1D DIRAC OPERATORS WITH SPECIAL PERIODIC POTENTIALS

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Abstract. For one-dimensional Dirac operators of the form

$$L y = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + v y, \quad v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

we single out a class $X$ of $\pi$-periodic potentials $v$ with the following properties:

(i) The smoothness of potentials $v$ is determined only by the rate of decay of related spectral gaps $\gamma_n = |\lambda^+_n - \lambda^-_n|$, where $\lambda^+_n$ are the eigenvalues of $L = L(v)$ considered on $[0, \pi]$ with periodic (for even $n$) or antiperiodic (for odd $n$) boundary conditions.

(ii) There is a Riesz basis in $L^2([0, \pi], \mathbb{C}^2)$ which consists of periodic (or antiperiodic) eigenfunctions and associated functions (at most finitely many).

In particular, the class $X$ contains the families of symmetric potentials $X_{sym}$ (defined by $Q = P$) and skew-symmetric potentials $X_{skew-sym}$ (defined by $Q = -P$), or more generally the families $X_t$, $t \in \mathbb{R} \setminus \{0\}$, defined by $Q = tP$. Finite-zone potentials belonging to $X_t$ are dense in $X_t$.

Another interesting example of potentials is given by

$$v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \quad \text{with} \quad P(x) = a e^{2ix} + be^{-2ix}, \quad Q(x) = A e^{2ix} + Be^{-2ix}.$$ 

If $a, b, A, B \in \mathbb{C} \setminus \{0\}$, then the system of root functions of $L_{Per^\pm}(v)$ consists eventually of eigenfunctions. Moreover, for $bc = Per^-$ this system is a Riesz basis in $L^2([0, \pi], \mathbb{C}^2)$ if $|aA| = |bB|$ (then $v \in X$), and it is not a basis if $|aA| \neq |bB|$. For $bc = Per^+$ the system of root functions is a Riesz basis (and $v \in X$) always.

1. Introduction

We consider one-dimensional Dirac operators of the form

$$L y = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + v(x) y, \quad v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

with periodic matrix potentials $v$ with $P, Q \in L^2([0, \pi], \mathbb{C}^2)$, subject to periodic ($Per^+$) or antiperiodic ($Per^-$) boundary conditions ($bc$):

$$Per^+: \quad y(\pi) = y(0); \quad Per^-: \quad y(\pi) = -y(0).$$

Our goal is to single out the class of potentials $v$ which are special in the sense that the periodic and antiperiodic boundary value problems (b.v.p.) have at most finitely many linearly independent associated functions and there is a Riesz basis in $L^2([0, \pi], \mathbb{C}^2)$ which consists of root functions. It turns out this is exactly the
class of potentials which smoothness could be determined only by the rate of
decay of related spectral gaps \( \gamma_n = |\lambda_n^+ - \lambda_n^-| \), where \( \lambda_n^\pm \) are the eigenvalues of
\( L = L(v) \) considered on \([0, \pi]\) with periodic (for even \( n \)) or antiperiodic (for odd \( n \)) boundary conditions.

Similar questions arise about the one-dimensional Schrödinger operator (e.g., see \cite{5, 4})
\[
Ly = -y'' + v(x)y
\]
with periodic potentials \( v \in L^2([0, \pi], \mathbb{C}) \), subject to periodic (\( \text{Per}^+ \)) or antiperiodic (\( \text{Per}^- \)) boundary conditions
\[
\text{Per}^\pm : \ y(\pi) = \pm y(0); \quad y'(\pi) = \pm y(0).
\]
Moreover, the methods we use to solve these questions were first developed for Schrödinger operators.

The spectra of self-adjoint Schrödinger and Dirac operators with periodic potentials on the real line \( \mathbb{R} \) are continuous and have gap–band structure: the segments of continuous spectrum alternate with spectral gaps or instability zones. The theory of Floquet and Lyapunov (e.g., see \cite{12, 24}) explains that the end points of spectral gaps are eigenvalues of the same differential operators but considered on a finite interval of length one period with periodic or antiperiodic boundary conditions.

The decay rate of spectral gaps depends on the smoothness of the potential, and vice versa. This phenomenon was first studied for the Schrödinger operator \( (1.3) \) with real periodic (say \( \pi \)-periodic) potentials \( v \in L^2([0, \pi]) \). Considered on \( \mathbb{R} \) it generates a self-adjoint operator in \( L^2(\mathbb{R}) \); its spectrum is continuous and consists of a sequence of intervals \([\lambda_0^+, \lambda_1^-], [\lambda_1^+, \lambda_2^-], [\lambda_2^+, \lambda_3^-], \ldots \), where \( \lambda_0^+ < \lambda_2^- \leq \lambda_2^+ < \lambda_4^- \leq \lambda_4^+ < \cdots \) are all eigenvalues of the periodic (b.v.p.) and \( \lambda_1^- \leq \lambda_1^+ < \lambda_3^- \leq \lambda_3^+ < \cdots \) are all eigenvalues of the antiperiodic b.v.p. generated by \( L \) on \([0, \pi]\).

H. Hochstadt \cite{18, 19} (see also \cite{23}) discovered a direct connection between the smoothness of \( v \) and the rate of decay of the lengths of spectral gaps \( \gamma_n = \lambda_n^+ - \lambda_n^- \):

\begin{itemize}
  \item[(A)] \( v \in C^\infty \), i.e., \( v \) is infinitely differentiable, then \( \gamma_n \) decreases more rapidly than any power of \( 1/n \).
  \item[(B)] If a continuous function \( v \) is a finite–zone potential, i.e., \( \gamma_n = 0 \) for large enough \( n \), then \( v \in C^\infty \).
\end{itemize}

In the mid-70's (see \cite{27}, \cite{33}) the latter statement was extended, namely, it was shown, for real \( L^2([0, \pi]) \)–potentials \( v \), that \( (B) \Rightarrow (A) \). E. Trubowitz \cite{42} has used the Gelfand–Levitan \cite{14} trace formula and Dubrovin equations \cite{10, 11} to explain, that a real \( L^2([0, \pi]) \)–potential \( v(x) = \sum_{k \in \mathbb{Z}} V(2k) \exp(2ikx) \) is analytic, i.e.,
\[
\exists A > 0 : \quad |V(2k)| \leq Me^{-A|k|},
\]
if and only if the spectral gaps decay exponentially, i.e.,
\[
\exists a > 0 : \quad \gamma_n \leq Ce^{-a|n|}.
\]
If the potential \( v \) is complex-valued then the Schrödinger operator \( L(v) \) is not self-adjoint and one cannot talk about spectral gaps. But for large enough \( n \in \mathbb{N} \) there are two periodic (if \( n \) is even) or antiperiodic (if \( n \) is odd) eigenvalues \( \lambda_n^\pm \) close to \( n^2 \), so one may consider "gaps" \( \gamma_n = |\lambda_n^+ - \lambda_n^-| \)

and ask whether the rate of decay of \( \gamma_n \) still determines the smoothness of the potential \( v \). The answer to this question is negative as the example of M. Gaşymov [13] shows: if \( v(x) = \sum_{k=0}^{\infty} v_k e^{2ikx}, \quad v \in L^2([0,\pi]) \)

then all eigenvalues of periodic and antiperiodic b.v.p. are of algebraic multiplicity 2, so \( \gamma_n = 0 \).

In [38] V. Tkachenko suggested to consider also the Dirichlet b.v.p. \( y(\pi) = y(0) = 0 \). For large enough \( n \) there is exactly one Dirichlet eigenvalue \( \mu_n \) close to \( n^2 \), so the deviation \( \delta_n = |\mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-)| \) is well defined. Using an adequate parametrization of potentials in spectral terms similar to Marchenko–Ostrovskii’s ones [25] [27] for self-adjoint operators, V. Tkachenko [38] [40] (see also [39]) characterized \( C^\infty \)-smoothness and analyticity in terms of \( \delta_n \) and differences between critical values of Lyapunov functions and \((-1)^n\).

T. Kappeler and B. Mityagin [20] [21] suggested a new approach to the study of spectral gaps and deviations based on Fourier analysis. Using the Lyapunov-Schmidt reduction method they showed that for large enough \( n \) the numbers \( z^\pm_n = \lambda^\pm_n - n^2 \) are the only roots in the unit disc of a quasi-quadratic equation coined by them as basic equation \( (z - \alpha_n(z))^2 = \beta^+_n(z)\beta^-_n(z), \quad |z| < 1, \)

where \( \alpha_n(z) = \alpha_n(z; v) \) and \( \beta^+_n(z) = \beta^+_n(z; v) \) depend analytically on \( z, \quad |z| < 1, \) and \( v \) but the dependance on \( v \) is suppressed in the notations. For large enough \( n \) the gaps \( \gamma_n \) and deviations \( \delta_n \) could be estimated from above in terms of \( \beta^+_n(z) \) and \( \beta^-_n(z) : \)

\( \exists C > 1: \quad \gamma_n \leq 2(|\beta^+_n(z)| + |\beta^-_n(z)|), \quad \delta_n \leq C(|\beta^+_n(z)| + |\beta^-_n(z)|), \quad |z| < 1. \)

Using (1.9), T. Kappeler and B. Mityagin estimated \( l^2 \)-weighted norms \( \gamma_n \) and \( \delta_n \) by the corresponding weighted Sobolev norms of \( v \). Let us recall that the smoothness of a potential \( v(x) = \sum_k v_k e^{2ikx} \) can be characterized by its Fourier coefficients in terms of appropriate weighted norms and spaces. Namely, if \( \omega = (\omega(k))_{k \in \mathbb{Z}}, \quad \omega(-k) = \omega(k) > 0, \quad \omega(0) = 1, \)
is a weight sequence (or weight), then the corresponding weighted Sobolev space is
\[ H(\omega) = \left\{ v : \|v\|_{\omega}^2 = \sum_{k \in \mathbb{Z}} |v_k|^2 (\omega(k))^2 < \infty \right\}, \]
and the corresponding weighted \( \ell^2 \) space is
\[ \ell^2(\omega, \mathbb{N}) = \left\{ x = (x_n) : \|x\|_{\omega}^2 = \sum_{n=1}^{\infty} |x_n|^2 (\omega(n))^2 < \infty \right\}. \]

Examples of weights:
(a) Sobolev weights: \( \omega_a(0) = 1, \omega_a(k) = |k|^a \) for \( k \neq 0 \);
(b) Gevrey weights: \( \omega_{b,\gamma}(k) = e^{b|k|^\gamma}, \ b > 0, \gamma \in (0,1) \);
(c) Abel (exponential) weights: \( \omega_A(k) = e^{A|k|}, A > 0 \).

