A note on a prey-predator model with constant-effort harvesting*

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Abstract. We study a prey-predator model based on the classical Lotka–Volterra system with Leslie–Gower and Holling IV schemes and a constant-effort harvesting. Our goal is twofold: to present the model proposed by Cheng and Zhang in 2021, pointing out some inconsistencies; to analyse the number and type of equilibrium points of the model. We end by proving the stability of the meaningful equilibrium point, according to the distribution of the eigenvalues.

Keywords: prey-predator model; equilibria; stability; computer algebra system; SageMath.

MSC 2020: 34C60; 34D20; 92D25.

1 Introduction

Prey-predator equations describe an ecological system of two linked species that depend on each other. One is the prey, which provides food for the other, the predator. Under some conditions, both prey and predator populations grow. Lotka (1880–1949) studied such equations in his book of 1925 [4]; Volterra (1860–1940) investigated them, independently [6]; and, for this reason, such prey-predator equations are also known as Lotka–Volterra equations. Recently, there has been a tremendous amount of research done in this area [1].

Here we consider a prey-predator model with Leslie–Gower and Holling IV schemes with constant-effort harvesting, proposed by Cheng and Zhang in 2021 [2]. The Cheng–Zhang model is given by

\[
\begin{align*}
\frac{dx}{dt} &= r_1 x \left(1 - \frac{x}{K}\right) - \frac{m x y}{b + x^2} - c_1 x, \\
\frac{dy}{dt} &= r_2 y \left(1 - \frac{y}{s x}\right) - c_2 y,
\end{align*}
\]

(1)

where \(x(t)\) and \(y(t)\) represent the size at time \(t\) of the prey and the predator populations, respectively; \(K\) denotes the environmental carrying capacity for the

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prey; \( m \) is the maximal predation rate; \( s \) measures the quality of the prey as food for the predator; \( b \) denotes the half-saturation constant, i.e., it measures the resources availability at which half of the maximum intake is reached; \( c_1 \) and \( c_2 \) measure the harvesting efforts; and \( r_1 \) and \( r_2 \) are the intrinsic growth rates of the prey and predators, respectively.

In the first equation of (1), a logistic model \( r_1 x \left( 1 - \frac{x}{K} \right) \) is used to describe the growth of the prey when there are no predators in an environment, which is limited by the carrying capacity \( K \); the simplified Holling IV response, given by the term \( \frac{mx}{b+x+y} \), describes the density of the prey attacked by the predators per unit of time; while \( c_1 x \) denotes the constant-effort harvesting of the prey.

In the second equation of (1), the Leslie–Gower function \( r_2 y \left( 1 - \frac{y}{sx} \right) \) is used to describe the growth of the predators and \( c_2 y \) represents the constant-effort harvesting of the predators.

In the past forty years, Computer Algebra Systems (CAS) have drastically changed the everyday practice of mathematics [3]. Here we use the free and open-source CAS SageMath [5] to give a simple and direct analysis of the prey-predator dynamical system. The obtained results show inaccuracies to the conclusions in [2] that may jeopardize the model. We conclude that SageMath is a wonderful tool to guarantee reproducible results and to avoid mistakes in the calculations.

2 An equivalent model

Let us consider the one-to-one scaling transformations

\[
\tau = r_1 t, \quad \bar{x} = \frac{x}{K}, \quad \bar{y} = \frac{my}{r_1 K^2},
\]

and the new quantities

\[
a := \frac{b}{K^2}, \quad \delta := \frac{r_2}{r_1}, \quad \beta := \frac{r_1 K}{sm}, \quad h_1 := \frac{c_1}{r_1}, \quad h_2 := \frac{c_2}{r_1},
\]

defined from the parameters of model (1).

Remark 1. The expressions of \( \beta \) and \( h_2 \) given in [2] have a typo.

One has from (2) that

\[
\frac{d\tau}{dt} = r_1 \frac{dx}{dt} \Leftrightarrow \frac{dx}{dt} = \frac{1}{r_1} \frac{d\tau}{dt},
\]

\[
\frac{d\bar{x}}{d\tau} = \frac{1}{K} \frac{dx}{dt} \Leftrightarrow \frac{dx}{dt} = K \frac{d\bar{x}}{d\tau},
\]

\[
\frac{d\bar{y}}{d\tau} = \frac{m}{r_1 K^2} \frac{dy}{dt} \Leftrightarrow \frac{dy}{dt} = \frac{r_1 K^2}{m} \frac{d\bar{y}}{d\tau}.
\]

