Collocation approximation by deep neural ReLU networks for parametric and stochastic PDEs with lognormal inputs

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Abstract

We obtained convergence rates of the collocation approximation by deep ReLU neural networks of solutions to elliptic PDEs with lognormal inputs, parametrized by $y$ from the non-compact set $\mathbb{R}^\infty$. The approximation error is measured in the norm of the Bochner space $L_2(\mathbb{R}^\infty, V, \gamma)$, where $\gamma$ is the infinite tensor product standard Gaussian probability measure on $\mathbb{R}^\infty$ and $V$ is the energy space. We also obtained similar dimension-independent results for the case when the lognormal inputs are parametrized on $\mathbb{R}^M$ with very large dimension $M$, and the approximation error is measured in the $\sqrt{g_M}$-weighted uniform norm of the Bochner space $L_\infty^{\sqrt{g_M}}(\mathbb{R}^M, V)$, where $g_M$ is the density function of the standard Gaussian probability measure on $\mathbb{R}^M$.

Keywords and Phrases: High-dimensional approximation; Collocation approximation; Deep ReLU neural networks; Parametric elliptic PDEs; Lognormal inputs.

Mathematics Subject Classifications (2010): 65C30, 65D05, 65D32, 65N15, 65N30, 65N35.

1 Introduction

Partial differential equations (PDEs) with parametric and stochastic inputs are a common model used in science and engineering. Stochastic nature reflects the uncertainty in various parameters presented in the physical phenomenon modelled by the equation. A central problem of computational uncertainty quantification is efficient numerical approximation for parametric and stochastic PDEs which has been of great interest and achieved significant progress in recent decades. There is a large number of non-deep-neural-network papers on this topic to mention all of them. We point out just some works [3, 4, 5, 7, 8, 9, 10, 11, 12, 14, 15, 24, 36, 61, 62] which are directly related to our
paper. In particular, collocation approximations which are based on a finite number of particular solvers to parametric and stochastic PDEs, were considered in [8, 9, 10, 14, 15, 18, 24, 61].

The approximation universality of neural networks has been achieved a basis understanding since the 1980’s ([6, 13, 25, 37]). Deep neural networks in recent years have been rapidly developed in theory and applications to a wide range of fields due to their advantage over shallow ones. Since their application range is getting wider, theoretical analysis discovering reasons of these significant practical improvements attracts special attention [2, 20, 44, 56, 57]. In recent years, there has been a number of interesting papers that addressed the role of depth and architecture of deep neural networks for non-adaptive and adaptive approximation of functions having a particular regularity [1, 22, 29, 32, 31, 42, 39, 51, 48, 59, 60]. High-dimensional approximations by deep neural networks have been studied in [43, 53, 16, 19], and their applications to high-dimensional PDEs in [23, 27, 28, 30, 33, 46, 52]. Most of these papers employed the rectified linear unit (ReLU) as the activation function of deep neural networks since the ReLU is a simple and preferable in many applications. The output of such a deep neural network is a continuous piece-wise linear function which is easily and cheaply computed. The reader can consult the recent survey papers [21, 47] for various problems and aspects of neural network approximation and bibliography.

Recently, a number of papers have been devoted to various problems and methods of deep neural network approximation for parametric and stochastic PDEs such as dimensionality reduction [58], deep neural network expression rates for generalized polynomial chaos expansions (gpc) of solutions to parametric elliptic PDEs [17, 49], reduced basis methods [38] the problem of learning the discretized parameter-to-solution map in practice [26], Bayesian PDE inversion [33, 34, 45], etc. Note that except [17] all of these papers treated parametric and stochastic PDEs with affine inputs on the compact set $\mathbb{I}^\infty := [-1, 1]^\infty$. The authors of paper [49] proved dimension-independent deep neural network expression rate bounds of the uniform approximation of solution to parametric elliptic PDE with affine inputs on $\mathbb{I}^\infty$ based on $n$-term truncations of the non-orthogonal Taylor gpc expansion. The construction of approximating deep neural networks relies on weighted summability of the Taylor gpc expansion coefficients of the solution which is derived from its analyticity. The paper [17] investigated non-adaptive methods of deep ReLU neural network approximation of the solution $u$ to parametric and stochastic elliptic PDEs with lognormal inputs on non-compact set $\mathbb{R}^\infty$. The approximation error is measured in the norm of the Bochner space $L_2(\mathbb{R}^\infty, V, \gamma)$, where $\gamma$ is the tensor product standard Gaussian probability on $\mathbb{R}^\infty$ and $V$ is the energy space. The approximation is based on an $m$-term truncation of the Hermite gpc of $u$. Under a certain assumption on $\ell_q$-summability ($0 < q < \infty$) for the lognormal inputs, it was proven that for every integer $n > 1$, one can construct a non-adaptive compactly supported deep ReLU neural network $\phi_n$ of size $\leq n$ on $\mathbb{R}^m$ with $m = O(n / \log n)$, having $m$ outputs so that the summation constituted by replacing Hermite polynomials in the $m$-term truncation by these $m$ outputs approximates $u$ with the error bound $O \left( \left( n / \log n \right)^{-1/q} \right)$. The authors of [17] also obtained some results on similar problems for parametric and stochastic elliptic PDEs with affine inputs, based on the Jacobi and Taylor gpc expansions.

In the present paper, we are interested in constructing deep ReLU neural networks for collocation approximation of the solution to parametric elliptic PDEs with lognormal inputs. We study the convergence rate of this approximation in terms of the size of deep ReLU neural networks.
Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Consider the diffusion elliptic equation

$$-\text{div}(a \nabla u) = f \quad \text{in} \quad D, \quad u|_{\partial D} = 0,$$

(1.1)

for a given right-hand side $f$ and diffusion coefficient $a$ as functions on $D$. Denote by $V := H^1_0(D)$ the so-called energy space of all those functions from the Sobolev space $H^1(D)$ that have compact support in $D$. Let $H^{-1}(D)$ be the dual space of $V$. Assume that $f \in H^{-1}(D)$ (in what follows this preliminary assumption always holds without mention). If $a \in L_{\infty}(D)$ satisfies the ellipticity assumption

$$0 < a_{\min} \leq a \leq a_{\max} < \infty,$$

by the well-known Lax–Milgram lemma, there exists a unique solution $u \in V$ to the (non-parametric) equation (1.1) in the weak form

$$\int_D a \nabla u \cdot \nabla v \, dx = \langle f, v \rangle, \quad \forall v \in V.$$

Partial differential equations with parametric and stochastic inputs are a common model used in science and engineering. For the equation (1.1), we consider diffusion coefficients having a parametrized form $a = a(y)$, where $y = (y_j)_{j \in \mathbb{N}}$ is a sequence of real-valued parameters ranging in the set $\mathbb{R}^\infty$. Denote by $u(y)$ the solution to the parametrized diffusion elliptic equation

$$-\text{div}(a(y) \nabla u(y)) = f \quad \text{in} \quad D, \quad u(y)|_{\partial D} = 0.$$

(1.2)

The resulting solution operator maps $y \in \mathbb{R}^\infty \mapsto u(y) \in V$. The goal is to achieve numerical approximation of this complex map by a small number of parameters with a guaranteed error in a given norm. Depending on the nature of the modeled object, the parameter $y$ may be either deterministic or random. In the present paper, we consider the so-called lognormal case when the diffusion coefficient $a$ is of the form

$$a(y) = \exp(b(y))$$

(1.3)

with $b(y)$ in the infinite-dimensional form:

$$b(y) = \sum_{j=1}^{\infty} y_j \psi_j, \quad y \in \mathbb{R}^\infty,$$

(1.4)

where the $y_j$ are i.i.d. standard Gaussian random variables and $\psi_j \in L_{\infty}(D)$. We also consider the finite-dimensional form when

$$b(y) = \sum_{j=1}^{M} y_j \psi_j, \quad y \in \mathbb{R}^M,$$

(1.5)

with finite but very large dimension $M$. Notice that for a fixed $y$ both the cases (1.4) and (1.5) of equation (1.2) satisfy the ellipticity assumption, and therefore there exists exists a unique solution $u(y) \in V$ to the equation (1.2) in the weak form. However, there is no the uniform ellipticity with respect to $y$ since $\mathbb{R}^\infty$ and $\mathbb{R}^M$ are not compact sets.
We briefly describe the main results of the present paper.

We investigate non-adaptive collocation methods of high-dimensional deep ReLU neural network approximation of the solution $u(y)$ to parametrized diffusion elliptic PDEs (1.2) with log-normal inputs (1.3) in the infinite-dimensional case (1.4) and finite-dimensional case (1.5). In the infinite-dimensional case (1.4), the approximation error is measured in the norm of the Bochner space $L_2(\mathbb{R}^\infty, V, \gamma)$, where $\gamma$ is the infinite tensor product standard Gaussian probability on $\mathbb{R}^\infty$. Assume that there exists an increasing sequence of positive numbers strictly larger than one $\rho = (\rho_j)_{j \in \mathbb{N}}$ such that for some $0 < q < 2$,

$$\left\| \sum_{j \in \mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} < \infty \quad \text{and} \quad \rho^{-1} = (\rho_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N}).$$

Then, given an arbitrary number $\delta$ with $0 < \delta < \min(1, 1/q - 1/2)$, for every integer $n > 1$, we can construct a deep ReLU neural network $\phi_n := (\phi_j)_{j=1}^m$ on $\mathbb{R}^m$ with $m = O(n^{1-\delta})$ of size at most $n$ and a sequence of points $Y_n := (y_j^m)_{j=1}^m \subset \mathbb{R}^m$ so that

(i) The deep ReLU neural network $\phi_n$ and sequence of points $Y_n$ are independent of $u$;
(ii) The output dimension of $\phi_n$ is $m = O(n^{1-\delta})$;
(iii) The depth of $\phi_n$ is $O(n^\delta)$;
(iv) The components $\phi_j$, $j = 1, \ldots, m$, of $\phi_n$ are deep ReLU neural networks on $\mathbb{R}^{m_j}$ with $m_j = O(n^\delta)$, having support of contained in the super-cube $[-T, T]^{m_j}$ with $T = O(n^{1-\delta})$;
(v) If $\Phi_j$ is the extension of $\phi_j$ to the whole $\mathbb{R}^\infty$ by $\Phi_j(y) = \phi_j((y_j^m)_{j=1}^m)$ for $y = (y_j, j \in \mathbb{N}) \in \mathbb{R}^\infty$, the collocation approximation of $u$ by the function

$$\Phi_n u := \sum_{j=1}^m u(y^j) \Phi_j,$$

which is based on the $m$ solvers $u((y^j)^m_{j=1})$ and the deep ReLU network $\phi_n$, gives the twofold error estimates

$$\|u - \Phi_n u\|_{L_2(\mathbb{R}^\infty, V, \gamma)} = O\left(m^{-(\frac{1}{q} - \frac{1}{2})}\right) = O\left(n^{-(1-\delta)}(\frac{1}{q} - \frac{1}{2})\right).$$

(1.6)

We also obtained similar results in manner of the items (i)–(v) in the finite-dimensional case (1.5) with the approximation error measured in the $\sqrt{g_M}$-weighted uniform norm of the Bochner space $L_2^{\sqrt{g}}(\mathbb{R}^M, V)$, where $g_M$ is the density function of the standard Gaussian probability measure on $\mathbb{R}^M$.

These results are derived from results on deep ReLU neural network collocation approximation of functions in Bochner spaces related to a general separable Hilbert space and standard Gaussian
probability measures based on weighted $\ell_2$-summabilities of the Hermite gpc expansion coefficients of functions (see Section 3 for details).

Notice that the error bound in $m$ in (1.6) is the same as the error bound of the collocation approximation of $u$ by the sparse-grid Lagrange gpc interpolation based on $m$ the same particular solvers $(u(y^j))_{j=1}^m$, which so far is the best known result [15, Corollary 3.1]. Moreover, the convergence rate $(1 - \delta)(1/q - 1/2)$ with arbitrarily small $\delta > 0$ in terms of the size of the deep ReLU network in the collocation approximation, is comparable with the convergence rate $1/q - 1/2$ with respect to the number of particular solvers in the collocation approximation by sparse-grid Lagrange gpc interpolation. This is a crucial difference between the results of the present paper and of [17] which proved the convergence rate of the deep ReLU network approximation of solutions to parametrized diffusion elliptic PDEs (1.2) with lognormal inputs (1.3) based on a different input information – the coefficients of Hermite gpc expansion in its finite truncations. Although that convergence rate is sharper than one in (1.6), in general, it is well-known that collocation approximations are more important, difficult and applicable than those using spectral information about the coefficients of an orthonormal expansion. The extension of the results (i)-(v) to the Bochner space $L_\infty^\gamma(\mathbb{R}^M, V)$ is also an important difference of our contribution comparing with [17].

We would like to emphasize that the motivation of this paper is to establish approximation results which should show posibilities of non-adaptive collocation approximation by deep ReLU neural networks and convergence rates of approximation for the parametrized diffusion elliptic equation (1.2) with lognormal inputs, and we do not consider the numerical aspect of the problem. The results themselves do not give a practically realizable approximation because they do not cover the approximation of the coefficients which are particular solvers at certain points of the spatial variables. Moreover, the approximant $\Phi_n u$ is not a real deep ReLU networks, but just a combination of these particular solvers and the components of a deep ReLU network. It would be interesting to investigate the problem of full deep ReLU neural network approximation of the solution $u$ to parametric and stochastic elliptic PDEs by combining the spatial and parametric domains based on fully discrete approximation in [3, 15]. This problem will be discussed in a forthcoming paper.

The paper is organized as follows. In Section 2, we present a necessary knowledge about deep ReLU neural networks. Section 3 is devoted to collocation methods of deep ReLU neural network approximation of functions in Bochner spaces $L_2(\mathbb{R}^\infty, X, \gamma)$ or in $L_2(\mathbb{R}^M, X, \gamma)$ related to a separable Hilbert space $X$ and the tensor product standard Gaussian probability measure $\gamma$. In Section 4, we apply the results in the previous section to the collocation approximation by deep ReLU neural networks of the solution $u$ to the parametrized elliptic PDEs (1.2) with lognormal inputs (1.3) on in the infinite case (1.4) and finite case (1.5).

Notation As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers and $\mathbb{N}_0 := \{s \in \mathbb{Z} : s \geq 0\}$. We denote $\mathbb{R}^\infty$ the set of all sequences $y = (y_j)_{j \in \mathbb{N}}$ with $y_j \in \mathbb{R}$. For a set $G$, we denote by $|G|$ the cardinality of $G$. If $a = (a_j)_{j \in J}$ is a sequence of positive numbers with any index set $J$, then we use the notation $a^{-1} := (a_j^{-1})_{j \in J}$. We use letters $C$ and $K$ to denote general positive constants which may take different values, and $C_{\alpha,\beta,...}$ and $K_{\alpha,\beta,...}$ when we want
to emphasize the dependence of these constants on $\alpha, \beta, \ldots$, or when this dependence is important in a particular situation.

For convenience to the reader, we list some specific notations and definitions which widely used in the present paper and indicate where they are introduced.