A weight \( \Omega \) is called submultiplicative if
\[ \Omega(k+m) \leq \Omega(k)\Omega(m), \ k, m \in \mathbb{Z}. \]

In [21], it was proved that if \( \Omega \) is a submultiplicative weight, then
\[ v \in H(\Omega) \Rightarrow (|\beta^+_n(z)| + |\beta^-_n(z)|) \in \ell^2(\Omega), \]
which implies (in view of (1.9))
\[ v \in H(\Omega) \Rightarrow (\gamma_n), (\delta_n) \in \ell^2(\Omega). \]

In [15] it was suggested to study the spectra of Dirac operators of the form (1.1) with periodic potentials in a similar way. If \( |n| \) is sufficiently large, then close to \( n \) there is one Dirichlet eigenvalue \( \mu_n \) and two periodic (for even \( n \)) or antiperiodic (for odd \( n \)) eigenvalues \( \lambda^+_n, \lambda^-_n \). So, with spectral gaps \( \gamma_n \) and deviations \( \delta_n \) defined by
\[ \gamma_n = |\lambda^+_n - \lambda^-_n|, \ \delta_n = |\mu_n - \frac{1}{2}(\lambda^+_n - \lambda^-_n)|, \ n \in \mathbb{Z}. \]
one may study the relationship between potential smoothness and the rate of decay of \( \gamma_n \) and \( \delta_n \). As in the case of Schrödinger operators, there is a basic equation
\[ (z - \alpha_n(z))^2 = \beta^+_n(z)\beta^-_n(z) \]
which characterizes when \( \lambda = z + n \) with \( |z| < 1/2 \) is a periodic or antiperiodic eigenvalue, and for large enough \( |n| \) the gaps \( \gamma_n \) and deviations \( \delta_n \) could be estimated from above in terms of \( \beta^+_n(z) \) and \( \beta^-_n(z) \) by (1.9) – see below Section 2 for details.

For Dirac potentials \( v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \), we say \( v \in H(\Omega) \) if \( P, Q \in H(\Omega) \). Then (1.12) holds for Dirac operators: for weights of the form \( \Omega(m) = |m|^a \omega(m) \) with \( a \in (0,1/4) \) and submultiplicative \( \omega \) it is proved in [16], and in full generality (for arbitrary submultiplicative weights \( \Omega \)) in [4, 5].

In [1, 2], respectively, the authors studied self-adjoint Schrödinger and Dirac operators (i.e., \( v \) is real-valued in the Schrödinger case and symmetric, \( \overline{Q} = P \),
in the Dirac case) and estimated the smoothness of potentials \( v \) by the rate of decay of \( \gamma_n \). For a wide classes of weights \( \Omega \) it was shown that

\[
(\gamma_n) \in \ell^2(\Omega) \Rightarrow v \in H(\Omega)
\]

by proving

\[
(|\beta^+_n(z)| + |\beta^-_n(z)|) \leq C\gamma_n, \quad |n| \geq n_0, \quad C = 2,
\]

and

\[
(|\beta^+_n(z)| + |\beta^-_n(z)|) \in \ell^2(\Omega) \Rightarrow v \in H(\Omega).
\]

In the non-self-adjoint case – see [2] for Schrödinger operators and [3, 5] for Dirac operators – we proved that

\[
(|\beta^+_n(z)| + |\beta^-_n(z)|) \leq C(\gamma_n + \delta_n), \quad |n| \geq n_0,
\]

where \( C \) is an absolute constant. Of course, (1.16) and (1.17) imply that

\[
(\gamma_n), (\delta_n) \in \ell^2(\Omega) \Rightarrow v \in H(\Omega).
\]

In the self-adjoint case deviations \( \delta_n \) are not important because the Dirichlet eigenvalue \( \mu_n \) is "trapped" between \( \lambda^-_n \) and \( \lambda^+_n \), so \( \delta_n \leq \gamma_n \).

Our aim in this paper is to study the class \( X \) of Dirac potentials \( v \) for which deviations are not essential in the sense that (1.15) holds with some constant \( C = C(v) \). A general criterion is given in Section 3 – see (3.1) and Proposition 8. It gives non-linear conditions for individual potentials. Sometimes the family of such potentials is a real linear space. We observe that an important example of such linear spaces is the one-parametric family

\[
X_t = \left\{ v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} : \quad Q = tP, \quad P \in L^2([0, \pi]) \right\}, \quad t \in \mathbb{R}, \quad t \neq 0.
\]

If \( t = +1 \) that is the space of symmetric potentials; if \( t = -1 \) then we get the space of skew-symmetric potentials.

For any real \( t \neq 0 \) we have the following analog of Theorem 58 in [5] (more general result is given in Theorem 10 below).

**Theorem 1.** Let

\[
L = L^0 + v(x), \quad L^0 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx}, \quad v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}
\]

be an \( X_t \)-periodic Dirac operator (i.e., \( P \) and \( Q \) are periodic \( L^2([0, \pi]) \)-functions such that \( Q(x) = tP(x) \)), and let \( \gamma = (\gamma_n)_{n \in \mathbb{Z}} \) be its gap sequence. If \( \Omega = (\Omega(n))_{n \in \mathbb{Z}} \) is a sub-multiplicative weight such that

\[
\frac{\log \Omega(n)}{n} \searrow 0 \quad \text{as} \quad n \to \infty,
\]

then

\[
\gamma \in \ell^2(\mathbb{Z}, \Omega) \Rightarrow v \in H(\Omega).
\]
If $\Omega$ is a sub–multiplicative weight of exponential type, i.e.,
\begin{equation}
\lim_{n \to \infty} \frac{\log \Omega(n)}{n} > 0,
\end{equation}
then there exists $\varepsilon > 0$ such that
\begin{equation}
\gamma \in \ell^2(\mathbb{Z}, \Omega) \Rightarrow v \in H(e^{\varepsilon|n|}).
\end{equation}

For skew-symmetric potentials (i.e., when $t = -1$) Theorem 1 is proved in [22] (see Theorem 1.2 and Theorem 1.3 there). See more comments about results and proofs in [22] in Section 6 below.

In Section 4 we explain that if $v \in X$ then the system of root functions of the operator $L_{Per+}(v)$ has at most finitely many linearly independent associated functions and there exists a Riesz basis in $L^2([0, \pi], C)$, which consists of root functions. Theorem 13, which is analogous to Theorem 1 in [9], gives a necessary and sufficient conditions for existence of such Riesz bases for a wide class of potentials in $X$.

A real-valued $v$ is called finite-zone potential if there are only finitely many $k$ such that $\lambda_k^+ < \lambda_k^-$. S. P. Novikov [37] raised the question on density of finite-zone potentials. In 1977 V. A. Marchenko published an article [26] without proofs, where he gave an explicit construction of a sequence of finite-zone potentials $v_n$ which converges to a given potential $v$. In [25] new, simplified proofs were given. To some extent they have been inspired by the works of T. V. Misyura [29, 30, 31, 32] on 1D Dirac operators with periodic matrix potentials.

She considered (in equivalent form) the Dirac operators
\begin{equation}
L = iJ \frac{d}{dx} + v, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ P \\ Q \\ 0 \end{pmatrix}, \quad P, Q \in L^2_{loc}(\mathbb{R}), \quad v(x+\pi) = v(x),
\end{equation}
with a symmetric matrix potential $v$, i.e.,
\begin{equation}
Q(x) = P(x).
\end{equation}

As in the case of Schrödinger operator, $L$ generates a self-adjoint operator in the space $L^2(\mathbb{R}; C^2)$ of $C^2$-vector functions; its spectrum is continuous and consists of a sequence of intervals $[\lambda_k^- \, \lambda_k^+]$, $k \in \mathbb{Z}$, where
\[ \cdots < \lambda_{k-1}^+ < \lambda_k^- \leq \lambda_k^+ < \lambda_{k+1}^- < \cdots \]
are eigenvalues of the periodic b.v.p. $Y(\pi) = Y(0)$, $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ if $k$ is even, and of the anti-periodic b.v.p. $Y(\pi) = -Y(0)$ if $k$ is odd. As in [27] the comb domains
\[ G = \{ z : \text{Im} \, z > 0 \} \setminus \bigcup_{k \in \mathbb{Z}} [0, h_k] \]
and their conformal mappings onto the upper half-plane are the essential tool in [31, 32]; there is an one-to-one correspondence between potentials $v = \begin{pmatrix} 0 \\ P \\ 0 \\ 0 \end{pmatrix}$ of the Dirac operators and sequences of real numbers $h = (h_k)_{k \in \mathbb{Z}}$, $h_k \geq 0$, $\sum h_k^2 < \cdots$
∞, and points \( \{k\pi + i\tilde{h}_k\}, |\tilde{h}_k| \leq h_k \). Finite-zone potentials were shown to correspond to sequences with \( h_k = 0 \) for \( |k| \geq N, 0 \leq N < \infty \).

If the potential with (1.24) and (1.25) corresponds to the sequences \((h_k)\) and \((k\pi + i\tilde{h}_k)\) then the truncated sequences \((h^N_k)\) and \((k\pi + i\tilde{h}^N_k)\), where

\[
\begin{align*}
  h^N_k &= \begin{cases} h_k & 0 \leq |k| \leq N \\ 0 & |k| > N \end{cases} \\
  \tilde{h}^N_k &= \begin{cases} \tilde{h}_k & 0 \leq |k| \leq N \\ 0 & |k| > N \end{cases}
\end{align*}
\]

correspond to the \((2N + 2)\)-zone potential \( v_N(x) = \begin{pmatrix} 0 & P_N(x) \\ P_N(x) & 0 \end{pmatrix} \) and

\[
\|P - P_N\|_{L^2([0,\pi])} \leq \|h - h^N\| \cdot (1 + 2\|h - h^N\|)C(\|h\|)
\]

where \( C(x) = 16\sqrt{\pi}(1 + \pi^2/2)e^{7x}, x > 0 \).

If the potential \( v \) in (1.24) is not symmetric then the methods of [28] and [31, 32] can not be applied directly.

V. A. Tkachenko [41] considered skew-symmetric potentials \( v(x) = i \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \).

In this class he proved that finite-zone skew symmetric potentials are dense.

(Of course, in the non-symmetric case the notion of finite-zone potential should be properly adjusted. A potential \( v \in (1.24) \) is finite-zone if for all but finitely many \( n \in \mathbb{Z} \)

\[
\lambda^+_n = \lambda^-_n = \mu_n,
\]

where \( \mu_n \) is a Dirichlet eigenvalue such that \( |\mu_n - n| < 1/4 \).

In 2000 B. Mityagin [34] suggested (at least in the Schrödinger-Hill case) an approach to construction of potentials with prescribed tails of their spectral gap sequences. In particular, if the tails are zero sequences one gets finite-zone potentials. (With more careful analysis of the eigenvalues of the operator \( L \) this approach leads to construction of potentials – both for Schrödinger-Hill and Dirac operators – whose eigenfunction expansions do not converge in \( L^2 \). For details see [5, Theorem 71 and Section 5.2].

It turns out that the same method works for Dirac operators as well. Following the scheme of [34] B. Grebert and T. Kappele [17] proved the density of finite-zone potentials in the spaces \( H(\Omega) \) (see Definition 2 in Section 2) under the restriction \( H(\Omega) \subset H^a, \exists a > 0 \), where \( H^a \) is a Sobolev space; in general, the density of finite-zone potentials in the spaces \( H(\Omega) \) was proved by P. Djakov and B. Mityagin (see an announcement in [35], and a complete proof in [5, Theorem 70]).

