We construct a model of an exchange economy in which agents trade assets contingent on an observable signal, the probability of which depends on public opinion. The agents in our model are replaced occasionally, and each person updates beliefs in response to observed outcomes. We show that the distribution of the observed signal is described by a quasi-nonergodic process and that people continue to disagree with each other forever. These disagreements generate large wealth inequalities that arise from the multiplicative nature of wealth dynamics, which makes successful bold bets highly profitable.

In standard macroeconomic models rational expectations can emerge in the long run, provided the agents’ environment remains stationary for a sufficiently long period. (Evans and Honkapohja 2013, 68)

I. Introduction

Our opening quote from Evans and Honkapohja encapsulates a commonly held view of macroeconomists—that rational expectations are a
justifiable assumption because, in a stationary environment and after a sufficiently long time, smart agents are able to learn about the probability distributions of the economic variables they care about. A stochastic process is a sequence of random variables; it is stationary if the unconditional probability of an element of the sequence is independent of the date at which it is observed, and it is ergodic if averages across possible realizations in a given period are equal to the time-series average of that variable over many different periods. When ergodicity holds, agents can reliably predict the future by averaging across events that have occurred in the past. Almost all stochastic macroeconomic models are assumed to be ergodic, and for this reason the argument summarized in our opening quote has proven persuasive to economists who have almost universally adopted the rational expectations assumption since it was introduced into macroeconomics by Robert Lucas 50 years ago (Lucas 1972).

In the real world, the random events that influence our lives are neither stationary nor ergodic. But although this observation is banal, it is not entirely obvious how to construct a model relevant to economics where ergodicity fails. In this paper, we propose a model to explain why agents fail to learn by exploiting the concept of quasi nonergodicity widely used in the physics literature to discuss the properties of glasses and “spin glasses” (Anderson 1989; Debenedetti and Stillinger 2001). Quasi nonergodicity occurs when a stochastic process is ergodic at very long time horizons but where ergodicity breaks down on a timescale at which realizations from the process might realistically be observed by a human agent.1 Although the probability distribution of observable variables is ergodic in the long run, as Keynes (1923) famously quipped, “in the long run we are all dead” (80).

To build a quasi-nonergodic process, we assume that agents learn the probability of a bivariate public signal that we refer to as public opinion. Public opinion is generated as the average probability over the subjective priors of all living agents. Public opinion generates an observable binary random variable that takes one of two values: zero or one. By observing a publicly observable sequence of zeros and ones, each individual forms a
subjective belief about the time-varying probability that next period’s realization will equal one.

When the agents in our model attempt to forecast public opinion, they are in effect making a forecast of the future beliefs of others. By combining this assumption with a model where the population of agents changes slowly over time, we induce a distribution over the probabilities of the binary signal that is quasi-nonergodic.

To ensure that no agent can exploit ergodicity by being sufficiently patient, we assume that newborn agents do not use the previous history of the public signal. Instead, they begin by making naive forecasts that become increasingly more sophisticated as agents accumulate observations on the signal over time. In this environment, we show that it is reasonable for agents to infer the probability of the public signal using a common constant-gain learning rule with gain parameter $\lambda$. We elaborate on this idea in section III.C and especially section IV.A, where we show that if all other agents learn with gain parameter $\lambda$, using the same learning rule as all other agents generates a forecast that has a negligible bias that is difficult or impossible to detect for almost all values of the time-varying probability of the public signal.

We endow our probabilistic world with a market that allows agents to trade two securities that are contingent on realizations of the public signal. At each date, agents solve an intertemporal optimization program to determine how much of each security they wish to hold and—because agents have different beliefs—they are willing to trade with each other.

Even though there is no cost to acquire information, traders in our model do not coordinate on the true probability. We show that the probability implied by market prices is a wealth-weighted average of individual subjective beliefs and not the unweighted average that in our model corresponds to the true probability. Some people accidentally benefit from the mismatch between these two probabilities and temporarily earn higher returns from trading in the asset markets. Interestingly, the resulting distribution of wealth is so unequal that the probability implied by market prices is influenced by the wealthiest agents and fails (even asymptotically) to reveal the true probability of the public signal. Because markets fail to aggregate private information correctly, market prices cannot be used by individuals to reveal the truth.

The behavior of our model depends on two parameters: $\delta$ and $\lambda$. The parameter $\delta$ represents the probability that a person dies and is replaced by a newborn. The parameter $\lambda$ represents the weight attached to the current signal when people update their beliefs. These two parameters are related to two important concepts: the ergodic time and the memory time.

The ergodic time is the time taken for sample averages to be good approximations to the moments of the ergodic distribution, and it is governed...
by $\delta^{-1}$. As $\delta^{-1}$ becomes large, the model becomes nonergodic. The memory time is the time over which sample averages are responsive to data, and it is governed by $\lambda^{-1}$. As $\lambda^{-1}$ increases, beliefs become unresponsive to new information. Interesting behavior occurs on timescales between these extremes.

The assumption that the stochastic process that generates observable signals is self-referential makes the evolution of the wealth distribution multiplicative, thereby generating a Pareto-tailed wealth distribution that displays large and empirically plausible wealth inequalities even though all agents receive the same nonstochastic endowment in every period. Interestingly, we are able to reproduce the empirical value of both the exponent of the Pareto tail and the Gini coefficient of real world wealth distributions using model parameters within a wide interval of reasonable values.\(^2\)

II. Literature Review

There is an extensive literature on self-fulfilling prophecies in rational expectations models. Early versions of this literature that rely on dynamic indeterminacy are discussed in Farmer’s (1999) textbook, and more recent models that display hysteresis and steady-state indeterminacy are reviewed in Farmer (2020b) and explored further in Farmer (2020a). The literature on self-fulfilling prophecies explains how beliefs drive economic fluctuations, but as with all rational expectations models, eventually everybody agrees with everybody else. Our current paper, in contrast, explains how a large number of agents interacting in a complete set of financial markets can continue to disagree forever.

Blume and Easley (2006) discuss two reasons why economists have been attracted to the rational expectations assumption. The first is that rational expectations may be a stable fixed point of an out-of-equilibrium learning mechanism. This explanation for rational expectations is the one followed by Evans and Honkapohja (2001) that we cited in our opening quote, and it is dismissed by Blume and Easley—rightly, in our view—because “positive results are delicate” and “robust results are mostly negative” (Blume and Easley 2006, 929).

The second route to rational expectations discussed by Blume and Easley is the market-selection hypothesis introduced by Friedman (1953) and Alchian (1970). According to this approach, “those who behave irrationally will be driven out of markets by those who behave as if they were rational” (Blume and Easley 2006, 930). Blume and Easley construct an economy

\(^2\) A Pareto tail refers to the ability of the Pareto distribution to approximate the density of a nonnegative random variable for values that are two or more standard deviations above the mean.
populated by infinitely lived agents with dynamically complete markets. They show that if there is a Bayesian learner, with the truth contained in the support of her prior, then all traders who survive will have asymptotically correct beliefs.

Following Blume and Easley (2006), an extensive literature builds on their main theme (Sandroni 2000; Cogley and Sargent 2008, 2009; Beker and Espino 2011). In contrast to this literature, we show that when new agents enter the model and when the stochastic process they are learning about is quasi-nonergodic, the economy never converges to a rational expectations equilibrium. In our setting, markets do not favor agents with accurate beliefs and prices fail to reveal the true underlying probabilities.

We are not the first to explore the topic of nonergodicity for economics. Brock and Durlauf (2001) have shown that interaction effects can trap the economy in a path-dependent state. Bouchaud (2013) has shown that the Random Field Ising model, which has proven useful in physics to understand interactions between particles, can fruitfully be adapted to understand non-market-based interactions between human beings. And Moran et al. (2020b) have shown that ergodicity breaking occurs in models of habit formation. Horst (2017, 1–6) reviews the literature on ergodicity and nonergodicity in economic models. In contrast to the literature on nonergodicity, we focus on a case where ergodicity is not strictly broken but where the timescale over which it applies may be longer than the lifetime of a human agent.

Our paper is related to a literature on the equilibria of environments in which agents have misspecified beliefs. Drawing on work in the statistics literature by Robert Berk (1966), Esponda and Pouzo (2016) define the concept of a Berk-Nash equilibrium in which each agent follows an optimal strategy given her beliefs. When beliefs are correctly specified, Berk-Nash equilibrium coincides with Nash equilibrium. Molavi, Tahbaz-Salehi, and Vedolin (2021) construct an asset-pricing model where agents have

In a paper related to ours, Borovička (2000) builds a model with two types of agents with distorted beliefs, but his model does not allow beliefs to adapt to changing information. In Massari (2019), an interesting scenario is presented where the market-selection mechanism fails in the sense that lucky traders become more wealthy than smart traders (as in our model) but prices still manage to remain efficient. Our model is also related to the “complex game” model of Galla and Farmer (2013), in which agents become trapped in chaotic trajectories that never converge.

Peters (2019) has pointed out that identifying time averages over a single trajectory with ensemble averages can lead to misleading conclusions and that special care should be devoted to the choice of an appropriate, process-dependent utility function. Our model illustrates a different facet of nonergodicity, where agents adapt their beliefs based on an observation window much shorter than the time needed to reach ergodicity.

Our model is a close cousin of Kirkman’s (1993) ant model, also known as the Moran (1958) model in the theory of population dynamics, for which results concerning the time taken to converge to the ergodic distribution were recently obtained by Moran et al. (2020a). Similar situations are encountered in business-cycle models with self-reflexive confidence effects (Morelli et al. 2020).
misspecified beliefs, and Molavi (2022) studies the implications of mis-
 specification of beliefs in three examples of common macro models. Both 
of these papers prove convergence to a set of beliefs that are closest to 
the truth in the sense that they minimize a Kullback-Leibler distance mea-
 sure. We also consider the implications of misspecified beliefs, but in 
our model the set of agents is changing over time. This feature is central 
to our demonstration that public opinion is quasi-nonergodic.

The closest precursor to our paper is Beker and Espino (2011). We 
modify their environment in two ways. First, the process that generates 
the states is self-referential and leads to a quasi-nonergodic process. Sec-
ond, we modify the environment to allow replacement of agents and we 
edow new agents with a random prior. Our work is similar to the discrete 
time stochastic extensions by Farmer, Nourry, and Venditti (2011) and 
Farmer (2018) of Blanchard’s (1985) perpetual youth model and the sto-
chastic continuous time version of that model in Gârleanu and Panageas 
(2015). The replacement of agents with new people with random priors is 
central to our demonstration that beliefs never converge.

Although we use the term “beauty contest,” our meaning is distinct from 
the work of Morris and Shin (2002) in which a beauty contest is modeled as 
a coordination game in which payoffs are interdependent. A related liter-
ature, following Angeletos and La’O (2013) and Benhabib, Wang, and 
Wen (2015), refers to “sentiment” to reflect a similar idea. In contrast to 
both of these papers, in our work individuals alive today try to guess what 
individuals who will be born in the future will think an asset will be worth in 
an environment where there is no fundamental uncertainty of any kind. 
Furthermore, in our model, in contrast to these alternative approaches, 
there exists a set of dynamically complete futures markets.

An important assumption that drives our results is that agents use 
constant-gain learning to update their beliefs, as in the work of Benhabib 
and Chetan (2014), Adam, Marcet, and Nicolini (2016), and Adam, Marcet, 
and Beutel (2017). Unlike those papers, we study a multiagent economy 
and link the true stochastic process to subjective beliefs through the obser-
vation of a public signal that depends on average beliefs. In our model, 
the event probability is time dependent, agents continue to disagree with 
one another forever, and the asymptotic wealth distribution is nontrivial 
and displays a Pareto tail. The random multiplicative growth mechanism 
that gives rise to this highly skewed wealth distribution is in the same fam-
ily of models as those considered in Bouchaud and Mézard (2000), 
Benhabib, Bisin, and Zhu (2011), and Benhabib and Bisin (2018).

III. A Two-Outcome, Self-Referential Model

We build up our argument in three stages. In stage 1 (this section and 
sec. IV), we describe a game in which agents form beliefs about a binary
outcome and we show that our game leads to a quasi-nonergodic process for the true belief. In stage 2 (secs. V–VII), we embed our agents in an endowment economy and allow them to trade Arrow securities contingent on the realization of the binary random variable. In stage 3 (sec. VI.C), we show that the contingent securities market can be replaced by debt and equity and that the equilibria of this more realistic version of our model is the same as the model in which agents trade Arrow securities. Section VIII derives the implications of our model for the wealth distribution.

