Constructive Wall-Crossing and Seiberg-Witten\footnote{This article is a conference proceeding contribution for \textit{Progress of Quantum Field Theory and String Theory}, Osaka, April 2012.}

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\textbf{Abstract}

We outline a comprehensive and first-principle solution to the wall-crossing problem in $D = 4 \ N = 2$ Seiberg-Witten theories. We start with a brief review of the multi-centered nature of the typical BPS states and recall how the wall-crossing problem thus becomes really a bound state formation/dissociation problem. Low energy dynamics for arbitrary collections of dyons is derived, from Seiberg-Witten theory, with the proximity to the so-called marginal stability wall playing the role of the small expansion parameter. We find that, surprisingly, the $\mathbb{R}^{3n}$ low energy dynamics of $n+1$ BPS dyons cannot be consistently reduced to the classical moduli space, $\mathcal{M}$, yet the index can be phrased in terms of $\mathcal{M}$. We also explain how an equivariant version of this index computes the protected spin character of the underlying field theory, where $SO(3)_J$ isometry of $\mathcal{M}$ turns out to be the diagonal subgroup of $SU(2)_L$ spatial rotation and $SU(2)_R$ R-symmetry. The so-called rational invariants, previously seen in the Kontsevich-Soibelman formalism of wall-crossing, are shown to emerge naturally from the orbifolding projection due to Bose/Fermi statistics.
Contents

1 Multi-Center Picture and Wall-Crossing 2
2 Excursion: Kontsevich-Soibelman and Rational Invariants 5
3 Low Energy Dynamics in the Strong Coupling Regime 7
4 Subtlety: $3n$ vs. $2n$ 9
5 $\mathcal{N} = 4$ Supersymmetry 11
6 Deformation to $\mathcal{M}_{n+1}$ and Index Theorem 13
7 Statistics, Rational Invariants, and Wall-Crossing Formula 16
8 Protected Spin Character and Equivariant Index 19
9 Conclusion and Beyond Multi-Center Picture 20
1 Multi-Center Picture and Wall-Crossing

Wall-crossing refers to phenomena where certain BPS states [1, 2] disappear or appear abruptly as parameters or vacuum moduli of a supersymmetric theory are continuously deformed. The places where this happens define the so-called marginal stability walls (MSW) that separate one domain from another in the parameter/vacuum moduli space. Within each such domain, the BPS spectrum is protected. In four dimensional supersymmetric theories, the phenomenon was first discovered in $SU(2)$ Seiberg-Witten theory [3, 4]: In the weak coupling limit the BPS spectrum is infinite, with the massive vector meson and certain infinite tower of dyons, while in a strongly coupled regime one finds only two particle-like BPS states, namely a monopole and a unit-electrically charged dyon [5]. Such phenomena turned out generic for BPS states (say, of higher supersymmetric theories) that preserve four or less supercharges.

Given the $SU(2)$ example with the MSW deep in the strongly coupled regime, one might be mislead to think that wall-crossing is inherently strong coupling phenomena. However, nothing can be further from the truth. An example where this can be seen most easily is the 1/4 BPS state of $N = 4$ super-Yang-Mills theories; these objects preserve the same number of supersymmetry as $N = 2$ BPS states and are also affected by wall-crossing. For those who prefer a stringy picture, one could equivalently consider a $(p,q)$ string web with end points at D3-branes [6]. With the latter, one finds various three-way junctions where three different types of $(p,q)$ strings meets in a supersymmetric fashion. The angles between strings at such junctions are determined entirely by balance of string tensions, or more precisely by BPS conditions, and so are independent of how long each string segments are. The latter means that one would inevitably encounter a MSW as a D3 is brought near a junction shortening a string segment. Passing the D3 through that junction violates the BPS condition, and a wall-crossing occurs. Because the underlying theory has $N = 4$ supersymmetry, one can choose to perform this analysis with arbitrarily small coupling, which clearly shows that the phenomenon must have a simple classical or semiclassical interpretation.

Understanding the wall-crossing of such states from the spacetime viewpoint, i.e., from $N = 4$ field theory viewpoint, came shortly afterward [7]. It was shown that a typical 1/4 BPS soliton is made up of two or more well-separated non-Abelian charge cores. Distances between these charge cores are determined entirely by classical balance of forces, such that, as vacuum moduli (equivalently, positions of D3) cross a MSW, at least one of these distances diverges. At quantum level, this translate to divergent size of the bound state wavefunction. With the bound state size no longer square normalizable, one can no longer interpret the BPS state in question as a single particle state. This is how BPS states disappear from the spectrum across an MSW.
All 1/4 BPS states are loose bound states of some simpler subset of BPS particles, and at MSW’s become classically destabilized [7, 8].

This multi-center picture and wall-crossing for 1/4 BPS states in $N = 4$ Yang-Mills theories was later extended to $N = 2$ theories [9, 10], where the same phenomenon was found: $N = 2$ BPS states are typically bound states of a simpler class of BPS particles, whose wavefunctions become nonnormalizable at MSW’s. An index theorem, counting the discontinuity, was also found and computed [11] for both $N = 4$ 1/4 BPS states and $N = 2$ BPS states. While this development could not rigorously address strongly coupled regimes, it was obvious that the intuitive wall-crossing based on the multi-center picture should hold universally and that the multi-centered nature of the BPS state should become semiclassically manifest when a relevant MSW is approached.

Rediscovery of the same multi-centered nature in the context of BPS black holes [12] allowed more systematic study of the wall-crossing phenomena. Two main results emerged from this and dominated the topic for a decade since. The first offers a simple sets of constraints for relative positions of charge cores. With charge $\gamma_A$ at $\vec{x}_A$, these positions are constrained as

$$\sum_{B \neq A} \frac{\langle \gamma_B, \gamma_A \rangle}{|\vec{x}_B - \vec{x}_A|} = 2 \text{Im} \left[ \zeta_T^{-1} Z(\gamma_A) \right],$$

where $Z(\gamma_A)$ is the central charge of $\gamma_A$ and $\zeta_T$ is the phase factor of the total central charge $Z_T = \sum_A Z(\gamma_A)$. This formula by itself does not really inform us how to decompose the total charge into $\sum_A \gamma_A$. Nevertheless, once the latter is known, this tightly constrains possible classical solutions. Note how the quantities here are expressed entirely in terms of charges and central charges, and other details of the underlying theories drop out. The simplicity of the formula is in part due to Abelian nature of black holes, but can be extended to non-Abelian field theory solitons, as will be one of main point of this talk, as long as we move near an MSW. The second is an index formula that count supersymmetric bound states of two charges [13, 14]

$$\Omega^-(\gamma_1 + \gamma_2) = (-1)^{|\langle \gamma_1, \gamma_2 \rangle| - 1} |\langle \gamma_1, \gamma_2 \rangle| \Omega^+(\gamma_1) \Omega^+(\gamma_2),$$

where $\pm$ in $\Omega^\pm$ refers to the two sides of a MSW. $\Omega$ is an index, called the 2nd helicity trace,

$$\Omega = -\frac{1}{2} \text{tr} \left( (-1)^{2J_3} (2J_3)^2 \right),$$

whose value is 1 for a half-hypermultiplet and $-2$ for a vector multiplet. More generally, for a BPS multiplet consisting of spin $j$ angular momentum multiplet times a half-hyper, the value is $(-1)^j (2j + 1)$. This so-called primitive wall-crossing formula has been observed in many studies, generalized to the so-called semi-primitive cases.
for $\gamma_1 + k\gamma_2$ states [14], and more recently elevated to the Kontsevich-Soibelman formalism [15, 16, 17, 18]. Despite successes of these later and more comprehensive work, much of the literature had remained mathematical, conjectural, and, from physics viewpoint, rather opaque.

