CLASSIFICATION OF RANK TWO LIE CONFORMAL ALGEBRAS

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Abstract. We give a complete classification (up to isomorphism) of Lie conformal algebras which are free of rank two as \( \mathbb{C}[\partial] \)-modules, and determine their automorphism groups.

Introduction

Lie conformal algebras, and their super versions, were first introduced rigorously by V. Kac [10] around 1996. Their structure theory, representation theory and cohomology theory have been studied extensively by V. Kac and his collaborators in [2, 4, 7, 8, 10]. A non-linear version of Lie conformal algebras was further studied in [1, 5, 6]. On one hand, Lie conformal algebras axiomatize the singular part of the operator product expansion (OPE) of chiral fields in a two-dimensional conformal field theory. On the other hand, they have been proved to be useful in the study of infinite-dimensional Lie algebras satisfying the locality property [11]. Lie conformal algebras are related to vertex algebras in a similar manner as Lie algebras are related to their universal enveloping algebras [10].

Besides the applications to other areas, Lie conformal algebras themselves are quite fascinating from a purely algebraic point of view. As algebraic objects, their definition has a lot of resemblance to that of Lie algebras. A Lie algebra is a vector space \( \mathfrak{g} \) endowed with a Lie bracket \([\cdot, \cdot]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}\) satisfying \([a, b] = -[b, a]\) (skew-symmetry) and \([a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0\) (Jacobi identity) for all \(a, b, c \in \mathfrak{g}\). In the case of Lie conformal algebras, several things need to be modified. First, the base ring is replaced by polynomial ring \( \mathbb{C}[\partial] \) in the indeterminate \( \partial \). The base space \( R \) is replaced by a module over \( \mathbb{C}[\partial] \) and Lie bracket is replaced by lambda-bracket \([\cdot, \cdot]: R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes R\), for some indeterminate \( \lambda \). Then the...
corresponding axioms of sesquilinearity, skew-symmetry and Jacobi identity for the lambda-bracket versions are required to be satisfied (see Definition 1.1).

The classification of rank two Lie conformal algebras can be considered as the parallel problem with the classification of three-dimensional complex Lie algebras [9]. A Lie conformal algebra $R$ is said to be finite if it is finitely generated as a $\mathbb{C}[\partial]$-module, and it is said to be of rank $n$ if it is free of rank $n$ as a $\mathbb{C}[\partial]$-module. As in the case of Lie algebras, the corresponding notions of simple, semisimple, nilpotent and solvable Lie conformal algebras can be defined. A complete classification of finite semisimple Lie conformal algebras was done in [4]. A super version of the classification was done soon after in [7].

Let $R$ be a $\mathbb{C}[\partial]$-module of rank $n$. Defining a Lie conformal algebra structure on $R$ is equivalent to finding a set of polynomials, which we will call the structure polynomials analogous to the structure constants in the case of Lie algebras, satisfying certain equations, i.e., the structure polynomial versions for skew-symmetry and Jacobi identity equations. In [10], V. Kac and M. Wakimoto classified the rank one Lie conformal algebras by giving explicit solutions of those equations. Further, V. Kac surmised in [10] that “It is clearly impossible to solve those equations directly for $n \geq 2$”. In this paper, we give a solution to the rank two case.

Our main result Theorem 2.21 may be stated briefly as follows. Let $R$ be a non-semisimple rank two Lie conformal algebra. Then

1. If $R$ is nilpotent, then $R$ is parametrized by a skew-symmetric polynomial $Q(\lambda, \partial)$, i.e. $Q(\lambda, \partial) = -Q(-\lambda - \partial, \partial)$. If $R$ is solvable but not nilpotent, then $R$ is parametrized by a non-zero polynomial $a(\lambda)$ (Proposition 2.3).

2. If $R$ is not solvable, then either $R$ is the direct sum of a rank one commutative Lie conformal algebra and the Virasoro Lie conformal algebra (Proposition 2.8), or it belongs to a class of Lie conformal algebras parametrized by three parameters $c, d, \text{ and } Q_c(\lambda, \partial)$, where $c, d \in \mathbb{C}$ are constants and $Q_c(\lambda, \partial)$ is a skew-symmetric polynomial. Moreover, $Q_c(\lambda, \partial) \neq 0$ only when $(c, d) \in \{(1, 0), (0, 0), (-1, 0), (-4, 0), (-6, 0)\}$. In these cases, we document the polynomials $Q_c(\lambda, \partial)$ explicitly in the table of Theorem 2.21.

The organization of the present paper is as follows. In §1 we recall the definition of a Lie conformal algebra and give some concrete examples. In §2 we give a complete classification of rank two Lie conformal algebras. We compute the automorphism groups of rank two Lie conformal algebras in §3. All vector spaces and tensor products are considered over the complex numbers $\mathbb{C}$.

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Postscript: After this paper was completed, in private communication with Prof. V. Kac, we were informed of his unpublished notes [12] (on work done with M. Wakimoto), which overlap part of our results. We would like to thank Prof. V. Kac for the information about [12] as well the references [3, 13], where related problems in the context of differential groups and differential algebraic Lie algebras were considered. However, our approach to the classification is different and is based on elementary arguments.

1. Preliminaries

Definition 1.1. A Lie conformal algebra ([10]) is a \( \mathbb{C}[\partial] \)-module \( R \) endowed with a \( \mathbb{C} \)-linear map \( R \otimes R \to \mathbb{C}[\lambda] \otimes R \), denoted by \( a \otimes b \mapsto [a \lambda b] \) and called \( \lambda \)-bracket, satisfying the following axioms

\[
[\partial a \lambda b] = -\lambda [a \lambda b], \quad [a \lambda \partial b] = (\lambda + \partial)[a \lambda b], \quad \text{(sesquilinearity)}
\]

\[
[a \lambda - \partial b] = -[a \lambda b], \quad \text{(skew-symmetry)}
\]

\[
[a \lambda [b \mu c]] - [b \mu [a \lambda c]] = [[a \lambda b] \lambda + a \lambda c], \quad \text{(Jacobi identity)}
\]

for all \( a, b, c \in R \).

The sesquilinearity implies that, for all \( a, b \in R \) and \( f(\partial), g(\partial) \in \mathbb{C}[\partial] \), we have

\[
[f(\partial) \lambda g(\partial) b] = f(-\lambda) g(\lambda + \partial)[a \lambda b], \quad \text{(1.1)}
\]

Given a Lie conformal algebra \( R \), for each \( n \in \{0, 1, 2, \cdots \} \), the \( n \)-th product is defined as a \( \mathbb{C} \)-linear product \( R \otimes R \to R \) and denoted by \( a \otimes b \mapsto a_{(n)} b \), which is encoded in the \( \lambda \)-bracket as:

\[
[a \lambda b] = \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} a_{(n)} b, \quad \text{(1.2)}
\]

where we use the notation \( \lambda^{(n)} = \frac{\lambda^n}{n!} \) and \( \lambda^{(n)} a_{(n)} b := \lambda^{(n)} \otimes a_{(n)} b \). Then we have the following equivalent definition.

Definition 1.2. A Lie conformal algebra ([10]) is a \( \mathbb{C}[\partial] \)-module \( R \) with a \( \mathbb{C} \)-linear product \( a_{(n)} b \) for each \( n \in \mathbb{Z}_+ \), such that the following axioms hold:

\[
\begin{align*}
\text{(C0)} & \quad a_{(n)} b = 0 \text{ for } n \geq 0, \\
\text{(C1)} & \quad \partial a_{(n)} b = -n a_{(n-1)} b, \quad a_{(n)} \partial b = \partial(a_{(n)} b) + n a_{(n-1)} b, \\
\text{(C2)} & \quad b_{(n)} a = \sum_{j \geq 0} (-1)^{n+j} \partial^{(j)} (a_{(n+j)} b), \quad \text{here } \partial^{(j)} = \frac{\partial^j}{j!}, \\
\text{(C3)} & \quad a_{(m)} (b_{(n)} c) - b_{(n)} (a_{(m)} c) = \sum_{j \geq 0} \binom{m}{j} (a_{(j)} b)(m+n-j)c,
\end{align*}
\]

for all \( a, b, c \in R \) and \( m, n \in \mathbb{Z}_+ \).
Definition 1.3. Let $R_1, R_2$ be two Lie conformal algebras. A Lie conformal algebra homomorphism $\varphi : R_1 \to R_2$ is a $\mathbb{C}[\partial]$-module homomorphism which preserves the $\lambda$-brackets, i.e., $\varphi([a, b]) = [\varphi(a), \varphi(b)]$ for all $a, b \in R_1$, where it is understood that $\varphi$ commutes with multiplication by $\lambda$.