Section 2: The symbols $W(\Phi), L(\Phi)$ and $\text{supp}(\Phi)$ denote the size, the depth and the support of the deep ReLU neural network $\Phi$, respectively; $\sigma(t) := \max\{t, 0\}$ is ReLU activation function.

Section 3.1: Denote by $F$ the set of all sequences of non-negative integers $s = (s_j)_{j \in \mathbb{N}}$ such that their support $\text{supp}(s) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set. Let $J$ denotes either $\infty$ or $M \in \mathbb{N}$; the set $U$ is defined in (3.2), the set $F$ in (3.6) and the set $N$ in (3.7); $\gamma$ and $\gamma_M$ are the standard Gaussian measures in $\mathbb{R}^{\infty}$ and $\mathbb{R}^M$, respectively. For $s \in F$, put $|s| := \sum_{j \in \mathbb{N}} s_j$ and $|s|_0 := |\text{supp}(s)|$. For $s, s' \in F$, the inequality $s' \preceq s$ means that $s'_j \leq s_j$, $j \in \mathbb{N}$. A set $\sigma = (\sigma_s)_{s \in F}$ with $\sigma_s \in \mathbb{R}$ is called increasing if $\sigma_{s'} \preceq \sigma_s$ for $s' \preceq s$. The Bochner space $\mathcal{L}(U, X)$ is defined in (3.5); the Bochner spaces $L_2(U, X, \gamma)$ and $L_{\gamma_M}^{\infty}(\mathbb{R}^M, X)$ are given by (3.3) and (3.4), respectively; In (3.9), $H_s$ is defined as the $s$th Hermite orthonormal polynomial and $v_s$ as the $s$th coefficient of the Hermite gpc expansion of $v$.

Section 3.2: $Y_m = (y_{m:k})_{k \in \pi_m}$ is the increasing sequence of the $m + 1$ roots of the Hermite polynomial $H_{m+1}$; $I_m$ is the Lagrange interpolation operator defined by (3.12); $\lambda_m$ is the Lebesgue constant defined by (3.13); $\Delta_s$ is the tensor product operator defined by (3.16); $I_A$ is the gpc interpolation operator defined by (3.18); the set $p(\theta, \lambda) := (p_s(\theta, \lambda))_{s \in F}$ is defined by (3.20); the set $A(\xi)$ is defined by (3.21) and the set $G(\xi)$ by (3.23).

2 Deep ReLU neural networks

In this section, we present some auxiliary knowledge on deep ReLU neural networks which will be used as a tool of approximation. As in [50], we will use such deep feed-forward neural networks that allows connections between neurons in a layer with neurons in any preceding layers (but not in the same layer). The ReLU activation function is defined by $\sigma(t) := \max\{t, 0\}, t \in \mathbb{R}$. We denote: $\sigma(x) := (\sigma(x_1), \ldots, \sigma(x_N))$ for $x = (x_1, \ldots, x_N) \in \mathbb{R}^d$.

Let us recall a standard definition of deep ReLU neural network and relevant terminology. Let $d, L \in \mathbb{N}$, $L \geq 2$, $N_0 = d$, and $N_1, \ldots, N_L \in \mathbb{N}$. Let $W^\ell = \left(w_{i,j}^\ell \right) \in \mathbb{R}^{N_\ell \times (\sum_{i=1}^{L-1} N_i)}$, $\ell = 1, \ldots, L$, be an $N_\ell \times (\sum_{i=1}^{\ell-1} N_i)$ matrix, and $b^\ell = (b_j^\ell) \in \mathbb{R}^{N_\ell}$. A ReLU neural network $\Phi$ (on $\mathbb{R}^d$) with input dimension $d$, output dimension $N_L$ and $L$ layers is called a sequence of matrix-vector tuples

$\Phi = ((W^1, b^1), \ldots, (W^L, b^L))$, 

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in which the following computation scheme is implemented:

\[ z^0 := x \in \mathbb{R}^d; \]
\[ z^\ell := \sigma \left( W^\ell (z^0, \ldots, z^{\ell-1})^T + b^\ell \right), \quad \ell = 1, \ldots, L - 1; \]
\[ z^L := W^L (z^0, \ldots, z^{L-1})^T + b^L. \]

We call \( z^0 \) the input and with an ambiguity we use the notation \( \Phi(x) := z^L \) for the output of \( \Phi \) which is an \( L \)-dimensional vector-function on \( \mathbb{R}^d \). In some places we identify a ReLU neural network with its output. We adopt the following terminology.

- The number of layers \( L(\Phi) = L \) is the depth of \( \Phi \);
- The number of nonzero \( w_{i,j}^\ell \) and \( b_j^\ell \) is the size of \( \Phi \) and denoted by \( W(\Phi) \);
- When \( L(\Phi) \geq 3 \), \( \Phi \) is called a deep ReLU neural network, and otherwise, a shallow ReLU neural network.
- If \( \Phi(x) = (\phi_j(x))_{j=1}^L \), the support of the deep ReLU neural network \( \Phi \) is defined as \( \bigcup_{j=1}^L \text{supp}(\phi_j) \) and denoted by \( \text{supp}(\Phi) \).

There are two basic operations which neural networks allow for. This is the parallelization of several neural networks and the concatenation of two neural networks. The reader can find for instance, in [32] (see also [21, 47]) for detailed decriptions as well as the following two lemmata on these operations.

**Lemma 2.1 (Parallelization)** Let \( N \in \mathbb{N} \), \( \lambda_j \in \mathbb{R}, j = 1, \ldots, N \). Let \( \Phi_j, j = 1, \ldots, N \) be deep ReLU neural networks with input dimension \( d \). Then we can explicitly construct a deep ReLU neural network denoted by \( \Phi \) so that

\[ \Phi(x) = \sum_{j=1}^N \lambda_j \Phi_j(x), \quad x \in \mathbb{R}^d. \]

Moreover, we have

\[ W(\Phi) \leq \sum_{j=1}^N W(\Phi_j) \quad \text{and} \quad L(\Phi) = \max_{j=1,\ldots,N} L(\Phi_j). \]

The deep ReLU neural network \( \Phi \) is called the parallelization of \( \Phi_j, j = 1, \ldots, N \).

**Lemma 2.2 (Concatenation)** Let \( \Phi_1 \) and \( \Phi_2 \) be two ReLU neural networks such that output layer of \( \Phi_1 \) has the same dimension as input layer of \( \Phi_2 \). Then, we can explicitly construct a ReLU neural network \( \Phi \) such that \( \Phi(x) = \Phi_2(\Phi_1(x)) \) for \( x \in \mathbb{R}^d \). Moreover we have

\[ W(\Phi) \leq 2W(\Phi_1) + 2W(\Phi_2) \quad \text{and} \quad L(\Phi) = L(\Phi_1) + L(\Phi_2). \]

The deep ReLU neural network \( \Phi \) is called the concatenation of \( \Phi_1 \) and \( \Phi_2 \).
The following lemma is a direct consequence of [49, Proposition 3.3].

**Lemma 2.3**  Let \( \ell \in \mathbb{N}^d \). For every \( \delta \in (0, 1) \), we can explicitly construct a deep ReLU neural network \( \Phi_P \) on \( \mathbb{R}^d \) so that

\[
\sup_{x \in [-1,1]^d} \left| \prod_{j=1}^d x_j^{\ell_j} - \Phi_P(x) \right| \leq \delta.
\]

Furthermore, if \( x_j = 0 \) for some \( j \in \{1, \ldots, d\} \) then \( \Phi_P(x) = 0 \) and there exists a constant \( C > 0 \) independent of \( \delta, d \) and \( \ell \) such that

\[
W(\Phi_P) \leq C|\ell|_1 \log(|\ell|_1 \delta^{-1}) \quad \text{and} \quad L(\Phi_P) \leq C \log |\ell|_1 \log(|\ell|_1 \delta^{-1}).
\]

For \( j = 0, 1 \), let \( \varphi_j \) be the continuous piece-wise linear functions with break points \( \{-2, -1, 1, 2\} \) and \( \text{supp}(\varphi_j) \subset [-2, 2] \) such that \( \varphi_0(x) = 1 \) and \( \varphi_1(x) = x \) if \( x \in [-1, 1] \).

**Lemma 2.4**  Let \( \ell \in \mathbb{N}^d \) and \( \varphi \) be either \( \varphi_0 \) or \( \varphi_1 \). For every \( \delta \in (0, 1) \), we can explicitly construct a deep ReLU neural network \( \Phi \) on \( \mathbb{R}^d \) so that

\[
\sup_{x \in [-2,2]^d} \left| \prod_{j=1}^d \varphi_j(x_j) - \Phi(x) \right| \leq \delta.
\]

Furthermore, \( \text{supp}(\Phi) \subset [-2,2]^d \) and there exists a constant \( C > 0 \) independent of \( \delta, d \) and \( \ell \) such that

\[
W(\Phi) \leq C(1 + |\ell|_1 \log(|\ell|_1 \delta^{-1})) \quad \text{and} \quad L(\Phi) \leq C \log |\ell|_1 \log(|\ell|_1 \delta^{-1}) \quad \text{for} \quad (2.1)
\]

**Proof.** Notice that the explicit forms of \( \varphi_j \) via the ReLU activation function are

\[
\varphi_0(x) = \sigma(x - 2) - 3\sigma(x - 1) + 4\sigma(x) - 3\sigma(x + 1) + \sigma(x + 2),
\]

and

\[
\varphi_1(x) = \sigma(x - 2) - 2\sigma(x - 1) + 2\sigma(x + 1) - \sigma(x + 2).
\]

This yields that \( \varphi_j \) can be realized exactly by a shallow ReLU neural network (still denoted by \( \varphi_j \)) with size \( W(\varphi_0) \leq 10 \) and \( W(\varphi_1) \leq 8 \). The network \( \Phi \) can be constructed as a concatenation of deep ReLU neural networks \( \{\varphi(x_j)\}_{j=1}^d \) and \( \Phi_P \). By the definitions of deep ReLU neural network and function \( \varphi \) we have

\[
z^1 = \{\varphi(x_j)\}_{j=1}^d \subset [-1,1]^d.
\]

Hence, the estimates (2.1) follow directly from Lemmata 2.2 and 2.3. \( \Box \)
3 Deep ReLU neural network approximation in Bochner spaces

In this section, we investigate collocation methods of deep ReLU neural network approximation of functions in Bochner spaces related to a Hilbert space $X$ and tensor product standard Gaussian probability measures $\gamma$. Functions to be approximated have the weighted $\ell_2$-summable Hermite gpc expansion coefficients (see Assumption (I) below). The approximation is based on the sparse-grid Lagrange gpc interpolation. We construct such methods and prove convergence rates of the approximation by them. The results obtained in this section will be applied to deep ReLU neural network collocation approximation of the solution of parametrized elliptic PDEs with lognormal inputs in the next section.

3.1 Tensor product Gaussian measures and Bochner spaces

Let $\gamma(y)$ be the standard Gaussian probability measure on $\mathbb{R}$ with the density

$$g(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad \text{i.e.,} \quad d\gamma(y) := g(y) \, dy.$$  

(3.1)

For $M \in \mathbb{N}$, the standard Gaussian probability measures $\gamma(y)$ on $\mathbb{R}^M$ can be defined by

$$d\gamma(y) := g_M(y) \, dy = \bigotimes_{j=1}^M g(y_j) \, dy_j, \quad y = (y_j)_{j=1}^M \in \mathbb{R}^M,$n

where $g_M(y) := \bigotimes_{j=1}^M g(y_j)$.

We next recall a concept of standard Gaussian probability measure $\gamma(y)$ on $\mathbb{R}^\infty$ as the infinite tensor product of the standard Gaussian probability measures $\gamma(y_j)$:

$$\gamma(y) := \bigotimes_{j\in\mathbb{N}} \gamma(y_j), \quad y = (y_j)_{j\in\mathbb{N}} \in \mathbb{R}^\infty.$n

The sigma algebra for $\gamma(y)$ is generated by the set of cylinders $A := \prod_{j\in\mathbb{N}} A_j$, where $A_j \subset \mathbb{R}$ are univariate $\gamma$-measurable sets and only a finite number of $A_i$ are different from $\mathbb{R}$. For such a set $A$, we have $\gamma(A) = \prod_{j\in\mathbb{N}} \gamma(A_j)$. (For details on infinite tensor product of probability measures, see, e.g., [35, pp. 429–435].)

In what follows, we use the joint notation: $J$ denotes either $\infty$ or $M \in \mathbb{N}$ and

$$U := \begin{cases} \mathbb{R}^M & \text{if } J = M, \\ \mathbb{R}^\infty & \text{if } J = \infty; \end{cases}$$

(3.2)

If $X$ is a separable Hilbert space, the standard Gaussian probability measure $\gamma$ on $U$ induces the Bochner space $L_2(U, X, \gamma)$ of $\gamma$-measurable mappings $v$ from $U$ to $X$, equipped with the norm

$$\|v\|_{L_2(U, X, \gamma)} := \left( \int_U \|v(\cdot, y)\|^2_X \, d\gamma(y) \right)^{1/2}.$$  

(3.3)
For a \( \gamma \)-measurable subset \( \Omega \) in \( U \) the spaces \( L_2(\Omega, X, \gamma) \) and \( L_2(\Omega, \gamma) \) is defined in the usual way.

In the case \( U = \mathbb{R}^M \), we introduce also the space \( L_2^\sqrt{g}(\mathbb{R}^M, X) \) as the set of all strongly \( \gamma \)-measurable functions \( v: \mathbb{R}^M \to X \) for which the \( \sqrt{g_M} \)-weighted uniform norm

\[
\|v\|_{L_2^\sqrt{g}(\mathbb{R}^M, X)} := \text{ess sup}_{y \in \mathbb{R}^M} \left( \|v(y)\|_X \sqrt{g_M(y)} \right) < \infty.
\]  

One may expect an infinite-dimensional version of this space. Unfortunately, we could not give a correct definition of space \( L_2^\sqrt{g}(\mathbb{R}^\infty, X) \) because there is no an infinite-dimensional counterpart of the weight \( g_M \). However, under certain assumptions (see Assumption (I) in Subsection 3.2), we can obtain some approximation results which do not depend on \( M \), in particular, when \( M \) are very large. We make use of the abbreviations: \( L_2^\sqrt{g}(\mathbb{R}^M, X) = L_2^\sqrt{g}(\mathbb{R}^M, \mathbb{R}) \) and \( L_2^\sqrt{g}(\mathbb{R}^\infty, X) = L_2^\sqrt{g}(\mathbb{R}, \mathbb{R}) \).

