THE STRUCTURE OF THE SOLUTIONS TO SEMILINEAR EQUATIONS AT A CRITICAL EXPONENT

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Dedicated to Vic Shapiro on the occasion of his retirement

Abstract. This paper is concerned with the structure of the solutions to subcritical elliptic equations related to the Matukuma equation. In certain cases the complete structure of the solution set is known, and is comparable to that of the original Matukuma equation. Here we derive sufficient conditions for a more complicated solution set consisting of; (i) crossing solutions for small initial conditions and large initial conditions; (ii) at least one open interval of slowly decaying solutions; and (iii) at least two rapidly decaying solutions. As a consequence we obtain multiplicity results for rapidly decaying, or minimal solutions.

1. Introduction

This article is concerned with the structure of the solutions to the semilinear elliptic equation

\[-\Delta u = f(|x|)u^p, x \in \mathbb{R}^n, \quad n \geq 3, \quad \frac{n+2}{n-2} > p > 1.\]  

(1.1)

It is assumed that \( f \) is positive and has asymptotic behavior \( f(|x|) \sim |x|^l, -2 < l < 0 \) as \( x \to \infty \). Among the important examples of related equations are the Matukuma equation

\[-\Delta u = \frac{1}{1+|x|^2}u^p, \quad x \in \mathbb{R}^3,\]

and the scalar curvature equation

\[-\Delta u = f(x)u^{\frac{n+2}{n-2}}.\]

There has been a substantial body of research into the question of the existence or non existence of positive solutions of (1.1) decaying to zero as \( |x| \to \infty \). Much of this research has focussed on the range of \( p \) values between the Sobolev critical exponent \( p^* = \frac{n+2}{n-2} \), and the critical exponent

\[p_* = \frac{n+2+2l}{n-2}.\]

(1.2)

The exponent \( p_* \) is critical with respect to the compact imbedding of weighted Sobolev spaces

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\[ H^1_n(R^n) \rightarrow L^q(R^n, f(|x|)dx), 0 < q < p^*, \]
(see [Eg]). For \( p > p^* \) variational methods are sufficient to prove the existence of positive, decaying solutions (see [Eg], [N-S]). For \( p = p^* \) those methods fail due to non compactness, and other techniques are required. This article will be primarily concerned with the structure of the set of solutions when \( p = p^* \).

When \( f \) decays rapidly to zero at infinity, e.g. \( l < -2 \), existence and nonexistence results for (1.1) were obtained in [N]. References for the case \(-2 < l < 0\), include [K-N-Y], [N-Y], [N-S], [Y-Y1] and [Y-Y2]. For \( 0 < f(0) < \infty \) and \( p \neq p^* \), the existence or nonexistence of ground states is determined by the decay rate \( l \). At the critical exponent, \( p = p^* \), the existence of positive, decaying solutions depends on higher order terms in the asymptotic expansion of \( f(r) \). In particular, the function

\[ (1.3) \quad h(r) = r(r^{-l}f(r))_r, \]

has been useful in obtaining more precise information about the structure of the set of solutions when \( 0 < f(0) \leq \infty \). The following examples illustrate the role played by \( p \) and \( h(r) \) in determining the existence or nonexistence of positive, entire solutions. We assume that \( f \) is radially symmetric and so the problem reduces to an ordinary differential equation. Since we are interested in nonnegative solutions we define \( u^+ = \text{max}\{u, 0\} \) and let \( u(r; \alpha) \) be the (unique) solution of the initial value problem

\[ (1.4) \quad u'' + \frac{n - 1}{r} u' + f(r)(u^+)^p = 0, \quad u(0; \alpha) = \alpha, \quad u'(0; \alpha) = 0. \]

**Example 1.1.** \( f(r) = r^l, -2 < l < 0 \).

- If \( 1 < p < p^* \), then \( u(r; \alpha) \) has a finite zero for every \( \alpha > 0 \).
- If \( p^* \leq p \) then \( u(r; \alpha) \) is a positive, decaying solution for every \( \alpha > 0 \).

In this case \( h(r) = 0 \), for \( r \geq 0 \).

**Example 1.2.** \( f(r) = O(r^{\sigma}) \) as \( r \rightarrow 0 \), \( \sigma > -2 \).

- If \( p = p^* \), \( f(r) \geq cr^l \) for large \( r \), and \( \int_0^R h(r) r^{n+l-1} dr \leq 0 \), for all \( R \geq 0 \), then for every \( \alpha > 0 \), \( u(r; \alpha) \) is a positive, entire solution satisfying \( \lim_{r \rightarrow \infty} u(r; \alpha) = 0 \).
- If \( p = p^* \), \( h(r) \neq 0 \), and \( \int_0^R h(r) r^{n+l-1} dr \geq 0 \) for all \( R > 0 \), then \( u(r; \alpha) \) has a finite zero for all \( \alpha > 0 \).

**Example 1.3.** Let \( \frac{n+2l}{n-2} < p < p^* \), and

\[ f(r) = \frac{(l + 2)(n - 2)}{(p - 1)^2} \left\{ \left( p - \frac{n + l}{n - 2} \right) + \left( l + 2 \frac{p}{n - 2} \right) \left( \frac{1}{1 + r^2} \right) \right\} (1 + r^2)^{l/2}. \]

- \( u(r; \alpha) \) has a finite zero for every sufficiently large \( \alpha > 0 \),
- \( u(r; 1) = (1 + r^2)^{-\frac{n+2l}{n-2}} \) is a positive, decaying solution,
- \( u(r; \alpha) \) has a finite zero for every sufficiently small \( \alpha > 0 \).
Example I is contained in Proposition 4.5 of [N-Y]. The results of Example II can be found in Theorems 9.2 and 9.3 of [K-N-Y], where it is assumed that \( \int_0^R h(r)r^{n+l-1}dr \) does not change sign. Example III shows the complexity of the solutions when \( h(r) \) and \( \int_0^R h(r)r^{n+l-1}dr \) are allowed to change sign (see also [N: Example 5.2], [N-S]). An important structure theorem for equation (1.1) is given in [K-Y-Y], where it is assumed that this integral changes sign exactly once.

