OPTIMISATION OF THE LOWEST ROBIN EIGENVALUE IN THE EXTERIOR OF A COMPACT SET, II: NON-CONVEX DOMAINS AND HIGHER DIMENSIONS

DAVID KREJČÍŘÍK AND VLADIMIR LOTOREICHIK

Abstract. We consider the problem of geometric optimisation of the lowest eigenvalue of the Laplacian in the exterior of a compact set in any dimension, subject to attractive Robin boundary conditions.

As an improvement upon our previous work [16], we show that under either a constraint of fixed perimeter or area, the maximiser within the class of exteriors of simply connected planar sets is always the exterior of a disk, without the need of convexity assumption. Moreover, we generalise the result to disconnected compact planar sets. Namely, we prove that under a constraint of fixed average value of the perimeter over all the connected components, the maximiser within the class of disconnected compact planar sets, consisting of finitely many simply connected components, is again a disk.

In higher dimensions, we prove a completely new result that the lowest point in the spectrum is maximised by the exterior of a ball among all sets exterior to bounded convex sets satisfying a constraint on the integral of a dimensional power of the mean curvature of their boundaries. Furthermore, it follows that the critical coupling at which the lowest point in the spectrum becomes a discrete eigenvalue emerging from the essential spectrum is minimised under the same constraint by the critical coupling for the exterior of a ball.

1. Introduction

1.1. Motivation and state of the art. Spectral optimisation problems constitute an intensively studied area of modern mathematics. In addition to important applications in physics, the attractiveness is certainly caused by the emotional impacts geometric shapes have over a person’s perception of the world. Moreover, the problems are typically easy to state but difficult to solve, leading thus to mathematically challenging interaction of differential geometry, operator theory, and partial differential equations. We refer to Henrot’s monographs [12, 13] for many results, open problems in this area of mathematics and further references.

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In this paper we are concerned with the optimisation of the lowest point \( \lambda_1^\alpha(\Omega) \) of the spectrum of the Robin Laplacian that is variationally characterised by the formula

\[
(1.1) \quad \lambda_1^\alpha(\Omega) := \inf_{u \in W^{1,2}(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^2 + \alpha \int_{\partial \Omega} |u|^2}{\int_{\Omega} |u|^2}.
\]

We are interested in extremal properties of \( \lambda_1^\alpha(\Omega) \) as regards the geometry of the smooth open set \( \Omega \subset \mathbb{R}^d \) and the value of the real parameter \( \alpha \).

If \( \Omega \) is bounded, then the infimum in (1.1) is achieved and \( \lambda_1^\alpha(\Omega) \) represents the lowest eigenvalue of the Robin Laplacian. For all positive \( \alpha \), it is then known that \( \lambda_1^\alpha(\Omega) \) is minimised by the ball among all domains of given volume \([6, 8, 9]\). For negative \( \alpha \), where it makes sense to look at a reverse optimisation, it was conjectured by Bareket in 1977 \([4]\) that \( \lambda_1^\alpha(\Omega) \) is now maximised by the ball among all domains of given volume (isochoric problem). This conjecture has been recently disproved by Freitas and one of the present authors \([10]\) by showing that spherical shells give larger values of \( \lambda_1^\alpha(\Omega) \) for all sufficiently large negative \( \alpha \). In the two-dimensional situation, however, it is true that \( \lambda_1^\alpha(\Omega) \) is maximised by the disk among all planar domains of given area provided that \( \alpha \) is negative and small \([10]\). Moreover, numerical simulations suggest \([3]\) that the conjecture actually does hold for all negative \( \alpha \) among the class of simply connected planar domains, but the proof constitutes a challenging open problem. Finally, it was shown in \([3]\) that, for all negative \( \alpha \), the eigenvalue \( \lambda_1^\alpha(\Omega) \) is maximised by the disk among all planar domains of given perimeter (isoperimetric problem).

The question of spectral optimisation is also natural to ask for unbounded sets. In our preceding paper \([16]\), we considered optimisation of \( \lambda_1^\alpha(\Omega^\text{ext}) \) with \( \Omega^\text{ext} := \mathbb{R}^d \setminus \overline{\Omega} \) being the exterior of a bounded open set \( \Omega \). The main result of \([16]\) reads as follows:

**Theorem 0** \(([16, \text{Thm. 1}])\). Let \( d = 2 \). For all negative \( \alpha \), we have

\[
\max_{\Omega \text{ convex}} \lambda_1^\alpha(\Omega^\text{ext}) = \lambda_1^\alpha(B_{R_1}^{\text{ext}}) \quad \text{and} \quad \max_{\Omega \text{ convex}} \lambda_1^\alpha(\Omega^\text{ext}) = \lambda_1^\alpha(B_{R_2}^{\text{ext}}).
\]

Here the maxima are taken over all convex, smooth, bounded planar open sets \( \Omega \) of a given perimeter \( c_1 > 0 \) or area \( c_2 > 0 \), respectively, and \( B_{R_1} \) and \( B_{R_2} \) are disks of perimeter \( |\partial B_{R_1}| = c_1 \) and area \( |B_{R_2}| = c_2 \).

Hence, contrary to the bounded setting, the exterior of the disk is the maximiser not only for the isoperimetric but also for the isochoric optimisation problem, at least among the class of exteriors of convex sets. The restriction to negative \( \alpha \) in Theorem 0 is due to the fact that \( \lambda_1^\alpha(\Omega^\text{ext}) = 0 \) for all \( \alpha \) positive and any bounded \( \Omega \), so the optimisation problems are not interesting for positive \( \alpha \). In fact (cf. \([16, \text{Prop. 1}]\)), the whole interval \([0, \infty)\) belongs to
the essential spectrum of the Robin Laplacian in the exterior of any compact set, for any \( \alpha \), and there is no other spectrum if \( \alpha \) is positive. On the other hand, for every bounded non-empty \( \Omega \) there exists a non-positive constant \( \alpha^*_\text{ext}(\Omega^\text{ext}) \) such that \( \lambda^*_1(\Omega^\text{ext}) \) is a negative discrete eigenvalue if, and only if, \( \alpha < \alpha^*_\text{ext}(\Omega^\text{ext}) \).

