Entropies of Sums of Independent Gamma Random Variables

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Abstract
We establish several Schur-convexity type results under fixed variance for weighted sums of independent gamma random variables and obtain nonasymptotic bounds on their Rényi entropies. In particular, this pertains to the recent results by Bartczak–Nayar–Zwara as well as Bobkov–Naumov–Ulyanov, offering simple proofs of the former and extending the latter.

Keywords
Entropy · Max-entropy · Gamma distribution · Weighted sums · Schur-convexity

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1 Introduction
Suppose $X_1, X_2, \ldots, X_n$ are independent, identically distributed (i.i.d.) square-integrable random variables, say with variance 1 and $a = (a_1, a_2, \ldots, a_n)$ is a unit vector in $\mathbb{R}^n$, $\sum_{k=1}^n a_k^2 = 1$, so that the variance of the sum $X_a = \sum_{k=1}^n a_k X_k$ does not depend on $a$ and also equals 1. For which vectors $a$, is the distribution of $X_a$ as close to the Gaussian distribution as possible? A natural way to quantify this vague question is to measure the distance to Gaussianity via relative entropy and ask about

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\[ \inf_a D(X_a \| G) \]. Here, \( D(X \| G) = h(G) - h(X) \) is the relative entropy of \( X \) with respect to a Gaussian random variable \( G \) of the same variance as \( X \), where

\[ h(X) = -\int_{\mathbb{R}} f \log f \]

is the Shannon entropy of a random variable \( X \) with density \( f \).

This question was raised in [13] and addressed for (symmetric) Gaussian mixtures, where the extremising sequence turns out to be simply \( a = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \). In a recent paper [2], Bartczak, Nayar and Zwara considered the case of gamma distribution and established that the same vector is extremal among all nonnegative vectors, that is whose all components are nonnegative. We refer to their paper for a comprehensive account of relevant reference and related problems. Their approach rests on the so-called method of interlacing densities (see also [14]). For the gamma distribution, this entails a rather technical and involved analysis for Bessel functions.

Our first goal in this paper is to offer an alternative approach. It turns out that for the gamma distribution, simple arguments involving moment generating functions allow to establish certain Schur-convexity type results. Those in particular give the main result of [2], as well as partially address Question 6 from [2] about moments.

Our second goal in this paper is to extend a recent result of [9], where Bobkov, Naumov and Ulyanov find a nonasymptotic expression for the maximum of the density of \( X_a = \sum a_k X_k \) with \( X_k \) having \( \Gamma(1/2) \) distribution, equivalently for the \( \infty \)-Rényi entropy of \( X_a \) in terms of \( a \). We extend this to \( \Gamma(\gamma) \) distribution with \( \gamma \geq 1/2 \). Such bounds have applications to Lévy’s concentration function, thus to anti-concentration inequalities (see, e.g. [5] as well as, e.g. the survey [33] for an exposition on anti-concentration).

Another piece of motivation to study such extensions is the fact that weighted sums of independent \( \Gamma(1/2) \) random variables emerge naturally from Gaussian quadratic forms, which was a starting point for both [2, 9].

In the next section, we recall the definition of Rényi entropy and formulate our results. The remaining part of this note will be devoted to their proofs.

## 2 Results

Let \( 0 \leq \alpha \leq \infty \). For a random variable \( X \) with density \( f \), we define its Rényi entropy of order \( \alpha \) as

\[ h_{\alpha}(X) = \frac{1}{1 - \alpha} \log \int f^\alpha, \]

see [35], understood as limits in the cases \( \alpha \in \{0, 1, \infty\} \), namely \( h_0(X) = \log |\text{supp}(f)| \), \( h_1(X) = h(X) \) (the Shannon entropy), \( h_\infty(X) = -\log \|f\|_\infty \). For notational convenience, we also introduce the functional

\[ M(X) = \|f\|_\infty. \]
Throughout, we let $\gamma > 0$ and let $X_1, X_2, \ldots$ be i.i.d. random variables with $\Gamma(\gamma)$ distribution, that is with density $\Gamma(\gamma)^{-1} x^{\gamma-1} e^{-x} 1_{(0,\infty)}(x)$ on $\mathbb{R}$.

### 2.1 Schur-convexity type results and entropies

Our first main result gives the Schur-concavity of centred weighted sums averaged against arbitrary completely monotone functions. For concise exposition on majorisation and Schur-convexity, we refer for instance to Chapter II of [3]. We recall that a function $\Phi : (0, +\infty) \to (0, +\infty)$ is completely monotone if it is a mixture of exponential functions, that is $\Phi(x) = \int_0^\infty e^{-tx} d\mu(t)$ for some nonnegative Borel measure $\mu$, equivalently (by Bernstein’s theorem) $(-1)^m \Phi^{(m)}(x) \geq 0$ for every $m = 0, 1, 2, \ldots$, see, e.g. [15].

