Size Limit of Superparamagnetic Inclusions in Dust Grains and Difficulty of Magnetic Grain Alignment in Protoplanetary Disks

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Abstract

Alignment of nonspherical grains with magnetic fields is an important problem, as it lays the foundation of probing magnetic fields with polarized dust thermal emissions. In this paper, we investigate the feasibility of magnetic alignment in protoplanetary disks (PPDs). We use an alignment condition that Larmor precession should be fast compared with the damping timescale. We first show that the Larmor precession timescale is some 3 orders of magnitude longer than the damping time for millimeter-sized grains under conditions typical of PPDs, making the magnetic alignment unlikely. The precession time can be shortened by superparamagnetic inclusions (SPIs), but the reduction factor strongly depends on the size of the SPI clusters, which we find is limited by the so-called “Néel’s relaxation process.” In particular, the size limit of SPIs is set by the so-called “anisotropic energy constant” of the SPI material, which describes the energy barrier needed to change the direction of the magnetic moment of an SPI. For the most common iron-bearing materials, we find maximum SPI sizes corresponding to a reduction factor of the Larmor precession timescale of order 103. We also find that reaching this maximum reduction factor requires fine-tuning on the SPI sizes. Lastly, we illustrate the effects of the SPI size limits on magnetic alignment of dust grains with a simple disk model, and we conclude that it is unlikely for relatively large grains of order 100 μm or more to be aligned with magnetic fields, even with SPIs.

Unified Astronomy Thesaurus concepts: Protoplanetary disks (1300); Interplanetary dust (821); Magnetic fields (994)

1. Introduction

The polarization of starlight was first observed in 1949 (Hiltner 1949). It was soon attributed to the alignment of dust grains in the foreground interstellar medium (ISM). Since then, the alignment of dust grains, especially with respect to magnetic fields, has had many developments. Many theories were developed to explain how dust grains are aligned with magnetic fields, such as the Davis–Greenstein mechanism (Davis & Greenstein 1951), hydrogen formation torque (Purcell 1979), radiative alignment torque (B-RAT; Dolginov & Mytrophanov 1976; Draine & Weinergartner 1997; Lazarian & Hoang 2007b), and, recently, mechanical alignment torque (B-MAT; Lazarian & Hoang 2007a; Hoang et al. 2018). We refer interested readers to Andersson et al. (2015) and references therein.

Superparamagnetism (SPM) was first introduced to the astronomical literature of grain alignment by Jones & Spitzer (1967). They pointed out that SPM can enhance magnetic relaxation, the process invoked by the Davis–Greenstein mechanism to dissipate oscillating magnetic moments and align grains, which was found to be insufficient to align regular paramagnetic dust grains with magnetic fields. Mathis (1986) adopted this theory with the assumption that grains containing any small superparamagnetic particle, the so-called superparamagnetic inclusions (SPIs), can be aligned with magnetic fields. Under this theory, the fact that bigger grains are better aligned is well explained, since bigger grains are more likely to contain SPIs. The Fe–Ni inclusions appear to present in interplanetary dust particles, and their spatial frequency supports Mathis’s theory in explaining the wavelength dependence of polarization (Goodman & Whittet 1995).

Magnetic nanoparticles and inclusions were also discussed recently by Draine & Hensley (2013), focusing on the impacts of such inclusions on the dust thermal emission and polarization.

Observationally, tracing magnetic fields with polarized thermal emission from grains aligned with magnetic fields is a classical and successful method. It is clear that grains in the diffuse ISM are aligned with the magnetic field from starlight polarization (Mathewson & Ford 1970). Recently, this picture received firm support from the Planck all-sky survey data (Planck Collaboration et al. 2015) and the interferometric polarimetry data (see Hull & Zhang 2019 and reference therein). There is no doubt that grains on scales larger than disks are aligned with magnetic fields.

In the past 6 yr, thanks to the improvement of interferometric polarimetry, especially with the Atacama Large Millimeter/submillimeter Array (ALMA), we have become able to resolve polarization maps down to the disk scale. The results have been surprising. Most systems show uniform polarization patterns, especially at shorter wavelengths, e.g., HL Tau (Stephens et al. 2014, 2017), IM Lup (Hull et al. 2018), DG Tau (Bacciotti et al. 2018), and HD 163296 (Dent et al. 2019), which are better explained with the self-scattering of dust grains than with magnetically aligned grains (Kataoka et al. 2015; Yang et al. 2016; Lin et al. 2020; Ohashi & Kataoka 2019). Cox et al.’s (2018) survey found a trend that the polarization is uniform on scales smaller than 100 au at the percent level, whereas the polarization is less organized on larger scales at a higher level (~5%). This is consistent with the picture that dust grains at disk scales are not aligned. There are also some systems showing complicated polarization features, such as the binary system BHB 07-11 (Alves et al. 2018), the southern part of HD 142527 (Kataoka et al. 2016; Ohashi et al. 2018), and HL Tau

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millimeter in size and gas density increases by many orders of magnitude compared with the ISM values, as we will see more quantitatively in the following section.

In theoretical works studying the dynamics of magnetic grain alignment, fast Larmor precession has been mostly assumed, so that the calculated torques are averaged over one precession before the dynamics of the dust grain is studied (e.g., Lazarian & Hoang 2007; Hoang et al. 2018). In this work, we use \( t_\text{L} < 0.1 t_\text{d} \) as the criterion for fast Larmor precession assumption. In this regime, the Larmor precession timescale will not be the limiting factor, and magnetic grain alignment becomes possible. We will come back to discuss this criterion and its caveats in more detail in Section 6.

2.2. Gaseous Damping Timescale

The random bombardment of gas particles on dust grains tends to misalign any ordered orientation of the dust grains. This happens roughly on a timescale of (Roberge et al. 1993)

\[
t_\text{d} = \frac{2\sqrt{\pi}}{5} \frac{\rho a}{n_g m_g v_{\text{th}}^2} = 3.54 \times 10^{12} s \times \left( \frac{\rho_s}{3 \text{ g cm}^{-3}} \right) \times \left( \frac{a}{0.1 \mu \text{m}} \right) \left( \frac{n_g}{20 \text{ cm}^{-3}} \right)^{-1} \left( \frac{T_g}{85 \text{ K}} \right)^{-1/2},
\]

where \( \rho_s \) is the mass density of the (solid) dust grain, \( a \) is the grain size, \( n_g \) is the number density of gas particles, and \( T_g \) is the gas temperature. This is a rather long timescale for the diffuse ISM conditions we used above. If we take a PPD with a total mass of 0.01 \( M_\odot \) as the normalization, uniformly distributed in a cylinder with 100 au as the radius and 10 au as the scale height, we have

\[
t_\text{d} = 2.6 \times 10^5 s \times \left( \frac{\rho_s}{3 \text{ g cm}^{-3}} \right) \left( \frac{a}{1 \text{ mm}} \right) \times \left( \frac{n_g}{5 \times 10^9 \text{ cm}^{-3}} \right)^{-1} \left( \frac{T_g}{25 \text{ K}} \right)^{-1/2},
\]

which is shorter than the typical dynamical timescales in PPDs.

