I. INTRODUCTION

Long ago, Dijkstra observed that “the quality of programmers is a decreasing function of the density of go to statements in the programs they produce” and famously argued that gotos should be abolished from high level programming languages \[1\]. If you are old (or geeky) enough to have ever used gotos, you know why: while they can simplify the task at hand and even make programs run faster, their repeated use quickly leads to “spaghetti code”. That’s a technical term for “incomprehensible mess”. The short term efficiency gain is paid for with a loss of understanding, which inevitably ends up causing errors.

In recent years, I have observed an analogous phenomenon in physics. Initially, I thought that it only occurred in isolated and particularly unfortunate cases (after all, finding oneself talking to me could be considered pretty unfortunate). As my sample has grown to include distinguished practitioners (or so I’m told), I have reluctantly been forced to accept that it is not so. The problem is ubiquitous.

With only months remaining before $10 billion worth of LHC start looking for the Higgs (or whatever actually breaks electroweak symmetry) it is my regrettable duty to inform you that many, probably even most of you do not understand gauge symmetry breaking. The culprit is none other than the “goto” of particle physics: the unitary gauge. Like the goto statement, the unitary gauge does simplify some tasks (proving unitarity, estimating mass spectra, introducing students to gauge symmetry breaking) it is my regrettable duty to tell you that many, probably even most of you do not understand gauge symmetry breaking. This is the essence of SSB: the theory (meaning the Lagrangian), has more than any individual ground state.

II. CLASSICAL THEORY

A. Goldstone model

The standard textbook example of spontaneous symmetry breaking (SSB) is the Goldstone model \[2\], defined by the Lagrangian

$$\mathcal{L}(\phi) = (\partial_{\mu} \phi)^{*} (\partial^{\mu} \phi) - V(\phi)$$

(1)

with “Mexican hat” potential

$$V(\phi) = \lambda \left[ \phi^{\dagger} \phi - \nu^{2} \right]^{2}$$

(2)

where $\phi = \phi(x)$ is a complex Lorentz scalar field and $\lambda$, $\nu$ are real, positive parameters. $V(\phi)$ has a degenerate “valley” of connected minima at $\phi^{\dagger} \phi = \nu^{2}/2$, i.e. a circle in the complex plane, and a local maximum at $\phi^{\dagger} \phi = 0$.

Solving the full equations of motion may be too hard for us, but we can at least obtain approximate solutions using perturbation theory. To do so, we must pick a stable point $\langle \phi \rangle$ about which small oscillations in $\phi$ will remain small. That means a point in the valley. Since gradient energy is minimized when $\phi(x)$ has the same value for all spacetime coordinates $x$, any constant $\phi$ in the valley minimizes total energy and therefore qualifies as a ground state (“vacuum”) of $\mathcal{L}(\phi)$. The circle $\phi^{\dagger} \phi = \nu^{2}/2$ is therefore known as the vacuum manifold of $\mathcal{L}(\phi)$. It has a global $U(1) \sim O(2)$ symmetry, i.e. it’s invariant under rotations of $\phi$ in the complex plane. Picking a particular vacuum breaks this symmetry. This is the essence of SSB: the theory (meaning the Lagrangian), has more symmetry than any individual ground state.

Let’s arbitrarily choose the point

$$\langle \phi \rangle = \frac{\nu}{\sqrt{2}}$$

(3)

on the real axis as the starting point for our perturbative expansion. To parameterize perturbations about it, introduce orthogonal field coordinates $\phi_{1}$ (along the real axis) and $\phi_{2}$ (along the imaginary axis):

$$\phi = \langle \phi \rangle + \frac{\phi_{1} + i \phi_{2}}{\sqrt{2}} = \frac{\nu + \phi_{1} + i \phi_{2}}{\sqrt{2}}$$

(4)

In terms of $\phi_{1}$ and $\phi_{2}$, the Lagrangian is

$$\mathcal{L}(\phi_{1}, \phi_{2}) = \frac{1}{2} (\partial_{\mu} \phi_{1})^{2} + \frac{1}{2} (\partial_{\mu} \phi_{2})^{2} - V(\phi_{1}, \phi_{2})$$

(5)
with potential

\[ V(\phi_1, \phi_2) = \frac{\lambda}{4} \left[ \phi_1^2 + \phi_2^2 + 2\nu \phi_1 \right]^2 \]

\[ = \lambda \nu^2 \phi_1^2 + \nu \lambda \phi_1 (\phi_1^2 + \phi_2^2) \]

\[ + \lambda \left( \frac{\phi_1^2 + \phi_2^2}{2} \right)^2 \] (6)

Note the first term: it gives \( \phi_1 \) a mass squared \( 2\lambda \nu^2 \). There is no term quadratic in \( \phi_2 \), so \( \phi_2 \) is massless.

The reason for this difference is that \( \phi_1 \) describes displacements against the restoring force of the potential, while \( \phi_2 \) describes displacements along the flat potential valley. The generalization of this observation is Goldstone’s theorem: spontaneous breaking of a continuous symmetry implies the existence of a massless field (a Goldstone boson; in supersymmetric theories, there are also Goldstone fermions – goldstinos – to match) for each flat direction. In mathematical terms, the number of such Goldstone bosons is the dimension of the coset space \( G/H \), where \( G \) is the full symmetry group of the vacuum manifold and \( H \) is the subgroup of \( G \) under which the vacuum remains invariant (if any).

The parametrization of Eq. (4) using Cartesian field coordinates \( \phi_1 \) and \( \phi_2 \) is the natural choice for a perturbative expansion, but ill suited to exploring \( \mathcal{L} \) beyond the infinitesimal neighborhood of a particular vacuum, i.e. to finding non-perturbative solutions. For that, it’s more convenient to use curvilinear coordinates which follow the geometry of the vacuum manifold. In the simple case of the Goldstone model, this means standard polar coordinates:

\[ \phi = \left( \langle \phi \rangle + \frac{\rho}{\sqrt{2}} \right) e^{i\theta} \]

\[ = \frac{\nu + \rho}{\sqrt{2}} e^{i\theta} \] (7)

The leading terms of a Taylor expansion of \( \phi \) in the radial and angular displacements \( \rho \) and \( \theta \) are

\[ \phi = \frac{\nu + \rho + i\nu \theta}{\sqrt{2}} \]

By comparison with Eq. (4), for small perturbations

\[ \phi_1 \simeq \rho \] (9)

\[ \phi_2 \simeq \nu \theta \] (10)

\( \phi_2 \) parameterizes the tangent of the vacuum manifold at \( \phi = \langle \phi \rangle \). In terms of \( \rho \) and \( \theta \), the Lagrangian is

\[ \mathcal{L}(\rho, \theta) = \frac{1}{2} (\partial_\mu \rho) (\partial^\mu \rho) + \frac{1}{2} (\rho + \nu)^2 (\partial_\mu \theta) (\partial^\mu \theta) - V(\rho) \]

with \( \theta \)-independent potential

\[ V(\rho) = \lambda [\rho (\nu + \rho/2)]^2 \]

\[ = \lambda \left[ \frac{\rho^4}{4} + \nu \rho^3 + \nu^2 \rho^2 \right] \] (12)

from which we again read off a mass squared \( 2\lambda \nu^2 \) for the radial excitation \( \rho \), while \( \theta \) remains massless.

### B. Higgs mechanism

The Goldstone model is turned into the simplest example of the Higgs mechanism [3][4] by upgrading its global U(1) symmetry to a local U(1) symmetry. The Goldstone Lagrangian is invariant under an identical rotation of \( \phi(x) \) at every spacetime point \( x \); the Higgs Lagrangian is invariant under an independent rotation at each spacetime point.

To construct it, introduce an Abelian gauge field \( A_\mu(x) \) with field strength tensor

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \] (13)

and Lagrangian

\[ \mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \] (14)

and substitute ordinary derivatives with covariant derivatives

\[ \partial_\mu \rightarrow D_\mu = \partial_\mu + ig A_\mu \] (15)

(\( g \) is the gauge coupling constant) in \( \mathcal{L}(\phi) \). The resulting total Lagrangian is invariant under the simultaneous local transformations

\[ \phi(x) \rightarrow \phi'(x) = e^{-i\omega(x)} \phi(x) \] (16)

\[ A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \frac{i}{g} \partial_\mu \omega(x) \] (17)

The variation in the gauge field \( A_\mu(x) \) exactly compensates the variation in \( \phi(x) \), i.e. if we apply the transformations of Eqs. (16) and (17) to the total Lagrangian and work through the algebra, we end up with exactly the same expression, apart from the trivial substitutions

\[ \phi(x) \rightarrow \phi'(x) \] (18)

\[ A_\mu(x) \rightarrow A'_\mu(x) \] (19)

This freedom to apply a local transformation implies that there are fewer independent equations of motion than degrees of freedom. The problem is of course the gauge field. Its Lagrangian is identical to that of a photon, with \( \phi \) in the role of a charged scalar, so what we have here is just scalar electrodynamics with an unusual \( V(\phi) \). We know that photons have only two independent degrees of freedom, the transverse polarization states, but \( A_\mu(x) \) has four. In scalar electrodynamics, we must therefore supplement the equations of motion with a gauge fixing condition before we can actually compute anything.

Following the standard textbook approach, we postpone that decision and choose a vacuum first. Choosing again \( \langle \phi \rangle = \nu/\sqrt{2} \) and expanding \( (\partial_\mu \phi)(\partial^\mu \phi) \rightarrow \)
In Cartesian coordinates, the new terms are

\[(D_\mu \phi)^\dagger (D^\mu \phi)\] yields

\[
(D_\mu \phi)^\dagger (D^\mu \phi) = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) + ig A_\mu ((\partial_\mu \phi)^\dagger \phi - \phi^\dagger (\partial^\mu \phi)) + g^2 A_\mu A^\nu \phi^\dagger \phi \tag{20}
\]

In Cartesian coordinates, the new terms are

\[(gv)^2 A_\mu A^\mu + ga_\mu (\phi_1 \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_1) + g^2 A_\mu A^\nu (\nu \phi_1 + (\phi^2_1 + \phi^2_2)/2) + gv A_\mu (\partial^\mu \phi_2) \tag{21}\]

In polar coordinates, they are

\[
(gv)^2 A_\mu A^\mu + ga_\mu \rho (\partial^\rho \phi_2) \tag{22}\]

The first term tells us that the gauge field \(A_\mu\) has picked up a mass squared \(2(gv)^2\). The remaining terms, except for the last one, describe interactions involving \(A_\mu, \phi_1 \sim \rho\) (massive as in the Goldstone case) and \(\phi_2 \sim \nu \theta\) (still massless).

The last term is problematic: it’s quadratic in the fields, like a mass term, but it mixes \(A_\mu\) and \(\phi_2 \sim \nu \theta\), suggesting that they are not independent. Counting degrees of freedom confirms this. We started out with a complex scalar (two real components) and a massless vector field (two more) for a total of four degrees of freedom. We ended up with two scalars and a massive vector field, which also has a longitudinal polarization state, for a total of \(2 + 3 = 5\) degrees of freedom. Since a simple change of coordinates can not affect the number of degrees of freedom, one of them is redundant.

