PONTRYAGIN-KREIN THEOREM: LOMONOSOV’S PROOF AND RELATED RESULTS

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To the memory of our dear friend and colleague Victor Lomonosov

Abstract. We discuss Lomonosov’s proof of the Pontryagin-Krein Theorem on invariant maximal non-positive subspaces, prove the refinement of one theorem from [22] on common fixed points for a group of fractional-linear maps of operator ball and deduce its consequences. Some Burnside-type counterparts of the Pontryagin-Krein Theorem are also considered.

1. Introduction and preliminaries

In 1944 L. S. Pontryagin, stimulated by actual problems of mechanics, published his famous paper [25] where it was proved that if an operator $T$ is selfadjoint with respect to a scalar product with finite number $k$ of negative squares then $T$ has invariant non-positive subspace of dimension $k$. The importance of results of this kind for stability of some mechanical problems was discovered by S. L. Sobolev in 1938, who proved the existence of non-positive eigenvectors in the case $k = 1$.

Before giving precise formulations we introduce some notations. By indefinite metric space we mean a linear space $H$ supplied with a semilinear form $[x, y]$ satisfying the following condition: $H$ can be decomposed in a direct sum of two subspaces $H_+, H_- (x = x_+ + x_-, \text{ for each } x \in H)$ in such a way that $H$ is a Hilbert space with respect to the form $(x, y) = [x_+, y_+] - [x_-, y_-]$.

The decomposition of this kind is not unique but the dimensions of the summands and the topology on $H$ do not depend on the choice of the decomposition. We assume in what follows that $\dim H_+ \geq \dim H_-$. If $\dim H_- = k < \infty$ then one says that $H$ is a Pontryagin space $\Pi_k$, otherwise $H$ is called a Krein space.$^\dagger$

A vector $x \in H$ is called positive (non-negative, negative, non-positive, neutral) if $[x, x] > 0$ (respectively $[x, x] \geq 0$, $[x, x] < 0$, $[x, x] \leq 0$, $[x, x] = 0$).

A subspace is positive (non-negative, non-positive, negative, neutral) if its non-zero elements are positive (respectively non-negative, non-positive, negative, neutral). For brevity we write MNPS for maximal non-positive subspaces.

Subspaces $H_1, H_2$ of $H$ form a dual pair if $H_1$ is positive, $H_2$ is negative and $H = H_1 + H_2$.

Sometimes it is convenient to start with a Hilbert space $H$ decomposed in the orthogonal sum of two subspaces $H = H_+ \oplus H_-$ and to set

$$[x, y] = (x_+, y_+) - (x_-, y_-).$$

Denoting by $P_+$ and $P_-$ the projections onto $H_+$ and $H_-$, respectively, set $J = P_+ - P_-$. Then one can write the relation between two "scalar products" in the form

$$[x, y] = (Jx, y) \text{ and } (x, y) = [Jx, y].$$

This notation determines the standard terminology. A space with indefinite metric is often called a J-space, a vector $x$ is $J$-orthogonal to a vector $y$ if $[x, y] = 0$. An operator $B$ (we consider only bounded linear operators) on $H$ is called $J$-adjoint to an operator $A$ if $[Ax, y] = [x, By]$, for all $x, y \in H$; we write $B = A^J$. If $A^J = A$
then $A$ is called $J$-selfadjoint; an equivalent condition is $[Ax, x] \in \mathbb{R}$, for all $x \in H$. If ${\text{Im}}([Ax, x]) \geq 0$ for all $x$, then $A$ is called $J$-dissipative.

Furthermore, $A$ is $J$-unitary if $A^* = A^{-1}$ (equivalently, $A$ is surjective and $[Ax, Ay] = [x, y]$, for $x, y \in H$); $A$ is $J$-expanding if $[Ax, Ax] \geq |x|^2$, for all $x \in H$.

In 1949 I. S. Iohvidov [14] constructed an analogue of Caley transform for indefinite metric spaces which allowed him to deduce from Pontryagin’s Theorem the existence of an invariant MNPS for $J$-unitary operators on $\Pi_k$-spaces. Then M. G. Krein [17], using absolutely different approach, proved that a $J$-unitary (and, more generally, $J$-expanding) operator $U$ in arbitrary indefinite metric space has an invariant MNPS, if its "corner" $P_-UP_+$ is compact. Clearly, this condition holds in $\Pi_k$-spaces. In 1964 Ky Fan [19] extended Krein’s Theorem to operators on Banach spaces preserving indefinite norms $\nu(x) = \| (1 - P)x \| - \|Px \|$ where $P$ is a projection of finite rank.

Now we have the following Pontryagin-Krein Theorem (hereafter PK-Theorem).

**Theorem 1.1.** Let an operator $A$ on a Krein space $H$ be $J$-dissipative and let $P_+AP_-$ be compact. Then there exists an MNPS invariant for $A$.

Note that the proof of Pontryagin’s result in [25] was very complicated and long. The Krein’s proof in [17] was short but far from elementary, because it was based on the Schauder-Tichonov fixed-point Theorem. Moreover Ky Fan, to prove his version of the PK-Theorem, previously obtained a more general fixed point theorem. We add that to deduce the result for $J$-dissipative operators from the Krein’s theorem about $J$-expanding operators, one needs to use Iohvidov’s Theory of Caley transformation for Krein spaces which is also very non-trivial.

In 1986 Victor Lomonosov in a talk at the Voronezh Winter School presented a proof of Theorem [11] which was extremely short and completely elementary; this proof was published in [12]. In Section 2 of our paper we present the Lomonosov’s proof in a complete form including the consideration of the finite-dimensional case. In Section 3 we consider the approach based on some fixed point theorems and discuss several results obtained on this way. In Section 4 we prove Theorem 4.1 which refines a theorem of M. Ostrovskii, V.S. Shulman and L. Turowska [22] about common fixed points for a group of fractional-linear maps of operator ball. This allows us to estimate the similarity degree for a bounded representation of a group on a Hilbert space which preserves a quadratic form with finite number of negative squares. In Section 5 we prove by using Theorem 4.1 that any bounded quasi-positive definite function on a group is a difference of two positive definite functions (this was known earlier only for amenable groups). In the final section we discuss Burnside type counterparts of PK-Theorem.

