Compact leaves of the foliation defined by the kernel of a \( T^2 \)-invariant presymplectic form

Asuka Hagiwara *

Abstract

We investigate the foliation defined by the kernel of an exact presymplectic form \( da \) of rank \( 2n \) on a \((2n + r)\)-dimensional closed manifold \( M \). For \( r = 2 \), we prove that the foliation has at least two leaves which are homeomorphic to a 2-dimensional torus, if \( M \) admits a locally free \( T^2 \)-action which preserves \( da \) and satisfies that the function \( \alpha(Z_2) \) is constant, where \( Z_1, Z_2 \) are the infinitesimal generators of the \( T^2 \)-action. We also give its generalization for \( r \geq 1 \).

1 Introduction

Let \( n \) and \( r \) be positive integers. A smooth manifold \( M \) of dimension \( 2n + r \) is said to be presymplectic if it carries a closed 2-form \( \omega \) such that the dimension of the kernel of \( \omega(x) \) is \( r \) for all \( x \in M \), that is, \( \omega \) is of constant rank \( 2n \). The 2-form \( \omega \) is called a presymplectic form on \( M \). We denote the kernel of \( \omega \) by \( \ker \omega \), which is an involutive distribution of dimension \( r \). Hence, by Frobenius’ theorem, \( \ker \omega \) defines an \( r \)-dimensional foliation \( \mathcal{F}_\omega \). The leaves of \( \mathcal{F}_\omega \) are integral manifolds of the distribution.

A special case of presymplectic manifolds is when a manifold \( M \) is \((2n + 1)\)-dimensional and carries a contact form \( \alpha \), that is, \( \alpha \) is a 1-form on \( M \) such that \( \alpha \wedge (d\alpha)^n \neq 0 \). In this case, the 2-form \( \omega := da \) is a presymplectic form on \( M \) and the foliation \( \mathcal{F}_\omega \) is 1-dimensional. The leaves of \( \mathcal{F}_\omega \) are integral curves of \( \ker \omega \), which are called characteristics. The Weinstein conjecture [9] asserts that there exists at least one closed characteristic on a closed contact manifold, which is one of the fundamental problem concerning dynamics on contact manifolds. This conjecture is still open, but has been proved in several cases. For instance, Viterbo [8] proved it for compact contact manifolds which are hypersurfaces of contact type in \((\mathbb{R}^{2n}, \omega_0)\), where \( \omega_0 = \sum_{j=1}^n dy_j \wedge dx_j \) is the standard symplectic form. Hofer [4] proved this conjecture
for $S^3$ and for overtwisted 3-dimensional contact manifolds, and after that Taubes \cite{7} solved it affirmatively in dimension 3.

In \cite{1}, Banyaga and Rukimbira raised the following question for presymplectic manifolds $(M, \omega)$, which generalizes the Weinstein conjecture.

**Question** (\cite{1} p.3902). When does the foliation $\mathcal{F}_\omega$ admits a compact leaf?

In addition, they focused their attention on the case where dim $\mathcal{F}_\omega = 1$ and proved the following theorem.

**Theorem 1** (Banyaga-Rukimbira, \cite{1}). Let $M$ be a $(2n + 1)$-dimensional oriented closed $C^\infty$-manifold with a 1-form $\alpha$ such that the 2-form $\omega := d\alpha$ has constant rank $2n$ everywhere. If there exists a locally free circle-action on $M$ which preserves $\omega$, then $\mathcal{F}_\omega$ has at least two closed leaves.

In particular, they pointed out that this result guarantees the existence of periodic orbits of the Hamiltonian system on hypersurfaces in $\mathbb{R}^{2n}$ which are not necessarily of contact type. However, they did not mention the case where dim $\mathcal{F}_\omega \geq 2$. For such a direction, it seems that there are few works which assume that $(M, \omega)$ is $r$-contact (see Remark in Section 3, and \cite{3} for details).

In this paper, we consider a higher dimensional version of Theorem 1 without assuming that $(M, \omega)$ is $r$-contact. We show the following result for the case where dim $\mathcal{F}_\omega = 2$.