2.3. Larmor Precession Timescale

The Larmor precession process is the precession of a magnetic moment around a magnetic field. A spinning dust grain possesses a magnetic moment due to the Barnett effect (Barnett 1915). Its magnetization is \( M = \chi \Omega / \gamma \) (Draine 2004), where \( \chi \) is the magnetic susceptibility, \( \Omega \) is the angular velocity, \( \gamma = g \mu_B / \hbar \) is the gyromagnetic ratio, and \( \mu_B \) is the Bohr magneton. The g-factor is about 2 for electrons.

For regular paramagnetic material, the magnetic susceptibility follows Curie’s Law (Morris 2001),

\[
\chi = \frac{n \mu^2}{3kT} = 10^{-3} \left( \frac{T}{25 \text{ K}} \right)^{-1},
\]

where \( n \) is the number density of magnetic units, \( \mu \) is the magnetic moment of each unit, and \( \chi \) is a dimensionless parameter that is on the order of unity for regular paramagnetic materials (Draine 1996; Lazarian 2007).

The structure of this paper is as follows. In Section 2, we introduce the basic timescales involved in the magnetic alignment process and give a simple comparison for regular paramagnetic materials. In Section 3, we introduce how SPIs can enhance the magnetic susceptibility and how the cluster size is limited by the relaxation process. In Section 4, we estimate the size limit of SPIs assuming a few commonly adopted forms of iron. In Section 5, we adopt a simple disk model and discuss the feasibility of magnetic alignment in PPDs. In Section 6, we discuss the caveats and uncertainties in this work, as well as its implications for several observed disks. We summarize our results in Section 7.

2. Basic Magnetic Alignment Theory

2.1. Criteria for Magnetic Alignment

Dust grain alignment with magnetic fields is a complicated process with many timescales involved. An outline of the process (regardless of alignment mechanism) is as follows (see, e.g., Lazarian 2007). Initially, a randomly spinning dust grain nutates about its angular momentum axis. Over a timescale \( t_{\text{int}} \), the principal axis of the dust grain becomes aligned with the angular momentum through some relaxation processes. The dust grain’s angular momentum also precesses around the external magnetic field on the Larmor precession timescale \( (t_L) \). At the end, some torques gradually force the angular momentum of the dust grain to be aligned with the magnetic field over a timescale of \( t_{\text{d}} \). At the same time, the random bombardment of gas particles tries to disturb the angular momentum over the gaseous damping timescale \( t_{\text{d}} \).

One typically considers that these three timescales must follow a hierarchical inequality—i.e., \( t_L < t_{\text{d}} < t_{\text{d}} \)—in order for grain alignment to proceed successfully. However, the alignment timescale \( t_{\text{d}} \) is considerably more complicated to compute than the other two timescales in general, because it depends on the specific torquing mechanism at play. For the purpose of this paper, we work with the insufficient but still necessary condition \( t_L < t_{\text{d}} \) to determine the conditions under which dust grains can align with ambient magnetic fields.\(^2\)

Even though \( t_L > t_{\text{d}} \) is unrealistic in most of the ISM, it becomes a strong possibility in PPDs as grains grow to a complex environment is very different from the diffuse ISM in many ways: higher density, bigger grain sizes, different temperature and radiation energy density, etc. A rough comparison given in Section 2 shows that the Larmor precession timescale can be some 3 orders of magnitude larger than the gaseous damping timescale, making magnetic alignment impossible (see also Tzaki et al. 2017). Hoang (2017) pointed out that if the dust grains contain large SPIs (on the order of \( 10^5 \) iron atoms each), magnetic alignment of millimeter dust grains can still be possible in PPDs. The enhancement from SPIs strongly depends on the size of each SPI. In this paper, we will focus on estimating the maximum size of SPIs and discuss how the size limit affects the magnetic alignment of dust grains in PPDs.

We summarize our results in Section 7.

\(^2\) Note that this criterion effectively ignored the suprathermal rotation, which will be discussed in Section 6.
The Larmor precession timescale is then

$$t_L = \frac{2\pi I}{\chi VB} = 4.3 \times 10^6 s \times \frac{\chi}{3 \text{ g cm}^{-3}}$$

$$\times \left( \frac{T_d}{25 \text{ K}} \right) \left( \frac{B}{5 \mu \text{G}} \right)^{-1} \left( \frac{a}{0.1 \mu \text{m}} \right)^2.$$ (4)

We can see that this timescale is some 6 orders smaller than the gaseous damping timescale for parameters appropriate for the diffuse ISM. But if we normalize the field strength to 5 mG, which is typical for a PPD with a $10^{-8} M_\odot$ yr$^{-1}$ accretion rate at tens of au scale (Bai 2011), and the grain size to 1 mm, we get

$$t_L = 4.3 \times 10^{11} s \times \frac{\chi}{3 \text{ g cm}^{-3}}$$

$$\times \left( \frac{T_d}{25 \text{ K}} \right) \left( \frac{5 \mu \text{G}}{B} \right)^{-1} \left( \frac{a}{1 \text{ mm}} \right)^2,$$ (5)

which is about $10^3 t_d$.

### 3. Superparamagnetic Inclusions

#### 3.1. Basic Picture

The SPIs are small (nano-sized) particles of ferromagnetic material.\(^3\) Within one such particle, all of the atoms are spontaneously magnetized and behave like a single large magnetic moment, the so-called “macrospin” (Bean & Livingston 1959). They are not big enough to create domain walls yet and are usually referred to as “single-domain particles.” In the absence of external magnetic fields, these macrospins are randomly oriented and behave like paramagnetic materials as an ensemble.