## 2. Lomonosov’s proof of PK-Theorem

As usual, $B(H_1, H_2)$ is the space of all bounded linear operators from $H_1$ to $H_2$, and $B(H) = B(H, H)$ is the algebra of all bounded linear operators on $H$. To any operator $W : H_- \to H_+$ there corresponds the graph-subspace $L_W = \{ x + Wx : x \in H_- \}$; it is easy to see that $L_W$ is maximal non-positive if and only if $W$ is contractive, that is $\|W\| \leq 1$. Conversely, each MNPS is of the form $L_W$, for some contraction $W \in B(H_-, H_+)$. It is not difficult to check that $L_W$ is invariant under an operator $A \in B(H)$ if and only if

$$WA_{11} + WA_{12}W - A_{21} - A_{22}W = 0,$$

where

$$A_{11} = P_-AP_-, \quad A_{12} = P_-AP_+, \quad A_{21} = P_+AP_-, \quad A_{22} = P_+AP_+.$$  \hfill (2.2)

Lomonosov in [12] introduced a "mixed" convergence ($M$-convergence) in $B(H)$: a sequence $\{A^{(k)}\}_{k=1}^\infty$ of operators $M$-converges to an operator $A$, if $A_{11}^{(k)} \to A_{11}$ and $(A_{22}^{(k)})^* \to (A_{22})^*$ in the strong operator topology (SOT), $A_{22}^{(k)} \to A_{22}$ in the weak operator topology (WOT) and $A_{12}^{(k)} \to A_{12}$ in norm.

**Theorem 2.1.** [12] Let a sequence $\{A^{(k)}\}_{k=1}^\infty$ of operators $M$-converge to an operator $A$. If each $A^{(k)}$ has an MNPS then $A$ has an MNPS.

**Proof.** It follows from our assumptions, that for each $k$, there is a contraction $W_k \in B(H_-, H_+)$ satisfying

$$W_kA_{11}^{(k)} + W_kA_{12}^{(k)}W_k - A_{21}^{(k)} - A_{22}^{(k)}W_k = 0.$$  \hfill (2.3)
Choosing a subsequence if necessary, one can assume that the sequence \( \{W_k\}_{k=1}^{\infty} \) WOT-converges to some contraction \( W \in \mathcal{B}(H_-, H_+) \). It follows easily from the definition of \( M \)-convergence that \( W \) satisfies (2.1). □

**Deduction of Theorem 1.1 from Theorem 2.1.** Denote by \( (P_{-}^{(k)})_{k=1}^{\infty} \) and \( (P_{+}^{(k)})_{k=1}^{\infty} \) increasing sequences of finite-dimensional projections such that \( P_{-}^{(k)} \) sot \( \rightarrow P_{-} \) and \( P_{+}^{(k)} \) sot \( \rightarrow P_{+} \), and set \( P^{(k)} = P_{-}^{(k)} + P_{+}^{(k)} \). Then the operators \( A^{(k)} = P^{(k)}A P^{(k)} \) are \( J \)-dissipative, finite-dimensional and \( M \)-converge to \( A \) (the condition \( \|A_{12}^{(k)} - A_{12}\| \to 0 \) follows from the compactness of \( A_{12} \)). To see that each \( A^{(k)} \) has an MNPS it suffices to show that any \( J \)-dissipative operator in a finite-dimensional indefinite metric space has an MNPS. □

The proof of the PK-Theorem in the finite-dimensional case was dropped in [12] as an easy one. In fact, the usual proof of this theorem for matrices (see e.g. [6]) is not simple and is not direct: it goes via study of \( J \)-expanding operators and application of Caley transform. To present Lomonosov’s result in the complete form we add a short direct proof for the finite-dimensional case which again uses Theorem 2.1.

**Completion of the proof of Theorem 1.1.** Let \( A \) be a \( J \)-dissipative operator on a finite-dimensional indefinite metric space \( H \). For each \( t > 0 \), the operator \( B = A + tJ \) satisfies the condition of strong \( J \)-dissipativity

\[ \text{Im} [Bx, x] > 0 \quad \text{if} \quad x \neq 0. \]

Since \( A + tJ \to A \) when \( t \to 0 \), Theorem 2.1 allows us to assume that \( A \) is strongly dissipative. In this case \( A \) has no real eigenvalues: if \( Ax = tx \), for some \( t \in \mathbb{R} \) and \( 0 \neq x \in H \), then \( |Ax, x| = t|x, x| \in \mathbb{R} \), a contradiction. Let us denote by \( H_{+} \) and \( H_{-} \) the spectral subspaces of \( A \) corresponding to sets \( C_{+} = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \) and \( C_{-} = \{ z \in \mathbb{C} : \text{Im} z < 0 \} \), respectively. We will show that subspaces \( H_{+} \) and \( H_{-} \) are positive and negative, respectively.

If an operator \( T \) is strongly \( J \)-dissipative then also \( -T^{-1} \) is strongly \( J \)-dissipative. Indeed,

\[ -\text{Im} [T^{-1}x, x] = \text{Im} [x, T^{-1}x] = \text{Im} [TT^{-1}x, T^{-1}x] > 0 \]

if \( x \neq 0 \). Since \( A - t1 \) is strongly \( J \)-dissipative, for each \( t \in \mathbb{R} \), we get that \( -(A - t1)^{-1} \) is strongly \( J \)-dissipative. Now, for each \( 0 \neq x \in H_{+} \), one has

\[ x = \frac{i}{\pi} \int_{-\infty}^{\infty} (A - t1)^{-1} x dt \]

whence

\[ [x, x] = \text{Re} [x, x] = -\text{Im} \left( \frac{1}{\pi} \int_{-\infty}^{\infty} [(A - t1)^{-1} x] dt \right) > 0. \]

Thus \( H_{+} \) is positive. Similarly, \( H_{-} \) is negative.

So \( H = H_{-} + H_{+} \) is the decomposition of \( H \) into the direct sum of a negative subspace and a positive subspace. It follows that \( H_{-} \) is an invariant MNPS.

We add that

– In works of T. J. Azizov, H. Langer, A. A. Shkalikov and other mathematicians Theorem 2.1 was extended to various classes of unbounded operators (see for example [29] and references therein);
– M. A. Naimark [20] proved that any commutative family \( Q \) of \( J \)-selfadjoint operators in a \( \Pi_{k} \)-space has a common invariant MNPS. It follows that the result holds for any commutative family \( Q \) of operators which is \( J \)-symmetric: \( T \in Q \) implies \( T^{2} \in Q \).

