DISTINGUISHED BASES AND STOKES REGIONS
FOR THE SIMPLE AND THE SIMPLE ELLIPTIC
SINGULARITIES

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Abstract. Isolated hypersurface singularities come equipped with a Milnor lattice, a \( \mathbb{Z} \)-lattice of finite rank, and a set of distinguished \( \mathbb{Z} \)-bases of this lattice. Usually these bases are constructed from one morsification and all possible choices of distinguished systems of paths. But what does one obtain if one considers all possible morsifications and one fixed distinguished system of paths? Looijenga asked this question 1974 for the simple singularities. He and Deligne found that one obtains a bijection between Stokes regions in a universal unfolding and the set of distinguished bases modulo signs. This allows to see the base space of the universal unfolding as an atlas of Stokes data. Here we reprove their result and extend it to the simple elliptic singularities. We use more conceptual arguments, moduli spaces of marked singularities (i.e. Teichmüller spaces for singularities), extensions of them to F-manifolds, and the actions of symmetries of singularities on the Milnor lattices and these moduli spaces. We use and extend results of Jaworski on the Lyashko-Looijenga maps for the simple elliptic singularities. The sections 2 and 3 give a survey on singularities and the associated objects which allows to read the paper independently of other sources.

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This paper has three parts. The first part is an introduction to isolated hypersurface singularities. It makes the paper readable independently of other sources, and it gives a basis for the other two parts.

The second part is an extension to the simple elliptic singularities of work which Looijenga and Deligne did 1974 for the simple singularities. This is the central part of the paper.

The third part is an extension and refinement of work of Jaworski 1986–1988 on the Lyashko-Looijenga maps for the simple elliptic singularities. The arguments are less conceptual and more computational and more laborious than the arguments in the second part. We need it in order to determine the sizes of certain finite sets which are in bijection by the second part.

An isolated hypersurface singularity is a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0. In order to see its topology, one chooses a good representative $f : Y \to T$ with $Y \subset \mathbb{C}^{n+1}$ a suitable neighborhood of 0 and $T \subset \mathbb{C}$ a small disk around 0. The Milnor lattice $Ml(f)$ is the (reduced for $n = 0$) middle homology $H_{n}^{(\text{red})}(f^{-1}(\tau), \mathbb{Z}) \cong \mathbb{Z}^{\mu}$ of a regular fiber $f^{-1}(\tau)$ of $f : Y \to T$ for some $\tau \in T \cap \mathbb{R}_{>0}$. Here $\mu \in \mathbb{Z}_{>0}$ is the Milnor number of $f$. The Milnor lattice comes equipped with a monodromy $M_h$, an intersection form $I$, a Seifert form $L$, and a set $B(f)$ of certain $\mathbb{Z}$-bases of $Ml(f)$, the distinguished bases. $M_h$ is a quasiunipotent automorphism, $I$ is a $(-1)^n$-symmetric bilinear form, $L$ is a unipotent bilinear form, and $L$ determines $M_h$ and $I$. The group $G_{Z}(f) := \text{Aut}(Ml(f), M_h, I, L) = \text{Aut}(Ml(f), L)$ will be important.

The distinguished bases are constructed as follows. First, one chooses a morsification $F^{(\text{mor})} : Y^{(\text{mor})} \to T$ of $f : Y \to T$, that is a deformation such that the one singularity of $f : Y \to T$ with Milnor number $\mu$ splits
into \( \mu A_1 \)-singularities \( x^{(j)}, j = 1, \ldots, \mu \), of \( F^{(\text{mor})} : Y^{(\text{mor})} \to T \) with pairwise different critical values \( u_j = F^{(\text{mor})}(x^{(j)}), j = 1, \ldots, \mu \), with \( |u_j| < \tau \). Second, one chooses a distinguished system of paths. That is a system of \( \mu \) paths \( \gamma_j, j = 1, \ldots, \mu \), from \( u_\sigma(j) \) to \( \tau \) for some permutation \( \sigma \in S_\mu \), which do not intersect except at \( \tau \) and which arrive at \( \tau \) in clockwise order. Third, one shifts from the \( A_1 \)-singularity above each value \( u_\sigma(j) \) the (up to the sign unique) vanishing cycle along \( \gamma_j \) to \( H_n^{(\text{red})}((F^{(\text{mor})})^{-1}(\tau), \mathbb{Z}) \) and then by a canonical isomorphism to \( M_l(f) \) and calls the image \( \delta_j \). The tuple \( \tilde{\delta} = (\delta_1, \ldots, \delta_\mu) \) turns out to be a \( \mathbb{Z} \)-basis of \( M_l(f) \) and is called a distinguished basis. One morsification, all possible choices of distinguished systems of paths and both possible choices \( \pm \delta_j \) of signs for each cycle give all distinguished bases.

The Stokes matrix of one distinguished basis \( \tilde{\delta} \) is the matrix \( S = (-1)^{(n-1)(n+2)/2}L(\tilde{\delta}^t, \tilde{\delta})^t \). It is an upper triangular integer matrix with 1’s on the diagonal. The following table gives some information on the sizes of the sets \( \mathcal{B}(f) \) and \( \{|\text{Stokes matrices}\|\} \).

\[
\begin{array}{ccc}
 f & |\mathcal{B}(f)| & |\{\text{Stokes matrices}\}| \\
 \text{simple singularity} & \text{finite} & \text{finite} \\
 \text{simple elliptic singularity} & \text{infinite} & \text{finite} \\
 \text{any other singularity} & \text{infinite} & \text{infinite}
\end{array}
\] (1.1)

The last line of it was proved only recently by Ebeling [Eb18]. The other two lines are explained for example in [Eb18] or in remark 7.2(i) below.

The simple singularities \( A_\mu (\mu \geq 1), D_\mu (\mu \geq 4), E_6, E_7 \) and \( E_8 \) and the simple elliptic singularities \( \tilde{E}_6, \tilde{E}_7 \) and \( \tilde{E}_8 \) are the first ones in Arnold’s lists [AGV85, ch. 15.1] of isolated hypersurface singularities. The simple singularities have no \( \mu \)-constant parameter. The simple elliptic singularities are 1-parameter families. See subsection 4.1 for normal forms for all of them.

Deligne [De74] characterized \( \mathcal{B}(f) \) and calculated the number \( |\mathcal{B}(f)| \) for the simple singularities. Yu [Yu90] [Yu96] [Yu99] derived from that the number \( |\{\text{Stokes matrices}\}| \) for the simple singularities. Kluitmann characterized \( \mathcal{B}(f) \) for the simple elliptic singularities. He calculated the number \( |\{\text{Stokes matrices}\}| \) for \( \tilde{E}_6 \) in [Kl83] and for \( \tilde{E}_7 \) in [Kl87], by huge combinatorial efforts. The number \( |\{\text{Stokes matrices}\}| \) for \( \tilde{E}_8 \) was not calculated before this paper. In corollary 7.3 we recover Kluitmann’s numbers for \( \tilde{E}_6 \) and \( \tilde{E}_7 \), and we give the number for \( \tilde{E}_8 \), by a completely different method. Our method combines a natural bijection in the second part with the calculation of three numbers in the third
part, the degrees of certain Lyashko-Looijenga maps for $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

The simple singularities $f = f(x_0, \ldots, x_n)$ have universal unfoldings

$$F^\text{alg}(x_0, \ldots, x_n, t_1, \ldots, t_\mu) = F(x, t) = F_t(x) = f(x) + \sum_{j=1}^{\mu} t_j m_j, \quad (1.2)$$

with $m_1, \ldots, m_\mu \in \mathbb{C}[x]$ the monomials in table (4.4) and with parameters $t \in M^\text{alg} := \mathbb{C}^\mu$.

1974 Looijenga [Lo74] and Lyashko (but his work was published only later in [Ly79] [Ly84]) considered the Lyashko-Looijenga map

$$LL^\text{alg} : M^\text{alg} \rightarrow M_{LL}^{(\mu)} := \{ y^{\mu} + \sum_{j=0}^{\mu-1} s_j y^j \mid (s_1, \ldots, s_\mu) \in \mathbb{C}^\mu \} \quad (1.3)$$

$$t \mapsto \prod_{j=1}^{\mu} (y - u_j) \text{ with } (u_1, \ldots, u_\mu) \text{ the critical values of } F_t^\text{alg}.$$ for the simple singularities. It is a branched covering of a finite degree $\text{deg } LL^\text{alg}$, see theorem 6.1 for details.

Looijenga posed the following problem: Consider a generic polynomial $p(y) = \prod_{j=1}^{\mu} (y - u_j) \in M_{LL}^{(\mu)}$. Then $F_t$ for any $t \in (LL^\text{alg})^{-1}(p(y))$ is a morsification of $f$ with the same critical values $u_1, \ldots, u_\mu$. Now fix one distinguished system of paths from $u_1, \ldots, u_\mu$ to $\tau$. Each morsification $F_t^\text{alg}$ with $t \in (LL^\text{alg})^{-1}(p(y))$ gives one distinguished basis $\delta = (\delta_1, \ldots, \delta_\mu)$ up to signs. One obtains a map

$$LD : (LL^\text{alg})^{-1}(p(y)) \rightarrow \mathcal{B}(f)/G_{\text{sign,} \mu}, \quad (1.4)$$

where the group $G_{\text{sign,} \mu} := \{ \pm 1 \}^\mu$ acts on $\delta = (\delta_1, \ldots, \delta_\mu)$ by sign changes. An easy argument in [Lo74] which uses that $LL^\text{alg}$ is a branched covering, shows that the map $LD$ is surjective. Looijenga asked whether $LD$ is injective. He proved this for the $A_\mu$-singularities. Then Deligne [De74] calculated $|\mathcal{B}(f)|$ for all simple singularities and showed $|\mathcal{B}(f)/G_{\text{sign,} \mu}| = \text{deg } LL^\text{alg} = |(LL^\text{alg})^{-1}(p(y))|$. This proved that $LD$ is a bijection for all simple singularities. Deligne’s letter [De74] to Looijenga is not published. The result that $LD$ is a bijection is stated in [Mi89], [Yu90] and below in theorem 7.1.

A central part of this paper is an extension of this result to the simple elliptic singularities. Here a universal unfolding of a single simple elliptic singularity is not good enough. In subsection 4.2 we present a
global family of functions

\[ F^{alg}(x_0, \ldots, x_n, t_1, \ldots, t_{\mu-1}, \lambda) = F^{alg}(x, t', \lambda) = F^{alg}_{t', \lambda}(x) \]
\[ = f_{\lambda}(x) + \sum_{j=1}^{\mu-1} t_j m_j, \]

with \( m_1, \ldots, m_{\mu-1} \in \mathbb{C}[x] \) the monomials in table (4.7), \( f_{\lambda}(x) \) the 1-parameter families in Legendre normal form in (4.2) of the simple elliptic singularities and with parameters \((t', \lambda) \in M^{alg} := \mathbb{C}^{\mu-1} \times (\mathbb{C} - \{0,1\})\). For each \( \lambda \in \mathbb{C} - \{0,1\} \), the family \( F^{alg} \) is (locally) a universal unfolding of \( f_{\lambda} \). The family \( F^{alg} \) is not completely canonical. But the family \( F^{mar} \) with parameter space \( M^{mar} := \mathbb{C}^{\mu-1} \times \mathbb{H} \), where \( \mathbb{H} \to \mathbb{C} - \{0,1\} \) is the universal covering, is canonical. This is made precise in theorem 4.3 in a way which uses marked singularities, the fact that the parameter space of each of the three families of marked simple elliptic singularities is \( M^{mar}_{\mu} \cong \mathbb{H} \) [GH17-1], and a thickening of this space to \( M^{mar} \). We obtain Lyashko-Looijenga maps \( LL^{alg} : M^{alg} \to M^{(\mu)}_{LL} \) and \( LL^{mar} : M^{mar} \to M^{(\mu)}_{mar} \).

Analogously to \( LD \) for the simple singularities, we obtain a Looijenga-Deligne map

\[ LD : (LL^{mar})^{-1}(p(y)) \to \mathcal{B}(f)/G_{\text{sign,\mu}} \]

for generic \( p(y) \in M^{(\mu)}_{LL} \). A main result of this paper is that this map is a bijection, see theorem 7.1. But our arguments are more involved than the arguments for the simple singularities. The surjectivity follows by the same easy argument in [Lo74], as soon as one has that the map

\[ LL^{alg} : M^{alg} - (\text{caustic} \cup \text{Maxwell stratum}) \]
\[ \to M^{(\mu)}_{LL} - (\text{discriminant } D^{(\mu)}_{LL}) \]

is a finite covering. This is a hard theorem of Jaworski [Ja86 Theorem 2] [Ja88, Proposition 1], see theorem 6.2.

As both sides of (1.6) are infinite, the injectivity of \( LD \) in (1.6) does not follow by a comparison of numbers. We need an action of \( G_{Z} \) on \( M^{mar} \) and on the middle homology bundle above \( M^{mar} - D^{mar} \). We need that \( M^{mar} \) is an F-manifold with Euler field. And we need that and how a Stokes matrix \( S \) of \( LD(t) \in \mathcal{B}(f)/G_{\text{sign,\mu}} \) for \( t \in (LL^{mar})^{-1}(p(y)) \) encodes the covering in (1.7). See the proof of theorem 7.1 for the details.

The bijection in (1.6) induces also a bijection

\[ (LL^{mar})^{-1}(p(y))/G_{Z} \to \{\text{Stokes matrices}\}/G_{\text{sign,\mu}}. \]
Both sides are finite, and the number $|(LL^{\text{mar}})^{-1}(p(y))/G_z|$ is in a simple way related to $\deg LL^{\text{alg}}$. Though Jaworski’s proofs in [Ja86] and [Ja88] that $LL^{\text{alg}}$ in (1.7) is a covering, do not allow to calculate its degree $\deg LL^{\text{alg}}$.

The main task in the third part of the paper is to extend Jaworski’s work and calculate $\deg LL^{\text{alg}}$. In theorem 6.3 we obtain an extension of $M^{\text{alg}}$ above $\mathbb{C} - \{0, 1\}$ to an orbibundle $M^{\text{orb}} \rightarrow \mathbb{P}^1$ such that $LL^{\text{alg}}$ extends to a holomorphic map $LL^{\text{orb}} : M^{\text{orb}} \rightarrow M^{(\mu)}_LL$ which is outside of the $\mu$-constant stratum (and its translates by the unit field) a branched covering and which maps (caustic) $\cup (\text{Maxwell stratum}) \cup \pi^{-1}_{\text{orb}}(\{0, 1, \infty\})$ to the discriminant $D^{(\mu)}_LL \subset M^{(\mu)}_LL$. Detailed information about $M^{\text{orb}}$ and $LL^{\text{orb}}$ allows us to calculate the degree $\deg LL^{\text{orb}} (= \deg LL^{\text{alg}})$.

The first part of the paper consists of section 2, the subsections 3.1 and 3.2 and the first three pages of section 4. Section 2 recalls classical data and facts around isolated hypersurface singularities, namely Milnor fibrations, Milnor lattices, universal unfoldings, the base spaces as F-manifolds with Euler fields, Lyashko-Looijenga maps, distinguished bases, Stokes matrices, and Thom-Sebastiani type results. Subsection 3.1 reviews results in [He02, Theorem 13.11 and Theorem 13.13] on symmetries of singularities. Subsection 3.2 reviews results in [He11, Theorem 4.3] on the moduli spaces $M^{\text{mar}}_\mu$ of marked singularities. The first three pages of section 4 give normal forms for the simple and the simple elliptic singularities and for the unfoldings.

The second part of the paper consists of the subsections 3.3 and 3.4, the latter part of section 4 and the sections 6 and 7. Subsection 3.3 describes a thickening $M^{\text{mar}} \supset M^{\text{mar}}_\mu$ of the moduli spaces of marked singularities. Theorem 4.3 in the latter part of section 4 proves the claims about this thickening in the cases of the simple and the simple elliptic singularities. Subsection 3.4 defines and discusses Looijenga-Deligne maps $LD$ in a general setting. Section 7 states and proves the main result theorem 7.1 that $LD$ is a bijection for each simple singularity and each family of simple elliptic singularities. Corollary 7.3 provides the finite numbers $|\{\text{Stokes matrices}\}|$. Section 6 states the old and new results on the Lyashko-Looijenga maps for the simple singularities (theorem 6.1 Lyashko and Looijenga) and the simple elliptic singularities (theorem 6.2 Jaworski, and theorem 6.3 new). The most beautiful formula in section 6 is formula (6.7) for $\deg LL^{\text{alg}}$ for the simple elliptic singularities.

The third part of the paper consists of the sections 5, 8, 9 and 10. Section 5 makes the general discussion of the symmetries of singularities in subsection 3.1 explicit in the cases of the simple and the simple
elliptic singularities. Section 8 follows Fulton’s book [Fu84] and extends some results there to the case of smooth cone bundles (definition 8.1). We need this for the proof of corollary 8.6 which is used in the proof of formula (6.7) in section 10. The long section 9 provides for the simple elliptic singularities the extension of $M^{\text{alg}} = \mathbb{C}^{\mu-1} \times (\mathbb{C} - \{0,1\})$ to $\lambda = 0$ such that $\text{LL}^{\text{alg}}$ extends well. Finding the right way to glue into $M^{\text{alg}}$ a fiber above $\lambda = 0$ was the most laborious part of this paper. Section 10 combines this with the symmetries in section 5 and provides the right extensions of $M^{\text{alg}}$ to $\lambda = 1$ and $\lambda = \infty$, and it uses corollary 8.6 to prove the formula (6.7) for $\deg \text{LL}^{\text{alg}}$ for the simple elliptic singularities.

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2. Topology and unfoldings of isolated hypersurface singularities

An isolated hypersurface singularity (short: singularity) is a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0. Such objects were studied intensively since the end of the 1960’ies. In this section, we review classical facts on their topology and their unfoldings and fix some notations. For the topology compare [AGV88] and [Eb07]. For the unfoldings compare [AGV85] and (especially for F-manifolds) [He02].

The Jacobi ideal of $f$ is the ideal $J_f := (\frac{\partial f}{\partial x^i}) \subset \mathcal{O}_{\mathbb{C}^{n+1}, 0}$, its Jacobi algebra is the quotient $\mathcal{O}_{\mathbb{C}^{n+1}, 0}/J_f$, its Milnor number is the finite number $\mu := \dim \mathcal{O}_{\mathbb{C}^{n+1}, 0}/J_f$.

2.1. Topology of singularities. A good representative of $f$ has to be defined with some care [Mi68],[AGV88],[Eb07]. It is $f : Y \to T$ with $Y \subset \mathbb{C}^{n+1}$ a suitable small neighborhood of 0 and $T \subset \mathbb{C}$ a small disk around 0. Then $f : Y' \to T'$ with $Y' = Y - f^{-1}(0)$ and $T' = T - \{0\}$ is a locally trivial $C^\infty$-fibration, the Milnor fibration. Each fiber has the homotopy type of a bouquet of $\mu$ $n$-spheres [Mi68].

Therefore the (reduced for $n = 0$) middle homology groups are $H^n_{\text{red}}(f^{-1}(\tau), \mathbb{Z}) \cong \mathbb{Z}^\mu$ for $\tau \in T'$. Each comes equipped with an intersection form $I$, which is a datum of one fiber, a monodromy $M_\tau$ and a Seifert form $L$, which come from the Milnor fibration, see [AGV88].
I.2.3] for their definitions (for the Seifert form, there are several conventions in the literature, we follow \cite{AGV88}). \( M_h \) is a quasirnipotent automorphism, \( I \) and \( L \) are bilinear forms with values in \( \mathbb{Z} \), \( I \) is \((-1)^n\)-symmetric, and \( L \) is unimodular. \( L \) determines \( M_h \) and \( I \) because of the formulas \cite{AGV88} I.2.3

\[
L(M_h a, b) = (-1)^{n+1} L(b, a), \quad (2.1)
\]

\[
I(a, b) = -L(a, b) + (-1)^{n+1} L(b, a). \quad (2.2)
\]

The lattices \( H_n(f^{-1}(\tau), \mathbb{Z}) \) for all Milnor fibrations \( f : Y' \to T' \) and then all \( \tau \in \mathbb{R}_{>0} \cap T' \) are canonically isomorphic, and the isomorphisms respect \( M_h \), \( I \) and \( L \). This follows from Lemma 2.2 in \cite{LR73}. These lattices are identified and called \textit{Milnor lattice} \( Ml(f) \). The group \( G_{\mathbb{Z}} \) is

\[
G_{\mathbb{Z}} = G_{\mathbb{Z}}(f) := \text{Aut}(Ml(f), L) = \text{Aut}(Ml(f), M_h, I, L), \quad (2.3)
\]

the second equality is true because \( L \) determines \( M_h \) and \( I \). We will use the notation \( Ml(f)_\mathbb{C} := Ml(f) \otimes_{\mathbb{Z}} \mathbb{C} \), and analogously for other rings \( R \) with \( \mathbb{Z} \subset R \subset \mathbb{C} \), and the notations

\[
Ml(f)_\lambda := \ker((M_h - \lambda \text{id})^\mu : Ml(f)_\mathbb{C} \to Ml(f)_\mathbb{C}) \subset Ml(f)_\mathbb{C},
\]

\[
Ml(f)_{1,\mathbb{Z}} := Ml(f)_1 \cap Ml(f) \subset Ml(f),
\]

\[
Ml(f)_{\neq 1} := \bigoplus_{\lambda \neq 1} Ml(f)_\lambda \subset Ml(f)_\mathbb{C},
\]

\[
Ml(f)_{\neq 1,\mathbb{Z}} := Ml(f)_{\neq 1} \cap Ml(f) \subset Ml(f).
\]

The formulas \ref{2.1} and \ref{2.2} show \( I(a, b) = L((M_h - \text{id})a, b) \). Therefore the eigenspace with eigenvalue 1 of \( M_h \) is the radical \( \text{Rad}(I) \subset Ml(f) \) of \( I \). By \ref{2.2} \( L \) is \((-1)^{n+1}\)-symmetric on the radical of \( I \).

2.2. Unfoldings of singularities. The notion of an unfolding of an isolated hypersurface singularity \( f \) goes back to Thom and Mather. An \textit{unfolding} of \( f \) is a holomorphic function germ \( F : (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}, 0) \) such that \( F|_{(\mathbb{C}^{n+1}, 0)} = f \) and such that \((M, 0)\) is the germ of a complex manifold. Its Jacobi ideal is \( J_F := (\frac{\partial F}{\partial x_1}) \subset \mathcal{O}_{\mathbb{C}^{n+1} \times M, 0} \), its critical space is the germ \((C, 0) \subset (\mathbb{C}^{n+1} \times M, 0) \) of the zero set of \( J_F \) with the canonical complex structure. The projection \((C, 0) \to (M, 0)\) is finite and flat of degree \( \mu \). A kind of Kodaira-Spencer map is the \( \mathcal{O}_{M, 0} \)-linear map

\[
a_C : \mathcal{T}_{M, 0} \to \mathcal{O}_{C, 0}, \quad X \mapsto \tilde{X}(F)|_{(C, 0)} \quad (2.4)
\]

where \( \tilde{X} \) is an arbitrary lift of \( X \in \mathcal{T}_{M, 0} \) to \((\mathbb{C}^{n+1} \times M, 0)\).
We will use the following notion of morphism between unfoldings. Let \( F_i : (\mathbb{C}^{n+1} \times M_i, 0) \to (\mathbb{C}, 0) \) for \( i \in \{1, 2\} \) be two unfoldings of \( f \) with projections \( \text{pr}_i : (\mathbb{C}^{n+1} \times M_i, 0) \to (M_i, 0) \). A morphism from \( F_1 \) to \( F_2 \) is a pair \( (\Phi, \varphi) \) of map germs such that the following diagram commutes,

\[
\begin{array}{ccc}
(\mathbb{C}^{n+1} \times M_1, 0) & \xrightarrow{\Phi} & (\mathbb{C}^{n+1} \times M_2, 0) \\
\downarrow \text{pr}_1 & & \downarrow \text{pr}_2 \\
(M_1, 0) & \xrightarrow{\varphi} & (M_2, 0)
\end{array}
\]

and

\[
\Phi|_{(\mathbb{C}^{n+1} \times \{0\}, 0)} = \text{id}, \quad F_1 = F_2 \circ \Phi
\]

hold. Then one says that \( F_1 \) is induced by \( (\Phi, \varphi) \) from \( F_2 \). An unfolding is versal if any unfolding is induced from it by a suitable morphism. A versal unfolding \( F : (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}, 0) \) is universal if the dimension of the parameter space \((M, 0)\) is minimal. Universal unfoldings exist by work of Thom and Mather. More precisely, an unfolding is versal if and only if the map \( a_C \) is surjective, and it is universal if and only if the map \( a_C \) is an isomorphism (see e.g. [AGV85, ch. 8] for a proof). Observe that \( a_C \) is surjective/an isomorphism if and only if its restriction to 0, the map

\[
a_{C,0} : T_0 M \to \mathcal{O}_{\mathbb{C}^{n+1},0}/J_f
\]

is surjective/an isomorphism. Therefore an unfolding

\[
F(x_0, \ldots, x_n, t_1, \ldots, t_\mu) = F(x, t) = F_i(x) = f(x) + \sum_{j=1}^\mu m_j t_i,
\]

with \((M, 0) = (\mathbb{C}^\mu, 0)\) with coordinates \( t = (t_1, \ldots, t_\mu) \) where \( m_1, \ldots, m_\mu \in \mathcal{O}_{\mathbb{C}^{n+1},0} \) represent a basis of the Jacobi algebra \( \mathcal{O}_{\mathbb{C}^{n+1},0}/J_f \), is universal.

2.3. F-manifolds. The base space of a universal unfolding is an F-manifold with Euler field.

**Definition 2.1.** [HM99] [He02] (a) An F-manifold is a complex manifold \( M \) together with a holomorphic commutative and associative multiplication \( \circ \) on its holomorphic tangent bundle \( TM \) and with a unit field \( e \in T_M \) such that the integrability condition

\[
\text{Lie}_{X \circ Y}(\circ) = X \circ \text{Lie}_Y(\circ) + Y \circ \text{Lie}_X(\circ)
\]

holds.
(b) Let \((M, \circ, e)\) be an F-manifold. An Euler field (of weight 1) is a
global holomorphic vector field \(E \in T_M\) with \(\text{Lie}_E(\circ) = \circ\).

F-manifolds are studied in [He02, ch. 2–5]. In the case of a universal
unfolding \(F : (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}, 0)\), its base space inherits from the
isomorphism \(a^{-1}_C : \mathcal{O}_{C,0} \to T_M\) a multiplication. It satisfies (2.10), and
\(e := a^{-1}_C([1])\) and \(E := a^{-1}_C([F])\) are the unit field and an Euler field
[He02, Theorem 5.3], so it is an F-manifold with Euler field.

We call a universal unfolding universal and not just semiuniversal,
because the morphism \(\varphi\) in (2.5) between the base spaces of any two
universal unfoldings is unique [He02, Theorem 5.4] (but \(\Phi\) on the total
spaces is not unique). Therefore the base space of a universal unfolding
is (with its structure as F-manifold with Euler field) unique up to
unique isomorphism.

The following result on decompositions of germs of F-manifolds is a
very instructive application of the integrability condition (2.10), and it
is especially telling in the case of isolated hypersurface singularities.

**Theorem 2.2. [He02, Theorem 2.11]** Let \((M, p)\) be the germ in \(p \in M\)
of an F-manifold. It is an elementary fact from commutative algebra
that the algebra \(T_pM\) decomposes into a direct sum \(\bigoplus_{k=1}^l (T_pM)_k\) of
irreducible local algebras (it is just the decomposition into simultaneous
generalized eigenspaces with respect to all (commuting!) multiplication
endomorphisms).

This decomposition extends uniquely to a decomposition \((M, p) = \prod_{k=1}^l (M_k, p_k)\) of germs of F-manifolds. These germs are irreducible
germs of F-manifolds. If \((M, p)\) has an Euler field, the Euler field
decomposes into a sum of Euler fields.

In the case of a good representative \(F : \mathcal{Y} \to T\) of a universal unfolding \(F\), for any \(t \in M\), the canonical decomposition from theorem 2.2
of \((M, t)\) into a product of germs of F-manifolds is a canonical decom-
position into a product of germs of base spaces of universal unfoldings
of the germs of \(F_t\) at all its critical points.

At generic \(t\), this is a decomposition into 1-dimensional F-manifolds,
and the eigenvalues \(u_1, \ldots, u_\mu\) of the Euler field form there local_coordinates, Dubrovin’s canonical coordinates. The Euler field has there the
shape \(E = \sum_{j=1}^\mu u_j e_j\) where \(e_j = \frac{\partial}{\partial u_j}\), the multiplication is given by
\(e_i \circ e_j = \delta_{ij} e_i\), the unit field is \(e = \sum_{j=1}^\mu e_j\), and the values \(u_1, \ldots, u_\mu\) are
the critical values of \(F_t\), i.e. the values of \(F_t\) at its critical points. Up
to isomorphism there is only one germ of a 1-dimensional F-manifold,
which is called \(A_1\). Then the germ \((M, t)\) at generic \(t\) is as a germ of
an F-manifold of the type \(A_1^\mu\).
2.4. Lyashko-Looijenga map. Looijenga [Lo74] was close to the notion of an F-manifold. He had already the canonical coordinates at generic points. And he and Lyashko [Ly79][Ly84] studied the Lyashko-Looijenga map and its behaviour near the caustic and the Maxwell stratum. For \( \mu \in \mathbb{Z}_{\geq 1} \) define

\[
M_{LL}^{(\mu)} = \{ y^\mu + \sum_{j=1}^{\mu} s_j y^{j-1} | (s_1, ..., s_\mu) \in \mathbb{C}^\mu \} \cong \mathbb{C}^\mu, \tag{2.11}
\]

\[
D_{LL}^{(\mu)} := \{ p(y) \in M_{LL}^{(\mu)} | p(y) \text{ has multiple roots} \}. \tag{2.12}
\]

\( D_{LL}^{(\mu)} \) is a hypersurface in \( M_{LL}^{(\mu)} \). Let \( F : (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}, 0) \) be a universal unfolding of a singularity \( f \). Let \( F : \mathcal{Y} \to T \) be a good representative of it with base space \( M \). Then its Lyashko-Looijenga map is the holomorphic map

\[
LL : M \to M_{LL}^{(\mu)}, \quad t \mapsto \prod_{j=1}^{\mu} (y - u_j), \quad \text{where } u_1, ..., u_\mu \tag{2.13}
\]

are the critical values of \( F_t \) (with multiplicities).

Define the caustic \( K_3 \subset M \) and the Maxwell stratum \( K_2 \subset M \) by

\[
K_3 := \{ t \in M | F_t \text{ has less than } \mu \text{ singularities} \}, \tag{2.14}
\]

\[
K_2 := \text{the closure in } M \text{ of the set } \{ t \in M | F_t \text{ has } \mu \text{ singularities, but less than } \mu \text{ critical values} \}. \tag{2.15}
\]

They are hypersurfaces in \( M \).

The Lyashko-Looijenga map \( LL \) restricts to a locally biholomorphic map \( LL : M - (K_3 \cup K_2) \to M_{LL}^{(\mu)} - D_{LL}^{(\mu)} \) (because the \( u_1, ..., u_\mu \) are local coordinates on \( M - K_3 \), it maps \( K_3 \cup K_2 \) to \( D_{LL}^{(\mu)} \), and it is a branched covering of order 3 respectively 2 at generic points of \( K_3 \) respectively \( K_2 \). All of this was proved by Lyashko [Ly79][Ly84] and Looijenga [Lo74]. Now it is an easy consequence of the F-manifold structure. At a generic point \( t \) of \( K_3 \), the germ of the F-manifold is of the type \( A_1^{\mu-2} \). Here \( A_2 \) is the first in the countable series of irreducible germs of massive F-manifolds [He02 ch. 4.2].

2.5. Distinguished bases and Stokes matrices of singularities. Good references for distinguished bases are [AGV88] and [Eb07]. We sketch their construction and properties.

