Multiplicty and ellipticity of closed characteristics on compact
star-shaped hypersurfaces in $\mathbb{R}^{2n}$

Huagui Duan$^1$,∗ Hui Liu$^2$†

$^1$ School of Mathematical Sciences and LPMC, Nankai University,
Tianjin 300071, People’s Republic of China
$^2$ Key Laboratory of Wu Wen-Tsun Mathematics, Chinese Academy of Sciences,
School of Mathematical Sciences, University of Science and Technology of China,
Hefei, Anhui 230026, People’s Republic of China

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Abstract

In this paper, we firstly generalize some theories developed by I. Ekeland and H. Hofer in [EkH] for closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$ to star-shaped hypersurfaces. As applications we use Ekeland-Hofer theory and index iteration theory to prove that if a compact star-shaped hypersurface in $\mathbb{R}^4$ satisfying some suitable pinching condition carries exactly two geometrically distinct closed characteristics, then both of them must be elliptic. We also conclude that the theory developed by Y. Long and C. Zhu in [LoZ] still holds for dynamically convex star-shaped hypersurfaces, and combining it with the results in [WHL], [LLW], [Wan3], we obtain that there exist at least $n$ closed characteristics on every dynamically convex star-shaped hypersurface in $\mathbb{R}^{2n}$ for $n = 3, 4$.

Key words: Compact star-shaped hypersurfaces, closed characteristics, Hamiltonian systems, Ekeland-Hofer theory, index iteration theory.

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1 Introduction and main result

Let $\Sigma$ be a $C^3$ compact hypersurface in $\mathbb{R}^{2n}$ strictly star-shaped with respect to the origin, i.e., the tangent hyperplane at any $x \in \Sigma$ does not intersect the origin. We denote the set of all such hypersurfaces by $\mathcal{H}_{st}(2n)$, and denote by $\mathcal{H}_{con}(2n)$ the subset of $\mathcal{H}_{st}(2n)$ which consists of all

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†Partially supported by NSFC (Nos.11401555, 11371339), Anhui Provincial Natural Science Foundation (No. 1608085QA01). E-mail: huiliu@ustc.edu.cn.
strictly convex hypersurfaces. We consider closed characteristics \((\tau, y)\) on \(\Sigma\), which are solutions of the following problem

\[
\begin{aligned}
\dot{y} &= J N_\Sigma(y), \\
y(\tau) &= y(0),
\end{aligned}
\]

(1.1)

where \(J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\), \(I_n\) is the identity matrix in \(\mathbb{R}^n\), \(\tau > 0\), \(N_\Sigma(y)\) is the outward normal vector of \(\Sigma\) at \(y\) normalized by the condition \(N_\Sigma(y) \cdot y = 1\). Here \(a \cdot b\) denotes the standard inner product of \(a, b \in \mathbb{R}^{2n}\). A closed characteristic \((\tau, y)\) is prime, if \(\tau\) is the minimal period of \(y\). Two closed characteristics \((\tau, y)\) and \((\sigma, z)\) are geometrically distinct, if \(y(R) \neq z(R)\). We denote by \(T(\Sigma)\) the set of geometrically distinct closed characteristics \((\tau, y)\) on \(\Sigma \in \mathcal{H}_{st}(2n)\). A closed characteristic \((\tau, y)\) is non-degenerate if 1 is a Floquet multiplier of \(y\) of precisely algebraic multiplicity 2; hyperbolic if 1 is a double Floquet multiplier of it and all the other Floquet multipliers are not on \(U = \{ z \in \mathbb{C} \mid |z| = 1 \}\), i.e., the unit circle in the complex plane; elliptic if all the Floquet multipliers of \(y\) are on \(U\). We call a \(\Sigma \in \mathcal{H}(2n)\) non-degenerate if all the closed characteristics on \(\Sigma\) together with all of their iterations are non-degenerate.

Fix a constant \(\alpha\) satisfying \(1 < \alpha < 2\) and define the Hamiltonian function \(H_\alpha : \mathbb{R}^{2n} \to [0, +\infty)\) by

\[
H_\alpha(x) = j(x)^\alpha, \quad \forall x \in \mathbb{R}^{2n},
\]

(1.2)

where \(j\) is the gauge function of \(\Sigma\), i.e., \(j(x) = \lambda\) if \(x = \lambda y\) for some \(\lambda > 0\) and \(y \in \Sigma\) when \(x \in \mathbb{R}^{2n} \setminus \{0\}\), and \(j(0) = 0\). Then \(H_\alpha \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \cap C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R})\) and \(\Sigma = H_\alpha^{-1}(1)\). It is well-known that the problem (1.1) is equivalent to the following given energy problem of the Hamiltonian system

\[
\begin{aligned}
\dot{y}(t) &= J H'_\alpha(y(t)), \\
y(\tau) &= y(0),
\end{aligned}
\]

(1.3)

Denote by \(T(\Sigma, \alpha)\) the set of all geometrically distinct solutions \((\tau, y)\) of the problem (1.3). Note that elements in \(T(\Sigma)\) and \(T(\Sigma, \alpha)\) are one to one correspondent to each other.

The study on closed characteristics in the global sense started in 1978, when the existence of at least one closed characteristic was first established on any \(\Sigma \in \mathcal{H}_{st}(2n)\) by P. Rabinowitz in [Rab] and on any \(\Sigma \in \mathcal{H}_{con}(2n)\) by A. Weinstein in [Wei] independently, since then the existence of multiple closed characteristics on \(\Sigma \in \mathcal{H}_{con}(2n)\) has been deeply studied by many mathematicians, for example, studies in [EkL], [Ekl], [Szu], [HWZ1], [LoZ], [WHL], [Wan2] and [Wan3] for convex hypersurfaces. For the star-shaped hypersurfaces, in [Gir] of 1984 and [BLMR] of 1985, \#\(T(\Sigma)\) \(\geq n\) for \(\Sigma \in \mathcal{H}_{st}(2n)\) was proved under some pinching conditions. In [Vit1] of 1989, C. Viterbo proved a generic existence result for infinitely many closed characteristics on star-shaped hypersurfaces. In [HuL] of 2002, X. Hu and Y. Long proved that \#\(T(\Sigma)\) \(\geq 2\) for \(\Sigma \in \mathcal{H}_{st}(2n)\) on which all the closed characteristics and their iterates are non-degenerate. In [HWZZ2] of 2003, H. Hofer, K. Wysocki, and E. Zehnder proved any non-degenerate compact star-shaped hypersurface has either two or infinitely closed characteristics, provided that all stable and unstable manifolds of the hyperbolic closed characteristics intersect transversally. Recently \#\(T(\Sigma)\) \(\geq 2\) was first proved
for every $\Sigma \in \mathcal{H}_{\text{st}}(4)$ by D. Cristofaro-Gardiner and M. Hutchings in [CGH] without any pinching or non-degeneracy conditions. Different proofs of this result can also be found in [GHHM], [LLo1] and [GiC].

I. Ekeland and H. Hofer in [EkH] provided a close relationship between the set of Maslov-type indices of closed characteristics and the set of even positive integers, which is the core in studying the multiplicity and ellipticity of the closed characteristics on compact convex hypersurfaces (cf. [LoZ]). Our main goal in this paper is to generalize the theory of Ekeland-Hofer to compact star-shaped hypersurfaces and as its applications we give some multiplicity and stability results of closed characteristics on compact star-shaped hypersurfaces.

For the stability of closed characteristics on $\Sigma \in \mathcal{H}_{\text{st}}(2n)$ we refer the readers to [LiL] and [LLo2]. Specially, in [LLo2], H. Liu and Y. Long proved that $\Sigma \in \mathcal{H}_{\text{st}}(4)$ and $\# \mathcal{T}(\Sigma) = 2$ imply that both of the closed characteristics must be elliptic provided that $\Sigma$ is symmetric with respect to the origin.

Let $n(y)$ be the unit outward normal vector of $\Sigma$ at $y$ and $d(y) := n(y) \cdot y$, i.e., the distance between the origin of $\mathbb{R}^{2n}$ and the tangent hyperplane to $\Sigma$ at $y$, then $d(y) > 0$ for all $y \in \Sigma$ since $\Sigma$ is strictly star-shaped. Let $d = \min \{d(y) : y \in \Sigma\}$ and $R = \max \{|y| : y \in \Sigma\}$. In this paper, we prove, under suitable pinching condition, the symmetric condition in Theorem 1.4 of [LLo2] can be dropped, i.e., the following theorem holds.

**Theorem 1.1.** Suppose that $\Sigma \in \mathcal{H}_{\text{st}}(4)$ satisfy $\# \mathcal{T}(\Sigma) = 2$ and $R^2 < 2d^2$. Then both of the closed characteristics are elliptic.

**Remark 1.2.** Note that the pinching condition on $\Sigma$ in Theorem 1.1 is only used to get a contradiction in the study of the Subcase 1.2 of Case 1 in the proof of Theorem 1.1.

In Definition 1.2 of [HWZ1], an interesting class of contact forms on $S^3$ which are called dynamically convex contact forms was introduced. Similarly, we give the following definition:

**Definition 1.3.** $\Sigma \in \mathcal{H}_{\text{st}}(2n)$ is called dynamically convex if any closed characteristic $(\tau, y)$ on $\Sigma$ has its Maslov-type index not less than $n$.

Note that from the proof of Theorem 3.4 of [HWZ1], for $n = 2$, the above definition coincides with that of [HWZ1]. Also from the Remark before Definition 3.6 of [HWZ1] and Corollary 1.2 of [LoZ], we know any $\Sigma \in \mathcal{H}_{\text{con}}(2n)$ is dynamically convex. As mentioned before Definition 1.2 of [HWZ1], “strictly convex” is not a symplectically invariant concept, thus a dynamically convex $\Sigma \in \mathcal{H}_{\text{st}}(2n)$ need not to be convex.

In this paper, we also prove the main results of [LoZ], [WHL], [Wan2], [Wan3] and [HuO] hold for dynamically convex star-shaped hypersurfaces which cover the works of these literatures. Specially, we have:

**Theorem 1.4.** Let $\Sigma \in \mathcal{H}_{\text{st}}(2n)$ be dynamically convex. Then $\# \mathcal{T}(\Sigma) \geq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$. If $\Sigma$ is nondegenerate, then $\# \mathcal{T}(\Sigma) \geq n$. If $\# \mathcal{T}(\Sigma) < +\infty$, then there exists at least two elliptic closed characteristic on $\Sigma$, and at least $\lfloor n/2 \rfloor$ closed characteristics possessing irrational mean indices.

**Theorem 1.5.** Let $\Sigma \in \mathcal{H}_{\text{st}}(2n)$ be dynamically convex. Then $\# \mathcal{T}(\Sigma) \geq n$ for $n = 3, 4$.  

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Remark 1.6. Note that J. Gutt and J. Kang in [GuK] proved that if $\Sigma \in \mathcal{H}_{st}(2n)$ is non-degenerate and dynamically convex, then there exist at least $n$ closed characteristics on such $\Sigma$, whose iterates’ Conley-Zehnder indices possess the same parity. Note that their index definition is slightly different from ours. Also recently, M. Abreu and L. Macarini in [AbM] gave a sharp lower bound for the number of geometrically distinct contractible periodic orbits of non-degenerate dynamically convex Reeb flows on prequantizations of symplectic manifolds that are not aspherical, which implies results of [GuK] (cf. Corollary 2.9 of [AbM]). We also mention that very recently, Y. Long, W. Wang and the authors in [DLLW] proved some sharp multiplicity results for non-degenerate star-shaped hypersurfaces under some index conditions, which are weaker than the convex or dynamically convex case. Our Theorem 1.4 and Theorem 1.5 give new multiplicity and stability results for the degenerate, dynamically convex, star-shaped hypersurfaces.

