A meshfree method for analysis of thermo-elastic problems with moving heat sources in welding

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Abstract. This manuscript presents the development and application of a meshfree procedure based on the finite pointset methods for thermo-elastic problems with moving heat sources, which are present in welding processes. The meshfree nature of this formulation gives the advantage of dealing with geometrical distortions and even fragmentations without the need of using computationally expensive remeshing approaches with a very simple implementation. A description of the implementation of this method and the solutions of some numerical examples are presented in order to show the potential of this formulation for dealing with thermoelasticity problems with moving heat sources and to introduce promising future fields of application.

1. Introduction

Different manufacturing and industrial processes such as welding, metal cutting, and grinding involve the movement of heat sources, which may cause several difficulties in the end products such as cracks due to thermal loads, micro-structures formation, residual stresses, or hardness. In order to prevent these kinds of problems during the materials processing, it is needed to know and control the temperature profiles since it is a key task to assure the quality of the finished products [1]. In welding, for example, the temperature distribution has a direct impact in the distortion and fatigue of the welded parts. This framework demands resorting to numerical modelling tools in order to achieve an appropriate performance in the manufacturing processes at the best possible cost. This is particularly severe when dealing with industrial case studies, which usually involve complex geometries and demanding conditions [2].

The Finite Element Method (FEM) is a universally accepted numerical technique for solving PDEs in different problems, and that is why during the last three decades numerical simulation tools in this field have used FEM and other mesh-based methods such as the Finite Difference Method (FDM). In order to get results with good accuracy in moving heat source problems, adaptive refinement strategies are commonly used around the position of the heat source. However, the lack of robust and efficient mesh generators for three-dimensional problems makes the numerical solution in these cases a very complex task. Furthermore, meshbased methods are not suitable for problems associated with very large mesh distortions or problems requiring constant remeshing during the solution process, since it implies
a high computational cost. Therefore, mesh-free methods have recently been proposed and developed as an alternative to overcome part of the difficulties arising when mesh-based methods are used [3].

In general, meshfree methods use a set of points to discretize the problem domain as well as its boundaries. Such points could be numerical interpolation points or mass particles, and most of them do not need some information about the connectivity between the points, thus they do not form a traditional mesh. Consequently, any problem directly emerged from the use of computational meshes could be avoided with this family of methods [4].

Different works have been carried out to numerically model thermomechanical phenomena with meshfree methods. For example, the element-free Galerkin method (EFG) has been used in order to analyze plates under thermal and mechanical loads [5], to model the crack interactions under mechanical and thermal loads [6], and for the thermomechanical flow of friction stir welding processes [7]. The local Petrov-Galerkin method has been applied to study heat conduction with residual stress due to welding and thermomechanical shock fracture [8–9] whilst the Radial Point Interpolation Method (RPIM) has been used in thermoelastic problems with moving heat sources and thermomechanical crack growths [10–11]. Finally, the particle finite element method (PFEM) has been applied in thermomechanical problems such as chip formation and metal cutting problems [12–13]. Nonetheless, despite the good stability and accuracy observed in these methods, their weak form makes them computationally expensive due to the mandatory use of background meshes. Therefore, strong-form meshfree methods are attractive alternatives.

Strong-form mesh-free methods that have been used successfully for these kinds of problems include the Local Radial Basis Function Collocation Method (LRBFCM) which has been applied to analyze thermomechanical problems in transient coupled thermoelasticity and Hot Rolling [14–15], as well as the Smoothed Particle Hydrodynamics (SPH) which has been used for the modelling of metal forging and theromechanical processes in break systems [16–17].

A truly strong-form meshfree method is the Finite Pointset Method (FPM) that was proposed by J. Kuhnert [18] to solve some of the problems in other meshless formulations as inconsistencies and tensile instabilities, as the ones observed in the SPH method [18]. This method has proven to be far superior to some other meshless methods and traditional meshbased methods in many computational mechanics and engineering fields, such as flows on manifolds [19], fluid-structure interactions [20], fluid transfer [21], fluid flow coupled with heat transfer considering phase changes [22] and recently, for solving static linear elasticity problems [23] and thermoelastic processes [24], among others. However, its use in thermo-elastic problems with moving heat sources in welding has not been studied. Therefore, a novel meshfree formulation based on the finite pointset method for thermo-elastic problems with moving heat sources is introduced in this work.

The article is organized as follows. Section 2 introduces the governing equations for static linear thermoelasticity, while the FPM formulation to solve the governing equations is presented in Section 3. Section 4 contains the numerical examples employed to verify the numerical implementation. The paper ends in Section 5 with the main conclusions of this work.

