Nematic quantum criticality in three-dimensional Fermi system with quadratic band touching

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We construct and discuss the field theory for tensorial nematic order parameter coupled to gapless four-component fermions at the quadratic band touching point in three (spatial) dimensions. Within a properly formulated epsilon-expansion this theory is found to have a quantum critical point, which describes the (presumably continuous) transition from the semimetal into a (nematic) Mott insulator. The latter phase breaks the rotational, but not the time-reversal symmetry, and may be relevant to materials such as gray tin or mercury telluride at low temperatures. The critical point represents the simplest quantum analogue of the familiar classical isotropic-to-nematic transition in liquid crystals. The properties and the consequences of this quantum critical point are discussed.

I. INTRODUCTION

Electronic systems that have their Fermi surface reduced to Fermi points have received plenty of attention lately. In particular, recent progress on the problem of interacting Dirac electrons, when the dispersion near the Fermi points is linear in momentum, has indicated that these systems suffer a quantum phase transition with increasing interactions into a gapped phase, described well by the relativistic field theory of the Gross-Neveu-Yukawa type \cite{1}. A weak long-range component of the Coulomb interaction appears to be an irrelevant perturbation at the quantum critical (QC) point, and the transition is essentially due to some of its short-range components becoming sufficiently large. When the dispersion near the Fermi point(s) is quadratic, on the other hand, the result is rather different. In the bilayer graphene, for example, it is one of many possible mass-gaps that opens up already at an infinitesimal interaction \cite{2}. The finite density of states that accompanies such a quadratic band touching (QBT) in two dimensions (2D) causes the long-range Coulomb interaction, loosely speaking, to be screened, and at the same time the non-interacting ground state to be unstable at weak short-range interaction \cite{3} \cite{4}.

The situation in three-dimensional (3D) systems with QBT is maybe more interesting. QBT arises naturally in many gapless semiconductors, such as gray tin, mercury telluride, or certain pyrochlore iridates \cite{5}, that feature band inversion due to the spin-orbit coupling. The density of states at the QBT point now vanishes, and the long-range nature of the electron-electron interaction must be taken into account. It has been argued by Abrikosov long ago \cite{6}, that the plain vanilla density-density Coulomb interaction in a 3D system with the QBT should turn the ground state into an example of a scale-invariant non-Fermi liquid (NFL). Such an exotic zero temperature phase would manifest itself in characteristic nontrivial power laws in temperature or frequency in various response functions of the system \cite{7}.

We have pointed out recently \cite{8}, on the other hand, that a 3D system with the chemical potential at the QBT and the Coulomb repulsion between the electrons may be unstable towards an insulating ground state with an anisotropic gap in the spectrum at low temperatures. The mechanism responsible for this instability was proposed to be the collision between the Abrikosov’s infrared stable NFL fixed point with another, QC point, which approaches it from the strong-coupling region as the spatial dimensionality of the system is taken to be decreasing from $d = 4$. The collision of fixed points has been studied as a mechanism behind several interesting instabilities in a variety of many-body systems in the past \cite{9} \cite{13}. Within the standard one-loop calculation it occurs here somewhat above and close to $d = 3$, when both the NFL and the QC fixed points become complex and disappear from the physical space of real couplings. As a result, the coupling constants in the theory run away towards the values at which spontaneous breaking of the rotational symmetry appears to be the most favorable instability. The system in its interacting ground state would effectively appear as if it were under, in this case dynamically generated, strain. Furthermore, in the materials with the rest of the band structure equivalent to that of gray tin or mercury telluride as well, the resulting insulating ground state, at least at the mean-field level, would be topologically nontrivial \cite{14}. It would therefore be a precious example of a topological Mott insulator \cite{15} \cite{18}.

In order to remove the Abrikosov’s NFL fixed point, however, the existence of which is guaranteed close to four spatial dimensions, from the physical real-valued space of couplings, it is necessary to have a QC point that would collide with it with the change of some parameter. Indeed, in a certain large-$N$ extension of the theory one can show that such a QC point does exist \cite{8}. At the physical value of $N = 1$, however, in the purely fermionic formulation of the problem the putative QC point lies at strong values of the short-range couplings in the relevant dimensions $3 \leq d < 4$. One may therefore question whether such a QC point is a genuine feature of the theory, and if it would, for example, survive if one went beyond the one-loop approximation. As we will see such reservations would not be entirely without grounds. Similar issue arises in the interacting system of linearly dispersing Dirac fermions \cite{19} \cite{20}. In this case, however, an alternative partially bosonized Gross-Neveu-Yukawa
formulation can be devised [21]. In this reformulation of the theory one finds a clearly identifiable upper critical dimension, which can be used to control the quantum critical point and compute its characteristics in perturbative fashion. The crucial ingredient, however, behind this fortunate outcome is the linearity of the Dirac quasiparticle spectrum, which allows the Lorentz symmetry, although absent at the level of the lattice Hamiltonian, to emerge dynamically at the QC point. In the systems with the QBT, on the other hand, such an enlarged symmetry is certainly not expected at low energies, and it is a priori not even clear what dynamics to assume for the order-parameter field that should couple to the fermions.

Furthermore, as already implicit in [6] and as will be discussed here at length, one readily finds that the minimal Hamiltonian with the QBT point in 3D requires the use of the maximal set of five four-dimensional mutually anticommuting Dirac matrices. This is not an accident, and the situation is the same in 4D, except that one there needs the maximal set of nine sixteen-dimensional Dirac matrices. Having no further anticommuting matrix left prohibits then the opening of an isotropic mass-gap in the insulating state, which is usually preferred in the systems with Dirac fermions [22, 23]. This leaves as the energetically next-best option the dynamical generation of the second-rank tensorial order parameter, which breaks the rotational and preserves the time-reversal symmetry. Such a nematic order parameter, as well known from the studies of liquid crystals [24], allows a cubic rotationally-invariant term, which is typically responsible for a discontinuous transition. This makes the existence of the QC point in this system seem additionally questionable.

Given these difficulties that appear to be inherent to the problem at hand, it is quite remarkable that together they conspire to allow the construction of the Gross-Neveu-Yukawa continuum field theory for the nematic transition in the system with QBT that has a perturbatively accessible QC point. We find that it precisely the presence of the cubic invariant for the nematic order parameter that implies the existence of the upper critical dimension in the theory. The apparent ambiguity in the order-parameter dynamics also turns out to be possible to parametrize with a coupling which is ultimately irrelevant at the QC point, found near the upper critical (spatial) dimension of four. The fluctuations of the fermions, when all integrated out up to the deep infrared, result in the emergence of a nonanalytic term in the bosonic potential that is of lower order by power counting than the cubic invariant. The nematic quantum transition from the semimetallic phase into the insulating phase with anisotropic gap, described by the above QC point, is therewith presumably continuous, at least on the level of mean-field theory—in contrast to the classical thermal isotropic-to-nematic transition in liquid crystals [24].

To the leading order, the QC point is characterized by the dynamical critical exponent $z = 2$, a nontrivial positive anomalous dimension of the order-parameter field, and a vanishing anomalous dimension for the fermions. The relative signs of the cubic-term coupling and the Yukawa coupling at the critical point are also such that the state with fully gapped fermions is favored in the ordered phase, as one would expect from energetics [8].

