Forcing and anti-forcing polynomials of a polyomino graph *

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Abstract

The forcing number of a perfect matching $M$ in a graph $G$ is the smallest number of edges inside $M$ that can not be contained in other perfect matchings. The anti-forcing number of $M$ is the smallest number of edges outside $M$ whose removal results in a subgraph with a single perfect matching, that is $M$. Recently, in order to investigate the distributions of forcing numbers and anti-forcing numbers, the forcing polynomial and anti-forcing polynomial were proposed, respectively. In this work, the forcing and anti-forcing polynomials of a polyomino graph are obtained. As consequences, the forcing and anti-forcing spectra of this polyomino graph are determined, and the asymptotic behaviors on the degree of freedom and the sum of all anti-forcing numbers are revealed, respectively.

Key words: Perfect matching; Forcing polynomial; Anti-forcing polynomial; Polyomino graph

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1 Introduction

A perfect matching in a graph is a set of independent edges that saturates each vertex. The perfect matching arose in many fields, such as a dimer covering in statistical physics [18], a Kekulé structure in quantum chemistry [7], and a solution for optimal assignment [24], etc. In 1985–1987 Klein and Randić [19,28] observed that a few number of fixed double bonds of a Kekulé structure can determine the whole Kekulé structure, and defined the least number of fixed double bonds as the innate degree of freedom of this Kekulé structure. This concept was extended to a perfect matching by Harary et al. [14] in 1991, and renamed as forcing number. From the opposite point of view, Vukičević and Trinajstić [36,37] proposed the anti-forcing number of a graph, that was extended to a single perfect matching in 2015 [22]. Generally, computing the forcing and anti-forcing numbers of a perfect matching are both NP-complete [1,10]. Recently, the forcing polynomial [46] and anti-forcing polynomial [15] were proposed to investigate the distributions of forcing and anti-forcing numbers of all perfect matchings, respectively. In this paper, we will calculate the forcing polynomial and anti-forcing polynomial of a polyomino graph, respectively.

Suppose graph $G$ has a perfect matching $M$. A subset $S \subseteq M$ is called a forcing set of $M$ if other perfect matchings in $G$ do not contain $S$. That is, $M$ is determined by an edge set inside of $M$. Thus the forcing number of $M$ is the size of a minimum forcing set, denoted by $f(G,M)$. The forcing spectrum $\text{Spec}_f(G)$ of $G$ is the collection of forcing numbers of all perfect matchings, denoted by $\text{Spec}_f(G)$. The minimum and maximum values in $\text{Spec}_f(G)$ are called the minimum and maximum forcing numbers of $G$, denoted by $f(G)$ and $F(G)$, respectively. Finding the minimum forcing number was proved to be a NP-complete problem [2], but the special structure of graph helps to compute it. The minimum forcing numbers of some special graphs were obtained, such as $2n \times 2n$ grids [27], hypercubes [13,30], $2m \times 2n$ toric grids [30], toric hexagonal systems [38], etc. On the other hand, the maximum forcing number was considered, such as cylindrical grids [2,17], $2m \times 2n$ toric grids [20], 4-8 lattices [17], etc. For a hexagonal system, Xu et al. [39] confirmed that the maximum forcing number is equal to the Clar number, that is true for polyomino graphs [17] and (4,6)-fullerenes [32] as well. The forcing spectra of some graphs were discussed, such as hypercubes [1], grid graphs [2], tubular boron-nitrogen fullerene graphs [16], hexagonal systems [42], etc. Especially, Randić, Vukičević and Gutman [29,34,35] calculated the forcing spectra of fullerenes $C_{60}$, $C_{70}$ and $C_{72}$, showing that the Kekulé structure contributing more to the stability of molecule often has larger innate degree of freedom. There are few results on forcing polynomial, Zhao and Zhang [46,49,51] considered some special hexagonal systems, and $2 \times n$ and $3 \times 2n$ grids. For more results, see [1,3,21,31,41,43,52,53].

Let $E(G)$ denote the edge set of a graph $G$, and let $M$ be a perfect matching of $G$. For a subset $S' \subseteq E(G) \setminus M$, if $G - S'$ (the graph obtained by removing all the edges of $S'$ from
$G$ has a single perfect matching, that is $M$, then $S'$ is named an anti-forcing set of $M$. In other words, $M$ is fixed by an edge set outside of $M$. The size of a minimum anti-forcing set is named the anti-forcing number of $M$, denoted by $af(G,M)$. The collection of anti-forcing numbers of all perfect matchings is called the anti-forcing spectrum of $G$, denoted by $Spec_{af}(G)$. The minimum and maximum values in $Spec_{af}(G)$ are called minimum and maximum anti-forcing numbers of $G$, denoted by $af(G)$ and $Af(G)$, respectively. Actually, as early in 1997 Li [23] had described the extremal hexagonal systems of minimal anti-forcing number 1. The minimum anti-forcing numbers of some chemical graphs were discussed, such as hexagonal systems [8, 9, 36, 37], catacondensed phenylene [48], fullerene graphs [40], etc. Recently, Lei et al. [22] proved that the maximum anti-forcing number is equal to Fries number for a hexagonal system, that is true for (4,6)-fullerenes [32] as well. Furthermore, Deng and Zhang [12] showed that the cyclomatic number is an attainable upper bound on the maximum anti-forcing number, and characterized the extremal graphs. Afterwards, Shi and Zhang [33] gave another achievable upper bound, and got the maximum anti-forcing numbers of hypercube and its two variants. The anti-forcing spectra of some hexagonal systems were confirmed to be continuous, such as monotonic constructable hexagonal systems [10], catacondensed hexagonal systems [11], etc. For anti-forcing polynomial, a few number of graphs are considered, such as extremal hexagonal systems with forcing number 1, and $2 \times n$ and $3 \times 2n$ grids.

In section 2, as preparations, some basic results on forcing and anti-forcing numbers are introduced, and some useful properties of the polyomino graph $G_n$ (see Fig. 1(a)) are discussed. In section 3, we get the forcing polynomial of $G_n$, as corollaries, $Spec_f(G_n) = [n,2n]$, and the asymptotic behavior of degree of freedom of $G_n$ is revealed. In section 4, we obtain the anti-forcing polynomial of $G_n$, as consequences, $Spec_{af}(G_n) = [n,3n]$, and the asymptotic behavior of the sum of all anti-forcing numbers is showed.

2 Preliminaries

A polyomino graph is a 2-edge-connected subgraph of the infinite plane grid such that the periphery of every interior face is a square. The polyomino graph is an important plane bipartite graphs, which is studied in many combinatorial problems [3, 6, 18, 26, 44]. The polyomino graph $G_n$ as shown in Fig. 1(a) consisting of $4n$ squares, that is a subgraph of $4 \times (2n+1)$ grids, and the vertices of $G_n$ are labeled by $u_0, v_0, u_i, v_i, w_i, z_i$, $i = 1, 2, \ldots, 2n$. The polyomino graph $H_n$ is obtained by removing the leftmost square $s_{1,1}$ from $G_n$ (see Fig. 1(b)), that is a subgraph of $4 \times 2n$ grids.

The number of perfect matchings of a graph $G$ is denoted by $\Phi(G)$. Let $G_0$ denote the null graph. Then $\Phi(G_0) = 1$. For a perfect matching $M$, a cycle $C$ is called an $M$-alternating
cycle if the edges of \( C \) appear alternately in \( M \) and \( E(C) \setminus M \). Let \( W \) be a set of vertices, and let \( G \oplus W \) denote the subgraph generated by deleting all the vertices of \( W \) and their incident edges from \( G \).

**Lemma 2.1.** For \( n \geq 2 \),

\[
\Phi(G_n) = 6\Phi(G_{n-1}) - 4\Phi(G_{n-2}),
\]

where \( \Phi(G_0) = 1 \) and \( \Phi(G_1) = 6 \).

**Proof.** Let \( M' \) be a perfect matching of \( H_n \). If \( u_1v_1 \in M' \), then \( w_1w_2, z_1z_2 \in M' \). Thus the restriction of \( M' \) on \( G_{n-1} = H_n \oplus \{u_1, v_1, w_1, w_2, z_1, z_2\} \) is a perfect matching of \( G_{n-1} \). If \( u_1v_1 \notin M' \), then \( s_{1,2} \) and \( s_{1,4} \) both are \( M' \)-alternating squares, and the restriction of \( M' \) on \( H_{n-1} = H_n \oplus \{w_1, w_2, u_1, u_2, v_1, v_2, z_1, z_2\} \) is a perfect matching of \( H_{n-1} \) (see Fig. 1(b)). Note that there are four different cases such that \( s_{1,2} \) and \( s_{1,4} \) both are \( M' \)-alternating. Therefore

\[
\Phi(H_n) = \Phi(G_{n-1}) + 4\Phi(H_{n-1}).
\]

Let \( M \) be a perfect matching of \( G_n \). If \( u_0v_0 \in M \), then the restriction of \( M \) on \( H_n = G_n \oplus \{u_0, v_0\} \) is a perfect matching of \( H_n \) (see Fig. 1(a)). If \( u_0v_0 \notin M \), then \( u_0u_1, v_0v_1, w_1w_2, z_1z_2 \in M \), and the restriction of \( M \) on \( G_{n-1} = G_n \oplus \{u_0, u_1, v_0, v_1, w_1, w_2, z_1, z_2\} \) is a perfect matching of \( G_{n-1} \). So \( \Phi(G_n) = \Phi(H_n) + \Phi(G_{n-1}) \), we have \( \Phi(H_n) = \Phi(G_n) - \Phi(G_{n-1}) \) and \( \Phi(G_{n-1}) = \Phi(G_{n-1}) - \Phi(G_{n-2}) \), substituting them into Eq. (2.2), then Eq. (2.1) is obtained. \( \square \)
Theorem 2.2.
\[ \Phi(G_n) = \frac{5 - 3\sqrt{5}}{10}(3 - \sqrt{5})^n + \frac{5 + 3\sqrt{5}}{10}(3 + \sqrt{5})^n. \] (2.3)

Proof. According to Lemma 2.1, the characteristic equation of recurrence formula (2.1) is \( x^2 - 6x + 4 = 0 \), with roots \( 3 \pm \sqrt{5} \). Suppose the general solution of formula (2.1) is \( \Phi(G_n) = \lambda_0(3 - \sqrt{5})^n + \lambda_1(3 + \sqrt{5})^n \), by the initial conditions \( \Phi(G_2) = 32 \) and \( \Phi(G_3) = 168 \), we can obtain \( \lambda_0 = \frac{5 - 3\sqrt{5}}{10} \) and \( \lambda_1 = \frac{5 + 3\sqrt{5}}{10} \), so Eq. (2.3) holds for \( n \geq 2 \). We can check that the equation also holds for \( n = 0, 1 \), so the proof is completed.

