Dualities between Scale Invariant and Magnitude Invariant Perturbation Spectra in Inflationary/Bouncing Cosmos

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We study the cosmological perturbation spectra using the dynamical equations of gauge invariant perturbations with a generalized blue/red-shift term, \( mH\ddot{\chi} \). Combined with the power-law index of cosmological background, \( \nu \), we construct a parameter space, \((\nu, m)\), to classify all possible perturbation spectra. The magnitude-invariant power spectra occupy a two dimensional region in the \((\nu, m)\) space. We find two groups of scale-invariant solutions, one of which are also magnitude invariant while the other not, in the expanding and contracting phases of cosmological evolution. We explore the implications of magnitude invariance to scale-invariance of an underlying spectrum, and unveil a novel duality between the scale-invariant solutions and the boundary of magnitude-invariant solutions: under two consecutive duality transformations, the scale-invariant solutions are mapped onto the boundary of magnitude invariant region. The physical origin of such a duality is yet to be studied. We present cosmological applications of our observations to a de-Sitter universe, an Ekpyrotic background, and a matter-dominated contraction. For the first two cases previously known results are re-derived; we also find for a matter-dominated contraction a solution with both scale-invariance as well as magnitude-invariance— in contrast to Wands’s scale-invariant but magnitude-variant spectrum.

PACS numbers: 11.10.Lm, 11.27.+d, 98.80.Cq

I. INTRODUCTION AND SUMMARY OF RESULTS

The single field slow-roll inflation model has been hailed a paradigm for its much lauded rendition of scale-invariant power spectrum that shows up in an array of modern cosmological observations. Generating scale-invariant power spectrum, however, has been shown not a monopoly of the slow-roll inflation scenario due to recent progress in model building. It is also possible to construct which generate scale-invariant power spectrum of cosmological perturbations.

In search of scale invariant solutions for the power spectrum of cosmological perturbations, compelling alternative cosmological models have been constructed and the possibilities of generating scale-invariant power spectrum investigated. For example, a contracting matter-dominated background was discovered by Wands [1] (see also, Finelli and Brandenberger [2]) to generate a scale-invariant spectrum. Khoury et al [3], by neglecting gravity back reaction in the dynamics of the fluctuations, found a scale-invariant power spectrum for an Ekpyrotic-type background (slowly contracting or expanding) [4]. If, however, gravity is taken into account, the spectrum of perturbations in an Ekpyrotic background with a single canonical scalar field becomes blue [5]. On the other hand, possible dualities of power spectra between backgrounds with seemingly different equations of state–in a particular gauge–have also been studied [6]. Power spectra with scale-invariance are more readily found for Ekpyrotic-like backgrounds once the long-held assumption that the equation of state of backgrounds be nearly constant is relaxed [7]. In this paper we shall, nevertheless, assume the constancy of the equation of state of the cosmological backgrounds during the period in which the primordial power spectra are generated.

The brilliant example discovered by Wands [1]—scale-invariant (with \( k \)-independence) power spectrum of cosmological perturbations generated during a matter-dominated contraction—lends hope to obtaining scale-invariant power spectrum in more general backgrounds other than de-Sitter. However the example given by Wands is not truly scale-invariant because of its time dependence: the amplitude of power spectrum increases on the effective horizon. The amplitude of each mode in the power spectrum depends on the moment at which the associated mode exits the effective horizon. The moment of exit is, in turn, determined by the mode’s \( k \) value. Altogether an implicit \( k \)-dependence creeps into the value of the Hubble parameter as the mode exits its horizon, \( H_*(k) \), rendering the power spectrum as a whole non-scale-invariant. A brief discussion concerning Wands’s result [1], can be found below.

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1 Wands’s result can be re-casted into a simplified form, \( P_{\delta\phi} \sim H^2(-k\eta)^{(3-2|\beta|)} \) with \( P_{\delta\phi}, H, k \) and \( \eta \) denoting a given power spectrum, the Hubble parameter, wave vector and conformal time respectively. And \( \beta = \pm \frac{3}{2} \) corresponds to an expanding de-Sitter background.
We pause here to clarify our terminologies. A power spectrum of density perturbations, \( P_\chi \), is conventionally defined as follows,
\[
P_\chi = \frac{k^3 |\chi|^2}{2\pi}. \tag{1}
\]
Since we are interested in the deviation from scale invariance we will suppress the three powers of \( k \) leading to scale invariance and concentrate on any deviations from it. For a gaussian spectrum we will simply write:
\[
P_\chi \equiv |\chi|^2. \tag{2}
\]
Scale-invariance (\( k \)-independence) of a power spectrum of \( \chi \) then implies,
\[
P_\chi \sim k^0 \eta^{2W(\nu,m)} \iff P_\chi \sim k^{-3} \eta^{2W(\nu,m)} \tag{3}
\]
where \( 2W(\nu,m) \neq 0 \), in general. Magnitude-invariance (\( \eta \)-independence) of a power spectrum of \( \chi \), likewise, implies
\[
P_\chi \sim k^{2L(\nu,m)+3} \eta^0 \iff P_\chi \sim k^{2L(\nu,m)} \eta^0, \tag{4}
\]
where \( 2L(\nu,m) \neq 0 \), in general. The two equations above also serve to define \( 2L(\nu,m) \) and \( 2W(\nu,m) \). The detailed derivations of these two indices from the dynamic equations are given below in Section II.

A power spectrum can be scale independent in a physical sense if and only if it is both time independent and scale independent:
\[
P_\chi \sim k^0 \eta^0 \iff P_\chi \sim k^{-3} \eta^0, \tag{5}
\]

We conform to the conventional usage of “scale-invariance” in denoting a power spectrum’s \( k \)-independence, and use “magnitude-invariance” to stress the lack of time evolution of the magnitudes. We would like to stress that only when a solution that is both “magnitude-invariant” and “scale-invariant” in the conventional sense can truly be a scale-independent power spectrum. In the following, we will see that the power spectrum generated in the single scalar field slow-roll inflation model is a genuine scale-independent power spectrum. Unfortunately, in the Wands’ case, the power spectrum is only scale-invariant but not magnitude-invariant.