The organization of the paper is as follows. In the next section we discuss the construction of the minimal isotropic QBT Hamiltonian in the form that most closely resembles the Dirac Hamiltonian, in general dimension. In Sec. III the Gross-Neveu-Yukawa continuum field theory for the nematic order parameter coupled to fermions is presented. We present the mean-field theory for the nematic quantum phase transition and discuss its order and the nature of the associated interacting ground state in Sec. IV. The structure of the renormalization group and the concomitant quantum critical point are discussed in Sec. V. In Sec. VI we interpret our findings. Concluding remarks are given in Sec. VII. Some nontrivial technical points necessary for the calculation are presented in four appendices.

II. QBT HAMILTONIANS IN DIFFERENT DIMENSIONS

We first discuss the construction of the minimal, rotationally invariant and particle-hole symmetric QBT Hamiltonian, in general spatial dimension $d$. We assume that in the momentum representation it has the form

$$ H = \sum_{i,j=1}^{d} G_{ij} p_i p_j, $$

with $G_{ij}$ as the matrix coefficients, which need to be determined. (For simplicity, we set the effective band mass to $2m = 1$.) Obviously, $G_{ij}$ must transform as the components of a second-rank symmetric tensor under rotations. Let us further assume that the anticommutator

$$ \{G_{oi}, G'_{ol}\} = \{G_{do}, G'_{ol}\} = 0, $$

where $G_{oi}$ is any of the diagonal elements $G_{ii}$, $G_{oi}$ is any of the off-diagonal element $G_{ij}$ with $i \neq j$, and $G_{of} \neq G'_{ol}$. Then

$$ H^2 = \sum_{i=1}^{d} G^2_{ii} p_i^4 + \sum_{i<j} p_i^2 p_j^2 (4G^2_{ij} + \{G_{ii}, G_{jj}\}). $$

If we normalize the diagonal elements so that all $G^2_{ii} = 1$, $H^2 = p^4$ provided that the following condition is satisfied:

$$ 4G^2_{of} + \{G_{oi}, G'_{ol}\} = 2. $$

Demanding further that the tensor $G_{ij}$ is traceless, the Hamiltonian $H$ would contain only the irreducible tensor $p_i p_j - \delta_{ij} p^2 / d$, and would be without the scalar term $\sim p^2$. The existence of such a scalar part would only introduce different curvatures of the upper and the lower
branches of the energy spectrum, and we omit it for the time being. We therefore set
\[ \sum_{i=1}^{d} G_{ii} = 0. \]
(5)
This, however, implies that, for arbitrary index \( k \),
\[ 0 = \{ G_{kk}, \sum_{i \neq k}^{d} G_{ii} \} = 2 + \sum_{i \neq k}^{d} \{ G_{kk}, G_{ii} \} \]
(6)
or, in other words that for any pair of diagonal elements
\[ \{ G_{d}, G_{d}' \} = \frac{2}{1 - d}. \]
(7)
When combined with Eq. (4) this in particular implies that off-diagonal elements are to be normalized as \( G_{d}^{2} = d/(2(d - 1)) \).
To construct the desired Hamiltonian \( H \) we therefore need
\[ (d^2 - d) + (d - 1) \]
mutually anticommuting Dirac matrices, for the off-diagonal (first) and the diagonal (second term) elements. Out of \( d - 1 \) Dirac matrices for the diagonal matrices, \( d \) matrices \( G_{ii} \) that satisfy Eq. (7) and square to unity can always be constructed.
For example:
(1) In \( d = 2 \) only two anticommuting matrices are needed, and therefore may be chosen as \( G_{12} = G_{21} = \sigma_1 \), and \( G_{11} = -G_{22} = \sigma_3 \). The Hamiltonian describes the band touching point in bilayer graphene, for example. Note that \( H \) is time-reversal symmetric, and the time-reversal operator is \( T = K \), the complex conjugation alone. Since \( T^2 = 1 \) this Hamiltonian can arise as a low-energy limit of a lattice Hamiltonian with spinless fermions hopping between sites \([23]\). Examples of such lattice Hamiltonians already exist in the literature\([3,4]\).
(2) In \( d = 3 \) one needs five Dirac matrices for the construction, so their minimal dimension is four. We can choose \( G_{12} = (\sqrt{3}/2)\gamma_2 \), \( G_{13} = (\sqrt{3}/2)\gamma_3 \), \( G_{23} = (\sqrt{3}/2)\gamma_4 \), and then for the diagonal elements
\[ G_{11} = -\frac{1}{2}\gamma_5 + \frac{\sqrt{3}}{2}\gamma_1, \]
(9)
\[ G_{22} = -\frac{1}{2}\gamma_5 - \frac{\sqrt{3}}{2}\gamma_1, \]
(10)
\[ G_{33} = \gamma_5. \]
(11)
The Hermitian Dirac matrices \( \gamma_a, a = 1, \ldots, 5 \) satisfy the Clifford algebra \( \{ \gamma_a, \gamma_b \} = 2\delta_{ab} \). With this particular choice the Hamiltonian can also be rewritten as
\[ H = \sum_{a=1}^{5} d_a(p)\gamma_a, \]
(12)
with \( d_a(p) = p^2\tilde{d}_a(\theta, \varphi) \) are proportional to five real spherical harmonics for the angular momentum of two; explicitly, \( \tilde{d}_1 + id_2 = (\sqrt{3}/2)\sin^2(\theta)e^{i\varphi} \), \( d_3 + id_4 = (\sqrt{3}/2)\sin(2\theta)e^{i\varphi} \), \( \tilde{d}_5 = (3\cos^2\theta - 1)/2 \), with \( \theta \) and \( \varphi \) as the spherical angles in the momentum space.
Note that among the five four-dimensional Dirac matrices we can always choose two (say \( \gamma_4 \) and \( \gamma_5 \)) as imaginary and the remaining three as real, so \( H \) is also time-reversal invariant, but now with (unique) \( T = \gamma_4\gamma_5 K \)[23]. Most importantly, \( T^2 = -1 \), and in three dimensions \( H \) inevitably describes particles with half-integer spin. In fact this “Luttinger Hamiltonian” is well known to arise from the spin-orbit coupling in gapless semiconductors such as gray tin, for example\([23,27]\).
Also, the Kramers’ theorem applies in this case and dictates that the spectrum is doubly degenerate at any momentum.
(3) For completeness, let us also display the solution for \( d = 4 \). For the off-diagonal elements we now need six mutually anticommuting matrices, and for the diagonal elements three more. The nine-component Clifford algebra has the unique irreducible representation being sixteen dimensional. We may then choose the off-diagonal elements as \( (G_{12}, G_{13}, G_{23}, G_{34}, G_{24}, G_{34}) = \sqrt{2}/3(\gamma_2, \gamma_3, \gamma_4, \gamma_6, \gamma_7, \gamma_8) \), and the diagonal elements as
\[ G_{11} = -\frac{1}{3}\gamma_9 - \sqrt{\frac{2}{3}}\gamma_3 + \sqrt{\frac{2}{3}}\gamma_1, \]
(13)
\[ G_{22} = -\frac{1}{3}\gamma_9 - \sqrt{\frac{2}{3}}\gamma_5 - \sqrt{\frac{2}{3}}\gamma_3, \]
(14)
\[ G_{33} = -\frac{1}{3}\gamma_9 + \sqrt{\frac{8}{3}}\gamma_5, \]
(15)
\[ G_{44} = \gamma_9. \]
(16)
Displaying \( H \) in the form equivalent to Eq. (12) would define the four-dimensional generalization of the \( \ell = 2 \) spherical harmonics. Note also that among the nine sixteen-dimensional Dirac matrices, four (say with indices \( a = 6, 7, 8, 9 \)) can be chosen to be purely imaginary, with the remaining five then as real\([28]\). The time-reversal operator that commutes with \( H \) exists, and is unique: \( T = \gamma_6\gamma_7\gamma_8\gamma_9 K \), but again \( T^2 = +1 \), and the minimal Hamiltonian in \( d = 4 \), similarly to \( d = 2 \) describes a spinless particle.
The solutions to the above conditions for the matrices \( G_{ij} \) can be found in all dimensions, with the properties of the minimal Hamiltonian under time reversal, for example, being strongly dimension dependent, as our examples already illustrate. Further details on the construction of the \( d \)-dimensional QBT Hamiltonian are provided in Appendix\([3]\) The construction can also be generalized to higher-order band touching, which would involve the higher-rank tensors and higher-angular-momentum spherical harmonics. Further elaboration of this point would be somewhat tangential to our main subject, and we leave it for another occasion.
III. THE GROSS-NEVEU-YUKAWA FIELD THEORY

