On the solution of the Hermite problem for cubic irrationalities

Nadir Murru

Department of Mathematics
University of Turin
Via Carlo Alberto 10, Turin, 10123, ITALY
e-mail: nadir.murru@unito.it

Abstract

In this paper, a solution of the Hermite problem is given for any cubic irrationality, i.e., a periodic representation is provided for any cubic irrationality. The solution is found only using an elementary approach based on the algebraic properties of cubic irrationalities and the properties of linear recurrent sequences. In particular, a periodic multidimensional continued fraction (with pre–period of length 2 and period of length 3) is proved convergent to a given cubic irrationality. This multidimensional continued fraction is derived from a modification of the Jacobi algorithm, which is proved periodic if and only if the inputs are cubic irrationals. Moreover, this representation provides simultaneous rational approximations for the real numbers.

1 Introduction

In 1839, Hermite (see the letters published in 1850, [19]) posed to Jacobi the problem of generalizing the construction of continued fractions to higher dimensions. In particular, he asked for a method of representing algebraic irrationalities by means of periodic sequences that can highlight algebraic properties and possibly provide rational approximations. Hermite especially focused the attention on cubic irrationalities.

Continued fractions completely solve this problem for quadratic irrationalities, but the problem for algebraic numbers of degree $\geq 3$ is still open. In 1868, Jacobi (see the VI volume of the book Ges. Werke published in 1891, [21]) built an algorithm that produces a generalization of the classic continued fractions. These multidimensional continued fractions, if periodic, converge to cubic irrationalities, but the vice versa has been never proved. In 1907, Perron [28] developed a generalization of this algorithm for any algebraic irrationalities (for a complete survey about the Jacobi–Perron
algorithm see [10] and [31]).

The study of periodic representations for algebraic numbers has interested many mathematicians. This is a very beautiful theoretical (and not only) question: can irrational numbers have a periodic representation? During the years many attempts have been performed. Several kinds of multidimensional continued fractions have been deeply studied in various works.

In [7], [8], [9], Bernstein studied and proved the convergence of the Jacobi–Perron algorithm for a vast class of algebraic irrationals. Further results on the Jacobi–Perron algorithm can be found in [29] and [37]. Many different algorithms are summarized and compared in [11] and [12]. Moreover, see the beautiful book of Schweiger [32] for a guide about multidimensional continued fractions.

The usual approach to the Hermite problem contemplates the research of functions whose iteration on algebraic irrationalities yields a periodical algorithm. In this sense, Tamura and Yasutomi [34], [35] recently presented a modified Jacobi–Perron algorithm. Similarly, the Jacobi–Perron algorithm has been modified using different functions, e.g., in [18], [39], [25], [33]. Further study can be found in [15], [20], [38].

A very interesting approach can be found in the works [16], [3], [13], where a multidimensional continued fraction related to triangle sequences is studied. Moreover, in [6] a generalization of the Minkowski question–mark function is developed. Finally, a completely different approach to multidimensional continued fractions can be found in [22].

All these algorithms, when periodic, provide sequences that converge to cubic irrationalities. However, none algorithm has been proved to become periodic when the input is a cubic irrational. Thus, not exists any algorithm that provides a periodic representation for a given cubic irrationality.

In this paper, a periodic representation for an irrational number satisfying the cubic equation \(x^3 - px^2 - qx - r = 0\), with \(p, q, r\) rational numbers, is given. This representation has been directly found by means of elementary techniques that only involve the algebraic properties of cubic irrationalities and the properties of linear recurrent sequences. Thus, an algorithm can be derived, such that a periodic sequence is provided when a cubic irrational is given in input.

In section 2, the fundamental properties of the multidimensional continued fraction, derived from the Jacobi algorithm, are presented. In section 3, the main case (\(\alpha\) root largest in modulus of \(x^3 - px^2 - qx - r\)) is treated. A periodic expansion that converges to the couple \((r/\alpha, \alpha)\) is shown. In section 4, all the remaining cases of cubic irrationalities satisfying a cubic equation \(x^3 - px^2 - qx - r = 0\) are treated. In this way, the Hermite problem is solved for any cubic irrational. The iteration on cubic irrationalities of the map in section 5 provides (according to the Jacobi algorithm) the periodic expansion found in the previous sections. Section 6 is devoted to the
conclusions.

2 Ternary continued fractions

The Jacobi algorithm associates a couple of integer sequences to a couple of real numbers by the following equations:

\[
\begin{align*}
    a_n &= \lfloor \alpha_n \rfloor \\
    b_n &= \lfloor \beta_n \rfloor \\
    \alpha_{n+1} &= \frac{1}{\beta_n - \lfloor \beta_n \rfloor} \\
    \beta_{n+1} &= \frac{\alpha_n - \lfloor \alpha_n \rfloor}{\beta_n - \lfloor \beta_n \rfloor},
\end{align*}
\]

\(n = 0, 1, 2, \ldots\), for any couple of real numbers \(\alpha = \alpha_0\) and \(\beta = \beta_0\). It follows that

\[
\begin{align*}
    \alpha_n &= a_n + \frac{\beta_{n+1}}{\alpha_{n+1}} \\
    \beta_n &= b_n + \frac{1}{\alpha_{n+1}}
\end{align*}
\]

Therefore, the real numbers \(\alpha\) and \(\beta\) are represented by the sequences \((a_n)_{n=0}^{\infty}\), \((b_n)_{n=0}^{\infty}\) as follows:

\[
\begin{align*}
    \alpha &= a_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{a_2 + \cdots}}}
    \quad \text{and} \\
    \beta &= b_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}
\end{align*}
\]

We call ternary continued fraction (as named, e.g., in [14] and [24], where these objects have been studied independently from the generating algorithm) such a couple of objects representing the numbers \(\alpha\) and \(\beta\) and we write

\[
(\alpha, \beta) = [\{a_0, a_1, a_2, \ldots\}, \{b_0, b_1, b_2, \ldots\}],
\]
where $a_i$’s and $b_i$’s are called \textit{partial quotients}.

\textbf{Remark 1.} Ternary continued fraction are also called bifurcating continued fraction as in [23] and [17].

Similarly to classical continued fractions, the notion of convergent is introduced as follows (for a complete survey of the Jacobi–Perron algorithm see [10]):

$$[\{a_0, a_1, \ldots, a_n\}, \{b_0, b_1, \ldots, b_n\}] = \left( \frac{A_n}{C_n}, \frac{B_n}{C_n} \right), \quad \forall n \geq 0$$

$$\lim_{n \to \infty} \frac{A_n}{C_n} = \alpha, \quad \lim_{n \to \infty} \frac{B_n}{C_n} = \beta$$

are the \textit{n–th convergents} of the ternary continued fraction (3), where $A_n, B_n, C_n$ satisfy the following recurrent relations

$$\begin{align*}
A_n &= a_n A_{n-1} + b_n A_{n-2} + A_{n-3} \\
B_n &= a_n B_{n-1} + b_n B_{n-2} + B_{n-3} \\
C_n &= a_n C_{n-1} + b_n C_{n-2} + C_{n-3}
\end{align*}$$

(4)

with initial conditions

$$\begin{align*}
A_{-2} &= 1, & A_{-1} &= 0, & A_0 &= a_0 \\
B_{-2} &= 0, & B_{-1} &= 1, & B_0 &= b_0 \\
C_{-2} &= 0, & C_{-1} &= 0, & C_0 &= 1
\end{align*}$$

Furthermore, a matricial approach is known. Indeed, it is easy to prove by induction that

$$\left( \begin{array}{ccc} a_0 & 1 & 0 \\ b_0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \ldots \left( \begin{array}{ccc} a_n & 1 & 0 \\ b_n & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} A_n & A_{n-1} & A_{n-2} \\ B_n & B_{n-1} & B_{n-2} \\ C_n & C_{n-1} & C_{n-2} \end{array} \right)$$

(5)

$$\Downarrow$$

$$\left( \begin{array}{c} A_n \\ B_n \\ C_n \end{array} \right) = \left[ \{a_0, \ldots, a_n\}, \{b_0, \ldots, b_n\} \right],$$

for $n = 0, 1, 2, \ldots$.