Choose a universal unfolding of \( f \), a good representative \( F : \mathcal{Y} \to T \) of it with base space \( M \), and a generic parameter \( t \in M \). Then \( F_t : Y_t \to T \) with \( T \subset \mathbb{C} \) the same disk as that for a Milnor fibration \( f : Y \to T \) above and \( Y_t \subset \mathbb{C}^{n+1} \) is a morsification of \( f \). It has \( \mu A_1 \)-singularities,
and their critical values \( u_1, \ldots, u_\mu \in T \) are pairwise different. Their numbering is also a choice. Now choose a value \( \tau \in T \cap \mathbb{R}_0 - \{u_1, \ldots, u_\mu\} \) and a distinguished system of paths. That is a system of \( \mu \) paths \( \gamma_j, j = 1, \ldots, \mu \), from \( u_j \) to \( \tau \) which do not intersect except at \( \tau \) and which arrive at \( \tau \) in clockwise order. Finally, shift from the \( A_1 \) singularity above each value \( u_j \) the (up to sign unique) vanishing cycle along \( \gamma_j \) to the Milnor lattice \( Ml(f) = H_n(f^{-1}(\tau), \mathbb{Z}) \), and call the image \( \delta_j \).

The tuple \( \delta = (\delta_1, \ldots, \delta_\mu) \) is a \( \mathbb{Z} \)-basis of \( Ml(f) \). All such bases are called distinguished bases. They form one orbit \( B(f) \) of an action of a semidirect product \( G_{\text{sign,} \mu} \rtimes \text{Br}_\mu \). Here \( \text{Br}_\mu \) is the braid group with \( \mu \) strings, see [AGV88] or [Eb07] for its action. The sign change group \( G_{\text{sign,} \mu} := \{\pm 1\}^\mu \) acts simply by changing the signs of the entries of the tuples \( (\delta_1, \ldots, \delta_\mu) \). The members of the distinguished bases are called vanishing cycles.

The matrix \( L(\delta^t, \delta) = (L(\delta_i, \delta_j))_{i,j=1,\ldots,\mu} \) of the Seifert form with respect to a distinguished basis \( \delta = (\delta_1, \ldots, \delta_\mu) \) is a lower triangular matrix with \( (-1)^{(n+1)(n+2)/2} \) on the diagonal. The Stokes matrix of the distinguished basis \( \delta \) is by definition the upper triangular matrix in \( M(\mu \times \mu, \mathbb{Z}) \)

\[
S := (-1)^{(n+1)(n+2)/2} \cdot L(\delta^t, \delta)^t
\]  

with 1’s on the diagonal. Then (2.1) and (2.2) give

\[
M_h(\delta) = \delta \cdot (-1)^{n+1} \cdot S^{-1} S^t,
\]

\[
I(\delta^t, \delta) = (-1)^{n(n+1)/2} \cdot (S + (-1)^n S^t).
\]

The Coxeter-Dynkin diagram of the distinguished basis \( \delta \) encodes \( S \) in a geometric way. It has \( \mu \) vertices which are numbered from 1 to \( \mu \). Between two vertices \( i \) and \( j \) with \( i < j \) one draws

- no edge if \( S_{ij} = 0 \),
- \( |S_{ij}| \) edges if \( S_{ij} < 0 \),
- dotted edges if \( S_{ij} > 0 \).

Coxeter-Dynkin diagrams of many singularities were calculated by A’Campo, Ebeling, Gabrielov and Gusein-Zade. Some of them can be found in [Ga74], [Eb83] and [Eb07]. Each Coxeter-Dynkin diagram of any singularity is connected. We will use this important result in lemma 2.3. There are three proofs of it, by Gabrielov [Ga74], Lazzeri [La73] and Lê [Le73].

The Picard-Lefschetz transformation on \( Ml(f) \) of a vanishing cycle \( \delta \) is

\[
s_\delta(b) := b - (-1)^{n(n+1)/2} \cdot I(\delta, b) \cdot \delta.
\]
For \( n \) even \( I(\delta, \delta) = (-1)^{n(n+1)/2} \cdot 2 \) and \( s_\delta \) is the identity on the subspace in \( ML(f) \) orthogonal to \( \delta \) and \( - \text{id} \) on \( \mathbb{Z} \cdot \delta \). For \( n \) odd \( s_\delta \) is unipotent with kernel of \( s_\delta - \text{id} \) of rank \( \mu - 1 \). In both cases \( s_\delta \) determines \( \delta \) up to the sign.

The monodromy \( M_h \) is
\[
M_h = s_{\delta_1} \circ \ldots \circ s_{\delta_\mu}
\] (2.20)
for any distinguished basis \( \delta = (\delta_1, ..., \delta_\mu) \).

Let us formulate a correspondence for later use.

**Lemma 2.3.** The orbit under \( G_{\text{sign}, \mu} \times \text{Br}_\mu \) of a tuple
\[
((u_1, ..., u_\mu), \text{a distinguished system of paths}, \text{a Stokes matrix} \ S)
\] (2.21)
where \( u_1, ..., u_\mu \in \mathbb{C} \) are pairwise different is equivalent to the isomorphism class of a \( \mathbb{Z} \)-lattice bundle of rank \( \mu \) above \( \mathbb{C} - \{u_1, ..., u_\mu\} \). The only automorphisms (which fix the basis \( \mathbb{C} - \{u_1, ..., u_\mu\} \)) of this \( \mathbb{Z} \)-lattice bundle are \( \pm \text{id} \).

**Proof:** If a morsification \( F_t \) with critical values \( u_1, ..., u_\mu \) and distinguished basis above the distinguished system of paths with Stokes matrix \( S \) exists, then the \( \mathbb{Z} \)-lattice bundle is up to isomorphism the flat extension to \( \mathbb{C} - \{u_1, ..., u_\mu\} \) of the middle homology bundle
\[
\bigcup_{\tau \in T - \{u_1, ..., u_\mu\}} H_n(F_t^{-1}(\tau), \mathbb{Z}).
\] (2.22)
If not, the \( \mathbb{Z} \)-lattice bundle is obtained from a case in (2.22) by a suitable deformation.

The vanishing cycle near \( u_j \) is up to its sign uniquely determined by its Picard-Lefschetz transformation.

Any automorphism of the \( \mathbb{Z} \)-lattice bundle maps each of these vanishing cycles to \( \pm \) itself. As the Coxeter-Dynkin diagram is connected, the only automorphisms of the \( \mathbb{Z} \)-lattice bundle are \( \pm \text{id} \). \( \square \)

### 2.6. Thom-Sebastiani type results

A result of Thom and Sebastiani compares the Milnor lattices and monodromies of the singularities \( f = f(x_0, ..., x_n), g = g(y_0, ..., y_m) \) and \( f + g = f(x_0, ..., x_n) + g(x_{n+1}, ..., x_{n+m+1}) \). There are extensions by Deligne for the Seifert form and by Gabrielov for distinguished bases. All results can be found in [AGVSS I.2.7]. They are restated here. There is a canonical isomorphism
\[
\Phi : ML(f + g) \xrightarrow{\cong} ML(f) \otimes ML(g),
\] (2.23)
with \( M_h(f + g) \cong M_h(f) \otimes M_h(g) \) (2.24)
and \( L(f + g) \cong (-1)^{(n+1)(m+1)} \cdot L(f) \otimes L(g) \) (2.25).
If $\delta = (\delta_1, \ldots, \delta_{\mu(f)})$ and $\gamma = (\gamma_1, \ldots, \gamma_{\mu(g)})$ are distinguished bases of $f$ and $g$ with Stokes matrices $S(f)$ and $S(g)$, then

$$\Phi^{-1}(\delta_1 \otimes \gamma_1, \ldots, \delta_1 \otimes \gamma_{\mu(g)}, \delta_2 \otimes \gamma_1, \ldots, \delta_2 \otimes \gamma_{\mu(g)}, \ldots, \delta_{\mu(f)} \otimes \gamma_1, \ldots, \delta_{\mu(f)} \otimes \gamma_{\mu(g)})$$

is a distinguished basis of $Ml(f + g)$, that means, one takes the vanishing cycles $\Phi^{-1}(\delta_i \otimes \gamma_j)$ in the lexicographic order. Then by (2.16) and (2.25), the matrix

$$S(f + g) = S(f) \otimes S(g) \quad (2.26)$$

(where the tensor product is defined so that it fits to the lexicographic order) is the Stokes matrix of this distinguished basis.

In the special case $g = x_{n+1}^2$, the function germ $f + g = f(x_0, \ldots, x_n) + x_{n+1}^2 \in O_{C^{n+2},0}$ is called stabilization or suspension of $f$. As there are only two isomorphisms $Ml(x_{n+1}^2) \to \mathbb{Z}$, and they differ by a sign, there are two equally canonical isomorphisms $Ml(f) \to Ml(f + x_{n+1}^2)$, and they differ just by a sign. Therefore automorphisms and bilinear forms on $Ml(f)$ can be identified with automorphisms and bilinear forms on $Ml(f + x_{n+1}^2)$. In this sense

$$L(f + x_{n+1}^2) = (-1)^n \cdot L(f) \quad \text{and} \quad M_h(f + x_{n+1}^2) = -M_h(f) \quad (2.27)$$

[AGV88 I.2.7], and $G_Z(f + x_{n+1}^2) = G_Z(f)$. The Stokes matrix $S$ does not change under stabilization.

3. MARKED SINGULARITIES AND THEIR SYMMETRIES

3.1. Symmetries of singularities. Here we will review results from [He02 13.1 and 13.2] on symmetries of singularities. A review with slightly simplified proofs was already given in [He11 ch. 6]. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a singularity, and let $F : \mathcal{Y} \to T$ be a good representative with base space $M \subset \mathbb{C}^u$ (with coordinates $t = (t_1, \ldots, t_\mu)$) of a universal unfolding $(\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}, 0)$. Let

$$\mathcal{R} := \{ \varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0) | \varphi \text{ biholomorphic} \}$$

be the group of all germs of coordinate changes, and let

$$\mathcal{R}^f := \{ \varphi \in \mathcal{R} | f \circ \varphi = f \} \quad (3.1)$$

be the group of symmetries of $f$. It is possibly $\infty$-dimensional, but the group $j_k \mathcal{R}^f$ of $k$-jets in $\mathcal{R}^f$ is an algebraic group for any $k \in \mathbb{Z}_{\geq 1}$. Let

$$R_f := j_1 \mathcal{R}^f / (j_1 \mathcal{R}^f)^0 \quad (3.2)$$

be the finite group of components of $j_1 \mathcal{R}^f$. It is easy to see that $R_f = j_k \mathcal{R}^f / (j_k \mathcal{R}^f)^0$ for any $k \in \mathbb{Z}_{\geq 1}$ [He02 Lemma 13.10]. Recall
the definition of $G_Z(f)$ in [23]. There is a natural homomorphism

$$\()_{\text{hom}} : \mathcal{R}^f \to G_Z(f), \varphi \mapsto (\varphi)_{\text{hom}}. \quad (3.3)$$

Let $\text{Aut}_M := \text{Aut}((M, 0), o, e, E)$ be the group of automorphisms of the germ $(M, 0)$ as a germ of an F-manifold with Euler field. It is a finite group because $M$ is a massive F-manifold [He02, Theorem 4.14]. We claim that there is also a natural homomorphism

$$()_M : \mathcal{R}^f \to \text{Aut}_M, \varphi \mapsto (\varphi)_M. \quad (3.4)$$

It arises as follows. $F \circ \varphi^{-1}$ is a universal unfolding of $f$ with the same base space $(M, 0)$ as the universal unfolding $F$. A morphism which induces $F \circ \varphi^{-1}$ by $F$ is given by a pair $(\Phi, (\varphi)_M)$ where $(\varphi)_M \in \text{Aut}_M$ and where $\Phi : (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}^{n+1} \times M, 0)$ is a biholomorphic map germ with

$$F \circ \varphi^{-1} = F \circ \Phi, \quad \Phi|_{(\mathbb{C}^{n+1} \times \{0\}, 0)} = \text{id}, \quad \text{pr}_M \circ \Phi = (\varphi)_M \circ \text{pr}_M. \quad (3.5)$$

Here $\Phi$ is not unique, but $(\varphi)_M$ is unique because $\text{Aut}_M$ is finite and the differential of $(\varphi)_M$ at $T_0M$ is determined by the action of $\varphi$ on the Jacobi algebra $\mathcal{O}_{\mathbb{C}^{n+1}, 0}/\mathcal{J}_f \cong T_0M$. The map $\Phi \circ \varphi$ satisfies

$$F \circ (\Phi \circ \varphi) = F \quad \text{and} \quad \text{pr}_M \circ (\Phi \circ \varphi) = (\varphi)_M \circ \text{pr}_M \quad (3.6)$$

and is an extension of the symmetry $\varphi$ of $f$ to a symmetry of $F$.

The following theorem is contained in [He02, Theorem 13.11] and is rewritten in [He11, Theorem 6.1].

**Theorem 3.1.** As above, let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated hypersurface singularity, and let $F : (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}, 0)$ be a universal unfolding with base space $(M, 0)$.

(a) The homomorphism $()_M : \mathcal{R}^f \to \text{Aut}_M$ factors through $R_f$ to a homomorphism $()_M : R_f \to \text{Aut}_M$. If $\text{mult} \ f \geq 3$ then $()_M : R_f \to \text{Aut}_M$ is an isomorphism and then $j_1 \mathcal{R}^f = R_f$. If $\text{mult} \ f = 2$ then $()_M : R_f \to \text{Aut}_M$ is surjective with kernel of order 2. If $f = g(x_0, \ldots, x_{n-1}) + x_n^2$ then the kernel is generated by the class of the symmetry $(x \mapsto (x_0, \ldots, x_{n-1}, -x_n))$.

(b) The homomorphism $()_{\text{hom}} : \mathcal{R}^f \to G_Z(f)$ factors through $R_f$ to an injective homomorphism $()_{\text{hom}} : R_f \to G_Z(f)$. Let $G^\text{smar}_R(f) \subset G_Z(f)$ denote its image. It contains $-\text{id}$ if and only if $\text{mult} \ f = 2$. If $f = g(x_0, \ldots, x_{n-1}) + x_n^2$ then $-\text{id} = (x \mapsto (x_0, \ldots, x_{n-1}, -x_n))_{\text{hom}}$.

(c) The homomorphism

$$()_M \circ ()^{-1}_{\text{hom}} : G^\text{smar}_R(f) \to \text{Aut}_M \quad (3.7)$$

is an isomorphism if $\text{mult} \ f \geq 3$. It is surjective with kernel $\{ \pm \text{id} \}$ if $\text{mult} \ f = 2$. 

Consider a singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and a good representative $F : \mathcal{Y} \to T$ of a universal unfolding with base space $M$. One can choose it such that any element of $R_f$ lifts to an automorphism of $F$. Consider the $\mathbb{Z}$-lattice bundle

$$
\bigcup_{(\tau, t) \in T \times M - D} H_n(F_t^{-1}(\tau), \mathbb{Z}).
$$

(3.8)

Definition/Lemma 3.2. (a) (Definition) We call the flat extension of the $\mathbb{Z}$-lattice bundle in (3.8) to $\mathbb{C} \times M - D$ the canonical $\mathbb{Z}$-lattice bundle of $M$.

(b) (Lemma) Any element of $\text{Aut}_M$ lifts to an automorphism of the canonical $\mathbb{Z}$-lattice bundle of $M$. The lift is unique up to $\pm 1$.

Proof: The surjectivity of the homomorphism $()_M : R_f \to \text{Aut}_M$ implies that any automorphism of $((M, 0), c, e, E)$ lifts to an automorphism of the unfolding and thus to an automorphism of the $\mathbb{Z}$-lattice bundle in (3.10). Because of lemma 2.3, the only automorphisms of the $\mathbb{Z}$-lattice bundle which fix $T \times M$, are $\pm 1d$. Therefore any element of $\text{Aut}_M$ has only two lifts, and they differ by $\pm 1$.

Part (b) justifies part (a): The bundle depends only on the isomorphism class of the germ $((M, 0), c, e, E)$. Instead of lemma 2.3, we could have used theorem 3.1 (c). But that would give only that any element of $\text{Aut}_M$ has two canonical lifts, which differ by $\pm 1$, not that they are the only lifts.

In the case of a quasihomogeneous singularity, the finite group of quasihomogeneous symmetries is a natural lift of $R_f$. This will be useful for the calculations in section 5.

Theorem 3.3. [He02, Theorem 13.13] [He11, Theorem 6.2] Let $f \in \mathbb{C}[x_0, ..., x_n]$ be a quasihomogeneous polynomial with an isolated singularity at 0 and weights $w_0, ..., w_n \in \mathbb{Q} \cap (0, \frac{1}{2}]$ and weighted degree 1. Suppose that $w_0 \leq ... \leq w_{n-1} < \frac{1}{2}$ (then $f \in \mathfrak{m}^3$ if and only if $w_n < \frac{1}{2}$). Let $G_w$ be the algebraic group of quasihomogeneous coordinate changes, that means, those which respect $\mathbb{C}[x_0, ..., x_n]$ and the grading by the weights $w_0, ..., w_n$ on it. Then

$$R_f \cong \text{Stab}_{G_w}(f).$$

Remark 3.4. Let $f \in \mathbb{C}[x_0, ..., x_n]$ be a quasihomogeneous polynomial with an isolated singularity at 0 and weights $w_0, ..., w_n \in (0, \frac{1}{2}]$ and weighted degree 1. Then

$$\varphi_1 := (x \mapsto (e^{2\pi i w_0}x_0, ..., e^{2\pi i w_n}x_n)) \in \text{Stab}_{G_w}(f)$$
satisfies $(\varphi_1)_{\text{hom}} = M_h$. Now let $F(x, t) = f(x) + \sum_{i=1}^{\mu} t_i m_i$ be a universal unfolding as in (2.9) with $m_i$ a weighted homogeneous polynomial of weighted degree $\deg_w m_i$. Then $\deg_w t_i := 1 - \deg_w m_i$,

$$\varphi_1 = (t \mapsto (e^{2\pi i \deg_w t_1}, ..., e^{2\pi i \deg_w t_\mu})), $$

and the pair $(\Phi_1, (\varphi_1)_M)$ with

$$\Phi_1 = ((x, t) \mapsto (x, (\varphi_1)_M)$$

induces $F \circ \varphi_1^{-1}$ by $F$, i.e. (3.6) holds:

$$F \circ (\Phi_1 \circ \varphi_1) = F, \quad \text{pr}_M \circ (\Phi_1 \circ \varphi_1) = (\varphi_1)_M \circ \text{pr}_M.$$  

Especially, $M_h \in \G_{s\text{mar}}^R(f)$ and $(\cdot)_M \circ (\cdot)_{\text{hom}}^{-1}(M_h) = (\varphi_1)_M$.

3.2. Marked singularities and their moduli spaces. In [He11] the notion of a marked singularity was defined and results from [He02] on moduli spaces of right equivalence classes of singularities were lifted to marked singularities. Here we recall the central notions and results.

**Definition 3.5.** Fix one reference singularity $f^{(0)} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$.

(a) Its $\mu$-homotopy class is the set of all singularities $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ for which a $C^\infty$-family $f_s$, $s \in [0, 1]$, of singularities with $\mu(f_s) = \mu(f^{(0)})$ and $f_0 = f^{(0)}$ and $f_1 = f$ exists.

(b) A marked singularity is a pair $(f, \pm \rho)$ with $f$ in the $\mu$-homotopy class of $f^{(0)}$ and $\rho : \text{Ml}(f, L) \to \text{Ml}(f^{(0)}, L)$ an isomorphism between Milnor lattices with Seifert forms. Here $\pm \rho$ means the set $\{\rho, -\rho\}$, so neither $\rho$ nor $-\rho$ is preferred.

(c) Two singularities $f_1$ and $f_2$ are right equivalent if a coordinate change $\varphi \in \mathcal{R}$ with $f_1 = f_2 \circ \rho$ exists. Notation: $f_1 \sim_{\mathcal{R}} f_2$.

Two marked singularities $(f_1, \pm \rho_1)$ and $(f_2, \pm \rho_2)$ are right equivalent if a coordinate change $\varphi \in \mathcal{R}$ with $f_1 = f_2 \circ \varphi$ and $\rho_1 = \varepsilon \cdot \rho_2 \circ (\varphi)_{\text{hom}}$ for some $\varepsilon \in \{\pm 1\}$ exists. Notation: $(f_1, \pm \rho_1) \sim_{\mathcal{R}} (f_2, \pm \rho_2)$.

(d) The moduli spaces $M_\mu(f^{(0)})$ and $M_\mu^\text{mar}(f^{(0)})$ are defined as the sets

$$M_\mu(f^{(0)}) := \{\text{the } \mu\text{-homotopy class of } f^{(0)}\}/ \sim_{\mathcal{R}},$$

$$M_\mu^\text{mar}(f^{(0)}) := \{\text{the marked singularities } (f, \pm \rho)\}/ \sim_{\mathcal{R}}.$$  

A central result in [He02] is that the moduli space $M_\mu(f^{(0)})$ has the structure of an analytic geometric quotient. In [He11] this result is extended to the space $M_\mu^\text{mar}(f^{(0)})$, and it is shown that $M_\mu^\text{mar}(f^{(0)})$ is locally isomorphic to a $\mu$-constant stratum. This is recalled in theorem 3.6 below.
Let \( f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \) be a singularity and let \( F : (\mathbb{C}^{n+1} \times M(f), 0) \to (\mathbb{C}, 0) \) be a universal unfolding of \( f \) with base space \((M(f), 0)\). Then the \( \mu \)-constant stratum is the germ \((S_\mu(f), 0) \subset (M(f), 0)\) of the subset
\[
S_\mu := \{ t \in M \mid F_t \text{ has only one singularity } x_0 \text{ and } F_t(x_0) = 0 \}.
\]
Here \( F : Y \to T \) is a good representative of the germ \( F \) with base space \( M \). Obviously \((S_\mu(f), 0)\) carries a natural structure as a reduced complex space germ. In [He02, Theorem] it is equipped furthermore with a natural complex structure, which is not necessarily reduced.

**Theorem 3.6.** Fix one reference singularity \( f^{(0)} \).

(a) ([He02, Theorem 13.11] and [He11, Theorem 4.3]) \( M_\mu(f^{(0)}) \) and \( M_\mu^{\text{mar}}(f^{(0)}) \) are in a natural way complex spaces. They can be constructed with the underlying reduced complex structures as analytic geometric quotients.

(b) For any \( f \) in the \( \mu \)-homotopy class of \( f^{(0)} \), the germ \((M_\mu(f^{(0)}), [f])\) is isomorphic with its canonical complex structure to the quotient \((S_\mu(f), 0)/\text{Aut}_{M(f)} \) (the action of \( \text{Aut}_{M(f)} \) on \((M(f), 0)\) restricts to an action on \((S_\mu(f), 0)\)).

For any marked singularity \((f, \pm \rho)\), the germ \((M_\mu^{\text{mar}}(f^{(0)}), [(f, \pm \rho)])\) is isomorphic with its canonical complex structure to the \( \mu \)-constant stratum \((S_\mu(f), 0)\).

(c) For any \( \chi \in G_Z(f^{(0)}) \), the map
\[
\chi_{\text{mar}} : M_\mu^{\text{mar}}(f^{(0)}) \to M_\mu^{\text{mar}}(f^{(0)}),
\]
\[
[(f, \pm \rho)] \mapsto [(f, \pm \chi \circ \rho)],
\]
is an automorphism of \( M_\mu^{\text{mar}}(f^{(0)}) \). The action
\[
G_Z(f^{(0)}) \times M_\mu^{\text{mar}}(f^{(0)}) \to M_\mu^{\text{mar}}(f^{(0)}),
\]
\[
(\chi, [(f, \pm \rho)]) \mapsto \chi_{\text{mar}}([(f, \pm \rho)]) = [(f, \pm \chi \circ \rho)],
\]
is a group action from the left. It is properly discontinuous. The quotient \( M_\mu^{\text{mar}}(f^{(0)})/G_Z(f^{(0)}) \) is the moduli space \( M_\mu(f^{(0)}) \) of unmarked singularities, with its canonical complex structure.

(d) (Definition) Recall the definition of \( G_\text{smar}^{\text{mar}}(f) \) in theorem 3.1 (b). Define
\[
G_\text{smar}^{\text{mar}}(f) := \{ \pm \psi \mid \psi \in G_\text{smar}^{\text{mar}}(f) \} \subset G_Z(f),
\]
Remark: By theorem 3.1 this is equal to \( G_\text{smar}^{\text{mar}}(f) \) if \( \text{mult}(f) = 2 \) and has \( G_\text{smar}^{\text{mar}}(f) \) as index 2 subgroup if \( \text{mult}(f) \geq 3 \).
(e) For any point \([(f, \pm \rho)] \in M^\text{mar}_\mu(f(0))\), the stabilizer of it in \(G_Z(f(0))\) is the finite group
\[
\rho \circ G^\text{mar}_R(f) \circ \rho^{-1} \subset G_Z(f(0)).
\]

\(3.15\)

Remarks 3.7. (i) In [He11] also the notion of a strongly marked singularity is defined. It is a pair \((f, \rho)\) with \(f, \rho\) and \(f(0)\) as in definition 3.5 (b). The moduli space \(M^\text{smar}_\mu(f(0))\) of strongly marked singularities behaves as well as \(M^\text{mar}_\mu(f(0))\) if the following holds: Either any singularity in the \(\mu\)-homotopy class of \(f(0)\) has multiplicity \(\geq 3\), or any singularity in the \(\mu\)-homotopy class of \(f(0)\) has multiplicity \(2\). We expect that to hold, but we don’t know it. If it does not hold then \(M^\text{smar}_\mu(f(0))\) is not Hausdorff, see [He11, Theorem 4.3 (e)]. We do not need strongly marked singularities here.

(ii) In [He11][GH17-1] and [GH17-2] the moduli space \(M^\text{mar}_\mu(f(0))\) for any singularity \(f(0)\) with modality \(\leq 2\) was studied. It turned out that almost all of them are connected, but not all, namely not those for \(f(0)\) in the subseries \(W^*_1,12r, S^*_1,10r, U_1,19r, E_{3,18r}, Z_{1,14r}, Q_{2,12r}, W_{1,12r}, S_{1,10r}\) of the eight bimodal series. This disproved the conjecture 3.2 (a) in [He11] that \(M^\text{mar}_\mu(f(0))\) is connected for any singularity. But for all other singularities \(f(0)\) with modality \(\leq 2\) \(M^\text{mar}_\mu(f(0))\) is connected. An equivalent statement to \(M^\text{mar}_\mu(f(0))\) connected is because of [He11, Theorem 4.4 (a)] that any element of \(G_Z(f(0))\) arises from geometry, namely it is \((\pm 1)\)-the transversal monodromy of a suitable \(\mu\)-constant family \(f_s, s \in [0,1]\), with \(f_0 = f_1 = f(0)\) (a \(C^\infty\) family of singularities \(f_s\) with \(\mu(f_s) = \mu(f(0))\)).

(iii) In the case of the ADE-singularities, which have modality 0, \(M^\text{mar}_\mu(f(0))\) is simply a point [He11]. In the case of the simple elliptic singularities, which have modality one and which are parametrized by elliptic curves, \(M^\text{mar}_\mu(f(0))\) is isomorphic to \(\mathbb{H}\) [GH17-1]. In both cases, the connectedness of \(M^\text{mar}_\mu(f(0))\) will be important in the proof of the main theorem in section 7.

3.3. A thickening of the moduli space of marked singularities. Fix one reference singularity \(f(0)\). By theorem 3.6 (b), locally at \([(f, \pm \rho)]\), the moduli space \(M^\text{mar}_\mu(f(0))\) is isomorphic to the \(\mu\)-constant stratum \((S_{\mu}(f), 0) \subset (M(f), 0)\) of the singularity \(f\). In [GH18] we will show the following.

Theorem 3.8. Fix one reference singularity \(f(0)\).
(a) The moduli space $M_{\mu}^{\text{mar}}(f^{(0)})$ of marked singularities can be extended globally to a $\mu$-dimensional F-manifold $M_{\mu}^{\text{mar}}(f^{(0)}) \supset M_{\mu}^{\text{mar}}(f^{(0)})$ with the following properties.

(i) For any point $[(f, \pm \rho)] \in M_{\mu}^{\text{mar}}(f^{(0)})$, a certain neighborhood $U_{[(f, \pm \rho)]}$ of it in $M_{\mu}^{\text{mar}}(f^{(0)})$ and an isomorphism $\psi_{[(f, \pm \rho)]: M(f) \to U_{[(f, \pm \rho)]}}$ of F-manifolds exist, where $M(f)$ is the base space of a good representative of a universal unfolding of $f$.

(ii) $M_{\mu}^{\text{mar}}(f^{(0)})$ is covered by these neighborhoods $U_{[(f, \pm \rho)]}$.

(iii) The action of $G_{\mathbb{Z}}(f^{(0)})$ on $M_{\mu}^{\text{mar}}(f^{(0)})$ extends to an action of $G_{\mathbb{Z}}(f^{(0)})$ on this F-manifold with Euler field, and the map

$$G_{\mathbb{Z}}(f^{(0)}) \to \text{Aut}(M_{\mu}^{\text{mar}}(f^{(0)}) \circ \sigma, e, E)$$

is surjective with kernel $\{\pm \text{id}\}$.

(b) Let $D_{\text{mar}} \subset \mathbb{C} \times M_{\mu}^{\text{mar}}(f^{(0)})$ be the discriminant

$$D_{\text{mar}} := \{(t, \tau) \in \mathbb{C} \times M_{\mu}^{\text{mar}}(f^{(0)}) \mid E_0 : T_t M \to T_{\tau} M \text{ has eigenvalue } \tau \}.$$

It is a hypersurface. $M_{\mu}^{\text{mar}}(f^{(0)})$ comes equipped with a $\mathbb{Z}$-lattice bundle $H_{\mathbb{Z}} \to (\mathbb{C} \times M_{\mu}^{\text{mar}}(f^{(0)}) - D_{\text{mar}})$ of rank $\mu$ with the following properties.

(i) For any point $[(f, \pm \rho)] \in M_{\mu}^{\text{mar}}(f^{(0)})$ and a good representative $F : S \to \mathcal{Y}$ of a universal unfolding of $f$ with base space $M(f)$ and the isomorphism $\psi_{[(f, \pm \rho)]: M(f) \to U_{[(f, \pm \rho)]}}$ as in (a)(i), this isomorphism lifts to an isomorphism from the canonical $\mathbb{Z}$-lattice bundle above $M(f)$ in definition 3.2 (a) to the restriction of $H_{\mathbb{Z}}$ to $\mathbb{C} \times U_{[(f, \pm \rho)]} - D_{\text{mar}}$. (Because of lemma 3.2 (b), this lift is unique up to $\pm 1$).

(ii) Let $r : M_{\mu}^{\text{mar}}(f^{(0)}) \to \mathbb{R}_{>0}$ be a $C^\infty$ function with $D_{\text{mar}} \subset \bigcup_{t \in M_{\mu}^{\text{mar}}(f^{(0)})} \Delta_{r(t)} \times \{t\}$. Then the restriction of $H_{\mathbb{Z}}$ to $\bigcup_{t \in M_{\mu}^{\text{mar}}(f^{(0)})} \mathbb{R}_{>r(t)} \times \{t\}$ has trivial monodromy, i.e. it is a trivial flat $\mathbb{Z}$-lattice bundle.

(iii) Write $t^{(0)} := [(f^{(0)}, \pm \text{id})] \in M_{\mu}^{\text{mar}}(f^{(0)})$. For any $t = [(f, \pm \rho)] \in M_{\mu}^{\text{mar}}(f^{(0)})$ and any small $\tau > 0$, the following diagram of isomorphisms commutes,

$$
\begin{array}{ccc}
H_{\mathbb{Z}}(\tau, t) & \cong & Ml(f) \\
\cong \downarrow & (i) & \cong \\
H_{\mathbb{Z}}(\tau, t^{(0)}) & \cong & \pm \rho \\
\end{array}
$$

$$
\begin{array}{ccc}
\cong \downarrow & (i) & \cong \\
\end{array}
$$
(iv) The action of \( G_Z(f(0)) \) on \( M^{mar}(f(0)) \) extends to an action on the \( \mathbb{Z} \)-lattice bundle \( H_Z \) (the action of \( G_Z(f(0)) \) on the first factor \( \mathbb{C} \) of \( \mathbb{C} \times M^{mar}(f(0)) \) is trivial by definition).

