THE CONDUCTOR DENSITY OF LOCAL FUNCTION FIELDS
WITH ABELIAN GALOIS GROUP

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Abstract. We give an exact formula for the number of $G$-extensions of local function fields $\mathbb{F}_q((t))$ for finite abelian groups $G$ up to a conductor bound. As an application we give a lower bound for the corresponding counting problem by discriminant.

1. Introduction

Let $F = \mathbb{F}_q((t))$ be the Laurent series field over the finite field with $q = p^f$ elements. Given a finite abelian group $G$, we are interested in analysing the function

$$Z(F, G; n) := |\{ E/F \text{ normal} : \text{Gal}(E/F) \cong G \text{ and } N(\mathfrak{F}(E/F)) \leq q^n \}|,$$

where $N(\mathfrak{F}(E/F))$ is the norm of the conductor.

Much progress has been made in analysing the asymptotic behaviour of similar counting functions, typically weighting by discriminant as in [12] over global fields and over global function fields (see [2]). Both papers leave the gap when we count abelian $p$-extensions over global fields in characteristic $p$. For this, there are results on counting global function fields by conductor and local function fields by discriminant ([6], [7]). In the number field situation Mäki [9] is counting by conductor.

For the local situation (characteristic 0 and $p$), there is Serre’s Mass Formula (see [11] Th. 2) which gives a strong summation formula over all totally ramified extensions over a fixed local field weighted by discriminant and the size of its automorphism group. Over $p$-adic fields, there are only finitely many extensions of a given degree. For a $p$-group $G$ there are infinitely many extensions with that group ([8] Th. 2.4.3], [10] Th. 7.5.10]) and we can show the following theorem.

Theorem 1. Let $G$ be a finite abelian $p$-group with exponent $\exp(G) = p^e$ and $\alpha_p(G) := \sum_{k=1}^{e} \left[ \frac{p-1}{p^k} \right] \text{rk}_{p^k}(G)$, where $\text{rk}_{p^k}(G) = \dim_{\mathbb{F}_p}(G^{p^{k-1}}/G^{p^k})$. Then there is a $p^e$-periodic function $\delta_G : \mathbb{Z} \to [-\alpha_p(G), 0]$ and a function $\varepsilon(G, q, n)$ such that

$$Z(F, G; n) = \frac{|G|}{|\text{Aut}(G)|} q^{\alpha_p(G)} q^{\delta_G(n)} \varepsilon(G, q, n)$$

and $Z(F, G; n) \sim \frac{|G|}{|\text{Aut}(G)|} q^{\alpha_p(G)} q^{\delta_G(n)}$.

Note, that the constants $\delta_G(n)$ and $\varepsilon(G, q, n)$ are defined in [5] and [7] and $\lim_{n \to \infty} \varepsilon(G, q, n) = 1$. Furthermore we prove that the constants $\alpha_p(G)$ and $\delta_G(n)$ are additive with respect to direct products, see Remark [9].
The function \( \delta_G \) shows an oscillation depending on the residue of \( n \) modulo \( \exp(G) \). Hence in order to get a convergence result of the form

\[
Z(F, G; n) \sim c(F, G) \cdot q^{n\alpha_p(G)},
\]

we need to restrict to an arithmetic progression modulo \( \exp(G) \). We remark that there is an oscillation even for the group \( G = C_p \), the cyclic group of order \( p \).

For a general abelian finite group \( G \) denote by \( G_p \) the coprime to \( p \)-part of \( G \). Then \( G = G_p \times G_p' \) and the existence of a solution to the inverse Galois problem only depends on \( G_p' \), while the \( p \)-part of \( G \) determines the asymptotical growth: If \( |G| \) is coprime to \( p \), there are at most finitely many extensions with Galois group \( G \). In Theorem 15 we give an exact formula for arbitrary finite abelian groups. The main work lies within \( p \)-part and Theorems 14. For \( \ell \neq p \), the \( \ell \)-rank of \( G \) is bounded by 2, and only the prime divisors of \( q - 1 \) contribute a possibly non-trivial factor.

Finally, we prove in Theorem 17 a non-trivial lower bound for the distribution of abelian local function field extensions counted by discriminant. We remark that our lower bound coincides with the asymptotic exponent given in Satz 2.1 in [6].

2. Higher Unit Groups

Let \( \mathbb{F}_q \) be a finite field with \( q = p^f \) elements and \( F = \mathbb{F}_q((t)) \) be the Laurent series ring over \( \mathbb{F}_q \). Let \( \mathcal{O}_F = \mathbb{F}_q[[t]] \) be the local ring with maximal ideal \( p = t\mathcal{O}_F \).