**Proposition 1.** Let \( \Omega \subset \mathbb{R}^d \) be an arbitrary non-empty smooth bounded open set. Then

\[
\begin{cases}
\alpha^*_\text{ext}(\Omega^\text{ext}) = 0 & \text{if } d = 1, 2, \\
\alpha^*_\text{ext}(\Omega^\text{ext}) < 0 & \text{if } d \geq 3.
\end{cases}
\]

The proof of the proposition for \( d = 1, 2 \) can be found in [16, Prop. 2]. The case \( d \geq 3 \) is established below with help of a Gagliardo-Nierenberg-Sobolev inequality. For the exteriors of balls, the critical coupling can be computed explicitly by using the explicit form of solutions of \(-\Delta u = \lambda u\) in terms of Bessel functions (see below).

**Proposition 2.** Let \( d \geq 2 \). Then

\[
\alpha^*_\text{ext}(B^\text{ext}_R) = -\frac{d - 2}{R}.
\]

As a consequence of Proposition 1, we see that \( \lambda^*_1(\Omega^\text{ext}) \) is a negative discrete eigenvalue for *all* negative \( \alpha \) only if \( d = 1, 2 \) and that is why Theorem 0 is restricted to the planar situation (the case \( d = 1 \) is trivial). At the same time, in [16, Sec. 5.3] we argue why the claim of Theorem 0 cannot hold in the same form for dimensions \( d \geq 3 \).

Since \( \Omega \) of Theorem 0 is assumed to be convex, it is necessarily connected. According to an example in [16, Sec. 5.1], the connectedness of \( \Omega \) is necessary in the above theorem.

The discussion in the two preceding paragraphs leads to the following natural questions related to Theorem 0:

1. *Can one replace the convexity assumption by connectedness?*

2. *Does a similar result hold in higher dimensions under other constraints?*

The objective of the present paper is to elaborate on these questions, which were left open in our previous work [16]. The present paper can be thus viewed as a natural continuation of [16], but it can be also read fully independently.

### 1.2. From convexity to connectedness.

In what follows, let \( \Omega \subset \mathbb{R}^d \) with \( d \geq 2 \) be a non-empty smooth bounded open set. The set \( \Omega \) is not necessarily connected, but we always assume that its exterior \( \Omega^\text{ext} \) is connected. The volume of and the area of the boundary for \( \Omega \) will be denoted by \( |\Omega| \) and \( |\partial \Omega| \), respectively. By \( N_\Omega \) we denote the number of connected components of \( \Omega \).

In the two-dimensional setting, our main result reads as follows.
Theorem 3. Let \( d = 2 \). For all negative \( \alpha \), we have
\[
\max_{|\partial \Omega| = c} \lambda_1^\alpha(\Omega^{\text{ext}}) \leq \lambda_1^\alpha(B_R^{\text{ext}}).
\]
Here the maximum is taken over all smooth, bounded open sets \( \Omega \) consisting of finitely many disjoint simply connected components and satisfying the relation \( \frac{|\partial \Omega|}{\lambda_1^\alpha} = c \) with given \( c > 0 \) and \( B_R \) is the disk of perimeter \( |\partial B_R| = c \).

The main improvement upon Theorem 0 consists in the replacement of exteriors of convex sets by exteriors of finite unions of simply connected (not necessarily convex) components. In Section 3.2 we use large coupling asymptotics (i.e. \( \alpha \to -\infty \)) to argue why the result of Theorem 3 is sharp even for a subclass of sets having prescribed fixed number of connected components. The proof of Theorem 3 relies on a usage of parallel coordinates as employed by Payne and Weinberger in [21] in order to get an upper bound on the principal Dirichlet eigenvalue on bounded domains (see [10, 16] for previous applications of the technique in the Robin problem).

In the special case of simply connected domains the statement of Theorem 3 implies the following important improvement upon Theorem 0.

Corollary 4. Let \( d = 2 \). For all negative \( \alpha \), we have
\[
\max_{|\partial \Omega|_c_1} \lambda_1^\alpha(\Omega^{\text{ext}}) = \lambda_1^\alpha(B_{R_1}^{\text{ext}}) \quad \text{and} \quad \max_{|\partial \Omega|_c_2} \lambda_1^\alpha(\Omega^{\text{ext}}) = \lambda_1^\alpha(B_{R_2}^{\text{ext}}).
\]
Here the maxima are taken over all smooth, bounded, simply connected open sets \( \Omega \) of a given perimeter \( c_1 > 0 \) or area \( c_2 > 0 \), respectively, and \( B_{R_1} \) and \( B_{R_2} \) are disks of perimeter \( |\partial B_{R_1}| = c_1 \) and area \( |B_{R_2}| = c_2 \).

1.3. Higher dimensions. Let us now pass to the discussion of our results in higher dimensions. To this aim we need to recall some geometric concepts. Let \( \kappa_1, \kappa_2, \ldots, \kappa_{d-1} \) denote the principal curvatures of \( \partial \Omega \); our convention is such that these functions are non-negative if \( \Omega \) is convex. The mean curvature of \( \partial \Omega \) is defined as the function
\[
M := \frac{\kappa_1 + \kappa_2 + \cdots + \kappa_{d-1}}{d-1}.
\]
For the notational convenience, we define also the number
\[
\mathcal{M}(\partial \Omega) := \frac{1}{|\partial \Omega|} \int_{\partial \Omega} M^{d-1}.
\]
Note that for the ball \( B_R \subset \mathbb{R}^d \) of radius \( R > 0 \) we have \( \mathcal{M}(\partial B_R) = R^{-(d-1)} \).

Now we are prepared to formulate our main result in higher dimensions.

Theorem 5. Let \( d \geq 3 \). For all negative \( \alpha \), we have
\[
\max_{\mathcal{M}(\partial \Omega) = c, \ \Omega \ \text{convex}} \lambda_1^\alpha(\Omega^{\text{ext}}) = \lambda_1^\alpha(B_R^{\text{ext}}).
\]
Here the maximum is taken over all convex, smooth, bounded open sets \( \Omega \) such that \( \mathcal{M}(\partial \Omega) = c \) with given \( c > 0 \) and \( B_R \) is the ball with \( \mathcal{M}(\partial B_R) = c \).
Since $M(\partial \Omega) = 2\pi/|\partial \Omega|$ for any simply connected planar domain, Theorem 5 remains valid in dimension two, where it follows from the isoperimetric result of Theorem 0. If $d = 3$, the integral $\int_{\partial \Omega} M^2$ in the numerator in (1.3) is sometimes referred to as the Willmore energy, due to Willmore’s demonstration that the sphere minimises this integral, cf. [27, Sec. 2].

