Abstract

The $h$-deformation of functions on the Grassmann matrix group $Gr(2)$ is presented via a contraction of $Gr_q(2)$. As an interesting point, we have seen that, in the case of the $h$-deformation, both R-matrices of $GL_h(2)$ and $Gr_h(2)$ are the same.
In recent years a new class of quantum deformations of Lie groups and algebras, the so-called $h$-deformation, has been intensively studied by many authors [1-9]. The $h$-deformation of matrix groups can be obtained using a contraction procedure. We start with a quantum plane and its dual and follow the contraction method of [9].

Consider the $q$-deformed algebra of functions on the quantum plane [10] generated by $x'$, $y'$ with the commutation rule

$$x'y' = qy'x'.$$  \hspace{1cm} (1)

Applying a change of basis in the coordinates of the (1) by use of the following matrix

$$g = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \quad f = \frac{h}{q - 1}$$  \hspace{1cm} (2)

one arrives at [9], in the limit $q \to 1$,

$$xy = yx + hy^2.$$  \hspace{1cm} (3)

We denote the quantum $h$-plane by $R_h(2)$.

Similarly, one gets the dual quantum $h$-plane $R^*_h(2)$ as generated by $\eta$, $\xi$ with the relations

$$\xi^2 = 0 \quad \eta^2 = h\eta\xi \quad \eta\xi + \xi\eta = 0.$$  \hspace{1cm} (4)

Let

$$\hat{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be a grassmann matrix in $Gr(2)$. All matrix elements of $\hat{A}$ are grassmann. We consider linear transformations with the following properties:

$$\hat{A} : R_h(2) \longrightarrow R^*_h(2) \quad \hat{A} : R^*_h(2) \longrightarrow R_h(2).$$  \hspace{1cm} (5)

The action on points of $R_h(2)$ and $R^*_h(2)$ of $\hat{A}$ is

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}.$$  \hspace{1cm} (6)

We assume that the entries of $\hat{A}$ commute with the coordinates of $R_h(2)$ and anti-commute with the coordinates of $R^*_h(2)$. As a consequence of the linear
transformations in (5) the vectors $\left( \frac{\eta}{\xi} \right)$ and $\left( \frac{x}{y} \right)$ should belong to $R_h^*(2)$ and $R_h(2)$, respectively, which impose the following $h$-anti-commutation relations among the matrix elements of $\hat{A}$:

\begin{align*}
\alpha\beta + \beta\alpha &= h(\alpha\delta + \beta\gamma) \quad \alpha\gamma + \gamma\alpha = 0 \\
\beta\gamma + \gamma\beta &= h(\delta\gamma - \gamma\alpha) \quad \beta\delta + \delta\beta = -h(\alpha\delta + \gamma\beta) \\
\alpha\delta + \delta\alpha &= h(\gamma\alpha - \delta\gamma) \quad \gamma\delta + \delta\gamma = 0 \\
\alpha^2 &= -h\gamma\alpha \quad \beta^2 = h(\beta\delta - \alpha\beta + h\alpha\delta) \quad \gamma^2 = 0 \quad \delta^2 = h\delta\gamma.
\end{align*}

These relations define the $h$-deformation of functions on the grassmann matrix group $Gr(2)$, $Gr_h(2)$.

Alternatively, the relations (7) can be obtained by the following similarity transformation [9]:

$$\hat{A}' = g\hat{A}g^{-1}$$

which in our case gives

\begin{align*}
\alpha' &= \alpha + \frac{h}{q-1}\gamma \quad \beta' = \beta + \frac{h}{q-1}(\delta - \alpha - \frac{h}{q-1}\gamma) \\
\gamma' &= \gamma \quad \delta' = \delta - \frac{h}{q-1}\gamma.
\end{align*}

and then taking the $q \to 1$ limit. Here $\alpha'$, $\beta'$, $\gamma'$ and $\delta'$ are generators of $Gr_q(2)$, which satisfy the following commutation relations [11,12]:

\begin{align*}
\alpha'\beta' + q^{-1}\beta'\alpha' &= 0 \quad \alpha'\gamma' + q^{-1}\gamma'\alpha' = 0 \\
\gamma'\delta' + q^{-1}\delta'\gamma' &= 0 \quad \beta'\delta' + q^{-1}\delta'\beta' = 0 \\
\alpha'\delta' + \delta'\alpha' &= 0 \quad \alpha'^2 = \beta'^2 = \gamma'^2 = \delta'^2 = 0 \\
\beta'\gamma' + \gamma'\beta' &= (q - q^{-1})\delta'\alpha'.
\end{align*}

Substituting (9) into (10) one gets the set of relations (7) above.

