On the stability of anti-de Sitter spacetime

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We present results from a detailed study of spherically symmetric Einstein-massless-scalar field dynamics with a negative cosmological constant in four to nine spacetime dimensions. This study is the first to examine dynamics in AdS beyond five dimensions and the gauge dependence of recently proposed perturbative methods. Using these perturbative methods, we provide evidence that the oscillatory divergence used to argue for instability of anti-de Sitter space by Bizoń et al. is a gauge-dependent effect in five spacetime dimensions. Interestingly, we find that this behavior appears to be gauge-independent in higher dimensions; however, understanding how this divergence depends on the initial data is more difficult. The results we present show that while much progress has been made in understanding the rich dynamics and stability of anti-de Sitter space, a clear route to the answer of whether or not it is stable still eludes us.

Introduction. Stability of de Sitter and Minkowski spacetimes under small perturbations was established in 1986 [1] and 1993 [2]. Following the Anti-de Sitter (AdS)/Conformal Field Theory (CFT) conjecture [3], the question of the stability of AdS became more interesting. Using the AdS/CFT conjecture it is possible to address the important question of thermalization and equilibrium of strongly coupled CFTs, which is dual to the question of whether or not small perturbations of AdS collapse to a black hole. The stability of AdS against arbitrarily small scalar field perturbations was first studied numerically in spherical symmetry [4] by Bizoń and Rostworowski in 2011 [5], where the authors suggested that a large class of perturbations eventually collapse to form a black hole even at arbitrarily small amplitude, $\epsilon$. However, in such simulations a finite $\epsilon$ must be used, leaving room for doubt as to whether arbitrarily small perturbations do actually form a black hole [6]. The probing of small-amplitude perturbations is aided by the recently proposed renormalization flow equations (RFEs) [7–9] for which any behavior observed at amplitude $\epsilon$ must be used, leaving room for doubt as to whether arbitrarily small perturbations do actually form a black hole [8]. The probing of small-amplitude perturbations is aided by the recently proposed renormalization flow equations (RFEs) [7–9] for which any behavior observed at amplitude $\epsilon$ must be used, leaving room for doubt as to whether arbitrarily small perturbations do actually form a black hole [8]. The probing of small-amplitude perturbations is aided by the recently proposed renormalization flow equations (RFEs) [7–9] for which any behavior observed at amplitude $\epsilon$ must be used, leaving room for doubt as to whether arbitrarily small perturbations do actually form a black hole [8]. The probing of small-amplitude perturbations is aided by the recently proposed renormalization flow equations (RFEs) [7–9] for which any behavior observed at amplitude $\epsilon$ must be used, leaving room for doubt as to whether arbitrarily small perturbations do actually form a black hole [8].

This rescaling symmetry was used by Bizoń et al. to argue for the instability of AdS$_4$ based on a divergence in the RFE solution for specific initial data [10]. However, it is suspected that this divergence is a gauge-dependent effect [11].

In this paper, we address the AdS stability question and the concerns of [11] by performing a detailed study of the RFEs and the nonlinear Einstein equations. Our study is the first to examine the gauge dependence of the RFEs and dynamics in AdS beyond five dimensions. Our numerical methods enable us to study the RFEs to a much higher accuracy than previous work, providing new insight into when the RFEs are no longer valid and the reasons they fail. With a new understanding of the RFEs we revisit AdS$_4$, finding agreement with previous work [11, 12] but strong contrast with what is observed in higher dimensions. Finally, we show that our results are largely robust against the choice of initial data and present evidence that the dynamics of AdS$_4$ are more intricate than in higher dimensions.

Model. We consider a self-gravitating massless scalar field in a spherically symmetric, asymptotically AdS spacetime in $d$ spatial dimensions. The metric in Schwarzschild-like coordinates is

$$ds^2 = \ell^2 \left[ -Ae^{-\delta t}dt^2 + A^{-1}dx^2 + \sin^2 \left( \frac{x}{\ell} \right) d\Omega_{d-1} \right],$$

(1)

where $d\Omega_{d-1}$ is the metric on $S^{d-1}$, $x/\ell \in [0, \pi/2]$, and $t/\ell \in [0, \infty)$. The areal radius is $R(x) = \ell \tan(x/\ell)$, and we henceforth work in units of the AdS scale $\ell$ (ie, $\ell = 1$).

