Scoring Interval Forecasts: Equal-Tailed, Shortest, and Modal Interval

Jonas R. Brehmer
University of Mannheim, Mannheim, Germany
Heidelberg Institute for Theoretical Studies, Heidelberg, Germany

Tilmann Gneiting
Heidelberg Institute for Theoretical Studies, Heidelberg, Germany
Karlsruhe Institute of Technology, Karlsruhe, Germany

October 23, 2020

Abstract

We consider different types of predictive intervals and ask whether they are elicitable, i.e. are unique minimizers of a loss or scoring function in expectation. The equal-tailed interval is elicitable, with a rich class of suitable loss functions, though subject to translation invariance, or positive homogeneity and differentiability, the Winkler interval score becomes a unique choice. The modal interval also is elicitable, with a sole consistent scoring function, up to equivalence. However, the shortest interval fails to be elicitable relative to practically relevant classes of distributions. These results provide guidance in interval forecast evaluation and support recent choices of performance measures in forecast competitions.

Key words and phrases. Elicitability, forecast evaluation, interval forecast, modal interval, predictive performance, scoring function

2020 MSC: 62C05; 91B06

1 Introduction

In situations where decision making relies on information about uncertain future quantities, it is desirable to not only have a single forecast value, i.e. a point forecast, but also information on the inherent uncertainty of the quantity of interest (Gneiting and Katzfuss, 2014). A particularly attractive, ubiquitously used way to achieve this is to require forecasters to report one or multiple predictive intervals, which are typically designed to contain the observation with specified nominal probability, as requested implicitly or explicitly in the Global Energy Forecasting Competition (Hong et al., 2016), the M4 and M5 Competitions (Makridakis et al., 2020; M Open Forecasting Center, 2020), and the COVID-19 Forecast Hub.
Consequently, methods for the comparative evaluation of interval forecasts are in strong demand. Likewise, researchers and practitioners need methods for choosing between different models for the generation of such intervals.

Technically, three types of predictive intervals have been proposed and used in the literature, two of which are based on the assumption of a nominal coverage probability $1 - \alpha$, where $\alpha \in (0, 1)$. The equal-tailed or central interval lies between the $\frac{\alpha}{2}$- and $(1 - \frac{\alpha}{2})$-quantiles, making it centered in terms of probability. The shortest interval has minimal length, subject to the interval covering the outcome with nominal probability of at least $1 - \alpha$. In contrast, the modal interval maximizes the probability of containing the outcome, subject to a fixed length. Early work on the evaluation problem for interval forecasts can be found in Aitchison and Dunsmore (1968), Winkler (1972), Casella et al. (1993), and Christoffersen (1998). Recently, Askanazi et al. (2018) have emphasized that tools for the comparative evaluation of equal-tailed intervals are readily available, whereas fundamental questions remain open for the shortest interval.

Consistent scoring (or loss) functions are well-established tools for quantifying predictive performance and comparing forecasts, see e.g. Dawid and Musio (2014) and Gneiting (2011) for reviews. In a nutshell, if we ask forecasters to report a certain functional of their predictive distributions, then a key requirement on the loss function is to be (strictly) consistent, in the sense that the expected loss or score is (uniquely) minimized if the directive asked for is followed. The functional is called elicitable if there is a strictly consistent scoring function. While consistent scoring functions have been in routine use for many distributional properties, such as means or quantiles (Gneiting, 2011), the existence problem for any given functional can be a challenge to tackle. For recent progress see Lambert et al. (2008), Heinrich (2014), Steinwart et al. (2014), Fissler and Ziegel (2016), and Frongillo and Kash (2020), among other works.

The remainder of the paper is structured as follows. Section 2 provides a short, technical introduction to the notions of elicitability and consistent scoring functions. The core of the paper is in Section 3, where we discuss the elicitability and properties of any consistent scoring functions for the equal-tailed, shortest, and modal intervals in detail. We show that the Winkler interval score arises as a unique choice for the equal-tailed interval under desirable further conditions, and we resolve a challenge raised by Askanazi et al. (2018), who state desiderata for loss functions tailored to the shortest interval, by showing that in practically relevant settings consistent scoring functions do not exist. Although conceptually different, the modal interval has a close connection to the shortest interval and, perhaps surprisingly, it has a unique consistent scoring function, up to equivalence. Section 4 concludes the paper with a discussion, where we support the choices of performance measures in the aforementioned forecast competitions. Proofs are
2 Consistent scoring functions and elicitation

Here we set up notation and provide general technical background and tools.

Let $Y$ be a random variable that takes values in a closed observation domain $O \subseteq \mathbb{R}$, and let $\mathcal{O}$ be the Borel $\sigma$-algebra on $O$. Let $\mathcal{F}$ be a class of probability measures on $(O, \mathcal{O})$ that represents the possible distributions for $Y$. Typically, the observation domain $O$ will either be the real line $\mathbb{R}$, or the set $\mathbb{N}_0$ of the nonnegative integers, corresponding to count data, which feature prominently in applications such as retail and epidemic forecasting.

A statistical property is a functional $T : \mathcal{F} \to 2^A$, where $2^A$ denotes the power set of the action domain $A \subseteq \mathbb{R}^n$ that contains all possible reports for $T$. The set $T(F) \subseteq A$ consists of all correct forecasts for $F \in \mathcal{F}$. Whenever $T(F)$ reduces to a single value $t \in A$, we use the intuitive notation $T(F) = \{t\}$. Moreover, we let $\mathbb{E}_F$ denote the expectation operator when $Y$ has distribution $F \in \mathcal{F}$. In the special case of an expectation of a derived binary variable we use the symbol $\mathbb{P}_F$ in customary ways. We identify probability distributions with their cumulative distribution functions (CDFs).

A measurable function $h : O \to \mathbb{R}$ is $\mathcal{F}$-integrable if $\mathbb{E}_F h(Y)$ is well-defined and finite for all $F \in \mathcal{F}$. Finally, a scoring function is a mapping $S : A \times O \to \mathbb{R}$ such that $S(x, \cdot)$ is $\mathcal{F}$-integrable for all $x \in A$.

**Definition 2.1.** A scoring function $S$ is **consistent** for a functional $T$ relative to the class $\mathcal{F}$ if

$$\mathbb{E}_F S(t, Y) \leq \mathbb{E}_F S(x, Y)$$

for all $F \in \mathcal{F}$, $t \in T(F)$, and $x \in A$. It is **strictly consistent** for $T$ if it is consistent for $T$ and equality in (1) implies that $x \in T(F)$. If there is a scoring function $S$ that is strictly consistent for $T$ relative to $\mathcal{F}$, then $T$ is called **elicitable**.

If a forecaster is faced with a penalty $S(x, y)$ for a forecast or report $x$ and outcome $y$, consistency of the scoring function $S$ for the functional $T$ ensures that any member of the forecaster’s set of true beliefs $T(F)$ minimizes the expected penalty. Since the ordering in (1) is not affected by scaling $S$ with a positive constant or adding a report-independent function, we say that the scoring function $S'$ is equivalent to $S$ if

$$S'(x, y) = cS(x, y) + h(y)$$

for some $c > 0$ and an $\mathcal{F}$-integrable function $h : O \to \mathbb{R}$.

A basic example of an elicitable functional is the mean or expectation functional $T(F) := \mathbb{E}_F Y$. If defined on the class of the distributions with finite second
moment, squared error, \( S(x, y) = (x - y)^2 \), is a strictly consistent scoring function for \( T \). As the mean functional is single-valued, it can be treated in the point-valued setting, which assumes that functionals map directly into the action domain \( \mathbb{A} \) (Fissler and Ziegel 2016). The following examples illustrate why interval forecasts call for the full set-valued framework of Gneiting (2011a), which assumes that functionals map into the power set \( 2^A \).

**Example 2.2** (quantiles and equal-tailed interval). For \( \alpha \in (0, 1) \) an \( \alpha \)-quantile of \( F \) is a point \( x \in \mathbb{R} \) that satisfies \( F(x-) \leq \alpha \leq F(x) \), where \( F(x-) := \lim_{y \uparrow x} F(y) \) denotes the left-hand limit of \( F \) at \( x \). The \( \alpha \)-quantile functional \( T_{\alpha}(F) := \{ x : F(x-) \leq \alpha \leq F(x) \} \) is set-valued, and it is elicitable relative to any class \( \mathcal{F} \). The strictly consistent scoring functions are equivalent to

\[
S_{\alpha}(x, y) = (1(y \leq x) - \alpha)(g(x) - g(y)),
\]

where \( g \) is \( \mathcal{F} \)-integrable and strictly increasing, see Gneiting (2011a,b) and references therein. The equal-tailed interval for \( F \) at level \( 1 - \alpha \) is defined via the quantiles at levels \( \alpha/2 \) and \( 1 - \alpha/2 \), respectively. Hence, unless both quantiles reduce to single points, there are multiple equal-tailed intervals at level \( 1 - \alpha \), making the equal-tailed interval a set-valued functional, too.