Definition 1.4. Let $R$ be a Lie conformal algebra, and $S \subseteq R$ be a $\mathbb{C}[\partial]$-submodule. Then $S$ is called a Lie conformal subalgebra of $R$ if $a_{(n)}b \in S$ for all $a, b \in S, n \in \mathbb{Z}_+$. It is called a Lie conformal ideal of $R$ if $a_{(n)}b \in S$ for all $a \in R, b \in S, n \in \mathbb{Z}_+$, and we denote it by $S \triangleleft R$. By the property of skew-symmetry, an ideal is always two-sided.

Lemma 1.5. Let $R$ be a Lie conformal algebra and $S, T$ be two ideals of $R$. The subspace $[S, T] := \text{span}_\mathbb{C}\{a_{(n)}b \mid a \in S, b \in T, n \in \mathbb{Z}_{\geq 0}\}$ is an ideal of $R$.

Proof. This can be easily seen by the Jacobi identity (C3) in Definition 1.2.

If $I \subseteq R$ is an ideal, then there is a canonical way to define a Lie conformal algebra structure on the quotient $R/I$. The derived algebra of $R$, which we denote by $R'$, is defined to be the $\mathbb{C}$-span of all elements of the form $a_{(n)}b$ with $a, b \in R, n \in \mathbb{Z}_+$. We denote by $R^{(1)} = R$ and define $R^{(n)} = (R^{(n-1)})'$ inductively for $n \geq 2$, then we have the derived algebra series, which are all ideals by Lemma 1.5:

$$R \supseteq R^{(2)} \supseteq \cdots \supseteq R^{(n)} \supseteq R^{(n+1)} \supseteq \cdots.$$

Let $R^n = \text{span}_\mathbb{C}\{a^1_{m_1}a^2_{m_2} \cdots a^n_{m_{n-1}} \mid a^i \in R, m_i \in \mathbb{Z}_+\}$, then we have $R^{(n)} \subseteq R^n$ and the following series of ideals

$$R \supseteq R^2 \supseteq \cdots \supseteq R^n \supseteq R^{n+1} \supseteq \cdots.$$

Definition 1.6. Let $R$ be a Lie conformal algebra. An element $a \in R$ is called central if $[a, b] = 0$ for all $b \in R$, and $R$ is called commutative if all elements of $R$ are central. If $R^{(n)} = 0$ for some $n \geq 1$, $R$ is called solvable and if $R^n = 0$ for some $n \geq 1$, $R$ is called nilpotent. A Lie conformal algebra is called simple if it is not commutative and has no proper ideals. It is called semisimple if it contains no non-zero solvable ideals.

Example 1.7. The Virasoro Lie conformal algebra is the rank one $\mathbb{C}[\partial]$-module

$$\text{Vir} = \mathbb{C}[\partial]L$$

with $[L, L] = (2\lambda + \partial)L$. It is the unique non-commutative Lie conformal algebra of rank one.

Example 1.8. Given a finite dimensional Lie algebra $\mathfrak{g}$, $\text{Cur } \mathfrak{g} := \mathbb{C}[\partial] \otimes \mathfrak{g}$ can be given a Lie conformal algebra structure by defining $[a, b] = [a, b]$ for $a, b \in \mathfrak{g}$, and it is called the current Lie conformal algebra associated to $\mathfrak{g}$.

Example 1.9. The semi-direct sum of Vir with $\text{Cur } \mathfrak{g}$ is the Lie conformal algebra $R = \text{Vir} \ltimes \text{Cur } \mathfrak{g}$ with $[L, L] = (2\lambda + \partial)L$, $[L, c] = (\lambda + \partial)c$, and $[a, b] = [a, b]$ for all $a, b, c \in \mathfrak{g}$.

Theorem 1.10 (3). Any finite semisimple Lie conformal algebra is a finite direct sum of Lie conformal algebras of the following types:
(i) the finite current Lie conformal algebra $\text{Cur}\ g$ with $g$ being simple,
(ii) the Virasoro Lie conformal algebra $\text{Vir}$,
(iii) the semidirect sum of (i) and (ii) as in Example 1.9.

The goal of this paper is to classify rank two Lie conformal algebras. Let $R = \mathbb{C}(\partial)X^1 \bigoplus \mathbb{C}(\partial)X^2$. To define a Lie conformal algebra structure on $R$, it is enough to define the $\lambda$-brackets on the basis elements $\{X^1, X^2\}$, namely,

$$[X_i^1, X_j^2] = Q_{i,j}^k(\lambda, \partial)X^k, \text{ where } Q_{i,j}^k(\lambda, \partial) \in \mathbb{C}[\lambda, \partial] \text{ and } i, j, k \in \{1, 2\}, \quad (1.3)$$

such that the skew-symmetry property for the pairs $(X^i, X^j)$ and the Jacobi identities for the triples $(X^i, X^j, X^k)$ are all satisfied. By sesquilinearity, we can then extend the $\lambda$-brackets to $R$. We will call the polynomials $Q_{i,j}^k(\lambda, \partial)$ the structure polynomials of $R$. They are similar to the structure constants of a Lie algebra under a chosen basis and completely determine the Lie conformal algebra structure on $R$.

Let us recall here the classification of rank one Lie conformal algebras following [10]. Let $R = \mathbb{C}(\partial)L$ be a rank one Lie conformal algebra, with $[L, L] = f(\lambda, \partial)L$ for some $f(\lambda, \partial)$. Then by skew-symmetry and Jacobi identity, we have

$$f(\lambda, \partial) = -f(-\lambda - \partial, \partial), \quad (1.4)$$

$$f(\mu, \lambda + \partial)f(\lambda, \partial) - f(\lambda, \mu + \partial)f(\mu, \partial) = f(\lambda, -\lambda - \mu)f(\lambda + \mu, \partial). \quad (1.5)$$

Let $\deg_{\partial} f(\lambda, \partial) = n$. If $n \geq 2$, an easy calculation shows that the degree with respect to $\partial$ on the left hand side of Equation (1.5) is $2n - 1$, while that on the right hand side is $n$, which is impossible since $2n - 1 \neq n$. So $n \leq 1$. If $n = 0$, skew-symmetry implies that $f = 0$, so we have the commutative Lie conformal algebra. If $n = 1$, then $f(\lambda, \partial) = (2\lambda + \partial)\alpha$ for some non-zero constant $\alpha \in \mathbb{C}$ by skew-symmetry. We can then do change of basis to set $\alpha = 1$, i.e., we have the Virasoro Lie conformal algebra. So a rank one Lie conformal algebra is either commutative or the Virasoro Lie conformal algebra.

**Definition 1.11.** A polynomial $f(x, y) \in \mathbb{C}[x, y]$ is said to be skew-symmetric if it satisfies the equation

$$f(x, y) = -f(-x - y, y).$$

Note that the polynomials $Q_{1,1}^1(\lambda, \partial), Q_{1,1}^2(\lambda, \partial), Q_{2,1}^1(\lambda, \partial)$ and $Q_{2,2}^2(\lambda, \partial)$ defined in Equation (1.3) are all skew-symmetric. Below we prove two lemmas on skew-symmetric polynomials that will play a crucial role in our classification.

**Lemma 1.12.** If $f(x, y) \in \mathbb{C}[x, y]$ is a skew-symmetric polynomial, then $f(x, y) = (2x + y)g(x^2 + xy, y)$ for some polynomial $g$.

**Proof.** It is easy to see that $(2x + y)f(x, y)$. Indeed, plugging $y = -2x$ in $f(x, y)$, we have $f(x, -2x) = -f(x, -2x)$ by skew-symmetry, i.e., $f(x, -2x) = 0$. Let $f(x, y) = (2x + y)g(x, y)$, then $g(x, y) = g(-x - y, y)$. Hence $g(x, y)$ is invariant under the transformation $x \mapsto -x - y, y \mapsto y$. Thus an easy application of the invariant theory implies that $g(x, y) \in \mathbb{C}[x^2 + xy, y]$. \qed
Corollary 1.13. Let \( f(x, y) \) be a skew-symmetric polynomial. If \( \deg_y f(x, y) = n \), then the coefficient of \( y^n \) is a polynomial in \( x \) of degree \( \leq n - 1 \).

