Chemotactic systems in the presence of conflicts: a new functional inequality

G. Wolansky
Department of Mathematics, Technion, Haifa 32000, israel

Abstract

The evolution of a chemotactic system involving a population of cells attracted to self-produced chemicals is described by the Keller-Segel system. In spatial dimension 2, this system demonstrates a balance between the spreading effect of diffusion and the concentration due to self-attraction. As a result, there exists a critical "mass" (i.e. total cell’s population) above which the solution of this system collapses in a finite time, while below this critical mass there is global existence in time. The existence of this critical mass is related to a functional inequality known as the Moser-Trudinger inequality.

An extension of the Keller-Segel model to several cells populations was considered before in the literature. Here we review some of these results and, in particular, consider the case of conflict between two populations, that is, when population one attracts population two, while, at the same time, population two repels population one. This assumption leads to a new functional inequality which generalizes the Moser-Trudinger inequality.

1 Introduction

The motivation of this paper is a non-local parabolic-elliptic system of the form

\[
\frac{\partial \rho}{\partial t} = \Delta \rho + \nabla \cdot \left[ \rho (\beta \nabla w - \alpha \nabla u) \right] ; \quad \Delta u + \rho = 0 ; \quad \Delta w + M_2 \frac{e^{\beta u - \gamma w}}{\int \! e^{\beta u - \gamma w}} = 0
\]

(1.1)

where all constants \( \alpha, \beta, \gamma \) are non-negative. The system (1.1) is defined on \( \Omega \times [0, \infty) \) where \( \Omega \) is a bounded planar domain. The no-flux boundary condition for \( \rho \) takes the form

\[
(\nabla \rho - \alpha \rho \nabla u + \beta \rho \nabla w) \cdot n = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty)
\]

(1.2)

where \( n \) is the normal to \( \partial \Omega \). In addition, \( u = w = 0 \) on \( \partial \Omega \times (0, \infty) \).

In addition, \( \rho, u, w \) satisfy the initial conditions at \( t = 0 \): \( u(0) = u^0 \in H^1_0, \ w(0) = w^0 \in H^1_0 \) and \( \rho(0) := \rho^0 \in L^1(\Omega) \) where \( \rho^0 \geq 0 \) on \( \Omega \). In particular, the no-flux boundary condition (1.2) implies, by a formal application of the divergence theorem, the conservation of mass:

\[
\int_\Omega \rho(x,t) \, dx = \int_\Omega \rho^0(x) \, dx := M_1 > 0.
\]

(1.3)

The function \( \rho \) corresponds, in the language of chemotaxis \([KS,H]\), to the density of a population of organisms (living cells, bacteria, slime molds or, perhaps, crowded human beings ...) which evolve in time without multiplication and mortality. The individuals of this population are moving on the planar domain \( \Omega \) by a combination of random walk and deterministic drift force along the gradient of self produced chemicals \( u \) and \( w \). We explain in Section 2 below the origin of the term "conflict" for (1.1).
Let us define now $F^M$ on $\Gamma_{M_1} \times \mathbb{H}_0^1$ (see (2.8)) as

$$F^M(\rho, w) := \int \rho \ln \rho + \frac{\alpha}{2} \int \rho \Delta^{-1} \rho + \left[ \frac{\gamma}{2} \int |\nabla w|^2 + M \ln \left( \int e^{-\gamma w - \beta \Delta^{-1}(\rho)} \right) \right]$$

where $\Delta^{-1}$ is the Green function for the Dirichlet Laplacian on $\Omega$. It follows that (1.1) can be written as

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left( \rho \nabla \delta F^M \right) , \quad \delta F^M \delta w = 0 .$$

(1.4)

A formal integration by parts yields

$$\frac{d}{dt} F^M(\rho(., t), w(., t)) = - \int \rho \left| \nabla \delta F^M \delta \rho \right|^2$$

(1.5)

so $F^M$ is monotone non-increasing along solutions of (1.1).

The steady states of (1.1) are solutions of the non-local Liouville system [CSW, Wa].

$$\Delta u + M_1 \frac{e^{\alpha u - \beta w}}{\int_{\Omega} e^{\alpha u - \beta w}} = 0 \quad ; \quad \Delta w + M_2 \frac{e^{-\gamma w + \beta u}}{\int_{\Omega} e^{-\gamma w + \beta u}} = 0 \quad , \quad u = w = 0 \quad \text{on} \quad \partial \Omega$$

(1.6)

where $\rho = M_1 \frac{e^{\alpha u - \beta w}}{\int_{\Omega} e^{\alpha u - \beta w}}$.

From the representation (1.4) it follows that any solution of (1.6) corresponds to a critical point of $F^M$ on $\Gamma_{M_1} \times \mathbb{H}_0^1$. In particular, the monotonicity (1.5) suggests that a local minimizer of $F^M$ on this domain corresponds to a stable steady state of (1.1). Thus, the question regarding the bound from below of $F^M$ on $\Gamma_{M_1} \times \mathbb{H}_0^1$ is interesting in that respect, as it is a necessary condition for the existence of a global minimizer on this domain, which is, evidently, a critical point, and thus stable steady state of (1.1).

If we substitute $\gamma = \beta = 0$ in $F^M$ we get, up to an irrelevant constant, the Free Energy functional

$$\rho \in \Gamma_M \mapsto F(\rho) := \int \rho \ln \rho + \frac{\alpha}{2} \int \rho \Delta^{-1}(\rho)$$

(1.7)

This functional is monotone non-increasing along solutions of the parabolic-elliptic Keller-Segel system for chemotaxis of a single component [T,W1,W2, BCC...] (see also [W4,W5] for application to self-gravitating systems)

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \nabla \cdot \left[ \rho (\alpha \nabla u) \right] \quad ; \quad \Delta u + \rho = 0 \quad ; \quad \int \rho \theta = M .$$

(1.8)

Note that (1.8) is obtained from the substitution $\gamma = \beta = 0$ in (1.1). This can be written as

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left( \rho \nabla \frac{\delta F}{\delta \rho} \right) \quad \text{on} \quad \Gamma_M .$$

(1.9)

The bound from below of $F$ on $\Gamma_M$ for $M \leq 8\pi/\alpha$ follows from the logarithmic HLS inequality [B, CL]:

$$\forall \rho \in \mathbb{L} \ln \mathbb{L}(D) , \quad \| \rho \|_1 \int_D |\rho| \ln |\rho| + (4\pi)^{-1} \int_D \int_D \rho(x) \ln |x - y| \rho(y) dxdy > -C(D)\| \rho \|_1 ,$$

(2.8)
for functions in a two dimensional bounded domain $D$. Using scaling and taking into account that $\Delta^{-1}(x,y) \approx (2\pi)^{-1} \ln |x-y|$, up to lower order terms, imply the bound from below on $\Gamma_M$ for $M \leq 8\pi/\alpha$.

This is a key inequality for the proof of global existence of (1.8) for $M < 8\pi/\alpha$ as well as the existence of solution to the nonlocal Liouville equation

$$\Delta u + \frac{M}{\int e^{\alpha u}} e^{\alpha u} = 0, \quad M < 8\pi/\alpha \quad (1.10)$$

in a bounded domain $\Omega$ [W2, KS1,L].

The parabolic elliptic Keller-Segel (1.8) is a limiting case of the parabolic parabolic system [CC]

$$\delta \frac{\partial \rho}{\partial t} = \Delta \rho - \nabla \cdot [\rho(\alpha \nabla u)] ; \quad \varepsilon \frac{\partial u}{\partial t} = \Delta u +\rho = 0, \quad \int \rho^0 = M, \quad u^0 \in H^1_0 \quad (1.11)$$

where $\delta = 1$ and $\varepsilon = 0$. Another, less known limit of (1.11) [W2, KS1] is $\varepsilon = 1, \delta = 0$:

$$\frac{\partial u}{\partial t} = \Delta u + \frac{M}{\int e^{\alpha u}} e^{\alpha u} \cdot \quad (1.12)$$

We observe that (1.12) is itself a gradient descend system on $H^1_0$ of the form

$$\frac{\partial u}{\partial t} = -\delta \frac{\partial H}{\partial u}$$

where

$$u \in H^1_0 \mapsto H^M(\alpha)(u) := \frac{1}{2} \int |\nabla u|^2 - M \ln \left( \int e^{\alpha u} \right).$$

A simple scaling shows that the bound from below of $H$ on $H^1_0$ where $M \leq 8\pi/\alpha$ follows from the Moser-Trudinger inequality

$$\frac{1}{2} \int |\nabla u|^2 - 8\pi \ln \left( \int e^{\alpha u} \right) > -C$$

for any $u \in H^1_0$ [RS, SW...]. This gives an alternative proof for the existence of solution to (1.10) for $M < 8\pi/\alpha$, as well as the global (in time) existence of (1.12) under the same condition [Bi].