This is where your favorite introductory field theory textbook points out that \(\phi_2 \sim \nu \theta\) can be made to vanish at every spacetime point \(x\) using the invariance of the total Lagrangian under the simultaneous local transformations of Eqs. (16) and (17). In polar coordinates, this is trivially easy to see: simply set the transformation parameter \(\omega(x)\) equal to the angular displacement \(\theta(x)\). The transformed field \(\phi'(x)\) will then be real, i.e. \(\phi_2' \sim \nu \theta' = 0:\)

\[
\phi'(x) = e^{-i\theta(x)} \phi(x) = e^{-i\theta(x)} (\nu + \rho(x)) e^{i\theta(x)} = \nu + \rho(x) \sqrt{2} \tag{23}\]

Provided that we also perform the transformation

\[
A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{g} \partial_\mu \theta(x) \tag{24}\]

the only effect on the Lagrangian, expressed in terms of \(\phi\) and \(A_\mu\), is to adorn the fields with primes according to Eqs. (15–19).

We conclude that gauge invariance allows us to impose the condition

\[
\theta(x) = 0 \Leftrightarrow \phi_2(x) = 0 \tag{25}\]

so as to remove the extra degree of freedom (known as a “would-be” Goldstone boson) and leave us with a Lagrangian written exclusively in terms of a massive \(A_\mu(x)\) (colloquially said to have “eaten” the would-be Goldstone boson in order to acquire a longitudinal component) and a massive \(\phi_1(x) \sim \rho(x)\). The latter is the simplest example of a Higgs boson. The condition of Eq. (23) (and its equivalents for larger symmetry groups) is the unitary gauge.

C. Can (maybe), not must!

The fact that we can impose the unitary gauge does not imply that we must do so, of course. There is a literally infinite number of conditions which may legitimately be used to remove the redundant degree of freedom, and you are free to use whichever is most convenient. Physical observables like energy density are by definition independent of this choice, but if you wish to directly compare solutions obtained in different gauges, all you have to do is transform them to a common gauge.

The main requirement on a gauge condition is that it be reversible: given an arbitrary field configuration \((A_\mu(x), \phi(x))\), a transformation must exist such that the transformed fields satisfy the condition \(\text{and}\) can be uniquely transformed back to the original configuration (see e.g. p. 7 in [3]). This ensures that no information is lost by applying the condition. If this requirement is not satisfied, strictly speaking the condition is not a gauge condition, but an unphysical constraint. It may still be useful in special situations, if it only cuts out a part of configuration space disjoint from that of the configurations under consideration, but it can not be used in full generality.

It is not hard to see that the unitary “gauge” is in fact such an unphysical constraint: it fails to be reversible at \(\phi(x) = 0\).

A famous example of the solutions living in the null space of the unitary gauge (its “blind spot”) is provided by the topologically stable vortices first described by Nielsen and Olesen [6]. Such a vortex is characterized by a non-trivial map from polar field coordinates \((\rho, \theta)\) to polar spatial coordinates \((r, \varphi)\)

\[
\phi(r, \varphi) = \rho(r) e^{i n \varphi} \tag{26}\]

and satisfies

\[
\lim_{r \to \infty} \rho(r) = 0 \tag{27}\]

\[
\lim_{r \to 0} \rho(r) = 0 \tag{28}\]
Extend it to a cylinder and you get a flux tube (or "string", not to be confused with fundamental ones) known to cosmologists as a cosmic string. The “winding number” is an integer (the Pontryagin index) specifying the number of times \( \phi \) goes around the potential valley as you walk along a single loop about spatial origin. The stability argument is simple: changing \( n \) requires the field to be lifted out of the potential valley and slide over the top of the Mexican hat, at an energy density cost \( \propto n \lambda^4 \).

If you try to transform a Nielsen-Olesen vortex to the unitary gauge throughout all space, you will inevitably run into trouble with Eq. (24) at \( r = 0 \): \( \theta \) is undefined there, so \( A'_\mu \) will be undefined too. The singularity at \( \phi(x) = 0 \) blinds the unitary gauge to this kind of solution. Nielsen and Olesen instead used the time-axial gauge \( A_0 = 0 \), and in so doing established the gauge of choice for non-perturbative work in gauge field theories with SSB \[1][2][3][4][5][6][7][8][9][10][11][12][13][14][15].

The lesson here is that while SSB is an intrinsically non-perturbative phenomenon, the unitary gauge is all about perturbation theory. There is nothing wrong with the Higgs Lagrangian expressed in terms of the original \( \phi \) field, i.e. before shifting field coordinates to a particular vacuum. The shift’s only purpose is to allow the use of perturbation theory. There is nothing wrong with a non-perturbative phenomenon, the unitary gauge is all you need.

\[ \Psi \rightarrow \Psi' = U \Psi \] (32)

To preserve gauge invariance, any matter field \( \Psi(x) \) added to the Lagrangian (fermion or scalar, at this point we don’t care) and coupling to \( W_{\alpha \mu} \) must transform according to

\[ \Psi \rightarrow \Psi' = U \Psi \] (32)

with

\[ U(x) = e^{-i \omega_a(x)} T_a \] (33)

and transformation parameters \( \omega_a(x) \). The covariant derivative

\[ D_\mu = \partial_\mu + igW_{\alpha \mu} T_a = \partial_\mu + igW_\mu \] (34)

compensates the variation in terms containing \( D_\mu \Psi \) (by making them transform like \( \Psi \), so that globally invariant combinations like \( \Psi U \partial_\mu \Psi \) become locally invariant after the substitution \( \partial_\mu \rightarrow D_\mu \) provided that the gauge fields undergo the simultaneous transformation

\[ W_{\alpha \mu} T_a \rightarrow W'_{\alpha \mu} T_a \]

\[ = U W_{\alpha \mu} T_a U^\dagger + \frac{i}{g} (\partial_\mu U) U^\dagger \] (35)

Note that we recover the Abelian case when \( T = 1 \).

The smallest non-Abelian Lie group is SU(2), with structure constants

\[ C_{abc} = \varepsilon_{abc} \] (36)

(the totally antisymmetric Levi-Civita symbol, with \( \varepsilon_{123} = 1 \)). In the fundamental (spinorial) representation,

\[ T_a = \frac{\tau_a}{2} \] (37)

where \( \tau_1, \tau_2, \tau_3 \) are the Pauli matrices

\[ \tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \] (38)

A scalar field in this representation must transform as an SU(2) doublet, i.e.

\[ \Phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} \rightarrow \Phi' = e^{-i \omega_a \tau_a} \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} \] (39)

The superscripts “+” and “0” are just labels (for now).

To induce SSB, we simply substitute this \( \Phi \) doublet into the Goldstone Lagrangian, Eq. (1), let \( \partial_\mu \rightarrow D_\mu \) and add the result to \( L_W \). In terms of the real \( \Phi \) components \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \)

\[ \phi^+ = \phi_3 + i \phi_4 \] (40)

\[ \phi^0 = \phi_1 + i \phi_2 \] (41)
the (Higgs) vacuum manifold is now a 3-sphere defined by
\[ \Phi^\dagger \Phi = (\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 + (\phi_4)^2 = \nu^2/2 \] (42)

There is actually more than we bargained for here: the symmetry of a 3-sphere is \( \text{O}(4) \sim \text{SU}(2) \times \text{SU}(2) \), twice the size of the \( \text{SU}(2) \) which we wish to break. This choice is made in anticipation of quantization. Classically, any symmetric potential will do, but if we want the quantized model to be renormalizable, the potential can be at most quartic in the fields (see 17 for a systematic survey of \( \text{SU}(N) \) and \( \text{O}(N) \) symmetry breaking patterns).

To estimate the perturbative mass spectrum, repeat the steps followed in the Abelian case. Shift the origin of our \( \Phi \) coordinates to
\[ \langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \nu \end{bmatrix} \] (43)

(i.e. \( \phi_0 \rightarrow \nu/\sqrt{2} + \phi_0 \)), expand the \((D_\mu \Phi) \dagger (D^\mu \Phi)\) term in the Higgs Lagrangian and read off the quadratic terms:
\[ \frac{1}{2} \left( \frac{g_\nu}{2} \right)^2 W_{\alpha\mu} W_{\alpha\mu} + \frac{\lambda \nu^2}{2} (\phi_0)^2 \] (44)

All three gauge bosons have become massive, along with the radial \( \Phi \) component in the chosen vacuum; but as expected, there are also terms mixing gauge bosons and derivatives of the angular \( \Phi \) components. They can again be removed by transforming to the unitary gauge: pass to “polar” field coordinates
\[ \Phi(x) = \frac{1}{\sqrt{2}} e^{\frac{i}{\kappa} \theta_a(x)} \begin{bmatrix} 0 \\ \nu + \rho(x) \end{bmatrix} \] (45)

and set the transformation parameters \( \omega_a(x) \) of Eq. 39 equal to the angular displacement \( \theta(x) \). Just as in the Abelian case, the exponentials cancel, and we are left with the real, radial component
\[ \Phi'(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \nu + \rho(x) \end{bmatrix} \] (46)

The would-be Goldstone bosons are gone, “eaten” by the gauge bosons, which have in turn been transformed according to Eq. 35.

It should now be evident by inspection of Eq. 35 that the analogous procedure will work for any semisimple Lie group, i.e. for any product of simple Lie groups. Historically, the electroweak sector of the Standard Model \([18, 19]\) was constructed this way, by Higgsing an existing gauge invariant Lagrangian, Glashow’s \( \text{U}(1) \times \text{SU}(2) \) model \([20]\).

E. Instantons and sphalerons

As we have already seen, the unitary gauge does not tell the whole story about the Abelian Higgs model. In the non-Abelian case, it reveals even less.

Consider again the Nielsen-Olesen vortex of Eq. (26). In topological terms, it is stable because a non-trivial map from polar spatial coordinates to polar field coordinates, i.e. from a circle (spatial infinity) to another circle (the vacuum manifold), can not be continuously deformed to the map from a circle to a single point; the maps belong to different homotopy classes. In the non-Abelian case, this stability argument need no longer hold. With more directions available in field space, there are more ways to deform a vortex without leaving the vacuum manifold: if the latter has the topology of a sphere rather than of a circle (i.e. if it is simply connected), there is nothing preventing a loop on it to be contracted to a single point, so classical stability of vortex solutions is no longer guaranteed. Such solutions can still occur as so-called embedded defects (solutions of a model based on a subgroup of the symmetry group under consideration \([21, 22]\) but to find out whether they are stable, you must study their dynamics in detail. Since \( \Phi \) going to zero at some point is a common property of defects, whether embedded or not, the unitary gauge is oblivious to their existence.

The problem is compounded by the non-trivial vacuum structure of YM theory, even without Higgs. Consider the gauge transformation of Eq. 35 applied to the trivial vacuum solution \( W_{\alpha\mu} = 0 \). Since a gauge transformation can not affect observable quantities (e.g. energy density) the resulting “pure gauge” configuration must be a vacuum solution too. But with a non-Abelian symmetry group at our disposal, we can clearly use Eq. (33) to create non-trivial maps between spacetime and internal (i.e. group) space: the smallest non-Abelian Lie group, \( \text{SU}(2) \), has three generators, so we have always at least one generator for each spatial dimension. For instance, with transformation parameters
\[ \omega_a(x) = \frac{n x a}{\sqrt{x^2 + \kappa^2}} \] (47)

(where \( \kappa \) is an arbitrary number and \( n \) is some integer) Eq. 33 maps each point \( U \) in group space to \( n \) points at spatial infinity (\( x^2 \rightarrow \infty \)). As with the Abelian vortices, two maps with different winding number \( n \) can not be continuously deformed into each other, i.e. they belong to different homotopy classes (unlike maps differing merely by \( \kappa \)). Substituting them into Eq. 35 therefore yields topologically distinct vacua, aptly called \( n \)-vacua, separated by energy barriers in configuration space. Since \( n \) can be any integer, the vacuum of pure YM theory is infinitely degenerate.