In this talk, we explain a new a universal approach to BPS states in Seiberg-Witten field theories [19], whereby we can derive wall-crossing formula in its full generality and entirely from the underlying $D = 4$ field theory. Combination of supersymmetry algebra and the proximity to an MSW turns out to give us enough control to constrain and solve low energy dynamics of generic Seiberg-Witten dyons, and furthermore can be connected to more general framework of quiver quantum mechanics [13]. This new method, in addition to providing a rather satisfactory and physical solution to the problem, also addresses two outstanding issues in the wall-crossing literature.

The first is whether and under what circumstance the relatively simple distance formula (1.1) can be extended for field theory solitons with non-Abelian cores. Old field theory dynamics of BPS dyons [8, 9, 10, 20] comes with similar distance formula for solitons; however, it treats magnetic and electric charges differently, which makes it ineffective for dyons with MSW located in the strongly coupled regime. In our new approach, we start by addressing this question; simply put, exactly the same distance formula works, but only if when we approach an MSW; this proximity to an MSW will also serve as a small parameter that controls the low energy approximation [21, 22] in the end.

The second concerns various issues surrounding the quiver quantum mechanics formulation [13]. Although very well motivated from wrapped D3 brane picture of BPS states, the subsequent analysis lead to two questions. One concerns what is the right index theorem to use in the so-called Coulomb phase. The other is why the so-called Higgs phase index agrees with the Coulomb phase only in some cases and not in others. Both of these questions turns out to be due to subtleties in the Coulomb phase of the quiver quantum mechanics. For those who are more familiar with this approach to wall-crossing, our work can be regarded as a physical derivation and subsequent study of the Coulomb phase viewpoint. We will end up addressing the first question head on, in this talk, while referring the second to another, more recent study.
2 Excursion: Kontsevich-Soibelman and Rational Invariants

Before we derive physical wall-crossing formulae, we wish to briefly digress to the algebraic formalism by Kontsevich and Soibelman [15]. For this we start with Lie algebra generators $e_\gamma$, one for each and every charge $\gamma$, such that their commutators are

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2},$$

with the Schwinger product $\langle \cdot, \cdot \rangle$. This is then exponentiated to operators

$$K_\gamma = \exp\left(\sum_{n=1}^{\infty} \frac{e_n \gamma}{n^2}\right).$$

Given the indices $\Omega(\gamma_i)$, one then proceeds to build a product of such operators as

$$\cdots K_{\gamma_1}^{\Omega(\gamma_1)} \cdots K_{\gamma_2}^{\Omega(\gamma_2)} \cdots,$$

which encodes the details of the theory in question.

As these operators do not generally commute, the ordering is crucial and dictated by the phase of central charges $Z(\gamma_i)$. While the phase is a periodic variable and does not usually define an ordering, there is a sense in which we may do so near an MSW. Since the phases of wall-crossing states all line up there, we can order at least these relative to their common phase at the MSW. Denoting again by the superscript $\pm$ the two sides of an MSW where phases of $Z(\gamma_1)$ and $Z(\gamma_2)$ line up, we have two different string of operators,

$$U^+ = \cdots K_{\gamma_1}^{\Omega^+(\gamma_1)} \cdots K_{\gamma_2}^{\Omega^+(\gamma_2)} \cdots,$$

and

$$U^- = \cdots K_{\gamma_2}^{\Omega^-(\gamma_2)} \cdots K_{\gamma_1}^{\Omega^-(\gamma_1)} \cdots.$$

There are also those states whose central charge phase does not line up with $\gamma_{1,2}$ etc; for these with very different phases from any associated with $n\gamma_1 + k\gamma_2$, we simply choose them to occupy the left end or the right end of these operator products, or more to the point, to occupy the same corner in $U^\pm$.

The KS conjecture states that the wall-crossing occurs to make sure that these two strings of operators are actually identical in the end, i.e., $U^+ = U^-$, even though the positions of individual $K$’s in $U^\pm$ are completely different. As positions of $K_{\gamma_{1,2}}$ are flipped, for instance, one must have different strings of $K_{n\gamma_1 + k\gamma_2}$’s between the two, in order for $U^+ = U^-$ to hold; this comparison determines $\Omega^-$, say, given
The equality of $U^\pm$ was later interpreted as a continuity condition of the so-called Darboux coordinates in compactified Seiberg-Witten theory [16], providing more physical motivation for the conjecture. Obviously, one important check of this KS formalism is a comparison to other wall-crossing formulae that are derived as a solution to the original physics problem, such as ours. Recently a complete agreement between the two was found [23] under the assumption that relevant $\gamma$’s belong to a single plane through the origin in the charge lattice. We will not repeat the proof, but instead point out one very important ingredient.

Let us note that $\Omega(k\gamma) \neq 0$ with $k \neq 1$ is a logical possibility. In fact, for BPS black holes, such non-primitive states would be routinely expected. Since $\gamma$ and $k\gamma$ share exactly the same central charge phase, (2.3) can be more precisely written as

$$\cdots \prod_{k_1} K_{k_1\gamma_1} \cdots \prod_{k_2} K_{k_2\gamma_2} \cdots ,$$

or equivalently as

$$\cdots V(\gamma_1) \cdots V(\gamma_2) \cdots ,$$

with $\gamma$’s now running only over the primitive charges; the operator $V(\gamma)$ knows about the actual spectrum as [24]

$$V^\pm(\gamma) \equiv \exp \left( \sum_{k=1}^{\infty} \bar{\Omega}^\pm(k\gamma) e_{k\gamma} \right), \quad \bar{\Omega}^\pm(\Gamma) \equiv \sum_{s|\Gamma} \frac{\Omega^\pm(\Gamma/s)}{s^2} .$$

The “rational invariant” $\bar{\Omega}$ is defined as a sum over positive integers $s$ such that $\Gamma/s$ is a well-defined integral charge. The KS wall-crossing formula then says,

$$U^+ = \cdots V^+(\gamma_1) \cdots V^+(p_1 \gamma_1 + p_2 \gamma_2) \cdots V^+(\gamma_2) \cdots = \cdots V^-(\gamma_2) \cdots V^-(p_1 \gamma_1 + p_2 \gamma_2) \cdots V^-(\gamma_1) \cdots = U^-$$

with the product running over primitive charges $\sum p_i \gamma_i$’s. From this, relationships between the two sets of rational invariants $\bar{\Omega}^\pm(\sum n_i \gamma_i)$ emerge, and one can decode in favor of the physical quantities, $\Omega^\pm(\sum n_i \gamma_i)$’s.