Motivated by the above, we consider in this paper the condition of bound from below for the functional $F^{M_2}$ on $\Gamma_{M_1} \times H^1_0$. Note that for $M_2 = 0$ and $\gamma = 0$, $F^M$ is just the Free Energy $F$.

It follows, then, that a new inequality for $F^{M_2} > -C$ on $\Gamma_{M_1} \times H^1_0$ is a generalization of the Logarithmic HLS inequality for the case $M_2 = 0$. Note also that if $\gamma = 0, M > 0$ then the last two nonzero terms of $F^M(-\Delta w)$ is, by integration by parts, just

$$-\frac{\alpha}{2} \int |\nabla w|^2 + M \ln \left( \int e^{\beta w} \right) \equiv -\alpha \overline{H^M/\alpha}(w),$$

3
which is related to the Moser-Trudinger inequality (with opposite sign, however). In fact, it is known that the Moser-Trudinger and Logarithmic HLS inequalities are equivalent. To see this, consider

\[(\rho, u) \in \Gamma_M \times H_0^1 \mapsto H(\rho, u) := \int_\Omega \rho \ln \rho + \frac{\alpha}{2} \int_\Omega |\nabla u|^2 - \alpha \int_\Omega \rho u \, .\]

and note that

\[F(\rho) = \inf_{u \in H_0^1} H(\rho, u) ; \quad H_M(u) := \inf_{\rho \in \Gamma_M} H(\rho, u) ,\]

so both logarithmic HLS and Moser-Trudinger inequalities follow from the bound \(H(\rho, u) > -C\) for \((\rho, u) \in \Gamma_{8\pi/\alpha} \times H_0^1\).

It is also interesting to note that \(H\) induces a gradient descend flow for the parabolic-parabolic Keller-Segel equation (1.11) via

\[\frac{\partial \rho}{\partial t} = \nabla \cdot \left( \rho \nabla \frac{\delta H}{\delta \rho} \right) ; \quad \frac{\partial u}{\partial t} = -\alpha^{-1} \frac{\delta H}{\delta u} ,\]

and that (1.8) (resp. (1.12)) are singular limits of (1.11) for \(\varepsilon = 0\) (resp. \(\delta = 0\)).

2 Multi-Component Chemotactic Systems

The general system of Chemotaxis for two components is a special case of the general system [W1]:

\[\delta_i \frac{\partial \rho_i}{\partial t} = \sigma_i \Delta \rho_i + \nabla \cdot [\rho_i (a_{ii} \nabla u_i + a_{ij} \nabla u_j)] \quad (2.1)\]

where \((i, j) \in \{1, 2\} \; i \neq j\), and

\[\varepsilon \frac{\partial u_i}{\partial t} = b_i \Delta u_i + \rho_i , \; i = 1, 2 \quad (2.2)\]

where \(\sigma_i > 0, b_i > 0\) are the diffusion coefficients, \(\delta_i, \varepsilon > 0\) and \(a_{i,j}\) are constants. Eq. (2.1, 2.2) are defined on \(\Omega \times [0, \infty)\) where \(\Omega \subset \mathbb{R}^2\), \(u_i\) are subjected to Dirichlet boundary condition \(u_i = 0\) on \(\partial \Omega \times [0, \infty)\) and initial data

\[u_i(\cdot, 0) = u_i^0 \in H_0^1(\Omega) . \quad (2.3)\]

\(\rho_i\) satisfy the no-flux boundary conditions

\[\hat{n} \cdot \{\sigma_i \nabla \rho_i + [\rho_1 (a_{ii} \nabla u_i + a_{ij} \nabla u_j)]\} = 0 \quad (2.4)\]

on \(\partial \Omega \times [0, \infty)\), where \(\hat{n}\) is the normal to \(\partial \Omega\). In addition, \(\rho_i\) satisfy the initial conditions at \(t = 0\): \(\rho_i(\cdot, 0) = \rho_i^0\), where \(\rho_i^0 \in L^1(\Omega)\) and \(\rho_i^0 \geq 0\) on \(\Omega\). In particular, the no-flux boundary conditions imply, by a formal application of the divergence theorem, the conservation of mass:

\[\int_\Omega \rho_i(x, t) dx = \int_\Omega \rho_i^0(x) dx := M_i . \quad (2.5)\]
The functions $\rho_i$ correspond to the densities of the two populations of organisms which evolve with time without multiplication and mortality. The individuals of these populations are moving on the planar domain $\Omega$ by a combination of random walk (corresponding to the diffusion coefficients $\sigma_i$), and deterministic drift forces along the gradient of self produced chemicals $u_i$.

Five of the constants in (2.1) can be eliminated by scaling $\rho_1, \rho_2, u_1, u_2$ and the time $t$. In particular, we can assume, with no loss of generality, that $\sigma_1 = \sigma_2 = b_1 = b_2 = 1$ and that $a_{12} = \pm a_{21} := \beta$. Let $a_{11} := -\alpha, a_{22} = \gamma$. 

**Assumption 2.1.** $\alpha \geq 0$ (self-attractive first population), $\gamma \geq 0$ (self repulsive second population) as well as $\beta > 0$ (first population is rejected by the second one).

We get

$$\delta_1 \frac{\partial \rho_1}{\partial t} = \Delta \rho_1 + \nabla \cdot [\rho_1 (\beta \nabla u_2 - \alpha \nabla u_1)] ; \quad \delta_2 \frac{\partial \rho_2}{\partial t} = \Delta \rho_2 + \theta \nabla \cdot [\rho_2 (\beta \nabla u_1 + \theta \gamma \nabla u_2)]$$ (2.6)

$$\varepsilon \frac{\partial u_i}{\partial t} = \Delta u_i + \rho_i , \quad i = 1, 2.$$ (2.7)

Here $\theta \in \{-1, 1\}$ corresponding to the choice of sign in $a_{21} = \pm \beta$. The case $\theta = 1$ is the conflict free case studied in [W1]. In that case the second population is rejected by the first one, while (since $\beta > 0$), the first population is rejected by the second one due to the first equation in (2.6).

Let us define now

$$\Gamma_M := \{ \rho \in L^1(\Omega), \rho \geq 0 \ \text{a.e.} \ \text{on} \ \Omega, \ \int_{\Omega} \rho \leq \infty, \ \int_{\Omega} \rho = M \}$$

$$\Gamma_{M_1,M_2} := \{ (\rho_1, \rho_2) ; \rho_1 \in \Gamma_{M_1}, \rho_2 \in \Gamma_{M_2} \} .$$ (2.8)

Let $H_{\theta} : \Gamma_{M_1,M_2} \times (H^1_0)^2 \rightarrow \mathbb{R} \cup \{\infty\}$:

$$H_{\theta}(\rho_1, \rho_2, u_1, u_2) :=
\int \rho_1 \ln \rho_1 + \theta \int \rho_2 \ln \rho_2 + \frac{\alpha}{2} \left( \int |\nabla u_1|^2 - 2 \int \nabla u_1 \cdot \nabla u_2 \right) - \frac{1}{2} \theta \gamma \left( \int |\nabla u_2|^2 - 2 \int \nabla u_2 \cdot \nabla u_1 \right) - \beta \left( \int \nabla u_1 \cdot \nabla u_2 - \int \rho_1 u_2 - \rho_2 u_1 \right)$$ (2.9)

The system (2.6, 2.7) subject to initial data (2.3, 2.5) takes the form

$$\delta_1 \frac{\partial \rho_1}{\partial t} = \nabla \cdot \left( \rho_1 \nabla H_{\theta} \right) ; \quad \delta_2 \frac{\partial \rho_2}{\partial t} = \theta \nabla \cdot \left( \rho_2 \nabla H_{\theta} \right) .$$ (2.10)

$$\varepsilon \frac{\partial u_1}{\partial t} = \frac{1}{\beta^2 + \alpha \gamma \theta} \left( \frac{\beta \delta H_{\theta}}{\delta u_2} - \theta \gamma \frac{\delta H_{\theta}}{\delta u_1} \right) , \quad \varepsilon \frac{\partial u_2}{\partial t} = \frac{1}{\beta^2 + \alpha \gamma \theta} \left( \beta \frac{\delta H_{\theta}}{\delta u_1} + \alpha \frac{\delta H_{\theta}}{\delta u_2} \right) .$$ (2.11)
2.1 Limit cases

i) The limit ε = 0 of system \(\text{[2.10], [2.11]}\) is reduced into the parabolic-elliptic system

\[
\frac{\partial \rho_1}{\partial t} = \Delta \rho_1 + \nabla \cdot [\rho_1 (\beta \nabla u_2 - \alpha \nabla u_1)] ; \quad \frac{\partial \rho_2}{\partial t} = \Delta \rho_2 + \theta \nabla \cdot [\rho_2 (\beta \nabla u_1 + \theta \gamma \nabla u_2)]
\]

\[
\Delta u_i + \rho_i = 0 , \quad i = 1, 2 .
\]

