GENERALIZED HODGE METRICS AND BCOV TORSION ON
CALABI-YAU MODULI

HAO FANG AND ZHIQIN LU

Abstract. We establish an unexpected relation among the Weil-Petersson metric, the
generalized Hodge metrics and the BCOV torsion. Using this relation, we prove that
certain kind of moduli spaces of polarized Calabi-Yau manifolds do not admit complete
subvarieties. That is, there is no complete family for certain class of polarized Calabi-Yau
manifolds. We also give an estimate of the complex Hessian of the BCOV torsion using
the relation. After establishing a degenerate version of the Schwarz Lemma of Yau, we
prove that the complex Hessian of the BCOV torsion is bounded by the Poincaré metric.

Contents

1. Introduction 1
2. Generalized Hodge Metrics 4
3. BCOV Torsion 9
Appendix A. Poincaré Metric and Generalized Hodge Metrics 15
References 19

1. INTRODUCTION

A Calabi-Yau manifold is a smooth Kähler manifold with trivial canonical bundle and
fundamental group. Moduli space of polarized Calabi-Yau manifolds (Calabi-Yau moduli)
is the object to study in Mirror Symmetry, hence the focal point of intensive studies
in areas of mathematical physics, algebraic geometry, differential geometry and number
theory. For the general reference of Mirror Symmetry and related topics, see the book of
Cox and Katz [9] and the recent survey paper of Todorov [23].

In this paper we study the differential geometry of Calabi-Yau moduli. The starting
point is the celebrated theorem of Yau, which establishes the existence of the Ricci flat
Kähler metric for a fixed polarization. Differential geometric objects of Calabi-Yau moduli
are usually constructed from global algebro-geometric or analytic properties of the Calabi-
Yau manifolds, for example, variation of Hodge structure, or spectral properties of the

Date: June 1, 2003.
1991 Mathematics Subject Classification. Primary: 53A30; Secondary: 32C16.
Key words and phrases. Calabi-Yau Manifold, analytic torsion, moduli space.
The first author is partially supported by a grant from the New York University Research Challenge
Fund Program; the second author is partially supported by the NSF grant DMS 0204667 and the Alfred
P. Sloan Research Fellowship.
Ricci-flat metrics. In this paper, we focus on the Weil-Petersson metric, the generalized Hodge metrics, the BCOV torsion and their relations.

From differential geometric point of view, Calabi-Yau moduli is amicable. It is smooth (or at worst with quotient singularities, so that it is a smooth Deligne-Mumford stack) and its local uniformization is an integral submanifold of the horizontal distribution of the classifying space of the variation of Hodge structure. The curvature of the first Hodge bundle is positive definite. Hence it defines the so-called Weil-Petersson metric. This metric initiates the study of the moduli space in terms of differential geometry. For computing the curvatures of the Weil-Petersson metrics (in more general settings) and its applications, see the works of Siu [20] and Schumacher [19].

In [13, 14], the second author introduced a new metric, the Hodge metric, on the Calabi-Yau moduli, mainly inspired from the theory of variation of Hodge structure. Both the Hodge metric and the Weil-Petersson metric are Kähler orbifold metrics. However, the Hodge metric enjoys better curvature properties: it has non-positive bisectional curvatures, and furthermore, its holomorphic sectional curvature and Ricci curvature are negative and bounded away from zero. This clearly is not the case for the Weil-Petersson metric, as shown in the example of Calabi-Yau quintics [7, page 65]. Thus, in terms of differential geometry, the Hodge metric is better than the Weil-Petersson metric on Calabi-Yau moduli.

As natural Kähler metrics on a given Calabi-Yau moduli, the Weil-Petersson metric and the Hodge metric are closely related. Both can be realized as curvature forms of various combinations of the Hodge bundles, in the sense of Griffiths [11]. There are explicit relations between the two metrics for K3 surfaces, Calabi-Yau three and four-folds (see Theorem 2.4). In higher dimensions, the concept of the Weil-Petersson geometry was introduced in [15].

However, in defining the Hodge metric, only those Hodge bundles of the middle dimensional primitive cohomology groups are considered. It turns out that considering the whole Hodge bundles would be more natural. This is indeed the case. The universal deformation space of Calabi-Yau manifolds can also be viewed as horizontal slices of the classifying space of any degree primitive cohomology groups, even though the period maps fail to be immersive in general. In this paper, we introduce pseudo-metrics on the classifying space for $H^k$ where $k$ may not be the (complex) dimension of the Calabi-Yau manifolds. These metrics are called generalized Hodge metrics. The positive definiteness of the generalized Hodge metrics is lost due to the possible degeneracy of the corresponding horizontal slices. Nevertheless, good “curvature” properties of the Hodge metric still hold for these generalized Hodge metrics. See Appendix A for the precise statements.

We now turn to the other geometric object that will be studied in this paper: the BCOV torsion. First introduced by and named after Bershadsky-Cecotti-Ooguri-Vafa [1, 2], the BCOV torsion is a smooth function on the Calabi-Yau moduli. It is defined as:

(1.1) \[ T = \prod_{1 \leq p,q \leq n} (\det \Delta'_{p,q})^{(-1)^{p+q}pq}, \]

where $\Delta_{p,q}$ is the $\overline{\partial}$-Laplace operator on $(p,q)$ forms with respect to the Ricci-flat metric on a fiber; $\Delta'_{p,q}$ represents the non-singular part of $\Delta_{p,q}$; the determinant is taken in the sense of zeta function regularization.
In physics literature, $T$ was first introduced as the stringy genus one partition function of $N = 2$ SCFT. It was computed using Physics Mirror Symmetry and was used to predict the number of holomorphic elliptic curves embedded in certain Calabi-Yau manifolds.

Due to its central role in the Physics Mirror Symmetry, we are interested in the analysis of the BCOV torsion mathematically. $T$ is a spectral invariant of the Ricci-flat metrics. The Weil-Petersson metric and the generalized Hodge metrics, which are defined by using the variation of Hodge structure, assume no apparent links to the function $T$. The main result of this paper is the following surprising relation between the Weil-Petersson metric and the generalized Hodge metrics by the BCOV torsion.

**Theorem 1.1.** Let $\omega_{ WP}$, $\omega_H$, and $\omega_H^i$ be the Kähler form of the Weil-Petersson metric, the Hodge metric and the generalized Hodge metrics, respectively (See §2 for the definition). Then

$$\sum_{i=1}^{n} (-1)^i \omega_H^i - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log T = \frac{\chi_{Z}}{12} \omega_{ WP},$$

where $\chi_{Z}$ is the Euler characteristic number of $Z$. In particular, if the Calabi-Yau manifold is primitive (See §3 for the definition), then

$$\omega_H = (-1)^n \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log T + \frac{\chi_{Z}}{12} \omega_{ WP} \right).$$

As the first application, we have the following

**Corollary 1.2.** If $N \subset M$ is a $k$-dimensional complete subvariety of $M$ where $M$ is the moduli space of a primitive Calabi-Yau manifold, then

$$\text{Vol}_H(N) = \left[ \frac{(-1)^n}{12} \chi_{Z} \right]^k \text{Vol}_{WP}(N),$$

where $\text{Vol}_H(N)$ and $\text{Vol}_{WP}(N)$ are the volumes of $N$ with respect to the Hodge and the Weil-Petersson metrics.

In Corollary 1.2, the BCOV torsion does not appear explicitly. Even in dimension 3 and 4, where the Hodge metric can be expressed explicitly by the Weil-Petersson metric and its Ricci curvature ([14], [15]), this volume identity is new. One of the notable consequences of the volume identity is the following

**Corollary 1.3.** Assume that a polarized Calabi-Yau manifold $Z$ is primitive, and that $(-1)^{n+1} \chi_{Z} > -24$. Let $M$ be the moduli space of $Z$. Then there exists no complete curve in $M$; hence, there exists no projective subvariety of $M$ (of positive dimensions). In particular, $M$ is not compact.