Let \( M \) be a perfect matching in graph \( G \), and \( c(M) \) denote the largest number of disjoint \( M \)-alternating cycles. Pachter and Kim [27] showed the following result by means of the minimax theorem on feedback set [25].

Theorem 2.3 [27]. Let \( M \) be a perfect matching of a plane bipartite graph \( G \). Then \( f(G, M) = c(M) \).

An \( M \)-resonant set with respect to a perfect matching \( M \) of a polyomino graph \( G \) is a collection of independent \( M \)-alternating squares. Let \( s(M) \) denote the size of a maximal \( M \)-resonant set. Then the maximum \( s(M) \) over all perfect matchings is called the Clar number of \( G \), denoted by \( cl(G) \).

Theorem 2.4 [47]. Suppose \( G \) is a polyomino graph that has a perfect matching. Then \( F(G) = cl(G) \).

Lemma 2.5 [45]. Let \( C \) be an \( M \)-alternating cycle with respect to a perfect matching \( M \) in a planar bipartite graph \( G \). Then there exists a face in the interior of \( C \) whose periphery is an \( M \)-alternating cycle.

Suppose \( M \) is a perfect matching of a polyomino graph, and \( \mathcal{A} \) is a collection of \( M \)-alternating cycles. Let \( I(\mathcal{A}) = \sum_{C \in \mathcal{A}} I(C) \), where \( I(C) \) is the number of squares in the interior of cycle \( C \).

Lemma 2.6. Let \( M \) be a perfect matching of \( G_n \). Then \( f(G_n, M) = s(M) \).

Proof. Let \( \mathcal{A} \) be a maximal set of disjoint \( M \)-alternating cycles such that \( I(\mathcal{A}) \) is as small as possible. Then \( \mathcal{A} \) is an \( M \)-resonant set. Otherwise, \( \mathcal{A} \) must contain a non-square cycle \( C \). By Lemma 2.5, there is an \( M \)-alternating square \( s \) in the interior of \( C \). Since each vertex is on the periphery of \( G_n \), \( \mathcal{A}' = (\mathcal{A} \setminus \{C\}) \cup \{s\} \) is another maximal set of disjoint \( M \)-alternating cycles, however, \( I(\mathcal{A}') < I(\mathcal{A}) \), a contradiction. Therefore \( |\mathcal{A}| \leq s(M) \). On the other hand, by Theorem 2.3 \( f(G_n, M) = |\mathcal{A}| \geq s(M) \). So \( f(G_n, M) = s(M) \).
Suppose that $M$ is a perfect matching in a graph $G$. A compatible $M$-alternating set $\mathcal{A}'$ is a collection of $M$-alternating cycles such that any two cycles of $\mathcal{A}'$ are disjoint, or overlap only on edges in $M$. Let $c'(M)$ denote the size of a maximal compatible $M$-alternating set.

**Theorem 2.7** [22]. Suppose that $M$ is a perfect matching in a planar bipartite graph $G$. Then $af(G, M) = c'(M)$.

For a planar bipartite graph, two cycles $C_1$ and $C_2$ in a compatible $M$-alternating set $\mathcal{A}'$ are called crossing if $C_1$ enters the interior of $C_2$ from the exterior of $C_2$. If any two cycles of $\mathcal{A}'$ are no crossing, then $\mathcal{A}'$ is called non-crossing.

**Lemma 2.8** [12,22]. Suppose $G$ is a planar bipartite graph that has a perfect matching $M$. Then there is a non-crossing compatible $M$-alternating set $\mathcal{A}'$ such that $af(G, M) = |\mathcal{A}'|$.}

![Fig. 2](chart.png)

Fig. 2. The illustrations of Lemma 2.9 where bold edges belong to a perfect matching.

The anti-forcing number of a perfect matching $M$ in $G_n$ is related with the number of substructures $L_k$ and $W_r$ as shown in Fig. 2. $L_k$ is a straight chain with $k$ squares, $1 \leq k \leq 2n - 1$ and $k$ is odd. The restriction of $M$ on $L_k$ is a perfect matching of $L_k$, and the periphery of $L_k$ is an $M$-alternating cycle which is compatible with $\frac{k-1}{2}$ $M$-alternating squares in its interior (see Fig. 2(a)). $W_r$ is isomorphic to $H_r$, the restriction of $M$ on $W_r$ is a perfect matching of $W_r$, and $W_r$ contains the substructure $L_{2r-1}$ such that the peripheries of $W_r$ and $L_{2r-1}$ are non-crossing compatible $M$-alternating cycles, and both of them are compatible with $r - 1$ $M$-alternating squares in their interiors (see Fig. 2(b)).

**Lemma 2.9.** Let $M$ be a perfect matching in $G_n$, and $\mathcal{A}'$ be a maximal non-crossing compatible $M$-alternating set such that $I(\mathcal{A}')$ is as small as possible. Then each $M$-alternating square must belong to $\mathcal{A}'$, and any non-square cycle of $\mathcal{A}'$ either is the periphery of an $L_k$ or the periphery of a $W_r$. 

6
Proof. Suppose $s$ is an $M$-alternating square that is not in $\mathcal{A}'$. Then $s$ is not compatible with a cycle $C \in \mathcal{A}'$. Since any two $M$-alternating squares are compatible with each other, $C$ is a non-square cycle. Therefore $\mathcal{A}'' = (\mathcal{A}' \setminus \{C\}) \cup \{s\}$ is another maximal non-crossing compatible $M$-alternating set, but $I(\mathcal{A}'') < I(\mathcal{A}')$, a contradiction. So each $M$-alternating square must belong to $\mathcal{A}'$.

Let $C \in \mathcal{A}'$ be a non-square cycle, $M(C) = M \cap E(C)$. Note that none of these vertical edges $w_iu_i$ and $v_jz_j$ for $1 \leq i, j \leq 2n$ belong to $M(C)$, otherwise $s_{1,2}$ or $s_{1,4}$ will be an $M$-alternating square that is not compatible with $C$ (see Fig. 1(a)). Since $C$ is $M$-alternating, $M(C)$ contains just two vertical edges $u_iv_i$ and $u_jv_j(0 \leq i < j \leq 2n)$ and $i + j$ is odd. If $i$ is even, then $C = u_iu_{i+1} \cdots u_jv_j \cdots v_{i}u_{i}$ is the periphery of an $L_{j-i}$. If $i$ is odd, then $C$ is the periphery of an $L_{j-i}$ or $C = u_iw_iu_{i+1}u_{i+1}u_{i+2}w_{i+2}w_{i+3}u_{i+3} \cdots \cdots u_j-1w_j-1w_jv_jz_jz_{j-1}v_{j-1}v_{j-2}z_{j-2}z_{j-3}v_{j-3} \cdots$ is the periphery of a $W_{[\frac{j-i}{2}]}$. □

Remark 1. The same as above, we can show that Lemmas 2.6 and 2.9 also hold for $H_n$.

3 Forcing polynomials

The forcing polynomial [46] of a graph $G$ is defined as follows:

$$F(G, x) = \sum_{M \in \mathcal{M}(G)} x^{f(G,M)},$$

(3.1)

where $\mathcal{M}(G)$ is the collection of all perfect matchings of $G$.

By Eq. (3.1), another expression is immediately obtained:

$$F(G, x) = \sum_{i=f(G)}^{F(G)} w(G,i)x^i,$$

where $w(G,i)$ is the number of perfect matchings with the forcing number $i$.

The forcing polynomial can count the number of perfect matchings with the same forcing number, in other words, the distribution of all forcing numbers is revealed. Moreover, $\Phi(G) = F(G,1)$, and the degree of freedom of $G$, $IDF(G) = \frac{d}{dx}F(G, x)|_{x=1}$, i.e. the sum of forcing numbers of all perfect matchings, which can estimate the resonance energy of a molecule [28]. Obviously, if $G$ is a null graph or a graph has a unique perfect matching, then $F(G, x) = 1$.

The following theorem give a recurrence relation for forcing polynomial of $G_n$.

Theorem 3.1. For $n \geq 3$,

$$F(G_n, x) = (4x^2 + 3x)F(G_{n-1}, x) - (8x^3 + 2x^2)F(G_{n-2}, x) + 4x^3F(G_{n-3}, x),$$

(3.2)

where $F(G_0, x) = 1, F(G_1, x) = 4x^2 + 2x, F(G_2, x) = 16x^4 + 12x^3 + 4x^2$.  