(2.12)

If we substitute \(u_i = -\Delta^{-1}(\rho_i)\) in \(H_\theta\) we get

\[
H_\theta(\rho_1, \rho_2) := \int_\Omega \rho_1 \ln \rho_1 + \frac{\alpha}{2} \int_\Omega \rho_2 \ln \rho_2 + \frac{\theta \gamma}{2} \int_\Omega \rho_1 \Delta^{-1}(\rho_1) - \frac{\theta \gamma}{2} \int_\Omega \rho_2 \Delta^{-1}(\rho_2) - \beta \int_\Omega \rho_2 \Delta^{-1}(\rho_1) .
\]

(2.13)

Then, \(\text{[2.12], [2.13]}\) takes the form

\[
\frac{\partial \rho_1}{\partial t} = \nabla \cdot \left( \rho_1 \nabla \frac{\delta H_{\theta}}{\delta \rho_1} \right) ; \quad \frac{\partial \rho_2}{\partial t} = \theta \nabla \cdot \left( \rho_2 \nabla \frac{\delta H_{\theta}}{\delta \rho_2} \right) .
\]

(2.14)

ii) The limit ε = δ = 2 = 0, \(\delta = 1\). Substitute \(\delta = 0\) in \(\text{[2.12]}\) and integrate to obtain

\[
\rho_2 = M_2 e^{-\theta\beta u_1 - \gamma u_2} / \int e^{-\theta\beta u_1 + \gamma u_2} .
\]

Hence

\[
\frac{\partial \rho_1}{\partial t} = \Delta \rho_1 + \nabla \cdot [\rho_1 (\beta \nabla u_2 - \alpha \nabla u_1)] ; \quad \Delta u_1 + \rho_1 = 0 ; \quad \Delta u_2 + M_2 e^{-\theta\beta u_1 - \gamma u_2} = 0 .
\]

(2.16)

Let us define now \(F_\theta^M\) on \(\Gamma_{M_1} \times \mathbb{H}_0^1\) as

\[
F_\theta^M(\rho, w) := \int_\Omega \rho \ln \rho + \frac{\alpha}{2} \int_\Omega \rho \Delta^{-1}(\rho) - \theta \left[ \frac{\gamma}{2} \int_\Omega |\nabla w|^2 + M \ln \left( \int e^{-\gamma w + \theta \beta \Delta^{-1}(\rho)} \right) \right] .
\]

Then, with \(\rho_1 := \rho, u_2 := w, \text{[2.16]}\) can be written as

\[
\frac{\partial \rho}{\partial t} = \nabla \cdot \rho \nabla \left( \frac{\delta F_\theta^M}{\delta \rho} \right) ; \quad \frac{\delta F_\theta^M}{\delta w} = 0 .
\]

(2.17)

iii) The limit \(\delta = 2 = 0\) for \(\text{[2.10], [2.11]}\) is reduced into

\[
\frac{\partial u_1}{\partial t} = \Delta u_1 + M_1 e^{\alpha u_1 - \beta u_2} \int_\Omega e^{\alpha u_1 - \beta u_2} ; \quad \frac{\partial u_2}{\partial t} = \Delta u_2 + M_2 e^{-\gamma u_2 - \theta \beta u_1} \int_\Omega e^{-\gamma u_2 - \theta \beta u_1} .
\]

(2.18)

If we substitute \(\text{[2.21]}\) in \(H_\theta\) and apply integration by parts, we get \(\overline{H}_\theta^{M_1, M_2}(u_1, u_2) + M_1 \ln M_1 + \theta M_2 \ln M_2\) where \(\overline{H}_\theta^{M_1, M_2}(u_1, u_2) := \)

\[
\frac{\alpha}{2} \int_\Omega |\nabla u_1|^2 - \frac{\gamma}{2} \int_\Omega |\nabla u_2|^2 - \beta \int_\Omega \nabla u_1 \cdot \nabla u_2 - M_1 \ln \left( \int e^{\alpha u_1 - \beta u_2} \right) - \theta M_2 \ln \left( \int e^{-\gamma u_2 - \theta \beta u_1} \right) .
\]

(2.19)

Then \(\text{[2.18]}\) takes the form

\[
\varepsilon \frac{\partial u_1}{\partial t} = \frac{1}{\beta^2 + \alpha \gamma \theta} \left( \beta \frac{\delta \overline{H}_\theta}{\delta u_1} - \theta \gamma \frac{\delta \overline{H}_\theta}{\delta u_1} \right) ; \quad \varepsilon \frac{\partial u_2}{\partial t} = \frac{1}{\beta^2 + \alpha \gamma \theta} \left( \beta \frac{\delta \overline{H}_\theta}{\delta u_2} + \alpha \frac{\delta \overline{H}_\theta}{\delta u_2} \right) .
\]

(2.20)
2.2 Steady states

Any critical point of \( H_\theta \) in \( \Gamma_{M_1,M_2} \times (\mathbb{H}_0^1)^2 \) is also an equilibrium solution of (2.6, 2.7). The variation of \( H_\theta \) with respect to \( \rho_i \) yields

\[
\ln \rho_1 - \alpha u_1 + \beta u_2 = \lambda_1 \quad ; \quad \theta (\ln \rho_2 + \gamma u_2) + \beta u_1 = \lambda_2
\]

where \( \lambda_i \) are the Lagrange multipliers associated with the constraints \( \int \rho_i = M_i \). Hence

\[
\rho_1 = M_1 \frac{e^{\alpha u_1 - \beta u_2}}{\int_{\Omega} e^{\alpha u_1 - \beta u_2}} \quad ; \quad \rho_2 = M_2 \frac{e^{-\gamma u_2 - \theta \beta u_1}}{\int_{\Omega} e^{-\gamma u_2 - \theta \beta u_1}} .
\]  

(2.21)

The variation of \( H_\theta \) with respect to \( (u_1, u_2) \in (\mathbb{H}_0^1)^2 \) yields

\[
\rho_i = -\Delta u_i .
\]  

(2.22)

Combining (2.21, 2.22) together we obtain the Liouville type system

\[
\Delta u_1 + M_1 \frac{e^{\alpha u_1 - \beta u_2}}{\int_{\Omega} e^{\alpha u_1 - \beta u_2}} = 0 \quad ; \quad \Delta u_2 + M_2 \frac{e^{-\gamma u_2 - \theta \beta u_1}}{\int_{\Omega} e^{-\gamma u_2 - \theta \beta u_1}} = 0 \quad , \quad u_1 = u_2 = 0 \quad \text{on} \quad \partial \Omega .
\]  

(2.23)

It can be verified directly that a solution \( (\rho_i, u_i) \) of (2.22, 2.23) yields a steady state solution of (2.6, 2.7).

Proposition 2.1. \((u_1, u_2)\) is a solution of the Liouville system (2.23) iff either \((u_1, u_2)\) is a critical point of \( \overline{H}^{M_1,M_2}_\theta \) in \((\mathbb{H}_0^1)^2\) or \((\rho_1, \rho_2)\), \(\rho_i = -\Delta u_i\) is a critical point of \( H_\theta \) in \( \Gamma_{M_1,M_2} \) or \((\rho_1, u_2)\) is a critical point of \( F^M \) in \( \Gamma_{M_1} \times \mathbb{H}_0^1 \).

3 The Case of Conflict

From now on we assume the case of conflict \( \theta = -1 \). The Liouville system (2.23) takes the form

\[
\Delta u_1 + M_1 \frac{e^{\alpha u_1 - \beta u_2}}{\int_{\Omega} e^{\alpha u_1 - \beta u_2}} = 0 \quad ; \quad \Delta u_2 + M_2 \frac{e^{-\gamma u_2 + \beta u_1}}{\int_{\Omega} e^{-\gamma u_2 + \beta u_1}} = 0 \quad , \quad u_1 = u_2 = 0 \quad \text{on} \quad \partial \Omega .
\]  

(3.1)

Here and thereafter we omit the index \( \theta \) from \( \overline{H}, \overline{H}_\theta \) and \( F^M \). In particular

\[
F^M(\rho, w) := \int \rho \ln \rho + \frac{\alpha}{2} \int \rho \Delta^{-1}(\rho) + \left[ \frac{\gamma}{2} \int |\nabla w|^2 + M \ln \left( \int e^{-\gamma \rho - \beta \Delta^{-1}(\rho)} \right) \right]
\]  

(3.2)

Lemma 3.1.

