A graph is called 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, we establish a local property of 1-planar graphs which describes the structure in the neighborhood of small vertices (i.e. vertices of degree no more than seven). Meanwhile, some new classes of light graphs in 1-planar graphs with the bounded degree are found. Therefore, two open problems presented by Fabrici and Madaras [The structure of 1-planar graphs, Discrete Mathematics, 307, (2007), 854–865] are solved. Furthermore, we prove that each 1-planar graph $G$ with maximum degree $\Delta(G)$ is acyclically edge $L$-choosable where $L = \max\{2\Delta(G) - 2, \Delta(G) + 83\}$.

Keywords: 1-planar graph; light graph; acyclic edge coloring.

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1. Introduction

In this paper, all graphs considered are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph $G$, respectively. Denote $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Let $d_G(v)$ (or $d(v)$ for simple) denote the degree of vertex $v \in V(G)$. A $k$-, $k^+$- and $k^-$-vertex is a vertex of degree $k$, at least $k$ and at most $k$, respectively. Any undefined notation follows that of Bondy and Murty [3].

A graph $G$ is 1-immersed into a surface if it can be drawn on the surface so that each edge is crossed by at most one other edge. In particular, a graph is 1-planar if it is 1-immersed into the plane (i.e. has a plane 1-immersion). The notion of 1-planar-graph was introduced by Ringel [10] in the connection with problem of the simultaneous coloring of adjacent/incidence of vertices and faces of plane graphs. Ringel conjectured that each 1-planar graph is 6-vertex colorable, which was confirmed by Borodin [4]. Recently, Albertson and Mohar [1] investigated the list vertex coloring of graphs which can be 1-immersed into a surface with positive genus. Borodin, et al. [5] considered the acyclic vertex coloring of 1-planar graphs and proved that each 1-planar graph is acyclically 20-vertex colorable. The structure of 1-planar graphs was studied in [6] by Fabrici and Madaras. They showed that the number of edges in a 1-planar graph $G$ is bounded by $4v(G) - 8$. This implies every 1-planar graph contains a vertex of degree at most 7. Furthermore, the bound 7 is the best possible because of the existence of a 7-regular 1-planar graph (see Fig.1 in [6]). In the same paper, they also derived the analogy of Kotzig theorem on light edges; it was proved that each 3-connected 1-planar graph $G$ contains an edge such that its endvertices are of degree at most 20 in $G$; the bound 20 is the best possible.

The aim of this paper is to exhibit a detailed structure of 1-planar graphs which generalizes the result that every 1-planar graph contains a vertex of degree at most 7 in Section 2. By using this structure, we answer two questions on light graphs posed by Fabrici and Madaras [6] in Section 3 and give a linear upper bound of acyclic edge chromatic number of 1-planar graphs in Section 4.

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2. Local structure of 1-planar graphs

To begin with, we introduce some basic definitions. Let $G$ be a 1-planar graph. In the following, we always assume $G$ has been drawn on a plane so that every edge is crossed by at most one another edge and the number of crossings is as small as possible (such a dawning is called to be proper). So for each pair of edges $x_1y_1, x_2y_2$ that cross each other at a crossing point $z$, their end vertices are pairwise distinct. Let $C(G)$ be the set of all crossing points and let $E_0(G)$ be the non-crossed edges in $G$. Then the associated plane graph $G^\times$ of $G$ is the plane graph such that $V(G^\times) = V(G) \cup C(G)$ and $E(G^\times) = E_0(G) \cup \{xzu, yzu \in E(G) \setminus E_0(G) \text{ and } z \text{ is the crossing point on } xy\}$. Thus the crossing points in $G$ become the real vertices in $G^\times$ all having degree four. For convenience, we still call the new vertices in $G^\times$ crossing vertices and use the notion $C(G^\times)$ to denote the set of crossing vertices in $G^\times$. A simple graph $G$ is triangulated if every cycle of length $\geq 3$ has an edge joining two nonadjacent vertices of the cycle. We say $GT$ is a canonical triangulation of a 1-planar graph $G$ if $GT$ is obtained from $G$ by the following operations.

Step 1. For each pair of edges $ab, cd$ that cross each other at a point $s$, add edges $ac, cb, bd$ and $da$ "close to $s"$, i.e. so that they form triangles $asc, csb, bsd$ and $dsa$ with empty interiors.

Step 2. Delete all multiple edges.

Step 3. If there are two edges that cross each other then delete one of them.

Step 4. Triangulate the planar graph obtained after the operation in Step 3 in any way.

Step 5. Add back the edges deleted in Step 3.

Note that the associated planar graph $G^\times_T$ of $GT$ is a special triangulation of $G^\times$ such that each crossing vertex remains to be of degree four. Also, each vertex $v$ in $G^\times_T$ is incident with just $d_{G^\times_T}(v)$ 3-faces. Denote $v_1, \ldots, v_d$ to be the neighbors of $v$ in $G^\times_T$ (in a cyclic order) and use the notations $v_i^+ = v_{i+1}$, $v_i^- = v_{i-1}$, where $d = d_{G^\times_T}(v)$ and $i$ is taken modulo $d$.