Proof. By skew-symmetry, we can write \( f(x, y) = (2x + y)g(x^2 + xy, y) \) for some polynomial \( g \), and \( \deg_y g(x^2 + xy, y) = n - 1 \). As \( \deg_y (x^2 + xy) = 1 \), we can write \( g(x^2 + xy, y) = \sum_{i+j \leq n-1} b_{i,j}(x^2 + xy)^i y^j \). Thus the coefficient of \( y^n \) in \( f(x, y) \) is \( \sum_{i=0}^{n-1} b_{i,n-1-i}x^i \), which is a polynomial in \( x \) of degree less than or equal to \( n - 1 \). \( \square \)

2. Classification of rank two Lie conformal algebras

2.1. The first step towards the classification. For a rank two Lie conformal algebra \( R \) with basis \( \{X^1, X^2\} \) and structure polynomials \( Q^k_{i,j}(\lambda, \partial) \), by the skew-symmetry and Jacobi identity axioms, the structure polynomials must satisfy the following equations:

\[
Q^k_{i,j}(\lambda, \partial) = -Q^k_{j,i}(-\lambda - \partial, \partial) \text{ for } i, j, k \in \{1, 2\}, \tag{2.1}
\]

\[
\sum_{s=1}^{2} \left( Q^s_{j,k}(\mu, \lambda + \partial)Q^s_{i,s}(\lambda, \partial) - Q^s_{i,k}(\lambda, \mu + \partial)Q^s_{j,s}(\mu, \partial) \right) = \sum_{s=1}^{2} Q^s_{i,j}(\lambda - \lambda - \mu)Q^s_{s,k}(\lambda + \mu, \partial) \tag{2.2}
\]

for all \( i, j, k, t \in \{1, 2\} \). Although the Jacobi identity is \( S_3 \)-symmetric, and we just need to consider Equation (2.2) for the triples \( (i, j, k) \) with \( i \leq j \leq k \), there are still 6 equations containing 12 polynomials in two variables of the same type as Equation (2.2). When the rank of \( R \) increases, the number of equations and polynomials increase drastically making it more difficult to solve those equations directly.

So it is important to simplify the calculations if we want to do the classification. From Theorem 1.10, we know that the only semisimple Lie conformal algebra of rank two is the direct sum of two Virasoro Lie conformal algebras. So we only need to consider the non-semisimple ones. The following key lemma plays a crucial role in simplifying the calculations in our classification.

Lemma 2.1. For a non-semisimple Lie conformal algebra \( R \) of rank two, there exists a basis, say \( \{A, B\} \), such that \([A, A]\) = 0, and \( \mathbb{C}[\partial]A \triangleleft R \) is an ideal.

Proof. Since \( R \) is not semisimple, there exists a non-zero solvable ideal. Let \( S \subseteq R \) be such an ideal. Then the derived series of \( S \) must terminate somewhere, i.e., \( S^{(n)} \neq 0 \) but \( S^{(n+1)} = 0 \) for some \( n \in \mathbb{Z}_+ \). Then \( I := S^{(n)} \) is commutative. By Lemma 1.15, all these \( S^{(i)} \)'s are ideals of \( R \). In particular, \( I \) is an ideal of \( R \). Since \( \mathbb{C}[\partial] \) is a principal ideal domain and \( I \) is a submodule of \( R \), we can find a basis of \( R \), say \( \{A, B\} \), such that \( I \) is generated by \( \{f(\partial)A, g(\partial)B\} \) for some \( f(\partial), g(\partial) \in \mathbb{C}[\partial] \).
where $f(\partial)$ is a non-zero polynomial and $f(\partial)$ divides $g(\partial)$. When $g(\partial) \neq 0$, i.e., $I$ is of rank two as a $\mathbb{C}[\partial]$-module, using Equation (1.4), it is straightforward to see that $R$ is commutative. When $g(\partial) = 0$, we have $I = \mathbb{C}[\partial]|f(\partial)|A$ is a rank one abelian ideal. Thus $\mathbb{C}[\partial]|A$ is an abelian ideal of $R$.

Let $\{A, B\}$ be a basis of $R$, such that $[A, A] = 0$ and $C[\partial]|A \triangleleft R$. Comparing the coefficients of $A$ in the Jacobi identity for the triple $(B, B, A)$, we have

$$Q_{B,A}^A(\mu, \lambda + \partial)Q_{B,A}^A(\lambda, \partial) - Q_{B,A}^A(\lambda, \mu + \partial)Q_{B,A}^A(\mu, \partial) = Q_{B,B}^B(\lambda, -\lambda - \mu)Q_{B,A}^A(\lambda + \mu, \partial),$$

(2.3)

where we assume that $[C_D] = Q_{C,D}^C(\lambda, \partial)A + Q_{C,D}^D(\lambda, \partial)B$ for $C, D \in \{A, B\}$. If $\deg Q_{B,A}^A(\lambda, \partial) = n > 1$, then the degree with respect to $\partial$ on the left hand side of Equation (2.3) is $2n - 1$, while that on the right hand side is $n$, which is impossible. So we may assume that $Q_{B,A}^A(\lambda, \partial) = a(\lambda) + b(\lambda)\partial$ for some polynomials $a(\lambda), b(\lambda) \in \mathbb{C}[\lambda]$. Since $\mathbb{C}[\partial]|A$ is an ideal, $\frac{R}{\mathbb{C}[\partial]|A} \cong \mathbb{C}[\partial]|B$ is a rank one Lie conformal algebra. Thus $[B_{\lambda}A] = \alpha(2\lambda + \partial)B \mod \mathbb{C}[\partial]|A$ for some scalar $\alpha$. However without loss of generality, it is enough to assume that $Q_{B,B}^B(\lambda, \partial) = (2\lambda + \partial)\alpha$ for $\alpha \in \{0, 1\}$.

**Lemma 2.2.** Let $R = \mathbb{C}[\partial]|A \oplus \mathbb{C}[\partial]|B$ be a rank two Lie conformal algebra with $[A, A] = 0, [B, A] = (a(\lambda) + b(\lambda)\partial)A$ and $Q_{B,B}^B(\lambda, \partial) = (2\lambda + \partial)\alpha$, for some polynomials $a(\lambda), b(\lambda) \in \mathbb{C}[\lambda]$ and $\alpha \in \{0, 1\}$. Then the following holds,

1. if $\alpha = 0$, then $b(\lambda) = 0$,
2. if $\alpha = 1$, then $b(\lambda) \in \{0, 1\}$.

Moreover, if $b(\lambda) = 0$, then we have $a(\lambda) = 0$; if $b(\lambda) = 1$, then $\deg a(\lambda) \leq 1$.

**Proof.** Plugging $Q_{B,A}^A(\lambda, \partial) = a(\lambda) + b(\lambda)\partial$ and $Q_{B,B}^B(\lambda, \partial) = (2\lambda + \partial)\alpha$ in Equation (2.3) yields

$$(a(\mu) + b(\mu)(\mu + \partial))(a(\lambda) + b(\lambda)\partial) - (a(\lambda) + b(\lambda)(\mu + \partial))(a(\mu) + b(\mu)\partial) = (\lambda - \mu)a(a(\lambda + \mu) + b(\lambda + \mu)\partial).$$

(2.4)

Comparing the coefficients of powers of $\partial$ in Equation (2.4), we get

$$b(\lambda)b(\mu) = b(\lambda + \mu)\alpha.$$  

(2.5)

$$b(\mu)a(\lambda)\lambda - b(\lambda)a(\mu)\mu = (\lambda - \mu)a(\lambda + \mu)\alpha.$$  

(2.6)

When $\alpha = 0$, we get $b(\lambda) = 0$. When $\alpha = 1$, Equation (2.5) ensures that $b(\lambda) = k$ is a constant and $k^2 = k$, i.e., $k \in \{0, 1\}$. When $b(\lambda) = 0$, Equation (2.6) implies that $a(\lambda) = 0$. When $b(\lambda) = 1$, Equation (2.6) implies that $\deg a(\lambda) \leq 1$.

