A new class of fuzzy spaces with classical limit

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Abstract

We present a new type of matrix regularization, which is based on matrix-valued functions defined on a cylinder. If non-commutative coordinates of a fuzzy space are defined by a regularization of such functions, we show that a classical limit for the fuzzy spaces exists, when the matrix-valued functions nearly commute. In this case, the classical limit of the fuzzy space is a manifold, which is composed of coordinate patches that are defined by the diagonal entries of the diagonalized matrix-valued functions. As applications, an interpolation for direct sums of fuzzy spaces is described and a fuzzy string vertex with classical limit, i.e. a surface interconnecting three circles, is constructed.

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1 Introduction

Non-commutative geometry is a promising candidate for describing quantized space-times, at least for providing effective actions, which go beyond the classical regime of general relativity [1]. In non-commutative geometry, the commutative algebra of functions on a manifold is replaced with a non-commutative algebra having extra structures that describe the geometry of the space. One approach for this are “spectral triples”, which encode the geometry of the non-commutative space in a functional analytic way [2].

We follow here a more physical approach, which is motivated by string theory and matrix models [4, 5, 6]. In this approach, which we call “fuzzy spaces”, the non-commutative geometry is defined analogously to an embedding of a classical manifold in a target space. In particular, a classical manifold can be defined via its coordinate functions, which embed it in a higher dimensional space, which is usually \( \mathbb{R}^n \). Other features of the geometry, such as a metric, then can be defined via a pull-back from the embedding space. For fuzzy spaces, the coordinate functions are replaced with coordinate matrices or more general with operators on a Hilbert space and it is then tried to derive objects such as a non-commutative metric, curvature etc., analogously to the corresponding objects in a classical geometry [3].

One way to define a fuzzy space is via algebraic relations of the non-commutative coordinates. Representations of the generated algebra then result in matrices for the non-commutative coordinates, which have the demanded properties. Such an approach is feasible for highly symmetric spaces, such as the fuzzy sphere [7] or the fuzzy torus [6], but becomes more and more complicated, in particular when the algebra relations or commutators become non-polynomial functions.

Another approach for defining fuzzy spaces is matrix regularization based on Fourier decomposition (see [10], for example). The algebra of functions on a manifold is decomposed into Fourier series and the Fourier series are truncated at a specific order. The Hilbert space basis of the Fourier series can be mapped to a basis of matrices and in such a way, every truncated function can be mapped to a matrix. This is the approach we will extend in the following chapters.

In chapter 2 we investigate the spaces introduced in [10], which we call “immersed cylinder fuzzy spaces”, since these spaces are matrix regularizations of a cylinder immersed in \( \mathbb{R}^n \). When it is possible to define the manifold to be regularized via only one coordinate patch in the form of a cylinder, i.e. a circle times an interval, the coordinate functions, which immerse the cylinder, can be decomposed into their Fourier series along the circle parameter. The results are Fourier coefficients for every coordinate function, which smoothly depend on the parameter of the interval. The entries of the coordinate matrices corresponding to the coordinate functions then can be defined by discretizing the Fourier coefficient, i.e. a matrix entry is the value of a Fourier coefficient at a specific coordinate of the interval, which coordinate depends on the indices of the matrix entry.

Chapter 3 relates to direct sums of fuzzy spaces. We show that direct sums of immersed cylinder fuzzy spaces can be related to matrix-valued functions on a cylinder. We furthermore investigate unitary transformations that naturally can be applied to matrix-valued functions. Such unitary transformations also can be regularized into unitary transformations of the fuzzy space. As the unitary transformations are already present in the classical regime, their regularizations are examples of transformations that do not correspond to...
smooth symplectomorphism.

In chapter 4 we generalize the construction of immersed cylinder fuzzy spaces to general matrix-valued functions. We show that when commuting matrix-valued functions are used, a classical limit can be defined. Usually, the classical limit of a fuzzy space is defined via a series of fuzzy spaces, which have the same properties, for example which are the representations of the same algebra, and which in a sense converge to a symplectic manifold, see [8] for a rigorous definition. For fuzzy spaces defined by matrix algebras, this is usually achieved by letting the size $N$ of the matrices grow to infinity.

For the type of fuzzy spaces developed in chapter 4, we show that in the large $N$ limit, the regularization of the product of two matrix-valued functions converges to the product of the quantization of these functions, when the two functions commute up to first order in $\frac{1}{N}$. Additionally, a generalization for a Poisson structure for matrix-valued functions is presented in this context. As will be shown, matrix-valued functions commuting up to first order can be interpreted as coordinate patches of a manifold.

In chapter 5, a general construction is provided, which makes it possible to interpolate two fuzzy spaces smoothly with each other. With this interpolation method, a fuzzy string vertex with classical limit, i.e. a surface that interconnects three circles with each other, is constructed. Such a string vertex can be used to construct surfaces of higher genus. For example, two string vertices can be concatenated into a torus. In such a way, fuzzy surfaces having as classical limit a surface of any genus can be constructed.

In chapter 6 we compare the new type of fuzzy spaces with already known fuzzy spaces, like the fuzzy cylinder and the fuzzy torus. The known fuzzy spaces are brought by a non-commutative coordinate transformation, i.e. by a mapping of the coordinate matrices into other matrices, into a form, which is suitable for a comparison. It is shown that these coordinate transformed fuzzy spaces have the same features and that also some kind of fuzzy string vertex is present.

## 2 Immersed cylinder fuzzy spaces

In the beginning, let us give a more rigorous definition of a fuzzy space, which is used herein. A fuzzy space is a set of $d$ Hermitian $N \times N$-matrices $\hat{X}^a$, $a = 1, \ldots, d$, which can be interpreted as the quantized embedding functions $x^a$ of a classical manifold immersed in $\mathbb{R}^d$.

Unitary transformations applied to the coordinate matrices and more general to matrices of the algebra generated by these matrices

$$\hat{X}^a \rightarrow \hat{U}\hat{X}^a\hat{U}^\dagger$$

(2.1)

correspond to symplectomorphic coordinate transformations. As a Hermitian matrix always can be diagonalized with a unitary transformation, we can fix this symmetry (at least partially up to permutations) be demanding that one of the matrices, such as the last one $\hat{X}^d$ is diagonal.

In the following we often will restrict to a three dimensional fuzzy space with three Hermitian $N \times N$-matrices $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ and assume that $\hat{Z}$ is diagonal.
2.1 Classical limit

The fuzzy spaces presented below are more precisely series of matrix algebras, numbered by the size $N$ of the matrices $\hat{X}_N^\alpha$. The entries of the matrices $\hat{X}_N^\alpha$ are determined from specific values of the Fourier components of a function $x^\alpha$ on a manifold immersed in $\mathbb{R}^d$. (see (2.6) and (4.2) below). These series, are matrix regularizations according to \cite{8}, i.e. the classical limit of these series of matrix algebras is the space of functions on the manifold immersed in $\mathbb{R}^d$.

The definition of a matrix regularization according to \cite{8} is the following: Let $Q_N$ be a series of linear maps from a space of functions on a manifold $M$ with functions $f$ (here functions on a submanifold of $\mathbb{R}^d$) to Hermitian $N \times N$-matrices $Q_N(f)$. Then the $Q_N$ are a matrix regularization, when

1. The matrices $Q_N(f)$ converge in matrix norm, i.e. for a function $f$

$$\lim_{N \to \infty} \|Q_N(f)\| < \infty$$  \hspace{1cm} (2.2)

(for example, $\|f\| = \max_{1<n<N} \sum_{n=1}^N |f_{nm}|$ for a matrix $\hat{f}$).

2. The matrix multiplication converges to the ordinary product of functions, i.e. for For two functions $f, g$

$$\lim_{N \to \infty} \|Q_N(f)Q_N(g) - Q_N(fg)\| = 0$$ \hspace{1cm} (2.3)

3. For differentiable functions $f, g$ the commutator converges to a Poisson bracket

$$\lim_{N \to \infty} \|-i\beta(N)[Q_N(f), Q_N(g)] - Q_N\{f, g\}\| = 0$$ \hspace{1cm} (2.4)

i.e. the manifold is a Poisson manifold. The function $\beta(N)$ should have the property to converge to a constant for $N \to \infty$.

Due to the last property, a matrix regularization can be seen as a special kind of quantization in the sense of quantum mechanics, at least, when the Poisson bracket is non-degenerate. In this case, the manifold becomes a symplectic manifold and functions on the symplectic manifold are mapped by the regularization to operators in a way that the commutator of two functions is nearly the regularization of the Poisson bracket induced by the symplectic structure.

We also introduce “convergence within border” and “equivalence within border” for matrices. To define this, let $\delta$ be small compared to $N$ and let $\hat{f}_\delta$ be the matrix, where the outer border of size $\delta$ has been set to 0, i.e. $\hat{f}_{\delta,nm} = \hat{f}_{nm}$ for $\delta < n, m < N - \delta$ and $\hat{f}_{\delta,nm} = 0$ for $n, m \leq \delta$ and $N - \delta \leq n, m$.

We say that a matrix $\hat{f}_N$ converges within the border $\delta$, when $\lim_{N \to \infty} \|\hat{f}_{N,\delta}\|$ converges. Equivalently, we also can use a weaker kind of matrix norm, which only accounts for matrix entries within the border, such as $\|\hat{f}\| = \max_{\delta<n<N-\delta} \sum_{n=\delta}^{N-\delta} |f_{nm}|$. Analogously, we say that two matrices $\hat{f}$ and $\hat{g}$ are equivalent within the border $\delta$, when $\hat{f}_\delta = \hat{g}_\delta$. With such definitions, we are able to account for fuzzy spaces that in a sense have a manifold with border as classical limit.
2.2 Generalized fuzzy cylinder

A simple example of a fuzzy space is the fuzzy cylinder (see [6, 11]). Generalizations of the fuzzy cylinder will play an important role in the following. The fuzzy cylinder is based on a mapping from functions to Toeplitz matrices, i.e. matrices in which each descending diagonal from left to right is constant.

A basis for Toeplitz matrices are \( N \times N \)-matrices \( \hat{e}_{a,N} \) with only 1 on the \( a \).th diagonal, i.e.

\[
\hat{e}_{a,N} = \sum_n |n\rangle\langle n+a|
\]  

where \( a \) can be an integer between \(-N+1\) and \( N-1\). For example, \( \hat{e}_0 \) is the unit matrix.

On the other hand let \( f(\varphi) = \sum f_n e^{in\varphi} \) be the Fourier series of the function \( f \). The mapping from functions to matrices via

\[
\hat{f} = Q_N(f) = \sum_{n=-N+1}^{N-1} f_n \hat{e}_{n,N}
\]  

preserves multiplication of functions for \( N \to \infty \), since \( Q_\infty(e^{in\varphi})Q_\infty(e^{im\varphi}) = \hat{e}_n \hat{e}_m = \hat{e}_{n+m} = Q_\infty(e^{i(n+m)\varphi}) \) and therefore

\[
Q_\infty(f)(Q_\infty(g)) = Q_\infty(fg)
\]  

for two functions \( f \) and \( g \). The linear operators \( Q_N \) are a matrix regularization as defined above. In the remaining, we will often indicate functions in the function space by its arguments, such as \( f(\varphi) \), and the corresponding regularized matrices with a hat, such as \( \hat{f} \).