**Theorem 1** For a completely monotone function $\Phi : (0, +\infty) \to (0, +\infty)$ and $c > 0$, the function

$$
(a_1, \ldots, a_n) \mapsto \mathbb{E} \Phi \left( c + \sum_{j=1}^n \sqrt{a_j} (X_j - \gamma) \right)
$$

is Schur-concave on the simplex $\{ a \in \mathbb{R}_+^n, \sum a_j < \frac{c^2}{\gamma^2 n} \}$.

We emphasise that the centring of the $X_j$ by its mean $\mathbb{E} X_j = \gamma$ is crucial for this result to hold. Without the centring, the resulting function is Schur-convex, as will follow from our proof.

**Theorem 2** For a completely monotone function $\Phi : (0, +\infty) \to (0, +\infty)$, the function

$$
(a_1, \ldots, a_n) \mapsto \mathbb{E} \Phi \left( \sum_{j=1}^n \sqrt{a_j} X_j \right)
$$

is Schur-convex on $\mathbb{R}_+^n$.

The main result of [2] follows as a corollary to Theorem 1.

**Corollary 3** (Bartczak–Nayar–Zwara, [2]) Provided that $\gamma n \geq 1$, we have for the Shannon entropy,

$$
\mathbf{h} \left( \sum_{j=1}^n \sqrt{a_j} X_j \right) \leq \mathbf{h} \left( \sum_{j=1}^n \frac{1}{\sqrt{n}} X_j \right),
$$

whenever $\sum a_j = 1$.

For general $\alpha$-Rényi entropies, we can deduce the same, but using Theorem 2 and imposing additional restrictions on the parameters $\alpha, \gamma$ and $n$. This can be compared with results for Gaussian mixtures (Theorem 8 in [13]) as well as sums of uniform random variables (Theorem 2 in [10]).
Corollary 4 Let $\alpha > 1$, $n\gamma < 1$ and $\sum_{j=1}^n a_j = 1$. We have,

$$h_\alpha \left( \sum_{j=1}^n \sqrt{a_j} X_j \right) \leq h_\alpha \left( \sum_{j=1}^n \frac{1}{\sqrt{n}} X_j \right). \tag{4}$$

We also have Schur-convexity for power functions with integral exponents. This relates to Question 6 from [2], except that here we are only able to handle centred moments of even order.

Theorem 5 For every positive integer $k$, the function

$$(a_1, \ldots, a_n) \mapsto \mathbb{E} \left( \sum_{j=1}^n \sqrt{a_j} (X_j - \gamma) \right)^k \tag{5}$$

is Schur-convex on $\mathbb{R}_+^n$.

2.2 Maximum density

Let $a_1 \geq \cdots \geq a_n > 0$, $\sum_{j=1}^n a_j = 1$. The main result of Bobkov–Naumov–Ulyanov from [9] asserts that when $\gamma = 1/2$ (i.e. when $X_j$ has the same distribution as $\frac{1}{2} Z^2$), we have

$$\frac{1}{2e^2 \sqrt{2\pi}} (1 - a_1)^{-1/4} \leq M \left( \sum_{j=1}^n \sqrt{a_j} X_j \right) \leq \frac{4}{\sqrt{\pi}} (1 - a_1)^{-1/4}, \tag{6}$$

Using their approach, we extend this to $\gamma \geq 1/2$. For $\gamma \geq 1$, $M$ is of the constant order, $\gamma^{-1/2}$ up to universal constants. For $\frac{1}{2} \leq \gamma < 1$, only the exponent in (6) has to be modified (and of course the universal constants).

Theorem 6 For $\gamma \geq 1$, we have

$$\frac{1}{\sqrt{12}} \gamma^{-1/2} \leq M \left( \sum_{j=1}^n \sqrt{a_j} X_j \right) \leq \gamma^{-1/2}. \tag{7}$$

For $\frac{1}{2} \leq \gamma < 1$, there are constants $c_\gamma$ and $C_\gamma$ for which we have

$$c_\gamma (1 - a_1)^{\gamma-1} \leq M \left( \sum_{j=1}^n \sqrt{a_j} X_j \right) \leq C_\gamma (1 - a_1)^{\gamma-1}. \tag{8}$$

The lower bound in fact holds for every $0 < \gamma < 1$ and we can take $c_\gamma = 0.003\gamma$. 

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Bounds (7) do not use the particular structure of the sum $\sum \sqrt{a_j}X_j$: the lower bound holds for all random variables, whereas the upper bound holds for all log-concave random variables (see Lemmas 12 and 13 below) with the constant 1 in front of $\gamma^{-1/2}$ being in fact optimal (attained for the one-sided exponential distribution, so when $n = 1$ and $\gamma = 1$). For $\gamma < 1$, we will prove slightly more general results, allowing to justify the following remark.

**Remark 7** For $\gamma < 1$, $M = +\infty$ regardless of $a$ as long as $n \leq \lfloor 1/\gamma \rfloor$ (e.g. see Lemma 15 below). For $\gamma < 1/2$, we only know the matching lower and upper bounds when $n = \lfloor 1/\gamma \rfloor + 1$ (see Remark 17 in the next section). The case of arbitrary $n$ has been elusive and we find it an interesting question.

