VARIABLE MASS THEORIES OF GRAVITY

M. Leclerc

Section of Astrophysics and Astronomy, Department of Physics
University of Athens
Panepistimopoules, 157 84 Zografou

Several attempts to construct theories of gravity with variable mass are considered. The theoretical impacts of allowing the rest mass to vary with respect to time or to an appropriate curve parameter are examined in the framework of Newtonian and Einsteiniand gravity theories. In further steps, scalar-tensor theories are examined with respect to their relation to the variation of the mass and in an ultimate step, an additional coordinate is introduced and its possible relation to the mass is examined, yielding a five dimensional space-time-matter theory.

I. INTRODUCTION

Many attempts have been proposed to generalize Einstein’s theory of gravity through the introduction of a variable rest mass. The motivation for doing so comes mainly from theoretical arguments, like scale invariance of the gravitational theory, additional scalar fields that emerge from string theories or additional degrees of freedom that arise in the framework of brane-world theories and could eventually be related to such a mass variation. In these theories, the rest mass $m$ is supposed to vary slowly with time. Usually, the variation is considered to occur at a rate related to the age of the universe. In particular, this means that the effects of this departure from classical general relativity only occur at cosmological scales and do not affect planetary motion or any other experiment carried out in the planetary system. These effects have been studied in the framework of several theories (see and references therein). In this article, we focus on the theoretical effects of the assumption of variable rest masses. Thus, instead of cosmological models, we consider the planetary system or even the gravitational field of point particles and examine the impact of the introduction of the concept of a variable rest mass. Even if there are no experimental changes, a small departure from general relativity may represent a severe change in the theoretical concepts. Especially, attention should be paid to the self-consistency and the covariance of the theory as well as to the very definition of the mass as a quantity that characterises the particles.

None of the considered models is essentially new and all of them has been considered with respect to several aspects in the literature. In this sense, the set of references is not complete and contains only those sources which were explicitly consulted. The aim of this article is merely to drive attention to some of the problems that one should have in mind when dealing with variable mass theories.

II. TIME DEPENDENT PARTICLE MASSES

A. Newtonian theory of gravity

In a first attempt, we generalize Newton’s theory of gravity by admitting a time dependence of the particle masses. The field equations and the equations of motion read

$$\frac{d}{dt}(m(t)\vec{v}) = -m(t)\vec{\nabla}\varphi$$

$$\Delta\varphi = 4\pi G \rho(\vec{x}, t),$$

where $G$ is treated as a constant. The point particle case is now described by a time dependent mass density

$$\rho(\vec{x}, t) = M(t)\delta(\vec{x} - \vec{x}(t)).$$

A test particle of mass $m$ in the field of a central body at rest at the origin with mass $M$ evolves as

$$\frac{d}{dt}(m(t)\vec{v}) = -\frac{GM(t)M(t)}{r^3} \vec{r}.$$ 

Although the orbital angular momentum $\vec{L} = m(\vec{r} \times \vec{v})$ is still conserved, the observable kinematical quantity $\vec{l} = \vec{r} \times \vec{v}$ is not. We have instead

$$\frac{d}{dt}\vec{l} = -\frac{\dot{m}}{m} \vec{l}. $$

The time dependence of the particle masses also destroys energy conservation. For the case of a constant central particle mass, $M = M_0$, we find

$$\frac{d}{dt}\left(\frac{1}{2} \dot{v}^2 - \frac{GM_0}{r}\right) = -\frac{\dot{m}}{m} \dot{v}.$$ 

The Newtonian equations do neither tell us about the time evolution of $m$ nor of $M$. In later sections, we try to remove this deficiency by treating the mass as a scalar field or as an additional dimension of spacetime, and thus as subject to the field equations.

An interesting idea, which is the realization of Mach’s principle in a very strong form, is to consider the rest
mass of the particles as entirely due to its gravitational energy (or rather its opposite) from all the other surrounding masses. Thus, if we suppose the energy to be of the Newtonian form, the mass could be expressed as (cf. [3])

$$mc^2 = \sum M \frac{GmM}{r_M},$$

where the sum is carried out over all the bodies of the universe. Note that this is a recursive definition, since the other masses, $M$, also depend in the same way of $m$. Without going into details and discussing this equation, we will directly apply it to the case of planetary motion.

Let $m$ be the mass of a planet, and $M$ the mass of the sun. Since the motion is confined at a small space region, we can suppose that the gravitational energy due to the interaction of $m$ or $M$ with all the other masses is constant, and that the variation of the masses is entirely due to the variation of the potential energy between $M$ and $m$.

Thus, we can write

$$mc^2 = m_0c^2 + \frac{mMG}{r} = m_0c^2 + \frac{m_0M_0G}{c^2r} \frac{G}{r} + \frac{m_0^2M_0^2G^2}{c^2r^2} + \ldots$$

The (Newtonian) Lagrangian is given by

$$L = \frac{m_0\vec{v}^2}{2} - m_0\phi,$$

with $\phi = -GM/r$. We now introduce our mass relation into this Lagrangian. The second term can be expanded as

$$\frac{mMG}{r} = (m_0 + \frac{m_0M_0G}{c^2r})(M_0 + \frac{m_0M_0G}{c^2r}) \frac{G}{r},$$

where terms of the order $\frac{1}{r}$ are neglected. This can be written as

$$\frac{mMG}{r} = -m_0\phi + \frac{m_0^3}{c^2M_0} \phi^2 + \frac{m_0^3}{c^2} \phi^2,$$

where $\phi$ is now defined as $\phi = -\frac{M_0G}{r}$. As to the first term, it yields

$$\frac{m\vec{v}^2}{2} = \frac{m_0\vec{v}^2}{2} + \frac{i^2}{2} \frac{m_0M_0G}{c^2r} = \frac{m_0\vec{v}^2}{2} - \frac{m_0\phi}{c^2} \vec{v}^2,$$

where terms of the order $\phi \vec{v}^2/c^2$ are neglected (since the velocity $\vec{v}$ is nonrelativistic).

The result is

$$L = \frac{m_0\vec{v}^2}{2} - m_0\phi - \frac{m_0^3}{c^2} \phi i^2 + \frac{m_0^3}{c^2} \phi^2 + \frac{m_0^3}{c^2M_0} \phi^2.$$

This is the Lagrangian that describes the motion of a planet $m_0$ for the considered mass variation, with an accuracy of 1 post-newtonian (1 PN) order. For planetary motion, the last term can of course be ignored, since $M_0 >> m_0$.