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They wrote (see [17]): "To prove Theorem 1.1 ... we follow the approach used in [34]: as a set-up we take the Fourier block decomposition introduced first for the Hill operator in [20, 21] and used out subsequently for the Zaharov–Shabat operators in [15, 16]. Unlike in [34] where a contraction mapping argument was used to obtain the density results for the Hill operator, we get a short proof of Theorem 1.1 by applying the inverse function theorem in a straightforward way. As in [34], the main feature of the present proof is that it does not involve any results from the inverse spectral theory."
We explain in Section 5 that the proof of Theorem 70 in [5] as it is written there covers not only the general and symmetric cases but a broad range of linear and nonlinear families of potentials; certainly, among them is the space of skew-symmetric potentials \( v = i \begin{pmatrix} 0 & P \\ -P^* & 0 \end{pmatrix} \).

The finite-zone potential density results announced in [35] and proved in [5] for general potentials and symmetric potentials \( (v^* = v) \) could be extended immediately for skew-symmetric potentials and \( X_t \)-potentials as well if one notices that all the (non-linear) operators \( \Phi_N, A_N \) (see below (5.8) and (5.9)) act in the space of general potentials \( \{ v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, P, Q \in L^2([0, \pi]) \} \), in such a way that both \( X_{sym} = \{ v \in X : Q(x) = \overline{P(x)} \} \) and \( X_{skew-sym} = \{ v \in X : P(x) = iR(x), Q(x) = i\overline{R(x)} \} \) and any \( X_t \) are invariant for these operators.

2. Preliminaries

The Dirac operator (1.1), considered on the interval \([0, \pi]\) with periodic Per\(^+\), antiperiodic Per\(^-\) and Dirichlet Dir boundary conditions \( bc \)

\[
Per^\pm : y(\pi) = \pm y(0), \quad Dir : y_1(0) = y_2(0), \quad y_1(\pi) = y_2(\pi),
\]

with \( y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \), gives a rise of three operators \( L_{bc}(v), bc = Per^\pm, Dir \). Their spectra are discrete; moreover, the following holds.

**Lemma 2.** (Localization Lemma.) The spectra of \( L_{bc}(v), bc = Per^\pm, Dir \) are discrete. There is an \( N = N(v) \) such that the union \( \cup_{|n| > N} D_n \) of the discs \( D_n = \{ z : |z - n| < 1/4 \} \) contains all but finitely many of the eigenvalues of \( L_{bc}, bc = Per^\pm, Dir \) while the remaining finitely many eigenvalues are situated in the rectangle \( R_N = \{ z : |Re z|, |Im z| \leq N + 1/2 \} \).

Moreover, for \( |n| > N \) the disc \( D_n \) contains one Dirichlet eigenvalue \( \mu_n \) and two (counted with algebraic multiplicity) periodic (if \( n \) is even) or antiperiodic (if \( n \) is odd) eigenvalues \( \lambda_n^-, \lambda_n^+ \) (where \( Re \lambda_n^- < Re \lambda_n^+ \) or \( Re \lambda_n^- = Re \lambda_n^+ \) and \( Im \lambda_n^- \leq Im \lambda_n^+ \)).

See details and more general results about localization of these spectra in [35, 36] and [5, Section 1.6.].

Now, in view of Lemma 2 for \( |n| > N(v) \) the spectral gaps

\[
\gamma_n = |\lambda_n^+ - \lambda_n^-|
\]

and deviations

\[
\delta_n = |\mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-)|
\]
are well-defined.
Moreover, the localization Lemma 2 allows us to apply the Lyapunov–Schmidt projection method and reduce the eigenvalue equation \( L \psi = \lambda \psi \) for \( \lambda \in D_n \) to an eigenvalue equation in the two-dimensional space \( E_n^0 = \{ L^0 \psi = n \psi \} \) (see [3] Section 2.4).

This leads to the following (see in [5] the formulas (2.59)–(2.80) and Lemma 30).

**Lemma 3.** Let

\[
P(x) = \sum_{k \in \mathbb{Z}} p(k)e^{ikx}, \quad Q(x) = \sum_{k \in \mathbb{Z}} q(k)e^{ikx},
\]

and let

\[
S^{11} = \sum_{\nu=0}^{\infty} S^{11}_{2\nu+1}, \quad S^{22} = \sum_{\nu=0}^{\infty} S^{22}_{2\nu+1}, \quad S^{12} = \sum_{\nu=0}^{\infty} S^{12}_{2\nu}, \quad S^{21} = \sum_{\nu=0}^{\infty} S^{21}_{2\nu},
\]

where

\[
S^{11}_{2\nu+1} = \sum_{j_0,j_1,\ldots,j_{2\nu} \neq n} p(-n-j_0)q(j_0+j_1)p(-j_1-j_2)q(j_2+j_3)\cdots q(j_{2\nu}+n) \over \mu (n-j_0+z)(n-j_1+z)\cdots(n-j_{2\nu}+z),
\]

\[
S^{22}_{2\nu+1} = \sum_{i_0,i_1,\ldots,i_{2\nu} \neq n} q(n+i_0)p(-i_0-i_1)q(i_1+i_2)p(-i_2-i_3)\cdots p(-i_{2\nu}-n) \over \mu (n-i_0+z)(n-i_1+z)\cdots(n-i_{2\nu}+z),
\]

\[
S^{12}_0 = \langle V e^2_n, e^1_n \rangle = p(-2n), \quad S^{21}_0 = \langle V e^1_n, e^2_n \rangle = q(2n),
\]

and, for \( \nu = 1, 2, \ldots, \)

\[
S^{12}_{2\nu} = \sum_{j_1,\ldots,j_{2\nu} \neq n} p(-n-j_1)q(j_1+j_2)p(-j_2-j_3)q(j_3+j_4)\cdots p(-j_{2\nu}-n) \over \mu (n-j_1+z)(n-j_2+z)\cdots(n-j_{2\nu}+z),
\]

\[
S^{21}_{2\nu} = \sum_{j_1,\ldots,j_{2\nu} \neq n} q(n+j_1)p(-j_1-j_2)q(j_2+j_3)p(-j_3-j_4)\cdots q(j_{2\nu}+n) \over \mu (n-j_1+z)(n-j_2+z)\cdots(n-j_{2\nu}+z).
\]

(a) For large enough \( |n| \) the series in (2.3)–(2.5) converge absolutely and uniformly if \( |z| \leq 1 \), so \( S^{ij}(n, z, p, q) \) are analytic functions of \( z \) for \( |z| < 1 \).

(b) The number \( \lambda = n+z, \; |z| < 1/4 \), is a periodic (for even \( n \)) or antiperiodic (for odd \( n \)) eigenvalue of \( L \) if and only if \( z \) is an eigenvalue of the matrix

\[
\begin{bmatrix}
S^{11} & S^{12} \\
S^{21} & S^{22}
\end{bmatrix}.
\]

(c) The number \( \lambda = n+z^*, \; |z| < 1/4 \), is a periodic (for even \( n \)) or antiperiodic (for odd \( n \)) eigenvalue of \( L \) of geometric multiplicity 2 if and only if \( z^* \) is an eigenvalue of the matrix

\[
\begin{bmatrix}
S^{11} & S^{12} \\
S^{21} & S^{22}
\end{bmatrix}
\]

of geometric multiplicity 2.

Moreover, (2.3)–(2.8) imply immediately the following.
Lemma 4. (a) For any potential functions $P,Q$
\[ S^{11}(n,z;p,q) = S^{22}(n,z;p,q), \quad S^{21}(n,z;p,q) = S^{12}(n,z;p,q), \]
\[ S^{2i}(n,z,tp;sq) = t^\nu s^{\nu+1} S^{2i}(n,z;p,q), \quad S^{ij}(n,z,tp;sq) = t^\nu s^{\nu+1} S^{ij}(n,z;p,q), \quad j = 1,2. \]
(b) If $Q(x) = cP(x)$, $c$ real, then (2.9)-(2.11) imply
\[ S^{21}(n,z;p,q) = cS^{12}(n,z;p,q), \quad S^{ij}(n,z;p,q) = S^{ji}(n,z;p,q). \]
(c) In the case of skew-symmetric potentials $c = -1$, so
\[ S^{21}(n,z) = -S^{12}(n,z). \]

We set for convenience
\[ \alpha_n(z) := S^{11}(n,z) \quad \beta^+_n(z) := S^{21}(n,z), \quad \beta^-_n(z) := S^{12}(n,z). \]

Next we summarize some basic properties of $\alpha_n(z)$ and $\beta^\pm_n(z)$.

Proposition 5. (a) The functions $\alpha_n(z)$ and $\beta^\pm_n(z)$ depend analytically on $z$ for $|z| \leq 1$ and the following estimates hold:
\[ |\alpha_n(v;z)|, |\beta^\pm_n(v;z)| \leq C \left( E_{|n|}(r) + \frac{1}{\sqrt{|n|}} \right) \quad \text{for } |n| \geq n_0, \ |z| \leq \frac{1}{2} \]
and
\[ \left| \frac{\partial \alpha_n}{\partial z}(v;z), \frac{\partial \beta^\pm_n}{\partial z}(v;z) \right| \leq C \left( E_{|n|}(r) + \frac{1}{\sqrt{|n|}} \right) \quad \text{for } |n| \geq n_0, \ |z| \leq \frac{1}{4}, \]
where $r = (r(m))$, $r(m) = \max\{|p(z)|, q(z)|\}$, $C = C(\|r\|)$, $n_0 = n_0(r)$ and
\[ E_m(r) = \left( \sum_{|n| \geq m} |r(n)|^2 \right)^{1/2}. \]
(b) For large enough $n$, the number $\lambda = n + z, z \in D = \{ \zeta : |\zeta| \leq 1/4 \}$, is an eigenvalue of $L_{Per+}$ if and only if $z \in D$ satisfies the basic equation
\[ (z - \alpha_n(z))^2 = \beta^+_n(z)v\beta^-_n(z,v), \]
(c) For large enough $n$, the equation (2.17) has exactly two roots in $D$ counted with multiplicity.

Proof. Part (a) is proved in [5, Proposition 35]. Lemma 3 implies Part (b). By (2.15), $\sup_D |\alpha_n(z)| \to 0$ and $\sup_D |\beta^\pm_n(z)| \to 0$ as $n \to \infty$. Therefore, Part (c) follows from the Rouché theorem. \qed
In view of Lemma 2 for large enough $|n|$ the numbers

\[ z_n^\pm = \frac{\lambda_n^+ + \lambda_n^-}{2} - n \]

are well defined. The following estimate from above of $\gamma_n$ follows from Proposition 6 (see [5, Lemma 40]).

**Lemma 6.** For large enough $|n|

\[ \gamma_n = |\lambda_n^+ - \lambda_n^-| \leq (1 + \delta_n)(|\beta_n^- (z_n^*)| + |\beta_n^+ (z_n^*)|) \]

with $\delta_n \to 0$ as $|n| \to \infty$.

