Hypergraphs with Spectral Radius at most \((r - 1)!\sqrt{2} + \sqrt{5}\)

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Abstract

In our previous paper, we classified all \(r\)-uniform hypergraphs with spectral radius at most \((r - 1)!\sqrt{2} + \sqrt{5}\), which directly generalizes Smith’s theorem for the graph case \(r = 2\). It is nature to ask the structures of the hypergraphs with spectral radius slightly beyond \((r - 1)!\sqrt{2} + \sqrt{5}\). For \(r = 2\), the graphs with spectral radius at most \(\sqrt{2} + \sqrt{5}\) are classified by [Brouwer-Neumaier, Linear Algebra Appl., 1989]. Here we consider the \(r\)-uniform hypergraphs \(H\) with spectral radius at most \((r - 1)!\sqrt{2} + \sqrt{5}\). We show that \(H\) must have a quipus-structure, which is similar to the graphs with spectral radius at most \(3\sqrt{2}\) [Woo-Neumaier, Graphs Combin., 2007].

1 Introduction

The spectral radius \(\rho(G)\) of a graph \(G\) is the largest eigenvalue of its adjacency matrix. The (simple undirected connected) graphs with small spectral radius have been well-studied in the literature. In 1970 Smith classified all connected graphs with spectral radius at most 2. The graphs \(G\) with \(\rho(G) < 2\) are simple Dynkin Diagrams \(A_n, D_n, E_6, E_7,\) and \(E_8\), while the graphs \(G\) with \(\rho(G) = 2\) simply extend Dynkin Diagrams \(A_n, D_n, E_6, E_7,\) and \(E_8\). Cvetković et al. [6] gave a nearly complete description of all graphs \(G\) with \(2 < \rho(G) < \sqrt{2} + \sqrt{5}\). Their description was completed by Brouwer and Neumaier [1]. Namely, \(E(1, b, c)\) for \(b = 2, c \geq 6\) or \(b \geq 3, c \geq 4\), \(E(2, 2, c)\) for \(c \geq 3\), and \(G_{1,a:b:1,c}\) for \(a \geq 3, c \geq 2, b > a + c\).

Wang et al. [25] studied some graphs with spectral radius close to \(3\sqrt{2}\). Woo and Neumaier [26] proved that any connected graph \(G\) with \(\sqrt{2} + \sqrt{5} < \rho(G) < \frac{3}{2}\sqrt{2}\) is one of the following graphs.

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1. If $G$ has maximum degree at least 4, then $G$ is a **dagger** (i.e., a tree obtained by attaching a path to a leaf vertex of the star $S_5$).

2. If $G$ is a tree with maximum degree at most 3, then $G$ is an **open quipu** (i.e., all the vertices of degree 3 lie on a path).

3. If $G$ contains a cycle, then $G$ is a **closed quipu** (i.e., a unicyclic graph with maximum degree at most 3 satisfies that all the vertices of degree 3 lie on a cycle).

Lan-Lu [11] proved that for any open quipu $G$ on $n$ vertices ($n \geq 6$) with spectral radius less than $\frac{1}{2}\sqrt{3}$, its diameter $D(G)$ satisfies $D(G) \geq (2n - 4)/3$, and for any closed quipu $G$ on $n$ vertices ($n \geq 13$) with spectral radius less than $\frac{1}{2}\sqrt{7}$, its diameter $D(G)$ satisfies $\frac{n}{3} < D(G) \leq \frac{2n-2}{3}$.

In this paper, we would like to study the $r$-uniform hypergraphs $H$ with small spectral radius. In our previous paper [27], we generalized Smith’s theorem to hypergraphs and classified all connected $r$-uniform hypergraphs with the spectral radius at most $\rho_r = (r-1)!\sqrt{7}$. The main method is using $\alpha$-normal labeling. Roughly speaking, we can label all “corners of edges” by some numbers in $(0, 1)$ such that for each vertex $v$ the sum of these numbers at $v$ is always equal to 1 while for each edge $f$ the product of these numbers at $f$ is always equal to $\alpha$. The detail of the definition of $\alpha$-normal labeling can be found in Section 2. If $H$ has a “consistent” $\alpha$-normal labeling, then $\rho(H) = (r-1)!\alpha^{-1/r}$.

As an important corollary, any $(r-1)$-uniform hypergraph $H'$ with $\rho(H') = (r - 2)!\alpha^{-1/(r-1)}$ can be extended to an $r$-uniform hypergraph $H$ with spectral radius $\rho(H) = (r - 1)!\alpha^{-1/r}$ by simply extending each edge by adding one new vertex. If $H$ is not extended from some $H'$, then $H$ is called **irreducible**. An $r$-uniform hypergraph is irreducible if and only if it contains an edge so that every vertex in this edge has degree greater than 1. We use the following convention: if the notation $H^{(r)}$ is a well-defined $r'$-uniform hypergraph, then for each $r > r'$, $H^{(r)}$ means the unique $r$-uniform hypergraph extended from $H^{(r')}$ by a sequence of extension described above.

From [27], we show all $r$-uniform hypergraphs $H$ with $\rho(H) = (r-1)!\sqrt{4}$ listed as follows:

**Extended from 2-graphs:** $C_n^{(r)}$, $D_n^{(r)}$, $E_6^{(r)}$, $E_7^{(r)}$, and $E_8^{(r)}$.

**Extended from 3-graphs:** $B_n^{(r)}$, $BD_n^{(r)}$, $C_2^{(r)}$, $S_4^{(r)}$, $F_{2,3,4}^{(r)}$, $F_{2,2,7}^{(r)}$, $F_{1,5,6}^{(r)}$, $F_{1,4,8}^{(r)}$, $F_{1,3,14}^{(r)}$, $G_{1,1,9:1,4}^{(r)}$, and $G_{1,1:6,1,3}^{(r)}$.

**Extended from 4-graphs:** $H_{1,1,1,2,2}^{(r)}$.

Similarly here are all $r$-uniform hypergraphs $H$ with $\rho(H) < (r-1)!\sqrt{4}$:

**Extended from 2-graphs:** $A_n^{(r)}$, $D_n^{(r)}$, $E_6^{(r)}$, $E_7^{(r)}$, and $E_8^{(r)}$.

**Extended from 3-graphs:** $D_n^{(r)}$, $B_n^{(r)}$, $B_n^{(r)}$, $BD_n^{(r)}$, $F_{2,3,3}^{(r)}$, $F_{2,2,7}^{(r)}$ (for $2 \leq j \leq 6$), $F_{1,4,j}^{(r)}$ (for $3 \leq j \leq 13$), $F_{1,5,j}^{(r)}$ (for $4 \leq j \leq 7$), $F_{1,5,5}^{(r)}$, and $G_{1,1:1:1,3}^{(r)}$ (for $0 \leq j \leq 5$).

**Extended from 4-graphs:** $H_{1,1,1,1,1}^{(r)}$, $H_{1,1,1,1,2}^{(r)}$, $H_{1,1,1,1,3}^{(r)}$, $H_{1,1,1,1,4}^{(r)}$.

The details of these hypergraphs can be found in the paper [27].

It is nature to ask what structures the hypergraphs with spectral radius slightly greater than $\rho_r$ can have. Since $(2, \sqrt{2} + \sqrt{5})$ is the next interesting interval for the spectral radius of graphs, naturally we consider all connected $r$-uniform hypergraphs $H$ with $\rho(H) \in ((r - 1)!\sqrt{4}, (r - 1)!\sqrt{2} + \sqrt{5})$. When $r = 2$, these graphs are $E_{1,b,c}$, $E_{2,2,2}$, and $G_{1,a:b:1,c}$ with $b > a + c$ as shown by Cvetković et al. [6] and Brouwer-Neumaier [1]. The structures of these hypergraphs are slightly more complicated for $r \geq 3$. For $k \geq 3$, a vertex is called a $k$-branching vertex if it is incident to $k$ edges while an edge
is called a $k$-branching edge if it contains no branching vertex but it is adjacent to exactly $k$ edges. (When $k = 3$, we simply say branching vertex/edge instead of 3-branching vertex/edge.) We have the following results.