To the same order, the general relativistic result (from the post-newtonian expansion of the two body system) has the form (see [3], §106)

$$L_{GR} = \frac{m_0\vec{v}^2}{2} - m_0\phi - \frac{3m_0^2}{c^2} \phi \vec{v}^2 - \frac{m_0^2}{c^2} \phi^2 - \frac{m_0^3}{c^2M_0} \phi^2,$$

where, in the expression from [3], we have set the velocity of the central planet $M_0$ to zero. (Actually, the going over from the 2 body problem to the one body problem, i.e. setting one velocity to zero, should be accompanied by the relation $M >> m$, i.e. the last term should also be dropped. We will do so in the following.)

As we have seen earlier in this section, the Newtonian energy is not a constant of motion any more. It is easy to see that the relation

$$\frac{m_0\vec{v}^2}{2} + m_0\phi = E_N$$

will suffer from first order corrections, i.e. we have $m_0\vec{v}^2/2 + m_0\phi = E_N + O(1)$ This, however, leads to the following relation:

$$\frac{m_0^2\vec{v}^2}{c^2} + \frac{m_0^2}{c^2} \phi^2 = \frac{m_0^2}{c^2} E_N\phi + O(2)$$

Using this relation, we can bring our 1 PN order Lagrangian into the form

$$L = \frac{m_0\vec{v}^2}{2} - m_0\phi - \frac{3m_0^2}{c^2} \phi \vec{v}^2 - \frac{m_0^2}{c^2} \phi^2 + 2\frac{m_0}{c^2} E_N \phi$$

$$= L_{GR} + \frac{2m_0}{c^2} E_N \phi.$$

In this form, the departure from general relativity is best seen. The last term is especially important for orbits near the sun. This, of course, is a general feature of post-newtonian corrections. As the Lagrangian $L_{GR}$ describes already all the observable effects in the planetary system (except for the light deflection, which cannot be subject to a post-newtonian extension, which is confined to the description of particles with $\vec{v} << c$), it is surprising, that by our straightforward concept, we got the right form of the corrections (even if the factors don’t match), using a non-relativistic scalar field theory.

As a last remark to the Newtonian theory, we should have in mind that, even if the above result would have been identical with the general relativistic expansion, the embedding of this theory into a special relativistic context would again destroy this matching. Indeed, we would have to bring the Newtonian equations into a covariant form, and the relation of the mass to the gravitational field would have to be changed. Problems arise, however, even before that!

### B. Special relativistic theory

In a next step, we introduce the concept of a variable mass into a Lorentz covariant theory. To begin with,
we consider the motion of a test particle in flat spacetime under the influence of an electromagnetic field. The equations of motions in special relativity follow from the Lagrangian
\[
L = \frac{1}{2} m u^i u^k + e A_i u^i.
\]
In order to conserve the Lorentz covariance, \( m \) cannot be, a priori, a function of the time coordinate \( t \) but should be regarded as dependent on the curve parameter \( \tau \) which is still to be interpreted. The Euler-Lagrange equations yield
\[
\frac{d}{d\tau} (m(\tau) u^i) = e F^i_k u^k.
\]
Contracting both sides with \( u^i \), we find a constant of motion \( m^2 u^i u^i \). Thus, we can write down a relation
\[
E^2 - c^2 p^2 = m_0^2 c^2 . \tag{1}
\]
Energy and momentum are defined with \( m(\tau) \) whereas the constant \( m_0 \) is defined through the above relation. If we relate \( \tau \) to proper time, i.e. the time measured by a comoving observer, we find in the comoving frame \( (u^\alpha = 0, \; u^0 = c) \):
\[
m(\tau) = m_0.
\]
Thus, the nonmoving particle has a constant mass. This seems to be a reasonable relation, even though this interpretation of \( \tau \) is not forced by the equations of motion. In general, we only have \( m u^0 = m_0 c \) for the particle at rest.

In terms of the observable kinematical quantity \( u^i \), the equations of motion read
\[
\dot{u}^i = \frac{e}{m} F^i_k u^k - \frac{\dot{m}}{m} u^i. \tag{2}
\]
The energy-momentum relation (1) allows us to eliminate \( m \) and \( \dot{m} \). The result is
\[
\dot{u}^i = \frac{e}{m_0 c^2} u^k u^k u^i + \frac{(u^0 u^i)}{u^k u^k} \frac{\dot{m}}{m} u^i.
\]
However, it is now obvious that another constant of motion is \( u^i u^i \), and hence \( m(\tau) = m_0 = \text{const} \). The mass cannot be variable with the above Lagrangian!

One may think of deriving the equations of motions from the following Lagrangian:
\[
L = \frac{1}{2} \dot{u} u^i + \frac{e}{m(\tau)} A_i u^i.
\]
However, the corresponding equations of motion
\[
\dot{u}^k = \frac{e}{m} F^k_i u^i - \frac{\dot{m}}{m m} A^k
\]
break the gauge invariance of the electromagnetic theory for variable \( m \), which is not the scope of our theory.

Finally, one may write down a parameter invariant Lagrangian of the form
\[
L_1 = m(\tau) \sqrt{\eta_{ik} u^i u^k} + e A_i u^i.
\]
This has the advantage of removing the ambiguity of whether we place the mass \( m \) on the first term or its inverse on the second term. Indeed, through the substitution \( d\lambda = d\tau/m \), the Lagrangian takes the form
\[
L_2 = \sqrt{\eta_{ik} u^i u^k} + \frac{e}{m(\lambda)} A_i u^i,
\]
where \( u^i \) is now defined as \( u^i = \frac{dx^i}{d\tau} \). Again, the second Lagrangian apparently breaks gauge invariance for variable \( m \). As it is equivalent to the first one, we can immediately conclude that \( m \) is a constant of motion. Indeed, \( L_1 \) yields the equations
\[
\frac{d}{d\tau} (m u^i) = e F^i_k u^k,
\]
which, contracted with \( \frac{m u^i}{\sqrt{u^i u^i}} \) leads to \( \frac{d}{d\tau} m = 0 \).

