In turn, their work was a generalization of an article by Finnell and Pouliot \[^{6}\] for \(SU(2)\). It may come as no surprise that our results agree with the prediction of the VYT and ADS superpotentials, and the WCI calculation. Nevertheless, it is satisfying, to have confirmation from another independent source. Our method is also
valid for a wider range of flavours than for the two superpotentials.

II. THE CALCULATION

As previously mentioned, starting with $\mathcal{N} = 2$ super Yang-Mills with $SU(N_c)$ gauge group and $N_f < 2N_c$ flavours, we add a mass term $\mu \text{Tr} \phi^2$ for the $\mathcal{N} = 1$ adjoint scalar field $\phi$. In terms of the resultant $\mathcal{N} = 1$ theory, the classical superpotential is given by

$$W_{\text{tree}} = \mu \text{Tr} \phi^2 + \sqrt{2} \text{Tr} \sum_{f=1}^{N_f} \tilde{Q}_f \phi Q^f + \text{Tr} \sum_{f=1}^{N_f} \tilde{Q}_f m_f Q^f,$$

where $Q^f$ and $\tilde{Q}_f$ are quark superfields in the fundamental representation of the $SU(N_c)$ gauge group, and $m_f$ are the quark masses for $f = 1, \ldots, N_f$. We focus on the case $N_f < 2N_c$ where the $\mathcal{N} = 2$ theory is asymptotically free, hence Seiberg-Witten curves can be used.

In the limit $\mu \to \infty$, the field $\phi$ decouples and can be integrated out to leave pure SQCD. Classically, when all $N_f$ quark masses are non-zero, the minima of the SQCD potential is given by $Q^f = \tilde{Q}_f = 0$.

The Seiberg-Witten curve for pure $\mathcal{N} = 2$ plus fundamental matter and $SU(N_c)$ gauge group is

$$y^2 = \frac{1}{4} \prod_{a=1}^{N_c} (x - \phi_a)^2 - \Lambda^{2N_c - N_f} \prod_{f=1}^{N_f} (x + m_f),$$

where $\phi = \text{diag} (\phi_1, \ldots, \phi_{N_c})$ and $\sum_{a=1}^{N_c} \phi_a = 0$.

We can in principle calculate $< \text{Tr} \phi^2 >$ for the $\mathcal{N} = 1$ theory described by $W_{\text{tree}}$ in (1), using the Seiberg-Witten curve. The mass term $\mu \text{Tr} \phi^2$ lifts the vacuum degeneracy and causes the moduli, $\phi_a$, to move to the point on the curve where the maximum number of massless magnetic monopoles condense. These $(N_c - 1)$ massless monopoles are responsible, via the Higgs mechanism, for giving a mass to the abelian gauge-fields belonging to the $\mathcal{N} = 2$ multiplet associated with $\phi_a$, as argued in [7, 8].

The maximum $(N_c - 1)$ massless magnetic monopoles correspond to $(N_c - 1)$ double zeros of the SW curve. So $\phi_a$ are chosen such that the curve is of the form

$$y^2 = F_2(x) H_{N_c - 1}^2(x),$$

where the subscript on $F$ and $H$ refers to the degree of the polynomial in $x$. The tuning of $\phi_a$ such that the curve is of the above form is known as performing a complete factorization of the curve.
In practice, the curve with fundamental matter is extremely difficult to factorize. In fact, it is only recently that the general solution has been found \cite{10}, by exploiting new powerful tools \cite{10} for calculating F-terms of the $\mathcal{N} = 1$ quantum superpotential. We will not need these techniques though, as in the limit $\mu \to \infty$, the curve (2) simplifies to a more manageable form, where the technique of Chebysev polynomials is applicable, as in \cite{3}.

First we write the $\mathcal{N} = 2$ dynamical scale $\Lambda_{\mathcal{N}=2}$ in terms of the scale on the curve $\Lambda$

$$
\Lambda^2 = 2^{-N_c/(N_c-\frac{1}{2}N_f)}\Lambda_{\mathcal{N}=2}^2.
$$

We then match the dynamical scales of the two theories in the usual way

$$
\Lambda_{\mathcal{N}=1}^{3N_c-N_f} = \mu^{N_c}\Lambda_{\mathcal{N}=2}^{2N_c-N_f}.
$$

Before taking the large $\mu$ limit, it is convenient to define new variables

$$
\hat{\phi}_a = \sqrt{\mu}\phi_a, \; \hat{x} = \sqrt{\mu}x, \; \hat{y}^2 = \mu^{N_c}y^2.
$$

In the limit $\mu \to \infty$, while keeping $\Lambda_{\mathcal{N}=1}$ constant, the Seiberg-Witten curve is given by

$$
\hat{y}^2 = \frac{1}{4} \prod_{a=1}^{N_c} (\hat{x} - \hat{\phi}_a)^2 - \frac{\Lambda_{\mathcal{N}=1}^{3N_c-N_f}}{2N_c} \det m,
$$

where $m = \text{diag} (m_1, \ldots, m_{N_f})$, and $\hat{y}^2$ can be factorized using Chebysev polynomials \cite{11}, just as if there’s no fundamental matter.

