On the Potential of Multi-Mode Antennas for Direction-of-Arrival Estimation

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Abstract—In this paper, we show that a multi-mode antenna (MMA) is an interesting alternative to a conventional phased antenna array for direction-of-arrival (DoA) estimation. By MMA we mean a single physical radiator with multiple ports, which excite different characteristic modes. In contrast to phased arrays, a closed-form mathematical model of the antenna response, like a steering vector, is not straightforward to define for MMAs. Instead one has to rely on calibration measurement or electromagnetic field (EMF) simulation data, which is discrete. To perform DoA estimation, array interpolation technique (AIT) and wavefield modeling (WM) are suggested as methods with inherent interpolation capabilities, fully taking antenna nonidealities like mutual coupling into account. We present a non-coherent DoA estimator for low-cost receivers and show how coherent DoA estimation and joint DoA and polarization estimation can be performed with MMAs. Utilizing these methods, we assess the DoA estimation performance of an MMA prototype in simulations for both 2D and 3D cases. The results show that WM outperforms AIT for high SNR. Coherent estimation is superior to non-coherent, especially in 3D, because non-coherent suffers from estimation ambiguities. In conclusion, DoA estimation with a single MMA is feasible and accurate.

Index Terms—characteristic modes, wavefield modeling, manifold separation, array interpolation technique, RSS, angle-of-arrival

I. INTRODUCTION

MULTI-mode antennas leverage the theory of characteristic modes (TCM), which was originally published in the 1970s [1], [2]. However it tended to be sidelined until the 2000s, when the need to fit antennas into compact handheld devices arose. It has then been realized that for efficient radiation, the dominating characteristic mode of the device chassis has to be excited [3]. The TCM has since then received an increasing amount of attention within the antenna community [4], as it allows to assess the radiation characteristics of defined shapes in terms of individual characteristic modes. An introduction to the concept of characteristic modes can be found in [5], [6]. With multiple-input, multiple-output (MIMO) communication systems becoming popular, the next step in the application of the TCM was to excite multiple characteristic modes. Our definition of a multi-mode antenna (MMA) is thus as a multiport antenna, where different characteristic modes are excited independently, see e.g. [7], [8]. Using MMAs allows to develop compact antennas for MIMO systems. Finally, multiple MMAs can be grouped into an array, forming a multi-mode, multi-element antenna which can serve at a base station for ultra-high data rates [9]. So far, design and application of MMAs was focused on communication applications only.

Contrarily, this paper highlights the usage of MMAs for positioning purposes, specifically for direction-of-arrival (DoA) estimation [10]–[12]. DoA is an essential part for numerous applications like robust global navigation satellite system (GNSS) receivers [13], multipath assisted positioning [14] and channel modeling [15]. While MMAs could be used for all of these applications, highlight two potential applications for DoA where MMAs are especially suited. First, fifth generation (5G) mobile networks are expected to provide high-throughput together with location information as a service [16], [17]. 5G is also envisaged to leverage location information to improve communication [18]. A wideband multi-mode, multi-element antenna, like the one from [9], could be applied at the basestation side. A second application where MMAs are well suited are multi-agent robotic systems [19], which are envisaged e.g. for terrestrial surveillance, disaster management and extra-terrestrial exploration. When it comes to small unmanned aerial vehicles (UAVs) like quadrocopters, stringent size, weight and shape constraints apply, making the design of multi antenna systems challenging. Here the TCM offers a handy tool to use the UV structure for radiation, see e.g. [20], [21]. With a single port antenna, it is only possible to obtain range information by measuring the signal time-of-flight (ToF). It has been shown that angular information, in addition to range information, is very valuable for autonomous navigation of multi-agent robotic systems, as it enables orientation estimation and makes positioning more robust [22]. Going one step further and applying the TCM to construct a multiport antenna, what we call MMA, would allow DoA estimation and thus make angular information available.

The design of antenna arrays for DoA estimation and beamforming is well known [23], while MMAs have not been widely considered for this purpose yet. The antenna response of an MMA cannot be simply described by a steering vector. The plethora of methods known from array signal processing and DoA estimation [24], [25] can thus not be directly applied to MMAs. Instead, one has to rely on either wavefield modeling (WM) and manifold separation or the array interpolation technique (AIT) to model the MMA response. Non-coherent [10] and coherent [11] DoA estimation for MMAs by WM, and coherent DoA estimation by AIT
In this paper we compare the AIT and WM approaches and extend the DoA estimation scheme to include polarization. Moreover we introduce a new non-coherent DoA estimator with reduced complexity and briefly analyze the differences between non-coherent and coherent DoA estimation in terms of ambiguities for a specific MMA design.

The aim of this paper is threefold. First we want to highlight the potential of MMAs for DoA estimation. Second we present suitable methods for determining the DoA with MMAs. And third we analyze the DoA estimation performance using electromagnetic field (EMF) simulation data from an MMA prototype. To this end we present two fundamentally different approaches how an MMA can be modeled for signal processing, see Section II. Both approaches take antenna nonidealities like mutual coupling into account. In Section III we introduce a non-coherent, i.e. received signal strength (RSS) measurement based DoA estimation scheme, aiming at low-cost and low-complexity receivers. In addition to the maximum likelihood (ML) estimator, we also develop a low-complexity alternative. We further present a coherent DoA estimator, which is the standard approach and suitable for e.g. navigation of multi-agent robotic systems or rough GNSS receivers. Finally we extend the DoA estimation approach to jointly estimate the polarization, increasing robustness in case the polarization is unknown. Joint DoA and polarization estimation is also useful for applications like channel modeling. The analysis is based on EMF simulation data from an MMA prototype, whose 3D power pattern can be seen in Figure 1. In Section IV we show the DoA estimation performance for 2D and 3D respectively. Finally in Section V we discuss pros and cons of the two presented antenna response models based on AIT and WM. We also talk about the choice of basis functions for the WM approach. Finally we give hints about the practical implementation of the proposed methods.

