Boundary value problems in dimensions seven, four and three related to exceptional holonomy

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Dedicated to Nigel Hitchin, for his 70th. birthday.

The variational point of view on exceptional structures in dimensions 6, 7 and 8 is one of Nigel Hitchin’s seminal contributions. One feature of this point of view is that it motivates the study of boundary value problems, for structures with prescribed data on a boundary. In this article we consider the case of 7 dimensions and “$G_2$-manifolds”. We will review briefly a general framework and then go on to examine in more detail symmetry reductions to dimensions 4 (in Section 2) and 3 (in Section 3). In the latter case we encounter an interesting variational problem related to the real Monge-Ampère equation and in Section 4 we describe a generalisation of this.

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1 The volume functional in 7 dimensions

Let $V$ be a 7-dimensional oriented real vector space. A 3-form $\phi \in \Lambda^3 V^*$ defines a quadratic form on $V$ with values in the real line $\Lambda^7 V^*$ by the assignment

$$v \mapsto (i_v \phi)^2 \wedge \phi.$$ 

The fixed orientation means that it makes sense to say that this form is positive definite, and in that case we call $\phi$ a positive 3-form. From the definition, a positive 3-form defines a conformal class of Euclidean structures on $V$ and the ambiguity of scale can be fixed by choosing the Euclidean structure so that $|\phi|^2 = 7$, using the standard induced metric on $\Lambda^3 V^*$.

Now let $M$ be an oriented 7-manifold. Applying the above in each tangent space, we have the notion of a positive 3-form $\phi \in \Omega^3(M)$ and such a form defines a metric $g_\phi$ and volume form $\nu_\phi \in \Omega^7(M)$. The first variation of the volume form with respect to a variation $\delta \phi$ in $\phi$ is given by

$$\delta \nu_\phi = \delta \phi \wedge \Psi.$$
where $\Psi = \Psi(\phi)$ is a 4-form determined by $\phi$, which can also be expressed as

$$3\Psi(\phi) = \star g_{\phi} \phi.$$ 

Suppose that $M$ is a closed 7-manifold and that $c \in H^3(M, \mathbb{R})$ is a cohomology class which can be represented by positive 3-forms, so we have a non-empty set $S_c \subset \Omega^3(M)$ of closed positive forms representing $c$. Hitchin’s idea ([5], [6]) is to consider the total volume

$$\text{Vol}(\phi) = \int_M \nu_\phi$$

as a functional on $S_c$. The first variation, with respect to a variation $\delta \phi = da$, in $\phi$ is

$$\int_M \Psi \wedge da = \int_M d\Psi \wedge a$$

so the Euler-Lagrange equation defining critical points is

$$d\Psi = 0.$$

By a well-known result of Fernández and Gray, the two equations $d\phi = 0$ and $d \star g_{\phi} \phi = 0$ imply that the $G_2$ structure defined by $\phi$ is torsion-free, or equivalently that the metric $g_\phi$ has holonomy contained in $G_2$. So, from this point of view, the search for these special structures can be divided into two stages:

- Identify manifolds $M$ and classes $c \in H^3(M)$ such that $S_c$ is non-empty;
- Study the variational problem for the volume functional on $S_c$.

The local theory of such critical points, with respect to small variations in $c$ and $\phi$, is well-understood. Hitchin proved that any critical point is a local maximum and in fact a strict local maximum modulo diffeomorphisms. The proof is an application of Hodge theory. One of the many interesting and fundamental questions in this area is whether it is a global maximum over the whole space $S_c$. Another standard fact (proved earlier by Bryant), is that critical points are stable with respect to variations in the cohomology class $c$: a critical point $\phi$ belonging to a class $c$ can be deformed to a critical point for nearby classes in $H^3(M)$. That is, the moduli space of $G_2$-structures is locally modelled on $H^3(M)$.

Now we introduce our boundary value problem. Let $M$ be a compact oriented 7-manifold with boundary an oriented 6-manifold $N = \partial M$. There is a similar notion of a positive 3-form $\rho$ on $N$: this is just the condition that at each point $p \in N$ the form $\rho$ can be extended to a positive form on $TM_p$. A basic algebraic fact is that such a positive 3-form in 6-dimensions is equivalent to a reduction to $SL(3, \mathbb{C})$, that is, to an almost-complex structure with a trivialisation of the “canonical line bundle”. Fix a closed positive 3-form $\rho$ on $N$. We assume that the class $[\rho] \in H^3(N)$ is in the image of the restriction map from $H^3(M)$. Define an enhancement of $\rho$ to be an equivalence class of
closed 3-forms on $M$ extending $\rho$, under the equivalence relation $\phi_1 \sim \phi_1 + da$ where $a$ vanishes on $N$. Thus the set of enhancements of $\rho$ is an affine space modelled on $H^3(M,N)$. Fix an enhancement $\hat{\rho}$ of $\rho$ and let $S_{\hat{\rho}}$ be the set of positive forms on $M$ in this equivalence class. Suppose that $S_{\hat{\rho}}$ is nonempty and consider the volume functional on this set, just as before. For a variation $\delta \phi = da$ with $a$ vanishing on the boundary the integration by parts (2) is still valid and the critical points are given by solutions of $d\Psi = 0$ just as before. That is, we are studying $G_2$-structures on $M$ with the given boundary value $\rho$ and in the given enhancement class. So we have the same two questions as before: identify enhanced boundary values $\hat{\rho}$ such that $S_{\hat{\rho}}$ is non-empty and then study the variational problem.

We will not enter into a proper discussion of the local theory of this boundary value problem (with respect to small variations in $\phi$ and $\hat{\rho}$) here, but we make two simple observations. For the first, we say that a $G_2$-structure $\phi$ on a manifold $M$ with boundary $N$ is a formal maximum of the volume functional if for any $a \in \Omega^2(M)$ whose restriction to the boundary vanishes we have

$$\frac{d^2}{dt^2} \text{Vol}(\phi + tda) \leq 0$$

at $t = 0$. In other words, the Hessian of the volume functional is non-negative.

**Proposition 1** Suppose that $M$ is the closure of a domain in a closed $G_2$-manifold $M^+$ with 3-form $\phi$. Let $\hat{\rho}$ be the enhanced boundary value given by the restriction of $\phi$ to $N = \partial M$ and $M$. Then $\phi|_{M}$ is a formal maximum of the volume functional on $S_{\hat{\rho}}$.

Let $f$ be a defining function for $\partial M = f^{-1}(0)$, positive on the interior of $M$. Then any 2-form $a$ on $M$ whose restriction to $\partial M$ is zero can be written as $a = b + \eta \wedge df$ where $b$ vanishes in $TM|_{\partial M}$. For small $\epsilon$, let $\chi_\epsilon : M \to \mathbb{R}$ be the composite of $f$ with a standard cut-off function, such that $\chi_\epsilon$ vanishes when $f \leq \epsilon$, is equal to 1 when $f \geq \epsilon$ and with $|d\chi_\epsilon| \leq C\epsilon^{-1}$. Set $a_\epsilon = \chi_\epsilon a$. Then

$$d(a_\epsilon) = d\chi_\epsilon \wedge b$$

satisfies a uniform $L^\infty$ bound, independent of $\epsilon$ (since $b$ is $O(\epsilon)$ on the support of $d\chi_\epsilon$). It follows that

$$\frac{d^2}{dt^2} \text{Vol}(\phi + tda) = \lim_{\epsilon \to 0} \frac{d^2}{dt^2} \text{Vol}(\phi + tda_\epsilon),$$

and the latter is non-positive since $a_\epsilon$ can be extended by zero over the closed manifold $M^+$ and then we can apply Hitchin’s result.