In this section, we will investigate the problem of deep ReLU neural network approximation of functions in \( L_2(\mathbb{R}^\infty, X, \gamma) \) or \( L_2(\mathbb{R}^M, X, \gamma) \) with the error measured in the norms of the space \( L_2^\sqrt{g}(\mathbb{R}^\infty, X) \) or of the space \( L_2^\sqrt{g}(\mathbb{R}^M, X) \), respectively. (Notice that these norms are the most important in evaluation of the error of collocation approximation of solutions of parametric and stochastic PDEs). It is convenient to us to incorporate these different approximation problems into unified consideration. Hence, in what follows, we use the joint notations:

\[
L(U, X) := \begin{cases} 
L_2^\sqrt{g}(\mathbb{R}^M, X) \quad & \text{if } U = \mathbb{R}^M, \\
L_2(\mathbb{R}^\infty, X, \gamma) \quad & \text{if } U = \mathbb{R}^\infty;
\end{cases}
\]  

\[
\mathcal{F} := \begin{cases} 
\mathbb{N}_0^M \quad & \text{if } U = \mathbb{R}^M, \\
\mathcal{F} \quad & \text{if } U = \mathbb{R}^\infty;
\end{cases}
\]

and

\[
\mathcal{N} := \begin{cases} 
\{1, \ldots, M\} \quad & \text{if } U = \mathbb{R}^M, \\
\mathbb{N} \quad & \text{if } U = \mathbb{R}^\infty.
\end{cases}
\]

Here \( \mathcal{F} \) is the set of all sequences of non-negative integers \( s = (s_j)_{j \in \mathbb{N}} \) such that their support \( \text{supp}(s) := \{j \in \mathbb{N} : s_j > 0\} \) is a finite set.

Let \( (H_k)_{k \in \mathbb{N}_0} \) be the Hermite polynomials normalized according to \( \int_{\mathbb{R}} |H_k(y)|^2 g(y) \, dy = 1 \). Then a function \( v \in L_2(U, X, \gamma) \) can be represented by the Hermite gpc expansion

\[
v(y) = \sum_{s \in \mathcal{F}} v_s H_s(y), \quad v_s \in X,
\]  

(3.8)

with

\[
H_s(y) = \bigotimes_{j \in \mathcal{N}} H_{s_j}(y_j), \quad v_s := \int_U v(y) H_s(y) \, d\gamma(y), \quad s \in \mathcal{F}.
\]  

(3.9)

Notice that \( (H_s)_{s \in \mathcal{F}} \) is an orthonormal basis of \( L_2(U, \gamma) := L_2(U, \mathbb{R}, \gamma) \). Moreover, for every \( v \in L_2(U, X, \gamma) \) represented by the series (3.8), Parseval’s identity holds

\[
\|v\|_{L_2(U,X,\gamma)}^2 = \sum_{s \in \mathcal{F}} \|v_s\|^2_X.
\]
For \( s, s' \in \mathcal{F} \), the inequality \( s' \leq s \) means that \( s'_j \leq s_j, j \in \mathcal{N} \). A set \( \sigma = (\sigma_s)_{s \in \mathcal{F}} \) with \( \sigma_s \in \mathbb{R} \) is called increasing if \( \sigma_{s'} \leq \sigma_s \) for \( s' \leq s \).

**Assumption (I)** For \( v \in L_2(U, X, \gamma) \) represented by the series (3.8), there exists an increasing set \( \sigma = (\sigma_s)_{s \in \mathcal{F}} \) of positive numbers such that for some \( q \) with \( 0 < q < 2 \),

\[
\left( \sum_{s \in \mathcal{F}} (\sigma_s \|v_s\|_X)^2 \right)^{1/2} \leq C_1 < \infty, \quad \text{with} \quad \|\sigma^{-1}\|_{\ell_q(\mathcal{F})} \leq C_2 < \infty, \tag{3.10}
\]

where the constants \( C_1 \) and \( C_2 \) are independent of \( J \).

Here and in what follows, "independent of \( J \)" means that \( C_1 \) and \( C_2 \) (and other constants) are independent of \( M \) when \( J = M \), since we are interested in convergence rates and other asymptotic properties which do not depend on \( M \) and which is based on Assumption (I).

**Lemma 3.1** For \( v \in L_2(U, X, \gamma) \) satisfying Assumption (I), the series (3.8) converges absolutely and therefore, unconditionally in \( L(U, X) \) to \( v \) and

\[
\sum_{s \in \mathcal{F}} \|v_s\|_X \leq C < \infty, \tag{3.11}
\]

where the constant \( C \) is independent of \( J \).

**Proof.** By applying the Hölder inequality from Assumption (I) we obtain

\[
\sum_{s \in \mathcal{F}} \|v_s\|_X \leq \left( \sum_{s \in \mathcal{F}} (\sigma_s \|v_s\|_X)^2 \right)^{1/2} \left( \sum_{s \in \mathcal{F}} \sigma_s^{-2} \right)^{1/2} \leq C \|\sigma^{-1}\|_{\ell_q(\mathcal{F})} < \infty.
\]

This proves (3.11). Hence, by the equality \( \|H_s\|_{L_2(\mathbb{R}^\infty)} = 1, \ s \in \mathcal{F} \), and the inequality \( \|H_s\|_{L_\infty(\mathbb{R}^\infty)} < 1, \ s \in \mathbb{N}_0^M \) (which follows from (A.9) in Appendix), the series (3.8) converges absolutely, and therefore, unconditionally to \( v \in L_2(U, X, \gamma) \) since by the Parseval’s identity it already converges to \( v \) in the norm of \( L_2(U, X, \gamma) \).

### 3.2 Sparse-grid Lagrange gpc interpolation

For \( m \in \mathbb{N}_0 \), let \( Y_m = (y_{m,k})_{k \in \pi_m} \) be the increasing sequence of the \( m + 1 \) roots of the Hermite polynomial \( H_{m+1} \), ordered as

\[
y_{m,-j} < \cdots < y_{m,1} < y_{m,0} = 0 < y_{m,1} < \cdots < y_{m,j} \quad \text{if} \quad m = 2j,
y_{m,-j} < \cdots < y_{m,-1} < y_{m,1} < \cdots < y_{m,j} \quad \text{if} \quad m = 2j - 1,
\]

where

\[
\pi_m := \begin{cases} \{-j, -j + 1, \ldots, -1, 0, 1, \ldots, j - 1, j\} & \text{if} \quad m = 2j; \\
\{-j, -j + 1, \ldots, -1, 1, \ldots, j - 1, j\} & \text{if} \quad m = 2j - 1.
\end{cases}
\]
(in particular, \( Y_0 = (y_0,0) \) with \( y_0,0 = 0 \)).

For a function \( v \) on \( \mathbb{R} \) taking values in a Hilbert space \( X \) and \( m \in \mathbb{N}_0 \), we define the Lagrange interpolation operator \( I_m \) by

\[
I_m(v) := \sum_{k \in \pi_m} v(y_m,k) L_{m,k}, \quad L_{m,k}(y) := \prod_{j \in \pi_m, j \neq k} \frac{y - y_m,j}{y_m,k - y_m,j},
\]

(3.12)

(in particular, \( I_0(v) = v(y_0,0)L_{0,0}(y) = v(0) \) and \( L_{0,0}(y) = 1 \)). Notice that \( I_m(v) \) is a function on \( \mathbb{R} \) taking values in \( X \) and interpolating \( v \) at \( y_m,k \), i.e., \( I_m(v)(y_m,k) = v(y_m,k) \). Moreover, for a function \( v: \mathbb{R} \to \mathbb{R} \), the function \( I_m(v) \) is the Lagrange polynomial having degree \( \leq m \), and that \( I_m(\varphi) = \varphi \) for every polynomial \( \varphi \) of degree \( \leq m \).

Let

\[
\lambda_m := \sup_{\|v\|_{L^\infty(\mathbb{R})} \leq 1} \|I_m(v)\|_{L^\infty(\mathbb{R})}
\]

(3.13)

be the Lebesgue constant. It was proven in [40, 41, 54] that

\[
\lambda_m \leq C(m + 1)^{1/6}, \quad m \in \mathbb{N},
\]

for some positive constant \( C \) independent of \( m \) (with the obvious inequality \( \lambda_0(Y_0) \leq 1 \)). Hence, for every \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \geq 1 \) independent of \( m \) such that

\[
\lambda_m \leq (1 + C_\varepsilon m)^{1/6 + \varepsilon}, \quad \forall m \in \mathbb{N}_0.
\]

(3.14)

We define the univariate operator \( \Delta_m \) for \( m \in \mathbb{N}_0 \) by

\[
\Delta_m := I_m - I_{m-1},
\]

with the convention \( I_{-1} = 0 \).

**Lemma 3.2** For every \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \) independent of \( m \) such that for every function \( v \) on \( \mathbb{R} \),

\[
\|\Delta_m(v)\|_{L^\infty(\mathbb{R})} \leq (1 + C_\varepsilon m)^{1/6 + \varepsilon} \|v\|_{L^\infty(\mathbb{R})}, \quad \forall m \in \mathbb{N}_0.
\]

(3.15)

whenever the norm in the right-hand side is finite.

**Proof.** From the assumptions we have that

\[
\|\Delta_m(v)\|_{L^\infty(\mathbb{R})} \leq 2C(1 + m)^{1/6} \|v\|_{L^\infty(\mathbb{R})}, \quad \forall m \in \mathbb{N}_0,
\]

which implies (3.15).

We will use a sparse-grid Lagrange gpc interpolation as an intermediate approximation in the deep ReLU neural network approximation of functions \( v \in L_2(U, X, \gamma) \). In order to have a correct definition of interpolation operator we have to impose some necessary restrictions on \( v \). Let \( \mathcal{E} \) be
a \gamma\text{-measurable subset in } U \text{ such that } \gamma(E) = 1 \text{ and } E \text{ contains all } y \in U \text{ with } |y|_0 < \infty \text{ in the case } U = \mathbb{R}^\infty, \text{ where } |y|_0 \text{ denotes the number of nonzero components } y_j \text{ of } y. \text{ For a given } E \text{ and Hilbert space } X, \text{ we define } L_2^E(U, X, \gamma) \text{ as the subspace in } L_2(U, X, \gamma) \text{ of all elements } v \text{ such that the point value } v(y) \text{ (of a representative of } v) \text{ is well-defined for all } y \in E. \text{ In what follows, } E \text{ is fixed.}

For \( v \in L_2^E(U, X, \gamma) \), we introduce the tensor product operator \( \Delta_s, s \in F \), by

\[
\Delta_s(v) := \bigotimes_{j \in N} \Delta_{s_j}(v),
\]

where the univariate operator \( \Delta_{s_j} \) is successively applied to the univariate function \( \bigotimes_{i < j} \Delta_{s_i}(v) \) by considering it as a function of variable \( y_j \) with the other variables held fixed. From the definition of \( L_2^E(U, X, \gamma) \) one can see that the operators \( \Delta_s \) are well-defined for all \( s \in F \). We define for \( s \in F \),

\[
I_s(v) := \bigotimes_{j \in N} I_{s_j}(v), \quad L_{s,k} := \bigotimes_{j \in N} L_{s_j,k_j}, \quad \pi_s := \prod_{j \in N} \pi_{s_j},
\]

(the function \( I_s(v) \) is defined in the same manner as \( \Delta_s(v) \)).

For \( s \in F \) and \( k \in \pi_s \), let \( E_s \) be the subset in \( F \) of all \( e \) such that \( e_j \) is either 1 or 0 if \( s_j > 0 \), and \( e_j \) is 0 if \( s_j = 0 \), and let \( y_{s;e,k} := (y_{s_j,k_j})_{j \in N} \in U \). Put \( |s|_1 := \sum_{j \in N} s_j \) for \( s \in F \). It is easy to check that the interpolation operator \( \Delta_s \) can be represented in the form

\[
\Delta_s(v) = \sum_{e \in E_s} (-1)^{|e|_1} I_{s-e}(v) = \sum_{e \in E_s} (-1)^{|e|_1} \sum_{k \in \pi_{s-e}} v(y_{s-e,k}) L_{s-e,k}.
\]

(3.17)

For a given finite set \( \Lambda \subset F \), we introduce the gpc interpolation operator \( I_\Lambda \) by

\[
I_\Lambda := \sum_{s \in \Lambda} \Delta_s.
\]

(3.18)

From (3.17) we obtain

\[
I_\Lambda(v) = \sum_{s \in \Lambda} \sum_{e \in E_s} (-1)^{|e|_1} \sum_{k \in \pi_{s-e}} v(y_{s-e,k}) L_{s-e,k}.
\]

(3.19)

A set \( \Lambda \subset F \) is called downward closed if the inclusion \( s \in \Lambda \) yields the inclusion \( s' \in \Lambda \) for every \( s' \in F \) such that \( s' \leq s \).

For \( \theta, \lambda \geq 0 \), we define the set \( p(\theta, \lambda) := (p_s(\theta, \lambda))_{s \in F} \) by

\[
p_s(\theta, \lambda) := \prod_{j \in N} (1 + \lambda s_j)^\theta, \quad s \in F,
\]

(3.20)

with abbreviations \( p_s(\theta) := p_s(\theta, 1) \) and \( p(\theta) := p(\theta, 1) \).
Let $0 < q < \infty$ and $\sigma = (\sigma_s)_{s \in \mathcal{F}}$ be a set of positive numbers. For $\xi > 0$, define the set

$$\Lambda(\xi) := \{s \in \mathcal{F} : \sigma_s^q \leq \xi\}. \quad (3.21)$$

By the formula (3.19) we can represent the operator $I_{\Lambda(\xi)}$ in the form

$$I_{\Lambda(\xi)}(v) = \sum_{(s,e,k) \in G(\xi)} (-1)^{|e|}v(y_{s-e;k})L_{s-e;k}, \quad (3.22)$$

where

$$G(\xi) := \{(s,e,k) \in \mathcal{F} \times \mathcal{F} \times \mathcal{F} : s \in \Lambda(\xi), e \in E_s, k \in \pi_{s-e}\}. \quad (3.23)$$

The following theorem gives an estimate for the error of the approximation of $v \in \mathcal{L}_2(U,X,\gamma)$ by the sparse-grid Lagrange gpc interpolation $I_{\Lambda(\xi)}v$ on the sampling points in the set $G(\xi)$, which will be used in the deep ReLU neural approximation in the next section.

**Theorem 3.1** Let $v \in \mathcal{L}_2(U,X,\gamma)$ satisfy Assumption (I) and let $\varepsilon > 0$ be a fixed number. Assume that $\|p(\theta/q,\lambda)^{-1}\|_{\ell_q(\mathcal{F})} \leq C < \infty$, where $\theta = 7/3 + 2\varepsilon$, $\lambda := C_\varepsilon$ is the constant in Lemma 3.2, and the constant $C$ is independent of $J$. Then for each $\xi > 1$, we have that

$$\|v - I_{\Lambda(\xi)}v\|_{\mathcal{L}(U,X)} \leq C\xi^{-(1/q-1/2)}, \quad (3.24)$$

where the constant $C$ in (3.24) is independent of $J$, $v$ and $\xi$.

A proof of this theorem is given in Appendix A.2.

**Corollary 3.1** Under the assumptions of Theorem 3.1, for each $n > 1$, we can construct a sequence of points $Y_{\Lambda(\xi_n)} := (y_{s-e;k})_{(s,e,k) \in G(\xi_n)}$ so that $|G(\xi_n)| \leq n$ and

$$\|v - I_{\Lambda(\xi_n)}v\|_{\mathcal{L}(U,X)} \leq Cn^{-(1/q-1/2)}, \quad (3.25)$$

where the constant $C$ in (3.25) is independent of $J$, $v$ and $n$.