**Theorem 2.** Let $M$ be a $(2n + 2)$-dimensional oriented closed $C^\infty$-manifold with an exact presymplectic form $\omega$, that is, there exists a 1-form $\alpha \in \Omega^1(M)$ such that the 2-form $\omega = d\alpha$ has rank $2n$ everywhere. Assume that $M$ admits a locally free $T^2$-action with the following conditions:

(i) The $T^2$-action preserves $\omega$,
(ii) The function $\alpha(Z_2)$ is constant on $M$,

where $Z_1, Z_2$ denote the infinitesimal generators of the $T^2$-action. Then

1. the 2-dimensional $C^\infty$-foliation $\mathcal{F}_\omega$ has at least two leaves which are homeomorphic to a 2-dimensional torus.
2. Moreover, if the function $\alpha(Z_1)$ is also constant, then $\mathcal{F}_\omega$ coincides with the foliation of the $T^2$-action and hence, all leaves of $\mathcal{F}_\omega$ are homeomorphic to a 2-dimensional torus.

We prove Theorem 2 in the next section. In Section 3 we provide examples of Theorem 2 and mention that a similar result holds for the general case where dim $\mathcal{F}_\omega = r$. 


2 Proof of Theorem 2

Let $\rho: M \times T^2 \to M$ be a locally free $T^2$-action on $M$ with the conditions (i) and (ii) above, and let $D: M \ni q \mapsto D_q = \text{span}\{Z_1(q), Z_2(q)\} \subset T_q M$ be the distribution determined by $\rho$, where $Z_1, Z_2$ denote the infinitesimal generators of $\rho$. The distribution $D$ is involutive and of dimension 2 since $\rho$ is locally free. Therefore, now $M$ has two foliations of codimension $2n$ which are determined by $\ker \omega$ and $D$. Note that, by definition, every leaf of $D$ is homeomorphic to a 2-dimensional torus $T^2$.

We define the diffeomorphism $s: M \to M$ by $s(x) := \rho(x, s)$ for $s \in T^2$ and the 1-form

$$\alpha_0 := \int_{T^2} (s^* \alpha) \, d\sigma,$$

where $\sigma$ is the Haar measure on $T^2$. Then $\alpha_0$ is $T^2$-invariant. Thus,

$$L_{Z_i} \alpha_0 = \lim_{t \to 0} \frac{(\phi^t_i)^* \alpha_0 - \alpha_0}{t} = 0, \quad i = 1, 2,$$

where $\phi^t_i = \rho(\cdot, \exp tZ_i)$ denotes the flow of $Z_i$, respectively. Moreover, define smooth functions $S_i: M \to \mathbb{R}$ by

$$S_i(x) := -\alpha_0(x)(Z_i(x)) = -i_{Z_i} \alpha_0(x), \quad i = 1, 2.$$

By the condition (i), we have

$$d \alpha_0 = \int_{T^2} (s^* d\alpha) \, d\sigma = \int_{T^2} (s^* \alpha) \, d\sigma = \omega,$$

and hence, due to the Cartan formula,

$$d S_i = -d i_{Z_i} \alpha_0 = -L_{Z_i} \alpha_0 + i_{Z_i} d \alpha_0 = i_{Z_i} \omega. \quad (1)$$

Since the group $T^2$ is commutative, we have $s \circ \phi^t_2(x) = \phi^t_2 \circ s(x)$ for all $x \in M$. Differentiating this equation in $t$, we obtain $ds(x)Z_2(x) = Z_2(s(x))$, so that

$$S_2(x) = -\int_{T^2} s^* \alpha(x)(Z_2(x)) \, d\sigma = -\int_{T^2} \alpha(s(x))(Z_2(s(x))) \, d\sigma.$$

Thus, by the condition (ii) the function $S_2$ is constant and so $dS_2 = 0$. Hence,

$$Z_2(q) \in \ker \omega(q), \quad \forall q \in M \quad (2)$$

by (1) for $i = 2$. To investigate the relation between $Z_1$ and $\ker \omega$, we shall take a special chart. Since $\ker \omega$ is involutive, by using Frobenius’ theorem [2, p.89, Theorem 1], for any point $p \in M$ we can choose the following chart $(\varphi, U)$ around $p$: for $q \in U$ we write

$$\varphi(q) = (x(q), z_1(q), z_2(q)), \quad x(q) = (x_1(q), \ldots, x_{2n}(q)),$$
then this chart satisfies that $\varphi(p) = (0, 0, 0)$ and

$$\xi_i := \frac{\partial}{\partial z_i}, \quad i = 1, 2$$

form a local frame of $\ker \omega$ on $U$. For $q \in U$ we put

$$Z_1(q) = \sum_{j=1}^{2n} X_j(x, z_1, z_2) \frac{\partial}{\partial x_j} + u_1(x, z_1, z_2) \frac{\partial}{\partial z_1} + u_2(x, z_1, z_2) \frac{\partial}{\partial z_2}$$

and prove the following

**Lemma 3.** The function $X_j$ is independent of $z_1, z_2$ for any $j = 1, \ldots, 2n$.

**Proof.** In $U$, since $\xi_i \in \ker \omega$, we have $L_{\xi_i} \omega = i_{\xi_i} d\omega + di_{\xi_i} \omega = 0$ and

$$L_{\xi_i}(i_{Z_j} \omega) = i_{\xi_i} d(i_{Z_j} \omega) + di_{\xi_i}(i_{Z_j} \omega) = i_{\xi_i} ddS_j - di_{Z_j} i_{\xi_i} \omega = 0 \quad (\because \text{(1)}).$$

