Probing new physics with the kaon decays $K \to \pi\pi E$

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Abstract

The most recent search for the rare kaon decay $K^+ \to \pi^+\nu\bar{\nu}$ by the NA62 experiment has produced evidence for it with a branching fraction consistent with the prediction of the standard model. The new result implies that in this decay, with the $\nu\bar{\nu}$ pair appearing as missing energy ($E$), the room for possible new physics is no longer sizable and that therefore its contributions to $s \to dE$ operators with parity-even $ds$ quark bilinears have become significantly constrained. Nevertheless, we point out that appreciable manifestations of new physics via operators with mainly parity-odd $ds$ quark bilinears may still occur in $K \to \pi\pi E$ modes, on which there are only minimal empirical details at present. We find in particular that new physics may enhance the branching fractions of $K^+ \to \pi^+\pi^0 E$ and $K_L \to \pi^0\pi^0 E$ to their current experimental upper limits and of $K_L \to \pi^+\pi^- E$ to the level of $10^{-5}$. Thus, quests for these decays in existing kaon facilities, such as NA62 and KOTO, could provide valuable information complementary to that gained from $K \to \pi E$. 
I. INTRODUCTION

One of the potentially promising avenues to discover new physics (NP) beyond the standard model (SM) is to look for processes that are expected to be very rare in the SM. An observation of such a process having a rate much greater than what the SM predicts would then be a compelling indication of NP effects. Among places where this may be realized are the flavor-changing neutral current (FCNC) decays of strange hadrons with missing energy ($\not{E}$), which are known to be dominated by short-distance physics [1–6] and arise mainly from the quark transition $s \to d/\not{E}$. In the SM, it proceeds from loop-suppressed diagrams [2] and the final state contains undetected neutrinos ($\nu\bar{\nu}$). Beyond the SM, there could be additional ingredients which alter the SM component and/or give rise to extra channels with one or more invisible nonstandard particles carrying away the missing energy.

Over the years hunts for $s \to d/\not{E}$ have focused the kaon modes $K \to \pi\nu\bar{\nu}$, leading mostly to limits on their branching fractions [7–10]. The efforts are ongoing in the KOTO [8] and NA62 [10] experiments. The former [8] has set $B(K_L \to \pi\nu\bar{\nu})_{\text{koto}} < 3.0 \times 10^{-9}$ at 90% confidence level (CL), exceeding but not far from the SM expectation [11] of $B(K_L \to \pi\nu\bar{\nu})_{\text{SM}} = (3.4 \pm 0.6) \times 10^{-11}$. On the other hand, very recently NA62 [12] has preliminarily reported $3.5\sigma$ evidence for the charged channel with $B(K^+ \to \pi^+\nu\bar{\nu})_{\text{NA62}} = (11.0_{-4.5}^{+4.0}\text{(stat)}\pm0.3\text{(syst)}) \times 10^{-11}$, which is in good agreement with the SM value [11] of $B(K^+ \to \pi^+\nu\bar{\nu})_{\text{SM}} = (8.4 \pm 1.0) \times 10^{-11}$ and more precise than the earlier E949 [7] finding of $B(K^+ \to \pi^+\nu\bar{\nu})_{\text{E949}} = (17.3_{-10.5}^{+11.5}) \times 10^{-11}$. As these measurements, notably the $K^+$ ones, have moved increasingly close to their SM predictions, the room for NP in $K \to \pi\not{E}$ has become quite small.

As it turns out, of the possible underlying $s \to d/\not{E}$ operators [13], these decays are sensitive to only a subset. Specifically, they can probe the operators having parity-even $ds$ quark bilinears but are not affected by the ones with exclusively parity-odd $ds$ bilinears [6, 13, 14]. However, the latter can contribute to kaon reactions with no or two pions, namely $K \to \not{E}$ and $K \to \pi\pi\not{E}$, as well as to analogous decays in the hyperon sector [6, 14]. This implies that, since at the moment there are precious few data on these processes [15], searches for them might still come up with substantial manifestations of NP or at least yield information about it complementary to that supplied by $K \to \pi\not{E}$ measurements.

In this paper, we adopt a model-independent approach to explore how large the branching fractions of the various $K \to \pi\not{E}$ modes might be, taking into account the available pertinent constraints. We assume especially that the invisibles comprise a pair of spin-1/2 fermions or spinless bosons, all of which are singlets under the SM gauge groups. The results of our study will hopefully motivate renewed attempts to pursue these decays as NP tests.

The organization of the rest of the article is the following. In Sec. II, we describe the quark-level operators responsible for the interactions of interest. In Sec. III, we derive the amplitudes for the aforementioned kaon decay modes and calculate their rates. We also write down the
corresponding numerical branching fractions. In Sec. IV, we compare the SM predictions for these transitions with their current data. In Sec. V, we address the allowed maximal branching fractions of $K \rightarrow \pi\pi\mathcal{E}$ due to NP and present our conclusions.

II. INTERACTIONS

Depending on the types of particles carrying away the missing energy, the effective $s \rightarrow d\mathcal{E}$ operators are generally subject to different sets of restrictions. If the invisible particles are SM neutrinos, which have charged-lepton partners because of the SM SU(2)$_L$-gauge invariance, the operators would have to face stringent restraints from lepton-flavor violation data. Since these do not apply if the invisibles are SM-gauge singlets, hereafter we consider a couple of cases involving them.

The missing energy is carried away by a spin-1/2 Dirac fermion $f$ and its antiparticle $\bar{f}$ in the first scenario, whereas it is due to a pair $\phi\bar{\phi}$ of complex spin-0 bosons in the second one.\footnote{In the recent literature covering the impact of NP on $K \rightarrow \pi\pi\mathcal{E}$, there are other possibilities for what carries away the missing energy. In particular, it could alternatively be due to a single particle such as a massless dark photon [17–19] or an invisible axion [20].}

At low energies, the relevant quark-level operators need to respect the strong and electromagnetic gauge symmetries and are obtainable from the literature [13, 16]. We can express the interaction Lagrangians as

$$L_f = -\left[\bar{d}\gamma^\nu s \bar{\tau}\gamma^\eta (C^{\alpha}_{\gamma} + \gamma_5 C^{\beta}_{\gamma}) f + \bar{d}s \bar{\tau} (C^{\alpha}_{\gamma} + \gamma_5 C^{\beta}_{\gamma}) f + \bar{d}\sigma^{\nu\mu}s \bar{\tau}\sigma_{\mu\nu} (C^{T}_{\gamma} + \gamma_5 C^{T}T f) f + \bar{d}\gamma^5 s (\bar{\tau}\gamma^5 \gamma_5 f) f \right] + \text{H.c.} \tag{1}$$

and

$$L_\phi = -\left[(C^{\alpha}_{\gamma} \bar{d}\gamma^\nu s + C^{\beta}_{\gamma} \bar{d}\gamma^5 s) i \left(\phi^i \gamma_5 \phi - \partial_\eta \phi^i \phi \right) + (C^{\alpha}_{\gamma} \bar{d}s + C^{\beta}_{\gamma} \bar{d}\gamma^5 s) \phi^i \phi \right] + \text{H.c.} \tag{2}$$

for the two scenarios, respectively, where the $C$, $\bar{C}$, and $c$ are generally complex coefficients, which have the dimension of inverse squared mass, except for $C^{S,p}_{\gamma}$, which are of inverse-mass dimension. These are free parameters in our model-independent approach and will be treated phenomenologically in our numerical work later on. In $L_f$, there are merely two tensor operators due to the identity $2i\sigma^a\alpha\gamma_5 = \epsilon^{a\omega\beta\psi}\sigma_{\beta\psi}$. We note that $L_f$ and $L_\phi$ could originate from effective Lagrangians that are invariant under all the SM gauge groups [13].