A. The Beauty Contest Game

We assume that \( N \) agents play a game in which each person must forecast the average belief of the other agents about the outcome of a sequence of binary random events \( \{ s_t \in S = \{0, 1\} \}_{t=1}^{\infty} \). This is a simple version of a game that Keynes ([1936] 1973) introduced in The General Theory to motivate his view that the stock market is driven by what he called “animal spirits.”

We represent the belief held at date \( t - 1 \) by agent \( i \) of the probability that \( s_t = 1 \) as \( \mathbb{P}_{i, t}(s = \{1\}) \), and we model the self-referential nature of beliefs by assuming that the true probability of the event, \( \mathbb{P}_t(s = \{1\}) \), is equal to the average belief,\(^6\)

\[
\mathbb{P}_t = \frac{1}{N} \sum_{i=1}^{N} \mathbb{P}_{i, t},
\]

where, throughout the paper, we drop the argument \( s = \{1\} \) after \( \mathbb{P} \) unless we explicitly need to distinguish the two outcomes.

One interpretation of our model is that people communicate with others on social networks and each person forms an opinion of what other people think by sampling those within her private network. In the limit, when everyone is connected to everyone else, there is a single value for the beliefs of others that equals the average belief over everyone in the population. We refer to \( \mathbb{P}_t \) as public opinion, and we refer to the event \( s_t \in \{0, 1\} \) as confidence. In the terminology of Cass and Shell (1983), confidence is a sunspot.

One possible interpretation of the public signal is the action of an influential journalist who writes an opinion piece in a widely read financial newspaper. That opinion piece can be optimistic—we interpret optimism as the event \( s_t = 1 \)—or pessimistic—we interpret pessimism as

\(^6\) More generally, one can consider a model where the true probability is a nonlinear, sigmoidal function of the average belief (see app. sec. A2). Many of the results discussed in the bulk of this paper are actually valid in a more general context, though with interesting twists.
the event $s_t = 0$. The probability that the journalist will write an optimistic article is equal to the average degree of optimism in the population as measured by public opinion.

In section VI.C, we provide an interpretation of our model in which the public signal triggers a common decision on the part of firms to pay dividends in period $t$. In the absence of self-referential effects, the payment or nonpayment of a dividend would be irrelevant to the value of the firm. In contrast, in our model the decision to pay a dividend triggers trades between agents in the asset markets.

### B. A Model Where Beliefs Are Nonergodic

In this section, we construct a model where people are infinitely lived least-squares learners, and we show that in this version of our model public opinion is described by a nonergodic stochastic process. We assume that people live forever and although they initially disagree they are exposed to a common sequence of the realizations of a binary signal. Each person’s prior is an independent random draw from a uniform measure on $[0, 1]$. The following equation describes how an individual’s belief would evolve if he were to assume that $\mathbb{P}$ is time invariant:

$$
\mathbb{P}_{i,t+1} = \mathbb{P}_{i,t} \left( 1 - \frac{1}{t} \right) + \frac{s_t}{t},
$$

(2)

$$
\mathbb{P}_{i,1} = z_{i,0},
$$

(3)

where $z_{i,0}$ represents an independent draw from a uniform measure on $[0, 1]$. Using the definition of $\mathbb{P}_t$ from equation (1), it follows that for large $N$, the evolution of $\mathbb{P}_t$ is given by the equation

$$
\mathbb{P}_{t+1} = \mathbb{P}_t \left( 1 - \frac{1}{t} \right) + \frac{s_t}{t},
$$

(4)

where $s_t = 1$ with probability $\mathbb{P}_t$ and zero otherwise. In this case, $\mathbb{P}_t$ converges to a number in $[0, 1]$, but that number is different for every realization of $\{\mathbb{P}_t\}_{t=1}^\infty$. This representation of our model is an economic analogue of the Pólya urn model, a stochastic process that is well known to be nonergodic (Pemantle 2007).

Before providing examples of data generated by equations (2)–(4), we first provide some definitions and state a result from stochastic process theory: the mean ergodic theorem.

**Definition 1.** A stochastic process is a sequence of random variables $\{x_t\}_{t \in \mathbb{N}}$. A stochastic process is stationary if the joint probability distribution of
\((x_t, x_{t_1}, \ldots, x_{t_k})\)

is the same as the probability distribution of
\((x_{t_1 + T}, x_{t_2 + T}, \ldots, x_{t_k + T})\)

for all \(t_1, t_2, \ldots, t_k, T \in \mathbb{Z}\).

Let \(F\) represent the forward shift operator, let \(\omega \in \Omega\) represent a draw from a stochastic sequence with \(\sigma\)-field \(\mathcal{F}\), let \(L^p(\mathcal{P})\) represent the space of \(\mathcal{F}\)-measurable functions on \(\Omega\), and let \(\mathcal{P}\) represent a probability measure on \((\Omega, \mathcal{F})\).\(^7\)

Given these definitions, we have the following mean ergodic theorem.

**Theorem 1 (Mean ergodic theorem).** Let \(p \in [1, \infty)\). Then, for any \(f \in L^p(\mathcal{P})\), the limit
\[
\lim_{T \to \infty} \frac{f(\omega) + f(F(\omega)) + \cdots + f(F^{T-1}(\omega))}{T} = g(\omega)
\]
exists in \(L^p(\mathcal{P})\). Further, the limit \(g(\omega)\) is given by the conditional expectation
\[
g(\omega) = \mathbb{E}_\mathcal{P}(f|\mathcal{I}),
\]
where \(\mathcal{I}\) represents the invariant \(\sigma\)-field defined as
\[
\mathcal{I} = \{ A \subseteq \Omega : FA = A \}.
\]

A stochastic process that satisfies the assumption of the mean ergodic theorem is said to be **ergodic for the mean**, and when the conditions of the theorem apply to a stochastic process \(\{x_t\}_{t \in \mathbb{N}}\), theorem 1 implies that sufficiently long time-series averages of \(x_t\) will converge to the mean of the marginal stationary distribution of \(\mathcal{P}\) at a point in time. Similar concepts can be used to define ergodicity of higher moments and ergodicity of measures.

To illustrate the practical implications of nonergodicity, figure 1 plots three different realizations of the process modeled by equations (2) and (3) for an economy with half a million people. The solid black lines are the values of \(\mathbb{P}_t\) at each date, and the gray-shaded areas enclose the 20th and 80th percentiles of the distribution of beliefs. Each run is initialized with 500,000 independent draws from a uniform distribution.

\(^7\) \(\Omega\) represents the space of sequences that take values in a measurable space \(\mathcal{X}\), with \(\sigma\)-algebra \(\mathcal{B}\), and \(\mathcal{F}\) represents the product \(\sigma\)-field. Define a measure \(\mathcal{P}\) on \((\Omega, \mathcal{F})\) that describes the evolution of a process \(\{x_t\}_{t \in \mathbb{N}}\) over time. Let \(p \in [1, \infty)\), and define \(L^p(\Omega, \mathcal{F}, \mathcal{P})\) as the space of equivalence classes
\[
[X] = \{ Y : X = Y \ \mathcal{P}\text{-almost everywhere} \}
\]
of \(\mathcal{F}\)-measurable functions \(X\) such that
\[
\mathbb{E}(|X|^p) < \infty.
\]
A remarkable feature of these plots is the rapid convergence of opinion; almost all disagreement vanishes after 75 rounds. But though people converge on a given belief quite rapidly, they converge to a different value of $P_t$ for every sequence of draws $\{s_t\}$. When people live forever and are least-squares learners, the stochastic process that governs the evolution of $P_t$ is nonergodic. Although this example is instructive, it is not very interesting as a theory of why trades take place in asset markets. Everyone’s belief eventually converges to the truth, and although the truth is itself a function of history, eventually people all agree with one another.

C. A Model Where People Disagree Forever

To generate a theory of permanent disagreement, we modify the model in two ways. First, we allow the set of decision makers to change over time by recognizing that people have finite lives. Second, we replace the assumption of least-squares learning with an alternative constant-gain learning algorithm in which people discount the faraway past.

Although one could potentially introduce one of these assumptions without the other, it is natural to make both assumptions together. Suppose,
for example, we were to assume that people die but that everyone continues to forecast the future conditional probability using observations beginning from the date they were born. In this case, more recent observations will have more weight in the construction of public opinion because the age distribution of the population will decline exponentially. This observation suggests that a more accurate conditional forecast can be obtained by placing greater weight on more recent observations and that the additional weight placed on more recent observations should be related to the probability of death. In other words, because agents are aware that the world changes, they adapt their learning rule accordingly.

This intuition is confirmed by numerical simulations. We conducted a series of simulation experiments in which a fraction of the population was assumed to be constant-gain learners and the remaining fraction was assumed to consist of least-squares learners. We found that for all fractions between zero and one, the average mean squared error of the constant-gain learners was substantially lower than that of the least-squares learners. We conclude that agents who place more weight on the recent past will attain a competitive advantage over least-squares learners and, appealing to that logic, we assume for the remainder of this paper that agents are constant-gain learners with the same constant-gain learning parameter.8

To make these ideas precise, we assume that people die with a probability $\delta$ that is independent of age and that when a person dies, she is replaced by a new person with belief $P_i = z$, where $z$ is a random variable drawn from a uniform measure on $[0, 1]$. We keep track of who lives and who dies by introducing a random vector $x_t \in X = \{0, 1\}^N$, where $x_{i,t} = 1$ with probability $1 - \delta$ and 0 with probability $\delta$. If a person who was alive in period $t-1$ survives into period $t$, then $x_{i,t} = 1$. If she dies, then $x_{i,t} = 0$. Under these assumptions, the evolution of the beliefs of the person with index $i$ is given by the expression

$$P_{i,t+1} = x_{i,t}[(1 - \lambda)P_{i,t} + \lambda z_t] + (1 - x_{i,t})z_{i,t},$$

where $\lambda \in (0, 1)$, $z_t \in Z = [0, 1]^N$, and each element of $z_t$ is an independent draw from a uniform distribution. The exact form of the distribution of $z_t$ is not important for any of our results. One could also assume that $z_{i,t}$ is a weighted sum of the average belief $P_i$ and a uniform random variable. Provided the weight of the latter is nonzero, this would not change the structure of the model at all, only the meaning of the parameters. A slightly different specification is to assume that people never die but make occasional observation errors—that is, mistake $s_t$ for $1 - s_t$ with a small rate $\delta$. Our results turn out to be robust against changes in the precise specification of the model.

8 See sec. IV.A for further discussion of this point.
Coming back to equation (5), the term in square brackets on the right-hand side represents the way that a person who is alive in two consecutive periods updates her belief. She uses constant-gain learning with gain parameter \( \lambda \), where a value of \( \lambda \) closer to one means that the person puts more weight on recent outcomes. This term is multiplied by \( x_{i,t} \) to reflect the fact that it applies only if person \( i \) survives into the period. The second term on the right-hand side of equation (5) is multiplied by \( 1 - x_{i,t} \). This reflects the assumption that if agent \( i \) dies, her position is filled by a newborn person who starts life with a random subjective belief, \( z_{i,t} \).

In the limit, as \( N \to \infty \) we can combine equations (1) and (5) to obtain the following expression for the public opinion:

\[
P_{t+1} = (1 - \delta)[(1 - \lambda)P_t + \lambda s_t] + \frac{\delta}{2}
\]  
(6)

Figure 2 plots three different realizations of the process modeled by equation (6) for a value of \( \delta = 0.02 \) and \( \lambda = 0.14 \). The solid blue line, the dashed lines, and the line marked by squares are the values of \( P_t \) for three different draws from the stochastic process, and the gray-shaded areas enclose the 20th and 80th percentiles of the distributions of beliefs. Our economy contains 500,000 people, and we initialized all three sequences with the same value: \( P_0 = 0.5 \).

These simulations demonstrate that knowledge of public opinion today provides very little information about the state of public opinion in the near future. All three trajectories begin at the same point, but they quickly diverge from each other.

D. Quasi Nonergodicity

In this section, we introduce the concept of quasi nonergodicity and explain how it helps us to understand the behavior of the stochastic sequences depicted in figure 2. Our goal is to find a way to express the idea that a stochastic process may be ergodic over very long time horizons, but ergodicity may be irrelevant for all practical purposes if the timescale over which convergence is achieved is longer than the life span of a human observer, given by \( \delta^{-1} \).

