Optimal Scheduling for Linear-Rate Multi-Mode Systems

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Abstract. Linear-Rate Multi-Mode Systems is a model that can be seen both as a subclass of switched linear systems with imposed global safety constraints and as hybrid automata with no guards on transitions. We study the existence and design of a controller for this model that keeps the state of the system within a given safe set for the whole time. A sufficient and necessary condition is given for such a controller to exist as well as an algorithm that finds one in polynomial time. We further generalise the model by adding costs on modes and present an algorithm that constructs a safe controller which minimises the peak cost, the average-cost or any cost expressed as a weighted sum of these two. Finally, we present numerical simulation results based on our implementation of these algorithms.

1 Introduction

Optimisation of electricity usage is an increasingly important issue because of the growing energy prices and environmental concerns. In order to make the whole system more efficient, not only the average electricity consumption should be minimised but also its peak demand. The energy produced during the peak times, typically occurring in the afternoon due to the heaters or air-conditioning units being switched on at the same time after people come back from work, is not only more expensive because the number of consumers outweighs the suppliers, but also the peaking power plants that provide the supply at that time are a lot less efficient. Therefore, the typical formula that is used for charging companies for electricity is a weighted average of its peak and average electricity demand [14]. Optimisation of the usage pattern of heating, ventilation and air-conditioning units (HVAC) not only can save electricity but also contribute to their longer lifespan, because they do not have to be used just as often.

In [9] Nghiem et al. considered a model of an organisation consisting of a number of decoupled zones whose temperature have to remain within a specified comfort temperature interval. Each zone has a heater with a number of possible output settings, but the controller can pick only one of them. That is, the heater can either be on in that one setting or it has to be off otherwise. A further restriction is that only some fixed number of heaters can be on at any time. The temperature evolution in each zone is governed by a linear differential equation whose parameters depend on the physical characteristics of the zone, the outside temperature, the heater’s picked setting and whether it is on or off. The aim is to find a safe controller, i.e. a sequence of time points at which to switch the heaters on or off, in order for the temperature in each zone to remain in its comfort interval which is given as the input. In the end, it was shown that a sufficient condition for such a controller to exist is whether a simple inequality holds. This fact can be used to minimise the peak number of heaters used
at the same time, but if heaters can have different costs, then this may not correspond to minimising the peak energy cost.

We strictly generalise the model in [9] and define linear-rate Multi-Mode Systems (MMS). The evolution of our system is the same, specifically it consists of a number of zones, which we will call variables, whose evolution do not directly influence each other. However, we do not assume that all HVAC units in the zones are heaters, so the system can cope with a situation when cooling is required during the day and heating during the night. Moreover, rather than having all possible combinations of settings allowed, our systems have a list of allowable joint settings for all the zones instead; we will call each such joint setting a mode. This allows to model specific behaviours, for instance, heat pumps, i.e. when the heat moves from one zone into another, and central heating that can only heat all the zones at the same time. Finally, we will be looking for the actual minimum peak cost without restricting ourselves to just one setting per heater nor the number of heaters being switched on at the same time, while keeping the running time polynomial in the number of modes. We also show how to find the minimum average-cost schedule and finally how to minimise the energy bill expressed as a weighted sum of the peak and the average energy consumption.

Related work. Apart from generalising the model in [9], MMSs can be seen both as switched linear systems (see, e.g. [5,12]) with imposed global safety constraints or as hybrid automata ([3,6]) with no guards on transitions. The analysis of switched linear systems typically focuses on several forms of stabilisation, e.g. whether the system can be steered into a given stable region which the system will never leave again. However, all these analyses are done in the limit and do not impose any constraints on the state of the system before it reaches the safe region. Such an analysis may suffice for systems where the constraints are soft, e.g. nothing serious will happen if the temperature in a room will briefly be too high or too low. However, it may not be enough when studying safety-critical systems, e.g. when cooling nuclear reactors. Each zone in an MMS is given a safe value interval in which the zone has to be at all times. This causes an interesting behaviour, because even if the system stabilises while staying forever in any single mode, these stable points may be all unsafe and therefore the controller has to constantly switch between different modes to keep the MMS within the safety set. For instance, a heater in a room has to constantly switch itself on and off as otherwise the temperature will become either too high or too low. On the other hand, even the basic questions are undecidable for hybrid automata (see, e.g. [7]) and therefore MMSs constitute its natural subclass with decidable and even tractable safety analysis.

In [2] we recently studied a different incomparable class of constant-rate Multi-Mode Systems where in each mode the state of a zone changes with a constant-rate as opposed to being governed by a linear differential equation as in linear-rate MMS. Specifically, in ever mode $m \in M$ the value of each variable $x_i$ after time $t$ increases by $c_{mi} \cdot t$ where $c_{mi} \in \mathbb{R}$ is the constant rate of change of $x_i$ in mode $m$. That model was a special case of linear hybrid automata ([3]), which has constant-rate dynamics and linear functions as guards on transitions. We showed a polynomial-time algorithm for both safe controllability and safe reachability questions, as well as finding optimal safe controllers in the generalised model where each mode has an associated cost per time unit.

There are many other approaches to reduce energy consumption and peak usage in buildings. One particularly popular one is model predictive control (MPC). In [11] stochastic MPC was used to minimize building’s energy consumption, while in [8] the peak electricity
demand reduction was considered. The drawback of using MPC in our setting is its high computational complexity and the fact it cannot guarantee optimality.

**Results.** The key contribution of the paper is an algorithm for constructing a safe controller for MMSs with any starting point in the interior of the safety set that we present in Section 3. Unlike in [9], we not only show a sufficient, but also necessary, condition for such a safe controller to exist. The condition is a system of linear inequalities that can be solved using polynomial-time algorithms for linear programming (see, e.g. [13]) and because that system does not depend on the starting state, we show that either all points in the interior of the safe set have a safe controller or none of them has one. Furthermore, we show that if there is a safe controller then there is a periodic one with the minimum dwell time, i.e. the smallest amount of time between two mode switches, being of polynomial-size. Such a minimum dwell time may be still too low for practical purposes. However, we prove that the problem of checking whether there is a safe controller with the minimum dwell time higher than 1 (or for any other constant) is PSPACE-hard. This means that any approximation of the largest minimum dwell time among all safe controllers is unlikely to be tractable.

In Section 4, we generalise the MMS model by associating cost per time unit with each mode and looking for a safe schedule that minimises the long-time average-cost. Similarly as before, if there is at least one safe controller, then the optimal cost do not depend on the starting point and there is always a periodic optimal controller. In order to prove that the controller that we construct has the minimum average-cost it is crucial that the condition found in Section 3 is both sufficient and necessary. Furthermore, in order to check whether there exists a safe controller with a peak cost at most $p$, it suffices to check safe controllability for the set of all modes whose cost do not exceed $p$. This allows us to find the minimum peak cost using binary search on $p$.

We show that all these periodic safe (and optimal) controllers can be constructed in time polynomial in the number of modes. However, if one considers the set of modes to be given implicitly as in [9] where each zone has a certain number of settings and all their possible combinations are allowed then the number of modes becomes exponential in the size of the input. We try to cope with the problem by performing a bottom-up binary search in order to avoid analysing large sets of modes and use other techniques to keep the running time manageable in practice.

Finally, we show how to find the minimum total cost calculated as a weighted sum of the peak cost and the average cost. The challenging part is that peak cost generally increases when the set of modes is expanded while the average-cost decreases. Therefore, the weighted cost may not be monotone in the size of set of modes and so binary search may not work and finding its minimum may require checking many possible subsets of the set of modes. However, we show a technique how one can narrow down this search significantly to make it practical even for a large sets of modes.

In the end, we conclude and point out some possible future work in Section 6.

2 Linear Multi-Mode Systems

Let us start by setting the notation. We write $\mathbb{N}$ for the set of natural numbers, $\mathbb{N}_{>0}$ for the set of positive integers and $\mathbb{Z}$ for the set of integers. For a set $X$, let $|X|$ denote the number of
elements in $X$. Also, we write $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ for the sets of all, non-negative and strictly positive real numbers, respectively. States of our system will be points in the Euclidean space $\mathbb{R}^n$ equipped with the standard Euclidean norm $\|\cdot\|$. By $\vec{x}, \vec{y}$ we denote points in this state space, by $\vec{f}, \vec{v}$ vectors, while $\vec{x}(i)$ and $\vec{f}(i)$ will denote the $i$-th coordinate of point $\vec{x}$ and vector $\vec{f}$, respectively. For $\diamond \in \{<,=,\geq,>\}$, we write $\vec{x} \diamond \vec{y}$ if $\vec{x}(i) \diamond \vec{y}(i)$ for all $i$. For a $n$-dimensional vector $\vec{v}$ by $\text{diag}(\vec{v})$ we denote a $n \times n$ dimensional matrix whose diagonal is $\vec{v}$ and the rest of the entries are 0. We can now formally define our model.

**Definition 1.** A linear-rate multi-mode system (MMS) is a tuple $\mathcal{H} = (M, N, A, B)$ where $M$ is a finite nonempty set of modes, $N$ is the number of continuous-time variables in the system, and $A : M \to \mathbb{R}_n^N, B : M \to \mathbb{R}^N$ give for each mode the coefficients of the linear differential equation that govern the dynamics of the system.

In all further computational complexity considerations, we assume that all real numbers are rational and represented in the standard way by writing down the numerator and denominator in binary. Throughout the paper we will write $a_i^m$ and $b_i^m$ as a shorthand for $A(m)(i)$ and $B(m)(i)$, respectively.

