A CONSTRUCTION OF ADMISSIBLE $A^{(1)}_1$–MODULES OF LEVEL $-\frac{4}{3}$.

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Abstract. By using generalized vertex algebras associated to rational lattices, we construct explicitly the admissible modules for the affine Lie algebra $A^{(1)}_1$ of level $-\frac{4}{3}$. As an application, we show that the $W(2,5)$ algebra with central charge $c = -7$ investigated in [A2] is a subalgebra of the simple vertex operator algebra $L(-\frac{4}{3} \Lambda_0)$.

1. Introduction

In this paper we shall present an explicit construction of irreducible representations of the vertex operator algebra $L(-\frac{4}{3} \Lambda_0)$ associated to the irreducible vacuum representation for the affine Lie algebra $A^{(1)}_1$ of level $-\frac{4}{3}$. Explicit realizations of certain irreducible highest weight representations play important role in the representation theory of affine Lie algebras and the associated vertex operator algebras. Integrable highest weight modules can be realized by using the theory of lattice vertex algebras or Clifford vertex superalgebras (cf. [DL], [F], [Fe], [FFR], [K2]). Admissible modules are a broad class of irreducible highest weight modules which contains integrable modules as a subclass. Admissible modules have rational levels and their characters are modular functions (cf. [KW], [Wak]). It was proved in [AM] and [DLM] that admissible $A^{(1)}_1$–modules of level $k$ can be viewed as modules for the simple vertex operator algebra $L(k\Lambda_0)$. Moreover, every $L(k\Lambda_0)$–module from the category $\mathcal{O}$ is a direct sum of admissible modules.

An explicit construction of admissible modules for the affine Lie algebra $C^{(1)}_n$ of level $-\frac{1}{2}$ was given in [FF] by using Weyl algebras. This construction was extended in [We] to the framework of vertex operator algebras. In the case $n = 1$ this construction gives the realization of four admissible $A^{(1)}_1$–modules. It is important to notice that the vertex algebras associated to Weyl algebras can be realized by using lattice vertex algebras (cf. [FMS], [Fr], [A1]).

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In the physics literature the admissible representations are also well-known. In particular, the Weyl vertex algebras correspond to the $\beta\gamma$–system. Quite recently, the work [LMRS] presented a very detailed study of the $\beta\gamma$–systems and its relation to the WZW–model $\widehat{su}(2)_{-\frac{1}{2}}$. Their construction was based on a $c = -2$ theory and a Lorenzian boson. M. Gaberdiel in [G] presented an interesting study of the WZW–model $\widehat{su}(2)_{-\frac{4}{3}}$ and analyzed the fusion products of admissible representations of level $-\frac{4}{3}$. He also showed that this theory contains logarithmic representations, i.e., the representations on which the action of $L(0)$ is not diagonalisable. His approach was based on the analysis of singular vectors and the concept of fusion product.

In the present paper we shall give an explicit construction of admissible representations of level $-\frac{4}{3}$. First we shall construct the simple vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$. Since the rank of the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$ is $-6$, it is natural to consider vertex algebras of rank $-7$ and rank one lattice vertex algebras. Since $c = -7$ is the central charge of the $(1, p)$ models for the Virasoro algebra (with $p = 3$), we can use the representation theory of the $(1, p)$ models. In [A2], we showed that the concept of generalized vertex algebras associated to rational lattices is useful for studying vertex algebras associated to the $(1, p)$ models. It turns out that this approach is useful for the present construction.

Let us now describe our construction in more details. We define the following lattices

$$\tilde{L} = \mathbb{Z}\gamma + \mathbb{Z}\delta, \quad \langle \gamma, \gamma \rangle = -\langle \delta, \delta \rangle = \frac{1}{6}, \quad \langle \gamma, \delta \rangle = 0,$$

$$L = \mathbb{Z}(\gamma + \delta) + \mathbb{Z}(\gamma - \delta).$$

Let $V_{\tilde{L}}$ and $V_L$ be the associated generalized vertex algebras. We consider the subalgebra $V_{2\gamma}$ as a module for the Virasoro algebra with central charge $-7$. As in [A2], we define the screening operator $Q = e_0^{-6\gamma}$. We prove, as one of our main results, that the subalgebra of the generalized vertex algebra $V_L$ generated by vectors

$$e = e^{3(\gamma - \delta)}, \quad h = 4\delta(-1), \quad f = -\frac{2}{9}Q e^{3(\gamma + \delta)},$$

is isomorphic to the simple vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$ (for details see Section [D]). Then we consider $V_L$ as a weak module for $L(-\frac{4}{3}\Lambda_0)$. Although $V_L$ is not completely reducible $L(-\frac{4}{3}\Lambda_0)$–module, it contains some irreducible submodules. Since every $L(-\frac{4}{3}\Lambda_0)$–module in the category $\mathcal{O}$ is completely reducible (cf. [AM]), every highest weight vector
in \( V_L \) generates an irreducible highest weight \( A_1^{(1)} \)–module. By using this fact, we identify admissible modules inside \( V_L \). We also identify infinitely many irreducible submodules of \( V_L \) which are not \( \mathbb{Z}_{\geq 0} \)–graded (cf. Theorem 4.3).