We consider next the continuum quantum action for the QBT fermions in \( d = 3 \) and at \( T = 0 \), coupled to the fluctuating nematic order parameter: \( S = \int d\tau d^d x L \), with the Lagrangian density

\[
L = L_\psi + L_{\phi\phi} + L_\phi,
\]

and with the individual terms defined as

\[
L_\psi = \bar{\psi} (\partial_\tau + \gamma_a d_a (-i \nabla)) \psi,
\]

\[
L_{\phi\phi} = g \phi_a \bar{\psi}^4 \gamma_a \psi,
\]

\[
L_\phi = \frac{1}{4} T_{ij} \left( c |\partial_\tau - \nabla^2 + r | T_{ji} + \lambda T_{ij} T_{jk} T_{ki} + O(T^4). \right)
\]

\[
\psi \text{ is the four-component Grassmann field, whereas } \phi_a \text{ is a real field. The summation over the repeated indices is now assumed, and } a = 1, \ldots, 5, \text{ and } i, j, k = 1, 2, 3.
\]

\( \gamma_a \) are the five mutually anticommuting four-dimensional Dirac matrices introduced earlier.

The real, symmetric, traceless tensor field \( T_{ij} \) is defined as

\[
T_{ij} = \phi_a \Lambda_{a,ij},
\]

where \( \Lambda_a \) are the five real, symmetric, three-dimensional Gell-Mann matrices. Their explicit form and important properties are discussed in Appendix A. Since the five spherical harmonics \( d_a(\vec{p}) \) transform as the components of the traceless symmetric tensor of rank two under rotations, the Lagrangian \( L \) will be invariant under rotations provided that the five components of the tensor \( T_{ij}, \phi_a, a = 1, \ldots, 5 \) do so as well. At the level of the quantum mechanical averages

\[
\langle \phi_a \rangle = -\frac{g}{r} \langle \bar{\psi} \gamma_a \psi \rangle,
\]

and finding \( \langle \phi_a \rangle \neq 0 \) signals spontaneous breaking of the rotational symmetry. The tensor \( T_{ij} \) can be understood as the nematic order parameter, in analogy with liquid crystals, where the identical object describes the finite temperature phase transition between the isotropic and anisotropic phases \[24\].

The above form of the Lagrangian \( L \) contains the minimal number of parameters, and the imaginary time, length, and the Grassmann and the real fields have been rescaled so that the coefficients in front of the first and the second terms in \( L_\psi \), and the second term in \( L_\phi \) are brought to unity. Besides the tuning parameter \( r \), this still leaves the coefficient in the first term in \( L_\phi \), \( c \), and the two interaction coupling constants: Yukawa coupling \( g \), and the cubic term self-interaction \( \lambda \). These have the engineering dimensions

\[
\text{dim}[g] = \text{dim}[\lambda] = \frac{6 - z - d}{2},
\]

whereas

\[
\text{dim}[c] = 2 - z.
\]

At the Gaussian fixed point \( \lambda = g = 0 \), the dynamical critical exponent is \( z = 2 \), and one finds that both couplings \( g \) and \( \lambda \) become relevant in the infrared simultaneously below \( d = 4 \). This observation allows one to formulate a perturbative approach to the problem of the infrared behavior as the expansion in the small parameter

\[
\epsilon = 4 - d,
\]

and search for possible non-Gaussian critical points in the theory. The terms \( O(T^4) \) in \( L_0 \) have for this reason been omitted as irrelevant to the leading order in \( \epsilon \).

A comment on the form of the term quadratic in \( T_{ij} \) in \( L_0 \) is in order. The first issue is the choice of the term \( T_{ij} |\partial_\tau | T_{ji} \sim |\omega| \phi_a \phi_a \) in Eq. \[20\], which is evidently nonanalytic in frequency. First, to have the dynamical critical exponent well defined at the Gaussian fixed point this term needs to be first order in frequency (i.e., in time derivative). Second, having simply \( T_{ij} |\partial_\tau | T_{ji} \) will not do, since the reality of the tensor \( T \) would make such a term vanish. Third, this term may be understood as deriving from the integration over the fermions, after the Fermi-Stratonovich field has been decoupled by the Hubbard-Stratonovich field in the nematic channel. Finally, we will find that at the critical point the coefficient of this term in fact vanishes, and ultimately \( c = 0 \). This term may therefore be understood as being necessary to include and keep at the intermediate stages of the calculation in order to treat the fluctuations of the fermions and order parameter on equal footing, but as ultimately irrelevant at the QC fixed point, in the technical (renormalization group) sense.

The second issue is the omission of yet another rotationally invariant term in \( L_\phi \),

\[
\sim \partial_i T_{ij} \partial_k T_{kj},
\]

which couples spatial rotations to internal rotations of the nematic order parameter, and is thus possible only when the dimension \( p \) of the tensor \( T_{ij} \) \( (i, j = 1, \ldots, p) \) is equal to the spatial dimension \( d \) (as is the case in our problem). We find that although of the same engineering dimension to the leading order in interactions, and as such we expect it to become irrelevant at the interacting critical point (see Appendix C). One can analogously justify the common omission of this term in the studies of the classical isotropic-to-nematic transition in three dimensions.

IV. MEAN-FIELD THEORY

Before we present the solution of the problem in the vicinity of the upper critical dimension, let us consider
the mean-field theory in which the fluctuations of the order-parameter field $\phi_a$ are neglected. This approximation can be justified by adding an additional “flavor” index to the fermions (e.g., by allowing more than one QBT point at the Fermi level) and taking the limit of large flavor number $N$ [8]. The mean-field theory is solved by minimizing the total energy

$$E_{\text{MF}}(\phi_1, \ldots, \phi_3) = \frac{r}{2} \phi_a \phi_a + 2 \int_0^\Lambda \frac{d\tilde{p}}{(2\pi)^3} \varepsilon(\tilde{p})$$

(27)

where $\varepsilon(\tilde{p})$ are the energy eigenvalues of the mean-field Hamiltonian $H_{\text{MF}}(\tilde{p}) = p^2 \tilde{a}_a(\theta, \varphi) \gamma_a + g \phi_a \gamma_a$ in the presence of constant $\phi_a$, viz.,

$$\varepsilon(p, \theta, \varphi) = -p^2 \sqrt{1 + 2\tilde{a}(\theta, \varphi) \frac{g \phi_a}{p^2} + \left(\frac{g \phi_a}{p^2}\right)^2}.$$  

(28)

$\Lambda$ denotes the UV momentum cutoff, $0 \leq |\tilde{p}| \leq \Lambda$. For convenience, and without loss of generality, let us assume $g > 0$. The first term in Eq. (27) represents the energy cost of a finite $\phi_a$. By contrast, the second term decreases with increasing order parameter, and thus involves the energy gain due to a (possible) ordering. It can be interpreted as the sum of the energies of the filled, doubly degenerate single-particle states in the ordered phase, with the Fermi level at the QBT. In the present model without the long-range Coulomb interaction and in $d = 3$ we expect the ordered state to be energetically favorable if the parameter $g^2/r$ exceeds a certain strong-coupling threshold. This threshold, however, may decrease substantially upon the inclusion of the long-range part of the Coulomb repulsion, and might even vanish completely [8].