This corollary is purely algebro-geometric. Primitive Calabi-Yau manifolds include interesting examples like Calabi-Yau three-folds and Calabi-Yau hyper-surfaces in projective spaces. It would be interesting to see a direct proof of the result without using differential geometry.

The second application of Theorem 1.1 is on the asymptotic behavior of the complex Hessian of the BCOV torsion.
Corollary 1.4. Let $\Delta$ and $\Delta^*$ be the unit disk and the punctured unit disk of $\mathbb{C}$ respectively. Let $(\Delta^*)^l \times \Delta^{m-l}$ be the parameter space of a family of Calabi-Yau manifolds. Then the BCOV torsion $T$, which is a smooth function on $(\Delta^*)^l \times \Delta^{m-l}$, satisfies

$$-C\omega_P < \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log T < C\omega_P,$$

where $C$ is a constant and $\omega_P$ is the Poincaré metric, defined as

\begin{equation}
\omega_P = \sum_{i=1}^{l} \sqrt{-1} \frac{1}{|z_i|^2(\log \frac{1}{|z_i|})^2} dz_i \wedge d\bar{z}_i + \sum_{i=l+1}^{m} \sqrt{-1} dz_i \wedge d\bar{z}_i.
\end{equation}

Using Theorem 1.1, the proof of the above corollary is reduced to the fact that the generalized Hodge metrics are bounded by the Poincaré metric. In the Appendix, we establish Theorem A.1, a degenerate version of the Schwarz Lemma of Yau [27], of which Corollary 1.4 is a direct consequence.

The relation among the Weil-Petersson metric, the generalized Hodge metric, and the BCOV torsion on Calabi-Yau moduli presented in this paper is quite delicate and our understanding of it is far from being complete. Because of the physics background of the BCOV torsion, it is very likely that some deeper relations can be used to explain the current coincidence. It is also expected that these constructions will produce new modular forms on various moduli spaces, as the previous works of Yoshikawa [29, 28] indicated.

We shall proceed to study the asymptotic behavior of the BCOV torsion near the boundary of Calabi-Yau moduli, and the BCOV prediction of counting the rational curves. The results will be the subject of an upcoming paper.

Acknowledgement. Both authors are indebted to G. Tian for introducing the papers [1, 2] to them and being interested in this work. They also thank J.-P. Bismut, S.-Y. A. Chang, P. Sarnak and P. Yang for their interest in this work. The first author also thanks A. Ching for discussion.

2. Generalized Hodge Metrics

Let $Z$ be a Calabi-Yau manifold and let $l$ be an ample line bundle over $Z$. The pair $(Z, l)$ is called a polarized Calabi-Yau manifold. The (coarse) moduli space $M$ exists and is constructed as follows: first, choose a large integer $k$ such that $l^k$ is very ample. In this way $Z$ is embedded into a complex projective space $\mathbb{C}P^N$. Let $\text{Hilb}(Z)$ be the Hilbert scheme of $Z$, which is a compact complex variety. The group $G = PSL(N + 1, \mathbb{C})$ acts on $\text{Hilb}(Z)$ and the moduli space $M$ is the quotient of the stable points of $\text{Hilb}(Z)$ by the group $G$.

By the smoothness theorem of Tian [22] (see also Todorov [21]), the deformation of the complex structures of Calabi-Yau manifold is unobstructed. That is, the universal deformation space (Kuranishi space) is smooth. On the other hand, due to the existence of finite automorphism, the moduli space for polarized Calabi-Yau manifolds may have quotient singularities. Thus in general, the moduli space $M$ is a complex orbifold, or a smooth Deligne-Mumford stack.
For the local geometry of the moduli space, the above possible singularities may never be a problem. We can always pass through a finite covering and assume that locally, the moduli space is a smooth complex manifold.

Let $Z$ be a generic polarized Calabi-Yau manifold. There exists the universal family $\mathcal{X}$ such that it is parametrized by the moduli space $\mathcal{M}$:

\begin{equation}
Z \xrightarrow{i} \mathcal{X} \xrightarrow{\pi} \mathcal{M},
\end{equation}

Assume that $U$ is an open neighborhood of a point $t \in \mathcal{M}$. By the Kodaira-Spencer deformation theory, there is an isomorphism

\begin{equation}
\iota : T_t U \cong H^1(Z_t, \Theta_t),
\end{equation}

where $Z_t$ is the fiber of $\mathcal{X} \xrightarrow{\pi} \mathcal{M}$ at $t$, and $\Theta_t$ is the holomorphic tangent bundle of $Z_t$.

Let $(t_1, \cdots, t_m)$ be a local holomorphic coordinate system of $\mathcal{M}$. Then $\iota(\frac{\partial}{\partial t_i}) \in H^1(Z_t, \Theta_t)$. We define a Hermitian inner product on $T_t U$ for $t \in \mathcal{M}$ as follows:

\begin{equation}
\left( \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right)_{WP} = \int_{Z_t} A^\alpha_{i\beta} : A^\dagger_{j\delta} g^{\delta\beta} g_{\alpha\gamma} dV_{Z_t},
\end{equation}

where $A_i = A^\alpha_{i\beta} \frac{\partial}{\partial t_i} \otimes d\bar{t}^\beta$, $(i = 1, \cdots, m)$ are the harmonic representation of $\iota(\frac{\partial}{\partial t_i})$. This inner product on each $T_t \mathcal{M}$ for $t \in \mathcal{M}$ gives a Hermitian metric on the moduli space $\mathcal{M}$, which is called the Weil-Petersson metric. Under the Weil-Petersson metric, $\mathcal{M}$ is a Kähler orbifold.

Let $\Omega$ be a (nonzero) holomorphic $(n, 0)$-form on $Z_t$. Define $\Omega \downarrow \iota(\frac{\partial}{\partial t_i})$ to be the contraction of $\Omega$ and $\iota(\frac{\partial}{\partial t_i})$. The Weil-Petersson metric can be re-written as (cf. [22]):

\begin{equation}
\left( \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right)_{WP} = -\frac{\int_{Z_t} \Omega \downarrow \iota(\frac{\partial}{\partial t_i}) \wedge \overline{\Omega \downarrow \iota(\frac{\partial}{\partial t_j})}}{\int_{Z_t} \Omega \wedge \overline{\Omega}}.
\end{equation}

The Weil-Petersson metric is the most natural metric on the moduli space. Unfortunately, it does not have a very good curvature property. In [13], another natural metric called the Hodge metric was defined. In this paper, we use the notations in [15] for the Hodge metric.

Recall that for an $n$-dimensional compact complex manifold $X$ with polarization, for any $0 \leq k \leq n$, we have the decomposition of the Hodge bundles

\[ H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}). \]

By the Lefschetz decomposition theorem, we can further decompose the Hodge bundles $H^{p,q}$ into its primitive parts as follows.
Define \( L : H^k(X, \mathbb{C}) \to H^{k+2}(X, \mathbb{C}) \) by \([\alpha] \to [\alpha \land \omega]\), where \( \omega \) is the curvature form of the ample line bundle over \( X \). Define the primitive cohomology group \( P^k(X, \mathbb{C}) \) to be the kernel of \( L^{n-k+1} \) on \( H^k(X, \mathbb{C}) \). Let \( P^{p,q} = P^k(X, \mathbb{C}) \cap H^{p,q}(X) \).

The Lefschetz decomposition theorem states that
\[
H^{p,q}(X) = P_{p,q} \oplus L(P^{p-1,q-1}) \oplus \cdots \oplus L^r(P^{p-r,q-r}),
\]
where \( r = \text{Min}(p, q) \).

