The Statistics of Cosmological Lyman-\(\alpha\) Absorption

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ABSTRACT

We study the effect of the non-Gaussianity induced by gravitational evolution upon the statistical properties of absorption in quasar (QSO) spectra. Using the generic hierarchical ansatz and the log-normal approximation we derive the analytical expressions for the one-point PDF as well as for the joint two-point probability distribution (2PDF) of transmitted fluxes in two neighbouring QSOs. These flux PDFs are constructed in 3D as well as in projection (i.e. in 2D). The PDFs are constructed by relating the lower-order moments, i.e. cumulants and cumulant correlators, of the fluxes to the 3D neutral hydrogen distribution which is, in turn, expressed as a function of the underlying dark matter distribution. The lower-order moments are next modelled using a generating function formalism in the context of a minimal tree-model for the higher-order correlation hierarchy. These different approximations give nearly identical results for the range of redshifts probed, and we also find a very good agreement between our predictions and outputs of hydrodynamical simulations. The formalism developed here for the joint statistics of flux-decrements concerning two lines of sight can be extended to multiple lines of sight, which could be particularly important for the 3D reconstruction of the cosmic web from QSO spectra (e.g. in the BOSS survey). These statistics probe the underlying projected neutral hydrogen field and are thus linked to “hot-spots” of absorption. The results for the PDF and the bias presented here use the same functional forms of scaling functions that have previously been employed for the modelling of other cosmological observation such as the Sunyaev-Zel’dovich effect.

1 INTRODUCTION

Ongoing Cosmic Microwave Background (CMB) experiments such as Planck\(^1\), the Atcama-Cosmology Telescope\(^2\) (ACT) and the South Pole Telescope\(^3\) (SPT) will pinpoint the cosmological parameters that describe the background geometry and dynamics of the Universe in an unprecedented detail. Along with very precise constraints on the structure of the Universe on the largest scales, smaller-scales observables will be crucial in order to further constrain the cosmological concordance model or find possible deviations from it. In particular, galaxy clustering (BOSS, 6dF, etc.) surveys and future weak lensing and clustering observations (e.g. EUCLID) could probe smaller scales and new redshift regimes. Spectroscopic surveys such as BOSS\(^4\) (and BIG BOSS) will also trace the large scale distribution of the baryonic matter in the Universe through the study of the flux distribution of the Lyman-\(\alpha\) absorption systems in a very large number of quasar (QSO) spectra.

The Lyman-\(\alpha\) “forest”, the many absorption features in QSO spectra, produced by intervening neutral hydrogen in the intergalactic medium (IGM) along the line-of-sight, is well known to be an important cosmological probe (for a recent review see Meiksin 2009). In the standard cosmological paradigm, the IGM consists of mildly non-linear gas, making up the cosmic web, that traces the dark matter and is photo-heated by

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using a lognormal model for the underlying mass distribution. The resulting PDFs for Figure 1. The PDF $p(F_\alpha)$ and bias $b(F_\alpha)$ of the flux $F_\alpha$ is plotted as a function of the flux $F_\alpha$. The PDF (left-panel) and the bias (right-panel) both are constructed using a lognormal model for the underlying mass distribution. The resulting PDFs for $\delta$ are next transformed into the flux PDFs using the fluctuating Gunn-Peterson approximation Eq. (8). We compare the results from the lognormal approximation (solid lines) against the one based on the Gaussian approximation (dashed lines). The values of $A(\zeta)$ and $\beta$ are constructed using the functional fit in Eq. (9) given by [Kim et al. (2007)] see text for more details. The bias changes signs at an intermediate flux value which depends on cosmology and hydrodynamical parameters that define the equation of state of the photo-ionized medium $A(\zeta)$ and $\beta$. A fiducial value for the variance $\sigma = 3$ was assumed for this plot.

a Ultra Violet (UV)-background. The Lyman-\(\alpha\) forest is thus the main probe of the IGM and it has been shown to arise naturally in hierarchical structure formation scenarios. Astrophysical effects produced by feedback from galaxies and/or AGNs do not seem to strongly affect the vast majority of the baryons in the cosmic web (McDonald et al. 2005; Theuns et al. 2002), thereby this can be used as a dark matter tracer. The relation between the Lyman-\(\alpha\) forest flux and the underlying matter field is a nonlinear one and it is generally expected that statistics of Lyman-\(\alpha\) are biased relative to the underlying dark matter distribution. The Lyman-\(\alpha\) forest has been studied using variety of analytical techniques such as the Zel'dovich approximation, which is valid in the quasilinear regime and often used in modelling of the nonlinear gravitational clustering (Doroshkevich & Shandarin 1977; Hui, Gnedin & Zhang 1997; McGill 1998; Matarrese & Mohavee 2002). In addition, the lognormal approximation (Coles & Jones 1991) is frequently used to model the statistics of Lyman-\(\alpha\) forest (Bi 1993; Gnedin & Hui 1996a; Bi & Davidson 1997; Roy Choudhury, Padmanabhan & Srianand 2001; Viel et al. 2002). Models based on the hierarchical or scaling ansatz (Balian & Schaeffer 1983; Bernardeau & Schaeffer 1992; 1999) for higher-order correlation functions have also been investigated in order to model the statistics of Lyman-\(\alpha\) forest (Valageas, Schaeffer & Silk 1999).

In addition to analytical modelling, hydrodynamical simulations have also played a very important role in this field (e.g. Cen et al. 1994; Gnedin & Hui 1998; Croft et al. 1998, 1999; Meiksin & White 2001) and support the simple analytical picture. Thus, analytical schemes, including the ones that we develop here, can be calibrated using numerical simulations and this in turn allows to explore a larger parameter space efficiently. However, numerical simulations are required to resolve the Jeans scale of the photo-ionized warm IGM, and this requirement typically means small box sizes that sample larger scales modes rather poorly. Indeed several semi-numerical prescriptions, which are not entirely based on hydrodynamical simulations, have also been developed to model Lyman-\(\alpha\) flux and recover the correlation function from observed data sets (Slosar et al. 2011).

The two most commonly used approaches in Lyman-\(\alpha\) studies are based either on decomposing the information encoded in the transmitted flux via Voigt profile fitting or treating the flux as a continuous field.

In the first approach, the shapes and clustering properties of absorption lines fitted by Voigt profiles have been investigated in variety of studies involving the temperature of the IGM (Schaye et al. 1999; McDonald et al. 2001), in order to constrain the reionization history [Theuns et al. 2002; Hui & Haimar 2003], to measure the matter power spectrum and cosmological parameters (Croft et al. 1999, Viel, Haenelt & Springel 2004; McDonald, Seljak & Burles 2004; McDonald & Miralda-Escude 1999; Rolinde et al. 2003; Coppolani et al. 2006; Guimaraes et al. 2007; Viel & Haehnelt 2006). By using the second set of methods, statistical properties of the flux such as the mean flux level, flux PDF, flux power spectrum (Viel, Haenelt & Springel 2004; Viel et al. 2008; Seljak, Slosar, McDonald 2004) and flux bi-spectrum are typically employed to explore flux statistics. For example, it has been shown that the mean-flux level can be used to constrain the amplitude of intergalactic UV background (Tytler et al. 2004; Bolton et al. 2005), while the flux PDF (McQuinn et al. 2005; Bolton, Oh, Furlanetto 2006) is sensitive to the thermal evolution of the IGM (see also Lidz et al. 2006). The flux power spectrum (Seljak, Slosar, McDonald 2004) contains useful information about the primordial, as well as gravity-induced (i.e. secondary) non-Gaussianity. The data typically used in these investigations consists mainly of two different sets of QSO spectra: the SDSS low
resolution low signal-to-noise spectra and UVES/VLT or HIRES/KECK high resolution spectra. The number of SDSS spectra is about a factor \( \sim 200 \) larger than that of high resolution samples though the later probes the smaller scales with greater accuracy.

More recently, it has been argued that BOSS-like QSO spectroscopic surveys could detect Baryon Acoustic Oscillations (BAO) signatures at high redshift (McDonald & Eisenstein 2007) and a sample of QSO pairs can constrain the geometry of the high redshift universe (McDonald 2003). Furthermore, analysis of coincident absorption lines in QSO pairs can also allow departures from the Hubble flow and non-gravitational effects to be measured (Rauch et al. 2005).

