Superfield T-duality rules in ten dimensions with one isometry

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Abstract

In this contribution we present the superfield T-duality rules relating type IIA and type IIB supergravity potentials for the case when both type IIA and type IIB superspaces have (at least) one isometry direction. We also give a brief review of T–duality and discuss the main steps of our approach to the derivation of the superfield T–duality rules, including the treatment of T-duality as an operation acting on differential forms rather than on the superspace coordinates.

1 Introduction

T–duality is a perturbative symmetry of closed string theory which appears when at least one of the spacetime directions is compact (see, e.g., [1]).

The string equations of motion may be linearized by fixing a conformal gauge for the worldsheet metric, \( \Box \tilde{X}^\mu(\tau, \sigma) = 0 \). Then the general solution describes independent motions of right–movers (\( \tilde{X}^\mu_R(\tau, \sigma) \)) and left–movers (\( \tilde{X}^\mu_L(\tau, \sigma) \))

\[
\Box \tilde{X}^\mu(\tau, \sigma) = 0 \Rightarrow \tilde{X}^\mu(\tau, \sigma) = \tilde{X}^\mu_L(\tau - \sigma) + \tilde{X}^\mu_R(\tau + \sigma) ;
\]

left movers : \( \tilde{X}^\mu_L(\tau - \sigma) = (\tau - \sigma)p^\mu_L + \text{oscillating terms} \),
right movers : \( \tilde{X}^\mu_R(\tau + \sigma) = (\tau + \sigma)p^\mu_R + \text{oscillating terms} \),

In flat noncompact spacetime the periodic boundary conditions in \( \sigma \) (\( \tilde{X}^\mu(\tau, \sigma) = \tilde{X}^\mu(\tau, \sigma + 2\pi) \)) implies \( p^\mu_L = p^\mu_R = 1/2p^\mu \). If some direction, say \( X^{25} \), is compact, \( X^{25} \sim X^{25} + 2\pi m, m \in \mathbb{Z} \), then, firstly, the momentum \( p^{25} = p^{25}_R + p^{25}_L \) becomes discrete \( p^{25} \in \mathbb{Z} \), and, secondly, the difference \( p^\mu_R - p^\mu_L \) is integer rather than vanishing as in the noncompact case. This integer, \( p^\mu_R - p^\mu_L = n^{25} \in \mathbb{Z} \) is called the ‘winding number’. It counts the number of times the string wraps the compact \( X^{25} \) direction. Thus, in this case,

\[
\tilde{X}^{25}(\tau, \sigma) = p^{25}\tau + n^{25}\sigma + \text{oscillating terms} , \quad p^{25}, n^{25} \in \mathbb{Z} ,
\]

and the T-duality is an operation which interchanges the momentum number \( p^{25} \in \mathbb{Z} \) and the winding number \( n^{25} \in \mathbb{Z} \),

\[
p^{25} \xleftarrow{T-\text{duality}} n^{25} .
\]
More precisely T-duality changes the sign of $\partial \tilde{X}_L^2$.

In open string theory the T-duality, as defined above, maps the Neumann boundary conditions in the isometry direction (corresponding to the string with free ends) into the Dirichlet boundary conditions (corresponding to the string endpoints attached to some hyperplane),

$$\text{Neumann boundary condition} \quad \leftrightarrow \quad \text{T-duality} \quad \leftrightarrow \quad \text{Dirichlet boundary condition}.$$  (4)

As the string theory implies gravity, the hyperplane where a string can have its endpoints cannot be treated as "frozen", but it is rather a dynamical object; this is identified as a so-called Dirichlet $p$–brane or $D_p$–brane [2]). Thus, in the case of spacetime with (at least) one isometry direction, T-duality should map a $D_p$–brane onto either a $D(p - 1)$–brane or a $D(p + 1)$–brane, depending on whether the isometry direction is tangential or orthogonal to the $D_p$–brane worldvolume $W^{p+1}$,

$$D_p \quad \leftrightarrow \quad \text{T-duality} \quad \leftrightarrow \quad D(p \pm 1).$$  (5)

T-duality is compatible with spacetime supersymmetry. In flat superspace T-duality can be defined as a map between type IIA and type IIB superstring models that transforms nontrivially one of the fermionic coordinate functions

$$\text{type IIA superstring} \quad \left( \begin{array}{c} \tilde{\theta}^{\alpha 1}(\tau, \sigma) \\ \tilde{\theta}^{\alpha 2}(\tau, \sigma) \end{array} \right) \quad \leftrightarrow \quad \text{T-duality} \quad \leftrightarrow \quad \left( \begin{array}{c} \tilde{\theta}^{\alpha 1}(\tau, \sigma) \\ \sigma_{\alpha \beta}^{\#} \tilde{\theta}^{\beta 2}(\tau, \sigma) \end{array} \right) \quad \text{type IIB superstring}$$  (6)