As the paper [GH18] is not yet available, we will prove this theorem for the cases of interest here, the simple and the simple elliptic singularities, directly in section 4. See theorem 4.3.

\( M^{mar}(f(0)) \) contains besides \( D^{mar} \) also two other hypersurfaces, the caustic \( K^{mar}_3 \) and the Maxwell stratum \( K^{mar}_2 \),

\[
K^{mar}_3 := \{ t \in M^{mar}(f(0)) | T_t M^{mar}(f(0)) \text{ decomposes into less than } \mu \text{ irreducible local algebras} \}, \tag{3.18}
\]

\[
K^{mar}_2 := \text{the closure of the set } \{ t \in M^{mar}(f(0)) - K^{mar}_3 | \text{some eigenvalues of } E \circ T_t M \text{ coincide} \}. \tag{3.19}
\]

3.4. A Looijenga-Deligne map for distinguished bases. Looijenga [Lo74] studied in the case of the simple singularities a relationship between distinguished bases and the base space of a universal unfolding. His idea carries over to the F-manifold \( M^{mar}(f(0)) \) in theorem 3.8 of an arbitrary \( \mu \)-homotopy class of singularities. We describe the idea here in this generality. In section 7 we will study it in the cases of the simple and the simple elliptic singularities. In [GH18] we will study it in the general case.

The set \( B(f) \) of distinguished bases of the Milnor lattice \( Ml(f) \) of a singularity \( f \) was constructed in subsection 2.5 by choosing one morsification \( F_t \) of \( f \) and considering all possible distinguished systems of paths. Following Looijenga, now we want to fix one distinguished system of paths and consider all possible morsifications. The following two definitions make this precise.

**Definition 3.9.** (a) Fix a tuple \( (u_1, ..., u_\mu) \subset \mathbb{C}^\mu \) with \( u_i \neq u_j \) for \( i \neq j \). The **good ordering** of it is the lexicographic ordering \( (u_{\sigma(1)}, ..., u_{\sigma(\mu)}) \) by (imaginary part, −real part). That means, the corresponding good permutation \( \sigma \in S_\mu \) is uniquely determined by

\[
\begin{align*}
   i < j &\iff \begin{cases} 
   \text{Im}(u_{\sigma(i)}) < \text{Im}(u_{\sigma(j)}) \quad \text{or} \\
   \text{Im}(u_{\sigma(i)}) = \text{Im}(u_{\sigma(j)}) \quad \text{and} \quad \text{Re}(u_{\sigma(i)}) > \text{Re}(u_{\sigma(j)})
\end{cases}
\end{align*}
\tag{3.20}
\]

(b) Fix a tuple \( (u_1, ..., u_\mu) \) as in (a) with good permutation \( \sigma \in S_\mu \), and fix additionally a \( \tau \in \mathbb{R}_{>0} \) with \( \tau > \max_i |u_i| \). A **good distinguished system of paths** is a distinguished system of paths \( \gamma_1, ..., \gamma_\mu \) such that \( \gamma_j \) goes from \( u_{\sigma(j)} \) to \( \tau \).

For a fixed tuple \( (u_1, ..., u_\mu, \tau) \) as above, all good distinguished systems of paths are homotopy equivalent with respect to a natural notion
of homotopy equivalence. And if $F_t$ is a morsification of a singularity $f$ with critical values $u_1, \ldots, u_\mu$, all good distinguished systems of paths give the same distinguished basis up to the action of the sign group $G_{\text{sign}, \mu}$.

**Definition 3.10.** Fix one reference singularity $f^{(0)}$.

(a) The set of **Stokes walls** within the space $M^{\text{mar}}(f^{(0)})$ in theorem 3.8 is the set

$$W_{\text{Stokes}} := \{ t \in M^{\text{mar}}(f^{(0)}) \mid \text{the eigenvalues } u_1, \ldots, u_\mu \text{ of } E \circ T_t M \text{ satisfy } \text{Im}(u_i) = \text{Im}(u_j) \text{ for some } i \neq j \}$$

The set $W_{\text{Stokes}}$ of Stokes walls is a real codimension 1 subvariety and contains $K_3^{\text{mar}} \cup K_2^{\text{mar}}$. The components of its complement $M^{\text{mar}}(f^{(0)}) - W_{\text{Stokes}}$ are called **Stokes regions**. Let $R_{\text{Stokes}}$ be the set of Stokes regions, and let $R_{\text{Stokes}}^0$ be the subset of those Stokes regions which are in the component $M^{\text{mar}}(f^{(0)})^0$ of $M^{\text{mar}}(f^{(0)})$ which contains $[(f^{(0)}, \pm \text{id})]$.

(b) The set $B^{\text{ext}}(f^{(0)})$ is the orbit of $B(f^{(0)})$ under the action of $G_Z$. It contains (all?) $Z$-bases $(\delta_1, \ldots, \delta_\mu)$ of $Ml(f^{(0)})$ whose elements $\delta_j$ are vanishing cycles and such that $s_{\delta_1} \circ \ldots \circ s_{\delta_\mu} = M_h$. It consists of $G_{\text{sign}, \mu} \rtimes Br_{\mu}$ orbits. One of these orbits is the set $B(f^{(0)})$ of distinguished bases.

(c) The **Looijenga-Deligne map** is the map

$$LD : R_{\text{Stokes}} \to B^{\text{ext}}(f^{(0)})/G_{\text{sign}, \mu}$$

which is defined as follows. For a Stokes region in $M^{\text{mar}}(f^{(0)})$, choose a point $t$ in it and a point $[(f, \pm \rho)] \in M^{\text{mar}}_{\mu}(f^{(0)})$ with $t \in U_{[(f, \pm \rho)]}$. Let $(u_1, \ldots, u_\mu)$ be the eigenvalues of $E \circ T_t M \to T_t M$ in the good ordering (definition 3.9). Consider a good distinguished system of paths from $(u_1, \ldots, u_\mu)$ to a value $\tau > \max \{|u_i|\}$ (definition 3.9 (b)). The usual construction of distinguished bases gives a distinguished basis in $B(f)$ up to the action of the sign group $G_{\text{sign}, \mu}$. Shift this basis with the isomorphism $\rho : Ml(f) \to Ml(f^{(0)})$ to an element of $B^{\text{ext}}(f^{(0)})/G_{\text{sign}, \mu}$.

**Remarks 3.11.** (i) We claim that $LD$ restricts to a map

$$LD : R_{\text{Stokes}}^0 \to B(f^{(0)})/G_{\text{sign}, \mu}.$$  

We prove this by a different description of the image $LD(t)$ for $t \in R_{\text{Stokes}}^0$. Choose $[(f, \pm \rho)] \in M^{\text{mar}}_{\mu}(f^{(0)})^0$ with $t \in U_{[(f, \pm \rho)]}$; choose $\tau > \max \{|u_i|\}$, and choose a good distinguished system of paths from $(u_1, \ldots, u_\mu)$ to $\tau$. One can move $t$ within $M^{\text{mar}}_{\mu}(f^{(0)})^0 - (K_3^{\text{mar}} \cup K_2^{\text{mar}})$ to a point in $U_{[(f^{(0)}, \pm \text{id})]} \subset M^{\text{mar}}_{\mu}(f^{(0))^0}$. Then the good distinguished
system of paths moves to some new distinguished system of paths. Now
the construction of distinguished bases for \( f(0) \) gives directly the class
of bases \( LD(t) \in \mathcal{B}(f(0))/G_{\text{sign},\mu} \). This follows with (3.17).

(ii) The action of \( G_Z \) on \( M_{\text{mar}}(f(0)) \) induces an action on \( R_{\text{Stokes}} \).
And \( R_{\text{Stokes}}^0 = R_{\text{Stokes}} \) if and only if \( M_{\text{mar}}(f(0)) \) is connected. The map
(3.22) is \( G_Z \)-equivariant. Therefore, if \( M_{\text{mar}}(f(0)) \) is connected, then
(3.23) and the definition of \( \mathcal{B}^{\text{ext}}(f(0)) \) give also \( \mathcal{B}^{\text{ext}}(f(0)) = \mathcal{B}(f(0)) \).

(iii) But if \( M_{\text{mar}}(f(0)) \) is not connected, we do not know whether
\( \mathcal{B}^{\text{ext}}(f(0)) = \mathcal{B}(f(0)) \) or \( \mathcal{B}^{\text{ext}}(f(0)) \supsetneq \mathcal{B}(f(0)) \) holds. The first open cases
are the subseries in remark 3.7 (ii) of the eight bimodal series.

(iv) Looijenga considered the map \( LD \) for the simple singularities
and proved that it is an isomorphism for the \( A_{\mu} \) singularities [Lo74].
Deligne [De74] proved the same for the \( D_{\mu} \) and \( E_{\mu} \) singularities. We will
reprove their results and extend them to the simple elliptic singularities
in section 7. We will study \( LD \) in the general case in [GH18].

4. Unfoldings of the simple and the simple elliptic
singularities

The first singularities in Arnold’s lists of isolated hypersurface singularities
[AGV85, ch 15.1] are the simple and the simple elliptic singularities. They are distinguished by many properties. Especially, they
possess universal unfoldings such that all members are defined globally
on \( \mathbb{C}^{n+1} \). We start by giving well known normal forms. Then we choose
universal unfoldings.

4.1. Normal forms for the simple and the simple elliptic
singularities. The first table lists normal forms from [AGV85] for the
simple singularities \( f : \mathbb{C}^{n+1} \to \mathbb{C} \),

\[
\begin{array}{cccc}
\text{name} & \mu & \nu & f(x_0, \ldots, x_n) \\
A_{\mu} & \geq 1 & \geq 0 & x_0^{\mu+1} + \sum_{i=1}^{\nu} x_i^2 \\
D_{\mu} & \geq 4 & \geq 1 & x_0^{\mu-1} + x_0 x_1^2 + \sum_{i=2}^{\nu} x_i^2 \\
E_6 & 6 & \geq 1 & x_0^4 + x_1^3 + \sum_{i=2}^{\nu} x_i^2 \\
E_7 & 7 & \geq 1 & x_0^3 x_1 + x_1^3 + \sum_{i=2}^{\nu} x_i^2 \\
E_8 & 8 & \geq 1 & x_0^5 + x_1^3 + \sum_{i=2}^{\nu} x_i^2 \\
\end{array}
\]

(4.1)

The simple elliptic singularities can be represented as 1-parameter families in different ways [Sa74, 1.9 and 1.11 [AGV85 ch 15.1]. We
choose the Legendre normal forms \( f = f_\lambda : \mathbb{C}^{n+1} \to \mathbb{C} \) from [Sa74 1.9]
in the following table with $\lambda \in \mathbb{C} - \{0, 1\}$,

| name | $\mu$  | $n$ | $f_\lambda(x_0, \ldots, x_n)$ |
|------|-------|----|-------------------------------|
| $E_6$ | 8     | $\geq 2$ | $x_1(x_1 - x_0)(x_1 - \lambda x_0) - x_0 x_2^2 + \sum_{i=3}^{n} x_i^2$ (4.2) |
| $\tilde{E}_7$ | 9     | $\geq 1$ | $x_0 x_1(x_1 - x_0)(x_1 - \lambda x_0) + \sum_{i=2}^{n} x_i^2$ |
| $\tilde{E}_8$ | 10    | $\geq 1$ | $x_1(x_1 - x_0^2)(x_1 - \lambda x_0^2) + \sum_{i=2}^{n} x_i^2$ |

4.2. Universal unfoldings. For the simple singularities, we reproduce the universal unfoldings which are given in $[Lo74]$. They are as follows,

$$F^\text{alg} : \mathbb{C}^{n+1} \times M^\text{alg} \to \mathbb{C} \quad \text{with} \quad M^\text{alg} = \mathbb{C}^{\mu},$$

$$F^\text{alg}(x_0, \ldots, x_n, t_1, \ldots, t_\mu) = F^\text{alg}(x, t) = F^\text{alg}_t(x) = f(x) + \sum_{j=1}^{\mu} t_j m_j$$

with $f = F^\text{alg}_0$ and $m_1, \ldots, m_\mu$ the monomials in the tables (4.4) and (4.5).

| name | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ | $m_8$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|
| $A_\mu$ | 1     | $x_0$ | $x_0^2$ | $x_0^3$ | $\ldots$ | $x_0^{\mu-1}$ |
| $D_\mu$ | 1     | $x_1$ | $x_0 x_1$ | $x_0^2 x_1$ | $\ldots$ | $x_0^{\mu-2}$ |

One checks easily that the monomials form a basis of the Jacobi algebra $\mathcal{O}_{\mathbb{C}^{n+1},0}/J_f$. Therefore the unfolding $F^\text{alg}$ is indeed universal (compare (2.2)).

For each of the three Legendre families of simple elliptic singularities, we give a global family of functions as follows,

$$F^\text{alg} : \mathbb{C}^{n+1} \times M^\text{alg} \to \mathbb{C} \quad \text{with} \quad M^\text{alg} = \mathbb{C}^{\mu-1} \times (\mathbb{C} - \{0, 1\}),$$

$$F^\text{alg}(x_0, \ldots, x_n, t_1, \ldots, t_{\mu-1}, \lambda) = F^\text{alg}_{t', \lambda}(x, t', \lambda)$$

$$= F^\text{alg}_{t', \lambda}(x) = f_\lambda(x) + \sum_{j=1}^{\mu-1} t_j m_j$$

with $f_\lambda = F^\text{alg}_{0, \lambda}$ and $m_1, \ldots, m_{\mu-1}$ the monomials in the table (4.7),

| name | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ | $m_8$ | $m_9$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $E_6$ | 1     | $x_0$ | $x_1$ | $x_2$ | $x_0 x_1$ | $x_1 x_2$ |
| $\tilde{E}_7$ | 1     | $x_0$ | $x_1$ | $x_0 x_1$ | $x_1^2$ | $x_0^2 x_1$ | $x_0 x_1^2$ |
| $\tilde{E}_8$ | 1     | $x_0$ | $x_0^2$ | $x_1$ | $x_0 x_1$ | $x_0^2 x_1$ | $x_1^2$ | $x_0 x_1^2$ |
Let $\lambda : \mathbb{H} \to \mathbb{C} - \{0, 1\}, t_\mu \mapsto \lambda(t_\mu)$, be the standard universal covering. For each of the three Legendre families of simple elliptic singularities, we will also consider the global family of functions

$$F^{mar} : \mathbb{C}^{n+1} \times M^{mar} \to \mathbb{C}, (x, t) \mapsto F^{alg}(x, t', \lambda(t_\mu)) \quad (4.8)$$

where $M^{mar} = \mathbb{C}^{n-1} \times \mathbb{H}$.

For the simple singularities, we set

$$M^{mar} = M^{alg} = \mathbb{C}^n, \quad \lambda := t_\mu, \quad F^{mar} = F^{alg}. \quad (4.9)$$

**Lemma 4.1.** Consider any of the three global families of functions in (4.7). At each point $(0, 0, \lambda) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n-1} \times (\mathbb{C} - \{0, 1\})$, the germ of the family $F^{alg}$ is a universal unfolding of $f_\lambda$. Also, at each point $(0, 0, t_\mu) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n-1} \times \mathbb{H}$, the germ of the family $F^{mar}$ in (4.8) is a universal unfolding of $f_{\lambda(t_\mu)}$.

**Proof:** It suffices to prove the statement for $F^{alg}$. Because of (2.9), it suffices to show that for any $\lambda \in \mathbb{C} - \{0, 1\}$ the monomials $m_1, \ldots, m_{n-1}$ together with the weighted homogeneous polynomial $\partial f_\lambda / \partial \lambda$ form a basis of the Jacobi algebra $\mathcal{O}_{\mathbb{C}^{n+1}, 0}/J_{f_\lambda}$. We carry out the least trivial case, which is the case $\tilde{E}_6$, and leave the cases $\tilde{E}_7$ and $\tilde{E}_8$ to the reader. In the case $\tilde{E}_6$, we work with the minimal number $n + 1 = 3$ of variables. The normalized weight system is $w = (w_0, w_1, w_2) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, and $\deg_w f_\lambda = 1$. As $f$ is quasihomogeneous, the Jacobi algebra is isomorphic to $\mathbb{C}[x]/J_{f_\lambda}^{\text{pol}}$ where $J_{f_\lambda}^{\text{pol}}$ denotes the Jacobi ideal in $\mathbb{C}[x_0, x_1, x_2] = \mathbb{C}[x]$. For $q \in \mathbb{Q}_{\geq 0}$ denote by $\mathbb{C}[x]^q$ the sub vector space of $\mathbb{C}[x]$ generated by the monomials of weighted degree $q$.

\[
\begin{align*}
\frac{\partial f_\lambda}{\partial x_0} &= -(\lambda + 1)x_1^2 + 2\lambda \cdot x_0x_1 - x_2^2 \quad \in J_{f_\lambda}^{\text{pol}} \cap \mathbb{C}[x]_{2/3}, \\
\frac{\partial f_\lambda}{\partial x_1} &= 3x_1^2 - 2(\lambda + 1) \cdot x_0x_1 + \lambda \cdot x_0^2 \quad \in J_{f_\lambda}^{\text{pol}} \cap \mathbb{C}[x]_{2/3}, \\
\frac{\partial f_\lambda}{\partial x_2} &= -2x_0x_2 \quad \in J_{f_\lambda}^{\text{pol}} \cap \mathbb{C}[x]_{2/3}, \\
\frac{\partial^2 f_\lambda}{\partial \lambda} &= x_0^2x_1 - x_0x_2^2 \quad \in \mathbb{C}[x]_1.
\end{align*}
\]

We have to show for $q \in \mathbb{Q}_{\geq 0}$

\[
(J_{f_\lambda}^{\text{pol}}) \cap \mathbb{C}[x]^q + \sum_{j : \deg_w m_j = q} \mathbb{C} \cdot m_j + \left\{ \begin{array}{ll} 0 & \text{if } q \neq 1, \\
\mathbb{C} \cdot \frac{\partial f_\lambda}{\partial \lambda} & \text{if } q = 1 \end{array} \right\} = \mathbb{C}[x](4.10)
\]

The only nontrivial cases are $q \in \{\frac{2}{3}, 1, \frac{4}{3}\}$.

The case $q = \frac{2}{3}$: $\mathbb{C}[x]_{2/3}$ is generated by the six monomials $x_0^2, x_0x_1, x_1^2, x_0x_2, x_1x_2, x_2^2$. Here $m_5 = x_0^2, m_6 = x_0x_1, m_7 = x_1x_2,$ and
\[ \frac{\partial f}{\partial x_2} = -2x_0x_2. \] Modulo these four monomials, \( \frac{\partial f}{\partial x_0} \) and \( \frac{\partial f}{\partial x_1} \) are congruent to \( -(\lambda + 1)x_1^2 - x_2^2 \) and \( 3x_1^2 \). Thus the left hand side of (4.10) contains also the monomials \( x_1^2 \) and \( x_2^2 \).

The case \( q = 1 \): \( \mathbb{C}[x]_1 \) is generated by the 10 monomials \( x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_0^2x_2, x_0x_1x_2, x_1^2x_2, x_0x_2^2, x_2^3 \). The partial derivatives \( x_0\frac{\partial f}{\partial x_2}, x_1\frac{\partial f}{\partial x_2}, x_2\frac{\partial f}{\partial x_2}, x_2\frac{\partial f}{\partial x_1} \) and \( x_2\frac{\partial f}{\partial x_0} \) generate the monomials \( x_0^2x_2, x_0x_1x_2, x_0x_2^2, x_1^2x_2, x_2^3 \). For \( \lambda \in \mathbb{C} - \{0, 1\} \), the polynomials \( x_0^2x_1 - x_0x_1^2 \) and \( x_0\frac{\partial f}{\partial x_0} \) (and \( x_0x_2^2 \)) generate the monomials \( x_0^3, x_1x_2 \) and \( x_0^2x_1 \). Modulo these seven monomials, the three remaining partial derivatives \( x_1\frac{\partial f}{\partial x_0}, x_0\frac{\partial f}{\partial x_1}, x_1\frac{\partial f}{\partial x_1} \) are congruent to \( -(\lambda + 1)x_1^2 - x_1x_2^2, \lambda x_0^3, 3x_1^3 \). Thus also the three remaining monomials \( x_0^3, x_1x_2, x_1x_2^2 \) are in the left hand side of (4.10).

The case \( q = \frac{4}{3} \): The ideal \( J^\text{pol}_{f_{\lambda}} \) contains \( \frac{\partial f}{\partial x_2} \), so \( x_0x_2 \), and \( x_2\frac{\partial f}{\partial x_1} \), so \( x_2^2x_2 \), and \( x_2\frac{\partial f}{\partial x_0} \), so \( x_2^3 \). Thus \( J^\text{pol}_{f_{\lambda}} \) contains all monomials in \( \mathbb{C}[x]_{4/3} \) which contain \( x_2 \). For \( g \in \{ x_0^2, x_0x_1, x_1^2 \} \), the intersection \( J^\text{pol}_{f_{\lambda}} \cap \mathbb{C}[x]_{4/3} \) contains \( g \cdot \frac{\partial f}{\partial x_0} \). For \( i \in \{ 0, 1 \} \)
\[
\begin{align*}
x_i x_1 & + \frac{1}{2} x_2 x_0 + \frac{3 \lambda + 1}{4} x_1 x_i \lambda (x_1^2 + 2\lambda x_0 x_1) \\
& = -3(\lambda - 1)^2 x_i x_1^3.
\end{align*}
\]
Therefore \( J^\text{pol}_{f_{\lambda}} \) contains \( x_1^4, x_0 x_1^3 \), and with \( g \cdot \frac{\partial f}{\partial x_1} \) also \( x_0 x_2^2, x_3^3 x_1, x_4 \).

This shows \( J^\text{pol}_{f_{\lambda}} \subset \mathbb{C}[x]_{4/3} \).

\textbf{Remarks 4.2.} (i) A priori, we do not see a reason why the monomials \( m_1, \ldots, m_{\mu - 1} \) can be chosen such that they and \( \frac{\partial f}{\partial x_0} \) generate \( \mathbb{C}[x]/J^\text{pol}_{f_{\lambda}} \) for each \( \lambda \in \mathbb{C} - \{0, 1\} \) simultaneously. It is a nice fact, but not crucial.

(ii) Thus the global family \( F^\text{alg} \) is nice, but it is not a unique global unfolding of the Legendre family in (4.2). Though the global family \( F^\text{mar} \) is a unique global unfolding of its restriction to \( t' = 0 \), which is a family over \( \mathbb{H} \) of functions on \( \mathbb{C}^{n+1} \) and which is the pull back by \( \lambda : \mathbb{H} \to \mathbb{C} - \{0, 1\} \) of the Legendre family in (4.2). The family \( F^\text{mar} \) is unique because \( \mathbb{H} \) is contractible.

(iii) For other 1-parameter families of simple elliptic singularities, sets \( m_1, \ldots, m_{\mu - 1} \) of monomials with the analogous property as in lemma 4.1 had been chosen in [Ja86 2.1] and in [MS16] in 2.1 around formula (2).

The next theorem is the special case of theorem 3.8 for the simple and simple elliptic singularities.
Theorem 4.3. Consider for a simple or a simple elliptic singularity the global family of functions \( F^\text{mar}_t \) above the space \( M^\text{mar} \) in \((4.8)\) respectively \((4.9)\). For any \( t \in M^\text{mar} \), the global Milnor number \( \mu^\text{global}(F^\text{mar}_t) := \sum_{x \in \text{Crit}(F^\text{mar}_t)} \mu(F^\text{mar}_t, x) \) is \( \mu \). The manifold \( M^\text{mar} \) is an F-manifold with Euler field \( E \). It is a thickening with the properties in theorem 3.8 of the moduli space \( M^\mu_\tau(f(0)) \). Here \( f(0) = f \) in the case of the simple singularities, and we choose \( f(0) = f_1/2 \) in the case of the simple elliptic singularities. The bundle \( H_Z \) is simply the bundle \( H_Z = \bigcup_{(\tau, t) \in C \times M^\text{mar} - D^\text{mar}} H_n((F^\text{mar}_t)^{-1}(\tau), \mathbb{Z}) \).

Proof: In all cases, \( f \) respectively \( f_\lambda \) is a quasihomogeneous polynomial of weighted degree 1 with respect to a weight system \( w = (w_0, ..., w_n) \in \mathbb{Q} \cap (0, 1/2] \). In the case of a simple singularity, all monomials \( m_1, ..., m_\mu \) have weighted degree in \( \mathbb{Q} \cap (0, 1) \). In the case of a simple elliptic singularity, all monomials \( m_1, ..., m_{\mu - 1} \) have weighted degree in \( \mathbb{Q} \cap (0, 1) \). Therefore in all cases, the highest part (with respect to the weighted degree) of \( \frac{\partial F^\text{mar}_t}{\partial x_i} \) is equal to \( \frac{\partial f}{\partial x_i} \) respectively \( \frac{\partial f_\lambda(t_\mu)}{\partial x_i} \). Denote \( \mathbb{C}[x]_{\leq q} := \{ f \in \mathbb{C}[x] | \deg_w f \leq q \} \) for \( q \in \mathbb{Q}_{\geq 0} \). In the case of the simple elliptic singularities, \((4.10)\) implies for \( q \in \mathbb{Q}_{\geq 0} \)

\[
(J_{\text{pol}}F^\text{mar}_t) \cap \mathbb{C}[x]_{\leq q} + \sum_{j: \deg_w m_j \leq q} \mathbb{C} \cdot m_j + \left\{ \begin{array}{ll} 0 & \text{if } q < 1 \\ \mathbb{C} \cdot \frac{\partial f_\lambda}{\partial t_\mu} & \text{if } q \geq 1 \end{array} \right\} = \mathbb{C}[x]_{\leq q}. \tag{4.11}
\]

And in the case of the simple singularities,

\[
(J_{\text{pol}}F^\text{mar}_t) \cap \mathbb{C}[x]_{\leq q} + \sum_{j: \deg_w m_j \leq q} \mathbb{C} \cdot m_j = \mathbb{C}[x]_{\leq q}. \tag{4.12}
\]

holds. This shows

\[
\dim \mathbb{C}[x]/J_{\text{pol}}F^\text{mar}_t = \mu. \tag{4.13}
\]

The left hand side is the global Milnor number \( \mu_{\text{global}}(F^\text{mar}_t) \).

Observe

\[
\frac{\partial F^\text{mar}_t}{\partial t_j} = \begin{cases} m_j & \text{for the ADE singularities}, \\ m_j & \text{for the } \tilde{E}_k \text{ cases with } j \leq \mu - 1, \\ \frac{\partial f_\lambda}{\partial t_\mu} & \text{for the } \tilde{E}_k \text{ cases with } j = \mu. \end{cases} \tag{4.14}
\]

This and \((4.11)\) and \((4.12)\) show that here the algebraic variant

\[
a^\text{alg,t}_C : T_{M^\text{mar},t} \to \mathbb{C}[x]/J_{\text{pol}}F^\text{mar}_t, \quad \frac{\partial}{\partial t_j} \mapsto \left( \frac{\partial F^\text{mar}_t}{\partial t_j} \right), \tag{4.15}
\]

holds.
(which is here for simplicity written pointwise) of the Kodaira-Spencer map in (2.4) is an isomorphism and equips $T_{M^{mar}, t}$ with a multiplication, a unit field vector and an Euler field vector. This gives $M^{mar}(f^{(0)})$ the structure of an F-manifold with Euler field. The proof of [He02, Theorem 5.3] works also here. The unit field $e$ and the Euler field $E$ are here

$$e = \frac{\partial}{\partial t_1}, \quad E = \sum_{j=1}^{\mu} (1 - \deg w m_j) t_j \frac{\partial}{\partial t_j}. \quad (4.16)$$

In the cases of the simple singularities, the weights $1 - w_j$ are all positive. The F-manifold structure on $M^{mar}$ can be obtained from that of the germ $(M^{mar}, 0)$ by using the flow of the Euler field and $\text{Lie}_E(\circ) = \circ$.

In the cases of the simple elliptic singularities, $1 - w_\mu = 0$, but all other weights $1 - w_j$ are positive. The F-manifold structure on $C^{\mu - 1} \times U$ for $U \subset \mathbb{H}$ a neighborhood of a point $t_\mu$ can be obtained from that on the germ $(M^{mar}, (0, t_\mu))$ by using the flow of the Euler field and $\text{Lie}_E(\circ) = \circ$.

A polynomial $g \in \mathbb{C}[x_0, ..., x_n] = \mathbb{C}[x]$ is tame in the sense of Broughton [Br88, Definition 3.1] if a compact neighborhood $U \subset \mathbb{C}^{n+1}$ exists such that $|(|(\frac{\partial g}{\partial x_0}, ..., \frac{\partial g}{\partial x_n})|)$ is bounded away from 0 on $\mathbb{C}^{n+1} - U$. He proved that $g$ is tame if and only of $\mu_{global}(g) = \mu_{global}(g + \sum_{i=0}^{n} x_i s_i)$ for any $(s_0, ..., s_n) \in \mathbb{C}^{n+1}$ [Br88, Proposition 3.1]. His main result is that for a tame polynomial $g$, the fiber $g^{-1}(\tau)$ for an arbitrary $\tau \in \mathbb{C}$ has the homotopy type of $\mu_{global}(g) - \sum_{x \in \text{Crit}(g^{-1}(\tau))} \mu(g, x)$ many $n$-spheres [Br88, Theorem 1.2].

This applies to all the polynomials $F^{mar}_t$. Because $\mu_{global}(F^{mar}_t) = \mu$ holds for all of them, and because the unfolding $F^{mar}$ comprises the unfolding by the terms $+ \sum_{i=0}^{n} x_i s_i$, they are all tame. The eigenvalues of $E_0 : T_{\tau} M \rightarrow T_{\tau} M$ are the critical values of $F_\tau$. Therefore a fiber $(F^{mar}_t)^{-1}(\tau)$ is smooth if and only if $(\tau, t) \in \mathbb{C} \times M^{mar} - \mathcal{D}^{mar}$. By Broughton such a fiber has the homotopy type of a bouquet of $\mu$ $n$-spheres. Therefore the middle homology groups of these fibers glue to a flat $\mathbb{Z}$-lattice bundle of rank $\mu$,

$$H_Z := \bigcup_{(\tau, t) \in \mathbb{C} \times M^{mar} - \mathcal{D}^{mar}} H_n((F^{mar}_t)^{-1}(\tau), \mathbb{Z}) \quad (4.17)$$

Many of the properties of $M^{mar}(f^{(0)})$ and $H_Z$ in theorem 3.8 are now clear: (a)(i)+(ii) and (b)(i) are obvious. (b)(ii) holds because $M^{mar}$ is simple connected. In the case of the simple singularities, (b)(iii) is empty, as $M^{mar}_\mu$ consists of a single point. In the case of the simple
elliptic singularities, (b)(iii) holds by the proof in [GH17-1, Theorem 6.1] that $M^\mu_{mar}$ is isomorphic to $\mathbb{H}$. There the markings on the points in $\mathbb{H}$ were defined essentially by the commutativity of the diagram (3.17).

It rests to prove (a)(iii) and (b)(iv). First we treat the simple singularities, where this is easier. As $M^\mu_{mar}(f)$ consists of only one point, the stabilizer of this point in $G_Z(f)$ is the whole group $G_Z(f)$, so (3.15) becomes

$$G_Z(f) = G^R_Z(f).$$

The homomorphism in theorem 3.1 (c) becomes a natural surjective homomorphism $G_Z(f) \to \text{Aut}_M$ with kernel $\{\pm \text{id}\}$. Because of the positive $\mathbb{C}^*$-action by the flow of the Euler field on $M^\mu_{mar}$,

$$\text{Aut}_M = \text{Aut}(M^\mu_{mar}, \circ, e, E).$$

By lemma 2.3, any such automorphism lifts to an up to $\pm 1$ unique automorphism of the canonical $\mathbb{Z}$-lattice bundle $H_Z$. The group of these automorphisms is $G_Z(f)$. This proves (a)(i) and (b)(iv) in theorem 3.8 for the simple singularities.