This paper is arranged as follows. In Sections 2 and 3, following the frameworks of [Vit1], [LLW] and [Eke], we establish a variational structure for closed characteristics on star-shaped hypersurfaces and prove some theories in [EkH] hold for star-shaped case, we omit most of the details of the proofs of the theories below and only point out differences from [Vit1], [LLW] and [Eke] when necessary. In Section 3.2, we further study the critical values obtained in Section 3.1 when the star-shaped hypersurface is suitably pinched, which we will use to prove Theorem 1.1 in Section 4. In Section 4, we also explain how to get Theorems 1.4 and 1.5 as another application of Ekeland-Hofer theory. In Section 5 (an appendix), we briefly review the equivariant Morse theory and the resonance identities for closed characteristics on compact star-shaped hypersurfaces in $\mathbb{R}^{2n}$ developed in [LLW], which are used in the proof of Theorem 1.1.

In this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^+$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, complex numbers and positive real numbers respectively. We define the function $[a] = \max \{k \in \mathbb{Z} \mid k \leq a \}$, $\{a\} = a - [a]$, and $E(a) = \min \{k \in \mathbb{Z} \mid k \geq a \}$. Denote by $a \cdot b$ and $|a|$ the standard inner product and norm in $\mathbb{R}^{2n}$. Denote by $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ the standard $L^2$ inner product and $L^2$ norm. For an $S^1$-space $X$, we denote by $X_{S^1}$ the homotopy quotient of $X$ by $S^1$, i.e., $X_{S^1} = S^\infty \times_{S^1} X$, where $S^\infty$ is the unit sphere in an infinite dimensional complex Hilbert space. In this paper we use $\mathbb{Q}$ coefficients for all homological and cohomological modules. By $t \to a^+$, we mean $t > a$ and $t \to a$.

2 A variational structure for closed characteristics on compact star-shaped hypersurfaces

In Section 2 and 3, we fix a $\Sigma \in \mathcal{H}_{st}(2n)$.

As in Sections V.2 and V.3 of [Eke], we consider the following fixed period problem

$$
\begin{align*}
\dot{x}(t) &= JH'_\alpha(x(t)), \\
x(0) &= x(1).
\end{align*}
$$

(2.1)

Then solutions of (2.1) are $x \equiv 0$ and $x = \tau^{- \frac{1}{2-\alpha}} y(\tau t)$, where $(\tau, y)$ is a solution of (1.3).
For technical reasons (to get Proposition 2.5 below), we need to further modify the Hamiltonian, more precisely, we follow Page 624 of [Vit2], and let \( \epsilon \) satisfy \( \epsilon < 2\pi \), we can construct a function \( H \), which coincides with \( H_\alpha \) on \( U_A = \{ x \mid H_\alpha(x) \leq A \} \) for some large \( A \), and with \( \frac{1}{2}\epsilon |x|^2 \) outside some large ball, such that \( \nabla H(x) \) does not vanish and \( H''(x) < \epsilon \) outside \( U_A \). As in Proposition 2.7 of [Vit2], we have the following result.

**Proposition 2.1.** For small \( \epsilon \), there exists a function \( H \) on \( \mathbb{R}^{2n} \) such that \( H \) is \( C^1 \) on \( \mathbb{R}^{2n} \), and \( C^3 \) on \( \mathbb{R}^{2n} \setminus \{0\} \), \( H = H_\alpha \) in \( U_A \), and \( H(x) = \frac{1}{2}\epsilon |x|^2 \) for \( |x| \) large, and the solutions of the fixed period system
\[
\begin{align*}
\dot{x}(t) &= JH'(x(t)), \\
x(0) &= x(1),
\end{align*}
\] (2.2)
are the same with those of (2.1), i.e., the solutions of (2.2) are \( x \equiv 0 \) and \( x = \tau^{-\frac{1}{\alpha}} y(\tau t) \), where \( (\tau, y) \) is a solution of (1.3).

Note that the condition (2.2) of Lemma 2.2 of [Vit2] is only used to get Theorem 7.1 of [Vit2], so the other statements in [Vit2] also hold for our choice of the Hamiltonian function.

As in [BLMR] (cf. Section 3 of [Vit2]), we can choose some large constant \( K \) such that
\[
H_K(x) = H(x) + \frac{1}{2}K|x|^2
\] (2.3)
is a strictly convex function, that is,
\[
(\nabla H_K(x) - \nabla H_K(y), x - y) \geq \frac{\epsilon}{2}|x - y|^2,
\] (2.4)
for all \( x, y \in \mathbb{R}^{2n} \), and some positive \( \epsilon \). Let \( H_K^* \) be the Fenchel dual of \( H_K \) defined by
\[
H_K^*(y) = \sup\{ x \cdot y - H_K(x) \mid x \in \mathbb{R}^{2n} \}.
\] (2.5)
The dual action functional on \( X = W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \) is defined by
\[
F_K(x) = \int_0^1 \left[ \frac{1}{2}(J\dot{x} - Kx, x) + H_K^*(-J\dot{x} + Kx) \right] dt.
\] (2.6)
Then we have

**Lemma 2.2.** (cf. Proposition 3.4 of [Vit2]) Assume \( K \not\in 2\pi\mathbb{Z} \), then \( x \) is a critical point of \( F_K \) if and only if it is a solution of (2.2).

As is well known, when \( K \not\in 2\pi\mathbb{Z} \), the map \( x \mapsto -J\dot{x} + Kx \) is a Hilbert space isomorphism between \( X = W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \) and \( E = L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \). We denote its inverse by \( M_K \) and the functional
\[
\Psi_K(u) = \int_0^1 \left[ -\frac{1}{2}(M_Ku, u) + H_K^*(u) \right] dt, \quad \forall u \in E.
\] (2.7)
Then \( x \in X \) is a critical point of \( F_K \) if and only if \( u = -J\dot{x} + Kx \) is a critical point of \( \Psi_K \). We have a natural \( S^1 \)-action on \( X \) or \( E \) defined by
\[
\theta \cdot u(t) = u(\theta + t), \quad \forall \theta \in S^1, \ t \in \mathbb{R}.
\] (2.8)
Clearly both of $F_K$ and $\Psi_K$ are $S^1$-invariant. For any $\kappa \in \mathbb{R}$, we denote by
\[
\Psi_K^\kappa = \{ u \in L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}) \mid \Psi_K(u) \leq \kappa \}.
\]
Obviously, this level set is also $S^1$-invariant.

**Definition 2.3.** (cf. p.628 of [Vit2]) Suppose $u$ is a nonzero critical point of $\Psi_K$. Then the formal Hessian of $\Psi_K$ at $u$ is defined by
\[
Q_K(v) = \int_0^1 (-M_Kv \cdot v + H_K''(u)v \cdot v)dt,
\]
which defines an orthogonal splitting $E = E_- \oplus E_0 \oplus E_+$ into negative, zero and positive subspaces. The index and nullity of $u$ are defined by $i_K(u) = \dim E_-$ and $\nu_K(u) = \dim E_0$ respectively.

Similarly, we define the index and nullity of $x = M_Ku$ for $F_K$, which are denoted by $i_K(x)$ and $\nu_K(x)$ respectively. Then we have
\[
i_K(u) = i_K(x), \quad \nu_K(u) = \nu_K(x),
\]
which follow from the definitions (2.6) and (2.7). The following important formula was proved in Lemma 6.4 of [Vit2]:
\[
i_K(x) = 2n\left(\frac{K}{2\pi} + 1\right) + i^v(x) \equiv d(K) + i^v(x),
\]
where the Viterbo index $i^v(x)$ does not depend on $K$, but only on $H$.

By the proof of Proposition 2 of [Vit1], we have that $v \in E$ belongs to the null space of $Q_K$ if and only if $z = M_Kv$ is a solution of the linearized system
\[
\dot{z}(t) = JH''(x(t))z(t).
\]
Thus the nullity in (2.10) is independent of $K$, which we denote by $\nu^v(x) \equiv \nu_K(u) = \nu_K(x)$.

Since $x$ is a solution of (2.1) corresponding to a solution $(\tau, y)$ of (1.3), we also denote $i^v(x)$ and $\nu^v(x)$ by $i^v(y)$ and $\nu^v(y)$ respectively, and define $i(y) := i^v(y)$ and $\nu(y) := \nu^v(y)$. By Theorem 2.1 of [HuL], we have:

**Lemma 2.4.** Suppose $\Sigma \in \mathcal{H}_{st}(2n)$ and $(\tau, y) \in T(\Sigma)$. Then we have
\[
i(y^m) = i(y, m) - n, \quad \nu(y^m) = \nu(y, m), \quad \forall m \in \mathbb{N},
\]
where $i(y, m)$ and $\nu(y, m)$ are the Maslov-type index and nullity of $(m\tau, y)$.

By Propositions 3.9, 4.1 of [Vit2] and the same proof of Proposition 2.12 of [LLW], we have:

**Proposition 2.5.** $\Psi_K$ satisfies the Palais-Smale condition on $E$, and $F_K$ satisfies the Palais-Smale condition on $X$, when $K \notin 2\pi\mathbb{Z}$. 
3 Fadell-Rabinowitz index theory for closed characteristics on star-shaped hypersurfaces

3.1 Critical values in the free case

Recall that for a principal $U(1)$-bundle $E \to B$, the Fadell-Rabinowitz index (cf. [FaR]) of $E$ is defined to be $\sup \{ k \mid c_1(E)^{k-1} \neq 0 \}$, where $c_1(E) \in H^2(B, \mathbb{Q})$ is the first rational Chern class. For a $U(1)$-space, i.e., a topological space $X$ with a $U(1)$-action, the Fadell-Rabinowitz index is defined to be the index of the bundle $X \times S^\infty \to X \times_{U(1)} S^\infty$, where $S^\infty \to CP^\infty$ is the universal $U(1)$-bundle.

For any $\kappa \in \mathbb{R}$, we denote by

$$\Psi^-_K = \{ u \in L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \mid \Psi_K(u) < \kappa \}. \quad (3.1)$$

Then as in P.218 of [Eke], we define

$$c_i = \inf \{ \delta \in \mathbb{R} \mid \hat{I}(\Psi^-_K) \geq i \}, i \in \mathbb{N}. \quad (3.2)$$

where $\hat{I}$ is the Fadell-Rabinowitz index given above.

For $i \geq d(K)/2 + 1$, where $d(K) = 2n([K/2\pi] + 1)$, $c_i$ is well defined. In fact, by Proposition 5.7 of [Vit2], there exists constant $c$ such that $\Psi^-_K$ is $S^1$-equivariant homotopy equivalent with a $(d(K) - 1)$ dimensional sphere. Then $\hat{I}(\Psi^-_K) = d(K)/2$. Hence for $i \geq d(K)/2 + 1$, $c_i \geq c$ is well defined.

Then similar to Proposition 3 in P.218 of [Eke], we have

**Proposition 3.1.** For $i \geq d(K)/2 + 1$, $c_i$ is a critical value of $\Psi_K$.

**Proof.** For the reader’s convenience, we sketch a brief proof here and refer to Sections V.2 and V.3 of [Eke] for related details.

By the proof of Theorem V.2.9 of [Eke], if we replace $L^\beta$ and $\psi$ by $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ and $\Psi_K$ respectively, the Theorem V.2.9 of [Eke] also works. Since the Fadell-Rabinowitz index $\hat{I}$ has the properties of monotonicity, subadditivity, continuity which are the only three properties of $I$ used in the proof of Proposition V.2.10 of [Eke], then the proof carries over verbatim of that of Proposition V.2.10 of [Eke].