Let us first consider the simplest case, where an SPI has no preferred direction for magnetization. This isothermal case is mathematically identical to the regular paramagnetic case, and we have the bulk magnetic susceptibility of an ensemble of identical SPIs as (Jones & Spitzer 1967)

$$\chi_{sp} = \frac{N \mu^2}{3kT},$$ (6)

where $N$ is the number density of SPIs inside the dust grain. Let $n_{tot}$ be the total number density of atoms, and assume that a fraction of $f_{sp}$ atoms are magnetic atoms embedded in SPIs. Let $\mu$ be the magnetic moment of the macrospins. If each SPI contains $N_f$ magnetic atoms, we have $N = f_{sp} n_{tot} / N_f$, and $\mu = N_f \mu_B$, where $p \mu_B$ is the averaged Bohr magneton for each magnetic atom. With these we have (Draine 1996)

$$\chi_{sp} = \frac{f_{sp} n_{tot} N_f (p \mu_B)^2}{3kT}$$

$$= 0.72 \times 10^{-2} N_f f_{sp} \left( \frac{n_{tot}}{10^{23} \text{ cm}^{-3}} \right) \left( \frac{p}{3} \right)^2 \left( \frac{T}{25 \text{ K}} \right)^{-1}.$$ (7)

Compared with Equation (3), we have

$$\chi = 7.2 N_c f_{sp} (p/3)^2.$$ (8)

In reality, SPIs will have a preferred direction for magnetization. For example, a prolate particle prefers to be magnetized along its long axis (Bean & Livingston 1959). This is called shape anisotropy. Another example is metallic iron, which has a cubic crystalline structure. Metallic iron has less energy when magnetized along one of the principal axes of its crystalline structure (the so-called “easy axis”; Dai & Qian 2017). This is called magnetocrystalline anisotropy.

Even though the energy would be different in the presence of anisotropy, the susceptibility remains the same in thermal equilibrium states. Let $K$ be the so-called “anisotropy constant,” such that $K V$ is the energy needed to change the direction of the magnetic moment. With a simple prescription, Bean & Livingston (1959) showed that the ensemble-averaged magnetic susceptibility remains the same for both of the limiting cases, when $K V \gg kT$ and when $K V \ll kT$. Thus, Equation (7) works even for anisotropic SPIs.

#### 3.2. Size Limit Determined by the Relaxation Process

Billas et al. (1994) showed that single-domain particles as small as $N_f = 30$ can show SPM. In this subsection, we will discuss what determines the maximum size for SPI, which is more important than the lower limit.

Néel (1949) first proposed that single-domain particles experience random Brownian-like motions that can change the orientation of its magnetic moment (see also Bean & Livingston 1959). This happens on a timescale (the so-called “Néel’s relaxation timescale”) of

$$t_N \equiv t_0 \exp \left( \frac{K V}{kT} \right),$$ (9)

where $t_0$ is called the “attempt timescale,” typically on the order of $10^{-9} \text{ s}$. Setting $t_N$ equal to the timescale of interest ($\tau$), we

\(^3\) In this work, we do not distinguish ferromagnetic material from ferrimagnetic material or even speromagnetic material. They all behave as macrospins, as discussed in the text, but maybe with a different number of effective Bohr magnetons per atom ($p$ in Equation (7)).
can define a critical blocking volume as

\[ V_{\text{cr}} = \frac{kT}{K} \ln \left( \frac{\tau}{t_0} \right). \]  

(10)

The typical dynamical timescale of a 100 au sized PPD is about 1000 yr, which yields \( \ln(\tau/t_0) \approx 45 \). We can see that the critical volume strongly depends on the temperature, and the cluster size measured at room temperature does not apply directly to an astronomical environment, which has not been considered before. The critical number of magnetic atoms can then be calculated, for iron-based ferromagnetic material, through

\[ N_{\text{cr}} = \frac{\rho V_{\text{cr}} f_{\text{Fe}}}{56m_p} = \frac{\rho f_{\text{Fe}} kT}{56m_p K} \ln \left( \frac{\tau}{t_0} \right). \]  

(11)

where \( f_{\text{Fe}} \) is the mass fraction of iron atoms in the ferromagnetic material constituting the SPIs, and \( 56m_p \) is the mass of an iron atom.

The magnetization of a dust grain with SPIs is illustrated with a simplified model in Figure 1. In this model, we consider two dust grains with uniformly sized SPIs. The left one has smaller SPIs, whereas the right one contains bigger SPIs. At \( t < 0 \), there is no external magnetic field. The magnetic moments of each SPI are randomly oriented such that both dust grains have no bulk magnetization. At \( t = 0 \), we turn on external magnetic fields and observe the magnetization of dust grains at \( \tau \), the dynamical timescale of our interest. We will find that all of the small SPIs in the left dust grain turn into the external magnetic field direction (with thermal fluctuation). The resulting magnetic susceptibility of the dust grain is the superparamagnetic susceptibility \( \chi_{\text{sp}}(N_{\text{cl}}) \) in Equation (7). In contrast, the large SPIs in the right dust grain do not have enough time to overcome the anisotropy energy barrier. Their magnetic moments are effectively “blocked” and do not contribute to the magnetization of the dust grain. The dependence of the magnetic susceptibility on the size of SPI clusters can be approximated as (see the Appendix for a more quantitative discussion)

\[ \chi = \begin{cases} 0, & N_{\text{cl}} > N_{\text{cr}} \\ \chi_{\text{sp}}(N_{\text{cl}}), & N_{\text{cl}} < N_{\text{cr}}. \end{cases} \]  

(12)

We can see that the maximum enhancement of the magnetic susceptibility for an ensemble of SPIs of equal sizes is

\[ \chi_{\text{max}} = \chi_{\text{sp}}(N_{\text{cr}}), \]  

(13)

which is achieved when all SPIs in a dust grain are of the same size with \( N_{\text{cr}} \) magnetic atoms.

3.3. Distribution of SPIs

So far we have only considered ensembles of SPIs with the same size. In order to understand the effects of a distribution of SPI sizes, we adopt a very simple power-law distribution with index \(-q\): \( dn(N_{\text{cl}})/dN_{\text{cl}} = CN_{\text{cl}}^{-q} \), where \( N_1 < N_{\text{cl}} < N_2 \) is the cluster size and \( C \) is an arbitrary constant to be determined from the iron abundance.