3. **Spectral gaps asymptotics and potential smoothness**

Let $X$ be the class of all Dirac potentials $v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$ such that

\[ \exists C, N > 0 : c^{-1}|\beta_n^\pm (z_n^*; v)| \leq |\beta_n^- (z_n^*; v)| \leq c|\beta_n^+ (z_n^*; v)|, \quad |n| > N. \]

**Lemma 7.** Suppose $v \in X$ and the set $M$ of all $n \in \mathbb{Z}$ such that $\beta_n^- (z_n^*; v) \neq 0$ and $\beta_n^+ (z_n^*; v) \neq 0$ is infinite. Let $K_n$ be the closed disc with center $z_n^*$ and radius $\gamma_n$, i.e. $K_n = \{ z : |z - z_n^*| \leq \gamma_n \}$. Then for all $n \in M$ with sufficiently large $|n|$ we have

\[ \frac{1}{2}|\beta_n^\pm (z_n^*; v)| \leq |\beta_n^\pm (z; v)| \leq 2|\beta_n^\pm (z_n^*; v)| \quad \forall z \in K_n, \]

where $c$ is the constant from (3.1).

**Proof.** In view of (2.16) in Proposition 5 if $z \in K_n$ then

\[ |\beta_n^\pm (z) - \beta_n^\pm (z_n^*)| \leq \varepsilon_n |z - z_n^*| \leq \varepsilon_n \gamma_n, \]

where $\varepsilon_n = C \left( \mathcal{E}_{|n|}(r) + \frac{1}{\sqrt{|n|}} \right) \to 0$ as $|n| \to \infty$. By Lemma 6 for large enough $|n|$ we have

\[ \gamma_n \leq 2 \left( |\beta_n^- (z_n^*)| + |\beta_n^+ (z_n^*)| \right). \]

Therefore,

\[ |\beta_n^\pm (z) - \beta_n^\pm (z_n^*)| \leq 2\varepsilon_n |\beta_n^- (z_n^*)| + |\beta_n^+ (z_n^*)| \leq 2\varepsilon_n (1 + c) |\beta_n^\pm (z_n^*)|, \]

which implies

\[ [1 - 2\varepsilon_n (1 + c)] |\beta_n^\pm (z)| \leq |\beta_n^\pm (z)| \leq [1 + 2\varepsilon_n (1 + c)] |\beta_n^\pm (z_n^*)|. \]

Since $\varepsilon_n \to 0$ as $|n| \to \infty$, (3.2) follows. \(\Box\)

**Proposition 8.** Suppose $v$ is a Dirac potential such that (3.1) holds. Then, for $|n| > N_0(v)$, the following two-sided estimates for $\gamma_n = |\lambda_n^+ - \lambda_n^-|$ hold:

\[ \frac{2\sqrt{c}}{1 + 4c} \left( |\beta_n^- (z_n^*; v)| + |\beta_n^+ (z_n^*; v)| \right) \leq \gamma_n \leq 2 \left( |\beta_n^- (z_n^*; v)| + |\beta_n^+ (z_n^*; v)| \right). \]
Proof. The estimate of $\gamma_n$ from above follows from Lemma 6.

In view of (3.1), $\beta^+_n(z_1^*; v)$ and $\beta^-_n(z_1^*; v)$ may vanish only simultaneously. Suppose that $\beta^+_n(z_1^*; v) \cdot \beta^-_n(z_1^*; v) \neq 0$ for infinitely many $n$ – for such $n$ we have $\gamma_n \neq 0$ due to Lemma 3(c). Then, by Lemma 49 in [5], there exists a sequence $\delta_n \downarrow 0$ such that, for large enough $|n|$, 

\[
\gamma_n \geq \left( \frac{2\sqrt{t_n} - \delta_n}{1 + t_n} \right) \left( |\beta^-_n(z_1^*)| + |\beta^+_n(z_1^*)| \right),
\]

where $\delta_n \to 0$ as $|n| \to \infty$ and 

\[
t_n = |\beta^+_n(z_1^*)|/|\beta^-_n(z_1^*)|, \quad z_1^* = \lambda_1^+ - n.
\]

In view of (3.2) in Lemma 7, for large enough $|n|$ we have $1/(4c) \leq t_n \leq 4c$. Therefore, by (3.4), 

\[
\gamma_n \geq \left( \frac{2\sqrt{4c} - \delta_n}{1 + 4c} \right) \left( |\beta^-_n(z_1^*)| + |\beta^+_n(z_1^*)| \right),
\]

which implies (since $\delta_n \to 0$) the left inequality in (3.3). This completes the proof. \hfill \Box

**Corollary 9.** If $v \in X$ then the operators $L_{\text{Per}^\pm}$ have at most finitely many eigenvalues of algebraic multiplicity 2 but geometric multiplicity 1.

**Proof.** Indeed, the estimate in (3.3) imply that for large enough $|n|$ the number $\lambda_n^* = \lambda_1^+ = \lambda_1^-$ is a double eigenvalue if and only if $\beta^+_n(z_1^*) = \beta^-_n(z_1^*) = 0$. But then, in view of (2.18), the number $z_1^*$ is a double root of the basic equation (2.17), so it is a double eigenvalue of the matrix 

\[
\begin{pmatrix}
\alpha_n(z_1^*) & \beta^-_n(z_1^*) \\
\beta^+_n(z_1^*) & \alpha_n(z_1^*)
\end{pmatrix}
\]

of geometric multiplicity 2 because the off-diagonal elements are zeros. By Lemma 3, the number $\lambda_n^* = z_1^* + n$ is a double eigenvalue of $L(v)$ of geometric multiplicity 2 (periodic for even $n$ or antiperiodic for odd $n$), so the corresponding two-dimensional invariant subspace consists of eigenvectors only. \hfill \Box

A sequence of positive numbers 

\[
\Omega(m), m \in \mathbb{Z}, \quad \Omega(-m) = \Omega(m),
\]

is called submultiplicative weight sequence (or submultiplicatve weight) if 

\[
\Omega(n + m) \leq \Omega(n)\Omega(m), \quad n, m \in \mathbb{Z}.
\]

For any submultiplicative weight we define the Hilbert sequence space

\[
\ell^2(\Omega) = \{(x_k)_{k \in \mathbb{Z}} : \sum_k |x_k|^2(\Omega(k))^2 < \infty\}
\]

and the functional space

\[
(3.5) \quad H(\Omega) = \{f = \sum f_k e^{2\pi ikx} : \sum_k |f_k|^2(\Omega(k))^2 < \infty\}.
\]
We consider also the weighted Hilbert space of potentials
\[ H_D(\Omega) = \left\{ v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} : \ P, Q \in H(\Omega) \right\}. \]

By [5, Theorem 41], if \( \Omega \) is a submultiplicative weight, then
\[ v \in H_D(\Omega) \Rightarrow (\gamma_n) \in \ell^2(\Omega). \]
The converse implication
\[ (\gamma_n) \in \ell^2(\Omega) \Rightarrow v \in H_D(\Omega) \]
holds in the self-adjoint case where \( Q(x) = P(x) \) under some additional assumptions on \( \Omega \) (see Theorem 58 in [5]) but fails in general (see however Theorem 68 in [5]). The following statement extends the validity of (3.8) to the case where \( v \in X \).

**Theorem 10.** Let
\[ L = L^0 + v(x), \quad L^0 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx}, \quad v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix} \]
be a Dirac operator with potential \( v \in X \) and let \( \gamma = (\gamma_n)_{n \in \mathbb{Z}} \) be its gap sequence. If \( \Omega = (\Omega(n))_{n \in \mathbb{Z}} \) is a sub–multiplicative weight such that
\[ \frac{\log \Omega(n)}{n} \searrow 0 \quad \text{as} \quad n \to \infty, \]
then
\[ \gamma \in \ell^2(\mathbb{Z}, \Omega) \Rightarrow v \in H(\Omega). \]
If \( \Omega \) is a sub–multiplicative weight of exponential type, i.e.,
\[ \lim_{n \to \infty} \frac{\log \Omega(n)}{n} > 0, \]
then there exists \( \varepsilon > 0 \) such that
\[ \gamma \in \ell^2(\mathbb{Z}, \Omega) \Rightarrow v \in H(e^{\varepsilon|n|}). \]

**Proof.** In view of Proposition 8 if \( \gamma = (\gamma_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{Z}, \Omega) \) then we have
\[ (|\beta_n^-(v, z_n^*)| + |\beta_n^+(v, z_n^*)|)_{|n| > N} \in \ell^2(\Omega). \]

In other notations,
\[ \gamma \in \ell^2(\mathbb{Z}, \Omega) \Rightarrow A_N(v) \in H(\Omega), \]
where (compare with [5] (3.52)–(3.54)) the nonlinear operators \( A_N \) are defined by
\[ A_N(v) = v + \Phi_N(v), \quad \Phi_n(v) = \begin{pmatrix} 0 & \Phi_{12}^N(v) \\ \Phi_{21}^N(v) & 0 \end{pmatrix} \]
with
\[ \Phi_{12}^N = \sum_{|n| > N} (\beta_n^-(z_n^*, v) - p(-n))e^{-2inx} \quad \text{and} \quad \Phi_{21}^N = \sum_{|n| > N} (\beta_n^+(z_n^*, v) - q(n))e^{2inx}. \]
Now Theorem 10 follows from [5, Lemma 48 and Proposition 57] in the same way as [5, Theorem 58] (self-adjoint case) – namely, by (3.13) and Lemma 48 there is a slowly growing weight $\Omega_1$ such that $A_N(v) \in H(\Omega \cdot \Omega_1)$, so Proposition 57 implies $v \in H(\Omega \cdot \Omega_1) \subset H(\Omega)$.

\[\square\]

4. Riesz bases

Let $H$ be a Hilbert space. A family of bounded finite-dimensional projections $\{P_\gamma : H \to H, \gamma \in \Gamma\}$ is called basis of projections if

\begin{align*}
(4.1) & \quad P_\alpha P_\beta = 0 \quad \text{if} \quad \alpha \neq \beta; \\
(4.2) & \quad x = \sum_{\gamma \in \Gamma} P_\gamma(x) \quad \forall x \in H,
\end{align*}

where the series converge in $H$.

If $(Q_\gamma)$ is a basis of orthogonal projections (i.e., $Q_\gamma^* = Q_\gamma$), the Pythagorian theorem implies $\sum_\gamma \|Q_\gamma x\|^2 = \|x\|^2$.

A family of projections $(P_\gamma, \gamma \in \Gamma)$ is called Riesz basis of projections if

\begin{equation}
(4.3) \quad P_\gamma = A Q_\gamma A^{-1}, \quad \gamma \in \Gamma,
\end{equation}

where $A : H \to H$ is an isomorphism and $(Q_\gamma, \gamma \in \Gamma)$ is a basis of orthogonal projections.

It is well known (see G-K) that a basis of projections $(P_\gamma, \gamma \in \Gamma)$ is a Riesz basis of projections if and only if there are constants $a, b > 0$ such that

\begin{equation}
(4.4) \quad a \|x\|^2 \leq \sum_\gamma \|P_\gamma x\|^2 \leq b \|x\|^2 \quad x \in H
\end{equation}

(equivalently, if and only if the family $\{P_\gamma, \gamma \in \Gamma\}$ is orthogonal with respect to an equivalent Hilbert norm).

