1 Principal axes of inertia (PAI) and moments of inertia (MI) of the cross-sectional imaging (CSI)

We use $\mathbf{ox} \mathbf{y}$ to show the rectangular coordinate system of the CSI’s COM. The CSI is consisted of finite surface area element of volume ($dV$). $I_x, I_y$ stand for moments of inertia (MI) on axis $x, y$ respectively, and $I_{xy}$ stands for product of inertia (PI), which is expressed by the equation:

$$
\begin{align*}
I_x & = \int y^2 \rho dA \quad (a) \\
I_y & = \int x^2 \rho dA \quad (b) \\
I_{xy} & = \int x y \rho dA \quad (c)
\end{align*}
$$

where $dA$ stands for area element, $\rho$ for its grey value, and $(x, y)$ for its position coordinate.

Let the coordinate system rotate $\phi$ around the CSI’s COM. A new coordinate system will be formed: $\mathbf{ox}_\phi, \mathbf{y}_\phi$. The relation between surface area element coordinates $\left(x, y \right)$ and those of $\left(x_\phi, y_\phi \right)$ is:

$$
\begin{align*}
x_\phi &= x \cos \phi + y \sin \phi \quad (a) \\
y_\phi &= -x \sin \phi + y \cos \phi \quad (b)
\end{align*}
$$

$I_x^\phi, I_y^\phi$ stand for the MI of axis $x_\phi, y_\phi$ respectively, which is expressed as:

$$
\begin{align*}
I_x^\phi & = \int x_\phi^2 \rho dA \quad (a) \\
I_y^\phi & = \int y_\phi^2 \rho dA \quad (b)
\end{align*}
$$

Substitute Eq 2(b) into 3(a), we will get:

$$
\begin{align*}
I_x^\phi &= \int \left( -x \sin \phi + y \cos \phi \right) \rho dA \\
&= \int \left( x^2 \sin^2 \phi - 2 x_\phi y_\phi \sin \phi \cos \phi + y_\phi^2 \cos^2 \phi \right) \rho dA \\
&= I_x - 2 I_{xy} \cos \phi + I_y \sin \phi \quad (4)
\end{align*}
$$

Substitute 2(a) into 3(b), we will get:

$$
\begin{align*}
I_y^\phi &= \int \left( x \cos \phi + y \sin \phi \right) \rho dA \\
&= \int \left( x_\phi^2 \cos^2 \phi + 2 x_\phi y_\phi \sin \phi \cos \phi + y_\phi^2 \sin^2 \phi \right) \rho dA \\
&= I_y - 2 I_{xy} \sin \phi + I_x \cos \phi \quad (5)
\end{align*}
$$

Add Eq 4 and Eq 5, we will get:

$$
\begin{align*}
I_x^\phi + I_y^\phi &= \int \left( x^2 \left( \sin^2 \phi + \cos^2 \phi \right) + y^2 \left( \cos^2 \phi + \sin^2 \phi \right) \right) \rho dA \\
&= \int \left( x^2 + y^2 \right) \rho dA \\
&= I_x + I_y \quad (6)
\end{align*}
$$
Eq 6 shows that when CSI rotates around its COM, its MI is invariable, which means Eq 6 is indeterminate. To indeterminate equation, we can set up an equation as follows:

\[ f(\alpha) = I_x^\alpha - I_y^\alpha \]  \hspace{1cm} (7)

Substitute Eq 4 and Eq 5 into Eq 7, we will get:

\[ f(\alpha) = \int \left( x^2 \sin^2 \varphi - \cos^2 \varphi \right) - 4xy \sin \varphi \cos \varphi - y^2 \left( \sin^2 \varphi - \cos^2 \varphi \right) \rho dA \]  \hspace{1cm} (8)

Since \( 2 \sin \varphi \cos \varphi = \sin 2\varphi \), \( \cos^2 \varphi - \sin^2 \varphi = \cos 2\varphi \), substitute these relations into Eq 8, we will get:

\[ f(\varphi) = \int \left( -x^2 \cos 2\varphi - 2xy \sin 2\varphi + y^2 \cos 2\varphi \right) \rho dA \]  \hspace{1cm} (9)

Let

\[ \frac{df(\varphi)}{d\varphi} = 0, \]

and we will get:

\[ \frac{df(\varphi)}{d\varphi} = \int \left( 2x^2 \sin 2\varphi - 4xy \cos 2\varphi - 2y^2 \sin 2\varphi \right) \rho dA = 0 \]  \hspace{1cm} (10)

