Massive spinning particles and the geometry of null curves

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Abstract

We study the simplest geometrical particle model associated with null paths in four-dimensional Minkowski space-time. The action is given by the pseudo-arclength of the particle worldline. We show that the reduced classical phase space of this system coincides with that of a massive spinning particle of spin \( s = \alpha^2 / M \), where \( M \) is the particle mass, and \( \alpha \) is the coupling constant in front of the action. Consistency of the associated quantum theory requires the spin \( s \) to be an integer or half integer number, thus implying a quantization condition on the physical mass \( M \) of the particle. Then, standard quantization techniques show that the corresponding Hilbert spaces are solution spaces of the standard relativistic massive wave equations. Therefore this geometrical particle model provides us with an unified description of Dirac fermions \((s = 1/2)\) and massive higher spin fields.

1 Introduction

The search for a classical particle model that under quantization yields the Dirac equation and its higher spin generalizations has a long history. By far the most popular approach is to supersymmetrize the standard relativistic particle model, whose action is given by the proper time of the particle’s path. It is

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not difficult to show that the system possessing $N = 2s$ extended supersymmetry corresponds, after quantization, to a massive spinning particle of spin $s$. Nevertheless, the search for geometrical particle models of purely bosonic character, which may lead to similar results, is interesting on its own. The reasons for this are quite simple, and they may be exemplified by Polyakov’s results on Fermi-Bose transmutation in three dimensions. In [1], Polyakov was able to show how, in the context of three-dimensional Chern–Simons theory, the presence of a torsion term in the effective action for a Wilson loop was responsible for the appearance of Dirac fermions in an otherwise apparently bosonic theory—thus opening the question of which is the natural counterpart, if any, in the four-dimensional case.

The purpose of this work is to show that there is a geometrical particle model, based purely on the geometry of null curves in four-dimensional Minkowski spacetime, that does the job, i.e., that under quantization yields the wave equations corresponding to massive spinning particles of arbitrary integer or half-integer spin. The action for such a model is given by the pseudo-arclength, the simplest among all the geometrical invariants associated with null curves.

A few words are due now in order to motivate, from a more ‘intuitive’ point of view, our approach to this problem.

How is that a particle model based on null curves, which should correspond a priori with massless particles, may be of relevance in the massive case? The main clue that this can be indeed the case comes from the Zitterbewegung associated with Dirac’s equation. It is a well-known result that a measurement of the instantaneous speed in the relativistic theory of Dirac must lead to a result of $\pm c$—that this is in no contradiction with Lorentz invariance is explained, for example, in [6]—hence making plausible that light-like paths may play a role in the corresponding particle system.

The second observation is a direct consequence of the one above: due to the null character of the path any sensible local action should depend on higher order derivatives. It is clear that a model for spinning particles should contain extra degrees of freedom in order to accommodate the ones associated with spin. Although some Lagrangian models have been introduced in the literature where the extra degrees of freedom are added by hand (see, e.g., [4] and references therein), it would be highly desirable if they were provided by the geometry itself. It is well known by now that higher derivative theories may provide the required phase space. As we already commented, that is the case in three dimensions, and moreover it has also already proven to work in four dimensions, for massless particles, with a Lagrangian given by the first curvature of the particle’s worldline [5]. Our choice has, then, been the obvious one, this is to consider the simplest geometrical invariant associated with null, or light-like, paths. Therefore, and with no further delay we will begin
to introduce the necessary geometrical background to work out the model at hand.

2 Frenet equations for null curves in $M^4$

Let us start by constructing the Frenet equations associated with null curves in four-dimensional Minkowski space-time. Our conventions for the signature of the metric are $(+, -, -, -)$, i.e., time-like vectors have a positive norm.