In the other direction, critical points are not always strict local maxima, modulo diffeomorphisms. To give an example of this, we define for $v \in \mathbb{R}^7$ with $|v| < 1/2$ the manifold-with-boundary $M_v \subset \mathbb{R}^7$ to be

$$M_v = \overline{B^7} \setminus (v + \frac{1}{2}B^7),$$
where $B^7$ is the open unit ball. Let $\varphi_0$ be the standard flat $G_2$ structure on $\mathbb{R}^7$ and let $\rho_v$ be its restriction to the boundary of $M_v$. In this case $H^3(M_v, \partial M_v) = 0$ so there is no extra enhancement data. We can choose diffeomorphisms $F_v : M_0 \to M_v$ such that the restriction to the boundaries pulls back $\rho_v$ to $\rho_0$. Then $F_v^*(\varphi_0)$ are critical points for the boundary value problem on $M_0$ which are not all equivalent, by diffeomorphisms of $M_0$, to $\varphi_0$.

## 2 Reduction to dimension 4.

In this section we consider an interesting reduction of the 7-dimensional theory to 4-dimensions, as follows. Take $M = X \times \mathbb{R}^3$ where $X$ is an oriented 4-manifold and consider 3-forms of the shape

$$\phi = \sum_{i=1}^{3} \omega_i d\theta_i - d\theta_1 d\theta_2 d\theta_3$$

where $\theta_i$ are co-ordinates on $\mathbb{R}^3$ and $\omega_i$ are 2-forms on $X$. The condition that $\phi$ is a positive 3-form goes over to the condition that $(\omega_i)$ form a “positive triple”, by which we mean that at each point they span a maximal positive subspace for the wedge product form on $\Lambda^2 T^*X$. More invariantly, we are considering positive forms $\phi$ which are preserved by the translation action of the $\mathbb{R}^3$ factor and such that the orbits are “associative” submanifolds. The condition that $\phi$ be closed goes over to the condition that the $\omega_i$ are closed 2-forms on $X$, making up a “hypersympletic” structure. These structures are of considerable interest in 4-dimensional geometry, see for example [3], [4].

Given such a triple $\omega = (\omega_i)$ we define a volume form $\chi$ on $X$ by the following procedure. Let $\chi_0$ be any volume form and define a matrix $(q_{ij})$ by

$$\chi_0 q^{ij} = \omega_i \wedge \omega^j.$$  

Now put

$$\chi = \det(q_{ij})^{1/3} \chi_0.$$  

It is clear that this is independent of the choice of $\chi_0$. The 7-dimensional volume form associated to $\phi$ is $-\chi d\theta_1 d\theta_2 d\theta_3$. All our constructions will be invariant under the action of $SL(3, \mathbb{R})$ on $\mathbb{R}^3$ so it will sometimes be clearer to introduce a 3-dimensional oriented vector space $W$ with fixed volume element and consider our data $\omega$ as an element of $W \otimes \Omega^2(X)$. Then a choice of co-ordinate system on $W$ gives the description as a triple $(\omega^1, \omega^2, \omega^3)$.

Given a positive triple $\omega_i$, we define a matrix $(\lambda^{ij})$ by

$$\omega^i \wedge \omega^j = \lambda^{ij} \chi.$$  

Thus $\det(\lambda^{ij}) = 1$, by the definition of $\chi$. Write $(\lambda_{ij})$ for the inverse matrix and set

$$\Theta_i = \sum_{j=1}^{3} \lambda_{ij} \omega^j.$$  

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The 4-form defined by \( \phi \) is

\[
\Psi = \sum_{\text{cyclic}} \Theta_i d\theta^j d\theta^k + \chi,
\]

where the notation means that \((ijk)\) runs over the three cyclic permutations of \((123)\). Thus the condition that a closed triple \((\omega^i)\) defines a \(G_2\) structure is

\[
d\Theta_i = 0 \quad \text{which is to say:}
\]

\[
\sum_{j=1}^{3} d\lambda_{ij} \wedge \omega^j = 0. \tag{6}
\]

These equations are obviously satisfied when the matrix \((\lambda^{ij})\) is constant on \(X\) and these solutions are the hyperkähler metrics. Of course we can produce these equations \((8)\) from a 4-dimensional reduction of Hitchin’s variation formualtion: the equations are the Euler-Lagrange equation for the functional

\[
\text{Vol}(\omega) = \int_X \chi \tag{7}
\]

on closed positive triples \(\omega\), with respect to exact variations of compact support.

It is well-known, and easy to show directly, that the only solutions of the equations \((6)\) on a compact 4-manifold are hyperkähler and this gives extra motivation for considering the boundary value problem. So let \(X\) be a 4-manifold with boundary \(Y\) and consider triples \(\mu = (\mu^1, \mu^2, \mu^3)\) of closed 2-forms on \(Y\) which form a basis for \(\Lambda^2 T^*Y\) at each point. In our more invariant set-up, \(\mu\) lies in \(W \otimes \Omega^2(Y)\). We define an enhancement \(\hat{\mu}\) in the obvious way, so the space of enhancements of a given \(\mu\) is an affine space modelled on \(W \otimes H^2(X, Y)\). Fix an enhancement \(\hat{\mu}\) and let \(S_{\hat{\mu}}\) be the set of closed positive triples on \(X\) in the given equivalence class. So the reduced versions of our questions are, first, whether this set is non-empty and, second, to study the variational problem given by the volume functional \((7)\).

Stokes’ Theorem implies that the integrals

\[
Q^{ij} = \int_X \omega^i \wedge \omega^j,
\]

are independent of the choice of \(\omega^i\) in a fixed enhancement class \(\hat{\mu}\). More invariantly, \(Q\) is a quadratic form on our vector space \(W\) and \(\det Q\) is defined, as a real number, using the fixed volume form on \(W\). This has two simple consequences.

**Proposition 2** If \(S_{\hat{\mu}}\) is non-empty then \(Q\) is positive definite and there is an upper bound

\[
\int_X \chi \leq \det Q,
\]

for \(\omega \in S_{\hat{\mu}}\) and \(\chi = \chi(\omega)\). Equality holds if and only if \(\omega\) is hyperkähler.
To see that $Q$ is positive definite it suffices, by change of basis, to see that $Q^{11} > 0$. But this clear since $\omega^1 \wedge \omega^1$ is positive pointwise on $X$. To establish the upper bound it suffices, by change of basis, to consider the case when $Q^{ij} = \delta^{ij}$. Recall that we write $\omega^i \wedge \omega^j = \lambda^{ij} \chi$ where $\det(\lambda) = 1$. Then we have the elementary inequality (the arithmetic-geometric mean inequality for the eigenvalues) $\text{Tr}(\lambda) \geq 3$. So

$$3 \int_X \chi \leq \int_X \text{Tr} \lambda \chi = \sum_i \int_X (\omega^i)^2 = 3.$$ 

Equality holds if and only if $\lambda^{ij} = \delta^{ij}$, which means that $\omega$ is hyperkähler.