By Eq 10, we will get:

\[ 2 \sin 2\varphi \int x^2 \rho dA - 4 \cos 2\varphi \int xy \rho dA - 2 \sin 2\varphi \int y^2 \rho dA = 0 \]  \hspace{1cm} (11)

By Eq 1 and Eq 11, we will get:

\[ 2 \sin 2\varphi I_y - 4 \cos 2\varphi I_{xy} - 2 \sin 2\varphi I_x = 0 \]  \hspace{1cm} (12)

Divide both sides of Eq 12 by \( 2 \cos 2\varphi \), and we will get:

\[ \tan 2\varphi = -\frac{2I_{xy}}{I_x - I_y} \]

The inverse function of Tangent is:

\[ \varphi = -\frac{1}{2} \arctan \left( \frac{2I_{xy}}{I_x - I_y} \right) \]  \hspace{1cm} (13)

Eq 13 shows that by only one rotation, we can position the CSI.

2 PAI and MI of the reconstruction of CSI

2.1 To rotate around axis x

We use \( oxyz \) to stand for the spatial rectangular coordinate system of the COM of the reconstructed CSI, which was consisted of finite elements of volume. \( I_x, I_y, I_z \) stand for the MI
of axis $x, y, z$ respectively.

Their MI is:

\[
\begin{align*}
I_x &= \int \left( y^2 + z^2 \right) \rho \, dV \\
I_y &= \int \left( x^2 + z^2 \right) \rho \, dV \\
I_z &= \int \left( x^2 + y^2 \right) \rho \, dV
\end{align*}
\]  \hspace{1cm} (14)

Their PI is:

\[
\begin{align*}
I_{xy} &= \int xy \rho \, dV \\
I_{yz} &= \int yz \rho \, dV \\
I_{zx} &= \int xz \rho \, dV
\end{align*}
\]  \hspace{1cm} (15)

where $dV$ stands for element of volume, $\rho$ for its grey value, and $(x, y, z)$ for its position coordinate.

Let the body coordinate system of the reconstructed CSI’s COM rotate $\alpha$ around $x$. Then a new coordinate system of $x_\alpha, y_\alpha, z_\alpha$ will be formed. The relation between body element of volume coordinate system of $(x_\alpha, y_\alpha, z_\alpha)$ and $(x, y, z)$ will be:

\[
\begin{align*}
x_\alpha &= x \\
y_\alpha &= y \cos \alpha - z \sin \alpha \\
z_\alpha &= y \sin \alpha + z \cos \alpha
\end{align*}
\]  \hspace{1cm} (16)

Substitute Eq 16(b) and 16(c) into the MI of $I_x^\alpha = \int \left( y_\alpha^2 + z_\alpha^2 \right) \rho \, dV$ relative to axis $x$, and we will get:

\[
I_x^\alpha = \int \left[ \left( y \cos \alpha - z \sin \alpha \right)^2 + \left( y \sin \alpha + z \cos \alpha \right)^2 \right] \rho \, dV
\]

\[
= \int \left( y^2 \cos^2 \alpha - 2yz \cos \alpha \sin \alpha + z^2 \sin^2 \alpha + y^2 \sin^2 \alpha + 2yz \sin \alpha \cos \alpha + z^2 \cos^2 \alpha \right) \rho \, dV
\]

\[
= \int \left( y^2 \left( \cos^2 \alpha + \sin^2 \alpha \right) + z^2 \left( \sin^2 \alpha + \cos^2 \alpha \right) \right) \rho \, dV
\]

\[
= \int \left( y^2 + z^2 \right) \rho \, dV
\]

\[
= I_x
\]

(17)

Eq 17 shows that when rotating axis $x$, MI relative to axis $x$ is invariable.

Substitute Eq 16(a), 16(b) and 16(c) into the sum of MI relative to axis $y$ and $z$, i.e.

\[
I_y^\alpha + I_z^\alpha = \int \left( x_\alpha^2 + z_\alpha^2 \right) \rho \, dV + \int \left( x_\alpha^2 + y_\alpha^2 \right) \rho \, dV ,
\]

and we will get:

\[
I_y^\alpha + I_z^\alpha = \int x_\alpha^2 \rho \, dV + \int z_\alpha^2 \rho \, dV + \int y_\alpha^2 \rho \, dV
\]

(18)
By Eq 17, Eq 18 can be expressed as:

\[ I_y^a + I_z^a = \int x^2 \rho dV + \int (z^2 + y^2) \rho dV + \int x^2 \rho dV \]

\[ = \int (x^2 + z^2) \rho dV + \int (x^2 + y^2) \rho dV \]

\[ = I_y + I_z \quad (19) \]

Eq 19 shows that by rotating axis \( x \), the MI relative to axis \( y \) and \( z \) is invariable. Together with Eq 17, by rotating axis \( x \), the MI of the reconstructed CSI is also invariable.