Therefore,

\[
\frac{dx}{dt} = \frac{K}{r_1} \frac{d\bar{x}}{d\tau} = K r_1 \frac{d\bar{x}}{d\tau}
\]
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and

\[
\frac{dy}{dt} = \frac{r_1 K^3}{m} \frac{dy}{dt} = \left(\frac{r_1 K}{m}\right)^2 \frac{dy}{dt}.
\]

This means that we can rewrite the first equation \(\frac{dx}{dt} = r_1 x \left(1 - \frac{x}{K}\right) - \frac{m x}{b + x^2} y - c_1 x\) of system (1) in the new variables as follows:

\[
K r_1 \frac{dx}{dt} = r_1 K x \left(1 - \frac{x}{K}\right) - \frac{m K x}{b + (K x)^2} \times \frac{r_1 K^2 y}{m} - c_1 K x,
\]

which is equivalent to

\[
\frac{dx}{dt} = x \left(1 - \frac{x}{a}\right) - x b + x^2 + c_1 x.
\]

Similarly, the second equation \(\frac{dy}{dt} = r_2 y \left(1 - \frac{y}{s x}\right) - c_2 y\) of system (1) is given in the new variables by

\[
\left(\frac{r_1 K}{m}\right)^2 \frac{dy}{dt} = \frac{r_2 r_1 K^2}{m} \left(1 - \frac{r_1 K K y}{m s x}\right) - c_2 r_1 K^2 y,
\]

which is equivalent to

\[
\frac{dy}{dt} = \frac{r_2}{r_1} y \left(1 - \frac{r_1 K y}{m s x}\right) - c_2 \frac{r_1 K y}{m},
\]

\[
\frac{dy}{dt} = \delta y \left(1 - \frac{\beta y}{x}\right) - h_2 y.
\]

We conclude that system (1) is given, in the new variables, as

\[
\begin{cases}
\frac{dx}{dt} = x \left(1 - \frac{x}{a}\right) - x b + x^2 + c_1 x, \\
\frac{dy}{dt} = \delta y \left(1 - \frac{\beta y}{x}\right) - h_2 y,
\end{cases}
\]

where \(a, \delta, \beta, h_1\) and \(h_2\) are the positive rescaled parameters given by (3).

Systems (1) and (4) are equivalent and we proceed by analysing (4).

Remark 2. The second equation of system (1) is wrongly written in [2] as

\[
\frac{dy}{dt} = y \left(\delta - \beta \frac{y}{x}\right) - h_2 y
\]

(cf. system (1.2) of [2]).
3 Equilibria and stability

In this section, we analyse the number and type of equilibria for system (4) and prove the stability of the meaningful equilibrium point. We make use of the free open-source mathematics software system SageMath [5].

We start by calculating the equilibrium points using the script

```python
var('x,y,a,h1,delta,h2,beta')
eq1 = x*(1-x) - (x/(a+x^2))*y - h1*x
eq2 = delta*y*(1-beta*(y/x))-h2*y
pretty_print(solve((eq1,eq2),(x,y)))
```

from which we obtain four possible equilibria: (0,0), (−h1+1,0), (−√−a,0), (√−a,0). As we are working with a prey-predator model, from the perspective of ecology, we are only interested in the pairs (x,y) such that x ≥ 0 and y ≥ 0. As a is a positive parameter and (0,0) means the extinction of both species, it is simple to understand that the only feasible equilibrium point we are interested in studying is $E = (1-h_1,0)$ with $0 ≤ h_1 < 1$. Indeed, if both populations are at 0 ($h_1 = 1$), then they will continue to be so indefinitely. Unfortunately, and in contrast with the classical Lotka–Volterra model, here we do not have a fixed point at which both populations sustain their non-zero numbers.

**Remark 3.** In [2] the authors claim the existence of a positive equilibrium $(x^*, y^*)$ of the system with $x^* > 0$ and $y^* > 0$. Unfortunately there is no such equilibrium, which means that the model (1) proposed by Cheng and Zhang in 2021 is not realistic.

To determine the type of equilibrium, we first calculate the Jacobian matrix $J$ evaluated at the point $(-h_1+1,0)$. We use the following SageMath script:

```python
j = jacobian((eq1,eq2),(x,y))
jac = j.substitute(x = -h1+1, y = 0)
jac
```

obtaining

$$\begin{pmatrix}
    h_1 - 1 & \frac{h_1-1}{(h_1-1)^2+a} \\
    0 & \delta - h_2
\end{pmatrix}.$$

The corresponding eigenvalues are obtained using the SageMath command

```python
jac.eigenvalues()
```

By doing so, we get the following two eigenvalues: $\delta - h_2$ and $h_1 - 1$. We have just proved the following result.

