Guaranteed Cost LQG Control of Uncertain Linear Quantum Stochastic Systems

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Abstract

In this paper, we formulate and solve a guaranteed cost control problem for a class of uncertain linear stochastic quantum systems. For these quantum systems, a connection with an associated classical (non-quantum) system is first established. Using this connection, the desired guaranteed cost results are established. The theory presented is illustrated using an example from quantum optics.

I. INTRODUCTION

The feedback control of quantum systems is an important emerging research area; e.g., see [1], [2], [5]–[7], [10], [13], [16]–[19]. However, most of the existing results in quantum feedback control do not directly address the issue of robustness with respect to parameter uncertainties in the quantum system model. In this paper, we study guaranteed cost control for a class of uncertain linear quantum systems. We consider quantum systems described by linear Heisenberg dynamics driven by quantum Gaussian noise processes, and controlled by a classical linear feedback controller. This class of systems includes examples from quantum optics with classical controllers implemented by standard analog or digital electronics. For such quantum systems, we address the issue of robust controller design by allowing for norm bounded uncertainties in the matrices defining the quantum model. Also, we consider the case of uncertainties in a quadratic

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Hamiltonian defining the quantum dynamics. Such uncertainties can represent uncertainty in the values of the physical parameters of the quantum system. Our results can also be extended to allow for linear unmodeled quantum dynamics subject to a certain $H^\infty$ norm bound constraint.

Guaranteed cost control involves constructing a controller such that the expected value of a quadratic cost functional satisfies a given bound for all possible values of the uncertain parameters in the model. The quadratic cost functional is determined to reflect the performance requirements of the quantum control system. This means that a controller can be constructed which addresses not only the issue of robustness but also the issue of performance.

A classical controller for a quantum systems takes measurement data obtained by monitoring the quantum system continuously in time. The controller determines the control actions which influence the dynamics of the quantum system in a feedback loop. The results in this paper provide a method for designing such classical controllers to achieve a guaranteed bound on a quadratic cost functional when the quantum system model is subject to uncertainty. Our results exploit simple computations of quantum expectations in Gaussian states which provide a link to an auxiliary classical system. This enables us to use established classical guaranteed cost control results to obtain corresponding quantum guaranteed cost control results.

The paper is organized as follows. Section II sets up a basic framework for describing the class of uncertain linear quantum systems under consideration. A quantum linear system is then related to an auxiliary classical (non-quantum) system. In Section III we present a guaranteed cost result for the auxiliary classical uncertain system and then use this to establish our main result which is a guaranteed cost result for the linear quantum uncertain system. An example from quantum optics is given in Section IV to illustrate our main results. Some conclusions are given in Section V.
II. PROBLEM FORMULATION AND PRELIMINARY RESULTS

A. Uncertain Linear Quantum System

We consider an uncertain linear quantum system described by the following non-commutative stochastic differential equations:

\[ dx = ([A + B_0\Delta C_0]x + [B_1 + B_0\Delta D_0]u) \, dt + B_0 \, dv; \quad x(0) = x_0, \]
\[ \mu = C_1 x + D_{12} u, \]
\[ dy = [C_2 + D_{20}\Delta C_0] x \, dt + [D_{22} + D_{20}\Delta D_0] u dt + D_{20} \, dv; \quad y(0) = 0. \]

(1)

Here \( v(t) \) vector of self-adjoint quantum noises with Ito table:

\[ dv(t)dv^T(t) = F_v dt, \]

(2)

where \( F_v \) is a non-negative Hermitian matrix which satisfies \( \frac{1}{2}(F_v + F_v^T) = I \); e.g., see [3], [14]. Note that it is straightforward to extend the results of this paper to allow for more general matrices \( F_v \); e.g., see [11]. The noise processes can be represented as operators on an appropriate Fock space; e.g., see [3], [14]. Also \( x(t) \) is vector of possibly non-commutative self-adjoint system variables which are operators defined on an appropriate Hilbert space. The components of \( x(t) \) represent physical properties of the system at time \( t \) (using the Heisenberg picture), such as position and momentum. The quantity \( \mu(t) \) is also a vector of possibly non-commutative self-adjoint variables corresponding to physical observables defining the desired performance objective. The classical quantities \( u(t) \) and \( y(t) \) represent control inputs and measurement outputs, respectively. Their components are self adjoint and commute among themselves and at different times. The matrices \( A, B_0, B_1, C_0, C_1, C_2, D_0, D_{12}, D_{22}, D_{20} \) are known constant real matrices of appropriate order and \( \Delta \) is an uncertain norm bounded real matrix satisfying

\[ \Delta^T \Delta \leq I. \]

(3)

We also consider an associated quadratic cost functional

\[ J(u(\cdot)) = \int_0^{t_f} \langle \mu^T(t)\mu(t) \rangle \, dt. \]

(4)

where the notation \( \langle \cdot \rangle \) represents expectation over all initial variables and noises. The interval \( [0, t_f] \) is the fixed time horizon. It is assumed that the initial condition \( x_0 \) is Gaussian, with
density operator $\rho$. We let $\tilde{x}_0 := \langle x_0 \rangle$ and

$$Y_0 := \frac{1}{2}((x_0 - \tilde{x}_0)(x_0 - \tilde{x}_0)^T + ((x_0 - \tilde{x}_0)(x_0 - \tilde{x}_0)^T)^T).$$

The system model (1) includes a wide range of quantum, classical, and quantum-classical stochastic uncertain systems.