By the main theorem of local class field theory, we get a one-to-one correspondence of abelian extensions \( E/F \) and norm groups \( U := N_{E/F}(E^\times) \) in \( F^\times \).

The conductor exponent \( c(U) \) of an open subgroup \( U \leq F^\times \) of finite index is the minimal natural number \( n \) such that \( 1 + p^n \leq U \). The conductor of an abelian extension \( E/F \) corresponding to \( U \) is

\[
\delta(E/F) = p^{c(U)}.
\]

**Theorem 2.** The mapping \( E \mapsto N_{E/F}(E^\times) \) defines a bijection between finite abelian extensions of \( F \) and open subgroups of \( F^\times \) of finite index.

Moreover the Galois group \( \text{Gal}(E/F) \) is isomorphic to the quotient group \( F^\times/U \).

For a proof, see [3, Theorem 6.2., p. 154].

Let \( G \) be a finite abelian group of exponent \( \exp(G) \). Recall \( F^\times \cong \mathbb{Z} \times \mathbb{F}_q^\times \times (1+p) \), see Hasse ([4, Ch. 15]). We define

\[
U_n := (1+p)/(1+p^n) \quad \text{and} \quad X_n := \mathbb{Z}/\exp(G)\mathbb{Z} \times \mathbb{F}_q^\times \times U_n.
\]

By class field theory the counting problem reduces to count the number of open subgroups \( U \leq F^\times \) with \( F^\times/U \) isomorphic to \( G \). The conductor bound \( N(\delta(E/F)) \leq q^n \) is equivalent to \( 1 + p^n \leq U \). Moreover, \( F^\times/U \cong G \) implies that \( \exp(G) \) annihilates \( F^\times/U \). So for our counting problem it is sufficient to consider the subgroups of \( F^\times \) containing

\[
\exp(G)\mathbb{Z} \times 1 \times (1+p^n)
\]

which correspond to the subgroups of \( X_n \).

By dualising, the number of subgroups of \( F^\times \) with quotient isomorphic to \( G \) is exactly the number of subgroups of \( X_n \) isomorphic to \( G \). Thus we reduce our counting problem to counting subgroups in certain finite abelian groups.

In establishing our desired formula, we first study higher unit groups, and consider formulas on subgroups of finite abelian groups depending on the \( p^k \)-ranks of the groups.
Definition 3. Let $G$ be a finite abelian group and $\ell \in \mathbb{F}$. Then

$$\text{rk}_\ell(G) := \dim_{\mathbb{F}_\ell}(G^{\ell^{-1}}/G^{\ell^k})$$

is the $\ell^k$-rank of $G$. If $\ell = p = \text{char}(F)$, we use the shorthand notation

$$r_k(G) := \text{rk}_p(G) \text{ and set } r_k(G) := r_k(G) - r_{k+1}(G).$$

Let now $G$ be a finite abelian $p$-group. A sequence of elements $(g_1,\ldots,g_n)$ is called a group-basis of $G$ if each element $g \in G$ has a unique representation

$$g = g_1^{i_1} \cdots g_n^{i_n}, \quad 0 \leq i_j < \text{ord}(g_j).$$

If $(g_1,\ldots,g_n)$ is a group-basis of $G$, then $r_k(G)$ is the number of generators with $\text{ord}(g_j) = p^k$, i.e., it is the number of cyclic factors of $G$ isomorphic to $C_{p^k}$.

Lemma 4. Let $(v_1,\ldots,v_t)$ be an $\mathbb{F}_p$-basis of $\mathbb{F}_q$. Then the following holds:
(a) $1 + p$ has a $\mathbb{Z}_p$-basis

$$\{1 + v_i t^k : k \in \mathbb{N}, p \nmid k, 1 \leq i \leq f\} \text{ and } \{1 + v_i t^k : 1 \leq i \leq f, k \leq n-1, p \nmid k\} \text{ is a group-basis of } U_n.$$

(b) For each $v \in \mathbb{F}_q^\times$ and $i \geq 1$ we have in $U_n$

$$\text{ord}(1 + vt^i) = p^{\lfloor \log_p(n/i) \rfloor}.$$

(c) For all $j \in \mathbb{N}$, $U_n[p^j]$ is generated by

$$\{1 + v_i t^k : 1 \leq i \leq f, p \nmid k, [n/p^j] \leq k \leq n - 1\}.$$

(d) For all $k \in \mathbb{N}$ we have

$$r_k(U_n) = f \left( \left\lfloor \frac{n-1}{p^k} \right\rfloor - \left\lfloor \frac{n-1}{p^{k+1}} \right\rfloor \right) \quad \text{and} \quad r_i(X_n) = r_i(U_n) + 1 \text{ for } i = 1,\ldots,e.$$

Proof. (a) [2] p. 227.