A controller of an MMS specifies a timed sequence of mode switches. Formally, a controller is defined as a finite or infinite sequences of timed actions, where a timed action $(m,t) \in M \times \mathbb{R}_{>0}$ is a tuple consisting of a mode and a time delay. We say that an infinite controller $\langle (m_1,t_1), (m_2,t_2), \ldots \rangle$ is Zeno if $\sum_{k=1}^{\infty} t_k < \infty$ and is periodic if there exists $l \geq 1$ such that for all $k \geq 1$ we have $(m_k,t_k) = (m_{(k \mod l)+1}, t_{(k \mod l)+1})$. Zeno controllers require infinitely many mode-switches within a finite amount of time, and hence, are physically unrealizable. However, one can argue that a controller that switches after $t_k = 1/k$ amount of time during the $k$-th timed action is also infeasible, because it requires the switches to occur infinitely frequently in the limit. Therefore, we will call a controller feasible if its minimum dwell time, i.e. the smallest amount of time between two mode switches, is positive. We will relax this assumption and allow for the modes that are not used at all by a feasible controller to occur in its sequence of time actions with timed delays equal to 0, but we still require any feasible controller to be non-Zeno. For a controller $\sigma = \langle (m_1,t_1), (m_2,t_2), \ldots \rangle$, we write $T_k(\sigma) \defeq \sum_{i=1}^k t_i$ for the total time elapsed up to step $k$ of the controller $\sigma$, $T_m^k(\sigma) \defeq \sum_{i \leq k, m_i = m} t_i$ for the total time spent in mode $m$ up to step $k$, and finally $t_{\min}(\sigma) = \inf\{k : t_k > 0\}$ defines the minimum dwell time of $\sigma$. For any non-Zeno controller $\sigma$ we have that $\lim_{k \to \infty} T_k(\sigma) = \infty$ and for any feasible controller $\sigma$ we also have $t_{\min}(\sigma) > 0$. Finally, for any $t \geq 0$ let $\sigma(t)$ denote the mode the controller $\sigma$ directs the system to be in at the time instance $t$. Formally, we have $\sigma(t) = m_k$ where $k = \min\{i : t \leq T_i(\sigma)\}$.

The state of MMS $\mathcal{H}$ initialized at a starting point $x_0$ under control $\sigma$ is a $N$-tuple of continuous-time variables $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))$ such that $\mathbf{x}(0) = x_0$ and $\dot{\mathbf{x}}(t) = B(\sigma(t)) - \text{diag}(A(\sigma(t)))\mathbf{x}(t)$ holds at any time $t \in \mathbb{R}_{\geq 0}$. It can be seen that if $\mathcal{H}$ is in mode $m$ during the entire time interval $[t_0, t_0 + t]$ then the following holds $x_i(t_0 + t) = b_i^m/a_i^m + (x_i(t_0) - b_i^m/a_i^m)e^{-a_i^m t}$. Notice that this expression is monotonic in $t$ and converges to $b_i^m/a_i^m$, because based on the definition of MMS we have $a_i^m > 0$ for all $m$ and $i$.

Given a set $S \subseteq \mathbb{R}^N$ of safe states, we say that a controller $\sigma$ is $S$-safe for MMS $\mathcal{H}$ initialized at $x_0$ if for all $t \geq 0$ we have $\mathbf{x}(t) \in S$. We sometimes say safe instead of $S$-safe if $S$ is clear from the context. In this paper we restrict ourselves to safe sets being
hyperrectangles, which can be specified by giving lower and upper bound value for each variable in the system. This assumption implies that controller \( \sigma \) is S-safe iff \( x(t) \in S \) for all \( t \in \{ T_k(\sigma) : k \geq 0 \} \), because each \( x_i(t) \) is monotonic when \( \mathcal{H} \) remains in the same mode and so if system is S-safe at two time points, the system is S-safe in between these two time points as well. This fact is crucial to the further analysis and allows us to only focus on S-safety at the mode switching time points of the controller. Formally, to specify any hyperrectangle \( S \), it suffices to give two points \( l, u \in \mathbb{R}^N \), which define the region as follows \( S = \{ x : l \leq x \leq u \} \). The fundamental decision problem for MMS that we solve in this paper is the following.

**Definition 2 (Safe Controllability).** Decide whether there exists a feasible S-safe controller for a given MMS \( \mathcal{H} \), a hyperrectangular safe set \( S \) given by two points \( l \) and \( u \) and an initial point \( x_0 \in S \).

The fact that \( a_{im}^m > 0 \) for all \( m \) and \( i \) make the system stable in any mode, i.e. if the system stays in any fixed mode forever, it will converge to an equilibrium point. However, none of these equilibrium points may be S-safe and as a result the controller may need to switch between modes in order to be S-safe. We present an algorithm to solve the safe controllability problem in Section 3 and later, in Section 4, we generalise the model to MMS with costs associated with modes and the aim being finding a feasible S-safe controller with the minimum average-cost, peak cost, or some weighted sum of these. As the following example shows, safe controllability can depend on the starting point if it lies on the boundary of the safe set. We will not analyse this special case and assume instead that the starting point belongs to the interior of the safe set.

It should be noted that the definition of MMS allows for an arbitrary switching between modes. Restricting the possible order the modes can be used in a timed sequence will be the subject of Corollary 1.

**Example 1.** Consider an apartment with two rooms and one heater. The heater can only heat one room at a time. When it is off, the room temperature converges to the outside temperature of 12°C, while if it is constantly on, the temperature of the room converges to 30°C. We assume the comfort temperature to be between 18°C and 22°C. The table below shows the coefficients \( b_{im}^m \) for all modes \( m \) and rooms \( i \), while all the \( a_{im}^m \)-s are assumed to be equal to 1. Intuitively, when heating room 1 and 2 half of the time each, the temperature in each room should oscillate around \((30°C + 12°C)/2 = 21°C\) and never leave the comfort zone assuming the switching occurs frequently enough. We will prove this intuition formally in Section 3. Therefore, as long as the temperature in one of the rooms is above 18°C at the very beginning, a safe controller exists. However, if the temperature in both rooms start at 18°C (a state which is safe), a safe controller does not exists, because in every mode the temperature has to drop in at least one of the rooms and so the state becomes unsafe under any control.

| Modes       | \( m_1 \) | \( m_2 \) | \( m_3 \) |
|-------------|-----------|-----------|-----------|
| \( b_{11}^m \) (Room 1) | 12        | 30        | 12        |
| \( b_{22}^m \) (Room 2) | 12        | 12        | 30        |
3 Safe Schedulability

Let us fix in this section a linear-rate MMS $\mathcal{M} = (M,N,A,B)$ and a safe set $S$ given by two points $\bar{t}, \underline{t} \in \mathbb{R}^N$, such that $\bar{t} < \underline{t}$ and $S = \{ \bar{x} : \bar{t} \leq \bar{x} \leq \underline{t} \}$. We call any vector $\vec{f} \in \mathbb{R}_+^M$ such that $\sum_{m \in M} \vec{f}(m) = 1$ a frequency vector. Also, let us define $F_i(\vec{f},\vec{y}) := \sum_{m \in M} \vec{f}(m)(b_i^m - a_i^m y)$.

Notice that for a fixed $\vec{f}$ and frequency vector $\vec{f}$, function $F_i(\vec{f},\vec{y})$ is continuous and strictly decreasing in $y$. Moreover, $F_i(\alpha \vec{f} + \beta \vec{g},\vec{y}) = \alpha F_i(\vec{f},\vec{y}) + \beta F_i(\vec{g},\vec{y})$. For a frequency vector $\vec{f}$, variable $x_i$ is called critical if $F_i(\vec{f},\bar{t}_i) = 0$ or $F_i(\vec{f},\underline{t}_i) = 0$ holds.

**Definition 3.** A frequency vector $\vec{f}$ is good if for every variable $x_i$, the following conditions hold: (I) $F_1(\vec{f},\bar{t}_i) \geq 0$, and (II) $F_1(\vec{f},\underline{t}_i) \leq 0$. A frequency vector $\vec{f}$ is implementable if it is good and for every variable $x_i$, we additionally have (III) if $F_i(\vec{f},\bar{t}_i) = 0$ then $\vec{f}(m) = 0$ for every $m \in M$ such that $b_i^m/a_i^m \neq \bar{t}_i$, and (IV) if $F_i(\vec{f},\underline{t}_i) = 0$ then $\vec{f}(m) = 0$ for every $m \in M$ such that $b_i^m/a_i^m \neq \underline{t}_i$.

**Theorem 1.** If there exists a feasible $S$-safe controller then there exists an implementable frequency vector.

**Proof.** Denote the feasible $S$-safe controller by $\sigma$. Let $f_i(k) = T_{i,k}(\sigma)/T_{i,k}(\sigma)$ be the fraction of the time spent by $\sigma$ in mode $m$ up to its $k$-th timed action; note that $f_i(k) \in [0,1]$, and $\sum_{m \in M} f_i(k) = 1$ for all $k$. Let us look at the sequence of vectors $\{\vec{f}_k \in [0,1]^M \}_{k=1}^\infty$, where we set $f_i(k) = \sum_{m \in M} f_i(m)$. Since this sequence is bounded, by the Bolzano-Weierstrass theorem, there exists an increasing integer sequence $j_1,j_2,\ldots$ such that $\lim_{k \to \infty} \vec{f}_{j_k}$ exists and let us denote this limit by $\vec{f}$. We prove by contradiction that $\vec{f}$ is an implementable frequency vector.

First, $\vec{f}$ is a frequency vector as a limit of a sequence of frequency vectors. So if it was not implementable we would have for some variable $x_i$ at least one of the following would hold: (I) $F_i(\vec{f},\bar{t}_i) < 0$, or (II) $F_i(\vec{f},\underline{t}_i) > 0$, or (III) $F_i(\vec{f},\bar{t}_i) = 0$ and the set $M' := \{ m \in M : \vec{f}(m) > 0 \}$ is bounded, or (IV) $F_i(\vec{f},\underline{t}_i) = 0$ and the set $\{ m \in M : \vec{f}(m) < 0 \}$ is bounded. We will consider only cases (I) and (III) as the other two are symmetric and their proofs are essentially the same.