This suggests that, if the KS formalism indeed provides a universal answer to the wall-crossing problem, one should very well expect that the rational invariants $\bar{\Omega}$ appear naturally in any sensible derivation of the wall-crossing formula from physics also, and in a manner that has nothing to do with details of dynamics. As we will see shortly, the answer to this lies in the quantum statistics.
3 Low Energy Dynamics in the Strong Coupling Regime

The first step is to consider a collection of $n + 1$ charge $\gamma_A$’s in Seiberg-Witten theory, and represent it as a semiclassical state. The BPS equations of the Seiberg-Witten theory is [25, 26, 27, 28]

$$\vec{F}_I - i \zeta_T^{-1} \vec{\nabla} \phi_I = 0, \quad \vec{F}_D^I - i \zeta^{-1} \vec{\nabla} \phi_D^I = 0,$$

where $F = B + iE$ with magnetic field $B$’s and electric field $E$’s, and $\phi$’s are unbroken part of the complex adjoint scalars. They are all labeled by the Cartan index $I = 1, 2, \ldots, r$. $F_D$’s are defined through the low energy $U(1)$ coupling matrix as

$$\vec{F}_D^I \equiv \tau^{IJ} \vec{F}_J, \quad \tau^{IJ} = \frac{\partial \phi_D^J}{\partial \phi_I}.$$

The pure phase factor $\zeta_T$ is determined by the supersymmetry left unbroken by the total charge, namely equals the phase factor of the total central charge, $Z(\gamma_T)/|Z(\gamma_T)|$. Let us begin with a core-probe approximation, where we split $\gamma_T = \gamma_h + \sum A \gamma_A$ and treat the latter $n$ as a fixed background. Real parts of $F$ and $F_D$ obeys a Bianchi/Gauss law

In this core-probe approximation, three quantities determine entire low energy dynamics of the probe dyon, $\gamma_h$. The first is the inertia function $f$ as in

$$\mathcal{L} = \frac{1}{2} f \left( \frac{d\vec{x}}{dt} \right)^2 + \cdots.$$

The inertia $f$ is generally position-dependent because the core dyons deform the scalar fields, which in turn deform the effective local value of the central charge. Solving (3.1) for a given spatial distribution of “core charges” $\gamma_A$ and constructing the relevant central charge function lead to [21]

$$Z_{\gamma_h}(\vec{x}) = \gamma_e H(\vec{x}) + \gamma_m H_D(\vec{x}) + f(\vec{x}) = |Z_{\gamma_h}(\vec{x})|,$$

with the electric part $\gamma_e$ and the magnetic part $\gamma_m$ of the probe charge vector $\gamma_h$.

Note that, in construction of $Z_h$, we do not include $\gamma_h$ as a charge source for $\phi$ and $\phi_D$; for bound state construction we only need to understand how charge centers affect other charge centers. Also $\phi$ and $\phi_D$ are approximate semiclassical configurations, as we dropped the non-Abelian part by starting with Seiberg-Witten description. However, this is good enough if individual charge centers are far apart from each other and look effectively point-like. The latter is guaranteed, on the other
hand, by moving very near an MSW where relevant charge centers separate far apart from one another.

The other two, more important quantities are the scalar potential and the vector potential, respectively, $K^2/2f$ and $W$ in

$$
\mathcal{L} = \frac{1}{2} f \left( \frac{d\vec{x}}{dt} \right)^2 - \frac{K^2}{2f} - \frac{d\vec{x}}{dt} \cdot \vec{\mathcal{W}} + \cdots. \quad (3.5)
$$

A remarkable fact is that these two are again determined entirely by the same central charge function $Z_{\gamma_h}$ as [21]

$$
dW = *dK, \quad K = \text{Im}[\zeta^{-1}Z_{\gamma_h}] = \text{Im}[\zeta^{-1}Z(\gamma_h)] - \sum_{A'} \frac{\langle \gamma_h, \gamma_{A'} \rangle / 2}{|\vec{x} - \vec{x}_{A'}|}. \quad (3.6)
$$

This means that the interaction is analogous to that of a unit electrically charge particle in a magnetic monopole of total magnetic flux $2\pi \langle \gamma_h, \gamma_{A'} \rangle$ while being also constrained by a radial potential with minimum at some finite radius.

This simple and universal low energy dynamics of a probe dyon is a direct consequence of (3.1), which can be trusted whenever one is near a marginal stability wall. Note how this probe-dynamics is entirely determined by the single central charge function $Z_{\gamma_h}(\vec{x})$. A little further away from such a wall, one actually find the potential to be

$$
|Z_h| - \text{Re}[\zeta^{-1}Z_{h}], \quad (3.7)
$$

but this reduces to

$$
K^2/2|Z_h| = (\text{Im}[\zeta^{-1}Z_h])^2/2|Z_h|, \quad (3.8)
$$

as we move near the marginal stability wall where phases of $Z(\gamma_h)$ and $Z(\gamma_c = \sum_{A'} \gamma_{A'})$ line up [21]. This approximation is more than good enough since we already know that BPS spectrum are continuous far away from marginal stability walls; for the purpose of BPS state counting, we are allowed to go as close to an MSW as we wish, as long as we do not cross it.

When we treat all charge centers on equal footing, the low energy dynamics can be quite complicated. The part least affected by this is the Lorentz force, thanks to its topological nature. The half-integral Dirac-quantized coefficient in $W$ encodes how one particle’s electric (magnetic) charge see the other particles’ quantized magnetic (electric) charge. Because of the Bianchi identity the basic shape of $W$ cannot be corrected either, and also such interactions arise only from sum of two-body interactions. Therefore, these one-derivative interaction terms are computed by adding up all pair-wise topological couplings, as

$$
-\frac{d\vec{x}}{dt} \cdot \vec{\mathcal{W}} \rightarrow -\frac{d\vec{x}_A}{dt} \cdot \vec{\mathcal{W}}_A, \quad (3.9)
$$
where [19]
\[ \vec{W}_A = \frac{1}{2} \sum_{B \neq A} \langle \gamma_A, \gamma_B \rangle \vec{W}^{\text{Dirac}}(\vec{x}_A - \vec{x}_B) . \] (3.10)

\( \vec{W}^{\text{Dirac}} \) is the Wu-Yang vector potential [29] of a 4\( \pi \) flux Dirac monopole.

It turns out that this also fixes the scalar potential completely; see section 5 for more detail. The answer is
\[ \frac{K_A^2}{2f} \rightarrow \frac{1}{2}(f^{-1})^{AB}K_A K_B \] (3.11)
with [19]
\[ K_A = \text{Im} \left[ \zeta^{-1} Z_A \right] = \text{Im} \left[ \zeta^{-1} Z(\gamma_A) \right] - \frac{1}{2} \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{|\vec{x}_A - \vec{x}_B|} . \] (3.12)

Here, \( f_{AB} \) is \((n+1)\)-particle version of the inertia function \( f \), i.e.,
\[ \frac{1}{2} f \left( \frac{d\vec{x}}{dt} \right)^2 \rightarrow \frac{1}{2} f_{AB} \frac{d\vec{x}_A}{dt} \cdot \frac{d\vec{x}_B}{dt} . \] (3.13)

\( f_{AB} \) cannot be computed precisely with the above approach; one would need to repeat the probe-core dynamics with the core dyons slowing moving, in order to derive these kinetic functions in their full generality. Fortunately for us, details of \( f_{AB} \) is not important as long as zero locus of \( K_A \) carves out a large manifold far away from the “origin,” \( \vec{x}_A = \vec{x}_B \). See section 6. For most field theory BPS states, this limit suffices for the wall-crossing physics.