Now we treat the simple elliptic singularities. Consider a group element $\chi \in G_Z(f(0))$, a point $(0, t_\mu) = [(f_{t\lambda_{t_\mu}}, \pm \rho)] \in M^\mu_{mar}(f(0)) \subset M^\mu_{mar}(f(0))$, and its image

$$\chi_{mar}(t) = [(f_{t\lambda_{t_\mu}}, \pm \chi \circ \rho)] = (0, \tilde{t}_\mu) = [(f_{t\lambda_{t_\mu}}, \pm \tilde{\rho})] \in M^\mu_{mar}(f(0))$$

under the action of $\chi_{mar}$ on $M^\mu_{mar}(f(0))$. Consider the isomorphism of F-manifolds with Euler fields

$$\psi_{(0, \tilde{t}_\mu)} : U_{(0, t_\mu)} \to U_{(0, \tilde{t}_\mu)}.$$

which (a)(i) in theorem 3.8 provides. We claim that these isomorphisms for varying $t_\mu$ glue to an automorphism of $M^\mu_{mar}(f(0))$ and that this lifts to an automorphism of $H_Z$ which restricts on the trivial $\mathbb{Z}$-lattice bundle above $\bigcup_{t \in M^\mu_{mar}(f(0))} \mathbb{R}_{>r(t)} \times \{t\}$ in theorem 3.8 (b)(ii) to $\chi$.

In several steps one sees that one local isomorphism in (4.20) extends to a global automorphism of $M^\mu_{mar}(f(0))$. First step: Its restriction to $M^\mu_{mar}(f(0))$ is well defined and given by $\chi_{mar}$. Second step: The local isomorphism in (4.20) extends to an automorphism of a suitable neighborhood of $M^\mu_{mar}(f(0))$ in $M^\mu_{mar}(f(0))$ because $M^\mu_{mar}(f(0)) = \mathbb{H}$ is contractible. Third step: With the $\mathbb{C}^*$-action by the flow of the Euler field on $M^\mu_{mar}(f(0))$, this extends to a global automorphism of $M^\mu_{mar}(f(0))$. 

Above the extension to $\mathbb{C} \times U(0, t_\mu) \to \mathbb{C} \times U(0, \tilde{t}_\mu)$ of the isomorphism in (4.20), one has an isomorphism of the corresponding restrictions of $H_Z$, because they are isomorphic to the canonical $\mathbb{Z}$-lattice bundles above $U(0, t_\mu)$ and $U(0, \tilde{t}_\mu)$ in definition/lemma 3.2. This isomorphism is unique up to $\pm 1$ by definition/lemma 3.2. The commuting diagram (3.17) tells that the restricted isomorphism from $H_Z$ above $\bigcup_{s \in U(0, t_\mu)} \mathbb{R}_{> r(s)} \times \{s\}$ to $H_Z$ above $\bigcup_{s \in U(0, \tilde{t}_\mu)} \mathbb{R}_{> r(s)} \times \{s\}$ is compatible with $\pm \chi$.

Because of the uniqueness up to $\pm 1$, all the local isomorphisms of restrictions of $H_Z$ to neighborhoods of points $(0, 0, t_\mu) \in \mathbb{C} \times M^{mar}(f(0))$ glue (possibly after changing some by $\pm \text{id}$) to one automorphism of $H_Z$. Its restriction to $\bigcup_{s \in M^{mar}} \mathbb{R}_{> r(s)} \times \{s\}$ is $\chi$.

Only now it becomes clear that the automorphism of $M^{mar}(f(0))$ restricts for any $s_\mu \in \mathbb{H}$ to the isomorphism in (4.20): Its restriction to $U(0, s_\mu)$ and the automorphism in (4.20) are in the same way compatible with $\pm \chi$, therefore they coincide if $U(0, s_\mu)$ and $U(0, t_\mu)$ overlap. Now (a)(iii) and (b)(iv) in theorem 3.10 are proved for the simple elliptic singularities. □

In the next section, we will be more concrete about the action of $G_Z(f(0))$ on $M^{mar}(f(0))$.

5. Symmetries of the simple and the simple elliptic singularities

In this section, we will write down concrete formulas for the action on $M^{mar}(f(0))$ of generating elements of $G_Z(f(0))$. We need these formulas for an explicit calculation of certain numbers in section 10. The formulas will also reprove a part of (a)(iii) and (b)(iv) in theorem 3.8 for the simple and the simple elliptic singularities. But we prefer to keep the entity of the conceptual arguments in the last part of the proof of theorem 4.3 than to drop some of them and mix the others with the concrete calculations below.

5.1. Symmetries of the simple singularities. We discussed the symmetries in the proof of theorem 4.3 in the paragraph which contains the formulas (4.18) and (4.19). In the case of a simple singularity $f$, $M^{mar}_\mu(f) = \{\text{pt}\}$, $G_Z(f) = G^{mar}_R(f)$, and

$$\text{Aut}(M^{mar}, \circ, e, E) = \text{Aut}_M \cong G_Z(f)/\{\pm \text{id}\}. \quad (5.1)$$
By the theorems 8.3 and 8.4 in Hell

\[ G_Z(f) = \{ \pm M^k_h | k \in \mathbb{Z} \} \times U \quad \text{(5.2)} \]

\[ U \cong \begin{cases} 
S_1 & \text{for } A_{16}, D_{2l+1}, E_6, E_7, E_8, \\
S_2 & \text{for } D_{2l} \text{ with } l \geq 3, \\
S_3 & \text{for } D_4.
\end{cases} \]

Remark 3.4 applies to \( f \) and \( F^{mar} = F^{alg} \) and gives

\[ (\cdot)_M \circ (\cdot)^{-1}_{hom}(M_h) = (t \mapsto (e^{2\pi i \deg w t_0}, ..., e^{2\pi i \deg w t_n})) \in \text{Aut}_M. \]

In all cases with \( U = S_1 \), this automorphism of \((M^{mar}, \circ, e, E)\) generates \( \text{Aut}_M \).

In the cases \( D_{2l} \), the coordinate change

\[ \varphi_2 = (x \mapsto (x_0, -x_1, x_2, ..., x_n)) \in \text{Stab}_G(w(f)) \subset \mathcal{R}^f \quad \text{(5.3)} \]

and \( \Phi_2 = (\text{id}_X, (\varphi)_M) \) satisfy

\[ (\varphi_2)_M = (t \mapsto (t_1, -t_2, t_3, ..., t_{\mu})) \quad \text{(5.4)} \]

\[ (\varphi_2)_{hom} \not\in \{ (\cdot)_M \circ (\cdot)^{-1}_{hom}(M^k_h) | k \in \mathbb{Z} \}, \]

\[ \Phi_2 \circ \varphi_2 = (\varphi_2, (\varphi_2)_M), \quad F = F \circ (\Phi_2 \circ \varphi_2), \quad \text{pr}_M \circ (\Phi_2 \circ \varphi_2) = (\varphi_2)_M \circ \text{pr}_M. \]

So, in the cases \( D_{2l} \) with \( l \geq 3 \), \((\varphi_2)_{hom}\) can be chosen as a generator of \( U \).

In the case \( D_4 \), \( U \) is generated by \((\varphi_2)_{hom}\) and \((\varphi_3)_{hom}\) where

\[ \varphi_3 := (x \mapsto (-1/2 x_0 + i/2 x_1, 3i/2 x_0 + 1/2 x_1, x_2, ..., x_n)) \quad \text{(5.5)} \]

\[ \in \text{Stab}_G(w(f)) \subset \mathcal{R}^f. \]

This follows from theorem 3.1, theorem 3.3, and the fact, which can be checked easily, that the group \( \text{Stab}_G \cong R_f \) is in the case \( D_4 \) with \( n = 1 \) generated by \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) where \( \varphi_1 \) is as in remark 3.4.

Only the unfolding morphism \((\Phi_3, (\varphi_3)_M)\) which induces \( F \circ \varphi_3^{-1} \) by \( F \), i.e., which satisfies (3.8),

\[ F \circ (\Phi_3 \circ \varphi_3) = F, \quad \text{pr}_M \circ (\Phi_3 \circ \varphi_3) = (\varphi_3)_M \circ \text{pr}_M, \quad \text{(5.6)} \]
is much more complicated than \((\Phi_1, (\varphi_1)_M)\) and \((\Phi_2, (\varphi_2)_M)\). It is given by

\[
(\Phi_3(x_0), \Phi_3(x_1)) = (x_0 + \frac{1}{4} t_4, x_1 + \frac{i}{4} x_1), \quad (5.7)
\]

\[
(\varphi_3)_M^{-1}(t_1, t_2, t_3, t_4) = (t_1 + \frac{i}{4} t_2 t_4 + \frac{1}{4} t_3 t_4 + \frac{1}{16} t_4^2, \quad \frac{1}{2} t_2 - \frac{i}{2} t_3 + \frac{i}{8} t_4,
\]

\[
\frac{3i}{2} t_2 + \frac{1}{2} t_3 + \frac{3}{8} t_4, \quad t_4). \quad (5.8)
\]

Here one calculates (5.8) with the ansatz (5.7) and

\[
F_t((\Phi_3 \circ \varphi_3)(x)) = F_{(\varphi_3)_M^{-1}}(x). \quad (5.9)
\]

For the simple elliptic singularities, we will encounter something similar, one coordinate change \(\varphi_3\) for which \(\Phi_3\) looks difficult.

**Remark 5.1.** For the simple singularities, it is rather obvious (and it will be shown in the proof of theorem 7.1) that \(\text{Aut}_M\) is the group of covering transformations of the covering

\[
\text{LL}^{\text{mar}} : M^{\text{mar}} - (\mathcal{K}_2^{\text{mar}} \cup \mathcal{K}_3^{\text{mar}}) \to M_{\text{LL}}^{(\mu)} - \mathcal{D}_{\text{LL}}^{(\mu)}
\]

in theorem 6.1.

This given, the results above (together with the shape of \(\{\pm M_k^h | k \in \mathbb{Z}\}\), see e.g. the theorems 8.3 and 8.4 in [He1], prove the main theorem in [L3] which describes this covering group. This theorem and the isomorphism \(\text{Aut}_M \cong G_\mathbb{Z}/\{\pm \text{id}\}\) have also been (re)proved in [Yu99, Theorem 1 and Theorem 2].

**5.2. Symmetries of the simple elliptic singularities.** The group \(G_\mathbb{Z} = G_\mathbb{Z}(f^{(0)})\) of the simple elliptic reference singularity \(f^{(0)} = f_{1/2}\) (see theorem 4.3) sits by theorem 3.1 in [GHI7] in an exact sequence

\[
1 \to (U_1^0 \rtimes U_2) \times \{\pm \text{id}\} \to G_\mathbb{Z}
\]

\[
\to \text{Aut}(Ml(f^{(0)})(-1)^n, L)/\{\pm \text{id}\} \to 1
\]

where

\[
\text{Aut}(Ml(f^{(0)})(-1)^n, L) \cong \text{SL}(2, \mathbb{Z}) \quad (5.11)
\]
and

$$U^0_1 \cong \{ (\alpha, \beta, \gamma) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z} \mid \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} \equiv 0 \mod \mathbb{Z} \}$$

\[
\begin{array}{ccc}
(p, q, r) & \tilde{E}_6 & \tilde{E}_7 & \tilde{E}_8 \\
U^0_1 & (3, 3, 3) & (4, 4, 2) & (6, 3, 2)
\end{array}
\]

By theorem 6.1 in [GH17-1], the action of $G_{\mathbb{Z}}$ on $M^{mar}_\mu$ pulls down to an action of the quotient $\text{Aut}(Ml(f^{(0)}(-1)^n, L)/\{ \pm \text{id} \}$ in the exact sequence (5.10) on $M^{mar}_\mu$, and by the isomorphisms (5.11) and $M^{mar}_\mu \cong \mathbb{H}$ this becomes the standard action of $\text{PSL}(2, \mathbb{Z})$ on $\mathbb{H}$.

The action of $\text{PSL}(2, \mathbb{Z})$ on $\mathbb{H}$ descends to the action of $S_3$ on $\mathbb{C} - \{0, 1\}$ where $S_3$ acts via

$$S_3 \cong \{ \lambda \mapsto \lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{\lambda - 1}{\lambda}, \frac{1}{\lambda - 1}, \frac{1}{1 - \lambda} \}. \quad (5.13)$$

This and theorem 3.6 (c), $M^{mar}_\mu / G_{\mathbb{Z}} \cong M_\mu$, reprove the well known fact that the orbits of this action of $S_3$ on $\mathbb{C} - \{0, 1\}$ give the right equivalence classes of one family of Legendre normals forms in table (4.2).

The kernel $(U^0_1 \times U_2) \times \{ \pm \text{id} \}$ in the exact sequence (5.10) acts on the fibers of the projection

$$\text{pr}^{mar}_\mu : M^{mar} = \mathbb{C}^{\mu-1} \times \mathbb{H} \to \mathbb{H}, \ t \mapsto t_\mu.$$ 

This action pulls down to an action on the fibers of the projection

$$\text{pr}^{alg}_\mu : M^{alg} = \mathbb{C}^{\mu-1} \times (\mathbb{C} - \{0, 1\}) \to \mathbb{C} - \{0, 1\}, \ (t', \lambda) \mapsto \lambda.$$

But the action of $G_{\mathbb{Z}}$ on $M^{mar}$ does not pull down to an action of a quotient of $G_{\mathbb{Z}}$ on $M^{alg}$, because the covering group $(\cong \Gamma(2)/\{ \pm 1_2 \} \subset \text{PSL}(2, \mathbb{Z}))$ of the coverings $M^{mar} \to M^{alg}$ and $\mathbb{H} \to \mathbb{C} - \{0, 1\}$ is not a normal subgroup of $G_{\mathbb{Z}}$.

The action of $G_{\mathbb{Z}}$ on $M^{mar}$ pulls only down to an action of a groupoid (see e.g. [ALR07] for the definition of a groupoid) on $M^{alg}$, whose quotient is $M^{mar}/G_{\mathbb{Z}}$. We will not delve into this. The groupoid structure comes from all isomorphisms $(M^{alg}, (t^{(1)}, \lambda^{(1)})) \to (M^{alg}, (t^{(2)}, \lambda^{(2)}))$ of germs of F-manifolds (they all underlie isomorphisms of universal unfoldings).

As global maps, many of these isomorphisms become multivalued. For the use in section 10, we make some of them explicit below. Before, we state a lemma on the relation between $M^{alg}$ and $M^{mar}/G_{\mathbb{Z}}$, which will be used in corollary 7.3.
Lemma 5.2. The map \( M^{\text{alg}} \rightarrow M^{\text{mar}}/G_{\mathbb{Z}} \) is finite and flat and has the following degree,

\[
\begin{align*}
\tilde{E}_6 & : \quad 6 \cdot 2 \cdot 3 \cdot 3^2 = 326, \\
\tilde{E}_7 & : \quad 6 \cdot 1 \cdot 4 \cdot 2^2 = 96, \\
\tilde{E}_8 & : \quad 6 \cdot 1 \cdot 6 \cdot 1^2 = 36.
\end{align*}
\]

(5.14)

Proof: Finiteness and flatness are clear. The degree is \(|S_3| \cdot |U^0_1| \cdot |U_2|\). By (5.12), this is the number in (5.14).

In section 10, we need to compare neighborhoods in \( M^{\text{alg}} = \mathbb{C}^{\mu-1} (\mathbb{C} - \{0, 1\}) \) of \( \mathbb{C}^{\mu-1} \times \{0\}, \mathbb{C}^{\mu-1} \times \{1\} \) and \( \mathbb{C}^{\mu-1} \times \{\infty\} \). For this, we give now multivalued maps \( \psi_2, \psi_3 : M^{\text{alg}} \rightarrow M^{\text{alg}} \) which underlie locally isomorphisms of unfoldings and which lift the automorphisms \( \lambda \mapsto \frac{1}{\lambda} \) and \( \lambda \mapsto 1 - \lambda \) of \( \mathbb{C} - \{0, 1\} \). In each case \( \tilde{E}_k, k \in \{6, 7, 8\} \), we will give two coordinate changes \( \varphi_2 \) and \( \varphi_3 \) and multivalued maps \( \psi_2, \psi_3 : M^{\text{alg}} \rightarrow M^{\text{alg}} \) with

\[
\begin{align*}
F^{\text{alg}}_{t', \lambda}((\Psi_i \circ \varphi_i)(x)) & = F^{\text{alg}}_{\psi_i(t', \lambda)}(x), \\
pr_{M}^{\text{alg}} \circ \psi_2 & = (\lambda \mapsto \frac{1}{\lambda}) \circ pr_{M}^{\text{alg}}, \\
pr_{M}^{\text{alg}} \circ \psi_3 & = (\lambda \mapsto 1 - \lambda) \circ pr_{M}^{\text{alg}}.
\end{align*}
\]

(5.15) (5.16) (5.17)

Then \( \Phi_i := (\Psi_i, \psi_i^{-1}) \) and \( \varphi_i \) satisfy (3.8).

The choice of \( \phi_2, \Psi_2 \) and \( \psi_2 \) is rather obvious, and there \( f_{\lambda} \circ \varphi_2 = f_{1/\lambda} \), (5.15), (5.16) and (3.8) are easy to check. Also the property of \( \varphi_3 \),

\[
f_{\lambda} \circ \varphi_3 = f_{1-\lambda}
\]

is easy to see. But \( \Psi_3 \) looks more difficult. It is determined by the requirement that \( F_{t', \lambda}^{\text{alg}}(\Psi_3 \circ \varphi_3)(x)) = F^{\text{alg}}_{\psi_3(\varphi_3(x), t', \lambda)}(\varphi_3(x), t', \lambda)) \) is an unfolding of \( f_{1-\lambda} \) only in the monomials in table (4.7). That means that the coefficients of the following monomials must vanish:

For \( \tilde{E}_6 \) : \( x_0 x_2, x_1^2 \) (automatic), \( x_2^2 \) (automatic),

For \( \tilde{E}_7 \) : \( x_0^3, x_1^3 \) (automatic),

For \( \tilde{E}_8 \) : \( x_0^5, x_0 x_1^3 \) (automatic), \( x_0^4 \) (automatic).

(5.18)

Having \( \Psi_3 \) and \( \varphi_3 \), \( \psi_3 \) is determined by (5.15). In the case of \( \tilde{E}_6 \) it takes two lines, in the case of \( \tilde{E}_7 \) it takes 11 lines, but in the case of \( \tilde{E}_8 \)
it would take 3 pages. There we do not write down $\psi_3$ completely, we write down only the part of it which is relevant in section 10.

The case $\tilde{E}_6$:

\[ \varphi_2(x_0, x_1, x_2) = (\lambda^{-1} x_0, x_1, \lambda^{1/2} x_2), \]  
\[ \Psi_2(x, t', \lambda) = x, \]  
\[ \psi_2(t', \lambda) = (t_1, \lambda^{-1} t_2, t_3, \lambda^{1/2} t_4, \lambda^{-2} t_5, \lambda^{-1} t_6, \lambda^{1/2} t_7, \lambda^{-1}). \]

\[ \varphi_3(x) = (-x_0, x_1 - x_0, ix_2), \]  
\[ \Psi_3(x, t', \lambda) = (x_0, x_1, x_2 - \frac{i}{2} t_7), \]  
\[ \psi_3(t', \lambda) = (t_1 + \frac{1}{2} t_4 t_7, -t_2 - t_3 - \frac{1}{4} t_7^2, t_3 + \frac{1}{2} t_7^2, it_4, t_5 + t_6, -t_6, it_7, 1 - \lambda). \]

The case $\tilde{E}_7$:

\[ \varphi_2(x) = (\lambda^{-3/4} x_0, \lambda^{1/4} x_1), \]  
\[ \Psi_2(x, t', \lambda) = x, \]  
\[ \psi_2(t', \lambda) = (t_1, \lambda^{-3/4} t_2, \lambda^{1/4} t_3, \lambda^{-3/2} t_4, \lambda^{-1/2} t_5, \lambda^{1/2} t_6, \lambda^{-5/4} t_7, \lambda^{-1/4} t_8, \lambda^{-1}). \]

\[ \varphi_3(x) = (-\xi x_0, \xi (x_1 - x_0)) \quad \text{with} \quad \xi = e^{2\pi i/8}, \]  
\[ \Psi_3(x, t', \lambda) = (x_0, x_1 - \frac{t_7 + t_8}{1 - \lambda}), \]  
\[ \psi_3(t', \lambda) = (\tilde{t}_1, ..., \tilde{t}_8, 1 - \lambda) \quad \text{with} \]  
\[ \tilde{t}_1 = t_1 + (-1) \frac{t_7 + t_8}{1 - \lambda} t_3 + \left( \frac{t_7 + t_8}{1 - \lambda} \right)^2 t_6, \]  
\[ \tilde{t}_2 = (-\xi) t_2 + (-\xi) t_3 + \xi \frac{t_7 + t_8}{1 - \lambda} t_5 + 2\xi \frac{t_7 + t_8}{1 - \lambda} t_6 + \xi \left( \frac{t_7 + t_8}{1 - \lambda} \right)^2 t_8, \]  
\[ \tilde{t}_3 = (\xi t_3 + (-2\xi) \frac{t_7 + t_8}{1 - \lambda} t_6, \]
\[ \tilde{t}_4 = \xi^2(t_4 + t_5 + t_6) + (-\xi^2)^2 \frac{t_7 + t_8}{1 - \lambda} + \frac{t_7 + t_8}{1 - \lambda} \]
\[ + \xi^2 (2 - \lambda) \left( \frac{t_7 + t_8}{1 - \lambda} \right)^2 , \]
\[ \tilde{t}_5 = (-\xi^2)^2 t_5 + (-2\xi^2)^2 t_6 + 2\xi^2 t_7 + t_8 + (-3\xi^2) \left( \frac{t_7 + t_8}{1 - \lambda} \right) , \]
\[ \tilde{t}_6 = \xi^2 t_6 , \]
\[ \tilde{t}_7 = \frac{\xi^3}{1 - \lambda}((-3 + \lambda)t_7 + (-2)t_8) , \]
\[ \tilde{t}_8 = \frac{\xi^3}{1 - \lambda}(3t_7 + (2 + \lambda)t_8) . \]

The case $\tilde{E}_8$:

\[ \varphi_2(x) = (\lambda^{-1/2} x_0, x_1) , \quad (5.31) \]
\[ \Psi_2(x,t',\lambda) = x , \quad (5.32) \]
\[ \psi_2(t',\lambda) = (t_1, \lambda^{-1/2} t_2, \lambda^{-1} t_3, t_4, \lambda^{-3/2} t_5 , \lambda^{-1/2} t_6, \lambda^{-1} t_7, t_8, \lambda^{-1/2} t_9, \lambda^{-1}) . \quad (5.33) \]

\[ \varphi_3(x) = (i x_0, x_1 - x_0^2) , \quad (5.34) \]
\[ \Psi_3(x,t',\lambda) = (x_0 + \frac{t_9}{2 (1 - \lambda)^2}, x_1 + \frac{t_7 + t_8}{1 - \lambda} + i \lambda \frac{t_9}{(1 - \lambda)^2} x_0 + \frac{1}{4} \frac{t_9^2 (4\lambda^2 - 2\lambda - 1)}{(1 - \lambda)^4} , \quad (5.35) \]
\[ \psi_3(t',\lambda) = (\tilde{t}_1,...,\tilde{t}_9,1-\lambda) \quad \text{with} \]
\[ \tilde{t}_1 = t_1 + \text{(a term in } \mathbb{C}[\lambda,t_2,...,t_9, \frac{t_7 + t_8}{1 - \lambda}, \frac{t_9}{1 - \lambda}^2]) , \]
\[ \tilde{t}_2 = it_2 + \text{(a term in } \mathbb{C}[\lambda,t_3,...,t_9, \frac{t_7 + t_8}{1 - \lambda}, \frac{t_9}{1 - \lambda}^2]) , \]
\[ \tilde{t}_3 = -t_3 - t_4 + \text{(a term in } \mathbb{C}[\lambda,t_5,...,t_9, \frac{t_7 + t_8}{1 - \lambda}, \frac{t_9}{1 - \lambda}^2]) , \]
\[ \tilde{t}_4 = t_4 + \text{(a term in } \mathbb{C}[\lambda,t_6,...,t_9, \frac{t_7 + t_8}{1 - \lambda}, \frac{t_9}{1 - \lambda}^2]) . \]
\[
\tilde{t}_5 = (-i)(t_5 + t_6) + \text{(a term in } \mathbb{C}[\lambda, t_7, t_8, t_9, \frac{t_7 + t_8}{1 - \lambda}, \frac{t_9}{(1 - \lambda)^2}]),
\]

\[
\tilde{t}_6 = it_6 + \text{(a term in } \mathbb{C}[\lambda, t_7, t_8, t_9, \frac{t_7 + t_8}{1 - \lambda}, \frac{t_9}{(1 - \lambda)^2}]),
\]

\[
\tilde{t}_7 = \frac{\lambda - 3}{\lambda - 1}t_7 + \frac{-2}{\lambda - 1}t_8 + \frac{6\lambda + 1}{2(1 - \lambda)^2}t_9^2,
\]

\[
\tilde{t}_8 = \frac{3}{\lambda - 1}t_7 + \frac{\lambda + 2}{\lambda - 1}t_8 + \frac{14\lambda^2 - 11\lambda - 2}{4(1 - \lambda)^4}t_9^2,
\]

\[
\tilde{t}_9 = i\frac{\lambda^2}{(1 - \lambda)^2}t_9.
\]

**Remarks 5.3.** (i) In the case of the minimal number of variables \((n = 1 \text{ for } \tilde{E}_7 \text{ and } \tilde{E}_8, \text{ and } n = 2 \text{ for } \tilde{E}_6)\), the subgroup \(U_0 \rtimes U_2\) of the kernel \((U_0 \rtimes U_2) \times \{\pm \text{id}\}\) in the exact sequence in (5.10) comes via 
\(\text{Stab}_{G}^{\mu}(f_{\lambda}) \cong R_f \xrightarrow{\text{hom}} G_{Z}\) from \(\text{Stab}_{G}^{\mu}(f_{\lambda})\) for generic \(\lambda\). This follows from the fact that the kernel of the exact sequence in (5.10) is the subgroup of \(G_{Z}\) which acts trivially on \(M_{\mu}^{\text{mar}} \cong \mathbb{H}\). In the cases of \(\tilde{E}_7\) and \(\tilde{E}_8\), the elements of \(\text{Stab}_{G}^{\mu}(f_{\lambda})\) for generic \(\lambda\) can be determined easily explicitly. In the case of \(\tilde{E}_6\), this is more difficult.

(ii) In any case, one can avoid at the beginning of this subsection the use of theorem 3.1 in [GH17-1], which gives the facts in (5.10)–(5.12) on \(G_{Z}\). One can recover these by the following steps:

1. Determine \(\text{Stab}_{G}^{\mu}(f_{\lambda})\) for generic \(\lambda\).
2. Use (i).
3. Show that \(\text{Stab}_{G}^{\mu}(f_{\lambda})\) for generic \(\lambda\) and \(\varphi_2\) and \(\varphi_3\) generate all quasihomogeneous coordinate changes which map each \(f_{\lambda}\) to some \(f_{\bar{\lambda}}\).

6. Lyashko-Looijenga maps for the simple and the simple elliptic singularities

6.1. Lyashko-Looijenga maps and their degrees. Lyashko-Looijenga maps in general were discussed in subsection 2.4. Here we
consider the Lyashko-Looijenga maps for the families of functions defined in section 4, the maps

\[ LL_{\text{alg}} : M_{\text{alg}} \to M_{LL}^{(\mu)} \]  \hspace{1cm} (6.1)

\[ LL_{\text{mar}} : M_{\text{mar}} \to M_{LL}^{(\mu)}, \text{ with} \]  \hspace{1cm} (6.2)

\[ t \in M_{\text{mar}} \mapsto \prod_{j=1}^{\mu} (y - u_j) \] with \( u_1, \ldots, u_\mu \) the critical values of \( F_{\text{mar}} \) (with multiplicities).

The caustic \( K_{\text{mar}}^3 \subset M_{\text{mar}} \) and the Maxwell stratum \( K_{\text{mar}}^2 \subset M_{\text{mar}} \) had been defined in (3.18) and (3.19). They are analytic hypersurfaces. The caustic \( K_{\text{alg}}^3 \subset M_{\text{alg}} \) and the Maxwell stratum \( K_{\text{alg}}^2 \subset M_{\text{alg}} \) are defined analogously. They are algebraic hypersurfaces as \( LL_{\text{alg}} \) is even an algebraic map.

By Looijenga [Lo74] and Lyashko [Ly79][Ly84], the map \( LL_{\text{alg}} \) restricts to a locally biholomorphic map 

\[ LL_{\text{alg}} : M_{\text{alg}} - (K_{\text{alg}}^3 \cup K_{\text{alg}}^2) \to M_{LL}^{(\mu)} - D_{LL}^{(\mu)}, \] it maps \( K_{\text{alg}}^3 \cup K_{\text{alg}}^2 \) to \( D_{LL}^{(\mu)} \), and it is a branched covering of order 3 respectively 2 at generic points of \( K_{\text{alg}}^3 \) respectively \( K_{\text{alg}}^2 \), and analogous statements hold for \( LL_{\text{mar}} \) and \( K_{\text{mar}}^3, K_{\text{mar}}^2 \subset M_{\text{mar}} \).

In the case of the simple and the simple elliptic singularities, we have the following more precise results.

Theorem 6.1 concerns the simple singularities and was proved by Looijenga [Lo74] and Lyashko [Ly79][Ly84]. The covering result in theorem 6.2 for the simple elliptic singularities is an achievement of Jaworski [Ja86, Theorem 2][Ja88, Proposition 1].

The refinement in theorem 6.3 of theorem 6.2 is a major result of this paper. It will be proved in section 10, which builds on the sections 5, 8 and 9. It reproves Jaworski’s result. But the main point is the degree \( \deg LL_{\text{alg}} \) for the simple elliptic singularities, which was not calculated before.

Though for the bijections in the main result theorem 7.1, we do not need the degree \( \deg LL_{\text{alg}} \). Theorem 6.2 and the analogous part of theorem 6.1 are sufficient.

**Theorem 6.1.** [Lo74][Ly79][Ly84] In the case of the simple singularities, \( LL_{\text{alg}} \) is a branched covering of degree

\[ \deg LL_{\text{alg}} = \prod_{j=1}^{\mu} \deg_{w_j} t_j^\mu. \]  \hspace{1cm} (6.3)
Here $\deg w t_j := 1 - \deg w m_j$. The degree $\deg LL^{alg}$ is given explicitly in table (6.4).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{name} & A_\mu & D_\mu & E_6 & E_7 & E_8 \\
\hline
\deg LL^{alg} & (\mu + 1)^{\mu - 1} & 2(\mu - 1)^{\mu} & 2^9 \cdot 3^4 & 2 \cdot 3^5 \cdot 5^7 & 2 \cdot 3^5 \cdot 7^7 \\
\hline
\end{array}
\] (6.4)

And the restriction $LL^{alg} : M^{alg} - (K_3^{alg} \cup K_2^{alg}) \to M_{LL}^{(\mu)} - D_{LL}^{(\mu)}$ is a covering.

**Theorem 6.2.** [Ja86][Ja88] In the case of the simple elliptic singularities, the restriction $LL^{alg} : M^{alg} - (K_3^{alg} \cup K_2^{alg}) \to M_{LL}^{(\mu)} - D_{LL}^{(\mu)}$ is a covering.