In the absence of other fields, transitions between \( n \)-vacua are classically forbidden. Upon quantization, tunneling transitions (instantons) become possible in principle, but remain far below observable rates in practice \([23, 24, 25, 26, 27]\). When you introduce \( \Phi \) however, non-contractible loops \([28]\) and spheres \([29]\) appear in configuration space, implying the existence of saddle point (i.e. unstable) static solutions perched on top of the barriers, along the minimum-energy paths between neigh-
boring n-vacua. Such solutions are known as sphalerons (Greek for “ready to fall”). Although they were first discovered in the SU(2) and U(1)×SU(2) Higgs models (where Nambu’s embedded vortices were also retroactively recognized as sphalerons), they have since turned up elsewhere too, from QCD to Einstein-Yang-Mills theory (YM coupled to gravity), showing that the primary role of Φ in enabling sphalerons may also provide an efficient mechanism for the production of a net B.

In the hot early universe, sphaleron transitions kept a B-violating channel open all the way down to T ∼ 100 GeV, so baryogenesis can not have happened much above the electroweak scale. Fortunately, under non-equilibrium conditions sphalerons may also provide an efficient mechanism for the production of a net B.

It should come as no great surprise that all these solutions feature points where Φ = 0, so they all live in the null space of the gauge. The problem has been known for a long time: gauge groups are compact, but the unitary gauge insists on topological triviality. A “unitary gauge-like” description of all physical degrees of freedom is therefore necessarily singular.

III. QUANTUM THEORY

Brief mention of instantons and baryon number non-conservation aside, everything we have done so far is strictly classical field theory, i.e. our fields have been ordinary functions. But as Feynman would say, “I’m not happy with all the analyses that go with just classical theory, because nature isn’t classical, dammit”. To make contact with the real world, we must quantize. In quantum field theory (QFT) this means turning the fields and their conjugate momenta into operators and imposing canonical commutation relations between them. Observables are then obtained as expectation values.

A. Effective action

Following Feynman, we introduce the action functional for a set of fields Φ(x) coupled to sources J(x) (all indices suppressed; each field has its own source)

$$\langle \mathcal{L} + J \Phi \rangle = \int d^4x \left[ \mathcal{L}(x) + J(x)\Phi(x) \right]$$

and write the transition amplitude between the vacuum in the infinitely far past and the vacuum in the infinitely far future as the path integral

$$W[J] = \langle 0, +\infty | 0, -\infty \rangle = N \int D\Phi e^{i(L + J\Phi)}$$

where N is a normalization constant. The expectation value of Φ can then be obtained by varying W[J] in J:

$$\frac{\delta W[J]}{\delta J(x)} = i\langle 0, +\infty | \Phi(x) | 0, -\infty \rangle_J$$

In practice it’s more convenient to work with the functional Z[J] (the generator of connected Green functions) satisfying

$$W[J] = e^{iZ[J]}$$

The closest relative to a classical field is the semiclassical (or “mean”) field

$$\Phi_{sc}(x) = \frac{\delta Z[J]}{\delta J(x)} = \frac{\langle 0, +\infty | \Phi(x) | 0, -\infty \rangle_J}{\langle 0, +\infty | 0, -\infty \rangle_J}$$

In the absence of sources, it reduces to the vacuum expectation value (VEV) of Φ(x); the condition for SSB in the quantum theory is therefore

$$\Phi_{sc}(x)_{J=0} \neq 0$$

If the functional dependence of Φ_{sc}(x) on J(x) is invertible, we can eliminate the explicit source dependence using the functional Legendre transform

$$\Gamma[\Phi_{sc}] = Z[J] - \langle J \Phi_{sc} \rangle$$

by which

$$J(x) = \frac{\delta \Gamma[\Phi_{sc}]}{\delta \Phi_{sc}(x)}$$

Without interactions, this expression reduces to the classical equations of motion for Φ. With interactions, there are corrections ∝ ℏ due to quantum fluctuations about the classical trajectory.

$$\Gamma[\Phi_{sc}]$$ is known as the effective action. It can be written as the multilocal expansion

$$\Gamma[\Phi_{sc}] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1...d^4x_n \cdot \Phi_{sc}(x_1)...\Phi_{sc}(x_n)\Gamma^{(n)}(x_1, ..., x_n)$$

where Γ^{(n)}(x_1, ..., x_n) are the proper vertices (one-particle irreducible Green functions) of the theory. Taylor expansion of Φ_{sc}(x_k) for k > 1 about x_1 yields the quasilocal form

$$\Gamma[\Phi_{sc}] = \left( \frac{1}{2} F(\Phi_{sc})\partial^\mu \Phi_{sc} \partial_\mu \Phi_{sc} - V(\Phi_{sc}) + ... \right)$$
where the functions $F(\Phi_{sc})$ and $V(\Phi_{sc})$ summarize our ignorance and "..." stands for higher order derivative terms. $V(\Phi_{sc})$ is the effective potential. If we require the vacuum to be translation invariant, the condition for SSB can be written

$$\frac{\partial V(\Phi_{sc})}{\partial \Phi_{sc}}\bigg|_{\Phi_{sc}=\nu\neq 0} = 0$$

(58)

i.e. the effective potential must have an asymmetric minimum, just like its classical counterpart.

Note that none of this involves perturbation theory. The shift of Eq. (3) is sometimes motivated with an analogy to elementary quantum mechanics, by anticipating the quantization of small oscillations about the bottom of the potential and their identification with particles, but as you can see, nothing in the formalism requires it. You are free to quantize the original, unshifted Lagrangian, and if you want to study non-perturbative phenomena, like SSB, that is what you should do. The vacuum manifold is then counterintuitively characterized by a non-vanishing occupation number, i.e. the vacua are not empty, but rather filled with the condensate of Eq. (53).

**B. Gauge fields**

When gauge fields are included, we are again confronted with the redundancy which they bring about. The path integral in Eq. (49) sums over all configurations, including those related by gauge transformations, and so becomes even more divergent than usual. The solution is to restrict the functional measure to only one representative from each gauge-equivalent class (Faddeev-Popov ansatz [43]). In practice, a factor $\text{det}(M_f)\delta(f(\chi))$ is introduced ($\chi$ now denotes both gauge and, optionally, matter fields). The function $f(\chi)$ goes to 0 only when the gauge condition is satisfied: $M_f = \delta f/\delta \omega$ is the functional Jacobi matrix of $f(\chi)$ which encodes its response to infinitesimal gauge transformations [2]. The requirement that the gauge condition be reversible translates to $\text{det}(M_f) \neq 0$.

When $M_f$ depends on the fields, it’s convenient to write

$$\text{det } M_f = e^{\text{Tr}(\ln M_f)}$$

(59)

and treat the exponential as additional terms in the action. This is done by introducing fictitious “ghost” fields. The full $\mathcal{L}$ going into Eq. (49) then consists of the original Lagrangian, the ghost Lagrangian and $\delta(f(\chi))$ in exponential form.

A simpler alternative is to use a gauge condition with field-independent $M_f$, so that $\text{det}(M_f)$ can be trivially factored out of the path integral. When the gauge condition is sufficiently simple for the Lagrangian to be written directly in terms of the independent variables, the $\delta(f(\chi))$ factor can be explicitly enforced and so does not appear as a separate gauge fixing term. A gauge with these properties is known as a physical gauge, since it only uses physical degrees of freedom.

The axial gauges $W^a_0 = 0$ and $W^3_0 = 0$ are physical gauges [4] (see [47] for a review). The naively defined unitary gauge is occasionally claimed to be one too, but since it produces gauge bosons with mass terms on the same form as those of the classical Maxwell-Proca Lagrangian for massive vector fields, it succumbs to the same problem upon quantization: a badly divergent perturbative expansion which can not be renormalized and so is useless for (most) practical calculations. From a modern perspective, it should come as no surprise that arbitrarily cutting out a part of configuration space spoils renormalizability (especially since the path integral runs over all field configurations, whether classical solutions or not), but the original work was done in the older canonical formalism. Historically [48], this is why the Weinberg-Salam model of electroweak interactions was not taken seriously until ’t Hooft’s proof of renormalizability in (initially a special case of) the $R_\xi$ gauges [49,50].

The $R_\xi$ gauges provide the only way to make sense of the unitary gauge at the quantum level that I know of. For the Abelian Higgs model, the gauge condition is

$$\partial_\mu A^\mu + \xi g \nu \phi_2 = 0$$

(60)

When it’s imposed, the Goldstone boson acquires a mass squared $\xi(g\nu)^2$. The $\xi$ parameter can take any value from 0 (Landau gauge, classically equivalent to the Lorenz gauge, featuring massless Goldstone bosons) through 1 (Feynman gauge, with Goldstone and gauge boson of equal mass) to the formal limit $\xi \rightarrow \infty$. The gauge boson propagator picks up an extra pole at the same mass as the Goldstone boson, so propagating Goldstone bosons damp out within a distance $\sim 1/(g\nu)$ [51] (essentially the same cancellation is seen in the axial gauge [52]). This gives the figure of speech that gauge bosons “eat” Goldstone bosons a much more direct interpretation than in the classical theory: they really do! The

\textsuperscript{2} See [44] and Section 9.1 in [147] for an introduction to the geometric interpretation of the Faddeev-Popov ansatz in terms of a hyperplane which intersects each gauge orbit once.

\textsuperscript{3} The Coulomb gauge famously fails to satisfy this requirement at the non-perturbative level in the non-Abelian case [45], and no gauge satisfies it when spacetime is a four-sphere, i.e. the hypersphere of five-dimensional Euclidean space [46]. While this may seem like a mathematical curiosity, it’s worth keeping in mind that which spacetime you’re working in can affect the validity of a gauge condition.

\textsuperscript{4} The time-axial gauge $W^a_0 = 0$ is a borderline case: it leaves a residual invariance under time-independent gauge transformations which is removed by explicitly imposing Gauss’ law. Since it commutes with the Hamiltonian, this need only be done at one point in time, e.g. on the initial conditions. See also Appendix A.
classically obscure transformation of Goldstone kinetic terms into gauge mass terms can now be understood as a consequence of Goldstone bosons being confined within a Compton wavelength of gauge bosons.

Gauge invariance implies that \( \xi \) must drop out from any calculation of physical observables, so leaving \( \xi \) unsigned and explicitly verifying that it does not enter the result provides a powerful error check. To make contact with the unitary gauge, note that the limit \( \xi \to \infty \) is equivalent to making the Goldstone bosons infinitely massive. Bringing an infinitely massive particle into existence requires an infinite amount of energy, so taking this limit effectively removes the Goldstone bosons from the theory (at least as long as gravity is ignored). In this sense, the limit \( \xi \to \infty \) corresponds to the unitary gauge, so the two are identified (a kind of argument known as “physicist’s mathematics”). But note that this is really very different from what we did in the classical theory.

The next time somebody tells you that “Goldstone bosons are just the longitudinal components of a massive gauge boson”, consider asking if gauge bosons really have infinite mass. That might prove entertaining.

C. Nielsen identities

Early work on SSB in QFT focused heavily on the effective potential, \( V(\Phi^{sc}) \) in Eq. (57), and understandably so: it is relatively easy to compute in a loop expansion (equivalent to an expansion in powers of \( \hbar \)) and lends itself to an intuitive interpretation as the quantum-corrected classical potential. In the absence of gauge interactions, it is also easily proved to be the expectation value of the energy density in the lowest energy state satisfying the constraint that \( \Phi^{sc} \) is spacetime independent \[58\]. Away from its minima, \( V(\Phi^{sc}) \) generally has an imaginary component, signalling that homogeneous field configurations are actually unstable, but this inconsistency was often ignored.