The generalized fuzzy cylinder is defined in the following way. Let \( x(\varphi) \) and \( y(\varphi) \) be two functions that define a closed curve in \( \mathbb{R}^2 \) and let

\[
x(\varphi) = \sum x_n e^{in\varphi} \quad \text{and} \quad y(\varphi) = \sum y_n e^{in\varphi}
\]

be their Fourier series. Then the matrices replacing the coordinate functions are defined via

\[
\hat{x}_N = Q(x) = \sum_{n=-N+1}^{N-1} x_n \hat{e}_{n,N} \\
\hat{y}_N = Q(y) = \sum_{n=-N+1}^{N-1} y_n \hat{e}_{n,N} \\
\hat{z}_N = \sum_{n=1}^{N} \beta \frac{n}{N} |n\rangle\langle n|
\]

Since \( \hat{x}_N \) and \( \hat{y}_N \) are Toeplitz-matrices, they commute, i.e. \([\hat{x}_N, \hat{y}_N] = 0\). When we further assume that \( x_n = 0 \) and \( y_n = 0 \) for \( n > \delta \), where \( \delta \) is small compared to \( N \), then the commutators with \( \hat{z}_N \) fulfill

\[
[\hat{z}_N, \hat{x}_N] = Q(-i\beta \partial_\varphi x(\varphi)), \quad [\hat{z}_N, \hat{y}_N] = Q(-i\beta \partial_\varphi y(\varphi))
\]

within the border \( \delta \). For \( N \to \infty \) the algebra generated by the matrices converge in the sense of (2.2), (2.3) and (2.4) within the border \( \delta \) to an algebra of functions on a
submanifold of \( \mathbb{R}^3 \), which is the product of the curve defined by \( x(\varphi) \) and \( y(\varphi) \) times the interval \([0, \beta]\). This manifold has the natural Poisson-bracket

\[
\{ f, g \} = \partial_\varphi f \partial_\varphi g - \partial_\varphi f \partial_\varphi g
\]  

(2.10)

It is also possible to start in (2.8) with \( \hat{\kappa}_\infty \) and \( \hat{\gamma}_\infty \) and use \( \hat{\zeta}_\beta = \sum_n \beta n \langle n \rangle \langle n \rangle \). When we let \( \beta \) got to 0, the classical limit is the curve defined by \( x(\varphi) \) and \( y(\varphi) \) times \( \mathbb{R} \). However, in this case, the definitions (2.2), (2.3) and (2.4) have to be extended to operators on a Hilbert space.

### 2.3 Immersed cylinder

For a further generalization of the fuzzy cylinder, we start again with a function defined by a Fourier series, which however depends on a second parameter

\[
f(q, \varphi) = \sum_{n \in \mathbb{N}} f_n(q) e^{in\varphi}
\]  

(2.11)

with \( q \in [q_1, q_2] \) a finite interval and \( \varphi \in [0, 2\pi] \). We note that, when the Fourier series is truncated at a specific \( n \), for example for \( |n| < \delta \) the function \( f \) is approximated by the truncated Fourier series. In the following it is assumed that \( f_n = 0 \) for \( |n| \geq \delta \) for a constant \( \delta \) that is small compared to \( N \).

A matrix regularization \( Q_N(f) \) for the function \( f \) is defined with the \( N \times N \)-matrix

\[
Q_N(f) = \hat{f} = \sum_{n,m=1}^N f_{m-n} (q(n, m)) \hat{e}_{nm}
\]  

(2.12)

where \( \hat{e}_{nm} = |n\rangle \langle m| \). This definition can be extended to “infinite dimensional” matrices, i.e. \( N = \infty \). Also with \( N = \infty \), criteria (2.2) is fulfilled, due to the \( \delta \)-condition. When \( f \) does not depend on \( q \), definition (2.12) reduces to the regularization (2.6) of the generalized fuzzy cylinder.

\( q(n, m) \) is a discretizing function, which assigns the matrix entry \( \hat{e}_{nm} \) to a point in the interval of \( q \). \( q(n, m) \) should have to following properties:

- It should fill out the interval for \( q \): \( q(0, 0) = q_1 \), \( q(N, N) = q_2 \) (as for large \( N \), \( q(0, 0) \approx q(1, 1) \)), we can take \( q(0, 0) \) instead of \( q(1, 1) \), which renders the formulas for the examples for \( q(n, m) \) below less complex).
- It should depend “smooth” on its arguments, i.e \( q(m, p) = q(m, n) + \frac{\beta}{N}(p - n) \) for \( p - n \) small compared to \( N \) and some constant \( \beta \) that can depend on \( m,n \) and \( p \). Analogously, \( q(p, n) = q(m, n) + \frac{\beta}{N}(p - m) \) for \( p - n \) small compared to \( N \).
- Examples are \( q(n, m) = \frac{q_2 - q_1}{2N}(m+n)+q_1 \) (in this case \( \beta = \frac{1}{2} \)) or \( q(n, m) = \frac{q_2 - q_1}{2N}n+q_1 \) (in this case \( \beta = 1 \) for the left argument and \( \beta = 0 \) for the right argument).

When the function \( f \) is real-valued, the resulting matrix \( Q_N(f) \) is Hermitian for a symmetric discretizing function \( q(m, n) = q(n, m) \), since for a real-valued function, the coefficients of the Fourier series have the property that \( f_n = f_{-n} \).
Analogously to \cite{10}, we can now evaluate the matrix product of two matrices \( \hat{f} \) and \( \hat{g} \) as defined in (2.12), which have one times differentiable Fourier coefficient functions \( f_n(q) \) and \( g_n(q) \)

\[
(\hat{f}\hat{g})_{nm} = \sum_{1<p<N} f_{p-n}(q(n,p)) g_{m-p}(q(p,m))
= \sum_{|n-p|<\delta,|p-m|<\delta} f_{p-n} \left( q(n,m) + \frac{\beta}{N}(p-m) \right) g_{m-p} \left( q(n,m) + \frac{\beta}{N}(p-n) \right)
= \sum_{|n-p|<\delta,|p-m|<\delta} f_{p-n} g_{m-p}
\quad + \frac{\beta}{N} \left( f_{p-n}(p-n) \frac{\partial g_{m-p}}{\partial q} + (p-m) \frac{\partial f_{p-n}}{\partial q} g_{m-p} \right) + \mathcal{O} \left( \frac{1}{N^2} \right)
\tag{2.13}
\]

where in the last step all functions are evaluated at \( q = q(n,m) \). Since

\[-i \frac{\partial f}{\partial \varphi} = \sum_n n f_n(q) e^{in\varphi} \tag{2.14}\]

the last equation is equivalent to

\[
Q_N(f)Q_N(g) = Q_N(fg) + i \frac{\beta}{N} Q \left( \frac{\partial f}{\partial \varphi} \frac{\partial g}{\partial q} - \frac{\partial g}{\partial \varphi} \frac{\partial f}{\partial q} \right) + \mathcal{O} \left( \frac{1}{N^2} \right)
= Q_N(fg) + i \frac{\beta}{N} Q_N(\{f,g\}) + \mathcal{O} \left( \frac{1}{N^2} \right)
\tag{2.15}
\]

It follows that the commutator of such regularized functions can be approximated by the Poisson bracket of the functions

\[
[Q_N(f),Q_N(g)] = i \frac{2\beta}{N} Q_N(\{f,g\}) + \mathcal{O} \left( \frac{1}{N^2} \right)
\tag{2.16}
\]

Thus, all three criteria (2.2), (2.3), (2.4) above are fulfilled. Note that the Poisson bracket of the classical limit is \( \{q,\varphi\} = -1 \), which also follows from \( [\hat{q},Q_N(e^{i\varphi})] = \frac{1}{N} Q_N(e^{i\varphi}) \).

When the coefficient functions \( f_n(q) \) and \( g_n(q) \) are only continuous, one can show that \( Q_N(f)Q_N(g) = Q_N(fg) + \mathcal{O} \left( \frac{1}{N}\right) \) and \( Q_N(f),Q_N(g) = \mathcal{O} \left( \frac{1}{N}\right) \), i.e. the criteria (2.2), (2.3) are still fulfilled.

From (2.14) one sees that the Fourier coefficients \( f_n \) for \( n \to \infty \) need to go to 0 faster than \( \frac{1}{n} \), which is the case for functions with \( f_n = 0 \) for \( |n| \geq \delta \). However, also other functions, like functions twice differentiable in \( \varphi \) and specific piece wise continuous functions (but not all) have this property. In (2.13) only the differentiability in \( q \) was used. As long as the Fourier coefficients go to 0 as fast such that (2.13) is fulfilled, (2.12) defines a matrix regularization. Thus, it is possible to start with functions \( x(q,\varphi) \) and \( y(q,\varphi) \), which do not define a closed curve for a specific \( q \), but which only have Fourier coefficients, which go to 0 fast enough. In this case, it would be better to talk of a fuzzy immersed strip. The same applies to the generalized cylinder as described in the previous section.
Up to now, we have ignored, what happens at the borders \( q = q_1 \) and \( q = q_2 \). Due to the discretizing function \( q(n, m) \), which maps values at the border of the \( q \)-interval to the border of the matrices, we have to look at \( n < \delta \) and \( N - n < \delta \). We see that formula (2.15) is only valid within the border \( \delta \). This problem can be avoided, when the coefficient functions \( f_n(q) \) and \( g_n(q) \) go to zero at the border, i.e. when \( f_n(q_1) = f_n(q_2) = 0 \). This can be seen by embedding a matrix of size \( N \) in a bigger zero matrix, resulting in a border of zeros surrounding the original matrix. The continuation of functions \( f_n(q) \), which do not go to zero at the border, into this border would result in non-continuous extended functions, which are not differentiable.

When we demand that \( f(q_1, \varphi) = f(q_2, \varphi) = 0 \), the submanifolds that can be regularized with \( Q_N \) have to be immersions of a cylinder into \( \mathbb{R}^d \) with ends shrunk to a point. When the submanifold has a border, the \( Q_N \) only converge within border.