**Example 2.3** (shortest and modal interval). Let \( \alpha, c \in (0, 1) \), and let \( F \) be the uniform distribution on the interval \([0, 1]\). Then every interval of the form \([x, x + \alpha]\), where \( x \in [0, \alpha] \), is a shortest interval at level \( 1 - \alpha \). Moreover, every interval of the form \([x, x + c]\), where \( x \in [0, 1 - c] \), is a modal interval at length \( c \).

A key characteristic of elicitable functionals is their behavior under convex combinations of distributions. The following proposition states the classical convex level sets (CxLS) result (Gneiting 2011a, Theorem 6; Wang and Wei 2020) together with the refined CxLS* property of Fissler et al. (2020, Proposition 3.3).

**Proposition 2.4** (convex level sets). Let \( T : \mathcal{F} \to 2^\mathbb{A} \) be an elicitable functional. If \( F_0, F_1 \in \mathcal{F} \) and \( \lambda \in (0, 1) \) are such that \( F_\lambda = \lambda F_1 + (1 - \lambda) F_0 \in \mathcal{F} \), then

(i) \( T(F_0) \cap T(F_1) \subseteq T(F_\lambda) \) (CxLS property);

(ii) \( T(F_0) \cap T(F_1) \neq \emptyset \Rightarrow T(F_0) \cap T(F_1) = T(F_\lambda) \) (CxLS* property).

If \( T \) is a single-valued functional, the properties coincide and are simply referred to as CxLS. The most relevant examples of functionals that do not have convex level sets and thus fail to be elicitable, are the risk measure Expected Shortfall (ES) and the variance (Gneiting 2011a). If \( \mathbb{A} \subseteq \mathbb{R} \) and certain regularity conditions hold, convex level sets are also sufficient for elicitation, as demonstrated by Steinwart et al. (2014). However, some statistical properties lack these conditions and
fail to be elicitable, even though they have the CxLS∗ property, such as the mode (Heinrich 2014) and tail functionals (Brehmer and Strokorb 2019). In such settings, the following result can be useful, which is a refined version of Theorem 3.3 of Brehmer and Strokorb (2019) that allows for set-valued functionals.

**Proposition 2.5.** Let $T : F \rightarrow 2^A$ be a functional, and let $F_0, F_1 \in F$ be such that $F_\lambda = \lambda F_1 + (1 - \lambda) F_0 \in F$ for all $\lambda \in (0, 1)$. If there are $t_0 \in T(F_0) \setminus T(F_1)$ and $t_1 \in T(F_1) \setminus T(F_0)$ such that for every $\lambda \in (0, 1)$ it holds that either $t_0 \in T(F_\lambda)$ and $t_1 \notin T(F_\lambda)$, or $t_1 \in T(F_\lambda)$ and $t_0 \notin T(F_\lambda)$, then $T$ is not elicitable.

Remarkably, the assertion of Proposition 2.5 overlaps with part (ii) of Proposition 2.4 in the sense that if $T(F_0) \cap T(F_1) \neq \emptyset$ and the conditions of Proposition 2.5 hold, then $T$ cannot have the CxLS∗ property and thus fails to be elicitable. If $T(F_0) \cap T(F_1) = \emptyset$, Proposition 2.5 provides a novel result, since Proposition 2.4(ii) does not address this situation.

The criteria for elicitation presented here will be key tools in what follows. Like the proofs for the subsequent section, the proof of Proposition 2.5 is deferred to the Appendix.

### 3 Types of intervals

We proceed to study equal-tailed, shortest, and modal intervals as functionals on suitable distribution classes $F$. Technically, we encode intervals via their lower and upper endpoints and use the action domain $A = A_0 := \{[a, b] : a, b \in O, a \leq b\}$.

This choice implies that the predictive intervals we consider are closed with endpoints in the observation domain $O$. The endpoint requirement leads to a natural and desirable reduction of the set of possible intervals for discrete data, such as in the case of count data, where the endpoints are required to be nonnegative integers. Closed intervals are compatible with the interpretation of the median as a ‘0% central prediction interval’. Moreover, in discrete settings an interval forecast might genuinely collapse to a single point, so closed intervals allow for a unified treatment of discrete and continuous distributions. Lastly, this setting is consistent with the extant literature, see e.g. Winkler (1972), Lambert and Shoham (2009, Section 7.6), and Askanazi et al. (2018). More general treatments lead to further complexity without recognizable benefits.

We denote the length of an interval $I$ as $\text{len}(I)$, and if $A' \subset A$ is a set of intervals that all have the same length, we refer to this common length as $\text{len}(A')$. The left- and right-hand limits of a function $h : \mathbb{R} \rightarrow \mathbb{R}$ at $x$ are denoted by $h(x-) := \lim_{y \uparrow x} h(y)$ and $h(x+) := \lim_{y \downarrow x} h(y)$, respectively.
3.1 Intervals with coverage guarantees

A standard principle for interval forecasts is that a correct report $I$ contains (or covers) the outcome with specified nominal probability of at least $1 - \alpha$, where $\alpha \in (0, 1)$. A guaranteed coverage interval (GCI) at level $\alpha$ under the predictive distribution $F$ is any element $[a, b] \in A$ satisfying $F(b) - F(a-) \geq 1 - \alpha$, and for all $\varepsilon > 0$

$$F(b - \varepsilon) - F(a-) \leq 1 - \alpha \quad \text{and} \quad F(b) - F((a+\varepsilon)-) \leq 1 - \alpha.$$  \hfill (3)

A GCI thus contains just as much probability mass as necessary, but is not as short as possible. For continuous distributions this definition reduces to the intuitive requirement $F(b) - F(a) = 1 - \alpha$. We write $\text{GCI}_\alpha(F)$ for the set of guaranteed coverage intervals at level $\alpha$ of $F$. An early theoretical treatment is in Proposition 7.6 of Lambert and Shoham (2009), according to which the $\text{GCI}_\alpha$ functional fails to be elicitable relative to the class of all distributions on the finite domain $O = \{1, \ldots, n\}$. Frongillo and Kash (2019, Section 4.2) apply tools of convex analysis to extend this result to more general classes of distributions.

It is straightforward to recover these findings by showing that the $\text{GCI}_\alpha$ functional lacks the CxLS$^*$ property. Specifically, let $\alpha \in (0, 1)$ and consider continuous distributions $F_0$ and $F_1$ that satisfy $F_0(b') - F_0(a') = F_1(b') - F_1(a') = 1 - \alpha$ for some $a' < b'$, whereas

$$F_0(b) - F_0(a) > 1 - \alpha \quad \text{and} \quad F_1(b) - F_1(a) < 1 - \alpha$$

for some $a < b$. Then for some $\lambda \in (0, 1)$ we have $[a, b] \in \text{GCI}_\alpha(F_\lambda)$, even though $[a, b] \not\in \text{GCI}_\alpha(F_0) \cap \text{GCI}_\alpha(F_1) \neq \emptyset$. Part (ii) of Proposition 2.4 thus implies that the $\text{GCI}_\alpha$ functional fails to be elicitable relative to classes $F$ that contain distributions of the type used here. A similar construction for discrete distributions is immediate.

Fissler et al. (2020) introduce a concept of guaranteed coverage without the length restriction (3), i.e. they consider the class of intervals $[a, b] \in A$ which satisfy $F(b) - F(a-) \geq 1 - \alpha$. Like $\text{GCI}_\alpha$, the corresponding set-valued functional fails to be elicitable (Fissler et al. 2020, Corollary 4.7).

In addition to lacking elicitation, the $\text{GCI}_\alpha$ functional has the unattractive feature that it fails to be unique for very many distributions, including, but not limited to, all continuous distributions. This motivates the imposition of additional constraints on the predictive intervals, as discussed now. Fissler et al. (2020) discuss still further types of prediction intervals.

3.2 Equal-tailed interval (ETI)

A straightforward way to pick an interval with nominal coverage at least $1 - \alpha$ under $F$ consists of choosing quantiles at level $\beta \in (0, \alpha)$ and $\beta + 1 - \alpha$ as the
lower and upper endpoint of the interval, respectively.

The ubiquitous choice is $\beta = \frac{a}{2}$, such that under a continuous $F$ the outcomes fall above or below the interval with equal probability of $\frac{a}{2}$. In general, an equal-tailed interval (ETI) at level $\alpha$ of $F$ is any member of

$$\text{ETI}_\alpha(F) := \{[a, b] \in A : a \in T_{\alpha/2}(F), b \in T_{1-\alpha/2}(F)\},$$

where $T_{\beta}(F) := \{x \in O : F(x-) \leq \beta \leq F(x)\}$ denotes the $\beta$-quantile functional. The literature also talks of the ‘central prediction interval’, see, e.g. Fissler et al. (2020). In the simplified situation where $F$ is strictly increasing, all quantiles are unique and thus $\text{ETI}_\alpha(F)$ reduces to a single interval.

