THE EXTRINSIC PRIMITIVE TORSION PROBLEM

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ABSTRACT. Let $P_k$ be the subgroup generated by $k$th powers of primitive elements in $F_r$. We show that $F_2/P_k$ is finite if and only if $k$ is 1, 2, or 3. We also fully characterize $F_2/P_k$ for $k = 2, 3, 4$. In particular, we give a faithful infinite nine dimensional representation of $F_2/P_4$.

Keywords. The Burnside Problem, Primitive elements, Characteristic subgroups

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1. INTRODUCTION

Let $G$ be a group such that the smallest cardinality of a generating set of $G$ is $r$. Recall that $g \in G$ is primitive if it is part of a generating set of $G$ with $r$ elements. Denote the rank $k$ free group by $F_k$. In this article, we tackle the following variants of the Burnside Problem:
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Question 1 (The Extrinsic (Restricted) Primitive Torsion Problems). Fix positive integers $r$ and $k$. Let $\Gamma$ be a (residually finite) image of $F_r$ so that the image of every $r$-primitive element in $F_r$ has order dividing $k$.

(a) Is $\Gamma$ necessarily finite?
(b) Is $\Gamma$ necessarily virtually nilpotent?
(c) Is $\Gamma$ necessarily virtually solvable?
(d) Is $\Gamma$ necessarily finitely presented?

This paper answers Question 1 in the cases $r = 2$ and $k \in \{2, 3, 4\}$. The answers to all questions in these cases are yes, except that when $r = 2$ and $k = 4$, the group $\Gamma$ need not be finite.

For $r = 2$ and $k \geq 5$, we show that the answer to Question 1(a) is no. We do not know the answers for parts (b), (c), or (d) for $r = 2$ and $k \geq 5$ and we do not consider the case $r > 2$ in this paper.

Let $P_{r,k} \subset F_r$ be the subgroup generated by $k$th powers of primitive elements in $F_r$. Observe that the answers to Question 1(a)-(d) is affirmative if and only if the respective answer to (a)-(d) is affirmative for $\Gamma = F_r/P_{r,k}$.

In this paper we concentrate on answering this question when $r = 2$. In fact we originally became interested in answering Question 1 in the case of $r = 2$ because of related geometric questions about square-tiled surfaces; see Appendix A. We summarize our results in following table where we use $P_k$ to denote $P_{2,k}$ and use $H(R)$ to denote be the Heisenberg group over a ring $R$.

| Subgroup | Index in $F_2$ | Quotient $G_k = F_2/P_k$ |
|----------|----------------|--------------------------|
| $P_2$    | 4              | The Klein four-group.     |
| $P_3$    | 27             | $H(\mathbb{Z}/3)$.        |
| $P_4$    | $\infty$      | Virtually a five dimensional image of $H(\mathbb{Z}) \times H(\mathbb{Z})$. |
| $P_5$    | $\infty$      | We conjecture virtually solvable. |
| $P_k$ with $k \geq 6$ | $\infty$ | We conjecture the quotient is not finitely presented. |

In resolving the cases $k = 4$ we show that $F_2/P_4$ is isomorphic to the matrix group generated by the following two matrices:

$$\text{diag}(1, -1, -i, -\bar{i}; 1, 1, 1, 1, 1),$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. $$

For $k \geq 5$, we develop tools for constructing and refining new infinite linear representations of $F_2/P_k$. These tools allow us to answer Question 1(a), and we hope they will be useful in future work.
Instead of speaking of primitivity in a free group, we can phrase an intrinsic version of Question 1. The rank of a group is the cardinality of a generating set of the group of minimal size.

**Question 2** (The Intrinsic (Restricted) Primitive Torsion Problems). Fix positive integers \( r \) and \( k \). Let \( \Gamma \) be a (residually finite) group of rank \( r \) such that every primitive element has order dividing \( k \). Which questions from Question 1 have affirmative answers?

The Primitive Torsion Problems are natural variants of the original *Bounded Burnside Problem*. There has been great progress in understanding these quotients arising from these problems, see for instance [CG17] for the state of the art.

**Question 3** (The Bounded Burnside Problem). Fix \( r, k \in \mathbb{Z} \). Let \( G \) be a group generated by \( r \) elements. Let \( B_k \) be the group in \( G \) generated by elements of the form \( g^k \) where \( g \in G \). Is \( G/B_k \) necessarily finite?

We note that when \( G/B_k \) is virtually solvable, the resulting group \( G/B_k \) is necessarily finite. Thus, our work recovers the well-known result that \( F \) is finite. If our conjecture that \( F_2/P_3 \) is virtually solvable is correct, then it follows that \( F_2/B_3 \) is finite, which is unknown.

There has been significant work on asymptotic versions of the Primitive Torsion Problem. In [KS16], Quantum Topological Field Theory answers Primitive Torsion Problems for sufficiently large \( k \) over surface groups.

2. **Normal generators for \( P_k \)**

2.1. **Primitive elements of \( F_2 \).** Let \( F_2 \) denote the free group \( \langle a, b \rangle \). The reader will recall or quickly observe the following facts about primitive elements of \( F_2 \):

1. If \( c \in F_2 \) is primitive then there is a \( \phi \in \text{Aut}(F_2) \) so that \( \phi(a) = c \).
2. If \( c \in F_2 \) is primitive then every element of its conjugacy class \( \{gcg^{-1} : g \in F_2\} \).

In particular, we will say a conjugacy class is *primitive* if it consists of primitive elements of \( F_2 \).

The observation that there is a short exact sequence

\[ 1 \rightarrow F_2 \rightarrow \text{Aut}(F_2) \rightarrow \text{GL}(2, \mathbb{Z}) \rightarrow 1. \]

dates back to Jakob Nielsen’s 1913 Thesis. Here the map \( F_2 \rightarrow \text{Aut}(F_2) \) sends an element of \( F_2 \) to its corresponding inner automorphism and thus \( \text{GL}(2, \mathbb{Z}) \) is isomorphic to the outer automorphism group \( \text{Out}(F_2) = \text{Aut}(F_2)/\text{Inn}(F_2) \). The map \( D: \text{Aut}(F_2) \rightarrow \text{GL}(2, \mathbb{Z}) \) may be defined by using the abelianization homeomorphism \( ab : F_2 \rightarrow \mathbb{Z}^2 \) defined so that \( a \rightarrow (1,0) \) and \( b \rightarrow (0,1) \). Then \( D(\phi) \in \text{GL}(2, \mathbb{Z}) \) is determined by the condition that \( D(\phi) \circ ab(g) = ab \circ \phi(g) \) for all \( g \in F_2 \).

An automorphism of \( F_2 \) either preserves the conjugacy class of the commutator \([a, b]\) or sends it to the conjugacy class of \([b, a]\). Thus there is a natural homomorphism \( \text{Aut}(F_2) \rightarrow C_2 \) where we identify \( C_2 \) with the permutation group of these conjugacy classes. We set \( \text{Aut}_+(F_2) \) to be the kernel which consists of automorphisms preserving the conjugacy class of the commutator \([a, b]\). We use \( \text{Aut}_-(F_2) \) to denote \( \text{Aut}(F_2) \setminus \text{Aut}_+(F_2) \).

The group \( \text{Out}_+(F_2) = \text{Aut}_+(F_2)/\text{Inn}(F_2) \) is isomorphic to \( \text{SL}(2, \mathbb{Z}) \) via the map \( D \) above. The following elements of \( \text{Aut}_+(F_2) \) have images in \( \text{Out}_+(F_2) \) which generate:

\[ \psi_0(a) = b, \; \psi_0(b) = b^{-1}a^{-1}; \; \psi_1(a) = b, \; \psi_1(b) = a^{-1}; \; \psi_2(a) = a, \; \psi_2(b) = ab. \]
We’ll use $\psi_0$, $\psi_1$, and $\psi_2$ to denote the outer automorphism classes of these elements. It may be observed that the following identities are satisfied:
\[
\psi_0 \circ \psi_2 = \psi_1, \quad \psi_0^3 = \psi_1^4 = 1, \quad [\psi_1^2, \psi_0] = [\psi_1^2, \psi_2] = 1.
\]
(2)

Recall that outer automorphisms act on conjugacy classes. We’ll use $[g]$ to denote the conjugacy class of $g \in F_2$. We have the following:

**Lemma 2.1** (Primitive conjugacy classes). An element $g \in F_2$ is primitive if and only if it lies in the conjugacy class $\psi([a])$ for some $\psi \in \text{Out}_+(F_2)$.

**Proof.** If $g \in F_2$ is primitive then by (1) above there is a $\psi \in \text{Aut}(F_2)$ so that $\psi(a) = g$. Then by possibly precomposing with the automorphism $\psi_- \in \text{Aut}^{-}(F_2)$ determined by $\psi_-(a) = a$ and $\psi_-(b) = b^{-1}$ we can assume that $\psi \in \text{Aut}_+(F_2)$. Let $\psi \in \text{Out}_+(F_2)$ be the class containing $\psi$. Then $[g] = [\psi([a])]$. The converse is clear since primitivity is a conjugacy invariant and invariant under automorphisms. □

It follows that the conjugacy classes of primitive elements are naturally identified with $\text{Out}_+(F_2)$ modulo the stabilizer of the conjugacy class $[a]$. This stabilizer is $\langle \psi_2 \rangle$.

The primitive conjugacy classes come naturally in pairs: if $g \in F_2$ is primitive, then we call the conjugacy classes $[g]$ and $[g^{-1}]$ opposites. We’ll denote the collection of unions of opposite pairs of conjugacy classes by $\mathcal{P}$. Opposites are related by the action of the central involution $\psi_2^2$ of $\text{Out}_+(F_2)$:

**Proposition 2.2.** If $[g]$ is a primitive conjugacy class then its opposite $[g^{-1}]$ is $\psi_2^2([g])$.

**Proof.** From the lemma above we have $[g] = \psi([a])$ for some $\psi \in \text{Out}_+(F_2)$. Since $\psi_2^2(a) = a^{-1}$ we have $\psi_2^2([a]) = [a^{-1}]$ and $[g^{-1}] = \psi \circ \psi_2^2([a])$. Since $\psi_2^2$ is central in $\text{Out}_+(F_2)$ we have $[g^{-1}] = \psi_2^2 \circ \psi([a]) = \psi_2^2([g])$. □

Since $\langle \psi_2 \rangle$ is the stabilizer of $[a]$ and $\psi_2^2$ acts as above, there is a bijective correspondence from the coset space
\[
\mathcal{C} = \text{Out}_+(F_2) / \langle \psi_2, \psi_2^2 \rangle \quad \text{to} \quad \mathcal{P} \quad \text{given by} \quad \psi([\psi_2, \psi_2^2]) \mapsto \psi([a]) \cup \psi([a^{-1}]).
\]
(3)

The group $\text{SL}(2, \mathbb{Z}) / \pm I$ has a well known action on the upper half plane by Möbius transformations with $-I$ acting trivially. Here the matrix
\[
\left( \begin{array}{cc} m_{11} & m_{12} \\ m_{21} & m_{22} \end{array} \right) \quad \text{acts by} \quad z \mapsto \frac{m_{11}z + m_{12}}{m_{21}z + m_{22}}.
\]
This is useful for organizing the pairs of primitive conjugacy classes. Observe that $\langle D(\psi_2), D(\psi_1) \rangle$ is the stabilizer in $\text{SL}(2, \mathbb{Z})$ of the point $\frac{1}{0}$. The $\text{SL}(2, \mathbb{Z})$ orbit of $\frac{1}{0}$ is $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{ \frac{1}{0} \}$. Thus, we have:

**Lemma 2.3.** There are bijections $\mathcal{C} : \hat{\mathbb{Q}} \to \mathcal{C}$ and $\mathcal{P} : \hat{\mathbb{Q}} \to \mathcal{P}$ compatible with (3) so that for any $\frac{p}{q} \in \hat{\mathbb{Q}}$ we have:

- The class $\mathcal{C}(\frac{p}{q})$ is the collection of $\psi \in \text{Out}_+(F_2)$ so that $D(\psi)(\frac{1}{0}) = \frac{p}{q}$.
- The union of the pair of conjugacy classes $\mathcal{P}(\frac{p}{q})$ consists of all primitive elements $g \in F_2$ so that $ab(g) = \pm(p, q)$ (where $p, q \in \mathbb{Z}$ are taken to be relatively prime).

The Farey triangulation $\mathcal{T}$ is an $\text{SL}(2, \mathbb{Z})$ invariant triangulation of the upper half plane with vertices in $\hat{\mathbb{Q}}$. We depict $\mathcal{T}$ in Figure 1. The group $\text{PSL}(2, \mathbb{Z})$ is the orientation preserving symmetry group of $\mathcal{T}$. It is useful to think of the three spaces $\hat{\mathbb{Q}}$, $\mathcal{C}$ and $\mathcal{P}$ as in bijective correspondence to the vertices in this triangulation.
2.2. Symmetries and images of primitive elements. Outer automorphisms of $F_2$ act on the normal subgroups of $F_2$. Having a power of a primitive element in a normal subgroup $N$ guarantees that some corresponding elements of $\text{Out}(F_2)$ stabilize $N$ and act trivially on $F_2/N$.

Lemma 2.4. Let $g \in F_2$ be primitive. Then there is a $p/q \in \hat{Q}$ so that one of the conjugacy classes in the pair $\mathcal{P}(p/q)$ contains $g$. Suppose $N \subset F_2$ is a normal subgroup containing $g^k$ for some $k \geq 1$. Then for any $\bar{\psi} \in \text{Out}_+(F_2)$ taken from the coset $\mathcal{C}(p/q)$ we have

$$\bar{\psi} \circ \bar{\psi}^k \circ \bar{\psi}^{-1}(N) = N.$$ 

Furthermore if we choose a representative $\psi \in \text{Aut}(F_2)$ from the outer automorphism class $\bar{\psi}$ then the induced automorphism of $F_2/N$ given by

$$hN \mapsto \psi \circ \psi^k \circ \psi^{-1}(h)N$$

is the trivial automorphism of $F_2/N$.

We remark that the outer automorphism $\bar{\psi} \circ \bar{\psi}^k \circ \bar{\psi}^{-1}$ discussed above is independent of the choice of $\bar{\psi}$ from $\mathcal{C}(p/q)$.

Proof. Assume $g$, $p/q$, $k$ and $N$ satisfy the hypotheses above. Select $\bar{\psi}$ as in the proposition and fix a representative $\psi$. Then $\psi([a] \cup [a^{-1}]) = [g] \cup [g^{-1}]$. Let $M = \psi^{-1}(N)$. Then $M$ is normal and contains $a^k$. After this change of coordinates it suffices to prove that $\psi^k(M) = M$ and that $hM \mapsto \psi^k(h)M$ is an inner automorphism of $F_2/M$.