(b) $U_n$ is a $p$-group of order $q^{n-1}$ as $(1 + p^i)/(1 + p^{i+1}) \cong \mathbb{F}_q$ for all $i \geq 1$. Let $i \leq n$ and put $\alpha := 1 + vt^i \in (1 + p)/(1 + p^\alpha)$ with $v \in \mathbb{F}_q^\times$ and $k \in \mathbb{N}$. Then:

$$1 + vt^k = 1 \iff v^{p^k} t^k \in p^n \iff ip^k \geq n \iff p^k \geq \frac{n}{i} k \geq \lfloor \log_p(n/i) \rfloor.$$

(c) This is (a) and (b) with $\lfloor \log_p(n/k) \rfloor \leq j \iff n/k \leq p^j \iff k \geq n/p^j$.

(d) By (a), $\mathcal{B} = \{1 + v_j t^i : 1 \leq i < n \text{ and } p \nmid i\}$ is a group-basis of $X_n$. Then

$$r_k(X_n) = |\{g \in \mathcal{B} : \text{ord}(g) \geq p^k\}|.$$

By (b) we have $\text{ord}(1 + v_j t^i) \geq p^k \iff ip^{k-1} < n \iff ip^{k-1} \leq n - 1$, hence

$$r_k(U_n) = f \cdot |\{i : i \leq \left\lfloor \frac{n-1}{p^{k-1}} \right\rfloor, p \nmid i\}| = f\left( \left\lfloor \frac{n-1}{p^{k-1}} \right\rfloor - \left\lfloor \frac{n-1}{p^k} \right\rfloor \right).$$

Note that $r_k(X_n) = r_k(U_n) + 1$ since $p \nmid |\mathbb{F}_q^\times|$. \qed
3. Monomorphisms and Automorphisms of finite abelian groups

For $G$ and $A$ finite abelian groups let $G_p = \{ g \in G : \text{ord}(g) = p^a \text{ for some } a \in \mathbb{N} \}$ be the $p$-Sylow subgroup of $G$ and let $G_{p'}$ be the coprime to $p$ part of $G$. We define

$$\text{Inj}(G, A) := \{ \phi : G \to A \text{ monomorphism} \}, \quad \alpha_G(A) := |\{ U \leq A : U \cong G \}|.$$

We immediately get

$$\alpha_G(A) \cdot |\text{Aut}(G)| = |\text{Inj}(G, A)|.$$

We start with the following reduction to $p$-groups.

**Lemma 5.** $|\text{Inj}(G, A)| = \prod_{\ell \in \mathbb{Z}} |\text{Inj}(G_{\ell}, A_{\ell})|$ and $\alpha_G(A) = \alpha_{G_p}(A_p) \cdot \alpha_{G_{p'}}(A_{p'})$.

**Proof.** Monomorphisms need to preserve the order of elements. \[\square\]

Thus it is sufficient to consider finite abelian $p$-groups. In the following $G$ and $A$ will be finite abelian $p$-groups with $\text{exp}(G) = p^r$. As in [6] we define

$$f_G(t_1, \ldots, t_e) := \prod_{k=1}^{e} p^{r_k+1(G) - 1} \prod_{j=r_k+1(G)} \neq (t_k - p^j).$$

**Lemma 6.** Let $t(A) := (p^{r_1(A)}, \ldots, p^{r_e(A)})$ for an abelian $p$-group $A$. Then:

1. $|\text{Inj}(G, A)| = f_G(t(A)) = \prod_{k=1}^{e} p^{r_k(A)r_k+1(G)} \prod_{j=0}^{r_k+1(G) - 1} (p^{r_k(A)} - p^{r_k+1(G) + j})$,
2. $|\text{Aut}(G)| = |\text{Inj}(G, G)| = f_G(t(G))$.

The formula goes back to works of Delsarte [1]. A proof can be found in [6], Lemma A.1 and Remark A.3., where we use $\tilde{r}_k(G) = r_k(G) - r_{k+1}(G)$.