If we denote by $x$ the embedding coordinate of the curve, the fact that the curve is null implies that $\dot{x}^2 = 0$. Let us denote by $e_+ = \dot{x}$ the tangent vector to the curve. From $e_+^2 = 0$ it follows that there is a space-like vector $e_1$ such that

$$\dot{e}_+ = \sigma e_1, \quad (1)$$

with $e_+ e_1 = 0$, and $e_1^2 = -1$. Although, a priori, a term proportional to $e_+$ may appear in the right hand side of the above equation it is always possible to reabsorb it by a redefinition of $e_1$.

It will now show convenient to choose a parametrization for which $\sigma = 1$. This is equivalent to demanding that $\ddot{x}\dot{x} = -1$, and this can always be achieved by a change of parametrization unless $\dddot{x} = 0$, this last case being trivial will be excluded from our considerations. This parametrization corresponds to choosing as our time parameter the pseudo-arclength, which is defined as

$$\sigma(t) = \int_{t_0}^{t} (-\dddot{x}(t')\dot{x}(t'))^{\frac{1}{2}} dt'.$$ \quad (2)

We now pass to obtain Frenet equations in this parametrization. From the definition of $e_1$ it follows that $\dot{e}_1 e_1 = 0$ and $\dot{e}_1 e_+ = 1$, hence

$$\dot{e}_1 = Ae_+ + f_- + C g_2, \quad (3)$$

where the two new vectors $g_2$ and $f_-$ are chosen to obey that $g_2^2 = -1$, $f_-^2 = 0$, $g_2 e_1 = g_2 e_+ = g_2 f_- = e_1 f_- = 0$, together with $e_+ f_- = 1$. From the fact that $(e_+, f_-, e_1, g_2)$ span the tangent space at any point on the curve the above equation trivially follows.

But notice that one can choose a basis $(e_+, e_-, e_1, e_2)$ with
\[ e_- = f_- + Ce_2 + \frac{1}{2}C^2e_+, \quad (4) \]
\[ e_2 = g_2 + Ce_+ \quad (5) \]

that, while preserving the same orthogonality relationships, simplifies the above equation to yield

\[ \dot{e}_1 = \kappa_1 e_+ + e_-, \quad (6) \]

with \( \kappa_1 = A - C^2/2 \). The remaining Frenet equations associated with the Frenet frame \((e_+, e_-, e_1, e_2)\) follow from the orthogonality relations, and the whole set reads

\[ \dot{x} = e_+, \quad (7) \]
\[ \dot{e}_+ = e_1, \quad (8) \]
\[ \dot{e}_1 = \kappa_1 e_+ + e_-, \quad (9) \]
\[ \dot{e}_- = \kappa_1 e_1 + \kappa_2 e_2, \quad (10) \]
\[ \dot{e}_2 = \kappa_2 e_+. \quad (11) \]

Notice that in this case there are only two independent curvature functions \( \kappa_1 \) and \( \kappa_2 \). This is intuitively obvious due to the extra constraint on null curves coming from \( \dot{x}\dot{x} = 0 \).

### 3 The classical model

We will consider the simplest geometrical action associated with null curves, this is, its pseudo-arclength:

\[ S = 2\alpha \int d\sigma. \quad (12) \]

It is convenient, although not strictly necessary, to write the action in first order form. One then can write

\[ S = \int dt \left( 2\alpha \sqrt{-\dot{e}_+\dot{e}_+} + p(\dot{x} - e_+ \sqrt{-\dot{e}_+\dot{e}_+}) - \lambda e_+^2 \right). \quad (13) \]

Notice that the equation of motion for the Lagrange multipliers \( p \) and \( \lambda \) imply that

\[ \sqrt{-\dot{e}_+\dot{e}_+} = (-\ddot{x}x)^{\frac{1}{2}}, \quad (14) \]
thus proving the equivalence of both actions.