Define
\[
Q(\eta_1, \eta_2) = \int \eta_1 \land \eta_2 \land \omega^{n-2k},
\]
for \( \eta_1, \eta_2 \in H^k(X, \mathbb{C}) \). Then \( Q \) extends to a bilinear form on \( H^*(X, \mathbb{C}) \). The Riemann-Hodge relations are
\[
\begin{align*}
(1) \quad Q(\eta_1, \eta_2) = 0, & \text{ if } \eta_1 \in P^{p_1,q_1}, \eta_2 \in P^{p_2,q_2}, \text{ but } p_1 + p_2 \neq q_1 + q_2; \\
(2) \quad (\sqrt{-1})^{q-p}Q(\eta_1, \bar{\eta}_1) > 0 & \text{ if } 0 \neq \eta_1 \in P^{p,q}.
\end{align*}
\]

The second Riemann-Hodge relation defines a Hermitian inner product on the primitive harmonic \((p, q)\) forms:
\[
\langle \eta_1, \eta_2 \rangle = (\sqrt{-1})^{q-p}Q(\eta_1, \bar{\eta}_1).
\]

When \( X \) is a polarized Calabi-Yau manifold \( Z \), the above inner product is equivalent to the inner product induced from the Ricci-flat metric(cf. [26, 10]):

\[
\textbf{Theorem 2.1.} \quad \text{Let } \phi \in H^{p,q}(Z), \ p \geq q \text{ and let } \phi = \phi_0 + L\phi_1 + \cdots + L^q\phi_q \text{ be the decomposition corresponding to (2.4). Then we have }
\]
\[
\langle \phi, \phi \rangle = (-1)^{\frac{1}{2}(p+q)(p+q+1)} \sum_{k=0}^{q} (-1)^k (n - p - q + 2k)! \int_M ||\phi_k||^2 dV_{\text{CY}},
\]
where \( || \cdot || \) is the metric induced from the Ricci-flat metric of \( Z \).

If we make the relative version of the above settings, we get bundles \( PR^q\pi_*\Omega^p_{X/M} \to \mathcal{M} \) in place of the cohomology groups. These bundles are called Hodge bundles. They are in fact the vector bundles of Kähler orbifolds (cf. [15]).

The Kodaira-Spencer map \( \partial_i : T\mathcal{M} \to H^1(Z_i, \Theta_i) \) gives a bundle map
\[
\frac{\partial}{\partial t_i} : PR^q\pi_*\Omega^p_{X/M} \to PR^k\pi_*\mathcal{C}/PR^q\pi_*\Omega^p_{X/M}
\]
for \( k \leq n \) by differentiation. In this way, we have a natural bundle map (of vector bundles over Kähler orbifold)
\[
T\mathcal{M} \to \bigoplus_{p+q=k} \text{Hom}(PR^q\pi_*\Omega^p_{X/M}, PR^k\pi_*\mathcal{C}/PR^q\pi_*\Omega^p_{X/M}).
\]
**Definition 2.2.** For each \( t \in \mathcal{M} \) and \( Z = Z_t \) with the polarized Ricci flat metric, Theorem 2.1 defines Hermitian metrics on the bundles \( \mathcal{P} \mathcal{R}_q \pi_\ast \Omega_\mathcal{P}^{p} \mathcal{X}/\mathcal{M} \rightarrow \mathcal{M} \). Let \( h_{P,k} \) be the pull back of the natural Hermitian metric on the bundle \( \oplus_{p+q=k} \text{Hom}(\mathcal{P} \mathcal{R}_q \pi_\ast \Omega_\mathcal{P}^{p} \mathcal{X}/\mathcal{M}) \rightarrow T\mathcal{M} \) for \( k \leq n \). We use \( \omega_{PH}^k \) to denote the corresponding Kähler forms for \( k \leq n \). According to (2.4), we define

\[
\omega_{H}^k = \omega_{PH}^k + \omega_{PH}^{k-2} + \cdots.
\]

We call both \( \omega_{H}^k \) and \( \omega_{PH}^k \) to be the generalized Hodge metrics.

**Remark 2.3.** The generalized Hodge metric is a generalization of the Hodge metric defined by the second author [14]. In fact, it is proved in [15] that \( \omega_{PH}^n = \omega_H \), the latter being the Hodge metric.

Because of the possible degeneration of the action (2.5), the generalized Hodge metric is only positive semi-definite; hence, it was only a pseudo-metric. However, it enjoys similar “curvature” properties of the Hodge metric proved by the second author [13]. The generalized Hodge metrics are bounded by the Poincaré metric. See Appendix A for more details.

We have the following relations between the Hodge metric and the Weil-Petersson metric:

**Theorem 2.4.** We use the above notations. The Hodge metric \( \omega_H \) defines a Kähler metric (i.e. \( d\omega_H = 0 \)). Furthermore, The bisectional curvature of \( h \) is non-positive and the holomorphic sectional curvature and the Ricci curvature are negative away from zero. In particular, we have

1. If \( n = 2 \), then \( \omega_H = 2\omega_{WP} \);
2. If \( n = 3 \), then \( \omega_H = (m+3)\omega_{WP} + \text{Ric} \omega_{WP} \); [13]
3. If \( n = 4 \), then \( \omega_H = (2m+4)\omega_{WP} + 2\text{Ric} \omega_{WP} \). [15]

where \( n = \dim Z \) and \( m = \dim \mathcal{M} \).

**Definition 2.5.** We call a Calabi-Yau manifold primitive if \( \omega_{H}^k = 0 \) for all \( k < n \).

A direct consequence of the above definition, equation (2.4) and Remark 2.3 is that \( \omega_{PH}^n = \omega_{H}^n = \omega_H \) when \( Z \) is primitive. Thus, the only interesting generalized Hodge metric is the original Hodge metric. The class of primitive Calabi-Yau manifolds includes many important examples.

**Example 2.6.** A Calabi-Yau three-fold is by definition simply connected. It is easy to see that the actions of \( T\mathcal{M} \) on lower degree Hodge bundles are trivial; hence, it is primitive. Similarly, a Calabi-Yau four-fold with vanishing \( h^{1,2} \) is also primitive.

**Example 2.7.** More generally, due to the hard Lefschetz theorem, all the Calabi-Yau hyper-surfaces and complete intersections in projective spaces are primitive.

We give the explicit formulae for the generalized Hodge metrics in the next proposition:

\[ \omega_{H}^k = \sqrt{-1} \frac{i}{2\pi} \omega_{PH}^k, \]
Proposition 2.8. Let $c_1(E)$ be the Ricci form of a vector bundle $E$. Then we have

\begin{equation}
\omega_{PH}^k = \sum_{0 \leq p \leq k} p c_1(PR^{k-p} \pi_* \Omega_{X/M}^p).
\end{equation}

\begin{equation}
\omega_H^k = \sum_{0 \leq p \leq k} p c_1(R^{k-p} \pi_* \Omega_{X/M}^p).
\end{equation}

for $k \leq n$.

**Proof.** Fixing a $k \leq n$, we define the Hodge bundles $F_k^0, \ldots, F_k^p$ to be

$$F_k^p = PR^0 \pi_* \Omega_{X/M}^k \oplus \cdots \oplus PR^{k-p} \pi_* \Omega_{X/M}^p$$

for $p = 0, \cdots, k$. Thus for $q = k - p$,

$$PR^q \pi_* \Omega_{X/M}^p = F_k^p / F_k^{p+1}.$$

In terms of the curvatures, we have

\begin{equation}
c_1(PR^q \pi_* \Omega_{X/M}^p) = c_1(F_k^p) - c_1(F_k^{p+1}).
\end{equation}

By the Abel summation formula, we have

\begin{equation}
\sum_{0 \leq p \leq k} p c_1(PR^{k-p} \pi_* \Omega_{X/M}^p) = c_1(F_k^0) + \cdots + c_1(F_k^k) + c_1(F_k^0).
\end{equation}

Each $F_k^p$ is a sub-bundle of the flat bundle $F_k^0 = PR^k \pi_* \mathbb{C}$. Let $t_1, \cdots, t_m$ be the local holomorphic coordinate of $M$ and let the bundle map

$$\frac{\partial}{\partial t_i} : F_k^p \rightarrow F_k^0 / F_k^p, 1 \leq i \leq m$$

be represented by the matrix

$$\frac{\partial \Omega_\alpha}{\partial t_k} = b_{k \alpha \mu} T_\mu,$$

where $\Omega_\alpha$ and $T_\mu$ are the basis of $F_k^p$ and $F_k^0 / F_k^p$, respectively. Then the first Chern class can be represented by

\begin{equation}
c_1(F_k^p) = \sqrt{-1} \sum_{\alpha, \mu} b_{k \alpha \mu} \bar{b}_{i \alpha \mu} dt_k \wedge d\bar{t}_i
\end{equation}

for $0 \leq p \leq k$. \eqref{eq:ph} follows from the definition of $\omega_{PH}^k$. \eqref{eq:om} follows from \eqref{eq:ph} and \eqref{eq:wo}. The proof is completed.