In the reference frame in which the tensor order parameter becomes diagonal,

$$(T_{ij}) = \begin{pmatrix} \phi_1 - \phi_0 & 0 & 0 \\ 0 & -\phi_1 - \phi_0 & 0 \\ 0 & 0 & 2\phi_0 \end{pmatrix},$$

(29)

we can write $(\phi_0) = (\phi \sin \xi, 0, 0, 0, \phi \cos \xi)$ with $\phi := \sqrt{\phi_a \phi_a} \geq 0$. Shifting the parameter $\xi$ by $\xi \rightarrow \xi + 2\pi/3$ corresponds to a cyclic permutation of the $x$-, $y$-, and $z$-axes. E.g., the state $(\phi_0) = (\phi(\sqrt{3}/2, 0, 0, 0, 1/2)$ for $\xi = \pi/3$ transforms into the state $(\phi_a) = (\phi(0, 0, 0, -1)$ for $\xi = \pi$ by permuting $(x, y, z) \rightarrow (y, z, x)$. We may thus restrict the range of $\xi$ to $0 \leq \xi < 2\pi/3$. Finding a finite $\phi \neq 0$ to be energetically favorable corresponds to a spontaneous breaking of the rotational symmetry. While for generic $\xi$ no continuous part of the symmetry is left intact, for $\xi \equiv 0 \mod 2\pi/3$ or $\xi \equiv \pi/3 \mod 2\pi/3$ only two generators of the O(3) are broken, with a residual O(2) symmetry resulting. The corresponding uniaxial states $(\phi_0) = (0, 0, 0, 0, 0)$ (modulo rotations) are characterized by a single director, in analogy to the uniaxial nematic phase in liquid crystals [24].

The energy in the present basis reads as

$$E_{\text{MF}}(\phi, \xi) = \frac{r}{2} \phi^2 - 2 (g \phi)^{5/2} \int_0^\Lambda \frac{d\tilde{p}}{(2\pi)^3} \varepsilon(\tilde{p}) \left( x^2 \sqrt{x^4 + 2x^2(\tilde{d}_1 \sin \xi + \tilde{d}_5 \cos \xi)} + 1 \right)$$

(30)

where we substituted $p/\sqrt{g \phi} \rightarrow x$ and abbreviated the angular integration as $\int d\Omega = \int_0^\pi \sin \vartheta \int_0^{2\pi} d\varphi$. The integral becomes finite for $\Lambda/\sqrt{g \phi} \rightarrow \infty$ when we add a suitably written zero (corresponding to the parts in $E_{\text{MF}}$ that are constant and quadratic in $\phi$, respectively) as

$$0 = -\frac{4\pi}{(2\pi)^3} \left( \frac{2}{5} \Lambda^5 + \frac{4}{5} \Lambda g^2 \phi^2 \right)$$

$$+ 2 (g \phi)^{5/2} \left( \frac{x^2 + 2}{5} \right).$$

(31)

The mean-field energy becomes (modulo irrelevant additive constants $\sim \Lambda^5$)

$$E_{\text{MF}}(\phi, \xi) = \frac{r'}{2} \phi^2 + t(\xi) (g \phi)^{5/2} + O(\phi^3)$$

(32)

with $r' = r - 8 \frac{4\ln 3}{3\pi g^2} g^2$ the curvature at the origin and with the coefficient of the nonanalytic term $\sim \phi^{5/2}$ as

$$t(\xi) = 2 \int_0^\infty dx \int \frac{d\Omega}{(2\pi)^3} \left[ x^4 + \frac{2}{5} \right]$$

$$- x^2 \sqrt{x^4 + 2x^2(\tilde{d}_1 \sin \xi + \tilde{d}_5 \cos \xi)} + 1$$

$$\approx \frac{4\pi}{(2\pi)^3} \left[ \frac{\pi}{8} + \frac{1}{2} \left( \frac{19}{30} - \frac{\ln 3}{8} - \frac{\pi}{8} \right) (1 - \cos 3\xi) \right].$$

(33)

The second line of Eq. (33) approximates the numerical quadrature within an error range of $\lesssim 0.5\%$ for generic $\xi$ and becomes exact for $\xi = 0$ and $\xi = \pi/3$. $t(\xi)$ is positive and bounded from below and above as $\frac{\pi}{16} \leq t(\xi) / (5g^2) \leq \frac{1}{16}$. The QC point at the critical coupling

$$\left( \frac{g^2}{r} \right)_c \approx \frac{5}{8} \frac{(2\pi)^3}{4\pi \Lambda},$$

(34)

when the curvature $r'$ of $E_{\text{MF}}(\phi, \xi)$ at $\phi = 0$ changes sign, thus corresponds to a continuous phase transition—in contrast to the discontinuous (at least on the mean-field level) classical isotropic-to-nematic transition in liquid crystals [24]. A similar such unconventional continuous phase transition has recently been found in a model describing the spontaneous breaking of time-reversal symmetry in the pyrochlore iridates [29].

$t(\xi)$ attains its unique minimum at $\xi = 0$. When $g^2/r > (g^2/r)_c$, the transition is thus into the state with the order parameter $(\phi_0) = (0, 0, 0, 0, \phi)$, $\phi > 0$, which
breaks the rotational $O(3)$ symmetry but leaves rotations about the $z$-axis intact. The spectrum of fermions in this state has a full, anisotropic ($\theta$-dependent) gap, with the minimal value at $\theta = \pi/2$ and $p^2 = g^2/2$ of $\sqrt{3}g\phi/2$. The system appears as if under (dynamically generated) uniaxial strain [31,30], and, for the systems with the band structure equivalent to that of α-Sn or HgTe, represents a topological Mott insulator [3]. We depict the mean-field energy $E_{MF}(\phi_5)$ for the $O(2)$-invariant states $(\phi_\alpha) = (0,0,0,0,\phi_5)$ for different values of the coupling $g^2/r$ in the vicinity of the critical coupling $(g^2/r)_c$. The unique absolute minimum of the potential is at zero or positive $g\phi_5$, corresponding to the isotropic state and uniaxial nematic fully gapped state, respectively. The transition into the latter phase for overcritical coupling is continuous.