The result of this section, namely that the mass of a test particle remains constant throughout its motion, holds true, of course, in the non-relativistic limit. The possibility of a variable mass obtained in the previous section is thus entirely due to the presence of a gravitational field, especially to the fact that the mass of the test particle occurs at both sides of the equations of motion. Having this in mind, one should conclude that the variation of the mass should be subject not to the kinematical evolution of the particles, but to the underlying theory of gravitation. Moreover, in the limit of vanishing gravitational fields, the particle masses should be found to be constant.

\section{General relativity}

The generalization to an arbitrary spacetime metric of the above considerations is straightforward. For a given metric, we find the equations of test particle motion by simply replacing the partial derivates through the covariant ones. The equation \( \frac{d^2 u^i}{d\tau^2} = 0 \) however will not be affected!

Next, consider the field equations of general relativity
\[
G_{ik} = \frac{8\pi G}{c^4} T_{ik}. \tag{2}
\]
It is known that the only spherical symmetric vacuum solution is the Schwarzschild solution, which can be written in the form
\[
ds^2 = (1 - \frac{R}{r}) c^2 dt^2 - (1 - \frac{R}{r})^{-1} dr^2 - r^2 d\Omega^2,
\]
where \( R \) is related to the mass of the central planet through \( R = \frac{2GM}{c^2} \). In solving the Einstein equations,
$R$ arises as a constant of integration. The identification with the mass $M$ is usually made by considering the Newtonian limit. However, strictly spoken, the Schwarzschild solution is not really a vacuum solution, but a solution corresponding to a stress-energy tensor of a point-like particle, i.e.

$$T^{ik}(\vec{x}, t) = \frac{M}{\sqrt{-g}} \frac{dx^i}{d\tau} \frac{dx^k}{dt} \delta^{(3)}(\vec{x} - \vec{x}(\tau)),$$

where $x^i(\tau)$ describes the worldline of the particle. In the comoving frame, the only equation involving the matter fields reads, for the particle at the spatial origin,

$$G_0^0 = \frac{8\pi G}{c^4} \frac{M^2}{\sqrt{-g}} \delta^{(3)}(\vec{x}).$$

From this, we see that the constant of integration is already fixed by the field equations. Now, we suppose that the central mass varies throughout its evolution, i.e. $M = M(\tau)$. In the comoving frame, the parameter $\tau$ of the particle coincides with coordinate time $ct$. Thus we have to solve Einstein’s equations with a time dependent energy-stress tensor.

In a first approximation, if we suppose that the time variation of $M$ is of negligible order (so that time derivatives appearing in $G_{ik}$ can be neglected), we can write the solution as

$$ds^2 = (1 - \frac{2GM(t)}{c^2r}) c^2dt^2 - (1 - \frac{2GM(t)}{c^2r})^{-1} dr^2 - r^2d\Omega^2.$$

It turns out, however, that the time dependent energy-stress tensor is not consistent with the underlying theory. Infact, as a consequence of the field equations (2), the matter also has to obey the relation

$$T^{ik}_{;k} = 0.$$

For the point particle energy-stress tensor, this equation is equivalent to the equation of motion obtained from the Lagrangian

$$L = M(\tau)g_{ik}u^iu^i,$$

and as we have seen, this leads to $M(\tau) = const$. (Note that the we have already partly used the above equations when we choose the rest frame of the particle, i.e. when we set $u^a = 0, \alpha = 1, 2, 3.$)

As a conclusion to this and the preceeding section, we can say that it is not possible, in a straightforward manner, to suppose that particle masses vary throughout the evolution on their worldlines. The reason for this can be seen in the relation $E^2 - c^2p^2 = const$ and its general relativistic generalization. In the Newtonian case, their is no such relation, and the mass evolution is left, a priori, undetermined.

### III. SCALAR-TENSOR THEORIES OF GRAVITY

A quite different approach to a varying mass theory is obtained by considering the mass not as a function of the parameter of the worldline but as a spacetime dependent quantity, i.e. a scalar field $m = m(x^i)$. In order to make contact to classical theories, it is convenient to write $m(x^i) = m_0 e^{\varphi(x^i)}$, where $m_0$ is a constant which can be identified as the mass of a particle for vanishing gravitational fields or the mass at some (cosmological) time $t_0$, depending on the model.

The free particle motion in flat spacetime is described by

$$L = m_0 e^{\varphi(x^i)} u^iu^i.$$  

(3)

It is tempting to interpret $\varphi(x^i)$ as the gravitational field and to look for appropriate field equations. It is indeed possible to describe planetary motion up to Newtonian order through the identification $\frac{\dot{x}}{c} = \varphi_{\text{newton}}$, but higher order corrections are not in agreement with experiment. Especially, no deflection of light rays can be deduced from (3), in agreement with the fact that the vacuum Maxwell theory is conformally invariant.

One should thus consider the scalar field as an additional field. The simplest extension of general relativity containing a scalar field is Brans-Dicke theory, which is based on the following action:

$$S = (16\pi)^{-1} \int d^4x \sqrt{-g} \left( \pi R - \omega \varphi^{-1} \partial_i \varphi \partial^i \varphi \right) + \int d^4x \sqrt{-g} \varepsilon_m.$$

The resulting field equations read

$$G_{ik} = 8\pi \varphi^{-1} T_{ik} + \omega \varphi^{-2}(\varphi_1,i \varphi_{,k} - \frac{1}{2} g_{ik} \varphi_1 \varphi_{,l}) + \varphi^{-1}(\varphi_{,ik} - g_{ik} \varphi_{,l}),$$

(4)

$$\varphi_{,i} = \frac{8\pi}{3 + 2\omega} \left[ T - 2 \frac{\partial T}{\partial \varphi} \right].$$

(5)

These equations contain the following identity, which can be interpreted as the equation of motion for the matter:

$$T^{ik}_{;k} - \frac{\partial T}{\partial \varphi} \varphi_{,i} = 0.$$

The presence of the scalar field now allows one to write down an energy stress tensor for a particle with variable mass:

$$T^{ik} = \frac{m(\varphi)}{\sqrt{-g}} \frac{dx^i}{d\tau} \frac{dx^k}{dt} \delta^{(3)}(\vec{x} - \vec{x}(\tau)).$$

(6)

The field equations with this energy-stress tensor can be solved using a post-Newtonian extension. Planetary
system observations are in agreement with the theory for \( \omega > 3000 \), without taking into account the variation of the mass. The results of general relativity are found for \( \omega \to \infty \) (identifying \( G = \varphi^{-1}(4 + 2\omega)/(3 + 2\omega) \)).