To understand why these statements are true, observe that $\psi^k$ acts so that

$$\psi^k(a) = a, \quad \psi^k(a^{-1}) = a^{-1}, \quad \psi^k(b) = a^kb \quad \text{and} \quad \psi^k(b^{-1}) = b^{-1}a^{-k}.$$ 

Let $h \in F_2$ and consider $h$ as a word in $\{a,a^{-1},b,b^{-1}\}$. From the above description of $\psi^k$ we see that $\psi^k(h)$ is formed from $h$ by inserting copies of $a^k$ and $a^{-k}$ into the word representing $h$. Let $n$ be the number of such insertions. Then we can write

$$h = \psi^k(h)g_1g_2\cdots g_n$$

where the $g_i$ are iteratively chosen to be a conjugate of either $a^{-k}$ or $a^k$ chosen respectively to remove an inserted copy of $a^k$ or $a^{-k}$. Since $a^k \in M$ and $M$ is normal each $g_i \in M$. Thus
Recall that $P_k \subset F_2$ is the subgroup generated by $k$-th powers of primitive elements of $F_2$. This subgroup is clearly characteristic, thus there is a well defined homomorphism

$$\varepsilon : \text{Aut}(F_2) \to \text{Aut}(F_2/P_k); \quad \varepsilon(\phi)(gP_k) = \phi(g)P_k.$$  

Inner automorphism of $F_2$ are sent by $\varepsilon$ to inner automorphisms of $F_2/P_k$, thus $\varepsilon$ induces a well defined map between outer automorphism groups

$$\bar{\varepsilon} : \text{Out}(F_2) \to \text{Out}(F_2/P_k).$$

The lemma above guarantees:

**Corollary 2.5.** The subgroup $O_k \subset \text{Out}_+(F_2)$ normally generated by $\psi_2^k$ is contained in $\ker \bar{\varepsilon}$.

**Proof.** We must show that for each $\psi_2 \in \text{Out}_+(F_2)$ we have $\psi_2 \circ \psi_2^k \circ \psi_2^{-1} \in \ker \bar{\varepsilon}$. Fixing $\psi$, we may find a $\frac{p}{q} \in \mathbb{Q}$ so that $\psi \in \mathcal{C}(\frac{p}{q})$. Choose a $g \in \mathcal{P}(\frac{p}{q})$. Then $g$ is primitive so that $g^k \in P_k$. Choose a representative $\psi \in \text{Aut}_+(F_2)$ of $\psi$. Lemma 2.4 guarantees that $\varepsilon(\psi \circ \psi_2^k \circ \psi_2^{-1})$ is the trivial automorphism of $F_2/P_k$ so that $\psi \circ \psi_2^k \circ \psi_2^{-1} \in \ker \bar{\varepsilon}$ as desired.

We remark that the image under $D$ of $\psi_2^k$ has the effect of “rotating” the Farey triangulation $\mathcal{F}$ by $k$ triangles about the vertex $\frac{1}{6}$. It follows that the triangulated space

$$\mathcal{F}_k = \mathcal{F}/D\mathcal{O}_k$$

is combinatorially the triangulation of a simply connected surface with $k$ triangles meeting at every vertex. So, $\mathcal{F}_k$ a triangulation of the sphere when $k \leq 5$, combinatorially equivalent to the tiling of the plane by Euclidean triangles when $k = 6$, and combinatorially equivalent to a tiling of the hyperbolic plane by regular triangles when $k \geq 7$.

The same mechanism can be used to shorten the list of group elements needed to normally generate $P_k$.

**Theorem 2.6.** Let $k \geq 2$. Let $\{\frac{p}{q_i} : i \in \Lambda\}$ be a subset of $\mathbb{Q}$ containing one representative of each preimage of a vertex of $\mathcal{F}_k$ under the covering map $\mathcal{F} \to \mathcal{F}_k$. For each $i$ choose a primitive element $g_i \in \mathcal{P}(\frac{p_i}{q_i})$ and an outer automorphism $\psi_i \in \mathcal{C}(\frac{p_i}{q_i})$. If

$$\{\psi_i \circ \psi_2^k \circ \psi_i^{-1} : i \in \Lambda\}$$

generates $\mathcal{O}_k$ then $P_k$ is normally generated by $\{g_i^k : i \in \Lambda\}$.

**Proof.** Fix the quantities above and assume all hypotheses are satisfied. Let $Q$ be the subgroup of $F_2$ normally generated by $\{g_i^k : i \in \Lambda\}$. Clearly $Q \subset P_k$ since each $g_i$ is primitive. We will show $P_k \subset Q$.

As a consequence of Lemma 2.4 we know that $\psi_i \circ \psi_2^k \circ \psi_i^{-1}$ stabilizes $Q$ for all $i \in \Lambda$. Then from the hypotheses we know each element of $\mathcal{O}_k$ stabilizes $Q$.

To show $P_k \subset Q$, it suffices to show that if $g \in F_2$ is primitive then $g^k \in Q$. Fix $g$. Then there is a $\frac{p}{q} \in \mathbb{Q}$ so that $\mathcal{P}(\frac{p}{q}) = [g] \cup [g^{-1}]$. From our hypothesis on $\{\frac{p_i}{q_i}\}$ we know there is an $i \in \Lambda$ and a $\psi_i \in \mathcal{O}_k$ so that $D\psi_i(\frac{p_i}{q_i}) = \frac{r}{q}$. Then $\psi_i([g_i] \cup [g_i^{-1}]) = [g] \cup [g^{-1}]$. By definition of $Q$ we know that the conjugacy classes $[g_i^k]$ and $[g_i^{-k}]$ are contained in $Q$. Since $Q$ is $\mathcal{O}_k$-invariant and $g^k \in \psi_i([g_i^k] \cup [g_i^{-k}])$ we have $g^k \in Q$ as desired.

The following describes a combinatorial way to find the generators:
Corollary 2.7. Fix \( k \geq 2 \). Let \( T \subset T_k \) be a tree in the 1-skeleton of \( T_k \) whose vertices are vertices of the triangulation and include all vertices of the triangulation. Let \( \bar{T} \) be a lift of \( T \) to \( \mathcal{T} \) and let \( \{ \bar{P}_q : i \in \Lambda \} \) be the vertices of \( \bar{T} \). Then \( P_k = (g_i)_{i \in \Lambda} \) where each \( g_i \in \mathcal{P}(\bar{P}_q) \) is chosen arbitrarily as in Theorem 2.6.

Proof. We must check the hypotheses of Theorem 2.6. Define \( \{ \bar{P}_q \} \) and \( \{ g_i \} \) as in the statement of the corollary and \( \{ \bar{P}_q \} \) as in Theorem 2.6. Since the vertices of \( T \) include all vertices of \( T_k \), we see that \( \{ \bar{P}_q \} \) contains one preimage of each vertex of \( T_k \). Let \( Q = \langle \bar{P}_q \circ \bar{P}_q \circ \bar{P}_q^{-1} \rangle \subset O_k \). We need to show \( Q = O_k \).

Associated to the chain of subgroups \( \{ 1 \} \subset Q \subset O_k \) is the sequence of spaces related by covering maps branched at the vertices of the triangulations:

\[ \mathcal{T} \to \mathcal{T}/DQ \to T_k. \]

Because these maps are branched at the vertices, paths through the vertices may not have unique lifts. Observe that proving \( Q = O_k \) is equivalent to proving that \( \mathcal{T}/DQ = T_k \) or that the triangulation \( \mathcal{T}/DQ = T_k \) has \( k \) triangles around each vertex.

Let \( T_0 \subset \mathcal{T}/Q \) denote the image of \( T \) under the covering map \( T \to \mathcal{T}/DQ \). Then \( T_0 \) is a tree because \( \pi(T_0) = T \). Observe that each vertex of \( T_0 \) is incident to \( k \) triangles because such a vertex is the image of some \( \bar{P}_q \in \bar{T} \) and the action of \( D(\bar{P}_q \circ \bar{P}_q \circ \bar{P}_q^{-1}) \) on \( \mathcal{T} \) rotates by \( k \) triangles about \( \bar{P}_q \). Thus it suffices to prove that every vertex of \( \mathcal{T}/Q \) is a vertex of the tree \( T_0 \). If this were not the case that there would be an edge of a triangle of \( \mathcal{T}/Q \) so that one vertex is a vertex of \( T_0 \) and the other is not. We will show this doesn’t happen.

A key observation is the following. Say that the link of a vertex of a triangulated surface is the union of the vertex with the interiors of incident edges and triangles. The link lifting observation is the observation that \( \pi \) restricted to the link of a vertex \( v_0 \in T_0 \subset \mathcal{T}_k \) is a bijection to the link of the image vertex \( v = \pi(v_0) \in T \subset T_k \) since both \( v_0 \) and \( v \) are incident to \( k \) triangles.

Now we return to the proof. Suppose \( e_0 = v_0w_0 \) be an oriented edge of a triangle of \( \mathcal{T}/Q \) initiating at a vertex \( v_0 \) of \( T_0 \). We will show that the terminating vertex \( w_0 \) is also a vertex of \( T_0 \). Let \( e = \bar{v}_0 \bar{w}_0 \) be \( (e_0) \). We break into two cases.

First, it could be that \( e \) is an edge of \( T \). Since \( v_0 \in T_0 \) by the link lifting observation we know that \( e \) has a unique lift to \( \mathcal{T}_k \) initiating at \( v_0 \). Since \( T_0 \) is a lift of \( T \) and \( e \) is an edge of \( T \), this means that \( e_0 \) must be an edge of \( T_0 \). Thus, \( w_0 \) is also a vertex of \( T_0 \) as desired.

Now suppose that \( e \) is not an edge of \( T \). Since \( T \) is a spanning tree, both \( v \) and \( w \) are vertices of \( T \). As \( T \) is a tree, there is a unique oriented path \( p \in T \) joining \( v \) to \( w \). Let \( v = p_0, p_1, \ldots, p_n = w \) be the sequence of vertices passed through by \( p \). We will inductively prove \( p \) has a unique lift to \( \mathcal{T}/Q \) starting at \( v_0 \). This involves checking that for each \( j \in \{ 1, \ldots, n \} \) there is a unique lift of the path \( p_0, \ldots, p_j \) denoted \( \tilde{p}_0, \ldots, \tilde{p}_j \) so that \( \tilde{p}_0 = v_0 \) and \( \pi(\tilde{p}_i p_{i+1}) = p_i p_{i+1} \) for \( i \in \{ 0, \ldots, j-1 \} \). This is true for \( j = 1 \) because \( v_0 \in T_0 \) using the unique lifting provided by the observation above. Now we will argue the inductive step. Suppose the lift is unique up through index \( j < n \). Then since \( p \) is a path in \( T \) and \( \pi(T_0) = T \), we must have that all vertices of the lift so far lie in \( T_0 \). From the link lifting observation we know that there is a unique lift of the next edge \( \tilde{p}_j p_{j+1} \) completing the inductive step.

Now observe that since \( T_k \) is a triangulation of a simply connected surface, \( p \cup e \) bounds a topological disk \( \Delta \). Since all vertices of \( \mathcal{T}_k \) lie in \( T \), no vertices are contained in the interior of \( T \). Since the vertex \( v_0 \in T_0 \) by the link lifting observation again, there is a unique lift \( \Delta \) of \( \Delta \) to \( \mathcal{T}/Q \) so that \( v \) lifts to \( v_0 \). From the previous paragraph, the path \( p \) in
the boundary of $\Delta$ lifts to a path $\tilde{p}$ in the boundary of $\tilde{\Delta}$ and contained in the tree $T_Q$. Again by the link lifting observation, edge $e$ in the boundary of $\Delta$ lifts to $e_Q$ in the boundary of $\tilde{\Delta}$. Thus, $e_Q$ joins the initial point $v_Q$ of $\tilde{p}$ to the terminal point $w_Q$ of $\tilde{p}$. Since $\tilde{p}$ is contained in $T_Q$ we see that $w_Q \in T_Q$ as desired. □

Conjecture 1. The normal generators for $P_k$ provided by Corollary 2.7 are a minimal set of normal generators. In particular, for $k \geq 6$, the group $F_2/P_k$ is not finitely presented.

2.3. Normal generators for $P_k$ with $k \leq 5$. We describe normal generators for $P_k$ when $k \leq 5$ because these are the cases where Corollary 2.7 yields a finite set of normal generators. These cases are finite because $\mathcal{F}_k$ of (4) is a triangulated sphere.

The case $k = 2$. The triangulated sphere $\mathcal{F}_2$ is the double of a triangle across its boundary. Below we depict a tree $T$ in an unfolding of $\mathcal{F}_2$. We have lifted $T$ to a tree $\tilde{T}$ in the Farey triangulation and labeled the vertices of $T$ by their lifts as elements of $\hat{Q}$. Following Theorem 2.6 and Corollary 2.7 we have converted these elements of $\hat{Q}$ to normal generators of $P_2$.

\[ \begin{array}{cc}
\text{Vertex} & \text{Generator of } P_2 \\
\infty & a^2 \\
0 & b^2 \\
1 & (ab)^2\end{array} \]

Proposition 2.8. The quotient $F_2/P_2$ is isomorphic to the Klein four-group.