By definition, the $\text{ETI}_\alpha$ functional is equivalent to the two-dimensional functional $(T_{\alpha/2}, T_{1-\alpha/2})$, such that forecasting equal-tailed intervals amounts to forecasting quantiles. As a result, the $\text{ETI}_\alpha$ functional is elicitable, and we can construct consistent scoring functions for it from the consistent scoring functions (2) for quantiles, as noted by Gneiting and Raftery (2007) and Askanazi et al. (2018). Specifically, if $w_1, w_2$ are nonnegative weights and $g_1, g_2 : O \to \mathbb{R}$ are non-decreasing $F$-integrable functions, then every $S : A \times O \to \mathbb{R}$ of the form

$$S([a, b], y) = w_1 \left( \mathbb{1}(y \leq a) - \frac{\alpha}{2} \right) \left( g_1(a) - g_1(y) \right) + w_2 \left( \mathbb{1}(y \leq b) - \left(1 - \frac{\alpha}{2}\right) \right) \left( g_2(b) - g_2(y) \right)$$

is a consistent scoring function for the $\text{ETI}_\alpha$ functional. Furthermore, $S$ is strictly consistent if $w_1, w_2 \in (0, \infty)$ and $g_1, g_2$ are strictly increasing. It is no substantial loss of generality to restrict attention to the class in (5), since essentially all strictly consistent scoring functions for $\text{ETI}_\alpha$ are equivalent to this form. This is due to the aforementioned fact that $\text{ETI}_\alpha$ can be interpreted as a vector of two quantiles, and under suitable regularity conditions, all strictly consistent scoring functions for vectors of quantiles are equivalent to a sum of scoring functions of the form (2), see Proposition 4.2(ii) of Fissler and Ziegel (2016, 2020).

The choice $w_1 = w_2 = \frac{2}{\alpha}$ and $g_1(x) = g_2(x) = x$ in (5) obtains the classical interval score (IS) of Winkler (1972), namely,

$$\text{IS}_\alpha([a, b], y) := (b - a) + \frac{2}{\alpha}(a - y) \mathbb{1}(y < a) + \frac{2}{\alpha}(y - b) \mathbb{1}(y > b),$$

which is strictly consistent relative to classes of distributions with finite first moment. This is the most commonly used scoring function for the $\text{ETI}_\alpha$ functional, and scaled or unscaled versions thereof have been employed implicitly or explicitly in highly visible, recent forecast competitions (Hong et al., 2016; Makridakis et al., 2020; M Open Forecasting Center, 2020; Ray et al., 2020).
The Winkler interval score (6) combines various additional, desirable properties of scoring functions on $\mathbb{O} = \mathbb{R}$, such as translation invariance, in the sense that for every $z, y \in \mathbb{R}$ and $a < b$

$$S([a - z, b - z], y - z) = S([a, b], y),$$

and positive homogeneity of order 1, in that for every $c > 0$, $y \in \mathbb{R}$, and $a < b$

$$S([ca, cb], cy) = cS([a, b], y).$$

Additionally, the score applies the same penalty terms to values falling above or below the reported interval, such that it is symmetric, in the sense that

$$S([a, b], y) = S([-b, -a], -y)$$

for $y \in \mathbb{R}$ and $a < b$.

Our next two results concern scoring functions on $\mathbb{O} = \mathbb{R}$ that are of the form (5) and share one or more of these often desirable additional properties. In particular, the next theorem demonstrates that either translation invariance or positive homogeneity and differentiability, combined with symmetry, suffice to characterize the Winkler interval score (6), up to equivalence. To facilitate the exposition, assumption (ii) identifies the action domain $A = \{(a, b) : a \leq b\}$ with the respective subset $\{(a, b)' \in \mathbb{R}^2 : a \leq b\}$ of the Euclidean plane.

**Theorem 3.1.** Let $S$ be of the form (5) with non-constant, non-decreasing functions $g_1$ and $g_2$. If $S$ is either

(i) translation invariant, or

(ii) positively homogeneous and differentiable with respect to $(a, b) \in A \subseteq \mathbb{R}^2$, except possibly along the diagonal,

then $g_1$ and $g_2$ are linear. In particular, if $S$ is symmetric and either (i) or (ii) applies, then $S$ is equivalent to $IS_\alpha$.

The first part of Theorem 3.1, which states the linearity of $g_1$ and $g_2$, continues to hold for asymmetric intervals, defined by choosing endpoints $a \in T_\beta(F)$ and $b \in T_{\beta+1-\alpha}(F)$ for $\beta \in (0, \alpha)$ in (4). However, the second statement does not apply, as non-constant consistent scoring functions for such intervals cannot be symmetric.

If only symmetry is required in (5), then the class of possible scoring functions for the equal-tailed interval is much larger than just the interval score. To characterize these functions take $I$ to be the class of all non-decreasing functions $g : \mathbb{R} \to \mathbb{R}$ with the property that $g(x) = \frac{1}{2}(g(x-) + g(x+))$ for $x \in \mathbb{R}$. In a
trivial deviation from Ehm et al. (2016) we define the elementary quantile scoring function as

\[ S_{\alpha,\theta}(x, y) = (1(y \leq x) - \alpha) \left( 1(\theta < x) + \frac{1}{2}1(\theta = x) - 1(\theta < y) - \frac{1}{2}1(\theta = y) \right), \]

which is the special case in (2) where \( g(x) = 1(\theta < x) + \frac{1}{2}1(\theta = x) \). Given any \( \theta \geq 0 \), we now define

\[ S_{\alpha,\theta}([a, b], y) = S_{\alpha/2,\theta}(a, y) + S_{1-\alpha/2,\theta}(b, y) \]

and refer to \( S_{\alpha,\theta} \) as the elementary symmetric interval scoring function. The following result shows that every symmetric scoring function of the form (5) arises as a mixture of elementary symmetric interval scoring functions. The Winkler interval score (6) emerges in the special case where the mixing measure \( \mu \) is proportional to Lebesgue measure.

**Theorem 3.2.** Let \( S \) be of the form (5) with non-constant, non-decreasing functions \( g_1, g_2 \in \mathcal{I} \). If \( S \) is symmetric, then it is of the form

\[ S([a, b], y) = \int_{[0, \infty)} S_{\alpha,\theta}([a, b], y) \, d\mu(\theta), \]

where \( \mu \) is a Borel measure on \([0, \infty)\), defined via \( d\mu(\theta) = dh(\theta) \) with \( h(\theta) = w_1(g_1(\theta) - g_1(-\theta)) \) for \( \theta \in [0, \infty) \).

The usual treatment considers distributions \( F \in \mathcal{F} \) with strictly increasing CDFs, such that all quantiles are unique. This ensures that the interval is truly equal-tailed, with \( \text{ETI}_\alpha(F) = [a, b] \) implying that \( \mathbb{P}_F(Y < a) = \mathbb{P}_F(Y > b) = \frac{\alpha}{2} \). When \( F \) admits a Lebesgue density, but some quantiles are not unique, this property continues to hold.

However, care is needed when interpreting equal-tailed intervals for discrete distributions. As a simple example, let \( \alpha = 0.2 \) and consider the distribution \( G \) on \( \mathbb{N}_0 \) that assigns probability 0.1, 0.4, 0.4, and 0.1 to 0, 1, 2, and 3, respectively. Since neither the \( \frac{\alpha}{2} \)- nor the \( (1 - \frac{\alpha}{2}) \)-quantile are unique, there are four possible equal-tailed intervals, as listed in Table 1. The distribution \( G \) illustrates that the coverage of an equal-tailed interval does not always equal \( 1 - \alpha \), and may differ among the valid intervals. Moreover, \([0, 3]\) is not a guaranteed coverage interval in the sense of Section 3.1 as it is unnecessarily long. A natural idea is to issue recommendations for such cases, e.g. ‘report the shortest available interval’ or ‘report the interval with the highest coverage’. However, consistent scoring functions for the \( \text{ETI}_\alpha \) functional cannot be used to ensure that forecasters follow such further guidelines, since by the definition of consistency, any valid report attains the same expected score.
Table 1: Properties of the four different intervals in ETI\(_\alpha(G)\), where \(\alpha = 0.2\). The expected penalty for an interval forecast \([a, b]\) is given by 
\[\mathbb{E}_G[\text{IS}\_\alpha([a, b], Y) \mathbb{1}(Y \notin [a, b])],\]
so that the expected score decomposes into length plus expected penalty. See text for details.

