BOUND FOR THE REGULARITY OF BINOMIAL EDGE IDEALS OF CACTUS GRAPHS

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Abstract. In this article, we obtain an upper bound for the regularity of the binomial edge ideal of a graph whose every block is either a cycle or a clique. As a consequence, we obtain an upper bound for the regularity of binomial edge ideal of a cactus graph. We also identify certain subclass attaining the upper bound.

1. Introduction

Let \( R = \mathbb{K}[x_1, \ldots, x_m] \) be the standard graded polynomial ring over an arbitrary field \( \mathbb{K} \) and \( M \) be a finitely generated graded \( R \)-module. Let

\[
0 \to \bigoplus_{j \in \mathbb{Z}} R(-p-j)\beta_{p,p+j}(M) \to \cdots \to \bigoplus_{j \in \mathbb{Z}} R(-j)\beta_{0,j}(M) \to M \to 0,
\]

be the minimal graded free resolution of \( M \), where \( R(-j) \) is the free \( R \)-module of rank 1 generated in degree \( j \). The number \( \beta_{i,j}(M) \) is called the \((i, j)\)-th graded Betti number of \( M \). The projective dimension of \( M \), denoted by projdim(\( M \)), is defined as

\[
\text{projdim}(M) := \max\{i : \beta_{i,i+j}(M) \neq 0\}
\]

and the Castelnuovo-Mumford regularity (or simply, regularity) of \( M \), denoted by reg(\( M \)), is defined as

\[
\text{reg}(M) := \max\{j : \beta_{i,i+j}(M) \neq 0\}.
\]

The Betti number \( \beta_{i,i+j}(M) \) is called an extremal Betti number if \( \beta_{i,i+j}(M) \neq 0 \) and for all pairs of integers \((k, l) \neq (i, j)\), \( \beta_{k,k+l}(M) = 0 \) where \( k \geq i \) and \( l \geq j \). If \( p = \text{projdim}(M) \) and \( r = \text{reg}(M) \), then \( M \) admits unique extremal Betti number if and only if \( \beta_{p,p+r}(M) \neq 0 \).

For a graph \( G \), on \([n]\), the binomial edge ideal of \( G \) is the ideal \( J_G = \langle x_i y_j - x_j y_i : \{i, j\} \in E(G) \rangle \subset S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n] \). There has been a lot of research in understanding the connection between algebraic properties of \( J_G \) and combinatorial properties of \( G \). In particular, researchers have been trying to understand the connection between algebraic invariants of \( J_G \) in terms of the combinatorial invariants of \( G \). In this paper, we deal with the regularity of \( J_G \).

It has been conjectured by Saeedi Madani and Kiani that for a graph \( G \), \( \text{reg}(S/J_G) \leq c(G) \), where \( c(G) \) denotes the number of maximal cliques in \( G \). This conjecture has been proved for only a few classes of graphs, see \cite{1,2,6,9,10,11,12,15}. A connected graph \( G \) is said to be a cactus graph if every block of \( G \) is either a cycle or an edge (see Section 2 for details). In this article, we obtain an improved upper bound for the regularity of cactus graph. For a cactus graph having a lot of cycles as blocks, it turns out that the invariant \( c(G) \) is much larger than the upper bound that we have obtained. We also prove that the

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upper bound is attained by a subclass of Cohen-Macaulay cactus graphs. In fact, we prove our results for a slightly larger class of graphs. In the next section we recall the necessary definitions and some of the results from the literature which are crucial to the proofs of main results. In Section 3, we prove the upper bound and in Section 4, we identify a class of graphs for which the regularity upper bound is attained.