**Remark 7.** We get another formula which is useful for asymptotic considerations:

$$|\text{Inj}(G, A)| = \prod_{k=1}^{e} p^{r_k(A)} \tilde{r}_k(G) \prod_{j=0}^{\tilde{r}_k(G) - 1} \left( 1 - \frac{p^{r_{k+1}(G) + j}}{p^{r_k(A)}} \right).$$

**Proof.** In [2], we use $t_k - p^j = t_k(1 - \frac{p^j}{t_k})$ and make an index shift to obtain [3] by plugging in $t_k = p^{r_k(A)}$. \[\square\]

We want to apply these formulas to the norm groups whose $p^k$-ranks involve ceiling operations. In the following we denote by $\{ \frac{n}{p^k} \} := \frac{n}{p^k} - \lfloor \frac{n}{p^k} \rfloor < 1$. Therefore we need the following lemma:

**Definition 8.** For a finite abelian $p$-group $G$ of exponent $p^e$ and $n \in \mathbb{N}$ we define

$$\delta(n, k) := \left\{ \frac{n}{p^k} \right\} - \left\{ \frac{n}{p^{k-1}} \right\} = \left\lfloor \frac{n}{p^k} \right\rfloor - \left\lfloor \frac{n}{p^{k-1}} \right\rfloor - \frac{(p-1)n}{p^k},$$

and

$$\alpha_p(G) := \sum_{k=1}^{e} \frac{p-1}{p^k} r_k(G) \text{ and } \delta_G(n) := -\alpha_p(G) + \sum_{k=1}^{e} \tilde{r}_k(G) (\delta(n-1, k)).$$

We immediately see that $\delta(n, k)$ is $p^k$-periodic and therefore $\delta_G(n)$ is $p^e$-periodic.

**Remark 9.** Let $G$ and $H$ be finite abelian $p$-groups of exponent $\leq p^e$. Then:

1. $\delta_G(n) = -\alpha_p(G) + \sum_{k=1}^{e} \tilde{r}_k(G) \left\{ \frac{p-1}{p^k} \right\}$ and $\delta_{G \times H}(n) = \delta_G(n) + \delta_H(n)$,
(b) \( \alpha_p(G) = \sum_{k=1}^{e} \frac{\hat{r}_k(G) p^{k} - 1}{p^{k}} \) and \( \alpha_p(G \times H) = \alpha_p(G) + \alpha_p(H) \),
(c) \( \delta_\mathcal{G}(1) = -\alpha_p(G) \leq \delta_\mathcal{G}(n) \leq 0 = \delta_\mathcal{G}(0) \).

Proof. We use an index shift and \( r_{e+1}(G) = 0 \):

\[
\sum_{k=1}^{e} \frac{\hat{r}_k(G) p^{k} - 1}{p^{k}} = \sum_{k=1}^{e} (r_k(G) - r_{k+1}(G)) \frac{p^{k} - 1}{p^{k}} = \sum_{k=1}^{e} r_k(G) \left( \frac{p^{k} - 1}{p^{k}} - \frac{p^{k-1} - 1}{p^{k-1}} \right)
= \sum_{k=1}^{e} r_k(G) \frac{p - 1}{p^{k}} = \alpha_p(G).
\]

Similarly, we get:

\[
\sum_{k=1}^{e} \hat{r}_k(G) \left\{ \frac{n - 1}{p^{k}} \right\} = \sum_{k=1}^{e} (r_k(G) - r_{k+1}(G)) \left\{ \frac{n - 1}{p^{k}} \right\}
= \sum_{k=1}^{e} r_k(G) \left\{ \frac{n - 1}{p^{k}} \right\} \left\{ \frac{n - 1}{p^{k-1}} \right\} = \sum_{k=1}^{e} r_k(G) \delta(n - 1, k) = \alpha_p(G) + \delta_\mathcal{G}(n).
\]

With \( \hat{r}_k(G \times H) = \hat{r}_k(G) + \hat{r}_k(H) \) for \( k \geq 1 \) this completes (a) and (b).

Finally \(-\alpha_p(G) = \delta_\mathcal{G}(1) \leq -\alpha_p(G) + \sum_{k=1}^{e} \hat{r}_k(G) \left\{ \frac{n - 1}{p^{k}} \right\} = \delta_\mathcal{G}(n) \leq 0 = \delta_\mathcal{G}(0) \).

Example 10. Let \( r \in \mathbb{N} \).
(a) If \( G = (C_p)^r \), then \( \alpha_p(G) = r \cdot \frac{p - 1}{p} \).
(b) If \( G = C_{p^r} \) is cyclic, then \( \alpha_p(G) = \sum_{k=1}^{r} \frac{p^{k} - 1}{p^{k}} = \frac{p^{r} - 1}{p^{r}} \).

4. PROOF OF THE MAIN THEOREM

In this section we prove Theorem 1. We prepare some formulas.

