Degeneration of the Julia set to singular loci of algebraic curves

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We show that, when a non-integrable rational map changes to an integrable one continuously, a large part of the Julia set of the map approach indeterminate points (IDP) of the map along algebraic curves. We will see that the IDPs are singular loci of the curves.

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I. INTRODUCTION

When we study the transition of a non-integrable map to an integrable one, we must know the behavior of the Julia set [1]. If there exists the Julia set, some orbits generated by the map are disturbed by the set and behave chaotic. On the other hand an integrable map has no Julia set, so that we can decide the behavior analytically for all initial points.

The Julia set is the closure of the set of repulsive periodic points of a map. It has been studied, for a long time, how it behaves and how it is created from integrable maps. There have been known some important results, such as the Poincaré-Birkhoff fixed point theorem and the KAM theorem [2, 3], which describe how the transition takes place. Since this

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phenomenon is quite singular, however, all results so far known are based on perturbation.

We would like to know analytically what happens at the transition point. How and where the Julia set disappears in an integrable limit. To explore the phenomenon it is convenient to study a map with a parameter which interpolates between these two regimes. We have been studying many rational maps in this way and have shown how periodic points of low periods move around as the parameter varies and where they approach in the integrable limit \[4\].

We have shown, in our previous works \[4, 6\], that the periodic points of an integrable map form a variety different for each period. We called such a variety an invariant variety of periodic points, or IVPP, because the variety is determined by the invariants of the map alone. Some of the periodic points of a non-integrable map were shown to approach IVPPs in the integrable limit. An interesting fact found in \[4\] was that there are many other points which move, instead of IVPP, to indeterminate points (IDP) of the map. In other words the IDP, where the denominator and the numerator of the map vanish simultaneously, are the source of the Julia set, such that the Julia set is created as the map becomes non-integrable.

The work of \[4\] was done by studying every path of a periodic point analytically as the value of the parameter changes, by using computer algebra. Although this method is sufficient to have information of the path one by one, it does not provide global information. In this note we would like to study the global feature of the transition. Namely we derive algebraic curves along which the periodic points approach the IDPs. We will show that a set of periodic points of each period are on one algebraic curve and move to the IDP simultaneously in the integrable limit. This means that the IDP is a singular locus of the curve. Since periodic points of all periods approach there altogether, a large part of the Julia set itself degenerate to the singular locus, if they do not approach IVPPs.

The singular locus of algebraic hypersurface has been one of the main subjects to study in mathematics \[5\]. When the characteristic is zero, like our case, the Hironaka theorem shows desingularization of the singular locus. It is interesting to mention that the singularity of an algebraic curve corresponds to the integrable limit of non-integrable rational maps. We can provide many examples which show this correspondence, although we present here only one of them.
II. PATHS OF PERIODIC POINTS TOWARD INTEGRABLE LIMIT

We consider, in this note, only one simple map to clarify our argument. We can present many other examples to support our results, but they are more or less similar. The map we consider is the following:

\[ (x, y) \to F_a(x, y) = \left( \frac{x}{1 - x - a}, \frac{y}{1 - y} \right) \tag{1} \]

where \( x, y \in \mathbb{C} \) and \( a \in \mathbb{C} \) is a continuous parameter. Notice that, when \( a = 0 \), this map has an IDP at \( (x, y) = (1, 1) \).

A. Integrable limit

When \( a = 0 \), the map \( F_0(x, y) \) is integrable. In fact, \( F_0(x, y) \) has the invariant

\[ r = xy \]

and reduces to the Möbius map

\[ x \to X = \frac{x - r}{1 - x}. \]

The periodic points of \( n \) period of the map (1) are obtained by solving the condition

\[ F_a^n(x, y) = (x, y), \quad n = 2, 3, 4, ... \tag{2} \]

When the map is integrable, however, there is a simple way to find them. Namely the singularity confinement enables us to generate IVPPs iteratively [4, 7]. To start with we choose an initial point at \( p(0) = (1, r) \), so that the map undergoes according to

\[ p(0) \to (\infty, 0) \to (-1, -r) \to \left( -\frac{1 + r}{2}, -\frac{2r}{1 + r} \right) \]

\[ \to \left( -\frac{1 + 3r}{3 + r}, -\frac{r(3 + r)}{1 + 3r} \right) \to \left( -\frac{1 + 6r + r^2}{4(1 + r)}, -\frac{4r(1 + r)}{1 + 6r + r^2} \right) \to \cdots. \tag{3} \]

