SPHERICAL AND GAUSSIAN SPIN GLASSES

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Abstract. We report some results on spin glass systems with soft spin, spherical and Gaussian. We analyse the Legendre variational structure linking the two models, in the spirit of equivalence of ensembles. At the end we discuss the existing link between these spin glasses and the analogical Hopfield model of neural networks.

1. Introduction, Motivations and Main Definitions

The duality between Gaussian measure and uniform measure on the sphere is nowadays a classical argument, common to probability and mathematical physics. It goes back traditionally to Poincaré and we remand to the paper [12] for a detailed mathematical and historical discussion. Roughly, the probabilistic idea is that the spherical measure of cylindrical sets approaches the Gaussian one when the dimension becomes infinite (see e.g. [19]). Physically this means that for a gas of non interacting particles, therefore with a fixed kinetic energy, the single particle velocity is distributed according to the Maxwell-Boltzmann statistics in the thermodynamic limit.

For spin systems, the intimate connection between Gaussian and spherical models has been noticed since their first systematic introduction by Berlin and Kac in [7]. In the present paper we are going to investigate this relation for spin glasses.

We will consider a system of $N$ soft spin $z_i \in \mathbb{R}$, $i = 1 \ldots N$ interacting throughout the mean field disordered Hamiltonian:

$$H_N(z, J) = -\frac{1}{\sqrt{N}} \sum_{(i,j)} J_{ij} z_i z_j.$$  \hfill (1)

In the whole paper we will make the following hypothesis on the disorder:

**H.** The random matrix $J_{ij}$ is in the Symmetric Wigner Ensemble (see for instance [1],[27] for definitions and properties). Therefore we have: $\mathbb{E}[J_{ij}] = 0$ and $\mathbb{E}[J_{ij} J_{hk}] = J^2 \delta_{ij} \delta_{hk}$ for a certain constant $J^2$. Moreover we assume that there is a $\theta > 0$ such that $\mathbb{E} \left[ e^{\theta \left( \frac{J_{ij}}{\sqrt{N}} \right)^2} \right] \leq \infty \ \forall \ i, j \ \text{uniformly}$.

Then $\left\{ J_{ij} / \sqrt{N} \right\}$ can be diagonalised and $\text{Spect}[J] := \text{Spect} \left[ \{ \frac{J_{ij}}{\sqrt{N}} \} \right] \in \mathbb{R}$. Furthermore

1. There is a $\bar{\lambda} > 0$ such that $\forall a > \bar{\lambda}$

$$P(|\lambda| \geq a) \leq C_1 e^{-\theta a^2 N},$$

for any $\lambda \in \text{Spect}[J]$ and a constant $C_1$;

2. The distribution of eigenvalues of $\left\{ J_{ij} / \sqrt{N} \right\}$ converges for $N \to \infty$ to the semicircle law

$$\rho(\lambda) = \frac{2\sqrt{\lambda^2 - \bar{\lambda}^2}}{\pi \bar{\lambda}^2}.$$ 

We will name the constant $\bar{\lambda} := \max \text{Spect}[J] = J\sqrt{2}$ the maximum eigenvalue of $\left\{ J_{ij} / \sqrt{N} \right\}$. 

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We will be concerned here about two kinds of distribution for the continuous spin $z$, namely the uniform distribution on the $N$ dimensional sphere in $\mathbb{R}^N$, centred in the origin with radius $R\sqrt{N}$, or spherical distribution $\sigma_N(z)$ and the Gaussian distribution $\gamma_N(z)$. We are going to state the equivalence of these two models in the meaning of equivalence of ensembles of statistical mechanics. Our interest in the topic comes from the theory of neural networks. In collaboration with A. Barra and F. Guerra we have proven that, in the case of Gaussian distributed patterns, the free energy of the Hopfield model can be written as a convex sum of the free energy of the Sherrington-Kirkpatrick (SK) model and a suitably defined Gaussian spin glass\footnote{Kirkpatrick (SK) model and a suitably defined Gaussian spin glass\cite{GIA}.}. This decomposition holds exactly in the high temperature regime and in the replica symmetric approximation. We stress that the same feature had been already observed in other bipartite spin glasses (see\footnote{Jones in\cite{GIA}] we have approached the problem from the nowadays usual perspective in spin glass theory, after the celebrated results by Guerra and Talagrand on the SK model (for which we refer to\cite{GIA} and\cite{GIA})): we have studied the Edward Anderson order parameter, i.e. the replicas overlap, in particular by using Guerra’s interpolation. The main achievement contained is that the broken replica symmetry (RSB) bound does not improve the replica symmetric (RS) one, that is:\footnote{In \cite{GIA} we have approached the problem from the nowadays usual perspective in spin glass theory, after the celebrated results by Guerra and Talagrand on the SK model (for which we refer to\cite{GIA} and \cite{GIA}): we have studied the Edward Anderson order parameter, i.e. the replicas overlap, in particular by using Guerra’s interpolation. The main achievement contained is that the broken replica symmetry (RSB) bound does not improve the replica symmetric (RS) one, that is:}

\begin{equation}
A_N^\beta(\beta, J, \lambda) = \frac{1}{N} \log \frac{1}{E} \sum_{\mathbf{z}} \exp\left(-\beta H_N(\mathbf{z}, J) - \frac{\lambda}{2} \sum_{i=1}^N \|z_i\|^2\right),
\end{equation}

such that $E Z(\beta, J, 0) = 1$ for Gaussian distributed disorder, with the associated quenched pressure

\begin{equation}
A_N = \frac{1}{N} \log Z_N^\beta(\beta, J, \lambda).
\end{equation}

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\begin{equation}
A_N^\beta(\beta, J, \lambda) = -\frac{1}{2} \log((1 - \beta^2 \bar{q})) + \frac{\beta^2 \bar{q}}{1 - \beta^2 \bar{q}} + \frac{\beta^2 \bar{q}^2}{4},
\end{equation}

with the RS order parameter given by $\bar{q} = 0$ for $\beta \leq 1 - \lambda$ and $\bar{q} = \frac{\lambda}{\lambda - 1}$ otherwise.

Originally, at least on the mathematical side, this model had been introduced and completely solved by Ben Arous, Dembo and Guionnet in\cite{GIA}. In this paper equilibrium as well as non equilibrium properties are analysed, the latter in the context of the aging phenomenon in the Langevin dynamics, via the construction of a large deviation principle. We present another strategy to obtain a part, the one about equilibrium, of the achievements there contained.

On the other hand, the spherical model of spin glass was introduced by Kosterlitz, Thouless and Jones in\cite{GIA}, where the authors gave the form of the free energy. It turns out to be RS and their method, although the proof passes over some mathematical details, is rigorous. Then Crisanti and Sommers have studied the $p$-spin case in\cite{GIA} and successively Talagrand has proved in all the details the validity of the general Crisanti-Sommers solution in\cite{GIA}.