Together with the uncertain quantum system (1), consider a classical (non-quantum) controller:

$$dx_K = A_K x_K dt + B_K dy; \quad x_K(0) = x_{K0}, \quad u = C_K x_K. \quad (5)$$

Here the controller initial state $x_{K0}$ is a fixed real vector. For a given value of the uncertainty matrix $\Delta$ satisfying (3), the quantum system (1) and the classical controller (5) together produce a closed loop classical-quantum system:

$$d\eta = \tilde{A}\eta dt + \tilde{B}\ dv; \quad \eta(0) = \eta_0, \quad \mu = \tilde{C}\eta, \quad (6)$$

where

$$\tilde{A} = \begin{bmatrix} A + B_0 \Delta C_0 & [B_1 + B_0 \Delta D_0]C_K \\ B_K[C_2 + D_{20}\Delta C_0] & A_K + B_K[D_{22} + D_{20}\Delta D_0]C_K \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_0 \\ B_K D_{20} \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} C_1 & D_{12}C_K \end{bmatrix}, \quad \eta = \begin{bmatrix} x \\ x_K \end{bmatrix}. \quad (7)$$

The guaranteed cost control problem under consideration involves constructing a classical output feedback controller of the form (5) and a cost bound $M > 0$, such that cost (4) corresponding to the closed loop system (6) satisfies the bound $J(u(\cdot)) \leq M$ for all uncertainty matrices $\Delta$ satisfying (3).

For a given value of the uncertainty matrix $\Delta$ satisfying (3), we now define the following matrix valued function of time associated with the closed loop system (6):

$$P(t) := \frac{1}{2}\langle \eta(t)\eta^T(t) + (\eta(t)\eta^T(t))^T\rangle. \quad (8)$$

Note that $P_0 := P(0) = \text{diag}(Y_0, 0)$. Using this definition, we can establish the following lemmas which provide a link between the cost associated with the classical-quantum closed loop system (6) and the cost associated with an auxiliary linear classical (non-quantum) closed loop system.
Lemma 1: The value of the cost (4) corresponding to the closed loop system (6) is given by

$$J(u(\cdot)) = \int_0^{t_f} \text{Tr}(\tilde{C}^T \tilde{C} P(t)) \, dt.$$  \hspace{1cm} (9)

Proof. We have

$$\langle \mu^T \mu \rangle = \langle \eta^T \tilde{C}^T \tilde{C} \eta \rangle = \frac{1}{2} \langle \text{Tr}(\tilde{C}^T \tilde{C} [\eta \eta^T + (\eta \eta^T)^T]) \rangle = \text{Tr}(\tilde{C}^T \tilde{C} P).$$

Hence, upon integration, it follows that the corresponding closed loop cost is given by (9). \hfill \blacksquare

Lemma 2: The matrix valued function $P(\cdot)$ defined by (8) satisfies the differential equation

$$\dot{P} = \tilde{A} P + P \tilde{A}^T + \frac{1}{2} \tilde{B} (F_v + F_v^T) \tilde{B}, \quad P(0) = P_0,$$ \hspace{1cm} (10)

where $P_0 = \text{diag}(Y_0, 0)$.

Proof. Using the quantum Ito rule (e.g., see Chapter III of [14]), it follows from the definition of $P(\cdot)$ in (8), that

$$dP = \frac{1}{2} [ (d\eta \eta^T) + (d\eta \eta^T)^T + (\eta \, d\eta^T) + (\eta \, d\eta^T)^T + (\tilde{B} F_v \tilde{B}^T + (\tilde{B} F_v \tilde{B}^T)^T) \, dt ]$$

$$= [ \tilde{A} P + P \tilde{A}^T + \frac{1}{2} \tilde{B} (F_v + F_v^T) \tilde{B} ] \, dt.$$

Therefore, $P(\cdot)$ satisfies the differential equation (10). \hfill \blacksquare

B. Auxiliary Classical Uncertain System

We now define the auxiliary classical uncertain system which will be used to solve the quantum guaranteed cost control problem defined above:

$$dx = ([A + B_0 \Delta C_0] x + [B_1 + B_0 \Delta D_0] u) \, dt + B_0 \, dw; \quad x(0) = x_0,$$

$$\mu = C_1 x + D_{12} u,$$

$$dy = [C_2 + D_{20} \Delta C_0] x \, dt + [D_{22} + D_{20} \Delta D_0] u dt + D_{20} \, dv; \quad y(0) = 0.$$ \hspace{1cm} (11)

Here, $x(t)$ is a real vector of state variables, $u(t)$ is the control input vector, $y(t)$ is the measured output vector, $w(t)$ is a classical Wiener process, and $x_0$ is a Gaussian random variable with mean $\tilde{x}_0$ and covariance matrix $Y_0$. Also, $\Delta$ is an uncertain norm bounded real matrix satisfying (3).