It is known ([Y-Y1]) that under our hypotheses all solutions of (1.4) with \( \alpha > 0 \), are of one of the following types:

1. \( u(r; \alpha) \) is a crossing solution, i.e. has a positive zero,
2. \( u(r; \alpha) \) is a slowly decaying solution, i.e.
   \[
   \lim_{r \to \infty} r^{n-2}u(r; \alpha) = \infty,
   \]
3. \( u(r; \alpha) \) is a rapidly decaying solution, i.e.
   \[
   \lim_{r \to \infty} r^{n-2}u(r; \alpha) < \infty.
   \]

In sections 3 and 4 we will give criteria for existence and non existence of positive, decaying solutions when both \( h(r) \) and \( \int_0^R h(r)\cdot r^{n+l-1}dr \) are allowed to change sign. Then in section 5 we prove that for \( p = p_* \), and under suitable hypotheses on \( h(r) \), the set of solutions of (1.4) consists of:

1. crossing solutions for sufficiently small or sufficiently large \( \alpha \);
2. at least one open interval of \( \alpha \) values for which \( u(r; \alpha) \) is slowly decaying;
3. at least two rapidly decaying solutions.

Thus under these hypotheses a structure comparable to that of Example III can be achieved for \( p = p_* \). A consequence of this result is that we prove the existence of multiple minimal, i.e. rapidly decaying solutions.

2. Main Results

The following hypotheses will be assumed throughout this paper.

1. \( f \in C(0, \infty), \ f(r) > 0 \quad \text{for } r > 0, \)
2. \( f(r) = O(r^l) \) as \( r \to \infty, -2 < l < 0, \)
3. \( f(r) = O(r^\sigma) \) as \( r \to 0, -2 < \sigma, \)
4. \( h(r) < -\delta_1 r^{-\beta}, \quad \text{for } r > r_2, \)
5. \( h(r) = O(r^\gamma), \quad \gamma > 0, \quad \text{as } r \to 0. \)
6. \( \int_0^R h(r)r^{n+l-1}dr < 0. \)
7. \( \int_0^\infty h(r)r^{n+l-1}dr > 0. \)
8. \( 0 < r_3 = \sup\{r > 0 : \int_0^r h(s)s^{n+l-1}ds \leq 0\}. \)
9. \( \gamma(2+l) > (\sigma-l)(n+l). \)
For $p \geq p_*$ the following two theorems give sufficient conditions for the existence of positive, entire solutions when $\alpha$ is sufficiently small.

**Theorem 1.** Assume that ($f_3$) and ($f_4$) are satisfied, $p \geq p_*$ and that $0 < \beta < n+1$. Then there exists an $\alpha_0 > 0$ such that for $0 < \alpha \leq \alpha_0$, the solution $u(r; \alpha)$ is a positive, entire solution which also satisfies $\lim_{r \to \infty} u(r; \alpha) = 0$.

**Example 2.1.** The function

$$f(r) = \left( c_1 + c_2 r^2 \right)^{\frac{n}{2}} \left( c_3 + c_4 r^2 \right)^{\frac{\beta}{2}}$$

$$-2 < \gamma + \nu < 0, \ c_1 c_4 \gamma + c_2 c_3 \nu > 0, \ c_1 > 0$$

satisfies ($f_1$)–($f_4$).

**Example 2.2.** The function

$$f(r) = (1 + (1 + r^2)^{-\frac{1}{2}}) (1 + r^2)^{-\frac{\gamma + \nu}{2}}$$

satisfies ($f_1$)–($f_4$). For $n = 3$ we have $l = -\frac{1}{2}, \ p_* = 4, \ \beta = \frac{1}{2}$, and $0 < \beta < n + 1$.

This example is not covered by [K-N-Y], Theorem 9.2, since $\int_0^R h(r)^{n+l} dr > 0$ for small $R$.

**Example 2.3.** The function

$$f(r) = \left( \frac{9}{8} + (1 + r^2)^{-1} \right) (1 + r^2)^{-\frac{\gamma + \nu}{2}}$$

satisfies ($f_1$)–($f_4$). For $n = 3$ we have $l = -\frac{3}{2}, \ p_* = 2, \ \beta = 2$, and $\beta > n + 1$. Since $\int_0^R h(r)^{n+l-1} dr > 0$ for all $R > 0$, it follows from Theorem 9.3 of [K-N-Y] that $u(r; \alpha)$ has a positive zero for all $\alpha > 0$.

The next result gives a variant of Theorem 1 in which the condition $\beta < n + l$ is replaced by an integral growth condition.

**Theorem 2.** Assume that ($f_3$), ($f_4$) and ($f_5$) are satisfied, and $p \geq p_*$. Then there exists an $\alpha_0 > 0$ such that for $0 < \alpha \leq \alpha_0$, the solution $u(r; \alpha)$ is a positive, entire solution which also satisfies $\lim_{r \to \infty} u(r; \alpha) = 0$.

The next two theorems give sufficient conditions for the non existence of positive solutions when $\alpha$ is sufficiently large, or sufficiently small.

**Theorem 3.** Assume that ($f_4$), ($f_6$) and ($f_9$) are satisfied, and $p \geq p_*$. Then there exist positive numbers $\gamma_*, \alpha_1$, such that for $\alpha > \alpha_1$ and $0 < \gamma < \gamma_*$ the solution $u(r; \alpha)$ has a positive zero.

**Remark 2.1.** For $p < p_*$ this result is proved in [N-Y, Theorem 2], without the additional hypothesis $\gamma < \gamma_*$.