where the isometry direction is identified with $\hat{z} = \hat{X}^9$ and $y = X^9$, respectively, and $\sigma_{\alpha \beta}^{\#} = \sigma_{\alpha \beta}^9$ is the $D = 10$ Majorana–Weyl gamma–matrix related to the isometry direction. Eq. (6) suggests that the T-duality should transform the fermionic coordinates of flat type IIA and type IIB superspaces with topology of the bosonic sector $\mathbb{R}^9 \otimes S^1$ (see, e.g. [20] and refs. therein)

$$\left( \begin{array}{c} \tilde{\theta}^{\alpha 1} \\ \tilde{\theta}^{\alpha 2} \end{array} \right) \quad \leftrightarrow \quad \text{T-duality} \quad \leftrightarrow \quad \left( \begin{array}{c} \theta^{\alpha 1} \\ \sigma_{\alpha \beta}^{\#} \theta^{\beta 2} \end{array} \right).$$  (7)

## 2 T-duality rules for bosonic fields

In 1987 Buscher found that the study of the string action in a bosonic background with isometries allows one to find an elegant field theoretical representation for T-duality as a relation between bosonic $NS–NS$ (Neveu-Schwarz–Neveu-Schwarz) fields of type IIA and type IIB supergravity theories:

$$\begin{align*}
NS–NS & \quad \text{type IIA} \\
\hat{g}_{\hat{\mu} \hat{\nu}}(\hat{x}), \hat{B}_{\hat{\mu} \hat{\nu}}(\hat{x}), \hat{\phi}(\hat{x}) \quad & \leftrightarrow \quad \text{T-duality} \\
& \quad \text{NS–NS} \quad \text{type IIB} \\
g_{\mu \nu}(x), \hat{B}_{\mu \nu}(x), \phi(x)
\end{align*}.$$  (8)

For the case of spacetime(s) with one isometry they have the form \footnote{We use the metric in the so-called 'Einstein frame', where the Einstein–Hilbert action does not include a dilaton factor, while the string action does. This results in the occurrence of the dilaton factors in Eqs. (9)–(11).}

$$\begin{align*}
e^{\frac{1}{2}} g_{yy} = \frac{1}{e^{\frac{1}{2}} \hat{g}_{\hat{z} \hat{z}}}, & \quad e^{\frac{1}{2}} g_{\hat{z} \hat{m}} = \frac{1}{e^{\frac{1}{2}} \hat{g}_{\hat{z} \hat{z}}} \hat{B}_{\hat{z} \hat{m}}.
\end{align*}$$  (9)
\[ e^{2\phi} g_{\tilde{m}\tilde{n}} = e^{2\phi} \tilde{g}_{\tilde{m}\tilde{n}} + \frac{1}{e^{2\phi} g_{zz}} \left( \tilde{B}_{m\tilde{z}} \tilde{B}_{n\tilde{z}} - e^{\phi} \tilde{g}_{m\tilde{n}} \tilde{g}_{\tilde{n}\tilde{z}} \right), \]

\[ e^{2\phi} = -\frac{1}{e^{2\phi} g_{zz}}, \]

\[ B_{m\tilde{n}} = \tilde{B}_{m\tilde{n}} + \frac{1}{g_{zz}} \left( \tilde{g}_{\tilde{m}\tilde{n}} \tilde{B}_{\tilde{z}\tilde{z}} - \tilde{g}_{m\tilde{n}} \tilde{B}_{\tilde{m}\tilde{z}} \right), \quad B_{\tilde{y}n} = \frac{1}{g_{zz}} \tilde{g}_{\tilde{y}\tilde{n}}. \]

Here \( \partial_{\tilde{z}} \) is along the isometry direction of the type IIA supergravity which is defined on the space with the coordinates \( \hat{x}^{\tilde{m}} = (x^{\tilde{m}}, \tilde{z}) \) (which may be called ‘type IIA spacetime’), \( \partial_{y} \) is along the isometry direction of type IIB supergravity which is defined on the spacetime with coordinates \( x^{\mu} = (x^{\tilde{m}}, y) \) (which may be called ‘type IIB spacetime’). Note that nine of the ten coordinates of ‘type IIA’ and ‘type IIB’ spacetimes are identified, \( \hat{x}^{\tilde{m}} = x^{\tilde{m}}\), \( \tilde{m} = 0, 1, \ldots, 8 \). Thus the intersection of type IIA and type IIB spacetimes gives a nine–dimensional spacetime,

\[ \mathcal{M}_{IIA}^{(10)} \cap \mathcal{M}_{IIB}^{(10)} = \mathcal{M}^{(9)} : \quad x^{\tilde{m}} = (x^{0}, x^{1}, \ldots, x^{8}). \]