**Proof.** Notice that this corollary was proven in [15, Corollary 3.1] for the case $U = \mathbb{R}^\infty$. By Lemma A.2 in Appendix $|G(\xi)| \leq C_q \xi$ for every $\xi > 1$. Hence, the corollary follows from Theorem 3.1 by selection of $\xi_n$ as the maximal number satisfying $|G(\xi_n)| \leq n$. \[ \square \]

### 3.3 Approximation by deep ReLU neural networks

In this section, we construct deep ReLU neural networks for collocation approximation of functions $v \in L_2(U,X,\gamma)$. We primarily approximate $v$ by the sparse-grid Lagrange gpc interpolation $I_{\Lambda(\xi)}v$. Under the assumptions of Lemma A.1(iii) in Appendix, $I_{\Lambda(\xi)}v$ can be seen as a function on $\mathbb{R}^m$, where $m := \min \{M, |K_\eta|\}$. In the next step, we approximate $I_{\Lambda(\xi)}v$ by its truncation $I_{\Lambda(\xi)}^\omega v$ on a sufficiently large super-cube

$$B_\omega^m := [-2\sqrt{\omega}, 2\sqrt{\omega}]^m \subset \mathbb{R}^m, \quad (3.26)$$
where the parameter $\omega$ depending on $\xi$ is chosen in an appropriate way. Finally, the function $I_{\Lambda(\xi)}^\xi v$ and therefore, $v$ is approximated by a function $\Phi_{\Lambda(\xi)} v$ on $\mathbb{R}^m$ which is constructed from a deep ReLU neural network. Let us describe this construction.

For convenience, we consider $\mathbb{R}^m$ as the subset of all $y \in U$ such that $y_j = 0$ for $j > m$. If $f$ is a function on $\mathbb{R}^m$ taking values in a Hilbert space $X$, then $f$ has an extension to $\mathbb{R}^{m'}$ with $m' > m$ and the whole $U$ which is denoted again by $f$, by the formula $f(y) = f\left((y_j)_{j=0}^m\right)$ for $y = (y_j)_{j=1}^m$ and $y = (y_j)_{j \in \mathcal{N}}$, respectively.

Suppose that deep ReLU neural networks $\phi_{s-e;k}$ on $\mathbb{R}^{\|\text{supp}(s)\|}$ are already constructed for approximation of the polynomials $L_{s-e;k}$, $(s,e,k) \in G(\xi)$. Then the network $\Phi_{\Lambda(\xi)} := (\phi_s)_{(s,e,k) \in G(\xi)}$ on $\mathbb{R}^m$ with $|G(\xi)|$ outputs which is constructed by parallelization, is used to construct an approximation of $I_{\Lambda(\xi)}^\xi v$ and hence of $v$. Namely, we approximate $v$ by

$$\Phi_{\Lambda(\xi)} v(y) := \sum_{(s,e,k) \in G(\xi)} (-1)^{|e_1|} v(y_{s-e;k}) \phi_{s-e;k}(y).$$

For the set $\Lambda(\xi)$, we introduce the following numbers:

$$m_1(\xi) := \max_{s \in \Lambda(\xi)} |s|_1,$$

and

$$m(\xi) := \max \{ j \in \mathcal{N} : \exists s \in \Lambda(\xi) \text{ such that } s_j > 0 \}.$$

In this section, we will prove our main results on deep ReLU neural network approximation of functions $v \in L_2^F(U,X,\gamma)$ with the error measured in the norm of the space $L_2(\mathbb{R}^\infty,X,\gamma)$ or of the space $L_\infty^\gamma(\mathbb{R}^M,X)$, which are incorporated into the following joint theorem.

Denote by $e^{\mathcal{F}} = (e^j)_{j \in \mathcal{N}} \in \mathcal{F}$ the element with $e^1 = 1$ and $e^j = 0$ for $j \neq i$.

**Theorem 3.2** Let $v \in L_2^F(U,X,\gamma)$ satisfy Assumption (I). Let $\theta$ be any number such that $\theta \geq 3/q$. Assume that the set $\sigma = (\sigma_s)_{s \in \mathcal{F}}$ in Assumption (I) satisfies $\sigma_{e^{i'}} \leq \sigma_{e^i}$ if $i' < i$, and that $\|p(\theta)\sigma^{-1}\|_{\ell_q(\mathcal{F})} \leq C < \infty$, where the constant $C$ is independent of $j$. Let $K_q$, $K_{q,\theta}$ and $C_q$ be the constants in the assumptions of Lemma A.1 and of Lemma A.2 in Appendix. Then for every $\xi > 2$, we can construct a deep ReLU neural network $\phi_{\Lambda(\xi)} := (\phi_{s-e;k})_{(s,e,k) \in G(\xi)}$ on $\mathbb{R}^m$ with

$$m := \begin{cases} \min \{ M, [K_q \xi] \} & \text{if } U = \mathbb{R}^M, \\ [K_q \xi] & \text{if } U = \mathbb{R}^\infty, \end{cases}$$

and a sequence of points $Y_{\Lambda(\xi)} := (y_{s-e;k})_{(s,e,k) \in G(\xi)}$ having the following properties.

(i) The deep ReLU neural network $\phi_{\Lambda(\xi)}$ and sequence of points $Y_{\Lambda(\xi)}$ are independent of $v$;

(ii) The output dimension of $\phi_{\Lambda(\xi)}$ are at most $[C_q \xi]$;

(iii) $W(\phi_{\Lambda(\xi)}) \leq C \xi^{1+2/\theta q} \log \xi$;

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(iv) \( L(\phi_\Lambda(\xi)) \leq C\xi^{1/\theta q}(\log \xi)^2 \);

(v) The components \( \phi_{s-e,k}, (s,e,k) \in G(\xi), \) of \( \phi_\Lambda(\xi) \) are deep ReLU neural networks on \( \mathbb{R}^{[\text{supp}(s)]} \) with \( |\text{supp}(s)| \leq K_{q,\theta} \xi^{16} \), having support contained in the super-cube \([-T, T]^{[\text{supp}(s)]} \), where \( T := 4\sqrt{K_{q,\theta} \xi} \);

(vi) The approximation of \( v \) by \( \Phi_\Lambda(\xi)v \) gives the error estimate
\[
\|v - \Phi_\Lambda(\xi)v\|_{L^2(U,X)} \leq C\xi^{-(1/q-1/2)}.
\]

Here the constants \( C \) are independent of \( J, v \) and \( \xi \).

Let us briefly draw a plan of the proof of this theorem. We will give a detailed proof for the case \( U = \mathbb{R}^\infty \) and then point out that the case \( U = \mathbb{R}^M \) can be proven in the same way with slight modification.

In what follows in this section, all definitions, formulas and assertions are given for the case \( U = \mathbb{R}^\infty \), and for \( \xi > 1 \), we use the letters \( m \) and \( \omega \) only for the notations
\[
m := \lfloor K_q \xi \rfloor, \quad \omega := \lfloor K_q \theta \xi \rfloor,
\]
where \( K_q \) and \( K_{q,\theta} \) are the constants defined in Lemma A.1 in Appendix. As mentioned above, we primarily approximate \( v \in L_2(\mathbb{R}^\infty, X, \gamma) \) by the gpc interpolation \( I_\Lambda(\xi)v \). In the next step, we approximate \( I_\Lambda(\xi)v \) by its truncation \( I_\Lambda^\omega(\xi)v \) on the super-cube \( B^m \), which will be constructed below. The final step is to construct a deep ReLU neural network \( \Phi_\Lambda(\xi) := (\phi_{s-e,k})(s,e,k) \in G(\xi) \) to approximate \( I_\Lambda^\omega(\xi)v \) by \( \Phi_\Lambda(\xi)v \) of the form (3.27).

For a function \( \varphi \) defined on \( \mathbb{R} \), we denote by \( \varphi^\omega \) the truncation of \( \varphi \) on \( B^1_\omega \), i.e.,
\[
\varphi^\omega(y) := \begin{cases} 
\varphi(y) & \text{if } y \in B^1_\omega \\
0 & \text{otherwise.}
\end{cases}
\]

If \( \text{supp}(s) \subset \{1, \ldots, m\} \), we put
\[
L^\omega_{s,k}(y) := \prod_{j=1}^m L^\omega_{s_j,k_j}(y_j), \quad y \in \mathbb{R}^m.
\]
We have \( L^\omega_{s,k}(y) = \prod_{j=1}^m L^\omega_{s_j,k_j}(y_j) \) if \( y \in B^m \), and \( L^\omega_{s,k}(y) = 0 \) otherwise. For a function \( v \in L^2_2(\mathbb{R}^\infty, X, \gamma) \), we define
\[
I^\omega_\Lambda(\xi)v := \sum_{(s,e,k) \in G(\xi)} (-1)^{|e|} v(y_{s-e,k}) L^\omega_{s-e,k}.
\]

Let the assumptions of Theorem 3.2 hold. By Lemma A.1(iii) in Appendix for every \( \xi > 2 \) we have \( m(\xi) \leq m \). Hence, for every \( (s,e,k) \in G(\xi) \), \( L_{s-e,k} \) and \( L^\omega_{s-e,k} \) and therefore, \( I_\Lambda(\xi)v \)
and \( I^\omega_{\Lambda(\xi)} v \) can be considered as functions on \( \mathbb{R}^m \). For \( g \in L_2(\mathbb{R}^m, X, \gamma) \), we have \( \|g\|_{L_2(\mathbb{R}^m, X, \gamma)} = \|g\|_{L_2(\mathbb{R}^{2\infty}, X, \gamma)} \) in the sense of extension of \( g \). We will make use of these facts without mention.

To prove Theorem 3.2 we will use some intermediate approximations for estimation of the approximation error as in (3.30). Suppose that the deep ReLU neural network \( \Phi_{\Lambda(\xi)} \) and therefore, the function \( \Phi_{\Lambda(\xi)} \) are already constructed. By the triangle inequality we have

\[
\|v - \Phi_{\Lambda(\xi)} v\|_{L_2(\mathbb{R}^{2\infty}, X, \gamma)} \leq \|v - I_{\Lambda(\xi)} v\|_{L_2(\mathbb{R}^{2\infty}, X, \gamma)} + \|I_{\Lambda(\xi)} v - I^\omega_{\Lambda(\xi)} v\|_{L_2(\mathbb{R}^{m} \setminus B_\beta^m, X, \gamma)} + \|I^\omega_{\Lambda(\xi)} v - \Phi_{\Lambda(\xi)} v\|_{L_2(\mathbb{R}^{m} \setminus B_\beta^m, X, \gamma)}. \tag{3.34}
\]

Hence the estimate (3.30) will be done via the bound \( C\xi^{-(1/q-1/2)} \) for every of the four terms in the right-hand side. The first term is already estimated as in Theorem 3.1. The estimates for the others will be carried out in the following lemmata (Lemmata 3.3–3.5). To complete the proof of Theorem 3.2 we have also to prove the bounds of the size and depth of \( \Phi_{\Lambda(\xi)} \) according to the items (iii) and (iv) which are given in Lemma 3.6 below.

For \( v \in L^\beta_x(\mathbb{R}^{2\infty}, X, \gamma) \) satisfying Assumption (1), by Lemma 3.1 the series (3.8) converges unconditionally in \( L_2(\mathbb{R}^{2\infty}, X, \gamma) \) to \( v \). Therefore, the formula (3.19) for \( \Lambda = \Lambda(\xi) \) can be rewritten as

\[
I_{\Lambda(\xi)} (v) = \sum_{s \in \Lambda(\xi)} \sum_{s' \in \mathcal{F}} v_{s'} \sum_{e \in E_s} (-1)^{|e|_1} \sum_{k \in \pi_{s-e}} H_{s'}(y_{s-e}; k) L_{s-e; k}. \tag{3.35}
\]

Hence, we also have by the definition (3.33)

\[
I^\omega_{\Lambda(\xi)} (v) = \sum_{s \in \Lambda(\xi)} \sum_{s' \in \mathcal{F}} v_{s'} \sum_{e \in E_s} (-1)^{|e|_1} \sum_{k \in \pi_{s-e}} H_{s'}(y_{s-e}; k) L_{s-e; k}. \tag{3.36}
\]

**Lemma 3.3** Under the assumptions of Theorem 3.2, for every \( \xi > 1 \), we have that

\[
\left\| I_{\Lambda(\xi)} v - I^\omega_{\Lambda(\xi)} v \right\|_{L_2(\mathbb{R}^{2\infty}, X, \gamma)} \leq C\xi^{-(1/q-1/2)}, \tag{3.37}
\]

where the constant \( C \) is independent of \( v \) and \( \xi \).

**Proof.** By the equality

\[
\left\| L_{s-e; k} - L_{s-e; k} \right\|_{L_2(\mathbb{R}^{2\infty}, X, \gamma)} = \left\| L_{s-e; k} \right\|_{L_2(\mathbb{R}^{m} \setminus B_\beta^m, X, \gamma)}, \quad \forall (s, e, k) \in G(\xi),
\]

and the triangle inequality, noting (3.35) and (3.36), we obtain

\[
\left\| I_{\Lambda(\xi)} v - I^\omega_{\Lambda(\xi)} v \right\|_{L_2(\mathbb{R}^{2\infty}, X, \gamma)} \leq \sum_{s \in \Lambda(\xi)} \sum_{s' \in \mathcal{F}} \left\| v_{s'} \right\|_X \sum_{e \in E_s} \sum_{k \in \pi_{s-e}} |H_{s'}(y_{s-e}; k)| \left\| L_{s-e; k} \right\|_{L_2(\mathbb{R}^{m} \setminus B_\beta^m, X, \gamma)}.
\]

Let \( (s, e, k) \in G(\xi) \) be given. Then we have

\[
L_{s-e; k} = \prod_{j=1}^m L_{s_j-e_j; k_j}(y_j), \quad y \in \mathbb{R}^m,
\]

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where $L_{s_j-e_j k_j}$ is a polynomial in variable $y_j$, of degree not greater than $m_1(\xi)$. Hence, applying Lemma A.7 in Appendix with taking account of (3.31) gives

$$
\| L_{s-e;k} \|_{L_2(\mathbb{R}^m; B^m_\gamma, \gamma)} \leq C \xi e^{-K_1 \xi} \| L_{s-e;k} \|_{L_2(\mathbb{R}^m, \gamma)}.
$$

From Lemmas A.3 and A.4 and Lemma A.1(ii) in Appendix we derive that

$$
\| L_{s-e;k} \|_{L_2(\mathbb{R}^m, \gamma)} = \prod_{j \in \mathbb{N}} \| L_{s_j-e_j;k_j} \|_{L_2(\mathbb{R}, \gamma)} \leq \prod_{j \in \mathbb{N}} e^{K_2(s_j-e_j)}
$$

$$
\leq \prod_{j \in \mathbb{N}} e^{K_2 s_j} = e^{K_2 |s|_1} \leq e^{K_2 m_1(\xi)} \leq e^{K_3 \xi^{1/\theta q}},
$$

and

$$
\sum_{k \in \mathbb{N}_{s-e}} |H_s'(y_{s-e;k})| \leq e^{K_4 |s|_1} \leq e^{K_4 m_1(\xi)} \leq e^{K_5 \xi^{1/\theta q}}. \tag{3.38}
$$

Summing up, we arrive at

$$
\| I_\Lambda(\xi) v - I_\Lambda^\omega(\xi) v \|_{L_2(\mathbb{R}^\infty, X, \gamma)} \leq C_1 \xi \exp \left( - K_1 \xi + (K_2 + K_5) \xi^{1/\theta q} \right) \sum_{s \in \Lambda(\xi)} \sum_{s' \in \mathbb{F}} \| v_{s'} \|_X \sum_{e \in E_s} 1
$$

$$
\leq C_1 \xi \exp \left( - K_1 \xi + K_6 \xi^{1/\theta q} \right) \sum_{s' \in \mathbb{F}} \| v_{s'} \|_X.
$$

Hence, by Lemma 3.1, Lemma A.2 in Appendix and the inequality $1/\theta q \leq 1/3$ we get

$$
\| I_\Lambda(\xi) v - I_\Lambda^\omega(\xi) v \|_{L_2(\mathbb{R}^\infty, X, \gamma)} \leq C_2 \xi^2 \exp \left( - K_1 \xi + K_6 \xi^{1/\theta q} \right) \leq C \xi^{-(1/\theta q - 1/2)}.
$$

The previous lemma gives the bound of the second term in the right-hand side of (3.34), i.e., the error bound for the approximation of sparse-grid Lagrange interpolation $I_\Lambda(\xi) v$ by its truncation $I_\Lambda^\omega(\xi) v$ on $B^m_\gamma$ for $v \in L_2(\mathbb{R}^\infty, X, \gamma)$. As the next step, we will construct a deep ReLU neural network $\phi_{\Lambda(\xi)} := (\phi_{s-e;k})(s, e, k) \in G(\xi)$ on $\mathbb{R}^m$ for approximating $I_\Lambda(\xi) v$ by the function $\Phi_{\Lambda(\xi)} v$ given as in (3.27), and prove the bound of the error as the third term in the right-hand side of (3.34).