The Spherical Model is defined as follows: be $z_i \in \mathbb{R}$, $i = 1...N$, be i.i.d. random (soft) spin $\mathcal{N}(0, 1)$ variables, and let them interact throughout the Hamiltonian\footnote{In \cite{GIA} we have approached the problem from the nowadays usual perspective in spin glass theory, after the celebrated results by Guerra and Talagrand on the SK model (for which we refer to\cite{GIA} and\cite{GIA}): we have studied the Edward Anderson order parameter, i.e. the replicas overlap, in particular by using Guerra’s interpolation. The main achievement contained is that the broken replica symmetry (RSB) bound does not improve the replica symmetric (RS) one, that is:}

\begin{equation}
Z_N^\beta(\beta, J, R) = \int_{\mathbb{R}^N} \sigma_N(z) e^{-\beta H_N(z, J)}.
\end{equation}

To this partition function it is associated the pressure:

\begin{equation}
A_N^\beta(\beta, R) = \frac{1}{N} \log Z_N^\beta(\beta, R).
\end{equation}

Usually in the letterature one finds $\lambda, R = 1$. The results of Crisanti, Sommers and Talagrand, along with later works on the subject (see for instance\cite{GIA}), deal with Gaussian disorder. On the other hand we will see that this assumption can be relaxed in the sense of hypothesis $H$ (see also\cite{GIA}). However the behaviour of these models seems to be much more sensible to the disorder
distribution with respect to, for instance, the SK model. We will discuss further this point at the end of the paper.

The work is organised as follows:
In Section 2 we will introduce Gaussian and spherical models defined on a spherical shell. These will be useful tool models in the rest of the paper, and some accessory results on them will be presented.
In Section 3 we will study the free energy of the spherical model: we will characterise it by a variational principle in the wake of the paper [18].
In Section 4 we will expound the main result of this work: the equivalence of spherical and Gaussian ensemble. The Legendre structure linking the two models will be analysed there.
Section 5 is devoted to use the previous achievements on the spherical free energy and on the spherical-Gaussian Legendre duality in order to obtain an explicit expression of the free energy for the Gaussian model as well.
We will discuss the relation with the analogical Hopfield model in Section 6.
Section 7 is left to concluding remarks.

2. Models on Spherical Shells

As a first step it is useful to introduce suitable restrictions of the model of our interest to a spherical shell. We define the \( \varepsilon \)-spherical shell of radius \( R \) as
\[
S_{R}^{\varepsilon} = \left\{ z_1, \ldots, z_n \in \mathbb{R}^N : R - \frac{\varepsilon}{2} < \|z\| \leq R + \frac{\varepsilon}{2} \right\},
\]
and of course \( \bigcup_R S_{R}^{\varepsilon} = \mathbb{R}^N \) \( \forall \varepsilon > 0 \). Thus we define the spherical shell partition function as
\[
Z_{sh}^{N,\varepsilon}(\beta, J, R_N) = \frac{1}{\varepsilon} \int_{S_{R_N}^{\varepsilon}} \frac{dz_1 \ldots dz_N}{S_N} e^{-\beta H_N(z, J)},
\]
where \( H_N(z, J) \) is given by (1) and \( R_N \) is a given \( N \)-sequence of radii. This turns out to be a fuzzy version of the spherical model, tuned such that, for the choice \( R_N = R \sqrt{N} \),
\[
\lim_{\varepsilon \to 0} Z_{sh}^{N,\varepsilon}(\beta, J, R_N) = Z_{sf}^N(\beta, J, R).
\]
We notice also that, once defined \( \delta^\varepsilon \) as a mollified projector on the sphere, we can also write
\[
Z_{sh}^{N,\varepsilon}(\beta, J, R) = \int_{\mathbb{R}^N} dz_1 \ldots dz_N \frac{\delta^\varepsilon(\|z\| = R)}{S_N} e^{-\beta H_N(z, J)}.
\]
In what follows we will take \( \delta^\varepsilon = \chi([R - \varepsilon/2, R + \varepsilon/2])/\varepsilon \), where \( \chi(\cdot) \) is the characteristic function of an interval in the radial coordinate.

To this partition function it is associated a pressure
\[
A_{sh}^{N,\varepsilon}(\beta, R_N) = \frac{1}{N} \mathbb{E} \log Z_{sh}^{N,\varepsilon},
\]
and we naturally have
\[
\lim_{\varepsilon \to 0} A_{sh}^{N,\varepsilon}(\beta, R_N) = A_{sf}^N(\beta, R_N).
\]
In the sequel, the existence of the thermodynamic limit for the pressure can be proved for both the spherical and the spherical shell model; furthermore, the convergence of the latter is also uniform in \( \varepsilon \), as stated by the following

**Lemma 1.** The \( N \to \infty \) and the \( \varepsilon \to 0 \) limits for the pressure of the spherical shell model can be taken in any order to obtain the one of the spherical model, viz.
\[
\lim_{N \to \infty} \lim_{\varepsilon \to 0} A_{sh}^{N,\varepsilon}(\beta, R_N) = \lim_{\varepsilon \to 0} \lim_{N \to \infty} A_{sh}^{N,\varepsilon}(\beta, R_N) = A_{sf}^N(\beta, R).
\]
Proof. By taking the $N \to \infty$ limit of (10) we obtain the first part of (11). On the other side, by the mean-value theorem of integration, we can write

$$A_N^{f, \epsilon}(\beta, R_N) = A_N^f(\beta, R_N, \epsilon),$$

for some $R_N, \epsilon \in [R_N - \epsilon/2, R_N + \epsilon/2]$. In this way, setting $R_N = R\sqrt{N}$, we can estimate the difference

$$\left| A_{N, \epsilon}^{f}(\beta, R_N) - A_N^f(\beta, R_N) \right| = \left| A_N^f(\beta, R_N, \epsilon) - A_N^f(\beta, R_N) \right|$$

$$= \frac{1}{N} \mathbb{E} \log \left( \frac{Z_N^{f}(\beta, J, R_N, \epsilon)}{Z_N^{f}(\beta, J, R_N)} \right).$$

Now, by using the properties of the spherical integral, we can turn the problem of integration on a different radius into a more treatable shift in temperature, i.e.

$$Z_N^f(\beta, J, R_N, \epsilon) = \int_{\|z\| = R_N} dz e^{-\beta H_N(z, J)} \int_{\|z\| = R_N} dz e^{-\beta^2 H_N(z, J)} \mathbb{E} \log \frac{Z_N^{f}(\beta, J, R_N, \epsilon)}{Z_N^{f}(\beta, J, R_N)}$$

such that

$$\frac{1}{N} \mathbb{E} \log \left( \frac{Z_N^{f}(\beta, J, R_N, \epsilon)}{Z_N^{f}(\beta, J, R_N)} \right) = \frac{1}{N} \mathbb{E} \log \omega_{R_N}^{f} \left( \exp(-\beta H N(z, J)) \right),$$

where we have isolated the $\epsilon$-dependence in the term $\frac{R_N^2}{R_N^2} - 1 = O(\epsilon/\sqrt{N})$. Finally, because of the extensive nature of the internal energy, i.e. $\sup (H_N) / N < \infty$, we obtain that

$$\left| A_{N, \epsilon}^{f}(\beta, R_N) - A_N^f(\beta, R_N) \right| = \left| A_N^f(\beta, R_N, \epsilon) - A_N^f(\beta, R_N) \right|$$

$$= O(\epsilon/\sqrt{N}) \xrightarrow{N \to \infty} 0.$$ \(\square\)

We notice that the second part of (11) holds already before taking the $\epsilon \to 0$ limit.

In the proof of the previous lemma we have also shown that, in the thermodynamic limit, the spherical shell model (on radius $R\sqrt{N}$ and thickness $\epsilon$) and a spherical model on any radius inside the shell, are both equivalent to a spherical model on radius $R\sqrt{N}$.