3. **Fixed points**

Let us return to Krein’s proof of the existence of invariant MNPS for \( J \)-unitary operators. It is clear that any \( J \)-unitary operator \( U \) maps any MNPS onto an MNPS. Using the bijection \( W \mapsto L_{W} \) between MNPS subspaces and contractions we see that \( U \) determines the map \( \phi_{U} \) from the closed unit ball \( B_{1}(H_{-}, H_{+}) \) of the space \( \mathcal{B}(H_{-}, H_{+}) \) into itself. It is easy to obtain the direct expression of \( \phi_{U} \) in terms of \( U \):

\[ \phi_{U}(W) = (U_{21} + U_{22}W)(U_{11} + U_{12}W)^{-1} \]
It was shown in [17] that if $U_{12}$ is compact then the map $\phi_U$ is WOT-continuous; since $B_1(H_-, H_+)$ is WOT-compact, the fixed-point theorem implies the existence of a contraction $W$ with $\phi_U(W) = W$. This means that $L_W$ is invariant with respect to $U$. We get the following result:

**Theorem 3.1.** [17] Let $U$ be a $J$-unitary operator on a Krein space $H = H_+ + H_-$. If the “corner” $U_{12}$ in the block-matrix of $U$ with respect to the decomposition $H = H_+ + H_-$ is compact then $U$ has an invariant MNPS.

This result can be reformulated independently of the choice of the decomposition $H = H_+ + H_-$ and without matrix terminology:

**Theorem 3.2.** If $J$-unitary operator $U$ on a Krein space $H$ is a compact perturbation of an operator that preserves a maximal negative subspace, then it has an invariant MNPS.

To prove this let $U = R + K$, where $K$ is compact, $R$ preserves a maximal negative subspace $L \subset H$. Let $M = L^\perp$, and let $P$ be the projection onto $L$ along $M$. Then

$$(1 - P)UP = (1 - P)RP + (1 - P)KP = (1 - P)KP$$

is a compact operator. But $(1 - P)U$ is the corner of the block-matrix $U$ with respect to the decomposition $H = L + M$. So, by Krein’s theorem, $U$ has an invariant MNPS.

Note that for $\Pi_k$-spaces the assumption of compactness of $U_{21}$ is automatically satisfied, so Krein’s Theorem implies that any $J$-unitary operator on a $\Pi_k$-space has an invariant MNPS.

The fractional-linear maps $\phi_U$ defined by [34] preserve the open unit ball $\mathcal{B} = \{X \in B(H_-, H_+) : \|X\| < 1\}$ and their restrictions to $\mathcal{B}$ form the group of all biholomorphic automorphisms of $\mathcal{B}$ (we refer to [1] or [16] for more information). So the existence of fixed points for such maps and families of such maps are of independent interest. After Naimark’s result it was natural to try to prove the existence of common fixed points for commutative sets of fractional-linear maps. Note that this does not follow directly from Naimark’s Theorem, because the maps $\phi_U$ and $\phi_V$ commute if and only if the operators $U$ and $V$ commute up to a scalar multiple: $UV = \lambda VU$, $\lambda \in \mathbb{C}$. The positive answer was obtained by J.W. Helton:

**Theorem 3.3.** [7] Let $H_1$, $H_2$ be Hilbert spaces and $\dim H_1 < \infty$. Then any commutative family of fractional-linear maps of the closed unit ball in $B(H_1, H_2)$ has a common fixed point.

This result implies Naimark’s Theorem, but the proof uses it. Another result of Helton [8] based on the consideration of fractional-linear maps states that a commutative group of $J$-unitary operators on a Krein space $H_1 \oplus H_2$ has an invariant maximal positive subspace if it contains a compact perturbation of an operator $A \oplus B$ with $\sigma(A) \cap \sigma(B) = \emptyset$. This extends the Naimark Theorem because the identity operator 1 in a $\Pi_k$-space is a compact perturbation of $J$.

The following result on fixed points of groups of fractional-linear maps was proved by M. Ostrovskii, V. S. Shulman and L. Turowska [22] [23] (see also [30] where the case $k = 1$ was considered).

**Theorem 3.4.** Let $\dim H_2 = k < \infty$ and let a group $\Gamma$ of fractional-linear maps of the open unit ball $\mathcal{B}$ in $B(H_2, H_2)$ have an orbit separated from the boundary $(\sup_{\phi \in \Gamma} \|\phi(K)\| < 1$, for some $K \in \mathcal{B}$). Then there is $K_0 \in \mathcal{B}$ such that $\phi(K_0) = K_0$, for all $\phi \in \Gamma$.

**Corollary 3.5.** Any bounded group of $J$-unitary operators in a $\Pi_k$-space has an invariant dual pair of subspaces.

We will obtain some related results in the next two sections.
the operators $U(g)$ are unitary, $V$ is an invertible operator. The infimum $c(\pi)$ of values $\|V\|\|V^{-1}\|$ for all possible $V$’s, is called the constant of similarity of $\pi$. It is obvious that a representation can be similar to a unitary one only if it is bounded:

$$\|\pi\| := \sup_{g \in G} \|\pi(g)\| < \infty;$$

By a quadratic form we mean a function $\Phi(x) = (Ax, x)$ on a Hilbert space $H$, where $A$ is an invertible self-adjoint operator on $H$. Changing the scalar product if necessary, one can reduce the situation to the case that

$$\Phi(x) = (P_1x, x) - (P_2x, x), \quad (4.1)$$

where $P_1$ and $P_2$ are projections with $P_1 + P_2 = 1$ (if a form is given as above then $P_1$ and $P_2$ are spectral projections of $A$ corresponding to the intervals $(-\infty, 0)$ and $(0, \infty)$. So we consider only forms given by (4.1). The number $\text{dim}(P_2H)$ is called the number of negative squares of $\Phi$.

A representation $\pi$ is said to preserve the form (4.1) if $\Phi(\pi(g)x) = \Phi(x)$, for all $x \in H, g \in G$.

**Theorem 4.1.** Any bounded representation $\pi$ preserving a form with finite number of negative squares is similar to a unitary representation. Moreover,

$$c(\pi) \leq 2\|\pi\|^2 + 1. \quad (4.2)$$

The first statement of the theorem was proved in [22]: to prove the inequality (4.2) we will repeat some steps of the proof in [22] adding necessary changes and estimations.

We begin with a general result on fixed points of groups of isometries.