The lattice construction enables us to investigate certain subalgebras of the vertex operator algebra \( L(-\frac{4}{3}\Lambda_0) \). By using structural results from [A2], we will show that the simple subalgebra \( \text{Ker}_{L(-\frac{4}{3}\Lambda_0)}h(0) \) of \( L(-\frac{4}{3}\Lambda_0) \) is isomorphic to the tensor product of the \( \mathcal{W}(2,5) \) algebra with central charge \(-7\) and the free boson vertex algebra \( M_{\delta}(1) \) (cf. Theorem 5.1). As a consequence, we get that the coset vertex algebra \( C(L(-\frac{4}{3}\Lambda_0), M_{\delta}(1)) \) (= the coset \( \frac{su(2)}{su(1)} \)) is isomorphic to the \( \mathcal{W}(2,5) \) algebra with central charge \(-7\).

2. **Vertex operator algebra \( L(k\Lambda_0) \)**

In this section we recall some basic facts about vertex operator algebras associated to affine Lie algebras (cf. [FZ], [Li1], [MP]).

Let \( \mathfrak{g} \) be a finite-dimensional simple Lie algebra over \( \mathbb{C} \) and let \( (\cdot, \cdot) \) be a nondegenerate symmetric bilinear form on \( \mathfrak{g} \). Let \( \mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+ \) be a triangular decomposition for \( \mathfrak{g} \). The affine Lie algebra \( \hat{\mathfrak{g}} \) associated with \( \mathfrak{g} \) is defined as \( \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \), where \( c \) is the canonical central element \([K1]\) and the Lie algebra structure is given by

\[
[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{n+m,0}c,
\]

\[
[d, x \otimes t^n] = nx \otimes t^n
\]

for \( x, y \in \mathfrak{g} \). We write \( x(n) \) for \( x \otimes t^n \) and identify \( \mathfrak{g} \) with \( \mathfrak{g} \otimes t^0 \). The Cartan subalgebra \( \mathfrak{h} \) and the subalgebras \( \mathfrak{g}_\pm, \mathfrak{h}_\pm \) of \( \hat{\mathfrak{g}} \) are defined as

\[
\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \hat{\mathfrak{h}}_\pm = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}], \quad \hat{n}_\pm = \mathfrak{n}_\pm \oplus \hat{\mathfrak{h}}_\pm.
\]

Let \( P = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}t \oplus \mathbb{C}d \) be upper parabolic subalgebra. For every \( k \in \mathbb{C}, k \neq -h^\vee \), let \( \mathcal{V}_k \) be 1–dimensional \( P \)–module such that the subalgebra \( \mathfrak{g} \otimes \mathbb{C}[t] + \mathbb{C}d \) acts trivially, and the central element \( c \) acts as multiplication with \( k \in \mathbb{C} \). Define the generalized Verma module \( N(k\Lambda_0) \) as

\[
N(k\Lambda_0) = U(\hat{\mathfrak{g}}) \otimes_{U(P)} \mathbb{C}v_k.
\]

Then \( N(k\Lambda_0) \) has a natural structure of a vertex operator algebra (VOA). The vacuum vector is \( 1 = 1 \otimes v_k \).

Let \( N^1(k\Lambda_0) \) be the maximal ideal in the vertex operator algebra \( N(k\Lambda_0) \). Then \( L(k\Lambda_0) = \frac{N(k\Lambda_0)}{N^1(k\Lambda_0)} \) is a simple vertex operator algebra.

Let \( M \) be any \( U(\hat{\mathfrak{g}}) \)–module. A weight vector \( v \in M \) is called a singular vector if \( \mathfrak{n}_+v = 0 \).
Let now $\mathfrak{g} = sl_2(\mathbb{C})$ with generators $e, f, h$ and relations $[h, e] = 2e, [h, f] = -2f, [e, f] = h$. Let $\Lambda_0, \Lambda_1$ be the fundamental weights for $\hat{\mathfrak{g}}$.

For $s \in \mathbb{Z}$, we define $H^s = -s^2$. Then $(H^s, h) = -s$. Define

$$\Delta_s(z) = z^{H^s(0)} \exp \left( \sum_{n=1}^{\infty} \frac{H^s(n)}{-n} (-z)^{-n} \right).$$

Applying the results obtained in [Li2] on $L(k\Lambda_0)$–modules we get the following proposition.

**Proposition 2.1.** Let $s \in \mathbb{Z}$ and let $\Delta_s(z)$ be defined as above. For any weak $L(k\Lambda_0)$–module $(M, Y_M(\cdot, z))$,

$$(\pi_s(M), Y_M^s(\cdot, z)) := (M, Y_M(\Delta_s(z)\cdot, z))$$

is a weak $L(k\Lambda_0)$–module. $\pi_s(M)$ is an irreducible weak $L(k\Lambda_0)$–module if and only if $M$ is an irreducible weak $L(k\Lambda_0)$–module.

By definition we have:

$$\Delta_s(z)e(-1)1 = z^{-s}e(-1)1,$$
$$\Delta_s(z)f(-1)1 = z^s f(-1)1,$$
$$\Delta_s(z)h(-1)1 = h(-1)1 - skz^{-1}1.$$

In other words, the corresponding automorphism $\pi_s$ of $U(\hat{\mathfrak{g}})$ satisfies the condition:

$$\pi_s(e(n)) = e(n-s), \quad \pi_s(f(n)) = f(n+s), \quad \pi_s(h(n)) = h(n) - sk\delta_{n,0}.$$ 