**Theorem 6.3.** In the case of the simple elliptic singularities, an extension $M^{orb}$

\[
(t', \lambda) \in M^{alg} \subset M^{orb} \\
\downarrow \quad \downarrow \pi^{orb} \\
\lambda \in \mathbb{C} - \{0, 1\} \subset \mathbb{P}^1
\] (6.5)

of $M^{alg}$ to an orbibundle above $\mathbb{P}^1 \supset \mathbb{C} - \{0, 1\}$ exists such that $LL^{alg}$ extends to a surjective holomorphic map $LL^{orb} : M^{orb} \to M^{(\mu)}_{LL}$ with the following properties. The two-dimensional subspace $M_0^{orb} \subset M^{orb}$ with

\[
M_0^{orb} = \text{(closure in } M^{orb} \text{ of } \{(t', \lambda) \in M^{alg} | t_2 = \ldots = t_{\mu - 1} = 0\}) \\
\cong \mathbb{C} \times \mathbb{P}^1
\]

(which is the $\mu$-constant stratum and its translates under the unit field) is mapped to the one-dimensional subspace $M_{LL, 0}^{(\mu)} \subset M_{LL}^{(\mu)}$ with

\[
M_{LL, 0}^{(\mu)} := \{ p(y) \in M_{LL}^{(\mu)} | p(y) = (y - u_1)^{\mu}, u_1 \in \mathbb{C} \} \cong \mathbb{C}.
\]

The restriction

\[
LL^{orb} : M^{orb} - M_0^{orb} \to M_{LL}^{(\mu)} - M_{LL, 0}^{(\mu)}
\] (6.6)

is a branched covering of degree

\[
\deg LL^{orb} = \deg LL^{alg} = \frac{\mu! \cdot \frac{1}{2} \cdot \sum_{j=2}^{\mu-1} \frac{1}{\deg_w t_j}}{\prod_{j=2}^{\mu-1} \deg_w t_j}. \quad (6.7)
\]

Here $\deg_w t_j := 1 - \deg_w m_j$. The degree $\deg LL^{alg}$ is given explicitly in table (6.8).

\[
\begin{array}{|c|c|c|}
\hline
\text{name} & \tilde{E}_6 & \tilde{E}_7 & \tilde{E}_8 \\
\hline
\deg LL^{alg} & 2^2 \cdot 3^{11} \cdot 5 \cdot 7 & 2^{18} \cdot 3 \cdot 5^3 \cdot 7 & 2^9 \cdot 3^{10} \cdot 7 \cdot 101 \\
\hline
\end{array}
\] (6.8)

And $LL^{orb}$ maps $K_3^{alg} \cup K_2^{alg} \cup \pi^{-1}_{orb}(\{0, 1, \infty\})$ to $D_{LL}^{(\mu)}$. 
Remark 6.4. (i) Let $N_{\text{Coxeter}}$ be the Coxeter number of an ADE root lattice, and $W$ its Weyl group. By [Bo68] $|W| = N_{\text{Coxeter}}^\mu \prod_{j=1}^\mu \deg_w t_j$. Therefore

$$\deg LL^{alg} = \frac{\mu!}{\prod_{j=1}^\mu \deg_w t_j} = \frac{\mu! \cdot N_{\text{Coxeter}}^\mu}{|W|}. \quad (6.9)$$

This was observed for example in [Yu90].

(ii) In order to make the tables (6.4) and (6.8) transparent, here we give the weights $\deg_w x_i = w_i$, the weights $\deg_w t_j$, in the ADE cases the Coxeter numbers $N_{\text{Coxeter}}^\mu$, and in the simple elliptic cases the number $\frac{1}{2} \sum_{j=2}^{\mu-1} \frac{1}{\deg_w t_j}$.

| $A_\mu$ | $N_{\text{Coxeter}}$ | $x_0$ | $x_1$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ | $t_7$ | $t_8$ |
|--------|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\mu + 1$ | $\frac{\mu + 1}{\mu - 1}$ | $\frac{\mu - 2}{2(\mu - 1)}$ | $1$ | $\frac{\mu}{\mu - 1}$ | $\frac{\mu - 2}{\mu - 1}$ | $\frac{\mu - 2}{\mu - 1}$ | $\frac{\mu - 3}{\mu - 1}$ | $\frac{\mu - 3}{\mu - 1}$ | $\frac{\mu - 4}{\mu - 1}$ | $\frac{\mu - 4}{\mu - 1}$ |
| $D_\mu$ | $2(\mu - 1)$ | $\frac{\mu + 1}{\mu - 1}$ | $\frac{\mu - 2}{2(\mu - 1)}$ | $1$ | $\frac{\mu}{\mu - 1}$ | $\frac{\mu - 2}{\mu - 1}$ | $\frac{\mu - 2}{\mu - 1}$ | $\frac{\mu - 3}{\mu - 1}$ | $\frac{\mu - 3}{\mu - 1}$ | $\frac{\mu - 4}{\mu - 1}$ | $\frac{\mu - 4}{\mu - 1}$ |

$E_6$ | 12 | $\frac{1}{3}$ | $\frac{1}{3}$ | 1 | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

$E_7$ | 18 | $\frac{2}{3}$ | $\frac{1}{3}$ | 1 | $\frac{7}{9}$ | $\frac{2}{3}$ | $\frac{5}{9}$ | $\frac{4}{9}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

$E_8$ | 30 | $\frac{5}{12}$ | $\frac{1}{3}$ | 1 | $\frac{4}{5}$ | $\frac{2}{3}$ | $\frac{3}{5}$ | $\frac{7}{10}$ | $\frac{7}{15}$ | $\frac{4}{15}$ | $\frac{1}{15}$ |

$\tilde{E}_6$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 1 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

$\tilde{E}_7$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | 1 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |

$\tilde{E}_8$ | $\frac{1}{6}$ | $\frac{1}{3}$ | 1 | $\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

6.2. Limit behaviour of $LL^{alg}$ after Jaworski. Jaworski’s proof of theorem 6.2 required an understanding of the limit behaviour of $LL^{alg}$ near $\lambda \in \{0, 1, \infty\}$. Here we will explain a result of him which concerns this limit behaviour. It will be crucial for the proof of the main theorem 7.1 in the case of the simple elliptic singularities.

The intersection form $I$ on the Milnor lattice $ML(f)$ of a simple elliptic singularity is positive semidefinite if $n \equiv 0(4)$, see e.g. [AGV88]. Consider the Stokes matrix $S$ of any distinguished basis as defined in (2.16). Because of (2.18), $S + S^t$ is positive semidefinite. Therefore all entries of $S$ are in $\{0, \pm 1, \pm 2\}$. Therefore for any two vanishing cycles $\delta_i$ and $\delta_j$ and any $n$ (not necessarily $n \equiv 0(4)$) $I(\delta_i, \delta_j) \in \{0, \pm 1, \pm 2\}$. 
Theorem 6.5. \[ LL_{\text{alg}} : M_{\text{alg}} - (K_{3}^{\text{alg}} \cup K_{2}^{\text{alg}}) \to M_{LL}^{(\mu)} - D_{LL}^{(\mu)} \] is a covering by theorem 6.2. Now consider a \((C^\infty\) or real analytic\) path

\[ r : [0, \varepsilon) \to M_{LL}^{(\mu)} \quad \text{with} \quad r((0, \varepsilon)) \subset M_{LL}^{(\mu)} - D_{LL}^{(\mu)}, \quad (6.10) \]

\[ r(0) \in D_{LL}^{(\mu), \text{reg}}. \]

Consider any lift \(\rho : (0, \varepsilon) \to M_{\text{alg}} - (K_{3}^{\text{alg}} \cup K_{2}^{\text{alg}})\) of the restriction of the path \(r\) to \((0, \varepsilon)\). For \(s \in (0, \varepsilon)\), denote by \(u_{1}(s), \ldots, u_{\mu}(s)\) the critical values of \(F_{\rho(s)}\). They are pairwise different. Because of \(r(0) \in D_{LL}^{(\mu), \text{reg}}\), precisely two of them will tend to one another if \(s \to 0\). We can suppose that they are numbered \(u_{i}(s)\) and \(u_{i+1}(s)\). Write \(\rho(s) = (t_{1}^{(s)}(s), \ldots, t_{\mu-1}^{(s)}(s), \lambda^{(s)})\) for \(s \in (0, \varepsilon)\).

For fixed \(s \in (0, \varepsilon)\), consider the \(\mathbb{Z}\)-lattice bundle

\[ \bigcup_{\tau \in \mathbb{C} - \{u_{1}(s), \ldots, u_{\mu}(s)\}} H_{n}(F_{\rho(s)}^{-1}(\tau), \mathbb{Z}). \]

Move the vanishing cycles at \(u_{i}(s)\) and \(u_{i+1}(s)\) along straight lines to the \(\mathbb{Z}\)-lattice

\[ H_{n}(F_{\rho(s)}^{-1}(\frac{s_{u(s)}+s_{u(s+1)}}{2}), \mathbb{Z}) \]

and call the images \(\delta_{i}(s), \delta_{i+1}(s)\). They are unique up to the sign. Then the following holds.

**Theorem 6.5.** \([Ja88\) Proposition 2]\]

\[ I(\delta_{i}(s), \delta_{i+1}(s)) = 0 \iff \rho \text{ extends to } 0 \text{ and } \rho(0) \in K_{2}^{\text{alg, reg}}, \quad (6.11) \]

\[ I(\delta_{i}(s), \delta_{i+1}(s)) = \pm 1 \iff \rho \text{ extends to } 0 \text{ and } \rho(0) \in K_{3}^{\text{alg, reg}}, \]

\[ I(\delta_{i}(s), \delta_{i+1}(s)) = \pm 2 \iff \lambda^{(s)} \text{ extends to } 0 \text{ and } \lambda^{(s)}(0) \in \{0, 1, \infty\}. \]

**Proof:** The statement in \([Ja88\) Proposition 2\] is slightly weaker. Therefore here we provide the additional arguments. Because of theorem 6.3 \(\lambda^{(s)}\) extends in any case to 0, and if \(\lambda^{(s)}(0) \notin \{0, 1, \infty\}\), then \(\rho\) extends to 0 and \(\rho(0) \in K_{3}^{\text{alg, reg}} \cup K_{2}^{\text{alg, reg}}\). Therefore exactly one of the three cases on the right hand side of \((6.11)\) holds. In the third case, Proposition 2 in \([Ja88\] applies and gives \(I(\delta_{i}(s), \delta_{i+1}(s)) = \pm 2\).

In the second case, \(F_{\rho(0)}\) has \(\mu - 2\) \(A_{1}\) singularities and one \(A_{2}\) singularity, with pairwise different critical values, and then \(\delta_{i}(s)\) and \(\delta_{i+1}(s)\) are a distinguished basis of the Milnor lattice of the \(A_{2}\) singularity. Then \(I(\delta_{i}(s), \delta_{i+1}(s)) = 1\).

In the first case, \(F_{\rho(0)}\) has \(\mu\) \(A_{1}\) singularities, and two of them have the same critical value, the others have pairwise different critical values. Then \(\delta_{i}(s)\) and \(\delta_{i+1}(s)\) are vanishing cycles of the two \(A_{1}\) singularities with the same critical value. Then \(I(\delta_{i}(s), \delta_{i+1}(s)) = 0\). \(\square\)

The situation for the simple singularities is analogous, but simpler. The Milnor lattice \(M(f)\) with intersection form of an ADE singularity is isomorphic to the ADE root lattice if \(n \equiv 0(4)\), see e.g. \([AGV88]\).
Consider the Stokes matrix $S$ of any distinguished basis as defined in (2.16). Because of (2.18), $S + S^t$ is positive definite. Therefore all entries of $S$ are in $\{0, \pm 1\}$. Therefore for any two vanishing cycles $\delta_i$ and $\delta_j$ and any $n$ (not necessarily $n \equiv 0(4)$) $I(\delta_i, \delta_j) \in \{0, \pm 1\}$.

Now consider a ($C^\infty$ or real analytic) path $r : [0, \varepsilon) \to M_{\text{LL}}^{(\mu)}$ as in (6.10), and consider any lift $\rho : (0, \varepsilon) \to M^{\text{alg}} - (K^{\text{alg}}_3 \cup K^{\text{alg}}_2)$ of the restriction of the path $r$ to $(0, \varepsilon)$. Because $LL^{\text{alg}} : M^{\text{alg}} \to M^{LL}_{\text{alg}}$ is a branched covering, $\rho$ extends to $0$, and $\rho(0) \in K^{\text{alg,reg}}_2 \cup K^{\text{alg,reg}}_3$.

For $s \in (0, \varepsilon)$, denote again by $u_1(s), \ldots, u_\mu(s)$ the critical values of $F_{\rho(s)}$. They behave for $s \to 0$ as above, and we obtain vanishing cycles $\delta_i(s)$ and $\delta_{i+1}(s)$ as above. Then the following holds.

**Lemma 6.6.**

\begin{align*}
I(\delta_i(s), \delta_{i+1}(s)) &= 0 \iff \rho(0) \in K^{\text{alg,reg}}_2, \\
I(\delta_i(s), \delta_{i+1}(s)) &= \pm 1 \iff \rho(0) \in K^{\text{alg,reg}}_3.
\end{align*}

The proof is a subset of the proof of theorem 6.5.

### 7. The main theorem, its proof and consequences

In subsection 3.4 we introduced for any reference singularity $f^{(0)}$ a Looijenga-Deligne map

$$LD : R_{\text{Stokes}} \to B^{\text{ext}}(f^{(0)})/G_{\text{sign,\mu}}.$$  \hfill (7.1)

Recall that $R_{\text{Stokes}}$ is the set of Stokes regions, which are the components of the complement of the Stokes walls $W_{\text{Stokes}}$ in $M^{\text{mar}}(f^{(0)})$, and $B^{\text{ext}}(f^{(0)})$ is the orbit under $G_Z$ of the set $B(f^{(0)})$ of distinguished bases of $f^{(0)}$. The map $LD$ is $G_Z$ equivariant.

In the case of a simple singularity $f^{(0)} = f$ or of a simple elliptic singularity $f^{(0)} = f_{1/2}/2$ (with $f_\lambda$ the Legendre normal form from (4.2)), $M^{\text{mar}}(f^{(0)})$ had been constructed in section 4.

The main theorem is as follows. For the simple singularities, the bijection (7.3) was proved in a different way in [Lo74] and [De74], see remark 7.2 (iv) below. Yu [Yu90, 6.3 Satz] built on this and proved the bijection (7.4) for the simple singularities.

**Theorem 7.1.** Consider a simple singularity $f^{(0)} = f$ or a simple elliptic singularity $f^{(0)} = f_{1/2}$. Then

$$R^{0}_{\text{Stokes}} = R^0_{\text{Stokes}}, \quad B^{\text{ext}}(f^{(0)}) = B(f^{(0)}).$$  \hfill (7.2)

The Looijenga-Deligne map

$$LD : R^0_{\text{Stokes}} \to B(f^{(0)})/G_{\text{sign,\mu}}.$$  \hfill (7.3)
and the induced quotient map

\[ LD/G : R^0_{\text{Stokes}}/G \to \{ \text{Stokes matrices} \}/G_{\text{sign,} \mu} \]  (7.4)

are bijections.

**Proof:** In [GH17-1] it was proved that the moduli space \( M^{\text{mar}}(f^{(0)}) \) is connected (see remark [GH17-1] (ii)). Therefore \( R_{\text{Stokes}} = R^0_{\text{Stokes}} \). Recall the argument in remark [3.11] (ii) for \( B^{\text{ext}}(f^{(0)}) = B(f^{(0)}) \): The map \( LD \) is \( G \)-equivariant, and remark [3.11] (i) shows that \( R^0_{\text{Stokes}} \) is mapped to \( B(f^{(0)}) \). Therefore \( B^{\text{ext}}(f^{(0)}) = B(f^{(0)}) \).

The Stokes matrix of a distinguished basis \( \delta \) of the Milnor lattice \( M_l(f^{(0)}) \) was defined in (2.16) as \( S = (-1)^{(n+1)(n+2)/2} \cdot L(\delta^t, \delta)^t \). Obviously the set of Stokes matrices can be identified with the quotient \( B(f^{(0)})/G \).

It suffices to prove that the map \( LD \) in (7.3) is a bijection. Then the quotient map \( LD/G \) in (7.4) is a bijection, too.

Looijenga’s argument [Lo74] that \( LD \) is surjective for the simple singularities, works because of Jaworski’s theorem [Jaw82] also for the simple elliptic singularities. The argument is as follows. Let \( U \in R^0_{\text{Stokes}} \) be any Stokes region, let \( t \in U \), and let \( \tilde{\delta} \) be the up to the action of \( G_{\text{sign,} \mu} \) unique distinguished basis which is constructed from the morsification \( F \) and the good distinguished system of paths in definition [3.9] (b). Then \( \tilde{\delta} \) is in \( LD(U) \). Let \( \gamma \) be any distinguished basis. It is the image of \( \tilde{\delta} \) under the action of a certain braid in \( B_{\mu} \) and possibly a sign change in \( G_{\text{sign,} \mu} \). The braid gives a (homotopy class of a) closed path in \( M^{(\mu)}_L - D^{(\mu)}_L \). The path has a unique lift to \( M^{\text{mar}} - (K_3^{\text{mar}} \cup K_2^{\text{mar}}) \) which starts at \( t \in U \) because the Lyashko-Looijenga map \( LL^\text{mar} : M^{\text{mar}} - (K_3^{\text{mar}} \cup K_2^{\text{mar}}) \to M^{(\mu)}_L - D^{(\mu)}_L \) is a covering by the theorems 6.1 and 6.2. Let \( \tilde{t} \) be the endpoint of this lift and let \( \tilde{U} \) be the Stokes region which contains \( \tilde{t} \). Then \( \gamma \in LD(\tilde{U}) \).

Therefore \( LD \) is surjective.

It rests to prove that \( LD \) is injective. Let \( U^{(1)} \) and \( U^{(2)} \) be two Stokes regions with \( LD(U^{(1)}) = LD(U^{(2)}) \). Because the Lyashko-Looijenga map \( LL^\text{mar} : M^{\text{mar}} - (K_3^{\text{mar}} \cup K_2^{\text{mar}}) \to M^{(\mu)}_L - D^{(\mu)}_L \) is a covering, both Stokes regions are mapped by \( LL^\text{mar} \) bijectively to the open subset

\[ \{ p(y) \in M^{(\mu)}_L \mid p(y) = \prod_{j=1}^{\mu} (y - u_j) \} \]  (7.5)

with \( \text{Im}(u_i) \neq \text{Im}(u_j) \) for \( i \neq j \).
of $M^{(\mu)}_L$. There is a unique isomorphism $\psi^U : U^{(1)} \to U^{(2)}$ which is compatible with $LL^{\text{mar}}$. Obviously it is an isomorphism of semisimple F-manifolds.

We claim that it extends to an automorphism $\psi^{\text{mar}} : M^{\text{mar}}(f^{(0)}) \to M^{\text{mar}}(f^{(0)})$. If this is true then the rest of the proof is an elegant application of theorem 4.3. Then $\psi^{\text{mar}}$ comes from an element $\psi \in G_Z(f^{(0)})$ (which is unique up to $\pm 1$). The element $\psi$ must map $LD(U^{(1)})$ to $LD(U^{(2)})$. As they coincide by assumption, $\psi = \pm \text{id}$. Thus $\psi^{\text{mar}} = \text{id}$ on $M^{\text{mar}}$, thus $U^{(1)} = U^{(2)}$.

It rests to show that $\psi^U$ extends an automorphism $\psi^{\text{mar}}$ of $M^{\text{mar}}(f^{(0)})$. Roughly, the reason is that the covering $LL^{\text{mar}} : M^{\text{mar}} - (K^{\text{mar}}_3 \cup K^{\text{mar}}_2) \to M^{(\mu)}_L - D^{(\mu)}_L$ with base point in $U^{(k)}$ is determined by the class of Stokes matrices modulo $G_{\text{sign},\mu}$ which are associated to the distinguished bases in $LD(U^{(k)})$. As this class coincides for $k = 1, 2$, a deck transformation $\psi^{\text{mar}} : M^{\text{mar}} - (K^{\text{mar},\text{sing}}_3 \cup K^{\text{mar},\text{sing}}_2) \to M^{\text{mar}} - (K^{\text{mar}}_3 \cup K^{\text{mar}}_2)$ exists, which extends $\psi^U$. It extends to $K^{\text{mar}}_3 \cup K^{\text{mar}}_2$ as there $LL^{\text{mar}}$ is generically branched of order 3 respectively 2.

More precisely, we can argue as follows. Let $t^{(k)} \in U^{(k)}$ be points with $LL^{\text{mar}}(t^{(1)}) = LL^{\text{mar}}(t^{(2)})$. Then $\psi^U(t^{(1)}) = t^{(2)}$. Choose a path within $M^{\text{mar}}(f^{(0)}) - (K^{\text{mar},\text{sing}}_3 \cup K^{\text{mar},\text{sing}}_2)$ from $t^{(1)}$ to any point in this space. We claim that $\psi^U$ extends from $U^{(1)}$ to a well defined map

$$\psi^{U, \text{path}} : U^{(1)} \cup (\text{a neighborhood of this path}) \to M^{\text{mar}} - (K^{\text{mar},\text{sing}}_3 \cup K^{\text{mar},\text{sing}}_2)$$

and that this is locally an isomorphism of F-manifolds. If the path does not meet $K^{\text{mar}}_3 \cup K^{\text{mar}}_2$, this is obvious. Now suppose that it meets $K^{\text{mar},\text{reg}}_3 \cup K^{\text{mar},\text{reg}}_2$. Let $\rho^{(1)}$ be the restriction of the path to a path from $t^{(1)}$ to a point $\tilde{t}^{(1)}$ just before the first meeting point with $K^{\text{mar},\text{reg}}_3 \cup K^{\text{mar},\text{reg}}_2$. Then

$$\psi^{U, \rho^{(1)}} : U^{(1)} \cup \{\text{a neighborhood of } \rho^{(1)}\} \to M^{\text{mar}} - (K^{\text{mar},\text{sing}}_3 \cup K^{\text{mar},\text{sing}}_2)$$

is well defined. Let $\rho^{(2)} := \psi^{U, \rho^{(1)}} \circ \rho^{(1)}$ be the image of $\rho^{(1)}$ under $\psi^{U, \rho^{(1)}}$. Then $\rho^{(2)}$ starts at $t^{(2)}$ and ends at $\tilde{t}^{(2)} := \psi^{U, \rho^{(1)}}(\tilde{t}^{(1)})$. Then $LL(\tilde{t}^{(1)}) = LL(\tilde{t}^{(2)})$.

Let $\tilde{U}^{(1)}$ and $\tilde{U}^{(2)}$ be the Stokes regions which contain $\tilde{t}^{(1)}$ and $\tilde{t}^{(2)}$. Then still $LD(\tilde{U}^{(1)}) = LD(\tilde{U}^{(2)})$, and also the associated Stokes matrices are equal up to the action of $G_{\text{sign},\mu}$.
Write $LL(\tilde{t}(1)) = LL(\tilde{t}(2)) = \prod_{i=1}^{\mu} (y - u_i)$ with $(u_1, \ldots, u_\mu)$ in good ordering (definition 3.9 (a)). By lemma 2.3, the $Z$-lattice bundles $\bigcup_{\tau \in C - \{u_1, \ldots, u_\mu\}} H_n(F^{-1}(\tilde{t}(k), Z)$ for $k = 1$ and $k = 2$ are isomorphic.

Near $\tilde{t}(k)$ the path $\rho^{(k)}$ is a lift of a path $r$ as in (6.10). By the construction before theorem 6.5 and lemma 6.6, we obtain vanishing cycle $\delta^{(k)}_i$ and $\delta^{(k)}_{i+1}$ in $H_n(F^{-1}(\tilde{t}(k)(u_i + u_{i+1}^2), Z)$. Because the $Z$-lattice bundles are isomorphic,

$I(\delta^{(1)}_i, \delta^{(1)}_{i+1}) = I(\delta^{(2)}_i, \delta^{(2)}_{i+1})$.

By theorem 6.5 and lemma 6.6, this is either 0 or $\pm 1$, and the first meeting point of the extension of $\rho^{(1)}$ is in $K^{mar}_2$ in the first case and in $K^{mar}_3$ in the second case, and $\rho^{(2)}$ extends to $K^{mar}_2$ in the first case and to $K^{mar}_3$ in the second case. Therefore the isomorphism $\psi^U$ extends to a local isomorphism of F-manifolds beyond this first meeting point.

Therefore (7.6) holds. As $K^{mar,sing}_3 \cup K^{mar,sing}_2$ has codimension two in $M^{mar}$, the extensions of $\psi^U$ to local isomorphisms of F-manifolds along all paths in $M^{mar} - K^{mar,sing}_3 \cup K^{mar,sing}_2$ glue to one global automorphism $\psi^{mar}$ of $M^{mar}$. □

Remarks 7.2. (i) In the case of a simple singularity, the sets $R^0_{Stokes}$, $B(f)/G_{sign,\mu}$ and $\{\text{Stokes matrices}\}/G_{sign,\mu}$ are all finite. $|R^0_{Stokes}| = \deg LL^{alg}$ is finite because the Lyashko-Looijenga map is algebraic. $B(f)$ and the quotient sets are finite because there are only finitely many vanishing cycles, they are the roots of the ADE lattice.

In the case of a simple elliptic singularity, the set $R^0_{Stokes}$ is not finite, because the universal covering $\lambda : \mathbb{H} \to \mathbb{C} - \{0, 1, \infty\}$ has infinite degree. The set $B(f_{1/2})$ of distinguished bases is not finite because the group $G_Z$ acts on it because of (7.2), and the group $G_Z$ is not finite [GH17-1].

The set of Stokes matrices is finite because the entries of each Stokes matrix are in $\{0, \pm 1, \pm 2\}$, see the beginning of subsection 6.2.

Ebeling [Eb18] showed that for all other singularities the set $B(f)$ and the set of Stokes matrices are infinite. Together this gives the picture in table (1.1).

(ii) Theorem 7.1 together with the degrees of $LL^{alg}$ in the theorems 6.1 and 6.3 and in the case of the simple singularities the number $|G_Z|$ allows now to calculate all finite numbers in (1.1). Corollary 7.3 gives the result.

(iii) All numbers in corollary 7.3 except the number of Stokes matrices for $E_8$ had already been determined. Deligne [De74] determined the number $|B(f)|$ for the simple singularities, Yu [Yu90] [Yu96] [Yu99] determined the number $|\{\text{Stokes matrices}\}|$ for the simple singularities.
Kluitmann determined the number $|\{\text{Stokes matrices}\}|$ for the simple elliptic singularities $\widetilde{E}_6$ [Kl83] and $\widetilde{E}_7$ [Kl87]. Deligne and Kluitmann worked directly with the braid group orbits $B(f)$. Their calculations are hard, especially those of Kluitmann. It is satisfying that theorem 7.1 together with $\deg LL_{\text{alg}}$ gives the same numbers $|\{\text{Stokes matrices}\}|$ for $\widetilde{E}_6$ and $\widetilde{E}_7$, and that they allow to find the missing number, the number $|\{\text{Stokes matrices}\}|$ for $\widetilde{E}_8$.

(iv) For the simple singularities, Deligne [De74] and Looijenga [Lo74] proved the bijection (7.3) by comparison of numbers. Looijenga proved that (7.3) is surjective (see the proof of theorem 7.1 for his argument) and calculated $|R^0_{\text{Stokes}}| = \deg LL_{\text{alg}}$. Deligne calculated $|B(f)|/G_{\text{sign, } \mu}$ and observed that it coincides with $|R^0_{\text{Stokes}}|$. Therefore (7.3) is a bijection. But for the simple elliptic singularities, both sides of (7.3) are infinite, and this argument does not work.

(v) For the simple singularities observe

$$|B(f)| = 2^\mu \cdot |B(f)/G_{\text{sign, } \mu}|,$$

(7.7)

For the simple and simple elliptic singularities observe

$$|\{\text{Stokes matrices}\}| = 2^{\mu-1} \cdot |\{\text{Stokes matrices}\}/G_{\text{sign, } \mu}|.$$  

(7.8)

The last equality holds because any Coxeter-Dynkin diagram is connected.

**Corollary 7.3.** For any simple singularity $|B(f)/G_{\text{sign, } \mu}| = \deg LL_{\text{alg}}$, and this number is given in table (6.4). The other numbers are as follows.

| $G_Z$ | $|\{\text{Stokes matrices}\}/G_{\text{sign, } \mu}|$ |
|-------|----------------------------------|
| $A_\mu$ | $2(\mu + 1)$ $(\mu + 1)^{\mu-2}$ |
| $D_4$ | 36 | 9 |
| $D_\mu, \mu \geq 5$ | $4(\mu - 1)$ $(\mu - 1)^{\mu-1}$ |
| $E_6$ | 24 | $2^7 \cdot 3^3 = 3456$ |
| $E_7$ | 18 | $2 \cdot 3^{10} = 118098$ |
| $E_8$ | 30 | $2 \cdot 3^4 \cdot 5^6 = 2531250$ |

(7.9)

In the case of the simple elliptic singularities

| $\deg(M_{\text{alg}} \to M_{\text{mar}}/G_Z)$ | $|\{\text{Stokes matrices}\}/G_{\text{sign, } \mu}|$ |
|-----------------|----------------------------------|
| $\widetilde{E}_6$ | $6 \cdot 2 \cdot 3 \cdot 3^2 = 326$ | $3^7 \cdot 5 \cdot 7 = 76545$ |
| $\widetilde{E}_7$ | $6 \cdot 1 \cdot 4 \cdot 2^2 = 96$ | $2^{13} \cdot 5^3 \cdot 7 = 7168000$ |
| $\widetilde{E}_8$ | $6 \cdot 1 \cdot 6 \cdot 1^2 = 36$ | $2^7 \cdot 3^8 \cdot 7 \cdot 101 = 593744256$ |

(7.10)

Here $\deg(M_{\text{alg}} \to M_{\text{mar}}/G_Z)$ means the generic degree.
**Proof:** First we consider the simple singularities. The bijection (7.3) gives
\[ \deg LL^{alg} = |R_{Stokes}^0| = |\mathcal{B}(f)/G_{sign,\mu}|. \]
The group $G_Z$ acts on $M^{alg} = M^{mar}$ with kernel $\{\pm \text{id}\}$. This and the bijection (7.4) give
\[ |\{\text{Stokes matrices}\}/G_{sign,\mu}| = |R_{Stokes}^0/G_Z| = 2 \cdot |R_{Stokes}^0|/|G_Z| = 2 \cdot \deg LL^{alg}/|G_Z|. \]
The values $|G_Z|$ can be found in [He11, Theorem 8.3 and Theorem 8.4]. Together with (6.4), this gives (7.9).

Now we consider the simple elliptic singularities. Obviously
\[ |R_{Stokes}^0/G_Z| = \deg(LL : M^{mar}/G_Z \to M^{(\mu)}_{LL}) = \frac{\deg LL^{alg}}{\deg(M^{alg} \to M^{mar}/G_Z)}. \]
Therefore the degree of the map $M^{alg} \to M^{mar}/G_Z$ in the second column of (7.10), the bijection (7.4) and the table (6.8) give the third column of (7.10). The degree $\deg(M^{alg} \to M^{mar}/G_Z)$ is calculated in lemma 5.2. \qed

8. Segre classes of smooth cone bundles

The calculation of the degrees $\deg LL^{alg}$ for the simple elliptic singularities in section 10 will use corollary 8.6 below. For this corollary, we have to extend some notions and results in [Fu84]. We do not need new ideas, just some new details. We follow closely this book.

In [Fu84, B.5] cones are defined in the category of algebraic schemes as follows. $X$ is an algebraic scheme. $S^* = \sum_{d \geq 0} S^d$ is a graded sheaf of $\mathcal{O}_X$-algebras such that the canonical map $\mathcal{O}_X \to S^0$ is an isomorphism, $S^1$ is a coherent $\mathcal{O}_X$-module, and $S^*$ is (locally) generated by $S^1$ as an $\mathcal{O}_X$-algebra.