In the following, we use $c(v)$ to denote the number of crossing vertices which are adjacent to $v$ in $G^\times_T$. Then we have the following observations. Since their proofs of them are trivial, we omit them here. In particular, the second observation uses the facts that $GT$ admits no multiple edge and the drawing of $GT$ minimizes the number of crossing.

Observation 1. For a canonical triangulation $G_T$ of a 1-planar simple graph $G$, we have

1. Any two crossing vertices are not adjacent in $G^\times_T$.
2. If $d_{G_T}(v) = 3$, then $c(v) = 0$.
3. If $d_{G_T}(v) = 4$, then $c(v) \leq 1$.
4. If $d_{G_T}(v) \geq 5$, then $c(v) \leq \frac{d_{G_T}(v)}{2}$.

Let $v \in V(G_T)$ and $u$ be a crossing vertex in $G^\times_T$ such that $wu \in E(G^\times_T)$. Then by the definitions of $u^+$ and $u^-$, we have $vu^+, vu^- \in E(G^\times_T)$. Furthermore, the path $u^-uu^+$ in $G^\times_T$ corresponds to the original edge $u^-u^+$ with a crossing point $u$ in $G_T$. Let $w$ be the neighbor of $v$ in $G_T$ so that $vw$ crosses $u^-u^+$ at $u$ in $G_T$. By the definition of $G^\times_T$, we have $wu^-, uu^+ \in E(G^\times_T)$. We call $w$ the mirror-neighbor of $v$ in $G_T$ and $u^-, uu^+$ the image-neighbors of $v$ in $G_T$. Other neighbors of $v$ in $G_T$ are called normal-neighbors. Sometimes when we say mirror vertex, image vertex and normal vertex, we refer to mirror neighbor, image neighbor and normal neighbor of $v$. The triangle $u^-uu^+$ in $G_T$ is called the mirror-triangle incident with $v$. Since the neighbors of $v$ in $G^\times_T$ can be listed in a cyclic order, via replacing the crossing vertex by the mirror vertex incident with it, the neighbors of $v$ in $G_T$ can be also listed in a cyclic order. Note that different crossing vertices are adjacent to different mirror vertices since multiple edges are forbidden in $G$. Let $v_1, \ldots, v_{d_{G_T}(v)}$ be the neighbors of $v$ in $G_T$ in a cyclic order. We define $\Omega(v_1v_j) = \{v_i, v_{i+1}, \ldots, v_{j-1}, v_j\}$, $v_i = v_{i+1}$ and $v_i = v_{i-1}$, where $i$ is taken modulo $d_{G_T}(v)$. Note that $G_T$ is a canonical triangulation of $G$. Then $v_1v_2 \cdots v_{d_{G_T}(v)}v_1$ is a cycle which is called the associated cycle of $v$, denoted by $C$. We call the
the vertex set this cyclic order). Denote \( d \) are of degree not to distinguish that

Let \( \Delta(G) \), \( k \) has been draw on a plane properly. In other words, we always assume shall also be a counterexample to the theorem. Hence in the following, without loss of generality, in the next, there is no necessity to distinguish the two notions neighbors.

Figure 2.1. Some definitions on a canonical triangulation \( G_T \) of a 1-planar graph \( G \)

path \( P_i = v_1^i v_2^i \cdots v_{2t_i-k+1}^i \) a segment of \( C \) if (a) the elements of \( \bigcup_{k=0}^{t_i} \{v_{2k+1}^i\} \) are image neighbors of \( v \); (b) the elements of \( \bigcup_{k=1}^{t_i} \{v_{2k}^i\} \) are mirror neighbors of \( v \); (c) the triangles in \( G_T \) of the form \( v_{2k-1}^i v_{2k}^i v_{2k+1}^i \) where \( 1 \leq k \leq t_i \) are mirror triangles incident with \( v \) and (d) \( v_{2t_i+1}^i \) and \( v_{2t_i+2}^i \) are not mirror neighbors of \( v \). Then scope of a segment \( P_i \) is defined to be the number of mirror triangles incident with \( v \) using vertices in \( V(P_i) \), denoted by \( S(P_i) \). Then we easily have \( S(P_i) = t_i \).

Now we show the main result in this section.

**Theorem 2.** Let \( G \) be a 1-planar simple graph. Then there exists a vertex \( v \) in \( G \) with exactly \( k \) neighbors \( v_1, v_2, \ldots, v_k \) satisfying \( d(v_1) \leq d(v_2) \leq \cdots \leq d(v_k) \) such that one of the following is true.