2. Preliminaries

Let us recall some basic definitions and notation from graph theory, which will be used throughout the article. Let $G$ be a simple graph on the vertex set $V(G) = [n] := \{1, \ldots, n\}$ and the edge set $E(G)$. A graph $G$ on $[n]$ is called a complete graph, if $\{i, j\} \in E(G)$ for all $1 \leq i < j \leq n$. We denote the complete graph on $n$ vertices by $K_n$. For $A \subseteq V(G)$, the induced subgraph of $G$ on the vertex set $A$, denoted by $G[A]$, is the graph such that for any pair of vertices $i, j \in A$, $\{i, j\} \in E(G[A])$ if and only if $\{i, j\} \in E(G)$. For a vertex $v \in V(G)$, $G \setminus v$ denotes the induced subgraph of $G$ on the vertex set $V(G) \setminus \{v\}$. For $U \subseteq V(G)$, $U$ is called a clique if the induced subgraph $G[U]$ is the complete graph. A vertex $v \in V(G)$ is said to be a simplicial vertex if there is only one maximal clique containing $v$. If $v$ is not a simplicial vertex, then $v$ is called an internal vertex. For a vertex $v \in V(G)$, let $N_G(v) := \{u \in V(G) : \{u, v\} \in E(G)\}$ denote the neighborhood of $v$ in $G$ and $G_v$ is the graph with the vertex set $V(G)$ and edge set $E(G_v) = E(G) \cup \{\{u, w\} : u, w \in N_G(v)\}$. A graph with the vertex set $[n]$ and the edge set $\{i, i+1 : 1 \leq i \leq n-1\}$ is called a path and it is denoted by $P_n$. A cycle on the vertex set $[n]$, denoted by $C_n$, is a graph with the edge set $\{i, i+1 : 1 \leq i \leq n-1\} \cup \{1, n\}$ for $n \geq 3$. A vertex $v \in V(G)$ is called a cut vertex if $G \setminus v$ has more number of components than that of $G$. A maximal connected subgraph of $G$, which has no cut vertex is called a block of $G$. A graph $G$ is called a block graph if every block of $G$ is a clique. For a block graph $G$, a block of $G$ is called a leaf of $G$ if that block contains at most one cut vertex. A connected graph $G$ is said to be a cactus graph if every block of $G$ is either a cycle or an edge. The block graph of $G$, denoted by $B(G)$, is the graph whose vertices are the blocks of $G$ and two vertices are adjacent whenever the corresponding blocks have a common cut vertex. The graph on the vertex set $[4]$ and with $E(G) = E(C_4) \cup \{(1, 3)\}$ is a diamond graph and is denoted by $D$. If $G_1(\neq K_m)$ and $G_2$ are two subgraphs of a graph $G$ such that $G_1 \cap G_2 = K_m$, $V(G_1) \cup V(G_2) = V(G)$ and $E(G_1) \cup E(G_2) = E(G)$, then $G$ is called the clique sum of $G_1$ and $G_2$ along the complete graph $K_m$, denoted by $G_1 \cup_K G_2$. If the clique sum is along an edge $e$, then we write $G_1 \cup_e G_2$ and if the clique sum is along a vertex $v$, then we write $G_1 \cup_v G_2$.

Throughout this article, $S$ denotes the polynomial ring over $\mathbb{K}$ in $2|V(G)|$ number of variables and in which the binomial edge ideal $J_G$ resides. For a graph $H$, we denote by $S_H$, the polynomial ring over $\mathbb{K}$ in $2|V(H)|$ number of variables and $J_H \subseteq S_H$. If $H$ is a subgraph of $G$, we would assume that $S_H$ is a subring of $S$. We say that $G$ is a Cohen-Macaulay graph if $S/J_G$ is Cohen-Macaulay.

Let $A = \mathbb{K}[x_1, \ldots, x_m]$, $A' = \mathbb{K}[y_1, \ldots, y_n]$ and $B = \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be polynomial rings. Let $I \subseteq A$ and $J \subseteq A'$ be homogeneous ideals. Then the minimal free resolution of $B/(I + J)$ can be obtained by the tensor product of the minimal free resolutions of $A/I$ and
Therefore, for all \( i, j \), we get:

\[
\beta_{i,i+j} \left( \frac{B}{I+J} \right) = \sum_{i_1+i_2 = i, j_1+j_2 = j} \beta_{i_1,i_1+j_1} \left( \frac{A}{I} \right) \beta_{i_2,i_2+j_2} \left( \frac{A'}{J} \right).
\]

The following lemma can be easily derived from the long exact sequence of Tor corresponding to given short exact sequence.

**Lemma 2.1.** Let \( R \) be a standard graded ring and \( M, N, P \) be finitely generated graded \( R \)-modules. If \( 0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0 \) is a short exact sequence with \( f, g \) graded homomorphisms of degree zero, then

(i) \( \text{reg}(M) \leq \max\{\text{reg}(N), \text{reg}(P) + 1\} \),
(ii) \( \text{reg}(M) = \text{reg}(N) \) if \( \text{reg}(N) > \text{reg}(P) \).

We recall the following lemma due to Ohtani.

**Lemma 2.2.** ([13, Lemma 4.8]) Let \( G \) be a graph on \( V(G) \) and \( v \in V(G) \) such that \( v \) is an internal vertex. Then \( J_G \) can be written as

\[
J_G = J_{G_v} \cap Q_v, \text{ where } Q_v = (x_v, y_v) + J_{G \setminus v}.
\]

Rinaldo proved that \( Q_v = \cap_{T \in \mathcal{C}(G), v \in T} P_T(G) \), [14, Corollary 1.1].

3. Regularity of Cactus Graph

In this section we consider graphs whose every block is either a cycle or a clique. For a graph \( G \), we denote by \( b(G) \) the number of blocks of \( G \) and by \( c(G) \) the number of cycles of length \( \geq 4 \). Let \( c_j(G) \) denote the number of cycles of length \( j \) in \( G \). Then we have \( c(G) = \sum_{k \geq 4} c_k(G) \). First, we prove a technical property that is required in the main proof.

**Lemma 3.1.** Let \( G \) be a graph with \( c(G) \geq 1 \) and such that each block of \( G \) is either a cycle or a clique. Then there exists a cycle \( C \) of length \( r(\geq 4) \) such that \( r-2 \) consecutive vertices of \( C \) are not part of any other cycles of length \( \geq 4 \) in \( G \).

