Abstract. We prove that the positive fragment of first-order intuitionistic logic in the language with two individual variables and a single monadic predicate letter, without functional symbols, constants, and equality, is undecidable. This holds true regardless of whether we consider semantics with expanding or constant domains. We then generalise this result to intervals \([\mathcal{QBL}, \mathcal{QKC}]\) and \([\mathcal{QBL}, \mathcal{QFL}]\), where \(\mathcal{QKC}\) is the logic of the weak law of the excluded middle and \(\mathcal{QBL}\) and \(\mathcal{QFL}\) are first-order counterparts of Visser’s basic and formal logics, respectively. We also show that, for most “natural” first-order modal logics, the two-variable fragment with a single monadic predicate letter, without functional symbols, constants, and equality, is undecidable, regardless of whether we consider semantics with expanding or constant domains. These include all sublogics of \(\mathcal{QKTB}, \mathcal{QGL}\), and \(\mathcal{QGrz}\)—among them, \(\mathcal{QK}, \mathcal{QT}, \mathcal{QKB}, \mathcal{QD}, \mathcal{QK4}\), and \(\mathcal{QS4}\).

Keywords: First-order intuitionistic logic, First-order modal logic, Undecidability, Two-variable fragment, Monadic fragment.

1. Introduction

While the (first-order) quantified classical logic \(\mathcal{QCl}\) is undecidable [6], it contains a number of rather expressive decidible fragments [3]. This has long stimulated interest in drawing the borderline between decidable and undecidable fragments of \(\mathcal{QCl}\) using a variety of criteria, in isolation or in combination, imposed on the language. One such criterion is the number and arity of predicate letters allowed in the language: while the monadic fragment is decidable [1], the fragment containing a single binary letter is not, as follows from [9]. Another criterion is the number of individual variables allowed in the language: while the two-variable fragment is decidable [10,17], the three-variable fragment is not [24].

Similar questions have long been of interest in (first-order) quantified intuitionistic and modal logics. For languages without restrictions on the
number of individual variables, Kripke [14] has shown that all “natural” quantified modal logics with two monadic predicate letters are undecidable, while Maslov et al. [15] and, independently, Gabbay [8] have shown that quantified intuitionistic logic with a single monadic predicate letter is undecidable.

The question of where the borderline lies in the intuitionistic and modal case when it comes to the number of individual variables allowed in the language has recently been investigated by Kontchakov et al. [12]. It is shown in [12] that two-variable fragments of quantified intuitionistic and all “natural” modal logics are undecidable. Moreover, it is established in [12] that, to obtain undecidability of two-variable fragments, in the intuitionistic case, it suffices to use two binary and infinitely many monadic predicate letters, while in the modal case, it suffices to use only (infinitely many) monadic predicate letters.

Two questions were raised in [12] concerning the languages combining restrictions on the number of individual variables and predicate letters: first, how many monadic predicate letters are needed to obtain undecidability of the two-variable fragments in the modal case, and second, whether it suffices to use monadic predicate letters to obtain undecidability of the two-variable fragment in the intuitionistic case.

In the present paper, we address both of the aforementioned questions. First, we show that for two-variable fragments of most modal logics considered in [12], it suffices to use a single monadic predicate letter to obtain undecidability. Second, we show that the positive fragment of quantified intuitionistic logic $Q_{\text{Int}}$ is undecidable in the language with two variables and a single monadic predicate letter. We also show that the latter result holds true for all logics in intervals $[Q_{BL}, Q_{KC}]$ and $[Q_{BL}, Q_{FL}]$, where $Q_{KC}$ is the logic of the weak law of the excluded middle and $Q_{BL}$ and $Q_{FL}$ are first-order counterparts of Visser’s basic and formal logics, respectively.

The paper is structured as follows. In Section 2, we prove undecidability results about modal logics. In Section 3, we do likewise for the intuitionistic and related logics. We conclude, in Section 4, by discussing how our results can be applied in settings not considered in this paper and pointing out some open questions following from our work.
2. Modal Logics

In this section, we prove undecidability results about two-variable fragments of quantified modal logics with a single monadic predicate letter. This is essentially achieved by adapting to the first-order language of Halpern’s technique [11] for establishing complexity results for single-variable fragments of propositional modal logics.

2.1. Syntax and Semantics

A (first-order) quantified modal language contains countably many individual variables; countably many predicate letters of every arity; Boolean connectives $\land$ and $\lnot$; modal connective $\Box$; and a quantifier $\forall$. Formulas as well as the symbols $\lor$, $\rightarrow$, $\exists$, and $\Diamond$ are defined in the usual way. We also use the following abbreviations:

\[
\Box^+ \varphi = \varphi \land \Box \varphi \text{ and } \Diamond^+ \varphi = \varphi \lor \Diamond \varphi.
\]

A Kripke frame is a tuple $\mathfrak{F} = \langle W, R \rangle$, where $W$ is a non-empty set (of worlds) and $R$ is a binary (accessibility) relation on $W$. A predicate Kripke frame is a tuple $\mathfrak{F}_D = \langle W, R, D \rangle$, where $\langle W, R \rangle$ is a Kripke frame and $D$ is a function from $W$ into a set of non-empty subsets of some set (the domain of $\mathfrak{F}_D$), satisfying the condition that $wRw'$ implies $D(w) \subseteq D(w')$. We call the set $D(w)$ the domain of $w$. We will also be interested in predicate frames satisfying the condition that $wRw'$ implies $D(w) = D(w')$; we refer to such frames as frames with constant domains.

A Kripke model is a tuple $\mathfrak{M} = \langle W, R, D, I \rangle$, where $\langle W, R, D \rangle$ is a predicate Kripke frame and $I$ is a function assigning to a world $w \in W$ and an $n$-ary predicate letter $P$ an $n$-ary relation $I(w, P)$ on $D(w)$. We refer to $I$ as the interpretation of predicate letters with respect to worlds in $W$.

An assignment in a model is a function $g$ associating with every individual variable $x$ an element of the domain of the underlying frame.

The truth of a formula $\varphi$ in a world $w$ of a model $\mathfrak{M}$ under an assignment $g$ is inductively defined as follows:

- $\mathfrak{M}, w \models^g P(x_1, \ldots, x_n)$ if $(g(x_1), \ldots, g(x_n)) \in I(w, P)$;
- $\mathfrak{M}, w \models^g \varphi_1 \land \varphi_2$ if $\mathfrak{M}, w \models^g \varphi_1$ and $\mathfrak{M}, w \models^g \varphi_2$;
- $\mathfrak{M}, w \models^g \neg \varphi_1$ if $\mathfrak{M}, w \not\models^g \varphi_1$;
\begin{itemize}
\item $\mathcal{M}, w \models^g \Box \varphi_1$ if $w R w'$ implies $\mathcal{M}, w' \models^g \varphi_1$, for every $w' \in W$;
\item $\mathcal{M}, w \models^g \forall x \varphi_1$ if $\mathcal{M}, w \models^g' \varphi_1$, for every assignment $g'$ such that $g'$ differs from $g$ in at most the value of $x$ and such that $g'(x) \in D(w)$.
\end{itemize}

Note that, given a Kripke model $\mathcal{M} = \langle W, R, D, I \rangle$ and $w \in W$, the tuple $\mathcal{M}_w = \langle D(w), I_w \rangle$, where $D_w = D(w)$ and $I_w(P) = I(w, P)$, is a classical predicate model.

We say that $\varphi$ is true at world $w$ of model $\mathcal{M}$ and write $\mathcal{M}, w \models \varphi$ if $\mathcal{M}, w \models^g \varphi$ holds for every $g$ assigning to free variables of $\varphi$ elements of $D(w)$. We say that $\varphi$ is true in $\mathcal{M}$ and write $\mathcal{M} \models \varphi$ if $\mathcal{M}, w \models \varphi$ holds for every world $w$ of $\mathcal{M}$. We say that $\varphi$ is true in predicate frame $\mathcal{F}$ and write $\mathcal{F} \models \varphi$ if $\varphi$ is true in every model based on $\mathcal{F}$. We say that $\varphi$ is true in frame $\mathcal{F}$ and write $\mathcal{F} \models \varphi$ if $\varphi$ is true in every predicate frame of the form $\mathcal{F}$. Finally, we say that a formula is true in a class of frames if it is true in every frame from the class.