It is now straightforward to develop the Hamiltonian formalism following Dirac’s description for singular Lagrangians. We will skip the details, which are completely standard. After a little work one arrives to a symplectic form

\[ \Omega = dp \wedge dx + dp_+ \wedge de_+, \]  

(15)

endowed with the following set of primary constraints

\[ \phi_1 = p_+^2 + (pe_+ - 2\alpha)^2, \]  

(16)

\[ \phi_2 = e_+^2, \]  

(17)

and secondary ones

\[ \phi_3 = p_+ e_+, \]  

(18)

\[ \phi_4 = pe_+ - \alpha, \]  

(19)

\[ \phi_5 = pp_+. \]  

(20)

The Hamiltonian is of the form

\[ H = v(p_+^2 + p^2e_+^2 + (pe_+ - 2\alpha)^2), \]  

(21)

where \( v \) should be regarded as an arbitrary function that is only fixed by choosing a particular parametrization. If one chooses to parametrize the path by pseudo-arclength one gets that \( v = -1/2\alpha. \)

There is only one first class constraint, the generator of reparametrizations. The dimension of the reduced phase space is therefore 10.

The equations of motion, in the pseudo-arclength parametrization, are given by

\[ \dot{x} = e_+, \]  

(22)

\[ \dot{e}_+ = -\frac{1}{\alpha}p_+, \]  

(23)

\[ \dot{p}_+ = \frac{1}{\alpha}p^2e_+ - p, \]  

(24)

\[ \dot{p} = 0. \]  

(25)
Consistency of these equations of motion with Frenet equations imply that \( p_+ = -\alpha e_1, \ p = \alpha e_- + p^2 e_+/2\alpha, \) together with
\[
\kappa_1 = \frac{1}{2\alpha^2} p^2. \quad (26)
\]
The solution of these equations of motion are particular examples of null helices [3].

The key observation, in order to arrive to a manageable expressions for the symplectic form on the constrained surface, is that it is possible to define a standard free coordinate \( X \) out from the canonical variables of our model. Notice that
\[
X = x - \frac{\alpha}{p^2} p_+. \quad (27)
\]
has the property that its time derivative is given by
\[
\dot{X} = \dot{x} - \frac{\alpha}{p^2} \dot{p}_+ = \frac{\alpha}{p^2} p. \quad (28)
\]
And from this it trivially follows that \( \ddot{X} = 0. \)

It is then natural to introduce the new coordinate
\[
E_+ = e_+ - \frac{\alpha}{p^2} p_+ \quad (29)
\]
so that the symplectic form \( \Omega \) takes the simple form
\[
\Omega = dp \wedge dX + dp_+ \wedge dE_+ \quad (30)
\]
on the constrained surface.

The constraints may be equally expressed in these variables, and they read
\[
\begin{align*}
p_+^2 + \alpha^2 &= 0, \quad (31) \\
p_+ E_+ &= 0, \quad (32) \\
E_+^2 + \frac{\alpha^2}{p^2} &= 0, \quad (33) \\
p p_{p+} &= 0, \quad (34) \\
p E_+ &= 0. \quad (35)
\end{align*}
\]
This constraint system suggests the introduction of the following complex coordinates

$$z = p_+ + i\sqrt{p^2} E_+.$$  \hfill (36)

In terms of $z$ the constraints simply read

$$z^2 = 0, \quad z\bar{z} + 2\alpha^2 = 0, \quad \text{and} \quad pz = 0.$$  \hfill (37)

Let us recall that the irreducible representations of the Poincaré algebra are labeled by the values of the two Casimirs $p^2$ and the square of the Pauli-Lubansky vector

$$S^\mu = \frac{1}{2} \epsilon^{\mu\rho\sigma} p_\rho M_{\sigma},$$  \hfill (38)

with $M_{\rho\sigma}$ the generator of Lorentz transformations. In our particular case the explicit expression for $S_\mu$ reads

$$S_\mu = \epsilon_{\mu\rho\sigma} p^\rho p^\sigma, \quad e_+$$  \hfill (39)

and one obtains that $S^2 = -\alpha^4$, while there is no restriction on the possible values of $p^2$. This implies that our phase space is not elementary, i.e., the Poincaré group does not act in a transitive way. In order to obtain irreducible representations of the Poincaré group under quantization—physical states should always be decomposable into irreducible representations of the Poincaré group—we will study the elementary phase spaces defined through the extra constraint

$$p^2 = M^2,$$  \hfill (40)

with $M$ a free parameter with dimensions of mass. A priori $M^2$ could be positive, negative or zero, we will only consider the first case, because the other two correspond to unphysical irreducible representations of the Poincaré group, i.e., tachyonic and continuous spin representations, respectively.