7
Proof. We divide $\mathcal{M}(G_n)$ in two subsets: $\mathcal{M}_1 = \{ M \in \mathcal{M}(G_n) | u_0v_0 \notin M \}$ and $\mathcal{M}_2 = \{ M \in \mathcal{M}(G_n) | u_0v_0 \in M \}$. If $M \in \mathcal{M}_1$, then $\{ u_0u_1, v_0v_1, w_1w_2, z_1z_2 \} \subseteq M$ and $s_{1,1}$ is an $M$-alternating square. Note that the restriction $M'$ of $M$ on $G_{n-1} = G_n \ominus \{ u_0, u_1, v_0, v_1, w_1, w_2, z_1, z_2 \}$ is a perfect matching of $G_{n-1}$. Let $\mathcal{A}'$ be a maximum $M'$-resonance set of $G_{n-1}$. Then $\mathcal{A}' \cup \{ s_{1,1} \}$ is a maximum $M$-resonance set of $G_n$ (see Fig. 1(a)). By Lemma 2.6 $f(G_n, M) = 1 + f(G_{n-1}, M')$. According to Eq. (3.1),

$$\sum_{M \in \mathcal{M}_1} x^{f(G_n, M)} = \sum_{M' \in \mathcal{M}(G_{n-1})} x^1 + f(G_{n-1}, M') = x \sum_{M' \in \mathcal{M}(G_{n-1})} x^{f(G_{n-1}, M')} = x F(G_{n-1}, x). \tag{3.3}$$

Now suppose $M \in \mathcal{M}_2$, we divide $\mathcal{M}_2$ in two subsets: $\mathcal{M}_{2,1} = \{ M \in \mathcal{M}_2 | u_1v_1 \in M \}$ and $\mathcal{M}_{2,2} = \{ M \in \mathcal{M}_2 | u_1v_1 \notin M \}$. If $M \in \mathcal{M}_{2,1}$, then the restriction $M''$ of $M$ on $G_{n-1} = G_n \ominus \{ u_0, u_1, v_0, v_1, w_1, w_2, z_1, z_2 \}$ is a perfect matching of $G_{n-1}$. Similarly,

$$\sum_{M \in \mathcal{M}_{2,1}} x^{f(G_n, M)} = \sum_{M'' \in \mathcal{M}(G_{n-1})} x^1 + f(G_{n-1}, M'') = x F(G_{n-1}, x). \tag{3.4}$$

If $M \in \mathcal{M}_{2,2}$, then $\{ s_{1,2}, s_{1,4} \}$ is an $M$-resonance set and the restriction $M'''$ of $M$ on $H_{n-1} = G_n \ominus \{ u_0, u_1, v_0, v_1, w_1, w_2, z_1, z_2, u_2, v_2 \}$ is a perfect matching of $H_{n-1}$ (see Fig. 1). Let $\mathcal{A}'''$ be a maximum $M'''$-resonance set of $G_n$. By Lemma 2.6 and Remark 1, $af(G_n, M) = 2 + |\mathcal{A}'''| = 2 + f(H_{n-1}, M''')$. Note that there are four independent subcases such that $\{ s_{1,2}, s_{1,4} \}$ is an $M$-resonance set, so

$$\sum_{M \in \mathcal{M}_{2,2}} x^{af(G_n, M)} = 4 \sum_{M''' \in \mathcal{M}(H_{n-1})} x^{2 + f(H_{n-1}, M''')} = 4x^2 F(H_{n-1}, x). \tag{3.5}$$

According to Eqs. (3.3), (3.4) and (3.5),

$$F(G_n, x) = \sum_{M \in \mathcal{M}(G_n)} x^{f(G_n, M)} = \sum_{M \in \mathcal{M}_1} x^{f(G_n, M)} + \sum_{M \in \mathcal{M}_2} x^{f(G_n, M)} = \sum_{M \in \mathcal{M}_1} x^{f(G_n, M)} + \sum_{M \in \mathcal{M}_{2,1}} x^{f(G_n, M)} + \sum_{M \in \mathcal{M}_{2,2}} x^{f(G_n, M)} = 2xF(G_{n-1}, x) + 4x^2 F(H_{n-1}, x). \tag{3.6}$$

On the other hand, we divide $\mathcal{M}(H_n)$ in two subsets: $\mathcal{N}_1 = \{ M \in \mathcal{M}(H_n) | u_1v_1 \in M \}$ and $\mathcal{N}_2 = \{ M \in \mathcal{M}(H_n) | u_1v_1 \notin M \}$. Further, $\mathcal{N}_1$ can be divided into two subsets $\mathcal{N}_{1,1} = \{ M \in$
\( \mathcal{N}_1|u_2v_2 \in M \) and \( \mathcal{N}_{1,2} = \{ M \in \mathcal{N}_1|u_2v_2 \notin M \} \). Similarly, \( \sum_{M \in \mathcal{N}_{1,1}} x^{f(H_n,M)} = xF(H_{n-1}, x) \), \( \sum_{M \in \mathcal{N}_{1,2}} x^{f(H_n,M)} = xF(G_{n-2}, x) \), and \( \sum_{M \in \mathcal{N}_2} x^{f(H_n,M)} = 4x^2F(H_{n-1}, x) \). Therefore

\[
F(H_n, x) = (4x^2 + x)F(H_{n-1}, x) + xF(G_{n-2}, x). \tag{3.7}
\]

By Eq. (3.7) minus Eq. (3.6), we can obtain

\[
F(H_n, x) - xF(H_{n-1}, x) = F(G_n, x) - 2xF(G_{n-1}, x) + xF(G_{n-2}, x),
\]

thus

\[
4xF(H_n, x) - 4x^2F(H_{n-1}, x) = 4xF(G_n, x) - 8x^2F(G_{n-1}, x) + 4x^2F(G_{n-2}, x). \tag{3.8}
\]

According to Eq. (3.6), \( 4x^2F(H_{n-1}, x) = F(G_n, x) - 2xF(G_{n-1}, x) \), substituting it into Eq. (3.8), we have

\[
4xF(H_n, x) = (4x + 1)F(G_n, x) - (8x^2 + 2x)F(G_{n-1}, x) + 4x^2F(G_{n-2}, x),
\]

so

\[
4xF(H_{n-1}, x) = (4x + 1)F(G_{n-1}, x) - (8x^2 + 2x)F(G_{n-2}, x) + 4x^2F(G_{n-3}, x),
\]

substituting it into Eq. (3.6), then Eq. (3.2) is obtained. \( \square \)

By Theorem 3.1, we can deduce an explicit expression as follow.

**Theorem 3.2.** \( F(G_n, x) = \)

\[
2^{2n}\sum_{m=0}^{2n} x^n \sum_{i=\max\left\{ \left\lfloor \frac{n}{4} \right\rfloor, m \right\} } \sum_{j=\max\left\{ \left\lfloor \frac{n}{2} \right\rfloor, m+n-2i \right\} } \sum_{k=m-(i+2j-n)}^{n-i} (-1)^{i+2j-n} 2^{n+2m-i}
\]

\[
\times i^{3-j-k} \binom{i}{j} \binom{j}{n-i-j} \binom{i-j}{k} \binom{i+2j-n}{m-k} - \sum_{i=\max\left\{ \left\lfloor \frac{n-i}{4} \right\rfloor, m \right\} } \sum_{j=\max\left\{ \left\lfloor \frac{n-i}{2} \right\rfloor, m+n-2i-1 \right\} } \sum_{k=m-(i+2j-n+1)}^{n-i-1} (-1)^{i+2j-n+1} 2^{n+2m-i+1}
\]

\[
\times i^{3-j-k} \binom{i}{j} \binom{j}{n-i-j-1} \binom{i-j}{k} \binom{i+2j-n+1}{m-k} \bigg) x^m. \tag{3.9}
\]

**Proof.** For convenience, let \( F_n := F(G_n, x) \). Then the generating function of the sequence \( \{ F_n \}_{n=0}^\infty \) is

\[
G(t) = \sum_{n=0}^{\infty} F_n t^n = F_0 t^0 + F_1 t^1 + F_2 t^2 + \sum_{n=3}^{\infty} F_n t^n.
\]

9
By Theorem 3.1

\[
G(t) = F_0 t^0 + F_1 t^1 + F_2 t^2 + (4x^2 + 3x) \sum_{n=3}^{\infty} F_{n-1} t^n - (8x^3 + 2x^2) \sum_{n=3}^{\infty} F_{n-2} t^n
\]

\[+4x^3 \sum_{n=3}^{\infty} F_{n-3} t^n\]

\[= F_0 t^0 + F_1 t^1 + F_2 t^2 + (4x^2 + 3x) t (\sum_{n=0}^{\infty} F_n t^n - F_0 t^0 - F_1 t^1)\]

\[-(8x^3 + 2x^2) t^2 (\sum_{n=0}^{\infty} F_n t^n - F_0 t^0) + 4x^3 t^3 (\sum_{n=0}^{\infty} F_n t^n)\]

\[= 1 - xt + ((4x^2 + 3x) t - (8x^3 + 2x^2) t^2 + 4x^3 t^3) G(t).\]

So

\[
G(t) = \frac{1 - xt}{1 - ((4x^2 + 3x) t - (8x^3 + 2x^2) t^2 + 4x^3 t^3)}\]

\[= (1 - xt) \sum_{i=0}^{\infty} ((4x^2 + 3x) t - (8x^3 + 2x^2) t^2 + 4x^3 t^3)^i\]

by the binomial theorem

\[
\sum_{i=0}^{\infty} ((4x^2 + 3x) t - (8x^3 + 2x^2) t^2 + 4x^3 t^3)^i
\]

\[= \sum_{i=0}^{\infty} x^i t^i \sum_{j=0}^{i} \binom{i}{j} (4x^2 t^2 - (8x^3 + 2x^2) t)(4x + 3)^{i-j}\]

\[= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} (-1)^j \binom{i}{k} \binom{j}{k} (4x + 3)^{i-j} (4x + 1)^{j-k} x^{i+j+k} t^i+j+k\]

\[= \sum_{n=0}^{\infty} \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} \sum_{j=\lceil \frac{n-i}{2} \rceil}^{n-i} (-1)^{i+2j-n} 2^{n-i} \binom{i}{j} \binom{j}{n-i-j} (4x + 3)^{i-j} (4x + 1)^{i+2j-n} x^n t^n.\]