Throughout the paper, we use the following notation:

- Vectors are written in bold lowercase letters and matrices in bold capital letters.
- $(\cdot)^T$, $(\cdot)^H$ stands for vector or matrix transpose and conjugate transpose.
- $[A]_{i,j}$ refers to the element in row $i$ and column $j$.
- $||A||$ is the Frobenius norm of matrix $A$.
- $A \odot B$ is the Hadamard-Schur product of matrices $A$ and $B$.
- $A \otimes B$ is the Kronecker product of matrices $A$ and $B$.
- $A^\dagger$ is the Moore-Penrose pseudo-inverse of matrix $A$.
- $\text{tr}\{A\}$ and $\det\{A\}$ are the trace and determinant of matrix $A$.
- $I_N$ is an $N \times N$ identity matrix.
- $1_N = [1,\ldots,1]^T$ is a vector of ones with length $N$.
- $E\{\cdot\}$, $\text{var}\{\cdot\}$ denote expectation and variance.
- $\text{cov}\{\cdot, \cdot\}$ is the covariance matrix.
- $\text{Re}\{\cdot\}$, $\text{Im}\{\cdot\}$ refer to real and imaginary part.
- $\lfloor x \rfloor$ is the floor function returning the greatest integer less than or equal to $x$.

To perform DoA estimation, we require a continuous, closed-form expression for the antenna response
\[
a_m(\theta, \phi) = \sqrt{g_m(\theta, \phi)} e^{j\Phi_m(\theta, \phi)},
\]
for antenna port $m = 1,\ldots,M$, antenna gain $g_m(\theta, \phi)$ and antenna phase response $\Phi_m(\theta, \phi)$ \cite{23}. Inclination angle $\theta$ and azimuth angle $\phi$ are visualized in Figure 2. The antenna response vector for $M$ ports is defined as
\[
a(\theta, \phi) = [a_1(\theta, \phi) \ldots a_M(\theta, \phi)]^T.
\]

Please note that $a(\theta, \phi)$ represents in general a non-linear vector function taking all effects into account. For MMAs, in contrast to ideal antenna arrays consisting of isotropic antennas, it is not straightforward to find an analytical expression. Instead, the starting point for determining $a(\theta, \phi)$ for an MMA are spatial samples of the antenna response given by
\[
e_q = [e_{q,1} \ldots e_{q,M}]^T
\]
for a specific sampling point $\{\theta_q, \phi_q\}$. For the entire sphere this extends to $E = [e_1,\ldots,e_Q]$ with $Q$ total sampling points. The spatial samples obtained by antenna calibration measurement or EMF simulation are inherently discrete, hence an interpolation strategy is needed. The goal is to find a closed-form expression for $a(\theta, \phi)$ such that
\[
a(\theta_q, \phi_q) \approx e_q, \quad \forall q \in \{1,\ldots,Q\}
\]
holds and $a(\theta, \phi)$ is continuous in $\theta$ and $\phi$. 

[Fig. 1. 3D MMA power pattern for right hand circular polarization (RHCP).]
A. Array Interpolation Technique

The idea of AIT is to model \( a(\theta, \phi) \) as
\[
a(\theta, \phi) = H a_{\text{ideal}}(\theta, \phi)
\]
being a linear transformation of the response of a virtual, ideal array \( a_{\text{ideal}}(\theta, \phi) \). The linear transformation is described by the interpolation matrix \( H \in \mathbb{C}^{M \times M} \). AIT was first proposed in [26] and has been extended in e.g. [27], [28]. Defining \( A_{\text{ideal}} = [a_{\text{ideal}}(\theta, \phi_1), \ldots, a_{\text{ideal}}(\theta, \phi_N)] \), the mapping can be found by solving the optimization problem
\[
\{ \hat{H}, a_{\text{ideal}}(\theta, \phi) \} = \arg \min_{H, a_{\text{ideal}}(\theta, \phi)} \left\| H^H A_{\text{ideal}} - E \right\|^2.
\]
(6)

Optimizing \( a_{\text{ideal}} \) means moving the elements of the virtual array to an optimum position. Although this can be done, their position is often chosen heuristically [29]. By that the optimization problem in (6) simplifies to an optimization of \( H \).

B. Wavefield Modeling and Manifold Separation

Another possibility to perform the interpolation is by building on a technique called wavefield modeling and manifold separation [31], [32]. The key finding here is that the antenna response vector is modeled as
\[
a(\theta, \phi) = G b(\theta, \phi) \in \mathbb{C}^M
\]
and can be decomposed into a product of the sampling matrix \( G \in \mathbb{C}^{M \times U} \), which is independent of the wavefield, i.e. the DoA, and the basis vector \( b(\theta, \phi) \in \mathbb{C}^U \), which is independent of the antenna [31]. This decomposition requires the \( U \) basis functions to be orthonormal on the antenna manifold \( \theta \in [-\pi, \pi) \) for 2D or \( \theta \in [0, \pi], \phi \in [0, 2\pi) \) for 3D respectively. For 2D we assume a cut through the x-z-plane. The antenna response vector \( a(\theta, \phi) \) must also be square integrable on the manifold. A suitable basis for 2D is given by the Fourier functions
\[
b(\theta) = \frac{1}{\sqrt{2\pi}} \left[ e^{i\theta} \cos \left( \frac{\pi \phi}{2} \right) \quad e^{i\phi} \right]^T.
\]
(14)