In other words, the Buscher rules (9)–(12) imply that (the bosonic fields) of type IIA and type IIB supergravity are defined on different ten–dimensional surfaces in an underlying 11 dimensional spacetime (cf. [4])

\[ \mathcal{M}^{(11)} : \quad X^{M} = (x^{\tilde{m}}, \tilde{z}, y) \equiv (\hat{x}^{\tilde{m}}, \tilde{z}), \quad \mathcal{M}_{IIA}^{(10)} \subset \mathcal{M}^{(11)} ; \quad \hat{x}^{\tilde{m}} = (x^{\tilde{m}}, \tilde{z}) , \quad \mathcal{M}_{IIB}^{(10)} \subset \mathcal{M}^{(11)} : \quad x^{\mu} = (x^{\tilde{m}}, y). \]

The field theoretical representation of T-duality was studied for the bosonic limit of supergravity in a number of papers (see, refs. in [4]). In particular, in [5] J. Simon studied the T–duality transform of the Dp–brane actions [10] in a purely bosonic supergravity background. In this way he reproduced the above Buscher rules and also derived the T-duality rules for the RR (Ramond–Ramond) gauge potential of type IIA and type IIB supergravity,

\[ \begin{align*}
\text{RR type IIA} & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua
\[ C^{(2n)}_{\tilde{m}_1 \ldots \tilde{m}_{2n}} = C^{(2n-1)}_{\tilde{m}_1 \ldots \tilde{m}_{2n-1}} + \frac{(2n-1)}{\tilde{g}_{\tilde{m}_1 \ldots \tilde{m}_{2n-2}}} \hat{g}_{\tilde{m}_{2n-1}} \hat{g}_{\tilde{m}_{2n}} \]
\[ C^{(2n)}_{\tilde{m}_1 \ldots \tilde{m}_{2n}} = C^{(2n+1)}_{\tilde{m}_{2} \tilde{m}_1 \ldots \tilde{m}_{2n}} + 2n \tilde{C}^{(2n-1)}_{\tilde{m}_1 \ldots \tilde{m}_{2n-1}} \hat{B}_{\tilde{m}_{2n}} + \frac{2n(2n-1)}{\tilde{g}_{\tilde{m}_1 \ldots \tilde{m}_{2n-2}}} \hat{B}_{\tilde{m}_{2n-1}} \hat{g}_{\tilde{m}_{2n}} \hat{g}_{\tilde{m}_{2n}} \]

for \( n = 1, 2, 3, 4 \), and
\[ C^{(10)}_{\tilde{m}_1 \ldots \tilde{m}_9} = C^{(9)}_{\tilde{m}_1 \ldots \tilde{m}_9} + \frac{9}{\tilde{g}_{\tilde{m}_1 \ldots \tilde{m}_8}} \tilde{C}^{(9)}_{\tilde{m}_1 \ldots \tilde{m}_9} \hat{g}_{\tilde{m}_9} \hat{g}_{\tilde{m}_{10}} \]

The T-duality rules for fermions

**type IIA**
\[ \hat{\psi}^{\alpha \dagger}_{\mu}(\hat{x}), \hat{\psi}_{\mu}^2(\hat{x}), \hat{\lambda}^1_{\alpha}(\hat{x}), \hat{\lambda}^{2\dagger}(\hat{x}) \]

**type IIB**
\[ \psi^{\alpha \dagger}_{\mu}(x), \psi_{\mu}^2(x), \lambda^1_{\alpha}(x), \lambda^{2\dagger}(x) \]

were recently obtained by Hassan by studying the map between supersymmetry transformations of type IIA and type IIB supergravity.

### 3 Towards the superfield T-duality rules

In superfield formulation of supergravity all the physical fields appear as leading \((\hat{\theta} = 0 \text{ or } \theta = 0)\) components of some superfields. Moreover, one may say that differential forms describing the physical fields, appear as the leading \((\hat{\theta} = 0, \hat{d} \theta = 0 \text{ or } \theta = 0, d\theta = 0)\) terms in superforms of the superspace formulation of \(D = 10\) type IIA and type IIB supergravity.

\[
\hat{e}^\alpha \equiv \hat{d} \hat{x}^\mu \hat{e}^{\alpha}_{\mu}(\hat{x}) = \hat{E}^\alpha |_{\hat{\theta} = 0 = d\hat{\theta} = 0}
\]
\[
\hat{d} \hat{x}^\mu \hat{\psi}^{\alpha \dagger}_{\mu}(\hat{x}) = \hat{E}^{\alpha 1} |_{\hat{\theta} = 0 = d\hat{\theta} = 0},
\]
\[
\hat{d} \hat{x}^\mu \hat{\psi}_{\mu}^2(\hat{x}) = \hat{E}^{\alpha 2} |_{\hat{\theta} = 0 = d\hat{\theta} = 0},
\]
\[
\frac{\hat{d} \hat{x}^\mu \wedge \hat{x}^{\nu}}{2!} \hat{B}_{\nu \mu}(\hat{x}) = \hat{B}_2(\hat{Z}) |_{\hat{\theta} = 0, d\hat{\theta} = 0},
\]
\[
\hat{\phi}(\hat{x}) = \hat{\Phi}(\hat{Z}) |_{\hat{\theta} = 0}
\]
\[
\frac{\hat{d} \hat{x}^\mu_1 \wedge \ldots \wedge \hat{x}^{\mu_{2n+1}}}{(2n+1)!} \hat{C}_{\mu_2 \ldots \mu_{2n+1} \mu_1}(\hat{x}) = \hat{C}_{2n+1}(\hat{Z}) |_{\hat{\theta} = 0 = d\hat{\theta} = 0},
\]