The first statement in the Proposition gives a potential obstruction to finding a positive triple with the gives enhanced boundary data. Consider for example the example when $Y = S^3$ and $X$ is the 4-ball. There is a well-known quadratic “Chern-Simons” form $Q_{CS}$ on the closed 2-forms on $S^3$ defined by

$$Q_{CS}(\mu) = \int_{S^3} a \wedge \mu,$$

where $a$ is any 1-form with $da = \mu$. The necessary condition on our boundary data in this case is that $\mu^i$ span a 3-dimensional positive subspace with respect to this form $Q_{CS}$.

3 Reduction to dimension 3

We specialise further, mimicking the Gibbons-Hawking construction of hyperkähler 4-manifolds. Thus we suppose that the 4-manifold $X$ is the total space of a principal $S^1$-bundle over a 3-manifold $U$, with the action generated by a vector field $\xi$, and consider closed positive triples $\omega$ which are invariant under the action. We assume that the action is Hamiltonian for each symplectic structure $\omega^i$, so we have Hamiltonian functions $h^i : X \to \mathbb{R}$ with

$$dh^i = i_{\xi} \omega^i,$$

and these functions are fixed by the circle action, so descend to $U$. More invariantly, writing $\omega \in \Omega^2(X) \otimes W$ we have $i_{\xi} \omega \in \Omega^1(X) \otimes W$ and this is the derivative of a map $h : X \to W$. The functions $h^i$ are then the components of $h$ with respect to a co-ordinate system $W = \mathbb{R}^3$. The definitions imply that $h$ induces a local diffeomorphism from $U$ to $W$, so for local calculations we can suppose that the base $U$ is a domain in $W$ and the functions $h^i$ can be identified with standard co-ordinates $x^i$ on $W$. One finds that the general form of such a triple is given by

$$\omega^i = \alpha \wedge dx^i + \sum_{\text{cyclic}} \sigma^{ij} dx^k dx^l,$$

where $(jkl)$ run over cyclic permutations, $\sigma = (\sigma^{ij})$ is a symmetric and positive definite matrix (a function of the co-ordinates $x^i$) and $\alpha$ is a connection 1-form on $X$. The condition that $\sigma$ is symmetric is the same as saying that
the connection is the obvious one defined by the metric induced by \( \omega^i \), with horizontal subspaces the orthogonal complement of \( \xi \).

We will now investigate the reduced \( G_2 \)-equations in this context. Write \( F \) for the curvature of the connection, so \( F = d\alpha \) and can be regarded as a 2-form on \( U \). We write

\[
F = \sum_{cyclic} F^i dx^i dx^k.
\]

Now, writing \( \partial_j \) for partial derivatives,

\[
d\omega^i = (F^i + \partial_j \sigma^{ij}) dx^1 dx^2 dx^3
\]

so the condition that we have a closed triple is that

\[
F^i = -\partial_j \sigma^{ij}.
\]

Now \( dF = 0 \), which is to say

\[
\sum_i \partial_i F^i = 0,
\]

and, at least locally, any closed 2-form specifies a connection, up to gauge equivalence. So, locally, we can eliminate the connection and curvature and closed triples correspond to matrix-valued functions \( \sigma^{ij} \) with

\[
\sum_{ij} \partial_i \partial_j \sigma^{ij} = 0. \tag{9}
\]

The volume form defined by the triple (8) is

\[
\chi = \det(\sigma)^{1/3} (\alpha dx^1 dx^2 dx^3)
\]

This implies that

\[
\lambda_{ij} = \sigma_{ij} (\det \sigma)^{1/3},
\]

where \( \sigma_{ij} \) denotes the matrix inverse to \( \sigma^{ij} \) as usual. Now the equation (6) is

\[
\sum_k \partial_k \lambda_{ij} dx^k \left( \alpha dx^j + \sum_{p,q,r \text{ cyclic}} \sigma^{jp} dx^q dx^r \right) = 0.
\]

Expanding this out we get two conditions

1. \( \partial_k \lambda_{ij} = \partial_j \lambda_{ik} \);
2. \( \sum_{jk} (\partial_k \lambda_{ij}) \sigma^{jk} = 0. \)

The first condition asserts, at least locally, that \( \lambda_{ij} \) is the Hessian of a function, \( u \) say:

\[
\lambda_{ij} = \partial_i \partial_j u.
\]
The second condition is implied by the first since
\[ \partial_k \lambda_{ij} \sigma^{jk} = -\partial_i \det(\lambda), \]
and the determinant of \( \lambda \) is 1 by construction. To sum up, the 3-dimensional reduction of the \( G_2 \) equations can be written locally as a pair of equations for two functions \( u, V \) on a domain in \( \mathbb{R}^3 \). First, the Monge-Ampère equation
\[ \det(\partial_i \partial_j u) = 1 \quad (10) \]
and second
\[ \sum \partial_i \partial_j (Vu^{ij}) = 0, \quad (11) \]
where \( u^{ij} \) is the inverse of the Hessian \( u_{ij} = \partial_i \partial_j u \). Given a pair \( u, V \) satisfying these equations, we set \( \sigma^{ij} = Vu^{ij} \) and the discussion above shows that all solutions arise in this way (locally). Notice that, given \( u \) the second equation is a linear equation for \( V \) and in fact is familiar as the linearisation of the Monge-Ampère equation at \( u \). Recall that the linearised operator \( \Box_u \) can be written in three different ways
\[ \Box_u f = \sum \partial_i \partial_j (fu^{ij}) = \sum \partial_i (\partial_j f u^{ij}) = \sum (\partial_i \partial_j f) u^{ij}, \]
using the identity \( \sum \partial_i u^{ij} = 0. \)