One final comment is in place: all of our results can be naturally interpreted in information theoretic language as Rényi entropy bounds. There has been significant amount of work devoted to such bounds, see for instance [4, 6–8, 23, 24, 26–28, 31, 34] for recent results and additional references. In particular, inequality (8) can be recast as a comparison between the (continuous) $\infty$-Rényi entropy of $\sum_{j=1}^n \sqrt{a_j}X_j$ and the (discrete) $\infty$-Rényi entropy of the coefficient vector $a = (a_1, \ldots, a_n)$: up to multiplicative constants, we have

$$1 - e^{-h_{\infty}(a)} \approx \gamma e^{\frac{2}{1-\gamma} h_{\infty}(\sum_{j=1}^n \sqrt{a_j}X_j)}.$$  

### 3 Proofs

We note for future use the formula for the moment generating function of a $\Gamma(\gamma)$ random variable $X$: for $t < 1$,

$$\mathbb{E}e^{tX} = (1 - t)^{-\gamma}. \quad (9)$$

#### 3.1 Proof of Theorems 1 and 2

We begin with a lemma.

**Lemma 8** Let $\gamma > 0$. The function $F(x_1, \ldots, x_n) = \prod_{j=1}^n e^{\gamma \sqrt{x_j}}(1 + \sqrt{x_j})^{-\gamma}$ is Schur-concave on $\mathbb{R}^n_+$, whereas the function $G(x_1, \ldots, x_n) = \prod_{j=1}^n (1 + \sqrt{x_j})^{-\gamma}$ is Schur-convex on $\mathbb{R}^n_+$.

**Proof** For function $F$, we have,

$$\frac{1}{F(x)} \frac{\partial F}{\partial x_k} = \frac{\partial}{\partial x_k} \log F = \gamma \frac{\partial}{\partial x_k} \left( \sqrt{x_k} - \log(1 + \sqrt{x_k}) \right) = \gamma \frac{1}{2\sqrt{x_k}} \left( 1 - \frac{1}{1 + \sqrt{x_k}} \right) = \gamma \frac{1}{2} \frac{1}{1 + \sqrt{x_k}}.$$
Thus, if \( x_k > x_l \), then
\[
\frac{\partial F}{\partial x_k} - \frac{\partial F}{\partial x_l} = \frac{\gamma}{2} F(x) \left( \frac{1}{1 + \sqrt{x_k}} - \frac{1}{1 + \sqrt{x_l}} \right) < 0.
\]

The Schur–Ostrowski criterion finishes the proof for \( F \). For function \( G \), the argument proceeds identically.

**Proof of Theorem 1** Since \( \Phi \) is completely monotone, there is a (nonnegative) Borel measure on \( \mu \) such that
\[
\Phi(x) = \int_0^\infty e^{-tx} d\mu(t).
\]
Thus, thanks to independence and (9),
\[
\mathbb{E} \Phi \left( c + \sum_{j=1}^n \sqrt{a_j} (X_j - \gamma) \right) = \int_0^\infty \left( \prod_{j=1}^n e^{\gamma t \sqrt{a_j}} (1 + t \sqrt{a_j}^{-\gamma}) \right) e^{-ct} d\mu(t).
\]
Lemma 8 finishes the proof.

**Remark 9** We emphasise that the factor \( e^{\gamma \sqrt{a_j}} \) appears as a result of centring the \( X_j \). This factor is crucial for function \( F \) from Lemma 8 to be Schur-concave, as without it, as we have seen, it is Schur-convex. Theorem 2 follows analogously.

### 3.2 Proof of Corollary 3

First note that applying Theorem 1 to \( \Phi(x) = x^{-q} \) with \( q \to 0^+ \) and using that \( \frac{x^{-q} - 1}{q} \downarrow - \log x \), as \( q \downarrow 0^+ \) for positive \( x \), we conclude that Theorem 1 also holds with \( \Phi(x) = - \log x \). To prove (3), fix \( c > \gamma \sqrt{n} \) and positive \( a_j \) with \( \sum_{j=1}^n a_j = 1 \). Recall that for an arbitrary probability density function \( g \), we have
\[
h \left( \sum_{j=1}^n \sqrt{a_j} X_j \right) = h \left( c + \sum_{j=1}^n \sqrt{a_j} (X_j - \gamma) \right) \leq \mathbb{E} \left[ - \log g \left( c + \sum_{j=1}^n \sqrt{a_j} (X_j - \gamma) \right) \right].
\]
Letting \( g \) be the density of \( \sum_{j=1}^n \frac{1}{\sqrt{n}} X_j \), that is
\[
g(x) = \frac{n^{(\gamma n - 1)/2}}{\Gamma(\gamma n)} x^{\gamma n - 1} e^{-x\sqrt{n}},
\]
we thus obtain from the first part that (note that we need $\gamma n - 1 \geq 0$)

$$h \left( \sum_{j=1}^{n} \sqrt{a_j} X_j \right) \leq \mathbb{E} \left[ -\log g \left( c + \sum_{j=1}^{n} \frac{1}{\sqrt{n}} (X_j - \gamma) \right) \right]$$

With $c \to \gamma \sqrt{n} +$, the right hand side becomes $h \left( \sum_{j=1}^{n} \frac{1}{\sqrt{n}} X_j \right)$.