In this perspective the dilatons \((\phi(x) \text{ and } \hat{\phi}(\hat{x}))\) appear as leading components of dilaton superfields \((\text{zero–form } \tilde{\Phi}(\tilde{Z}) \text{ and } \Phi(Z))\) and the spin 1/2 fermions are leading components of the covariant Grassmann derivatives of these dilaton superfields

\[
\dot{\lambda}_{\alpha_1}(\hat{x}) \propto \nabla_{\alpha_1} \Phi(\hat{Z}) |_{\hat{\theta} = 0}
\]
\[
\dot{\lambda}_{\alpha_2}(\hat{x}) \propto \nabla_{\alpha_2} \Phi(\hat{Z}) |_{\hat{\theta} = 0}
\]
\[
\dot{\lambda}_\alpha(x) \propto \nabla_\alpha \Phi(Z) |_{\theta = 0}
\]
\[
\dot{\lambda}_\alpha(x) \propto \nabla_\alpha \Phi(Z) |_{\theta = 0},
\]

The problem was to find the complete set of T–duality rules relating the above superfield supergravity potentials, or, equivalently, related superforms

**type IIA SUGRA**

**type IIB SUGRA**
duality rules for Ramond–Ramond (RR) superfield potentials and fermionic supervielbeins \( \kappa \) were found by studying the relation between complete type IIA and type IIB superstring

Where, following \[10\], we collected all the RR superforms of type IIA /IIB supergravity and required significant efforts to extract the transformation rules for the fermionic coordinate functions \( \tilde{\theta} \) of Romans massive type IIA supergravity \[11\] was studied up to quadratic order in the refs. in \[4\]). In particular, in \[14\] the T-duality map of type IIA super string into the type IIB supergravity in the formal sum of all odd/even differential forms on superspace,

\[
\begin{align*}
\text{type IIA:} & \quad \hat{C}_{2n+1} = \frac{1}{(2n+1)} d\hat{Z}^{M_{2n+1}} \wedge \ldots \wedge d\hat{Z}^{M_{1},M_{2n+1}} \hat{C}^{(2n+1)}_{M_{1},M_{2n+1}}(\hat{Z}) , \\
\text{type IIB:} & \quad C_{2n} = \frac{1}{2n} dZ^{M_{2n}} \wedge \ldots \wedge dZ^{M_{1},M_{2n}} C^{(2n)}_{M_{1},M_{2n}}(Z) .
\end{align*}
\]

In nineties there appeared a number of papers addressing this problem (see \[7\] and refs. in \[3\]). In particular, in \[14\] the T-duality map of type IIA superstring into the type IIB one and then back to type IIA massive superstring (superstring in the background of Romans massive type IIA supergravity \[11\]) was studied up to quadratic order in the fermionic coordinate functions \( \tilde{\theta}(\tau, \sigma) \). In \[9\] the T–duality rules for \textit{NS–NS superfields} and fermionic supervielbeins

\[
\begin{align*}
\text{type IIA SUGRA} & \quad \Phi(\hat{Z}) \\
\text{type IIB SUGRA} & \quad \Phi(Z)
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
\hat{E}^a &= d\hat{Z}^M E^a_M(\hat{Z}) \\
\hat{B}_2(\hat{Z}) &= \frac{d\hat{Z}^M \hat{Z}^N}{2!} \hat{B}_{MN}(\hat{Z})
\end{pmatrix} & \quad \leftrightarrow \quad \\
\begin{pmatrix}
E^a &= dZ^M E^a_M(Z) \\
B_2(Z) &= \frac{dZ^M \hat{Z}^N}{2!} \hat{B}_{MN}(Z)
\end{pmatrix} .
\end{align*}
\]

were found by studying the relation between complete type IIA and type IIB superstring actions and their \( \kappa \)–symmetries. However, this approach did not allow to find the T–duality rules for Ramond–Ramond (RR) \textit{superfield potentials}

\[
\begin{align*}
\hat{C}(\hat{Z}) = \bigoplus_{n=0}^4 \hat{C}_{2n+1}(\hat{Z}) & \quad \leftrightarrow \quad C(Z) = \bigoplus_{n=0}^5 C_{2n}(Z)
\end{align*}
\]

and required significant efforts to extract the transformation rules for the \textit{components} of the RR field strengths from Bianchi identities.
4 Derivation of superfield T-duality rules (19).