To approach this idea, we first rewrite the stochastic process defined by equation (6) as a sequence of probability measures, \( \{P\} \), generated by the transition operator \( T \):

\[
T\{P\} = \int_0^1 dP \mathbb{P}(P) \left[ P d\left( P' - (1 - \delta)[(1 - \lambda)P + \lambda] - \frac{\delta}{2} \right) \right]
\]  
(7)
Here d(·) is the Dirac delta function and the symbols \( P \) and \( P' \) refer to probabilities in consecutive periods.9

The measure \( P(P) \) represents the probability that \( P \in A \) for any set \( A \subset [0, 1] \). For given \( P \), \( s = 1 \) with probability \( P \) and \( s = 0 \) with probability \( 1 - P \). The Dirac delta function assigns a value to \( P' \) for each of these two outcomes, and the measure \( P \) assigns a probability to each of the possible values of \( P \). Integrating over all of these possible values (weighted by \( P \)) generates the next period’s probability measure \( P' = T[P] \).

For this definition of \( P \), the probability densities at dates \( t \) and \( t + 1 \) are related by the equation

\[
P_{t+1} = TP_t,
\]

and the density at \( t \) is related to the initial measure \( P_0 \) by the expression

\[
P_t = P_0 P_t.
\]

---

9 A more usual notation is \( \delta(·) \) for the Dirac delta function. We use \( d(·) \) to avoid confusion with \( \delta \), which we reserve for the age-invariant probability of death.
where $T^t$ is the $t$th iterate of the operator $T$. Notice that $P$ defines a probability density over probabilities. Complex systems are often defined as probabilistic systems for which probabilities are themselves unknown and must be described with probabilities, as argued by Parisi (2007) and—in an economic context—by Bouchaud (2019).

Armed with this representation of the system, we are ready to introduce two preliminary concepts—distance from equilibrium and ergodic time—that we will use to define our central concept: quasi nonergodicity.

**Definition 2 (Distance from equilibrium).** Let the initial value of a random variable be $x_0 = x_{t=0} \in \Xi$, corresponding to an initial distribution $P_0(x)$ with unit mass localized on $x_0$. The distribution of $x_t$ at time $t$ is obtained from $P_0$ as $P_t = T^t P_0$. The similarity between the conditional distribution $P_t$ and the stationary distribution $P_\infty$ can be characterized by a distance $D \in [0,1]$ defined as (Boyd, Diaconis, and Xiao 2004)

$$D(P_t, P_\infty|_x) = \sup_S \left| \int_S dP_t - \int_S dP_\infty \right|,$$

where $S \in B$ is any subset of $\Xi$.

The argument of the sup operator measures the difference in mass that $P_t$ and $P_\infty$ attribute to any subset of the space $\Xi$, and the distance between the two measures, represented by $D$, is small when $P_t$ and $P_\infty$ assign similar probabilities to all possible subsets of $\Xi$.

Next, we need a way to measure how long it takes for a sequence of probability distributions to converge to an invariant measure. That requirement leads us to define the concept of ergodic time.

**Definition 3 (Ergodic time).** An ergodic stationary stochastic process $\{x_t\}_{t \in \mathbb{N}}$ has ergodic time $T_\epsilon(\epsilon)$ if

$$\int dP_\infty(x_0) D(P_{\epsilon T_\epsilon}, P_\infty|_x) \leq \epsilon.$$

In words, for a given confidence level $\epsilon$, the ergodic time $T_\epsilon(\epsilon)$ is the time beyond which the difference between the conditional distribution and the stationary distribution of the random variable $x$ is no greater than $\epsilon$. Finally, we are ready to introduce our central concept.

**Definition 4 (Quasi nonergodicity).** If $T_\epsilon(\epsilon)$ is larger than some large time $K$, we say that the stochastic process $\{x_t\}_{t \in \mathbb{N}}$ is $K - \epsilon$—quasi-nonergodic.

In the rest of the paper, we drop the terms $K$ and $\epsilon$ and refer simply to quasi nonergodicity. Informally, $\epsilon$ is a small number and $K$ is greater than the life span of a typical observer.
IV. Characterizing the Invariant Measure

A. Properties

In this section, we characterize the properties of the invariant measure as $N \to \infty$ and the period length $\Delta t \to 0$. We refer to this as the large $N$ continuous time limit.

Introducing the change of variable $u = \mathbb{P} - (1/2)$, we show in appendix A that in the large $N$ continuous time limit, $\mathcal{P}_t(u)$ converges to a symmetric beta distribution with parameter $\alpha = \delta / \lambda^2$,

$$\mathcal{P}_\alpha(u) = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \left( \frac{1}{4} - u^2 \right)^{\alpha - 1}.$$  

(8)

This distribution is hump-shaped for $\alpha > 1$—this is the case where $\delta > \lambda^2$—and U-shaped when $\alpha < 1$—this is the case where $\delta < \lambda^2$. In our baseline calibration, we choose $\alpha = 1$, which coincides with $\delta = \lambda^2$, but all of our results are robust to variations in $\alpha$ in a wide range of values between $\alpha = 0.5$ (for which $\mathcal{P}_\alpha(u)$ is U-shaped) and $\alpha = 2$ (for which it is hump-shaped).\(^{10}\)

The properties of this invariant measure depend on two parameters: $\delta$ and $\lambda$. The parameter $\delta$ is closely related to the ergodic time. Indeed, Moran et al. (2020a) have shown that if we fix $\alpha$ and take $\delta \to 0$, then $T_e(\epsilon)$ is of order $\delta^{-1}$. The parameter $\lambda$ has a similar interpretation in terms of the memory time.

**Definition 5 (Memory time).** Let person $i$ use the rule

$$\mathbb{P}_{i,t+1} = (1 - \lambda)\mathbb{P}_{i,t} + \lambda s_i$$

(9)

to forecast future values of $\mathbb{P}_T$ for $T > t$. The memory time, $T_m(\epsilon)$, is the number of periods after which the observation $s_i$ has weight less than or equal to $\epsilon$. It is defined by the expression

$$(1 - \lambda)^{T_m(\epsilon)} = \epsilon.$$  

It follows from this definition that, holding $\epsilon$ fixed, $T_m(\epsilon)$ is of order $\lambda^{-1}$.

Define the $i$th person’s degree of disagreement, $\mathbb{D}_{i,t}$, as the difference between the belief of agent $i$ and the average belief across all members of the population. In symbols,

$$\mathbb{D}_{i,t} = \mathbb{P}_{i,t} - \mathbb{P}_t.$$  

(10)
When we fix $\alpha$ and take $\lambda \to 0$, we are able to obtain an exact expression for the stochastic evolution of $D_{i,t}$ in the continuous time limit,

$$
D_{i,t+1} = x_{i,t} \left[ (1 - \lambda)D_{i,t} + \delta \left( (1 - \lambda)P_t + \lambda(x_t - \frac{1}{2}) \right) \right] + (1 - x_{i,t}) \left[ z_t - (1 - \delta) \left( (1 - \lambda)P_t + \lambda s_t + \frac{\delta}{2} \right) \right].
$$

(11)

In appendix B, we show that the unconditional expectation of $D_i$ converges to zero almost surely and that, in the large $N$/small $\lambda$ limit, its variance is given by the expression

$$
\mathbb{V}[D_i] = \left[ \frac{\lambda}{2 + (\alpha - 1)\lambda} \right] \frac{\alpha(\alpha + 2)}{6(2\alpha + 1)} + O(\lambda^3), \quad \alpha = \frac{\delta}{\lambda^2}.
$$

(12)

The variance of $D_i$ is a measure of disagreement between agents in the unconditional limiting distribution. For fixed $\lambda$, the disagreement tends to zero as $\delta \to 0$. In this case, people never die and are never replaced by agents holding fresh opinions. Correspondingly, $P_\alpha(u)$ has mass points at zero and one: everybody agrees in the limiting distribution that either $P = 0$ or $P = 1$. If $\alpha = \delta/\lambda^2$ is small but not equal to zero, public opinion switches between these two mass points with a frequency that vanishes asymptotically as $\alpha \to 0$.

In the opposite case, as $\alpha \to \infty$, $P_\alpha(u)$ has a single mass point at $u = 1/2$ and $\mathbb{V}[D_i]$ converges to the variance of the distribution of initial beliefs. When this distribution is uniform, as we assumed to derive equation (12), $\mathbb{V}[D_i] \to (1/12)$. In this case, agents never agree because they die much faster than they forget their initial beliefs. Indeed, $\lambda^2 \to 0$ means that the memory time $T_m$ (defined above) is infinite.

In our simulations, we chose a time interval of 1 week and set $\delta = 3.9 \times 10^{-4}$. These choices imply that life expectancy, averaged over people of all ages, is approximately 50 years, which accords well with crude estimates from US actuarial tables. For this fixed value of $\delta$, the standard deviation of disagreements is plotted as a function of $\alpha$ in figure 3, for $\alpha \in (0.2, 2)$. When $\alpha = 1$, the standard deviation of $D_i$ is approximately 4%, corresponding to a memory time $T_m$ (defined above) of 50 weeks.

Figure 3 demonstrates that although the standard deviation of $D$ varies with $\alpha$, it remains within a very small range of approximately 3.5%–4.5%. This level of disagreement is the same order of magnitude as that reported by Sunstein et al. (2021) for the dispersion of the estimates of experts using common information. As we show in our simulations, it is large enough to generate substantial discrepancies between the market price and the true price and a “fat” power-law right tail of the wealth distribution when people make bets based on their subjective beliefs.
B. Can Some Agents Learn Better Than Others?

In this section, we explore the question, Is it reasonable to use constant-gain learning with gain parameter \( \lambda \), given that everyone else in the economy is using the same forecast mechanism? We use the word “reasonable” because the use of constant-gain learning in this environment is clearly not optimal since the individual learning rule, given by equation (5),

\[
P_{i,t+1} = (1 - \lambda)P_{i,t} + \lambda s_t,
\]

is different from the evolution of the true probability, given by equation (6),

\[
P_{i,t+1} = (1 - \delta)((1 - \lambda)P_t + \lambda s_t) + \frac{\delta}{2}.
\]

The difference between the individual learning rule and the true evolution of public opinion arises because agents using naive constant-gain learning neglect to account for the arrival of new agents at rate \( \delta \).

Consider the problem of a single individual, born at date \( j \), who observes the sequence \( \{s_t\}_{t=j}^T \). The optimal Bayesian forecast of \( P_t \) is the solution to a nonlinear filtering problem where \( s_t \) provides a noisy signal of
the hidden state variable \( P_t \).\(^{11}\) It follows that constant-gain learning is not the best that an arbitrary observer with limitless computational power could achieve. But in the real world, people do not have limitless computational power and it may be sufficient to use a simpler rule that has a low predictive error.

So how bad is the constant-gain learning rule? Consider the following representation of this rule, which we refer to as the R-estimator:

\[
R_{t+1} = \sum_{j=0}^{t} (1 - \lambda)^j \lambda s_j + (1 - \lambda)^{t+1} R_0.
\]

The following discussion is based on an approximation that is valid for time periods in the interval \( \lambda^{-1} \ll t \ll \delta^{-1} \), which, for the case \( \alpha = 1 \), is between 1 and 50 years. On timescales, where \( \lambda^{-1} \ll t \), there are enough data to form estimates of \( P_t \), and for timescales where \( t \ll \delta^{-1} \), \( P_t \) is approximately constant.

On these timescales, the R-estimator is conditionally biased, with a bias given by the expression

\[
E[R_t - P_t | P_0] \approx \frac{\delta}{\lambda} \left( P_0 - \frac{1}{2} \right), \quad P_0 := P_{t=0},
\]

where the expectation is taken with respect to the true conditional time-varying probability that \( s_t = 1 \). This bias term reflects the fact that an agent who uses constant-gain learning neglects the mean-reverting force toward \( P = 1/2 \) that is induced by the birth of new agents with priors centered on \( P_{t,i} = 1/2 \).\(^{12}\)

Could this bias be detected by the agent over time periods of order \( \lambda^{-1} \)—that is, over lengths of time consistent with the learning window? For that to happen, the agent would need to observe a series of binary outcomes that were statistically implausible given her current belief about the value of \( P_t \). But for most of the range of \( P_t \), the variance of the R-estimator is large relative to its bias. The standard deviation of the R-estimator is approximated by the expression

\[
\text{SD}[R_t] \approx \sqrt{\frac{\lambda}{2} P_0 (1 - P_0)}.
\]

\(^{11}\) Although this problem is superficially similar to the problem of forecasting the state in the linear state space model, it is complicated by the facts that the variance of \( \{P_t\} \) is time varying and that the shocks to the state equation and the measurement equation are correlated. The optimal Bayesian forecast could be found using nonlinear methods, such as the particle filter, but applying methods of this kind are costly.