**Theorem 1.** Consider an irreducible connected 3-uniform hypergraph $H$. If the spectral radius of $H$ satisfies $\rho(H) \leq 2\sqrt{2} + \sqrt{5}$, then no vertex (of $H$) can have degree more than three, no edge can incident to more than 3 other edges, each branching vertex is not incident to any branching edges. Moreover, $H$ belongs to one of the following two categories:

**Open 3-quipu:** $H$ is a hypertree with all branching vertices and all branching edges lying on a path. Moreover, there are at most 2 branching vertices. A branching vertex cannot lie between two branching edges, or between a branching edge and another branching vertex.

**Closed 3-quipu:** $H$ contains a cycle $C$ and no branching vertices. All branching edges lie on $C$, and any branching edge can be only attached by a path.

**Theorem 2.** Suppose that $H$ is an irreducible 4-uniform hypergraphs with $\rho(H) \leq 6\sqrt{2} + \sqrt{5}$. Then $H$ is a hypertree with no vertex (of $H$) having degree more than three and no edge incident to more than 4 other edges. The hypergraph $H$ belongs to one of the following two categories:

**Open 4-quipu:** $H$ is a hypertree with all branching vertices and all branching edges lying on a path. Moreover, there are at most two 3-branching vertices (or two 4-branching edges). A 4-branching edge (or a branching vertex) cannot lie between two 3-branching edges, or between a 3-branching edge and another branching edge. In addition, each 4-branching edge is attached by three path of length 1, 1, and $k$ ($k = 1, 2, 3$) respectively.

**4-dagger:** $H$ is obtained by attaching 4-paths of length $i, j, k, l$ to a 4-branching edge. Denote this hypergraph by $H_{i,j,k,l}^{(4)}$ with $i \leq j \leq k \leq l$. Then $H$ must be one of the following hypergraphs $H_{1,2,2,2}^{(4)}, H_{1,2,2,3}^{(4)}, H_{1,1,4,4}^{(4)}, H_{1,1,4,5}^{(4)}$, and $H_{1,1,k,l}^{(4)}$ ($1 \leq k \leq 3$, and $k \leq l$).

**Theorem 3.** For $r = 5$, there is only one irreducible 5-uniform hypergraph $H$ with $\rho(H) \leq (r - 1)!\sqrt{2} + \sqrt{5}$; namely the five edge-star as shown below.
For $r \geq 6$, all $r$-uniform hypergraphs $H$ with $\rho(H) \leq (r-1)! \sqrt{2 + \sqrt{5}}$ are reducible.

2 Notation and Lemmas

Let us review some basic notation about hypergraphs. An $r$-uniform hypergraph $H$ is a pair $(V, E)$ where $V$ is the set of vertices and $E \subset \binom{V}{r}$ is the set of edges. The degree of vertex $v$, denoted by $d_v$, is the number of edges incident to $v$. If $d_v = 1$, we say $v$ is a leaf vertex. A walk on a hypergraph $H$ is a sequence of vertices and edges: $v_0e_1v_1e_2\ldots v_le_l$ satisfying that both $v_{i-1}$ and $v_i$ are incident to $e_i$ for $1 \leq i \leq l$. The vertices $v_0$ and $v_l$ are called the ends of the walk. The length of a walk is the number of edges on the walk. A walk is called a path if all vertices and edges on the walk are distinct. The walk is closed if $v_l = v_0$. A closed walk is called a cycle if all vertices and edges in the walk are distinct. A hypergraph $H$ is called connected if for any pair of vertex $(u, v)$, there is a path connecting $u$ and $v$. A hypergraph $H$ is called a hypertree if it is connected, and acyclic. A hypergraph $H$ is called simple if every pair of edges intersects at most one vertex. In fact, any non-simple hypergraph contains at least a 2-cycle: $v_1F_1v_2F_2v_1$, i.e., $v_1, v_2 \in F_1 \cap F_2$. A hypertree is always simple.

The spectral radius $\rho(H)$ of an $r$-uniform hypergraph $H$ is defined as
\begin{equation}
\rho(H) = r! \max_{x \in \mathbb{R}_n^\geq \neq 0} \frac{\sum_{i_1, i_2, \ldots, i_r \in E(H)} x_{i_1}x_{i_2}\ldots x_{i_r}}{\sum_{i=1}^n x_i^r}.
\end{equation}

Here $\mathbb{R}_n^\geq$ denote the set of points with nonnegative coordinates in $\mathbb{R}^n$. This is a special case of $p$-spectral norm for $p = r$. The general $p$-spectral norm has been considered by various authors (see [2, 5, 10, 17]). The following lemma has been proved in several papers.

Lemma 1. [5, 10, 17] If $G$ is a connected $r$-uniform hypergraph, and $H$ is a proper subgraph of $G$, then $\rho(H) < \rho(G)$.

In our previous paper [27], we discovered an efficient way to compute the spectral radius $\rho(H)$, in particular when $H$ is a hypertree. The idea is using the method of the $\alpha$-normal labelling (or weighed matrix).

Definition 1 (See [27]). A weighted incidence matrix $B$ of a hypergraph $H$ is a $|V| \times |E|$ matrix such that for any vertex $v$ and any edge $e$, the entry $B(v, e) > 0$ if $v \in e$ and $B(v, e) = 0$ if $v \notin e$.

Definition 2 (See [27]). A hypergraph $H$ is called $\alpha$-normal if there exists a weighted incidence matrix $B$ satisfying
1. \( \sum_{e : v \in e} B(v, e) = 1 \), for any \( v \in V(H) \).
2. \( \prod_{v \in e} B(v, e) = \alpha \), for any \( e \in E(H) \).

Moreover, the incidence matrix \( B \) is called consistent if for any cycle \( v_0 e_1 v_1 e_2 \ldots v_l (v_l = v_0) \)

\[ \prod_{i=1}^{l} \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = 1. \]

In this case, we call \( H \) consistently \( \alpha \)-normal.

The following important lemma was proved in [27].

**Lemma 2** (See Lemma 3 of [27]). Let \( H \) be a connected \( r \)-uniform hypergraph. Then the spectral radius of \( H \) is \( \rho(H) \) if and only if \( H \) is consistently \( \alpha \)-normal with \( \alpha = ((r - 1)!/\rho(H))^r \).

Often we need compare the spectral radius with a particular value.

**Definition 3** (See [27]). A hypergraph \( H \) is called \( \alpha \)-subnormal if there exists a weighted incidence matrix \( B \) satisfying

1. \( \sum_{e : v \in e} B(v, e) \leq 1 \), for any \( v \in V(H) \).
2. \( \prod_{v \in e} B(v, e) \geq \alpha \), for any \( e \in E(H) \).

Moreover, \( H \) is called strictly \( \alpha \)-subnormal if it is \( \alpha \)-subnormal but not \( \alpha \)-normal.

We have the following lemma.

**Lemma 3** (See Lemma 4 of [27]). Let \( H \) be an \( r \)-uniform hypergraph. If \( H \) is \( \alpha \)-subnormal, then the spectral radius of \( H \) satisfies

\[ \rho(H) \leq (r - 1)!\alpha^{\frac{1}{r}}. \]

Moreover, if \( H \) is strictly \( \alpha \)-subnormal then \( \rho(H) < (r - 1)!\alpha^{\frac{1}{r}} \).

**Definition 4** (See [27]). A hypergraph \( H \) is called \( \alpha \)-supernormal if there exists a weighted incidence matrix \( B \) satisfying

1. \( \sum_{e : v \in e} B(v, e) \geq 1 \), for any \( v \in V(H) \).
2. \( \prod_{v \in e} B(v, e) \leq \alpha \), for any \( e \in E(H) \).

Moreover, \( H \) is called strictly \( \alpha \)-supernormal if it is \( \alpha \)-supernormal but not \( \alpha \)-normal.