Let us look at case (I). Denote $c := F_i(\vec{f},\bar{t}_i) < 0$. Let $\chi_i(t)$ be equal to $1$ if $\sigma(t) = m$ and let it be 0 otherwise. Notice that $x_i(T_{i,k})$, the value of the variable $x_i$ after the $j$-th timed action of $\sigma$, is equal to $x_i(0) + \int_0^{T_{i,k}} x_i(t) dt = x_i(0) + \sum_{m \in M} \int_0^{T_{i,k}} (b_i^m - a_i^m x_i(t)) x_i(t) dt$ $\leq x_i + \sum_{m \in M} T_{i,k} (b_i^m - a_i^m \bar{t}_i)$, because if the system is $S$-safe, then for every mode $m$ we have $b_i^m - a_i^m x_i \leq b_i^m - a_i^m \bar{t}_i$. From the definition of $\vec{f}$, for any $\varepsilon > 0$ we can pick $K$ such that for all $k > K$ and $m \in M$ we have $|\vec{f}_{j_k}(m) - \vec{f}(m)| < \varepsilon$. So

$$x_i(T_{i,k}) \leq x_i + \sum_{m \in M} T_{i,k} (b_i^m - a_i^m \bar{t}_i) = x_i + T_{i,k} \sum_{m \in M} \vec{f}_{j_k}(m)(b_i^m - a_i^m \bar{t}_i)$$

$$= x_i + T_{i,k} \left( \sum_{m \in M} \vec{f}(m) + (\vec{f}_{j_k}(m) - \vec{f}(m)) \right) (b_i^m - a_i^m \bar{t}_i)$$

$$= x_i + T_{i,k} \sum_{m \in M} \vec{f}(m)(b_i^m - a_i^m \bar{t}_i) + \varepsilon |M| c_{\max}$$
where $c_{\text{max}} := \max_{m \in M} |b_i^m - a_i^m T_i| \geq |E_i(f, \tilde{T})| > 0$.

If we now set $\epsilon$ to be $-\epsilon/(2|M|c_{\text{max}})$, which is $> 0$, then $x_i(T_k) \leq \pi_i + \frac{1}{2} T_k c$, and so $\lim_{k \to \infty} x_i(T_k) = -\infty$, because $\sigma$ is non-Zeno and $c < 0$. This is a contradiction with the assumption that $\sigma$ is $S$-safe, i.e. $x_i(t) \geq \tilde{T}$ for all $t \geq 0$.

Now let us move on to case (III). Let $a_{\text{max}} := \max_{m \in M'} |a_i^m|$, $c_{\text{min}} := \min_{m \in M'} |b_i^m - a_i^m T_i|$ and $t_{\text{min}} := t_{\text{min}}(\sigma)$. Of course $c_{\text{min}} > 0$, because $b_i^m/a_i^m \neq \tilde{T}_i$ for $m \in M'$ and $t_{\text{min}} > 0$, because $\sigma$ is feasible. Let $\gamma := \frac{c_{\text{min}}}{2t_{\text{min}} a_{\text{max}}}$, which is $> 0$.

**Lemma 1.** For at least half of the time duration of every timed action of $\sigma$ which uses mode $\in M'$, $x_i(t) \geq \tilde{T}_i + \gamma$ holds.

The proof of Lemma 1 can be found in the appendix. We can now proceed similarly as in case (I). We have that $x_i(T_k)$ is equal to $x_i(0) + \int_0^{T_k} \dot{x}_i(t) dt = x_i(0) + \sum_{m \in M} \int_0^{T_k} (b_i^m - a_i^m T_i) \chi_m(t) dt \leq \pi_i + \sum_{m \in M} T_k^m (b_i^m - a_i^m T_i) + \sum_{m \in M} -\frac{1}{2} T_k^m a_i^m \gamma$, because using Lemma 1 we know that for at least half of the time spent in any mode $m \in M'$ we have $b_i^m - a_i^m T_i \leq b_i^m - a_i^m (\tilde{T}_i + \gamma)$ and for the other half and any other $m \in M \setminus M'$ we have $b_i^m - a_i^m x_i(t) \leq b_i^m - a_i^m 2 \tilde{T}_i$. Again, from the definition of $f$, for any $\epsilon > 0$ we can pick $K$ such that for all $k > K$ and $m \in M$ we have $|f_i^m(m) - f(m)| < \epsilon$ and so

\[
x_i(T_k) \leq \pi_i + \sum_{m \in M} T_k^m (b_i^m - a_i^m T_i) + \frac{1}{2} T_k a_i^m \gamma
\]

\[
= \pi_i + T_k \sum_{m \in M} T_k^m (b_i^m - a_i^m T_i) - \frac{1}{2} T_k a_i^m \gamma
\]

\[
\leq \pi_i + T_k \sum_{m \in M} \left( f_i^m(m) + \frac{1}{2} T_k \gamma \right) a_i^m - \frac{1}{2} T_k a_i^m \gamma
\]

\[
\leq \pi_i + T_k \sum_{m \in M} \left( f_i^m(m) + \frac{1}{2} T_k \gamma \right) a_i^m - \frac{1}{2} T_k a_i^m \gamma
\]

\[
\leq \pi_i + T_k \sum_{m \in M} \left( f_i^m(m) + \frac{1}{2} T_k \gamma \right) a_i^m.
\]

If we now set $\epsilon := \frac{1}{2} T_k \gamma \sum_{m \in M} (f_i^m(m) + \gamma M'|a_{\text{max}}| \gamma a_{\text{max}} - \frac{1}{2} \gamma \sum_{m \in M'} f_i^m(m))$, then $\lim_{k \to \infty} x_i(T_k) = -\infty$, because $\gamma \sum_{m \in M'} f_i^m(m) > 0$ and $\sigma$ is non-Zeno. This is a contradiction with the assumption that $\sigma$ is $S$-safe.

Similarly we can show that neither case (II) nor case (IV) can hold which finishes the proof that $f$ is an implementable frequency vector.

**Theorem 2.** If there exists an implementable frequency vector then there exists a periodic $S$-safe controller for any initial state in the interior of the safety set.

**Proof.** Let $f$ be the implementable frequency vector. We first remove from $M$ all modes $m$ such that $f(m) = 0$. We claim that the following periodic controller $\sigma = (\langle m_k, t_k \rangle)_{k=1}^{\infty}$ with period $|M|$ is $S$-safe for sufficiently small $\varepsilon$: $m_k = (k \mod |M|) + 1$ and $t_k = f(m_k) \cdot \varepsilon$. As we already know it suffices to check $S$-safety of the system at time points $T_k$ for all $k$. We will focus here on checking just the lower bound, $\bar{x}(T_k) \geq \tilde{T}$, because the estimations concerning
the upper bound are very similar. Note that for any variable $x$, we have

$$x(T_i) = \frac{b_{0i}}{d_i} + \left(x_0(i) - \frac{b_{0i}}{d_i}\right) e^{-\alpha_{i} t_i},$$

and further by induction we get

$$x(T_k) = \frac{b_{mk}}{d_i} + \sum_{n=1}^{k-1} \left(\frac{b_{mn}}{d_i} - \frac{b_{m+1,n}}{d_i}\right) e^{-\Sigma_{j=k-n+1}^{m} \alpha_{j} t_j} + \left(x_0(i) - \frac{b_{0i}}{d_i}\right) e^{-\Sigma_{j=1}^{m} \alpha_{j} t_j}$$

(1)

Now, because $f$ is implementable, if $F_i(T_i, \tilde{T}_i) = 0$ then it has to be $b_{mn}/a_m = \tilde{T}_i$ for all $m$. In such a case, it is easy to see from equation (1) that $x(T_k) > \tilde{T}_i$ for all $k$, because $x_0(i) > \tilde{T}_i$.

Therefore, we can assume $F_i(T_i, \tilde{T}_i) > 0$. Let $\tilde{x}_i \equiv x(T_i)$ for all $l \in \mathbb{N}$, $\Sigma_i \equiv \Sigma_{j=k}^{m} a_j t_j = s \cdot \sum_{j=k}^{M} a_j f_j(m_j)$, $\alpha(s) \equiv e^{-\Sigma_{j=1}^{M} \alpha_j \tilde{T}_i}$, and

$$\beta(s) \equiv \frac{d_i}{a_i} + \sum_{n=1}^{M-1} \left(\frac{b_{mn}}{d_i} - \frac{b_{m+1,n}}{d_i}\right) e^{-\Sigma_{j=1}^{M-n} \alpha_j t_j}$$

Notice that since $\sigma$ is periodic with period $|M|$ from equation (1) we can deduce $\tilde{x}_i + (1 - \alpha(s)) \geq \tilde{T}_i$. The last condition is equivalent to $\beta(s) - \tilde{T}_i + \alpha(s) \tilde{T}_i > 0$, because $1 - \alpha(s) > 0$. It is well-known that $1 - x \leq e^{-x} \leq 1 - x + x^2$ for all $x > 0$. Let us also denote $d := \tilde{T}_i + |\pi| + 2 \cdot \max_{m \in M} |b_{mn}/a_m|$. Notice that for all $m, m' \in M$ we have $|b_{mn}/a_m - b_{m'n'}/a_{m'}| \leq d$, $|\tilde{T}_i - b_{mn}/a_m| \leq d$, as well as $|\tilde{x}_0(i) - b_{mn}/a_m| \leq d$. Therefore, $\beta(s) - \tilde{T}_i + \alpha(s) \tilde{T}_i = \left(\frac{b_{0i}}{d_i} - \tilde{T}_i\right) + (\sum_{n=1}^{M-1} \left(\frac{b_{mn}}{d_i} - \frac{b_{m+1,n}}{d_i}\right) e^{-\Sigma_{j=1}^{M-n} \alpha_j t_j} + (\tilde{T}_i - \frac{b_{0i}}{d_i}) e^{-\Sigma_{j=1}^{M} \alpha_j t_j} \geq \left(\frac{b_{0i}}{d_i} - \tilde{T}_i\right) + (\sum_{n=1}^{M} \left(\frac{b_{mn}}{d_i} - \frac{b_{m+1,n}}{d_i}\right) e^{-\Sigma_{j=1}^{M-n} \alpha_j t_j} + (\tilde{T}_i - \frac{b_{0i}}{d_i}) (1 - (\Sigma_{j=1}^{M} \alpha_j t_j)^2) = \sum_{j=1}^{M} \tilde{T}_i f_j(m_j) a_j t_j - d \cdot (\Sigma_{j=1}^{M} \alpha_j t_j)^2 \geq s \sum_{j=1}^{M} \tilde{T}_i f_j(m_j) a_j t_j - d \cdot (\Sigma_{j=1}^{M} \alpha_j t_j)^2 = s \sum_{j=1}^{M} \tilde{T}_i f_j(m_j) a_j t_j$.

