Existence of an observation window of finite width for continuous-time autonomous nonlinear systems

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Abstract

In this paper, the relationship between notions of observability for continuous-time nonlinear system related to distinguishability, observability rank condition and K-function has been investigated. It is proved that an autonomous nonlinear system that is observable in both distinguishability and rank condition sense permits an observation window of finite width, and it is possible to construct a K-function related to observability for such system.

keywords nonlinear systems, observability, distinguishability, K-function

1 Introduction

The state estimation problem is one of the most fundamental problems in control system theory. The intrinsic property of the system that makes the state estimation problem feasible is observability, and this property has been extensively studied in past decades.

For linear systems, the notion of observability is firmly established\[12\]. Contrary, for nonlinear systems, several non-equivalent definitions of observability have been proposed, and by using these definitions, the state estimation problem of nonlinear systems have been extensively studied \[1, 3, 4, 6, 9, 10, 13\]. However, the relations between several different notions of observability have not been fully understood (although there are several established facts\[3, 6, 9\].)

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Recently, the author has proved that a discrete-time nonlinear systems whose observation map is injective and the Jacobian of the observation map is of full
rank satisfies a seemingly stronger condition called uniform observability, and by assuming these conditions, it is possible to construct a $K$-function related to observability. This paper is an attempt to establish corresponding results for continuous-time systems.

The scope of this paper is limited to autonomous nonlinear systems, and we deal with three typical definitions of observability for nonlinear systems which are related to distinguishability, rank condition and $K$-function, respectively (precise definitions are given later.) Roughly speaking, we prove that distinguishability together with the observability rank condition implies that there is a ‘observation window’ (the sequence of past output as a function of time) of finite width which determines the initial state uniquely, and it is possible to construct a $K$-function related to observability.

2 Main Results

In this paper, we consider an autonomous nonlinear system of the form

$$\begin{align*}
\dot{x} &= f(x), \\
y &= h(x),
\end{align*}$$

where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}^p$ is the output. The functions $f(x)$ and $h(x)$ are assumed to be of compatible dimensions, and smooth up to required order. The solution of (1) is assumed to be unique, and the solution initialized at $t = 0$ by $x_0$ is denoted by $\varphi(t, 0, x_0)$. The set of permissible initial conditions of (1) is assumed to be compact, and is denoted by $\Omega$. Note that the state space itself is not compact.

Next, we introduce the definitions of observability considered in this paper. Unfortunately, there is no general agreement on how to name these properties. Hence, we temporally call them ‘D-observability’, ‘R-observability’, and ‘K-observability’ (to be defined below) for brevity.

Definition 1 A pair of initial states $(x_1, x_2)$ of (1) with $x_1 \neq x_2$ is said to be an indistinguishable pair if $\forall t \geq 0$, $h(\varphi(t, 0, x_1)) = h(\varphi(t, 0, x_2))$.

Definition 2 The system (1) is said to be D-observable (with respect to $\Omega$) if there is no indistinguishable pair in the set $\Omega$.

Definition 3 The system (1) is said to be R-observable (with respect to $\Omega$) if $\exists N > 0$, the Jacobian of the map $H(x) = (h(x), L_f h(x), \ldots, L_f^{N-1} h(x))$ is of full rank on $\Omega$, where $L_f h = \frac{\partial h}{\partial x} f$, and $L_f^k h = L_f (L_f^{k-1} h)$.

Definition 4 A function $\alpha : D \rightarrow [0, \infty)$ (where $D$ is either $[0, \infty)$, $[0, a)$ or $[0, a]$ with $a > 0$) is said to be a $K$-function if it is continuous, $\alpha(0) = 0$, and is strictly increasing.
Definition 5. The system (1) is said to be K-observable if
\[ \exists T > 0, \forall x_1, x_2 \in \Omega, \int_0^T \| h(\varphi(t, 0, x_1)) - h(\varphi(t, 0, x_2)) \|^2 dt \geq \alpha(|x_1 - x_2|), \]
where \( \alpha(\cdot) \) is a K-function and \( \| \cdot \| \) denotes the Euclid norm of a vector.

It is a known fact that, if a system is R-observable at a point \( x_0 \), then it is D-observable on a neighborhood of \( x_0 \) [3, 9], but it is not always possible to extend the result to the whole of \( \Omega \). On the other hand, D-observability does not imply R-observability, as the following example shows.