Proof. We have

\[ F_2/P_2 = \langle a, b | a^2, b^2, (ab)^2 \rangle. \]

Clearly $a$ and $b$ have order two in this quotient. We have $[a, b] = (ab)^2 = 1$ since $a = a^{-1}$ and $b = b^{-1}$. □

The case $k = 3$. The triangulated sphere $\mathcal{F}_3$ is a tetrahedron. Below we depict a tree $T$ in an unfolded copy of the tetrahedron. We have lifted $T$ to a tree $\tilde{T}$ in the Farey triangulation and labeled the vertices of $T$ by their lifts as elements of $\hat{Q}$. Following Theorem 2.6 and Corollary 2.7 we have converted these elements of $\hat{Q}$ to normal generators of $P_3$.

\[ \begin{array}{cc}
\text{Vertex} & \text{Generator of } P_3 \\
\infty & a^3 \\
0 & b^3 \\
1 & (ab)^3 \\
-1 & (ab^{-1})^3\end{array} \]

Proposition 2.9. The quotient $F_2/P_3$ is isomorphic to $UT(3, 3)$, the group of upper triangular matrices with entries in $\mathbb{Z}/3\mathbb{Z}$ and with ones along the diagonal.
Proof. In the group $F_2/P_3 = \langle a, b : a^3, b^3, (ab)^3, (ab^{-1})^3 \rangle$, we have:

\[
\begin{align*}
[a, [a, b]] &= a^{-1}(b^{-1}a^{-1}ba)a(a^{-1}b^{-1}ab) \\
&= a^{-1}b^{-1}a^{-1}bab^{-1}ab \\
&= (a^{-1}b^{-1})^2b^2ab^{-1}ab \\
&= b(ab^{-1})^3ba^{-1}ab \\
&= b^2a^{-1}ab = b^3 \\
&= 1.
\end{align*}
\]

Further, since $P_3$ is characteristic, we get $[b, [a, b]] = 1$. It follows that $[a, b]$ is central, thus $[a, b]^3 = [a^3, b] = 1$ via commutator identities. Thus, $F_2/P_3 \cong UT(3, 3)$, since $UT(3, 3) = H(\mathbb{Z}/3\mathbb{Z}) = \langle a, b \mid a^3, b^3, [a, [a, b]], [b, [a, b]] \rangle$. □

The case $k = 4$. The triangulated sphere $\mathcal{S}_4$ is an octahedron. Below we depict a tree $T$ in an unfolded copy of the octahedron. We have lifted $T$ to a tree $\tilde{T}$ in the Farey triangulation and labeled the vertices of $T$ by their lifts as elements of $\hat{Q}$. Following Theorem 2.6 and Corollary 2.7, we have converted these elements of $\hat{Q}$ to normal generators of $P_4$.

| Vertex | Generator of $P_4$ |
|--------|-------------------|
| $\infty$ | $a^4$ |
| 0 | $b^4$ |
| 1 | $(ab)^4$ |
| $-1$ | $(ab^{-1})^4$ |
| 2 | $(a^2b)^4$ |
| $\frac{1}{2}$ | $(ab^2)^4$ |

The case $k = 5$. The triangulated sphere $\mathcal{S}_5$ is an icosahedron. Below we depict a tree $T$ in an unfolded copy of the icosahedron. We have lifted $T$ to a tree $\tilde{T}$ in the Farey triangulation and labeled the vertices of $T$ by their lifts as elements of $\hat{Q}$. Following Theorem 2.6 and Corollary 2.7, we have converted these elements of $\hat{Q}$ to normal generators of $P_5$. 
3. Characteristic representations

3.1. Definition and a criterion. We say that a homomorphism \( \rho : F_2 \to \text{GL}(n, \mathbb{C}) \) is a characteristic representation if for any \( \psi \in \text{Aut}(F_2) \) there is a \( \Psi \in \text{Aut}(\text{GL}(n, \mathbb{C})) \) so that

\[
\Psi \circ \rho \circ \psi^{-1}(g) = \rho(g) \quad \text{for all } g \in F_2.
\]

The following should be clear:

**Proposition 3.1.** The kernel of a characteristic representation is a characteristic subgroup of \( F_2 \).

Recall from §2.1 that \( \text{Aut}(F_2) = \text{Aut}_+ (F_2) \cup \text{Aut}_- (F_2) \). Our automorphisms of \( \text{GL}(n, \mathbb{C}) \) will have one of two forms corresponding to this partition. If \( M \in \text{GL}(n, \mathbb{C}) \) then we define

\[
\Psi_M, \overline{\Psi}_M \in \text{Aut}(\text{GL}(n, \mathbb{C})) \quad \text{by} \quad \Psi_M(X) = MXM^{-1} \quad \text{and} \quad \overline{\Psi}_M(X) = M\overline{X}M^{-1}.
\]
The group of inner automorphisms is \( \text{Inn} \left( (\text{GL}(n, \mathbb{C})) \right) = \{ \Psi_M : M \in \text{GL}(n, \mathbb{C}) \} \) and the collection of conjugate-inner automorphisms is \( \text{Inn} \left( (\text{GL}(n, \mathbb{C})) \right) = \{ \Psi_M : M \in \text{GL}(n, \mathbb{C}) \} \). The union \( \text{Inn}(\text{GL}(n, \mathbb{C})) \cup \text{Inn}(\text{GL}(n, \mathbb{C})) \) is a subgroup of \( \text{Aut}(\text{GL}(n, \mathbb{C})) \).

We say \( \rho : F_2 \to \text{GL}(n, \mathbb{C}) \) is a oriented characteristic representation if the following two statements hold:

- For each \( \psi \in \text{Aut}_+(F_2) \) there is an \( \Psi_M \in \text{Inn}(\text{GL}(n, \mathbb{C})) \) so that \( \boxed{5} \) holds.
- For each \( \psi \in \text{Aut}_-(F_2) \) there is an \( \Psi_M \in \text{Inn}(\text{GL}(n, \mathbb{C})) \) so that \( \boxed{5} \) holds.

Observe that oriented characteristic representations are examples of characteristic representations. We will be working exclusively with oriented characteristic representations.

Based on properties of the tensor product, it can be observed:

**Proposition 3.2.** If \( \rho_1 : F_2 \to \text{GL}(n_1, \mathbb{C}) \) and \( \rho_2 : F_2 \to \text{GL}(n_2, \mathbb{C}) \) are oriented characteristic representations then so is their tensor product \( \rho_1 \otimes \rho_2 : F_2 \to \text{GL}(n_1n_2, \mathbb{C}) \) and so is the complex-conjugate representation \( \overline{\rho} \).

We will now give an elementary method to prove that a homomorphism \( \rho \) is an oriented characteristic representation. We single out elements \( \psi_1, \psi_2 \in \text{Aut}_+(F_2) \) and \( \psi_- \in \text{Aut}_-(F_2) \) whose images in \( \text{Out}(F_2) \) generate:

\[
\begin{align*}
\psi_1(a) &= b, \quad \psi_1(b) = a^{-1}; \\
\psi_2(a) &= a, \quad \psi_2(b) = ab; \\
\psi_-(a) &= a^{-1}, \quad \psi_-(b) = b.
\end{align*}
\]

(7)

We have the following criterion for checking if a representation is oriented characteristic:

**Proposition 3.3.** Let \( \rho : F_2 \to \text{GL}(n, \mathbb{C}) \) be a homomorphism. Then \( \rho \) is an oriented characteristic representation if and only if the following statements are satisfied:

1. There is an \( M_1 \in \text{GL}(n, \mathbb{C}) \) so that \( M_1 = \rho(a)M_1\rho(b) \) and \( M_1\rho(a) = \rho(b)M_1 \).
2. There is an \( M_2 \in \text{GL}(n, \mathbb{C}) \) so that \( M_2\rho(a) = \rho(a)M_2 \) and \( M_2\rho(b) = \rho(ab)M_2 \).
3. There is an \( M_- \in \text{GL}(n, \mathbb{C}) \) so that \( M_- = \rho(a)\overline{\rho(a)} \) and \( M_-\overline{\rho(b)} = \rho(b)M_- \).

We remark that the equations in the respective statements above are simple algebraic manipulations of \( \boxed{5} \) in the special cases where \( (\psi, \Psi) \) is taken to be one the pairs \( (\psi_1, \Psi_{M_1}) \), \((\psi_2, \Psi_{M_2})\) or \((\psi_-, \overline{\Psi}_{M_-})\) and \( g \) is restricted to be one of the generators \( a \) or \( b \) of \( F_2 \). Thus the "only if" direction is clear.

**Proof of "if" direction.** Assume statements (1), (2) and (3) of the proposition hold. We must prove statements (+) and (−) of the definition of oriented characteristic definition. Let

\[
\begin{align*}
\Delta_+ &= \{ (\psi, \Psi) \in \text{Aut}_+(F_2) \times \text{Inn}(\text{GL}(n, \mathbb{C})) : \boxed{5} \text{ holds} \} \quad \text{and} \\
\Delta_- &= \{ (\psi, \Psi) \in \text{Aut}_-(F_2) \times \text{Inn}(\text{GL}(n, \mathbb{C})) : \boxed{5} \text{ holds} \}.
\end{align*}
\]

Observe that \( \Delta_+ \cup \Delta_- \) is a group. If statement \( \boxed{5} \) holds for some collection of pairs in \( \Delta_+ \cup \Delta_- \) then it also holds for the generated subgroup. We must prove that the projection of \( \Delta_+ \cup \Delta_- \) to \( \text{Aut}(F_2) \) is surjective.

First consider the inner automorphisms of \( F_2 \), which have the form \( \psi_h(g) = hgh^{-1} \) for some \( h \in F_2 \). By manipulating \( \boxed{5} \) it can be observed that \( (\psi_h, \Psi_{\rho(g)}) \in \Delta_+ \) for all \( h \).

Now consider \( \psi_1 \) and \( \psi_2 \). Observe that \( \boxed{5} \) holds for all \( g \in F_2 \) if and only if it holds for the generators \( a \) and \( b \). By manipulating \( \boxed{5} \) in each case, it follows that \( (\psi_1, \Psi_{M_1}), (\psi_2, \Psi_{M_2}) \in \Delta_+ \). The elements \( \psi_1 \) and \( \psi_2 \) together with the inner automorphisms generate \( \text{Aut}_+(F_2) \), so \( \text{Aut}_+(F_2) \) is in the image of the projection of \( \Delta_+ \).

Similarly consider \( \psi_- \). Again by considering \( \boxed{5} \) in this case, we see that \( (\psi_-, \overline{\Psi}_{M_-}) \in \Delta_- \). The collection \( \{ \psi_- \} \cup \text{Aut}_+(F_2) \) generates \( \text{Aut}(F_2) \), so it must be that \( \text{Aut}(F_2) \) is in the image of the projection of \( \Delta_+ \cup \Delta_- \) as desired. \( \square \)
3.2. Some characteristic representations with finite image. We will now give some finite oriented characteristic representations.

We define \( \rho_2 : F_2 \to \text{GL}(3, \mathbb{C}) \) by

\[
\rho_2(a) = \text{diag}(-1,-1,1) \quad \text{and} \quad \rho_2(b) = \text{diag}(1,-1,-1).
\]  

(8)

**Proposition 3.4.** The image \( \rho_2(F_2) \) is homomorphic to the Klein four group, \( C_2 \times C_2 \). The representation \( \rho_2 \) is oriented characteristic.

**Proof.** The image \( \rho(F_2) \) can easily be seen to consist of four elements: \( \rho_2(a) \), \( \rho_2(b) \), the identity and \( \rho(ab) = \text{diag}(-1,1,-1) \). By inspection the image is homomorphic to the Klein four group. By an elementary calculation it can be observed that the statements of Proposition 3.3 are satisfied when the choices of

\[
M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

and \( M_- = I \) are made. \( \square \)

For odd numbers \( k \geq 3 \) define \( \rho_k : F_2 \to \text{GL}(k, \mathbb{C}) \) by

\[
\rho_k(a) = \text{diag}(1, \omega, \omega^2, \ldots, \omega^{k-1}) \quad \text{and} \quad \rho_k(b) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots \end{pmatrix},
\]

(9)

where \( \omega = e^{2\pi i/k} \). Here \( \rho_k(b) \) is a permutation matrix of order \( k \).

**Proposition 3.5.** The image \( \rho_k(F_2) \) is homomorphic to the Heisenberg group \( H(Z/kZ) \). The representation is oriented characteristic: It satisfies the hypotheses of Proposition 3.3 with the matrices \( M_1 \) and \( M_2 \) taken so that

\[
(M_1)_{i,j} = \omega^{(i-1)(j-1)} \quad \text{for} \ i, j \in \{1, \ldots, k\},
\]

so that \( M_2 \) is the diagonal matrix with entries \( (M_2)_{i,i} = \omega^{-(i-1)(k-2)} \) and so that \( M_- = I \).

**Proof.** To see the image is the Heisenberg group, recall that

\[
H(Z/kZ) = \left\langle a, b \big| a^k, b^k, [a, b]^k, [a, [a, b]], [b, [a, b]] \right\rangle.
\]

First we will check that \( \rho_k \) factors through \( H(Z/kZ) \). It should be clear that \( a^k \) and \( b^k \) lie in \( \ker \rho_k \). By computation we see \( \rho_k([a, b]) = \omega^{-1}I \). Thus \( [a, b] \) is central in the image and \( [a, b]^k \in \ker \rho_k \). This shows that the image \( \rho_k(F_2) \) is isomorphic to a quotient of \( H(Z/kZ) \). The image must be isomorphic to \( H(Z/kZ) \) because the homomorphism restricts to an isomorphism of the center of \( H(Z/kZ) \).

The statements of Proposition 3.3 for the matrices \( M_1 \), \( M_2 \) and \( M_- \) listed can be verified by a direct computation. \( \square \)

Observe that the images of \( \rho_k \) are matrices with entries in \( Z[\omega] \). Later we will need the following observation:

**Proposition 3.6.** Fix an odd \( k \geq 3 \). Let \( M_{k,k} \) denote the additive group of \( k \times k \) matrices with entries in \( Z[\omega] \). The subgroup of \( M_{k,k} \) generated by \( \{\rho_k(g) : g \in F_2\} \) has finite index.
Proof. Let $E_{i,j}$ denote the matrix with a 1 in the entry in row $i$ and column $j$ but with all other entries equal to zero. It suffices to show that $k\omega^nE_{i,j}$ is in the generated subgroup for all $i,j \in \{1, \ldots ,k\}$ and all $n \in \{0, \ldots ,k-1\}$. By direct computation we observe
\[
kE_{1,1} = \sum_{i=0}^{k-1} \rho_k(a^i).
\]
Utilizing the action of $\rho_k(b)$ as a permutation matrix we can then see
\[
E_{i,j} = \rho_k(b^{j-i}) \cdot E_{1,1} \cdot \rho_k(b^{i-j})
\]
so that $kE_{i,j}$ is in this generated subgroup as well. Finally to get the powers of $\omega$ observe that $\rho_k([b,a]) = \omega I$.

\[\square\]

Corollary 3.7. For odd $k \geq 3$, the representation $\rho_k$ is irreducible.

Proof. Any subspace of $\mathbb{C}^k$ which is invariant under $\rho_k$ must be mapped into itself by all elements of the subgroup of $M_{k,k}$ generated by $\{\rho_k(g): g \in F_2\}$. The previous proposition implies that the there is no such non-zero proper subspace. □

3.3. Improving characteristic representations. We will now explain a process which can take an oriented characteristic representation $\rho : F_2 \to \text{GL}(n, \mathbb{C})$ and produce a new oriented characteristic representation $\hat{\rho} : F_2 \to \text{GL}(\tilde{n}, \mathbb{C})$ where $\tilde{n} \geq n$ and hopefully the $\ker\hat{\rho}$ is strictly smaller than $\ker\rho$.