For 3D the spherical harmonic functions
\[
Y^m_l(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(|l|+m)!}} P^m_l(\cos(\theta)) e^{im\phi},
\]
(15)

with degree \( l \in \{0, \ldots, L\} \) for maximum degree \( L \) and order \( m \in \{-l, \ldots, l\} \) fulfill the orthonormality property [33]. Please note that we use \( l \) and \( m \) here to be consistent with the literature, \( m \) is not to be confused with the antenna index utilized in the rest of this paper. \( P^m_l(\cdot) \) is the associated Legendre polynomial, see [13] in Appendix A. Defining \( Y_u(\theta, \phi) \) analogous to \( Y^m_l(\theta, \phi) \) with the enumeration \( u = (l+1)l + m + 1 \) for \( u = 1, \ldots, U \), we can form a basis
\[
b(\theta, \phi) = [Y_1(\theta, \phi) \quad \ldots \quad Y_U(\theta, \phi)]^T.
\]
(16)
Another choice would be the 2D Fourier functions
\[
\begin{align*}
\mathbf{b}(\theta) &= \frac{1}{\sqrt{2\pi}} \begin{bmatrix} e^{i\theta_1} & \ldots & e^{i\theta_{N-1}} \end{bmatrix}^T, \\
\mathbf{b}(\phi) &= \frac{1}{\sqrt{2\pi}} \begin{bmatrix} e^{i\phi_1} & \ldots & e^{i\phi_{N-1}} \end{bmatrix}^T. 
\end{align*}
\]
leading to the basis vector
\[
\mathbf{b}(\theta, \phi) = \mathbf{b}_\theta(\theta) \otimes \mathbf{b}_\phi(\phi).
\]

We chose this definition, essentially limiting \( U \) to square numbers, to unify the definition of \( U \) for (16) and (18). In practice, this limitation is not necessary and the number of coefficients in \( \theta \) and \( \phi \) domain may also be different. The approach (18) is likewise called effective aperture distribution function (EADF) [34]. The 2D Fourier functions are orthonormal on the torus, not on the sphere, i.e. the data has to be expanded to be periodic in both inclination and azimuth [32]. We provide a short discussion on the choice of basis functions in Section V.

In [31] it is shown that when \( u \) is increased, \(|G|_{m,u}\) decays superexponentially for \( u > 2\pi\lambda_c^{-1}R_s \), where \( R_s \) is the radius of the smallest sphere enclosing the antenna. Therefore a finite order \( U \) is sufficient to obtain an accurate representation of the antenna response vector. In practice, \( U \) can be adjusted according to the noise floor of the calibration measurements [34]. Using \( \mathbf{E} \), we can determine the sampling matrix \( \mathbf{G} \) for a given basis \( \mathbf{B} = [\mathbf{b}(\theta_1, \phi_1), \ldots, \mathbf{b}(\theta_Q, \phi_Q)] \) by least squares
\[
\hat{\mathbf{G}} = (\mathbf{BB}^H)^{-1} \mathbf{BE}^T.
\]

Once \( \hat{\mathbf{G}} \) has been found, the interpolation can be performed by (13). For basis functions (14) and (18), (19) is equivalent to performing a Fourier transform.

### C. Antenna Characteristics

To illustrate the interpolation process, we show 2D cuts of both power, Figure 3, and phase patterns, Figure 4 of our MMA prototype for a fixed polarization. The figures present both discrete EMF simulation data and interpolated patterns with AIT and WM. For AIT we assume a virtual ULA with \( \lambda_c/4 \) spacing and four elements. The manifold is divided into \( 30^\circ \) sectors with \( 15^\circ \) overlap, yielding 11 matrices \( \mathbf{H}^{(e)} \in \mathbb{C}^{4 \times 4} \), i.e. 44 weighting factors per antenna port. As basis for WM we use Fourier functions (14) with \( U = 75 \) coefficients per port. The basis for our analysis is noise-free data obtained by EMF simulation, therefore we choose a large \( U \) to achieve exact interpolation. When measurement data from an anechoic chamber is used, \( U \) can be significantly reduced, as measurement data is always noisy. It can be seen from Figures 3 and 4 that both AIT and WM accurately interpolate in power and phase domain. For AIT, a slight deviation for low elevations, i.e. \( |\theta| > 70^\circ \), is visible. In Section IV-A both approaches are compared in terms of their impact on DoA estimation. Figure 1 shows the interpolated power pattern of the MMA in 3D. Obviously the different ports of the MMA have distinct characteristics. These are utilized by the signal processing schemes presented in the next section to estimate the DoA of incoming signals.

### III. DOA ESTIMATION WITH MULTI-MODE ANTENNAS

In this section we derive estimators for non-coherent and coherent DoA estimation as well as joint DoA and polarization estimation. Non-coherent means that the receiver has only knowledge of RSS measurements, i.e. power of the received signal. We also provide a fundamental limit in terms of the Cramér-Rao bound (CRB) for every estimator. All equations are given for 3D, i.e. azimuth and inclination. Simplification to 2D is straightforward. Using the non-coherent approach, only a single signal can be estimated. Coherent estimation allows to distinguish between multiple overlaying signals [35]. How many signal parameters can be uniquely identified depends on the number of antenna ports, the antenna response, and the correlation of the signals. If the signals are uncorrelated and any subset of \( P \) antenna response vectors is linearly independent, \( P < M \) signals can be identified.