For $s \in \mathbb{N}_0$, we represent the univariate interpolation polynomial $L_{s;k}$ in the form of linear combination of monomials:

$$
L_{s;k}(y) =: \sum_{\ell=0}^{s} b^s_{\ell} y^\ell. \tag{3.39}
$$

From (3.39) for each $(s, e, k) \in G(\xi)$ we have

$$
L_{s-e;k} = \sum_{\ell=0}^{s-e} b^s_{\ell} y^\ell. \tag{3.40}
$$
where the summation $\sum_{\ell=0}^{s-e}$ means that the sum is taken over all $\ell$ such that $0 \leq \ell \leq s - e$, and

$$b_{\ell}^{s-e:k} = \prod_{j=1}^{m} b_{\ell_j}^{\delta_j - e_j, k_j}, \quad y^\ell = \prod_{j=1}^{m} y_j^{\ell_j}.$$  

Indeed, we have

$$L_{s-e:k} = \prod_{j=1}^{m} L_{s_j-e_j,k_j}(y_j) = \prod_{j=1}^{m} \sum_{\ell_j=0}^{s_j-e_j} b_{\ell_j}^{s_j-e_j,k_j} y_j^{\ell_j}$$

$$= \sum_{\ell=0}^{s-e} \left( \prod_{j=1}^{m} b_{\ell_j}^{s_j-e_j,k_j} \right) y^\ell = \sum_{\ell=0}^{s-e} b_{\ell}^{s-e:k} y^\ell.$$  

By (3.36) and (3.40) we get for every $y \in B_\omega^m$,

$$I_\Lambda(x)(v)(y) = \sum_{(s,e,k) \in G(\xi)} (-1)^{|e|} v(y_{s-e:k}) \sum_{\ell=0}^{s-e} b_{\ell}^{s-e:k} \left( \frac{y_j}{2\sqrt{\omega}} \right)^{\ell_j} \prod_{j \in \text{supp}(\ell)} \left( \frac{y_j}{2\sqrt{\omega}} \right)^{\ell_j}.$$  

(3.41)

Let $\ell \in \mathbb{F}$ be such that $0 \leq \ell \leq s - e$. By definition we have $\text{supp}(\ell) \subset \text{supp}(s)$. By changing variables

$$x = \frac{y}{2\sqrt{\omega}}, \quad x \in \text{supp}(s),$$

we have

$$\prod_{j \in \text{supp}(\ell)} \left( \frac{y_j}{2\sqrt{\omega}} \right)^{\ell_j} = \prod_{j \in \text{supp}(\ell)} \varphi_{1}^{\ell_j} \left( \frac{y_j}{2\sqrt{\omega}} \right) \prod_{j \in \text{supp}(s) \setminus \text{supp}(\ell)} \varphi_{0} \left( \frac{y_j}{2\sqrt{\omega}} \right) = h_{\ell}^{s-e}(x), \quad y \in B_\omega^{|\text{supp}(s)|},$$  

(3.42)

where

$$h_{\ell}^{s-e}(x) := \prod_{j \in \text{supp}(\ell)} \varphi_{1}^{\ell_j} (x_j) \prod_{j \in \text{supp}(s) \setminus \text{supp}(\ell)} \varphi_{0} (x_j),$$  

(3.43)

$\varphi_0$ and $\varphi_1$ are the piece-wise linear functions defined before Lemma 2.4. We put

$$B_s := \max_{e \in B_s, k \in s-e} \max_{0 \leq \ell \leq s-e} |b_{\ell}^{s-e:k}|,$$  

(3.44)

and

$$\delta^{-1} := \xi^{1/2 - 1/2} \sum_{s \in \Lambda(\xi)} e^{|s|} p_a(2) \left( 2\sqrt{\omega} \right)^{|s|} B_s,$$  

(3.45)

where $K$ is the constant in Lemma A.3 in Appendix. Hence, by applying Lemma 2.4 to the product in the left-hand side of (3.43), for every $(s,e,k) \in G(\xi)$ and $\ell$ satisfying $0 < \ell \leq s - e$, there exists a deep ReLU neural network $\phi_{\ell}^{s-e}$ on $\mathbb{R}^{|\text{supp}(s)|}$ with $\text{supp}(\phi_{\ell}^{s-e}) \subset [-2,2]^{|\text{supp}(s)|}$ such that

$$\sup_{y \in B_\omega^{|\text{supp}(s)|}} \left| \prod_{j \in \text{supp}(s)} \left( \frac{y_j}{2\sqrt{\omega}} \right)^{\ell_j} - \phi_{\ell}^{s-e} \left( \frac{y}{\sqrt{\omega}} \right) \right| \leq \sup_{y \in B_\omega^{|\text{supp}(s)|}} \left| h_{\ell}^{s-e} \left( \frac{y}{2\sqrt{\omega}} \right) - \phi_{\ell}^{s-e} \left( \frac{y}{2\sqrt{\omega}} \right) \right| \leq \delta,$$  

(3.46)
and
\[ \text{supp} \left( \phi^s_{b,e} \left( \frac{\cdot}{2 \sqrt{\omega}} \right) \right) \subset B_{b, \omega}^{\text{supp}(s)}. \] (3.47)

Also, from Lemma 2.4 and the inequalities \(|\ell| + |\text{supp}(s) \setminus \text{supp}(\ell)| \leq |s| \leq \delta^{-1}\) one can see that
\[ W \left( \phi^s_{b,e} \right) \leq C \left( 1 + |s| \left( \log |s| + \log \delta^{-1} \right) \right) \leq C \left( 1 + |s| \log \delta^{-1} \right) \] (3.48)
and
\[ L \left( \phi^s_{b,e} \right) \leq C \left( 1 + \log |s| \left( \log |s| + \log \delta^{-1} \right) \right) \leq C \left( 1 + \log |s| \log \delta^{-1} \right). \] (3.49)

We define the deep ReLU neural network \( \phi_{s,e,k} \) on \( \mathbb{R}^{\text{supp}(s)} \) by
\[ \phi_{s,e,k}(y) := \sum_{\ell \in \mathbb{Z}} b_{\ell}^{s,e,k} (2 \sqrt{\omega})^{|\ell|} \phi^s_{\ell} \left( \frac{y}{2 \sqrt{\omega}} \right), \quad y \in \mathbb{R}^{\text{supp}(s)}, \] (3.50)
which is the parallelization deep ReLU neural network of the component deep ReLU neural networks \( \phi^s_{\ell} \left( \frac{\cdot}{2 \sqrt{\omega}} \right) \). From (3.47) it follows
\[ \text{supp} \left( \phi_{s,e,k} \right) \subset B_{b, \omega}^{\text{supp}(s)}. \] (3.51)

According to the above convention, for \((s, e, k) \in G(\xi)\), in some places without mention we identify the functions \( \phi^s_{b,e} \) and \( \phi_{s,e,k} \) on \( \mathbb{R}^{\text{supp}(s)} \) with their extensions on \( \mathbb{R}^m \) or on \( \mathbb{R}^\infty \) due to the inclusions \( \text{supp}(s) \subset \{1, ..., m\} \subset \mathbb{N} \).

We define \( \Phi_{\Lambda(\xi)} := (\phi_{s,e,k})_{(s,e,k) \in G(\xi)} \) as the deep ReLU neural network on \( \mathbb{R}^m \) which is realized by parallelization of \( \phi_{s,e,k}, \ (s, e, k) \in G(\xi) \). Consider the approximation of \( I_{\Lambda(\xi)}^\omega v \) by the function \( \Phi_{\Lambda(\xi)} v \) where for convenience we recall
\[ \Phi_{\Lambda(\xi)} v(y) := \sum_{(s,e,k) \in G(\xi)} (-1)^{|e|_1} v(y_{s,e,k}) \phi_{s,e,k}(y). \] (3.52)

**Lemma 3.4** Under the assumptions of Theorem 3.2, for every \( \xi > 1 \), we have
\[ \left\| I_{\Lambda(\xi)}^\omega v - \Phi_{\Lambda(\xi)} u \right\|_{L_2(B^m_{\omega}, X, \gamma)} \leq C \xi^{-1/q - 1/2}, \] (3.53)
where the constant \( C \) is independent of \( v \) and \( \xi \).

**Proof.** According to Lemma 3.1 the series (3.8) converges unconditionally to \( v \). Hence, for every \( y \in B^m_{\omega} \), we have by (3.36)
\[ I_{\Lambda(\xi)}^\omega v(y) = \sum_{s \in \Lambda(\xi)} \sum_{s' \in F} v_{s'} \sum_{e \in E_s} (-1)^{|e|_1} \sum_{k \in \pi_{s-e}} H_{s'}(y_{s,e,k}) \sum_{\ell = 0}^{s-e} b_{\ell}^{s-e,k}(2 \sqrt{\omega})^{|\ell|_1} \prod_{j \in \text{supp}(s)} \left( \frac{y_j}{2 \sqrt{\omega}} \right)^{\ell_j}, \] (3.54)
and by (3.52)
\[ \Phi_{\Lambda(\xi)}^\omega(y) = \sum_{s \in \Lambda(\xi)} \sum_{s' \in F} v_{ss'} \sum_{e \in E_s} (-1)^{[e]} \sum_{k \in \pi_{s-e}} H_s'(y_{s-e;k}) \sum_{\ell=0}^{s-e} b_{\ell}^{s-e;k} (2\sqrt{\omega})^{[\ell]} \phi_{\ell}^{s-e} \left( \frac{y}{2\sqrt{\omega}} \right). \]

(3.55)

From these formulas and (3.46) we derive the inequality
\[ \left\| I_{\Lambda(\xi)}^\omega - \Phi_{\Lambda(\xi)}^\omega \right\|_{L_2(B_0^\infty, X, \gamma)} \leq \sum_{s \in \Lambda(\xi)} \sum_{s' \in F} \|v_{s'}\|_X \sum_{e \in E_s} \sum_{k \in \pi_{s-e}} |H_s'(y_{s-e;k})| \sum_{\ell=0}^{s-e} b_{\ell}^{s-e;k} (2\sqrt{\omega})^{[\ell]} \delta. \]

(3.56)

We have by (3.44)
\[ \sum_{\ell=0}^{s-e} b_{\ell}^{s-e;k} \leq B_s \prod_{j \in \text{support}(s-e)} s_j \leq p_s(1)B_s, \]

and by Lemma A.3 in Appendix
\[ \sum_{e \in E_s} \sum_{k \in \pi_{s-e}} |H_s'(y_{s-e;k})| \leq \sum_{e \in E_s} e^{K|s-e|} \leq 2|s|_0 e^{K|s|_1} \leq p_s(1)e^{K|s|_1}. \]

(3.57)

This together with (3.56), Lemma 3.1 and (3.45) yields that
\[ \left\| I_{\Lambda(\xi)}^\omega - \Phi_{\Lambda(\xi)}^\omega \right\|_{L_2(B_0^\infty, X, \gamma)} \leq \sum_{s \in \Lambda(\xi)} \sum_{s' \in F} \delta B_s p_s(1) \sum_{e \in E_s} \sum_{k \in \pi_{s-e}} |H_s'(y_{s-e;k})| \leq \sum_{s' \in F} \|v_{s'}\|_X \delta \sum_{s \in \Lambda(\xi)} e^{K|s|_1} p_s(2) (2\sqrt{\omega})^{[s]} B_s \leq C\xi^{-1/2}. \]

In the previous lemma, we proved the bound of the third term in the right-hand side of (3.34), i.e., the error bound for the approximation of $I_{\Lambda(\xi)}^\omega v$ by the function $\Phi_{\Lambda(\xi)}^\omega$ for $v \in L_2(\mathbb{R}^\infty, X, \gamma)$. As the last step in the error estimation, we will establish the bound for the fourth term in the right-hand side of (3.34).

**Lemma 3.5** Under the assumptions of Theorem 3.2, for every $\xi > 1$, we have
\[ \|\Phi_{\Lambda(\xi)}^\omega\|_{L_2((\mathbb{R}^m \setminus B_0^\infty), X, \gamma)} \leq C\xi^{-1/2}, \]

(3.58)

where the constant $C$ is independent of $v$ and $\xi$.

**Proof.** We use the formula (3.55) to estimate the norm $\|\Phi_{\Lambda(\xi)}^\omega\|_{L_2((\mathbb{R}^m \setminus B_0^\infty), X, \gamma)}$. We need the following auxiliary inequality
\[ \left| \phi_{\ell}^{s-e} \left( \frac{y}{2\sqrt{\omega}} \right) \right| \leq 2, \quad \forall y \in \mathbb{R}^m. \]

(3.59)
Due to (3.47), it is sufficient to prove this inequality for \( y \in B_{4\omega}^{\supp(s)} \). Considering the right-hand side of (3.45), we have
\[
\sum_{s \in \Lambda(\xi)} e^{K|s|1} p_s(2) (2\sqrt{\omega})^{|s|1} B_s \geq e^{K|0|1} p_0(2) (2\sqrt{\omega})^{|0|1} B_0 = 1 \tag{3.60}
\]
With the definition (3.45), this yields that \( \delta \leq 1 \). On the other hand, by the definition of \( h_{\ell}^{s-e} \),
\[
\sup_{y \in B_{4\omega}^{\supp(s)}} \left| h_{\ell}^{s-e} \left( \frac{y}{2\sqrt{\omega}} \right) \right| \leq 1
\]
From the last two inequalities, (3.46) and the triangle inequality we derive (3.59) for \( y \in B_{4\omega}^{\supp(s)} \).