We are particularly interested in the scenario where \( N_1 < N_{\text{cr}} < N_2 \). With Equation (12), the magnetic susceptibility of this ensemble of SPIs can be calculated as

\[ \chi = \frac{1}{B} \int_{N_1}^{N_2} \frac{dn(N_{\text{cl}})}{dN_{\text{cl}}} (\mu_z) dN_{\text{cl}} = \int_{N_1}^{N_2} CN_{\text{cl}}^{-q} N_{\text{cl}}^2 \mu_z^2 dN_{\text{cl}} = \frac{Cp^2 \mu_B^2}{3(3 - q)kT} N_0^{3-q} |N_{\text{cr}}|, \]  

(14)

where \( (\mu_z) \) is the averaged magnetic moment along the magnetic field direction (see the Appendix for more detail). With some arithmetic of finding the constant \( C \), we can express our results as

\[ \chi = \chi_{\text{max}} \times \begin{cases} 2 - q \left( \frac{N_{\text{cl}}}{N_{\text{cr}}} \right)^{2-q}, & q < 2; \\ q - 2 \left( \frac{N_{\text{cr}}}{N_{\text{cl}}} \right)^{q-2}, & 2 < q < 3. \end{cases} \]  

(15)

We can see that the end magnetic susceptibility of this ensemble is always smaller than \( \chi_{\text{max}} \). The reduction factor is roughly the ratio of the number density of iron atoms within SPIs with sizes close to \( N_{\text{cr}} \) to the total number density of iron atoms within SPIs. Because of this reduction factor, reaching the maximum value of \( \chi_{\text{sp}} = \chi_{\text{max}} \) implicitly assumes that all SPIs have sizes close to the critical size \( N_{\text{cr}} \).

4. Estimate the Critical Sizes of SPIs

The critical size of an SPI is an important quantity for determining how big of an enhancement it will have on the magnetic susceptibility of the host dust grain. In this section, we perform estimates of SPI critical sizes in terms of the number of magnetic atoms \( N_{\text{cr}} \) for various materials that might plausibly be contained in astrophysical dust grains.

As seen in Equation (10), the anisotropy constant \( K \) determines the critical volume, when the temperature is fixed. In nature, there are two important contributions to \( K \): the shape anisotropy and the magnetocrystalline anisotropy. In this work, we will ignore the first one and assume spherical SPIs. Including the shape anisotropy will increase the anisotropy constant and decrease the size estimates given below. In other words, our estimates are conservative upper limits.

The critical volume is defined in Equation (10), and values at 25 K are reported here. We assume room temperature densities of stoichiometric materials. Besides the critical number of magnetic atoms in one SPI \( N_{\text{cr}} \), we also report the magnetic susceptibility \( \chi_{\text{max}} \), taking \( f_{\text{sp}} = 0.1 \) and \( n_{\text{tot}} = 10^{23} \text{ cm}^{-3} \).

4.1. Fe3O4 (Magnetite)

At low temperature \( T < 120 \text{ K} \), magnetite has a monoclinic structure (Izumi et al. 1982). The magnetic anisotropy energy was determined by Abe et al. (1976) as

\[ E_a = K_a \alpha_a^2 + K_b \alpha_b^2 + K_{ab} \alpha_a^4 + K_{bb} \alpha_b^4 + K_{ab} \alpha_a^2 \alpha_b^2 + K_{aa} \alpha_{11}^4, \]  

(16)

with \( K_a = 25.2, K_b = 3.7, K_{ab} = 2.1, K_{aa} = 1.8, K_{bb} = 2.4, \) and \( K_{ab} = 7.0 \text{ in} 10^5 \text{ erg cm}^{-3} \), and all \( \alpha \) are directional cosines (see Abe et al. 1976 for their definitions). The easy axes are along
Table 1
Anisotropy Constant $K$, Effective Number of Bohr Magneton $\rho$, Critical Volume $V_{cr}$ (Equation (10)), Critical Cluster Size $N_{cr}$, and Reduced Magnetic Susceptibility $\chi_{max}$ for Various Iron-bearing Ferromagnetic Materials that May Exist in Dust Grains

| Material      | $K$ (erg cm$^{-3}$) | $\rho$ | $V_{cr}$ (cm$^3$) | $N_{cr}$ | $\chi_{max}$ |
|---------------|---------------------|--------|-------------------|---------|--------------|
| Fe, FeO      | $6.1 \times 10^4$   | 1.18   | $2.5 \times 10^{-19}$ | $1.0 \times 10^9$ | $1.1 \times 10^4$ |
| $\gamma - \text{Fe}_2\text{O}_3$ | $1.9 \times 10^5$   | 1.25   | $8.2 \times 10^{-19}$ | $3.0 \times 10^9$ | $3.7 \times 10^3$ |
| Fe            | $1.35 \times 10^5$  | 3      | $1.15 \times 10^{-18}$ | $9.7 \times 10^9$ | $7.0 \times 10^4$ |

Note. See text for discussion and references.

the (001) and (001) directions. The magnetization can change from one easy axis to another through saddle points along the (010) or (010) directions, and the energy barrier is $K = 6.1 \times 10^7$ erg cm$^{-3}$. This translates to a critical volume of $V_{cr} = 2.5 \times 10^{-19}$ cm$^3$. Li et al. (2007) found a spin density of 3.54$\mu_B$ formula$^{-1}$ at 10 $K$ along the [100] direction, which we will use in this paper. For every iron atom, we have $\rho = 1.18$ Bohr magnetons. Taking $\rho \approx 5.17$ g cm$^{-3}$, we get $N_{cr} = 1.0 \times 10^4$ and $\chi_{max} = 1.1 \times 10^3$.

4.2. $\gamma - \text{Fe}_2\text{O}_3$ (Maghemite)

Maghemite ($\gamma - \text{Fe}_2\text{O}_3$) is ferrimagnetic iron oxide with similar structures to magnetite at room temperature. Its formula is often supposed to be (Fe$^{3+}$)$_6$(Fe$^{3+}$)$_{13/3}$ Fe$^{3+}$O$_4$, where Fe represents a vacancy. A perfect crystal has 3.33$\mu_B$ formula$^{-1}$. Hence, we have $\rho = 1.25$. In bulk samples, the moments were usually found to be about 87%–94% of this value (Coey & Khalafalla 1972). Pisane et al. (2017) fitted the effective magnetic anisotropy as a function of particle size with three terms. The leading term (and the dominating term for big particles) corresponds to $K = 1.9 \times 10^7$ erg cm$^{-3}$. This translates into a critical volume of $V_{cr} = 8.2 \times 10^{-19}$ cm$^3$. Taking $\rho = 4.9$ g cm$^{-3}$, we have $N_{cr} = 3.0 \times 10^4$ and $\chi_{max} = 3.7 \times 10^3$.