Note that here we can’t get $c_i \neq 0$ and prove Proposition 3.5 below, because it depends on Proposition 3.4 and the identity (3.10) below, but we should firstly prove Proposition 3.1 and 3.3 in order to get the identity (3.10) by the method of Lemma V.3.8 of [Eke].

**Definition 3.2.** Suppose $u$ is a nonzero critical point of $\Psi_K$, and $\mathcal{N}$ is an $S^1$-invariant open neighborhood of $S^1 \cdot u$ such that $\text{crit}(\Psi_K) \cap (\Lambda_K(u) \cap \mathcal{N}) = S^1 \cdot u$. Then the $S^1$-critical modules of $S^1 \cdot u$ is defined by

$$C_{S^1 \cdot u}(\Psi_K, S^1 \cdot u) = H_q((\Lambda_K(u) \cap \mathcal{N})_{S^1}, ((\Lambda_K(u) \setminus S^1 \cdot u) \cap \mathcal{N})_{S^1}), \quad (3.3)$$

where $\Lambda_K(u) = \{ w \in L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \mid \Psi_K(w) \leq \Psi_K(u) \}$.

Comparing with Theorem 4 in P.219 of [Eke], we have the following
Proposition 3.3. For every $i \geq d(K)/2 + 1$, there exists a point $u \in L^2(R/Z, R^{2n})$ such that

\[
\Psi'_K(u) = 0, \quad \Psi_K(u) = c_i, \quad (3.4)
\]

\[
C_{S^1, 2(i-1)}(\Psi_K, S^1 \cdot u) \neq 0. \quad (3.5)
\]

Proof. By Lemma 8 in P.206 of [Eke], we can use Theorem 1.4.2 of [Chal] in the equivariant form to obtain

\[
H_{S^1, *}(\Psi_K^{c_i+\epsilon}, \Psi_K^{c_i-\epsilon}) = \bigoplus_{\Psi_K(u) = c_i} C_{S^1, *}(\Psi_K, S^1 \cdot u), \quad (3.6)
\]

for $\epsilon$ small enough such that the interval $(c_i - \epsilon, c_i + \epsilon)$ contains no critical values of $\Psi_K$ except $c_i$.

Similar to P.431 of [EkH], we have

\[
H^{2(i-1)}(\Psi_K^{c_i+\epsilon}, \Psi_K^{c_i-\epsilon}) \xrightarrow{q^*} H^{2(i-1)}((\Psi_K^{c_i+\epsilon})_{S^1}) \xrightarrow{p^*} H^{2(i-1)}((\Psi_K^{c_i-\epsilon})_{S^1}), \quad (3.7)
\]

where $p$ and $q$ are natural inclusions. Denote by $f : (\Psi_K^{c_i+\epsilon})_{S^1} \to CP^\infty$ a classifying map and let $f^\pm = f|_{(\Psi_K^{c_i+\epsilon})_{S^1}}$. Then clearly each $f^\pm : (\Psi_K^{c_i+\epsilon})_{S^1} \to CP^\infty$ is a classifying map on $(\Psi_K^{c_i+\epsilon})_{S^1}$. Let $\eta \in H^2(CP^\infty)$ be the first universal Chern class.

By definition of $c_i$, we have $\hat{I}(\Psi_K^{c_i-\epsilon}) < i$, hence $(f)^* (\eta^{i-1}) = 0$. Note that $p^* (f^+)^* (\eta^{i-1}) = (f^-)^* (\eta^{i-1})$. Hence the exactness of (3.7) yields a $\sigma \in H^{2(i-1)}((\Psi_K^{c_i+\epsilon})_{S^1}, (\Psi_K^{c_i-\epsilon})_{S^1})$ such that $q^* (\sigma) = (f^+)^* (\eta^{i-1})$ and $\hat{I}(\Psi_K^{c_i+\epsilon}) \geq i$, we have $(f^+)^* (\eta^{i-1}) \neq 0$. Hence $\sigma \neq 0$, and then

\[
H^{2(i-1)}((\Psi_K^{c_i+\epsilon})_{S^1}, (\Psi_K^{c_i-\epsilon})_{S^1}) \neq 0.
\]

Thus the proposition follows from (3.6) and the universal coefficient theorem.

Now we define two numbers $\gamma^+_\alpha(\Sigma)$ and $\gamma^-_\alpha(\Sigma)$ by:

\[
\gamma^+_\alpha(\Sigma) = \limsup_{i \to \infty} \left[ \left( -c_i \right)^{\frac{2-\alpha}{\alpha}} \right]^{-1},
\]

\[
\gamma^-_\alpha(\Sigma) = \liminf_{i \to \infty} \left[ \left( -c_i \right)^{\frac{2-\alpha}{\alpha}} \right]^{-1},
\]

and we set

\[
\gamma^+(\Sigma) = \frac{\alpha}{4} (1 - \frac{\alpha}{2})^{\frac{2-\alpha}{\alpha}} \gamma^+_\alpha(\Sigma), \quad (3.8)
\]

\[
\gamma^-(\Sigma) = \frac{\alpha}{4} (1 - \frac{\alpha}{2})^{\frac{2-\alpha}{\alpha}} \gamma^-_\alpha(\Sigma). \quad (3.9)
\]

Then by the proofs of Lemma V.3.8 of [Eke] and Proposition 2.8 of [LLW], noticing that when $\Sigma$ is convex, the Viterbo index $i^\nu(y)$ and nullity $\nu^\nu(y)$ are the same as Ekeland index and nullity, we have

\[
\gamma^+(\Sigma_R) = \gamma^-(\Sigma_R) = \frac{\pi R^2}{2n}, \quad (3.10)
\]

where $\Sigma_R$ is the sphere of radius $R$ in $R^{2n}$.

Proposition 3.4. We have $0 < \gamma^-(\Sigma) \leq \gamma^+(\Sigma) < +\infty$ for any $\Sigma \in \mathcal{H}_{st}(2n)$. 

8
Proof. Since $\Sigma$ is star-shaped, there exist some $0 < r < R$ such that
\[ R^{-\alpha}|x|^\alpha \leq H_\alpha(x) \leq r^{-\alpha}|x|^\alpha. \] (3.11)
We denote the modified Hamiltonian functions of $R^{-\alpha}|x|^\alpha$ and $r^{-\alpha}|x|^\alpha$ in Proposition 2.1 by $H_R(x)$ and $H_r(x)$ respectively, and we can also choose the functions to satisfy
\[ H_R(x) \leq H(x) \leq H_r(x), \forall x \in \mathbb{R}^{2n}, \] (3.12)
where $H(x)$ is the modified Hamiltonian function of $H_\alpha(x)$.

Denote by $\Psi^r_{\alpha}$ and $\Psi^R_{\alpha}$ the corresponding dual action functionals defined in (2.7) associated with the Hamiltonians $H_r$ and $H_R$ respectively, then by (3.12) we have
\[ \Psi^r_{\alpha} \leq \Psi_{\alpha} \leq \Psi^R_{\alpha}. \] (3.13)

Define
\[
\begin{align*}
\gamma^+(\Sigma_r) & \leq \gamma^+(\Sigma) \leq \gamma^+(\Sigma_R), \\
\gamma^-(\Sigma_r) & \leq \gamma^-(\Sigma) \leq \gamma^-(\Sigma_R). 
\end{align*}
\]
(3.14)
(3.15)

Hence by (3.10), (3.14) and (3.15), we obtain $\frac{\pi^2}{2n} \leq \gamma^-(\Sigma) \leq \gamma^+(\Sigma) \leq \frac{\pi R^2}{2n}$.

**Proposition 3.5.** If $c_i = c_j$ for some $d(K)/2 + 1 \leq i < j$, then there are infinitely many geometrically distinct closed characteristics on $\Sigma$.

**Proof.** Note that by Proposition 3.4, we have $c_i \neq 0, i \geq d(K)/2 + 1$. Then by the same proof of Proposition V.3.3 of [Eke], we prove our proposition.

Since every solution $(\tau, y) \in T(\Sigma, \alpha)$ gives rise to a sequence $\{z^y_m\}_{m \in \mathbb{N}}$ of solutions of the given period-1 problem (2.1), and a sequence $\{u^y_m\}_{m \in \mathbb{N}}$ of critical points of $\Psi_K$ defined by
\[
\begin{align*}
\hat{z}^y_m(t) &= (m\tau)^{-\alpha-1} y(m\tau), \\
\hat{u}^y_m(t) &= -J(m\tau)^{-\alpha-1} \dot{y}(m\tau) + K(m\tau)^{-\alpha-1} y(m\tau) 
\end{align*}
\] (3.16) (3.17)

From the proof of Proposition 2.8 of [LLW], we know that $\Psi_K(u^y_m)$ is independent of $K$, together with (V.3.45) of [Eke], it follows that
\[ \Psi_K(u^y_m) = -(1 - \frac{\alpha}{2}) \left( \frac{2}{\alpha} k A(y) \right)^{-\frac{\alpha}{2}}, \] (3.18)
where the action of a closed characteristic $(\tau, y)$ is defined by
\[ A(y) = \frac{1}{2} \int_0^\tau (Jy \cdot \dot{y}) dt. \]
Corollary 3.6. We have \( \lim_{i \to \infty} c_i = 0 \) and for every \( i \in \mathbb{N} \), there exists \((\tau, y) \in \mathcal{T}(\Sigma, \alpha)\) and \( m \in \mathbb{N} \) such that

\[
\Psi_K(u_m^y) = 0, \quad \Psi_K(u_m^y) = c_i + d(K)/2, \tag{3.19}
\]

\[
i^v(y^m) \leq 2(i - 1) \leq i^v(y^m) + \nu^v(y^m) - 1, \tag{3.20}
\]

where \( u_m^y \) is defined as in (3.17).

Definition 3.7. We call \((\tau, y) \in \mathcal{T}(\Sigma, \alpha)\) is \( i \)-essential if there exists some \( m \in \mathbb{N} \) such that (3.19), (3.20) hold. It is essential if it is \( i \)-essential for some \( i \in \mathbb{N} \). We denote by \( C \) the family of essential closed characteristics on \( \Sigma \).

Theorem 3.8. We have \( [1/\gamma^+(\Sigma), 1/\gamma^-(\Sigma)] \subseteq \) the closure of \( \{i(y)_{A(y)} | y \in C\} \), where \( i(y) \equiv \lim_{m \to \infty} \frac{i(y^m)}{m} \) is the mean index of \((\tau, y)\).

The proof of Theorem 3.8 relies on the following:

Lemma 3.9. There exists constant \( d \), which only depends on \( \Sigma \), such that whenever \( y \in C \) is \( i \)-essential for some \( i \in \mathbb{N} \), we have

\[
\left| \frac{1}{C\alpha A(y)} - (d(K)/2 + i)\{c_d(K)/2 + i\}_{\alpha} \right| \leq d|c_d(K)/2 + i|_{\alpha}^{2-\alpha},
\]

where \( C\alpha = \frac{1}{\alpha}(1 - \frac{2}{\alpha})^{2-\alpha} \).

Proof. By Theorem 10.1.1, 10.1.2 of [Lon2] and Lemma 2.4, the Viterbo index \( i^v(y) \) has the property of Proposition I.5.21 of [Eke], note that Theorem V.1.4 of [Eke] also holds for star-shaped hypersurfaces, then our lemma follows by the same proof of Lemma V.3.12 of [Eke].

Proof of Theorem 3.8. From Lemma 3.9 instead of Lemma V.3.12 of [Eke], our theorem follows by the same proof of Theorem V.3.11 of [Eke].