1. \((C1) k \leq 2,\) \n2. \((C2) k = 3 \) with \( d(v_1) \leq 35,\) \n3. \((C3) k = 4 \) with \( d(v_1) \leq 19 \) and \( d(v_2) \leq 35,\) \n4. \((C4) k = 5 \) with \( d(v_1) \leq 14, \) and \( d(v_2) \leq 19 \) and \( d(v_3) \leq 35,\) \n5. \((C5) k = 6 \) with \( d(v_1) \leq 11, \) and \( d(v_2) \leq 14, \) and \( d(v_3) \leq 19 \) and \( d(v_4) \leq 35,\) \n6. \((C6) k = 7 \) with \( d(v_1) \leq 8, \) and \( d(v_2) \leq 11, \) and \( d(v_3) \leq 14, \) and \( d(v_4) \leq 19 \) and \( d(v_5) \leq 35.\)

**Proof.** The theorem is proved by contradiction. Let \( G \) be a simple 1-planar graph with a fixed embedding in the plane, and suppose \( G \) is a counterexample to the theorem. Note that if we add a new edge \( e \) between two nonadjacent vertices in \( G \) so that \( G + e \) is still 1-planar graph, then \( G + e \) shall also be a counterexample to the theorem. Hence in the following, without loss of generality, we always assume \( G \) is 2-connected and \( G = G_T \), where \( G_T \) is a canonical triangulation of \( G \) that has been draw on a plane properly. In other words, \( G \) is just a canonical triangulation of itself. So in the next, there is no necessity to distinguish the two notions \( G \) and \( G_T \), and when we say \( G^x \), we also refer to \( G_T^x \). On the other hand, by the definition of associated plane graph, one can observe that \( d_{G^x}(v) = d_G(v) \) when \( v \) is not a crossing vertex. So in the detail proof below, we either need not to distinguish \( d_{G^x}(v) \) and \( d_G(v) \) when \( v \) is a vertex of \( G \), in which case we only use the notation \( d(v) \) to represent both \( d_{G^x}(v) \) and \( d_G(v) \).

For a fixed vertex \( v \) of \( G \), we define \( n_i(v) \) \( (n_i \) in short) to be the number of neighbors of \( v \) in \( G \) which are of degree \( i \). For a vertex set \( S \subseteq V(G) \), let \( n_i(S) = \sum_{v \in S} n_i(v) \). Denote \( n_i^+(v) = \sum_{v \in \Delta(G)} n_k(v) \) and \( d = d(v) \). For a subgraph \( H \) of \( G \), \( n_i(H) \) represents the number of \( i \)-vertices contained in \( H \). Let \( C \) be the associated cycle of \( v \). Suppose \( C \) has \( n \) segments, denoted by \( P_1, \ldots, P_n \) (in this cyclic order). Denote \( t = \sum_{k=1}^{n} S(P_k) \). Let \( M_i = \Omega(v_{2k+1}^i v_{2k+2}^i) \) \( (i = 1, 2, \ldots, n) \). We call the vertex set \( \bigcup_{k=1}^{n} M_k \{v_{2k+1}^i, v_{2k+2}^i\} \) the interval of the associated cycle \( C \). For each segment
Let \( P_i \) be a graph \( G_i \) so that \( V(G_i) = V(P_i) \) and \( E(G_i) = \bigcup_{k=1}^{t_i} \{v_{2k-1}^i v_{2k}^i \} \bigcup E(P_i) \). Let \( H_i = G_i \setminus \{v_{2t_i}^i, v_1^i \} \). By the definition of \( P_i \), we have \( H_i \subseteq G_i \subseteq G \). (see Fig. 2.1).

A triangle \( abc \) in \( G \) is light if \( \max \{d(a), d(b), d(c)\} \leq 7 \). Otherwise we say that \( abc \) is heavy. Note that for the vertex \( v \) described above, there are \( t \) mirror triangles incident with \( v \) (recall the definition of the parameter \( t \)). Now, suppose \( (t - x) \) of them are heavy and \( x \) of them are light. We divide all the light mirror triangles incident with \( v \) into three classes.

Class I. Triangles in the form \( abc \) such that \( a \) is mirror vertex and \( b, c \) are image vertices with \( d(a) \leq 5 \) and \( \min \{d(b), d(c)\} \geq 6 \).

Class II. Triangles in the form \( abc \) such that \( a \) is mirror vertex and \( b, c \) are image vertices with \( \min \{d(b), d(c)\} \leq 5 \).

Class III. Triangles in the form \( abc \) such that \( \min \{d(a), d(b), d(c)\} \geq 6 \).

Denote the number of triangles belonging to Class I, II and III by \( N_i \), \( N_j \) and \( N_k \) respectively.

Claim 1. \( n_8^+ \geq \lceil \frac{t-x}{2} \rceil + \lceil \frac{x}{2} \rceil \).

Since each heavy mirror triangle incident with \( v \) covers at least one 8-vertex, there are at least \( \lceil \frac{t-x}{2} \rceil \) 8-vertices contained in heavy mirror triangles. And this lower bound is reachable only if each heavy mirror triangle covers exactly one 8-vertex such that each pair of incident mirror triangles share one common 8-vertex.

For each Class II light mirror triangle \( abc \) such that \( a \) is mirror vertex and \( b, c \) are image vertices with \( d(b) \leq 5 \), since (C1)-(C4) are forbidden in \( G \), we have \( d(b) = 5 \) and another three neighbors of \( b \) are all 36-vertices. Let \( C \) be the associated cycle of \( v \). Then \( 1 \leq |N_C(b) - \{a, c\}| \leq 2 \). If \( |N_C(b) - \{a, c\}| = 1 \), let \( p \in N_C(b) - \{a, c\} \). Then \( p \) is a normal vertex. So \( p \) could be incident with at most two image vertices of degree no more than five. If \( |N_C(b) - \{a, c\}| = 2 \), let \( N_C(b) - \{a, c\} = \{p, q\} \). Then \( bpq \) is a heavy mirror triangle with two 8-vertices. In either case, we would account at least \( \lceil \frac{x}{2} \rceil \) new 8-vertices which are not counted in the above step.