Fix $\rho$ for this subsection. We will consider deformations of $\rho$ into the affine group $\text{Aff}(n) = \mathbb{C}^n \times \text{GL}(n, \mathbb{C})$ where the product in $\text{Aff}(n)$ is given by
\[
(v,M) \cdot (w,N) = (v + Mw, MN).
\]
Let $\pi_1 : \text{Aff}(n) \to \mathbb{C}^n$ and $\pi_2 : \text{Aff}(n) \to \text{GL}(n, \mathbb{C})$ be the natural projections (noting that $\pi_1$ is not a homomorphism). We will say that an affable representation $\hat{\rho} : F_2 \to \text{Aff}(n)$ is a homomorphism for which $\pi_2 \circ \hat{\rho} = \rho$. We use $\mathcal{A}$ to denote the collection of all affable representations. Observe:

Proposition 3.8. The collection $\mathcal{A}$ is a vector space over $\mathbb{C}$ when endowed with the operations of addition and scalar multiplication defined so that
\[
\hat{\rho}_1 + \hat{\rho}_2(g) = (\pi_1 \circ \hat{\rho}_1 (g) + \pi_1 \circ \hat{\rho}_2 (g), \rho(g)) \quad \text{and}
\]
\[
\lambda \hat{\rho}_1(g) = (\lambda \pi_1 \circ \hat{\rho}_1 (g), \rho(g))
\]
for all $\hat{\rho}_1, \hat{\rho}_2 \in \mathcal{A}$, all $\lambda \in \mathbb{C}$ and all $g \in F_2$. In particular, for any $g$ the map $\text{eval}_g : \mathcal{A} \to \mathbb{C}^n$ defined by $\text{eval}_g(\hat{\rho}) = \pi_1 \circ \hat{\rho} (g)$ is linear.

Discussion of proof. The operations are clearly linear in nature, but it must be checked that $\hat{\rho}_1 + \hat{\rho}_2$ and $\lambda \hat{\rho}_1$ define group homomorphisms (assuming $\hat{\rho}_1$ and $\hat{\rho}_2$ are group homomorphisms). We leave this elementary check to the reader. □

Proposition 3.9. Recall $a$ and $b$ denote the generators of $F_2$. The map $\text{eval}_a \times \text{eval}_b : \mathcal{A} \to \mathbb{C}^n \times \mathbb{C}^n$ is a vector space isomorphism.

Proof. It should be clear that this defines a homomorphism between vector spaces by definition of the operations in Proposition 3.8. It is an isomorphism because the images of the generators determine the homomorphism; the inverse map sends $(a,b)$ to the homomorphism determined by the following images of the generators of $F_2$:
\[
a \mapsto (a, \rho(a)) \quad \text{and} \quad b \mapsto (b, \rho(b)).
\]

\[\square\]
Let \( \text{conj} : \mathbb{C}^n \times \mathcal{A} \to \mathcal{A} \) be the action defined by post-conjugation by \( \mathbb{C}^n \subset \text{Aff}(n) \):

\[
\text{conj}_v(\hat{\rho})(g) = (v, I) \cdot \hat{\rho}(g) \cdot (-v, I) \quad \text{for all } g \in F_2,
\]

where \( I \) denotes the identity element of \( \text{GL}(n, \mathbb{C}) \). When \( \mathcal{A} \) is viewed as isomorphic to \( \mathbb{C}^{2n} \), we see that each \( \text{conj}_v \) acts by translation on \( \mathcal{A} \) (i.e., \( \text{conj}_v(\hat{\rho}) = \hat{\rho} \) does not depend on \( \hat{\rho} \)).

**Proposition 3.10.** For each \( v \in \mathbb{C}^n \), each \( \hat{\rho} \in \mathcal{A} \) and each \( g \in F_2 \) we have

\[
(\text{conj}_v(\hat{\rho}) - \hat{\rho})(g) = \left( (I - \rho(g))v, \rho(g) \right).
\]

We call \( \text{conj}_v(\hat{\rho}) - \hat{\rho} \) the translation vector of \( \text{conj}_v \).

**Proof.** This follows from the computation in \( \text{Aff}(n) \):

\[
\text{conj}_v(\hat{\rho})(g) = (v, I) \cdot (\pi_1 \circ \hat{\rho}(g) \cdot \rho(g)) \cdot (-v, I) = (v + \pi_1 \circ \hat{\rho}(g) + \rho(g)(-v), \rho(g)).
\]

\( \square \)

Let \( \sim \) denote the equivalence relation on \( \mathcal{A} \) where

\[
\hat{\rho}_1 \sim \hat{\rho}_2 \quad \text{if there is a } v \in \mathbb{C}^n \text{ so that } \text{conj}_v(\hat{\rho}_1) = \hat{\rho}_2.
\]

**Corollary 3.11.** The quotient \( \mathcal{A} / \sim \) is a vector space with operations induced by those of \( \mathcal{A} \).

**Proof.** It needs to be observed that the operations of addition and scalar multiplication induce well defined actions on \( \mathcal{A} / \sim \). This follows from linearity of the translation vector of \( \text{conj}_v \) in \( v \in \mathbb{C}^n \).

\( \square \)

Fixing a homomorphism \( \rho : F_2 \to \text{GL}(n, \mathbb{C}) \), we define:

\[
\tilde{\Delta}_+ = \{(M, \psi) \in \text{GL}(n, \mathbb{C}) \times \text{Aut}_+(F_2) : M \cdot [\rho \circ \psi^{-1}(g)] \cdot M^{-1} = \rho(g) \text{ for all } g \in F_2 \},
\]

\[
\tilde{\Delta}_- = \{(M, \psi) \in \text{GL}(n, \mathbb{C}) \times \text{Aut}_-(F_2) : M \cdot [\rho \circ \psi^{-1}(g)] \cdot M^{-1} = \rho(g) \text{ for all } g \in F_2 \}.
\]

(15)

Here the \( \circ \) in the definition of \( \tilde{\Delta}_- \) denotes complex conjugation. Observe \( \tilde{\Delta}_+ \) is a subgroup of \( \text{GL}(n, \mathbb{C}) \times \text{Aut}_+(F_2) \). The disjoint union \( \tilde{\Delta} = \tilde{\Delta}_+ \sqcup \tilde{\Delta}_- \) also forms a group, though the group operation needs adjustment. We define:

\[
(M, \psi) \cdot (M', \psi') = \begin{cases} 
(MM', \psi \circ \psi') & \text{if } (M, \psi) \in \tilde{\Delta}_+, \\
(MM', \psi \circ \psi') & \text{if } (M, \psi) \in \tilde{\Delta}_-.
\end{cases}
\]

Observe that \( \rho \) is an oriented characteristic representation if and only if the projection of \( \tilde{\Delta} \) to \( \text{Aut}(F_2) \) is surjective.

We view \( \text{GL}(n, \mathbb{C}) \) as a subgroup of \( \text{Aff}(n) \). In addition, the act of conjugation induces a homomorphism of \( \text{Aff}(n) \).

We use \( \text{GL}(\mathcal{A}) \) to denote the group of complex-linear automorphisms of \( \mathcal{A} \) and \( \overline{\text{GL}}(\mathcal{A}) \) to denote the conjugate-linear maps. We have the following:

**Lemma 3.12.** There is a homomorphism \( N : \tilde{\Delta} \to \text{GL}(\mathcal{A}) \cup \overline{\text{GL}}(\mathcal{A}) \) so that

(+) If \( (M, \psi) \in \tilde{\Delta}_+ \) and \( \hat{\rho} \in \mathcal{A} \) then \( N_{M, \psi} \in \text{GL}(\mathcal{A}) \) and

\[
N_{M, \psi}(\hat{\rho})(g) = M \cdot (\hat{\rho} \circ \psi^{-1}(g)) \cdot M^{-1} \quad \text{for all } g \in F_2.
\]

(–) If \( (M, \psi) \in \tilde{\Delta}_- \) and \( \hat{\rho} \in \mathcal{A} \) then \( N_{M, \psi} \in \overline{\text{GL}}(\mathcal{A}) \) and

\[
N_{M, \psi}(\hat{\rho})(g) = M \cdot (\overline{\rho} \circ \psi^{-1}(g)) \cdot M^{-1} \quad \text{for all } g \in F_2.
\]
Furthermore each $N_{M,\psi}$ sends $\sim$-equivalence classes to $\sim$-equivalence and so there is an induced homomorphism $N^{\sim} : \Delta \rightarrow \text{GL}(\mathcal{A}/\sim) \cup \text{GL}(\mathcal{A}/\sim)$.

Proof. Since $M \in \text{GL}(n, \mathbb{C})$ and $\psi \in \text{Aut}(F_2)$, it should be clear that the definitions provided for $N_{M,\psi}(\hat{\rho})$ give a homomorphism $F_2 \rightarrow \text{Aff}(n)$. Writing $\hat{\rho}(g) = (\pi_1 \circ \hat{\rho}(g), \rho(g))$ (using affability of $\hat{\rho}$) we see that when $(M, \psi) \in \tilde{\Delta}_{\pm}$ we have

$$N_{M,\psi}(\hat{\rho})(g) = \left( M \cdot \pi_1 \circ \hat{\rho} \circ \psi^{-1}(g), M \cdot (\rho \circ \psi^{-1}(g)) \cdot M^{-1} \right) \quad (16)$$

with the last step given by definition of $\tilde{\Delta}_{\pm}$ in (15). To see linearity observe that $\pi_1 \circ \hat{\rho} \circ \psi^{-1}(g)$ varies linearly in $\hat{\rho}$ by Proposition 3.8 and we are simply postcomposing with the linear action of $M \in \text{GL}(n, \mathbb{C})$. Similarly if $(M, \psi) \in \hat{\Delta}_{-},\tilde{\Delta}_{+}$, we have

$$N_{M,\psi}(\hat{\rho})(g) = \left( M \cdot \pi_1 \circ \hat{\rho} \circ \psi^{-1}(g), M \cdot (\rho \circ \psi^{-1}(g)) \cdot M^{-1} \right) \quad (17)$$

Observe that $N_{M,\psi}$ is conjugate-linear in this case.

Now we must check that the linear action respects $\sim$-equivalence classes. Suppose $\hat{\rho}_1 \sim \hat{\rho}_2$. By Proposition 3.10 this is true if and only if there is a $v \in \mathbb{C}^n$ such that

$$(\hat{\rho}_1 - \hat{\rho}_2)(g) = \left( (I - \rho(g))v, \rho(g) \right) \quad \text{for all } g \in F_2. \quad (18)$$

Fix such a $v$ and let $\hat{\rho}_V \in \mathcal{A}$ be defined as in the right side of (18). Then by linearity or conjugate-linearity of $N_{M,\psi}$ we have

$$N_{M,\psi}(\hat{\rho}_1) - N_{M,\psi}(\hat{\rho}_2) = N_{M,\psi}(\hat{\rho}_V).$$

By (16) if $(M, \psi) \in \tilde{\Delta}_{+}$ we have

$$N_{M,\psi}(\hat{\rho}_V)(g) = \left( M \cdot (I - \rho \circ \psi^{-1}(g))v, \rho(g) \right) = \left( (I - \rho(g))Mv, \rho(g) \right),$$

where we are using the identity $M \cdot (\rho \circ \psi^{-1}(g)) \cdot M^{-1} = \rho(g)$ again in the second step. Similarly if $(M, \psi) \in \hat{\Delta}_{-}$ we have

$$N_{M,\psi}(\hat{\rho}_V)(g) = \left( M \cdot (I - \rho \circ \psi^{-1}(g))v, \rho(g) \right) = \left( (I - \rho(g))Mv, \rho(g) \right),$$

Now fix $(M, \psi)$. Observe from the above that (18) holds if and only if (18) holds with $N_{M,\psi}(\hat{\rho}_V)$ replacing $\hat{\rho} - \hat{\rho}_2$ and for $Mv$ or $M\bar{v}$ replacing $v$ (depending if $(M, \psi) \in \tilde{\Delta}_{+}$ or $(M, \psi) \in \hat{\Delta}_{-}$). 

It will be useful later to note that inner automorphisms act trivially on $\mathcal{A}/\sim$.

**Proposition 3.13.** Let $\psi_h \in \text{Aut}(F_2)$ denote the inner automorphism $g \mapsto g h h^{-1}$. Then $(\rho(h), \psi_h) \in \Delta$ and $N^{\sim}_{\rho(h),\psi_h}$ acts trivially on $\mathcal{A}/\sim$.

**Proof.** The fact that $(\rho(h), \psi_h) \in \Delta$ follows from the definition of $\Delta$ and the fact that $\rho$ is a homomorphism. Fix any $\hat{\rho} \in \mathcal{A}$ and observe $\hat{\rho} \circ \psi^{-1}_h(g) = \hat{\rho}(h^{-1}) \hat{\rho}(g) \hat{\rho}(h)$. Choose $v, w \in \mathbb{C}^n$ so that $\hat{\rho}(g) = (v, \rho(g))$ and $\hat{\rho}(h) = (w, \rho(h))$. Then $\hat{\rho}(h)^{-1} = (-\rho(h)^{-1}w, \rho(h)^{-1})$ so that

$$\hat{\rho} \circ \psi^{-1}_h(g) = (-\rho(h)^{-1}w, \rho(h)^{-1)) \cdot (v, \rho(g)) \cdot (w, \rho(h)) \quad (19)$$

By definition,

$$N_{\rho(h),\psi_h}(\hat{\rho})(g) = \rho(h) \circ (\hat{\rho} \circ \psi^{-1}_h(g)) \cdot \rho(h)^{-1}$$
By combining with the above we see
\[ [N_{\rho(h), w}(\hat{\rho}) - \hat{\rho}](g) = \left( (\rho(g) - I)w, \rho(g) \right), \]
so \( N_{\rho(h), w}(\hat{\rho}) = \text{conj}_w(\hat{\rho}) \) by Proposition 3.10 and thus \( N_{\rho(h), w}(\hat{\rho}) \sim \hat{\rho} \).

Fix an integer \( k \geq 2 \). Recall \( P_k \subset F_2 \) denotes the set of \( k \)-th powers of primitive elements of \( F_2 \). Assume \( P_k \subset \ker \rho \). The collection of \( k \)-affable representations is
\[ \mathcal{A}_k = \{ \hat{\rho} \in \mathcal{A} : P_k \subset \ker \hat{\rho} \}. \]

As a consequence of Proposition 3.8, \( \mathcal{A}_k \) is a linear subspace of \( \mathcal{A} \): it is the intersection of the kernels of the linear maps \( \text{eval}_p \) taken over all primitive \( p \in F_2 \).

We have:

**Proposition 3.14.**

1. Each \( \sim \)-equivalence class is either contained in or disjoint from \( \mathcal{A}_k \).
2. For each \( (M, \psi) \in \hat{\Delta} \), \( \mathcal{A}_k \) is invariant under \( N_{M, \psi} \).

**Proof.** Since \( P_k \subset F_2 \) is characteristic, \( \mathcal{A}_k \) is invariant under any \( f : \mathcal{A} \to \mathcal{A} \) so that \( \ker f(\hat{\rho}) \) differs from \( \ker \hat{\rho} \) by an automorphism of \( F_2 \). This holds for \( f \) replaced by \( \text{conj} \) and \( N_{M, \psi} \), which cover the respective cases of the proposition.

