Error Exponent Bounds for the Bee-Identification Problem

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Abstract—Consider the problem of identifying a massive number of bees, uniquely labeled with barcodes, using noisy measurements. We formally introduce this “bee-identification problem”, define its error exponent, and derive efficiently computable upper and lower bounds for this exponent. We show that joint decoding of barcodes provides a significantly better exponent compared to separate decoding followed by permutation inference. For low rates, we prove that the lower bound on the bee-identification exponent obtained using typical random codes (TRC) is strictly better than the corresponding bound obtained using a random code ensemble (RCE). Further, as the rate approaches zero, we prove that the upper bound on the bee-identification exponent meets the lower bound obtained using TRC with joint barcode decoding.

I. INTRODUCTION

Consider a group of \( m \) different bees, in which each bee is tagged with a unique barcode for identification purposes in order to understand interaction patterns in honeybee social networks [1]. Assume that a camera is employed to picture the beehive to study the interactions among bees. The image output (see Fig. 1) can be considered as a noisy and unordered set of \( m \) barcodes. We formally pose the problem of bee-identification from a beehive image as an information-theoretic problem (Sec. I-B).

The bee-identification problem has applications in identification of warehouse products (labeled with unique RFID barcodes) using wide-area sensors. Other applications include package-distribution to recipients from a batch of deliveries with noisy address labels, and similar “bipartite matching” settings. It also has potential applications in identification of the mapping between signals and their meaning in “alien communication” with extraterrestrials, and also in learning communication protocols among robots, via the use of pilot signals going through the alphabet.

We consider the scenario where the barcode for each bee is represented as a binary vector of length \( n \), and the bee barcodes are collected in a codebook \( C \) comprising \( m \) rows and \( n \) columns, with each row corresponding to a bee barcode. As shown in Fig. 2 the channel first permutes the rows of \( C \) with a random permutation \( \pi \) to produce \( \pi C \). The entries of \( \pi C \) are then subjected to noise (corresponding to a binary symmetric channel (BSC) with crossover probability \( p \)), and the channel output is denoted \( \tilde{C}_\pi \). We assume that the decoder has knowledge of codebook \( C \), and its task is to recover the row-permutation \( \pi \) introduced by the channel. Note that the permutation \( \pi \) directly ascertains the identity of all the bees.

A. Related Work

In a related work motivated by an Internet of Things (IoT) setting, the identification of users in strongly asynchronous massive access channels was studied [2]. The identification of the underlying distributions of a set of observed sequences (where each sequence is generated i.i.d. by a distinct distribution) was analyzed in [3]. The bee-identification problem, on the other hand, allows codebooks where all barcode sequences are generated using the same underlying distribution.

In another related work [4], the fundamental limits of data storage via unordered DNA molecules was investigated. Here, a DNA molecule corresponds to an \( \ell \)-length sequence over an alphabet of size 4, and the information is written onto \( m \) DNA molecules stored in an unordered way. The storage capacity results in [4] were extended to noisy settings in [5] where the channel adds noise and randomly permutes the \( m \) DNA molecules used to store information. The capacity results are obtained under the scenario where the length, \( \ell \), of each DNA molecule grows with \( m \). Although the effective channel in [5] is closely related to the bee-identification channel in Fig. 2 we
note that the fundamental problem in [3] is to quantify the data storage capacity, while the main issue in the bee-identification problem is the identification of the row-permutation induced by the channel.

Data communication over permutation channels with impairments was analyzed in [6]. The authors of [6] presented bounds on the size of optimal codes over a finite input alphabet, when the channel randomly permutes the letters of the input sequence in addition to causing impairments such as insertions, deletions, and substitutions. The effective channel for the bee-identification problem (see Fig. 2) differs from the communication channel in [6] in two aspects: (i) The input to the channel in the bee-identification problem is the entire codebook, not just a codeword belonging to the codebook. (ii) The channel in Fig. 2 only permutes the rows of the codebook, but does not permute the letters within a row.

B. Bee-Identification Problem Formulation

The channel output is a row-permutated and noisy version of the codebook. If \( \pi \) denotes a given permutation of \( m \) letters, then the channel first permutes the \( m \) rows of codebook \( C \), based on \( \pi \), to produce \( \tilde{C}_\pi \) (see Fig. 2). Therefore, if \( j = \pi(i) \) and the \( i \)-th row of codebook \( C \) is denoted \( c_i = [c_{i,1}, c_{i,2} \cdots c_{i,n}] \), then the \( j \)-th row of \( \tilde{C}_\pi \) is equal to \( c_{\pi(j)} \). The channel then applies noise on the permuted codebook \( \tilde{C}_\pi \) to produce \( \tilde{C}_\pi \), where noise is modeled by a BSC with crossover probability \( p \), denoted BSC\((p)\), with \( 0 < p < 0.5 \). If \( j = \pi(i) \), and \( \tilde{c}_{\pi(i)} \) denotes the \( j \)-th row of \( \tilde{C}_\pi \), then

\[
\Pr\{\tilde{c}_{\pi(i)}|c_i, \pi\} = p^{d_i(1-p)^{n-d_i}}, \quad 1 \leq i \leq m, \\
\Pr\{\tilde{C}_\pi|C, \pi\} = \prod_{i=1}^m \Pr\{\tilde{c}_{\pi(i)}|c_i, \pi\} = \prod_{i=1}^m p^{d_i(1-p)^{n-d_i}},
\]

(1)

where \( d_i \triangleq d_H(\tilde{c}_{\pi(i)}, c_i) \) denotes the Hamming distance between vectors \( \tilde{c}_{\pi(i)} \) and \( c_i \). Let \( \mathcal{M} \triangleq \{1,2,\ldots,m\} \), and let the decoder correspond to a function \( \nu \) which takes \( \tilde{C}_\pi \) as an input and produces a map \( \nu : \mathcal{M} \rightarrow \mathcal{M} \) where \( \nu(k) \) corresponds to the index of the transmitted codeword which produced the received word \( \tilde{c}_k \), for \( 1 \leq k \leq m \). In effect, the bee-identification problem is that the decoder has to recover the row-permutation \( \pi \) introduced by the channel, by using the knowledge of codebook \( C \) and the channel output \( \tilde{C}_\pi \).

C. Bee-Identification Error Exponent

The indicator for the bee-identification error is defined as

\[
D(\phi(\tilde{C}_\pi), \pi^{-1}) = D(\nu, \pi^{-1}) \triangleq \begin{cases} 
1, & \text{if } \nu \neq \pi^{-1}, \\
0, & \text{if } \nu = \pi^{-1}.
\end{cases}
\]

For a given codebook \( C \) and decoding function \( \phi \), the expected bee-identification error probability over the BSC\((p)\) is

\[
D(C, p, \phi) \triangleq \mathbb{E}_\pi \left[ \mathbb{E} \left[ D(\phi(\tilde{C}_\pi), \pi^{-1}) \right] \right],
\]

(2)

where the inner expectation is over the distribution of \( \tilde{C}_\pi \) given \( C \) and \( \pi \) (see [1]), and the outer expectation is over a uniform distribution of \( \pi \) over all \( m \)-letter permutations. Note that \( D(C, p, \phi) \) can be equivalently expressed as

\[
D(C, p, \phi) = \Pr \{ \phi(\tilde{C}_\pi) \neq \pi^{-1} \} = \Pr \{ \nu \neq \pi^{-1} \}. \quad (3)
\]

For a given \( R > 0 \), let the number of barcodes \( m \) scale exponentially with blocklength \( n \) as \( m = 2^{nR} \). Now, for given values of \( n \) and \( R \), define the minimum expected bee-identification error probability as

\[
D_n(n, R, p) \triangleq \min_{C, \pi} D(C, p, \phi),
\]

(4)

where the minimum is over all codebooks \( C \) of size \( 2^{nR} \times n \), and all decoding functions \( \phi \).

Define, \( E_{D_n}(n, R, p) \), the exponent corresponding to the minimum expected bee-identification error probability, as

\[
E_{D_n}(n, R, p) = \lim_{n \rightarrow \infty} \frac{-1}{n} \log D_n(n, R, p).
\]

(5)

We introduce some notation that is used in the rest of the paper. We will denote \( f(n) \triangleq g(n) \) when \( \lim_{n \rightarrow \infty} \frac{1}{n} \log f(n)/g(n) = 0 \). Similarly, we write \( f(n) \preceq g(n) \) if \( \limsup_{n \rightarrow \infty} \frac{1}{n} \log f(n)/g(n) \leq 0 \).