At the point where there is a maximum number of double zero’s, the curve becomes

$$
\hat{y}^2 = \frac{\Lambda_{\mathcal{N}=1}^{3N_c-N_f}}{2N_c} \det m (T^{2}_{N_c}(\xi) - 1),
$$

where

$$
\xi = (\frac{1}{2N_c\Lambda_{\mathcal{N}=1}^{3N_c-N_f}} \det m)^{\frac{1}{2N_c}} \frac{e^{-2\pi ik/2N_c}}{2N_c} \; k = 1, \ldots, N_c,
$$

where $T_{N_c}(\xi)$ is a Chebysev polynomial of the first kind, as described in appendix A and we have used equation (A5) to normalize $\xi$. So the rescaled moduli are

$$
\hat{\phi}_a = \omega_a (2^{N_c}\Lambda_{\mathcal{N}=1}^{3N_c-N_f} \det m)^{-\frac{1}{2N_c}} e^{2\pi ik/2N_c},
$$

where $\omega_a$ is defined in equation (A2). Note that $\sum_a \hat{\phi}_a = 0$, since $\sum_a \omega_a = 0$. Furthermore

$$
<\text{Tr} \hat{\phi}^2> = \lim_{\mu \to \infty} \mu <\text{Tr} \phi^2> = N_c (\Lambda_{\mathcal{N}=1}^{3N_c-N_f} \det m)^{\frac{1}{2N_c}} e^{2\pi ik/N_c},
$$

where
where we have used that \( \sum_{a=1}^{N_c} \omega_a \omega_a = N_c/2 \).

Using the Konishi anomaly equations \[12\]

\[
\frac{1}{16\pi^2} < \text{Tr} \lambda \lambda > + \frac{1}{N_c} < \text{Tr} q^f \phi q^f > = \frac{\mu}{N_c} < \text{Tr} \phi^2 >, \quad (12)
\]

\[
< \text{Tr} q^f \bar{q}_g > + \sqrt{2} (m^{-1})^f_g < \text{Tr} \tilde{q}_g \phi q^f > = \frac{(m^{-1})^f_g}{16\pi^2} < \text{Tr} \lambda \lambda >, \quad (13)
\]

where \( \lambda \) is the gaugino field, and \( q^f \) and \( \bar{q}_g \) the lowest components of the quark superfields \( Q^f \) and \( \tilde{Q}_g \), we calculate the gaugino and meson condensates. Note that in the limit \( \mu \to \infty \), using equation \( 5 \), \( \Lambda_{N=2} \) becomes zero and hence so does \( \phi_a \). Thus, terms of the form \( < \text{Tr} q \phi \bar{q} > \) can be ignored.

Finally, the gaugino and meson vacuum condensates are given by

\[
\frac{1}{16\pi^2} < \text{Tr} \lambda \lambda > = (\Lambda_{N=1}^{3N_c-N_f} \det m) \frac{1}{N_c} e^{2\pi ik/N_c}, \quad (14)
\]

\[
< \text{Tr} q^f \bar{q}_g > = (m^{-1})^f_g (\Lambda_{N=1}^{3N_c-N_f} \det m) \frac{1}{N_c} e^{2\pi ik/N_c}, \quad (15)
\]

respectively, where \( k = 1 \ldots N_c \).

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APPENDIX A: THE CHEBYSEV POLYNOMIALS

The Chebysev polynomials of the first and second kind are \( T_n(\xi) = \cos(n \cos^{-1} \xi) \) and \( U_n(\xi) = [\sin((n+1) \cos^{-1} \xi)]/\sqrt{1-\xi^2} \) respectively. They obey the relation

\[
T_n^2(\xi) - 1 = (\xi^2 - 1)U_{n-1}^2(\xi) \quad (A1)
\]

The zero’s of \( T_n \) are given by

\[
\omega_a = \cos \left( \frac{\pi}{n} (a - \frac{1}{2}) \right) \quad a = 1, \ldots, n, \quad (A2)
\]

and of \( U_n \) by

\[
\zeta_a = \cos \left( \frac{\pi}{n+1} a \right) \quad a = 1, \ldots, n, \quad (A3)
\]
Thus,
\[
\sum_{a=1}^{n} \omega_a = 0, \quad \sum_{a=1}^{n} \omega_a \omega_a = n/2. \tag{A4}
\]

It is useful to know the explicit factorized form of \( T_n \) and \( U_n \)

\[
T_n(\xi) = \begin{cases} 
\frac{1}{2} \prod_{a=1}^{n/2} (4\xi^2 - 4\omega_a^2) & \text{(n even)} \\
\frac{1}{2} (2\xi) \prod_{a=1}^{(n-1)/2} (4\xi^2 - 4\omega_a^2) & \text{(n odd)}
\end{cases}
\tag{A5}
\]

\[
U_n(\xi) = \begin{cases} 
\prod_{a=1}^{n/2} (4\zeta^2 - 4\omega_a^2) & \text{(n even)} \\
(2\xi) \prod_{a=1}^{(n-1)/2} (4\zeta^2 - 4\omega_a^2) & \text{(n odd)}
\end{cases}
\tag{A6}
\]

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