The matrices defining the classical uncertain system (11) are the same as the matrices defining the quantum uncertain system (1). Associated with this classical uncertain system is the quadratic cost functional $\hat{J}(u(\cdot)) := \int_0^{t_f} E[\mu^T(t) \mu(t)] \, dt$. 

July 29, 2008 DRAFT
For a given value of the uncertainty matrix $\Delta$ satisfying (3), the classical system (11) and the classical controller (5) together produce a closed loop classical system:

\[
d\zeta = \tilde{A}\zeta \, dt + \tilde{B} \, dv; \quad \eta(0) = \eta_0,
\]

\[
\mu = \tilde{C}\zeta,
\]

where $\zeta := \begin{bmatrix} x \\ x_K \end{bmatrix}$ is a real vector of state variables for the closed loop system and the matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are defined as in (7).

**Lemma 3:** For a given uncertainty matrix $\Delta$ satisfying (3), the value of the cost $J(u(\cdot))$ corresponding to the quantum classical closed loop system (6) is the same as the value of the cost $\hat{J}(u(\cdot))$ corresponding to the classical closed loop system (12).

**Proof.** Let $Q(t) := E\zeta(t)\zeta^T(t)$. Clearly $Q(0) = P_0$. As in Lemma 1, one can now show that $\hat{J}(u(\cdot)) = \int_0^T \text{Tr}(\tilde{C}^T\tilde{C}Q(t)) \, dt$. Also, in a similar fashion to the proof of Lemma 2 (but using the classical Ito rule instead of the quantum Ito rule), it follows that $Q(\cdot)$ also satisfies the differential equation (10). Thus $Q(\cdot) \equiv P(\cdot)$, and hence $\hat{J}(u(\cdot)) = J(u(\cdot))$. □

From this lemma, we immediately obtain the following result.

**Lemma 4:** Suppose a classical controller of the form (5) is a guaranteed cost controller for the classical uncertain system (11) such that the closed loop system (12) satisfies $\hat{J}(u(\cdot)) \leq M$ for all uncertain matrices $\Delta$ satisfying (3). Then this classical controller is also a guaranteed cost controller for the quantum uncertain system (1) such that the closed loop system (6) satisfies $J(u(\cdot)) \leq M$ for all uncertain matrices $\Delta$ satisfying (3).

### III. THE MAIN RESULT

It follows from the results of the previous section that we can find a guaranteed cost controller for the quantum uncertain system (1) by constructing a guaranteed cost controller for the classical uncertain system (11). This leads to the main result of this paper which is Theorem 2 given in Subsection III-B. However, to obtain this result, we first consider the classical case.

**A. Guaranteed Cost Control of the Classical Uncertain System**

In order to construct a suitable guaranteed cost controller for the classical system (11), we will use a result which is derived from the minimax LQG results of [15]. In order to present this result, we first require some assumptions and notation.
Assumption 1: For simplicity, we assume \( C_1 = \begin{bmatrix} R^{1/2} \\ 0 \end{bmatrix} \), and \( D_{12} = \begin{bmatrix} 0 \\ G^{1/2} \end{bmatrix} \). With this simplification, the expression for the cost becomes

\[
\hat{J}(u(\cdot)) = \int_0^{t_f} [x^T R x + u^T G u] \, dt.
\]

(13)

Also, we assume there exists a \( d_0 > 0 \) such that \( \Gamma = D_{20} D_{20}^T \geq d_0 I \).

Notation. For \( \tau > 0 \), we define the matrices

\[
R_{\tau} = R + \tau C_0^T C_0, \quad G_{\tau} = G + \tau D_0^T D_0, \quad \Upsilon_{\tau} = \tau C_0^T D_0.
\]

Assumption 2: There exists a \( \tau > 0 \) such that the following three conditions hold:

1) The Riccati differential equation

\[
\dot{Y} = (A - B_0 D_{20}^T \Gamma^{-1} C_2) Y + Y (A - B_0 D_{20}^T \Gamma^{-1} C_2)^T - Y (C_2^T \Gamma^{-1} C_2 - \frac{1}{\tau} R_{\tau}) Y \\
+ B_0 (I - D_{20}^T \Gamma^{-1} D_{20}) B_0^T,
\]

(14)

has a symmetric solution \( Y(\cdot) : [0, t_f] \rightarrow \mathbb{R}^{n \times n} \) satisfying \( Y(0) = Y_0 \) and, there exists a \( c_0 > 0 \), such that \( Y(t) \geq c_0 I \), for all \( 0 \leq t \leq t_f \).

