STABLE 1:2 RESONANT PERIODIC ORBITS IN ELLIPTIC THREE-BODY SYSTEMS

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ABSTRACT

The results of an extensive numerical study of the periodic orbits of planar, elliptic restricted three-body planetary systems consisting of a star, an inner massive planet, and an outer massless body in the external 1:2 mean-motion resonance are presented. Using the method of differential continuation, the locations of the resonant periodic orbits of such systems are identified, and through an extensive study of their phase-parameter space it is found that the majority of the resonant periodic orbits are unstable. For certain values of the mass and the orbital eccentricity of the inner planet, however, stable periodic orbits can be found. The applicability of such studies to the 1:2 resonance of the extrasolar planetary system GJ 876 is also discussed.

Subject headings: celestial mechanics — minor planets, asteroids — planetary systems — solar system: general

1. INTRODUCTION

It is well known that orbital or mean-motion resonances are of great importance in the dynamics of planetary and satellite systems. Resonances drive the dynamical evolution of such systems by creating regions of stability and instability in orbital phase space. Mean-motion resonances are usually associated with orbits whose geometrical configurations are periodically repeated. Such orbits, known as resonant periodic orbits (RPOs), are of particular interest since they define the structure of the associated resonance. The stability of an RPO implies that it may be a potential location for harboring an actual body. Unstable RPOs, on the other hand, provide pathways for dramatic orbital evolution of bodies and can also be used to explain chaotic motions (Varadi & Kaula 1999).

Until a few years ago, mean-motion resonances were found only in our solar system. Recent success of the precise radial velocity searches in detecting more than 100 extrasolar planets has extended the applicability of resonances to a broader context, several parsecs beyond the boundaries of our solar system. Among these planetary systems, GJ 876 is a three-body system with two planets in a 1:2 resonance (Marcy et al. 2001), and 47 UMa has two planets that seem to be in a near 2:5 commensurability (Rivera & Haghighipour 2003). Because of the universal importance and applicability of resonances, it is of great value to study RPOs and investigate their dynamical stability.

In this paper, we follow Varadi (1999) and use the method of differential continuation to identify the regions of the phase-parameter space of planar, elliptic restricted three-body systems that correspond to stable orbits of the outer body while in the 1:2 resonance. We study a system with an inner massive planet (hereafter, the major planet) and an outer zero-mass particle.

The basic properties of RPOs are discussed in § 2. A review of our methodology is given in § 3. Section 4 presents the results of our numerical study, and in § 5 we discuss the implications of these results for the extrasolar planetary system of GJ 876. Section 6 presents the concluding remarks.

2. PROPERTIES OF RESONANT PERIODIC ORBITS

Since the problem of resonances is typically approached through perturbation theory and averaging methods and not through RPOs, we briefly discuss what RPOs are and how they are related to perturbation theory. Since many of the technical details have been studied in previous works (see, for instance, Hadjidemetriou 1993, and the references therein), we attempt to provide only a brief overview of the main issues.

The present work deals with RPOs in the elliptic restricted three-body systems. The massless body is on an RPO if its orbit is periodic in time. For an exterior \( p : q \) mean-motion resonance, the major planet completes \( q \) orbits, while the massless particle revolves around the central mass \( p \) times and returns to the starting point in the phase space. While the major planet moves on an unperturbed Keplerian orbit, the massless particle’s orbit is perturbed by the major planet. Since the configuration of all bodies returns to the starting one, there are no long-term drifts in any of the orbital elements of the massless particle, except for its mean longitude. All orbital elements are stationary, except for the effects of traversing the RPO.

The most widely known RPOs are the ones associated with the 1:1 resonance. In the general three-body problem with arbitrary masses, the bodies form an equilateral triangle whose orientation and size change in time while each body moves on an ellipse within the same plane (Arnold 1988a; Danby 1992; Boccaletti & Pucacco 1996). In a restricted circular three-body system, i.e., when the mass of the third body is zero and the ellipses have zero eccentricities, the RPOs are the Lagrangian points in a frame corotating with the second body. The mean longitude of the
third body, in this case, is offset by 60° relative to that of the second body. While in the case of the circular restricted three-body system, the stability of Lagrangian RPOs can be determined by analytical means (Danby 1992), in the elliptical case one has to resort to numerical computations (Danby 1964).