**Theorem 1.** If $h_1 ∈ ]0,1]$ and $\delta ∈ ]0,h_2]$, then the equilibrium point $(1-h_1,0)$ is a sink.
4 Numerical simulations

Now we use SageMath to plot some solutions of the prey-predator model and illustrate the fact that for different initial values, if one chooses the parameters according with Theorem 1 then the solutions of the model converge to the equilibrium point \((1 - h_1, 0)\). For this purpose, we start by importing some libraries for the numerical integration of the non-linear system (4) and the visualization of its solutions:

```python
import numpy as np
from scipy import integrate
from scipy.integrate import odeint
import matplotlib.pyplot as plt
```

Then we define model (4) with

\[
a := 0.2, \quad \delta := 0.3, \quad \beta := 0.8, \quad h_1 := 0.4, \quad h_2 := 0.6,
\]

as follows:

```python
def model (z,t):
    x = z[0]
    y = z[1]
    dxdt = x*(1-x) - (x/(0.2+x^2))*y - 0.4*x
    dydt = y*0.3*(1 - 0.8*(y/x)) - 0.6*y
    dzdt = [dxdt, dydt]
    return dzdt
```

Next, we compute the solutions of the system for initial conditions \((\bar{x}(0), \bar{y}(0))\) given by \((3, 1), (2, 4)\) and \((4, 3)\):

```python
z0 = [3,1]
z1 = [2,4]
z2 = [4,3]
n = 5
for i in range(1,n):
    tspan = [t[i-1],t[i]]
```
Finally, we plot the solutions with

```python
plt.figure(figsize=(10,4))
plt.subplot(1,2,1)
plt.plot(t,x,'b-', label='x(t), initial value (3,1)', alpha=0.5)
plt.plot(t,xx,'r--', label='x(t), initial value (2,4)', alpha=0.5)
plt.plot(t,xxx,'g:', label='x(t), initial value (4,3)', alpha=1)
plt.ylabel('x(t)')
plt.xlabel('time')
plt.legend(loc='best')

plt.subplot(1,2,2)
plt.plot(t,y,'b-', label='y(t), initial value (3,1)', alpha=0.5)
plt.plot(t,yy,'r--', label='y(t), initial value (2,4)', alpha=0.5)
plt.plot(t,yyy,'g:', label='y(t), initial value (4,3)', alpha=1)
plt.ylabel('y(t)')
plt.xlabel('time')
plt.legend(loc='best')
```

obtaining Figure 1.

We can also illustrate the stability of the meaningful equilibrium point of the prey-predator system by plotting the phase portrait of (4). For that, we use random initial conditions through the following script:

```python
a,h1,delta,beta,h2 = 0.2,0.4,0.1,0.8,0.6

def dX_dt(X, t=0):
    return [X[0]*(1-X[0]) - (X[0]/(a+X[0]^2))*X[1] - h1*X[0],
            X[1]*(delta-beta*(X[1]/X[0])) - h2*X[1]]

def g(x,y):
```

```
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Fig. 1: Solutions of system (4) with $a = 0.2$, $h_1 = 0.4$, $\delta = 0.3$, $\beta = 0.8$, $h_2 = 0.6$ and different initial conditions $(x(0), y(0))$: (3, 1), (2, 4) and (4, 3).

```python
v = vector(dX_dt([x, y]))
return v/v.norm()

var('x,y')
v = plot_vector_field(g(x,y), (x,0.58,0.63), (y,-0.01,0.02), axes_labels = ('x', 'y'))

t = srange(0, 15, .01)
X = integrate.odeint(dX_dt, [0.63,0.01], t)
```
The obtained phase portrait is displayed in Figure 2.

Fig. 2: Phase portrait for system (4) with $a = 0.2$, $h_1 = 0.4$, $\delta = 0.3$, $\beta = 0.8$ and $h_2 = 0.6$.

5 Conclusion

In this paper we considered a prey-predator model with constant harvesting effort previously studied in [2]. We remarked several typos and inconsistencies in [2], showing detailed computations for each of them. We trust that the analysis of the stability of the equilibrium points may be helpful to new researchers in the field. As future work, we plan to modify the proposed model in order to make possible the co-existence equilibrium point.
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