AN INSIGHT ON THE FRACTAL POWER LAW FLOW: FROM A HAUSDORFF VECTOR CALCULUS PERSPECTIVE

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Abstract. In the article we suggest the Hausdorff vector calculus based on the Chen Hausdorff calculus for the first time. The Gauss-Ostrogradsky-like, Stokes-like, and Green-like theorems, and Green-like identities are obtained in the framework of the Hausdorff vector calculus. The formula is proposed as a mathematical tool to describe the real world problems for the fractal power-law flow equations with the anomalous diffusion equation. A conjecture for the fractal power-law flow equations analogous to the Smale’s 15th Problem (one of the Millennium Prize Problems for the Navier–Stokes equations) is also addressed.

1. Introduction

The Hausdorff derivative involving the fractal geometry with the Hausdorff measure, proposed by Chinese mathematician Wen Chen [1, 2], has played an important role in the treatment for the mathematical model for the anomalous diffusion process. The Hausdorff integral was suggested in 2018 by Chen and coauthors to develop the three-dimensional diffusion model for fractal porous media [3]. The Hausdorff calculus was used to the mathematical models in the real world problems for the fractal power law [4]. As is known, the Chen Hausdorff calculus is connected with the anomalous transport in porous media [5] and the fractional calculus [6].

The fractal vector calculus involving the fractal media was considered by many researchers from the different perspectives. The fractal vector calculus involving the local fractional calculus was suggested in 2009 and published in 2012 [7]. The fractal vector calculus with the fractional calculus was suggested [8]. The vector calculus with the fractal metric by using the fractional integrals was used to handle the fractional wave problem [9]. The fractal vector calculus was considered to model the fluid flows in fractally permeable reservoirs by using the theory of the Hausdorff derivative involving the Euclidean volume of the pore space scales [10].

The vector calculus with respect to monotone functions based on the Leibniz derivative and Stieltjes-Riemann integral [11, 12] was proposed in 2020 by author

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to suggest the PDEs arising in heat conduction [13] and fluid flows [14]. By using the vector calculus with respect to monotone increasing function, the vector power-law calculus was considered to deduce the PDEs in the power-law fluid flow [15]. The vector power-law calculus is called the Hausdorff vector calculus if the power-laws has the same value. The main target of the paper is to suggest the theory of the Hausdorff vector calculus to suggest the real world problems for the fractal power-law flow with the anomalous diffusion equation by using the Chen Hausdorff calculus. The structure of the paper is designed as follows. In Section 2, we introduce the theory of the Chen Hausdorff calculus. In Section 3, we propose the theory of the Hausdorff vector calculus. In Section 4, we present the system of the partial differential equations arising in fractal power-law flow. In Section 5, we suggest the anomalous diffusion equation arising in the real theory of the turbulent fluid motion. Finally, we draw the conclusion in Section 6.

2. THE THEORY OF THE CHEN HAUSDORFF CALCULUS

Let $\mathbb{R}$ be the set of the real numbers.

Suppose that $\varpi (t) = t^\mu$ is the power-law function, which derived from the Hausdorff measure [1], where $0 < \mu \leq 1$ is the fractal dimension, and $t \in \mathbb{R}$.

Let
\[ (1) \quad \Phi (t) = (\Phi \circ \varpi) (t) = \Phi (\varpi (t)), \]
where $\Phi (\varpi)$ is the background function, which is continuous.

We denote the sets of the composite functions by
\[ (2) \quad \Im = \{ \Phi (t) : \Phi (\varpi (t)) = \varpi (t) = t^\mu \}. \]

Let $\Xi \in \Im$.

- **The Chen Hausdorff derivative**

The Chen Hausdorff derivative of the function $\Xi (t)$ is defined as [1]
\[ (3) \quad C D_t^{(1)} \Xi (t) = \frac{t^{1-\mu} d \Xi (t)}{\mu dt}. \]

If (3) holds, then $\Xi (t)$ is the Hausdorff differential function for $t \in \mathbb{R}$, i.e.,
\[ \varphi = \left\{ g (t) : g (t) = C D_t^{(1)} \Xi (t) \right\}. \]

The properties for the Chen Hausdorff derivative is given as follows:

(A1) The sum and difference rules for the Chen Hausdorff derivative:
\[ (4) \quad C D_t^{(1)} (\Xi_1 (t) \pm \Xi_2 (t)) = C D_t^{(1)} \Xi_1 (t) \pm C D_t^{(1)} \Xi_2 (t), \]
where $\Xi_1 \in \varphi$ and $\Xi_2 \in \varphi$. 

(A2) The constant multiple rule for the Chen Hausdorff derivative:

\[ CD_t^{(1)} (\alpha \Xi (t)) = \alpha CD_t^{(1)} \Xi (t) , \]

where \( \Xi \in \wp \) exists and \( \alpha \) is a constant.

(A3) The product rule for the Chen Hausdorff derivative:

\[ CD_t^{(1)} (\Xi_1(t) \cdot \Xi_2(t)) = \Xi_2(t) CD_t^{(1)} \Xi_1(t) + \Xi_1(t) CD_t^{(1)} \Xi_2(t) , \]

where \( \Xi_1 \in \wp \) and \( \Xi_2 \in \wp \).

(A4) The quotient rule for the Chen Hausdorff derivative:

\[ CD_t^{(1)} \left( \frac{\Xi_1(t)}{\Xi_2(t)} \right) = \frac{\Xi_2(t) CD_t^{(1)} \Xi_1(t) - \Xi_1(t) CD_t^{(1)} \Xi_2(t)}{\Xi_2(t) \Xi_2(t)} , \]

where \( \Xi_1 \in \wp \) and \( \Xi_2 \in \wp \) exist, and \( \Xi_2(t) \neq 0 \).

(A5) The chain rule for the Chen Hausdorff derivative:

\[ CD_t^{(1)} \Lambda (t) = \frac{d\Lambda (\Xi)}{d\Xi} CD_t^{(1)} \Xi (t) , \]

where \( \Lambda (t) = \Lambda (\Xi (t)) = (\Lambda \circ \Xi)(t) \), \( d\Lambda (\Xi) / d\Xi \) and \( \Xi \in \wp \).