### 3.3 Proof of Corollary 4

Suppose $\sum a_j = 1$, let $f$ be the density of $\sum \sqrt{a_j} X_j$ and $g$ be the density of $\sum X_j / \sqrt{n}$. Our goal is to show that $\int f^\alpha \geq \int g^\alpha$. By Hölder’s inequality,

$$\left( \int f^\alpha \right)^{\frac{1}{\alpha}} \left( \int g^\alpha \right)^{\frac{\alpha - 1}{\alpha}} \geq \int f g^{\alpha - 1}.$$ 

Note that the right hand side reads $\mathbb{E} \Phi(\sum \sqrt{a_j} X_j)$ with

$$\Phi(x) = g(x)^{\alpha - 1} = \left( \Gamma(n \gamma)^{-1} (x \sqrt{n})^{n \gamma - 1} e^{-x \sqrt{n}} \right)^{\alpha - 1}$$

which is completely monotone as a product of two completely monotone functions (hence the assumptions, to have $(n \gamma - 1)(\alpha - 1) < 0$ and $\alpha - 1 > 0$). It remains to apply Theorem 2 to the sequence $(a_j)$ which always majorises the constant sequence $(\frac{1}{n})$.

**Remark 10** The application of Hölder’s inequality in the form of a variational formula for the Renyi entropy, as in the proof of Corollary 4, has been recently used in a number of information theoretic contexts and can be probably traced back to [36].

### 3.4 Proof of Theorem 5

First we prove a lemma about centred integral moments of a single summand.

**Lemma 11** For every positive integer $k$, $\mathbb{E} (X_1 - \gamma)^k \geq 0$.

**Proof** Rephrasing the lemma, it suffices to prove that the power-series expansion of the moment generating function $\mathbb{E} e^{t(X_1-\gamma)}$ has nonnegative coefficients. Invoking (9) and using that $-\log(1-t) = \sum_{k=1}^{\infty} \frac{t^k}{k}$, we obtain

$$\mathbb{E} e^{t(X_1-\gamma)} = \exp \{ \gamma (-t - \log(1-t)) \} = \exp \left\{ \gamma \sum_{k=2}^{\infty} \frac{t^k}{k} \right\}.$$ 

Since the power series expansion of $\exp$ has positive coefficients, the proof is complete. □
Proof of Theorem 5 Let $S = \sum_{j=1}^{n} \sqrt{a_j} (X_j - \gamma)$. Consider for sufficiently small positive $t$,

$$\mathbb{E}e^{tS} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}S^k = \prod_{j=1}^{n} \exp \left\{ -\gamma \left( t \sqrt{a_j} + \log(1 - t \sqrt{a_j}) \right) \right\}.$$ 

Call the right hand side $F$. Fix two indices $i \neq j$ and observe that $(\frac{\partial}{\partial a_j} - \frac{\partial}{\partial a_i}) \mathbb{E}S^k$ is the Taylor coefficient of $(\frac{\partial}{\partial a_j} - \frac{\partial}{\partial a_i}) F$ at $t^k$. On the other hand,

$$\frac{\partial F}{\partial a_j} = F \cdot \left( -\gamma \left( \frac{t}{2 \sqrt{a_j}} - \frac{t}{2 \sqrt{a_j}(1 - t \sqrt{a_j})} \right) \right) = F \cdot \frac{\gamma t^2}{2(1 - t \sqrt{a_j})}.$$ 

Thus,

$$\left( \frac{\partial}{\partial a_j} - \frac{\partial}{\partial a_i} \right) F = F \cdot \frac{\gamma t^3}{2(1 - t \sqrt{a_j})(1 - t \sqrt{a_i})}(\sqrt{a_j} - \sqrt{a_i}).$$ 

For $a_j > a_i$, the power-series expansion of the right hand side has nonnegative coefficients ($F$ has, by Lemma 11, and plainly so does $(1 - t \sqrt{a_j})^{-1}$). Combining this with the Schur–Ostrowski criterion finishes the argument. 

3.5 Proof of Theorem 6

We assume throughout that $a_1 \geq a_2 \geq \ldots$. For the proofs, we recall several lemmas. The first one is classical and goes back to Moriguti.

Lemma 12 (Moriguti, [32]) For every random variable $X$, $M(X) \geq \frac{1}{\sqrt{12}} \frac{1}{\sqrt{\text{Var}(X)}}$.

The second lemma is a reverse bound for log-concave random variables, that is having densities of the form $e^{-V}$ for a convex function $V : \mathbb{R} \to (-\infty, +\infty]$.

Lemma 13 (Fradelizi [16], Bobkov–Chistyakov, [5]) For every log-concave random variable $X$, $M(X) \leq \frac{1}{\sqrt{\text{Var}(X)}}$.