\(^{12}\) Note that the unconditional mean of \( P_0 \) is equal to 1/2, which implies that \( E[R_T - P_T | P_0] \to 0 \) as \( T \to \infty \). In words, the \( T \)-step ahead R-estimator is asymptotically unbiased.
To detect the bias, the standard deviation of the estimator must be small relative to the bias. This condition is represented by the inequality

\[ \sqrt{\frac{\lambda}{2} \mid P_0(1 - P_0) \mid} \ll \lambda \mid P_0 - \frac{1}{2} \mid, \]

where we have used the fact that \( \delta = \alpha \lambda^2 \) and the special case \( \alpha = 1 \). These two terms have a component that depends on \( P_t \), and a component that depends on \( \lambda \). In figure 4, we set \( \delta = 3.9 \times 10^{-4} \) and plot the absolute bias of the R-estimator as the solid line and its standard deviation as the dashed line. Both plots are functions of \( P_t \). This figure makes clear that, except for a sliver of values close to either of the extreme possible values of \( P_t \), the bias of \( R_t \) is swamped by its standard deviation.13

This result holds because, most of the time, the assumption that \( P_t \) is a random walk is a good approximation of the truth. But when \( P_t \) gets close to the boundaries, an observer will begin to observe more mean-reverting values than she would consider to be statistically plausible. This anomalous

13 We have also constructed this figure for values of \( \alpha = 0.5 \) and \( \alpha = 2 \). The results are qualitatively identical to those we report in fig. 4, reflecting the fact that our results are not sensitive to the value of \( \alpha \) for a large range of values that includes our chosen parameterization of \( \alpha = 1 \).
behavior occurs when \( P_0 < \Delta \) or \( P_0 > 1 - \Delta \), where \( \Delta \) represents a thin sliver of width \( \frac{\delta^2}{2\lambda^3} = O(\lambda) \). Apart from these rare situations, an observer, using the R-estimator, would not be able to distinguish the small bias in her estimate from measurement noise.\(^{14}\)

In conclusion, using a simple constant-gain learning estimator leads to a negligible bias that is difficult or impossible to detect most of the time. Of course, some smart agents could be aware that the death probability is non-zero and account for it in their update rule. However, this would not necessarily make them more successful in the securities market that we set up in the next section.\(^{15}\)

V. Heterogeneous Beliefs in a Market Economy

We have built a model to describe the evolution of public opinion. But what happens if people trade with other people with different beliefs? To answer that question, we construct an endowment economy where each person is endowed with \( \epsilon \) units of a nonstorable commodity in every period in which she is alive. We further assume that people trade a complete set of Arrow securities, indexed to the exogenous state, which we represent by \( \sigma \). We use the adjective “exogenous” to distinguish the vector \( \sigma \) from a vector of endogenous states that we introduce in section V.C.

We locate our agents in a market economy and allow them to make trades on all publicly observable events. These events include not only the binary signal that we refer to as public opinion but also the realization of who lives and who dies in every period. This complication introduces \( 2^N \) new markets since every agent must, in a complete markets economy, trade life insurance contracts contingent on the survival of everyone alive.

Much of section V involves the introduction of notation to deal with these additional life-insurance markets. Our main results refer to the large \( N \) limit, and in this case, there is no aggregate risk from the mortality of individual agents. Although this means that the large \( N \) results are much cleaner, we need the finite \( N \) machinery to compute the difference between true probabilities and market-implied, wealth-weighted probabilities. It is this distinction (which does not disappear in the large \( N \) limit) that drives our main results.

A. The Definition of the Exogenous State

The exogenous state has three elements. The first element, \( s \in S \equiv \{0, 1\} \), is the realization of a public signal. The second element, \( x \in X \equiv \{0, 1\}^N \),

---

\(^{14}\) In principle, this bias could be reduced by choosing a slightly larger value of \( \lambda \)—i.e., a slightly faster rule. But this increases the mean-square error of the estimator. The trade-off between the two would again lead to a negligible improvement of the bias of order \( O(\lambda) \).

\(^{15}\) See the detailed discussion of this important point in sec. VIII.B.
is a vector that differentiates newborns from survivors. The third element, \( z \in \mathbb{Z} \equiv [0, 1]^N \), encodes the conditional probabilities of newborns. Putting these pieces together, we have that \( \sigma \equiv \{s, x, z\} \in \Sigma = S \times X \times Z \). We use a prime to denote the state in period \( t + 1 \).

At each date, people trade a complete set of Arrow securities that depend not only on \( s' \) but also on the realizations of \( x' \), which encodes who lives and who dies. There are \( 2^N \) possible realizations of \( x' \) where the \( i \)th element of \( x' \) equals \( 1 \) if person \( i \) survives and \( 0 \) if she dies. The \( \sigma' = (s', x') \) security costs \( Q(\sigma' | \sigma) \) commodities at date \( t \) and pays one commodity at date \( t + 1 \) if and only if state \( \sigma' \) occurs. We assume that everybody has different beliefs, represented by \( P_i(\sigma') \), and that the state at period \( t + 1 \) is \( \sigma' = (s', x') \).

This completes our definition of the exogenous state. In the next subsection, we define the objectives and constraints of individual agents and derive a set of rules that represents their behavior in an exchange economy.

B. A Model of Rational Choice

We assume that agents maximize the discounted expected utility of the logarithm of consumption. This assumption implies that our agents choose to spend a fixed fraction of wealth in each period on the consumption good. The novel aspect of our approach is the decision rule we derive, which shows how agents allocate their wealth to the two Arrow securities. This decision rule depends on their subjective beliefs, which evolve in the manner described in section III.C.

First, we break wealth into two components: human wealth and financial wealth. The human wealth of person \( i \) is defined by the recursion

\[
H_i(\sigma) = \epsilon + \sum_{\sigma'} Q(\sigma' | \sigma) x'_i H_i(\sigma').
\]  

(13)

Next, we define the financial wealth of person \( i \), \( a_i(\sigma) \), to be the value of Arrow securities brought into period \( t \). The total wealth of person \( i \) is the sum of human wealth and financial wealth

\[
W_i(\sigma) = H_i(\sigma) + a_i(\sigma).
\]  

(14)

Each period, the agent faces the following budget equation:

\[
\sum_{\sigma'} x'_i(\sigma') Q(\sigma' | \sigma) a_i'(\sigma') + c_i(\sigma) = a_i(\sigma) + \epsilon.
\]  

(15)

The right-hand side of equation (15) represents a person’s available resources at date \( t \). The left-hand side represents the ways those resource can be allocated—to consumption or to the accumulation of a bundle.

\[16\] We generate this vector for all \( i \), including survivors from the previous period. Notice, however, that \( z \) enters the model only when multiplied by \( 1 - x \), which is zero for survivors.
of Arrow securities that will be available for consumption or saving in the subsequent period.

We model the consumption and asset allocations of each person as the unique solution to the following maximization problem.

**Problem 1.**

\[
V_i[W_i(\sigma)] = \max_{W(\sigma)} \left[ \log c_i(\sigma) + \beta \sum_{\sigma'} \mathbb{P}_i(\sigma') x_i(\sigma') V_i[W_i'(\sigma')] \right], \tag{16}
\]

such that

\[
\mathbb{P}_i(\sigma') = x_i[(1 - \lambda)\mathbb{P}_i(\sigma) + \lambda s] + (1 - x_i')z_i', \tag{17}
\]

and

\[
\sum_{\sigma} x_i(\sigma') Q(\sigma'|\sigma) W_i(\sigma') + c_i(\sigma) \leq W_i(\sigma). \tag{18}
\]

In section III.C, we derived an expression for the evolution of person \(i\)'s beliefs. Equation (17) reproduces that equation using the definition of \(\rho\) and replacing time subscripts with prime notation.

Equation (18) is derived by combining equations (13) and (15) with the assumption that agents must remain solvent. The quantity \(V_i[W_i(\sigma)]\) represents the maximum attainable utility given wealth \(W_i(\sigma)\), \(c_i(\sigma)\) represents date \(t\) consumption, and \(\beta\) represents the common discount rate. Following common usage, we refer to the consumption decision that solves problem 1 as the *policy function* and to the maximum attainable utility as a function of wealth as the *value function*.

**Proposition 1.** The policy function and the value function for problem 1 are given by equations (19) and (20):

\[
c_i(\sigma) = [1 - \beta(1 - \delta)] W_i(\sigma), \tag{19}
\]

\[
V_i[W_i(\sigma)] = \frac{1}{1 - \beta(1 - \delta)} \log[W_i(\sigma)] + B, \tag{20}
\]

where \(B\) is a constant that can be computed but its value is irrelevant for our purposes. The wealth of the person with label \(i\) evolves according to

\[
W_i(\sigma') = x_i \left[ \frac{\beta \mathbb{P}_i(\sigma')}{Q(\sigma'|\sigma)} W_i(\sigma) \right] + (1 - x_i') H_i(\sigma'), \tag{21}
\]

where \(H_i(\sigma)\) is defined by the recursion equation (13).

The first term on the right-hand side of equation (21) is the wealth evolution equation for person \(i\) if she survives into period \(t + 1\). The second term on the right-hand side of the equation resets person \(i\)'s wealth to \(H_i(\sigma')\) if she dies and is replaced by a newborn. For a proof of proposition 1, see appendix C.
C. Definition of Equilibrium

We have constructed a theory of individual choice. According to this theory, people’s decisions are a function of the exogenous state and the stochastic process for prices. In this section, we construct an equilibrium theory where prices are determined by setting the excess demands for goods and Arrow securities in every period to zero. First, we define a new object: the endogenous state.

The endogenous state has two elements. The first element, \( P \in \mathbb{P} \equiv [0, 1]^N \), is a vector of subjective conditional probabilities with generic element \( P_i \). The second element, \( W \in \mathbb{W} \equiv \mathbb{R}^N \), is a vector of wealth positions with generic element \( W_i \). Putting these pieces together, the endogenous state is represented by \( y \equiv \{P, W\} \in \mathbb{Y} \equiv \mathbb{P} \times \mathbb{W} \).

Next, we derive a function \( G(\cdot) \) to explain how the endogenous state evolves through time. Our approach is a relatively standard application of recursive equilibrium theory (Stokey, Lucas, and Prescott 1989). Our innovation, over conventional dynamic stochastic general equilibrium models, is to provide a self-referential theory of learning in which the economy does not converge to a rational expectations equilibrium.

We begin with a definition of recursive equilibrium.

**Definition 6 (Recursive equilibrium).** A recursive equilibrium is a price function \( Q: \Sigma \rightarrow \mathbb{Q} \equiv [0, 1]^{2N} \) and a state evolution function \( G: \mathbb{Y} \times \Sigma \times \mathbb{Q} \rightarrow \mathbb{Y} \) with the following properties:

1) The state evolution function, \( G \), is given by equations (17) and (21). This function determines the evolution of the vector of beliefs, \( P \), and the vector of wealth positions, \( W \).
2) When the Arrow security prices are given by \( Q(\sigma'|\sigma) \) and when \( y' = G(y; \cdot) \), the implied consumption plan solves problem 1.
3) The goods market clears for all \( \sigma' \) where \( c_i(\sigma') \) solves problem 1:

\[
\sum_{i=1}^{N} c_i(\sigma') = N\epsilon. \quad (22)
\]

4) The Arrow securities markets clear for all \( \sigma' \) where \( a_i(\sigma') = W_i(\sigma') - H_i(\sigma') \):

\[
\sum_{i=1}^{N} a_i(\sigma') = 0. \quad (23)
\]

In proposition 2, we show that, in equilibrium, human wealth is a number that does not depend on the state, and we derive an expression for the equilibrium price function \( Q(\sigma'|\sigma) \).

**Proposition 2.** In a recursive equilibrium:

1) Individual human wealth \( H_i \) is independent of \( \sigma \) and is the same for all agents. It is given by the expression
The price of an Arrow security is given by

\[ Q(\sigma|\sigma) = \beta \sum_{t=1}^{N} P_i(\sigma') x_i W_i(\sigma), \quad (25) \]

where \( N(\sigma') = \sum x' \) gives the number of surviving agents at time \( t + 1 \) and \( N(\sigma')H \) represents aggregate human wealth.

For a proof of proposition 2, see appendix D. In the next section, we show how the pricing function, \( Q(\sigma|\sigma) \), depends on the assumptions about the information structure and the number of agents.