When the target space is \( \mathbb{R}^3 \), a two-dimensional immersed submanifold can be parametrized by the three functions

\[
\begin{align*}
  x(q, \varphi) &= \sum_{n \in \mathbb{N}} x_n(q)e^{in\varphi} \\
  y(q, \varphi) &= \sum_{n \in \mathbb{N}} y_n(q)e^{in\varphi} \\
  z(q, \varphi) &= z(q)
\end{align*}
\]

The fuzzy space defined by matrix regularization of these functions with \( N \times N \)-matrices is then

\[
\begin{align*}
  \hat{x} &= \sum_{n,m=1}^{N} x_{m-n}(q(n, m)) \hat{e}_{nm} \\
  \hat{y} &= \sum_{n,m=1}^{N} y_{m-n}(q(n, m)) \hat{e}_{nm} \\
  \hat{z} &= \sum_{n=1}^{N} z(q(n, n)) \hat{e}_{nn}
\end{align*}
\]

### 2.4 Example: from circle to 8

In this section we give an example based on the fuzzy immersed cylinder that later on will be generalized to a fuzzy manifold having the structure of a string vertex. Here, we construct the part of the vertex, which at one end with respect to the \( z \)-direction has a circular cross-section and transforms along the \( z \)-direction into a cross-section similar to an 8. The center crossing of the eight is the point, where the string vertex splits into two cylinders.

To define the fuzzy "circle-to-eight"-space let us start with two coordinate functions in polar coordinates, i.e. \( x(z, \varphi) = r(z, \varphi) \cos \varphi \) and \( y(z, \varphi) = r(z, \varphi) \sin \varphi \) with \( r(z, \varphi) = r_1(z) + r_2(z) \cos 2\varphi \). This is equivalent to
Figure 1: From circle to eight

\[ x(z, \varphi) = \left( r_1(z) + \frac{r_2(z)}{2} \right) \cos \varphi + \frac{r_2(z)}{2} \cos 3\varphi \]
\[ y(z, \varphi) = \left( r_1(z) - \frac{r_2(z)}{2} \right) \sin \varphi + \frac{r_2(z)}{2} \sin 3\varphi \]  

(2.19)

With \( r_1(z) = 1 \) and \( r_2 \) continuously increasing between \( z_1 \) and \( z_2 \) and \( r_2(z_1) = 0 \) and \( r_2(z_2) = 1 \) the corresponding immersed manifold has a transition from a circle to a 8 as shown to the left of Fig. 1.

To fix the functions \( r_1 \) and \( r_2 \) between \( z_1 \) and \( z_2 \), we need a continuously increasing, differentiable function \( h(q) \) with \( h(q < -1) = 0 \) and \( h(q > 1) = 1 \). For the figures and matrices in this article, this function is always represented with a cubic spline interpolating the points \((-1, 0), (-0.5, 0.1), (0, 0.5), (0.5, 0.9) \) and \((1, 1)\) with further support points at \( h = 0 \) and \( h = 1 \) outside of \(-1 \) and \( 1 \). The spline \( h \) with it support points (and the function \( r_2 \), which is here equivalent to \( h \)) is shown to the right of Fig. 1.

A matrix regularization according to (2.12) for the manifold shown in Fig. 1 is

\[ \hat{z} = \sum_n \frac{n}{N} \hat{e}_{nn} \]  

(2.20)

\[ \hat{x} = \sum_n \left( 1 + \frac{1}{2} h(-1 + \frac{2n}{N}) \right) \frac{1}{2} (\hat{e}_{n,n+1} + \hat{e}_{n+1,n}) + \frac{1}{2} h(-1 + \frac{2n}{N}) \frac{1}{2} (\hat{e}_{n,n+3} + \hat{e}_{n+3,n}) \]  

(2.21)

\[ \hat{y} = \sum_n \left( 1 - \frac{1}{2} h(-1 + \frac{2n}{N}) \right) \frac{i}{2} (\hat{e}_{n,n+1} - \hat{e}_{n+1,n}) + \frac{1}{2} h(-1 + \frac{2n}{N}) \frac{i}{2} (\hat{e}_{n,n+3} - \hat{e}_{n+3,n}) \]  

(2.22)

Here, \( z = 1 + 2q \) and \( q \) is discretized with \(-1 + \frac{2n}{N} \). Again, as the ends of the circle-to-eight manifold are not closed, the algebra defined by these matrices only converges within border in the sense of (2.2) to (2.4). Completely converging matrices can be formed by providing the ends with “caps”, for example by fading out the Fourier coefficients in an interval larger than \([-1, 1]\) and extending the matrices correspondingly.
Fig. 2 shows a visualization of the matrices $\hat{x}$, $\hat{y}$ and $\hat{z}$ with $N = 30$ in the form of “dot matrix diagrams”. Every matrix is represented by a matrix of circles. Every circle represents a matrix entry and has a radius proportional to the root of the absolute value of the respective matrix entry. Matrix entries equal to 0 are depicted with a small point, such that the rows and columns can be better identified. We will use these dot matrix diagrams in the following for more complicated examples. They represent an easy way to visually compare properties of matrices with each other.

3 Direct sums

For two manifolds (that can be immersed in the same space), the space of functions on the two manifolds is the direct sum of the function spaces of the manifolds. Thus, if one wants to compose a fuzzy space of two other fuzzy spaces, the corresponding operation is the direct sum of matrices. When $X_1^a$ and $X_2^a$ are two sets of matrices that are matrix regularizations of two submanifolds in the same space, the matrices $X^a = X_1^a \oplus X_2^a$ represent a matrix regularization of both submanifolds:

$$X^a = \begin{pmatrix} X_1^a \\ X_2^a \end{pmatrix} \quad (3.1)$$

In the following, we will see that a matrix regularization of a direct sum of fuzzy spaces can be seen as a matrix regularization of diagonal matrix-valued functions. By generalizing to arbitrary matrix-valued functions, we are able to describe a new class of fuzzy spaces in the next chapter.

3.1 Matrix regularizations of direct sums

In general, let us assume that we have two (or more general $S$) matrices $\hat{f}_a$, which are regularizations of the functions $f_a(q, \varphi)$. The direct product of these matrices is the matrix $\hat{F}$ with $F_{ab,nm} = f_{a,nm}\delta_{ab}$, where we have introduced a double-index notation with $a, b =$
1...S and i, j = 1...N. For example, with S = 2

\[
\hat{F} = \begin{pmatrix}
  f_{1,11} & f_{1,12} & \cdots \\
  f_{1,21} & f_{1,22} & \cdots \\
  \vdots & \vdots & \ddots \\
  f_{2,11} & f_{2,12} & \cdots \\
  f_{2,21} & f_{2,22} & \cdots \\
  \vdots & \vdots & \ddots 
\end{pmatrix} = \sum_{n,m} \left( f_{1,nn} \hat{e}_{nm} f_{2,nm} \hat{e}_{nm} \right) \tag{3.2}
\]

Due to the invariance of a fuzzy space with respect to permutations (being a special kind of unitary transformation), we also can exchange the role of the “inner” and the “outer” indices

\[
\hat{F}' = \hat{P} \hat{F} \hat{P} = \begin{pmatrix}
  f_{1,11} & f_{1,12} & \cdots \\
  f_{1,21} & f_{1,22} & \cdots \\
  \vdots & \vdots & \ddots \\
  f_{2,11} & f_{2,12} & \cdots \\
  f_{2,21} & f_{2,22} & \cdots \\
  \vdots & \vdots & \ddots 
\end{pmatrix} = \sum_{i,j} \left( f_{1,nn} \hat{e}_{nm} \right) \hat{e}_{nm} \tag{3.3}
\]

where \(\hat{P}\) is a suitable permutation. The matrix \(\hat{F}'\) can be seen as a regularization of a matrix-valued function \(F\)

\[
\hat{F}' = Q(F) \quad \text{with} \quad F(q,\varphi) = \begin{pmatrix}
  f_1(q,\varphi) \\
  f_2(q,\varphi)
\end{pmatrix} \tag{3.4}
\]

From now on, we will consider \(\hat{F}'\) and \(\hat{F}\) as equivalent. In summary, we have transformed the regularization of the two real-valued functions \(f_1\) and \(f_2\), which are defined on the cylinder (or more general strip) \([q_l, q_r] \times [0, 2\pi]\) to a regularization of one matrix-valued function \(F\) defined on this cylinder.

Due to (3.3), the borders of the matrices \(\hat{f}_a\) also have been shifted to the border of the matrix \(\hat{F}\). Thus, when we have matrices converging in border then also their direct product permuted according to (3.3) converges in border.

### 3.2 Constant unitary transformations

The coordinate functions \(x_1^\alpha(q, \varphi)\) and \(x_2^\alpha(q, \varphi)\) used for defining the matrices of two fuzzy spaces by regularization can be composed into matrices \(X^\alpha(q, \varphi)\) according to (3.4). When a constant unitary transformation \(\hat{U}\), i.e. a unitary transformation independent of \(q\) and \(\varphi\) is applied to the matrix-valued functions \(X^\alpha(q, \varphi)\), this is equivalent to applying the unitary transformation

\[
\hat{U} = \sum_n U \hat{e}_{nn} \tag{3.5}
\]

(which can be seen as the regularization of the matrix \(U\)) to the corresponding regularization \(X^\alpha\) (3.1). In this case, multiplication of matrices commutes with regularization

\[
Q(U^\dagger FU) = \hat{U}^\dagger \hat{F} \hat{U} \tag{3.6}
\]
On the other hand, when we start with a set of general matrix-valued functions $X^\alpha(q, \varphi) \in M_N(\mathbb{C})$, such as

$$X^\alpha(q, \varphi) = \left( \begin{array}{cc} X^\alpha_{11}(q, \varphi) & X^\alpha_{12}(q, \varphi) \\ X^\alpha_{21}(q, \varphi) & X^\alpha_{22}(q, \varphi) \end{array} \right)$$

and the matrices $X^\alpha(q, \varphi)$ are simultaneously diagonalizable with a constant unitary transformation, we can associate a direct sum of fuzzy spaces to the matrix regularization. Due to the matrix form of the defining functions $X^\alpha(q, \varphi)$, a new class of symmetry is present, namely constant $U(n)$-transformations, which can be seen as a generalization of the trivial $U(1)$ symmetry of single-valued functions. After defining a regularization for general matrix-valued functions, we will also see that this symmetry can be generalized to coordinate dependent $U(n)$-transformations.

An interesting case is, when one of the matrix-valued functions $Z(q, \varphi)$ is proportional to the unit matrix, i.e. $Z_{ab} = z(q, \varphi)\delta_{ab}$. Fuzzy spaces based on direct sums have this property, when the same diagonal matrix for the $z$-coordinate is used for both spaces. In this case, a residual symmetry remains that can be used to transform the other matrices $\hat{X}^\alpha$. Every unitary transformation $\hat{U}$ based on a constant unitary transformations leaves the matrix $\hat{Z}$, see (3.3), invariant.