Substitute Eq 16(a) and 16(c) into \( I_y^a = \int (x^2 + z^2) \rho dV \), and we will get:

\[ I_y^a = \int (x^2 + (y \sin \alpha + z \cos \alpha)) \rho dV \]

\[ = \int (x^2 + y^2 \sin^2 \alpha + 2yz \sin \alpha \cos \alpha + z^2 \cos^2 \alpha) \rho dV \quad (20) \]

Substitute Eq 16(a) and 16(b) into \( I_z^a = \int (x^2 + y^2) \rho dV \), and we will get:

\[ I_z^a = \int (x^2 + (y \cos \alpha - z \sin \alpha)^2) \rho dV \]

\[ = \int (x^2 + y^2 \cos^2 \alpha - 2yz \sin \alpha \cos \alpha + z^2 \sin^2 \alpha) \rho dV \quad (21) \]

Set up the following equation:

\[ f(\alpha, \beta, \gamma)_a = I_y^a - I_z^a \quad (22) \]

By Eq 20 and 21, Eq 22 can be expressed as:

\[ f(\alpha, \beta, \gamma)_a = \int (y^2 (\sin^2 \alpha - \cos^2 \alpha) + 4yz \sin \alpha \cos \alpha + z^2 (\cos^2 \alpha - \sin^2 \alpha)) \rho dV \]

\[ = \int (y^2 \cos 2\alpha + 2yz \sin 2\alpha + z^2 \cos 2\alpha) \rho dV \quad (23) \]

Since \( 2 \sin \alpha \cos \alpha = \sin 2\alpha \), \( \cos^2 \alpha - \sin^2 \alpha = \cos 2\alpha \), Eq 23 can be expressed as:

\[ f(\alpha, \beta, \gamma)_a = \int (-y^2 \cos 2\alpha + 2yz \sin 2\alpha + z^2 \cos 2\alpha) \rho dV \quad (24) \]

Let

\[ \frac{\partial f(\alpha, \beta, \gamma)_a}{\partial \alpha} = 0 \]

Change Eq 24 to

\[ \frac{\partial f(\alpha, \beta, \gamma)_a}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left( \int (-y^2 \cos 2\alpha + 2yz \sin 2\alpha + z^2 \cos 2\alpha) \rho dV \right) \]

\[ = \int (2y^2 \sin 2\alpha + 4yz \cos 2\alpha - 2z^2 \sin 2\alpha) \rho dV = 0 \quad (25) \]

Hence,

\[ \sin 2\alpha \int y^2 \rho dV + 2 \cos 2\alpha \int yz \rho dV - \sin 2\alpha \int z^2 \rho dV = 0 \quad (26) \]

Since

\[ \sin 2\alpha \int x^2 \rho dV - \sin 2\alpha \int x^2 \rho dV = 0 \quad (27) \]

Substitute Eq 27 into Eq 26, and we will get:
\[
\sin 2\alpha \int y^2 \rho dV + \sin 2\alpha \int x^2 \rho dV + 2\cos 2\alpha \int yz \rho dV - \sin 2\alpha \int z^2 \rho dV - \sin 2\alpha \int x^2 \rho dV = 0
\]

(28)

By Eq 14, Eq 28 can be expressed as:
\[
\sin 2\alpha I_z + 2\cos 2\alpha I_{yz} - \sin 2\alpha I_y = 0 \quad (29)
\]

Divide both sides of Eq 27 by \( \cos 2\alpha \), and we will get:
\[
\tan 2\alpha I_z + 2I_{yz} - \tan 2\alpha I_y = 0 \quad (30)
\]

Next, we will get:
\[
\tan 2\alpha = \frac{2I_{yz}}{I_y - I_z} \quad (31)
\]

Then, get the inverse function of Eq 31:
\[
\alpha = \frac{1}{2} \arctan \left( \frac{2I_{yz}}{I_y - I_z} \right) \quad (32)
\]

2.2 To rotate around axis \( y \)

After rotating around axis \( x \) at \( \alpha \), \( ox, oy, oz \) are used to stand for the spatial rectangular coordinate system of the reconstructed CSI’s COM, which is consisted of finite elements of volume. \( I_{xx}, I_{yy}, I_{zz} \) stand for the MI of axis \( x, y, z \) respectively.