The third lemma is a straightforward extension of Lemma 3 from [9]. It relies on the Fourier inversion formula to derive bounds on densities, allowing to leverage independence. This technique has been particularly fruitful in geometric questions concerning sharp bounds on volumes of sections of $\ell_p$-balls, perhaps pioneered by Hensley in his paper [17] on the cube, see also Ball’s celebrated work [1] as well as Koldobsky’s works [18, 19] for an in-depth general treatment, with the topic enjoying significant recent activity, see, e.g. [11, 12, 20–22, 25, 29, 30].
Lemma 14  If $a_1 \leq \frac{1}{m}$ for a positive integer $m$, then the characteristic function $\phi$ of $\sqrt{a_1}X_1 + \cdots + \sqrt{a_n}X_n$ satisfies

$$|\phi(t)| \leq (1 + t^2/m)^{-m\gamma/2}, \quad t \in \mathbb{R}. \quad (10)$$

Moreover, if $m\gamma > 1$,

$$M \left( \sqrt{a_1}X_1 + \cdots + \sqrt{a_n}X_n \right) \leq \frac{\sqrt{m}\Gamma \left( \frac{m\gamma-1}{2} \right)}{2\sqrt{\pi}\Gamma \left( \frac{m\gamma}{2} \right)}.$$  \quad (11)

Proof  The characteristic function of $X_1$ is

$$\mathbb{E}e^{itX_1} = (1 - it)^{-\gamma}, \quad t \in \mathbb{R},$$

with choosing, say the principal branch. Thus,

$$\phi(t) = \prod_{j=1}^{n}(1 - i \sqrt{a_j}t)^{-\gamma}$$

and

$$\log |\phi(t)| = -\frac{\gamma}{2} \sum_{j=1}^{n} \log(1 + a_j t^2).$$

To finish the proof of (10), we find the maximum of the convex function

$$(a_1, \ldots, a_n) \mapsto -\frac{1}{2} \sum_{j=1}^{n} \log(1 + a_j t^2)$$

over the domain $D = \{(a_1, \ldots, a_n), a_1, \ldots, a_n \geq 0, a_1 + \cdots + a_n = 1\} \cap [0, 1/m]^n$. We can either follow [9] verbatim and examine its extreme points, or, alternatively, it is clear that an arbitrary vector in $D$ is majorised by the vector $(\frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0)$ (with $\frac{1}{m}$ repeated $m$-times and 0 repeated $n - m$ times), so the lemma follows from the Schur-convexity of this function.

To see (11), we apply the Fourier inversion formula and (10),

$$M \left( \sqrt{a_1}X_1 + \cdots + \sqrt{a_n}X_n \right) \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\phi(t)| dt \leq \frac{1}{2\pi} \int_{\mathbb{R}} (1 + t^2/m)^{-m\gamma/2} dt \leq \frac{\sqrt{m}\Gamma \left( \frac{m\gamma-1}{2} \right)}{2\sqrt{\pi}\Gamma \left( \frac{m\gamma}{2} \right)}.$$

$\square$
We will also need a simple point-wise bound on the density of the sum $\sqrt{a_1}X_1 + \cdots + \sqrt{a_n}X_n$.

**Lemma 15** The density $p$ of $\sqrt{a_1}X_1 + \cdots + \sqrt{a_n}X_n$ satisfies

$$\frac{1}{\Gamma(n\gamma)}(a_1 \cdots a_n)^{-\gamma/2}x^{n\gamma-1}e^{-x/\sqrt{a_n}} \leq p(x) \leq \frac{1}{\Gamma(n\gamma)}(a_1 \cdots a_n)^{-\gamma/2}x^{n\gamma-1}.$$

**Proof** Fix $x > 0$. By independence, convolving the densities of $\sqrt{a_j}X_j$ yields

$$p(x) = \Gamma(\gamma)^{-n}(a_1 \cdots a_n)^{-\gamma/2}x^{n\gamma-1}$$

$$\int_{t_1, \ldots, t_{n-1} > 0, \ t_1 + \cdots + t_{n-1} < x} \left[ (t_1 \ldots t_{n-1})^{\gamma-1}(x - t_1 - \cdots - t_{n-1})^{\gamma-1} \right. \left. \cdot \exp \left\{ -\frac{t_1}{\sqrt{a_1}} - \cdots - \frac{t_{n-1}}{\sqrt{a_{n-1}}} - \frac{x - t_1 - \cdots - t_{n-1}}{\sqrt{a_n}} \right\} \right] dt_1 \ldots dt_{n-1}. \tag{12}$$

Changing each $t_j$ to $xt_j$ gives

$$p(x) = \Gamma(\gamma)^{-n}(a_1 \cdots a_n)^{-\gamma/2}x^{n\gamma-1}$$

$$\cdot \int_{t_1, \ldots, t_{n-1} > 0, \ t_1 + \cdots + t_{n-1} < 1} \left[ (t_1 \ldots t_{n-1})^{\gamma-1}(1 - t_1 - \cdots - t_{n-1})^{\gamma-1} \right. \left. \cdot \exp \left\{ -x \left( \frac{t_1}{\sqrt{a_1}} + \cdots + \frac{t_{n-1}}{\sqrt{a_{n-1}}} + \frac{1 - t_1 - \cdots - t_{n-1}}{\sqrt{a_n}} \right) \right\} \right] dt_1 \ldots dt_{n-1}.$$}