Therefore, any non-semisimple rank two Lie conformal algebra $R$ has a basis $\{A, B\}$, such that $[A, A] = 0, [B, A] = (a(\lambda) + b(\lambda)\partial)A$ for some polynomials $a(\lambda), b(\lambda) \in \mathbb{C}[\lambda]$, and $[B, B] = Q(\lambda, \partial)A + (2\lambda + \partial)\alpha B$ for some skew-symmetric polynomial $Q(\lambda, \partial)$ and for $\alpha \in \{0, 1\}$. Lemma 2.2 implies that we only need to consider the following three cases:
Case 1: $\alpha = 0$, $b(\lambda) = 0$.
Case 2: $\alpha = 1$, $a(\lambda) = b(\lambda) = 0$.
Case 3: $\alpha = 1$, $b(\lambda) = 1$ and $a(\lambda) = c\lambda + d$ for some $c, d \in \mathbb{C}$.

We divide Case 3 into two subcases: 3a: $d \neq 0$ and 3b: $d = 0$.

Note that for all these three cases, the Jacobi identities for the triples $(A, A, A)$, $(A, A, B)$ and $(A, B, B)$ are easily verified. Moreover, the coefficients of $B$ in both sides of the Jacobi identity for the triple $(B, B, B)$ are equal. So we only need to consider the coefficients of $A$ in the Jacobi identity for the triple $(B, B, B)$, which gives us the following equation:

$$\alpha(\lambda - \mu)Q(\lambda + \mu, \partial) - Q(\lambda, -\lambda - \mu)(a(-\lambda - \mu - \partial) + b(-\lambda - \mu - \partial)\partial)
= Q(\mu, \lambda + \partial)(a(\lambda) + b(\lambda)\partial) + \alpha(2\mu + \lambda + \partial)Q(\lambda, \partial)
- Q(\lambda, \mu + \partial)(a(\mu) + b(\mu)\partial) - \alpha(2\lambda + \mu + \partial)Q(\mu, \partial). \quad (2.7)$$

Hence the classification of non-semisimple rank two Lie conformal algebras is equivalent to the classification of the quadruples $(a(\alpha), b(\lambda), \alpha, Q(\lambda, \partial))$, such that $a(\alpha), b(\lambda)$, and $\alpha$ satisfy the conditions in one of the cases described above, and the skew-symmetric polynomial $Q(\lambda, \partial)$ satisfies Equation $(2.7)$.

2.2. Case 1. $(\alpha = 0, b(\lambda) = 0)$

Proposition 2.3. Let $R = \mathbb{C}[\partial]A \bigoplus \mathbb{C}[\partial]B$ be a rank two Lie conformal algebra satisfying $[A, A] = 0$, $[B, A] = a(\lambda)A$, and $[B, B] = Q(\lambda, \partial)A$. If $a(\lambda) = 0$, then $Q(\lambda, \partial)$ may be any skew-symmetric polynomial. If $a(\lambda) \neq 0$, then we can find a new basis $(A, C')$, such that $[A, A] = [C, C'] = 0$ and $[C, A] = a(\lambda)A$.

Proof. When $a(\lambda) = 0$, $A$ is a central element in $R$ and $R' \subseteq \mathbb{C}[\partial]A$. So all the Jacobi identities are trivially satisfied and the only constraint for $Q(\lambda, \partial)$ is that it should be skew-symmetric.

When $a(\lambda) \neq 0$, we show that by a suitable change of basis we can kill $Q(\lambda, \partial)$. If $Q(\lambda, \partial) \neq 0$, we assume that $Q(\lambda, \partial) = \sum_{i \geq 0} f_i(\lambda)\partial^i$ and $f_n(\lambda) \neq 0$. Then Equation $(2.7)$ gives us

$$a(\mu) \sum_{i \geq 0} f_i(\lambda)(\mu + \partial)^i - a(\lambda) \sum_{i \geq 0} f_i(\mu)(\lambda + \partial)^i$$

$$= a(-\lambda - \mu - \partial) \sum_{i \geq 0} f_i(\lambda)(-\lambda - \mu)^i. \quad (2.8)$$

Comparing the degrees with respect to $\partial$ on both sides of Equation $(2.8)$, we have $\deg a(\lambda) \leq \deg Q(\lambda, \partial)$. If $\deg a(\lambda) < n$, then the coefficient of $\partial^n$ on the left side of Equation $(2.8)$ must be zero, which implies that $f_n(\mu)a(\lambda) = f_n(\lambda)a(\mu)$.

So $f_n(\lambda) = ka(\lambda)$ for some constant $k \in \mathbb{C}^\times$. For $B' = B - k\partial^n A$, $\{A, B\}$ forms a new basis and $[B, A] = a(\lambda)A$, $[B', B'] = Q(\lambda, \partial)A$, where

$$Q(\lambda, \partial) = Q(\lambda, \partial) + (-\lambda)^n a(-\lambda - \partial)k - (\lambda + \partial)^n a(\lambda)k.$$
It is clear that $\deg_\partial Q'(\lambda, \partial) \leq n$. Moreover, the coefficient of $\partial^n$ in $Q'(\lambda, \partial)$ is $f_n(\lambda) - ka(\lambda) = 0$, i.e., $\deg_\partial Q'(\lambda, \partial) < \deg_\partial Q(\lambda, \partial)$. By induction, we can assume that $\deg a(\lambda) = n = \deg_\partial Q(\lambda, \partial)$.

Since $Q(\lambda, \partial)$ is skew-symmetric, Corollary 2.7 implies that the coefficient of $\partial^m$ in $Q(\lambda, \partial)$ is a polynomial in $\lambda$ of degree less than or equal to $m - 1$. Let us assume that $f_m(\lambda) = \sum_{i=0}^{m-1} b_i \lambda^i$. If $p(\partial) = \partial^m + \sum_{i=0}^{m-1} p_i \partial^i$, and $C = B + p(\partial)A$, then $[C_\lambda A] = a(\lambda)A$ and

$$Q^A_{C,C}(\lambda, \partial) = Q(\lambda, \partial) - p(-\lambda)a(-\lambda - \partial) + p(\lambda + \partial)a(\lambda).$$

Note that $\deg_\partial Q^A_{C,C}(\lambda, \partial) \leq m$. If we write $a(\lambda) = \sum_{i=0}^m a_i \lambda^i$ with $a_m \neq 0$, then the coefficient of $\partial^m$ in the polynomial $Q^A_{C,C}(\lambda, \partial)$ is

$$\sum_{i=0}^{m-1} (b_i + (-1)^{m+i+1} a_m p_i + a_i) \lambda^i. \quad (2.10)$$

Since $a_m \neq 0$, given $a_i, b_i$, we can choose $p_i = (-1)^{m+i+1} a_i + b_i$ such that the coefficient of $\partial^m$ in $Q^A_{C,C}(\lambda, \partial)$ becomes zero, which implies that $\deg_\partial Q^A_{C,C}(\lambda, \partial) < m$.

But if $Q^A_{C,C}(\lambda, \partial)$ is non-zero, it must also satisfy Equation (2.8), i.e., $\deg a(\lambda) \leq \deg Q^A_{C,C}(\lambda, \partial)$. Hence $Q^A_{C,C}(\lambda, \partial) = 0$ and in the new basis $\{A, C\}$, we have $[A_\lambda A] = [C_\lambda A] = 0, [C_\lambda A] = a(\lambda)A$.

**Remark 2.4.** It is clear that $R$ is solvable but not nilpotent when $[A_\lambda A] = [B_\lambda B] = 0$ and $[B_\lambda A] = a(\lambda)A$ for some non-zero polynomial $a(\lambda)$. So we denote it by $R_{sol}(a(\lambda))$. When $[A_\lambda A] = [B_\lambda A] = 0$ and $[B_\lambda B] = Q(\lambda, \partial)A$ for some skew-symmetric polynomial $Q(\lambda, \partial)$, $R$ is nilpotent. So we denote it by $R_{nil}(Q(\lambda, \partial))$.

We will call the $\lambda$-brackets in Proposition 2.3 the normalized $\lambda$-brackets.

**Remark 2.5.** As in Case 2 and Case 3, we have $B \in R^{(n)}$ for any $n$, Proposition 2.3 classifies all solvable rank two Lie conformal algebras.

**Definition 2.6.** Two polynomials $f$ and $g$ are said to be associated if $f = kg$ for some $k \in \mathbb{C}^\times$, which we denote by $f \sim g$.