Similarly,

\[
x t \sum_{i=0}^{\infty} ((4x^2 + 3x) t - (8x^3 + 2x^2) t^2 + 4x^3 t^3)^i
\]

\[= \sum_{n=1}^{\infty} \sum_{i=\lceil \frac{n+1}{2} \rceil}^{n-1} \sum_{j=\lceil \frac{n-i+1}{2} \rceil}^{n-i-1} (-1)^{i+2j-n+1} 2^{n-i-1} \binom{i}{j} \binom{j}{n-i-j-1} (4x + 3)^{i-j}\]

\[\cdot (4x + 1)^{i+2j-n+1} x^n t^n.\]
Therefore

\[(1 - xt) \sum_{i=0}^{\infty} ((4x^2 + 3x)t - (8x^3 + 2x^2)t^2 + 4x^3t^3)^i \]

\[= 1 + \sum_{n=1}^{\infty} x^n \left( \sum_{i=\lfloor \frac{n}{2} \rfloor}^{n} \sum_{j=\lfloor \frac{n-i}{2} \rfloor}^{n-i} (-1)^{i+2j-n} 2^{n-i} \binom{i}{j} \binom{j}{n-i-j} (4x + 3)^{i-j} \cdot (4x + 1)^{i+2j-n} \right. \]

\[\cdot \left( \sum_{i=\lfloor \frac{n-1}{3} \rfloor}^{n-1} \sum_{j=\lfloor \frac{n-i-1}{2} \rfloor}^{n-i-1} (-1)^{i+2j-n+1} 2^{n-i-1} \binom{i}{j} \binom{j}{n-i-j-1} (4x + 3)^{i-j} \cdot (4x + 1)^{i+2j-n+1} \right)^{t^n} \]

\[= 1 + \sum_{n=1}^{\infty} F_n t^n, \]

which implies that

\[F_n = x^n \left( \sum_{i=\lfloor \frac{n}{2} \rfloor}^{n} \sum_{j=\lfloor \frac{n-i}{2} \rfloor}^{n-i} (-1)^{i+2j-n} 2^{n-i} \binom{i}{j} \binom{j}{n-i-j} (4x + 3)^{i-j} (4x + 1)^{i+2j-n} \right. \]

\[\left. - \sum_{i=\lfloor \frac{n-1}{3} \rfloor}^{n-1} \sum_{j=\lfloor \frac{n-i-1}{2} \rfloor}^{n-i-1} (-1)^{i+2j-n+1} 2^{n-i-1} \binom{i}{j} \binom{j}{n-i-j-1} (4x + 3)^{i-j} \cdot (4x + 1)^{i+2j-n+1} \right) \] \quad (3.10)

Moreover,

\[\left(4x + 3\right)^{i-j} (4x + 1)^{i+2j-n} \]

\[= \sum_{m=0}^{2i+j-n} \sum_{k=(i+2j-n)}^{m} 3^{i+j-k} 4^m \binom{i-j}{k} \binom{i+2j-n}{m-k} x^m, \]
Finally, we can obtain Eq. (3.9) by substituting Eqs. (3.11) and (3.12) into Eq. (3.10).

Corollary 3.3. Following corollary.

Similarly,

\[
\sum_{i=\left[\frac{n}{2}\right]}^{n-1} \sum_{j=\left[\frac{n+1}{2}\right]}^{n-i-1} (-1)^{i+2j-n} 2^{n-i-1} \binom{i}{j} \binom{j}{n-i-j} (4x + 3)^{i-j} (4x + 1)^{i+2j-n+1} \]

\[
= \sum_{m=0}^{n-1} \sum_{i=\left[\frac{n-1}{2}\right]}^{n-i-1} \sum_{j=\left[\frac{n-1}{2}\right]}^{n-i-1} \sum_{k=m-(i+2j-n+1)}^{m} (-1)^{i+2j-n+1} 2^{n+2m-i-1} 3^{i-j+k} \binom{i}{j} \binom{j}{i-j-1} \binom{i-j}{k} \binom{i+2j-n+1}{m-k} x^m. \tag{3.12}
\]

Finally, we can obtain Eq. (3.9) by substituting Eqs. (3.11) and (3.12) into Eq. (3.10). \[\square\]

By Theorem 3.2, \(x^i\) is a term of \(F(G_n, x)\) for \(i = n, n + 1, \ldots, 2n\), thus we have the following corollary.

Corollary 3.3.

1. \(f(G_n) = n, F(G_n) = 2n\);
2. \(\text{Spec}_f(G_n) = [n, 2n]\).

In the following we calculate the degree of freedom of \(G_n\), and reveal its asymptotic behavior with respect to \(\Phi(G_n)\).

Theorem 3.4.

\[
\text{IDF}(G_n) = -4 + \frac{50 + 22\sqrt{5}}{25}(3 - \sqrt{5})^n + \frac{13 - 3\sqrt{5}}{10}n(3 - \sqrt{5})^n + \frac{50 - 22\sqrt{5}}{25}(3 + \sqrt{5})^n + \frac{13 + 3\sqrt{5}}{10}n(3 + \sqrt{5})^n. \tag{3.13}
\]
Proof. According to Theorem 3.1
\[
\frac{d}{dx} F(G_n, x) = (8x + 3)F(G_{n-1}, x) + (4x^2 + 3x)\frac{d}{dx} F(G_{n-1}, x) - (24x^2 + 4x)F(G_{n-2}, x) \\
- (8x^3 + 2x^2)\frac{d}{dx} F(G_{n-2}, x) + 12x^2F(G_{n-3}, x) + 4x^3\frac{d}{dx} F(G_{n-3}, x).
\]
For convenience, let \( IDF_n := IDF(G_n) \). Then
\[
IDF_n = \frac{d}{dx} F(G_n, x) \bigg|_{x=1}
= 7IDF_{n-1} - 10IDF_{n-2} + 4IDF_{n-3} + 11\Phi_{n-1} - 28\Phi_{n-2} + 12\Phi_{n-3}.
\]
By Lemma 2.1
\[
IDF_n = 7IDF_{n-1} - 10IDF_{n-2} + 4IDF_{n-3} + 11\Phi_{n-1} - 3(6\Phi_{n-2} - 4\Phi_{n-3}) - 10\Phi_{n-2}
= 7IDF_{n-1} - 10IDF_{n-2} + 4IDF_{n-3} + 8\Phi_{n-1} - 10\Phi_{n-2}.
\]
Therefore
\[
IDF_{n+1} = 7IDF_n - 10IDF_{n-1} + 4IDF_{n-2} + 8\Phi_n - 10\Phi_{n-1},
\]
and
\[
IDF_{n+2} = 7IDF_{n+1} - 10IDF_n + 4IDF_{n-1} + 8\Phi_{n+1} - 10\Phi_n.
\]
Note that
\[
8\Phi_{n+1} - 10\Phi_n = 8(6\Phi_n - 4\Phi_{n-1}) - 10(6\Phi_{n-1} - 4\Phi_{n-2}) \\
= 6(8\Phi_n - 10\Phi_{n-1}) - 4(8\Phi_{n-1} - 10\Phi_{n-2}) \\
= 6(7IDF_n - 10IDF_{n-1} + 4IDF_{n-2} + 8\Phi_n - 10\Phi_{n-1}) \\
- 4(7IDF_{n-1} - 10IDF_{n-2} + 4IDF_{n-3} + 8\Phi_{n-1} - 10\Phi_{n-2}) \\
- 6(7IDF_n - 10IDF_{n-1} + 4IDF_{n-2}) \\
+ 4(7IDF_{n-1} - 10IDF_{n-2} + 4IDF_{n-3}) \\
= 6IDF_{n+1} - 46IDF_n + 88IDF_{n-1} - 64IDF_{n-2} + 16IDF_{n-3},
\]
so
\[
IDF_{n+2} = 13IDF_{n+1} - 56IDF_n + 92IDF_{n-1} - 64IDF_{n-2} + 16IDF_{n-3}. \tag{3.14}
\]
The characteristics equation of Eq. (3.14) is \( x^5 - 13x^4 + 56x^3 - 92x^2 + 64x - 16 = 0 \),
whose roots are \( x_0 = 1, x_1 = x_2 = 3 - \sqrt{5}, x_3 = x_4 = 3 + \sqrt{5} \). Suppose the general solution of (3.14) is
\[
IDF_n = \lambda_0 + \lambda_1(3 - \sqrt{5})^n + \lambda_2n(3 - \sqrt{5})^n + \lambda_3(3 + \sqrt{5})^n + \lambda_4n(3 + \sqrt{5})^n.
\]
According to the initial values \( IDF_4 = 5948, IDF_5 = 38908, IDF_6 = 244348, IDF_7 = 1492092, \) and \( IDF_8 = 8926204 \), we can obtain \( \lambda_0 = -4, \lambda_1 = \frac{50 + 22\sqrt{5}}{25}, \lambda_2 = \frac{13 - 3\sqrt{5}}{10}, \lambda_3 = \frac{50 - 22\sqrt{5}}{25}, \) and \( \lambda_4 = \frac{13 + 3\sqrt{5}}{10} \). Therefore Eq. (3.13) holds for \( n \geq 4 \). In fact, (3.13) is true for \( n = 0, 1, 2, 3 \) as well, the proof is completed.
By Theorems 2.2 and 3.4, we have the following corollary.

Corollary 3.5

\[ \lim_{n \to \infty} \frac{IDF_n}{n\Phi_n} = -\frac{5 + 6\sqrt{5}}{5}. \]

4 Anti-forcing polynomials

The anti-forcing polynomial \([15,22]\) of a graph \(G\) is defined as below:

\[ Af(G, x) = \sum_{M \in \mathcal{M}(G)} x^{af(G,M)}. \] (4.1)

Thus the following expression is immediate:

\[ Af(G, x) = \sum_{i=af(G)} u(G,i)x^i, \]

where \(u(G,i)\) is the number of perfect matchings with the anti-forcing number \(i\).

The anti-forcing polynomial can count the number of perfect matchings with the same anti-forcing number, that is, the distribution of all anti-forcing numbers is showed. If \(G\) is a null graph or a graph with a unique perfect matching, then \(Af(G,x) = 1\). Obviously, \(\Phi(G) = Af(G,1)\), and the sum of the anti-forcing numbers of all perfect matchings is \(\frac{d}{dx} Af(G,x) \big|_{x=1}\).

In the following, a recurrence formula of anti-forcing polynomial of \(G_n\) is showed. First, some useful local lemmas on \(H_n\) and \(G_n\) are discussed.