Another definition of RPOs can be formulated using perturbation theory. In order to study the dynamics within an exterior $p : q$ mean-motion resonance, one usually introduces resonant angle variables

$$\sigma = q\lambda' - p\lambda - (q - p)\varpi'$$

and

$$\nu = q\lambda' - p\lambda - (q - p)\varpi,$$

where $\lambda$ denotes mean longitude and $\varpi$ is the longitude of pericenter. The primed variables refer to the massless particle and the unprimed ones refer to the major planet. In a restricted three-body system, either circular or elliptic, $\varpi$ is constant and $\lambda$ is a linear function of time. As the major planet completes $q$ orbits, $\lambda$ changes by $2q\pi$. The corresponding change in $\lambda'$ is approximately $2q\pi$, which implies that both $\sigma$ and $\nu$ are slow-varying quantities. In addition, these angles are invariant with respect to rotating the frame of rectangular coordinates.

Consider the resonant or secular part of the Hamiltonian of an elliptic restricted three-body system associated with a mean-motion resonance after averaging over the non-resonant combination of mean longitudes. The averaged Hamiltonian can be expressed in terms of $\sigma$ and $\nu$ and their conjugate action variables. The RPOs of such a system are the equilibria of this averaged Hamiltonian at which $\sigma$, $\nu$, and their conjugate action variables are constant. For an RPO, $\varpi'$ is also constant since it can be expressed in terms of $\sigma$, $\nu$, and $\varpi$. In other words, an RPO represents both exact mean-motion and also secular (or corotation) resonances with the major planet. The effects of traversing an RPO, i.e., changes in the orbital elements on the orbital timescale, are absorbed in the transformation of variables used to drive the averaged Hamiltonian of the system, although the transformation itself is not usually computed.

We note that it can be difficult to use the averaged Hamiltonian to locate RPOs for large eccentricities due to the slow convergence of expansions (Ferraz-Mello 1994). Our approach is based on the direct integration of the full dynamical equations in rectangular coordinates. The same direct approach has been used extensively in the case of the circular restricted three-body problem (see Celletti et al. 2002, and references therein).

In case of a restricted circular three-body system, it is customary to study the system in a frame corotating with the major planet (Celletti et al. 2002). In this frame, the equilibria of the averaged Hamiltonian are periodic orbits, although they are not necessarily periodic in a nonrotating frame. Such periodic orbits have been studied in detail by Henon (1997). A major difference between the circular and elliptic restricted three-body systems is that in the circular case the averaged Hamiltonian does not depend on $\nu$. As a consequence, the action variable conjugate to $\nu$ is a first integral. This action variable is a function of the massless particle’s orbital semimajor axis and eccentricity. The properties of the equilibria of the averaged Hamiltonian, such as their stability, change as this action variable is varied.

Among the numerous works in this area, we cite Beaugé (1994), who computed the locations of equilibria in the $1 : 2$ resonance and found the so-called asymmetric solutions for which $\sigma$ is different from $0°$ or $180°$. As mentioned above, the equilibria of the averaged Hamiltonian are not necessarily RPOs in the rectangular coordinate frame since the angle $\nu$ is free to vary.

An RPO represents exact resonance in the sense that all orbital elements are stationary except for variations due to traversing the RPO (see Varadi & Kaula 1999, for an example). Since RPOs in the elliptic restricted three-body problem are periodic orbits of a time-dependent dynamical system, their stability is studied with the help of Floquet theory (Danby 1964; Bennett 1965; Arnold 1988b). A stable RPO is the center of librations around it, and the periods of those librations can be determined through numerical computations, as we do in the present work. Unstable RPOs are associated with stable and unstable manifolds, which are separatrix-like features in the phase space, and their structure can explain the nature of chaos, e.g., in the central part of the $3 : 1$ resonance (Varadi & Kaula 1999).