A family of vectors $\{f_\gamma, \gamma \in \Gamma\}$ is called a basis in $H$ if

\begin{equation}
(4.5) \quad x = \sum_{\gamma \in \Gamma} c_\gamma(x) f_\gamma \quad \forall x \in H,
\end{equation}

where the series converge in $H$ and the scalars $c_\gamma(x)$ are uniquely determined.

Obviously, if $(f_\gamma)$ is a basis in $H$ then the system of one-dimensional projections $P_\gamma(x) = c_\gamma(x) f_\gamma$ is a basis of projections in $H$, and vice versa, every basis of one dimensional projections can be obtained in that way from some basis of vectors.

A system of vectors $\{f_\gamma, \gamma \in \Gamma\}$ is called Riesz basis in $H$ if it has the form

\begin{equation}
(4.6) \quad f_\gamma = A e_\gamma, \quad \gamma \in \Gamma,
\end{equation}

where $A$ is an isomorphism $A : H \to H$ and $e_\gamma, \gamma \in \Gamma$ is an orthonormal basis in $H$. 
A basis \( \{ f_\gamma, \gamma \in \Gamma \} \) is a Riesz basis if and only if there are constants \( a, b, c, C > 0 \) such that
\[
(4.7) \quad c \leq \| f_\gamma \| \leq C \quad \forall \gamma \in \Gamma, \quad a \| x \|^2 \leq \sum_\gamma (c_\gamma(x))^2 \leq b \| x \|^2, \quad x \in H
\]
(equivalently, if and only if the family \( \{ f_\gamma, \gamma \in \Gamma \} \) is orthogonal with respect to an equivalent Hilbert norm and \( 0 < \inf \| f_\gamma \|, \sup \| f_\gamma \| < \infty \).

**Lemma 11.** Let \( (P_\gamma, \gamma \in \Gamma) \) be a Riesz basis of two-dimensional projections in a Hilbert space \( H \), and let \( f_\gamma, g_\gamma \in \text{Ran} P_\gamma, \gamma \in \Gamma \) are linearly independent unit vectors. Then the system \( \{ f_\gamma, g_\gamma, \gamma \in \Gamma \} \) is a Riesz basis if and only if
\[
(4.8) \quad \kappa := \sup |\langle f_\gamma, g_\gamma \rangle| < 1.
\]

**Proof.** If the system \( \{ f_\gamma, g_\gamma, \gamma \in \Gamma \} \) is a Riesz basis in \( H \), then
\[
x = \sum_\gamma (f_\gamma^*(x)f_\gamma + g_\gamma^*(x)g_\gamma), \quad x \in H,
\]
where \( f_\gamma^*, g_\gamma^* \) are the conjugate functionals. In view of (4.7), the one-dimensional projections
\[
P_\gamma^1(x) = f_\gamma^*(x)f_\gamma, \quad P_\gamma^2(x) = g_\gamma^*(x)g_\gamma
\]
are uniformly bounded. On the other hand, it is easy to see that
\[
\| P_\gamma^1 \|^2 \geq (1 - |\langle f_\gamma, g_\gamma \rangle|^2)^{-1}, \quad \| P_\gamma^2 \|^2 \geq (1 - |\langle f_\gamma, g_\gamma \rangle|^2)^{-1},
\]
so (4.8) holds.

Conversely, suppose (4.8) holds. Then we have for every \( \gamma \in \Gamma \)
\[
(1 - \kappa) (|f_\gamma^*(x)|^2 + |g_\gamma^*(x)|^2) \leq \| P_\gamma(x) \|^2 \leq (1 + \kappa) (|f_\gamma^*(x)|^2 + |g_\gamma^*(x)|^2)
\]
which implies, in view of (4.4),
\[
\frac{a}{1 + \kappa} \| x \|^2 \leq \sum_\gamma (|f_\gamma^*(x)|^2 + |g_\gamma^*(x)|^2) \leq \frac{b}{1 - \kappa} \| x \|^2.
\]
Therefore, (4.8) holds, which means that the system \( \{ f_\gamma, g_\gamma, \gamma \in \Gamma \} \) is a Riesz basis in \( H \).

In view of Lemma 2, the Dirac operators (1.1) with \( L^2 \)-potentials
\[
v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}, \quad P, Q \in L^2([0, \pi]),
\]
considered on \([0, \pi]\) with periodic or antiperiodic boundary conditions have discrete spectra, and the Riesz projections
\[
(4.9) \quad S_N = \frac{1}{2\pi i} \int_{\partial R_N} (z - L_{P\text{er}^\pm})^{-1}dz, \quad P_n = \frac{1}{2\pi i} \int_{|z-n|=\frac{1}{N}} (z - L_{P\text{er}^\pm})^{-1}dz
\]
are well–defined for \( |n| \geq N \) if \( N \) is sufficiently large.

By [7] Theorem 3),
\[
(4.10) \quad \sum_{|n| > N} \| P_n - P_n^0 \|^2 < \infty,
\]
where $P_n^0$, $n \in \mathbb{Z}$, are the Riesz projections of the free operator. Moreover, the Bari–Markus criterion implies (see Theorem 9 in [7]) that the spectral Riesz decompositions

$$
(4.11) \quad f = S_N f + \sum_{|n| > N} P_n f, \quad \forall f \in L^2([0, \pi], \mathbb{C}^2),
$$

converge unconditionally. In other words, $\{S_N, P_n, |n| > N\}$ is a Riesz projection basis in the space $L^2([0, \pi], \mathbb{C}^2)$.

Each of the projections $P_n$, $|n| > N$, is two-dimensional, and if $v \in X$ then for large enough $N$ each two-dimensional block $\text{Ran} P_n$ consists of eigenfunctions only. In the next theorem, we show that if $v \in X$, then it is possible to build a Riesz basis of eigenfunctions in $H = \bigoplus_{|n| > N} \text{Ran}(P_n)$ by "splitting" two-dimensional blocks $\text{Ran}(P_n)$.

**Theorem 12.** If $v \in X$, i.e., if there is $c > 0$ such that for sufficiently large $|n|$ (where $n$ is even if $bc = \text{Per}^+$ or odd if $bc = \text{Per}^-$)\n
$$
(4.12) \quad c^{-1}|\beta_n^+(z_n^*; v)| \leq |\beta_n^+(z_n^*; v)| \leq c|\beta_n^+(z_n^*; v)|,
$$

then there exists a Riesz basis in $L^2([0, \pi], \mathbb{C}^2)$ which consists of eigenfunctions and at most finitely many associated functions of the operator $L_{\text{Per}^\pm}(v)$.

**Remark.** To avoid any confusion, let us emphasize that in Theorem 12 two independent theorems are stacked together: one for the case of periodic boundary conditions $\text{Per}^+$ (where we consider only even $n$), and another one for the case of antiperiodic boundary conditions $\text{Per}^-$ (where we consider only odd $n$).

**Proof.** Let $N$ be chosen so large that the formula (4.3) in Proposition 8 holds for $|n| > N$ (with a constant $c$ coming from (1.12)), and the range $\text{Ran}(P_n)$ consists of eigenfunctions only. In view of Corollary 9 such choice of $N$ is possible. Moreover, we may assume without loss of generality that $N$ is so large that the estimates (3.2) in Lemma 7 holds for $|n| > N$.

We have the following two cases:

(a) $\beta_n^-(z_n^* n) = \beta_n^+(z_n^*) = 0$;

(b) $\beta_n^-(z_n^*) \neq 0$, $\beta_n^+(z_n^*) \neq 0$.

In Case (a) it follows from (3.3) that $\gamma_n = 0$, so $\lambda_n^* = n + z_n^*$ is a double eigenvalue of geometric multiplicity two. In this case we choose eigenfunctions $f(n), g(n) \in \text{Ran}(P_n)$ so that

$$
(4.13) \quad \|f(n)\| = \|g(n)\| = 1, \quad \langle f(n), g(n) \rangle = 0.
$$

In Case (b) we have $\gamma_n \neq 0$ by Proposition 8 so $\lambda_n^-$ and $\lambda_n^+$ are simple eigenvalues. Now we choose corresponding eigenvectors $f(n), g(n) \in \text{Ran}(P_n)$ so that

$$
(4.14) \quad \|f(n)\| = \|g(n)\| = 1, \quad L_{\text{Per}^\pm}(v)f(n) = \lambda_n^- f(n), \quad L_{\text{Per}^\pm}(v)g(n) = \lambda_n^+ g(n).
$$

In view of (4.11), to prove the theorem it is enough to show that the system of eigenfunctions $\{f(n), g(n), |n| > N\}$ (where $n$ is even for $bc = \text{Per}^+$ and odd
for $bc = Per^-$) is a Riesz basis in the space $H = \bigoplus_{|n| > N} \text{Ran}(P_n)$. In view of (4.8) in Lemma 11 it is enough to check that

$$\sup_{|n| > N} |\langle f(n), g(n) \rangle| < 1.$$ 

Obviously, we need to consider only $n$ falling into Case (b). Let $M$ be the set of all (even for $bc = Per$ or odd for $bc = Per^-$) $n$ such that $|n| > N$ and (b) holds. Next we show that

$$\sup_M |\langle f(n), g(n) \rangle| < 1.$$ 

By Lemma 7 the quotient $\eta_n(z) = \beta_n^-(z)/\beta_n^+(z)$ is a well defined analytic function on a neighborhood of the disc $K_n = \{ z : |z - z_n^-| \leq \gamma_n \}$. Moreover, in view of (3.2) and (4.12), we have

$$\frac{1}{4c} \leq |\eta_n(z)| \leq 4c \quad \text{for} \quad n \in M, z \in K_n.$$ 

Since $\eta_n(z)$ does not vanish in $K_n$, there is an appropriate branch $\text{Log}$ of $\log z$ defined on a neighborhood of $\eta_n(K_n)$. We set

$$\text{Log} (\eta_n(z)) = \log |\eta_n(z)| + i\varphi_n(z);$$

then

$$\eta_n(z) = \beta_n^-(z)/\beta_n^+(z) = |\eta_n(z)| e^{i\varphi_n(z)}$$

so the square root $\sqrt{\beta_n^-(z)/\beta_n^+(z)}$ is a well defined as analytic function on a neighborhood of $K_n$ by

$$\sqrt{\beta_n^-(z)/\beta_n^+(z)} = \sqrt{|\eta_n(z)| e^{i\varphi_n(z)}}.$$

Now the basic equation (2.17) splits into the following two equations

$$z = \zeta_n^+(z) := a(n, z) + \beta_n^+(z) \sqrt{\beta_n^-(z)/\beta_n^+(z)},$$

$$z = \zeta_n^-(z) := a(n, z) - \beta_n^+(z) \sqrt{\beta_n^-(z)/\beta_n^+(z)}.$$ 

For large enough $n$, each of the equations (4.19) and (4.20) has exactly one root in the disc $D = \{ z : |z| < 1/4 \}$. Indeed, in view of (2.16),

$$\sup_{|z| \leq 1/2} |d\zeta_n^+/dz| \to 0 \quad \text{as} \quad n \to \infty.$$ 