**Theorem 4.** Assume that ($f_4$) and ($f_7$) are satisfied, $p \geq p_*$ and that there exists an $r_2$ such that $h(r) \geq 0$, for $r_2 < r < \infty$. Then there exists an $\alpha_0 > 0$ such that for $\alpha \in (0, \alpha_0)$ the solution $u(r; \alpha)$ has a positive zero.

**Remark 2.2.** A stronger version of this result is found in [Y-Y2, Theorem 3].
The solution satisfies the integral equation

\[ \phi(0; \alpha_*)^{p_*+1} \int_0^a k(s) ds + \phi(b; \alpha_*)^{p_*+1} \int_a^b k(s) ds \]

\[ + \phi(b; \alpha_*)^{p_*+1} \int_b^{r_*} k(s) ds < -\delta^2, \]

where \( \gamma_* \) is given in theorem 3 and \( \phi(r; \alpha) \) is the solution of the equation

\[ -(r^{n-1} \phi_r)_r = r^{n-1} r^l \phi^{p_*}, \phi(0) = \alpha, \quad \phi_r(0) = 0. \]

Then there exists a positive function \( f \in C(0, \infty) \), with \( f(r) = O(r^l) \) as \( r \to \infty \), \( -2 < l < 0 \), and positive numbers \( \alpha_0 < \alpha_1 < \alpha_* < \alpha_2 < \alpha_3 \) such that the solutions \( u(r; \alpha) \) of equation (1.4) are of the following type:

\[ (a) \quad 0 < \alpha < \alpha_0 \Rightarrow u(r; \alpha) \text{ as a positive zero}, \]

\[ (b) \quad u(r; \alpha_*) \text{ as a positive, slowly decaying solution}, \]

\[ (c) \quad \alpha_3 < \alpha < \infty \Rightarrow u(r; \alpha) \text{ as a positive zero}, \]

\[ (d) \quad u(r; \alpha_1) \text{ and } u(r; \alpha_2) \text{ are rapidly decaying solutions}. \]

It is not difficult to construct examples which satisfy the hypotheses (2.1).

3. Existence of positive, decaying solutions

The basic existence theorem for solutions to the initial value problem can be found in [N-Y]. In establishing Theorem 1 we will require several preliminary lemmas. The first is contained in Proposition 4.1 of [N-Y] and is also found in [N].

Lemma 3.1. There exists a unique solution \( u(r; \alpha) \in C((0, \infty) \cap C^2(0, \infty)) \) of (1.4). The solution satisfies the integral equation

\[ u(r) = \alpha - \frac{1}{n-2} \int_0^r \left\{ 1 - \left( \frac{s}{r} \right)^{n-2} \right\} sf(s)(u^+(s))^p ds, \]

and also the following conditions:

\[ (a) \quad \lim_{r \to \infty} r^{n-1} u_r(r) = 0(b) \quad u_r(r) = - \int_0^r \left( \frac{s}{r} \right)^{n-1} f(s)(u^+(s))^p ds \leq 0, \quad \text{for } r > 0, \]

\[ (c) \quad u \text{ is non-increasing on } [0, \infty). \]
Lemma 3.2. ([N-Y], Prop. 4.3 and (6.8)) For $R > 0$ the solution $u(r; \alpha)$ satisfies the Pohozaev type identity

\begin{equation}
\frac{n-2}{2} R^{n-1} u(R; \alpha) u'(R; \alpha) + \frac{1}{2} R^n u'(R; \alpha)^2 + \frac{1}{p+1} R^n f(R)(u^+(R; \alpha))^{p+1} = \frac{1}{p+1} \int_0^R \left\{ -\frac{n-2}{2} (p-p^*) r^{-l} f(r) + h(r) \right\} r^{n+l-1} (u^+(r; \alpha))^{p+1} dr.
\end{equation}

Define $r_\alpha$ by

\begin{equation}
r_\alpha = \inf \left\{ r > 0 : u(r; \alpha) = \frac{\alpha}{2} \right\}.
\end{equation}

Lemma 3.3. $r_\alpha$ satisfies the following estimates:

\begin{enumerate}
\item[(3.5a)] $r_\alpha = O(\alpha^{1-p})$ as $\alpha \to 0$,
\item[(3.5b)] $r_\alpha = O(\alpha^{1-p^*})$ as $\alpha \to \infty$.
\end{enumerate}

Proof. By Proposition 4.1 of [N-Y] and (f2) we have

\begin{align*}
\frac{n-2}{2} \alpha &= \int_0^{r_\alpha} \left\{ 1 - \left( \frac{s}{r_\alpha} \right)^{n-2} \right\} sf(s)(u^+(s; \alpha))^p ds \\
&\leq \alpha^p \int_0^{r_\alpha} sf(s) ds.
\end{align*}

By Lemma 6.1 of [N-Y] $\lim_{\alpha \to 0} r_\alpha = \infty$, and therefore

\begin{align*}
\frac{n-2}{2} \alpha^{1-p} &\leq \int_0^{r_\alpha} sf(s) ds + \int_1^{r_\alpha} sf(s) ds \\
&\leq c + \int_1^{r_\alpha} s^{1+l} ds \leq cr_\alpha^{2+l},
\end{align*}

which implies

\begin{equation}
C \alpha^{1-p^*} \leq r_\alpha \text{ as } \alpha \to 0.
\end{equation}

By a similar estimate we can show

\begin{equation}
C \alpha^{1-p^*} \geq r_\alpha \text{, as } \alpha \to 0,
\end{equation}

and this proves 3.5a.