**Proof.** We prove our assertion by induction on the number of blocks of \( G \). If \( b(G) = 1 \), then we are through. Suppose now that \( b(G) \geq 2 \). It follows from [13, Theorem 3.5] that \( B(G) \) is block graph. Set \( b(B(G)) = s \). Let \( B_1, \ldots, B_s \) be the blocks of \( B(G) \) and assume that \( B_s \) is a leaf of \( B(G) \). There exist blocks \( H_1, \ldots, H_l \), for some \( l \geq 2 \), such that \( H = H_1 \cup \cdots \cup H_l \), for some \( v \in V(G) \), is an induced subgraph of \( G \) corresponding to the block \( B_s \). Since \( B_s \) is a leaf of \( B(G) \), it has only one cut-vertex. This implies that there exists an \( i \in [l] \) such that \( (V(H_j) \setminus v) \cap (V(G) \setminus V(H)) = \emptyset \) for any \( j \neq i \), i.e., for \( j \neq i \), the vertices of \( V(H_j) \), except \( v \), are not part of any other blocks of \( G \). If for some \( j \neq i \), \( H_j = K_m \), then set \( G' \) to be the induced subgraph on \( (V(G) \setminus V(H_j)) \cup \{v\} \), i.e., the graph obtained by removing the block \( H_j \) from \( G \). Since \( b(G') = b(G) - 1 \), by induction there exists a cycle \( C \) of length \( r(\geq 4) \) such that \( r-2 \) consecutive vertices of \( C \) are not part of the other cycles of length \( \geq 4 \) in the graph \( G' \). Since \( G' \) is an induced subgraph of \( G \) and \( V(C) \cap V(H_j) \subseteq \{v\} \), we see that \( C \) is a cycle in \( G \) with the required property. Suppose \( H_j \) is not a complete graph for any \( j \neq i \), then taking \( C = H_j \) for some \( j \neq i \), we see that the cycle \( C \) satisfies the required property.

\( \square \)
We now prove the main theorem of this article, an upper bound for the regularity of binomial edge ideal of a graph whose each block is either a cycle or a clique.

**Theorem 3.2.** Let $G$ be a graph such that each block of $G$ is either a cycle or a clique. Then

$$\text{reg}(S/J_G) \leq c'(G) + \sum_{k \geq 4} (k - 2)c_k(G),$$

where $c'(G)$ is the number of maximal cliques except the edges of any cycle of length $\geq 4$ in $G$.

**Proof.** We prove our assertion by induction on $C(G)$. If $C(G) = 0$, then $G$ is a block graph, and hence the assertion follows from [2 Theorem 3.9]. Assume that $C(G) \geq 1$. Then by Lemma 3.1, there exists a cycle $C$ of length $r \geq 4$ such that $r - 2$ consecutive vertices of $C$ are not part of any other cycles of length $\geq 4$ in $G$. Suppose $V(C) = \{v_1, \ldots, v_r\}$ and $v_1, \ldots, v_{r-2}$ are not part of any other cycles of length $\geq 4$. Now, we bring in an intermediate class of graphs which is a clique sum of $G$ with a complete graph $K_m$, for some $m \geq 2$. Let $H = G \cup_{\{v_1, v_2\}} K_m$ for $m \geq 2$. Note that $H = G$ if $m = 2$.

**Claim**: $\text{reg}(S_H/J_H) \leq c'(H) + \sum_{k \geq 4} (k - 2)c_k(H)$.

We prove this claim for all $m \geq 2$ and finally deduce the required assertion on $G$ by taking $m = 2$. We proceed by induction on $r$. Suppose now that $r = 4$. Since $v_1$ is an internal vertex of $H$, by Lemma 2.2, we can write $J_H = J_{H_{v_1}} \cap Q_{v_1}$ with $Q_{v_1} = (x_{v_1}, y_{v_1}) + J_{H_{v_1} \setminus v_1}$, where $H_{v_1} \setminus v_1$ is a graph whose every block is either a cycle or a clique with $C(H_{v_1} \setminus v_1) = C(G) - 1$. Therefore, by induction and (1),

$$\text{reg}(S_H/Q_{v_1}) \leq c'(H_{v_1} \setminus v_1) + \sum_{k \geq 4} (k - 2)c_k(H_{v_1} \setminus v_1) \leq c'(H_{v_1}) + \sum_{k \geq 4} (k - 2)c_k(H_{v_1}),$$

where the last inequality follows because $c'(H_{v_1} \setminus v_1) \leq c'(H) + 2$. Note that $J_{H_{v_1}} + Q_{v_1} = (x_{v_1}, y_{v_1}) + J_{H_{v_1 \setminus v_1}}$. Let $n_{v_1} = |N_{H}(v_1)|$, $B' = C_3 \cup \{v_2, v_4\} K_{n_{v_1} + 1}$ and $B'' = C_3 \cup \{v_2, v_4\} K_{n_{v_1}}$. Then it can be seen that the block containing $v_1$ in $H_{v_1}$ (resp. $H_{v_1} \setminus v_1$) is $B'$ (resp. $B''$) and all the other blocks of $H_{v_1}$ (resp. $H_{v_1} \setminus v_1$) are blocks of $H$. We show that

$$\text{reg}(S_H/J_{H_{v_1}}) \leq c'(H_{v_1}) + \sum_{k \geq 4} (k - 2)c_k(H_{v_1}) \quad \text{and} \quad (2)$$

$$\text{reg}(S_H/((x_{v_1}, y_{v_1}) + J_{H_{v_1 \setminus v_1}})) \leq c'(H_{v_1} \setminus v_1) + \sum_{k \geq 4} (k - 2)c_k(H_{v_1} \setminus v_1). \quad (3)$$