The curvature computation is a natural generalization of the similar result in [11], where only the middle dimensional primitive Hodge structure was considered.

**Remark 2.9.** The Weil-Petersson metric is the curvature of the first Hodge bundle:

$$\omega_{WP} = c_1(R^0 \pi_*(\Omega_{X/M}^0))$$
by (2.3), (2.14). Thus by the above equation, Remark 2.3 and Proposition 2.8, the Weil-Petersson metric, the Hodge metric and the generalized Hodge metrics are the Ricci curvatures of the combination of the Hodge bundles.

With the above interpretation of the metrics, the following is obvious:

**Corollary 2.10.** Using the above notations, for \( n \geq 2 \), we have
\[
(2.15) \quad \omega_H \geq 2\omega_{WP}.
\]

**Proof.** By Remark 2.9 and Serre Duality, we get
\[
(2.16) \quad \omega_{WP} = c_1(R^0\pi_* (\Omega^n_{X/M})) = -c_1(R^n\pi_* (O)).
\]

By (2.12) and the fact that \( F^0_n \) is flat, we have
\[
(2.17) \quad c_1(F^1_n) = c_1(R^n\pi_* (O)) = \omega_{WP}.
\]

Also, since \( F_{n+1}^n = 0 \),
\[
(2.18) \quad c_1(F^n_n) = c_1(R^0\pi_* (\Omega^n_{X/M})) = \omega_{WP}.
\]

According to (2.14), \( c_1(F^n_p) \geq 0 \) for all \( p \). Hence, when \( n \geq 2 \), by (2.10), (2.13), (2.17) and (2.18), we have
\[
(2.19) \quad \omega_H \geq c_1(F^n_n) + c_1(F^1_n) = 2\omega_{WP}.
\]
The proof is finished.

\[\square\]

3. **BCOV Torsion**

BCOV torsion was first defined by Bershadsky-Ceccotti-Ooguri-Vafa in their study of the Physics Mirror Symmetry. It was constructed as the partition function for the N=2 SCFT. In their breakthrough works [1, 2], the torsion was determined using the Mirror Symmetry Conjecture. One amazing consequence is that, given the local expansion of the BCOV torsion, they were able to give a prediction of counting numbers of embedded elliptic curves of all degrees in a given Calabi-Yau manifold. The prediction matches all known low degree cases. Furthermore, they also discussed the higher genus cases based on genus one computation. Notice that a similar prediction of counting rational curves for quintics, which invoked intensive mathematical research, was first made by Candelas et. al. [7]. Through the fundamental works of Kontsevich, Givental, Lian-Liu-Yau and many others (see [9] for a complete reference), the prediction has been mathematically verified. It is thus of crucial interest to understand the BCOV torsion in terms of algebraic and differential geometry.

First, we give the following

**Definition 3.1.** The BCOV torsion of a Calabi-Yau manifold is
\[
(3.1) \quad T = \prod_{1 \leq p,q \leq n} (\det \Delta'_{p,q})^{(-1)^{p+q}pq},
\]
where \( \Delta_{p,q} \) is the \( \overline{\partial} \)-Laplace operator on \( (p,q) \) forms with respect to the Ricci-flat metric on a fiber; \( \Delta'_{p,q} \) represents the non-singular part of \( \Delta_{p,q} \); the determinant is taken in the sense of zeta function regularization.
The BCOV torsion is an analytic torsion in the sense of Ray and Singer [17]. To see this, we define a holomorphic coefficient vector bundle over a polarized Calabi-Yau manifold \( Z \),

\[
E = \bigoplus_{p=1}^{n} (-1)^p p \Omega^p(X/M).
\]

\( E \) inherits a natural Hermitian metric induced from the metric on the relative tangent bundle. According to [12], there exists a corresponding determinant line bundle over \( M \), which is defined to be

\[
\lambda = \bigwedge_{0 \leq p, q \leq n} (\det(H^{p,q}(Z, E, \bar{\partial}))(-1)^{p+q} p,
\]

where \( H^{p,q}(Z, E, \bar{\partial}) = R^q \pi_* \Omega^p(X/M) \) are holomorphic vector bundles over \( M \); and we identify the cohomology groups with the corresponding harmonic forms with respect to the natural induced metrics on various spaces.

There are two natural metrics defined on \( \lambda \). The usual \( L^2 \) metric is defined by the harmonic forms, and the Quillen metric is given by

\[
\| \cdot \|_Q = \| \cdot \|_{L^2} T.
\]

Following the comprehensive studies on the determinant line bundles and the associated Quillen metrics in [4, 5, 6], where local index techniques are extensively used, people have made many progress in this direction with applications in many branches of mathematics. See [3] for more references and survey.

The formulation that we have in (3.2), (3.3) and (3.4) make it possible to study BCOV torsion in the framework developed in [4, 5, 6]. First, we verify the following:

**Lemma 3.2.** \( \mathfrak{X} \) satisfies the local Kählerian property defined in [4] and [5].

**Proof.** Assume that \( U \) is a coordinate open set of \( M \).\(^4\) Let \( h_t \) be the Hermitian metric of \( L \) over the fiber \( \pi^{-1}(t) \), where \( L \) is the polarization of \( Z \). The form \( \sqrt{-1} \partial \bar{\partial} \log h_t \), when restricted on \( \pi^{-1}(t) \), is positive. Let \( t_1, \ldots, t_m \) be local holomorphic coordinate system of \( U \). Then

\[
\frac{\sqrt{-1}}{2\pi} a \sum_i dt_i \wedge d\bar{t}_i + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h_t
\]

gives the local Kähler metrics on \( \pi^{-1}(U) \) for \( a \in \mathbb{R} \) large enough.

\( \Box \)

The following Lemma of Beshadsky-Cecotti-Ooguri-Vafa [1, 2] characterizes the geometry of the coefficient bundle \( E \).

\(^4\)If \( U \) contains a quotient singularity of \( M \). Then \( \mathfrak{X}|_U \) is Kählerian in the following sense: let \( (\hat{U}, p, U) \) be the local uniformization system. That is, \( \hat{U} \) is a finite cover of \( U \) and \( \hat{U} \) is homeomorphic to the unit ball of \( \mathbb{C}^m \). \( U = \hat{U}/G \) by a finite group \( G \). Let \( \hat{X}_\| \) be the pull back of \( \mathfrak{X}|_U \) through the map \( p : \hat{U} \to U \). Then we say \( \mathfrak{X}|_U \) is Kählerian if \( \hat{X}_\| \) is and the Kähler metric is \( G \)-invariant. The proof of the lemma will go through in this case.
Lemma 3.3. For $E$ associated with the induced metric from that of $X$, as forms on $X$ we have
\begin{equation}
Td(T^{1,0}X/M)ch(E) = -c_{n-1} + \frac{n}{2}c_1 - \frac{1}{12}c_1c_n,
\end{equation}
where $c_i = c_i(T^{1,0}X/M)$.