If we now set $s := \frac{-F_i(T_{mk})}{d_i |M| (\sum_{j=1}^{M} f_j(m_j))^2}$, then the last expression will be $\geq 0$. Notice that this bound does not depend on the order of the modes in the period nor on the starting state. So if for any $k < |M|$ we repeat this estimation for the initial point $x(T_k)$, controller $\sigma'(t) := \sigma(t + T_k)$ and exactly the same $s$, the value of $x(T_{mk}|M|+k - T_k)$ under control $\sigma'$ will also monotonically converge to some value $\geq \tilde{T}_i$ as $l \to \infty$. Therefore, as long as $x(T_k)$ is a $S$-safe for all $k < |M|$ for the just selected $s$, all states of the system that follow will be $S$-safe as well. Now, if we repeat the same analysis for the upper bound $\tilde{T}_i$ then we would get an expression

$s := \frac{-E_i(f_{mk})}{d_i |M| (\sum_{j=1}^{M} f_j(m_j))^2}$, so it suffices to set $s$ to be the minimum of these two.

Now, to find $s$ such that the system is $S$-safe for the first $|M|$ steps, we can estimate $x(T_k)$ to be $\geq \tilde{x}_0(i) - T_k \max_m |b_{mn} / a_m (\tilde{T}_0)|$ and $\leq \tilde{x}_0(i) + T_k \max_m |b_{mn} / a_m (\tilde{T}_0)|$. We have $T_k \leq s$
for \( k < |M| \) from the definition of \( \sigma \) and so it suffices to set \( s := \min\{\pi_i - x_0(i), x_0(i) - \ell_i\} / \max_m |b^m_i - a^m_i x_0(i)| \) if \( \max_m |b^m_i - a^m_i x(0)| \neq 0 \) and otherwise set \( s \) to an arbitrary high value in order for the variable \( x_i \) to be \( S \)-safe in the first \( |M| \) steps.

Finally, if we pick the minimum value from these estimates on \( s \) over all possible variables \( x_i \), we will guarantee that the system is both \( S \)-safe in the first \( |M| \) steps as well as after that, because \( x_i(T_{|M|+k}) \) will monotonically converge for every fixed \( i \) and \( k < |M| \) to a safe state as \( l \to \infty \).

**Theorem 3.** Algorithm \[\] returns in polynomial time a \( S \)-safe feasible controller from \( x_0 \) if there exists one.

**Proof.** We first need the following lemma whose proof can be found in the appendix.

**Lemma 2.** Either there is a variable which is critical for all good frequency vectors or there is a good frequency vector in which no variable is critical.

Now, let \( \sigma \) be the controller returned by Algorithm \[\]. Notice that the frequency vector \( \vec{f} \) the controller \( \sigma \) is based on is implementable, because \( \vec{f} \) satisfies the constraints at line 11 which imply the conditions (I) and (II) of \( \vec{f} \) being implementable and from Lemma 2 it follows that all modes that could violate the conditions (III) and (IV) were removed in the loop between lines 5–10. Moreover, constant \( s \) used in the construction of \( \sigma \) is exactly the same as the one used in Theorem 2 which guarantees \( \sigma \) to be \( S \)-safe.

On the other hand, from Theorem 1, if there exists a feasible \( S \)-safe controller then there also exists an implementable frequency vector \( \vec{f} \). Such a vector will satisfy the constraints of being good at line 2 of the algorithm. In the loop between the lines 5–10 all variables that are critical in \( \vec{f} \) are first checked whether they satisfy conditions (III) and (IV), and they will satisfy them because \( \vec{f} \) is implementable, and after that these critical variables are removed. Finally, \( \vec{f} \) consisting of just the remaining variables will satisfy the constraints at line 7 of being implementable with no critical variables. Therefore, Algorithm \[\] will always return a controller if there exists a \( S \)-safe one.

It is easy to see that Algorithm \[\] runs in polynomial time, because at least one critical variable is removed in each iteration of the loop between lines 5–10 one iteration checks at most \( N \) remaining variables, and each such a check requires calling a linear programming solver which runs in polynomial time. Finally, steps 4 and 13 of Algorithm \[\] are achievable, because if a linear program has a solution then it has a solution of polynomial size (see, e.g. \[\]). This shows that the size of the returned controller is always polynomial.

Notice that the controller returned by Algorithm \[\] has a polynomial-size minimum dwell time. We do not know whether finding a safe controller with the largest possible dwell time is decidable, nor is checking whether such a minimum dwell time can be greater than \( \geq 1 \). We now show that the last problem is PSPACE-hard, so it is unlikely to be tractable. The details of the proof are in the appendix. Finally, this also implies PSPACE-hardness of checking whether a safe controller exists in the case the system is controlled using a digital clock, i.e. when all timed delays have to be a multiple of some given sampling rate \( \Delta > 0 \).

**Theorem 4.** For a given MMS \( H \), hyperrectangular safe set \( S \) described by two points \( \vec{\pi}, \vec{\ell} \), starting point \( x_0 \in S \), checking whether there exits a \( S \)-safe controller with minimum dwell time \( \geq 1 \) is PSPACE-hard.
Algorithm 1: Finds a $S$-safe feasible controller from a given $x_0 \in S$.

**Input:** MMS $\mathcal{M}$, two points $\bar{l}$ and $\bar{u}$ that define a hyperrectangle $S = \{x : \bar{l} \leq x \leq \bar{u}\}$ and an initial point $x_0 \in S$ such that $\bar{l} < x_0 < \bar{u}$.

**Output:** NO if no $S$-safe feasible controller exists from $x_0$, and a periodic such controller, otherwise.

1. $I := M$;
2. Check whether the following linear program is satisfiable for some frequency vector $\vec{f}$:

\[
F_i(\vec{f}, \bar{l}_i) \geq 0 \text{ for all } i \in I
\]
\[
F_i(\vec{f}, \bar{u}_i) \leq 0 \text{ for all } i \in I.
\]

if no satisfying assignment exists then

3. return NO

Let $\vec{f}^*$ be any frequency vector of polynomial size that satisfies conditions in step 2

repeat

foreach $j \in I$ do

4. Check whether the following linear program is satisfiable for some frequency vector $\vec{f}$:

\[
F_i(\vec{f}, \bar{l}_i) \geq 0 \text{ for all } i \in I \setminus \{j\}
\]
\[
F_i(\vec{f}, \bar{u}_i) \leq 0 \text{ for all } i \in I \setminus \{j\}
\]
\[
F_j(\vec{f}, \bar{l}_j) > 0 \text{ and } F_j(\vec{f}, \bar{u}_j) < 0.
\]

if no satisfying assignment exists then

5. If $F_j(\vec{f}^*, \bar{l}_i) = 0$, remove all modes for which $b_m / a_m \neq \bar{l}_i$ and otherwise remove all modes for which $b_m / a_m \neq \bar{u}_i$.

6. Remove $j$ from $I$.

until no mode was removed from $M$ in this iteration;

7. Check whether the following linear program is satisfiable for any frequency vector $\vec{f}$:

\[
F_i(\vec{f}, \bar{l}_i) > 0 \text{ for all } i \in I
\]
\[
F_i(\vec{f}, \bar{u}_i) < 0 \text{ for all } i \in I.
\]

if no satisfying assignment exists or $M = \emptyset$ then

8. return NO

Let $\vec{f}_*$ be any frequency vector of polynomial size that satisfies conditions in step 11

9. Let

\[
s := \min_{i \in I} \left( \frac{\min \{x_0(i) - \bar{l}_i, \bar{u}_i - x_0(i)\}}{\max_m |b_m^\alpha - a_m^\alpha x_0(i)|} \right) \left( \frac{\min(F_i(\vec{f}_*, \bar{l}_i), F_i(\vec{f}_*, \bar{u}_i))}{(|\bar{l}_i| + |\bar{u}_i| + 2 \cdot \max_m |b_m^\alpha / a_m^\alpha|)(\sum_m a_m^\alpha \vec{f}_*(m))^2} \right).
\]

10. return the following periodic controller with period $|M|$: $m_k = (k \mod |M|) + 1$ and $t_k = \vec{f}_*(m_k) \cdot s$. 

11. }
Proof. (Sketch) The proof is similar to the PSPACE-hardness proof in \cite{2} of the discrete-time reachability in constant-rate MMS that reduces from the acceptance problem for linear bounded automata (LBAs), but our reduction is a lot more involved, because of the differences in the dynamics of the system. For instance, we deal with a decision problem for the minimum dwell time of a safe continuous-time controller instead of a discrete-time one. Also, unlike for constant-rate MMS, there is no possibility to keep the value of any variable constant over time regardless of its current value. To overcome this problem, we will take advantage of the fact that for every LBA and input word, there exists an exponential upper bound on the number of steps this LBA can take before the input word is accepted. \hfill \Box

Notice that the periodic controller returned by Algorithm $1$ just cycles forever over the set of modes in some fixed order which can be arbitrary. This allows us to extend the model by specifying an initial mode $m_0$ and a directed graph $G \subseteq M \times M$, which specifies for each mode which modes can follow it. Formally, we require any controller $\langle (m_1,t_1),(m_2,t_2),\ldots \rangle$ to satisfy $(m_i,m_{i+1}) \in G$ for all $i \geq 1$ and $m_1 = m_0$. The proof is in the appendix.

Corollary 1. Deciding whether there exists a feasible $S$-safe controller for a given MMS $\mathcal{H}$ with a mode order specification graph $G$, initial mode $m_0$, a hyperrectangular safe set $S$ given by two points $\bar{l}$ and $\bar{u}$ and an initial point $\bar{l} < x_0 < \bar{u}$ can be done in polynomial time.