\[
\inf_{\rho_1 \in \Gamma_{M_1}} \sup_{\rho_2 \in \Gamma_{M_2}} H(\rho_1, \rho_2) = \inf_{\rho \in \Gamma_{M_1}} \inf_{w \in \mathbb{H}_0^1} F^M_2(\rho, w) + M_2 \ln M_2 .
\]
Proof. First note that
\[ \frac{1}{2} \int \rho \Delta^{-1}(\rho) = \inf_{w \in \mathcal{H}_0^1} \frac{1}{2} \int |\nabla w|^2 + \int \rho w. \]

\[ H(\rho_1, \rho_2) := \int \rho_1 \ln \rho_1 - \int \rho_2 \ln \rho_2 + \frac{\alpha}{2} \int \rho_1 \Delta^{-1}(\rho_1) + \frac{\gamma}{2} \int \rho_2 \Delta^{-1}(\rho_2) - \beta \int \rho_2 \Delta^{-1}(\rho_1). \]

\[ \leq \int \rho_1 \ln \rho_1 - \int \rho_2 \ln \rho_2 + \frac{\alpha}{2} \int \rho_1 \Delta^{-1}(\rho_1) + \gamma \left( \frac{1}{2} \int |\nabla w|^2 - \int \rho_2 w \right) - \beta \int \rho_2 \Delta^{-1}(\rho_1) = H(\rho_1, \rho_2, w), \]

and \[ \inf_{w \in \mathcal{H}_0^1} H(\rho_1, \rho_2, w) = H(\rho_1, \rho_2). \] A direct calculation shows that
\[ \sup_{\rho_2 \in \Gamma_{M_2}} \left\{ -\int \rho_2 \ln \rho_2 - \gamma \int \rho_2 w - \beta \int \rho_2 \Delta^{-1}(\rho_1) \right\} = M_2 \ln \left( e^{-\gamma w - \beta \Delta^{-1}(\rho_1)} \right) + M_2 \ln M_2. \]

In particular,
\[ \sup_{\rho_2 \in \Gamma_{M_2}} H(\rho, \rho_2, w) = F^{M_2}(\rho, w) + M_2 \ln M_2. \]

Hence \( \inf_{w \in \mathcal{H}_0^1} F^{M_2}(\rho, w) + M_2 \ln M_2 = \)
\[ \inf_{w \in \mathcal{H}_0^1} \sup_{\rho_2 \in \Gamma_{M_2}} H(\rho, \rho_2, w) = \sup_{\rho_2 \in \Gamma_{M_2}} \inf_{w \in \mathcal{H}_0^1} H(\rho, \rho_2, w) = \sup_{\rho_2 \in \Gamma_{M_2}} H(\rho, \rho_2) \]
\[ \square \]

Lemma 3.2. For any \((u_1, u_2) \in (\mathcal{H}_0^1)^2\)
\[ \overline{H}^{M_1, M_2}(u_1, u_2) = \inf_{\rho_1 \in \Gamma_{M_1}} \sup_{\rho_2 \in \Gamma_{M_2}} H(\rho_1, \rho_2, u_1, u_2) \equiv \sup_{\rho_2 \in \Gamma_{M_2}} \inf_{\rho_1 \in \Gamma_{M_1}} H(\rho_1, \rho_2, u_1, u_2). \] (3.3)

If, in addition, \(\alpha \gamma \geq \beta^2\) then for any \((\rho_1, \rho_2) \in \Gamma_{M_1, M_2}\)
\[ \overline{H}(\rho_1, \rho_2) = \inf_{u_1, u_2 \in \mathcal{H}_0^1} H(\rho_1, \rho_2, u_1, u_2). \] (3.4)

Proof. The equalities in (3.3) follow from the definition of \(\overline{H}\) (2.19) which use (2.21). Indeed, (2.21) are the unique minimizer (maximizer) of \(H\) as a function of \(\rho_1\) (\(\rho_2\)) where \(u_1, u_2\) are fixed, since \(H\) is strictly convex in \(\rho_1\) and strictly concave in \(\rho_2\).

To get (3.4) note that \(\alpha \gamma > \beta^2\) implies that \(H\) is strictly convex, jointly in \((u_1, u_2)\), and the only minimizer is \(\Delta u_i + \rho_i = 0\), namely \(u_i = -\Delta^{-1}(\rho_i), i = 1, 2\). Then (3.4) follows directly from definition (2.14). The case of equality \(\alpha \gamma = \beta^2\) follows from a simple limit argument. \[ \square \]

Lemma 3.3. If \(\alpha \gamma \geq \beta^2\) then
\[ \inf_{u_1, u_2 \in \mathcal{H}_0^1} \overline{H}^{M_1, M_2}(u_1, u_2) = \inf_{\rho \in \Gamma_{M_1}} F^{M_2}(\rho, w) + M_2 \ln M_2. \]

8
Proof. Since \( H \) is jointly convex in \((u_1, u_2)\) and concave in \( \rho_2 \) it follows, by the minmax theorem, that
\[
\sup_{\rho_2 \in \Gamma_{M_2}} \inf_{u_1, u_2 \in \mathcal{H}_0^1} H(\rho_1, \rho_2, u_1, u_2) = \inf_{u_1, u_2 \in \mathcal{H}_0^1} \sup_{\rho_2 \in \Gamma_{M_2}} H(\rho_1, \rho_2, u_1, u_2) \tag{3.5}
\]
So
\[
\inf_{\rho_1 \in \Gamma_{M_1}} \sup_{\rho_2 \in \Gamma_{M_2}} H(\rho_1, \rho_2) = \inf_{\rho_1 \in \Gamma_{M_1}} \inf_{\rho_2 \in \Gamma_{M_2}} \sup_{u_1, u_2 \in \mathcal{H}_0^1} H(\rho_1, \rho_2, u_1, u_2) = \inf_{u_1, u_2 \in \mathcal{H}_0^1} \sup_{\rho_1 \in \Gamma_{M_1}} H(\rho_1, \rho_2, u_1, u_2) = \inf_{u_1, u_2 \in \mathcal{H}_0^1} \prod^{M_1, M_2}_{\rho_1}(u_1, u_2) \tag{3.6}
\]
where the first equality from (3.4), the second one from (3.5), the third one is trivial and the last one follows from (3.3).

Lemma 3.4. If \( \alpha \gamma > \beta^2 \) then \( \prod^{M_1, M_2}_{\rho_1} \) is a Lyapunov functional for (2.20), that is
\[
\frac{d}{dt} \prod^{M_1, M_2}_{\rho_1}(u_1(\cdot, t), u_2(\cdot, t)) \leq 0
\]
where \((u_1, u_2)\) is a solution of (2.20) in \( C^1(\mathbb{R}^+; (\mathbb{H}_0^1(\Omega))^2) \). The above equality is strict unless \((u_1, u_2)\) is a steady state of this system.

Proof. of Lemma 3.4
From (2.11) we get
\[
\frac{d}{dt} \prod^{M_1, M_2}_{\rho_1} = \delta_{u_1} \prod^{M_1, M_2}_{\rho_1} \frac{\partial u_1}{\partial t} + \delta_{u_2} \prod^{M_1, M_2}_{\rho_1} \frac{\partial u_2}{\partial t}
\]
\[
= -\frac{\varepsilon}{\alpha \gamma - \beta^2} \left[ \left( \beta \delta_{u_1} \prod^{M_1, M_2}_{\rho_1} + \gamma \delta_{u_1} \prod^{M_1, M_2}_{\rho_1} \right) \delta_{u_2} \prod^{M_1, M_2}_{\rho_1}
\]
\[
+ \left( \beta \delta_{u_2} \prod^{M_1, M_2}_{\rho_1} + \alpha \delta_{u_2} \prod^{M_1, M_2}_{\rho_1} \right) \delta_{u_2} \prod^{M_1, M_2}_{\rho_1} \right]
\]
\[
\leq -\frac{\varepsilon}{\alpha \gamma - \beta^2} \left[ \gamma \| \delta_{u_1} \prod^{M_1, M_2}_{\rho_1} \|_2^2 + \alpha \| \delta_{u_2} \prod^{M_1, M_2}_{\rho_1} \|_2^2 - 2 \beta \langle \delta_{u_1} \prod^{M_1, M_2}_{\rho_1}, \delta_{u_2} \prod^{M_1, M_2}_{\rho_1} \rangle \right]
\]
\[
= -\frac{\varepsilon}{\alpha \gamma - \beta^2} \left( \| \delta_{u_1} \prod^{M_1, M_2}_{\rho_1} \|_2, \| \delta_{u_2} \prod^{M_1, M_2}_{\rho_1} \|_2 \right) \left( \frac{\alpha}{-\beta} \right) \left( \frac{-\beta}{\gamma} \right) \left( \| \delta_{u_1} \prod^{M_1, M_2}_{\rho_1} \|_2 \right) \left( \| \delta_{u_2} \prod^{M_1, M_2}_{\rho_1} \|_2 \right) \tag{3.7}
\]
where we used Cauchy-Schwartz inequality. Since the quadratic form is positive definite, the last term is non-positive and, in fact, negative unless
\[
\delta_{u_1} \prod^{M_1, M_2}_{\rho_1} = \delta_{u_2} \prod^{M_1, M_2}_{\rho_1} = 0.
\]
In a similar (and more direct) way we prove
Lemma 3.5. $F^{M_2}$ is a Lyapunov functional for (2.10), that is
\[
\frac{d}{dt} F^{M_2}(\rho_1(\cdot,t),u_2(\cdot,t)) \leq 0
\]
where $(\rho_1,u_2)$ is a solution of (2.10) in $C^1([0,\infty);\Gamma_{M_1}\times\mathbb{H}^1_0)$. The above equality is strict unless $(\rho_1,u_2)$ is a steady state of this system.