Now we want to set up our boundary value problem in this context. We suppose that \( U \) is a 3-manifold with boundary \( \Sigma \) and the circle bundle extends to the boundary, so that \( Y = \partial X \) is a circle bundle over \( \Sigma \). (The extension of the circle bundle over \( U \) means that it must be a trivial bundle, but we do not have a canonical trivialisation.) We want to consider triples \( \mu^i \) of closed 2-forms on \( Y \), as before, invariant under the circle action and such that the action is “Hamiltonian”, i.e. there are circle-invariant functions \( h^i_Y \) on \( Y \) with
\[ dh^i_Y = i_\xi \mu^i, \]
These functions give a map \( h^i_Y : \Sigma \to W = \mathbb{R}^3 \) and it follows from the definitions that this is an immersion. Now we encounter a potential obstruction of a differential topological nature to the existence of an invariant closed positive triple on \( X \) with these boundary values: the immersion \( h^i_Y \) must extend to an immersion of \( U \) in \( \mathbb{R}^3 \). But let us suppose here for simplicity that \( h^i_Y \) is an embedding of \( \Sigma \) as the boundary of a domain in \( \mathbb{R}^3 \). Then for any extension of the \( \rho^i \) over \( X \), of the kind considered above, the map \( h \) must be a diffeomorphism from \( U \) to this domain. Thus we can simplify our notation by taking \( U \) to be a domain in \( \mathbb{R}^3 \) with smooth boundary \( \Sigma \). To avoid complication, we suppose that \( U \) is simply connected, so that \( \Sigma \) is diffeomorphic to a 2-sphere. Thus our PDE problem is to solve the equations (10) and (11) for functions \( u, V \) on \( U \subset \mathbb{R}^3 \) and the remaining task is to identify the boundary conditions on \( \Sigma = \partial U \) defined by a triple \( \mu^i \). (The assumption that \( U \) is simply connected means that the above local analysis of solutions applies globally on \( U \).)
The differential geometric analysis of invariant triples \( \mu = (\mu^i) \) is complicated by the fact that there is no natural connection on the circle bundle \( \pi: Y \to \Sigma \). But the analysis has a simple conclusion which can be expressed in terms of certain distributions, or currents, which we call layer currents. In this analysis it will be important to keep track of the full \( SL(3, \mathbb{R}) \)-invariance of the set-up so we work in the 3-dimensional vector space \( W \) with fixed volume element. So we have an embedding of \( \Sigma \) in \( W \) as the boundary of a domain \( U \) and a triple \( \mu \) is a section of \( W \otimes \Lambda^2 T^* Y \).

We define a layer current supported on \( \Sigma \) to be a linear map from functions on \( W \) to \( \mathbb{R} \) of the form

\[
L_{\theta_1, \theta_2, v}(f) = \int_{\Sigma} (\nabla_v f) \theta_1 + f \theta_2.
\]

(12)

where \( \theta_1, \theta_2 \) are 2-forms on \( \Sigma \) with \( \theta_1 > 0 \) and \( v \) is an outward-pointing normal vector field along \( \Sigma \)—a section of the tangent bundle of \( W \) restricted to \( \Sigma \) which is complementary to the tangent bundle of \( \Sigma \). Of course this depends only on the restriction of \( f \) to the first formal neighbourhood of \( \Sigma \), in particular it is defined for a function \( f \) on \( U \) which is smooth up to the boundary.

The point is that the same functional \( L \) can be defined by different data \((\theta_1, \theta_2, v)\). First, it is obvious that for any positive function \( g \) on \( \Sigma \) the data \((g\theta_1, \theta_2, g^{-1}v)\) defines the same current. Second, if \( w \) is a tangential vector field on \( \Sigma \) we have

\[
\int_{\Sigma} (\nabla_w f) \theta_1 = \int_{\Sigma} \tilde{\theta}_2 f
\]

(13)

where \( \tilde{\theta}_2 = -d(i_w \theta_1) \). It follows that a given layer current \( L \) of this mind can be represented using any normal vector field, for appropriate \( \theta_1, \theta_2 \). Let \( \nu^*_\Sigma = TW/T\Sigma \) be the normal bundle of \( \Sigma \) in \( W \). The fixed volume element on \( W \) gives an isomorphism

\[
\nu^*_\Sigma = \Lambda^2 T^* \Sigma.
\]

(14)

Let \([v]\) denote the image of \( v \) in \( \nu^*_\Sigma \). The product

\[
H_L = [v], \mu_2 \in (\Lambda^2 T^* \Sigma)^\otimes 2
\]

is independent of the choice of data \((v, \mu_1, \mu_2)\) used to represent \( L \); we call \( H_L \) the primary invariant of \( L \). For a function \( f \) which vanishes on \( \Sigma \) the derivative \( df \) along \( \Sigma \) is defined as a section of \( \nu^*_\Sigma \) and for such functions we have

\[
L(f) = \int_{\Sigma} H df,
\]

(15)

where we use the isomorphism \( \nu^*_\Sigma = \Lambda^2 T^* \Sigma \) and the pairing with \( H \) yields a 2-form \( H df \) on \( \Sigma \).

Now let \( \mu \in W \otimes \Omega^2(Y) \) be a closed \( S^1 \)-invariant triple on the circle bundle \( Y \) over \( \Sigma \) such that the inclusion \( \Sigma \to W \) is the Hamiltonian map for the action.
Let \( y \) be a point of \( Y \) and \( \epsilon \in W^* \) be a co-normal to \( \Sigma \) at \( x = \pi(y) \), i.e. an element of \( W^* \) vanishing on \( (T\Sigma)_x \subset W \). Then we have a map

\[
\epsilon : (\Lambda^2 T^* Y)_y \otimes W \to (\Lambda^2 T^* Y)_y,
\]

and it follows from the definitions that \( \epsilon(\mu) \) lies in the image of the pull-back map \( \pi^* : (\Lambda^2 T^* \Sigma)_x \to (\Lambda^2 T^* Y)_y \). Thus we have a unique element \( h \in (\Lambda^2 T^* \Sigma)_x \) with \( \pi^*(h) = \epsilon(\mu) \). Multiplying \( \epsilon \) by a factor \( \kappa \) clearly multiplies \( h \) by \( \kappa \) so, using again the isomorphism (14), we get a well-defined section \( H\mu \) of \( (\Lambda^2 T^* \Sigma)^{\otimes 2} \), independent of the choice of \( \epsilon \). We call \( H\mu \) the primary invariant of the triple \( \mu \).

Next choose a normal vector field \( v \) along \( \Sigma \). At a point \( y \in Y \) we transpose \( \mu(y) \) to give a map

\[
\hat{\mu} : W^* \to (\Lambda^2 T^* Y)_y = TY \otimes \Lambda^3 T^* Y.
\]

The annihilator of \( v(\pi(y)) \) is a 2-dimensional subspace of \( W^* \) and it follows from the definitions that the image of this subspace under \( \hat{\mu} \) defines a 2-dimensional subspace of \( TY \) transverse to the \( S^1 \)-orbit. In other words the choice of normal vector field \( v \) defines a connection on the \( S^1 \) bundle \( \pi : Y \to \Sigma \): in fact giving a connection is equivalent to giving a complementary bundle to \( T\Sigma \subset W \). Let \( \Phi \) be the curvature of this connection, a 2-form on \( \Sigma \) and define a current \( \mathcal{L}^\mu v \) by

\[
\mathcal{L}^\mu v(f) = \int_\Sigma (H\mu[v]^{-1}) \nabla_v f + \Phi f. \tag{16}
\]

Here \( H\mu[v]^{-1} \) is the 2-form given by the pairing of \( [v]^{-1} \in \nu^* = (\Lambda^2 T\Sigma)^{-1} \) with \( H\mu \in (\Lambda^2 T^* \Sigma)^{\otimes 2} \).