2. Mathematical Model

The mathematical governing equations for static linear thermoelasticity considered in this work are the Navier-Cauchy equations coupled with the transient heat equation which can be written as:

\[ (\lambda + \mu) \nabla (\nabla \cdot u) + \mu \nabla^2 u - \nabla \left[ \beta (T - T_{\text{ref}}) \right] = 0 \]

\[ \rho c \frac{\partial T}{\partial t} = \nabla (kT) + G \]

For a two-dimensional problem and considering an implicit Euler scheme for the temporal discretization of the transient heat equation, these equations could be expressed as follows:
\[(\lambda + \mu) \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\partial \left[ \beta (T - T_0) \right]}{\partial x} = 0 \]  
\[(\lambda + \mu) \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\partial \left[ \beta (T - T_0) \right]}{\partial y} = 0 \]

\[\rho c T^{n+1} - \Delta t \left[ \frac{\partial}{\partial x} \left( k \frac{\partial T^{n+1}}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T^{n+1}}{\partial y} \right) \right] = \rho c T^n + \Delta t G, \]

where \(\lambda, \mu\) denotes the Lamé coefficients, \(\rho\) is the density, \(k\) is the thermal conductivity, \(c\) is the specific heat, \(G\) is the intensity of the heat source, \(\Delta t\) is the time step, and \(T^n\) denotes the temperature at time \(t^n\). The coefficient \(\beta\) is given by \(\beta = (3\lambda + 2\mu)\gamma\), where \(\gamma\) denotes the linear coefficient of thermal expansion. The considered moving heat source in this work is of Gaussian type and it is defined as follows:

\[G = G_0 e^{-\frac{r^2}{2\sigma^2}}, \]

where \(G_0\), \(r\) and \(\sigma\) denote the maximum energy power, the radial distance to the center, and the radius of the heat source, respectively.

The boundary conditions considered in this work are essential or Dirichlet denoted as \(u = u_0\) or some prescribed temperature \(T = T_0\) on some portions of the defined boundary. In a similar manner, traction or natural boundary condition are denoted as \(\sigma n = q\), as well as heat flux denoted by \((k(x)\nabla T) \cdot n = q\), prescribed on a given boundary. \(\sigma\) is the stress tensor which could be expressed as:

\[\sigma = [\lambda \nabla \varepsilon - \beta (T - T_0)] I + 2 \mu \varepsilon\]

\[\varepsilon = \frac{\nabla u + (\nabla u)^T}{2},\]

where \(I\) and \(\varepsilon\) are the identity and strain tensors, respectively. \(t_0 = (t_0, t_0)^T\) is the traction vector and \(t = (n_1, n_2)^T\) denotes the unit normal vector.

### 3. FPM Approach for Thermo-Elastic Problems

In this section, a short description of FPM discretization for thermo-elastic problems is described. We consider the Taylor expansions for the two approximating solution functions of the coupled system:

\[u = u + \sum_{k=1}^{2} \frac{\partial u}{\partial x_k} (x_{k,j} - x_k) + \frac{1}{2} \sum_{j,k=1}^{2} \frac{\partial^2 u}{\partial x_k \partial x_j} (x_{k,i} - x_k)(x_{j,i} - x_j) + e_{1,i}\]

\[v = v + \sum_{k=1}^{2} \frac{\partial v}{\partial x_k} (x_{k,i} - x_k) + \frac{1}{2} \sum_{j,k=1}^{2} \frac{\partial^2 v}{\partial x_k \partial x_j} (x_{k,i} - x_k)(x_{j,i} - x_j) + e_{2,i}\]

\[T = T + \sum_{k=1}^{2} \frac{\partial T}{\partial x_k} (x_{k,i} - x_k) + \frac{1}{2} \sum_{j,k=1}^{2} \frac{\partial^2 T}{\partial x_k \partial x_j} (x_{k,i} - x_k)(x_{j,i} - x_j) + e_{3,i}\]

where \(e_{1,i}\), \(e_{2,i}\) and \(e_{3,i}\) are the truncation errors of the Taylor series expansion. Together with these equations, (3), (4), and (5) with the corresponding boundary conditions should be also considered. \(u = u_0 = (u_0, v_0)\) or \(T = T_0\) in case of Dirichlet boundary conditions for displacement and temperature, or Neumann boundary conditions which is the most involved case explained next. If \(x \in \Gamma_n\), a linear system of \(m + 6\) equations is obtained which in terms of the truncation error can be written as:

\[e = Ma - b,\]
where
\[
\tilde{M} = \begin{pmatrix}
M & 0 & 0 \\
0 & M & 0 \\
0 & 0 & M \\
J_1 & J_2 & J_3
\end{pmatrix}
\]
and
\[
e = (e_{1,1}, e_{1,m}, e_{2,1}, \ldots, e_{2,m}, e_{3,1}, \ldots, e_{3,m}, 0, 0, 0, 0, 0, 0)'
\]
\[
\begin{pmatrix}
u_1, \ldots, u_m, v_1, \ldots, v_m, T_1, \ldots, T_m, 0, 0, \rho c T^n + \Delta G, t_{0,1} - \beta n_1 T_{ref}, t_{0,2} - \beta n_2 T_{ref}, q_0
\end{pmatrix}'
\]
\[
a = (a_u, a_v, a_T)
\]
\[
M
\]
is of the same form as in the original FPM discretization for elliptic PDEs. Moreover, matrices \(J_1, J_2\) and \(J_3\) are defined as follows:
\[
J_1 = \begin{pmatrix}
0 & 0 & 0 & 2\mu + \lambda & 0 & \mu \\
0 & 0 & 0 & 0 & \lambda + \mu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & (2\mu + \lambda)n_i & \mu n_i & 0 & 0 & \mu \\
0 & \lambda n_i & \mu n_i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
\[
J_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & \lambda + \mu & 0 \\
0 & 0 & 0 & \mu & 0 & 2\mu + \lambda \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu n_i & \lambda n_i & 0 & 0 & 0 \\
0 & \mu n_i & (2\mu + \lambda)n_i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
and
\[
J_3 = \begin{pmatrix}
0 & -\beta & 0 & 0 & 0 & 0 \\
0 & 0 & -\beta & 0 & 0 & 0 \\
0 & 0 & 0 & k & 0 & k \\
-\beta n_i & 0 & 0 & 0 & 0 & 0 \\
-\beta n_i & 0 & 0 & 0 & 0 & 0 \\
\rho c & -n_k \Delta t & -n_k \Delta t & 0 & 0 & 0
\end{pmatrix}
\]
Then, the corresponding solution for the linear system can be written as:
\[
a = \left(\tilde{M}' W \tilde{M}\right)^{-1} (\tilde{M}' W) b
\]
Therefore, for the computation of the displacements and temperature fields, only the first, seventh and thirteenth components in \(a\) are really needed. Consequently if:
\[
q_i = (q_{i,1}, q_{i,2}, \ldots, q_{i,18})'
\]
is the $i - th$ row in $\left( M^T W \tilde{M} \right)^{-1}$, equation (8) could be worked out to form the following linear system for the elements of the displacement vector $u$ and $T$:

$$u_j - \sum_{i=1}^{m_j} W_{ij} \left[ q_{k,1} + h_{1,i} q_{k,2} + h_{2,i} q_{k,3} + \frac{1}{2} h_{1,i}^2 q_{k,4} + h_{1,i} h_{2,i} q_{k,5} + \frac{1}{2} h_{2,i}^2 q_{k,6} \right] u_i +$$

$$+ \left[ \lambda n_1 q_{k,2} + \mu n_2 (q_{k,3} + q_{k,8}) + \lambda n_1 q_{k,9} - \beta n_1 q_{k,13} \right] (t_{0,i} - \beta n_1 T_{ref})$$

$$+ \left[ \lambda n_2 q_{k,2} + \mu n_1 (q_{k,3} + q_{k,8}) + (2\mu + \lambda) n_2 q_{k,9} - \beta n_2 q_{k,13} \right] (t_{0,i} - \beta n_1 T_{ref})$$

$$+ \left[ n_1 q_{k,14} + n_2 q_{k,15} \right] q_0 + \left[ \rho c q_{k,13} - n_1 k q_{k,16} - n_2 k q_{k,18} \right] (\rho c T^* + \Delta T)$$

for the component $u$. A similar equation could be obtained for the component $v$ and $T$ on the $j-th$ point. This is valid for all $u_j, v_j, T_j \in \Gamma_n$ where $k = 1, 7, 13$. Furthermore, a similar linear system is obtained for internal domain points where in such case the last three rows in $M^T$ and elements in $b$ must be deleted.

Finally, using this last formulation for the displacement and temperature fields, a coupled sparse linear system, $K\Phi = f$, is obtained which is numerically solved with iterative methods.