| Interval | Coverage | Expected IS\(_\alpha\) | Length | Expected Penalty |
|----------|----------|------------------------|--------|------------------|
| [1, 2]   | 0.8      | 3                      | 1      | 2                |
| [0, 2]   | 0.9      | 3                      | 2      | 1                |
| [1, 3]   | 0.9      | 3                      | 2      | 1                |
| [0, 3]   | 1.0      | 3                      | 3      | 0                |

3.3 Shortest interval (SI)

Instead of defining an interval at the coverage level \(1 - \alpha\) via fixed quantiles, the shortest of these intervals is often sought. Specifically, a *shortest interval* (SI) at level \(\alpha\) of \(F\) is any member of the set

\[\text{SI}\_\alpha(F) := \arg \min_{[a, b] \in A} \left\{ b - a : F(b) - F(a-) \geq 1 - \alpha \right\}. \tag{7}\]

The shortest interval is never longer than an equal-tailed interval, and in general the two types of intervals differ from each other. To see this we follow Askanazi et al. (2018, Appendix) and consider a distribution \(F\) on \(O = [0, \infty)\) with strictly decreasing Lebesgue density, so that \(\text{SI}\_\alpha(F) = [0, T_{1-\alpha}(F)]\), whereas \(\text{ETI}\_\alpha(F) = [T_{\alpha/2}(F), T_{1-\alpha/2}(F)]\) with a lower endpoint that is strictly positive. However, for distributions with a symmetric, strictly unimodal Lebesgue density the two types of intervals are both unique and agree with each other. If a distribution has multiple shortest intervals, then neither of them needs to be an equal-tailed interval.

As noted in Askanazi et al. (2018), loss functions that have been proposed for interval forecasts fail to be strictly consistent for the SI\(_\alpha\) functional, since they are usually tailored to the ETI\(_\alpha\) functional. The question whether the SI\(_\alpha\) functional is elicitable thus remains unanswered, and Askanazi et al. (2018) formulate desiderata for possible scoring functions. A first result in this direction is discussed in Section 4.2 of Frongillo and Kash (2019), who show that the SI\(_\alpha\) functional fails to be elicitable relative to classes \(\mathcal{F}\) that contain piecewise uniform distributions. In the following we show non-elicitability for more general classes of distributions, and we also treat discrete distributions on \(\mathbb{N}_0\). We start by studying level sets.

**Proposition 3.3** (convex level sets).

(i) The functional SI\(_\alpha\) has the CxLS property.
(ii) If the class $\mathcal{F}$ consists of distributions with continuous CDFs only, then $\text{SI}_\alpha$ has the CxLS$^*$ property.

The next example shows that the CxLS$^*$ property can be violated for discrete distributions.

**Example 3.4.** Let $\alpha \in (0, \frac{1}{2})$, and let $k \geq 1$ be an integer. Let $\varepsilon \in (0, \frac{2}{3})$ and $\delta \in (0, \varepsilon)$. Let $F_0$ and $F_1$ be probability distributions on $\mathbb{N}_0$ that assign mass $\varepsilon + \delta$ to $k - 1$ and mass $1 - \alpha - \varepsilon$ to $k$. Furthermore, $F_0$ and $F_1$ assign mass $\varepsilon + \delta$ and $\varepsilon - \delta$, respectively, to $k + 1$. This partial specification of $F_0$ and $F_1$ implies that

$$\text{SI}_\alpha(F_0) = \{[k-1, k], [k, k+1]\} \quad \text{and} \quad \text{SI}_\alpha(F_1) = \{[k-1, k]\},$$

and for $\lambda \in [0, \frac{1}{2}]$ we have $\text{SI}_\alpha(F_\lambda) = \text{SI}_\alpha(F_0) \supseteq \text{SI}_\alpha(F_1)$. Therefore, $\text{SI}_\alpha$ does not have the CxLS$^*$ property relative to any convex class $\mathcal{F}$ that includes $F_0$ and $F_1$.

The restrictions on $\alpha$, $\varepsilon$, and $\delta$ in Example 3.4 ensure that the distributions $F_0$ and $F_1$ are well-defined, unimodal, and satisfy (8). To construct such distributions for general $\alpha \in (0, 1)$, we choose $\varepsilon$ and $\delta$ suitably small and ‘spread’ the probability mass outside of $\{k-1, k, k+1\}$ such that $k$ is the unique mode and (8) holds. We thus obtain the following result.

**Theorem 3.5.** Let $k \geq 1$ be an integer, and let $\mathcal{F}$ be a class of probability measures on $\mathbb{N}_0$ that contains all unimodal distributions with mode $k$. Then the $\text{SI}_\alpha$ functional is not elicitable relative to $\mathcal{F}$.

We turn to classes of distributions with Lebesgue densities, so that the $\text{SI}_\alpha$ functional has the CxLS$^*$ property, and a more refined analysis proves useful. First we take up an example in Section 4.2 of [Frongillo and Kash (2019)].

**Example 3.6.** Given $\alpha \in (0, \frac{3}{5})$, we define distributions $F_0$ and $F_1$ via the piecewise uniform densities

$$f_0(x) = (1 - \alpha) \mathbb{1}_{[0,1]}(x) + \frac{\alpha}{3} \mathbb{1}_{[2,5]}(x) \quad \text{and} \quad f_1(x) = \frac{1 - \alpha}{2} \mathbb{1}_{[0,2]}(x) + \frac{\alpha}{3} \mathbb{1}_{[2,5]}(x),$$

so that $\text{SI}_\alpha(F_0) = [0, 1]$ and $\text{SI}_\alpha(F_1) = [0, 2]$, respectively. As $\text{SI}_\alpha(F_\lambda) = [0, 2]$ for all $\lambda \in (0, 1)$, we conclude from Proposition 2.5 that the $\text{SI}_\alpha$ functional fails to be elicitable relative to convex classes of distributions that contain $F_0$ and $F_1$.

As noted, Example 3.6 applies in situations where the class $\mathcal{F}$ includes distributions with piecewise uniform densities. As this assumption may be restrictive in practice, we proceed to demonstrate non-elicitability based on substantially more flexible criteria.
**Condition 3.7.** The distribution $F$ admits a Lebesgue density, and there are numbers $a < b$ and $\varepsilon > 0$ such that $\text{SI}_\alpha(F) = [a, b]$, $F(b) = F(b + \varepsilon)$, and if $\beta < \alpha$ then $\text{len}(\text{SI}_\beta(F)) > \text{len}(\text{SI}_\alpha(F)) + \frac{1}{2}\varepsilon$.

Loosely speaking, this condition requires that there are ‘gaps’ on the right- and left-hand side of the shortest interval at level $\alpha$, while every shortest interval for a level $\beta < \alpha$ is notably longer than the one at level $\alpha$.

**Theorem 3.8.** If the class $\mathcal{F}$ contains the location-scale family of a distribution satisfying Condition [3.7] along with its finite mixtures, then the $\text{SI}_\alpha$ functional is not elicitable relative to $\mathcal{F}$.

A related result concerning the non-elicitability of $\text{SI}_\alpha$ is given in Theorem 4.16(i) of Fissler et al. (2020). The main difference to Theorem 3.8 is that Fissler et al. (2020) consider a different class $\mathcal{F}$ and allow for scoring functions which take values in the extended real numbers $\mathbb{R} \cup \{-\infty, \infty\}$.

Although Condition 3.7 might seem technical, suitable distributions $F$ can be constructed under rather weak assumptions. For instance, assume $\alpha < \frac{1}{2}$, and let the class $\mathcal{F}$ contain some compactly supported distribution, along with the respective location-scale family, and all finite mixtures thereof. Then constructing an $F$ that satisfies Condition 3.7 is straightforward. A more restrictive requirement is the identity $F(b) = F(b + \varepsilon)$, as it rules out distributions with strictly positive densities. The existence of strictly consistent scoring functions relative to classes of distributions of this type, including but not limited to the important case of the finite mixture distributions with Gaussian components, remains an open problem.

We conclude this subsection by considering limit cases of the shortest interval functional. For $\alpha \to 0$ the set $\text{SI}_\alpha(F)$ reduces to a single member, namely, the interval $[\text{ess inf}(F), \text{ess sup}(F)]$, where ess inf and ess sup denote the essential infimum and essential supremum, respectively. This functional is not elicitable in our setting (Brehmer and Strokorb, 2019), however, when allowing for infinite scores, strictly consistent scoring functions become available (Fissler et al., 2020, Proposition 4.13). For $\alpha \to 1$ we need to distinguish two cases. If the elements in $\mathcal{F}$ admit strictly unimodal densities with respect to Lebesgue measure, then $\text{SI}_\alpha$ tends to the mode, which fails to be elicitable (Heinrich, 2014), see also the discussion in Section 3.4. For discrete distributions on $\mathbb{N}_0$ the minimal interval length zero can be attained so that as $\alpha \to 1$ the members of the set $\text{SI}_\alpha(F)$ eventually comprise the single point intervals to which $F$ assigns positive probability. This limit functional does not have the CxLS$^*$ property, thus it fails to be elicitable by Proposition 2.4.
3.4 Modal interval (MI)

In stark contrast to shortest and equal-tailed intervals, we turn to a type of interval that seeks to maximize coverage, subject to constraints on length.