4 Subtlety: 3n vs. 2n

The low energy dynamics of such \((n+1)\) dyons include a scalar potential, \((f^{-1})^{AB}K_A K_B/2\), so it sounds natural to reduce it further to the classical moduli space, \( K_A = 0 \). Of \( 3n + 3 \) position coordinates, three center of mass coordinates would decouple, so the reduction would localize the relative part of the dynamics with \( 3n \) coordinates to \( 2n \) coordinates. It would seem that, for the purpose of finding ground states, one have an option to simplify the problem as,
\[ \mathbb{R}^{3n+3} \Rightarrow \mathbb{R}^3_{\text{c.o.m.}} \times \mathcal{M}_{n+1} , \] (4.1)
where \( \mathcal{M}_{n+1} \equiv \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{n+1} | K_A = 0 \}/\mathbb{R}^3_{\text{c.o.m.}} \). Among other things, this would immediately imply that supersymmetric ground state exists only if \( \mathcal{M}_{n+1} \) is not empty. The latter require the signs of \( \text{Im} \left[ \zeta^{-1} Z_A \right] \) be appropriately correlated with those of \( \langle \gamma_A, \gamma_B \rangle \). MSW’s are located where one or more of \( \text{Im} \left[ \zeta^{-1} Z(\gamma_A) \right] \)’s vanish.

9
However, this naive truncation is unjustified at quantum level, and cannot be taken at the face value. Before seeing why this is the case, it is worthwhile to note how it had failed in the past. Consider a two-particle case. As mentioned in the first section, this problem has been addressed by many different approaches including that of Denef [13], and the correct answer for the number of ground states is $|\langle \gamma_1, \gamma_2 \rangle|$ times individual degeneracies of the two constituent particles. On the other hand, $\mathcal{M}_2$ is a two-sphere threaded by the magnetic flux $2\pi \langle \gamma_1, \gamma_2 \rangle$. Since one is supposed to find BPS states that belong to $D = 4 N = 2$ theory, she should expect four (real) unbroken supercharges control the low energy dynamics. A two-sphere is not a hyperKähler manifold, so the best we can expect for the reduced dynamics with four real supercharges is a nonlinear sigma model onto $\mathcal{M}_2 = S^2$ with $\mathcal{N}_C = 2$ complex supersymmetry. However, this is a well-known quantum mechanics problem, whose ground state counting gives $|\langle \gamma_1, \gamma_2 \rangle| + 1$, instead. The correct answer $|\langle \gamma_1, \gamma_2 \rangle|$, on the other hand, is known to arise if one assumes smaller supersymmetry, which generates even more questions.

The reason for the failure can be understood easily with the same two-particle example. Note that, near $\mathcal{M}_2$, the massive radial mode $\delta r \equiv |\vec{x}_1 - \vec{x}_2| - R_{12}$ enters the potential as

$$\frac{K^2}{2f} \approx \frac{1}{2f(R_{12})} \left( \frac{\langle \gamma_1, \gamma_2 \rangle/2}{(R_{12})^2} \right)^2 \delta r^2,$$

where $R_{12} \equiv \langle \gamma_1, \gamma_2 \rangle/2\text{Im}[\zeta^{-1}Z_1] = \langle \gamma_2, \gamma_1 \rangle/2\text{Im}[\zeta^{-1}Z_2]$. The other, angular directions appear massless, and thus are deemed to be lower energy degrees of freedom.

This reasoning would have held if we were either dealing with classical mechanics or with higher dimensional quantum field theory. For quantum mechanics, a massgap can arise not only from massive potential but also when the target is of finite volume. A canonical example is the one-dimensional potential well of width $l$, whose lowest energy eigenvalue goes like $\sim 1/l^2$. Here the classical moduli space is $S^2$ of radius $R_{12}$, and the angular momentum part of the wavefunction contributes to the quantum energy,

$$\frac{1}{2f(R_{12})} \cdot \vec{L} \cdot \vec{L} - \left( \frac{\langle \gamma_1, \gamma_2 \rangle/2}{(R_{12})^2} \right)^2$$

with the angular momentum operator $\vec{L}$. Because of the magnetic flux in the background, both $\vec{L}$ gets shifted by the amount $\langle \gamma_1, \gamma_2 \rangle/2 \times (\vec{x}_1 - \vec{x}_2)/|\vec{x}_1 - \vec{x}_2|$, and the lowest eigenvalue for $\vec{L} \cdot \vec{L}$ is $|\langle \gamma_1, \gamma_2 \rangle/2|^2 + |\langle \gamma_1, \gamma_2 \rangle/2|$. The massgap associated with these classically massless directions are thus

$$\frac{|\langle \gamma_1, \gamma_2 \rangle/2|}{2f(R_{12}) \times (R_{12})^2}.$$
Note that this massgap equals exactly the ground state energy of the $\delta r$ harmonic oscillator (4.2) with $f(R_{12})$ being the inertia for $\delta r$. In short [19], at quantum level, there is no further natural reduction of relative dynamics on $\mathbb{R}^3$ to the classical moduli space $\mathcal{M}_2 = S^2$.

One can easily trace this equality among massgaps, to the identity

$$d\mathcal{W} = \ast d\mathcal{K}, \quad (4.5)$$

which, as we will see below, is due to $\mathcal{N} = 4$ supersymmetry of the low energy quantum mechanics. As was already stated in (3.10) and (3.12), such a close relationship between the scalar potential and the vector potential is not limited to two particle problem but holds true for arbitrary $(n + 1)$ particle problem. There again, $\mathcal{N} = 4$ supersymmetry demands it. Therefore, despite the very useful classical picture of charge centers at fixed mutual distances, one is not allowed to formulate the low energy dynamics on the classical moduli space, $\mathcal{M}_{n+1}$, spanned by such solutions.

5 $\mathcal{N} = 4$ Supersymmetry

Supersymmetrization of the low energy dynamics can be performed most economically in the $\mathcal{N} = 1$ superspace notation. This choice is also convenient because, later, we end up mathematically deforming the dynamics to a $\mathcal{N} = 1$ nonlinear sigma model for facilitating index computation.

Denote by $\vec{x}^A$ the position vector of the charge $\gamma_A$ center in the real space $\mathbb{R}^3$. The collective coordinate degrees of freedom for each charge center can be put into four $\mathcal{N} = 1$ superfields,

$$\Phi^{Aa} = x^{Aa} - i\theta\psi^{Aa}, \quad \Lambda^A = i\lambda^A + i\theta b^A, \quad (5.1)$$

with an auxiliary field $b^A$ introduced for the fermionic superfield $\Lambda^A$. The Lagrangian that supersymmetrizes the low energy dynamics of section 3 can be written as [21, 19]

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \quad (5.2)$$

where

$$\mathcal{L}_1 = \int d\theta \left( i\mathcal{K}_A(\Phi)\Lambda^A - i\mathcal{W}_{Aa}(\Phi) D\Phi^{Aa} \right), \quad (5.3)$$

encodes the scalar and the vector potentials; the kinetic term $\mathcal{L}_0$ will be discussed later. One can show

$$\delta_\epsilon \int dt \mathcal{L}_1 = 0 \quad (5.4)$$

11
under $\mathcal{N} = 4$ supersymmetry transformation rules, with $\psi^A \equiv \lambda^A$,

$$
\begin{align*}
\delta_\epsilon x^A &= i \eta^{mn}_{am} \epsilon^n \psi^A, \\
\delta_\epsilon \psi^A_m &= \eta^a_{mn} \epsilon^n \dot{x}^A_a + \epsilon_m b^A, \\
\delta_\epsilon b^A &= -i \epsilon_m \psi^{Am},
\end{align*}
$$

(5.5)

provided that

$$
\partial_{Aa} K_B = \frac{1}{2} \epsilon_{abc} (\partial_{Ab} \mathcal{W}_{Bc} - \partial_{Bc} \mathcal{W}_{Ab}),
$$

(5.6)

and

$$
\begin{align*}
\epsilon_{abc} \partial_{Ab} \partial_{Bc} K_C &= 0, \\
\partial_{Aa} \partial_{Ba} K_C &= 0.
\end{align*}
$$

(5.7)

If these constraints are not met, $\mathcal{L}_1$ would be invariant under the one manifest supersymmetry, which corresponds to $\epsilon^4$ in (5.5).