Followed from the above discussion it is obvious that we should check the time evolution of each individual mode in a power spectrum to see if its magnitude varies with time. To this end we extend the usual blue/red-shift term, \( 3H \dot{\chi} \) in the dynamical equation for the gauge invariant perturbations \( \chi \) to a more general form, \( mH \dot{\chi} \). This generalized blue/red-shift is found in many a cosmological model, e.g. the bounce universe models \(^2\), the multi-field models with non-standard kinetic term and non-trivial potential \(^3\) etc.. Adopting a general blue/red-shift term enables us to unveil the \( (\nu,m) \) parameter space of power spectra, where \( \nu \) is the power law index of the scale factor of the background defined as \( a = \eta^\nu \) with \( \eta \) being the conformal time. Each point in the \( (\nu,m) \) space hence corresponds to one cosmological model with the perturbation modes evolving according to, \( mH \dot{\chi} \), and the background according to, \( a = \eta^\nu \).

As explained in detail in Section II, power spectra can be classified by a two dimensional parameter space shown below in Fig. 1 with \( \nu \) being the power law index (horizontal axis) and \( m \) the red/blue-shift index (vertical axis). It is then easy to notice the following four salient features:

1. We find, to our surprise, that the magnitude-invariant solutions occupy a two dimensional region in the \( (\nu,m) \) parameter space, rather than some one-dimensional curves as one would naively expect. Dynamically, \( \nu \) and \( m \) serve as two independent and free parameters for the equation of motion for the perturbations obeying (Eq. 6) and (Eq. 7), respectively. Therefore, the \( \eta \)-dependence and \( k \)-dependence of the spectrum of \( \chi \) are completely determined by a point in the \( (\nu,m) \) space as given by \( P_\chi \sim k^{2L(\nu,m)+3} \eta^{2W(\nu,m)} \). The power spectra satisfy the relations \( 2L(\nu,m)+3 = 0 \) (Eq. 20) and \( 2W(\nu,m) = 0 \) (Eq. 19) are, respectively, scale-invariant and magnitude-invariant.

and a matter-dominated contraction, respectively. Naively for both cases \( \beta = \pm \frac{1}{2} \), the \( P_\delta(k) \) are \( k \)-independent. However, \( H \) should be measured at the moment at which a given perturbation mode exits its effective horizon and hence picking up a dependence in time. For an expanding de-Sitter background, the Hubble parameter, \( H \), is the same for each mode as it exits its effective horizon; while in a matter-dominated contraction, the value of the Hubble parameter varies with time and therefore each perturbation mode takes on a different value of \( H_*(k) \) as the mode exits its effective horizon. In other words, a \( k \)-dependence enters the final expression of power spectrum through \( H_*(k) \) in Wands’ dual solution in a matter-dominated contraction.

\(^2\) See [17] for a review.

\(^3\) A general formalism for multi-scalar-fields models with second-order actions of cosmological perturbations can be found in [15].
FIG. 1: The $(\nu, m)$ parameter space for classifying scale invariant and magnitude invariant solutions of the power spectra of density perturbations. $\nu$ is the power law index (horizontal axis) and $m$ is the red/blue-shift index (vertical axis). The shaded region includes all magnitude-invariant solutions satisfying $(m - 1) \nu - 1 < 0$ whose boundaries are defined by $(m - 1) \nu - 1 = 0$ and drawn with thin dash lines. The purple dot-dash lines obeying $(m - 1) \nu = -2$ give physically scale-invariant solutions. Another set of scale invariance solutions given by $(m - 1) \nu = 4$ (violet solid lines) have Fourier modes varying with time and therefore are not truly scale-invariant in a physical sense.

Generically, one would expect each of these two relations lead to one, or more, one dimensional curve in the $(\nu, m)$ parameter space, and these curves have one or several intersection points which can give rise to some scale-invariant and magnitude-invariant solutions. From this point of view, the magnitude-invariant and scale-invariant solutions should distribute sparsely in the $(\nu, m)$ space. However, thanks to a remarkable duality of the equation of motion for $\chi$, the magnitude-invariant relation, $2W(\nu, m) = 0$, is an inequality of $\nu$ and $m$ rather than an equation of constraint ((Eq. 27) and (Eq. 29)). Mathematically, an inequality often allows a larger set of solutions. In this case, the solution space (Eq. 29) of magnitude-invariant power spectra occupies a two dimensional region. In Fig. 1, the magnitude-invariant region is the shaded area, and its boundaries, $(m - 1) \nu - 1 = 0$, are depicted by a pair of thin dash lines.

Theoretically, this novel property implies that the probability of getting a magnitude-invariant solution by choosing a random value of $(\nu, m)$ is nearly $\frac{1}{2}$—in contrast to a measure zero set as previously expected. Moreover, given the two curves of the scale-invariant solutions lie in the two dimensional magnitude-invariant region, the number of scale-invariant and magnitude-invariant solutions are infinite rather than finite. A substantial portion
of the cosmological models can henceforth produce a magnitude-invariant power spectra. And with careful choice
of \( \nu \) and \( m \) producing a power spectrum with both scale-invariance and magnitude-invariance is no longer a
monopoly of the slow-roll inflation scenario!