The properties for the Chen Hausdorff derivative is presented as follows:

(9) \( CD_t^{(1)} 1 = 0 \),

(10) \( CD_t^{(1)} \varpi (t) = 1 \),

(11) \( CD_t^{(1)} \varpi^n (t) = n \varpi^{n-1} (t) = nt^{\mu(n-1)} \),

(12) \( CD_t^{(1)} e^{\beta t} = \beta e^{\beta t} \),

(13) \( CD_t^{(1)} \ln (t^\mu) = \frac{1}{t^\mu} \),

(14) \( CD_t^{(1)} s^{\mu} = (\ln s) s^{\mu} \),

(15) \( CD_t^{(1)} \log_s (t^\mu) = \frac{1}{t^\mu \ln s} \),

(16) \( CD_t^{(1)} e^{\Xi (t)} = e^{\Xi (t)} CD_t^{(1)} \Xi (t) \),

(17) \( CD_t^{(1)} \log_s \Xi (t) = \frac{1}{\ln s} CD_t^{(1)} \Xi (t) \),

(18) \( CD_t^{(1)} \ln \Xi (t) = \frac{CD_t^{(1)} \Xi (t)}{\Xi (t)} \).
(19) \[ CD_t^{(1)} s^{\Xi(t)} = \left( \ln s \right) s^{\Xi(t)} \cdot CD_t^{(1)} \Xi(t), \]
where \( \beta \) is the constant and

(20) \[ e^{\beta t} = \sum_{n=0}^{\infty} \frac{(\beta^n t^n \mu)}{n!}, \]
is the Kohlrausch-Williams-Watts function [12].

**The Chen Hausdorff integral**

Let \( \xi \in \mathcal{S} \).

The Chen Hausdorff integral of the function \( \xi(t) \) in the interval \([a, b]\) is defined as [3]

(21) \[ C_a I_t^{(1)} \xi(t) = \mu \int_a^b \xi(t) \, t^{\mu-1} \, dt. \]

If (3) holds, then \( \Xi(t) \) is the Hausdorff integral in the interval \([a, b]\), i.e.,

\[ \mathcal{N} = \left\{ h(t) : h(t) = \mu \int_a^b \xi(t) \, t^{\mu-1} \, dt \right\}. \]

The properties for the Chen Hausdorff integral is presented as follows:

(B1) The sum and difference rules for the Chen Hausdorff integral:

(22) \[ C_a I_t^{(1)} (\xi_1(t) \pm \xi_2(t)) = C_a I_t^{(1)} \xi_1(t) \pm C_a I_t^{(1)} \xi_2(t), \]
where \( \xi_1 \in \mathcal{N} \) and \( \xi_2 \in \mathcal{N} \).

(B2) The first fundamental theorem of the Chen Hausdorff integral:

(23) \[ \Xi(t) - \Xi(a) = C_a I_t^{(1)} \left( CD_t^{(1)} \Xi(t) \right) \]

(B3) The mean value theorem for the topology integral:

(24) \[ C_a I_t^{(1)} \xi(t) = \xi(l) \left( \varpi(t) - \varpi(a) \right) \]
where \( a < t < l < b \), and \( \xi \in \mathcal{N} \).

(B4) The second fundamental theorem of the Chen Hausdorff integral:

(25) \[ \xi(t) = CD_t^{(1)} \left( C_a I_t^{(1)} \xi(t) \right) \]
where \( \xi \in \mathcal{N} \).

(B5) The net change theorem for the Chen Hausdorff integral:

(26) \[ \Xi(b) - \Xi(a) = T_a I_t^{(1)} \left( TD_t^{(1)} \Xi(t) \right) \]
where \( \Xi \in \wp \).
(B6) The integration by parts for the Chen Hausdorff integral:

\[
C_a^I t \left( \xi_2 (t) C D_t^{(1)} \xi_1 (t) \right) = [\xi_1 (t) \cdot \xi_2 (t)]_a^b - C_a^I t \left( \xi_1 (t) C D_t^{(1)} \xi_2 (t) \right),
\]

where \([\xi_1 (t) \cdot \xi_2 (t)]_a^b = \xi_1 (b) \cdot \xi_2 (b) - \xi_1 (a) \cdot \xi_2 (a)\), \(\xi_1 \in \mathbb{N}\) and \(\xi_2 \in \mathbb{N}\).

The indefinite Chen Hausdorff integral of the function \(\xi (t)\) is defined as

\[
C_t^I \xi (t) = \mu \int \xi (t) t^{\mu - 1} dt = \Xi (t) + \chi,
\]

which implies that

\[
C_t^D \left( C_t^I \xi (t) \right) = C_t^D \Xi (t) = \xi (t)
\]

and

\[
\Xi (t) - \Xi (a) = C_t^I \xi (t) - C_a^I \xi (t) = C_t^I \left( C_t^D \Xi (t) \right),
\]

where \(\chi\) is the constant.

The sum and difference rules for the indefinite Chen Hausdorff integral implies that

\[
C_t^I (\xi_1 (t) \pm \xi_2 (t)) = C_t^I \xi_1 (t) \pm C_t^I \xi_2 (t),
\]

where \(\xi_1 \in \mathbb{N}\) and \(\xi_2 \in \mathbb{N}\).

The properties for the Chen Hausdorff integral is presented as follows:

(28) \(C_t^I 1 = \varpi (t) + \chi = t^\mu + \chi\),

(29) \(C_t^I n \varpi^n (t) = \varpi^n (t) + \chi\),

(30) \(C_t^I \left( \frac{1}{\ln s} \cdot \frac{C_t^D \Xi (t)}{\Xi (t)} \right) = \log_s \Xi (t) + \chi\),

(31) \(C_t^I \frac{1}{\varpi (t)} = \ln \varpi (t) + \chi\),

(32) \(C_t^I \left( \frac{1}{\ln s} \cdot \frac{1}{\varpi (t)} \right) = \log_s \varpi (t) + \chi\),

(33) \(C_t^I (\ln s) s^{\mu} = s^{\mu} + \chi\),

(34) \(C_t^I \left( e^{\Xi (t)} C_t^D \Xi (t) \right) = e^{\Xi (t)} + \chi\),

(35) \(C_t^I \left( \frac{\Xi (t)}{\Xi (t)} C_t^D \Xi (t) \right) = |\Xi (t)| + \chi\),
\[ C I_t^{(1)} \left( \frac{C D_t^{(1)} \Xi(t)}{\Xi(t)} \right) = \ln \Xi(t) + \chi, \]

\[ C I_t^{(1)} (e^{\beta t}) = \beta e^{\beta t} + \chi, \]

\[ C I_t^{(1)} \left[ (\ln s) s^{\Xi(t)} \cdot C D_t^{(1)} \Xi(t) \right] = s^{\Xi(t)} + \chi, \]

where \( \chi \) is the constant.

**The Chen Hausdorff partial derivatives**

We now consider the coordinate system, expressed by

\[ i x^\mu + j y^\mu + k z^\mu = (x^\mu, y^\mu, z^\mu), \]

where \( i, j \) and \( k \) denote the unit vectors in the Cartesian coordinate system.