\textbf{Remark 2.} A ternary continued fraction (2) can converge to a couple of real numbers although the partial quotients are not obtained by the Jacobi algorithm. Thus, it is possible to study convergence of ternary continued fractions independently from the origin of the partial quotients. In sections 3 and 4, we find the partial quotients $a_i$’s and $b_i$’s such that the ternary continued fraction (2) converges to a given cubic irrational. In Section 5, these partial quotients are obtained by the Jacobi algorithm (1) by means of two functions $f_{\alpha}^z, g_{\alpha}^z$ that substitute the role of the floor function.
In [2], the authors studied the convergence of ternary continued fractions with rational partial quotients, finding infinitely many periodic representations for every cubic root.

**Theorem 1.** The periodic ternary continued fraction
\[
\left\{ \frac{a_0}{b_0}, \frac{a_1}{b_1}, \ldots, \frac{c_0}{d_0}, \frac{c_1}{d_1}, \ldots \right\}, \left\{ \frac{z}{d}, \frac{3dz}{d^2 + 3z^2}, \frac{3z}{d} \right\}, \left\{ 0, -\frac{z^2}{d}, -\frac{3z^2}{d^2 + 3z^2}, -\frac{3z^2}{d^2 + 3z^2} \right\}
\] (6)
converges for every integer \( z \neq 0 \) to the couple of irrationals \((\sqrt[3]{d^2}, \sqrt[3]{d})\), for \( d \) integer not cube.

**Remark 3.** A ternary continued fraction with rational partial quotients is clearly determined by sequences of integer numbers. Indeed, given
\[
\left[ \frac{a_0}{b_0}, \frac{a_1}{b_1}, \ldots, \frac{c_0}{d_0}, \frac{c_1}{d_1}, \ldots \right],
\]
where \( a_i, b_i, c_i, d_i \) integer numbers, for \( i = 0, 1, 2, \ldots \), then
\[
\begin{pmatrix}
a_0 b_0 & b_0 d_0 & 0 \\
c_0 b_0 & 0 & b_0 d_0 \\
b_0 d_0 & 0 & 0
\end{pmatrix}
\cdots
\begin{pmatrix}
a_n d_n & b_n d_n & 0 \\
c_n b_n & 0 & b_n d_n \\
b_n d_n & 0 & 0
\end{pmatrix}
\]
(7)

\[
\begin{pmatrix}
d_0 s_n & d_0 b_n d_n s_{n-1} & d_0 b_n d_n b_{n-1} d_{n-1} s_{n-2} \\
b_0 d_0 s'_{n-1} & b_0 b_n d_n s_{n-1} & b_0 b_n d_n b_{n-1} d_{n-1} s'_{n-2} \\
b_0 d_0 d_1 s''_{n-1} & b_0 d_0 d_1 b_n d_n s''_{n-1} & b_0 d_0 d_1 b_n d_n b_{n-1} d_{n-1} s''_{n-2}
\end{pmatrix}
\]
\[
\updownarrow
\]
\[
\left[ \frac{a_0}{b_0}, \ldots, \frac{a_n}{b_n} \right], \left( \frac{c_0}{d_0}, \ldots, \frac{c_n}{d_n} \right) = \left( \frac{s_n}{b_0 d_1 s''_{n-1}}, \frac{s'_n}{d_0 s''_{n-1}} \right) = \left( \frac{A_n}{C_n}, \frac{B_n}{C_n} \right),
\]
where
\[
\begin{align*}
& s_0 = a_0, \quad s_1 = a_0 a_1 d_1 + b_0 b_1 c_1, \quad s_2 = a_2 d_2 s_1 + b_2 b_1 c_2 d_0 s_0 + b_2 b_1 b_0 d_2 d_1 \\
& s'_{n+1} = c_0, \quad s'_1 = a_1 c_0 + b_1 d_0, \quad s'_2 = b_1 b_2 c_0 a_1 + a_2 c_0 d_2 + a_2 b_1 d_0 d_2 \\
& s''_0 = 1, \quad s''_1 = a_1, \quad s''_2 = b_1 b_2 c_2 + a_1 a_2 d_2 \\
\end{align*}
\]
and
\[
\begin{align*}
& s_n = a_n d_n s_{n-1} + b_n b_{n-1} c_n d_{n-1} s_{n-2} + b_n b_{n-1} d_{n-1} b_{n-2} d_n d_{n-1} s_{n-3} \\
& s'_{n+1} = a_n d_n s'_{n-1} + b_n b_{n-1} c_n d_{n-1} s'_{n-2} + b_n b_{n-1} d_{n-1} b_{n-2} d_n d_{n-1} s'_{n-3} \\
& s''_n = a_n d_n s''_{n-1} + b_n b_{n-1} c_n d_{n-1} s''_{n-2} + b_n b_{n-1} d_{n-1} b_{n-2} d_n d_{n-1} s''_{n-3}
\end{align*}
\]
\(, \ \forall n \geq 3.\)

For these results see [2] and [26]. Thus, a ternary continued fraction with rational partial quotients can be represented by matrices with integer entries like
\[
\begin{pmatrix}
a_0 d_1 & b_0 d_1 & 0 \\
a_1 c_1 & b_1 d_1 & 0 \\
b_0 d_1 & 0 & 0
\end{pmatrix},
\]
which play the same role of the matrices used in [5].
The periodic expansion of Theorem 1 has been found starting from the development of

\[(z + \sqrt[3]{d})^n = \nu_n^{(0)} + \nu_n^{(1)} \sqrt[3]{d} + \nu_n^{(2)} \sqrt[3]{d^2}, \quad (8)\]

for every integer \(z \neq 0, d \) integer not cube, and where \( \nu_n^{(0)}, \nu_n^{(1)}, \nu_n^{(2)} \) are polynomials such that

\[\lim_{n \to \infty} \frac{\nu_n^{(0)}}{\nu_n^{(2)}} = \sqrt[3]{d^2}, \quad \lim_{n \to \infty} \frac{\nu_n^{(1)}}{\nu_n^{(2)}} = \sqrt[3]{d}\]

These ratios generalize the Rédei rational functions [30]. Indeed Rédei rational functions arise from the development of

\[(z + \sqrt{d})^n = N_n(d, z) + D_n(d, z) \sqrt{d},\]

for every integer \(z \neq 0, d \) integer not square, and where

\[N_n(d, z) = \sum_{k=0}^{[n/2]} \binom{n}{2k} d^k z^{n-2k}, \quad D_n(d, z) = \sum_{k=0}^{[n/2]} \binom{n}{2k+1} d^k z^{n-2k-1}.\]

The Rédei rational functions are defined as

\[Q_n(d, z) = \frac{N_n(d, z)}{D_n(d, z)}, \quad \forall n \geq 1.\]

The Rédei rational functions are very interesting and useful tools in number theory. Indeed they are permutation functions of finite fields (see, e.g., [23]) and they can be also used in order to generate pseudorandom sequences [26] or to construct a public key cryptographic system [27]. Moreover in [4] and [1], Rédei rational functions are connected to periodic continued fractions with rational partial quotients convergent to square roots. In [2], Rédei rational functions have been generalized in order to obtain periodic representations of cubic roots. In the next section, we will propose a different generalization of the Rédei rational functions in order to construct periodic ternary continued fractions convergent to cubic irrationalities.