Hence, we have \( n_8^+ \geq \lceil \frac{t-x}{2} \rceil + \lceil \frac{x}{2} \rceil \).

Claim 2. There is an integer \( \mu \geq 0 \) such that \( n_6 + n_7 = x + x_2 - j + \mu \).

Since each Class I light mirror triangle contains two vertices either of degree 6 or of degree 7 and each Class III light mirror triangle contains three vertices either of degree 6 or of degree 7, we deduce that \( n_6 + n_7 \geq i + 2x_2 = x + x_2 - j \).

Claim 3. \( n_5 \leq d - \frac{t+x-j}{2} - x_2 - \mu - n_3 - n_4 \).

Since (C1) is forbidden, we have \( \delta(G) \geq 3 \) and \( n_5 = d - n_3 - n_4 - n_6 - n_7 - n_8^+ \). Note that \( x + x_2 = x \). We deduce from Claim 1 and Claim 2 that \( n_5 \leq d + \frac{t+x-j}{2} - x_2 - \mu - n_3 - n_4 \).

Claim 4. \( 2n_3 + n_4 + t + x \leq d \).

Recall the definitions of \( G_i, H_i \) and \( M_i \) where \( i = 1, 2, \ldots, n \). Each \( G_i \) contains \( t_i \) mirror triangles incident with \( v \). Suppose \( \alpha_i \) of them are light and \( \beta_i \) of them are heavy. Then \( \alpha_i + \beta_i = t_i \). Since (C3) is forbidden, no light mirror triangle contains 4-vertex. So the neighbors of \( v \) in \( G \) with degree 4 are all contained either in the heavy mirror triangles or in the interval of \( C \). Note that all the image vertices contained in \( H_i \) are of degree at least five (see Figure 2.1), so if there is a 4-vertex in \( H_i \), then it must be a mirror vertex. In view of this, one can easily claim that \( H_i \) contains at most \( \beta_i \) 4-vertices. Furthermore, if \( |M_i| = 2 \), \( d(v_1^i + 1) = 4 \) and \( d(v_2^i + 1) = 4 \), then \( v_2^i \) and \( v_1^i + 1 \) are both heavy. Then one can similarly claim that \( H_j \) contains at most \( \beta_j - 1 \) 4-vertices, where \( j = i + 1 \).

By (2) of Observation 2 and the definition of \( G_i \), if there are 3-vertices on \( C \), they must be on the interval. Suppose there are \( \gamma_i \) 3-vertices in \( M_i \) where \( \gamma_i \geq 0 \). If \( |M_i| \geq 3 \), then \( M_i \) contains at least \( 2\gamma_i + 1 \) non-4-vertices since (C2) and (C3) are forbidden. Here, note that neither \( v_4^i \) nor
$v_i^{i+1}$ can be 3-vertex by (2) of Observation 1 since each image vertex is adjacent to a crossing vertex in $G^\times$. So $n_4(M_t) \le |M_t| - (2\gamma_i + 1)$ if $|M_t| \ge 3$. If $|M_t| = 2$, then $\gamma_i = 0$. So the above inequation on $n_4(M_t)$ holds unless $d(v_i^{i+1}) = 4$ and $d(v_i^{2i+1}) = 4$. In this special case, this inequation becomes $n_4(M_t) \le |M_t| - (2\gamma_i + 1) + 1$ indeed, but on the other hand, we have $n_4(H_t) \le \beta_t - 1$ and $n_4(H_{t-1}+1) \le \beta_t - 1 - 1$ by the former arguments. Note that $N_{G^\times}(v) = V(G) = \bigcup_{t=1}^n(V(H_t) \cup V(M_t))$ and $|\bigcup_{t=1}^n M_t| = d - \sum_{k=1}^n (2k_t - 1) = d + n + 2\sum_{k=1}^n (\alpha_k + \beta_k) = d + n - 2t + 2t - 2(t - x) = d + n - 2t$. So we can deduce that $n_4 = \sum_{k=1}^n n_4(H_t) + \sum_{k=1}^n n_4(M_t) \le \sum_{k=1}^n \beta_k + \sum_{k=1}^n |M_t| - \sum_{k=1}^n (2\gamma_k + 1) = (t - x) + (d + n - 2t) - (2n + n) = d - x - t - 2n$. Hence, we have $2n + n + t + x \le d$.