Since the word “libration” is widely used in many different contexts, it is worth clarifying how librations around RPOs are related to other types of librations. It is customary to talk about libration even when such motion is present only in some of the dynamical variables but not all. For instance, there can be libration in certain variables while others are slowly varying or even have instabilities on long timescales, as in the case of the $3 : 1$ resonance (Varadi & Kaula 1999). Furthermore, a stable RPO is a center of librations only in topological sense, and large-amplitude librations may have apparent centers shifted from the RPO. For example, for the case of the $1 : 1$ resonance in the three-dimensional, circular restricted three-body problem, Namouni & Murray (2000) have shown that even for moderate eccentricities and inclinations the centers of librations in relative mean longitudes; i.e., in the angle $\sigma$, are shifted from $60°$. The orbits they considered have slow but large amplitude variations in the other variables. Conversely, De la Barre, Kaula, & Varadi (1996) have demonstrated that in the elliptic restricted three-body problem the center of libration in the longitude of pericenter varies with the amplitude of libration in relative mean longitude.

There has been a great deal of recent activity on RPOs. Celletti et al. (2002) carried out a systematic study of the stability of RPOs in the circular restricted three-body problem. The general three-body problem was considered in two articles by Michtchenko & Ferraz-Mello (2001) and Beaugé, Ferraz-Mello, & Michtchenko (2003) using new types of expansions. We leave detailed comparisons with our results for the future. Perhaps the works most relevant to the present article are the ones by Hadjidemetriou (2002) and Kotoulas & Hadjidemetriou (2002). We comment on them in the discussion section.

3. METHODOLOGY

Our method of identifying an RPO consists of two steps: search and continuation. To describe this process, we start by briefly reviewing the condition for an orbit to be periodic. Let us consider a planar, restricted three-body system with a massless particle in an exterior resonance with a major planet. If $x$ denotes the four-dimensional vector of the initial position and velocity of the outer particle, then after a
certain number of orbits of the major planet, the particle’s position and velocity will be given by another vector \( \mathbf{F}(x) \). The orbit is periodic if

\[
\mathbf{F}(x(\epsilon), \epsilon) - x(\epsilon) = 0 ,
\]

(3)

where \( \epsilon \) is a parameter of the system. For the systems considered here, \( \epsilon \) can be taken to be the ratio of the mass of the major planet to that of the central star. The mapping \( \mathbf{F} \) is found by numerically integrating the equations of motion. Solution vectors \( x \) of equation (3) represent RPOs of the system.

Equation (3) can be differentiated with respect to \( \epsilon \) to obtain

\[
\frac{dx}{d\epsilon} = -\left( \frac{d\mathbf{F}}{dx} - \mathbf{I} \right)^{-1} \frac{d\mathbf{F}}{d\epsilon} ,
\]

(4)

where \( \mathbf{I} \) is the identity matrix. Equation (4) involves the variational equations of the dynamics of the system. It leads to a complicated system of differential equations in which the independent variable is \( \epsilon \) (Varadi 1999).

Equation (3) can, in principle, be solved by Newton’s method. Because the latter converges only for some vectors \( x \), due to the nonlinear nature of equation (3), many initial vectors have to be tried. For a chosen value of \( \epsilon \), we select random initial conditions, integrate the dynamical equations, and if the left-hand side of equation (3) happens to be small, we apply Newton’s method to search for a nearby solution of this equation. We then apply differential continuation (Keller 1977), i.e., solve equation (4), to follow the changes in the RPO due to varying \( \epsilon \).

We start our search for 1:2 resonant periodic orbits by choosing the semimajor axis of the outer body near its resonant value with the inner planet. The semimajor axis of the inner planet is assumed to be equal to unity. Many possible configurations are considered in which the other orbital parameters of the outer body are chosen randomly. The inner planet is placed at either the peri- or the apocenter without any loss of generality. We integrate the system for two orbits of the inner planet and form the differences between the initial and the final position and velocity vectors of the outer body. When these differences are small, we apply Newton’s method to further reduce them. This process is repeated until these differences become so small that the orbit can be considered to be an RPO.