In the case $s = -1$ we get

$$\pi_{-1}(L(k\Lambda_0)) = L(k\Lambda_1).$$

It is also important to notice the following important property:

$$\pi_{s+t}(M) \cong \pi_s(\pi_t(M)), \quad (s, t \in \mathbb{Z}).$$

In particular,

$$M \cong \pi_0(M) \cong \pi_s(\pi_{-s}(M)).$$

**Remark 2.1.** Modules of type $\pi_s(M)$ have played an important role in the fusion rules analysis (cf. [G]).

3. **Vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$**

First we study the vertex operator algebra $N(-\frac{4}{3}\Lambda_0)$. Since level $-\frac{4}{3}$ is admissible, this vertex operator algebra contains a unique maximal ideal $N^1(-\frac{4}{3}\Lambda_0)$ which is generated by one singular vector $v_{\text{sing}}$ (cf.
This vector can be written in the PBW basis of \(N(-\frac{4}{3}\Lambda_0)\) as follows:

\[
v_{s\text{ing}} = f_{s\text{ing}} \mathbf{1}
\]

where \(f_{s\text{ing}} \in U(\hat{\mathfrak{g}})\) is defined by

\[
f_{s\text{ing}} = e(-1)\phi_{-\frac{4}{3}} + \frac{1}{3} e(-3) - \frac{1}{2} h(-1)e(-2) + \frac{1}{2} h(-2)e(-1),
\]

and

\[
\phi_{-\frac{4}{3}} = \frac{3}{4} (e(-1)f(-1) + f(-1)e(-1) + \frac{1}{2} h(-1)^2).
\]

Note that \(\omega_{\text{Sug}} = \phi_{-\frac{4}{3}} \mathbf{1}\) is the usual Virasoro element in the vertex operator algebra \(N(-\frac{4}{3}\Lambda_0)\) obtained by using the Sugawara construction (cf. [K2], [FZ], [MP]).

Set

\[
L(-\frac{4}{3}\Lambda_0) = \frac{N(-\frac{4}{3}\Lambda_0)}{N1(-\frac{4}{3}\Lambda_0)}.
\]

Then \(L(-\frac{4}{3}\Lambda_0)\) is a simple vertex operator algebra of rank \(-6\). The representation theory of this vertex operator algebra is very interested.

It is important to determine which \(\hat{\mathfrak{g}}\)-modules of level \(-\frac{4}{3}\) have the structure of a module for the vertex operator algebra \(L(-\frac{4}{3}\Lambda_0)\). This problem can be solved inside certain categories of \(\hat{\mathfrak{g}}\)-modules (see [AM], [DLM]).

Applying the general result obtained in Theorem 3.5.3 of [AM] to the case when \(k = -\frac{4}{3}\) we get the following result.

**Theorem 3.1.** [AM] The set

\[
\{L(-\frac{4}{3}\Lambda_0), L(-\frac{2}{3}\Lambda_0 - \frac{2}{3}\Lambda_1), L(-\frac{4}{3}\Lambda_1)\}
\]

provides all irreducible \(L(-\frac{4}{3}\Lambda_0)\)-modules from the category \(\mathcal{O}\). Every \(L(-\frac{4}{3}\Lambda_0)\)-module from the category \(\mathcal{O}\) is completely reducible.

Applying the automorphism \(\pi_s, s \in \mathbb{Z}\), on \(L(-\frac{4}{3}\Lambda_0)\)-modules from the category \(\mathcal{O}\) one can get a family of weak \(L(-\frac{4}{3}\Lambda_0)\)-modules which in general don’t belong to the category \(\mathcal{O}\). These modules can be identified by using the following lemma.

**Lemma 3.1.**

(1) Assume that \(M\) is a weak \(L(-\frac{4}{3}\Lambda_0)\)-module which is generated by the vector \(v_s\) \((s \in \mathbb{Z})\) such that:

\[
e(n-s)v_s = f(n+s)v_s = 0 \quad (n \geq 0),
\]

\[
h(n)v_s = \delta_{n,0}(-\frac{4}{3}s)v_s \quad (n \geq 0).
\]
Then
\[ M \cong \pi_{-s}(L(-\frac{4}{3}\Lambda_0)).\]

(2) Assume that \( M \) is a weak \( L(-\frac{4}{3}\Lambda_0) \)-module which is generated by the vector \( v'_s \) (\( s \in \mathbb{Z} \)) such that:

\[
e(n - s)v'_s = f(n + 1 + s)v'_s = 0 \quad (n \geq 0),
\]

\[
h(n)v'_s = \delta_{n,0}(-\frac{2}{3} - \frac{4}{3}s)v'_s \quad (n \geq 0).
\]

Then
\[ M \cong \pi_{-s}(L(-\frac{2}{3}\Lambda_0 - \frac{2}{3}\Lambda_1)).\]

Proof. (1) We consider the weak \( L(-\frac{4}{3}\Lambda_0) \)-module \( \pi_s(M) \). By construction, we have that \( \pi_s(M) \) is a highest weight \( \hat{\gamma} \)-module with the highest weight \( -\frac{4}{3}\Lambda_0 \). Since every \( L(-\frac{4}{3}\Lambda_0) \)-module from the category \( \mathcal{O} \) is completely reducible, we have that \( \pi_s(M) \cong L(-\frac{4}{3}\Lambda_0) \), which implies that
\[ M \cong \pi_{-s}(\pi_s(M)) \cong \pi_{-s}(L(-\frac{4}{3}\Lambda_0)),\]

and the statement (1) holds. The proof of (2) is similar. \( \square \)

4. Lattice construction of \( L(-\frac{4}{3}\Lambda_0) \) and its modules

In this section we shall present a lattice construction of the vertex operator algebra \( L(-\frac{4}{3}\Lambda_0) \). This construction will use certain ideas and results obtained in \([A2]\). Our results in Section 5 will show that this approach can be applied because the vertex operator algebra \( L(-\frac{4}{3}\Lambda_0) \) contains the \( \mathcal{W}(2,5) \)-algebra investigated in \([A2]\).