V. RG FLOW EQUATIONS

In order to show the existence of the nematic QC point beyond the mean-field theory we include next the effects of the bosonic fluctuations. To this end we perform the standard Wilson’s renormalization group calculation, in which both the order parameter and the fermionic fields with the momenta within the momentum shell $[\Lambda/b, \Lambda]$ and with all Matsubara frequencies are integrated out [32]. We depict the mean-field energy $E_{MF}(\phi_5)$ for different values of the coupling $g^2/r$ in Fig. 1, illustrating the continuous nature of the transition and the energetically favored minimum at $g\phi_5 > 0$ [31].

\[
\frac{d\lambda}{d\ln b} = \frac{1}{2} (6 - d - z - 3\eta_\phi)\lambda - \frac{18}{\pi} \lambda^3 - \sqrt{\frac{35}{3}} g^3. \quad (37)
\]

Here, we have rescaled the couplings as $g^2\Lambda^{d+z+2n_\phi}S_d/(2\pi)^d \rightarrow g^2$ and $\lambda^2\Lambda^{d+z+3n_\phi}S_d/(2\pi)^d \rightarrow \lambda^2$ with $S_d$ the surface area of the $(d-1)$-sphere. The Eq. (35) is in fact exact, as follows from the nonanalyticity of the time-derivative term in $L_\phi$. The order parameter’s and the fermion’s anomalous dimensions, and the dynamical critical exponent are to the leading order

\[
\begin{align*}
\eta_\psi &= \frac{4}{5} F(c) g^2, \\
\eta_\phi &= \frac{17}{15} g^2 + \frac{14}{3\pi} \frac{\lambda^2}{c}, \\
\lambda &= 2 + 5G(c)g^2 - \eta_\psi.
\end{align*}
\]  

The anomalous dimensions and the dynamical exponent are chosen so that the coefficients in $L_\phi$ remain unity after the mode elimination, which forces the remaining coefficient $c$ in $L_\phi$ then to flow. The functions $F(c)$, $G(c)$, and $H(c)$ are the result of the one-loop frequency integrals, and are defined as

\[
\begin{align*}
F(c) &= \frac{\pi (1 - 3c^2) + c(5 + 6c^2 + c^4 + 6\ln c - 2c^2 \ln c)}{\pi (1 + c^2)^3}, \\
G(c) &= \frac{c(-\ln c - 1 + \pi c - c^2(1 - \ln c))}{\pi (1 + c^2)^2}, \\
H(c) &= \frac{\pi(2 + c^2) + c(1 + c^2 + 5\ln c + 3c^2 \ln c)}{\pi (1 + c^2)^3}.
\end{align*}
\]

Small perturbations out of the critical surface are relevant in the sense of the RG, and governed by the flow equation

\[
\frac{dr}{d\ln b} = (2 - \eta_\psi)r - \frac{8}{5} g^2 + \frac{84}{\pi} \frac{\lambda^2}{c} \frac{1}{1 + r}, \quad (44)
\]

where we have rescaled $r\Lambda^{d+z-2} \rightarrow r$. In order to arrive at Eqs. (35)-[44], we have kept the general counting of dimensions in the couplings, but have performed the angular integrations directly in $d = 3$ spatial dimensions. For details on the computation we refer to Appendix C. In Appendix D, we present the analogous derivation of the RG flow for the theory near $d = 4$ with nine-component order-parameter field $\phi_\alpha$ and $16 \times 16$ gamma matrices $\gamma_\alpha$, $\alpha = 1, \ldots, 9$.

The mean-field result from the previous section can be recovered by neglecting all bosonic fluctuations (corresponding formally to taking the limit of large $c$ in Eqs. (35)-(44), or by reintroducing the flavor number $N$ and taking the limit of large $N$). The flow equation for the coupling $g^2/r$ then becomes

\[
\frac{d(g^2/r)}{\ln b} = (d-4)g^2_r + \frac{8}{5} \left(\frac{g^2}{r}\right)^2, \quad (45)
\]
which in \( d = 3 \) has the zero exactly at the mean-field critical coupling \( (g^2/\nu_0) = 5/8 \), c.f. Eq. (34) and the coupling rescalings below Eqs. (37) and (38).

To show that there exists a stable (quantum critical) fixed point of the equations also at \( N = 1 \) we introduce new variables

\[
u = \frac{\lambda}{\epsilon^{1/2}}, \quad \nu = \frac{g}{\epsilon^{1/6}}. \quad (46)
\]

In terms of the new variables we can rewrite the flow equations as

\[
du \quad d\ln b = \frac{1}{2} \left( \epsilon - 2\eta_\phi \right) u - \frac{18}{\pi} u^3 - \frac{\sqrt{3}}{35} \nu^3, \quad (47)
\]

\[
dv \quad d\ln b = \frac{1}{2} \left( \epsilon + \frac{4 - 2z}{3} - 2\eta_\phi \right) v + \frac{3}{5} \epsilon^{1/3} H(c) \nu^3, \quad (48)
\]

where we also displayed the small parameter \( \epsilon = 4 - d \).

After this change of variables, the stable fixed point is readily found to lie at \( c = 0 \). Since \( F(0) = 1, G(0) = 0, \) and \( H(0) = 2 \), the fixed point leads to the critical

\[
z = 2 + \mathcal{O}(\epsilon^2), \quad \eta_\psi = \mathcal{O}(\epsilon^2), \quad \eta_\phi = \frac{3}{2} \epsilon + \mathcal{O}(\epsilon^2), \quad (49)
\]

and it is located at the values of \( u \) and \( v \) that satisfy the equations:

\[
\frac{3}{2} \epsilon = \frac{14u^2}{3\pi}, \quad \frac{\sqrt{3}}{35} \nu^3 = u \left( -\epsilon - \frac{18}{\pi} u^2 \right). \quad (50)
\]

The last equation, in particular, implies that at the fixed point \( \text{sign}(v) = -\text{sign}(u) \), whereas the first one leaves the sign of \( u \) undetermined. We find the following finite fixed-point values

\[
u^*_\pm = \pm \sqrt[9]{\frac{9}{28}} \epsilon, \quad v^*_\pm = \pm \left( \frac{475}{4} \right)^{1/3} \left( \frac{3\pi}{7} \right)^{1/6} \epsilon. \quad (51)
\]

As the partition function is invariant under the simultaneous sign change of \( g \) and \( \lambda \), so are the flow equations. Thus, the two fixed points at \( (\nu^*_+, v^*_+) \) are physically equivalent. It is easy to check that this fixed point is indeed critical, i.e., with no other unstable directions except for the direction of the tuning parameter \( r \). From the flow of \( r \) we find the exponent \( \nu \) that governs the scaling of the correlation length \( \xi \sim |\delta|^{-\nu} \), with \( \delta \) denoting the deviation from the critical point, as

\[
1/\nu = 2 + \frac{51}{2} \epsilon + \mathcal{O}(\epsilon^2). \quad (52)
\]

Notably, the correction to the mean-field exponent \( 1/\nu = 2 \) is positive, in contrast to the QC points in Dirac fermion systems that are described by the \( z = 1 \) Gross-Neveu universality classes [21, 33].

We have plotted the RG flow in the \( u-v \) plane for \( c = 0 \) in Fig. 2 showing besides the unstable Gaussian (G) and stable fermionic (F) fixed points also the purely bosonic fixed point (B) at \( v = 0 \) and finite \( u \neq 0 \). B is unstable in the direction of \( v \), in analogy to the bosonic Wilson-Fisher fixed point in Dirac fermion systems [21].

In the calculation with four-dimensional tensor order parameter we find that the bosonic fixed point B disappears, in full analogy to the \( p = 4 \) critical point in the field theory of the classical isotropic-to-nematic phase transition in liquid crystals [34]. In contrast, the fermionic fixed point (F in Fig. 2) survives for any dimension \( p \) of the tensor field, with changing only its stability properties at larger values of \( p \) (see Appendix D).

VI. INTERPRETATION

At the mean-field level, the model features a continuous (quantum) phase transition, described by the large-\( N \) fixed point of the Gross-Neveu-Yukawa field theory. It therefore seems natural to associate the identified fixed point also for \( N = 1 \) with a continuous nematic quantum phase transition. One should note, however, that near the upper critical dimension and at small \( N \) the parameter \( c \) in \( L_\phi \) becomes small at the fixed point, emphasizing the significance of purely bosonic fluctuations. The result of the mean-field theory may thus as well be overturned in the physical limit, and the possibility that at small \( N \) the nature of the transition differs from the mean-field picture cannot be excluded with certainty. We believe, nonetheless, that even in a scenario with a discontinuous quantum phase transition the above critical fixed point would still retain its physical significance: such a situation arises, for example, in the related classical Ginzburg-Landau-Wilson theory for the (presumably
discontinuous) thermal isotropic-to-nematic transition in liquid crystals, which also exhibits a critical fixed point in the related \(\epsilon = 6 - d\) expansion [33]. A plausible interpretation of the latter is that it describes the disappearance of the energy barrier between the high- and low-temperature phases, and the ultimate instability of the metastable symmetric phase.