Now by the same proof of Theorem V.3.15 of [Eke], we obtain

Theorem 3.10. If \( C \) is finite. Then we have

\[
\gamma(\Sigma) \equiv \gamma^+(\Sigma) = \gamma^-(\Sigma),
\]

\[
\hat{i}(y) = \frac{1}{\gamma(\Sigma)}, \quad \forall y \in C,
\]

\[
\sum_{y \in C} \frac{1}{i(y)} \geq \frac{1}{2}. \tag{3.21}
\]

By (3.21), we have

Corollary 3.11. If there is a closed characteristic on \( \Sigma \in \mathcal{H}_{st}(2n) \) whose mean index is greater than 2, then there exist at least two closed characteristics on \( \Sigma \).

3.2 Critical values in the pinched case

In this subsection, we prove under suitable pinching condition, the critical values \( c_{i+d(K)/2} \) found in Subsection 3.1 correspond to \( n \) distinct closed characteristics for \( 1 \leq i \leq n \).
Let \( n(y) \) be the unit outward normal vector of \( \Sigma \) at \( y \) and \( d(y) := n(y) \cdot y \), i.e., the distance between the origin of \( \mathbb{R}^{2n} \) and the tangent hyperplane to \( \Sigma \) at \( y \), then \( d(y) > 0 \) for all \( y \in \Sigma \) since \( \Sigma \) is strictly star-shaped. Let \( d = \min \{ d(y) : y \in \Sigma \} \), \( R = \max \{ |y| : y \in \Sigma \} \). Then we have

**Theorem 3.12.** Suppose that \( \Sigma \) satisfies the pinching condition \( R^2 < 2d^2 \), then the critical values \( c_{i+d(K)/2} \) found in Proposition 3.1 and Corollary 3.6 correspond to at least \( n \) distinct closed characteristics for \( 1 \leq i \leq n \).

**Proof.** We carry out our proof in two steps:

**Step 1.** We have

\[
c_{i+d(K)/2} \leq -(1 - \frac{\alpha}{2})(\frac{\alpha}{2\pi R^2})^{\frac{2}{2-\alpha}}, \quad \forall 1 \leq i \leq n.
\] (3.22)

In fact, when \( \Sigma = \Sigma_R \) is the sphere of radius \( R \) in \( \mathbb{R}^{2n} \), by the proofs of Lemma V.3.8 of [Eke] and Proposition 2.8 of [LLW], noticing that when \( \Sigma \) is convex, the Viterbo index \( i^v(y) \) and nullity \( \nu^v(y) \) are the same as Ekeland index and nullity, we obtain that the corresponding critical values \( c_{i+d(K)/2} \) found in Proposition 3.1 and Corollary 3.6 satisfy

\[
c_{i+d(K)/2}^R = -(1 - \frac{\alpha}{2})(\frac{\alpha}{2\pi d^2})^{\frac{2}{2-\alpha}}, \quad \forall 1 \leq i \leq n,
\]

which, together with (3.2) and (3.13), yields (3.22).

**Step 2.** We have

\[
c_{i+d(K)/2} \geq (1 - \frac{\alpha}{2})(\frac{\alpha}{2\pi d^2})^{\frac{2}{2-\alpha}}, \quad \forall 1 \leq i \leq n.
\] (3.23)

In fact, when we replace \( r \) in the proof of Theorem V.1.4 of [Eke] by \( d \), then Theorem V.1.4 of [Eke] holds for star-shaped hypersurface \( \Sigma \), i.e., for every closed characteristic \((\tau, y)\) on star-shaped hypersurface \( \Sigma \), there holds \( A(\tau, y) \geq \pi d^2 \), which, together with (3.18), yields (3.23).

Now, combining (3.22)-(3.23), (3.18) and \( R^2 < 2d^2 \), by Proposition 3.5 we obtain that the critical values \( c_{i+d(K)/2} \) correspond to at least \( n \) distinct closed characteristics for \( 1 \leq i \leq n \).

4 Proofs of Theorems 1.1 and 1.4-1.5

In this section, we prove Theorems 1.1 and 1.4-1.5.

**Lemma 4.1.** (cf. Proposition 6.2 of [LLo2]) Let \( \Sigma \in \mathcal{H}_{\text{st}}(4) \) satisfy \( \#\mathcal{T}(\Sigma) = 2 \). Denote the two geometrically distinct prime closed characteristics by \( \{ (\tau_j, y_j) \}_{1 \leq j \leq 2} \). If \( i(y_j) \geq 0, j = 1, 2 \), then both of the closed characteristics are elliptic.

**Proof of Theorem 1.1.** Let \( \Sigma \in \mathcal{H}_{\text{st}}(4) \) satisfy \( \#\mathcal{T}(\Sigma) = 2 \) and \( R^2 < 2d^2 \), we denote by \( \{ (\tau_1, y_1), (\tau_2, y_2) \} \) the two geometrically distinct prime closed characteristics on \( \Sigma \), and by
\( \gamma_j \equiv \gamma_{y_j} \) the associated symplectic paths of \((\tau_j, y_j)\) for \(1 \leq j \leq 2\). Then by Lemma 3.3 of [Hil], (cf. also Lemma 15.2.4 of [Lon2]), there exist \(P_j \in \text{Sp}(4)\) and \(M_j \in \text{Sp}(2)\) such that
\[
\gamma_j(\tau_j) = P_j^{-1}(N_1(1, 1) \circ M_j)P_j, \quad \text{for } j = 1, 2. \tag{4.1}
\]

Note that by Section 9 of [Vit2], we know that there exists at least one non-hyperbolic closed characteristic on \(\Sigma\) and it is certainly elliptic when \(n = 2\). In the following, we prove Theorem 1.1 by contradiction. Without loss of generality, we assume that \((\tau_1, y_1)\) is elliptic and \((\tau_2, y_2)\) is hyperbolic.

For these two closed characteristics, we have the following properties:

**Claim 1.** The closed characteristics \((\tau_1, y_1), (\tau_2, y_2)\) satisfy
\[
(i) \quad i(y_2^m) = m(i(y_2) + 3) - 3 \quad \text{and} \quad \nu(y_2^m) = 1, \quad \forall \ m \in \mathbb{N}, \quad \text{and thus} \quad \hat{i}(y_2) = i(y_2) + 3.
\]
\[
(ii) \quad \hat{i}(y_2) > 0.
\]
\[
(iii) \quad \hat{i}(y_1) \in \mathbb{Q}.
\]
\[
(iv) \quad \text{If } i(y_2) \text{ is even, then } i(y_2^2) - i(y_2) \in 2\mathbb{Z} - 1, \quad \hat{\chi}(y_2) = \frac{1}{2}, \quad \text{and } i(y_2) \geq -2.
\]

In fact, by Theorem 8.3.1 of [Lon2], we have \(i(y_2, m) = m(i(y_2, 1) + 1) - 1, \quad \forall \ m \in \mathbb{N}\). Together with Lemma 2.4, we obtain (i).

We claim \(\hat{i}(y_2) \neq 0\). In fact, because \(y_2\) is hyperbolic, \(y_2^m\) is non-degenerate for every \(m \geq 1\). Thus if \(\hat{i}(y_2) = 0\), we then have \(i(y_2^m) = i(y_2, m) - 2 = -3\) for all \(m \geq 1\). Then the Morse-type number satisfies \(m_{-3} = +\infty\). But then \(\hat{i}(y_1)\) must be positive by Theorem 5.6, and contributions of \(\{y_2^m\}\) to every Morse-type number thus must be finite. Then the Morse inequality yields a contradiction and proves the claim (cf. the proof below (9.3) of [Vit2] for details).

If \(\hat{i}(y_2) < 0\), by (5.21) we obtain
\[
\frac{\hat{\chi}(y_2)}{\hat{i}(y_2)} = 0. \tag{4.2}
\]

But because \((\tau_2, y_2)\) is hyperbolic, by (5.22) we have \(\hat{\chi}(y_2) \neq 0\), which contradicts to (4.2) and proves (ii).

If \((\tau_1, y_1)\) and its iterates are all non-degenerate, since \((\tau_1, y_1)\) is elliptic, then \(\hat{i}(y_1)\) must be irrational by Corollary 8.3.2 of [Lon2] and then so is \(\frac{\hat{\chi}(y_1)}{i(y_1)}\), because \(\hat{\chi}(y_1)\) is rational and nonzero by (5.23). Then by (5.23) of Theorem 5.6, the other closed characteristic \((\tau_2, y_2)\) must possess an irrational mean index \(\hat{i}(y_2)\), which contradicts to the second identity in (i), and thus \(\hat{i}(y_1)\) must be rational, which proves (iii).

If \(i(y_2)\) is even, then \(i(y_2^2) - i(y_2) = i(y_2) + 3 \in 2\mathbb{Z} - 1\) by (i), which together (5.22) implies \(\hat{\chi}(y_2) = \frac{1}{2}\). Then \(i(y_2) \geq -2\) follows from (i) and (ii).

The proof of Claim 1 is complete.

By (iii) of Claim 1, we only need to consider the following four cases according to the classification of basic norm forms of \(\gamma_1(\tau_1)\). In the following we use the notations from Definition 1.8.5 and Theorem 1.8.10 of [Lon2], and specially we let \(R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\) with \(\theta \in \mathbb{R}\), and use
To denote the symplectic direct sum of two symplectic matrices \( M \) and \( N \) as in pages 16-17 of [Lon2].

**Case 1.** \( \gamma_1(\tau_1) \) can be connected to \( N_1(1, 1)\Diamond N_1(-1, b) \) within \( \Omega^0(\gamma_1(\tau_1)) \) with \( b = 0 \) or \( \pm 1 \).

In this case, by Theorems 8.1.4 and 8.1.5 of [Lon2], and Lemma 2.4, we have

\[ i(y_1, 1) \quad \text{and} \quad i(y_1) \quad \text{are even}. \tag{4.3} \]

By Theorem 1.3 of [Lon1], we have

\[ i(y_1, m) = m(i(y_1, 1) + 1) - 1, \quad \text{for} \quad b = 1; \]
\[ i(y_1, m) = m(i(y_1, 1) + 1) - 1 - \frac{1 + (-1)^m}{2}, \quad \text{for} \quad b = 0, -1. \]

By Lemma 2.4, we obtain

\[ i(y_1^m) = m(i(y_1) + 3) - 3, \quad \text{for} \quad b = 1; \tag{4.4} \]
\[ i(y_1^m) = m(i(y_1) + 3) - 3 - \frac{1 + (-1)^m}{2}, \quad \text{for} \quad b = 0, -1. \tag{4.5} \]

Then in both cases we obtain

\[ \hat{i}(y_1) = i(y_1) + 3. \tag{4.6} \]

Next we separate our proof in two subcases according to the parity of \( i(y_2) \).

