A Parameterized View on Multi-Layer Cluster Editing∗

Jiehua Chen1, Hendrik Molter2, Manuel Sorge1, and Ondřej Suchý3

1Dept. Industrial Engineering and Management, Ben-Gurion University of the Negev, Beer Sheva, Israel, jiehua.chen2@gmail.com, sorge@post.bgu.ac.il
2Institut für Softwaretechnik und Theoretische Informatik, TU Berlin, Germany, h.molter@tu-berlin.de
3Faculty of Information Technology, Czech Technical University in Prague, Prague, Czech Republic, ondrej.suchy@fit.cvut.cz

September 27, 2017

Abstract

In classical Cluster Editing we seek to transform a given graph into a disjoint union of cliques, called a cluster graph, using the fewest number of edge modifications (deletions or additions). Motivated by recent applications, we propose and study Cluster Editing in multi-layer graphs. A multi-layer graph consists of a set of simple graphs, called layers, that all have the same vertex set. In Multi-Layer Cluster Editing we aim to transform all layers into cluster graphs that differ only slightly. More specifically, we allow to mark at most d vertices and to transform each layer of the multi-layer graph into a cluster graph with at most k edge modifications per layer such that, if we remove the marked vertices, we obtain the same cluster graph in all layers.

Multi-Layer Cluster Editing is NP-hard and we analyze its parameterized complexity. We focus on the natural parameters “max. number d of marked vertices”, “max. number k of edge modifications per layer”, “number n of vertices”, and “number ℓ of layers”. We fully explore the parameterized computational complexity landscape for those parameters and their combinations. Our main results are that Multi-Layer Cluster Editing is FPT with respect to the parameter combination (d, k) and that it is para-NP-hard for all smaller or incomparable parameter combinations. Furthermore, we give a polynomial kernel with respect to the parameter combination (d, k, ℓ) and show that for all smaller or incomparable parameter combinations, the problem does not admit a polynomial kernel unless NP ⊆ coNP/poly.

Keywords: Cluster Editing, Multi-Layer Graphs, Fixed-Parameter Algorithms, Polynomial Kernels, Parameterized Complexity

1 Introduction

The NP-hard Cluster Editing problem, also known as Correlation Clustering, models the following clustering task. Given a set of objects and their binary similarity relations, partition the objects into parts, minimizing the similarities between objects in different parts and non-similarities between objects in the same part. In graph-theoretic terms, given a graph G and an integer k, we want to edit, that is, add or delete, at most k edges in G such that we obtain a disjoint union of cliques, also called a cluster graph. Cluster Editing was introduced by Ben-Dor, Shamir, and Yakhini 3 and Bansal, Blum, and Chawla 1 in biological and machine-learning contexts and has been studied extensively by the corresponding communities. The edited edges can be thought of as noise that obfuscates the inherent cluster structure, where the noise may stem from measurement errors, for example. Cluster Editing

∗OS supported by grant 17-20065S of the Czech Science Foundation. JC and MS supported by the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement number 631163.11 and Israel Science Foundation (grant no. 551145/14). This work was initiated at the research retreat of the TU Berlin Algorithmics and Computational Complexity group held in Boiensdorf (Baltic Sea), in April 2017.
has since also become one of the best-studied parameterized problems, see Becker and Baumbach \cite{6} for a survey. Notably, parameterized approaches to Cluster Editing were also successful in practice \cite{7}.

In recent years, the multi-layered nature of data in many applications is becoming more and more relevant \cite{6, 8, 14, 16}. For example, useful information about individuals may be represented in their social interactions, geographic closeness, common interests or activities \cite{14}. An example from biology is the neural network of Caenorhabditis elegans in which neurons can be connected by either chemical links or ionic channels \cite{5}. To represent and analyze such data, researchers commonly use multi-layer graphs \cite{14}, which are collections of ordinary graphs on the same vertex set, called layers. Multi-layer graphs enable us to take into account the different aspects of the data modeled in each layer. The field of clustering on multi-layer graphs, while still in its infancy \cite{16}, was already focus of much research (see surveys \cite{14, 16}). Indeed, crucial information may be lost, if we instead aggregate all layers into one graph \cite{2, 12, 16}.

Here we introduce a natural discrete model that lifts the established Cluster Editing problem to multi-layer graphs. The challenge in multi-layer clustering is to recover the cluster structure inherent in each layer, while also determining the overlap of communities between layers \cite{4}. A natural approach is thus as follows. To ensure that the cluster structure that we recover by editing edges is sufficiently reflected in each layer, we specify a maximum budget \(k\) and allow in each layer to edit at most \(k\) edges. To determine the overlap of communities between layers, we specify an upper bound \(d\) on the number of entities that may switch communities between layers—the (recovered) communities represented by the remaining entities have to be the same across all layers. Formally, the computational problem that we study is as follows.

\textbf{Multi-Layer Cluster Editing}

\textbf{Input:} \(\ell\) graphs \(G_1 = (V, E_1), \ldots, G_\ell = (V, E_\ell)\), and two integers \(k, d\).

\textbf{Question:} Is there a vertex subset \(D \subseteq V\) with \(|D| \leq d\) and \(\ell\) edge modification sets \(M_1, \ldots, M_\ell \subseteq \binom{V}{2}\) such that

1. for each \(i \in [\ell]\) we have that \(|M_i| \leq k\),
2. for each \(i \in [\ell]\) the graph \(G'_i = (V, E_i \oplus M_i)\) is a cluster graph, and
3. for all \(i, j \in [\ell]\) we have that \(G'_i[V \setminus D] = G'_j[V \setminus D]\)\(\).

Herein, \(\oplus\) denotes the symmetric difference, that is, \(A \oplus B = (A \setminus B) \cup (B \setminus A)\) and \([\ell]\) denotes \(\{1, \ldots, \ell\}\) for \(\ell \in \mathbb{N}\).

We study \textbf{Multi-Layer Cluster Editing} from a parameterized algorithms point of view. Our motivation is threefold. First, we think that it is a natural model for (correlation) clustering in multi-layer graphs. Indeed, many works for multi-layer clustering strive to find a consensus clustering among all layers \cite{12, 13, 19}. If \(d = 0\) in \textbf{Multi-Layer Cluster Editing}, then we check whether the input multi-layer graph conforms to a consensus clustering, up to noise. However, it is intuitive, that such a consensus clustering does not always exist. For example, following the data analyzed by Kim et al. \cite{14}, a researcher may be part of different communities with respect to the work, lunch, facebook, friend, and coauthor relationships (corresponding to different layers), while for others these communities are similar. This motivates us to partition the vertex set into a consensus cluster part \(V \setminus D\) and a “fluctuating” part \(D\) in \textbf{Multi-Layer Cluster Editing}.

Second, given the success of parameterized approaches for \textbf{Cluster Editing} \cite{3, 5}, we think that they are suitable to attack also \textbf{Multi-Layer Cluster Editing} and that this problem is a natural candidate to extend the toolkit of parameterized algorithms and apply it to the new challenges arising in the emerging field of multi-layer graphs.

Third, our techniques apply to temporal graphs. These are multi-layer graphs in which the layers are equipped with a linear order, modeling time-stamped communication in a social network, for example. Berger-Wolf and Tantipathananandh \cite{20} studied a cluster editing problem for temporal graphs that is closely related to \textbf{Multi-Layer Cluster Editing}. The main difference is that, instead of marking vertices in \(D\) over all layers, they mark vertices for each layer individually, allowing them to move between clusters between the layer in which they are marked and the successor layer\(\footnote{Multi-layer graphs are known by a multitude of other names including multi-dimensional networks, multiplex networks, and edge-colored multigraphs \cite{14}.}\). This translates to a modified Condition (3) which applies only to consecutive layers \(i, j = i + 1\). We believe that studying \textbf{Multi-Layer Cluster Editing} constitutes the natural first step towards the multivariate analysis of the somewhat more complex problem of Berger-Wolf and Tantipathananandh.

\footnote{In addition, Berger-Wolf and Tantipathananandh’s \cite{21} objective function translates to minimizing \(d + k\).}
Our Results. We completely classify Multi-Layer Cluster Editing in terms of fixed-parameter tractability and existence of polynomial-size problem kernels with respect to the parameters “max. number \(d\) of marked vertices”, “max. number \(k\) of edge modifications per layer”, “number \(n\) of vertices”, and “number \(\ell\) of layers”, and all of their combinations. Multi-Layer Cluster Editing is para-NP-hard for all parameter combinations colored in red. It is FPT for all parameter combinations that are colored yellow or green and admits a polynomial kernel for all parameter combinations colored green. It does not admit a polynomial kernel for all parameter combinations that are colored yellow unless \(\text{NP} \subseteq \text{coNP}/\text{poly}\).

Notation. We use standard notation from parameterized complexity [10] and graph theory [11]. We use \(n\) to denote the number of vertices in the given vertex set, i.e., unless stated otherwise we have that \(n = |V|\). Similarly, \(\ell\) is the number of layers in the input multi-layer graph. We call \(G_i\) the \(i\)-th layer or layer \(i\). We call the vertices in \(D\) marked and we call \(k\) the budget of a layer. We say that a tuple \((M_1, \ldots, M_\ell, D)\) of edge modification sets and marked vertices \(D\) is a solution if it satisfies Conditions (1) to (3) of Multi-Layer Cluster Editing.

2 Hardness of Multi-Layer Cluster Editing

Our problem is contained in \(\text{NP}\) since we can verify in polynomial time whether a given subset of vertices and some edges sets fulfill the three requirements given in the definition of our problem question. Thus, in all proofs for \(\text{NP}\)-completeness, we omit the proof for \(\text{NP}\) containment and only show the hardness part.

Since Cluster Editing is \(\text{NP}\)-complete [1], we immediately get \(\text{NP}\)-hardness for Multi-Layer Cluster Editing.

Observation 1. Multi-Layer Cluster Editing is \(\text{NP}\)-complete for \(d = 0\) and \(\ell = 1\).

By a polynomial-time reduction from Vertex Cover (given a simple graph \(G\) and an integer \(s\), decide whether there is a size-at-most-\(s\) vertex cover, i.e., a subset of at most \(s\) vertices which are jointly incident to all edges) which is \(\text{NP}\)-complete on graphs with maximum vertex degree three [12], we obtain...
that Multi-Layer Cluster Editing is NP-hard for a constant number of layers even if no edge modifications are allowed. This means that also the marking of vertices is a computationally hard task.

**Proposition 1.** Multi-Layer Cluster Editing is NP-complete for \(k = 0\) and \(\ell = 4\).

**Proof.** We reduce from Vertex Cover on graphs with maximum (vertex) degree three. Let \((G, s)\) be an instance of Vertex Cover, where \(G = (V, E)\) has maximum degree three. By Vizing’s Theorem \([21]\), we know that \(G\) is 4-edge-colorable and we can compute a proper 4-edge-coloring in polynomial time \([17]\).