Since the point \( p(n) = (x^{(n)}, y^{(n)}) \), after \( n \) steps of the map, must diverge at the periodic points of period \( n - 1 \), the denominator of \( x^{(n)} \) vanishes there. For example, from (3), we see that the periodic points of period 3 is on the line specified by \( r + 3 = 0 \). In this way we find that the IVPP of period \( n \) is given by the curve,

\[ \left\{ x, y \mid \gamma^{(n)}(x, y) = 0 \right\}, \quad \gamma^{(n)}(x, y) := xy + \tan \frac{\pi}{n}, \quad n = 3, 4, 5, ... \]
We notice that there is no IVPP of period 2 when $a = 0$.

**B. Fixed points**

There are two fixed points of $F_a(x, y)$ at

$$(x, y) = (0, 0) \text{ and } (-a, 0),$$

Although the fixed points are not considered being periodic in general, they form the line $y = x$ in the integrable limit in this particular map.

**C. Period 2 points**

When $a \neq 0$, we must solve the periodicity condition (2) to find the period 2 points. They are at $(x, y) = (x_j^{(2)}, y_j^{(2)}), \ j = 1, 2,$ where

$$
(x_1^{(2)}, y_1^{(2)}) = \left( 1 - \frac{a}{2} + \sqrt{\frac{a(a^2 - 4)}{a - 4}}, \ 1 + \frac{a(2 - a)}{4} - \frac{a}{2} \sqrt{\frac{a(a^2 - 4)}{a - 4}} \right),
$$

$$(x_2^{(2)}, y_2^{(2)}) = \left( 1 - \frac{a}{2} - \sqrt{\frac{a(a^2 - 4)}{a - 4}}, \ 1 + \frac{a(2 - a)}{4} + \frac{a}{2} \sqrt{\frac{a(a^2 - 4)}{a - 4}} \right). \quad (4)
$$

Now we recall that, in the $a = 0$ case, there is no IVPP of period 2, while the period 2 points of (4) exist at $a = 0$. But the point $(x, y) = (1,1)$, where the period 2
points of (4) approach, is exactly the IDP of the map $F_0(x, y)$. Therefore all period 2 points approach the IDP in this case, and none of them move to IVPP, in the integrable limit.

The explicit expression of the points like (4) is not easy to find as the period $n$ becomes large. It will be more convenient to present the polynomial function $K^{(n)}_a(x)$ from which we can derive $x^{(n)}_j$ by solving $K^{(n)}_a(x) = 0$, and another polynomial function $L^{(n)}_a(x)$ so that $y^{(n)}_j$ is given by $y^{(n)}_j = L^{(n)}_a(x^{(n)}_j)$. In the period 2 case we obtain

$$K^{(2)}_a(x) = (a - 4)x^2 + (a - 2)(a - 4)x - 2(a - 2)(a - 1),$$
$$L^{(2)}_a(x) = \left(1 - \frac{a}{2}\right)(x + a),$$

from which we find (4) immediately. This is the way we studied the behavior of periodic points in our previous papers [4, 6, 7].

Now we want to know the paths of the periodic points along which they move as $a$ changes and approaches in the limit $a = 0$. We can do it if we eliminate the parameter $a$ from $K^{(2)}_a(x) = 0$ and $y - L^{(2)}_a(x) = 0$ of (5). The result we obtain is the algebraic curve $G^{(2)}(x, y) = 0$, with

$$G^{(2)}(x, y) = (2 - x)^2(1 - y)^2 - 3(1 - x)(1 - y)(2 - x) + (1 - x)^2(2 - x + xy).$$

This curve certainly passes the IDP $(1, 1)$, hence $G^{(2)}(1, 1) = 0$, as we can check easily. From (5) we see that it corresponds the integrable limit $a = 0$. To see the paths of period 2 points, we can draw the curve on the $(x, y)$ real plane. We find a curve in FIG. 1.