Because of the null character of $z$ it will show convenient to introduce a spinor parametrization of the reduced phase space. Given an arbitrary complex four-vector $y$ it can be rewritten in spinor coordinates as follows:

$$y^{A\bar{A}} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} y^0 + y^3 \\ y^1 - iy^2 \\ y^1 + iy^2 \\ y^0 - y^3 \end{array} \right)$$  \hfill (41)
so that \( \det(y^{A\dot{A}}) = \frac{1}{2}g_{\mu\nu}y^\mu y^\nu \).

Because of the two-to-one local isomorphism between \( SL(2, \mathbb{C}) \) and the identity component of the Lorentz group, one such Lorentz transformation on \( y \) is equivalently represented by the action of an \( SL(2, \mathbb{C}) \) matrix acting on the undotted indices and its complex conjugate matrix on the dotted ones. Raising and lowering of indices is mimicked in spinor language by contraction with the invariant antisymmetric tensors \( (\epsilon^{A\dot{B}}) = (\epsilon_{A\dot{B}}) \), with \( \epsilon^{01} = +1 \), and analogous expressions for the dotted indices.

The null character of \( z \) now implies that
\[
z^{A\dot{A}} = \sqrt{2}\alpha \xi^{A}\bar{\eta}^{\dot{A}},
\]
and from the second of our constraints it must follow that \( \xi^{A}\eta_{A} = 1 \), or equivalently that \( \xi \) and \( \eta \) form a spinor basis. Notice, though, that this does not completely fix the spinors \( \xi \) and \( \eta \) because one still has the freedom to rescale both as follows
\[
\xi^{A} \rightarrow a\xi^{A} \quad \text{and} \quad \eta^{A} \rightarrow \frac{1}{a}\eta^{A},
\]
with \( a \) an arbitrary (nonzero) real number. This residual freedom will be fixed as follows. Let us consider the remaining constraint \( pz = 0 \). In spinor coordinates it reads
\[
p^{A\dot{A}}\xi_{A}\bar{\eta}_{\dot{A}} = 0
\]
or equivalently
\[
p^{A\dot{A}}\xi_{A} = \Lambda\sqrt{p^{2}\bar{\eta}^{\dot{A}}} = \Lambda M\bar{\eta}^{\dot{A}},
\]
with \( \Lambda \) an arbitrary (nonzero) real number. One then may fix completely the freedom to rescale the spinors by setting \( \Lambda = \pm 1/\sqrt{2} \), and then one may finally write the only remaining constraint in the following form
\[
p^{A\dot{A}}\xi_{A}\bar{\xi}_{\dot{A}} = \pm \frac{M}{\sqrt{2}}
\]

A direct computation now shows that the symplectic form in spinor coordinates reads
\[
\Omega = dp \wedge dX \pm i\frac{\sqrt{2}s}{M}(p_{A\dot{A}}d\xi^{A} \wedge d\bar{\xi}^{\dot{A}} + \bar{\xi}^{\dot{A}}d\xi^{A} \wedge dp_{A\dot{A}}) + \ c.c.,
\]
where \(c.c.\) stands for complex conjugate of the previous term, and \(s\) is a dimensionless parameter defined as
\[
s = \frac{\alpha^2}{M}.
\] (48)

### 4 The quantum theory

One may now check (see [2]) that the reduced phase space can be identified with the coadjoint orbit of the Poincaré group associated with a representation of mass \(M\) and spin \(s\). Quantization is then a standard exercise whose solution can be found, for example, in the excellent book on geometric quantization by Woodhouse [2]. Nevertheless, because of completeness, and the desire to simplify things a little to the less mathematically oriented reader, we will now sketch how the quantization procedure may be carried away in spinor coordinates. In particular, we will show how Dirac equation may be obtained in the \(s = 1/2\) case.