We now define the function by

\[ \psi = \Pi (x^\mu, y^\mu, z^\mu). \]

The Chen partial derivatives of the function \( \psi = \Pi (x^\mu, y^\mu, z^\mu) \) are defined as

\[ C \partial_x^{(1)} \psi = \left( \frac{x^{1-\mu}}{\mu} \frac{\partial}{\partial x} \right) \psi, \]

\[ C \partial_y^{(1)} \psi = \left( \frac{y^{1-\mu}}{\mu} \frac{\partial}{\partial x} \right) \psi, \]

\[ C \partial_z^{(1)} \psi = \left( \frac{z^{1-\mu}}{\mu} \frac{\partial}{\partial x} \right) \psi. \]

The total differential of the function \( \psi = \Pi (x^\mu, y^\mu, z^\mu) \) is defined as

\[ d\psi = \mu \left( x^{\mu-1} C \partial_x^{(1)} \psi \right) dx + \mu \left( y^{\mu-1} C \partial_y^{(1)} \psi \right) dy \]

\[ + \mu \left( z^{\mu-1} C \partial_z^{(1)} \psi \right) dz. \]

Thus, the Chen Hausdorff derivative with respect to the time \( t \) reads

\[ \frac{d\psi}{dt} = \mu \left( x^{\mu-1} C \partial_x^{(1)} \psi \right) \frac{dx}{dt} + \mu \left( y^{\mu-1} C \partial_y^{(1)} \psi \right) \frac{dy}{dt} \]

\[ + \mu \left( z^{\mu-1} C \partial_z^{(1)} \psi \right) \frac{dz}{dt}. \]

**The Chen gradient**
The Chen gradient in the Cartesian coordinate system is defined as \[ \nabla^\mu = \mu i (x^{\mu-1}) C \partial_x^{(1)} + j (y^{\mu-1}) C \partial_y^{(1)} + k (z^{\mu-1}) C \partial_z^{(1)}. \]

From (44) we arrive at
\[ d\psi = \nabla^\mu \psi \cdot d\mathbf{r} = \nabla^\mu \psi \cdot \mathbf{n} \, dr, \]
in which
\[ dr = \mathbf{n} \, dr = dx + dy + dz, \]
where \( \mathbf{n} \) is the unit normal, and \( dr \) is a distance measured along the normal \( \mathbf{n} \).

- **The Hausdorff directional derivative**

The Hausdorff directional derivative of the function \( \psi = \Pi(x, y, z) \), denoted by \( \nabla_n \psi \), is defined as
\[ \frac{d\psi}{dr} = \nabla^\mu \psi \cdot \mathbf{n} = \partial_n \psi, \]
where \( d\psi/dr \) is the rates of change of \( \psi \) along the normal \( \mathbf{n} \), respectively.

- **The Laplace-Chen operator**

The Laplace-Chen operator, denoted as \( \nabla^\mu \cdot \nabla^\mu = \nabla^{2\mu} \), of the scalar field \( \psi \) is defined as \[1, 4\]
\[ \nabla^{2\mu} \psi \]
\[ = \mu^2 \left( x^{\mu-2} \frac{\partial_x}{x} \right)^2 \psi + \mu^2 \left( y^{\mu-2} \frac{\partial_y}{y} \right)^2 \psi \]
\[ + \mu^2 \left( z^{\mu-1} \frac{\partial_z}{z} \right)^2 \psi \]
\[ = \mu^2 x^{2\mu-2} \frac{\partial_x^2}{x} \psi + \mu^2 y^{2\mu-2} \frac{\partial_y^2}{y} \psi \]
\[ + \mu^2 z^{2\mu-2} \frac{\partial_z^2}{z} \psi. \]

The properties for the Laplace-Chen operator are presented as
\[ (\nabla^\mu \cdot \nabla^\mu) \psi = \nabla^{2\mu} \psi, \]
\[ \nabla^\mu (\psi \Theta) = \psi \nabla^\mu \Theta + \Theta \nabla^\mu \psi, \]
\[ \nabla^\mu \cdot (\Theta \nabla^\mu \psi) = \Theta \nabla^{2\mu} \psi + \nabla^\mu \psi \cdot \nabla^\mu \Theta, \]
where \( \psi \) and \( \Theta \) are the fractal scalar fields.

Here, (51) is discovered by Chen \[1\].
3. The theory of the Hausdorff vector calculus

The element of the vector line

\[ \ell = \ell (x, y, z) = \tilde{\ell} (x^\mu, y^\mu, z^\mu) \]

is expressed in the form

\[ dl = m\,d\ell \]
\[ = \mu (ix^{\mu-1}dx + jy^{\mu-1}dy + kz^{\mu-1}dz) \]

and

\[ d\ell = |dl| \]
\[ = \mu \sqrt{x^{2\mu-2} (dx)^2 + y^{2\mu-2} (dy)^2 + z^{2\mu-2} (dz)^2}, \]

where \( m \) is the vector with \( |m| = 1 \).

The Hausdorff arc length is represented in the form:

\[ \ell = \int_a^b \sqrt{\frac{1}{x^{2\mu-2}} \frac{dx}{dt}^2 + \frac{1}{y^{2\mu-2}} \frac{dy}{dt}^2 + \frac{1}{z^{2\mu-2}} \frac{dz}{dt}^2} \, dt. \]

- The line Hausdorff integral of the fractal vector field

The line Hausdorff integral of the fractal vector field

\[ T = T (x, y, z) = \tilde{T} (x^\mu, y^\mu, z^\mu) \]

along the curve

\[ L = L (x, y, z) = \tilde{L} (x^\mu, y^\mu, z^\mu), \]

denoted by \( L \), is defined as

\[ L = \int_L T (x, y, z) \cdot dl, \]

where the element of the vector line is

\[ dl = \mu (ix^{\mu-1}dx + jy^{\mu-1}dy + kz^{\mu-1}dz). \]

By using (62), we show that

\[ \int_L T \cdot dl = \int_L T (x, y, z) \cdot dl = \int_{L(t)} T \cdot \frac{dl}{dt} \, dt, \]

where

\[ \frac{dl}{dt} = \mu (ix^{\mu-1}dx/dt + jy^{\mu-1}dy/dt + kz^{\mu-1}dz/dt). \]
Thus, from (58), (58) can be presented as follows:

\[(61) \int L T \cdot d\mathbf{l} = \mu \int L T x^{\mu-1} dx + T_y y^{\mu-1} dy + T_z z^{\mu-1} dz.\]

The vector field \( T = T(x, y, z) \) in

\( L = L(x, y, z) = \tilde{L}(x^\mu, y^\mu, z^\mu) \)

is said to be conservative if

\[(62) \oint_L T \cdot d\mathbf{l} = 0.\]

**The double Hausdorff integral of the fractal scalar field**

The double Hausdorff integral of the fractal scalar field

\( M = M(x, y) = \tilde{M}(x^\mu, y^\mu) \)

on the region \( S(x, y) = \tilde{S}(x^\mu, y^\mu) \), denoted by \( A(M) \), is defined as

\[(63) A(M) = \iint_S M(x, y) dS,\]

where

\[(64) dS = \mu^2 x^{\mu-1} y^{\mu-1} dx dy.\]

With the aid of from (63) and (64) we may see that

\[(65) \iint_S M(x, y) dS = \mu^2 \int_a^b \left[ \int_c^d M(x, y) x^{\mu-1} dx \right] y^{\mu-1} dy = \mu^2 \int_a^b \left[ \int_c^d M(x, y) y^{\mu-1} dy \right] x^{\mu-1} dx,\]

where \( x \in [a, b] \) and \( y \in [c, d] \).