Their MI will be:
\[
\begin{align*}
I_{xx}^a &= \int \left( y_a^2 + z_a^2 \right) dV \\
I_{yy}^a &= \int \left( x_a^2 + z_a^2 \right) dV \\
I_{zz}^a &= \int \left( x_a^2 + y_a^2 \right) dV
\end{align*} \quad (33)
\]

Their PI will be:
\[
\begin{align*}
I_{xy}^a &= \int x_a y_a \rho dV \\
I_{xz}^a &= \int y_a z_a \rho dV \\
I_{yz}^a &= \int x_a z_a \rho dV
\end{align*} \quad (34)
\]

where \( dV \) stands for element of volume, \( \rho \) for the volume’s grey value, and \( (x_a, y_a, z_a) \) for the position coordinate of the element of volume.

Let the body coordinate system of the reconstructed CSI’s COM rotate \( \beta \) around \( y \). Then a new coordinate system of \( ox_{ab}, oy_{ab}, oz_{ab} \) will be formed. The relation between body element of
volume coordinate system of \((x_{\alpha\beta}, y_{\alpha\beta}, z_{\alpha\beta})\) and \((x_\alpha, y_\alpha, z_\alpha)\) will be:

\[
\begin{align*}
  x_{\alpha\beta} &= x_\alpha \cos \beta + z_\alpha \sin \beta & (a) \\
  y_{\alpha\beta} &= y_\alpha & (b) \\
  z_{\alpha\beta} &= -x_\alpha \sin \beta + z_\alpha \cos \beta & (c)
\end{align*}
\]

Substitute Eq 35(a) and 35(c) into the MI of \(I_{yy}^{\alpha\beta} = \int (x_{\alpha\beta}^2 + z_{\alpha\beta}^2) \rho dV\) relative to axis \(y\), and we will get:

\[
I_{yy}^{\alpha\beta} = \int \left((x_\alpha \cos \beta + z_\alpha \sin \beta)^2 + (-x_\alpha \sin \beta + z_\alpha \cos \beta)^2\right) \rho dV
\]

\[
= \int \left(x_\alpha^2 \cos^2 \beta + 2x_\alpha z_\alpha \cos \beta \sin \beta + z_\alpha^2 \sin^2 \beta + x_\alpha^2 \sin^2 \beta - 2x_\alpha z_\alpha \sin \beta \cos \beta + z_\alpha^2 \cos^2 \beta\right) \rho dV
\]

\[
= \int \left(x_\alpha^2 + z_\alpha^2\right) \rho dV = I_{yy}^\alpha
\]

(36)

Eq 36 shows that when rotating axis \(y\), the MI relative to axis \(y\) is invariable.

Substitute Eq 35(a), 35(b) and 35(c) into the sum of the MI relative to axis \(x\) and \(z\), i.e.

\[
I_{xx}^{\alpha\beta} + I_{zz}^{\alpha\beta} = \int (y_{\alpha\beta}^2 + z_{\alpha\beta}^2) \rho dV + \int (x_{\alpha\beta}^2 + y_{\alpha\beta}^2) \rho dV, \text{ and we will get:}
\]

\[
I_{xx}^{\alpha\beta} + I_{zz}^{\alpha\beta} = \int y_\alpha^2 \rho dV + \int (x_\alpha^2 + z_\alpha^2) \rho dV + \int y_\alpha^2 \rho dV
\]

(37)

By Eq 36, Eq 37 can be expressed as:

\[
I_{xx}^{\alpha\beta} + I_{zz}^{\alpha\beta} = \int y_\alpha^2 \rho dV + \int (x_\alpha^2 + z_\alpha^2) \rho dV + \int y_\alpha^2 \rho dV
\]

\[
= I_{xx}^\alpha + I_{zz}^\alpha
\]

(38)

Eq 38 shows that by rotating around axis \(y\), the MI relative to axis \(x\) and \(z\) is invariable. Together with Eq 36, by rotating axis \(y\), the MI of the reconstructed CSI is invariable.