The complete set of superfield T–duality rules were obtained in [4]. They follow from the relation between the complete \( \kappa \)–symmetric actions for Dirichlet superbranes in type IIA and type IIB supergravity backgrounds and subsequent study of the exchange between the type IIA and IIB superspace supergravity constraints. Namely, in the first stage, the comparison of the type IIA super–D\((p + 1)\)–brane and type IIB super–D\(p\)–brane actions [10] which are known to be related by T–duality [1, 2], provides the T–duality transformation rules for the bosonic superforms of type IIB resp. IIA supergravity (including all RR superforms), Eqs. (24), (26). Then, in a second stage, substituting these rules into the superspace torsion constraints and the constraints on NS–NS field strengths of type IIA and type IIB supergravities [15, 16, 10], one can derive the T–duality rules for the remaining (fermionic) supervielbein forms.

It turns out that the T-duality transformation rules for the bosonic superforms, which can be obtained from the comparison of the super–D\(p\)–brane actions (i.e. by the superfield generalization of the method of Ref. [5]), can be reproduced as well by a straightforward superfield (superform) generalization of the final results of Ref. [5].

In [4] we used such a shortcut. By substituting the NS–NS T–duality rules thus obtained into the superspace torsion constraints and into the constraints on NS–NS field strengths [15, 16, 10], we have derived the T–duality rules for fermionic supervielbein forms (in Einstein frame). Finally, we described in [4] the verification of the consistency of the complete set of T–duality rules thus obtained with the superspace constraints for RR superform field strengths [10].

4.1 T–duality as an operation acting on differential forms and underlying superspace \( \mathcal{M}^{(11\mid32)} \).

Our approach [4] treats T–duality as an operation which acts on differential forms in superspace rather than on the superspace coordinates. This treatment of T-duality allows one to identify all but one bosonic and all the fermionic coordinates of curved type

Such a simple possibility to reproduce the superfield results form the component ones can be regarded as a reflection of the existence of the ‘rheonomic’ (group manifold) approach to supergravity [17] which allows to lift the component equations (written in terms of differential forms on spacetime) to the superspace equations for superforms. This is also natural in a view of recent observation [18] that superfield description of the dynamical supergravity–superbrane interacting system (still hypothetical for \( D = 10, 11 \)) is gauge equivalent to a more simple dynamical system described by the sum of the standard (component) supergravity action and the action for pure bosonic brane (the pure bosonic limit of the original superbrane action).

5Such a possibility is guaranteed by (super)diffeomorphism invariance of (superspace super)gravity, i.e. by its gauge symmetry under arbitrary changes of local coordinate system (in superspace). (Super)diffeomorphism invariance allows one to replace any coordinate transformations by the equivalent transformations of the supergravity (super)fields (see, e.g., [19] and refs. therein). However, such ‘picture changing’ allows to overcome the problems that blocked the way to superfield T–duality rules (see e.g. [20]).
IIA and type IIB superspace \(^6\), i.e.

\[
\text{type IIA} : \quad \mathcal{M}_{IIA}^{(10|32)} : \quad \tilde{Z}^M = (\tilde{Z}^\hat{M}, \tilde{z}) ,
\]

\[
\text{type IIB} : \quad \mathcal{M}_{IIB}^{(10|32)} : \quad Z^M = (\tilde{Z}^\hat{M}, y) .
\]

In other words, the intersection of curved type IIA and type IIB superspaces, \(\mathcal{M}_{IIA}^{(10|32)}\) and \(\mathcal{M}_{IIB}^{(10|32)}\), defines some \(D = 9, N = 2\) superspace \(\mathcal{M}^{(9|32)}\) (cf. (13))

\[
\mathcal{M}_{IIA}^{(10|32)} \cap \mathcal{M}_{IIB}^{(10|32)} = \mathcal{M}^{(9|32)}
\]

\[
\mathcal{M}^{(9|32)} : \quad \tilde{Z}^\hat{M} \equiv \left( \tilde{X}^\hat{m}, \theta^\mu \right) , \quad \hat{m} = 0, \ldots, 8 , \quad \mu = 1, \ldots, 32 .
\]

Moreover, this point of view makes transparent that type IIA and type IIB theories with isometries \(\partial_{\tilde{z}}\) and \(\partial_y\) can be defined on the hypersurfaces \(\tilde{z} = 0\) and \(y = 0\) of an underlying superspace \(\mathcal{M}^{(11|32)}\) with 11 bosonic and 32 fermionic coordinates,

\[
\mathcal{M}^{(11|32)} : \quad (\tilde{Z}^\hat{M}, y, \tilde{z}) .
\]