The evolution of the scalar field $\psi$ is governed by the nonlinear system

$$\Phi_t = (Ae^{-\delta} \Phi)_x, \quad \Pi_{,t} = \frac{(Ae^{-\delta} \tan^{-1} x \Phi)_x}{\tan^{-1} x},$$

(2)

where $\Pi = A^{-1} e^{\delta} \psi_t$ is the conjugate momentum and $\Phi = \psi_{,x}$ is an auxiliary variable. The metric functions are solved for from

$$\delta_x = - \sin x \cos (\Pi^2 + \Phi^2)$$

(3)

$$A_x = \frac{d - 2 + 2 \sin^2 x}{\sin x \cos x} (1 - A) - \sin x \cos x (\Phi^2 + \Pi^2).$$

(4)

See [13, 14] for a detailed discussion of the code we use to solve this system. For asymptotic flatness at the origin we require $A(x = 0, t) = 1$.

Novel results beyond spherical symmetry were recently presented by Dias and Santos [15].
We are particularly interested in perturbations about \( \text{AdS}_{(d+1)} \) whose evolution at zeroth order is governed by \( \hat{L} = -(\tan^{d-1} x) \partial_x (\tan^{d-1} x \partial_x) \) (this can be seen by setting \( A = 1, \delta = 0 \) and \( \Pi = \psi, t \) in Eq. (2)). The eigenmodes of \( \hat{L} \) are given in terms of Jacobi polynomials,

\[
e_j(x) = \kappa_j \cos^d(x) P_j^{(d/2-1,d/2)}(\cos 2x)
\]

with eigenvalues \( \omega_j = d + 2j \) and where \( \kappa_j = 2 \sqrt{\Gamma(j+d-1)!/\Gamma(j+d/2)!} \).

Recently much attention has been given to the renormalization flow or two-time framework equations \([7,12]\). A detailed study of \( \text{AdS}_4 \) was presented in \([12]\), while \([10]\) investigated \( \text{AdS}_5 \). To study the RFEs, a “slow time” \( \tau = \epsilon^2 t \) is introduced, and dynamics on very short time scales can be thought of as being averaged over. The scalar field perturbation is expanded as \( \psi(x,t) = \sum_{i=0}^{\infty} A_i \cos(\omega_i t + B_i) e_i(x) \), where \( A_i(\tau) \) and \( B_i(\tau) \) are time-dependent coefficients. The evolution of \( A_i \) and \( B_i \) is given by the RFEs \([9]\)

\[
-\frac{dA_i}{dt} = \sum_{i+j+k+l \neq (k,l)} \frac{S_{ijkl}}{2\omega_i} A_i A_j A_k \sin(B_l - B_i - B_j),
\]

\[
-\frac{dB_i}{dt} = \sum_{i+j+k+l \neq (k,l)} \frac{S_{ijkl}}{2\omega_i} A_i A_j A_k \cos(B_l - B_i - B_j) + \frac{T_i}{2\omega_i} A_i^2 + \sum_{i \neq l} \frac{R_{il}}{2\omega_i} A_i^2,
\]

where \( \{i,j\} \neq \{k,l\} \) means both \( i \) and \( j \) are not equal to \( k \) or \( l \), and the coefficients \( T_i, R_{il} \) and \( S_{ijkl} \) are given by integrals over the eigenmodes in appendix A of \([9]\) and by recursion relations in \([11]\). The gauge dependence of the coefficients is discussed in \([11]\).

In our numerical computations we typically truncate the RFEs \([6,7]\) at \( l_{\text{max}} = 399 \), giving a good balance between computational cost and accuracy, and refer to this system as the truncated RFEs (TRFEs). We note that the evolutions dominate the computational cost, not the construction of \( T_i, R_{il} \) and \( S_{ijkl} \), which we have computed to \( l_{\text{max}} \geq 700 \).

**Results.** We present results from a detailed study of the TRFEs and fully nonlinear numerical evolutions in four to nine spacetime dimensions. For concreteness we focus on two-mode initial data of the form

\[
\psi(x,0) = \epsilon (e_0(x) + \kappa e_1(x))/d
\]

but have also studied Gaussian initial data given by

\[
\Pi(x,0) = \epsilon \exp\left(-\frac{\tan^2(x)}{\sigma^2}\right), \quad \psi(x,0) = 0.
\]

In the evolutions presented here we choose \( \kappa = d/(d+2) \), which has been studied extensively in \( \text{AdS}_4 \) \([7,12,14]\) and in \( \text{AdS}_5 \) \([10]\) using the ITG. A logarithmic divergence in the time derivative of the phases, \( dB_i/dt \), was observed in \([10]\). This is consistent with an asymptotic analysis of the equations in the ITG; however, the terms leading to the logarithmic divergence appear to be absent in the BTG\[11\]. We will address this in detail below.