3 The main case

Let \(\alpha\) be a real root of the polynomial \(x^3 - px^2 - qx - r\), with \(p, q, r \in \mathbb{Q}\). Let us consider

\[(z + \alpha^2)^n = \mu_n^{(0)} + \mu_n^{(1)} \alpha + \mu_n^{(2)} \alpha^2, \quad (9)\]

for \(z\) integer number not zero and where the polynomials \(\mu_n^{(i)}\) depends on \(p, q, r, z\) and we will call it Cerruti polynomials, since when \(\alpha = \sqrt[3]{d}\) they

6
are the polynomials \( \nu_n^{(i)} \) introduced the first time in [2]. Let \( N \) be the following fundamental matrix

\[
N = \begin{pmatrix} z & r & pr \\ 0 & q + z & pq + r \\ 1 & p & p^2 + q + z \end{pmatrix}.
\]

(10)

Its characteristic polynomial is

\[
x^3 - \text{Tr}(N)x^2 + \frac{1}{2}(\text{Tr}(N)^2 - \text{Tr}(N^2))x - \det(N),
\]

i.e.,

\[
x^3 - (p^2 + 2q + 3z)x^2 + (q^2 - 2pr + 2p^2 z + 4q z + 3z^2)x - (r^2 + q^2 z - 2p r z + p^2 z^2 + 2q z^2 + z^3).
\]

In the following, we set \( I_1(N) = \frac{1}{2}(\text{Tr}(N)^2 - \text{Tr}(N^2)) \).

**Theorem 2.** Let \( N \) and \( \mu_n^{(i)} \) be the matrix (11) and the Cerruti polynomials above defined.

1. The characteristic polynomial of \( N \) has roots

\[
z + \alpha_1^2, \quad z + \alpha_2^2, \quad z + \alpha_3^2,
\]

where \( \alpha_1, \alpha_2, \alpha_3 \) are the roots of \( x^3 - px^2 - qx - r \).

2.

\[
N^n = \begin{pmatrix} \mu_n^{(0)} & r \mu_n^{(2)} & r \mu_n^{(1)} + pr \mu_n^{(2)} \\ \mu_n^{(1)} & \mu_n^{(0)} + q \mu_n^{(2)} & (pq + r) \mu_n^{(2)} + q \mu_n^{(1)} \\ \mu_n^{(2)} & p \mu_n^{(1)} + \mu_n^{(0)} & \mu_n^{(1)} + (p^2 + q) \mu_n^{(2)} \end{pmatrix}
\]

**Proof.** 1. Considering that

\[
\alpha_1 \alpha_2 \alpha_3 = r, \quad \alpha_1 + \alpha_2 + \alpha_3 = p, \quad \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_1 \alpha_3 = -q,
\]

we have

\[
\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = p^2 + 2q, \quad \alpha_1^2 \alpha_2 + \alpha_2^2 \alpha_3 + \alpha_3^2 \alpha_1 = q^2 - 2pr.
\]

Moreover, expanding the polynomial \((x - (z + \alpha_1^2))(x - (z + \alpha_2^2))(x - (z + \alpha_3^2))\), it is easy to see that the coefficient of \( x^2 \), the coefficient of \( x \), and the constant term are

\[
-(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 3z) = -\text{Tr}(N)
\]

\[
\alpha_1^2 \alpha_2^2 + \alpha_2^2 \alpha_3^2 + \alpha_3^2 \alpha_1^2 + 2z(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + 3z^2 = I_1(N)
\]

\[
-\alpha_1^2 \alpha_2^2 \alpha_3^2 - z(\alpha_1^2 \alpha_2^2 + \alpha_2^2 \alpha_3^2 + \alpha_3^2 \alpha_1^2) - z^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) - z^3 = -\det(N),
\]

respectively.
2. By definition of Cerruti polynomials \([9]\), it follows that \(\mu_{n}^{(i)}\)'s, for \(i = 0, 1, 2\), are linear recurrent sequences of degree 3 whose characteristic polynomial is the minimal polynomial of \(z + \alpha^2\) (where \(\alpha\) real root of \(x^3 - px^2 - qx - r\)), i.e., the characteristic polynomial of \(N\).

Thus, we only have to check the initial conditions. We start from
\[
(z + \alpha^2)^0 = 1,
\]
i.e.
\[
\mu_0^{(0)} = 1, \quad \mu_0^{(1)} = 0, \quad \mu_0^{(2)} = 0
\]
and since
\[
N^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
the initial condition for \(n = 0\) is satisfied. Considering \(z + \alpha^2\), it follows that
\[
\mu_1^{(0)} = z, \quad \mu_1^{(1)} = 0, \quad \mu_1^{(2)} = 1.
\]
Thus,
\[
N = \begin{pmatrix} z & r & pr \\ 0 & q + z & pq + r \\ 1 & p & p^2 + q + z \end{pmatrix} = \begin{pmatrix} \mu_1^{(0)} & r\mu_1^{(2)} & r\mu_1^{(1)} + pr\mu_1^{(2)} \\ \mu_1^{(1)} & \mu_1^{(0)} + q\mu_1^{(2)} & (pq + r)\mu_1^{(2)} + q\mu_1^{(1)} \\ \mu_1^{(2)} & \mu_1^{(1)} + p\mu_1^{(2)} & \mu_1^{(0)} + p\mu_1^{(1)} + (p^2 + q)\mu_1^{(2)} \end{pmatrix}.
\]

Finally,
\[
(z + \alpha^2)^2 = z^2 + 2\alpha^2 z + \alpha^4 = z^2 + pr + (pq + r)\alpha + (p^2 + q + 2z)\alpha^2
\]
and
\[
N^2 = \begin{pmatrix} pr^2 + 2pqr + r^2 + 2prz \\ pq + r & p^2q + q^2 + pr + 2gz + z^2 \\ p^2 + q + 2z & p^3 + 2pq + r + 2pz & p^4 + 3p^2q + q^2 + 2pr + 2p^2z + 2qz + z^2 \end{pmatrix}.
\]