2) The Riccati differential equation

\[
- \dot{X} = X (A - B_1 G_{\tau}^{-1} T_{\tau}^T) + (A - B_1 G_{\tau}^{-1} \Upsilon_{\tau})^T X + R_{\tau} - \Upsilon_{\tau} G_{\tau}^{-1} \Upsilon_{\tau}^T \\
- X (B_2 G_{\tau}^{-1} B_2^T - \frac{1}{\tau} B_0 B_0^T) X,
\]

(15)

has a symmetric nonnegative definite solution \( X(\cdot) : [0, t_f] \rightarrow \mathbb{R}^{n \times n} \) with \( X(t_f) = 0 \).

3) For every \( 0 \leq t \leq t_f \), the spectral radius of the matrix \( Y(t) X(t) \) is less than \( \tau \).

Theorem 1: Suppose that the classical uncertain system [III] is such that Assumptions [I] and [II] are satisfied. Then the controller [5] defined by the matrices

\[
A_K = A + \frac{1}{\tau} Y R_r - B_K C_2 + (B_1 + \frac{1}{\tau} Y \Upsilon_r) C_K - B_K D_{22} C_K,
\]

\[
B_K = (Y C_2^T + B_0 D_{20}^T \Gamma^{-1}), \quad C_K = -G_{\tau}^{-1} (B_1^T X + \Upsilon_{\tau}^T) (I - \frac{1}{\tau} Y X)^{-1},
\]

and the initial condition \( x_{K0} = \ddot{x}_0 \), is a guaranteed cost controller for the classical uncertain system [III], and the associated closed loop value of the cost satisfies the bound \( \hat{J}(u(\cdot)) \leq V_{r} \) for all uncertainty matrices \( \Delta \) satisfying [3] where

\[
2 V_{r} = \dot{\ddot{x}}_0^T X(0) (I - \frac{1}{\tau} Y_0 X(0))^{-1} \ddot{x}_0 \\
+ \int_0^{t_f} \text{Tr} \left[ Y R_{\tau} + B_K (Y C_2^T + B_0 D_{20}^T)^T X (I - \frac{1}{\tau} Y X)^{-1} \right] \, dt.
\]

(16)
Proof. First note the classical uncertain system (11), can be re-written in the form
\[
\begin{align*}
\frac{dx}{dt} &= (Ax + B_1u + B_0\xi) \, dt + B_0 \, dw, \\
\mu &= C_1x + D_{12}u, \\
z &= C_0x + D_0u, \\
\frac{dy}{dt} &= (C_2x + D_{22}u + D_{20}\xi) \, dt + D_{20} \, dw,
\end{align*}
\]
(17)
where \(\xi = \Delta z\). Also, \(\|\xi\| = \|\Delta z\| \leq \|z\|\). This yields the Stochastic Integral Quadratic Constraint:
\[
E \int_0^{t_f} \|\xi(t)\|^2 \, dt \leq E \int_0^{t_f} \|z(t)\|^2 \, dt.
\]
(18)
It now follows that the uncertainty in the classical uncertain system (11) is a special case of the uncertainty in the minimax optimal control result Theorem 8.4.1 of [15]. From this result the required guaranteed cost control result follows. □

B. Guaranteed Cost Control of the Quantum Uncertain System

Combining Theorem 1 and Lemma 4, we immediately obtain the following result which is the main result of the paper.

**Theorem 2:** Suppose that the quantum uncertain system (1) is such that Assumptions 1 and 2 are satisfied. Then the controller (5) defined by the matrices
\[
\begin{align*}
A_K &= A + \frac{1}{\tau}YR_\tau - B_KC_2 + (B_1 + \frac{1}{\tau}YY_\tau)C_K - B_KD_{22}C_K, \\
B_K &= (YC_2^T + B_0D_{20}^T)\Gamma^{-1}, \\
C_K &= -G_{\tau}^{-1}(B_1^TX + Y^T)(I - \frac{1}{\tau}YX)^{-1},
\end{align*}
\]
and the initial condition \(x_{K0} = \tilde{x}_0\), is a guaranteed cost controller for the quantum uncertain system (11), and the associated closed loop value of the cost satisfies the bound \(J(u(\cdot)) \leq V_\tau\) for all uncertainty matrices \(\Delta\) satisfying (3) where
\[
2V_\tau = \tilde{x}_0^TX(0)(I - \frac{1}{\tau}Y_0X(0))^{-1}\tilde{x}_0 \\
+ \int_0^{t_f} \text{Tr} \left[ YR_\tau + B_K(YC_2^T + B_0D_{20}^T) \right] X(I - \frac{1}{\tau}YX)^{-1} dt.
\]
(19)