One should start by choosing a polarization, or in simpler words one should demand the wave function to depend only on half of the canonical coordinates. Our choice will be that the wave function will depend on \(p\) and \(\xi\).

In order to quantize the system we should implement the remaining first class constraints as conditions on the wave function. The first constraint \(p^2 = M^2\) implies that the wave function has support on the mass hyperboloid. In order to implement the second one (46) notice that, roughly speaking, \(p_{AA} \xi^A\) and \(\xi\) are conjugate variables in the induced symplectic form, therefore under quantization
\[
p_{AA} \xi^A \rightarrow \frac{\partial}{\partial \xi^A}
\] (49)

up to some numerical factors. A careful computation shows that (46) implies at the quantum level that
\[
\xi^A \frac{\partial \psi(p, \xi)}{\partial \xi^A} = 2s \psi(p, \xi),
\] (50)
i.e., the wave function is a homogeneous function of \(\xi\) of degree \(2s\). Then single-valuedness of our wave function under \(\xi \rightarrow \exp(2\pi i)\xi\) requires \(s\) to be an integer or half-integer number. Therefore one has that
\[
\psi(p, \xi) = \psi_{A_1 A_2 \ldots A_{2s}}(p) \xi^{A_1} \xi^{A_2} \ldots \xi^{A_{2s}},
\] (51)
where $\psi_{A_1A_2\ldots A_{2s}}(p)$ has support on the mass hyperboloid. It is now straightforward to check that its Fourier transform
\begin{equation}
\varphi_{A_1A_2\ldots A_{2s}}(x) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{H_M^s} \psi(p)_{A_1A_2\ldots A_{2s}} e^{-ipx} d\tau, \tag{52}
\end{equation}

with $H_M^s$ the positive energy branch of the mass hyperboloid, and $d\tau$ its invariant measure, yield positive frequency solutions of the massive wave equation
\begin{equation}
\Box \varphi_{A_1A_2\ldots A_{2s}}(x) + M^2 \varphi_{A_1A_2\ldots A_{2s}}(x) = 0. \tag{53}
\end{equation}

Notice that, for example, for $s = 1/2$ this equation has the same physical content that the Dirac equation. This is so because the dotted component of the four component Dirac spinor, which will be denoted by $\chi_{\dot{A}}$, may be defined through the relation
\begin{equation}
\nabla_{A\dot{A}} \varphi^A = \frac{M}{\sqrt{2}} \chi_{\dot{A}}. \tag{54}
\end{equation}

This together with (53) implies that
\begin{equation}
\nabla_{A\dot{A}} \chi_{\dot{A}} = \frac{M}{\sqrt{2}} \varphi_A, \tag{55}
\end{equation}

thus being equivalent to the standard Dirac equation for $(\varphi, \chi)$.

Although this short discussion about the quantization of the system cannot make justice to this extensive topic, the interested reader may fill the gaps with the help of the current literature on the subject.

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**References**

[1] A.M. Polyakov, *Mod. Phys. Lett.* A3 (1988) 325.
[2] N.M.J. Woodhouse, *Geometric Quantization* (Oxford Science Publications, Oxford, 1991).

[3] W. B. Bonnor, *Tensor, N.S.* 20 (1969) 229.

[4] M. V. Atre, A. P. Balachandran, T.R. Govidarajan, *Int. J. Mod. Phys.* A2 (1987) 453.

S. L. Lyakhovich, A. A. Sharapov, K. M. Shekhter, *Massive spinning particle in any dimension. I Integer spins.*, (hepth/9805020).

[5] M.S. Plyushchay, *Mod. Phys. Lett.* A4 (1989) 837.

E. Ramos and J. Roca, *Nucl. Phys.* B477 (1996) 606.

[6] P.A.M. Dirac, *The principles of quantum mechanics* (Oxford, Clarendon, 4th ed. 1958).