4 Optimal Schedulability

In this section we extend our results on $S$-safe controllability of MMS to a model with costs per time unit on modes. We will call this model priced linear-rate multi-mode systems. The aim is to find an $S$-safe controller with the minimum cost where the cost is either defined as the peak cost, the (long-time) average cost or a weighted sum of these.

Definition 4. A priced linear-rate multi-mode system (MMS) is a tuple $\mathcal{H} = (M,N,A,B,\pi)$ where $(M,N,A,B)$ is a MMS and $\pi: M \to \mathbb{R}_\geq 0$ is a cost function such that $\pi(m)$ characterises the cost per-time unit of staying in mode $m$.

We define the (long-time) average cost of an infinite controller $\sigma = \langle (m_1,t_1),(m_2,t_2),\ldots \rangle$ as the long-time average of the cost per time-unit over time, i.e.

$$\text{AvgCost}(\sigma) \overset{\text{def}}{=} \limsup_{k \to \infty} \frac{\sum_{i=1}^{k} \pi(m_i) \cdot t_i}{\sum_{i=1}^{k} t_i}.$$  

For the results to hold it is crucial that limsup is used in this definition instead of liminf. In the case of minimising the average cost, it is more natural to minimise its lim sup anyway, which intuitively is its reoccurring maximum value. On the other hand, the peak cost is simply defined as $\text{PeakCost}(\sigma) \overset{\text{def}}{=} \sup_{k,t_i>0} \pi(m_k)$.

We will try to answer the following question for priced MMS.

Definition 5 (Optimal Controllability). Given a priced MMS $\mathcal{H}$, a hyperrectangular safe set $S$ defined by two points $\bar{l}$ and $\bar{u}$, an initial point $x_0 \in S$ such that $\bar{l} < x_0 < \bar{u}$, and constants $\mu_{\text{avg}}, \mu_{\text{peak}} \geq 0$, find an $S$-safe controller $\sigma$ with the minimum value of $\mu_{\text{avg}} \text{AvgCost}(\sigma) + \mu_{\text{peak}} \text{PeakCost}(\sigma)$.  






The following example shows that such a weighted cost does not always increase with the increase in the peak cost.

Example 2. The table below shows the values of $b^m_i$ for each mode $m \in M = \{m_1, m_2, m_3\}$ and variable $x_i$ as well as the cost of each mode. We assume that all $a^m_i$-s are equal to 1. The safe value interval for each variable is $[0, 1]$, i.e. $\bar{l}_i = 0$, $\bar{u}_i = 1$ for all $i$.

| Modes   | $m_1$ | $m_2$ | $m_3$ | $m_4$ |
|---------|-------|-------|-------|-------|
| $b^m_1$ | -1    | 2     | -1    | 5     |
| $b^m_2$ | -1    | -1    | 2     | 5     |
| $\pi$ (cost) | 0     | 3     | 3     | 4     |

One can compute that the optimal average cost of any $S$-safe controller which uses only modes from $M' = \{m_1, m_2, m_3\}$ is equal to 2 and that average cost is achieved when the frequency of each mode from $M'$ is equal to $\frac{1}{3}$. At the same time, the peak cost of that controller is 3. On the other hand, there is a $S$-safe controller for the whole set of modes $M$ with peak cost 4 and average cost just $\frac{5}{6}$, when the frequency of mode $m_1$ is $\frac{5}{6}$ and the frequency of mode $m_4$ is $\frac{1}{6}$. If we assume that the weighted cost of a controller $\sigma$ is $\text{PeakCost}(\sigma) + \text{AvgCost}(\sigma)$, then clearly the second controller has a lower weighted cost although it has a higher peak cost.

The algorithm that we define is designed to cope with systems where the set of modes is large and given implicitly like in [9], where the input is a list of heaters with different output levels and energy costs. Each heater is placed in a different zone and any possible on/off combination of the heaters gives us a different mode in our setting, which leads to exponentially many modes in the size of the input. The cost of a mode is the sum of the energy cost of all heaters switched on in that particular mode. We try to deal with this setting by using binary search and a specific narrowing down technique to consider only the peak costs for which the weighted cost can be optimal. Unfortunately our algorithm will not run in time polynomial in the number of heaters, but the techniques used can reduce the running time in practice. If we assume that modes are given explicitly as the input then there is a much simpler algorithm which runs in polynomial time and is presented in Appendix E as Algorithm 3. Let us now fix a MMS with costs $\mathbf{H} = (M, N, A, B, \pi)$, the safe set $S$ and a starting point $x_0$ in the interior of $S$.

Theorem 5. Algorithm 2 finds a $S$-safe feasible periodic controller that optimises the weighted cost defined by the peak and average cost coefficients $\mu_{\text{peak}}$ and $\mu_{\text{avg}}$.

Proof. Let $M_{\leq p}$ denote the set of modes with cost at most $p$. First, to find the minimum peak cost among all $S$-safe controllers we can first order all the modes according to their costs and then the algorithm makes a binary search on the possible peak cost $p$, i.e. guesses initial $p$ and checks whether $M_{\leq p}$ has a $S$-safe controller; if it does not then it doubles the value of $p$ and if it does then halves the value of $p$. In may be best to start with a small value of $p$ first, because the bigger $p$ is, the bigger is the set of modes and the slower is checking its feasibility.

Second, to find the minimum average cost among all $S$-safe controllers, notice that the average cost of the periodic controller returned by Algorithm 1 based on the frequency vector $\hat{f}_s$ is $\sum_{m \in M} \hat{f}_s(m)\pi(m)$. Therefore, if we find an implementable frequency vector which
Algorithm 2: Finds an optimal $\mathcal{S}$-safe feasible controller from a given $\bar{x}_0 \in \mathcal{S}$.

**Input:** A priced MMS $\mathcal{M}$, two points $\bar{t}$ and $\bar{\pi}$ that define a hyperrectangle $\mathcal{S} = \{\bar{x}: \bar{t} \leq \bar{x} \leq \bar{\pi}\}$ and an initial point $\bar{x}_0 \in \mathcal{S}$ such that $\bar{t} < \bar{x}_0 < \bar{\pi}$, and constants $\mu_{\text{avg}}$ and $\mu_{\text{peak}}$ which define the weighted cost of a controller.

**Output:** NO if no $\mathcal{S}$-safe feasible controller exists from $\bar{x}_0$, and an periodic such controller $\sigma$ for which $\mu_{\text{peak}} \text{PeakCost}(\sigma) + \mu_{\text{avg}} \text{AvgCost}(\sigma)$ is minimal, otherwise.

1. min-size := 1;
2. repeat
   3.   min-size := 2 · min-size;
   4.   Pick minimal $p$ such that $M_{\leq p}$, the set of all modes with cost at most $p$, has size at least min-size.
   5.   Call Algorithm 1 for the set of modes $M_{\leq p}$.
3. until min-size < $|M|$ and the call returned NO;
4. if the last call to Algorithm 1 returned NO then
   5.   return NO.
5. Perform a binary search to find the minimal $p$ such that $M_{\leq p}$ is feasible using the just found upper bound on the minimal feasible set of modes.
6. Modify Algorithm 1 by adding the objective function $\text{Minimise } \sum_{m \in M} \bar{f}_m \pi(m)$ to the linear program at line 11. Let $\text{OptAvgCost}(M')$ be the value of this objective when Algorithm 1 is called for the set of modes $M'$.
7. $p' := p' + \frac{p_{\text{peak}} \text{OptAvgCost}(M_{\leq p})}{\mu_{\text{peak}}}$
8. repeat
   9.   until $p'$ decreases;
10. Pick a peak value $p^* \in [p, p']$ for which $\mu_{\text{peak}} p^* + \mu_{\text{avg}} \text{OptAvgCost}(M_{\leq p^*})$ is the smallest.
11. return the periodic controller returned by the modified version of Algorithm 1, called for the set of modes $M_{\leq p^*}$.

minimises that value, then we will also find a safe controller with the minimum average-cost among all periodic safe controllers. This can be easily done by adding the objective $\text{Minimise } \sum_{m \in M} \bar{f}_m \pi(m)$ to the linear program at line 11 of Algorithm 1. However, using similar techniques as in Theorem 1 we can show that no other controller can have a lower average-cost. The key observation is the fact that $\text{AvgCost}(\sigma) = \limsup_{k \to \infty} \sum_{m \in M} \bar{f}_k \pi(m) \geq \limsup_{k \to \infty} \sum_{m \in M} \bar{f}_k \pi(m) = \sum_{m} \bar{f}_m \pi(m)$ where just like in the proof of Theorem 1 the frequency of being in mode $m$ up to the $k$-th timed action and $(j_k)_{k \in \mathbb{N}}$ defines a subsequence of $\bar{f}_k$ that converges for every $m$. The second inequality holds because a lim sup of a subsequence is at most equal to the lim sup of the whole sequence.

For any set of modes $M' \subseteq M$, let $\text{OptAvgCost}(M')$ denote the minimum average cost when only modes in $M'$ can be used. Now, if $\mu_{\text{peak}} = 0$ then it suffices to compute the optimal average cost for the whole set of modes to find the minimum weighted cost. Otherwise, to find a safe controller with the minimum value of $\mu_{\text{peak}} \text{PeakCost}(\sigma) + \mu_{\text{avg}} \text{AvgCost}(\sigma)$ the algorithm first finds a feasible set of modes with the minimum peak cost and let us denote that peak cost by $p_{\text{min}}$. If $\mu_{\text{avg}} = 0$ then this suffices. Otherwise, observe that from the definition the cost of each mode is always nonnegative and so the average cost has to
be as well. Even if we assume that the average cost is equal to 0 for some larger set of modes with peak cost $p$, the weighted cost will at least be equal to $\mu_{\text{peak}} p$ as compared to $\mu_{\text{peak}} p_{\text{min}} + \mu_{\text{avg}} \text{OptAvgCost}(M \leq p_{\text{min}})$, which gives us an upper bound on the maximum value of $p$ worth considering to be $p' = p_{\text{min}} + \frac{\mu_{\text{avg}}}{\mu_{\text{peak}}} (\text{OptAvgCost}(M \leq p_{\text{min}}))$. But now we can check the actual value of $\text{OptAvgCost}(M \leq p')$ instead of assuming it is 0 and calculate again a new bound on the maximum peak value worth considering and so on. To generate modes on-the-fly in the order of increasing costs, we can use Dijkstra algorithm with a priority queue.