Summarizing the results above, we see that \( \mathcal{A}_k / \sim \) has the structure of a vector space. The action of \( N_{M, \psi} \) for \( (M, \psi) \in \hat{\Delta} \) is well defined on \( \mathcal{A}_k / \sim \).

Choose any subspace \( \mathcal{F} \subset \mathcal{A}_k / \sim \) which is invariant under the action of \( N_{M, \psi} \) for \( (M, \psi) \in \hat{\Delta} \). Ideally we would take \( \mathcal{F} = \mathcal{A}_k / \sim \) to get the largest invariant space possible. Later in the proof of Theorem 3.17 we do not prove that our choice of \( \mathcal{F} \) is all of \( \mathcal{A}_k / \sim \).

Let \( m = \dim \mathcal{F} \). Choose \( \hat{\rho}_1, \ldots, \hat{\rho}_m \in \mathcal{A}_k \) so that the images in \( \mathcal{A}_k / \sim \) form a basis for \( \mathcal{F} \). In block matrix form we define
\[ \hat{\rho} : F_2 \to \text{GL}(n + m, \mathbb{C}); \quad g \mapsto \left( \begin{array}{cc} \rho(g) & Q(g) \\ 0 & I \end{array} \right) \in \text{GL}(n + m, \mathbb{C}) \]
where \( Q(g) = \left( \begin{array}{cccc} \pi_1 \circ \hat{\rho}_1(g) & \pi_1 \circ \hat{\rho}_2(g) & \ldots & \pi_1 \circ \hat{\rho}_m(g) \end{array} \right) \).

Here each \( \pi_1 \circ \hat{\rho}_i(g) \) is interpreted as the \( i \)-th column vector of \( Q(g) \). Then:

**Theorem 3.15.** Assume \( \rho : F_2 \to \text{GL}(n, \mathbb{C}) \) is an oriented characteristic representation with \( P_k \subset \ker \rho \). Define \( \mathcal{F}, m, \hat{\rho}_1, \ldots, \hat{\rho}_m \) and \( \hat{\rho} \) as above. Then \( \hat{\rho} \) is also an oriented characteristic representation with \( P_k \subset \ker \hat{\rho} \). Furthermore, there is a short exact sequence of the form
\[ 1 \to \mathbb{Z}^d \to F_2 / \ker \rho \to F_2 / \ker \hat{\rho} \to 1 \]
where \( d \geq 0 \) is the rank of the abelian image \( \hat{\rho}(\ker \rho) \).

**Proof.** First we will show that \( \hat{\rho} \) is a group homomorphism. Considering the block form of the image, observe that it suffices to understand the top right block (since we are given that \( \rho \) is a homomorphism). Checking that \( \hat{\rho}(g_1 g_2) = \hat{\rho}(g_1) \hat{\rho}(g_2) \) then reduces to checking that
\[ Q(g_1 g_2) = Q(g_1) + \rho(g_1) Q(g_2). \]

Checking this for column \( i \) amounts to checking that
\[ \pi_1 \circ \hat{\rho}_i(g_1 g_2) = \pi_1 \circ \hat{\rho}_i(g_1) + \rho(g_1) \cdot \pi_1 \circ \hat{\rho}_i(g_2), \]
which is true because \( \hat{\rho}_i \) is a homomorphism to \( \text{Aff}(n) \) which has product rule as in (10).
Observe that $P_k \subset \hat{\rho}$ since $P_k \subset \ker \rho$ and each $\hat{\rho}_i \in \mathcal{A}_k$. From (20) and by definition of $\mathcal{A}_k$ we see that $\hat{\rho}(g^i) = I$ for each primitive $g \in F_2$ guaranteeing that $P_k \subset \ker \hat{\rho}$.

Now consider the statement involving the short exact sequence. Observe we have the exact sequence

$$1 \to \hat{\rho}(\ker \rho) \to \hat{\rho}(F_2) \to \rho(F_2) \to 1.$$  

(21)

The group $\hat{\rho}(\ker \rho)$ is free and abelian because for each $g \in \ker \rho$ we have

$$\hat{\rho}(g) = \begin{pmatrix} I & Q(g) \\ 0 & I \end{pmatrix}.$$

Thus the exact sequence in the statement follows from (21) by noting that $\hat{\rho}(\ker \rho)$ is isomorphic to $\mathbb{Z}^d$, that $\hat{\rho}(F_2)$ is isomorphic to $F_2/\ker \hat{\rho}$ and that $\rho(F_2)$ is isomorphic to $F_2/\ker \rho$.

It remains to show that $\hat{\rho}$ is an oriented characteristic representation. Choose any $\psi \in \text{Aut}(F_2)$. Let $s \in \{+, -\}$ be such that $\psi \in \text{Aut}_B(F_2)$. Define

$$\hat{\rho}_0 : F_2 \to \text{GL}(n + m, \mathbb{C}) \quad \text{by} \quad \hat{\rho}_0(g) = \begin{cases} \hat{\rho} \circ \psi^{-1}(g) & \text{if } s = + \\ \hat{\rho} \circ \psi^{-1}(g) & \text{if } s = - . \end{cases}$$

We need to show that $\hat{\rho}_0$ is conjugate by an element of $\text{GL}(m + n, \mathbb{C})$ to $\hat{\rho}$. We will demonstrate this by applying a sequence of conjugations.

First since $\rho$ is an oriented characteristic representation, there is a matrix $M \in \text{GL}(n, \mathbb{R})$ such that $(M, \psi) \in \hat{\Delta}_s$. This guarantees that either

$$M \cdot [\rho \circ \psi^{-1}(g)] \cdot M^{-1} = \rho(g) \quad \text{or} \quad M \cdot \rho \circ \psi^{-1}(g) \cdot M^{-1} = \rho(g)$$

(22)

for all $g \in F_2$ depending on the sign $s$. Define $\hat{\rho}_1$ to be a conjugate of $\hat{\rho}_0$ formed as follows:

$$\hat{\rho}_1(g) = \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \cdot \hat{\rho}_0(g) \cdot \begin{pmatrix} M^{-1} & 0 \\ 0 & I \end{pmatrix}.$$  

Recall from Lemma 3.12 there is an $N_{M, \psi}$ in $\text{GL}(\mathcal{A})$ or $\overline{\text{GL}(\mathcal{A})}$ (depending on $s$) describing the conjugation action on $\mathcal{A}$. Let $\text{Col}_i(X)$ denote the $i$-th column of a matrix $X$. By (22) and description of $N_{M, \psi}$ we have

$$\hat{\rho}_1(g) = \begin{pmatrix} \rho(g) & Q_1(g) \\ 0 & I \end{pmatrix} \quad \text{where} \quad \text{Col}_i(Q_1(g)) = \pi_1(N_{M, \psi}(\hat{\rho}_i)(g))$$

for $i \in \{1, \ldots, m\}$.

Let $\hat{\mathcal{I}} \subset \mathcal{A}_k$ denote the preimage of $\mathcal{I}$ under the quotient map $\mathcal{A}_k \to \mathcal{A}_k/\sim$; this is the union of the equivalence classes in $\mathcal{I}$. Recall that $\mathcal{I} \subset \mathcal{A}_k/\sim$ is $N_{M, \psi}$-invariant. In addition $\sim$-equivalence classes are permuted by $N_{M, \psi}$; see Lemma 3.12. It follows that $\hat{\mathcal{I}}$ is $N_{M, \psi}$-invariant. We therefore have that $N_{M, \psi}(\hat{\rho}_i)$ in $\hat{\mathcal{I}}$ for $i \in \{1, \ldots, m\}$. Let $\hat{\mathcal{I}}_L = \text{span}\{\hat{\rho}_1, \ldots, \hat{\rho}_m\}$. From our choice of $\hat{\rho}_1, \ldots, \hat{\rho}_m$ the space $\hat{\mathcal{I}}_L$ is a lift of $\mathcal{I}$; i.e., the restriction to the quotient map $\mathcal{A}_k \to \mathcal{A}_k/\sim$ gives an isomorphism $\hat{\mathcal{I}}_L \to \mathcal{I}$. Recall that the $\sim$-equivalence classes are $\text{conj}_s$ orbits; see (14). Thus for each $i \in \{1, \ldots, m\}$ there is a unique vector $v_i$ so that

$$\text{conj}_s \circ N_{M, \psi}(\hat{\rho}_i) \in \hat{\mathcal{I}}_L.$$  

We now define a homomorphism $\hat{\rho}_2 : F_2 \to \text{GL}(n + m, \mathbb{C})$ conjugate to $\hat{\rho}_1$ by

$$\hat{\rho}_1(g) = \begin{pmatrix} I & V \\ 0 & I \end{pmatrix} \cdot \hat{\rho}_1(g) \cdot \begin{pmatrix} I & -V \\ 0 & I \end{pmatrix} \quad \text{where} \quad V(g) = \begin{pmatrix} v_1 & \ldots & v_m \end{pmatrix}.$$
By definition of $\text{conj}$ in (12) we see that

$$\hat{\rho}_2(g) = \begin{pmatrix} \rho(g) & Q_2(g) \\ 0 & I \end{pmatrix} \quad \text{where} \quad \text{Col}_1(Q_2(g)) = \pi_1(\text{conj}_{\psi} \circ N_{M,\psi}(\hat{\rho}_i))$$

for all $i \in \{1, \ldots, m\}$.

Recall that $N_{M,\psi}$ induces an automorphism $\tilde{N}_{M,\psi}$ of $\mathcal{F}$ and this automorphism leaves $\mathcal{F}$ invariant. Recall that the map $p : \mathcal{F}_L \to \mathcal{F}$ is an isomorphism. Thus there is an $\tilde{N} \in \text{GL}(\mathcal{F}_L) \cup \text{GL}(\mathcal{F}_L)$ so that $\hat{\rho} \circ \tilde{N} = N_{M,\psi} \circ p$. Then in particular we have

$$\text{conj}_{\psi} \circ N_{M,\psi}(\hat{\rho}_i) = \tilde{N}(\hat{\rho}_i) \quad \text{for all} \quad i \in \{1, \ldots, m\}.$$ 

As a consequence we see that $\{\text{conj}_{\psi} \circ N_{M,\psi}(\hat{\rho}_i) : i = 1, \ldots, m\}$ is a basis for $\mathcal{F}_L$. Thus there is a matrix $R = (R_{i,j}) \in \text{GL}(m, \mathbb{C})$ so that

$$\hat{\rho}_i = \sum_{j=1}^m R_{i,j} \text{conj}_{\psi} \circ N_{M,\psi}(\hat{\rho}_j) \quad \text{for all} \quad i \in \{1, \ldots, m\}.$$ 

Then by linearity of the evaluation maps (see Proposition 3.8), for each $g \in F_2$ we have

$$\pi_1(\hat{\rho}_j(g)) = \sum_{i=1}^m R_{i,j} \pi_1(\text{conj}_{\psi} \circ N_{M,\psi}(\hat{\rho}_i)(g)) \quad \text{for all} \quad j \in \{1, \ldots, m\}. \quad (23)$$

Equivalently, we have $Q_2(g) \cdot R = Q(g)$ for all $g$, where $Q(g)$ is the top right submatrix of $\rho(g)$; see (20). We define the conjugate $\hat{\rho}_3$ of $\hat{\rho}_2$ by

$$\hat{\rho}_3(g) = \begin{pmatrix} I & 0 \\ 0 & R^{-1} \end{pmatrix} \cdot \hat{\rho}_1(g) \cdot \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} \rho(g) & Q_2(g)R \\ 0 & 1 \end{pmatrix} = \hat{\rho}(g).$$

Since $\hat{\rho}_3 = \hat{\rho}$ we have produced the desired conjugacy. \qed

3.4. Case $k = 6$. We define $\rho_6$ to be the tensor product $\rho_2 \otimes \rho_3$ where $\rho_2$ and $\rho_3$ are defined as in (8) and (9). Then $\rho_6$ may be thought of as a homomorphism $F_2 \to \text{GL}(9, \mathbb{C})$. Letting $\omega = e^{\pi i/3}$, we have the presentation:

$$\rho_6(a) = \text{diag}(-1, -\omega, -\omega^2; -1, -\omega, -\omega^2; 1, \omega, \omega^2),$$

$$\rho_6(b) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Applying the improvement algorithm of Theorem 3.15 to $\rho_6$ can be shown by calculation to give rise to the representation $\hat{\rho}_6 : F_2 \to \text{GL}(12, \mathbb{C})$ defined as block matrices as

$$\hat{\rho}_6(a) = \begin{pmatrix} \rho_6(a) & 0 \\ 0 & I \end{pmatrix}$$

and

$$\hat{\rho}_6(b) = \begin{pmatrix} \rho_6(b) & 0 \\ 0 & I \end{pmatrix}.$$
We will not present the computational proof that $\tilde{\rho}_6$ arises from $\rho_6$ by applying Theorem 3.15 in the case $I = A_6$ in this case. However we will demonstrate that it is an oriented characteristic representation:

**Proposition 3.16.** The homomorphism $\tilde{\rho}_6$ is an oriented characteristic representation. The kernel $\ker \tilde{\rho}_6$ contains $P_6$ and is infinite index in $F_2$. Furthermore, there is a short exact sequence of groups of the form

$$1 \to \mathbb{Z}^{18} \to F_2/\ker \tilde{\rho}_6 \to C_2 \times C_2 \times H(\mathbb{Z}/3\mathbb{Z}) \to 1.$$ 

**Proof.** To see the representation is oriented characteristic, observe that the criterion of Proposition 3.3 is satisfied with the choices of matrices $M_1 = I$ and $M_2$ as below:

$$M_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \omega^2 & \omega & \omega^2 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Thus $\ker \tilde{\rho}_6$ is a characteristic subgroup of $F_2$ by Proposition 3.1. Observe that $\tilde{\rho}_6(a^6) = I$, so that $P_6 \subset \ker \tilde{\rho}_6$.

The top left $9 \times 9$ block is isomorphic to the representation $\rho_6 = \rho_2 \otimes \rho_3$, where these representations were taken from [3.2]. It follow from the block description of the matrices

$$\rho_6(b) = \begin{pmatrix} \rho_6(b) & B \end{pmatrix}$$

where $B = \begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}$. 