Let $\mathcal{M} = \langle W, R, D, I \rangle$ be a model, $w \in W$, and $a_1, \ldots, a_n \in D(w)$. Let $\varphi(x_1, \ldots, x_n)$ be a formula whose free variables are among $x_1, \ldots, x_n$. We write $\mathcal{M}, w \models \varphi[a_1, \ldots, a_n]$ to mean $\mathcal{M}, w \models^g \varphi(x_1, \ldots, x_n)$, where $g(x_1) = a_1, \ldots, g(x_n) = a_n$.

Given a propositional normal modal logic $L$, let $QL$ be $QCl \oplus L$ where $\oplus$ is the operation of closure under (predicate) substitution, modus ponens, generalization, and necessitation. Of particular interest to us are the quantified counterparts $QGL$, $QGrz$, and $QKTB$ of propositional logics $GL$, $QGrz$, and $KTB$. We recall that $GL$ is the logic of Kripke frames whose accessibility relation is irreflexive, transitive, and contains no infinite ascending chains, while $Grz$ is the logic of frames whose accessibility relation is reflexive, transitive, antisymmetric, and does not contain infinite ascending chains of pairwise distinct worlds (in other words, the accessibility relation on the frames for $Grz$ is the reflexive closure of the one on the frames for $GL$). We also recall that $QGL$ and $QGrz$ are Kripke-incomplete [16,20], but are valid on all the frames for $GL$ and $Grz$, respectively. Thus, for technical reasons—namely, to avoid being distracted with Kripke-completeness—we define logics $QGL^{sem}$ and $QGrz^{sem}$ as the sets of quantified formulas true in all the frames of $GL$ and $Grz$, respectively. What is important for us is that $QGL \subseteq QGL^{sem}$ and $QGrz \subseteq QGrz^{sem}$. Lastly, we recall that $KTB$ is the logic of Kripke frames whose accessibility relation is reflexive and symmetric and that $QKTB$ is complete with respect to this class of frames.
Given a logic $L$ and a closed formula $\varphi$ in the language of $L$, we say that $\varphi$ is $L$-satisfiable if $\neg \varphi \notin L$. If $L$ is complete with respect to a class $\mathfrak{C}$ of frames, $L$-satisfiability of $\varphi$ amounts to $\varphi$ being true at a world of a model based on a frame in $\mathfrak{C}$.

We now turn to addressing the question, raised in [12], of how many monadic predicate letters are needed in the language of quantified modal logics to obtain undecidability of their two-variable fragments. Using suitable adaptations of a technique originally proposed in [11], and further refined in [4,22], and [23], for propositional languages, we show that all sublogics of $\text{QGL}$, $\text{QGrz}$, and $\text{QKTB}$ are undecidable in the language with a single monadic predicate letter.

### 2.2. Sublogics of $\text{QGL}$ and $\text{QGrz}$

It is established in [12], Theorem 3, that two-variable fragments of a wide variety of quantified modal logics in the language with infinitely many monadic predicate letters are undecidable.

To that end, it is shown in [12] how, given an instance $T$ of an undecidable tiling problem [2], one can effectively compute a formula $\xi_T$ containing only monadic predicate letters and two individual variables such that $T$ tiles $\mathbb{N} \times \mathbb{N}$ if, and only if, $\xi_T$ is satisfiable in a logic $L$ such that $L$ is the set of quantified formulas valid on all the frames of a propositional logic valid on a frame containing a world that can see all worlds from an infinite set $V_1$, each of which can in its turn see infinitely many worlds from an infinite set $V_2$ disjoint from $V_1$. Note that the logics $\text{QK}$, $\text{QGL}^{sem}$, and $\text{QGrz}^{sem}$ satisfy the above condition; indeed, for $\text{GL}$ and $\text{Grz}$, we can take $V_1$ and $V_2$ to be infinite anti-chains of irreflexive and reflexive worlds, respectively. As formulas $\xi_T$ are computed in the same way for all of $\text{QK}$, $\text{QGL}^{sem}$, and $\text{QGrz}^{sem}$, it follows that if $T$ does not tile $\mathbb{N} \times \mathbb{N}$, then $\xi_T$ is not satisfiable in any of them, and if $T$ tiles $\mathbb{N} \times \mathbb{N}$, then $\xi_T$ is satisfiable in each of them. Let $L$ be a logic such that $\text{QK} \subseteq L \subseteq \text{QGL}^{sem}$. If $T$ does not tile $\mathbb{N} \times \mathbb{N}$, then $\neg \xi_T \in \text{QK}$, and thus $\neg \xi_T \not\in L$. If, on the other hand, $T$ tiles $\mathbb{N} \times \mathbb{N}$, then $\neg \xi_T \not\in \text{QGL}^{sem}$, and thus $\neg \xi_T \not\in L$. This gives us a reduction of (the complement of) the undecidable tiling problem to $L$ using formulas with only monadic predicate letters and two individual variables. Thus, every logic in $[\text{QK}, \text{QGL}]$ is undecidable in the language with a single monadic letter and two individual variables; a similar argument can be made for logics in $[\text{QK}, \text{QGrz}]$. 
As formulas $\xi_T$ are computed in the same way for all the logics in $[QK, QGL]$ and $[QK, QGrz]$, all formulas $\neg \xi_T$ corresponding to “bad” instances of the tiling problem [2] make up an undecidable fragment, $F$, that belongs to every logic in $[QK, QGL]$ and $[QK, QGrz]$. In the rest of this section, we effectively embed $F$, using an embedding $e$ that does not increase the number of individual variables in a formula, into a fragment, $F^e$, containing a single monadic predicate letter and belonging to every logic in $[QK, QGL]$ and $[QK, QGrz]$. To that end, given a formula $\varphi$, we effectively construct, using $e$, the formula $\varphi^e$, such that $\varphi \in F$ if, and only if, $\varphi^e \in F^e$; as $\varphi^e$ contains the same number of individual variables as $\varphi$, our main result in this section immediately follows.

Let $\varphi$ be a (closed) formula containing monadic predicate letters $P_1, \ldots, P_n$. Let $P_{n+1}$ be a monadic predicate letter distinct from $P_1, \ldots, P_n$ and let $B = \forall x P_{n+1}(x)$. Define an embedding $\cdot'$ as follows:

$P_i(x)' = P_i(x), \text{ where } i \in \{1, \ldots, n\}$;

$(\neg \phi)' = \neg \phi'$;

$(\phi \land \psi)' = \phi' \land \psi'$;

$(\forall x \phi)' = \forall x \phi'$;

$(\Box \phi)' = \Box (B \rightarrow \phi')$.

**Lemma 2.1.** Let $L \in \{QK, QGL^{sem}, QGrz^{sem}\}$. Then, $\varphi$ is $L$-satisfiable if, and only if, $B \land \varphi'$ is $L$-satisfiable.

**Proof.** Assume that $\mathcal{M}, w_0 \models \varphi$, for some $\mathcal{M}$ based on a frame for $L$ and some $w_0$. Let $\mathcal{M}'$ be a model that extends $\mathcal{M}$ by setting $I(w, P_{n+1}) = D(w)$, for every $w \in W$. Then, $\mathcal{M}', w_0 \models B \land \varphi'$. Conversely, assume that $\mathcal{M}, w_0 \models B \land \varphi'$, for some $\mathcal{M}$ based on a frame for $L$. Let $\mathcal{M}'$ be a submodel of $\mathcal{M}$ with $W' = \{w : \mathcal{M}, w \models B\}$. Then, $\mathcal{M}', w_0 \models \varphi$. Note that, for every logic $L$ in the statement of the Lemma, $\mathcal{M}'$ is based on a frame for $L$. ■

**Remark 2.2.** In view of the proof of Lemma 2.1, if $B \land \varphi'$ is satisfied in a model $\mathcal{M}$, we can assume, without a loss of generality, that $B$ is true in $\mathcal{M}$.