The SDSS-III/BOSS survey aims at identifying and observing more than 160,000 QSO over \( \sim 10,000 \) square degrees within a redshift range \( z = (2.15 - 3.5) \). This survey is primarily design to studying baryonic acoustic oscillations by performing a full 3D sampling of the matter density. Such studies will also provide an unprecedented opportunity to study the clustering statistics using projected Lyman-\( \alpha \) flux decrements of QSOs.

In this paper, we will consider the Lyman-\( \alpha \) flux decrement in two dimensions (2D) which is related the projected density of neutral hydrogen. The statistical study of projected Lyman-\( \alpha \) flux decrement can be performed using one- or two-point PDF or their lower order moments. The PDFs contain information of cumulants or their correlators to an arbitrary order and can be constructed using well-established machinery of hierarchical ansatz (Munshi, Coles, Melott 1999a,b; Munshi, Melott, Coles 1999c). Several authors have recently studied the lower-order cumulants of Lyman-\( \alpha \) forests and cross-correlated them against weak lensing convergence as well as to the CMB sky (Vallinotto et al. 2009; Valentino et al. 2011). The results presented here are complementary to such studies as we take into account the lower order moments to an arbitrary order not just for the one-point cumulants but also for their two-point counterparts or cumulant correlators. The cumulant correlators are the two-point analogues of one-point cumulants and are already in use in different areas of cosmology, e.g. in analysis of galaxy surveys (Szapudi & Szalay 1997). The lowest in the two-point hierarchy is the two-point correlation function. In the context of Lyman-\( \alpha \) studies, the two-point correlation function has already been introduced in studies involving two neighbouring line of sights (Viel et al 2002; D’Odorico et al 2002). Our study generalizes these results to probe non-Gaussian correlation functions involving multiple line of sight. We model the statistics of underlying neutral hydrogen distribution using lognormal distribution as well as an extension of perturbation theory approach (Valageas & Munshi 2004) in 3D. Predictions from these models are then tested using hydrodynamical simulations at three different redshifts (\( z = 2, 3, 4 \)). We use these results to build and test statistical description of the projected flux distribution. The results presented here can be generalized to cross-correlation studies involving external data sets and Lyman-\( \alpha \) flux distribution. The correlation functions (CCs) are equivalent to their Fourier (harmonic) space counterpart, the multispectra, recently introduced by (Munshi & Heavens 2009).

The plan of this paper is as follows. In §2 we introduce the notations and define the relevant quantities such as the transmitted flux and its relation to the underlying density contrast. In §3 we give details of the simulations that were used in our study. In §4 details of modelling of the 3D flux are presented. In §5.1 we derive the lower order statistics for the flux in terms of that of the underlying density contrast. In §5.2 we provide a very brief introduction to the hierarchical ansatz which we use to model the statistics of underlying density contrast. In §5.3 we provide derivation of the PDF and the bias associated with the flux distribution and finally §6 is left for discussion of results. We also provide a brief appendix introducing the lognormal approximation as well as the hierarchical ansatz.

## 2 NOTATION

We will be using the following form of the Robertson-Walker line element for the background geometry of the universe:

\[
ds^2 = -c^2 dt^2 + a^2(t)(dr^2 + d\alpha^2(r)(d\phi^2 + \sin^2 \theta d\theta^2))
\]

The angular position on the surface of the sky is specified by the unit vector \( \hat{\Omega} = (\theta, \phi) \). The scale factor of the Universe is given by \( a(t) \). We have denoted the comoving angular diameter distance by \( d_A(r) \) where \( r \) denotes the comoving radial distance to redshift \( z \); \( d_A(r) = K^{-1/2} \sinh((\pm K)^{1/2} r) \) for positive curvature, \( d_A(r) = \pm (1 - \Omega_M)^{-1/2} \sinh((\pm K)^{1/2} r) \) for negative curvature and \( r \) for the flat universe. For a present value of of \( H_0 \) and \( \Omega_M \) we have \( K = (\Omega_M - 1)H_0^2 \). The Hubble constant is denoted by \( H_0 \). The comoving radial distance \( r(z) \) at a redshift \( z \) can be expressed through the following LOS integration, with \( z' \) playing the role of intermediate redshift along the line of sight:

\[
r(z) = \int_0^z \frac{dz'}{H_0 \sqrt{\Omega_M (1 + z')^3 + \Omega_K (1 + z')^2 + \Omega_A}}.
\]

Throughout, in our calculation, we will adopt \( H_0 = 100 h \) km\( \text{pc}^{-1} \text{Mpc}^{-1} \) with \( h = 0.7 \) and \( \sigma_8 = 0.87 \). Here \( \Omega_M, \Omega_A \) and \( \Omega_K \) are dark-matter, vacuum-energy and curvature contribution to cosmic density \( \Omega_M + \Omega_A + \Omega_K = 1 \). For a flat Universe \( \Omega_K = 0 \).

## 3 SIMULATIONS

We use simulations run with the parallel hydrodynamical (TreeSPH) code GADGET-2 based on the conservative ‘entropy-formulation’ of SPH (Springel et al 2005). These consist of a cosmological volume with periodic boundary conditions filled with an equal number of dark matter and
The PDF of the density contrast $p(\tilde{\delta})$ as a function of the density contrast $\tilde{\delta}$. The left ($z=2$), middle ($z=3$) and right ($z=4$) panels correspond to different redshifts $z$ studied. Two different analytical models are displayed the lognormal approximation (long-dashed) and the model proposed by Valageas & Munshi (2004) (short- and long-dashed) denoted as VM. The solid lines correspond to the results from simulations. Notice that the analytical results and simulations for positive (over-dense) $\delta$ values of the PDF agree extremely well for all three redshifts investigated in this work. The PDF was estimated using $128^3$ grid and can resolve PDF as low as $O(10^{-6})$. However the PDF becomes increasingly dominated by the presence or absence of rare large overdensities.

Figure 3. The PDF of density contrast $p(\tilde{\delta})$ as a function of the density contrast $\tilde{\delta}$. The left ($z=2$), middle ($z=3$) and right ($z=4$) panels correspond to different redshift $z$. Two different analytical models are displayed the lognormal approximation (long-dashed) and the model proposed by Valageas & Munshi (2004) (short- and long-dashed) denoted as VM. The PDF is constructed on a grid of $128^3$. The lowest probability that can be estimated on this grid is $O(10^{-6})$. However higher $\tilde{\delta}$ tails of PDFs are increasingly dominated by the presence (or absence) of rare over (under)dense objects.

gas particles. Radiative cooling and heating processes were followed for a primordial mix of hydrogen and helium. We assumed a mean Ultraviolet Background similar to that proposed by Haardt & Madau (1996) produced by quasars and galaxies as given by with helium heating rates multiplied by a factor 3.3 in order to better fit observational constraints on the temperature evolution of the IGM. This background gives a hydrogen ionisation rate $\Gamma_{12} \sim 1$ at the redshifts $z = 2 - 4$ of interest here (Bolton et al. 2008). The star formation criterion is a very simple, one that converts all the gas particles into stars whose temperature falls below $10^5$ K and whose density contrast is larger than 1000 (it has been shown that the star formation criterion has a negligible impact on flux statistics). More details can be found in Viel, Haenelt & Springel (2004).

The cosmological reference model corresponds to a ‘fiducial’ $\Lambda$CDM Universe with parameters, at $z = 0$, $\Omega_m = 0.3$, $\Omega_{\Lambda} = 0.7$, $\Omega_b = 0.05$, $n_s = 1$, and $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$ and $\sigma_8 = 0.85$. The initial conditions are generated at $z = 49$ using as linear matter power the one extracted using the CAMB software. We have used $2 \times 512^3$ dark matter and gas particles in a volume of linear size $60 h^{-1}$ Mpc$^{-1}$ and the gravitational softening was set to $4 h^{-1}$ kpc in comoving units for all particles. The mass per particle is $1.1 \times 10^8 M_{\odot} h^{-1}$ and $1.8 \times 10^7 M_{\odot} h^{-1}$ for DM and gas particles, respectively. This simulation should have sufficient resolution to properly reproduce most of flux statistics at least at $z \leq 3$ and marginally at $z = 4$: the flux power should be in fact converged with this setup and also (marginally) the flux PDF. However, since we are dealing with projected quantities, we believe that numerical convergence issues are less severe than for other small scale flux observables.