**Lemma 2.7.** The solvable Lie conformal algebras $R_{sol}(a(\lambda))$ and $R_{sol}(a'(\lambda))$ are isomorphic if and only if $a(\lambda) \sim a'(\lambda)$. The nilpotent Lie conformal algebras $R_{nil}(Q(\lambda, \partial))$ and $R_{nil}(Q'(\lambda, \partial))$ are isomorphic if and only if $Q(\lambda, \partial) \sim Q'(\lambda, \partial)$.

**Proof.** Let $R_1$ denote $R_{sol}(a(\lambda))$ or $R_{nil}(Q(\lambda, \partial))$ and $R_2$ denote $R_{sol}(a'(\lambda))$ or $R_{nil}(Q'(\lambda, \partial))$, respectively, with basis elements $\{A, B\}$ and $\{A', B'\}$ satisfying the normalized $\lambda$-brackets. It is clear that if $a(\lambda) = ka'(\lambda)$, then the map $A \mapsto A', B \mapsto kB'$ gives an isomorphism between $R_{sol}(a(\lambda))$ and $R_{sol}(a'(\lambda))$, and if $Q(\lambda, \partial) = kQ'(\lambda, \partial)$, then the map $A \mapsto A', B \mapsto \sqrt{k}B'$ gives an isomorphism between $R_{nil}(Q(\lambda, \partial))$ and $R_{nil}(Q'(\lambda, \partial))$.

For the converse, assuming that $\varphi$ is an isomorphism between $R_1$ and $R_2$, we have $\varphi(R_1') \cong R_2'$. In both cases, we have

$$\{0\} \subsetneq R_1' \subseteq \mathbb{C}[\partial]A, \quad \{0\} \subsetneq R_2' \subseteq \mathbb{C}[\partial]A'.$$
So \( \varphi(A) = f(\partial)A' \) for some polynomial \( f(\partial) \). But \( \{\varphi(A), \varphi(B)\} \) forms a basis of \( R_2 \), so we have \( f(\partial) = s \) and \( \varphi(B) = tB + g(\partial)A \) for some constants \( s, t \in \mathbb{C}^\times \) and some polynomial \( g(\partial) \). When \( R_1 = R_{\text{nil}}(a(\lambda)) \) and \( R_2 = R_{\text{nil}}(a'(\lambda)) \), as \( \varphi([B_\lambda A]) = [\varphi(B)\lambda\varphi(A)] \), we get \( a(\lambda) = ta'(\lambda) \). When \( R_1 = R_{\text{nil}}(Q(\lambda, \partial)) \) and \( R_2 = R_{\text{nil}}(Q'(\lambda, \partial)) \), as \( \varphi([B_\lambda B]) = [\varphi(B)\lambda\varphi(B)] \), we get \( Q(\lambda, \partial) = \frac{t^2}{s}Q'(\lambda, \partial) \).

\[ \square \]

2.3. Case 2. \((\alpha = 1, a(\lambda) = b(\lambda) = 0)\)

**Proposition 2.8.** Let \( R = \mathbb{C}[\partial]A \bigoplus \mathbb{C}[\partial]B \) be a rank two Lie conformal algebra satisfying \([A_\lambda A] = [B_\lambda A] = 0\), and \([B_\lambda B] = (2\lambda + \partial)B + Q(\lambda, \partial)A\). Then by a suitable change of basis we can set \( Q(\lambda, \partial) = 0\).

**Proof.** If \( Q(\lambda, \partial) \neq 0 \), then by skew-symmetry, we can assume that \( Q(\lambda, \partial) = (2\lambda + \partial)\sum_{i \geq 0} f_i(\partial)\partial^i \) with \( f_n(\lambda) \neq 0 \). From Equation (2.11), we get

\[
(2\mu + \lambda + \partial)(2\lambda + \partial) \sum_{i \geq 0} f_i(\lambda)\partial^i - (2\lambda + \mu + \partial)(2\mu + \partial) \sum_{i \geq 0} f_i(\mu)\partial^i
= (\lambda - \mu)(2\lambda + 2\mu + \partial) \sum_{i \geq 0} f_i(\lambda + \mu)\partial^i.
\]

(2.11)

Setting \( \mu = 0 \) in Equation (2.11), we have

\[
\sum_{i \geq 0} f_i(\lambda)\partial^i = \sum_{i \geq 0} f_i(0)\partial^i,
\]

i.e., \( f_i(\lambda) = f_i(0) \) for all \( i \). If \( B' = B + \sum_{i \geq 0} f_i(0)\partial^i A \), then \([B'_\lambda B'] = (2\lambda + \partial)B' \) and \([A_\lambda A] = [B'_\lambda A] = 0\).

\[ \square \]

**Lemma 2.9.** Let \( R = \mathbb{C}[\partial]A \bigoplus \mathbb{C}[\partial]B \) be a rank two Lie conformal algebra satisfying \([A_\lambda A] = 0, [B_\lambda A] = (c\lambda + d + \partial)A\), and \([B_\lambda B] = (2\lambda + \partial)B + Q(\lambda, \partial)A\). Then \( R \) has no non-trivial central element.

**Proof.** Let \( X = f(\partial)A + g(\partial)B \) be central in \( R \). As \([X_\lambda A] = g(-\lambda)[B_\lambda A] = 0 \) and \([X_\lambda B] = f(-\lambda)[A_\lambda B] = 0 \), we have that \( f(\partial) = g(\partial) = 0 \). Hence \( X = 0 \).

\[ \square \]

**Remark 2.10.** It is clear that there exists a non-trivial central element in the Lie conformal algebras considered in Case 2. Hence the Lie conformal algebras considered in Case 2 and those in Case 3 are not isomorphic.
2.4. **Case 3a.** \( (\alpha = 1, b(\lambda) = 1, a(\lambda) = c\lambda + d \text{ for } c \in \mathbb{C}, d \in \mathbb{C}^\times) \)

Assuming that \( Q(\lambda, \partial) = \sum_{i=0}^{m} f_i(\lambda) \partial^i \), from Equation (2.7) in Case 3 we have

\[
(c\lambda + c\mu + (c - 1)\partial - d) \sum_{i=0}^{m} f_i(\lambda)(-\lambda - \mu)^i + (\lambda - \mu) \sum_{i=0}^{m} f_i(\lambda + \mu) \partial^i
\]

\[
= (c\lambda + d + \partial) \sum_{i=0}^{m} f_i(\mu)(\lambda + \partial)^i + (2\mu + \lambda + \partial) \sum_{i=0}^{m} f_i(\lambda) \partial^i
\]

\[
- (c\mu + d + \partial) \sum_{i=0}^{m} f_i(\lambda)(\mu + \partial)^i - (2\mu + \partial + \partial) \sum_{i=0}^{m} f_i(\mu) \partial^i. \quad (2.12)
\]

By skew-symmetry, it is clear that if \( \deg_{\partial} Q(\lambda, \partial) \leq 1 \), then \( Q(\lambda, \partial) = (2\lambda + \partial)k \) for some constant \( k \in \mathbb{C} \). It is easy to see that such a polynomial always satisfies Equation (2.12). Hence we may assume that \( \deg_{\partial} Q(\lambda, \partial) = m \geq 2 \). Comparing the coefficients of \( \partial^m \) in both sides of Equation (2.12), we get

\[
(\lambda - \mu)f_m(\lambda + \mu) = ((c + m - 2)\mu + d)f_m(\mu) - ((c + m - 2)\mu + \mu - \lambda + d)f_m(\lambda). \quad (2.13)
\]

When \( \deg_{\partial} Q(\lambda, \partial) = m \geq 3 \), comparing the coefficients of \( \partial^{m-1} \) in both sides of Equation (2.12), we get

\[
(\lambda - \mu)f_{m-1}(\lambda + \mu) = \left(cm + \binom{m}{2}\right) \lambda^2 f_m(\mu) + ((c + m - 3)\lambda - \mu) f_{m-1}(\mu)
\]

\[
- \left(cm + \binom{m}{2}\right) \mu^2 f_m(\lambda) - ((c + m - 3)\mu - \lambda) f_{m-1}(\lambda)
\]

\[
+ d(f_m(\mu)m\lambda - f_m(\lambda)m\mu + f_{m-1}(\mu) - f_{m-1}(\lambda)). \quad (2.14)
\]

**Proposition 2.11.** Let \( R = \mathbb{C}[\partial]A \oplus \mathbb{C}[\partial]B \) be a rank two Lie conformal algebra satisfying \( [A, A] = 0, [B, A] = (c\lambda + d + \partial)A \) and \([B, B] = Q(\lambda, \partial)A + (2\lambda + \partial)B \). If \( d \neq 0 \), then by a suitable change of basis we can set \( Q(\lambda, \partial) = 0 \).