**Lemma 4** (See Lemma 5 of [27]). Let \( H \) be an \( r \)-uniform hypergraph. If \( H \) is strictly and consistently \( \alpha \)-supernormal, then the spectral radius of \( H \) satisfies

\[ \rho(H) > (r - 1)!\alpha^{\frac{1}{r}}. \]

Note that if \( H \) is consistently \( \alpha \)-normal and \( H \) is extended from \( H' \), then so is \( H' \). This implies the following corollary.

**Corollary 1.** For any \( r \geq 3 \) and \( \alpha \in (0, 1) \), if \( H \) extends \( H' \), then \( \rho(H) = (r - 1)!\alpha^{-1/r} \) (or \( \rho(H) < (r - 1)!\alpha^{-1/r} \)) if and only if \( \rho(H') = (r - 2)!\alpha^{-1/(r-1)} \) (or \( \rho(H') < (r - 2)!\alpha^{-1/(r-1)} \)) respectively.
Definition 5. Given two $r$-uniform hypergraphs $H_1$ and $H_2$, a homomorphism from $H_1$ to $H_2$ is a map $f: V(H_1) \rightarrow V(H_2)$ which preserves the edges. If $f$ derives an injective map, also denoted by $f$, from $E(H_1)$ to $E(H_2)$, then $f$ is called a sub-homomorphism. In this case, we also say $H_1$ is a sub-homomorphic type of $H_2$.

Every subhypergraph is a subhomomorphic type. The reverse statement is not true. Consider the following example. Suppose that $v_1$ and $v_2$ are two vertices of $H$ which are not contained in any common edge. We can form a new hypergraph $H'$ from $H$ by identifying $v_1$ and $v_2$ into a fat vertex, called $x$. Now the map $f: V(H) \rightarrow V(H')$ by sending both $v_1$ and $v_2$ into $x$ and mapping other vertices itself. Then $f$ is a sub-homomorphism. The following lemma generalizes Lemma 1.

Lemma 5. Suppose $H_1$ and $H_2$ are two connected $r$-uniform hypergraphs. If $H_1$ is a sub-homomorphic type of $H_2$, then we have
\[
\rho(H_1) \leq \rho(H_2)
\]
and the equality holds if and only if $H_1$ is isomorphic to $H_2$.

Proof. Let $f: V(H_1) \rightarrow V(H_2)$ be the sub-homomorphism. Setting $\alpha = \left(\frac{(r-1)!}{\rho(H_2)}\right)^r$, by Lemma 2, $H_2$ is consistently $\alpha$-normal and let $B_2$ be the incident matrix. We can define an incident matrix $B_1$ of $H_1$ as follows:
\[
B_1(v, e) = B_2(f(v), f(e)) \quad \text{for any } v \in V(H_1) \text{ and } e \in E(H_1).
\]

For any fixed $e \in E(H_1)$, we have
\[
\prod_{v \in e} B_1(v, e) = \prod_{v' \in f(e)} B_2(v', f(e)) = \alpha.
\]

For any fixed $v \in E(H_1)$, the set $\{e \in E(H_1): v \in e\}$ is a subset of $\{e \in E(H_1): f(v) \in f(e)\}$. Since $f(e)$ is uniquely determined by $e$, the latter set is one-to-one corresponding to the set $\{e' \in E(H_2): f(v) \in e'\}$. This observation implies
\[
\sum_{e: v \in e} B_1(v, e) \leq \sum_{e': f(v) \in e'} B_2(f(v), e') = 1.
\]

Therefore, $H_1$ is $\alpha$-subnormal. It implies $\rho(H_1) \leq \rho(H_2)$. When the inequality holds, $f(H_1) = H_2$ (otherwise $\rho(H_1) \leq \rho(f(H_1)) < \rho(H_2)$), and for any $v \in V(H_1)$ and $e \in E(H_1)$, $v \in e$ if and only if $f(v) \in f(e)$. This implies that $f$ must be an injective map, (otherwise, we have $f(v_1) = f(v_2)$), then we can find an edge $e_1$ containing $v_1$. Since $f$ is a homomorphism, $v_2$ is not in $e_1$, but $f(v_2) = f(v_1) \in f(e_1)$. Contradiction.) Hence, $f$ is an isomorphism.

Often, we need to calculate the limit of the spectral radius of a sequence of hypergraphs. The following lemma is helpful.

Lemma 6. For any fixed $\beta \in (0, \frac{1}{4})$, let $f_\beta(x) = \frac{\beta}{1-x}$ and $f_\beta^n(x) = f(f_\beta^{n-1}(x))$ for $n \geq 2$.

1. If $0 < x \leq \frac{1-\sqrt{1-4\beta}}{2}$, then $f_\beta^n(x)$ is increasing with respect to $n$, and $\lim_{n \to \infty} f_\beta^n(x) = \frac{1-\sqrt{1-4\beta}}{2}$. Moreover, when $x = \frac{1-\sqrt{1-4\beta}}{2}$, $f_\beta^n(x) = \frac{1-\sqrt{1-4\beta}}{2}, \forall n \geq 1$.

2. If $\frac{1-\sqrt{1-4\beta}}{2} < x < \frac{1+\sqrt{1-4\beta}}{2}$, then $f_\beta^n(x)$ is decreasing with respect to $n$, and $\lim_{n \to \infty} f_\beta^n(x) = \frac{1+\sqrt{1-4\beta}}{2}$. If $x = \frac{1+\sqrt{1-4\beta}}{2}$, then $f_\beta^n(x) = \frac{1+\sqrt{1-4\beta}}{2}, \forall n \geq 1$. If $x = \frac{1+\sqrt{1-4\beta}}{2}$, then $f_\beta^n(x) = \frac{1+\sqrt{1-4\beta}}{2}, \forall n \geq 1$. If $x = \frac{1+\sqrt{1-4\beta}}{2}$, then $f_\beta^n(x) = \frac{1+\sqrt{1-4\beta}}{2}, \forall n \geq 1$. If $x = \frac{1+\sqrt{1-4\beta}}{2}$, then $f_\beta^n(x) = \frac{1+\sqrt{1-4\beta}}{2}, \forall n \geq 1$.
Proof. We first prove item 1. Since $0 < x \leq \frac{1-\sqrt{1-4\beta}}{2}$, the function $f_\beta(x) = \frac{\beta}{1-x}$ attains its maximum when $x = \frac{1-\sqrt{1-4\beta}}{2}$. So, $0 < f_\beta(x) \leq \frac{1-\sqrt{1-4\beta}}{2}$, Similarly, $f_\beta^n(x) = \frac{\beta}{1-f_\beta(x)}$ attains its maximum when $f_\beta(x) = \frac{1-\sqrt{1-4\beta}}{2}$, so we get $0 < f_\beta^n(x) \leq \frac{1-\sqrt{1-4\beta}}{2}$, With the same way, we get $0 < f_\beta^n(x) \leq \frac{1-\sqrt{1-4\beta}}{2}$, for all $n \geq 3$. On the other hand, if $0 < f_\beta^n(x) < \frac{1-\sqrt{1-4\beta}}{2}$, we can easily check that $f_\beta^n(x) - f_\beta^{n-1}(x) = \frac{1-f_\beta^{-1}(x)}{1-f_\beta^{-1}(x)} - f_\beta^{-1}(x) = \frac{-f_\beta^{-1}(x)+f_\beta^{-1}(x)^2}{1-f_\beta^{-1}(x)} > 0$ for all $n \geq 2$. So, $f_\beta^{n-1}(x) < f_\beta^n(x)$ for all $n \geq 2$. Thus, we get $\lim_{n \to \infty} f_\beta^n(x) = f_\beta(x)$, and by $f_\beta^n(x) = \frac{\beta}{1-f_\beta^{-1}(x)}$, we get $f_0(x) = \frac{1-\sqrt{1-4\beta}}{2}$. The proof of item 2 is very similar to the proof of item 1, so we omit the proof here.