Define first the following lattice:
\[ \tilde{L} = \mathbb{Z}\gamma + \mathbb{Z}\delta, \quad \langle \gamma, \gamma \rangle = -\langle \delta, \delta \rangle = \frac{1}{6}, \quad \langle \gamma, \delta \rangle = 0.\]

Let \( \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \tilde{L} \). Extend the form \( \langle \cdot, \cdot \rangle \) on \( \tilde{L} \) to \( \mathfrak{h} \). Let \( \hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus Cc \) be the affinization of \( \mathfrak{h} \). Set \( \hat{\mathfrak{h}}^+ = \mathfrak{h} \otimes t\mathbb{C}[t] \); \( \hat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}] \). Then \( \hat{\mathfrak{h}}^+ \) and \( \hat{\mathfrak{h}}^- \) are abelian subalgebras of \( \hat{\mathfrak{h}} \). Let \( U(\hat{\mathfrak{h}}^-) = S(\hat{\mathfrak{h}}^-) \) be the universal enveloping algebra of \( \hat{\mathfrak{h}}^- \). Let \( \lambda \in \mathfrak{h} \). Consider the induced \( \hat{\mathfrak{h}} \)-module
\[ M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}} \otimes \mathbb{C}[t, t^{-1}] \oplus Cc)} \mathbb{C}\lambda \simeq S(\hat{\mathfrak{h}}^-) \] (linearly),

where \( \mathfrak{h} \otimes t\mathbb{C}[t] \) acts trivially on \( \mathbb{C} \), \( \mathfrak{h} \) acting as \( \langle \alpha, \lambda \rangle \) for \( \alpha \in \mathfrak{h} \) and \( c \) acts on \( \mathbb{C} \) as multiplication by 1. We shall write \( M(1) \) for \( M(1, 0) \). For \( \alpha \in \mathfrak{h} \) and \( n \in \mathbb{Z} \) write \( \alpha(n) = \alpha \otimes t^n \). Set \( \alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n)z^{-n-1} \).
Then $M(1)$ is a vertex algebra which is generated by the fields $\alpha(z)$, $\alpha \in \mathfrak{h}$, and $M(1, \lambda)$, for $\lambda \in \mathfrak{h}$, are irreducible modules for $M(1)$.

As in [DL], [GL] (see also [FLM], [K2]), we have the generalized vertex algebra

$$V_L = M(1) \otimes \mathbb{C}[\tilde{L}],$$

where $\mathbb{C}[\tilde{L}]$ is a group algebra of $\tilde{L}$ with generators $e^\gamma$ and $e^\delta$. For $v \in V_L$ let $Y(v, z) = \sum_{s \in \mathbb{Z}} v_s z^{s-1}$ be the corresponding vertex operator (for precise formulae see [DL]).

Recall some useful facts which hold in the generalized vertex algebra $V_L$.

Assume that $\alpha_1, \alpha_2 \in \tilde{L}$, $\langle \alpha_1, \alpha_2 \rangle = r \in \mathbb{Z}$. Then

$$Y(e^{\alpha_1}, z) e^{\alpha_2} = \sum_{n \in \mathbb{Z}} e^{\alpha_1 n} e^{\alpha_2} z^{-n-r-1}, \quad (4.5)$$

$$e^{\alpha_1 n} e^{\alpha_2} = 0 \quad \text{for every } n \in \mathbb{Z}_{\geq 0}. \quad (4.6)$$

Clearly, $M(1)$ is a vertex subalgebra of $V_L$.

Let $M_\gamma(1)$ (resp. $M_\delta(1)$) be a vertex subalgebra of $M(1)$ generated by the field $\gamma(z)$ (resp. $\delta(z)$). $M_\gamma(1)$ (resp. $M_\delta(1)$) is a subalgebra of the generalized vertex algebra $V_{Z\gamma}$ (resp. $V_{Z\delta}$).

Define the following Virasoro element in $M_\gamma(1) \subset V_{Z\gamma} \subset V_L$:

$$\omega_\gamma = 3 \gamma(\gamma^{-2}) - 2 \gamma(-2). \quad (4.7)$$

Then $\omega_\gamma$ spans a subalgebra of $M_\gamma(1)$ isomorphic to the Virasoro vertex operator algebra $L_{Vir}(-7, 0)$ with central charge $-7$.

As in [A2], we define the following screening operators:

$$Q = e_0^{-6\gamma}, \quad \tilde{Q} = e_0^{2\gamma}, \quad (4.8)$$

and the following subalgebras of $M(1)$:

$$M(1) = \text{Ker}_{M(1)} \tilde{Q}, \quad M_\gamma(1) = \text{Ker}_{M(1)} \tilde{Q}. \quad (4.9)$$

Then

$$H = e_0^{-6\gamma} e^{6\gamma} \in \overline{M(1)} \subset \overline{M(1)}$$

is a primary vector of conformal weight 5.

**Remark 4.1.** The generalized vertex algebra $V_L$ is larger than the algebras considered in [A2] in the case when $p = 3$. More precisely, if we set $\alpha = -6\gamma$ and $\beta = -2\gamma$, then the subalgebras $V_{Z\alpha}$ and $V_{Z\beta}$ of $V_L$ are isomorphic to the (generalized) vertex operator algebras considered in [A2]. Since

$$Q = e_0^\alpha, \quad \tilde{Q} = e_0^\beta, \quad \omega_\gamma = \frac{3}{4} \beta(-1)^2 + \beta(-2), \quad H = e_0^\alpha e^{-\alpha},$$
we have that expressions for vectors and operators defined by relations (4.7)-(4.9) coincide with the definition of the same objects in Section 2 of \[A2\] when \(p = 3\).