We select $128^2$ grid points in the $x - y$ plane and we interpolate along lines of sight parallel to the $z -$ axis using a Cloud-In-Cell (CIC)
Figure 4. We show the projected transmitted Lyman–α flux (top panels) and projected gas density (bottom panels) in the simulated volume at $z=2,3,4$ (left, middle and right columns, respectively). The gas density is extracted at $128^3$ grid points using a CIC or cloud-in-cell interpolation scheme: the values shown in the bottom panels are the projected $\tilde{\delta}$ along $128^3$ lines of sight (along the $z$-axis). The growth of the cosmic structures can be appreciated from high to low redshift. In the upper panels we show the mean value $\langle F_{\alpha} \rangle$ along the same $128^2$ lines of sight, the values at the three redshift are all normalized to the $\langle F_{\alpha}(z = 2) \rangle$ value and the spectra are simulated by using the exact definition of the transmitted flux and not approximations. Since the mean flux is a strongly evolving function of redshift, the growth of cosmic structure in Lyman–α flux is more difficult to interpret than in the corresponding density slices and simulations and/or models are needed. In the upper panels, voids correspond to regions of high-transmissivity, while dense regions produce absorptions. At $z = 2$ the $\langle F_{\alpha} \rangle$ values fluctuate by about 10% around the mean, while at $z = 4$ this value becomes 20% (since the mean transmitted flux is about two times smaller at this redshift). The simulation refers to a hydrodynamical run with $2 \times 512^3$ gas and DM particles in a $60^3$ (Mpc $h^{-1}$)$^3$ periodic comoving volume.

algorithm. The Lyman–α flux is also computed in redshift space along the same lines of sight by using the exact definition of transmitted flux and not the Fluctuating Gunn-Peterson Approximation. Peculiar velocities, neutral hydrogen fraction and gas temperature are calculated along the lines of sight and a Voigt profile is used to obtain the mock quasar spectra. The density and transmitted flux are then projected along the simulated lines of sight in the $z$–direction.

4 THE LYMAN–α FLUX AND THE IGM IN 3D

The fluctuating Gunn-Peterson approximation (Gunn & Peterson 1965) allows us to relate the transmitted flux $F_{\alpha}(z, \Omega)$ along a line of sight at a direction $\Omega \equiv (\theta, \phi)$ at a redshift $z$ with the fluctuation in neutral hydrogen density contrast $\tilde{\delta}$:

$$F_{\alpha}(z, \Omega) = \exp \left[ -A(z)(1 + \tilde{\delta}(z, \Omega))^{\beta} \right];$$

$$A(z) \equiv 0.0023(1 + z)^{3.65}; \quad \beta \equiv 2 - 0.7(\gamma - 1); \quad \gamma \equiv 1.3.$$  \hspace{1cm} (3)  \hspace{1cm} (4)

The parameters $A(z)$ and $\beta$ are two redshift-dependent functions relating the flux fluctuations to the dark matter overdensities. The parameter $A$ is related to the level of mean transmitted flux which is accurately measured (e.g. Kim et al. (2007)). It also depends on inputs from baryonic physics such as the baryonic fraction, IGM temperature, photo-ionization rate as well as background cosmological parameters. The power-law index $\gamma$ of the IGM temperature-density relation determines the value of the parameter $\beta$ and is relatively independent of redshift $z$ (Hui & Gnedin 1997; Kim et al. 2007) (this is of course assuming fluctuations in temperature due to e.g. reionization play a sub-dominant role). The allowed range of values for $\gamma$ is $\gamma = (0.7 - 1.5)$ we will use $\gamma = 1.3$ for our calculation, which is a good approximation to the actual value of the hydrodynamical simulation.
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4.1 PDF and Bias in 3D

We use the lognormal distribution to model the distribution of $\tilde{\delta}$; the resulting flux PDF is obtained via a change of variable:

$$p(F_\alpha) = p(\tilde{\delta}) \left| \frac{d\tilde{\delta}}{dF_\alpha} \right|; \quad p(F_\alpha^{(1)}, F_\alpha^{(2)}) = p(\tilde{\delta}_1, \tilde{\delta}_2) \left| \frac{d\tilde{\delta}_1}{dF_\alpha^{(1)}} \right| \left| \frac{d\tilde{\delta}_2}{dF_\alpha^{(2)}} \right|.$$  \hspace{1cm} (5)

In our notation $\tilde{\delta}_{(i)} = \tilde{\delta}(\Omega_i, z_i)$ and a similar notation is adopted for the fluxes along different lines of sight $F_\alpha^{(i)} = F_\alpha(\Omega_i, z_i)$. For 3D studies we have compared flux from neighbouring lines of sight at the same redshift. We have compared the PDFs using this approach for three different redshifts $z = 2, 3, 4$. The PDFs for $p(\tilde{\delta})$ are constructed using lognormal prescriptions (Eq. (B1) and Eq. (B4)).

The lognormal distribution is also used in the analysis based on the fitted column density distribution of absorption lines. In such studies the lognormal distribution is introduced to relate $\tilde{\delta}$ and number density of neutral hydrogen $n_{\text{HI}}$. In relating the flux distribution and $\tilde{\delta}$ on the other hand we assume that $\tilde{\delta}$ traces the underlying dark matter distribution which can be described well by lognormal distribution. The lognormal distribution here is being used to model the effect of gravity induced non-linearity on PDFs of matter density contrast $\delta$. A similar but somewhat different approach was used by Viel et al. (2002) who used lognormal approximation to map the density field to flux. In our case we directly map the PDF of the flux using simply a change of variable Eq.(4). The construction of the PDF of $\tilde{\delta}$ is done using lognormal approximation as well as using extensions of perturbative methods Valageas & Munshi 2004. These methods give near identical results. The hierarchical ansatz (to be introduced later; see Appendix-B) will be used too for the construction of one-point and two-point PDFs. It is interesting to point out that it has also been used by Valageas, Schaeffer & Silk 1999 to predict the clustering Lyman-α absorbers. It was used to construct an unifying model from the clustering of low column-density clouds to the collapsed dense damped systems. The methods that we present here is complementary to these studies as we directly probe the statistics of the transmitted flux.

4.2 Numerical Results in 3D

In Figure (2) we have plotted the PDF of $p(\tilde{\delta})$ as a function of $\tilde{\delta}$. The PDF was computed by binning the data on a $128^3$ grid. This allows us to resolve the PDF down to $O(10^{-3})$. Different panels show the PDF for three different redshifts. We also show the results from different analytical calculations. The two-point PDF or 2PDF denotes the joint PDF of $\tilde{\delta}$ at two different points. Typically the 2PDF can be written as a product of two 1PDFs at these two-point and additional correction terms. The corrective terms represents the contribution due to correlation. In general in the large separation limit when the two-point correlation is weak compared to the variance (i.e. $\xi_{12}^p < \xi_{12}^\alpha; \xi_{12}^\alpha = (\delta(x_1)\delta(x_2))$ is the two-point correlation function and $\xi_2$ is its volume average), the 2PDF takes the following form in 3D:

$$p(\tilde{\delta}_1, \tilde{\delta}_2)d\tilde{\delta}_1d\tilde{\delta}_2 = p(\tilde{\delta}_1)p(\tilde{\delta}_2)[1 + b(\tilde{\delta}_1)\xi_{12}^\alpha b(\tilde{\delta}_2)]d\tilde{\delta}_1d\tilde{\delta}_2.$$ \hspace{1cm} (6)

This equation describes the two-point PDF (or 2PDF) $p(\tilde{\delta}_1, \tilde{\delta}_2)$ of the underlying density contrast $\tilde{\delta}$ in terms of PDFs $p(\tilde{\delta})$ and bias $b(\tilde{\delta})$. An identical relation also holds for $\tilde{\delta}$ above the Jeans scale. In general the (differential) bias $b(\eta)$ is difficult to estimate. To increase signal-to-noise it is useful to compute the integrated bias beyond a threshold (Bernardeau 1994):

$$b(\delta_0) \equiv \left[ \int_{\delta_0}^{\infty} p(\tilde{\delta})b(\tilde{\delta})d\tilde{\delta} \right] = \frac{1}{\xi_2} \left[ \int_{\delta_0}^{\infty} \frac{p(\tilde{\delta}_1, \tilde{\delta}_2)d\tilde{\delta}_1d\tilde{\delta}_2}{\int_{\delta_0}^{\infty} p(\tilde{\delta})d\tilde{\delta}^2} - 1 \right]^{1/2}.$$ \hspace{1cm} (7)

For our numerical estimation of bias associated with the $\tilde{\delta}$ we have used Eq. (7) to compute the cumulative bias associated with $\tilde{\delta}$. In Figure (3) we have presented the numerical results for $b(> \delta_0)$ against the theoretical expectations from lognormal distribution. As expected the lognormal distribution is quite accurate in prediction of the numerical results for over dense regions. We show results for three different redshift relevant to high redshift Lyman-α studies. Notice that the high $\delta$ tail of the PDF is reproduced well by the lognormal approximation Eq. (15). The departure at lower $\delta$ is partly related to the smoothing introduced in hydrodynamical simulation during interpolation. Indeed we only use the leading order term $\xi_{12}^p/\xi_2^\alpha$ in our calculation of bias from the lognormal distribution. These results are valid in the large separation limit. Additional corrective terms can be included if necessary.