\(\hat{\rho}_6(a)\) and \(\hat{\rho}_6(b)\) that we get an exact sequence as described but with \(\mathbb{Z}^{18}\) replaced by of \(\hat{\rho}_6(\ker \rho_6)\) (which is a free abelian group because matrices in \(\hat{\rho}_6(\ker \rho_6)\) have a block form with copies of the identity matrix along the diagonal and a zero matrix in the lower left block). To figure out the rank observe that \(C_2 \times C_2 \times H(\mathbb{Z}/3\mathbb{Z})\) has a presentation of the form

\[
\ker \rho_6 = \langle \langle d^6, b^6, [a, b]^3, [a, [a, b]], [b, [a, b]] \rangle \rangle.
\]

We already know that \(d^6, b^6 \in \ker \hat{\rho}_6\) and can check that \([a, b]^3 \in \ker \hat{\rho}_6\). Thus the abelian image \(\hat{\rho}_6(\ker \rho_6)\) is generated by elements of the form

\[
\hat{\rho}_6(g[a, [a, b]]g^{-1}) \text{ and } \hat{\rho}_6(g[b, [a, b]]g^{-1})
\]

with \(g \in F_2\). Since \(\hat{\rho}_6(\ker \rho_6)\) is abelian, these conjugacy classes only depend on the image \(\rho_6(g)\), which reduces us no more than 216 generators. This reduces the computation of the rank of \(\hat{\rho}_6(\ker \rho_6)\) to a finite computation which can be done on the computer. We computed rank \(\hat{\rho}_6(\ker \rho_6) = 18\) so that \(\hat{\rho}_6(\ker \rho_6) \cong \mathbb{Z}^{18}\). \(\square\)

\section{Odd \(k \geq 5\) Let \(k \geq 5\) be odd and define \(\rho_k : F_2 \to \text{GL}(k, \mathbb{C})\) as in \((9)\). We define \(\omega = e^{\frac{2\pi}{k}}\).

We will define an extension \(\hat{\rho}_k : F_2 \to \text{GL}(k, \mathbb{C})\). Let \(B\) denote the \(k \times \frac{k-3}{2}\) matrix whose column vectors are given by

\[
b_j = e_{j+1} - e_{k-j} \quad \text{for integers } j \text{ with } 1 \leq j \leq \frac{k-3}{2},
\]

where \(e_i\) denotes the standard basis vector with a 1 in position \(i\). We define \(\hat{\rho}_k\) in block form by

\[
\hat{\rho}_k(a) = \begin{pmatrix} \rho_k(a) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{\rho}_k(b) = \begin{pmatrix} \rho_k(b) & B \\ 0 & I \end{pmatrix}.
\]

\section{Theorem 3.17.} For each odd \(k \geq 5\) the homomorphism \(\hat{\rho}_k\) is an oriented characteristic representation. The kernel \(\ker \hat{\rho}_k\) contains \(P_k\) and is infinite index in \(F_2\). Furthermore, there is a short exact sequence of groups of the form

\[
1 \to \mathbb{Z}^d \to F_2/\ker \hat{\rho}_k \to H(\mathbb{Z}/k\mathbb{Z}) \to 1
\]

where \(d = k \cdot \frac{k-3}{2} \cdot [\mathbb{Q}(\omega) : \mathbb{Q}]\).

\section{Proof.} We will show that \(\hat{\rho}_k\) is derivable from \(\rho_k\) as described by Theorem 3.15. The theorem then implies that \(\hat{\rho}_k\) is an oriented characteristic representation and that \(\hat{P}_k \subset \ker \hat{\rho}_k\).

Verifying the Theorem applies requires working through 3.3. Let \(\mathcal{A}\) denote the collection of \(\rho_k\)-affine representations as in 3.3. Recall that \(\text{eval}_a \times \text{eval}_b\) gives an isomorphism \(\mathcal{A} \to \mathbb{C}^k \times \mathbb{C}^k\); see Proposition 3.9. We’ll find it useful to use coordinates provided by the inverse map

\[
R = (\text{eval}_a \times \text{eval}_b)^{-1} : \mathbb{C}^k \times \mathbb{C}^k \to \mathcal{A}.
\]

The image of \((a, b)\) is defined as in 11.

The subgroup \(\mathbb{C}^n \subset \text{Aff}(k)\) acts on \(\mathcal{A}\) by conjugation. For \(v \in \mathbb{C}^k\) we used \(\text{conj}_v\) to denote this action. By Proposition 3.10 we know that \(\text{conj}_v\) acts by translation and this translation vector depends linearly on \(v\). Let \(\mathcal{T} \subset \mathcal{A}\) denote these translation vectors. By applying the formula in Proposition 3.10 to the standard basis vectors \(e_1, \ldots, e_k \in \mathbb{C}^k\) and our particular \(\rho_k\), we see

\[
\mathcal{T} = \text{span}_\mathbb{C} \left( \{ R(0, e_1 - e_k) \} \cup \{ R(1 - \omega^{-j})e_j - e_{j-1} : 2 \leq j \leq k \} \right).
\]
In particular, for each \( \hat{\rho} \in \mathcal{A} \) there is a unique \( \mathbf{v} \in \mathbb{C}^k \) so that
\[
\text{conj}_\mathcal{A}(\hat{\rho}) = R(c_1 e_1, c_2 e_2 + c_3 e_3 + \ldots + c_k e_k)
\] for some choice of \( c_1, \ldots, c_k \in \mathbb{C} \). (26)

This gives a standard representative for each conjugacy class. Let \( \mathcal{S} \subset \mathcal{A} \) denote those representations which can be written in the form on the right side of (26).

Now consider the subspace \( \mathcal{S}_L \subset \mathcal{A} \) consisting of those \( \hat{\rho} \) so that \( \mathcal{P}_L \subset \hat{\rho} \). Define
\[
\mathcal{S}_L = \text{span}_{\mathbb{C}} \{ R(0, \mathbf{b}_j) : 1 \leq j \leq \frac{k-3}{2} \} \subset \mathcal{S},
\] where the vectors \( \mathbf{b}_j \) are defined as in (24). We define \( \mathcal{S} = \mathcal{S}_L/\sim \) where \( \sim \) denotes equivalence relation defined by conj orbits as in (14). We claim:

1. \( \mathcal{S} \) is \( N_{M, \psi} \)-invariant for all \((M, \psi) \in \widetilde{\Delta} \).
2. \( \mathcal{S} \subset \mathcal{S}_L \).

This will verify the hypotheses of Theorem 3.15 providing a new oriented characteristic representation \( \hat{\rho}_L \) so that \( \mathcal{P}_L \subset \ker \hat{\rho}_L \). Let
\[
\hat{\rho}_j = R(0, \mathbf{b}_j) \quad \text{for} \quad j \in \{1, \ldots, \frac{k-3}{2}\}.
\]

We obtain the matrix representation for \( \hat{\rho}_L \) using (20).

Claim (1) is a consequence of the group theoretic structure of \( N^\sim : \widetilde{\Delta} \to \text{GL}(\mathcal{A}/\sim) \). Namely, that the image \( N^\sim(\widetilde{\Delta}) \) is a quotient of \( \text{GL}(2, \mathbb{Z}) \). To see this first observe that \( N(M, \psi) \) only depends on \( \psi \). This follows from the fact that \( N(M, \psi) \) is trivial whenever \( \psi \) is the identity automorphism. From (15) \( (M, \psi) \in \Delta \) with \( \psi \) the identity if and only if \( M \) commutes with each \( \hat{\rho}(g) \). Since \( \hat{\rho} \) is irreducible (Corollary 3.7) only the center of \( \text{GL}(k, \mathbb{C}) \) commutes with all of \( \hat{\rho}(F_2) \). But by definition of \( N \) in Lemma 3.12 we see that \( N(M, \psi) = N(\mathbb{C}M, \psi) \) for all \( (M, \psi) \in \Delta \) and all \( z \in \mathbb{C} \). We have shown that \( N(M, \psi) \) only depends on \( \psi \), so it follows that \( N^\sim(M, \psi) \) only depends on \( \psi \). By Proposition 3.13 it then follows that \( N^\sim(M, \psi) \) is trivial whenever \( \psi \) is an inner automorphism of \( F_2 \). Thus the image \( N^\sim(M, \psi) \) only depends on the outer automorphism class of \( \psi \), i.e., \( N^\sim \) factors through \( \text{Out}(F_2) \cong \text{GL}(2, \mathbb{Z}) \) as claimed.

Let \( \psi_1 \in \text{Aut}(F_2) \) be as in (7). Note that the outer automorphism class of \( \psi_1^2 \) represents \(-I\) in the identification of \( \text{Out}(F_2) \) with \( \text{GL}(2, \mathbb{Z}) \). Let \( M_1 \in \text{GL}(k, \mathbb{C}) \) be the matrix in Proposition 3.5 so that \((M_1, \psi_1) \in \Delta \). Then the matrix \( N^\sim(M_1, \psi_1)^2 \) is central in the image \( N^\sim(\widetilde{\Delta}) \). It follows that the eigenspaces of \( N^\sim(M_1, \psi_1)^2 \) are \( N^\sim(\widetilde{\Delta}) \)-invariant. Thus we can show Claim (1) by proving that \( \mathcal{S} \) is a right eigenspace for \( N^\sim(M_1, \psi_1)^2 \).

We will now find a basis of right eigenvectors for \( N^\sim(M_1, \psi_1)^2 \) to show that \( \mathcal{S} \) is a right eigenspace. We will work with the subspace \( \mathcal{S} \subset \mathcal{A} \) as defined below (26), because this includes exactly one representation from each \( \sim \)-equivalence class. Let \( N_1^2 : \mathcal{S} \to \mathcal{S} \) be the map which applies \( N(M_1, \psi_1) \) twice and then projects along \( \sim \)-equivalence classes back into \( \mathcal{S} \). We will show that a list of eigenvalues and eigenvectors of \( N_1^2 \) is given by:

(a) The vectors \( \hat{\rho}_j = R(0, \mathbf{b}_j) \) are eigenvectors with eigenvalue \( k \) for \( j \in \{1, \ldots, \frac{k-3}{2}\} \).
(b) The vectors \( R(\mathbf{0}, \mathbf{e}_{j+1}) + \mathbf{e}_{j-1}) \) are eigenvectors with eigenvalue \( -k \) for \( j \in \{1, \ldots, \frac{k-1}{2}\} \).
(c) The vector \( R(\mathbf{0}, \mathbf{e}_{1,1}) \) is an eigenvector with eigenvalue \( -k \).
(d) The vectors \( R(\mathbf{e}_1, \mathbf{0}) \) and \( R(\mathbf{0}, \mathbf{e}_k) \) are eigenvectors with an eigenvalue of \( -k \).

The reader will observe that the vectors listed above span \( \mathcal{S} \) and the eigenspace formed by the span of the eigenvectors in case (a) coincides with \( \mathcal{S}_L \). Thus by proving these statements we will have verified Claim (1).
Before proving (a)-(d) we need to understand the action of $N(M_1, \psi_1)$. Let $(a, b) \in \mathbb{C}^n \times \mathbb{C}^n$ and \( \hat{\rho} = R(a, b) \). Since \( \psi_i^{-1}(a) = b^{-1} \) and \( \psi_i^{-1}(b) = a \) and \( \hat{\rho}(b^{-1}) = (\rho(b^{-1}), \rho(b^{-1})) \), we have by definition of $N$: 

\[
N(M_1, \psi_1)(\hat{\rho})(a) = M_1 \cdot (\rho(b^{-1}), \rho(b^{-1})) = \rho_1(b^{-1}) \cdot M_1^{-1} = (M_1 \rho(b^{-1})b, \rho(a)),
\]

\[
N(M_1, \psi_1)(\hat{\rho})(b) = M_1 \cdot (a, \rho(a)) \cdot M_1^{-1} = (M_1a, \rho(b)).
\]

We make several useful calculations using this. First we compute $N(M_1, \psi_1)(R(0, e_i))$ for $i \in \{1, \ldots, k\}$. From the above and definition of $M_1$ as the matrix with entries $(M_1)_{i,j} = \omega^{(i-1)(j-1)}$ we have 

\[
N(M_1, \psi_1)(R(0, e_i)) = R(-M_1, \rho(b^{-1})e_i, 0) = R(-M_1e_{i+1}, 0)
\]

Applying $N(M_1, \psi_1)$ one more time yields 

\[
N(M_1, \psi_1)^2(R(0, e_i)) = R(-M_1, \rho(b^{-1})e_i, 0) = R(0, e_{i+1}).
\]

Applying $N(M_1, \psi_1)$ one more and using (25) yields 

\[
N(M_1, \psi_1)^2(R(e_i, 0)) = \sum_{i=1}^k N(M_1, \psi_1)(R(0, e_i)) = \sum_{i=1}^k R(-M_1, \rho(b^{-1})e_i, 0) = R(-M_1e_{i+1}, 0).
\]

We return to proving the statements (a)-(d). The reader will observe that statement (c) from (29) with $i = \frac{k+1}{2}$ with no projection needed. Statements (a) and (b) also follow from (29) in a similar way. Now consider (d). The fact that $R(e_1, 0)$ is an eigenvector follows from (31). The final case of $R(0, e_k)$ is slightly more subtle because we must consider the projection. By (29), we have 

\[
N(M_1, \psi_1)^2(R(0, e_k)) = R(0, -\omega k e_1) \sim R(0, -k e_k),
\]

where we have applied the observation that $R(0, e_1 - e_k) \in \mathcal{F}$; see (25).

Finally we need to prove Claim (2) that $\mathcal{F} \subset \mathcal{A}_k / \sim$. From the above we know that $\mathcal{F}$ is $N^\sim$-invariant, so it suffices to prove that $\hat{\rho}(a^k) = 1$ for each $a \in \mathcal{F}_k$. We clearly have this since each $\hat{\rho} = R(0, v)$ for some $v \in \mathbb{C}^n$, see (27). This proves Claim (3) and proves that we have met the hypotheses of Theorem 3.15.

The rank $d$ from the Theorem coincides with the rank of $\hat{\rho}_k(\ker \rho_k)$ by Theorem 3.15. We will show that this rank is as large as possible: as large as the rank of the additive group of $k \times \frac{k+2}{2}^2$ matrices with entries in $\mathbb{Z}[\omega]$. To see this, it suffices to find a $g \in \ker \rho_k$ so that the top right block $Q(g)$ (in the notation of (20)) has linearly independent columns. Then given any $h \in F_2$ we know $ghh^\sim \in \ker \rho_k$ and $Q(ghh^\sim) = \rho_k(h) \cdot Q(g)$. Thus the result will follow from the statement from Proposition 3.15 guaranteeing that the additive group of matrices generated by $\rho_k(F_2)$ is finite index inside the group of all $k \times k$ matrices with entries in $\mathbb{Z}[\omega]$.