The results from \[A2\] applied to the case \(p = 3\) give the following result:

**Theorem 4.1.** \[A2\]

1. \([\tilde{Q}, \tilde{Q}] = 0\),
2. \(\text{Ker} M_{\gamma}(1) \approx L^{\text{vir}}(-7,0)\),
3. The vertex operator algebra \(\tilde{M}_{\gamma}(1)\) is generated by \(\omega = \omega_{\gamma} - 3\delta(-1)^2\).

This theorem has the following consequence:

**Corollary 4.1.**

1. The vertex operator algebra \(\text{Ker} M_{\gamma}(1) \approx L^{\text{vir}}(-7,0) \otimes M_{\delta}(1)\) is generated by \(\delta(-1), \omega_{\gamma}\) and \(H\).
2. The vertex operator algebra \(\tilde{M}(1) \approx \tilde{M}_{\gamma}(1) \otimes M_{\delta}(1)\) is generated by \(\delta(-1), \omega_{\gamma}\) and \(H\).

Define

\[\omega = \omega_{\gamma} - 3\delta(-1)^2 = 3\gamma(-1)^2 - 2\gamma(-2) - 3\delta(-1)^2.\]

Then \(\omega\) is a Virasoro element which generates the subalgebra \(L^{\text{vir}}(-6,0)\).

Set

\[L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}.\]

Define the following vectors in \(V_L\):

\[(4.10)\quad e = e^{3(\gamma - \delta)},\]
\[(4.11)\quad h = 4\delta(-1),\]
\[(4.12)\quad f = -\frac{2}{9}Qe^{3(\gamma + \delta)} = -(4\gamma(-1)^2 - \frac{2}{3}\gamma(-2))e^{-3(\gamma - \delta)}.\]

Then \(e, h, f\) are primary vectors of conformal weight 1 for the Virasoro algebra, i.e.,

\[L(n)e = \delta_{n,0}e, \quad L(n)h = \delta_{n,0}h, \quad L(n)f = \delta_{n,0}f \quad (n \geq 0).\]

We are interested in the subalgebra generated by \(e, f\) and \(h\). For this purpose we define the following lattice:

\[L = \mathbb{Z}(\gamma + \delta) + \mathbb{Z}(\gamma - \delta).\]

Clearly, \(V_L\) is a subalgebra of \(V_L\) which contains vectors \(e, f\) and \(h\). It is important to notice that

\[(4.13)\quad \langle 3(\gamma \pm \delta), L \rangle \subset \mathbb{Z},\]
i.e., $3(\gamma + \delta)$ and $3(\gamma - \delta)$ are elements of the dual lattice of $L$ (see also Remark 4.2 below).

The following lemma is a consequence of (4.13) and the Jacobi identity in generalized vertex algebras (see [DL], [GL], [S]).

**Lemma 4.1.** Assume that $a, b \in \text{span}_\mathbb{C}\{e, f, h\}$ and $v \in V_L$. Then $Y(a, z)v = \sum_{n \in \mathbb{Z}} a_n v z^{-n-1}$, and for every $m, n \in \mathbb{Z}$ the following commutator formula holds:

$$[a_n, b_m] v = \sum_{i=0}^{\infty} \binom{n}{i} (a_i b)_{n+m-i} v.$$

(4.14)

For $a \in \text{span}_\mathbb{C}\{e, f, h\}$ set $a(n) = a_n$ and $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$. Lemma 4.1 shows that the field $a(z)$ is in fact the restriction of the vertex operator $Y(a, z)$ on $V_L$.

**Theorem 4.2.** The vectors $e$, $f$ and $h$ span a subalgebra of the generalized vertex algebra $V_L$ isomorphic to the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$. Moreover, $V_L$ is a weak module for the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$.

**Proof.** First we notice that for every $n \in \mathbb{Z}$, $n \geq 0$, the following relations in the generalized vertex algebra $V_L$ hold:

$$
e(n)e = 0,$$
$$f(n)f = \frac{2}{81} Q^2(e_n^{3(\gamma+\delta)} e^{3(\gamma+\delta)}) = 0,$$
$$h(n)h = 0 \quad (n \neq 1),$$
$$h(1)h = -\frac{8}{3} 1,$$
$$h(n)e = 2\delta_{n,0} e,$$
$$h(n)f = -2\delta_{n,0} f,$$
$$e(n)f = 0 \quad \text{for every } n \geq 2,$$
$$e(1)f = -\frac{4}{3} 1,$$
$$e(0)f = h.$$

By using these relations and commutator formula (4.14) we conclude that the components of the fields $e(z), f(z), h(z)$ satisfy the commutation relations for the affine Lie algebra $\hat{\mathfrak{g}}$ of level $-\frac{4}{3}$. So $V_L$ is a $U(\hat{\mathfrak{g}})$-module of level $-\frac{4}{3}$. By construction, we have that $V_L$ is a weak module for the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$. Let $S$ be the subalgebra of $V_L$ generated
by vectors $e, f$ and $h$, i.e.,

$$S = \text{span}_C\{u_{n_1} \cdots u_{n_r} | u^1, \ldots, u^r \in \{e, f, h\}, n_1, \ldots, n_r \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}\}.$$  

Then $S$ is isomorphic to a certain quotient of the vertex operator algebra $N(-\frac{1}{3} \Lambda_0)$. The Virasoro element $\omega$ belongs to the subalgebra $S$ because

$$\phi_{-\frac{4}{3}} \mathbf{1} = \omega = 3 \gamma(-1)^2 - 2 \gamma(-2) - 3 \delta(-1)^2.$$  