4.3. Metallic Iron

The leading term in the magnetic anisotropy energy of metallic iron has the form of $K_1(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$, where $\alpha_i$, $i = 1, 2, 3$ are directional cosines. At temperatures below 100 K, $K_1 \approx 5.4 \times 10^5$ erg cm$^{-3}$ (Dai & Qian 2017). The energy barrier to change the magnetization from one easy axis to another is $K = (1/4)K_1 = 1.35 \times 10^7$ erg cm$^{-3}$. Hence, $V_{cr} = 1.15 \times 10^{-18}$ cm$^3$. We will follow Draine (1996) and take $\rho = 3$, which was inspired by Billas et al.’s (1994) work showing that small clusters of iron have $3\mu_B$ atom$^{-1}$. Taking $\rho = 7.87$ g cm$^{-3}$, we have $N_{cr} = 9.7 \times 10^4$ and $\chi_{max} = 7.0 \times 10^4$.

4.4. Other Forms of Iron and Summary

Hematite ($\alpha - \text{Fe}_2\text{O}_3$) is another possible form of iron that is more stable than maghemite ($\gamma - \text{Fe}_2\text{O}_3$), discussed above. It is, however, antiferromagnetic, and a perfect crystal can be essentially considered nonmagnetic. In reality, some defects may exist to contribute to the magnetization, but it should be negligible compared with the other ferromagnetic materials discussed above. The same goes for FeO. For a summary of the magnetic properties of the iron oxides, we refer interested readers to Cornell & Schwertmann (2003), especially their Table 6.2. We do not consider sulfuric iron in this work.

All of the results discussed above are summarized in Table 1. We can see that the maximum cluster size and the enhancement of magnetic susceptibility strongly depends on the form of iron. Even though $\chi_{max}$ on the order of $10^3$ is still possible with metallic iron, it is less likely to exist in real dust grains, as it can easily be oxidized. We suggest that $10^3$ is a more realistic enhancement factor $\chi_{max}$ on the magnetic susceptibility through SPIs.

5. Magnetic Alignment in PPDs

In Section 2, we gave rough estimates of timescales relevant for magnetic alignment and compared them to motivate this study. In this section, we perform a more detailed study of whether magnetic alignment is feasible in a fiducial PPD model.

5.1. Disk Model

As an illustration, we adopt the well-known Chiang & Goldreich (1997) model. It is a passive disk with a minimum-mass solar nebula (MMSN) density profile $\Sigma = (r/au)^{-3/2}\Sigma_0$, with $\Sigma_0 = 10^3$ g cm$^{-2}$ (Weidenschilling 1977). It has a superheated surface layer and a cooler interior region, where most grown millimeter-sized dust grains reside. The temperature in the interior region and the scale height in the model under both hydrostatic and radiative equilibrium were fitted as

$$T = \begin{cases} 150 K \times \left(\frac{r}{1 \text{ au}}\right)^{-3/7} & 0.4 \text{ au} < r < 84 \text{ au} \\
21 K & 84 \text{ au} < r < 100 \text{ au} \end{cases}$$

and

$$H = \begin{cases} 0.17 \left(\frac{r}{1 \text{ au}}\right)^{2/7} & 0.4 \text{ au} < r < 84 \text{ au} \\
0.59 \left(\frac{r}{84 \text{ au}}\right)^{1/2} & 84 \text{ au} < r < 100 \text{ au}. \end{cases}$$

At a given radius, the midplane density is used to calculate the timescale, which is $\Sigma/\sqrt{2\pi H}$. We adopt a mean molecular weight of 2.3 and assume the temperature to be the same for gas and dust.

For the magnetic field structure, we adopt the estimate from Bai (2011),

$$B = 1.0 \times \left(\frac{M}{10^{-8} M_{\odot} \text{ yr}^{-1}}\right)^{1/2} \left(\frac{r}{1 \text{ au}}\right)^{-11/8},$$

where $\dot{M}$ is the mass accretion rate, which is assumed to be $10^{-8} M_{\odot} \text{ yr}^{-1}$, typical for classical T Tauri stars (see Dutrey et al. 2004 and references therein).
5.2. Timescale Comparison

As discussed in Section 2.1, we are especially interested in two conditions: \( t_L = t_d \) and \( 0.1 t_d \). These two conditions define three different regimes:

1. \( t_L > t_d \), no magnetic alignment;
2. \( 0.1 t_d < t_L < t_d \), more complicated dynamical study needed; and
3. \( t_L < 0.1 t_d \), magnetic alignment is possible with fast Larmor precession.

From Equations (2) and (5),

\[
\frac{n_c}{t_d} = 1.65 \times 10^3 \left( \frac{\hat{\chi}}{a_{\text{mm}}} \right)^{-1} \left( \frac{T}{25 \text{ K}} \right)^{3/2} \times \left( \frac{n_\mu}{5 \times 10^9 \text{ cm}^{-3}} \right) \left( \frac{B}{5 \text{ mG}} \right)^{-1},
\]

where \( a_{\text{mm}} \equiv (a/1 \text{ mm}) \). We can see that \( \hat{\chi} \) and grain size \( a \) are degenerate. Due to this degeneracy, the timescale ratio is calculated as a function of radius in the PPD, and the dimensionless factor \( \hat{\chi}/a_{\text{mm}} \). The results are plotted in Figure 2.

First of all, we can see that grains in the inner disk are harder to align compared with those in the outer disk. This is because the gas density power-law index (−2.8 in our model within 84 au) is usually more negative than the magnetic field power-law index (−1.4 in our model). As we decrease the radius, the gaseous damping timescale increases faster than the Larmor precession timescale. In order to achieve fast Larmor precession (\( t_L < 0.1 t_d \)) at a radius of 30 au, which is the typical resolution of ALMA polarization observations of the nearest star-forming regions, we need roughly \( \hat{\chi}/a_{\text{mm}} > 10^4 \).

This result can be easily translated into more meaningful statements after fixing \( \hat{\chi} \). Figure 3 shows the largest grain sizes with fast Larmor precession (\( t_L/t_d < 0.1 \)) at each radius in the disk. If one takes \( \hat{\chi} = 10^3 \), the upper limit we suggested in Section 4, we get \( a = 100 \mu\text{m} \). This means that grains with sizes of 100 \( \mu\text{m} \) or bigger are unlikely to be aligned with the magnetic field within 30 au of the central star, even with the aid of SPI. The situation is even worse for regular paramagnetic material (\( \hat{\chi} = 1 \)), where we need \( a > 0.1 \mu\text{m} \) to have fast Larmor precession.

The situation is a lot better in the outer disk at scales larger than 100 au. We will only need \( \hat{\chi}/a_{\text{mm}} \sim 10^3 \). Alignment of 1 mm dust grains becomes marginally possible if they contain large SPI clusters (\( \hat{\chi} > 10^3 \)).