Now we apply the discharging method to the associated planar graph $G^\times$ of $G$. Since $G^\times$ is a planar graph and $\sum_{v \in V(G)}(d_{G^\times}(v) - 4) = \sum_{v \in V(G)}(d_{G^\times}(v) - 4) = \sum_{f \in F(G^\times)}(d_{G^\times}(v) - 4) = \sum_{f \in F(G^\times)}(d_{G^\times}(v) - 4) = -8$. By Euler’s formula, we have

$$\sum_{v \in V(G)} (d_{G^\times}(v) - 4) + \sum_{f \in F(G^\times)} (d_{G^\times}(f) - 4) = -8.$$  

Now we define $c(x)$ to be the initial charge of $x \in V(G) \cup F(G^\times)$. Let $c(x) = d_{G^\times}(v) - 4$ for each vertex $v \in V(G)$ and let $c(f) = d_{G^\times}(f) - 4$ and for each face $f \in F(G^\times)$. It follows that $\sum_{x \in V(G) \cup F(G^\times)} c(x) = -8$. We now redistribute the initial charge $c(x)$ and form a new charge $c'(x)$ for each $x \in V(G) \cup F(G^\times)$ by discharging method. Since our rules only move charge around, and do not affect the sum, we have $\sum_{x \in V(G) \cup F(G^\times)} c'(x) = \sum_{x \in V(G) \cup F(G^\times)} c(x) = -8$. A 3-face in $G^\times$ is special if it is incident with one crossing vertex. Our discharging rules are defined as follows:

- (R1) Each non-special 3-face in $G^\times$ receives $\frac{1}{4}$ from each vertex incident with it;
- (R2) Each special 3-face in $G^\times$ receive $\frac{1}{4}$ from each non-crossing vertex incident with it;
- (R3) Each vertex $v$ in $G$ with $9 \le d(v) \le 11$ sends $\frac{1}{27}$ to each adjacent 7-vertex in $G$;
- (R4) Each vertex $v$ in $G$ with $12 \le d(v) \le 14$ sends $\frac{1}{15}$ to each adjacent 7-vertex and $\frac{1}{6}$ to each adjacent 6-vertex in $G$;
- (R5) Each vertex $v$ in $G$ with $15 \le d(v) \le 19$ sends $\frac{1}{15}$ to each adjacent 7-vertex, $\frac{1}{5}$ to each adjacent 6-vertex and $\frac{1}{12}$ to each adjacent 5-vertex in $G$;
- (R6) Each vertex $v$ in $G$ with $20 \le d(v) \le 35$ sends $\frac{1}{12}$ to each adjacent 7-vertex, $\frac{1}{4}$ to each adjacent 6-vertex, $\frac{1}{3}$ to each adjacent 5-vertex and $\frac{5}{12}$ to each adjacent 4-vertex in $G$;
- (R7) Each vertex $v$ in $G$ with $d(v) \ge 36$ sends $\frac{1}{6}$ to each adjacent 7-vertex, $\frac{1}{3}$ to each adjacent 6-vertex, $\frac{5}{6}$ to each adjacent 5-vertex, $\frac{5}{6}$ to each adjacent 4-vertex and $\frac{5}{3}$ to each adjacent 3-vertex in $G$.

Let $f$ be a face of $G^\times$. Then $d_{G^\times}(f) = 3$. If $f$ is non-special, then by (R1), $c'(f) = c(f) + 3 \times \frac{1}{4} = 0$. If $f$ is special, then by Observation 1 $f$ is incident with two non-crossing vertices. By (R2), we have $c'(f) = c(f) + 2 \times \frac{1}{4} = 0$.

Let $v$ be a vertex of $G$. Since (C1) is forbidden, we have $d(v) \ge 3$. Suppose $v$ is a $d$-vertex and has $d$ neighbors $v_1, \ldots, v_d$ in $G$ where $d(v_1) \le \cdots \le d(v_d)$. In the following, we show $c'(v) \ge 0$ for each such a vertex.

Suppose $d = 3$. Since (C2) is forbidden, $v$ is adjacent three 36$^+$-vertices. Note that $v$ is not incident with any special 3-face by (2) of Observation 1. By (R1) and (R7), we have $c'(v) \ge c(v) - 3 \times \frac{1}{4} + 3 \times \frac{2}{3} = 0$.

Suppose $d = 4$. Since (C3) is forbidden, $v$ is incident with at most two special 3-faces. If $d(v_1) \ge 20$, then by (R1) and (R7), we have $c'(v) \ge c(v) - 2 \times \frac{1}{2} - 2 \times \frac{2}{3} + 4 \times \frac{5}{12} = 0$. If $d(v_1) \le 19$, then $d(v_2) \ge 36$ since (C3) is forbidden. So by (R7) we have $c'(v) \ge c(v) - 2 \times \frac{1}{2} - 2 \times \frac{1}{3} + 3 \times \frac{5}{6} = 0$. 

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Suppose $d = 5$. If $d(v_1) \geq 15$, then by (R1), (R2), (R5) and (4) of Observation 1, we have $ch'(-v) \geq ch(v) - 4 \times \frac{1}{2} - \frac{1}{2} + 5 \times \frac{1}{6} = 0$. So we may assume $d(v_1) \leq 14$. If $d(v_2) \geq 20$, then by (R1), (R2) and (R6), we have $ch'(-v) \geq ch(v) - 4 \times \frac{1}{2} - \frac{1}{2} + 4 \times \frac{1}{3} = 0$. So we may assume $d(v_2) \leq 19$. Then $d(v_3) \geq 36$ for otherwise (C4) occurs. In this case, by (R1), (R2) and (R7), we also have $ch'(-v) \geq ch(v) - 4 \times \frac{1}{2} - \frac{1}{2} + 3 \times \frac{1}{3} = 0$.