Now, given a monadic predicate letter $P$, we inductively define the following sequence of formulas:
\[
\delta_1(x) = P(x) \land \Diamond (\neg P(x) \land \Diamond \Box^+ P(x));
\]

\[
\delta_{m+1}(x) = P(x) \land \Diamond (\neg P(x) \land \Diamond \delta_m(x)).
\]

Using formulas from this sequence, define, for every \(k \in \{1, \ldots, n + 1\}\), the formula

\[
\alpha_k(x) = \delta_k(x) \land \neg \delta_{k+1}(x) \land \Diamond \Box^+ \neg P(x).
\]

We now define models associated with formulas \(\alpha_k(x)\). For every \(k \in \{1, \ldots, n + 1\}\), let \(F_k = \langle W_k, R_k \rangle\) be a Kripke frame where \(W_k = \{w^0_k, \ldots, w^{2k}_k\} \cup \{w^*_k\}\) and \(R_k\) is the transitive closure of the relation \(\{\langle w^i_k, w^{i+1}_k \rangle : 0 \leq i < 2k\} \cup \{\langle w^0_k, w^*_k \rangle\}\). For every such \(k\), let \(M_k = \langle W_k, R_k, D, I \rangle\) be a model with constant domains and let \(a\) be an individual in the domain of every \(M_k\) (other than that, the relationship between the domains of \(M_k\)'s is immaterial at this point). We say that \(M_k\) is \(a\)-suitable if

\[
M_k, w \models P[a] \iff w = w^i_k, \text{ for } i \in \{0, \ldots, k\}.
\]

**Lemma 2.3.** Let \(a\) be an individual in the domain of the models \(M_1, \ldots, M_{n+1}\) and let \(M_1, \ldots, M_{n+1}\) be \(a\)-suitable. Then,

\[
M_k, w \models \alpha_m[a] \iff k = m \text{ and } w = w^0_k.
\]

**Proof.** Straightforward. \(\Box\)

**Remark 2.4.** Notice that the statement of Lemma 2.3 holds true if we replace the accessibility relations in \(M_1, \ldots, M_{n+1}\) with their reflexive closures.

Now, for every \(\alpha_k(x)\), where \(k \in \{1, \ldots, n + 1\}\), define

\[
\beta_k(x) = \neg P(x) \land \Diamond \alpha_k(x).
\]

Let \(\varphi^*\) be the result of replacing in \(\varphi'\) of \(P_k(x)\) with \(\beta_k(x)\), for every \(k \in \{1, \ldots, n + 1\}\).

**Lemma 2.5.** Let \(L \in \{\text{QK, QGL}^{sem}, \text{QGrz}^{sem}\}\). Then, \(B \land \varphi'\) is \(L\)-satisfiable if, and only if, \(\forall x \beta_{n+1}(x) \land \varphi^*\) is \(L\)-satisfiable.

**Proof.** The right-to-left direction follows from the closure of \(L\) under predicate substitution. For the other direction, suppose that \(B \land \varphi'\) is \(\text{QK}\)-satisfiable. Let \(M = \langle W, R, D, I \rangle\) be a model such that \(M, w_0 \models B \land \varphi'\), for some \(w_0 \in W\). In view of Remark 2.2, we may assume, without a loss of generality, that \(M \models B\).
For every \( w \in W \) and every frame \( \mathfrak{F}_k \) \( 1 \leq k \leq n+1 \), let \( \mathfrak{F}^w_k = \langle \{ w \} \times W_k, R^w_k \rangle \) be an isomorphic copy of \( \mathfrak{F}_k \). For every \( w \in W \) and \( k \in \{ 1, \ldots, n+1 \} \), add \( \{ w \} \times W_k \) to \( W \) to obtain the set \( W^* \). Define the relation \( R^* \) on \( W^* \) as follows:

\[
R^* = R \cup \bigcup \{ R^w_k \cup \{ (w, (w, w^0_k)) \} : w \in W, 1 \leq k \leq n+1 \}.
\]

Thus, for every \( w \in W \), we make the roots of frames \( \mathfrak{F}^w_1, \ldots, \mathfrak{F}^w_{n+1} \) accessible from \( w \). Next, for every \( u \in W^* \) let

\[
D^*(u) = \begin{cases} D(u), & \text{if } u \in W, \\ D(w), & \text{if } u \in \{ w \} \times W_k. \end{cases}
\]

Finally, for every \( u \in W^* \) and every \( a \in D^*(u) \), let

\[
\langle a \rangle \in I^*(u, P) \iff u = (w, w^2_k), \text{ for some } w \in W, k \in \{ 1, \ldots, n+1 \}, \text{ and } i \in \{ 0, \ldots, k \}; \text{ and } \mathfrak{M}, w \models P_k[a].
\]

Let \( \mathfrak{M}^* = \langle W^*, R^*, D^*, I^* \rangle \). It immediately follows from Lemma 2.3 that, for every \( w \in W \), every \( a \in D(w) \), and every \( k \in \{ 1, \ldots, n+1 \} \),

\[
\mathfrak{M}, w \models P_k[a] \iff \mathfrak{M}^*, w \models \beta_k[a].
\]

We can then show that, for every \( w \in W \), every subformula \( \psi(x_1, \ldots, x_m) \) of \( \varphi \), and every \( a_1, \ldots, a_m \in D(w) \),

\[
\mathfrak{M}, w \models \psi'[a_1, \ldots, a_m] \iff \mathfrak{M}^*, w \models \psi'[a_1, \ldots, a_m],
\]

where \( \psi'(x_1, \ldots, x_m) \) is obtained by substituting \( \beta_1(x), \ldots, \beta_{n+1}(x) \) for \( P_1(x), \ldots, P_{n+1}(x) \) in \( \psi'(x_1, \ldots, x_m) \).

The proof proceeds by induction. We only consider the modal case, leaving the rest to the reader. In this case, \( \psi'(x_1, \ldots, x_m) = \Box(\forall x P_{n+1}(x) \rightarrow \chi'(x_1, \ldots, x_m)) \) and \( \psi^*(x_1, \ldots, x_m) = \Box(\forall x \beta_{n+1}(x) \rightarrow \chi^*(x_1, \ldots, x_m)) \).

If \( \mathfrak{M}^*, w \not\models \psi^*[a_1, \ldots, a_m] \), then there exists \( w' \in W^* \) with \( w R^* w' \) such that \( \mathfrak{M}^*, w' \models \forall x \beta_{n+1}(x) \) and \( \mathfrak{M}^*, w' \not\models \chi^*[a_1, \ldots, a_m] \). The condition \( \mathfrak{M}^*, w' \models \forall x \beta_{n+1}(x) \) guarantees that \( w' \in W \); therefore, we may apply the inductive hypothesis to conclude that \( \mathfrak{M}, w' \not\models \chi'[a_1, \ldots, a_m] \). The other direction is straightforward.

Thus, \( \mathfrak{M}^*, w_0 \models \forall x \beta_{n+1}(x) \land \varphi^* \), i.e., \( \forall x \beta_{n+1}(x) \land \varphi^* \) is \( \text{QK} \)-satisfiable.

For \( \text{QGL}^{\text{sem}} \) and \( \text{QGrz}^{\text{sem}} \), the proof is similar. The only difference is that, when defining the model \( \mathfrak{M}^* \), instead of \( R^* \) mentioned above, we take as the accessibility relations its transitive, and its reflexive and transitive, closure, respectively.
We can now prove our main result in this section.

**Theorem 2.6.** Let $L$ be a logic such that $\mathsf{QK} \subseteq L \subseteq \mathsf{QGL}$ or $\mathsf{QK} \subseteq L \subseteq \mathsf{QGrz}$. Then, $L$ is undecidable in the language with two individual variables and a single monadic predicate letter.