6. Discussion

6.1. Caveats and Uncertainties in This Work

The above discussions have relied on a number of simplifying assumptions. In this section, we enumerate these assumptions and discuss the implications that relaxing them could have for our results.

6.1.1. Suprathermal Rotation

Our first assumption is that the dust grain rotational motions are distributed thermally, meaning that we have effectively ignored the possibility of suprathermal rotation. Purcell (1979) first pointed out that the dust grains can be spun up to suprathermal motion (rotation energy much larger than thermal energy) by torques arising from the formation of molecular hydrogen on the surface of the dust grains. A suprathermally rotating grain would have a larger gaseous damping timescale, which would make grain alignment easier. However, this hydrogen formation torque is unlikely to work for large millimeter-sized dust grains, since the rotation energy decreases with the increasing number of hydrogen formation sites. In PPDs, more plausible mechanisms for producing suprathermal rotation in millimeter-sized dust grains are B-RATs (Lazarian & Hoang 2007b) and mechanical torques from differential motion with the surrounding gas (B-MAT; Lazarian & Hoang 2007a; Hoang et al. 2018). Indeed, both Lazarian & Hoang (2007b) and Hoang et al. (2018) showed that grains can be aligned toward the so-called “high-J” attractors with angular momenta that are factors of several tens to hundreds of times larger than the thermal angular momentum. However, Hoang et al. (2018) also showed that the B-MAT goes as \( 1/\sqrt{N_{\text{facet}}} \), with \( N_{\text{facet}} \) being the number of facets on the grain surface. This cancellation effect may prevent the B-MAT from working for large millimeter-sized grains. Such grains may also have
limited helicity, which is crucial for both the B-RAT and the B-MAT to operate, so it is possible that this cancellation effect applies to B-RAT as well. More detailed study of the suprathermal rotation of large millimeter-sized grains in dense environments is needed to better understand how they might modify our results.

6.1.2. Alignment Condition

Our second assumption is that of fast Larmor precession, coupled with the lack of a precise understanding regarding what constitutes “fast enough” precession. Previous work on magnetic grain alignment has focused on two limiting cases. In the first limit, that of fast Larmor precession ($t_L \ll t_d$), torques are taken to have values that are averaged over the precession cycle prior to investigating grain dynamics (e.g., B-RAT work in Lazarian & Hoang 2007b and B-MAT work in Hoang et al. 2018). In the other limit, that of slow Larmor precession ($t_L \gg t_d$), the effects of magnetic fields are completely ignored (e.g., k-RAT work in Lazarian & Hoang 2007b and k-MAT work in Hoang et al. 2018). However, as we can see from Figure 2, it is not unreasonable to expect that the ratio $t_L/t_d$ may fall in the range 0.1–1 in PPD environments, in which case, we can neither assume precession-averaged torques nor ignore the magnetic field completely. Whether dust grains can be aligned magnetically in this regime is thus not currently clear, and this uncertainty limits the predictive power of magnetic alignment theories. Solving this problem will require studying the grain dynamics in three dimensions (i.e., without averaging over Larmor precession) for each potential alignment mechanism.

6.1.3. Form of Iron

Our third assumption is the iron abundance. In deriving $\hat{\chi}_{\text{max}}$, we took $f_{sp} = 0.1$. Draine (1996) suggested that about 10% of atoms in a grain are iron atoms. Here $f_{sp} = 0.1$ means that all iron atoms are in the form of SPIs, and no iron atoms are in other forms, such as silicate. This optimistic assumption can easily be wrong and reduce $\hat{\chi}_{\text{max}}$ by 1 order of magnitude or more, making magnetic alignment harder. We also see that the maximum cluster size depends on the detailed form of iron. Even for ferromagnetic materials, the critical cluster size can vary by orders of magnitude. A chemical study of dust compositions may help resolve this uncertainty.

6.1.4. Disk Model

Our fourth assumption is the specific form of the PPD disk model. The MMSN model, by construction, contains only the minimum amount of mass required to form our solar system. It is thus likely to have less mass than a real PPD. Increasing the mass and density of a PPD would increase the gaseous damping timescale, which would make magnetic grain alignment more difficult.

6.2. Effects of Temperature Dependence on Cluster Sizes

In Section 5, we used $\hat{\chi}$ as a free parameter to discuss the possibility of magnetic alignment in PPDs, even though $\hat{\chi}_{\text{max}}$ has a temperature dependence. In this section, we discuss the implications of this temperature dependence.

One may be tempted to set $\hat{\chi}$ at each radius in the disk to the local $\hat{\chi}_{\text{max}}$ defined by its local temperature. However, doing so would mean that the SPI sizes in dust grains are changing at each location in the disk, in such a way as to maximize the effects of SPM. This behavior is not physical, both because dust grains at different radii should have similar origins and because the sizes of SPIs in any given dust grain are unlikely to change as it migrates to other locations of the disk.

However, the temperature dependence of maximum cluster sizes may still have an impact on the magnetic susceptibility of dust grains as a function of disk radius. If the dust grains contain SPIs with sizes larger than the critical $N_c$ (see Equation (11)), then it is possible that these SPIs will be blocked in the low-temperature regions of the disk while contributing heavily to the magnetic susceptibility in the high-temperature regions of the disk. Such a model would predict a higher degree of magnetic alignment near the center of the disk. It is interesting to note that this behavior is opposite that expected from the usual Curie’s Law (Equation (3)).

In addition to its explicit linear dependence on temperature, the critical volume has another implicit temperature dependence through the anisotropic constant $K$. For metallic iron, this dependence can be safely ignored, as it changes very slowly and smoothly within the temperature range 0–300 K (decreasing by roughly only 15% across the range; Dai & Qian 2017). For magnetite ($\text{Fe}_3\text{O}_4$), however, the situation is considerably more complicated. The first complication is that magnetite undergoes a Verwey transition (Verwey 1939; Walz 2002) and thus changes crystal structure at a temperature of around 120 K (Lizumi et al. 1982); the structure changes from cubic spinel above this temperature to monoclinic below it. Our results for magnetite above (Section 4) would thus need to be revisited for environments with a temperature higher than 120 K. A second complication is that magnetite has a so-called “isotropic point” near 130 K. Around this temperature, the cubic anisotropy constant changes signs, such that the easy axis changes from the cubic diagonals ($T > 130$ K) to the cubic edges ($T < 130$ K); in the immediate vicinity of $T \approx 130$ K, the anisotropy constant is close to zero. A near-zero anisotropy constant permits an arbitrarily large SPI cluster size. If dust grains contain large clusters of magnetite, they may be blocked at temperatures other than $\approx 130$ K, in which case our model would predict a ring of magnetically aligned dust grains only in the region of the disk where $T \approx 130$ K. However, we note that for typical PPDs, such a high temperature is reached only on the au scale (see Equation (18)), which is unlikely to be resolved by ALMA in polarized emission.