Consider the case $\alpha \to \infty$.

\begin{align*}
\frac{n-2}{2} \alpha &= \int_0^{r_\alpha} \left\{ 1 - \left( \frac{s}{r_\alpha} \right)^{n-2} \right\} sf(s)(u(s; \alpha)^+)^p ds \\
&\geq \left( \frac{\alpha}{2} \right)^p C \int_0^{r_\alpha} sf(s) ds, \\
\alpha^{1-p} &\geq \int_0^{r_\alpha} sf(s) ds,
\end{align*}

which implies $r_\alpha \to 0$ as $\alpha \to \infty$. (3.5b) can then be deduced. \qed
Proof of Theorem 1. By \((f_4)\) \(h(r) > 0\) on an interval \((0, \epsilon)\). Define the quantities

\[
\begin{align*}
    r_0 &= \inf \{r > 0 : h(r) < 0\}, \\
    r_1 &= \sup \{r > 0 : h(r) > 0\}, \\
    \delta_2 &= \int_0^{r_1} |h(r)| r^{n+l-1} dr, \\
    k &= \left( \frac{2-(p+1)}{n+l-\beta} \right) \left( 1 - 2^{-(n+l-\beta)} \right) > 0.
\end{align*}
\]

Then \(r_1\) exists by \((f_3)\), and by \((f_4)\) \(0 < r_0 \leq r_1 < \infty\). We apply the identity (3.3). By (3.5a) we may choose \(\alpha_0\) so small that

\[
\frac{r_0^{n+l-\beta}}{\alpha_0} \geq \frac{\delta_2}{k\delta_1}.
\]

For some \(\alpha \in (0, \alpha_0]\) assume that \(u(r; \alpha)\) satisfies the conditions

\[
u(r; \alpha) > 0, \quad 0 \leq r < R, \quad u(R; \alpha) = 0.
\]

The left hand side of (3.3) is \(> 0\), because \(u'(R; \alpha) \neq 0\). Also, \(R > r_\alpha\) by the definition of \(r_\alpha\), and by Lemma (3.3) we may assume \(r_1 < \frac{r_\alpha}{2}\). Then

\[
\int_0^R h(r) r^{n+l-1} (u^+(r; \alpha))^{p+1} dr = \int_0^{r_1} + \int_{r_1}^{r_\alpha} + \int_{r_\alpha}^R h(r) r^{n+l-1} (u^+(r; \alpha))^{p+1} dr \leq \delta_2 \alpha^{p+1} + \int_{r_\alpha}^R h(r) r^{n+l-1} (u^+(r; \alpha))^{p+1} dr \leq \delta_2 \alpha^{p+1} - \delta_1 \int_{r_\alpha}^R r^{n+l-1-\beta} (u^+(r; \alpha))^{p+1} dr \leq \alpha^{p+1} (\delta_2 - k\delta_1 r_\alpha^{n+l-\beta}) \leq 0.
\]

This contradiction implies that \(u(r; \alpha)\) is positive on \([0, \infty)\). The condition

\[
\lim_{r \to \infty} u(r; \alpha) = 0
\]

is a consequence of [N; Theorem 3.10].

Example 3.1. If

\[
f(r) = A_0 r^l + o(r^l), \quad r \to \infty,
\]

then the hypotheses of Theorem 1 are satisfied.

Proof. It is easy to verify that in this case \(\beta = 1 < n + l\).
Lemma 3.4. Define \( r_{\alpha,k} \) by

\[
r_{\alpha,k} = \inf \left\{ r > 0 : u(r;\alpha) = \frac{\alpha}{k} \right\}, \quad k > 1.
\]

Then \( r_{\alpha,k} \) has asymptotic behavior

\[
\lim_{\alpha \to 0} r_{\alpha,k} = \infty.
\]

Proof. Following the proof of Lemma 3.3 we have

\[
u(r_{\alpha,k};\alpha) = \frac{\alpha}{k} = \alpha - \frac{1}{n-2} \int_{0}^{r_{\alpha,k}} \left\{ 1 - \left( \frac{s}{r_{\alpha,k}} \right)^{n-2} \right\} s f(s)(u^+(s;\alpha))^p ds,
\]

\[
\alpha \left( \frac{k-1}{k} \right) = \frac{1}{n-2} \int_{0}^{r_{\alpha,k}} \left\{ 1 - \left( \frac{s}{r_{\alpha,k}} \right)^{n-2} \right\} s f(s)(u^+(s;\alpha))^p ds
\]

\[
\leq \frac{1}{n-2} \int_{0}^{r_{\alpha,k}} s f(s)(u^+(s;\alpha))^p ds
\]

\[
\leq \left( \frac{\alpha}{k} \right)^p \int_{0}^{r_{\alpha,k}} s f(s) ds,
\]

\[
\alpha^{1-p} \leq C \int_{0}^{r_{\alpha,k}} s f(s) ds.
\]

Therefore

\[
\alpha^{1-p} \leq C \int_{0}^{r_{\alpha,k}} s f(s) ds
\]

\[
= C \int_{0}^{r_{\alpha,k}} s^{l+1} ds
\]

\[
= C r_{\alpha,k}^{l+2}.
\]

It follows that

\[
\alpha^{\frac{1}{1+2}} \leq C r_{\alpha,k}
\]

and since \( \frac{1}{1+2} < 0 \), we have the stated result. 

It is possible to obtain more precise estimates, analogous to Lemma 3.3, however such estimates will not be needed.

