Characters of $\hat{sl}(4)_k$ fusion algebra at non-rational level

P. Furlan*,** and V.B. Petkova†

* Dipartimento di Fisica Teorica dell'Università di Trieste and 
** Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Trieste, 
Strada Costiera 11, 34100 Trieste, Italy 
† Institute for Nuclear Research and Nuclear Energy, 
Tzarigradsko Chaussee 72, 1784 Sofia, Bulgaria

Abstract

We construct the fusion ring of a quasi-rational $\hat{sl}(4)_k$ WZNW theory at generic level $k \notin \mathbb{Q}$. It is generated by commutative elements in the group ring $\mathbb{Z}[\hat{W}]$ of the extended affine Weyl group $\hat{W}$ which extend polynomially the formal characters of finite dimensional representations of $sl(4)$.

1. Introduction

The WZNW models at generic (non-rational) level provide examples of quasi-rational conformal field theories (Q-RCFT). These are theories described by an infinite discrete spectrum of representations of the chiral algebra, here $\mathfrak{g} = \hat{sl}(n)_k$, and a fusion rule producing a finite number of terms. The study of these quasi-rational fusion rings is motivated by the fact that they determine, upon “quantisation”, the fusion rules of the corresponding RCFT, described by the fractional level admissible representations of $\mathfrak{g}$ [1]. For the latter there is no sensible Verlinde formula at hand, see the two fully worked out $\hat{sl}(n)_k$ examples so far, $n=2$ [2], [3], [4] and $n=3$ [5], [6], [7]. The quasi-rational fusion rings and their characters are also important as part of the data of more general CFT with a continuum spectrum, on manifolds with or without boundaries; see, e.g., [8] and references therein for the simplest example of generic level $\hat{sl}(2)_k$ theory.
Consider the “preadmissible” set of representations labelled by the highest weights

\[
\{ \Lambda = \tilde{y} \cdot (\lambda' - \lambda (k + n)) \mid k \notin \mathbb{Q} \} \quad \tilde{y} \in \tilde{W}, \ \lambda', \lambda \in P_+, \text{ s.t.,}
\]

\[
\langle \lambda, \alpha_i \rangle \delta + \tilde{y}(\alpha_i) \in \Delta_{+}^{\text{re}}, \ i = 1, 2, \ldots n - 1 \}
\]

where \( \tilde{y} \cdot \lambda \) is the shifted action of the Weyl group \( \tilde{W} \) of the horizontal algebra \( \mathfrak{g} = \mathfrak{sl}(n) \), \( P_+ = \oplus_i \mathbb{Z}_{\geq 0} \Lambda_i \) is the chamber of integral dominant weights, \( \Lambda_i \) being the fundamental weights of \( \mathfrak{g} \), and \( \Delta_{+}^{\text{re}} \) is the set of real positive roots of \( \mathfrak{g} \). The “preintegrable” subset \( (\tilde{y} = 1, \lambda = 0, \Lambda = \lambda' \in P_+) \) has fusion ring coinciding with the representation ring of the finite dimensional irreducible representations (irreps) of \( \mathfrak{g} \). Its structure constants are given by the classical Weyl - Steinberg formula, which can be “quantised” to recover the fusion rule multiplicities of the integer level integrable representations \[9\], \[10\], \[11\]. The main ingredient in both the classical and the “quantised” versions of this formula is the multiplicity of states of finite dimensional irreps of \( \mathfrak{g} \), encoded in their formal characters. However all these classical data have no direct meaning for the second, labelled by non-integer highest weights of \( \mathfrak{g} \), subseries of \( (1.1) \) \( (\lambda' = 0) \), we shall deal with below.

Remarkably in the simplest \( \hat{\mathfrak{sl}}(2)_k \) case the fusion characters for the subseries \( \lambda' = 0 \) turn out to be given by the formal characters of finite dimensional irreps of the super-algebra \( \mathfrak{osp}(2|1) \). Accordingly the quasi-rational fusion ring of the \( \hat{\mathfrak{sl}}(2)_k \) representations \[1.1\] coincides with a product of the representation rings of \( \mathfrak{osp}(2|1) \) and \( \mathfrak{sl}(2) \). Their quantised rational counterpart inherits this “hidden” \( \mathbb{Z}_2 \) - graded structure, first noticed in \[4\]. The group \( \mathbb{Z}_2 \) is the Weyl group \( \tilde{W} \) of \( \mathfrak{sl}(2) \) and the next truly nontrivial case \( n = 3 \) exhibits a \( \tilde{W} \) - graded algebraic structure too. The specific character formulae established for \( n = 3 \) do not extend, however, to \( n \geq 4 \), that is why it is important to study the simplest next case \( n = 4 \) by methods admitting in principle a generalisation to arbitrary \( n \).