6.3. Applications to Observed Disks

6.3.1. Small Grains Near the Surface of the AB Aur Disk

Li et al. (2016) used mid-infrared polarimetry to observe the disk around AB Aur at a wavelength of about 10.3 μm. The polarization was interpreted as arising from dust grains aligned with poloidal magnetic fields.

At this wavelength, dust grains with sizes on the order of about a micron are most important for emission. Such small dust grains are lifted more easily by turbulence and thus reach higher above the disk midplane than (sub)millimeter grains. Both the dust grains and their environment are thus different from what we have discussed in Section 5. For the sake of

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4 The cluster size may still be limited by the domain size—which is usually larger than the blocking size and is ignored in this work—as well as other sources of anisotropy, such as the shape anisotropy.
example, suppose the grains are lifted to twice the scale height, where the gas density is an order of magnitude smaller than in the disk midplane. The magnetic field is also likely to be different in this region, but we will assume the same value as in Equation (19). Given these conditions, we derive the criterion for fast Larmor precession to be \( \dot{\chi}/a_{\text{mm}} = 10^3 \) at 30 au (Li et al. 2016 has a 50 au resolution). For micron-sized grains, SPIs are thus unnecessary to achieve magnetic alignment because \( \dot{\chi} = 1 \) is sufficient to guarantee fast Larmor precession. Mid-IR polarimetry probing the small dust grains at the disk surface should therefore be capable of studying magnetic field structures in PPDs.

### 6.3.2. Class 0 Systems

Class 0 disks represent the earliest stages of planet formation. They are likely to have stronger magnetic fields (Yen et al. 2017) and may be more massive than their later-stage counterparts (Tobin et al. 2020). If we adopt a representative accretion rate, \( 10^{-4} M_\odot \text{yr}^{-1} \), we would increase the strength of the magnetic field by 1 order of magnitude, compared with the value adopted in Equation (19). It is possible that class 0 disks are also more massive than class I/II disks. For example, Tobin et al. (2020) reported that the mass of class 0 and class I disks has a mean dust mass of 25.9 and 14.9 \( M_\odot \), respectively. If we assume that the mass of the class 0 disk is the same as the one adopted in Section 5—hence the density is also the same—then applying the fast Larmor precession criterion yields \( \dot{\chi}/a_{\text{mm}} = 10^3 \). Valdivia et al. (2019) inferred that the dust grains in class 0 sources have grown to at least 10 \( \mu m \). Such dust grains require at least \( \dot{\chi} = 10 \) to have fast Larmor precession, which requires minimal SPI enhancement to achieve. At the same time, 10 \( \mu m \) dust grains are less efficient at producing polarization through self-scattering at (sub)millimeter wavelengths, avoiding a known confounding effect for studying magnetic fields (Kataoka et al. 2015). Class 0 disks are thus reasonable targets for detecting magnetic fields through spatially resolved (sub)millimeter polarimetry.

### 7. Summary

In this paper, we discussed the feasibility of magnetic alignment of dust grains with SPIs in PPDs. The major results are summarized as follows.

1. Under Néel’s relaxation theory, we show that there exists a critical size of SPIs within dust grains. The SPIs larger than this critical size cannot respond to external magnetic fields and do not contribute to the magnetic susceptibility of the grain. There is thus a maximum enhancement that SPIs can provide to a grain’s magnetic susceptibility and therefore a corresponding maximum reduction factor for the Larmor precession timescale of the grain.

2. We explore the effect on magnetic susceptibility for a dust grain containing an ensemble of SPIs having a power-law size distribution. We find that if the ensemble contains SPIs bigger than the threshold \( N_{\text{cr}} \), the grain as a whole will be unlikely to achieve the maximum magnetic susceptibility \( \chi_{\text{max}} \) (Equation (13)) without fine-tuning (i.e., all SPIs in the dust grain would need to have sizes close to the maximum cluster size). This is because SPIs larger than the critical size do not respond to external magnetic fields and do not contribute to the ensemble magnetic susceptibility (Equation (12)).

3. We estimate the maximum sizes for SPIs composed of several plausible ferromagnetic materials, given their magnetic anisotropy constant at the temperature of astronomical interest (25 K). Our results are tabulated in Table 1. We suggest that \( 10^2 \) is a more realistic upper limit for the SPI enhancement factor \( \dot{\chi} \). This value is 2 orders of magnitude smaller than that obtained from previous work, implying that magnetic alignment is more difficult than previously thought.

4. We explore the feasibility of magnetic grain alignment in the disk midplane of an MMSN model for a PPD. We find that (1) magnetic alignment is impossible unless SPIs are ubiquitous throughout the disk, even if the dust grains are as small as 10 \( \mu m \), and (2) it is difficult to align grains larger than 100 \( \mu m \) even with SPIs, particularly at small orbital radii where the high gas density leads to short damping timescales.

We conclude that large millimeter-sized dust grains in the midplanes of PPDs are unlikely to be aligned with the ambient magnetic fields (see Figure 3). An important implication of this finding is that observations that are primarily sensitive to the emission from this population of grains—such as (sub)millimeter-wavelength polarimetric observations of class I/II PPDs—are unlikely to be tracing the disk magnetic field structure. We suggest instead that observations of disk surfaces—probed by mid-infrared polarimetry—and the early class 0 PPDs are better suited to studying the magnetic field structures in PPDs.

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### Appendix

**A Simple Model for SPI Dynamics**

In order to see how the magnetic susceptibility of an ensemble of identical SPIs changes as a function of their volume, we use a simplified model with a three-axis assumption: all of the SPIs have their easy axes along one of an arbitrary set of Cartesian axes, and they are split evenly among three axes. Also, we are interested in the regime where \( V > kT/K \), so the equilibrium state has an averaged magnetic moment of \( \mu^2 B/kT \) along the easy axis and zero along the perpendicular directions. Initially, all of the SPIs have their magnetic moments randomly distributed, so that there are equal numbers of \( \mu \) along the six directions (\( \pm x, \pm y, \text{ and } \pm z \)).