If $d \geq 3$ and $\alpha \geq \alpha_\ast(\Omega^\text{ext})$, the inequality $\lambda_1^\ast(\Omega^\text{ext}) \leq \lambda_1^\ast(B^\text{ext}_R)$ of Theorem 5 by itself is just a trivial statement, for in this case $\lambda_1^\ast(\Omega^\text{ext}) = 0$ is the lowest point of the essential spectrum. However, from this inequality immediately follows an interesting optimisation result for the critical coupling.

**Corollary 6.** Let $d \geq 3$. We have

$$\min_{M(\partial \Omega) = c} \min_{\Omega \text{ convex}} M^2 \alpha_\ast(\Omega^\text{ext}) = \alpha_\ast(\Omega^\text{ext}) = \alpha_\ast(B^\text{ext}_R).$$

Here the minimum is taken over all convex, smooth, bounded open sets $\Omega$ such that $M(\partial \Omega) = c$ with given $c > 0$ and $B_R$ is the ball with $M(\partial B_R) = c$.

For the proof of Theorem 5 we push forward the approach of [16] based on the parallel coordinates. This method requires the convexity assumption, as otherwise the parallel coordinates on $\Omega^\text{ext}$ are not well defined.

### 1.4. Organisation of the paper.

This paper is organised as follows. In Section 2 we provide an operator-theoretic framework for the Robin eigenvalue problem in the exterior of a compact set. Namely, we recall basic spectral properties from [16] and obtain some new ones. In particular, we prove Proposition 1 for $d \geq 3$ and Proposition 2. The two-dimensional case is discussed in Section 3, in which we prove Theorem 3 and its Corollary 4. Moreover, we argue why the result of Theorem 3 is sharp for domains with fixed number of connected components. Finally, in Section 4 we define higher order mean curvatures and use them to prove Theorem 5 on the higher-dimensional case. We conclude Section 4 by a discussion of a connection between Theorem 5 and large coupling asymptotics.

### 2. The spectral problem in the exterior of a compact set

Throughout this section, $\Omega$ is an arbitrary bounded open set in $\mathbb{R}^d$ with $d \geq 2$. While $\Omega$ is not assumed to be connected, a standing assumption is that the exterior $\Omega^\text{ext}$ is connected. We also assume that the boundary $\partial \Omega$ is of class $C^\infty$. Finally, $\alpha$ stands for an arbitrary negative real number.

We are interested in the eigenvalue problem

$$\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega^\text{ext}, \\
\frac{\partial u}{\partial n} = \alpha u & \text{on } \partial \Omega^\text{ext},
\end{cases}$$

where $n$ is the outer unit normal to $\Omega$. As usual, we understand (2.1) as the spectral problem for the self-adjoint operator $-\Delta^\Omega_0^\text{ext}$ in $L^2(\Omega^\text{ext})$ associated
via the first representation theorem [14, Thm. VI. 2.1] with the closed, densely defined, symmetric, and lower semi-bounded quadratic form

\[
Q_\alpha^{\text{ext}}[u] := \int_{\Omega^{\text{ext}}} |\nabla u|^2 + \alpha \int_{\partial \Omega} |u|^2, \quad \mathcal{D}(Q_\alpha^{\text{ext}}) := W^{1,2}(\Omega^{\text{ext}}).
\]

The boundary term is understood in the sense of traces \(W^{1,2}(\Omega^{\text{ext}}) \hookrightarrow L^2(\partial \Omega)\) and represents a relatively bounded perturbation of the Neumann form \(Q_0^{\text{ext}}\) with the relative bound equal to zero. Since \(\Omega\) is smooth, the operator domain of \(-\Delta_\alpha^{\text{ext}}\) consists of functions \(u \in W^{2,2}(\Omega^{\text{ext}})\), which satisfy the Robin boundary conditions from (2.1) in the sense of traces and the operator acts as the distributional Laplacian (see, e.g., [5, Thm. 3.5] for the \(W^{2,2}\)-regularity).

We call \(-\Delta_\alpha^{\text{ext}}\) the Robin Laplacian in \(\Omega^{\text{ext}}\). By the minimax principle, the number \(\lambda_1^{\text{ess}}(\Omega^{\text{ext}})\) defined in (1.1) coincides with the lowest point in the spectrum of \(-\Delta_\alpha^{\text{ext}}\).

Let us now recall some qualitative spectral properties of \(-\Delta_\alpha^{\text{ext}}\). Since the boundary \(\partial \Omega\) is bounded, the essential spectrum coincides with the essential spectrum of the Laplacian in the whole space (cf. [16, Prop. 1]):

\[
\sigma_{\text{ess}}(-\Delta_\alpha^{\text{ext}}) = [0, \infty).
\]

If \(\alpha\) were non-negative, then the spectrum of \(-\Delta_\alpha^{\text{ext}}\) would be exhausted by the essential spectrum and thus \(\lambda_1^{\text{ess}}(\Omega^{\text{ext}}) = 0\). Since \(\alpha\) is assumed to be negative, however, there may exist negative discrete eigenvalues, depending on the largeness of \(\alpha\) as regards the dimension. For low dimensions \(d = 1, 2\) (cf. [16, Prop. 2]), the number \(\lambda_1^{\text{ess}}(\Omega^{\text{ext}})\) is negative whenever \(\alpha\) is negative and represents therefore the lowest discrete eigenvalue of \(-\Delta_\alpha^{\text{ext}}\). For dimensions \(d \geq 3\), the coupling parameter \(\alpha\) must be sufficiently negative to make the discrete spectrum exist. This claim is the content of Proposition 1 above that we prove now.