The ratio of the mass of the inner planet to the mass of the central star \( (\mu) \) and also the orbital eccentricity of the inner planet \( (e) \) are the parameters of our system. During the process of differential continuation, we vary \( \mu \) and \( e \) and solve equation (4) to obtain new RPOs along a path in \((\mu, e)\) space. To identify the regions of the phase-parameter space of the system that correspond to stable RPOs, we change \( \mu \) and \( e \) systematically and criss-cross the \((\mu, e)\) space with paths of continuation along which the stability of RPOs is computed.

The stability of RPOs is examined by computing the eigenvalues of the monodromy matrix \( d\mathbf{F}/dx \) (Danby 1964; Golub & Van Loan 1996). Stable RPOs are centers of libration for nearby orbits. Unstable RPOs are associated with separatrix-like features (Guckenheimer & Holmes 1983). The eigenvalues of the monodromy matrix provide the frequency of libration associated with stable RPOs and also the Lyapunov exponent of the outer body when the RPO is unstable. The system is linearly (spectrally) stable if these eigenvalues are on the unit circle in the complex plane. There is usually a separation of timescales between fast and slow dynamics, and both can correspond to either stable or unstable motions. The fast dynamics usually consists of motions that are also present in the case of a circular orbit for the major planet. When this fast motion is stable, we call it libration. The slow dynamics is associated with the effects of orbital eccentricity of the major planet. We call stable motions in this case secular librations and unstable ones secular instabilities. An RPO is unstable when either of the two motions is unstable.

4. NUMERICAL RESULTS

We studied the orbit of the outer body of our three-body system for different values of the mass ratio \( \mu \) and also different values of the orbital eccentricity of the inner planet \( e \). To describe the shape and the orientation of these orbits, we use

\[
h = e \cos \varpi , \quad k = e \sin \varpi ,
\]

(5)

\[
h' = e' \cos \varpi' , \quad k' = e' \sin \varpi' ,
\]

(6)

where \( e' \) is the orbital eccentricity of the outer body. In order to completely specify an RPO, one needs to determine the values of \( h', k' \), the semimajor axis, and the mean longitude of an RPO at a given orbital phase of the inner planet.

We performed an extensive initial numerical search for 1:2 resonant periodic orbits for several mass ratios and orbital eccentricities of the inner planet. Once an RPO was found, we used its orbital parameters for starting the continuation process. Figure 1 shows the overall results of the continuation process for starting values of 0.001 and 0.1 for \( \mu \) and \( e \), respectively. The value of \( \mu \) was kept constant during continuation. For all branches shown in this figure, \( \varpi = 0 \) or \( 180^\circ \) corresponding to \( k = 0 \) and \( h = e \) or \(-e\). For the purpose of discussing the relationships between these

![Fig. 1.—Structure of the continuation branches for different values of the orbital eccentricity of the outer body. Each branch represents the results of the continuation process started from an RPO with the given values of \( h' \) and \( k' \). The mass ratio and the orbital eccentricity of the inner planet at the start of the continuation were \( \mu = 0.001 \) and \( e = 0.1 \), respectively. During the continuation, the value of \( \mu \) was kept constant.](image)
It is worthwhile to show them for both \( h < 0 \) and \( h > 0 \). It is also worth mentioning that any branch for \( h > 0 \) can be transformed into a branch for \( h < 0 \) by rotating the rectangular coordinate system by \( 180^\circ / \pi \), accompanied with changing the sign of \( h_0 \). For the same orbit of the inner planet, there can be several RPOs with different orbital elements on different branches. These branches are labeled according to the initial values of their \( h_0 \) and \( k_0 \) quantities. This method of labeling enables one to readily calculate the orbital eccentricity and the longitude of the pericenter of the outer body at the start of the continuation process. For instance, the branch labeled as \( h_0 = 52 \) and \( k_0 = 16 \) represents the result of continuation started from a resonant periodic orbit with \( h_0 = 0.52 \) and \( k_0 = 0.16 \). Such an orbit has an eccentricity equal to 0.544 and a longitude of pericenter equal to 171°.