In the same way we can define the Gaussian model on the shell, as a cut off:

$$Z_{N, \epsilon}^{gsh}(\beta, J, \lambda, R_N) = \frac{1}{\mathbb{E}} \int_{S_{R_N}^{g}} d\beta \cdots d\lambda \cdots e^{-\beta H_N(z, J) - \beta^2 \lambda \|z\|^2 + \lambda(\lambda - 1) \|z\|^2 / 2},$$

$$A_{N, \epsilon}^{gsh}(\beta, \lambda, R_N) = \frac{1}{N} \mathbb{E} \log Z_{N, \epsilon}^{gsh}.$$ (16)

As in the previous case we have that

$$Z_{N, \epsilon}^{gsh}(\beta, J, \lambda, R_N) = \int_{R_N} d\beta \cdots d\lambda \cdots e^{-\beta H_N(z, J) - \beta^2 \lambda \|z\|^2 + \lambda(\lambda - 1) \|z\|^2 / 2},$$

(17)

Let us take now both a spherical and a Gaussian model on the same spherical shell centred on $R_N = R\sqrt{N}$, for a certain real number $R$. Therefore we can think of fixing the terms $\|z\|^2 = R^2 N$ in the Boltzmannfaktor of the Gaussian model, with a small error $O(\epsilon)$. Of course we do not need to do the same for the spherical one, in virtue of the previous lemma. It is

$$A_{N, \epsilon}^{gsh}(\beta, \lambda, R\sqrt{N}) - A_{N, \epsilon}^{sh}(\beta, R\sqrt{N}) = -\frac{\beta^2}{4} R^4 + \frac{\lambda - 1}{2} R^2 + \frac{1}{N} \log \left( \frac{S_N}{\sqrt{2\pi}} \right) + O(\epsilon)$$

$$= -\frac{\beta^2}{4} R^4 + \frac{\lambda - 1}{2} R^2 + \log R + \frac{1}{2} + O(\epsilon),$$

since $\lim_{N} \frac{1}{N} \log \left( \frac{S_N}{\sqrt{2\pi}} \right) = \log R + \frac{1}{2}$. Thus we have proven the following:
Proposition 1. The spherical shell models are related in the following way:

$$\lim_{\varepsilon \to 0} A_{N,\varepsilon}^{sh} (\beta, \lambda, R \sqrt{N}) = -\frac{\beta^2 R^4}{4} + \frac{(\lambda - 1) R^2}{2} + A_N^f (\beta, R).$$

(18)

3. Free Energy: Spherical Model

In this section we deepen our study of the free energy properties of the spherical model, by deriving the explicit expression that appeared for the first time in [18]. In primis we diagonalise the interaction, as it is usual for this kind of models [18][6][13][9], in virtue of their rotational symmetry:

$$H_N = -\sum_i \lambda_i z_i^2.$$  

(19)

Our first result is a bound on the annealed pressure, and so, a fortiori, on the quenched one. This is defined as:

$$A_N^f (\beta, R) := \frac{1}{N} \log \mathbb{E} Z_N^f,$$

(20)

and of course by Jensen inequality it is $A_N^f (\beta, R) \leq A_N^f (\beta, R)$ uniformly in $N$. It will result helpful to define the events

$$B_a := \{ \lambda_i < a, i = 1, ..., N\}$$

for fixed $N$ (that we omit in the notations) and each $a \geq \bar{\lambda}$, and its complementary $B_a^c$. In the following we will denote with $I_B$ the indicator function of the set $B$. We have

Proposition 2. It is

$$\limsup_N A_N^f \leq A_N^f \leq \frac{1}{2} (\beta \lambda R^2)^2.$$  

(21)

Proof. Let us fix any $a > \bar{\lambda}$. It is

$$\mathbb{E}[e^{\beta \sum_i \lambda_i z_i^2}] = \mathbb{E}[e^{\beta \sum_i \lambda_i z_i^2} I_{B_a}] + \mathbb{E}[e^{\beta \sum_i \lambda_i z_i^2} I_{B_a^c}].$$

For the first addendum on the r.h.s. we easily get the bound

$$\mathbb{E}[e^{\beta \sum_i \lambda_i z_i^2} I_{B_a}] \leq e^{\beta R^2 a N}.$$  

(22)

For the second one we have

$$\mathbb{E}[e^{\beta \sum_i \lambda_i z_i^2} I_{B_a^c}] = \int_{e^{\beta R^2 a N}}^{+\infty} dX \rho (e^{\beta \sum_i \lambda_i z_i^2} \geq X)$$

$$= \int_{e^{\beta R^2 a N}}^{+\infty} dX \rho (\beta \sum_i \lambda_i z_i^2 \geq \log X)$$

$$\leq \int_{e^{\beta R^2 a N}}^{+\infty} dX \rho \left( \max_i \lambda_i \geq \frac{\log X}{\beta R^2 N} \right).$$

Now, since it is $\frac{\log X}{\beta R^2 N} > \bar{\lambda}$ in the integration domain, we obtain

$$\mathbb{E}[e^{\beta \sum_i \lambda_i z_i^2} I_{B_a^c}] \leq C_1 \int_{e^{\beta R^2 a N}}^{+\infty} dX e^{-c_2 \left( \frac{\log X}{\beta R^2 N}\right)^2}$$

$$= C_1 e^{\beta R^2 a^2 N/4} \int_{1}^{+\infty} dY e^{-c_2 (Y - \beta a R^2 \sqrt{N})^2}.$$  

Therefore, neglecting some term vanishing in the limit, we readily get for every $a > \bar{\lambda}$

$$A_N^f (\beta, R) \leq \lim_{N} \frac{1}{N} \log \left( e^{\beta R^2 a N} + e^{\beta R^2 a^2 N/4} \right)$$

$$= \max \left( \beta R^2 a, \frac{\beta^2 R^4 a^2}{4} \right),$$

and hence the inequality is satisfied also by taking the infimum on $a > \bar{\lambda}$, whence (21) follows. \[\square\]
Now we prove a lemma that will be helpful in the rest of the section. Merely the result is that, because of hypothesis (H), only the bounded part of the spectrum of the interaction contributes to the thermodynamics.

**Lemma 2.** It is
\[ E[A_N^s f(\beta, R) I_{B_c}] = O(e^{-N}). \] (23)

**Proof.** At first we note that the gradient of the pressure before expectation is a bounded function:
\[
\frac{1}{N} \partial_{\lambda_i} \log Z_N^s f = \frac{1}{N} \sum_{i=1}^{N} \omega(z_i^2) \leq R^2, \quad N \text{ uniformly.} \] (24)

We have
\[
(\mathbb{E}[A_N^s f I_{B_c}])^2 \leq P(B_c^*) (\mathbb{E}[A_N^s]) + (\mathbb{E}[A_N^s f])^2
\leq P(B_c^*) (\|\nabla A_N^s\|_2^2 + (\mathbb{E}[A_N^s f])^2),
\]
where we have exploited Poincaré inequality w.r.t. the empirical measure of the eigenvalues. Due to the previous proposition and (24), we obtain the assert by the decay required in (H). \(\square\)

Because of this last property we have immediately the subsequent

**Corollary 1.** The following a priori bound holds
\[
\limsup_{N} A_N^s f(\beta, R \sqrt{N}) \leq \beta \bar{\lambda} R^2. \] (25)

**Proof.** For Lemma 2 we can consider only particular realisations of the disorder such that the spectrum remains bounded almost surely. Hence the partition function can be easily bounded by
\[ Z_N^s f(\beta, R \sqrt{N}) \leq e^{\beta \bar{\lambda} R^2 N}. \]

Therefore
\[
\frac{1}{N} \mathbb{E} \left[ \log Z_N^s f(\beta, R \sqrt{N}) \sup_{i=1,\ldots,N} |\lambda_i| \leq \bar{\lambda} \right] \leq \bar{\lambda} R^2,
\]
and (25) easily follows. \(\square\)

This bound, albeit rather course, refines the annealed one for small temperature. However we can do much better than that: we are going to give an explicit expression for the pressure in the thermodynamic limit.