**Proposition 3** The layer current \( \mathcal{L}^\mu v \) is independent of the choice of normal vector field \( v \) so can be written as \( \mathcal{L}\mu \). Two triples \( \mu, \mu' \) are equivalent by \( S^1 \)-equivariant diffeomorphisms if and only if \( \mathcal{L}\mu = \mathcal{L}\mu' \).

If we change \( v \) by multiplication by a positive function then we do not change the connection and hence we do not change the integral of \( \Phi f \). The other term in the integrand is also unchanged because the scalings of \( [v]^{-1} \) and \( \nabla_v \) cancel.

So to prove the first statement it suffices to consider changing \( v \) to \( v + w \) where \( w \) is a tangential vector field on \( \Sigma \). Using the formula (13), we have to show that the connection changes by the addition of the 1-form \( a = i_w \theta_1 \). To see this we work in co-ordinates at a given point on \( \Sigma \), taking \( v = \partial_1 \) and the tangent space of \( \Sigma \) spanned by \( \partial_2, \partial_3 \). Write \( \theta_1 = Gdx^2 dx^3 \) at the given point. If \( \alpha \) is the connection 1-form on \( Y \) defined by \( v \) then it follows from the definitions that, over this point,

\[
\mu^1 = Gdx^2 dx^3 , \mu^2 = \alpha \wedge dx^2 , \mu^3 = \alpha \wedge dx^3.
\]

If \( w = w^2 \partial_2 + w^3 \partial_3 \) at this point the annihilator of \( v + w \) in \( W^* \) is spanned by \( dx^2 - w^2 dx^1, dx^3 - w^3 dx^1 \) and this maps to the 2-dimensional subspace in
\[ \Lambda^2 T^* Y \] spanned by
\[ \alpha \wedge dx^2 - w^2 G dx^2 dx^3, \quad \alpha \wedge dx^3 - w^3 G dx^2 dx^3 \]
which corresponds to the 2-dimensional subspace in \( TY \) spanned by
\[ \partial_3 - G w^2 \xi, \quad -\partial_2 - G w^3 \xi. \]
This is the anhilliator of the 1-form \( \alpha + a \) where \( a = G w^2 dx^3 - G w^3 dx^2 \) which is the contraction \( \iota_{\omega}\theta_1 \) as required.

The second statement of the proposition follows easily from the fact that, since \( \Sigma \) is simply connected, a connection is determined up to gauge equivalence by its curvature.

So far we have considered our structures over the surface \( \Sigma \subset W \). Now let \( \sigma \) be a matrix-valued function over \( U \subset W \), as before, defining a triple \( \omega \) on \( X \). Then for any smooth function \( f \) on \( U \) we have
\[ \int_U \sum \sigma^{ij} \partial_i \partial_j f - \sum (\partial_i \partial_j \sigma^{ij}) f = \mathcal{L}_\sigma(f), \quad (17) \]
where \( \mathcal{L}_\sigma \) is the layer current supported on \( \Sigma \) defined by
\[ \mathcal{L}_\sigma f = \int_\Sigma \sum \sigma^{ij} \partial_i f - (\sum \partial_i \sigma^{ij}) f. \quad (18) \]
(To clarify notation: in (17) we suppress the volume form on \( W \) which defines our measure and in (18) the integrand is written as a vector field, which defines a 2-form on \( \Sigma \) by contraction with the 3-dimensional volume form as in (14).)

Then we have:

**Proposition 4** The boundary value of the triple \( \omega \) corresponding to \( \sigma \) is equivalent to the triple \( \mu \) on \( Y \) if and only if \( \mathcal{L}_\sigma = \mathcal{L}_\mu \).

To see this, regard the inverse matrix \( \sigma_{ij} \) as a Riemannian metric on \( U \). The orthogonal complement with respect to this metric defines a normal vector field \( v_\sigma \) over \( \Sigma \) and hence a connection on \( Y \to \Sigma \). We know that \( \sigma \) defines a connection on the circle bundle \( X \to U \) with curvature given by \( F_i = -\partial_j \sigma^{ij} \).

The Proposition amounts to the fact that the restriction of this connection to \( Y \to \Sigma \) is the same as the connection defined by \( v_\sigma \), which we leave for the reader to check.

To illustrate the nature of this boundary condition consider an example where \( \Sigma \) is locally given by the plane \( x^1 = 0 \) and take \( \partial_1 \) as normal vector field. Then \( \mathcal{L}_\mu \) is locally represented by 2-forms
\[ \theta_1 = G_1 dx^2 dx^3, \quad \theta_2 = G_2 dx^2 dx^3, \]
where \( G_i \) are functions of \( x^2, x^3 \). That is, for functions \( f \) supported in this region
\[ \mathcal{L}(f) = \int_{x^1 = 0} \left( G_1 \frac{\partial f}{\partial x^1} + G_2 f \right) dx^2 dx^3. \]
Now if $\sigma$ is defined over $U$ we have, for such functions $f$,$$
abla(\sigma(f)) = \int_{x^1=0} (\sigma^{11} \frac{\partial f}{\partial x^1} + (\sigma^{12} \frac{\partial f}{\partial x^2} + \sigma^{13} \frac{\partial f}{\partial x^3}) - (\partial_i \sigma^{1i}) f) \ dx^2 dx^3.$$
Integrating by parts, the sum of the second and third terms is
$$-\int_{x^1=0} f(\partial_1 \sigma^{11} + 2\partial_2 \sigma^{12} + 2\partial_3 \sigma^{13}) dx^2 dx^3.$$

Our boundary conditions are
- $\sigma^{11} = G_1$,
- $\partial_1 \sigma^{11} + 2\partial_2 \sigma^{12} + 2\partial_3 \sigma^{13} = -G_2$.

Notice that if $f$ is an affine-linear function then $\nabla(f)$ vanishes for any $\sigma$ on $U$. This is connected to the following identities on the boundary:
- For a circle bundle $Y \rightarrow \Sigma$ with Chern class $d$ and any invariant triple $\mu$ on $Y$, the value of functional $\nabla(1) = 2\pi d$ (Here 1 denotes the constant function).
- Suppose $d = 0$, so $Y$ is diffeomorphic to $S^1 \times \Sigma = S^1 \times S^2$ and there is a lift $[\Sigma] \in H_2(Y)$. Then for any invariant triple $\rho$$\nabla(x^1) = \int_{[\Sigma]} \mu^i$.

Again, we leave the proofs as exercises for the interested reader.

Putting all this together, we can formulate the dimensionally-reduced version of our general boundary value problem as follows. The functional (9) clearly reduces to the functional
$$\text{Vol}(\sigma) = \int_U (\det \sigma)^{1/3}. \quad (19)$$

**Variational Problem I**

Given a (simply connected) domain $U \subset \mathbb{R}^3$ with smooth boundary $\Sigma$ and a layer current $\mathcal{L}$ on $\Sigma$, find the critical points of the volume functional (19) over all $\sigma = (\sigma^{ij})$ on $U$ satisfying
- (A) $\sum \partial_i \partial_j \sigma^{ij} = 0$,
- (B) $\nabla = \mathcal{L}$.