Lemma 4.1. Let \(n\) be a positive integer, and \(0 \leq k \leq n - 1\), and let \(\mathcal{M}_{2k+1}(H_n) = \{M \in \mathcal{M}(H_n) | u_{2k+1}v_{2k+1} \in M, \text{ and } u_jv_j \notin M \text{ for } j \leq 2k\}\). Then

\[ \sum_{M \in \mathcal{M}_{2k+1}(H_n)} x^{af(H_n,M)} = (x^3 + 3x^2)^k x^2 Af(G_{n-k-1}, x). \]

Proof. Up to isomorphism, \(H_n\) can be split into two subsystems \(H_k\) (\(H_0\) is an empty graph) on the left and \(H_{n-k}\) on the right by deleting edges \(u_{2k}v_{2k+1}\) and \(v_{2k}u_{2k+1}\) from \(H_n\) (see Fig. 1(b)). Let \(M \in \mathcal{M}_{2k+1}(H_n)\). Then the restrictions \(M'\) and \(M''\) of \(M\) on \(H_k\) and \(H_{n-k}\) are perfect matchings of \(H_k\) and \(H_{n-k}\), respectively. Let \(A'\) and \(A''\) be maximum non-crossing compatible \(M'\)-alternating set and \(M''\)-alternating set of \(H_k\) and \(H_{n-k}\) with \(I(A')\) and \(I(A'')\) as small as possible, respectively. Note that vertical edges \(u_jv_j \notin M\) for \(j \leq 2k\), by Remark 1, \(A = A' \cup A''\) is a maximum non-crossing compatible \(M\)-alternating set of \(H_n\). According to Lemma 2.8,

\[ af(H_n, M) = |A'| + |A''| = af(H_k, M') + af(H_{n-k}, M''). \] (4.2)
For \( af(H_k, M') \), let \( N_i \) be the substructure consisting of three squares \( s_{i,2}, s_{i,3} \) and \( s_{i,4} \) \((i = 1, 2, \ldots, k)\). Then the restriction of \( M' \) on \( N_i \) is a perfect matching of \( N_i \). Recall that \( u_{2i-1}v_{2i-1}, u_{2i}v_{2i} \not\in M' \), there are two cases to be considered. If \( u_{2i-1}v_{2i}, v_{2i-1}v_{2i} \in M' \), then \( s_{i,2}, s_{i,3} \) and \( s_{i,4} \) all belong to \( A' \). So the substructure \( N_i \) contributes 3 to \( af(H_k, M') \). If one of \( u_{2i-1}u_{2i} \) and \( v_{2i-1}v_{2i} \) is not in \( M' \), then there are three possible cases such that only \( s_{i,2} \) and \( s_{i,4} \) are \( M' \)-alternating squares. Hence \( N_i \) contributes 2 to \( af(H_k, M') \). Suppose there are \( r(0 \leq r \leq k) \) substructures \( N_i \) contributing 3 to \( af(H_k, M') \), then \( af(H_k, M') = 3r + 2(k - r) \). Let \( \mathcal{M}'(H_k) = \{ M' \in \mathcal{M}(H_k) | u_i v_i \not\in M', i = 1, 2, \ldots, 2k \} \). Then

\[
\sum_{M' \in \mathcal{M}'(H_k)} x^{af(H_k, M')} = \sum_{r=0}^{k} \binom{k}{r} 3^{k-r} x^{2(k-r)} = (x^3 + 3x^2)^k. \tag{4.3}
\]

For \( af(H_{n-k}, M'') \), we recall that \( u_{2k+1}v_{2k+1} \in M'' \), so there must be a vertical edge \( u_{2m+2}v_{2m+2} \in M'' \) \((k \leq m \leq n - 1)\) such that every vertical edge \( u_j v_j (2k + 1 < j < 2m + 2) \) is not in \( M'' \). As a consequence, the fragment of \( H_{n-k} \) between the two vertical edges \( u_{2k+1}v_{2k+1} \) and \( u_{2m+2}v_{2m+2} \) forms the substructure \( W_{m-k+1} \), which also contains the substructure \( L_{2(m-k)+1} \) (see Fig. [3]). Note that \( H_{n-k} \ominus \{ u_{2k+1}, v_{2k+1}, w_{2k+1}, w_{2k+2}, z_{2k+1}, z_{2k+2} \} \) is isomorphic to \( G_{n-k-1} \), and the restriction \( M'' \) of \( M'' \) on \( G_{n-k-1} \) is a perfect matching of \( G_{n-k-1} \). Let \( A'' \) be a maximum non-crossing compatible \( M'' \)-alternating set of \( G_{n-k-1} \) with \( I(A'') \) as small as possible, and let \( P(W_{m-k+1}) \) and \( P(L_{2(m-k)+1}) \) be peripheries of \( W_{m-k+1} \) and \( L_{2(m-k)+1} \), respectively. By Remark 1, \( A'' \cup \{ P(W_{m-k+1}), P(L_{2(m-k)+1}) \} \) is a maximum non-crossing compatible \( M'' \)-alternating set of \( H_{n-k} \). By Lemma 2.8 \( af(H_{n-k}, M'') = 2 + |A''| = 2 + af(G_{n-k-1}, M'') \). Let \( M''(H_{n-k}) = \{ M'' \in \mathcal{M}(H_{n-k}) | u_{2k+1} v_{2k+1} \in M'' \} \). Then

\[
\sum_{M'' \in M''(H_{n-k})} x^{af(H_{n-k}, M'')} = \sum_{M'' \in \mathcal{M}(G_{n-k-1})} x^{2+af(G_{n-k-1}, M'')} = x^2 Af(G_{n-k-1}, x). \tag{4.4}
\]

By Eqs. (4.2), (4.3) and (4.4),

\[
\sum_{M \in \mathcal{M}_{2k+1}(H_n)} x^{af(H_n, M)} = \sum_{M' \in \mathcal{M}'(H_k)} x^{af(H_k, M')} \sum_{M'' \in \mathcal{M''}(H_{n-k})} x^{af(H_{n-k}, M'')} = (x^3 + 3x^2)k x^2 Af(G_{n-k-1}, x).
\]

The following lemma is the case of Eq. (4.3) for \( k = n \).
Lemma 4.2. Let $\mathcal{M}_0(H_n) = \{M \in \mathcal{M}(H_n) | u_jv_j \not\in M \text{ for } j = 1, 2, \ldots, 2n \}$. Then
\[
\sum_{M \in \mathcal{M}_0(H_n)} x^{af(H_n,M)} = (x^3 + 3x^2)^n.
\]

Lemma 4.3. Let $n$ be a positive integer, and $0 \leq k \leq n - 1$, $\mathcal{M}^{u_0v_0}_{2k+1}(G_n) = \{M \in \mathcal{M}(G_n) | u_0v_0, u_{2k+1}v_{2k+1} \in M, \text{ and } u_jv_j \not\in M \text{ for } j = 1, 2, \ldots, 2k \}$. Then
\[
\sum_{M \in \mathcal{M}^{u_0v_0}_{2k+1}(G_n)} x^{af(G_n,M)} = ((x^3 + 3x^2)^k + (x - 1)x^{3k})x^2Af(G_{n-k-1}, x).
\]

Proof. Let $M \in \mathcal{M}^{u_0v_0}_{2k+1}(G_n)$. Then $M' = M \setminus \{u_0v_0\}$ is a perfect matching of $H_n = G_n \ominus \{u_0, v_0\}$, and $M'$ belongs to $\mathcal{M}_{2k+1}(H_n)$ (defined in Lemma 4.1). Let $\mathcal{A}'$ be a maximum non-crossing compatible $M'$-alternating set of $H_n$ with $I(\mathcal{A}')$ as small as possible. There are two cases to be considered. If $u_{2i+1}u_{2i+2}$ and $v_{2i+1}v_{2i+2}$ belong to $M$ for all $i = 0, 1, \ldots, k - 1$, then there is a substructure $L_{2k+1}$ ($L_1$ is the square $s_{1,1}$) between the vertical edges $u_0v_0$ and $u_{2k+1}v_{2k+1}$, whose periphery $P(L_{2k+1})$ is an $M$-alternating cycle (see Fig. 1(a)). By Lemma 2.9 and Remark 1, $\mathcal{A}' \cup \{P(L_{2k+1})\}$ is a maximum compatible $M$-alternating set of $G_n$. By Lemma 2.8, $af(G_n, M) = 1 + |\mathcal{A}'| = 1 + af(H_n, M')$. Suppose there exists a $j (0 \leq j \leq k - 1)$ such that $u_{2j+1}u_{2j+2}$ or $v_{2j+1}v_{2j+2}$ does not belong to $M$, then $\mathcal{A}'$ can be a maximum compatible $M$-alternating set of $G_n$, so $af(G_n, M) = af(H_n, M')$. Analogously,
\[
\sum_{M \in \mathcal{M}^{u_0v_0}_{2k+1}(G_n)} x^{af(G_n,M)} = \sum_{r=0}^{k-1} \binom{k}{r} 3^{k-r} x^{2(k-r)} x^{3r} + 3^{k+1})x^2Af(G_{n-k-1}, x)
\]
\[
= ((x^3 + 3x^2)^k + (x - 1)x^{3k})x^2Af(G_{n-k-1}, x).
\]

\[\Box\]

Lemma 4.4. Let $\mathcal{M}^{u_0v_0}(G_n) = \{M \in \mathcal{M}(G_n) | u_0v_0 \in M, u_jv_j \not\in M \text{ for } j = 1, 2, \ldots, 2n \}$. Then
\[
\sum_{M \in \mathcal{M}^{u_0v_0}(G_n)} x^{af(G_n,M)} = (x^3 + 3x^2)^n.
\]

Proof. Let $M \in \mathcal{M}^{u_0v_0}(G_n)$. Then $M' = M \setminus \{u_0v_0\} \in \mathcal{M}_0(H_n)$. On the other hand, if $M' \in \mathcal{M}_0(H_n)$, then $M = M' \cup \{u_0v_0\} \in \mathcal{M}^{u_0v_0}(G_n)$. So $|\mathcal{M}^{u_0v_0}(G_n)| = |\mathcal{M}_0(H_n)|$. Let $\mathcal{A}'$ be a maximum non-crossing compatible $M'$-alternating set of $H_n$ with $I(\mathcal{A}')$ as small as possible. By Lemma 2.9 and Remark 1, $\mathcal{A}'$ also is a maximum compatible $M$-alternating set of $G_n$. By Lemma 2.8, $af(G_n, M) = af(H_n, M')$. According to Lemma 4.2,
\[
\sum_{M \in \mathcal{M}^{u_0v_0}(G_n)} x^{af(G_n,M)} = \sum_{M' \in \mathcal{M}_0(H_n)} x^{af(H_n,M')} = (x^3 + 3x^2)^n.
\]

\[\Box\]
Lemma 4.5. Let $\mathcal{M}_0(G_n) = \{ M \in \mathcal{M}(G_n) | u_0v_0 \not\in M \}$. Then
\[
\sum_{M \in \mathcal{M}_0(G_n)} x^{af(G_n,M)} = xA\!f(G_{n-1},x).
\]