Note that for the $t_j$ from the integral’s domain,

$$0 \leq \frac{t_1}{\sqrt{a_1}} + \cdots + \frac{t_{n-1}}{\sqrt{a_{n-1}}} + \frac{1 - t_1 - \cdots - t_{n-1}}{\sqrt{a_n}}$$

$$= \frac{1}{\sqrt{a_n}} + \sum_{j=1}^{n-1} t_j \left( \frac{1}{\sqrt{a_j}} - \frac{1}{\sqrt{a_n}} \right) \leq \frac{1}{\sqrt{a_n}}$$

(recalling that $a_j \geq a_n$). The resulting estimates on $\exp\{\ldots\}$ in the integrand give the desired bounds on $p$, where the factor $\frac{1}{\Gamma(n\gamma)}$ comes from

$$\Gamma(\gamma)^{-n}\int_{t_1, \ldots, t_{n-1} > 0, \ t_1 + \cdots + t_{n-1} < 1} (t_1 \ldots t_{n-1})^{\gamma-1}(1 - t_1 - \cdots - t_{n-1})^{\gamma-1} dt_1 \ldots dt_{n-1}$$

$$= \Gamma(\gamma)^{-n}B(\gamma, \gamma)B(2\gamma, \gamma) \cdots B((n-1)\gamma, \gamma) = \frac{1}{\Gamma(n\gamma)}.$$

\[\square\]

We move to the proof of Theorem 6. First we assume that $\gamma \geq 1$. 

\[\square\] Springer
Proof of 7, the lower bound. It immediately follows from Lemma 12 since we have, \( \text{Var}(\sum \sqrt{a_j} X_j) = \gamma. \)

\[ \square \]

Proof of 7, the upper bound. It immediately follows from Lemma 13 since for \( \gamma \geq 1, \Gamma(\gamma) \) random variables are log-concave and sums of independent log-concave random variables are log-concave.

\[ \square \]

Now we assume that \( \gamma < 1. \) The upper bound in (8) as well as Remark 7 follow from the following upper bound.

Theorem 16 Fix a positive integer \( k \) and let \( \frac{1}{k+1} \leq \gamma < \frac{1}{k}. \) Then

\[ M \left( \sum_{j=1}^{n} \sqrt{a_j} X_j \right) \leq C_\gamma (a_1 \ldots a_k)^{-\gamma/2} (1 - a_1 - \cdots - a_k)^{\frac{ky-1}{2}} \]

(13)

with the right hand side understood as \( +\infty \) when \( n \leq k. \) Constant \( C_\gamma \) depends only on \( \gamma. \)

Remark 17 In particular, when \( k = 1, \) this gives the upper bound in (8). Unfortunately, when \( \gamma < \frac{1}{2}, \) that is \( k \geq 2, \) bound (13) is not optimal: consider for instance the case when \( a_1 = \cdots = a_n = \frac{1}{n} \) with large \( n. \) However, note that when \( n = k + 1, \) bound (13) is matched from below by Lemma 15 which gives that in this case

\[ M \left( \sum_{j=1}^{k+1} \sqrt{a_j} X_j \right) \geq c_\gamma (a_1 \ldots a_k)^{-\gamma/2} a_{k+1}^{\frac{ky-1}{2}} \]

with \( c_\gamma = \frac{(k+1)^{\gamma-1}(k+1)^{\gamma-1} e^{(k+1)\gamma-1} \Gamma((k+1)\gamma)}{\Gamma(k+1)} \), justifying Remark 7.

Proof of 13 In the course of the proof, the value of \( C_\gamma \) may change from line to line. Note that when \( n \leq k, \) by Lemma 15, the maximum of the density is \( +\infty \) (because the exponent at \( x \) is negative). Thus, we can assume that \( n \geq k + 1. \)

Case \( n = k + 1. \) From (12), after changing the variables (scaling each \( t_i \) by \( \sqrt{a_{k+1}} x) \) we have,

\[ p(\sqrt{a_{k+1}} x) = \Gamma(\gamma)^{-k-1} (a_1 \ldots a_k)^{-\gamma/2} a_{k+1}^{\frac{ky-1}{2}} x^{(k+1)\gamma-1} \]

\[ \cdot \int_{t_1, \ldots, t_k > 0, t_1 + \cdots + t_k < 1} (t_1 \ldots t_k)^{\gamma-1} (1 - t_1 - \cdots - t_k)^{\gamma-1} \]

\[ \cdot \exp \left\{ -x \left( \sqrt{\frac{a_{k+1}}{a_1}} t_1 + \cdots + \sqrt{\frac{a_{k+1}}{a_k}} t_k + 1 - t_1 - \cdots - t_k \right) \right\} \] \] dt_1 \ldots dt_k. \]