III. DECAY AMPLITUDES AND RATES

To examine the amplitudes for the kaon decays of concern, we need the mesonic matrix elements of the quark portions of the operators in Eqs. (1) and (2). They can be estimated with the aid of
flavor-SU(3) chiral perturbation theory at leading order \([6, 13, 21]\). For \(K_{L,S} \to \bar{E}\), the relevant hadronic matrix elements are

\[
\langle 0 | \bar{d} \gamma^\alpha \gamma_5 s | K^0 \rangle = \langle 0 | \bar{s} \gamma^\alpha \gamma_5 d | K^0 \rangle = -i f_K p_K^\alpha ,
\]

\[
\langle 0 | \bar{d} \gamma_5 s | K^0 \rangle = \langle 0 | \bar{s} \gamma_5 d | K^0 \rangle = i B_0 f_K ,
\]

with \(f_K \approx 156 \text{ MeV} \([15]\) denoting the kaon decay constant, \(p_X\) being the momentum of \(X\), and \(B_0 = m_N^2/(\bar{m} + m_s) \approx 2.0 \text{ GeV}\) involving the average kaon mass and the light-quark mass combination \(\bar{m} + m_s \approx 124 \text{ MeV}\) at a renormalization scale of 1 GeV, while for \(K \to \pi \bar{E}\),

\[
\langle 0 \bar{d} \gamma^\alpha s | K^- \rangle = p_K^\alpha + p_s^\alpha , \quad \langle 0 \bar{d}s | K^- \rangle = B_0 ,
\]

\[
\langle 0 \bar{d} s^\alpha s | K^- \rangle = 2i a_T (p_K^\alpha p_K^\alpha - p_s^\alpha p_s^\alpha) ,
\]

where \(a_T\) is a constant. Assuming isospin symmetry and making use of charge conjugation, we further have \(\langle 0 | \bar{d}(\gamma^n, 1, \sigma^{\nu s}) s | K^0 \rangle = \langle 0 | \bar{s}(\gamma^n, 1, -\sigma^{\nu s}) d | K^0 \rangle = -\langle 0 \bar{d}(\gamma^n, 1, \sigma^{\nu s}) s | K^- \rangle /\sqrt{2}\).

For \(K \to \pi \pi \bar{E}\), we find \([6, 13, 19]\)

\[
\langle \pi^0(p_0) \pi^-(p_-) | \bar{d}(\gamma^n, 1) \gamma_5 s | K^- \rangle = \frac{i \sqrt{2} \eta_{\pi K}}{f_K} \left[ (p_0^\alpha - p_-^\alpha, 0) + \frac{p_0^\alpha - p_-^\alpha}{m_K^2 - q^2} (\bar{q}^\alpha, -B_0) \right] ,
\]

\[
\langle \pi^+(p_+) \pi^-(p_-) | \bar{d}(\gamma^n, 1) \gamma_5 s | K^0 \rangle = \frac{2i \eta_{\pi K}}{f_K} \left[ (p_+^\alpha, 0) + \frac{p_+^\alpha q_\alpha}{m_K^2 - q^2} (\bar{q}^\alpha, -B_0) \right] ,
\]

\[
\langle \pi^+(p_+) \pi^-(p_-) | \bar{s}(\gamma^n, 1) \gamma_5 d | K^0 \rangle = \frac{2i \eta_{\pi K}}{f_K} \left[ (p_-^\alpha, 0) + \frac{p_-^\alpha q_\alpha}{m_K^2 - q^2} (\bar{q}^\alpha, -B_0) \right] ,
\]

\[
\langle \pi^0(p_1) \pi^0(p_2) | \bar{d}(\gamma^n, 1) \gamma_5 s | K^0 \rangle = \frac{i \eta_{\pi K}}{f_K} \left[ (p_1^\alpha + p_2^\alpha, 0) + \frac{(p_1^\alpha + p_2^\alpha) q_\alpha}{m_K^2 - q^2} (\bar{q}^\alpha, -B_0) \right] ,
\]

\[
\langle \pi^0(p_0) \pi^-(p_-) | \bar{d}\sigma_{\nu s} s | K^- \rangle = \frac{i \sqrt{2} \eta_{\pi K}}{f_K} \epsilon_{\eta \mu \nu \tau} \left[ 4 p_-^\mu p_0^\tau + (p_0^\tau - p_-^\alpha) \bar{q}^\tau \right] ,
\]

\[
\langle \pi^+(p_+) \pi^-(p_-) | \bar{d}\sigma_{\nu s} s | K^0 \rangle = \frac{2i \eta_{\pi K}}{f_K} \epsilon_{\eta \mu \nu \tau} (2 p_-^\alpha + \bar{q}^\mu) p_+^\tau ,
\]

\[
\langle \pi^+(p_+) \pi^-(p_-) | \bar{s}\sigma_{\nu s} d | K^0 \rangle = \frac{2i \eta_{\pi K}}{f_K} \epsilon_{\eta \mu \nu \tau} p_+^\tau (2 p_+^\tau + \bar{q}^\tau) ,
\]

\[
\langle \pi^0(p_1) \pi^0(p_2) | \bar{d}\sigma_{\nu s} s | K^0 \rangle = -\langle \pi^0(p_1) \pi^0(p_2) | \bar{s}\sigma_{\nu s} d | K^0 \rangle = \frac{i \eta_{\pi K}}{f_K} \epsilon_{\eta \mu \nu \tau} \bar{q}^\mu (p_1^\tau + p_2^\tau) ,
\]

where \(\bar{q} = p_K - p_0 - p_- = p_K - p_+ - p_- = p_K - p_1 - p_2\). Although the matrix elements in Eqs. (4)-(6) generally involve momentum-dependent form factors, to investigate the NP influence on these processes in this study we do not need a high degree of precision and therefore can ignore form-factor effects. We also neglect \(\langle \pi^0 \pi^+ | \bar{s}\gamma^n d | K^{+, 0} \rangle\), and their charge conjugates, which arise from small contributions derived from the anomaly Lagrangian, which occurs at next-to-leading order in the chiral expansion \([3, 13]\).
We now apply these matrix elements to kaon decays induced by $\mathcal{L}_t$ in Eq. (1), taking the $f$ mass to vanish, i.e. $m_f = 0$. Thus, with the approximate relations $\sqrt{2} K_{L,S} = K^0 \pm \bar{K}^0$, we obtain the amplitudes for $K_{L,S} \to f \bar{f}$ to be

$$\mathcal{M}_{K_{L,S} \to f \bar{f}} = i \sqrt{2} B_0 f_K \bar{u}_f \left( \tilde{S}_{K_{L,S}f} + \gamma_5 \tilde{P}_{K_{L,S}f} \right) v_{\bar{f}},$$

(7)

from which follow the decay rates

$$\Gamma_{K_{L,S} \to f \bar{f}} = \frac{B_0^2 f_K^2 m_{K^0}}{4\pi} \left| \tilde{S}_{K_{L,S}f} \right|^2 \left| \tilde{P}_{K_{L,S}f} \right|^2,$$

(8)

where

$$\tilde{S}_{K_L f} = i \text{Im} \tilde{c}_f, \quad \tilde{P}_{K_L f} = \text{Re} \tilde{c}_f, \quad \tilde{S}_{K_S f} = -\text{Re} \tilde{c}_f, \quad \tilde{P}_{K_S f} = -i \text{Im} \tilde{c}_f.$$

(9)

We see that $K_{L,S} \to f \bar{f}$ cannot probe $c^{V,A,S,P,T}_{f}$ and $\tilde{c}^{V,A}_{f}$.