Proof. Let $M \in \mathcal{M}_0(G_n)$. Then $u_0u_1,v_0v_1,w_1w_2,z_1z_2 \in M$, $s_{1,1}$ is an $M$-alternating square, and the restriction $M'$ of $M$ on $G_{n-1} = G_n \ominus \{u_0,u_1,v_0,v_1,w_1,w_2,z_1,z_2\}$ is a perfect matching of $G_{n-1}$, see Fig. 1(a). Let $A'$ be a maximum non-crossing compatible $M'$-alternating set of $G_{n-1}$ with $I(A')$ as small as possible. By Lemma 2.9, $A' \cup \{s_{1,1}\}$ is a maximum compatible $M$-alternating set of $G_n$. By Lemma 2.8, $a\!f(G_n,M) = 1 + a\!f(G_{n-1},M')$. Therefore
\[
\sum_{M \in \mathcal{M}_0(G_n)} x^{af(G_n,M)} = \sum_{M' \in \mathcal{M}(G_{n-1})} x^{1+af(G_{n-1},M')} = xA\!f(G_{n-1},x).
\]

Lemma 4.6. Let $n$ be a positive integer. Then
\[
A\!f(H_n,x) = x^2A\!f(G_{n-1},x) + (x^3 + 3x^2)A\!f(H_{n-1},x).
\]

Proof. We can divide $\mathcal{M}(H_n)$ into two subsets: $\mathcal{M}_1(H_n) = \{ M \in \mathcal{M}(H_n) | u_1v_1 \in M \}$ and $\mathcal{M}_1(H_n) = \{ M \in \mathcal{M}(H_n) | u_1v_1 \not\in M \}$ (see Fig. 1(b)). In fact, $\mathcal{M}_1(H_n)$ is the case of Lemma 4.1 for $k = 0$. So
\[
\sum_{M \in \mathcal{M}_1(H_n)} x^{af(H_n,M)} = x^2A\!f(G_{n-1},x).
\]

Let $M \in \mathcal{M}_1(H_n)$. Then $s_{1,2}$ and $s_{1,4}$ have to be $M$-alternating squares, and the restriction of $M$ on $H_{n-1} = H_n \ominus \{w_1,w_2,u_1,u_2,v_1,v_2,z_1,z_2\}$ is a perfect matching of $H_{n-1}$. Similar to Lemma 4.1
\[
\sum_{M \in \mathcal{M}_1(H_n)} x^{af(H_n,M)} = (x^3 + 3x^2)A\!f(H_{n-1},x).
\]

By Eq. (4.1),
\[
A\!f(H_n,x) = \sum_{M \in \mathcal{M}(H_n)} x^{af(H_n,M)} = \sum_{M \in \mathcal{M}_1(H_n)} x^{af(H_n,M)} + \sum_{M \in \mathcal{M}_1(H_n)} x^{af(H_n,M)}
\]
\[
= x^2A\!f(G_{n-1},x) + (x^3 + 3x^2)A\!f(H_{n-1},x)
\]

\]

Theorem 4.7. For $n \geq 3$,
\[
A\!f(G_n,x) = (3x^3 + 3x^2 + x)A\!f(G_{n-1},x) - (2x^6 + 6x^5 - x^4 + 3x^3)A\!f(G_{n-2},x)
\]
\[
+ (x^7 + 3x^6)A\!f(G_{n-3},x),
\]

where $A\!f(G_0,x) = 1$, $A\!f(G_1,x) = 2x^3 + 3x^2 + x$ and $A\!f(G_2,x) = 4x^6 + 9x^5 + 15x^4 + 3x^3 + x^2$.  

17
Proof. By Lemmas 4.3, 4.4 and 4.5, \( \mathcal{M}(G_n) \) can be divided into \( n + 2 \) subsets: \( \mathcal{M}_{0}^{u_{0}}(G_n) \), \( \mathcal{M}_{0}(G_n) \) and \( \mathcal{M}_{2k+1}^{u_{0}}(G_n), k = 0, 1, 2, \ldots, n - 1 \). By Eq. (4.1),

\[
Af(G_n, x) = \sum_{M \in \mathcal{M}(G_n)} x^{a_{f}(G_n, M)}
\]

\[
= \sum_{k=0}^{n-1} \left( \sum_{M \in \mathcal{M}_{2k+1}^{u_{0}}(G_n)} x^{a_{f}(G_n, M)} + \sum_{M \in \mathcal{M}_{0}(G_n)} x^{a_{f}(G_n, M)} \right)
\]

\[
= \sum_{k=0}^{n-1} \left( x^3 + 3x^2 \right)^k x^2 Af(G_{n-k-1}, x) + x^2(x-1) \sum_{k=0}^{n-1} x^{3k} Af(G_{n-k-1}, x)
\]

\[
+ xAf(G_{n-1}, x) + (x^3 + 3x^2)^n. \tag{4.5}
\]

By Lemmas 4.1 and 4.2,

\[
Af(H_n, x) = \sum_{M \in \mathcal{M}(H_n)} x^{a_{f}(H_n, M)}
\]

\[
= \sum_{k=0}^{n-1} \left( \sum_{M \in \mathcal{M}_{2k+1}(H_n)} x^{a_{f}(H_n, M)} + \sum_{M \in \mathcal{M}_{0}(H_n)} x^{a_{f}(H_n, M)} \right)
\]

\[
= \sum_{k=0}^{n-1} \left( x^3 + 3x^2 \right)^k x^2 Af(G_{n-k-1}, x) + (x^3 + 3x^2)^n. \tag{4.6}
\]

Substituting Eq. (4.6) into (4.5), then

\[
Af(G_n, x) = Af(H_n, x) + xAf(G_{n-1}, x) + x^2(x-1) \sum_{k=0}^{n-1} x^{3k} Af(G_{n-k-1}, x).
\]

So

\[
x^3 Af(G_n, x) = x^3 Af(H_n, x) + x^4 Af(G_{n-1}, x)
\]

\[
+ x^2(x-1) \sum_{k=0}^{n-1} x^{3(k+1)} Af(G_{n-k-1}, x) \tag{4.7}
\]

and

\[
Af(G_{n+1}, x) = Af(H_{n+1}, x) + xAf(G_n, x) + x^2(x-1) \sum_{k=0}^{n} x^{3k} Af(G_{n-k}, x). \tag{4.8}
\]

Subtracting Eq. (4.7) from (4.8),

\[
Af(G_{n+1}, x) = (2x^3 - x^2 + x)Af(G_n, x) - x^4 Af(G_{n-1}, x)
\]

\[
+ Af(H_{n+1}, x) - x^3 Af(H_n, x). \tag{4.9}
\]
According to Lemma 4.6,

\[ Af(H_{n+1}, x) - x^3 Af(H_n, x) = x^2 Af(G_n, x) + 3x^2 Af(H_n, x), \]

therefore

\[
Af(G_{n+1}, x) = (2x^3 - x^2 + x)Af(G_n, x) - x^4 Af(G_{n-1}, x) \\
+ x^2 Af(G_n, x) + 3x^2 Af(H_n, x) \\
= (2x^3 + x)Af(G_n, x) - x^4 Af(G_{n-1}, x) + 3x^2 Af(H_n, x).
\]

This implies that

\[
3x^2 Af(H_n, x) = Af(G_{n+1}, x) - (2x^3 + x)Af(G_n, x) + x^4 Af(G_{n-1}, x),
\]

and

\[
3x^2 Af(H_{n-1}, x) = Af(G_n, x) - (2x^3 + x)Af(G_{n-1}, x) + x^4 Af(G_{n-2}, x).
\]

By Lemma 4.6

\[
3x^2 Af(H_n, x) = 3x^4 Af(G_{n-1}) + (x^3 + 3x^2)3x^2 Af(H_{n-1}, x).
\]

Substituting Eqs. (4.10) and (4.11) into (4.12), we can obtain the following recurrence formula

\[
Af(G_{n+1}, x) = (3x^3 + 3x^2 + x)Af(G_n, x) - (2x^6 + 6x^5 - x^4 + 3x^3)Af(G_{n-1}, x) \\
+ (x^7 + 3x^6)Af(G_{n-2}, x),
\]

the proof is completed.

According to Theorem 4.7 we can obtain the following expression.

**Theorem 4.8.**

\[
Af(G_n, x) = R_n(x) + Q_n(x) + 3x^{n+1} + x^n,
\]

where \( R_n(x) \) and \( Q_n(x) \) are listed in the Appendix.