Less easily ignored was Jackiw’s observation that in gauge theories, \( V(\Phi^{sc}) \) is gauge dependent \[54\]. Physical quantities cannot depend on the choice of gauge, so apparently \( V(\Phi^{sc}) \) could not be an energy density in gauge theories. Not in every gauge anyway. But maybe in a special one?

In \[53\], Dolan and Jackiw performed the change to polar field coordinates (for the Abelian Higgs model) inside the path integral (without the singularities retained in \[41\], so implicitly assuming the absence of vortices), rewrote the resulting functional Jacobian as a ghost term and argued that the resulting “unitary Lagrangian” is “the unique description of the physical dynamics of the system from which the gauge degrees of freedom have been removed by a functional integration”. Happily, a one-loop calculation of \( V(\Phi^{sc}) \) yielded a finite result which could be reproduced in the \( R_\xi \) gauges, provided that the unitary limit \( \xi \to \infty \) was taken before sending the momentum cutoff in the loop integral \( \to \infty \).

Of course, \( V(\Phi^{sc}) \) is by definition a static quantity, not a dynamic one, and if there is any context where the unitary gauge might be expected to yield a finite answer, despite the exclusion of topologically non-trivial configurations, it should be one involving spacetime independent configurations only. But the “unitary Lagrangian” of \[52\] is not the classical one: the new ghost field term arising in the functional measure is now understood to be equivalent to a quartically divergent, non-polynomial Higgs self-coupling \[57\].

Even so, the suggestion that this “unitary Lagrangian” enjoys a unique status was quickly put to rest by Nielsen \[57\] and, independently, by Fukuda and Kugo \[58\]. Working in the Fermi gauges

\[
f(A^\mu) = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2
\]

Nielsen showed that gauge invariance implies a simple differential equation which relates the dependence of the effective potential on \( \xi \) and \( \Phi^{sc} \), respectively:

\[
\left[\frac{\partial}{\partial \xi} + C(\xi, \Phi^{sc}) \frac{\partial}{\partial \Phi^{sc}}\right] V(\xi, \Phi^{sc}) = 0
\]

The function \( C(\xi, \Phi^{sc}) \) can be determined order by order in the loop expansion. Essentially, this means that a variation in \( \xi \) is always compensated by one in \( \Phi^{sc} \), keeping the value of the effective potential invariant to each order of the expansion. The generalization of this equation to the non-Abelian case and to successively more general subclasses of the \( R_\xi \) gauges was carried out in \[59\] \[60\] \[61\] \[62\]. The result is now collectively known as the Nielsen identities.

In \[58\], a generalization of this approach was used to show that the full effective action (effective potential plus derivative terms) is gauge invariant at stationary points (so solutions of the effective field equations are gauge invariant) to each order of the loop expansion. The effective action can be computed in any (workable) gauge, the energy density computed at its stationary points is gauge invariant, the stationary points with the smallest energy densities are the true vacua, and if the fields at such a stationary point are not spacetime dependent, it follows that their energy density is given by the effective potential. The last condition is satisfied by a wide class of gauges (dubbed “good gauges” by Fukuda and Kugo), including Coulomb, axial, Fermi and \( R_\xi \).

On the other hand, the effective action (and potential) is generally not gauge invariant away from its stationary points (“off shell”). As first pointed out by Vilkovisky, this is due to the dependence of the couplings by the unitary gauge” was born.
between fields and external sources in Eq. (18) on the chosen field parametrization \[63\]. It is possible to write down Nielsen-like identities which enforce invariance under field reparametrizations and to construct effective actions which satisfy them, and which can therefore be identified with energy density also off shell \[64\]. In particular, the so-called Vilkovisky-DeWitt effective action for YM theory is just the ordinary effective action evaluated in the covariant background field gauge \[44\] \[65\]: each field is written as the sum \(W_\mu + Q_\mu\) of a background part \(W_\mu\) (neither gauge fixed, coupled to a source or path integrated over) and of a quantum part \(Q_\mu\) (coupled to a source and path integrated over as usual) whose covariant derivative with respect to \(W_\mu\) is required to vanish:

\[
\partial_\mu Q_\mu + gC_{abc} W_{\nu a} Q^c_\nu = 0
\] (63)

Like all covariant gauge conditions, Eq. \[63\] requires the introduction of ghosts. If on-shell gauge invariance is all you need, an axial condition on \(Q_\mu\) will do the job without them.

### D. Finite temperature

When reading old papers from the golden age of QFT, roughly mid-60s to mid-70s, it is hard to miss the shifting view of gauge symmetries and SSD: from neat mathematical tricks allowing the construction of renormalizable theories to physical reality. The watershed event was the realization that spontaneously broken symmetries are restored at high temperature \[67\] \[68\] \[69\]. As Weinberg put it, “if a gauge symmetry becomes unbroken for sufficiently high temperature, then it is difficult to doubt its reality” \[69\].

To see how this comes about, consider again the effective action. Using the conventionally defined path integral of Eq. \[49\], the proper vertices in Eq. \[56\] are the vacuum-to-vacuum expectation values of time-ordered field operators:

\[
\Gamma^{(n)}(x_1, \ldots, x_n) = \langle 0, +\infty | T[\Phi(x_1) \cdots \Phi(x_n)] | 0, -\infty \rangle
\] (64)

In plain English, they describe sequences of scattering events starting in empty space in the infinitely far past and ending in empty space in the infinitely far future. If the system under study is not empty space, this is not an adequate model. For instance, in a heat bath in equilibrium at inverse temperature \(\beta\), the probability of a scattering event between energy eigenstates \(|\Phi_n\rangle\) with energy \(E_n\) follows the Boltzmann distribution

\[
P_n = \frac{e^{-\beta E_n}}{\sum_m e^{-\beta E_m}}
\] (65)

Given a complete, orthonormal set of \(|\Phi_n\rangle\), we should then use the finite-temperature proper vertices

\[
\Gamma^{(n)}_{\beta}(x_1, \ldots, x_n) = \sum_l e^{-\beta E_l} \langle \Phi_l, +\infty | T[\Phi(x_1) \cdots \Phi(x_n)] | \Phi_l, -\infty \rangle
\]

or, dropping the orthogonality requirement, the more general

\[
\Gamma^{(n)}_{\beta}(x_1, \ldots, x_n) = \text{Tr} e^{-\beta H} [\Phi(x_1) \cdots \Phi(x_n)]
\]

where the trace runs over any complete set of states and \(H\) is the Hamiltonian. By comparison with the usual expression for the transition amplitude from initial state \(|\Phi_i\rangle\) to final state \(|\Phi_f\rangle\),

\[
\langle \Phi_f | e^{-tH} | \Phi_i \rangle = \int \mathcal{D}\Phi e^{-\int_0^t dt \int d^3x L}
\]

we can therefore handle the (equilibrium) finite temperature case by performing the formal substitution \(t \rightarrow \beta\) and restricting the functional integration to field configurations periodic in \(\beta\) (antiperiodic for fermions) \[70\] \[71\] \[72\]. The generating functional \(W[J]\) of Eq. \[49\] then becomes the partition function of statistical mechanics.

Using this technique to compute the effective potential \(V(\Phi_{sc})\) at finite temperature reveals that it picks up a positive mass term \(\propto 1/\beta^2\). At a sufficiently small \(\beta\) (the inverse critical temperature), this thermal mass term becomes dominating and turns \(\Phi_{sc} = 0\) into the global potential minimum. The symmetry is then restored, i.e. \(\Phi\) is equally likely to be found anywhere on the zero temperature vacuum manifold.

The argument that the “unitary Lagrangian” of \[52\] enjoys a unique status was dealt its first blow in this context (and by its own originators). When the one-loop comparison with the \(R_\xi\) gauges was extended to finite temperature in \[68\], the critical temperatures did not match, and the \(R_\xi\) result was identified as the correct one.

The discrepancy was attributed to the non-renormalizability of the unitary gauge, but this was counterintuitive: the compactification of the time dimension which turns QFT into equilibrium thermal field theory replaces loop integrals with sums over discrete frequency modes, each described by its own three-dimensional theory, and lower dimensional theories have better high energy behavior than higher dimensional ones. Indeed, renormalization at zero temperature is always sufficient to remove all infinities, so why did problems crop up in the finite, thermal loop contributions and not in the zero temperature part?

This became known as “the unitary gauge puzzle”. The key to the solution is the observation that while the value of the effective potential is gauge invariant at stationary points (in “good gauges”), its curvature (used to obtain the effective Higgs mass in \[68\]) is not: to compute
the latter necessarily involves going off shell \[73\], where the identification of \(V(\Phi_{sc})\) with energy density no longer holds. Even so, it took several iterations \(74\), \(75\), \(76\) (and a couple of decades) to recognize that the correct critical temperature can in fact be extracted order by order from any self-consistent perturbative expansion, even (!) one based on the “unitary Lagrangian” of \[57\].

Needless to say, the high temperature symmetric phase is in the null space of the unitary gauge. There is also a problem common to all gauges featuring unphysical degrees of freedom: the traces of Eq. (67) run over all of state space, but there is no reason why ghosts etc. should be in thermal equilibrium with the heat bath. The starting point for thermal gauge field theory is therefore necessarily a physical gauge \[77\].

**E. Effective field theories**

The temperature dependence of \(V(\Phi_{sc})\) is a reminder that physics is not scale invariant; the world looks and acts differently at different energy scales. Fortunately, you don’t need detailed knowledge of physics at the scales of grand unification or quantum gravity to understand physics at the electroweak scale, you don’t need detailed knowledge of physics at the electroweak scale to understand nuclear physics, and you don’t need detailed knowledge of nuclear physics to understand chemistry.

In QFT, this independence of lower energy phenomena from higher energy ones is formally known as the Appelquist-Carazzone decoupling theorem \[78\]. What it says is that massive fields effectively decouple at low energy: given a renormalizable Lagrangian containing both massless and massive fields, you can describe its low energy behaviour with a renormalizable Lagrangian written in terms of the massless fields only. The massive fields only contribute to the low energy Lagrangian through the renormalization of its couplings and fields.

More generally, you can eliminate heavy fields from a Lagrangian which also contains light fields by encoding their effects in (generally non-renormalizable) interaction terms involving the light fields only. The resulting effective field theory (EFT) is valid at energies below the masses of the eliminated fields \[79\]. The whole edifice of standard model extensions – technicolor, supersymmetry, grand unified theories (GUTs), supergravity and string theory with its infinite tower of massive excitations – implicitly depends on the suppression by powers of energy over mass of the effective interaction terms induced by the high energy extensions.

In principle, constructing an effective field theory from a more fundamental one is straightforward. Take the effective action of Eq. (50). Fourier transform it and do all momentum integrals involving the heavy fields. Split the remaining integrals in two parts, one for momenta going up to your cutoff (the mass of the lightest eliminated field or less), one for momenta above the cutoff, and do the latter too (alternatively, gauge-fix and integrate out the heavy fields only, leaving any local invariance specific to the light fields unbroken until the need actually arises to gauge-fix them too \[80\]). Transform back, and you are left with a non-local action on the form of Eq. (50), written in terms of the light fields only and valid down to distance scales \(\sim 1/(\text{momentum cutoff})\).