For example, we can consider the unitary transformation

$$U = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right)$$

which has the property that it mixes the diagonal and off-diagonal elements of the matrix-valued function $F = \left( \begin{array}{cc} f_{11} & f_{12} \\ f_{21} & f_{22} \end{array} \right)$

$$U^\dagger F U = \frac{1}{2} \left( \begin{array}{cc} (f_{11} + f_{22}) + (f_{12} + f_{21}) & (f_{22} - f_{11}) + (f_{12} - f_{21}) \\ (f_{22} - f_{11}) - (f_{12} - f_{21}) & (f_{11} + f_{22}) - (f_{12} + f_{21}) \end{array} \right)$$

Such a transformation will below become important for two symmetric fuzzy spaces, which have the property that $x^1_1(q, \varphi) = -x^2_2(q, \varphi)$ for the coordinate functions except $z(q, \varphi)$. In this case, the transformation results in a swap between diagonal entries and off-diagonal entries.

$$F_I = U^\dagger \left( \begin{array}{cc} -f \\ f \end{array} \right) U = \left( \begin{array}{cc} f \\ f \end{array} \right)$$

We will call this “interlacing” of fuzzy spaces.

### 3.3 Example: Two cylinders

We will apply the preceding discussion to the direct sum of two generalized fuzzy cylinders. In particular, the part of the string vertex, which follows the circle-to-eight manifold (2.19), consists of two cylinders, which move away from each other with increasing $z$. In chapter 5, the fuzzy circle-to-eight space will be connected with the two interlaced fuzzy cylinders, which are defined here.

Firstly, more generally, we consider two cylinders, which are aligned along a center $((x_i,0(z)), y_i,0(z))$ ($i = 1, 2$ enumerates the two cylinders) and which have two $z$-depend
radii $r_{i,x}(z)$, $r_{i,y}(z)$ along the $x$-direction and the $y$-direction. A matrix regularization for each cylinder is

\[
\hat{x}_i = \sum_n x_{i,0} (q_{n,n}) \hat{e}_{nn} + \frac{1}{2} r_{i,x} (q_{n,n+1}) \hat{e}_{n,n+1} + \frac{1}{2} r_{i,x} (q_{n+1,n}) \hat{e}_{n+1,n} + \frac{1}{2} r_{i,x} (q_{n+1,n}) \hat{e}_{n+1,n}
\]

\[
\hat{y}_i = \sum_n y_{i,0} (q_{n,n}) \hat{e}_{nn} + \frac{1}{2} r_{i,y} (q_{n,n+1}) \hat{e}_{n,n+1} + \frac{1}{2} r_{i,y} (q_{n+1,n}) \hat{e}_{n+1,n} + \frac{1}{2} r_{i,y} (q_{n+1,n}) \hat{e}_{n+1,n}
\]

\[
\hat{z} = \sum_n q_{n,n} \hat{e}_{nn}
\]

The direct sum (3.1) of these two fuzzy spaces is

\[
\hat{X} = \begin{pmatrix} \hat{x}_1^1 & \hat{x}_2^1 \\ \hat{x}_1^2 & \hat{x}_2^2 \end{pmatrix}, \quad \hat{Y} = \begin{pmatrix} \hat{y}_1^1 & \hat{y}_2^1 \\ \hat{y}_1^2 & \hat{y}_2^2 \end{pmatrix}, \quad \hat{Z} = \begin{pmatrix} q & q \\ q & q \end{pmatrix}
\]

(3.12)

The $2 \times 2$-matrix-valued functions (3.4) for regularization are

\[
X(q, \varphi) = \begin{pmatrix} x_{1,0}(q) + r_{1,x}(q) \cos \varphi & x_{2,0}(z) + r_{2,x}(z) \cos \varphi \\ y_{1,0}(q) + r_{1,y}(q) \sin \varphi & y_{2,0}(q) + r_{2,y}(q) \sin \varphi \end{pmatrix}
\]

\[
Y(q, \varphi) = \begin{pmatrix} q & q \\ q & q \end{pmatrix}
\]

(3.13)

When the two cylinders are symmetric with respect to the $z$-axis, this implies $x_0(q) = -x_{1,0}(q) = x_{2,0}(q)$ and $r_{x/y}(q) = -r_{1,x/y}(q) = r_{2,x/y}(q)$. The interlaced form (3.10) of the matrices becomes

\[
X_1(q, \varphi) = \begin{pmatrix} x_{o}(q) + r_{x}(q) \cos \varphi \\ r_{y}(q) \sin \varphi \end{pmatrix}
\]

\[
Y_1(q, \varphi) = \begin{pmatrix} q & q \\ q & q \end{pmatrix}
\]

(3.14)

Fig. 3 shows dot matrix diagrams of matrix regularizations with $N = 15$ for the two cylinders. The first row of diagrams shows the matrices $\hat{X}, \hat{Y}$ and $\hat{Z}$ for the pure direct sum of the two fuzzy cylinders, according to (3.2). The second row shows the matrices $\hat{X}', \hat{Y}'$ and $\hat{Z}'$ after a permutation according to (3.3). The third row shows the interlaced matrices according to (3.14) adjoined with (3.5) based on (3.8). If one compares Fig. 2 above for the circle-to-eight fuzzy space with the third row of Fig. 3, one sees, that the same diagonals of the matrices only have entries different from 0. In chapter 5 we will develop a method, how these two matrices for one coordinate can be smoothly interconnected with each other.

4 Fuzzy spaces based on matrix-valued functions

In the previous chapter we have seen that the direct sum of two or more fuzzy spaces, each of which is defined by a regularization of coordinate functions, can be described
Figure 3: Visualization for two cylinder matrices: direct sum, z-ordered and interlaced
with a regularization of diagonal matrix-valued functions \((3.4)\). In the following, we will generalize to fuzzy spaces defined by a regularization of arbitrary matrix-valued functions.

To give a precise definition of such a regularization, let \(F_{ab}(q, \varphi)\) be a matrix-valued function with \(a, b = 1 \ldots S\), which can be decomposed into a Fourier series

\[
F_{ab}(q, \varphi) = \sum_n F_{ab,n}(q)e^{in\varphi}
\]  

Again \(q \in [q_1, q_2]\) is in a finite interval and \(\varphi \in [0, 2\pi]\). A \(N\)-dimensional matrix regularization \(Q_N(F) = \hat{F}\) of \(F\) is then

\[
\hat{F}_{ab,nm} = F_{ab,m-n}(q(n, m))
\]

with a discretizing function \(q\) as defined above. Note that the matrix \(\hat{F}\) is actually a square matrix of dimensions \(N\) times \(S\). When the discretizing function \(q\) is symmetric and the function \(F(q, \varphi)\) is Hermitian, then the matrix \(\hat{F}\) is also Hermitian. With this definition, as in the single-valued case, criteria \((2.2)\) is fulfilled for matrix-valued functions.

### 4.1 Semi-classical limit with matrix-valued functions

In the following, we assume that the Fourier coefficients \(F_{ab,n}(q)\) of \(F\) are one times differentiable with respect to \(q\). This always can be achieved for a regularization, by interpolation the points \(q(n, n)\) with a one times differentiable function. Analogously to \((2.12)\), we assume that the \(F_{ab,n}(q)\) are again 0 for \(|n| \geq \delta\), where \(\delta\) is a constant that is small compared to \(N\).

As in the case of the immersed cylinder (see \((2.13)\)), we now can evaluate the matrix product of \(\hat{F}\) with a further matrix \(\hat{G}\), which is a matrix regularization of a matrix-valued function \(G_{ab}(q, \varphi)\).

\[
(\hat{F}\hat{G})_{ab,nm} = \sum_{1 \leq c \leq S} F_{ac, np} G_{cb, pm} = \sum_{1 \leq c \leq S, 1 \leq p \leq N} F_{ac, p-n}(q(n, p)) G_{cb, m-p}(q(p, m))
\]

\[
= \sum_{1 \leq c \leq S, |n-p| < \delta, |p-m| < \delta} F_{ac, p-n}(q(n, m) + \frac{\beta}{N}(p-m)) G_{bc, m-p}(q(n, m) + \frac{\beta}{N}(p-n))
\]

\[
= \sum_{1 \leq c \leq S, |n-p| < \delta, |p-m| < \delta} F_{ac, p-n}G_{cb, m-p} + \frac{\beta}{N} \frac{\partial F_{ac, p-n}}{\partial q} (p-m) G_{cb, m-p}
\]

\[
+ \frac{\beta}{N} F_{ac, p-n} (p-n) \frac{\partial G_{cb, m-p}}{\partial q} + \mathcal{O} \left( \frac{1}{N^2} \right)
\]

\[
= Q \left( \sum_{1 \leq c \leq S} F_{ac} G_{cb} + \frac{i\beta}{N} \left( \frac{\partial F_{ac}}{\partial q} \frac{\partial G_{cb}}{\partial \varphi} - \frac{\partial F_{ac}}{\partial \varphi} \frac{\partial G_{cb}}{\partial q} \right) \right) + \mathcal{O} \left( \frac{1}{N^2} \right) \quad (4.3)
\]
where in the second but last two step all functions are evaluated at \( q = q(n,m) \) and \( Q \) is the regularization mapping, i.e. \( \tilde{F} = Q(F) \). As in the immersed cylinder case, for solely continuous functions, the result is \( \tilde{F}G = Q(FG) + O\left(\frac{1}{N}\right) \). When the Fourier coefficient functions are not \( 0 \) at \( q_1 \) and \( q_2 \), then the considerations apply again, which have been made with respect to the immersed cylinder relating to convergence in border.

From (4.3) it follows that also (2.3) is fulfilled. However, in the case of single-valued functions, from (2.3) it is a trivial result that the commutator of two regularized functions converges to \( 0 \) for large \( N \), i.e.

\[
\lim_{N \to \infty} \|[Q_N(f), Q_N(g)]\| = 0
\] (4.4)

since the functions commute. This is not longer the case for matrix-valued functions. However, when we want that the regularized matrices converge to a space of functions on a manifold, its is necessary that (4.4) is fulfilled. According to (4.3) this is only the case, when the commutator of the two matrix-valued functions vanishes for \( N \) going to infinity, i.e. \( [F(q, \varphi), G(q, \varphi)] = O\left(\frac{1}{N}\right) \). If one finds \( d \) matrix-valued functions \( X^\alpha(q, \varphi) \), which have the property that

\[
[X^\alpha(q, \varphi), X^\beta(q, \varphi)] = O\left(\frac{1}{N}\right)
\] (4.5)

then the \( N \times S \)-dimensional matrices \( Q_N(X^\alpha) \) defined by (4.2), converge to a function space. When the commutator (4.5) vanishes completely, then even a property analogous to (2.4) is present.

(4.2) is really a generalization of (3.4), since it suffices that the matrix-valued functions (4.5) nearly commute. Since nearly commuting matrices can be nearly diagonalized, one sees that there can be contributions to the \( O\left(\frac{1}{N}\right) \)-part of (4.3), which are not based on a Poisson bracket.