Suppose $d = 6$. If $d(v_1) \geq 12$, then by (R1), (R2) and (R4), we have $ch'(-v) \geq ch(v) - 6 \times \frac{1}{2} + 6 \times \frac{1}{6} = 0$. So we may assume $d(v_1) \leq 11$. If $d(v_2) \geq 15$, then by (R1), (R2) and (R5), we have $ch'(-v) \geq ch(v) - 6 \times \frac{1}{2} + 5 \times \frac{1}{6} = 0$. So we may assume $d(v_2) \leq 14$. If $d(v_3) \geq 20$, then by (R1), (R2) and (R6), we have $ch'(-v) \geq ch(v) - 6 \times \frac{1}{2} + 4 \times \frac{1}{3} = 0$. So we may assume $d(v_3) \leq 19$. Then $d(v_4) \geq 36$ for otherwise (C5) occurs. In this case, by (R1), (R2) and (R7), we also have $ch'(-v) \geq ch(v) - 6 \times \frac{1}{2} + 3 \times \frac{1}{3} = 0$.

Suppose $d = 7$. If $d(v_1) \geq 9$, then by (R1), (R2), (R3) and (4) of Observation 1, we have $ch'(-v) \geq ch(v) - 6 \times \frac{1}{2} + 7 \times \frac{1}{6} = 0$. So we may assume $d(v_1) \leq 8$. If $d(v_2) \geq 12$, then by (R1), (R2) and (R4), we have $ch'(-v) \geq ch(v) - 6 \times \frac{1}{2} + \frac{1}{6} + 6 \times \frac{1}{15} = 0$. So we may assume $d(v_2) \leq 11$. If $d(v_3) \geq 15$, then by (R1), (R2) and (R5), we have $ch'(-v) \geq ch(v) - 6 \times \frac{1}{2} - \frac{1}{3} + 5 \times \frac{1}{6} = 0$. So we may assume $d(v_3) \leq 14$. If $d(v_4) \geq 20$, then by (R1), (R2) and (R6), we have $ch'(-v) \geq ch(v) - 6 \times \frac{1}{2} + \frac{1}{4} + \frac{1}{15} = 0$. So we may assume $d(v_4) \leq 19$. Then $d(v_5) \geq 36$ for otherwise (C6) occurs. In this case, by (R1), (R2) and (R7), we also have $ch'(-v) \geq ch(v) - 6 \times \frac{1}{2} - \frac{1}{3} + 3 \times \frac{1}{3} = 0$.

Suppose $d = 8$. Then by (R1)-(R8), we have $ch'(-v) \geq ch(v) - 8 \times \frac{1}{2} = 0$.

Suppose $9 \leq d \leq 11$. Recall that $t$ is the number of mirror triangles incident with $v$. So $c(v) = t$. Note that $t \leq \frac{d}{2}$ and $n_7 \leq d$. By (R1), (R2) and (R3), we have $ch'(-v) \geq ch(v) - \frac{24}{2} - \frac{d-2t-\frac{9}{2}}{3} \geq \frac{10}{2}d - 4 > 0$.

Suppose $12 \leq d \leq 14$. Note that $t \leq \frac{d}{2}$ and $n_6 + n_7 \leq d$. By (R1),(R2) and (R4), we have $ch'(-v) \geq ch(v) - \frac{24}{2} - \frac{d-2t-\frac{9}{2}}{6} - \frac{9}{15} \geq \frac{1}{2}d - 4 \geq 0$.

Suppose $15 \leq d \leq 19$. Note that $t \leq \frac{d}{2}$. By (R1),(R2), (R5) and Claims 2, 3, we have $ch'(-v) = ch(v) - \frac{23}{12} - \frac{d-2t-\frac{9}{2}}{3} - \frac{9}{6} - \frac{n_9 - n_7}{15} \geq \frac{2}{3}d - 4 - \frac{1}{2}x + x + x - t + \mu - \frac{1}{9}(d - x - t - x - t + x - x - t + x - x) \geq \frac{2}{3}d - 4 - \frac{1}{2}t \geq \frac{2}{3}d - 4 \geq 0$.

Suppose $20 \leq d \leq 35$. Note that $t \leq \frac{d}{2}$. By (R1),(R2), (R2) and Claims 2, 3, 4, we have $ch'(-v) = ch(v) - \frac{23}{12} - \frac{d-2t-\frac{9}{2}}{3} - \frac{9}{6} - \frac{9}{6} - \frac{n_9 - n_7}{12} \geq \frac{2}{3}d - 4 - \frac{1}{2}x + x + x - t + \mu - \frac{1}{9}(d - 2x - t + x - t + x - t + x - t + x) \geq \frac{2}{3}d - 4 - \frac{1}{2}t \geq \frac{2}{3}d - 4 \geq 0$.