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The crude bound \( \sum \sqrt{\frac{a_k+1}{a_j}} t_j \geq 0 \) yields

\[
p(\sqrt{a_k+1}x) \leq \Gamma(\gamma)^{-k-1}(a_1 \ldots a_k)^{-\gamma/2} a_{k+1}^{k\gamma-1} x^{(k+1)\gamma-1} \]

\[
\cdot \int_{t_1 \ldots t_k > 0, \sum t_j < 1} \left[ (t_1 \ldots t_k)^{\gamma-1} \left(1 - \sum t_j\right)^{\gamma-1} \cdot \exp \left\{-x \left(1 - \sum t_j\right)\right\} \right] \, dt_1 \ldots dt_k.
\]

Using

\[
x^{(k+1)\gamma-1} \exp \left\{-x \left(1 - \sum t_j\right)\right\} \leq L_{\gamma} \left(1 - \sum t_j\right)^{1-(k+1)\gamma},
\]

where \( L_{\gamma} = \sup_{x > 0} x^{(k+1)\gamma-1} e^{-x} = ((k+1)\gamma - 1)^{(k+1)\gamma-1} e^{-(k+1)\gamma-1} \), we obtain the desired bound

\[
\|p\|_{\infty} \leq C_{\gamma} (a_1 \ldots a_k)^{-\gamma/2} a_{k+1}^{k\gamma-1},
\]

with

\[
C_{\gamma} = \Gamma(\gamma)^{-k-1} L_{\gamma} \int_{t_1 \ldots t_k > 0, \sum t_j < 1} (t_1 \ldots t_k)^{\gamma-1} \left(1 - \sum t_j\right)^{-k\gamma} \, dt_1 \ldots dt_k
\]

which is finite because \( k\gamma < 1 \).

\emph{Case } \( n \geq k+2 \). If \( a_1 \leq \frac{1}{k+2} \), then (11) applied to \( m = k+2 \) gives

\[
M \left( \sum_{j=1}^{n} \sqrt{a_j X_j} \right) \leq C_{\gamma} \leq C_{\gamma} (a_1 \ldots a_k)^{-\gamma/2} (1 - a_1 - \cdots - a_k)^{k\gamma-1},
\]

since \( a_1 \ldots a_k \leq 1, 1 - a_1 - \cdots - a_k \leq 1 \), where \( C_{\gamma} \) only depends on \( \gamma \). Now we assume that \( a_1 > \frac{1}{k+2} \), write

\[
\sum_{j=1}^{n} \sqrt{a_j X_j} = \sqrt{\alpha \eta} + \sqrt{1 - \alpha \xi}
\]

with

\[
\alpha = \sum_{j=1}^{k} a_j
\]

and

\[
\eta = \sum_{j=1}^{k} \sqrt{\frac{a_j}{\alpha}} X_j, \quad \xi = \sum_{j=k+1}^{n} \sqrt{\frac{a_j}{1 - \alpha}} X_j.
\]
We break the argument into two further cases depending on whether we can guarantee that \( \xi \) has a bounded density (using Lemma 14).

**Case** \( \frac{a_{k+1}}{1-\alpha} \leq \frac{1}{k+2} \). Here, necessarily the number of summands in \( \xi \) is at least \( k+2 \) (by comparing the largest coefficient to the average). Let \( g \) be the density of \( \xi \). By (11) applied with \( m = k+2 \), we get \( \|g\|_\infty \leq C_\gamma \). Moreover, if we let \( f \) be the density of \( \eta \), we know by Lemma 15 that

\[
f(x) \leq \frac{1}{\Gamma(k\gamma)} a^{k\gamma/2}(a_1 \ldots a_k)^{-\gamma/2}x^{k\gamma-1}, \quad x > 0.
\]

Thus for the density \( p \) of \( \sum_{j=1}^n \sqrt{a_j} X_j \), we obtain

\[
p(x) = \int_0^x \frac{1}{\sqrt{\alpha(1-\alpha)}} f\left(\frac{t}{\sqrt{\alpha}}\right) g\left(\frac{x-t}{\sqrt{1-\alpha}}\right) \, dt
\]

\[
\leq \frac{1}{\Gamma(k\gamma)\sqrt{1-\alpha}} (a_1 \ldots a_k)^{-\gamma/2} \int_0^x t^{k\gamma-1} g\left(\frac{x-t}{\sqrt{1-\alpha}}\right) \, dt.
\]

Changing \( x \) to \( \sqrt{1-\alpha} x \) and \( t \) to \( \sqrt{1-\alpha} t \), we get

\[
p(\sqrt{1-\alpha} x) \leq \frac{1}{\Gamma(k\gamma)} (a_1 \ldots a_k)^{-\gamma/2} (1-\alpha)^{k\gamma/2} \int_0^x t^{k\gamma-1} g(x-t) \, dt.
\]

It remains to observe that the resulting integral is bounded,

\[
\int_0^x t^{k\gamma-1} g(x-t) \, dt \leq \|g\|_\infty \int t^{k\gamma-1} \, dt + \int_{t>1} g(x-t) \, dt \leq \frac{\|g\|_\infty}{k\gamma} + 1.
\]