In practice, this “top-down” program may not be possible to carry out, even approximately, either because you don’t know the fundamental Lagrangian or because there is no workable approximation scheme. In such cases, you may still be able to create an EFT by systematically writing down all interaction terms (up to some cutoff dimension) involving light fields only and respecting all known symmetries. The schoolbook example of this approach is chiral perturbation theory of low energy QCD, a perturbative expansion in masses and momenta of quark bound states, small on the hadron mass scale \(\sim 1\ \text{GeV} \[81\] \[82\] \[83\]. In this case, the fundamental Lagrangian is known, but ordinary perturbation theory breaks down at low energy due to the growth of the effective coupling (confinement).

Fortunately, this problem does not occur in Higgsed YM theories: small couplings stay small also at low energy, and the Appelquist-Carazzone theorem ensures the existence of an EFT written in terms of the massless fields only (if any). Since corrections to the tree level terms are \(\propto\) powers of couplings over energy, integrating out the massive gauge bosons from a weakly coupled theory induces negligible effects at energies \(\ll\) the symmetry breaking scale. In this low energy/long distance limit, the EFT reduces to the classical Lagrangian with all massive gauge bosons set to zero\(^6\).

More generally, the classical theory should provide a good approximation to the finite temperature dynamics of modes with large occupation number, i.e. for fields with mass \(\ll\) temperature \[84\] \[85\] \[86\].

**F. Non-linear sigma models as EFTs**

Consider the low energy limit of the original U(1) \(\sim\) O(2) Goldstone model. A look at Eqs. (11)-(12) is enough to tell that the effective Lagrangian must be

\[
\mathcal{L}(\theta) = \frac{1}{2} \nu^2 \partial_\mu \theta \partial^\mu \theta
\]

i.e. that of a free, massless scalar field (the Goldstone boson) taking values on a circle in internal space (since \(\theta\) is an angle, so \(\theta = 0\) and \(\theta = 2\pi\) are identified; the circle is of course the bottom of the Mexican hat potential). There can’t be any \(\rho\) particles on shell when the energy density is \(\ll\) the rest mass of a \(\rho\) per Compton volume.

\(^6\) I emphasize this because I have occasionally noticed some confusion on this point, especially among cosmologists, who sometimes seem to take classical Lagrangians a little too literally.
i.e. \( \sim \lambda^2 \nu^4 \), and quantum corrections at energy \( E \) can’t be worse than \( \sim \lambda^2 E/\nu \).

The factor \( \nu^2 \) in front of the derivative term is the “stiffness” of \( \theta \). Make it larger and it costs more energy to lift \( \theta \) from the ground state (an arbitrarily chosen, constant \( \theta \) value). Note that massless is not synonymous with “cheap”; when we speak of “low energy EFT”, we really mean “low energy density EFT” (just like we really mean “Lagrangian density” when we say “Lagrangian” in field theory). This is why the derivative term of a massless field belongs in a low energy EFT even if it is associated with a very large energy scale: unlike a mass term, a derivative term has a continuous spectrum (unless Lorentz symmetry is broken, e.g. by a spatially periodic potential a la Kronig-Penney) so arbitrarily small values are allowed.

There is more interesting physics in Eq. (69) than meets the eye, all due to the identification of massless fields with “cheap”; when we speak of “low energy EFT”, we really mean “low energy EFT” (just like we really mean “Lagrangian density” when we say “Lagrangian” in field theory). The low energy limit is then given by the Lagrangian

\[
\mathcal{L}(\vec{n}) = \frac{1}{2} \nu^2 \partial_\mu \vec{n} \cdot \partial^\mu \vec{n}
\]

where \( \vec{n} \) is an \( N \)-dimensional vector constrained to take values on the \( (N - 1) \)-sphere, i.e. satisfying \( |\vec{n}| = 1 \). Mathematicians call this a wave map, physicists a non-linear sigma model (NLSM).

For \( N = 3 \), \( \vec{n} \) is a 3-vector taking values on an ordinary unit sphere (a 2-sphere). In terms of standard spherical coordinates \((\vartheta, \varphi)\),

\[
\begin{align*}
n_1 &= \sin(\vartheta) \cos(\varphi) \\
n_2 &= \sin(\vartheta) \sin(\varphi) \\
n_3 &= \cos(\vartheta)
\end{align*}
\]

and the Lagrangian reads

\[
\mathcal{L}(\vartheta, \varphi) = \frac{1}{2} \nu^2 \left[ \partial_\mu \partial^\mu \vartheta + \sin^2(\vartheta) \partial_\mu \varphi \partial^\mu \varphi \right]
\]

Add a quartic derivative term and you get either the Skyrme model \[54\], a precursor to modern chiral perturbation theory, or the Skyrme-Faddeev model \[55\], a candidate low energy EFT for pure SU(2) YM theory. The extra terms are introduced to defeat Derrick’s theorem, a simple scaling argument which rules out the existence of finite-energy static solutions to scalar theories with quadratic gradient terms in more than two spatial dimensions \[56\]. On its own, the NLSM can only have time-dependent three-dimensional solutions with finite energy.

Note the non-linear term (the “NL” in NLSM), which suggests richer dynamics than in O(2), but also note that the O(2) case is recovered when either \( \varphi \) or \( \vartheta \) is constant. This is easily seen to generalize: higher O(N) models embed lower ones.

The construction of O(3) NLSM solutions is greatly simplified by the stereographic projection to the complex plane (extended with a point at infinity to represent the north pole)

\[
u(\vartheta, \varphi) = \tan(\vartheta/2) e^{i\varphi}
\]

after which

\[
\mathcal{L}(u) = 2\nu^2 \frac{\partial_\mu u^\mu \partial^\mu u}{(1 + u^\mu u^\mu)^2}
\]

In this form, the model is known as CP\(^1\) (Complex Projective, one dimension). Writing \( u = p + i q \) with real \( p \) and \( q \), the equations of motion are

\[
\begin{align*}
\partial_\mu \partial^\mu p + 2p \partial^\mu q \partial_\mu q - \partial^\mu p \partial_\mu p & = 0 \\
\partial_\mu \partial^\mu q + 2q \partial^\mu p \partial_\mu p - \partial^\mu q \partial_\mu q & = 0
\end{align*}
\]

If \( u \) is analytic in the \((x, y)\) plane, the Cauchy-Riemann equations guarantee that \( p \) and \( q \) satisfy both the Laplace \((\nabla^2 p = 0)\) and the eikonal \((|\nabla p|^2 = 0)\) equation and therefore also the equations of motion in \((x, y)\). A separable factor which satisfies the wave equation along \( z \), i.e. \( u(t \pm z) \), also works, so you can easily build either static solutions constant in \( z \) (e.g. \( u = (x + iy)^n \)) or vortex winding \( n \) times around the \((x, y)\) plane or wave packets of arbitrary three-dimensional shape moving up and down the \( z \) axis.

But the real news relative to the O(2) case is the existence of nontrivial maps from the 2-sphere in field space to the 2-sphere at spatial infinity, as in

\[
\begin{align*}
u & = \left( \frac{x + iy}{r + z} \right)^n \\
r & = \sqrt{x^2 + y^2 + z^2}
\end{align*}
\]

(see \[90\] for more). When \( n = 1 \), this is known as a hedgehog. In line with Derrick’s theorem, its energy is proportional to volume, but like the global vortex, it has a gauged counterpart with finite energy: the monopole first described by ’t Hooft and Polyakov \[91\][92].

The next step up the NLSM ladder is O(4), i.e. a 4-vector taking values on a unit 3-sphere. Its new feature is the existence of knot-like maps from the internal 3-sphere.
to the spatial 2-sphere ("textures" to cosmologists). Unlike vortices and hedgehogs, those do not have a singular core corresponding to $\Phi_{sc} = 0$ in the full Goldstone model. Derrick’s theorem guarantees that they too must be unstable: they shrink until their gradient energy density becomes too large for the low energy EFT to handle. In the full model, the collapse ends with $\Phi_{sc}$ sliding over the top of the potential, allowing the knot to unwind. This never happens given suitably "small" initial conditions, however [92]. Topologically trivial configurations (sometimes referred to as "non-topological textures") can therefore keep evolving indefinitely.

Hadron physicists know all about the O(4) NLSM. When the vector components are light mesons (bound pairs of up and down quarks), it corresponds to the leading order terms of chiral perturbation theory. The "sigma" in "sigma model" comes from the nuclear isospin singlet which acquires a VEV (the "Higgs particle"); the three independent vector components (the Goldstone bosons) are the pions. Vortices, hedgehogs and even polyhedral maps have all been described in this context, and it has been suggested that bubbles of a Disoriented Chiral Condensate (DCC) rotated away from its value in our QCD vacuum may be possible to create in collider experiments [93][94][95][96][97][98][99][100].

In three spatial dimensions, the map from internal 3-sphere to spatial 2-sphere is the last one with separate homotopy classes, so higher O(N) NLSMs do not add qualitatively new kinds of solutions to those found at N = 4. This also implies that the O(4) NLSM can provide a reasonable approximation to the dynamics of the general N > 4 case [101].

IV. COSMOLOGY

If you subscribe to standard hot big bang cosmology, the relevance of high temperature symmetry restoration is obvious. Immediately after the bang, the universe was filled with a high temperature plasma in near thermal equilibrium. As it cooled, $V(\Phi_{sc})$ gradually changed shape, from harmonic to quartic, and $\Phi_{sc}$ eventually rolled or tunnelled from $\Phi_{sc} = 0$ to a randomly picked, asymmetric minimum\(^7\). Given a particle horizon, i.e., a finite maximum radius within which light (or anything else) could have traveled, this phase transition could not be coordinated over all space, so causally disconnected regions picked vacua independently of each other.

A. Domains and dark energy

This mechanism was recognized already by Weinberg in his seminal paper on high temperature symmetry restoration [69], which introduced the now familiar analogy with domain formation in ferromagnetism and went on to ask: "Does the universe consist of domains, in which symmetries are broken in equivalent but different directions? If so, what happens when a particle or an observer travels from one domain to another?"

The second part of Weinberg’s question was quickly (but, as it turns out, only partially) answered by A. Everett, who considered the fate of light crossing between domains in the standard electroweak model [103] and found it to depend on the width of the boundary: "If the transition between domains is smooth, with a transition region whose thickness is large compared with a wavelength of the incident radiation, there is no reflection and one observes a transmitted wave identical to the incident wave." A thin boundary, on the other hand, would result in total reflection. After the discovery in 1998 of a luminosity deficit from high redshift Type Ia supernovae (SNe Ia) [104][105][106], his description of the intermediate case, a domain boundary of width comparable to wavelength, seems almost prophetic: "A distant object viewed through such a domain boundary would appear less bright than it should. If one knew independently the distance and brightness of the object, say by knowing its cosmological red-shift and the brightness of objects similar to it, then the existence of such semi-transparent domain walls would be detectable." This was published in 1974!

But there is a problem with Everett’s derivation which would not have been lost on Feynman: it’s a purely classical treatment of the electroweak boson sector (linearized to boot). The decay to fermions of virtual weak bosons traversing a wide boundary is not considered. For a macroscopic boundary width and an incident photon packing enough energy to create at least a neutrino pair, the negligible lifetime $\sim 10^{-25}$ s of weak bosons (and the absence of reflection at tree level) clearly makes this the dominating effect. The problem then reduces to projecting the photon (i.e. massless) state in the source domain onto the photon state in the destination domain, and the residual luminosity for domains related by SU(2) parameters $\vec{\omega} = [\omega_1, \omega_2, \omega_3]$ is easily found to be $[102]$

$$\ell(\vec{\omega}) = \sin^2(\theta_W) \frac{\omega_1^2 \cos(|\vec{\omega}|) + \omega_2^2}{|\vec{\omega}|^2} + \cos^2(\theta_W)$$

where $\omega_1^2 = \omega_1^2 + \omega_2^2$ and $\theta_W$ is the Weinberg angle, $\sin^2 \theta_W \approx 0.2216$. One or two electroweak domain boundaries between us and high redshift SNe Ia would explain their luminosity deficit without new physics, and may also provide the missing energy density required by

\(^7\) This does not imply that the finite temperature effective potential provides a good description of the dynamics of the transition from symmetric to broken symmetry phase. $V(\Phi_{sc})$ is computed for spacetime independent $\Phi_{sc}$ and (at finite temperature) using the equilibrium partition function. Spacetime independent, equilibrium configurations only occur at extrema of the potential, so $V(\Phi_{sc})$ can only be used to determine from which critical temperature a phase transition is allowed, not to study its dynamics. See [102] and [83].
the apparent flatness of the universe, so obviously everybody but me hates the idea.

