THREE-DIMENSIONAL REP-TILES

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Abstract. A 3D rep-tile is a compact 3-manifold $X$ in $\mathbb{R}^3$ that can be decomposed into finitely many pieces, each of which are similar to $X$, and all of which are congruent to each other. In this paper we classify all 3D rep-tiles up to homeomorphism. In particular, we show that a 3-manifold is homeomorphic to a 3D rep-tile if and only if it is the exterior of a connected graph in $S^3$.

1. Introduction

A set $X \subset \mathbb{R}^n$ with non-empty interior is an $n$-dimensional $k$-index rep-tile if there are sets $X_1, X_2, \ldots, X_k$ with disjoint interiors and with $X = \bigcup_{i=1}^{k} X_i$ that are mutually congruent and similar to $X$. Rep-tiles have been well-studied since 1963, when they were introduced by Gardner and Golomb [11], [12]. Much of the study of rep-tiles has focused on 2-dimensional rep-tiles from the perspective of fractal geometry. There are robust methods for constructing rep-tiles [4]. However, these methods usually result in rep-tiles that are not piecewise linear manifolds. Various authors have explored the basic topological properties of rep-tiles. For example, a rep-tile $X$ has a hole if the complement of the closure of some component of the interior of $X$ has a bounded component. John Conway asked if there exists a 2-dimensional rep-tile with a hole, and Grünbaum answered this question in the affirmative by providing a 36-index rep-tile example [9] (also see [14]). The topology of 2-dimensional rep-tiles is limited. In particular, Bandt and Wang [3] and Luo et al. [17] proved that if the interior of a 2-dimensional rep-tile is connected, then the rep-tile is a topological disk.

In this paper we apply 3-manifold techniques to study the topology of 3-dimensional rep-tiles. To this end, we restrict to 3D rep-tiles, 3-dimensional $k$-index rep-tiles that are piecewise linear embeddings of compact connected 3-manifolds in $\mathbb{R}^3$ with $k > 1$. Examples of 3D rep-tiles that are homeomorphic to the 3-ball are prevalent. For example, a cube is an 8-index 3D rep-tile homeomorphic to a 3-ball, see Figure 2. Additionally, various authors have investigated which tetrahedra are 3D rep-tiles [16], [18], [15], [13]. However, examples homeomorphic to other 3-manifolds have been more challenging to generate. In 1998,
Goodman-Strauss asked if there exists a 3D rep-tile with a hole. This was answered by Van Ophysen, who generated an example homeomorphic to a solid torus. See Figure 1 for an example of a 3D rep-tile homeomorphic to a solid torus. Furthermore, a description of an infinite family of 3D rep-tiles homeomorphic to any genus \( n \) handlebody is given in [19]. In [5], the authors use a computer search to generate an 8-index 3D rep-tile homeomorphic to a solid torus. This was the unique 8-index 3D rep-tile not homeomorphic to the 3-ball among the one million examples they generated.

In this paper we classify all 3D rep-tiles up to homeomorphism. In particular, we give a method of constructing a 3D rep-tile of any suitable homeomorphism type.

**Theorem 1.1.** A set \( X \) in \( \mathbb{R}^3 \) is a 3D rep-tile if and only if \( X \) is homeomorphic to the exterior of a finite connected graph in \( S^3 \).

The concept of self-affine tile is related to that of rep-tile. A self-affine tile is defined by a finite collection of contractions (not necessarily similarities) that are affine translates of a single linear contraction. In [8], the authors generate the first examples of 3-dimensional self-affine tiles which are topological 3-manifolds with boundary. In particular, they show that every handlebody is homeomorphic to a 3-dimensional self-affine tile. However, the embeddings they generate are not piecewise linear and not 3D rep-tiles. They conjecture that all 3-dimensional self-affine tiles which are topological 3-manifolds are homeomorphic to handlebodies. We wonder if the examples of rep-tiles which are not homeomorphic to handlebodies generated in the current paper can be modified to provide counterexamples to the conjecture posed in [8].

Rep-tiles are a model for topological self-replication in that a rep-tile is an object that can be decomposed into finitely many copies each of which is homeomorphic to the original. Previous models of self-replication in low-dimensional topology have focused on idempotents (i.e. morphisms with the property that \( f \circ f = f \)) in the appropriate topological category. Such idempotents are manifolds that can be decomposed along embedded surfaces into two copies of themselves. In particular, idempotents have been classified in the Temperley-Lieb category [1], the tangle category [7], and the (2+1)-cobordism category [6]. Models based on idempotents inherently model 1-to-2 self-replication. In contrast, 3D rep-tiles model 1-to-many self-replication. See Figure 2.