The variation of the masses leads to further effects, none of which have been observed up to now however. Generally, the scalar field, entering the equations of motions and the field equations in a different manner, may violate the weak equivalence principle. The equations of motion depend on \( m(\varphi) \) and thus on the particle’s structure. Of special interest is the emission of dipole gravitational radiation from binary systems which is not present in general relativity and could be subject to future experiments.

It should however be emphasized that the variable mass entering the above energy-stress tensor is not a direct consequence of the theory. It should not be applied to elementary particles but only to extended bodies. It was actually introduced in the framework of hydrodynamics for Brans-Dicke theory (Nordtvedt effect, cf. \[3\]). The quantity \( m \) varies because of the presence of the field \( \varphi \), which contributes to the internal energy of the body. The tensor (6) is just a convenient way to parametrize the matter by just one field \( m(\varphi) \) and treat it as a point particle. A more physical interpretation is to use the identification \( G = \varphi^{-1}(4 + 2\omega)/(3 + 2\omega) \) (see \[3\] and references therein). The variation of \( G \) then leads to the variation of the total energy which we described by \( m \).

Another remark is that the field equations do not determine the function \( m(\varphi) \). This is of course not surprising, since we have treated the body as a point particle. To find this function, one has to look for inner solutions of the body under investigation. It then turns out that the bodies can conveniently be parametrized by their sensitivity \( s \), which is defined as

\[
s = -\frac{\partial(\ln m)}{\partial(\ln G)},
\]

with \( G = \varphi^{-1}(4 + 2\omega)/(3 + 2\omega) \) and the models give values between \( s = 0.1 \) and \( 0.3 \) for neutron stars of masses around \( 1, 4 M_\odot \) and \( s = 0.5 \) for black holes \[3\].

The general remarks of this section hold true for every scalar-tensor theory based on general relativity. The scalar field can always be interpreted as a variation in the mass or as a variation in the gravitational constant, at least in a certain limit, i.e. if the departures from general relativity are supposed to be small (which has to be the case for the theory to be in agreement with experiment). The interpretation as variation of the gravitational constant is however more straightforward. Differences occur when we couple the theory to other fields and/or when we consider elementary particles, for instance the Dirac equation. In the latter case, a variable mass of the elementary particle would lead to many difficulties we would have to investigate and for which we do not have an experimental justification (see section IV.F).

IV. VARIABLE MASS THROUGH A FIFTH DIMENSION

A. Introduction

Finally, we take a look at theories who try to impose the concept of variable masses in general relativity through the use of a fifth dimension. The five dimensional space is called space-time-mass and the metric is taken as \([3, 4]\)

\[
dS^2 = g_{AB}dx^A dx^B = g_{ik}(x^m, x^4)dx^i dx^k + \varepsilon \varphi^2(x^m, x^4)(dx^4)^2, \quad (7)
\]

where \( \varepsilon = \pm 1 \), depending on which signature we choose. We take the convention that capital indices \( A, B, \ldots \) run from 0 to 4 whereas the indices \( i, k, l, \ldots \) run from 0 to 3. The mixed components \( g_{4i} \) have been transformed to zero through an appropriate coordinate transformation.

The action is taken to be

\[
S = \int R \sqrt{-g} \, d^5x
\]

and the resulting (vacuum) field equations read

\[
G_{AB} = 0. \quad (8)
\]

\( R \) and \( G_{AB} \) are constructed from the five dimensional metric \( g_{AB} \) in the usual way.

B. Spherically symmetric solution

Before we interpret these equations and eventually generalize them to include matter fields, we begin our discussion by considering the following exact spherically symmetric solution of (8)

\[
dS^2 = (1 - \frac{a}{r})c^2 dt^2 - \frac{dr^2}{1 - a/r} - r^2 d\Omega^2 + \varepsilon (dx^4)^2. \quad (9)
\]

Here, \( a \) arises as a constant of integration.

The idea of space-time-mass theory is to identify the fifth coordinate with the mass through the definition \( x^4 = Gm/c^2 \) (see \[3\]). To be in agreement with experiment, the variation of the mass, \( dx^4 \) has to be very small (at least for systems of dimensions of the solar system), and the above solution thus differs only slightly from the Schwarzschild metric if we identify \( a \) with \( 2GM_0/c^2 \), where \( M_0 \) is the mass of the central planet. One should however avoid to write \( x^4 = Gm/c^2 \). A coordinate cannot be a constant of integration of the solution of the

---

1 This choice should rely on experimental data. Since no experiment is known that contradicts GR, the wisest choice would consist in setting \( \varepsilon = 0 \).
field equations. The solution has to be valid in all 5 dimensional spacetime (except for singularities), and thus for all \((x^m, x^4)\). In other words, the coordinate \(x^4\) is related to the mass of a test particle in the above metric, and not to the (constant) mass of the source planet. The coordinate \(x^4\) is not constant, all that can be said is that the metric \(g_{AB}\) does not depend on it. The quite different manners in which enter the masses of the test particle and of the source may eventually be removed if one considers the fully 2-body equations, which requires however that we solve the field equations with matter.

C. Geodesic motion and matter fields

Since we deal with a five dimensional theory, it is natural to require that test particles will follow five dimensional geodesics, which follow from the Lagrangian

\[
L = g_{AB} u^A u^B, \tag{10}
\]

where \(u^A = \text{d}x^A/\text{d}S\). These equations can easily be reparametrized with the four dimensional parameter \(ds = c\,\text{d}t\). The question which parameter is the more physical one and what is their relation to proper time has been extensively discussed in the framework of Kaluza-Klein theory (see \[CC\] for a detailed discussion). More fundamental however is the fact, that we cannot write instead of (10)

\[
L_2 = mg_{AB} u^A u^B,
\]

because \(m\) being a coordinate, \(L_2\) is not a scalar function. The question thus arises how we shall describe a particle under the influence of both a gravitational and an electromagnetic field for instance. Since we know from general relativity that the equations of motion follow directly from the field equations, we have to discuss the coupling of matter to the gravitational field in order to answer this question.