**Theorem 3.** Let \(\alpha\) be a real root largest in modulus of \(x^3 - px^2 - qx - r\) and let \(\alpha_2, \alpha_3\) be the remaining roots. Let \(\mu_{n}^{(i)}\) be the Cerruti polynomials \([9]\), then
\[
\lim_{n \to \infty} \frac{\mu_{n}^{(0)}}{\mu_{n}^{(2)}} = \frac{r}{\alpha}, \quad \lim_{n \to \infty} \frac{\mu_{n}^{(1)}}{\mu_{n}^{(2)}} = \alpha - p,
\]
for any integer \(z\) such that \(z + \alpha^2\) larger in modulus than \(z + \alpha_2^2, z + \alpha_3^2\) and \(\mu_{n}^{(2)} \neq 0\).
Proof. Let
\[ \beta_1 = z + \alpha_2^2, \quad \beta_2 = z + \alpha_2^2, \quad \beta_3 = z + \alpha_3^2 \]
be the roots of the characteristic polynomial of \( N \). By the Binet formula
\[
\begin{align*}
\mu_n^{(0)} &= a_1 \beta_1^n + a_2 \beta_2^n + a_3 \beta_3^n \\
\mu_n^{(1)} &= b_1 \beta_1^n + b_2 \beta_2^n + b_3 \beta_3^n \\
\mu_n^{(2)} &= c_1 \beta_1^n + c_2 \beta_2^n + c_3 \beta_3^n
\end{align*}
\]
where the coefficients \( a_i, b_i, c_i \) can be obtained by initial conditions, solving the system
\[
\begin{align*}
a_1 + a_2 + a_3 &= 1 \\
a_1 \beta_1 + a_2 \beta_2 + a_3 \beta_3 &= z, \\
a_1 \beta_1^2 + a_2 \beta_2^2 + a_3 \beta_3^2 &= pr + z^2
\end{align*}
\]
and similar systems for the \( b_i \)'s and \( c_i \)'s. Since \( \beta_1 \) is larger in modulus than \( \beta_2, \beta_3 \), we are only interested in
\[
\begin{align*}
a_1 &= \frac{\beta_2 \beta_3 - (\beta_2 + \beta_3) + pr + z^2}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)} \\
b_1 &= \frac{(\beta_1 - \beta_2)(\beta_1 - \beta_3)}{2z + p^2 + q - (\beta_2 + \beta_3)} \\
c_1 &= \frac{2z + p^2 + q - (\beta_2 + \beta_3)}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)}
\end{align*}
\]
Now,
\[
\lim_{n \to \infty} \frac{\mu_n^{(1)}}{\mu_n^{(2)}} = \frac{b_1}{c_1} = \frac{pq + r}{2z + p^2 + q - (2z + \alpha_2^2 + \alpha_3^2)} = \frac{pq + r}{\alpha^2 - q}
\]
moreover
\[
(\alpha^2 - q)(\alpha - p) = \alpha^3 - p\alpha^2 - q\alpha + pq = pq + r
\]
and it is proved that
\[
\lim_{n \to \infty} \frac{\mu_n^{(1)}}{\mu_n^{(2)}} = \alpha - p.
\]
The proof that \( \lim_{n \to \infty} \frac{\mu_n^{(0)}}{\mu_n^{(2)}} = \frac{r}{\alpha} \) is left to the reader. \( \square \)

Remark 4. Clearly, it is always possible to find integers \( z \) satisfying the condition of Theorem 3.
Theorem 4. Let $\alpha$ be a real root largest in modulus of $x^3 - px^2 - qx - r$ and $N$ the matrix defined in \(\text{[10]}\), then

$$\begin{align*}
\{z, \frac{2z + p^2 + q}{pq + r}, \frac{(pq + r)\text{Tr}(N)}{\text{det}(N)}, \text{Tr}(N), \frac{\text{Tr}(N)}{pq + r}\}, \\
\{p, \frac{-z^2 + qz + p^2z - pr}{pq + r}, \frac{-I_1(N)}{\text{det}(N)}, \frac{-(pq + r)I_1(N)}{\text{det}(N)}, \frac{-I_1(N)}{pq + r}\} = (\frac{r}{\alpha}, \alpha)
\end{align*}$$

(11)

for any integer $z$ satisfying the hypothesis of Theorem 3.

Proof. By Theorem 3, it is sufficient to prove that

$$\frac{A_n}{C_n} = \frac{\mu_{n+1}^{(0)}}{\mu_{n+1}^{(2)}}, \quad \frac{B_n}{C_n} = \frac{\mu_{n+1}^{(1)}}{\mu_{n+1}^{(2)}} + p, \quad \forall n \geq 0,$$

where $A_n, B_n, C_n$ are the convergents of the ternary continued fraction satisfying Eqs. (11). First of all we prove by induction that

$$A_n = \frac{\mu_{n+1}^{(0)}}{(pq + r)^k \text{det}(N)^{\frac{n+1}{3}}}, \quad \forall n \geq 0,$$

where

$$k = \begin{cases} 1 & n \equiv 1 \pmod{3} \\ 0 & \text{otherwise} \end{cases}$$

The inductive basis is straightforward to prove, indeed

$$A_0 = z, \quad A_1 = \frac{pr + z^2}{pq + r}, \quad A_2 = \frac{p^3r + 2pqr + r^2 + 3prz + z^3}{r^2 + q^2z - 2prz + p^2z^2 + 2qz^2 + z^3}$$

Now, for $n \geq 3$, we consider the cases

$$n \equiv 0 \pmod{3}, \quad n \equiv 1 \pmod{3}, \quad n \equiv 2 \pmod{3}.$$

Let us consider $n \equiv 0 \pmod{3}$, then

$$A_n = \text{Tr}(N)A_{n-1} - \frac{(pq + r)I_1(N)}{\text{det}(N)}A_{n-2} + A_{n-3},$$

by inductive hypothesis we have

$$A_n = \text{Tr} \left( \frac{\mu_{n+1}^{(0)}}{\text{det}(N)^{\frac{n+1}{3}}} \right) - \frac{(pq + r)I_1(N)}{\text{det}(N)} \frac{\mu_{n-1}^{(0)}}{\text{det}(N)^{\frac{n-1}{3}}} + \frac{\mu_{n-2}^{(0)}}{\text{det}(N)^{\frac{n-2}{3}}},$$
Since $n \equiv 0 \pmod{3}$, we have
\[
\left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{n+1}{3} \right\rfloor, \quad \left\lfloor \frac{n-1}{3} \right\rfloor = \left\lfloor \frac{n-2}{3} \right\rfloor = \left\lfloor \frac{n+1}{3} \right\rfloor - 1.
\]

Using the recurrence relation for the Cerruti polynomials, we obtain
\[
A_n = \frac{\text{Tr}(N)\mu^{(0)}_{n+1} - I_1(N)\mu^{(0)}_{n-1} + \det(N)\mu^{(0)}_{n-2}}{\det(N)^{\frac{n+1}{3}}} = \frac{\mu^{(0)}_{n+1}}{(pq + r)^k \det(N)^{\frac{n+1}{3}}}.
\]

Let us consider $n \equiv 1 \pmod{3}$, then
\[
A_n = \frac{\text{Tr}(N)}{pq + r}A_{n-1} - \frac{I_1(N)}{pq + r}A_{n-2} + A_{n-3}
\]
and
\[
\left\lfloor \frac{n+1}{3} \right\rfloor = \left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{n-1}{3} \right\rfloor = \left\lfloor \frac{n-2}{3} \right\rfloor + 1.
\]

Thus, we easily obtain
\[
A_n = \frac{\text{Tr}(N)\mu^{(0)}_{n+1} - I_1(N)\mu^{(0)}_{n-1} + \det(N)\mu^{(0)}_{n-2}}{(pq + r) \det(N)^{\frac{n+1}{3}}} = \frac{\mu^{(0)}_{n+1}}{(pq + r)^k \det(N)^{\frac{n+1}{3}}}.
\]

Similarly when $n \equiv 2 \pmod{3}$. In a similar way, it is possible to prove that
\[
C_n = \frac{\mu^{(2)}_{n+1}}{(pq + r)^k \det(N)^{\frac{n+1}{3}}}, \quad \forall n \geq 0.
\]

Consequently,
\[
\lim_{n \to \infty} \frac{A_n}{C_n} = \lim_{n \to \infty} \frac{\mu^{(0)}_{n+1}}{\mu^{(2)}_{n+1}} = \frac{r}{\alpha}
\]

Finally, let us consider the sequence $(B_n)_{n=0}^{\infty}$. We prove by induction that
\[
B_n = \frac{\mu^{(1)}_{n+1} + p\mu^{(2)}_{n+1}}{(pq + r)^k \det(N)^{\frac{n+1}{3}}}, \quad \forall n \geq 0.
\]