The algebra (10) is associative under multiplication and the relations in (10) may be also expressed in a tensor product form [11,12]

$$R_q\hat{A}'_1\hat{A}'_2 = -\hat{A}'_2\hat{A}'_1R_q$$

(11)
where
\[
R_q = \begin{pmatrix}
q + q^{-1} & 0 & 0 & 0 \\
0 & 2 & q^{-1} - q & 0 \\
0 & q - q^{-1} & 2 & 0 \\
0 & 0 & 0 & q + q^{-1}
\end{pmatrix}.
\]  
(12)

Here, since the matrix elements of \( \hat{A}' \) are all grassmann, for the conventional tensor products
\[
\hat{A}'_1 = \hat{A}' \otimes I \quad \text{and} \quad \hat{A}'_2 = I \otimes \hat{A}'
\]  
(13)
one can write (no-grading)
\[
(\hat{A}_1)^{ij}_{kl} = \hat{A}^i_k \delta^j_l \quad (\hat{A}_2)^{ij}_{kl} = \delta^i_k \hat{A}'^j_l
\]  
(14)
where \( \delta \) denotes the Kronecker delta. Note that in the limit \( q \rightarrow 1 \) the matrix \( R_q \) becomes twice the 4x4 unit matrix. Notice also that although the algebra (10) is an associative algebra of the matrix entries of \( \hat{A} \), \( R_q \) does not satisfy the quantum Yang-Baxter equation (QYBE)
\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\]
Thus the Yang-Baxter equation is not a necessary condition for associativity [see the paragraph after (19) for other remarks]. It is obvious that a change of basis in the \( R_h(2) \) leads to the similarity transformation
\[
\hat{A} = g^{-1} \hat{A}' g
\]  
(15)
for the quantum grassmann group and the following similarity transformation for the corresponding \( R \)-matrix
\[
R_{h,q} = (g \otimes g)^{-1} R_q (g \otimes g).
\]  
(16)
If we define the \( R \)-matrix \( R_h \) as
\[
R_h = \lim_{q \rightarrow 1} R_{h,q}
\]  
(17)
we get (after dividing by 2)
\[
R_h = \begin{pmatrix}
1 & -h & h & h^2 \\
0 & 1 & 0 & -h \\
0 & 0 & 1 & h \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]  
(18)
Substituting (9) and (16) into (11) we arrive at the $q \to 1$ limit

$$R_h \hat{A}_1 \hat{A}_2 = -\hat{A}_2 \hat{A}_1 R_h.$$ (19)

An other interesting point is that, although the $R$-matrices of $GL_q(2)$ and $Gr_q(2)$ are different, in the case of the $h$-deformation, the $R$-matrices of $GL_h(2)$ and $Gr_h(2)$ are the same [see Ref. 9, for the $R$-matrix $R_h$ of $GL_h(2)$]. In the limit $q \to 1$ both the $R$-matrices of $GL_q(2)$ and $Gr_q(2)$ become the same 4x4 unit matrix. Although the $R$-matrix $R_q$ of $Gr_q(2)$ does not satisfy the QYBE, the $R$-matrix $R_h$ of $Gr_h(2)$ satisfies the QYBE.

Since the entries of $\hat{A}$ are all grassmann, a proper inverse can not exist. However, the left and right inverses of $\hat{A}$ can be constructed:

$$\hat{A}_L^{-1} = \begin{pmatrix} \delta + h \gamma & \beta + h \alpha \\ -\gamma & -\alpha \end{pmatrix},$$ (20)

$$\hat{A}_R^{-1} = \begin{pmatrix} -\delta & \beta + h \delta \\ -\gamma & \alpha + h \gamma \end{pmatrix}. $$ (21)

It is now easy to show that

$$\hat{A}_L^{-1} \hat{A} = \Delta_L$$ (22)

$$\hat{A} \hat{A}_R^{-1} = \Delta_R$$ (23)

where

$$\Delta_L = \beta \gamma + \delta \alpha \quad \Delta_R = \gamma \beta + \alpha \delta.$$ (24)

In this case at least formally, $\Delta_L$ and $\Delta_R$ may be considered as the left and right quantum (dual) determinants, respectively. Note that one can write

$$\Delta_L \hat{A}_R^{-1} = \hat{A}_L^{-1} \Delta_R.$$ (25)

**Final Remarks.** We known that all the matrix elements of $\hat{A}$ are grassmann (odd or fermionic) if $\hat{A}$ is a grassmann matrix, i.e., it belongs to $Gr(2)$. Now let $\hat{A}$ and $\hat{A}'$ be any two anti-commuting (i.e., any element of grassmann matrices whose elements $\hat{A}$ anti-commutes with any element of $\hat{A}'$ ) satisfy (10). Then, all the matrix elements of a product $A = \hat{A} \hat{A}'$ are bosonic (or even) since
the elements of the matrix product of two grassmann matrices are all bosonic. It can also be verified that the matrix elements of \( A \) satisfy \( q \)-commutation relations of \( GL_q(2) \), i.e., for

\[
A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

\[ab = qba\quad ac = qca\quad bc = cb\]  

(26)

etc. That is, if

\[\hat{A}, \hat{A}' \in Gr_q(2) \implies A = \hat{A}\hat{A}' \in GL_q(2).\]

In view of these facts, we can say that, there may be no coproduct of the form \( \Delta(\hat{A}) = \hat{A} \otimes \hat{A} \). For, this coproduct is invariant under the \( q \)-commutation relations (26) of \( GL_q(2) \). These facts also prevent the existence of a coproduct of the form \( \Delta(\hat{A}) = \hat{A}^{t_2} \otimes \hat{A} \) where \( t_2 \) is an involution acting on the elements of \( \hat{A} \). Hence a construction of the coproduct along the lines of Ref. 13 is also not possible.

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