Example 1. Consider a 1-dimensional system
\[ \dot{x} = x^3, \]
\[ y = h(x) = x^3 \]
This system is D-observable because it is possible to directly calculate \( x \) from \( y \) \( (x = y^{1/3}) \), but is not R-observable at \( x = 0 \), because \( h(x) = x^3 \), \( L_fh(x) = 3x^2 \), and inductively, \( L_k^2 h(x) = 1 \cdot 3 \cdot \cdots \cdot (2k-1)x^{2k+1} \), and hence their derivatives vanish at \( x = 0 \).

It is desirable that the width of the ‘observation window’ (the time interval that the output of the system is stored in order to determine the initial state uniquely) is finite. In this sense, K-observability is convenient, and has been widely adopted in works on moving horizon state estimation [1, 2]. If the system (1) is K-observable, then for \( x_1, x_2 \in \Omega \) with \( x_1 \neq x_2 \), \( \exists T > 0, \forall t \in [0, T], h(\varphi(t, 0, x_1)) \neq h(\varphi(t, 0, x_2)) \), hence (1) is D-observable. Then, a natural question arises: do systems that are D-observable always permit an observation window of finite width? Unfortunately, the answer is negative, which is given in the following example.

Example 2. Consider a 1-dimensional system
\[ \dot{x} = x \]
\[ y = h(x) = \begin{cases} 0 & x < M \\ x - M & x \geq M \end{cases} \]
where \( M \) is a positive constant. If the initial condition is zero, then the output is identically zero. For an initial condition \( x_0 > 0 \), \( x(t) = \exp(t)x_0 \), and hence the output is identically zero for \( t < \ln(M/x_0) \) and is \( \exp(t)x_0 - M \) for \( t \geq \ln(M/x_0) \). Hence, the zero initial condition and \( x_0 \) cannot be distinguished until \( t = \ln(M/x_0) \), and hence as the initial condition gets smaller, the required width of the observation window tends to infinity.

One may argue that the reason for making the width of the observation window infinite is the non-differentiability of the output function, but this is not the case. For example, by replacing the output function \( h(x) \) of (4) with
\[ h(x) = \begin{cases} 0 & x \leq M \\ \exp[-1/(x-M)] & x > M \end{cases} \]
a similar conclusion holds.

Thus far, we have seen that there are gaps between D-observability, R-observability and K-observability, and a D-observability system does not always permit an observation window of finite width. In the following, we show that, if \( \mathbb{I} \) is D-observable as well as R-observable, then there exists an observation window of finite width, and it is possible to construct a \( \mathcal{K} \)-function corresponding to Definition \( \mathbb{I} \) and hence \( \mathbb{I} \) is K-observable.

**Theorem 6** If \( \mathbb{I} \) is D-observable as well as R-observable for the initial condition set \( \Omega \), then there is a finite \( T > 0 \) such that \( \forall x_1, x_2 \in \Omega \) with \( x_1 \neq x_2 \), \( \exists t : 0 \leq t \leq T, \varphi(t, 0, x_1) \neq \varphi(t, 0, x_2) \).

**Proof.** We first prove that

\[
\forall x \in \Omega, \exists N(x), \forall z_1, z_2 \in N(x) \text{ with } z_1 \neq z_2, \\
\forall T > 0, \exists t : 0 \leq t \leq T, h(\varphi(t, 0, z_1)) \neq h(\varphi(t, 0, z_2))
\]

(5)

by contradiction, where \( N(x) \) denotes an open neighborhood of \( x \). Suppose that \( \text{(5)} \) is false, that is,

\[
\exists x \in \Omega, \forall N(x), \exists z_1, z_2 \in N(x) \text{ with } z_1 \neq z_2, \\
\exists T > 0, \forall t : 0 \leq t \leq T, h(\varphi(t, 0, z_1)) = h(\varphi(t, 0, z_2)).
\]

(6)

Then, \( h(\varphi(t, 0, z_1)) - h(\varphi(t, 0, z_2)) \) is identically zero as a function of \( t \). Hence, for all \( k \geq 0 \), \( \frac{d^k}{dt^k} h(\varphi(t, 0, z_1)) = \frac{d^k}{dt^k} h(\varphi(t, 0, z_2)) \), hence \( H(\varphi(t, 0, z_1)) = H(\varphi(t, 0, z_2)) \), and by letting \( t = 0 \), \( H(z_1) = H(z_2) \). On the other hand, because the Jacobian of \( H \) is of full rank, there is a neighborhood of \( x \) in which \( H \) is injective. By choosing such neighborhood \( N(x) \) (recall that \( N(x) \) is arbitrary), it follows that \( H(z_1) \neq H(z_2) \) because \( z_1 \neq z_2 \), hence a contradiction has been obtained. Therefore, \( \text{(5)} \) is false and hence \( \text{(6)} \) is true.