The steps $n = 0, 1, 2$ can be directly checked. Let us consider $n \geq 3$ and $n \equiv 2 \pmod{3}$ (similarly the formula can be proved when $n \equiv 0 \pmod{3}$ and $n \equiv 1 \pmod{3}$). We have
\[
B_n = \frac{(pq + r)\text{Tr}(N)}{\det(N)}B_{n-1} - \frac{I_1(N)}{\det(N)}B_{n-2} + B_{n-3} =
\]

\[
= \frac{(pq + r)\text{Tr}(N)}{\det(N)} \frac{\mu^{(1)}_{n+1} + p\mu^{(2)}_{n+1}}{(pq + r) \det(N)^{\frac{n+1}{3}}} - \frac{I_1(N)}{\det(N)} \frac{\mu^{(1)}_{n-1} + p\mu^{(2)}_{n-1}}{(pq + r) \det(N)^{\frac{n-1}{3}}} + \frac{\mu^{(1)}_{n-2} + p\mu^{(2)}_{n-2}}{(pq + r) \det(N)^{\frac{n-2}{3}}}
\]

\[
= \frac{\mu^{(1)}_{n+1} + p\mu^{(2)}_{n+1}}{(pq + r) \det(N)^{\frac{n+1}{3}}} - \frac{I_1(N)}{\det(N)} \frac{\mu^{(1)}_{n-1} + p\mu^{(2)}_{n-1}}{(pq + r) \det(N)^{\frac{n-1}{3}}} + \frac{\mu^{(1)}_{n-2} + p\mu^{(2)}_{n-2}}{(pq + r) \det(N)^{\frac{n-2}{3}}}
\]
Since
\[
\frac{n + 1}{3} = \frac{n}{3} + 1 = \frac{n - 1}{3} + 1 = \frac{n - 2}{3} + 1
\]
the formula is proved. Thus,
\[
\lim_{n \to \infty} B_n C_n = \lim_{n \to \infty} \frac{\mu^{(1)}_{n+1}}{\mu^{(2)}_{n+1}} + p = \alpha.
\]

The previous theorem provides a periodic representation for all the cubic irrationalities \(\alpha\), such that \(\alpha\) is the root greatest in modulus of its minimal polynomial. This representation is a ternary continued fraction (2) of period 3 whose partial quotients are given by (11). Clearly, this expansion leads to periodic sequences of integer numbers that represent cubic irrationals. This fact is highlighted by using the matricial approach. In particular, the irrationalities \(\left(\frac{r}{\alpha}\right)\) are represented by a periodic product of matrices (5) whose entries are rational numbers, or equivalently by a periodic product of matrices (7) whose entries are integer numbers.

**Example 1.** Let us consider the cubic polynomial \(x^3 - 5x^2 + x - 3\), having a real root
\[
\alpha = \frac{1}{3}(5 + \sqrt[3]{44} + \sqrt[3]{242}) \simeq 4.9207,
\]
greater in modulus than the complex roots. Theorem 4 provides periodic representations of \(\alpha\). If we choose, e.g., \(z = 5\) (which satisfies conditions of Theorem 3), we have
\[
N = \begin{pmatrix}
5 & 3 & 15 \\
0 & 4 & -2 \\
1 & 5 & 29
\end{pmatrix}
\]
and
\[
\left(\frac{3}{\alpha}\right) = \left\{ [5, -17, \frac{19}{141}, 38, -19], [5, 65, \frac{23}{47}, -46, 47, 138] \right\}.
\]
Moreover, we can use (7) in order to express the cubic irrationality \(\alpha\) as a periodic product of matrices with integer entries
\[
\begin{pmatrix}
5 & 1 & 0 \\
5 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-17 & 1 & 0 \\
65 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-893 & 6627 & 0 \\
-3243 & 0 & 6627 \\
6627 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1786 & 47 & 0 \\
46 & 0 & 47 \\
47 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-19 & 1 & 0 \\
138 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]
This periodic representation provides rational approximations for \(\alpha\). For example, the convergents of the ternary continued fraction can be evaluated by using (11) and we obtain rational approximations of \(\frac{3}{\alpha}\) and \(\alpha\), respectively:
\[
\left(\frac{3}{\alpha}\right) = (5, 1.1764, 0.6929, 0.6221, 0.6115, ...)
\]
\[
\alpha = (5, 1.1764, 0.6929, 0.6221, 0.6115, ...)
\]
(5, 84, 1251, 36651, 535575, 17, 254, 7447, 108838) = (5, 4.9412, 4.9252, 4.9216, 4.9208, ...)

These rational approximations can be obviously obtained by the matricial representation. For example if we set $A$ for the matrix of the pre-period (i.e., the matrix product of the two matrices of the pre-period) and $P$ for the matrix of the period (i.e., the matrix product of the three matrices of the period), then

$$AP = \begin{pmatrix} -147028831 & 10234297 & 388784 \\ -1183085175 & 80962059 & 2763459 \\ -240423142 & 16450423 & 561086 \end{pmatrix}$$

and

$$\frac{-147028831}{-240423142} = \frac{66559}{108838}.$$ 

Example 2. Let us consider the cubic polynomial $3x^3 - 12x^2 - 4x + 1$. We can apply Theorem 4 with $p = 4$, $q = 4/3$, $r = -1/3$. Using $z = 1$, we obtain

$$(-\frac{1}{3\alpha}, \alpha) = [(1, \frac{58}{15}, \frac{975}{218}, \frac{65}{3}, \frac{13}{3}), \{4, -\frac{59}{15}, \frac{403}{218}, -\frac{2015}{218}, \frac{403}{45}\}],$$

that is a periodic representation of $\alpha$ root greatest in modulus of $3x^3 - 12x^2 - 4x + 1$.

If we choose, e.g., $z = -1$, we obtain a different periodic ternary continued fraction convergent to $\alpha$:

$$(-\frac{1}{3\alpha}, \alpha) = [(-1, \frac{46}{15}, \frac{47}{8}, \frac{47}{3}, \frac{13}{15}), \{4, 3, \frac{269}{120}, \frac{269}{24}, \frac{269}{45}\}],$$

where $\alpha \simeq 4.29253$. These periodic representations leads to a periodic product of matrices with integer entries convergent to the irrational $\alpha$.

4 The remaining cases

When $\alpha$ is the root largest in modulus of a general cubic polynomial $x^3 - px^2 - qx - r$, a periodic representation for $\alpha$ is provided by Theorem 4. In this section, we treat all the remaining cases of cubic irrationalities.

If $\alpha$ is the root smallest in modulus of $x^3 - px^2 - qx - r$, Theorem 4 does not work. In this case, we consider the reflected polynomial $x^3 + \frac{q}{r}x^2 + \frac{p}{r}x - \frac{1}{r}$ whose roots are $\frac{1}{\alpha}$, $\frac{1}{\alpha_2}$, $\frac{1}{\alpha_3}$. In this way $\frac{1}{\alpha}$ is the root greatest in modulus of $x^3 + \frac{q}{r}x^2 + \frac{p}{r}x - \frac{1}{r}$ and by Theorem 4 we get a periodic representation for the couple $(\frac{1}{r}, \alpha)$. Successively, a periodic representation for $\alpha$ can be derived. We need the following
Theorem 5. Let \([\{a_0, a_1, a_2, a_3, a_4\}, \{b_0, b_1, b_2, b_3, b_4\}]\) be a periodic ternary continued fraction that converges to a couple of real number \((\alpha, \beta)\), then the periodic ternary continued fraction
\[
[(ra_0, ra_1, \frac{a_2}{r^2}, ra_3, ra_4), \{ \frac{b_0}{r}, r^2 b_1, \frac{b_2}{r}, r^2 b_3, \frac{b_4}{r^2} \}]
\]
(12) converges to the couple of real number \((r\alpha, \frac{1}{r\beta})\), for \(r\) rational number.