Then $C := \text{Spec}(S^*)$ with the projection $C \to X$ is a cone. Its fibers are affine and come equipped with a $\mathbb{C}^*$-action. The bundle of $\mathbb{C}^*$-orbits is $P(C) := \text{Proj}(S^*)$. The projection $p_C : P(C) \to X$ is proper. The rational functions on $C$ which are homogeneous of degree $d$ induce the line bundle $\mathcal{O}_{P(C)}(d)$.

If $f : Y \to X$ is a morphism, then the pull-back $f^*C = C \times_X Y$ is the cone on $Y$ defined by the sheaf $f^*S^*$ of $\mathcal{O}_Y$-algebras. If $C_1$ and $C_2$ are two cones on $X$ defined by $S_1^*$ and $S_2^*$, their direct sum $C_1 \oplus C_2$ is the cone on $X$ defined by the graded sheaf $S_1^* \otimes S_2^*$. 
Now suppose that the cone $C$ is pure dimensional. Then its Segre class is by \cite{Fu84} Example 4.1.2

$$s(C) = (p_C)_* \left( \sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [P(C)] \right) \in A_* X. \quad (8.1)$$

Here $A_* P(C)$ and $A_* X$ are the spaces of cycles modulo rational equivalence \cite{Fu84} 1.3], $[P(C)] \in A_* P(C)$, $(p_C)_*: A_* P(C) \to A_* X$ is the push-forward \cite{Fu84} 1.4], $\mathcal{O}(1)$ is the canonical line bundle on $P(C)$, and the Chern class $c_1(\mathcal{O}(1))$ is understood in the operational sense, as a map

$$c_1(\mathcal{O}(1)) \cap : A_k P(C) \to A_{k-1} P(C)$$

\cite{Fu84} 3.2].

We are interested in the (more special and more general) situation where the base $X$ is pure dimensional, the fibers are smooth, and the fibration $C \to X$ is locally trivial, but where the condition that $S^1$ generates $S^*$ is not necessarily satisfied. The following definition fixes this situation.

**Definition 8.1.** For some $n \in \mathbb{Z}_{\geq 1}$, let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 1}^n$ with $a_1 \leq a_2 \leq \ldots \leq a_n$. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the $\mathbb{C}$-algebra with the grading $R = R^* = \sum_{d \geq 0} R^d$ such that $x_i \in R^{a_i}$. Let $X$ be an algebraic pure dimensional scheme. Let $S^* = \sum_{d \geq 0} S^d$ be a graded sheaf of $\mathcal{O}_X$-algebras such that there is a covering of $X$ by open affine charts $U_i, i \in I$, and there are isomorphisms $S|_{U_i} \cong \mathcal{O}_{U_i} \otimes R$ of graded $\mathcal{O}_{U_i}$-algebras.

Then $C := \text{Spec}(S^*)$ is a smooth cone bundle (smooth because the fibers of $C \to X$ are smooth, bundle because $C \to X$ is locally trivial).

By the next lemma, a smooth cone bundle comes equipped with a chain of smooth cone subbundles and quotients, which are vector bundles.

**Lemma 8.2.** The situation in definition \text{8.1} is kept. For $k \in \mathbb{Z}$ with $a_1 \leq k \leq a_n + 1$, define $I_k \subset S^*$ as the sheaf of homogeneous ideals generated by $S^1 + \ldots + S^{k-1}$ (so $I_{a_1} = \{0\}$), define $S^*_k := S^*/I_k$ as the quotient sheaf of $\mathcal{O}_X$-algebras with the induced grading, and define $S^*_{k, \text{sub}} \subset S^*_k$ as the subring sheaf of $S^*_k$ generated by $S^d_k$ (obviously $S^d_{k, \text{sub}} = 0$ if $k \not| d$). Define $S^*_k$ essentially as $S^*_{k, \text{sub}}$, but with the new grading $S^d_{k} := S^d_{k, \text{sub}}$.

Then the $C_k := \text{Spec}(S^*_k)$ are smooth cone bundles on $X$ and form a chain

$$(\text{zero section}) = C_{a_n+1} \subset C_{a_n} \subset C_{a_{n-1}} \subset \ldots \subset C_{a_1+1} \subset C_{a_1} = C. \quad (8.2)$$
Proposition 8.4. \(4.1 \] 4.1 \]

The cones \(C_{(k)} := \text{Spec}(S^*_k)\) are vector bundles on \(X\) with rank \(C_{(k)} = |\{i \mid k = a_i\}|\) (so many of them may be 0, and \(\sum_{k=1}^{n} \text{rank } C_{(k)} = \text{rank } C\)). The smooth cone bundle \(C_{k+1}\) is the kernel of the projection \(\text{pr}_k : C_k \rightarrow C_{(k)}\) (from the inclusion \(S^*_{k,\text{sub}} \hookrightarrow S^*_k\)).

The proof is clear.

Given a smooth cone bundle \(C \rightarrow X\), we want to define a Segre class. The rational functions of degree \(d\) on \(C\) induce a sheaf \(\mathcal{O}_{P(C)}(d)\). But \(S^*\) is in general not generated by \(S^1\), therefore \(\mathcal{O}_{P(C)}(1)\) is not necessarily invertible. For example if \(\gcd(a_1, ..., a_n) > 1\) then \(S^d = 0\) and \(\mathcal{O}_{P(C)}(d) = 0\) if \(\gcd(a_1, ..., a_n) \not| d\). But for certain larger \(d\), the sheaf \(\mathcal{O}_{P(C)}(d)\) is good enough.

**Definition 8.3.** The situation in definition 8.1 is kept. Choose \(d \in \text{lcm}(a_1, ..., a_n) \cdot \mathbb{Z}_{\geq 1}\), and define a Segre class

\[
s^{(d)}(C) := (p_C)_* \left( \sum_{i \geq 0} \left( \frac{c_1(\mathcal{O}(d))}{d} \right)^i \cap \frac{[P(C)]}{\gcd(a_1, ..., a_n)} \right) \in A^Q_k X, \quad (8.3)
\]

here \(A^Q_k X = A_k X \otimes \mathbb{Q}\), \(A^Q_k P(C) = A_k P(C) \otimes \mathbb{Q}\), and \(c_1(\mathcal{O}(d)) \cap : A^Q_k XP(C) \rightarrow A^Q_{k-1} P(C)\), \((p_C)_* : A^Q_k P(C) \rightarrow A^Q_k X\).

Part (b) in the following proposition generalizes [Fu84, Proposition 4.1 (a)].

**Proposition 8.4.** The situation in definition 8.1 is kept.

(a) \(s^{(d)}(C)\) is independent of the choice of \(d\) and is called \(s^{(scb)}(C)\). If \(a_1 = ... = a_n = 1\), then \(C\) is a vector bundle and \(s^{(scb)}(C)\) is the classical Segre class in [Fu84, Example 4.1.2].

(b) \(s^{(scb)}(C) = \frac{1}{a_1...a_n} \cdot \prod_{k=1}^{a_n} c_1^{\text{pol}}(C_{(k)})^{-1} \cap [X], \quad (8.4)\)

here \(C_{(k)}\) is the vector bundle associated to \(C\) in lemma 8.2, and \(c_1^{\text{pol}}(C_{(k)}) = c_0 + tc_1 + t^2c_2 + ... \) is its Chern polynomial (in \(8.4\) the variable \(t\) is replaced by the number \(\frac{1}{k}\)).

**Proof:** (a) The independence will follow from the formula (8.4) in (b). If \(a_1 = ... = a_n = 1\), then \(C\) is a vector bundle, \(\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}\), \(c_1(\mathcal{O}(d)) = d \cdot c_1(\mathcal{O}(1))\), and the definition (8.3) agrees with [Fu84, Example 4.1.2].

(b) This will be proved by induction on the dimension \(n\) of the fibers of the smooth cone bundle. We carry out the first step of the induction. Part of it is close to [Fu84, Example 4.1.5].
By the splitting construction [Fu84, Proof of Theorem 3.2], there is a flat morphism \( f : Y \to X \) such that \( f^* : A_*X \to A_*Y \) is injective and \( f^*C(a_1) \) has a filtration by subbundles

\[
f^*C(a_1) = E_r \supset E_{r-1} \supset ... \supset E_1 \supset E_0 = 0 \tag{8.5}
\]

with line bundle quotients \( L_i = E_i/E_{i-1} \) (and \( r = \{ i \mid a_i = a_1 \} \) = \( \text{rk} C(a_1) \)).

The smooth cone bundle \( F := f^*(C) \) in \( Y \) contains the smooth cone bundle \( G := \ker(F \to L_r) \) in codimension one, where \( F \to L_r \) is the composition of the projections \( f^* \text{pr}_{a_1} : F \to E_r \) and \( E_r \to L_r \). Denote by \( \mathcal{L}^* \) the graded sheaf of \( \mathcal{O}_X \)-algebras with \( \mathcal{L}^r \). The projection \( F \to L_r \) with kernel \( G \) corresponds to an embedding \( \mathcal{L}^* \hookrightarrow f^*S^\bullet_{\text{sub}} \subset f^*S^\bullet \). Denote by \( p_F : P(F) \to Y \) the projection.

The line bundle \( (p_F)^*L^{d/a_1}_r \otimes \mathcal{O}_{P(F)}(d) \) has a global section \( \sigma \): If \( U \subset Y \) is an open affine chart and \( f^*S^\bullet|_U = \mathcal{O}_U \otimes \mathcal{R}_e \) is a trivialization and \( \mathcal{L}^*|_U \cong \mathcal{O}_U \otimes \mathbb{C}[x_1] \), then \( \sigma|_U \) is \( (p_F)^*c^d/a_1 \otimes x_1^{d/a_1} \). Its zero-scheme in \( P(F) \) may be identified with \( P(G) \) with multiplicity \( \frac{d}{a_1} \cdot \frac{\gcd(a_1, \ldots, a_n)}{\gcd(a_2, \ldots, a_n)} \cdot [P(G)] = c_1((p_F)^*L^{d/a_1}_r \otimes \mathcal{O}_{P(F)}(d)) \cap [P(F)] \tag{8.6} \)

Here \( \gcd(a_1, \ldots, a_n) \) and \( \gcd(a_2, \ldots, a_n) \) are the sizes of the kernels of the \( \mathbb{C}^* \)-actions on \( F \) and \( G \).

Now we want to calculate \( s^{(d)}(G) \) in terms of \( s^{(d)}(F) \) and the value at \( t = \frac{1}{a_1} \) of the Chern polynomial \( c^1_{\text{pol}}(L_r) \). For this observe that the closed embedding \( i : P(G) \hookrightarrow P(F) \) is proper. The formula \( i^*\mathcal{O}_{P(F)}(d) = \mathcal{O}_{P(G)}(d) \) and the projection formula for Chern classes
\[ \frac{1}{a_1} \cdot s^{(d)}(G) \]

\[ = \frac{1}{a_1} \cdot (p_G)_* \left( \sum_{i \geq 0} \left( \frac{c_1(O_{P(G)}(d))}{d} \right)^i \cap \frac{[P(G)]}{\gcd(a_2, \ldots, a_n)} \right) \]

\[ = \frac{1}{d} \cdot (p_F)_* \left( \sum_{i \geq 0} \left( \frac{c_1(O_{P(F)}(d))}{d} \right)^i \cap c_1((p_F)^*L_r^{\otimes d/a_1} \otimes O_{P(F)}(d)) \right) \]

\[ = (p_F)_* \left( \sum_{i \geq 0} \left( \frac{c_1(O_{P(F)}(d))}{d} \right)^i \cap \frac{1}{a_1}c_1((p_F)^*L_r) + \frac{1}{d}c_1(O_{P(F)}(d)) \right) \]

\[ = (p_F)_* \left( \sum_{i \geq 0} \left( \frac{c_1(O_{P(F)}(d))}{d} \right)^i \cap c_{1/a_1}^{pol}(L_r) \cap \frac{[P(F)]}{\gcd(a_1, \ldots, a_n)} \right) \]

\[ = c_{1/a_1}^{pol}(L_r) \cap s^{(d)}(F). \]

In the second to last equality, the term \((p_F)_* \left( (c_0(p_F)^*(L_r) \cap \frac{[P(F)]}{\gcd(a_w, \ldots, a_n)} \right)\) was added. This term vanishes, as \([P(F)] \in A_{\text{dim}P(F)} \) is mapped by \((p_F)_*\) to \(A_{\text{dim}P(F)}Y\), which is zero because \(\text{dim } P(F) = \text{dim } P(G) + 1 \geq \text{dim } Y + 1\). Therefore

\[ s^{(d)}(F) = \frac{1}{a_1} \cdot c_{1/a_1}^{pol}(L_r)^{-1} \cap s^{(d)}(G). \quad (8.7) \]
By induction and the product formula $c_t^{\text{pol}}(L_r) \cdot c_t^{\text{pol}}(E_{r-1}) = c_t^{\text{pol}}(f^*C(a_1))$ we obtain

$$s^{(d)}(F) = \frac{1}{a_1^r} \cdot \prod_{j=1}^r c_{1/a_1 j}^{\text{pol}}(L_j)^{-1} \cap s^{(d)}(f^*C(a_1+1))$$

$$= \frac{1}{a_1} \cdot c_{1/a_1}^{\text{pol}}(f^*C(a_1))^{-1} \cap s^{(d)}(f^*C(a_1+1))$$

$$= \frac{1}{a_1 a_1 \cdots a_n} \cdot \prod_{k=a_1}^{a_n} c_{1/k}^{\text{pol}}(f^*C(k))^{-1} \cap [Y]$$

$$= \frac{1}{a_1 a_1 \cdots a_n} \cdot f^* \left( \prod_{k=a_1}^{a_n} c_{1/k}^{\text{pol}}(C(k))^{-1} \right) \cap [X]. (8.8)$$

Now the injectivity of $f^*: A_* X \to A_* Y$ gives

$$s^{(d)}(C) = \frac{1}{a_1 a_1 \cdots a_n} \cdot \prod_{k=a_1}^{a_n} c_{1/k}^{\text{pol}}(C(k))^{-1} \cap [X]. (8.9)$$

It rests to settle the beginning of the induction. Consider the case $n = 1$ and choose $d \in a_1 \cdot \mathbb{Z}_{\geq 1}$. Then $P(C) = X$, $p_C = \text{id}$, $C = C(a_1) \cong \mathcal{O}_{P(C)}(-a_1)$, and

$$\sum_{i \geq 0} \left( \frac{c_1(\mathcal{O}_X(d))}{d} \right)^i = \left( 1 - \frac{1}{d} c_1(\mathcal{O}_X(d)) \right)^{-1} = \left( 1 + \frac{1}{d} c_1(\mathcal{O}_X(-d)) \right)^{-1}$$

$$= \left( 1 + \frac{1}{a_1} c_1(\mathcal{O}_X(-a_1)) \right)^{-1} = c_{1/a_1}^{\text{pol}}(C(a_1))^{-1}.$$  

Therefore

$$s^{(d)}(C) = (p_C)_* \left( \sum_{i \geq 0} \left( \frac{c_1(\mathcal{O}_P(C))}{d} \right)^i \cap \frac{[P(C)]}{a_1} \right)$$

$$= \frac{1}{a_1} \cdot (c_{1/a_1}^{\text{pol}}(C(a_1)))^{-1} \cap [X]. \quad \square$$

As in [Fu84], a variety means a reduced and irreducible scheme. A smooth cone bundle $C \to X$ on a variety is also a variety. In the following $X \subset C$ means the embedding as zero section.

Suppose that $C_1 \to X_1$ and $C_2 \to X_2$ are smooth cone bundles on complete varieties $X_1$ and $X_2$ with $\dim C_1 = \dim C_2$ and $\dim X_1 \geq \dim X_2$, and suppose that $f: C_1 \to C_2$ is a $\mathbb{C}^*$-equivariant proper morphism such that $f^{-1}(X_2) = X_1$ and the restriction $f^{C-X}: C_1$
$X_1 \to C_2 - X_2$ is finite. Then $f$ and the restrictions $f^{C-X}$ and $f^X : X_1 \to X_2$ are surjective, and $f$ has a finite degree $\deg f = [K(C_1) : K(C_2)]$, which is the number of preimages of a generic point in $C_2 - X_2$. In [Fu84, Definition 1.4] a degree $\int_X^*(\sigma_{\text{scb}})_{(C_1)} = \deg f \cdot \int_X^*(\sigma_{\text{scb}})_{(C_2)}$, which is the number of preimages of a generic point in $C_2 - X_2$. In [Fu84, Definition 1.4] a degree $\int_X^*(\sigma_{\text{scb}})_{(C_1)} = \deg f \cdot \int_X^*(\sigma_{\text{scb}})_{(C_2)}$.

**Proposition 8.5.** In the situation just described

\[ f^X \cdot \sigma \text{scb}_{(C_1)} = \deg f \cdot \sigma \text{scb}_{(C_2)}, \quad (8.10) \]

\[ \int_{X_1} \sigma \text{scb}_{(C_1)} = \deg f \cdot \int_{X_2} \sigma \text{scb}_{(C_2)}. \quad (8.11) \]

**Proof:** Denote by $\mathbf{a} = (a_1, \ldots, a_{n_1})$ respectively by $\mathbf{v} = (v_1, \ldots, v_{n_2})$ the weights of $C_1$ respectively $C_2$. Denote $\bar{w} := \gcd(a_i), \bar{v} := \gcd(v_i), d_1 := \text{lcm}(a_i), d_2 := \text{lcm}(v_i)$. Because $f$ is $\mathbb{C}^*$-equivariant and does not map $C_1$ to $X_2$, $\bar{w}$ divides $\bar{v}$. The map $f^{C-X}$ induces a finite (and surjective) morphism $f^{PC} : P(C_1) \to P(C_2)$ with

\[ \deg f^{PC} = \left( \frac{\bar{v}}{\bar{w}} \right)^{-1} \cdot \deg f. \]

Furthermore

\[ (f^{PC})^* \mathcal{O}_{P(C_2)}(d) = \mathcal{O}_{P(C_1)}(d), \]

\[ (f^{PC})_* [P(C_1)] = \deg f^{PC} \cdot [P(C_2)]. \]

Choose $d \in \text{lcm}(d_1, d_2) \cdot \mathbb{Z}_{\geq 1}$. Then

\[ \mathcal{O}_{P(C_1)}(d) = \mathcal{O}_{P(C_1)}(d_i)^{\otimes d/d_i}, \quad \frac{1}{d_i} c_1(\mathcal{O}_{P(C_1)}(d_i)) = \frac{1}{d} c_1(\mathcal{O}_{P(C_1)}(d)). \]
Therefore
\[ f_*^X s^{(d)}(C_1) = f_*^X \circ (p_{C_1})_* \left( \sum_{i \geq 0} \left( \frac{c_1(\mathcal{O}_{P(C_1)}(d))}{d} \right)^i \cap \left[ \frac{P(C_1)}{\tilde{w}} \right] \right) \]
\[ = (p_{C_2})_* \circ (f_{P_{C}})_* \left( \sum_{i \geq 0} \left( \frac{c_1((f_{P_{C}})^*\mathcal{O}_{P(C_2)}(d))}{d} \right)^i \cap \left[ \frac{P(C_1)}{\tilde{w}} \right] \right) \]
\[ = (p_{C_2})_* \left( \sum_{i \geq 0} \left( \frac{c_1(\mathcal{O}_{P(C_2)}(d))}{d} \right)^i \cap \left[ \frac{P(C_1)}{\tilde{w}} \right] \right) \]
\[ = \deg f \cdot (p_{C_2})_* \left( \sum_{i \geq 0} \left( \frac{c_1(\mathcal{O}_{P(C_2)}(d))}{d} \right)^i \cap \left[ \frac{P(C_2)}{\tilde{w}} \right] \right) \]
\[ = \deg f \cdot s^{(d)}(C_2). \]

With the functoriality \( \int_{X_1} \alpha = \int_{X_2} f_*^X \alpha \) [Fu84, Definition 1.4], we obtain
\[ \int_{X_1} s^{(scb)}(C_1) = \deg f \cdot \int_{X_2} s^{(scb)}(C_2). \quad \square \]

If \( \int_{X_2} s^{(scb)}(C_2) \neq 0 \), then (8.11) can be used to calculate \( \deg f \). We will use it in the following case.

**Corollary 8.6.** Keep the situation in and before proposition 8.5. Suppose additionally that \( X_1 \) is a smooth complete curve and \( X_2 \) is a point.

The weights of \( C_1 \) are denoted \((a_1, \ldots, a_{n_1})\), the weights of \( C_2 \) are denoted \((b_1, \ldots, b_{n_2})\). The vector bundles on \( X_1 \) associated to \( C_1 \) in lemma 8.2 are denoted by \( C_{1,(k)} \), \( a_1 \leq k \leq a_n \). Then \( n_2 = n_1 + 1 \),

\[ \int_{X_2} s^{(scb)}(C_2) = \frac{1}{b_1 \ldots b_{n_2}} > 0, \quad (8.12) \]
\[ \int_{X_1} s^{(scb)}(C_1) = \frac{1}{a_1 \ldots a_{n_1}} \cdot \left( -\sum_{k=a_1}^{a_n} \frac{1}{k} \deg C_{1,(k)} \right), \quad (8.13) \]
\[ \deg f = \frac{b_1 \ldots b_{n_2}}{a_1 \ldots a_{n_1}} \cdot \left( -\sum_{k=a_1}^{a_n} \frac{1}{k} \deg C_{1,(k)} \right). \quad (8.14) \]

**Proof:** This follows with proposition 8.4 (b) and proposition 8.5 \( \square \)
9. Extension to $\lambda = 0$ of the Lyashko-Looijenga map for the simple elliptic singularities

Here we will do the first and biggest step in the proof of theorem 6.3. The $\lambda$-parameter space $\mathbb{C} - \{0, 1\}$ contains the punctured disk $\Delta^* = \Delta - \{0\}$, where $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$. Define $c := 3$ for $E_6$ and $E_8$ and $c := 2$ for $E_7$, and define the $c$-fold coverings

$$
\rho_{\text{naive}} : \mathbb{C}^{n-1} \times \Delta^* \to \mathbb{C}^{n-1} \times \Delta^*, \quad (t', \kappa) \mapsto (t', \kappa^c) = (t', \lambda),
$$

$$
R_{\text{naive}} : \mathbb{C}^{n+1} \times \mathbb{C}^{n-1} \times \Delta^* \to \mathbb{C}^{n+1} \times \mathbb{C}^{n-1} \times \Delta^*, \quad (x, t', \kappa) \mapsto (x, t', \kappa^c) = (x, t', \lambda).
$$

We will glue fibers above $\kappa = 0$ into $\mathbb{C}^{n+1} \times \mathbb{C}^{n-1} \times \Delta^*$ and $\mathbb{C}^{n-1} \times \Delta^*$ such that $F_{\text{alg}} \circ R_{\text{naive}}$, its critical space and its Lyashko-Looijenga map extend well to $\kappa = 0$. This is the content of the following theorem 9.1 and its long proof.

**Theorem 9.1.** Consider for each of the three families of simple elliptic singularities and their unfoldings the following spaces and maps.

For $E_6$:

$$(x, y, s, \kappa) = (x_0, ..., x_n, y_0, y_1, y_2, s_1, ..., s_7, \kappa) \in \mathbb{C}^{n+1} \times \mathbb{C}^3 \times \mathbb{C}^7 \times \Delta = \mathbb{C}^{n+11} \times \Delta^*,
Y := \{(x, y, s, \kappa) \in \mathbb{C}^{n+11} \times \Delta \mid x_0(x_1 + s_5) = \kappa y_0, (x_1 + s_5)y_0 = \kappa y_1, x_0x_2 = \kappa^2 y_2\},
pr_\mu : Y \to \mathbb{C}^7 \times \Delta, (x, y, s, \kappa) \mapsto (s, \kappa),$$

$$
\rho : \mathbb{C}^7 \times \Delta^* \to \mathbb{C}^7 \times \Delta^*, \quad (s, \kappa) \mapsto (s_1, \kappa^2 s_2 + \kappa s_5 s_6 - s_5^2, s_3, s_4, s_7, \kappa^3),
$$

$$
R : Y \cap \mathbb{C}^{n+11} \times \Delta^* \to \mathbb{C}^{n+8} \times \Delta^*, \quad (x, y, s, \kappa) \mapsto (\kappa^{-2} x_0, x_1, ..., x_n, \rho(s, \kappa)).
$$

For $E_7$:

$$(x, y, s, \kappa) = (x_0, ..., x_n, y, s_1, ..., s_8, \kappa) \in \mathbb{C}^{n+1} \times \mathbb{C} \times \mathbb{C}^8 \times \Delta = \mathbb{C}^{n+10} \times \Delta^*,
Y := \{(x, y, s, \kappa) \in \mathbb{C}^{n+10} \times \Delta \mid x_0x_1 = \kappa y\},
pr_\mu : Y \to \mathbb{C}^8 \times \Delta, (x, y, s, \kappa) \mapsto (s, \kappa),$$

$$
\rho : \mathbb{C}^8 \times \Delta^* \to \mathbb{C}^8 \times \Delta^*, \quad (s, \kappa) \mapsto (s_1, \kappa s_2 + \kappa s_5 s_6 - s_5^2, s_3, s_4, s_7, \kappa^3),
$$

$$
R : Y \cap \mathbb{C}^{n+10} \times \Delta^* \to \mathbb{C}^{n+7} \times \Delta^*, \quad (x, y, s, \kappa) \mapsto (\kappa^{-2} x_0, x_1, ..., x_n, \rho(s, \kappa)).
$$
\[ \rho : \mathbb{C}^8 \times \Delta^* \rightarrow \mathbb{C}^8 \times \Delta^*, \]
\[ (s, \kappa) \mapsto (s_1, \kappa s_2, s_3, \kappa^2 s_4, s_5, s_6, \kappa s_7, s_8, \kappa^2), \quad (9.5) \]
\[ R : Y \cap \mathbb{C}^{n+10} \times \Delta^* \rightarrow \mathbb{C}^{n+9} \times \Delta^*, \]
\[ (x, y, s, \kappa) \mapsto (\kappa^{-1}x_0, x_1, ..., x_n, \rho(s, \kappa)). \quad (9.6) \]

For \( \widetilde{E}_8 \):
\[ (x, y, s, \kappa) = (x_0, ..., x_n, y, s_1, ..., s_9, \kappa) \]
\[ \in \mathbb{C}^{n+1} \times \mathbb{C} \times \mathbb{C}^9 \times \Delta = \mathbb{C}^{n+11} \times \Delta^*, \]
\[ Y := \{(x, y, s, \kappa) \in \mathbb{C}^{n+11} \times \Delta \mid (x_0 - \frac{1}{2}s_9)x_1 = \kappa y\}. \quad (9.7) \]
\[ \text{pr}_\mu : Y \rightarrow \mathbb{C}^9 \times \Delta, (x, y, s, \kappa) \mapsto (s, \kappa), \]
\[ \rho : \mathbb{C}^9 \times \Delta^* \rightarrow \mathbb{C}^9 \times \Delta^*, \]
\[ (s, \kappa) \mapsto (s_1, \kappa s_2, \kappa^2 s_3, \]
\[ s_4 - \frac{1}{2}\kappa^{-1}s_6 s_9 - \frac{1}{4}\kappa^{-1}s_7 s_9^2 - \frac{1}{16}\kappa^{-1}s_9^4, \kappa^3 s_5, \]
\[ s_6, \kappa s_7, s_8 = \frac{1}{4}\kappa^{-2} s_9^2, \kappa^{-1} s_9, \kappa^3) \].

For \( \mathbb{E}_7 \):
\[ R : Y \cap \mathbb{C}^{n+11} \times \Delta^* \rightarrow \mathbb{C}^{n+10} \times \Delta^*, \]
\[ (x, y, s, \kappa) \mapsto (\kappa^{-1}x_0, x_1, ..., x_n, \rho(s, \kappa)). \quad (9.9) \]

(a) The \( \mathbb{C}^* \)-action on \( \mathbb{C}^{n+11} \times \Delta \) for \( \widetilde{E}_6 \) and \( \widetilde{E}_8 \) and on \( \mathbb{C}^{n+10} \times \Delta \) for \( \mathbb{E}_7 \) with the following weights restricts to a \( \mathbb{C}^* \)-action on \( Y \),
\[ \deg_w x_i = w_i, \ \deg_w s_i = \deg_w t_i, \ \deg_w \kappa = \deg_w \lambda = 0, \]
for \( \mathbb{E}_7 \) and \( \mathbb{E}_8 \):
\[ \deg_w y = w_0 + w_1, \]
for \( \mathbb{E}_6 \):
\[ \begin{cases} \deg_w y_0 = w_0 w_1, \ \deg_w y_1 = w_0 + 2 w_1, \\ \deg_w y_2 = w_0 + 2 w_2. \end{cases} \quad (9.10) \]

The map \( R \) is \( \mathbb{C}^* \)-equivariant with respect to this \( \mathbb{C}^* \)-action and the natural \( \mathbb{C}^* \)-action on the image space with coordinates \( (x, t', \lambda) \).

(b) The maps \( \rho \) and \( R \) are coverings of degree \( c \). Especially, for each fixed \((s, \kappa) \in \mathbb{C}^{n-1} \times \Delta^* \),
\[ R : \text{pr}_\mu^{-1}((s, \kappa)) \xrightarrow{\sim} \mathbb{C}^{n+1} \times \{\rho(s, \kappa)\}. \quad (9.11) \]

(c) The pull back \( F^{\text{alg}} \circ R \) extends from \( Y \cap \mathbb{C}^{n+(10 \text{ or } 11)} \times \Delta^* \) holomorphically to \( \kappa = 0 \), that means, to a function \((F^{\text{alg}} \circ R)^{\text{ext}} : Y \rightarrow \mathbb{C}\).

(d) Let \( C^{\text{alg}} \) := \( \{(x, t', \lambda) \in \mathbb{C}^{n+1} \times M^{\text{alg}} \mid \frac{\partial F^{\text{alg}}}{\partial t'} = \frac{\partial F^{\text{alg}}}{\partial \lambda} = 0\} \) be the critical space of the unfolding \( F^{\text{alg}} \). Consider the closure \( R^{-1}(C^{\text{alg}}) \) in
of the pull back by $R$ of $C^{\text{alg}} \cap \mathbb{C}^{n+1} \times \mathbb{C}^{\mu-1} \times \Delta^*$. The restriction
\[ \text{pr}_\mu : R^{-1}(C^{\text{alg}}) \to \mathbb{C}^{\mu-1} \times \Delta, \quad (x, y, s, \kappa) \mapsto (s, \kappa) \quad (9.12) \]
is finite and flat of degree $\mu$.

(e) The composition $LL^{\text{alg}} \circ \rho : \mathbb{C}^{\mu-1} \times \Delta^* \to M^{(\mu)}_{LL}$ of the Lyashko-Looijenga map $LL^{\text{alg}}$ with $\rho$ extends holomorphically to $\mathbb{C}^{\mu-1} \times \Delta$. The restriction
\[ (LL^{\text{alg}} \circ \rho)^{\text{ext}} : \mathbb{C}^{\mu-1} \times \Delta - \{(s, \kappa) \mid s_2 = \ldots = s_{\mu-1} = 0\} \to M^{(\mu)}_{LL, 0} \]
is finite and flat onto its image. And $\mathbb{C}^{\mu-1} \times \{0\}$ is mapped by $(LL^{\text{alg}} \circ \rho)^{\text{ext}}$ to $D^{(\mu)}_{LL}$.

The rest of this section is devoted to the proof of this theorem.

**Proof:** (a) This follows from comparison of the formulas $(9.1)$ to $(9.9)$ with the weights in remark 6.4 (ii) and in $(9.10)$.

(b) The definition of $Y$ shows that for $\kappa \in \Delta^*$ ($x_0, \ldots, x_n$) serve as coordinates on $\text{pr}_\mu^{-1}((s, \kappa))$ and that this is isomorphic to $\mathbb{C}^{n+1}$.