2. There are two groups of scale-invariant solutions, one of which is magnitude-invariant but the other changes
with time. According to the scale-invariant relation \( 2L(\nu, m) + 3 = 0 \), we have two groups of scale-invariant
solutions, \((m - 1)\nu = -2 \) (purple dot-dash lines) and \((m - 1)\nu = 4 \) (violet solid lines). The scale-invariant
solutions, \((m - 1)\nu = -2 \), are also magnitude-invariant. Only solutions, which are both scale-invariant and
magnitude invariant, are truly independent of \( k \). They therefore lie inside of the magnitude-invariant region
(shaded purple), the boundary (the dash-dot lines) of which obeys \((m - 1)\nu - 1 = 0 \) in Fig. 1. The other group
of scale-invariant solutions, \((m - 1)\nu = 4 \) (violet solid lines), are magnitude-non-invariant. Cosmologically,
the time-dependence of their power spectrum render them implicitly \( k \)-dependent, so that they are not truly
scale-invariant. In Fig. 1 they lie on the pair of solid blue lines but outside of the magnitude-invariant region.

3. A novel duality between the scale-invariant solutions and the boundaries of the magnitude-invariant solutions
can be detected from a closer inspection of Fig. 1. We notice that the fourth vertex of a rectangle, the three other
vertices of which lie on the scale-invariant solutions curves, is located on the boundary of magnitude-invariant
region. For instance, the fourth vertex of the rectangle with other three vertexes being \((-1, 3)\), \((2, 3)\) and \((2, 0)\)
(marked pale blue), is located on the \( \nu < 0 \) branch of the boundary \((m - 1)\nu - 1 = 0 \). This in turn implies the
existence of a duality, in the same light of that found by Wands [1], between the scale-invariant solutions and
the boundary of the magnitude-invariant region.

Generically, starting from a scale-invariant and magnitude-invariant solution, under two consecutive duality
transformations defined in Section IV we can obtain an accompanying scale-invariant and magnitude-invariant
solution. This scale-invariant and magnitude-invariant solution and its accompanying solution, therefore, deter-
mine a solution of the boundary of the magnitude-invariant region uniquely. Moreover, going through all scale-
invariant and magnitude-invariant solutions, we are able to determine the whole boundary of the magnitude-
invariant region. Though the physical origin of such a duality is yet to be explored, we show the existence of
this duality with explicit calculation in the Section IV.

4. Several distinguished solutions, corresponding to several important cosmological models, are included in the
\((\nu, m)\) parameter space. Studying them case by case, we find that the three points, \((-1, 3)\), \((2, 3)\) and \((2, 0)\),
correspond to, in order, the well-known single field slow-roll inflation model, Wands’ matter-dominated contrac-
tion model and an Ekpyrotic model with negligible gravitational back-reaction. They are, in the same order,
scale-invariant and magnitude-invariant, scale-invariant and magnitude-non-invariant, and magnitude-invariant
and scale-non-invariant, and are consistent with previously known results. We can easily predict the existence
of a matter-dominated contraction solution, \((0, 2)\), with both scale-invariance as well as magnitude-invariance.
This solution is nothing but the power spectrum of the tachyon field in the coupled tachyon-scalar bounce
universe model [14] proposed earlier.

This paper is organized as follows. In Section II we compute the expression of power spectrum of density pertur-
bations in terms of \( k \) and \( \eta \) with general blue/red-shift term \( mH\chi \). In Section III we derive the magnitude-invariant
and scale-invariant solutions, and discuss the general duality of \((\nu, m)\) parameter space. In Section IV we give a proof
to the duality between scale-invariant solution and the boundaries of magnitude-invariant solutions. In Section VI
we apply our general results to a few examples of interest, namely, the de-Sitter background, the Ekpyrotic (slowly
contracting/expanding) background, and finally a matter-dominated contracting background. And, last but not least,
we summarize our findings and discuss their implications to cosmological model building in Section VII.

II. THE DYNAMICAL EQUATIONS FOR DENSITY PERTURBATIONS

For our purpose \( \chi \) is taken to be a gauge invariant variable related to the linear density perturbations of an
underlying cosmological model. The equation of motion for each Fourier mode, \( \chi_k \), of, \( \chi \), then takes the following
form,

\[
\ddot{\chi}_k + mH\dot{\chi}_k + \frac{k^2}{a^2}\chi_k = 0. 
\]  

(6)

The cosmological background is, meanwhile, evolving by a power-law

\[
a \propto \eta^\nu, 
\]

(7)
where $a$ being the scale factor and $\eta$ the conformal time. We will henceforth call, $\nu$, the “power-law index” of the underlying cosmological background. It is then obvious that $\eta \to 0$ corresponds to era in which perturbations could exit their “effective” horizon while $\eta \to \infty$ corresponds to perturbations re-entering the “effective” horizon. In this epoch with $\eta \leq 0$ an expanding phase is described by $\nu < 0$ while the contracting phase is described by $\nu > 0$.

Therefore the solutions of $\chi$ for a given cosmological model are expected to locate on the two dimensional parameter space $(\nu, m)$. We can also interpret the $(\nu, m)$ in terms of the “effective” equation of state of $\chi$ and that of the cosmological background, $(\omega_b, \omega_\chi)$, which are related in a flat FLRW background by

$$\nu = \frac{2}{3(1 + \omega_b) - 2}$$

and

$$m = 3(1 + \omega_\chi).$$

The physical implication of our starting point is that the “effective” Equation of State of the perturbations as well as those of the cosmological background fields would be different, for instance, in a multi-field cosmological model, in models with nonlinear kinetic terms [8][9][10], or in models with quintessence [11][12][13] and so on. Of course the best known example is still the slow-roll inflation model which we will analyze in detail as a concrete application to our proposal. We will use it as a reference point to compare other models whenever possible.

With the following change of variables (we use $u$ instead of $u_k$ as there is no ambiguity in meaning.):

$$\chi_k \equiv a^{-\frac{1}{2}(m-1)}u$$

allowing us to simplify (Eq.6)

$$u'' + (k^2 - \frac{2\gamma}{\eta^2})u = 0$$

where

$$\gamma = \frac{1}{2} \left[ \frac{(m-3)(m-1)}{4} \eta^2 - \frac{1-m}{2} \nu(\nu-1) \right]$$

For non-zero $\gamma$, when $\eta \to 0$, $u$ of different wave vectors will gradually get out of their “effective horizon” defined by

$$\left| \frac{2\gamma}{\eta^2} \right| = k^2.$$  

The general solution for $u$ then takes the familiar form

$$u = C_j(\gamma)\eta^\frac{1}{2}J_n(k\eta) + C_y(\gamma)\eta^\frac{1}{2}Y_n(k\eta)$$

where $J_n$ and $Y_n$ are the $n$-th order Bessel functions, of first and second kind, respectively, with

$$n \equiv \frac{1}{2} \sqrt{1 + 8\gamma},$$

with $C_j$ and $C_y$ being the coefficients that could depend on $\gamma$ but not on $k\eta$.