Let \(E_i \subseteq E\) be the set of edges colored with color \(i \in \{1, 2, 3, 4\}\) for an arbitrary but fixed 4-edge-coloring of \(G\). Define \(I' = (G_1 = (V, E_1), G_2 = (V, E_2), G_3 = (V, E_3), G_4 = (V, E_4), 0, s)\) to be an instance of Multi-Layer Cluster Editing. As already argued, \(I'\) can be constructed from \((G, s)\) in polynomial time. We show that \((G, s)\) is a yes-instance of Vertex Cover if and only if \(I'\) is a yes-instance of Multi-Layer Cluster Editing.

(\(\Rightarrow\)) Let \(S \subseteq V\) be a vertex cover of \(G\) with \(|S| \leq s\). We first mark all vertices of \(S\). Note that the graph in each layer consists of isolated edges since two adjacent edges cannot have the same color in a proper edge-coloring. Hence for every \(i \in \{1, 2, 3, 4\}\), graph \(G_i\) is a cluster graph. By definition, \(G[V \setminus S]\) is edgeless. Therefore \(G_i[V \setminus S] = G_j[V \setminus S]\) for all \(i, j \in \{1, 2, 3, 4\}\), implying that \(I\) is a yes-instance of Multi-Layer Cluster Editing.

(\(\Leftarrow\)) Let \(D \subseteq V\) be a set of marked vertices with \(|D| \leq s\). Note that \(E_i \cap E_j = \emptyset\) for all \(i, j \in \{1, 2, 3, 4\}\) with \(i \neq j\) since every edge in \(E\) is colored with exactly one color. Since \(k = 0\) we are not allowed to make any edge modification and hence for each edge \((u, v) \in E\) at least one of the endpoints has to be marked. It follows that \(D\) is a vertex cover for \(G\) of size at most \(s\).

Modifying the reduction in the proof of [Proposition 1], to introduce one layer for each edge in the graph of the Vertex Cover instance, we obtain a parameterized reduction from Independent Set which is \(W[2]\)-hard with respect to the solution size. This gives the following corollary.

**Corollary 1.** Multi-Layer Cluster Editing is \(W[1]\)-hard with respect to the parameter \((n - d)\) for \(k = 0\).

**Proof.** We reduce from Independent Set parameterized by the solution size. Note that this is equivalent to Vertex Cover parameterized by the dual of the solution size. We can use a reduction that is very similar to the reduction used in the proof of [Proposition 1]. The main difference is that we cannot restrict the input graphs to graphs with maximum degree 3 since Independent Set parameterized by the solution size is FPT on graphs with bounded maximum degree. However, by Vizing’s Theorem \([21]\), we can still edge color the graph with one more color than its maximum degree in polynomial time. As we again introduce one layer for each color, this increases the number of layers in the reduction to one plus the maximum degree of the input graph. The rest of the proof is analogous.

### 3 An FPT Algorithm for Multi-Layer Cluster Editing

In this section, we present an FPT algorithm for Multi-Layer Cluster Editing with respect to the combined parameter \((k, d)\).

**Theorem 1.** Multi-Layer Cluster Editing is FPT with respect to the number \(k\) of edge modifications per layer and number \(d\) of marked vertices combined. It can be solved in time \(k^{O(kd)} \cdot O(n^2 \cdot d \cdot \ell)\).

We describe a recursive search-tree algorithm (see Algorithm 1) which takes the following data as input:

- An instance \(I\) of Multi-Layer Cluster Editing consisting of a multi-layer graph \(G_1, \ldots, G_\ell = (V, E_1), \ldots, (V, E_\ell)\) and two integers \(k\) and \(d\).
- A constraint \(P = (D, (M_i)_{i \in [\ell]}, B)\), consisting of a set of marked vertices \(D \subseteq V\), edge modification sets \(M_1, \ldots, M_\ell \subseteq \binom{\binom{V}{2}}{d}\), and a set \(B \subseteq \binom{\binom{V}{2}}{2}\) of permanent vertex pairs.

The algorithm follows the greedy localization approach in which we make some decisions greedily, which we possibly revert through branching later on. The greedy decisions herein give us some structure that we can exploit to keep the search-tree size small. The edge modification sets \(M_i\) represent both the greedy decisions and those that we made through branching. The set \(B\) contains only those made by branching.
Throughout the algorithm, we try to maintain a good constraint which intuitively means that the constraint can be turned into a solution (if one exists).

**Definition 1** (Good Constraint). Let $I$ be an instance of Multi-Layer Cluster Editing. A constraint $P = (D, M_1, \ldots, M_\ell, B)$ is good for $I$ if there is a solution $S = (M_1^*, \ldots, M_\ell^*, D^*)$ such that (i) $D \subseteq D^*$, (ii) there is no $\{u, v\} \in B$ such that $u \in D^*$, and (iii) for all $i \in [\ell]$ we have $M_i \cap B = M_i^* \cap B$. We also say that $S$ witnesses that $P$ is good.

The following is easy to see.

**Observation 2.** For any yes-instance $I = (G_1, \ldots, G_\ell, d, k)$ of Multi-Layer Cluster Editing, we have that $P_0 = (D = \emptyset, M_1 = \emptyset, \ldots, M_\ell = \emptyset, B = \emptyset)$ is a good constraint.

We also call the above constraint $P_0$ trivial. The initial call of our algorithm is with the input instance of Multi-Layer Cluster Editing together with the trivial constraint $P_0$.

Our algorithm uses a number of different branching rules to search for a solution to our Multi-Layer Cluster Editing input instance:

**Definition 2** (Branching Rule). A branching rule takes as input an instance $I$ of Multi-Layer Cluster Editing and a constraint $P$ and returns a set of constraints $P^{(1)}, \ldots, P^{(c)}$.

When a branching rule is applied, the algorithm invokes a recursive call for each constraint returned by the branching rule and returns true if at least one of the recursive calls returns true; otherwise, it returns false. For that to be correct, whenever a branching rule is invoked with a good constraint, at least one of the constraints returned by the branching rule has to be a good constraint as well. We say that a branching rule is safe if it has this property.

In the following, we introduce the branching rules used by the algorithm and prove that each of them is safe. This together with Observation 2 will allow us to prove by induction that the algorithm eventually finds a solution for the input instance of Multi-Layer Cluster Editing if it is a yes-instance. To make the description of the branching rules more readable, we introduce four types of non-marked vertex pairs. Say that a vertex pair $\{u, v\} \in (V \setminus D)_2$ is
- settled if $\{u, v\} \in E_i \oplus M_i$ for all $i \in \ell$ or $\{u, v\} \notin E_i \oplus M_i$ for all $i \in [\ell]$ (edge always present or never present),
- frequent if $|\{i \mid (\{u, v\} \subseteq E_i \oplus M_i)\}| \geq \frac{2\ell}{4}$ (edge almost always present),
- scarce if $|\{i \mid (\{u, v\} \subseteq E_i \oplus M_i)\}| \leq \frac{\ell}{4}$ (edge almost never present), and
- unsettled otherwise, that is, $\frac{\ell}{4} < |\{i \mid (\{u, v\} \subseteq E_i \oplus M_i)\}| < \frac{2\ell}{4}$ (edge sometimes present).

Note that, by definition, if a vertex pair falls in one of the above categories, both of the vertices in that pair are not marked.

Our aim with the first two rules is to settle all pairs in $(V \setminus D)_2$. In order to achieve our desired running time bound, we can only afford to exhaustively search through all unsettled vertex pairs:

**Branching Rule 1.** If there is an unsettled vertex pair $\{u, v\} \in (V \setminus D)_2$, then output the following up to four constraints:

1. For all $i \in [\ell]$, put $M_i^{(1)} = M_i \cup \{\{u, v\}\} \setminus E_i \cup B \cup \{\{u, v\}\}$. (Add the edge corresponding to the vertex pair in all layers where it is not present and mark it as permanent.)
2. For all $i \in [\ell]$, put $M_i^{(2)} = M_i \cup \{\{u, v\}\} \cap E_i \cup B \cup \{\{u, v\}\}$. (Remove the edge corresponding to the vertex pair from all layers where it is present and mark it as permanent.)
3. If there is no $x \in V \setminus D$ such that $\{u, x\} \in B$, then $D^{(3)} = D \cup \{u\}$, the rest stays the same. (Mark the first vertex in the vertex pair.)
4. If there is no $x \in V \setminus D$ such that $\{v, x\} \in B$, then $D^{(4)} = D \cup \{v\}$, the rest stays the same. (Mark the second vertex in the vertex pair.)

**Lemma 1.** [Branching Rule 1] is safe.

**Proof.** Let the input constraint $P = (D, M_1, \ldots, M_\ell, B)$ be good and let $S = (M_1^*, \ldots, M_\ell^*, D^*)$ be a solution to the input instance that witnesses that $P$ is good. If $\{u, v\} \cap D^* = \emptyset$, then by Condition 3 on solutions either $\{u, v\} \subseteq E_i \oplus M_i^*$ for all $i \in [\ell]$ or $\{u, v\} \notin E_i \oplus M_i^*$ for all $i \in [\ell]$. Hence, one of the first two constraints is good. Otherwise, if $\{u, v\} \cap D^* \neq \emptyset$, then one of the last two constraints is good. 

\[ \square \]
The following Greedy Rule deals with all frequent and scarce vertex pairs. It only produces one constraint and hence no branching occurs in that sense. For formal reasons it is nevertheless useful to treat the Greedy Rule as a special case of a branching rule. Note that the algorithm also invokes a recursive call with the output constraint of this rule. The rule greedily adds the edge corresponding to a frequent vertex pair in all layers where it is not present and removes edges corresponding to scarce vertex pairs in all layers where it is present. Intuitively, the Greedy Rule is safe, because all of its decisions can be reverted later on.

**Greedy Rule.** If there is a frequent or a scarce vertex pair \(\{u, v\} \in \binom{V}{2}^D\), then return one of the following two constraints:

- If \(\{u, v\}\) is frequent, then for all \(i \in [\ell]\) put \(M_i^{(1)} = M_i \cup \{\{u, v\}\} \cap \mathcal{E}_i\), the rest stays the same. (Add the edge corresponding to the vertex pair in all layers where it is not present.)
- If \(\{u, v\}\) is scarce, then for all \(i \in [\ell]\) put \(M_i^{(1)} = M_i \cup \{\{u, v\}\} \cap \mathcal{E}_i\), the rest stays the same. (Remove the edge corresponding to the vertex pair from all layers where it is present.)