Therefore,

$$i_{[\xi_i, Z_j]} \omega = L_{\xi_i} i_{Z_j} \omega - i_{Z_j} L_{\xi_i} \omega = 0$$

and namely, there exist functions $\lambda_1^i, \lambda_2^i : U \to \mathbb{R}$ such that

$$[\xi_i, Z_1] = \lambda_1^i \xi_1 + \lambda_2^i \xi_2.$$

On the other hand,

$$[\xi_i, Z_1] = \left[\frac{\partial}{\partial z_i}, Z_1\right] = \left[\frac{\partial}{\partial z_i}, \sum_{j=1}^{2n} X_j \frac{\partial}{\partial x_j}\right] + \left[\frac{\partial}{\partial z_i}, u_1 \frac{\partial}{\partial z_1}\right] + \left[\frac{\partial}{\partial z_i}, u_2 \frac{\partial}{\partial z_2}\right]$$

$$= \sum_{j=1}^{2n} \frac{\partial X_j}{\partial z_i} \frac{\partial}{\partial x_j} + \frac{\partial u_1}{\partial z_i} \frac{\partial}{\partial z_1} + \frac{\partial u_2}{\partial z_i} \frac{\partial}{\partial z_2}.$$

Therefore, $\partial X_j/\partial z_i = 0$.

By this lemma, we have

$$Z_1(q) = \sum_{j=1}^{2n} X_j(x) \frac{\partial}{\partial x_j} + u_1(x, z_1, z_2) \frac{\partial}{\partial z_1} + u_2(x, z_1, z_2) \frac{\partial}{\partial z_2} \quad \text{for } q \in U.$$

Now, we assume that $p \in M$ is a critical point of $S_1$. By (1) for $i = 1$, we have $Z_1(p) \in \ker \omega(p)$. Because $\varphi(p) = (0, 0, 0)$, we have

$$Z_1(p) = \sum_{j=1}^{2n} X_j(0) \frac{\partial}{\partial x_j} + u_1(0, 0, 0) \frac{\partial}{\partial z_1} + u_2(0, 0, 0) \frac{\partial}{\partial z_2}.$$
so that
\[ X_j(0) = 0, \quad j = 1, \ldots, 2n. \]  

(3)

We denote by \( \mathcal{F}_\omega(p) \) the leaf of \( \mathcal{F}_\omega \) passing through the point \( p \). Let \( q \in \mathcal{F}_\omega(p) \cap U \). Then there exist \( t_1, t_2 \in \mathbb{R} \) such that \( \varphi(q) = (0, t_1, t_2) \), since
\[ T_q(\mathcal{F}_\omega(p)) = \ker \omega(q) = \text{span} \left\{ \left( \frac{\partial}{\partial z_1} \right)_q, \left( \frac{\partial}{\partial z_2} \right)_q \right\}. \]

Thus, by (3), we obtain
\[ Z_1(q) = \sum_{j=1}^{2n} X_j(0) \frac{\partial}{\partial x_j} + u_1(0, t_1, t_2) \frac{\partial}{\partial z_1} + u_2(0, t_1, t_2) \frac{\partial}{\partial z_2} \]
\[ = u_1(0, t_1, t_2) \frac{\partial}{\partial z_1} + u_2(0, t_1, t_2) \frac{\partial}{\partial z_2} \in \ker \omega(q). \]

(4)

It follows from (2) that
\[ \ker \omega(q) = \mathcal{D}_q, \quad \forall q \in \mathcal{F}_\omega(p) \cap U, \]