Remark 3.1. According to Proposition 2.7, Lemma 3.3 and Lemma 3.4, the steady states of the system (2.16) and (2.18) for $\theta = -1$ are the solutions of (3.1).

Definition 3.1. In the domain $(M_1,M_2) \in \mathbb{R}^2$, the set where $F^{M_2}$ is unbounded from below on $\Gamma_{M_1}$ is denoted by $\Lambda$. The set where $F^{M_2}$ is bounded from below on $\Gamma_{M_1}$ is $\overline{\Lambda}$.

In the case where $\Omega$ is a disc $\{|x| \leq R\} \subset \mathbb{R}^2$ we denote $\Gamma^R_M \subset \Gamma_M$ the set of all radial functions in $\Gamma_M$. Then $\overline{\Delta^R}$ (resp. $\overline{\Delta^R}$) is defined as above for $F^{M_2}$ restricted to $\Gamma^R_{M_1}$.

Lemma 3.6. If $(M_1,M_2)$ is an interior point of $\overline{\Lambda}$ then there exists a minimizer of $F^{M_2}$ on $\Gamma_{M_1} \times \mathbb{H}^1_0$. If, moreover, $\alpha \gamma > \beta^2$ then there exists a minimizer of $H^{M_1,M_2}$ on $(\mathbb{H}^1_0)^2$.

Proof. Let $q \in (0,1)$, $\tilde{\gamma} = \gamma/q$, $\tilde{\beta} := \beta/q$. Let $\overline{\Gamma}$ defined according to Definition 3.1 with respect to $\overline{F}$, where
\[
\overline{F}^{M}(\rho,w) := \int \rho \ln \rho + \frac{\alpha}{2} \int \rho \Delta^{-1}(\rho) + \frac{\tilde{\gamma}}{2} \int |\nabla w|^2 + M \ln \left( \int e^{-\tilde{\gamma}w - \tilde{\beta} \Delta^{-1}(\rho)} \right).
\]

We can find such $q$ for which $(\tilde{M}_1,\tilde{M}_2) := (qM_1,q^{-1}M_2) \in \overline{\Gamma}$. Set $\rho := q\tilde{\rho}$, $w = q\tilde{w}$. Then
\[
\overline{F}^{\tilde{M}_2}(\tilde{\rho},\tilde{w}) := q^{-1} \int \rho \ln \rho + \frac{\alpha}{2}\frac{q^2}{q^2} \int \rho \Delta^{-1}(\rho) + \frac{\gamma}{2q^2} \int |\nabla w|^2 + \tilde{M}_2 \ln \left( \int e^{-(\tilde{\gamma}/q)w - \tilde{\beta}q^{-1} \Delta^{-1}(\rho)} \right)
\]
\[= q^{-1} \left[ \int \rho \ln \rho + \frac{\alpha}{2q} \int \rho \Delta^{-1}(\rho) + \frac{\gamma}{2} \int |\nabla w|^2 + M_2 \ln \left( \int e^{-\gamma w - \beta \Delta^{-1}(\rho)} \right) \right] + q^{-1} M_1 \ln q
\]
\[= q^{-1} \left[ F^{M_2}(\rho,w) - \frac{\alpha}{2}(1-q^{-1}) \int \rho \Delta^{-1}(\rho) \right] + q^{-1} M_1 \ln q.
\]

Since $\overline{F}^{\tilde{M}_2}(\tilde{\rho},\tilde{w}) > C$ for some $C \in \mathbb{R}$ independent of $\tilde{\rho},\tilde{w} \in \Lambda_{\tilde{M}_1} \times \mathbb{H}^1_0$ by assumption, it follows
\[
F^{M_2}(\rho,w) \geq qC - M_1 \ln q + \frac{\alpha}{2}(1-q^{-1}) \int \rho \Delta^{-1}(\rho) \quad \text{(3.8)}
\]
for any $(\rho,w) \in \Lambda_{M_1} \times \mathbb{H}^1_0$. Let now $(\rho^n,w^n)$ be a minimizing sequence for $F^{M_2}$ in $\Gamma_{M_1} \times \mathbb{H}^1_0$. From (3.8), and since $q \in (0,1)$ we conclude that $\int \rho^n \Delta^{-1}(\rho^n)$ is bounded uniformly from below. Since
\[
\frac{\gamma}{2} \int |\nabla w^n|^2 + M_2 \ln \left( \int e^{-\gamma w^n - \beta \Delta^{-1}(\rho^n)} \right) \geq \frac{\gamma}{2} \int |\nabla w^n|^2 + M_2 \ln \left( \int e^{-\gamma w^n} \right)
\]
is bounded from below as well, we obtain that \( \int \rho^n \ln \rho^n \) is bounded from above. Let \( \bar{\rho} \) be a weak limit of \( \rho^n \) in the Zygmund space \( L_\ln L \). Then

\[
\int \bar{\rho} \ln \bar{\rho} \leq \liminf_{n \to \infty} \int \rho^n \ln \rho^n.
\]

On the other hand, \( \Delta^{-1} \) is a compact operator from \( L_\ln L \) to its dual space \( L_{\exp} \), composed of all functions \( w \) for which \( e^{\lambda |w|} \) is integrable for some \( \lambda(w) > 0 \). Hence \( w^n := -\Delta^{-1}(\rho^n) \) admits a strongly convergent subsequence in \( L_{\exp} \), whose limit is \( \bar{\omega} = -\Delta^{-1}(\bar{\rho}) \). Hence

\[
\lim_{n \to \infty} \int \rho^n \Delta^{-1}(\rho^n) = \int \bar{\omega} \Delta^{-1}(\bar{\rho}).
\]

It follows that \( \bar{\rho} \in \Gamma_{M_1} \) satisfies

\[
\inf_{\rho \in \Gamma_{M_1}, u \in H^1_0} F_{M_2}(\rho, u) = \inf_{u \in H^1_0} F_{M_2}(\bar{\rho}, u).
\]

Finally, the existence of a minimizer \( \bar{u} \) for

\[
\gamma \int \frac{1}{2} |\nabla u|^2 + M_2 \ln \left( \int e^{-\gamma u - \beta \Delta^{-1}(\rho)} \right) := F_{M_2}(\bar{\rho}, u) - \gamma \int \bar{\rho} \ln \bar{\rho} - \frac{\alpha}{2} \int \bar{\rho} \Delta^{-1}(\bar{\rho})
\]

on \( H^1_0 \) follows from the a-priori bound on the \( \|u\|_{H^1_0} \) norm of the minimizing sequence, and from the compact embedding of \( H^1_0 \) in \( L_{\exp} \).

The proof for \( \Omega_{M_1, M_2} \) in case \( \alpha \gamma > \beta^2 \) is easier, and is left to the reader.

3.1 Objectives

Our object is to characterize the sets \( \overline{\Omega} \) and \( \underline{\Omega} \).

We note that the second part in the definition of \( F_{M_2} \), namely

\[
H_{M_2}^{\gamma}(\rho) := \inf_{w \in H^1_0} \frac{\gamma}{2} \int |\nabla w|^2 + M_2 \ln \left( \int e^{-\gamma w - \beta \Delta^{-1}(\rho)} \right) \quad (3.9)
\]

is bounded from below. Indeed, since \( \Delta^{-1}(\rho) \leq 0 \) by the maximum principle, it follows that

\[
\frac{\gamma}{2} \int |\nabla w|^2 + M_2 \ln \left( \int e^{-\gamma w - \beta \Delta^{-1}(\rho)} \right) \geq \frac{\gamma}{2} \int |\nabla w|^2 + M_2 \ln \left( \int e^{-\gamma w} \right)
\]

for any \( w \in H^1_0 \). The last expression is bounded from below on \( H^1_0 \) for any \( M_2 > 0 \). Hence \( H_{M_2}^{\gamma}(\rho) > -C \) independent of \( \rho \) so \( F_{M_2}(\rho) \) is bounded from below by the Free energy (1.7) which, in turn, is bounded from below on \( \Gamma_{M_1} \) for \( M_1 \leq 8\pi/\alpha \). Hence, we expect that \( \overline{\Omega} \) contains \( M_1 \leq 8\pi/\alpha \) for any \( M_2 \geq 0 \). We also expect that for any \( M_1 > 8\pi/\alpha \), \( (M_1, M_2) \in \overline{\Omega} \) if \( M_2 \) is large enough.