**Subcase 1.** \( i(y_2) \) is even.

By (iv) of Claim 1, we have \( i(y_2) \geq 0 \) or \( i(y_2) = -2 \). We continue our proof in two steps according to the value of \( i(y_2) \):

**Step 1.** \( i(y_2) \geq 0. \)

In this step, by (i) of Claim 1 we have

\[ \hat{i}(y_2) \geq 3, \tag{4.7} \]

which together with (iv) of Claim 1 implies

\[ \frac{\hat{\chi}(y_2)}{i(y_2)} \leq \frac{1}{6}. \tag{4.8} \]

Combining (4.8) with Theorem 5.6, we obtain

\[ \hat{i}(y_1) > 0, \quad \frac{\hat{\chi}(y_1)}{i(y_1)} = \frac{1}{2} - \frac{\hat{\chi}(y_2)}{i(y_2)} \geq \frac{1}{3}. \tag{4.9} \]

Note that by Proposition 5.4 and the form of \( \gamma_1(\tau_1) \), we have \( K(y_1) = 2 \). Thus by (5.21) and (4.9), we obtain

\[ 0 < \hat{\chi}(y_1) = \frac{1 + (-1)^{i(y_2)}(k_0(y_1^2) - k_1(y_1^2) + k_2(y_1^2))}{2}. \tag{4.10} \]
Since at most one of \( k_l(y_l^2) \)s for \( 0 \leq l \leq 2 \) can be non-zero by (iv) of Remark 5.5, we obtain

\[
(-1)^{i(y_l^2)} + l k_l(y_l^2) \geq 0, \quad \text{for } l = 0, 1, 2. \tag{4.11}
\]

When \( m \) is odd, we have \( \nu(x_1^m) = 1 \) by the assumption on \( \gamma_1(\tau_1) \). In this case, because \( i(y_1) \) is even by (14.3), we have \( i^\nu(x_1^m) = i(y_1^m) = m(i(y_1) + 1) - 3 \) is even, and then

\[
\beta(x^m) = (-1)^{i^\nu(x_1^m) - i^\nu(x_1)} = 1,
\]

where we denote by \( x_j \) the critical point of \( F_{a,K} \) corresponding to \( y_j \) for \( j = 1 \) and 2. Thus by (15.17) of Proposition 5.2 for every odd \( m \in \mathbb{N} \), we obtain

\[
C_{S^1, d(K) + k}(F_{a,K}, S^1 \cdot x_1^m) = Q, \quad \text{if } k = i(y_1^m), \tag{4.12}
\]

\[
C_{S^1, d(K) + k}(F_{a,K}, S^1 \cdot x_1^m) = 0, \quad \text{if } k \neq i(y_1^m), \tag{4.13}
\]

where (4.13) holds specially when \( k \in 2\mathbb{Z} - 1 \).

When \( m \) is even, we consider two cases: (A-1) for \( b = 0, -1 \) with (4.5); (B-1) \( b = 1 \) with (4.4).

**A-1** \( m \) is even, \( b = 0 \) or \(-1\), and (4.5) holds.

In this case, \( i(y_1^2) \) is even by (4.5). Therefore by (4.10)-(4.11) we obtain

\[
k_1(y_1^2) = 0, \quad \hat{\chi}(y_1) = \frac{1 + (k_0(y_1^2) + k_2(y_1^2))}{2} > 0. \tag{4.14}
\]

Because \( K(y_1) = 2 \), we then obtain

\[
C_{S^1, d(K) + 2k-1}(F_{a,K}, S^1 \cdot x_1^m) = 0, \quad \forall k \in \mathbb{Z}, m \in 2\mathbb{N}. \tag{4.15}
\]

Therefore, when \( b = 0, -1 \), from (4.12), (4.13) and (4.15) we obtain

\[
C_{S^1, d(K) + 2k-1}(F_{a,K}, S^1 \cdot x_1^m) = 0, \quad \forall k \in \mathbb{Z}, m \in \mathbb{N}. \tag{4.16}
\]

**B-1** \( m \) is even, \( b = 1 \), and (4.4) holds.

In this case, \( i(y_1^2) \) is odd by (4.4). Therefore by (4.10)-(4.11) we obtain

\[
k_0(y_1^2) = k_2(y_1^2) = 0, \quad 0 < \hat{\chi}(y_1) = \frac{1 + k_1(y_1^2)}{2}. \tag{4.17}
\]

Because \( K(y_1) = 2 \), we then obtain

\[
C_{S^1, d(K) + 2k-1}(F_{a,K}, S^1 \cdot x_1^m) = 0, \quad \forall k \in \mathbb{Z}, m \in 2\mathbb{N}. \tag{4.18}
\]

Therefore when \( b = 1 \), from (4.12), (4.13) and (4.18), we obtain

\[
C_{S^1, d(K) + 2k-1}(F_{a,K}, S^1 \cdot x_1^m) = 0, \quad \forall k \in \mathbb{Z}, m \in \mathbb{N}. \tag{4.19}
\]

In summary, from (4.16) and (4.19), for any case we have

\[
C_{S^1, d(K) + 2k-1}(F_{a,K}, S^1 \cdot x_1^m) = 0, \quad \forall k \in \mathbb{Z}, m \in \mathbb{N}. \tag{4.20}
\]
Note that in Subcase 1.1, \(i(y_2)\) is even and \((\tau_2, y_2)\) is hyperbolic, then by (5.17) of Proposition 5.2, we obtain
\[
C_{S^1, d(K) + 2k - 1}(F_{a,K}, S^1 \cdot x_2^m) = 0, \quad \forall k \in \mathbb{Z}, m \in \mathbb{N}.
\] (4.21)
Combining (4.20) and (4.21), we have \(m_{2q-1} = 0\) for every \(q \in \mathbb{Z}\) and \(U(t) \equiv 0\) in (5.28). Here and below in this Section \(m_i\) denotes the coefficient of \(t^i\) of \(M(t) = \sum_{i \in \mathbb{Z}} m_i t^i\) in (5.28). Then
\[
\sum_{i \in \mathbb{Z}} m_i t^i = \frac{1}{1 - t^2} = \sum_{i \in \mathbb{N}} t^{2i-2}.
\]
Thus \(i(y_j) \geq 0\) for \(j = 1, 2\) by Proposition 5.2. By Lemma 4.1 we know that the two closed characteristics are elliptic, which contradicts to our assumption.

**Step 1.2.** \(i(y_2) = -2\).

In this step, by (i) of Claim 1 we have
\[
\hat{i}(y_2) = 1,
\] (4.22)
which together with (iv) of Claim 1 implies
\[
\frac{\hat{x}(y_2)}{i(y_2)} = \frac{1}{2}.
\] (4.23)
We continue in two cases: (A-2) \(i(y_1) = -2\); (B-2) \(i(y_1) \leq -4\) or \(i(y_1) \geq 0\).

(A-2) \(i(y_1) = -2\) holds.

In this case, by (4.6) we have
\[
\hat{i}(y_1) = 1,
\] (4.24)
Combining (4.22), (4.24) with Theorem 5.6, we obtain
\[
\frac{\hat{x}(y_1)}{i(y_1)} + \frac{\hat{x}(y_2)}{i(y_2)} = \frac{1}{2},
\] (4.25)
which together with (4.24) implies
\[
\hat{x}(y_1) = 0.
\] (4.26)
Since \(i(y_1) = -2\) and \(i(y_2) = -2\) in this case, there hold \(i(c_k^m) \geq -2, \forall m \geq 1, k = 1, 2\). Then by (5.10)-(5.17) of Proposition 5.2, (4.14)-(4.15) and (i) of Claim 1, we obtain \(m_{-2} \geq 2\) and \(m_q = 0\) for \(q < -2\), which together with (5.28) gives
\[
m_{-1} = u_{-2} + u_{-1} = m_{-2} + u_{-1} \geq 2.
\] (4.27)
On the other hand, by (i) of Claim 1, \(i(y_2^m) = m - 3\), then by (5.17) of Proposition 5.2, we get
\[
C_{S^1, d(K) - 1}(F_{a,K}, S^1 \cdot x_2^m) = 0, \quad \forall m \in \mathbb{N},
\] (4.28)
which, together with (4.27), implies that \( y_1^m \) has contribution to \( m = -1 \) for some \( m \in \mathbb{N} \) and \( y_2^m \) has no contribution to \( m = -1 \) for all \( m \in \mathbb{N} \). Note that by (4.3)-(4.5) and Proposition 5.2, then we have 
\[
m_{-1} = k_0(y_1^2) \geq 2 \quad \text{when} \quad b = 1 \quad \text{and} \quad (4.3) \quad \text{holds, or} \quad m_{-1} = k_1(y_1^2) \geq 2 \quad \text{when} \quad b = 0 \quad \text{or} \quad -1, \quad \text{and} \quad (4.5) \quad \text{holds. Together with (iv) of Remark 5.5, (4.27) and (5.21), it yields}
\]
\[
\hat{\chi}(y_1) = \frac{1 + (-1)^i(y_1^2)(k_0(y_1^2) - k_1(y_1^2) + k_2(y_1^2))}{2} \neq 0,
\]
which contradicts to (4.26).

(B-2) \( i(y_1) \leq -4 \) or \( i(y_1) \geq 0 \) holds.

In this case, by (4.4)-(4.5) and Proposition 5.2, \( y_1^m \) has no contribution to \( m_{-1} \) and \( m_{-3} \) for all \( m \in \mathbb{N} \). Note that \( i(y_2^m) = m - 3 \) in Step 2, then by Proposition 5.2, \( y_2^m \) has no contribution to \( m_{-1} \) and \( m_{-3} \) for all \( m \in \mathbb{N} \). Thus \( m_{-1} = m_{-3} = 0 \), which together with (5.28) gives
\[
m_{-2} = 0.
\]
(4.29)

Since \( i(y_2) = -2 \), then by Proposition 5.2, we have \( C_{S_1, d(K)-2}(F_{a,K}, \ S^1 \cdot x_2) = Q \) which implies \( m_{-2} > 0 \), it contradicts to (4.29).

**Subcase 1.2.** \( i(y_2) \) is odd.

In this subcase, by (i)-(ii) of Claim 1, we have \( i(y_2) \geq -1 \) and it is odd. When \( i(y_2) \geq 1 \), then by the same proof of Subcase 1.1 of Theorem 1.4 in [LLo2], we get a contradiction. Thus we can assume that \( i(y_2) = -1 \). Then by (i) of Claim 1, we have \( i(y_2^m) = 2m - 3 \), it together with Proposition 5.2 gives \( C_{S_1, d(K)+2i-2}(F_{a,K}, \ S^1 \cdot x_2^m) = 0, \quad \forall \ m \geq 1, \ i = 1, 2 \). On the other hand, by Proposition 3.3 and Theorem 3.12, we have \( C_{S_1, d(K)+2i-2}(\Psi_K, \ S^1 \cdot y_2^m) \neq 0 \) for \( i = 1 \) or \( i = 2 \), where \( y_2^m \) is the critical point of \( \Psi_K \) corresponding to \( y_2 \). By the same proof of Proposition 3.6 of [Wan1], we have \( C_{S_1, d(K)+2i-2}(\Psi_K, \ S^1 \cdot y_2^m) \cong C_{S_1, d(K)+2i-2}(F_{a,K}, \ S^1 \cdot x_2), \) it is a contradiction.

**Case 2.** \( \gamma_1(\tau_1) \) can be connected to \( N_1(1,1) \circ R(\theta) \) within \( \Omega^0(\gamma_1(\tau_1)) \) with some \( \theta \in (0, \pi) \cup (\pi, 2\pi) \) and \( \theta/\pi \in \mathbb{Q} \).

In this case, we have always \( K(y_1) \geq 3 \) by the definition of \( \theta \). By Theorems 8.1.4 and 8.1.7 of [Lon2] and Lemma 2.4 we obtain
\[
i(y_1, 1) \quad \text{and} \quad i(y_1) \quad \text{are even.} \tag{4.30}
\]

By Theorem 1.3 of [Lon1] (i.e., Theorem 8.3.1 of [Lon2]), we have
\[
i(y_1, m) = mi(y_1, 1) + 2E(m\frac{\theta}{2\pi} - 2), \quad \forall \ m \geq 1,
\]
which, together with Lemma 2.4, yields
\[
i(y_1^m) = m(i(y_1) + 2) + 2E(m\frac{\theta}{2\pi} - 4), \quad \forall \ m \geq 1. \tag{4.31}
\]
Then
\[
i(y_1) = i(y_1) + 2 + \frac{\theta}{\pi}. \tag{4.32}
\]
We have two subcases according to the parity of $i(y_2)$.