Besides desperation for any sign of new physics, why did nobody follow up Everett’s work in 1998? A possible explanation is that it had simply been forgotten. Another is the popularity of inflation: when the supernova news broke, the first thought to cross the mind of most physicists probably was “Oh, so it’s still going on!” (it certainly was my first thought).

Yet another reason might have been Everett’s use of the term “domain wall”, which had since become synonymous with “topologically stable two-dimensional defect passing through Φ_{sc} = 0”. The standard electroweak model notoriously fails to satisfy Kibble’s group-theoretic criteria for the existence of domain walls, and so must any sensible alternative to it, since a domain wall network would quickly and catastrophically collapse to a massive, single wall dominating its horizon volume.

In hindsight, it may seem strange that transients (i.e. not necessarily stable configurations) interpolating between different values of Φ_{sc} on the vacuum manifold (rather than passing through Φ_{sc} = 0) were not considered in this context. That they existed in the early universe, and may have played an important cosmological role as seeds of large-scale magnetic fields, was already well understood. But by now you should have no problem guessing the (erroneous) argument against their survival to late times: transforming such configurations to the unitary gauge turns them into collections of weak gauge bosons, which are unstable and decay in a microphysical time.

One way to see that this cannot be right is to recognize the implicit assumption that only the Higgs vacuum manifold is degenerate. The transformation to the unitary gauge would then leave the gauge bosons with only one vacuum to “fall” to, so no coordination across space would be needed to complete the relaxation to the vacuum. That this assumption is incorrect in the standard electroweak model has been elegantly demonstrated by Lepora and Kibble, who showed that the vacuum manifold of the electroweak gauge bosons is actually a squashed hypersphere. The unitary gauge conceals this degeneracy in a “spaghetti code” mess of interaction terms which I won’t even try to reproduce here (see e.g. Ch. 14 in [119]).

More generally, even disregarding that Goldstone bosons are not “just longitudinal components of massive gauge bosons” in a consistently quantized theory, it is obvious from the vantage point of a physical gauge that the argument must be wrong, since it assigns a privileged role to an arbitrarily chosen point on the vacuum manifold. It is isomorphic to somebody in New York claiming that people in London are not standing vertically and so must all be falling over: after all, they are not aligned with the Empire State Building!

Since Earth’s (idealized) surface and the vacuum manifold are both Riemannian and symmetric, it is of course always possible to connect any two points on either one by a curve featuring a continuous sequence of tangent spaces, each having equivalent but different definitions of “vertical” (or of “gauge boson X”). Your ability to stand vertically at your current location is sufficient to know that you could create a chain of people connecting any two cities on Earth, all standing vertically according to their own, local definition. Likewise, the existence of a stable gauge boson in a single vacuum is sufficient to know that you could create interpolating configurations between any two points on the vacuum manifold using only stable gauge bosons. Massless gauge bosons are necessarily stable (below some fermion pair production threshold), so if your theory has them, you are all set. Nambu gave us a convenient sufficient criterion for their existence: the broken symmetry group must be semisimple (as in the standard electroweak model) or have rank ≥ 2. In other words, the micropysical decay argument fails in any realistic theory.

Microphysical stability does not imply classical stability of macroscopic configurations, of course. What it does imply is that the realignment of Φ_{sc} after the initial, random choice of vacua in causally disconnected regions must be resolved according to the effective field equations. In a cosmological model with finite particle horizons, this is enough to guarantee the existence of electroweak (and maybe other) domains. Their boundaries, whether classically stable or not, might be possible to inflate beyond the current horizon with a late period of exponential expansion, but that would require new physics at the electroweak scale (or lower) and risk diluting away any net baryon number created at the electroweak phase transition.

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8 To answer the inevitable question: no, I was not aware of it when I wrote [107], nor does Penrose seem to have been when he made the remarks [108] which got me thinking about this problem.

9 A related problem is posed by topologically stable monopoles, which occur in any gauge theory featuring SSB of a simple group to a semisimple group containing a U(1) factor, like the standard electroweak group. This was one of the original selling points of inflation: if the standard model descends from a simple GUT by SSB, the energy density of a single GUT monopole per horizon volume at the symmetry breaking transition would smother the universe by many orders of magnitude. Inflation solves the problem by diluting monopoles and other defects left over from GUT symmetry breaking to acceptable levels.

10 Yes, I was once summarily dismissed with this argument by a supposedly top journal.

11 An esteemed editor at another top journal recently informed me that the unitary gauge can always be applied using partially overlapping, flat charts proves that Earth is flat. Yes, flat-earthers are funny.
B. Global and local textures

In the early 90s, textures, topological [122] and not [123], were considered serious candidate sources of the primordial density perturbations which seeded the large scale structure of the universe. More recently, it has been suggested that the large anomalous cold spot found by WMAP in the cosmic microwave background (CMB) is due to a texture which originated at a symmetry breaking scale $8.7 \times 10^{15}$ GeV (intriguingly close to the hypothetical GUT symmetry scale $\sim 10^{16}$ GeV [124]) and eventually collapsed at redshift $z \sim 6$, a billion years after the big bang [125].

The textures in question have all been global, of the $O(N > 3)$ Goldstone model → NLSM variety. This runs counter to the commonly held view that continuous symmetries should be local (i.e. gauged), and it runs into serious difficulties with quantum gravity (string theory in particular does not seem to allow any global continuous symmetries; see e.g. p. 255 in [126]). The reason for this particular does not seem to allow any global continuous symmetries with quantum gravity (string theory). However, the reason why $\Phi$ should have vanishing gradient energy at every point in space is clear. Gauging the scale 8

A scalar field $\Phi$ constrained to the vacuum manifold can be written

$$\Phi(x) = U(x)\Phi_0$$  \hspace{1cm} (82)

where $U(x)$ is a symmetry transformation and $\Phi_0$ is an arbitrarily chosen reference point on the manifold. If $\Phi$ is gauged, with covariant derivative $D_\mu$ given by Eq. (34), the gauge field can “fall” to

$$W_\mu = \frac{i}{g} (\partial_\mu U) U^{-1}$$  \hspace{1cm} (83)

at every point in space. This makes $D_\mu \Phi = 0$, so the gradient energy of $\Phi$ is zero. Since $W_\mu$ is a pure gauge configuration, its energy also vanishes and the configuration stops evolving. Naively, this relaxation process should complete on the time scale of the gauge interaction, i.e. in a microphysical time, so any gauged texture will be gone long before it can have observable consequences.

It is not hard to see where this argument comes from. If we start from the trivial vacuum $\Phi(x) = \Phi_0$, $W_\mu(x) = 0$ and apply the gauge transformation $U(x)$ according to Eqs. (32) and (35), we obtain Eqs. (82) and (83). A gauge transformation can not affect physical observables, so the energy density must remain zero.

In other words, $U(x)$ is just the inverse of the transformation used to impose the unitary gauge, so this is just the usual microphysical decay argument in light disguise. How does it go wrong? Let me count the ways.

One: As often pointed out by Vachaspati, there is no reason why $\Phi$ should have vanishing gradient energy at finite temperature. By dimensionality and equipartition of energy alone, given an inverse temperature $\beta$ we should expect $D_\mu \Phi \sim 1/\beta^2$.

Two: At first sight it would seem that this argument must work for any field configuration. But you already know that domain walls, Nielsen-Olesen vortices and 't Hooft-Polyakov monopoles are stable. Clearly, the argument fails for them. Why? An easily seen reason is that they have cores where $\Phi(x) = 0$, which can not be written according to Eq. (82) (i.e. they are in the null space of the unitary gauge). Try setting $U = 0$ and the $U^{-1}$ in Eq. (83) blows up. Stated another way, $U$ fails to be unitary at the core of such a defect: $U^\dagger \neq U^{-1}$. This clearly does not apply to textures, which have no singular core, but topological textures nevertheless owe their existence to the existence of maps in different homotopy classes. Transformations from one homotopy class to another are by definition discontinuous, so even if the $U^{-1}$ in Eq. (83) won’t blow up, the $\partial_\mu U$ will. Gauging the theory lets you move the discontinuity from $\Phi$ to $W_\mu$, but does not eliminate it.

Three: Even topologically trivial $\Phi$ configurations carry conserved quantities (energy, momentum, gauge currents, all in derivative terms) which must go elsewhere, i.e. to fermions, upon relaxation. Fermions can only be produced effectively while the energy and charges within the Compton volume of a fermion pair (e.g. electron + anti-neutrino for electroweak interactions) are ≥ the total mass and charges of such a pair (on shell). Once $\Phi$ gradients fall below this threshold, dissipation to fermions becomes exponentially suppressed. The evolution of $\Phi$ and $W_\mu$ then becomes a Hamiltonian flow, but need not stop.

Combine the first and third point and you can estimate the initial size beyond which charged $\Phi$ configurations should have been safe from dissipation: the horizon scale when $\beta$ crossed above the Compton length of the lightest charged fermion. For electroweak interactions, that’s $\sim 10^3 km$ [107]. If such configurations then simply tracked overall metric expansion, their corresponding minimum size today would be $\sim$ one lightyear. Apply the same redshift to their energy density and you inevitably land right at the current “dark energy” scale, $10^{-5} eV$.

Four: As Nambu pointed out long ago, $D_\mu \Phi = 0$ does not guarantee vanishing energy when there are linear combinations of commuting generators which annihilate $\Phi$, i.e. when massless gauge fields remain after SSB [21].

For a heuristic understanding of this point, consider the standard electroweak model, which conveniently has the same scalar sector as the $O(4)$ Goldstone model: setting the photon $A_\mu(x) = 0$ along with the massive gauge bosons and applying some $U(x)$ will certainly yield a gauge-equivalent vacuum according to Eqs. (82) and (83). But thanks to the absence of a mass term, any finite, constant $A_\mu$ is an equally valid vacuum. Starting from such a configuration, i.e. from $W_\mu \propto 1 + \gamma_3$ (see [107]) Eq. (35) picks up an additional term not contained in Eq. (83). The common term does not depend on the gauge fields, so evidently the same $U(x)$ yields different vacua for different choices of initial constant $A_\mu$. This is a simple example of the degeneracy of the gauge sector’s vacuum [118]. The implication is that just as for the...
scalars, an independent choice of vacuum at each point in space will not generally result in a global energy minimum. That requires coordination, so the usual causality bound (the horizon) applies.

In a slightly more mathematical language, for any “good gauge” (in the sense of Fukuda and Kugo) the lowest energy configurations are spacetime independent, so the energy density in a sufficiently small neighborhood of such a minimum (in field space) can be approximated by an ordinary Taylor expansion. Since there are no linear terms, the first non-trivial term is the Hessian matrix of second order derivatives in the fields, i.e. the mass matrix. For the minimum to be non-degenerate, the discriminant (the determinant of the Hessian) must be positive definite. If there is at least one massless field, this condition is not satisfied: the minimum is degenerate and SSB ensues.