#### A. Non-Coherent DoA Estimation

The sampled signal \( r(n) = [r_1(n), \ldots, r_M(n)]^T \) received at the \( M \) ports of the MMA is given by
\[
r(n) = \mathbf{a}(\theta, \phi)s(n) + \mathbf{w}(n),
\]
where \( \mathbf{a}(\theta, \phi) \) is the antenna response vector and \( s(n) \) is the signal vector.
where \( a(\theta, \phi) \) is the antenna response vector, \( s(n) \) is the arriving baseband signal and \( w(n) \sim \mathcal{C}\mathcal{N}(0, \sigma^2 I_M) \) is i.i.d. white circular symmetric Gaussian noise with covariance matrix \( \sigma^2 I_M \). Assuming stationarity, the RSS estimate of port \( m \), time-averaged over \( N \) samples, can be calculated by

\[
\hat{r}_m = \frac{1}{N} \sum_{n=1}^{N} |r_m(n)|^2.
\]

(21)

For the first receiver type under consideration, being non-coherent, we assume that only RSS measurements \( \hat{r} = [\hat{r}_1, ..., \hat{r}_M]^T \) instead of the actual received signals \( r(n) \) are available. In Appendix B we show that for large \( N \), the RSS measurements \( \hat{r} \) can be well approximated by a Gaussian distribution \( \hat{r} \sim \mathcal{N}(\tilde{\mu}, \Sigma) \) with mean

\[
\tilde{\mu} = E\{\hat{r}\} = g(\theta, \phi)\hat{s} + \mu M \sigma^2,
\]

(22)

where \( g(\theta, \phi) = [g_1(\theta, \phi), ..., g_M(\theta, \phi)]^T \) is the antenna gain vector, \( \hat{s} = \frac{1}{N} \sum_{n=1}^{N} |s(n)|^2 \) is the signal power and \( \sigma^2 = \frac{1}{N} \sum_{n=1}^{N} E\{\|w_m(n)\|^2\} \) the noise power. The entries of the diagonal covariance matrix \( \Sigma = \text{cov}\{\hat{r}, \hat{r}\} \) are given by

\[
|\Sigma|_{m,m} = N^{-1} \sigma^4 + 2 N^{-1} \sigma^2 \hat{s} g_m(\theta, \phi).
\]

(23)

Equations (22) and (23) do not contain the antenna response, but only the antenna gain. Instead of performing the expansion based on the complex antenna response vector \( \theta \), we can expand the real antenna gain vector

\[
g(\theta, \phi) = GB(\theta, \phi) \in \mathbb{R}^M.
\]

(24)

The basis \( B(\theta, \phi) \) for Fourier functions is defined analogously to (14) or (18), where the negative coefficients are fixed to the complex conjugate of the positive ones. For spherical harmonic functions, the real-valued form for defining the basis \( B(\theta, \phi) \) and its derivatives can be found in Appendix B. We define the signal-to-noise ratio (SNR) with respect to an isotropic antenna with unit gain, i.e.

\[
\text{SNR} = \frac{\hat{s}}{\sigma^2}.
\]

(25)

First we consider the general case where both signal power \( \hat{s} \) and noise power \( \sigma^2 \) are unknown. The set of parameters to be estimated is then defined by

\[
\zeta = [\theta \ \phi \ \hat{s} \ \sigma^2]^T.
\]

(26)

Neglecting the constant terms, the log-likelihood function is given by

\[
\ln p_r(\hat{r}|\zeta) = -\ln \left( \det\{\Sigma\}\right) - (\hat{r} - \tilde{\mu})^T \Sigma^{-1} (\hat{r} - \tilde{\mu}),
\]

(27)

which leads to the non-coherent ML estimator (NC-ML)

\[
\hat{\zeta} = \arg \max_{\zeta} \ln p_r(\hat{r}|\zeta).
\]

(28)

The variance of any unbiased estimator is lower bounded by the CRB [36]. The CRB for the non-coherent case (NC-CRB) is given by

\[
\begin{align*}
\text{var} \{\hat{\theta}\} & \ge \text{CRB}(\hat{\theta}) = [I(\zeta)]_{1,1}, \\
\text{var} \{\hat{\phi}\} & \ge \text{CRB}(\hat{\phi}) = [I(\zeta)]_{2,2}.
\end{align*}
\]

(29a, 29b)

with the elements of the Fisher information matrix \( I(\zeta) \in \mathbb{R}^{4\times4} \) defined as [36]

\[
[I(\zeta)]_{i,j} = \frac{\partial \tilde{\mu}^T}{\partial \zeta_i} \Sigma^{-1} \frac{\partial \tilde{\mu}}{\partial \zeta_j} + \frac{1}{2} \text{tr} \left\{ \Sigma^{-1} \frac{\partial \Sigma}{\partial \zeta_i} \Sigma^{-1} \frac{\partial \Sigma}{\partial \zeta_j} \right\}.
\]

(30)

However solving a non-linear optimization problem with four unknowns is unfavorable for a low-cost and low-complexity receiver. Therefore we present a reduced complexity (RC) alternative to the ML estimator (28). In practice, the noise power \( \sigma^2 \) can often be estimated separately, e.g. from unoccupied time-division multiple access (TDMA) slots. The unknowns then reduce to

\[
\zeta' = [\theta \ \phi \ \hat{s} ]^T.
\]

(31)

Neglecting the log-term in (27) and maximizing the other term we arrive at

\[
\hat{\zeta}' = \arg \min_\zeta ||\hat{r}' - g(\theta, \phi)^T \hat{s}||^2,
\]

(32)

with \( \hat{r}' = \hat{r} - \hat{s}^2 \). Following the principle from [37], we plug in the least-squares estimate \( \hat{s} = g(\theta, \phi)^T \hat{r}' \) and obtain the RC estimator (NC-RC)

\[
\{\hat{\theta}, \hat{\phi}\} = \arg \min_{(\theta, \phi)} || (I_M - g(\theta, \phi)g(\theta, \phi)^T ) \hat{r}' ||^2
\]

(33)

where the term in round brackets is idempotent. The complexity has been reduced from four to two unknowns. The CRB for the noncoherent RC estimator (NC-RC-CRB) is also given by (29), but with the reduced unknown vector (31). For an efficient implementation, \( (I_M - g(\theta, \phi)g(\theta, \phi)^T ) \) can be precomputed for a \( \theta-\phi \) grid with required accuracy.