Therefore, for large enough $n$ each of the functions $\zeta_n^\pm$ is a contraction on the disc $K_n$, which implies that each of the equations (4.19) and (4.20) has at most one root in the disc $K_n$. On the other hand, by Lemma 2 for large enough $n$ the basic equation has two simple roots in $K_n$, which implies that each of the equations (4.19) and (4.20) has exactly one root in the disc $K_n$. 
For large enough \( n \), let \( z_1(n) \) (respectively \( z_2(n) \)) be the only root of the equation (4.19) (respectively (4.20)) in the disc \( D \). Of course, we have either \( z_1(n) = \lambda_n^+ - n \), \( z_2(n) = \lambda_n^+ - n \) or \( z_1(n) = \lambda_n^+ - n \), \( z_2(n) = \lambda_n^- - n \). Therefore,

\[
|z_1(n) - z_2(n)| = \gamma_n = |\lambda_n^+ - \lambda_n^-|.
\]

We set

\[
f^0(n) = P_n^0 f(n), \quad g^0(n) = P_n^0 g(n).
\]

From (4.10) it follows that \( \|P_n - P_n^0\| \to 0 \). Therefore,

\[
\|f(n) - f^0(n)\| = \|(P_n - P_n^0)f(n)\| \leq \|P_n - P_n^0\| \to 0
\]

and \( \|g(n) - g^0(n)\| \to 0 \), \( \|f(n) - f^0(n), g(n) - g^0(n)\| \to 0 \). Since \( \|f(n)\|^2 = \|f^0(n)\|^2 + \|f(n) - f^0(n)\|^2 \) and \( \langle f(n), g(n) \rangle = \langle f^0(n), g^0(n) \rangle + \langle f(n) - f^0(n), g(n) - g^0(n) \rangle \), we get

\[
\|f^0(n)\|, \|g^0(n)\| \to 1, \quad \limsup_{n \to \infty} \langle f(n), g(n) \rangle = \limsup_{n \to \infty} \langle f^0(n), g^0(n) \rangle.
\]

Then, by [5, Lemma 21] (see formula (2.4)), \( f^0(n) \) is an eigenvector of the matrix

\[
\begin{pmatrix}
\alpha_n(z_1) & \beta_n^+(z_1) \\
\beta_n^-(z_1) & \alpha_n(z_1)
\end{pmatrix}
\]

corresponding to its eigenvalue \( z_1 = z_1(n) \), i.e.,

\[
\begin{pmatrix}
\alpha_n(z_1) - z_1 & \beta_n^+(z_1) \\
\beta_n^-(z_1) & \alpha_n(z_1) - z_1
\end{pmatrix}
\begin{pmatrix}
1 \\
f^0(n)
\end{pmatrix} = 0.
\]

Therefore, \( f^0(n) \) is proportional to the vector \( \left(1, \frac{z_1 - \alpha_n(z_1)}{\beta_n^+(z_1)} \right)^T \). Taking into account (4.17), (4.18) and (4.19) we obtain

\[
f^0(n) = \frac{\|f^0(n)\|}{\sqrt{1 + |\eta_n(z_1)|}} \begin{pmatrix} 1 \\ \sqrt{\eta_n(z_1)} e^{\frac{i}{2} \varphi(z_1)} \end{pmatrix}.
\]

In an analogous way, from (4.17), (4.18) and (4.20) it follows

\[
g^0(n) = \frac{\|g^0(n)\|}{\sqrt{1 + |\eta_n(z_2)|}} \begin{pmatrix} 1 \\ -\sqrt{\eta_n(z_2)} e^{\frac{i}{2} \varphi(z_2)} \end{pmatrix}.
\]

Now, (4.24) and (4.25) imply

\[
\langle f^0(n), g^0(n) \rangle = \|f^0(n)\| \|g^0(n)\| \frac{1 - \sqrt{|\eta_n(z_1)| |\eta_n(z_2)|} e^{i \psi_n}}{\sqrt{1 + |\eta_n(z_1)|} \sqrt{1 + |\eta_n(z_2)|}},
\]

where

\[
\psi_n = \frac{1}{2} (\varphi_n(z_1(n)) - \varphi_n(z_2(n))).
\]

Next we explain that

\[
\psi_n \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \( \varphi_n = \text{Im} (\text{Log} \eta_n) \) we obtain, taking into account (4.21),

\[
|\varphi_n(z_1(n)) - \varphi_n(z_2(n))| \leq \sup_{|z_1, z_2|} \left| \frac{d}{dz} (\text{Log} \eta_n) \right| \cdot \gamma_n,
\]

where \([z_1, z_2]\) denotes the segment with end points \( z_1 = z_1(n) \) and \( z_2 = z_2(n) \).
By (2.16) in Proposition 5 and (3.2) in Lemma 7 we estimate
\[ \frac{d}{dz} (\log \eta_n) = \frac{1}{\beta_n^-(z)} \frac{d\beta_n^-(z)}{dz} - \frac{1}{\beta_n^+(z)} \frac{d\beta_n^+(z)}{dz}, \quad z \in [z_1, z_2], \]
as follows:
\[ \left| \frac{d}{dz} (\log \eta_n) \right| \leq \frac{\varepsilon_n}{|\beta_n^-(z_n^*)|} + \frac{\varepsilon_n}{|\beta_n^+(z_n^*)|} \]
where \( \varepsilon_n = C \left( \mathcal{E}_{|n|}(r) + \frac{1}{\sqrt{|n|}} \right) \to 0 \) as \( n \to \infty \). Therefore, from (4.12) and (3.3) it follows
\[ |\varphi_n(z_1(n)) - \varphi_n(z_2(n))| \leq 4(1 + c) \cdot \varepsilon_n \to 0, \]
i.e., (4.27) holds.

From (4.26) it follows
(4.28) \[ |\langle f^0(n), g^0(n) \rangle|^2 = \|f^0(n)\|^2 \|g^0(n)\|^2 \cdot \Pi_n, \]
with
(4.29) \[ \Pi_n = \frac{1 + |\eta_n(z_1)| |\eta_n(z_2)| - 2 \sqrt{|\eta_n(z_1)| |\eta_n(z_2)|} \cos \psi_n}{(1 + |\eta_n(z_1)|)(1 + |\eta_n(z_2)|)} \]
If (4.12) holds, then (4.27) implies \( \cos \psi_n > 0 \) for large enough \( n \), so taking into account that \( \|f^0(n)\|, \|g^0(n)\| \leq 1 \), we obtain by (4.16)
\[ |\langle f^0(n), g^0(n) \rangle|^2 \leq \Pi_n \leq \frac{1 + |\eta_n(z_1)| |\eta_n(z_2)|}{(1 + |\eta_n(z_1)|)(1 + |\eta_n(z_2)|)} \leq \delta < 1 \]
with
\[ \delta = \sup \left\{ \frac{1 + xy}{(1 + x)(1 + y)} : \frac{1}{4c} \leq x, y \leq 4c \right\} = \frac{1 + 16c^2}{(1 + 4c)^2} \]
Now (4.23) implies that (4.15) holds, hence the system of normalized eigenfunctions and associated functions is a (Riesz) basis in \( L^2([0, \pi]) \). The proof is complete. \( \square \)

In fact, Theorem 12 says that (4.12) is a sufficient condition which guarantees
(i) the system of root functions of \( L_{P\epsilon^\pm}(v) \) is complete and has at most finitely many linearly independent associated functions;
(ii) there exists a Riesz basis in \( L^2([0, \pi], \mathbb{C}^2) \) which consists of root functions of the operator \( L_{P\epsilon^\pm}(v) \).

Besides the case \( v \in X_t \) (see the next section for definition of the class of potentials \( X_t \)) it seems difficult to verify the condition (4.12). Moreover, since the points \( z_n^* \) are not known in advance, in order to check (4.12) one has to consider the values of the functions \( \beta_n^\pm(z) \) for all \( z \) close to 0.

In the next theorem we consider potentials \( v \) such that for large enough \( |n| \)
(4.30) \[ \beta_n^-(0) \neq 0, \quad \beta_n^+(0) \neq 0 \]
and
(4.31) \[ \exists d > 0 : \quad d^{-1} |\beta_n^+(0)| \leq |\beta_n^+(z)| \leq d |\beta_n^+(0)| \quad \forall z \in K_n = \{ z : |z - z_n^*| \leq \gamma_n \} \]
(notice that $K_n$ consists of one point only if $\gamma_n = 0$). Then (i) holds, and moreover, the condition (4.12) is necessary and sufficient for existence of Riesz bases consisting of root functions of the operator $L_{Per}(v)$.

**Theorem 13.** Suppose $v$ is a Dirac potential such that (4.30) and (4.31) hold. Then

(a) the system of root functions of $L_{Per}(v)$ is complete and has at most finitely many linearly independent associated functions;

(b) if

$$0 < a := \liminf \frac{\beta_n^+(0)}{\beta_n^-(0)}, \quad b := \limsup \frac{\beta_n^+(0)}{\beta_n^-(0)} < \infty,$$

where $n$ is even if $bc = Per^+$ or odd if $bc = Per^-$, then there exists a Riesz basis in $L^2([0, \pi], \mathbb{C}^2)$ which consists of root functions of the operator $L_{Per}(v)$;

(c) if (4.32) fails, then there is no basis in $L^2([0, \pi], \mathbb{C}^2)$ consisting of root functions of the operator $L_{Per}(v)$.

**Remark.** Although the conditions (4.30)–(4.32) look too technical there is – after [5, 6] – a well elaborated technique to evaluate these parameters and check these conditions. To compare with the case of Hill operators with trigonometric polynomial coefficients – see [8, 9].

**Proof.** By Proposition 5, the basic equation

$$\left( z - \alpha_n(z) \right)^2 = \beta_n^+(z)\beta_n^-(z),$$

has exactly two roots (counted with multiplicity) in the disc $D = \{ z : |z| < 1/4 \}$. Therefore, a number $\lambda = n + z$ with $z \in D$ is a periodic or antiperiodic eigenvalue of algebraic multiplicity two if and only if $z \in D$ satisfies the system of two equations (4.33) and

$$2(z - \alpha_n(z)) \frac{d}{dz} (z - \alpha_n(z)) = \frac{d}{dz} \left( \beta_n^+(z)\beta_n^-(z) \right).$$

In view of [7, Theorem 9], the system of root functions of the operator $L_{Per}(v)$ is complete, so Part (a) of the theorem will be proved if we show that there are at most finitely many $n$ such that the system (4.33), (4.34) has a solution $z \in D$.

Suppose $z^* \in D$ satisfies (4.33) and (4.34); then it follows $z^* \in K_n$. By (2.16), for each $z \in D$

$$\left| \frac{d\alpha_n}{dz}(z) \right| \leq \varepsilon_n, \quad \left| \frac{d\beta_n^+}{dz}(z) \right| \leq \varepsilon_n \quad \text{with} \quad \varepsilon_n \to 0 \quad \text{as} \quad |n| \to \infty.$$

In view of (4.35), the equation (4.34) implies

$$2 |z^* - \alpha_n(z^*)| (1 - \varepsilon_n) \leq \varepsilon_n \left( |\beta_n^+(z^*)| + |\beta_n^-(z^*)| \right).$$

By (4.33),

$$|z^* - \alpha_n(z^*)| = |\beta_n^+(z^*)\beta_n^-(z^*)|^{1/2},$$
so it follows, in view of (4.31),
\[ 2(1 - \varepsilon_n) \leq \varepsilon_n \left( \frac{\beta_n^+(z^*)}{\beta_n^-(z^*)} \right)^{1/2} + \left| \frac{\beta_n^-(z^*)}{\beta_n^+(z^*)} \right|^{1/2} \leq 2d\varepsilon_n. \]

Since \( \varepsilon_n \to 0 \) as \( |n| \to \infty \), the latter inequality holds for at most finitely many \( n \), which completes the proof of (a).