Next, fix \( x \in \Omega \), and let \( N_{\text{loc}}(x) \) be an open neighborhood of \( x \) in which

\[
\forall z_1, z_2 \in N(x) \text{ with } z_1 \neq z_2, \forall T > 0, \exists t : 0 \leq t \leq T, h(\varphi(t, 0, z_1)) \neq h(\varphi(t, 0, z_2))
\]

(7)

holds. Because \( \Omega \) is compact and \( N_{\text{loc}}(x) \) is open, \( \Omega \setminus N_{\text{loc}}(x) \) is compact. For each \( z \in \Omega \setminus N_{\text{loc}}(x) \), by D-observability, \( \exists t_z > 0, h(\varphi(t_z, 0, x)) \neq h(\varphi(t, 0, z)) \). Because \( h \) and \( \varphi \) are continuous, there is an open neighborhood \( O_z \) of \( x \) and an open neighborhood \( G_z \) of \( z \) for which \( h(\varphi(t_z, 0, O_z)) \cap h(\varphi(t_z, 0, G_z)) = \emptyset \). Let \( W_z = (z, t_z, O_z, G_z) \) be the tuple satisfying this condition, and consider the set

\[
\{ W_z : z \in \Omega \setminus N_{\text{loc}}(x) \}.
\]

(8)

Because \( \{ G_z : z \in \Omega \setminus N_{\text{loc}}(x) \} \) covers \( \Omega \setminus N_{\text{loc}}(x) \) and \( \Omega \setminus N_{\text{loc}}(x) \) is compact, for some \( L > 0 \), there is a finite subcollection

\[
\{ W_z^{(1)}, \ldots, W_z^{(L)} \}
\]
of (5) (let us rewrite \( W_z^{(i)} = (z^{(i)}, t_z^{(i)}, O_z^{(i)}, G_z^{(i)}) \) for which \( \{G_z^{(1)}, \ldots, G_z^{(L)}\} \) covers \( \Omega \setminus N_{loc}(x) \). Let \( V_x = \left( \bigcap_{i=1}^{L} O_z^{(i)} \right) \cap N_{loc}(x) \),

\[ T_x = \max\{t_z^{(1)}, \ldots, t_z^{(L)}\}. \tag{9} \]

\( V_x \) is an open neighborhood of \( x \), because it is a finite intersection of open sets containing \( x \). Let \( M_x = (x, V_x, T_x) \) be the tuple corresponding to above construction, and consider the set

\[ \{ M_x : x \in \Omega \}. \tag{10} \]

Because \( \Omega \) is compact, for some \( J > 0 \), there is a finite subcollection \( \{ M_x^{(1)}, \ldots, M_x^{(J)} \} \) of (11) (let us rewrite \( M_x^{(j)} = (x^{(j)}, V_x^{(j)}, T_x^{(j)}) \) for which \( \{V_x^{(1)}, \ldots, V_x^{(J)}\} \) covers \( \Omega \). Let

\[ T = \max\{T_x^{(1)}, \ldots, T_x^{(J)}\}. \tag{11} \]

Then, all pairs of initial conditions \((x_1, x_2)\) with \( x_1 \neq x_2 \) are distinguishable for some \( t \) with \( 0 \leq t \leq T \). For, by construction, there is a \( V_x^{(j)} \) such that \( x_1 \in V_x^{(j)} \). If \( x_2 \) is in \( N_{loc}(x^{(j)}) \), then by (10), \( x_1 \) and \( x_2 \) are distinguishable. Otherwise, \( x_2 \in \Omega \setminus N_{loc}(x^{(j)}) \). Because \( \Omega \setminus N_{loc}(x^{(j)}) \) is covered by corresponding \( \{G_z^{(1)}, \ldots, G_z^{(L)}\} \), by (10) and (11), \( x_1 \) and \( x_2 \) are distinguishable at some \( t \) with \( 0 \leq t \leq T \).

Next, we prove that a system that is D-observable as well as R-observable on \( \Omega \) is K-observable.

**Theorem 7** If the system (1) is D-observable as well as R-observable on \( \Omega \), then it is K-observable on \( \Omega \).