**Proof.** Let \( Af_n := Af(G_n, x) \). Then the generating function of the sequence \( \{Af_n\}_{n=0}^\infty \) is

\[
P(t) = \sum_{n=0}^\infty Af_n t^n = 1 + Af_1 t + Af_2 t^2 + \sum_{n=3}^\infty Af_n t^n.
\]

By Theorem 4.7

\[
P(t) = 1 + Af_1 t + Af_2 t^2 + (3x^3 + 3x^2 + x) \sum_{n=3}^\infty Af_{n-1} t^n \\
-(2x^6 + 6x^5 - x^4 + 3x^3) \sum_{n=3}^\infty Af_{n-2} t^n + (x^7 + 3x^6) \sum_{n=3}^\infty Af_{n-3} t^n \\
= 1 - x^3 t + ((3x^3 + 3x^2 + x)t - (2x^6 + 6x^5 - x^4 + 3x^3)t^2 + (x^7 + 3x^6)t^3)P(t).
\]
Therefore,

\[ P(t) = \frac{1 - x^3 t}{1 - ((3x^3 + 3x^2 + x)t - (2x^6 + 6x^5 - x^4 + 3x^3)t^2 + (x^7 + 3x^6)t^3)} \]

\[ = (1 - x^3 t) \sum_{i=0}^{\infty} ((3x^3 + 3x^2 + x)t - (2x^6 + 6x^5 - x^4 + 3x^3)t^2 + (x^7 + 3x^6)t^3)^i \]

\[ = (1 - x^3 t) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i-j-k} \binom{i}{j} \binom{i-j}{k} (3x^3 + 3x^2 + x)^j (x^7 + 3x^6)^k \]

\[ \cdot (2x^6 + 6x^5 - x^4 + 3x^3)^{i-j-k} t^{2i-j+k}. \]

Let \( n = 2i - j + k. \) Then

\[ P(t) = 1 + \sum_{n=1}^{\infty} \left( \sum_{i=\lceil \frac{n}{2} \rceil}^{\lfloor \frac{n}{3} \rfloor} \sum_{j=\max\{0,2i-n\}}^{\lfloor \frac{3i-n}{2} \rfloor} (-1)^{3i-2j-n} \binom{i}{j} \binom{i-j}{n-2i+j} (3x^3 + 3x^2 + x)^j \]

\[ \cdot (x^7 + 3x^6)^{n-2i+j} (2x^6 + 6x^5 - x^4 + 3x^3)^{3i-2j-n} + \]

\[ \sum_{i=\lfloor \frac{n-1}{2} \rfloor}^{\lfloor \frac{n-1}{3} \rfloor} \sum_{j=\max\{0,2i-n+1\}}^{\lfloor \frac{3i-n+1}{2} \rfloor} (-1)^{3i-2j-n+2} \binom{i}{j} \binom{i-j}{n-2i+j-1} x^3 (3x^3 + 3x^2 + x)^j \]

\[ \cdot (x^7 + 3x^6)^{n-2i+j-1} (2x^6 + 6x^5 - x^4 + 3x^3)^{3i-2j-n+1} t^n \]

\[ = 1 + \sum_{n=1}^{\infty} A_f_n t^n. \]

Therefore,

\[ A_f_n = \sum_{i=\lceil \frac{n}{2} \rceil}^{\lfloor \frac{n}{3} \rfloor} \sum_{j=\max\{0,2i-n\}}^{\lfloor \frac{3i-n}{2} \rfloor} (-1)^{3i-2j-n} \binom{i}{j} \binom{i-j}{n-2i+j} (3x^3 + 3x^2 + x)^j \]

\[ \cdot (x^7 + 3x^6)^{n-2i+j} (2x^6 + 6x^5 - x^4 + 3x^3)^{3i-2j-n} + \]

\[ \sum_{i=\lfloor \frac{n-1}{2} \rfloor}^{\lfloor \frac{n-1}{3} \rfloor} \sum_{j=\max\{0,2i-n+1\}}^{\lfloor \frac{3i-n+1}{2} \rfloor} (-1)^{3i-2j-n+2} \binom{i}{j} \binom{i-j}{n-2i+j-1} x^3 (3x^3 + 3x^2 + x)^j \]

\[ \cdot (x^7 + 3x^6)^{n-2i+j-1} (2x^6 + 6x^5 - x^4 + 3x^3)^{3i-2j-n+1}. \]

Furthermore, we can derive the Eq. (4.13) by using the binomial theorem, more details are listed in the Appendix.

By Theorem 4.8, \( x^i \) is a term of \( A_f(G_n, x) \) for \( i = n, n+1, \ldots, 3n \), thus we have the following corollary.

**Corollary 4.9.**
(1) \( af(G_n) = n, Af(G_n) = 3n; \)

(2) \( \text{Spec}_{af}(G_n) = [n, 3n]. \)

By Theorem 4.7, we can obtain the exact expression of the sum over all anti-forcing number of perfect matchings of \( G_n \).

**Theorem 4.10.**

\[
\left. \frac{d}{dx} Af(G_n, x) \right|_{x=1} = -3 + \frac{150 + 67\sqrt{5}}{100} (3 - \sqrt{5})^n + \frac{29 - 10\sqrt{5}}{20} n(3 - \sqrt{5})^n \\
+ \frac{150 - 67\sqrt{5}}{100} (3 + \sqrt{5})^n + \frac{29 + 10\sqrt{5}}{20} n(3 + \sqrt{5})^n. \quad (4.14)
\]

**Proof.** Recall that \( \Phi(G_n) = Af(G_n, 1) \), for convenience, let \( \Phi_n := \Phi(G_n) \) and \( A_n := \left. \frac{d}{dx} Af(G_n, x) \right|_{x=1} \).

By Theorem 4.7,

\[
A_n = 7A_{n-1} - 10A_{n-2} + 4A_{n-3} + 16\Phi_{n-1} - 47\Phi_{n-2} + 25\Phi_{n-3}.
\]

According to Lemma 2.1,

\[
4A_n = 28A_{n-1} - 40A_{n-2} + 16A_{n-3} + 64\Phi_{n-1} - 188\Phi_{n-2} + 100\Phi_{n-3} \\
= 28A_{n-1} - 40A_{n-2} + 16A_{n-3} + 64\Phi_{n-1} - 25(6\Phi_{n-2} - 4\Phi_{n-3}) - 38\Phi_{n-2} \\
= 28A_{n-1} - 40A_{n-2} + 16A_{n-3} + 39\Phi_{n-1} - 38\Phi_{n-2}.
\]

So

\[
4A_{n+1} = 28A_n - 40A_{n-1} + 16A_{n-2} + 39\Phi_n - 38\Phi_{n-1}
\]

and

\[
4A_{n+2} = 28A_{n+1} - 40A_n + 16A_{n-1} + 39\Phi_{n+1} - 38\Phi_n. \quad (4.15)
\]

Note that

\[
39\Phi_{n+1} - 38\Phi_n = 39(6\Phi_n - 4\Phi_{n-1}) - 38(6\Phi_{n-1} - 4\Phi_{n-2}) \\
= 6(39\Phi_n - 38\Phi_{n-1}) - 4(39\Phi_{n-1} - 38\Phi_{n-2}) \\
= 6(28A_n - 40A_{n-1} + 16A_{n-2} + 39\Phi_n - 38\Phi_{n-1}) \\
- 4(28A_{n-1} - 40A_{n-2} + 16A_{n-3} + 39\Phi_{n-1} - 38\Phi_{n-2}) \\
- 6(28A_n - 40A_{n-1} + 16A_{n-2}) + 4(28A_{n-1} - 40A_{n-2} + 16A_{n-3}) \\
= 24A_{n+1} - 184A_n + 352A_{n-1} - 256A_{n-2} + 64A_{n-3},
\]

substituting it into Eq. (4.15),

\[
A_{n+2} = 13A_{n+1} - 56A_n + 92A_{n-1} - 64A_{n-2} + 16A_{n-3}.
\]
This recurrence relation is the same as (3.14), by initial values $A_4 = 7721, A_5 = 50541, A_6 = 317565, A_7 = 1939901$ and $A_8 = 11608381$, we can prove Eq. (4.14) for $n \geq 4$. Actually, Eq. (4.14) also holds for $n = 0, 1, 2, 3$, so the proof is completed.

By Theorems 2.2 and 4.10 we can obtain the following asymptotic behavior.

**Corollary 4.11.**

$$\lim_{n \to \infty} \frac{A_n}{n \Phi_n} = \frac{5 + 37\sqrt{5}}{40}.$$

**References**

[1] P. Adams, M. Mahdian, E.S. Mahmoodian, On the forced matching numbers of bipartite graphs, Discrete Math. 281 (2004) 1–12.

[2] P. Afshani, H. Hatami, E.S. Mahmoodian, On the spectrum of the forced matching number of graphs, Australas. J. Combin. 30 (2004) 147–160.

[3] C. Berge, C. Chen, V. Chvátal, C.S. Seow, Combinatorial properties of polyominoes, Combinatorica 1 (1981) 217–224.

[4] Z. Che, Z. Chen, Forcing on perfect matchings-A survey, MATCH Commun. Math. Comput. Chem. 66 (2011) 93–136.

[5] Z. Che, Z. Chen, Conjugated circuits and forcing edges, MATCH Commun. Math. Comput. Chem. 69 (2013) 721–731.

[6] E.J. Cockayne, Chessboard domination problems, Discrete Math. 86 (1990) 13–20.

[7] S. J. Cyvin, I. Gutman, Kekulé structures in benzenoid hydrocarbons (Lecture notes in chemistry 46), Springer Verlag, Berlin, 1988.

[8] H. Deng, The anti-forcing number of hexagonal chains, MATCH Commun. Math. Comput. Chem. 58 (2007) 675–682.

[9] H. Deng, The anti-forcing number of double hexagonal chains, MATCH Commun. Math. Comput. Chem. 60 (2008) 183–192.

[10] K. Deng, H. Zhang, Anti-forcing spectra of perfect matchings of graphs, J. Comb. Optim. 33 (2017) 660–680.

[11] K. Deng, H. Zhang, Anti-forcing spectrum of any cata-condensed hexagonal system is continuous, Front. Math. China 12 (2017) 19–33.

[12] K. Deng, H. Zhang, Extremal anti-forcing numbers of perfect matchings of graphs, Discrete Appl. Math. 224 (2017), 69–79.

[13] A.A. Diwan, The minimum forcing number of perfect matchings in the hypercube, Discrete Math. 342 (2019) 1060–1062.
[14] F. Harary, D. Klein, T. Živković, Graphical properties of polyhexes: perfect matching vector and forcing, J. Math. Chem. 6 (1991) 295–306.

[15] H.K. Hwang, H. Lei, Y. Yeh, H. Zhang, Distribution of forcing and anti-forcing numbers of random perfect matchings on hexagonal chains and crowns (preprint, 2015).

[16] X. Jiang, H. Zhang, On forcing matching number of boron-nitrogen fullerenes, Discrete Appl. Math. 159 (2011) 1581–1593.

[17] X. Jiang, H. Zhang, The maximum forcing number of cylindrical grid, toroidal 4-8 lattice and Klein bottle 4-8 lattice, J. Math. Chem. 54 (2016) 18–32.

[18] P.W. Kasteleyn, The statistics of dimers on a lattice I: the number of dimer arrangements on a quadratic lattice, Phys. 27 (1961), 1209–1225.