For $K \to \pi f \bar{f}$, we express the amplitude as

$$\mathcal{M}_{K \to \pi f \bar{f}} = \bar{u}_f \left( S_{K_{\pi f}} + P_{K_{\pi f}} \gamma_5 \right) v_{\bar{f}}.$$

(10)

For $K^- \to \pi^- f \bar{f}$ and $K_L \to \pi^0 f \bar{f}$, the $S$ and $P$ terms are

$$S_{K^- \pi^- f} = 2 \hat{\psi}_K \tilde{c}^0_f + B_0 c^s_f + 4a_T p_K \cdot (p_f - p_{\bar{f}}) c^f_T,$$

$$P_{K^- \pi^- f} = 2 \hat{\psi}_K \tilde{c}^0_f + B_0 c^p_f + 4a_T p_K \cdot (p_f - p_{\bar{f}}) c^f_T,$$

$$S_{K_L \pi^0 f} = -2i \hat{\psi}_K \text{Im} c^0_f - B_0 \text{Re} c^s_f - 4ia_T p_K \cdot (p_f - p_{\bar{f}}) \text{Im} c^f_T,$$

$$P_{K_L \pi^0 f} = -2i \hat{\psi}_K \text{Im} c^0_f - iB_0 \text{Im} c^p_f - 4ia_T p_K \cdot (p_f - p_{\bar{f}}) \text{Re} c^f_T.$$

(11)

These lead to the differential rates with respect to the invariant mass squared, $s$, of the $f \bar{f}$ pair

$$\frac{d\Gamma_{K^- \to \pi^- f \bar{f}}}{ds} = \frac{\lambda_{K^- \pi^- f}^{1/2}}{192\pi^3 m_{K^-}^3} \left[ \lambda_{K^- \pi^-} \left( \left| c^0_f \right|^2 + \left| c^f_T \right|^2 \right) + \frac{3}{2} B_0 \left( \left| c^s_f \right|^2 + \left| c^p_f \right|^2 \right) s \right],$$

$$\frac{d\Gamma_{K_L \to \pi^0 f \bar{f}}}{ds} = \frac{\lambda_{K_L \pi^0 f}^{1/2}}{192\pi^3 m_{K^0}^3} \left[ \lambda_{K_L \pi^0} \left( \left| \text{Im} c^0_f \right|^2 + \left| \text{Im} c^f_T \right|^2 \right) + \frac{3}{2} B_0 \left( \left| \text{Re} c^s_f \right|^2 + \left| \text{Re} c^p_f \right|^2 \right) s \right],$$

(12)

where

$$\lambda_{XY} = \mathcal{K}(m_X^2, m_Y^2, s), \quad \mathcal{K}(x, y, z) = (x - y - z)^2 - 4yz.$$

(13)

Evidently, $K \to \pi f \bar{f}$, unlike $K \to f \bar{f}$, are sensitive to $c^{V,A,S,P,T}_{f}$, but not to $\tilde{c}^{V,A,S,P}_{f}$. 
For $K^- \to \pi^0\pi^- f\bar{f}$ and $K_L \to (\pi^+\pi^- , \pi^0\pi^0) f\bar{f}$, we find
\[
\mathcal{M}_{K^- \to \pi^0\pi^- f\bar{f}} = i\frac{\sqrt{2}}{f_K} \bar{u}_f \left[ (\not\! p_0 - \not\! p_\perp) (\not\! c_\perp + \gamma_5 \not\! c_\perp^\prime) - \frac{B_0}{K^2} (\not\! c_\perp^2 + \gamma_5 \not\! c_\perp^\prime + p_\perp^0 - p_\perp^\alpha) \hat{q}_\alpha \right. \\
\left. + 2i a_T \left[ 4p_\perp^2 p_\perp^0 + (p_\perp^2 - p_\perp^\alpha) \hat{q}_\alpha \right] \sigma_{\alpha\nu} (\gamma_5 \sigma^\nu + \sigma^\nu) \right] v_f ,
\]
\[
\mathcal{M}_{K_L \to \pi^+\pi^- f\bar{f}} = i\frac{\sqrt{2}}{f_K} \bar{u}_f \left[ \not\! p_\perp (\not\! c_\perp^\prime + \gamma_5 \not\! c_\perp^\prime) + \frac{B_0}{K^2} (\not\! c_\perp^\prime - \gamma_5 \not\! c_\perp^\prime) p_\perp^\alpha \hat{q}_\alpha \right. \\
\left. + \not\! p_\perp (\not\! c_\perp^\prime + \gamma_5 \not\! c_\perp^\prime) - \frac{B_0}{K^2} (\not\! c_\perp^\prime + \gamma_5 \not\! c_\perp^\prime) p_\perp^\alpha \hat{q}_\alpha \right. \\
\left. + 2i a_T p_\perp^\alpha (2p_\perp^\alpha + \hat{q}_\perp) \sigma_{\alpha\nu} (\gamma_5 \sigma^\nu - \sigma^\nu) \right. \\
\left. + 2i a_T (2p_\perp^\alpha + \hat{q}_\perp) p_\perp^\alpha \sigma_{\alpha\nu} (\gamma_5 \sigma^\nu + \sigma^\nu) \right] v_f ,
\]
\[
\mathcal{M}_{K_L \to \pi^0\pi^0 f\bar{f}} = i\frac{\sqrt{2}}{f_K} \bar{u}_f \left[ \not\! p_\perp (\text{Re} \not\! c_\perp^\prime + \gamma_5 \text{Re} \not\! c_\perp^\prime) - \frac{B_0}{K^2} (i \text{Im} \not\! c_\perp^\prime + \gamma_5 \text{Re} \not\! c_\perp^\prime) (p_\perp^0 \hat{q}_\alpha - \hat{s}) \right. \\
\left. + 2i a_T \hat{q}_\perp p_\perp^\alpha \sigma_{\alpha\nu} (i \gamma_5 \text{Im} \not\! c_\perp^\prime + \text{Re} \not\! c_\perp^\prime) \right] v_f ,
\]
where
\[
\hat{q} = p_\perp + p_\perp , \quad \hat{s} = \hat{q}^2 , \quad \hat{K}^2 = m_K^2 - \hat{s} ,
\]
with $m_K$ in $\hat{K}$ being the average kaon mass. We then arrive at the double differential rates
\[
\frac{d^2 \Gamma_{K^- \to \pi^0\pi^- f\bar{f}}}{d\hat{s} d\hat{c}} = \frac{\beta^2 \lambda_{K^-}^{1/2} m_{K^-}^2}{2(4\pi)^5 f_K^2} \left\{ \left( \lambda_{K^-} + 12 \hat{s} \hat{\varsigma} \right) \frac{|\bar{\not\! c}_\perp^{\nu\prime} + \hat{s}|^2 + |\bar{\not\! c}_\perp^{\nu}|^2}{9 m_{K^-}^2} + \lambda_{K^-} B_0^2 \hat{s} \frac{|\bar{\not\! c}_\perp^{\nu\prime} + \hat{s}|^2}{6 \hat{K}^4 m_{K^-}^4} \right. \\
\left. + 2a_T^2 \left[ \lambda_{K^-} (\hat{s} + 4 \hat{\varsigma}) + 12 \hat{\varsigma} (m_{K^-}^2 - \hat{\varsigma}^2) \right] \frac{|\bar{\not\! c}_\perp^{\nu\prime}|^2 + |\bar{\not\! c}_\perp^{\nu}|^2}{9 m_{K^-}^4} \right\} ,
\]
\[
\frac{d^2 \Gamma_{K_L \to \pi^+\pi^- f\bar{f}}}{d\hat{s} d\hat{c}} = \frac{\beta^2 \lambda_{K^0}^{1/2} m_{K^0}^2}{2(4\pi)^5 f_K^2} \left\{ \left( \lambda_{K^0} + 12 \hat{s} \hat{\varsigma} \right) \frac{(|\text{Im} \not\! c_\perp^{\nu\prime} + \hat{s}|)^2 + (|\text{Im} \not\! c_\perp^{\nu}|)^2}{9 m_{K^0}^4} + \lambda_{K^0} B_0^2 \hat{s} \frac{(|\text{Im} \not\! c_\perp^{\nu\prime} + \hat{s}|)^2 + (|\text{Im} \not\! c_\perp^{\nu}|)^2}{6 \hat{K}^4 m_{K^0}^4} \right. \\
\left. + \lambda_{K^0} \frac{(|\text{Re} \not\! c_\perp^{\nu\prime} + \hat{s}|)^2 + (|\text{Re} \not\! c_\perp^{\nu}|)^2}{3 \hat{\varsigma}^2 m_{K^0}^2} + (\lambda_{K^0} + 4 \hat{\varsigma}) B_0^2 \hat{s} \frac{|\text{Re} \not\! c_\perp^{\nu\prime} + \hat{s}|^2 + (|\text{Re} \not\! c_\perp^{\nu}|)^2}{2 \hat{\varsigma} \hat{K}^4 m_{K^0}^4} \right. \\
\left. + 2a_T^2 \left[ \lambda_{K^0} (\hat{s} + 4 \hat{\varsigma}) + 12 \hat{\varsigma} (m_{K^0}^2 - \hat{\varsigma}^2) \right] \frac{(\text{Re} \not\! c_\perp^{\nu\prime})^2 + (|\text{Re} \not\! c_\perp^{\nu}|)^2}{9 m_{K^0}^4} \right. \\
\left. + 2 \lambda_{K^0} a_T^2 \hat{s} \frac{(|\text{Re} \not\! c_\perp^{\nu\prime}|)^2 + (|\text{Re} \not\! c_\perp^{\nu}|)^2}{3 \hat{\varsigma}^2 m_{K^0}^2} \right\} ,
\]
\[
\frac{d^2 \Gamma_{K_L \to \pi^0\pi^0 f\bar{f}}}{d\hat{s} d\hat{c}} = \frac{\beta^2 \lambda_{K^0}^{1/2} m_{K^0}^2}{4(4\pi)^5 f_K^2} \left\{ \lambda_{K^0} \frac{(|\text{Re} \not\! c_\perp^{\nu\prime} + \hat{s}|)^2 + (|\text{Re} \not\! c_\perp^{\nu}|)^2}{3 m_{K^0}^4} + (\lambda_{K^0} + 4 \hat{\varsigma}) B_0^2 \hat{s} \frac{(|\text{Im} \not\! c_\perp^{\nu\prime} + \hat{s}|)^2 + (|\text{Im} \not\! c_\perp^{\nu}|)^2}{2 \hat{K}^4 m_{K^0}^4} \right. \\
\left. + 2 \lambda_{K^0} a_T^2 \hat{s} \frac{(|\text{Im} \not\! c_\perp^{\nu\prime}|)^2 + (|\text{Im} \not\! c_\perp^{\nu}|)^2}{3 m_{K^0}^4} \right\} ,
\]
For obtaining the rates, the \( \hat{s} \) and \( \zeta \) integration ranges are \( 0 \leq \hat{s} \leq (m_{K^-K^0} - 2m_\pi)^2 \) and \( 4m_\pi^2 \leq \zeta \leq (m_{K^-K^0} - \hat{s}^{1/2})^2 \) for the \( K^- \) and \( K_L \) channels, respectively. In the formulas above for the mode with the \( \pi^0\pi^- (\pi^+\pi^- \text{ or } \pi^0\pi^0) \) pair, \( m_\pi \) refers to the isospin-average (charged or neutral) pion mass. The corresponding expressions for the \( K_S \to (\pi^+\pi^-, \pi^0\pi^0) \bar{f} \bar{f} \) rates equal their \( K_L \) counterparts in Eq. (16) except that the imaginary (real) parts of the coefficients are replaced by their real (imaginary) parts. Clearly, \( K \to \pi \pi \bar{f} \bar{f} \), as opposed to \( K \to \pi \bar{f} \bar{f} \), can probe \( C^{V,A,S,1}_f \) besides \( C^{T,1}_f \) but not \( C^{V,A,S,3}_f \) in our approximation of the hadronic matrix elements.