**Proof.** Given a formula $\varphi$ with two individual variables and only monadic predicate letters, let $e(\varphi) = \forall x \beta_{n+1}(x) \land \varphi^*$. Let $\neg \xi_T$ be a formula corresponding to a “bad” instance $T$ of the tiling problem [2]. Due to Lemmas 2.1 and 2.5, $\neg \xi_T \in \mathsf{QK}$ if, and only if, $\neg e(\xi_T) \in \mathsf{QK}$; likewise, $\neg \xi_T \in \mathsf{QGL}^{sem}$ if, and only if, $\neg e(\xi_T) \in \mathsf{QGL}^{sem}$. As noticed at the beginning of this section, all such $\neg \xi_T$ make up an undecidable fragment, $F$, belonging to every logic in $[\mathsf{QK}, \mathsf{QGL}^{sem}]$ and $[\mathsf{QK}, \mathsf{QGrz}^{sem}]$. Therefore, every such logic contains an undecidable fragment $F^e = \{ \neg e(\xi_T) : \neg \xi_T \in F \}$ made up of formulas with two individual variables and a single monadic predicate letter. The statement of the Theorem follows.

**Corollary 2.7.** $\mathsf{QK}$, $\mathsf{QT}$, $\mathsf{QD}$, $\mathsf{QK4}$, $\mathsf{QS4}$, $\mathsf{QGL}$, and $\mathsf{QGrz}$ are undecidable in the language with two individual variables and a single monadic predicate letter.

**Remark 2.8.** Theorem 2.6 and Corollary 2.7 hold true if we replace every logic $L$ mentioned in their statements with $L \oplus \mathbf{bf}$, where $\mathbf{bf} = \forall x \Box P(x) \rightarrow \Box \forall x P(x)$; adding $\mathbf{bf}$ to $L$ forces us to consider only predicate frames for $L$ with constant domains.

We conclude this section by noticing that the results obtained herein are quite tight. In has been shown in [26], Theorem 5.1, that for logics $\mathsf{QK}$, $\mathsf{QT}$, $\mathsf{QK4}$, and $\mathsf{QS4}$, adding—on top of the restriction to at most two individual variables and a single monadic predicate letter—the very slight restriction that modal operators apply only to formulas with at most one free individual variable results in decidable fragments. As noticed in [26], the same holds true for the other logics mentioned in Corollary 2.7.

### 2.3. Sublogics of $\mathsf{QKTB}$

We now prove results similar to those established in the preceding section for logics in the interval $[\mathsf{QK}, \mathsf{QKTB}]$, where $\mathsf{QKTB}$ is the predicate logic of reflexive and symmetric frames. In so doing, we use an adaptation of a technique used in [23] for proving results about computational complexity of finite-variable fragments of sublogics of the propositional logic $\mathsf{KTB}$. 
We proceed as in the previous section right up to the point where formulas \( \alpha_k \) and models \( M_k \) are defined. Then, we define the formulas \( \alpha_k \) as follows. First, let
\[
\Box^0 \varphi = \varphi, \quad \Box^{\leq 0} \varphi = \varphi,
\]
\[
\Box^{n+1} \varphi = \Box \Box^n \varphi, \quad \Box^{\leq n+1} \varphi = \Box^{\leq n} \varphi \land \Box^n \varphi,
\]
\[
\Diamond^n \varphi = \neg \Box^n \neg \varphi, \quad \Diamond^{\leq n} \varphi = \neg \Box^{\leq n} \neg \varphi.
\]
Next, inductively define, for every \( k \in \{1, \ldots, n+1\} \), the following sequence of formulas:
\[
\delta(x) = \Box^+ P(x);
\]
\[
\delta_k^i(x) = \Box^{\leq i} P(x) \land \Diamond^{i+1} P(x) \land \Diamond^{i+2} \delta(x);
\]
\[
\delta_k(x) = \Box^{\leq i} P(x) \land \Diamond^{i+1} P(x) \land \Diamond^{2i+3} \delta_{i+1}(x), \quad \text{where } 1 \leq i < k,
\]
and let, for every \( k \in \{1, \ldots, n+1\} \),
\[
\alpha_k(x) = P(x) \land \Diamond^{2} \delta_1^k(x).
\]
Now we define models \( M_k \) associated with formulas \( \alpha_k \). Given an individual \( a \) and \( k \in \{1, \ldots, n+1\} \), a model \( M_k \), whose domain contains \( a \), looks as follows. For brevity, we call some worlds \( a \)-worlds; if a world is not an \( a \)-world, we call it an \( \bar{a} \)-world. The model is a chain of worlds whose root, \( r_k \), is an \( a \)-world. The root is part of a pattern of worlds, described below, which is in turn succeeded by three final \( a \)-worlds. The pattern looks as follows: a single \( a \)-world is followed by \( 2i+1 \) \( \bar{a} \)-worlds, for \( 1 \leq i \leq k \). Thus the chain looks as follows: the root (an \( a \)-world), then three \( \bar{a} \)-worlds, then an \( a \)-world, then five \( \bar{a} \)-worlds, then an \( a \)-world, \ldots, then an \( a \)-world, then \( 2k+1 \) \( \bar{a} \)-worlds, then three \( a \)-worlds. The accessibility relation between the worlds of \( M_k \) is both reflexive and symmetric.

We say that \( M_k \) is \( a \)-suitable if
\[
M_k, w \models P[a] \iff w \text{ is an } a \text{-world}.
\]
We can, then, prove the following analogue of Lemma 2.3.

**Lemma 2.9.** Let \( a \) be an individual in the domain of the models \( M_1, \ldots, M_{n+1} \) and let \( M_1, \ldots, M_{n+1} \) be \( a \)-suitable. Then,
\[
M_k, w \models \alpha_m[a] \iff k = m \text{ and } w = r_k.
\]
Proof. Straightforward.

As before, let
\[ \beta_k(x) = \neg P(x) \land \Diamond \alpha_k(x), \]
and let \( \varphi^* \) be the result of replacing in \( \varphi' \) of \( P_k(x) \) with \( \beta_k(x) \), for every \( k \in \{1, \ldots, n+1\} \).

We can then prove the following analogue of Lemma 2.5:

Lemma 2.10. Let \( L \in \{\text{QK, QKTB}\} \). Then, \( B \land \varphi' \) is \( L \)-satisfiable if, and only if, \( \forall x \beta_{n+1}(x) \land \varphi^* \) is \( L \)-satisfiable.

Proof. Analogous to the proof of Lemma 2.5, with the observation that the truth status of formulas \( \alpha_k \) is not changed at the worlds of the models \( M_k \) once they get attached to the model \( M \) satisfying the formula \( B \land \varphi' \) to obtain the model \( M^* \) satisfying the formula \( \forall x \beta_{n+1}(x) \land \varphi^* \), even though their roots can now see the worlds of \( M \) due to the symmetry of the accessibility relation of \( M^* \). For a detailed argument showing that the truth status of formulas \( \alpha_k \) in \( M^* \) at worlds from \( M_k \) is not affected, we refer the reader to the proof of Lemma 3.9 in [23].

Then, using an argument analogous to the one used in the proof of Theorem 2.6, we obtain the following:

Theorem 2.11. Let \( L \) be a logic such that \( \text{QK} \subseteq L \subseteq \text{QKTB} \). Then, \( L \) is undecidable in the language with two individual variables and a single monadic predicate letter.

Corollary 2.12. \( \text{QKB} \) and \( \text{QKTB} \) are undecidable in the language with two individual variables and a single monadic predicate letter.
3. Intuitionistic and Related Logics

We now consider logics closely related to the quantified intuitionistic logic $\text{QInt}$.

3.1. Syntax and Semantics

The (first-order) quantified intuitionistic language contains countably many individual variables; countably many predicate letters of every arity; propositional constants $\bot$ ("falsehood") and $\top$ ("truth"); propositional connectives $\land$, $\lor$, and $\rightarrow$; and quantifiers $\exists$ and $\forall$. Formulas are defined in the usual way; when parentheses are left out, $\land$ and $\lor$ are understood to bind tighter than $\rightarrow$. We also use the following abbreviations: $\square \varphi = \top \rightarrow \varphi$, $\square^0 \varphi = \varphi$, and $\square^{n+1} \varphi = \square \square^n \varphi$.