One can see from Figure 1 that certain branches appear to be connected, either through so-called turning points or via circular connections. To explain these cases, we recall that any continuation branch corresponds to a periodic orbit that has been continued from an initial RPO, for different values of the mass and orbital eccentricity of the inner body. The continuation process can break down at points where the matrix \( J = dF/dx - I \) becomes singular. When two branches of continuation meet at a point on the \( h' - h \) plane where \( h \neq 0 \), they are said to be connected through a turning point. Turning point is a general term indicating that a branch changes its direction, or in other words, at a turning point, the tangent line to the continuation branch will be vertical (Keller 1977). The two branches \( h'15k'00 \) and \( h'59k'00 \) are connected via a turning point. Figure 2 shows the changes of different orbital parameters of the outer body for these two branches.

On the line \( h = 0 \), matrix \( J \) is always singular. This is due to the fact that the differential equations for the outer planet become time-independent. In this case, the variational equations transport the equations of motion invariantly along the orbit (Siegel & Moser 1971; Varadi 1999). A circular connection occurs when two branches of continuation meet on the \( h = 0 \) line. Branches \( h'004k'00 \) and \( h'15k'00 \), \( h'59k'00 \) and \( h' - 67k'00 \), and also \( h' - 05k' - 57 \) and \( h'52k'16 \) are connected in this way. We also have to note that the matrix above can also become singular at bifurcation points, which is discussed in the context of stability analysis.

The continuation process can sometimes pass through singularities without major difficulties. We use a standard multistep integrator, namely, the explicit Adams scheme, usually at order 8 (Gear 1971), to solve equation (4). When a singularity falls between the actual points computed along a continuation path, the integration can proceed, at least in some cases, without being affected by the singularity. This happens by chance and not by design. An example is the smooth connection between \( h'004k'00 \) and \( h'15k'00 \) in Figure 1. Eventually there is a singularity of some type through which continuation is not possible.

As mentioned above, periodic orbits of the outer body are labeled by their orbital elements at the pericenter.
The parameterized equations of motion of the outer body are still calculated at the time of the pericenter passage of the inner planet, and because we are dealing with the 1:2 external resonance, we integrate the equations of motion of the outer body for half of the orbital period of the inner planet and then rotate the coordinate system in such a way that the pericenter of the inner planet is located in the positive direction of the x-axis. During this time, the outer body travels only one-fourth of its orbit. As a result of this realignment, on the other hand, there will be two sets of orbital parameters associated with the outer body. We ensure that these two sets describe orbits unambiguously by limiting the values of the mean longitude of the outer body, \( \lambda' \). In this study \(-90^\circ < \lambda' < 90^\circ \). In order to correctly identify which branches are connected through the circular case, i.e., \( h = 0 \), we realign all orbits along the continuation branches. Figure 3 illustrates this for two branches \( h'/04k'/00 \) and \( h'/15k'/00 \). The branch labeled \( h'/15k'/00 \) has a portion with \( h' > 0 \) for \( h > 0 \) (see Fig. 1). After rotating the rectangular coordinate system by \( 180^\circ \), we obtain another portion with \( h' < 0 \) for \( h < 0 \). On the other hand, we were able to continue the \( h'/004k'/00 \) branch from \( h > 0 \) to \( h < 0 \). After realigning the \( h < 0 \) portion of \( h'/15k'/00 \), we obtained the same orbits as those on the branch \( h'/004k'/00 \). This is illustrated in Figure 3. The change in the value of \( h' \) after realignment is typically small.

Figure 1 provides an overall view of different continuation branches for a specific value of the inner planet’s mass. When the latter is also varied, the branches become leaves parameterized by the mass and the eccentricity of the inner planet. In order to explore these leaves, the \( (\mu, e) \) space is criss-crossed with continuation paths. Along these paths, \( \mu \) and \( e \) are assumed to depend on a common independent variable. The derivatives of \( \mu \) and \( e \) with respect to the independent variable are prescribed to yield the desired continuation path. Since the actual variable used in the continuation is not \( e \) but the velocity of the inner planet at peri- or apocenter, the paths will have a slight curvature (Varadi 1999).