For the processes induced by \( \mathcal{L}_\phi \) in Eq. (2), we also take the \( \phi \) mass to vanish, \( m_\phi = 0 \). Accordingly, the amplitudes for \( K_{L,S} \to \phi \bar{\phi} \) are [14]

\[
\mathcal{M}_{K_L \to \phi \bar{\phi}} = -\sqrt{2} B_0 f_K \text{Im } c^p_\phi, \quad \mathcal{M}_{K_S \to \phi \bar{\phi}} = i\sqrt{2} B_0 f_K \text{Re } c^p_\phi
\]  

For the three-body modes, we find [14]

\[
\mathcal{M}_{K^- \to \pi^- \phi \bar{\phi}} = 2 f_+ p_K \cdot (p - \bar{p}) c^v_\phi + B_0 f_0 c^s_\phi, \\
\mathcal{M}_{K_L \to \pi^0 \phi \bar{\phi}} = -2 i f_+ p_K \cdot (p - \bar{p}) \text{Im } c^v_\phi \quad - B_0 f_0 \text{Re } c^s_\phi,
\]

with \( p (\bar{p}) \) standing for the momentum of \( \phi (\bar{\phi}) \), and consequently

\[
\frac{d\Gamma_{K^- \to \pi^- \phi \bar{\phi}}}{ds} = \frac{\lambda_{K^-\pi^-}^{1/2}}{768 \pi^3 m_{K^-}^3} \left( \lambda_{K^-\pi^-} f_+^2 |c^v_\phi|^2 + 3 B_0^2 f_0^2 |c^s_\phi|^2 \right), \\
\frac{d\Gamma_{K_L \to \pi^0 \phi \bar{\phi}}}{ds} = \frac{\lambda_{K_L\pi^0\phi}^{1/2}}{768 \pi^3 m_{K_L}^3} \left[ \lambda_{K_L\pi^0\phi} f_+^2 (\text{Im } c^v_\phi)^2 + 3 B_0^2 f_0^2 (\text{Re } c^s_\phi)^2 \right],
\]

where \( \hat{s} = (p + \bar{p})^2 \). For \( K \to \pi \pi \phi \bar{\phi} \), we derive

\[
\mathcal{M}_{K^- \to \pi^0 \pi^- \phi \bar{\phi}} = \frac{i\sqrt{2}}{f_K} \left[ c^h_\phi \left( p^0_+ - p^-_+ \right) \cdot \left( p \bar{p} \right) \eta - B_0 c^p_\phi \frac{\left( p^0_+ - p^-_+ \right) \eta}{K^2} \right], \\
\mathcal{M}_{K_L \to \pi^+ \pi^- \phi \bar{\phi}} = \frac{i\sqrt{2}}{f_K} \left[ c^h_\phi \left( p^0_+ + c^h_\phi p^-_+ \right) \cdot \left( p \bar{p} \right) \eta - B_0 \frac{\left( c^p_\phi p^0_+ - c^p_\phi p^-_+ \right) \eta}{K^2} \right], \\
\mathcal{M}_{K_L \to \pi^0 \pi^0 \phi \bar{\phi}} = \frac{i\sqrt{2}}{f_K} \left[ \text{Re } c^h_\phi \left( p^1_+ + p^2_+ \right) \cdot \left( p \bar{p} \right) \eta - i B_0 \text{Im } c^p_\phi \frac{\left( p^1_+ + p^2_+ \right) \eta}{K^2} \right],
\]

(22)
TABLE I: Summary of which coefficients in Eqs. (1) and (2) contribute to the various FCNC kaon decays with missing energy carried away by Dirac spin-1/2 fermions, $f\bar{f}$, or by spin-0 bosons, $\phi\phi$, if their masses are negligible, $m_{\tau,\phi} \approx 0$.