VI. Equilibrium Behavior under Two Different Assumptions

Next, we study the evolution of asset prices and the wealth distribution under two different assumptions. First, in section VI.A we assume that \( P_i(\sigma') = P(\sigma') \) for all \( i \). We call this the \textit{common knowledge economy}, and we refer to the outcome of this version of our model as a rational expectations equilibrium. In Section VI.B, we allow beliefs to differ, and we ask and answer the question, Do markets reveal enough information for the economy to converge to a rational expectations equilibrium? We call this the \textit{heterogeneous beliefs economy}.

A. The Common Knowledge Economy

When beliefs about the probability of \( s' \) are common and \( s' \) and \( x' \) are independent, we can write equation (25) for \( Q(\sigma|\sigma) \) as follows:

\[ Q(\sigma|\sigma) = \beta P(\sigma') \theta(x'), \quad (26) \]

where, \( \theta(x') \) represents the commonly held probability for the vector of survival outcomes \( x' \) and \( P(\sigma') \) represents the commonly held probability for the next state being \( s' = 1 \) or \( s' = 0 \). Furthermore, we have used the equality \( \sum_{i=1}^{N} x_i' = N(\sigma') \) to define the variable \( \theta(x') \) as follows:

\[ \theta(x') = 1 + \sum_{i=1}^{N} a_i(\sigma) x_i'. \]

The term \( \theta(x') \) corrects Arrow security prices for mortality risk, and we need to keep track of this term in our simulations to ensure that asset markets clear. This term disappears in the large \( N \) limit because each
cohort is perfectly insured. As $N \to \infty$, $\theta(x') \to 1$ and we obtain the limiting expression\(^\text{17}\)

$$Q(\alpha'|\sigma) = \beta P(s') p(x').$$

Consider next the following expression for the evolution of wealth over time that follows from equations (21) and (26) and the fact that $H$ is state independent and common to all individuals:

$$W_i(\alpha') = x\left[\frac{W_i(\sigma)}{\theta(x')}\right] + (1 - x') H. \quad (28)$$

In the large $N$ limit, there is no aggregate mortality risk, and in this case we obtain the following expression for $W_i(s)$:

$$W_i(s) = x_i W_i(s) + (1 - x_i) H. \quad (29)$$

Equation (29) implies that in the large $N$ economy the wealth of the person with index $i$, contingent on her survival, is time invariant.

In a finite population, the variable $\theta(x')$ plays a nontrivial role. Suppose, for example, that in period 1 there are two people. One person has positive financial assets equal to $a$ and the other has negative financial assets equal to $-a$. In that economy, the rich person consumes more than the poor person for as long as they are both alive. But if one person dies and is replaced by a new person with wealth $H$, all debts are canceled and the economy enters an absorbing state with an egalitarian wealth distribution. The wealth reallocation that occurs as a consequence of mortality risk is encoded into the random variable $\theta(x')$.

B. The Heterogeneous Belief Economy

Next, we turn to the case where people have different beliefs. In this case, $P_i(\alpha')$ can no longer be factored out of the summation in equation (25), and instead of equation (26) we obtain the following expression for the price of an Arrow security:

$$Q(\alpha'|\sigma) = \beta p(x') \left(\frac{\sum_{i=1}^{N} P_i(\alpha') W_i(\sigma) x_i}{N(\sigma) H}\right) \equiv \beta P_{\text{imp}}(\alpha') p(x'), \quad (30)$$

where $P_{\text{imp}}(\alpha')$ is defined as the probability of state $\alpha'$ that would be inferred from market prices if market participants believed that they were living in a common knowledge economy. We henceforth refer to $P_{\text{imp}}(\alpha')$ as the implied probability.

\(^{17}\) Notice that $\text{plim}_{N \to \infty} N^{-1} \sum a_i(\sigma) x_i = 0$, using market clearing and assuming that $\text{plim}_{N \to \infty} N^{-2} \sum a_i = 0$, which turns out to be true provided $\delta$ remains fixed as $N \to \infty$. Hence, $\text{plim}_{N \to \infty} \theta(x') = 1$. 
Since who dies and who survives is independent from both wealth and beliefs one has, in the large $N$ limit,$^{18}$

$$
\lim_{N \to \infty} \left( \frac{\sum_{i=1}^{N} \mathbb{P}_i(s') W_i(s) x_i}{N(s') H} \right) = \lim_{N \to \infty} \left( \frac{\sum_{i=1}^{N} \mathbb{P}_i(s) W_i(s)}{NH} \right),
$$

where we distinguish $N$—which refers to the number of people in state $\sigma$ at date $t$—from $N(\sigma')$—which is the number of survivors in state $\sigma'$ at date $t + 1$.

In the large $N$ limit, $\mathbb{P}_{\text{imp}}(s')$ depends on the future realization of $s$ but not on the mortality state. It is given by the expression

$$
\mathbb{P}_{\text{imp}}(s') = \frac{\sum_{i=1}^{N} \mathbb{P}_i(s') W_i(s)}{NH}.
$$

The quantity $\mathbb{P}_{\text{imp}}(s')$ is the wealth-weighted average probability, and it differs from the true probability, $\mathbb{P}(s')$, which is the unweighted average of individual subjective probabilities.

Using the definition of $\mathbb{P}_{\text{imp}}(s')$, the analogue of equation (29) for the heterogeneous belief case is given by

$$
W_i'(s') = x_i' \frac{\mathbb{P}_i(s')}{\mathbb{P}_{\text{imp}}(s')} W_i(s) + (1 - x_i') H.
$$

The parameters $\mathbb{P}_i$ and $W_i$ are strongly coupled by the dynamics of individual wealth accumulation (eq. [33]), and because of this strong coupling we cannot split $\mathbb{P}_{\text{imp}}(s')$ into the product of $\mathbb{P}(s')$ and $\lim_{N \to \infty} \sum_i (W_i(\sigma)/N)$, even asymptotically, as we did in the common knowledge economy. This failure of independence generates fat tails in the wealth distribution, and it implies that the implied probability, $\mathbb{P}_{\text{imp}}(s')$, and the true probability, $\mathbb{P}(s')$, can differ even in the large $N$ limit.$^{19}$

### C. Debt and Equity in the Heterogeneous Belief Economy

We have derived explicit trading rules for agents who buy and sell Arrow securities. But there is no reason to restrict ourselves to securities of this kind, and the same equilibrium we described above can be supported by any set of securities with independent payoffs that span the space of possible outcomes. In this subsection, we show that in the large $N$ limit, an

---

$^{18}$ Here we use the fact that if $\eta_i$ and $\xi_i$ are independent random variables, then

$$
\lim_{N \to \infty} N^{-i} \sum_{i=1}^{N} \eta_i \xi_i = \lim_{N \to \infty} N^{-i} \sum_{i=1}^{N} \eta_i \left( N^{-1} \sum_{i=1}^{N} \xi_i \right),
$$

and we choose $\eta_i = \frac{\mathbb{P}_i W_i}{\mathbb{P}_{\text{imp}}(s')}$ and $\xi_i = x_i'$.

$^{19}$ We explore the implications for the wealth distribution in sec. VIII.
equilibrium can be supported by a security that pays one commodity in both states—we call this security debt—and a security that pays \( d \) units if \( s = \{1\} \) and zero otherwise—we call this security equity.

The assumption that \( N \) is large allows us to ignore fluctuations in the annuities markets and to concentrate on trades contingent on disagreement over the realization of \( s' \). This signal could be any mechanism for the revelation of public opinion. What is important for our interpretation is that firms choose to pay dividends only if \( s' = 1 \).

**Proposition 3.** For the large \( N \) economy, equilibrium can be supported by trades in debt and equity. Debt is a security that costs \( Q_t \) units of commodities at date \( t \) and pays one commodity at date \( t + 1 \) in both states. Equity is a security that costs \( p_{E,t} \) units of commodities at date \( t \) and pays \( d \) in state \( s'_{t+1} = \{1\} \) and zero in state \( s'_{t+1} = \{0\} \), where

\[
p_{E,t} = \frac{d\beta}{2} \left[ \frac{2\overline{P}_{imp,t} - 1}{1 - \beta(1 - \delta)} + \frac{1}{1 - \beta} \right], \tag{34}
\]

\[
Q_t = \beta. \tag{35}
\]

For a proof of proposition 3, see appendix E.

**D. What Drives Our Results?**

In the work of Beker and Espino (2011), agents all eventually agree. We make two deviations from the Beker-Espino environment. First, we assume that agents die and are replaced by new agents. Second, we assume that the process that agents learn about is both self-referential and nonfundamental.

The assumption that people die is essential to our result that the economy fails to converge to a rational expectations equilibrium. The assumption that the stochastic process for \( s_t \) is self-referential is secondary but important. The defining property of this process is that of quasi nonergodicity.

If agents never die, there is no advantage to the use of constant-gain learning. If, as would be optimal in the infinite-lives environment, everyone were to use least-squares learning, agents would asymptotically converge to the truth and they would all agree, as in Beker and Espino, although convergence would be slow and the final belief would be history dependent.

One could assume that the process for \( s_t \) is exogenous but quasi-nonergodic. It might, for example, be generated by the equation

\[
P_{t+1} = (1 - \delta_t)[(1 - \lambda_t)P_t + \lambda_t s_t] + \frac{\delta_t}{2}, \tag{36}
\]
where \( \lambda_1 \) and \( \delta_1 \) are not necessarily equal to \( \lambda \) and \( \delta \). If agents were to learn about this process using constant-gain learning with gain parameter \( \lambda = \lambda_1 \), and if the parameter \( \delta_1 \) were by chance equal to the death probability \( \delta \), the model in which the stochastic process for \( P \) is exogenous would be indistinguishable from the model we have presented here.

It is plausible that a three-parameter model in which \( \delta \) and \( \delta_1 \) were different but where everybody had correctly learned the parameter \( \lambda_1 \) would display behavior similar to the model we have described here. We have not explored that variant of our main theme. The importance of our interpretation of equation (36), as a self-referential process driven by social interactions, is that it provides a microfounded theory for the assumption that agents must learn about a quasi-nonergodic process. In the absence of our interpretation of this equation, one would need to find some alternative economic explanation for what we think is an attractive feature of our work—the idea that the value of stocks today depends on guessing what others think that stocks will be worth in the future. In our model, the interaction of market and nonmarket forces generates a microfounded model of why people disagree that has implications for both the wealth distribution and excess volatility in the asset markets. Both of those implications are pursued further below.

VII. Results from a Numerical Simulation

In this section, we illustrate the implications of our results by reporting some statistics for simulated data in a calibrated version of our model. We simulated an economy with one million agents for 300 years, and we chose the period length to be 1 week. We normalized the weekly endowment to 1, and we chose the annual discount rate to be 0.97, which in an endowment economy corresponds to an equilibrium annual real interest rate of 3%. These are relatively uncontroversial choices.

In figure 5, we graph some data from a single simulation of this calibrated version of our model when agents have a life expectancy of 50 years and for a memory time of 1 year. Importantly, for our calibration, \( \alpha = 1 \) and the invariant measure is uniform. We checked that our reported results, especially those concerning the wealth distribution, are robust to values of \( \alpha \in [0.5, 2] \) (see, e.g., fig. 3 for the level of disagreement generated by the model). We suspect that our main results are relatively insensitive to the choice of \( \alpha \) even outside of this range.\(^{20}\)

The top left panel of figure 5 graphs the invariant measure \( P_{\alpha}(P) \). The other three panels present some key data for a single simulation.

\(^{20}\) Only by taking \( \alpha \) to 10 were we able to break one of our key results—that the wealth distribution has fat tails.
of 300 years of weekly data. The top right panel shows the percentage difference between \( P(s^t) \) and \( P_{imp}(s^t) \). This difference is a measure of how wrong the market can be as a measure of the true probability. For much of the sample this difference is less than 1%, but there are times when this deviation exceeds \( \geq 15\% \). Such large discrepancies are quite remarkable in view of the size of the market (one million participants) and are the consequence of the emergent wealth inequalities in our model.

The bottom right panel shows the time-series behavior of the price-earnings ratio using the formula derived in section VI.C. To compute this series, we normalized the dividend payment to \( 1/52 \) to make the units comparable to an expected weekly dividend payment. This series has many characteristics in common with the price dividend ratio in US data for realized values of the S&P. It wanders randomly over a bounded interval, and sometimes it moves substantially in a short period of time. The bottom left panel shows the distribution of the log of wealth. In section VIII, we explore the properties of this distribution further and
show that it shares many characteristics in common with empirical wealth distributions in Western economies.