We observe that (Eq.12) yields the condition:

$$1 + 8\gamma = [(m-1)\nu - 1]^2 \geq 0,$$

and thus in the limit $k\eta \to 0$ the leading order contribution to $u$ becomes

$$u \sim C(\gamma)\eta^\frac{1}{2}(k\eta)^{-\frac{1}{2}\sqrt{1+8\gamma}}.$$  

Substituting (Eq.7) and (Eq.10) into (Eq.17), we have

$$\chi_k \sim C(\gamma)\eta^{-\frac{1}{2}(m-1)\nu}[(m-1)\nu-(1-\sqrt{1+8\gamma})]k^{-\frac{1}{2}\sqrt{1+8\gamma}} \sim C(\gamma)\eta^W(\nu,m)k^L(\nu,m)$$

where $W(\nu,m)$ and $L(\nu,m)$ are two-dimensional functions. However, these expressions become singular at $\nu = 0$ when $m = 1$, or $\nu = 1/2$ when $m = 3$. For large wave vectors $k\eta \to 0$ the solutions take the form

$$u \sim C(\gamma)\eta^\frac{1}{2}(k\eta)^{-\frac{1}{2}\sqrt{1+8\gamma}}.$$
where we have introduced the magnitude-index, $W(\nu, m)$,

$$W(\nu, m) \equiv \frac{1}{2} \left[ (m - 1) \nu - \left( 1 - \sqrt{1 + 8\gamma} \right) \right],$$  

and the scale-index, $L(\nu, m)$,

$$L(\nu, m) \equiv -\frac{1}{2} \sqrt{1 + 8\gamma}.$$  

### III. THE SCALE-INVARINACE AND MAGNITUDE-INVARINACE CONDITIONS FOR POWER SPECTRA OF COSMOLOGICAL PERTURBATIONS

In this section, we discuss the conditions under which a power spectrum of cosmological perturbations $\chi$ could be scale invariant and/or magnitude invariant. A power spectrum $P_\chi$ is conventionally defined as following,

$$P_\chi \equiv \frac{k^3|\chi|^2}{2\pi}.$$  

And in this paper, we sometimes use

$$P_\chi \equiv |\chi|^2$$  

as well to concentrate on the deviation from usual scale invariant spectrum. A power spectrum written in terms of the underlying variables, $k$ and $\eta$, becomes:

$$P_\chi \sim k^{2L(\nu, m)+3} \eta^{2W(\nu, m)}, \quad \text{i.e.} \quad P_\chi \sim k^{2L(\nu, m)} \eta^{2W(\nu, m)},$$  

Consequently the scale-invariance ($k$-independence) of a power spectrum of $\chi$ implies

$$P_\chi \sim k^0 \eta^{2W(\nu, m)}, \quad \text{i.e.} \quad P_\chi \sim k^{-3} \eta^{2W(\nu, m)},$$  

and the magnitude-invariance ($\eta$-independence) implies

$$P_\chi \sim k^{2L(\nu, m)+3} \eta^0, \quad \text{i.e.} \quad P_\chi \sim k^{2L(\nu, m)} \eta^0.$$  

Finally a physically scale-invariant (both $k$ and $\eta$ independent) power spectrum is the one satisfies

$$P_\chi \sim k^0 \eta^0, \quad \text{i.e.} \quad P_\chi \sim k^{-3} \eta^0.$$  

### A. Magnitude Invariance condition

The magnitude-index defined by (Eq.19) can be expressed as

$$W(\nu, m) = -\frac{1}{2} \left\{ (m - 1) \nu - \left( 1 - \sqrt{1 + 8\gamma} \right) \right\}$$  

by substituting (Eq.12) into (Eq.19). Magnitude-invariance then implies:

$$W(\nu, m) = 0.$$  

Therefore the spectra of cosmological perturbations, $\chi$, whose values of $(\nu, m)$ obey

$$(m - 1) \nu - 1 \leq 0$$  

are magnitude-invariant solutions. These magnitude-invariant power spectra span a two dimensional magnitude-invariance region, shaded violet in Fig, and the boundary of magnitude-invariance region is given by $m = \frac{1}{\nu} + 1$, as delineated by a pair of grey dashed lines. In term of the equations of state, $\omega_b$ and $\omega_\chi$, the boundaries are expressed as

$$\omega_\chi = \frac{1}{2} (\omega_b - 1).$$  

B. Scale Invariance condition

Likewise, from (Eq. 25), we obtain the scale-invariance condition,

\[ 2L(\nu, m) + 3 = 0 \implies \gamma = 1, \quad (31) \]

and the solutions are

\[ m = \begin{cases} 
\frac{4}{\nu} + 1 \\
-\frac{2}{\nu} + 1 
\end{cases} \quad (32) \]

both combinations of \( \nu \) and \( m \) lead to scale-invariant spectra of \( \chi \).

The scale-invariant solutions from \( m = -\frac{2}{\nu} + 1 \) branches–indicated by purple dot-dash lines in Fig. 1–also satisfy the magnitude invariance condition

\[ (m - 1) \nu - 1 \leq 0. \quad (33) \]

Therefore we conclude that \( m = -\frac{2}{\nu} + 1 \) branch gives physically scale-invariant solutions. But the scale-invariant solutions on the \( m = \frac{4}{\nu} + 1 \) branches–indicated by solid violet lines in Fig. 1–are not truly scale-invariant solutions as the magnitudes of their Fourier modes change with conformal time, \( \eta \). This is corroborated by the fact that the pair of lines lie outside the magnitude invariant region.