Preliminary results of the present work were announced in \[12\] and to make the paper self-contained we repeat in the next section some of the introductory material there. Our main new result is the explicit formula in section 3 for the characters shown to generate a consistent fusion ring.

2. General setting

Consider the subset \( \tilde{W}^{(+)_{\}} \) of the extended affine Weyl group \( \tilde{W} = \tilde{W} \times t_P = W \rtimes A \)

\[
\tilde{W}^{(+)_{\}} := \{ y \in \tilde{W} \mid y(\alpha_i) \in \Delta_{+}^{\text{re}} \text{ for }\forall i = 1, \ldots, n - 1 \} . \]

Here \( t_P \) is the subgroup of translations in the weight lattice \( P \) of \( \mathfrak{g} = \mathfrak{sl}(n) \), and \( A \) is the cyclic subgroup of \( \tilde{W} \), generated by \( \gamma = \prod_{i} w_1 w_2 \ldots w_{n-1} \), which keeps invariant the set of simple roots \( \Pi = \{ \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \} \) of \( \mathfrak{g} \).
Let \( k \notin \mathbb{Q} \). The subset \( \tilde{W}^{(+)} \) is a fundamental domain in \( \tilde{W} \) with respect to the right action of \( \tilde{W} \). The set of weights \( \tilde{W}^{(+)} \cdot k\Lambda_0 \) (corresponding to the subset \( \mathcal{L}' = 0 \) of \( \{1, i\} \), \( y = \tilde{y} t_{-\lambda} \)) or, equivalently, the subset \( \tilde{W}^{(+)} \subset \tilde{W} \) itself, labels the highest weights \( \Lambda \) of maximally reducible Verma modules of \( g \). Indeed for \( \Lambda = y \cdot k\Lambda_0 \) and \( \beta = y(\alpha) \), s.t. \( y \in \tilde{W}^{(+)} \), the Kac-Kazhdan singular vector criterion holds true for any positive root \( \alpha \) of \( \tilde{g} \). Here \( \Lambda_0 \) is the fundamental weight of \( g \) dual to the affine root \( \alpha_0 \) and the Kac-Kazhdan reflections are identified with the right action of \( \tilde{W} \) on \( \tilde{W} \), i.e., \( w_{y(\alpha)} \cdot \Lambda = y_{\alpha} \cdot k\Lambda_0 \).

The factorisation of the submodules generated by these singular vectors imposes restrictions on the possible three-point couplings which determine the fusion rules in the CFT. Solving completely the corresponding null vector decoupling equations is a difficult problem (see also the Discussion below). Instead we shall construct recursively a fusion ring building on and extending the approach in [6]. We associate with any \( y \in \tilde{W}^{(+)} \) a formal “character”, an element of the group ring \( \mathbb{Z}[\tilde{W}] \) of \( \tilde{W} \)

\[
\chi_y = \sum_{z \in \tilde{W}, zy^{-1} \in W} m^y_z z, \quad y \in \tilde{W}^{(+)},
\]

(2.2)

extended to \( \tilde{W} \) by

\[
\chi_{yw} := \det(w) \chi_y, \quad y \in \tilde{W}^{(+)}, \quad w \in \tilde{W}.
\]

(2.3)

The sum in (2.2) is required to run over elements in \( \tilde{W} \) with length not exceeding the length of the “highest weight” \( y \) and \( m^y_z \) are integer nonnegative multiplicities, yet to be determined. The characters (2.2) are generalisations of the formal characters of finite dimensional irreps of \( \tilde{g} \),

\[
\bar{\chi}_\lambda = \sum_{\mu \in \Gamma_\lambda} m^\lambda_\mu \ t_{-\mu} \in \mathbb{Z}[t_{-P}], \quad \bar{\chi}_{w \cdot \lambda} = \det(w) \bar{\chi}_\lambda, \quad w \in \tilde{W}, \ \lambda \in P_+.
\]

(2.4)