(5)

since the dimensions of \( \ker \omega \) and \( \mathcal{D} \) are the same. In order to verify that \( \ker \omega \) coincides with \( \mathcal{D} \) on the leaf \( \mathcal{F}_\omega(p) \), we cover \( \mathcal{F}_\omega(p) \) by a set of charts \( \{(U_a, \varphi_a)\}_{a \in A} \), where each \( (U_a, \varphi_a) \) is a chart as we chose above. For \( (p \neq \hat{p}) \in \mathcal{F}_\omega(p) \) we take a chart \( (\hat{U}, \hat{\varphi}) \in \{(U_a, \varphi_a)\}_{a \in A} \) around \( \hat{p} \) which satisfies \( \mathcal{F}_\omega(p) \cap U \cap \hat{U} \neq \emptyset \). We write
\[ \hat{\varphi}(q) = (y(q), w_1(q), w_2(q)), \quad y(q) = (y_1(q), \ldots, y_{2n}(q)) \]
for \( q \in \hat{U} \), then \( \hat{\varphi}(\hat{p}) = (0, 0, 0) \) and \( \partial/\partial w_1, \partial/\partial w_2 \) form a local frame of \( \ker \omega \) on \( \hat{U} \).

With this setting, by applying Lemma 3, we have
\[ Z_1(q) = \sum_{j=1}^{2n} \hat{X}_j(0) \frac{\partial}{\partial y_j} + \hat{u}_1(y, w_1, w_2) \frac{\partial}{\partial w_1} + \hat{u}_2(y, w_1, w_2) \frac{\partial}{\partial w_2} \]
in \( \hat{U} \) as well. Since \( T_q(\mathcal{F}_\omega(p)) = \ker \omega(q) \), we have \( \hat{\varphi}(q) = (0, w_1(q), w_2(q)) \) for all \( q \in \mathcal{F}_\omega(p) \cap \hat{U} \). Therefore, for \( q \in \mathcal{F}_\omega(p) \cap U \cap \hat{U} \),
\[ Z_1(q) = \sum_{j=1}^{2n} \hat{X}_j(0) \frac{\partial}{\partial y_j} + \hat{u}_1(0, w_1(q), w_2(q)) \frac{\partial}{\partial w_1} + \hat{u}_2(0, w_1(q), w_2(q)) \frac{\partial}{\partial w_2} \]
holds. From (3), we obtain \( \hat{X}_j(0) = 0, \quad j = 1, \ldots, 2n. \) Thus, for all \( q \in \mathcal{F}_\omega(p) \cap \hat{U} \) we have
\[ Z_1(q) = \hat{u}_1(0, w_1(q), w_2(q)) \frac{\partial}{\partial w_1} + \hat{u}_2(0, w_1(q), w_2(q)) \frac{\partial}{\partial w_2} \in \ker \omega(q), \]

(5)
so that ker \( \omega(q) = D_q \). Repeating this argument we see that

\[
\ker \omega(q) = D_q, \quad \forall q \in F_w(p).
\]

(6)

Therefore, \( F_w(p) \) is not only an integral manifold of ker \( \omega \), but also that of \( D \) passing through \( p \). On the other hand, the \( T^2 \)-orbit \( T^2(p) \) of \( p \) is nothing but the maximal connected integral manifold of \( D \) passing through \( p \). Hence, by [5, p.172, Theorem 1], \( F_w(p) \) is an open submanifold in \( T^2(p) \). Actually, we claim the following

**Lemma 4.** \( F_w(p) = T^2(p) \) holds. In particular, \( F_w(p) \) is homeomorphic to \( T^2 \).

**Proof.** Since \( F_w(p) \) is a nonempty open set in a connected space \( T^2(p) \), it suffices to show that \( F_w(p) \) is a closed set in \( T^2(p) \). We denote by \( \overline{F_w(p)} \) the closure of \( F_w(p) \) in \( T^2(p) \). Let \( q_0 \in \overline{F_w(p)} \). We take the following chart \((U, \varphi)\) around \( q_0 \) as we chose above: for \( q \in U \) we write

\[
\varphi(q) = (x(q), z_1(q), z_2(q)), \quad x(q) = (x_1(q), \ldots, x_{2n}(q)),
\]

then \( \varphi(q_0) = (0, 0, 0) \), and \( \partial/\partial z_1, \partial/\partial z_2 \) form a local frame of ker \( \omega \) on \( U \). By Frobenius’ theorem [21, p.89, Theorem 1], the slice

\[
S_0 := \{ q \in U \mid x_1(q) = 0, \ldots, x_{2n}(q) = 0 \}
\]

is a connected integral manifold of ker \( \omega \) passing through \( q_0 \).