In the past literature, the common way to face the problem relied on a direct calculation of the partition function. Several techniques can be implemented for this task: the original one by Berlin-Kac (see [7], appendix B) and a variant by Montroll [20] make use essentially of Riemann steepest descendent method; in addition, another method developed by Von Neumann in [28] certainly deserves to be mentioned.

We will prefer a more modern perspective, presenting a different and purely variational proof, that captures in our opinion the two essential aspects of the model, to wit: only the largest eigenvalue determines the form of the free energy; thermodynamics naturally forces the equilibrium configurations of the system on the surface of the sphere, even if we allow them to stay in the whole space.

More precisely we are going to prove the following

**Theorem 1.** The pressure of Spherical model is given by the following variational principle:
\[
A^s f(\beta, R) = \min_{q \geq \beta \lambda} \left( qR^2 - \frac{1}{2} \int \rho(\lambda) \log(q - \beta \lambda) - \log R - \frac{1}{2} \frac{1}{2} \log 2 \right). \] (26)
Proof. We consider for simplicity only realisation of the disorder such that the spectrum is contained in an interval $[-\lambda, \lambda]$ with full probability. Then for every $q > \beta \lambda$ we have

$$Z_{N}^{f}(\beta) = e^{q R^{2}N} \int_{\mathbb{R}^{N}} d\sigma_{N}(z) e^{-\sum_{i}^{N} (q - \beta \lambda_{i}) z_{i}^{2}}$$

$$\leq e^{q R^{2}N} \frac{(2\pi)^{\frac{N}{2}}}{SN} \int_{\mathbb{R}^{N}} \frac{dN z}{(2\pi)^{\frac{N}{2}}} e^{-\sum_{i}^{N} (q - \beta \lambda_{i}) z_{i}^{2}}$$

$$= e^{q R^{2}N} \frac{(2\pi)^{\frac{N}{2}}}{SN} e^{-\frac{1}{2} \sum_{i}^{N} 2(q - \beta \lambda_{i})}$$

and so

$$\limsup_{N} A_{N}^{f} (\beta) \leq q R^{2} - \frac{1}{2} \int \rho(\lambda) \log(q - \beta \lambda) - \log R - \frac{1}{2} - \frac{1}{2} \log 2 := \tilde{A}(q).$$

(27)

We note that for $q > \beta \lambda$

$$\partial_{q}^{2} \tilde{A}(q) = \frac{1}{2} \int d\lambda \frac{\rho(\lambda)}{(q - \beta \lambda)^{2}} > 0,$$

and so the functional $\tilde{A}(q)$ is uniformly convex (independently on $R$). Furthermore we can explicitly verify that $\tilde{A}(q)$ is actually continuous for $q \to \beta \lambda$. All that permits to improve the previous bound as

$$\limsup_{N} A_{N}^{f} (\beta) \leq \min_{q \geq \beta \lambda} \tilde{A}(q),$$

and, since the functional is uniformly convex in $q$, there is a unique point $\bar{q}$ where the minimum is attained.

The reverse bound is slightly less direct. Let us consider for each $\varepsilon > 0$ the spherical shell around the radius $R \sqrt{N}$ (as defined in the previous section). Since $S_{\varepsilon} \cup S_{\varepsilon}^{c} = \mathbb{R}^{N}$, it is

$$\varepsilon Z_{\varepsilon N} = e^{q R^{2}N} \frac{(2\pi)^{\frac{N}{2}}}{SN} \int_{\mathbb{R}^{N}} \frac{dN z}{(2\pi)^{\frac{N}{2}}} e^{-\sum_{i}^{N} (q - \beta \lambda_{i}) z_{i}^{2}} - e^{q R^{2}N} \frac{(2\pi)^{\frac{N}{2}}}{SN} \int_{S_{\varepsilon}^{c}} \frac{dN z}{(2\pi)^{\frac{N}{2}}} e^{-\sum_{i}^{N} (q - \beta \lambda_{i}) z_{i}^{2}}.$$

In primis we note that we can use the Chernoff bound to estimate the second addendum. In fact, once introduced the free parameters $\mu, \eta > 0$ we have

$$\int_{S_{\varepsilon}^{c}} \frac{dN z}{(2\pi)^{\frac{N}{2}}} e^{-\sum_{i}^{N} (q - \beta \lambda_{i}) z_{i}^{2}} \leq \exp \left[ N \left( \frac{\mu (R^{2} - \varepsilon)}{N} - \frac{1}{2} \sum_{j} \log(q - \beta \lambda_{j} + \mu) - \frac{1}{2} \log 2 \right) \right]$$

$$+ \exp \left[ N \left( -\eta (R^{2} + \varepsilon) - \frac{1}{2} \sum_{j} \log(q - \beta \lambda_{j} - \eta) - \frac{1}{2} \log 2 \right) \right].$$

Thereby the r.h.s. of the last inequality is $o(e^{-N})$ if $\frac{\varepsilon}{N} \to \infty$ when $N \to \infty$, so it does not contribute to thermodynamics. Therefore we can neglect $\varepsilon$ growing at those scales, by setting $\varepsilon = \varepsilon N$, $\varepsilon$ positive and independent by $N$. Thus, by a straightforward computation, we obtain in the limit $N \to \infty$

$$\liminf_{N} A_{\varepsilon N}^{sh} \geq \max \left( \tilde{A}(q), A_{N}^{f}(\mu; q), A_{N}^{f}(\eta; q) \right),$$

(28)

with

$$A_{N}^{f}(\mu; q) = (\mu + q) R^{2} - \tilde{\varepsilon} \mu - \frac{1}{2} \int d\lambda \rho(\lambda) \log(q - \beta \lambda + \mu) - \frac{1}{2} - \frac{1}{2} \log 2 - \log R;$$

$$A_{N}^{f}(\eta; q) = (q - \eta) R^{2} - \tilde{\varepsilon} \eta - \frac{1}{2} \int d\lambda \rho(\lambda) \log(q - \beta \lambda - \eta) - \frac{1}{2} - \frac{1}{2} \log 2 - \log R.$$

(29)

(30)

Our aim is to show that for some $q \tilde{A}(q)$ is greater than these other two quantities. We have

$$d_{1}(q; \mu) := \tilde{A}(q) - A_{N}^{f}(\mu; q) = -\mu (R^{2} - \tilde{\varepsilon}) + \frac{1}{2} \int d\lambda \rho(\lambda) \log \left( \frac{q - \beta \lambda + \mu}{q - \beta \lambda} \right);$$

$$d_{2}(q; \mu) := \tilde{A}(q) - A_{N}^{f}(\eta; q) = \eta (R^{2} + \tilde{\varepsilon}) + \frac{1}{2} \int d\lambda \rho(\lambda) \log \left( \frac{q - \beta \lambda - \eta}{q - \beta \lambda} \right).$$

(31)

(32)
Let us regard for instance to $d_1(q; \mu)$ as a function of $\mu$: it is continuous and derivable, it vanishes in $\mu = 0$ and it goes to $-\infty$ for $\mu \to +\infty$. So it can assume positive values (in particular a positive maximum) if and only if the derivative in $\mu = 0$ is positive, that is

$$0 < -(R^2 - \tilde{\varepsilon}) + \frac{1}{2} \int d\lambda \frac{\rho(\lambda)}{q - \beta \lambda} = \tilde{\varepsilon} - \partial_q \tilde{A}(q), \quad (33)$$

where we have used that in the $\mu$ derivative of $d_1(q; \mu)$ it appears exactly the derivative $\partial_q \tilde{A}(q)$ (see the explicit analysis below).