The finite set \( G_y = \{ z \in \tilde{W} \mid m^y_z \neq 0 \} \) generalises the weight diagram \( \Gamma_\lambda \). With a notion of a generalised weight diagram, consider a formula for the fusion rule multiplicities, generalising the classical Weyl - Steinberg formula for \( \bar{\chi}_\lambda \)

\[
\chi_x \chi_y = \sum_{z \in G_x} m^x_z \chi_{zy} = \sum_{z \in \tilde{W}^{(+)}} N^z_{x,y} \chi_z,
\]

(2.5)

\[
N^z_{x,y} = \sum_{w \in \tilde{W}} \det(w) \ m^{x}_{zw y^{-1}}.
\]

(2.6)

The second equality in (2.5) with the multiplicities given in (2.6) is derived as for the usual \( sl(n) \) characters, using the symmetry in (2.3) and the fact that \( \tilde{W}^{(+)} \) is a fundamental domain in \( \tilde{W} \).
Introduce a map \( \iota \) of \( \tilde{W} \) into the root lattice \( Q \) of \( g \):

\[
\iota: \tilde{W} \ni y = \bar{y}t_{-\lambda} \mapsto n\lambda + \bar{y}^{-1} \cdot 0 \in Q.
\] (2.7)

It has the properties

\[
\iota(xy) = \bar{y}^{-1}(\iota(x)) + \iota(y),
\]

\[
\iota(yw) = w^{-1} \cdot \iota(y), \quad w \in \tilde{W},
\]

\[
\iota(ax) = \iota(x), \quad a \in A,
\]

and the set \( \tilde{W}^{(+)} \) is expressed alternatively as \( \tilde{W}^{(+)} = \{ y \in \tilde{W} | \iota(y) \in P_+ \} \).

In the \( n = 3 \) case the coefficients \( m_{y}^{z} \) in (2.2) are given as

\[
m_{z}^{y} = \overline{m}_{\iota(z)}^{(y)},
\] (2.9)

\( \overline{m}_{\iota(z)}^{(y)} \) being as in (2.4) the multiplicity of the weight \( \mu = \iota(z) \) of the representation of \( sl(3) \) of highest weight \( \lambda = \iota(y) \). Similarly the fusion coefficients (2.6) are expressed in terms of the structure constants \( \overline{N}_{\iota(y)}^{(z)} \) of the \( sl(3) \) character ring

\[
N_{x,y}^{z} = \overline{N}_{\iota(x) \iota(y)}^{(z)}.
\] (2.10)

The generalised weight diagrams \( G_{y} \) are determined by (2.9) and thus have the structure of the weight diagrams \( \Gamma_{\iota(y)} \) of triality zero \( sl(3) \) representations, with multiplicities preserved, but with the weights \( \mu \notin \text{Im}(\iota) \) excluded. The same type of formulae hold in the simpler \( sl(2) \) case where \( |G_{y}| = |\Gamma_{\iota(y)}| \).

However as discussed in [12] the definition of the generalised weight diagram based on (2.9), and hence (2.10), has to be modified in the higher rank cases, since it is not consistent with the Weyl - Steinberg formula (2.5) in general. The multiplicities in (2.2) are only restricted by the inequality \( m_{z}^{y} \leq \overline{m}_{\iota(z)}^{(y)} \).

3. Fusion character ring

We denote by \( \overline{W} \) the character ring of finite dimensional irreps of \( sl(n) \) generated by the formal classical characters \( \overline{\chi}_{\lambda} \). They commute with any \( w \in \tilde{W} \) because of the invariance of the classical weight multiplicities \( \overline{m}_{w(\mu)}^{\lambda} = \overline{m}_{\mu}^{\lambda}, w \in \tilde{W} \).

Let \( x = t_{-\nu}x \in \tilde{W}^{(+)} \). Guided by the \( n = 2, 3 \) examples we first introduce the following combinations of classical \( sl(n) \) characters \( \overline{\chi}_{\lambda} \) times powers of the generator \( \gamma \) of \( A \), parametrised by weights \( b \in Q \),

\[
\overline{\chi}_{x}^{(b)} = \det(\overline{x}) \sum_{i=0}^{n-1} \gamma^{i} \overline{\chi}_{\nu_{-i}(x) + b + \overline{\gamma}^{-i}}.
\] (3.1)
where $\overline{\chi}_n := 0$. These elements of the group ring $\overline{W}[A]$ are covariant under $A$

$$\overline{\chi}^{(b)}_{ax} = a\overline{\chi}^{(b)}_x, \quad \forall a \in A.$$  