Proof. Let \(\frac{A_n}{C_n}\) and \(\frac{B_n}{C_n}\) be the \(n\)–th convergents of the ternary continued fraction of \((\alpha, \beta)\). Let \(\tilde{A}_n\) and \(\tilde{C}_n\) be the \(n\)–th convergents of the ternary continued fractions (12), then
\[
\tilde{A}_n = r^{k_1} A_n, \quad \tilde{B}_n = r^{k_2} B_n, \quad \tilde{C}_n = r^{k_3} C_n, \quad \forall n \geq 0
\]
where
\[
k_1 = \begin{cases} 
0, & n \equiv 2 \pmod{3} \\
1, & n \equiv 0 \pmod{3} \\
2, & n \equiv 1 \pmod{3}
\end{cases}, \quad
k_2 = \begin{cases} 
-1, & n \equiv 2 \pmod{3} \\
0, & n \equiv 0 \pmod{3} \\
1, & n \equiv 1 \pmod{3}
\end{cases}
\]
\[
k_3 = \begin{cases} 
-2, & n \equiv 2 \pmod{3} \\
-1, & n \equiv 0 \pmod{3} \\
0, & n \equiv 1 \pmod{3}
\end{cases}
\]
It is straightforward to check these identities for \(n = 0, 1, 2\). Let us proceed by induction, considering an integer \(m \equiv 0 \pmod{3}\). Then
\[
\tilde{A}_m = r a_3 \tilde{A}_{m-1} + \frac{b_3}{r} \tilde{A}_{m-2} + \tilde{A}_{m-3} = r a_3 A_{m-1} + b_3 r A_{m-2} + r A_{m-3} = r A_m.
\]
Similarly when \(m \equiv 1 \pmod{3}\) and \(m \equiv 2 \pmod{3}\), and for the sequences \(\tilde{B}_n\) and \(\tilde{C}_n\).

Thus, if \(\alpha\) is the root smallest in modulus of \(x^3 - px^2 - qx - r\), by Theorem 4 we get the periodic ternary continued fraction of \((\frac{\alpha}{r}, \frac{1}{r\alpha})\) and by Theorem 5 we get the periodic ternary continued fraction of \((\alpha, \frac{1}{r\alpha})\).

Example 3. Let us consider the cubic polynomial \(x^3 - 2x^2 + x + 1\). It has one real root \(\alpha\) whose modulus is smaller than the modulus of the complex roots. Theorem 4 does not work on this polynomial, but we can consider the reflected polynomial \(x^3 + x^2 - 2x + 1\) whose roots are the inverse roots of
Thus, \( \frac{1}{a} \) is the real root of \( x^3 + x^2 - 2x + 1 \) largest in modulus and we can apply Theorem \([3]\). Posing, e.g., \( z = 5 \), we obtain
\[
(-\alpha, \frac{1}{\alpha}) = \left[ \left\{ 5, -\frac{13}{3}, -\frac{20}{87}, 20, -\frac{20}{3} \right\}, \left\{ -1, 13, -\frac{127}{261}, -\frac{127}{87}, -\frac{127}{3} \right\} \right].
\]
Finally, we multiply by -1 this ternary continued fraction and by Theorem \([5]\) we obtain
\[
(\alpha, \frac{1}{-\alpha}) = \left[ \left\{ -5, \frac{13}{3}, -\frac{20}{87}, -20, \frac{20}{3} \right\}, \left\{ 1, 13, \frac{127}{261}, \frac{127}{87}, \frac{127}{3} \right\} \right],
\]
i.e., we found a periodic representation for \( \alpha \) root smallest in modulus of \( x^3 - 2x^2 + x + 1 \).

Now, we are able to determine a periodic representation for any cubic irrational that is the root largest or smallest in modulus of a cubic polynomial \( x^3 - px^2 - qx - r \).

Finally, we treat the last case, i.e., \( \alpha \) is the intermediate root of a cubic polynomial having three real roots. Let \( \alpha_1, \alpha_2, \alpha_3 \) be the real root of \( x^3 - px^2 - qx - r \) such that \( |\alpha_3| < |\alpha_2| < |\alpha_1| \). Using previous techniques we can get periodic expansions for \( \alpha_1 \) and \( \alpha_3 \). Moreover, a rational number \( k \) can be ever found such that \( \alpha_2 \pm k \) is the root largest or smallest in modulus of its minimal polynomial \( (x - (\alpha_1 \pm k))(x - (\alpha_2 \pm k))(x - (\alpha_3 \pm k)) \). The coefficients of \( (x - (\alpha_1 \pm k))(x - (\alpha_2 \pm k))(x - (\alpha_3 \pm k)) \) can be derived from the coefficients of \( x^3 - px^2 - qx - r \) (see, e.g., Th. 10 and Cor. 11 \([5]\)). Thus, by Theorem \([4]\) we know the periodic ternary continued fraction of \( (r \alpha_2 \pm k, \alpha_2 \pm k) \). Then it is immediate to obtain the periodic expansion of \( (r \alpha_2 \pm k, \alpha_2 \pm k) \) (see \([2]\)).

**Example 4.** Let us consider the Ramanujan cubic polynomial \( x^3 + x^2 - 2x - 1 \) with three real roots. We can determine their periodic expansions. The roots of this polynomial are quite famous (see, e.g., \([40]\)) and it is well-known that they are
\[
\alpha_1 = 2 \cos \frac{2\pi}{7}, \quad \alpha_2 = 2 \cos \frac{4\pi}{7}, \quad \alpha_3 = 2 \cos \frac{8\pi}{7},
\]
where
\[
\alpha_1 \simeq 1.24698, \quad \alpha_2 \simeq -0.445042, \quad \alpha_3 \simeq -1.80194.
\]
Thus, from Theorem \([4]\) for \( z = 3 \), we obtain
\[
(\frac{1}{\alpha_3}, \alpha_3) = \left[ \left\{ 3, -9, \frac{2}{13}, 14, -14 \right\}, \left\{ -1, 19, \frac{9}{13}, \frac{9}{13}, 63 \right\} \right].
\]
Considering the polynomial $x^3 + 2x^2 - x - 1$ and $z = 1$, we obtain

$$(\alpha_2, \frac{1}{\alpha_2}) = \begin{bmatrix} 1, & -7, & -\frac{9}{13}, & -9, & \{-2, & -\frac{20}{13}, & 20\} \end{bmatrix}.$$ 

Finally, we can consider the minimal polynomial of

$$\alpha_1 + 1, \quad \alpha_2 + 1, \quad \alpha_3 + 1$$

that is $x^3 - 2x^2 - x + 1$, whose root largest in modulus is $\alpha_1 + 1$. For $z = 2$, by Theorem 4, we obtain

$$\left(\frac{-1}{\alpha_1 + 1}, \alpha_1 + 1\right) = \begin{bmatrix} 2, & \frac{12}{43}, & 12, & 12, \end{bmatrix}, \begin{bmatrix} 2, & -16, & -\frac{41}{43}, & -\frac{41}{43} \end{bmatrix}$$

and

$$\left(\frac{-1}{\alpha_1 + 1}, \alpha_1 \right) = \begin{bmatrix} 2, & \frac{12}{43}, & 12, & 12, \end{bmatrix}, \begin{bmatrix} 1, & -16, & -\frac{41}{43}, & -\frac{41}{43} \end{bmatrix}.$$ 

5 The periodic algorithm

An approach to the Hermite problem contemplates the research of a function whose iteration on algebraic irrationalities provides a periodical algorithm. The partial quotients of the ternary continued fraction (11) can be derived from the Jacobi algorithm (1) using two functions $f^\alpha_z, g^\alpha_z$ instead of the floor function.