D. Our Contributions

The “bee-identification problem” is introduced and the corresponding bee-identification exponent \( E_{D_n}(n, R, p) \) is analyzed in this paper. In particular, we provide the following explicit bounds on this exponent.

- A lower bound on \( E_{D_n}(n, R, p) \) using a random code ensemble (RCE) with independent barcode decoding (Sec. [II-A]) and joint barcode decoding (Sec. [II-B]).
- A lower bound on \( E_{D_n}(n, R, p) \) using typical random codes (TRC) with independent barcode decoding (Sec. [III-A]) and joint barcode decoding (Sec. [III-B]).
- An upper bound on \( E_{D_n}(n, R, p) \) which is applicable to all possible codebook designs (Sec. [IV]).

We show that joint decoding of barcodes provides a significantly better exponent compared to separate decoding followed by learning the permutation. For low rates, we prove that the lower bound obtained using TRC is strictly better than the corresponding bound obtained using RCE. Further, as the rate approaches zero, we prove that the upper bound meets the lower bound obtained using TRC with joint barcode decoding.

II. Random Code Ensemble

In this section, we present lower bounds on \( E_{D_n}(n, R, p) \) using an RCE [7]. Let \( \mathcal{E}(n, R) \) denote the set of all binary matrices with \( m = 2^{nR} \) rows and \( n \) columns. Assume that codebook \( C \) is uniformly distributed over \( \mathcal{E}(n, R) \). It is immediate from the definition of \( D_n(n, R, p) \) (3) that

\[
D_n(n, R, p) \preceq \frac{1}{\mathcal{E}(n, R)} \sum_{C \in \mathcal{E}(n, R)} D(C, p, \phi),
\]

(6)

where the expression on the right denotes the average performance using RCE. We proceed by quantifying this expression when the decoding function \( \phi \) corresponds to: (i) independent barcode decoding (Sec. [II-A]), and (ii) joint barcode decoding.
The critical rate given by the Gilbert-Varshamov (GV) distance \[7\] is defined as the value where 
\[
l\lim \inf_{n \to \infty} \frac{1}{n} \log \left( \frac{d_{GV}(C, \nu)}{p} \right) = 0
\]
and the definition of \(E_t(R, p)\), we get
\[
E_2(R, p) \geq |E_t(R, p) - R|^+.
\]

Now, using explicit numerical computation, it can be shown that \(R_0(p) \leq 2R_c(p)\). The proof is complete by combining \[13\], \[14\], and noting that \(|E_t(R, p) - R|^+ = 0\) when \(R \geq R_c(p)\) because \(E_t(R, p)\) is a decreasing function of \(R\). Q.E.D.

The lower bound on \(E_2(R, p)\) given by \(11\) was obtained by applying a na"ïve decoding strategy where each barcode was decoded independently. In the next subsection, we analyze the bee-identification exponent using joint barcode decoding.

**B. Joint Decoding of Barcodes**

Let \(S_m\) denote the set of permutations of \(\{1, \ldots, m\}\). For joint maximum likelihood (ML) decoding of barcodes, the decoding function \(\nu\) takes the noisy row-permutated codebook \(\tilde{C}_\nu\) as input, and produces permutation \(\nu = \rho^{-1}\) as output, where \(\rho = \arg \min_{\sigma \in S_m} d_H(\tilde{C}_\nu, C_{\sigma})\), and \(d_H(\tilde{C}_\nu, C_{\sigma}) \triangleq |(i, j) : \tilde{C}_\nu(i, j) \neq C_{\sigma}(i, j), 1 \leq i \leq m, 1 \leq j \leq n|\). We aim to provide bounds on \(P(1 - \nu = 1) = P(\rho \neq \pi)\).

For any two permutations \(\sigma_1, \sigma_2 \in S_m\), the sets of distances \(|d_H(\tilde{C}_{\sigma_1}, C_{\sigma})|_{\sigma \in S_m}\) and \(|d_H(\tilde{C}_{\sigma_2}, C_{\sigma})|_{\sigma \in S_m}\) are equal. Therefore, the performance of the joint ML decoder is independent of the channel permutation \(\pi\), and we assume, without loss of generality, that the permutation induced by the channel is the identity permutation, denoted \(\pi_0\).

For a given codebook \(C\) at the transmitter, let \(\tilde{C}_{\pi_0}\) denote the received noisy codebook at the output of the effective channel, and for \(\sigma \in S_m\) with \(\sigma \neq \pi_0\), we define
\[
\Pr(\pi_0 \to \sigma) = \Pr \left\{ d_H(\tilde{C}_{\pi_0}, C_{\sigma}) \leq d_H(\tilde{C}_{\pi_0}, C_{\pi_0}) \right\},
\]
where the event \(\{\pi_0 \to \sigma\}\) is said to occur if \(d_H(\tilde{C}_{\pi_0}, C_{\sigma}) \leq d_H(\tilde{C}_{\pi_0}, C_{\pi_0})\). From \[5\], we have
\[
D(C, p, \phi) = \Pr \left\{ \bigcup_{\sigma \in S_m, \sigma \neq \pi_0} \{\pi_0 \to \sigma\} \right\} \leq \sum_{\sigma \in S_m, \sigma \neq \pi_0} \Pr(\pi_0 \to \sigma),
\]
where \[15\] follows from the union bound. Now define
\[
P_{BCE,\sigma} \triangleq \frac{1}{|C(\pi_0, R)|} \sum_{\sigma \in S_m} \Pr(\pi_0 \to \sigma),
\]
which denotes the probability of the event \(\{\pi_0 \to \sigma\}\), averaged over the ensemble of random binary codebooks. Using \[6\], \[15\], and \[16\], we get
\[
D(n, R, p) \leq \sum_{\sigma \in S_m, \sigma \neq \pi_0} P_{BCE,\sigma}.
\]

Now consider two codewords \(c_1, c_2\) at distance \(d\) from each other. Given that \(c_1\) is transmitted over BSC\((p)\), the probability that the Hamming distance of the received word from \(c_2\) is not more than its distance from \(c_1\) is \[7\]
\[
\Pr(\{c_1 \to c_2\}) \leq 2^{-d \cdot \alpha}\,
\]
where
\[ \alpha_p \triangleq -\log \sqrt{4p(1-p)}. \] (18)

Therefore, for a given codebook \( C = C_{\pi_0} \) and permutation \( \sigma \in S_m \) with \( \sigma \neq \pi_0 \), if \( d_\sigma \triangleq d_H(C_{\pi_0}, C_\sigma) \), then
\[ \Pr \{ \pi_0 \rightarrow \sigma \} \geq 2^{-d_\sigma \alpha_p}. \] (19)

In the following, we quantify \( P_{\text{RCE}, \sigma} \) for different \( \sigma \in S_m \), via (16) and (19).

1) \( \sigma \) is a transposition: We first consider the case where \( \sigma \) is a transposition, i.e., a permutation that interchanges only two indices. For indices \( i, j \), with \( 1 \leq i < j \leq m \), the Hamming distance between codewords \( c_i \) and \( c_j \) in a random codebook satisfies \[ \Pr \{ d_H(c_i, c_j) = d \} \leq 2^{-n(1-H(d/n))}. \] (20)

When \( \sigma \) is a transposition, then \( d_H(C_{\pi_0}, C_{(i \ j)}) = 2d \). Thus, it follows from (20) that \[ \Pr \{ d_H(C_{\pi_0}, C_{(i \ j)}) = 2d \} \leq 2^{-n(1-H(d/n))}. \] Further, when \( d_H(C_{\pi_0}, C_{(i \ j)}) = 2d \), we have \[ \Pr \{ \pi_0 \rightarrow (i \ j) \} \geq 2^{-2 \alpha_p}. \]

Therefore, the probability \( P_{\text{RCE}, (i \ j)} \) can be characterized using (16), (19), and (20) as
\[ P_{\text{RCE}, (i \ j)} \leq \sum_{d=0}^{n} 2^{-n(1-H(d/n))} + 2^{-2 \alpha_p}. \] (21)

When \( n \to \infty \), the sum in (21) is dominated by the minimum of \( n + 1 \) exponents. If \( \delta = d/n \) is treated as a continuous variable, then the exponent \( E_2(\delta) \triangleq 1 - H(\delta) + 2 \delta \alpha_p \) is a convex function with a unique minimum at \( \delta = \delta_p \) where
\[ \delta_p \triangleq \frac{4p(1-p)}{1 + 4p(1-p)}. \] (22)