**Lemma 2.** The [Greedy Rule](#) is safe.

**Proof.** Let the input constraint \(P = (D, M_1, \ldots, M_\ell, B)\) be good and let \(S = (M_1^*, \ldots, M_\ell^*, D^*)\) be a solution for the input instance that witnesses that \(P\) is good and denote \(G_i^* = (V, E_i \oplus M_i^*)\) for all \(i \in [\ell]\). Note that neither \(D\) nor \(B\) is changed in the constraint output by the Greedy Rule. Hence, trivially, Conditions (i) and (ii) of being good are satisfied. For Condition (iii), we claim that no set \(M_i \cap B\) changes, implying the condition. For a contradiction, assume \(\{u, v\} \in M_i \cap B\). Since \(S\) witnesses \(P\) being good, we have that \(M_j \cap B = M_j^* \cap B\) for all \(j \in [\ell]\) and we have that there is no \(\{u', v'\} \in B\) such that \(u' \in D^*\). This implies that \(\{u, v\}\) is settled, since otherwise there is a \(j \in [\ell]\) such that \(\{u, v\} \notin M_j^* \cap B\) which would imply that \(G_j^*[V \setminus D^*] \neq G_j^*[V \setminus D]\), which is impossible because \(S\) is a solution. Thus, the Greedy Rule is not applicable to \(\{u, v\}\), a contradiction. \(\square\)

After the above two rules have been applied exhaustively, all pairs in \(\binom{V}{2}^D\) are settled. With the following rule we edit the subgraphs induced by all non-marked vertices into cluster graphs. This branching rule represents a well-known rule from the classical CLUSTER EDITING with the addition that we also branch on marking vertices.

**Branching Rule 2.** If there is an induced \(P_3 = (\{u, v\}, \{v, w\})\) in \(G_i^*[V \setminus D]\) for some \(i \in [\ell]\), where \(G_i^* = (V, E_i \oplus M_i)\), then return the following up to six constraints:

1. If \(\{u, v\} \notin B\), then for all \(i \in [\ell]\) put \(M_i^{(2)} = M_i \oplus \{\{u, v\}\}\), \(D^{(1)} = D\), and \(B^{(1)} = B \cup \{\{u, v\}\}\). (Remove the edge corresponding to the vertex pair from all layers and mark it as permanent.)
2. If \(\{v, w\} \notin B\), then for all \(i \in [\ell]\) put \(M_i^{(2)} = M_i \oplus \{\{v, w\}\}\), \(D^{(2)} = D\), and \(B^{(2)} = B \cup \{\{v, w\}\}\). (Remove the edge corresponding to the vertex pair from all layers and mark it as permanent.)
3. If \(\{u, w\} \notin B\), then for all \(i \in [\ell]\) put \(M_i^{(3)} = M_i \oplus \{\{u, w\}\}\), \(D^{(3)} = D\), and \(B^{(3)} = B \cup \{\{u, w\}\}\). (Add the edge corresponding to the vertex pair in all layers and mark it as permanent.)
4. For each \(x \in \{u, v, w\}\): If there is no \(y \in V \setminus D\) such that \(\{x, y\} \in B\), then return a constraint with \(D^{(4)} = D \cup \{x\}\), the rest stays the same. (Mark vertices of the \(P_3\) that are not part of permanent vertex pairs. This gives up to three constraints.)

If none of the above possibilities apply, then reject the current branch. \(^3\)

**Lemma 3.** [Branching Rule 2](#) is safe.

**Proof.** Let the input constraint \(P = (D, M_1, \ldots, M_\ell, B)\) be good and let \(S = (M_1^*, \ldots, M_\ell^*, D^*)\) be a solution for the input instance witnessing that \(P\) is good. By Condition (ii) of solutions it holds that, for all \(i \in [\ell]\), graph \(G_i^*[V \setminus D^*]\) does not contain any \(P_3\) as an induced subgraph, where \(G_i^* = (V, E_i \oplus M_i^*)\). Hence, if there is some \(i \in [\ell]\) and three vertices \(u, v, w\) that induce a \(P_3\) in \(G_i^*[V \setminus D]\), where \(G_i^* = (V, E_i \oplus M_i)\), then there are two cases:

- In the first case, one of \(u, v, w\) is also in \(D^*\), say \(v \in D^*\). Note that, then, \(v\) cannot be part of any permanent vertex pair, by the definition of good constraints. Thus, the constraint that puts \(v \in D\) output in the fourth part of [Branching Rule 2](#) is good.

\(^3\)This technically does not fit the definition of a branching rule but we can achieve the same effect by returning trivially unsatisfiable constraints such as a constraint with \(|D^{(1)}| > \ell\).
The second case is that \( u, v, w \in V \setminus D^* \). Then, since \( G^*_i[V \setminus D^*] \) is a cluster graph, at least one of the vertex pairs formable from \( u, v, w \) is modified by \( S_i \), that is, in \( M_i^* \). Say \( \{u, v\} \in M_i^* \). By Condition 3 of solutions, \( \{u, v\} \) is settled. Note that \( \{u, v\} \) cannot be permanent since otherwise we already have that \( \{u, v\} \in M_i \) by the definition of a good constraint. Thus the constraint which adds \( \{u, v\} \) to \( M_i \) and makes it permanent is good. Hence, the rule is safe.

The next rule keeps the sets of edge modifications \( M_i \) free of marked vertices. Pairs in \( M_i \) can become marked if vertices of vertex pairs processed by the Greedy Rule are marked by other branching rules further down the search tree. Like the Greedy Rule, it only produces one constraint and hence no branching occurs, so it is also a degenerate branching rule and we treat it as such. Note that the algorithm also invokes a recursive call with the output constraint of this rule.

**Clean-up Rule.** If there is an \( i \in [\ell] \) such that there is a \( \{u, v\} \in M_i \) with \( u \in D \), then return a constraint with \( M_i^{(1)} = M_i \setminus \{\{u, v\}\} \), the rest stays the same.

**Lemma 4.** The **Clean-up Rule** is safe.

*Proof.* Let the input constraint \( P = (D, M_1, \ldots, M_\ell, B) \) be good and let \( S = (M_1^*, \ldots, M_\ell^*, D^*) \) be a solution for the input instance witnessing that \( P \) is good. Note that permanent vertex pairs cannot contain marked vertices by the definition of constraints. It follows that the Clean-up Rule does not add or remove permanent vertex pairs from any set \( M_i \); this implies Condition (iii) of being good. Furthermore, it does not change the sets \( D \) and \( B \), implying Conditions (i) and (ii). Hence, the Clean-up Rule is safe.

The next rule tries to repair any budget violations that might occur. Since with the Greedy Rule we greedily make decisions and do not exhaustively search through the whole search space, we expect that some of the choices were not correct. This rule will then revert these choices. Also, to have a correct estimate of the sizes of the current edge modification sets, this rule requires that the Clean-up Rule is not applicable. For technical reasons, it also requires Branching Rule 1 and the Greedy Rule not to be applicable.

**Branching Rule 3.** If there is an \( M_i \) for some \( i \in [\ell] \) with \( |M_i| > k \), then if \( |M_i \setminus B| \leq k + 1 \), let \( M_i' = M_i \setminus B \), otherwise, take any \( M_i' \subseteq M_i \setminus B \) with \( |M_i'| = k + 1 \) and return the following constraints:

1. For each \( \{u, v\} \in M_i' \) return a constraint in which for all \( j \in [\ell] \) we put \( M_j^{(1)} = M_j \cup \{\{u, v\}\} \), \( D^{(1)} = D \) and \( B^{(1)} = B \cup \{\{u, v\}\} \).
2. For each \( \{u, v\} \in M_i' \):
   - If there is no \( x \in V \setminus D \) such that \( \{x, v\} \in B \), then return a constraint with \( D^{(1)} = D \cup \{u\} \), \( B^{(1)} = B \) and \( 1 \leq j \leq \ell : M_j^{(1)} = M_j \setminus \{\{u, v\}\} \).
   - If there is no \( x \in V \setminus D \) such that \( \{x, v\} \in B \), then return a constraint with \( D^{(1)} = D \cup \{v\} \), \( B^{(1)} = B \) and \( 1 \leq j \leq \ell : M_j^{(1)} = M_j \setminus \{\{u, v\}\} \).

If \( M_i' = \emptyset \), then reject the current branch.

**Lemma 5.** If Branching Rule 1, the Greedy Rule and the Clean-up Rule are not applicable, then Branching Rule 3 is safe.

*Proof.* Let the input constraint \( P = (D, M_1, \ldots, M_\ell, B) \) be good and let \( S = (M_1^*, \ldots, M_\ell^*, D^*) \) be a solution for the input instance witnessing that \( P \) is good.

Since the Greedy Rule and Branching Rule 1 are not applicable, we have that all vertex pairs in \( (V \setminus D^*)^2 \) are settled. Since the Clean-up Rule is not applicable, no edge modification set contains marked vertices. Now if there is an \( M_i \) with \( |M_i| > k \), then there clearly is a \( \{u, v\} \in M_i \) with \( \{u, v\} \notin M_i^* \). Since \( P \) is a good constraint, we also have that \( \{u, v\} \notin M_i \cap B \).

If \( |M_i \setminus B| > k \), then it is easy to see that for each \( M_i' \subseteq M_i \setminus B \) with \( |M_i'| = k + 1 \) there is at least one vertex pair \( \{u, v\} \in M_i' \) such that \( \{u, v\} \notin M_i^* \); otherwise we would have that \( |M_i^*| > k \). This holds in particular for the \( M_i \) chosen by the branching rule. The branching rule creates constraints for each possible vertex pair in \( M_i' \) that could be removed from \( M_i \), there is particularly one output constraint where \( \{u, v\} \) is removed from \( M_i \). If \( \{u, v\} \cap D^* = \emptyset \), then by Condition 3 on solutions either \( \{u, v\} \in E_i \oplus M_i^* \) for all \( i \in [\ell] \) or \( \{u, v\} \notin E_i \oplus M_i^* \) for all \( i \in [\ell] \). However, since all \( \{u, v\} \) is settled,
we also have that \( \{u, v\} \in E_i \oplus M_i \) for all \( i \in [\ell] \) or \( \{u, v\} \notin E_i \oplus M_i \) for all \( i \in [\ell] \) and furthermore, \( \{u, v\} \in E_i \oplus M_i \) if and only if \( \{u, v\} \notin E_i \oplus M^*_i \). Since we have that \( \{u, v\} \in E_i \oplus M_i \) if and only if \( \{u, v\} \notin E_i \oplus M \cup \{\{u, v\}\} \), one of the constraints in the first case is good.

Otherwise at least one of its endpoints is marked in \( S \) and the one of the constraints in the second case is good.