**Remarks.** It is possible to extend Theorem 1 to the case in which the norm bounded uncertain matrix \(\Delta\) in the classical uncertain system (11) is replaced by a stable \(H^\infty\) norm bounded transfer function matrix \(\Delta(s)\); e.g., see Section 2.4.3 of [15]. This enables Theorem 2 to be
extended to allow for linear unmodeled dynamics in the quantum system (1). In this case, the linear unmodeled quantum dynamics would be defined by quantum stochastic differential equations corresponding to matrices obtained from a state space realization of the uncertain transfer function $\Delta(s)$.

Note that if the quantum system (1) has no uncertainty, $C_0 = 0$, $D_0 = 0$, and we let $\tau \to \infty$, then Theorem 2 reduces to a result on quantum LQG control; e.g., see [7], [8].

C. Application to Quantum Uncertain Systems with Uncertainty in the Hamiltonian Matrix

Rather than describing a linear quantum system in terms of a quantum stochastic differential equation such as in (1), a quantum system can also be described in terms of a quadratic Hamiltonian $H = x(0)^T R_0 x(0)$ and a coupling operator $L = \Lambda x(0)$; e.g., see [8], [11]. Here $R_0$ is an $n \times n$ real matrix and $\Lambda$ is an $N_w \times n$ complex matrix. Then as in [8], [11], the dynamics of the quantum system can be described as follows:

$$x_k(t) = U(t)^* x_k(0) U(t), \quad k = 1, 2, \ldots, n,$$
$$y_l(t) = U(t)^* w_l(t) U(t), \quad l = 1, 2, \ldots, n_y,$$
$$dU = \left( -iH dt - \frac{1}{2} L^\dagger L dt + [-L^\dagger, L^T] \Gamma \, dw \right) U, \quad U(0) = I,$$

where the variables $x_k$ are the system variables, the variables $y_l$ are the output variables and $U(t)$ is an adapted process of unitary operators. Then as in [11], the nominal matrices $A, B_0, B_1, C_2, D_{20}$ in (1) are given by

$$A = 2 \Theta \left( R_0 + \Im(\Lambda^\dagger \Lambda) \right),$$
$$\begin{bmatrix} B_0 & B_1 \end{bmatrix} = 2\Theta \left( -\Lambda^\dagger \Lambda^T \right) \Gamma,$$
$$C_2 = P_{N_y}^T \begin{bmatrix} \Sigma_{N_y} & 0_{N_y \times N_w} \\ 0_{N_y \times N_w} & \Sigma_{N_y} \end{bmatrix} \begin{bmatrix} \Lambda + \Lambda^# \\ -i\Lambda + i\Lambda^# \end{bmatrix},$$
$$\begin{bmatrix} D_{20} & 0_{n_y \times n_u} \end{bmatrix} = P_{N_y}^T \begin{bmatrix} \Sigma_{N_y} & 0_{N_y \times N_w} \\ 0_{N_y \times N_w} & \Sigma_{N_y} \end{bmatrix} \begin{bmatrix} I_{n_y \times n_y} & 0_{n_y \times (n_w-n_y)} \end{bmatrix},$$

where $N_w = \frac{n_w}{2}$, $N_y = \frac{n_y}{2}$, $\Sigma_{N_y} = \begin{bmatrix} I_{N_y \times N_y} & 0_{N_y \times (N_w-N_y)} \end{bmatrix}$, $\Theta = \frac{1}{2i} \left( x(0)x(0)^T - (x(0)x(0)^T)^T \right)$, $P_{N_y}$ is the $n_y \times n_y$ permutation matrix satisfying

$$P_{N_y} \begin{bmatrix} a_1 & a_2 & \cdots & a_{n_y} \end{bmatrix}^T = \begin{bmatrix} a_1 & a_3 & \cdots & a_{n_y-1} & a_2 & a_4 & \cdots & a_{n_y} \end{bmatrix}^T.$$
\[ \Gamma = P_{N_w} \text{diag}_{N_w} \left( \frac{1}{2} \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \right) \], and \( \Lambda^\# \) is obtained by taking the adjoint of each of the components of \( \Lambda \).