**Proof of Proposition 1, case \(d \geq 3\).** Let \(\phi\) be any test function from \(C_0^\infty(\Omega^{\text{ext}})\), a space dense in the form domain \(\mathcal{D}(Q_\alpha^{\text{ext}})\).

By the Gagliardo-Nirenberg-Sobolev-type inequality for exterior domains proven in [18, Prop. 5.1], there exists a positive constant \(C\) such that,

\[
\int_{\Omega^{\text{ext}}} |\nabla \phi(x)|^2 \, dx \geq C \left( \int_{\Omega^{\text{ext}}} |\phi(x)|^{\frac{2d}{d-2}} \, dx \right)^{\frac{d-2}{2}}.
\]

Let us introduce an auxiliary weight function \(\mathbb{R}^d \ni x \mapsto \omega(x) := \exp(-|x|)\). By the H"older inequality, it follows

\[
\int_{\Omega^{\text{ext}}} |\nabla \phi(x)|^2 \, dx \geq C \frac{\int_{\Omega^{\text{ext}}} |\phi(x)|^2 \omega(x) \, dx}{\left( \int_{\Omega^{\text{ext}}} \omega(x)^{\frac{2d}{d-2}} \, dx \right)^{\frac{d-2}{2}}} = \frac{Cd^2}{4(\Gamma(d)sd)^{2\frac{d-2}{d}}} \int_{\Omega^{\text{ext}}} |\phi(x)|^2 \omega(x) \, dx,
\]

where \(\Gamma\) denotes the Euler Gamma-function and \(sd := |\partial B_1| = \frac{2\pi^{d/2}}{\Gamma(d/2)}\) stands for the area of the unit sphere. Note that the obtained bound is a Hardy-type inequality for the Neumann Laplacian in \(\Omega^{\text{ext}}\).
Now, let $B_R \subset \mathbb{R}^d$ be an open ball of a sufficiently large radius $R > 0$, so that $\overline{\Omega} \subset B_R$. Furthermore, we define the domain $\Omega_{\text{ext}}^0 := \Omega_{\text{ext}} \cap B_R$. Notice that the function $\omega$ is uniformly positive in $\Omega_{\text{ext}}^0$. Hence, the estimate above yields that there exists a positive constant $c$ such that,

$$
\int_{\Omega_{\text{ext}}^0} |\nabla \phi(x)|^2 \, dx \geq c \left( \int_{\Omega_{\text{ext}}^0} |\nabla \phi(x)|^2 \, dx + \int_{\Omega_{\text{ext}}^0} |\phi(x)|^2 \, dx \right).
$$

Eventually, the trace theorem [19, Thm. 3.38] implies that there exists another positive constant $c'$ such that,

$$
\int_{\Omega_{\text{ext}}^0} |\nabla \phi(x)|^2 \, dx \geq c' \int_{\partial \Omega} |\phi(x)|^2 \, d\sigma(x).
$$

Since $\phi$ is an arbitrary function from the form core of $-\Delta_{\Omega_{\text{ext}}^0}$, this inequality and (2.3) show that $\lambda_{\text{ext}}^0(\Omega_{\text{ext}}^0) = 0$ for all $\alpha \geq -c'$. Consequently, the critical constant $\alpha_*(\Omega_{\text{ext}}^0)$ for which a discrete eigenvalue emerges from the essential spectrum must be negative.

To show that $\alpha_*(\Omega_{\text{ext}}^0)$ is actually finite and that the discrete spectrum exists for all $\alpha < \alpha_*(\Omega_{\text{ext}}^0)$, it is enough to notice the validity of the form ordering $Q_{\text{ext}}^{\text{form}}[\phi] \leq Q_{\text{ext}}^{\text{form}}[\phi]$ for $\alpha_1 \leq \alpha \leq \alpha_2$ and the continuity of $\alpha \mapsto Q_{\text{ext}}^{\text{form}}[\phi]$. Consequently, the minimax principle then implies that $\alpha \mapsto \lambda_{\text{ext}}^0(\Omega_{\text{ext}}^0)$ is non-decreasing and continuous. Hence, the desired claim follows. \hfill $\square$

For $d = 2$ it was shown in [16, Prop. 5] that, for all negative $\alpha$,

$$
2.4 \quad R \mapsto \lambda_{\text{ext}}^0(B_R) \text{ is strictly decreasing.}
$$

Obviously, because of Proposition 1, the same monotonicity result cannot hold for all negative $\alpha$ if $d \geq 3$. In the higher dimensions, a monotonicity of the critical coupling $\alpha_*(B_R)$ for the exteriors of the balls $B_R$ follows from Proposition 2 above that we prove now.

**Proof of Proposition 2.** In view of the radial symmetry of the problem, the eigenfunction $u_1 \in W^{1,2}(B_R^{\text{ext}})$ of the Robin Laplacian in the exterior of the ball $B_R$ corresponding to its lowest eigenvalue $\lambda_{\text{ext}}^0(B_R^{\text{ext}}) < 0$, if it exists, must necessarily be radially symmetric as well. Using this simple observation we see that $\lambda_{\text{ext}}^0(B_R^{\text{ext}}) = -k^2 < 0$ if, and only if, the following ordinary differential spectral problem

$$
2.5 \quad \begin{cases} 
- r^{-(d-1)} [r^{d-1} \psi'(r)]' = -k^2 \psi(r) & \text{for } r \in (R, \infty), \\
\psi'(R) = \alpha \psi(R), \\
\lim_{r \to \infty} \psi(r) = 0,
\end{cases}
$$

possesses a solution $(\psi, k)$ with $\psi \neq 0$ and $k > 0$; cf. [10, Sec. 3]. Observe that the general solution of the differential equation in (2.5) with $k > 0$ is given by

$$
\psi(r) = r^{-\nu} [C_1 K_{\nu}(kr) + C_2 I_{\nu}(kr)], \quad C_1, C_2 \in \mathbb{C}, \quad \nu := \frac{d-2}{2},
$$
where \( K_\nu (\cdot) \) and \( I_\nu (\cdot) \) are modified Bessel functions of the second kind and order \( \nu \). Taking into account the required decay at infinity from (2.5) and using the asymptotic behaviour of \( K_\nu (x) \) and \( I_\nu (x) \) for large \( x \), see [1, Sec. 9.7.2], we conclude that \( C_2 = 0 \). Thus, the expression for \( \psi \) simplifies to

\[
\psi' (r) = -C_1 \nu r^{-\nu-1} K_\nu (kr) + C_1 kr^{-\nu} K'_\nu (kr) .
\]