Let $H' = H_{v_1}$. Since $v_2$ is an internal vertex in $H'$, by Lemma 2.2, $J_{H'} = J_{H'_{v_2}} \cap Q_{v_2}$ with $Q_{v_2} = (x_{v_2}, y_{v_2}) + J_{H'_{v_2}}$, where $H'_{v_2}$ is a graph such that each block of $H'_{v_2}$ is either a cycle or a clique and $C(H'_{v_2}) = C(G) - 1$. Therefore, by induction and (1), we have

$$\text{reg}(S_H/Q_{v_2}) \leq c'(H'_{v_2}) + \sum_{k \geq 4} (k - 2)c_k(H'_{v_2}) \leq c'(H') + \sum_{k \geq 4} (k - 2)c_k(H'),$$

where the second inequality follows since $c'(H'_{v_2}) \leq c'(H')$ and $\sum_{k \geq 4} (k - 2)c_k(H'_{v_2}) = \sum_{k \geq 4} (k - 2)c_k(H')$. The block containing $v_2$ in $H'_{v_2}$ is $K_{n_{v_2} + 1}$, where $n_{v_2} = N_{H'}(v_2)$ and all other blocks of $H'_{v_2}$ are blocks of $H'$. Therefore, $C(H'_{v_2}) = C(G) - 1$. By induction,

$$\text{reg}(S_H/J_{H'_{v_2}}) \leq c'(H'_{v_2}) + \sum_{k \geq 4} (k - 2)c_k(H'_{v_2}) \leq c'(H') - 1 + \sum_{k \geq 4} (k - 2)c_k(H'),$$

where the second inequality follows since $c'(H'_{v_2}) \leq c'(H')$ and $\sum_{k \geq 4} (k - 2)c_k(H'_{v_2}) = \sum_{k \geq 4} (k - 2)c_k(H')$. Therefore, $C(H'_{v_2}) = C(G) - 1$. By induction,

$$\text{reg}(S_H/J_{H'_{v_2}}) \leq c'(H'_{v_2}) + \sum_{k \geq 4} (k - 2)c_k(H'_{v_2}) \leq c'(H') - 1 + \sum_{k \geq 4} (k - 2)c_k(H'),$$

where the second inequality follows since $c'(H'_{v_2}) \leq c'(H')$ and $\sum_{k \geq 4} (k - 2)c_k(H'_{v_2}) = \sum_{k \geq 4} (k - 2)c_k(H')$. Therefore, $C(H'_{v_2}) = C(G) - 1$. By induction,
where the second inequality follows since \( c'(H'_{v_2}) \leq c'(H') - 1 \) and \( \sum_{k \geq 4} (k - 2) c_k(H'_{v_2}) = \sum_{k \geq 4} (k - 2) c_k(H') \). Note that \( J_{H'_{v_2}} + Q_{v_2} = (x_{v_2}, y_{v_2}) + J_{H'_{v_2}v_2} \). Therefore, by [12, Corollary 2.2] and (11),

\[
\text{reg}(S_H/(x_{v_2}, y_{v_2}) + J_{H'_{v_2}}) \leq \text{reg}(S_H/J_{H'_{v_2}}) \leq c'(H') - 1 + \sum_{k \geq 4} (k - 2) c_k(H').
\]

We consider the following short exact sequence:

\[
0 \rightarrow \frac{S_H}{J_{H'}} \rightarrow \frac{S_H}{J_{H'_{v_2}}} \oplus \frac{S_H}{Q_{v_2}} \rightarrow \frac{S_H}{J_{H'_{v_2}} + Q_{v_2}} \rightarrow 0.
\]

By applying Lemma [2.1] on the above short exact sequence, we get that

\[
\text{reg}(S_H/J_{H'}) \leq c'(H') + \sum_{k \geq 4} (k - 2) c_k(H').
\]

Since \( c'(H') \leq c'(H) + 2 \) and \( \sum_{k \geq 4} (k - 2) c_k(H') = \sum_{k \geq 4} (k - 2) c_k(H) - 2 \), we have

\[
\text{reg}(S_H/J_{H'}) \leq c'(H) + \sum_{k \geq 4} (k - 2) c_k(H).
\]

Now we prove (3). If \( n_{v_1} = 2 \), then \( H_{v_1} \setminus v_1 \) is a graph whose blocks are either a cycle or a clique with \( C(H_{v_1} \setminus v_1) = C(G) - 1 \). Therefore by induction and (11),

\[
\text{reg}(S_H/(x_{v_1}, y_{v_1}) + J_{H_{v_1}v_1}) \leq c'(H_{v_1} \setminus v_1) + \sum_{k \geq 4} (k - 2) c_k(H_{v_1} \setminus v_1).
\]