Our first question is now whether the set $\mathcal{S}_\mathcal{L}$ of matrix-valued functions $\sigma$ satisfying (A), (B) above is non-empty. The integral formula gives an immediate constraint on the boundary data: if $\mathcal{S}_\mathcal{L}$ is not empty then $\mathcal{L} f \geq 0$ for all convex functions $f$ on $U$ (with equality if and only if $f$ is affine linear). We also have
Proposition 5 If a solution to the Variational Problem I exists it is an absolute maximum of the volume functional on $S_L$.

This follows immediately from the facts that both conditions (A),(B) are linear in $\sigma$ and the function $(\det \sigma)^{1/3}$ is concave.

Let $f$ be a convex function on $U$ satisfying the Monge-Ampère equation $\det(f_{ij}) = 1$. Then for any $\sigma$ we have, pointwise on $U$,

$$\det(\sigma)^{1/3} = (\det \sigma_{ij} \det(f_{ij}))^{1/3} \leq \frac{1}{3} \sum \sigma^{ij} f_{ij}. \quad (20)$$

So if $\sigma$ satisfies the conditions (1),(2) of Variational Problem I we have, integrating over $U$ and using the definition of $L_\sigma$,

$$\text{Vol}(\sigma) \leq \frac{1}{3} \int_U \sum \sigma^{ij} f_{ij} = \frac{1}{3} L_\sigma f. \quad (21)$$

(Our previous bound, in Proposition 2, arises by taking quadratic functions $f$.) These bounds furnished by solutions of the Monge-Ampère equation, lead to a dual formulation of the variational problem, which incorporates the boundary conditions in a simple way. Write $MA(U)$ for the set of convex solutions of the Monge-Ampère equation on $U$, smooth up to the boundary.

Variational problem II

Given a (simply connected) domain $U \subset \mathbb{R}^3$ with smooth boundary $\Sigma$ and a layer current $L$ on $\Sigma$, minimise $L(f)$ over all $f \in MA(U)$.

Proposition 6 The variational problems I,II are equivalent in the sense that for $u \in MA(U)$ we can find a positive function $V$ such that $\sigma^{ij} = Vu^{ij}$ is a solution of variational problem I if and only $u$ is a solution of variational problem II.

In one direction, equality holds in (21) if and only if $u_{ij}$ is a multiple of the inverse of $\sigma^{ij}$. We know that a solution to the variational problem I has the form $\sigma_{ij} = Vu^{ij}$ where $u$ satisfies the Monge-Ampère equation, so taking $f = u$ equality holds in (21), and it follows that $u$ minimises $L(f)$ over $MA(U)$.

In the other direction, suppose that $u \in MA(U)$ is an extremum of the functional $\mathcal{L}$ (it will follow from the discussion below that $u$ is in fact a minimum and is unique up to the addition of an affine-linear function). The Euler-Lagrange equation is $\mathcal{L}(W) = 0$ for all solutions $W$ of the linearised equation $\Box u W$ over $U$. We can solve the Dirichlet problem for this linearised equation, to find $V$ such that $\Box_u V = 0$ and so that if $\sigma^{ij} = Vu^{ij}$ the primary invariant of $L_\sigma$ is equal to that of $L$. If we chose a co-normal $\nu^*$ to $\Sigma$ this is just saying that $\sum Vu^{ij} \nu^*_i \nu^*_j$ is a prescribed function on $\Sigma$, which for fixed $u$ is just prescribing $V$ on $\Sigma$. Then it follows from the previous discussion that $L_\sigma(W) = 0$ for all solutions $W$ of the linearised equation. Since $L_\sigma$ and $L$ have the same primary invariant so the difference can be written as

$$(\mathcal{L}_\sigma - \mathcal{L})(f) = \int_\Sigma \Theta f,$$
for a 2-form $\Theta$ on $\Sigma$. For any function $f$ on $\Sigma$ we can solve the Dirichlet problem for $\square u$ with boundary value $f$ and so

$$\int_\Sigma \Omega f = 0$$

for all $f$. This implies that $\Omega = 0$ so $\mathcal{L}_\sigma = \mathcal{L}$ and we have solved the variational problem $I$.

Modifying our problem, we can obtain a decisive existence result. Rather than fixing the full boundary data $\mu$ we just fix the primary invariant $H^\mu$. Given a positive $H \in \Gamma(\Lambda^2 T^* \Sigma)^{\otimes 2}$ we write $\mathcal{C}_H$ for the set of $\sigma$ over $U$ satisfying $\sigma_{ij}^{ij} = 0$ and with the primary invariant of $\mathcal{L}_\sigma$ equal to $H$. As above, in terms of a co-normal $\nu^*$ this amounts to prescribing $\nu^*_i \nu^*_j \sigma^{ij}$ on the boundary.

**Proposition 7** If $U$ is strictly convex there is a unique critical point of the volume functional on $\mathcal{C}_H$ and this is an absolute maximum.

The uniqueness and the fact that a critical point is an absolute maximum follows from concavity, just as before. For the existence, we first solve (invoking [2]) the Dirichlet problem for the Monge-Ampère equation to get a function $u \in MA(U)$ with $u = 0$ on $\Sigma$. Now solve the Dirichlet problem for the linearised equation to find a function $V$ with $\square_u = 0$ in $U$ and such that $Vu^{ij} \nu^*_i \nu^*_j$ is the prescribed function on the boundary and write $\sigma^{ij} = Vu^{ij}$. We claim that this $\sigma$ is a critical point of the volume functional on $\mathcal{C}_H$. Let $\tau^{ij}$ be an infinitesimal variation within $\mathcal{C}_H$. In other words, $\sum \partial_i \partial_j \tau^{ij} = 0$ in $U$ and on the boundary $\sum \tau^{ij} \nu^*_i \nu^*_j = 0$. Then the variation in the volume functional is

$$3d\text{Vol} = \int_U \sum u_{ij} \tau^{ij} = \int_\Sigma \sum \tau^{ij} u - \sum \tau^{ij} \partial_j u. \quad (22)$$

The first term on the right hand side of (22) vanishes since $u$ vanishes on $\Sigma$. In the second term, the derivative of $u$ along $\Sigma$ vanishes, so there is only a contribution from the normal derivative of $u$ and the integrand is a multiple of $\sum \tau^{ij} \nu^*_i \nu^*_j$, so this also vanishes.

4 Further remarks

4.1 Singularities

It seems unlikely that the variational problems I, II always have solutions, even given the constraints we have found. To see this we consider the well-known singular solutions of the Monge-Ampère equation, going back to Pogolov. With co-ordinates $x^1, x^2, x^3$ set $r = \sqrt{(x^1)^2 + (x^2)^2}$ and consider functions $u$ of the form $u = f(x^1)r^{4/3}$. Then one finds that

$$\det(u_{ij}) = \frac{64}{27} f \left( \frac{ff''}{3} - (f')^2 \right),$$

for a 2-form $\Theta$ on $\Sigma$. For any function $f$ on $\Sigma$ we can solve the Dirichlet problem for $\square_u$ with boundary value $f$ and so

$$\int_\Sigma \Omega f = 0$$

for all $f$. This implies that $\Omega = 0$ so $\mathcal{L}_\sigma = \mathcal{L}$ and we have solved the variational problem $I$.