Remark 11. Let \( n \in \mathbb{N} \). Then:

\[
\prod_{k=1}^{e} p^{r_k(G) r_k(X_n)} = |G|^q^{n \alpha_p(G)} q^{\delta_\mathcal{G}(n)}.
\]

Proof. For all \( k = 1, \ldots, e \) we have

\[
r_k(X_n) = 1 + f \left( \left\lfloor \frac{n - 1}{p^{k} - 1} \right\rfloor - \left\lfloor \frac{n - 1}{p^{k}} \right\rfloor \right) \equiv 1 + f \frac{p - 1}{p^{k}} (n - 1) + f \delta(n - 1, k).
\]

We get:

\[
\sum_{k=1}^{e} r_k(G) r_k(X_n) \equiv \sum_{k=1}^{e} r_k(G) + f \sum_{k=1}^{e} r_k(G) \frac{p - 1}{p^{k}} (n - 1) + f \sum_{k=1}^{e} r_k(G) \delta(n - 1, k)
= \log_p(G) + fn \alpha_p(G) + f \delta_\mathcal{G}(n).
\]

Note that \( q = p^f \).

Theorem 12. Let \( G \) be a finite abelian \( p \)-group with exponent \( \exp(G) = p^e \). Let \( \alpha_p(G) \) and \( \delta_\mathcal{G}(n) \) as defined in (5) where \( \delta_\mathcal{G}(\cdot) \) is \( p^e \)-periodic. Let \( F = \mathbb{F}_q((t)) \) and

\[
e(G, q, n) := \prod_{k=1}^{e} \prod_{j=0}^{E_k(G) - 1} \left( 1 - \frac{p^{r_k(G) + j - 1}}{q^{(p - 1)(n - 1)/p^k + \delta(n - 1, k)}} \right).
\]
Then we have: 
(a) \(Z(F, G; n) = \frac{|G|}{|\text{Aut}(G)|} q^{n_{G}(G)} q^{\delta_{G}(n)} \varepsilon(G, q, n)\).
(b) \(Z(F, G; n) \sim \frac{|G|}{|\text{Aut}(G)|} q^{n_{G}(G)} q^{\delta_{G}(n)}\) and \(\lim_{n \to \infty} \varepsilon(G, q, n) = 1\).
(c) For \(x_n = n \cdot p^r\), i.e. \(x_n \equiv 0 \mod p^r\) we have \(Z(F, G; x_n) \sim \frac{|G|}{|\text{Aut}(G)|} q^{x_n \alpha_p(G)}\).

Proof:

\[
Z(F, G; n) = \alpha_G(X_n) \equiv \frac{|\text{Inj}(G, X_n)|}{|\text{Aut}(G)|} = \frac{1}{|\text{Aut}(G)|} \prod_{k=1}^{e} p^{r_k(G) r_k(X_n)} \prod_{j=0}^{r_k(G)-1} \left(1 - \frac{p^{r_{k+1}(G)+j}}{p^{r_k(X_n)}}\right)
\]

Rem. \[\prod_{k=1}^{e} \frac{|G| q^{n_{G}(G)} q^{\delta_{G}(n)} \varepsilon(G, q, n)}{|\text{Aut}(G)|} = \frac{|G| q^{\delta_{G}(n)} \varepsilon(G, q, n)}{|\text{Aut}(G)|} \]

Using \(|\delta(n-1, k)| < 1\) we get \(\lim_{n \to \infty} \varepsilon(G, q, n) = 1\) for all \(k \geq 1\) and we proved (b). Using Remark 3 (c) we see \(\delta(0) = 0\) which gives (c). \(\square\)

Example 13. (a) Let \(G = (C_p)^r\) be elementary abelian. Then \(\alpha_p(G) = r^p - 1\) and

\[
\delta_G(n) = -\alpha_p(G) + r \cdot \left\{ \frac{n-1}{p} \right\} = \begin{cases} 0, & p \mid n \\ r\left\{ \frac{n}{p} \right\} - 1, & p \nmid n. \end{cases}
\]

Hence, if \(p\) does not divide \(n\) we get

\[
Z(F, G; n) = \frac{|G|}{|\text{Aut}(G)|} q^{n_{G}(G)} q^{r\left\{ \frac{n}{p} \right\}}^{-1} \prod_{j=0}^{r-1} \left(1 - \frac{p^{j+1}}{q^{(p-1)(n-1)/p + \left\{ \frac{n-1}{p} \right\} + (n-1)}\right).
\]

(b) Let \(G = C_{p^r}\) be cyclic. Then \(\alpha_p(G) = \frac{p^r - 1}{p^r}\) and

\[
\delta_G(n) = -\alpha_p(G) + \left\{ \frac{n-1}{p^r} \right\} = \begin{cases} 0, & p^r \mid n \\ \left\{ \frac{n}{p^r} \right\} - 1, & p^r \nmid n. \end{cases}
\]

Hence, if \(p^r\) does not divide \(n\) we get

\[
Z(F, G; n) = \frac{|G|}{|\text{Aut}(G)|} q^{n_{G}(G)} q^{r\left\{ \frac{n}{p^r} \right\}}^{-1} \left(1 - \frac{p^{r-1}}{q^{(p-1)(n-1)/p^r + \left\{ \frac{n-1}{p^r} \right\} + (n-1)}\right).
\]

5. Arbitrary finite abelian groups

We now consider an arbitrary abelian group \(G\) and fix a prime number \(\ell \in \mathbb{P}\) with \(p \neq \ell\). We denote by \(G\ell\) the \(\ell\)-Sylow subgroup of \(G\).