**The volume Hausdorff integral of the fractal scalar field**

The volume Hausdorff integral of the fractal scalar field

\( N = N(x, y, z) = \tilde{N}(x^\mu, y^\mu, z^\mu) \)

is defined as

\[(66) V(N) = \iiint_\Omega N(x, y, z) dV,\]
where
\[ dV = \mu^3 x^{\mu-1} y^{\mu-1} z^{\mu-1} dx dy dz \]
and
\[ \Omega = \Omega (x, y, z) = \tilde{\Omega} (x^\mu, y^\mu, z^\mu) . \]
By using (66), we may show
\[ \int_\Omega N (x, y, z) dV \]
\[ = \mu^3 \int_a^b z^{\mu-1} dz \int_c^d y^{\mu-1} dy \int_a^b N (x, y, z) x^{\mu-1} dx \]
\[ = \mu^3 \int_a^b x^{\mu-1} dx \int_c^d z^{\mu-1} dz \int_a^b N (x, y, z) y^{\mu-1} dy \]
\[ = \mu^3 \int_c^d y^{\mu-1} dy \int_a^b x^{\mu-1} dx \int_a^b N z^{\mu-1} dz , \]
where \( x \in [a, b], y \in [c, d] \) and \( z \in [\alpha, \beta] . \)

**The surface Hausdorff integral of the fractal vector field**
The surface Hausdorff integral of the fractal vector field
\[ \mathbf{W} = \mathbf{W} (x, y, z) = \tilde{\mathbf{W}} (x^\mu, y^\mu, z^\mu) \]
is defined as
\[ \int_\mathbf{S} \mathbf{W} (x, y, z) \cdot d\mathbf{S} = \int_\mathbf{S} \mathbf{W} (x, y, z) \cdot \mathbf{n} dS , \]
where \( \mathbf{n} = d\mathbf{S} / dS \) is the unit normal vector to the surface
\[ \mathbf{S} = \mathbf{S} (x, y, z) = \mathbf{s} (x^\mu, y^\mu, z^\mu) . \]
Suppose that
\[ \mathbf{n} = \frac{d\mathbf{S}}{|d\mathbf{S}|} = d\mathbf{S} / dS , \]
\[ dS = |d\mathbf{S}| , \]
and
\[ d\mathbf{S} = \mu^2 i y^{\mu-1} z^{\mu-1} dy dz + \mu^2 j x^{\mu-1} z^{\mu-1} dx dz + \mu^2 k x^{\mu-1} y^{\mu-1} dx dy . \]
Then, by (69), Eq.(68) can be represented in the form:
\[ \int_\mathbf{S} \mathbf{W} (x, y, z) \cdot d\mathbf{S} \]
\[ = \mu^2 \int_\mathbf{S} W_x y^{\mu-1} z^{\mu-1} dy dz + \mu^2 W_y x^{\mu-1} z^{\mu-1} dx dz \]
\[ + \mu^2 W_z x^{\mu-1} y^{\mu-1} dx dy . \]
where
\[
\mathbf{W} = \mathbf{W}(x, y, z) = \tilde{\mathbf{w}}(x^\mu, y^\mu, z^\mu) = iW_x + jW_y + kW_z.
\]

The flux of the fractal vector field \( \mathbf{W} = \mathbf{W}(x, y, z) \) across the surface \( d\mathbf{S} \), denoted by \( Q \), is defined as

\[
(71) \quad Q = \iint_{\mathbf{S}} \mathbf{W}(x, y, z) \cdot d\mathbf{S}.
\]

Let \( Q = 0 \). Then we have from (71) that

\[
(72) \quad \iint_{\mathbf{S}} \mathbf{W}(x, y, z) \cdot d\mathbf{S} = 0.
\]

- **The Hausdorff divergence of the fractal vector field**

The Hausdorff divergence of the fractal vector field \( \psi \) is defined as

\[
(73) \quad \nabla^\mu \cdot \mathbf{W} = \lim_{\Delta V_m \to 0} \frac{1}{\Delta V_m} \iint_{\Delta \mathbf{S}_m} \mathbf{W} \cdot d\mathbf{S},
\]

where the volume \( V \) is divided into a large number of small subvolumes \( \Delta V_m \) with surfaces \( \Delta \mathbf{S}_m \), \( \mathbf{W} \) is a Hausdorff differentiable vector field, and \( d\mathbf{S} \) is an element of the surface \( \mathbf{S} \) bounding the solid \( \Omega \).

In the coordinate system (39), (73) can be represented as [1]

\[
(74) \quad \nabla^\mu \cdot \mathbf{W} = \mu \left( x^{\mu-1} \partial_x^{(1)} W_x + y^{\mu-1} \partial_y^{(1)} W_y + z^{\mu-1} \partial_z^{(1)} W_z \right),
\]

where
\[
\mathbf{W} = \mathbf{W}(x, y, z) = \tilde{\mathbf{W}}(x^\mu, y^\mu, z^\mu) = iW_x + jW_y + kW_z.
\]

- **The Hausdorff curl of the fractal vector field**

The Hausdorff curl of the fractal vector field \( \mathbf{W} \) is defined as

\[
(75) \quad \nabla^\mu \times \mathbf{W} = \lim_{\Delta V_m \to 0} \frac{1}{\Delta V_m} \iint_{\Delta \mathbf{S}_m} \mathbf{W} \times d\mathbf{S},
\]

where the volume \( V \) is divided into a large number of small subvolumes \( \Delta V_m \) with surfaces \( \Delta \mathbf{S}_m \), \( \mathbf{W} \) is a Hausdorff differentiable vector field, and \( d\mathbf{S} \) is an element of the surface \( \mathbf{S} \) bounding the solid \( \Omega \).