As a consequence,

Proposition 3.4. The Quillen metric is the potential of $\chi_Z/12$ times the Weil-Petersson metric.
\begin{equation}
c_1(\lambda, \|\cdot\|_Q) = \frac{\chi_Z}{12} \omega_{WP}.
\end{equation}

Proof. This also appears in [2]. We include the proof here for completeness. First notice that, by using the Ricci-flat metric on the relative tangent bundle and Remark 2.3, we have
\begin{equation}
c_1(T^{1,0}X/M) = -c_1(H^{n,0}) = -\omega_{WP}.
\end{equation}
Then, by Lemma 3.2, a direct application of the family Grodenthick-Riemann-Roch theorem proved by Bismut-Gillet-Soule [1], shows that, for $E$ in $\mathfrak{e}_i$,
\begin{equation}
c_1(\lambda, \|\cdot\|_Q) = \left[ \int_Z Td(T^{1,0}X/M)ch(E) \right]^{(1,1)}.
\end{equation}
By Lemma 3.3 and the Gauss-Bonnet formula, we have:
\begin{equation}
c_1(\lambda, \|\cdot\|_Q) = \left[ \int_Z \frac{1}{12} \omega_{WP} c_n(T^{1,0}X/M) \right]^{(1,1)} = \frac{\chi_Z}{12} \omega_{WP}.
\end{equation}

The next proposition relates the curvature of $\lambda$ with respect to the $L^2$-metrics to the generalized Hodge metrics.

Proposition 3.5. Using the notations in the above and in the previous section, we have
\begin{equation}
c_1(\lambda, \|\cdot\|_{L^2}) = \sum_{i=1}^n (-1)^i \omega_{H^i}.
\end{equation}

Proof. This is due to Proposition 2.8. It is an easy computation to show that
\begin{equation}
c_1(\lambda, \|\cdot\|_{L^2}) = \sum_{0 \leq p,q \leq n} (-1)^{p+q} pc_1(R^q\pi_*\Omega^p(X/M)) = \sum_{k=1}^n (-1)^k \omega_{H^k}.
\end{equation}
The proof is complete.

Proof of Theorem 1.1 Using the relation of the $L^2$ metric, the Quillen metric and the BCOV torsion, (1.2) follows from (3.4), Proposition 3.4, and Proposition 3.5. The equation (1.3) follows from (1.2) and Remark 2.3. Theorem 1.1 is the explicit relation between two kinds canonically defined Kähler metrics: the Weil-Petersson metric and the generalized Hodge metrics, on the moduli space.
The surprising fact is that the bridge is the BCOV torsion, a spectral invariant of Ricci-flat metrics, which is also of its own significance in physics literature.

**Proof of Corollary 1.4.** The global Poincaré metric defined at the beginning of Appendix A is asymptotic to the Poincaré metric in (1.3). So they are equivalent on $(\Delta^*)^l \times \Delta^{m-l}$. The corollary is thus follows from Theorem 1.1, Corollary 2.10 and Theorem A.1.

Asymptotic expansion of $T$ near points of maximal monodromy degeneration will give the prediction for counting elliptic curves in the Calabi-Yau manifolds. The above result determines the asymptotic behavior of the BCOV torsion up to a (possibly multi-valued) pluri-harmonic function.

**Proof of Corollary 1.2.** If $N$ is a smooth submanifold in the smooth part of $M$, the corollary follows from the ordinary Stokes theorem. In general, we generalize the Stokes theorem into the singular case. Let’s first define the Hodge and the Weil-Petersson volumes on $N$.

Let $\mu$ be a smooth $2k$-form of the orbifold $M$, where $k = \dim N$. Then we can define a measure on $N$ as follows:

Let $x \in N \subset M$ and let $(\hat{U}, p, U)$ be a local uniformization of $M$ at $x$; i.e., $p : \hat{U} \to U = \hat{U}/G$ for a finite group $G$. Let $V = U \cap N$. Let $\tilde{N} \subset \hat{U}$ be the pre-image of $N$ under $p$. For $y \in V$, define $n(y)$ to be the multiplicity of the map $\tilde{N} \to N$. We can define a measure $\mu$ on $V$ by

$$
\mu(V) = \int_{\hat{V}} \frac{1}{n(y)} p^*(\mu)|_{\tilde{N}},
$$

where $\hat{V}$ is the pre-image of $V$ under $p$. The generalized Hodge metrics, the Weil-Petersson metric, and $\sqrt{-1/2\pi}\partial\bar{\partial}\log T$, through this definition, define the corresponding measures on $N$. In particular, the Hodge and the Weil-Petersson volumes are defined.

The function $n(y)$ defined above is in fact independent of the choice of the local uniformization so it is a global function on $N$. It is a constant on some Zariski open set $N'$ of $N$. Because of this, it is sufficient to prove that

$$(3.12) \quad \text{Vol}_H(N') = \left[\frac{(-1)^n}{12} \chi_Z\right]^k \text{Vol}_{WP}(N').$$

Thus the corollary follows from the following version of Stokes theorem:

**Lemma 3.6.** Let $\eta$ be a smooth $2k-1$ form on the orbifold $M$. Then $d\eta$ defines a measure $\mu(\eta)$ on $N$. We have

$$
\int_{N_{\text{reg}}} \mu(\eta) = 0,
$$

where $N_{\text{reg}}$ is the smooth part of $N$.

**Proof.** Let $N'' = N - N_{\text{reg}} \cap N'$. Let $\rho$ be a smooth function such that: (1). $\rho = 0$ if $\text{dist}(x, N'') < \varepsilon$; (2). $\rho = 1$ if $\text{dist}(x, N'') > 2\varepsilon$; and $|\nabla\rho| \leq 3/\varepsilon$.

By the ordinary Stokes Theorem,

$$
\int_N d(\rho \eta) = 0.
$$
Thus we have
\[ \int_{N_{\text{reg}} \cap N'} \rho \eta + \int_N \eta = 0. \]
Since \( N'' \) is compact, its Hausdorff measure is finite. Thus
\[ \left| \int_N \eta \right| \leq C \cdot \frac{3}{\varepsilon} \cdot \pi \varepsilon^2 (H(N'')) \to 0. \]
Let \( n(y) = \text{const} \) on \( N' \). Then
\[ 0 \leftarrow \text{const} \cdot \int_{N_{\text{reg}} \cap N'} \rho \eta \to \int_N \mu(\eta) \]
This completes the proof. \( \square \)

**Proof of Corollary 1.3.** Suppose \( N \) is a complete curve of \( M \). Then by Corollary 1.2,
\[ \text{Vol}_H(N) = \frac{(-1)^n}{12} \chi_Z \text{Vol}_{WP}(N). \]
By the assumption, we thus have
\[ \text{Vol}_H(N) < 2 \text{Vol}_{WP}(N), \]
unless \( N \) is of dimension 0. However, the above inequality contradicts to Corollary 2.10. \( \square \)

**Remark 3.7.** All the Calabi-Yau three-folds are primitive. By the Physics Mirror Symmetry, it is conjectured that all the Calabi-Yau three-folds occur in pairs (mirror pair); and the Euler characteristics of paired Calabi-Yau three-folds are differed only by signs. Hence, Corollary 1.3 claims that “more than half” of moduli of polarized Calabi-Yau three-folds contains no complete subvariety.

We proceed to discuss the obstruction to the existence of complete curves in the Calabi-Yau moduli, in some more specific situations.

**Remark 3.8.** For a primitive Calabi-Yau four-fold \( Z \), let \( N \) be a complete curve in the moduli space \( M \). By Theorem 2.4, we have
\[ \int_N \omega_H = (2m + 4) \int_N \omega_{WP} + 2 \int_N \text{Ric}(\omega_{WP}). \]
By Theorem 1.1, we have
\[ \int_N \omega_H = \frac{\chi_Z}{12} \int_N \omega_{WP}. \]
If \( \chi_Z > 24(m + 2) \), from (3.13) and (3.14), we have
\[ \int_N \text{Ric}(\omega_{WP}) \geq 0. \]
On the other side, the Ricci curvature of \( \omega_H \) is negative, so
\[ \int_N \text{Ric}(\omega_H) < 0. \]
The above inequality contradicts to (3.15) because $\text{Ric}(\omega_H) - \text{Ric}(\omega_{WP}) = -\sqrt{-1} \partial \bar{\partial} \log (\frac{\omega_H}{\omega_{WP}})$, and thus using Lemma 3.6, the integration of the two Ricci curvatures are the same. This is a contradiction. Thus if $\chi_Z > 24(m+2)$, there is no complete curve in $\mathcal{M}$. The similar result in the case of Calabi-Yau threefold is trivial and the similar results for high dimensional Calabi-Yau manifolds are still unknown.