If we define
\[ R_1(p) \triangleq 1 - \log(1 + 4p(1-p)), \] (23)

then it can be verified that \( E_2(\delta_p) = R_1(p) \), and it follows from (21) that when \( \sigma \) is a transposition, we have
\[ P_{\text{RCE}, \sigma} \leq 2^{-n(1-\log(1+4p(1-p)))} = 2^{-n R_1(p)}. \] (24)

2) \( \sigma \) is a product (composition) of disjoint transpositions: We now consider the case where \( \sigma = \sigma_1 \sigma_2 \), where \( \sigma_1 \) and \( \sigma_2 \) are disjoint transpositions with \( \sigma_1 = (i \ j) \) and \( \sigma_2 = (i \ j) \). As the code words in a random codebook are independent, then using (20), we have \[ \Pr \{ \{ d_H(c_i, c_j) = d_1 \} \cap \{ d_H(c_i, c_j) = d_2 \} \} \leq \prod_{i=1}^{2} 2^{-n(1-H(d_i/n))}. \] Further, if \( d_H(c_i, c_j) = d_1 \) and \( d_H(c_i, c_j) = d_2 \), then \( d_H(C_{\pi_0}, C_\sigma) = 2(d_1 + d_2) \), and \[ \Pr \{ \pi_0 \rightarrow \sigma \} \geq 2^{-2(d_1 + d_2) \alpha_p}. \]

Therefore, if \( \sigma \) is a product of two disjoint transpositions, then
\[ P_{\text{RCE}, \sigma} \leq \sum_{d_1, d_2} 2^{-n(H(d_1/n) + 2(d_1/n) \alpha_p)} \]
\[ = \frac{2}{\alpha_p} \left( \sum_{d_1=0}^{n} 2^{-n(1-H(d_1/n) + 2(d_1/n) \alpha_p)} \right) \]
\[ \geq 2^{-2n(1-\log(1+4p(1-p)))} = 2^{-n R_1(p)}. \]

In general, when \( \sigma \) is a product of \( s \) disjoint transpositions, the above argument can be readily extended to show that
\[ P_{\text{RCE}, \sigma} \leq 2^{-s n R_1(p)}. \] (25)

Now, define
\[ \lambda_p \triangleq \min \left\{ \frac{2R_0(p)}{3}, \frac{R_1(p)}{2} \right\}, \]
where \( R_0(p) \) and \( R_1(p) \) are defined in (12) and (23), respectively. As \( 2 \lambda_p \leq R_1(p) \), it follows from (25) that
\[ P_{\text{RCE}, \sigma} \leq 2^{-n^2 \lambda_p}. \] (26)

We remark that when \( \sigma \) is just a transposition, then from (24) we have \( P_{\text{RCE}, \sigma} \leq 2^{-n^2 R_1(p)} \leq 2^{-n^2 \lambda_p} \), which is only a special case of (26) with \( s = 1 \).

3) \( \sigma \) is a \( k \)-cycle with \( k > 2 \): Let \( \sigma \in S_m \) be a \( k \)-cycle \( (i_1 i_2 \cdots i_k) \) where \( i_{l+1} = \sigma(i_l) \) for \( 1 \leq l \leq k-1 \), and \( i_k = \sigma(i_1) \). We will apply the following proposition towards characterizing \( P_{\text{RCE}, \sigma} \).

**Proposition 1.** Let \( \mathcal{F}_n \) denote the space of all \( n \)-length binary vectors. Let \( c_1, c_2, \ldots, c_k \) be \( k > 2 \) i.i.d. random vectors, uniformly distributed over \( \mathcal{F}_n \), and let \( d_1, d_2, \ldots, d_{k-1} \) be given non-negative integers. Then the following holds
\[ \Pr \left\{ \bigcap_{i=1}^{k-1} \{ d_H(c_i, c_{i+1}) = d_i \} \right\} \leq \prod_{i=1}^{k-1} 2^{-n(1-H(d_i/n))}. \] (27)

**Proof:** See Appendix A.

For a given codebook \( C \), if \( d_H(c_i, c_{i+1}) = d_i \) for \( 1 \leq l \leq k-1 \), and \( d_H(c_k, c_1) = d_k \), then \( d_H(C_{\pi_0}, C_\sigma) = \sum_{i=1}^{k} d_i \), and we have
\[ \Pr \{ \pi_0 \rightarrow \sigma \} \leq 2^{-\left( \sum_{i=1}^{k} d_i \right) \alpha_p}. \] (28)

Further, if codebook \( C \) is uniformly distributed over \( \mathcal{F}(n, R) \),
\[ \Pr \left\{ \bigcap_{i=1}^{k-1} \{ d_H(c_i, c_{i+1}) = d_i \} \right\} \leq 2^{-n\left( \sum_{i=1}^{k-1} (1-H(d_i/n)) \right)} \]
\[ \leq 2^{-n\left( \sum_{i=1}^{k-1} \frac{1}{k-1} \right)} \leq 2^{-n \left( \sum_{i=1}^{k-1} d_i \right) \alpha_p}. \] (29)

Combining (28) and (29),
\[ P_{\text{RCE}, \sigma} \leq \sum_{0 \leq d_l \leq n, \ 1 \leq l \leq k} 2^{-n\left( \sum_{i=1}^{k} (1-H(d_i/n)) + \left( \sum_{i=1}^{k-1} d_i \right) \alpha_p \right)} \]
\[ = \sum_{d_k=0}^{n} 2^{-d_k \alpha_p} \left( \prod_{l=1}^{k-1} \sum_{d_l=0}^{n} 2^{-n(1-H(d_l/n) + (d_l/n) \alpha_p)} \right) \]
\[ = \sum_{d_k=0}^{n} 2^{-n(1-H(d_k/n) + (d_k/n) \alpha_p)} \]
\[ = \sum_{d_k=0}^{n} 2^{-n(1-H(d_k/n) + (d_k/n) \alpha_p)} \] (30)

When \( n \to \infty \), the sum \( \sum_{d_l=0}^{n} 2^{-n(1-H(d_l/n) + (d_l/n) \alpha_p)} \) is dominated by the minimum of \( n + 1 \) exponents. If \( \delta = d_l/n \) is treated as a continuous variable, then the exponent \( E_1(\delta) \triangleq 1 - H(\delta) + \delta \alpha_p \) is a convex function with a unique minimum at \( \delta = \delta_p \), where
\[ \delta_p \triangleq \frac{\sqrt{4p(1-p)}}{1 + \sqrt{4p(1-p)}}. \] (31)
We have
\[ E_1(\hat{p}) = 1 - \log(1 + \sqrt{4p(1-p)}) = R_0(p), \]
and it follows from (30) that
\[ P_{\text{RCE},\sigma} \leq 2^{-n(k-1)\left(1-\log(1+\sqrt{4p(1-p)})\right)} = 2^{-n(k-1)R_0(p)}. \tag{32} \]
As \( 2k/3 \leq k-1 \) for \( k > 2 \), we have \( k\lambda_{\text{R}} \leq 2kR_0(p)/3 \leq (k-1)R_0(p) \), and it follows from (32) that
\[ P_{\text{RCE},\sigma} \leq 2^{-nk\lambda_{\text{R}}}. \tag{33} \]
The above equation has been derived for the case where \( \sigma \) is a \( k \)-cycle with \( k > 2 \). However, a transposition is just a \( k \)-cycle with \( k = 2 \), and from the remark following (26), it follows that (33) holds even for \( k = 2 \).