| Decay modes | $K \to f\bar{f}$ | $K \to \pi f\bar{f}$ | $K \to \pi\pi f\bar{f}$ | $K \to \phi\phi$ | $K \to \pi\phi\phi$ | $K \to \pi\pi\phi\phi$ |
|-------------|------------------|------------------|------------------|-----------------|-----------------|------------------|
| Coefficients | $c_{t}^{v}$, $\tilde{c}_{t}^{p}$ | $c_{t}^{v}$, $c_{t}^{A}$, $c_{t}^{P}$, $c_{t}^{S}$, $c_{t}^{T}$ | $c_{t}^{v}$, $\tilde{c}_{t}^{v}$, $\tilde{c}_{t}^{p}$, $c_{t}^{T}$, $c_{t}^{P}$ | $c_{\phi}^{p}$ | $c_{\phi}^{p}$, $c_{\phi}^{S}$ | $c_{\phi}^{A}$, $c_{\phi}^{P}$ |

from which we arrive at the double-differential rates

\[
\frac{d^{2}\Gamma_{K\to\pi^{0}\pi^{-}\phi\bar{\phi}}}{d\hat{s}d\hat{\zeta}} = \frac{\beta^{3}\lambda_{K}^{2} m_{K^{-}}}{24(4\pi)^{5} f_{K}^{2}} \left[ \frac{\lambda_{K^{-} + 12\hat{s}\hat{\zeta}}}{3 m_{K^{-}}^{2}} |c_{\phi}|^{2} + \frac{\tilde{\lambda}_{K^{-} - B_{0}^{2}}}{K^{2} m_{K^{-}}^{2}} |c_{\phi}^{p}|^{2} \right],
\]

\[
\frac{d^{2}\Gamma_{K_{L}\to\pi^{0}\pi^{-}\phi\bar{\phi}}}{d\hat{s}d\hat{\zeta}} = \frac{\beta^{3}\lambda_{K_{0}}^{2} m_{K_{0}}^{2}}{24(4\pi)^{5} f_{K}^{2}} \left[ \frac{\beta_{g}^{2}\tilde{\lambda}_{K_{0}}^{2} + 12\hat{s}\hat{\zeta}}{3 m_{K_{0}}^{2}} (\text{Im } c_{\phi}^{A})^{2} + \frac{\beta_{g}^{2}\tilde{\lambda}_{K_{0}}^{2} B_{0}^{2}}{K^{2} m_{K_{0}}^{2}} (\text{Re } c_{\phi}^{p})^{2} \right] + \frac{\tilde{\lambda}_{K_{0}}^{2}}{m_{K_{0}}^{2}} (\text{Re } c_{\phi}^{p})^{2} + \frac{3(\tilde{\lambda}_{K_{0}}^{2} + 4\hat{s}\hat{\zeta}) B_{0}^{2}}{K^{2} m_{K_{0}}^{2}} (\text{Im } c_{\phi}^{p})^{2},
\]

\[
\frac{d^{2}\Gamma_{K_{L}\to\pi^{0}\pi^{0}\phi\bar{\phi}}}{d\hat{s}d\hat{\zeta}} = \frac{\beta^{3}\lambda_{K_{0}}^{2} m_{K_{0}}^{2}}{48(4\pi)^{5} f_{K}^{2}} \left[ \frac{\lambda_{K_{0}}^{2}}{m_{K_{0}}^{2}} (\text{Re } c_{\phi}^{p})^{2} + \frac{3(\tilde{\lambda}_{K_{0}}^{2} + 4\hat{s}\hat{\zeta}) B_{0}^{2}}{K^{2} m_{K_{0}}^{2}} (\text{Im } c_{\phi}^{p})^{2} \right].
\]

As the last paragraph shows, $K_{L,S} \to \phi\phi$ are sensitive exclusively to $c_{\phi}^{p}$, whereas $K \to \pi\pi\phi\bar{\phi}$ can probe solely the two parity-odd couplings, $c_{\phi}^{A}$ and $c_{\phi}^{p}$, in our approximation of the mesonic matrix elements. By contrast, the $K \to \pi\phi\phi$ amplitudes depend on the parity-even coefficients, $c_{\phi}^{v}$ and $c_{\phi}^{S}$, but are independent of $c_{\phi}^{A,P}$.

In Table I, we list the contributions of the different constants in Eqs. (1) and (2) to the kaon decays of interest according to the preceding discussion. We remark that for $f$ having a Majorana nature, instead of Dirac one, $\overline{\mathbf{T}}\gamma^{\mu}\mathbf{f} = 0$, causing the $c_{t}^{v}$ and $\tilde{c}_{t}^{v}$ parts to disappear. Moreover, for $\phi$ being a real field, rather than complex one, the $c_{\phi}^{A}$ terms would be absent.

For later convenience, from the $K \to \overline{E}$ and $K \to \pi\pi\overline{E}$ rate formulas above, here we write down the corresponding numerical branching fractions in terms of the coefficients, employing the central values of the measured kaon lifetimes and meson masses from Ref. [15] as well as $a_{T} = 0.658(23) \text{ GeV}$ from lattice QCD work [22]. Thus, Eq. (8) translates into

\[
B(K_{L} \to f\bar{f}) = 2.91 \left[ (\text{Im } c_{t}^{p})^{2} + (\text{Re } c_{t}^{p})^{2} \right] 10^{14} \text{ GeV}^{4},
\]

\[
B(K_{S} \to f\bar{f}) = 5.09 \left[ (\text{Re } c_{t}^{p})^{2} + (\text{Im } c_{t}^{p})^{2} \right] 10^{11} \text{ GeV}^{4},
\]

(24)
and Eq. (16) yields

\[
B(K^- \to \pi^0\pi^- f\bar{f}) = \left[ 6.28 \left( |\tilde{c}_t^\gamma|^2 + |\tilde{c}_t^\phi|^2 \right) + 2.01 \left( |\tilde{c}_t|^2 + |\tilde{c}_t^\rho|^2 \right) \right] + 6.59 \left( |\tilde{c}_t^T|^2 + |\tilde{c}_t^{P,T}|^2 \right) \times 10^5 \text{GeV}^4,
\]

\[
B(K_L \to \pi^+\pi^- f\bar{f}) = \left\{ 0.267 \left[ (\text{Im} \tilde{c}_t^\gamma)^2 + (\text{Im} \tilde{c}_t^\phi)^2 \right] + 0.086 \left[ (\text{Re} \tilde{c}_t^\gamma)^2 + (\text{Im} \tilde{c}_t^\rho)^2 \right] \right. \\
+ 1.35 \left[ (\text{Re} \tilde{c}_t^\gamma)^2 + (\text{Re} \tilde{c}_t^\phi)^2 \right] + 2.37 \left[ (\text{Im} \tilde{c}_t^\rho)^2 + (\text{Re} \tilde{c}_t^\gamma)^2 \right] \right. \\
\left. + 0.286 \left[ (\text{Re} \tilde{c}_t^T)^2 + (\text{Re} \tilde{c}_t^{P,T})^2 \right] + 0.010 \left[ (\text{Im} \tilde{c}_t^T)^2 + (\text{Re} \tilde{c}_t^{P,T})^2 \right] \right\} \times 10^7 \text{GeV}^4,
\]

\[
B(K_L \to \pi^0\pi^0 f\bar{f}) = \left\{ 0.848 \left[ (\text{Re} \tilde{c}_t^\gamma)^2 + (\text{Re} \tilde{c}_t^\phi)^2 \right] + 1.60 \left[ (\text{Im} \tilde{c}_t^\rho)^2 + (\text{Re} \tilde{c}_t^\gamma)^2 \right] \right. \\
+ 0.007 \left[ (\text{Im} \tilde{c}_t^T)^2 + (\text{Re} \tilde{c}_t^{P,T})^2 \right] \right\} \times 10^7 \text{GeV}^4.
\]