Substitute Eq 35(b) and 35(c) into \(I_{xx}^{\alpha\beta} = \int (y_{\alpha\beta}^2 + z_{\alpha\beta}^2) \rho dV, \text{ and we will get:}

\[
I_{xx}^{\alpha\beta} = \int \left(y_\alpha^2 + (-x_\alpha \sin \beta + z_\alpha \cos \beta)^2\right) \rho dV
\]

\[
= \int \left(y_\alpha^2 + x_\alpha^2 \sin^2 \beta - 2x_\alpha z_\alpha \sin \beta \cos \beta + z_\alpha^2 \cos^2 \beta\right) \rho dV
\]

(39)

Substitute Eq 35(a) and 35(b) into \(I_{zz}^{\alpha\beta} = \int (x_{\alpha\beta}^2 + y_{\alpha\beta}^2) \rho dV, \text{ and we will get:}

\[
I_{zz}^{\alpha\beta} = \int \left((x_\alpha \cos \beta + z_\alpha \sin \beta)^2 + y_\alpha^2\right) \rho dV
\]

\[
= \int \left(x_\alpha^2 \cos^2 \beta + 2x_\alpha z_\alpha \sin \beta \cos \beta + z_\alpha^2 \sin^2 \beta + y_\alpha^2\right) \rho dV
\]

(40)

Set up the following equation:

\[
f(\alpha, \beta, \gamma)_\beta = I_{zz}^{\alpha\beta} - I_{xx}^{\alpha\beta}
\]

(41)
By Eq 39 and 40, Eq 41 can be expressed as:

\[
 f(\alpha, \beta, \gamma) = \int \left( x^2_{\alpha} (\cos^2 \beta - \sin^2 \beta) + 4 x_{\alpha} z_{\alpha} \sin \beta \cos \beta - z^2_{\alpha} (\cos^2 \beta - \sin^2 \beta) \right) \rho dV
\]

(42)

Since \( 2 \sin \beta \cos \beta = \sin 2\beta \), \( \cos^2 \beta - \sin^2 \beta = \cos 2\beta \), Eq 42 can be expressed as:

\[
 f(\alpha, \beta, \gamma) = \int \left( x^2_{\alpha} \cos 2\beta + 2 x_{\alpha} z_{\alpha} \sin 2\beta - z^2_{\alpha} \cos 2\beta \right) \rho dV
\]

Let

\[
 \frac{\partial f(\alpha, \beta, \gamma)}{\partial \beta} = 0
\]

Since

\[
 \frac{\partial f(\alpha, \beta, \gamma)}{\partial \beta} = \int \left( -2 x^2_{\alpha} \sin 2\beta + 4 x_{\alpha} z_{\alpha} \cos 2\beta + 2 z^2_{\alpha} \sin 2\beta \right) \rho dV
\]

Hence

\[
 -\sin 2\beta \int x^2_{\alpha} \rho dV + 2 \cos 2\beta \int x_{\alpha} z_{\alpha} \rho dV + \sin 2\beta \int z^2_{\alpha} \rho dV = 0
\]

(45)

Since

\[
 \sin 2\beta \int y^2_{\alpha} \rho dV - \sin 2\beta \int y^2_{\alpha} \rho dV = 0
\]

(46)

Substitute Eq 46 into Eq 45, and we will get:

\[
 -\sin 2\beta \int x^2_{\alpha} \rho dV - \sin 2\beta \int y^2_{\alpha} \rho dV + 2 \cos 2\beta \int x_{\alpha} z_{\alpha} \rho dV + \sin 2\beta \int z^2_{\alpha} \rho dV + \sin 2\beta \int y^2_{\alpha} \rho dV = 0
\]

(47)

By Eq 33, Eq 47 can be expressed as:

\[
 -\sin 2\beta I^z_{zz} + 2 \cos 2\beta I^x_{zz} + \sin 2\beta I^x_{zz} = 0
\]

(48)

Divide both sides of Eq 48 by \( \cos 2\beta \), and we will get:

\[
 -\tan 2\beta I^z_{zz} + 2 \beta I^x_{zz} + \tan 2\beta I^x_{zz} = 0
\]

(49)

Next, we will get:

\[
 \tan 2\beta = -\frac{2 I^x_{zz}}{I^z_{zz} - I^x_{zz}}
\]

(50)

Then, get the inverse function of Eq 45:

\[
 \beta = -\frac{1}{2} \arctan \left( \frac{2 I^x_{zz}}{I^z_{zz} - I^x_{zz}} \right)
\]

(51)
2.3 To rotate around axis $z$

After rotating around axis $x$ at $\alpha$ and then around axis $y$ at $\beta$, $ox_{\alpha\beta}, y_{\alpha\beta}, z_{\alpha\beta}$ are used to stand for the spatial rectangular coordinate system of the reconstructed CSI’s COM, which is consisted of finite elements of volume. $I_{x\beta}^\alpha, I_{y\beta}^\alpha, I_{z\beta}^\alpha$ stand for the MI of axis $x_{\alpha\beta}, y_{\alpha\beta}, z_{\alpha\beta}$ respectively.