B. Coherent DoA Estimation

The signal model for coherent DoA estimation,

\[
r(n) = A(\theta, \phi)s(n) + w(n),
\]

(34)

is based on (20), but is more general since it covers not only one but \( P \) signals \( s(n) = [s_1(n), ..., s_P(n)]^T \) arriving from different angles \( \{\theta_1, \phi_1\}, ..., \{\theta_P, \phi_P\} \), i.e. we have \( \theta = [\theta_1, ..., \theta_P]^T, \phi = [\phi_1, ..., \phi_P]^T \). The antenna response vector \( a(\theta, \phi) \) then becomes a matrix,

\[
A(\theta, \phi) = [a(\theta_1, \phi_1) \ ... \ a(\theta_P, \phi_P)]
\]

(35)

Based on \( N \) received signal samples we can calculate the sample covariance matrix

\[
\hat{R}_r = \frac{1}{N} \sum_{n=1}^{N} r(n)r^H(n).
\]

(36)

From the array processing literature [36] it is known that the log-likelihood function, ignoring constant terms, is given by

\[
\ln p_r(r|\theta, \phi) = - N M \ln(\sigma^2)
\]

(37)

\[
- \frac{1}{\sigma^2} \sum_{n=1}^{N} ||r(n) - A(\theta, \phi)s(n)||^2,
\]
which leads to the coherent ML estimator (C-ML)

$$\{\hat{\theta}, \hat{\phi}\} = \arg \min_{\{\theta, \phi\}} \text{Re}\{\text{tr}\{\Pi^*_A \hat{R}_r}\}\}, \quad (38)$$

with the projector onto the noise subspace $\Pi^*_A = \mathbb{I}_M - A'(\theta, \phi)A(\theta, \phi)$. In [39] it is shown that the CRB matrix for the coherent case (C-CRB) can be calculated as

$$\text{CRB} \left( \frac{\hat{\theta}^T}{\hat{\phi}^T} \right) = \frac{\sigma^2}{2N} \text{Re}\{D^H \Pi^*_A D \odot Z^T R_s Z\}^{-1}, \quad (39)$$

with $D = \left[ \frac{\partial a(\theta_1, \phi_1)}{\partial \theta_1} \ldots \frac{\partial a(\theta_P, \phi_P)}{\partial \theta_P} \right] \left[ \frac{\partial a(\theta_1, \phi_1)}{\partial \phi_1} \ldots \frac{\partial a(\theta_P, \phi_P)}{\partial \phi_P} \right] \quad (40)$

and the selection matrix $Z = [\mathbb{I}_P \mathbb{I}_P]$ and $R_s = \frac{1}{N} \sum_{n=1}^N s(n)s^H(n)$. Depending on which method is used, the derivatives of steering vectors (7) and (10) for AIT or Fourier [14] and 2D Fourier functions [18] for WM are trivial. For WM with spherical harmonics, derivatives of [15] are provided in Appendix A for convenience of the reader.

C. Joint DoA and Polarization Estimation

Different parameterizations describing polarization parameters of electromagnetic waves exist. We use the auxiliary angle $\gamma$ with $0 \leq \gamma \leq \frac{\pi}{2}$ and the polarization phase $\beta$ with $-\pi \leq \beta < \pi$ as parameters of the polarization ellipse. As our MMA prototype has different polarizations on different ports, we can apply methods from diversely polarized array processing [41], [45]. We define partial antenna response vectors for a single signal and antenna port $m$

$$a_{co,m}(\theta, \phi) = \sqrt{g_{co,m}(\theta, \phi)} e^{j\Phi_{co,m}(\theta, \phi)}, \quad (41a)$$

$$a_{cross,m}(\theta, \phi) = \sqrt{g_{cross,m}(\theta, \phi)} e^{j\Phi_{cross,m}(\theta, \phi)}, \quad (41b)$$

where $a_{co,m}(\theta, \phi)$ is the antenna response when being illuminated by a wave with the reference polarization with DoA $\{\theta, \phi\}$, while $a_{cross,m}(\theta, \phi)$ results from a wave with orthogonal polarization. Correspondingly $g_{co,m}(\theta, \phi)$ and $g_{cross,m}(\theta, \phi)$ are the partial gains and $\Phi_{co,m}(\theta, \phi)$ and $\Phi_{cross,m}(\theta, \phi)$ the partial phase responses. Forming the partial antenna response vectors

$$a_{co}(\theta, \phi) = [a_{co,1}(\theta, \phi) \ldots a_{co,M}(\theta, \phi)]^T, \quad (42a)$$

$$a_{cross}(\theta, \phi) = [a_{cross,1}(\theta, \phi) \ldots a_{cross,M}(\theta, \phi)]^T, \quad (42b)$$

the polarimetric antenna response vector is given by

$$a(\theta, \phi, \gamma, \beta) = \sin(\gamma) e^{j\beta} a_{co}(\theta, \phi) + \cos(\gamma) a_{cross}(\theta, \phi). \quad (43)$$