(3.2)

We recall the expressions for the generalised characters for the $n = 2, 3$ cases for which $\tilde{W}^{(+)} = A t_+ t_+$ and $\tilde{W}^{(+)} = A t_+ t_+ \cup A w_0 t_+$ respectively \[6\]

$$\chi_x = \overline{\chi}^{(0)}_x = \det(\varpi) \left( \overline{\chi}_\nu + \gamma \overline{\chi}_{\nu - \gamma} \right), \quad n = 2,$$

$$\chi_x = \overline{\chi}^{(0)}_x + \chi_{w_0} \overline{\chi}^{(0)}_x, \quad \chi_{w_0} = 2 + w_0 + w_1 + w_2, \quad n = 3.$$  

(3.3)

The square of the $A$-invariant combination $F := w_0 + w_1 + w_2 = aFa^{-1}$ in (3.3) lies in $\overline{W}[A]$.

We now turn to the $sl(4)$ case. The fundamental chamber $\tilde{W}^{(+)}$ is alternatively represented as $\tilde{W}^{(+)} \equiv U t_+$, where

$$U = \{ A, A w_0, A w_{10}, A w_{30}, A w_{310}, A w_{2310} \}$$  

(3.4)

is a subset of $\tilde{W}$; its projection $\overline{U}$ onto the subgroup $\overline{W}$ gives the right cosets of $\overline{A}$. The group $A$ defines an automorphism of $W$, $w_\alpha \rightarrow \gamma w_\alpha \gamma^{-1} = w_\gamma(\alpha), \alpha \in \Pi$, with $\gamma(\alpha_j) = \alpha_{j+1}$ for $j = 0, 1, 2, \ldots n - 1$, identifying $\alpha_n \equiv \alpha_0$. Using this we define $A$-invariants in the group algebra of $W$, $F_y = F_{a_ya^{-1}} = aF_ya^{-1}, \forall a \in A$,

$$F_{rst\ldots} \equiv F_{w_{rst\ldots}} := \frac{1}{l_{w_{rst\ldots}}} \sum_{a \in A} a w_{rst\ldots}a^{-1}, \quad w_{rst\ldots} \in W,$$  

(3.5)

where $l_w$ takes the value 1 or 2 if the sum over $A$ contains 4 or 2 different terms, respectively; e.g., $F_0 = w_0 + w_1 + w_2 + w_3$, $F_{13} = w_{13} + w_{02} = F_{20}$. As it is clear from (2.8), the terms in a given $F_y$ have their $\iota$ images in an orbit of the cyclic subgroup $\overline{A}$ of $\overline{W}$. In general $F_x \neq F_y F_x$, but e.g., the three elements $Y_0 := F_0, Y_{30} := F_{30} + F_{13}, Y_{10} := F_{10} + F_{13}$, commute between themselves.

We shall introduce now a finite set of formal characters $\chi_y, y \in W^{(+)} := \tilde{W}^{(+)} \cap W$, as in (2.2), for all of which we will adopt the definition (2.9). In employing the map (2.7) and comparing with the standard $sl(4)$ weight diagrams one can use the recursive formula for the multiplicity of a weight $\mu$ (see, e.g., [13])

$$\overline{m}_\mu = - \sum_{\overline{w}_1 \in W, \overline{w}_1 \neq \overline{w}} \det(\overline{w}) \overline{m}_{\mu + \overline{w}_1},$$  

(3.6)

with the weights in the r.h.s. strictly greater than $\mu$. Using (3.6) we have for $y \in W^{(+)}$ and of length $l(y) \leq 3$

$$\chi_{w_0} = 3 + F_0 \equiv 3 + Y_0, \quad \iota(w_0) = (1, 0, 1),$$

$$\chi_{w_{10}} = 3 + 2F_0 + F_{13} + F_{10} \equiv 3 + 2Y_0 + Y_{10}, \quad \iota(w_{10}) = (0, 1, 2),$$

$$\chi_{w_{30}} = 3 + 2F_0 + F_{13} + F_{30} \equiv 3 + 2Y_0 + Y_{30}, \quad \iota(w_{30}) = (2, 1, 0),$$