**Remark 3.9.** When the dimension of $\mathcal{M}$ is 2, if there exists a complete curve $C \subset \mathcal{M}$, then

$$\int_C \text{Ric}_{WP} = \chi_C + C.C,$$

where $C.C$ is the self-intersection number. Let $\omega' = \omega_{WP}|_C$. In local coordinate, let

$$\omega' = \frac{\sqrt{-1}}{2\pi} h_{1\bar{1}} dt_1 \wedge d\bar{t}_1,$$

and let

$$\omega_{WP} = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^2 h_{ij} dt_i \wedge d\bar{t}_j.$$  

Assume that at a point $x$ $h_{ij} = \delta_{ij}$. Then we have

$$\partial_1 \bar{\partial}_1 \log h_{1\bar{1}} = \frac{\partial^2 h_{1\bar{1}}}{\partial t_1 \partial \bar{t}_1} - h_{1\bar{1}}^{-1} \left| \frac{\partial h_{1\bar{1}}}{\partial t_1} \right|^2.$$  

Since

$$h_{1\bar{1}}^{-1} \left| \frac{\partial h_{1\bar{1}}}{\partial t_1} \right|^2 \leq h_{ij} \frac{\partial h_{i\bar{j}}}{\partial t_1} \cdot \frac{\partial h_{1\bar{1}}}{\partial t_1},$$

we have

$$\partial_1 \bar{\partial}_1 \log h_{1\bar{1}} \geq R_{111\bar{1}},$$

where $R_{ijk\bar{l}}$ is the curvature tensor of $\omega_{WP}$. By the (generalized) Strominger formula (cf. [15, Theorem 3.1], [21], and [25]), we have

$$R_{112\bar{2}} \leq 1.$$  

Thus we have the following

$$\text{Ric}(\omega') \leq \text{Ric}(\omega_{WP}) + 1.$$  

Comparing the above equation with (3.16), we have

$$C.C \geq -1.$$  

This would be a new obstruction to the existence of complete curves in these cases.

Notice that the results we proved above are only on the smooth moduli. It is very interesting to see the corresponding results for the compactified moduli. This kind of global results will be achieved by finer local analysis of the BCOV torsion near the boundary of the moduli.
APPENDIX A. POINCARÉ METRIC AND GENERALIZED HODGE METRICS

In this Appendix, we prove that the generalized Hodge metrics are bounded by the Poincaré metric. The main result of this Appendix, Theorem A.1, is a degenerate version of Yau’s Schwarz Lemma (cf. [27]) from Kähler manifolds to Hermitian manifolds.

Calabi-Yau moduli $\mathcal{M}$ is quasi-projective. The smooth part $\mathcal{M}_{\text{reg}}$ of $\mathcal{M}$ allows a compactification $\overline{\mathcal{M}}$, where $\mathcal{M}$ is a compact smooth manifold such that $\mathcal{M} \setminus \mathcal{M}_{\text{reg}}$ is a divisor of normal crossing. For the pair of manifolds $(\mathcal{M}, \mathcal{M}_{\text{reg}})$, we define a Kähler metric $\omega_{GP}$ called the global Poincaré metric as follows: in a neighborhood $(\Delta^*)^l \times \Delta^{m-l}$ of $x \in \mathcal{M} \setminus \mathcal{M}$, the global Poincaré metric $\omega_{GP}$ is asymptotically the Poincaré metric $\omega$ defined in (1.5):

$$\omega_{GP} \sim \omega = \sum_{i=1}^{l} \sqrt{-1} \frac{1}{|z_i|^2 (\log |z_i|^2)^2} dz_i \wedge d\bar{z}_i + \sum_{i=l+1}^{m} \sqrt{-1} dz_i \wedge d\bar{z}_i.$$

The global Poincaré metric $\omega_{GP}$ is a complete Kähler metric on $\mathcal{M}$ whose Ricci curvature is bounded from below. For the detailed construction of $\omega_{GP}$, see [16].

The main result of this section is the following:

**Theorem A.1.** Using the above notations, we have

$$\omega_{PH^k} \leq C \omega_{GP}$$

for $0 \leq k \leq n$ and a constant $C$ depending on the lower bound of $\text{Ric}(\omega_{GP})$, dimension of the Calabi-Yau manifolds and dimension of the moduli space $\mathcal{M}$.

We use $H^{p,q}$ to denote the bundle $PR^n_{\pi_\ast \Omega_{\mathcal{X}/\mathcal{M}}}$ for $0 \leq p, q \leq n$. Let $F^p_k = H^{k,0} \oplus \cdots \oplus H^{k-p}$ for $p = 0, \cdots, k$. We also assume that $H^{k+2, -2} = H^{k+1, -1} = H^{-1, k+1} = H^{-2, k+2} = 0$ and $F^k_k = 0$, for the sake of simplicity.

First we try to express the generalized Hodge metrics in local coordinates. Fix $k \leq n$, $p \leq k$ and $q = k - p$. Let $\{\Omega_{p,i}\}$, $i = 1, \cdots, h^{p,q}$ be a local frame of $H^{p,q}$.

**Definition A.2.** Let $(t_1, \cdots, t_m)$ be a holomorphic local coordinate at a point of $\mathcal{M}_{\text{reg}}$. We define $D_\alpha \Omega_{p,i} \in H^{p-1, q+1}$ to be the projection of $\partial_\alpha \Omega_{p,i} = \frac{\partial}{\partial t_\alpha} \Omega_{p,i}$ to $H^{p-1, q+1}$ with respect to the bilinear form $Q(\ ,\ )$ in [25].

For simplicity, we shall use $(\ ,\ )$ in stead of the bilinear form $Q$ in [25]. With the above notation,

(A.1) $$(g_p)_{ij} = \langle \Omega_{p,i}, \Omega_{p,j} \rangle = (\sqrt{-1})^{q-p} \langle \Omega_{p,i}, \Omega_{p,j} \rangle$$

is the Hermitian metric matrix of $H^{p,q}$ for $p = 0, \cdots, k$. It is thus easy to see that

**Proposition A.3.** The generalized Hodge metric matrix for the local coordinate system $(t_1, \cdots, t_m)$ with respect to $PH^k$, defined in Definition 2.2, is

(A.2) $$h_{\alpha \beta} = \sum_{p=0}^{k} (\sqrt{-1})^{q-p+2} g_p^{ij} (D_\alpha \Omega_{p,i}, D_\beta \Omega_{p,j}),$$

where $(g_p^{ij})$ is the inverse of $(g_p)_{ij}$.