Notice that the IDP $(1, 1)$ is shifted to the origin in this graph. From this picture it is apparent that the IDP is the singular locus of the curve (6). In fact we can convince ourselves that the multiplicity of the curve at the point $(1, 1)$ is two. Since $K^{(2)}_a(x)$ and $L^{(2)}_a(x)$ are smooth functions of $a$, two periodic points must approach IDP along this curve at the same time, continuously as $a$ becomes small.
By solving (2) for $n = 3$, we obtain the functions $K_a^{(3)}(x)$ and $L_a^{(3)}(x)$, which are given by

$$K_a^{(3)}(x) = 6(a - 2)x^9 + 3(a^2 - 13a + 12 + a^3)x^8 + 3(a - 2)(3a^3 - 11a^2 + 11a - 24)x^7$$
$$+ (a - 2)(6a^4 - 83a^3 + 289a^2 - 543a + 456)x^6$$
$$+ (516a^4 - 26a^5 - 2189a^3 + 4494a^2 - 7a^6 - 4686a + 1944)x^5$$
$$+ (124a^5 - 12a^7 + 6960a - 8940a^2 + 5722a^3 - 1780a^4 + 66a^6 - 2088)x^4$$
$$+ (63a^7 + 9335a^2 - 6a^8 - 5310a - 8069a^3 - 142a^4 - 538a^5 + 3530a^4 + 1104)x^3$$
$$- (a - 1)(a^2 - 3a + 3)(a^6 - 16a^5 + 27a^4 + 170a^3 - 547a^2 + 462a - 48)x^2$$
$$+ (a^2 - 3a + 3)(2a^5 - 9a^4 - 23a^3 + 108a^2 - 60a - 36)(a - 1)^2x$$
$$- (a^2 - 3a + 3)(a^3 + 2a^2 - 9a - 12)(a - 1)^4$$

$$L_a^{(3)}(x) = \frac{[a^2 - 3a + 3](3x^4 + 2(4a - 7)x^3 + (a - 2)(7a - 8)x^2$$
$$+(a - 1)(2a^2 - 9a + 6)x - (a - 1)^3) + 2x^3(1 - x)(a - 2]}{x(x + a - 3)(3x^3 + 3(2a - 3)x^2 + (3a^2 - 10a + 5)x - 2a^2 + 1 + a)}.$$  

From this result we see that there are nine period 3 points. It is already not easy to know how each of the nine points moves as the parameter $a$ changes. Nevertheless we can say that, as $a$ becomes small, three of them approach to the IVPP curve $xy + 3 = 0$ and the rest move toward the IDP $(1, 1)$. We can convince ourselves if we notice that $K_a^{(3)}(x)$ factorizes
to

\[ K_0^{(3)}(x) = (x - 1)^6(x^3 + 3x^2 - 9x - 3) \]

at \( a = 0 \).

We have studied, in our previous work \( 4 \), the behavior of the 3 point Lotka-Volterra map by using this method. The best we could do was to investigate the behavior of period 2 points. The new method we used in the previous subsection, however, suggests that we might get much more information if we can eliminate the parameter \( a \) from \( K_a^{(3)}(x) = 0 \) and \( y - L_a^{(3)}(x) = 0 \). In fact we can use the elimination program of computer algebra to obtain an algebraic curve \( G^{(3)}(x, y) = 0 \) in this way, where

\[
G^{(3)}(x, y) = x^3(x - 1)^5y^9 + x^2(3x^4 - 12x^3 + 15x^2 - 16x + 15)(x - 1)^3y^8 \\
- x(x - 1)(16x^7 - 81x^6 + 103x^5 + 109x^4 - 396x^3 + 392x^2 - 165x + 18)y^7 \\
- (-486 + 713x^3 - 13x^{10} + 2394x^5 + 68x^9 - 1568x^6 + 1044x - 220x^8) \\
+ 717x^2 - 1864x^4 + x^{11} + 640x^7)y^6 \\
+ (-2059x^4 + 344x^9 + 5x^{11} + 1278x + 2493x^7 + 3570x^5 - 891 - 3542x^6 \\
- 46x^3 - 52x^2 - 1182x^8 - 58x^{10})y^5 \\
- (-4025x^6 + 2493x^5 - 594 - 92x^{10} - 497x^4 + 8x^{11} + 3393x^7 \\
- 412x^2 - 462x^3 + 402x + 498x^9 - 1656x^8)y^4 \\
+ (1436x^5 + 351x^9 + 150x^2 - 171 + 2129x^7 + 283x^4 - 2619x^6 - 1067x^8 \\
+ 568x^3 - 24x - 64x^{10} + 4x^{11})y^3 \\
+ 2(126x^3 - 3x^2 - 5x^4 - 68x^9 - 287x^5 + 9 + 8x^{10} + 216x^8 + 429x^6 \\
+ 374x^7 + 9x)y^2 + 4x^3(47x^4 - 9 + 5x^6 + 15x^2 - 26x^5 - 38x^3 - 6x)y \\
- 8x^6(-3x + x^2 + 3).)
\]

If we plot the result we obtain a curve, as we show in Fig.2 and Fig.3.