5
\[ \chi_{w_{130}} = 7 + 5F_0 + 4F_{13} + 2F_{30} + 2F_{10} + (F_{121} + F_{130} + F_{213}) \]
\[ = 7 + 5Y_0 + 2Y_{10} + 2Y_{30} + Y_{130}, \quad \iota(w_{130}) = (1, 2, 1), \]
\[ \chi_{w_{230}} = 1 + F_0 + F_{13} + F_{30} + F_{230} \equiv 1 + Y_0 + Y_{30} + \gamma \chi_{A_1}, \quad \iota(w_{230}) = (4, 0, 0), \]
\[ \chi_{w_{210}} = 1 + F_0 + F_{13} + F_{10} + F_{210} \equiv 1 + Y_0 + Y_{10} + \gamma^3 \chi_{A_3}, \quad \iota(w_{210}) = (0, 0, 4), \]
of dimension 7, 17, 17, 63, 15, 15, respectively (here \( (a_1, a_2, a_3) = \sum_i a_i \bar{A}_i \)). Being linear combinations of \( A \)-invariant elements they commute with any \( a \in A \). To each of these characters we associate a weight diagram which can be identified with a finite subset of the Cayley graph of \( W \) (see [12] for a schematic drawing of the latter). In agreement with (2.6) one obtains by a direct computation
\[ \chi_{w_0} \chi_{w_0} = 1 + 2 \chi_{w_0} + \chi_{w_{10}} + \chi_{w_{30}}, \]
\[ \chi_{w_0} \chi_{w_{10}} = \chi_{w_0} + 2 \chi_{w_{10}} + \chi_{w_{130}} + \chi_{w_{210}}, \]
\[ \chi_{w_0} \chi_{w_{30}} = \chi_{w_0} + 2 \chi_{w_{30}} + \chi_{w_{130}} + \chi_{w_{230}}, \]
which serve as algebraic relations restricting the set of characters
\[ \mathcal{F} = \{\chi_{w_0}, \chi_{w_{10}}, \chi_{w_{30}}, \chi_{w_{210}}, \chi_{w_{230}}\}. \]

Further fusions recover the characters of length 4, in particular the character \( \chi_{w_{2130}} \) of dim 177, with \( \iota(w_{2130}) = (2, 2, 2) \),
\[ \chi_{w_{2130}} = \chi_{w_{10}} \chi_{w_{30}} - \left(1 + 2 \chi_{w_0} + \chi_{w_{10}} + \chi_{w_{30}} + \chi_{w_{130}}\right) \]
\[ = 11 + 9F_0 + 8F_{13} + 4F_{10} + 4F_{30} + 3F_{121} + 3F_{130} + 3F_{213} + 2F_{230} + 2F_{210} \]
\[ + (F_{10} + F_{30} + F_{1213} + F_{1232} + F_{1321} + F_{2321} + F_{10213} + F_{2130}) \]
\[ =: 11 + 9Y_0 + 4Y_1 + 4Y_3 + 3Y_{130} + 2\gamma \chi_{A_1} + 2\gamma^3 \chi_{A_3} + Y_{2130}. \]

This is the simplest example in which formula (2.9) fails. The expression obtained by fusion corresponds to a weight diagram that is a subset of the one determined by (2.9) and (3.6). It can be summarised by the rules: i) delete all elements longer than the highest weight element; ii) decrease the multiplicity of \( w_{ijk...} \), determined from (2.9), by the complement to 4 of the number of different elementary reflections appearing in \( w_{ijk...} \). E.g. for \( w_{2321} \) the multiplicity (2.9) is decreased by 1 since three of the four reflections appear, while for \( w_{0213} \) it is left unchanged; the multiplicity of the identity is decreased by 4.
One obtains by a direct computation the relations

\[
\begin{align*}
Y_0^2 &= 4 + Y_{10} + Y_{30}
\end{align*}
\]

\[
\begin{align*}
Y_{10}^2 &= 2 + 3Y_{30} - Y_{10} + Y_0 \gamma^2 \chi_{\Lambda_3} + \gamma^2 \chi_{\Lambda_2} \\
Y_{10}^2 &= 2 + 3Y_{10} - Y_{30} + Y_0 \gamma \chi_{\Lambda_1} + \gamma^2 \chi_{\Lambda_2} \\
Y_0 Y_{10} &= 2Y_0 + Y_{130} + \gamma^3 \chi_{\Lambda_3} \\
Y_0 Y_{30} &= 2Y_0 + Y_{130} + \gamma \chi_{\Lambda_1} \\
Y_{10} Y_{30} &= 6 + Y_{2130},
\end{align*}
\]