Proof of Theorem 2. Define

\[
r^0 = \sup \{ r > 0 : \int_{0}^{r} h(s)s^{n+l-1} ds \geq 0 \},
\]

\[
\Omega^+_0 = \{ r \in [0,r^0] : h(r) \geq 0 \},
\]

\[
\Omega^-_0 = \{ r \in [0,r^0] : h(r) \leq 0 \}.
\]

By \((f_4)\) and \((f_5)\) \( r^0 > 0 \). For \( \epsilon \in (0,1) \) it is a consequence of Lemma 3.5 that there exists an \( \alpha_0(\epsilon) \) such that for \( \alpha \in (0,\alpha_0(\epsilon)) \) the conditions

\[
(i) \quad (1-\epsilon)^{\frac{1}{n+l}} \alpha \leq u(r;\alpha) \leq \alpha, \quad \text{for } 0 \leq r \leq r^0,
\]

\[
(ii) \quad r_\alpha > 2r^0,
\]

\[
(iii) \quad \int_{r^0}^{2r^0} h(r)r^{n+l-1} dr < 2^{p+1}\epsilon \int_{\Omega^-_0} h(r)r^{n+l-1} dr
\]
are satisfied. We will show that there exists an \( \alpha_0 \) such that
\[
\alpha \in (0, \alpha_0) \Rightarrow \int_0^R h(r)r^{n+l-1}(u^+(r; \alpha))^{p_\ast+1}dr < 0, \quad \text{for all } R > r^0.
\]
In fact, for \( R > r^0 \),
\[
\int_0^R h(r)r^{n+l-1}(u^+(r; \alpha))^{p_\ast+1}dr
\leq \left( \int_{\Omega^-_0} + \int_{\Omega^+_0} + \int_{r^0}^R \right) h(r)r^{n+l-1}(u^+(r; \alpha))^{p_\ast+1}dr
\leq \alpha^{p_\ast+1} \int_0^{r^0} h(r)r^{n+l-1}dr - \epsilon \alpha^{p_\ast+1} \int_{\Omega^-_0} h(r)r^{n+l-1}dr
\quad + \int_{r^0}^R h(r)r^{n+l-1}dr + \int_{r^0}^R h(r)r^{n+l-1}(u^+(r; \alpha))^{p_\ast+1}dr
\leq -\epsilon \alpha^{p_\ast+1} \int_{\Omega^-_0} h(r)r^{n+l-1}dr + \int_{r^0}^R h(r)r^{n+l-1}(u^+(r; \alpha))^{p_\ast+1}dr.
\]
Then for \( R > r^0 \),
\[
-\epsilon \alpha^{p_\ast+1} \int_{\Omega^-_0} h(r)r^{n+l-1}dr + \int_{r^0}^R h(r)r^{n+l-1}(u^+(r; \alpha))^{p_\ast+1}dr
\leq -\epsilon \alpha^{p_\ast+1} \int_{\Omega^-_0} h(r)r^{n+l-1}dr + \int_{r^0}^{2r^0} h(r)r^{n+l-1}(u^+(r; \alpha))^{p_\ast+1}dr
\leq -\epsilon \alpha^{p_\ast+1} \int_{\Omega^-_0} h(r)r^{n+l-1}dr + \left( \frac{\alpha}{2} \right)^{p_\ast+1} \int_{r^0}^{2r^0} h(r)r^{n+l-1}dr
\leq \alpha^{p_\ast+1} \left\{ -\epsilon \int_{\Omega^-_0} h(r)r^{n+l-1}dr + 2^{-\frac{(p_\ast+1)}{2}} \int_{r^0}^{2r^0} h(r)r^{n+l-1}dr \right\} < 0,
\]
by (3.11iii).

As in the proof of Theorem 1, an application of the Pohozaev identity (3.3) leads to a contradiction, and once again the limit condition follows from [N; Theorem 3.10].

4. Nonexistence of positive, decaying solutions

Our proofs of the nonexistence of positive, decaying solutions are based on the Pohozaev identity. Define \( w(r; \alpha) = r^{\frac{n-2}{2}}u(r; \alpha) \), and let \( u(r; \alpha) \) be a positive, entire solution which decays to zero as \( r \to \infty \). According to Theorem 2.2 of [K-N-Y] the following variant of the Pohozaev identity applies.

\[
\begin{align*}
(4.1) & \quad -R^2 w''(R; \alpha)w(R; \alpha) - Rw(R; \alpha)w'(R; \alpha) + R^2 w'(R; \alpha)^2 \\
& \quad - \left( \frac{l+2}{n+1} \right) R^n f(R)u(R; \alpha)^{p+1} \\
& = \int_0^R h(r)r^{n+l-1}(u(r; \alpha)^+)^{p+1}dr.
\end{align*}
\]
The function $w$ is bounded and eventually monotone, so by Lemma 4.1 of [K-N-Y] there exists a sequence $\{R_j\}$ such that as $j \to \infty$, 

\[(4.2) \quad R_j \to \infty, \quad w(R_j; \alpha) \to C, \quad R_j w'(R_j; \alpha) \to 0, \quad R_j^2 w''(R_j; \alpha) \to 0.\]

Evaluating (4.1) on the sequence $\{R_j\}$ we see that the lim sup as $j \to \infty$ of the left hand side of (4.1) is $\leq 0$ for $p \geq p_*$. Evaluation of the right hand side will require several preliminary results.

**Lemma 4.1.** Assume that $p = p_*$, and that $(f_4)$ and $(f_9)$ are satisfied. Let $u(r; \alpha)$ be a solution of (1.4). Then for $\gamma > 0$ sufficiently small, 

\[(4.3) \quad \lim_{\alpha \to \infty} \int_0^{r_0} h(r)r^{n+l-1}(u(r; \alpha)^+)^{p+1}dr = \infty.\]

**Proof.** According to Lemma 3.3 $\lim_{\alpha \to \infty} r_\alpha = 0$, so by $(f_4)$ for sufficiently large $\alpha$, 

\[
\int_0^{r_0} h(r)r^{n+l-1}(u(r; \alpha)^+)^{p+1}dr > \left(\frac{\alpha}{2}\right)^{p+1} \int_0^{r_0} h(r)r^{n+l-1}dr \geq c \left(\frac{\alpha}{2}\right)^{p+1} \int_0^{r_0} r^{n+l-1+\gamma}dr = \frac{c}{(n+l+\gamma)(2p+1)\alpha^{p+1}r_\alpha^{n+l+\gamma}}.
\]

By Lemma 3.3 this is $\geq \alpha^q$, for 

\[
q = -\{l^2 + (n + \gamma - \sigma)l + (2\gamma - \sigma n)\} = -\{\gamma(2 + l) - \sigma(n + l) + l(n + l)\} = -\{\gamma(2 + l) - (\sigma - l)(n + l)\},
\]

and $\gamma(2 + l) - (\sigma - l)(n + l) < 0$ by $(f_9)$. In fact, it is $< 0$ for $\gamma < \frac{(\sigma-l)(n+l)}{(2+l)}$. Therefore the conclusion of the lemma follows. Note that a sufficient condition is $\sigma > l$ and $\gamma$ sufficiently small. \[\square\]

We will also require the following lemma, which gives an apriori bound for positive solutions on $[r_0, \infty)$, for some $r_0 > 0$. It is based on Theorem 3.10 of [N]. Similar results can be found in [B-L], [E-R].