The task is to count the number of open subgroups \(U \leq F^x\) such that \(F^x/U \cong G\ell\). Then the extension given by \(U\) is at most tamely ramified, so the conductor exponent is \(\leq 1\) and as such \(1 + p \leq U\). Hence we can consider \(G\ell\) as a quotient of \(\mathbb{Z} \times \mathbb{F}_q^x \cong \mathbb{Z} \times C_{q-1}\). Obviously, the only possible quotients isomorphic to \(\ell\)-groups are groups of the form \(C_{a\ell^a} \times C_{b\ell^b}\), where \(a \geq b\) and \(\ell^b \mid q - 1\). In the following
remark we only consider situations, where \( G \) is a subgroup of \( A \) and we assume that the exponent of \( A \) equals the exponent of \( G \).

**Remark 14.**
Let \( G = C_{t^a} \times C_{t^b} \) and \( A = C_{r^c} \times C_{r^d} \) with \( a \geq d \geq b \).
(a) If \( a = b \) or \( d = 0 \), then \( \alpha_G(A) = \alpha_G(G) = 1 \).
(b) If \( a > b \) then
\[
\alpha_G(A) = \begin{cases} 
\ell^{d-b}, & a > d, \\
(\ell + 1)^{d-b-1}, & a = d.
\end{cases}
\]

**Proof.** We write \( r_k(A) := \text{rk}_k(A) \) and \( r_k(G) := \text{rk}_k(G) \) in this proof.
(a) If \( a = b \) or \( d = 0 \), then we get \( G = A \) and therefore \( \alpha_G(A) = \alpha_G(G) = 1 \).
(b) By (1) and Lemma (a) we have
\[
\alpha_G(A) = \prod_{k=1}^{a} \ell^{r_{k+1}(G)r_k(A)} \prod_{j=0}^{r_k(G)-1} \left( \ell^{r_k(G)} - \ell^{r_{k+1}(G)+j} \right) = \frac{(\ell^{r_k(A)} - \ell)(\ell^{r_k(A)} - 1)}{\ell - 1} \prod_{k=1}^{a} \ell^{r_{k+1}(G)(r_k(A) - r_k(G))}.
\]

As \( r_k(G) = r_k(A) \) for \( k \leq b \) we have \( \ell^{r_k(G)} - \ell = \ell^2 - \ell \). Moreover:
\[
\prod_{k=1}^{a} \ell^{r_{k+1}(G)(r_k(A) - r_k(G))} = \prod_{k=b+1}^{a-1} \ell^{(r_k(A)-1)} = \ell^{\min(a-1,d)-b} \quad \text{and}
\]
\[
\frac{\ell^{r_k(A)} - 1}{\ell - 1} = \begin{cases} 
\ell + 1, & r_a(A) = 2 \iff d = a \\
1, & r_a(A) = 1 \iff b \leq d < a.
\end{cases}
\]

Note in the following theorem that \( X_1 = \mathbb{Z}/\exp(G)\mathbb{Z} \times \mathbb{Z}/(q - 1)\mathbb{Z} \). We still use the notation \( G_{\ell^p} \) for the prime to \( p \)-part of \( G \).

**Theorem 15.** Let \( G \) be a finite abelian group and \( F = \mathbb{F}_q((t)) \) with \( q = p^f \).
(a) \( G \) is realisable as a Galois group over \( F \) if and only if \( G_{\ell^p}^{q-1} \) is cyclic for all prime numbers \( \ell \nmid p \).
(b) If \( G \) is realisable then for all \( n \geq 1 \) we have
\[
Z(F,G;\nu) = Z(F,G_{\ell^p};\nu) \cdot \prod_{\ell | (q-1)} \alpha_{G|_{\ell^p}}(C_{|\ell^p|} \times C_{q-1}) \leq \frac{(q - 1)q}{2} Z(F,G_{\ell^p};\nu).
\]

**Proof.** We use Lemma
\[
Z(F,G;\nu) = \alpha_G(X_{\nu}) = \prod_{\ell \nmid p} \alpha_{G_{\ell}}(X_{\nu}) = \alpha_{G_\nu}(X_{\nu}) \cdot \prod_{\ell \nmid p} \alpha_{G_{\ell}}(X_{1}).
\]