That these $\mathcal{N} = 4$ constraints are solved by (3.10) and (3.12) should be obvious to readers. As noted already in section 3, $\mathcal{W}_A$ are of topological nature and cannot be corrected by higher order effects. When obtaining $K_A$’s from $\mathcal{W}_A$’s via (5.6) and (5.7), the only extra input needed is the asymptotic values, $K_A(\infty) = \text{Im} \left[ \zeta^{-1}_A Z_A \right]$. However, these are tied to the energy cost when we separate $\gamma_A$ dyon center to spatial infinity and is determined unambiguously from the superalgebra. Therefore, despite the fact that we are dealing with dyons in strongly coupled theories, the interaction Lagrangian with one or less time derivative, $\mathcal{L}_1$, is fixed without any ambiguity at all. The small control parameters are the inverse of the classical distances between centers, which are in turn held small by the proximity to MSW’s.

The kinetic term $\mathcal{L}_0$ is a little more involved. The simplest way to find general form of $\mathcal{L}_0$ is to note that the collection $\{\Phi^A, \Phi^A_2, \Phi^A_3, \Lambda^A\}$ for each $A$ can be thought of as dimensional reduction of a $D = 4$ $N = 1$ vector superfield [30, 31]. In the Wess-Zumino gauge of the latter, $x^a$’s come from the spatial part of the vector gauge field, the fermions from the gaugino, and the auxiliary field $b$ from the $D = 4$ auxiliary field. As such, $N = 1$ the superspace of the latter can be used as $\mathcal{N} = 4$ superspace here. Let us again display $\mathcal{N} = 1$ form of such a general $\mathcal{N} = 4$ $\mathcal{L}_0$ as, [32]

$$
\mathcal{L}_0 = \int d\theta \dfrac{i}{2} g_{AaBb} D\Phi^A a \partial_t \Phi^B_b - \dfrac{1}{2} k_{AB} \Lambda^A D \Lambda^B - i k_{AaB} \Phi^A_2 a \Lambda^B + \cdots.
$$

(5.8)

where the ellipsis denotes four cubic terms that we omit here for the sake of simplicity. $\mathcal{L}_0$ is also invariant under the four supersymmetries we listed above,

$$
\delta_\epsilon \int dt \mathcal{L}_0 = 0
$$

(5.9)
with off-shell $b^A$’s, provided that various coefficient functions derive from a single real function $L(x)$ of $3n + 3$ variables as

\[
\begin{align*}
  g_{AaBb}(\Phi) &= \left( \delta_a^e \delta_b^f + \epsilon_{c}^{e} \epsilon_{a}^{cf} \right) \partial_{Ae} \partial_{Bf} L(\Phi), \\
  h_{AB}(\Phi) &= \delta_{ab} \partial_{Aa} \partial_{Bb} L(\Phi), \\
  k_{AaB}(\Phi) &= \epsilon_{ef}^{a} \partial_{Ae} \partial_{Bf} L(\Phi), \\
  \vdots
\end{align*}
\] (5.10)

Figuring out the precise form of the function $L$ for $n + 1$ charge centers requires further work. However, for the purpose of deriving wall-crossing formula for field theory BPS states, the asymptotic form of $L$ should suffice, as we argue below. In this limit, $L$ encodes only the masses of individual charge centers as

\[
L \approx -\frac{1}{2} \sum_{A} |Z(\gamma_A)| \vec{x}_A \cdot \vec{x}_A .
\] (5.11)

### 6 Deformation to $\mathcal{M}_{n+1}$ and Index Theorem

Now that we supersymmetrized the low energy dynamics, the discussion of section 5 extends easily to fermions as well. Of four fermions for each $A$, one pair acquires mass from the bilinear coupling to $dK_A$ while the other pair become massive via such a coupling to $dW_A$. With the tight constraint between $dK_A$’s and $dW_A$’s, it is clear that the fermions cannot be divided into massive “radial” and massless “angular” modes, either. In principle, one could proceed with the above Lagrangian and compute relevant indices in $\mathbb{R}^{3n}$. However, it turns out that the dichotomy between classically massive and classically massless direction can be salvaged yet, simplifying the index computations and bringing us to the classical moduli space $\mathcal{M}_{n+1}$ after all.

The key observation is that, in defining a supersymmetric index (i.e. the difference between the number of bosonic and the number of fermionic ground states), we only need one supercharge and one chirality operator. Although the full dynamics has $\mathcal{N} = 4$ supersymmetry, only one supercharge is needed for the computation. Furthermore, supersymmetric index is a topological quantity and insensitive to “small” deformations of dynamics. As we formulated the low energy dynamics of dyons in terms of $\mathcal{N} = 1$ superspace, we may as well keep the one manifest supersymmetry and deform the dynamics as we see fit, without affecting the index at all. Therefore, the same index can be computed from a different dynamical system, say, the one with the potential part of Lagrangian given as [19]

\[
\mathcal{L}_1^\xi = \int d\theta \ (i\xi K_A(\Phi) A^A - i\mathcal{W}_{Aa}(\Phi) D\Phi^{Aa}) ,
\] (6.1)
for an arbitrary real positive number $\xi$. This clearly breaks $\mathcal{N} = 4$ down to $\mathcal{N} = 1$ but has the advantage of making the “radial” direction far heavier than angular directions.

The equality of massgaps between the two sectors is disrupted by $\xi$; radial massgaps are now $\xi$ times larger than the angular massgaps. With $\xi \to \infty$, then, $n$ radial modes become infinitely heavy and can be decoupled from $2n$ modes along $\mathcal{M}_{n+1}$. Similarly the $2n$ fermions coupled $\xi dK$’s will become infinitely heavy, leaving behind the other $2n$ coupled to $dW$’s. It has been shown rigorously [19] that this deformation leaves behind an $\mathcal{N} = 1$ nonlinear sigma model with the Lagrangian onto $\mathcal{M}_{n+1}$

$$\mathcal{L}_{\text{for index only}}^{\mathcal{N}=1}(\mathcal{M}_{n+1}) = \frac{1}{2} g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu + \frac{i}{2} g_{\mu\nu} \psi^\mu \bar{\psi}^\nu + \cdots - A_\mu \dot{z}^\mu + \cdots,$$

(6.2)
schematically, where $A$ is an external Abelian gauge field on $\mathcal{M}_{n+1}$,

$$dA = \mathcal{F} \equiv d \left( \sum_A \mathcal{W}_{Adx^A} \right) \bigg|_{\mathcal{M}_{n+1}}$$

(6.3)
and the induced metric $g$ on $\mathcal{M}_{n+1}$ from the ambient $\mathbb{R}^{3n}$.