We proceed with two technical lemmas.
Lemma A.4.  
(A.3) \[ \bar{\partial} D_\alpha \Omega_{p,i} = g^{j}_{p} < \partial_\beta \Omega_{p,j}, \Omega_{p,l} > \Omega_{p,l}. \]

**Proof.** We first claim that \( \bar{\partial} D_\alpha \Omega_{p,i} \in H^{p,q} \). To see this, let \( \Omega_1 \in F^{p}_{k+1} \). Then
(A.4) \[ (\bar{\partial} D_\alpha \Omega_{p,i}, \Omega_1) = -(D_\alpha \Omega_{p,i}, \bar{\partial} \Omega_1) = 0. \]

On the other hand, if \( \Omega_2 \in H^{p-1,q+1} \), we have the decomposition
\[ \partial_\alpha \Omega_{p,i} = D_\alpha \Omega_{p,i} + B \]
for \( B \in H^{p,q} \). Furthermore,
\[ \partial B \in F^{p}_{k}. \]
Thus,
(A.5) \[ \bar{\partial} D_\alpha \Omega_{p,i} \in F^{p}_{k}. \]
Combining (A.4), (A.5), we have \( \bar{\partial} D_\alpha \Omega_{p,i} \in H^{p,q} \). Writing \( \bar{\partial} D_\alpha \Omega_{p,i} \) as the linear combination of \( \Omega_{p,i} \)'s, we get (A.3). \[ \square \]

Lemma A.5. If \( A \) is a local section of \( H^{p,q} \) and \( B \) is a local section of \( H^{p-1,q+1} \), then
\[ < D_\alpha A, \bar{\partial} B > = < A, \bar{\partial} \alpha B >. \]

**Proof.** This follows from a straightforward computation:
\[ < D_\alpha A, \bar{\partial} B > = (\sqrt{-1})^{q-p+2}(\partial_\alpha A, \bar{\partial} B) = -(\sqrt{-1})^{q-p+2}(A, \bar{\partial} \alpha B) = < A, \bar{\partial} \alpha B >. \]

**Proof of Theorem A.1.** The generalized Hodge metrics are only semi-positive definite but not positive definite. If they were positive definite, then using the similar method as in [13], we should have been able to prove that the holomorphic sectional curvatures of the metrics were negative and bounded away from zero, and the holomorphic bisectional curvatures of the metrics were nonpositive. Thus we could have used Yau’s Schwarz Lemma [27] to get the conclusion. The contribution of this Appendix is that we prove the same result even if the generalized Hodge metrics fail to be positive definite.

We assume all the notations in the previous sections. Write the global Poncaré metric in local coordinates as:
\[ \omega_{GP} = \frac{\sqrt{-1}}{2\pi} \tau_{\alpha \beta} dt_\alpha \wedge dt_\beta. \]
Let \( -C_1 \) be the lower bound of the Ricci curvature of \( \omega_{GP} \) for some constant \( C_1 > 0 \). We define a smooth function
\[ f = \sum \tau^{\alpha \beta} h_{\alpha \beta} \]
on \( \mathcal{M} \). \( f \) is nonnegative. If \( f \) is bounded, then \( \omega_{PH^k} \) is bounded by \( \omega_{GP} \).

For the rest of the Appendix, we assume that at the given point \( x \), \( \tau_{ij} = \delta_{ij} \) and \( d\tau_{ij} = 0. \) Then at \( x \),
\[ \Delta f \geq -C_1 f + \frac{\partial^2}{\partial t_\gamma \partial t_\gamma} h_{\alpha \bar{\alpha}}, \]
where $\Delta$ is the Laplacian on $\mathcal{M}$ with respect to $\omega_{G\mathcal{P}}$.

We assume that at the point $x$, the frames $\Omega_{p,i}$ are chosen so that $(g_p)_{ij} = \delta_{ij}$, and $\frac{\partial}{\partial t_{\alpha}}(g_p)_{ij} = 0$ for $p = 0, \cdots, k$ and $\alpha = 1, \cdots, m$. Then a straightforward computation gives

$$
\frac{\partial^2}{\partial t_{\gamma}\partial t_{\gamma}} h_{\alpha\alpha} = \sum_p (\sqrt{-1})^{q-p+2}(-(R_p)_{j\bar{i}j\bar{\gamma}})(D_{\gamma}\Omega_{p,i}, \overline{D_{\alpha}\Omega_{p,j}}) \\
+ \sum_p (\sqrt{-1})^{q-p+2}(\partial_{\gamma}D_{\alpha}\Omega_{p,i}, \overline{D_{\gamma}D_{\alpha}\Omega_{p,i}}) \\
+ \sum_p (\sqrt{-1})^{q-p+2}(\overline{\partial_{\gamma}D_{\alpha}\Omega_{p,i}}, \overline{D_{\gamma}D_{\alpha}\Omega_{p,i}}) \\
+ \sum_p (\sqrt{-1})^{q-p+2}(D_{\alpha}\Omega_{p,i}, \overline{\partial_{\gamma}D_{\alpha}\Omega_{p,i}}),
$$

(A.7)

where $(R_p)_{j\bar{i}j\bar{\gamma}}$ is the curvature tensor of $g_p$ for $p = 0, \cdots, k$. By Lemma A.4, we have

$$
(\partial_{\gamma}\partial_{\gamma}D_{\alpha}\Omega_{p,i}, \overline{D_{\alpha}\Omega_{p,i}}) = -(\partial_{\gamma}D_{\alpha}\Omega_{p,i}, \overline{D_{\gamma}D_{\alpha}\Omega_{p,i}}); \quad \text{(A.8)}
$$

$$
(D_{\alpha}\Omega_{p,i}, \partial_{\gamma}D_{\alpha}\Omega_{p,i}) = -(\partial_{\gamma}D_{\alpha}\Omega_{p,i}, \partial_{\gamma}D_{\alpha}\Omega_{p,i}). \quad \text{(A.9)}
$$

Inserting the above two equations into (A.7), we have

$$
\frac{\partial^2}{\partial t_{\gamma}\partial t_{\gamma}} h_{\alpha\alpha} = \sum_p (\sqrt{-1})^{q-p+2}(-(R_p)_{j\bar{i}j\bar{\gamma}})(D_{\gamma}\Omega_{p,i}, \overline{D_{\alpha}\Omega_{p,j}}) \\
+ \sum_p (\sqrt{-1})^{q-p+2}(\partial_{\gamma}D_{\alpha}\Omega_{p,i}, \overline{D_{\gamma}D_{\alpha}\Omega_{p,i}}) \\
- \sum_p (\sqrt{-1})^{q-p+2}(\overline{\partial_{\gamma}D_{\alpha}\Omega_{p,i}}, \overline{D_{\gamma}D_{\alpha}\Omega_{p,i}}).
$$

(A.10)

By [11] page 33, Proposition 4], the curvature of $(g_p)_{ij}$ is

$$
(R_p)_{j\bar{i}j\bar{\gamma}} = (\sqrt{-1})^{q-p}(D_{\gamma}\Omega_{p,i}, \overline{D_{\alpha}\Omega_{p,j}}) - (\sqrt{-1})^{q-p}(\overline{\partial_{\gamma}\Omega_{p,i}}, \partial_{\gamma}\Omega_{p,j}). \quad \text{(A.11)}
$$

Let

$$
\partial_{\gamma}D_{\alpha}\Omega_{p,i} = A_{p\alpha\gamma i} + B_{p\alpha\gamma i}, \quad \text{(A.12)}
$$

where $A_{p\gamma ai} \in H^{p-2,q+2}$ and $B_{p\gamma ai} \in H^{p-1,q+1}$. Then

$$
\frac{\partial^2}{\partial t_{\gamma}\partial t_{\gamma}} h_{\alpha\alpha} = -\sum_p (\sqrt{-1})^{q-p+2}(R_p)_{j\bar{i}j\bar{\gamma}}(D_{\gamma}\Omega_{p,i}, \overline{D_{\alpha}\Omega_{p,j}}) \\
- |A_{p\alpha\gamma i}|^2 + |B_{p\alpha\gamma i}|^2 - \sum_p (\sqrt{-1})^{q-p+2}(\overline{\partial_{\gamma}D_{\alpha}\Omega_{p,i}}, \overline{\partial_{\gamma}D_{\alpha}\Omega_{p,i}}).
$$

(A.13)

By Lemma A.4 and Lemma A.5 let

$$
D_{\alpha}\Omega_{p,i} = (A_{\alpha}^p)_{i}l\Omega_{p+1,l},
$$
for matrices $A_p^\alpha = (A_p^\alpha)_{il}$. Then we have

$$\bar{\partial}_\alpha \Omega_{p,i} = (A_p^{\alpha-1})_{it}(A_p^\alpha)_{is}(A_p^\alpha)_{js}.$$  