This can be seen explicitly by working out the low energy EFT of the standard electroweak model in an axial gauge. You can start from scratch or you can get a head start by using the gauged NLSM (GNLSM), which was written down long ago (see Ch. 2 in [131] for a review). Choosing the second route, the Goldstone field matrix in polar field coordinates $\theta_a$ is

$$\Sigma = e^{i\theta_a \tau_a/2} = \cos(\theta/2) + i \frac{\theta_a \tau_a \sin(\theta/2)}{\theta} \tag{84}$$

with $\theta = \sqrt{(\theta_1)^2 + (\theta_2)^2 + (\theta_3)^2}$, and

$$\begin{bmatrix} \phi_0^+ & \phi^+ \\ -\phi^+ & \phi_0^+ \end{bmatrix} = \frac{\nu}{\sqrt{2}} \Sigma \tag{85}$$

The covariant derivative acting on $\Sigma$ is

$$D_\mu = \partial_\mu + i \frac{g_W}{2} W_\mu^a \tau_a - i \frac{g}{2} B_\mu \tau^3 \tag{86}$$

and the full Lagrangian is

$$\mathcal{L}_{\text{GNLSM}} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W_\mu W^\mu W_{\mu\nu} \tag{87}$$

You can easily convince yourself that it is equivalent to the standard electroweak boson Lagrangian with the radial Higgs degree of freedom clamped to its VEV (just write out all terms explicitly and compare). It has been used as is to compute the one-loop thermal effective action for an electroweak plasma at temperatures below the mass of a (heavy) Higgs and above that of the weak gauge bosons, but we are interested in the real low energy limit, where the massive gauge bosons can not be excited either and must be set to zero along with radial Higgs excitations (so there is no assumption about the Higgs mass here, other than that it is much larger than the interaction energies under consideration). We therefore need to isolate the linear combination of gauge fields with zero mass, i.e. the photon, as defined in an arbitrary vacuum $\bar{\theta} = [\theta_1, \theta_2, \theta_3]$.

In the basis $[B_\mu, W_\mu^1, W_\mu^2, W_\mu^3]$, the $\mathcal{L}_{\text{GNLSM}}$ terms quadratic in $B_\mu$ and $W^\mu_a$ give rise to the mass matrix

$$\frac{\nu^2}{2} \begin{bmatrix} g_B^2 & g_{B g W} \Theta_1 & g_{B g W} \Theta_2 & -g_{B g W} \Theta_3 \\ g_{B g W} \Theta_1 & g_W^2 & 0 & 0 \\ g_{B g W} \Theta_2 & 0 & g_W^2 & 0 \\ -g_{B g W} \Theta_3 & 0 & 0 & g_W^2 \end{bmatrix} \tag{88}$$

where we have introduced the convenient auxiliary quantities

$$\begin{align*}
\Theta_1 &= \left[ \theta_1 \theta_2 (\cos(\theta) - 1) + \theta_2 \sin(\theta) \right] / \theta^2 \\
\Theta_2 &= \left[ \theta_2 \theta_3 (\cos(\theta) - 1) - \theta_1 \sin(\theta) \right] / \theta^2 \\
\Theta_3 &= \left[ (\theta_1^2 + \theta_2^2) \cos(\theta) + \theta_3^2 \right] / \theta^2
\end{align*} \tag{89}$$

satisfying $(\Theta_1)^2 + (\Theta_2)^2 + (\Theta_3)^2 = 1$. The eigenvalues of Eq. (88) are the tree level masses squared of photon, $W^\pm$ and $Z^0$. The two degenerate eigenstates can be orthogonalized to obtain

$$\begin{align*}
A_\mu &\propto [g_W / g_B, -\Theta_1, -\Theta_2, \Theta_3] \\
W^1_\mu &\propto [0, -\Theta_1, \Theta_2, 0] \\
W^2_\mu &\propto [0, \Theta_1, \Theta_2, \Theta_3] \\
Z_\mu &\propto [-g_B / g_W, -\Theta_1, -\Theta_2, \Theta_3]
\end{align*} \tag{90}$$

To eliminate the massive bosons, invert Eqs. (92)-(95) and set $W^1_\mu = W^2_\mu = Z_\mu = 0$ (note that given orthogonal eigenstates, inversion amounts to normalizing them, assembling them in a column matrix and transposing). The result is

$$\begin{align*}
B_\mu &= A_\mu \cos(\theta_W) \\
W^1_\mu &= -A_\mu \Theta_1 \sin(\theta_W) \\
W^2_\mu &= -A_\mu \Theta_2 \sin(\theta_W) \\
W^3_\mu &= A_\mu \Theta_3 \sin(\theta_W)
\end{align*} \tag{91}$$

To obtain the low energy EFT of the electroweak boson sector to leading order, substitute Eqs. (90)-(99) into $\mathcal{L}_{\text{GNLSM}}$ and find

$$\mathcal{L}_{\text{EFT}} = -\frac{\nu^2}{8} \left[ \partial_\mu \partial_\nu \Theta^\mu \right] + \frac{4 \sin^2(\theta/2)}{\theta^2} \left( \frac{\partial_\mu \theta}{\theta} \times \partial_\nu \theta \right)^2 - \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{\sin^2(\theta_W)}{4} (A_\mu \partial_\nu \Theta_n - A_n \partial_\nu \Theta_\mu)^2 \tag{100}$$

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12 Note that the scalar product of Eq. (82) (properly normalized) for $\bar{\theta} = 0$ and $\bar{\theta} = 0$ is Eq. (31).
The first row in Eq. (100) is just the plain O(4) NLSM in polar field coordinates, the second row is the Maxwell Lagrangian, the third row couples them $\propto \sin^2(\theta_W)$, acting as an effective photon mass term when $\vec{\theta}$ is not constant. Everett’s classical result is now easily recovered by noting that a photon can penetrate a region of varying magnetism, as it must. Note that $\nu \simeq 246.3$ GeV makes the electroweak vacuum very stiff; you would need $\sim 10^{11}$ joule to “melt” a cubic centimeter of it and create the electroweak equivalent of a DCC. In cold war terms, that’s $\sim 10^{25}$ megatons of TNT, enough to completely vaporize Earth, journals included, a couple billion times. Being immersed in our vacuum, such a configuration would also immediately snap back to it (unless dynamically stable, which might be theoretically possible). It is only on astronomical scales that random $\vec{\theta}$ gradients can be shallow enough to be long-lived.

Specializing to the time-axial gauge $W_0 = 0$, the electric and magnetic fields are

$$E = -\partial_0 \vec{A} \quad (101)$$

$$B = \nabla \times \vec{A} \quad (102)$$

Varying $\mathcal{L}_{EFT}$ in $A_0$ yields the Gauss constraint

$$\nabla \cdot \vec{E} = \partial_0 \theta^T \mathbf{H} \cdot \vec{A} \quad (103)$$

(a plane in $\vec{A}$ space) where the matrix $\mathbf{H}$ has components

$$H_{ab} = \sin^2(\theta_W) \frac{\partial \Theta_c}{\partial \theta_a} \frac{\partial \Theta_c}{\partial \theta_b} \quad (104)$$

This completely fixes the gauge, leaving $\vec{A}$ with only two independent degrees of freedom. The energy density becomes a manifestly non-negative sum of quadratic forms,

$$\rho = \frac{1}{2} \left( \vec{E}^2 + \vec{B}^2 \right) + \frac{1}{2} \partial_0 \theta^T \left( \nu^2 \mathbf{G} + \vec{A}^2 \mathbf{H} \right) \partial_0 \vec{\theta} + \frac{\nu^2}{2} \partial_m \vec{\theta}^T \mathbf{G} \partial_m \vec{\theta} + \frac{1}{2} \left( \epsilon_{jkl} A_k \partial_0 \vec{\theta} \right)^T \mathbf{H} \left( \epsilon_{jmn} A_m \partial_n \vec{\theta} \right) \quad (105)$$

where $\mathbf{G}$, with components

$$G_{ab} = \left( \frac{\delta_{ab} \delta_{cc}}{2} + \frac{1 - \cos \theta}{\theta^2} \epsilon_{ced} \epsilon_{ceb} \right) \partial_d \theta_c \partial_e \theta_b \quad (106)$$

is the 3-sphere metric, with eigenvalues 1/4 (and doubly degenerate $0 \leq (1 - \cos(\theta))/(2\theta^2) \leq 1/4$. $\mathbf{H}$ also has no negative eigenvalues (but zeros along all axes of $\vec{\theta}$ space), so by the spectral theorem, $\vec{A} \neq 0$ can only increase $\rho$ for a given $\vec{\theta}$.

Since the low energy EFT of electroweak interactions (valid whether the minimal model is the correct UV completion or not) contains the O(4) NLSM, any solution obtained in the latter, textures included, is also a valid electroweak solution on macroscopic scales. It should be obvious that this result generalizes to larger symmetry groups. In particular, any viable GUT must embed the standard model and so must contain at least nine massless gauge fields (eight gluons and a photon, but there may be more; “hidden” sectors which only couple to the standard model fields via gravity are commonplace e.g. in string phenomenology).

Usual unitary gauge shenanigans aside, the one intelligent concern which has been raised in this context, echoing the discussion in [127], is that the coupling between photon and $\vec{\theta}$ (the third row in Eq. (100)) may significantly affect the dynamics of electroweak textures, invalidating numerical simulation results obtained in the plain O(4) NLSM. While only simulations based on the full EFT will tell for sure, it should be noted that GUTs tend to have complicated Higgs sectors (even the original, minimal SU(5) model needs two Higgs multiplets, one in the adjoint to obtain the standard model group, one in the fundamental representation to break electroweak symmetry) and that the NLSM of a GUT could have viable subgroups which do not couple directly to the photon. To the extent that the O(4) NLSM is a passable representative of the general $N \geq 4$ case, it should also be a reasonable approximation of the dynamics of textures arising in such subgroups.

Finally, I can’t resist pointing out that the kind of energy released by the collapse of an electroweak texture could conceivably match that of even the largest gamma ray bursts (GRBs), $\sim 10^{47}$ joule (equivalent to one cubic meter of electroweak scale plasma). That may be something worth thinking about when confronted with “shots in the dark” not easily accomodated by the standard massive stellar collapse model, like GRB 070125, and one more reason to undertake dynamic simulations of $\mathcal{L}_{EFT}$.

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13. A truly stunning objection raised against $\mathcal{L}_{EFT}$ is that it is less symmetric than the full electroweak Lagrangian, so “important terms are missing”. Yes, Virginia, the low energy EFT has less symmetry than the full theory! It’s called SSB, you may have heard of it.

14. Even if you are uncomfortable with the time-axial gauge at the quantum level, this is not a problem after the EFT has been obtained using e.g. the covariant background field gauge. In the present case this is a non-issue, since we are staying at tree level and disregarding quantum corrections as negligible at low energy.

15. Note the emergence of two natural metrics in field space, one ($G_{ab}$) associated with the scalar sector and one ($H_{ab}$) with the gauge sector, in line with [113].
V. CONCLUSION

The unitary “gauge” is strictly speaking not a gauge, but an unphysical constraint which excludes an important part of configuration space and conceals the full symmetry of the theory. It is oblivious to non-perturbative solutions like sphalerons and topological defects, even at the classical level, and is undefined in the high temperature symmetric phase. Upon quantization, it yields a non-renormalizable and therefore mostly useless perturbative expansion. Identifying it with the singular limit $\xi \to \infty$ of the $R_\xi$ gauges, equivalent to making Goldstone bosons infinitely massive, saves renormalizability, but only at the cost of introducing unphysical degrees of freedom which can not be assumed to be in thermal equilibrium with physical ones.