Specifically, given any \( c > 0 \), a modal interval (MI) of length \( 2c \) of \( F \) is any member of the set

\[
\text{MI}_c(F) = \arg \max_{[a,b] \in A} \{ F(b) - F(a-) : b - a \leq 2c \}.
\]

(9)

If \( F \) has a strictly unimodal Lebesgue density, then the modal interval shrinks towards the mode as \( c \to 0 \). For distributions on \( \mathbb{N}_0 \) the modal interval even agrees with the mode if \( c < \frac{1}{2} \). This connection and the fact that the length of a modal interval is fixed, suggest that the \( \text{MI}_c \) functional can be interpreted as a location statistic, whereas the shortest and equal-tailed intervals contain information on both location and spread.

In what follows, separate discussions for classes \( \mathcal{F} \) of continuous and discrete distributions will be warranted. For distributions on \( \mathbb{N}_0 \), the length of the modal interval will effectively be \( \lfloor 2c \rfloor \), since expanding it further cannot add probability mass. In this situation it is convenient to consider \( c \geq 0 \), substitute \( 2c = k \) where \( k \in \mathbb{N}_0 \), and encode the interval via its lower endpoint functional \( l_k \), so that \( \text{MI}_{k/2}(F) = \{ [x, x + k] : x \in l_k(F) \} \). Then

\[
S(x, y) = -1(x \leq y \leq x + k)
\]

(10)

is a strictly consistent scoring function for the functional \( l_k \) on the class of all distributions on \( \mathbb{N}_0 \). In particular, the \( l_k \) and \( \text{MI}_{k/2} \) functionals are elicitable. In the special case \( k = 0 \), \( l_0 \) is the mode functional and (10) becomes \( S(x, y) = -1(x = y) \), the familiar zero-one or misclassification loss. Lambert and Shoham (2009) and Gneiting (2017) demonstrate that for distributions with finitely many outcomes, zero-one loss is essentially the only consistent scoring function for the mode functional. We extend this result to all integers \( k \geq 0 \), showing that \( k \)-zero-one-loss (10) is essentially the only strictly consistent scoring function for the \( l_k \) and \( \text{MI}_{k/2} \) functionals.

**Theorem 3.9.** Let \( k \geq 0 \) be an integer, and let \( \mathcal{F} \) be a class of probability measures on \( \mathbb{N}_0 \) that contains all distributions with finite support. Then any scoring function that is strictly consistent for the \( l_k \) functional relative to the class \( \mathcal{F} \) is equivalent to \( k \)-zero-one-loss (10).

For distributions with Lebesgue densities we encode \( \text{MI}_c \) via its midpoint functional \( m_c \) so that \( \text{MI}_c(F) = \{ [x - c, x + c] : x \in m_c(F) \} \), where \( c > 0 \). Under this convention

\[
S(x, y) := -1(x - c \leq y \leq x + c)
\]

(11)
is a strictly consistent scoring function for \( m_c \) on the class of distributions with Lebesgue densities, whence \( m_c \) and MI\(_c\) are elicitable. In the limit as \( c \to 0 \), the scoring function (11) becomes zero almost everywhere and thus cannot be strictly consistent for any functional. Heinrich (2014) shows that there are no alternative scoring functions, so the mode fails to be elicitable relative to sufficiently rich classes of distributions with densities. Further aspects are treated in [Dearborn and Frongillo (2020)].

The following theorem demonstrates, perhaps surprisingly, that \( c \)-zero-one-loss (11) is essentially the only strictly consistent scoring function for the \( m_c \) and MI\(_c\) functionals.

**Theorem 3.10.** Let \( c > 0 \), and let \( \mathcal{F} \) be a class of probability measures on \( \mathbb{R} \) that contains all distributions with Lebesgue densities on bounded support. Then any scoring function that is strictly consistent for the \( m_c \) functional relative to \( \mathcal{F} \) is almost everywhere equal to a scoring function which is equivalent to \( c \)-zero-one-loss (11).

We complete this section by connecting modal and shortest intervals. While these are conceptually different types of intervals, a comparison of (7) and (9) shows that the SI\(_\alpha\) and MI\(_c\) functionals relate via their defining optimization problems. Specifically, the SI\(_\alpha\)(\( F \)) functional is a solution to the constrained optimization problem

\[
\min_{[a,b] \in \mathcal{A}} (b - a) \quad \text{such that} \quad F(b) - F(a-) \geq 1 - \alpha,
\]

while the MI\(_c\) functional is a solution to

\[
\max_{[a,b] \in \mathcal{A}} \left( F(b) - F(a-) \right) \quad \text{such that} \quad b - a \leq 2c.
\]

Consequently, if either \( \text{len}(\text{SI}_\alpha(F)) = 2c \) or \( \mathbb{P}_F(Y \in \text{MI}_c(F)) = 1 - \alpha \), one condition implies the other, and MI\(_c\)(\( F \)) = SI\(_\alpha\)(\( F \)) holds. It remains unclear whether this connection can be exploited to construct strictly consistent scoring functions for the SI\(_\alpha\) functional on suitably restrictive, special classes of distributions.

### 4 Discussion

A central task in interval forecasting is the evaluation of competing forecast methods or models, a problem that is often addressed by using scoring or loss functions. For each method or model, and for each forecast case, the empirical loss is computed. Losses are then averaged over forecast cases, and methods with lower mean loss or score are preferred. However, for this type of comparative evaluation to be
decision theoretically justifiable, the loss function needs to be strictly consistent for the predictive interval at hand.

Of the three types of predictive intervals discussed in this paper, the equal-tailed and modal intervals are elicitable, and we have discussed the available strictly consistent scoring functions. For the popular equal-tailed interval, a rich family of suitable functions is available, and our findings support the usage of the Winkler interval score \([6]\), well in line with implementation decisions in forecast competitions. In contrast, the shortest interval functional fails to be elicitable relative to classes of distributions of practical relevance. In this way, we resolve the questions raised by Askanazi et al. (2018) concerning the existence of suitable loss functions for the shortest interval in the negative. Importantly, there is no obvious way of setting incentives for forecasters to report their true shortest intervals. Equal-tailed intervals are preferable due to their elicitation, in concert with other considerations, such as the intuitive connection to quantiles and equivariance under strictly monotone transformations (Askanazi et al., 2018, p. 961).

The modal interval admits a unique strictly consistent scoring function relative to comprehensive classes of both discrete and continuous distributions, up to equivalence. This appears to be a rather special situation, as functionals studied in the extant literature either fail to be elicitable, or admit rich classes of genuinely distinct consistent scoring functions (Fissler and Ziegel, 2016; Frongillo and Kash, 2019; Gneiting, 2011a; Steinwart et al., 2014). It would be of great interest to gain an understanding of conditions under which consistent scoring functions are essentially unique.

As illustrated, interval forecasts are best suited for continuous distributions, and may exhibit counter-intuitive properties in discrete settings. In particular, in the discrete case it may be unavoidable that the coverage probability of a perfect forecast exceeds the nominal level \(1 - \alpha\). This raises problems when assessing interval calibration with the methods of Christoffersen (1998), since asymptotically the null hypothesis of frequency calibration will then be rejected even under perfectly correct forecasts. Modifying the null hypothesis to nominal coverage greater than or equal to \(1 - \alpha\) is not a remedy, since such a test does not have any power against forecast intervals with too high coverage. Consequently, tests for correct forecast specification as in Christoffersen (1998) can be problematic when data fail to be well-approximated by continuous distributions, such as in the case of retail sales. Fortunately, comparative evaluation via consistent scoring functions remains valid and unaffected (Czado et al., 2009; Kolassa, 2016).