Suppose $d \geq 36$. Note that $t \leq \frac{d}{2}$. By (R1),(R2), (R7) and Claims 2, 3, 4, we have $ch'(-v) = ch(v) - \frac{23}{12} - \frac{d-2t-\frac{9}{2}}{3} - \frac{9}{6} - \frac{9}{6} - \frac{n_9 - n_7}{12} \geq \frac{2}{3}d - 4 - \frac{1}{2}x + x + x - t + \mu - \frac{1}{9}(d - 2x - t + x - t + x - t + x) \geq \frac{2}{3}d - 4 - \frac{1}{2}t \geq \frac{2}{3}d - 4 \geq 0$.

By the above arguments, we have $\sum_{x \in V(G)} ch'(x) \geq 0$, a contradiction. Hence we have proved the theorem.

3. LIGHT GRAPHS IN 1-PLANAR GRAPHS OF THE BOUNDED DEGREE

Let $\mathcal{H}$ be a family of graphs and let $\mathcal{H}$ be a connected graph. Let $\phi(H, \mathcal{H})$ be the smallest integer with the property that each graph $G \in \mathcal{H}$ contains a subgraph $K \cong H$ such that $\max_{x \in V(K)} \{\text{deg}(x)\} \leq \phi(H, \mathcal{H})$. If such an integer does not exist, we write $\phi(H, \mathcal{H}) = +\infty$. We say that the graph $H$ is light in the family $\mathcal{H}$ if $\phi(H, \mathcal{H}) < +\infty$. By $\mathcal{L}(\mathcal{H})$, we denote the set of light graphs in the family $\mathcal{H}$.

In the next, $P_k$ denotes a path with $k$ vertices and $S_k$ denotes a star with maximum degree $k$. We use the notation $P_3^\delta$ for the family of all 1-planar graphs of minimum degree at least $\delta$. In [6], Fabrici and Madaras showed that $\{P_1, P_2\} \subseteq \mathcal{L}(P_3^1) \subseteq \{P_1, P_2, P_3\}$ and $\{P_1, P_2, P_3\} \subseteq \mathcal{L}(P_3^1) \subseteq \{P_1, P_2, P_3, P_4, S_3\}$ and posed a few of open problems. Two of them are stated as follows.
Is $P_3 \in \mathcal{L}(P_3^1)$ true?
Is $P_4, S_3 \in \mathcal{L}(P_3^1)$ true?

In this section, we partially answer these two questions by applying the results in Section 2.

**Theorem 3.** Let $G$ be a simple 1-planar graph with minimum degree $\delta \geq 4$. Then $G$ contains a 3-path with all vertices of degree at most 35.

**Proof.** By Theorem 2 $G$ contains one of the configuration in \{(C3),(C4),(C5),(C6)\} described in Section 2. In each case, we will find a path $uvw$ in $G$ such that $\max\{d(u), d(v), d(w)\} \leq 35$. 

Similarly we can prove an analogous theorem.

**Theorem 4.** Let $G$ be a simple 1-planar graph with minimum degree $\delta \geq 5$. Then $G$ contains a 3-star with all vertices of degree at most 35.

Hence we have the following many corollaries.

**Corollary 5.** $P_3 \in \mathcal{L}(P_4^1)$.

**Corollary 6.** $\mathcal{L}(P_4^1) = \{P_1, P_2, P_3\}$.

**Corollary 7.** $S_3 \in \mathcal{L}(P_5^1)$.

4. ACYCLIC EDGE COLORING OF 1-PLANAR GRAPHS

A mapping $c$ from $E(G)$ to the sets of colors $\{1, \cdots, k\}$ is called a proper $k$-edge coloring of $G$ provided any two adjacent edges receive different colors. The edge chromatic number $\chi'(G)$ is the minimum number of colors needed to color the edges of $G$ properly. A proper $k$-edge coloring $c$ of $G$ is called an acyclic $k$-edge-coloring of $G$ if there are no bichromatic cycles in $G$ under the coloring $c$. The smallest number of colors such that $G$ has an acyclic edge coloring is called the acyclic chromatic number of $G$, denoted by $\chi'_a(G)$. Acyclic edge coloring was introduced by Alon et al. \cite{2}, and they presented a linear upper bound on $\chi'_a(G)$. It was proved that $\chi'_a(G) \leq 16\Delta(G)$ by Molloy and Reed \cite{9}. For planar graph $G$, A. Fiedorowicz et al. \cite{7} proved that $\chi'_a(G) \leq 2\Delta(G) + 29$. Recently, Hou et al. \cite{8} gave a better upper bound. They showed that $\chi'_a(G) \leq \max\{2\Delta(G) - 2, \Delta(G) + 22\}$ holds for each planar graph. Let $\phi$ be an edge coloring of $G$. For any vertex $v \in V(G)$, we define $\phi(v) = \{\phi(uv) | u \in N(v)\}$. In this section, we consider the acyclic edge coloring of 1-planar graphs.

**Theorem 8.** Let $G$ be a 1-planar simple graph. Then $\chi'_a(G) \leq \max\{2\Delta(G) - 2, \Delta(G) + 83\}$.

**Proof.** The theorem is proved by contradiction. Let $L$ stand for $\max\{2\Delta(G) - 2, \Delta(G) + 83\}$. Suppose $G$ is a minimum counterexample to the theorem. Then $G$ is 2-connected and then $\delta(G) \geq 2$.