In Section 2 we introduce useful 3-manifold and lattice terminology. In Section 3 we prove the backward direction of Theorem 1.1. In Section 4 we prove the forward direction of Theorem 1.1.
Figure 1. An example of a polycube and a 3D rep-tile homeomorphic to \( D^2 \times S^1 \). It is constructed by removing the two cubes corresponding to \([-1,0] \times [0,1] \times [0,2]\) from \([-2,2] \times [-2,2] \times [0,2]\) and then adding two cubes corresponding to \([-1,0] \times [-1,0] \times [2,4]\).

2. Preliminaries

We begin with a discussion of useful classes of manifolds and submanifolds. A handlebody is a compact orientable 3-manifold with boundary homeomorphic to the closed regular neighborhood of a connected finite graph embedded in a 3-manifold. The genus of the boundary of a handlebody uniquely determines the homeomorphism class of the handlebody. We say a handlebody is genus \( g \) if its boundary is homeomorphic to a genus \( g \) surface.

Definition 2.1. Given manifold \( M \) with submanifold \( N \), the exterior of \( N \) in \( M \) is \( E_M(N) := M \setminus \eta(N) \) where \( \eta(N) \) is an open tubular neighborhood of \( N \) in \( M \). When \( M \) is understood to be \( S^3 \), we will write the exterior as \( E(N) \).
Here we introduce terminology related to cubic lattices in $\mathbb{R}^3$ that will be relevant to our construction. We will denote the standard embedding of the unit cubic lattice in $\mathbb{R}^3$ as

$$Z^3 = \bigcup_{(i,j) \in \mathbb{Z}^2} (\{i\} \times \{j\} \times \mathbb{R}) \cup (\{i\} \times \mathbb{R} \times \{j\}) \cup (\mathbb{R} \times \{i\} \times \{j\}).$$

Suppose $\lambda > 0$ and let $f_\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ denote the scaling function given by $f(x) = \lambda x$. Let $Z^3_\lambda = f(Z^3)$.

Sometimes it will be useful to refer to the 3-complex structure induced by each of the cubic lattices mentioned above on $\mathbb{R}^3$ (i.e. the decomposition $\mathbb{R}^3$ into the union of vertices, edges, squares and cubes). Given the lattice $Z^3_\lambda$, the corresponding 3-complex structure on $\mathbb{R}^3$ will be denoted $C(Z^3_\lambda)$. Ultimately, our construction of 3D rep-tiles will be based on polycubes. A polycube is a subset of $\mathbb{R}^3$ that is similar to a finite union of 3-cells in $C(Z^3)$. See Figure 1 for an example of a polycube consisting of 32 cubes and homeomorphic to $D^2 \times S^1$.

3. Graph exteriors as 3D rep-tiles

In [2] Adams shows that for any 3-dimensional compact submanifold $M$ of $\mathbb{R}^3$ with a single boundary component, a 3-ball can be decomposed into four congruent tiles, each homeomorphic to $M$. The following theorem generalizes Adams’ results and then applies the generalization
to 3D rep-tiles. In particular, the following proof implies that for any 3-dimensional compact submanifold $M$ of $\mathbb{R}^3$ with a single boundary component, a cube can be decomposed into two congruent tiles such that each is a polycube and each is homeomorphic to $M$. For example, two copies of the polycube in Figure 1 tile the $4 \times 4 \times 4$ cube. We can see this by rotating the polycube in Figure 1 by $\pi$ about the line that passes through the points $A$ and $B$. This observation implies that this polycube is a rep-tile since each of its 32 cubes can be decomposed into a total of 64 polycubes which are all congruent to each other and similar to the original.

**Theorem 3.1.** If $M$ is the exterior of a connected graph in $S^3$, then $M$ is homeomorphic to a 3D rep-tile.

**Proof.** Let $M = E(\Gamma)$ for some connected graph $\Gamma$ embedded in $S^3$. We begin by showing that $E(\Gamma)$ is homeomorphic to a cube-with-holes (i.e. the exterior of a collection of properly embedded arcs in a cube). Preform edge contractions on $\Gamma$ to produce an embedded graph $\Gamma_1$ with a single vertex and $n$ edges. Note that $E(\Gamma)$ is homeomorphic to $E(\Gamma_1)$. See Figure 3 for an example of an embedded graph in $S^3$ with a single vertex. Let $V$ be a 3-ball corresponding to a small closed regular neighborhood of the vertex of $\Gamma_1$. Let $B$ be the 3-ball $S^3 \setminus int(V)$. The 3-ball $B$ meets in $\Gamma_1$ in a collection of $n$ properly embedded arcs $\alpha_1, \alpha_2, ..., \alpha_n$. Note that $E_B(\bigcup_{i=1}^{n} \alpha_i)$ is homeomorphic to $E(\Gamma_1)$. Label the boundary of each arc by $\partial \alpha_i = \{a_i, b_i\}$. Identify the three ball $B$ with the unit cube $C = [0,1] \times [0,1] \times [0,1]$ in $\mathbb{R}^3$ via a homeomorphism that takes the set $\{a_1, a_2, ..., a_n\}$ to the interior of the square $[0,1] \times [0,1] \times \{0\}$ and takes the set $\{b_1, b_2, ..., b_n\}$ to the interior of the square $[0,1] \times [0,1] \times \{1\}$. See Figure 4. Through an abuse of notation, we continue to refer to the image of the arcs properly embedded in $C$ as $\alpha_1, \alpha_2, ..., \alpha_n$. Thus, $E_C(\bigcup_{i=1}^{n} \alpha_i) \cong E_B(\bigcup_{i=1}^{n} \alpha_i)$.