Modifying our problem, we can obtain a decisive existence result. Rather than fixing the full boundary data $\mu$ we just fix the primary invariant $H^\mu$. Given a positive $H \in \Gamma(\Lambda^2 T^* \Sigma)^{\otimes 2}$ we write $\mathcal{C}_H$ for the set of $\sigma$ over $U$ satisfying $\sigma_{ij}^{ij} = 0$ and with the primary invariant of $\mathcal{L}_\sigma$ equal to $H$. As above, in terms of a co-normal $\nu^*$ this amounts to prescribing $\nu^*_i \nu^*_j \sigma^{ij}$ on the boundary.

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The first term on the right hand side of (22) vanishes since $u$ vanishes on $\Sigma$. In the second term, the derivative of $u$ along $\Sigma$ vanishes, so there is only a contribution from the normal derivative of $u$ and the integrand is a multiple of $\sum \tau^{ij} \nu^*_i \nu^*_j$, so this also vanishes.
so we can find smooth functions \( f \) on an interval, say \((-\epsilon, \epsilon)\) with \( f'' > 0 \) and such that \( u \) satisfies the Monge-Ampère equation. Fix such a function \( f \) and let \( \eta \) be the vector field

\[
\eta = 2x^1 \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^2} - x^3 \frac{\partial}{\partial x^3}.
\]

This vector field generates volume preserving transformations, so \( V = \nabla_{\eta}u \) satisfies the linearised equation \( \Box u V = 0 \) and if we define \( \sigma^{ij} = Vu^{ij} \) we get a singular solution of our reduced \( G_2 \) equations (provided that \( V > 0 \)). Suppose that, near the origin, \( \Sigma \) is given by the co-ordinate plane \( x^1 = 0 \) and let our boundary data be given locally by the layer current

\[
\int G_1 \partial f \partial x^1 + G_2 f dx^2 dx^3,
\]

as above, for smooth functions \( G_i(x^2, x^3) \). As we saw above, the boundary condition is given by

\[
\sigma^{11} = G_1, \quad \partial_1 \sigma^{11} + 2 \partial_2 \sigma^{12} + 2 \partial_3 \sigma^{13} = -G_2.
\]

One can compute that

\[
V = (2x^1 f' - \frac{4}{3} f) r^{4/3},
\]

\[
\sigma^{11} = \frac{16 f^2}{9} (2x^1 f' - \frac{4}{3} f),
\]

\[
\sum_i \partial_i \sigma^{1i} = \frac{16 f^2}{9} (2x^1 f' - \frac{4}{3} f)' - \frac{16}{3} f f'.
\]

Thus \( \sigma^{11}, \sum_i \partial_i \sigma^{1i} \) are smooth functions of \( x^1 \) so we get a singular solution of our boundary value problem with smooth boundary data \( G_1, G_2 \).

### 4.2 Connection with the Apostolov-Salamon construction

In the discussion above we have passed from 7 dimensions to 3 dimensions by first imposing translational symmetry in 3 variables to get down to 4 dimensions and then imposing a circle action to pass from 4 to 3. We can achieve the same end by imposing the circle action first, to get a reduction to 6 dimensions, and then studying translation invariant solutions. The material in this subsection was explained to the author by Lorenzo Foscolo.

\( G_2 \) structures on a 7-manifold \( M \) invariant under a free circle action have been studied by Apostolov and Salamon \cite{ApostolovSalamon} and others. The quotient space \( N \) has an induced \( SU(3) \) structure, that is to say a 2-form \( \omega \) and a complex 3-form \( \Omega \) equivalent at each point to the standard structures on \( \mathbb{C}^3 \) (with complex 3-form \( dz_1 dz_2 dz_3 \)). The \( G_2 \) structure on \( M \) can be written as

\[
\phi = \alpha \wedge \omega + V^{3/4} \text{Re} \Omega
\]  

(23)
where we identify forms on $N$ with their lifts to $M$ and $V$ is a smooth positive function on $N$. (In fact $V^{-1/2}$ is the length of generator of the circle action in the metric $g_\phi$.) The 1-form $\alpha$ is a connection form on the circle bundle $M \to N$. Now one finds that

$$*g_\phi = -V^{1/4}\alpha \wedge \Im \Omega + \frac{V}{2} \alpha^2.$$  

Thus if $F = d\alpha$ is the curvature of the connection the conditions to be satisfied for a torsion free $G_2$-structure are:

$$d\omega = 0, \quad F \wedge \omega + d(V^{3/4}\Re \Omega) = 0$$  

and

$$d(V^{1/4}\Re \Omega) = 0, \quad dV \wedge \omega^2 = 2V^{1/4}F \wedge \Im \Omega.$$  

Now let $W$ be a 3-dimensional real vector space as before, and set $N = U \times W^*$ where $U$ is an open set in $W$. Take standard co-ordinates $x^i$ on $W$ and $\theta_i$ on $W^*$ so there is a standard symplectic form

$$\omega = \sum dx^i \wedge d\theta_i.$$  

Let $u$ be a convex function on $U$ and define complex 1-forms

$$\epsilon_a = d\theta_a + i \sum u_{ab} dx^b.$$  

It is well-known that these define a complex structure compatible with $\omega$ and with holomorphic 3-form $\epsilon_1 \epsilon_2 \epsilon_3$. If $u$ satisfies the Monge-Ampère equation $\det u_{ij} = 1$ then this is a Calabi-Yau structure. If $V$ is a positive function on $U$ we modify this by taking

$$\epsilon'_a = V^{-1/4}d\theta_a + iV^{1/4} \sum u_{ab} dx^b,$$

and

$$\Omega = \epsilon'_1 \epsilon'_2 \epsilon'_3.$$  

This complex 3-form is also algebraically compatible with $\omega$. One checks that if $u$ satisfies the Monge-Ampère equation and $F$ is the 2-form

$$F = -\sum_{j,k,l \text{ cyclic}} (\partial_i) V u^{ij} dx^k dx^l$$

then $V, F, \Omega, \omega$ satisfy the equations (24),(25). The condition that $F$ is closed, so arises as the curvature of a connection, is the equation $\square_a V = 0$.