As a $U(\mathfrak{g})$-module, $S$ a cyclic module generated by the vacuum vector $\mathbf{1}$. Next we consider element $f_{\text{sing}} \cdot \mathbf{1} \in S$. The definition of the action of $\hat{\mathfrak{g}}$ on $V_L$ gives that

$$f_{\text{sing}} \cdot \mathbf{1} = e(-1)\mathbf{1} + \frac{1}{3} e(-3) \mathbf{1} - \frac{1}{2} h(-1) e(-2) \mathbf{1} + \frac{1}{2} h(-2) e(-1) \mathbf{1}$$

$$= e^{3(\gamma-\delta)} - \frac{1}{3} e^{3(\gamma-\delta)} \mathbf{1} - 2 \delta(-1) e^{3(\gamma-\delta)} \mathbf{1} + 2 \delta(-2) e^{3(\gamma-\delta)} \mathbf{1}$$

$$= \frac{4}{3} e^{3(\gamma-\delta)} \mathbf{1} - 9 (\gamma(-1) - \delta(-1))^2 e^{3(\gamma-\delta)} \mathbf{1} - 6 \delta(-1) (\gamma(-1) - \delta(-1)) e^{3(\gamma-\delta)} + 2 \delta(-2) e^{3(\gamma-\delta)}$$

$$+ (3 \gamma(-1)^2 - 2 \gamma(-2) - 3 \delta(-1)^2) e^{3(\gamma-\delta)}$$

$$= \frac{2}{3} \left( (3 \gamma(-1) - 3 \delta(-1)^2)^2 + 3 \gamma(-2) - 3 \delta(-2) \right) e^{3(\gamma-\delta)}$$

$$+ (-9 (\gamma(-1) - \delta(-1))^2 - 6 \delta(-1) (\gamma(-1) - \delta(-1)))$$

$$+ 2 \delta(-2) + 3 \gamma(-1)^2 - 2 \gamma(-2) - 3 \delta(-1)^2) e^{3(\gamma-\delta)}$$

$$= 0.$$  

Since $f_{\text{sing}} \cdot \mathbf{1} = 0$, we conclude that $S$ is a certain quotient of

$$\frac{N(-\frac{4}{3} \Lambda_0)}{U(\mathfrak{h})^\text{sing}} = L(-\frac{4}{3} \Lambda_0).$$  

The simplicity of $L(-\frac{4}{3} \Lambda_0)$ implies that $S = U(\mathfrak{h}). \mathbf{1} \cong L(-\frac{4}{3} \Lambda_0)$. Therefore, $V_L$ is a weak $L(-\frac{4}{3} \Lambda_0)$-module. 

**Remark 4.2.** The vertex operator algebra $L(-\frac{4}{3} \Lambda_0)$ can be embedded into certain vertex algebras. Let us describe these algebras. Define

$$L_0 = \mathbb{Z}(3 \gamma - 3 \delta) + \mathbb{Z}(3 \gamma + 3 \delta).$$

Then $L_0$ is the dual lattice of $L$ in $\mathfrak{h}$. $L_0$ is an even lattice and therefore the subalgebra $V_{L_0}$ of $V_L$ has the structure of a vertex algebra (cf. [DL]). Moreover, $V_L$ is a $V_{L_0}$-module. Since $e, f, h \in V_{L_0}$, we have that the vertex operator algebra $L(-\frac{4}{3} \Lambda_0)$ is a subalgebra of $V_{L_0}$. It is also interesting to notice that $e, f, h \in \text{Ker} V_{L_0} \tilde{Q}$. So $L(-\frac{4}{3} \Lambda_0)$ is also a subalgebra of the vertex algebra $\text{Ker} V_{L_0} \tilde{Q}$. 
On the other hand, we also want to construct explicitly other \( L(-\frac{4}{3} \Lambda_0) \)-modules from the category \( \mathcal{O} \). It turns out that for this purpose the larger algebra \( V_L \) is more suitable.

Now we want to identify certain \( L(-\frac{4}{3} \Lambda_0) \)-submodules of \( V_L \). We have the following result.

**Theorem 4.3.** For every \( s \in \mathbb{Z} \) we have:

1. \[ U(\hat{g}).e^{2s\delta} \cong \pi_{-s}(L(-\frac{4}{3} \Lambda_0)). \]

2. \[ U(\hat{g}).e^{-\gamma+(2s+1)\delta} \cong \pi_{-s}(L(-\frac{2}{3} \Lambda_0 - \frac{2}{3} \Lambda_1)). \]

**Proof.** Define \( v_s = e^{2s\delta} \) and \( v'_s = e^{-\gamma+(2s+1)\delta} \). Since \( V_L \) is a weak \( L(-\frac{4}{3} \Lambda_0) \)-module, we have that for every \( s \in \mathbb{Z} \) the submodules \( U(\hat{g}).v_s \) and \( U(\hat{g}).v'_s \) have the structure of a weak module for the vertex operator algebra \( L(-\frac{4}{3} \Lambda_0) \).