Analogously $d_2(q; \eta)$ is zero in the origin and it approaches $+\infty$ for $\eta \to +\infty$. Thus it is always positive provided that $\partial_q d_2(q; \eta)|_{\eta=0} \geq 0$, i.e.

$$0 \leq (R^2 + \tilde{\varepsilon}) + \frac{1}{2} \int d\lambda \frac{\rho(\lambda)}{q - \beta \lambda} = \tilde{\varepsilon} + \partial_q \tilde{A}(q). \quad (34)$$

Conditions (33) and (34) have to be satisfied together, therefore we seek a $\tilde{q}$ such that for every $\tilde{\varepsilon} > 0$ it is

$$-\tilde{\varepsilon} \leq \partial_q \tilde{A}(q)|_{q=\tilde{q}} < \tilde{\varepsilon}.$$ 

This simply means that $\tilde{q} = \bar{q}$, viz. the unique stationary point of $\tilde{A}(q)$. With this choice of $q$, relation (28) gives

$$\liminf_{N} A_{s,b, N}^{\beta}(\beta) \geq \min_{q \geq \beta \lambda} \tilde{A}(q) - \tilde{\varepsilon} \text{ uniformly,} \quad (35)$$

that concludes the proof. 

Let us look more thoroughly to $\tilde{A}(q)$. The first derivative reads

$$\partial_q \tilde{A}(q) = R^2 - \frac{1}{2} \int d\lambda \frac{\rho(\lambda)}{q - \beta \lambda};$$

hence we see at once that for $\beta < (\bar{\lambda} R^2)^{-1}$ the derivative changes sign from negative to positive in a point $\tilde{q}$ given by the equation

$$\sqrt{\left(\frac{\tilde{q} R^2}{2}\right)^2 - \left(\frac{\beta \bar{\lambda} R^2}{2}\right)^2} = \frac{\tilde{q} R^2}{2} - \frac{\beta \bar{\lambda}^2 R^4}{2}, \quad (36)$$

which is solved by $\tilde{q} = \frac{\frac{1}{\beta \bar{\lambda}^2} + 1}{\frac{1}{\beta \bar{\lambda}^2}} (1 + \bar{\lambda}^2 \lambda^2 R^4)$.

On the other hand for $\beta > \frac{1}{\bar{\lambda} R^2}$ the derivative is always a positive function (equation (36) is never satisfied): this means that the minimum is attained in the extremum of the interval of definition, i.e. $\bar{q} = \beta \bar{\lambda}$. Thus we have that the critical point is defined by $\beta_c = \frac{1}{\lambda R^2}$ as a singular point of the minimiser function of $\tilde{A}(q)$.

Let us recall that the free energy is defined as $f^{s f} := -\frac{1}{\beta} A^{s f}$. We can exhibit explicitly its expression by a direct calculation:

**Proposition 3.** The free energy of the spherical model has a discontinuity in its third derivative at the point $\beta_c := \frac{1}{\lambda R^2}$. It is

$$f^{s f}(\beta, R, \lambda) = -\frac{1}{\beta} \left( \frac{1}{4} \left( \frac{\beta}{\beta_c} \right)^2 \right) \quad \beta \leq \beta_c, \quad (37)$$

$$f^{s f}(\beta, R, \lambda) = -\frac{1}{\beta} \left( \frac{\beta}{\beta_c} - \frac{1}{2} \log \left( \frac{\beta}{\beta_c} \right) - \frac{3}{4} \right) \quad \beta \geq \beta_c. \quad (38)$$

Finally it is important to establish the self averaging property of the pressure of the spherical model:

**Proposition 4.** We have that $\frac{1}{N} \log Z_N^{s f} \to A^{s f}(\beta, R)$ for $N \to \infty$ a.s.
**Theorem 2.** The Gaussian model is related to the spherical model through the following variational principle:

\[ A^{q}(\beta, \lambda) = \max_{R \in (0, \infty)} \left( A^{sf}(\beta, R) - \frac{\beta^2 R^4}{4} + \frac{(\lambda - 1) R^2}{2} + \log R + \frac{1}{2} \right). \]

To achieve a proof of this statement, we need some preliminary results. At first we present an analogous of the annealed bound for the Gaussian model:

**Proposition 5.** The following bound holds:

\[ \limsup N A^{q}_{sf}(\beta, \lambda) \leq A^{q}_{A}(\beta, \lambda) \leq \max \left( \frac{1}{2} \left( 2 + \frac{\lambda}{\beta} \right)^2, -\frac{1}{2} \log(1 - \lambda) \right). \]

**Proof.** We proceed exactly as in the proof of Proposition 2: at first we use Jensen inequality to exchange the logarithm and the expectation with respect to the quenched disorder; then we compute

\[ E[Z^{q}_{N}(\beta, \lambda) | \mathcal{I}_{B_{\delta}}] \leq \exp \left[ \frac{N}{2} \left( 2 + \frac{\lambda}{\beta} \right)^2 \right]. \]

For the second addendum we can repeat line by line the calculation of Proposition 2, so obtaining

\[ E[Z^{q}_{N}(\beta, \lambda) | \mathcal{I}_{B_{\delta}}] \leq \exp \left[ \frac{N}{2} \left( 2 + \frac{\lambda}{\beta} \right)^2 \right]. \]

Recollecting the contribution given by (32) and (33), we have (31) in the limit \( N \to \infty \).

Then we want to show that the Gibbs measure of the Gaussian Model is concentrated inside a sphere of radius growing as \( \sqrt{N} \). To this purpose, fixed two arbitrary numbers \( \delta > 0 \) and \( \bar{R} > 0 \), we set

\[ T_{N}(\delta, R) := \{ z_{1}, \ldots, z_{N} \in \mathbb{R}^{N} : \| z \|^{2} \geq R^{2} N^{1+\delta} \}, \]

\[ Z^{q}_{N}|_{\| z \|^{2} \geq R^{2} N^{1+\delta}} := \int_{T_{N}(\delta, R)} d z_{1} \ldots d z_{N} \frac{e^{-\| z \|^{2}/2}}{(2\pi)^{N/2}} e^{(-\beta H_{N}(z,J) - \frac{\beta^2}{4N} \| z \|^{4} + \frac{1}{2} \| z \|^{2})}. \]

Now we are ready to establish the subsequent
Proposition 6. Let us fix arbitrarily $R > 0$. It is for every $\delta > 0$

$$Z^g_{||z||^2 \geq R^2 N^{1+\delta}} = O \left( e^{-N^{1+\delta}} \right).$$

(46)

Proof. We proceed to prove (46) by exploiting Markov inequality. We start by defining for a certain event $\Omega \subseteq \mathbb{R}^N$

$$\pi^{N,\lambda}_0(2 \delta) := \frac{1}{2^N} \int_\Omega \gamma_N(z) \exp \left( \beta \sum_{i} N \lambda_i z_i^2 + \lambda ||z||^2 - \frac{\beta}{4N} ||z||^4 \right).$$

(47)