Thus, the boundary condition from (2.5) yields the requirement

\[
-\nu R^{-\nu-1} K_\nu (kR) + k R^{-\nu} K'_\nu (kR) - \alpha R^{-\nu} K_\nu (kR) = 0 .
\]

With the aid of the identity \( K'_\nu (x) = -K_{\nu+1} (x) + \frac{\nu}{x} K_\nu (x) \) (see [1, Sec. 9.6.26]), this scalar equation simplifies to

\[
(2.6) \quad \alpha = \frac{\nu}{R} + k \frac{K'_{\nu+1} (kR)}{K_\nu (kR)} = \frac{\nu}{R} + k \frac{K_{\nu+1} (kR)}{K_\nu (kR)} = -k R \frac{K_{\nu+1} (kR)}{K_\nu (kR)} \frac{1}{R} .
\]

Introduce now \( f(x) := \frac{x K_{\nu+1} (x)}{K_\nu (x)} \). The function \( f \) is clearly continuous on \([0, \infty)\) and the equation (2.6) rewrites as \(-\alpha R = f(kR)\). Using the identities \( 2K'_\nu = -K_{\nu+1} - K_{\nu-1} \) and \( K_{\nu+1} (x) - K_{\nu-1} (x) = \frac{2\nu}{x} K_\nu (x) \) (see [1, Sec. 9.6.26]), we find after lengthy but elementary computations that (here the derivative is with respect to \( x \) and for brevity we omit the arguments of the functions)

\[
f' = \frac{K_{\nu+1}}{K_\nu} + \frac{x}{2K_\nu} \left( K_\nu K_{\nu+1}' - K'_\nu K_{\nu+1} \right)
= \frac{K_{\nu+1}}{K_\nu} + \frac{x}{2K_\nu} \left( K_{\nu-1} K_{\nu+1} + K_{\nu+1}^2 - K_{\nu}^2 \right)
= \frac{K_{\nu+1}}{K_\nu} + \frac{x}{2K_\nu} \left( K_{\nu+1} - K_{\nu}^2 \right)
+ \frac{x}{2K_\nu} \left( K_{\nu+1} - K_{\nu} \right) \left( K_{\nu+1} - K_{\nu} \left( K_{\nu+1} + \frac{2(\nu+1)}{x} K_{\nu+1} \right) \right)
= -2 \frac{K_{\nu+1}}{K_\nu} + \frac{x}{2K_\nu} \left( K_{\nu+1} - K_{\nu}^2 \right) = \frac{x}{2K_\nu} \left[ \frac{2\nu}{x} K_\nu K_{\nu+1} + K_{\nu+1}^2 - K_{\nu}^2 \right]
= \frac{x}{2K_\nu} \left( K_{\nu+1} - K_{\nu+1} K_{\nu+1} + K_{\nu+1}^2 - K_{\nu}^2 \right) = x \left[ \frac{K_{\nu+1} K_{\nu+1}}{K_{\nu+1}^2} - 1 \right] > 0 ,
\]

where the last step follows from the inequality in [25, Thm. 8]. We have thus shown that \( f' > 0 \) and hence the function \( f \) is strictly increasing. In view of the asymptotic expansion \( K_\nu (x) \sim \frac{\Gamma (\nu)}{\sqrt{x}} \left( \frac{2}{x} \right)^\nu \) as \( x \to 0^+ \) (see [1, Sec. 9.6.9]), we obtain

\[
(2.7) \quad \lim_{x \to 0^+} f(x) = \frac{2^{\nu+1} \Gamma (\nu + 1)}{\Gamma (\nu + 1)} \left( \frac{2^\nu \Gamma (\nu)}{2} \right)^{-1} = 2 \frac{\Gamma (\nu + 1)}{\Gamma (\nu)} = 2 \nu .
\]

Using \( K_\nu (x) \sim \left( \frac{x}{\pi} \right)^{1/2} e^{-x} \) as \( x \to \infty \) (see [1, Sec. 9.7.2]), we also find

\[
(2.8) \quad \lim_{x \to \infty} f(x) = \infty .
\]
Finally, combining monotonicity of \( f \) and the limits (2.7), (2.8), we conclude that the algebraic equation \(-\alpha R = f(kR)\) has at least one (and exactly one) solution \( k > 0 \) if, and only if, \( \alpha < -\frac{2}{\pi R} = -\frac{\sqrt{2}}{\pi R} \).

3. Planar domains

3.1. Proof of Theorem 3. Let \( N := N_\Omega \in \mathbb{N} \) be the number of connected components of the domain \( \Omega = \bigcup_{n=1}^{N} \Omega_n \subset \mathbb{R}^2 \), where \( \Omega_n \subset \mathbb{R}^2 \) are bounded, smooth, simply connected domains and \( \overline{\Omega_n} \cap \overline{\Omega_m} = \emptyset \) if \( n \neq m \).

Let \( \kappa_n: \partial \Omega_n \to \mathbb{R} \) be the curvature of the curve \( \partial \Omega_n \), with the sign convention that \( \kappa_n \) is non-positive if \( \Omega_n \) is convex. By [15, Cor. 2.2.2], we have \( \int_{\partial \Omega_n} \kappa_n = -2\pi \) for all \( n = 1, \ldots, N \). According to the constraint in the formulation of the theorem, we also have \( \frac{|\partial \Omega|}{c} = |\partial B_R| = c > 0 \).

Let \( \rho: \Omega^\text{ext} \mapsto (0, +\infty) \) be the distance function from the boundary \( \partial \Omega \) of \( \Omega \). Furthermore, we define one more auxiliary function by

\[
A: [0, \infty) \to [0, \infty), \quad A(r) := |\{x \in \Omega^\text{ext} : \rho(x) < r \}|.
\]

Note that the value \( A(r) \) is simply the area of the sub-domain of \( \Omega^\text{ext} \) which consists of the points located at a distance less than \( r \) from the boundary \( \partial \Omega \). According to [23, Prop. A.1], the function \( A(r) \) is locally Lipschitz continuous and thus differentiable almost everywhere.