Let us say that a metric space $(X, d)$ is ball-compact if a family of balls

$$E_{a, r} = \{x \in X : d(a, x) \leq r\}$$

has non-void intersection provided each its finite subfamily has non-void intersection (see [33]).

A subset $M \subset X$ is called ball-convex if it is the intersection of a family of balls. The compactness property extends from balls to ball-convex sets: if $(X, d)$ is ball-compact, then a family $\{M_\lambda : \lambda \in \Lambda\}$ of ball-convex subsets of $X$ has non-void intersection if each its finite subfamily has non-void intersection.

The diameter of a subset $M \subset X$ is defined by

$$\text{diam}(M) = \sup\{d(x, y) : x, y \in M\}. \quad (4.3)$$

A point $a \in M$ is called diametral if

$$\sup\{d(a, x) : x \in M\} = \text{diam}(M).$$

A metric space $X$ is said to have normal structure if every ball-convex subset of $X$ with more than one element has a non-diametral point.

**Lemma 4.2.** Suppose that a metric space $(X, d)$ is ball-compact and has normal structure. If a group $\Gamma$ of isometries of $(X, d)$ has a bounded orbit $O$, then it has a fixed point $x_0$. Moreover, $x_0$ belongs to the intersection of all ball-convex subsets containing $O$.

**Proof.** The family $\Phi$ of all balls containing $O$ is non-void. Since $O$ is invariant under $\Gamma$, the family $\Phi$ is also invariant: $g(E) \in \Phi$, for each $E \in \Phi$. Hence the intersection $M_1$ of all elements of $\Phi$ is a non-void $\Gamma$-invariant ball-convex set; moreover, it follows easily from the definition that $M_1$ is the intersection of all ball-convex subsets containing $O$.

Thus the family $\mathcal{M}$ of all non-void $\Gamma$-invariant ball-convex subsets of $M_1$ is non-void. Therefore the intersection of a decreasing chain of sets in $\mathcal{M}$ belongs to $\mathcal{M}$ and, by Zorn Lemma, $\mathcal{M}$ has minimal elements. Our aim is to prove that any minimal element $M$ of $\mathcal{M}$ consists of one point.

Assuming the contrary, let $\text{diam}(M) = \alpha > 0$. Since $(X, d)$ has normal structure, $M$ contains a non-diametral point $a$. It follows that $M \subset \{x \in X : d(a, x) \leq \delta\}$ for some $\delta < \alpha$. Set

$$D = \bigcap_{b \in M} E_{b, \delta}.$$

The set $D$ is non-void because $a \in D$. Furthermore, $D$ is ball-convex by definition. To see that $D$ is a proper subset of $M$ take $b, c \in M$ with $d(b, c) > \delta$, then $c \notin E_{b, \delta}$, hence $c \notin D$. 

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Since $\Gamma$ is a group of isometric transformations and $M$ is invariant under each element of $\Gamma$, $D$ is $\Gamma$-invariant. We get a contradiction with the minimality of $M$.

Thus $M = \{x_0\}$, for some $x_0 \in M_1$. \hfill $\square$

Let now $H_1$ and $H_2$ be Hilbert spaces, $\text{dim} \ H_2 < \infty$. We denote by $\mathfrak{B}$ the open unit ball of the space $B(H_2, H_1)$ of all linear operators from $H_2$ to $H_1$.

For each $A \in \mathfrak{B}$, we define a transformation $\mu_A$ of $\mathfrak{B}$ (a M"{o}bius transformation) by setting

$$\mu_A(X) = (1 - AA^*)^{-1/2}(A + X)(1 + A^*X)^{-1}(1 - A^*A)^{1/2}.$$  \hfill (4.4)

It can be easily checked that $\mu_A(0) = A$ and $\mu_A^{-1} = -\mu_A$, for each $A \in \mathfrak{B}$.

We set

$$\rho(A, B) = \tanh^{-1}(|\mu_A(B)|).$$  \hfill (4.5)

It was proved in \cite{22} Theorem 6.1 that the space $(\mathfrak{B}, \rho)$ is ball-compact and has a normal structure. It can be also verified that $\rho$ coincides with the Carathéodory distance $c_\mathfrak{B}$ in $\mathfrak{B}$. Therefore all biholomorphic maps of $\mathfrak{B}$ preserve $\rho$. Applying Lemma 4.3 we get the following statement.

**Lemma 4.3.** If a group of biholomorphic transformations of $\mathfrak{B}$ has an orbit contained in the ball $r\mathfrak{B} = \{X \in \mathfrak{B}(H_2, H_1) : \|X\| \leq r\}$, where $r < 1$, then it has a fixed point $K \in r\mathfrak{B}$.

As we know, biholomorphic transformations of $\mathfrak{B}$ are just fractional-linear transformations corresponding to J-unitary operators in $H = H_1 + H_2$ with the indefinite scalar product $[x, y] = (P_1x, y) - (P_2x, y)$.

Let us denote by $T$ the group of all fractional-linear transformations of $\mathfrak{B}$. Note that $T$ contains all M"{o}bius maps. Indeed it can be easily checked that $\mu_A = \phi_{M_A}$ where $M_A$ is the J-unitary operator with the matrix

$$\begin{pmatrix}
1_H - A^*A & A^*(1_K - AA^*)^{-1/2} \\
A(1_H - A^*A)^{-1/2} & (1_K - AA^*)^{-1/2}
\end{pmatrix}$$

Since $\mu_A(0) = A$ we see that $T$ acts transitively on $\mathfrak{B}$.

**Lemma 4.4.** Let $U$ be a J-unitary operator on a $\Pi_k$-space $H$, $\phi_U$ the corresponding fractional-linear map and $A = \phi_U(0)$. Let $C = \|U\|$ and $r = \|A\|$. Then

$$C \leq \sqrt{(1 + r)(1 - r)^{-1}}.$$  \hfill (4.6)

and

$$r \leq \sqrt{(C^2 - 1)/(C^2 + 1)}.$$  \hfill (4.7)

**Proof.** Let $V = M_A^{-1}U$, then $\phi_V(0) = (\mu_A)^{-1}(A) = 0$ so the J-unitary operator $V$ preserves subspaces $H_1$ and $H_2$; it follows that $V$ is a unitary operator on $H$. Thus $\|U\| = \|MAV\| = \|MA\|$, so it suffices to prove the inequalities (4.6) and (4.7) for $U = M_A$.