D. Induced matter theory

There are two ways of introducing matter fields to the five dimensional theory. One is to complete the action with an adequate matter Lagrangian density, i.e. to write

\[
S = \int \left(-\frac{\epsilon^4}{16\pi G} R + \mathcal{L}_m\right) \sqrt{-g} \text{d}^5 x
\]

which leads to

\[
G_{AB} = \frac{8\pi G}{c^4} T_{AB} \tag{11}
\]

where \(T_{AB}\) is the five dimensional generalization of the energy-stress tensor and \(G\) is the coupling constant. From the theoretical point of view, this yields a theory that is covariant under the complete \(O(4,1)\) or \(O(3,2)\) (depending on which signature we choose) as well as under the five dimensional translational group. The five coordinates are treated equally and the relation to physical 4d spacetime has to be found from physical arguments.

The second way is the induced matter approach. In this theory, the 5 dimensional vacuum equations are split into a 4+1 form and interpreted as 4 dimensional equations for gravitational fields plus matter fields (see the review article \[11\] for instance). Indeed, using the metric (7), the equation \(G_{AB} = 0\) can be written

\[
\epsilon_{ik} \frac{\partial}{\partial x^i} \left(\frac{\epsilon \phi^2}{2} \right) + \frac{1}{4} g_{ik} (g^{lm} g_{tm} (1/2 g^{lm} y_{tm} y_{ik})
\]

\[
+ \frac{1}{4} g_{ik} (g^{lm} y_{tm} + (g^{lm} y_{tm})^2) \right) \tag{12}
\]

and the additional equation

\[
\epsilon \phi^2 i; i = \frac{1}{4} g^{lm} y_{tm} - \frac{1}{2} g^{lm} y_{tm} + \frac{\epsilon \phi}{2} g^{lm} y_{tm}. \tag{13}
\]

The tensor \(G_{ik}\) as well as the covariant derivatives are formed with the 4 dimensional part of the metric, \(g_{ik}\). The dot means derivation with respect to \(x^4\). The right hand side of (12) is now identified as \(8\pi G T_{ik}\). In this way, four dimensional matter arises from five dimensional vacuum. It is claimed that these equations recover all the equations of state commonly used in astrophysics and in cosmology. Apart from the fact that it would require at least some imagination to identify the right hand side of equation (12) with the energy-stress tensor of a point-particle for instance, another remark is worthwhile to be pointed out. \(G_{ik}\) being constructed exactly as in general relativity, the 4 dimensional Bianchi identity \(G^{ik} ; i = 0\) also holds true. Thus we are left with the equation \(T_{ik} = 0\) (for this strange \(T_{ik}\), the right hand side of (12)), which is the equation of motion for the (4 dimensional) matter fields. This, however, corresponds, in the point-particle case, to the 4 dimensional geodesic equation.

More generally, \(G_{ik}\) being a four dimensional tensor, i.e. transforming covariantly under the Poincaré group, the same has to be true for the right hand side of (12). Thus, we are actually led back to a effective four dimensional theory, which is not an extension of general relativity, but rather a constraint version, since the energy-stress tensor has to be of the particular form (r.h.s. of (12)) and in addition, the constraint equation (13) has to be fulfilled.

Especially, the identification of \(x^4\) with \(m\) does not seem to make much sense in this case. It should however be noticed in favour of this interpretation that if \(g_{ik}\) does not depend on \(x^4\), the effective 4d energy-stress tensor has to be traceless, and thus describes a radiation-like equation of state, i.e. massless particles.
As a conclusion, we remark that in the induced matter approach, we are led back to 4 dimensional equations of motion. The physical meaning of the five dimensional metric remains unclear and the relation of the fifth coordinate to the particle masses is doubtful.

E. Five dimensional general relativity

There is still another argument against the induced matter approach. The solution (9), which is a vacuum solution of the 5d Einstein equations, suffers from a singularity at the spatial origin. This is quite unusual for real vacuum solutions (which are wavelike) and should be related to some boundary conditions due to the matter distribution. This can be done if we include matter fields in the way of equation (11).

For our specific solution (9), it is easy to see that $G_{ik} = \delta_{ik} G$ and that $g_{ik}$ is just the Schwarzschild solution, which satisfies

$$G^{ik} = \frac{8\pi G}{c^4} \frac{M_0}{\sqrt{-g_{ij}}} \frac{dx^i dx^k}{d\tau} \delta^{(3)}(\vec{x} - \vec{x}(\tau)).$$

Thus, in order to write down the equations $G^{AB} = (8\pi G/c^4) T^{AB}$, we have to find a five dimensional tensor whose spacetime components are of the form

$$T^{ik} = \frac{G}{\Gamma} \frac{M_0}{\sqrt{-g_{ij}}} \frac{dx^i dx^k}{d\tau} \delta^{(3)}(\vec{x} - \vec{x}(\tau)).$$

Note that, except from the fact that $g = \delta^{(4)} \delta$, the right hand side of this equation contains the 4 dimensional parameter $\tau$ as well as a 3 dimensional mass density $M_0 \delta^{(3)}(\vec{x} - \vec{x}(\tau))$.

If this extension can be done in an appropriate way, the equation $T^{ik} A_k = 0$ reduces to the five dimensional geodesic equation (for the particle of constant mass $M_0$), i.e. to the extremization of the five dimensional line element $dS$. In this case, the five dimensional space has a real physical meaning and the deviations from general relativity (due to non-vanishing $dx^A/dS$) can be discussed, even though the identification of $x^A$ and $m$ remains still doubtful.