For the last equation we use that \( \ell \neq p \) and the fact that \( X_{\nu}/X_{1} \) is a \( p \)-group.
If \( G \) is realisable we get for \( \ell \neq p \) that \( G_{\ell} \) is a quotient of \( \mathbb{Z} \times \mathbb{Z}/(q - 1)\mathbb{Z} \) and therefore \( G_{\ell}^{q-1} \) has to be cyclic. Note that for \( \ell \nmid p(q - 1) \) we get by Remark (14) that \( \alpha_{G_{\ell}}(X_{1}) = 1 \). It remains to show the estimate in (b). We have
\[
\prod_{\ell | (q-1)} \alpha_{G_{\ell}}(X_{1}) \leq \prod_{\ell | (q-1)} \ell^{r_{\ell}(q-1)-1}(\ell + 1) = (q - 1) \prod_{\ell | (q-1)} \frac{\ell + 1}{\ell}
\]
bounded discriminant. We define for abelian 
be the counting function of local function field extensions with Galois group 
the counting problem weighted by discriminant. Let 
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Theorem 16. We use the local version of the conductor-discriminant theorem, see [5, Thm. 7.15].
Theorem 17. Let 
Then we have
which has trivial conductor. Thus:
By Lemma 4(c), we have
The idea of the proof of Theorem 17 is contained in [7, Ch. 2].
Proof. Let n be the conductor exponent and U be the norm group of 
Using 
we have for 

Mk := \{\chi \text{ character of } G : G[p^{k-1}] \leq \text{Ker}(\chi) \land G[p^k] \not\leq \text{Ker}(\chi)\}.
By Lemma 4(c), we have \(c(U_n[p^{k-1}]) = \lfloor n/p^{k-1}\rfloor\) for all \(\chi \in Mk\).
Then we have \(|M_k| = |G/G[p^{k-1}]| - |G/G[p^{k}]| = |G[p^{k-1}]| - |G[p^{k}]|\). Moreover, \(\sum_{k=1}^\epsilon |M_k| = |G| - 1\) and the \(M_k\) are disjoint – we only miss the trivial character which has trivial conductor. Thus:

\[
N(D(E/F)) \leq \prod_{k=1}^\epsilon q^{n/p^{k-1}} = \prod_{k=1}^\epsilon q^{n/p^{k-1}|M_k|} = \prod_{k=1}^\epsilon q^{n/p^{k-1}|(G[p^{k-1}] - |G[p^k]|)}
\]

\[
\leq \prod_{k=1}^\epsilon q^{n/p^{k-1}+1)((G[p^{k-1}] - |G[p^k]|)} = q^{\sum_{k=0}^{\epsilon-2} p^{-k} n/|G[p^k]|}|G|^{-1} = q^{n\rho(G)} q^{|G|^{-1}}.
\]

Note that the last estimate can be easily improved.

6. APPLICATION TO COUNT BY DISCRIMINANTS

The asymptotic behaviour weighted by conductor gives interesting insights to the counting problem weighted by discriminant. Let \(G\) be a finite group and
\(D(F,G; n) = |\{E/F : \text{Gal}(E/F) \cong G, N(D(E/F)) \leq q^n\}|\)
be the counting function of local function field extensions with Galois group \(G\) and bounded discriminant. We define for abelian \(p\)-groups \(G\) of exponent \(p^\epsilon\) :

\[
\beta_p(G) := \frac{\alpha_p(G)}{\rho(G)}, \quad \text{where } \rho(G) := \sum_{k=0}^{\epsilon-1} \frac{1}{p^k} \left( |G[p^k]| - |G[p^{k+1}]| \right).
\]

We use the local version of the conductor-discriminant theorem, see [4 Thm. 7.15].

Theorem 16. Let \(K'/K\) be a finite abelian extension of local fields, then
\(D(K'/K) = \prod_\chi \mathfrak{d}(\chi),\)
where the product is taken over all characters \(\chi\) of \(\text{Gal}(K'/K)\) and \(\mathfrak{d}(\chi) = q^{c(\text{Ker}(\chi))}\).

The idea of the proof of Theorem 17 is contained in [7 Ch. 2].

Theorem 17. Let \(F = \mathbb{F}_q((t))\), \(G\) be a finite abelian \(p\)-group and \(n \in \mathbb{N}\).
(a) Let \(E/F\) be a normal extension with Galois group \(G\) and \(N(\mathfrak{d}(E/F)) = q^n\).
Then
\[N(D(E/F)) \leq N(\mathfrak{d}(E/F))^{\rho(G)} q^{G^{-1}} = q^{n\rho(G)} q^{|G|^{-1}}.
\]
(b) There exists a constant \(\gamma(F,G) > 0\) such that
\(D(F,G; n) \geq \gamma(F,G) \cdot q^{n\beta_p(G)}\).