It is important to remember that the reduction applies to the relative part of the dynamics. The free center of mass part $\mathbb{R}^3$ remains intact with its own $\mathcal{N} = 4$ supersymmetry, and supplies the obligatory half-hypermultiplet structure to bound states that may emerge from the relative dynamics. Among other things, this also implies that the second helicity trace is computed in the relative dynamics as the usual Witten index with $(-1)^{2J_3}$ as the chirality operator, whose precise definition in the quantum mechanics need a bit more of clarification. Anyhow, since the surviving supersymmetry clearly reduces to the real supersymmetry on $\mathcal{M}_{n+1}$ twisted by the magnetic field $\mathcal{F}$, and since the fermionic partners are real, we come to the Dirac index,

$$\text{Tr} \left( (-1)^{F_{\mathcal{M}_{n+1}}} e^{-\beta Q^2} \right) = \int_{\mathcal{M}_{n+1}} \text{Ch}(\mathcal{F}) \hat{A}(\mathcal{M}_{n+1}) ,$$

with the Chern character $\text{Ch}$ and A-roof genus $\hat{A}$. This index theorem is the most basic ingredient that will eventually compute $\Omega \left( \sum_A \gamma_A \right)$, although there are more things to do before we can connect to actual bound state spectrum and the wall-crossing formulae.

The first thing to clear up is the chirality operator $(-1)^{F_{\mathcal{M}_{n+1}}}$, which in this equation is defined so that we have the canonical index formula. What is not immediate, however, is how this chirality operator is related to $(-1)^{2J_3}$. The answer to this has the universal form, [19]

$$(-1)^{2J_3} \rightarrow (-1)^{\sum_{A>B} \langle \gamma_A \gamma_B \rangle + n} (-1)^{F_{\mathcal{M}_{n+1}}}$$

(6.4)
bringing us to

\[
I_{n+1}(\{\gamma_A\}) \equiv (-1)^{\sum_{A>B}^{A \neq B} \left< \gamma_A, \gamma_B \right> + n} \int_{\mathcal{M}_{n+1}} Ch(\mathcal{F}) \hat{A} (\mathcal{M}_{n+1}) . \tag{6.5}
\]

as the right index theorem.

Second, as one approaches an MSW, the zero locus of the potential, \(\mathcal{K}_A = 0\), would expand to the asymptotic region, where the ambient space \(\mathbb{R}^{3n}\) near such large \(\mathcal{M}_{n+1}\) would be flat for all intent and purpose. Recall that there is no discontinuity before one reaches MSW; spectrum should be independent of how close we get to the MSW, or equivalently independent of how large \(\mathcal{M}_{n+1}\) becomes as long as the latter remains finite. With \(g\) induced from the flat \(\mathbb{R}^{3n}\), \(\mathcal{M}_{n+1}\) turns out to carry a trivial \(A\)-roof genus and the above collapses to a symplectic volume, [19]

\[
I_{n+1}(\{\gamma_A\}) = (-1)^{\sum_{A>B}^{A \neq B} \left< \gamma_A, \gamma_B \right> + n} \int_{\mathcal{M}_{n+1}} e^{\mathcal{F}/2\pi} , \tag{6.6}
\]

simplifying the problem of evaluation enormously. For two particle problems, in particular, this translates to

\[
\pm \left< \gamma_1, \gamma_2 \right> , \tag{6.7}
\]

which, as we mentioned already, is the right index for primitive wall-crossing. In particular, this without the wrong +1 shift that would have resulted from naive truncation of section 3.

Third, we should take care to include the degeneracies, or the indices \(\Omega(\gamma_A)\), of individual charge centers as well. In the above derivation, an implicit assumption was that such internal degeneracies of constituent particles do not interfere with the bound state formation. As such, they will simply contribute a multiplicative factor to the bound state counting. The index \(\Omega^{-}\), counted from the second helicity trace introduced in section 1, should be computed from \(I\) as

\[
\Omega^{-} \left( \sum_A \gamma_A \right) = I_{n+1}(\{\gamma_A\}) \prod \Omega(\gamma_A) . \tag{6.8}
\]

\(\Omega(\gamma_A)\) has no \(\pm\) superscript here since by construction we are considering \(\gamma_A\)'s that are point-like on both sides of MSW; such states would have the same intrinsic index on the two sides of the wall, since wall-crossing affects loose bound states. By the way, it could happen, in principle, that + side has states of total charge \(\sum_A \gamma_A\) but of different origin. In such cases, the left hand side is to be understood as \(\Omega^{-} - \Omega^{+}\).

The final issue, which is of more fundamental nature, comes from the fact that we treated \(\gamma_A\)'s as all distinguishable. In most wall-crossing problems, however, we end
up counting bound states of type \( n \gamma_1 + k \gamma_2 \) for positive integers \( n \) and \( k \). Statistics is thus a vital issue. In next section, we will see how this gets incorporated into the problem, whereby the rational invariants seen in the Kontsevich-Soibelman formalism of section 2 will re-emerge.

7 Statistics, Rational Invariants, and Wall-Crossing Formula

Statistics can be imposed on top of the index problem by inserting a projection operator that symmetrizes or anti-symmetrizes wavefunctions under permutations of identical particles. Given each permutation group \( S(k_i) \) of \( k_i \) identical particles, we need to insert the projection operator \([19]\]

\[
P(k_i) = \frac{1}{k_i!} \sum_{\sigma \in S(k_i)} (\pm 1)^{|\sigma|} \sigma
\]

for bosonic and fermionic constituent particles, respectively. As we expect typically more than one species of particles involved, we should insert such projections for each species. On most of configuration space, this projection will act freely and the net result would be a division of the target volume by \( 1/k_i! \).

However, there are fixed submanifolds spanned by configurations where, say, \( k_1 \) identical particles are moving together. This generates an additive contribution. Carving out a tubular neighborhood of such submanifolds, we can see that the dynamics along this submanifold will look exactly as before except that, instead of \( k_i \) identical particles of charge \( \gamma_i \), we see \( (k_i - k_1) \) identical particles of charge \( \gamma_i \) plus a single additional particle of charge \( k_1 \gamma_i \). For the extra contribution from such fixed submanifold, we must now insert a projection operator associated with \( S(k_i - k_1) = S(k_i)/S(k_1) \) permutation group, and repeat the exercise. From this general discussion, it is clear that the index with the projector \( P_i \)'s inserted can be iteratively decomposed to many index problems, each of which has the same total charge but smaller number of charge centers.