**Case 1.** $\delta(G) = 2$

Let $d(v) = 2$ and $N(v) = \{v_1, v_2\}$. Suppose $v_1v_2 \not\in E(G)$. By the minimality of $G$, the graph $G' = (G \setminus v) \cup \{v_1v_2\}$ has an acyclic $L$-edge coloring $\phi$ with color set $C$. Let $\tau(vv_1) = \phi(v_1v_2)$ and $\tau(vv_2) \in C \setminus \{\phi(v_2) \cup \tau(vv_1)\}$. For the edge $e \in E(G') - \{v_1v_2\}$, we remain $\tau(e) = \phi(e)$. Note that $|C \setminus \{\phi(v_2) \cup \tau(vv_1)\}| > 0$. Then $\tau$ is an acyclic $L$-edge coloring of $G$, a contradiction. So $v_1v_2 \in E(G)$. Let $G' = G \setminus v$. Then $G'$ has an acyclic $L$-edge coloring $\phi$ with color set $C$. Now we let $\tau(vv_1) \in S_1 = C \setminus \{\phi(v_1) \cup \phi(v_2)\}$. Since $|C| \geq 2\Delta(G) - 2$ and $|\phi(v_1) \cup \phi(v_2)| \leq 2\Delta(G) - 3$, $|S_1| \geq 1$. Now we color $vv_2$ by $\tau(vv_2) \in S_2 = C \setminus \{\phi(v_2) \cup \tau(vv_1)\}$. It is easy to see that $|S_2| > 0$. For the edge $e \in E(G')$, we also remain $\tau(e) = \phi(e)$. Then $\tau$ is again an acyclic $L$-edge coloring of $G$, a contradiction.

**Case 2.** $\delta(G) \geq 3$

In this case, $G$ has one of the five configurations \{(C2),(C3),(C4),(C5),(C6)\} which are described in Theorem 2. Let $c_1 = 8$, $c_2 = 11$, $c_3 = 14$, $c_4 = 19$ and $c_5 = 35$. Suppose $G$ contains the $(d - 1)$-th configuration, where $d \in \{3, 4, 5, 6, 7\}$. 

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If \( v_{d-1}v_d \notin E(G) \), let \( G' = (G\setminus v) \cup \{v_{d-1}v_d\} \). Otherwise, let \( G' = G\setminus v \). Then \( G' \) has an acyclic \( L \)-edge coloring \( \phi \) with color set \( C \). If \( v_{d-1}v_d \notin E(G) \), let \( \tau(v_{d-1}) = \phi(v_{d-1}v_d) \). Otherwise, let \( S_{d-1} = \phi(v_{d-1}) \cup \phi(v_d) \). Now we color \( vv_{d-1} \) by a color \( \tau(vv_{d-1}) \in C\setminus S_{d-1} \). Note that \( |C| \geq 2\Delta(G) - 2 \) and \( S_{d-1} \leq 2\Delta(G) - 3 \), we have \( |C\setminus S_{d-1}| > 0 \). Let \( S_d = \phi(v_1) \cup \cdots \cup \phi(v_{d-2}) \cup \phi(v_d) \) and \( S_i = \bigcup_{k=1}^{d-1} \phi(v_k) \) where \( 1 \leq i \leq d - 2 \). Then we color \( vv_d, vv_1, vv_2, \ldots, vv_{d-2} \) in turn as follows. Let \( \tau(vv_d) \in T_d = C \setminus \{S_d \cup \tau(vv_{d-1})\} \). If \( \tau(vv_d) \notin \phi(v_1) \), let \( \tau(vv_1) \in T_1 = C \setminus \{(S_1 \setminus \tau(vv_{d-1})) \cup \{	au(vv_{d-1}), \tau(vv_d)\}\} \). Otherwise we let \( \tau(vv_1) \in T_1 = C \setminus \{S_1 \cup \tau(vv_{d-1}), \tau(vv_d)\} \). At last, for each \( 2 \leq i \leq d - 2 \), we let \( \tau(vv_i) \in T_i = C \setminus \{S_i \cup \tau(vv_{i-1}), \tau(vv_{i-1}), \tau(vv_{d-1}), \tau(vv_d)\} \). For the edge \( e \in E(G') \), we still remain \( \tau(e) = \phi(e) \). Note that \( |T_{d-2}| \geq \cdots \geq |T_1|, |T_1| \geq |T_1| \) and \( \min\{|T_1|, |T_d| \} \geq L - (\sum_{k=1}^{d-2} (c_i - 1) + \Delta(G)) > 0 \). So this coloring \( \tau \) does exist. It is easy to check that \( \tau \) is proper and acyclic. So we have constructed a new coloring \( \tau \) which is an acyclic \( L \)-edge coloring of \( G \), a contradiction. This completes the proof of Theorem 8. \( \square \)

**Remark.** The proof of Theorem 8 above does not use recolorings, therefore, it actually yields a more general result as follows.

**Theorem 9.** Every simple 1-planar graph \( G \) is acyclically edge \( L \)-choosable where \( L = \max\{2\Delta(G) - 2, \Delta(G) + 83\} \).

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