Using the transformation given in Eq. (5), we can express the pdf $p(F_\alpha)$ and bias $b(< F_\alpha)$ in terms of the pdf $p(\delta_0)$ and bias for the density $b(> \delta_0)$:

$$b(\delta_0) \equiv \frac{1}{\sqrt{\xi_2}} \left[ \int_{\delta_0}^{\infty} \frac{\int_{\delta_0}^{\infty} p(\tilde{\delta}_1, \tilde{\delta}_2)d\tilde{\delta}_1d\tilde{\delta}_2}{\int_{\delta_0}^{\infty} p(\tilde{\delta})d\tilde{\delta}^2} - 1 \right]^{1/2} = \frac{1}{\sqrt{\xi_2}} \left[ \int_{\delta_0}^{F_\alpha} \int_{\delta_0}^{F_\alpha} p(\tilde{\delta}_1, \tilde{\delta}_2) d\tilde{\delta}_1 d\tilde{\delta}_2 \right]^{1/2} \equiv \frac{\sqrt{\alpha}}{\xi_2} b(< F_\alpha).$$ \hspace{1cm} (8)

The results of such calculation for the PDF is shown in Figure (5). There is no accurate model for bias in the highly nonlinear regime. The lognormal approximation is accurate in the quasi-linear regime and starts to fail in the highly nonlinear regime. The approach taken by Valageas & Munshi (2004) can be extended to the construction of bias. The results of integrated bias for the flux as defined in Eq. (16) and are plotted in Figure (6).
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PDF of flux $p(F_\alpha)$

![PDF of flux $p(F_\alpha)$](image)

Figure 5. The PDF of the flux $p(F_\alpha)$ as a function of the flux $F_\alpha$. The left ($z = 2$), middle ($z = 3$) and right ($z = 4$) panels correspond to different redshift $z$. Two different analytical models are displayed the lognormal approximation (long-dashed) and the Gaussian approximation (short and long-dashed). The flux PDF is constructed from the PDF of $\tilde{\delta}$ using Eq. (4). The results from the simulation are also depicted (solid lines).

Bias associated with flux $b(<F_\alpha)$

![Bias associated with flux $b(<F_\alpha)$](image)

Figure 6. The bias of the flux $b(<F_\alpha)$ as a function of the flux $F_\alpha$. The left ($z = 2$), middle ($z = 3$) and right ($z = 4$) panels correspond to different redshift $z$. The solid lines correspond to direct estimates from 3D flux maps. The dashed lines are from maps that are generated from the density maps.

found that including a power law mapping $(1 + \delta) \rightarrow (1 + \delta)\Delta$, where $\Delta$ is a constant of order unity, can fit the numerical results, if it is applied before the exponential transformation.

5 PROJECTED FLUX DECREMENT

We are interested in scales larger than the comoving Jeans length. At around redshift $z = 3$ which is about $1h^{-1}\text{Mpc}$. The fractional flux $\delta F_\alpha(z, \hat{\Omega})$ in this limit can be related to the underlying dark matter density distribution $\delta$, using Eq. (4) by the following expression:

$$\delta F_\alpha(z, \hat{\Omega}) \equiv (F_\alpha(z, \hat{\Omega}) - \langle F_\alpha(z, \hat{\Omega}) \rangle)/\langle F_\alpha(z, \hat{\Omega}) \rangle = -A(z)\beta(z)\hat{\delta}(\hat{\Omega}, z) \approx -A(z)\beta(z)b(z)\hat{\delta}(\hat{\Omega}, z).$$  (9)

The relation assumes that IGM traces the dark matter distribution at larger scales (above Jeans length). We have incorporated an additional redshift dependent bias factor $b(z)$ in relating $\delta_{\text{IGM}}(z)$ and $\delta(z)$ for generality. The results presented here can be generalised for arbitrary redshift dependent bias. In our calculations we will set $b(z) = 1$. The factors $A(z)$ and $\beta$ have already been introduced above; see e.g. Hui & Gnedin (1997) and McDonald (2003) for a discussion on the temperature-density relation which is well described by a power law.

The integrated flux decrement $\delta F_\alpha(\hat{\Omega})$ towards a particular direction $\hat{\Omega}$ can be expressed through a line-of-sight integration as a projected
density contrast:

$$\delta \mathcal{F}_\alpha (\hat{\Omega}) = \int_{r_i}^{r_q} dr \frac{\omega_{\alpha}(r)}{dA(r)} \cdot \frac{d^4 k}{(2\pi)^4} \exp(irk_\parallel + i d_{A}(r) \theta_{12} \cdot k_\perp) \delta (k, r).$$

The range of comoving distances probed by the Lyman-α spectrum extends from \(r_i\) to the quasar distance \(r_q\). We will define a parameter \(\delta \mathcal{F}^{\text{min}}_{\alpha} = \int_{r_i}^{r_q} \omega_{\alpha}(r) \cdot dr\) that is the minimum value of \(\delta \mathcal{F}_\alpha\) for a given \(r_i\) and \(r_q\). The weight \(\omega_{\alpha}(z)\) that acts as a kernel for projection can be expressed in terms of previously introduced functions \(w(z) = A(z) \beta(z)\). Few comments are in order. The assumption of quasilinear theory (e.g. the lognormal model) is expected to be valid above the Jeans length. This is where our results are expected to be valid (Eisenstein & Hu 1998). The relationship between the flux and the underlying mass distribution has been verified in numerous studies (Mcquinn et al. 2009; Bi & Davidson 1997; Croft et al. 2002; Viel et al. 2002; Saittta et al. 2008). It is also important to note that the simple linear relationship that is often used in deriving many analytical results in the context of Lyman-α studies can be modified in the presence of any non-gravitational process such as fluctuations in the level of ultraviolet radiation or temperature fluctuation. Such non-gravitational effects are harder to model analytically. Next we will use these expressions to model the lower order statistics of \(\delta \mathcal{F}_\alpha (\hat{\Omega})\).

### 5.1 Lower order Cumulants and Cumulant Correlators in Projection

We will use both cumulants and cumulant correlators in our studies. The one-point cumulants can be estimated from a single LOS where as the higher order moments of the field relate to the flux decrements or the bias associated with the flux decrements. Just as the normalised cumulants can specialise them by assuming a particular form for the higher order multispectra to make further progress. Just as the normalised cumulants and their correlators to construct the PDF and the bias associated with the flux decrements or \(\delta \mathcal{F}_\alpha\). We will use these results to show that it is possible to compute the complete probability distribution function (PDF) of \(\delta \mathcal{F}_\alpha\) from the underlying dark matter PDF. The details of the analytical results presented here can be found in Munshi & Jain 2000, 2001.