**Case** \( \frac{a_{k+1}}{1-\alpha} \geq \frac{1}{k+2} \). Plainly,

\[
M\left(\sum_{j=1}^n \sqrt{a_j} X_j\right) \leq M\left(\sum_{j=1}^{k+1} \sqrt{a_j} X_j\right) = \frac{1}{\sqrt{A}} M\left(\sum_{j=1}^{k+1} \frac{\sqrt{a_j}}{A} X_j\right)
\]

with \( A = \sum_{j=1}^{k+1} a_j \). By the case \( n = k+1 \), i.e. (14),

\[
M\left(\sum_{j=1}^{k+1} \frac{\sqrt{a_j}}{A} X_j\right) \leq C_\gamma (a_1 \ldots a_k)^{-\gamma/2} A^{k\gamma/2} (a_{k+1}/A)^{k\gamma-1/2},
\]

thus

\[
M\left(\sum_{j=1}^n \sqrt{a_j} X_j\right) \leq C_\gamma (a_1 \ldots a_k)^{-\gamma/2} a_{k+1}^{k\gamma-1/2}
\]

\[
\leq C_\gamma (k+2)^{-\gamma/2} (a_1 \ldots a_k)^{-\gamma/2} (1-\alpha)^{k\gamma-1/2}.
\]
Proof of 8, the lower bound  
We assume that $0 < \gamma < 1$ and denote $Z = \sum \sqrt{a_j} X_j$. The argument from [9] can be repeated almost verbatim. We include it for completeness.

Case $a_1 \leq \frac{1}{2}$. Since $\text{Var}(Z) = \gamma$, Lemma 12 yields $M(Z) \geq \frac{1}{2} \sqrt{3\gamma}$ so if $a_1 \leq 1/2$ then $M(Z) \geq c_\gamma (1 - a_1)^{\gamma - \frac{1}{2}}$ with $c_\gamma = (2^{\frac{3-\gamma}{2}} \sqrt{3\gamma})^{-1}$.

Case $a_1 \geq 1/2$. Let $\xi = \sum_{j=2}^{n} \sqrt{a_j} X_j$, so that $Z = \sqrt{a_1} X_1 + \sqrt{1 - a_1} \xi$. Note that $\xi$ is independent of $\sqrt{a_1} X_1$ so the density $f_Z$ of $Z$ is given by the convolution of the densities $f_{\sqrt{a_1} X_1}$ and $f_{\sqrt{1 - a_1} \xi}$.

We have,

$$f_Z(x) = \frac{1}{\Gamma(\gamma) \sqrt{a_1(I - a_1)}} \int_{0}^{x} \frac{(x - t)^{\gamma - 1}}{\sqrt{a_1}} \exp \left(-\frac{x - t}{\sqrt{a_1}}\right) f_\xi \left(\frac{t}{\sqrt{1 - a_1}}\right) dt,$$

and, applying this for $x \sqrt{1 - a_1},$

$$f_Z(x \sqrt{1 - a_1}) = \frac{(1 - a_1)^{\gamma - 1}}{\Gamma(\gamma) a_1^{\gamma/2}} \int_{0}^{x} \frac{(x - t)^{\gamma - 1}}{\sqrt{a_1}} \exp \left(-\frac{\sqrt{1 - a_1}}{\sqrt{a_1}}(x - t)\right) f_\xi(t) dt.$$

We will use this identity for $x = \mathbb{E}\xi + 2$, lower bounding the expression in the right hand side by integrating on the interval $I = (\max(\mathbb{E}\xi - 2, 0), \mathbb{E}\xi + 2)$. Note that $x - t \leq 4$ for every $t \in I$. It follows that

$$M(Z) \geq \frac{(1 - a_1)^{\gamma - 1}}{\Gamma(\gamma) a_1^{\gamma/2}} 4^{\gamma - 1} \exp \left(-\frac{4\sqrt{1 - a_1}}{\sqrt{a_1}}\right) \cdot \mathbb{P}(\xi \in I).$$

The assumption $a_1 \geq 1/2$ yields $\frac{1 - a_1}{a_1} \leq 1$. Since $\text{Var}(\xi) = \gamma$, we get by Chebyshev’s inequality that

$$\mathbb{P}(\xi \in I) = 1 - \mathbb{P}(|\xi - \mathbb{E}\xi| \geq 2) \geq 1 - \frac{1}{4} \text{Var}(\xi) = 1 - \frac{\gamma}{4}.$$

Putting these together and the trivial bound $a_1^{\gamma/2} \leq 1$, we get the lower bound $M(Z) \geq c_\gamma (1 - a_1)^{\gamma - \frac{1}{2}}$ with $c_\gamma = \frac{4^{\gamma}(4 - \gamma)}{4^{4}e^{4}\Gamma(\gamma)}$.

Combining the two cases together, the lower bound in (8) holds with

$$c_\gamma = \min \left\{ \left(2^{\frac{3-\gamma}{2}} \sqrt{3\gamma}\right)^{-1}, \frac{4^{\gamma}(4 - \gamma)}{4^{4}e^{4}\Gamma(\gamma)} \right\} \geq \min \left\{ \frac{1}{2^{3/2} \sqrt{3}}, \frac{3\gamma}{4^{2}e^{2}} \right\} > 0.003\gamma,$$

since $\frac{1}{\Gamma(\gamma)} \geq \gamma$.  
\[\square\]
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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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