4) General \( \sigma \in S_m \) with \( \sigma \neq \pi_0 \): It is well known that any permutation \( \sigma \neq \pi_0 \) can be written as a product (composition) of \( t \) disjoint cycles, for \( t \geq 1 \) [9]. Consider a given \( \sigma \) which is a product of \( t \) disjoint cycles of length \( k_1, \ldots, k_t \), respectively, where \( k_i \geq 1 \) for \( 1 \leq i \leq t \). Then, we can extend the result in (33) to obtain
\[ P_{\text{RCE},\sigma} \leq 2^{-n(\sum_{i=1}^{t} k_i)\lambda_{\text{R}}}. \tag{34} \]

5) Putting it all together: For \( 1 \leq j \leq m \), if we define
\[ \Sigma_j \triangleq \{ \sigma \in S_m : |\{i : \sigma(i) \neq i, 1 \leq i \leq m\}| = j\}, \tag{35} \]
\[ P_{\text{RCE},\Sigma_j} \triangleq \sum_{\sigma \in \Sigma_j} P_{\text{RCE},\sigma}, \tag{36} \]
then (57) can be equivalently expressed as
\[ D(n, R, p) \leq \sum_{j=2}^{m} P_{\text{RCE},\Sigma_j}. \tag{37} \]

Note that the set \( \Sigma_1 \) is empty, as the Hamming distance between two distinct permutations is at least two. The set \( \Sigma_2 \) consists of all transpositions and \( |\Sigma_2| = \binom{m}{2} = 2^{m}(2R) \). For all \( \sigma \in \Sigma_2 \), the value of \( P_{\text{RCE},\sigma} \) is given by (24), and combining this with (36), we get
\[ P_{\text{RCE},\Sigma_2} \leq 2^{-n(R_1(p)-2R)}, \quad 0 \leq R \leq R_1(p)/2. \tag{38} \]

For a given \( j > 2 \), if \( \sigma \in \Sigma_j \), then from (34) it follows that \( P_{\text{RCE},\sigma} \leq 2^{-nj\lambda_{\text{R}}} \). For \( j \geq 2 \), the size of the set \( \Sigma_j \) satisfies \( |\Sigma_j| = \prod_{i=0}^{j-1} (m-i) \leq 2^{njR} \). If we define \( \beta \triangleq 2^{-n(\lambda_{\text{R}}-R)} \), then for \( j \geq 3 \) and \( R < \lambda_{\text{R}} \), we have \( P_{\text{RCE},\Sigma_j} \leq \beta^j \), and
\[ \sum_{j=3}^{m} P_{\text{RCE},\Sigma_j} \leq \sum_{j=3}^{\infty} \beta^j = \frac{\beta^3}{1-\beta} \leq \beta^3 = 2^{-3n(\lambda_{\text{R}}-R)}. \tag{39} \]

Combining (37), (38), and (39), for \( R < \lambda_{\text{R}} \),
\[ D(n, R, p) \leq 2^{-n(R_1(p)-2R)} + 2^{-n(3\lambda_{R}-3R)}. \tag{40} \]

The next theorem presents an explicit lower bound for \( E_D(R, p) \) when the decoder jointly decodes all the barcodes using a maximum likelihood approach.

**Theorem 2.** We have
\[ E_D(R, p) \geq |\eta(p)|^+, \tag{41} \]
where \( \eta(p) \triangleq \min \{ R_1(p) - 2R, 2R_0(p) - 3R \} \).

**Proof:** If \( R < \lambda_{\text{R}} \), then \( R_1(p) \geq 2\lambda_{\text{p}} > 2R \). Therefore, from (40) it follows that if \( R < \lambda_{\text{R}} \), then \( E_D(R, p) \) is lower bounded by \( \min \{ R_1(p) - 2R, 3\lambda_{\text{R}} - 3R \} = \eta(p) \). Further, note that \( \eta(p) > 0 \) if and only if \( R < \lambda_{\text{R}} \).

The following proposition shows that the lower bound (obtained using joint decoding of barcodes) is strictly better than the bound given by (11) (obtained with independent decoding of barcodes) in the interval where it is positive.

**Proposition 2.** When \( R_0(p) > 2R \) and \( 0 < p < 0.5 \), then we have the strict inequality
\[ \eta(p) > R_0(p) - 2R. \]

**Proof:** When \( 0 < p < 0.5 \), we have \( 0 < 4p(1-p) < \sqrt{4p(1-p)} < 1 \), and hence \( R_1(p) > R_0(p) \). If \( R_0(p) > 2R \), then \( 2R_0(p) - 3R = 2(R_0(p) - 2R) + R > R_0(p) - 2R \). The proof is complete by combining these observations with the definition of \( \eta(p) \).

Note that \( \eta(p) = 0 \) for \( R \geq 0.5 \), because in this case \( \eta(p) \leq R_1(p) - 2R \leq R_1(p) - 1 \leq 0 \). In the following section, we present improved lower bounds on \( E_D(R, p) \) by analyzing typical random codebooks.

**III. TYPICAL RANDOM CODE**

TRCs are known, in general, to provide higher error exponents than RCE over a BSC [7], [10]. Roughly speaking, TRCs are characterized by the property that their relative minimum distance is at least \( \delta_{\text{GV}}(2R) \). Formally, for indices \( 1 \leq i < j \leq m = 2^nR \), and \( \epsilon_n = o(1) \), the Hamming distance between codewords \( c_i \) and \( c_j \) in a TRC satisfies [7]
\[ \Pr \{ d_H(c_i, c_j) = d \} \begin{cases} \geq 2^{-n(1-H(\delta))}, & |\frac{1}{2} - \delta | \leq \frac{1}{2} - \delta_+, \\ \geq \frac{1}{2} - \delta_-, & \frac{1}{2} - \delta_+ \leq \frac{1}{2} - \delta_-, \end{cases} \tag{42} \]
where \( \delta = d/n \), \( \delta_+ = \delta_{\text{GV}}(2R) + \epsilon_n \), and \( \delta_- = \delta_{\text{GV}}(2R) - \epsilon_n \).

Let \( \mathcal{C}_{\text{TRC}}(n, R) \) denote the set of all codebooks of size \( 2^nR \times n \), with the property that the Hamming distance between a pair of codewords \( c_i \) and \( c_j \) satisfies the relation \( n\delta_- < d_H(c_i, c_j) < n(1-\delta_-) \) for all \( i \neq j \). Note that if codebook \( C \) is uniformly distributed over \( \mathcal{C}_{\text{TRC}}(n, R) \), then the Hamming distance between a pair of distinct codewords satisfies (42). It is immediate from (4) that
\[ D(n, R, p) \leq \frac{1}{|\mathcal{C}_{\text{TRC}}(n, R)|} \sum_{C \in \mathcal{C}_{\text{TRC}}(n, R)} D(C, \rho, \phi), \tag{43} \]
where the expression on the right denotes the average performance using TRCs.

In this section we provide lower bounds on the barcode-identification exponent \( E_D(R, p) \) using TRCs. The case where each barcode is decoded independently is analyzed in Sec. III-A, while joint barcode decoding is analyzed in Sec. III-B. It is shown that these lower bounds on \( E_D(R, p) \) using TRCs outperform the corresponding bounds for RCEs when the rate is smaller than a certain threshold.
A. Independent Decoding of Barcodes

With independent barcode decoding, the decoder picks \( \hat{c}_j \), the \( j \)-th row of \( C_\pi \), and then assigns \( \nu(j) = \arg \min_k d_H(\hat{c}_j, c_k) \), for \( 1 \leq j \leq m \). From the union bound, we have \( D(C, p, \phi) \leq \sum_{j=1}^{m} \Pr\{\nu(j) \neq \pi^{-1}(j)\} \), and using \( (43) \) we get

\[
D(n, R, p) \leq \sum_{j=1}^{m} \left( \sum_{C \in \mathcal{E}_{\pi}(n, R)} \Pr\{\nu(j) \neq \pi^{-1}(j)\} \right).
\]

(44)

Let \( P_{\pi}(n, R, p) \triangleq \sum_{C \in \mathcal{E}_{\pi}(n, R)} \Pr\{\nu(j) \neq \pi^{-1}(j)\} \). Note that \( P_{\pi}(n, R, p) \) is independent of the index \( j \) due to the symmetry resulting from averaging over codebooks uniformly distributed over \( \mathcal{E}(n, R) \). For \( i = \pi^{-1}(j) \), the expression for \( P_{\pi}(n, R, p) \) corresponds to the probability of error when the \( i \)-th codeword is transmitted. From \( (43) \), we get

\[
D(n, R, p) \leq mP_{\pi}(n, R, p).
\]

(45)

The following theorem uses \( (45) \) to present an explicit lower bound on \( E_D(R, p) \) when the rate is smaller than a certain threshold.