Let, for brevity, $S = (1 + A^*A)(1 - A^*A)^{-1}$ and $T = (1 + AA^*)(1 - AA^*)^{-1}$. For any $z = x_1 + x_2 \in H_1 + H_2$, a direct calculation gives

$$\|MAz\|^2 = (Sx_1, x_1) + (Tx_2, x_2) + 4\text{Re}(((1 - AA^*)^{-1}Ax_1, x_2)).$$

Recall that in our notations $\|A\| = r$, $\|MA\| = C$. Since

$$\|S\| = \|T\| = (1 + r^2)(1 - r^2)^{-1}$$

and

$$\|(1 - AA^*)^{-1}A\| = \|(1 - AA^*)^{-1}AA^*(1 - AA^*)^{-1}\|^{1/2}$$

$$= r(1 - r^2)^{-1},$$

we get

$$\|MAz\|^2 \leq (1 + r^2)(1 - r^2)^{-1}(\|x_1\|^2 + \|x_2\|^2) + 4r(1 - r^2)^{-1}\|x_1\|\|x_2\|$$

$$\leq (1 + r^2)(1 - r^2)^{-1}(\|x_1\|^2 + \|x_2\|^2) + 2r(1 - r^2)^{-1}(\|x_1\|^2 + \|x_2\|^2)$$

$$= (1 + r)(1 - r)^{-1}\|z\|^2,$$

which proves (4.6).
On the other hand, for \( x \in H_2 \), we have
\[
\|M_A x\|^2 = (AA^*(1 - AA^*)^{-1}x, x) + ((1 - AA^*)^{-1}x, x)
\]
whence
\[
\sqrt{(1 + r^2)/(1 - r^2)} = \|\sqrt{T}\| \leq \|M_A\| = C.
\]
This shows that the inequality\footnote{\(4.7\)} holds.

\[\square\]

The proof of (4.2) in Theorem 4.1. Now recall that by the assumptions of theorem we have a bounded group \( \{\pi(g) : g \in G\} \) of operators on a Hilbert space \( H \) preserving the form \( \Phi \) given by (4.1). Introducing the indefinite scalar product \( [x, y] = (P_1 x, y) - (P_2 x, y) \) on \( H \), we convert \( H \) into a \( \Pi_k \)-space:
\[H = H_1 + H_2, \text{ where } H_i = P_i H.\]

Since \( \Phi(x) = [x, x] \), all operators \( \pi(g) \) are \( J \)-unitary. Let \( \Gamma = \{\phi_{\pi(g)} : g \in G\} \), the corresponding group of fractional-linear transformations of the open unit ball \( \mathcal{B} \) of \( \mathcal{B}(H_2, H_1) \), and consider the \( \Gamma \)-orbit \( O \) of the point \( 0 \in \mathcal{B} \).

For \( g \in G \), the inequality \( \|\pi(g)\| \leq \|\pi\| \), Lemma 4.4 and monotonicity of the function \( t \mapsto \sqrt{(t^2 - 1)/(t^2 + 1)} \) imply that
\[
\|\phi_{\pi(g)}(0)\| \leq R := \sqrt{\|\pi\|^2 - 1}/(\|\pi\|^2 + 1),
\]
so \( O \subset R\mathcal{B} \). By Lemma 4.3 there is an operator \( K \in R\mathcal{B} \) such that \( \phi_{\pi(g)}(K) = K \), for all \( g \in G \).

Let \( V = M_K \) and \( U(g) = V \pi(g) V^{-1} \) for each \( g \in G \). Then \( U(g) \) is \( J \)-unitary and
\[
\phi_{U(g)}(0) = \mu_K \circ \phi_{\pi(g)}(0) = \mu_K \left( \phi_{\pi(g)}(K) \right) = \mu_K (K) = 0.
\]
Therefore \( U(g) \) preserves \( H_1 \) and \( H_2 \). Since \( (x, y) = [x_1, y_1] - [x_2, y_2] \), where \( x_i = P_i x \in H_i \), \( y_i = P_i y \in H_i \), \( i = 1, 2 \), we see that
\[
(U(g)x, U(g)y) = [U(g)x_1, U(g)y_1] - [U(g)x_2, U(g)y_2] = [x_1, y_1] - [x_2, y_2]
= (x, y),
\]
for all \( x, y \in H \). Thus \( U(g) \) is a unitary operator in \( H \). We proved that \( \pi \) is similar to a unitary representation; moreover, by Lemma 4.4
\[
c(\pi) \leq \|V\|\|V^{-1}\| = \|M_K\|\|M_{-K}\| \leq \sqrt{(1 + R)(1 - R)^{-1}}^2
= (1 + R)(1 - R)^{-1}.
\]
Since \( R = \sqrt{\|\pi\|^2 - 1}/(\|\pi\|^2 + 1) \), we get that
\[
c(\pi) \leq \|\pi\|^2 + 1 + \sqrt{\|\pi\|^4 - 1} < 2\|\pi\|^2 + 1,
\]
which completes the proof.

\[\square\]

The fact that our estimate of the similarity degree does not depend on the number of negative squares leads to the conjecture that the result extends to representations preserving forms with infinite number of negative squares. We shall see now that this is not true.

It is known (see \cite{24}) that for some groups there exist bounded representations which are not similar to unitary ones (there is a conjecture that all non-amenable groups have such representations). Let \( \pi \) be such a representation of a group \( G \) on a Hilbert space \( H \). We define a representation \( \tau \) of \( G \) on \( \mathcal{H} = H \oplus H \) by setting
\[
\tau(g) = \begin{pmatrix} \pi(g) & 0 \\ 0 & \pi(g^{-1})^* \end{pmatrix}.
\]
Clearly \( \tau \) is bounded. Moreover, it is not similar to a unitary representation because otherwise \( \pi \), being its restriction to an invariant subspace, would be similar to a restriction of a unitary representation, which is again unitary.
The space $\mathcal{H}$ is a Krein space with respect to the inner product $[x_1 \oplus y_1, x_2 \oplus y_2] = (x_1, y_2) + (y_1, x_2)$. Indeed, $\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_-$, where the subspaces $\mathcal{H}_+ = \{x \oplus + x : x \in H\}$ and $\mathcal{H}_- = \{x \oplus (-x) : x \in H\}$ are respectively positive and negative. It remains to check that the form $\Phi(x \oplus y) = [x \oplus y, x \oplus y]$ is preserved by operators $\tau(g)$:

$$
[\tau(g)(x \oplus y), \tau(g)(x \oplus y)] = ([\pi(g)x, \pi(g^{-1})^*y] + [\pi(g^{-1})^*y, \pi(g)x])
= (x, y) + (y, x) = [x \oplus y, x \oplus y].
$$