**Lemma 4.2.** Let $u(r; \alpha)$ be a positive, entire solution of (1.1), with $p \leq p_*$, and assume that there exist positive numbers $r_0, C_1$ such that 

\[f(r) \geq C_1 r^l, \quad -2 < l < 0, \quad \text{for } r > r_0.\]

Then there exists a positive constant $C$, independent of $\alpha$, such that if $u(r; \alpha)$ is a positive, entire solution then 

\[(4.4) \quad u(r; \alpha) \leq Cr^{\frac{2-\alpha}{2}}, \quad \text{for } r \geq r_0, \quad 0 < \alpha < \infty.\]
Proof. By Green’s identity in any ball of radius $R > 0$,

$$
\int_{B_R} f(|x|) u(|x|; \alpha)^p \, dx = - \int_{B_R} \Delta u(|x|; \alpha) \, dx
= \int_{\partial B_R} u'(r; \alpha) \, ds
= -\omega_n R^{n-1} u'(R; \alpha).
$$

By (4.4) we may assume that

$$
f(r) \geq C_1 r^l, \quad \text{for } r \geq \frac{r_0}{2}.
$$

Then for $R \geq \frac{r_0}{2}$, and using the monotonicity of $u(r; \alpha)$,

$$
\int_{B_R} f(|x|) (u^+(|x|; \alpha))^p \, dx \geq \int_{B_R - B_{\frac{r_0}{2}}} f(|x|) (u^+(|x|; \alpha))^p \, dx \geq C_1 |x|^l (u^+(|x|; \alpha))^p \, dx \geq C_1 u(R; \alpha)^p \int_{\frac{r_0}{2}}^{R} r^{n+l-1} \, dr \geq \frac{C_1}{n+l} u(R; \alpha)^p \left\{ R^{n+l} - \left( \frac{r_0}{2} \right)^{n+l} \right\}
$$

Therefore for $R \geq \frac{r_0}{2}$,

$$
\int_{B_R} f(|x|)(u^+ (|x|; \alpha))^p \, dx \geq \frac{C_1}{2(n+l)} R^{n+l} u(R; \alpha)^p,
$$

and by (4.5)

$$
-\omega_n R^{n-1} u'(R; \alpha) \geq \frac{C_1}{2(n+l)} R^{n+l} u(R; \alpha)^p,
$$

$$
-\frac{u'(r; \alpha)}{u(r; \alpha)^p} \geq \frac{C_1}{\omega_n} R^{l+1} \quad \text{for } r \geq \frac{r_0}{2}.
$$

Since $l > -2$, an integration from $\frac{r_0}{2}$ to $R$ gives

$$
-\int_{\frac{r_0}{2}}^{R} \frac{u'(r; \alpha)}{u(r; \alpha)^p} \, dr
= \frac{u(r; \alpha)^{1-p}}{1-p} \left|_{\frac{r_0}{2}}^{R} \right.
= \frac{1}{p-1} \left\{ u(R; \alpha)^{1-p} - u \left( \frac{r_0}{2}; \alpha \right)^{1-p} \right\}
$$

and therefore

$$
\frac{1}{p-1} \left\{ u(r; \alpha)^{1-p} - u \left( \frac{r_0}{2}; \alpha \right)^{1-p} \right\} \geq \frac{C_1}{\omega_n(l+2)} r^{l+2},
$$
\[ u(r; \alpha)^{1-p} \geq \left( \frac{C_1}{\omega_n(l+2)} \right)^{r^{l+2}} + u \left( \frac{r_0}{2}; \alpha \right)^{1-p} (p-1) \geq \frac{C_1 (p-1)}{\omega_n(l+2)} r^{l+2}, \]

and

\[ u(r; \alpha) \leq \left( \frac{C_1 (p-1)}{\omega_n(l+2)} \right)^{\frac{1}{1-p}} r^{\frac{l+2}{1-p}}, \text{ for } r \geq r_0, \]

and since \( p \leq p_* \), the lemma is proved.

**Proof of Theorem 3.** We will evaluate the right hand side of (4.1) using Lemmas 4.1 and 4.2. Note that \( r_0 \) is defined in (3.10).

\[
\left| \int_{r_0}^{R_j} h(r)r^{n+l-1}(u(r; \alpha)^+)^{p+1}dr \right| \leq
\int_{r_0}^{R_j} |h(r)|r^{n+l-1}(u(r; \alpha)^+)^{p+1}dr \leq
\int_{r_0}^{\infty} |h(r)|r^{n+l-1}r^{\frac{2-p}{p+1}(p+1)}dr \leq
C\delta \int_{r_0}^{\infty} r^{n+l-1+\frac{2-p}{p+1}(p+1)-\beta}dr \leq
C\delta \int_{r_0}^{\infty} r^{-(1+\beta)}dr = C_0,
\]

by (f_6) and Lemma 4.2. By the hypotheses \( 0 < \gamma < \gamma_* \) and Lemma 4.1, there exists an \( \alpha_1 > 0 \) such that \( \alpha > \alpha_1 \Rightarrow \int_{r_0}^{\infty} h(r)r^{n+l-1}(u(r; \alpha)^+)^{p+1}dr > C_0 \). For \( \alpha > \alpha_1 \) by (4.2) we can choose a sequence \( \{R_j\} \), with \( R_j \to \infty \), such that for \( j > j(\alpha) \) the left hand side of (4.1) is < \( C_0 \), which proves the theorem.