An interesting technique for analyzing solutions to the TRFEs is the analyticity-strip method \([10,14]\). This method involves fitting the spectrum \( A_l \) to

\[
A_l = C(t) \Gamma^{-\gamma(t)} e^{-\rho(t)l}
\]

for \( l \gg 1 \). The analyticity radius \( \rho(t) \) should be interpreted as the distance between the real axis and the nearest singularity in the complex plane. When \( \rho \) becomes zero the TRFEs have evolved to a singular spectrum. We denote the time when the spectrum becomes singular by \( t_\star \) (or \( \tau_\star \) in slow time) and in \( d > 3 \) stop our evolutions of the TRFEs when \( t \) is slightly larger than \( t_\star \). All fits use data from simulations done with \( l_{\text{max}} = 399 \) and omit the lowest and highest twenty modes to reduce errors from truncation. For concreteness we present results in \( \text{AdS}_9 \) but observe qualitatively identical behavior for \( d > 3 \). The spectrum for initial data \([5]\) in \( \text{AdS}_9 \) is shown in Fig. 1. At \( \tau = 1.367 \times 10^{-2} \) the spectrum is already singular, so we show it only for completeness.

In Fig. 2 we plot \( \rho(t) \) for both the ITG and BTG for \( \text{AdS}_9 \). We observe that the spectrum becomes singular at approximately the same \( t_\star \) in both the BTG and the ITG, independent of the dimension being studied, suggesting

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\[2\] See Eq. (2.2) of \([15]\) for more details.
FIG. 2. $\rho(\tau)$ for the $l_{\text{max}} = 399$ AdS$_5$ evolution in the ITG and BTG. In both gauges the spectrum becomes singular at $\tau \approx 1.356 \times 10^{-2}$, suggesting this behavior is gauge-independent.

this behavior is gauge-independent. Interestingly, in our study of $d > 3$ we find that the spectrum becomes singular at approximately the same time that a black hole forms in the nonlinear theory, at least for the initial data studied. We will discuss this further in the context of AdS$_5$ below.

We study $dB_l/dt$ for several different values of $l$ up to $l = 300$ (far below $l_{\text{max}}$ to minimize mode truncation errors) in the ITG and BTG to see if the logarithmic divergence observed in [10] is gauge-dependent as suggested by the asymptotic analysis in [11]. In Fig. 3 we plot $dB_{250}/dt$ for AdS$_5$ and AdS$_9$ in the ITG and BTG. To assess the presence of a logarithmic divergence we fit $a \ln(\bar{t} - t) + b$ to $dB_l/dt$. In AdS$_5$ a logarithmic divergence is observed in the ITG and $\bar{t}$ is within $\sim 0.3\%$ of $t_*$. We estimate the error in $\bar{t}$ to be $\lesssim 0.5\%$. In the BTG $\bar{t}$ is approximately $2\%$ larger than $t_*$. However, evolutions carried out beyond $t_*$ no longer exhibit divergent behavior in $dB_l/dt$, making it difficult to interpret the significance of $\bar{t}$ in the BTG in AdS$_5$. This suggests the divergence in $dB_l/dt$ coinciding with $\rho$ going to zero is a gauge-dependent effect in AdS$_5$.

For $d > 4$ the situation is more interesting and difficult to assess. We find that in the BTG in AdS$_6$ $\ell$ and $t_*$ differ by less than $1\%$, in AdS$_8$ by less than $0.2\%$, and by $\sim 0.1\%$ in AdS$_9$. While the divergent behavior appears more prominently in the ITG, it is clearly also present in the BTG (see Fig. 3). This behavior disagrees with the asymptotic analysis of the equations discussed in [11], where it is suggested that the logarithmic behavior arises from terms that are absent from the coefficients $T_l$ and $R_{il}$ in the BTG. The improving agreement between $\ell$ and $t_*$ with increasing dimension demonstrates that understanding the origin of the divergence in $dB_l/dt$ is more complicated than initially thought and that attempting to connect the divergent behavior to critical phenomena or thermalization would be a premature conclusion.