The last rule, Branching Rule 4, requires that all other rules are not applicable. In this case the non-marked vertices induce the same cluster graph in every layer. Branching Rule 3 checks whether in every layer it is possible to turn the whole layer (including the marked vertices) into a cluster graph such that the cluster graph induced by the non-marked vertices stays the same and the edge modification budget is not violated in any layer.

**Branching Rule 4.** For all \( 1 \leq i \leq \ell \) we use \( M_i \) to denote the set of all possible edge modifications where each edge is incident to at least one marked vertex, that turn \( G^*_i = (V, E_i \oplus M_i) \) into a cluster graph. More specifically, we have

\[
M_i = \{ M \subseteq \binom{V}{2} \mid \forall e \in M : e \cap D = \emptyset \land G''_i = (V, E_i \oplus (M \cup M)) \text{ is a cluster graph} \}.
\]

If there is an \( 1 \leq i \leq \ell \) such that \( \min_{M \in M_i} |M| > k - |M_i| \) then let \( M'_i = M_i \setminus B \) and return the following constraints:

1. For each \( \{u, v\} \in M'_i \) return a constraint in which for all \( j \in [\ell] \) we put \( M''_j = M_j \oplus \{\{u, v\}\} \), \( D^{(j)} = D \cup \{u, v\} \), and \( B^{(j)} = B \cup \{\{u, v\}\} \).
2. For each \( \{u, v\} \in M'_i \):
   - If there is no \( x \in V \setminus D \) such that \( \{u, x\} \in B \), then return a constraint with \( D^{(j)} = D \cup \{u\} \), \( B^{(j)} = B \), and \( 1 \leq j \leq \ell \) such that \( M''_j = M_j \setminus \{\{u, v\}\} \).
   - If there is no \( x \in V \setminus D \) such that \( \{v, x\} \in B \), then return a constraint with \( D^{(j)} = D \cup \{v\} \), \( B^{(j)} = B \), and \( 1 \leq j \leq \ell \) such that \( M''_j = M_j \setminus \{\{u, v\}\} \).

If \( M'_i = \emptyset \), then reject the current branch.

**Lemma 6.** If the Greedy Rule, the Clean-up Rule, and Branching Rules 2, 3 and 4 are not applicable, then Branching Rule 4 is safe.

**Proof.** Let the input constraint \( P = (D, M_1, \ldots, M_\ell, B) \) be good and let \( S = (M^*_1, \ldots, M^*_\ell, D^*) \) be a solution for the input instance witnessing that \( P \) is good. Since the Greedy Rule and Branching Rule 4 are not applicable, we have that all vertex pairs in \( (V \setminus D)^2 \) are settled. Since Branching Rule 2 is not applicable, we have that for all \( i, j \in [\ell] \) we have that \( G^*_i[V \setminus D] = G^*_j[V \setminus D] \) and \( G^*_i[V \setminus D] \) is a cluster graph, where \( G^*_i = (V, E_i \oplus M_i) \). Furthermore, we have that \( |M_i| \leq k \) for all \( i \in [\ell] \), otherwise Branching Rule 3 would be applicable, and that each \( M_i \) does not contain vertex pairs with marked vertices, otherwise the Clean-up Rule would be applicable.

For each layer \( i \), Branching Rule 4 checks the minimum number of edge modifications involving at least one marked vertex to turn \( G^*_i \) into a cluster graph. Since \( G^*_i[V \setminus D] \) is already a cluster graph, this number always exists. Since \( M_i \) does not contain vertex pairs with marked vertices, if the number of edge modifications needed is too large, i.e. larger than \( k - |M_i| \), there is at least one non-permanent, settled vertex pair in \( M_i \) that is not in \( M^*_i \). If follows from an analogous argumentation to the one in the proof of Lemma 5 that the rule is safe.

To prove correctness of the algorithm, we first argue that, whenever the algorithm outputs true, then the input instance of MULTILAYER CLUSTER EDITING was indeed a yes-instance. This follows in a straightforward manner from the fact, that if the algorithm outputs true, then none of the branching rules are applicable.

**Lemma 7.** Given an instance \( I \) of MULTILAYER CLUSTER EDITING, if Algorithm 1 outputs true on input \( I \) and the trivial partial solution \( P_0 \), then \( I \) is a yes-instance.

**Proof.** Let \( I \) be the input instance of MULTILAYER CLUSTER EDITING. If the algorithm outputs true, then there is a constraint \( P = (D, M_1, \ldots, M_\ell, B) \) such that for all \( e \in M_i \) we have that \( e \cap B = \emptyset \) and
Algorithm 1: Multi-Layer Cluster Editing

Input:

- A set of graphs $G_1, \ldots, G_\ell = (V, E_1), \ldots, (V, E_\ell)$.
- Two integers $k$, and $d$.
- A set of marked vertices $D$.
- Edge modification sets $M_1, \ldots, M_\ell$.
- A set $B \subseteq \binom{V \setminus D}{2}$ of permanent vertex pairs.

1. If $|D| > d$ or there is an $i \in [\ell]$ such that $|M_i \cap B| > k$ then return false
2. Apply the first applicable rule in the following ordered list: Branching Rule 1, Greedy Rule
   Branching Rule 2, Clean-up Rule, Branching Rule 3, and Branching Rule 4
3. Return true

$|D| \leq d$, and none of the branching rules are applicable. In the following we show that then there is a solution $S = (M_1', \ldots, M_\ell', D')$ for $I$ such that $D' = D$. Let $G''_{\ell} = (V, E_{\ell} \oplus M_{\ell})$ for all $i \in [\ell]$.

First of all, for all $i \in [\ell]$ we have that $|M_i| \leq k$, otherwise Branching Rule 3 would apply. Also, for all $i,j \in [\ell]$ we have that $G''_i[V \setminus D] = G''_j[V \setminus D]$, otherwise either Branching Rule 1 or the Greedy Rule would apply. Furthermore, we have that for all $i \in [\ell]$ we have that $G''_i[V \setminus D]$ is a cluster graph, otherwise Branching Rule 2 would apply.

It remains to show that there are $M_i' \subseteq \binom{V}{2}$ for all $i \in [\ell]$ such that $G_i' = (V, E_i \oplus (M_i \cup M_j'))$ is a cluster graph and $|M_i \cup M_i'| \leq k$. Since Branching Rule 4 is not applicable, we know that these sets $M_i'$ exist: take $M_i' = \arg \min_{M \subseteq M_i} |M|$ for the definition of $M_i$ of Branching Rule 4. Then we have that $|M_i' | \leq k - |M_i|$ and hence $|M_i \cup M_i' | \leq k$.

It remains to show that, whenever the input instance $I$ of the algorithm is a yes-instance, then the algorithm outputs true. To this end, we define the quality of a good constraint and show that the algorithm increases the quality until it eventually finds a solution.

**Definition 3** (Quality of a constraint). Let $I = (G_1, \ldots, G_\ell, k, d)$ be an instance of Multi-Layer Cluster Editing. The quality $\gamma_I(P)$ of a constraint $P = (D, M_1, \ldots, M_\ell, B)$ for $I$ is $\gamma_I(P) = |D| + |B| - |\{u, v\} \in \binom{V}{2} \mid \{u, v\}$ is frequent or scarce$|$. |

**Lemma 8.** Let $P$ be a good constraint for a yes-instance of Multi-Layer Cluster Editing. If applicable, each of the Greedy Rule and Branching Rules 1, 2, and 4 return a good constraint with strictly increased quality in comparison to $P$.

**Proof.** We show the claim individually for each of the rules. We consider each of the possible returned constraints $P'$ and show that, assuming that $P'$ is good, then the quality of $P'$ is strictly larger than $P$.

It is easy to see that the Greedy Rule decreases the number of frequent or scarce vertex pairs by one.

Next, we consider Branching Rule 1. Observe that if $I$ be a yes-instance of Multi-Layer Cluster Editing and $P = (D, M_1, \ldots, M_\ell, B)$ a good constraint for $I$, then we have that all vertex pairs in $B$ are settled. Otherwise there would be a contradiction to the fact that in a good constraint there is no $\{u, v\} \in B$ such that $u$ needs to be marked and that for all $i \in [\ell]$ we have that the edge modifications in $M_i \cup B$ can all be kept in a solution. Hence, in the first two cases, the branching rule increases $|B|$. In the other two cases, the branching rule increases $|D|$.

Next, we consider Branching Rule 2. In the first three cases, the branching rule increases $|B|$. In the remaining cases, the branching rule increases $|D|$. Lastly, we consider Branching Rules 3 and 4. In each case, the branching rules increase $|B|$.

Next we show that the notion of quality of a good constraint is indeed a measure that allows us to argue that the algorithm eventually produces a solution (if it exists).

**Lemma 9.** Let $I$ be a yes-instance of Multi-Layer Cluster Editing, then there is a constant $c_I \geq 0$ such that for every good constraint $P$ we have that $\gamma_I(P) \leq c_I$ and there is at least one good constraint $P_{\max}$ with $\gamma_I(P_{\max}) = c_I$. Furthermore, for any good constraint $P'_{\max}$ with $\gamma_I(P'_{\max}) = c_I$, we have that Algorithm 4 outputs true on input $I$ and $P_{\max}$.
Proof. We first show the first part of the statement. Let \( I = (G_1, \ldots, G_\ell, k, d) \) be a yes-instance of Multi-Layer Cluster Editing. By Definition 1, for any good constraint \( P = (D, M_1, \ldots, M_\ell, B) \) there has to be a solution \( S = (M_1', \ldots, M_\ell', D^*) \) for \( I \) with \( D \subseteq D^* \), there is no \( (u, v) \in B \) such that \( u \in D^* \), and for all \( i \in [\ell] \) we have that \( M_i \cap B = M_i' \cap B \).

Fix a solution \( S = (M_1', \ldots, M_\ell', D^*) \), let \( P_S \) be the set of all good constraints \( P = (D, M_1, \ldots, M_\ell, B) \) with \( D \subseteq D^* \), there is no \( (u, v) \in B \) such that \( u \in D^* \), and \( M_i \cap B = M_i' \cap B \) for all \( i \in [\ell] \). It is easy to see that for good constraints in \( P_S \) the quality is maximized when \( D = D^* \) and \( B = (V \setminus D^*)^2 \). Note that this also implies that \( M_i = M_i' \cap B \) for all \( i \in [\ell] \) and that there are no frequent or scarce vertex pairs.

Let \( S_i \) be the set of all solutions for a given yes-instance \( I \) of Multi-Layer Cluster Editing. Then we have that
\[
\gamma_i(P) = \max_{S \in S_i} P \in P_S \text{ that each branching rule increases the quality of the good constraint.}
\]
and this already yields that this maximum is reached by at least one good constraint.