Let us fix $R, \delta > 0$. Using Chernoff bound we get

$$\pi^{N,\lambda}_0(2 \delta) \leq e^{-\frac{\beta}{2} R^2 N^{1+\delta}} \mathbb{E}_\pi \left[ e^{\mu ||z||^2} \right],$$

(48)

for every $\mu > 0$; in particular we choose $\mu \leq \lambda$. In addition we have

$$\log \mathbb{E}_\pi \left[ e^{\mu ||z||^2} \right] = \log \int_{T_N(\delta, R)} dz_1...dz_N \frac{e^{-||z||^2/2}}{(2\pi)^{N/2}} e\left(-\beta H_N(z,J) - \frac{\gamma^2}{4} ||z||^4 + \frac{1}{2} ||z||^2 \right) - N \log Z_N^g(\beta, \lambda)$$

$$= N[\delta(\beta, \lambda - \mu) - \delta(\beta, \lambda)].$$

Therefore

$$Z^g_{||z||^2 \geq R^2 N^{1+\delta}} \leq \exp \left[ -\mu R^2 N^{1+\delta} + N \delta(\beta, \lambda - \mu) \right]$$

$$= \exp \left[ -N^{1+\delta} (\mu R^2 - N^{-\delta} \delta(\beta, \lambda - \mu)) \right]$$

$$\leq \exp \left[ -N^{1+\delta} \max_{\mu \in (0, \lambda)} (\mu R^2 - N^{-\delta} \delta(\beta, \lambda - \mu)) \right],$$

and, because of the annealed bound (41), we obtain (46).

\[ \square \]

Finally we can proceed with the

Proof of Theorem 2. We will show that the r.h.s. of (40) is actually an upper and a lower bound for $A^u_N(\beta, \lambda)$ in the limit $N \to \infty$.

We start with the lower bound. Let us define $\tilde{R}$ as the radius where is reached the maximum of (40) and

$$S^e_{\tilde{R}} = \left\{ z_1, ..., z_n \in \mathbb{R}^N : R^\sqrt{N} - \frac{\lambda}{2} < ||z|| \leq R^\sqrt{N} + \frac{\lambda}{2} \right\}.$$

(49)

We have that

$$Z^g_{N}(\beta, \lambda; J) \geq \int_{S^e_{\tilde{R}}} dz_1...dz_N \frac{e^{-||z||^2/2}}{(2\pi)^{N/2}} e\left(-\beta H_N(z,J) - \frac{\gamma^2}{4} ||z||^4 + \frac{1}{2} ||z||^2 \right)$$

$$\geq \int_{S^e_{\tilde{R}}} dz_1...dz_N \frac{e^{-||z||^2/2}}{(2\pi)^{N/2}} e\left(-\beta H_N(z,J) - \frac{\gamma^2}{4} ||z||^4 + \frac{1}{2} ||z||^2 \right)$$

$$\geq e^{N\left(-\frac{\lambda^2}{4} \tilde{R}^2 - \frac{\lambda R^4}{4} \tilde{R} + \frac{1}{2} \log S^e_{\tilde{R}} - \frac{1}{2} \log(2\pi)\right)} \left( \int_{S^e_{\tilde{R}}} dz_1...dz_N e\left(-\beta H_N(z,J)\right) \right)$$

and then $\forall \varepsilon$

$$A^u_N(\beta, \lambda) \geq \frac{\lambda - 1}{2} R^2 - \frac{\beta^2}{4} R^4 + \frac{1}{N} \log S^e_{\tilde{R}} - \frac{1}{2} \log(2\pi) + A^u_{N,2}(\beta, \tilde{R}^\sqrt{N}).$$

(50)

We take the limit over $N$ on the left, and just the limit for on the right. By using Lemma 1 and the computation of $\frac{1}{N} \log S^e_{\tilde{R}} - \frac{1}{2} \log(2\pi) \to \log \tilde{R} + \frac{1}{2}$, we eventually obtain the r.h.s. of (40) as a lower bound:

$$\liminf_{N} A^u_N(\beta, \lambda) \geq \max_{R \in (0, \infty)} \left(A^f(\beta, R) - \frac{\beta^2 R^4}{4} + \frac{(\lambda - 1) R^2}{2} + \log R + \frac{1}{2}\right).$$

(51)

In order to get the reverse one, for an arbitrary $\delta > 0$ we decompose $\mathbb{R}^N = T_N(\delta) \cup T^c_N(\delta)$, where we recall the definition of $T_N(\delta)$ in (44). In virtue of Proposition 6 the integration over $T_N(\delta)$ does not give any thermodynamic contribution to the free energy. Consequently, for simplicity, in
what follows we will consider as phase space just $T_N^c(\delta)$. If we then look at a generic partition of $T_N^c(\delta)$ into $N^\delta/2\varepsilon$ shells of thickness $2\varepsilon$ as in the definition (49) we can estimate

$$Z_N^g \leq \frac{N^\delta}{2\varepsilon} \max_{R \in [0, N^\delta]} \left( \int_{S_R^N} dz_1 \ldots dz_N e^{-||z||^2/2} \left( \frac{1}{(2\pi)^{N/2}} e^{-\beta H_N(z,J)} - \frac{\pi^2}{4} \log S_R^N - \frac{1}{2} \log(2\pi) \right) \right)$$

and thus

$$A_N^g \leq \frac{\delta}{N} \log \left( \frac{N}{2\varepsilon} \right) + \max_{R \in [0, N^\delta]} \left( \frac{(\lambda - 1)}{2} R^2 - \frac{\beta^2}{4} R^4 + \frac{1}{N} \log S_R^N - \frac{1}{2} \log(2\pi) + N^{N/2} (\beta, R \sqrt{N}) + o_N(1; \varepsilon) \right),$$

where we have used the monotonicity of $\limsup$ and Theorem 4. Again we take the infinite volume limit: the $\limsup$ on the left and the limit on the right. Here we note that it is possible to exchange the limit with the max, since the functional converges uniformly in $R$ in each bounded subset of $\mathbb{R}$ and tends to $-\infty$ as $R \to \infty$ for all $N$. Thus we get the reverse inequality:

$$\limsup_{N} A_N^g(\beta, \lambda) \leq \max_{R \in (0, \infty)} \left( A^f(\beta, R) - \frac{\beta^2 R^4}{4} + \frac{(\lambda - 1) R^2}{2} + \log R + \frac{1}{2} \right).$$

The assert follows from equations (51), (52).