We carry out this calculation for $g = [a^{-1}, b^{-1}] = aba^{-1}b^{-1}a^{-1}bab^{-1}$ in $\ker(\rho_k)$. The columns of $Q(g)$ are given by $\pi_1 \circ \hat{\rho}_j(g)$ for $j \in \{1, \ldots, \frac{k+2}{2}\}$. It may be computed that 

\[
\hat{\rho}_j(aba^{-1}b^{-1}) = ((\omega^j - \omega^{-1})e_{j+1} - (\omega^{-1} - \omega^{-j-1})e_{k-j}, \omega^{-1}I),
\]
Proposition 3.19. The homomorphism \( \hat{\rho}_j(a^{-1}bab^{-1}) = ((\omega^{-j} - \omega^j) e_{j+1} - (\omega^j - \omega^{j+1}) e_{k-j}, \omega I) \),
\[ \hat{\rho}_j(g) = ((\omega^j - 1)(1 - \omega^{-j}) (e_{j+1} - e_{k-j}), I). \]
The coefficient \((\omega^j - 1)(1 - \omega^{-j})\) is never zero for the range of \( j \) under consideration. Also the vectors are linearly independent since the positions of non-zero entries never coincide. Thus the above argument gives us the rank we claimed. \( \square \)

3.6. Case \( k = 4 \). We define \( \rho_4 : F_2 \to \text{GL}(2, \mathbb{C}) \) by
\[ \rho_4(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \rho_4(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Proposition 3.18. The image \( \rho_4(F_2) \) is isomorphic to the quaternion group \( Q \) of order eight. The representation \( \rho_4 \) is oriented characteristic and \( P_4 \subset \ker \rho_4 \). The additive subgroup of \( 4 \times 4 \) matrices generated by the image of \( \rho_4 \) consists of those matrices with entries in \( \mathbb{Z}[i] \) of the form
\[ M_{x,y} = \begin{pmatrix} x & -y \\ y & \bar{x} \end{pmatrix}. \]

Proof. The elements of the image can be enumerated and the image can be observed to be isomorphic to \( Q \). Such an enumeration should also make the final statement clear. To see \( \rho_4 \) is oriented characteristic observe it satisfies Proposition 3.3 with the choice of matrices
\[ M_1 = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_- = I. \]
\( \square \)

Let \( \tilde{\rho}_4 : F_2 \to \text{GL}(4, \mathbb{C}) \) be defined by
\[ \tilde{\rho}_4(a) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{\rho}_4(b) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

This representation was produced by following the argument of Theorem 3.18 with \( \mathcal{F} = \mathcal{A}_4 / \sim \), though we will not prove this. We do show:

Proposition 3.19. The homomorphism \( \tilde{\rho}_4 \) is an oriented characteristic representation. The kernel of \( \tilde{\rho}_4 \) contains \( P_4 \) and is infinite index in \( F_2 \). We have
\[ \tilde{\rho}_4(\ker \rho_4) = \left\{ \begin{pmatrix} 1 & 0 & z & -\bar{w} \\ 0 & 1 & w & \bar{z} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : (w, z) \in \Lambda \right\} \]
where \( \Lambda \subset \mathbb{Z}[i]^2 \) is the index two subgroup generated by \((-1, 1), (-i, -i), (-1, -1) \) and \((0, i+1) \). Thus there is a short exact sequence of groups of the form
\[ 1 \to \mathbb{Z}^4 \to F_2 / \ker \rho_4 \to Q \to 1. \]

Proof. To see \( \tilde{\rho} \) is oriented characteristic apply Proposition 3.3 with \( M_- = I \),
\[ M_1 = \begin{pmatrix} 2 & 2i & i-1 & -i-1 \\ 2i & 2 & -i+1 & -i-1 \\ 0 & 0 & 2i-2 & 0 \\ 0 & 0 & 2i-2 & 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}. \]
Then to see that \( P_4 \subset \ker \bar{\rho}_4 \), it suffices to observe that \( \bar{\rho}_4(a^4) = I \).

Define \( \gamma : \mathbb{Z}[i]^2 \to \text{GL}(4, \mathbb{C}) \) by

\[
\gamma(z, w) = \begin{pmatrix}
1 & 0 & z & \bar{w} \\
0 & 1 & w & \bar{z} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (32)

The Proposition claims that \( \bar{\rho}_4(\ker \rho_4) = \gamma(\Lambda) \). Recall that the quaternion group has a presentation of the form

\[ Q = \langle a, b | a^4 = b^4 = a^2b^2 = ab^{-1}ab = 1 \rangle. \]

Since \( \bar{\rho}_4(a^4) = \bar{\rho}_4(b^4) = I \), it follows that \( \bar{\rho}_4(\ker \rho_4) \) is generated by images under \( \bar{\rho}_4 \) of conjugates of \( a^2b^2 \) and \( ab^{-1}ab \). We compute

\[ \bar{\rho}_4(a^2b^2) = \gamma(-1, 1) \quad \text{and} \quad \bar{\rho}_4(ab^{-1}ab) = \gamma(0, i + 1). \]

Now we will consider \( \bar{\rho}_4(ga^2b^2g^{-1}) \) for \( g \in F_2 \). Let \( P \) be the top right \( 2 \times 2 \) submatrix \( \bar{\rho}_4(a^2b^2) \) above. Conjugates \( \bar{\rho}_4(ga^2b^2g^{-1}) \) have top right submatrix given by \( \rho_4(g) \cdot P \). Thus \( \bar{\rho}_4(\ker \rho_4) \) contains all the matrices \( M_{x,y} \) where \( M_{x,y} \) is in the additive group generated by \( \rho_4(g) \) which was described by Proposition \( 3.18 \) in terms of a vector \( (x, y) \in \mathbb{Z}[i]^2 \).

We have

\[
M_{x,y} = \begin{pmatrix}
x & -\bar{y} \\
y & \bar{x}
\end{pmatrix} \cdot \begin{pmatrix}
-1 & -1 \\
1 & -1
\end{pmatrix} = \begin{pmatrix}
-x - \bar{y} & -x + \bar{y} \\
-\bar{x} + y & -\bar{x} - y
\end{pmatrix}.
\] (33)

Varying \( (x,y) \) over \( \{(1, 0), (i, 0), (0, 1), (0, i)\} \) gives generators for the normal subgroup of \( \bar{\rho}_4(F_2) \)

\[ N_1 = \langle \bar{\rho}_4(ga^2b^2g^{-1}) | g \in F_2 \rangle. \]

Namely we see that

\[ N_1 = \gamma(\Lambda_1) \quad \text{where} \quad \Lambda_1 = \langle (-1, 1), (-i, -i), (-1, -1), (i, -i) \rangle \subset \mathbb{Z}[i]^2. \]

A similar calculation shows that the normal subgroup

\[ N_2 = \langle \bar{\rho}_4(ga^{-1}abg^{-1}) | g \in F_2 \rangle \quad \text{is given by} \]

\[ N_2 = \gamma(\Lambda_2) \quad \text{where} \quad \Lambda_2 = \langle (0, i + 1), (0, 1 - i), (-1 - i, 0), (-1 + i, 0) \rangle \subset \mathbb{Z}[i]^2. \]

A simple calculation shows that

\[ \langle \Lambda_1, \Lambda_2 \rangle = \langle (-1, 1), (-i, -i), (-1, -1), (0, i + 1) \rangle \]

which is a subgroup of \( \mathbb{Z}[i]^2 \) with index two. Observe that \( \Lambda = \langle \Lambda_1, \Lambda_2 \rangle \) and from the discussion above we have \( \bar{\rho}_4(\ker \rho_4) = \gamma(\Lambda) \).

The short exact sequence follows from the fact that \( \gamma(\Lambda) \) is a free abelian group of rank four. \[ \Box \]

Given \( \bar{\rho}_4 \) and \( \rho_4 \) as above we may consider the tensor product \( \bar{\rho}_4^{4} = \mathcal{P}_{\mathbb{C}} \otimes \rho_4 \), which is also an oriented characteristic representation by Proposition \( 3.2 \). We have \( \ker \bar{\rho}_4^{4} = \ker \bar{\rho}_4 \).

We can view \( \bar{\rho}_4^{4} \) as a homomorphism to \( \text{GL}(8, \mathbb{C}) \).

Let \( \bar{\rho}_4 : F_2 \to \text{GL}(9, \mathbb{C}) \) so that be the homomorphism defined so that

\[ \bar{\rho}_4(a) = \text{diag}(1, -1, -i, -i; -1, i, i, 1), \] (34)
The top left 8 × 8 submatrices of images of \( \tilde{\rho}^4 \) realize \( \tilde{\rho}^4' \). The representation \( \tilde{\rho}^4 \) was found by applying the approach of Theorem 3.15 to \( \tilde{\rho}^4 \) but we will not prove this. We have:

**Proposition 3.20.** The homomorphism \( \tilde{\rho}^4 \) is an oriented characteristic representation. The kernel of \( \tilde{\rho}^4 \) contains \( P_4 \). Furthermore, there is a short exact sequence of groups of the form

\[
1 \rightarrow \mathbb{Z}^d \rightarrow F_2 / \ker \tilde{\rho}^4 \rightarrow F_2 / \ker \tilde{\rho}^4 \rightarrow 1
\]

where \( d \geq 1 \).

It will follow from later work that \( \ker \tilde{\rho}^4 = P_4 \) and that \( d = 1 \) in the statement above. See Theorem 4.2.

**Proof.** That \( \tilde{\rho} \) is oriented characteristic follows from Proposition 3.3 with

\[
M_1 = \begin{pmatrix}
2 & 2i & i - 1 & -i - 1 & -2i & 2 & i + 1 & i - 1 \\
2i & 2 & -i + 1 & -i - 1 & 2 & -2i & -i - 1 & i - 1 \\
0 & 0 & 2i - 2 & 0 & 0 & 2i + 2 & 0 & 2 \\
0 & 0 & 0 & -2i - 2 & 0 & 0 & 0 & 2i - 2 \\
-2i & 2 & i + 1 & -i - 1 & 2 & 2i & i - 1 & -i - 1 \\
2 & -2i & -i - 1 & i - 1 & 2i & 2 & -i + 1 & -i - 1 \\
0 & 0 & 2i + 2 & 0 & 0 & 2i - 2 & 0 & -2 \\
0 & 0 & 0 & 2i - 2 & 0 & 0 & 0 & -2i - 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix},
\]

\( M_2 = \text{diag}(1, -i, -i, 1, 1, 1, 1, 1) \)

and \( M_{-} = I \). Again we have \( P_4 \subset \ker \tilde{\rho}_4 \) because \( \ker \tilde{\rho}_4(a^4) = I \).

It may be observed that the upper left 8 × 8 submatrix represents \( \overline{\rho}_4 \otimes \rho_4 \). Thus \( F_2 / \ker \tilde{\rho}_4 \) is a \( \mathbb{Z}^d \) extension of \( \ker \tilde{\rho}_4 \) for some \( d \geq 0 \). We compute

\[
\tilde{\rho}_4([a, b]^2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

(36)

Thus \([a, b]^2\) lies in \( \ker \tilde{\rho}_4 \) and its image generates a copy of \( \mathbb{Z} \) in \( \tilde{\rho}_4(F_2) \). This shows \( d \geq 1 \).
For this section let \( G = F_2 / P_4 \). The following proposition tells us that this group is virtually \( \mathbb{Z} \times \mathbb{Z} \times H(\mathbb{Z}) \). We will then use this result to prove the representation \( \hat{\rho}_4 \) is faithful.

**Proposition 4.1.** Let \( N_p \) be the subgroup of \( G \) generated by

\[
E^{-2}, \quad A^4D^2, \quad AE^{-2}A^{-4}BA^{-1}B^{-1}, \quad \text{and} \quad A^6CA^{-1}C^{-1}BAB^{-1}A^{-1},
\]

where

\[
A = ba^{-1}b^{-1}a, \quad B = b^{-1}aba^{-1}, \quad C = b^{-1}a^{-1}ba,
\]
\[
D = a^2b(a^{-1}b^{-1})^2a^{-1}ba, \quad \text{and} \quad E = b^{-1}(ab)^2a^{-3}b^{-1}a.
\]

Then \( N_p \) is a 5-dimensional torsion-free nilpotent subgroup of index 2\(^{12} \) in \( F_2 / P_4 \) that is isomorphic to \( H(\mathbb{Z}) \times H(\mathbb{Z}) \) with one non-trivial added relator.

**Proof.** Let \( G_2 \) be the second term of the derived series for \( G \) described in terms of the relations provided by the table on page 10. Using GAP, we compute the following presentation for \( G_2 \):

\[
\langle F_1, F_2, F_3, F_4, F_5 : \\
F_1^{-1}F_1^{-1}F_3F_1 = F_2^{-1}F_1^{-1}F_2F_3 = F_2^{-1}F_4F_3F_4^{-1} = 1, \\
F_1^{-1}F_2F_1F_2^{-1} = F_4F_3F_4^{-1}F_5 = F_5^{-1}F_2F_3F_4^{-1} = 1, \\
F_4F_1^{-1}F_4^{-1}F_1F_3^{-1} = F_4^{-1}F_4F_3F_4^{-1}F_1F_3^{-1} = 1, \\
F_4F_1^{-1}F_4^{-1}F_1F_3^{-1} = F_3^{-1}F_2F_1^{-1}F_4F_2F_1^{-1} = 1, \\
F_5^{-1}F_3F_5^{-1}F_3 = F_5^{-1}F_3F_5^{-1}F_3^{-1} = 1, \\
F_5^{-1}F_2^{-1}F_3^{-1}F_5^{-1}F_1^{-1}F_3^{-1} = F_2F_1F_4^{-1}F_3F_2^{-1}F_1^{-1} = 1 \rangle
\]

From this presentation we see that \( G_2 \) satisfies the following:

(1) first homology of \( G_2 \) is \( \mathbb{Z} / 4 \times \mathbb{Z}^4 \).

(2) \( G_2 \) has index 1024.

Let \( N_p \) be the group generated by \( F_1, F_3, F_4, F_5 \). Using GAP, we see that \( N_p \) has the desired index and has the desired generators. Moreover, we see that \( N_p \) has a presentation of the form:

\[ N_p = \langle a_1, a_2, a_3, a_4 : R \rangle, \]

where

\[ R = \{ [a_1, a_2], [a_3, a_4], [a_1, a_3], [a_2^2a_3, a_1^{-1}a_2], [a_2, a_4], [a_4a_3, a_1^{-2}a_2], [a_4a_3, (a_1^{-1}a_2)^{-1}a_4(a_1^{-1}a_2)] \}. \]

This is a quotient of the right-angled Artin group \( F_2 \times F_2 \) with the three added relations

\[ [a_1^2a_3, a_1^{-1}a_2], [a_4a_3, a_1^{-2}a_2], [a_4a_3, (a_1^{-1}a_2)^{-1}a_4(a_1^{-1}a_2)]\].

Viewing the group as \( F_2 \times F_2 = \langle a_1, a_4 \rangle \times \langle a_2, a_3 \rangle \), we can simplify the relations to:

\[ ([a_2^2, a_1^{-1}][a_3, a_2]), ([a_4, a_1^{-2}][a_3, a_2]), \gamma := ([a_4, a_1a_4a_1^{-1}], 1) \].