The second part of the statement follows from Lemma 8 and the safeness of the branching rules. Note that the order in which rules are applied (see Algorithm 1) ensures safeness for all branching rules \( \sum_{i \leq \ell} |M_i \cap B| \leq |\sum_{i \leq \ell} |M_i \cap B| \leq \ell \cdot \ell \cdot k \). Furthermore, by Lemma 7 we have that each branching rule increases the quality of the good constraint. Since \( P_{\text{max}} \) has maximum quality, there is no good constraint with a higher quality. Hence, no branching rule is applicable, otherwise we would have a contradiction to the safeness of the rule. It follows that the algorithm outputs true. \( \square \)

Now we have all the tools to show the correctness of Algorithm 1. Lemma 7 ensures that we only output true if the input is actually a yes-instance and Lemma 8 together with the safeness of all branching rules ensures that if the input is a yes-instance, the algorithm outputs true.

Corollary 2 (Correctness of Algorithm 1). Given an instance \( I \) of Multi-Layer Cluster Editing, Algorithm 1 outputs true on input \( I \) and the trivial good constraint \( P_0 \) if and only if \( I \) is a yes-instance.

Proof. We have that if Algorithm 1 outputs true on input \( I \) and the trivial good constraint \( P_0 \), then \( I \) is a yes-instance. This follows from Lemma 7. It remains to show the other direction.

Let \( I \) be a yes-instance of Multi-Layer Cluster Editing. By Observation 2 we have that \( P_0 \) is a good constraint. Note that the order in which rules are applied (see Algorithm 1) ensures safeness for all branching rules (Lemmas 1 and 2) and 3. Furthermore, by Lemma 4 we have that all branching rules except the Clean-up Rule strictly increase the quality of a good constraint. It is easy to see that the Clean-up Rule does not decrease the quality of a good constraint and that it can be applied at most \( \ell \cdot \left| \left( \frac{\ell}{2} \right) \right| \) times before either one of the other rules apply or the algorithm terminates. The quality of \( P_0 \) is at least \(-\left| \left( \frac{\ell}{2} \right) \right| \), hence the algorithm eventually reaches a good constraint with quality \( c_1 \) (or outputs true earlier). By Lemma 9 the algorithm then outputs true. \( \square \)

It remains to show that Algorithm 1 has the claimed running time upper-bound. We can check that all branching rules create at most \( O(k) \) recursive calls. The differentiation between unsettled, frequent and scarce vertex pairs ensures that the edge modification sets in sufficiently many layers increase for the search tree to have depth of at most \( O(k + d) \). The time needed to apply a branching rule is dominated by Branching Rule 4 where we essentially have to solve classical Cluster Editing in every layer.

Lemma 10. The running time of Algorithm 1 is in \( k^{O(k+d)} \cdot O(n^2 \cdot \ell) \).

Proof. We follow the following straightforward approach to bound the running time of Algorithm 1. First, we bound the size of the search tree, and then the computation spent in each node of the search tree.

The search tree is spanned by the non-degenerate branching rules. To bound the depth of the search tree, we show that each branching rule increases either \( |D| \) by exactly one or it increases \( \sum_{i \leq \ell} |M_i \cap B| \). If \( |D| > d \) or \( \sum_{i \leq \ell} |M_i \cap B| > \ell \cdot k \), then the algorithm terminates (Line 1). In the first two cases, Branching Rule 1 increases \( \sum_{i \leq \ell} |M_i \cap B| \) by at least \( \frac{\ell}{2} \) since the vertex pair that is modified in unsettled. In the case of Branching Rule 2, it is important to note, that if it is applicable, then the Greedy Rule was not applicable since it appears earlier in Algorithm 1. Hence, in the first three cases \( \sum_{i \leq \ell} |M_i \cap B| \) increases by \( \ell \) if the modified vertex pair was originally settled, and \( \frac{\ell}{2} \) if the vertex pair was originally frequent or scarce, since in that case a modification decreases \( |M_i| \) for at most \( \frac{\ell}{2} \) different
layers $i$ and increases $|M_i|$ for at least $\frac{3}{2}$ different layers $i$. By a similar argument, also Branching Rules 3 and 4 increase $\sum_{1 \leq i \leq \ell} |M_i \cap B|$ by at least $\frac{3}{4}$. Hence, we can upper-bound the depth of the search tree with $3k + d$. It is not difficult to check that the number of children of each node in the search tree is asymptotically upper-bounded by $3k + 3$. It follows that the size of the whole search tree is in $k^{O(k+d)}$.

The Greedy Rule and the Clean-up Rule play a special role. Note that if they are applicable, they do not create branches. Also, in both cases it is easy to check that an application of the rule cannot make an earlier rule applicable in the recursive call. Hence, these rules are essentially applied in a loop until they are not applicable any more. The Greedy Rule can be applied at most $\binom{\ell}{2} \in O(n^2)$ times consecutively and the Clean-up Rule can be applied at most $|E| \in O(n^2 \cdot \ell)$ times consecutively.

Lastly, we analyze for each rule, how much time is needed to check whether the rule is applicable and if so compute the constraints it outputs. It is not difficult to check that the algorithm needs $O(n^2 \cdot \ell)$ time to check whether Branching Rule 1 is applicable and output the constraints, same for the Greedy Rule. To check the applicability of Branching Rule 2, the algorithm needs to check whether there is a layer containing an induced $P_3$. This can be done in $O(n + m)$ time, where $m$ is the maximum number of edges in a layer. Hence, overall we need $O((n + m) \cdot \ell)$ time to check whether Branching Rule 2 is applicable and in this time we can also compute the output constraints. For the Clean-up Rule we need $O(n^2 \cdot \ell)$ time to check whether it is applicable and to output the new constraint. In the case of Branching Rule 3 we need $O(n^2 \cdot \ell)$ time to check whether it is applicable and to output the constraints. For the last rule, Branching Rule 4, we essentially need to solve Cluster Editing on each layer to check whether the rule is applicable. This can be done in $O((2^k + n + m) \cdot \ell)$ time. In the same time, we can also compute the constraints. Hence, overall, the algorithm has running time $k^{O(k+d)} \cdot O(n^2 \cdot \ell)$.

Remark. It is not difficult to see that Multi-Layer Cluster Editing can also be solved in $n^{O(n)} \cdot O(\ell)$ time, which is incomparable to the running time of Algorithm 1 since $k$ might be as large as $\Omega(n^2)$: First guess the marked vertices. Then guess how many clusters (i.e. disjoint cliques) there are in the modified graph induced by the non-marked vertices, and for every non-marked vertex, guess to which cluster it belongs. Now for every layer, independently! guess whether the layer contains an induced $P_3$ or not. This can be done in $O(n^2)$ time. In the same time, we can also compute the output constraints. Finally check, whether such a solution can be obtained by at most $k$ modifications per layer.

4 Kernelization of Multi-Layer Cluster Editing

In this section we investigate the kernelizability of Multi-Layer Cluster Editing for different combinations of the four parameters as introduced in Section 1. More specifically, we identify the parameter combinations for which Multi-Layer Cluster Editing admits a polynomial kernel, and then we identify the parameter combination for which no polynomial kernels exist, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$.

4.1 A Polynomial Kernel for Multi-Layer Cluster Editing

We start with presenting a polynomial kernel for the parameter combination $(k, d, \ell)$. Formally, we prove the following theorem.

**Theorem 2.** Multi-Layer Cluster Editing admits a polynomial kernel with respect to the parameter combination $(k, d, \ell)$. In particular, the problem admits a kernel of size $O(\ell^3 \cdot (k+d)^4)$ that can be computed in $O(\ell \cdot n^3)$ time.

We provide several reduction rules that subsequently modify the instance and we assume that if a particular rule is to be applied, then the instance is reduced with respect to all previous rules, that is, all previous rules were already exhaustively applied. For each rule we immediately prove its correctness, that is, the produced instance is a yes-instance if and only if the original instance is. However, we leave the analysis of the running time of testing whether particular reduction rule applies and of applying the rule until all the rules are presented.

To keep track of the budget in the individual layers we introduce the following intermediate problem.

---

4The proof is folklore and proceeds roughly as follows. Find the connected components of the input graph. Next, determine whether there are two nonadjacent vertices $u, v$ in a connected component. If so, then find an induced $P_3$ along a shortest path between $u$ and $v$. Otherwise, there is no induced $P_3$. Nonadjacent vertices in a connected component can be checked for in $O(\text{deg}(v))$ time summed over each vertex $v$ in that component.
Multi-Layer Cluster Editing with Separate Budgets

**Input:** \( \ell \) graphs \( G_1 = (V, E_1), \ldots, G_\ell = (V, E_\ell) \) and \( \ell + 1 \) integers \( k_1, \ldots, k_\ell, d \).

**Question:** Is there a vertex subset \( D \subseteq V \) with \( |D| \leq d \) and \( \ell \) edge modification sets \( M_1, M_2, \ldots, M_\ell \subseteq \binom{V}{2} \) such that

1. for each \( 1 \leq i \leq \ell \) we have that \( |M_i| \leq k_i \),
2. for each \( 1 \leq i \leq \ell \) the graph \( G'_i = (V, E_i \oplus M_i) \) is a cluster graph,
3. and for all \( 1 \leq i, j \leq \ell \) we have that \( G'_i[V \setminus D] = G'_j[V \setminus D] \)?

We first transform the input instance of Multi-Layer Cluster Editing to an equivalent instance of Multi-Layer Cluster Editing with Separate Budgets by letting \( k_i = k \) for every \( i \in [\ell] \). Then we apply all our reduction rules to Multi-Layer Cluster Editing with Separate Budgets.

Finally, we show how to transform the resulting instance of Multi-Layer Cluster Editing with Separate Budgets to an equivalent instance of Multi-Layer Cluster Editing with just a small increase of the vertex set.

Through the presentation, let \( (G_1 = (V, E_1), \ldots, G_\ell = (V, E_\ell), k_1, \ldots, k_\ell, d) \) be the current instance. We let \( k = \max\{k_i \mid i \in [\ell]\} \).

The first rule formalizes the obvious constraint on the solvability of the instance. We omit a proof of correctness for this rule.

**Reduction Rule 1.** If there is a layer \( i \in [\ell] \) such that \( k_i < 0 \), then answer NO.

**Observation 3.** Reducet Rule 1 is correct.

The next two rules represent well known rules for classical Cluster Editing applied to the individual layers of the multi-layer graph.

**Reduction Rule 2.** If there is a layer \( i \in [\ell] \) and an edge \( \{u, v\} \in E_i \) in layer \( i \) such that \( G_i \) contains at least \( k_i + 1 \) different induced \( P_3 \)s each of which contains the edge \( \{u, v\} \), then remove \( \{u, v\} \) from \( E_i \) and decrease \( k_i \) by one.