By (3.59) and Lemma A.7 in Appendix,
\[
\left\| \phi_{\ell}^{s-e} \left( \frac{\cdot}{2\sqrt{\omega}} \right) \right\|_{L_2(\mathbb{R}^m \setminus B_{\omega}^{\gamma})} \leq 2 \left\| 1 \right\|_{L_2(\mathbb{R}^m \setminus B_{\omega}^{\gamma})} \leq C_1 m \exp(-K_1 \omega).
\]
This together with (3.55) implies that
\[
\left\| \Phi_\lambda(\xi) v \right\|_{L_2(\mathbb{R}^m \setminus B_{\omega}^{\gamma}, X, \gamma)} \leq \sum_{s \in \Lambda(\xi)} \sum_{s' \in F} \| v_{s'} \|_X \sum_{e \in E_s} \sum_{k \in \pi_s-e} |H_s'(y_{s-e}; k)| \sum_{\ell=0}^{s-e} |b_{\ell}^{s-e; k}| (2\sqrt{\omega})^{\ell_1} \left\| \phi_{\ell}^{s-e} \left( \frac{\cdot}{2\sqrt{\omega}} \right) \right\|_{L_2(\mathbb{R}^m \setminus B_{\omega}^{\gamma})}
\leq C_1 m \exp(-K_1 \omega) \sum_{s \in \Lambda(\xi)} (2\sqrt{\omega})^{|s|1} \sum_{s' \in F} \| v_{s'} \|_X \sum_{e \in E_s} \sum_{k \in \pi_s-e} |H_s'(y_{s-e}; k)| \sum_{\ell=0}^{s-e} |b_{\ell}^{s-e; k}|.
\]
By a tensor product argument from Lemma A.6 in Appendix and the inequality \( s - e \leq s \) for \( e \in E_s \), we deduce the estimates
\[
\sum_{\ell=0}^{s-e} |b_{\ell}^{s-e; k}| \leq e^{K_2|s|1} \leq e^{K_2|s|1} |s|_1^{s|1|}, \tag{3.61}
\]
which and (3.57) give
\[
\sum_{k \in \pi_{s-e}} |H_s'(y_{s-e}; k)| \sum_{\ell=0}^{s-e} |b_{\ell}^{s-e; k}| \leq \sum_{k \in \pi_{s-e}} |H_s'(y_{s-e}; k)| e^{K_2|s|1} |s|_1^{s|1|} \leq p_s(1) e^{K_2|s|1} |s|_1^{s|1|}. \tag{3.62}
\]
This in combining with (3.31), (3.38), Lemma 3.1 allows us to continue the estimation as
\[
\left\| \Phi_\lambda(\xi) v \right\|_{L_2(\mathbb{R}^m \setminus B_{\omega}^{\gamma}, X, \gamma)} \leq C_1 m \exp(-K_1 \omega) \sum_{s' \in F} \| v_{s'} \|_X \sum_{s \in \Lambda(\xi)} (2\sqrt{\omega})^{|s|1} p_s(1) e^{K_2|s|1} |s|_1^{s|1|}
\leq C_2 m \exp(-K_1 \omega) \sum_{s \in \Lambda(\xi)} (2\sqrt{\omega})^{|s|1} p_s(1) e^{K_2|s|1} |s|_1^{s|1|} \tag{3.63}
\leq C_2 \xi \exp(-K_1 \xi) \left( C_3 \xi (1/2) m_1(\xi) e^{K_2 m_1(\xi)} |m_1(\xi)|^{m_1(\xi)} \right) \sum_{s \in \Lambda(\xi)} p_s(1).
\]
By the assumption of Theorem 3.2 \( \|p(\theta)\sigma^{-1}\|_{\ell_q(F)} \leq C < \infty \) for some \( \theta \geq 3/q \), we derive that
\[
\|p(3/q)\sigma^{-1}\|_{\ell_q(F)} \leq \|p(\theta)\sigma^{-1}\|_{\ell_q(F)} \leq C < \infty.
\]
Applying Lemma A.1(i) in Appendix gives
\[
\sum_{s \in \Lambda(\xi)} p_s(1) \leq \sum_{s \in \Lambda(\xi)} p_s(3) \leq C.
\]
Hence by (3.63) and Lemma A.1(ii) in Appendix we have that
\[
\|\Phi_{\Lambda(\xi)} v\|_{L_2(\mathbb{R}^m \setminus B_m^\omega, X, \gamma)} \leq C_2 \xi \exp(-K_1 \xi) \left( C_3 \xi^{1/2} \right)^{K_q,\theta}^{1/\theta q} e^{K_3 K_q,\theta \xi^{1/\theta q}} \left( K_q,\theta \xi^{1/\theta q} \right)^{K_q,\theta \xi^{1/\theta q}} C_4 \xi
\]
\[
\leq C_5 \xi^2 \exp(-K_1 \xi + K_4 \xi^{1/\theta q} \log \xi + K_5 \xi^{1/\theta q}).
\]
Since \( 1/\theta q \leq 1/3 \), we obtain
\[
\|\Phi_{\Lambda(\xi)} v\|_{L_2(\mathbb{R}^m \setminus B_m^\omega, X, \gamma)} \leq C \xi^{-(1/\theta q - 1/2)}.
\]

To complete the proof of Theorem 3.2, we have to establish the bounds of the size and depth of the deep ReLU neural network \( \phi_{\Lambda(\xi)} \) as in (iii) and (iv).

**Lemma 3.6** Under the assumptions of Theorem 3.2, the input dimension of \( \phi_{\Lambda(\xi)} \) is at most \( [K_q \xi] \), for every \( \xi > 1 \), the output dimension of \( \phi_{\Lambda(\xi)} \) at most \( [C_q \xi] \),
\[
W(\phi_{\Lambda(\xi)}) \leq C \xi^{1 + 2/\theta q} \log \xi,
\]
and
\[
L(\phi_{\Lambda(\xi)}) \leq C \xi^{1/\theta q} (\log \xi)^2,
\]
where the constants \( C \) are independent of \( v \) and \( \xi \).

**Proof.** The input dimension of \( \phi_{\Lambda(\xi)} \) is not greater than \( m(\xi) \) which is at most \( [K_q \xi] \) by Lemma A.1(iii) in Appendix. The output dimension of \( \phi_{\Lambda(\xi)} \) is the number \( |G(\xi)| \) which is at most \( [C_q \xi] \) by Lemma A.2 in Appendix.

By Lemmas 2.1 and 2.4 and (3.48) the size of \( \phi_{\Lambda(\xi)} \) is estimated as
\[
W(\phi_{\Lambda(\xi)}) \leq \sum_{(s,e,k) \in G(\xi)} W(\phi_{s-e;k}) \leq \sum_{s \in \Lambda(\xi)} \sum_{e \in E_s} \sum_{k \in \pi_{s-e}} \sum_{\ell=0}^{s-e} W(\phi_{s-e}^{s-e}) \leq C_1 \sum_{s \in \Lambda(\xi)} \sum_{e \in E_s} \sum_{k \in \pi_{s-e}} \sum_{\ell=0}^{s-e} (1 + |s| \log \delta^{-1}),
\]

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where we recall,

\[ \delta^{-1} := \xi^{1/q-1/2} \sum_{s \in \Lambda(\xi)} e^{K_1 |s|_1 p_s(2)} (2\sqrt{\omega})^{|s|_1} B_s, \]

\[ B_s := \max_{e \in E_s, k \in \pi_{s-e}} \max_{0 \leq \ell \leq s-e} |\phi^{s-e}_{\ell,k}|. \]

From (3.61) it follows that

\[ B_s \leq \max_{e \in E_s, k \in \pi_{s-e}} \sum_{\ell=0}^{s-e} |\phi^{s-e}_{\ell,k}| \leq \exp \left( K_2 \xi^{1/\theta q} \log \xi \right), \]

which by Lemma A.1(i) in Appendix implies

\[ \delta^{-1} \leq \xi^{1/q-1/2} \exp \left( K_2 \xi^{1/\theta q} \log \xi \right) \sum_{s \in \Lambda(\xi)} p_s(2) \]

\[ \leq C_2 \xi^{1/q+1/2} \exp \left( K_3 \xi^{1/\theta q} \log \xi \right) \leq C_2 \exp \left( K_3 \xi^{1/\theta q} \log \xi \right). \]

Hence,

\[ \log(\delta^{-1}) \leq K_4 \xi^{1/\theta q} \log \xi. \quad (3.68) \]

and consequently,

\[ (1 + |s|_1 \log \delta^{-1}) \leq \left( 1 + |s|_1 K_4 \xi^{1/\theta q} \log \xi \right) \leq C_2 \xi^{2/\theta q} \log \xi. \]

From (3.66)–(3.67) and Lemma A.2 in Appendix we obtain the desired bound of the size of \( \phi_{\Lambda(\xi)} \):

\[ W(\phi_{\Lambda(\xi)}) \leq C_2 \xi^{2/\theta q} \log \xi \sum_{s \in \Lambda(\xi)} \sum_{e \in E_s} \sum_{k \in \pi_{s-e}} \sum_{\ell=0}^{s-e} 1 \]

\[ \leq C_2 \xi^{2/\theta q} \log \xi \sum_{(s,e,k) \in G(\xi)} p_s(1) \leq C_3 \xi^{1+2/\theta q} \log \xi. \]

By using Lemma 2.1, (3.49), (3.68) and Lemma A.1(ii) in Appendix, we prove that the depth of \( \phi_{\Lambda(\xi)} \) is bounded as in (3.65):

\[ L \left( \phi_{\Lambda(\xi)} \right) \leq \max_{(s,e,k) \in G(\xi)} L(\phi_{s-e};k) \leq \max_{(s,e,k) \in G(\xi)} \max_{0 \leq \ell \leq s-e} L(\phi^{s-e}_{\ell}) \]

\[ \leq C_4 \max_{(s,e,k) \in G(\xi)} \max_{0 \leq \ell \leq s-e} \left( 1 + \log |s|_1 \log \delta^{-1} \right) \]

\[ \leq C_4 \max_{s \in \Lambda(\xi)} \left( 1 + \log \left( K_0 \delta \xi^{1/\theta q} \right) \left( K_3 \xi^{1/\theta q} \log \xi \right) \right) \leq C_5 \xi^{1/\theta q} (\log \xi)^2. \]
We are now in a position to give a formal proof of Theorem 3.2.

Proof. [Proofs of Theorem 3.2] From (3.34), Theorem 3.1 and Lemmata 3.3 – 3.5, for every \( \xi > 2 \), we deduce that

\[
\| v - \Phi_{\Lambda(\xi)v} \|_{L_2(\mathbb{R}^\infty, X, \gamma)} \leq C\xi^{-(1/q-1/2)}.
\]

The claim (vi) is proven. The claim (i) follows directly from the construction of the deep ReLU neural network \( \Phi_{\Lambda(\xi)} \) and the sequence of points \( Y_{\Lambda(\xi)} \), the claim (ii) from Lemma A.2, the claims (iii)–(iv) from Lemma 3.6 and the claim (v) from Lemma A.1(ii) in Appendix and (3.51). Thus, Theorem 3.2 is proven for the case when \( U = \mathbb{R}^\infty \).

The case \( U = \mathbb{R}^M \) can be proven in the same way with a slight modification. Counterparts of all definitions, formulas and assertions which have been used in the proof of the case \( U = \mathbb{R}^\infty \), are true for the case \( U = \mathbb{R}^M \). In the proof of this case, in particular, the used equality \( \| H_s \|_{L_2(\mathbb{R}^\infty)} = 1 \), \( s \in \mathbb{F} \), is replaced by the inequality \( \| H_s \|_{L_2(\mathbb{R}^M)} < 1 \), \( s \in \mathbb{N}_0^M \).

4 Application to parametrized elliptic PDEs

In this section, we apply the results in the previous section to the deep ReLU neural network approximation of the solution \( u \) to the parametrized elliptic PDEs (1.2) with lognormal inputs (1.3). This is based on the weighted \( \ell_2 \)-summability of the series \( (\| u_s \|_V)_{s \in \mathbb{F}} \) in following lemma which has been proven in [4, Theorems 3.3 and 4.2].

Lemma 4.1 Assume that there exist a number \( 0 < q < \infty \) and an increasing sequence \( \rho = (\rho_j)_{j \in \mathbb{N}} \) of numbers strictly larger than 1 such that \( \| \rho^{-1} \|_{\ell_q(\mathbb{N})} \leq C < \infty \) and

\[
\left\| \sum_{j \in \mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} \leq C < \infty,
\]

where the constants \( C \) are independent of \( J \). Then we have that for any \( \eta \in \mathbb{N} \),

\[
\sum_{s \in \mathbb{F}} (\sigma_s \| u_s \|_V)^2 \leq C < \infty \quad \text{with} \quad \sigma_s^2 := \sum_{\| s' \|_{\ell_\infty(\mathbb{F})} \leq \eta} \left( \binom{s'}{s} \prod_{j \in \mathbb{N}} \rho_j^{2s'_j} \right),
\]

where the constant \( C \) is independent of \( J \).

The following two lemmata are proven in [15, Lemmata 5.2 and 5.3].

Lemma 4.2 Let the assumptions of Lemma 4.1 hold. Then the solution map \( y \mapsto u(y) \) is \( \gamma \)-measurable and \( u \in L_2(U, V, \gamma) \). Moreover, \( u \in L_2^E(U, V, \gamma) \) where

\[
E := \left\{ y \in \mathbb{R}^\infty : \sup_{j \in \mathbb{N}} \rho_j^{-1} |y_j| < \infty \right\}
\]
having $\gamma(\mathcal{E}) = 1$ and containing all $y \in \mathbb{R}^\infty$ with $|y|_0 < \infty$ in the case when $U = \mathbb{R}^\infty$.

**Lemma 4.3** Let $0 < q < \infty$, $\rho = (\rho_j)_{j \in \mathcal{N}}$ be a sequence of positive numbers such that $\|\rho^{-1}\|_{\ell_q(\mathcal{N})} \leq C < \infty$, where the constant $C$ is independent of $J$. Let $\theta$ be an arbitrary nonnegative number and $p(\theta) = (p_s(\theta))_{s \in \mathcal{F}}$ the set given as in (3.20). For $\eta \in \mathbb{N}$, let the set $\sigma = (\sigma_s)_{s \in \mathcal{F}}$ be defined as in (4.1). Then for any $\eta > 2(\theta + 1)^q$, we have

$$
\|p(\theta)\sigma^{-1}\|_{\ell_q(\mathcal{F})} \leq C < \infty,
$$

where the constant $C$ is independent of $J$.