In spatial dimensions $d > 3$ we observe a direct cascade of energy to higher modes without any inverse cascades, suggesting the initial data is far from a quasi-periodic solution[12]. In Fig. 4 we show the upper envelope of $\Pi^2(x = 0, t)$, which is proportional to the Ricci scalar at the origin, for several different values of $l_{\text{max}}$ and different values of $\epsilon$ for nonlinear evolutions in AdS$_9$. There is good agreement between the nonlinear and TRFE solutions and the agreement improves with increasing dimensionality, at least for $\Pi^2(x = 0, t)$. This may be related to the eigenmodes having larger values at $x = 0$ in higher dimensions. For example, we find that in AdS$_9$ $\epsilon_{250}(x = 0)$ is $\sim 10^4$ times larger than in AdS$_5$.

We now turn to the case of two-mode equal-energy data, Eq. (9), in AdS$_5$. This case has been studied extensively using numerical relativity and the TRFEs[5, 7]. It was suggested in [12] that this solution
orbits a quasi-periodic solution of the same temperature. For these evolutions we use $l_{\text{max}} = 99, 199, 299,$ and 395 to test convergence and to understand how the analyticity radius depends on mode truncation. We compare $\Pi^2(x = 0, t)$ with the nonlinear evolution in Fig. 3. The 299 and 395 mode evolutions are almost indistinguishable until the third increase in $\Pi^2$. This suggests that the agreement between the TRFE and nonlinear solutions would improve if a lower amplitude nonlinear evolution were studied, similar to what is observed in Fig. 4. The flattening of the top 30% of modes is in contrast to what is observed in AdS$_4$ where increasing $\tau$ and $t_*$ increases $\rho$ by $\sim 1\%$ so it is possible that using $l_{\text{max}} > 400$ would resolve the discrepancies. Similar to the two-mode data, agreement of $\bar{t}$ in the BTG and ITG and between $\bar{t}$ and $t_*$ improves with increasing dimensionality. Nevertheless, even for AdS$_5$ we do not find agreement of $\bar{t}$ in the BTG and ITG or between $\bar{t}$ and $t_*$. Interestingly, in AdS$_4$ the analyticity radius does not cross zero, at least up to $l_{\text{max}} = 399$, even though this data collapses in nonlinear evolutions. However, increasing $l_{\text{max}}$ still decreases $\rho$, so the possibility of $\rho$ crossing zero with sufficiently many modes is not ruled out. This suggests that at least in AdS$_4$, and possibly all dimensions, the TRFE solution spectrum not becoming singular is insufficient to guarantee that the nonlinear evolution will never collapse to a black hole.

FIG. 5. The upper envelope of $\Pi^2(x = 0, t)$ for two-mode data for evolutions in AdS$_4$. Plotted are solutions to the TRFEs (dash lines) for different values of $l_{\text{max}}$, and the nonlinear evolution (solid line). The TRFE solutions with $l_{\text{max}} = 299$ and 395 differ only marginally until the third increase in $\Pi^2$.

FIG. 6. Analyticity radius $\rho$ for two-mode equal-energy data in AdS$_4$ using different $l_{\text{max}}$. The minimum of $\rho$ increases with $l_{\text{max}}$, suggesting the evolutions are still suffering from mode truncation to some extent.
Conclusions. In summary, our study is the first to examine the gauge dependence of the RFEs and dynamics in AdS beyond five dimensions. Our numerical methods allow us to test the RFEs to a much higher accuracy than previous studies, providing new insight into when the equations no longer accurately approximate the Einstein equations. We provide evidence that the oscillatory singularity of the RFEs used to argue for the instability of AdS$_5$ in [10] is a gauge-dependent effect in five dimensions and that this behavior is independent of initial data. However, the oscillatory singularity appears to be gauge-independent for dimensions greater than five, which is in disagreement with an analysis of the RFEs in the ultraviolet limit[11]. Interestingly, we find that in higher dimensions the singular behavior of the RFEs occurs at the same time that a black hole forms in the full nonlinear theory. In agreement with [12], we do not observe an oscillatory singularity in AdS$_4$. Additionally, we find that at least in AdS$_4$, the spectrum becoming singular is not indicative of black hole formation in the full theory but rather that the truncated RFEs are no longer valid. While our results aid in understanding the validity and behavior of the RFEs, they also show that even though much progress has been made in understanding the (in)stability of AdS, a clear route to answering the question of stability of AdS still eludes us.

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