We now consider the case in which the matrix defining the Hamiltonian is subject to uncertainty with a specific structure and show that this leads to an uncertain linear quantum system of the form (I). Suppose that the quadratic Hamiltonian is of the form \( H = x(0)^T R x(0) \) where

\[ R = R_0 + \nu [-\Lambda^\dagger \Lambda^T] \Gamma_0 \Delta C_0, \Delta = \begin{bmatrix} 0_{ny \times n} \\ \hat{\Delta} \end{bmatrix}, \tag{20} \]

and \( \hat{\Delta} \) is a real \((n_w - n_u - n_y) \times n\) uncertain matrix satisfying \( \hat{\Delta} \hat{\Delta}^T \leq I \). Here \( C_0 \) is an arbitrary (but fixed) \( n \times n \) matrix, and \( \Gamma_0 \) is the matrix consisting of the first \( n_w - n_u \) columns of \( \Gamma \). In this case, it is straightforward to verify that this leads to a linear quantum uncertain system of the form (I) where the matrices \( A, B_0, B_1, C_0, C_2, D_{20} \) are defined as above.

IV. ILLUSTRATIVE EXAMPLE

In this section, we consider an example which illustrates the use of Theorem 2. We consider an optical cavity resonantly coupled to three optical channels as shown in Figure 1; e.g., see [9] and [11].

![An optical cavity (plant).](image)

Fig. 1. An optical cavity (plant).
The annihilation operator $a$ for this cavity system (representing a standing wave) evolves in time according to the equations

$$
\begin{align*}
\dot{a} &= -\frac{\gamma}{2} a \, dt - \sqrt{\kappa_1} dW - \sqrt{\kappa_2} dV - \sqrt{\kappa_3} dU, \\
\dot{Y} &= \sqrt{\kappa_1} a \, dt + dW.
\end{align*}
$$

Here $\gamma = \kappa_1 + \kappa_2 + \kappa_3$. The system (21) can be written in real quadrature form as follows (e.g., see [11]):

$$
\begin{align*}
\dot{x} &= A x \, dt + B_1 u \, dt + B_0 \begin{bmatrix} dW \\ dV \end{bmatrix}, \\
\dot{y} &= C_2 x \, dt + D_{20} \begin{bmatrix} dW \\ dV \end{bmatrix}.
\end{align*}
$$

Here $a = (x_1 + i x_2)/2$, $y = y_1 = Y + Y^*$, $V = (v_1 + i w_2)/2$, $W = (w_1 + i w_2)/2$, $U = (u_1 + i u_2)/2,$

$$
\begin{align*}
\dot{u} &= u dt = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dt, \\
\dot{v} &= \begin{bmatrix} d v_1 \\ d v_2 \end{bmatrix}, \\
\dot{w} &= \begin{bmatrix} d w_1 \\ d w_2 \end{bmatrix}, \\
A &= -\frac{\gamma}{2} I, \\
B_0 &= -\begin{bmatrix} \sqrt{\kappa_1} I_2 & \sqrt{\kappa_2} I_2 \end{bmatrix}, \\
B_1 &= -\sqrt{\kappa_3} I, \\
C_2 &= \begin{bmatrix} \sqrt{\kappa_1} & 0 \end{bmatrix}, \\
D_{20} &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T.
\end{align*}
$$

The quantum noises $v, w$ have Hermitian Ito matrices defined as follows:

$$
\begin{align*}
\dot{v}(t)\dot{v}^T(t) &= F_v dt, \\
\dot{w}(t)\dot{w}^T(t) &= F_w dt, \\
F_v &= F_w = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}.
\end{align*}
$$

Now suppose an uncertain parameter $\delta$ is introduced into the linear quantum system (22) corresponding to a perturbation in the parameter $\kappa_2$. Then linear quantum system (22) becomes:

$$
\begin{align*}
\dot{x} &= (A - \delta \frac{\gamma}{2} I) x \, dt + B_1 u \, dt + B_0(\delta) \begin{bmatrix} dW \\ dV \end{bmatrix}, \\
\dot{y} &= C_2 x \, dt + D_{20} \begin{bmatrix} dW \\ dV \end{bmatrix}.
\end{align*}
$$

(23)
where \( B_0(\delta) := -\left[ \sqrt{\kappa_1} I \quad \sqrt{\kappa_2 + \delta} I \right] \). We assume that the absolute value of the uncertain parameter \( \delta \) is bounded as \(|\delta| \leq \delta_0\) where \( \delta_0 \leq 2\sqrt{1 + \kappa_2} \). Now let

\[
\Delta = \frac{\delta}{2} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{\kappa_2 + \delta_0}} I \end{bmatrix}, \quad C_0 = I, \quad D_0 = 0
\]

and observe that \( B_0(\delta_0)\Delta C_0 = -\frac{\delta}{2} I, \quad D_{20}\Delta C_0 = 0, \quad B_0(\delta_0)\Delta D_0 = 0, \quad D_{20}\Delta D_0 = 0, \) and \( \Delta^T\Delta \leq I \). From this, it follows that the above model for the cavity is a special case of the following linear quantum uncertain system

\[
\begin{align*}
\dot{x} &= (\left[ A + B_0(\delta_0)\Delta C_0 \right] x + \left[ B_1 + B_0(\delta_0)\Delta D_0 \right] u) \ dt + B_0(\delta) \begin{bmatrix} dw \\ dv \end{bmatrix}, \\
\dot{y} &= (\left[ C_2 + D_{20}\Delta C_0 \right] x + D_{20}\Delta D_0 u) \ dt + D_{20} \begin{bmatrix} dw \\ dv \end{bmatrix},
\end{align*}
\]