We are now in position to formulate our main results on collocation deep ReLU neural network approximation of the solution $u$ to parametric elliptic PDEs with lognormal inputs.

**Theorem 4.1** Under the assumptions of Lemma 4.1, let $0 < q < 2$. Then, given an arbitrary number $\delta > 0$, for every integer $n > 2$, we can construct a deep ReLU neural network $\phi_{\Lambda(\xi_n)} := \phi_{n-1;e,k}(s,e,k)_{s \in \mathcal{F}}$ of the size $W(\phi_{\Lambda(\xi_n)}) \leq n$ on $\mathbb{R}^m$ with

$$
m := \begin{cases}
\min \left\{ M, \left\lfloor K \left( \frac{n}{\log n} \right)^{\frac{1}{1+q}} \right\rfloor \right. & \text{if } U = \mathbb{R}^M, \\
\left\lfloor K \left( \frac{n}{\log n} \right)^{\frac{1}{1+q}} \right\rfloor & \text{if } U = \mathbb{R}^\infty,
\end{cases}
$$

and a sequence of points $Y_{\Lambda(\xi_n)} := (y_{n-1;e,k}(s,e,k))_{s \in \mathcal{F} \subset \mathcal{G}(\xi_n)}$ having the following properties.

(i) The deep ReLU neural network $\phi_{\Lambda(\xi_n)}$ and sequence of points $Y_{\Lambda(\xi_n)}$ are independent of $u$;

(ii) The output dimension of $\phi_{\Lambda(\xi_n)}$ is at most $\left\lfloor K \left( \frac{n}{\log n} \right)^{\frac{1}{1+q}} \right\rfloor$;

(iii) $L(\phi_{\Lambda(\xi_n)}) \leq C_\delta \left( \frac{n}{\log n} \right)^{\frac{1}{1+q}} (\log n)^2$;

(iv) The components $\phi_{n-1;e,k}(s,e,k)_{s \in \mathcal{F} \subset \mathcal{G}(\xi_n)}$ of $\phi_{\Lambda(\xi_n)}$ are deep ReLU neural networks on $\mathbb{R}^{m_s}$ with $m_s \leq C_\delta n^\delta$, having support contained in the super-cube $[-T,T]^{m_s}$, where $T := C_\delta \left( \frac{n}{\log n} \right)^{\frac{1}{1+q}}$;

(v) The approximation of $u$ by $\Phi_{\Lambda(\xi_n)}u$ defined as in (3.27), gives the error estimate

$$
\|u - \Phi_{\Lambda(\xi_n)}u\|_{L(U,V)} \leq C \left( \frac{n}{\log n} \right)^{-\frac{1}{1+q}} \left( \frac{1}{q} - \frac{1}{2} \right).
$$

Here the constants $C$, $K$ and $C_\delta$ are independent of $J$, $u$ and $n$. 

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Proof. To prove the theorem we apply Theorem 3.2 to the solution \( u \). Without loss of generality we can assume that \( \delta \leq 1/6 \). We take first the number \( \theta := 2/\delta q \) satisfying the inequality \( \theta \geq 3/q \), and then choose a number \( \eta \in \mathbb{N} \) satisfying the inequality \( \eta > 2(\theta + 1)/q \). By using Lemmata 4.1–4.3, one can check that \( u \in L_2^q(U,V,\gamma) \) satisfies the assumptions of Theorem 3.2 for \( X = V \) and the set \((\sigma_s)_{s \in \mathcal{F}}\) defined as in (4.1), where \( \mathcal{E} \) is the set defined in Lemma 4.2. For a given integer \( n > 2 \), we choose \( \xi_n > 2 \) as the maximal number satisfying the inequality \( C_{\xi_n}^{1+\delta} \log \xi_n \leq n \), where \( C \) is the constant in the claim (ii) of Theorem 3.2. It is easy to verify that there exist positive constants \( C_1 \) and \( C_2 \) independent of \( n \) such that

\[
C_1 \left( \frac{n}{\log n} \right)^{\frac{1}{1+\delta}} \leq \xi_n \leq C_2 \left( \frac{n}{\log n} \right)^{\frac{1}{1+\delta}}.
\]

From Theorem 3.2 with \( \xi = \xi_n \) we deduce the desired results. \( \square \)

**Theorem 4.2** Under the assumptions of Lemma 4.1, let \( 0 < q < 2 \) and \( \delta_q := \min(1,1/q - 1/2) \). Then, given an arbitrary number \( \delta \in (0, \delta_q) \), for every integer \( n > 1 \), we can construct a deep ReLU neural network \( \phi_{\Lambda(\xi_n)} := (\phi_{s-e,k})_{(s,e,k) \in G(\xi_n)} \) of the size \( W(\phi_{\Lambda(\xi_n)}) \leq n \) on \( \mathbb{R}^m \) with

\[
m := \begin{cases} 
\min \{ M, \lfloor Kn^{1-\delta} \rfloor \} & \text{if } U = \mathbb{R}^M, \\
\lfloor Kn^{1-\delta} \rfloor & \text{if } U = \mathbb{R}^\infty,
\end{cases}
\]

and a sequence of points \( Y_{\Lambda(\xi_n)} := (y_{s-e,k})_{(s,e,k) \in G(\xi_n)} \) having the following properties.

(i) The deep ReLU neural network \( \phi_{\Lambda(\xi_n)} \) and sequence of points \( Y_{\Lambda(\xi_n)} \) are independent of \( u \);

(ii) The output dimension of \( \phi_{\Lambda(\xi_n)} \) are at most \( \lfloor Kn^{1-\delta} \rfloor \);

(iii) \( L(\phi_{\Lambda(\xi_n)}) \leq C_\delta n^\delta \);

(iv) The components \( \phi_{s-e,k}, (s,e,k) \in G(\xi_n), \) of \( \phi_{\Lambda(\xi_n)} \) are deep ReLU neural networks on \( \mathbb{R}^{m_s} \) with \( m_s \leq C_\delta n^\delta \), having support contained in the super-cube \( [-T,T]^{m_s} \), where \( T := C_\delta n^{1-\delta} \);

(v) The approximation of \( u \) by \( \Phi_{\Lambda(\xi_n)} u \) defined as in (3.27), gives the error estimates

\[
\| u - \Phi_{\Lambda(\xi_n)} u \|_{L(U,V)} \leq C m^{-\left(\frac{1}{q} - \frac{1}{2}\right)} \leq C_{\delta n^{-\left(1-\delta\right)}}^{-\left(\frac{1}{q} - \frac{1}{2}\right)}.
\]

Here the constants \( K, C \) and \( C_\delta \) are independent of \( J, u \) and \( n \).

Let us compare the collocation approximation of \( u \) by the function

\[
\Phi_{\Lambda(\xi_n)} u := \sum_{(s,e,k) \in G(\xi_n)} (-1)^{|e|_1} u(y_{s-e,k}) \phi_{s-e,k},
\]

(4.4)
generated from the deep ReLU neural network $\phi_{\Lambda(\xi_n)}$ as in Theorem 4.2, and the collocation approximation of $u$ by the sparse-grid Lagrange gpc interpolation

$$I_{\Lambda(\xi_n)}u := \sum_{(s,e,k) \in G(\xi_n)} (-1)^{|e|} u(y_{s-e,k}) L_{s-e,k}. \quad (4.5)$$

Both the methods are based on $m$ the same particular solvers $(u(y_{s-e,k}))_{(s,e,k) \in G(\xi_n)}$. From Corollary 3.1 one can see that under the assumptions of Theorem 4.2, there holds the error bound in $m$ for the last approximation:

$$\|u - I_{\Lambda(\xi_n)}u\|_{\mathcal{L}(U,V)} \leq Cm^{-\left(\frac{1}{q} - \frac{1}{2}\right)},$$

which is the same as that in (4.3) for the first approximation since by the construction the parameter $m$ in (4.3) can be treated as independent.

After the present paper and the paper [17] appeared in ArXiv website, we have been informed about the paper [50] on some problems similar to the problems considered in [17] in a private communication with its authors.

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A Appendix

A.1 Auxiliary lemmata

Lemma A.1 Let $\theta \geq 0$ and $0 < q < \infty$. Let $\sigma = (\sigma_s)_{s \in \mathcal{F}}$ be a set of numbers strictly larger than 1. Then we have the following.

(i) If $\|p\left(\frac{q}{2}\right)\sigma^{-1}\|_{\ell_q(\mathcal{F})} \leq K < \infty$, where the constant $K$ is independent of $J$, then

$$\sum_{s \in \Lambda(\xi)} p_s(\theta) \leq K\xi \quad \forall \xi > 1. \quad (A.1)$$

In particular, if $\|\sigma^{-1}\|_{\ell_q(\mathcal{F})}^q \leq K_q < \infty$, where the constant $K_q \geq 1$ and is independent of $J$, then the set $\Lambda(\xi)$ is finite and

$$|\Lambda(\xi)| \leq K_q\xi \quad \forall \xi > 1. \quad (A.2)$$
(ii) If \( \| p(\theta)\sigma^{-1}\|_{\ell_q(F)}^{1/\theta} \leq K_{q,\theta} < \infty \), where the constant \( K_{q,\theta} \) is independent of \( J \), then
\[
m_1(\xi) \leq K_{q,\theta} \xi^{\frac{1}{\theta}} \quad \forall \xi > 1.
\] (A.3)

(iii) If \( \sigma_{e^i} \leq \sigma_{e^{i'}} \) for \( i' < i \), and if \( \|\sigma^{-1}\|_{\ell_q(F)}^q \leq K_q < \infty \), where the constant \( K_q \geq 1 \) and is independent of \( J \), then
\[
m(\xi) \leq K_q \xi \quad \forall \xi > 1.
\] (A.4)

Proof. The claim (ii) and (iii) were proven in [17, Lemmata 3.2 and 3.3] for the case \( F = F \). The case \( F = N_0^n \) can be proven in a similar way. Let us prove the claim (i). Indeed, we have for every \( \xi > 1 \),
\[
\sum_{s \in \Lambda(\xi)} p_s(\theta) \leq \sum_{s \in F: \sigma_s^{-q} \xi \geq 1} p_s(\theta) \xi \sigma_s^{-q} \leq \xi \sum_{s \in F} p_s(\theta) \sigma_s^{-q} \leq C \xi.
\]

Lemma A.2 Let \( \theta \geq 0 \), \( 0 < q < \infty \) and \( \xi > 1 \). Let \( \sigma = (\sigma_s)_{s \in F} \) be a set of numbers strictly larger than 1. If \( \| p\left(\frac{\theta+2}{q}\right)\sigma^{-1}\|_{\ell_q(F)} \leq C < \infty \), where the constant \( C \) is independent of \( J \), then there holds
\[
\sum_{(s,e,k) \in G(\xi)} p_s(\theta) \leq C \xi \quad \forall \xi > 1.
\] (A.5)

In particular, if and \( \| p\left(\frac{2}{q}\right)\sigma^{-1}\|_{\ell_q(F)}^q \leq C_q < \infty \), where the constant \( C_q \) is independent of \( J \), then
\[
|G(\xi)| \leq C_q \xi \quad \forall \xi > 1.
\]

Proof. We have for every \( \xi > 1 \),
\[
\sum_{(s,e,k) \in G(\xi)} p_s(\theta) = \sum_{s \in \Lambda(\xi)} \sum_{e \in E_s} \sum_{k \in \pi_{s-e}} p_s(\theta) \leq \sum_{s \in \Lambda(\xi)} \sum_{e \in E_s} |\pi_{s-e}| \tag{A.6}
\leq \sum_{s \in \Lambda(\xi)} p_s(\theta)|E_s| p_s(1) = \sum_{s \in \Lambda(\xi)} p_s(\theta + 1)2^{||s||} \leq \sum_{s \in \Lambda(\xi)} p_s(\theta + 2) \tag{A.7}
\leq \sum_{s \in F: \sigma_s^{-q} \xi \geq 1} p_s(\theta + 2) \xi \sigma_s^{-q} \leq \xi \sum_{s \in F} p_s(\theta + 2) \sigma_s^{-q} \leq C \xi.
\]

Lemma A.3 We have for any \( s, s' \in F \),
\[
\sum_{k \in \pi_s} |H_{s'}(y_{s,k})| \leq e^{K|s|}, \tag{A.8}
\]
where the constant \( K \) is independent of \( J \) and \( s, s' \).
Proof. From Cramér’s bound we deduce that (see, e.g., [15, Lemma 3.2])

\[
|H_s(y)\sqrt{g(y)}| < 1, \quad \forall y \in \mathbb{R}, \forall s \in \mathbb{N}_0,
\]  

or, equivalently,

\[
|H_s(y)| < (2\pi)^{1/4}e^{y^2/4}, \quad \forall y \in \mathbb{R}, \forall s \in \mathbb{N}_0.
\]  

Let \(s, s' \in \mathcal{F}\) and \(k \in \pi_s\) be given. Notice that for the univariate Hermite polynomials, \(H_0 = 1, H_{2s+1}(0) = 0\) and \(|H_{2s}(0)| \leq 1\) for \(s \in \mathbb{N}_0\). Hence, we have by (A.10),

\[
|H_{s'}(y_{s;k})| \leq \prod_{j \in \text{supp}(s') \cap \text{supp}(s)} |H_{s_j}(y_{s_j,k_j})| \leq \prod_{j \in \text{supp}(s)} (2\pi)^{1/4}e^{y_{s_j,k_j}^2/4}.
\]  

Therefore,

\[
\sum_{k \in \pi_s} |H_{s'}(y_{s;k})| \leq \sum_{k \in \pi_s} \prod_{j \in \text{supp}(s)} (2\pi)^{1/4}e^{y_{s_j,k_j}^2/4} = \prod_{j \in \text{supp}(s)} (2\pi)^{1/4} \sum_{k_j \in \pi_{s_j}} e^{y_{s_j,k_j}^2/4}.
\]  

The inequalities [55, (6.31.19)] yield that

\[
|y_{s,k}| \leq K_1 \frac{|k|}{\sqrt{s}}, \quad \forall k \in \pi_s, \forall s \in \mathbb{N}.
\]  

Consequently,

\[
(2\pi)^{1/4} \sum_{k_j \in \pi_{s_j}} e^{y_{s_j,k_j}^2/4} \leq 2(2\pi)^{1/4} \sum_{k_j = 0}^{[s_j/2]} \exp \left( \frac{K_1 k_j^2}{4s_j} \right) \leq e^{Ks_j}, \quad \forall s_j \in \mathbb{N}.
\]  

This allows us to finish the proof of the lemma as

\[
\sum_{k \in \pi_s} |H_{s'}(y_{s;k})| \leq \prod_{j \in \text{supp}(s)} e^{Ks_j} = e^{K|s|_1}.
\]

\[\square\]