Similarly, from Eqs. (19) and (23) we have

\[
B(K_L \to \phi \bar{\phi}) = 5.87 \times 10^{14} \text{GeV}^2 \left| c_\phi^p \right|^2,
\]

\[
B(K_S \to \phi \bar{\phi}) = 1.03 \times 10^{12} \text{GeV}^2 \left| c_\phi^p \right|^2
\]

and

\[
B(K^- \to \pi^0\pi^- \phi \bar{\phi}) = \left( 0.0157 \left| c_\phi^A \right|^2 + 1.38 \left| c_\phi^p \right|^2 \text{GeV}^{-2} \right) \times 10^7 \text{GeV}^4,
\]

\[
B(K_L \to \pi^+\pi^- \phi \bar{\phi}) = \left[ 0.0669 \left( \text{Im} c_\phi^A \right)^2 + 5.88 \left( \text{Re} c_\phi^p \right)^2 \text{GeV}^{-2} \right. \\
+ 0.339 \left( \text{Re} c_\phi^A \right)^2 + 103 \left( \text{Im} c_\phi^p \right)^2 \text{GeV}^{-2} \right] \times 10^7 \text{GeV}^4,
\]

\[
B(K_L \to \pi^0\pi^0 \phi \bar{\phi}) = \left[ 0.212 \left( \text{Re} c_\phi^A \right)^2 + 64.3 \left( \text{Im} c_\phi^p \right)^2 \text{GeV}^{-2} \right] \times 10^7 \text{GeV}^4,
\]

respectively.

### IV. SM PREDICTIONS AND EMPIRICAL INFORMATION

As mentioned earlier, the latest NA62 measurement on \( K^+ \to \pi^+\nu\bar{\nu} \) has turned up evidence for it that is fully consistent with the SM expectation \[12\]. In view of Table I, this implies that the couplings \( c_\phi^{A,S,P,T} \) and \( c_\phi^T \) originating from possible NP cannot by sizable anymore.\(^2\) To explore how much the other coefficients shown in Table I may be affected by NP to amplify the \( K \to \bar{E} \) and \( K \to \pi\pi\bar{E} \) rates with respect to their SM values, we need to know the latter.

\(^2\) A preliminary report by KOTO \[23\] has revealed that their most recent data contain a couple of \( K_L \to \pi^0\nu\bar{\nu} \) events suggesting an anomalously high rate, which still needs confirmation from further measurements. If this anomaly persists in the future, it may be due to NP, as discussed in e.g. \[24, 25\] and the references therein, but its effects would not be large enough to modify our conclusions for \( K \to \pi\pi\bar{E} \).
In the SM, our processes of interest arise at short distance from effective $dsνl\bar{ν}_l$ interactions, with $l = e, \mu, \tau$, described by [2]

$$\mathcal{L}^{\text{SM}}_{sdνν} = -\frac{α_s G_F}{\sqrt{8} π sin^2 θ_w} \sum_{i = e, μ, τ} \left(V_{td}^* V_{ts} X_t + V_{cd}^* V_{cs} X_c^l\right) \bar{d} γ_l(1 - γ_5) s \bar{ν}_l(1 - γ_5) ν_l + \text{H.c.}, \quad (28)$$

where $α_s = 1/128$, $G_F$ is the Fermi constant, $sin^2 θ_w = 0.231$, $V_{qq'}$ are Cabibbo-Kobayashi-Maskawa (CKM) matrix elements, $X_t = 1.481$ from $t$-quark loops, and $X_c^e = X_c^μ ≈ 1.2 × 10^{-3}$ and $X_c^τ ≈ 8 × 10^{-4}$ are $c$-quark contributions [2]. Applying the notation of Eq. (1) to $\mathcal{L}^{\text{SM}}_{sdνν}$, we then have $c^V_{ν_l} = -c^A_{ν_l} = -c^A_{ν_l} = c^A_{ν_l} = α_s G_F (λ_t X_t + λ_c X^l_c) / (\sqrt{8} π sin^2 θ_w) ≈ -3(3 + 0.9i) × 10^{-11}/\text{GeV}^2$ and $c^{\tilde{S},p,π,ν}_ν = \tilde{c}^{\tilde{S},p}_ν = 0$.

Accordingly, in light of Eqs. (8) and (9) we see that $B(K_{L,S} \rightarrow ν\bar{ν})_{\text{SM}} = 0$, given that the neutrinos are massless in the SM. However, supplementing it with nonzero neutrino masses and taking their biggest one from the direct limit $m_{ν_e}^{\exp} < 18.2 \text{ MeV}$ [15] would instead lead to the maximal values [6] $B(K_L \rightarrow ν\bar{ν})_{\text{SM}} ≃ 1 × 10^{-10}$ and $B(K_S \rightarrow ν\bar{ν})_{\text{SM}} ≃ 2 × 10^{-14}$.

As for the four-body channels, employing Eq. (25) we obtain $B(K^- → π^0 π^- ν\bar{ν})_{\text{SM}} ≃ 4 × 10^{-15}$ and $B(K_L → (π^+ π^−, π^0 π^0) ν\bar{ν})_{\text{SM}} ≃ (8, 5) × 10^{-14}$. These are in rough agreement with more refined evaluations in the literature [3, 4, 26, 27]:

$$B(K^- → π^0 π^- ν\bar{ν})_{\text{SM}} ≃ 1.2 × 10^{-14}, \quad B(K_L → π^+ π^- ν\bar{ν})_{\text{SM}} ≃ 2.8 × 10^{-13},
B(K_L → π^0 π^0 ν\bar{ν})_{\text{SM}} ≃ 1.5 × 10^{-13}. \quad (29)$$

with the most recent CKM matrix elements [15]. The estimates for $K_S → π π ν\bar{ν}$ are about three orders of magnitude less than their $K_L$ counterparts. The two sets of $K^-$ and $K_L$ numbers above indicate the level of uncertainties in our $K → π π \bar{E}$ predictions in the next section.

On the experimental side, only two of these modes have been looked for [15], with negative outcomes which led to the limits [28, 29]

$$B(K^- → π^0 π^- ν\bar{ν})_{\text{exp}} < 4.3 × 10^{-5}, \quad B(K_L → π^0 π^0 ν\bar{ν})_{\text{exp}} < 8.1 × 10^{-7} \quad (30)$$

both at 90% CL. These exceed the corresponding SM numbers in Eq. (29) by several orders of magnitude. As regards $K_{L,S} → \bar{E}$, there have been no direct searches for them yet. Nevertheless, from the existing data [15] on the visible decay channels of $K_{L,S}$ one can extract indirect upper bounds on their invisible branching fractions [30]. Thus, one can infer [19]

$$B(K_L → \bar{E}) < 1.8 × 10^{-3}, \quad B(K_S → \bar{E}) < 7.1 × 10^{-4} \quad (31)$$

at the 2σ level, which are far away from the aforesaid $B(K_{L,S} → ν\bar{ν})_{\text{SM}}$ values. Comparing Eqs. (30)-(31) with Eqs. (24)-(27), as well as Table I, we conclude that currently there remains potentially plenty of room for NP to enhance the rates of these decays via $c^{V,A,S,p}_φ$ and $c^{A,p}_φ$. 