VIII. Exploring the Empirical Wealth Distribution

While our model is constructed in such a way that no agent is better informed than any other, some agents are—purely by chance—temporarily much more successful than others. This allows these agents to accumulate wealth through the multiplicative process described in equation (33), reproduced below:

\[ W_i'(s') = x'_i \frac{p_i(s')}{p_{\text{imp}}(s')} W_i(s) + (1 - x'_i)H. \]

Multiplicative wealth processes of this form are well known to generate important wealth inequalities.\(^{21}\)

In figure 6, we graph the Lorenz curve for the time average of 250 equally spaced samples of the wealth distribution in our simulated data.\(^{22}\) The Lorenz curve is a graphical representation of inequality that plots the cumulative percentage of wealth on the y-axis against the percentile of the population on the x-axis. One popular index of inequality is the Gini coefficient, which is equal to twice the area between the 45-degree line and the Lorenz curve.

For our numerical data, the Gini coefficient is equal to 0.7. A value of zero would represent a completely equal distribution, and a value of one would represent a distribution where one person owns everything.

Table 1 reports data from a selection of countries. This table shows that a Gini coefficient of 0.7 is well within the bounds of empirical data, which varies between a low of 0.55 for China in 2008 and a high of 0.85 for the United States in 2019.

To explore the nature of the wealth distribution further, we define \( F(W) \) to represent the cumulative distribution function (CDF) of wealth and define \( G(W) \equiv 1 - F(W) \) to represent the complementary CDF. In figure 7, we plot \( \log G(W) \) against \( \log(W) \) for values of \( \log(W) \) greater than zero. This figure reveals a power-law tail of the form \( G(W) \sim W^{-\mu} \), and a regression of \( \log(G(W)) \) on \( \log(W) \) for the linear portion of the plot provides an estimate of the tail index of \( \mu = 1.4 \). Note that \( G(W) \sim W^{-\mu} \) corresponds to a probability distribution function (PDF) \( \varphi(W) \sim W^{-1-\mu} \). A person who is neither a borrower nor a lender has zero

\(^{21}\) Examples of papers in the literature that study multiplicative wealth dynamics include Kesten (1973), Bouchaud and Mézard (2000), Gabaix (2009), Benhabib, Bisin, and Zhu (2011), Gabaix et al. (2016), and Benhabib and Bisin (2018).

\(^{22}\) For large \( T \), our sample histogram will converge to the ergodic wealth distribution. There will still be some variability in a sample of 300 years, but our experiments with different random draws suggest that this variability is not too large.
financial assets, and her net worth would be equal to the discounted present value of her labor income. For our calibration, this number, which we refer to as human wealth, is equal to 1,032 weeks of income. In the common knowledge economy, the wealth distribution would be egalitarian, the Gini coefficient would be zero, and everyone would have wealth equal to $H$. Instead, in our economy there is considerable inequality.

A person at the 50th percentile of the wealth distribution is a net borrower who has total wealth equal to 39% of human wealth. In contrast, a person at the 99th percentile in the wealth distribution has total wealth equal to 892% of human wealth, and a person at the 99.9th percentile has total wealth equal to 4,999% of human wealth. Wealth becomes highly concentrated because market prices do not reflect average beliefs. Instead, they reflect wealth-weighted beliefs. In equilibrium, wealth and

---

23 Human wealth is defined by the expression $H = 1/(1 - \beta(1 - \delta))$. For our calibration, the weekly discount rate is $0.97^{1/52}$ and the survival probability, $(1 - \delta)$, is equal to $3.9 \times 10^{-4}$. This leads to a value of $H = 1,031$ measured in weeks of income.
market prices are correlated in a way that leads to a self-reinforcing mechanism whereby a few people, by chance, get lucky and become very rich.

To explore the dependence of our results on $\alpha$, we recomputed the data reported in figure 5 for two alternative values of $\alpha$. For both simulations, we held $\delta$ constant and selected values of $\lambda$ that set $\alpha$ to 0.5 and 2. For $\alpha = 2$, we find a Gini coefficient of 0.69 and a tail slope coefficient of 1.4. This calibration has a hump-shaped invariant measure and a memory time of 72 weeks. For $\alpha = 0.5$, we find a Gini coefficient of 0.72 and,

\begin{table}
\centering
\caption{Wealth Ginis for a Selection of Countries in 2008 and 2019}
\begin{tabular}{lcc}
\hline
Country & 2008 & 2019 \\
\hline
China & .55 & .7 \\
United Kingdom & .7 & .75 \\
Italy & .7 & .77 \\
France & .73 & .7 \\
Switzerland & .74 & .87 \\
United States & .8 & .85 \\
\hline
\end{tabular}
\footnotesize{Source.—Wikipedia (https://en.wikipedia.org/wiki/List_of_sovereign_states_by_wealth_equality).
}
\end{table}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{Estimate of the tail parameter in 1,000 years of simulated monthly data.}
\end{figure}
once again, a tail slope coefficient (to one significant digit) of $\mu = 1.4$. This calibration has a U-shaped invariant measure and a memory time of 36 weeks. We infer from these robustness checks that our results are insensitive to variations in $\alpha$ for a substantial range of plausible parameters. In fact, as we explain below, $\mu$ converges to a nondegenerate value $>1$ when $\delta \to 0$.

A. The Behavior of Wealth in the Large $N$ Limit

We can learn quite a bit about the dynamics of wealth by analyzing the properties of equation (33). Using this equation, one may derive the following expression for the average return for agent $i$ between dates $t$ and $t + 1$, conditional on surviving:\footnote{Equation (37) follows since}

$$R_i = \mathbb{E} \left[ \frac{W_i'}{W_i} - 1 \right] = \frac{(P - P_{imp})(P_i - P_{imp})}{P_{imp}(1 - P_{imp})}.$$  \hspace{1cm} (37)

Several interesting conclusions can be drawn from equation (37). First, in the common knowledge economy where $P = P_{imp}$, agents cannot expect to make money on average, even temporarily.

Second, when an agent’s belief $P_i$ is larger than the market probability $P_{imp}$, her expected gain is positive if the actual probability $P$ is also greater than $P_{imp}$ and negative otherwise. In fact, provided the sign of $P_i - P_{imp}$ is the same as that of $P - P_{imp}$, the instantaneous expected gain is larger when the bet is bolder, albeit with a larger variance (see eq. [38]).

Finally, since agents are assumed to act on the assumption that their estimate of the probability is an unbiased estimate of the true probability, they also believe that their trades will be profitable on average and proportional to $(P_i - P_{imp})^2$. In other words, they expect to make a larger profit the further their belief is from the probability implied by the market price. This implies that there is no incentive for agents to align their beliefs with the observable implied probability, since this would reduce their subjective expected profit. Everybody in this economy believes that they know more than the market—indeed, a most common feature of the real world!

\footnote{Equation (37) follows since}

$$R_i = \mathbb{E} \left[ \frac{W_i'}{W_i} - 1 \right] = \left[ P \times \frac{P_i}{P_{imp}} + (1 - P) \times \frac{1 - P_i}{1 - P_{imp}} - 1 \right]$$

$$= \frac{(P - P_{imp})(P_i - P_{imp})}{P_{imp}(1 - P_{imp})}.$$
In equation (38), we derive an expression for the average of the square of the relative change of wealth for surviving agents:

$$E \left[ \left( \frac{W'_i}{W_i} - 1 \right)^2 \right] = \frac{(P(1 - P_{\text{imp}})^2 + (1 - P)P_{\text{imp}}^2)(P_i - P_{\text{imp}})^2}{P_{\text{imp}}^2(1 - P_{\text{imp}})^2}. \quad (38)$$

One sees from this equation that “bold beliefs,” corresponding to a large difference between $P_i$ and the market probability $P_{\text{imp}}$, lead to a larger variance of gains. Equation (38) explains why our model generates large wealth inequalities. For surviving agents, the wealth dynamic is a multiplicative random process with a time- and agent-dependent variance. This multiplicative process is reset to one at a Poisson rate $\delta$—that is, when an agent dies.

Multiplicative random processes with reset have been widely studied in the literature, and it is known that such processes lead to a stationary distribution with a power-law tail with a PDF $\varphi(W)$ and a complementary CDF $G(W)$ of the form

$$\varphi(W) \sim W \to \infty W^{-\mu}, \quad G(W) \sim W \to \infty W^{-\mu}, \quad (39)$$

where the exponent $\mu$ depends on the parameters of the problem. In appendix F, we discuss how $\mu$ can be approximately computed, and we find in particular that $\mu > 1$ whenever $\delta > 0$.

In conclusion, wealth inequalities in our model arise from the multiplicative nature of wealth dynamics, which makes successful bold bets highly profitable. Unsuccessful bold bets, however, are ruinous and lead the person who makes such bets into poverty. People who agree with the market belief have a low expected subjective gain from trading. People who disagree may become either spectacularly rich or spectacularly poor.

25 Equation (38) follows from

$$E \left[ \left( \frac{W'_i}{W_i} - 1 \right)^2 \right] = \left[ \frac{P}{P_{\text{imp}}} - 1 \right]^2 + (1 - P) \left( \frac{1 - P_i}{1 - P_{\text{imp}}} - 1 \right)^2.$$

26 See the citations in n. 21.

27 Random variables with a Pareto tail can be sorted into three classes depending on the value of the tail parameter $\mu$. A Pareto-tailed distribution is well defined for all positive $\mu$, but when $0 < \mu \leq 1$, the mean and all higher moments do not exist. When $1 < \mu \leq 2$, the mean exists but the variance and higher moments do not exist, and for $\mu > 2$, the distribution has a finite mean and a finite variance. In our example, as in the data, we find a value of $\mu$ between 1 and 2, which implies that the wealth distribution has a finite first moment but all higher-order moments are not well defined.

28 The limit $\delta \to 0$ is interesting since the wealth distribution has a Pareto tail even as $\delta \to 0$. Using eq. (F5) in app. F and taking $\delta$ to zero for fixed $\lambda$, the Pareto exponent $\mu$ converges to $(1/2)(1 + \sqrt{1 + 8\lambda})$, which equals 1.25 when $\lambda = 0.14$. In the limit, two effects cancel each other out as the limit of $\delta/\sigma^2$ converges to one. Disagreements tend to disappear (see eq. [12]) and thus mispricings vanish (i.e., $(P_{\text{imp}} - P)^2 \sim \delta$), but at the same time lucky agents can benefit from these mispricings for a longer time and $T \sim \delta^{-1}$. 
B. The Kelly Criterion

In section II, we discussed the market-selection hypothesis, which is the claim that the agents who survive will be those who hold beliefs that are closest to the truth. It is equivalent to the assertion that those agents who dominate the asset markets will be those who maximize the growth of their wealth, and the Blume and Easley (2006) formulation of this hypothesis implies that all surviving agents will hold common beliefs that converge asymptotically to the rational expectation.

The investment strategy that maximizes the growth rate of wealth was studied by Kelly (1956), and it is widely referred to as the Kelly criterion. In the context of our model, the Kelly criterion amounts to maximizing the quantity

\[ E \left[ \frac{\frac{\lambda}{2 + (\alpha - 1)\lambda} \frac{\alpha(\alpha + 2)}{6(2\alpha + 1)} + O(\lambda^3)}{\frac{\lambda}{2 + (\alpha - 1)\lambda}} \right]. \]

This term is defined by equation (12),

\[ v \equiv \sqrt{\mathbb{E}[D_i]} = \left[ \frac{\lambda}{2 + (\alpha - 1)\lambda} \right] \frac{\alpha(\alpha + 2)}{6(2\alpha + 1)} + O(\lambda^3), \]

and for \( v \) much smaller than unity we have the following expansion for the log of the growth rate of agent \( i \)'s wealth:

\[ \log \left( \frac{W_i'}{W_i} \right) \approx \left( \frac{W_i'}{W_i} - 1 \right) - \frac{1}{2} \left( \frac{W_i'}{W_i} - 1 \right)^2 + \ldots. \] (40)

Taking expectations of equation (40), using equations (37) and (38), one finds the following approximation to the first order in \( v \):

\[ E \left[ \log \frac{W_i'}{W_i} \right] \approx \frac{\left[ (\mathbb{P} - \mathbb{P}_{imp})(\mathbb{P}_i - \mathbb{P}_{imp}) - (1/2)(\mathbb{P}_i - \mathbb{P}_{imp})^2 \right]}{\mathbb{P}_{imp}(1 - \mathbb{P}_{imp})} + o(v). \] (41)

Equation (40) implies that when the market-implied probability \( \mathbb{P}_{imp} \) is equal to the true probability \( \mathbb{P} \), any belief \( \mathbb{P}_i \neq \mathbb{P} \) leads to a negative growth rate for \( W_i \). Any agent \( i \) who continues to hold an inaccurate belief of this kind will be wiped out in the long run. This is the content of the market-selection hypothesis.