A Kripke frame is a tuple $\mathfrak{F} = \langle W, R \rangle$, where $W$ is a non-empty set (of worlds) and $R$ is a binary (accessibility) relation on $W$ that is reflexive, anti-symmetric, and transitive.

A Kripke model $\mathfrak{M} = \langle W, R, D, I \rangle$ is defined as in the modal case, except that the interpretation function $I$ satisfies the additional condition that $wRw'$ implies $I(w, P) \subseteq I(w', P)$. An assignment is defined as in the modal case.

The truth of a formula $\varphi$ in a world $w$ of a model $\mathfrak{M}$ under an assignment $g$ is inductively defined as follows:

- $\mathfrak{M}, w \not\models_g \bot$;
- $\mathfrak{M}, w \models_g \top$;
- $\mathfrak{M}, w \models_g P(x_1, \ldots, x_n)$ if $\langle g(x_1), \ldots, g(x_n) \rangle \in I(w, P)$;
- $\mathfrak{M}, w \models_g \varphi_1 \land \varphi_2$ if $\mathfrak{M}, w \models_g \varphi_1$ and $\mathfrak{M}, w \models_g \varphi_2$;
- $\mathfrak{M}, w \models_g \varphi_1 \lor \varphi_2$ if $\mathfrak{M}, w \models_g \varphi_1$ or $\mathfrak{M}, w \models_g \varphi_2$;
- $\mathfrak{M}, w \models_g \varphi_1 \rightarrow \varphi_2$ if $wRw'$ and $\mathfrak{M}, w' \models_g \varphi_1$ imply $\mathfrak{M}, w' \models_g \varphi_2$, for every $w' \in W$;
- $\mathfrak{M}, w \models_g \exists x \varphi_1$ if $\mathfrak{M}, w \models_{g'} \varphi_1$, for some assignment $g'$ that differs from $g$ at most in the value of $x$ and such that $g'(x) \in D(w)$;
- $\mathfrak{M}, w \models_g \forall x \varphi_1$ if $\mathfrak{M}, w' \models_{g'} \varphi_1$, for every $w' \in W$ such that $wRw'$ and every assignment $g'$ such that $g'$ differs from $g$ in at most the value of $x$ and such that $g'(x) \in D(w')$.

Truth in models, frames, and classes of frames is defined as in the modal case. $\text{QInt}$ is the set of formulas true in all frames.
We also consider some logics closely related to $\mathbf{QInt}$. First, $\mathbf{QKC}$ is the quantified counterpart of the propositional logic $\mathbf{KC} = \mathbf{Int} + \neg p \lor \neg \neg p$. Semantically, $\mathbf{QKC}$ is characterized by the frames that satisfy the (convergence) condition that $wRv_1$ and $wRv_2$ imply the existence of a world $u$ such that $v_1Ru$ and $v_2Ru$.

Second, we consider quantified counterparts of Visser's basic propositional logic $\mathbf{BPL}$ and formal propositional logic $\mathbf{FPL}$ [25]: $\mathbf{BPL}$ and $\mathbf{FPL}$ are logics in the intuitionistic language whose modal companions are $\mathbf{K4}$ and $\mathbf{GL}$—that is, given the Gödel's translation $t$ of the intuitionistic language into the modal one (see, for example, [5], § 3.9), $\mathbf{BPL} = t^{-1}(\mathbf{K4})$ and $\mathbf{FPL} = t^{-1}(\mathbf{GL})$. Therefore, we define their quantified counterparts as logics $\mathbf{QBL} = T^{-1}(\mathbf{QK4})$ and $\mathbf{QFL} = T^{-1}(\mathbf{QGL})$, where $T$ is the extension of $t$ with the following clauses: $T(\exists x \varphi) = \exists x T(\varphi)$; and $T(\forall x_1 \ldots \forall x_n \varphi) = \Box \forall x_1 \ldots \forall x_n T(\varphi)$, where $\varphi$ does not begin with a universal quantifier. To give the semantic account of $\mathbf{QBL}$ and $\mathbf{QFL}$, we use Kripke frames and models as defined for $\mathbf{QInt}$, except that the accessibility relation is now only required to be anti-symmetric and transitive. The relation $M, w \models^g \varphi$ is defined as in the intuitionistic case, with the following modification for the universal quantifiers:

- $M, w \models^g \forall x_1 \ldots \forall x_n \varphi_1$, where $\varphi_1$ does not begin with a universal quantifier, if $M, w' \models^{g'} \varphi_1$, for every $w' \in W$ such that $wRw'$ and every assignment $g'$ such that $g'$ differs from $g$ in at most the values of $x_1, \ldots, x_n$ and such that $g'(x_1), \ldots, g'(x_n) \in D(w')$.

This clause is required to make, in the absence of reflexivity of the accessibility relation, the formula $\forall x \forall y \varphi$ equivalent to the formula $\forall y \forall x \varphi$. Then, $\mathbf{QBL}$ is sound (and complete) with respect to all such frames, while $\mathbf{QFL}$ is sound (but not complete) with respect to the subclass where the converse of the accessibility relation is well-founded (i.e., with respect to the frames of the logic $\mathbf{GL}$). For technical reasons, namely to avoid being distracted with Kripke-completeness, we define the logic $\mathbf{QFL}^{sem}$ as the set of formulas valid in all frames where the converse of the accessibility relation is well-founded; all that matters to us is that $\mathbf{QFL} \subseteq \mathbf{QFL}^{sem}$.

### 3.2. Undecidability Results

We now address the question, raised in [12], of whether it suffices to use only monadic predicate letters to obtain undecidability of the two-variable fragment $\mathbf{QInt}(2)$ of $\mathbf{QInt}$. We show that, in fact, it suffices to use a single monadic predicate letter to obtain undecidability of $\mathbf{QInt}(2)$. We do so by
suitably adapting the technique used in [21] to (polynomially) reduce satisfiability in propositional intuitionistic logic \( \text{Int} \) to satisfiability in the fragment of \( \text{Int} \) with only two propositional variables. As the technique from [21] requires that we work with positive formulas, we first show that the positive monadic fragment of \( \text{QInt}(2) \) is undecidable. We note here that transitioning from the propositional language to the first-order one, we “strengthen” the result from [21] in the following sense: while in the propositional case, (the positive fragment of) \( \text{Int} \) is polynomially reducible to its two-variable subfragment, in the first-order case, we (polynomially) reduce (the positive fragment of) \( \text{QInt}(2) \) to its subfragment containing a single predicate letter.\(^1\)

Working with the positive fragment of \( \text{QInt} \) also allows us to extend our results to the interval \([\text{QInt}, \text{QKC}]\), as all logics in this interval share the positive fragment. Moreover, a modification of this construction allows us to obtain analogous results for logics in \([\text{QBL}, \text{QFL}]\).

It is proven in [12] that \( \text{QInt}(2) \) is undecidable by reducing the following undecidable tiling problem [2] to the complement of \( \text{QInt}(2) \): given a finite set \( T \) of tile types that are tuples of colours \( t = (\text{left}(t), \text{right}(t), \text{up}(t), \text{down}(t)) \), decide whether \( T \) tiles the grid \( \mathbb{N} \times \mathbb{N} \) in the sense that there exists a function \( \tau : \mathbb{N} \times \mathbb{N} \to T \) such that, for every \( i, j \in \mathbb{N} \), we have \( \text{up}(\tau(i, j)) = \text{down}(\tau(i, j + 1)) \) and \( \text{right}(\tau(i, j)) = \text{left}(\tau(i + 1, j)) \). The results in this section build on this proof.