The projected index theorem goes as, with the two representative terms discussed above shown explicitly,

\[
\mathcal{I}_P \left( \sum_i k_i \gamma_i \right) = \text{Tr} \left( (-1)^{F_{M+1}} e^{-\beta Q^2} \prod P(k_i) \right)
\]

\[
= \frac{1}{\prod_i k_i!} \mathcal{I}_{n+1} \left( \{ \cdots, \gamma_i, \gamma_i, \cdots, \gamma_i, \gamma_i, \cdots \} \right)
\]
\[
\Delta(k_i^1) \quad \text{(7.2)}
\]
where each and every \( \mathcal{I} \) computes bound states of the same total charge and

\[
\Delta(p) = \pm 1/p^2 
\]  

(7.3)

is a universal factor associated with \( p \) coincident identical (bosonic/fermionic) particles. The origin of \( \Delta \) will be explained below. Once we accept the latter, it should be clear how terms in the above sum are generally constructed. For each and every partition

\[
k_i = \sum_{\alpha} k_i^\alpha, \quad k_i^\alpha \geq 1 ,
\]

(7.4)
treat \( k_i^\alpha \gamma_i \) as if it is an individual particle and compute \( \mathcal{I} \) for this reduced dynamics with less number of particles. Then the additive contribution to \( \mathcal{I}_P \) from such a partition is

\[
\prod_{i,\alpha} \Delta(k_i^\alpha) \quad \mathcal{I}(\{k_i^1 \gamma_i, k_i^2 \gamma_i, \cdots \}) .
\]

(7.5)
The denominator is clearly the volume-reducing factor from the residual permutation symmetry, and we again have the numerical factors \( \Delta(k_i^\alpha) \) for each and every coincident identical particles. \( \mathcal{I}_P (\sum_i k_i \gamma_i) \) is a sum of such terms over all possible partitions of \( \{k_i\} \).

For \( \mathcal{I}_P \), then, it remains to derive the multiplicative factors \( \Delta(k_i^\alpha) \). In the above, we evaluated the projected index \( \mathcal{I}_P \) by decomposing it according to how the permutation groups act. Each additive contribution arise from carving out an infinitesimally thin tubular neighborhood around a fixed submanifold where \( k_i^\alpha \) of \( \gamma_i \) charge centers coincides and move together. In this tubular neighborhood, however, there are also directions associated with separating \( k_i^\alpha \gamma_i \) into \( k_i^\alpha \) number of \( \gamma_i \)’s. They are the fibre directions of the normal bundle in \( \mathcal{M} \) of such a fixed submanifold. On top of \( \mathcal{I}(\{k_1^1 \gamma_1, k_1^2 \gamma_1, \cdots \}) \), then, one should expect to see a multiplicative factor associated with these directions.

Since these \( k_i^\alpha \gamma_i \)’s are identical and of the same charge, there are no interaction of type \( \mathcal{L}_1 \); Therefore, each \( \Delta(k_i^\alpha) \) should be computed from free \( k_i^\alpha \) particle dynamics, except that the projector \( \mathcal{P}(k_i^\alpha) \) should be inserted to correctly take into account the statistics. One might think that, since free dynamics cannot lead to a bound
state, so the corresponding factor $\Delta(k_i^\alpha)$ should vanish. However, when computing the above index $I_P$, we are actually computing the so-called bulk term which captures continuum contributions as well; for the full index this is good enough because $\mathcal{M}_{n+1}$ is a compact manifold and for $\Delta(k_i^\alpha)$ computation, the same limit should be used consistently since this part of computation is embedded in the index computation on the same compact $\mathcal{M}_{n+1}$.

Interestingly, exactly such a quantity $\Delta(p)$ was computed some fifteen years ago, in the context of supersymmetric Yang-Mills quantum mechanics [33, 34]. The problem back then was whether or not identical (and wrapped) D-branes can have a threshold bound state; a particular case of this with the maximal supersymmetry is the famous D0 bound state problem [35]. $\Delta(p)$ appeared there as a defect term, coming from a continuum contribution from $p$ identical and unbound particles, and had the universal form

$$\Delta^{YMQM}(p) = + \frac{1}{p^2}, \quad (7.6)$$

for any allowed supersymmetry. The $+$ sign shows up because the bosonic statistics is built-in for the Yang-Mills quantum mechanics, via the Weyl group. Repeating the exercises with the fermionic statistics allowed, we finds (7.3).

We must go one step further and figure out how the indices $\Omega(\gamma_i)$’s of the constituent particles affects the computation of $\Delta$’s, which lead to [19]

$$\Omega^- \left( \sum_i k_i \gamma_i \right) = \frac{\prod_i \Omega(\gamma_i)^{k_i}}{\prod_i |S(k_i)|} \times I(\{\gamma_1, \gamma_1, \ldots\})$$

$$\vdots$$

$$+ \frac{\prod_{i,o} \Omega(\gamma_i)/(k_i^\alpha)^2}{\prod_i |S(k_i)/(S(k_i^1) \times S(k_i^2) \times \cdots)|} \times I(\{k_1^1 \gamma_1, k_1^2 \gamma_1, \ldots\})$$

$$\vdots$$

$$= \Omega^{-} \left( \sum_i k_i \gamma_i \right)$$

$$\times \prod_i \frac{\Omega(\gamma_i)^{k_i}}{|S(k_i)|} \times I(\{\gamma_1, \gamma_1, \ldots\})$$

where one last ingredient we used is that the positive and the negative $\Omega(\gamma_i)$ imply, respectively, the fermionic and the bosonic statistics for $\gamma_i$. This computes the bound state index of charge $\sum_i k_i \gamma_i$ made from $\sum_i k_i$ such charge centers.

For the most general wall-crossing formula, we may allow the logical possibility that such a BPS state can be built differently, when other constituent charge centers are available, say, $\Omega^+(2\gamma_1) \neq 0$ or $\Omega^+(2\gamma_1 + 3\gamma_2) \neq 0$, etc. Adding up all such contributions yet again, we find that the final answer is a sum over all possible partition, $\{\gamma_K\}$,

$$\gamma_T \equiv \sum_A \gamma_A = \sum_K \gamma_K,$$  \quad (7.8)
where $\gamma_A$, some of which can be indistinguishable, are all primitive. By a partition, we mean that each $\gamma_K$ is a nonnegative integral linear combination of $\gamma_A$'s. Each partition generates an additive contribution so that the final index on the - side of MSW is

$$\Omega^-(\sum_A \gamma_A) = \cdots + \prod_K \frac{\bar{\Omega}(\gamma_K)}{|S(\{\gamma_K\})|} \times I(\{\gamma_K\}) + \cdots$$

(7.9)

with the residual permutation group, $S(\{\gamma_K\})$, and the rational invariants,

$$\bar{\Omega}(\gamma_K) = \sum_{s|\gamma_K} \frac{\Omega(\gamma_K/s)}{s^2}.$$  

(7.10)

To be more precise, one must take care to keep track of flavor charges in $\gamma$'s as well, to avoid potential ambiguities in this formula.

8 Protected Spin Character and Equivariant Index

A more general index that keeps track of global quantum numbers of states, beyond counting degeneracies, is known and dubbed the protected spin character (PSC) [36],

$$\Omega_{PSC}(y) = -\frac{1}{2} \text{tr} \left( (-1)^{2J_3} (2J_3)^2 y^{2J_3+2I_3} \right).$$

(8.1)

$J_3$ and $I_3$ are generators of, respectively, the little group $SU(2)_L$ and the R-symmetry $SU(2)_R$. Of $SU(2)_R \times U(1)_R$ R-symmetry, the latter factor is “spontaneously broken” by any given BPS state, as the central charge phase rotates under $U(1)_R$. How such an equivariant generalization descends to the low energy quantum mechanics deserves a brief explanation, as it also have caused some confusion in the past.