(24)

where \( \Delta^T\Delta \leq I \). Furthermore, in order to convert this into a quantum uncertain system of the form (1) note that \( B_0(\delta)B_0(\delta)^T \leq B_0(\delta_0)B_0(\delta_0)^T \) for all \( \delta \) such that \(|\delta| \leq \delta_0\). Hence, we can increase the size of the noise in this uncertain system to obtain the following quantum uncertain system of the form (1):

\[
\begin{align*}
\dot{x} &= (\left[ A + B_0(\delta_0)\Delta C_0 \right] x + \left[ B_1 + B_0(\delta_0)\Delta D_0 \right] u) \ dt + B_0(\delta_0) \begin{bmatrix} dw \\ dv \end{bmatrix}, \\
\dot{y} &= (\left[ C_2 + D_{20}\Delta C_0 \right] x + D_{20}\Delta D_0 u) \ dt + D_{20} \begin{bmatrix} dw \\ dv \end{bmatrix},
\end{align*}
\]

(25)

Thus, if we can construct a controller which leads to a guaranteed cost upper bound on the cost functional

\[
J(u(\cdot)) = \int_0^{t_f} \langle x^T R x + u^T Gu \rangle \ dt,
\]

(26)

for this quantum uncertain system, then this controller will lead to the same upper bound on the closed loop value of the cost functional (26) when applied to the model of our cavity.

We now apply Theorem 2 to the quantum uncertain system (25) and cost functional (26), taking \( \kappa_1 = \kappa_2 = \kappa_3 = 2, \quad R = G = I, \quad t_f = 100, \) and \( \delta_0 = 1 \). For the long time horizon being considered, the solutions to the Riccati differential equations (14), (15) can be approximated by the solutions to the corresponding algebraic Riccati equations. Solving these Riccati equations for different values of the parameter \( \tau > 0 \), we find that the cost bound \( V_\tau \) defined in (19) is
minimized with $\tau = 1.41$. For this value of $\tau$, we obtain the cost bound of $V_\tau = 322.1$. The corresponding Riccati solutions are

$$X = \begin{bmatrix} 0.455 & 0 \\ 0 & 0.455 \end{bmatrix}; \quad Y = \begin{bmatrix} 1.267 & 0 \\ 0 & 1.361 \end{bmatrix}. $$

Also, the corresponding controller matrices are

$$A_K = \begin{bmatrix} -2.908 & 0 \\ 0 & -2.297 \end{bmatrix}; \quad B_K = \begin{bmatrix} 0.377 \\ 0 \end{bmatrix}, \quad C_K = \begin{bmatrix} 1.088 & 0 \\ 0 & 1.148 \end{bmatrix}. $$

This classical controller can be implemented using standard electronic devices. The closed loop quantum-classical system is illustrated in Figure 2.

![Diagram of an optical cavity controlled by a classical system](image)

**Fig. 2.** An optical cavity (plant) controlled by a classical system (controller $K$, implemented using standard electronics). The quadrature measurement is achieved by homodyne photo-detection (HD), and the control actions are applied via an optical modulator (Mod).

We now consider a modification to the above example to provide an example of a quantum system with uncertainty in the Hamiltonian matrix such as considered in subsection [III-C].
example consists of a cavity with uncertainty in the detuning parameter. The cavity detuning corresponds to a mismatch $\Omega$ between the resonant frequency of the optical cavity and the frequency of input field. In this case, equation (21) describing the cavity is modified by replacing the term $-\frac{\gamma}{2}$ by the term $-\frac{\gamma}{2} + i\Omega$. Thus, the coefficient matrices are the same as in the previous case except the matrix $A$ is now given by $A = \begin{bmatrix} \frac{-\gamma}{2} & -\Omega \\ \Omega & \frac{-\gamma}{2} \end{bmatrix}$.