Now we notice that the pressure of the spherical model $A^f(\beta, R)$ depends in fact on $R^2$:

$$Z_N^f(\beta, R) = \int_{||z||^2 = R^2} \frac{dz_1 \ldots dz_N}{S_N} e^{-\beta \sum_i \lambda_i z_i^2}$$

$$= \int_{||z||^2 = R^2} \frac{dz_1 \ldots dz_N}{S_N} e^{-\beta R^2 \sum_i \lambda_i z_i^2}$$

$$= Z_N^f(\beta R^2, 1),$$

with the simple change of variables $R\hat{z} = z$. So we have

$$A^f(\beta, R) = A^f(\beta R^2, 1),$$

and

$$A^f(\beta, \lambda) = \sup_{R^2 \in (0, \infty)} \left( A^f(\beta R^2, 1) - \frac{\beta^2 R^4}{4} + \frac{(\lambda - 1) R^2}{2} + \log R + \frac{1}{2} \right).$$

Now, we want to find the dual relation of the (53), expressing the spherical pressure in terms of a variational principle involving the Gaussian pressure. To this aim we have to rewrite (53) as an ordinary Legendre transformation: we put $\rho := \frac{R^2}{2}$ and we have that

$$A^f(\beta, \lambda) = \sup_{\rho \in (0, \infty)} \left( \lambda \rho + A^f(2\beta \rho, 1) - \rho - \beta^2 \rho^2 + \frac{1}{2} \log(2\rho) + \frac{1}{2} \right) = \mathcal{L} g(\beta, \rho),$$

where

$$g(\beta, \rho) = -A^f(2\beta \rho, 1) + \rho + \beta^2 \rho^2 - \frac{1}{2} \log(2\rho) - \frac{1}{2}$$

and $\mathcal{L} g(\beta, \rho)$ indicates the Legendre transformation

$$\mathcal{L} g(\beta, \rho)(\lambda) = \sup_{\rho \in (0, \infty)} (\lambda \rho - g(\beta, \rho)).$$

Since the function $g(\beta, \rho)$ is strictly convex in $\rho$ (it is two times differentiable with positive and continuous second derivative), the Legendre transformation is well-defined and involutive, in the sense that $\mathcal{L}^2 = I$, to wit $A^f(\beta, \lambda)$ is convex in $\lambda$ and

$$\mathcal{L} A^f(\beta, \lambda)(\rho) = \sup_{\lambda \in (-\infty, \infty)} (\rho \lambda - A^f(\beta, \lambda)) = g(\beta, \rho).$$
Bearing in mind definition (55) we have that
\[ A^s_f(2\beta \rho, 1) = \inf_{\lambda \in (\infty, \infty)} \left( -\rho \lambda + A^s(\beta, \lambda) + \rho + \beta^2 \rho^2 - \frac{1}{\beta} \log(2\rho) - \frac{1}{2} \right) . \] (58)

We finally can recollect our results in the subsequent Proposition 7. The following Legendre duality relates the pressures of the spherical and Gaussian spin glasses:
\[
\begin{cases}
A^s_f(\beta, R) = \min_{\lambda \in (-\infty, \infty)} \left( A^s(\beta, \lambda) + \frac{\beta^2 R^4}{4} + \frac{(1-\lambda) R^2}{2} - \log(R) - \frac{1}{2} \right), \\
A^s(\beta, \lambda) = \max_{R \in (0, \infty)} \left( A^s_f(\beta, R^2) - \frac{\beta^2 R^4}{4} + \frac{(\lambda-1) R^2}{2} + \frac{1}{2} \log R^2 + \frac{1}{2} \right).
\end{cases}
\] (59)

As a last remark, let us go back to (54). We can find the optimal value for \( \rho \) simply by taking the derivative. It is
\[ \rho(\lambda - 1) - 2\beta \rho^2 + \frac{1}{2} + \rho \partial_\rho A^s_f(2\beta \rho, 1) = 0 . \]

Now we define \( \beta' = 2\beta \rho \), and rephrasing the previous relation in terms of \( \beta' \) we get
\[ \beta' \frac{\lambda - 1}{2\beta} - \beta' \lambda + \frac{1}{2} + \beta' \partial_{\beta'} A^s_f(\beta', 1) = 0 . \]

The derivative of the pressure is minus the internal energy: \( u^s_f(\beta', 1) = u^s_f(\beta, R) = -\partial_{\beta'} A^s_f(\beta', 1) \); therefore we conclude
\[ \frac{\lambda - 1}{2\beta} - \beta \lambda = u^s_f(\beta', 1) - \frac{1}{\beta'} = u^s_f(\beta, R) - \frac{1}{\beta R^2} . \] (60)

5. Free Energy: Gaussian Model

Via the Legendre principle proven in the last section, we can now recover the explicit form of the pressure of the Gaussian model. First we recall that the pressure of the spherical model, made explicit in terms of \( R^2 \), reads (see Proposition 3)
\[ A^s_f(\beta, R) = A^s_f(\beta R^2, 1) = \begin{cases} \frac{\beta^2 R^4}{4}, & \beta R^2 \leq 1 \\
\beta R^2 - \frac{1}{\beta} \log \beta R^2 - \frac{3}{4}, & \beta R^2 \geq 1. \end{cases} \] (61)

Thus, according to Theorem 2, the pressure of the Gaussian model is the maximum over \( R^2 \) of the function
\[ W(\beta, R^2) := \begin{cases} \frac{(\lambda-1) R^2}{2} + \frac{1}{2} \log(R^2) + \frac{1}{4}, & \beta R^2 < 1 \\
\beta R^2 - \frac{\beta^2 R^4}{4} + \frac{(\lambda-1) R^2}{2} - \frac{1}{2} \log \beta - \frac{1}{4}, & \beta R^2 > 1, \end{cases} \] (62)

which is continuous in \( R^2 \in (0, \infty) \), goes to \( -\infty \) at the extremes of the interval and is concave in \( R^2 \). Hence it must have a finite unique maximum. Therefore we optimise in \( R^2 \):
\[ \frac{\partial}{\partial R^2} f(\beta, R) = \begin{cases} \frac{(\lambda - 1)}{2R^2}, & \beta R^2 \leq 1 \\
\beta - \frac{\beta^2 R^4}{2} + (\lambda - 1), & \beta R^2 \geq 1. \end{cases} \] (63)

and
\[ \frac{\partial}{\partial R^2} f(\beta, R) = 0 \Leftrightarrow \begin{cases} R^2 = \frac{1}{\beta}, & \beta R^2 \leq 1 \\
R^2 = \frac{2}{\beta} \frac{(\lambda - 1)}{2}, & \beta R^2 \geq 1. \end{cases} \] (64)

Because of the concavity of \( f \), for each value of \( (\beta, \lambda) \), only one critical point \( \hat{R}^2 \) can exists: if \( \beta < (1 - \lambda), \hat{R}^2 = 1/(1 - \lambda) \), otherwise \( \hat{R}^2 = 2\beta + (\lambda - 1)/\beta^2 \). Again we are concerned about the free energy, defined by \( f^s := -\frac{1}{\beta} A^s \); definitely we obtain
\[ \text{Proposition 8.} \text{ The free energy of the Gaussian spin glass has the following explicit expression} \]
\[ f^s(\beta, \lambda) = \begin{cases} \frac{1}{\beta} \log(1 - \lambda), & \beta < 1 - \lambda \\
\frac{1}{\beta} \log(\beta) - \frac{\beta^2}{2} - \frac{\beta}{4}, & \beta \geq 1 - \lambda \end{cases} \] (65)

with \( q(\beta, \lambda) := \frac{\beta - (1 - \lambda)}{\beta^2} \).
It is worthwhile to remark that the r.h.s. of equation (65) coincides the RS approximation exhibited in [4]. This enables us to complete the picture, by identifying $\bar{q}$ with the Edward-Anderson order parameter of the model: we have that replica symmetry holds in the whole phase diagram and the transition is between a high temperature phase and a RS one. It is witnessed by the value of the overlap that is fixed to zero for $\beta < 1 - \lambda, \lambda < 1$ and to $\bar{q}$ otherwise. All that is true provided we deal with random interactions in the Wigner ensemble with a sub-Gaussian tail, according to the hypothesis H.

We stress that our approach, reminiscent of the earliest works in spin glasses, albeit apparently more general, does not give naturally this picture. Practically, we do not know a priori the significance of the minimiser $\bar{q}$. We need the scheme of [4] for giving a complete interpretation to our results in terms of the correct order parameter (i.e. the overlap).