For a Lipschitz continuous, compactly supported \( \phi: [0, \infty) \to \mathbb{C} \), we introduce the compositions \( u := \phi \circ A \circ \rho: \Omega^\text{ext} \to \mathbb{C} \) and \( \psi := \phi \circ A: [0, \infty) \to \mathbb{C} \). Hence, \( u \) and \( \psi \) are Lipschitz continuous and compactly supported in \( \Omega^\text{ext} \) and in \( [0, \infty) \), respectively. In particular, we have \( u \in W^{1,2}(\Omega) \) and

\[
\|\nabla u\|_{L^2(\Omega; \mathbb{C}^2)}^2 = \int_0^\infty \left| \phi'(A(r)) \right|^2 |A'(r)|^2 \, dr = \int_0^\infty |\psi'(t)|^2 A'(t) \, dt,
\]

\[
\|u\|_{L^2(\Omega)}^2 = \int_0^\infty \left| \phi(A(r)) \right|^2 A'(r) \, dr = \int_0^\infty |\psi(t)|^2 A'(t) \, dt,
\]

\[
\|u\|_{L^2(\partial \Omega)}^2 = |\partial \Omega| |\phi(0)|^2 = |\partial \Omega| |\psi(0)|^2.
\]

The last formula of the above three is almost obvious. The first and the second formulae are consequences of the co-area formula and their complete derivation can be found in [23, App. 1] (see also [10, Sec. 4]).

Furthermore, we observe using geometric isoperimetric inequality and [23, Prop. A.1 (iv)] (see also [26]) that

\[
(4\pi |\Omega|)^{1/2} \leq A'(r) = |\{x \in \Omega^\text{ext} : \rho(x) = r \}| \leq |\partial \Omega| - r \sum_{n=1}^{N} \int_{\partial \Omega_n} \kappa_n(s) \, ds
\]

\[
= |\partial \Omega| + 2\pi Nr.
\]

Note that any \( \psi \in C_0^\infty([0, \infty)) \) can be represented as a composition \( \phi \circ A \) with a Lipschitz continuous, compactly supported \( \phi = \psi \circ A^{-1}: [0, \infty) \to \mathbb{C} \), where \( A^{-1} \) stands for the function inverse to \( A \). Thus, by the minimax
3.1. Large coupling for a union of identical disjoint disks. The result of Theorem 3 can be viewed as an inequality

\[
\lambda_1^\alpha(\Omega^{ext}) \leq \inf_{\psi \in C_0^\infty(\mathbb{R}^+, [0, \infty)) \setminus \{0\}} \frac{\int_0^\infty |\psi'(t)|^2 (|\partial \Omega| + 2\pi N t) \, dt + \alpha |\partial \Omega| |\psi(0)|^2}{\int_0^\infty |\psi(t)|^2 (|\partial \Omega| + 2\pi t) \, dt} = \inf_{\psi \in C_0^\infty(\mathbb{R}^+, [0, \infty)) \setminus \{0\}} \frac{\int_0^\infty |\psi'(t)|^2 (|\partial B_R| + 2\pi t) \, dt + \alpha |\partial B_R| |\psi(0)|^2}{\int_0^\infty |\psi(t)|^2 (|\partial B_R| + 2\pi t) \, dt} = \lambda_1^\alpha(B_R^{ext}),
\]

where in the last step we implicitly employed that \( \lambda_1^\alpha(B_R^{ext}) < 0 \) and that the eigenfunction of the Robin Laplacian \(-\Delta^{B_R^{ext}}\) corresponding to \( \lambda_1^\alpha(B_R^{ext}) \) is radially symmetric. \( \square \)

3.2. Large coupling for a union of identical disjoint disks. The result of Theorem 3 can be viewed as an inequality

\[
\lambda_1^\alpha(\Omega^{ext}) \leq \lambda_1^\alpha(B_R^{ext}), \quad \forall \alpha < 0,
\]

for \( \Omega \subset \mathbb{R}^2 \) being a smooth, bounded open set consisting of \( N \in \mathbb{N} \) disjoint simply connected components and satisfying the relation \( \frac{|\partial \Omega|}{N} = |\partial B_R| = c > 0 \). In this section we argue why the inequality (3.1) cannot be improved for a subclass of domains with prescribed number of connected components.

To this aim fix a positive number \( r \) and a discrete set of points \( X := \{x_n\}_{n=1}^N \subset \mathbb{R}^2 \) such that \( |x_n - x_m| > r \) for any \( n \neq m \). Let us consider the planar set

\[
\Omega := \bigcup_{n=1}^N B_r(x_n),
\]

where \( B_r(x_n) \) denotes the open disk of radius \( r \) centered at \( x_n \). Clearly, we have \( N_\Omega = N \), \( c = \frac{|\partial \Omega|}{N} = 2\pi r \), and the curvature of \( \partial \Omega \) equals \(-1/r\) pointwise. Using the large coupling asymptotics given in [20, Cor. 1.4], we have

\[
\lambda_1^\alpha(\Omega^{ext}) = -\alpha^2 - \frac{\alpha}{r} + o(\alpha), \quad \alpha \to -\infty,
\]

\[
\lambda_1^\alpha(B_R^{ext}) = -\alpha^2 - \frac{\alpha}{R} + o(\alpha), \quad \alpha \to -\infty.
\]

We conclude that for \( R > r \) the “reversed” inequality \( \lambda_1^\alpha(\Omega^{ext}) > \lambda_1^\alpha(B_R^{ext}) \) holds for all \( \alpha < 0 \) with \( |\alpha| \) large enough. Hence, the inequality (3.1) is in general not valid for \( |\partial B_R| > c \) for a subclass of domains with exactly \( N \) connected components.
3.3. Proof of Corollary 4. The optimisation result under the fixed perimeter constraint immediately follows from Theorem 3. In order to show the optimisation result under fixed area constraint, we first observe that, by the isoperimetric result, \( \lambda_1^\alpha(\Omega) \leq \lambda_1^\alpha(B^\text{ext}_{R_2}) \) if \( |\partial \Omega| = |\partial B_{R_1}| \). Next, note that by the geometric isoperimetric inequality for the ball \( B_{R_1} \) satisfying \( |\Omega| = |B_{R_1}| \), we have \( R_1 \leq R_2 \) and thus by the strict monotonicity (2.4) we get, for any negative \( \alpha \),

\[
\lambda_1^\alpha(\Omega) \leq \lambda_1^\alpha(B^\text{ext}_{R_2}) \leq \lambda_1^\alpha(B^\text{ext}_{R_1}).
\]