[19] D. Klein, M. Randić, Innate degree of freedom of a graph, J. Comput. Chem. 8 (1987) 516–521.

[20] S. Kleinerman, Bounds on the forcing numbers of bipartite graphs, Discrete Math. 306 (2006) 66–73.

[21] F. Lam, L. Pachter, Forcing number for stop signs, Theor. Comput. Sci. 303 (2003) 409–416.

[22] H. Lei, Y. Yeh, H. Zhang, Anti-forcing numbers of perfect matchings of graphs, Discrete Appl. Math. 202 (2016) 95–105.

[23] X. Li, Hexagonal systems with forcing single edges, Discrete Appl. Math. 72 (1997) 295–301.

[24] L. Lovász, M. Plummer, Matching Theory, Annals of Discrete Mathematics, Vol 29. Amsterdam: North-Holland, 1986.

[25] C.L. Lucchesi, D.H. Younger, A minimax theorem for directed graphs, J. Lond. Math. Soc. 17 (1978) 369–374.

[26] A. Motoyama, H. Hosoya, King and domino polyominals for polyomino graphs, J. Math. Phys. 18 (1997) 1485–1490.

[27] L. Pachter, P. Kim, Forcing matchings on square grids, Discrete Math. 190 (1998) 287–294.

[28] M. Randić, D. Klein, in Mathematical and Computational Concepts in Chemistry, ed. by N. Trinajstić (Wiley, New York, 1985), pp. 274–282.

[29] M. Randić, D. Vukičević, Kekulé structures of fullerene C_{70}, Croat. Chem. Acta 79 (2006) 471–481.

[30] M.E. Riddle, The minimum forcing number for the torus and hypercube, Discrete Math. 245 (2002) 283–292.
[31] L. Shi, H. Zhang, Forcing and anti-forcing numbers of (3,6)-fullerenes, MATCH Commun. Math. Comput. Chem. 76 (2016) 597–614.

[32] L. Shi, H. Wang, H. Zhang, On the maximum forcing and anti-forcing numbers of (4,6)-fullerenes, Discrete Appl. Math. 233 (2017) 187–194.

[33] L. Shi, H. Zhang, Tight upper bound on the maximum anti-forcing numbers of graphs, Discrete Math. Theor. Comput. Sci. 19(3) (2017) #9.

[34] D. Vukičević, I. Gutman, M. Randić, On instability of fullerene $C_{72}$, Croat. Chem. Acta 79 (2006) 429–436.

[35] D. Vukičević, M. Randić, On Kekulé structures of buckminsterfullerene, Chem. Phys. Lett. 401 (2005) 446–450.

[36] D. Vukičević, N. Trinajstić, On the anti-forcing number of benzenoids, J. Math. Chem. 42 (2007) 575–583.

[37] D. Vukičević, N. Trinajstić, On the anti-Kekulé number and anti-forcing number of cata-condensed benzenoids, J. Math. Chem. 43 (2008) 719–726.

[38] H. Wang, D. Ye, H. Zhang, The forcing number of toroidal polyhexes, J. Math. Chem. 43 (2008) 457–475.

[39] L. Xu, H. Bian, F. Zhang, Maximum forcing number of hexagonal systems, MATCH Commun. Math. Comput. Chem. 70 (2013) 493–500.

[40] Q. Yang, H. Zhang, Y. Lin, On the anti-forcing number of fullerene graphs, MATCH Commun. Math. Comput. Chem. 74 (2015) 681–700.

[41] F. Zhang, X. Li, Hexagonal systems with forcing edges, Discrete Math. 140 (1995) 253–263.

[42] H. Zhang, K. Deng, Forcing spectrum of a hexagonal system with a forcing edge, MATCH Commun. Math. Comput. Chem. 73 (2015) 457–471.

[43] H. Zhang, D. Ye, W.C. Shiu, Forcing matching numbers of fullerene graphs, Discrete Appl. Math. 158 (2010) 573–582.

[44] H. Zhang, F. Zhang, Perfect matchings of polyomino graphs, Graphs Combin. 13 (1997) 295–304.

[45] H. Zhang, F. Zhang, Plane elementary bipartite graphs, Discrete Appl. Math. 105 (2000) 291–311.

[46] H. Zhang, S. Zhao, R. Lin, The forcing polynomial of catacondensed hexagonal systems, MATCH Commun. Math. Comput. Chem. 73 (2015) 473–490.

[47] H. Zhang, X. Zhou, A maximum resonant set of polyomino graphs, Discuss. Math. Graph Theory 36 (2016) 323–337.

[48] Q. Zhang, H. Bian, E. Vumar, On the anti-kekulé and anti-forcing number of cata-condensed phenylenes, MATCH Commun. Math. Comput. Chem. 65 (2011) 799–806.
[49] S. Zhao, H. Zhang, Forcing polynomials of benzenoid parallelogram and its related benzenoids, Appl. Math. Comput. 284 (2016) 209–218.

[50] S. Zhao, H. Zhang, Anti-forcing polynomials for benzenoid systems with forcing edges, Discrete Appl. Math. 250 (2018) 342–356.

[51] S. Zhao, H. Zhang, Forcing and anti-forcing polynomials of perfect matchings for some rectangle grids, J. Math. Chem. 57 (2019) 202–225.

[52] X. Zhou, H. Zhang, Clar sets and maximum forcing numbers of hexagonal systems, MATCH Commun. Math. Comput. Chem. 74 (2015) 161–174.

[53] X. Zhou, H. Zhang, A minimax result for perfect matchings of a polyomino graph, Discrete Appl. Math. 206 (2016) 165–171.
Appendix

\[
\sum_{i=\left\lfloor \frac{n}{2} \right\rfloor}^{n} \sum_{j=\max\{0,2i-n\}}^{\left\lfloor \frac{3i-n}{2} \right\rfloor} (-1)^{3i-2j-n} \binom{i}{j} \binom{i-j}{n-2i+j} (3x^3 + 3x^2 + x^j)(x^7 + 3x^6)^{n-2i+j}(2x^6 + 6x^5 - x^4 + 3x^3)^{3i-2j-n}
\]

\[= x^n \sum_{q=0}^{2n} \sum_{i=\max\left\{n-q,\left\lceil \frac{n}{2} \right\rceil \right\}}^{n} \sum_{j=\max\{0,2i-n\}}^{\min\{q+3i-2n,\frac{3i-n}{2}\}} \sum_{k=\max\{0,\frac{2+4j-4i}{2} \}}^{\min\{j,q+3i-j-2n\}} \sum_{l=r=\max\{0,q+4j-k-4i\}}^{\min\{\alpha,q+3i-k-r-j-2n\}} \sum_{s=m=\max\{0,\gamma+n-j-q\}}^{\min\{\beta,s,\frac{\gamma+6i-5j-3s-n-q}{2}\}} \binom{i}{j} \binom{i-j}{k} \binom{k-\alpha}{r} \binom{\beta-s}{m} \binom{\beta-s-l}{s} \binom{\beta-s-l}{\theta} (-1)^{b+l} 2^{b-s-l} 3^{k+i-j-m-l-\theta} x^q
\]

\[= x^n \sum_{q=0}^{2n} \sum_{i=\max\left\{n-q,\left\lceil \frac{n}{2} \right\rceil \right\}}^{n} \sum_{j=\max\{0,2i-n\}}^{\min\{q+3i-2n,\frac{3i-n}{2}\}} \sum_{k=\max\{0,\frac{2+4j-4i}{2} \}}^{\min\{j,q+3i-j-2n\}} \sum_{l=r=\max\{0,q+4j-k-4i\}}^{\min\{\alpha,q+3i-k-r-j-2n\}} \sum_{s=m=\max\{0,\gamma+n-j-q\}}^{\min\{\beta,s,\frac{\gamma+6i-5j-3s-n-q}{2}\}} \binom{i}{j} \binom{i-j}{k} \binom{k-\alpha}{r} \binom{\beta-s}{m} \binom{\beta-s-l}{s} \binom{\beta-s-l}{\theta} (-1)^{b+l} 2^{b-s-l} 3^{k+i-j-m-l-\theta} x^q + 3x^{n+1} + x^n
\]

\[= R_n(x) + 3x^{n+1} + x^n
\]

\[
\sum_{i=\left\lceil \frac{n}{2} \right\rceil}^{n-1} \sum_{j=\max\{0,2i-n+1\}}^{\left\lfloor \frac{3i-n+1}{2} \right\rfloor} (-1)^{3i-2j-n+1} \binom{i}{j} \binom{i-j}{n-2i+j-1} x^3(3x^3 + 3x^2 + x^j)(x^7 + 3x^6)^{n-2i+j-1}(2x^6 + 6x^5 - x^4 + 3x^3)^{3i-2j-n+1}
\]

\[= x^n \sum_{q=2}^{n-1} \sum_{i=\max\left\{n-q+1,\left\lceil \frac{n}{2} \right\rceil \right\}}^{n-1} \sum_{j=\max\{0,2i-n+1\}}^{\min\{q+3i-2n,\frac{3i-n+1}{2}\}} \sum_{k=\max\{0,\frac{3i-n+1}{2}\}}^{\min\{j,q+3i-j-2n\}} \sum_{l=r=\max\{0,q+4j-k-4i-2\}}^{\min\{\alpha,q+3i-k-r-j-2n\}} \sum_{s=m=\max\{0,\gamma+n-j-q+1\}}^{\min\{\beta+1,s,\frac{\gamma+6i-5j-3s-n-q+3}{2}\}} \binom{i}{j} \binom{i-j}{k} \binom{k-\alpha-1}{r} \binom{\alpha-1}{m} \binom{\beta+1}{s} \binom{\beta-s+1}{l} \binom{\beta-s-l+1}{\theta-2} (-1)^{b+l+2} 2^{b-s+l+1} 3^{k+i-j-m-l-\theta-2} x^q
\]

\[= Q_n(x)
\]

Where \( n \geq 1, \alpha = n - 2i + j, \beta = 3i - 2j - n, \gamma = k + m + r, \theta = q + 3j + 2s + l - k - r - m - 3i. \)