Lemma 7. Let $f_\beta(x) = \frac{\beta}{1-x}$ and $f_\beta^n(x) = f(f_\beta^{n-1}(x))$ for $n \geq 2$, then for any positive integer $n$, and any real $\beta \in (0, \frac{1}{4})$, there exists a unique $x \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ such that $f_\beta^n(x) = 1 - x$.

Proof. Consider the set $F$ of functions $f$ satisfying

1. $f$ is an increasing continuous function in $(\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$.

2. Both $\frac{1-\sqrt{1-4\beta}}{2}$ and $\frac{1+\sqrt{1-4\beta}}{2}$ are fixed points of $f$.

We claim that for any $f \in F$ there exists a unique $x \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ such that $f(x) = 1 - x$. This is because $g(x) := f(x)+x$ is a strictly increasing and continuous function in $(\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ and

$$g\left(\frac{1-\sqrt{1-4\beta}}{2}\right) = 1 - \sqrt{1-4\beta} < 1, \quad g\left(\frac{1+\sqrt{1-4\beta}}{2}\right) = 1 + \sqrt{1-4\beta} > 1.$$ 

It suffices to show $f_\beta^n(x) \in F$ for any positive integer $m$. This can be proved by induction on $m$. For $m = 1$, $f_\beta(\beta) = f_\beta(\beta) \in F$ can be easily verified. Now we assume $f_\beta^m \in F$. Note both $f_\beta$ and $f_\beta^m$ map $(\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ to $(\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ increasingly and continuously to itself. So is their composition, $f_\beta \circ f_\beta^m = f_\beta^{m+1}$. We finished the proof.

Lemma 8. Let the following graph denote $F_{m,n,k}^{(3)}$,

and the spectral radius of $F_{m,n,k}^{(3)}$ be $\rho(F_{m,n,k}^{(3)})$. Then, when $m, n, k \to \infty$,

$$\lim_{m, n, k \to \infty} \rho(F_{m,n,k}^{(3)}) = 2 \sqrt{2 + \sqrt{5}}.$$ 

Proof. We label this graph as follows

\[ \text{Diagram of graph} \]

...
Let $\beta$ be a real number in $(0, \frac{1}{2})$, chosen later. Set $z_1 = 1 - x_n = 1 - f_\beta^{n-1}(\beta)$, $z_2 = 1 - f_\beta^{k-1}(\beta)$, and $z_3 = 1 - f_\beta^{m-1}(\beta)$. Now let $\beta_{m,n,k}$ be the solution of

\[z_1z_2z_3 = \beta.\]

We get a $\beta_{n,m,k}$-normal labeling. Thus $\rho(F_{m,n,k}^{(3)}) = 2\beta_{n,m,k}^{-1/3}$. By the first item of Lemma 6 for a fixed $\beta$, note that all $z_i$’s decreasingly approach to $\frac{1 + \sqrt{1 - 43}}{2}$. We conclude that $\beta_{m,n,k}$ are decreasing functions of each $m$, $n$, and $k$. The limit $\lim_{m,n,k \to \infty} \beta_{m,n,k}$ must exist and is the solution of

\[
\left(\frac{1 + \sqrt{1 - 43}}{2}\right)^3 = \beta.
\]

By simple calculus, we get this limit $\beta = \sqrt{5} - 2$. By Lemma 2 we get $\lim_{m,n,k \to \infty} \rho(F_{m,n,k}^{(3)}) = 2\sqrt{2 + \sqrt{5}}$.

Taking $\rho' = (r - 1)!\sqrt{2 + \sqrt{5}}$, we have the following lemma.

**Lemma 9.** For $r \geq 3$, let $H$ be an $r$-uniform hypergraph with spectral radius $\rho(H) \leq \rho'_r$. If $H$ is not simple, then $H = C_2^{(r)}$ (i.e., the hypergraph consists of two edges sharing two common vertices).

**Proof.** In [27], we have shown that $\rho_r(C_2^{(r)}) = (r - 1)!\sqrt{2 + \sqrt{5}} < \rho'_r$.

Since $H$ is not simple, $H$ contains two edges $F_1$ and $F_2$ sharing $s$ vertices for some $s \geq 2$.

If $s \geq 3$, call the subgraph consisting of the two edges $F_1, F_2$, $C_{s+}^{(r)}$. Define a weighted incident matrix $B$ of $C_{s+}^{(r)}$ as follows: for any vertex $v$ and edge $e$ (called the other edge $e'$),

\[
B(v, e) = \begin{cases} 
1 & \text{if } v \in e \cap e', \\
1 & \text{if } v \in e \setminus e', \\
0 & \text{otherwise.}
\end{cases}
\]

It is easy to check that when $s \geq 3$ we have $(\frac{1}{2})^s < 0.1251 < \beta$, so $C_{s+}^{(r)}$ is consistently $\beta$-supernormal and thus $\rho(H) \geq \rho(C_{s+}^{(r)}) > \rho'_r$. Contradiction!

Thus, $F_1$ and $F_2$ can only share 2-common vertices. Since $H$ is connected and $H \neq C_2^{(r)}$, there is a third edge $F_3$ having non-empty intersection with $F_1 \cup F_2$. Since identifying the vertices will not change the sub-homomorphic type, we can only consider the two sub-homomorphic types: $C_{2+}^{(r)}$ and $C_{2+}^{(r)}$. Here both the hypergraphs $C_{2+}^{(r)}$ and $C_{2+}^{(r)}$ consist of three edges $F_1, F_2, F_3$ where $|F_1 \cap F_2| = 2$ and $|F_3 \cap (F_1 \cup F_2)| = 1$. The difference is that in $C_{2+}^{(r)}$, $F_3 \cap (F_1 \cup F_2) \in F_1 \cap F_2$ while in $C_{2+}^{(r)}$, $F_3 \cap (F_1 \cup F_2) \in F_1 \Delta F_2$ the symmetric difference of $F_1$ and $F_2$. The below are the figures of $C_{2+}^{(3)}$ and $C_{2+}^{(3)}$. 

...
To draw the contradiction, it is sufficient to show $\rho_r(C_{2+}^{(r)}) > \rho'_r$ and $\rho_r(C'_{2+}^{(r)}) > \rho'_r$ (this implies $\rho(H) > \rho'_r$ by Lemma 3). Observe that $C_{2+}^{(r)}$ is extended from $C_{2+}^{(3)}$ and $C'_{2+}^{(r)}$ is extended from $C'_{2+}^{(3)}$. We only need to show that both $C_{2+}^{(3)}$ and $C'_{2+}^{(3)}$ are consistently strict $\beta$-supernormal. We label the two hypergraphs as follows:

In $C_{2+}^{(3)}$, we set the labels $y_1 = \beta$, $y_2 = y_3 = \frac{1-\beta}{2}$, and $y_4 = y_5 = \frac{\beta}{1-\beta}$. Since $y_4 + y_5 \approx 1.2361 > 1$, this is a consistently $\beta$-supernormal labelling.

In $C'_{2+}^{(3)}$, we set $x_1 = \beta$, $x_2 = 1-\beta$, $x_3 = x_6 = \sqrt{\frac{\beta}{1-\beta}}$, and $x_4 = x_5 = \sqrt{\beta}$. Since $x_3 + x_4 = x_5 + x_6 \approx 1.0418 > 1$, this is a consistently $\beta$-supernormal labelling.

3 Proof of Theorem 1

Proof. It suffices to consider irreducible hypergraphs. Assume that $H$ is an irreducible 3-uniform hypergraph with $\rho(H) \leq 2\sqrt{2 + \sqrt{5}}$. We need to show that $H$ has certain forbidden structures. The idea is to show these forbidden subgraphs have some (consistently, if not a hypertree) $(\sqrt{5} - 2)$-supernormal labelings. To simplify our notation, we write $\beta = \sqrt{5} - 2$ in this proof. By Lemma 9 when $r = 3$, we only need to consider $H$ is simple.

Case 1. If $\exists v \in V(H)$, such that $d_v \geq 5$, then $H$ contains $S_{5}^{(3)}$ that has been labeled as follows.