Scale-invariant solutions (Eq. 32) can also be expressed in terms of equations of state of the background, \( \omega_b \), and of the field, \( \omega_\chi \) as follows:

\[ \omega_\chi = \omega_b \quad (34) \]

and

\[ \omega_\chi = -(1 + \omega_b). \quad (35) \]

IV. DUALITY BETWEEN THE SCALE-INVARIANT SOLUTIONS AND THE BOUNDARY OF THE MAGNITUDE-INVARIANCE REGION

In this section, we will unveil a novel duality between scale-invariant solutions and the boundary of the magnitude-invariance region.

For each \( m \), there are two scale-invariant solutions, \((\nu, m)\) and \((\hat{\nu}, \hat{m})\), satisfying the two scale-invariance conditions (Eq. 32).

\[ m = \frac{4}{\nu} + 1, \quad \text{or}, \quad m = -\frac{2}{\hat{\nu}} + 1. \quad (36) \]

Two solutions having the same value of \( m \) implies the dynamics equation of the perturbations \( \chi \) for these two solutions are the same. Therefore we call the duality between these two scale-invariant solutions as “Iso-Perturbation Scale-Invariance (IPSI)” duality, and a transformation between them “Iso-Perturbation Scale-Invariance” transformation.

Similarly, for each \( \nu \), there are two scale-invariant solutions, \((\nu, m)\) and \((\nu, \hat{m})\), satisfying the either one of scale-invariance conditions (Eq. 32).

\[ m = \frac{4}{\nu} + 1, \quad \text{or}, \quad \hat{m} = -\frac{2}{\nu} + 1. \quad (37) \]

The same value of \( \nu \) implies that the perturbations \( \chi \) for these two solutions evolve in the same background, we call the duality between these two scale-invariant solutions as “Iso-Background Scale-Invariance (IBSI)” and the transformation between them as “Iso-Background Scale-Invariance” transformation.

With the IPSI and IBSI dualities at hand, we are able to show the existence of a subtle duality between the scale-invariant solutions and the boundary of the magnitude-invariance region. We summarize our findings in the following theorem.
Theorem 1 The boundary of the magnitude-invariance region can be determined uniquely by from scalar-invariant solutions by two Scale-Invariance transformations.

Proof: Without loss of generality, we start from the point \((\nu_1, m_1)\) on the \(\nu < 0\) quadrant of the scalar-invariant and magnitude-invariant branch of \(m = -\frac{5}{2} + 1\), say \((-1, 3)\) for concreteness. We shall label the IPSI duality transformation as \(D_{IPS1}\) and IBSI duality transformation as \(D_{IBS1}\). Since \(D_{IPS1}\) and \(D_{IBS1}\) may not be commutative, we act both \(D_{IBS1}D_{IPS1}\) and \(D_{IPS1}D_{IBS1}\) on the \((\nu, m_1)\) respectively at following. In the \(D_{IPS1}D_{IBS1}\) case, acting \(D_{IBS1}\) on \((\nu_1, m_1)\) gives,

\[ D_{IBS1}(\nu_1, m_1) = (\nu_1, m_2), \]

taking \((-1, 3)\) to \((-1, -3)\), landing on another scale-invariant solution on the

\[ m_2 = \frac{4}{\nu_1} + 1 \]

branch, which violates the magnitude-invariance condition, \((m - 1)\nu - 1 \leq 0\). Continue acting with \(D_{IPS1}\) on \((\nu_1, m_2)\),

\[ D_{IPS1}(\nu_1, m_2) = (\nu_2, m_2) \]

maps it to another the scale-invariant solution \((0.5, -3)\) on the \(\nu > 0\) branch of the

\[ m_2 = -\frac{2}{\nu_2} + 1 \]

curve which lies properly inside the shaded region of magnitude invariant solutions. Therefore \((\nu_2, m_2)\) satisfies both the scale-invariance as well as the magnitude-invariance conditions.

Both \((\nu_1, m_1)\) and \((\nu_2, m_2)\) (These points are \((-1, 3)\) and \((0.5, -3)\) in the example we give.) are scale-invariant and magnitude-invariant in this case. And more intriguingly the point \((\nu_2, m_1)\) satisfies the boundary condition of the magnitude-invariance region:

\[ m_1 = \frac{1}{\nu_2} + 1 \]

hence lies on the grey line of the shaded region. Therefore we can map out the boundary of magnitude-invariant solutions in the \((\nu > 0, m > 0)\) quadrant by acting with consecutive Iso-Background and Iso-Perturbation duality transformations, \(D_{IPS1}D_{IBS1}\), on the solutions lying on the purple dash dot line—both scalar-invariant and magnitude-invariant—in the \((\nu < 0, m > 0)\) quadrant. The above observation also hints at a duality between a scale-invariant solution \((-1, -3)\) to a magnitude invariant solution \((0.5, 3)\) which warrants further study for model building purposes.

We now turn our attention to the \(D_{IBS1}D_{IPS1}\) case, i.e. one Iso-Perturbation transformation followed by one Iso-Background transformation. Acting \(D_{IPS1}\) on \((\nu_1, m_1)\)

\[ D_{IPS1}(\nu_1, m_1) = (\tilde{\nu}_2, m_1) , \]

we obtain the \((\tilde{\nu}_2, m_1)\) (mapping \((-1, 3)\) to \((2, 3)\)) which is also a scale-invariant solution lying on the \(\nu > 0\) branch of

\[ m_1 = \frac{4}{\tilde{\nu}_2} + 1 , \]

but violates magnitude-invariance condition, \((m - 1)\nu - 1 \leq 0\), and hence lying outside of the purple shaded region.