Lemma A.4 We have for any \(s \in \mathbb{N}\) and \(k \in \pi_s\),

\[
\|L_{s;k}\|_{L_2(\mathbb{R},\gamma)} \leq e^{Ks},
\]  

and

\[
\|L_{s;k}\|_{L_\infty(\mathbb{R})} \leq e^{Ks},
\]

where the constants \(K\) are independent of \(s\) and \(k \in \pi_s\).
Proof. Notice that $L_{s;k}$ is a polynomial having $s$ single zeros $\{y_{s,j}\}_{j \in \pi_s, j \neq k}$, and that $L_{s;k}(y_{s;k}) = 1$. Moreover, there is no any zero in the open interval $(y_{s;k-1}, y_{s;k})$ and

$$L_{s;k}(y_{s;k}) = \max_{y \in [y_{s;k-1}, y_{s;k}]} L_{s;k}(y) = 1.$$ 

Hence,

$$|L_{s;k}(y)| \leq 1, \quad \forall y \in [y_{s;k-1}, y_{s;k+1}]. \tag{A.17}$$

Let us estimate $|L_{s;k}(y)|$ for $y \in \mathbb{R} \setminus (y_{s;k-1}, y_{s;k+1})$. From the definition one can see that

$$L_{s;k}(y) := \prod_{k' \in \pi_s, k' \neq k} \frac{y - y_{s;k'}}{y_{s;k} - y_{s;k'}} = A_{s;k}(y - y_{s;k})^{-1} H_{s+1}(y), \tag{A.18}$$

where

$$A_{s;k} := ((s + 1)!)^{1/2} \prod_{k' \in \pi_s, k' \neq k} (y_{s;k} - y_{s;k'})^{-1}. \tag{A.19}$$

From the inequalities [55, (6.31.22)]

$$\frac{\pi \sqrt{2}}{\sqrt{2s + 3}} \leq d_s \leq \frac{\sqrt{10.5}}{\sqrt{2s + 3}} \tag{A.20}$$

for the minimal distance $d_s$ between consecutive $y_{s;k}$, $k \in \pi_s$, we have that

$$|y - y_{s;k}|^{-1} \leq d_s^{-1} \leq \frac{\sqrt{2s + 3}}{\sqrt{10.5}} < \sqrt{s}, \quad \forall y \in \mathbb{R} \setminus (y_{s;k-1}, y_{s;k+1}),$$

and for any $s \in \mathbb{N}$ and $k, k' \in \pi_s$ with $k' \neq k$,

$$|y_{s;k} - y_{s;k'}|^{-1} \leq C \frac{\sqrt{s}}{|k - k'|}. \tag{A.21}$$

which yield for any $y \in \mathbb{R} \setminus (y_{s;k-1}, y_{s;k+1})$,

$$|y - y_{s;k}|^{-1} |A_{s;k}| \leq \sqrt{s} ((s + 1)!)^{1/2} \prod_{k' \in \pi_s, k' \neq k} |y_{s;k} - y_{s;k'}|^{-1} \leq \sqrt{s} C^s ((s + 1)!)^{1/2} s^{s/2} \frac{1}{k!(s - k)!}$$

$$\leq \sqrt{s} C^s \left( \frac{s}{k} \right) ((s + 1)!)^{1/2} \frac{s^{s/2}}{s!} \leq \sqrt{s} (2C)^s \frac{(s + 1)!}{s!} \leq e^{K_1 s}. \tag{A.22}$$

In the last step we used the Stirling’s approximation for factorial. Thus, we have proven that

$$|L_{s;k}(y)| \leq e^{K_1 s} |H_{s+1}(y)|, \quad \forall y \in \mathbb{R} \setminus (y_{s;k-1}, y_{s;k+1}). \tag{A.23}$$

With $I_{s;k} := [y_{s;k-1}, y_{s;k+1}]$, from the last estimate and (A.17) we prove (A.15):

$$\|L_{s;k}\|_{L_2(\mathbb{R}, \gamma)}^2 = \|L_{s;k}\|_{L_2(I_{s,k}, \gamma)}^2 + \|L_{s;k}\|_{L_2(\mathbb{R}\setminus I_{s,k}, \gamma)}^2$$

$$\leq 1 + e^{2K_1 s} \|H_s\|_{L_2(\mathbb{R}, \gamma)}^2 = 1 + e^{2K_1 s} \leq e^{2K s}.$$ 

The inequality (A.16) can be proven similarly by using (A.9).
Lemma A.5 Assume that $p$ and $q$ are polynomials on $\mathbb{R}$ in the form

$$p(y) := \sum_{k=0}^{m} a_k y^k, \quad q(y) := \sum_{k=0}^{m-1} b_k y^k,$$

(A.24)

and that $p(y) = (y - y_0)q(y)$ for a point $y_0 \in \mathbb{R}$. Then we have

$$|b_k| \leq \sum_{k=0}^{m-1} |a_k|, \quad k = 0, ..., m - 1.$$  

(A.25)

Proof. From the definition we have

$$\sum_{k=0}^{m} a_k y^k = -b_0 y_0 + \sum_{k=0}^{m-1} (b_{k-1} - b_k y_0) y^k + b_{m-1} y^m.$$  

(A.26)

Hence we obtain

$$0 = a_0 + b_0 y_0; \quad b_k = a_{k+1} + b_{k+1} y_0, \quad k = 1, ..., m - 2; \quad b_{m-1} = a_m.$$  

(A.27)

From the last equalities one can see that the lemma is trivial if $y_0 = 0$. Consider the case $y_0 \neq 0$. If $|y_0| \leq 1$, from (A.27) we deduce that

$$b_k = \sum_{j=k+1}^{m} a_j y_0^{j-k-1}.$$  

(A.28)

and, consequently,

$$|b_k| \leq \sum_{j=k+1}^{m} |a_j||y_0|^{j-k-1} \leq \sum_{j=0}^{m} |a_j|.$$  

(A.29)

If $|y_0| > 1$, from (A.27) we deduce that

$$b_k = -\sum_{j=0}^{k} a_j y_0^{-(k+1-j)},$$  

(A.30)

and, consequently,

$$|b_k| \leq \sum_{j=0}^{k} |a_j||y_0|^{-(k+1-j)} \leq \sum_{j=0}^{m} |a_j|.$$  

(A.31)

□

Lemma A.6 Let $b_{\ell}^{s;k}$ be the polynomial coefficients of $L_{s;k}$ as in the representation (3.39). Then we have for any $s \in \mathbb{N}_0$ and $k \in \pi_s$,

$$\sum_{\ell=0}^{s} |b_{\ell}^{s;k}| \leq e^{Ks} s!,$$

(A.32)

where the constant $K$ are independent of $s$ and $k \in \pi_s$. 

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Proof. For $s \in \mathbb{N}_0$, we represent the univariate Hermite polynomial $H_s$ in the form

$$H_s(y) := \sum_{\ell=0}^{s} a_{s,\ell} y^\ell. \quad (A.33)$$

By using the well-known equality

$$H_s(y) = s! \sum_{\ell=0}^{\lfloor s/2 \rfloor} (-1)^\ell \frac{y^{s-2\ell}}{(s-2\ell)!} 2^\ell, \quad (A.34)$$

one can derive that

$$\sum_{\ell=0}^{s} |a_{s,\ell}| \leq s!. \quad (A.35)$$

From (A.18) we have

$$A_{s;k} H_{s+1}(y) = (y - y_{s;k}) L_{s;k}(y), \quad (A.36)$$

where $A_{s;k}$ is given as in (A.19). By Lemma A.5, (A.35) and (A.22), we obtain

$$\sum_{\ell=0}^{s} |b_{s,k}^{s+1}| \leq \sum_{\ell=0}^{s} A_{s;k} \sum_{\ell'=0}^{s+1} |a_{s+1,\ell'}| \leq e^{Ks} s!. \quad (A.37)$$

Lemma A.7 Let $\varphi(y) = \prod_{j=1}^{m} \varphi_j(y_j)$ for $y \in \mathbb{R}^m$, where $\varphi_j$ is a polynomial in the variable $y_j$ of degree not greater than $\omega$ for $j = 1, \ldots, m$. Then there holds

$$\|\varphi\|_{L_2(\mathbb{R}^m \setminus B_m^\omega, \gamma)} \leq C m \exp(-K\omega) \|\varphi\|_{L_2(\mathbb{R}^m, \gamma)}, \quad (A.38)$$

and

$$\|\varphi\|_{L_2^\gamma(\mathbb{R}^m \setminus B_m^\omega)} \leq C m \exp(-K\omega) \|\varphi\|_{L_2^\gamma(\mathbb{R}^m)}, \quad (A.39)$$

where the constants $C$ and $K$ are independent of $\omega$, $m$ and $\varphi$.

Proof. The inequality (A.38) was proven in [17, Lemma 3.3]. The inequality (A.39) can be proven in a similar way with a slight modification.

A.2 Proof of Theorem 3.1

Proof. This theorem was proven in [15, Corollary 3.11] for the case $U = \mathbb{R}^\infty$. Let us prove it for the case $U = \mathbb{R}^M$. By Lemma 3.1 the series (3.8) converges unconditionally in the space $L_2(\mathbb{R}^M, X, \gamma)$.
to \(v\). Observe that \(I_{\Lambda(\xi)} H_s = H_s\) for every \(s \in \Lambda(\xi)\) and \(\Delta_s H_{s'} = 0\) for every \(s \not\in s'\). Hence for the downward closed set \(\Lambda(\xi) \subset \mathbb{N}_0^M\), we can write

\[
I_{\Lambda(\xi)} v = I_{\Lambda(\xi)} \left( \sum_{s \in \mathbb{N}_0^M} v_s H_s \right) = \sum_{s \in \mathbb{N}_0^M} v_s I_{\Lambda(\xi)} H_s = S_{\Lambda(\xi)} v + \sum_{s \notin \Lambda(\xi)} v_s I_{\Lambda(\xi) \cap R_s} H_s,
\]

where \(R_s := \{s' \in \mathbb{N}_0^M : s' \leq s\}\) and

\[
S_{\Lambda(\xi)} v := \sum_{s \in \Lambda(\xi)} v_s H_s
\]

for \(v \in L_2(\mathbb{R}^M, X, \gamma)\) represented by the Hermite gpc expansion (3.8). This implies

\[
\|v - I_{\Lambda(\xi)} v\|_{L_2^\gamma(\mathbb{R}^M, X)} \leq \|v - S_{\Lambda(\xi)} v\|_{L_2^\gamma(\mathbb{R}^M, X)} + \sum_{s \notin \Lambda(\xi)} \|I_{\Lambda(\xi) \cap R_s} H_s\|_{L_2^\gamma(\mathbb{R}^M)}.
\]

(A.40)

Therefore, to prove the lemma it is sufficient to show that each term in the right-hand side is bounded by \(C\xi^{-(1/q - 1/2)}\). The bound of the first term can be obtained from the Cauchy–Schwaz inequality and (A.9):

\[
\|v - S_{\Lambda(\xi)}\|_{L_2^\gamma(\mathbb{R}^M, X)} \leq \sum_{\sigma_s > \xi^1/q} \|v_s\|_X \|H_s\|_{L_2^\gamma(\mathbb{R}^M)} \leq \sum_{\sigma_s > \xi^1/q} \|v_s\|_X
\]

\[
\leq \left( \sum_{\sigma_s > \xi^1/q} (\sigma_s \|v_s\|_X)^2 \right)^{1/2} \left( \sum_{\sigma_s > \xi^1/q} \sigma_s^{-2} \right)^{1/2} \leq C \left( \sum_{\sigma_s > \xi^1/q} \sigma_s^{-q} \sigma_s^{-(2-q)} \right)^{1/2}
\]

\[
\leq C\xi^{-(1/q - 1/2)} \left( \sum_{s \in \mathbb{N}_0^M} \sigma_s^{-q} \right)^{1/2} \leq C\xi^{-(1/q - 1/2)}.
\]

(A.41)

Let us prove the bound of the second term in the right-hand side of (A.40). We have that

\[
\|I_{\Lambda(\xi) \cap R_s} H_s\|_{L_2^\gamma(\mathbb{R}^M)} \leq \sum_{s' \in \Lambda(\xi) \cap R_s} \|\Delta_{s'}(H_s)\|_{L_2^\gamma(\mathbb{R}^M)}.
\]

(A.42)

We estimate the norms inside the right-hand side. For \(s \in \mathbb{N}_0^M\) and \(s' \in \Lambda(\xi) \cap R_s\), we have \(\Delta_{s'}(H_s) = \prod_{j=1}^M \Delta_{s'_j}(H_{s_j})\). From Lemma 3.2 and (A.9) we deduce that

\[
\|\Delta_{s'_j}(H_{s_j})\|_{L_2^\gamma(\mathbb{R})} \leq (1 + C\epsilon s_j^{1/6+\epsilon}) \|H_{s_j}\|_{L_2^\gamma(\mathbb{R})} \leq (1 + C\epsilon s_j^{1/6+\epsilon}),
\]

and consequently,

\[
\|\Delta_{s'}(H_s)\|_{L_2^\gamma(\mathbb{R}^M)} = \prod_{j=1}^M \|\Delta_{s'_j}(H_{s_j})\|_{L_2^\gamma(\mathbb{R})} \leq p_{s'}(\theta_1, \lambda) \leq p_s(\theta_1, \lambda),
\]

(A.43)
where $\theta_1 = 1/6 + \varepsilon$ and recall that $\lambda = C_\varepsilon$. Substituting $\|\Delta_{s'}(H_s)\|_{L^\infty_M(\mathbb{R}^M)}$ in (A.42) by the right-hand side of (A.43) gives that

$$
\|I_{\Lambda(\xi) \cap R_s} H_s\|_{L^\infty_M(\mathbb{R}^M)} \leq \sum_{s' \in \Lambda(\xi) \cap R_s} p_s(\theta_1, \lambda) \leq |R_s| p_s(\theta_1, \lambda) \leq p_s(1,1) p_s(\theta_1, \lambda) \leq p_s(\theta/2, \lambda).
$$

By using of the last estimates and the assumption $\|p(\theta/q, \lambda)\|_\sigma^{-1} \leq C < \infty$ with a positive constant $C$ independent of $M$, we derive the bound of the second term in the right-hand side of (A.40):

$$
\sum_{s \in \Lambda(\xi)} \|I_{\Lambda(\xi) \cap R_s} H_s\|_{L^\infty_M(\mathbb{R}^M)} \leq C \sum_{s \in \Lambda(\xi)} \|v_s\||X p_s(\theta/2, \lambda) \leq C \left( \sum_{\sigma_s > \xi^{1/q}} (\sigma_s \|v_s\||X)^2 \right)^{1/2} \left( \sum_{\sigma_s > \xi^{1/q}} p_s(\theta/2, \lambda)^2 \sigma_s^{-2} \right)^{1/2} \leq C \left( \sum_{\sigma_s > \xi^{1/q}} p_s(\theta/2, \lambda)^2 \sigma_s^{-q} \sigma_s^{-(2-q)} \right)^{1/2} \leq C \xi^{-(1/q-1/2)} \left( \sum_{s \in \mathbb{N}_0^M} p_s(\theta, \lambda) \sigma_s^{-q} \right)^{1/2} \leq C \xi^{-(1/q-1/2)},
$$

which together with (A.40) and (A.41) proves the theorem.

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