In addition to the cumulants their correlators are important as they are related to the two-point PDF (or 2PDF) and hence the bias associated with the higher flux regions:

$$\langle \delta \mathcal{F}_\alpha (\hat{\Omega}_1) \delta \mathcal{F}_\beta (\hat{\Omega}_1) \rangle_c = \int_{r_i}^{r_q} dr \frac{\omega_{\alpha}(r)}{dA(r)} \cdot \frac{d^4 l_1}{(2\pi)^4} \ldots \frac{d^4 l_{p+q}}{(2\pi)^4} B^{(p)}_{\alpha}(l_1) \ldots B^{(q)}_{\beta}(l_{p+q}) \cdot \text{Σ}_{l_i=0}. \tag{14}$$

A special case of this equation corresponds to the expression for the two-point correlation function \(\langle \delta \mathcal{F}_\alpha (\hat{\Omega}_1) \delta \mathcal{F}_\beta (\hat{\Omega}_1) \rangle\). Tare generic. We will specialise them by assuming a particular form for the higher order multispectra to make further progress. Just as the normalised cumulants \(S_p\) we can define the normalised cumulant correlators \(C_{pq} = \langle \delta \mathcal{F}_\alpha (\hat{\Omega}_1) \delta \mathcal{F}_\beta (\hat{\Omega}_1) \rangle / \langle \delta \mathcal{F}_\alpha (\hat{\Omega}_1) \rangle \langle \delta \mathcal{F}_\beta (\hat{\Omega}_1) \rangle\) We will use the normalised cumulants or the \(S_p\) parameters and the normalised cumulant correlators to construct the PDF and the bias associated with the flux decrements or \(\delta \mathcal{F}_\alpha\). We will denote the normalised cumulants and their correlators of density by \(S_p\) and \(C_{pq}\). The corresponding quantities for the flux decrements will be denoted using a superscript \(\alpha\) i.e \(S^\alpha_p\) and \(C^\alpha_{pq}\).
5.2 A Minimal Tree-Model

The spatial length scales, corresponding to the small angular scales, that are relevant in our discussion, are in the highly non-linear regime. Assuming a tree model for the matter correlation hierarchy in the highly non-linear regime, one can write the general form of the $N$th order correlation function $\xi^{(N)}_s$ as a product of two-point correlation function $\xi^{(2)}_s$ (Groth & Peebles 1977; Davis & Peebles 1977; Fry 1984; Balian & Schaeffer 1989; Szapudi & Szalay 1993, 1997). In Fourier space such an ansatz means that the hierarchy of multispectra can be written as sums of products of the matter power-spectrum:

$$B_3(k_1, k_2, k_3)\sum_{k_i=0} = Q_3(P_3(k_1)P_3(k_2) + P_3(k_2)P_3(k_3) + P_3(k_3)P_3(k_1))$$

$$T_5(k_1, k_2, k_3, k_4)\sum_{k_i=0} = R_5s P_5(k_1)P_5(|k_1 + k_2 + k_3 + k_4|) + \text{cyclic. perm.}$$

The subscript $\sum_{k_i=0}$ represents a Dirac’s $\delta$ function $\delta_D[\sum_{i=1}^5 k_i]$. Different hierarchical models differ in the way they specify the amplitudes $Q_3, R_5$, etc. The results that we will derive are valid for generic hierarchical models. Using Eq. (15) and Eq. (16) we can write:

$$\langle \delta F_3^3 \rangle_c = (3Q_3)C_3[(\mathcal{F}_0^\alpha)^3] = S_0^\alpha \langle \delta F_2^2 \rangle_c$$

$$\langle \delta F_4^4 \rangle_c = (12R_4 + 4R_5)C_4[(\mathcal{F}_0^\alpha)^4] = S_0^\alpha \langle \delta F_2^2 \rangle_c$$

Where we have defined the following quantities:

$$C_p[(\mathcal{J}_0^\alpha(r))^p] = \int_{r_{\min}}^{r_{\max}} \frac{\omega_p'(r)}{d_A(r)} (\mathcal{J}_0^\alpha(r))^{p-1} dr; \quad [\mathcal{J}_0^\alpha(r)] = \int \frac{d^3l}{(2\pi)^3} P_b \left( \frac{l}{d_A(r)} \right).$$

These lower order moments are next used to construct the void probability function (VPF) which also acts as a generating function for the normalised $S_0^\alpha$ parameters for flux-decrement $\delta F_\alpha(\Omega)$. The probability distribution function (PDF) can be constructed from the knowledge of the VPF as will be detailed in the next section.

5.3 Probability Distribution Function of Transmitted Flux

Before we discuss our techniques to construct the PDF for the projected transmitted flux a few explanations regarding the notation and its link to the entire hierarchical paradigm are in order. Although our primary aim is to construct the PDF it is usually easier to construct the moment generating function $\phi(y)$ that will be introduced later. The lower order normalised cumulants (also known as the $S_p$ parameters) up to an arbitrary order can be recovered, if necessary, using a Taylor expansion of $\phi(y)$. The function $h(x)$ is related to $\phi(x)$ through an inverse Laplace transform, i.e., a knowledge of $S_p$ parameters to an arbitrary order can be used to construct the function $h(x)$. The function $h(x)$ on the other hand is simply a scaled PDF and the real PDF can be extracted from it using the definition of scaling variable $x$ (to be defined later).

The construction of PDF for $\delta F_\alpha$ involves additional steps. First the generating function $\Phi^\alpha(y)$ for the lower order normalised cumulants $S_p^\alpha$ is expressed in terms of the underlying $\phi(y)$. Next we define a new scaling variable for $\delta F_\alpha$ which is denoted by $\eta$. We show that, under certain simplifying approximation, the cumulant generating function for $\eta$ i.e. $\Phi^\eta(y)$ is same as the underlying cumulant generating function $\phi(y)$. Once this
proved, it is easy see that the scaled PDFs $H^n(y)$ and $h(y)$ too are same as they are simply the Laplace transform of $\Phi^0(y)$ and $\phi(y)$ respectively. This technique we use has already been used to derive similar relations in the context of weak lensing statistics (Munshi & Jain 2004, 2001; Valageas 2000; Bernardeau & Valageas 2000).

The generating function $\Phi^0(y)$ (or the Void Probability Function, i.e. VPF) for the lower order normalised cumulants $S_p^n$ for the flux-decrement $\delta F_\alpha^\alpha(\bar{\Omega})$ can be expressed as:

$$\Phi^0(y) = \sum_{p=1}^{\infty} \frac{S_p^n}{p!} y^p = y + \sum_{p=2}^{\infty} \frac{\langle \delta F_\alpha^0(\bar{\Omega}) \rangle_c^p}{\langle \delta F_\alpha^2(\bar{\Omega}) \rangle_c} y^p.$$  \hfill (19)

Our aim is to relate the VPF $\Phi^0(y)$ and $\phi(y)$ where $\phi^0(y)$ is the VPF of the underlying matter distribution. The exact functional form of $\phi(y)$ is fixed by choosing a specific hierarchical ansatz. We will keep our discussion here completely general without any reference to any specific form of the correlation hierarchy. Using Eq. (17) we can write:

$$\Phi^0(y) = \int_{r_i}^{r_i} \frac{1}{\langle \delta F_\alpha^2 \rangle_c} \phi \left[ \frac{\langle \delta F_\alpha^1 \rangle_c}{\langle \delta F_\alpha^0 \rangle_c} \right] \phi \left[ \frac{\langle \delta F_\alpha^0 \rangle_c}{\langle \delta F_\alpha^2 \rangle_c} y \right]; \quad \phi(y) = \sum_{p=1}^{\infty} \frac{S_p}{p!} y^p.$$  \hfill (20)

Using the VPF of the underlying mass distribution $\phi(x)$, the above expression can be rewritten in a more compact form:

$$\Phi^0(y) = \int_{r_i}^{r_i} dr \frac{\omega_\alpha(r)}{\omega_\alpha(0)} \phi \left[ \frac{\omega_\alpha(r)}{\omega_\alpha(0)} \phi \left[ \frac{\omega_\alpha(0)}{\omega_\alpha(r)} y \right] \right]; \quad \phi(y) = \sum_{p=1}^{\infty} \frac{S_p}{p!} y^p.$$  \hfill (21)

We will define a variable $\eta$ that will make the analysis simpler. The VPF for $\eta$, denoted $\Phi^\eta(y)$, can now be expressed in terms of the VPF of the underlying mass distribution $\phi(y)$ by the following expression:

$$\Phi^\eta(y) = \frac{1}{\langle \delta F_\alpha^2 \rangle_c} \int_{r_i}^{r_i} dr \frac{\omega_\alpha(r)}{\omega_\alpha(0)} \phi \left[ \frac{\omega_\alpha(r)}{\omega_\alpha(0)} \phi \left[ \frac{\omega_\alpha(0)}{\omega_\alpha(r)} y \right] \right]; \quad \eta = \langle \delta F_\alpha - \delta F_\alpha^0 \rangle / \langle \delta F_\alpha^0 \rangle.$$  \hfill (22)

The scaling properties of the PDF in an hierarchical model are encoded in a scaling function. The scaling function $H^n(x)$ is related to the PDF of $x$, $p(x)$, as the scaling function $h(x)$ is associated with the PDF of the underlying mass distribution $\delta$. The following relation relates $H^n(x)$ and the scaling function $h(x)$.