To evaluate \( \underline{\Omega} \) we only have to indicate a sequence \( \rho_j \in \Gamma_{M_1} \) for which \( F_{M_2}(\rho_j) \to -\infty \). The evaluation of \( \overline{\Omega} \) is more subtle. At this stage we can only investigate \( \overline{\Omega}_R \).
3.2 Main results

Let the unit disc \( D_1 := |z| \leq 1 \) be our domain. Let

\[
\Lambda(M_1, M_2) := 2(M_1 - M_2) - \frac{\alpha M_1^2}{4\pi} + \frac{\beta M_1 M_2}{2\pi} - \frac{\gamma M_2^2}{4\pi}.
\] (3.10)

Note that

\[
\Lambda(M_1, M_2) = M_2 \left(-2 + \frac{\beta M_1 - \gamma M_2}{2\pi}\right) + \left(2M_1 - \frac{\alpha M_1^2}{4\pi} + \frac{\gamma M_2^2}{4\pi}\right) := \Lambda_1(M_1, M_2) + \Lambda_2(M_1, M_2).
\]

**Theorem 1.** If both \( \Lambda(M_1, M_2) < 0 \) and \( \Lambda_2(M_1, M_2) < 0 \) then \( (M_1, M_2) \in \Lambda \).

**Theorem 2.** If either \( M_1 \leq 8\pi/\alpha \) or \( \Lambda(M_1, M_2) > 0 \) and \( \Lambda_1(8\pi/\alpha, M_2) \geq 0 \) then \( (M_1, M_2) \in \overline{\Lambda}^R \).

In the case \( \gamma = 0 \), Theorems 1 and 2 give an (almost) complete description. Indeed, in that case \( \Lambda_2(M_1, M_2) = 0 \) iff \( M_1 = 8\pi/\alpha \). We can, then, distinguish two cases:

**Corollary 3.1.** If \( \Lambda_1(8\pi/\alpha, M_2) < 0 \) then \( \overline{\Lambda}^R \) contains the interval \( \{M_2\} \times (0, 8\pi/\alpha) \) and \( \Lambda \) contains \( \{M_2\} \times (8\pi/\alpha, \infty) \).

If \( \Lambda_1(8\pi/\alpha, M_2) > 0 \) then \( \Lambda \) contains the interval \( \{M_2\} \times (M_0, \infty) \) and \( \overline{\Lambda}^R \) contains \( \{M_2\} \times (0, M_0) \), where \( M_0 \) is the larger root of \( M \to \Lambda(M, M_2) = 0 \).

From Lemma 3.6 and Theorem 2 we obtain

**Theorem 3.** If either \( M_1 < 8\pi/\alpha \) or \( \Lambda(M_1, M_2) > 0 \) and \( \Lambda_1(8\pi/\alpha, M_2) > 0 \) then the Liouville system \( \{3.1\} \) in a disc domain \( D_R := \{|x| \leq R\} \subset \mathbb{R}^2 \) admits a radially symmetric solution.

4 Proofs

**Remark 4.1.** From here forward we denote \((f, g) := \int_{D_1} f(x)g(x)dx\) for a pair of integrable functions \(f, g\) on \(D_1\).

**Proof.** of Theorem 1
We may assume that \( \Omega \) is the unit disc \( D_1 \) and \( \rho = \rho(|z|) \). For \( \psi \geq 1 \) set \( \rho^\psi(r) = \psi^2 \rho(\psi r) \) if \( r \in [0, 1/\psi] \), \( \rho^\psi(r) = 0 \) if \( r \in (1/\psi, 1] \). Then, for \( \rho \in \Gamma_M \),

\[
\begin{align*}
u^\psi(r) := \begin{cases} u(\psi r) - (2\pi)^{-1} M_1 \ln(1/\psi) & \text{if } 0 \leq r \leq 1/\psi \\ -(2\pi)^{-1} M_1 \ln(r) & \text{if } 1 \geq r \geq 1/\psi \end{cases} \\
u^\psi(r) := \begin{cases} w(\psi r) + (2\pi)^{-1} M_2 \ln(1/\psi) & \text{if } 0 \leq r \leq 1/\psi \\ (2\pi)^{-1} M_2 \ln(r) & \text{if } 1 \geq r \geq 1/\psi \end{cases}
\end{align*}
\]

Note that under this scaling \( \Delta^{-1} u^\psi = -\rho^\psi \) if \( \Delta^{-1} u = -\rho \). Also \( u^\psi \in \mathbb{H}^1_0 \) if \( u \in \mathbb{H}^1_0 \). We obtain:

\[
\int_{D_1} \rho^\psi \ln \rho^\psi = 2M_1 \ln \psi + \int_{D_1} \rho \ln \rho.
\] (4.1)
\[(\Delta^{-1} \rho^\psi, \rho^\psi) = (\Delta^{-1} \rho, \rho) - \frac{M_2^2}{2\pi} \ln \psi. \quad (4.2)\]

\[
\int_{D_1} |\nabla w^\psi|^2 = \int_{D_1} |\nabla w|^2 + \frac{M_2^2}{2\pi} \ln \psi \quad (4.3)
\]

and

\[
\int_{D_1} e^{\beta u^\psi + \gamma w^\psi} = 2\pi \left( e^{(\gamma M_2 - \beta M_1) \ln(1/\psi) / 2\pi} \int_0^{1/\psi} r e^{\beta u(r) + \gamma w(r)} + \int_{1/\psi}^{1} r^{1+(\gamma M_2 - \beta M_1) / 2\pi} dr \right)
\]

\[
= \left[ 2\pi \int_0^{1} r e^{\beta u(r) + \gamma w(r)} + O(1) \right] \psi^{-2 + \left(\beta M_1 - \gamma M_2\right) / 2\pi} + O(1) \quad (4.4)
\]

It follows from \((3.2, 4.1-4.3)\) that, if \(\Lambda_1(M_1, M_2) \equiv -2 + (\beta M_1 - \gamma M_2) / 2\pi \geq 0\) then, for \(\psi \to \infty\)

\[
F^{M_2}(\rho^\psi, w^\psi) = F^{M_2}(\rho, w) + O(1) + \left[ 2(M_1 - M_2) - \frac{\alpha M_1^2}{4\pi} + \frac{\beta M_2 M_1}{2\pi} - \frac{\gamma M_2^2}{4\pi} \right] \ln \psi \equiv F^{M_2}(\rho, w) + O(1) + \Lambda(M_1, M_2) \ln \psi \quad (4.5)
\]

Letting \(\psi \to \infty\) we obtain a blow-down sequence for \(F^{M_2}(\rho^\psi, w^\psi)\) where \((\rho^\psi, w^\psi) \in \Gamma_{M_1} \times \mathbb{H}_0^1\), provided \(\Lambda(M_1, M_2) < 0\).

If \(\Lambda_1(M_1, M_2) < 0\) then \(F^{M_2}(\rho^\psi, w^\psi)\) is unbounded from below on \(\Gamma_{M_1}\) via \((4.5)\). If \(\Lambda_1(M_1, M_2) < 0\) then the second alternative \((4.6)\) holds and \(F^{M_2}\) is unbounded from below on \(\Gamma_{M_1}\) as well.

For the proof of Theorem 2 we shall need the following auxiliary lemma:

**Lemma 4.1.** For \(\psi \in (0, 1)\), assume there is a solution \(v = v(r)\) of

\[
r^{-1}(rv_r)_r = r^{-\beta M/2\pi} e^{\gamma v} \quad (4.7)
\]

on the interval \([\psi, 1]\) satisfying: \(v_r(1) = M_2 / 2\pi\).