In our model, the entry of new agents causes the market-selection hypothesis to fail. It is the wealthy agents who determine the market price, and even though these agents would be wiped out if everyone lived forever, in the finite lived environment the smart agents die before they have had time to benefit from their more accurate subjective beliefs. The richest agents in any given period are not the smartest ones but rather those who have made bold successful bets. The market price is determined by rich people who were right in the past, whereas those people with beliefs that are closer to the current truth will only become rich in the future.
IX. Conclusion

We have constructed a theory of beliefs in which people exchange information through both market and nonmarket interactions. Nonmarket interaction generates an aggregate signal that reflects average public opinion. Market exchange through the purchase and sale of financial assets allows people to bet on their beliefs. Importantly, market prices reveal information about wealth-weighted beliefs $P_{\text{imp}}$, but it is unweighted beliefs $P$ that generate the public signal.

One is led to the question, Why do people continue to bet with each other when these bets are highly risky? The answer we propose is that everyone in our economy thinks that the market is wrong and that by betting they will be able to make money on average. They do not use the implied probability revealed by the markets to improve their estimate of $P$, since this trading strategy would be (subjectively) suboptimal. Quite remarkably, the coupled dynamics of individual wealth and beliefs leads to a fat-tailed distribution of wealth. The richest agents at a given instant in time are not necessarily the smartest ones but rather those who have made bold, successful bets in the past. Since those agents tend to dominate the market, the implied probability $P_{\text{imp}}$ cannot be used to learn the true probability $P$. 29

Why are there no Warren Buffets who invest for the very long run by guessing that the probability of a successful outcome will be equal to the mean $P = 1/2$ of the invariant distribution? Our answer is that this would be the case only if we lived forever and could afford to be strict Bayesian learners, but the world that we live in is far better approximated by observing recent realizations than by relying on an unconditional long-run ergodic measure. We believe that our quasi-nonergodic model aptly illustrates what Keynes had in mind when he wrote that “in the long run we are all dead.”

Appendix A

The Continuous Time Limit

A1. Derivation of Equation (8)

Introducing a change of variable $u$ such that $P = (1/2) + u$, one can convert equation (7) into

$$
(1 - \delta)^2 P_{t+1}(u) = \frac{1 - \delta - \hat{\lambda}}{2} \left[ P_t \left( \frac{u - \hat{\lambda}/2}{1 - \delta} \right) + P_t \left( \frac{u + \hat{\lambda}/2}{1 - \delta} \right) \right] + u \left[ P_t \left( \frac{u - \hat{\lambda}/2}{1 - \delta} \right) - P_t \left( \frac{u + \hat{\lambda}/2}{1 - \delta} \right) \right] \tag{A1}
$$

29 Introducing a wealth tax (or an inheritance tax) tends to reduce inequalities and in our model helps prices reveal private beliefs; see app. E.
where $\lambda = \lambda(1 - \delta)$ and $\delta = \delta + \dot{\lambda}$. Note that this equation preserves the symmetry $\mathcal{P}(u) = \mathcal{P}(u)$ (i.e., $\mathcal{P} \to 1 - \mathcal{P}$), valid for all times.

In the following analysis, we assume long memory ($\lambda \ll 1$) and long lifetimes ($\delta \ll 1$) by focusing on the limit where $\lambda, \delta \to 0$ with $\delta = \alpha \lambda^2$ for fixed $\alpha = O(1)$.

Expanding equation (A1) to order $\lambda^3$ yields

$$
\Delta_t = \delta [uQ'] + \frac{\lambda^2}{2} \left[ \left( \frac{1}{4} - u^2 \right) Q'' - 2\lambda \delta [u^2 Q]'' \right] - \frac{\lambda^3}{2} \left[ \left( \frac{u}{12} - \frac{u^3}{3} \right) Q'' - u^2 Q' + \frac{5}{12} Q' \right] + O(\lambda^4),
$$

(A2)

where primes denote derivatives with respect to $u$, $\mathcal{P}(u) = (1 - \delta)Q(u) - Q_1(u)$, and $\Delta_t = Q_{t+1}(u) - Q_t(u)$. Note that the last two terms of equation (A2) are of order $\lambda^3$, and we will neglect them in the following approximation.

In the small $\delta, \lambda$ limit, equation (A2) converges to the following continuous time Fokker-Planck equation for $\mathcal{P}$:

$$
\frac{1}{\lambda^2} \frac{\partial \mathcal{P}}{\partial t} = \alpha [u\mathcal{P}'] + \frac{1}{2} \left[ \left( \frac{1}{4} - u^2 \right) \mathcal{P}'' \right].
$$

(A3)

This equation coincides with the continuous time description of Kirman’s (1993) ant-recruitment model, for which a great deal is known (for recent results and references, see Moran et al. 2020a).

In particular, the stationary distribution $\mathcal{P}_*$ is described by the following second-order differential equation:

$$
\alpha [u\mathcal{P}'] + \frac{1}{2} \left[ \left( \frac{1}{4} - u^2 \right) \mathcal{P}'' \right] = 0.
$$

(A4)

The solution to this equation is given by

$$
\mathcal{P}_*(u) = \Gamma(2\alpha) \left( \frac{1}{4} - u^2 \right)^{-\alpha-1},
$$

(A5)

which corresponds to equation (8) in the main text.

A2. Generalization: Nonlinear Feedback

The Fokker-Planck equation (A3) corresponds to the following stochastic differential equation:

$$
d\mathcal{P} = -\delta(\mathcal{P} - \frac{1}{2}) \, dt + \lambda \sqrt{\mathcal{P}(1 - \mathcal{P})} \, dW_t, \tag{A6}
$$

where $W_t$ represents a Wiener noise. More generally, one can consider a sigmoidal feedback term $\mathcal{F}(\mathcal{P})$ mapping the average belief onto the true probability,

$$
\mathcal{P}_{t+1} = \mathcal{F}(\mathcal{P}_t), \tag{A7}
$$

with $\mathcal{F}(\mathcal{P}) = \mathcal{P}$ throughout the main part of the paper and in the above section. In this case, one obtains as a stochastic differential equation
\[ d\mathbb{P} = -\partial_x \mathcal{V}(\mathbb{P}) \, dt + \lambda \sqrt{\mathbb{P}(1 - \mathbb{P})} \, dW, \]  
where we have introduced a “potential function” \( \mathcal{V}(x) \) such that

\[ \partial_x \mathcal{V}(x) = \delta(x - \frac{1}{2}) + \lambda(x - \mathcal{F}(x)). \]  

For definiteness, consider a sigmoidal function \( \mathcal{F}(x) \) defined as

\[ \mathcal{F}(x) = \frac{1}{2} \left( 1 + \tan \left( x - \frac{1}{2} \right) \right). \]  

The corresponding potential \( \mathcal{V}(x) \) is then given by

\[ \mathcal{V}(x) = \frac{1}{2} \left( \delta + \lambda \right) u^2 - \frac{\lambda}{2\xi} \log \cosh \xi u; \quad u = x - \frac{1}{2}. \]  

For small \( \xi \), \( \mathcal{V}(x) \) has a unique minimum corresponding to \( x = 1/2 \). For \( \xi > \xi_c = 2(1 + \delta/\lambda) \), \( \mathcal{V}(x) \) has two minima \( x^* < 1/2 \) and \( 1 - x^* > 1/2 \) and one maximum at \( x = 1/2 \).

In the absence of the Wiener noise term, the dynamics of \( x \) would simply be “rolling down” the potential slopes, selecting one of the minima of \( \mathcal{V}(x) \) (corresponding to the stable solutions of \( \mathcal{F}(x) = x \)).

In the presence of noise and for \( \xi > \xi_c \), the dynamics become a succession of long phases where \( \mathbb{P}_t \) remains close to either \( x^* \) or \( 1 - x^* \), separated by rapid switches from one minimum to the other. The time \( \tau_x \) needed to “climb up the hill” separating the two minima, however, can be very long when \( \lambda \to 0 \).

In fact, this time can be rather accurately computed by changing variables from \( \mathbb{P} \) to \( \phi \) where \( \mathbb{P} = (1 + \sin \phi)/2 \), which allows one to get rid of the factor \( \sqrt{\mathbb{P}(1 - \mathbb{P})} \) in front of the Wiener noise (see, e.g., Moran et al. 2020a). Using a standard approach (e.g., Hänggi, Talkner, and Borkovec 1990), one can then show that

\[ \tau_x \sim \lambda^{-1} e^{\Gamma/\lambda}, \quad (\lambda \to 0), \]

where \( \Gamma \) can be fully computed (at least numerically) for any potential \( \mathcal{V}(x) \). The exponential dependence of \( \tau_x \) in \( \lambda \) implies that (a) there is a strong separation of timescales in such models and (b) the precise value of \( \tau_x \) is unknowable in practice, as it is highly sensitive on the detailed value of the parameters of the model. Hence, agents cannot be assumed to use the same learning rule. Since these switches can be interpreted as “crashes,” in our simple model the probability of such crashes is unknowable, much as the trajectories of a chaotic system are unknowable (for a related discussion, see Morelli et al. 2020).

### Appendix B

**Dispersion of Opinions**

Taking the expectation of equation (11) over the realization of \( s_n \) one gets

\[ E[\mathcal{D}_{u+1}] = (1 - \delta) \left[ (1 - \lambda) E[\mathcal{D}_u] + \delta \left( \mathbb{P}_t - \frac{1}{2} \right) \right] + \delta(1 - \delta) \left[ \frac{1}{2} - \mathbb{P}_t \right], \]  

where
which shows that $E[D_{i,t}/C_{138}]$ tends to zero when $t \to \infty$. 

Now let us square equation (11) before taking the average over $s$. One now gets

$$E[D_{i,t}^2/C_{138}] = (1 - \delta)(1 - \lambda)^2 E[D_{i,t}^2] + \delta^2 \left[ \left( \frac{P_{t}}{2} - \frac{1}{2} \right)^2 \right] \tag{B3}$$

Now taking further the expectation over the distribution $P$ of the probability $P$ and using

$$E_P[P^2] = \frac{1 + \alpha}{2(1 + 2\alpha)}, \quad \alpha = \frac{\delta}{\lambda^2}, \tag{B4}$$

we obtain (in the limit $\delta, \lambda \to 0$, with $\alpha$ fixed)

$$E^*[D_{i,t}^2] = (1 - \delta)(1 - \lambda)^2 E^*[D_{i,t}^2] + \frac{\delta^2}{6(1 + 2\alpha)} + O(\delta^2), \tag{B5}$$

where $E^*$ represents an expectation over both $s$ and $P$.

Hence, in the stationary state where $E^*[D_{i,t}^2]$ is independent of $t$, one finds

$$E^*[D_i^2] \approx \frac{\delta}{6(1 - (1 - \delta)(1 - \lambda)^2)} \frac{2 + \alpha}{1 + 2\alpha} \tag{B6}$$

and hence the result equation (12).

Appendix C

Solving the Individual Optimization Problem

We conjecture that the value function has the form

$$A \log W_i(\sigma) + B, \tag{C1}$$

for unknown constants $A$ and $B$. Substituting from equation (18) for $c_i(\sigma)$ in equation (16) and taking derivatives with respect to $W_i(\sigma')$ leads to the following Euler equation:

$$\frac{x_i(\sigma')Q(\sigma'|\sigma)}{c_i(\sigma)} = \frac{A_\beta P_i(\sigma')x_i(\sigma')}{W_i(\sigma')}, \tag{C2}$$

which holds state by state. Using the envelope condition $A c_i(\sigma) = W_i(\sigma)$, which holds at every date and in every state, we can write equation (C2) as

$$x_i(\sigma')Q(\sigma'|\sigma)W_i(\sigma') = \beta P_i(\sigma')x_i(\sigma')W_i(\sigma). \tag{C3}$$
Combining the budget equation—equation (18), which holds with equality—with equation (C3) leads to the expression

$$
\sum_{\sigma'} \beta_{\text{P},(\sigma')} x_{i}(\sigma') W_{i}(\sigma) + \frac{W_{i}(\sigma)}{A} = W_{i}(\sigma).
$$

(C4)

Because \(x'\) is independent of \(x'\),

$$
\sum_{\sigma'} P_{i}(\sigma') x_{i}(\sigma') = \sum_{\sigma'} P_{i}(\sigma') x_{i}(\sigma') = 1 - \delta,
$$

(C5)

and thus by canceling terms and rearranging equation (C3) we arrive at the following value for \(A\):

$$
A = \frac{1}{1 - \beta(1 - \delta)}.
$$

(C6)

The constant \(B\) does not affect the solution and can be solved for by plugging the value of \(A\) into the expression

$$
A \log(W_{i}) + B = \log\left(\frac{W_{i}}{A}\right) + \beta(1 - \delta)[A \log(W_{i}) + B]
$$

and equating the coefficients on the constant terms.