By the symmetry, we only label one branching. We can check $5\beta \approx 1.1803 > 1$, so, by Lemma 11 and Lemma 4 we get $\rho(H) > \rho'_r$. Thus we can assume that every vertex in $H$ has degree at most 4.

If $\exists v \in V(H)$, such that $d_v = 4$, and $H$ contains graph $S_{4}^{(3)}$ that has been labeled as follows,
where \( x_1 = \beta, \ x_2 = 1 - \beta, \ x_3 = \frac{\beta}{2}, \ x_4 = x_5 = x_6 = \beta \). We can check that \( x_3 + x_4 + x_5 + x_6 \approx 1.0172 > 1 \), so, by Lemma 11 and Lemma 12 we get \( \rho(H) > \rho(S_4^{(3)}) > \rho_3' \). Thus, since \( \rho(S_4^{(3)}) = \rho_3 \) and \( \rho(S_4^{(3)}) > \rho_3' \), so if \( H \) is irreducible, we can assume that every vertex in \( H \) has degree at most 3.

**Case 2.** The hypergraph \( H \) contains a cycle, saying \( C_n^{(3)} \). Since \( \rho(C_n^{(3)}) = 2\sqrt{3} \) (see [27]), we may assume \( H \) contains at least one edge \( F \) not on the cycle \( C_n^{(3)} \) (but attached to \( C_n^{(3)} \)). First we prove that \( F \) can be only attached to the cycle through a branching edge, not a branching vertex, otherwise, \( H \) contains a sub-homomorphic type \( C_n^{(3)}+ \) shown as follows:

![Diagram](image1)

This graph is reducible and can be extended from the following 2-graph \( C_{n+}^{(2)} \):

![Diagram](image2)

The graph \( C_{n+}^{(2)} \) is not in the list of Brouwer and Neumaier (see Page 1). Thus, \( \rho(C_{n+}^{(2)}) > \sqrt{2 + \sqrt{5}} \). Applying Corollary 11 we get \( \rho(C_{n+}^{(3)}) > 2\sqrt{2 + \sqrt{5}} \). Contradiction!

Thus, \( F \) must be attached to the cycle through a branching edge. Considering that we walk away from the cycle through this edge \( F \), we have the following subcases.

1. Eventually, the path leaving at \( F \) reaches a branching vertex. In this subcase, \( H \) contains the following sub-homomorphic type \( C_{n+}^{(3)} \):

![Diagram](image3)

By Lemma 11 there exists a \( x_1 \in (\frac{1-\sqrt{1-4\beta^2}}{2}, \frac{1+\sqrt{1-4\beta^2}}{2}) \) satisfying \( f_\beta^n(x_1) = 1 - x_1 \). Now \( x_n = f_\beta^n(x_1) = 1 - x_1 \). (This symmetry guarantees the labeling is consistent.) So, \( z_1 = 1 - x_n = x_1 \).

We set \( y_1 = y_2 = \beta, \ y_i = f_\beta^{i-2}(2\beta) \) for \( 3 \leq i \leq m \). Since \( \frac{1-\sqrt{1-4\beta^2}}{2} < 2\beta < \frac{1+\sqrt{1-4\beta^2}}{2} \), by Lemma 11 we get that \( y_i \) is decreasing and the limit goes to \( \frac{1-\sqrt{1-4\beta^2}}{2} \). In particular, \( y_m > \frac{1-\sqrt{1-4\beta^2}}{2} \). It implies \( z_2 = 1 - y_m < \frac{1+\sqrt{1-4\beta^2}}{2} \). Therefore, we have

\[
x_1 \cdot z_1 \cdot z_2 < \left( \frac{1 + \sqrt{1-4\beta^2}}{2} \right)^3 = \beta.
\]

Thus, \( C_{n+}^{(3)} \) is consistently \( \beta \)-supernormal. So, we have \( \rho(H) \geq \rho(C_{n+}^{(3)}) > \rho_3 \). Contradiction!
2. Eventually, the path leaving at $F$ reaches a branching edge. In this subcase, $H$ contains the following sub-homomorphic type $\mathcal{C}_{n+}^{\mu(3)}$:

![Diagram of $\mathcal{C}_{n+}^{\mu(3)}$]

This is very similar to the previous subcase. By Lemma 7, there exists a $x_1 \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ satisfying $f_{\beta}^m(x_1) = 1 - x_1$. Now $x_n = f_{\beta}^m(x_1) = 1 - x_1$. (This symmetry guarantees the labeling is consistent.) So, $z_1 = 1 - x_n = x_1$. We set $k_1 = k_2 = \beta$, $y_1 = \frac{\beta}{(1-\beta)^2}$, and $y_i = f_{\beta}^{i-1}(y_1)$ for $2 \leq i \leq m$. Since $\frac{1-\sqrt{1-4\beta}}{2} < \frac{\beta}{(1-\beta)^2} < \frac{1+\sqrt{1-4\beta}}{2}$, by Lemma 8 we get that $y_i$ is decreasing and the limit goes to $\frac{1-\sqrt{1-4\beta}}{2}$. In particular, $y_m > \frac{1-\sqrt{1-4\beta}}{2}$. It implies $z_2 = 1 - y_m < \frac{1+\sqrt{1-4\beta}}{2}$. Therefore, we have

$$x_1 \cdot z_1 \cdot z_2 < \left(\frac{1 + \sqrt{1-4\beta}}{2}\right)^3 = \beta.$$ 

Thus, $\mathcal{C}_{n+}^{\mu(3)}$ is consistently $\beta$-supernormal. So, we have $\rho(H) \geq \rho(\mathcal{C}_{n+}^{\mu(3)}) > \rho'_3$. Contradiction!

3. Eventually, the path leaving at $F$ returns to the cycle. In this subcase, $H$ contains subgraph $\Theta(m_1, m_2, m_3)$, which can be obtained by connecting three pairs of vertices between two branching edges using three paths of lengths $m_1$, $m_2$, and $m_3$ respectively.

![Diagram of $\Theta(m_1, m_2, m_3)$]

By Lemma 7 for $i = 1, 2, 3$, there exists a $x_i \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ satisfying $f_{\beta}^{m_i}(x_i) = 1 - x_i$. We label $x_1$, $x_2$, and $x_3$ on the $\Theta(m_1, m_2, m_3)$ and extend these labels on path $P_{m_i}$ naturally. The definition of $x_i$ makes the labelings on $P_{m_i}$ symmetric and this symmetry guarantees the labeling is consistent. Note

$$x_1 x_2 x_3 < \left(\frac{1 + \sqrt{1-4\beta}}{2}\right)^3 = \beta.$$ 

This is consistently $\beta$-supernormal and this implies

$$\rho(H) \geq \rho(\Theta(m_1, m_2, m_3)) > \rho'_3.$$ 

Contradiction!

4. This is the remaining subcase: $H$ contains a cycle $C$ with several path attached to $C$. So $H$ is a closed quipu as stated in the theorem.
Case 3. We assume that $H$ is a hypertree, and let the following partial hypergraphs denote $H^{(3)}_1$ and $H^{(3)}_2$ that correspond to the branching vertex and the branching edge structure respectively.

In graph $H^{(3)}_1(n)$, we set $x_1 = x_2 = \beta$, $y_1 = \frac{\beta}{1 - 2\beta} = f_\beta(2\beta)$, $y_n = f_\beta^n(2\beta)$. Since $2\beta \in (\frac{1 - \sqrt{1 - 4\beta}}{2}, \frac{1 + \sqrt{1 - 4\beta}}{2})$, by Lemma 6, we get that $y_n = f_\beta^n(2\beta) > 1 - \sqrt{1 - 4\beta}$.