We also studied the phase-parameter space of the system for all different branches of Figure 1 in search of regions corresponding to stable resonant periodic orbits. Figure 4 shows a criss-crossing of the \( (\mu, e) \) space with continuation paths of the RPO designated as \( h'--05k'--05 \). The system, in this case, reveals a region of stability with a distinct boundary from its large region of instability. The frequencies of libration of the system, computed from the eigenvalues of the monodromy matrix for different values of the mass ratio \( \mu \) and the orbital eccentricity of the inner planet, are shown in Figures 5, 6, and 7. The orbit is linearly stable if all plotted values are positive and it is unstable otherwise. As shown in Figure 5, the orbit of the outer planet is stable along the horizontal line of \( \mu = 0.001 \) in Figure 4, for all values of the orbital eccentricity of the outer planet less than 0.576. Along the vertical line of \( e = 0.5 \) in Figure 4, the system is stable for all values of the mass ratio less than 0.0014 (Fig. 6). Figure 7 shows frequencies of libration for the continuation branch in which the orbital eccentricity and the mass ratio of the inner planet increase in steps of 0.01 and 0.001, respectively. The continuation path corresponding to this figure has been denoted by X mark in Figure 4. As shown in Figure 7, the orbit of the outer planet is stable for the values of the mass ratio \( \mu \) less than 0.005, and it becomes unstable for its larger values.

Figure 4 also shows that at higher values of the eccentricity of the inner planet, the stability of the system requires lower mass ratios, implying that the RPOs of a system with a more massive inner planet in a 1:2 exterior resonance are most stable when the orbit of the inner planet is closer to circular. The actual boundaries of stable regions can be different for different resonances and different leaves of the

![Graph of h’ vs. h for two circularly connected branches](image)
same resonance. For instance, the case of the $2:3$ external resonance has a small wedge of instability (Varadi 1999). Figures 8a and 8b show two different RPOs of a system in $1:2$ mean-motion resonance for the mass ratio of 0.001 and different values of the inner planet’s orbital eccentricity. It is worth noting that for these stable orbits, the pericenter directions of the inner and outer planets are not aligned. Such asymmetric librations have also been reported by Nesvorný & Roig (2001) in their study of stable RPOs of the $1:2$ mean-motion resonance in the trans-Neptunian region.

5. 1:2 RESONANCE OF GJ 876

In this section we present an application of the numerical results of §4 to the extrasolar planetary system of GJ 876. It is of utmost importance to keep in mind that the numerical results presented in this study have been obtained by assuming a massless particle in an exterior $1:2$ resonance with a massive planet. The extrasolar planetary system GJ 876 is a three-body system of an M4 dwarf with a $0.32 \pm 0.05$ solar mass and a pair of planets at 0.13 and 0.21 AU. The inner planet of this system has a minimum mass of 0.6 Jupiter masses with an orbital period equal to 30.12 $\pm$ 0.02 days. The minimum mass and the orbital period of the outer planet are 1.89 Jupiter masses and $61.02 \pm 0.03$ days, respectively (Marcy et al. 2001). Since the outer planet of this system is not massless, the application of our results to the dynamics of GJ 876 is not entirely plausible. In order to employ an analysis such as the one presented in this paper to study the dynamics of GJ 876, it is necessary to approximate this planetary system by an interior resonance. In our initial search, we were unable to find any stable RPO for an interior $2:1$ resonance. On the other hand, since our results have been derived using a massless particle on an exterior resonant orbit, we are unable to say whether GJ 876 is more closely approximated by an exterior or interior resonance. Our inability to find stable orbits in the interior resonance suggests that the exterior resonance may be a more plausible case for providing stable configurations in which systems with finite-mass planets such as GJ 876 will be found. In this
case the external resonance may provide an approximate picture of the dynamical stability of the system and the corresponding ranges of the eccentricity and mass ratio of the inner planet.