Assume that \( n \geq 0 \). Since \( \langle 4\delta, 2s\delta \rangle = -\frac{4}{3}s \), we have

\[ h(n)v_s = -\frac{4}{3}s \delta_{n,0} v_s. \]

Since \( \langle 3\gamma - 3\delta, 2s\delta \rangle = s \), relation (4.6) implies that

\[ e(n-s)v_s = 0. \]

Since \( \langle 3\gamma + 3\delta, 2s\delta \rangle = -s \) and \( Qv_s = 0 \), we have that

\[ f(n+s)v_s = -\frac{2}{9}(Qe^{3\gamma+3\delta})_{n+s}v_s \]
\[ = -\frac{2}{9}Qe^{3\gamma+3\delta}_{n+s}v_s + \frac{2}{9}e^{3\gamma+3\delta}_{n+s}Qv_s \]
\[ = 0. \]

In this way we have verified that vector \( v_s \) satisfies conditions (3.1) and (3.2) and Lemma 3.1 implies that

\[ U(\hat{g}).v_s \cong \pi_{-s}(L(-\frac{4}{3} \Lambda_0)). \]

This proves assertion (1).

Since \( \langle 4\delta, -\gamma + (2s + 1)\delta \rangle = -\frac{2}{3} - \frac{4}{3}s \), we have

\[ h(n)v'_s = (-\frac{2}{3} - \frac{4}{3}s) \delta_{n,0} v'_s. \]

Since \( \langle 3\gamma - 3\delta, -\gamma + (2s + 1)\delta \rangle = s \), relation (4.6) implies that

\[ e(n-s)v'_s = 0. \]
Since $\langle 3\gamma + 3\delta, -\gamma + (2s + 1)\delta \rangle = -s - 1$ and $Qv'_s = 0$, we have that

$$f(n + 1 + s)v'_s = -\frac{2}{9}(Qe^{3\gamma + 3\delta})_{n+1+s}v'_s$$

$$= -\frac{2}{9}Qe^{3\gamma + 3\delta}v'_s + \frac{2}{9}e^{3\gamma + 3\delta}Qv'_s$$

$$= 0.$$ 

So vector $v'_s$ satisfies conditions (3.3) and (3.4) and by Lemma 3.1 we have that

$$U(\hat{g}).v'_s \approx \pi_{-s}(L(-\frac{2}{3}\Lambda_0 - \frac{2}{3}\Lambda_1)),$$

and assertion (2) holds. □

This theorem has the following consequence.

**Corollary 4.2.** The vectors $e^{-\gamma + \delta}, e^{2\delta}$ are singular vectors for the action of $\hat{g}$. Moreover, we have

$$U(\hat{g}).e^{-\gamma + \delta} \cong L(-\frac{2}{3}\Lambda_0 - \frac{2}{3}\Lambda_1), \quad U(\hat{g}).e^{2\delta} \cong L(-\frac{4}{3}\Lambda_1).$$

The generalized vertex algebra $V_L$ contains some other interesting $L(-\frac{4}{3}\Lambda_0)$-submodules. Let us give one example.

**Example 4.1.** Define $\tilde{E} = U(\hat{g}).e^{-2\gamma}$. Since $g \otimes t\mathbb{C}[t].e^{-2\gamma} = 0$, we have that the $\hat{g}$-module $\tilde{E}$ is $\mathbb{Z}_{\geq 0}$-graded. Moreover, the top level $E = U(\hat{g}).e^{-2\gamma}$ is an irreducible $U(\hat{g})$-module which is neither highest nor lowest weight with respect to $g$. Therefore, the $\hat{g}$-module $\tilde{E}$ is not in the category $\mathcal{O}$, but it is a $\mathbb{Z}_{\geq 0}$-graded $L(-\frac{4}{3}\Lambda_0)$-module on which $h(0)$ and $L(0)$ act semisimply. By using Zhu’s algebra theory (cf. [Z]), modules of this type were investigated in [AM].

5. **Vertex subalgebras of $L(-\frac{4}{3}\Lambda_0)$**

In Section 4 we proved that the vertex algebra $M(1)$ contains the subalgebra $\overline{M(1)}$ which is a vertex operator algebra of rank $-6$. This vertex operator algebra is obtained as kernel of the screening operator $\tilde{Q}$. In this section we will show that $\overline{M(1)}$ actually lies in the vertex operator algebra $L(-\frac{4}{3}\Lambda_0) \subset V_L$. This result will imply that the coset vertex operator algebra

$$C(L(-\frac{4}{3}\Lambda_0), M_\delta(1)) \cong \overline{M_\gamma(1)}.$$ 

Since $\overline{M_\gamma(1)}$ is a simple vertex operator algebra generated by Virasoro element $\omega^\gamma$ and primary vector $H$ of conformal weight 5, we can say that this coset vertex operator algebra (or the coset $\frac{\mathfrak{su}(2)}{u(1)}$ in the
The terminology of \([\text{BEHHH}]\) is the \(\mathcal{W}(2, 5)\) algebra with central charge \(-7\).

The operator \(h(0)\) acts semisimply on \(L(-\frac{4}{3}\Lambda_0)\) and

\[ L(-\frac{4}{3}\Lambda_0) = \bigoplus_{i \in \mathbb{Z}} M^{(i)}, \text{ where } M^{(i)} = \{v \in L(-\frac{4}{3}\Lambda_0) \mid h(0)v = 2iv\}. \]

**Proposition 5.1.** The vertex operator subalgebra \(M^{(0)} = \text{Ker}_L(-\frac{4}{3}\Lambda_0)h(0)\) is simple. As a \(M^{(0)}\)-module, \(L(-\frac{4}{3}\Lambda_0)\) is isomorphic to the direct sum \(L(-\frac{4}{3}\Lambda_0) = \bigoplus_{i \in \mathbb{Z}} M^{(i)}\), and each \(M^{(i)}\) is a simple \(M^{(0)}\)-module.

**Proof.** Clearly, \(M^{(0)} = \text{Ker}_L(-\frac{4}{3}\Lambda_0)h(0)\) is a vertex operator subalgebra of \(L(-\frac{4}{3}\Lambda_0)\). For \(u \in M^{(i)}\) and \(n \in \mathbb{Z}\) we have that \(u_n M^{(j)} \subset M^{(i+j)}\).