All these difficulties can be avoided by adopting a physical gauge, as is routinely done in non-perturbative numerical simulations. Carrying out the effective field theory program in such a gauge makes the transition from symmetric high temperature phase to broken low temperature phase explicit and suggests that several high profile problems in cosmology may have a common denominator: the overly simplified picture of gauge symmetry breaking painted by the unitary gauge.
The exponentiated action is not affected by symmetry transformations, so for each orbit around field space (e.g. the bottom of the Mexican hat potential) it can be factored out of the integral, reducing the integrand to

\[ Q(\Phi) e^{iJ(\Phi)} \]  

(A2)

For vanishing external source \( J = 0 \), if \( Q(\Phi) \) averages to 0 over an orbit, the integral over that orbit will vanish; if this holds for all orbits, \( Q \) must also vanish. The hard part is proving that \( Q \to 0 \) for finite \( J \to 0 \), which is not immediately obvious since the path integral runs over an infinite-dimensional space\(^{16}\).

Putting the path integral on the lattice turns it into a finite number of nested, ordinary integrals, one for each lattice site and field. Taking the continuum limit then reveals that the argument fails for global symmetries (i.e. \( Q \) can remain finite for arbitrarily small, finite \( J \)), but apparently does hold for local symmetries\(^{17}\). In particular, this implies that the vacuum expectation value of a Higgs field must vanish: \( \Phi_{vac} = \langle \Phi \rangle = 0 \).

What should we make of this? Elitzur’s own assessment was clear: for SSB to occur, the gauge symmetry must be explicitly broken by imposing a gauge condition which leaves some global symmetry unbroken (either the gauge symmetry restricted to spacetime independent transformations or some other symmetry). It is the global symmetry that is spontaneously broken. This is in line with the standard (well informed) interpretation of the Higgs mechanism, and with the classical view that the theory is undefined until the gauge is completely fixed.

Remember the “good gauges” of Fukuda and Kugo\(^ {58} \); only after a gauge condition is imposed can you know whether a given field configuration is a vacuum. The usual textbook presentation does things the other way around; first a “vacuum” is picked, then the unitary gauge is imposed. The validity of this procedure can only be verified \textit{a posteriori}, and as we have seen, it does not pass the check at the non-perturbative level. Indeed, the unitary gauge and the \( R_2 \) gauges (for generic \( \xi \)) break the gauge symmetry both locally and globally, unlike most common gauges. Neither is therefore adequate for studying SSB (as opposed to studying perturbations

\(^{16}\) From a cosmological perspective, nitpicking about finite “external” sources may strike you as nonsensical until you realize that as far as you are concerned, everything now entering your particle horizon is an external source, quantum and thermal fluctuations about a vanishing expectation value included.

\(^{17}\) A major loophole is that the continuum limit is unlikely to exist for theories which are not asymptotically free. Theories with a U(1) product group, like the standard electroweak model, are not; that’s one reason to view them as low energy EFTs derived from something more fundamental. If the underlying theory is a simple GUT, the continuum limit may exist. If the underlying theory is something else, maybe involving a fundamental length scale, all bets are off.
on a given, constant Higgs background, a.k.a. particle physics).

So far so good. But what if you impose a gauge condition and \( \langle \Phi \rangle \) still vanishes?

In \[137\] it was argued that the residual invariance of the time-axial gauge under time-independent gauge transformations implies (for the Abelian Higgs model, in the canonical operator formalism) \( \langle \Phi \rangle = 0 \). This result was obtained for “physical” states of infinite norm, which may give you some pause (are field expectation values well-defined?), but the same conclusion was reached again by a different route in \[138\], for arbitrary symmetry groups, using the lattice-regularized path integral (in Euclidean space, subject to the usual provisos about the validity of analytic continuation to Minkowski spacetime and the existence of a continuum limit)\(^{18}\).

If \( \Phi \) were a physical observable, this would mean that repeated independent measurements must yield random values with average 0 and variance \( \langle \Phi \Phi\rangle \), and we would have to conclude that \( \Phi \) is in a superposition state smeared out symmetrically over each gauge orbit (an idea first floated in \[139\]). But since \( \Phi \) is an unobservable, gauge dependent quantity, all we can say (if we believe this result) is that \( \langle \Phi \rangle \) is not a good order parameter in the time-axial gauge, i.e. that it is not useful for the determination of the presence and nature of SSB in that particular gauge (in \[139\], it was argued that \( \langle \Phi \rangle \) is a good order parameter in the Landau gauge only).

While the vanishing of \( \langle \Phi \rangle \) in the time-axial gauge may thus be nothing more than a quirk of that (incomplete) gauge condition, in \[138\] it was conjectured to be a physical disorder effect caused by a “gas” of instantons (and maybe other topologically non-trivial configurations) of microphysical size, with average separation on the order of the inverse Higgs mass. Elitzur notwithstanding, the gauge symmetry would then also remain unbroken under global transformations.

In the electroweak context, there is an obvious problem with this picture: contrary to observation, neither photons (as ordinarily defined) nor left-handed fermions would propagate freely on everyday distance scales, which are \( \gg \) the inverse Higgs mass (right-handed fermions would, since they are weak isospin singlets). The only way out of this dilemma is another conjecture, first publicized in \[140\] but attributed to Susskind: all fields which are not singlets under weak isospin, i.e. the SU(2) subgroup of the standard electroweak group \( U(1) \times SU(2) \), are confined to singlet bound states, like quarks and gluons under color SU(3) of QCD. What we see as freely propagating particles are either such singlet bound states or fundamental singlets. In particular, the left-handed fermions observable at low energy all consist of a fundamental left-handed fermion bound to a fundamental Higgs boson. The observable “Higgs particle” is actually a bound pair of fundamental Higgs + anti-Higgs.

The confinement conjecture was, once again, motivated by lattice results. When simple models with Higgs fields in the fundamental representation were put on the lattice and their parameters were varied (in the “frozen Higgs”, i.e. GNLSM approximation), no indication was found of a phase boundary (discontinuities in physical quantities) separating the confinement regime (small Higgs VEV, large gauge coupling) and the Higgs regime (large Higgs VEV, small gauge coupling) \[141\]. No qualitative difference therefore seems to exist between the two regimes, in the same sense that no qualitative difference exists between water vapor and liquid water\(^{19}\). But there are significant quantitative differences, first detailed in \[142\].

For the confinement conjecture to work, the non-Abelian interaction must be strongly coupling. Electromagnetism is not strongly coupling, so there can be no significant weak gauge boson contribution to the photon, i.e. no significant mixing between U(1) and SU(2) gauge bosons. The electric charges of observable particles are then just their U(1) hypercharges, and the photon does not interact significantly with the SU(2) instanton gas. Since the three massive gauge bosons are an almost pure SU(2) triplet, they must have almost identical masses, significantly larger than the standard ones due to the strong SU(2) coupling (\( \sim 125 \, GeV \)). At low energy, the particle spectrum is the same as in the standard, weakly coupled electroweak model, but the composite nature of quarks and leptons starts showing up around 80 GeV, the mass of the standard \( W^{\pm} \) bosons. At higher energies, there is a complicated spectrum of hadron-like bound states.

This scenario was all but killed in 1983 by the experimental detection of the \( W^{\pm} \) bosons at 80 GeV and of the \( Z \) boson at 91 GeV, as predicted by the standard, weakly coupled electroweak model. Subsequently, precision electroweak measurements have shown no sign of composite structure up to energies well in excess of 100 GeV, also ruling out attempts to save electroweak confinement through additional assumptions about its non-perturbative dynamics \[140\].

Backtracking, the demise of the electroweak confinement conjecture reduces the instanton gas suggestion of \[138\] to an interesting thought experiment. It may still be relevant to other theories, but not to the only example of the Higgs mechanism actually known (?) to be

\(^{18}\) Keep in mind that \( \langle \Phi \rangle = 0 \) does not imply \( \Phi = 0 \). Quantum fluctuations alone guarantee that \( \langle \Phi^\dagger \Phi \rangle \) can not vanish, even in the symmetric phase.

\(^{19}\) Subsequent work showed that this property, dubbed “complementarity”, can be lost when the radial Higgs degree of freedom is allowed to fluctuate \[142\] and when fermions are included \[143\] [144]. These days, invoking complementarity therefore requires additional assumptions about unknown non-perturbative dynamics, in particular the absence of spontaneous chiral symmetry breaking.
realized in nature.

What’s left? Out on the fringes of the arXiv, you can still occasionally see (presumably well-meaning) arguments that SSB is impossible in QFT on fundamental quantum mechanical grounds, because a symmetric Hamiltonian cannot produce an asymmetric state starting from a symmetric one. By the same logic, it is clearly impossible to find out whether Schrödinger’s cat is dead or alive, or to measure the \( J_x \) angular momentum component of an electron known to have \( J_z = 1/2 \). You may also be told that a wave function prepared on one side of a double well potential will evolve to a symmetric shape, and that by analogy, classically stable topological defects can unwind without passing through energetically forbidden configurations, courtesy of quantum tunneling. If that happens, inquire about the rate of such events in realistic field theories, then compare the answer with the age of the universe.

Yet another argument which you may come across is that the universe cannot be likened to open thermodynamic systems, where SSB demonstrably occurs, because unlike such systems it is isolated and thus not subject to random external disturbances. I don’t know about the universe of those who write such things, but the universe which I can see around me is a ball which has been growing at the speed of light for the past 14 billion years (give or take a few), and for all that time, its boundary in every direction has been sampling a heat bath hot enough to melt the electroweak vacuum, and who knows what else.

That works for me.

APPENDIX B: BOOK TIPS

Georgi once started off a nice review of effective field theories [79] with the following piece of advice about old literature on almost any subject: “Ignore it! With rare exceptions, old papers are difficult to read because the issues have changed over the years.” I respectfully disagree. There is often much more to the original papers than what you’ll find in streamlined textbook presentations. Read them, and you may discover hidden or forgotten gems.

But textbooks obviously have their place, and there are some which you may find particularly useful if field theory is not your day job. Since I am an old guy, I know mostly old books. Don’t worry though. You could have slept through the past quarter century without missing much if any new particle theory worth knowing about.

If you are going to get only one book, get Cheng & Li [147]. It starts with basic QFT, thoroughly works through the whole standard model and does not stop before it has also told you about technicolor and SU(5). The introductions to SSB and to path integral quantization of gauge theories are excellent. Journal editors should read them religiously at least once every full moon.

The most accessible QFT textbook I know of remains Ryder [148]. Among other things, it features the best introduction to non-perturbative aspects.

Another excellent QFT textbook which also covers the standard electroweak model and SU(5) is Bailin & Love [149]. Unusually for a textbook, it introduces finite temperature field theory and high temperature symmetry restoration. It is also a good place to start learning about anomalies, which you need to understand if you want to understand the role of sphalerons in baryogenesis.

More advanced discussions of path integral quantization, SSB and field theory at finite temperature can be found in Rivers [150].

For an introduction to supersymmetry and string theory, or just for something more recent, consider Dine [151], but understand that it’s a very compressed presentation of essentially all fundamental physics, the kind of book best enjoyed when you already know the subject. If you don’t, see it as a starting point for your literature searches.

Hat off to Dine for being secure enough in his physicist-hood to openly admit that he finds symmetry breaking “a puzzling notion in gauge theories” (p. 17).

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