Their MI will be:

$$
I_{x\beta}^\alpha = \int (y_{\alpha\beta}^2 + z_{\alpha\beta}^2) \rho dV
$$

$$
I_{y\beta}^\alpha = \int (x_{\alpha\beta}^2 + z_{\alpha\beta}^2) \rho dV
$$

$$
I_{z\beta}^\alpha = \int (x_{\alpha\beta}^2 + y_{\alpha\beta}^2) \rho dV
$$

Their PI will be:

$$
I_{x\beta}^\alpha = \int x_{\alpha\beta} y_{\alpha\beta} \rho dV
$$

$$
I_{y\beta}^\alpha = \int y_{\alpha\beta} z_{\alpha\beta} \rho dV
$$

$$
I_{z\beta}^\alpha = \int x_{\alpha\beta} z_{\alpha\beta} \rho dV
$$

where $dV$ stands for element of volume, $\rho$ for the volume’s grey value, and $(x_{\alpha\beta}, y_{\alpha\beta}, z_{\alpha\beta})$ for the position coordinate of the element of volume.

Let the body coordinate system of the reconstructed CSI’s COM rotate $\gamma$ around $z$. Then a new coordinate system of $ox_{\alpha\beta\gamma}, y_{\alpha\beta\gamma}, z_{\alpha\beta\gamma}$ will be formed. The relation between body element of volume coordinate system of $(x_{\alpha\beta\gamma}, y_{\alpha\beta\gamma}, z_{\alpha\beta\gamma})$ and $(x_{\alpha\beta}, y_{\alpha\beta}, z_{\alpha\beta})$ will be:

$$
x_{\alpha\beta\gamma} = x_{\alpha\beta} \cos \gamma - y_{\alpha\beta} \sin \gamma \quad (a)
$$

$$
y_{\alpha\beta\gamma} = x_{\alpha\beta} \sin \gamma + y_{\alpha\beta} \cos \gamma \quad (b)
$$

$$
z_{\alpha\beta\gamma} = z_{\alpha\beta} \quad (c)
$$

Substitute Eq 54(a) and 54(b) into the MI of $I_{z\beta\gamma}^\alpha = \int (x_{\alpha\beta\gamma}^2 + y_{\alpha\beta\gamma}^2) \rho dV$ relative to axis $z$, and we will get:

$$
I_{z\beta\gamma}^\alpha = \int ((x_{\alpha\beta} \cos \gamma - y_{\alpha\beta} \sin \gamma)^2 + (x_{\alpha\beta} \sin \gamma + y_{\alpha\beta} \cos \gamma)^2) \rho dV
$$

$$
= \int (x_{\alpha\beta}^2 \cos^2 \gamma - 2x_{\alpha\beta} y_{\alpha\beta} \cos \gamma \sin \gamma + y_{\alpha\beta}^2 \sin^2 \gamma + x_{\alpha\beta}^2 \sin^2 \gamma + 2x_{\alpha\beta} y_{\alpha\beta} \sin \gamma \cos \gamma + y_{\alpha\beta}^2 \cos^2 \gamma) \rho dV
$$

$$
= \int (x_{\alpha\beta}^2 (\cos^2 \gamma + \sin^2 \gamma) + y_{\alpha\beta}^2 (\sin^2 \gamma + \cos^2 \gamma)) \rho dV
$$

$$
= \int (x_{\alpha\beta}^2 + y_{\alpha\beta}^2) \rho dV
$$

$$
= I_{z\beta}^\alpha
$$
(55)

Eq 55 shows that when rotating around axis z, the PI relative to axis z is invariable.

Substitute Eq 54(a), 54(b) and 54(c) into the sum of PI, i.e.

$$I_{xx}^{ab} + I_{yy}^{ab} = \int (x_{ab}^2 + z_{ab}^2) \alpha dV + \int (x_{ab}^2 + z_{ab}^2) \alpha dV$$

relative to axis x and axis z, and we will get:

$$I_{xx}^{ab} + I_{yy}^{ab} = \int z_{ab}^2 \alpha dV + \int (x_{ab}^2 + y_{ab}^2) \alpha dV + \int z_{ab}^2 \alpha dV$$

$$= \int (y_{ab}^2 + z_{ab}^2) \alpha dV + \int (x_{ab}^2 + z_{ab}^2) \alpha dV$$

$$= I_{xx}^{ab} + I_{yy}^{ab}$$

Eq 56 shows that when rotating around axis z, the PI relative to axis x and axis y is invariable.