It follows from equation (C3) that for all \(x_{i}(\sigma') = 1\) (i.e., those who survive),

$$
W_{i}(\sigma') = \beta \frac{P_{i}(\sigma')}{Q(\sigma' | \sigma)} W_{i}(\sigma).
$$

(C8)

This establishes the first term on the right-hand side of equation (21). If \(x_{i}(\sigma') = 0\), the newborn with index \(i\) has wealth \(H\) by assumption. This establishes the second term on the right-hand side of equation (21).

**Appendix D**

**Establishing the Properties of Equilibrium**

From equation (13), we have the following equation for human wealth:

$$
H_{i}(\sigma) = e + \sum_{\sigma'} Q(\sigma' | \sigma) x_{i}' H_{i}(\sigma').
$$

(D1)

From the definition of total wealth, we have that \(W_{i}(\sigma') - H_{i}(\sigma') = a_{i}(\sigma')\), where \(a_{i}(\sigma')\) represents the amount of Arrow security held by agent \(i\) that pays one unit if \(\sigma'\) is realized. Assuming market clearing means that for each \(\sigma'\),

$$
\sum_{i=1}^{N} a_{i}(\sigma') = 0, \quad \forall \sigma',
$$

(D2)

and hence, using equation (C8), we have that

$$
\sum_{i=1}^{N} W_{i}(\sigma') = N(\sigma') H_{i}(\sigma') = \beta \frac{1}{Q(\sigma' | \sigma)} \sum_{i=1}^{N} P_{i}(\sigma') W_{i}(\sigma).
$$

(D3)
Rearranging this equation and factoring $P_i(s')$ as $p(x')P_i(s')$ gives the following expression for the pricing kernel:

$$Q(s'|\sigma) = \beta p(x') \frac{\sum_{i=1}^{N} P_i(s') x'_i W_i(\sigma)}{N(\sigma') H(\sigma')},$$

(D4)

which establishes equation (25) from proposition 2.

Replacing equation (D4) in equation (D1) and reversing the order of summation gives

$$H(\sigma) = \varepsilon + \sum_{i=1}^{N} W_i(\sigma) \sum_{\sigma'} \left\{ \frac{\beta}{N(\sigma') H(\sigma')} P_i(s') p(x') x' H(\sigma') \right\}. \quad \text{(D5)}$$

Next, cancel $H(\sigma')$ from top and bottom:

$$H(\sigma) = \varepsilon + \beta \sum_{i=1}^{N} W_i(\sigma) \sum_{x'} \left\{ \frac{\beta p(x') x'}{N(x')} \right\} \sum_{\sigma'} P_i(s'). \quad \text{(D6)}$$

Using the facts that $P_i(s') = 1$, $\Sigma_{x'} \{(p(x') x')/(N(x'))\} = 1 - \delta$ and $\Sigma_{x,s} W_i(\sigma) = H(\sigma)$, this expression simplifies to

$$H(\sigma) = \varepsilon + \beta H(\sigma)(1 - \delta), \quad \text{(D7)}$$

or

$$H(\sigma) = \frac{\varepsilon}{1 - \beta(1 - \delta)}, \quad \text{(D8)}$$

which established equation (24) in proposition 2.

Appendix E

Proof of Proposition 3

We now seek an expression for the price of a security that pays a dividend $d$ every time $s_i = \{1\}$. This is given by the expression

$$p_{t\sigma}(s) = \sum_{\sigma'} Q(s'|\sigma)[d \delta_{\varepsilon,1} + p_{t\sigma'}(s')], \quad \text{(E1)}$$

where $s' = (x', s')$ represents tomorrow’s state, with $x'$ encoding who survives and who dies, and $\delta_{\varepsilon,1}$ is the index function, which equals one when $s' = 1$ and zero otherwise. Iterating equation (E1) gives the following infinite series:

$$p_{t\sigma}(s) = d \sum_{\sigma'} Q(s'|\sigma) \delta_{\varepsilon,1} + d \sum_{\sigma''} Q(s'|\sigma) Q(s''|\sigma') \delta_{\varepsilon,1} + \cdots, \quad \text{(E2)}$$

where, from equation (30),

$$Q(s'|\sigma) = \beta p(x') \left( \sum_{i=1}^{N} P_i(s') W_i(\sigma) x'_i \right) \frac{1}{N(\sigma') H(\sigma')} \quad \text{(E3)}$$
As we have shown in the main text, for large $N$ this object converges to

$$Q(\sigma'|\sigma) = \beta p(x') p_{\text{imp}}(s'),$$

where

$$p_{\text{imp}}(s') \equiv \frac{1}{NH} \sum_{i=1}^{N} p_i(s') W_i(s).$$

Hence,

$$\sum_{\sigma} Q(\sigma'|\sigma) Q(\sigma'|\sigma) \delta_{\sigma,1} \equiv \beta p_{\text{imp}},$$

where we recall that dropping the argument $s$ implicitly means $s = \{1\}$. The first contribution to $p_b$ is thus simply

$$d(\beta p_{\text{imp}}).$$

Now let us turn to the second term, which takes the form

$$\sum_{\sigma'} Q(\sigma'|\sigma) Q(\sigma'|\sigma) = \frac{\beta p(x'' \mid \sigma) N(\sigma)}{\beta p(x' \mid \sigma)} \sum_{\sigma} \sum_{x} x(x' \mid \sigma) p_{\text{imp}}(s' \mid s') W_i(s') Q(\sigma'|\sigma).$$

Expressing $W_i(s')$ thanks to equation (33), the right-hand side reads

$$\beta \sum_{\sigma} \sum_{x} x(x' \mid \sigma) p(x' \mid \sigma) p_{\text{imp}}(s' \mid s') W_i(s) x(x') p_{\text{imp}}(s' \mid s')(1 - x(x')) H Q(\sigma'|\sigma)$$

(E6)

where the first term corresponds to surviving agents in the next time step and the second term corresponds to dying agents who are replaced with newborn agents with wealth $H$.

Consider the two terms of equation (E6) in turn. The first term contains a factor $x(x' \mid \sigma) x(x')$ that equals one if an agent survives for both of the next two periods and zero otherwise. We now use the update rule of agents’ beliefs to compute $p_{\text{imp}}(s' \mid \sigma)$. One finds, for $s' = \{1\}$, that

$$p_{\text{imp}}(1|1) = (1 - \lambda) p_j + \lambda; \quad p_{\text{imp}}(1|0) = (1 - \lambda) p_j,$$

where we recall that $p_j := p_j(1)$. Hence,

$$\sum_{s'} p_{\text{imp}}(s') = [(1 - \lambda) p_j + \lambda] p_j + [(1 - \lambda) p_j] (1 - p_j) = 1.$$

In words, conditional on survival, the agent’s belief is a martingale. Conditioning on $s' = \{1\}$, one has

$$\sum_{x, x' \in \{1\}} \beta p(x' \mid \sigma) \sum_{s} x(x' \mid s') p_{\text{imp}}(s' \mid s') p(x' \mid s') W_i(s) = NH \beta (1 - \delta)^2 p_{\text{imp}},$$

In the large $N$ limit, $N(\sigma) = N(1 - \delta)$ and this term gives a contribution to $p_b(\sigma)$ equal to

$$d(\beta (1 - \delta) p_{\text{imp}}).$$
Let us now look at the second term. Because of the $1 - x_j(x')$ term, we are looking at states of the world where agent $j$ has died and is replaced by a new agent with an idiosyncratic probability of the next state $P_j(s'' = \{1\})$ equal to $z$, which is uniformly distributed between zero and one, with no memory of the past. Therefore, the sum over $\sigma'$ can be taken independently of the future and gives

$$
\sum_{x', j \in \{1\}} p(x') x_j(x'') P_j(s'' = \{1\}) \sum_{x, \sigma}(1 - x_j(x')) Q(\sigma'|\sigma) = \beta \delta (1 - \delta) \mathbb{E}[z].
$$

Hence, we find that dying agents give a contribution to $p_{\text{E}}(\sigma)$ equal to

$$
d\beta^2 \delta \frac{1}{2},
$$

where we have replaced $\mathbb{E}[z]$ by $1/2$ and again used the fact that $N(\sigma'') \approx N(1 - \delta)$ when $N \gg 1$.

Generalizing to all $\ell \geq 1$ time steps in the future, each agent $j$ can either survive $\ell$ times with probability $(1 - \delta)^\ell$ or die at least once with probability $1 - (1 - \delta)^\ell$. In the first case, his or her belief is a martingale. In the second case, the last death cuts all dependence from the past. The calculation above can thus be generalized to give a contribution to $p_{\text{E}}(\sigma)$ equal to

$$
d\beta^\ell \left[ (1 - \delta)^{\ell-1} \mathbb{P}_{\text{imp}} + (1 - (1 - \delta)^{\ell-1}) \frac{1}{2} \right].
$$

Summing over $\ell$ yields our final result for the price of equity in our economy:

$$
p_{\text{E}} = d\beta \frac{2 \mathbb{P}_{\text{imp}} - 1}{1 - \beta (1 - \delta)} + \frac{1}{1 - \beta}.
$$

(E8)

If agents never die, we recover

$$
p_{\text{E}} = d\beta \frac{\mathbb{P}_{\text{imp}}}{1 - \beta},
$$

as expected. If agents die at every time step, then $\mathbb{P}_{\text{imp}} = 1/2$ and one also recovers the expected result.

Appendix F

Multiplicative Random Process with Reset

Consider the simplest case where, conditioned on survival, returns are independent and identically distributed (i.i.d.) random variables—that is,

$$
W_t = \begin{cases} 
W_t(1 + \eta) & \text{with probability } 1 - \delta, \\
1 & \text{with probability } \delta,
\end{cases}
$$

(F1)

where $\eta$ represents the date $t$ element of a sequence of i.i.d. random variables with zero mean and variance equal to $\sigma^2$. For this simple case, the sequence of conditional probability measures $q(W)$ obeys the operator equation,
\[ \varphi(W) = (1 - \delta) \int dW \varphi(W) \int d\eta p(\eta) d(W' - W(1 + \eta)) + \delta d(W' - 1), \quad (F2) \]

where \( d \) represents Dirac’s delta function. For large \( W \), this equation delivers a power-law tail, with an exponent \( \mu \) that is implicitly defined by the self-consistency condition

\[ 1 = (1 - \delta) \int d\eta p(\eta)(1 + \eta)^\mu. \quad (F3) \]

In the limit when \( \delta \) and \( \sigma^2 \) are small, the solution for \( \mu \) is approximated by the expression

\[ \mu = \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8\delta}{\sigma^2}} \right]. \quad (F4) \]

For the wealth process considered in this paper, however, the \( \eta \) are correlated in time (since agent \( i \) will consistently make or lose money as long as the sign of \( \bar{\mathcal{P}}_i(t) - \bar{\mathcal{P}}(t) \) is constant—i.e., during a time \( \sim \lambda^{-1} \)) and its variance is time dependent (see eqq. [37], [38]).

A simplified analysis assumes that \( \eta \) is constant during a time \( \lambda^{-1} \). This provides the following approximation for \( \mu \) in this case:

\[ \mu \approx \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8\delta\lambda}{\sigma^2}} \right], \quad \sigma^2 = \mathbb{E}[\bar{\sigma}^2(t)]. \quad (F5) \]

Note that \( \mu \geq 1 \) from this formula, meaning that the wealth distribution always has a finite mean when \( \delta > 0 \).

A way to decrease wealth inequalities is to introduce a wealth tax. If at each time step a small fraction \( \varphi \) of the wealth of each individual is levied and redistributed across the economy, the value of \( \mu \) in the simple i.i.d. model above changes to

\[ \mu = \varphi + \sqrt{\varphi^2 + 2\delta\sigma^2} \]

Hence, as expected, increasing \( \varphi \) increases \( \mu \) and decreases both the Gini coefficients, thereby making markets more efficient in the sense that the difference between \( \bar{\mathcal{P}} \) and \( \bar{\mathcal{P}}_{\text{imp}} \) is reduced.

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