In graph $H^{(3)}_2(n)$, we set $x_1 = x_2 = \beta$, $h_1 = h_2 = 1 - \beta$, $h_3 = \frac{\beta}{(1 - \beta)^2}$. We can check that $h_3 \in (\frac{1 - \sqrt{1 - 4\beta}}{2}, \frac{1 + \sqrt{1 - 4\beta}}{2})$. Since $q_n = f_\beta^n(h_3)$, and thus by Lemma 6, we get $q_n > 1 - \sqrt{1 - 4\beta}$.

To show $H$ must be an open quipu as stated in the theorem, we need exclude the following structures. First, suppose that there is a branching vertex in the middle of $H$, and $H$ contains the following subgraph,

where $G_1$ and $G_2$ are chosen from $H^{(3)}_1(n)$ and $H^{(3)}_2(n)$ (for some $n \geq 0$) and pieces are glued through red nodes. We can get

$$z_1 + z_2 + \beta > \frac{1 - \sqrt{1 - 4\beta}}{2} + \frac{1 - \sqrt{1 - 4\beta}}{2} + \beta = 1.$$  

This is a supernormal labeling of this subgraph. Thus, $\rho(H) > \rho'_3$. Contradiction!

If $H$ contains one branching edge, whose all three branches are not paths, then $H$ contains the following subgraph,

where $K_1$, $K_2$ and $K_3$ are chosen from $H^{(3)}_1(n)$ and $H^{(3)}_2(n)$ (for some $n \geq 0$) and pieces are glued through red nodes. Similar to the previous case, for $i = 1, 2, 3$, by Lemma 6 we can get $z_i < \frac{1 + \sqrt{1 - 4\beta}}{2}$. Thus,

$$z_1 \cdot z_2 \cdot z_3 < \left( \frac{1 + \sqrt{1 - 4\beta}}{2} \right)^3 = \beta.$$  

This is a supernormal labeling of this subgraph. So, we have $\rho(H) > \rho'_3$. Contradiction! Therefore $H$ must be an open quipu as stated in the theorem.
4 Proof of Theorem

Proof. Let $H$ be an irreducible 4-uniform hypergraph with $\rho(H) \leq \rho'_4 = 6\sqrt{2 + \sqrt{5}}$. If $H$ is not simple, then it must be $C^{(4)}_2$ by Lemma 9. Now we consider $H$ is simple.

Case 1. $H$ contains a cycle $C$. Since $H$ is irreducible, it also has an edge $F$ which contains no leaf vertex. We consider the following two subcases.

1. The edge $F$ is on the cycle $C$. The $H$ contains the following sub-isomorphic type:

By Lemma 7 there exists a $x_1 \in (\frac{1-\sqrt{1-4\beta^2}}{2}, \frac{1+\sqrt{1-4\beta^2}}{2})$ satisfying $f_{1}^n(x_1) = 1 - x_1$. Now $x_n = f_{1}^n(x_1) = 1 - x_1$. (This symmetry guarantees the labeling is consistent.) So, $z_1 = 1 - x_n = x_1$.

We set $y_1 = y_2 = \beta$, $z_2 = z_3 = 1 - \beta$, and we can check that $x_1 \cdot z_1 \cdot z_2 \cdot z_3 < \left(\frac{1+\sqrt{1-4\beta^2}}{2}\right)^2 \cdot (1 - \beta)^2 \approx 0.2229 < \beta$, and thus $C^{(4)}_n$ is $\beta$-supernormal. So we have $\rho(H) \geq \rho(C^{(4)}_n) > \rho'_4$.

2. If $F$ is not on $C$, there is a path connecting $F$ to $C$. Thus, $H$ has the following sub-homomorphic type:

As above, there exists a $x_1 \in (\frac{1-\sqrt{1-4\beta^2}}{2}, \frac{1+\sqrt{1-4\beta^2}}{2})$ and $z_1 = x_1$. We set $x_1 = x_2 = x_3 = \beta$, $q_1 = \frac{\beta}{(1-\beta)^m}$, and we can check $q_1 \in (\frac{1-\sqrt{1-4\beta^2}}{2}, \frac{1+\sqrt{1-4\beta^2}}{2})$. We set $q_m = f_{m-1}^n(q_1)$, and thus by Lemma 6 we get $q_m$ decreases with $m$, and when $m \to \infty$, we get $q_m > \frac{1-\sqrt{1-4\beta^2}}{2}$. So $z_2 = 1 - q_m < \frac{1+\sqrt{1-4\beta^2}}{2}$. We can check that $x_1 \cdot z_1 \cdot z_2 < \left(\frac{1+\sqrt{1-4\beta^2}}{2}\right)^3 = \beta$, and thus $C^{(4)}_n$ is $\beta$-supernormal. So we have $\rho(H) \geq \rho(C^{(4)}_n) > \rho'_4$.

Case 2. $H$ is a hypertree but not a 4-dagger. To get the open quipu structures, we need forbid certain subhypergraphs.

The following partial hypergraphs $H^{(4)}_1(n)$ and $H^{(4)}_2(n,j)$ (for $j = 0, 1, 2, 3$) correspond to the branching vertex and the branching edge structure respectively.
Claim (a): Both $H_1^{(4)}(n)$ and $H_2^{(4)}(n,j)$ (for $j = 0, 1, 2, 3$) admit a $\beta$-supernormal labeling such that the label at the corner of the red vertex is greater than $\frac{1 - \sqrt{1 - 4\beta}}{2}$.

**Proof of Claim (a):** We will label the partial graphs so that the $\beta$-normal properties hold except at the corner of the red vertex. In graph $H_1^{(4)}(n)$, we set $x_1 = x_2 = \beta$, $y_1 = \frac{f_1}{\beta} = f_1(2\beta)$, $y_n = f_n^{(2\beta)}$. Since $2\beta \in \left(\frac{1 - \sqrt{1 - 4\beta}}{2}, \frac{1 + \sqrt{1 - 4\beta}}{2}\right)$, by Lemma 6, we get that $y_n = f_n^{(2\beta)} > \frac{1 - \sqrt{1 - 4\beta}}{2}$.

In graph $H_2^{(4)}(n,j)$, we set $x_1 = x_2 = c_1 = \beta$, $h_1 = h_2 = 1 - \beta$, $c_j = f_j^{(1-\beta)}(\beta)$. When $j = 0$, we have $h_3 = 1$ and $h_4 = \frac{\beta}{h_1 h_2 h_3} = \frac{\beta}{(1-\beta)^2}$. When $j = 1$, we have $h_3 = 1 - \beta$ and $h_4 = \frac{\beta}{h_1 h_2 h_3} = \frac{\beta}{(1-\beta)^3}$. When $j = 2$, we have $h_3 = 1 - c_2 = \frac{1 - 2\beta}{1 - \beta}$ and $h_4 = \frac{\beta}{h_1 h_2 h_3} = \frac{\beta}{(1-\beta)(1-2\beta)}$. When $j = 3$, we set $h_3 = 1 - c_3 = \frac{1 - 3\beta + \beta^2}{1 - 2\beta}$ and $h_4 = \frac{\beta}{h_1 h_2 h_3} = \frac{\beta(1-2\beta)}{(1-\beta^3)(1-2\beta)}$. We can check directly that for all $j = 0, 1, 2, 3$, the value $h_4 \in \left(\frac{1 - \sqrt{1 - 4\beta}}{2}, \frac{1 + \sqrt{1 - 4\beta}}{2}\right)$. Since $q_n = f_n^{(h_4)}$, and thus by Lemma 6, we get $q_n > \frac{1 - \sqrt{1 - 4\beta}}{2}$.

To show $H$ must be an open quipu as stated in the theorem, we need exclude the following structures.

1. We first show that all branching vertices and branching edges lie on the same path denoted by $P$. Otherwise, $H$ contains the following subhypergraph.

   ![Diagram](image)

   Where $U_1$, $U_2$ and $U_3$ are chosen from $H_1^{(4)}(n)$ (for some $n \geq 0$) and $H_2^{(4)}(n,j)$ (for some $n \geq 0$ and $j = 0, 1, 2, 3$) and pieces are glued through red nodes.