This picture shows us many important information, which we can not guess from the algebraic expression of \( G^{(3)}(x, y) \). The IDP \((1, 1)\) is again the singular locus of the curve. The multiplicity of the point is six, so that six period 3 points approach the singularity in the integrable limit.
We can use the computer algebra to solve the periodicity condition (2) in the case \( n = 4 \) as well and obtain \( K_\alpha^{(4)}(x) \) and \( L_\alpha^{(4)}(x) \). Since their expressions are too big to present here, we report only the special case \( a = 0 \) of \( K_\alpha^{(4)}(x) \). It is given by

\[
K_0^{(4)}(x) = (x - 1)^{12}(x^4 + 4x^3 - 6x^2 - 4x + 1)
\]
from which we can see that twelve out of sixteen points of period 4 move to the IDP (1, 1) and four other points approach the IVPP of period 4.

In order to find the paths of the periodic points as \( a \) becomes small, we must derive the algebraic curve by the elimination of \( a \) from \( K_4^{(4)}(x) = 0 \) and \( L_4^{(4)}(x) = y \). The formula of the curve \( G^{(4)}(x, y) = 0 \), thus obtained, is the 27th degree in \( x \) and the 22nd degree in \( y \). It is given in Appendix. If we draw the curve, we again confirm that the IDP (1, 1) is the singular locus of the curve. In the period 4 case we can check that this is a 12-ple point, hence twelve lines cross at this point, as we see in Fig.4 and Fig.5.

![FIG. 4: Path of period 4 points.](image)

![FIG. 5: Details of Fig.4 near IDP](image)
III. REMARKS

Some remarks are in order.

1. We have presented the behavior of periodic points in this note only those of small number of period. Nevertheless we are certain that all other periodic points of higher periods behave similar, because of the IVPP theorem \[4, 6\]. In particular the periodic points will approach the IDP \((1, 1)\) as \(a\) becomes zero, if they do not move to the IVPPs. Since this is true for all periods, we can say that a large part of the Julia set approach the IDP. This is the phenomenon we found already in \[4\], but we could not understand it well. Now the reason why this happens is clear because the IDP is a singular locus of the curve of each period. The loci of the curves of all periods degenerate there altogether.

2. The Julia set is understood generally that it is produced via destruction of invariant tori by a perturbation. The Poincaré-Birkhoff fixed point theorem as well as the KAM theorem are based on this point of view \[2, 3\]. This corresponds to the part of the Julia set which approach to the IVPPs in our analysis. Since we know now that a large part of the Julia set is produced also from the IDPs of a rational map, we must change our view.

3. In this note we have studied a simple rational map, so that we can manage all formula explicitly. In particular the IDP of this map is a point. If we study higher dimensional maps, we should have a set of IDPs which might form a hypersurface. In fact the IDPs of \(d\) dimensional Lotka-Volterra map form a variety of \(d - 2\) dimensional hypersurface. Therefore we should have higher dimensional singular locus of higher dimensional algebraic varieties. We will report some results in our forthcoming paper.
The function $G^{(4)}(x, y)$ is as follows:

$$G^{(4)}(x, y) = (8816x^{12} + 256x^{10} - 13857x^{17} - 33909x^{15} - 23x^{21} - 2240x^{11} + 5271x^{18} + 233x^{20} + 25683x^{16} - 1375x^{19} + x^{22} - 20596x^{13} + 31740x^{14})y^{22} + (490827x^{15} - 396294x^{16} + 26416x^{11} - 105480x^{18} - 442001x^{14} - 119x^{22} + 5x^{23} + 1268x^{21} + 237030x^{17} + 34544x^{19} - 114896x^{12} - 8074x^{20} - 288x^{9} - 1808x^{10} + 278870x^{13})y^{21} + (2807x^{22} + 11104x^{8} - 2x^{25} - 394x^{23} + 273248x^{10} + 1197014x^{12} - 1963200x^{13} - 79856x^{9} + 40x^{24} - 633472x^{11} + 2529495x^{16} - 2971035x^{15} + 803945x^{18} - 1642281x^{17} + 80059x^{20} - 16653x^{21} - 293843x^{19} + 2703024x^{14})y^{20} + (18x^{25} + 1695854x^{11} + 92029x^{21} + 88736x^{8} - 388x^{24} + 7472120x^{15} - 21794x^{22} + 4259305x^{13} + 4701916x^{17} + 963340x^{19} - 320012x^{20} + 18408x^{9} - 644680x^{10} - 6165979x^{14} - 24640x^{7} - 240959x^{18} - 6883117x^{16} - 2825251x^{12} + 3730x^{23})y^{19} + (3398840x^{8} - x^{26} + 980318x^{19} - 10510320x^{14} - 5198624x^{9} + 11110019x^{13} - 1214032x^{7} + 809x^{24} + 7506580x^{17} - 158679x^{21} + 191392x^{6} - 9693x^{23} - 3809547x^{18} - 8577042x^{12} + 82868x^{20} + 55769x^{22} + 10050359x^{15} - 17x^{25} - 9738850x^{16} + 3600742x^{10} + 2239109x^{11})y^{18} + (48136688x^{11} - 46259480x^{12} - 44009978x^{10} - 104615268x^{17} - 563680x^{5} - 190x^{25} + 137427935x^{16} + 29195734x^{9} + 4343801x^{20} - 112009280x^{15} + x^{27} - 8926428x^{8} + 53115198x^{18} + 23046203x^{13} + 2481104x^{6} - 11x^{26} + 149650x^{22} + 3914x^{24})y^{17} + (1092544x^{4} + 188973x^{23} - 20840938x^{8} - 1204091x^{22} + 57627698x^{19} - 173645515x^{11} + 463350900x^{12} - 5294816x^{5} + 5013937x^{21} + 10036456x^{6} - 1539x^{25} - 157838037x^{18} - 5479904x^{7} - 14x^{27} + 293x^{26} + 60783280x^{9} + 226949466x^{15} + 192896610x^{14} - 17631661x^{20} - 516416271x^{13} - 10879x^{24} + 315090997x^{17} - 402584676x^{16} - 32082813x^{10})y^{16}
\[-1950208x^3 + 6687808x^4 - 78187648x^6 + 151096242x^7 - 8407000x^{21} + 1566120x^{22} + 55274x^{23} + 605411253x^{10} - 51206220x^{19} + 118483207x^{18}
+ 30729419x^{15} - 255344719x^{17} + 327759324x^{16} - 424750858x^9 - 1208014178x^{12}
+ 1593873742x^{13} + 82x^{27} - 2086x^{26} + 21189x^{25} - 96900x^{24} + 7241576x^5
- 4712192x^8 + 80889113x^{11} - 914704386x^{14} + 23562014x^{20}\]
\[ (+(-35939904 + 64769184x + 11553224x^2 - 378347964x^3 + 266721942x^4 \\
+1146474658x^7 - 245870933x^{21} + 649653470x^{20} + 65910006x^{22} - 12401664x^{23} \\
+1033130062x^{10} - 1189766722x^{19} + 1416378080x^{18} - 1843803742x^{15} \\
-98675451x^{17} + 669723226x^{16} - 1246384040x^{9} + 1545608278x^{12} \\
-3868999655x^{13} - 72x^{27} + 5152x^{26} - 124290x^{25} + 1574659x^{24} - 92229270x^{6} \\
+262022404x^{5} - 142305796x^{8} + 215331583x^{11} + 850801995x^{14})y^{10} \\
+(-37974512 - 41989448x - 131341496x^{2} + 251838780x^{3} - 1956260354x^{4} \\
-101728022x^{4} + 325964322x^{6} - 552148272x^{7} + 134646500x^{21} + 40523573x^{20} \\
-31332231x^{22} - 514714x^{24} + 4998235x^{23} + 1017113492x^{10} + 829836329x^{19} \\