**Definition 1.** Let $\alpha$ and $\mathbb{Q}(\alpha)$ be a root of $x^3 - px^2 - qx - r$ and the algebraic extension of $\mathbb{Q}$, respectively. We define the linear functions $f^\alpha_z, g^\alpha_z : \mathbb{Q}(\alpha) \rightarrow \mathbb{Q}$, for $z \in \mathbb{Z}$, such that

1. $f^\alpha_z(q) = g^\alpha_z(q) = q, \quad \forall q \in \mathbb{Q}$
2. $f^\alpha_z\left(\frac{r}{\alpha}\right) = g^\alpha_z\left(\frac{r}{\alpha}\right) = z$
3. $f^\alpha_z(\alpha) = g^\alpha_z(\alpha) = p$
4. $f^\alpha_z(\alpha^2) = 2z + p^2 + 2q, \quad g^\alpha_z(\alpha^2) = z + p^2 + q$

The ternary continued fraction (11) is obtained from the following algorithm:

$$\begin{align*}
a_n &= f^\alpha_z(x_n) \\
b_n &= g^\alpha_z(y_n) \\
x_{n+1} &= \frac{1}{y_n - b_n} \\
y_{n+1} &= \frac{x_n - a_n}{y_n - b_n}
\end{align*}$$ 

(13)
for \( n = 0, 1, 2, \ldots \) and \( x_0 = \frac{r}{\alpha}, y_0 = \alpha \), where \( \alpha \) root of the polynomial
\( x^3 - px^2 - qx - r \).

Let us start to use the algorithm (13) with inputs \((\frac{r}{\alpha}, \alpha)\), we immediately have
\[ a_0 = z, \quad b_0 = p. \]

Now, we evaluate \( x_1 \) and \( y_1 \):
\[ x_1 = \frac{1}{\alpha - p}, \quad y_1 = \frac{r}{\alpha - p}. \]

We need to manipulate \( x_1 \) and \( y_1 \) in order to find the values of \( f_\alpha(x_1) \) and \( g_\alpha(y_1) \). In particular, we will often use that \( \alpha^3 = p\alpha^2 + q\alpha + r \). We have
\[ x_1 = \frac{1}{\alpha - p} \cdot \frac{\alpha^2 - q}{\alpha^2 - q} = \frac{\alpha^2 - q}{pq + r} \]
\[ y_1 = \frac{r}{\alpha - p} \cdot \frac{\alpha^2 - q}{\alpha^2 - q} = \frac{r\alpha - z\alpha^2 + qz - qr}{pq + r}. \]

Now we can apply the properties of \( f_\alpha \) and \( g_\alpha \) and we obtain
\[ a_1 = f_\alpha(x_1) = \frac{2z + p^2 + q}{pq + r}, \quad b_1 = g_\alpha(y_1) = -\frac{z^2 + qz + p^2z - pr}{pq + r}. \]

Let us continue with
\[ x_2 = \frac{1}{y_1 - b_1} = \frac{pq + r}{(\frac{r}{\alpha} - z)(\alpha^2 - q) + (z^2 + qz + p^2z - pr)(\alpha^2 - q)} = \frac{(pq + r)(\alpha^2 + z)}{\det(N)}, \]
where the last identities follow by using \( \alpha^3 = p\alpha^2 + q\alpha + r \). Moreover,
\[ y_2 = (x_1 - a_1) \cdot x_2 = \frac{\alpha^2 - q(\alpha^2 + z) - (2z + p^2 + q)(\alpha^2 + z)}{\det(N)} \]
and
\[ a_2 = \frac{(pq + r)(3z + p^2 + 2q)}{\det(N)} = \frac{(pq + r)\text{Tr}(N)}{\det(N)}, \quad b_2 = -\frac{I_1(N)}{\det(N)}. \]

Then,
\[ x_3 = \frac{\det(N)}{(\alpha^2 - q)(\alpha^2 + z) - (2z + p^2 + q)(\alpha^2 + z) + I_1(N)\alpha^2 + z} = \frac{\det(N)(\alpha^2 + z)}{\det(N)} = \alpha^2 + z. \]
\[ y_3 = \frac{(pq + r)((\alpha^2 + z)^2 - \text{Tr}(N)(\alpha^2 + z))}{\det(N)} \]

and
\[ a_3 = 3z + 2q + p^2 = \text{Tr}(N), \quad b_3 = -\frac{(pq + r)I_1(N)}{\det(N)} \]

Finally,
\[ x_4 = \frac{\det(N)}{(pq + r)((\alpha^2 + z)^2 - \text{Tr}(N)(\alpha^2 + z) + I_1(N))} \cdot \frac{\alpha^2 + z}{\alpha^2 + z} = \frac{\alpha^2 + z}{pq + r} \]

where the last identity is obtained recalling that \( \alpha^2 + z \) is the root of the characteristic polynomial of \( N \).

\[ y_4 = (\alpha^2 + z - \text{Tr}(N)) \cdot \frac{\alpha^2 + z}{pq + r} \]

from which
\[ a_4 = \frac{\text{Tr}(N)}{pq + r}, \quad b_4 = -\frac{I_1(N)}{pq + r} \]

Now we check that \( x_5 = x_2 \) and \( y_5 = y_2 \):
\[ x_5 = \frac{pq + r}{(\alpha^2 + z)^2 - \text{Tr}(N)(\alpha^2 + z) + I_1(N)} = \frac{(pq + r)(\alpha^2 + z)}{\det(N)} = x_2 \]
\[ y_5 = \frac{\alpha^2 + z - \text{Tr}(N)}{pq + r} \cdot \frac{(pq + r)(\alpha^2 + z)}{\det(N)} = y_2. \]

### 6 Conclusions

The Hermite problem has been solved for cubic irrationalities providing a periodic representation via ternary continued fraction with rational partial quotients. A periodic representation involving integer numbers can be directly derived from it. The periodic ternary continued fraction can be obtained from a modification of the classical Jacobi algorithm. In particular a family of algorithms based on Eqs. (13) can be derived. These algorithms become periodic when the input is a couple \((r, \alpha)\), for any cubic irrationality \( \alpha \). The ternary continued fraction \((r, \alpha)\) is very manageable, since the pre-period has length 2, the period has length 3, and it provides simultaneous rational approximations. Of course, many questions and further developments remain open:

- Generalization of Theorem 4 to any algebraic irrationality is the natural development of the present work. Indeed, it seems possible to generalize the whole method presented in this paper.
• The convergence’s rate of the ternary continued fraction \((11)\) has not been studied in the present paper.

• Different values of \(z\) give different ternary continued fractions \((11)\) convergent to the same couple of irrationals. It could be interesting to study the role of \(z\) in order to determine the values for which \((11)\) converges as quickly as possible.

• Functions \(f_\alpha^z, g_\alpha^z\) have been only used in order to determine a periodic algorithm. A deep study of these functions could be interesting.

• The problem of the periodicity of the original Jacobi algorithm is still open. It would be possible to use the present work in order to solve this question.

• Comparison of the present algorithm with other multidimensional continued fractions has not been performed. Moreover, the results of this paper could be used in order to prove or disprove periodicity of other multidimensional continued fractions.

• Cerruti polynomials \((9)\) appear to be very interesting and they could be applied in different fields of number theory. Indeed, they are a generalization of the Rédei rational functions. Since Rédei rational functions are very useful in several fields of number theory, Cerruti polynomials could have many different applications. A deep study of Cerruti polynomials and their properties could be really interesting.