Now assume that \( n_{v_1} \geq 3 \). Set \( H'' = H_{v_1} \setminus v_1 \). Replacing \( H' \) by \( H'' \) in the proof of (2), and obtain

\[
\text{reg}(S_H/(x_{v_1}, y_{v_1}) + J_{H_{v_1}v_1}) \leq c'(H_{v_1} \setminus v_1) + \sum_{k \geq 4} (k - 2) c_k(H_{v_1} \setminus v_1).
\]

For \( H' \setminus v_1 \), it can be observed that \( c'(H' \setminus v_1) \leq c'(H) + 1 \) and \( \sum_{k \geq 4} (k - 2) c_k(H') = \sum_{k \geq 4} (k - 2) c_k(H) - 2 \). Therefore,

\[
\text{reg}(S_H/(x_{v_1}, y_{v_1}) + J_{H'v_1}) \leq c'(H) - 1 + \sum_{k \geq 4} (k - 2) c_k(H).
\]

Now the claim follows from Lemma [2.1] applied on the short exact sequence:

\[
0 \rightarrow \frac{S_H}{J_{H'}} \rightarrow \frac{S_H}{J_{H'_{v_1}}} \oplus \frac{S_H}{Q_{v_1}} \rightarrow \frac{S_H}{J_{H'_{v_1}} + Q_{v_1}} \rightarrow 0.
\]

Assume that \( r \geq 5 \). Since \( v_1 \) is an internal vertex of \( H \), by Lemma [2.2] we write \( J_H = J_{H_{v_1} \cap Q_{v_1}} \) with \( Q_{v_1} = (x_{v_1}, y_{v_1}) + J_{H' \setminus v_1} \), where \( J_{H_{v_1} \cap Q_{v_1}} = (x_{v_1}, y_{v_1}) + J_{H_{v_1} \setminus v_1} \). Since \( H' \setminus v_1 \) is a graph such that every block of \( H \setminus v_1 \) is either a cycle or a clique and \( C(H \setminus v_1) = C(G) - 1 \), by induction and (11),

\[
\text{reg}(S_H/Q_{v_1}) \leq c'(H \setminus v_1) + \sum_{k \geq 4} (k - 2) c_k(H \setminus v_1) \leq c'(H) + \sum_{k \geq 4} (k - 2) c_k(H),
\]

where the last inequality follows because \( \sum_{k \geq 4} (k - 2) c_k(H \setminus v_1) + r - 2 = \sum_{k \geq 4} (k - 2) c_k(H) \) and \( c'(H \setminus v_1) \leq c'(H) + r - 2 \). Let \( n_{v_1} = |N_H(v_1)|, B' = C_{r-1} \cup_{\{v_2,v_3\}} K_{n_{v_1}+1} \) and \( B'' = C_{r-1} \cup_{\{v_2,v_3\}} K_{n_{v_1}} \). Then, the block containing \( v_1 \) in \( H_{v_1} \) is \( B' \) and all the other blocks of \( H_{v_1} \) are blocks of \( H \). Note that both the graphs \( H_{v_1} \) and \( H_{v_1} \setminus v_1 \) have a cycle of length \( r - 1 \).
whose \( r - 3 \) vertices are not part of any other cycles. Hence by induction on \( r \) and using (1), we get

\[
\text{reg}(S_H/J_{H_{v_1}}) \leq c'(H_{v_1}) + \sum_{k \geq 4} (k - 2)c_k(H_{v_1}) \quad \text{and}
\]

\[
\text{reg}(S_H/((x_{v_1}, y_{v_1}) + J_{H_{v_1}})) \leq c'(H_{v_1} \setminus v_1) + \sum_{k \geq 4} (k - 2)c_k(H_{v_1} \setminus v_1).
\]

Note also that \( \sum_{k \geq 4} (k - 2)c_k(H_{v_1}) = \left[ \sum_{k \geq 4} (k - 2)c_k(H) \right] - 1 \) and \( c'(H_{v_1}) \leq c'(H) + 1 \). Therefore,

\[
\text{reg}(S_H/J_{H_{v_1}}) \leq c'(H) + \sum_{k \geq 4} (k - 2)c_k(H).
\]

For the graph \( H_{v_1} \setminus v_1 \), we notice that \( \sum_{k \geq 4} (k - 2)c_k(H_{v_1} \setminus v_1) = \left[ \sum_{k \geq 4} (k - 2)c_k(H) \right] - 1 \) and \( c'(H_{v_1} \setminus v_1) \leq c'(H) \). Therefore,

\[
\text{reg}(S_H/((x_{v_1}, y_{v_1}) + J_{H_{v_1}})) \leq c'(H) + \left[ \sum_{k \geq 4} (k - 2)c_k(H) \right] - 1.
\]

Hence it follows from Lemma 2.1 applied on the following short exact sequence

\[
0 \rightarrow \frac{S_H}{J_{H_{v_1}}} \rightarrow \frac{S_H}{J_{H_{v_1}}} \oplus \frac{S_H}{J_{Q_{v_1}}} \rightarrow \frac{S_H}{J_{H_{v_1} + Q_{v_1}}} \rightarrow 0.
\]

that \( \text{reg}(S_H/J_{H_{v_1}}) \leq c'(H) + \sum_{k \geq 4} (k - 2)c_k(H) \). This completes the proof of the Claim.