In many ways, interval forecasts can be seen as an intermediate stage in the ongoing, transdisciplinary transition from point forecasts to fully probabilistic or distribution forecasts (Askanazi et al., 2018). Indeed, probabilistic forecasts in the form of predictive distributions are the gold standard, as they allow for full-fledged
decision making and well-understood, powerful evaluation methods are available (Dawid, 1986; Gneiting et al., 2007; Gneiting and Katzfuss, 2014). Generally, probabilistic forecasts can be issued in a number of distinct formats, ranging from the use of parametric distributions, such as in the Bank of England Inflation Report (Clements, 2004), to Monte Carlo samples from predictive models, as well as simultaneous quantile forecasts at pre-specified levels, such as in the Global Energy Forecasting Competition 2014 (Hong et al., 2016), the M5 Competition (M Open Forecasting Center, 2020) and the COVID-19 Forecast Hub (Ray et al., 2020). If the quantile levels requested are symmetric about the central level of $\frac{1}{2}$, the collection of quantile forecasts corresponds to a family of equal-tailed predictive intervals. Predictive performance can then be assessed via weighted or unweighted averages of scaled or unscaled versions of the Winkler interval score (6). The theoretical results presented here support this widely used practice.

Appendix: Proofs

Proof of Proposition 2.5

Let $t_0, t_1$ be as stated and set $F_\lambda := \lambda F_1 + (1 - \lambda)F_0$. Suppose that $S$ is a strictly consistent scoring function for $T$. Linearity of expectations in the measure yields

$$\mathbb{E}_{F_\lambda} [S(t_0, Y) - S(t_1, Y)] = \lambda \mathbb{E}_{F_1} [S(t_0, Y) - S(t_1, Y)] + (1 - \lambda) \mathbb{E}_{F_0} [S(t_0, Y) - S(t_1, Y)],$$

where the first difference is positive, while the second difference is negative. Consequently, $\mathbb{E}_{F_\lambda} S(t_0, Y) = \mathbb{E}_{F_\lambda} S(t_1, Y)$ for some $\lambda \in (0, 1)$. Since either $t_0 \in T(F_\lambda)$ and $t_1 \notin T(F_\lambda)$, or $t_1 \in T(F_\lambda)$ and $t_0 \notin T(F_\lambda)$, we arrive at a contradiction.

Proof of Theorem 3.1

Let $S$ be a scoring function of the form (5). Let $y, z \in \mathbb{R}$, $a < b$ and choose $b = y$. Then translation invariance of $S$ gives

$$-w_1 \frac{\alpha}{2}(g_1(a) - g_1(y)) = S([a, y], y) = S([a - z, y - z], y - z) = -w_1 \frac{\alpha}{2}(g_1(a - z) - g_1(y - z)),$$

and rearranging yields $g_1(a) - g_1(y) = g_1(a - z) - g_1(y - z)$ for $a, y, z \in \mathbb{R}$. Choose $y = 0$ and define $\hat{g}(x) := g_1(x) - g_1(0)$ to obtain $\hat{g}(a - z) = \hat{g}(a) + \hat{g}(-z)$ for $a, z \in \mathbb{R}$. Thus $g$ obeys Cauchy’s functional equation, and since $\hat{g}$ is non-constant.
and non-decreasing, we get \( g_1(x) = \gamma x + g_1(0) \) for some \( \gamma > 0 \). For \( g_2 \) we apply the same arguments, to complete the proof of part (i).

Let \( y \in \mathbb{R}, a < b \) and choose \( b = y \). If \( S \) is positively homogeneous then for all \( c > 0 \)

\[
-w_1 c^\frac{\alpha}{2} (g_1(a) - g_1(y)) = c S([a, y], y) = S([ca, cy], cy) = -w_1 c^\frac{\alpha}{2} (g_1(ca) - g_1(cy)),
\]

and thus \( c(g_1(a) - g_1(y)) = g_1(ca) - g_1(cy) \) for \( a, y \in \mathbb{R} \) and \( c > 0 \). Choose \( y = 0 \) and define \( \tilde{g}(x) := g_1(x) - g_1(0) \) to obtain \( c\tilde{g}(a) = \tilde{g}(ca) \) for \( c > 0 \) and \( a \in \mathbb{R} \), as in Section C of the Supplementary Material for Nolde and Ziegel (2017). Since \( \tilde{g} \) is non-constant, non-decreasing, and differentiable, \( g_1(x) = \gamma x + g_1(0) \) for some \( \gamma > 0 \). Using the same arguments for \( g_2 \) we complete the proof of part (ii).

Now suppose \( S \) is also symmetric and \( g_2(x) = \rho x + g_2(0) \) for some \( \rho > 0 \). Then the same reasoning as in the proof of Theorem 3.2 shows that \( w_1 \gamma = w_2 \rho \), which proves the equivalence to IS\(_\alpha\).

**Proof of Theorem 3.2**

Let \( S \) be a scoring function of the form \( (5) \) and let \( a, b, y \in \mathbb{R} \) with \( a < b \) and \( b = y \). Then the symmetry of \( S \) gives

\[
-w_1 \frac{\alpha}{2} (g_1(a) - g_1(y)) = S([a, y], y) = S([-y, -a], -y) = w_2 \frac{\alpha}{2} (g_2(-a) - g_2(-y)),
\]

and rearranging yields \( w_1 (g_1(a) - g_1(y)) = w_2 (g_2(-y) - g_2(-a)) \) for \( a, y \in \mathbb{R} \). For \( x, y, \theta \in \mathbb{R} \), define the function

\[
f(x, y, \theta) := 1(\theta < x) + \frac{1}{2} 1(\theta = x) - 1(\theta < y) - \frac{1}{2} 1(\theta = y),
\]

which satisfies \( f(-y, -x, \theta) = f(x, y, -\theta) \) for \( x, y, \theta \in \mathbb{R} \). Recall that \( \mathcal{I} \) is the class of non-decreasing functions \( g : \mathbb{R} \to \mathbb{R} \) such that \( g(x) = \frac{1}{2}(g(x-)+g(x+)) \) for \( x \in \mathbb{R} \). For all \( g \in \mathcal{I} \) and \( y < x \)

\[
\int f(x, y, \theta) \, d\mu_g(\theta) = \frac{1}{2}(g(x+) - g(y-)) + \frac{1}{2}(g(x-) - g(y+)) = g(x) - g(y),
\]

where \( \mu_g \) is the Borel measure on \( \mathbb{R} \) induced by \( g \). If we define the measures \( \mu_1 = w_1 \mu_{g_1} \) and \( \mu_2 = w_2 \mu_{g_2} \), then the first part of the proof implies

\[
\int f(x, y, \theta) \, d\mu_2(\theta) = w_2 (g_2(x) - g_2(y))
\]
\[ = w_1(g_1(-y) - g_1(-x)) \]
\[ = \int f(-y, -x, \theta) \, d\mu_1(\theta) = \int f(x, y, -\theta) \, d\mu_1(\theta) \]

for \( y < x \), and the proof is completed by defining \( \mu \) via \( \mu((y, x]) = \mu_1((y, x]) + \mu_1([-x, -y]) \).

**Proof of Proposition 3.3**

Let \( F_0, F_1 \in \mathcal{F} \), and suppose that \([a, b] \in SL_\alpha(F_0) \cap SL_\alpha(F_1)\). Set \( F_\lambda := \lambda F_1 + (1 - \lambda)F_0 \) and note that for all \( \lambda \in (0, 1) \) and all \( s, t \in \mathbb{R} \) we have

\[ F_\lambda(t) - F_\lambda(s-) = \lambda (F_1(t) - F_1(s-)) + (1 - \lambda) (F_0(t) - F_0(s-)). \]  

(12)

In particular, \( F_\lambda(b) - F_\lambda(a-) \geq 1 - \alpha \) and \([a, b] \in SL_\alpha(F_\lambda)\), as otherwise \( (12) \) yields a contradiction to our initial assumption. This proves part (i).

Now let \( F_0, F_1 \in \mathcal{F} \) have continuous CDFs. Since \((s, t) \mapsto F(t) - F(s)\) is a continuous function for all \( F \in \mathcal{F} \), we must have \( F(b) - F(a) = 1 - \alpha \) for every \([a, b] \in SL_\alpha(F)\). Suppose \([a', b'] \in SL_\alpha(F_0) \cap SL_\alpha(F_1)\), as otherwise there is nothing to show, and let \( \lambda \in (0, 1) \) and \([a, b] \in SL_\alpha(F_\lambda)\) be given. By the first part of the proof

\[ \text{len}(SL_\alpha(F_1)) = \text{len}(SL_\alpha(F_0)) = b' - a' = b - a. \]  

(13)

Furthermore, \( F_\lambda(b) - F_\lambda(a) = 1 - \alpha \) and we see from \( (12) \) that \( F_0(b) - F_0(a) \geq 1 - \alpha \) or \( F_1(b) - F_1(a) \geq 1 - \alpha \) must hold. Suppose the first of these two inequalities is satisfied. Then equality must hold since the strict inequality \( F_0(b) - F_0(a) > 1 - \alpha \) would contradict \( (13) \). This yields \([a, b] \in SL_\alpha(F_0)\) and via \( (12) \) we obtain \( F_1(b) - F_1(a) = 1 - \alpha \). Taken together this gives \([a, b] \in SL_\alpha(F_1) \cap SL_\alpha(F_0)\), which proves part (ii).