The following three statements show that $\rho$ and $R$ are coverings of degree $c$. The last component of $\rho$ is $\rho_\mu(s, \kappa) = \kappa^c$. Each other component $\rho_i$ has a nonvanishing linear term in $s_i$ (and in the case of $\tilde{E}_6$ the linear terms of $\rho_5$ and $\rho_6$ are $\kappa^3 s_5$ and $\kappa s_6 - 2 s_5$). The map $R$ restricts to a linear isomorphism $R : \text{pr}_\mu^{-1}((s, \kappa)) \to \mathbb{C}^{n+1} \times \{\rho((s, \kappa))\}$ for $(s, \kappa) \in \mathbb{C}^{\mu-1} \times \Delta^*$.

(c) The pull back $F^{\text{alg}} \circ R = (f_\lambda(x) + \sum_{i=1}^{\mu-1} m_i t_i) \circ R$ can be written as follows.

For $\tilde{E}_6$:
\[ F^{\text{alg}} \circ R = \kappa^3 \kappa^{-4} x_0^2 x_1 - (\kappa^3 + 1) \kappa^{-2} x_0 x_1^2 + x_1^3 - \kappa^{-2} x_0 x_2^2 + \sum_{i=3}^{n} x_i^2 \]
\[ + s_1 + \kappa^{-2} x_0(\kappa^2 s_2 + \kappa s_5 s_6 - s_5^2) + x_1 s_3 + x_2 s_4 \]
\[ + \kappa^{-4} x_0^2 \kappa^3 s_5 + \kappa^{-2} x_0 x_1(\kappa s_6 - 2 s_5) + x_1 x_2 s_7 \]
\[ = \kappa^{-1} x_0(x_0 + s_6)(x_1 + s_5) - \kappa^{-2} x_0(x_1 + s_5)^2 - \kappa x_0 x_1^2 + x_1^3 \]
\[ - \kappa^{-2} x_0 x_2^2 + \sum_{i=3}^{n} x_i^2 + s_1 + x_0 s_2 + x_1 s_3 + x_2 s_4 + x_1 x_2 s_7 \]
\[ = (x_0 + s_6)y_0 - y_1 - \kappa x_0 x_1^2 + x_1^3 - y_2 + \sum_{i=3}^{n} x_i^2 \]
\[ + s_1 + x_0 s_2 + x_1 s_3 + x_2 s_4 + x_1 x_2 s_7. \quad (9.14) \]
For $\tilde{E}_7$:

$$F^{alg} \circ R = \kappa^2 \kappa^{-3} x_0^3 x_1 - (\kappa^2 + 1) \kappa^{-2} x_0^2 x_1 + \kappa^{-1} x_0 x_1^2 + \sum_{i=2}^{n} x_i^2$$

$$+ s_1 + \kappa^{-1} x_0 \kappa s_2 + x_1 s_3 + \kappa^{-2} x_0^2 \kappa^2 s_4$$

$$+ \kappa^{-1} x_0 x_1 s_5 + x_1^2 s_6 + \kappa^{-2} x_0^2 x_1 \kappa s_7 + \kappa^{-1} x_0 x_1^2 s_8$$

$$= x_0^2 y - (\kappa^2 + 1) y^2 + x_1^2 y + \sum_{i=2}^{n} x_i^2 \quad (9.15)$$

$$+ s_1 + x_0 s_2 + x_1 s_3 + x_0^2 s_4 + y s_5 + x_1^2 s_6 + x_0 y s_7 + x_1 y s_8.$$  

For $\tilde{E}_8$:

$$F^{alg} \circ R = \kappa^3 \kappa^{-4} x_0^4 x_1 - (\kappa^3 + 1) \kappa^{-2} x_0^2 x_1 + x_1^2 + \sum_{i=2}^{n} x_i^2$$

$$+ s_1 + \kappa^{-1} x_0 \kappa s_2 + \kappa^{-2} x_0^2 \kappa^2 s_3$$

$$+ x_1 (s_4 - \frac{1}{2} \kappa^{-1} s_6 s_9 - \frac{1}{4} \kappa^{-1} s_7 s_6 - \frac{1}{16} \kappa^{-1} s_9)$$

$$+ \kappa^{-3} x_0^3 \kappa s_5 + \kappa^{-1} x_0 x_1 s_6 + \kappa^{-2} x_0^2 x_1 \kappa s_7$$

$$+ x_1^2 (s_8 - \frac{1}{4} \kappa^{-2} s_9) + \kappa^{-1} x_0 x_1^2 \kappa^{-1} s_9$$

$$= (x_0^3 + x_0^2 \frac{1}{2} s_9 + x_0 \frac{1}{4} s_6^2 + \frac{1}{8} s_9 + (x_0 + \frac{1}{2} s_9) s_7 + s_6) y$$

$$- \kappa x_0^2 x_1^2 - y^2 + x_1^3 + \sum_{i=2}^{n} x_i^2 \quad (9.16)$$

$$+ s_1 + x_0 s_2 + x_0^2 s_3 + x_1 s_4 + x_0^3 s_5 + x_1^2 s_8.$$  

In all three cases, the terms after the last equality sign are in $\mathbb{C}[x, y, s, \kappa]$.

(d) We call $y$-relations the elements

$$\begin{align*}
&x_0 (x_1 + s_5) - \kappa y_0, \ (x_1 + s_5) y_0 - \kappa y_1, \quad \text{for } \tilde{E}_6, \\
&x_0 (x_1 + s_5)^2 - \kappa^2 y_1, x_0 x_1^2 - \kappa^2 y_2
\end{align*}$$

$$x_0 x_1 - \kappa y \quad \text{for } \tilde{E}_7, \quad (9.17)$$

$$(x_0 - \frac{1}{2} s_9) x_1 - \kappa y \quad \text{for } \tilde{E}_8$$

in $\mathbb{C}[x, y, s, \kappa, \kappa^{-1}]$ and in $\mathbb{C}[x, y, s, \kappa]$. The compositions $\frac{\partial F^{alg}}{\partial x_i} \circ R$ of the partial derivatives of $F^{alg}$ with $R$ are in $\mathbb{C}[x, y, s, \kappa, \kappa^{-1}]$. We consider
the following ideals,

\[
I_0 := \left( \frac{\partial F^{alg}}{\partial x_i} \circ R, y\text{-relations} \right) \subset \mathbb{C}[x, y, s, \kappa, \kappa^{-1}],
\]

\[
I_1 := I_0 \cap \mathbb{C}[x, y, s, \kappa],
\]

\[
I_2 := \{ g(x, y, s, 0) \mid g(x, y, s, \kappa) \in I_1 \} \subset \mathbb{C}[x, y, s], \tag{9.18}
\]

\[
I_3 := \{ g(x, y, 0) \mid g(x, y, s) \in I_2 \} \subset \mathbb{C}[x, y].
\]

We will calculate generating elements of these ideals. Then we will show \( \dim \mathbb{C}[x, y]/I_3 = \mu \). This is sufficient for (d) because of the following. \( C^{alg} \subset \mathbb{C}^{n+1} \times M^{alg} \) and \( R^{-1}(C^{alg}) \subset Y \) are invariant under the \( \mathbb{C}^* \)-actions. Therefore it is sufficient to show that the restriction of (9.12) to \( s = 0 \) is finite and flat of degree \( \mu \). This holds above \( \Delta^* \). For \( \kappa = 0 \) is equivalent to \( \dim \mathbb{C}[x, y]/I_3 = \mu \).

\( I_1 \) determines \( R^{-1}(C^{alg}) \subset Y \), and \( I_2 \) determines \( R^{-1}(C^{alg}) \subset Y \cap \mathbb{C}^{n+10} \times \{0\} \). The information below on \( I_2 \) will also be useful in the proof of part (e).

**The case \( \tilde{E}_6 \):**

\[
\frac{\partial F^{alg}}{\partial x_0} \circ R = 2\kappa x_0 x_1 - (\kappa^3 + 1)x_1^2 - x_2^2 + (\kappa^2 s_2 + \kappa s_5 s_6 - s_5^2 - (\kappa^3 + 1)\kappa - 2s_5)
\]

\[
+ 2\kappa x_0 s_5 + x_1(\kappa s_6 - 2s_5)
\]

\[
y\text{-relations} \equiv \begin{array}{l}
2\kappa^2 y_0 - \kappa^3 x_1^2 - (x_1 + s_5 - \kappa s_6)(x_1 + s_5)
\end{array}
\]

\[
- x_2^2 + \kappa^2 s_2.
\]

\[
\frac{\partial F^{alg}}{\partial x_0} \circ R \& y\text{-relations} \Rightarrow (x_1 + s_5)^2 + x_2^2 \in I_2,
\]

\[
x_0 \cdot \frac{\partial F^{alg}}{\partial x_0} \circ R \& y\text{-relations} \Rightarrow 2x_0 y_0 - y_1 + y_0 s_6 - y_2 + x_0 s_2 \in I_2.
\]

\[
\frac{\partial F^{alg}}{\partial x_1} \circ R = \kappa^{-1} x_0^2 - 2(\kappa^3 + 1)\kappa^{-2} x_0 x_1 + 3x_1^2 + s_3 + \kappa^{-2} x_0(\kappa s_6 - 2s_5) + x_2 s_7
\]

\[
y\text{-relations} \equiv \kappa^{-1} x_0^2 - 2\kappa x_0 x_1 - 2\kappa^{-1} y_0 + 3x_1^2 + s_3 + \kappa^{-1} x_0 s_6 + x_2 s_7.
\]
\[
\frac{\partial F^{\text{alg}}}{\partial x_1} \circ R \text{ & } y\text{-relations} \Rightarrow x_0^2 - 2y_0 + x_0s_6 \in I_2,
\]
\[
(x_1 + s_5) \cdot \frac{\partial F^{\text{alg}}}{\partial x_1} \circ R \text{ & } y\text{-relations} \Rightarrow x_0y_0 - 2y_1 + 3(x_1 + s_5)x_1^2
\]
\[
+ (x_1 + s_5)s_3 + y_0s_6 \quad + (x_1 + s_5)x_2s_7 \in I_2. \quad (9.19)
\]
\[
\frac{\partial F^{\text{alg}}}{\partial x_2} \circ R = -2\kappa^{-2}x_0x_2 + s_4 + x_1s_7.
\]
\[
x_2 \cdot \frac{\partial F^{\text{alg}}}{\partial x_2} \circ R \text{ & } y\text{-relations} \Rightarrow -2y_2 + x_2s_4 + x_1x_2s_7 \in I_2.
\]
\[
x_2 \cdot \frac{\partial F^{\text{alg}}}{\partial x_1} \circ R \text{ & } \frac{\partial F^{\text{alg}}}{\partial x_2} \circ R \text{ & } x_0 \cdot \frac{\partial F^{\text{alg}}}{\partial x_2} \circ R
\]
\[
\Rightarrow -(x_1 + s_5)(s_4 + x_1s_7) + 3x_1^2x_2 + x_2s_3 + x_2^2s_7 \in I_2.
\]
\[
y\text{-relation } x_0(x_1 + s_5) - \kappa y_0 \Rightarrow x_0(x_1 + s_5) \in I_2.
\]

This gives the following \( n + 2 \) elements of \( I_2 \). The first three elements express \( y_0, y_1 \) and \( y_2 \) in terms of \( (x, s) \), the last element is calculated from these three elements and from \((9.19)\).

\[-2y_0 + x_0(x_0 + s_6), \ -2y_2 + x_2(s_4 + x_1s_7), \]
\[-y_1 - y_2 + (2x_0 + s_6)y_0 + x_0s_2, \ x_3, ..., x_n, \ x_0(x_1 + s_5), \ x_0x_2, \]
\[(x_1 + s_5)^2 + x_2^2, \ 3x_1^2x_2 - (x_1 + s_5)(s_4 + x_1s_7) + x_2s_3 + x_2^2s_7, \]
\[-\frac{1}{2}x_0(x_0 + s_6)(3x_0 + s_6) - 2x_0s_2 + 3x_1^2(x_1 + s_5) \]
\[+ x_2(s_4 + x_1s_7) + (x_1 + s_5)s_3 + (x_1 + s_5)x_2s_7. \quad (9.20)\]

Restriction to \( s = 0 \) gives the following \( n + 2 \) elements of \( I_3 \).

\[-2y_0 + x_0^2, \ y_2, \ -y_1 + 2x_0y_0, \ x_3, ..., x_n, \ x_0x_1, \ x_0x_2, \]
\[x_1^2 + x_2^2, \ x_1x_2, \ -x_0^3 + 2x_1^3. \quad (9.21)\]

Therefore the monomials

\[1, \ x_0, \ x_1, \ x_2, \ x_0^2, \ x_1^2, \ x_1x_2, \ x_0^3 \]

generate the quotient \( \mathbb{C}[x, y]/I_3 \). As this quotient cannot have dimension less than 8, it has dimension 8, the elements in \((9.21)\) generate \( I_3 \), and the elements in \((9.21)\) generate \( I_2 \).
The case $\tilde{E}_7$:

$\frac{\partial F^\text{alg}}{\partial x_0} \circ R = 3x_0^2x_1 - 2(\kappa^2 + 1)\kappa^{-1}x_0x_1^2 + x_1^3$

$+ \kappa s_2 + 2\kappa x_0s_4 + x_1s_5 + 2x_0x_1s_7 + x_1^2s_8$

$\equiv \partial F^\text{alg} \circ R \& y$-relation

$y$-relation

$\frac{\partial F^\text{alg}}{\partial x_0} \circ R \circ y \Rightarrow -2x_1y + x_1^3 + x_1s_5 + x_1^2s_8 \in I_2,$

$x_0 \cdot \frac{\partial F^\text{alg}}{\partial x_0} \circ R \& y$-relation

$3x_0^2y - 2y^2 + x_1^2y + x_0s_2 + 2x_0^2s_4$

$+ y s_5 + 2x_0ys_7 + x_1ys_8 \in I_2.$

$\frac{\partial F^\text{alg}}{\partial x_1} \circ R = \kappa^{-1}x_0^3 - 2(\kappa^2 + 1)\kappa^{-2}x_0^2x_1 + 3\kappa^{-1}x_0x_1^2$

$+ s_3 + \kappa^{-1}x_0s_5 + 2x_1s_6 + \kappa^{-1}x_0^2s_7 + 2\kappa^{-1}x_0x_1s_8$

$\equiv \kappa^{-1}x_0^3 - 2(\kappa^2 + 1)\kappa^{-1}x_0y + 3x_1y$

$+ s_3 + \kappa^{-1}x_0s_5 + 2x_1s_6 + \kappa^{-1}x_0^2s_7 + 2ys_8.$

$\frac{\partial F^\text{alg}}{\partial x_1} \circ R \& y$-relation

$x_0^3 - 2x_0y + x_0s_5 + x_0^2s_7 \in I_2,$

$x_1 \cdot \frac{\partial F^\text{alg}}{\partial x_1} \circ R \& y$-relation

$x_0^2y - 2y^2 + 3x_1^2y + x_1s_3 + y s_5$

$+ 2x_1^2s_6 + x_0ys_7 + 2x_1ys_8 \in I_2.$

$y$-relation $x_0x_1 - \kappa y_0 \Rightarrow x_0x_1 \in I_2.$

This gives the following $n + 4$ elements of $I_2$.

$x_2, ..., x_n, x_0x_1, -2x_0y + x_0^3 + x_0s_5 + x_0^2s_7,$

$-2x_1y + x_1^3 + x_1s_5 + x_1^2s_8,$

$-4y^2 + (4x_0^2 + 4x_1^2 + 3x_0s_7 + 3x_1s_8 + 2s_5)y$

$+ x_0s_2 + x_1s_3 + 2x_0^2s_4 + 2x_1^2s_6,$

$(2x_0^2 - 2x_1^2 + x_0s_7 - x_1s_8)y + x_0s_2 - x_1s_3 + 2x_0^2s_4 - 2x_1^2s_6.$
Restriction to $s = 0$ gives the following $n + 4$ elements of $I_3$.

\[\begin{align*}
x_2, \ldots, x_n, x_0 x_1, -2x_0 y + x_0^3, -2x_1 y + x_1^3, \\
y^2 + (x_0^2 + x_1^2)y, (x_0^2 - x_1^2)y.
\end{align*}\] (9.23)

Therefore the monomials

\[1, x_0, x_1, x_0^2, x_1^2, y, x_0^3, x_1^3, x_0^4\]

generate the quotient $\mathbb{C}[x, y]/I_3$. As this quotient cannot have dimension less than 9, it has dimension 9, the elements in (9.23) generate $I_3$, and the elements in (9.22) generate $I_2$.

The case $\tilde{E}_8$:

\[
\frac{\partial F^{\text{alg}}}{\partial x_0} \circ R \quad \equiv \quad 4x_0^3 x_1 - 2(\kappa^3 + 1)\kappa^{-1} x_0 x_1^2 + \kappa s_2 + 2\kappa x_0 s_3 \\
+ 3\kappa x_0^2 s_5 + x_1 s_6 + 2x_0 x_1 s_7 + \kappa^{-1} x_1^2 s_9
\]

\[
\frac{\partial F^{\text{alg}}}{\partial x_0} \circ R \quad \circledast \quad 4\kappa x_0^2 y + 2\kappa x_0 y s_9 + \kappa y s_9^2 + \frac{1}{2}x_1 s_9 - 2\kappa^2 x_0 x_1^2 \\
- 2x_1 y + \kappa s_2 + 2\kappa x_0 s_3 + 3\kappa x_0^2 s_5 + x_1 s_6 \\
+ 2\kappa y s_7 + x_1 s_7 s_9.
\]

\[
\frac{\partial F^{\text{alg}}}{\partial x_0} \circ R \quad \circledast \quad \text{y-relation}
\]

\[
\Rightarrow \quad \frac{1}{2}x_1 s_9^3 - 2x_1 y + x_1 s_6 + x_1 s_7 s_9 \in I_2,
\]

\[
(x_0 - \frac{1}{2}s_9) \cdot \frac{\partial F^{\text{alg}}}{\partial x_0} \circ R \quad \circledast \quad \text{y-relation}
\]

\[
\Rightarrow \quad 4x_0^3 y + 2y^2 + (x_0 - \frac{1}{2}s_9)(s_2 + 2x_0 s_3 + 3x_0^2 s_5) \\
+ y s_6 + 2x_0 y s_7 \in I_2.
\]
\[
\frac{\partial F_{\text{alg}}}{\partial x_1} \circ R = \kappa^{-1} x_0^4 - 2(\kappa^3 + 1)\kappa^{-2} x_0^2 x_1 + 3x_1^2 \\
+ \left( s_4 - \frac{1}{2} s_6 s_9 - \frac{1}{4} s_7 s_9^2 - \frac{1}{16} \kappa^{-1} s_9^4 \right) \\
+ \kappa^{-1} x_0 s_6 + \kappa^{-1} x_0^2 s_7 + 2x_1 (s_8 - \frac{1}{4} \kappa^{-2} s_9^2) \\
+ 2\kappa^{-2} x_0 x_1 s_9
\]

\[
y\text{-relation } \equiv \kappa^{-1} x_0^4 - 2\kappa x_0^2 x_1 - 2\kappa^{-1} x_0 y + 3x_1^2 \\
(s_4 - \frac{1}{2} s_6 s_9 - \frac{1}{4} s_7 s_9^2 - \frac{1}{16} \kappa^{-1} s_9^4) \\
+ \kappa^{-1} x_0 s_6 + \kappa^{-1} x_0^2 s_7 + 2x_1 s_8 + \kappa^{-1} y s_9.
\]

\[
\frac{\partial F_{\text{alg}}}{\partial x_1} \circ R \& y\text{-relation } \Rightarrow x_0^4 - 2x_0 y + y s_9 - \frac{1}{2} s_6 s_9 - \frac{1}{4} s_7 s_9^2 \\
- \frac{1}{16} s_9^4 + x_0 s_6 + x_0^2 s_7 \in I_2,
\]

\[
x_1 \cdot \frac{\partial F_{\text{alg}}}{\partial x_1} \circ R \& y\text{-relation } \Rightarrow (x_0^3 + x_0^2 \frac{1}{2} s_9 + x_0 \frac{1}{4} s_9^2 + \frac{1}{8} s_9^3)y \\
-2y^2 + 3x_1^3 + x_1 s_4 + y s_6 + (x_0 + \frac{1}{2} s_9)y s_7 + 2x_1^2 s_8 \in I_2.
\]

\text{y-relation } (x_0 - \frac{1}{2} s_9)x_1 - \kappa y_0 \Rightarrow (x_0 - \frac{1}{2} s_9)x_1 \in I_2.

This gives the following \( n + 4 \) elements of \( I_2 \).

\[
x_2, \ldots, x_n, (x_0 - \frac{1}{2} s_9)x_1, x_1(-2y + s_6 + s_7 s_9 + \frac{1}{2} s_9^3),
\]

\[
y(-2y + s_6 + 2x_0 s_7 + 4x_0^3) + (x_0 - \frac{1}{2} s_9)(s_2 + 2x_0 s_3 + 3x_0^2 s_5),
\]

\[
(x_0 - \frac{1}{2} s_9)(-2y + s_6 + (x_0 + \frac{1}{2} s_9)s_7 + (x_0^2 + x_0^2 \frac{1}{2} s_9 + x_0 \frac{1}{4} s_9^2 + \frac{1}{8} s_9^3)),
\]

\[
y(-2y + s_6 + (x_0 + \frac{1}{2} s_9)s_7 + (x_0^3 + x_0^2 \frac{1}{2} s_9 + x_0 \frac{1}{4} s_9^2 + \frac{1}{8} s_9^3)) \\
+ x_1(3x_1^2 + s_4 + 2x_1 s_8).
\] (9.24)

Restriction to \( s = 0 \) gives the following \( n + 4 \) elements of \( I_3 \).

\[
x_2, \ldots, x_n, x_0 x_1, x_1 y, -y^2 + 2x_0^3 y, \\
-2x_0 y + x_0^4, -2y^2 + x_0^3 y + 3x_1^3.
\] (9.25)
Therefore the monomials
\[ 1, x^0, x_0^2, x_1, x_0^3, y, x_0^4, x_1^2, x_0^5, x_6^0 \]
generate the quotient \( \mathbb{C}[x,y]/I_3 \). As this quotient cannot have dimension less than 10, it has dimension 10, the elements in (9.25) generate \( I_3 \), and the elements in (9.24) generate \( I_2 \).

(e) The critical space \( C^{alg} \) of \( F^{alg} \) is everywhere smooth as \( F^{alg} \) is everywhere locally a universal unfolding. But the closure \( R^{-1}(C^{alg}) \subset Y \) is not smooth above \( \mathbb{C}^{\mu-1} \times \{0\} \). Denote by
\[ C^0 := R^{-1}(C^{alg}) \cap \text{pr}_{\mu}^{-1}(\mathbb{C}^{\mu-1} \times \{0\}) \]
the restriction of it which lies above \( \mathbb{C}^{\mu-1} \times \{0\} \). It will turn out below that
\[ C^{0,red} := (C^0 \text{ with the reduced complex structure}) \]
is in each of the three cases a union of four smooth components,
\[ C^{0,red} = C_1^{0,red} \cup C_2^{0,red} \cup C_3^{0,red} \cup C_4^{0,red}. \]
The first three components appear in \( C^0 \) with their reduced structure, \( C_4^{0,red} \) appears in \( C^0 \) with a nonreduced structure with multiplicity two. The following table (9.27) collects facts, which will be proved below in a case-by-case discussion. \( A\text{-to-}(9.26) \) means the answer to the following question (9.26).

|     | \( C_1^{0,red} \) | \( C_2^{0,red} \) | \( C_3^{0,red} \) | \( C_4^{0,red} \) |
|-----|-----------------|-----------------|-----------------|-----------------|
| \( E_6 \) | \( \text{deg pr}_{\mu} \mid_{C_i^{0,red}} \) | 2 | 2 | 2 | 1 |
| A-to-(9.26) | no | yes, \( A_2 \) | yes, \( A_2 \) | no |
| \( E_7 \) | \( \text{deg pr}_{\mu} \mid_{C_i^{0,red}} \) | 3 | 3 | 1 | 1 |
| A-to-(9.26) | yes, \( A_3 \) | yes, \( A_3 \) | no | no |
| \( E_8 \) | \( \text{deg pr}_{\mu} \mid_{C_i^{0,red}} \) | 5 | 2 | 1 | 1 |
| A-to-(9.26) | yes, \( A_5 \) | yes, \( A_2 \) | no | no |

The Lyashko-Looijenga map \( LL^{alg} \circ \rho \) maps \( (s, \kappa) \in \mathbb{C}^{\mu-1} \times \Delta^* \) to the tuple of the symmetric polynomials (with suitable signs) in the values of \( F^{alg} \circ R \) on \( R^{-1}(C^{alg}) \cap \mathbb{C}^{\mu+1} \times \{(s, \kappa)\} \).

Because of (c), \( F^{alg} \circ R \) extends to \( \kappa = 0 \). Because of (d), \( R^{-1}(C^{alg}) \) extends to \( \kappa = 0 \). Therefore the map \( LL^{alg} \circ \rho \) extends to a holomorphic
map \((LL_{alg} \circ \rho)^{ext} : \mathbb{C}^{\mu - 1} \times \Delta \rightarrow M_{LL}^{(\mu)}\). The table (9.27) shows that in each of the three cases
\[
\sum_{i=1}^{4} \deg \text{pr}_{\mu} |_{C_{i}^{0,\text{red}}} = \mu - 1.
\] (9.28)

Therefore \((LL_{alg} \circ \rho)^{ext}\) maps \(\mathbb{C}^{\mu - 1} \times \{0\}\) to \(D_{LL}^{(\mu)}\).

It rests to show that the map in (9.13) is finite and flat onto its image. Above \(\mathbb{C}^{\mu - 1} \times \Delta^*\) this holds. Therefore in order to prove it, it rests to show
\[
(LL_{alg})^{ext}(s, 0) = 0 \implies s = 0.
\] (9.29)

This will be shown below in the case-by-case discussion.

The case \(\tilde{E}_6\): (9.20) shows that \(C_{0,\text{red}}^{0,\text{red}}\) has the following four components \(C_{i}^{0,\text{red}}, i \in \{1, 2, 3, 4\}\). The components are given in terms of functions which vanish on them. In each case, they contain the first three functions in (9.20), which express \(y_0, y_1\) and \(y_2\) in terms of \((x, s)\).

Of course, \(x_3, \ldots, x_n\) vanish on all four components.

\(C_{1}^{0,\text{red}}:\) the first three functions in (9.20), \(x_1 + s_5, x_2, (x_0 + s_6)(3x_0 + s_6) + 4s_2\) (and generically \(x_0 \neq 0\)).

\(C_{2}^{0,\text{red}}\) and \(C_{3}^{0,\text{red}}:\) the first three functions in (9.20),
\[
x_0, x_2 - \varepsilon \cdot i \cdot (x_1 + s_5)\]
with \(\varepsilon = 1\) for \(C_{2}^{0,\text{red}}\) and \(\varepsilon = -1\) for \(C_{3}^{0,\text{red}}\),
\[
3x_1^2 + \varepsilon i(2x_1s_7 + s_4 + s_5s_7) + s_3
\]
(and generically \(x_1 + s_5 \neq 0, x_2 \neq 0\)).

\(C_{4}^{0,\text{red}}:\) the first three functions in (9.20),
\[
x_0, x_1 + s_5, x_2.
\] (9.32)

Obviously, each \(C_{i}^{0,\text{red}}\) is smooth, and \(\deg \text{pr}_{\mu} |_{C_{i}^{0,\text{red}}}\) is as claimed in table (9.27).

It rests to prove (9.29). The restriction of \((F_{alg} \circ R)^{ext}\) (which was calculated in the proof of (c)) to \(C_{i}^{0,\text{red}}\) is as follows:
\[
(F_{alg} \circ R)^{ext}|_{C_{1}^{0,\text{red}}} = -\frac{1}{2}(x_0 + s_6)x_0^2 + s_1 - s_3s_5 - s_5^3,
\] (9.33)
\[
(F_{alg} \circ R)^{ext}|_{C_{j}^{0,\text{red}}} = x_1^3 + s_1 + x_1s_3 + \varepsilon i(x_1 + s_5)s_4
\]
\[
+ \varepsilon ix_1(x_1 + s_5)s_7\quad\text{for }\ j \in \{2, 3\},
\] (9.34)
\[
(F_{alg} \circ R)^{ext}|_{C_{4}^{0,\text{red}}} = s_1 - s_3s_5 - s_5^3.
\] (9.35)
Consider a parameter \( s \in \mathbb{C}^7 \) with \((LL_{\text{alg}} \circ R)^{\text{ext}}(s, 0) = 0\). We want to show \( s = 0 \). \([9.33]\) gives \( s_1 - s_3s_5 - s_5^2 = 0 \). \([9.33]\) and \([9.30]\) give the existence of a number \( x_0 \in \mathbb{C} \) with \((x_0 + s_6)(3x_0 + s_6) + 4s_2 = 0\) and \((x_0 + s_6)x_0 = 0\). The first quadratic polynomial has a double zero if and only if \( 12s_2 - s_6^2 = 0 \), and then the double zero is \( x_0 = -\frac{2}{3}s_6 \). It is a zero of \((x_0 + s_6)x_0\) only if \( s_6 = 0 \), and then \( s_2 = 0 \). In the case \( 12s_2 - s_6^2 \neq 0 \), we must have 

\[(x_0 + s_6)(3x_0 + s_6) + 4s_2 = 3(x_0 + s_6)x_0, \quad \text{thus again} \quad s_6 = 0, \, s_2 = 0.\]

So, \([9.33]\) gives in any case \( s_6 = 0 \) and \( s_2 = 0 \).

Now consider \( j \in \{2, 3\} \) and \([9.34]\). It motivates the definition of the unfolding

\[ G_j(x_1, s_1, s_3, s_4, s_5, s_7) := x_1^3 + s_1 + x_1s_3 + \varepsilon i(x_1 + s_5)s_4 + \varepsilon ix_1(x_1 + s_5)s_7 \]

in the variable \( x_1 \) and with parameters \( s_1, s_3, s_4, s_5, s_7 \) of the \( A_2 \)-singularity \( x_1^3 \). The derivative \( \frac{\partial G_j}{\partial x_1} \) is in the ideal which defines \( C_j^{\text{red}} \), so

\[ \text{Crit}(G_j) \cong C_j^{\text{red}} \text{ and } G_j|_{\text{Crit}(G_j)} \cong (F_{\text{alg}} \circ R)^{\text{ext}}|_{C_j^{\text{red}}}. \]

Denote the Lyashko-Looijenga map of \( G_j \) by \( LL_{G_j} \). Then \( LL_{G_j}(s_1, s_3, s_4, s_5, s_7) = 0 \). The unfolding \( G_j \) is induced by the universal unfolding

\[ G_{A_2}(z, t_1, t_2) = z^3 + t_1 + zt_2 \]

via the morphism \((\Phi^{(j)}, \varphi^{(j)})\) with \( G_{A_2} \circ \Phi^{(j)} = G_j \) and

\[ z = \Phi_1^{(j)}(x_1, s) = x_1 + \frac{1}{3}\varepsilon is_7, \]

\[ t_1 = \varphi_1^{(j)}(s) = s_1 + \varepsilon is_4s_5 + \frac{1}{27}\varepsilon is_7^3 \]

\[ - \frac{1}{3}\varepsilon i(s_3 + \varepsilon is_4 + \varepsilon is_5s_7 + \frac{1}{3}is_7^2)s_7, \]

\[ t_2 = \varphi_2^{(j)}(s) = s_3 + \varepsilon is_4 + \varepsilon is_5s_7 + \frac{1}{3}s_7^2. \]

Then \( LL_{G_j} = LL_{A_2} \circ \varphi^{(j)} \) where \( LL_{A_2} \) is the Lyashko-Looijenga map of the universal unfolding \( G_{A_2} \). The map \( LL_{A_2} \) is a finite branched covering and has value 0 only at 0. Therefore \( \varphi_2^{(j)}(s) = \varphi_1^{(j)}(s) = 0 \).