**Theorem 3.** We have

\[
E_D(R, p) \geq \alpha_p \delta_{GV}(2R), \quad 0 \leq R \leq R_{TRC}(p),
\]

(46)

where \( \alpha_p \) is defined in \( (18) \), and

\[
R_{TRC}(p) \triangleq 0.5 \left( 1 - H\left( \frac{\sqrt{4p(1-p)}}{1 + \sqrt{4p(1-p)}} \right) \right).
\]

(47)

**Proof:** It is known that for \( 0 \leq R \leq R_{TRC}(p) \), the error exponent using a TRC over BSC(p), defined as \( E_{TRC}(R, p) \triangleq \lim_{n \to \infty} (1/n) \log \{ 1/P_{\pi}(n, R, p) \} \), is given by \( (7) \)

\[
E_{TRC}(R, p) = \alpha_p \delta_{GV}(2R) + R.
\]

(48)

Using the fact that \( m = 2n^R \), and combining \( (5) \), \( (45) \), with the definition of \( E_{TRC}(R, p) \), we get

\[
E_D(R, p) \geq |E_{TRC}(R, p) - R|^+.
\]

(49)

The proof is completed by applying \( (43) \) in \( (49) \). \( \square \)

It is well known that \( E_{TRC}(R, p) > E_1(R, p) \) for \( 0 \leq R < R_{TRC}(p) \). This implies that the lower bound on \( E_D(R, p) \) for TRC given by \( (46) \) is strictly better than the corresponding bound for RCE given by \( (11) \) when \( 0 \leq R < R_{TRC}(p) \). The next subsection provides a more refined lower bound on \( E_D(R, p) \) by analyzing joint decoding of barcodes using TRCs.

B. Joint Decoding of Barcodes

With joint barcode decoding, the decoder takes the noisy row-permuted codebook \( C_\pi \) as input, and produces the permutation \( \nu = \rho^{-1} \) as output, where \( \rho = \arg \min_{\pi \in \mathcal{S}_m} d_H(C_\pi, C_\phi) \). As in Sec. II-B, we assume, without loss of generality, that the permutation induced by the channel is the identity permutation \( \pi_0 \), and for a given codebook \( C \), we have \( D(C, p, \phi) \leq \sum_{\sigma \in \mathcal{S}_m, \sigma \neq \pi_0} \Pr\{\pi_0 \to \sigma\} \). We now define

\[
P_{\pi_0}(n, R, p) \triangleq \mathbb{E}[\Pr\{\pi_0 \to \sigma\}],
\]

(50)

where the expectation is over a uniform distribution of codebook over \( \mathcal{E}(n, R) \). Then we have

\[
D(n, R, p) \leq \mathbb{E}[D(C, p, \phi)] \\
\leq \sum_{\sigma \in \mathcal{S}_m, \sigma \neq \pi_0} P_{\pi_0}(n, R, p).
\]

(51)

In the following, we quantify \( P_{\pi_0}(n, R, p) \) for different \( \sigma \in \mathcal{S}_m \), in order to bound \( D(n, R, p) \) via \( (51) \).

1) \( \sigma \) is a transposition: When \( \sigma = (i \ j) \) is the permutation that only transposes indices \( i \) and \( j \), and \( d_H(c_i, c_j) = d \), then \( d_H(C_{\pi_0}, C_{(i \ j)}) = 2d \), and we have

\[
\Pr\{\pi_0 \to (i \ j)\} = 2^{-2d_{\alpha}}.
\]

(52)

When \( C \) is uniformly distributed \( \mathcal{E}(n, R) \), and \( n(\delta_{GV}(2R) - \epsilon_n) \leq d < n(1 - \delta_{GV}(2R) + \epsilon_n) \), then we have

\[
\Pr\{d_H(C_{\pi_0}, C_{(i \ j)}) = 2d\} = \Pr\{d_H(c_i, c_j) = d\} \leq 2^{-n(1 - H(d/n))},
\]

(53)

where \( (53) \) follows from \( (42) \). As \( \epsilon_n = o(1) \), by combining \( (50) \), \( (52) \), and \( (53) \), we get

\[
P_{\pi_0}(n, R, p) \triangleq \sum_{d=n \delta_{GV}(2R)} 2^{-n(1 - H(d/n)) + 2d/n \alpha_p}.
\]

(54)

When \( n \to \infty \), the sum in \( (54) \) is dominated by the minimum of \( 1 + n(1 - 2\delta_{GV}(2R)) \) exponents. If \( \delta = d/n \) is treated as a continuous variable, then the exponent \( E_2(\delta) = 1 - H(\delta) + 2\delta \alpha_p \) is a convex function of \( \delta \) with a unique minimum at \( \delta_p \) defined in \( (22) \). If we define

\[
\hat{\delta}_p \triangleq 0.5(1 - H(\delta_p)),
\]

(55)

then for \( 0 \leq R \leq \hat{\delta}_p \), we have

\[
\delta_{GV}(2R) \geq 2\delta_{GV}(2\hat{\delta}_p) = \delta_p.
\]

The exponent \( E_2(\delta) \) increases monotonically in \( \delta \) if \( \delta \geq \delta_p \), and therefore when \( 0 \leq R \leq \hat{\delta}_p \), the exponent in \( (54) \) is minimized when \( d/n = \delta_{GV}(2R) \). As \( E_2(\delta_{GV}(2R)) = 2\delta_{GV}(2R) + 2R \), we have

\[
P_{\pi_0}(n, R, p) \triangleq 2^{-n(2\alpha_p \delta_{GV}(2R) + 2R)}, \quad 0 \leq R \leq \hat{\delta}_p.
\]

(56)

2) \( \sigma \) is a k-cycle: We now consider the case where \( \sigma \) is a k-cycle with \( k \geq 3 \). We will apply the following proposition towards characterizing \( P_{\pi_0}(n, R, p) \).

**Proposition 3.** Let codebook \( C \) be uniformly distributed over \( \mathcal{E}(n, R) \), and let \( c_{1}, c_{2}, \ldots, c_{k} \) be distinct rows in \( C \). For \( 1 \leq l \leq k-1 \), let \( d_l \) satisfy \( n\delta_{GV}(2R) \leq d_l \leq n(1 - \delta_{GV}(2R)) \). Then the following holds

\[
\Pr\{\pi_0 \to \sigma^{(l)} \} \leq \prod_{l=1}^{k-1} 2^{-n(1 - H(d_l/n))}.
\]

(57)