Defining $\gamma = [\gamma_1, \ldots, \gamma_P]^T$ and $\beta = [\beta_1, \ldots, \beta_P]^T$ for $P$ arriving signals, we can construct the antenna response matrix $A(\theta, \phi, \gamma, \beta) = [a(\theta_1, \phi_1, \gamma_1, \beta_1) \ldots a(\theta_P, \phi_P, \gamma_P, \beta_P)]$ and extend the signal model (34) to

$$r(n) = A(\theta, \phi, \gamma, \beta) s(n) + w(n). \quad (44)$$

Similar to (38), the polarimetric ML estimator (P-ML) is given by

$$\{\hat{\theta}, \hat{\phi}, \hat{\gamma}, \hat{\beta}\} = \arg \min_{\{\theta, \phi, \gamma, \beta\}} \text{Re}\{\text{tr}\{\Pi^*_A \hat{R}_r\}\}. \quad (45)$$

It is assumed that the polarization parameters are stationary during the observation time. The CRB matrix (39) is extended for the polarimetric CRB (P-CRB) to

$$\text{CRB} \left[ \frac{\hat{\theta}}{\hat{\phi}} \frac{\hat{\gamma}}{\hat{\beta}} \right] = \frac{\sigma^2}{2N} \text{Re}\{D^H \Pi^*_A D \odot Z^T R_s Z\}^{-1}. \quad (46)$$

Setting $a_p = a(\theta_p, \phi_p, \gamma_p, \beta_p)$,

$$D = \left[ \ldots \frac{\partial a_p}{\partial \theta_p} \ldots \frac{\partial a_p}{\partial \phi_p} \ldots \frac{\partial a_p}{\partial \gamma_p} \ldots \frac{\partial a_p}{\partial \beta_p} \right] \quad (47)$$

is defined analogously to (40) for $p = 1, ..., P$ and $Z = [\mathbb{I}_P \mathbb{I}_P \mathbb{I}_P]$. For $P = 1$.

The estimators introduced in this section, as well as their corresponding error bounds in terms of the CRB, are summarized in Table I. Each of them requires the application of either the AIT, see Section II-A, or WM, see Section II-B.

| Signal model   | Estimator | CRB       |
|---------------|-----------|-----------|
| Non-coherent  | NC-ML     | NC-CRB    |
| Coherent      | C-ML      | C-CRB     |
| Polarimetric  | P-ML      | P-CRB     |

IV. PERFORMANCE ANALYSIS

A. 2D DoA Estimation

Simulations have been performed to assess the DoA estimation performance using the MMA prototype from [9] with the aim to compare AIT and WM and the different estimation schemes. The received signals are generated based on the original EMF simulation data with a 5° grid. The
number of samples used is always \( N = 1000 \) and we evaluate the DoA estimation performance in terms of
\[
\sqrt{\text{MSE}(\theta)} = \sqrt{\frac{1}{N_{mc}} \sum_{n_{mc}=1}^{N_{mc}} (\hat{\theta}_{n_{mc}} - \theta)^2}
\]
for \( N_{mc} = 1000 \) Monte Carlo runs.

First we want to analyze the DoA estimation performance using AIT, see Section II-A, and WM, see Section II-B, as models for the MMA response vector. Figure 5 shows the root-mean-square error (RMSE) for coherent and non-coherent estimation with both models depending on the DoA \( \theta \). Using AIT, the RMSE for non-coherent and coherent estimation is close to the CRB for the main beam of the antenna. For lower elevations however, i.e. \( \theta \geq 30^\circ \), AIT suffers from additional errors due to a model mismatch between true and approximated MMA response. Applying WM, the RMSE is always equal to the CRB. Therefore, when WM is applied with a sufficient number of coefficients, it is able to perfectly interpolate the antenna response at the provided spatial sampling points.

In Figure 6 we show the RMSE for coherent and non-coherent estimation with AIT and WM, averaged over \( \theta \in [-90^\circ, 90^\circ] \). The received signals are again based on the original EMF data. In the lower SNR regime, AIT and WM show similar performance. Contrarily for high SNR, WM attains the respective CRB, whereas using AIT, the RMSE does not drop below an error floor of 0.5° for non-coherent and 0.3° for coherent estimation. As stated already in the last paragraph, AIT suffers from additional errors in the high SNR domain due to a model mismatch between true and approximated MMA response. From here on we present only results using WM, as it achieves exact interpolation with the employed number of coefficients. Nevertheless AIT is also a suitable approach for many applications, see the discussion in Section V.

Figure 7 shows the RMSE for 2D DoA estimation with the non-coherent, i.e. RSS based, and coherent estimators. For the given SNR of 20 dB, the non-coherent ML estimator attains the corresponding CRB. The CRB, however, i.e. the achievable estimator performance, depends strongly on \( \theta \). For the simulation of the RC estimator, \( \hat{\sigma}^2 \) was estimated from \( N = 1000 \) samples where no signal was present, corresponding to e.g. an unoccupied TDMA slot. The RMSE of the RC is mostly close to the corresponding CRB; except around \( \theta = 0^\circ \). There the approximations leading from (27) to (32) cause additional errors, likely due to large gain differences between the ports. The coherent ML estimator fully attains the CRB and the performance is relatively constant over \( \theta \) compared to the non-coherent ML estimator.