5 Numerical Simulations

![Fig. 1.](image)

**Fig. 1.** Comparison of temperature evolution under optimal and lazy control in an organisation consisting of two zones. The safe temperature is between $18^\circ\text{C}$ and $22^\circ\text{C}$. On the left, a periodic controller with the minimum peak cost which was then optimised for the minimal average-cost. On the right, the behaviour of the lazy controller. The $y$-axis is temperature in $^\circ\text{C}$ and the $x$-axis measures time in hours. The optimal controller used 3 modes and its minimum dwell time was 43 seconds. On the other hand, the lazy controller used 5 different modes and its minimum dwell time was 180 seconds.

We have implemented Algorithms 1 and 2 using a basic implementation of the simplex algorithm as their underlying linear program solver in Java. The tests were run on Intel Core i5 1.7 GHz with 1GB memory available. The examples are based on the model of an organisation with decoupled zones as in [9] and were randomly generated with exactly the same parameters as described there. We implemented also a simple lazy controller to compare its peak and average energy consumption to our optimal one. Simply asking the lazy controller to let the temperature oscillate around the minimum comfort temperature in each room is risky and causes high peak costs, so our “lazy” controller uses a different
approach. It switches any heater to its minimum setting if its zone has reached a temperature in the top 5% of its allowable value range. On the other hand, if the temperature in a zone is in the bottom 5% of its allowable value range, then the lazy controller finds and switches its heater to the minimum setting that will prevent the temperature in that zone dropping any further. However, before it does that, it first checks whether there are any zones with their temperature above 10% of their allowable value range and switches them off first. This tries to minimise the number of heaters being switched on at the same time and thus also tries to minimise the peak cost.

We have tested our systems for an organisation with eight zones and each heater having six possible settings, which potentially gives $6^8 > 10^6$ possible modes. Zones parameters and their settings were generated using the same distribution as described in [9] and the outside temperature was set to $10^\circ C$. The simulation of the optimal and the lazy controller was performed with a time step of three minutes and the duration of nine hours.

First, in Figure 1 we can compare the difference in the behaviour of the optimal controller as compared to the lazy one in the case of just two zones. In the case of the optimal controller, we can see that the temperature in each zone stabilises around the lower safe bound by using a constant switching between various modes. On the other hand, for the lazy controller the temperature oscillates between the lower and upper safe value, which wastes energy. The peak cost was 15 kW for the optimal controller and 18.43 kW for the lazy one, while the average energy usage was 13.4 kW and 15.7 kW, respectively. This gives 23% savings in the peak energy consumption and 17% savings in the average energy consumption. Note that any safe controller cannot use more than 16.9 kW of energy on the average, because otherwise it would exceed the upper comfort temperature for one of the rooms, so the maximum possible savings in the average energy consumption cannot exceed 26%. For a building with eight rooms, the running time of our algorithm was between less than a second to up to a minute with an average 40 seconds, depending on how many modes were necessary to ensure safe controllability of the system. The lazy controller was found to have on the average 40% higher peak cost than the optimal controller and 15% higher average-cost. In the extreme cases it had 70% higher peak cost and 22% higher average-cost. Again, the reason why the lazy controller did better in the average energy consumption than the peak consumption is that the comfort zone is so narrow and any safe controller cannot waste too much energy without violating the upper comfort temperature in one of the rooms.

6 Conclusions

We have proposed and analysed a subclass of hybrid automata with dynamics governed by linear differential equations and no guards on transitions. This model strictly generalises the models studied by Nghiem et al. in [9] in the context of peak minimisation for energy consumption in buildings. We gave a sufficient and necessary condition for the existence of a controller that keeps the state of the system within a given safe set at all times as well as an algorithm that finds such a controller in polynomial time. We also analysed an extension of this model with costs per time unit associated with modes and gave an algorithm that constructs a safe controller which minimises the peak cost, the average cost or any cost expressed as a weighted sum of these two. Finally, we implemented some of these algorithms and showed how they perform in practice.
From the practical point of view, the future work will involve turning the prototype implementation of the algorithms in this paper into a tool. Our model can be extended by adding disturbances and interactions between zones to the dynamics of the model like in [10]. This, however, would further complicate the already complicated formula given for the switching frequency of each mode of the safe controller as defined in Algorithm [1]. The special cases that could be looked at are the initial state being on the boundary of the safe set and checking whether Theorem [1] also holds for all non-Zeno controllers not just for controllers with a positive minimum dwell time. An interesting problem left open is the decidability of finding a safe controller with the minimum dwell time above a fixed constant.

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Appendix

A Proof of Lemma 1

We first prove the following proposition.

Proposition 1. If $\mathcal{H}$ is in the same mode $m$ during the time interval $[t_0, t_0 + t]$ we have that $x_i(t_0) + t(b^n - a^n_i x_i(t_0 + t)) \leq x_i(t_0) + t(b^n - a^n_i x_i(t_0))$ holds, where $\text{sgn}$ is the signum function.

Proof. Recall that $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(x) = 0$ if $x = 0$. Notice that $\dot{x}_i(t) = b_i - a_i x_i(t)$ attains its minimum and maximum at the ends of the time interval $[t_0, t_0 + t]$, because $x_i$ is monotone in $t$. Therefore, if $\dot{x}_i(t_0) > 0$, i.e. $\text{sgn}(\dot{x}_i(t_0)) = 1$ and $x_i$ is increasing in $t$, then for all $c \in [t_0, t_0 + t]$ we have $\dot{x}_i(c) \leq \dot{x}_i(t_0)$ and $\dot{x}_i(c) \geq \dot{x}_i(t_0 + t)$. From the mean value theorem, we know that for some $c \in [t_0, t_0 + t]$ we have $x_i(t_0 + t) = x_i(t_0) + t \dot{x}_i(c)$; the inequality follows and we proceed similarly in the other cases.

Lemma 1. For at least half of the time duration of every timed action of $\sigma$ which uses mode $m$, $x_i(t) \geq \tilde{t}_i + \gamma$ holds.

Proof. Recall that $a_{\max} := \max_{m \in M'} |a^n_m|$, $c_{\min} := \min_{m \in M'} |b^n - a^n_m \tilde{t}_i|$ and $t_{\min} := t_{\min}(\sigma)$. Of course $c_{\min} > 0$, because $b^n - a^n_m \tilde{t}_i \neq 0$ for $m \in M'$ and $t_{\min} > 0$, because $\sigma$ is feasible. Let $\gamma := \frac{a_{\max}}{2^{t_{\min}(\sigma)}}$, which is $> 0$.

Let us consider the $k$-th timed action of $\sigma$ such that $m_k \in M'$. Of course we have $t_k \geq t_{\min}$. Notice that $\gamma < \frac{1}{t_{\min}} c_{\min}$, because $t_{\min} = \frac{a_{\min}}{a_{\max}}$ and also, easy calculations show $\gamma < \frac{a_{\min}}{a_{\max}}$.

If $x_i(T_{k-1}) \geq \tilde{t}_i + \gamma$ and $x_i(T_k) \geq \tilde{t}_i + \gamma$, then we are done, because $x_i$ is monotonically increasing in the time interval $[T_{k-1}, T_k]$ and so $x_i(t) \geq \tilde{t}_i + \gamma$ would hold for the whole $k$-th timed action. We will estimate the longest amount of time the system can be in the same mode while $x_i(t) \in [\tilde{t}_i, \tilde{t}_i + \gamma]$ holds. First, notice that for every $m \in M'$ we either have $b^n - a^n_m x > 0$ for all $x \in [\tilde{t}_i, \tilde{t}_i + \gamma]$ or it is $b^n - a^n_m x < 0$ for all $x \in [\tilde{t}_i, \tilde{t}_i + \gamma]$. Otherwise, there would be $x \in [\tilde{t}_i, \tilde{t}_i + \gamma]$ such that $b^n - a^n_m x = 0$ and so $b^n - a^n_m \tilde{t}_i = a^n_m (x - \tilde{t}_i) \leq a_{\max} \gamma < a_{\max} c_{\min} = c_{\min}$; a contradiction with the definition of $c_{\min}$. Because we just showed that the process cannot converge to any point in the interval $[\tilde{t}_i, \tilde{t}_i + \gamma]$, either it reaches the lower or upper boundary of this interval or it runs out of the allocated amount of time $t_k$.

Now, assume that $b^n - a^n_m x < 0$ holds in that interval, i.e. the value of $x_i(t)$ is decreasing in $t$. The amount of time $x_i(t) \in [\tilde{t}_i, \tilde{t}_i + \gamma]$ holds is the greatest if the value of variable $x_i$ starts at $\tilde{t}_i + \gamma$ and ends at $\tilde{t}_i$. Using Proposition 1 we can estimate this time to be at most $\gamma/(b^n - a^n_m) < \frac{1}{2} t_{\min} c_{\min}/c_{\min} = \frac{1}{2} t_{\min}$, which means that during the remaining time equal to $t_k - \frac{1}{2} t_{\min}$, the value of $x_i(t)$ stays above $\tilde{t}_i + \gamma$.

Finally, if $b^n - a^n_m x > 0$ holds in that interval, then we can again estimate using Proposition 1 the time $x_i(t) \in [\tilde{t}_i, \tilde{t}_i + \gamma]$ can hold to be at most $\gamma/(b^n - a^n_m) < \gamma/(c_{\min} - a^n_m) = 1/(2^{t_{\min}(\sigma)}/c_{\min} - a^n_m) < \frac{1}{2} t_{\min}$. Therefore, again the amount of time the value of $x_i$ stays outside of the interval $[\tilde{t}_i, \tilde{t}_i + \gamma]$ is greater than the amount of time spent inside of it.
B Proof of Lemma

Lemma. Either there is a variable which is critical for all good frequency vectors or there is a good frequency vector in which no variable is critical.