We start off by proving that the positive fragment of \( \text{QInt}(2) \) containing two binary and an unlimited number of unary predicate letters, as well as two propositional variables, is undecidable. This is achieved by eliminating the constant \( \bot \) from the formulas used in the proof of undecidability of \( \text{QInt}(2) \) from [12]. For most formulas from [12], all we do is replace \( \bot \) with a propositional variable \( q \). The resultant formulas are listed below for the reader’s convenience; for ease of reference, we preserve the numbering from [12]:

\[
\forall x \bigvee_{t \in T} \big( P_t(x) \land \bigwedge_{t' \neq t} (P_{t'}(x) \to q) \big), \quad (1)
\]

\[
\bigwedge_{\text{right}(t) \neq \text{left}(t')} \forall x \forall y \big( H(x, y) \land P_t(x) \land P_{t'}(y) \to q \big), \quad (2)
\]

\[
\bigwedge_{\text{up}(t) \neq \text{down}(t')} \forall x \forall y \big( V(x, y) \land P_t(x) \land P_{t'}(y) \to q \big), \quad (3)
\]

\[
\forall x \exists y \big( H(x, y) \land \forall x \exists y \big( V(x, y) \big) \big), \quad (4)
\]

\(^1\)In light of [19], the reduction of \( \text{Int} \) to its single-variable fragment would imply that the complexity classes \( \text{P} \) and \( \text{PSPACE} \) are equivalent.
∀x∀y(V(x,y) ∨ (V(x,y) → q)),
∀x∀y[V(x,y) ∧ ∃x(Δ(x) ∧ H(y,x)) → ∀y(H(x,y) → ∀x(Δ(x) → V(y,x)))]  \tag{6}
\]

Let ψ+T be the conjunction of formulas (1) through (6). Then, define

\[ \varphi^+_T = \psi^+_T \rightarrow ((\exists x(Δ(x) → q)) → p) → p), \]

where p is a propositional variable distinct from q.

\textbf{Lemma 3.1.} \( \varphi_T^+ \notin \mathbb{QInt}(2) \) if, and only if, T tiles \( \mathbb{N} \times \mathbb{N} \).

\textbf{Proof.} The proof is a minor modification of the proof of Theorem 1 from [12], with q essentially playing the role that “falsehood” plays in [12].

For the left to right direction, we observe that, given a model \( \mathcal{M} \) and a world \( w \) such that \( \mathcal{M},w \not\models \varphi_T^+ \), as well as an arbitrary \( d \in D(w) \), there exists a world \( u \) in \( \mathcal{M} \) with \( wRu \) such that \( \mathcal{M},u \models Δ[d] \) and \( \mathcal{M},u \not\models q \). This is a straightforward consequence of the fact that \( \mathcal{M},w \not\models (\exists x(Δ(x) → q)) → p → p \).

Given this, the argument from [12] applies.

For the other direction, the model falsifying \( \varphi_T^+ \) is different from the one used in [12] only in the evaluation of p and q. Thus, we use the same frame and interpretation of predicate letters as in [12], and additionally make q false at every world of the model and make p false at \( w_0 \) and true at every other world.

Since \( \varphi_T^+ \) is a positive formula, this immediately gives us the following:

\textbf{Corollary 3.2.} The positive fragment of \( \mathbb{QInt} \) with two individual variables, two binary predicate letters, an unlimited number of unary predicate letters, and two propositional variables is undecidable.

We now show how, drawing on an idea of Kripke’s for modal logics [14], one can, in the positive fragment of \( \mathbb{QInt} \), simulate binary predicate letters using monadic predicate letters and propositional variables. As this does not increase the number of individual variables in a formula, it will allow us to eliminate binary predicate letters from the formula \( \varphi_T^+ \).

\textbf{Lemma 3.3.} Let \( χ \) be a positive formula in \( \mathbb{QInt} \) containing an occurrence of a binary predicate letter \( Q \), and let \( Q_1 \) and \( Q_2 \) be unary predicate letters, and \( r \) and \( s \) be propositional variables, not occurring in \( χ \). Let \( χ' \) be the result of uniformly replacing every subformula of \( χ \) of the form \( Q(x,y) \) with \( (Q_1(x) \land Q_2(y) \rightarrow r) \lor s \). Then, \( χ \in \mathbb{QInt} \) if, and only if, \( χ' \in \mathbb{QInt} \).

\textbf{Proof.} The left-to-right direction follows from the closure of \( \mathbb{QInt} \) under substitution. For the other direction, assume that there exist \( \mathcal{M} = (W,R,Δ,I) \) and \( w_0 \in W \) such that \( \mathcal{M},w_0 \not\models χ \). We modify \( \mathcal{M} \) to
obtain a model $\mathcal{M}'$ falsifying $\chi'$ as follows. For every $w \in W$ and every $a, b \in D(w)$ such that $\mathcal{M}, w \not\models Q[a, b]$, add to $W$ a world $w_{a, b}$ with $wR'w_{a, b}$ and let

$$
\begin{align*}
\mathcal{M}', w_{a, b} &\not\models r; \\
\mathcal{M}', w_{a, b} &\models s; \\
\mathcal{M}', w_{a, b} &\models Q_1[d] \iff d = a; \\
\mathcal{M}', w_{a, b} &\models Q_2[d] \iff d = b;
\end{align*}
$$

and let all the predicate letters different from $Q_1$ and $Q_2$ and occurring in $\chi'$ be universally true at every such world; likewise for propositional variables different from $r$ and $s$. Also, let $\mathcal{M}', w \not\models s$.

Then we can show that $\mathcal{M}, w \models \theta[a_1, \ldots, a_m]$ if, and only if, $\mathcal{M}', w \models \theta'[a_1, \ldots, a_m]$, for every subformula $\theta$ of $\chi$, every $w \in W$, and every $a_1, \ldots, a_m \in D(w)$, where $\theta'$ is the result of substituting in $\theta$ every occurrence of $Q(x, y)$ with $(Q_1(x) \land Q_2(y) \rightarrow r) \lor s$. The proof is by induction on $\theta$.

For the base case, first note that if $\mathcal{M}, w \not\models Q[a, b]$, then the presence in $\mathcal{M}'$ of the world $w_{a, b}$ guarantees that $\mathcal{M}', w \not\models (Q_1[a] \land Q_2[b] \rightarrow r) \lor s$; on the other hand, if $\mathcal{M}, w \models Q[a, b]$, then $\mathcal{M}', w \models (Q_1[a] \land Q_2[b] \rightarrow r) \lor s$, as $\mathcal{M}, u \not\models Q_1[a]$ or $\mathcal{M}, u \not\models Q_2[b]$, for every $u$ with $wR'u$.

The cases for $\theta = \theta_1 \lor \theta_2$, $\theta = \theta_1 \land \theta_2$, and $\theta = \exists x \theta_1$ are straightforward.

Let $\theta = \theta_1 \rightarrow \theta_2$. Assume that $\mathcal{M}', w \not\models \theta'[a_1, \ldots, a_m]$. Then, $\mathcal{M}', u \models \theta'_1[a_1, \ldots, a_m]$ and $\mathcal{M}', u \not\models \theta'_2[a_1, \ldots, a_m]$, for some $u \in W$ with $wR'u$. If we could apply the inductive hypothesis to $u$, we would be done. To see that we can, notice that $\theta'_2$ is build out of atomic formulas and the formula $(Q_1(x) \land Q_2(y) \rightarrow r) \lor s$, all of which are true under every assignment in every $w' \in W' - W$, using only $\land$, $\lor$, $\exists$, and $\forall$. Therefore, $\theta'_2$ is true in every $w' \in W' - W$ under every assignment; hence, $u \in W$ and the inductive hypothesis is, therefore, applicable. Thus, $\mathcal{M}, w \not\models \theta[a_1, \ldots, a_m]$.

The other direction is straightforward.

The case $\theta = \forall x \theta_1$ is similarly argued.

Now, let $\xi^+_T$ be the result of replacing in $\varphi^+_T$ of

$$
\begin{align*}
H(x, y) \text{ with } (H_1(x) \land H_2(y) \rightarrow r_1) \lor s_1; \\
V(x, y) \text{ with } (V_1(x) \land V_2(y) \rightarrow r_2) \lor s_2.
\end{align*}
$$

In view of Lemma 3.3, $\xi^+_T \not\in \text{QInt}(2)$ if, and only if, $T$ tiles $\mathbb{N} \times \mathbb{N}$. As we can replace $\xi^+_T$ a propositional variable such as $q$ with, say, $\exists x Q(x)$, we immediately obtain the following:
Theorem 3.4. The positive monadic fragment of $\mathbf{QInt}$ with two individual variables is undecidable.