In Figure 8 the RMSE is averaged over \( \theta \in [-90^\circ, 90^\circ] \) and plotted versus SNR. According to Figure 3 the MMA power pattern is relatively symmetric with respect to \( \theta = 0^\circ \), thus the discrimination between positive and negative values of \( \theta \) is handicapped. Figure 8 shows that the non-coherent estimator for \( \theta \in [-90^\circ, 90^\circ] \) has much higher errors in the low SNR regime compared to the case \( \theta \in [0^\circ, 90^\circ] \). Above
approximately 13 dB SNR both curves are close to the CRB. The coherent estimator attains its CRB above SNR = 7 dB. For lower SNR, the difference between \( \theta \in [-90^\circ, 90^\circ] \) and \( \theta \in [0^\circ, 90^\circ] \) is much smaller compared to the non-coherent case. The coherent estimator can efficiently use the phase response of the MMA, see Figure 4, to distinguish between positive and negative \( \theta \) and thus suffers less from estimation ambiguities.

Figure 9 shows the RMSE averaged over \( \theta = [-90^\circ, 90^\circ] \) versus the SNR for the non-coherent ML and RC estimator. For the RC estimator, \( \hat{\sigma}^2 \) was again estimated from \( N = 1000 \) noise samples. The performance of ML and RC estimator is similar, however ML is close to its CRB, whereas RC is not. The RC can even slightly outperform ML due to better knowledge of \( \sigma^2 \), but as Figure 7 shows this highly depends on \( \theta \).

**B. 3D DoA Estimation**

For the 3D section we use WM with 2D Fourier functions \( (18) \) as basis and \( U = 5182 \) coefficients after periodic expansion to generate the received signals, which allows to show results for a finer \( \theta \) and \( \phi \) grid of \( 1^\circ \) compared to the original EMF simulation data with \( 5^\circ \) grid. In Figure 10 the simulated RMSE and the corresponding CRB for \( \theta \) and \( \phi \), i.e. 3D, non-coherent DoA estimation are shown. Similar to Figure 7 for the 2D case, the CRB shown in Figure 10a for 3D varies depending on \( \theta \) and \( \phi \). Comparing the simulated RMSE of the non-coherent ML estimator in Figure 10b with its corresponding CRB in Figure 10a we see that for \( \theta < 60^\circ \), the CRB is mostly attained, except for some distinct points which are visible as red or black dots in the plot. For larger \( \theta \), excessive estimation errors occur. This can be explained by the antenna power pattern, see Figures 11 and 3. For \( \theta \) approaching \( 90^\circ \), the antenna gain is very low, leading to a low SNR.

Figure 11 shows RMSE and CRB for 3D coherent DoA estimation. As expected, the CRB in Figure 11a is lower and more uniform compared to the non-coherent case in Figure 10a. Visually comparing the plots for the simulated RMSE of the coherent estimator, see Figure 11b with its corresponding CRB, see Figure 11a we observe that the two figures appear almost identical. This implies that the RMSE of the coherent estimator meets the CRB. Only for very low elevations, i.e. \( \theta \) close to \( 90^\circ \), we see darker colors in Figure 11b. Here the estimator RMSE is higher than the CRB. As before, the reason can be found in the antenna power pattern.

Figure 12 shows the log-likelihood functions for the coherent \( (37) \) and non-coherent estimator \( (27) \) for a fixed DoA.
This paper addresses the question how multi-mode antennas (MMAs) can be used for DoA estimation. We define an MMA as a multiport antenna, where different characteristic modes are excited independently. MMAs have so far been designed and investigated only for communications, while their potential for positioning has not been leveraged. To enable DoA estimation with MMAs, we present two suitable ways, based on either array interpolation technique (AIT) or wavefield modeling (WM). Both fully take antenna nonidealities like mutual coupling into account. We further show how non-coherent, i.e. RSS based, coherent and joint DoA and polarization estimation can be carried out. Based on EMF simulation data, we perform extensive simulations in both 2D and 3D to assess the expected performance. We compare AIT and WM in terms of DoA estimation performance and show that WM has an advantage in the high SNR regime. For low-cost and low-complexity receivers, non-coherent DoA estimation based on RSS measurements is possible. However it suffers from estimation ambiguities, especially in the 3D case, and thus requires a relatively high SNR for accurate results. The standard coherent approach does not suffer from this problem and performs better. The coherent receiver achieves sub-degree accuracy for a 2D scenario with an SNR above 5 dB, whereas the non-coherent one requires at least 14 dB. As the investigated MMA prototype features diverse polarizations, we also show that the polarization parameters of the incoming wave can be estimated. In conclusion, DoA estimation with MMAs is both feasible and accurate. MMAs thus offer an appealing alternative to conventional antenna arrays, especially in applications with tight shape constraints.
Appendix A

Legendre Polynomials and Derivatives of Complex Spherical Harmonics

The spherical harmonics \( Y_l^m(\theta, \phi) \) with degree \( l \) and order \( m \) can be calculated with the associated Legendre polynomial \( P_l^m(x) \) by (48).

\[
P_l^m(x) = (-1)^m \frac{(1-x^2)^{m/2} d^m}{dx^m} P_l(x)
\]

and the Legendre polynomial

\[
P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l.
\]

The partial derivatives of the spherical harmonics with respect to \( \theta \) and \( \phi \) are

\[
\frac{\partial}{\partial \theta} Y_l^m(\theta, \phi) = m \cot(\theta) Y_l^m(\theta, \phi) + \sqrt{(l - m)(l + m + 1)} e^{-j\phi} Y_{l+1}^{m+1}(\theta, \phi),
\]

\[
\frac{\partial}{\partial \phi} Y_l^m(\theta, \phi) = j m Y_l^m(\theta, \phi).
\]