5. Quasi-Positive Definite Functions

Recall that a function $\phi$ on a group $G$ is positive definite (PD, for brevity) if $\phi(g^{-1}) = \overline{\phi(g)}$, for $g \in G$, and the matrices $A_n = (\phi(g_i^{-1} g_j))_{i,j=1}^{n}$ have no negative eigenvalues, for all $n \in \mathbb{N}$ and all $n$-tuples $g_1, ..., g_n \in G$. In other words, the quadratic forms $\sum_{i,j=1}^{n} \phi(g_i^{-1} g_j)z_i z_j$ are positive for all $n \in \mathbb{N}$. A famous theorem of Bochner [2] states that all such functions can be described as matrix elements of unitary representations:

$$
\phi(g) = (\pi(g)x, x),
$$

where $\pi$ is a unitary representation of $G$ in a Hilbert space $H$ and $x \in H$.

We say that $\phi$ is PD of finite type if the corresponding representation is finite-dimensional. It could be proved that $\phi$ is PD of finite type if and only if it satisfies the condition

$$
\phi(g^{-1} h) = \sum_{i=1}^{m} a_i(g) \overline{a_i(h)} \quad \text{for all } g, h \in G,
$$

where $a_i$ are some functions on $G$. For example, the function $\cos x$ is PD of finite type on $\mathbb{R}$.

A function $\phi$ on a group $G$ is called quasi-positive definite (QPD hereafter) if $\phi(g^{-1}) = \overline{\phi(g)}$, for $g \in G$, and there is $k \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$ and any $n$-tuple $g_1, ..., g_n \in G$, the matrix $(\phi(g_i^{-1} g_j))_{i,j=1}^{n}$ has at most $k$ negative eigenvalues. In other words, the quadratic form $\sum_{i,j=1}^{n} \phi(g_i^{-1} g_j)z_i z_j$ should have at most $k$ negative squares.

The study of QPD functions was initiated by M. G. Krein [18] motivated by applications to probability theory — in particular, to infinite divisible distributions and, more generally, to stochastic processes with stationary increments. Other applications of theory of QPD functions are related to moment problems, Toeplitz forms and other topics of functional analysis, see [27, 28] and references therein.

It is easy to see that the difference $a(g) - b(g)$ of two PD functions is a QPD function if $b$ is of finite type. Clearly such QPD functions are bounded. The following theorem shows that all bounded QPD functions are of this type.

**Theorem 5.1.** Every bounded QPD function $\phi$ can be written in the form

$$
\phi(g) = \phi_1(g) - \phi_2(g),
$$

where $\phi_1$ is a PD function and $\phi_2$ is a PD function of finite type.

**Proof.** There is a standard way to associate with $\phi$ a $J$-unitary representation of $G$ on a $\Pi_k$-space. Let $W$ be the linear space of all finitely supported functions on $G$; we define an indefinite scalar product $[\cdot, \cdot]$ on $W$ by setting

$$
[f_1, f_2] = \sum_{g \in G} f_1(g) \overline{f_2(h)} \phi(g^{-1} h). \quad (5.1)
$$

For each $g \in G$ we define an operator $T_g$ on $W$ by setting $T_g f(h) = f(g^{-1} h)$. It is easy to check that the operators $T_g$ preserve $[\cdot, \cdot]$, that is, $[T_g f_1, T_g f_2] = [f_1, f_2]$, for all $f_1, f_2$. Clearly, the map $g \mapsto T_g$ is a representation of $G$ on $W$.

Defining by $\varepsilon_g$, for $g \in G$, the function on $G$ equal 1 at $g$ and 0 at other elements, we see that the matrix $(\phi(g_i^{-1} g_j))_{i,j=1}^{n}$ is the Gram matrix for the family $\varepsilon_{g_1}, ..., \varepsilon_{g_n}$. Since the linear span of vectors $\varepsilon_g$ coincides with $W$, the condition ”$\phi$ is QPD” implies that the dimension of any negative subspace of $W$ does not exceed $k$. It follows that $W = W_1 + W_-$, where $W_1$ is a positive subspace, $H_-$ is negative and $\dim H_- = k$. Denoting by $H_+$ the completion of $W_1$ with respect to the scalar product $[\cdot, \cdot]_{|W_1}$, we get a $\Pi_k$-space $H = H_+ + H_-$. It is not difficult to show that operators $T_g$ extend to bounded $J$-unitary operators $U(g)$ on $H$. It follows easily from the definition that $\phi(g) = [U(g)f, f]$, where $f$ is the image of $\varepsilon_g$ in $H$. 

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Since $\phi$ is bounded, the representation $U$ is bounded (see for example [28] Theorem 3.2]). By Corollary 6.5 there is a decomposition $H = K_+ + K_-$ where $K_+$ is positive, $K_-$ is negative, and both subspaces are invariant for operators $U(g)$. In other words, the operators $U(g)$ commute with the projection $P$ on $K_+$. Setting $f_+ = Pf$, $f_- = (1 - P)f$, we get

$$\phi(g) = [U(g)f, f] = [U(g)f_+, f_+] + [U(g)f_-, f_-] = (U(g)f_+, f_+) - (U(g)f_-, f_-) = \phi_1(g) - \phi_2(g),$$

which is what we need because the functions $\phi_1$ and $\phi_2$ are PD, and $\phi_2$ is of finite type. \hfill \Box

For amenable groups the result was proved by K. Sakai [28].

6. J-symmetric algebras and Burnside-type theorems

As in linear algebra, after proving the existence of a nontrivial invariant subspace (IS, for brevity) for a single operator, one looks for conditions under which a family of operators has a common IS. Since the lattice $\text{Lat}(E)$ of invariant subspaces of a family $E \subset \mathcal{B}(H)$ coincides with $\text{Lat}(A(E))$, where $A(E)$ is the algebra generated by $E$, it is reasonable to restrict ourself by study of non-positive invariant subspaces for algebras (more precisely, for $J$-symmetric operator algebras in a $\Pi_k$-space $H$). Thus one may rewrite the Naimark’s Theorem in the form: all commutative $J$-symmetric algebras in $H$ have invariant MNPS.

For algebras of operators in a finite-dimensional space, the problem of existence of invariant subspaces was completely solved by W. Burnside [3]: the only algebra that has no IS is the algebra of all operators.