Then we act \(D_{IBS1}\) on \((\tilde{\nu}_2, m_1)\),

\[ D_{IBS1}(\tilde{\nu}_2, m_1) = (\tilde{\nu}_2, \tilde{m}_2) \]

to \((\tilde{\nu}_2, \tilde{m}_2)\) (mapping \((2, 3)\) to \((2, 0)\)), which satisfies both the scale-invariance and magnitude-invariance conditions,

\[ \tilde{m}_2 = -\frac{2}{\tilde{\nu}_2} + 1 \quad \text{and} \quad (m - 1)\nu - 1 \leq 0 , \]

respectively. Both \((\nu_1, m_1)\) and \((\tilde{\nu}_2, \tilde{m}_2)\) (in our example \((-1, 3)\) and \((2, 0)\)) are scale-invariant and magnitude-invariant. The point \((\nu_1, \tilde{m}_2)\) lies on the boundary of magnitude-invariant solutions,

\[ \tilde{m}_2 = \frac{1}{\nu_1} + 1 . \]

Going along the dot-dash line in the \((\nu < 0, m > 0)\) quadrant, acting with \(D_{IPS1}D_{IBS1}\), we can map out the boundary of magnitude-invariant solutions in the \((\nu < 0, m < 0)\) quadrant of the \((\nu, m)\) parameter space.

Clearly seen from the above analysis the IBSI duality and IPSI duality transformations do not commute. \(D_{IPS1}D_{IBS1}\) and \(D_{IBS1}D_{IPS1}\) acting on the scalar-invariant solutions will map out for us, respectively, the upper and lower boundaries of magnitude-invariant solutions Fig. [I].
V. GENERAL DUALITY

In this section, we are studying the general duality, which leave \( k \)-dependence and/or \( \eta \)-dependence of power spectrum unchanged, in the \( (\nu, m) \) parameter space. The expressions of the scale-index, \( L(\nu, m) \) (Eq. 25), and the magnitude-index, \( W(\nu, m) \) (Eq. 27), in terms of \( \nu \) and \( m \) are

\[
L(\nu, m) = -\frac{1}{2} |(m - 1) \nu - 1|,
\]

and

\[
W(\nu, m) = -\frac{1}{2} \{[(m - 1) \nu - 1] + |(m - 1) \nu - 1|\}.
\]

Then a power spectrum with \( (\tilde{\nu}, \tilde{m}) \), which satisfies

\[
|(m - 1) \nu - 1| = |(\tilde{m} - 1) \tilde{\nu} - 1|,
\]

will have same \( k \)-dependence as a power spectrum with \( (\nu, m) \). According to (Eq. 27), a power spectrum with \( (\tilde{\nu}, \tilde{m}) \), satisfying

\[
[(m - 1) \nu - 1] + |(m - 1) \nu - 1| = [(\tilde{m} - 1) \tilde{\nu} - 1] + |(\tilde{m} - 1) \tilde{\nu} - 1|,
\]

has same \( \eta \)-dependence with a power spectrum with \( (\nu, m) \).

Furthermore if \( (m - 1) \nu - 1 \) and \( (\tilde{m} - 1) \tilde{\nu} - 1 \) are either both positive or both negative simultaneously, then they can also satisfy the last requirement (Eq. 29). When this is the case, a power spectrum with \( (\tilde{m} - 1) \tilde{\nu} - 1 \) will have the same \( k \)-dependence as well as \( \eta \)-dependence with a power spectrum with \( (\nu, m) \). Note, however, that when they are both positive they can neither be scale-invariant nor magnitude-invariance. We are now ready to proceed to a detailed case study.

VI. CASE STUDY

In this section, we will relate our results to the several important cosmological models–putting into context to help illustrate the power of dualities. This also helps in the search of new cosmological models which can produce scale-invariant solutions.

A. de-Sitter Inflation Background

In de-Sitter cosmological background, we have \( \nu = -1 \). According to the scale invariance condition, (Eq. 32) there are two power spectra satisfying the condition yielding two scale-invariant solutions \( (\nu, m) = (-1, 3) \) and \( (\nu, m) = (-1, -3) \). Furthermore, the \((-1, 3)\) case is also magnitude-invariant following from the magnitude-invariance condition (Eq. 29). \( P_\chi(-1, -3) \propto \eta^0 \). On the other hand the \((-1, -3)\) case is not magnitude-non-invariant as \( P_\chi(-1, -3) \propto \eta^{-3} \).

An alert reader would have recognized \((-1, 3)\) as the celebrated single field slow-roll inflation model. The evolution of its linear perturbations of scalar (the inflaton), \( \delta \phi \) (in spatially-flat slicing), and the tensor mode \( h_{+/\times} \) are governed by (Eq. 6) and (Eq. 7) with \( (\nu, m) \simeq (-1, 3) \). Therefore, according to above analysis, the power spectrum of \( h_{+/\times} \) and \( \delta \phi \) should be nearly scale-invariant and magnitude-invariant. The tensor mode of perturbation \( h_{+/\times} \) is gauge-invariant, but \( \delta \phi \) is not gauge-invariant. Fortunately, \( \delta \phi \) relates to the gauge-invariant quantity \( \zeta \), which is related to the scalar perturbations by a factor \( \frac{aH}{\phi} \).

\[
\zeta = -\frac{aH}{\phi} \delta \phi .
\]  

(51)

So the power spectrum of \( \zeta \) is also nearly scale-invariant: the conversion factor gives only a small contribution to the running of spectral index.