$$H^n(x) = \frac{1}{\langle \delta F_\alpha^2 \rangle_c} \int_{r_i}^{r_i} w_\alpha(r) x^{w_\alpha(r)} dr \left[ \frac{d_\alpha(r) / \langle \delta F_\alpha^2 \rangle_c}{J_\alpha(r) / \langle \delta F_\alpha^0 \rangle_c w_\alpha(r)} \right]^2 h \left[ \frac{x(\delta F_\alpha^0) / d_\alpha(r)}{w_\alpha(r) / \langle \delta F_\alpha^0 \rangle_c J_\alpha(r)} \right].$$  \hfill (23)

The above relation is derived using the following definition of $H^n(x)$ in terms of the moment generating function or VPF $\Phi^\eta(y)$ (Balian & Schaeffer 1989) as the moment generating function $\Phi^\eta(y)$ and the scaled PDF $H^n(x)$ are linked through an inverse Laplace transform:

$$H^n(x) = - \int_{-\infty}^{\infty} \frac{dy}{2\pi i} \exp(xy) \Phi^\eta(y).$$  \hfill (24)

A similar result holds for $h(x)$ and $\phi(y)$. Using an approximate form for the integrals we recover the following relation:

$$p_\alpha(\delta F_\alpha) = p(\eta)/\langle \delta F_\alpha^0 \rangle.$$  \hfill (25)

In terms of the cumulants this means $S_p^n = S_p^\eta / \langle \delta F_\alpha^0 \rangle^{p-2}$. It is instructive to note that the final result is independent of any detailed modelling. We will use a lognormal PDF as well as the PDF constructed using the hierarchical ansatz to model $p(\eta)$.

It is important to note that the function $h(x)$ is fundamental to all scaling analysis. It was initially introduced to study the clustering of galaxies later was used in diverse cosmological studies (e.g. Valageas, Schaeffer & Silk 1999); e.g. the mass function of collapsed objects that includes clusters, groups and galaxies can all be described by the scaling function $h(x)$. The multiplicity function $\eta(M, z)$ at a redshift $x$ for collapsed objects of mass $M$ can be expressed via: $\eta(M, z) dM/M = \langle \rho/M \rangle x h(x) dx$. Here the scaling function $x$ can be expressed as $x = [1 + \delta(M, z)]/\xi_2$ and $\rho$ is the mean physical density of the Universe. Thus, scaling arguments can also provide an alternative to the usual mass function calculation based on the Press-Schechter formalism. These issues have been discussed extensively in the literature (e.g. Valageas & Schaeffer 2003). The volume averaged two-point correlation function $\xi_2$ is evaluated at a corresponding relevant scale. It is interesting to notice that the functional form for the scaling function $h(x)$ depends only on the form for initial power spectrum and its qualitative behaviour can be predicted using very general arguments. Two different asymptotes are particularly well understood; (1) $x \ll 1$: $h(x) \propto x^{-2}$ and (2) $x \gg 1$: $h(x) \propto x^{\omega_2-1} \exp(-x/x_s)$ with $\omega_2 \sim 0.5$ and $\omega_s \sim -3/2$. More detailed discussions about some of the salient features are reported in Appendix-A.

The numerical results for the 2D PDF of $\eta$ are presented in Figure 1. The results are depicted for three redshifts, $z = 2, 3, 4$. These results were computed using a grid of $512^2$ LOS drawn from a $512^3$ simulation box. The underlying cosmology is same as the ones adopted for 3D studies. The resulting PDFs computed using this grid are stable down to $O(10^{-4})$. Notice that the projected PDF is closer to a Gaussian in form than its 3D counterpart.
Using the expression Eq. (32) derived using the hierarchical ansatz we can make further progress:

\[ C_{pq}^0 = \langle \delta F_{\alpha}^c(\hat{\Omega}_1) \delta F_{\alpha}^c(\hat{\Omega}_2) \rangle_c / \langle \delta F_{\alpha}^2(\hat{\Omega}_1) \rangle_c \]

Notice that, by construction, \( C_{11} = 1 \) which is due to the particular normalization that is generally adopted for \( C_{pq} \). At the lower order the cumulant correlators can be expressed in terms of the hierarchical amplitudes (i.e. \( Q_3, R_0, R_6, S_a, S_b, S_c \)):

\[
\langle \delta F_{\alpha}^2(\hat{\Omega}_1) \delta F_{\alpha}^c(\hat{\Omega}_2) \rangle_c = 2Q_3C_3[\mathcal{P}^3_{\alpha} J_{12}^3] = C_{11}^2C_4[\mathcal{P}^4_{\alpha} J_{12}^4] = C_{11}^2 \langle \delta F_{\alpha}^2(\hat{\Omega}_1) \delta F_{\alpha}^c(\hat{\Omega}_2) \rangle_c.
\]

The expressions for the cumulant correlators given above apply in the large separation limit, i.e. \( \langle \delta F_{\alpha}^c(\hat{\Omega}_1) \delta F_{\alpha}^c(\hat{\Omega}_2) \rangle_c \ll \langle \delta F_{\alpha}^2(\hat{\Omega}_1) \rangle_c \). Though higher-order correction terms can be computed using the framework of the hierarchical ansatz, in practice the large separation limit is reached very quickly even for nearby lines of sight. A detailed discussion of the tree-amplitudes can be found in [Munshi, Melott, Coles (1999c)]. In general the cumulant correlators can be expressed as:

\[
\langle \delta F_{\alpha}^p(\hat{\Omega}_1) \delta F_{\alpha}^q(\hat{\Omega}_2) \rangle_c \equiv C_{pq}^p \langle \mathcal{P}^{p+q}_{\alpha} J_{12}^{p+q-2} \rangle_c = C_{pq}^p / \langle \delta F_{\alpha}^2(\hat{\Omega}_1) \delta F_{\alpha}^c(\hat{\Omega}_2) \rangle_c.
\]

Using these expressions we can construct the joint two-point PDF or the bias. To do so we introduce the generating function \( \beta_{\alpha}^{(2)}(y_1, y_2) \) for the normalised cumulant correlators:

\[
\beta_{\alpha}^{(2)}(y_1, y_2) = \sum_{p, q} \frac{y_1^p y_2^q}{p! q!} C_{pq}^p \sum_{p, q} \frac{1}{p! q!} \langle \delta F_{\alpha}^p(\hat{\Omega}_1) \delta F_{\alpha}^q(\hat{\Omega}_2) \rangle_c^p q^2 \langle \delta F_{\alpha}^2(\hat{\Omega}_1) \delta F_{\alpha}^c(\hat{\Omega}_2) \rangle_c^{p+q-2}.
\]

Using the expression Eq. (32) derived using the hierarchical ansatz we can make further progress:

\[
\beta_{\alpha}^{(2)}(y_1, y_2) = \sum_{p, q} C_{pq}^0 \frac{y_1^p y_2^q}{p! q!} \langle \delta F_{\alpha}^2(\hat{\Omega}_1) \delta F_{\alpha}^c(\hat{\Omega}_2) \rangle_c^{p+q-2} \int_0^\infty dr \frac{d^2A(r)}{d^2A_c(r)} \frac{\omega_0^p(r) \omega_0^q(r)}{\omega_{\alpha}^p(r) \omega_{\alpha}^q(r)} \langle J_0(\hat{\Omega}_1) \rangle^{p+q-1} J_{12}(r).
\]
Next we introduce the generating function $\beta_n^{(2)}(y_1, y_2)$ for the normalized cumulant correlators. While $\beta_n^{(2)}(y_1, y_2)$ is associated with $C_{pq}$ the generating function $\beta_n^{(1)}(y)$ is the generating function for $C_{pq}$. In 3D the hierarchical ansatz ensures the factorization property $C_{pq} = C_p C_q$.

$$\beta_n^{(2)}(y_1, y_2) = \int_{r_1}^{r_2} \frac{d^2 r_2}{|\delta F_{\alpha}^{\min}|^2} J_{\beta_2}^{(2)}(r_2) \beta_n^{(1)}(y_2) \left( \frac{\omega_n(r_2)}{\delta F_{\alpha}^{\min}(r_2)} \right) J_{\beta_1}^{(1)}(r_1) \beta_n^{(1)}(y_1)$$