This system can be considered as an open quantum harmonic oscillator with noise input $[w \ v \ u]^T$, Hamiltonian matrix $R = \frac{-\Omega}{2}I$ and coupling matrix $\Lambda = \begin{bmatrix} \sqrt{\kappa_1} & \sqrt{\kappa_1}t \\ \sqrt{\kappa_2} & \sqrt{\kappa_2}t \\ \sqrt{\kappa_3} & \sqrt{\kappa_3}t \end{bmatrix}$. In this example, we consider an uncertainty in the frequency mismatch $\Omega$. Indeed, a perturbation in $\Omega$ to $\Omega = \Omega_0 + \Omega_e$ with $|\Omega_e| \leq \epsilon_0$ corresponds to a perturbation $R = R_0 + E$ in the Hamiltonian matrix where $R_0 = \frac{-\Omega_0}{2}I$ and $E = \frac{-\Omega_e}{2}I$. Furthermore, this perturbation is of the form (20) with $\Delta = \begin{bmatrix} 0 & -\Omega_e \\ \Omega_e & 0 \\ 0 & 0 \end{bmatrix}$ and $C_0 = \frac{\epsilon_0}{\sqrt{\kappa_2}}I$.

The corresponding guaranteed cost controller was calculated in the case when $\Omega_0 = 0$ and $\epsilon_0 = 1$ keeping all other system parameters the same as in the previous example. In this case, a parameter value of $\tau = 0.9$ gave the minimal closed loop cost bound $V_{\tau} = 126$ and the associated controller matrices are $A_K = \begin{bmatrix} -2.067 & 0 \\ 0 & -2.336 \end{bmatrix}$; $B_K = \begin{bmatrix} 0.202 \\ 0 \end{bmatrix}$, $C_K = \begin{bmatrix} 0.519 & 0 \\ 0 & 0.521 \end{bmatrix}$.

This controller could also be implemented as in Figure 2.

V. CONCLUSIONS

In this paper, we presented a theory for synthesizing classical guaranteed cost controllers for a class of uncertain linear quantum stochastic systems. The theory was illustrated using some simple examples from quantum optics.

REFERENCES

[1] M. A. Armen, K. J. Au, J. K. Stockton, A. C. Doherty and H. Mabuchi, Adaptive homodyne measurement of optical phase, Phy. Rev. A, 89 (13), 2002.

[2] V. P. Belavkin, On the theory of controlling observable quantum systems, Automation and Remote Control, 44(2): 178-188, 1983.
[3] V. P. Belavkin, continuous non-demolition observation, quantum filtering and estimation, In Quantum aspects of Optical Communication, Vol. 45, Lecture Notes in Physics, pp. 131-145, Springer, Berlin, 1991.

[4] L. Bouten, R. Van Handel and M. R. James, An introduction to quantum filtering, arxiv.org/math.OC/0601741, 2006.

[5] E. Brown and H. Rabitz, Some mathematical and algorithmical challenges in the control of quantum dynamics phenomena, J. Mathematical Chemistry, 31(1): 17-63, 2002.

[6] C. D’Helton and M. R. James, Stability, gain, and robustness in quantum feedback networks, Phy. Rev. A, to appear. quant-ph/0511140 2006.

[7] A. C. Doherty and K. Jacobs, Feedback-control of quantum systems using continuous state estimation, Phy. Rev. A, 60: 2700, 1999, quant-ph/9812004. 2006.

[8] S. C. Edwards and V. P. Belavkin, Optimal quantum feedback control via quantum dynamic programming, quant-ph/0506018 University of Nottingham, 2005.

[9] C. W. Gardiner and P. Zoller, Quantum Noise, Springer, Berlin, 2000.

[10] J. M. Geremia, J. K. Stockton and H. Mabuchi, Real-time quantum feedback control of atomic spin-squeezing, Science, 304: 270-273, April 2004.

[11] M. R. James, H. I. Nurdin and I. R. Petersen, $H^\infty$ Control of Linear Quantum Stochastic systems, to appear in IEEE Transactions on Automatic Control.

[12] S. Lloyd, Coherent quantum feedback, Phy. Rev. A, 62: 022108, 2000.

[13] H. I. Nurdin, M. R. James, and I. R. Petersen. Quantum LQG control with quantum mechanical controllers. In Proceedings of the 17th IFAC World Congress, Seoul, Korea, July 2008. To Appear, also see arXiv:0711.2551v1 [quant-ph].

[14] K. R. Parthasarathy, An Introduction to Quantum Stochastic Calculus, Birkhauser, 1992.

[15] I. R. Petersen, V. Ugrinovskii and A. V. Savkin, Robust Control Design Using $H^\infty$ Methods, Springer, 2000.

[16] H. Wiseman, Quantum theory of continuous feedback, Phy. Rev. A, 49(3): 2133-2150, 1994.

[17] M. Yanagisawa and H. Kimura, Transfer function approach to quantum control, Part I: Dynamics of quantum feedback systems, IEE Trans. Automatic Control, 48(12):2107-2120, 2003.

[18] M. Yanagisawa and H. Kimura, Transfer function approach to quantum control, Part II: Control Concepts and applications, IEE Trans. Automatic Control, 48(12):2107-2121, 2132.

[19] H. Zhang and H. Rabitz, Robust optimal control of quantum molecular systems in the presence of disturbances and uncertainties, Phys. Rev. A, 49(4):2241-2254, 1994.