**Claim.** The slice \( S_0 \) is an open submanifold in \( T^2(p) \) which contains \( q_0 \).

**Proof of Claim.** For \( q \in U \), by Lemma 3 we have

\[
Z_1(q) = \sum_{j=1}^{2n} X_j(x) \frac{\partial}{\partial x_j} + u_1(x, z_1, z_2) \frac{\partial}{\partial z_1} + u_2(x, z_1, z_2) \frac{\partial}{\partial z_2}.
\]

First we prove that \( Z_1(q_0) \notin \ker \omega(q_0) \). Arguing by contradiction we assume that \( Z_1(q_0) \notin \ker \omega(q_0) \), that is, \( X_j(x(q_0)) = X_j(0) \neq 0 \) for some \( j \in \{1, \ldots, 2n\} \). Since the function \( X_j \) is smooth on \( U \), there exists a neighborhood \( U' \subset U \) of \( q_0 \) such that \( X_j(x(q)) \neq 0 \) for any \( q \in U' \). Because \( q_0 \in \overline{F_w(p)} \), the intersection \( F_w(p) \cap U' \) is nonempty, so that for \( q \in F_w(p) \cap U' \) it holds that \( X_j(x(q)) \neq 0 \). This is a contradiction to (4). Hence, we have \( X_j(0) = 0 \) for all \( j = 1, \ldots, 2n \).

Therefore, for all \( q \in S_0 \), we can write \( \varphi(q) = (0, z_1(q), z_2(q)) \) and we have

\[
Z_1(q) = u_1(0, z_1(q), z_2(q)) \frac{\partial}{\partial z_1} + u_2(0, z_1(q), z_2(q)) \frac{\partial}{\partial z_2} \in \ker \omega(q).
\]

Thus, by (4) we obtain \( D_q = \ker \omega(q) \) for all \( q \in S_0 \). It follows from \( T_q S_0 = \ker \omega(q) \) that \( S_0 \) is a connected integral manifold of \( D \) through \( q_0 \). On the other hand, \( T^2(p) \) is the maximal connected integral manifold of \( D \) which contains \( q_0 \). Therefore, by [5, p.172, Theorem 1], we see that \( S_0 \) is an open submanifold in \( T^2(p) \) which contains \( q_0 \).  \( \square \)
By this claim, \( S_0 \) is an open neighborhood of \( q_0 \) in \( T^2(p) \). Because \( q_0 \in F_\omega(p) \), the intersection \( S_0 \cap F_\omega(p) \) is nonempty. Thus, \( F_\omega(p) \) is the leaf of \( \ker\omega \) passing through a point \( p_0 \in S_0 \cap F_\omega(p) \) as well. Due to Frobenius’ theorem [2, p.89, Theorem 1], \( S_0 \) is a connected integral manifold of \( \ker\omega \) which contains the point \( p_0 \). Hence, by [3] p.172, Theorem 1], \( S_0 \) is also an open submanifold in \( F_\omega(p) \). Consequently, \( q_0 \in S_0 \subset F_\omega(p) \) and so we have \( F_\omega(p) = F_\omega(p) \).

Finally, we check the existence of critical points of the function \( S_1 \) on \( M \). If \( S_1 \) is not constant, then it has at least two critical points \( p_{\max}, p_{\min} \in M \) corresponding to a maximum and a minimum, respectively, since \( M \) is closed. Moreover, \( p_{\max} \) and \( p_{\min} \) are on different leaves of \( F_\omega \). This follows from the following. By (1) and (2), we have

\[
L_{Z_1}S_1 = i_{Z_1}dS_1 + di_{Z_1}S_1 = i_{Z_1}i_{Z_1}\omega = 0
\]

and

\[
L_{Z_2}S_1 = i_{Z_2}dS_1 + di_{Z_2}S_1 = i_{Z_2}i_{Z_1}\omega = -i_{Z_1}i_{Z_2}\omega = 0,
\]

so that \( S_1 \) is constant along \( T^2(p) = \{ \varphi_s^* \circ \varphi_t^*(p) \mid s, t \in \mathbb{R} \} \). Therefore, \( T^2(p_{\max}) \cap T^2(p_{\min}) = \emptyset \) and hence, by Lemma 4, we obtain \( F_\omega(p_{\max}) \cap F_\omega(p_{\min}) = \emptyset \). If \( S_1 \) is constant on \( M \), then \( dS_1(p) = 0 \) for all \( p \in M \), so that \( \ker\omega = D \) on \( M \). Hence, from the uniqueness of foliations, \( F_\omega \) coincides with the foliation defined by \( D \) and therefore, each leaf of \( F_\omega \) is homeomorphic to \( T^2 \). Consequently, in any case, we see that \( F_\omega \) has at least two leaves which are homeomorphic to \( T^2 \). In particular, if \( \alpha(Z_1) \) is constant, then, by a similar calculation for \( S_2 \), we see that \( S_1 \) is constant on \( M \). Thus, we complete the proof of Theorem 2.