\( \square \)

4. Domains in higher space-dimensions

4.1. Higher order mean curvatures. Let \( \Omega \subset \mathbb{R}^d \), with \( d \geq 3 \), be a bounded smooth domain. Recall that \( \kappa_1, \kappa_2, \ldots, \kappa_{d-1} \) denote the principal curvatures of \( \partial \Omega \). They are defined locally as eigenvalues of the Weingarten tensor \( \mathcal{W} := dn \), where \( n \) is the outer unit normal to \( \Omega \) in our convention. Consequently, the principal curvatures are non-negative if \( \Omega \) is convex.

Given \( j \in \{0, 1, \ldots, d-1\} \), let \( M_j \) be the \( j \)-th order mean curvature, normalised so that the symmetric function of the principal curvatures \( F_{\partial \Omega} : \partial \Omega \times [0, \infty) \to \mathbb{R} \) can be expanded as follows:

\[
F_{\partial \Omega}(s, t) := \prod_{j=1}^{d-1} (1 + t \kappa_j(s)) = \sum_{j=0}^{d-1} \binom{d-1}{j} M_j(s) t^j.
\]

Thus, \( M_0 = 1, M_1 = M \) is the usual mean curvature introduced already in (1.2), and \( M_{d-1} = \prod_{j=1}^{d-1} \kappa_j \) is the Gauss-Kronecker curvature. While the principal curvatures \( \kappa_1, \ldots, \kappa_{d-1} \) are defined only locally, the invariants \( M_1, \ldots, M_{d-1} \) are globally defined functions.

The averaged \( j \)-th-order integral of the mean curvature is defined by

\[
\mathcal{M}_j(\partial \Omega) := \frac{1}{|\partial \Omega|} \int_{\partial \Omega} M_j(s) \, d\sigma(s),
\]

for \( j \in \{1, \ldots, d-1\} \). By [24, §4.2] we have \( \mathcal{M}_{d-1}(\partial \Omega) = \frac{\pi^{d/2}}{|\partial \Omega|} \) for any convex set \( \Omega \), where \( s_d \) as usual denotes the area of the unit sphere in \( \mathbb{R}^d \). Note also that \( \mathcal{M}_j(\partial B_R) = R^{-j} \) for all \( j = 1, \ldots, d-2 \).

4.2. Proof of Theorem 5. Let \( \Omega \subset \mathbb{R}^d, d \geq 3 \), be a bounded, convex smooth domain and let \( B_R \subset \mathbb{R}^d \) be a ball such that \( \mathcal{M}(\partial \Omega) = \mathcal{M}(\partial B_R) = R^{-(d-1)} \). In the case that \( \lambda_1^\alpha(B^\text{ext}_{R}) = 0 \) the claim obviously holds and we assume without loss of generality that \( \alpha < 0 \) and \( R > 0 \) are such that \( \lambda_1^\alpha(B^\text{ext}_{R}) < 0 \). We split the proof into four steps.

Step 1. For any \( j \in \{1, \ldots, d-2\} \), Maclaurin’s inequality [11, Ineq. 52] yields that \( M_j \leq M^j \) holds pointwise. Hence, for \( j \in \{1, \ldots, d-2\} \) we get
using Jensen’s inequality applied to the concave function \([0, \infty) \ni x \mapsto x^{\frac{1}{m-1}}\)

\[
\mathcal{M}_j(\partial \Omega) = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} M_j(s) \, d\sigma(s)
\]

\[
\leq \frac{1}{|\partial \Omega|} \int_{\partial \Omega} (M(s))^j \, d\sigma(s)
\]

\[
\leq \left( \frac{1}{|\partial \Omega|} \int_{\partial \Omega} (M(s))^{d-1} \, d\sigma(s) \right)^{\frac{1}{d-1}}
\]

\[
= (\mathcal{M}(\partial \Omega))^{\frac{1}{d-1}} = (\mathcal{M}(\partial B_R))^{\frac{1}{d-1}} = \mathcal{M}_j(\partial B_R).
\]

In particular, we have shown that

\[
\mathcal{M}_1(\partial \Omega) \leq \mathcal{M}_1(\partial B_R) = \frac{1}{R}.
\]

**Step 2.** In this step we show that \(|\partial \Omega| \geq |\partial B_R|\). Using [7, Eq. 17 in §19.3, Rem. 19.3.4] and the Alexandrov-Fenchel inequality [7, §20.2, Eq. 20] we obtain

\[
\mathcal{M}_1(\partial \Omega) \geq \left( \frac{s_d}{|\partial \Omega|} \right)^{\frac{1}{d-1}}.
\]

Hence, the inequality \(\mathcal{M}_1(\partial \Omega) \leq R^{-1}\) (shown in Step 1) yields

\[
|\partial \Omega| \geq s_d R^{d-1} = |\partial B_R|.
\]

**Step 3.** Integrating the symmetric functions \(F_{\partial \Omega}\) and \(F_{\partial B_R}\) of the principal curvatures defined as in (4.1) over \(\partial \Omega\) and \(\partial B_R\), respectively, we obtain Steiner-type polynomials

\[
P_{\partial \Omega}(t) := \int_{\partial \Omega} \frac{F_{\partial \Omega}(s,t)}{|\partial \Omega|} \, d\sigma(s) = 1 + \sum_{j=1}^{d-2} \left( \frac{M_j(\partial \Omega)}{|\partial \Omega|} \right) t^j + \frac{s_d t^{d-1}}{|\partial \Omega|},
\]

\[
P_{\partial B_R}(t) := \int_{\partial B_R} \frac{F_{\partial B_R}(s,t)}{|\partial B_R|} \, d\sigma(s) = 1 + \sum_{j=1}^{d-2} \left( \frac{M_j(\partial B_R)}{|\partial B_R|} \right) t^j + \frac{s_d t^{d-1}}{|\partial B_R|}.
\]