Proof. Recall that $F_i(\tilde{x}, y) := \sum_{m \in M} \tilde{f}(m)(b^m_i - a^m_i)$ and for a frequency vector $\tilde{f}$, variable $x_i$ is called critical if $F_i(\tilde{f}, \tilde{i}) = 0$ or $F_i(\tilde{f}, \tilde{u}_i) = 0$ holds. Finally, a frequency vector $\tilde{f}$ is good if for every variable $x_i$ the following conditions hold (I) $F_i(\tilde{f}, \tilde{i}) \geq 0$, and (II) $F_i(\tilde{f}, \tilde{u}_i) \leq 0$.

Now, let us assume that there is no variable which is critical for all good frequency vectors. If so, for each variable $x_i$ we can find a good frequency vector $\tilde{f}_i$ for which $x_i$ is not critical. But if we consider the frequency vector $\tilde{f} = \frac{1}{|N|} \sum_i \tilde{f}_i$, then no variable can be critical in $\tilde{f}$, because $F_i(\frac{1}{|N|} \sum_i \tilde{f}_i, \tilde{i}) = \frac{1}{|N|} \sum_i F_i(\tilde{f}_i, \tilde{i}) \geq \frac{N-1}{N} \tilde{i} + \frac{1}{|N|} F_i(\tilde{f}_i, \tilde{i}) > \tilde{i}$ and also $F_i(\frac{1}{|N|} \sum_i \tilde{f}_i, \tilde{u}_i) = \frac{1}{|N|} \sum_i F_i(\tilde{f}_i, \tilde{u}_i) \leq \frac{N-1}{N} \tilde{u}_i + \frac{1}{|N|} F_i(\tilde{f}_i, \tilde{u}_i) < \tilde{u}_i$, which also proves that such defined frequency vector $\tilde{f}$ would be good.

C Proof of Theorem

Theorem. For a given MMS $\mathcal{H}$, hyperrectangular safe set $S$ described by two points $\tilde{I}, \tilde{u}$, starting point $\tilde{T}_0 \in S$, checking whether there exists a $S$-safe controller with minimum dwell time $\geq 1$ is PSPACE-hard.

Proof. As mentioned before, the proof is similar to the PSPACE-hardness proof in [2] of the discrete-time reachability in constant-rate MMS, which reduces from the acceptance problem for linear bounded automata (LBAs), so we first formally define LBAs.

An LBA $\mathcal{A}$ is a tuple $(\Sigma, Q, q_0, q_A, \delta)$, where $\Sigma$ is a finite alphabet, $Q$ is a finite set of states, $q_0 \in Q$ is the initial state, and $q_A \in Q$ is a distinguished accepting state, and $\delta \subseteq Q \times \Sigma \times Q \times \Sigma \times \{-1, 0, +1\}$ is the transition relation. We can assume the alphabet $\Sigma$ to be the binary alphabet $\{0, 1\}$. Let us explain the interpretation of the elements of the transition relation. Let $\tau = (q, a, q', b, D) \in \delta$ be a transition. If LBA $\mathcal{A}$ is in state $q \in Q$ and its (read/write) tape head reads character $a$, then it writes character $b$ at the current cell and moves its head in the direction $D$ (left if $D = -1$, right if $D = +1$, and unchanged in $D = 0$), and it changes the state to $q'$. Let $w \in \Sigma^L$ be an input word. Without loss of generality we assume that the LBA uses exactly $L$ tape cells, which hold the whole input word of size $L$ at the very beginning. Hence configuration of the LBA can be written as $(q, p, b_0b_1 \ldots b_{L-1})$ where $q$ is the current state, $p$ is the position of head such that $0 \leq p < L$, and $b_0b_1 \ldots b_{L-1}$ is the current contents of the tape. Notice that such an LBA has only $|Q| \cdot L \cdot |\Sigma|^L$ different configurations and so if LBA does not enter the accepting state $q_A$ for a given input word after that many steps then it never will.

We show a reduction from the acceptance problem for LBAs to the problem of the existence of a safe controller with its minimum dwell time $\geq 1$ for linear-rate MMSs. For a given LBA $\mathcal{A}$ and input word $w = b_0b_1 \ldots b_{L-1}$, we define LBA $\mathcal{H}_d = (M, N, A, B)$ where there is one variable $x_{q, p}$ for each state $q \in Q$ and head position $0 \leq p < L$, one variable $x_i$ for each input cell $b_i$ where $0 \leq i < L$, and one variable $x_{p, \tau}$ for each head position $0 \leq p < L$ and direction $\tau \in \{-1, 0, +1\}$. The safety condition $S$ simply requires that the value of all these variables at all time belong to the interval $[-1, 1]$. Recall that $T_d(\sigma) = \sigma_{18}$.
\[ \sum_{i=1}^{k} t_{i} \] is the total time elapsed up to step \( k \) of the controller \( \sigma \) of \( \mathcal{H}_x \). A configuration \((q', p', b_0 b_1 \ldots b_{L-1})\) of machine \( \mathcal{A} \) at step \( k \) is encoded in the variables of \( \mathcal{H}_x \) in a way that \( x_{q,p}(T_k(\sigma)) > 0 \) iff \( q = q' \) \& \( p = p' \) and we have \( x_{q,p}(T_k(\sigma)) < 0 \) otherwise; and also for all \( 0 \leq i < L \) we have \( x_i(T_k(\sigma)) > 0 \) iff the input cell \( b_i = 1 \) and we have \( x_i(T_k(\sigma)) < 0 \) iff \( b_i = 0 \). There is also a special variable \( x_{A} \) which deals with the case when the input word is accepted and special variable \( x_{F} \) which as the only variable has a different safety interval \([-1, -0.9]\).

We will now construct a gadget used in our reduction. We say that variable \( x_v \) (where \( v \in \{i : 0 \leq i < L\} \cup \{(q, p) : q \in Q, 0 \leq p < L\}\)) in mode \( m \) is of (i) type \( 1 \) if \( b_v^m = 2 \) and \( a_v^m = 1 \), (ii) type \(-1\) if \( b_v^m = -2 \) and \( a_v^m = 1 \), (iii) type \( 0 \) if \( b_v^m = 0 \) and \( a_v^m = 1/(11 \cdot |Q| : L \cdot |\Sigma|) \).

Assume that the current mode is \( m \), time is \( t \) and the value of variable \( x_v \) is safe, i.e. \( x := x_v(t) \in [-1, 1] \). Notice that if the type of this variable is \( 1 \) in mode \( m \), then after time \( t \) its value becomes \( 2 + (x - 2)e^{-t} \) which belongs to the safe set if \( t = 1 \) and \( x \leq 2 - e \approx -0.718 \), but for \( t \geq 1.1 > \ln 3 \) its value is never safe. Similarly for type \(-1\), the new value is safe for \( t = 1 \) and \( x \geq e - 2 \approx 0.718 \), but after time \( t \geq 1.1 > \ln 3 \) it is never safe. Finally, for a variable \( x_v \) of type \( 0 \), we can compute that the relative change in the value of this variable after time \( t \leq 1.1 \cdot |Q| : L \cdot |\Sigma| \) to be \( |e^{-a_v^m t}x - x| / |x| = 1 - e^{-a_v^m t} \leq 1 - e^{-\frac{1}{11}} < 0.1 \), i.e. its value does not change by more than 10%. Moreover, a constant switching between a mode of type \( 1 \) and \(-1\) for some variable while spending in each mode amount of time \( t = 1 \) results in a trajectory that converges to \( \frac{4e^{-1}e^{-2}e^{-2}}{1 - e^{-2}} \approx -0.924 \) on odds steps and \( \approx 0.924 \) on even steps independently of the starting point. Therefore, assuming the initial value of a variable is either \( 1 \) or \(-1\), only modes of type \( 1 \) or \(-1\) are used, and the system is safe at all time, the closest this variable can get to value \( 0 \) is after the first step of length \( 1 \). That value is \( 2 - 3e^{-1} \approx 0.896 \) for a variable that starts at \(-1 \) and \(-0.896 \) for a variable that starts at \( 1 \).

If we now allow that variable to switch to type \( 0 \) as well, then the closest such a process can get to \( 0 \) is to let the just computed value decay towards \( 0 \) by using modes where it has type \( 0 \) only. Its absolute value after time \( t \leq 1.1 \cdot |Q| : L \cdot |\Sigma| \) would be still \( > 0.896 + 0.9 \approx 0.8 \) which is \( > 0.718 \). Assuming the number of timed actions in controller \( \sigma \) does not exceed \( |Q| : L \cdot |\Sigma| \), the minimum dwell time of each action is \( \geq 1 \) and each mode has at least one variable of a nonzero type then we have the following. A variable can remain safe in two consecutive timed actions if and only if its type changes from \( 1 \) to \( 0 \) or \(-1\), from \(-1 \) to \( 0 \) or \( 1 \), and from \( 0 \) to \( 0 \) or to type \(-d\) where \( d \) was the last nonzero type this variable had before \( 0 \). If we interpret the value of a variable above \( 0.718 \) as \( 1 \) and below \(-0.718 \) as \( 0 \), then we can look at each timed action in a mode of type \( 1 \), \(-1\), and \( 0 \) as adding \( 1 \), subtracting \( 1 \), or keeping the value of that binary value constant, respectively.

Now, each transition \( \tau = (a, a', q', b, D) \in \delta \) and head position \( p \) is simulated using two modes \( M_{p, \tau} \) and \( M'_{p, \tau} \). Mode \( M_{p, \tau} \) checks whether the letter in the \( p \)-th cell is \( a \in \Sigma = \{0, 1\} \), while the mode \( M'_{p, \tau} \) changes the content of the \( p \)-th cell to \( b \in \{0, 1\} \), and moves the head to a new position. The rates of various variables in these modes are set in such a manner that a schedule is safe if and only if it respects the transition structure of \( \mathcal{A} \). The main features of the construction are the following.