Now let us apply the magnetic field \( B \) along the \( z \)-axis at \( t = 0 \). The Barnett equivalent field (for the thermal angular velocity of a 10 \( \mu m \) dust grain) \( H_\text{B} = \Omega/\gamma \sim 10^{-7} \text{ Oe} \), and the corresponding magnetic energy is at most on the order of \( 10^2 \mu_\text{B} H_\text{B}/k \sim 10^{-7} \text{ K} \), which is very small compared with the thermal energy (on the order of 10 K). We conclude that the magnetic field cannot overcome the anisotropy energy on its own, which is even bigger than the thermal energy. We need the Brownian-like process Néel proposed, in which each magnetic moment tries to change its orientation every \( t_0 \). Since
the SPIs with easy axes along the x- or y-direction cannot respond to the magnetic field along the z-direction, we focus on those with easy axes along the z-axis and define $f_+$ and $f_-$ as the fraction of particles along $z+$ and $z-$, respectively. The energies for these two states are $-\mu B$ and $\mu B$, but there is an energy barrier of $KV$ if the magnetic moment tries to switch to the opposite state. As such, the probability of one successful transition is

$$
\begin{align*}
P(z \rightarrow z^-) &= \exp\left(-\frac{KV + \mu B}{kT}\right), \\
P(z \rightarrow z^+) &= \exp\left(-\frac{KV - \mu B}{kT}\right).
\end{align*}
$$

The dynamical equation for $(f_+, f_-)$ is then

$$
\frac{d}{dt} \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \frac{1}{t_N} \begin{pmatrix} -\exp\left(-\frac{KV + \mu B}{kT}\right) & \exp\left(-\frac{KV - \mu B}{kT}\right) \\ \exp\left(-\frac{KV + \mu B}{kT}\right) & -\exp\left(-\frac{KV - \mu B}{kT}\right) \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}.
$$

The solution to this equation is

$$
\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \exp\left(\frac{t}{t_N} \mathcal{M}\right) \begin{pmatrix} f_{+0} \\ f_{-0} \end{pmatrix},
$$

where $t_N$ is Néel’s relaxation timescale, and $\mathcal{M}$ is a matrix defined as $\mathcal{M} \equiv \begin{pmatrix} [-e^{-\beta}, e^{-\beta}], [e^{-\beta}, -e^{-\beta}] \end{pmatrix}$, with $\beta \equiv \mu B/kT$.

It is possible to explicitly calculate the above matrix exponential with the aid of the following matrix transformation:

$$
\mathcal{M} = \mathcal{P} \begin{pmatrix} 0 & 0 \\ 0 & -2 \cosh \beta \end{pmatrix} \mathcal{P}^{-1},
$$

with $\mathcal{P} \equiv \begin{pmatrix} [e^{\beta}, 1], [e^{-\beta}, -1] \end{pmatrix}$. Equation (A3) can then be rewritten as

$$
\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \mathcal{P} \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-\frac{2t}{t_N} \cosh \beta\right) \end{pmatrix} \mathcal{P}^{-1} \begin{pmatrix} f_{+0} \\ f_{-0} \end{pmatrix}.
$$

Now let us consider the two limiting cases. For $t \gg t_N$, the exponential in Equation (A5) is basically zero. We have

$$
\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \frac{f_{+0} + f_{-0}}{2 \cosh \beta} \begin{pmatrix} e^\beta \\ e^{-\beta} \end{pmatrix}.
$$

This means that the end state is independent of the initial state and the system has reached equilibrium (we always have $f_{+0} + f_{-0} = 1$). In this case, the averaged magnetic moment along the magnetic field direction is $\langle \mu z \rangle = f_+ \mu - f_- \mu = \mu \tanh \beta \approx \mu^2 B/kT$. The magnetic susceptibility of this ensemble is thus the same as the superparamagnetic magnetic susceptibility $\chi_{sp}$ in Equation (7) (remember that two-thirds of SPIs have perpendicular easy axes and do not contribute to the magnetic susceptibility).

![Figure 4. Averaged magnetic moment along the magnetic field direction $\langle \mu \rangle$ of SPIs with a magnetic moment $\mu$ and Néel’s relaxation timescale $t_N$ as a function of time $t$ vs a hyperbolic function of time $t$ vs $t_N$.](image)

For $t \ll t_N$, we have (this is easier to derive directly from Equation (A3))

$$
\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \approx \begin{pmatrix} t + \frac{t}{t_N} \mathcal{M} \end{pmatrix} \begin{pmatrix} f_{+0} \\ f_{-0} \end{pmatrix} = \begin{pmatrix} f_{+0} + \frac{t}{t_N} f_{-0} \exp(-\beta) \\ f_{-0} \exp(-\beta) - f_{+0} \exp(\beta) \end{pmatrix}.
$$

From the end result, we can see that a fraction of $(t/t_N)\exp(-\beta)$ SPIs have changed from the $+$ state to the $-$ state, whereas a fraction of $(t/t_N)\exp(\beta)$ SPIs have changed from the $-$ state to the $+$ state. In this case, for an initial condition $f_{+0} = f_{-0} = 0.5$, we get $\langle \mu \rangle = f_+ \mu - f_- \mu = (2t/t_N)\exp(\beta) \sinh \beta \approx (2t/t_N)\mu^2 B/kT$.

The accurate solution, together with the above two asymptotic solutions, is plotted in Figure 4. Here $\mu$ linearly increases to its equilibrium value. However, since Néel’s relaxation timescale increases exponentially with the volume $V$, the magnetic susceptibility of the ensemble of SPIs changes with volume $V$ very sharply across the critical volume defined in Equation (10), which, to a good approximation, can be summarized as follows:

$$
\chi = \begin{cases} 0, & V > V_c \\
\chi_{sp}, & V < V_c.
\end{cases}
$$

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References

Abe, K., Miyamoto, Y., & Chikazumi, S. 1976, JPSJ, 41, 1894
Alves, F. O., Girart, J. M., Padovani, M., et al. 2018, A&A, 616, A56
Anderson, B. G., Lazarian, A., & Vaillancourt, J. E. 2015, ARA&A, 53, 501
Bacciotti, F., Girart, J. M., Padovani, M., et al. 2018, ApJ, 865, L12
Bai, X.-N. 2011, ApJ, 739, 50
Bean, C. P., & Livingston, J. D. 1959, JAP, 30, S120
Billas, I. M., Château, A., & de Heer, W. A. 1994, Sci, 265, 1682
Chiang, E. I., & Goldreich, P. 1997, ApJ, 490, 368
Coey, J. M. D., & Khalafalla, D. 1972, Phys. Stat. Sol. (a), 11, 229

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