If \(\frac{\beta M - \gamma M_2}{2\pi} - 2 > 0\) then, near \(\psi = 0\),

\[
\int_{\sqrt{\psi}}^1 r |v_r|^2 dr + \left(\frac{M_2}{2\pi}\right)^2 \ln(\sqrt{\psi}) = o(\ln \psi) .
\]
Proof. Under the change of variables: \( r \to e^{-t} \) we get that (4.7) is transformed to
\[
\hat{v}_{tt} = e^{(\beta M/2\pi-2)t+\gamma \hat{v}}, \quad 0 \leq t \leq \ln(1/\psi)
\]
for \( \hat{v}(t) := v(e^{-t}) \). The end point \( r = 1 \) are transformed into \( t = 0 \) and
\[
\hat{v}_t(0) = -M_2/2\pi
\]
Setting now
\[
\bar{v}(t) := \hat{v}(t) + \gamma^{-1}(\beta M/(2\pi) - 2)t
\]
we get
\[
\bar{v}_{tt} = e^{\gamma \bar{v}}
\]
and
\[
\bar{v}_t(0) = \gamma^{-1} \left[ \frac{\beta M - \gamma M_2}{2\pi} - 2 \right].
\]
From (4.9) it follows that \( |\bar{v}_t|^2/2 - \gamma^{-1} e^{\gamma \bar{v}} := E \) is an invariant, so
\[
\bar{v}_t = \pm \sqrt{2(E + \gamma^{-1} e^{\gamma \bar{v}})}.
\]
for some constant \( E \). The assumption \( \frac{\beta M - \gamma M_2}{2\pi} - 2 > 0 \) implies the + sign above, so
\[
\bar{v}_t = \sqrt{2(E + \gamma^{-1} e^{\gamma \bar{v}})}
\]
with
\[
E = -\gamma^{-1} e^{\gamma \bar{v}(0)} + \frac{1}{2\gamma^2} \left[ \frac{\beta M - \gamma M_2}{2\pi} - 2 \right]^2.
\]
Evidently, the solution of (4.10) is monotone increasing and blow up at some finite point \( T > 0 \), depending on the initial data \( \bar{v}(0) \). Note that \( T \to \infty \) if \( \bar{v}(0) \to -\infty \). Hence, for the solution to exist on the interval \( [0, \ln(1/\psi)] \), \( \bar{v}(0) \to -\infty \) as \( \psi \to 0 \). The explicit solution of (4.10) for \( E > 0 \) is
\[
\bar{v}(t) = \ln(2\gamma E) + \sqrt{2Et} - 2\ln \left( 1 - \Gamma e^{\sqrt{2}Et} \right)
\]
where \( \Gamma \) is uniquely determined by \( \bar{v}(0) \) using (4.11). Since, by assumption, \( \bar{v} \) exists on the interval \( [0, \ln(1/\psi)] \), it follows that \( 1 - \Gamma e^{\sqrt{2}E \ln(1/\psi)} > 0 \). In particular, \( \Gamma := \Gamma(\psi) < e^{\sqrt{2}E \ln \psi} \to 0 \) as \( \psi \to 0 \). From this we obtain that \( \Gamma e^{\sqrt{2}Et} < 1/\sqrt{T} \) if \( t \in [0, \ln \sqrt{1/\psi}] \). From (4.12)
\[
\bar{v}_t(t) = 2\sqrt{E} - \frac{2\sqrt{2\pi} e^{\sqrt{2}Et}}{1 - \Gamma e^{\sqrt{2}Et}}.
\]
Hence, on the interval \( t \in [0, \ln \sqrt{1/\psi}] \),
\[
\bar{v}_t(t) - 2\sqrt{E} = O(\sqrt{T}) = O(\psi^{E/\sqrt{2}}) \]
From (4.11) and \( \bar{v}(0) \to -\infty \) for \( \psi \to 0 \) we get

\[
\bar{v}_t(t) = \gamma^{-1} \left[ \frac{\beta M - \gamma M_2}{2\pi} - 2 \right] + o(1)
\]

for \( \psi << 1 \) uniformly in \( t \in [0, \ln \sqrt{\psi}] \). Using (4.8)

\[
\hat{v}_t(t) = \bar{v}_t(t) - \gamma^{-1} \left[ \frac{\beta M}{2\pi} - 2 \right] = M + o(1).
\]

Returning to the variables \( r = e^{-t}, \ v(r) = \hat{v}(t) \) we get

\[
rv_r(r) = \frac{M}{2\pi} + o(1)
\]

uniformly on the interval \( [\sqrt{\psi}, 1] \), so

\[
\int_{\sqrt{\psi}}^1 r|v_r(r)|^2 \, dr = - \left( \frac{M}{2\pi} \right)^2 \ln \sqrt{\psi} + o(|\ln(\psi)|).
\]

Proof. of Theorem 2

Assume \( M_1 > 8\pi/\alpha \). Since \( \Lambda_2(8\pi/\alpha) \geq 0 \) and, under the condition of the Theorem, \( \Lambda_1(8\pi/\alpha, M_2) > 0 \) it follows that \( \Lambda(8\pi/\alpha, M_2) > 0 \). Since \( \Lambda(M_1, M_2) > 0 \) by assumption and \( M \to \Lambda_1(M, M_2) \) is linear with non-negative slop \( (\beta \geq 0) \), then both \( \Lambda(s, M_2) > 0 \) and \( \Lambda_1(s, M_2) > 0 \) for \( 8\pi/\alpha \leq s \leq M_1 \).

Let \( \delta := (M_1 - 8\pi/\alpha)/n \) where \( n \) is so large, for which

\[
\min_{8\pi/\alpha \leq s \leq M_1} \Lambda(s, M_2) > \frac{\delta M_1}{\pi}. \tag{4.13}
\]

We shall prove that if \( F^{M_2} \) is unbounded from below on \( \Gamma_s^R \) where \( s \in [8\pi/\alpha + \delta, M] \) then it is still unbounded from below on \( \Gamma_{s-\delta}^R \). Iterating this argument \( n \) times we obtain that \( F^{M_2} \) is unbounded from below on \( \Gamma_{8\pi/\alpha}^R \) and get a contradiction.

So let \( s \) in this interval and \( \{\rho_j, w_j\} \in \Gamma_s \times \mathbb{H}_0^1 \) a blow down sequence (e.g. \( F^{M_2}(\rho_j, w_j) < -j \)). Choose \( \psi_j \in (0, 1) \) such that \( \int_{D_{\psi_j}} \rho_j = s - \delta \). Set \( \frac{\rho_j}{\delta} \in \Gamma_{s-\delta} \) which is the restriction of \( \rho_j \) to \( D_{\psi_j} \) that is:

\[
\rho_j(r) = \rho_j(r) \quad \text{for } r \in [0, \psi_j], \quad \rho_j(r) = 0 \quad \text{for } r \in (\psi_j, 1].
\]

Note that potential \( u_j = -\Delta^{-1} \rho_j \) satisfies

\[
u_j(r) = \frac{s - \delta}{2\pi} \ln \left( \frac{1}{r} \right) \quad \text{for } 1 \geq r \geq \psi_j. \tag{4.14}
\]

Set also

\[
\Delta u_j = \frac{M_2}{\int e^{\gamma u_j + \beta w_j} e^{\gamma u_j + \beta w_j}}. \tag{4.15}
\]
Using (4.14) in (4.15) we observe that $w_j$ is equal to a solution $v$ of (4.7), where $M = s - \delta$ and $\psi = \psi^j$. From Lemma 4.1 we get

$$\frac{\gamma}{2} \int_{|\nabla w_j|^2 \leq 1} |\nabla w_j|^2 \geq -\frac{\gamma M^2}{4\pi} \ln \sqrt{\psi} - o(|\ln \psi|) \ .$$

(4.16)

We now estimated the difference $F^M_2(\rho_j) - F^M_2(\rho_j)$. Since the function $s \to -s \ln s$ is bounded from above by $e^{-1}$ it follows that

$$\int_{D_1} p_j \ln p_j - \int_{D_1} \rho_j \ln \rho_j = - \int_{D_1 - D_\psi} \rho_j \ln \rho_j \leq \pi e^{-1} \ .$$

(4.17)

Since $\Delta^{-1} \leq 0$ on $D_1$ and $p_j \leq \rho_j$ then $\Delta^{-1} p_j \geq \Delta^{-1} \rho_j$ (same reasoning can be applied via the maximum principle). Then, since $\beta > 0$ we obtain by (3.9)

$$H^M_2(\rho_j) \leq H^M_2(\rho_j) \ .$$

(4.18)

Finally, to estimate the difference of the energies $(\rho_j, \Delta^{-1} \rho_j) - (\rho_j, \Delta^{-1} \rho_j)$ we observe

$$(\rho_j, \Delta^{-1} \rho_j) - (\rho_j, \Delta^{-1} \rho_j) = 2(\rho_j - \rho_j, \Delta^{-1} \rho_j) + (\rho_j - \rho_j, \Delta^{-1} \rho_j) \ .$$

(4.19)

Since $\rho_j - \rho_j$ is supported in the ring $1 \geq r \geq \psi_j$ we obtain from (4.14)

$$2(\rho_j - \rho_j, \Delta^{-1} \rho_j) \geq \frac{\delta (s - \delta)}{2\pi} \ln \psi_j \ .$$

(4.20)

The second term is the (negative) energy due to a mass concentrated in the ring $r \in [\psi_j, 1]$. It is maximized if the mass is concentrated in the inner circle $r = \psi_j$. The potential induced by the mass $\delta$ concentrated on this circle is just $\delta/(2\pi) \ln \psi_j$, so the energy is bounded from below by $\delta^2/(2\pi) \ln \psi_j$. Hence

$$(\rho_j, \Delta^{-1} \rho_j) - (\rho_j, \Delta^{-1} \rho_j) \geq \frac{\delta}{2\pi} (2s - \delta) \ln \psi_j \ .$$

(4.21)

To summarize (4.17), (4.21)

$$F^M_2(\rho_j) - F(\rho_j) \leq \pi e^{-1} - \frac{\delta}{4\pi} (2s^2 - \delta^2) \ln \psi_j \ .$$

(4.22)

Set now $\rho^\psi_j(r) = \psi_j \rho(\sqrt{\psi_j} r)$ for $r \in [0, 1]$ \footnote{Note that $\rho^\psi_j$ is supported on the disc of radius $\sqrt{\psi}$.} Evidently, $\rho^\psi_j \in \Gamma_{s-\delta}$ as well.