During the past 2 years, several authors have studied the dynamical stability of GJ 876 and have shown that like any other multibody system, the long-term stability of GJ 876 is dependent on the initial values of its orbital parameters (Rivera & Lissauer 2001; Hadjidemetriou 2002; Ji, Li, & Liu 2002). Marcy et al. (2001) were the first to indicate the epoch dependence of the stability of GJ 876. Rivera & Lissauer (2001) have shown that at low inclinations, for the best fit to the radial velocity data of GJ 876, this system is stable for at least 100 Myr. They have also shown that for higher values of the mass of the inner planet, the system becomes unstable in a very short time. A result that was also obtained by Ji et al. (2002), who discovered that for orbital elements obtained from a dynamical fit by Laughlin & Chambers (2001) and with $\sin i = 0.55$, where $i$ is the inclination of the plane of the orbits of the two planets with respect to the plane of the sky, 75% of their cases became unstable within 100–1000 years. Hadjidemetriou (2002) has also studied the stability of 1:2 resonant periodic orbits within the context of extrasolar planets and has shown the configuration dependence of the stability of such systems. Also, in a recent paper, Beauge & Michtchenko (2003) have presented an analytical model for studying the dynamics of a highly eccentric three-body system with application to the resonance capture and stability of GJ 876.

In case of an exterior 1:2 resonance, such dependence of the stability of an RPO on the values of the orbital parameters of the inner planet is quite evident from the results presented in § 4. As we mentioned in that section, in the majority of the systems that we studied, the RPOs were unstable. Despite such instability, as shown in Figure 4, we were able to identify a set of initial orbital parameters, among the RPOs of Figure 1, that correspond to stable orbits. Figure 4 indicates that in a hypothetical scenario in which the perturbation due to the outer planet of GJ 876 is neglected, the orbit of the outer body will be stable when the mass ratio $\mu$ is less than 0.007 corresponding to the values of the eccentricity of the inner planet less than 0.1, and when it is larger than 0.0007 for highly eccentric orbits ($e \simeq 0.65$). Interestingly, in all stable orbits reported by Marcy et al. (2001), Rivera & Lissauer (2001), Hadjidemetriou (2002),
and Ji et al. (2002) for the real GJ 876 planetary system, \( \mu \) and \( \epsilon \) are well within the stable region of Figure 4.

To investigate to what extent this figure could be indicative of the stability of the outer planet of the actual GJ 876 planetary system, we numerically integrated the three-body system of GJ 876 using the SWIFT symplectic integrator of Levison & Duncan (1994) for different values of the mass ratio \( \mu \). The initial orbital parameters of the system are given in Table 1. These parameters were obtained assuming an 84° inclination for the orbital plane of the inner planet. Such an assumption is necessary to ensure coplanar orbits for both planets since astrometric measurements restrict the inclination of the outer planet’s orbit to 84° with respect to the plane of the sky (Benedict et al. 2002).

As shown in Table 1, the initial orbital eccentricity of the inner planet is approximately 0.225. From the stable region of Figure 4 and for a hypothetical case where the outer planet of GJ 876 is massless, this corresponds to an upper limit of 0.004 for the mass ratio \( \mu \). The results of our numerical integrations of the three-body system of GJ 876, however, indicate that this system becomes unstable in less than 1600 years when \( \mu \) becomes larger than 0.008. The increase of the upper limit of \( \mu \) from 0.004 to 0.008 for stable orbits can be attributed to the perturbative effect of the outer planet.