![Figure 3](image-url)

**Figure 3.** A graph with a single vertex embedded in $S^3$.

Next, we show the cube-with-holes we just generated is homeomorphic to a polycube. Note that for every positive integer $m$, $C(\mathbb{Z}_m^3)$
induces a cell structure on $C$. Moreover, by choosing $m$ sufficiently large and possibly preforming an small proper isotopy of $\cup_{i=1}^{n} \alpha_i$, we can assume $\cup_{i=1}^{n} \alpha_i$ is disjoint from the $Z_3$ and, if $A$ is the union of all 3-cells in $C$ with non-trivial intersection with $\cup_{i=1}^{n} \alpha_i$, then $A$ is isotopic to a closed tubular neighborhood of $\cup_{i=1}^{n} \alpha_i$ in $C$. Let $i(A)$ denote the interior of all cells in $C$ with non trivial intersection with $\cup_{i=1}^{n} \alpha_i$. Then $i(A)$ is isotopic to an open tubular neighborhood of $\cup_{i=1}^{n} \alpha_i$ in $C$ and $C \setminus i(A)$ is a polycube homeomorphic to $E_C(\cup_{i=1}^{n} \alpha_i) \cong E(\Gamma)$. See Figure 5.

We now modify $C \setminus i(A)$ to generate a polycube homeomorphic to $C \setminus i(A)$ which tiles the cube using two congruent copies. Let $C_1 = [0, 1] \times [-1, 0] \times [0, 1]$, $C_2 = [-1, 0] \times [-1, 0] \times [0, 1]$ and $C_3 = [-1, 0] \times [0, 1] \times [0, 1]$. Note that $(C \setminus i(A)) \cup C_1 \cup C_2 \cup C_3$ is again a polycube homeomorphic to $E(\Gamma)$. Let $r : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation of $\pi$ about the line parameterized by $< t, 0, 1 >$. Let $X = (C \setminus i(A)) \cup (C_1 \cup C_2 \cup C_3) \cup r(A)$. The polycube $X$ is homeomorphic to $(C \setminus i(A)) \cup C_1 \cup C_2 \cup C_3$ since it is the boundary connected sum of $(C \setminus i(A)) \cup C_1 \cup C_2 \cup C_3$ with a collection of $n$ 3-balls. See Figure 6. Hence, $X \cong E(\Gamma)$. By construction, $X \cup r(X) = [-1, 1] \times [-1, 1] \times [0, 2]$ and int($r(X)$) $\cap$
Figure 5. The exterior of the graph in Figure 3 represented as a polycube.

\[ \text{int}(X) = \emptyset. \] Hence, two congruent copies of \( X \) with disjoint interiors tile the cube \([-1, 1] \times [-1, 1] \times [0, 2] \). Since \( X \) is a polycube consisting of \( 4m^3 \) cubes of side length \( \frac{1}{m} \), then \( X \) is a 3-dimensional \( 8m^3 \)-index rep-tile.

\[ \square \]

4. Classification

In order to show that every 3D rep-tile is homeomorphic to the exterior of a connected graph, we will require the following reimbedding theorem due to Fox.

**Theorem 4.1.** [10] Every compact connected 3-dimensional sub-manifold \( X \) of \( S^3 \) can be reimbedded in \( S^3 \) so that the exterior of the image of \( X \) is a union of handlebodies, i.e. regular neighborhoods of embedded graphs.

The idea behind the following proof is to show that if a 3D reptile \( X = \bigcup_{i=1}^{k} X_i \) has two boundary components, then a “minimal” boundary component over all \( X_i \) must also be a boundary component for \( X \). This leads to a contradiction.

**Theorem 4.2.** If \( X \) is a 3D rep-tile, then \( \partial X \) is connected.
Figure 6. A 3D rep-tile homeomorphic to the exterior of the graph in Figure 3.

Proof. Suppose $X$ is a 3D rep-tile such that $\partial X$ is not connected. Then $X = \bigcup_{i=1}^{k} X_i$ such that, $X_i$ is congruent to $X_j$ for any $i, j \in \{1, ..., k\}$, $\text{int}(X_i) \cap \text{int}(X_j) = \emptyset$ if $i \neq j$, and $X_1$ is similar to $X$. Moreover, each $X_i$ has at least two connected boundary components.