In the case of commuting matrix-valued coordinate functions, these can be diagonalized with a coordinate dependent unitary transformation \( U(q, \varphi) \). However, in general, \( Q(U(q, \varphi)X^\alpha(q, \varphi)U(q, \varphi)^*) \neq U\tilde{F}U^\dagger \) and it is not obvious that the fuzzy space based on the original matrix-valued functions is equivalent to the fuzzy space based on the diagonalized matrix-valued functions.

### 4.2 One diagonalized matrix

As in the case of the immersed cylinder (2.17), we can consider the case, where one of the regularized coordinate matrices, such as \( \tilde{X}^d = \tilde{Z} \) is diagonal. This always can be achieved by a unitary transformation on the fuzzy space, i.e. after regularization. From (4.2) it can be seen that when the regularized matrix \( \tilde{Z} \) is diagonal, then the corresponding matrix-valued function \( Z(q, \varphi) \) has to be diagonal and independent of \( \varphi \), i.e. \( Z_{ab}(q, \varphi) = z_a(q)\delta_{ab} \).

The commutator with a general matrix-valued function \( F(q, \varphi) \) is

\[
[Z(q), F(q, \varphi)]_{ab} = (z_a(q) - z_b(q)) F_{ab}(q, \varphi)
\] (4.6)

When the differences of the diagonal entries of the function \( Z(q) \) are of order \( \frac{1}{N} \), the commutator of \( Z(q) \) with every other matrix-valued function \( F(q, \varphi) \) is of the same order. Thus, a fuzzy space, which is defined by matrix-valued functions \( X^\alpha(q, \varphi) \) has a classical
limit, when the diagonal matrix-valued function $Z(q)$ has entries, which differ only by order $\frac{1}{N}$, and when the commutators of the other $X^\alpha(q, \varphi)$ are also of this order.

An example with matrices of size $2 \times 2$, which will become important in the following is

$$
Z = \begin{pmatrix}
q & \frac{\beta(q)}{N} \\
q + \frac{\beta(q)}{N} & q
\end{pmatrix} \tag{4.7}
$$

$$
X^\alpha = \begin{pmatrix}
x^\alpha(q, \varphi) & x^\alpha(q, \varphi) \\
x^\alpha(q, \varphi) & x^\alpha(q, \varphi)
\end{pmatrix}
= \begin{pmatrix}
r^\alpha(q, \varphi)e^{-i\psi(q, \varphi)} & r^\alpha(q, \varphi)e^{i\psi(q, \varphi)} \\
r^\alpha(q, \varphi)e^{i\psi(q, \varphi)} & r^\alpha(q, \varphi)e^{-i\psi(q, \varphi)}
\end{pmatrix}
$$

where $\beta(q)$ is a function of order 1 and the $r^\alpha$ and $\psi$ are real-valued functions. For example, two interlaced mirror symmetric fuzzy spaces (3.10), such as two interlaced cylinders, have the form of such an “off-diagonal fuzzy space”. In the next section we will show that this is also the case for the circle-to-eight space (2.19), when it is considered as a regularization based on $2 \times 2$-matrix valued functions.

The $X^\alpha(q, \varphi)$ of (4.7) commute with each other, since

$$
\left[ \begin{pmatrix} f & g \\ \bar{g} & \bar{f} \end{pmatrix}, \begin{pmatrix} f & g \\ \bar{g} & \bar{f} \end{pmatrix} \right] = \begin{pmatrix} fg - \bar{g}\bar{f} & -(f\bar{g} - g\bar{f}) \\ -(f\bar{g} - g\bar{f}) & fg - \bar{g}\bar{f} \end{pmatrix} \tag{4.8}
$$

The right-hand side vanishes, when $f\bar{g} = g\bar{f}$, which is the case for the $x^\alpha$ defined in (4.7).

However, the matrix-valued functions $Z(q)$ and $X^\alpha(q, \varphi)$ of (4.7) are in general not simultaneously diagonalizable. The corresponding matrix regularization does not split into a direct sum of immersed cylinders. On the other hand, in the limit $N \to \infty$ the functions $Z(q)$ and $X^\alpha(q, \varphi)$ are commuting matrices, which can be diagonalized simultaneously and therefore can be seen as the direct sum of two submanifolds.

### 4.3 Immersed cylinder based on matrix-valued functions

We will now show that the immersed fuzzy cylinder as defined in (2.8) can be seen as fuzzy space defined by $2 \times 2$-matrix valued functions.

In general let $f(q, \varphi)$ be a function and let $\hat{f}$ be its (single-valued) matrix regularization, i.e. $\hat{f}_{ij}(q) = f_{j-i}(q(i, j))$. We can now ask the question, whether there is a matrix regularization of a matrix-valued function $F(q, \varphi)$, which results in the same matrix $\hat{f}$. To construct such a function $F(q, \varphi)$, we can decompose the matrix $\hat{f}$ into $2 \times 2$ cells $\hat{F}_{ij}$

$$
\hat{F}_{ij} = \begin{pmatrix}
f_{2(i-1)}(q(2i, 2j)) & f_{2(j-1)+1}(q(2i, 2j+1)) \\
f_{2(i-1)+1}(q(2i+1, 2j)) & f_{2(j-1)}(q(2i+1, 2j+1))
\end{pmatrix} \tag{4.9}
$$

and compare this with the Fourier series of the matrix-valued function $F(q, \varphi) = \sum_n F_n e^{in\varphi}$. For $q(i, j) = \frac{i+j}{2N}$ we see that the matrix $\hat{f}$ also is a regularization of

$$
F(q, \varphi) = \begin{pmatrix}
f_{2n}(2q) & f_{2n+1}(2q + \frac{1}{2N}) \\
f_{2n-1}(2q + \frac{1}{2N}) & f_{2n}(2q + \frac{1}{N})
\end{pmatrix} e^{in\varphi} \tag{4.10}
$$

This results in

$$F(q, \varphi) = \ldots$$
of double anti-periodic coordinate functions do not contribute to the classical limit. Thus, a fuzzy space defined as a regularization interlaced form of two fuzzy spaces. As is shown below, see (4.18), the phase factors seen as the direct sum of two fuzzy spaces defined by 

\[
\frac{1}{2} \left( \begin{array}{cc}
\frac{f(2q, \frac{\varphi}{2}) + f(2q, \frac{\varphi + 2\pi}{2})}{2} & (f(2q + \frac{1}{2N}, \frac{\varphi}{2}) - f(2q + \frac{1}{2N}, \frac{\varphi + 2\pi}{2})) e^{-i\varphi} \\
(f(2q + \frac{1}{2N}, \frac{\varphi}{2}) - f(2q + \frac{1}{2N}, \frac{\varphi + 2\pi}{2})) e^{i\varphi} & f(2q + \frac{1}{2N}, \frac{\varphi}{2}) + f(2q + \frac{1}{2N}, \frac{\varphi + 2\pi}{2})
\end{array} \right)
\]

(4.11)
i.e. the diagonal elements are the double periodic part of \(f\) and the off-diagonal elements are the phase-shifted double anti-periodic part of \(f\).

For coordinate functions \(x^\alpha(q, \varphi)\), which are double anti-periodic, i.e. \(x^\alpha(q, \frac{\varphi}{2} + \pi) + x^\alpha(q, \frac{\varphi}{2}) = 0\), (such as those for the circle-to-eight manifold (2.19)), the corresponding matrix-valued coordinate functions (4.11) reduce to

\[
X^\alpha(q, \varphi) = \left( \begin{array}{c}
x^\alpha(2q + \frac{1}{2N}, \frac{\varphi}{2}) e^{i\varphi} \\
x^\alpha(2q + \frac{1}{2N}, \frac{\varphi}{2}) e^{-i\varphi}
\end{array} \right)
\]

(4.12)
i.e. \(X^\alpha(q, \varphi)\) have the form (4.7) with \(\psi(q, \varphi) = -\frac{\varphi}{2}\).

For a function \(z(q)\), which does not depend on \(\varphi\), the result is

\[
Z(q) = \left( \begin{array}{c}
z(2q) \\
z(2q + \frac{1}{N})
\end{array} \right)
\]

(4.13)
The function \(Z(q)\) has the property that the difference of its entries is of order \(\frac{1}{N}\). Thus, the matrix-valued functions \(X^\alpha(q, \varphi)\) and \(Z(q)\) are an “off-diagonal fuzzy space” (4.7) as described in the previous section.

Note that all matrix-valued functions \(X^\alpha(q, \varphi)\) and \(Z(q)\) depend on \(N\). However in the large \(N\) limit, the term \(\frac{1}{N}\) can be set to zero and the matrix regularization of the \(X^\alpha(q, \varphi)\) and \(Z(q)\) splits into a direct sum. The functions \(X^\alpha(q, \varphi)\) and \(Z(q)\) can be seen as the interlaced form of two fuzzy spaces. As is shown below, see (4.18), the phase factors \(e^{\pm i\varphi} \) do not contribute to the classical limit. Thus, a fuzzy space defined as a regularization of double anti-periodic coordinate functions \(x^\alpha(q, \varphi)\) and \(z(q)\) in the large \(N\) limit can be seen as the direct sum of two fuzzy spaces defined by \(\pm x^\alpha(2q, \frac{\varphi}{2})\) and \(z(2q)\).

### 4.4 Non-constant unitary transformations

We already have considered the case of a constant unitary transformation of a matrix-valued function. Such a constant unitary transformation (4.5) corresponds to a unitary transformation in the matrix regularization, which is the \(N\)-fold direct product of this constant unitary transformation with itself. One can know examine the relationship between unitary transformations \(U(q, \varphi)\) applied on the side of the matrix-valued functions, i.e. before regularization, and unitary transformations on the side of the fuzzy space, i.e. after regularization.

In general, when the unitary transformation \(U(q, \varphi)\) is coordinate dependent, regularization (4.2) and multiplication do not commute. In particular, when \(\hat{U} = Q_N(U)\) is the matrix regularization of \(U(q, \varphi)\) \(\hat{U}\) need not be unitary and \(\hat{U}\) need not be its inverse.