**Lemma 11.** Reducet Rule 2 is correct.

**Proof.** Let \( I = (G_1, \ldots, G_\ell, k_1, \ldots, k_\ell, d) \) be the original instance and \( \tilde{I} = (G_1, \ldots, \tilde{G}_i, \ldots, G_\ell, k_1, \ldots, \tilde{k}_i, \ldots, k_\ell, d) \) be the instance after the application of the rule, where \( G_i \equiv (V, E_i), \tilde{G}_i \equiv (V, \tilde{E}_i), \tilde{E}_i = E_i \setminus \{\{u, v\}\} \) and \( \tilde{k}_i = k_i - 1 \). If \( \tilde{I} \) is a yes-instance, then \( I \) is also a yes-instance with the same solution as the one for \( \tilde{I} \).

For the converse, assume that \( S = (D, M_1, \ldots, M_\ell) \) is a solution for \( I \) and let \( G'_i = (V, E_i \oplus M_i) \). We claim that \( S \) is also a solution for \( \tilde{I} \). Since the input multi-layer graphs in \( I \) and \( \tilde{I} \) only differ by one edge \( \{u, v\} \), suppose towards a contradiction that \( G'_i \) still contains \( \{u, v\} \), meaning that \( \{u, v\} \in M_i \).

By the assumptions of the rule we know that there are \( k_i + 1 \) vertices \( w_1, \ldots, w_{k_i+1} \) such that for each \( i \in [k_i+1] \) the induced subgraph \( G_i[\{u, v, w_j\}] \) is a \( P_3 \), which has to be destroy to obtain a cluster graph. Since \( \{u, v\} \in E_i \cap M_i \), in order to destroy all \( P_3 \)s, for each \( j \in [k_i+1] \) we have to either add the absent edge to or delete an existing edge \( e \) (with \( e \neq \{u, v\} \)) to the induced subgraph \( G[\{u, v, w_j\}] \). However, since for two different layers \( j_1, j_2 \in [k_i+1] \) the pair \( \{u, v\} \) is the only pair of vertices shared between \( \{u, v, w_{j_1}\} \) and \( \{u, v, w_{j_2}\} \), we have to modify at least \( k_i + 1 \) edges, a contradiction to \( |M_i| \leq k \). Hence, \( G'_i \) does not contain \( \{u, v\} \) and \( S \) is also a solution to \( \tilde{I} \).

**Reduction Rule 3.** If there is a layer \( i \in [\ell] \) and a pair \( \{u, v\} \in V \) of vertices with \( \{u, v\} \notin E_i \) (a non-edge) in layer \( i \) such that \( G_i \) contains at least \( k_i + 1 \) different induced \( P_3 \)s each of which involves both \( u \) and \( v \), then add \( \{u, v\} \) to \( E_i \) and decrease \( k_i \) by one.

**Lemma 12.** Reducet Rule 3 is correct.

**Proof.** The proof is almost the same as for Lemma 11, the obvious difference is that we assume \( \tilde{E}_i = E_i \cup \{\{u, v\}\} \). Also in the second implication, supposing that \( M_i \) does not contain \( \{u, v\} \) leads to a contradiction.

As with the classical Cluster Editing we can bound the number of vertices involved in a \( P_3 \) in each layer. Let \( R_i \subseteq V \) be the set of the vertices \( v \) that appear in some induced \( P_3 \) in \( G_i \), and let \( R = \bigcup_{i=1}^\ell R_i \).
Reduction Rule 4. If there is a layer \( i \in [\ell] \) such that \( |R_i| > k_i^2 + 2k_i \), then answer NO.

Lemma 13. **Reduction Rule 4** is correct.

Proof. Suppose towards a contradiction that \( |R_i| > k_i^2 + 2k_i \) and \( I = (G_1, \ldots, G_\ell, k_1, \ldots, k_\ell, d) \) is a yes-instance. Let \((D, M_1, \ldots, M_\ell)\) be a solution to \( I \) and define \( G'_i = (V, E(G_i) \cup M_i) \). For each modified edge \( \{u, v\} \in M_i \) denote by \( R_{uv} \), the set of vertices \( w \) such that the induced subgraph \( G_i[\{u, v, w\}] \) is a \( P_3 \). Since the instance is reduced with respect to Reduction Rules 2 and 3, for each modified edge \( \{u, v\} \in M_i \) we have \( |R_{uv}| \leq k_i \). Since \( G'_i \) is a cluster graph and, thus, does not contain \( P_3 \) as an induced subgraph, we know that \( R_i \subseteq \bigcup_{\{u, v\} \in M_i} (\{u, v\} \cup R_{uv}) \). It follows that \( |R_i| \leq k_i(2 + k_i) = k_i^2 + 2k_i \) — a contradiction. 

As a major difference to Cluster Editing for a single layer, we cannot simply remove the vertices that are not involved in any \( P_3 \) since we require the cluster graphs in individual layers not to differ too much. Let us first make a folklore observation.

Observation 4. If a connected component \( C \) of a graph has at least three vertices and is not complete, then every vertex of \( C \) appears in some induced \( P_3 \).

Proof. Consider an arbitrary vertex \( u \in V(C) \). If \( u \) is adjacent to \( v \) for every \( v \in V(C) \setminus \{u\} \), then there must be some pair \( \{x, y\} \subseteq V(C) \setminus \{u\} \) of vertices such that \( \{x, y\} \notin E(C) \) since the component is not complete. Then, \( C[\{u, x, y\}] \) is a \( P_3 \).

Otherwise \( u \) is not adjacent to some vertex \( v \in V(C) \setminus \{u\} \). Then let \( P \) be the shortest path between \( u \) and \( v \). This path has at least three vertices and each three consecutive vertices of this path induce a subgraph which is a \( P_3 \). 

Next we show that the vertices in the clusters that do not change can be freely removed.

Reduction Rule 5. If there is a subset \( A \subseteq V \setminus R \) such that for each layer \( i \in [\ell] \), the subset \( A \) is the vertex set of a connected component of \( G_i \), then remove \( A \) (and the corresponding edges) from every \( G_i \).

Lemma 14. **Reduction Rule 5** is correct.

Proof. Let \( I = (G_1, \ldots, G_\ell, k_1, \ldots, k_\ell, d) \) be the original instance and \( \tilde{I} = (\tilde{G}_1, \ldots, \tilde{G}_\ell, k_1, \ldots, k_\ell, d) \) be the instance after the application of the rule, where for each \( i \in [\ell] \), we have \( \tilde{G}_i = G_i[V(G_i) \setminus A] \). Since \( A \cap R = \emptyset \) we have that \( A \) induces a complete subgraph in each layer \( i \in [\ell] \). Moreover, for each layer \( i \in [\ell] \), the set \( A \) is the vertex set of a connected component of graph \( G_i \). Thus, \( G_i[A] \) is a complete connected component in \( G_i \), meaning that \( E(G_i) \setminus A = E(\tilde{G}_i) \cup (A \choose 2) \).

Let \( (\tilde{D}, \tilde{M}_1, \ldots, \tilde{M}_\ell) \) be a solution to \( \tilde{I} \) and let \( \tilde{D} = D \cup A \) and \( \tilde{M}_i = M_i \cap (V \setminus A) \) for every \( i \in [\ell] \). Then \( (\tilde{D}, \tilde{M}_1, \ldots, \tilde{M}_\ell) \) forms a solution to \( I \).

Conversely, let \( \tilde{S} = (\tilde{D}, \tilde{M}_1, \ldots, \tilde{M}_\ell) \) be a solution to \( \tilde{I} \). We claim that \( \tilde{S} \) is also a solution to \( I \). Indeed, each \( G'_i = (V, E(G_i) \cup \tilde{M}_i) \) is a cluster graph (note that \( A \) is the vertex set of a complete connected component in \( G_i \)), \( |\tilde{M}_i| \leq k_i \) and for all \( 1 \leq i, j \leq \ell \) we have that \( E(G_i) \cup \tilde{M}_i \cap (\tilde{V} \choose 2) = E(G_j) \cup \tilde{M}_j \cap (\tilde{V} \choose 2) \) since \( E(\tilde{G}_i) \cup \tilde{M}_i \cap (\tilde{V} \choose 2) = E(\tilde{G}_j) \cup \tilde{M}_j \cap (\tilde{V} \choose 2) \).

The next rule allows us to reduce vertices that appear in exactly the same clusters, if there are many.

Reduction Rule 6. If there is a set \( A \subseteq V \setminus R \) with \( |A| \geq k + d + 3 \) such that for every layer \( i \in [\ell] \) it holds that all vertices of \( A \) are in the same connected component of \( G_i \), then select an arbitrary \( v \in A \) and remove \( v \) from every \( G_i \).

Lemma 15. **Reduction Rule 6** is correct.

For the proof of this and subsequent lemmata we find the following observation handy.

Observation 5. Let \( G \) be a complete graph on at least \( k + 2 \) vertices and let \( H \) be a cluster graph such that \( V(G) = V(H) \). If \( H \) is not complete, then \( G \) and \( H \) differ in at least \( k + 1 \) edges, i.e., \( |E(G) \cup E(H)| \geq k + 1 \).
Proof. The statement is obviously true for \( k \leq 0 \), let us assume that \( k > 0 \). Since \( H \) is a cluster graph which is not complete, it must have several connected components. Let \( X \) be the smallest of these connected components and \( Y = V(G) \setminus X \). The set \( \Delta(E(G), E(H)) \) must contain at least all the edges between \( X \) and \( Y \), hence \( |\Delta(E(G), E(H))| \geq |X| \cdot |Y| \).

If \( |X| \geq \frac{k+2}{4} \), then also \( |Y| \geq \frac{k+2}{4} \) since \( X \) is the smallest component. But then \( |X| \cdot |Y| \geq \left( \frac{k+2}{4} \right)^2 = \frac{k^2 + 4k + 4}{16} \geq k + 1 \).

If \( |X| < \frac{k+2}{4} \), then let us denote \( x = |X| \) and we have \( |Y| \geq k + 2 - x \). We know that \( |X| \cdot |Y| \geq x \cdot (k + 2 - x) \). The function \( f(x) = x \cdot (k + 2 - x) \) is increasing for \( x < \frac{k+2}{2} \) with \( f(1) = k + 1 \), hence \( |X| \cdot |Y| \geq k + 1 \), finishing the proof.

\( \square \)

Proof of **Lemma 15.** Let \( I = (G_1, \ldots, G_t, k_1, \ldots, k_t, d) \) be the original instance and \( \bar{I} = (\bar{G}_1, \ldots, \bar{G}_t, k_1, \ldots, k_t, d) \), where \( \bar{G}_j = G_j[V(G_j) \setminus \{v\}] \) be the instance after the application of the rule.