(3.11)

which imply that all five elements \(Y_0, Y_{10}, Y_{30}, Y_{130}, Y_{2130}\) commute. Because of the first three equalities the product of any two of these elements is expressed as a linear combination of the same elements plus the identity, with coefficients in the ring \(\mathbb{W}[A]\). Alternatively the relations (3.11) can be rewritten as

\[
\begin{align*}
Y_{10} + Y_{30} &= Y_0^2 - 4 =: P_2(Y_0) \\
2Y_{130} &= Y_0^3 - 8Y_0 - (\gamma^3 \chi_{\Lambda_3} + \gamma \chi_{\Lambda_1}) =: P_3(Y_0) \\
2Y_{2130} &= Y_0^4 - 10Y_0^2 - Y_0(\gamma^3 \chi_{\Lambda_3} + \gamma \chi_{\Lambda_1}) + 8 - 2\gamma^2 \chi_{\Lambda_2} =: P_4(Y_0) \\
(Y_{10} - Y_{30})(\gamma^3 \chi_{\Lambda_3} - \gamma \chi_{\Lambda_1}) &= -Y_0^5 + 12Y_0^3 + 2Y_0^2(\gamma^3 \chi_{\Lambda_3} + \gamma \chi_{\Lambda_1}) \\
&+ 4Y_0(\gamma^2 \chi_{\Lambda_2} - 6) =: P_5(Y_0),
\end{align*}
\]

where \(P_k(Y_0)\) are \(k\)-order polynomials of \(Y_0\), and furthermore, \(Y_0\) satisfies a 6-order polynomial equation,

\[
Y_0^6 - 12Y_0^4 - 2cY_0^3 - 4Y_0^2(\gamma^2 \chi_{\Lambda_2} - 6) + c^2 - 4(1 + \chi_{g}) = 0
\]

(3.13)

where \(c = \gamma^3 \chi_{\Lambda_3} + \gamma \chi_{\Lambda_1}\). The relations (3.11) suggest that the formal characters we look for are given as linear combinations of the six elements \(Y_g\), \(g \in \mathcal{U}_1 := \{1, w_0, w_{10}, w_{30}, w_{130}, w_{2130}\} \subset \mathcal{U}\), with coefficients in the ring \(\mathbb{W}[A]\). We define

\[
\begin{align*}
\chi_x := \sum_{g \in \mathcal{U}_1} \chi_g \sum_{b} c_{g,b} \chi_x^{(b)} &= (\chi_x^{(0)} - \chi_x^{(\theta + \alpha_2)}) + \chi_0 (\chi_x^{(2\theta)} + \chi_x^{(\theta)}) \\
&+ \chi_{10} \chi_x^{(2\theta - \alpha_1)} + \chi_{30} \chi_x^{(2\theta - \alpha_3)} + \chi_{130} \chi_x^{(\theta + \alpha_2)} + \chi_{2130} \chi_x^{(\theta)}
\end{align*}
\]

(3.14)

Choosing \(x = g\) for \(g \in \mathcal{U}_1\) (3.14) reproduces the formulae for the characters in (3.7), (3.10). The values of the shifts \(b\) are recovered from each of these five basic characters demanding that the first classical character of the quadruplet (3.1) is an identity. This gives \(b_g = 0\) for the identity \(g = 1\), while \(b_g = -\bar{g} \cdot (-\theta)\) for \(g = \bar{g} t_{-\theta}\), i.e., \(b_g = 2\theta, 2\theta - \alpha_1, 2\theta - \alpha_3, \theta + \alpha_2, \theta\), respectively. In these checks one has to use repeatedly the symmetry (2.4) of the classical characters to cancel abundant
terms. Similarly one finds $\chi_a = a$ for any $a \in A$. The proposed expression \((3.14)\) is justified by the

**Lemma:** The following Pieri-type formulae hold true for the characters defined in \((3.14)\) and any $f \in F$:

\[
\chi_f \chi_x = \sum_{w \in G_f} \chi_{wx}.
\]  
\[(3.15)\]