Let \(0 \neq v \in M^{(j)}\). Since \(L(-\frac{4}{3}\Lambda_0)\) is a simple vertex operator algebra, by Corollary 4.2 of [DM] we have that

\[ (5.15) \quad L(-\frac{4}{3}\Lambda_0) = \text{span}_C \{u_n v \mid u \in L(-\frac{4}{3}\Lambda_0), n \in \mathbb{Z}\}. \]

This relation implies that

\[ (5.16) \quad M^{(j)} = \text{span}_C \{u_n v \mid u \in M^{(0)}, n \in \mathbb{Z}\}. \]

So \(M^{(j)}\) is a simple \(M^{(0)}\)-module. In particular, \(M^{(0)}\) is a simple vertex operator algebra. \(\square\)

We shall now identify the subalgebra \(M^{(0)}\).

**Theorem 5.1.** The vertex operator algebra \(M^{(0)} = \text{Ker}_L(-\frac{4}{3}\Lambda_0)h(0)\) is isomorphic to \(\overline{M(1)} \cong M_\gamma(1) \otimes M_\delta(1)\). In particular, \(\overline{M(1)}\) and \(\overline{M_\gamma(1)}\) are simple vertex operator algebras.

**Proof.** By using the lattice construction of \(L(-\frac{4}{3}\Lambda_0)\) from Section \(\S\), one can easily see that \(M^{(0)}\) is actually a subalgebra of the vertex operator algebra \(\overline{M(1)}\). Since \(L(-\frac{4}{3}\Lambda_0) \subset \text{Ker}_{\overline{V_L}}\tilde{Q}\), we have that

\[ (5.17) \quad M^{(0)} \subseteq \text{Ker}_{M(1)}\tilde{Q} = \overline{M(1)}. \]

We shall now prove the other inclusion. By Corollary \(\S.1\) we have that the vertex operator algebra \(\overline{M(1)}\) is generated by vectors \(\delta(-1), \omega^\gamma\) and \(H\). It suffices to prove that these generators belong to \(L(-\frac{4}{3}\Lambda_0)\).

Clearly,

\[ (5.18) \quad \delta(-1) = \frac{1}{4} h \in L(-\frac{4}{3}\Lambda_0). \]
The fact that $\omega = \omega^\gamma - 3\delta(-1)^2$ is the Virasoro element in $L(-\frac{4}{3}\Lambda_0)$ implies that
\begin{equation}
\omega^\gamma = \omega + \frac{3}{16} h(-1)^2 1 \in L(-\frac{4}{3}\Lambda_0). \tag{5.19}
\end{equation}
Since $\text{Ker}_{M(1)} Q$ is generated by $\delta(-1)$ and $\omega^\gamma$, relations (5.18) and (5.19) imply that
\begin{equation}
\text{Ker}_{M(1)} Q \subset M^{(0)}. \tag{5.20}
\end{equation}
Now we shall prove that $H \in M^{(0)}$. Define vector
\[ v = -9e(-4)f = 2e^{3\gamma-3\delta}Qe^{3\gamma+3\delta} \in M^{(0)}. \]
Since $Q^2 e^{3\gamma-3\delta} = Q^2 e^{3\gamma+3\delta} = 0$, we have
\[ Qv = 2(Qe^{3\gamma-3\delta})_{-4}(Qe^{3\gamma+3\delta}) = Q^2(e^{3\gamma-3\delta}e^{3\gamma+3\delta}) = Q^2 e^{6\gamma}. \]
This implies that $Q(v - Qe^{6\gamma}) = 0$. Since
\[ v - Qe^{6\gamma} \in \text{Ker}_{M(1)} Q \subset M^{(0)} \]
we conclude that
\begin{equation}
H = Qe^{6\gamma} \in M^{(0)}. \tag{5.21}
\end{equation}
Now relation (5.17)-(5.21) imply that $M^{(0)} = \overline{M^{(1)}}$.

The simplicity of $M^{(1)}$ and $\overline{M^{(1)}}$ follows from the fact that $M^{(0)}$ is a simple vertex operator algebra (cf. Proposition 5.1). \qed

By the lattice construction, we have that the subalgebra of $L(-\frac{4}{3}\Lambda_0)$ generated by the field $h(z)$ is isomorphic to $M^{(1)}$. Theorem 5.1 enables us to describe the following vertex operator algebra.

**Corollary 5.1.** The coset vertex operator algebra
\[ C(L(-\frac{4}{3}\Lambda_0), M^{(1)}) = \{ v \in L(-\frac{4}{3}\Lambda_0) \mid h(n)v = 0, \ n \geq 0 \} \]
is isomorphic to the vertex operator algebra $\overline{M^{(1)}}$ (= $W(2, 5)$ algebra with central charge $-7$).

**Remark 5.1.** It was proved in [KR] that the subalgebra $\text{Ker}_{L(-\frac{4}{3}\Lambda_0)} h(0)$ is isomorphic to the vertex algebra $W_{1+\infty}$ with central charge $c = -1$. In [WN], Wang proved that the vertex algebra $W_{1+\infty}$ with central charge $-1$ is isomorphic to the tensor product of the $W(2, 3)$ algebra with central charge $-2$ and the free boson vertex algebra. Our result shows that the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$ contains a subalgebra of similar type. The difference is that the $W(2, 3)$ algebra with central charge $-2$ is replaced by the $W(2, 5)$ algebra with central charge $-7$. 
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