An interesting feature of the identified fixed point is worth pointing out. In the reference frame where the nematic tensor would become diagonal [Eq. (25)], the bosonic part of the Lagrangian for uniform order parameter \((\phi_a) = (\phi \sin \xi, 0, 0, 0, \phi \cos \xi), \phi > 0\), becomes

\[
L_\phi = \frac{r}{2} \phi^2 + \frac{2\lambda}{\sqrt{3}} \cos(3\xi)\phi^3 + b\phi^4 + O(\phi^5),
\]

(53)

where we have displayed the unique symmetry-allowed quartic term as well. At intermediate steps of the RG the Lagrangian is analytic in \(\phi\), and the nonanalytic term \(\sim \phi^{5/2}\) will only emerge in the deep infrared, when all modes are integrated out. During this process it may in general receive contributions from the flow of all higher-order terms. If we focus for simplicity only on the leading cubic and quartic invariants and consider (without loss of generality) the fixed point at \(\lambda/e^{1/2} < 0\), we find that the effective quantum potential at the fixed point is minimized for \(\xi = 0\). If this remains true up to the infrared, when the \(\phi^{5/2}\) term in \(L_\phi\) emerges, it indicates that the interacting ground state for strong coupling has (also for small \(N\)) the uniaxial form with \(\phi_0 > 0\) and \(\phi_1 = 0\). The fate of fermions in this state depends crucially then on the sign of the remaining Yukawa coupling \(\gamma\). If \(\gamma > 0\), the combination \(\gamma\phi_0 > 0\), and we recover the mean-field ground state with the spectrum of fermions having the full, anisotropic gap (c.f. Sec. [IV]). If, on the other hand, \(\gamma < 0\) and \(\gamma\phi_0 < 0\), the spectrum has two gapless points in the vicinity of which the energy dispersion becomes linear [33].

We see however, that the leading term in the flow equation for \(\lambda\) is \(\gamma^3\), so that a negative self-interaction \(\lambda\) is generated only by a positive Yukawa coupling \(\gamma\). This is reflected in the fixed-point location, at which the signs of the two couplings are inevitably opposite. Incidentally, this feature is also responsible for the stability of the fixed point. Also, even if we start the RG flow at microscopic couplings \(\lambda\) and \(\gamma\) of the same sign, we always flow to a regime in which \(\gamma\phi_0 < 0\), at least in the vicinity of the critical surface (see Fig. 2). We thus find that the consistent theory in the infrared has the fermions fully gapped in the broken symmetry phase, in agreement with the mean-field result.

VII. CONCLUSIONS

In sum, we constructed the field theory of the fermions with the chemical potential at the point of quadratic band touching in three spatial dimensions coupled to the second-rank tensorial nematic order parameter. We argued that this field theory has an upper critical dimension of four, and that it possesses a perturbatively accessible quantum critical point in the vicinity of four dimensions. The critical point governs the (presumably continuous) transition between the semimetal to the fully, but anisotropically gapped Mott insulator. The existence of the critical point in the theory supports the scenario of the “fixed-point collision” [8], according to which the QC point and the Abrikosov’s NFL fixed point, in presence of long-range tail of the Coulomb interaction, which we have here suppressed, collide and then disappear from the real space of physical couplings, leaving behind the runaway flow.

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Appendix A: Generalized real Gell-Mann matrices

For completeness, let us review the construction of the generalized Gell-Mann matrices in \(d\) dimensions (the generators of \(SU(\hat{d})\)) [35]. They can be classified into three groups. The first group is given by the real, diagonal, and traceless matrices

\[
\hat{u}_l = \sqrt{\frac{2}{l(l + 1)}} \sum_{j=1}^l |\langle j | \langle j | - | l + 1 \rangle \langle l + 1 | \rangle, \quad \text{(A1)}
\]

where \(1 \leq l \leq d - 1\) and \(|1\), \ldots, \(|d\) denote the (standard) orthonormal basis vectors in \(\mathbb{R}^d\), \(\langle i | j \rangle = \delta_{ij}\). The second group are \(d(d-1)/2\) real symmetric matrices that have nonvanishing elements only on the off-diagonal, namely the matrices \(\hat{u}_{jk}\) with ones in the \(jk\)-th and \(kj\)-th entries and zero otherwise

\[
\hat{u}_{jk} = |j \rangle \langle k| + |k \rangle \langle j|, \quad \text{where} \quad 1 \leq j < k \leq d. \quad \text{(A2)}
\]

The third group are \(d(d-1)/2\) imaginary matrices which can be constructed similarly to \(\hat{u}_{jk}\). However, for the purposes of the present work we only need the real Gell-Mann matrices of the first and second group.

In \(d = 2\), this construction gives \(\hat{w}_1 = -\sigma_3\) and \(\hat{u}_{12} = \sigma_1\). In \(d = 3\), we recover the standard (modulo name and sign conventions) \(3 \times 3\) real Gell-Mann matrices:

\[
\Lambda_1 = -\hat{w}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \hat{u}_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\Lambda_3 = \hat{u}_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Lambda_4 = \hat{u}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]
\[ \Lambda_5 = \hat{w}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \]  

In \( d = 4 \), we find
\[ \begin{align*} 
\Lambda_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & 
\Lambda_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & 
\Lambda_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_5 &= \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & 
\Lambda_6 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & 
\Lambda_8 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\Lambda_9 &= \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. 
\]  

In general dimension \( d \), the \((d^2 - d)/2 + (d - 1)\) off-diagonal and diagonal, respectively, real matrices \( \Lambda_a \) form an orthogonal set:
\[ \text{Tr}(\Lambda_a \Lambda_b) = 2\delta_{ab}, \] (A5)
and together with the unit matrix, they form a basis in the space of real symmetric \( d \)-dimensional matrices. We can therefore write the matrix element of any symmetric matrix \( M \) as
\[ M_{ij} = \frac{1}{d} \delta_{ij} M_{kk} + \frac{1}{2} M_{lm} \Lambda_{a,ml} \Lambda_{a,ij}, \] (A6)

or equivalently, as
\[ \frac{1}{2} (\delta_{ij} \delta_{mj} + \delta_{ij} \delta_{mi}) M_{lm} = \left( \frac{1}{d} \delta_{ij} \delta_{lm} + \frac{1}{2} \Lambda_{a,ml} \Lambda_{a,ij} \right) M_{lm}. \] (A7)

From here we deduce an important relation:
\[ \Lambda_{a,ml} \Lambda_{a,ij} = \delta_{ij} \delta_{mj} + \delta_{ij} \delta_{mi} - \frac{2}{d} \delta_{ij} \delta_{lm}, \] (A8)

which we use in the computation of the RG flow equations.