Using suitable conjugations, we further simplify the relations to:

\[ ([a_1, a_4^2], [a_3, a_2]), ([a_4^2, a_4], [a_4, a_2]), \gamma := ([a_4, a_1a_4a_1^{-1}], 1) \].

Then \( N_p \) is the group \( F_2 \times F_2 / K \) where \( K \) is the normal subgroup generated by the elements above.
The last relator gives \([a_1, a_4]\) and \(a_4\) commute. By the two other relators, we have
\([(a_1, a_2^3), 1] = ([a_1^2, a_4], 1)\). This equality is equivalent to
\([a_1, a_4][a_1, a_4]^{a_1} = [a_1, a_4][a_1, a_4]\).
Hence, \([(a_1, a_4), 1]\) is central in \(\frak P\).

Let \(H_1\) be the image of \(F_2 \times 1\) in \(\frak N\) and \(H_2\) the image of \(1 \times F_2\). Since \([(a_1, a_4), 1]\) is
central, the groups \(H_1\) and \(H_2\) are both quotients of \(UT(3, \mathbb{Z})\) (in fact, they are both
isomorphic to \(UT(3, \mathbb{Z})\)). It follows that \(N_\rho\) must be \(UT(3, \mathbb{Z}) \times UT(3, \mathbb{Z})\) with a relation
identifying the square of a central generator of \(UT(3, \mathbb{Z}) \times 1\) with one of \(1 \times UT(3, \mathbb{Z})\).
By projecting onto the second factor, we see that \(N_\rho\) has infinite center. Thus, \(N_\rho\) is 5-
dimensional and torsion-free, as desired.

Recall the definition of \(\tilde{\rho}_4 : F_2 \to \text{GL}(9, \mathbb{C})\) described by (34) and (35). From Proposition
3.20 \(P_4 \subset \ker \tilde{\rho}_4\) thus we can consider \(\tilde{\rho}_4\) to be a homomorphism from \(G = G_2/P_4\) to
\(\text{GL}(9, \mathbb{C})\).

**Theorem 4.2.** The representation \(\tilde{\rho}_4 : F_2/P_4 \to \text{GL}(9, \mathbb{C})\) is faithful. We have \(d = 1\) in
Proposition 3.20.

**Proof.** We claim that \(\tilde{\rho}_4(N_\rho)\) has index \(2^{12}\) inside of \(\tilde{\rho}_4(G)\), that \(\tilde{\rho}_4(N_\rho)\) is torsion free and
that the dimension of \(\tilde{\rho}_4(N_\rho)\) has equal dimension. The conclusion follows since we already know \(\tilde{\rho}_4(G)\) is a quotient of \(G\) and no non-trivial quotient of \(N_\rho\) has equal dimension.

Define
\[g_0 = E^{-2}, \ g_1 = A^4D^2, \ g_2 = AE^{-2}A^{-4}BA^{-1}B^{-1}, \ \text{and} \ g_3 = A^9CA^{-1}C^{-1}BAB^{-1}A^{-1},\]
where \(A, B, C, D,\) and \(E\) are defined as in Proposition 4.1. It is a computation that \(g_i \in \ker \tilde{\rho}_4\)
for \(i \in \{1, 2, 3\}\). Observe \(\tilde{\rho}_4(\ker \rho_4)\) is torsion-free since \(F_2/\ker \tilde{\rho}_4\) is a \(\mathbb{Z}^4\)-extension of
\(F_2/\ker \rho_4\) and \(F_2/\ker \tilde{\rho}_4\) is a \(\mathbb{Z}^d\)-extension of \(F_2/\ker \rho_4\) by Propositions 3.19 and 3.20
respectively. Thus in particular \(\tilde{\rho}_4(N_\rho)\) is torsion-free as claimed.

Also observe that \(d = 1\) in Proposition 3.20. This is because \(\dim \tilde{\rho}_4(\ker \rho_4) = 4 + d\)
from the description above and \(\tilde{\rho}_4(\ker \rho_4) \geq 5\) since we know \(d \geq 1\) from Proposition
3.20. Recalling that \(\tilde{\rho}_4(\ker \rho_4)\) contains \(\tilde{\rho}_4(N_\rho)\) with finite index and that \(\dim N_\rho = 5\), we see
that \(\tilde{\rho}_4(\ker \rho_4) \leq 5\). Thus we must have \(\tilde{\rho}_4(\ker \rho_4) = 5\) and \(d = 1\). It also follows that
\(\dim \tilde{\rho}_4(N_\rho) = 5\) as claimed above.

Finally, we will check that \(\tilde{\rho}_4(G) : \tilde{\rho}_4(N_\rho) = 2^{12}\). Note that it suffices to prove that
\([\tilde{\rho}_4(G) : \tilde{\rho}_4(N_\rho)] \geq 2^{12}\) since index can not grow under group homomorphisms.

First observe that \([\rho_4(G) : \rho_4(N_\rho)] = 2^3\) since \(N_\rho \subset \ker \rho_4\) and \(\rho_4(G)\) is isomorphic to the
quaternion group.

Define \(\gamma : \mathbb{Z}[i]^2 \to \text{GL}(4, \mathbb{C})\) as in (32). By Proposition 3.19 \(\tilde{\rho}_4(\ker \rho_4) = \gamma(\Lambda)\) where
\(\Lambda \subset \mathbb{Z}[i]^2\) is a subgroup of index two. We compute
\[
\begin{align*}
\tilde{\rho}_4(g_0) & = \gamma(-2i - 2i + 2), & \tilde{\rho}_4(g_1) & = \gamma(2i - 2i + 2) \\
\tilde{\rho}_4(g_2) & = \gamma(4, 0), & \tilde{\rho}_4(g_3) & = \gamma(0, 4i).
\end{align*}
\]

Thus \(\tilde{\rho}_4(N_\rho) = \gamma(\Lambda')\) where
\(\Lambda' = \langle (-2 - 2i, -2i + 2), (2i - 2i + 2), (4, 0), (0, 4i) \rangle\).

Observe that \([\mathbb{Z}[i]^2 : \Lambda'] = 2^7\) and thus \([\Lambda : \Lambda'] = 2^6\). It follows that
\([\rho_4(\ker \rho_4) : \tilde{\rho}_4(N_\rho)] = 2^6\) and \([\rho_4(F_2/P_4) : \tilde{\rho}_4(N_\rho)] = 2^{6+3}\).

Now recall that \(F_2/\ker \tilde{\rho}_4\) was a \(\mathbb{Z}\) extension of \(F_2/\ker \rho_4\). We have \([a, b]^2 \in \ker \rho_4\) but
\(\tilde{\rho}_4([a, b]^2) \neq I\) (see (36)). Consider images under \(\tilde{\rho}_4\) of commutators of the generators of
$N_p$. We compute
\[ \tilde{\rho}_4([g_0, g_1]) = \tilde{\rho}_4([g_2, g_3]) = 1. \]
For other pairs of generators of $N_p$ we have:
\[ \tilde{\rho}_4([g_2, g_0]) = \tilde{\rho}_4([g_3, g_0]) = \tilde{\rho}_4([g_1, g_2]) = \tilde{\rho}_4([g_3, g_1]) = \tilde{\rho}_4([a, b]^2)^8. \]
Thus the copy of $\mathbb{Z}$ in $\tilde{\rho}_4(G)$ provided by Proposition 3.20 contains $\tilde{\rho}_4(N_p \cap \ker \tilde{\rho}_4)$ with index at least $2^3$. Consequently, $[\tilde{\rho}_4(F_2/P_4) : \tilde{\rho}_4(N_p)] \geq 2^{3+6+3}$ as required for our argument above. □

APPENDIX A. RELATION TO SQUARE TILED SURFACES

A translation surface is a surface equipped with an atlas of coordinate charts to the plane so that the transition functions are restrictions of translations.

Let $\mathbb{T}$ denote the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathbb{T}^o = \mathbb{T} \setminus \{0\}$ be the once punctured torus. A square-tiled surface is a cover of $\mathbb{T}^o$ endowed with the pullback translation structure. Here we allow the cover to be finite or infinite. See [Zor06] for a survey discussing translation surfaces including square-tiled surfaces.

Fix a translation surface $S$. Given a vector $(u, v) \in \mathbb{R}^2$ the straight-line flow determined by $(u, v)$ is the flow $F^t : S \to S$ given in local coordinates by
\[ F^t(x, y) = (x, y) + t(u, v). \]
The straight line flow of a point will not be defined for all time if under the projection to $\mathbb{T}$ the flow hits the puncture at 0. We call such a straight-line trajectory singular.

Let $(u, v) \in \mathbb{Z}^2$ and assume $u$ and $v$ are relatively prime. Then the straight-line flow determined by $(u, v)$ on the torus $\mathbb{T}$ is periodic with all points having period one. Let $S$ be a square tiled surface. For a positive integer $k$ we say $S$ is $k$-periodic if for all relatively prime $(u, v) \in \mathbb{Z}^2$, the every non-singular straight-line trajectory determined by $(u, v)$ is periodic with period dividing $k$.

We take $\left(\frac{1}{2}, \frac{1}{2}\right)$ to be the basepoint of $\mathbb{T}^o$ and say that a square tiled surface with basepoint is a square tiled surface with basepoint $s$ so that the covering map to $\mathbb{T}^o$ maps $s$ to $\left(\frac{1}{2}, \frac{1}{2}\right)$. If $S_1$ and $S_2$ are two square tiled surfaces with basepoints $s_1$ and $s_2$ respectively and $\pi_i : S_i \to \mathbb{T}^o$ are the associated covering maps we say that $S_1$ covers $S_2$ if there is a covering map $\pi : S_1 \to S_2$ so that $\pi(s_1) = \pi(s_2)$ and $\pi_2 \circ \pi_1 = \pi_1$.

This paper originated with the following observation:

**Proposition A.1.** For any $k \geq 1$ there is a $k$-periodic square tiled surface with basepoint $U_k$ so that $U_k$ covers any other $k$-periodic square-tiled surface with basepoint.

We call $U_k$ the universal $k$-periodic square-tiled surface.

Covering space theory associates a square tiled surface $S$ with basepoint to a subgroup $\Gamma_S$ of the fundamental group $\pi_1(\mathbb{T}^o, \left(\frac{1}{2}, \frac{1}{2}\right))$. Note that this fundamental group is isomorphic to the free group $F_2$. For purposes of this appendix consider $\pi_1(\mathbb{T}^o, \left(\frac{1}{2}, \frac{1}{2}\right))$ to be the same as $F_2$. Following Herrlich we call $S$ characteristic if $\Gamma_S$ is a characteristic subgroup of $F_2$. Characteristic square-tiled surfaces $S$ are maximally symmetric: they have a deck group acting transitively on the lifts of any point of $\mathbb{T}^o$ and the affine action of $\text{GL}(2, \mathbb{Z})$ on $S$ stabilizes $S$.

Some finite characteristic square-tiled surfaces which are $k$-periodic are have attained an almost mythical status in the subject of translation surfaces serving up numerous counterexamples in the field. Especially famous are the fantastically named eierlegende Wollmilchsau discovered independently in [For05] and [HS08] and the ornithorynque first described...
in [FM08]. These surfaces were further studied in and studied further in [PMZ11] and [MWS14]. If this article were written more geometrically the Heisenberg origamis studied by Herrlich in [Her06] would play a leading role.

Two facts combine to give a proof of Proposition A.1:

1. From basic covering space theory, the square-tiled surface with basepoint $S_1$ covers the square-tiled surface $S_2$ with basepoint if and only if $\Gamma_{S_2} \subset \Gamma_{S_1}$.

2. A conjugacy classes in $\mathbb{F}_2$ represents a homotopy class of curves containing closed geodesics on $T$ if and only if the conjugacy class consists of primitive elements in $\mathbb{F}_2$. This observation dates back to Jakob Nielsen’s 1913 Thesis.

It follows that a square-tiled surface with basepoint $S$ is $k$-periodic if and only if it is covered by the square tiled surface $U_k$ defined so that $\Gamma_{U_k} = P_k$ where $P_k \subset \mathbb{F}_2$ denotes the subgroup generated by $k$-th powers of primitive elements as in this paper.

From work in this paper we obtain a understanding of $U_1, \ldots, U_4$:

1. We have $U_1 = \mathbb{F}_2$.

2. The surface $U_2$ is $(\mathbb{R}/2\mathbb{Z})^2$ punctured at the integer points.

3. The surface $U_3$ is the Heisenberg origami denoted $O_{3,3}$ in [Her06] jointly discovered by Herrlich, Möller and Weitze-Schmithuesen.

The eierlegende Wollmilchsau mentioned above is the square-tiled surface $W$ so that $\Gamma_W$ is the kernel of the surjective homomorphism $\mathbb{F}_2 \rightarrow \mathbb{Q}$ where $\mathbb{Q}$ is the quaternionic group. The surface $W$ is 4-periodic. From our understanding in this paper of $P_4$ and in particular knowledge of the representation $\tilde{\rho}_4$ of $\mathbb{I}$ which is faithful by Theorem 4.2, we see:

Theorem A.2. The surface $U_4$ is an infinite area square tiled surface and is a torsion-free 5-dimensional nipotent cover of the eierlegende Wollmilchsau.

It is particularly interesting that $U_4$ is a geometrically natural example of an infinite nilpotent cover of a compact translation surface, because some methods are available to study the dynamics of the straight-line flow on such a surface; see for instance [Con09].

It is a consequence of [Hoo15, Theorem G.3, Remark 4.1] and $\text{GL}(2, \mathbb{Z})$-invariance of $U_4$ that:

Corollary A.3. There is a dense subset $E$ of the unit circle in $\mathbb{R}^2$ with Hausdorff dimension larger than $\frac{1}{2}$ so that for any $(u, v) \in E$ the straight-line flow determined by $(u, v)$ on $U_4$ is ergodic.

As a consequence of the universality of $U_4$ it follows that the straight-line flow determined by each $(u, v) \in E$ is ergodic on each 4-periodic square tiled surface. This motivates:

Question 4. Is it the straight-line flow determined by $(u, v)$ ergodic on $U_4$ whenever $\frac{v}{u} \notin \mathbb{Q}$?

The kernels of the representations $\tilde{\rho}_k$ for odd $k \geq 5$ determine characteristic $k$-periodic origamis $O_k$ which are infinite free abelian covers of the Heisenberg origamis of Herrlich. The conclusions of Corollary A.3 then hold for the surfaces $O_k$ and we similarly wonder Question 4 in these cases.

This paper shows that $P_k$ is infinite index in $\mathbb{F}_2$ when $k \geq 4$ and it follows that for $k \geq 4$ the surface $U_k$ is infinite. Virtual nilpotence of $\mathbb{F}_2/P_k$ is necessary to apply the [Hoo15, Theorem G.3] so an affirmative answer to Question 1(b) in a case of $r = 2$ and $k \geq 5$ would extend Corollary A.3 to cover the corresponding $U_k$.

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