V. NP EXPECTATIONS AND CONCLUSIONS

Based on the considerations made in the previous section, we hereafter entertain the possibility that among the couplings listed in the table NP manifests itself exclusively via $\tilde{c}^{V,A,S,P}_\phi$ or $c^{P}_\phi$ and demand that they fulfill the conditions
\[
\mathcal{B}(K_L \to \bar{E})_{NP} < 1.8 \times 10^{-3}, \quad \mathcal{B}(K_S \to \bar{E})_{NP} < 7.1 \times 10^{-4}, \quad (32)
\]
\[
\mathcal{B}(K^- \to \pi^0\pi^-\bar{E})_{NP} < 4 \times 10^{-5}, \quad \mathcal{B}(K_L \to \pi^0\pi^0\bar{E})_{NP} < 8 \times 10^{-7}. \quad (33)
\]
Moreover, as remarked earlier, the same underlying interactions induce analogous decays of hyperons $(\Lambda, \Sigma^+, \Xi^0, \Xi^-, \Omega^-)$ and their data turn out to supply additional constraints. Although these transitions have never been searched for, they are among the hyperons’ yet-unobserved modes whose branching fractions have maxima which can be inferred from the available data on the observed channels [15]. These upper bounds have been estimated in Ref. [19], which we adopt here to impose
\[
\mathcal{B}(\Lambda \to n\bar{E})_{NP} < 1.4 \times 10^{-2}, \quad \mathcal{B}(\Sigma^+ \to p\bar{E})_{NP} < 8.0 \times 10^{-3},
\]
\[
\mathcal{B}(\Xi^0 \to \Lambda\bar{E}, \Sigma^0\bar{E})_{NP} < 3.4 \times 10^{-4}, \quad \mathcal{B}(\Xi^- \to \Sigma^\pm\bar{E})_{NP} < 8.3 \times 10^{-4},
\]
\[
\mathcal{B}(\Omega^- \to \Xi^-\bar{E})_{NP} < 1.6 \times 10^{-2}. \quad (34)
\]
The expressions for the corresponding hyperon rates in terms of $\tilde{c}^{V,A,S,P}_\phi$ ($c^{P}_\phi$) have been derived in Ref. [6] ([14]). To illustrate the ramifications that may arise for the various $K \to \pi\pi\bar{E}$ modes if NP occurs in these couplings, we can look at several simple examples.

If it impacts $\tilde{c}^{A,S}_\phi$ alone, we learn from Eqs. (24)-(25) that these parameters need to satisfy the kaon restraints specified in the last paragraph and that Eq. (32) is stricter than Eq. (33). Then, assuming $\tilde{c}^{A,S}_\phi$ to be complex, from their allowed ranges we find the maximal values
\[
\mathcal{B}(K^- \to \pi^0\pi^-\bar{f}f) < 2.8 \times 10^{-10}, \quad \mathcal{B}(K_L \to \pi^0\pi^0\bar{f}f) < 1.3 \times 10^{-9},
\]
\[
\mathcal{B}(K_L \to \pi^0\pi^0\bar{f}f) < 9.9 \times 10^{-11}, \quad \mathcal{B}(K_S \to \pi^0\pi^0\bar{f}f) < 5.8 \times 10^{-11},
\]
\[
\mathcal{B}(K_S \to \pi^0\pi^0\bar{f}f) < 3.9 \times 10^{-11}. \quad (35)
\]
If now only $\tilde{c}^{V,A}_\phi$ are influenced by NP, it is clear from Eq. (24) that Eq. (32) is no longer relevant but Eq. (33) still matters. In this case, if these couplings are real, the $K_L \to \pi^0\pi^0\bar{E}$ requirement is the stronger and yields $(\text{Re} \tilde{c}_\phi^V)^2 + (\text{Re} \tilde{c}_\phi^A)^2 < 9.4 \times 10^{-14}$ GeV$^{-4}$, which translates into
\[
\mathcal{B}(K^- \to \pi^0\pi^-\bar{f}f) < 5.9 \times 10^{-8}, \quad \mathcal{B}(K_L \to \pi^+\pi^-\bar{f}f) < 1.3 \times 10^{-6},
\]
\[
\mathcal{B}(K_L \to \pi^0\pi^0\bar{f}f) < 8 \times 10^{-7}, \quad \mathcal{B}(K_S \to \pi^+\pi^-\bar{f}f) < 4.4 \times 10^{-10},
\]
\[
\mathcal{B}(K_S \to \pi^0\pi^0\bar{f}f) = 0. \quad (36)
\]
\[\text{It is worth pointing out that a scenario with this kind of enhancement can be realized in a model involving two scalar leptoquarks and bring about substantial rates of the aforementioned FCNC hyperon decays with missing energy [31].}\]
For the preceding two instances the hyperon limitations in Eq. (34) are unimportant. In contrast, if $c_{\pi}^{V,A}$ are permitted to be complex, the above requisite on their real parts still applies, but Eq. (34) needs to be taken into account as well, leading to $(\text{Im} \, \tilde{c}_{f}^{V,A})^2 + (\text{Im} \, \tilde{c}_{f}^{A})^2 < 1.4 \times 10^{-11} \text{ GeV}^{-4}$, and so these mostly much bigger results may be achieved:

\begin{align}
B(K^- \rightarrow \pi^0 \pi^- f \bar{f}) &< 9.0 \times 10^{-6}, & B(K_L \rightarrow \pi^+ \pi^- f \bar{f}) &< 3.9 \times 10^{-5}, \\
B(K_L \rightarrow \pi^0 \pi^0 f \bar{f}) &< 8 \times 10^{-7}, & B(K_S \rightarrow \pi^+ \pi^- f \bar{f}) &< 3.4 \times 10^{-7}, \\
B(K_S \rightarrow \pi^0 \pi^0 f \bar{f}) &< 2.1 \times 10^{-7}, & &
\end{align}

(37)

only the second bound in Eq. (33) being saturated as in Eq. (36).

If NP solely enters $c_{\phi}^{P}$, which can be complex, from Eqs. (26) and (32) we obtain

\begin{align}
B(K^- \rightarrow \pi^0 \pi^0 \phi \bar{\phi}) &< 9.6 \times 10^{-9}, & B(K_L \rightarrow \pi^+ \pi^- \phi \bar{\phi}) &< 4.4 \times 10^{-8}, \\
B(K_L \rightarrow \pi^0 \pi^0 \phi \bar{\phi}) &< 2.0 \times 10^{-9}, & B(K_S \rightarrow \pi^+ \pi^- \phi \bar{\phi}) &< 1.2 \times 10^{-9}, \\
B(K_S \rightarrow \pi^0 \pi^0 \phi \bar{\phi}) &< 7.8 \times 10^{-10}. & &
\end{align}

(38)

If NP resides exclusively in $c_{\phi}^{A}$ instead, the situation turns out to be analogous to that reflected by Eqs. (36)-(37), as may be inferred from Eqs. (27) and (33), and especially for $c_{\phi}^{A}$ being complex we have $(\text{Re} \, c_{\phi}^{A})^2 < 3.8 \times 10^{-13} \text{ GeV}^{-4}$ and $(\text{Im} \, c_{\phi}^{A})^2 < 5.7 \times 10^{-11} \text{ GeV}^{-4}$ from the $K_L$ and $\Xi^0$ conditions in Eqs. (33) and (34), respectively, and consequently

\begin{align}
B(K^- \rightarrow \pi^0 \pi^0 \phi \bar{\phi}) &< 9.0 \times 10^{-6}, & B(K_L \rightarrow \pi^+ \pi^- \phi \bar{\phi}) &< 3.9 \times 10^{-5}, \\
B(K_L \rightarrow \pi^0 \pi^0 \phi \bar{\phi}) &< 8 \times 10^{-7}, & B(K_S \rightarrow \pi^+ \pi^- \phi \bar{\phi}) &< 3.4 \times 10^{-7}, \\
B(K_S \rightarrow \pi^0 \pi^0 \phi \bar{\phi}) &< 2.1 \times 10^{-7}, & &
\end{align}

(39)

with the numbers being identical to those in Eqs. (37).