### 3 Examples and Remarks

We first provide an example of Theorem 2.

**Example 5.** We consider the case of a submanifold of codimension 2 of a symplectic manifold \((\mathbb{R}^6, d\lambda)\), where

\[
\lambda = \frac{1}{2} \sum_{j=1}^{3} (y_j dx_j - x_j dy_j)
\]

is the Liouville form. Then \( d\lambda = \omega_0 \) is the standard symplectic form on \( \mathbb{R}^6 \). We define functions \( G_i : \mathbb{R}^6 \to \mathbb{R}, i = 1, 2, \) by

\[
G_1 := (x_1^2 + y_1^2)^2 + x_2^2 + y_2^2, \quad G_2 := x_2^2 + y_2^2 + x_3^2 + y_3^2
\]

and for real numbers \( c_1 > c_2 > 0 \) we put

\[
M := \{(x, y) \in \mathbb{R}^6 \mid G_1 = c_1, \ G_2 = c_2\}.
\]
Since the gradients
\[ \nabla G_1 = \begin{pmatrix} 4x_1(x_1^2 + y_1^2) \\ 4y_1(x_1^2 + y_1^2) \\ 2x_2 \\ 2y_2 \\ 0 \\ 0 \end{pmatrix}, \quad \nabla G_2 = \begin{pmatrix} 0 \\ 0 \\ 2x_2 \\ 2y_2 \\ 2x_3 \\ 2y_3 \end{pmatrix} \]
are linearly independent on \( M \), we see that \( M \) is a 4-dimensional closed submanifold of \( \mathbb{R}^6 \). Because \( \{G_1, G_2\} = 0 \), we deduce that \( \dim(\ker (\omega_0|_M)) = 2 \) and \( \omega_0|_M \) is a presymplectic form of constant rank 2 (see \([6, \text{p.} 27]\)). We define a \( T^2 \)-action \( \rho \) on \( M \) as follows. We put \( z_j = x_j + \sqrt{-1}y_j \) and for \((e^{\sqrt{-1}2\pi s_1}, e^{\sqrt{-1}2\pi s_2}) \in T^2, s_1, s_2 \in \mathbb{R}, \) we set
\[
\begin{align*}
  f_{s_1} : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} &\mapsto \begin{pmatrix} e^{\sqrt{-1}2\pi s_1} & 0 & 0 \\ 0 & e^{\sqrt{-1}2\pi s_1} & 0 \\ 0 & 0 & e^{\sqrt{-1}2\pi s_1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \\
g_{s_2} : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} &\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\sqrt{-1}2\pi s_2} & 0 \\ 0 & 0 & e^{\sqrt{-1}2\pi s_2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix},
\end{align*}
\]
and define
\[
\rho(e^{\sqrt{-1}2\pi s_1}, e^{\sqrt{-1}2\pi s_2})(z) := f_{s_1} \circ g_{s_2}(z) = g_{s_2} \circ f_{s_1}(z), \quad z = (z_1, z_2, z_3) \in M.
\]
Then \( \rho(e^{\sqrt{-1}2\pi s_1}, e^{\sqrt{-1}2\pi s_2})(z) \in M \) and we can easily check that
\[
\rho(e^{\sqrt{-1}2\pi s_1}, e^{\sqrt{-1}2\pi s_2})^* \lambda = \lambda
\] (7)
and \( \rho \) is a free action preserving \( d\lambda \). The infinitesimal generators of \( \rho \) are given by
\[
\begin{align*}
  Z_1 &= \frac{d}{ds_1} f_{s_1}(x, y) \bigg|_{s_1=0} = \sum_{j=1}^3 \left( -2\pi y_j \frac{\partial}{\partial x_j} + 2\pi x_j \frac{\partial}{\partial y_j} \right), \\
  Z_2 &= \frac{d}{ds_2} g_{s_2}(x, y) \bigg|_{s_2=0} = \sum_{j=2}^3 \left( -2\pi y_j \frac{\partial}{\partial x_j} + 2\pi x_j \frac{\partial}{\partial y_j} \right),
\end{align*}
\]
and therefore, the function
\[
\lambda(Z_1) = -\pi \sum_{j=1}^3 (x_j^2 + y_j^2) = -\pi (x_1^2 + y_1^2 + c_2)
\]
is nonconstant on \( M \) and the function
\[
\lambda(Z_2) = -\pi (x_2^2 + y_2^2 + x_3^2 + y_3^2) = -\pi c_2
\]
8
is constant on $M$. It follows that $(M, \omega_0|_M)$ satisfies the conditions (i), (ii) of Theorem 2 but not the additional condition in (2) of Theorem 2. Thus, the foliation $\mathcal{F}_{\omega_0|_M}$ has at least two leaves which are homeomorphic to $T^2$.