As an immediate consequence, we obtain an upper bound for the regularity of cactus graph.

**Corollary 3.3.** Let \( G \) be a cactus graph. Then

\[
\text{reg}(S/J_G) \leq c'(G) + \sum_{k \geq 4} (k - 2)c_k(G).
\]

4. **Regularity of Cohen-Macaulay Cactus Graph**

In this section we obtain a class of cactus graph for which the upper bound we proved in the last section is attained. We begin by computing the regularity of certain classes of graphs which are required in the main theorem.

**Lemma 4.1.** Let \( k, m_1 \geq 3 \) and \( m_2 \geq 2 \). Let \( G = C_k \cup_e K_{m_1} \cup_v K_{m_2} \) with \( v \in e \). If \( m_2 = 2 \), then \( \text{reg}(S/J_G) = k - 1 \) and if \( m_2 \geq 3 \), then \( \text{reg}(S/J_G) = k \).

**Proof.** Let \( N_{G_k}(v) = \{u, w\} \) and \( e = \{u, v\} \). If \( m_2 = 2 \), then the assertion follows from the proof [7, Theorem 4.1]. Suppose now that \( m_2 \geq 3 \). Then using Lemma 2.2, we get that \( J_G = J_{G_v} \cap Q_v \) with \( Q_v = (x_v, y_v) + J_{G \setminus v} \), where \( G \setminus v \) is the graph with two components \( K_{m_1 - 1} \cup_u P_{k - 1} \) and \( K_{m_2 - 1} \). By [3, Theorem 3.1] and (1), \( \text{reg}(S/Q_v) = k \). Note that \( G_v = C_{k-1} \cup e' K_{m_1+m_2} \), where \( e' = \{u, w\} \). Also, \( J_{G_v} + Q_v = (x_v, y_v) + J_H \), where \( H \) is obtained by deleting the vertex \( v \) from \( G_v \), i.e., \( H = C_{k-1} \cup e' K_{m_1+m_2-1} \). By [7, Theorem 3.12] and (1), we have \( \text{reg}(S/J_{G_v}) = \text{reg}(S/(J_{G_v} + Q_v)) = k - 2 \). As \( v \) is not a simplicial vertex, we consider the following short exact sequence:

\[
0 \rightarrow \frac{S}{J_G} \rightarrow \frac{S}{J_{G_v}} \oplus \frac{S}{Q_v} \rightarrow \frac{S}{J_{G_v} + Q_v} \rightarrow 0.
\]

(4)
Therefore, it follows from Lemma \[2.1\] and the above short exact sequence that \( \text{reg}(S/J_G) = k \). 

\[ \Box \]

**Lemma 4.2.** Let \( k \geq 4 \) and \( m_1, m_2 \geq 2 \). Let \( G = C_k \cup u K_{m_1} \cup v K_{m_2} \) for some \( \{u, v\} \in E(C_k) \). If \( m_1 = m_2 = 2 \), then \( \text{reg}(S/J_G) = k - 1 \), otherwise \( \text{reg}(S/J_G) = k \).

**Proof.** If \( m_1 = m_2 = 2 \), then the assertion follows from \[7\], Proposition 3.14. Suppose \( m_1 \geq 3 \) or \( m_2 \geq 3 \). We assume that \( m_2 \geq 3 \). Let \( G = C_k \cup u K_{m_1} \cup v K_{m_2}, m_1 \geq 2, m_2 \geq 3 \) and \( N_{C_1}(v) = \{u, w\} \). Note that \( G_v = C_{k-1} \cup v K_{m_2+2} \cup u K_{m_1}, \) where \( e = \{u, w\} \) and \( Q_v = (x_v, y_v) + J_{K_{m_2+1}} + J_H \), where \( H = K_{m_1} \cup u P_{k-1} \). Also, \( J_{G_v} + Q_v = (x_v, y_v) + J_{H'} \), where \( H' \) is obtained by deleting the vertex \( v \) from \( G_v \), i.e., \( H' = C_{k-1} \cup v K_{m_2+1} \cup u K_{m_1} \). By \[8\], Theorem 3.1 and \[11\], \( \text{reg}(S/Q_v) = k \). By Lemma \[4.1\] we get that if \( m_1 = 2 \), then \( \text{reg}(S/J_{G_v}) = \text{reg}(S/(J_{G_v} + Q_v)) = k - 2 \), otherwise \( \text{reg}(S/J_{G_v}) = \text{reg}(S/(J_{G_v} + Q_v)) = k - 1 \). Hence, the assertion follows from Lemma \[2.1\] and the short exact sequence \[4.1\]. 