For $\rho^\psi_j$ as above, (4.11) (4.2) imply

$$\int_{D_1} \rho^\psi_j \ln \rho^\psi_j = 2M_1 \ln \sqrt{\psi_j} + \int_{D_1} \rho_j \ln \rho_j \ .$$

(4.23)
The potential corresponding to $\lambda_j^{\psi_j}$ is $\bar{w}_j^{\psi_j}(r) = w_j(\sqrt{\psi_j}r) - w_j(\sqrt{\psi_j})$ where $w_j$ is the minimizer of (2.19) with $\rho_j^{\psi_j}$ substituted for $\rho$ (analogously, the solution of (2.10) where $w_j^{\psi_j}$ substituted for $u$). Then (4.4) (with $\sqrt{\psi_j}$ replacing $\psi^{-1}$) takes the form

$$
\int_{D_1} e^{\beta u_j^{\psi_j} + \gamma w_j^{\psi_j}} = 2 \pi e^{(\beta(s - \delta) \ln \sqrt{\psi_j}/2\pi - \gamma w_j(\sqrt{\psi_j}))} \int_0^1 e^{\beta u_j^{\psi_j} + \gamma w_j^{\psi_j}}(\sqrt{\psi_j}r) dr
$$

$$
= 2 \pi e^{(\beta/(2\pi)(s - \delta)) \ln \sqrt{\psi_j} - \gamma w_j(\sqrt{\psi_j})} \int_0^1 e^{\beta u_j^{\psi_j} + \gamma w_j^{\psi_j}(r)} dr
$$

$$
\leq 2 \pi e^{(\beta(s - \delta)/(2\pi)) \ln \sqrt{\psi_j} - \gamma w_j(\sqrt{\psi_j})} \int_0^1 e^{\beta u_j^{\psi_j} + \gamma w_j(r)} dr
$$

(4.25)

It follows that

$$
M_2 \ln \left( \int_{D_1} e^{\beta u_j^{\psi_j} + \gamma w_j^{\psi_j}} \right) \leq M_2 \ln \left( \int_{D_1} e^{\beta u_j^{\psi_j} + \gamma w_j^{\psi_j}} \right) + M_2 \left[ (s - \delta)/(2\pi) - 2 \right] \ln \sqrt{\psi_j} - \gamma w_j(\sqrt{\psi_j}) + \ln(2\pi). \tag{4.26}
$$

Recall that $w_j$ is the solution of (2.10) (with $w_j^{\psi_j}$ substituted for $u$). In any case, $\Delta w_j \geq 0$ on $D_1$, $w_j(1) = 0$ and $\int_{D_1} \Delta w_j = M_2$. It follows that $w_j(r) \geq (M_2/2\pi) \ln(r)$ for any $r \in (0, 1]$. In particular

$$
[M_2(\beta(s - \delta)/(2\pi) - 2] \ln \sqrt{\psi_j} - \gamma w_j(\sqrt{\psi_j}) \leq M_2 \left( \frac{\beta(s - \delta)}{2\pi} - \frac{\gamma M_2}{2\pi} - 2 \right) \ln \sqrt{\psi_j} \tag{4.27}
$$

Next,

$$
\frac{\gamma}{2} \int_{D_1} \left| \nabla w_j^{\psi_j} \right|^2 = \pi \gamma \int_0^1 \left| \frac{dw_j^{\psi_j}}{dr} \right|^2 r dr = \pi \gamma \int_0^1 \left| w_j(t(\sqrt{\psi_j}r)) \right|^2 \psi_j dr dr =
$$

$$
\pi \gamma \int_0^{\sqrt{\psi_j}} \left| w_j^j(r) \right|^2 r dr = \pi \gamma \int_0^1 \left| w_j^j(r) \right|^2 r dr - \gamma \pi \int_0^{\sqrt{\psi_j}} \left| w_j^j(r) \right|^2 r dr =
$$

$$
\frac{\gamma}{2} \int_{D_1} \left| \nabla w_j \right|^2 + \frac{\gamma M_2^2}{4\pi} \ln \sqrt{\psi_j} + o(\ln \psi_j) \tag{4.28}
$$

where we used $\Lambda_1(s - \delta, M_2) > 0$, (4.16) in the last inequality. Using (4.24) (4.25) (4.26) (4.27) (4.28) and the definition of $\Lambda$ (4.10)

$$
F_{M_2}(\lambda_j^{\psi_j}) \leq F_{M_2}(\lambda_j) + [\Lambda(s - \delta, M_2) + o(1)] \ln \sqrt{\psi_j}.
$$

and from (4.23)

$$
F_{M_2}(\lambda_j^{\psi_j}) \leq F_{M_2}(\lambda_j) + \left( \Lambda(s - \delta, M_2) - \frac{s}{2}(2s^2 - \delta^2) + o(1) \right) \ln \sqrt{\psi_j} + \pi e^{-1}.
$$

17
Since $\psi_j \in (0,1)$ it follows by (4.13) that $F_{M^2}(\rho^{\psi_j}) \leq F_{M^2}(\rho_j) + \pi e^{-1}$. In particular $\lim_{j \to \infty} F_{M^2}(\rho^{\psi_j}) = -\infty$ if $\lim_{j \to \infty} F_{M^2}(\rho_j) = -\infty$. Recalling $\rho_j \in \Gamma_s$ while $\rho^{\psi_j} \in \Gamma_{s-\delta}$ we obtain the result.

\begin{flushright}
\qed
\end{flushright}

References

[B] Beckner, W.: Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. Ann. of Math. (2) 138, 213242 (1993)

[BCC] Blanchet, A., Carlen, E.A. and Carrillo, J.A.: Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model J. Funct. Anal. 262 (2012), no. 5, 2142230.

[Bi] Biler, P., Karch, G.: Blowup of solutions to generalized Keller-Segel model, J. Evol. Equ. 10 (2010), no. 2, 247262.

[CC] Calvez, V., Corrias, L. The parabolic-parabolic Keller-Segel model in $R^2$, Commun. Math. Sci. 6:417447 (2008)

[CL] Carlen, E., Loss, M.: Competing symmetries, the logarithmic HLS inequality and Onofri inequality on $S^n$. Geom. Funct. Anal. 2, 90104 (1992)

[CSW] Chipot, M.; Shafrir, I.; Wolansky, G. On the solutions of Liouville systems. J. Differential Equations 140 (1997), no. 1, 59105.

[H] Horstmann, Dirk. Generalizing the Keller-Segel model: Lyapunov functionals, steady state analysis, and blow-up results for multi-species chemotaxis models in the presence of attraction and repulsion between competitive interacting species. J. Nonlinear Sci. 21 (2011), no. 2, 231-270

[KS] Keller, E. F. and Segel, L. A. Traveling bends of chemotactic bacteria. J. Theor. Biol. 30, 235-248 (1971)

[KS1] Kavallaris, N., Suzuki, T.: On the finite-time blow-up of a non-local parabolic equation describing chemotaxis. Differential Integral Equations 20 (2007), no. 3, 293308.

[L] Lin, C.S: Liouville systems of mean field equations, Milan J. Math. 79 (2011), no. 1, 819

[RS] Ricciardi, T. and Suzuki, T.: Duality and best constant for a Trudinger-Moser inequality involving probability measures. J. Eur. Math. Soc. (JEMS) 16 (2014), no. 7, 13271348

[SW] Shafrir, I and Wolansky, G.: The logarithmic HLS inequality for systems on compact manifolds. J. Funct. Anal. 227 (2005), no. 1, 200226.

[S] Suzuki, T.: Free Energy and Self-Interacting Particles, Birkhäuser Boston, Boston, 2005.

[Wa] Wang, G.: Moser-Trudinger inequalities and Liouville systems. C. R. Acad. Sci. Paris Sr. I Math. 328 (1999), no. 10, 895900.
[W1] Wolansky, G. *Multi-components chemotactic system in the absence of conflicts.* European J. Appl. Math. 13 (2002), no. 6, 641-661.

[W2] Wolansky, G. *A critical parabolic estimate and application to nonlocal equations arising in chemotaxis.* Appl. Anal. 66 (1997), no. 3-4, 291-321.

[W3] Wolansky, G.: *On the evolution of self-interacting clusters and applications to semilinear equations with exponential nonlinearity.* Festschrift on the occasion of the 70th birthday of Shmuel Agmon. J. Anal. Math. 59 (1992), 251-272.

[W4] Wolansky, G.: *On steady distributions of self-attracting clusters under friction and fluctuations,* Arch. Rational Mech. Anal. 119 (1992), no. 4, 355-391.