**Proof.** Firstly, if \( \deg_{\partial} Q(\lambda, \partial) \geq 2 \), then we show that by a suitable change of basis we may decrease its degree with respect to \( \partial \). Let \( Q(\lambda, \partial) = \sum_{i=0}^{m} f_i(\lambda) \partial^i \) with \( f_m(\lambda) \neq 0 \) and \( m \geq 2 \). Setting \( \lambda = 0 \) in Equation (2.13), we get

\[
f_m(\mu)d - ((c + m - 2)\mu + d)f_m(0) = 0,
\]

i.e., \( f_m(\mu) \frac{f_m(0)}{d} - ((c + m - 2)\mu + d) \) is of degree less or equal to 1.

For \( B' = B - k\partial^mA \), we have \([B'_A] = [B_A] \) and

\[
[B'A'] = (2\lambda + \partial)B' + Q^4_{B', B}(\lambda, \partial)A
\]

where

\[
Q^4_{B', B}(\lambda, \partial) = (2\lambda + \partial)k\partial^m + Q(\lambda, \partial)
\]

\[
- (k(-\lambda)^m(c\lambda + c\partial - d - \partial) - k(\lambda + \partial)^m(c\lambda + d + \partial)).
\]
It is straightforward to see that \( \deg_{\partial} Q^{A}_{B', B'}(\lambda, \partial) \leq m \). Rewriting \( Q^{A}_{B', B'}(\lambda, \partial) = \sum_{i \geq 0} g_i(\lambda) \partial^i \), we get
\[
g_m(\lambda) = f_m(\lambda) - k[(c + m - 2)\lambda + d].
\]
Choosing \( k = \frac{f_m(0)}{d} \), we conclude that \( g_m(\lambda) = 0 \), i.e., \( \deg_{\partial} Q^{A}_{B', B'}(\lambda, \partial) < \deg_{\partial} Q(\lambda, \partial) \). This procedure can be continued until we have \( \deg_{\partial} Q(\lambda, \partial) = 1 \). By induction, we may assume that \( Q(\lambda, \partial) = (2\lambda + \partial)k \) for some constant \( k \).

Setting \( C = B - \frac{k}{d} \partial A \), we get \( [\lambda A] = (c\lambda + d + \partial)A \) and \( [\lambda C] = (2\lambda + \partial)C \).

2.5. **Case 3b.** \( (\alpha = 1, b(\lambda) = 1, a(\lambda) = c\lambda \text{ for } c \in \mathbb{C}) \)

In this subsection, we assume that \( R = \mathbb{C}[\partial]A \bigoplus \mathbb{C}[\partial]B \) satisfying \([\lambda A] = 0, [B A] = (c\lambda + \partial)A \) and \([B B] = Q(\lambda, \partial)A + (2\lambda + \partial)B \). We write \( Q(\lambda, \partial) = \sum_{n \geq 0} Q_n(\lambda, \partial) \) where \( Q_n(\lambda, \partial) \) is the homogeneous component of \( Q(\lambda, \partial) \) of degree \( n \) as a polynomial in \( \lambda \) and \( \partial \). Note that Equation (2.7) is homogeneous in Case 3b in the sense that \( Q(\lambda, \partial) \) satisfies Equation (2.7) if and only if \( Q_n(\lambda, \partial) \) does so for all \( n \). Also, \( Q(\lambda, \partial) \) is skew-symmetric if and only if \( Q_n(\lambda, \partial) \) is skew-symmetric for all \( n \). Thus \( Q(\lambda, \partial) \) satisfies Equation (2.7) and the skew-symmetry if and only if \( Q_n(\lambda, \partial) \) does so for all \( n \).

We show that after suitable changes of basis, we can kill almost all \( Q_n(\lambda, \partial) \) except a few special values of \( n \) depending on the parameter \( c \). We also determine \( Q_n(\lambda, \partial) \) explicitly for those special values which we discuss in two cases, \( n < 4 \) and \( n \geq 5 \). Note that \( Q_0(\lambda, \partial) \equiv 0 \) by skew-symmetry. When \( n \leq 4 \), by skew-symmetry, the polynomials \( Q_n(\lambda, \partial) \) must be of the following forms (see Lemma 1.12),
\[
\begin{align*}
Q_1(\lambda, \partial) &= \alpha_1(2\lambda + \partial), \\
Q_2(\lambda, \partial) &= \alpha_2(2\lambda + \partial)\partial, \\
Q_3(\lambda, \partial) &= (2\lambda + \partial)(\alpha_3 \partial^2 + \beta_3(\lambda^2 + \lambda\partial)), \\
Q_4(\lambda, \partial) &= (2\lambda + \partial)(\alpha_4 \partial^3 + \beta_4(\lambda^2 + \lambda\partial)\partial),
\end{align*}
\]
for \( \alpha_1, \alpha_2, \alpha_3, \beta_3, \alpha_4, \beta_4 \in \mathbb{C} \).

For \( B' = B + k \partial^{m-1}A \), where \( k \in \mathbb{C} \) and \( m \geq 1 \), \( \{A, B'\} \) forms a new basis of \( R \) satisfying \([\lambda A] = 0, [B' A] = (c\lambda + \partial)A \) and \([B' B'] = (2\lambda + \partial)B' + Q'(\lambda, \partial)A \) such that
\[
Q'(\lambda, \partial) - Q(\lambda, \partial)
= k[(-\lambda)^{m-1}(c(\lambda + \partial) - \partial) + (\lambda + \partial)^{m-1}(c\lambda + \partial) - (2\lambda + \partial)\partial^{m-1}]
= k[c + m - 3]\lambda\partial^{m-1} + k(m - 1)(c + (m - 2)/2)\lambda^2\partial^{m-2} + \cdots,
\]
where “…” stands for some homogeneous polynomial in \( \lambda, \partial \) of degree \( m \) involving only powers of \( \partial \) which are strictly less than \( m - 2 \). Note that \( Q'(\lambda, \partial) \) differs from \( Q(\lambda, \partial) \) only in the homogeneous component of degree \( m \).
Lemma 2.12. Let $R = \mathbb{C}[\partial]A \bigoplus \mathbb{C}[\partial]B$ be a rank two Lie conformal algebra satisfying $[A, A] = 0, [B, A] = (c + \partial)A$ and $[B, B] = Q(\lambda, \partial)A + (2\lambda + \partial)B$. If we write $Q(\lambda, \partial) = \sum_{n \geq 1} Q_n(\lambda, \partial)$ where $Q_n(\lambda, \partial)$ is the homogeneous component of degree $n$ of $Q(\lambda, \partial)$, then

1. $Q_1(\lambda, \partial) = \alpha_1(2\lambda + \partial)$ for some $\alpha_1 \in \mathbb{C}$. Moreover, if $c \neq 1$, then by a suitable change of basis we can set $\alpha_1 = 0$.
2. $Q_2(\lambda, \partial) = \delta_{c, 0} \alpha_2(2\lambda + \partial)\partial$ for some $\alpha_2 \in \mathbb{C}$.
3. $Q_3(\lambda, \partial) = (2\lambda + \partial)(\delta_{c, 1} \alpha_3 \partial^2 + \beta_3(\lambda^2 + \lambda \partial))$ for some $\alpha_3, \beta_3 \in \mathbb{C}$. Moreover, if $c \neq 0$, by a suitable change of basis we can set $\beta_3 = 0$.
4. $Q_4(\lambda, \partial) = \beta_4(2\lambda + \partial)(\lambda^2 + \lambda \partial)\partial$, i.e., $\alpha_4 \equiv 0$ in the general form of $Q_4(\lambda, \partial)$ in Equation (2.13). Moreover, if $c \neq -1$, by a suitable change of basis we can set $\beta_4 = 0$.

Proof. The general forms of $Q_n(\lambda, \partial)$ are given in Equations (2.16)–(2.18). The coefficients of $\alpha_2, \alpha_3, \beta_3$ are straightforward calculations.