**Proof of Theorem 3.8**

We proceed by constructing suitable convex combinations as in Example 3.6. Specifically, let \( F_0 \) satisfy Condition 3.7 and without loss of generality assume that \( SL_\alpha(F_0) = [0, b] \) for some \( b > 0 \). For instance, if \( \alpha \in (0, 1/2) \) a valid choice is \( F_0 = \alpha G_1 + (1 - \alpha)G_0 \), where \( G_0 \) and \( G_1 \) are absolutely continuous distributions with support \([0, b]\) and \([2b, 3b]\), respectively. Define \( F_1 \) via

\[ F_1(x) := F_0 \left( \frac{b}{b + \frac{1}{2} \varepsilon} x \right) \]

and set \( F_\lambda := \lambda F_1 + (1 - \lambda)F_0 \). We proceed to show that \([0, b + \frac{1}{2} \varepsilon] \in SL_\alpha(F_\lambda)\) for all \( \lambda \in (0, 1]\), which allows us to apply Proposition 2.5 and conclude non-elicitability.
Clearly, $\text{SL}_0(F_1) = [0, b + \frac{1}{2}\varepsilon]$, and since $F_0(b) = F_0(b + \varepsilon)$ it holds that $F_\lambda(b + \frac{1}{2}\varepsilon) - F_\lambda(0) = 1 - \alpha$ for $\lambda \in (0, 1)$. For a contradiction, suppose there are $\lambda \in (0, 1)$ and $a_\lambda \leq b_\lambda$ with $F_\lambda(b_\lambda) - F_\lambda(a_\lambda) \geq 1 - \alpha$ and $b_\lambda - a_\lambda < b + \frac{1}{2}\varepsilon$. Since $\text{SL}_0(F_1) = [0, b + \frac{1}{2}\varepsilon]$ it cannot be true that $F_1(b_\lambda) - F_1(a_\lambda) \geq 1 - \alpha$ and so $F_0(b_\lambda) - F_0(a_\lambda) > 1 - \alpha$ must hold, for a contradiction to the final part of Condition 3.7. Consequently, $\text{SL}_0(F_\lambda) = [0, b + \frac{1}{2}\varepsilon]$ for all $\lambda \in (0, 1]$, and the proof is complete.

**Proof of Theorem 3.9**

Let $k \geq 0$ be an integer, and suppose that $S$ is a strictly consistent scoring function for the functional $l_k$ relative to $\mathcal{F}$. To facilitate the presentation, we introduce the alternative notation $S(M, y)$ for $S(x_M, y)$, where $x_M \in \mathbb{N}_0$ denotes the lower endpoint of an interval $M \in \mathbb{A}$, with $\mathbb{A} = \{[x, x + k] : x \in \mathbb{N}_0\}$. We proceed in three steps.

**Step 1** We show that $S$ is of the form

$$S(x, y) = g(x, y)1(x \leq y \leq x + k) + h(y)$$

for functions $g : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ and $h : \mathbb{N}_0 \to \mathbb{R}$.

To this end, let $M_0, M_1 \in \mathbb{A}$ such that $M_0 \cap M_1 = \emptyset$. For a contradiction, suppose that the mapping $\varphi : \mathbb{N}_0 \to \mathbb{R}$ defined via $\varphi(y) = S(M_0, y) - S(M_1, y)$ is non-zero on $U := (M_0 \cup M_1)^c \cap \mathbb{N}_0$. We first treat the case where $\varphi(y) = c$ for all $y \in U$ and some $c \in \mathbb{R} \setminus \{0\}$. If $c > 0$ let $F_0$ be the uniform distribution on $M_0$ and for all $n \in \mathbb{N}$ let $F_n$ be the uniform distribution on some set $U_n \subset U$ with $\lvert U_n \rvert = 2nk$. If we define $G_n := \frac{1}{n} F_0 + (1 - \frac{1}{n}) F_n$, then $\text{MI}_{k/2}(G_n) = \text{MI}_{k/2}(F_0) = M_0$ for all $n \in \mathbb{N}$. Since $\int \varphi(y) dG_n(y) \to c > 0$ for $n \to \infty$, we obtain a contradiction to the strict consistency of $S$. A similar argument applies if $c < 0$. Consequently, $\varphi$ cannot be constant on $U$, i.e. there are $i_0, i_1 \in U$ such that $\varphi(i_0) \neq \varphi(i_1)$.

Now set $I := \{i_0, i_1\}$. As the class $\mathcal{F}$ contains all distributions with finite support, we can find probability measures $F_0, F'_0, F_1 \in \mathcal{F}$ that satisfy the following three conditions:

(i) There exists a $\lambda^* \in (0, 1)$ such that for $F_\lambda := \lambda F_1 + (1 - \lambda) F_0$ and $F'_\lambda := \lambda F'_1 + (1 - \lambda) F'_0$

$$\text{MI}_{k/2}(F_\lambda) = \text{MI}_{k/2}(F'_\lambda) = \begin{cases} M_0, & \lambda < \lambda^*, \\ M_1, & \lambda > \lambda^*. \end{cases}$$

(ii) $F_0$ and $F'_0$ coincide outside of $I$.

(iii) $\int_I \varphi(y) dF_0(y) \neq \int_I \varphi(y) dF'_0(y)$.
To see this, define $F_0$ and $F'_0$ via the probabilities $F_0(\{j\}) = F'_0(\{j\}) = 1/(k+2)$ for $j \in M_0$ and
\[
F_0(\{i_0\}) = F'_0(\{i_1\}) = \frac{1}{2(k+2)} + \varepsilon, \quad \text{and} \quad F_0(\{i_1\}) = F'_0(\{i_0\}) = \frac{1}{2(k+2)} - \varepsilon
\]
(15)
for some $\varepsilon \in (0, 1/(2(k+2)))$. Condition (ii) is immediate and (iii) follows from the fact that $\varphi(i_0) \neq \varphi(i_1)$. Moreover, letting $F_1$ be the uniform distribution on $M_1$ ensures (i).

Consider the integrated score difference
\[
\Delta(F, G, \lambda) := \int (S(M_0, y) - S(M_1, y)) \, d(\lambda G + (1-\lambda)F)(y),
\]
which is linear in $\lambda \in [0, 1]$. The strict consistency of $S$ in concert with (i) yields
$\Delta(F_0, F_1, 0) < 0$, $\Delta(F'_0, F_1, 0) < 0$, and $\Delta(F_0, F_1, 1) = \Delta(F'_0, F_1, 1) > 0$. Since $\Delta(F_0, F_1, 0) \neq \Delta(F'_0, F_1, 0)$ by (ii) and (iii), the linear mappings $\lambda \mapsto \Delta(F_0, F_1, \lambda)$ and $\lambda \mapsto \Delta(F'_0, F_1, \lambda)$ must have distinct roots. This implies that one of the two mappings does not vanish at $\lambda^*$, in contradiction to the consistency of $S$. Consequently, $\varphi = 0$ on $U$ such that we can conclude $S(M_0, y) = S(M_1, y)$ for all $y \in (M_0 \cup M_1)^c$. By varying the disjoint intervals $M_0, M_1 \in A$, we obtain that for all $y \in \mathbb{N}_0$ the values $S(M, y)$ are the same for all $M \in A$ with $y \notin M$. This yields that there exists a function $h : \mathbb{N}_0 \to \mathbb{R}$ such that $S$ is of the form (14).

**Step 2** Now we prove that $y \mapsto g(x, y)$ is constant on $[x, x+k]$. As before, we use the notation $g(M, y)$ for $g(x_M, y)$, where $x_M \in \mathbb{N}_0$ is the lower endpoint of $M \in A$. For $k = 0$ there is nothing to show, so let $k > 0$. For a contradiction, suppose there is an $M_0 \in A$ such that $y \mapsto g(M_0, y)$ is not constant on $M_0$, i.e. there are $i_2, i_3 \in M_0$ such that $g(M_0, i_2) \neq g(M_0, i_3)$. This ensures that we can choose an interval $M_1 \in A$, with $M_1 \cap M_0 = \emptyset$, and distributions $F_0, F'_0, F_1 \in \mathbb{F}$ that satisfy conditions (i), (ii), and (iii) in Step 1, for $I = \{i_2, i_3\}$. For example, we can choose $F_0$ and $F'_0$ by using the uniform distribution on $M_0$ and modifying it at $i_2$ and $i_3$ as in (15), while ensuring $M_1$ is separated from $M_0$ by a sufficiently large gap. As in Step 1 we obtain $\Delta(F_0, F_1, 0) \neq \Delta(F'_0, F_1, 0)$ such that the mappings $\lambda \mapsto \Delta(F_0, F_1, \lambda)$ and $\lambda \mapsto \Delta(F'_0, F_1, \lambda)$ have distinct roots. This is a contradiction to the consistency of $S$ and proves that $y \mapsto g(M_0, y)$ is constant on $M_0$. We can thus replace $g(x, y)$ in (14) by $\tilde{g}(x)$ for some function $\tilde{g} : \mathbb{N}_0 \to \mathbb{R}$.