It is however not straightforward to find the 5 dimensional energy-stress tensor that describes a point particle (or some other mass distribution). If we require the equations of motion to be 5d geodesics, the action has to be of the form

$$S_m = \int q g_{AB} u^A u^B dS,$$

where $q$ is some constant and $\mu = \frac{dx^A}{d\tau}$. Hence, the Lagrangian density is of the form

$$\mathcal{L}_m = \frac{\mu}{\sqrt{-g}} g_{AB} \frac{dx^A}{dS} \frac{dx^B}{dt}$$

and the corresponding energy-stress tensor is given by

$$T^{AB} = \frac{\mu}{\sqrt{-g}} \frac{dx^A}{dS} \frac{dx^B}{dt}.$$

The impact of the introduction of a fifth dimension is best seen in the Newtonian limit of the field equations. If we suppose that the matter distribution $\mu$ is described in the comoving frame, i.e. if we set $u^\alpha = 0$, $\alpha = 1, 2, 3$ and $u^4 = 0$, and consider the first order approximation $g_{00} = 1 + 2\varphi/c^2$, $g_{AB} = \eta_{AB}$ else, the equations, up to terms of higher order in $\varphi$, reduce to

$$\Delta \varphi \pm \frac{\partial^2 \varphi}{\partial(x^4)^2} = 4\pi \Gamma \mu. \quad (14)$$

The sign on the left hand side again depends on the signature of the metric. Note that there are no time derivates, since they are of higher order (due to the $c$ that occurs in $x^0 = ct$). Apart from the derivates with respect to $x^4$, the main difference from the Newtonian field equations lies in the definition of the matter distribution $\mu$. As a generalization of the equation $\rho = dm/dV$, the quantity $\mu$ was defined as a four dimensional density $\mu = dq/dV dx^4$. This cannot be avoided if we ask for the energy-stress tensor to transform covariantly.

The quantity $q$ thus plays the role of the conserved source charge (which in general relativity is $M$) and in the point particle case, it is the quantity that characterizes the particle (apart from its spin and its electric charge). Since we do not know any other quantities that describe a point particle, $q$ should, at least in some classical limit, reduce to the mass. This, in turn, makes it very difficult to interpret the fifth coordinate as the mass. Indeed, if we try to describe a point particle with a definite mass, the density should take the form

$$\mu = q \delta^{(3)}(\vec{x} - \vec{x}(\tau)) \delta(m - m(\tau)),$$

where we write $x^4 = m$, omitting the conversion factor for simplicity. (Note that $x(\tau)$ and $m(\tau)$ are actually constant in the frame we have chosen.) But certainly, with this matter distribution, the solution will not be of the form $\varphi = m/r$, because of the singularity not only at $r = r(\tau) = 0$ but also at $m = m(\tau) = m_0$ in the mass distribution. Having in mind however that the Newtonian solution is linear in $m$, the equations (14) take the form

$$\Delta \varphi = 4\pi \Gamma \mu,$$

and if we compare this with the Newtonian equation $\Delta \varphi = 4\pi G \rho$ we can identify

$$\mu = G \frac{\Gamma}{\rho}, \quad (15)$$
or in the particle case

\[ \mu = \frac{G}{\Gamma} m_0 \delta^{(3)}(\vec{x} - \vec{x}(\tau)). \]

This leads to the correct result, its interpretation is however doubtful. If we integrate over 3d space, we find

\[ \frac{dq}{dm} = \int \mu \, d^3x = \frac{G}{\Gamma} m_0. \]

The meaning of this equation is that the charge \( q \) depends homogeneously on the mass coordinate \( m \), or in other words, no definite mass can be associated with the particle.

We conclude that we cannot interpret \( x^4 \) as being related to the mass. In general, we retain the fact that we can find the Newtonian solution in the case where \( \varphi \) does not depend on \( x^4 \) and if we set \( \mu \) as in equation (15).

This can easily be generalized to the full equations (11) and the result is again the metric (9), with the difference that we can now justify the existence of the singularity at \( r = 0 \). In this special case, nothing new is gained, except from the additional equation of motion for test particles in this metric, \( u^4 = \text{const} \). In particular, this means that we have \( u_i u^i = \text{const} \) as well as \( u_A u^A = \text{const} \), which could be of specific interest in the case of massless particles (light deflection). However, for the moment, we don’t have neither a theoretical interpretation nor an experimental result that allows us to fix the additional initial condition \( u^4(S = 0) \).

We close this section with some general remarks. One may wonder if the conclusions we have got at in both the case of the induced matter approach and in the five dimensional theory with additional matter Lagrangian are not actually confined to the special solution (9) on which we based our discussion. Indeed, even in the spherical symmetric case, there exist other solutions (see [7] for instance, where a generalization of Birkhoff’s theorem in the context of space-time-mass theory is discussed). The main conclusions remain however valid for every solution: In the induced matter approach, the (4d) Bianchi identity leads to 4 dimensional equations of motion, and in the theory with additional matter fields, the 3d mass density \( \rho = dm/dV \) has to be replaced by a 4d density \( \mu = dq/dV dx^4 \) (or more generally, a 5 dimensional description of matter fields is needed).

The first is of course no problem. It leads us back to an effective 4d theory. The physical meaning of the 5d space has however to be carefully investigated, especially with respect to transformation that mix the space-time coordinates with the fifth coordinate. If we take the interpretation from Kaluza-Klein theory to identify the mixed components with the electromagnetic potential, we have to analyse carefully the field equations and the equations of motion in the case of a non-compactified fifth dimension.

As to the second theory, it seems difficult to find a decent five dimensional description for the matter field. In classical, i.e. compactified Kaluza-Klein theory, the five dimensional quantities can always be related to a four dimensional quantity through the integration over the compactified dimension. So, for instance, one could refine the mass density \( \rho \) as \( \int \mu \, dx^4 \). In our, non-compactified theory, this is however not possible, since it leads in general to infinite results (this can already be seen from equation (15), whose right hand side cannot be integrated in this way). On a more fundamental level, the use of equations (11) require a general 5d description of the matter fields, i.e. a 5d description of particle physics and quantum fields.

F. Mass as a fifth momentum component

Consider the special relativistic energy-momentum relation

\[ E^2 - c^2 \vec{p}^2 = m^2 c^4 \]  \hspace{1cm} (16)

If one intends to introduce variable rest masses through a fifth dimension, this equation suggests to interpret \( m \) not as the fifth coordinate, but rather as a fifth component of the momentum, i.e. to write

\[ E^2 - p^2 - m^2 = p_A p^A = 0, \]  \hspace{1cm} (17)

where for simplicity, we have set \( c = 1 \). Note that the above substitution has been made possible by the fact, that the mass occurs explicitly only once in equation (16). If one repeats the steps that lead from (16) to the Dirac equation, the equation (17) yields

\[ i\gamma^A \partial_A \psi = 0, \]

with the corresponding Lagrangian density

\[ \mathcal{L} = \bar{\psi}(i\gamma^A \partial_A)\psi. \]  \hspace{1cm} (18)

In order for the momentum \( p_A = i \partial_A \) to satisfy equation (17), the matrices \( \gamma^A \) have to fulfill the following anticommutation relation:

\[ \{ \gamma^A, \gamma^B \} = 2\eta^{AB}, \]

where the signature of the five dimensional Minkowski metric is now clear from (17). It is easily seen that the Dirac matrices \( \gamma^4 \) together with the matrix \( \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \) will do this.