Together with Eq 55, by rotating axis z, the MI of the reconstructed CSI is invariable.

Substitute Eq 54(b) and 54(c) into $I_{xx}^{ab} = \int (x_{ab}^2 + z_{ab}^2) \alpha dV$, and we will get:

$$I_{xx}^{ab} = \int \left( x_{ab}^2 \sin^2 \gamma + y_{ab}^2 \cos \gamma + z_{ab}^2 \right) \alpha dV$$

$$= \int \left( x_{ab}^2 \sin^2 \gamma + 2x_{ab} y_{ab} \sin \gamma \cos \gamma + y_{ab}^2 \cos^2 \gamma + z_{ab}^2 \right) \alpha dV$$

(57)

Substitute Eq 54(a) and 54(b) into $I_{yy}^{ab} = \int (x_{ab}^2 + z_{ab}^2) \alpha dV$, and we will get:

$$I_{yy}^{ab} = \int \left( x_{ab}^2 \cos^2 \gamma - y_{ab}^2 \sin \gamma \right) \alpha dV$$

$$= \int \left( x_{ab}^2 \cos^2 \gamma - 2x_{ab} y_{ab} \sin \gamma \cos \gamma + y_{ab}^2 \sin^2 \gamma + z_{ab}^2 \right) \alpha dV$$

(58)

Set up the following equation:

$$f(\alpha, \beta, \gamma) = I_{xx}^{ab} - I_{yy}^{ab}$$

(59)
By Eq 57 and 58, Eq 59 can be expressed as:

\[ f(\alpha, \beta, \gamma) = \int \left( -x_{\alpha\beta}^2 \cos^2 \gamma - \sin^2 \gamma + 4x_{\alpha\beta}x_{\alpha\beta} \sin \gamma \cos \gamma + y_{\alpha\beta}^2 \cos^2 \gamma - \sin^2 \gamma \right) dV \]  

(60)

Since \( 2 \sin \gamma \cos \lambda = \sin 2\gamma \), \( \cos^2 \gamma - \sin^2 \gamma = \cos 2\gamma \), Eq 60 can be expressed as:

\[ f(\alpha, \beta, \gamma) = \int \left( -x_{\alpha\beta}^2 \cos 2\gamma + 2x_{\alpha\beta}y_{\alpha\beta} \sin 2\gamma + y_{\alpha\beta}^2 \cos 2\gamma \right) dV \]  

(61)

Let

\[ \frac{\partial f(\alpha, \beta, \gamma)}{\partial \gamma} = 0 \]

Since

\[ \frac{\partial f(\alpha, \beta, \gamma)}{\partial \gamma} = \frac{\partial}{\partial \gamma} \left( \int \left( -x_{\alpha\beta}^2 \cos 2\gamma + 2x_{\alpha\beta}y_{\alpha\beta} \sin 2\gamma + y_{\alpha\beta}^2 \cos 2\gamma \right) dV \right) \]

(62)

Hence

\[ \sin 2\gamma \int x_{\alpha\beta}^2 dV + 2 \cos 2\gamma \int x_{\alpha\beta} y_{\alpha\beta} \rho dV - \sin 2\gamma \int y_{\alpha\beta}^2 \rho dV = 0 \]

(63)

Since

\[ \sin 2\gamma \int z_{\alpha\beta}^2 dV - \sin 2\gamma \int z_{\alpha\beta}^2 dV = 0 \]

(64)

Substitute Eq 64 into Eq 63, we will get:

\[ \sin 2\gamma \int x_{\alpha\beta}^2 dV + \sin 2\gamma \int z_{\alpha\beta}^2 dV + 2 \cos 2\gamma \int x_{\alpha\beta} y_{\alpha\beta} \rho dV - \sin 2\gamma \int y_{\alpha\beta}^2 \rho dV - \sin 2\gamma \int z_{\alpha\beta}^2 \rho dV = 0 \]

(65)

By Eq 52, Eq 65 can be expressed as:

\[ \sin 2\gamma I_{\alpha\beta}^{xy} + 2 \cos 2\gamma I_{\alpha\beta}^{xy} - \sin 2\gamma I_{\alpha\beta}^{xy} = 0 \]

(66)