**Figure 1.** Problem configuration.

### 4. Assessment Case Studies

In order to test the numerical performance of this FPM formulation, two case studies are considered in this section.

**4.1. Heat Source Moving Along a Straight Line**

The first benchmark problem considered here corresponds to a heat source moving along a straight line according to which the kinematic relation $x(t) = [4t/5000 + 1,2]^T$ for $0 \leq t \leq 5000$. The problem configuration is shown in Figure 1.
The material employed in this example corresponds to an aluminum alloy A356.0-T6 with density \( \rho = 2670 \text{ kg/m}^3 \), thermal conductivity \( k = 151 \text{ W/mK} \), specific heat \( c = 963 \text{ J/kgK} \), linear coefficient of thermal expansion \( \gamma = 21.4 \times 10^{-6} \text{ 1/K} \), Young’s modulus, and Poisson’s ratio \( E = 72.4 \text{ GPa} \) and \( \nu = 0.33 \), respectively. The problem domain was discretized with approximately 9500 points with a mean spacing distance of 0.006 m and the time step was chosen as \( \Delta t = 10 \text{ s} \). The intensity of the heat source was considered as \( G_0 = 10000000 \text{ W} \), with a radius of \( r_0 = 0.05 \text{ m} \), and the initial temperature was taken as \( T = 200 \text{ K} \).
Aiming at validating the FPM for this kind of problems, the results obtained with it were compared with similar ones provided by FEM. A comparison of the temperature, displacement magnitude and von Mises stress fields are given in figures 2, 3 and 4 at two different times. As can be seen these figures, both methods predicted all the involved fields, with a very well correlated values along the domain.

Figure 4. Von Mises stress contours.

Figure 5. Temperature variation along the heat source path.

Figure 6. Displacement magnitude variation along the heat source path.
Figure 7. Von Mises stress variation along the heat source path.

For further assessment, the spatial variations of the fields considered in the previous figures were plotted along the heat source path and the results are shown in figures 5, 6 and 7. As shown, the numerical results predicted by FPM match very well the ones provided by the FEM counterpart. These results suggest that FPM is reliable for the analysis of thermo-elastic problems with moving heat sources in welding.

Figure 8. Problem configuration.

4.2. Heat Source Moving along a Curved Path

In order to confirm the results observed in the previous example, a second numerical benchmark is considered here. In this problem, a heat source moves along a curved path according to $x(t) = \left[3t/5000 + 1 \ (3t/5000)+1\right]^2/9 - 2(3t/5000 + 1)/9 + 10/9$ for $0 \leq t \leq 5000$. The problem configuration is shown in Figure 8.
The material employed in this example corresponds to steel with density $\rho = 7833 \text{ kg/m}^3$, thermal conductivity $k = 54 \text{ W/mK}$, specific heat $c = 465 \text{ J/kgK}$, linear coefficient of thermal expansion $\gamma = 11.5 \times 10^{-6} \text{ 1/K}$, Young’s modulus, and Poisson’s ratio $E = 200 \text{ GPa}$ and $\nu = 0.29$, respectively. The problem domain was again discretized with approximately 9500 points with a mean spacing distance of 0.006 m and the time step was chosen as $\Delta t = 10 \text{ s}$. The intensity of the heat source was considered as $G_0 = 5000000 \text{ W}$, with a radius of $r_0 = 0.05 \text{ m}$, and the initial temperature was taken as $T = 200 \text{ K}$.
A comparison of the temperature, displacement magnitude and von Mises stress fields are given in figures 9, 10 and 11 at two different times. Similarly, to the results observed in the previous example, both methods predicted similar values for the considered fields.

One more time, the spatial variations of the involved fields were plotted along the heat source path and the results are shown in figures 12, 13 and 14. These pictures indicate the robustness of FPM formulation for the solution of this kind of problems since its numerical prediction matches very well with the FEM solution. This confirms the results observed for the previous example. Therefore, it can be concluded that FPM is suitable and reliable for thermo-elastic problems with moving heat sources.
5. Conclusion
The very good results obtained in the numerical examples strongly suggest that the numerically implemented Finite Pointset Method approach is suitable and feasible for thermo-elastic problems with moving heat sources, which are present in welding processes. It is stable and has enough accuracy to capture the physical behavior observed in the governed processes. Moreover, the effectiveness and simplicity in incorporating the boundary conditions as well as their totally mesh-free Lagrangian nature make these approaches a promising numerical tool in this field. Therefore, it depicts a rich source of research opportunities in problems involving the movement of heat sources.

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