We now embed the positive monadic fragment of $\mathbf{QInt}(2)$ into its subfragment containing formulas with only one monadic predicate letter, suitably adapting the technique from [21]. As this embedding does not introduce any fresh variables, our main result in this section immediately follows.

We start by defining the frame $\mathcal{F} = \langle W, R \rangle$. This frame, depicted in Figure 1, is made up of “levels” of worlds. The three top-most levels are depicted at the top of Figure 1: the top-most level contains worlds $d_1$, $d_2$, and $d_3$; level 0, worlds $a_0^1$, $a_0^2$, $b_0^1$, and $b_0^2$; level 1, worlds $a_1^1$, $a_1^2$, $a_1^3$, $b_1^1$, $b_1^2$, and $b_1^3$. The successive levels are defined inductively. Assume that level $k$ has been defined and that it contains worlds $a_k^1, \ldots, a_k^n, b_k^1, \ldots, b_k^n$. For every $i, j \in \{2, \ldots, n\}$, the level $k + 1$ contains the world $a_{k+1}^i$ such that $a_{m+1}^i R b_k^j$, $a_{m+1}^i R a_i^k$, and $a_{m+1}^i R b_j^k$, as well as the world $b_{m+1}^i$ such that $b_{m+1}^i R a_1^k$, $b_{m+1}^i R a_i^k$, and $b_{m+1}^i R b_j^k$. Let $\mathcal{M}$ be a model with constant domains, say $Z$, based on $\mathcal{F}$ (without a loss of generality, we can assume that $Z$ contains at least three individuals) and let $a \in Z$. We say that $\mathcal{M}$ is $a$-suitable if, for some $b \in Z$ such that $b \neq a$, the following hold: $I(d_2, P) = \{\langle c \rangle : c \in Z \text{ and } c \neq a\}$; $I(d_3, P) = \{\langle a \rangle, \langle b \rangle\}$; $I(b_0^1, P) = \{\langle b \rangle\}$; $I(w, P) = \emptyset$, for $w \notin \{d_2, d_3, b_0^1\}$.

We now define formulas, of one free variable $x$, that correspond to the worlds of an $a$-suitable model in the sense that each formula fails at a world $w$ of the model, with $a$ assigned to $x$, exactly when $w$ can see the world...
corresponding to the formula. For these formulas, we use notation that makes clear which worlds they correspond to; thus, formula $D_i$ corresponds to world $d_i$, $A_i^k$ to $a_i^k$, and $B_i^k$ to $b_i^k$. First, we define formulas for the three top-most levels:

\[
\begin{align*}
D_1 &= \exists x \ P(x), \\
D_2(x) &= \exists x \ P(x) \rightarrow \ P(x), \\
D_3(x) &= P(x) \rightarrow \forall x \ P(x),
\end{align*}
\]

\[
\begin{align*}
A_1^0(x) &= A_0^0(x) \land A_0^0(x) \rightarrow B_1^0(x) \lor B_2^0(x), \\
A_2^0(x) &= A_1^0(x) \land B_1^0(x) \rightarrow A_2^0(x) \lor B_2^0(x), \\
A_3^0(x) &= A_1^0(x) \land B_2^0(x) \rightarrow A_2^0(x) \lor B_1^0(x), \\
B_1^0(x) &= D_2(x) \rightarrow D_1 \lor D_3(x), \\
B_2^0(x) &= D_3(x) \rightarrow D_1 \lor D_2(x), \\
B_3^0(x) &= D_1 \rightarrow D_2(x) \lor D_3(x),
\end{align*}
\]

Now, assume that the formulas for level $k$ have been defined and define

\[
\begin{align*}
A_{m}^{k+1}(x) &= A_{m}^{k}(x) \rightarrow B_{1}^{k}(x) \lor A_{j}^{k}(x) \lor B_{j}^{k}(x), \\
B_{m}^{k+1}(x) &= B_{1}^{k}(x) \rightarrow A_{i}^{k}(x) \lor A_{j}^{k}(x) \lor B_{j}^{k}(x),
\end{align*}
\]

where $m$ is uniquely determined for every pair $i, j \in \{2, \ldots, n_k\}$, where $n_k$ is the maximal index for formulas of level $k$.

**Lemma 3.5.** Let $\mathcal{M} = \langle W, R, D, I \rangle$ be an $a$-suitable model and let $w \in W$. Then,

\[
\mathcal{M}, w \not\models A_{m}^{k}[a] \iff wRa_{m}^{k}, \text{ and } \mathcal{M}, w \not\models B_{m}^{k}[a] \iff wRb_{m}^{k}.
\]

**Proof.** Induction on $k$. $\blacksquare$

Now, let $\varphi$ be a positive formula containing monadic predicate letters $P_1, \ldots, P_n$ (we may assume $n \geq 2$). For each $i \in \{1, \ldots, n\}$, define

\[
\alpha_i(x) = A_i^n(x) \lor B_i^n(x).
\]

Finally, let $\varphi^*$ be the result of substituting, for every $i \in \{1, \ldots, n\}$, of $\alpha_i(x)$ for $P_i(x)$ into $\varphi$.

**Lemma 3.6.** $\varphi \in \boldsymbol{QInt}$ if, and only if, $\varphi^* \in \boldsymbol{QInt}$. 

**Proof.** The right-to-left direction follows from the closure of $\boldsymbol{QInt}$ under predicate substitution. For the other direction, assume that $\mathcal{M}_\varphi, w_0 \not\models \varphi$ for some $\mathcal{M}_\varphi = \langle W_\varphi, R_\varphi, D_\varphi, I_\varphi \rangle$ and $w_0 \in W_\varphi$. (We may assume without a loss of generality that the domain of $\mathcal{M}_\varphi$ contains at least three individuals;
we use this fact in the construction of $\mathcal{M}^*$ below.) We need to construct a model $\mathcal{M}^*$ falsifying $\varphi^*$ at some world.

First, for every $w \in W_\varphi$ and $a \in D_\varphi(w)$, consider an $a$-suitable model $\mathcal{M}_a^w = \langle W_a^w, R_a^w, D_a^w, I_a^w \rangle$, based on a copy of the frame $\mathcal{F}$ defined above, where $D_a^w(u) = D_\varphi(w)$, for every $u \in W_a^w$. To obtain the frame $\mathcal{F}^*$, first, append to $\mathcal{F}_\varphi = \langle W_\varphi, R_\varphi \rangle$, for every $w \in W_\varphi$ and $a \in D_\varphi(w)$, frames of all such models $\mathcal{M}_a^w$; in addition, let $wR^*a_i^n$ and $wR^*b_i^n$, for $a_i^n$ and $b_i^n$ belonging to $\mathcal{M}_a^w$, exactly when $\mathcal{M}_\varphi, w \not\models P_i[a]$, for $i \in \{1, \ldots, n\}$. Define $D^*$ to agree with $D_\varphi$ on $W_\varphi$ and to agree with $D_a^w$ on $W_a^w$, for every $w \in W_\varphi$ and $a \in D_\varphi(w)$. To finish off the definition of the model $\mathcal{M}^* = \langle W^*, R^*, D^*, I^* \rangle$, define $I^*(u, P)$ to agree with $I_a^w(u, P)$ at the worlds in the appended models and to be $\emptyset$ at the worlds from $W_\varphi$. We can now show that $\mathcal{M}_\varphi, w \models \psi[a_1, \ldots, a_m]$, if and only if, $\mathcal{M}^*, w \models \psi^*[a_1, \ldots, a_m]$, for every $w \in W_\varphi$ and every subformula $\psi$ of $\varphi$.