Of course, Legendre duality permits us to transfer all these considerations to the spherical spin glass as well.

6. Connection to Neural Networks

In this section we finally examine the relation between the Gaussian and spherical models and the (analogue) Hopfield model of neural networks. We begin with a brief but necessary introduction, then we pass to inspect a direct application of our results to the Hopfield model and we terminate with some heuristics.

The analogue Hopfield Model is defined as follows: consider $N$ Ising spin $\sigma_i$ interacting via the Hamiltonian

$$H_N = -\frac{1}{N} \sum_{\mu=1}^{K} \sum_{i,j} \xi_{i,j}^\mu \sigma_i \sigma_j$$

where $\xi_i^\mu$ are $K$ quenched patterns of the $N$-spin system with $\lim_N K/N = \alpha \in \mathbb{R}^+$, representing here the disordered interaction. One is interested as usual to the pressure of the model defined as

$$A_H(\beta, \alpha) = \lim_{N,K} \frac{1}{N,K} \mathbb{E} \log \sum_{\sigma} e^{-\beta H_N}.$$ 

Differently to the original formulation by Hopfield in his celebrated paper [17], where $\xi_i^\mu$ were i.i.d. Bernoulli $\pm 1$ r.v., in the analogue version we will adopt $\xi_i^\mu$ to be $\mathcal{N}(0,1)$ i.i.d r.v. The two models are in fact believed to be formally different [15], even though no precise relation has been specified among them so far. We remand the reader to [8] and [26] for an exhaustive account on the topic.

Recently, it has been shown in [5] that, as a general feature of bipartite spin glasses (see also [3]), the pressure of the analogue Hopfield Model, at least in the high temperature phase and in RS approximation, can be written as a convex combination of the pressure of a SK model and the one of a Gaussian model, calculated at different suitable temperatures. More precisely, once defined

$$\beta_1 := \frac{\sqrt{\alpha} \beta}{1 - \beta(1 - \bar{q}_H)},$$

$$\beta_2 := 1 - \beta(1 - \bar{q}_H),$$

where $\bar{q}_H := \frac{1}{N} \sum_i \sigma_i \sigma'_i$ is the ordinary order parameter of the networks, evaluated in the RS approximation, we have

**Theorem** (see [5]). Fixed $\beta_1$ and $\beta_2$ as in (67) and (68), the replica symmetric approximation of the quenched pressure of the analogue neural networks can be linearly decomposed as follows:

$$A_{NN}^{RS}(\beta) = A_{SK}^{RS}(\beta_1) - \frac{1}{4} \beta_1^2 + \alpha A_{Gauss}(\beta_2, \beta).$$

This theorem is essentially the main reason why the Gaussian model has been investigated in [4]. Now we can reinterpret this result in terms of the relation between the Gaussian and the spherical Model. Indeed, we note that the radius of the spherical model has a definite meaning in the context of neural networks, being the self-overlap of the Gaussian spins, that is known to be self averaging. Moreover, as shown for instance in [2], it is possible to relate the internal energy of
the analog neural networks, self-averaging as well for thermodynamic reason, to the sole Gaussian spin self-overlap \( p_{11} := \frac{1}{N} \sum_{i=1}^{K} z_i^2 \) as
\[
\lim_{N \to \infty} \frac{\langle H_N \rangle}{N} = -\frac{1}{2} \sum_{\mu=1}^{K} \langle m_{\mu}^2(\sigma) \rangle = \frac{\alpha}{2\beta} (1 - \langle p_{11} \rangle),
\]
where \( m_{\mu} = \frac{1}{N} \sum_{i=1}^{N} \xi_{i\mu} \sigma_i \), the Mattis magnetisations, are fundamental quantities of the theory. Therefore we can modify the previous theorem, by rephrasing it in terms of the SK and spherical models:

**Theorem 3.** Fixed \( \beta_1 \) and \( \beta_2 \) as in (67) and (68), the replica symmetric approximation of the quenched pressure of the analogical neural networks can be linearly decomposed as follows
\[
A_{NN}^{RS}(\beta) = A_{SK}^{RS}(\beta_1) - \frac{1}{4} \beta_1^2 + \alpha A^f(\beta_2, \sqrt{p_{11}}) + \frac{\alpha}{2} (\beta - 1) \langle p_{11} \rangle - \frac{\alpha \beta_2}{4} \langle p_{11}^2 \rangle + \frac{\alpha}{2} (1 + \log \langle p_{11} \rangle).
\]

Here we stress that, even though the r.h.s. of (69), (71) are in fact unaffected by the choice of the randomness, according to hypothesis \( H \), the decomposition by itself has been proven only in the case of Gaussian disorder. We do not know in particular if it can be extended exactly to the original Hopfield model with \( \pm 1 \) patterns. However the relation between the two models is a very involved subject, and we are not going to discuss it further.

However, it is perhaps useful the observation that we can obtain, by (60), an explicit form of \( p_{11} \) as optimal radius, as a function of \( \tilde{q}_H \), since we can calculate the internal energy of the spherical model. But, even in a RSB scenario for the analogical Hopfield model, whatever could be the form assumed by \( \tilde{q}_H \), we know that \( p_{11} \) has to conserve replica symmetry. This maybe suggests that some of the SK-spherical spin glass decomposition has to survive beyond RS approximation.

Anyway we are far to state any precise result on this last part and we let it to future developments. In conclusion, the invariance properties under rotations of the Hopfield model, and so its connection with rotationally invariant spin glasses, are indubitably suggestive. They have been certainly already investigated, but probably they can still be useful tools in order to shed more light into its mathematical structure.

### 7. Conclusions and Perspectives

In this paper we have analysed the Gaussian and spherical spin glass models, along with the relation between them. In particular we have pointed out precisely the expected duality in terms of Legendre structure, or equivalence of ensembles. Our work consequently permits to deal deliberately with one or other model in further studies.

We have already discussed singularly some properties of the two models emerging by our analysis, together with the link with neural networks. Consequently in what follows we will mainly connect our work to the existing literature, indicating some possible development directions.

Many features of the spherical model suggest that it can be used as a toy model in order to test the methods to be employed in the hardest \( \pm 1 \) spin models. This has been in turn the case for the SK model: we record that the spherical model has been used in the early mathematical theory in [10] in order to get a bound of the free energy, and then the strategy used by Talagrand in [25] to prove the Crisanti-Sommers solution of the spherical \( p \)-spin is similar in the main steps to the one which successfully brought to the Parisi free energy (see [14], [26]).

On the other hand, as remarked also by Talagrand in the Introduction of [25], these kind of continuous models present difficulties exactly where the \( \pm 1 \) spin analogue are more treatable. A good example of that is the cavity method, that is really natural in the SK model, but quite involved in the spherical case (and so in the Gaussian one). This is essentially due to the fact that we are not dealing with a product measure (we refer to [21] for a detailed analysis). Even in the perspective to apply the results to neural networks, it would be important to deepen this point. Another interesting viewpoint starts by the early observation by Stanley (on ferromagnetic systems, in [24]) that the spherical model can be thought as the limit of a Heisenberg model of infinite...
dimensionality. For spin glasses the limit has been investigated in [9], and in our opinion a further study of the topic may be interesting. All those, along with the connections with the neural networks problem already discussed, are possible research directions, moving by spherical symmetric spin glass models. There are probably many others, beyond spin glasses, in the theory of disordered systems, as rotationally invariant models are a vast field of research, paradigmatic in many branches of mathematical sciences.

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