#### 4.4.1 \(q\)-dependent unitary transformation

In the case, when \(U(q)\) depends only on \(q\), then it follows from (4.2) that \(Q_N(U)^\dagger = Q_N(U^\dagger)\) is the inverse of \(Q_N(U)\). At least in this case, in the large \(N\) limit, the manifold
defined by the matrix-valued coordinate functions, when present, stays the same, since

\[ Q_N(U)Q_N(F)Q_N(U^\dagger) = Q_N(UFU^\dagger) + O\left(\frac{1}{N}\right) \]  

(4.14)

according to \(4.3\). For example, the pure \(q\)-dependent diagonal unitary transformation

\[ U(q) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\psi(q)} & e^{i\psi(q)} \\ e^{-i\psi(q)} & e^{-i\psi(q)} \end{pmatrix} \]  

(4.15)

applied to an off-diagonal matrix-valued function \(F(q, \varphi)\), such as the \(X^\alpha(q, \varphi)\) in \(4.7\), results in a further phase shift of the off-diagonal elements

\[ U(q) \left( \frac{f(q, \varphi)}{f(q, \varphi)} \right) U(q) = \left( \begin{array}{c} e^{-i((\psi_1(q) - \psi_2(q))f(q, \varphi))} \end{array} \right) \]  

(4.16)

Thus, the phase of \(4.7\), when only \(q\)-dependent as in \(4.12\) can be removed with a suitable unitary transformation in the large \(N\) limit. In particular, The matrix-valued function

\[ U(q) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\psi(q)} \\ e^{-i\psi(q)} & 1 \end{pmatrix} \]  

(4.17)

applied to the \(X^\alpha(q, \varphi)\) of \(4.7\) with only \(q\)-dependent \(\psi(q)\) results in

\[ U(q)X^\alpha(q, \varphi)U^\dagger(q) = \begin{pmatrix} -r^\alpha(q, \varphi) \\ 0 \\ r^\alpha(q, \varphi) \end{pmatrix} \]  

(4.18)

and leaves \(Z(q)\) invariant. Only the magnitudes \(r^\alpha(q, \varphi)\) of the functions \(x^\alpha(q, \varphi)\) define the classical limit of the fuzzy space. The phase can be seen as pure non-commutative artifact.

### 4.4.2 Block transformations

A further interesting example arises in the application of unitary block matrices \(\hat{U}\), which are based on constant unitary matrices that are applied only to some of the matrix blocks of the regularized matrices, such as

\[ \hat{U} = \sum_{n>n_0} U\hat{e}_{nm} \]  

(4.19)

where \(n_0\) is some number between 1 and \(N\) and \(U\) is a unitary \(S \times S\)-matrix. This unitary transformation \(\hat{U}\) can be seen as the regularization of the matrix-valued function \(U(q) = \theta(q)U\) where \(\theta\) is a step function, which is 0 below a \(q_0 = q(n_0, n_0)\) \((q\) is the discretizing function) and which is 1 above.

For example, let \(\hat{F}\) be the regularization of the matrix-valued function \(F(q, \varphi) =\)
\[ \sum_n F_n(q)e^{i\nu q}, \text{ i.e.} \]
\[ \hat{F} = \begin{pmatrix}
  F_0 & F_1 & F_2 & F_3 \\
  F_{-1} & F_0 & F_1 & F_2 \\
  F_{-2} & F_{-1} & F_0 & F_1 \\
  F_{-3} & F_2 & F_{-1} & F_0 \\
  \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}, \quad \hat{U} = \begin{pmatrix}
  1 & & & \\
  & 1 & & \\
  & & U & \\
  & & & \cdots \\
\end{pmatrix} \tag{4.20} \]

where we have omitted the arguments of the \( S \times S \)-matrices \( F_i \). Then
\[ \hat{U}^\dagger \hat{F} \hat{U} = \begin{pmatrix}
  \cdots & F & F_1 & F_2U & F_3U \\
  \cdots & F_{-1} & F_0 & F_1U & F_2U \\
  \cdots & U^\dagger F_{-2} & U^\dagger F_{-1} & U^\dagger F_0U & U^\dagger F_1U \\
  \cdots & U^\dagger F_{-3} & U^\dagger F_2 & U^\dagger F_{-1}U & U^\dagger F_0U \\
  \vdots & \vdots & \vdots & \vdots & \cdots \\
\end{pmatrix} \tag{4.21} \]

One sees that the upper left part is not transformed at all, that the lower right part is completely transformed with \( U \), and that the off-diagonal parts are only transformed on one side. Later we will encounter such a transformation in an example.

## 5 Interpolation of fuzzy spaces

We have seen that the matrix-valued functions, which define both the circle-to-eight fuzzy space and the interlaced cylinder fuzzy space, have the form (4.7), i.e. the \( Z(q) \)-coordinate function is diagonal and the \( X^\alpha(q, \varphi) \)-coordinate functions are off-diagonal. In the following we will show that it is possible to interpolate the corresponding matrices along the \( q \)-coordinate such that also the part, which depends on the coordinate functions of both spaces has the form (4.7). In this way, we arrive at a fuzzy space that has a defined classical limit everywhere along the \( q \)-coordinate.

Based on this, we explicitly construct a fuzzy space that in the classical limit corresponds to a string vertex, i.e. a submanifold that has three ends formed like cylinders, which are interconnected by a branching.

### 5.1 Extension of circle-to-eight by two cylinders (classical case)

In this section, we construct a classical string vertex from two coordinate patches. We hope that this will render the construction based on matrix-valued functions in the next section more clear.

In general, we have two manifolds that overlap in a middle part between \( q_2 \) and \( q_3 \) along the \( q \)-coordinate. The first manifold (the circle-to-eight manifold) is defined between \( q_1 \) and \( q_3 \) and the second manifold (the two cylinders) are defined between \( q_2 \) and \( q_4 \).
As described above, the circle-to-eight (2.19) is defined by

\[ x(q, \varphi) = r(q, \varphi) \cos \varphi \quad y(q, \varphi) = r(q, \varphi) \sin \varphi \quad z(q, \varphi) = q \]  

(5.1)

with

\[ r(q, \varphi) = r_1(q) + r_2(q) \cos 2\varphi \]  

(5.2)

such that \( r_2(q_1) = 0 \) and \( r_1(q_2) = r_2(q_3) = r_1 \) (see (2.19) above). We assume that the functions \( r_1 \) and \( r_2 \) stay constant between \( q_2 \) and \( q_3 \). The functions \( x \) and \( y \) are double anti-periodic, i.e.

\[ x(q, \frac{\varphi}{2}) = -x(q, \frac{\varphi}{2} + \pi) \quad y(q, \frac{\varphi}{2}) = -y(q, \frac{\varphi}{2} + \pi) \]  

(5.3)

This is due to the fact that the submanifold defined by \( x, y \) and \( z \) is axis-symmetric to the \( z \)-axis. The parts of the submanifolds for \( \varphi \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \) and \( \varphi \in \left[ \frac{5\pi}{2}, -\frac{\pi}{2}, \frac{5\pi}{2} \sim \frac{\pi}{2} \right] \) are symmetric. Between \( q_2 \) and \( q_3 \), the “crossing” of the eight is located at \( \varphi = \frac{\pi}{2} \) and \( \varphi = -\frac{\pi}{2} \).

We can define two other coordinate patches for the circle-to-eight manifold. In particular, the right half with \( \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) is defined via \( x_R(q, \varphi) = x(q, \frac{\varphi}{2} + \pi) \) and \( y_R \) is defined respectively. For the left half, the function \( y_L = y_R \) is the same and the function \( x_L = -x_R \) is the negative of the one of the right half. Between \( q_2 \) and \( q_3 \), the Fourier coefficients \( x_{Rm}, y_{Rm}, x_{Lm}, \) and \( y_{Rm} \) of these functions for \( m < -1 \) and \( m > 1 \) can be faded out and the Fourier coefficients for \( m = -1, 0, 1 \) can be continuously adjusted such they go over to the ones of two cylinders as in (3.13), which are defined between \( q_3 \) and \( q_4 \).

More general, let \( x^\alpha_1 \) be the coordinate functions of one of the halves of the first manifold between \( q_1 \) and \( q_3 \) defined by double anti-periodic functions and let \( x^\alpha_2 \) be the coordinate functions for a cylinder between \( q_2 \) and \( q_4 \). Then, the middle part between \( q_2 \) and \( q_3 \) can be defined by interpolation

\[ x^\alpha(q, \varphi) = \theta_1(q)x^\alpha_1(q, \frac{\varphi}{2} + \frac{\pi}{2}) + \theta_2(q)x^\alpha_2(q, \varphi) \]

where \( \theta_1 \) and \( \theta_2 \) are two continuous functions with \( \theta_1(q < q_2) = 1, \theta_1(q > q_3) = 0, \theta_2(q < q_2) = 0 \) and \( \theta_2(q > q_3) = 1 \). Correspondingly, the other half can be defined. When appropriately choosing the parameters for \( x^\alpha_2 \), i.e. the part with the two cylinders, the resulting manifold will have the form of a string vertex.

Even more general, we can include a \( q \)-dependent coordinate transformation in the interpolation.

\[ x^\alpha(q, \varphi) = \theta_1(q)x^\alpha_1(q, \frac{\varphi}{2} + \pi + \pi \beta(q)) + \theta_2(q)x^\alpha_2(q, \varphi + \pi \beta(q)) \]  

(5.4)

with a continuous function \( \beta \) that is equal to \(-\frac{1}{2}\) in the part of the space with the circle-to-eight, i.e. for \( q < q_2 \) and that is 0 in the pure double cylinder part, i.e. for \( q > q_3 \). The function \( \beta \) has the advantage that there is no need for applying a coordinate transformation on the parts outside of the transition part between \( q_2 \) and \( q_3 \) and the Fourier coefficients of \( x^\alpha_1 \) and \( x^\alpha_2 \) outside of the transition part need not be adapted.

We will carry over this construction in the following to fuzzy spaces of the form (4.7), where \( Z(q) \) is diagonal and the \( X^\alpha(q, \varphi) \) are off-diagonal.
5.2 Interpolating between off-diagonal fuzzy spaces

As just mentioned, we start with two fuzzy spaces as defined in (4.7). We assume that the first fuzzy space has off-diagonal functions \( x_\alpha = f_\alpha^1 e^{-i\frac{\varphi}{2}} \), which are defined for \( q \in [q_1, q_3] \), and that the second fuzzy space has off-diagonal functions \( x_\alpha^2 = f_\alpha^2 e^{i\beta(q)} \), which are defined for \( q \in [q_2, q_4] \), where \( q_2 < q_3 \). The first fuzzy space is of the form (4.12), such as the circle-to-eight space, and the second space is of the form (3.10) such as two interlaced cylinders.

In correspondence with (5.4), an interpolation between the off-diagonal functions \( x_\alpha^1 \) and \( x_\alpha^2 \) is introduced by

\[
x_\alpha(q, \varphi) = \left( \theta_1(q)f_\alpha^1(q, \varphi) + \theta_2(q)f_\alpha^2(q, \varphi + \pi \beta(q)) \right) e^{i\alpha(q)\varphi + i\gamma(q)} \tag{5.5}
\]

with the functions \( \theta_1, \theta_2 \) and \( \beta \) as in the previous section. The continuous function \( \alpha \) has the same properties as \( \beta \) and is used for a continuous transition of the phase factor. The continuous function \( \gamma \) will be used for adapting a final phase shift of the function \( x_\alpha \), which does not change the classical limit (see (4.16)).