If (4.32) holds, then by Theorem 12 there exists a Riesz bases in \( L^2([0, \pi], \mathbb{C}^2) \) which consists of root functions of the operator \( L_{Per^\pm}(v) \), i.e., (b) holds.

Next, we show that if (4.32) fails then there is no bases in \( L^2([0, \pi], \mathbb{C}^2) \) which consists of root functions of the operator \( L_{Per^\pm}(v) \).

By (a) and Lemma 2 for large enough \( |n| \), say \( |n| > N \) there are two simple (periodic for even \( n \) and antiperiodic for odd \( n \)) eigenvalues \( \lambda_n^- \) and \( \lambda_n^+ \) close to \( n \). Let us choose corresponding unit eigenfunctions \( f(n) \) and \( g(n) \), i.e.,
\[ ||f(n)|| = ||g(n)|| = 1, \quad L_{Per^\pm}(v)f(n) = \lambda_n^- f(n), \quad L_{Per^\pm}(v)g(n) = \lambda_n^+ g(n). \]

The same argument as in the proof of Theorem 12 shows that there is a bases in \( L^2([0, \pi], \mathbb{C}^2) \) which consists of root functions of the operator \( L_{Per^\pm}(v) \) if and only if
\[ \sup\{|\langle f(n), g(n) \rangle | : |n| > N \} < 1, \]
where we consider even \( n \) for periodic boundary conditions \( bc = Per^+ \) or odd \( n \) for antiperiodic boundary conditions \( bc = Per^- \).

By Lemma (4.31) the quotient \( \eta_n(z) = \beta_n^-(z)/\beta_n^+(z) \) is a well defined analytic function on a neighborhood of the disc \( \bar{D} \) which does not vanishes on \( \bar{D} \). Therefore, there is an appropriate branch (depending on \( n \)) \( \log \) of \( \log z \) defined in a neighborhood of \( \eta_n(\bar{D}) \). We set
\[ \log(\eta_n(n)) = \log |\eta_n(z)| + i\varphi_n(z); \]
then (4.17) holds.

Further we follow the proof of Theorem 12 after formula (4.17). With \( f^0(n) \) and \( g^0(n) \) given by (1.22) the formulas (4.23)–(4.26) and (4.28), (4.29) hold. In view of (4.30) and (4.31), if (4.32) fails then
\[ \liminf_{K_n} \left( \inf_{K_n} |\eta_n(z)| \right) = 0 \quad \text{or} \quad \limsup_{K_n} \left( \sup_{K_n} |\eta_n(z)| \right) = \infty. \]

By (4.23), it follows \( \limsup \Pi_n = 1 \), so (4.27) and (4.28) imply
\[ \limsup\{|\langle f(n), g(n) \rangle | : |n| > N \} = 1, \]
i.e., (4.37) fails. Therefore, if (4.32) fails there is no bases in \( L^2([0, \pi], \mathbb{C}^2) \) consisting of root functions of the operator \( L_{Per^\pm}(v) \), i.e., (c) holds. This completes the proof. \( \square \)

**Example 14.** If \( a, b, A, B \in \mathbb{C} \setminus \{0\} \) and
\[ v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \quad \text{with} \quad P(x) = ae^{2ix} + be^{-2ix}, \quad Q(x) = Ae^{2ix} + Be^{-2ix}, \]
then the system of root functions of $L_{\text{Per}^\pm}(v)$ consists eventually of eigenfunctions.

Moreover, for $bc = \text{Per}^-$ this system is a Riesz basis in $L^2([0,\pi],[C^2]$ if $|aA| = |bB|$, and it is not a basis if $|aA| \neq |bB|$.

For $bc = \text{Per}^+$ the system of root functions is a Riesz basis always.

Let us mention that if $bc = \text{Per}^+$ then it is easy to see by (2.14), (2.6) and (2.8) that

$$\beta_n^+(0) = A_n^{n+1} a^{-\frac{n-1}{2}} 4^{-n+1} \left( \frac{n-1}{2} \right)! \left( 1 + O(1/\sqrt{|n|}) \right),$$

(4.40)

$$\beta_n^-(0) = b_n^{n+1} B_n^{-\frac{n-1}{2}} 4^{-n+1} \left( \frac{n-1}{2} \right)! \left( 1 + O(1/\sqrt{|n|}) \right).$$

(4.41)

Proofs of (4.40), (4.41) and similar asymptotics, related to other trigonometric polynomial potentials and implying Riesz bases existence or non-existence, will be given elsewhere (see similar results for the Hill-Schrödinger operator in [8, 9]).

5. Density of finite zone potentials in the class $X_t$

Consider the classes of Dirac potentials

$$X_t = \left\{ v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, \quad Q(x) = tP(x), \quad P, Q \in L^2([0,\pi]) \right\}, \quad t \in \mathbb{R} \setminus \{0\}.$$  

(5.1)

If $t = 1$ we get the class $X_1$ of symmetric Dirac potentials (which generate self-adjoint Dirac operators), and $X_{-1}$ is the class of skew-symmetric Dirac potentials. In this section we show that

$$X_t \subset X \quad \forall t \in \mathbb{R} \setminus \{0\},$$

(5.2)

and prove that finite-zone $X_t$-potentials are dense in $X_t$ for real $t \neq 0$.

**Lemma 15.** (a) The Dirac operators $L_{\text{Per}^\pm}$ with potentials $v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$ and $v_c = \begin{pmatrix} 0 & cP \\ \frac{1}{c}Q & 0 \end{pmatrix}, \quad c \in \mathbb{C} \setminus \{0\}$, are similar. Therefore, $Sp(L_{\text{Per}^\pm}(v_c))$ does not depend on $c$.

(b) $(L_{\text{Per}^\pm}(v))^* = L_{\text{Per}^\pm}(v^*)$, $v^* = \begin{pmatrix} 0 & \overline{Q} \\ P & 0 \end{pmatrix}$.

(c) If $t \neq 0$ is real and $v \in X_t$ then $v^* = v_t$, so

$$Sp[(L_{\text{Per}^\pm}(v))^*] = Sp(L_{\text{Per}^\pm}(v)).$$

(5.3)

**Proof.** Let $C = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$; then $C^{-1} = \begin{pmatrix} 1/c & 0 \\ 0 & 1 \end{pmatrix}$, and we have

$$CL(v)C^{-1} = iCJD C^{-1} + Cv C^{-1} = iJD + v_c = L(v_c).$$
Moreover, if \( G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \) satisfies periodic (or antiperiodic) boundary conditions, \( CG = \begin{pmatrix} cg_1 \\ cg_2 \end{pmatrix} \) satisfies the same boundary conditions, and vice versa. Thus, the operators \( L_{\text{Per}}^{\pm}(v_c) \) and \( L_{\text{Per}}^{\pm}(v) \) are similar.

Part (b) is standard.

Since \( v^* = \begin{pmatrix} 0 \\ Q \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{i}Q \\ 0 \end{pmatrix} = v_t \), (5.3) follows from Part (a).

\[ \square \]

If \( v \in X_t \) and \( c \neq 0 \) is real, then

\[ v_c = CvC^{-1} = \begin{pmatrix} 0 & cP \\ \frac{1}{i}P & 0 \end{pmatrix} \in X_{t/c^2}. \]

This observation and (5.3) lead to the following specification of Lemma 2 for potentials in the classes \( X_t \).

**Lemma 16.** (a) If \( v \in X_t \) with \( t > 0 \), then \( L_{\text{Per}}^{\pm}(v) \) is similar to a self-adjoint operator, so \( \text{Sp}(L_{\text{Per}}^{\pm}(v)) \subset \mathbb{R} \).

(b) If \( v \in X_t \) with \( t < 0 \), then there is an \( N = N(v) \) such that for \( |n| > N \) either

(i) \( \lambda^-_n \) and \( \lambda^+_n \) are simple eigenvalues and \( \lambda^+_n = \lambda^-_n \), \( \text{Im} \lambda^\pm_n \neq 0 \)

or (ii) \( \lambda^+_n = \lambda^-_n \) is a real eigenvalue of algebraic and geometric multiplicity 2.

**Proof.** In view of Lemma 15 and (5.4), considered with \( c = \sqrt{|t|} \), in case (a) the operator \( L_{\text{Per}}^{\pm}(v) \) is similar to a self-adjoint operator \( L_{\text{Per}}^{\pm}(v_1) \) with \( v_1 \in H_1 \).

The same argument shows that in case (b) we need to consider only the skew-symmetric case \( t = -1 \). By Lemma 2 there is an \( N = N(v) \) such that for \( |n| > N \) the disc \( D_n = \{ z : |z - n| < 1/4 \} \) contains exactly two (counted with algebraic multiplicity) periodic (for even \( n \)) or antiperiodic (for odd \( n \)) eigenvalues of the operator \( L_{\text{Per}}^{\pm} \). By (5.3) in Lemma 15 if \( \lambda \in D_n \) with \( \text{Im} \lambda \neq 0 \) is an eigenvalue of \( L_{\text{Per}}^{\pm} \) then \( \overline{\lambda} \in D_n \) is also an eigenvalue of \( L_{\text{Per}}^{\pm} \) and \( \overline{\lambda} \neq \lambda \), so \( \lambda \) and \( \overline{\lambda} \) are simple, i.e., (i) holds.

Suppose \( \lambda \in D_n \) is a real eigenvalue. We are going to show that \( \lambda \) is of geometric multiplicity two, i.e., (ii) holds.

Let \( \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \) be a corresponding (nonzero) eigenvector, i.e.,

\[ L \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda L \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \]

Passing to conjugates we obtain

\[ L \begin{pmatrix} \overline{w}_2 \\ -\overline{w}_1 \end{pmatrix} = \lambda L \begin{pmatrix} \overline{w}_2 \\ -\overline{w}_1 \end{pmatrix}, \]
i.e., \((\frac{w_2}{w_1}, -\frac{w_1}{w_2})\) is also an eigenvector corresponding to the same eigenvalue \(\lambda\). But 
\[
\left\langle \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} -w_1 \\ w_2 \end{pmatrix} \right\rangle = 0,
\]
so these vector-functions are orthogonal, and therefore, linearly independent. This completes the proof of Lemma 16. □

**Proposition 17.** Suppose \(v \in X_t\) with \(t \neq 0\) real. Then there is \(N = N(v)\) such that

\[
z_n^* = \frac{1}{2}(\lambda_n^- + \lambda_n^+) - n\quad \text{is real for } |n| > N.
\]

Moreover, for every real \(t \neq 0\)

\[
\beta_n^+(z_n^*, v) = t \cdot \beta_n^-(z_n^*, v),
\]

which implies \(v \in X\), i.e.,

\[
X_t \subset X.
\]

**Proof.** Suppose \(v \in X_t\) with \(t \neq 0\) real. Lemma 16 implies (5.5) immediately. In view of (2.14) and (5.5), it follows from Part (b) of Lemma 4, formula (2.12), that (5.6) holds. In view of (3.1) we obtain \(v \in X\), which completes the proof. □

In view of Theorem 12 and (5.7) we have

**Corollary 18.** If \(v \in X_t\) then there is a Riesz basis in \(L^2([0, \pi], \mathbb{C}^2)\) which consists of eigenfunctions and at most finitely many associated functions of the operator \(L_{\text{Per}}(v)\).