**Proof:** See Appendix B. \( \square \)
Define $d_0 \triangleq n\delta_{GV}(2R)$ and let $d_0 \leq d_l \leq n - d_0$, for $1 \leq l \leq k$. As $\epsilon_\eta$ in Prop. 3 can be made arbitrarily small, we have
\[
\Pr \left\{ \left( \bigcap_{l=1}^{k-1} \{ d_{H}(c_{i_l}, c_{i_{l+1}}) = d_l \} \right) \right\} \leq \Pr \left\{ \bigcap_{l=1}^{k-1} \{ d_{H}(c_{i_l}, c_{i_{l+1}}) = d_l \} \right\},
\]
where $(a)$ follows from (57). Further, for a given codebook $C$, given that $\sigma = (i_1 i_2 \cdots i_k)$ and $d_{H}(c_{i_l}, c_{i_{l+1}}) = d_l$ for $1 \leq l \leq k - 1$, and $d_{H}(c_{i_k}, c_{i_1}) = d_k$, we have $d_{H}(C_{\sigma_0}, C_{\sigma}) = \sum_{l=1}^{k} d_l$, and therefore
\[
\Pr \{ \pi_0 \rightarrow \sigma \} \leq 2^{-n\delta_{GV}(2R)}.
\]
Combining (58) and (59), we have
\[
P_{TRC, \sigma} \leq \sum_{d_0 \leq d_l \leq n - d_0} 2^{-n(\sum_{l=1}^{k} d_l (d_l/n) \alpha_p + (\sum_{l=1}^{k-1} (1 - H(d_l/n)))},
\]
where, for $1 \leq l \leq k - 1$, we have
\[
\zeta_l \triangleq \sum_{d_0 \leq d_l \leq n - d_0} 2^{-n(1 - H(d_l/n) + (d_l/n) \alpha_p)}, \quad \text{and} \quad \eta_k \triangleq \sum_{d_0 \leq d_l \leq n - d_0} 2^{-d_k \alpha_p} = 2^{-d_0 \alpha_p} = 2^{-n\delta_{GV}(2R) \alpha_p}. \tag{61}
\]
The function $E_1(\delta) = 1 - H(\delta) + \delta \alpha_p$ is a convex function of $\delta$, and has a unique minimum that occurs at $\delta_p$ defined in (51). From (49) we observe that $R_{TRC}(p) = 0.5(1 - H(\delta_p))$. Thus, if $R \leq R_{TRC}(p)$ then we have $\delta_{GV}(2R) \geq \delta_p$. Further, $E_1(\delta)$ is an increasing function of $\delta$ for $\delta \geq \delta_p$, and so if $R \leq R_{TRC}(p)$, the exponent in (61) is minimized when $d_l/n = \delta_{GV}(2R) = d_0/n$. As $E_1(\delta_{GV}(2R)) = 2R + \alpha_p \delta_{GV}(2R)$, we have
\[
\zeta_l \leq 2^{-n(2R + \alpha_p \delta_{GV}(2R))}, \quad 0 \leq R \leq R_{TRC}(p). \tag{62}
\]
Combining (60), (61), and (63), we get
\[
P_{TRC, \sigma} \leq 2^{-n(2(k - 1)R + k_0 \alpha_p \delta_{GV}(2R))}, \quad 0 \leq R \leq R_{TRC}(p). \tag{64}
\]
where $\sigma$ is a $k$-cycle with $k > 2$. As $k < 2(k - 1)$ for $k > 2$, it follows from (64) that
\[
P_{TRC, \sigma} \leq 2^{-n(kR + \alpha_p \delta_{GV}(2R))}, \quad 0 \leq R \leq R_{TRC}(p). \tag{65}
\]
Recall that $\delta_p$ and $\tilde{R}_p$ are given by (22) and (55), respectively. As $x/(1 + x)$ is an increasing function of $x$, and $0 < p < 0.5$, it follows that $\delta_p < \delta_p < 0.5$, which implies that $R_{TRC}(p) < \tilde{R}_p$. Now a transposition is simply a $k$-cycle with $k = 2$, and so by comparing (56) with (65), we note that the relation given by (65) holds even when $k = 2$. 3) $\sigma$ is a product (composition) of two disjoint cycles: We now consider the case where $\sigma = \sigma_1 \sigma_2$, where $\sigma_1$ and $\sigma_2$ are disjoint cycles of length $k_1$ and $k_2$, respectively. Let $\sigma_1 = (i_1 i_2 \cdots i_{k_1})$ and $\sigma_2 = (i_{k_1+1} i_{k_1+2} \cdots i_{k_1+k_2})$. If $d_0 \leq d_l \leq n - d_0$ for $1 \leq l \leq k + k_2$, then from (57), the probability $\Pr \{ \bigcap_{l=1}^{k+k_2} \{ d_{H}(c_{i_l}, c_{i_{l+1}}) = d_l \} \}$
\[
\leq \eta_k \prod_{l=1}^{k+k_2} \zeta_l \leq 2^{-n(\sum_{l=1}^{k+k_2} (1 - H(d_l/n)))}, \tag{69}
\]
Further, for a given codebook $C$, with $d_{H}(c_{i_l}, c_{i_{l+1}}) = d_l$ for $1 \leq l \leq k$, the exponent in (69) is minimized when $d_l/n = \delta_{GV}(2R) = d_0/n$, as $E_1(\delta_{GV}(2R)) = 2R + \alpha_p \delta_{GV}(2R)$, we have $d_{H}(C_{\sigma_0}, C_{\sigma}) = \sum_{l=1}^{k+k_2} d_l$, and therefore
\[
\Pr \{ \pi_0 \rightarrow \sigma \} \leq 2^{-n(\sum_{l=1}^{k+k_2} (1 - H(d_l/n)))} \times 2^{-n(\sum_{l=1}^{k+k_2} (1 - H(d_l/n)))}. \tag{70}
\]
Combining (60) and (67), we can upper bound $P_{TRC, \sigma}$ by
\[
\sum_{d_0 \leq d_l \leq n - d_0} 2^{-n(\sum_{l=1}^{k+k_2} (1 - H(d_l/n)))} \times 2^{-n(\sum_{l=1}^{k+k_2} (1 - H(d_l/n)))}. \tag{68}
\]
The above expression can be equivalently written as
\[
(\eta_k)^2 (\zeta_l) (\tilde{R}_p^{k_1+k_2 - 2}), \tag{69}
\]
where $\zeta_l$ and $\eta_k$ are given by (61) and (62), respectively. Now, applying (62), (65) in (69) for $0 \leq R \leq R_{TRC}(p)$, we get
\[
P_{TRC, \sigma} \leq 2^{-n(2(k_1+k_2-2)R + (k_1+k_2)\alpha_p \delta_{GV}(2R))}, \tag{70}
\]
where $\sigma = (i_1 i_2 \cdots i_{k_1}) (i_{k_1+1} i_{k_1+2} \cdots i_{k_1+k_2})$. As $k_1 \geq 2$ and $k_2 \geq 2$, we have $2(k_1 + k_2 - 2) \geq k_1 + k_2$, and therefore
\[
P_{TRC, \sigma} \leq 2^{-n((k_1+k_2)(R + \alpha_p \delta_{GV}(2R)))}, \quad 0 \leq R \leq R_{TRC}(p). \tag{71}
\]
4) General $s \in S_m$ with $s \neq \pi_0$: If permutation $\sigma$ is a product of $r$ disjoint cycles of length $k_1, \ldots, k_r$, respectively, then similar to (65), (71), we have for $0 \leq R \leq R_{TRC}(p)$,
\[
P_{TRC, \sigma} \leq 2^{-n((\sum_{l=1}^{r} k_l)(R + \alpha_p \delta_{GV}(2R)))}. \tag{72}
\]
5) Putting it all together: For $1 \leq j \leq m$, if we define $P_{TRC, \Sigma_j} \triangleq \sum_{\sigma \in \Sigma_j} P_{TRC, \sigma}$, where $\Sigma_j$ is given by (35), then (51) can be equivalently expressed as
\[
D(n, R, p) \leq \sum_{j=2}^{m} P_{TRC, \Sigma_j}. \tag{73}
\]
If $\sigma$ is a product of $r$ disjoint cycles of length $k_1, \ldots, k_r$, respectively, and $s = \sum_{l=1}^{r} k_l$, then $\sigma$ belongs to the set $\Sigma_s$, and $P_{TRC, \sigma}$ is given by (72). Equivalently, for a given $j \geq 2$, if $s \in S_m$ belongs to the set $\Sigma_j$, then we have
\[
P_{TRC, \sigma} \leq 2^{-n(j(R + \alpha_p \delta_{GV}(2R)))} \quad 0 \leq R \leq R_{TRC}(p). \tag{74}
\]
The size of $\Sigma_j$ satisfies $|\Sigma_j| < \prod_{i=1}^{j-1} (m - i) \leq 2^{njR}$. Therefore, for $0 \leq R \leq R_{TRC}(p)$, we have

$$P_{TRC,\Sigma_j} = \sum_{\sigma \in \Sigma_j} P_{TRC,\sigma} \leq 2^{-n(j(R+\alpha_p2\delta_{GV}(2R)))} 2^{njR} = 2^{-n(jo_p2\delta_{GV}(2R))}. \quad (75)$$

Now, if we define $\beta \triangleq 2^{-n(o_p2\delta_{GV}(2R))}$, then $$(75)$$ can be equivalently expressed as $P_{TRC,\Sigma_j} \leq \beta^j$. Finally, from $$(73)$$ and $$(75)$$, we get for $0 \leq R \leq R_{TRC}(p)$,

$$D(n, R, p) \leq \sum_{j=2}^{m} \frac{\beta^j}{1 - \beta} \leq \beta^2 = 2^{-n(2\delta_{GV}(2R)\alpha_p)}. \quad (76)$$

The following theorem encapsulates the main result of this subsection on bounding the bee-identification exponent, $E_D(R, p)$, using joint decoding for TRC.

**Theorem 4.** We have

$$E_D(R, p) \geq 2\delta_{GV}(2R) \alpha_p, \quad 0 \leq R \leq R_{TRC}(p). \quad (77)$$

**Proof:** Follows from $$(5)$$ and $$(76)$$.

We note that the above lower bound for $E_D(R, p)$ using TRCs with joint barcode decoding is twice the corresponding bound obtained using independent barcode decoding (see $$(46)$$). The following proposition shows that the lower bound given by Thm. 3 using TRC is strictly better than corresponding bound using RCE (see Thm. 2) for $0 \leq R < R_{TRC}(p)$.

**Proposition 4.** The lower bound on $E_D(R, p)$ in $$(77)$$ obtained for TRC is strictly better than the corresponding bound in $$(41)$$ obtained for RCE when $0 \leq R < R_{TRC}(p)$.

**Proof:** It is known that $E_{TRC}(R, p) > E_c(R, p)$ when $0 \leq R < R_{TRC}(p)$. Further, using explicit numerical computation, it can be shown that $2R_0(p) > R_1(p) + 2R_{TRC}(p)$. Therefore, it follows that for $0 \leq R < R_{TRC}(p)$, we have

$$2\delta_{GV}(2R) \alpha_p = 2(E_{TRC}(R, p) - R) > 2(E_c(R, p) - R) = 2(R_0(p) - 2R) \geq R_1(p) - 2R + 2(R_{TRC}(p) - R) > R_1(p) - 2R \geq \eta_p(R).$$

The next section presents an explicit upper bound for $E_D(R, p)$ which applies to all possible codebook designs.