Note that every closed connected surface in $\mathbb{R}^3$ separates. Hence, any such surface $G$ is the boundary of some finite volume region $R_G \subset \mathbb{R}^3$. Let $F$ be a connected boundary component of some $X_m$ such that $R_F$ has the least volume over all finite volume regions bounded by any connected boundary component of any $X_i$.

Suppose the interior of $R_F$ has nontrivial intersection with $X$. Then there exists some $X_n$ embedded in $R_F$. Since $X_n$ has at least two boundary components and $R_F$ has one boundary component, then $X_n \neq R_F$. So, some boundary component of $X_n$ bounds a finite volume region which is a proper subset of $R_F$ and has volume strictly less than $R_F$. This is a contradiction to the minimality of the volume of $R_F$. Thus, the interior of $R_F$ is disjoint from $X$. This implies that $F$ is a boundary component of $X$.

However, if $X = \bigcup_{i=1}^{k} X_i$ is a rep-tile and $v$ is the volume of $R_F$, then the least volume of any finite volume region bounded by a connected boundary component of $X$ is $kv$. Since $F$ is a boundary component of $X$, then $kv = v$. Since $v > 0$, then $k = 1$, a contradiction. Hence, $\partial X$ must be connected.

Theorems 4.1 and 4.2 immediately imply the following result.
Theorem 4.3. Every 3D rep-tile is homeomorphic to the exterior of a connected graph.

Theorem 1.1 follows from Theorems 3.1 and 4.3.

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References

[1] Samson Abramsky, Temperley-Lieb algebra: from knot theory to logic and computation via quantum mechanics, Mathematics of quantum computation and quantum technology, 2007, pp. 515–558.
[2] Colin C. Adams, Knotted tilings, The mathematics of long-range aperiodic order (Waterloo, ON, 1995), 1997, pp. 1–8. MR1460017
[3] C. Bandt and Y. Wang, Disk-like self-affine tiles in $\mathbb{R}^2$, Discrete Comput. Geom. 26 (2001), no. 4, 591–601. MR1863811
[4] Christoph Bandt, Self-similar sets. V. Integer matrices and fractal tilings of $\mathbb{R}^n$, Proc. Amer. Math. Soc. 112 (1991), no. 2, 549–562. MR1036982
[5] Christoph Bandt and Dmitry Mekhontsev, Computer geometry: rep-tiles with a hole, Math. Intelligencer 42 (2020), no. 1, 1–5. MR4068835
[6] Ryan Blair and Ricky Lee, Self-replicating 3-manifolds, arXiv:2107.04528.
[7] Ryan Blair and Joshua Sack, Idempotents in tangle categories split, J. Knot Theory Ramifications 28 (2019), no. 5, 1950025, 9. MR3943699
[8] Gregory R. Conner and Jörg M. Thuswaldner, Self-affine manifolds, Adv. Math. 289 (2016), 725–783. MR3439698
[9] HT Croft, KJ Falconer, and RK Guy, Unsolved problems in geometry, Springer, 1991 (English).
[10] Ralph H. Fox, On the imbedding of polyhedra in 3-space, Ann. of Math. (2) 49 (1948), 462–470. MR26326
[11] Martin Gardner, On rep-tiles, polygons that can make larger and smaller copies of themselves, Scientific Amer. 208 (1963), 154–164.
[12] S. W. Golomb, Replicating figures in the plane, Math.Gaz. 48 (1964), 403–412.
[13] Herman Haverkort, No acute tetrahedron is an 8-reptile, Discrete Math. 341 (2018), no. 4, 1131–1135. MR3764364
[14] Francis Jordan and Sze-Man Ngai, Reptiles with holes, Proc. Edinb. Math. Soc. (2) 48 (2005), no. 3, 651–671. MR2171191
[15] Jan Kynčl and Zuzana Patáková, On the nonexistence of k-reptile simplices in $\mathbb{R}^3$ and $\mathbb{R}^4$, Electron. J. Combin. 24 (2017), no. 3, Paper No. 3.1, 44. MR3691518
[16] Anwei Liu and Barry Joe, On the shape of tetrahedra from bisection, Math. Comp. 63 (1994), no. 207, 141–154. MR1240660
[17] Jun Luo, Hui Rao, and Bo Tan, Topological structure of self-similar sets, Fractals 10 (2002), no. 2, 223–227. MR1910665
[18] Jiří Matoušek and Zuzana Safernová, *On the nonexistence of k-reptile tetrahedra*, Discrete Comput. Geom. 46 (2011), no. 3, 599–609. MR2826971

[19] G. van Ophuysen, *Problem 19*, Tagungsbericht 20 (1997).