Let \( (D, M_1, \ldots, M_t) \) be a solution to \( I \), and for every \( i \in [t] \) define \( \bar{D} = D \setminus \{v\} \) and \( \bar{M}_i = M_i \cap (V \setminus \{v\}) \).

Then \( (\bar{D}, \bar{M}_1, \ldots, \bar{M}_t) \) forms a solution to \( \bar{I} \).

Conversely, let \( \bar{S} = (\bar{D}, \bar{M}_1, \ldots, \bar{M}_t) \) be a solution to \( \bar{I} \). Let \( w \) be an arbitrary vertex of \( A \setminus (\bar{D} \cup \{v\}) \) (note that since \( |A| \geq k + d + 3, |\bar{D}| \leq d \) and \( k \geq 0 \), the set \( A \setminus (\bar{D} \cup \{v\}) \) is not empty). We will construct a solution for \( I \) such that after applying the solution \( v \) is a true twin of \( w \) in every layer, i.e., we will put \( v \) into the same clusters as \( w \). Formally, for each layer \( i \in [t] \), we define \( \bar{E}_i' = E(\bar{G}_i) \cup \bar{M}_i \), \( E_i' = \bar{E}_i' \cup \{\{v, x\} \mid \{x, w\} \in \bar{E}_i'\} \cup \{\{v, w\}\} \) and \( M_i = E_i \oplus E_i' \). We claim, that \( (\bar{D}, \bar{M}_1, \ldots, \bar{M}_t) \) is a solution to \( I \).

First, each \( G_i' = (V, E_i') \) is a cluster graph. If there are two layers \( i, j \in \{1, \ldots, t\} \) such that \( G_i' \cap D \neq G_j' \setminus D \), then without loss of generality we can assume that there is some \( x \in V \setminus (D \cup \{v\}) \) such that \( \{v, x\} \notin E_i' \) but \( \{v, x\} \notin E_j' \). But then \( \{w, x\} \in \bar{E}_i' \) and \( \{w, x\} \notin \bar{E}_j' \) — a contradiction, since neither \( w \) nor \( x \) is in \( \bar{D} \).

Finally, let us show that for each layer \( i \in [t] \) we have that \( |M_i| \leq |\bar{M}_i| \leq k_i \). To this end, we first observe that all vertices of \( A \setminus \{v\} \) are in the same component of \( (V \setminus \{v\}, \bar{E}_i') \). Since \( A \cap R = \emptyset \) and all vertices of \( A \) are in the same connected component of \( \bar{G}_i \) we have that \( \bar{G}_i[A \setminus \{v\}] \). Hence, if \( A \setminus \{v\} \) does not induce a complete subgraph in \( (V \setminus \{v\}, \bar{E}_i') \), then by Observation 5 we can conclude that \( M_i \) contains at least \( k + 1 \) edges, as \( |A \setminus \{v\}| \geq k + d + 2 \geq k + 2 \)

Now for every \( x \in V \setminus A \) and every \( i \in [t] \) we have that \( \{v, x\} \notin E_i' \) if and only if \( \{u, x\} \notin E_i' \) for every \( u \in A \) as otherwise the induced subgraph \( G_i[\{v, u, x\}] \) would be a \( P_3 \), contradicting \( A \cap R = \emptyset \). Similarly, for every \( x \in V \setminus A \) and every \( i \in [t] \) we have that \( \{v, x\} \notin E_i' \) if and only if \( \{u, x\} \notin E_i' \) for every \( u \in A \), since \( \{v, x\} \notin E_i' \) is a cluster graph and \( E_i' \) is constructed in this way. It follows that if \( \{x, v\} \in M_i \) for some \( x \in V \setminus A \), then \( \{x, u\} \in \bar{M}_i \) for every \( u \in A \setminus \{v\} \) and \( |\bar{M}_i| \geq k + 1 \geq k_i + 1 \) — a contradiction. Hence \( \{x, v\} \notin M_i \) for every \( x \in V \setminus \{v\} \) and thus \( M_i \subseteq \bar{M}_i \).

\( \square \)

The next rule shows that the remaining clusters in a yes-instance cannot be too large.

**Reduction Rule 7.** If there is a layer \( i \in [t] \) and a connected component \( A \) of \( G_i \) with \( |A| \cap R | \geq k + 2d + 3 \), then answer \( \text{NO} \).

**Lemma 16.** **Reduction Rule 7** is correct.

**Proof.** Suppose towards a contradiction that \( A \) is a connected component of \( G_i \) for some layer \( i \in [t] \) with \( |A| \cap R | \geq k + 2d + 3 \), and \( G_1, \ldots, G_t, k_1, \ldots, k_t, d \) is a yes-instance. Let \( (D, M_1, \ldots, M_t) \) be a solution to the instance. For every layer \( j \in [t] \), define \( E_j' = E(\bar{G}_j) \cup M_j \) and \( G_j' = (V, E_j') \). Let \( A' = A \setminus (D \cup R) \) and note that \( |A'| \geq k + d + 3 \). We claim that for every \( i' \in [t] \), all vertices of \( A' \) in the same connected component of \( G_{i'} \), contradicting the instance being reduced with respect to **Reduction Rule 6**.

Since \( A' \cap R = \emptyset \) and all vertices of \( A \) are in the same connected component of \( G_i \), by Observation 4 we have that \( G_i[A'] \) is complete. Hence, if \( G_i[A'] \) is not complete, then by Observation 5 \( M_i \) contains at least \( k + 1 \geq k_i + 1 \) edges, as \( |A'| \geq k + 2 \) — a contradiction. Therefore \( G_i[A'] \) is complete. For every \( j \in [t] \), since \( G_j[V \setminus D] = G_j[V \setminus D], \) the graph \( G_j[A'] \) is complete. Now again, if \( G_j[A'] \) is not complete for some layer \( j \in [t] \), then again by Observation 5 \( M_j \) contains at least \( k + 1 \geq k_j + 1 \) edges — a contradiction.
Now we are ready to introduce our final rule bounding the number of vertices in the instance.

**Reduction Rule 8.** If $|V| > \ell \cdot (k^2 + 2k + d \cdot (k + 2d + 2) + 2k)$, then answer NO.

**Lemma 17.** Reduction Rule 8 is correct.

**Proof.** Suppose towards a contradiction that $|V| > \ell \cdot (k^2 + 2k + d \cdot (k + 2d + 2) + 2k)$ and $(G_1, \ldots, G_\ell, k_1, \ldots, k_\ell, d)$ is a yes-instance. Let $(D, M_1, \ldots, M_\ell)$ be a solution to the instance. For each $i \in [\ell]$, define $E_i' = E_i \cup M_i$ and $G_i' = (V_i, E_i')$. Let us denote by $S = \bigcup_{i=1}^\ell \bigcup_{(u,v) \in M_i} \{(u,v)\}$ the set of vertices adjacent to any modification. Obviously, $|S| \leq \ell \cdot 2k$.

For every layer $i \in [\ell]$ and every $x \in D$ let us denote by $Q'_x \subseteq V_x \setminus R$ the set of vertices from $V_x \setminus R$ in the same connected component of $G_i'$ as the vertex $x$ and $Q = \bigcup_{i=1}^\ell \bigcup_{x \in D} Q'_x$. Since the instance is reduced with respect to Reduction Rule 7, we know that $|Q'_x| \leq k + 2d + 2$ and thus $|Q| \leq \ell \cdot (k + 2d + 2)$.

Note also that $|R| \leq \ell \cdot (k^2 + 2k)$, since the instance is reduced with respect to Reduction Rule 3.

Now since $|V| > \ell \cdot (k^2 + 2k + d \cdot (k + 2d + 2) + 2k)$, $|R| \leq \ell \cdot (k^2 + 2k)$, $|Q| \leq \ell \cdot (k + 2d + 2)$, and $|S| \leq \ell \cdot 2k$, the set $V' = V \setminus (Q \cup R \cup S)$ is not empty. Let $u$ be an arbitrary vertex from $V'$. Since the instance is reduced with respect to Reduction Rule 5, we know that there are two distinct layers $i, j \in [\ell]$ and a vertex $v$ such that $u$ and $v$ are in the same connected component of $G_i$ and in different connected components of $G_j$. Since $v$ is not in $S$, we know that the same holds for the graphs $G_i'$ and $G_j'$. However, since $v$ is not in $Q$ neither in $R$, we have that neither $u$ nor $v$ is in $D$. But then $G_i'[V \setminus D]$ and $G_j'[V \setminus D]$ are different — a contradiction. 

**Reduction Rule 5** effectively bounds the size of the reduced instance to polynomial in $k$, $d$, and $\ell$.

It remains to transform the resulting instance of Multi-Layer Cluster Editing with Separate Budgets to an equivalent instance of Multi-Layer Cluster Editing. To this end we introduce new vertex set $A$ of size exactly $2k + 2$ to $V$ and to each $E_i$ introduce all edges from $\binom{A}{2}$. Then, for each $i \in \{1, \ldots, \ell\}$ we remove $k - k_i$ arbitrary edges between vertices of $A$ from $E_i$ and set $k_i = k$.

If $\{u, v\}$ is an edge removed in this step, then $u$ and $v$ had $2k$ common neighbors in $A$ and by at most $k - 1$ other edge removals they could loose at most $k - 1$ of them. Hence, Reduction Rule 3 would apply to each pair of vertices from $A$ with an edge removed. Applying Reduction Rule 3 exhaustively and then Reduction Rule 5 would revert all the changes made. Hence, the constructed instance is equivalent to the one obtained after exhaustive application of all the reduction rules.

The constructed instance can be turned into an equivalent instance of Multi-Layer Cluster Editing in an obvious way.

Since no rule increases $k$, $d$, or $\ell$, $|V| = O(\ell \cdot (k + d)^2)$, the resulting instance can be described using $O(\ell^3 \cdot (k + d)^4)$ bits and it is equivalent to the original instance, it remains to show that the kernelization can be performed in polynomial time. Then Theorem 2 follows.

**Lemma 18.** The kernelization can be done in $O(\ell \cdot n^3)$ time.

**Proof.** If $n < k^2$, then we can output the original instance as the kernel. Let us assume that $k^2 \leq n$.

We can check whether Reduction Rule 1 applies in $O(\ell)$ time on the beginning and in constant time whenever any later rule changes the budget. Applying the rule takes constant time.

For each layer $i \in \{1, \ldots, \ell\}$ in time $O(n^3)$ we can compute for each pair of vertices in how many induced subgraphs isomorphic to $P_3$ it appears and classify the pairs according to that count. Then we apply Reduction Rules 2 and 3 to the pairs which appear in many $P_3$’s. Each application takes $O(n)$ time and at the same time we can update the counts for affected pairs. Hence, these reduction rules can be exhaustively applied to one layer in $O(n^3)$ time. Also in the same time we can determine the sets $R_i$ and eventually apply Reduction Rule 3. Since the later rules only delete vertices or answer NO, no application of a later rule can create an opportunity to apply Reduction Rule 2 or 3. Hence, these reduction rules can be exhaustively applied to the instance in $O(\ell \cdot n^3)$ time.