Thus, by interpreting \( m \) as a fifth momentum component, the four dimensional Dirac equation is replaced by

\[ \text{References} \]

2 Especially, just as in the case of the Dirac equation, (16) is not independent of the mass, as are classical equations of motion for free particles

3 Note that \( \gamma^5 \) differs by a factor \( i \) from the usual definition, in order to satisfy \( (\gamma^5)^2 = -1 \). (In the remaining of this section, the indices \( A, B, \ldots \) take the values \( 0, 1, 2, 3, 5 \)).
a five dimensional massless Dirac equation. In the Lagrangian, the mass term \( m\psi \) has been replaced by the term \( \bar{\psi}(i\gamma^5\partial_5)\psi \), or in an explicit representation (see [12] for instance),

\[
m \to \left( \begin{array}{cc}
-\partial_5 & 0 \\
0 & \partial_5
\end{array} \right).
\]

Noether’s theorem now yields a conserved current

\[
J^A = \bar{\psi}\gamma^A\psi, \quad \partial_A J^A = 0.
\]

Integration over 3-dimensional space does, however, not lead to a conserved charge. We find instead:

\[
\frac{d}{dt} \int J^0 d^3x - \frac{d}{dx^5} \int J^5 d^3x = 0,
\]

where we have assumed that \( J^0 \) does not depend on \( x^5 \) and vice-versa. If we take the usual interpretation for the first term as the time derivate of the (unit) charge, the second term represents a violation of charge conservation.

As a final remark to the Dirac equation, we note that the \( \text{U}(1) \) gauge invariance is preserved in the same manner as in the 4 dimensional case to a 5 dimensional gauge theory through the extension of \( \partial_A \to \partial_A + i e A^5 \). We will try to identify \( A^5 \) with the gravitational potential.

We know that in the 4 dimensional case, to the Lagrangian (18) corresponds a classical Lagrangian \( L = u_i u^i \). In the 5 dimensional case, this leads to a problem, because we cannot simply divide equation (16) by \( m^2 \) without breaking the covariance of the theory. (This is the reason why we began this section with the Dirac equation and not with the classical theory.) To solve this problem, we have to reintroduce a constant mass \( m_0 \) which we interpret as the restmass far from gravitational fields and we write

\[
m = m_0 u^5.
\]

Thus, \( u^5 \) can be interpreted as the variation of the unit mass. Further, we define \( u^i = p^i / m_0 \). We can now write down the following Lagrangian for free particle motion:

\[
L = \eta_{AB} u^A u^B = u_i u^i - (u^5)^2.
\]

Note that due to (17), this Lagrangian is actually zero (in analogy to (18), a 5d Dirac equation for massless particles). This Lagrangian now satisfies the properties we have requested at the end of section II.B, namely that in the absence of the gravitational field, the mass has to be found to be constant. Indeed, the solution of the equations of motion are just \( u^A = \text{const} \), and \( u^5 \) is set to one because of \( m = m_0 u^5 \).

Now, we proceed introducing a gravitational potential. If we suppose that the fifth component of the gauge potential \( A_5 \) is responsible for the gravitational interaction, we can write

\[
L = \frac{1}{2} u_A u^A - \varphi u^5,
\]

where \( A_5 \sim \varphi \) (the coupling constants have been absorbed into \( \varphi \)). Since \( L \) is of the form \( u_A u^A / 2 - A_A u^A \), in analogy to the EM case, \( u^A A^5 \) is still constant (and hence equal to zero). If we suppose that \( \varphi \) is a function of \( r \) only, we have, in addition, the following constants of motion (in spherical coordinates \( r, \psi, \vartheta \)):

\[
l = r^2 \dot{\psi}, \quad d = t, \quad \varepsilon = u^5 + \varphi,
\]

where we have used the Euler-Lagrange equation of the coordinate \( \theta \) to set \( \vartheta = \pi/2 \) (planar motion). The dot denotes differentiation with respect to the (5 dimensional) curve parameter. With these constants, the relation \( u_A u^A = 0 \) can be written in the form

\[
r^2 = d^2 - \varepsilon^2 + 2\varepsilon \varphi - \frac{l^2}{r^2} - \varphi^2.
\]

In a parameter independent form, introducing \( \rho = 1/r \), this can be written as

\[
\left( \frac{d\rho}{d\psi} \right)^2 = \frac{d^2 - \varepsilon^2}{l^2} + \frac{2\varphi \varepsilon}{l^2} - \frac{1}{r^2} - \frac{\varphi^2}{l^2}.
\]

It is tempting to set \( \varepsilon = 1 \), because in the limit \( r \to \infty \), \( u^5 \) should tend to 1. This, however, cannot be done, because we consider planetary motion, i.e. confined motion, and the limit \( r \to \infty \) has no physical meaning in this case. (In analogy, the constant \( a = t(1-2R/r) \) that arises in general relativity (Schwarzschild metric) cannot be set to one. This would lead only to geodesics with zero (newtonian) energy.) On the other hand, our constant \( d = t \) can of course be set to one, it is just a rescaling of the curve parameter.