A graph \( G \) is said to be a decomposable graph if \( G \) can be written as a clique sum of two subgraphs along a simplicial vertex i.e., \( G = G_1 \cup v G_2 \), where \( v \) is a simplicial vertex of \( G_1 \) and \( G_2 \). If \( G \) is not decomposable, then it is called an indecomposable graph. It follows from \[8\], Theorem 3.1 that to find the regularity, it is enough to consider \( G \) to be an indecomposable graph. So, for the rest of the section, we assume that \( G \) is an indecomposable graph. We now study the regularity of binomial edge ideal of Cohen-Macaulay cactus graphs. Let \( G \) be a graph such that \( B(G) \) is a path of length \( l - 1 \). Let \( V(B(G)) = \{B_1, \ldots, B_l\} \). If \( B_i \) is a graph on \( m_i \) vertices, then set \( V(B_i) = \{v_{i1}, \ldots, v_{imi}\} \) and \( B_i \cap B_{i+1} = \{w_i\} \). Also, we choose the order of vertices in \( V(B_i) \) in such a way that \( v_{imi} = w_i = v_{i+1} \).

In \[14\], Theorem 2.2, Rinaldo characterized Cohen-Macaulay cactus graph. Let \( G \) be an indecomposable Cohen-Macaulay cactus graph whose blocks are \( B_1, \ldots, B_l \). Then it follows from \[14\], Lemma 2.3 that either \( G \in \{K_2, C_3\} \) or \( G \) satisfies the following conditions:

1. \( B_1, B_l \in \{C_3, K_2\} \),
2. \( B_2 = B_{l-1} = C_4 \),
3. \( B_i \in \{C_3, C_4\} \) for \( 3 \leq i \leq l - 2 \) and if \( B_i = C_3 \) then \( B_{i+1} = C_4 \), and
4. there are exactly two cut points in \( C_4 \) and they are adjacent.

Our goal in this article is to compute the regularity of binomial edge ideals of such classes of graphs. We compute the regularity of a slightly more general class of graphs.

**Theorem 4.3.** Let \( G \) be a graph such that \( B(G) \) is a path of length \( l - 1 \) for some \( l \geq 3 \). Also let \( B_1 = K_{m_1}, B_l = K_{m_l}, B_2 = B_{l-1} = C_4 \) with \( m_1 \geq 2, m_l \geq 3 \) and \( B_i \in \{C_4, K_{m_j} : m_j \geq 3\} \) for \( 3 \leq i \leq l - 2 \). Further assume that there are exactly two cut points in each \( C_4 \) in \( G \) and they are adjacent. Then \( \text{reg}(S/J_G) = 2c_4(G) + c'(G) \), where \( c_4(G) \) is the number of \( C_4 \)'s in \( G \) and \( c'(G) \) is the number of maximal cliques except the edges of \( C_4 \)'s in \( G \).

**Proof.** We proceed by induction on \( l \). If \( l = 3 \), then \( G = K_{m_1} \cup v_{21} C_4 \cup v_{24} K_{m_3} \), and hence, the assertion follows from Lemma \[4.2\] considering \( k = 4 \). Assume that \( l \geq 4 \). Since \( v_{23} \) is not a simplicial vertex, by Lemma \[2.2\] we get the following short exact sequence:

\[
0 \rightarrow \frac{S}{J_G} \rightarrow \frac{S}{J_{G_{v_{23}}}} \oplus \frac{S}{Q_{v_{23}}} \rightarrow \frac{S}{J_{G_{v_{23}}} + Q_{v_{23}}} \rightarrow 0.
\]

Set \( r = 2c_4(G) + c'(G) \). First, we show that \( \text{reg}(S/J_{G_{v_{23}}}) = r \). Note that \( E(G_{v_{23}}) = E(G) \cup \{v_{22}, v_{24}\} \), and so the second block in \( G_{v_{23}} \) is a diamond graph \( D \). Except for the second block, all the blocks in \( G \) and \( G_{v_{23}} \) are the same. Here, \( G_{v_{23}} \) is a decomposable
Theorem 3.1, induction and (1), reg($S$) from [14, Proposition 2.3] that if $T_n^{(6)}$ in homological degree $n - 1$ and $H_{v_{22}}$ satisfies induction hypothesis, we have $S_H/J_{H_{v_{22}}} = r - 2$. It follows from [14, Proposition 2.3] that if $T \in \mathcal{E}(H)$ and $v_{22} \in T$, then $v_{24} \notin T$. So, $Q_{v_{22}} = (x_{v_{22}}, y_{v_{22}}, x_{v_{24}}, y_{v_{24}}) + J_{H/\{v_{22}, v_{24}\}}$.

Note that $H_{v_{22}} = (x_{v_{22}}, y_{v_{22}}, x_{v_{24}}, y_{v_{24}}) + J_{H/\{v_{22}, v_{24}\}}$ + $(x_{v_{21}}, y_{v_{21}} - x_{v_{23}}, y_{v_{23}})$. It can be seen that $H \setminus \{v_{22}, v_{24}\} = B_3 \cup \{v_{21}, v_{23}\}$. Now to find the regularity of $S/Q_{v_{22}}$, $H \setminus \{v_{22}, v_{24}\}$ is obtained. By deleting the vertex $w_2 = v_{24}$ from $B_3$.