In terms of the reduced flux $\eta$ the generating function for the joint cumulant correlator can be written as:

$$\beta_n^{(2)}(y_1, y_2) = \int_{r_1}^{r_2} \frac{d^2 r_2}{|\delta F_{\alpha}^{\min}|^2} J_{\beta_2}^{(2)}(r_2) \frac{\omega_n(r_2)}{\delta F_{\alpha}^{\min}(r_2)} \beta_n^{(1)}(y_2) \left( \frac{\omega_n(r_2)}{\delta F_{\alpha}^{\min}(r_2)} \right) J_{\beta_1}^{(1)}(r_1) \beta_n^{(1)}(y_1).$$

These equations are generic and exact. They only depend on the hierarchical ansatz for the higher order correlation functions. To simplify further we replace the relevant integrals by the following approximate results:

$$\beta_n^{(2)}(y_1, y_2) \approx \beta_n^{(1)}(y_1) \beta_n^{(1)}(y_2).$$

From the generating function $\beta_n^{(1)}(y)$ it is possible to construct the scaling function $b_n(x)$ that encodes all the information about the bias $b(\eta)$. The functions $b_n(x)$ can be recovered using the following integrals in the complex $y$ plane (Balian & Schaeffer 1989):

$$b_n(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dy \beta_n^{(1)}(y) \exp(xy).$$

Finally, using the definition of $\eta$ we can write the 2PDF of the flux decrement $\delta F_{\alpha}$ for two different neighbouring lines of sight:

$$p_{\alpha}(\delta F_{\alpha}(\Omega_1), \delta F_{\alpha}(\Omega_2)) = p_{\alpha}(\delta F_{\alpha}(\Omega_1)) p_{\alpha}(\delta F_{\alpha}(\Omega_2)) (1 + b_n(\delta F_{\alpha}(\Omega_1)) \xi_{12} b_n(\delta F_{\alpha}(\Omega_2)))$$

The bias $b_n$ for flux-decrement and the bias associated with the underlying mass distribution $\eta$ are related by the following expression:

$$b_n(\delta F_{\alpha}) = b_n(\eta)/|\delta F_{\alpha}^{\min}|.$$
6 CONCLUSIONS

The diffuse intergalactic medium acts as a significant reservoir of baryons at low to intermediate redshift (z < 5), which can be probed via the absorption lines in the spectrum of distant QSOs. The Lyman-α absorption lines along the line of sights are due to a vast range of completely different classes of object. These objects include under-dense neutral hydrogen clouds, halos of large and over-dense systems which may have already reached virial equilibrium, as well as UV heated systems that are strongly coupled to their environments. Previous studies have mainly dealt with the problem of detailed modelling of number of these objects as a function of their clustering and internal properties. Using self-consistent scaling models, which have a long history and were initially employed in galaxy clustering statistics, several authors have computed the column density distribution of Lyman-α absorption systems not only for the low column density Lyman-α forest systems but also for Lyman-limit systems and the damped absorption systems. In addition to the hierarchical modelling, lognormal approximation is also quite successful in reproducing the clustering statistics of Lyman-α absorption lines. In this study we showed the approach taken by the lognormal approximation and the hierarchical ansatz generates near identical results. While previous studies focussed only on one-point statistics we extend these results to the two-point distributions and their lower order moments.

In addition to the statistics of absorption lines, the statistics of the transmitted flux play an important complementary role in Lyman-α studies. The flux can be treated as a continuous field. Various statistics which are often employed in analysing the flux include the LOS power spectrum measurements, estimation of bispectrum and more recently the entire PDF. The flux PDF contains the information regarding cumulants to an arbitrary order. Clearly the flux statistics and the column density distributions are related. One major goal in this study was to unify these two pictures in the context of the hierarchical ansatz or scaling models as well as the lognormal approximation.

The hierarchical model is primarily valid at smaller scales where the correlation functions assumes a hierarchical form. Gravitational clustering is known to develop such a form of hierarchy both in the perturbative (quasi-linear) as well as in the highly non-linear regime. The hierarchical ansatz can be used to describe the mass functions and the bias associated with collapsed objects using the scaling function $h(x)$ and $b(x)$; the variable $x$ is a scaling variable. Previous studies that have employed the hierarchical ansatz have shown that the column density distribution of absorption systems can be described using the same functional form for $h(x)$ and $b(x)$ that are often used for wide ranging studies from the galaxy clustering to thermal Sunyaev-Zel’dovich effects or X luminosity of clusters of galaxies. We showed that the same analytical framework can also be used to understand the statistics of QSO flux measurements, thereby providing a unified statistical approach to what at first sight appear to be very different observables.

We have approached the modelling of the flux PDF in two different ways. For the 3D analysis, the PDF of the neutral hydrogen density contrast $\delta$ is modelled according to the hierarchical ansatz as well as lognormal distribution. We find very good agreement with the numerical simulations for the entire range of redshifts that we have considered. The predictions from both these models are almost identical and differ only marginally in the less interesting under-dense regions. We also employ a modified version of scale invariant hierarchical approximation which was developed recently (Valageas & Munshi 2004). This particular approximation provides a very accurate model for the clustering of $\delta$. Using the fluctuating Gunn-Peterson approximation we map the $\delta$ PDF to that of the transmitted flux $F_x$. We find reasonable approximation for the allowed range of parameters $A(z)$ and $\beta$ that define the Gunn-Peterson approximation. Next we consider the projected or the 2D distribution of flux. For projected statistics, we start with linking the cumulants and cumulant correlators for the flux and the underlying neutral hydrogen density contrast $\delta$. Then $\delta$ is taken to be a tracer of the underlying density contrast $\delta$, modelled statistically using the hierarchical ansatz. Next it is shown that, under certain simplifying assumptions, the PDF of $\delta$ and PDF of a suitably defined reduced flux decrement are linked through a very simple relation. Tests against numerical simulations shows good agreement. Results were obtained for the entire bias functions that act as a generating function for the cumulant correlators. The scaling function $h(x)$ and the bias function $b(x)$ are also known to describe the number density and bias of over-dense objects respectively.

The formalism developed here can also be used to probe the higher order cross-correlation statistics involving Lyman-α flux decrement and weak lensing convergence of CMB maps or those extracted from convergence maps constructed using the weak gravitational lensing of optical galaxies.

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APPENDIX A: HIERARCHICAL ANSATZ: A VERY BRIEF REVIEW

There is no complete analytical model for the evolution of gravitational clustering in the nonlinear regime and the Eulerian and Lagrangian perturbation theories (Bernardeau et al. 2002) are often used to describe clustering in the quasi-linear regime. The halo model is extensively used and is very successful in modeling fully evolved structure formation. A parallel approach, depends on the hierarchical nature of the correlation...
functions that develop during the nonlinear regime. It has a long history in describing clustering dark matter distribution as well as the collapsed objects. Additional assumptions regarding virialization and hydrodynamical equilibrium can be used in association to make specific predictions regarding diverse cosmological phenomenon from luminosity of X-ray clusters (Valageas & Schaeffer 2000), column density distribution of neutral hydrogen (Valageas, Schaeffer & Silk 2002), redshift evolution bias (Valageas, Silk & Schaeffer 2001), mass and luminosity distribution of galaxies and clusters (Valageas & Schaeffer 1999), cosmic microwave background (CMB) secondaries such as the thermal and kinetic Sunyaev and Zel’dovich effect (Munshi et al. 2011; Valageas, Balbi & Silk 2001), reheating and reionization of the Universe (Valageas & Schaeffer 1999; Valageas, Schaeffer & Silk 2002). Weak lensing observables have already been studied in this framework (Valageas 2001; Bernardeau & Valageas 2000; Munshi & Jain 2000, 2001; Valageas & Schaeffer 1998).

The hierarchical ansatz (Balian & Schaeffer 1989) is remarkably successful at making approximate calculations of the entire PDF and bias of the density field, improving significantly upon the order-by-order analysis of other approaches (Bernardeau et al. (2002) for a detailed review). The hierarchical ansatz in the highly non-linear regime depends on assuming a specific correlation hierarchy, on the other hand in the quasi-linear regime it can be linked to the gravitational dynamics using perturbative analysis.