   From Claim (a), we have $z_1 \cdot z_2 \cdot z_3 \cdot 1 < (\frac{1 + \sqrt{1 - 4\beta}}{2})^3 \cdot 1 = \beta$. So, this subhypergraph is $\beta$-supernormal. It implies $\rho(H) > \rho_4$.

2. Now we show that any branch vertex must lie at the end of that path $P$. Otherwise, $H$ contains the following subhypergraph.

   ![Diagram](image)

   Where $U_4$ and $U_5$ are chosen from $H_1^{(4)}(n)$ (for some $n \geq 0$) and $H_2^{(4)}(n,j)$ (for some $n \geq 0$ and $j = 0, 1, 2, 3$) and pieces are glued through red nodes. From Claim (a), we have $z_1 + z_2 + z_3 > \frac{1 - \sqrt{1 - 4\beta}}{2} + \frac{1 - \sqrt{1 - 4\beta}}{2} + \beta = 1$. So, this subhypergraph is $\beta$-supernormal. It implies $\rho(H) > \rho_4$.
3. Now we show that any branch edge must also lie at the end of that path $P$. Otherwise, $H$ contains the following subhypergraph.

\[
\begin{align*}
\beta & \quad z_1 \\
\beta & \quad z_2 \\
\beta & \quad z_3 \\
\beta & \quad z_4
\end{align*}
\]

where $U_6$ and $U_7$ are chosen from $H^4_1(n)$ (for some $n \geq 0$) and $H^4_2(n,j)$ (for some $n \geq 0$ and $j = 0, 1, 2, 3$) and pieces are glued through red nodes. We have

\[
z_1 \cdot z_2 \cdot z_3 \cdot z_4 < \left( \frac{1 + \sqrt{1 - 4\beta^2}}{2} \right)^2 \cdot (1 - \beta)^2 \approx 0.2229 < \beta.
\]

This subhypergraph is $\beta$-supernormal. Thus we have $\rho(H) > \rho'_r$. Contradiction.

4. It remains to show that each 4-branching edge is attached by three paths of length 1, 1, and $k$ ($k = 1, 2, 3$) respectively if it is not a 4-dagger. Otherwise, it contains one of the following two hypergraphs as a subhypergraph.

\[
\begin{align*}
\beta & \quad x_1 \\
\beta & \quad x_2 \\
\beta & \quad y_1 \\
\beta & \quad y_2
\end{align*}
\]

\[
\begin{align*}
\beta & \quad q_1 \\
\beta & \quad q_2 \\
\beta & \quad q_3 \\
\beta & \quad q_4
\end{align*}
\]

where $U_8$ and $U_9$ are chosen from $H^4_1(n)$ (for some $n \geq 0$) and $H^4_2(n,j)$ (for some $n \geq 0$ and $j = 0, 1, 2, 3$) and pieces are glued through red nodes.

For the left hypergraph, we set $x_1 = y_1 = y_3 = \beta$, $x_2 = y_2 = \frac{\beta}{1 - \beta}$, and $z_1 = z_2 = \frac{1 - 2\beta}{1 - \beta}$, and $z_4 < \frac{1 + \sqrt{1 - 4\beta^2}}{2}$ (from Claim (a)). Thus, the product of labels on the branching edge is

\[
z_1 \cdot z_2 \cdot z_3 \cdot z_4 < \left( \frac{1 + \sqrt{1 - 4\beta^2}}{2} \right) \cdot \left( \frac{1 - 2\beta}{1 - \beta} \right)^2 \cdot (1 - \beta) \approx 0.2254 < \beta.
\]

For the right hypergraph, we set $q_1 = x_1 = x_2 = \beta$, $q_i = f^{-1}_\beta(\beta)$ ($i = 2, 3, 4$), $z_1 = 1 - q_4 = \frac{1 - 4\beta + 3\beta^2}{1 - 3\beta + \beta^2}$, $z_2 = z_3 = 1 - \beta$, and $z_4 < \frac{1 + \sqrt{1 - 4\beta^2}}{2}$ (from Claim (a)).

Thus, the product of labels on the branching edge is

\[
z_1 \cdot z_2 \cdot z_3 \cdot z_4 < \frac{1 + \sqrt{1 - 4\beta^2}}{2} \cdot \frac{1 - 4\beta + 3\beta^2}{1 - 3\beta + \beta^2} \cdot (1 - \beta)^2 \approx 0.2314 < \beta.
\]

Thus the both hypergraphs above are $\beta$-supernormal. Thus we have $\rho(H) > \rho'_r$. Contradiction. Therefore $H$ must be an open quipu as stated in the theorem.
It is easy to verify that $g$ also easy to verify that those 4-daggers are $\beta$-normal properties hold except the product of the labels at the branching edge. Not that the product of the labels at the branching edge, denoted by $g(i, j, k, l)$, is given by

$$g(i, j, k, l) = f_\beta^{j-1}(\beta)f_\beta^{j-1}(\beta)f_\beta^{k-1}(\beta)f_\beta^{l-1}(\beta).$$

It is easy to verify that $g(i, j, k, l) < \beta$ for $(i, j, k, l) = (2, 2, 2, 2), (1, 2, 2, 4), (1, 2, 3, 3), (1, 1, 5, 5), (1, 1, 4, 6)$. $H$ cannot contain those 4-daggers as a subhypergraph. Therefore, $H$ must be one of the following hypergraphs $H_{1,2,2,2}^{(4)}, H_{1,2,2,3}^{(4)}, H_{1,1,4,4}^{(4)}, H_{1,1,4,5}^{(4)}$, and $H_{1,1,k,l}^{(4)} (1 \leq k \leq 3, \text{and } k \leq l)$. It is also easy to verify that those 4-daggers are $\beta$-subnormal. So this is a complete list of 4-daggers with $\rho(H) < \rho'_r$.

$\square$

5 Proof of Theorem 3

Proof. Let the edge-star $S_r^{(r)}$ be the $r$-uniform hypergraph consisting of $r+1$ edges: $F_0 = \{v_1, v_2, \ldots, v_r\}, F_1, \ldots, F_r$, where each $F_i \cap F_0 = \{v_i\}$ for $1 \leq i \leq r$, and $F_i \cap F_j = \emptyset$ for $1 \leq i \leq j \leq r$. (See the picture of $S_5^{(5)}$ at Theorem 3)

We first show that $\rho_r(S_r^{(r)}) > \rho'_r$ for $r \geq 6$. This can be done by assigning $B(v_i, F_i) = \beta$ and $B(v_i, F_0) = 1 - \beta$, for $1 \leq i \leq r$. Note that the product of labels on $F_0$ is

$$(1 - \beta)^r < \beta$$

for all $r \geq 6$. Thus, $S_r^{(r)}$ is $\beta$-supernormal. If there is an irreducible $r$-uniform hypergraph $H$ with $\rho(H) \leq \rho'_r$ for $r \geq 6$, then $H$ contains a sub-homomorphic type $S_r^{(r)}$. By Lemma 3, we have $\rho(H) \geq \rho(S_r^{(r)}) > \rho'_r$, contradiction.

The same argument shows that $S_5^{(5)}$ is $\beta$-subnormal. Let $H$ be an irreducible 5-uniform hypergraph $H$ with $\rho(H) \leq \rho_5'$. If $H$ is not $S_5^{(5)}$, $H$ contains one of the following sub-homomorphic types $S_5^{(5)}$, and $S_{5^+}^{(5)}$.

For $S_5^{(5)}$, we can label the corner of the only identified vertex not on the branching edge by $\frac{1}{7}$, and set $x_1 = x_2 = 1 - 2\beta, x_3 = x_4 = x_5 = 1 - \beta$. We can check that the product of labels on the branching edge is

$$x_1x_2x_3x_4x_5 = (1 - 2\beta)^2(1 - \beta)^3 \approx 0.1242 < \beta.$$ 

For $S_{5^+}^{(5)}$, we can set $y_1 = 1 - f_\beta(\beta) = \frac{1 - 2\beta}{1 - \beta}, y_2 = y_3 = y_4 = y_5 = 1 - \beta$. We can check that the product of labels on the branching edge is

$$y_1y_2y_3y_4y_5 = (1 - 2\beta)(1 - \beta)^4 \approx 0.1798 < \beta.$$ 

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Thus, both $S_{5}^{(5)}$ and $S_{5+}^{(5)}$ are consistently $\beta$-supernormal. This implies that $\rho(H) > \rho_{5}^{'5}$, contradiction. Thus $H$ must be the five edge-star.