In view of Proposition 17, (5.5) and (5.6), for sufficiently large \(N\) the nonlinear operators (compare with [5, (3.52)-(3.54)])

\[
A_N(v) = v + \Phi_N(v), \quad \Phi_n(v) = \begin{pmatrix} 0 & \Phi_{12}^N \\ \Phi_{21}^{12}(v) & 0 \end{pmatrix}
\]

where

\[
\Phi_{12}^N = \sum_{|n| > N} (\beta_n^-(v, z_n^*) - p(-n))e^{-2inx} \quad \text{and} \quad \Phi_{21}^{12} = \sum_{|n| > N} (\beta_n^+(v, z_n^*) - q(n))e^{2inx},
\]

are well-defined, and

\[
v \in X_t \quad \Rightarrow \quad \Phi_N(v), A_N(v) \in X_t
\]

as well. Therefore, all constructions and proofs of [5, Section 3.4] for symmetric (self-adjoint) potentials become valid for any \(X_t\)-potential.

Moreover, in [5, Theorem 70] the density of finite-zone potentials is first proved for general Dirac potentials, and then the \(A_N\)-invariance of the space symmetric potentials is used (see Remark 56 therein) to derive that the symmetric finite-zone potentials are dense in any weighted space of symmetric potentials. So, without any need to repeat or reproduce hard analysis we can claim the following analog of [5, Theorem 70].
Theorem 19. If $Ω$ is a submultiplicative weight and $X_t(Ω) = X_t \cap H_D(Ω)$ is the corresponding Sobolev space of $X_t$-Dirac potentials, then the finite-zone $X_t$-potentials are dense in $X_t(Ω)$.

For skew-symmetric potentials (i.e., when $t = -1$) Theorem 19 is proved in [22] (see Corollary 1.1 there). See more comments about [22] in Appendix.

6. Appendix: remarks on the paper [22]

Presumably, T. Kappeler, F. Serier and P. Topalov (the authors of [22]) have not noticed that $X_{skew-sym}$ potentials are invariant for all operators $Φ_N$, $A_N$, etc. in [5], and they rewrote all technical constructions, lemma by lemma, inequality by inequality from [4] or [5] to justify analogs of [5, Theorems 58, 70] for skew-symmetric potentials. But such copying is done in [22] without specifying which lemmas and inequalities are rewritten and without explaining that the entire architecture of [4, 5] is reproduced.

Moreover, the main results of [22] follow immediately from Lemma 49 and Theorem 68 in [5] (or, from Lemmas 48, 49 and Proposition 57 in [5]) but this fact is not mentioned. Appendix is aimed to cover these gaps in [22], at least partially.

1. First, let us recall [5, Theorem 68].

Theorem 68 in [5]. If

$$L = L^0 + v(x), \quad L^0 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx}, \quad v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}$$

is a periodic Dirac operator with $L^2$-potential (i.e., $P$ and $Q$ are periodic $L^2([0,1])$-functions), then, for $|n| > n_0(v), n ∈ \mathbb{Z}$, the operator $L$ has, in the disk of center $n$ and radius $r = 1/4$, exactly two (counted with their multiplicity) periodic (for even $n$), or anti-periodic (for odd $n$) eigenvalues $λ^+_n$ and $λ^-_n$, and one Dirichlet eigenvalue $µ_n$.

Let

$$\Delta_n = |λ^+_n - λ^-_n| + |λ^+_n - µ_n|, \quad |n| > n_0;$$

then, for each sub–multiplicative weight $Ω$,

$$v ∈ H(Ω) \Rightarrow (Δ_n) ∈ ℓ^2(Ω).$$

Conversely, if $Ω = (Ω(n))_{n∈\mathbb{Z}}$ is a sub–multiplicative weight such that

$$\frac{\log Ω(n)}{n} \searrow 0 \quad \text{as} \quad n → ∞,$$

then

$$(Δ_n) ∈ ℓ^2(Ω) \Rightarrow v ∈ H(Ω).$$

If $Ω$ is a sub–multiplicative weight of exponential type, i.e.,

$$\lim_{n→∞} \frac{\log Ω(n)}{n} > 0$$
then
\[(\Delta_n) \in L^2(\Omega) \Rightarrow \exists \varepsilon > 0 : v \in H(\varepsilon|n|).\]

The main results in [22] – see Theorems 1.2 and 1.3 there – follow from [5, Theorem 68] because for skew-symmetric potentials \(v\)
\begin{equation}
\Delta_n \leq K\gamma_n, \quad |n| \geq N_1(v),
\end{equation}
where \(K\) is an absolute constant. Indeed, by [5, Theorem 66] we have for any \(L^2\) potential
\begin{equation}
\frac{1}{144}(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|) \leq \Delta_n \leq 54(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|), \quad |n| \geq N(v).
\end{equation}
On the other hand, since
\begin{equation}
|\beta_n^-(z_n^*, v)| = |\beta_n^+(z_n^*, v)| \quad \text{for skew-symmetric} \quad v,
\end{equation}
[5, Lemma 49] implies easily
\begin{equation}
\gamma_n \geq D(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|),
\end{equation}
where \(D\) is an absolute constant. Therefore, (6.7) holds for skew-symmetric potentials.

2. The authors of [22] did not say anything about the relation between [5, Theorem 68] and their main results but they explained (see p. 2087 in [22]) that they wrote a big portion in their 40 page long paper to fill a gap in the paper [5]. They write: ”The proof of Lemma 36 in [5] has a gap on p. 710 as Lemma 32 in [5] cannot be applied to the expression \(\Sigma_4(n)\) given by (2.117) of [5]. However it turns out that the method developed in [4] can be applied.”

Maybe in [5] not everything is explained letter-by-letter (which is common in mathematical research papers) but without any extra effort the expression \(\Sigma_4(n)\) given by (2.117) of [5] can be estimated by Lemma 32. Indeed, we have
\begin{equation}
\Sigma_4(n) = \langle \hat{V}\hat{D}\hat{T}^2(1 - \hat{T}^2)^{-1}\hat{T}\hat{V}e_n^1, e_n^2\rangle = \sum_{i,j \neq n} \frac{r(n + i)r(n + j)}{|n - i||n - j|} h_{ij}^{21}(n)
\end{equation}
with
\begin{equation}
h = \hat{T}^2(1 - \hat{T}^2)^{-1}\hat{V}.
\end{equation}
Therefore, if \(\Sigma_4(n)\) is written as (6.11) with \(h\) given by (6.12) then Lemma 32 immediately yields the inequality (2.122) on page 710 of [5].

3. Lemma 49 in [5] is essentially [2, Lemma 12] (its formulation and proof are the same for Dirac and Hill-Schrödinger operators). It plays a crucial role in getting estimates of \(\gamma_n\) from below in terms of \((|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|)\) (see [5, Section 3.2]) which leads to two-sided estimates of \(\gamma_n\) (see [5, Theorem 50]). In Section 7 of [22] (titled ”lower bound for \(\gamma_n\)”), Lemma 7.1 is almost identical to Lemma 49 in [5] or [2, Lemma 12] but the authors did not give any credit to the papers [2, 5].
Let us mention also that with (6.10) proven for skew-symmetric potentials the main results of [22] follow from [5, Lemma 48 and Proposition 57] in the same way as [5, Theorem 58] (self-adjoint case) – namely, if \((\gamma_n) \in \ell^2(\Omega)\), then by (6.10) and Lemma 48 there is a slowly growing weight \(\Omega_1\) such that \(A_N(v) \in H(\Omega \cdot \Omega_1)\), so [5, Proposition 57] implies \(v \in H(\Omega \cdot \Omega_1) \subset H(\Omega)\).

4. Theorems 1.1 and 4.1 in [22] claim (6.2), respectively, for skew-symmetric and arbitrary potentials \(v\). Of course, Theorem 1.1 is a partial case of Theorem 4.1 but the proof in both cases is the same. The authors of [22] write on p. 2075: "To make the paper self-contained we include for convenience of the reader a proof of Theorem 1.1." Although it is written in the introduction that, in the generality stated, Theorem 1.1 (or Theorem 4.1) is proven in [4, 5], it is not said that the proof of Theorem 1.1 in [22] is copied from there. In particular:

(i) Lemma 2.2 on p. 2083 in [22] (which is crucial for the proof of Lemmas 4.1 and 4.2, and therefore for the whole paper) reproduces Lemma 2 in [4] and its proof – see pp. 144–146 there – but no credit is given to [4].

(ii) Proposition 4.1 in [22] (which also provides the crucial a priori estimate in the proof of Theorems 1.2 and 1.3 [22]) reproduces Lemma 36 in [5].

(iii) Corollary 4.1, p. 2096 in [22] and its inequality is essentially the same as pp. 163–164 in [4], in particular Inequality (4.18).

5. Proposition 5.1, p. 2100 in [22], reproduces – with all essential steps in the proof – Lemma 55, p. 729 in [5]. One semantic remark, however, should be made. Instead of straightforward application of Banach-Caccioppoli contraction principle – as it has been done in [34] and explained and further used in [17] – the authors of [22] prefer to talk about Implicit Function Theorem for analytic diffeomorphisms. But an abstract approach – Fixed Point Theorem or Implicit Function Theorem – works only if some hard analysis is done, and the authors of [22] copy such analysis from [4, 5].

6. Proposition 6.2, p. 2103 in [22], reproduces [5, Proposition 57] but this fact is not mentioned in [22].

7. This comparative analysis could go on and on but at the end we have to mention ONE case when [22] give a (funny) credit to [5], see p. 2108, lines 1–4 from the bottom:

"To prove Theorem 1.2 and Theorem 1.3, we want to apply Theorem 8.1. The following lemma which can be found in [5, Lemma 48] allows to get rid of the weight function \(w\) in Theorem 8.1.

Lemma 9.1. If \(z = (z_k)_{k \in \mathbb{Z}}\), then there exists an unbounded slowly increasing weight \(w = (w(k))_{k \in \mathbb{Z}}\) such that \(z \in h^w\)."

Of course, the above statement is a version of a well known Calculus exercise: if a series of positive numbers \(\sum c_k\) converges then there is a sequence \(D_k \to \infty\) such that the series \(\sum c_k D_k\) converges as well.

This is the only lemma for which the authors of [22] give credit to [5] although they rewrote tens of other lemmas and inequalities and their proofs without
saying anything about the origin. But quite paradoxically this "credit" is very revealing. It shows that [22] borrow the entire structure of the proof from [4, 5] so even such a minor item as an elementary lemma (Lemma 48 in [5]) could not be omitted; without that brick the proofs of their main results would not exist.

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