**IV. UPPER BOUND ON THE BEE-IDENTIFICATION EXPONENT**

This section presents an upper bound on the bee-identification exponent $E_D(R, p)$. Towards this, we define the following optimum minimum distance metrics

$$d^*(n, R) \triangleq \max_{C \in \mathcal{C}(n, R)} \min_{e_i, e_j \in C} d_H(e_i, e_j),$$

$$\delta^*(n, R) \triangleq \min_{e_i \neq e_j} d_H(e_i, e_j),$$

$$\delta^*(R) \triangleq \limsup_{n \to \infty} \delta^*(n, R).$$

For any given codebook $C \in \mathcal{C}(n, R)$, we show that there exists a set $\mathcal{I}_C$ consisting of pairs of codeword indices $(i, j)$, $i \neq j$, with the following properties:

(i) If $(i, j) \in \mathcal{I}_C$, then $d_H(e_i, e_j) \leq d_H(e_i, e_j) \leq d^*(n, R - \frac{1}{n}).$

(ii) If $(i, j) \in \mathcal{I}_C$ and $(i, j) \in \mathcal{I}_C$, then $i \neq i, i \neq j$ and $j \neq i, j \neq j$.

(iii) Size of set $\mathcal{I}_C$ is at least $m/4$.

A set satisfying the above properties can be constructed iteratively as follows.

- **Step 1:** For a given codebook $C \in \mathcal{C}(n, R)$, initialize $\mathcal{I}_C$ to be the empty set and let $T = C$.

- **Step 2:** As $T$ contains at least $m/2$ codewords, there exists $e_i, e_j \in T$, with $i \neq j$, satisfying $d_H(e_i, e_j) \leq d_{\mathcal{H}}(e_i, e_j) \leq d^*(n, R - \frac{1}{n}).$ Include the pair $(i, j)$ to $\mathcal{I}_C$, and let $T = T \setminus \{e_i, e_j\}$.

- **Step 3:** If $|\mathcal{I}_C| < m/4$, then go to Step 2, else stop.

Let the receiver employ ML decoding, and interpret each pair $(i, j) \in \mathcal{I}_C$ as a transposition $\sigma = (i, j)$ that interchanges indices $i$ and $j$. Let $A(i, j)$ denote the error event that the receiver incorrectly decodes the channel induced permutation to transposition $(i, j)$ (instead of the identity permutation $\pi_0$). i.e. $A(i, j) = \{\pi_0 \to (i, j)\}$. Then, the bee-identification error probability $D(C, p, \phi)$ can be lower bounded as

$$D(C, p, \phi) \geq \Pr \left\{ \bigcup_{(i,j) \in \mathcal{I}_C} A(i,j) \right\} \quad (78)$$

Using de Caen’s lower bound on the probability of a union $$(11)$$, the expression on the right side in $$(78)$$ can itself be lower bounded by

$$\sum_{(i,j) \in \mathcal{I}_C} \Pr\{A(i,j)\}^2 \sum_{(i,j) \in \mathcal{I}_C} \Pr\{A(i,j) \cap A(i,j)\} \frac{1}{\sum_{(i,j) \in \mathcal{I}_C} \Pr\{A(i,j)\}^2 \Pr\{A(i,j)\} \Pr\{A(i,j)\}} (79)$$

where $$(a)$$ follows because events $A(i,j)$ and $A(i,j)$ are independent when $(i, j) \neq (i, j)$.
The proof is completed by using (82) and (83).

\[ \lim_{R \to 0} E_D(R, p) = \alpha_p. \]

**Proof:** As \( \lim_{R \to 0} \delta_{LP}(R) = 0.5 \), we have from (84) that

\[ \lim_{R \to 0} E_D(R, p) \leq \lim_{R \to 0} 2\delta_{LP}(R)\alpha_p - R = \alpha_p. \]  

On the other hand, we have \( \lim_{R \to 0} \delta_{GV}(R) = 0.5 \) and so it follows from (77) that

\[ \lim_{R \to 0} E_D(R, p) \geq \lim_{R \to 0} 2\delta_{GV}(2R)\alpha_p = \alpha_p. \]

The proof is completed by using (86) and (87).

The above corollary shows that the lower bound on \( E_D(R, p) \) given by (77), and the upper bound on \( E_D(R, p) \) given by (84) become tight as \( R \to 0 \).

**Theorem 5.** We have

\[ E_D(R, p) \leq |\delta^*(R)\alpha_p - R|^+ \leq |\delta_{LP}(R)\alpha_p - R|^+. \]  

**Proof:** Follows immediately from (82) and (83).

The following corollary shows that \( E_D(R, p) \) can be explicitly characterized with a rather simple expression when rate \( R \) tends to zero.

**Corollary 1.** We have

\[ \lim_{R \to 0} E_D(R, p) = \alpha_p. \]  

**Proof:** As \( \lim_{R \to 0} \delta_{LP}(R) = 0.5 \), we have from (84) that

\[ \lim_{R \to 0} E_D(R, p) \leq \lim_{R \to 0} 2\delta_{LP}(R)\alpha_p - R = \alpha_p. \]

The above corollary shows that the lower bound on \( E_D(R, p) \) given by (77), and the upper bound on \( E_D(R, p) \) given by (84) become tight as \( R \to 0 \).

**V. A Numerical Example**

Fig. 3 plots different bounds for the bee-identification exponent \( E_D(R, p) \). The explicit lower bound for RCE with independent decoding (ID) (respectively, joint decoding (JD)) is given by (11) (respectively, (41)). The performance with JD is seen to be much better than with ID. When \( 0 \leq R < R_{TRC} \), the explicit lower bound for TRC with ID (resp., JD) is given by (46) (resp., (77)). As shown in Prop. 4 the lower bound obtained using TRC with joint decoding is better than the corresponding bound using RCE. The upper bound is given by (83) and holds for all possible codebook designs. Further, as shown in Cor. 1 it is observed from Fig. 3 that \( \lim_{R \to 0} E_D(R, p) = \alpha_p = 2.33 \) for \( p = 0.01 \).
an appropriate choice of the distance threshold parameter for declaring an erasure.

The work in this paper may be extended by considering different variants of the bee-identification error metric, for instance, where error is flagged only when the fraction of incorrectly decoded barcodes exceeds a threshold. Another interesting scenario for future analysis is the problem formulation where some of the m rows in codebook C are deleted, due to some bees being outside the hive when taking the picture.

APPENDIX A
PROOF OF PROP. [1]

Proof: Let \( \gamma_{k-1}, \gamma_{k-1} \in \mathbb{F}_{2^n} \), and \( \Delta \triangleq \gamma_{k-1} \oplus \gamma_{k-1} \), where \( \oplus \) denotes modulo-2 addition. Then, \( \Pr\{d_H(\gamma_{k-1}, c_k) = d_{k-1}\} = \Pr\{d_H(\gamma_{k-1}, c_k + \Delta) = d_{k-1}\} \) (i) \( = \Pr\{d_H(\gamma_{k-1}, c_k) = d_{k-1}\} \), where (i) follows from the fact that for a given \( \Delta \), the distribution of \( c_k + \Delta \) is same as the distribution of \( c_k \). This implies that \( \Pr\{d_H(c_k, c_k) = d_{k-1}\} = \Pr\{d_H(c_k-1, c_k) = d_{k-1}\} \) (ii) \( = \Pr\{d_H(c_k-1, c_k) = d_{k-1}\} \). Then \( \Pr\{\bigcap_{i=1}^{k-1}\{d_H(c_i, c_{i+1}) = d_i\}\} \) can be expressed as

\[
\sum_{\gamma_i, \ldots, \gamma_{k-1} \in \mathbb{F}_{2^n}} \left( \Pr\left\{ \bigcap_{i=1}^{k-1}\{d_H(c_i, c_{i+1}) = d_i\} \right\} \right) \times \Pr\left\{ \bigcap_{i=1}^{k-1}\{d_H(c_i, c_{i+1}) = d_i\} \right\},
\]

= \[
\sum_{\gamma_i, \ldots, \gamma_{k-1} \in \mathbb{F}_{2^n}} \left( \Pr\left\{ \bigcap_{i=1}^{k-1}\{d_H(c_i, c_{i+1}) = d_i\} \right\} \right) \times \Pr\{d_H(c_k-1, c_k) = d_{k-1}\},
\]