**Subcase 2.1.** $i(y_2)$ is odd.

For this case, as the same proof of Case 2 of Theorem 1.4 in [LLo2], we can get a contradiction.

**Subcase 2.2.** $i(y_2)$ is even.

Next we continue our proof in two steps according to the value of $i(y_1) \in 2\mathbb{Z}$ by (4.30).

**Step 2.1.** $i(y_1) \leq -4$.

In this step, by (4.32), we have $\hat{i}(y_1) < 0$, together with Theorem 5.6, we obtain

$$\frac{\hat{x}(y_2)}{\hat{i}(y_2)} = \frac{1}{2},$$

which implies

$$\hat{x}(y_1) = 0.$$  \hspace{1cm} (4.35)

By (4.33) and (iv) of Claim 1, we get $\hat{i}(y_2) = 1$, which, together with (i) of Claim 1, implies

$$i(y_2) = -2.$$  \hspace{1cm} (4.36)

By (5.21), we have

$$\hat{x}(y_1) = \frac{K(y_1) - 1 + k_0(y_1^{K(y_1)}) - k_1(y_1^{K(y_1)}) + k_2(y_1^{K(y_1)})}{K(y_1)},$$

which, together with (iv) of Remark 5.5 and (4.35), yields

$$k_1(y_1^{K(y_1)}) = K(y_1) - 1 > 0, \quad k_0(y_1^{K(y_1)}) = k_2(y_1^{K(y_1)}) = 0.$$  \hspace{1cm} (4.38)

Since $i(y_1^{mK(y_1)}) \leq (-2 + \frac{\theta}{\pi})mK(y_1) - 4 < -4$, $\forall m \in \mathbb{N}$, then by Proposition 5.2 and (4.38) we know that $y_1^{mK(y_1)}$ has no contribution to $m_{-1}$ and $m_{-3}$. On the other hand, note that $i(y_1^m)$ is even, it follows from Proposition 5.2 that $y_1^m$ has no contribution to $m_{-1}$ and $m_{-3}$ for $m \neq 0 \bmod K(y_1))$.

In addition, $y_2^m$ also has no contribution to $m_{-1}$ and $m_{-3}$ since $y_2$ is hyperbolic and $i(y_2) \in 2\mathbb{Z}$. Hence we obtain $m_{-1} = m_{-3} = 0$, which, together with (5.28), yields $m_{-2} = 0$. But by (4.36) and Proposition 5.2, $y_2$ contributes 1 to $m_{-2}$. So we get a contradiction.

**Step 2.2.** $i(y_1) \geq -2$.

In this step, note that $i(y_1)$ and $i(y_2)$ are even, we have either $i(y_j) \geq 0$ for $1 \leq j \leq 2$, or $i(y_j) = -2$ for some $1 \leq j \leq 2$, if the former holds, then by Lemma 4.1, $y_1$ and $y_2$ are elliptic which
contradicts to our assumption. Thus we can assume that \( i(y_j) = -2 \) for some \( 1 \leq j \leq 2 \). Then by Proposition 5.2 we have

\[
m_{-2} \geq 1.
\] (4.39)

Note that \( i(y_j) \geq -2 \) for \( 1 \leq j \leq 2 \) by (iv) of Claim 1. By (i) of Claim 1 and (4.31) we have

\[
i(y_j^m) \geq -2, \quad \forall \ m \geq 1, \quad j = 1, 2.
\] (4.40)

Thus we have

\[
C_{S^1, d(K)-q}(F_{a,K}, S^1 \cdot x_j^m) = 0, \quad \forall \ m \in \mathbb{N}, \ q \geq 3, \ j = 1, 2.
\]

Hence we have

\[
m_{-q} = 0, \quad \forall \ q \geq 3,
\]

which, together with (4.39) and (5.28), yields

\[
m_{-1} = u_{-1} + u_{-2} \geq u_{-2} + u_{-3} = m_{-2} \geq 1.
\] (4.41)

Note that \( y_2^m \) has no contribution to \( m_{2k-1} \) for \( k \in \mathbb{Z} \) since \( y_2 \) is hyperbolic and \( i(y_2) \in 2\mathbb{Z} \), so some \( y_1^m \) must have contribution to \( m_{-1} \). Also note that, in this case, \( C_{S^1, d(K)+2i-1}(F_{a,K}, S^1 \cdot x_1^m) = 0 \) for any \( i \in \mathbb{Z} \) and \( m \neq 0 \) (mod \( K(y_1) \)). Therefore \( m_{-1} \) only can be contributed by iterates \( y_1^{K(y_1)} \). This implies \( i(y_1^m) \leq i(y_1^{K(y_1)}) = -2, \ \forall \ 1 \leq m \leq K(y_1) - 1 \). Thus by (4.40) and Proposition 5.2, we have

\[
i(y_1^m) = -2, \quad \forall \ 1 \leq m \leq K(y_1) - 1, \quad m_{-1} = k_1(y_1^{K(y_1)}).
\] (4.42)

By (4.41)-(4.42), we have \( k_1(y_1^{K(y_1)}) = m_{-1} \geq m_{-2} \geq K(y_1) - 1 \), which together with (4.37) yields

\[
\hat{\chi}(y_1) \leq 0.
\] (4.43)

Noticing that \( i(y_j) \geq -2 \) for \( 1 \leq j \leq 2 \), then by (ii) of Claim 1 and (4.32) we have \( \hat{i}(y_j) > 0 \), which, together with (4.43) and Theorem 5.6, yields

\[
\frac{\hat{\chi}(y_2)}{i(y_2)} = \frac{1}{2} - \frac{\hat{\chi}(y_1)}{i(y_1)} \geq \frac{1}{2}.
\] (4.44)

On the other hand, by (i) and (iv) of Claim 1, we have

\[
\frac{\hat{\chi}(y_2)}{i(y_2)} = \frac{1}{2(i(y_2) + 3)} \leq \frac{1}{2},
\]

which together with (4.44) implies

\[
\hat{\chi}(y_1) = 0, \quad i(y_2) = -2.
\] (4.45)

Then \( y_2^m \) contributes exactly 1 to \( m_{-2} \), which, together with (4.42), yields \( m_{-2} = K(y_1) \). Thus by (4.37), (4.41)-(4.42), we obtain \( \hat{\chi}(y_1) < 0 \), which contradicts to (4.45).
Case 3. $\gamma_1(\tau_1)$ can be connected to $N_1(1,1) \circ N_1(1,b)$ within $\Omega^0(\gamma_1(\tau_1))$ with $b = 0$ or 1.

In this case, we have $K(y_1) = 1$ by Proposition 5.4, $i(y_1,1)$ and then $i(y_1)$ is even by Theorem 8.1.4 of [Lon2] and Lemma 2.4. By Theorem 8.3.1 of [Lon2], we obtain $i(y_1,m) = m(i(y_1,1) + 2) - 2$ for all $m \in \mathbb{N}$. Thus by Lemma 2.4 we have

$$i(y_1) = 1$$

Then we can assume that $\hat{i}(y_1) \neq 0$, and $\{y_1^m\}_{m \in \mathbb{N}}$ has contributions to the Morse-type numbers $\{m_q\}_{q \in \mathbb{Z}}$, which implies that exactly one of $k_l(y_1^m)$ for $0 \leq l \leq 2$ is nonzero by (iv) of Remark 5.5.

In fact, if $y_1^m$ has no contribution to any Morse-type number $m_q$, by the proof of Theorem 1.1 of [LLo1] we obtain three closed characteristics, which contradicts to our assumption. If $\hat{i}(y_1) = 0$ and $\{y_1^m\}_{m \in \mathbb{N}}$ has contributions to the Morse type numbers $\{m_q\}_{q \in \mathbb{Z}}$, then $i(y_1^m) = -4$ by (4.46) and exactly one of $k_l(y_1^m)$ for $0 \leq l \leq 2$ is nonzero by (iv) of Remark 5.5, then as the proof of (i) of Claim 1, we can get a contradiction.

Next we consider two subcases according to the parity of $i(y_2)$.

Subcase 3.1. $i(y_2)$ is odd.

For this case, as the same proof of Case 3 of Theorem 1.4 in [LLo2], we can get a contradiction.

Subcase 3.2. $i(y_2)$ is even.

In this case, we note that $\hat{i}(y_1) \neq 0$. If $\hat{i}(y_1) < 0$, by Theorem 5.6, we have

$$\hat{\chi}(y_1) = 0.$$ \hfill (4.46)

Then $\hat{\chi}(y_1) = 0$. But exactly one of $k_l(y_1^m)$ for $0 \leq l \leq 2$ is nonzero, which together with (5.21) implies

$$\hat{\chi}(y_1) = k_0(y_1) - k_1(y_1) + k_2(y_1) \neq 0,$$

which is a contradiction. Thus

$$\hat{i}(y_1) > 0.$$ \hfill (4.47)

Then by Theorem 5.6 and (i) and (iv) of Claim 1, we have

$$\frac{\hat{\chi}(y_1)}{\hat{i}(y_1)} = \frac{1}{2} \frac{\hat{\chi}(y_2)}{\hat{i}(y_2)} \geq 0.$$

Then there holds $\hat{\chi}(y_1) \geq 0$ by (4.47). By (5.21), we get

$$k_0(y_1) - k_1(y_1) + k_2(y_1) = \hat{\chi}(y_1) \geq 0,$$

which, together with (iv) of Remark 5.5, yields $k_1(y_1) = 0$. Then by Proposition 5.4 we know that $y_1^m$ has no contribution to $m_{2q-1}$ for all $q \in \mathbb{Z}$. Note that $i(y_2)$ is even and $y_2$ is hyperbolic, then
by Proposition 5.2, \( y_2^{2n} \) also has no contribution to \( m_{2q-1} \) for all \( q \in \mathbb{Z} \). Thus \( m_{2q-1} = 0 \) for every \( q \in \mathbb{Z} \), which implies \( U(t) \equiv 0 \) in (5.28), then

\[
\sum_{i \in \mathbb{Z}} m_i t^i = \frac{1}{1 - t^2} = \sum_{i \in \mathbb{N}} t^{2i-2}.
\]  

(4.48)

Thus \( i(y_2) \geq 0 \) by Proposition 5.2 and (4.48). Note that \( i(y_1) \geq 0 \) or \( i(y_1) = -2 \) by (4.46)-(4.47) and the fact that \( i(y_1) \) is even. If \( i(y_1) \geq 0 \), then by Lemma 4.1 we know that the two closed characteristics are elliptic, which contradicts to our assumption. Thus we suppose \( i(y_1) = -2 \). By (i) of Claim 1, it is impossible that \( y_2^{2n} \) contributes 1 to every Morse-type number \( m_q \) for \( q \in 2\mathbb{N}_0 \). Noticing that exactly one of \( k_l(y_1^m) \) for \( 0 \leq l \leq 2 \) is nonzero, by (4.48) we have \( k_2(y_1^m) = 1 \), which implies that \( \{y_1^m\}_{m \in \mathbb{N}} \) contributes exactly 1 to every Morse-type number \( m_q \) for \( q \in 2\mathbb{N}_0 \), but \( y_2 \) also has contribution to some Morse-type number \( m_q \), which contradicts to (4.48).

**Case 4.** \( \gamma_1(\tau_1) \) can be connected to \( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \diamond \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \) within \( \Omega^0(\gamma_1(\tau_1)) \).