**Proof of Theorem 4.** Let

\[ h^+(r) = \max\{0, h(r)\}, \]
\[ h^-(r) = \min\{0, h(r)\}. \]

By (f_7) and (f_8)

\[ 0 < r_3 = \sup\{r > 0 : \int_0^r h(s)s^{n+l-1}ds \leq 0\}. \]

We choose \( k, \epsilon, \alpha \) and \( R_j \) as follows:

(a) let \( 0 < k < 1 \), and let \( \epsilon > 0 \) be so small that

\[ \epsilon \int_{r_0}^{\pi_2} h^+(r)r^{n+l-1}dr < (1-k) \int_{r_0}^{2} h(r)r^{n+l-1}dr, \]

(b) let \( \alpha \) be so small that

\[ r_\alpha > r_2, \quad \text{and} \quad (1-\epsilon)\frac{1}{p+1} \alpha \leq u(r; \alpha) \leq \alpha, \quad \text{for } 0 \leq r \leq r_2, \]

(c) choose \( R_j > r_2 \) so large that

\[ L(R_j, \alpha) < k\alpha^{p+1} \int_{r_0}^{2} h(r)r^{n+l-1}dr, \]
where $L(R_j, \alpha)$ is the left hand side of (4.1), and assume that $u(r; \alpha) > 0$ for $0 \leq r \leq R_j$. Then $\int_0^{R_j} h(r)r^{n+l-1}dr = 0$, and the right hand side of (4.1) is

$$(4.8) \quad \int_0^{R_j} h(r)r^{n+l-1}(u^+(r; \alpha))^{p+1}dr$$

$$= \left\{ \int_0^{r_0} + \int_{r_0}^{r_2} + \int_{r_2}^{R_j} \right\} h(r)r^{n+l-1}(u^+(r; \alpha))^{p+1}dr$$

$$\geq \alpha^{p+1} \left\{ (1 - \epsilon) \int_0^{r_0} h(r)r^{n+l-1}dr + (1 - \epsilon) \int_{r_0}^{r_2} h^+(r)r^{n+l-1}dr + \int_{r_0}^{r_2} h^-(r)r^{n+l-1}dr \right\}$$

$$\geq \alpha^{p+1} \left\{ (1 - \epsilon) \int_0^{r_0} h(r)r^{n+l-1}dr + (1 - \epsilon) \int_{r_0}^{r_2} h^+(r)r^{n+l-1}dr + \int_{r_0}^{r_2} h^-(r)r^{n+l-1}dr \right\}$$

$$+ (1 - \epsilon) \int_{r_0}^{r_2} h^-(r)r^{n+l-1}dr + \epsilon \int_{r_0}^{r_2} h^-(r)r^{n+l-1}dr + (1 - \epsilon) \int_{r_0}^{r_2} h^+(r)r^{n+l-1}dr$$

$$+ (1 - \epsilon) \int_{r_2}^{R_j} h^-(r)r^{n+l-1}dr + \epsilon \int_{r_2}^{R_j} h^-(r)r^{n+l-1}dr \right\}$$

$$= \alpha^{p+1} \left\{ (1 - \epsilon) \int_0^{r_0} h(r)r^{n+l-1}dr + \epsilon \int_{r_0}^{r_2} h^-(r)r^{n+l-1}dr \right\}$$

$$\geq k \int_0^{r_2} h(r)r^{n+l-1}dr,$$

which contradicts (4.7iii).

\[ \square \]

**Remark 4.1.** If the condition $\int_0^{r} h(s)s^{n+l-1}ds \geq 0$ is satisfied for all $r > 0$, then the conclusion follows from [K-N-Y], Theorem 9.3.

### 5. Proof of Theorem 5

The proof is based on Theorems 3 and 4. We will construct a specific function $f(r)$ such that for $p = p_*$ the corresponding problem (1.4) has solutions with properties analogous to Example III, Section 1. This result demonstrates the complexity of the structure of the solution set which can occur when the hypothesis $r_H \leq r_G$ of [Y-Y1] is not satisfied.

**Lemma 5.1.** Under the hypotheses (2.1) the set of slowly decaying solutions of (1.4) is open.

**Proof.** By Lemma 2.6 of [K-Y-Y] a sufficient condition for the openness of slowly decaying solutions is that there exist an $r_2 > 0$ such that