There is an alternative definition of (75) as follows:
The Hausdorff curl of the fractal vector field \( W \) is defined as
\[
(\nabla^\mu \times W) \cdot n = \lim_{\Delta S_m \to 0} \frac{1}{\Delta S_m} \oint_{\Delta L_m} W \cdot dl,
\]
where \( W \) is a Hausdorff differentiable vector field, \( dl \) is the element of the vector line, \( \Delta S_m \) is a small surface element perpendicular to \( n \), \( \Delta L_m \) is the closed curve of the boundary of \( \Delta S_m \), and \( n \) are oriented in a positive sense.

Similarly, in the coordinate system \((39)\), Eqs.\((75)\) and \((76)\) can be rewritten as \([1]\)
\[
\nabla^\mu \times W = \begin{pmatrix}
i \\
\mu x^{\mu-1} C \frac{\partial (1)}{\partial x} \\
\mu x^{\mu-1} C \frac{\partial (1)}{\partial y} \\
\mu x^{\mu-1} C \frac{\partial (1)}{\partial z}
\end{pmatrix} W_x W_y W_z,
\]
where
\[
W = W(x, y, z) = \tilde{W}(x^\mu, y^\mu, z^\mu) = iW_x + jW_y + kW_z.
\]

- **The Gauss-Ostrogradsky-like theorem for the fractal vector field**

By using \((73)\), we obtain the Gauss-Ostrogradsky-like theorem for the fractal vector field \( W \).

Now, we show that
\[
\oint_S W dS = \oint_S W dS
\]
\[
= \mu^2 \oint_S (iW_x + jW_y + kW_z)
\]
\[
\cdot (iy^{\mu-1} z^{\mu-1} dydz + jx^{\mu-1} z^{\mu-1} dxdz + kx^{\mu-1} y^{\mu-1} dxdy)
\]
\[
= \mu^2 \oint_S [W_x y^{\mu-1} z^{\mu-1} dydz + W_y x^{\mu-1} z^{\mu-1} dxdz]
\]
\[
+ \mu^2 \oint_S W_z x^{\mu-1} y^{\mu-1} dxdy.
\]

From \((73)\) the Gauss-like theorem for the fractal vector field \( W \) states that
\[
\iiint_\Omega \nabla^\mu \cdot W dV = \oint_S W \cdot ndS,
\]
where \( W \) is a Hausdorff differentiable vector field, \( dV \) denotes an element of volume \( \Omega \), \( n \) is the unit outward normal to \( S \), and \( dS \) is an element of the surface area of the surface \( S \) bounding the solid \( \Omega \).

With \((78)\) we have
\[
dS = ndS,
\]
we obtain an alternative form of (79) as follows:

\[
\int_{\Omega} \nabla^\mu \cdot \mathbf{W} \, dV = \oint_{\partial \Omega} \mathbf{W} \cdot d\mathbf{S}.
\]

It is easy to see that (80) becomes the Gauss-Ostrogradsky theorem [16] due to Gauss [17] and Ostrogradsky [18] if \(\mu = 1\).

- **The Stokes-like theorem for the fractal vector field**

By using (76), we present the Stokes-like theorem for the fractal vector field. We now consider that

\[
\oint_{L} \mathbf{W} \, dl = \mu \oint_{L} (iW_x + jW_y + kW_z)
\]

\[
\cdot (ix^{\mu-1}dx + jy^{\mu-1}dy + kz^{\mu-1}dz)
\]

\[
= \mu \oint_{L} (W_xx^{\mu-1}dx + W_yy^{\mu-1}dy + W_zz^{\mu-1}dz).
\]

From (76) the Stokes-like theorem for the fractal vector field \(\mathbf{W}\) states that

\[
\oint_{S} (\nabla^\mu \times \mathbf{W}) \cdot \mathbf{n} \, dS = \oint_{L} \mathbf{W} \cdot dl,
\]

where \(\mathbf{W}\) is a Hausdorff differentiable vector field, \(S\) denotes an open, two sided curve surface, \(L\) represents the closed contour bounding \(S\), and \(dl\) denotes the element of the vector line.

Taking \(dS = ndS\), we have from (52) that

\[
\oint_{S} (\nabla^\mu \times \mathbf{W}) \cdot \mathbf{n} \, dS = \oint_{L} \mathbf{W} \cdot dl.
\]

It is shown that (83) becomes the Stokes theorem due to Stokes [19] when \(\mu = 1\).

- **The Green-like theorem for the fractal vector field**

Let us consider

\[
\nabla^\mu \times \mathbf{T}
\]

\[
= \begin{pmatrix}
    i & j & k \\
    \mu x^{\mu-1}C \partial_x^{(1)} & \mu y^{\mu-1}C \partial_y^{(1)} & 0 \\
    T_x & T_y & 0
\end{pmatrix}
\]

\[
= k \begin{pmatrix}
    \mu x^{\mu-1}C \partial_x^{(1)} T_y - \mu y^{\mu-1}C \partial_y^{(1)} T_x \\
    T_y & T_x
\end{pmatrix}
\]

\[
= k \mu \begin{pmatrix}
    x^{\mu-1}C \partial_x^{(1)} T_y - y^{\mu-1}C \partial_y^{(1)} T_x \\
    T_y & T_x
\end{pmatrix}
\]
where
\[ T = T(x, y) = \tilde{T}(x', y') = iT_x + jT_y. \]

Hence, by (84) we show that
\[
\iint_S (\nabla^\mu \times T) \cdot dS = \mu \iint S \left( x^{\mu - 1}C \partial_x^{(1)}T_y - y^{\mu - 1}C \partial_y^{(1)}T_x \right) dS,
\]
where
\[
\iint_S (\nabla^\mu \times T) \cdot dS = \iint S \left[ \mu k \left( x^{\mu - 1}C \partial_x^{(1)}T_y - x^{\mu - 1}C \partial_y^{(1)}T_x \right) \right] \cdot \left[ i\mu^2 y^{\mu - 1}z^{\mu - 1}dydz + j\mu^2 x^{\mu - 1}z^{\mu - 1}dxdz + kx^{\mu - 1}y^{\mu - 1}dxdy \right]
\]
\[ = \mu \iint S \left( x^{\mu - 1}C \partial_x^{(1)}T_y - y^{\mu - 1}C \partial_y^{(1)}T_x \right) dS,
\]
\[ dS = \mu^2 x^{\mu - 1}y^{\mu - 1}dxdy,
\]
and
\[ dS = i\mu^2 y^{\mu - 1}z^{\mu - 1}dydz + j\mu^2 x^{\mu - 1}z^{\mu - 1}dxdz + k\mu^2 x^{\mu - 1}y^{\mu - 1}dxdy. \]