### Table 1

| Parameters | Inner Planet | Outer Planet |
|------------|--------------|--------------|
| \( a \) (AU) | 1.298 | 2.083 |
| \( e \) (deg) | 0.225 | 0.012 |
| \( i \) (deg) | 84 | 84 |
| \( \Omega \) (deg) | 25 | 25 |
| \( \omega \) (deg) | 352 | 354 |
| \( M \) (deg) | 167 | 261 |
| \( m_{\text{min}} \) | 0.577 | 1.89 |

6. DISCUSSION

We presented the results of a systematic search for stable resonant periodic orbits of planar, elliptic, and restricted three-body systems in the exterior 1:2 resonance. Our approach toward the understanding of orbital resonances provides fundamental information on their structures. Stable periodic orbits are centers of librations whose periods are also obtained, at least for small amplitudes, from our stability computations.

Using the method of differential continuation, we examined the phase-parameter space of the system for different values of the mass ratio and the orbital eccentricity of the inner planet. In the majority of the cases, the RPOs were unstable. We were, however, able to show that stable resonant periodic orbits can be found when the mass and the orbital eccentricity of the inner planet are within a certain range (Fig. 4).

We approached the problem of finding RPOs through a large-scale search of the phase-parameter space of elliptic restricted three-body systems. An alternative is to continue orbits in the circular problem that are periodic in a frame corotating with the major planet and happen to be periodic in nonrotating frame. Kotoulas & Hadjidemetriou (2002) did so and found what appear to be the branches \( h'/59k'00, h'/15k'00, h'/004k'00, \) and \( h'/76k'00 \) in our study. They also restricted their analysis to a fixed value of \( \mu \). Compared to their results, we have found additional branches and determined their stability for large ranges of the mass ratio and orbital eccentricity of the inner planet. Note that our \( \epsilon \) is their \( \epsilon' \). Hadjidemetriou (2002) considered periodic orbits in the general three-body system in a rotating frame and concluded that a planetary system in the 1:2 resonance is unstable when the mass of the outer planet is smaller than that of the inner planet. We surmise that his analysis did not include the stable region of branch \( h'/05k'47 \). On the other hand, our initial search for RPOs also covered the internal 2:1 resonance, where the massless body was placed between the major planet and the central star. We have not found any RPOs, either stable or unstable, in this case. Tsiganis, Varvoglis, & Hadjidemetriou (2002) also noted this and called it a topological defect and argued that it might explain the presence of slow chaos.

The apparent lack of RPOs in the internal 2:1 resonance also raises questions regarding the same resonance in the full three-body system, i.e., when the mass of the outer body is nonzero. It is important to mention that such lack of RPOs does not exclude the existence of stable 2:1 interior resonant periodic orbits in the general three-body system. In a restricted three-body system in an exterior 1:2 resonance, as the mass of the outer body increases and the mass of the inner planet decreases, the system passes from an external resonance to an internal resonance. Along the way, where the neither of the bodies are massless, one should find the limits of masses for which stable RPOs exist. We tried to identify such limits for a general three-body system in 1:2 resonance by applying our results to the extrasolar planetary system GJ 876. Although the application of the results to GJ 876 is not entirely plausible, since this system is more resembled by a 2:1 interior resonance, one can use our results as an starting point for analyzing the smooth transition from an exterior 1:2 to an interior 2:1 resonance. For instance, from Figure 4 and for a hypothetical system where the mass of the outer planet of GJ 876 is ignored and the orbital elements of its inner planet are similar to the actual ones given in Table 1, the upper limit for the mass ratio \( \mu \) is approximately 0.004. However, numerical integration of the actual system of GJ 876 indicates that, considering astrometric measurements, the limit of the mass corresponding to stable RPOs for coplanar orbits is given by \( \mu \leq 0.008 \). Such difference in the upper limit of the mass ratio \( \mu \) can be attributed to the mass of the outer planet.

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Table 1: Orbital Parameters of GJ 876

| Parameters | Inner Planet | Outer Planet |
|------------|--------------|--------------|
| \( a \) (AU) | 1.298 | 2.083 |
| \( e \) (deg) | 0.225 | 0.012 |
| \( i \) (deg) | 84 | 84 |
| \( \Omega \) (deg) | 25 | 25 |
| \( \omega \) (deg) | 352 | 354 |
| \( M \) (deg) | 167 | 261 |
| \( m_{\text{min}} \) | 0.577 | 1.89 |
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