## 5 Superfield T–duality rules

The superfield T–duality rules are simplest when written in terms of the supervielbein forms adapted to the isometry, i.e. obeying the superfield generalization of the Kaluza–Klein type ansatz familiar from dimensional reductions of supergravity theories [21]:

\[
\text{type IIA} : \quad \hat{E}^{\hat{a}} = (\hat{E}^{\hat{a}}, \hat{E}^\#) , \quad \hat{E}^{\hat{a}} = \hat{E}^{\hat{a}}(-) = d\tilde{Z}^\hat{M} \hat{E}^a_M (\tilde{Z}) ,
\]

\[
\hat{a} = 0, \ldots, 9 , \quad a = 0, \ldots, 9 , \quad \hat{a} = 0, \ldots, 8
\]

\[
\text{type IIB} : \quad E^a = (E^\tilde{z}, E^*) , \quad E^\tilde{z} = E^\tilde{z}(-) = d\tilde{Z}^\hat{M} E^\tilde{z}_M (\tilde{Z}) ,
\]

\[
i.e. \quad \hat{E}^a_\tilde{z} = 0 , \quad E^a_\tilde{z} = 0 , \quad \text{and all the nonvanishing component-superfields depending only on the coordinates } \tilde{Z}^\hat{M} \text{ of nine-dimensional superspace } \mathcal{M}^{(9|32)} , \text{ Eq. } (29).
\]

T-duality rules for NS-NS superfields read [4]

\[
e^{\hat{\Phi}(\tilde{z})} E^a_M = e^{\hat{\Phi}(\tilde{z})} E^a_M , \quad e^{\hat{\Phi} E^*_\tilde{z}} = \frac{1}{e^{\hat{\Phi} E^*_\tilde{z}}} , \quad e^{\hat{\Phi} E^*_M} = \frac{\hat{B}_{\tilde{z}M}}{e^{\hat{\Phi} E^*_\tilde{z}}} ,
\]

\[
e^{\Phi(\tilde{z})} = e^{\Phi(\tilde{z})} ,
\]

\[
^{\frac{1}{8}} \quad \text{The possibility of identification of the fermionic coordinates of curved type IIA and type IIB superspaces, } \hat{\theta}^\mu = \theta^\mu , \text{ Eqs. (27), (28), would not seem surprising if one remembers that the fermionic coordinates of a general curved superspace do not carry any chirality. Their indices } \mu \text{ and } \mu \text{ are not the spinor indices of the Lorentz group; } \theta^\mu \text{ and } \theta^\mu \text{ are rather transformed by the general superdiffomorphism symmetry. The chirality, which is used to distinguish the type IIA and type IIB is a characteristic of the fermionic supervielbein 1–forms } (\hat{E}^{\alpha a}, \hat{\bar{E}}_a^2, \hat{\bar{E}}^\alpha_2) \text{ which do carry } SO(1, 9) \text{ spinor indices. Only in the flat superspace limits, when one takes the fermionic supervielbein to be derivatives of the fermionic coordinates, the chiral structure, together with the definite spinor representation of the Lorentz group, applies to the fermionic coordinates of flat superspace.}
\]
\[ B_{y\bar{M}} = \frac{\hat{E}_{z}^\#}{E_{z}^\#}, \quad B_{\bar{M}N} = \hat{B}_{M\bar{N}} - \frac{2}{E_{z}^\#} \hat{B}_{z[M} \hat{E}_{\bar{N}]}^\#. \] (36)

The T–duality rules for fermionic supervielbeins are

\[ e^{-\frac{1}{8} \Phi} \frac{E_{y}^{\beta_1}}{E_{z}^{\gamma}} = e^{-\frac{1}{8} \Phi} \left( \frac{\hat{E}_{z}^{\gamma}}{E_{z}^{\gamma}} + i \frac{3}{8} \hat{a}^{\beta \gamma} \nabla_{\gamma} \hat{\Phi} - i \frac{3}{8} \hat{\Phi} \nabla_{\gamma} \ln \left( e^{\frac{1}{2} \hat{E}_{z}^{\gamma}} \right) \right), \] (37)

\[ e^{-\frac{1}{8} \Phi} \frac{E_{y}^{\beta_2}}{E_{z}^{\gamma}} = e^{-\frac{1}{8} \Phi} \left( \frac{\hat{E}_{z}^{\gamma}}{E_{z}^{\gamma}} + i \frac{3}{8} \hat{a}^{\beta \gamma} \nabla_{\gamma} \hat{\Phi} - i \frac{3}{8} \hat{\Phi} \nabla_{\gamma} \ln \left( e^{\frac{1}{2} \hat{E}_{z}^{\gamma}} \right) \right), \] (38)

\[ e^{\frac{1}{8} \Phi} E^{\beta_1[-]} = e^{\frac{1}{8} \Phi} \left( \hat{E}^{\beta_1[-]} - \frac{1}{8} \hat{E}^{\beta} \hat{a}^{\beta \gamma} \nabla_{\gamma} \ln \left( e^{\frac{1}{2} \hat{E}_{z}^{\gamma}} \right) \right), \] (39)

\[ e^{\frac{1}{8} \Phi} E^{\beta_2[-]} = e^{\frac{1}{8} \Phi} \hat{a}^{\beta \gamma} \left( \hat{E}^{\beta_1[-]} - \frac{1}{8} \hat{E}^{\beta} \hat{a}^{\beta \gamma} \nabla_{\gamma} \ln \left( e^{\frac{1}{2} \hat{E}_{z}^{\gamma}} \right) \right), \] (40)