In this case, by (7), the function $S_1$ defined in the proof of Theorem 2 is given by

$$S_1 = - \int_{T^2} \left( p(e^{\sqrt{-1}2\pi s_1} e^{\sqrt{-1}2\pi s_2}) \lambda \right)(Z_1) \ d\sigma = - \int_{T^2} \lambda(Z_1) \ d\sigma = -\lambda(Z_1).$$

Due to the compactness of $M$, there exist $p_{\max}, p_{\min} \in M$ such that $S_1(p_{\max})$ is a maximal value and $S_1(p_{\min})$ is a minimal value of $S_1$. By the former part of the last paragraph of the proof of Theorem 2, the function $S_1$ is constant along the leaves $\mathcal{F}_{\omega_0|_M}(p_{\max})$ and $\mathcal{F}_{\omega_0|_M}(p_{\min})$. In fact, by using the method of Lagrange multipliers, we see that $S_1$ has only two critical submanifolds and $S_1$ takes the maximal value $S_1(p_{\max}) = \pi(\sqrt{c_1} + c_2)$ on

$$\mathcal{F}_{\omega_0|_M}(p_{\max}) = \{(x, y) \in M \mid x_1^2 + y_1^2 = \sqrt{c_1}, \ x_2^2 = y_2^2 = 0, \ x_3^2 + y_3^2 = c_2\}$$

and the minimal value $S_1(p_{\min}) = \pi(\sqrt{c_1} - c_2 + c_2)$ on

$$\mathcal{F}_{\omega_0|_M}(p_{\min}) = \{(x, y) \in M \mid x_1^2 + y_1^2 = \sqrt{c_1} - c_2, \ x_2^2 + y_2^2 = c_2, \ x_3 = y_3 = 0\}.$$

Moreover, we can easily check that

$$Z_1(q) \notin \ker \omega(q) \quad \text{for} \ q \in M \setminus (\mathcal{F}_{\omega_0|_M}(p_{\max}) \cup \mathcal{F}_{\omega_0|_M}(p_{\min})). \quad (8)$$

Another example can be found in [6, p.24], which satisfies the condition in (2) of Theorem 2.

**Example 6.** We also consider $(\mathbb{R}^6, \omega_0 = d\lambda)$, where $\lambda$ is the Liouville form. Define functions on $\mathbb{R}^6$ by

$$G_1 := x_1^2 + y_1^2, \quad G_2 := \sum_{j=1}^{3} (x_j^2 + y_j^2).$$

For real numbers $c_2 > c_1 > 0$, we set

$$M := \{(x, y) \in \mathbb{R}^6 \mid G_1 = c_1, \ G_2 = c_2\}$$

then, as in Example 5, $M$ is a 4-dimensional closed manifold with a presymplectic form $\omega_0|_M$ of constant rank 2. We shall define a $T^2$-action on $M$. Put $z_j = x_j + \sqrt{-1}y_j$.

For $(e^{\sqrt{-1}2\pi s_1}, e^{\sqrt{-1}2\pi s_2}) \in T^2, \ s_1, s_2 \in \mathbb{R}$, we set

$$f_{s_1} : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} e^{\sqrt{-1}2\pi s_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix},$$

$$g_{s_2} : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} e^{\sqrt{-1}2\pi s_2} & 0 & 0 \\ 0 & e^{\sqrt{-1}2\pi s_2} & 0 \\ 0 & 0 & e^{\sqrt{-1}2\pi s_2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$
and define
\[ \rho(e^{\sqrt{-1} \pi s_1}e^{\sqrt{-1} \pi s_2})(z) := f_{s_1} \circ g_{s_2}(z) = g_{s_2} \circ f_{s_1}(z), \quad z = (z_1, z_2, z_3) \in M. \]

We see that this action is free and preserves \( \omega_0 \). Similar to Example 5, we denote the infinitesimal generators of \( \rho \) by \( Z_1, Z_2 \). Then the functions \( \lambda(Z_1), \lambda(Z_2) \) are constant on \( M \). Thus, the foliation \( F_{\omega_0|M} \) coincides with the foliation of the \( T^2 \)-action \( \rho \) and therefore, all leaves of \( F_{\omega_0|M} \) are homeomorphic to \( T^2 \).

Similar to the proof of Theorem 2, we have the following result for arbitrary \( r \geq 1 \).