For $B' = B + (k_0 + k_1 \partial + k_2 \partial^2 + k_3 \partial^3)A$, $\{A, B'\}$ forms a new basis of $R$ with $[A, A] = 0, [B'_A, A] = (c + \partial)A$ and $[B'_A B'] = (2\lambda + \partial)B' + Q'(\lambda, \partial)A$ such that $Q'(\lambda, \partial) = Q(\lambda, \partial) + (2\lambda + \partial)[k_0(c - 1) + k_2 c(\lambda^2 + \lambda \partial) + k_3(c + 1)(\lambda^2 + \lambda \partial)\partial]$.

If we write $Q'(\lambda, \partial) = \sum_{n \geq 1} Q_n'(\lambda, \partial)$, where $Q_n'(\lambda, \partial)$ is the homogeneous component of degree $n$ of $Q'(\lambda, \partial)$, then $Q_n'(\lambda, \partial) = Q_n(\lambda, \partial)$ for $n \geq 5$ and $n = 2$.

If $c \neq 1$, we may choose $k_0$ suitably to get $Q_n'(\lambda, \partial) = 0$. If $c \neq 0$, we may choose $k_2$ suitably to set $\beta_3 = 0$ in the general form of $Q_n'(\lambda, \partial)$ given in Equation (2.17). If $c \neq -1$, $k_3$ may be suitably chosen to set $\beta_4 = 0$ in the general form of $Q_n'(\lambda, \partial)$ given in Equation (2.18). \hfill \square

Next we consider the case $n \geq 5$. If we write $Q_n(\lambda, \partial) = \sum_{i=0}^{n} b_i \lambda^{n-i} \partial^i$. Then Equation (2.27) for $Q_n(\lambda, \partial)$ gives us

\[
(\lambda - \mu) \sum_{i=0}^{n} b_i (\lambda + \mu)^{n-i} \partial^i + (c\lambda + c\mu + (c - 1)\partial) \sum_{i=0}^{n} b_i \lambda^{n-i} (-\lambda - \mu)^i \\
= (c\lambda + \partial) \sum_{i=0}^{n} b_i \mu^{n-i} (\lambda + \partial)^i + (2\mu + \lambda + \partial) \sum_{i=0}^{n} b_i \lambda^{n-i} \partial^i \\
- (c\mu + \partial) \sum_{i=0}^{n} b_i \lambda^{n-i} (\mu + \partial)^i - (2\lambda + \mu + \partial) \sum_{i=0}^{n} b_i \mu^{n-i} \partial^i. \tag{2.20}
\]

Lemma 2.13. If $n \geq 4$, then $Q_n(0, \partial) = 0$, i.e., $\deg Q_n(\lambda, \partial) \leq n - 1$. Moreover, if $n$ is even, we have $Q_n(\lambda, 0) = 0$.

Proof. Let $Q_n(\lambda, \partial) = \sum_{i=0}^{n} b_i \lambda^{n-i} \partial^i$. Setting $\lambda = 0$ in Equation (2.20), we get

\[
(c\mu + (c - 1)\partial)b_n(\mu)^n = (2\mu + \partial)b_n \partial^n - (c\mu + \partial)b_n(\mu + \partial)^n. \tag{2.21}
\]
As \( n \geq 4 \), comparing the coefficients of \( \mu^2 \partial^{n-1} \) and \( \mu^{n-1} \partial^2 \) in both sides of Equation (2.21), we get

\[
\left( \frac{n}{2} \right) + cn = \left( c \left( \frac{n}{2} \right) + n \right) b_n = 0.
\]

But \( \left( \frac{n}{2} \right) + cn \) and \( c \left( \frac{n}{2} \right) + n \) can not be zero at the same time when \( n \geq 4 \). Hence \( b_n = 0 \), i.e., \( Q_n(0, \partial) = 0 \).

Since \( Q_n(\lambda, \partial) \) is a skew-symmetric polynomial, by Lemma 2.12, we can write

\[
Q_n(\lambda, \partial) = (2\lambda + \partial) \sum_{i \geq 0} k_i (\lambda^2 + \lambda \partial i) \partial^{n-1-2i}
\]

where \( k_i \in \mathbb{C} \). If \( n \) is even, then \( n - 1 - 2i > 0 \). Hence \( Q_n(\lambda, 0) = 0 \) if \( n \) is even. \( \square \)

The following lemma will be very important in the sequel.

**Lemma 2.14.** If \( Q_n(\lambda, \partial) \neq 0 \) and \( n \geq 4 \), then \( \deg \partial Q_n(\lambda, \partial) \geq n - 3 \). Moreover, \( \deg \partial Q_n(\lambda, \partial) = n - 2 \) or \( n - 3 \) only if \( c + n - 3 = 0 \).

**Proof.** Let us assume that \( \deg \partial Q_n(\lambda, \partial) = m \) and \( Q_n(\lambda, \partial) = \sum_{i \geq 0} b_i \lambda^{n-i} \partial^i \) such that \( b_m \neq 0 \). Since \( Q_n(\lambda, \partial) \) is skew-symmetric, by Corollary 1.13, the coefficient of \( \partial^m \) in \( Q_n(\lambda, \partial) \) is of degree less than or equal to \( m - 1 \), i.e., \( n - m \leq m - 1 \). Hence \( m \geq \left\lfloor \frac{n+1}{2} \right\rfloor \) which implies that \( m \geq 2 \) since \( n \geq 4 \). By Equation (2.13), we have

\[
b_m b_{n-m}^m(c, \lambda, \mu) = 0,
\]

where \( h_j^m(c, \lambda, \mu) := (c+m-2)\lambda - \mu j^i - ((c+m-2)\mu - \lambda) j^i - (\lambda - \mu)(\lambda + \mu)^i \). By straightforward calculations, we have \( h_1^m(c, \lambda, \mu) = 0 \) for all \( c \) and \( h_j^m(c, \lambda, \mu) \neq 0 \) for all \( j \geq 4 \). Moreover, we have

\[
h_0^m(c, \lambda, \mu) = 0 \quad \text{if and only if} \quad c + m - 2 = 0, \quad (2.22)
\]

\[
h_2^m(c, \lambda, \mu) = 0 \quad \text{if and only if} \quad c + m - 1 = 0, \quad (2.23)
\]

\[
h_3^m(c, \lambda, \mu) = 0 \quad \text{if and only if} \quad c + m = 0. \quad (2.24)
\]

As \( b_m b_{n-m}^m(c, \lambda, \mu) = 0 \) and \( b_m \neq 0 \), Equations (2.22)–(2.24) imply that \( n - m \leq 3 \), i.e., \( \deg \partial Q_n(\lambda, \partial) = m \geq n - 3 \) which proves the first part of the lemma.

If \( m = n - 2 \), then \( c + m - 1 = 0 \), i.e., \( c + n - 3 = 0 \) by Equation (2.23).

If \( m = n - 3 \), then \( c + m = 0 \), i.e., \( c + n - 3 = 0 \) by Equation (2.24). \( \square \)

**Corollary 2.15.** Let \( n \geq 4 \) and \( n \neq 3 - c \). Then by a suitable change of basis we can kill \( Q_n(\lambda, \partial) \).

**Proof.** By Lemma 2.13, we have \( \deg \partial Q_n(\lambda, \partial) \leq n - 1 \). We show that we can kill off the \( \lambda \partial^{n-1} \) term in \( Q_n(\lambda, \partial) \). If \( B' = B + k \partial^{n-1} A \), then \( [B_\lambda' B'] = (2\lambda + \partial) B' + Q'(\lambda, \partial) A \). Write \( Q'(\lambda, \partial) = Q_\ell'(\lambda, \partial) \), where \( Q_\ell'(\lambda, \partial) \) is the homogeneous component of degree \( \ell \) in \( Q'(\lambda, \partial) \). By Equation (2.19), we have \( Q'_\ell(\lambda, \partial) = Q_\ell(\lambda, \partial) \) for \( \ell \neq n \), and

\[
Q'_n(\lambda, \partial) = Q_n(\lambda, \partial) + k(c + n - 3) \lambda \partial^{n-1} + ..., \quad (2.25)
\]
where “...” stands for some polynomial whose degree with respect to $\partial$ is less than $n - 1$. Since $n \neq 3 - c$, we may choose $k$ properly to kill off the $\lambda \partial^{n-1}$ term in $Q_n(\lambda, \partial)$, i.e., $\deg_\partial Q_n(\lambda, \partial) \neq n - 1$. Then Lemma 2.14 implies that $Q'_n(\lambda, \partial) = 0$ as $c + n \neq 3$.