Moreover, we have
\[
\oint_T \cdot dl = \oint_T (iT_x + jT_y) \cdot \left[ \mu (ix^{\mu - 1}dx + jy^{\mu - 1}dy) \right]
\]
\[ = \mu \oint_T (T_x x^{\mu - 1}dx + T_y y^{\mu - 1}dy)
\]
\[ = \oint_T T_x dl_x + T_y dl_y,
\]
in which
\[ dl = \mu (ix^{\mu - 1}dx + jy^{\mu - 1}dy) = idl_x + idl_y,
\]
where
\[ dl_x = \mu x^{\mu - 1}dx\]
and
\[ dl_y = \mu y^{\mu - 1}dy. \]
From (83), (85) and (86) the Green-like theorem for the fractal vector field $T$ states

$$
\mu \left( \oint_L T_x x^{\mu-1} dx + T_y y^{\mu-1} dy \right)
= \mu^3 \iint_S \left( x^{\mu-1}C \partial_x^{(1)} T_y - y^{\mu-1}C \partial_y^{(1)} T_x \right) x^{\mu-1} y^{\mu-1} dxdy,
$$

which is equal to

$$\oint_L T_x dl_x + T_y dl_y
= \mu \iint_S \left( x^{\mu-1}C \partial_x^{(1)} T_y - y^{\mu-1}C \partial_y^{(1)} T_x \right) dS,$$

where $S$ is the domain bounded by the contour $L$, and $T = T_x i + T_y j$.

When $\mu = 1$, (88) leads to the Green theorem [16] due to Green [19].

- **The Green-like identities for the fractal vector field**

Taking $X = \Theta \nabla^{\mu} \psi$ and $Y = \psi \nabla^{\mu} \Theta$, we show that

$$\nabla^{\mu} \cdot X = \nabla^{\mu} \cdot (\Theta \nabla^{\mu} \psi) = \Theta \nabla^{2\mu} \psi + \nabla^{\mu} \psi \cdot \nabla^{\mu} \Theta$$

and

$$\nabla^{\mu} \cdot Y = \nabla^{\mu} \cdot (\psi \nabla^{\mu} \Theta) = \psi \nabla^{2\mu} \Theta + \nabla^{\mu} \psi \cdot \nabla^{\mu} \Theta,$$

where $\psi$ and $\Theta$ are the fractal scalar fields.

With the use of (80), we may have

$$\iiint_{\Omega} \nabla^{\mu} \cdot X dV = \iint_S X \cdot dS$$

and

$$\iiint_{\Omega} \nabla^{\mu} \cdot Y dV = \iint_S Y \cdot dS.$$

By using (89) and (91) we have that

$$\iiint_{\Omega} \nabla^{\mu} \cdot (\Theta \nabla^{\mu} \psi) dV
= \iiint_{\Omega} (\Theta \nabla^{2\mu} \psi + \nabla^{\mu} \psi \cdot \nabla^{\mu} \Theta) dV
= \iint_S (\Theta \nabla^{\mu} \psi) \cdot dS.$$
By using (90) and (92) we show that

\[
\int_{\Omega} \int_{\Omega} \int_{\Omega} \nabla^\mu \cdot (\psi \nabla^\mu \Theta) \, dV
= \int_{\Omega} \int_{\Omega} (\psi \nabla^2 \Theta + \nabla^\mu \psi \cdot \nabla^\mu \Theta) \, dV
= \oint_S (\psi \nabla^\mu \Theta) \cdot dS.
\]

(94)

Making use of (93), we give

\[
\int_{\Omega} \int_{\Omega} \int_{\Omega} (\Theta \nabla^2 \psi + \nabla^\mu \psi \cdot \nabla^\mu \Theta) \, dV = \oint_S (\Theta \nabla^\mu \psi) \cdot dS,
\]

which leads to

(95)

\[
\int_{\Omega} \int_{\Omega} \int_{\Omega} (\Theta \nabla^2 \psi + \nabla^\mu \psi \cdot \nabla^\mu \Theta) \, dV = \oint_S \Theta \partial^\mu_n \psi dS,
\]

where

\[
\oint_S (\Theta \nabla^\mu \psi) \cdot dS
= \oint_S (\Theta \nabla^\mu \psi) \cdot n dS
= \oint_S \Theta \partial^\mu_n \psi dS.
\]

Here, (95) is the Green-like identity of first type.

With the aid of (94) we show

\[
\int_{\Omega} \int_{\Omega} \int_{\Omega} (\psi \nabla^2 \Theta + \nabla^\mu \psi \cdot \nabla^\mu \Theta) \, dV = \oint_S (\psi \nabla^\mu \Theta) \cdot dS,
\]

which leads to

(96)

\[
\int_{\Omega} \int_{\Omega} \int_{\Omega} (\psi \nabla^2 \Theta + \nabla^\mu \psi \cdot \nabla^\mu \Theta) \, dV = \oint_S \psi \partial^\mu_n \Theta dS,
\]

where

\[
\oint_S (\psi \nabla^\mu \Theta) \cdot dS
= \oint_S (\psi \nabla^\mu \Theta) \cdot n dS
= \oint_S \psi \partial^\mu_n \Theta dS.
\]

By using (95) and (96), we give

\[
\int_{\Omega} \int_{\Omega} \int_{\Omega} \nabla^\mu \cdot (\Theta \nabla^2 \psi - \psi \nabla^2 \Theta) \, dV
= \oint_S (\Theta \nabla^\mu \psi - \psi \nabla^\mu \Theta) \cdot dS
\]
which implies that

\[
\int_{\Omega} \nabla^{\mu} \cdot (\Theta \nabla^{2\mu} \psi - \psi \nabla^{2\mu} \Theta) \, dV = \oint_{\mathcal{S}} (\Theta \partial_{\mu} \psi - \psi \partial_{\mu} \Theta) \, dS.
\]

(97)

Here, (97) is the Green-like identity of second type.

When \( \mu = 1 \), Eqs. (95) and (97) yield the Green-like identities [16] due to Green [20].

The theory of the Hausdorff vector calculus is the special case of the theory of the general vector calculus [13, 14, 15] by using the idea of the Gibbs’s expression [21].

4. Modelling the fractal power-law flow

We now consider the coordinate system, expressed in the form:

\[
(t^\mu, x^\mu, y^\mu, z^\mu) = t^\mu + i x^\mu + j y^\mu + k z^\mu
\]

where \( i, j \) and \( k \) are the unit vectors in the Cartesian coordinate system.

- **The material Hausdorff derivative**

Let us consider

\[
\varphi = \varphi (t, x, y, z) = \tilde{\varphi} (t, x^\mu, y^\mu, z^\mu)
\]

be the fractal fluid field.