In Table II we collect our findings in Eqs. (35)-(39) and the associated coefficients. We note that in the cases seen in this table with the largest branching fractions the corresponding predictions

| Contributing coefficients | $K^- \rightarrow \pi^0 \pi^- f \bar{f}$ | $K_L \rightarrow \pi^+ \pi^- f \bar{f}$ | $K_L \rightarrow \pi^0 \pi^0 f \bar{f}$ | $K_S \rightarrow \pi^+ \pi^- f \bar{f}$ | $K_S \rightarrow \pi^0 \pi^0 f \bar{f}$ |
|----------------------------|----------------------------------------|----------------------------------------|----------------------------------------|----------------------------------------|----------------------------------------|
| $\tilde{c}_{f}^{S,P}$      | $2.8 \times 10^{-10}$                  | $1.3 \times 10^{-9}$                  | $9.9 \times 10^{-11}$                  | $5.8 \times 10^{-11}$                  | $3.9 \times 10^{-11}$                  |
| $\text{Re} \, c_{f}^{V,A}$ | $5.9 \times 10^{-8}$                   | $1.3 \times 10^{-6}$                  | $8 \times 10^{-7}$                     | $4.4 \times 10^{-10}$                  | $0$                                    |
| $\tilde{c}_{f}^{V,A}$      | $9.0 \times 10^{-6}$                   | $3.9 \times 10^{-5}$                  | $8 \times 10^{-7}$                     | $3.4 \times 10^{-7}$                  | $2.1 \times 10^{-7}$                  |
| $c_{\phi}^{P}$             | $9.6 \times 10^{-9}$                   | $4.4 \times 10^{-8}$                  | $2.0 \times 10^{-9}$                   | $1.2 \times 10^{-9}$                  | $7.8 \times 10^{-10}$                  |
| $c_{\phi}^{A}$             | $9.0 \times 10^{-6}$                   | $3.9 \times 10^{-5}$                  | $8 \times 10^{-7}$                     | $3.4 \times 10^{-7}$                  | $2.1 \times 10^{-7}$                  |
for their hyperon counterparts are also large and might therefore be within the sensitivity reach of the BESIII experiment [32, 33], as discussed in Refs. [6, 14, 31].

To conclude, motivated by the latest NA62 measurement on $K^+ \to \pi^+\nu\bar{\nu}$, which is in good agreement with the SM and consequently implies stringent constraints on NP which might be hiding in $K \to \pi E$, we have explored how other types of FCNC kaon decays with missing energy might shed additional light on potential NP in the underlying $s \to d E$ transition. We have argued that there are other operators contributing to $s \to d E$ which are not restricted by $K \to \pi E$ and accordingly could still significantly affect $K \to E$ and $K \to \pi\pi E$, on which the empirical details are currently meager. We have demonstrated especially that the branching fractions of $K \to \pi\pi E$ could yet be amplified far higher than their SM expectations, to levels which might be within the reaches of ongoing or near-future experiments, such as KOTO and NA62. Our results, which are illustrated with the instances summarized in Table II, will hopefully help stimulate new quests for these decays as NP probes. Lastly, we have pointed out that similar kinds of enhancement would occur in the hyperon sector, which may be detectable by BESIII and thus could provide complementary information.

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[1] L. Littenberg and G. Valencia, “Rare and radiative kaon decays,” Ann. Rev. Nucl. Part. Sci. 43, 729 (1993) [hep-ph/9303225].
[2] G. Buchalla, A.J. Buras, and M.E. Lautenbacher, “Weak decays beyond leading logarithms,” Rev. Mod. Phys. 68, 1125 (1996) [hep-ph/9512380].
[3] C.Q. Geng, I.J. Hsu, and Y.C. Lin, “CP conserving and violating contributions to $K_L \to \pi^+\pi^-\nu\bar{\nu}$”, Phys. Rev. D 50, 5744 (1994) [hep-ph/9406313].
[4] C.Q. Geng, I.J. Hsu, and Y.C. Lin, “Study of long distance contributions to $K \to n\pi\nu\bar{\nu}$”, Phys. Rev. D 54, 877 (1996) [hep-ph/9604228].
[5] V. Cirigliano, G. Ecker, H. Neufeld, A. Pich, and J. Portoles, “Kaon Decays in the Standard Model,” Rev. Mod. Phys. 84, 399 (2012) [arXiv:1107.6001 [hep-ph]].
[6] J. Tandean, “Rare hyperon decays with missing energy,” JHEP 1904, 104 (2019) [arXiv:1901.10447 [hep-ph]].
[7] A.V. Artamonov et al. [E949 Collaboration], “New measurement of the $K^+ \to \pi^+\nu\bar{\nu}$ branching ratio”, Phys. Rev. Lett. 101, 191802 (2008) [arXiv:0808.2459 [hep-ex]].
[8] J.K. Ahn et al. [KOTO Collaboration], “Search for the $K_L \to \pi^0\tau\bar{\tau}$ and $K_L \to \pi^0X^0$ decays at the J-PARC KOTO experiment,” Phys. Rev. Lett. 122, no. 2, 021802 (2019) [arXiv:1810.09655 [hep-ex]].
[9] E. Cortina Gil et al. [NA62 Collaboration], “First search for $K^+ \to \pi^+\nu\bar{\nu}$ using the decay-in-flight technique,” Phys. Lett. B 791, 156 (2019) [arXiv:1811.08508 [hep-ex]].
[10] E. Cortina Gil et al. [NA62 Collaboration], “An investigation of the very rare $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ decay,” arXiv:2007.08218 [hep-ex].

[11] A.J. Buras, D. Buttazzo, J. Girrbach-Noe, and R. Knegjens, “$K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$ in the Standard Model: status and perspectives,” JHEP 1511, 033 (2015) [arXiv:1503.02693 [hep-ph]].

[12] R. Marchevski, “Evidence for the decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ from the NA62 experiment at CERN”, talk at the 40th International Conference on High Energy Physics (ICHEP 2020), Prague, Czech Republic, 30 July - 5 August 2020.

[13] J.F. Kamenik and C. Smith, “FCNC portals to the dark sector,” JHEP 1203, 090 (2012) [arXiv:1111.6402 [hep-ph]].

[14] G. Li, J.Y. Su, and J. Tandean, “Flavor-changing hyperon decays with light invisible bosons,” Phys. Rev. D 100, no. 7, 075003 (2019) [arXiv:1905.08759 [hep-ph]].

[15] P.A. Zyla et al. [Particle Data Group], “Review of Particle Physics,” PTEP 2020, no. 8, 083C01 (2020).

[16] A. Badin and A.A. Petrov, “Searching for light Dark Matter in heavy meson decays,” Phys. Rev. D 82, 034005 (2010) [arXiv:1005.1277 [hep-ph]].

[17] M. Fabbrichesi, E. Gabrielli, and B. Mele, “Hunting down massless dark photons in kaon physics”, Phys. Rev. Lett. 119, no. 3, 031801 (2017) [arXiv:1705.03470 [hep-ph]].

[18] J.Y. Su and J. Tandean, “Searching for dark photons in hyperon decays,” Phys. Rev. D 101, no. 3, 035044 (2020) [arXiv:1911.13301 [hep-ph]].

[19] J.Y. Su and J. Tandean, “Kaon decays shedding light on massless dark photons,” Eur. Phys. J. C 80, 824 (2020) [arXiv:2002.04623 [hep-ph]].

[20] J. Martin Camalich, M. Pospelov, P.N.H. Vuong, R. Ziegler, and J. Zupan, “Quark Flavor Phenomenology of the QCD Axion,” Phys. Rev. D 102, no. 1, 015023 (2020) [arXiv:2002.04623 [hep-ph]].
[31] J.Y. Su and J. Tandean, “Exploring leptoquark effects in hyperon and kaon decays with missing energy,” arXiv:1912.13507 [hep-ph].

[32] H.B. Li, “Prospects for rare and forbidden hyperon decays at BESIII”, Front. Phys. (Beijing) 12, no. 5, 121301 (2017) [arXiv:1612.01775 [hep-ex]]; (Erratum) 14, 64001 (2019).

[33] M. Ablikim et al., “Future Physics Programme of BESIII”, Chin. Phys. C 44, no. 4, 040001 (2020) [arXiv:1912.05983 [hep-ex]].