In this case, we have \( K(y_1) = 1 \) by Proposition 5.4, \( i(y_1,1) \) and then \( i(y_1) \) is odd by Theorem 8.1.4 of [Lon2] and Lemma 2.4. By Theorem 8.3.1 of [Lon1], we have \( i(y,m) = m(i(y,1) + 1) - 1 \) for all \( m \in \mathbb{N} \). Thus by Lemma 2.4, we obtain

\[
i(y_1^m) = m(i(y_1) + 3) - 3, \quad \forall m \in \mathbb{N}, \quad \hat{i}(y_1) = i(y_1) + 3.
\]  

(4.49)

Then as Case 3, we can suppose that \( \hat{i}(y_1) \neq 0 \) and exactly one of \( k_l(y_1^m) \) for \( 0 \leq l \leq 2 \) is nonzero.

We have two subcases according to the parity of \( i(y_2) \).

**Subcase 4.1.** \( i(y_2) \) is odd.

For this case, as the same proof of Case 4 of Theorem 1.4 in [LLo2] we can get a contradiction.

**Subcase 4.2.** \( i(y_2) \) is even.

As Subcase 3.2 we have

\[
\hat{i}(y_1) > 0, \quad \hat{\chi}(y_1) \geq 0.
\]  

(4.50)

Then by (5.21), we have

\[
-k_0(y_1) + k_1(y_1) = \hat{\chi}(y_1) \geq 0,
\]

which, together with (iv) of Remark 5.5, implies \( k_0(y_1) = 0 \). Then by Proposition 5.4 we know that \( y_1^m \) has no contribution to \( m_{2q-1} \) for all \( q \in \mathbb{Z} \). Note that \( i(y_2) \) is even and \( y_2 \) is hyperbolic, then by Proposition 5.2, \( y_2^{2n} \) also has no contribution to \( m_{2q-1} \) for all \( q \in \mathbb{Z} \). Thus \( m_{2q-1} = 0 \) for every \( q \in \mathbb{Z} \), which implies \( U(t) \equiv 0 \) in (5.28), then we have

\[
\sum_{i \in \mathbb{Z}} m_i t^i = \frac{1}{1 - t^2} = \sum_{i \in \mathbb{N}} t^{2i-2}.
\]  

(4.51)
Thus $i(y_2) \geq 0$ by Proposition 5.2 and (4.51). Note that $i(y_1) \geq 1$ or $i(y_1) = -1$ by (4.49)-(4.50) and the fact that $i(y_1)$ is odd. If $i(y_1) \geq 1$, then by Lemma 4.1 we know that the two closed characteristics are elliptic, which contradicts to our assumption. Thus we suppose $i(y_1) = -1$. By (i) of Claim 1, it is impossible that $y_2^m$ contributes 1 to every Morse-type number $m_q$ for $q \in 2\mathbb{N}_0$. Noticing that exactly one of $k_1(y_1^m)$ for $0 \leq l \leq 1$ is nonzero, by (4.51) we have $k_1(y_1^m) = 1$ and $\{y_1^m\}_{m \in \mathbb{N}}$ contributes exactly 1 to every Morse-type number $m_q$ for $q \in 2\mathbb{N}_0$, but $y_2$ also has contribution to some Morse-type numbers $m_q$, which contradicts to (4.51).

The proof of Theorem 1.1 is complete.

In the following, we explain why Theorems 1.4 and 1.5 hold.

**Proof of Theorem 1.4.** By Lemma 2.4 and Corollary 3.6, we have a similar result as Theorem 1.6 of [LoZ], and by Theorem 3.10, Lemma 3.1 of [LoZ] holds for star-shaped hypersurfaces. Note that all the proofs of [LoZ] are based on the fact that every closed characteristic $(\tau, y)$ on the hypersurface $\Sigma$ in $\mathbb{R}^{2n}$ satisfies $i(y) \geq n$ and Theorem 1.6, Lemma 3.1 of [LoZ]. Hence for dynamically convex star-shaped case, all the theories of [LoZ] hold. Then combining it with Theorem 1.1 of [Wan2] and Theorem 1.1 of [HuO], we get the desired results.

**Proof of Theorem 1.5.** Note that all the proofs of [WHL] and [Wan1] rely on the resonance identity in Theorem 1.2 of [WHL], the periodic property of critical modules in Proposition 3.13 of [WHL], and the results in [LoZ]. Since we have extended the theories of [WHL] to star-shaped case in [LLW], and all the theories of [LoZ] hold for dynamically convex star-shaped hypersurfaces by Theorem 1.4, then the main results of [WHL] and [Wan3] hold for dynamically convex star-shaped case, i.e., Theorem 1.5 holds.

5 Appendix

In the section, we briefly review the equivariant Morse theory and the resonance identities for closed characteristics on compact star-shaped hypersurfaces in $\mathbb{R}^{2n}$ developed in [LLW]. Now we fix a $\Sigma \in \mathcal{H}_{st}(2n)$ and assume the following condition:

(F) There exist only finitely many geometrically distinct prime closed characteristics $\{(\tau_j, y_j)\}_{1 \leq j \leq k}$ on $\Sigma$.

Let $\hat{\sigma} = \inf_{1 \leq j \leq k} \sigma_j$ and $T$ be a fixed positive constant. Then by Section 2 of [LLW], for any $a > \frac{T}{\hat{\sigma}}$, we can construct a function $\varphi_a \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ which has 0 as its unique critical point in $[0, +\infty)$. Moreover, $\varphi'(t)$ is strictly decreasing for $t > 0$ together with $\varphi(0) = 0 = \varphi'(0)$ and $\varphi''(0) = 1 = \lim_{t \to 0^+} \varphi'(t)$. More precisely, we define $\varphi_a$ and the Hamiltonian function $\tilde{H}_a(x) = a\varphi_a(j(x))$ via Lemma 2.2 and Lemma 2.4 in [LLW]. The precise dependence of $\varphi_a$ on $a$ is explained in Remark 2.3 of [LLW].

For technical reasons we want to further modify the Hamiltonian, we define the new Hamiltonian function $H_a$ via Proposition 2.5 of [LLW] and consider the fixed period problem

\begin{align*}
\begin{cases}
\dot{x}(t) &= JH'_a(x(t)), \\
x(0) &= x(T).
\end{cases}
\end{align*}

(5.1)
Lemma 6.4 of [Vit2]: which follow from the definitions (5.4) and (5.5). The following important formula was proved in where the index
respectively. Similarly, we define the index and nullity of
which defines an orthogonal splitting
by
for all
Then
Then
is a strictly convex function, that is,
for all
and some positive
Let
be the Fenchel dual of
defined by
The dual action functional on
is defined by
Then
satisfies the Palais-Smale condition and
is a critical point of
if and only if it is a solution of (5.1). Moreover,
< 0 and it is independent of
for every critical point
< 0 of
When
, the map
is a Hilbert space isomorphism between
and
. We denote its inverse by
and the functional
Then
and only if
is a critical point of
Moreover,
< 0 and it is independent of
for every critical point
< 0 of
When
, the map
is a Hilbert space isomorphism between
and
. We denote its inverse by
and the functional
Then
is a critical point of
if and only if
is a critical point of
Suppose
is a nonzero critical point of
Then the formal Hessian of
at
is defined by
which defines an orthogonal splitting
 into negative, zero and positive subspaces. The index and nullity of
are defined by
and
respectively. Similarly, we define the index and nullity of
for
, we denote them by
and
. Then we have
which follow from the definitions (5.4) and (5.5). The following important formula was proved in Lemma 6.4 of [Vit2]:
where the index
does not depend on
, but only on
.
By the proof of Proposition 2 of [Vit1], we have that \( v \in E \) belongs to the null space of \( Q_{a,K} \) if and only if \( z = M_K v \) is a solution of the linearized system

\[
\dot{z}(t) = J H''_a(x(t)) z(t).
\]

(5.9)

Thus the nullity in (5.7) is independent of \( K \), which we denote by \( \nu^v(x) \equiv \nu_K(u) = \nu_K(x) \).

By Proposition 2.11 of [LLW], the index \( i^v(x) \) and nullity \( \nu^v(x) \) coincide with those defined for the Hamiltonian \( H(x) = j(x)^\alpha \) for all \( x \in \mathbb{R}^{2n} \) and some \( \alpha \in (1,2) \). Especially \( 1 \leq \nu^v(x) \leq 2n - 1 \) always holds.

We have a natural \( S^1 \)-action on \( X \) or \( E \) defined by

\[
\theta \cdot u(t) = u(\theta + t), \quad \forall \theta \in S^1, \ t \in \mathbb{R}.
\]

(5.10)

Clearly both of \( F_{a,K} \) and \( \Psi_{a,K} \) are \( S^1 \)-invariant. For any \( \kappa \in \mathbb{R} \), we denote by

\[
\Lambda_{a,K}^\kappa = \{ u \in L^2(\mathbb{R}/(T \mathbb{Z}); \mathbb{R}^{2n}) \mid \Psi_{a,K}(u) \leq \kappa \},
\]

and

\[
X_{a,K}^\kappa = \{ x \in W^{1,2}(\mathbb{R}/(T \mathbb{Z}); \mathbb{R}^{2n}) \mid F_{a,K}(x) \leq \kappa \}.
\]

(5.11)

(5.12)

For a critical point \( u \) of \( \Psi_{a,K} \) and the corresponding \( x = M_K u \) of \( F_{a,K} \), let

\[
\Lambda_{a,K}(u) = \Lambda_{a,K}^{\Psi_{a,K}(u)} = \{ w \in L^2(\mathbb{R}/(T \mathbb{Z}); \mathbb{R}^{2n}) \mid \Psi_{a,K}(w) \leq \Psi_{a,K}(u) \},
\]

and

\[
X_{a,K}(x) = X_{a,K}^{F_{a,K}(x)} = \{ y \in W^{1,2}(\mathbb{R}/(T \mathbb{Z}); \mathbb{R}^{2n}) \mid F_{a,K}(y) \leq F_{a,K}(x) \}.
\]

(5.13)

(5.14)

Clearly, both sets are \( S^1 \)-invariant. Denote by \( \text{crit}(\Psi_{a,K}) \) the set of critical points of \( \Psi_{a,K} \). Because \( \Psi_{a,K} \) is \( S^1 \)-invariant, \( S^1 \cdot u \) becomes a critical orbit if \( u \in \text{crit}(\Psi_{a,K}) \). Note that by the condition (F), the number of critical orbits of \( \Psi_{a,K} \) is finite. Hence as usual we can make the following definition.

**Definition 5.1.** Suppose \( u \) is a nonzero critical point of \( \Psi_{a,K} \), and \( N \) is an \( S^1 \)-invariant open neighborhood of \( S^1 \cdot u \) such that \( \text{crit}(\Psi_{a,K}) \cap (\Lambda_{a,K}(u) \cap N) = S^1 \cdot u \). Then the \( S^1 \)-critical modules of \( S^1 \cdot u \) are defined by

\[
C_{S^1,q}(\Psi_{a,K}, S^1 \cdot u) = H_q((\Lambda_{a,K}(u) \cap N)_{S^1}, ((\Lambda_{a,K}(u) \setminus S^1 \cdot u) \cap N)_{S^1}).
\]

Similarly, we define the \( S^1 \)-critical modules \( C_{S^1,q}(F_{a,K}, S^1 \cdot x) \) of \( S^1 \cdot x \) for \( F_{a,K} \).