In $O(\ell \cdot n^2)$ time we can compute the graphs $G_i = (V, \bigcup_{j=1}^{i-1} E_j)$ and $G_\cup = (V, \bigcup_{j=1}^{\ell} E_j)$. Then Reduction Rule 5 applies to all connected components of $G_\cup$ not containing vertices of $R$ that are also connected components of $G_i$. All of these applications can be recognized in $O(n^2)$ time and all of them together applied in $O(\ell \cdot n^2)$ time. No application of a later rule can create an opportunity to apply Reduction Rule 6.

Reduction Rule 6 applies to each connected component of $G_i$, which has the appropriate number of vertices not in $R$. All of these applications can be recognized in $O(n^2)$ time and all of them together
applied in $O(\ell \cdot n^2)$ time. Since later rules only answer NO, no application of a later rule can create an opportunity to apply \textbf{Reduction Rule 6}.

We can check whether the rule applies in $O(\ell \cdot n^2)$ for \textbf{Reduction Rule 7} and in constant time for \textbf{Reduction Rule 8} and apply any of them in constant time.

Hence the reduction rules can be exhaustively applied in $O(\ell \cdot n^3)$ time, the final reduction back to \textsc{Multi-Layer Cluster Editing} takes $O(k^2) = O(n)$ time and the result follows. \hfill \Box

### 4.2 A Kernel Lower Bound for Multi-Layer Cluster Editing

In the remainder of this section we argue that for all parameter combinations that are smaller or incomparable to $(k, d, \ell)$, \textsc{Multi-Layer Cluster Editing} does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP/poly}$. More specifically, we show the following theorem. Note that \textsc{Multi-Layer Cluster Editing} is para-$\text{NP}$-hard for all parameter combinations that are incomparable to $n$ and smaller than $(k, d, \ell)$.

**Theorem 3.** \textsc{Multi-Layer Cluster Editing} does not admit a polynomial kernel with respect to the number $n$ of vertices, unless $\text{NP} \subseteq \text{coNP/poly}$.

Notice that, if the clustering is allowed to change completely in each layer, that is, $d = n$, then our task reduces to edit each layer into a cluster graph with $k$ modifications each. That is, each layer represents its own, independent instance of \textsc{Cluster Editing}. This naturally yields an AND-cross-composition \cite{8, 10} from \textsc{Cluster Editing}, proving \textbf{Theorem 3}.

We need the following notation for the proof. An equivalence relation $R$ on the instances of some problem $L$ is a polynomial equivalence relation if

(i) one can decide for each two instances in time polynomial in their sizes whether they belong to the same equivalence class, and

(ii) for each finite set $S$ of instances, $R$ partitions the set into at most $(\max_{x \in S} |x|)^{O(1)}$ equivalence classes.

An AND-cross-composition of a problem $L \subseteq \Sigma^*$ into a parameterized problem $P$ (with respect to a polynomial equivalence relation $R$ on the instances of $L$) is an algorithm that takes $\ell$ $R$-equivalent instances $x_1, \ldots, x_\ell$ of $L$ and constructs in time polynomial in $\sum_{i=1}^\ell |x_i|$ an instance $(x, k)$ of $P$ such that

(i) $k$ is polynomially upper-bounded in $\max_{1 \leq i \leq \ell} |x_i| + \log(\ell)$ and

(ii) $(x, k) \in P$ if and only if $x_i \in L$ for every $i \in \{1, \ldots, \ell\}$.

If an $\text{NP}$-hard problem $L$ AND-cross-composes into a parameterized problem $P$, then $P$ does not admit a polynomial-size kernel, unless $\text{NP} \subseteq \text{coNP/poly}$ \cite{8, 10}, which would cause a collapse of the polynomial-time hierarchy to the third level.

**Proof of Theorem 3** We provide an AND-cross-composition from classical \textsc{Cluster Editing}. We define relation $R$: Two instances $(G_1, k_1)$ and $(G_2, k_2)$ are equivalent under $R$ if and only if $k_1 = k_2$ and $|V(G_1)| = |V(G_2)|$. Clearly, $R$ is a polynomial equivalence relation.

Now let $(G_1, k_1), \ldots, (G_\ell, k_\ell)$ be $R$-equivalent instances of \textsc{Cluster Editing}. Then there is an integer $k \in \mathbb{N}$ such that $k = k_i$ for every $i \in \{1, \ldots, \ell\}$. Moreover, since the names of the vertices are not important for the problem and $|V(G_i)| = |V(G_j)|$ for every $i, j \in \{1, \ldots, \ell\}$, we can assume without loss of generality that there is a set $V$ such that $V = V(G_i)$ for every $i \in \{1, \ldots, \ell\}$. Hence, $(G_1, \ldots, G_\ell, k, d)$, where $d = |V|$, is a valid instance of \textsc{Multi-Layer Cluster Editing}.

This instance can be constructed in polynomial time and no extra vertices are added, hence $|V|$ is upper-bounded by a maximum size of an input instance. Furthermore, as we are allowed to mark all vertices, it follows directly from the definition of \textsc{Multi-Layer Cluster Editing}, that $(G_1, \ldots, G_\ell, k, d)$ is a yes-instance if and only if for every $i \in \{1, \ldots, \ell\}$ it is possible to turn $G_i$ into a cluster graph by at most $k$ edge modifications.

Since \textsc{Cluster Editing} is $\text{NP}$-hard \cite{11} and we AND-cross-composed it into \textsc{Multi-Layer Cluster Editing} parameterized by $n = |V|$, the result follows. \hfill \Box

### 5 Outlook

We have fully explored the parameterized complexity of \textsc{Multi-Layer Cluster Editing} with respect to the natural parameters “max. number $d$ of marked vertices”, “max. number $k$ of edge modifications
per layer”, “number $n$ of vertices”, and “number $\ell$ of layers”. However, we believe that both the running time of the FPT-algorithm and the size of the polynomial kernel leave room for improvement. Also, lower bounds on running time and kernel size for this problem would further help to understand its computational complexity.

There are also a number of natural generalizations for the problem that are worth considering. It might be especially interesting put further constraints on the marking of vertices, like giving marked vertices a weight corresponding to the number of different clusters they are part of in different layers. In future work, we also plan to consider temporal versions of this problem, where there is a linear order given over the layers and each layer is interpreted as a time step.

References

[1] Nikhil Bansal, Avrim Blum, and Shuchi Chawla. Correlation clustering. *Machine Learning*, 56:89–113, 2004. 1 3 16

[2] Matteo Barigozzi, Giorgio Fagiolo, and Giuseppe Mangioni. Identifying the community structure of the international-trade multi-network. *Physica A: Statistical Mechanics and its Applications*, 390(11):2051–2066, 2011. 2

[3] Amir Ben-Dor, Ron Shamir, and Zohar Yakhini. Clustering gene expression patterns. *Journal of Computational Biology*, 6(3-4):281–297, 1999. 1

[4] Michele Berlingerio, Michele Coscia, and Fosca Giannotti. Finding redundant and complementary communities in multidimensional networks. In *Proceedings of the 20th ACM International Conference on Information and Knowledge Management (CIKM 2011)*, pages 2181–2184. ACM, 2011. 2

[5] S. Boccaletti, G. Bianconi, R. Criado, C. I. del Genio, J. Gómez-Gardeñes, M. Romance, I. Sendiña-Nadal, Z. Wang, and M. Zanin. The structure and dynamics of multilayer networks. *Physics Reports*, 544(1):1–122, 2014. 2

[6] Sebastian Böcker and Jan Baumbach. Cluster Editing. In *Proceedings of the 9th Conference on Computability in Europe (CiE 2013)*, Lecture Notes in Computer Science, pages 33–44. Springer, 2013. 2 14

[7] Sebastian Böcker, Sebastian Briesemeister, and Gunnar W. Klau. Exact algorithms for Cluster Editing: Evaluation and experiments. *Algorithmica*, 60(2):316–334, 2011. 2

[8] Hans L Bodlaender, Bart MP Jansen, and Stefan Kratsch. Kernelization lower bounds by cross-composition. *SIAM Journal on Discrete Mathematics*, 28(1):277–305, 2014. 10

[9] Robert Bredereck, Christian Komusiewicz, Stefan Kratsch, Hendrik Molter, Rolf Niedermeier, and Manuel Sorge. Assessing the computational complexity of multi-layer subgraph detection. In *Proceedings of the 10th International Conference on Algorithms and Complexity (CIAC 2017)*, pages 128–139. Springer, 2017. 2

[10] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015. 3 15

[11] Reinhard Diestel. *Graph Theory*. Springer, 5th edition, 2016. 3

[12] X. Dong, P. Frossard, P. Vandergheynst, and N. Nefedov. Clustering With Multi-Layer Graphs: A Spectral Perspective. *IEEE Transactions on Signal Processing*, 60(11):5820–5831, 2012. 2

[13] Michael R. Garey and David S. Johnson. *Computers and Intractability—A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, 1979. 3

[14] Jungeun Kim and Jae-Gil Lee. Community detection in multi-layer graphs: A survey. *SIGMOD Rec.*, 44(3):37–48, 2015. 2
[15] Jungeun Kim, Jae-Gil Lee, and Sungsu Lim. Differential Flattening: A Novel Framework for Community Detection in Multi-Layer Graphs. *ACM Transactions on Intelligent Systems and Technology*, 8(2):27:1–27:23, 2016.

[16] Mikko Kivelä, Alex Arenas, Marc Barthelemy, James P. Gleeson, Yamir Moreno, and Mason A. Porter. Multilayer networks. *J Complex Netw.*, 2(3):203–271, 2014.

[17] Jayadev Misra and David Gries. A constructive proof of Vizing’s theorem. *Information Processing Letters*, 41(3):131–133, 1992.

[18] Andrea Tagarelli, Alessia Amelio, and Francesco Gullo. Ensemble-based community detection in multilayer networks. *Data Mining and Knowledge Discovery*, 31(5):1506–1543, 2017.

[19] W. Tang, Z. Lu, and I. S. Dhillon. Clustering with Multiple Graphs. In *Proceedings of the Ninth IEEE International Conference on Data Mining (ICDM ’09)*, pages 1016–1021, 2009.

[20] C. Tantipathananandh and T. Y. Berger-Wolf. Finding Communities in Dynamic Social Networks. In *Proceedings of the IEEE 11th International Conference on Data Mining (ICDM ’11)*, pages 1236–1241, 2011.

[21] Vadim G Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz.*, 3:25–30, 1964.