The total Hausdorff differential of the fractal scalar field \( \varphi \) is expressed by:

\[
d\varphi = \left( \mu x^\mu - 1C \partial^{(1)}_{x} \varphi \right) dx + \left( \mu y^\mu - 1C \partial^{(1)}_{y} \varphi \right) dy
\]

\[
+ \left( \mu z^\mu - 1C \partial^{(1)}_{z} \varphi \right) dz + \partial^{(1)}_{t} \varphi dt,
\]

(98)

which implies that

\[
\frac{d\varphi}{dt} = \left( \mu x^\mu - 1C \partial^{(1)}_{x} \varphi \right) \frac{dx}{dt} + \left( \mu y^\mu - 1C \partial^{(1)}_{y} \varphi \right) \frac{dy}{dt}
\]

\[
+ \left( \mu z^\mu - 1C \partial^{(1)}_{z} \varphi \right) \frac{dz}{dt} + \partial^{(1)}_{t} \varphi.
\]

(99)

The material Hausdorff derivative for the fractal fluid density \( \varphi \) is defined as [15]:

\[
\frac{D \varphi}{Dt} = \partial^{(1)}_{t} \varphi + \nu \cdot \nabla^{\mu} \varphi
\]

(100)

where

\[
\nabla^{\mu} \varphi = \mu \left( i x^\mu - 1C \partial^{(1)}_{x} \varphi + j y^\mu - 1C \partial^{(1)}_{y} \varphi + k z^\mu - 1C \partial^{(1)}_{z} \varphi \right)
\]

and \( \nu = (\partial x/\partial t, \partial y/\partial t, \partial z/\partial t) = i v_x + j v_y + k v_z \) is the velocity vector.

When \( \mu = 1 \), Eq. (100) leads to the material derivative due to Stokes [22].

- **The transport theorem for the fractal power-law fluid**
By using (100), the transport theorem for the fractal power-law fluid $G$ is expressed as

$$
\frac{D}{Dt} \iiint_{\Omega(t)} G dV = \iiint_{\Omega(t)} \left( \partial_t^{(1)} G + v \cdot \nabla^\mu G \right) dV,
$$

which yields that

$$
\frac{D}{Dt} \iiint_{\Omega(t)} G dV = \iiint_{\Omega(t)} \partial_t^{(1)} G dV + \iint_{S(t)} G v \cdot dS,
$$

because

$$
\iiint_{\Omega(t)} v \cdot \nabla^\mu G dV = \iint_{S(t)} G (v \cdot n) dS = \iint_{S(t)} G v \cdot dS,
$$

where $S(t)$ is the surface of $\Omega(t)$, $n$ is the unit normal to the surface, $v$ is the velocity vector, and $G = G(t, x, y, z) = \tilde{G}(t, x^\mu, y^\mu, z^\mu)$ is the fractal power-law fluid.

When $\mu = 1$, Eqs. (101) and (102) yield the Reynolds transport theorem due to Reynolds [23].

- **The conservation of the mass for the fractal power-law fluid**

The mass of the fractal power-law fluid is defined as

$$
\iiint_{\Omega(t)} \rho dV = M
$$

where $\rho = \rho(t, x, y, z) = \tilde{\rho}(t, x^\mu, y^\mu, z^\mu)$ and $M = M(t, x, y, z) = \tilde{M}(t, x^\mu, y^\mu, z^\mu)$.

The conservation of the mass for the fractal power-law fluid is represented in the form:

$$
\partial_t^{(1)} \rho + v \cdot \nabla^\mu \rho = 0
$$

because

$$
\frac{D}{Dt} \iiint_{\Omega(t)} \rho dV = \iiint_{\Omega(t)} \left( \partial_t^{(1)} \rho + v \cdot \nabla^\mu \rho \right) dV = 0,
$$

where $v$ is the velocity vector.

Let $v$ be a constant. Then we have

$$
\partial_t^{(1)} \rho + \nabla^\mu \cdot (v \rho) = 0
$$

which is derived from

$$
\frac{D}{Dt} \iiint_{\Omega(t)} \rho dV = \iiint_{\Omega(t)} \left[ \partial_t^{(1)} \rho + \nabla^\mu (v \cdot \rho) \right] dV = 0.
$$

When $\mu = 1$, Eqs. (105) and (106) are the expressions for the classical conservation of the mass due to Euler [24] and Lagrange [25].

- **The velocity gradient tensor for the fractal power-law fluid**
We now consider the velocity gradient tensor for the fractal power-law fluid, expressed in the form
\[
\nabla^\mu \cdot v = \frac{1}{2} (\varsigma + \tau) + \frac{1}{2} (\varsigma - \tau) = \eta + \frac{1}{2} (\varsigma - \tau)
\]
in which
\[
\varsigma = \nabla^{(D_1, D_2, D_3)} \cdot v = 0,
\]
where the strain tensor for the fractal power-law fluid is defined as
\[
\eta = \frac{(\varsigma + \tau)}{2}
\]
with velocity gradient \( \varsigma = \nabla^\mu \cdot v \) and \( \tau = v \cdot \nabla^\mu \).

The stress tensor for the fractal power-law fluid is defined as
\[
H = -p I + 2 \varepsilon \eta,
\]
where \( \varepsilon \) are the shear moduli of viscosity, and \( I \) is the unit tensor.

When \( \mu = 1 \), the velocity gradient tensor for the fractal power-law fluid implies the Cauchy strain tensor due to Cauchy [26].

**The conservation of the momentums for the fractal power-law fluid**
The conservation of the momentums for the fractal power-law fluid reads
\[
\frac{D}{D_t} \iiint_{\Omega(t)} \rho v dV = \iiint_{\Omega(t)} b dV + \iiint_{S(t)} H \cdot dS,
\]
where \( b \) is the specific body force.