(A.11)

Thus from (A.11), (A.13), in terms of the matrices $A_p^\alpha$, we have

\[
\frac{\partial^2}{\partial t_\gamma \partial t_\gamma} h_{\alpha \bar{\alpha}} = \sum_{p, \alpha, \gamma, i} (A_p^\alpha)_{it}(A_p^\alpha)_{is}(A_p^\alpha)_{js}
\]

(A.14)

\[
- \sum_{p, \alpha, \gamma, i} (A_p^{\alpha-1})_{it}(A_p^{\alpha-1})_{is}(A_p^\alpha)_{js}
\]

\[
- \sum_{p, \alpha, \gamma, i} |\sum_t (A_p^\alpha)_{it}(A_p^{\alpha+1})_{ts}|^2 + \sum_{p, \alpha, \gamma, i} |\sum_s (A_p^\alpha)_{is}(A_p^\alpha)_{js}|^2
\]

\[
+ |B_{p\alpha\gamma i}|^2
\]

(A.15)

The $H^{p-2,q+2}$ part $A_{p\gamma ai}$ of $\partial_\alpha D_\gamma \Omega_{p,i}$ is the same as the $H^{p-2,q+2}$ part of $\partial_\alpha \partial_\gamma \Omega_{p,i}$. Thus

$$A_{p\gamma ai} = A_{p\alpha i}$$

for $1 \leq \alpha, \gamma \leq m$ and $p = 0, \cdots, k$. In terms of the matrices $A_p^\alpha$, we have

$$A_p^\gamma A_p^{\alpha+1} = A_p^\alpha A_p^\gamma$$

for $1 \leq \alpha, \gamma \leq m$ and $p = 0, \cdots, k$. Using these commutative relations of the matrices $A_p^\alpha$, from (A.14), we have

\[
\sum_{\alpha, \gamma} \frac{\partial^2}{\partial t_\gamma \partial t_\gamma} h_{\alpha \bar{\alpha}} = \sum_{p, \alpha, \gamma, i} |B_{p\alpha\gamma i}|^2
\]

(A.15)

\[
+ \sum_{p, \alpha, \gamma} \text{Tr} \left[ \left( (A_p^{\alpha-1})^T A_p^{\alpha-1} - A_p^\alpha (A_p^\alpha)^T \right) (A_p^{\alpha-1})^T A_p^{\alpha-1} - A_p^\alpha (A_p^\alpha)^T \right]^T \right].
\]

Since the last term of the above expression is zero if and only if $A_p^\alpha \equiv 0$ for any $\alpha$ and $p$, there exists an $\varepsilon > 0$, such that

$$\frac{\partial^2}{\partial t_\gamma \partial t_\gamma} h_{\alpha \bar{\alpha}} \geq \varepsilon \sum_p \text{Tr}(A_p^\alpha (A_p^\alpha)^T) \geq \varepsilon |h_{\alpha \bar{\alpha}}|^2.$$  

(A.16)

Finally, from (A.6) and (A.16), we have

$$\Delta f \geq \varepsilon m t^2 - C_1 f.$$  

Since the Ricci curvature of $\omega_{WP}$ is lowerly bounded and since $f$ is nonnegative, the generalized maximum principal (cf. [8]) gives

$$f \leq mC_1/\varepsilon.$$  

and the theorem is thus proved.
H. Fang and Z. Lu

REFERENCES

[1] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa. Holomorphic anomalies in topological field theories. Nuclear Phys. B, 405(2-3):279–304, 1993.

[2] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa. Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. Comm. Math. Phys., 165(2):311–427, 1994.

[3] J.-M. Bismut. Local index theory and higher analytic torsion. In Proceedings of the International Congress of Mathematicians, Vol. 1 (Berlin, 1998), pages 143–162 (electronic), 1998.

[4] J.-M. Bismut, H. Gillet, and C. Soulé. Analytic torsion and holomorphic determinant bundles. III. Quillen metrics on holomorphic determinants. Comm. Math. Phys., 115(2):301–351, 1988.

[5] J.-M. Bismut, H. Gillet, and C. Soulé. Analytic torsion and holomorphic determinant bundles. II. Direct images and Bott-Chern forms. Comm. Math. Phys., 115(1):79–126, 1988.

[6] J.-M. Bismut, H. Gillet, and C. Soulé. Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion. Comm. Math. Phys., 115(1):49–78, 1988.

[7] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. In S.Y. Yau, editor, Essays in mirror manifolds, pages 31–95. International Press, 1992.

[8] S. Cheng and S. Yau. On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation. Comm. Pure Appl. Math., 33:507–544, 1980.

[9] D. A. Cox and S. Katz. Mirror symmetry and algebraic geometry, volume 68 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999.

[10] H. Fang and Z. Lu. Some linear algebra of the $SL(2, C)$ representation. private notes.

[11] P. Griffiths, editor. Topics in Transcendental Algebraic Geometry, volume 106 of Ann. Math Studies. Princeton University Press, 1984.

[12] F. F. Knudsen and D. Mumford. The projectivity of the moduli space of stable curves. I. Preliminaries on “det” and “Div”. Math. Scand., 39(1):19–55, 1976.

[13] Z. Lu. On the geometry of classifying spaces and horizontal slices. Amer. J. Math., 121:177–198, 1999.

[14] Z. Lu. On the Hodge metric of the universal deformation space of Calabi-Yau threefolds. J. Geom. Anal., 11(1):103–118, 2001.

[15] Z. Lu and X. Sun. Weil-Petersson Geometry of the Moduli Spaces of Calabi -Yau Manifolds. accepted by Journal de l’Institut Mathematique de Jussieu, 2002.

[16] Z. Lu and X. Sun. The Weil-Petersson volume of the moduli space of Calabi-Yau manifolds. preprint, 2002.

[17] D. B. Ray and I. M. Singer. Analytic torsion for complex manifolds. Ann. of Math. (2), 98:154–177, 1973.

[18] Y. Ruan. String geometry and topology of orbifolds. AG/0011149, 2000.

[19] G. Schumacher. The curvature of the Petersson-Weil metric on the moduli space of Kähler-Einstein manifolds. Ancona, Vincenzo(ed.) et al., Complex analysis and geometry. Unif. Ser. Math., 339–354. Plenum, New York, 1993.

[20] Y. T. Siu. Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class. Contributions to several complex variables. Aspects Math., E9, 261–298. Vieweg, Braunschweig, 1986.

[21] A. Strominger. Special Geometry. Comm. Math. Phy., 133:163–180, 1990.

[22] G. Tian. Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Peterson-Weil metric. In S.-T. Yau, editor, Mathematical aspects of string theory, volume 1, pages 629–646. World Scientific, 1987.

[23] A. N. Todorov. Introduction to Weil-Petersson Geometry of the moduli space of CY manifolds. preprint.

[24] A. N. Todorov. The Weil-Petersson geometry of the moduli space of SU($n \geq 3$) (Calabi-Yau) manifolds. I. Comm. Math. Phys., 126(2):325–346, 1989.

[25] C.-L. Wang. Curvature properties of the Calabi-Yau moduli. to appear in Documenta Mathematica.

[26] R. O. Wells, Jr. Differential analysis on complex manifolds, volume 65 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1980.
[27] S.-T. Yau. A general Schwarz lemma for Kähler manifolds. *Amer. J. Math.*, 100:197–203, 1978.
[28] K.-I. Yoshikawa. Discriminant of theta divisors and Quillen metrics. *J. Differential Geom.*, 52(1):73–115, 1999.
[29] K.-I. Yoshikawa. K3 surfaces with involutions, equivariant analytic torsion, and automorphic forms on the moduli space. to appear in *Invent. Math*.

_E-mail address_, Hao Fang: haofang@cims.nyu.edu
_E-mail address_, Zhiqin Lu: zlu@math.uci.edu