In the single scalar field slow-roll model, we have \( (\tilde{\nu}, \tilde{m}) = (-1, 3) \) and \( \delta m = 0 \). Then the primordial spectrum of tensor mode, \( h_{+/\times} \), becomes

\[
P_h \sim k^{-|((\tilde{m} - 1) \tilde{\nu} + \delta \phi + (\tilde{m} - 1) \delta \phi)|} \sim k^{-3+2\delta \phi}.
\]  

(52)
We obtain the primordial spectral index $n_T$ of the tensor mode

$$n_T \equiv \frac{d \ln (k^3 P_h)}{d \ln k} = 2\delta\nu$$

(53)

The slow-roll parameter, $\epsilon$, is defined as,

$$\epsilon \equiv \frac{d}{d\eta} \left( \frac{1}{H} \right) = \frac{\nu + 1}{\nu} \simeq -\delta\nu$$

(54)

Substituting (Eq.54) into (Eq.53), we re-derive the well-known relation between the $n_T$ and $\epsilon$ for the slow-roll inflation model

$$n_T = -2\epsilon .$$

(55)

In above analysis we do not need to know at which exact time the $k$ mode exit the horizon since such information have been encoded into (Eq.18) through (Eq.12).

For the single field slow-roll inflation model, the factor $-\frac{aH}{\dot{\phi}}$ is related to the slow parameter $\epsilon$,

$$\left(-\frac{aH}{\dot{\phi}}\right)^2 \propto \epsilon^{-1} = (-\delta\nu)^{-1}.$$  

(56)

Therefore, we obtain the primordial spectra index of scalar mode $n_s$,

$$n_s - 1 \equiv \frac{d \ln (k^3 P_\zeta)}{d \ln k} = \frac{d \ln (k^3 P_{\delta\phi})}{d \ln k} - \frac{d \ln (-\delta\nu)}{d \ln k} = 2\delta\nu - \frac{\delta\nu'}{\delta\nu}\eta$$

(57)

With the (Eq.53) and (Eq.57), the non-zero values of primordial spectral indices, $n_T$ and $n_s - 1$, reflect the small departure of the cosmological background of the single field slow-roll inflation from a perfectly de-Sitter cosmos.

We remark here that $\delta m = 0$ for the single scalar field slow roll inflation since its kinetic term is simply $X = \frac{1}{2} \dot{\phi}^2$. For a more general kinetic term, $P(X)$, the $\delta m$ could not be neglected. In that case, the $\delta m$ will appear in the expression of $n_T$ and $n_s$. In other words precise measurements $n_T$ and $n_s$ can be used to resolve the present degeneracy of cosmological models.

**B. Slowly Expanding/Contracting Backgrounds**

An Ekpyrotic/cyclic scenario requires a slowly expanding or contracting background [3] which can be characterized by $|\nu| \ll 1$. We will now study the power spectrum of cosmological perturbations in such a background.

The potential of the single scalar field, in this scenario, takes the following form:

$$V(\phi) = -V_0 \exp(-\sqrt{\frac{2}{p} \phi M}) ,$$

(58)

with $M \equiv \sqrt{\frac{1}{8\pi G}}$ as usual; because $p \ll 1$, the potential very steep. The equation of motion governing the cosmological perturbations [5] is,

$$\ddot{\xi} + 3H \dot{\xi} + \frac{k^2}{a^2} \xi = 0$$

(59)

where $\xi \equiv -\sqrt{\frac{2}{3M}} \delta\phi$; meanwhile the cosmological background obeys

$$a \propto t^p \rightarrow a \propto \eta^{\frac{1}{1-p}} .$$

(60)

One then gets $(\nu, m) = (\frac{p}{1-p}, 3)$, giving a power spectrum of $\xi$,

$$P_\xi \propto \eta^0 k^{-\frac{1}{1-p}} .$$

(61)
The spectrum \( P_\xi \) is time-independent (i.e. magnitude-invariant) as expected. For \( 0 < p \ll 1 \),

\[
P_\xi \sim k^{-1+\frac{2p}{1-p}} \sim k^{-1}.
\]

Therefore, \( P_\xi \) is not scale-invariant, but it is red-tilted in comparison with a scale-invariant spectrum \( P \sim k^{-3} \). These results are in accordance with the previously analysis [5]. With the scale invariance conditions (Eq. 32) the scale-invariant power spectrum cannot be generated in a slowly expanding/contracting background unless \( |m| \) is very large \( (m \sim -(2p)^{-1}) \), which is made trivially obvious in our language.

C. Matter-dominated contracting background

In a matter-dominated contraction, we have \( \nu = 2 \). According to (Eq. 32) there are two scale-invariant solutions, \((\nu, m) = (2, 3)\) and \((\nu, m) = (2, 0)\). The first solution, \((\nu, m) = (2, 3)\), corresponds to the scale-invariant spectrum observed by Wands [1]. The spectrum of perturbations takes the following form,

\[
P \propto k^{-3}\eta^{-6}.
\]

However the magnitude of such a spectrum is amplified during the contraction process. The time dependence would make the power spectrum implicitly \( k \)-dependent at the moment of horizon-crossing and ever since. And the strongly blue-shifted power spectrum implies instabilities in the subsequent cosmological evolution. This type of spectra often arise in the matter-dominated phase of contraction in a number of cosmological models, e.g. [19], loosely called "matter-bounce" models.

The second solution, \((\nu, m) = (2, 0)\), is not only scale-invariant but also magnitude-invariant. The power spectrum in this case takes the following form:

\[
P \propto k^{-3}\eta^0.
\]

which is the power spectrum of perturbations due to the tachyon field during the contracting phase of coupled-scalar-tachyon bounce cosmos (CSTBC) [13].