Proof. Let \(n\) be the conductor exponent and \(U\) be the norm group of \(E^\times\). Using \(G = F^\times/U\), we have for \(k = 1, \ldots, \epsilon\) : 

\[M_k := \{\chi \text{ character of } G : G[p^{k-1}] \leq \text{Ker}(\chi) \land G[p^k] \not\leq \text{Ker}(\chi)\}.
\]

By Lemma 4(c), we have \(c(U_n[p^{k-1}]) = \lfloor n/p^{k-1}\rfloor\) for all \(\chi \in M_k\).

Then we have \(|M_k| = |G/G[p^{k-1}]| - |G/G[p^{k}]| = |G[p^{k-1}]| - |G[p^{k}]|\). Moreover, 
\(\sum_{k=1}^\epsilon |M_k| = |G| - 1\) and the \(M_k\) are disjoint – we only miss the trivial character which has trivial conductor. Thus:

\[
N(D(E/F)) \leq \prod_{\text{char. of Gal}(E/F)} \prod_{k=0}^{\epsilon-1} N(\mathfrak{d}(\chi)) = \prod_{k=1}^\epsilon q^{n/p^{k-1}} = \prod_{k=1}^\epsilon q^{n/p^{k-1}|M_k|} = \prod_{k=1}^\epsilon q^{n/p^{k-1}|(G[p^{k-1}] - |G[p^k]|)}
\]

\[
\leq \prod_{k=1}^\epsilon q^{n/p^{k-1}+1)((G[p^{k-1}] - |G[p^k]|)} = q^{\sum_{k=0}^{\epsilon-2} p^{-k} n/|G[p^k]|}|G|^{-1} = q^{n\rho(G)} q^{|G|^{-1}}.
\]
Hence \( D(F; G; n) \geq Z(F; G; n) \). We set \( \tilde{n} := \lfloor (n - |G| + 1)/\rho(G) \rfloor \).

By Theorem 12 there exists a constant \( C > 0 \) such that
\[
Z(F; G; \tilde{n}) \geq C q^{\alpha_p(G)}.
\]

Hence in total
\[
D(F; G; n) \geq D(F; G; n) \geq Z(F; G; \tilde{n}) \geq C q^{\alpha_p(G)}.
\]

Finally, using index shifts we can easily show that the constant \( \beta_p(G) \) coincides with the asymptotic exponent given in Satz 2.1 in [6] (denoted \( a_p(G) \) in his work).

References

[1] S. Delsarte. Fonctions de Möbius sur les groupes abéliens finis. Ann. of Math. (2), 49:600–609, 1948.
[2] J. Ellenberg and A. Venkatesh. Counting extensions of function fields with bounded discriminant and specified Galois group. In Geometric Methods in Algebra and Number Theory, volume 235 of Progress in Mathematics, pages 151–168. Birkhäuser, 2005.
[3] I. B. Fesenko and S. V. Vostokov. Local fields and their extensions, volume 121 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, second edition, 2002. With a foreword by I. R. Shafarevich.
[4] H. Hasse. Number theory, volume 229. Springer-Verlag, Berlin-New York, 1980. Translated from the third German edition and with a preface by Horst Günter Zimmer.
[5] K. Iwasawa. Local class field theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986. Oxford Mathematical Monographs.
[6] T. Lagemann. Asymptotik wild verzweigter abelscher Funktionenkörper. Dissertationsschrift, Technische Universität Berlin, 2010. Logos-Verlag, ISBN 978-3-8325-2710-5.
[7] T. Lagemann. Distribution of Artin-Schreier-Witt extensions. J. Number Theory, 148:288–310, 2015.
[8] A. Ledet. Brauer type embedding problems, volume 21 of Fields Institute Monographs. American Mathematical Society, Providence, RI, 2005.
[9] S. Mäki. The conductor density of abelian number fields. J. London Math. Soc. (2), 47(1):18–30, 1993.
[10] J. Neukirch, A. Schmidt, and K. Wingberg. Cohomology of number fields, volume 323. Springer-Verlag, Berlin, second edition, 2008.
[11] J.-P. Serre. Une “formule de masse” pour les extensions totalement ramifiées de degré donné d’un corps local. C. R. Acad. Sci. Paris Sér. A-B, 286(22):A1031–A1036, 1978.
[12] D. Wright. Distribution of discriminants of abelian extensions. Proc. London Math. Soc., 58:17–50, 1989.