We fix \( a \) and let \( u_K \neq 0 \) be a critical point of \( \Psi_{a,K} \) with multiplicity \( \text{mul}(u_K) = m \), that is, \( u_K \) corresponds to a closed characteristic \( (\tau, y) \subset \Sigma \) with \( (\tau, y) \) being \( m \)-iteration of some prime closed characteristic. Precisely, we have \( u_K = -J \dot{x} + K x \) with \( x \) being a solution of (5.1) and \( x = \rho y(\frac{\tau'}{p}) \) with \( \frac{\tau'}{p} = \frac{\tau}{aT} \). Moreover, \( (\tau, y) \) is a closed characteristic on \( \Sigma \) with minimal period \( \frac{1}{m} \). By Lemma 2.10 of [LLW], we construct a finite dimensional \( S^1 \)-invariant subspace \( G \) of \( L^2(\mathbb{R}/(T \mathbb{Z}); \mathbb{R}^{2n}) \) and a functional \( \psi_{a,K} \) on \( G \). For any \( p \in \mathbb{N} \) satisfying \( p \tau < aT \), we choose \( K \) such that \( pK \notin \frac{2\pi}{aT} \mathbb{Z} \), then the \( p \)-th iteration \( u_{pK}^p \) of \( u_K \) is given by \( -J \dot{x}^p + pK x^p \), where \( x^p \) is the unique solution of (5.1) corresponding to \( (p\tau, y) \) and is a critical point of \( F_{a,pK} \), that is, \( u_{pK}^p \) is the critical point of \( \Psi_{a,pK} \) corresponding to \( x^p \). Denote by \( g_{pK}^p \) the critical point of \( \psi_{a,pK} \) corresponding to \( u_{pK}^p \) and let

\[
\tilde{\Lambda}_{a,K}(g_K) = \{ g \in G \mid \psi_{a,K}(g) \leq \psi_{a,K}(g_K) \}.
\]
We make the following definition:

\[ \text{mul}(\beta(\forall l \in \text{point of } K) \text{ at } \Phi) \]

Here \( K \) is a critical point of \( F_{a,K} \). Then we have

**Proposition 5.2.** (cf. Proposition 4.2 of [LLW]) For any \( p \in \mathbb{N} \), we choose \( K \) such that \( pK \notin \frac{2\pi}{\tau} \mathbb{Z} \). Let \( u_K \neq 0 \) be a critical point of \( \Psi_{a,K} \) with \( \text{mul}(u_K) = 1 \), \( u_K = -Jx + Kx \) with \( x \) being a critical point of \( F_{a,K} \). Then for all \( q \in \mathbb{Z} \), we have

\[ C_{S^1, q}(\Psi_{a,pK}, S^1 \cdot u_{pK}) = \begin{cases} 0, & \text{if } q < i_{pK}(u_{pK}) \text{ or } q > i_{pK}(u_{pK}) + \nu_{pK}(u_{pK}) - 1 \\ \text{Q}, & \text{otherwise} \end{cases} \]

In particular, if \( u_{pK} \) is non-degenerate, i.e., \( \nu_{pK}(u_{pK}) = 1 \), then

\[ C_{S^1, q}(\Psi_{a,pK}, S^1 \cdot u_{pK}) = \begin{cases} 0, & \text{if } q = i_{pK}(u_{pK}) \text{ and } \beta(x^p) = 1 \\ \text{Q}, & \text{otherwise} \end{cases} \]

We make the following definition:

**Definition 5.3.** For any \( p \in \mathbb{N} \), we choose \( K \) such that \( pK \notin \frac{2\pi}{\tau} \mathbb{Z} \). Let \( u_K \neq 0 \) be a critical point of \( \Psi_{a,K} \) with \( \text{mul}(u_K) = 1 \), \( u_K = -Jx + Kx \) with \( x \) being a critical point of \( F_{a,K} \). Then for all \( l \in \mathbb{Z} \), let

\[ k_{l+1}(u_{pK}) = \dim \left( H_l(W(g_{pK}^a) \cap \tilde{A}_{a,K}(g_{pK}^p), (W(g_{pK}^a) \setminus \{g_{pK}^p\}) \cap \tilde{A}_{a,K}(g_{pK}^p)) \right)^{\beta(x^p) \mathbb{Z}_p} \]

\[ k_l(u_{pK}) = \dim \left( H_l(W(g_{pK}^a) \cap \tilde{A}_{a,K}(g_{pK}^p), (W(g_{pK}^a) \setminus \{g_{pK}^p\}) \cap \tilde{A}_{a,K}(g_{pK}^p)) \right)^{\beta(x^p) \mathbb{Z}_p} \]

Here \( k_l(u_{pK}) \)'s are called critical type numbers of \( u_{pK} \).

By Theorem 3.3 of [LLW], we obtain that \( k_l(u_{pK}) \) is independent of the choice of \( K \) and denote it by \( k_l(x^p) \), here \( k_l(x^p) \)'s are called critical type numbers of \( x^p \).

We have the following properties for critical type numbers:

**Proposition 5.4.** (cf. Proposition 4.6 of [LLW]) Let \( x \neq 0 \) be a critical point of \( F_{a,K} \) with \( \text{mul}(x) = 1 \) corresponding to a critical point \( u_K \) of \( \Psi_{a,K} \). Then there exists a minimal \( K(x) \in \mathbb{N} \) such that

\[ \nu^v(x^{p+K(x)}) = \nu^v(x^p), \quad i^v(x^{p+K(x)}) - i^v(x^p) \in 2\mathbb{Z}, \quad \forall p \in \mathbb{N}, \]

\[ k_l(x^{p+K(x)}) = k_l(x^p), \quad \forall p \in \mathbb{N}, l \in \mathbb{Z}. \]

We call \( K(x) \) the minimal period of critical modules of iterations of the functional \( F_{a,K} \) at \( x \).

For every closed characteristic \( (\tau, y) \) on \( \Sigma \), let \( aT > \tau \) and choose \( \varphi_\alpha \) as above. Determine \( \rho \)
uniquely by \( \frac{\varphi_\alpha(\rho)}{\rho} = \frac{\tau}{aT} \). Let \( x = \rho y(\frac{aT}{\tau}) \). Then we define the index \( i(\tau, y) \) and nullity \( \nu(\tau, y) \) of \( (\tau, y) \) by

\[ i(\tau, y) = i^v(x), \quad \nu(\tau, y) = \nu^v(x). \]
Then the mean index of \((\tau, y)\) is defined by

\[
\hat{i}(\tau, y) = \lim_{m \to \infty} \frac{i(m\tau, y)}{m}.
\]

(5.20)

Note that by Proposition 2.11 of \[LLW\], the index and nullity are well defined and are independent of the choice of \(a\).

For a closed characteristic \((\tau, y)\) on \(\Sigma\), we simply denote by \(y^m = (m\tau, y)\) the \(m\)-th iteration of \(y\) for \(m \in \mathbb{N}\). By Proposition 3.2 of \[LLW\], we can define the critical type numbers \(k_l(y^m)\) of \(y^m\) to be \(k_l(x^m)\), where \(x^m\) is the critical point of \(F_{a,K}\) corresponding to \(y^m\). We also define \(K(y) = K(x)\).

**Remark 5.5.** (cf. Remark 4.10 of \[LLW\]) Note that \(k_l(y^m) = 0\) for \(l \notin [0, \nu(y^m) - 1]\) and it can take only values 0 or 1 when \(l = 0\) or \(l = \nu(y^m) - 1\). Moreover, the following facts are useful:

(i) \(k_0(y^m) = 1\) implies \(k_l(y^m) = 0\) for \(1 \leq l \leq \nu(y^m) - 1\).

(ii) \(k_{\nu(y^m) - 1}(y^m) = 1\) implies \(k_l(y^m) = 0\) for \(0 \leq l \leq \nu(y^m) - 2\).

(iii) \(k_l(y^m) \geq 1\) for some \(1 \leq l \leq \nu(y^m) - 2\) implies \(k_0(y^m) = k_{\nu(y^m) - 1}(y^m) = 0\).

(iv) In particular, only one of the \(k_l(y^m)\)’s for \(0 \leq l \leq \nu(y^m) - 1\) can be non-zero when \(\nu(y^m) \leq 3\).

For a closed characteristic \((\tau, y)\) on \(\Sigma\), the average Euler characteristic \(\hat{\chi}(y)\) of \(y\) is defined by

\[
\hat{\chi}(y) = \frac{1}{K(y)} \sum_{1 \leq m \leq K(y)} (-1)^{i(y^m)} k_l(y^m).
\]

(5.21)

\(\hat{\chi}(y)\) is a rational number. In particular, if all \(y^m\)’s are non-degenerate, then by Proposition 5.4 we have

\[
\hat{\chi}(y) = \begin{cases} 
(-1)^{i(y^2)}, & \text{if } i(y^2) - i(y) \in 2\mathbb{Z}, \\
\frac{(-1)^{i(y)}}{2}, & \text{otherwise}.
\end{cases}
\]

(5.22)

We have the following mean index identities for closed characteristics.

**Theorem 5.6.** Suppose that \(\Sigma \in \mathcal{H}_{ad}(2n)\) satisfies \(\# \mathcal{T}(\Sigma) < +\infty\). Denote all the geometrically distinct prime closed characteristics by \(\{(\tau_j, y_j)\}_{1 \leq j \leq k}\). Then the following identities hold

\[
\sum_{1 \leq j \leq k \atop i(y_j) > 0} \frac{\hat{\chi}(y_j)}{i(y_j)} = \frac{1}{2},
\]

(5.23)

\[
\sum_{1 \leq j \leq k \atop i(y_j) < 0} \frac{\hat{\chi}(y_j)}{i(y_j)} = 0.
\]

(5.24)

Let \(F_{a,K}\) be a functional defined by (5.4) for some \(a, K \in \mathbb{R}\) sufficiently large and let \(\epsilon > 0\) be small enough such that \([-\epsilon, 0)\) contains no critical values of \(F_{a,K}\). For \(b\) large enough, The normalized Morse series of \(F_{a,K}\) in \(X^{-\epsilon} \setminus X^{-b}\) is defined, as usual, by

\[
M_a(t) = \sum_{\nu \geq 0, 1 \leq j \leq p} \dim C_S^{1, q}(F_{a,K}, S^1 \cdot v_j) t^{q - d(K)},
\]

(5.25)

where we denote by \(\{S^1 \cdot v_1, \ldots, S^1 \cdot v_p\}\) the critical orbits of \(F_{a,K}\) with critical values less than \(-\epsilon\). The Poincaré series of \(H_{S^1 \ast}(X, X^{-\epsilon})\) is \(t^{d(K)}Q_a(t)\), according to Theorem 5.1 of \[LLW\], if we set \(Q_a(t) = \sum_{k \in \mathbb{Z}} q_k t^k\), then

\[
q_k = 0 \quad \forall \; k \in \mathbb{Z}.
\]

(5.26)
where $I$ is an interval of $\mathbb{Z}$ such that $I \cap [i(\tau, y), i(\tau, y) + \nu(\tau, y) - 1] = \emptyset$ for all closed characteristics $(\tau, y)$ on $\Sigma$ with $\tau \geq aT$. Then by Section 6 of \cite{LLW}, we have

$$M_a(t) - \frac{1}{1 - t^2} + Q_a(t) = (1 + t)U_a(t),$$

(5.27)

where $U_a(t) = \sum_{i \in \mathbb{Z}} u_it^i$ is a Laurent series with nonnegative coefficients. If there is no closed characteristic with $\hat{i} = 0$, then

$$M(t) - \frac{1}{1 - t^2} = (1 + t)U(t),$$

(5.28)

where $M(t) = \sum_{i \in \mathbb{Z}} m_it^i$ denotes the limit of $M_a(t)$ as $a$ tends to infinity, $U(t) = \sum_{i \in \mathbb{Z}} u_it^i$ denotes the limit of $U_a(t)$ as $a$ tends to infinity and possesses only non-negative coefficients. Specially, suppose that there exists an integer $p < 0$ such that the coefficients of $M(t)$ satisfy $m_p > 0$ and $m_q = 0$ for all integers $q < p$. Then (5.28) implies

$$m_{p+1} \geq m_p.$$

(5.29)

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