With such definitions, the resulting space defined by

\[
X_\alpha = \begin{pmatrix} x_\alpha & \bar{x_\alpha} \\ \bar{x_\alpha} & Z \end{pmatrix} = \theta_1 Z_1 + \theta_2 Z_2 \tag{5.6}
\]

is equal to the first space for \( q < q_2 \) and equal to the second space for \( q > q_3 \). Importantly, between \( q_2 \) and \( q_3 \), the matrix-valued functions \( X_\alpha \) and \( Z \) are also of the form (4.7), i.e. their commutators vanish to first order and thus the regularization (4.2) has a classical limit. Due to our considerations above, we know that the manifold of the classical limit is defined by the magnitude of the functions \( x_\alpha \), i.e. by (5.4).

After regularization (4.2), the part of the coordinate matrices \( \hat{X}_\alpha \) belonging to \( q < q_2 \) (the left upper part) looks like the two left diagrams of Fig. 2. For interlaced cylinders, the part belonging to \( q > q_3 \) (the lower right part) looks like the two lower left diagrams of Fig. 3. With the interpolation (5.5), the part in between is filled out. One sees that due to the interlacing of the second space, the rows of Fourier coefficients of the first space can go over to the rows of Fourier coefficients of the second space in a smooth way. Due to the interpolation, a bifurcation of the Fourier coefficient functions of the first space into the double periodic Fourier coefficient functions of the second space takes place.

We now determine the Fourier coefficients of the middle part of the coordinate matrices \( \hat{X}_\alpha \) with \( q_2 < q < q_3 \). In the following we will omit the label \( \alpha \) and will concentrate on one of the coordinates. With

\[
f_1(q, \varphi) = \sum_n f_{1,n}(q)e^{in\varphi} \tag{5.7}
\]

\[
f_2(q, \varphi) = \sum_n f_{2,n}(q)e^{in\varphi} \tag{5.8}
\]

where for the first fuzzy space, the Fourier coefficients \( f_{1,n} \) of the respective immersed fuzzy cylinder (2.11) have been used, the Fourier coefficients of the interpolating functions
become
\[
f_m(q) = e^{i\gamma(q)} \sum_n \left( \theta_1(q) f_{1,n}(q) \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i(n-m+\frac{1}{2}+\alpha(q)) \varphi + i\pi(\frac{1}{2}+\beta(q)) n} \
+ \theta_2(q) f_{2,n}(q) \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i(n-m+\alpha(q)) \varphi + i\pi\beta(q) n} \right) \]
\[
= e^{i\alpha(q) \pi + i\gamma(q)} \frac{1}{\pi} \sum_n \left( \theta_1(q) \sin \pi \left( \frac{1}{2} + \alpha(q) \right) \frac{f_{1,n} e^{i\pi(\frac{1}{2}+\beta(q)) n}}{n-m+\frac{1}{2}+\alpha(q)} \
+ \theta_2(q) \sin \pi\alpha(q) \frac{f_{2,n} e^{i\pi\beta(q) n}}{n-m+\alpha(q)} \right) \tag{5.9}
\]
As (like for ordinary Fourier series)
\[
\lim_{\alpha \to -\pi} \frac{1}{\pi} \sum_n \left( \frac{f_{1,n}}{n-m+\frac{1}{2}+\alpha} \sin \pi \left( \frac{1}{2} + \alpha(q) \right) \right) = f_{1,n} \delta_{nm} \tag{5.10}
\]
\[
\lim_{\alpha \to -\pi} \frac{1}{\pi} \sum_n \left( \frac{f_{2,n}}{n-m+\alpha} \sin \pi\alpha(q) \right) = f_{2,n} \delta_{nm} \tag{5.11}
\]
at the the points \( q = q_2 \) and \( q = q_3 \), the Fourier coefficients \( f_m \) converge to the respective ones of the first and second fuzzy space, when one chooses \( \lambda = -\pi\alpha \) at these points. We will choose \( \lambda = -\pi\alpha \) for all \( q \).

Furthermore, we are free to choose \( \beta = \alpha \), which results in
\[
f_m(q) = \frac{1}{\pi} \sum_n \left( \theta_1(q) \cos \pi\alpha(q) \frac{f_{1,n} e^{i\pi\left(\frac{1}{2}+\alpha(q)\right)n}}{n-m+\frac{1}{2}+\alpha(q)} + \theta_2(q) \sin \pi\alpha(q) \frac{f_{2,n} e^{i\pi\alpha(q)n}}{n-m+\alpha(q)} \right) \tag{5.12}
\]
When \( f_{1,n}(q) = 0 \) and \( f_{2,n}(q) = 0 \) for \( |n| > \delta \), which we have assumed for (4.3), we can estimate for \( m > \delta \)
\[
|f_m(q)| \leq \frac{1}{\pi} \sum_{|n| \leq \delta} \left( \left( \frac{f_{1,n}}{n-m+\frac{1}{2}+\alpha(q)} \right)^2 + \left( \frac{f_{2,n}}{n-m+\alpha(q)} \right)^2 \right) \leq \frac{C \sum_{|n| \leq \delta} \left( \frac{1}{n-m} \right)^2}{m^2} < \frac{C(2\delta + 1)}{m^2} \tag{5.13}
\]
with \( C = \frac{2}{\pi} \max_{|n| \leq \delta} \{|f_{1,n}, f_{2,n}|\} \). We see that in the interpolation part, the Fourier coefficients \( f_m(q) \) for raising \( m \) go at least quadratically to 0. This was to be expected, since the interpolated function \( x^\alpha \) (5.5) has the same properties (such as differentiability, etc.) as the functions \( x_1^\alpha \) and \( x_2^\alpha \) being interpolated, when the functions \( \theta_1, \theta_2 \) and \( \alpha \) are sufficiently smooth. Therefore, it is possible to truncate the Fourier series \( f_m(q) \) at a specific \( m < \delta \), for a \( \delta > \delta \), which is also small compared to \( N \). This \( \delta \) can become larger with growing \( N \) approximating the manifold better and better and (4.3) stays valid.

The Fourier coefficient \( \tilde{f}_n \) of the complex conjugate of a function is in general the complex conjugate of the Fourier coefficient for \( -n \), i.e. \( \tilde{f}_n = \overline{f_{-n}} \). In the present case
\[ f_{1,n} = \tilde{f}_{2n+1} \text{ of a real valued function } \tilde{f} \text{ and therefore } \tilde{f}_{1,n} = \tilde{f}_{1,-n} = \tilde{f}_{2n+1} = \tilde{f}_{2(n-1)+1} = \tilde{f}_{1,n-1}. \text{ In summary} \]

\[ \tilde{f}_m(q) = -\frac{1}{\pi} \sum_n \left( \theta_1(q) \cos \pi \alpha(q) \frac{f_{1,n} - e^{i\pi(\frac{1}{2} + \alpha(q))}}{n - m - \frac{1}{2} - \alpha(q)} + \theta_2(q) \sin \pi \alpha(q) \frac{f_{2,n} e^{i\pi\alpha(q)n}}{n - m - \alpha(q)} \right) \]

If we additionally choose

\[ \theta_1(q) = -\lambda(q) \sin \pi \alpha(q) \quad \theta_2(q) = \lambda(q) \cos \pi \alpha(q) \]

a very compact form for the Fourier coefficients can be achieved. These functions \(\theta_1\) and \(\theta_2\) fulfill the demanded properties, if \(\lambda(q)\) is a continuous function, which is equal to 1 outside of the interval \([q_2, q_3]\) (For the “\(\cdot\)” in front of the sine function note that \(-\frac{1}{2} \leq \alpha \leq 0\). If we further demand that a fuzzy space interpolated with itself stays the same, it must be that \(\theta_1 + \theta_2 = 1\) and \(\lambda\) is

\[ \lambda(q) = \frac{1}{\cos \pi \alpha(q) - \sin \pi \alpha(q)} \]

We arrive at

\[ f_m(q) = \frac{1}{\pi} \cos \pi \alpha(q) \frac{\sin \pi \alpha(q)}{\sin \pi \alpha(q)} \sum_n \left( -\frac{f_{1,n} e^{i\pi(\frac{1}{2} + \alpha(q))}}{n - m + \frac{1}{2} + \alpha(q)} + \frac{f_{2,n} e^{i\pi\alpha(q)n}}{n - m + \alpha(q)} \right) \]

5.3 A fuzzy vertex with classical limit

To give an explicit example, we numerically calculate the matrix regularization of (5.12) for the circle-to-eight example interpolated with two cylinders. We define for the circle-to-eight part (see (2.19))

\[ \tilde{x}_{1,m} = \frac{r_2}{4} e^{-2i\pi(\frac{1}{2} + \alpha)} + \frac{1}{2} (r_1 + \frac{r_2}{2}) e^{-i\pi(\frac{1}{2} + \alpha)} + \frac{1}{2} (r_1 + \frac{r_2}{2}) e^{-i\pi(\frac{1}{2} + \alpha)} + \frac{r_2}{4} e^{i\pi(\frac{1}{2} + \alpha)} \]

\[ \tilde{y}_{1,m} = \frac{-ir_2}{2} e^{-2i\pi(\frac{1}{2} + \alpha)} + \frac{i}{2} (r_1 - \frac{r_2}{2}) e^{-i\pi(\frac{1}{2} + \alpha)} + \frac{i}{2} (r_1 - \frac{r_2}{2}) e^{-i\pi(\frac{1}{2} + \alpha)} + \frac{ir_2}{2} e^{i\pi(\frac{1}{2} + \alpha)} \]

and for the interlaced cylinder part (see (3.14))

\[ \tilde{x}_{2,m} = \frac{r e^{-i\pi\alpha}}{1 - m + \alpha} + \frac{x_0(q)}{-m + \alpha} + \frac{r e^{i\pi\alpha}}{1 - m + \alpha} \]

\[ \tilde{y}_{2,m} = \frac{-ir e^{-i\pi\alpha}}{-1 - m + \alpha} + \frac{ire^{i\pi\alpha}}{1 - m + \alpha} \]

Then

\[ x_m = \frac{1}{\pi} (\theta_1 \tilde{x}_{1,m} + \theta_2 \tilde{x}_{2,m}) \]

\[ y_m = \frac{1}{2} (\theta_1 \tilde{y}_{1,m} + \theta_2 \tilde{y}_{2,m}) \]
With (4.11) and (5.6), the $z$-coordinate becomes

$$Z(q, \varphi) = \left( \begin{array}{c} \theta_1 2q + \theta_2 q \\ \theta_1 (2q + \frac{1}{N}) + \theta_2 q \end{array} \right)$$

For the numerically example, shown in Fig. 4, we have chosen $r_1 = 1$, $r_2 = 1$, $x_o(q) = 0.7 + 0.3q$, $\alpha(q) = \frac{1}{2} (h(2q - 3) - 1)$, $\theta_2(q) = h(2q - 3)$ and $\theta_1(q) = 1 - \theta_2(q)$ with the spline $h$ as defined in chapter 2. These parameters and functions result in a manifold for the classical limit, which does not intersect with itself (see the right, bottom picture of Fig. 4). The two upper pictures and the left lower picture of Fig. 4 show a visualization of $\hat{X}$, $\hat{Y}$ and $\hat{Z}$. These matrices have been generated with the discretizing function $q(n, m) = 4n + 2m$ for $2N = 60$. In the diagrams for $\hat{X}$, $\hat{Y}$ and $\hat{Z}$, matrix entries with norm smaller than 0.1 are depicted with a small dot.