where

\[ \hat{E}^{\alpha_1[-]} = d\hat{Z}^{\hat{M}} \left( E_{M}^{\alpha_1} - \frac{\hat{E}_{z}^{\alpha_1}}{E_{z}^{\alpha_1}} \right), \quad \hat{E}^{\alpha_2[-]} = d\hat{Z}^{\hat{M}} \left( E_{M}^{\alpha_2} - \frac{\hat{E}_{z}^{\alpha_2}}{E_{z}^{\alpha_2}} \right), \] (41)

\[ E^{\alpha_1[-]} = d\hat{Z}^{\hat{M}} \left( E_{M}^{\alpha_1} - \frac{E_{N}^{\alpha_1}}{E_{N}^{\alpha_1}} \right), \quad E^{\alpha_2[-]} = d\hat{Z}^{\hat{M}} \left( E_{M}^{\alpha_2} - \frac{E_{N}^{\alpha_2}}{E_{N}^{\alpha_2}} \right). \] (42)

One may rewrite Eqs. (39), (40) in terms of \( E^{\alpha_1(-)} = d\hat{Z}^{\hat{M}} E_{M}^{\alpha_1}, \ C^{\alpha_2(-)} = d\hat{Z}^{\hat{M}} E_{M}^{\alpha_2} \) and their type IIA counterparts. However, the notion of \([-\cdot]\) components of differential forms, which implies separation of the terms proportional to \( E^{\alpha_1 \cdot} \) and \( E^{\alpha_2 \cdot} \) (see Eqs. (32), (33)), rather than the ones proportional to \( dy \) and \( d\hat{z} \), is quite suggestive. In particular, with this notation, the fermionic T–duality transformation rules (37)–(40) do not involve any contribution from the NS–NS superfields enclosed in \( B_{2} \) and \( \hat{B}_{2} \). Such contributions appear when one rewrites (39), (40) in terms of \( E^{\alpha_1(-)} \) etc. (see (31)), but, taking in mind the original form of Eqs. (39), (40), one immediately clarifies their origin in the T–duality rules for bosonic superfields \( \hat{E}_{M}^{\#} (\hat{Z}) \) and \( \hat{E}_{M}^{\#} (\hat{Z}) \), Eqs. (34), (35).

The above set of T–duality rules from (31) coincides with the ones from (31) after passing to the so–called String frame and to the supervielbeins which are not adapted to isometries (see (31) for a detailed comparison).

However, our approach allowed us to derive as well the T-duality rules for the RR superform potentials (31). They are

\[ C^{(0)} = \hat{C}^{(1)} \]

\[ C^{(2n)}_{M_{1}...M_{2n-1}} = -\hat{C}^{(2n-1)}_{M_{1}...M_{2n-1}} + \frac{2n-1}{E_{z}^{\#}} \hat{E}_{z[M_{1}} \hat{C}^{(2n-1)}_{z]M_{2}...M_{2n-1}} \]

\[ C^{(2n)}_{M_{1}...M_{2n}} = \hat{C}^{(2n+1)}_{z[M_{1}...M_{2n}} + 2n \hat{B}_{z[M_{1}} \hat{C}^{(2n-1)}_{M_{2}...M_{2n}]} + \frac{2n-2n}{E_{z}^{\#}} \hat{B}_{z[M_{1}} \hat{E}_{z]M_{2}} \hat{C}^{(2n-1)}_{z]M_{3}...M_{2n}} \] (43)

### 5.1 Superfield T–duality rules in differential form notation.

The above Eqs. (43) may be collected in compact expressions for the formal sums of all type IIA and all type IIB superforms, (22) and (23).

\[ C = i_{z} \hat{C} + (dy + i_{z} \hat{B}_{2}) \wedge \left( \hat{C}(-) - \frac{\hat{E}_{z}^{\#}(-)}{E_{z}^{\#}} \wedge i_{z} \hat{C} \right), \] (44)
Here the contractions $i_z$ and the ‘minus’ components $\hat{C}^{(-)}$ are defined by

\[
i_z\hat{\Omega}_q := \frac{1}{(q-1)!}d\hat{Z}^{\hat{M}_q-1} \wedge \ldots \wedge d\hat{Z}^{\hat{M}_1}\hat{\Omega}_{z\hat{M}_1\ldots\hat{M}_{q-1}}(\hat{Z}),
\]

\[
\hat{\Omega}_q^{(-)} := \frac{1}{q!}d\hat{Z}^{\hat{M}_q} \wedge \ldots \wedge d\hat{Z}^{\hat{M}_1}\hat{\Omega}_{\hat{M}_1\ldots\hat{M}_{q}}(\hat{Z})
\]

(46) (47)

for any $q$–forms $\hat{\Omega}_q = \frac{1}{q!}d\hat{Z}^{\hat{M}_q} \wedge \ldots \wedge d\hat{Z}^{\hat{M}_1}\hat{\Omega}_{\hat{M}_1\ldots\hat{M}_{q}}(\hat{Z})$ in type IIA superspace; in the same way one defines $i_y$ and $(-)$ components in the type IIB case.