**Theorem 7.** Let \( M \) be a \((2n + r)\)-dimensional oriented closed \( C^\infty \)-manifold with an exact presymplectic form \( \omega \), that is, there exists a 1-form \( \alpha \in \Omega^1(M) \) such that the 2-form \( \omega = d\alpha \) has rank \( 2n \) everywhere. Assume that \( M \) admits a locally free \( T^r \)-action with the following conditions:
(i) The \( T^r \)-action preserves \( \omega \),
(ii) The functions \( \alpha(Z_i) \), \( i = 2, \ldots, r \), are constant on \( M \),
where \( Z_1, Z_2, \ldots, Z_r \) denote the infinitesimal generators of the \( T^r \)-action. Then the \( r \)-dimensional \( C^\infty \)-foliation \( F_\omega \) has at least two leaves which are homeomorphic to an \( r \)-dimensional torus. Moreover, if the function \( \alpha(Z_1) \) is also constant, then \( F_\omega \) coincides with the foliation of the \( T^r \)-action and hence, all leaves of \( F_\omega \) are homeomorphic to an \( r \)-dimensional torus.

Theorem 7 also gives a partial answer to Question in Section 1. If \( r = 1 \), then Theorem 7 agrees with Theorem 1.

**Remark.** We emphasize that in Theorem 7 (and 2) we do not assume that \((M, \omega)\) is \( r \)-contact, which means that \( M \) carries \( r \) linearly independent non-vanishing 1-forms \( \alpha_1, \ldots, \alpha_r \) with a splitting \( TM = \mathcal{R} \oplus (\cap_{i} ker \alpha_i) \) satisfying that \( d\alpha_i|_{\cap_{i} ker \alpha_i} \) is non-degenerate and \( ker d\alpha_i = \mathcal{R} \) for every \( i \). In our context, \( \mathcal{R} \) corresponds to \( ker \omega \). Finamore [3, Theorem 3.23] proved that if a closed presymplectic manifold \((M, \omega)\) is \( r \)-contact with a special metric, then the \( r \)-dimensional foliation defined by \( \mathcal{R} \) has at least two leaves which are homeomorphic to an \( r \)-dimensional torus. In the case where \( M \) is \( r \)-contact, by definition, \( M \) admits a locally free \( \mathbb{R}^r \)-action with the infinitesimal generators \( R_1, \ldots, R_r \in \mathcal{R} \). On the other hand, in Theorem 7, \( M \) does not always satisfy that \( Z_1 \in ker \omega \), as \( \delta \) in Example 5 shows. Thus, Theorem 3.23] and Theorem 7 are independent results.

**Acknowledgement**

This paper is a part of the author's Master thesis presented in February 2022 at Ibaraki University. I would like to thank my supervisor, Prof. Hiroshi Iriyeh for suggesting the problem, enlightening discussions and helpful comments.
References

[1] A. Banyaga, P. Rukimbira, *On characteristics of circle invariant presymplectic forms*, Proc. Amer. Math. Soc. 123 (1995), no. 12, 3901–3906. doi:10.2307/2161922

[2] C. Chevalley, *Theory of Lie groups*, Princeton, 1946.

[3] D. Finamore, *Contact foliations and generalised Weinstein conjectures*, doi:10.48550/arXiv.2202.07622

[4] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. 114 (1993), 515–563. doi:10.1007/BF01232679

[5] Y. Matsushima, *Differentiable manifolds*, Translated from the Japanese by E. T. Kobayashi. Pure and Applied Mathematics, 9. Marcel Dekker, Inc., New York, 1972. vii+303 pp.

[6] J. Moser, *A fixed point theorem in symplectic geometry*, Acta Math. 141 (1978), no. 1-2, 17–34. doi:10.1007/BF02545741

[7] C. Taubes, *The Seiberg-Witten equations and the Weinstein conjecture*, Geom. Topol. 11(4) (2007), 2117–2202. doi:10.2140/gt.2007.11.2117

[8] C. Viterbo, *A proof of the Weinstein conjecture in \( \mathbb{R}^{2n} \)*, Ann. Inst. H. Poincaré, Anal. Non linéaire, 4(4) (1987), 337–357. doi:10.1016/S0294-1449(16)30363-8

[9] A. Weinstein, *On the hypothesis of Rabinowitz’s periodic orbit theorems*, J. Diff. Equ. 33 (1979), 353–358. doi:10.1016/0022-0396(79)90070-6

Present Address:
Numata Girls’ High School,
Higashikurauchimachi 753-3, Numata, Gumma, 378-0043, Japan.
e-mail: hagiwara-asuka@edu-g.gsn.ed.jp