For infinite-dimensional Hilbert spaces, the problem is unsolved: it is unknown if there exists an algebra $A \subset \mathcal{B}(H)$ which is not WOT-dense in $\mathcal{B}(H)$. The first answer was given by R. S. Ismagilov [9]: a $J$-symmetric operator algebra has an invariant subspace if and only if it is not $J$-symmetric. The proof is quite complicated and uses the striking theorem of J. Cuntz [4] about $C^*$-equivalent Banach $*$-algebras.

The following Burnside-type result is more closely related to the Pontryagin-Krein Theorem: it describes $J$-symmetric algebras that have no non-positive invariant subspaces. To formulate it let us consider a Hilbert space $E$ and the direct sum $H = \oplus_{i=1}^n E_i$ of $n \leq \infty$ copies of $E$. Let $B(E)^{(n)}$ be the algebra of all operators on $H$ of the form $T \oplus T \oplus ...$, where $T \in B(E)$. On each summand $E_i = E$ in $H$ we choose a projection $P_i$ with $0 \leq \dim P_i E = k_i < \dim E$, assuming that $\sum k_i = k < \infty$, and set $P = P_1 \oplus P_2 \oplus ...$. Then $H$ is a $\Pi_k$-space with respect to the inner product $[x, y] = (Jx, y)$. The algebra $B(E)^{(n)}$ is clearly $J$-symmetric; $J$-symmetric algebras of this form are called model algebras.

**Theorem 6.1.** A WOT closed $J$-symmetric algebra $A$ on a $\Pi_k$-space $H$ does not have non-positive invariant subspaces if and only if it is a direct $J$-orthogonal sum of a $W^*$-algebra on a Hilbert space and a finite number of model algebras.
The proof can be easily deduced from [10, Theorem 13.7] that gives a description of all algebras that have no neutral invariant subspaces. To describe norm-closed $J$-symmetric algebras without non-positive invariant subspaces one should replace in Theorem 6.3 a W*-algebra by a C*-algebra and model algebras $\mathcal{B}(E)(n)$ by the algebras $A^{(n)}$, where $A \subset \mathcal{B}(E)$ is a C*-algebra containing $\mathcal{K}(E)$.

Another natural version of the problem is to describe Banach $*$-algebras with the property that all their $J$-symmetric representations in a $\Pi_k$-space have MNPS. It is shown in [10, Theorem 19.4] that this property is equivalent to the absence of irreducible $\Pi_k$-representations; let us denote by $(K)$ the class of all Banach $*$-algebras that possess it.

It follows from Naimark’s Theorem that $(K)$ contains all commutative algebras. On the other hand Theorem 6.1 implies that any Banach algebra, generated by a bounded subgroup of unitary elements belongs to $(K)$. This implies that $(K)$ contains all C*-algebras (this was proved earlier in [31]).

Recall that a Banach $*$-algebra $A$ is Hermitian if all its selfadjoint elements have real spectra. Let us say that $A$ is almost Hermitian if the elements with real spectra are dense in the space of all selfadjoint elements. It is proved in [16, Corollary 20.6] that all almost Hermitian algebras belong to $(K)$; this result has applications to the study of unbounded derivations of C*-algebras (see [10]).

It is known that the group algebras $L^1(G)$ of locally compact groups are not Hermitian for some $G$ (the Referee kindly informed us about a recent result of Samei and Wiersma [26] which states that $L^1(G)$ is not Hermitian if $G$ is not amenable). It is not known if all algebras $L^1(G)$ are almost Hermitian. Nevertheless all $L^1(G)$ belong to $(K)$; moreover, the following result holds.

**Theorem 6.2.** If $G$ is a locally compact group then any $J$-symmetric representation of $L^1(G)$ on a $\Pi_k$-space $H$ has invariant dual pair of subspaces.

We begin the proof of this theorem with a general statement which is undoubtedly known but it is difficult to give a precise reference.

Recall that the essential subspace for a representation $D$ of an algebra $A$ on a Banach space $X$ is the closure of the linear span $D(A)X$ of all vectors $D(a)x$, where $a \in A$, $x \in X$. If the essential subspace for $D$ coincides with $X$ then $D$ is called essential.

**Lemma 6.3.** Let $L$ be an ideal of a Banach algebra $A$, and $D : L \to \mathcal{B}(X)$ be a bounded essential representation of $L$ in a Banach space $X$. If $L$ has a bounded approximate identity $\{u_n\}$, then $D$ extends to a bounded representation $\tilde{D}$ of $A$ in $X$, and $\|\tilde{D}\| \leq C\|D\|$ where $C = \sup_n \|u_n\|$.

**Proof.** Let us show that

$$\left\| \sum_{i=1}^{n} D(ab_i)x_i \right\| \leq C\|D\|\|a\| \left\| \sum_{i=1}^{n} D(b_i)x_i \right\|,$$

for any $a \in A, b_i \in L, x_i \in X$. Indeed,

$$\left\| \sum_{i=1}^{n} D(au_nb_i)x_i \right\| = \left\| \sum_{i=1}^{n} D((au_n)b_i)x_i \right\| = \left\| D(au_n) \left( \sum_{i=1}^{n} D(b_i)x_i \right) \right\| \leq \|D(au_n)\| \left\| \sum_{i=1}^{n} D(b_i)x_i \right\| \leq C\|D\| \|a\| \left\| \sum_{i=1}^{n} D(b_i)x_i \right\|,$$

and it remains to note that

$$\left\| \sum_{i=1}^{n} D(ab_i)x_i - \sum_{i=1}^{n} D(au_nb_i)x_i \right\| \to 0 \text{ when } n \to \infty.$$
Now we may define a map \( T_a \) on the space \( D(L)X \) by setting

\[
T_a \left( \sum_{i=1}^{n} D(b_i)x_i \right) = \sum_{i=1}^{n} D(ab_i)x_i \quad \text{for all } b_i \in L \text{ and } x_i \in X.
\]

By the above, \( T_a \) is a well defined linear operator on \( D(L)X \) and

\[
\|T_a\| \leq C\|D\|\|a\|.
\]

Denoting by \( \tilde{D}(a) \) the closure of \( T_a \), we obtain an operator on \( X \) with

\[
\|\tilde{D}(a)\| \leq \|C\|\|D\|\|a\|.
\]

It is easy to see that the map \( \tilde{D} : a \rightarrow \tilde{D}(a) \) is a representation of \( A \) on \( X \), extending \( D \). \( \Box \)

Now we need a result about \( J \)-symmetric representations of *-algebras. Recall that a closed subspace \( L \) of an indefinite metric space \( H \) is non-degenerate if \( L \cap L^\perp = 0 \).