- In mode \( M_{p, \tau} \) the type of variable \( x_p \) is \(-1\) if \( a = 1 \) and is \( 1 \) otherwise; the type of all other variables is \( 0 \). This mode checks whether the character at head position \( p \) is \( a \).
- In mode \( M'_{p, \tau} \) the type of variable \( x_p \) is \( 0 \) if \( a \neq b \). If \( a = b \) then the type of variable \( x_p \) is \( 1 \). If \( D = -1 \) (\( D = +1 \)) then variable \( x_{q,p} \) has type \(-1\) and \( x_{q',p-1} \) has type \( 1 \) (\( x_{q',p+1} \)
has type 1). While if $D = 0$ then $x_{q,p}$ has type 0 for all $q \in Q$ and $0 \leq p < n$. The type of all other variables is 0.

- To make sure that mode $M_{p,\tau}$ is immediately followed by mode $M'_{p,\tau}$ in every safe run, the type of the variable $x_{p,\tau}$ is 1 in mode $M_{p,\tau}$, and -1 in mode $M'_{p,\tau}$, while it is of type 0 in every other mode.

- The special variable $x_T$ has type 0 in every mode and safe set $[-1, -0.9]$, so the system is safe as long as the decay from its initial value $-1$ is not greater than 10%, which does not happen before $1.1 \cdot |Q| \cdot L \cdot |\Sigma|^{|t|}$ amount of time has elapsed. This guarantees that MMS $\mathcal{H}_{s_d}$ becomes unsafe once we cannot guarantee that it follows the transitions of LBA $\mathcal{A}$ exactly.

- For each head position $0 \leq p < L$ we have two special modes $M_{A,p}$ and $M'_{A,p}$, which deals with the case when LBA $\mathcal{A}$ enters the accepting state $q_A$. In $M_{A,p}$ variable $x_{q_A,p}$ has type -1, variable $x_A$ has type 1, and all other variables $x_v$ have a special safe type $S$ such that $b_v^{M_{A,p}} = -1$ and $a_v^{M_{A,p}} = 1$. On the other hand, in $M'_{A,p}$ variable $x_{q_A,p}$ has type 1, $x_A$ has type -1, and all other variables have type $S$. Notice that once the system enters mode $M_{A,p}$ it can keep switching between modes $M_{A,p}$ and $M'_{A,p}$ forever while being safe. This is because, as it was pointed out before when all timed actions have delay $t = 1$, the values of variables $x_{q_A,p}$ and $x_A$ in the limit keep switching between $\approx -0.924$ and $\approx 0.924$ and all other variables, which have type $S$, will converge from above to $-1$; which belongs to the safe set of all of them.

Note that each of the constructed modes has at least one variable of a nonzero type. Let the initial state $s_0$ of $\mathcal{H}_{s_d}$ be such that $x_i(0) = 1$ if the $i$-th input character $b_i = 1$, $x_{q_0,0}(0) = 1$ (i.e. the initial state of $\mathcal{A}$ is $(q_0, 0)$), and for all other variables we have $x_v(0) = -1$. Notice that if the LBA $\mathcal{A}$ accepts the input word then there exists a $S$-safe controller in MMS $\mathcal{H}_{s_d}$ from the initial state $x_0$, which at some point enters mode $M_{A,p}$ for some head position $p$ and keep switching between $M_{A,p}$ and $M'_{A,p}$. This has to happen before the value of variable $x_T$ becomes too close to 0 to violate its safety condition; until that moment $\mathcal{H}_{s_d}$ models precisely the configurations of LBA $\mathcal{A}$ and its transitions. On the other hand, there is a safe feasible controller only if $M_{A,p}$ is entered at some point, because otherwise variable $x_T$ will violate its safety condition eventually. So if a safe feasible controller for $\mathcal{H}_{s_d}$ with minimum dwell time $\geq 1$ exists, then LBA $\mathcal{A}$ has to enter the accepting state $q_A$ within its first $|Q| \cdot L \cdot |\Sigma|^{|t|}$ timed actions. This shows that $\mathcal{A}$ accepts the input iff $\mathcal{H}_{s_d}$ has a safe feasible controller with minimum dwell time $\geq 1$. $\square$

D Proof of Corollary 1

**Corollary 1** Deciding whether there exists a feasible $S$-safe controller for a given MMS $\mathcal{H}$ with a mode order specification graph $G$, initial mode $m_0$, a hyperrectangular safe set $S$ given by two points $\bar{1}$ and $\bar{\pi}$ and an initial point $\bar{1} < \bar{x}_0 < \bar{\pi}$ can be done in polynomial time.

**Proof.** Recall that a controller $(\langle m_1, t_1 \rangle, \langle m_2, t_2 \rangle, \ldots)$ respects the mode order specification graph $G$ with initial mode $m_0$ if for all $i \geq 1$ we have $(m_i, m_{i+1}) \in G$ and $m_1 = m_0$. Notice that the system $\mathcal{H}$ under any feasible $S$-safe controller will eventually end up in one of the strongly connected components (SCC) of the graph $G$ reachable from $m_0$, because the controller is non-Zeno and each timed action takes only a finite amount of time. Also note
that the safe controller returned by Algorithm 1 returns a periodic controller which cycles over all the modes given to it in exactly the same order as they were passed. Therefore, we can make sure the controller returned satisfies the mode order specification $G$ by passing the mode sequence in a particular order. For an SCC $C$ of $G$ consisting of modes $C = \langle m'_1, m'_2, \ldots, m'_k \rangle$ that sequence of modes, denoted by $\rho_C$, is as follows: it starts at $m'_1$, then follows any path of modes in $G$ to $m'_2$, ..., then any path of modes to $m'_k$, and finally any path of modes to $m'_1$; all these paths exist because $C$ is an SCC. The sequence of modes $\rho_C$ can repeat some modes, but it satisfies the mode order specification graph $G$, each mode of $C$ occurs at least once and no mode outside $C$ occurs along $\rho_C$. It is quite easy to see that there is a feasible $S$-safe controller for an initial state in the interior of the safe set for the set of modes $C$ iff there is one for the sequence of modes $\rho_C$.

Now, for each SCC $C$ of $G$ reachable from the initial mode $m_0$ we check using Algorithm 1 whether there is a feasible $S$-safe controller for the mode sequence $\rho_C$ and initial point $\bar{x}_0$. If there is no such SCC then there is no feasible $S$-safe controller which respects the mode order specification from $\bar{x}_0$ either, because while using such a controller the system $H$ has to eventually repeat only modes from a single SCC of $G$. On the other hand, if there is such an SCC $C$, then we construct a feasible $S$-safe controller from $\bar{x}_0$ as follows. First, we find any path in $G$ from $m_0$ to the very first mode in the mode sequence $\rho_C$. We create a finite timed action sequence based on this path where the time delay of each mode is set to such a small value that when starting at $\bar{x}_0$ the system will still remain within the safe set $S$ at the very end of it. Such a value always exists when the initial point of $H$ is in the interior of the safe set. To be precise, it suffices to set it to $\min_{i \in I} \left( \frac{\min(\tau_0(i) - l_i, u_i - \tau_0(i))}{\max(\|b_i - a_i\|, |\tau_0(i)|)} \right)$. Let the point reached at the end of this finite timed sequence be $\bar{x}_1 \in S$ and, because the coordinates of that point are likely to be irrational, let $\bar{x}_l$ and $\bar{x}_u$ be any two points with rational coordinates such that $\bar{l} < \bar{x}_l \leq \bar{x}_1 \leq \bar{x}_u < \bar{u}$ holds. In other words, $\bar{x}_l$ and $\bar{x}_u$ are simply some polynomial size lower and upper bounds on the coordinates of the point $\bar{x}_1$. Notice that the feasible controller for the mode sequence $\rho_C$ that we found earlier may not be safe when the system starts at $\bar{x}_1$ instead of $\bar{x}_0$, because the value of $s$ may need to be smaller for the system to remain safe. The new value of $s$ should be the minimum of the value of $s$ for the mode sequence $\rho_C$ when the initial point is $\bar{x}_1$ and when it is $\bar{x}_u$. Finally, once we combine the finite timed action sequence, which starts at the mode $m_0$ and $\bar{x}_0$, with the feasible controller for the mode sequence $\rho_C$, which is safe for any initial point between $\bar{x}_l$ and $\bar{x}_u$, we will get a feasible $S$-safe controller that respects the mode order specification graph $G$. 

Notice that if one would like to extend the model and allow the system to remain in the same mode forever, instead of forcing it to constantly switch between modes, it suffices to add in $G$ an edge from each mode to itself.
### E  Simpler Algorithm for Finding an Optimal Controller

**Algorithm 3:** Finds an optimal $S$-safe feasible controller from a given $x_0 \in S$.

**Input:** A priced MMS $\mathcal{H}$, two points $\bar{l}$ and $\bar{u}$ that define a hyperrectangle $S = \{x : \bar{l} \leq x \leq \bar{u}\}$, and an initial point $x_0 \in S$ such that $\bar{l} < x_0 < \bar{u}$, and constants $\mu_{\text{avg}}$ and $\mu_{\text{peak}}$ which define the cost of a controller.

**Output:** NO if no $S$-safe feasible controller exists from $x_0$, and an periodic such controller $\sigma$ for which $\mu_{\text{peak}} \text{PeakCost}(\sigma) + \mu_{\text{avg}} \text{AvgCost}(\sigma)$ is minimal, otherwise.

1. Modify Algorithm 1 by adding the objective function $\text{Minimise } \sum_{m \in M'} \vec{f}_m \pi(m)$ to the linear program at line 11. Let $\text{OptAvgCost}(M')$ be the value of this objective when Algorithm 1 is called for the set of modes $M'$.
2. Let $P = \{\pi(m) : m \in M\}$ be the set of all different costs of modes of $\mathcal{H}$. (Notice that only these costs can be potential peak costs.) For a given $p$ let $M_{\leq p}$ denote the set of modes with cost at most $p$.
3. Iterate over $p \in P$ and find the one with the smallest value of $\mu_{\text{peak}} p + \mu_{\text{avg}} \text{OptAvgCost}(M_{\leq p})$ and denote it by $p^*$. 
4. **return** the periodic controller returned by the modified version of Algorithm 1 called for the set of modes $M_{\leq p^*}$.