### Appendix B: QBT Hamiltonian in \( d \) dimensions

We can construct the general QBT Hamiltonian \( H = G_{ij} p_i p_j \) in \( d \) dimensions with the help of \((d^2 - d)/2 + (d - 1) = (d + 2)(d - 1)/2\) gamma matrices \( \gamma_a \). They have dimension \( d_2 = 2(d + 2)(d - 1)/4 \) with \( \lfloor \ldots \rfloor \) denoting the floor function. The relationship between the \( G_{ij} \) and the gamma matrices \( \gamma_a \), \( a = 1, \ldots, (d + 2)(d - 1)/2 \) are given by the real and symmetric (generalized) \( d \times d \) Gell-Mann matrices \( \Lambda_a \) as
\[ G_{ij} = \sqrt{\frac{d}{2(d - 1)}} \Lambda_{a,ij} \gamma_a. \] (B1)

Together with the Clifford algebra \( \{ \gamma_a, \gamma_b \} = 2\delta_{ab} \) and Eq. (A8), this immediately gives \( H^2 = p^4 \), as expected. In any dimension, we can thus write the Hamiltonian in the form
\[ H = d_a (\vec{p}) \gamma_a, \quad a = 1, \ldots, \frac{1}{2} (d + 2)(d - 1), \] (B2)

with
\[ d_a (\vec{p}) = p^2 \tilde{d}_a (\Omega) = \sqrt{\frac{d}{2(d - 1)}} p_a \Lambda_{a,ij} p_j. \] (B3)

This defines the real hyperspherical harmonics \( \tilde{d}_a (\Omega) \) for angular momentum of two in general dimension, with \( \Omega \) denoting the spherical angles on the \((d - 1)\)-sphere in \( \vec{p} \)-space.

### Appendix C: Computation of RG flow equations

Let provide details on the computation of the RG flow equations \([35][40]\). In the perturbative expansion, after integrating out the high-energy modes with momenta in the thin shell \([\Lambda/b, \Lambda] \), we arrive at the effective action for the low-energy modes
\[ S_c = \int_0^{\Lambda/b} \frac{d \vec{k}}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d \omega}{2\pi} \left[ \psi^\dagger \left( b^{n_i} i \omega + b^{n_e} \right) \left( \tilde{k} \gamma_a \right) \psi + \frac{1}{2} \phi_a \left( c \mid \omega \mid + b^{n_e} \tilde{k}^2 + (r + \delta r) \right) \phi_a + (\beta + \delta \beta) k_k k_j \Lambda_{a,kl} \Lambda_{b,lj} \phi_a \phi_b \right] + \int_0^{\Lambda/b} \frac{d \vec{k}_1}{(2\pi)^d} \int_0^{\Lambda/b} \frac{d \vec{k}_2}{(2\pi)^d} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d \omega_1}{2\pi} \frac{d \omega_2}{2\pi} \left[ (g + \delta g) \left( \phi_a \psi^\dagger \gamma_a \psi \right) + (\lambda + \delta \lambda) \Lambda_{a,ij} \Lambda_{b,kl} \Lambda_{c,li} \phi_a \phi_b \phi_c \right] \] (C1)

where we have included the second momentum term \( \sim k_k k_T T_i T_j \) (third line in Eq. [C1]) for generality. The anomalous dimensions \( \eta_1, \eta_\psi, \) and \( \eta_\gamma \) and the coupling renormalizations \( \delta r, \delta \beta, \delta g, \) and \( \delta \lambda \) are determined by evaluating the corresponding one-loop diagrams, as depicted in Fig. [3] \( \eta_1 \) and \( \eta_\gamma \) are given by the fermion-boson loop in Fig. [5](a), expanded to first order in external frequency \( \omega \) and second order in external momentum \( k \), respectively. \( \eta_\psi \) has two contributions, given by the diagrams in Figs. [5](b)–(c), expanded to second order.
in external momentum. The constant parts of these diagrams determine the shift of the tuning parameter \( r \). Neither diagram gives a contribution linear in the external frequency. The boson- and fermion-loop diagrams in Figs. 3(d)–(e), respectively, renormalize the bosonic self-interaction \( \lambda \). In order to evaluate these diagrams, we continually make use of the identities in Eqs. (A8) and (B3) derived above. For instance, for the evaluation of the fermion loop in Fig. 3(c) we need the following angular integral over the \((d-1)\)-sphere in \( \vec{p} \)-space

\[
\int d\Omega_d(\vec{p})d_b(\vec{p})d_c(\vec{p}) = \left( \frac{d}{2(d-1)} \right)^{3/2} \int d\Omega p_i p_j p_k p_t p_m p_n \Lambda_{a,i} \Lambda_{b,j} \Lambda_{c,mn} = \frac{4S_d}{(2d-1)(d-1)(d+2)(d+4)} \text{Tr}(\Lambda_a \Lambda_b \Lambda_c) p^6, \tag{C2}
\]

where \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) is the surface area of the \((d-1)\)-sphere and \( \Omega \) again denotes the spherical angles on the sphere. The evaluation of the triangle diagram in Fig. 3(f), which renormalizes the Yukawa vertex \( g \), is straightforward when making use of the orthogonality of the real spherical harmonics

\[
\int d\Omega d_a(\vec{p})d_b(\vec{p}) = \frac{2S_d}{(d+2)(d-1)} p^4 \delta_{ab}, \tag{C3}
\]

In order to bring the cutoff in \( S_\varphi \) back to \( \Lambda \) we shift the momenta \( b\vec{k} \rightarrow \vec{k} \) and frequencies \( b^2 \omega \rightarrow \omega \) with suitable dynamical exponent \( z \). The coefficients of the momentum terms \( \sim k^2 \) in the fermionic and bosonic propagators in the first and second line of Eq. (C1), respectively, can be fixed to one if we renormalize the fields as

\[
b^{-2(2+d+z-\eta_\varphi)/2} \psi \rightarrow \psi, \quad b^{-2(2+d+z-\eta_\varphi)/2} \phi \rightarrow \phi. \tag{C4}
\]

However, then only one of the frequency terms can be fixed. We choose the fermionic term \( \sim i\omega \), which is done by setting \( z = 2 + \eta_\varphi - \eta_\omega \). The low-energy action \( S_\varphi < \) is hence brought back into the same form as before integrating out the momentum shell if the couplings are renormalized as

\[
\frac{dc}{d\ln b} = (2 - z - \eta_\varphi)c, \quad \frac{dg}{d\ln b} = \frac{1}{2} \left( 6 - d - z - \eta_\varphi - 2\eta_\omega \right) g + \frac{\partial \delta g}{\partial \ln b}, \quad \frac{d\lambda}{d\ln b} = \frac{1}{2} \left( 6 - d - z - 3\eta_\omega \right) \lambda + \frac{\partial \delta \lambda}{\partial \ln b}, \quad \frac{dr}{d\ln b} = (2 - \eta_\varphi)r + \frac{\partial \delta r}{\partial \ln b}. \tag{C5}
\]

If we rescale the couplings as

\[
g^2 \Lambda^d z + \eta_\omega \rightarrow g \Lambda^d z, \quad \lambda^2 \Lambda^d + 3\eta_\omega \rightarrow \lambda \Lambda^d, \quad r \Lambda^d \rightarrow r, \tag{C6}
\]

the new low-energy action \( S_\varphi \) is unchanged. We then only one of the frequency terms can be fixed. We choose the fermionic term \( \sim i\omega \), which is done by setting \( z = 2 + \eta_\varphi - \eta_\omega \). The low-energy action \( S_\varphi \) is hence brought back into the same form as before integrating out the momentum shell if the couplings are renormalized as

\[
\frac{dc}{d\ln b} = (2 - z - \eta_\varphi)c, \quad \frac{dg}{d\ln b} = \frac{1}{2} \left( 6 - d - z - \eta_\varphi - 2\eta_\omega \right) g + \frac{\partial \delta g}{\partial \ln b}, \quad \frac{d\lambda}{d\ln b} = \frac{1}{2} \left( 6 - d - z - 3\eta_\omega \right) \lambda + \frac{\partial \delta \lambda}{\partial \ln b}, \quad \frac{dr}{d\ln b} = (2 - \eta_\varphi)r + \frac{\partial \delta r}{\partial \ln b}. \tag{C5}
\]