Now \( \varphi_2 \pm \varphi_2 = 0 \) and \( \varphi_1 \pm \varphi_2 = 0 \) give 

\[ s_3 + \frac{1}{3}s_7^2 = 0, \quad s_4 + s_5s_7 = 0, \quad s_1 = 0, \quad s_4s_5 + \frac{1}{27}s_7^3 = 0. \]

Together with

\[ s_1 - s_3s_5 - s_5^2 = 0 \quad \text{and} \quad s_6 = 0, \quad s_2 = 0, \]
this gives \( s = 0 \). Now \( (9.29) \) is proved in the case \( \tilde{E}_6 \).

The case \( \tilde{E}_7 \): \( (9.22) \) shows that \( C_{0,\text{red}} \) has the following four components \( C_{0,\text{red}}^i, i \in \{1, 2, 3, 4\} \). The components are given in terms of functions which vanish on them. Of course, \( x_2, \ldots, x_n \) vanish on all four components.

\[ C_{0,\text{red}}^1 : \quad -2y + x_0^2 + s_5 + x_0 s_7, \quad x_1, \]
\[ (2x_0 + s_7)y + s_2 + 2x_0 s_4 \]
\[ \text{(and generically } x_0 \neq 0). \]
\[ C_{0,\text{red}}^2 : \quad -2y + x_1^2 + s_5 + x_1 s_8, \quad x_0, \]
\[ (2x_1 + s_8)y + s_3 + 2x_1 s_6 \]
\[ \text{(and generically } x_1 \neq 0). \]
\[ C_{0,\text{red}}^3 : \quad y, \quad x_0, \quad x_1. \]
\[ C_{0,\text{red}}^4 : \quad y - \frac{1}{2} s_5, \quad x_0, \quad x_1. \]

Obviously, each \( C_{0,\text{red}}^i \) is smooth, and \( \deg \text{pr}_\mu |_{C_{0,\text{red}}^i} \) is as claimed in table \( (9.27) \).

It rests to prove \( (9.29) \). The restriction of \( (F^{\text{alg}} \circ R)^{\text{ext}} \) to \( C_{0,\text{red}}^i \) is as follows:

\[ (F^{\text{alg}} \circ R)^{\text{ext}} |_{C_{0,\text{red}}^1} = x_0^2 y - y^2 + s_1 + x_0 s_2 + x_0^2 s_4 + y s_5 + x_0 y s_7, \]
\[ (F^{\text{alg}} \circ R)^{\text{ext}} |_{C_{0,\text{red}}^2} = x_1^2 y - y^2 + s_1 + x_1 s_3 + x_1^2 s_6 + y s_5 + x_1 y s_8, \]
\[ (F^{\text{alg}} \circ R)^{\text{ext}} |_{C_{0,\text{red}}^3} = s_1, \]
\[ (F^{\text{alg}} \circ R)^{\text{ext}} |_{C_{0,\text{red}}^4} = s_1 + \frac{1}{4} s_5^2. \]

Consider a parameter \( s \in \mathbb{C}^8 \) with \( (LL^{\text{alg}} \circ R)^{\text{ext}}(s, 0) = 0 \). We want to show \( s = 0 \). \( (9.32) \) and \( (9.33) \) give \( s_1 = s_5 = 0 \). This and \( (9.40) \) motivate the definition of the unfolding

\[ G(x_0, y, s_2, s_4, s_7) := x_0^2 y - y^2 + x_0 s_2 + x_0^2 s_4 + x_0 y s_7 \]

of the \( A_3 \)-singularity in the parameters \( s_2, s_4, s_7 \). The derivatives \( \frac{\partial G}{\partial x_0} \) and \( \frac{\partial G}{\partial y} \) are in the ideal which defines \( C_{0,\text{red}}^1 |_{s_1=s_5=0} \), so

\[ \text{Crit}(G) \cong C_{0,\text{red}}^1 |_{s_1=s_5=0} \quad \text{and} \quad G |_{\text{Crit}(G)} \cong (F^{\text{alg}} \circ R)^{\text{ext}} |_{C_{0,\text{red}}^1 |_{s_1=s_5=0}}. \]

Denote the Lyashko-Looijenga map of \( G \) by \( LL_G \). Then \( LL_G(s_2, s_4, s_7) = 0 \).
The unfolding $G$ is induced by the universal unfolding

$$G_{A_3}(z, y_1, t_1, t_2, t_3) = z^2 y_1 - y_1^2 + t_1 + z t_2 + x z^2 t_3$$

via the morphism $(\Phi, \varphi)$ with $G_{A_3} \circ \Phi = G$ and

$$z = \Phi_1(x_0, y, s) = x_0 + \frac{1}{2} s_7,$$

$$y_1 = \Phi_1(x_0, y, s) = y + \frac{1}{8} s_7^2,$$

$$t_1 = \varphi_1(s) = -\frac{1}{2} s_2 s_7 + \frac{1}{4} s_4 s_7^2 + \frac{1}{64} s_7^4,$$

$$t_2 = \varphi_2(s) = s_2 - s_4 s_7,$$

$$t_3 = \varphi_3(s) = s_4 - \frac{1}{8} s_7^2.$$}

Then $LL_G = LL_{A_3} \circ \varphi$ where $LL_{A_3}$ is the Lyashko-Looijenga map of the universal unfolding $G_{A_3}$. The map $LL_{A_3}$ is a finite branched covering and has value 0 only at 0. Therefore $0 = \varphi_1(s) = \varphi_2(s) = \varphi_3(s)$. This gives $s_2 = s_4 = s_7 = 0$.

(9.36) & (9.37) and (9.40) & (9.41) are symmetric with respect to

$$(x_0, y, s_1, s_2, s_4, s_5, s_7) \longleftrightarrow (x_1, y, s_1, s_3, s_6, s_5, s_8).$$

Therefore also $s_3 = s_6 = s_8 = 0$. This gives $s = 0$. Now (9.29) is proved in the case $\tilde{E}_7$.

The case $\tilde{E}_8$: (9.24) shows that $C_{i, \text{red}}^{0}$ has the following four components $C_{i, \text{red}}^{0}$, $i \in \{1, 2, 3, 4\}$. The components are given in terms of functions which vanish on them. Of course, $x_2, ..., x_n$ vanish on all four components.

$$C_{1, \text{red}}^{0} : -2y + (x_0^3 + x_0^2 \frac{1}{2} s_9 + x_0 \frac{1}{4} s_5 + \frac{1}{8} s_9^3) + s_6 + (x_0 + \frac{1}{2} s_9) s_7, x_1, y(s_7 + 3x_0^2 + x_0 s_9 + \frac{1}{4} s_5^2) + s_2 + 2x_0 s_3 + 3x_0^2 s_5$$

(and generically $x_0 - \frac{1}{2} s_9 \neq 0$). (9.44)

$$C_{2, \text{red}}^{0} : x_0 - \frac{1}{2} s_9, -2y + s_6 + s_7 s_9 + \frac{1}{2} s_9^3, 3x_1^2 + s_4 + 2x_1 s_8$$

(and generically $x_1 \neq 0$). (9.45)

$$C_{3, \text{red}}^{0} : x_0 - \frac{1}{2} s_9, x_1, y.$$ (9.46)

$$C_{4, \text{red}}^{0} : x_0 - \frac{1}{2} s_9, x_1, -2y + s_6 + s_7 s_9 + \frac{1}{2} s_9^3.$$ (9.47)
Obviously, each \( C_i^{0,\text{red}} \) is smooth, and \( \deg \rho_i \big|_{C_i^{0,\text{red}}} \) is as claimed in table (9.27).

It rests to prove (9.29). The restriction of \((F^{\text{alg}} \circ R)^{\text{ext}}\) to \( C_i^{0,\text{red}} \) is as follows:

\[
(F^{\text{alg}} \circ R)^{\text{ext}} \big|_{C_i^{0,\text{red}}} = y \left[ \left( x_0^3 + x_0^2 \frac{1}{2} s_9 + x_0 \frac{1}{4} s_9^2 + \frac{1}{8} s_9^3 \right) + s_6 \right] + (x_0 + \frac{1}{2} s_9) s_7 - y^2 + s_1 + x_0 s_2 + x_0^2 s_3 + x_0^3 s_5, 
\]

\[
(F^{\text{alg}} \circ R)^{\text{ext}} \big|_{C_1^{0,\text{red}}} = y (s_6 + s_7 s_9 + \frac{1}{2} s_9^3) - y^2 + x_1^3 + s_1 + \frac{1}{2} s_2 s_9 + \frac{1}{4} s_3 s_9^2 + x_1 s_4 + \frac{1}{8} s_5 s_9^3 + x_1^2 s_8, 
\]

\[
(F^{\text{alg}} \circ R)^{\text{ext}} \big|_{C_2^{0,\text{red}}} = s_1 + \frac{1}{2} s_2 s_9 + \frac{1}{4} s_3 s_9^2 + \frac{1}{8} s_5 s_9^3, 
\]

\[
(F^{\text{alg}} \circ R)^{\text{ext}} \big|_{C_4^{0,\text{red}}} = \frac{1}{4} (s_6 + s_7 s_9 + \frac{1}{2} s_9^3)^2 + (s_1 + \frac{1}{2} s_2 s_9 + \frac{1}{4} s_3 s_9^2 + \frac{1}{8} s_5 s_9^3). 
\]

Consider a parameter \( s \in \mathbb{C}^9 \) with \((LL^{\text{alg}} \circ R)^{\text{ext}}(s, 0) = 0\). We want to show \( s = 0 \). (9.50) and (9.51) give

\[
0 = s_1 + \frac{1}{2} s_2 s_9 + \frac{1}{4} s_3 s_9^2 + \frac{1}{8} s_5 s_9^3, \quad 0 = s_6 + s_7 s_9 + \frac{1}{2} s_9^3. 
\]

This and (9.48) motivate the definition of the unfolding

\[
G_1(x_0, y, s_2, s_3, s_5, s_7, s_9) := y \left[ \left( x_0^3 + x_0^2 \frac{1}{2} s_9 + x_0 \frac{1}{4} s_9^2 - \frac{3}{8} s_9^3 \right) + (x_0 - \frac{1}{2} s_9) s_7 \right] - y^2 - (\frac{1}{2} s_2 s_9 + \frac{1}{4} s_3 s_9^2 + \frac{1}{8} s_5 s_9^3) + x_0 s_2 + x_0^2 s_3 + x_0^3 s_5
\]

of the \( A_5 \)-singularity \( y x_0^3 - y^2 \) in the parameters \( s_2, s_3, s_5, s_7, s_9 \). The derivatives \( \frac{\partial G}{\partial x_0} \) and \( \frac{\partial G}{\partial y} \) are in the ideal which defines \( C_1^{0,\text{red}} \) with (9.32), so

\[
\text{Crit}(G_1) \supseteq C^{0,\text{red}} \big|_s \text{ with (9.52)} \quad \text{and} \quad G_1 \big|_{\text{Crit}(G_1)} \supseteq (F^{\text{alg}} \circ R)^{\text{ext}} \big|_{C_1^{0,\text{red}}} \big|_s \text{ with (9.48),}
\]

Denote the Lyashko-Looijenga map of \( G_1 \) by \( LL_{G_1} \). Then \( LL_{G_1}(s_2, s_3, s_5, s_7, s_9) = 0 \).

The unfolding \( G_1 \) is induced by the universal unfolding

\[
G_{A_5} = y z^3 - y_1^2 + t_1 + z t_2 + z^2 t_3 + z^3 t_4 + z y_1 t_5
\]
via the morphism \((\Phi^{(1)}, \varphi^{(1)})\) with \(G_{A_5} \circ \Phi^{(1)} = G_1\) and

\[
\begin{align*}
z &= \Phi_{1}^{(1)}(x_0, y, s) = x_0 + \frac{1}{6}s_9, \\
y_1 &= \Phi_{2}^{(1)}(x_0, y, s) = y + \frac{1}{2}(s_7s_9 + \frac{11}{27}s_9^3), \\
t_1 &= \varphi_{1}^{(1)}(s) = -\frac{2}{3}s_7s_9 + \frac{11}{27}s_9^3, \\
t_2 &= \varphi_{2}^{(1)}(s) = s_2 - \frac{1}{3}s_3s_9 + \frac{1}{12}s_5s_9^2, \\
t_3 &= \varphi_{3}^{(1)}(s) = s_3 - \frac{1}{2}s_5s_9, \\
t_4 &= \varphi_{4}^{(1)}(s) = s_5, \\
t_5 &= \varphi_{5}^{(1)}(s) = s_7 + \frac{1}{6}s_9^2.
\end{align*}
\]

Then \(LL_G = LL_{A_5} \circ \varphi^{(1)}\) where \(LL_{A_5}\) is the Lyashko-Looijenga map of the universal unfolding \(G_{A_5}\). The map \(LL_{A_5}\) is a finite branched covering and has value 0 only at 0. Therefore

\[
0 = \varphi_{1}^{(1)}(s) = \varphi_{2}^{(1)}(s) = \varphi_{3}^{(1)}(s) = \varphi_{4}^{(1)}(s) = \varphi_{5}^{(1)}(s).
\]

This gives

\[
s_2 = s_3 = s_5 = s_7 = s_9 = 0, \text{ and with } (9.52) \; s_1 = s_6 = 0. \quad (9.53)
\]

This and (9.49) motivate the definition of the unfolding

\[
G_2(x_1, y, s_4, s_8) := -y^2 + x_1^2 + x_1s_4 + x_1^2s_8
\]

of the \(A_2\)-singularity \(-y^2 + x_1^2\) in the parameters \(s_4\) and \(s_8\). The derivatives \(\frac{\partial G_2}{\partial x_1}\) and \(\frac{\partial G_2}{\partial y}\) are in the ideal which defines \(C_2^{0, \text{red}}\) with (9.52), so

\[
\text{Crit}(G_2) \cong C_2^{0, \text{red}}_{s \text{ with } (9.53)} \quad \text{and} \quad G_2|_{\text{Crit}(G_2)} \cong (F_{\text{alg}} \circ R)^{\text{ext}}|_{C_2^{0, \text{red}}_{s \text{ with } (9.53)}}.
\]

Denote the Lyashko-Looijenga map of \(G_2\) by \(LL_{G_2}\). Then

\[
LL_{G_2}(s_4, s_8) = 0.
\]

The unfolding \(G_2\) is induced by the universal unfolding

\[
G_{A_2}(z, y_1, t_1, t_2) = -y_1^2 + z^3 + t_1 + zt_2
\]
via the morphism $(\Phi^{(2)}, \varphi^{(2)})$ with $G_{A_2} \circ \Phi^{(2)} = G_2$ and

$$z = \Phi_1^{(2)}(x_1y, s) = x_1 + \frac{1}{3}s_8,$$

$$y_1 = \Phi_2^{(2)}(x_0, y, s) = y,$$

$$t_1 = \varphi_1^{(2)}(s) = -\frac{1}{3}s_4s_8 + \frac{2}{27}s_8^3,$$

$$t_2 = \varphi_2^{(2)}(s) = s_4 - \frac{1}{3}s_8^2.$$

Then $LL_{G_2} = LL_{A_2} \circ \varphi^{(2)}$ where $LL_{A_2}$ is the Lyashko-Looijenga map of the universal unfolding $G_{A_2}$. The map $LL_{A_2}$ is a finite branched covering and has value 0 only at 0. Therefore

$$0 = \varphi_1^{(2)}(s) = \varphi_2^{(2)}(s), \quad \text{so} \quad 0 = s_4 = s_8.$$

This gives $s = 0$. Now (9.29) is proved in the case $\tilde{E}_8$. This finishes the proof of theorem 9.1.

□

10. Degree of the Lyashko-Looijenga map $LL^{alg}$ for the simple elliptic singularities

This section is devoted to the proof of theorem 6.3. The main work has already been done in the sections 9, 5 and 8. The maps $\rho$ in theorem 9.1 tell how to glue into $M^{alg} = \mathbb{C}^{\mu - 1} \times (\mathbb{C} - \{0, 1\})$ a fiber above $\lambda = 0$. This and the maps $\psi_2$ and $\psi_3$ in subsection 5.2 tell how to glue into $M^{alg}$ fibers above $\lambda = 1$ and $\lambda = \infty$. Corollary 8.6 together with the maps $\psi_2, \psi_3$ and $\rho$ allows to calculate the degree of $LL^{alg}$.

The maps $\psi_2$ in (5.21), (5.27) and (5.33) and the maps $\rho$ in (9.2), (9.5) and (9.8) contain the following fractional powers of $\lambda$,

$$\psi_2: \lambda^{1/2} \quad \psi_3: \lambda^{1/4} \quad \rho: \lambda^{1/2} \quad \kappa = \lambda^{1/c} = \lambda^{1/3} \quad \kappa = \lambda^{1/c} = \lambda^{1/2} \quad \kappa = \lambda^{1/c} = \lambda^{1/3} \quad (10.1)$$

Therefore we consider coverings of $\mathbb{C} - \{0, 1\}$ and of $M^{alg}$ which are of order $2c$ at respectively above $\lambda \in \{0, 1, \infty\}$. Denote by $\mathbb{P}^1(2c, 2c, 2c)$ the orbifold $\mathbb{P}^1$ with orbifold points 0, 1 and $\infty$ which all have multiplicity $2c$. Because of $2 - 3(1 - \frac{1}{2c}) = -1 + \frac{3}{2c} < 0$ it is a hyperbolic orbifold, so a good orbifold. By a classical theorem of Fox [Sc83, Theorem 2.5], a finite orbifold covering $p_X : X \to \mathbb{P}^1(2c, 2c, 2c)$ with $X$ a manifold exists. It is a branched covering of order $2c$ at each preimage of 0, 1 and $\infty$ and a covering everywhere else. Denote $N^{alg} := \mathbb{C}^{\mu - 1} \times (X - p_X^{-1}(\{0, 1, \infty\}))$, and denote by $p_{alg} :=$
are smooth cone bundles with the weights $(a_1, a_2, \ldots, a_{\mu-1}) = (\deg_w t_{\mu-1}, \deg_w t_{\mu-2}, \ldots, \deg_w t_1) \cdot d$. Here
\[ d := 3 \text{ for } \tilde{E}_6, \quad d := 4 \text{ for } \tilde{E}_7, \quad d := 6 \text{ for } \tilde{E}_8, \] (10.2)
is chosen so that all weights $d \cdot \deg t_i$ are integers. Now $N^{\alg}$ will be extended to a smooth cone bundle $N^{\orb} \to X$, i.e. fibers above the points in $p_X^{-1}(\{0, 1, \infty\})$ will be glued into $N^{\alg}$.

Let $\delta_0 : \Delta \to X$ be an isomorphism from the unit disk $\Delta$ to a neighborhood of any point in $p_X^{-1}(0)$ with $p_X \circ \delta_0(z) = z^{2c}$. Glue $\mathbb{C}^{\mu-1} \times \Delta$ into $N^{\alg}$ with the map
\[ \mathbb{C}^{\mu-1} \times \Delta^* \to N^{\alg}, \]
\[ (t', z) \mapsto ((\rho_1, \ldots, \rho_{\mu-1})(t', z^2), \delta_0(z)). \] (10.3)

Let $\delta_1 : \Delta \to X$ be an isomorphism from the unit disk $\Delta$ to a neighborhood of any point in $p_X^{-1}(1)$ with $p_X \circ \delta_1(z) = 1 - z^{2c}$. Glue $\mathbb{C}^{\mu-1} \times \Delta$ into $N^{\alg}$ with the map
\[ \mathbb{C}^{\mu-1} \times \Delta^* \to N^{\alg}, \]
\[ (t', z) \mapsto (((\psi_3)_1, \ldots, (\psi_3)_{\mu-1})((\rho_1, \ldots, \rho_{\mu-1})(t', z^2), z^{2c}), \delta_1(z)). \] (10.4)

Let $\delta_\infty : \Delta \to X$ be an isomorphism from the unit disk $\Delta$ to a neighborhood of any point in $p_X^{-1}(\infty)$ with $p_X \circ \delta_\infty(z) = z^{-2c}$. Glue $\mathbb{C}^{\mu-1} \times \Delta$ into $N^{\alg}$ with the map
\[ \mathbb{C}^{\mu-1} \times \Delta^* \to N^{\alg}, \]
\[ (t', z) \mapsto (((\psi_2)_1, \ldots, (\psi_2)_{\mu-1})((\rho_1, \ldots, \rho_{\mu-1})(t', z^2), z^{2c}), \delta_\infty(z)). \] (10.5)

This is a univalued map although $\psi_2$ contains $\lambda^{1/2}$ (in the cases $\tilde{E}_6$ and $\tilde{E}_8$) and $\lambda^{1/4}$ (in the case $\tilde{E}_7$), by setting $\lambda^{1/2} \circ z^{2c} = z^c$ and $\lambda^{1/4} \circ z^4 = z$.

The resulting manifold $N^{\orb}$ is a smooth cone bundle above $X$ with weights $(a_1, a_2, \ldots, a_{\mu-1}) = (\deg_w t_{\mu-1}, \deg_w t_{\mu-2}, \ldots, \deg_w t_1) \cdot d$ because $M^{\alg}$ and $N^{\alg}$ are smooth cone bundles with these weights and all involved maps are $\mathbb{C}^*$-equivariant with respect to the natural $\mathbb{C}^*$-actions. The covering group of the covering $p^{\alg} : N^{\alg} \to M^{\alg}$ extends to an automorphism group of $N^{\orb}$. The quotient of $N^{\orb}$ by this group is an orbibundle $M^{\orb}$ above $\mathbb{P}^1$ which extends $M^{\alg} \to \mathbb{C} - \{0, 1\}$. Let $p^{\orb} : N^{\orb} \to M^{\orb}$ be the quotient map.

Recall the definition of $M^{\orb}_0 \subset M^{\orb}$ in theorem 6.3 and define $N^{\orb}_0 := p^{\orb \cdot 1}(M^{\orb}_0)$. We claim that $LL^{\alg} \circ p^{\alg} : N^{\alg} \to M^{\mu}_{LL}$ extends
to a holomorphic map $LL^\text{orb}_N : N^\text{orb} \to M_{LL}^{(\mu)}$, that the restriction

$$LL^\text{orb}_N : N^\text{orb} - N^\text{orb}_0 \to M_{LL}^{(\mu)} - M_{LL,0}^{(\mu)}$$ (10.6)

is a branched covering of a finite degree, and that $LL^\text{orb}_N$ maps the fibers of $N^\text{orb}$ above the points in $p_X^{-1}(\{0, 1, \infty\})$ to $D_{LL}^{(\mu)}$. Near the fibers of $N^\text{orb}$ above the points in $p_X^{-1}(0)$, this follows from theorem 9.1 (e). Near the fibers of $N^\text{orb}$ above the points in $p_X^{-1}(\{1, \infty\})$, this follows again from theorem 9.1 (e) and from the fact that $\psi_3$ and $\psi_2$ are locally isomorphisms of F-manifolds with Euler fields and thus

$$LL^\text{alg}(t', \lambda) = LL^\text{alg}(\psi_2(t', \lambda)) = LL^\text{alg}(\psi_3(t', \lambda))$$ (10.7)

for $(t', \lambda) \in \mathbb{C}^{\mu-1} \times \Delta^* \subset M^\text{alg} = \mathbb{C}^{\mu-1} \times (\mathbb{C} - \{0, 1\})$.

$LL^\text{alg}$ inherits the good properties from $LL^\text{alg} \circ p^\text{alg}$. It extends to a holomorphic map $LL^\text{orb} : M^\text{orb} \to M_{LL}^{(\mu)}$, the restriction in (10.6) is a branched covering, and $\pi_0^\text{orb}^{-1}(\{0, 1, \infty\})$ is mapped to $D_{LL}^{(\mu)}$.

It rests to determine the degree of $LL^\text{alg}$. Of course, $\deg LL^\text{orb}_N = \deg LL^\text{alg} \cdot \deg p^\text{alg}$.

The tuple $(LL^\text{orb}_N, N^\text{orb}, M_{LL}^{(\mu)})$ satisfies almost the properties of the tuple $(f, C_1, C_2)$ in corollary 8.6 but not completely.

The affine group $G_{\mathbb{A}_1} = (\mathbb{C}, +)$ acts freely on $N^\text{orb}$ and $M_{LL}^{(\mu)}$ as follows, and $LL^\text{orb}_N$ is equivariant with respect to these actions. We have to divide out these actions. The action of $G_{\mathbb{A}_1}$ on $N^\text{orb}$ comes from the lift to $N^\text{alg}$ and extension to $N^\text{orb}$ of the action on $M^\text{alg}$,

$$G_{\mathbb{A}_1} \times M^\text{alg} \to M^\text{alg}, \quad (s, t', \lambda) \mapsto (t_1 + s, t_2, \ldots, t_{\mu-1}, \lambda).$$ (10.8)

The action of $G_{\mathbb{A}_1}$ on $M_{LL}^{(\mu)}$ is given by

$$G_{\mathbb{A}_1} \times M_{LL}^{(\mu)} \to M_{LL}^{(\mu)}, \quad (s, p(y)) \mapsto p(y - s).$$ (10.9)

The quotient triple $(LL^\text{orb}_N, N^\text{orb}, M_{LL}^{(\mu)})/G_{\mathbb{A}_1}$ satisfies the properties of the triple $(f, C_1, C_2)$ in corollary 8.6.

$C_1 := N^\text{orb}/G_{\mathbb{A}_1}$ is a smooth cone bundle with weights $(a_1, \ldots, a_{\mu-2}) = (\deg_w t_{\mu-1}, \ldots, \deg_w t_2) \cdot d$ and basis $X_1 := X$ of dimension 1. $C_2 := M_{LL}^{(\mu)}/G_{\mathbb{A}_1}$ is a smooth cone bundle with weights $(b_1, b_2, \ldots, b_{\mu-1}) = (2, 3, \ldots, \mu - 1) \cdot d$ with basis $X_2 = (a \text{ point})$. And $f := LL^\text{orb}_N/G_{\mathbb{A}_1}$
satisfies the properties in the situation before proposition 8.5. Therefore by corollary 8.6

\[ \deg LL^\text{orb}_N = \deg f = \frac{b_1 \cdots b_{\mu-1}}{a_1 \cdots a_{\mu-2}} \left( - \sum_{k=a_1}^{a_{\mu-2}} \frac{1}{k} \cdot \deg C_{1,(k)} \right) \]

\[ = \frac{2 \cdot 3 \cdot \cdots \cdot \mu}{\prod_{i=2}^{\mu-1} \deg w t_i} \left( - \sum_{k=a_1}^{a_{\mu-2}} \frac{d}{k} \cdot \deg C_{1,(k)} \right) \]

For the proof of formula (6.7) it rests to show

\[ \frac{-\deg C_{1,(k)}}{\deg p_{\text{alg}}} = \frac{1}{2} \cdot |\{ j \mid a_j = k\}|. \] (10.12)

A basis of trivial global sections of the trivial smooth cone bundle \( \mathbb{C}^{\mu-1} \times X \supset N^\text{alg} \) and the glueing maps (10.3), (10.4) and (10.5) give a global meromorphic section in the determinant bundle \( \deg C_{1,(k)} \) of each vector bundle \( C_{1,(k)} \). The sum of the orders of zeros and poles of this section is \( \deg C_{1,(k)} \). In fact, we can read of \( -\deg C_{1,(k)}/\deg p_{\text{alg}} \) directly from the sum of the orders of \( \lambda \) in those parts of \( \rho, \psi_3 \) and \( \psi_2 \) which correspond to \( C_{1,(k)} \). Here \( \rho \) is used three times, \( \psi_3 \) and \( \psi_2 \) are each used one times. The following tables collect the relevant data from the formulas (9.2), (9.5), (9.8), (5.24), (5.30), (5.36), (5.21), (5.27) and (5.33).

### The case \( \tilde{E}_6 \):

| \( k \) | involved \( t_i \) | \( \rho : \text{order of } \lambda \text{ in } (9.2) \) | \( \psi_3 : \text{order of } \lambda \text{ in } (5.24) \) |
|---|---|---|---|
| 1 = a_1 = a_2 = a_3 | \( t_7, t_6, t_5 \) | 0 + \( \frac{1}{3} \) + 1 | 0 + 0 + 0 |
| 2 = a_4 = a_5 = a_6 | \( t_4, t_3, t_2 \) | 0 + 0 + \( \frac{2}{3} \) | 0 + 0 + 0 |

| \( k \) | \( \psi_2 : \text{order of } \lambda \text{ in } (5.21) \) | \( -\deg C_{1,(k)}/\deg p_{\text{alg}} \) |
|---|---|---|
| 1 | \( \frac{1}{2} - 1 - 2 \) | 3 \cdot \frac{4}{3} + 0 - \frac{5}{2} = \frac{3}{2} |
| 2 | \( \frac{1}{2} + 0 - 1 \) | 3 \cdot \frac{2}{3} + 0 - \frac{1}{2} = \frac{3}{2} |
The case $\tilde{E}_7$:

\[
\begin{array}{c|c|c|c}
 k & \text{involved } t_i & \rho : \text{order of } \lambda & \psi_3 : \text{order of } \lambda \\
1 = a_1 = a_2 & t_8, t_7 & 0 + \frac{1}{2} & 1 \text{ (see } 10.13) \\
2 = a_3 = a_4 = a_5 & t_6, t_5, t_4 & 0 + 0 + 1 & 0 + 0 + 0 \\
3 = a_6 = a_7 & t_3, t_2 & 0 + \frac{1}{2} & 0 + 0 \\
\end{array}
\]

\[
k \psi_2 : \text{order of } \lambda \text{ in } (5.27) \quad - \deg C_{1,(k)}/ \deg p_{\text{alg}}
\]

\[
\begin{array}{c|c|c}
k & \psi_2 & -\deg C_{1,(k)}/ \deg p_{\text{alg}} \\
1 & -\frac{3}{4} - \frac{2}{4} & 3 \cdot \frac{1}{2} + 1 - \frac{3}{2} = 1 \\
2 & \frac{1}{2} - \frac{1}{2} - \frac{3}{2} & 3 \cdot 1 + 0 - \frac{3}{2} = \frac{3}{2} \\
3 & \frac{1}{4} - \frac{3}{4} & 3 \cdot \frac{1}{2} + 0 - \frac{1}{2} = 1 \\
\end{array}
\]

\[
det \begin{pmatrix} -3 + \lambda & -2 \\ 3 & 2 + \lambda \end{pmatrix} = -\lambda(1 - \lambda). \quad (10.13)
\]

The case $\tilde{E}_8$:

\[
\begin{array}{c|c|c|c|c}
 k & \text{involved } t_i & \rho : \text{order of } \lambda & \psi_3 : \text{order of } \lambda \\
1 = a_1 & t_9 & \frac{1}{3} & 2 \\
2 = a_2 = a_3 & t_8, t_7 & 0 + \frac{1}{3} & 1 \text{ (see } 10.13) \\
3 = a_4 = a_5 & t_6, t_5 & 0 + 1 & 0 + 0 \\
4 = a_6 = a_7 & t_4, t_3 & 0 + \frac{2}{3} & 0 + 0 \\
5 = a_8 & t_2 & \frac{1}{3} & 0 \\
\end{array}
\]

\[
k \psi_2 : \text{order of } \lambda \text{ in } (5.33) \quad - \deg C_{1,(k)}/ \deg p_{\text{alg}}
\]

\[
\begin{array}{c|c|c}
k & \psi_2 & -\deg C_{1,(k)}/ \deg p_{\text{alg}} \\
1 & -\frac{1}{2} & 3 \cdot \frac{1}{2} + 2 - \frac{1}{2} = \frac{3}{2} \\
2 & 0 - 1 & 3 \cdot \frac{1}{3} + 1 - 1 = 1 \\
3 & -\frac{1}{2} - \frac{3}{2} & 3 \cdot 1 + 0 - 2 = 1 \\
4 & 0 - 1 & 3 \cdot \frac{2}{3} + 0 - 1 = 1 \\
5 & -\frac{1}{2} & 3 \cdot \frac{1}{3} + 0 - \frac{1}{2} = \frac{1}{2} \\
\end{array}
\]

In all cases (10.12) holds. This and (10.11) show (6.7). This completes the proof of theorem 6.3.

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