Appendix B

Real Spherical Harmonics and Their Derivatives

The real version of the spherical harmonics, which can be applied in [16] for the non-coherent signal model described in Section III-A, are given by

\[
Y_l^m(\theta, \phi) = \begin{cases} 
\sqrt{\frac{2l+1}{4\pi}} \frac{d^m}{dx^m} P_l^m(\cos(\theta)) & m > 0 \\
\frac{N_l^m}{P_l^m(\cos(\theta))} & m = 1 \\
\sqrt{\frac{2l+1}{4\pi}} \frac{d^m}{dx^m} \sin(m\phi) P_l^m(\cos(\theta)) & m < 0,
\end{cases}
\]

with degree \( l = 0, \ldots, L \), order \( m = -l, \ldots, l \) and \( P_l^m(.) \) given by (48). The normalization factor \( N_l^m \) is defined as

\[
N_l^m = \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!}.
\]

The derivative of the real spherical harmonics with respect to \( \theta \) is given by

\[
\frac{\partial}{\partial \theta} Y_l^m(\theta, \phi) = \begin{cases} 
\sqrt{\frac{2l+1}{4\pi}} \frac{d^{m+1}}{dx^{m+1}} P_l^m(\cos(\theta)) & m > 0 \\
\frac{N_l^m}{P_l^m(\cos(\theta))} \frac{\partial P_l^m(\cos(\theta))}{\partial \theta} & m = 1 \\
\sqrt{\frac{2l+1}{4\pi}} \frac{d^{m+1}}{dx^{m+1}} \sin(m\phi) P_l^m(\cos(\theta)) & m < 0.
\end{cases}
\]

The derivative of the real spherical harmonics with respect to \( \phi \) is given by

\[
\frac{\partial}{\partial \phi} Y_l^m(\theta, \phi) = \begin{cases} 
\sqrt{\frac{2l+1}{4\pi}} (-m) \sin(m\phi) P_l^m(\cos(\theta)) & m > 0 \\
0 & m = 1 \\
\sqrt{\frac{2l+1}{4\pi}} (-m) \cos(m\phi) P_l^m(\cos(\theta)) & m < 0.
\end{cases}
\]

Appendix C

Proof that RSS Measurements \( \hat{r} \) are Approximately Gaussian Distributed

Here we show that the RSS measurements \( \hat{r} = [\hat{r}_1, \ldots, \hat{r}_M]^T \), with \( \hat{r}_m \) given by (21), can be approximated by a Gaussian distribution with mean (22) and covariance matrix (23). For clarity we use scalar notation, the subscript \( m \) refers to the \( m \)-th element of the respective vector. Defining \( r_{m,n} = \Re \{r_m(n)\} \) and \( r_{m,i} = \Im \{r_m(n)\} \), the sum of the squared magnitude of the received signal,

\[
\hat{r}_m = \sum_{n=1}^N |r_{m,n}|^2 = \sum_{n=1}^N r_{m,n}^2 + r_{m,i}^2(n) \sim \chi^2(2N, \Lambda, \sigma^2 / 2)
\]

follows a noncentral \( \chi^2 \) distribution [46] with 2N degrees of freedom. The noncentrality parameter can be derived as

\[
\Lambda = \sum_{n=1}^N (\Re \{r_m(n)\}^2 + \Im \{r_m(n)\}^2)
\]

= \sum_{n=1}^N (\Re \{a_m(\theta, \phi)s(n)\}^2 + \Im \{a_m(\theta, \phi)s(n)\}^2)
\]

= \sum_{n=1}^N (|a_m(\theta, \phi)|^2 |s(n)|^2)
\]

= \sum_{n=1}^N g_m(\theta, \phi) |s(n)|^2
\]

= N \mu_m(\theta, \phi) \bar{s}.

The probability density function (PDF) of the noncentral \( \chi^2 \) distribution is given by

\[
p_{\chi^2}(x) = \frac{1}{\sigma^2} \left( \frac{x}{\sigma^2} \right)^{\frac{N-2}{2}} e^{-\frac{x}{\sigma^2}} I_N \left( \frac{2 \sqrt{Ax}}{\sigma^2} \right),
\]

where \( I_N(.) \) is the modified Bessel function of the first kind, see [33]. Since \( \hat{r}_m \) is just a scaled version of that, its distribution can be obtained by transformation \( p_{\hat{r}_m}(x) = N p_{\chi^2}(N x) \).

By inserting (57), we obtain the PDF

\[
p_{\hat{r}_m}(x) = \frac{N}{\sigma^2} \left( \frac{x}{g_m(\theta, \phi) \bar{s}} \right)^{\frac{N-2}{2}} e^{-\frac{N g_m(\theta, \phi) \bar{s} x}{\sigma^2}} I_N \left( \frac{2N \sqrt{g_m(\theta, \phi) \bar{s}}} {\sigma^2} \right).
\]

The mean and variance can be derived as

\[
\bar{\mu}_m = \E[\hat{r}_m] = N^{-1} \E[\hat{r}_m] = N^{-1}(N \sigma^2 + \Lambda)
\]

= \sum_{n=1}^N g_m(\theta, \phi) \bar{s} + \sigma^2,
\]

\[
\sigma^2_m = \var \{\hat{r}_m\} = N^{-2} \sum_{n=1}^N g_m(\theta, \phi) \bar{s}.
\]

For a growing number of samples \( N \), [49] approaches a Gaussian distribution \( \bar{r}_m \sim N(\bar{\mu}_m, \sigma^2_m) \) due to the central limit theorem. The approximation is reasonable for \( N > 25 \).
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