However, then only one of the frequency terms can be fixed. We choose the fermionic term \( \sim i\omega \), which is done by setting \( z = 2 + \eta_\varphi - \eta_\omega \). The low-energy action \( S_\varphi \) is hence brought back into the same form as before integrating out the momentum shell if the couplings are renormalized as

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\frac{dc}{d\ln b} = (2 - z - \eta_\varphi)c, \quad \frac{dg}{d\ln b} = \frac{1}{2} \left( 6 - d - z - \eta_\varphi - 2\eta_\omega \right) g + \frac{\partial \delta g}{\partial \ln b}, \quad \frac{d\lambda}{d\ln b} = \frac{1}{2} \left( 6 - d - z - 3\eta_\omega \right) \lambda + \frac{\partial \delta \lambda}{\partial \ln b}, \quad \frac{dr}{d\ln b} = (2 - \eta_\varphi)r + \frac{\partial \delta r}{\partial \ln b}. \tag{C5}
\]

If we rescale the couplings as

\[
g^2 \Lambda^d z + \eta_\omega \rightarrow g \Lambda^d z, \quad \lambda^2 \Lambda^d + 3\eta_\omega \rightarrow \lambda \Lambda^d, \quad r \Lambda^d \rightarrow r, \tag{C6}
\]

the new low-energy action \( S_\varphi \) is unchanged. We then only one of the frequency terms can be fixed. We choose the fermionic term \( \sim i\omega \), which is done by setting \( z = 2 + \eta_\varphi - \eta_\omega \). The low-energy action \( S_\varphi \) is hence brought back into the same form as before integrating out the momentum shell if the couplings are renormalized as

\[
\frac{dc}{d\ln b} = (2 - z - \eta_\varphi)c, \quad \frac{dg}{d\ln b} = \frac{1}{2} \left( 6 - d - z - \eta_\varphi - 2\eta_\omega \right) g + \frac{\partial \delta g}{\partial \ln b}, \quad \frac{d\lambda}{d\ln b} = \frac{1}{2} \left( 6 - d - z - 3\eta_\omega \right) \lambda + \frac{\partial \delta \lambda}{\partial \ln b}, \quad \frac{dr}{d\ln b} = (2 - \eta_\varphi)r + \frac{\partial \delta r}{\partial \ln b}. \tag{C5}
\]

However, then only one of the frequency terms can be fixed. We choose the fermionic term \( \sim i\omega \), which is done by setting \( z = 2 + \eta_\varphi - \eta_\omega \). The low-energy action \( S_\varphi \) is hence brought back into the same form as before integrating out the momentum shell if the couplings are renormalized as

\[
\frac{dc}{d\ln b} = (2 - z - \eta_\varphi)c, \quad \frac{dg}{d\ln b} = \frac{1}{2} \left( 6 - d - z - \eta_\varphi - 2\eta_\omega \right) g + \frac{\partial \delta g}{\partial \ln b}, \quad \frac{d\lambda}{d\ln b} = \frac{1}{2} \left( 6 - d - z - 3\eta_\omega \right) \lambda + \frac{\partial \delta \lambda}{\partial \ln b}, \quad \frac{dr}{d\ln b} = (2 - \eta_\varphi)r + \frac{\partial \delta r}{\partial \ln b}. \tag{C5}
\]
Appendix D: Flow equations for four-dimensional tensor field

We finally discuss the flow equations and fixed-point structure when evaluating the angular integral directly at the upper critical dimension \( d = 4 \) with the nine \( 16 \times 16 \) gamma matrices \( \gamma_n \), the 16-component Dirac fermion \( \psi \), and the four-dimensional tensor field \( T_{ij} \), \( i, j = 1, \ldots, 4 \) with its irreducible components \( \phi_n \), \( n = 1, \ldots, 9 \). The computation of the one-loop diagrams in Fig. 3 now gives

\[
\frac{dc}{d\ln b} = (2 - z - \eta_c) c, \tag{D1}
\]

\[
\frac{dg}{d\ln b} = \frac{1}{2} (6 - d - z - \eta_g - 2\eta_c) g + \frac{7}{9} \tilde{H}(c) g^3, \tag{D2}
\]

\[
\frac{d\lambda}{d\ln b} = \frac{1}{2} (6 - d - z - 3\eta_g) \lambda + \frac{36 \lambda^3}{\pi c} - \frac{1}{9} \sqrt{\frac{2}{3}} g^3, \tag{D3}
\]

with the anomalous dimensions

\[
\eta_c = \frac{7}{6} F(c) g^2, \tag{D4}
\]

\[
\eta_g = \frac{49}{9} g^2 + \frac{24 \lambda^2}{\pi c}, \tag{D5}
\]

\[
z = 2 + 9 G(c) g^2 - \eta_c. \tag{D6}
\]

\( F(c) \) and \( G(c) \) are given in Eqs. (41)–(42) in the main text and

\[
\tilde{H}(c) = \frac{\pi(4 + 3\epsilon^2) + c(1 + \epsilon^2 + 9 \ln(1 + c e^2 + 7 e^2 \ln(c))}{\pi(1 + e^2)^3}. \tag{D7}
\]

The only qualitative and universal difference to the computation in \( d = 3 \) is the sign of the \( \lambda^3 \)-term in \( d\lambda/d\ln b \), which eliminates the (unstable) purely bosonic fixed point (B in Fig. 2) at \( g = 0 \). This is in full analogy to the Ginzburg-Landau-Wilson theory for the classical nematic phase transition in liquid crystals, which exhibits a fixed point if and only if the dimension \( p \) of the tensor order parameter is \( p < p_c \) with \( p_c = 4 \) to leading order in the related \( \epsilon = 6 - d \) expansion [34]. However, the existence of the fermionic fixed point (F in Fig. 2) remains unaffected by this, and we find the nontrivial solution for \( c \to 0 \)

\[
\lambda_{\pm} = \pm \frac{1}{4} \sqrt{\frac{\pi}{2}}, \quad g^*_\pm = \pm \frac{(6075\pi)^{1/6}}{2\sqrt{2}} \sqrt{\epsilon}, \tag{D8}
\]

where albeit \( \lambda^*_+ \) and \( g^*_+ \) now have the same sign. Examination of the stability matrix shows that the fermionic fixed point now exhibits a second relevant direction \( \sim \lambda \). This again reflects the fact that for the four-dimensional tensor order parameter there is no purely bosonic fixed point at \( g = 0 \) and \( \lambda \neq 0 \) and the flow in the direction of \( \lambda \) is unbounded. In agreement with the discussion of the classical nematic phase transition [34] we thus believe that the physical situation in \( d = 3 \) is more accurately described by the calculation directly in \( d = 3 \) as presented in the main text, which gives the stable fermionic fixed point with \( g^* \) and \( \lambda^* \) being of opposite sign. In any case, we find for the \( d = 4 \) calculation the same values for the anomalous dimensions at the fermionic fixed point as before [c.f. Eq. (49)]

\[
\eta_c = 0 + O(\epsilon^2), \quad \eta_g = \frac{3}{2} \epsilon + O(\epsilon^2), \quad z = 2 + O(\epsilon^2). \tag{D9}
\]
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