**Lemma 6.4.** Let a *-algebra \( L \) have a bounded approximate identity \( \{u_n\} \), and let \( D \) be a \( J \)-symmetric representation of \( L \) on a Krein space \( H \). Then the essential subspace \( H_0 = \overline{D(L)H} \) of \( D \) is non-degenerate, and \( H_0^\perp \subset \ker D(L) \).

**Proof.** Let \( K = H_0 \cap H_0^\perp \). For any \( x \in H \), \( y \in H_0^\perp \) and \( a \in L \) we have \( [x, D(a)y] = [D(a^*)x, y] = 0 \) whence \( D(a)y = 0 \). We proved that \( H_0^\perp \subset \ker D(L) \).

On the other hand, since \( K \subset H_0 \), then for each \( y \in K \) and each \( \varepsilon > 0 \) there is \( z \in D(A)H \) with \( \|z - y\| < \varepsilon \). Note that \( \|D(u_n)z - z\| \to 0 \) when \( n \to \infty \), because \( D(u_n)D(a)x = D(u_na)x \to D(a)x \). Since \( D(u_n)y = 0 \), we get that

\[
\|z\| = \lim \|D(u_n)(z - y)\| \leq C\|D\|\varepsilon,
\]

where \( C = \sup_n \|u_n\| \). Therefore

\[
\|y\| \leq \|z\| + \|y - z\| \leq \varepsilon(1 + C\|D\|).
\]

Since \( \varepsilon \) can be arbitrary we conclude that \( y = 0 \). Thus \( K = 0 \) and \( H_0 \) is non-degenerate. \( \Box \)

**The proof of Theorem 6.3**. Let now \( H \) be a \( \Pi L \)-space and \( D : L^1(G) \to B(H) \) be a continuous \( J \)-symmetric representation. It is known that \( L^1(G) \) has a bounded approximate identity \( \{u_n\} \) (moreover \( \|u_n\| = 1 \), for all \( n \)) so, by Lemma 6.4 \( H \) decomposes in \( J \)-orthogonal sum of subspaces \( H = H_0 + H_0^\perp \), where \( H_0 \) is the essential subspace for \( D \).

The algebra \( L^1(G) \) is an ideal of the *-algebra \( M(G) \) of all finite measures on \( G \); we will denote the involution in \( M(G) \) by \( \mu \mapsto \mu^\ast \) and the product by \( \mu \star \nu \). Applying Lemma 6.3 to the restriction of \( D \) to \( H_0 \), we have that there is a representation \( \tilde{D} \) of \( M(G) \) on \( H_0 \) extending \( D \). To check that \( \tilde{D} \) is \( J \)-symmetric, it suffices to check the equality \( [\tilde{D}(\mu)x, y] = [x, \tilde{D}(\mu^\ast)y] \), for \( x \) of the form \( D(f)z \), where \( f \in L^1(G) \), \( z \in H_0 \). In this case we have

\[
[\tilde{D}(\mu)x, y] = [\tilde{D}(\mu)D(f)z, y] = [D(\mu \ast f)z, y] = [z, D(f^\ast \mu^\ast)y]
\]

\[
= [z, D(f^\ast)\tilde{D}(\mu^\ast)y] = [D(f)z, \tilde{D}(\mu^\ast)y]
\]

\[
= [x, \tilde{D}(\mu^\ast)y].
\]

For each \( g \in G \), we denote by \( \delta_g \) the point measure in \( g \). Setting \( \pi(g) = \tilde{D}(\delta_g) \), we obtain a \( J \)-unitary representation of \( G \). Indeed, since \( (\delta_g)^{-1} = \delta_g^{-1} \), we have

\[
\pi(gh) = \tilde{D}(\delta_{gh}) = \tilde{D}(\delta_g \ast \delta_h) = \tilde{D}(\delta_g)\tilde{D}(\delta_h) = \pi(g)\pi(h),
\]

and

\[
\pi(g)^\ast = (\tilde{D}(\delta_g))^\ast = \tilde{D}(\delta_g^{-1}) = \pi(g)^{-1}.
\]

Since \( \|\delta_g\| = 1 \),

\[
\|\pi(g)\| \leq \|\tilde{D}\|,
\]

so \( \pi \) is bounded.
Let us check that the representation $\pi$ is strongly continuous. Since $\pi$ is bounded, it suffices to verify that the function $g \mapsto \pi(g)x$ is continuous for $x$ in a dense subset of $H_0$. So we may take $x = D(f)y$, for some $f \in L^1(G)$, $y \in H_0$. Since the map $g \mapsto (\delta_g * f)(h) = f(g^{-1}h)$ from $G$ to $L^1(G)$ is continuous, we get that

$$\pi(g)x = \pi(g)D(f)y = D(\delta_g * f)y$$

continuously depends on $g$.

Applying Corollary 3.5, we find an invariant dual pair of subspaces $K_+, K_-$ of $H_0$ invariant for all operators $\pi(g)$. To see that these subspaces are invariant for $D(L^1(G))$, let us denote by $W$ the representation of $L^1(G)$ generated by $\pi$:

$$W(f) = \int_G f(g)\pi(g)dg.$$  

Clearly $K_+$ and $K_-$ are invariant for all operators $W(f)$, and we have only to show that $W(f) = D(f)$, for all $f \in L^1(G)$.

Since $\pi(g)(D(f)x) = D(\delta_g * f)x$, for all $x \in H$ and $f \in L^1(G)$, we have

$$W(u)D(f)x = \int_G u(g)\pi(g)D(f)xdg = \int_G u(g)D(\delta_g * f)xdg = D\left(\int_G u(g)(\delta_g * f)dg\right)x = D(u * f)x$$

for each $u \in L^1(G)$. Since vectors of the form $(D(f)x)$ generate $H$, we conclude that $W(u) = D(u)$.

As we know, the restrictions of all operators $D(f)$, $f \in L^1(G)$, to $H_0^+$ are trivial. So we may choose any dual pair $N_+, N_-$ of $H_0^+$ and, setting $H_+ = K_+ + N_+$, $H_- = K_- + N_-$, we will obtain a dual pair in $H$ invariant for $D(L^1(G))$. \hfill \square

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