Divide both sides of Eq 66 by \( \cos 2\gamma \), and we will get:

\[ \tan 2\gamma I_{\alpha\beta}^{xy} + 2 I_{\alpha\beta}^{xy} - \tan 2\gamma I_{\alpha\beta}^{xy} = 0 \]

(67)

Next, we will get:

\[ \tan 2\gamma = \frac{2 I_{\alpha\beta}^{xy}}{I_{\alpha\beta}^{xy} - I_{\alpha\beta}^{xy}} \]

(68)

Then, get the inverse function of Eq 68:

\[ \gamma = \frac{1}{2} \arctan \left( \frac{2 I_{\alpha\beta}^{xy}}{I_{\alpha\beta}^{xy} - I_{\alpha\beta}^{xy}} \right) \]

(69)
3 The order of rotation of the reconstructed CSI and its PAI and MI

3.1 Rotation of $x \rightarrow y \rightarrow z$

By Eq 16, Eq 35 can be expressed as:

\[
\begin{align*}
  x_{xy} &= x \cos \beta + (y \sin \alpha + z \cos \alpha) \sin \beta \quad (a) \\
  y_{xy} &= y \cos \alpha - z \sin \alpha \quad (b) \\
  z_{xy} &= -x \sin \beta + (y \sin \alpha + z \cos \alpha) \cos \beta \quad (c)
\end{align*}
\]

By Eq 54, Eq 70 can be expressed as:

\[
\begin{align*}
  x_{xy} &= (x \cos \beta + y \sin \alpha \sin \beta + z \cos \alpha \sin \beta) \cos \gamma - (y \cos \alpha - z \sin \alpha) \sin \gamma \quad (a) \\
  y_{xy} &= (x \cos \beta + y \sin \alpha \sin \beta + z \cos \alpha \sin \beta) \sin \gamma + (y \cos \alpha - z \sin \alpha) \cos \gamma \quad (b) \\
  z_{xy} &= -x \sin \beta + y \sin \alpha \cos \beta + z \cos \alpha \cos \beta \quad (c)
\end{align*}
\] (71)

3.2 Rotation of $y \rightarrow z \rightarrow x$

By Eq 54, Eq 35 can be expressed as:

\[
\begin{align*}
  x_{yz} &= (x \cos \beta + z \sin \beta) \cos \gamma - y \sin \gamma \quad (a) \\
  y_{yz} &= (x \cos \beta + z \sin \beta) \sin \gamma + y \cos \gamma \quad (b) \\
  z_{yz} &= -x \sin \beta + z \cos \beta \quad (c)
\end{align*}
\]

By Eq 16, Eq 72 can be expressed as:

\[
\begin{align*}
  x_{yz} &= (x \cos \beta + z \sin \beta) \cos \gamma - y \sin \gamma \quad (a) \\
  y_{yz} &= (x \cos \beta \sin \gamma + z \sin \beta \sin \gamma + y \cos \gamma) \cos \alpha - (z \cos \beta - x \sin \beta) \sin \alpha \quad (b) \\
  z_{yz} &= (x \cos \beta \sin \gamma + z \sin \beta \sin \gamma + y \cos \gamma) \sin \alpha + (z \cos \beta - x \sin \beta) \cos \alpha \quad (c)
\end{align*}
\] (73)

3.3 Rotation of $z \rightarrow x \rightarrow y$

By Eq 16, Eq 54 can be expressed as:

\[
\begin{align*}
  x_{zx} &= x \cos \gamma - y \sin \gamma \quad (a) \\
  y_{zx} &= (x \sin \gamma + y \cos \gamma) \cos \alpha - z \sin \alpha \quad (b) \\
  z_{zx} &= (x \sin \gamma + y \cos \gamma) \sin \alpha + z \cos \alpha \quad (c)
\end{align*}
\]

By Eq 35, Eq 74 can be expressed as:

\[
\begin{align*}
  x_{zy} &= (x \cos \gamma - y \sin \gamma) \cos \beta + (x \sin \gamma \sin \alpha + y \cos \gamma \sin \alpha + z \cos \alpha) \sin \beta \quad (a) \\
  y_{zy} &= (x \sin \gamma \cos \alpha + y \cos \gamma \cos \alpha - z \sin \alpha \cos \beta \quad (b) \\
  z_{zy} &= -(x \cos \gamma - y \sin \gamma) \sin \beta + (x \sin \gamma \sin \alpha + y \cos \gamma \sin \alpha + z \cos \alpha) \cos \beta \quad (c)
\end{align*}
\]

(75)

Eq 71, 73 and 75 show that with different rotation order, the results of the positioning of bone in vivo differ.