6 Constructing open quipus and closed quipus with $\rho(H) \leq (r - 1)! \sqrt[2]{2 + \sqrt[5]{5}}$

In this paper, we give a description of the connected $r$-uniform hypergraphs with spectral radius at most $(r - 1)! \sqrt[2]{2 + \sqrt[5]{5}}$: they are extended from the irreducible ones listed in Theorems [13] and the 2-graphs listed by Cvetković et al [6] and Brouwer-Neumaier [1]. This is not a complete description for $r \geq 3$, but rather a coarse description. The scenario is similar to the results of Woo and Neumaier on the graphs with spectral radius at most $\frac{3}{2} \sqrt[2]{2}$ (see [26]). Our method is very different from the linear algebra method used by Woo and Neumaier. In fact, it is possible to simply the proof of Woo-Neumaier’s result using our new method but we will omit it here.

In the rest of this section, we will construct many examples with $\rho(H) \leq (r - 1)! \sqrt[2]{2 + \sqrt[5]{5}}$. This shows that the descriptions in Theorem 1-3 are somewhat tight.

The 4-daggers are completely classified so no construction is needed. We only need to construct closed 3- quipus, open 3-quipus and open 4-quipus first. The idea is to present some partial hypergraphs, which can be glued together to form a hypergraph with $\rho(H) \leq (r - 1)! \sqrt[2]{2 + \sqrt[5]{5}}$. A partial $r$-uniform hypergraph is an $r$-uniform hypergraph together with (one or two) designated vertex/vertices. A partial hypergraph $H$ is called $\alpha$-subnormal if there exists a weighted incidence matrix $B$ satisfying

1. $\prod_{v \in e} B(v, e) \geq \alpha$, for any $e \in E(H)$.
2. $\sum_{e: v \in e} B(v, e) \leq \frac{1}{2}$, for any designated vertex $v$,
3. $\sum_{e: v \in e} B(v, e) \leq 1$, for any non-designated vertex.

Lemma 10. Consider the following partial hypergraphs $G_{1}^{(3)}(m, k_{1}, k_{2})$, $G_{2}^{(2)}(m, k)$, and $G_{3}^{(4)}(t, k)$ (with designated vertices colored in red). We have

1. For any $m \geq 1$, there exists a $k_{0}$ such that for any $k_{1}, k_{2} \geq k_{0}$, $G_{1}^{(3)}(m, k_{1}, k_{2})$ is $(\sqrt[5]{5} - 2)$-subnormal.
2. For any $m \geq 1$, there exists a $k_{0}$ such that for any $k \geq k_{0}$, $G_{2}^{(2)}(m, k)$ is $(\sqrt[5]{5} - 2)$-subnormal.
3. For any $t = 1, 2, 3$, there exists a $k_{t}$ such that for any $k \geq k_{t}$, $G_{3}^{(4)}(t, k)$ is $(\sqrt[5]{5} - 2)$-subnormal.
Proof. We label the corner of the designated vertices by $\frac{1}{2}$ and the corner of other leaf-vertices by 1. We try to maintain the properties that the product of all labels in one edge is $\beta$ and the sum of all labels at one vertex is 1 except at the branching vertex or at the branching edge. We get the labels of the three partial graphs as follows

Now we consider the first partial hypergraph $G_1^{(3)}$. Using the function $f_\beta$, we have $x_1 = 1 - f_{\beta}^{m-1}(\beta)$, $x_2 = 1 - f_{\beta}^{k_1-1}(2\beta)$, and $x_3 = 1 - f_{\beta}^{k_2-1}(2\beta)$. The product of the labels on the central branching edges, denoted by $g(m, k_1, k_2)$, satisfies

$$g(m, k_1, k_2) = x_1x_2x_3 = (1 - f_{\beta}^{m-1}(\beta))(1 - f_{\beta}^{k_1-1}(2\beta))(1 - f_{\beta}^{k_2-1}(2\beta)).$$

By Lemma 6, $1 - f_{\beta}^{m-1}(\beta) > \frac{1 + \sqrt{1 - 4\beta}}{2}$, and $\lim_{k_1 \to \infty}(1 - f_{\beta}^{k_1-1}(2\beta)) = \lim_{k_2 \to \infty}(1 - f_{\beta}^{k_2-1}(2\beta)) = \frac{1 + \sqrt{1 - 4\beta}}{2}$ since $2\beta \in \left(\frac{1 - \sqrt{1 - 4\beta}}{2}, \frac{1 + \sqrt{1 - 4\beta}}{2}\right)$. Thus,

$$\lim_{k_1, k_2 \to \infty} g(m, k_1, k_2) > \left(\frac{1 + \sqrt{1 - 4\beta}}{2}\right)^3 = \beta.$$

There exists a $k_0$ such that for $k_1, k_2 \geq k_0$, $g(m, k_1, k_2) > \beta$. I.e., $G_1^{(3)}$ is $\beta$-subnormal.

Similar argument works for the graph $G_2^{(2)}$. We have $y_1 = f_{\beta}^{m-1}(\beta)$ and $y_2 = f_{\beta}^{k_1-1}(2\beta)$. The sum of the labels at the branching vertex is

$$\beta + y_1 + y_2 = \beta + f_{\beta}^{m-1}(\beta) + f_{\beta}^{k_1-1}(2\beta).$$

Note that the limit of this sum as $k$ goes to the infinity satisfies

$$\lim_{k \to \infty} (\beta + f_{\beta}^{m-1}(\beta) + f_{\beta}^{k_1-1}(2\beta)) < \beta + \frac{1 - \sqrt{1 - 4\beta}}{2} + \frac{1 - \sqrt{1 - 4\beta}}{2} = 1.$$

Thus, there exists a $k_0 = k_0(m)$ such that for any $k \geq k_0$, we get $y_1 + y_2 + \beta < 1$. So $G_2^{(2)}$ is $\beta$-subnormal.

In graph $G_3^{(4)}(t, k)$, we have $z_1 = z_2 = 1 - \beta$, $z_3 = 1 - f_{\beta}^{t-1}(\beta)$, $z_4 = 1 - f_{\beta}^{k-1}(2\beta)$. The product of the labels at the branching edge is

$$z_1z_2z_3z_4 = (1 - \beta)^2(1 - f_{\beta}^{t-1}(\beta))(1 - f_{\beta}^{k-1}(2\beta)).$$
For each $t = 1, 2, 3$, it is easy to check

$$(1 - \beta)^2 (1 - f_{\beta}^{t-1}(\beta)) \frac{1 + \sqrt{1 - 4\beta^2}}{2} < \beta.$$ 

There exists a $k_t$ such that for any $k \geq k_t$, $G_3^{(4)}$ is $\beta$-subnormal.

The extension also works for partial hypergraphs: add one vertex to each edge while keep the designated vertices being designated. Observe that if a partial hypergraph $H$ is $\alpha$-subnormal then so is the extension of $H$. For any $r \geq 4$, we can extend $G_1^{(3)}(m, k_1, k_2)$ to $G_1^{(r)}(m, k_1, k_2)$, $G_2^{(2)}(m, k)$ to $G_2^{(r)}(m, k)$, and $G_3^{(4)}(t, k)$ to $G_3^{(r)}(t, k)$, glue $G_1^{(r)}$, $G_2^{(r)}$ and $G_3^{(r)}$ together via the designated vertices, and get a new graph $H$ that is still $(\sqrt{5} - 2)$-subnormal. We can get many examples of $H$ with $\rho(H) < \rho'.$

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