### A1 Highly Non-linear Regime

The PDF \( p(\delta) \) and the bias \( b(\delta) \) can both be constructed from the knowledge of the VPF \( \phi(y) = \sum_{p=1} S_p y^p/p! \) and its two-point analog \( \tau(y) = \sum_{i} C_{p1} y^p/p! \). Where the parameters \( S_p \) and \( C_{p1} \) are normalized cumulants and cumulant correlators for the density field. The PDF is related to the scaling function \( h(x) \) in the text of the paper \( p(\delta) = h(x)/\xi_2^2 \) with \( x = (1+\delta)/\xi_2 \).

\[
p(\delta) = \int_{-\infty}^{\infty} \frac{dy}{2\pi i} \exp \left[ \frac{(1+\delta) y - \phi(y)}{\xi_2} \right]; \quad b(\delta) p(\delta) = \int_{-\infty}^{\infty} \frac{dy}{2\pi i} \tau(y) \exp \left[ \frac{(1+\delta) y - \phi(y)}{\xi_2} \right]. \tag{A1}
\]

The modelling of \( \phi(y) \) and \( \tau(y) \) needs a detailed knowledge of the entire correlation hierarchy. The detailed knowledge of the entire correlation hierarchy is encoded in the vertex generating function \( G(\tau) \). Typically for large values of \( y \) the PDF exhibits a power law \( \phi(y) = ay^{1-\omega} \). There are no theoretical estimates of \( \omega \) and it is generally estimated from numerical simulations. The parameter typically takes a value \( \omega = 0.3 \) for CDM like spectra. For small but negative values of \( y \) the functions \( \phi(y) \) and \( \tau(y) \) develops a singularity in the complex plane which is described by the following parametrization.

\[
\phi(y) = \phi_s - a_s \Gamma(\omega_s)(y - y_s)^{-\omega_s} ; \quad \tau(y) = \tau_s - b_s (y - y_s)^{-\omega_s - 1}.
\tag{A2}
\]

The singularity structure of \( \phi(y) \) and \( \tau(y) \) depends on the nature of the vertex generating function \( G(\tau) \) and its behaviour near the singularity \( \tau_s \):

\[
a_s = \frac{1}{\Gamma(-1/2)} G'(\tau_s) G''(\tau_s) \left[ \frac{2G'(\tau_s) G''(\tau_s)}{G'''(\tau_s)} \right]^{3/2}; \quad b_s = \left[ \frac{2G'(\tau_s) G''(\tau_s)}{G'''(\tau_s)} \right]^{1/2}.
\tag{A3}
\]

On the other hand the parameters \( \omega \) and \( a \) can be described in terms of a parameter \( y_s \) which in turn describes the exponential decay of the PDF for large density contrast \( \delta \):

\[
\omega = k_s/(k_s + 2) ; \quad a = \frac{k_s + 2}{2} k_s^{-k_s/2} ; \quad -\frac{1}{y_s} = x_s = \frac{1}{k_s (k_s + 1)} k_s^{k_s + 1}.
\tag{A4}
\]

The PDF and the bias thus has two distinct regimes that are dictated by the two asymptotes. For intermediate values of \( \delta \) the PDF shows a power law behaviour. The PDF and the bias are given by the following expression:

\[
\xi_2^{\frac{1}{1-\omega}} << 1 + \delta << \xi_2 ; \quad p(\delta) = \frac{a_s}{\xi_2^2} \frac{1 - \omega}{\xi_2} \left( \frac{1 + \delta}{\xi_2} \right)^{-\omega - 2} ; \quad b(\delta) = \left( \frac{\omega}{2\pi} \right)^{1/2} \frac{\Gamma(\omega)}{\Gamma(1 + \omega)/\xi_2} \left( \frac{1 + \delta}{\xi_2} \right)^{-(1-\omega)/2}.
\tag{A5}
\]

For large values of \( \delta \) the PDF on the other hand shows an exponential behaviour:

\[
1 + \delta >> \xi_2 ; \quad p(\delta) = \frac{a_s}{\xi_2^2} \frac{1 + \delta}{\xi_2} \exp \left( - \frac{1 + \delta}{x_s \xi_2} \right) ; \quad b(\delta) = - \frac{1}{\xi_2} \left( \frac{1}{G'(\tau_s)} \right) \left( \frac{1 + \delta}{\xi_2} \right).
\tag{A6}
\]

At very small values of \( \delta \) the PDF shows an exponential decay which is described only by the parameter \( \omega \):

\[
1 + \delta << \xi_2 ; \quad p(\delta) = a^{-1/(1-\omega)} \xi_2^{(1-\omega)/2} \exp \left[ - \omega \left( \frac{1}{1 - \omega} \right)^{(1-\omega)/\omega} \right] ; \quad b(\delta) = - \left( \frac{2\omega}{\xi_2} \right)^{1/2} \left( \frac{1 - \omega}{z} \right)^{-(1-\omega)/2\omega}.
\tag{A7}
\]

The range of \( \delta \) for which the power law regime is valid depends on the value of \( \xi_2 \). For smaller values of \( \xi_2 \) the power law regime is less pronounced.
A2 Quasi-linear Regime

In the quasi-linear regime the parameters that describe the generating functions can be directly linked with gravitational dynamics (Bernardeau 1992, 1994). The PDF and bias in the intermediate power-law regime in the quasi-linear regime is given by the following result:

\[
p(\beta) d\beta = \frac{1}{G_\beta(\tau)} \left[ 1 - \frac{\tau G'_{\beta}(\tau)/G_{\beta}(\tau)}{2\pi \xi_2} \right]^{1/2} \exp \left( -\frac{\tau^2}{2\xi_2} \right) d\tau; \quad b(\beta) = -\left( \frac{k_0}{\xi_2} \right) \left[ (1 + G_{\beta}(\tau))^{1/k} - 1 \right]; \tag{A8}
\]

\[
G_{\beta}(\tau) = G(\tau) - 1 = \delta. \tag{A9}
\]

For the large density contrast the exponential decay take the following form:

\[
p(\beta) d\beta = \frac{3a_s \sqrt{\xi_2}}{4\sqrt{\pi}} \delta^{-5/2} \exp \left[ -\frac{\delta}{\xi_2} - \left| \alpha_{\beta} \right| \right] d\beta; \quad b(\beta) = -\frac{1}{G'(\tau_s)} \frac{(1 + \delta)}{\xi_2}. \tag{A10}
\]

These results ignore the loop corrections to tree-level perturbation theory but take into account correlation hierarchy to an arbitrary order.

**APPENDIX B: LOGNORMAL DISTRIBUTION**

The evolution of the PDF \( p(\delta) \) of density field \( \delta \) due to gravitational clustering has been studied extensively in many cosmological context. The lognormal distribution (Hamilton 1988; Coles & Jones 1991; Bouchet et al. 1993; Kofman et al. 1994) is known to accurately reproduce the results from numerical simulation in the quasi-linear regime and has been used to model both one- and two-point PDF. In general the lognormal distribution can provide a good statistical description of a random variable if it can be modeled as a product of many other random variables (Kayo, Taruya, Suto 2001; Taruya et al. 2002).

\[
p(\delta) d\delta = \frac{1}{\sqrt{2\pi} \Sigma} \exp \left[ -\frac{\Lambda_i^2}{2\Sigma} \right] d\delta; \quad \Sigma = \ln(1 + \sigma^2); \quad \Lambda_i = \ln((1 + \delta_i)/(1 + \sigma^2)); \tag{B1}
\]

\[
p(\delta_1, \delta_2) d\delta_1 d\delta_2 = \frac{1}{2\pi \sqrt{\Sigma^2 - X_{12}^2}} \exp \left[ -\frac{\Sigma(\Lambda_1^2 + \Lambda_2^2) - 2X_{12} \Lambda_1 \Lambda_2}{2(\Sigma^2 - X_{12}^2)} \right] d\delta_1 d\delta_2; \tag{B2}
\]

\[
\Lambda_i = \ln((1 + \delta_i)/(1 + \sigma^2)); \quad X_{12} = \ln(1 + \xi_{12}) \tag{B3}
\]

In the lowest order in \( \xi_{12} \) the lognormal model takes a factorized form. In this limit a bias function \( b(\beta) \) can be defined.

\[
p(\delta_1, \delta_2) = p(\delta_1) p(\delta_2) [1 + b(\delta_1) \xi_{12} b(\delta_2)]; \quad b(\delta_i) = \Lambda_i / \Sigma. \tag{B4}
\]

The lognormal distribution has also been used to model many cosmological observation from galaxy surveys to weak lensing data.