**Step 3** It remains to be shown that $\tilde{g}$ reduces to a negative constant. To this end, consider $M_0 \in A$ and $M_1 \in A$ and assume that $\tilde{g}(M_0) < \tilde{g}(M_1)$. Due to the specific form of (14) we have
\[
\mathbb{E}_F [S(M_0, Y) - S(M_1, Y)] = \tilde{g}(M_0)\mathbb{P}_F(Y \in M_0) - \tilde{g}(M_1)\mathbb{P}_F(Y \in M_1)
\]
20
for all $F \in \mathcal{F}$. However, due to the strict consistency of $S$ this expression must be negative if $M_0 \in \text{MI}_{k/2}(F)$ and positive if $M_1 \in \text{MI}_{k/2}(F)$, for the desired contradiction. Therefore $\tilde{g}$ reduces to a constant, and using once more the consistency of $S$, we see that this constant is negative. The proof is complete.

**Proof of Theorem 3.10**

We sketch this proof only, as it proceeds in the very same three steps as the proof of Theorem 3.9. Specifically, let $c > 0$, and let $S$ be a strictly consistent scoring function for the functional $m_c$ relative to $F$. In Step 1, we show that $S$ is almost everywhere of the form

$$S(x, y) = g(x, y)\mathbb{1}(x - c \leq y \leq x + c) + h(y)$$

for $\mathcal{F}$-integrable functions $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$. In Step 2 we prove that $g$ reduces to a function $\tilde{g}$ in the variable $x$ only, and in Step 3 we demonstrate that $\tilde{g}$ reduces to a negative constant. The technical details are analogous to those in the above proof of Theorem 3.9, with the only difference that the set $I$ is now an interval and the statements hold Lebesgue almost everywhere.

**Acknowledgments**

The authors are grateful for support by the Klaus Tschira Foundation. Jonas Brehmer gratefully acknowledges support by the German Research Foundation (DFG) through Research Training Group RTG 1953. We thank two anonymous referees, Francis Diebold, and Tobias Fissler for thoughtful comments and suggestions.

**References**

Aitchison, J. and Dunsmore, I. R. (1968). Linear-loss interval estimation of location and scale parameters. *Biometrika*, 55:141–148.

Askanazi, R., Diebold, F. X., Schorfheide, F., and Shin, M. (2018). On the comparison of interval forecasts. *Journal of Time Series Analysis*, 39:953–965.

Brehmer, J. R. and Strokorb, K. (2019). Why scoring functions cannot assess tail properties. *Electronic Journal of Statistics*, 13:4015–4034.

Casella, G., Hwang, J. T. G., and Robert, C. (1993). A paradox in decision-theoretic interval estimation. *Statistica Sinica*, 3:141–155.
Christoffersen, P. F. (1998). Evaluating interval forecasts. *International Economic Review*, 39:841–862.

Clements, M. P. (2004). Evaluating the Bank of England density forecasts of inflation. *Economic Journal*, 114:844–866.

Czado, C., Gneiting, T., and Held, L. (2009). Predictive model assessment for count data. *Biometrics*, 65:1254–1261.

Dawid, A. P. (1986). Probability forecasting. In Kotz, S., Johnson, N. L., and Read, C. B., editors, *Encyclopedia of Statistical Sciences*, volume 7, pages 210–218. John Wiley & Sons, Inc., New York.

Dawid, A. P. and Musio, M. (2014). Theory and applications of proper scoring rules. *Metron*, 72:169–183.

Dearborn, K. and Frongillo, R. (2020). On the indirect elicitation of the mode and modal interval. *Annals of the Institute of Statistical Mathematics*, 72:1095–1108.

Ehm, W., Gneiting, T., Jordan, A., and Krüger, F. (2016). Of quantiles and expectiles: Consistent scoring functions, Choquet representations and forecast rankings. *Journal of the Royal Statistical Society. Series B. Statistical Methodology*, 78:505–562.

Fissler, T., Frongillo, R., Hlavinová, J., and Rudloff, B. (2020). Forecast evaluation of quantiles, prediction intervals, and other set-valued functionals. Preprint, https://arxiv.org/abs/1910.07912v2.

Fissler, T. and Ziegel, J. F. (2016). Higher order elicibility and Osband’s principle. *The Annals of Statistics*, 44:1680–1707.

Fissler, T. and Ziegel, J. F. (2020). Erratum: Higher order elicibility and Osband’s principle. Preprint, https://arxiv.org/abs/1901.08826v2.

Frongillo, R. and Kash, I. A. (2019). General truthfulness characterizations via convex analysis. Preprint, https://arxiv.org/abs/1211.3043v4.

Frongillo, R. and Kash, I. A. (2020). Elicitation complexity of statistical properties. Preprint, https://arxiv.org/abs/1506.07212v3.

Gneiting, T. (2011a). Making and evaluating point forecasts. *Journal of the American Statistical Association*, 106:746–762.

Gneiting, T. (2011b). Quantiles as optimal point forecasts. *International Journal of Forecasting*, 27:197–207.
Gneiting, T. (2017). When is the mode functional the Bayes classifier? *Stat*, 6:204–206.

Gneiting, T., Balabdaoui, F., and Raftery, A. E. (2007). Probabilistic forecasts, calibration and sharpness. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 69:243–268.

Gneiting, T. and Katzfuss, M. (2014). Probabilistic forecasting. *Annual Review of Statistics and Its Application*, 1:125–151.

Gneiting, T. and Raftery, A. E. (2007). Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association*, 102:359–378.

Heinrich, C. (2014). The mode functional is not elicitable. *Biometrika*, 101:245–251.

Hong, T., Pinson, P., Fan, S., Zareipour, H., Troccoli, A., and Hyndman, R. J. (2016). Probabilistic energy forecasting: Global energy forecasting competition 2014 and beyond. *International Journal of Forecasting*, 32:896–913.

Kolassa, S. (2016). Evaluating predictive count data distributions in retail sales forecasting. *International Journal of Forecasting*, 32:788–803.

Lambert, N. S., Pennock, D. M., and Shoham, Y. (2008). Eliciting properties of probability distributions. In *Proceedings of the 9th ACM Conference on Electronic Commerce*, EC ’08, pages 129–138.

Lambert, N. S. and Shoham, Y. (2009). Eliciting truthful answers to multiple-choice questions. In *Proceedings of the 10th ACM Conference on Electronic Commerce*, EC ’09, pages 109–118.

M Open Forecasting Center (2020). The M5 competition: Competitor’s Guide. Available at [https://mofc.unic.ac.cy/m5-competition/](https://mofc.unic.ac.cy/m5-competition/).

Makridakis, S., Spiliotis, E., and Assimakopoulos, V. (2020). The M4 competition: 100,000 time series and 61 forecasting methods. *International Journal of Forecasting*, 36:54–74.

Nolde, N. and Ziegel, J. F. (2017). Elicitability and backtesting: Perspectives for banking regulation. *The Annals of Applied Statistics*, 11:1833–1874.

Ray, E. L., Wattanachit, N., Niemi, J., Kanji, A. H., House, K., Cramer, E. Y., Bracher, J., Zheng, A., Yamana, T. K., Xiong, X., Woody, S., Wang, Y., Wang, L., Walraven, R. L., Tomar, V., Sherratt, K., Sheldon, D., Reiner, R. C.,
Prakash, B. A., Osthus, D., Li, M. L., Lee, E. C., Koyluoglu, U., Keskinocak, P., Gu, Y., Gu, Q., George, G. E., España, G., Corsetti, S., Chhatwal, J., Cavany, S., Biegel, H., Ben-Nun, M., Walker, J., Slayton, R., Lopez, V., Biggerstaff, M., Johansson, M. A., Reich, N. G., and COVID-19 Forecast Hub Consortium (2020). Ensemble forecasts of Coronavirus Disease 2019 (COVID-19) in the U.S. Preprint, https://www.medrxiv.org/content/10.1101/2020.08.19.20177493v1.

Steinwart, I., Pasin, C., Williamson, R., and Zhang, S. (2014). Elicitation and identification of properties. Journal of Machine Learning Research: Workshop and Conference Proceedings, 35:1–45.

Wang, R. and Wei, Y. (2020). Risk functionals with convex level sets. Mathematical Finance, 30:1337–1367.

Winkler, R. L. (1972). A decision-theoretic approach to interval estimation. Journal of the American Statistical Association, 67:187–191.