By using (101) and (102), we have
\[
\frac{D}{D_t} \iiint_{\Omega(t)} \rho v dV = \iiint_{\Omega(t)} \varphi^{(1)}_t (\rho v) dV + \iiint_{S(t)} (\rho v) v \cdot dS
\]
and
\[
\frac{D}{D_t} \iiint_{\Omega(t)} \rho v dV = \iiint_{\Omega(t)} \left[ \varphi^{(1)}_t (\rho v) + v \cdot \nabla^\mu (\rho v) \right] dV.
\]

With (80) we present
\[
\iiint_{S(t)} H \cdot dS = \iiint_{\Omega(t)} \nabla^\mu \cdot H dV.
\]
From (112) we have
\[
\iiint_{\Omega(t)} \left[ \varphi^{(1)}_t (\rho v) + v \cdot \nabla^\mu (\rho v) \right] dV = \iiint_{\Omega(t)} b dV + \iiint_{\Omega(t)} \nabla^\mu \cdot H dV,
\]
which leads to
\[
\iiint_{\Omega(t)} \left[ \varphi^{(1)}_t (\rho v) + v \cdot \nabla^\mu (\rho v) - \nabla^\mu \cdot H - b \right] dV = 0.
\]
From (116) it follows that
\[
\partial_t^{(1)} \left( \rho v \right) + v \cdot \nabla^\mu \left( \rho v \right) - \nabla^\mu \cdot H - \mathbf{b} = 0,
\]
which leads to
\[
\rho \left( \partial_t^{(1)} v + v \cdot \nabla^\mu v \right) - \nabla^\mu \cdot H - \mathbf{b} = 0.
\]
From (111) and (118) we show that
\[
\nabla^\mu \cdot H = -\nabla^\mu p + \varepsilon \nabla^2 \nabla^\mu v
\]
such that
\[
\rho \left( \partial_t^{(1)} v + v \cdot \nabla^\mu v \right) = -\nabla^\mu p + \varepsilon \nabla^2 \nabla^\mu v + \mathbf{b}.
\]
From (120) we get the system of the power-law flow, given by the the fractal power-law equations
\[
\rho \left( \partial_t^{(1)} v + v \cdot \nabla^\mu v \right) = -\nabla^\mu p + \varepsilon \nabla^2 \nabla^\mu v + \mathbf{b},
\]
and
\[
\nabla^\mu \cdot v = 0.
\]
When \( \mu = 1 \), Eqs. (121) and (122) become the system of the Navier-Stokes equations for the fluid due to Navier [27] and Stokes [22]. The power-law flow in the real world problems is the special case of the theory of the general and power-law flow in the real world problems studied in [13, 14, 15].

Let \( F = F (t, x, y, z) = \tilde{F} (t, x^\mu, y^\mu, z^\mu) \) be a curve. If there exist all Chen Hausdorff derivatives of the curve \( F \) in the space domain \( x^\mu \times y^\mu \times z^\mu \in \mathbb{R}^{3\mu} \) and all derivatives of the curve \( F \) in the time domain \( t \in \mathbb{R} \), then \( F \) is smooth in the space-time domain, where \( 0 < \mu \leq 1 \).

**Conjecture** Do the fractal power-law equations (121) and (122) on a fractal domain \( \Omega (t) \) in \( \mathbb{R}^{3\mu} \) have a unique smooth solution for all time \( t \geq 0 \)?

It is easy to see that **Conjecture** is analogous to the Smale’s 15th Problem [28]. When we take \( \mu = 1 \), **Conjecture** becomes the Smale’s 15th Problem [28] or one of the Millennium Prize Problems for the Navier–Stokes equations [29].

**5. Modelling the anomalous diffusion equation**

When \( p \) is a constant, we have from (119) that
\[
\nabla^\mu \cdot H = \varepsilon \nabla^2 \nabla^\mu v.
\]
which implies that

\[
\oint_{S(t)} H \cdot dS = \iiint_{\Omega(t)} \nabla^\mu \cdot Hv dV = \varepsilon \iiint_{\Omega(t)} \nabla^2 \mu v dV = \varepsilon \oint_{S} (\nabla^\mu v) \cdot dS.
\]  

(124)

Thus, from (124), we get

\[
\oint_{S(t)} H \cdot dS = \iiint_{\Omega(t)} \nabla^\mu \cdot Hv dV = \varepsilon \iiint_{\Omega(t)} \nabla^2 \mu v dV = \varepsilon \oint_{S} (\nabla^\mu v) \cdot dS.
\]  

(125)

By using (112) we show

\[
\iiint_{\Omega(t)} \left[ \partial_t^{(1)} (\rho v) + v \cdot \nabla^\mu (\rho v) \right] dV = \iiint_{\Omega(t)} b dV + \iiint_{\Omega(t)} \nabla^\mu \cdot Hv dV
\]

(126)

which is equal to

\[
\iiint_{\Omega(t)} \left[ \partial_t^{(1)} (\rho v) + v \cdot \nabla^\mu (\rho v) \right] dV = \iiint_{\Omega(t)} b dV + \varepsilon \iiint_{\Omega(t)} \nabla^2 \mu v dV.
\]  

(127)

From (127) we have

\[
\iiint_{\Omega(t)} \left[ \partial_t^{(1)} (\rho v) + v \cdot \nabla^\mu (\rho v) - b - \varepsilon \nabla^2 \mu v \right] dV = 0
\]

(128)

such that

\[
\partial_t^{(1)} (\rho v) + v \cdot \nabla^\mu (\rho v) - \varepsilon \nabla^2 \mu v = b.
\]  

(129)
When $b = 0$, we show that

\[(130) \quad \partial_t^{(1)} (\rho v) + v \cdot \nabla^\mu (\rho v) = \varepsilon \nabla^{2\mu} v, \]

which implies that

\[(131) \quad \partial_t^{(1)} v + v \cdot \nabla^\mu v = \vartheta \nabla^{2\mu} v, \]

where $\vartheta = \varepsilon / \rho$.

In one-dimensional case, we have from (131) that

\[(132) \quad \partial_t^{(1)} v + v \mu x^{\mu-1} C \partial_x^{(1)} v = \vartheta \mu^2 x^{2\mu-2} C \partial_x^{(2)} v, \]

where $v = v(t, x) = \tilde{v}(t, x^\mu)$.

When we neglect the nonlinear term in (131), the linear anomalous diffusion equation is expressed in the form:

\[(133) \quad \partial_t^{(1)} v = \vartheta \mu^2 x^{2\mu-2} C \partial_x^{(2)} v, \]

where $v = v(t, x) = \tilde{v}(t, x^\mu)$.

Eq. (132) is an anomalous diffusion equation in the real theory of the turbulent fluid motion.

When we take $\mu = 1$, Eq. (132) becomes the Burgers diffusion equation due to Burgers [30].

6. Conclusion

In our work we showed the theory of the Hausdorff vector calculus with use of the Chen Hausdorff calculus. We discussed the Gauss-Ostrogradsky-like, Stokes-like, and Green-like theorems, and Green-like identities for the fractal field. By using the Hausdorff vector calculus we obtained the theory of the fractal power-law flow analogous to the Navier-Stokes and Burgers diffusion equations. A conjecture for the fractal power-law flow equations analogous to the Smales 15th Problem (one of the Millennium Prize Problems for the Navier-Stokes equations) has been also addressed. This plays an important role in the study of the anomalous and complex behaviors of the flows in the real world problems.

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