### 4.3 A general class of equations and LeBrun’s construction

Our variational problem I, and its dual formulation II have natural extensions. We can clearly replace $\mathbb{R}^3$ by $\mathbb{R}^n$, but more interestingly we can consider a class
of different functionals. Let $W$ be an $n$-dimensional real vector space and write $s_+^2(W) \subset s^2(W)$ for the cone of positive definite quadratic forms on $W$. Let $\nu$ be a smooth positive concave function on $s_+^2(W)$ which is homogeneous of degree 1, so $\nu(k\sigma) = k\nu(\sigma)$. This means that $\nu$ is not strictly concave but we suppose that the kernel of the Hessian of $\nu$ has dimension 1 everywhere which means that $\log \nu$ is strictly concave. The basic example, which we have discussed in the case $n = 3$, is to take $\nu(\sigma) = (\det \sigma)^{1/n}$. In the general case we consider functions $\sigma$ on a simply-connected domain $U \subset W$ taking values in $s_+^2(W)$, which we express in terms of a basis as $\sigma = (\sigma_{ij})$. We consider the functional

$$I = \int \nu(\sigma)$$

subject to the constraint $\sum \partial_i \partial_j \sigma^{ij} = 0$. Initially we consider variations of this integral with respect to compactly-supported variations of $\sigma$, and later we consider boundary conditions.

We regard $D\nu$ as a map from $s_+^2(W)$ to $s^2(W^*)$. The homogeneity of $\nu$ implies that this map is constant on rays in $s_+^2(W)$, so it cannot map onto an open set in $s^2(W^*)$. Let $\hat{L}$ be the Legendre transform of $L = \log \nu$. So $\hat{L}$ is a concave function on some open subset $\Omega$ of $s^2(W^*)$ and $\hat{N} = \hat{L}^{-1}(1)$ is a smooth hypersurface in $\Omega$. The homogeneity of $\nu$ implies that the image of $D\nu$ is exactly $\hat{V}$ and for $v \in \hat{N}$ the pre-image $(D\nu)^{-1}(v)$ is a single ray in $s_+^2(W)$. Set $N = \nu^{-1}(1) \subset s_+^2(W)$. Then the restriction of $D\nu$ gives a diffeomorphism from $N$ to $\hat{N}$ and we write $\psi : \hat{N} \rightarrow N$ for the inverse.

**Proposition 8** The integral $I$ is stationary, with respect to compactly-supported variations in $\sigma$ satisfying the constraint $\sum \partial_i \partial_j \sigma^{ij} = 0$, if and only if there is a function $u$ on $U$ such that

$$\partial_i \partial_j u = (D\nu)(\sigma).$$

In one direction, if there is such a function $u$ and if $\tau$ is a compactly supported variation of $\sigma$ with $\partial_i \partial_j \tau^{ij} = 0$ then

$$\delta I = \int \langle D\nu(\sigma), \tau \rangle = \int \langle \partial_i \partial_j u \rangle \tau^{ij} = \int u \partial_i \partial_j \tau^{ij} = 0.$$

The other direction follows easily from the fact that we can generate solutions of $\partial_i \partial_j \tau^{ij} = 0$ from an arbitrary tensor $h^{iaj}$ which is skew symmetric in $i, a$ via the formula

$$\tau^{ij} = \partial_a h^{iaj} + \partial_a h^{iai}.$$

Now the general local solution of our variational problem is obtained as follows. First solve the equation for a function $u$ that $u_{ij} = \partial_i \partial_j u$ lies in $\hat{N}$; that is

$$\hat{L}(u_{ij}) = 1. \quad (26)$$

Now set $\lambda = (\lambda^{ij}) = \psi(u_{ij})$. We know that $\sigma = V\psi(u_{ij})$ for some positive function $V$ and the remaining equation to solve is the linear equation for $V$

$$\partial_i \partial_j (\lambda^{ij} V) = 0. \quad (27)$$
We compare this with the linearisation of the nonlinear equation (26) at \( u \). The derivative of \( \hat{L} \) at a point \( v \in N \) is given by \( \psi(v) \), so the linearised equation is
\[
\sum \lambda^{ij} \partial_i \partial_j V = 0.
\]
(28)

In general this is not the same as the equation (27), but the two equations are adjoint in that the formal adjoint \( \halo{\ast} \) of the operator \( \halo{u}(V) = \sum \lambda^{ij} \partial_i \partial_j V \) is \( \halo{\ast}(V) = \sum \partial_i \partial_j (\lambda^{ij} V) \).

In the case when \( \nu(\sigma) = (\det \sigma)^{1/n} \) the derivative of \( \log \nu \) is the map \( \sigma \mapsto n^{-1} \sigma^{-1} \) and we recover the previous set-up. In this special case the linearised equation is self-adjoint, i.e \( \halo{\ast} = \halo{u} \).

We can now introduce a boundary value problem on a domain \( U \subset W = \mathbb{R}^n \) with smooth boundary \( \Sigma \) and with a given layer current \( \mathcal{L} \) supported on \( \Sigma \), extending the definitions from \( n = 3 \) in the obvious way. If \( u \) solves the nonlinear equation \( \hat{L}(u_{ij}) = 1 \) on \( U \) then for any \( \sigma \) satisfying the boundary conditions and \( \sum \partial_i \partial_j \sigma^{ij} = 0 \) we have an inequality
\[
\int_U \nu(\sigma) \leq \mathcal{L}(u),
\]
and we get a dual variational problem as before.

Claude LeBrun pointed out to the author that there are some striking similarities between the variant of the Gibbons-Hawking construction studied in the previous section and another variant introduced by him in [7], constructing Kähler surfaces of zero scalar curvature. While this does not exactly fit into the general framework above, we will outline how it can be treated in a similar fashion.

We consider a triple of forms \( \omega^i \) with \( \omega^1 \) a Kähler form and \( \omega^2, \omega^3 \) the real and imaginary parts of a holomorphic 2-form. This means that we restrict attention to matrix-valued functions \( \sigma \) which are diagonal, with \( \sigma^{11} = a \) and \( \sigma^{22} = \sigma^{33} = b \) for positive functions \( a, b \). The condition \( \partial_i \partial_j \sigma^{ij} = 0 \) is then
\[
a_{11} + b_{22} + b_{33} = 0,
\]
(29)
(writing \( a_{11} = \partial_1 \partial_{\bar{1}} a \) etc.). The variational formulation, generating the zero scalar curvature equation, comes from the “Mabuchi functional”, which in this situation is given by
\[
I = \int a(\log(a/b) - 1),
\]
(30)
and the function \( \nu(a, b) = a(\log(a/b) - 1) \) is homogeneous of degree 1. The condition that \( I \) is stationary with respect to compactly supported variations satisfying the constraint (29) is that there is a function \( u \) with
\[
u_{11} = \log(a/b) \quad u_{22} + u_{33} = -(a/b).
\]
In other words, \( u \) satisfies the nonlinear equation
\[
e^{u_{11}} + u_{22} + u_{33} = 0.
\]
(31)
Given such a function $u$ we set $b = V$ and $a = e^{u_{11}}V$ and the equation (29) is the linear equation for $V$:

$$(e^{u_{11}}V)_{11} + V_{22} + V_{33} = 0.$$  \hspace{1cm} (32)

This is again the adjoint of the linearisation of the nonlinear equation (31). To relate this to LeBrun’s set-up we put $U = u_{11}$ so

$$(e^{U})_{11} + U_{22} + U_{33} = 0;$$

$$(e^{U}V)_{11} + V_{22} + V_{33} = 0,$$

and these are the equations, for functions $U, V$, obtained by LeBrun.

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