**Case 1.** If $B_3 = K_{m_3}$, then $H \setminus \{v_{22}, v_{24}\} = K_{m_3-1} \cup \{v_{21}, v_{23}\}$. Hence, by induction and (1), $S_H/J_{Q_{v_{22}}} = r - 3$ and $\text{reg}(S_H/(J_{H_{v_{22}}} + Q_{v_{22}})) = r - 2$.

**Case 2.** If $B_3 = C_4$, then $H \setminus \{v_{22}, v_{24}\} = P_3 \cup \{v_{21}, v_{23}\}$. Therefore, by [8, Theorem 3.1], induction and (1), $S_H/J_{Q_{v_{22}}} = r - 3$ and $\text{reg}(S_H/(J_{H_{v_{22}}} + Q_{v_{22}})) = r - 2$. Hence it follows from Lemma 2.4 and the short exact sequence (6) that $\text{reg}(S_H/J_{H_{v_{22}}}) \leq r - 1$.

Now we prove that $r - 1 \leq \text{reg}(S_H/J_H)$. Set $|V(H)| = n$. By [14, Theorem 2.1], $S_H/J_{H_{v_{22}}}$, $S_H/(J_{H_{v_{22}}} + Q_{v_{22}})$ and $\text{reg}(S_H/J_{H_{v_{22}}})$ are Cohen-Macaulay. Therefore, $\beta_{n-1,n-1+r-3}(S_H/J_{H_{v_{22}}})$ and $\beta_{n-1,n+r-2}(S_H/(J_{H_{v_{22}}} + Q_{v_{22}}))$ are the unique extremal Betti numbers. We consider the long exact sequence of Tor corresponding to the short exact sequence (6) in homological degree $n$ and in graded degree $n + r - 2$:

$$0 \to \text{Tor}_n^S(S_H/J_{H_{v_{22}}} + Q_{v_{22}}, \mathbb{K}) \to \text{Tor}_n^S(S_H/J_{H_{v_{22}}} + Q_{v_{22}}, \mathbb{K}) \to \cdots$$

which implies that $\beta_{n-1,n+r-2}(S_H/J_H) = 0$ and hence, $r - 1 \leq \text{reg}(S_H/J_H)$. Therefore, $\text{reg}(S_H/J_H) = r - 1$. By [8, Theorem 3.1] and the claim, we get $\text{reg}(S_H/J_{G_{v_{23}}}) = r$.

Now we show that $\text{reg}(S_H/J_{Q_{v_{22}}}) \leq r - 2$ and $\text{reg}(S_H/(Q_{v_{22}} + J_{G_{v_{23}}})) \leq r - 1$. It follows from [14, Proposition 2.3] that if $T \in \mathcal{E}(G)$ and $v_{23} \in T$, then $v_{21} \in T$ and $v_{24} \notin T$. Thus, $Q_{v_{23}} = (x_{v_{21}}, y_{v_{21}}, x_{v_{23}}, y_{v_{23}}) + J_{K_{m_1-1} + H'}$, where $H' = (B_3 \cup \{v_{21}, v_{23}\} \cup \{v_{21}, v_{23}\} \cup \{v_{21}, v_{23}\} \cup \{v_{21}, v_{23}\} \cup \{v_{21}, v_{23}\})$. Therefore, by [8, Theorem 3.1], $\text{reg}(S_H/(J_{G_{v_{23}}} + Q_{v_{23}})) = \text{reg}(S_H/J_{G_{v_{23}}}) + 1 \leq r - 1$. Hence, it follows from Lemma 2.4 and the short exact sequence (6) that $\text{reg}(S_H/J_G) = r$.

As an immediate consequence, we obtain a class of cactus graphs for which the upper bound obtained in Corollary 3.3 is attained.

**Corollary 4.4.** If $G$ is a Cohen-Macaulay indecomposable cactus graph such that $B_1 = C_3$ or $B_1 = C_3$, then $\text{reg}(S/H_G) = 2c_4(G) + c'(G)$.
Below, we give examples of two graphs. The graph $G_1$ is an example of a graph for which $\text{reg}(S_{G_1}/J_{G_1}) < 2c_4(G_1) + \ell'(G_1)$. The graph $G_2$ shows that the assumption $B_1 = C_3$ or $B_1 = C_3'$ is not a necessary one for the equality. If one inputs the binomial edge ideals corresponding to these two graphs into any of the computational commutative algebra packages (we used Macaulay2, \[3\]) and compute the regularity, then we get $\text{reg}(S_{G_1}/J_{G_1}) = 6 = \text{reg}(S_{G_2}/J_{G_2})$. It may be noted that $2c_4(G_1) + \ell'(G_1) = 7$ and $2c_4(G_2) + \ell'(G_2) = 6$.

![Graphs G1 and G2](image)

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