(\( \text{iii} \)) \[= \sum_{\gamma_i, \ldots, \gamma_{k-1} \in \mathbb{F}_{2^n}} \left( \Pr\left\{ \bigcap_{i=1}^{k-1}\{d_H(c_i, c_{i+1}) = d_i\} \right\} \right) \times \Pr\{d_H(c_k-1, c_k) = d_{k-1}\},
\]

= \[
\left( \Pr\left\{ \bigcap_{i=1}^{k-2}\{d_H(c_i, c_{i+1}) = d_i\} \right\} \right) \times \Pr\{d_H(c_k, c_k) = d_{k-1}\},
\]

(88)

where \( 1_{\{\}} \) denotes the indicator function, and (\( \text{iii} \)) follows from (i). Recursively applying (88), we get

\[\Pr\left\{ \bigcap_{i=1}^{k-1}\{d_H(c_i, c_{i+1}) = d_i\} \right\} = \prod_{i=1}^{k-1} \Pr\{d_H(c_i, c_{i+1}) = d_i\}.\]

Now, (24) follows from the fact that \( \Pr\{d_H(c_i, c_{i+1}) = d_i\} = 2^{n(1-H(d_i/n))} \) when \( c_i \) and \( c_{i+1} \) are uniformly distributed over \( \mathbb{F}_{2^n} \).

APPENDIX B
PROOF OF PROP. [3]

Proof: For \( 1 \leq i \leq m = 2^{nR} \), let \( c_i \) denote the \( i \)-th row of codebook \( C \). Let \( \mathbb{F}_{2^n} \) denote the space of all \( n \)-length binary vectors, and let \( \gamma_i \in \mathbb{F}_{2^n} \) for \( 1 \leq i \leq m \). Let \( Q_{\text{TRC}} \{\bigcap_{i=1}^{m}\{c_i = \gamma_i\}\} \) (respectively, \( Q_{\text{RCE}} \{\bigcap_{i=1}^{m}\{c_i = \gamma_i\}\} \)) denote the probability \( \Pr\{\bigcap_{i=1}^{m}\{c_i = \gamma_i\}\}\). If \( C \) is uniformly distributed over \( \mathcal{C}_{\text{TRC}}(n, R) \) (respectively, over \( \mathcal{C}(n, R) \)). If we define

\[\alpha_n \triangleq \sum_{(\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathcal{C}_{\text{TRC}}(n, R)} Q_{\text{RCE}}\left\{ \bigcap_{i=1}^{m}\{c_i = \gamma_i\} \right\} ,\]

then we have

\[Q_{\text{TRC}}\left\{ \bigcap_{i=1}^{m}\{c_i = \gamma_i\} \right\} = \frac{1}{\alpha_n} Q_{\text{RCE}}\left\{ \bigcap_{i=1}^{m}\{c_i = \gamma_i\} \right\} 1_{\{\gamma_1, \gamma_2, \ldots, \gamma_m\} \in \mathcal{C}_{\text{TRC}}(n, R)} ,\]

(89)

where \( 1_{\{\}} \) denotes the indicator function. Further, let \( Q_{\text{TRC}}\left\{\bigcap_{i=1}^{k-1}\{d_H(c_i, c_{i+1}) = d_i\}\right\} \) (respectively, \( Q_{\text{RCE}}\left\{\bigcap_{i=1}^{k-1}\{d_H(c_i, c_{i+1}) = d_i\}\right\} \)) denote the probability \( \Pr\{\bigcap_{i=1}^{k-1}\{d_H(c_i, c_{i+1}) = d_i\}\} \) when codebook \( C \) is uniformly distributed over \( \mathcal{C}_{\text{TRC}}(n, R) \) (respectively, over \( \mathcal{C}(n, R) \)). Now, we have

\[Q_{\text{TRC}}\left\{ \bigcap_{i=1}^{k-1}\{d_H(c_i, c_{i+1}) = d_i\} \right\} \]

= \[
\sum_{\gamma_i \in \mathbb{F}_{2^n}, 1 \leq i \leq m} Q_{\text{TRC}}\left\{ \bigcap_{i=1}^{m}\{c_i = \gamma_i\} \right\} 1_{\{\gamma_1, \gamma_2, \ldots, \gamma_m\} \in \mathcal{C}_{\text{TRC}}(n, R)} ,\]

(\( \text{a} \)) \[= \frac{1}{\alpha_n} \sum_{\gamma_i \in \mathbb{F}_{2^n}, 1 \leq i \leq m} \sum_{1 \leq i \leq m} \sum_{1 \leq i \leq m} Q_{\text{RCE}}\left\{ \bigcap_{i=1}^{m}\{c_i = \gamma_i\} \right\} 1_{\{\gamma_1, \gamma_2, \ldots, \gamma_m\} \in \mathcal{C}_{\text{TRC}}(n, R)} ,\]

(\( \text{b} \)) \[\leq \frac{1}{\alpha_n} \sum_{\gamma_i \in \mathbb{F}_{2^n}, 1 \leq i \leq m} \sum_{1 \leq i \leq m} \sum_{1 \leq i \leq m} Q_{\text{RCE}}\left\{ \bigcap_{i=1}^{m}\{c_i = \gamma_i\} \right\} 1_{\{\gamma_1, \gamma_2, \ldots, \gamma_m\} \in \mathcal{C}_{\text{TRC}}(n, R)} ,\]

(\( \text{c} \)) \[= \prod_{i=1}^{k-1} 2^{-n(1-H(d_i/n))} ,\]

where (a) follows from (89), (b) follows from the fact that \( \alpha_n \to 1 \) as \( n \to \infty \), and (c) follows from Prop. [1].

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REFERENCES

[1] T. Gernert, V. D. Rao, M. Middendorf, H. Dankowicz, N. Goldenfeld, and G. E. Robinson, “Automated monitoring of behavior reveals bursty interaction patterns and rapid spreading dynamics in honeybee social networks,” Proc. Nat. Acad. Sci. U.S.A., vol. 115, no. 7, pp. 1433–1438, Feb. 2018.
[2] S. Shahi, D. Tuninetti, and N. Devroye, “The strongly asynchronous massive access channel,” Jul. 2018, arXiv:1807.09934 [cs.IT].
[3] S. Shahi, D. Tuninetti, and N. Devroye, “On identifying a massive number of distributions,” in Proc. 2018 IEEE Int. Symp. Inf. Theory, Jun. 2018, pp. 331–335.

[4] R. Heckel, I. Shomorony, K. Ramchandran, and D. N. C. Tse, “Fundamental limits of DNA storage systems,” in Proc. 2017 IEEE Int. Symp. Inf. Theory, Jun. 2017, pp. 3130–3134.

[5] I. Shomorony and R. Heckel, “Capacity results for the noisy shuffling channel,” Feb. 2019. [arXiv:1902.10832] [cs.IT].

[6] M. Kovačević and V. Y. F. Tan, “Codes in the space of multisets – coding for permutation channels with impairments,” IEEE Trans. Inform. Theory, vol. 64, no. 7, pp. 5156–5169, Jul. 2018.

[7] A. Barg and G. D. Forney, “Random codes: minimum distances and error exponents,” IEEE Trans. Inform. Theory, vol. 48, no. 9, pp. 2568–2573, Sep. 2002.

[8] R. G. Gallager, Information Theory and Reliable Communication. New York: John Wiley and Sons, 1968.

[9] I. Herstein, Topics In Algebra, 2nd ed. John Wiley and Sons, New York, 1975.

[10] N. Merhav, “Error exponents of typical random codes,” IEEE Trans. Inform. Theory, vol. 64, no. 9, pp. 6223–6235, Sep. 2018.

[11] D. de Caen, “A lower bound on the probability of a union,” Discrete Mathematics, vol. 169, no. 1, pp. 217 – 220, May 1997.

[12] R. McEliece, E. Rodemich, H. Rumsey, and L. Welch, “New upper bounds on the rate of a code via the Delsarte-MacWilliams inequalities,” IEEE Trans. Inform. Theory, vol. 23, no. 2, pp. 157–166, Mar. 1977.

[13] S. Litsyn, “New upper bounds on error exponents,” IEEE Trans. Inform. Theory, vol. 45, no. 2, pp. 385–398, Mar. 1999.

[14] G. D. Forney, Jr., “Generalized minimum distance decoding,” IEEE Trans. Inform. Theory, vol. 12, no. 2, pp. 125–131, Apr. 1966.