It can be seen that the diagonal rows of matrices $\hat{X}$ and $\hat{Y}$ smoothly go over from the four rows of the circle-to-eight space into the four double periodic rows of the two cylinders. In the interpolating part, further diagonal rows are present, which, however, have rather small norms, as expected from (5.13).

Since the ends of the string vertex are open, the matrix entries of the commutators of the matrices for $N \to \infty$ only go to 0 within borders. Again, the ends of the string vertex can be closed by letting all functions $f_m(q)$ go to 0 at the ends. This also can be done by multiplying the matrix-valued functions (5.6) with a function, which goes continuously to 0 at $q_1$ and $q_4$.

We also have calculated the commutators of the numerically determined matrices for $N$ up to 120 and verified that the maximal values of the norm of the matrix entries of these commutators between borders, i.e. for $5 < n < N - 5$ decreases with raising $N$.

### 5.4 Fuzzy surfaces with higher order genus

With the fuzzy vertex, higher order genus surfaces can be concatenated:

Firstly, the functions $f_m(q)$ can be concatenated with mirrored functions $f_m(qE - q)$ to generate a surface, in which a cylinder splits into two cylinders that surrounds a hole and again merge into one cylinder. The $\hat{X}$ and $\hat{Y}$ matrices of this fuzzy one-hole space would have an upper part looking like the matrices shown in Fig. 4 and a lower part, which is mirrored at the $n + m = 0$ axis of these matrices.

As described above, such a one-hole surface can be deformed at the ends by letting the functions $f_m(q)$ go to 0 at the ends to form a (deformed) torus. In this way we have found a deformed fuzzy torus with classical limit for $N \to \infty$.

For surfaces with genus $g$, $g$ one-hole surfaces can be concatenated to a surface with $g$ holes and the ends of the result can be closed. Thus, fuzzy spaces with a classical limit having an arbitrary genus can be constructed. A similar result was obtained in [9] in a more algebraic way.

### 6 Comparison with known fuzzy spaces

After presenting one of the main results in the last chapter, we compare the new type of fuzzy spaces with already known fuzzy spaces. In the next two sections, we transform a fuzzy cylinder and a fuzzy torus by a coordinate transformation into a form, in which we
Figure 4: Circle-to-eight extended by two cylinders
expect to find fuzzy string vertices within the resulting matrices. We will see that these matrices are composed of parts, which are rather similar to the ones constructed in the previous chapter.

6.1 Coordinate transformed fuzzy cylinder

We start with an ordinary, classical cylinder aligned along the $z$-axis and parametrized by $x(z, \varphi) = \cos \varphi$ and $x(z, \varphi) = \cos \varphi$. A coordinate transformation for deforming the cylinder into a U-shaped surface is

$$z' = \alpha z^2 - x \quad y' = y \quad x' = z$$

(6.1)

Fig. 5 shows this surface for $\alpha = \frac{1}{3}$ and it can be seen that cross-sections orthogonal to the $z$-axis (from the bottom up) are a circle transitioning into a 8, which goes over into two circles. A string vertex is part of the deformed cylinder. One would expect that a correspondingly transformed fuzzy cylinder shows an analogous behavior.

A corresponding fuzzy cylinder can be defined by the Hermitian $N \times N$ matrices $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ with entries

$$\hat{Z} = \sum_{n=1}^{N} 5 \left( -1 + \frac{2n}{N} \right) \hat{e}_{nn} \quad \hat{X} = \sum_{n=1}^{N} \frac{1}{2} (\hat{e}_{n,n+1} + \hat{e}_{n+1,n}) \quad \hat{Y} = \sum_{n=1}^{N} \frac{i}{2} (\hat{e}_{n,n+1} - \hat{e}_{n+1,n})$$

(6.2)

see (3.11) or [6], for example. As mentioned several times, the fuzzy cylinder as defined here converges only within border. However in the following, we are only interested in the middle part of the cylinder, which will be deformed to a "U".

We apply now the same coordinate transformation (6.1) to the fuzzy cylinder

$$\hat{Z}' = \alpha \hat{Z}^2 - \hat{X}$$

(6.3)

and diagonalize the matrix $\hat{Z}$ with a unitary matrix $\hat{P}$. With this we define

$$\hat{Y}' = \hat{P}^\dagger \hat{Y} \hat{P} \quad \hat{X}' = \hat{P}^\dagger \hat{X} \hat{P}$$

(6.4)

We have performed this transformation numerically for $N = 40$ and $\alpha = 5/8$. A solver was used, which calculated real eigenvectors for the eigenvalues of $\hat{Z}'$. In such a
way, a unitary transformation could be assembled that guaranted that the matrix entries of $\hat{X}'$ are real and the ones of $\hat{Y}'$ are imaginary. We furthermore ordered the resulting matrices according to increasing eigenvalues of $\hat{Z}'$.

Fig. 6 shows the result with dot matrix diagrams for $\hat{X}', \hat{Y}'$ and $\hat{Z}'$. The $2 \times 2$-matrix structure is visible very well. Comparing the diagrams with Fig. 2, 3 and 4, it is clear that they represent a fuzzy space, which has Fig. 5 as classical limit. In the lower right part, where the $\hat{Z}$-values become ambiguous (i.e. the part of the double cylinder), the transformation $\hat{P}$ is not any more unique. A block transformation \( (4.21) \) from the two interlaced cylinders to two ordinary cylinders had been produced by the solver accidently.

In summary, the coordinate transformed fuzzy cylinder can be seen as a regularization of a matrix-valued function.

### 6.2 Projected fuzzy Clifford torus

Another very well known space is the fuzzy Clifford torus \([10, 6]\). It can be defined with 4 Hermitian $N \times N$ matrices

\[
\hat{X}^1 = \sum_{n=1}^{N} \frac{a}{2} (\hat{e}_{n,n+1} + \hat{e}_{n+1,n}) \quad \hat{Y}^1 = \sum_{n=1}^{N} \frac{ia}{2} (\hat{e}_{n,n+1} - \hat{e}_{n+1,n}) \\
\hat{X}^2 = \sum_{n=1}^{N} b \cos \left( \frac{2\pi n}{N} \right) \hat{e}_{nn} \quad \hat{Y}^2 = \sum_{n=1}^{N} b \sin \left( \frac{2\pi n}{N} \right) \hat{e}_{nn} 
\]

which can be seen as a regularization of the Clifford torus embedded in a 4 dimensional space. As in the case of the fuzzy cylinder, we apply a coordinate transformation to the Clifford torus to project it into 3 dimensional space. The matrix analogue of the coordinate transformation is

\[
\hat{X}^I = \sum_{n=1}^{N} \frac{1}{1 + \hat{X}_4^4} \hat{e}_{nn} \\
\hat{Y} = \hat{X}^I \hat{X}^2 \hat{X} = \hat{X}^I \hat{X}^3 \hat{Z} = \hat{X}^I \hat{X}^1
\]

Again, the matrix $\hat{Z}$ is then diagonalized and $\hat{X}$ and $\hat{Y}$ are transformed as in \( (6.4) \) to arrive at the projected fuzzy Clifford torus.
We have numerically performed this transformations with $a = 1$ and $b = 2$. Fig. 7 shows the result for the matrices $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ for $N = 40$. A $2 \times 2$ matrix structure is again visible and by comparing the diagrams with the previous ones, one would expect that the fuzzy space is composed of two caps that are connected with each other via two string vertices, which is the case for an ordinary torus.

6.3 Graph-based fuzzy vertex

In the end we will use the formalism developed so far for finding a classical limit for the fuzzy spaces defined in [11], which was actually the main motivation for this work. The building block of these fuzzy spaces was the so-called fuzzy vertex with matrices $\hat{X}$ and $\hat{Y}$, which are both of the form

$$
\hat{F} = \begin{pmatrix}
\ddots & \ddots & & & \\
& 0 & r^A & & \\
& r^A & 0 & r^A & \\
& & & & \\
\end{pmatrix}
\begin{pmatrix}
r & r \\
-x^B & r^B \\
-x^B & r^B \\
\ddots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
r & r \\
-x^B & r^B \\
-x^B & r^B \\
\ddots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
r & r \\
-x^B & r^B \\
-x^B & r^B \\
\ddots & \ddots \\
\end{pmatrix}
$$

(6.9)

while the matrix $\hat{Z}$ is diagonal and is composed of $2 \times 2$ blocks being a multiple of the identity in the lower right part (which part is indicated by the lines in $\hat{F}$). The dots indicate that there can be a number of equal entries. However, the matrices $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ are finite dimensional. After a unitary $2 \times 2$ block transformation (4.19) with (3.8) for the lower right part, which leaves the diagonal matrix $\hat{Z}$ unchanged, the result is
\[
\hat{F}' = \begin{pmatrix}
\ddots & r^A \\
& \ddots & r^A \\
& & \ddots & r^A \\
& & & \ddots & r'^P
\end{pmatrix} \begin{pmatrix}
r^A & x^B & r^B \\
x^B & r^B & x^B \\
r^B & x^B & \ddots
\end{pmatrix}
\]

(6.10)

with \( r' = \sqrt{2}r \). When we set \( r' = r^A \), the matrix \( \hat{F}' \) is of the form (4.2) and can be seen as matrix regularization of a \( 2 \times 2 \) matrix-valued function with suitable functions. The upper left part, which is a fuzzy cylinder, and the lower right part, which are two interlaced fuzzy cylinders, also can be interpolated with each other analogously to section 5.3. We can choose functions \( \theta_1, \theta_2 \) and \( \alpha \) in such a way that they change only for the \( q \)-values between the matrix entry of \( r' \) and the matrix entry of the first \( x^B \) of the right lower part of \( \hat{F}' \). With such functions, after regularization with the same \( N \) the matrix \( \hat{F}' \) is reproduced. On the other hand with growing \( N \), there exists a series of matrices of arbitrary size, which has a string vertex as classical limit. Thus, each fuzzy space defined in [11] is a member of a series of matrix algebras with classical limit. This classical limit has the same topology as the corresponding zero-modes surface.

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Everyone who is interested in the computer programs that have been used for calculating the matrices and their visualizations is invited to contact me. I will then send him the SageMath [12] notebooks.

Finally, I will thank my little daughter for bringing my back to earth. One Sunday breakfast we talked about the calculations I do and I started to try to explain her the basics of topology with the standard examples of cups with handles and donuts. Then I tried to explain her the creation of fuzzy surfaces of higher genus by gluing of donuts. She then said: “I understand, but why should one glue donuts?”

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