The first two relations are proved by a direct but tedious computation comparing the products in the l.h.s. with the r.h.s. of \((3.15)\). It is based on the polynomial relations \((3.11)\) satisfied by the invariants $Y_g$. One has also to use the classical characters multiplication tables of the fundamental characters $\chi_{\bar{\Lambda}_i}, i = 1, 2, 3$ and $\chi_\theta$, which extend to the multiplication rules

\[
\chi_\lambda \chi_x = \sum_{\mu \in \Gamma_\lambda} \chi_{t-\mu} x.
\]  
\[(3.16)\]

The third relation is recovered from the second by the symmetry $w_1 \leftrightarrow w_3$. The proof for the last two characters $\chi_{w_{230}}, \chi_{w_{210}}$ uses the fact that they are expressed in terms of the first three characters in \((3.9)\) (cf. \((3.7)\)) and the fundamental classical characters $\chi_{\bar{\Lambda}_i}, i = 1, 2$. □

Formulae \((3.15)\) and \((3.14)\) hold for generic $x$, sufficiently far from the walls of the chamber $\tilde{W}^{(+)}$, otherwise cancellations occur, as e.g., in the examples \((3.8)\). Using \((3.2)\) the “fundamental” multiplication formulae \((3.15)\) are extended to include the "simple currents" corresponding to the elements $a = \chi_a$ of $A$

\[
\chi_a \chi_x = \chi_{ax}.
\]  
\[(3.17)\]

Having the explicit formula \((3.14)\) one can compute the multiplicities in \((2.2)\). Furthermore we recall that the following proposition was proved in \[12\] under the assumption that \((3.15)\) holds true:

**Proposition:** For any $y \in \mathcal{W}^{(+)}$ there is a formal character $\chi_y$ obtained recursively, using \((3.15)\), as a polynomial of the commuting “fundamental” characters in \((3.9)\).

Using \((3.17)\) the Proposition is extended to $y \in \tilde{\mathcal{W}}^{(+)}$. Combined with associativity it furthermore allows to extend \((3.15)\), \((3.17)\) to the multiplication of arbitrary two characters as in the first equality in \((2.5)\), and hence to confirm formula \((2.6)\) for the fusion multiplicities. The proof is a straightforward generalisation of the second proof of Lemma 4.5 in \[3\]. What however remains to be proved in general is the non-negativity of the multiplicities $N_{x,y}^z$ in \((2.6)\); so far we have checked it on numerous examples.
4. Discussion

Extending the results of [6], we have found a consistent $\tilde{sl}(4)_k$ fusion ring generated by the formal characters (3.14). To interpret it as the fusion ring of the related quasi-rational WZNW field theory one has to show that the (shifted) generalised weight diagrams of the generating characters in (3.15), (3.7) are consistent with the solution of the equations expressing the decoupling of the corresponding Verma module singular vectors. As in [5], [7] we can use the standard functional realisation of the representations of $sl(4)$, in which the generators are represented by differential operators in 6 variables, see e.g., [13]. The resulting systems of partial differential equations are however rather involved and we have checked the simplest of them, corresponding to the “fundamental” representation labelled by $w_{230} = \gamma t_{-\Lambda_1}$: the 15 points of the generalised weight diagram in (3.7) are confirmed. We have also partially checked the multiplication rule of the generator $\chi_{w_0}$, choosing a particular target representation for which the system of equations simplifies: once again the generalised weight diagram in (3.7) consisting of 4 points of multiplicity 1 and one point of multiplicity 3 is confirmed.

In the rational case $k + 4 = 4/p$ ($p$ - odd) the roots of the equation (3.13) determine $Y_0$ (and hence all five generators expressed by the polynomials $P_k(Y_0)$) in terms of the integrable representations fusion ring characters $\chi^{(p)}(\mu)$ at level $p$. In principle this should allow the “quantisation” of the general characters in (3.14), as it has been achieved in the $sl(3)$ case in [6]. Similarly we can use (3.13) in order to evaluate the formal characters (3.14) on the Cartan subgroup of $SU(4)$. In particular their values at the identity give the dimensions, alternatively obtained by sending all $z \to 1$ in (2.2).

The method is expected to apply algorithmically to any $n$ starting with the analogues of the set (3.4) and determining the coefficients $c_{a,b}$ in the analogue of (3.14) from the “fundamental” fusions generalising (3.13). The 6-order polynomial will be replaced by a $(n - 1)!$-order polynomial. The non-trivial problem that remains is to find a universal formula for the weight multiplicities in (2.3), extending (2.3), which in particular would allow to prove the non-negativity of the structure constants in (2.6).

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