-1036638499x^{18} - 858876830x^{15} + 402699508x^{17} + 861801359x^{16} \\
-1329186418x^{9} - 5306469228x^{12} + 5385746848x^{13} - 732x^{26} + 30148x^{25} \\
-112441894x^{5} + 820000812x^{8} + 1790817058x^{11} )y^{9} \\
+(-25913976 + 10793432x + 90539488x^{2} - 121110249x^{14} \\
-91478756x^{3} + 942164x^{4} + 103915466x^{6} + 186707816x^{7} - 48061450x^{21} \\
+169140349x^{20} + 9315454x^{22} - 1176908x^{23} - 727043350x^{10} \\
-40442803x^{19} + 600405225x^{18} + 1437591457x^{15} - 318328630x^{17} \\
-679751531x^{16} + 2176833146x^{9} + 5093465542x^{12} - 3191570706x^{13} \\
-2804x^{25} + 86768x^{24} - 30060728x^{5} - 1246953650x^{8} - 2993945246x^{11} )y^{8} \\
+(11940584 + 3293712x - 37952592x^{2} + 11302896x^{3} + 12930744x^{4} - 178584834x^{6} \\
-123180198x^{7} + 10936264x^{21} - 47103882x^{20} - 1641007x^{22} + 144020x^{23} \\
-176324386x^{10} + 602080461x^{14} + 134914901x^{19} - 245425110x^{18} - 755767973x^{15} \\
+209205466x^{17} + 189135314x^{16} - 1348734748x^{9} - 2575626848x^{12} \\
+898925588x^{13} - 5596x^{24} + 70801624x^{5} + 102443616x^{8} + 2310495792x^{11} )y^{7} \]
\[(3862248x^3 - 84294803x^{17} + 104166188x^7 - 4489028x^4 + 755984650x^{12} - 3764504 + 68848621x^{18} - 301032512x^{14} - 503516084x^8 + 359635802x^{10} - 30361454x^{19} + 95239240x^6 - 6608x^{23} + 148260x^{22} - 1470444x^{21} + 212451793x^{15} - 997934348x^{11} - 64770610x^{13} - 6521550x^{16} + 8424977x^{20} - 45955316x^5 + 429526986x^9 + 9186216x^2 - 3666424x)y^6 + (259154032x^{11} - 165956678x^{10} + 97348x^{21} + 17621652x^5 - 4748x^{22} + 78297118x^{14} - 991048x^2 - 34967968x^{15} + 802280 - 62009120x^9 - 1714808x^3 - 132764454x^{12} + 21896808x^{17} - 30379682x^{13} + 831356x^4 + 1345800x - 870582x^{20} + 4357898x^{19} + 150628008x^8 - 13023482x^{18} - 10925854x^{16} - 53565568x^7 - 27540100x^6)y^5 + (40960x^{20} - 4471024x^5 - 51360x^2 - 26776688x^8 - 1996x^{21} + 3888022x^{16} + 10683404x^{13} + 14490878x^{12} - 3699380x^{17} - 209152x^4 - 578048x^9 - 110672 - 263920x - 13076026x^{14} - 42943952x^{11} + 4159888x^6 + 227600x^3 + 2813736x^{15} - 344788x^{19} + 15613788x^7 + 1527780x^{18} + 38852534x^{10})y^4 + (5009038x^{11} - 18294272x^{19} - 29513x^{23} + 39692303x^{14} - 794239x^{21} - 2090192x^7 - 2507336x^7 - 897118x^{12} + 2726588x^8 + 8928 + 1249024x^9 - 90772x^{18} + 10864x^{19} - 5415958x^{10} + 189702x^{15} - 1959908x^{13} + 27616x + 361440x^{17} - 163728x^6 - 3648x^3 + 23872x^2 - 444x^{20} - 675918x^{16} + 56800x^4 + 745312x^5 + 1418766x^{14})y^3 + (529396x^{10} - 7552x^4 - 73280x^5 - 132528x^9 + 5648x^{16} - 36392x^{12} - 320 - 63464x^{14} - 15380x^{17} + 227776x^{13} + 1692x^{18} - 40x^{19} + 195920x^7 - 72388x^{15} - 1024x^3 - 425036x^{11} - 1216x - 35520x^6 - 182424x^8 - 1728x^2)y^2 + (8608x^9 + 18824x^{11} + 3136x^5 + 12448x^{12} - 1568x^{16} - 12056x^{13} + 320x^4 - 4960x^7 + 5424x^{15} + 14288x^8 + 3584x^6 + 120x^{17} - 3088x^{14} - 37848x^{10})y + 1200x^{10} - 80x^{11} - 80x^{15} - 640x^9 - 560x^{12} + 80x^{13} - 800x^8 + 240x^{14} \]
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