Further, using the inequalities \(\mathcal{M}_j(\partial \Omega) \leq \mathcal{M}_j(\partial B_R), j = 1, \ldots, d-2, \) and \(|\partial B_R| \leq |\partial \Omega|\) (shown in Steps 1 and 2) we obtain

\[
P_{\partial \Omega}(t) \leq P_{\partial B_R}(t), \quad \forall \ t > 0.
\]

**Step 4.** Next, we parameterise \(\Omega^{\text{ext}}\) by means of the parallel coordinates

\[
\mathcal{L}: \partial \Omega \times (0, \infty) \to \Omega^{\text{ext}} : \{ (s,t) \mapsto s + n(s) t \},
\]

where \(n\) is the outer unit normal to \(\Omega\) as above. Notice that \(\mathcal{L}\) is indeed a global diffeomorphism because of the convexity and smoothness assumptions; cf. [16, Sec. 4]. The metric induced by (4.3) acquires a block form

\[
d\mathcal{L}^2 = g \circ (I + t \mathcal{W})^2 + dt^2,
\]
where \( g \) is the Riemannian metric of \( \partial \Omega \) and \( W = du \) is the Weingarten tensor introduced above, cf. [17, Sec. 2] for details. Hence, \( \Omega_{ext} \) can be identified with the product manifold \( \partial \Omega \times (0, \infty) \) equipped with the metric (4.4). Consequently, the Hilbert space \( L^2(\Omega_{ext}) \) can be identified with

\[
\mathcal{H} := L^2(\partial \Omega \times (0, \infty), F_{\partial \Omega}(s, t) \, ds \, dt)
\]

via the unitary transform

(4.5) \[ U : L^2(\Omega_{ext}) \to \mathcal{H} : \{ u \mapsto u \circ \mathcal{L} \}. \]

It is natural to introduce the unitarily equivalent operator \( H_\alpha := U(\Delta_{\Omega}^{ext})U^{-1} \), associated with the transformed quadratic form \( h_\alpha[v] := Q_{\Omega_{ext}}^{ext}[U^{-1}v], D(h_\alpha) = U(W^{1,2}(\Omega_{ext})). \) Applying the minimax principle [22, Sec. XIII.1] for the quadratic form \( h_\alpha \) on functions in \( C_0^\infty(\partial \Omega \times [0, \infty)) \) dependent on \( t \)-variable only we get

\[
\lambda_\alpha^0(\Omega_{ext}) \leq \inf_{\psi \in C_0^\infty([0, \infty)) \setminus \{0\}} \frac{\int_{\partial \Omega} \int_0^\infty |\psi(t)|^2 F_{\partial \Omega}(s, t) \, dt \, ds(s) + \alpha |\partial \Omega||\psi(0)|^2}{\int_{\partial \Omega} \int_0^\infty |\psi(t)|^2 F_{\partial \Omega}(s, t) \, dt \, ds(s)}
\]

\[
= \inf_{\psi \in C_0^\infty([0, \infty)) \setminus \{0\}} \frac{\int_0^\infty |\psi(t)|^2 P_{\partial \Omega}(t) \, dt + \alpha |\psi(0)|^2}{\int_0^\infty |\psi(t)|^2 P_{\partial \Omega}(t) \, dt}
\]

\[
\leq \inf_{\psi \in C_0^\infty([0, \infty)) \setminus \{0\}} \frac{\int_0^\infty |\psi(t)|^2 P_{\partial B_R}(t) \, dt + \alpha |\psi(0)|^2}{\int_0^\infty |\psi(t)|^2 P_{\partial B_R}(t) \, dt}
\]

\[
= \lambda_\alpha^0(B_R^{ext}),
\]

where we have applied the inequality (4.2) and used in between that \( \lambda_\alpha^0(B_R^{ext}) < 0 \) and that the eigenfunction of \( -\Delta_{\Omega_{ext}}^{B_R} \) corresponding to \( \lambda_\alpha^0(B_R^{ext}) \) is radially symmetric.

4.3. A connection with large coupling asymptotics. The result of Theorem 5 can be seen as the inequality

(4.6) \[ \lambda_\alpha^0(\Omega_{ext}) \leq \lambda_\alpha^0(B_R^{ext}), \quad \forall \alpha < 0, \]

for \( \Omega \subset \mathbb{R}^d, \ d \geq 3 \), being a smooth, bounded convex open set satisfying the relation \( \mathcal{M}(\partial \Omega) = \mathcal{M}(\partial B_R) \) with \( \mathcal{M}(\cdot) \) defined as in (1.3). Assuming that \( \Omega \) and \( B_R \) are not congruent, the constraint \( \mathcal{M}(\partial \Omega) = \mathcal{M}(\partial B_R) \) and the main result of [2] imply that

\[
\delta := \min_{s \in \partial \Omega} M(s) - R^{-1} < 0.
\]

Hence, by [20, Cor. 1.4] we have

\[
\lambda_\alpha^0(B_R^{ext}) - \lambda_\alpha^0(\Omega_{ext}) = \alpha \delta (d - 1) + o(\alpha), \quad \alpha \to -\infty.
\]

Informally speaking, the “gap” in the isoperimetric inequality (4.6) grows asymptotically linearly in \( |\alpha| \) as \( \alpha \to -\infty \).
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DEPARTMENT OF MATHEMATICS, FACULTY OF NUCLEAR SCIENCES AND PHYSICAL ENGINEERING, CZECH TECHNICAL UNIVERSITY IN PRAGUE, TROJANOVA 13, 12000 PRAGUE 2, CZECH REPUBLIC
E-mail address: david.krejcirik@fjfi.cvut.cz

DEPARTMENT OF THEORETICAL PHYSICS, NUCLEAR PHYSICS INSTITUTE, CZECH ACADEMY OF SCIENCES, 25068 ŘEŽ, CZECH REPUBLIC
E-mail address: lotoreichik@ujf.cas.cz