Just this compact form of the fermionic T–duality rules allowed us to check [4] the consistency of the full set of the T–duality rules with the complete set of superfield supergravity constraints [15, 16, 10], including the ones for the RR superfield strengths.

6 A brief conclusion

More details and a discussion can be found in the original paper [4]. Our approach can be also extended to the more complicated $SO(n, n)$ T–duality provided the superfield generalization of the Kaluza–Klein type ansatz for the dimensional reduction down to $d = 10 - n$ dimensions is elaborated for this case. The results of our study clarify the relationship of T–duality with superfield formulations of supergravity and, as we hope, might provide new insights in M–theory.

References

[1] J. Polchinski, Superstring Theory, V.1,2. CUP, 1998.
[2] J. Dai, R. G. Leigh and J. Polchinski, *Mod. Phys. Lett.* A4 (1989) 2073; R. G. Leigh, *Mod. Phys. Lett.* A4 (1989) 2767; J. Polchinski, *Phys. Rev. Lett.* 75 (1995) 4724.

[3] T. Buscher, *Phys.Lett.* B194 (1987) 59; *Phys.Lett.* B201 (1988) 466; A. Giveon, M. Porrati and E. Rabinovici, *Phys.Rept.* 244 (1994) 77-202.

[4] Igor Bandos and Bernard Julia, *JHEP* 0308 (2003) 032 [arXiv:hep-th/0303075].

[5] J. Simon, *Phys.Rev.* D61 047702 (2000) [arXiv:hep-th/9812095].

[6] P. Meessen and T. Ortin, Nucl. Phys. B 541 (1999) 195 [arXiv:hep-th/9806120].

[7] W. Siegel, *Phys.Rev.* D47 (1993) 5453-5459 [arXiv:hep-th/9302036]; *Phys.Rev.* D48 (1993) 2826-2837. [arXiv:hep-th/9305073].

[8] S. F. Hassan, *Nucl.Phys.* B568 (2000) 145-161 [hep-th/9907152]; *Nucl.Phys.* B583 (2000) 431-453 [arXiv:hep-th/9912236].

[9] B. Kulik and R. Roiban, *T-duality of the Green-Schwarz superstring*, JHEP 0209, 007 (2002) [arXiv:hep-th/0012010].

[10] M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell and A. Westerberg, *Nucl.Phys.* B490 (1997) 179-201 [arXiv:hep-th/9611159].

[11] L. J. Romans, Phys. Lett. B 169, 374 (1986).

[12] G. Dall’Agata, K. Lechner and D. P. Sorokin, Class. Quant. Grav. 14, L195 (1997) [arXiv:hep-th/9707044]; G. Dall’Agata, K. Lechner and M. Tonin, JHEP 9807, 017 (1998) [arXiv:hep-th/9806140].

[13] P. Pasti, D. P. Sorokin and M. Tonin, Phys. Lett. B 352, 59 (1995) [arXiv:hep-th/9503182]; Phys. Rev. D 52, 4277 (1995) [arXiv:hep-th/9506109]; Phys. Rev. D 55, 6292 (1997) [arXiv:hep-th/9611100].

[14] M. Cvetič, H. Lü, C.N. Pope and K.S. Stelle, *Nucl.Phys.* B573 (2000) 149-176 [arXiv:hep-th/9907202].

[15] P.S. Howe and P.C. West, *Nucl. Phys.* B238 (1984) 181.

[16] J.L. Carr, S.J. Gates and R.N. Oerter, *Phys.Lett.* B189 (1987) 68; S. Bellucci, S.J. Gates, B. Radak and Sh. Vashakidze, *Mod. Phys. Lett.* A4 (1989) 1985.

[17] Y. Neeman and T. Regge, *Riv. Nuovo Cim.* 1 (1978) 1; L. Castellani, R. D’Auria and P. Fré, *Supergravity and superstrings, a geometric perspective*, v. 2, World Scientific, 1991, and references therein.

[18] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo and J. Lukierski, *Phys. Rev.* D67 (2003) 065003 [arXiv:hep-th/0207139]; *Phys. Rev.* D68 (2003) 046004 [arXiv:hep-th/0301255]; [arXiv:hep-th/0211065].

[19] I.A. Bandos, J.A. de Azcárraga and J.M. Izquierdo, *Phys. Rev.* D65 (2002) 105010 [arXiv:hep-th/0112207].

[20] K. Kamimura and J. Simon, *Nucl.Phys.* B585 (2000) 219-252 [arXiv:hep-th/0003211].

[21] M.J. Duff, P.S. Howe, T. Inami, K.S. Stelle, *Phys.Lett.* B191 (1987) 70.