The proof proceeds by induction on $\psi$. The only case we explicitly consider here is $\psi = \psi_1 \rightarrow \psi_2$, leaving the rest to the reader. Assume $\mathcal{M}^*, w \not\models \psi^*[a_1, \ldots, a_m]$. Then, $\mathcal{M}^*, u \models \psi^*[a_1, \ldots, a_m]$ and $\mathcal{M}^*, u \not\models \psi^*_2[a_1, \ldots, a_m]$, for some $u \in W^*$ with $wR^*u$. If we could apply the inductive hypothesis to $u$, we would be done. To see that we can, notice that $\psi^*_2$ is built out of formulas of the form $A_i^n(x) \lor B_i^n(x)$ using only $\land, \lor, \rightarrow, \exists$, and $\forall$. As, in view of Lemma 3.5, formulas $A_i^n(x) \lor B_i^n(x)$ are true at every world of $\mathcal{M}^*$ that lies outside of $W_\varphi$ and is accessible from $W_\varphi$, we conclude that $u \in W_\varphi$, and the inductive hypothesis is, therefore, applicable. Thus, $\mathcal{M}_\varphi, w \not\models \psi[a_1, \ldots, a_m]$. The other direction is straightforward.

We conclude that $\mathcal{M}^*, w_0 \not\models \varphi^*$ and, thus, $\varphi^* \not\in \text{QInt}$. 

As the construction of $\varphi^*$ from $\varphi$ did not introduce any fresh individual variables, we have the following:

**Theorem 3.7**. The positive fragment of $\text{QInt}$ with two individual variables and a single predicate letter is undecidable.

We now extend the argument presented above to the logics in the intervals $[\text{QBL}, \text{QKC}]$ and $[\text{QBL}, \text{QFL}]$.

First, to establish the undecidability of the two-variable fragments of logics whose semantics might contain irreflexive worlds, we need to slightly modify formulas (1) through (6) listed above. Therefore, we define $\psi^*_T$ to be the conjunction of $\psi^*_T$ and the following formula:

$$\forall x \forall y (H(x, y) \lor (H(x, y) \rightarrow q)), \quad (5a)$$
and define
\[ \varphi^*_T = \psi^*_T \rightarrow [(\exists x (D(x) \rightarrow \Box^5 q) \rightarrow p) \rightarrow \Box p]. \]

This enables us to prove, using the tiling problem described above, that \( T \) tiles \( \mathbb{N} \times \mathbb{N} \) if and only if \( \varphi^*_T \not\in L(2) \), where \( L \in \{QBL, QFL^{sem}\} \). We leave the details of the proof to the reader. As the construction of \( \varphi^*_T \) is uniform for both logics, it follows that the claim holds for every \( L \in \{QBL, QFL^{sem}\} \). Notice that the same proof also works for logics in \( \{QBL, QKC\} \). We simulate binary predicate letters by monadic ones as for \( QInt \). We now show how to simulate all monadic predicate letters with a single one.

For the interval \([QBL, QKC]\), notice that if we add to the model \( M^* \) built in the proof of Lemma 3.6 a world \( d \) accessible from every element of \( W^* \) and such that \( I^*(d, P) = D(d) \), the resultant model is a model of every logic in the interval \( \{QBL, QKC\} \). Thus, we have the following:

**Theorem 3.8.** Let \( L \) be a logic in the interval \([QBL, QKC]\). Then, the positive fragment of \( L \) with two individual variables and a single predicate letter is undecidable.

We next consider the interval \([QBL, QFL^{sem}]\). In this case, we need to make a more substantial modification to the frame \( \mathcal{F} \), as the semantics of \( QFL^{sem} \) prohibits the existence of reflexive worlds. We then proceed as follows. First, add to \( W \) worlds \( \bar{d}_2 \) and \( \bar{d}_3 \) with \( d_2R\bar{d}_2 \) and \( d_3R\bar{d}_3 \). Second, for every \( k \geq 0 \), do the following: for every world \( a_i^k \), add to \( W \) the world \( \bar{a}_i^k \) and, for every world \( b_i^k \), add to \( W \) the world \( \bar{b}_i^k \), also, let \( a_i^kRa_i^k \) and \( b_i^kR\bar{b}_i^k \), for every \( k \) and \( i \). Lastly, whenever in \( \mathcal{F} \) we had \( a_i^{k+1}Ra_j^k \) or \( a_i^{k+1}R\bar{b}_j^k \), let \( a_i^{k+1}Ra_j^k \) and \( \bar{a}_i^{k+1}R\bar{b}_j^k \); also, whenever we had \( b_i^{k+1}Ra_j^k \) or \( b_i^{k+1}R\bar{b}_j^k \), let \( b_i^{k+1}Ra_j^k \) and \( \bar{b}_i^{k+1}R\bar{b}_j^k \). We then define \( a \)-suitable models so that
\[ I(\bar{d}_2, P) = I(d_2, P), \quad I(\bar{d}_3, P) = I(d_3, P), \]
and for every \( k \) and \( i \),
\[ I(\bar{a}_i^k, P) = I(a_i^k, P) \]
and
\[ I(\bar{b}_i^k, P) = I(b_i^k, P). \]
In essence, we created “doubles” for the worlds \( d_2 \), \( d_3 \), \( a_i^k \), and \( b_i^k \), which serve to evaluate formulas whose main connective is \( \rightarrow \) or \( \forall \) at the worlds whose doubles they are. Then, \( a \)-suitable models satisfy the condition in the statement of Lemma 3.5, and the model \( M^* \) built in the proof of Lemma 3.6 becomes a model of every logic in \( \{QBL, QFL^{sem}\} \). As \( QFL \subseteq QFL^{sem} \), we have the following:

**Theorem 3.9.** Let \( L \) be a logic in the interval \([QBL, QFL]\). Then, the positive fragment of \( L \) with two individual variables and a single predicate letter is undecidable.

**Remark 3.10.** Note that the results of this section hold true if we only consider frames with constant domains.
4. Discussion

As already noticed, the results presented in the present paper concerning sublogics of QGL and QGrz are quite tight; as shown in [26], for all “natural” sublogics of QGL and QGrz—including QK, QT, QD, QK₄, QS₄, QGL, and QGrz—adding to the restriction to two individual variables and a single monadic predicate letter considered in Section 2 a minor restriction that the modal operators only apply to formulas with at most one free variable, results in decidable fragments of those logics. It is not difficult to notice that the results analogous to those obtained in Section 2 can be obtained for quasi-normal logics such as QS (Solovay’s logic) and Lewis’s QS₁, QS₂, and QS₃ [7].

A notable exception in our consideration of modal logics is QS₅, whose two-variable monadic fragment was shown to be undecidable in [12]. While it is not difficult to extend our results to the multimodal version of QS₅—we need to modify the construction used for sublogics QKTB by substituting a succession of two steps along distinct accessibility relations for a single step along a single accessibility relation in the frames of a-suitable models—nor is it difficult to show, by encoding the tiling problem used in [12], that the two-variable fragment of QS₅ with two monadic predicate letters and infinitely many propositional symbols is undecidable, the case of QS₅ remains elusive.

We conjecture that the fragment of QS₅ with two variables and a single monadic predicate letter is decidable.

On the other hand, it is relatively straightforward to show that the two-variable fragment of QS₅ with a single monadic predicate letter and an infinite supply of individual variables is undecidable. Indeed, let SIB be the first-order theory of a symmetric irreflexive binary relation S; it is well-known that SIB is undecidable [13,18]. We can then simulate S(x, y) as □(¬P(x) ∨ ¬P(y)) and show that, if a quantified modal logic L is valid on a frame containing a world that can see infinitely many worlds, then L is undecidable in the language with a single monadic predicate letter (and infinitely many individual variables). This observation covers all modal logics considered in [12], but not covered by the results of Section 2, including QS₅, QGL₃, and QGrz₃.

By contrast, we can say nothing about superintuitionistic logics not included in the interval [QInt, QKC], as our proof relies on the fact that we are working with the positive fragment of those logics. It is not essential to our proof that formulas $A^k(x)$ and $B^k_i(x)$ be positive; however, by discarding their positivity we would weaken, rather than strengthen, our results.
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