For the true low energy dynamics involving $3n+3$ bosonic coordinates, the descent is actually straightforward. Bosons $x^{Aa}$’s and fermions $\psi^Am$ transform naturally under $SO(4) = SU(2)_+ \times SU(2)_-$, as $(3,1)$ and $(2,2)$, respectively. Since $x^{Aa}$’s encode the positions of charge centers, they rotate as vectors under $SU(2)_L$ but must be invariant under $SU(2)_R$. From this it is then clear that $SU(2)_+ = SU(2)_L$ and $SU(2)_- = SU(2)_R$. As such, PSC descend to a quantum mechanical equivariant index,

$$\Omega_{PSC}(y) \rightarrow \Omega(y) = \text{tr} \left( (-1)^{2J_3} y^{2J_3+2I_3} \right),$$

(8.2)

verbatim, with $J_3$ and $I_3$ now understood to be those of the low energy dynamics. We further factored out the free center of mass part of Hilbert space, which effectively
drops \( -(2J_3)^2/2 \) and instead traces only over the relative part of the dynamics. When \( y = 1 \), this is precisely the index we computed above.

In deforming the dynamics for index computation, down to the nonlinear sigma model on \( \mathcal{M}_{n+1} \), one must be a little more careful, as the process does not preserve \( SO(4) \) global symmetry. Again this can be illustrated easily with the minimal example of two particle system, where \( \mathcal{M}_2 = S^2 \). The latter manifold admits only one isometry group, call it \( SO(3)_J \), whose origin in \( SO(4) = SU(2)_L \times SU(2)_R \) should be clarified. Recall that, after the deformation and taking the limit \( \xi \to \infty \), one finds a nonlinear sigma model onto \( \mathcal{M}_2 = S^2 \). In nonlinear sigma models, bosonic fluctuations \( \delta z^\mu \) and fermionic fluctuations \( \psi^\mu \) transform in the same manner under coordinate transformations and also under isometry. This means that \( SO(3)_J \) transform the surviving \( 2n \) bosonic angles and \( 2n \) fermionic partners by the same rule. Each originate from \((3,1) \) and \((2,2) \) of \( SO(4) \), and clearly this implies that the surviving isometry is the diagonal subgroup. That is,

\[
J_a = J_a + I_a. \tag{8.3}
\]

The same is easily seen to be true of general \((n + 1)\) particle problems. Furthermore, it is obvious from (5.5) that this \( J \) commute with the single manifest \( \mathcal{N} = 1 \) supersymmetry, associated with \( \epsilon^4 \), of \( L_1^\xi \) in (6.1). The latter is also what becomes the supersymmetry of \( \mathcal{M}_{n+1} \) nonlinear sigma model.

Therefore, PSC of the field theory reduces to the \( \mathcal{M}_{n+1} \) index as

\[
\Omega_{\text{PSC}}(y) \to \Omega(y) = \text{tr}_{\mathcal{M}_{n+1}} \left( (-1)^{2J_3} y^{2J_3} \right), \tag{8.4}
\]

where \((-1)^{2J_3}\) still makes sense as a chirality operator, even though \( SU(2)_L \) is broken by \( \xi \neq 1 \). (See section 6 for more on this chirality operator.) This is the usual equivariant index for the nonlinear sigma model, up to an overall sign of the chirality operator that we took care to fix in section 6, so we finally matched PSC of field theory to equivariant index of \( \mathcal{M}_{n+1} \). Extension of wall-crossing formula of previous section to such an equivariant version is straightforward and well-established in mathematics literatures. We will refer readers to Maschots et.al. \[24, 37\] for detailed exposition on these equivariant quantities in the current context, as well as for explicit evaluations.

### 9 Conclusion and Beyond Multi-Center Picture

In this talk, we reviewed how the multi-center picture of BPS dyons leads to an intuitive understanding of wall-crossing, and outlined how one derives low energy dynamics for such semiclassical objects even in strongly coupled regime. We then derived general wall-crossing formula for \( D = 4 \; \mathcal{N} = 2 \) supersymmetric field theories.
Along the process, we clarified when the low energy dynamics may be used, how it should be formulated as $\mathbb{R}^{3n+3}$ quantum mechanics rather than $\mathcal{M}_{n+1}$ nonlinear sigma model, how the field theory index descends to those of the low energy dynamics, and finally why the Dirac index on $\mathcal{M}_{n+1}$ is the relevant one despite the wrong supersymmetry and the wrong dynamics. Bose/Fermi statistics of the constituent particles are shown to be incorporated in the wall-crossing formula via rational invariants of Kontsevich-Soibelman, which was used later to show equivalence of the latter’s proposal and our physically derived one [23].

Similar low energy dynamics had appeared in the past as the so-called Coulomb phase picture of the quiver quantum mechanics [13]. These quiver theories arise naturally as low energy theories of D3 branes wrapped on 3-cycles in Calabi-Yau compactified type IIB theory; through various dualities, they are potentially capable of capturing dynamics of large classes of BPS states with four preserved supercharges. While our starting point is very different from this, one could regard the low dynamics we found as an alternate derivation, in the field theory limit, of Coulomb phase dynamics of relevant quiver quantum mechanics. Furthermore, the latter part of our analysis applies to more general Coulomb phase dynamics such as those for black holes: deformation of $\mathcal{N} = 4$ relative dynamics to $\mathcal{N} = 1$ $\mathcal{M}_{n+1}$ nonlinear sigma model, Dirac index on $\mathcal{M}_{n+1}$ as the basic counting quantity, statistics via rational invariants, and the resulting wall-crossing formulae are all straightforwardly applicable to general quiver theories.

Sometimes, however, the multi-center BPS state picture, inherent to the Coulomb phase description and typical for field theory BPS states, is known to miss a large class of states in the quiver theory. One early question in this topic was whether or not the exponential degeneracy of BPS black hole might be explained from such multi-center pictures, but it was soon realized that, with a given quiver, one sometimes finds states of large degeneracy which appear completely missing in the usual Coulomb phase description. In fact, it is easy to construct examples where exponentially large number of these extra states appear [14] in the Higgs phase instead; the degeneracy in the Coulomb phase in those examples are at most powerlike.

This tells us that there are more to BPS state counting than wall-crossing phenomena know about; In the Higgs phase, one finds more comprehensive ground state space. Furthermore, index counting there is also more straightforward in that subtleties we encountered in the Coulomb phase are absent. However, these advantages come at the cost of losing the simple and intuitive multi-center picture. Wall-crossing occurs also in the Higgs phase, with exactly the same discontinuity as in the Coulomb phase; what has been missing is a way to distinguish, among the Higgs phase states, the counterparts of the wall-crossing multi-center Coulomb phase states from those non-wall-crossing states. Recently, exactly such a method was devised [38, 39]. The
proposal classifies Higgs phase ground states into two types with geometrically distinct origins, and identifies one class as the counterpart of wall-crossing states. The other, expected to be non-wall-crossing, must be then naturally an invariant of the quiver quantum mechanics insensitive to continuous change of parameters. Both of these claims have been tested extensively [40, 41]. This new handle will hopefully provide even powerful and versatile methods to address BPS states, in particular including a large class of BPS black holes and microstates thereof.

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