For the CSTB model, in the Newtonian gauge, \( g_{\mu\nu} = \text{diag}\{-1 - 2\psi, a^2(1 + 2\psi)\} \delta_{ij}\), the equations of motion governing the perturbations for scalar field \( \phi \) and tachyon field \( T \) take the following forms:

\[
-\delta\ddot{\phi} + 2\psi\dot{\phi} - k^2a^{-2}\delta\phi + \left(-3H\delta\phi - 4\dot{\phi}\dot{\psi} + 6H\psi\phi\right) - \left(m^2_\phi + 2\lambda T^2\right)\delta\phi = 0,
\]

and

\[
\dot{\delta}T - 2\psi\ddot{T} + k^2\delta T a^{-2} + (2\psi\dot{T}^2 - 2\dot{\phi}\ddot{T}) \left(-\frac{1}{\sqrt{2}} + 3H\dot{T} + 2\lambda\phi^2 T\frac{\sqrt{1-T^2}}{V(T)}\right) + (1 - \dot{T}^2)\left[4\dot{\psi}\dot{T} + 3H\dot{T} - 6H\psi\dot{T}\right] + (1 - \dot{T}^2)\left[2\lambda\sqrt{1-T^2}\frac{\sqrt{1-T^2}}{V(T)} \phi \left(2T\delta\psi + \frac{\sqrt{2} + 1}{\sqrt{2}}\phi\delta T + \psi\ddot{T}\phi - \dot{T}\phi\delta T\right)\right] = 0;
\]

where \( \delta\phi \) and \( \delta T \) are, respectively, the perturbations of the scalar field, \( \phi \), and the tachyon field, \( T \).

During the phase of tachyon matter domination in the CSTB model, the background fields \( \phi \) and \( T \) oscillate swiftly, see [13] for details 4. Such a dynamical attractor behavior ensures that

\[
\langle \phi \rangle = \langle \dot{\phi} \rangle = \langle \ddot{\phi} \rangle = \langle T \rangle = \langle 1 - \dot{T}^2 \rangle = \left\langle -\frac{1}{\sqrt{2}} + 3H\dot{T} + 2\lambda\phi^2 T\frac{\sqrt{1-T^2}}{V(T)} \right\rangle = 0
\]

where \( \langle A \rangle \) stands for the average of \( A \) in a relative long period. Using the above conditions to simplify the general perturbation equations of the tachyon and the scalar field we obtain

\[
\ddot{\delta}\phi + k^2a^{-2}\delta\phi + 3H\dot{\delta}\phi + \left(m^2_\phi + 2\lambda T^2\right)\delta\phi = 0
\]

\[4 \text{ An analytic study of the dynamics of two coupled scalar fields can be found in [15].} \]
The effective mass square of $\delta \phi$, $M_{\text{eff}} = m^2 \phi + 2 \lambda T^2$, is very large during the contracting phase with tachyon matter domination. Therefore, the power spectrum of scalar field perturbation is highly suppressed, and its contribution can be neglected safely. With (Eq. 69), we find that the effective value of $m$ for the tachyon field perturbation vanishes due to an accidental cancelation (Eq. 67). And in the CBST model, the contracting background is also matter-dominated and hence $\nu = 2$. Therefore the power spectrum of tachyon field perturbations in the matter-dominated contracting phase is none other but the solution $(\nu, m) = (2, 0)$, leading to

$$P_{\delta T} \propto k^{-3} \eta^0,$$

which is both scale-invariant and magnitude-invariant.

VII. DISCUSSION AND CONCLUSION

In this paper we study the power spectra of cosmological perturbations using their equations of motion with a general blue/red-shift term, $mH \dot{\chi}$. The implicit time-dependence of a power spectrum is uncovered and its consequent influence on the scale dependence of the power spectrum is discussed case by case. Only if a power spectrum is both scale-invariant as well as magnitude-invariant (time-independent), can it be truly scale-invariant at all time. Otherwise, the power spectrum would have implicit $k$-dependence, rendering it non-scale-invariant.

Combined the power-law index of cosmological background, $\nu$, with the blue/red-shift parameter, $m$, we can construct a parameter space, $(\nu, m)$, to classify perturbation spectra. In the $(\nu, m)$ space, the magnitude-invariant solutions are found to occupy a two dimensional region rather than one dimensional curves as one may have naively expected–implying that a large portion of cosmological models could have magnitude-invariant power spectra. We, moreover, obtain two groups of scale-invariant solutions: one of them is magnitude-invariant and the other is magnitude-non-invariant. The number of truly scale-invariant solutions is infinite rather than precisely few–a conviction commonly held.

In the process, we also unveil a duality between the region of the scale-invariant solutions and the boundary of magnitude-invariant region: under two consecutive duality transformations, the scale-invariant solutions are mapped onto the boundary of magnitude invariant region. The physical origin of such a duality is quite worthy of further studies. In the last part of the paper we present a few cosmological applications of our general analysis, to a de-Sitter universe, an Ekpyrotic cosmos, and a matter-dominated contraction. Previously known results confirmed our general analysis. A new power spectrum that is both scale-invariant as well as magnitude-invariant is uncovered by our method in a matter-dominated contracting background. This new solution to a matter-dominated contracting background was found previously in the coupled scalar-tachyon bouncing universe model [14].

One point worth re-iterated here: our result is valid for all perturbation spectra produced by scalar quantum fluctuations with the equation of motion, $\ddot{\chi}_k + mH \dot{\chi}_k + \frac{k^2}{a^2} \chi_k = 0$. For density perturbations produced by other mechanisms such as thermal fluctuations [20, 21], holographic principle [22–24] and possibly others, e.g. [25], our analysis is, however, not applicable. The reason is that this above equation of motion describing cosmological perturbations is not the most general possible formulation of density perturbations produced by a given cosmological model. It hence calls for a systematic way to simplify the dynamical equations of cosmological perturbations in various gauges to the above form. An attempt in this direction is underway [26].

We take the liberty to conjecture that more cosmological models, whose the equations of motion for the cosmological perturbations have a general blue/red-shift term $mH \dot{\chi}$, can be constructed in multitude in the near future. We also hope that this phenomenological work would perhaps give some hints from a slightly more fundamental prospective in the search of alternative cosmological models with scale-invariant power spectra that can withstand the the test of time, that is scale invariance without implicit or explicit time dependence.

VIII. ACKNOWLEDGMENTS

Useful discussions with Yifu Cai, Konstantin Savvidy, Henry Tye and Lingfei Wang are gratefully acknowledged. This research project has been supported in parts by 985 Projects Grants from Chinese Ministry of Education, by the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD) as well as by the Swedish
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