$$\frac{1}{p + 1} r^{l-1}(r^{-l} f(r))_r \leq 0, \quad r > r_2.$$

This is equivalent to $h(r) \leq 0$ for $r > r_2$, and is implied by (2.1a). \[ \square \]
Now let \( \phi(r; \alpha) \) be the solution of the equation
\[
-(r^{n-1}\phi_r)_r = r^{n-1}r^l\phi^p, \\
\phi(0) = \alpha, \quad \phi_r(0) = 0.
\]
The solutions \( \phi(r; \alpha) \) are all rapidly decaying, and are given explicitly by
\[
\phi(r; \alpha) = \alpha \left( 1 + \frac{2\alpha^{p-1}}{(p + 1)(n - 2)^2} r^{(n - 2)(p - 1)/2} \right)^{-\frac{1}{p-1}}.
\]
Consider equation (4.1) with \( f \) defined by
\[
f(r) = r^l + \epsilon(p_\ast + 1) \int_0^r s^{-(n+1)}k(s)ds,
\]
and define \( h_\epsilon(r) = r(r^{-l}f(r))_r = \epsilon(p_\ast + 1)r^{-(n+l-1)}k(r) \). We will compare the solutions of (5.1) with those of equation (1.4) with \( f = f_\epsilon \) given by (5.3). By the assumptions (2.1) the function \( f(r) \) satisfies \( (f_4), (f_6), (f_7) \) and \( (f_9) \), and clearly \( h_\epsilon(r) \geq 0 \) for \( r \geq c \). It follows from Theorems 3 and 4 that for each \( \epsilon > 0 \) there are positive numbers \( \alpha_0(\epsilon), \alpha_1(\epsilon) \) which satisfy conditions (2.3a) and (2.3c). We will show that for \( \epsilon \) sufficiently small the family of solutions also satisfies (2.3b), i.e. are slowly decaying. In fact, for \( \alpha_\ast > 0 \) the solution \( u(r; \alpha_\ast) \) converges uniformly on the interval \([0, c]\) to the solution \( \phi(r; \alpha_\ast) \) as \( \epsilon \to 0 \). Therefore there exists an \( r_\ast > 0 \) such that for \( r > r_\ast \),
\[
P(r; u(r; \alpha_\ast)) = \frac{1}{p_\ast + 1} \int_0^r s^{(n+l-1)}h_\epsilon(s)(u^+(s; \alpha_\ast))^{p_\ast + 1}ds
\leq \epsilon \left\{ \int_0^a k(s)(u^+(s; \alpha_\ast))^{(p_\ast + 1)}ds + \int_a^b k(s)(u^+(s; \alpha_\ast))^{(p_\ast + 1)}ds \right. \\
+ \int_b^r k(s)(u^+(s; \alpha_\ast))^{(p_\ast + 1)}ds \right\}
\leq \epsilon u(0; \alpha_\ast)^{(p_\ast + 1)} \int_0^a k(s)ds + \epsilon u(b; \alpha_\ast)^{(p_\ast + 1)} \int_a^b k(s)ds
\quad + \epsilon u(b; \alpha_\ast)^{(p_\ast + 1)} \int_b^c k(s)ds
\quad + \epsilon \left\{ \phi(0; \alpha_\ast)^{(p_\ast + 1)} \int_0^a k(s)ds + \phi(b; \alpha_\ast)^{(p_\ast + 1)} \int_a^b k(s)ds \right. \\
\left. + \phi(b; \alpha_\ast)^{(p_\ast + 1)} \int_b^c k(s)ds \right\} + o(\epsilon).
\]
For \( \epsilon \) sufficiently small, by (2.1d) this is \( \leq -c^2 \). The existence of a slowly decaying solution follows from [Y-Y1, Lemma 2.5].

From the existence of a slowly decaying solution we can conclude, by the open property of the slowly decaying solutions under hypotheses 2.1, and the corresponding property for crossing solutions, that there also exist at least two rapidly decaying solutions, \( u(r; \alpha_1) \) and \( u(r; \alpha_2) \), with \( \alpha_1 \in (\alpha_0, \alpha_\ast) \) and \( \alpha_2 \in (\alpha_\ast, \alpha_3) \).

**Remark 5.1.** Since \( p = p_\ast \) the solution \( u(r; \alpha_\ast) \) is slowly decaying by the Pohozaev identity.
References

[B-L] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) pp 349-381

[B-E] G. Bianchi and H. Egnell, An ODE approach to the equation $\Delta u + K(x)|u|^{n-2}u = 0$ in $\mathbb{R}^n$, Math. Z. 210 (1992) pp 137-166

[B-L] H. Berestycki and P.L. Lions, Nonlinear scalar field equation I: Existence of a ground state, Arch. Rational Mech. Anal. 82 (1983) pp 313-345

[E-R] Allan L. Edelson and Adolfo J. Rumbos, Linear and semilinear eigenvalue problems in $\mathbb{R}^n$, Commun. in Partial Diff. Equations 18 No. 1 & 2 (1993) pp215-240

[Eg] Henrik Egnell, Asymptotic results for finite energy solutions of semilinear elliptic equations, Jour. of Differential Equations 98 (1992) pp 34-56

[F-L-N] D.G. de Figueiredo, P.L. Lions, and R.D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. Pures Appl. 61 (1982) pp 41-63

[G-N-N] B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, Adv. in Math. Supplementary Stud. 7A (1981) pp 369-402

[K-N] T. Kusano and M. Naito, Positive entire solutions of superlinear elliptic equations, Hiroshima Math. Jour. 16 (1986) pp 361-366

[K-N-Y] N. Kawano, W.-M. Ni, and S. Yotsutani, A generalized Pohozaev identity and its applications, J. Math. Soc. Japan 42 No. 3 (1990) pp 541-563

[K-Y-Y] Nichiro Kawano, Eiji Yanagida and Shoji Yotsutani, Structure theorems for positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^n$, Funkcialaj Ekvacioj 36 (1993) pp 557-579

[L-N] Y. Li and W.M. Ni, On conformal scalar curvature in $\mathbb{R}^n$, Duke Math. J. 57 (1988) pp 895-924

[N] W.M. Ni, On the elliptic equation $\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$, its generalizations, and applications in geometry, Indiana Univ. Math. Jour. 31 (1982) pp 493-529

[N-S] E.S. Noussair and C.A. Swanson, Solutions of Matukuma’s equation with finite total mass, Indiana University Mathematics Journal 38, No. 3 (Fall 1989)

[N-Y] W.M. Ni and S. Yotsutani, Semilinear elliptic equations of Matukuma-type and related topics, Japan Journal of Applied Mathematics 5 No. 1 (1988) pp 1-32

[Y-Y1] E. Yanagida and S. Yotsutani, Classification of the structure of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^n$, Arch. Rational Mech. Anal. 124 (1993) pp 239-259

[Y-Y2] E. Yanagida and S. Yotsutani, Existence of positive radial solutions to $\Delta u + K(|x|)u^p = 0$, in $\mathbb{R}^n$, Journal of Differential Equations 115 (1995) pp 477-502

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