Neglecting the higher order term \( -\varphi^2 \), the above equation can be identified with the Newtonian equation of motion if we set \( \varphi = -\varphi_N \), with \( \varphi_N = -GM/r \) the Newtonian potential. To compare the higher order corrections, we remind that the spherically symmetric post-newtonian extension of general relativity, to an order that describes all the observational effects, leads to the equation

\[
\left( \frac{d\rho}{d\psi} \right)^2 = \frac{a^2 - 1}{l^2} - \frac{(4a^2 - 2)\varphi_N}{l^2} - \frac{1}{r^2} + \frac{6a^2 \varphi_N^2}{l^2}.
\]

The main difference lies in the sign of the correction term \( \sim \varphi_N \), which is opposite to the one in (22). It is possible to force equation (22) into the form (23) if we extend our potential \( \varphi \) as \( \varphi = \alpha + \beta \varphi_N + \gamma \varphi_N^2 + \ldots \). The Newtonian
limit leads, as we have seen, to $\beta = -1$. The positive factor in front of the $\varphi^2_N$ term requires that $\gamma > 0$. This, however, means that $\varphi$, and hence also $\psi^5$, which is related to $\varphi$ through $\varepsilon = u^5 + \varphi$, will annulate at some values $r$ and can be represented (at least up to the considered order) with periodic functions ($u^5 \sim \sin(a\varphi_N) \sim \sin(b/r)\ldots$) which does not fit the interpretation of $u^5$ as the variation of the unit mass. Note that if the last sign in (22) were opposite, we could bring the equation into the form (23) with a relation of the form $\varphi - \varepsilon \sim u^5 \sim \exp(-\varphi_N)$ which would be similar to the mass definition in section (II.B), i.e. the interpretation of the mass as the opposite of the gravitational energy. This would however require a different signature of the 5d metric.

The conclusion of these considerations is that the fifth component of the potential $A_A$ cannot be, in this way, interpreted as gravitational potential. The question remains what it represents instead. This can only be answered after we have properly defined a conserved charge (if this is possible at all). The problem is actually the same as the one discussed in section (IV.E). It is not possible, in a straightforward way, to describe matter using four dimensional charges in the context of a five dimensional theory.

On the other hand, even in the induced matter approach, we were dealing with a Lagrangian of the form $L = u_A u^A$, and to this, some five dimensional Dirac equation should correspond, with or without mass term. So, even if this approach did not lead to fundamental problems on the level of general relativity, it may do so on the quantum theory level.

Actually, the considerations of the last section are very close to the induced matter approach, since we have interpreted the matter term $\bar{\psi}m\psi$ as fifth component of the Dirac equation, and achieved a five dimensional massless equation. Note, however, that Noether’s theorem applied on the Lagrangian (18) yields the following energy-stress tensor

$$T^{AB} = i\bar{\psi}\gamma^A \partial^B \psi,$$

which is traceless (because of the Dirac equation), but not zero.

To the second approach of section (IV.E), where five dimensional matter is explicitly introduced, would correspond a Dirac Lagrangian of the form

$$\mathcal{L} = \bar{\psi}(i\gamma^A \partial_A - m)\psi,$$

where $m$ is now a constant and the meaning of the fifth component remains open. It yields the same energy-stress tensor (which is not traceless in this case, because of the mass term in the Dirac equation).

V. CONCLUSIONS

The considerations of the last section, but also those of the Newtonian case, lead us to conclude that we cannot use the variation of the mass (or more generally, an additional dimension) to describe the gravitational interaction, i.e. to replace curved spacetime by a higher dimensional flat space. So, as we have pointed out earlier, the variation of the mass has to be considered as an additional degree of freedom. Further, we came to the conclusion, that in the absence of gravitational fields, the masses are found to be constant. In other words, the additional degree of freedom is directly related to the gravitational fields. These properties are all included in a five dimensional general relativity theory (see metric (7)), if we interpret the mass as the fifth moment component. The classical limit is found in the case were $g_{ik}$ in (7) is independent of $x^4$ (we can then remove the field $\varphi$ through the transformation $dx^4 \rightarrow \varphi dx^4$).

The unsolved problem is the question of the matter description. In the induced matter approach, the fact that $g_{ik}$ does not depend on $x^4$ means that we are dealing with 4d vacuum solutions (see (12)), which is unsatisfactory for the description of the field of a planet or of a point particle. More realistic descriptions should thus include an $x^4$ dependency or a five dimensional energy-stress tensor, which leads however to other difficulties.

A general problem of five dimensional theories is the fact, that the conserved current density does not lead to a conserved charge (or mass). A relation of the form $\partial_A J^A = 0$ leads (if $\partial_0 J^0 = \partial_x J^0 = 0$) to

$$\frac{d}{dt} \int J^0 dx^3 \frac{d}{dx^5} \int J^5 dx^3 = 0.$$

In order to get a charge that is conserved throughout time, we have to integrate not over 3 dimensional space, but over a four dimensional space, i.e. we have

$$\frac{d}{dt} \int J^0 dx^3 dx^5 = 0.$$

This does, however, not correspond to the notion of the total charge contained in three dimensional space. Especially, such a relation makes it rather difficult to suppose a direct relation between $m$ and $x^5$ (or $p^5$).

These problems suggest that before we introduce an additional dimension into a gravitational theory, we have to establish a well defined five dimensional elementary particle description.

Finally, we remark that all these problems are absent in scalar-tensor theories of the kind of Brans-Dicke theory. This is due to the fact that the only field that couples to matter in this approach is the metric, whereas the scalar field only contributes to the energy density. Thus, the scalar field reacts on matter only indirectly through its contribution to spacetime curvature, i.e. its influence on the metric. Hence, a consistent Dirac equation in the framework of this theory just consists in the usual generally covariant form of this equation (with the help of the spin-connection). This observation is an additional hint to the fact that we should regard, in this theory, the coupling constant as variable, not the rest mass.
[1] J. Ponce de Leon, arXiv:gr-qc/0111011 (2001)
[2] P. S. Wesson, G. Rel. Grav. 16, 2 (1984)
[3] R. Booth, arXiv:gr-qc/0203065 (2002)
[4] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, Pergamon Press, Oxford 1975
[5] F. B. Estabrook, Ap. J., 158 (1969)
[6] C. M. Will, Ap. J., 611 (1971)
[7] D. M. Eardley, Ap. J., 196 (1975)
[8] C. M. Will and H. W. Zaglauer, Ap. J., 346 (1989)
[9] V. D. Gladush, arXiv:gr-qc/0106079 (2001)
[10] J. Ponce de Leon, arXiv:gr-qc/0104008 (2001)
[11] J. M. Overduin and P. S. Wesson, arXiv:gr-qc/9805018 (1998)
[12] M. Kaku, Quantum Field Theory, Oxford University Press, New York (1993)