• The present work could be applied on the study of cubic fields and cubic units.

7 Acknowledgment

I would like to thank to Prof. Cerruti for the continuous support of this work and for introducing me to the beauty of mathematics and its problems, like the Hermite problem.

I would like to thank Prof. Ferrarese and Prof. Roggero for their helpful suggestions.

I am grateful to my friends Marianna, Roberta, Simone, and Vanni for the helpful discussions on this work and to my family.

References

[1] M. Abrate, S. Barbero, U. Cerruti, N. Murru, Periodic representations and rational approximations of square roots, Accepted for publication on Journal of Approximation Theory, 2013.
[2] M. Abrate, S. Barbero, U. Cerruti, N. Murru, Periodic representations for cubic irrationalities, The Fibonacci Quarterly, Vol. 50, Issue 3, 252–264, 2012.

[3] S. Assaf, L. C. Chen, T. Cheslack–Postava, B. Cooper, A. Diesl, T. Garrity, M. Lepinski, A. Schuyler, A dual approach to triangle sequences: a multidimensional continued fraction algorithm, Integers: Electronic Journal of Combinatorial Number Theory, 5, A08, 2005.

[4] S. Barbero, U. Cerruti, N. Murru, Solving the Pell equation via Rédei rational functions, The Fibonacci Quarterly, Vol. 48, 348–357, 2010.

[5] S. Barbero, U. Cerruti, N. Murru, Transforming recurrent sequences by using the Binomial and Invert operators, Journal of Integer Sequences, Vol. 13, Article 10.7.7, 2010.

[6] O. R. Beaver, T. Garrity, A two–dimensional Minkowsky ?(x) function, Journal of Number Theory, 107, 105–134, 2004.

[7] L. Bernstein, Periodical continued fractions for irrationals n by Jacobi’s algorithm, J. Reine Angew. Math., 213, 31–38, 1964.

[8] L. Bernstein, Periodicity of Jacobi’s algorithm for a special case of cubic irrationals, J. Reine Angew. Math., 213, 137–147, 1964.

[9] L. Bernstein, New infinite classes of periodic Jacobi–Perron algorithms, Pacific Journal of Mathematics, Vol. 16, No. 3, 439–469, 1965.

[10] L. Bernstein, The Jacobi–Perron algorithm – its theory and application, Lectures Notes in Mathematics, Vol. 207, 1971.

[11] A. J. Brentjes, Multi–dimensional continued fraction algorithms, Mathematical Centre Tracts, Amsterdam, 1981.

[12] A. D. Bryuno, V. I. Parusnikov, Comparison of various generalizations of continued fractions, Mathematical notes, Vol. 61, No. 3, 278–286, 1997.

[13] K. Dasaratha, L. Flapan, T. Garrity, C. Lee, C. Mihaila, N. Neumann–Chun, S. Peluse M. Stoffregen, Cubic irrationals and periodicity via a family of multi–dimensional continued fraction algorithms, Available online at: [http://www.arxiv.org/abs/1208.4244v2] 2013.

[14] P. H. Daus, Normal ternary continued fraction expansions for the cube roots of integers, American Journal of Mathematics, Vol. 44, No. 4, 279–296, 1922.
[15] H. R. P. Ferguson, R. W. Forcade, Generalization of the Euclidean algorithm for real numbers to all dimensions higher than two, Bull. Amer. Math. Soc. 1, No. 6, 912–914, 1979.

[16] T. Garrity, On periodic sequences for algebraic numbers, Journal of Number Theory, Vol. 88, 86–103, 2001.

[17] A. Gupta, A. Mittal, Bifurcating continued fractions, Available online at: [http://front.math.ucdavis.edu/math.GM/0002227], 2000.

[18] M.D. Hendy, N.S. Jeans, The Jacobi–Perron algorithm in integer form, Mathematics of Computation, Vol. 36, No. 154, 565–574, 1981.

[19] C. Hermite, Extraits de lettres de M. Ch. Hermite a M. Jacobi sur differents objets de la theorie des nombres, J. Reine Angew. Math. 40, 286, 1850.

[20] S. Ito, J. Fujii, H. Higashino, S. Yasutomi, On simultaneous approximation to \((\alpha, \alpha^2)\) with \(\alpha^3 + k\alpha - 1 = 0\), Journal of Number Theory, 99, 255–283, 2003.

[21] C. G. J. Jacobi, Ges. Werke, Vol. VI, 385–426, Berlin Academy, 1891.

[22] O. N. Karpenkov, Constructing multidimensional periodic continued fractions in the sense of Klein, Mathematics of Computation, Vol. 78, No. 267, 1687–1711, 2009.

[23] R. Lidl, G. L. Mullen, G. Turnwald, Dickson polynomials, Pitman Monogr. Surveys Pure appl. Math. 65, Longman, 1993.

[24] D. N. Lehmer, On Jacobi’s extension of the continued fraction algorithm, National Academy of Sciences, Vol. 4, No. 12, 360–364, 1918.

[25] G. Martin, The unreasonable effectualness of continued function expansions, J. Aust. Math. Soc., Vol. 77, 305–319, 2004.

[26] N. Murru, Approximations of irrationalities by using linear recurrent sequences, PhD Thesis, University of Turin, 2011.

[27] R. Nobauer, Cryptanalysis of the Rédei scheme, Contributions to General Algebra, 3, 255–264, 1984.

[28] O. Perron, Grundlagen fur eine theorie des Jacobischen kettenbruchalgorithmus, Math. Ann. 64, 1–76, 1907.

[29] N. S. Raju, Periodic Jacobi–Perron algorithms and fundamental units, Pacific Journal of Mathematics, Vol. 64, No. 1, 241–251, 1976.

[30] L. Rédei, Über eindeutige umkehrbare polynome in endlichen korpen, Acta Sci. Math. (Szeged), 11, 85–92, 1946.
[31] F. Schweiger, *The metrical theory of Jacobi–Perron algorithm*, Lectures Notes in Mathematics, Vol. 334, Springer–Verlag, Berlin, 1973.

[32] F. Schweiger, *Multidimensional continued fractions*, Oxford: Oxford University Press, 2000.

[33] F. Schweiger, *Brun meets Selmer*, Integers, 13, A17, 2013.

[34] J. Tamura, S. Yasutomi, *A new multidimensional continued fraction algorithm*, Mathematics of Computation, **78.268**, 2209–2222, 2009.

[35] J. Tamura, S. Yasutomi, *A new algorithm of continued fractions related to algebraic number fields of degree ≤ 5*, Integers, 11B, A16, 2011.

[36] A. Topuzoglu, A. Winterhof, *Topics in Geometry, Coding Theory and Cryptography*, Algebra and Applications, Vol. 6, 135–166, 2006.

[37] T. P. Vaughan, *A note on the Jacobi–Perron algorithm*, Pacific Journal of Mathematics, Vol. 72, No. 1, 261–271, 1977.

[38] Q. Wang, K. Wang, Z. Dai, *On optimal simultaneous rational approximation to (ω, ω^2)^τ with ω being some kind of cubic algebraic function*, Journal of Approximation Theory, **148**, 194–210, 2007.

[39] H. C. Williams, G. W. Dueck, *An analogue of the nearest integer continued fraction for certain cubic irrationalities*, Mathematics of computation, Vol. 42, No. 166, 683–705, 1984.

[40] R. Witula and D. Slota, *New Ramanujan – Type Formulas and Quasi–Fibonacci Numbers of Order 7*, Journal of Integer Sequences, Vol. 10, Article 07.5.6, 2007.