**Proof.** Instead of constructing a K-function that satisfies (2) directly, we construct an increasing function of \( |z_1 - z_2| \) that is positive if \( |z_1 - z_2| \neq 0 \). We assume that \( T \) of (2) is sufficiently large and \( \forall x_1, x_2 \in \Omega \) with \( x_1 \neq x_2 \), \( \exists t : 0 \leq t \leq T, h(\varphi(t, 0, x_1)) \neq h(\varphi(t, 0, x_1)) \). The existence of such \( T \) is assured by Theorem 3. For \( r > 0 \), let \( D_r = \{(x_1, x_2) \in \Omega \times \Omega : |x_1 - x_2| \geq r\} \). Define

\[ \eta(t, x_1, x_2) = |h(\varphi(t, 0, x_1)) - h(\varphi(t, 0, x_2))|^2. \tag{12} \]

For all \((x_1, x_2)\) in \( D_r \), \( \exists t_{x_1, x_2} : 0 \leq t_{x_1, x_2} \leq T \) such that

\[ \eta(t_{x_1, x_2}, x_1, x_2) = c_{x_1, x_2} > 0. \]

Because \( \eta \) is continuous, there is an open interval \( I_{x_1, x_2} = (t_{x_1, x_2} - \delta_{x_1, x_2}, t_{x_1, x_2} + \delta_{x_1, x_2}) \) and an open neighborhood \( U_{x_1, x_2} \) of \((x_1, x_2)\) such that \( \forall (t, w_1, w_2) \in I_{x_1, x_2} \times \cup_{x_1, x_2}, \eta(t, w_1, w_2) \geq c_{x_1, x_2} / 2 \). Let \( Y_{x_1, x_2} = (t_{x_1, x_2}, x_1, x_2, I_{x_1, x_2}, U_{x_1, x_2}, c_{x_1, x_2}) \) be the tuple that satisfy these conditions. Because \( \{U_{x_1, x_2} : (x_1, x_2) \in D_r\} \) covers \( D_r \), \( D_r \) is compact, there is a finite subcover. Let the subset of the tuple corresponding to this finite subcover be \( \{Y^{(1)}, \ldots, Y^{(K)}\} \). Let us rewrite
\( Y^{(k)} = (t^{(k)}, x_1^{(k)}, x_2^{(k)}, I^{(k)}, U^{(k)}) \), and \( I^{(k)} = (t^{(k)} - \delta^{(k)}, t^{(k)} + \delta^{(k)}) \). For any \((w_1, w_2) \in D_r\), there is a \( U^{(k)} \) such that \((w_1, w_2) \in U^{(k)}\). Hence,

\[
\int_0^T \eta(t, w_1, w_2) dt \geq \int_{t^{(k)} - \delta^{(k)}}^{t^{(k)} + \delta^{(k)}} \eta(t, w_1, w_2) dt \geq c^{(k)} \delta^{(k)}.
\]

Therefore, by letting \( \gamma_r = \min\{c^{(1)} \delta^{(1)}, \ldots, c^{(K)} \delta^{(K)}\} \), we obtain

\[
\int_0^T \eta(t, x_1, x_2) dt \geq \gamma_r > 0
\]

for \((x_1, x_2) \in D_r\). Let

\[
\alpha_0(r) = \inf_{(x_1, x_2) \in D_r} \int_0^T \eta(t, x_1, x_2) dt.
\]

Then, \( \alpha_0(r) \) is an increasing function of \( r \), and for \( r > 0 \), \( \alpha_0(r) > 0 \). Note that \( \alpha_0(r) \) may not be strictly increasing, and may be discontinuous. However, by Lemma 5 of [5], it is possible to construct a \( K \)-function that satisfy \( \alpha(r) \leq \alpha_0(r) \). Hence, for \((x_1, x_2) \in \Omega \times \Omega \) with \(|x_1 - x_2| = r\),

\[
\int_0^T \eta(t, x_1, x_2) dt \geq \alpha_0(r) \geq \alpha(r) = \alpha(|x_1 - x_2|),
\]

and the desired \( K \)-function has been constructed. \( \square \)

3 Conclusion

In this paper, we have shown that an autonomous nonlinear system which is D-observable and R-observable always permits an observation window of finite width, and it is actually K-observable as well. A theoretical construction of corresponding \( K \)-function has been provided as well. In many researches related to nonlinear observability, the existence of an observation window of finite width and a \( K \)-function are assumed \textit{a priori}. The significance of this paper is the proof that they are consequences of D-observability and R-observability. It is also to be noted that our result is purely existential, and no practical method of obtaining the width of the observation window and the \( K \)-function have been provided.

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