Thus for $n \geq 5$, we only need to consider the special case where $n = 3 - c$.

Lemma 2.16. If $n = 3 - c \geq 5$, then $\deg_\partial Q_n(\lambda, \partial) \neq n - 1$. Moreover, by a suitable change of basis we can kill off the $\lambda^2 \partial^{n-2}$ term in $Q_n(\lambda, \partial)$.

Proof. If $\deg_\partial Q_n(\lambda, \partial) = n - 1$, then $Q_n(\lambda, \partial) = \sum_{i \geq 0} b_i \lambda^{n-i} \partial^i$ for some $b_{n-1} \neq 0$. As $n - 1 \geq 3$, by Equation (2.14), we have (where in our case, $m = n - 1$, $f_m(\lambda) = b_{n-1} \lambda$, $f_m(\lambda) = b_{n-2} \lambda^2$ and $d = 0$)

$$
\left( cm + \frac{m(m - 1)}{2} \right) \lambda^2 b_{n-1} - \left( cm + \frac{m(m - 1)}{2} \right) \mu^2 b_{n-1} = 0.
$$

(2.26)

But Equation (2.26) is satisfied only when $c = -1$ and $n = 4$. Hence we have $\deg_\partial Q_n(\lambda, \partial) \neq n - 1$.

For $B'' = B + k \partial^{n-1} A$, we have $[B'_n B''_n] = (2 \lambda + \partial) B' + Q'(\lambda, \partial) A$ for some polynomial $Q'(\lambda, \partial)$. If we write $Q'(\lambda, \partial) = \sum_{\ell \geq 0} Q'_\ell(\lambda, \partial)$, where $Q'_\ell(\lambda, \partial)$ is the homogeneous component of degree $\ell$ of $Q'(\lambda, \partial)$, then by Equation (2.19), we have

$$
Q'_n(\lambda, \partial) = Q_n(\lambda, \partial) + k(n - 1) \left( c + \frac{n - 2}{2} \right) \lambda^2 \partial^{n-2} + ...,
$$

where “...” stands for some polynomial whose degree with respect to $\partial$ is less than $n - 2$. Hence we may choose $k$ properly to kill off the $\lambda^2 \partial^{n-2}$ term in $Q_n(\lambda, \partial)$. □

By Lemma 2.16 if $n = 3 - c \geq 5$ then we may assume that $\deg_\partial Q_n(\lambda, \partial) \leq n - 3$. Moreover, by Lemma 2.14 if $Q_n(\lambda, \partial) \neq 0$, then $\deg_\partial Q_n(\lambda, \partial) = n - 3$.

Remark 2.17. When $n = 3 - c \geq 5$, if there exists a skew-symmetric and homogeneous polynomial $Q_n(\lambda, \partial)$ of degree $n$ satisfying Equation (2.7), such that $\deg_\partial Q_n(\lambda, \partial) = n - 3$, then it is unique up to scalar multiples. Indeed, if there are two such polynomials, after multiplying by a scalar, we can assume that their leading terms $\lambda^3 \partial^{n-3}$ have the same coefficients. Their difference is still skew-symmetric and satisfies Equation (2.7), but is of degree less than or equal to $n - 4$ with respect to $\partial$, hence must be zero by Lemma 2.14.

Corollary 2.18. If $c = -2, -3$, then by a suitable change of basis we can kill $Q_{3-c}(\lambda, \partial)$.

Proof. By Lemma 2.16 if $Q_{3-c}(\lambda, \partial) \neq 0$, we can assume that $\deg_\partial Q_{3-c}(\lambda, \partial) = -c$. Then the coefficient of $\lambda^3 \partial^{-c}$ in $Q_{3-c}(\lambda, \partial)$ is non-zero, which contradicts Corollary 1.13 as $3 \geq -c$.

Lemma 2.19. Let $n = 3 - c$. Assume that $n \geq 6$ if $n$ is even and $n \geq 11$ if $n$ is odd. If $\deg_\partial Q_n(\lambda, \partial) \leq n - 3$, then $Q_n(\lambda, \partial) = 0$. 

Proof. □
Proof. Let \( Q_n(\lambda, \partial) = \sum_{i=0}^{n-3} b_i \lambda^{n-i} \partial^i \). Comparing the coefficients of \( \lambda^2 \partial^k \mu^{n-1-k} \) in both sides of Equation (2.20), we have,

\[
\binom{n-k}{1} - \binom{n-k}{2} b_k = \left( c \binom{k+1}{1} + \binom{k+1}{2} \right) b_{k+1},
\]

(2.27)

for \( 0 \leq k \leq n-3 \) because the only terms containing the monomial \( \lambda^2 \partial^k \mu^{n-1-k} \) in Equation (2.20) are \( (\lambda - \mu) \sum_{i=0}^{n-k} b_i (\lambda + \mu)^{n-i} \partial^i \) and \( (c \lambda + \partial) \sum_{i=0}^{n} b_i \mu^{n-i} (\lambda + \partial)^i \).

When \( n \geq 6 \) is even, we have \( b_0 = 0 \) by Lemma 2.13. Note that for \( k \leq n-3 \), we always have \( c \binom{k+1}{1} + \binom{k+1}{2} \neq 0 \). So \( b_k = 0 \) by Equation (2.27), i.e., \( Q_n(\lambda, \partial) = 0 \).

Now let \( n = 3 - c \geq 11 \) be odd. Comparing the coefficients of \( \lambda^4 \mu^{n-3} \) on both sides of Equation (2.20), we get

\[
b_0 \left( \binom{n}{3} - \binom{n}{4} \right) + c(1-c)b_{n-3} = cb_3.
\]

(2.28)

Since \( c \binom{k+1}{1} + \binom{k+1}{2} \neq 0 \) for \( k \leq n-3 \), Equation (2.27) gives us

\[
b_{k+1} = \frac{\Pi_{i=0}^{k}(n-i) \left( 3 + i - n \right)}{\Pi_{i=0}^{k}(i+1) \left( 6 - 2n + i \right)} b_0 = \frac{\Pi_{i=0}^{k}(n-i)(3 + i - n)}{\Pi_{i=0}^{k}(i+1)(6 - 2n + i)} b_0.
\]

In particular,

\[
b_{n-3} = \frac{\Pi_{i=0}^{n-4}(n-i)(3 + i - n)}{\Pi_{i=0}^{n-4}(i+1)(6 - 2n + i)} b_0,
\]

and

\[
b_3 = \frac{\Pi_{i=0}^{n-4}(n-i)(3 + i - n)}{\Pi_{i=0}^{n-4}(i+1)(6 - 2n + i)} b_0 = \frac{n(n-1)(n-2)}{3!} \frac{5 - n}{4(7 - 2n)} b_0.
\]

Note that

\[
\frac{\Pi_{i=0}^{n-4}(n-i)}{\Pi_{i=0}^{n-4}(i+1)} = \frac{n(n-1)(n-2)}{3!}, \quad \frac{\Pi_{i=0}^{n-4}(3 + i - n)}{\Pi_{i=0}^{n-4}(6 - 2n + i)} = \frac{(n-3)! \times (n-3)!}{(2n-6)!},
\]

and

\[
\binom{n}{3} - \binom{n}{4} = \frac{n(n-1)(n-2)(7-n)}{3! \times 4}.
\]

Replacing \( b_3 \) and \( b_{n-3} \) by the above expressions in Equation (2.28) and then dividing both sides of Equation (2.28) by \( \frac{n(n-1)(n-2)}{3!} \), we get

\[
b_0 \left( \frac{7-n}{4} + (3-n)(n-2) \frac{(n-3)! \times (n-3)!}{(2n-6)!} + \frac{(n-3)(5-n)}{4(7 - 2n)} \right) = 0.
\]
Theorem 2.21. Let \( Q \) be a general form for \( \partial \). Then Equation (2.27) implies that \( b_k = 0 \) for all \( k \), i.e., \( Q_n(\lambda, \partial) = 0 \).

Lemma 2.20. If \( c = -4 \), then \( Q_2(\lambda, \partial) = (2\lambda + \partial)(\lambda^2 + \lambda\partial)^3 \) satisfies Equation (2.20). If \( c = -6 \), then \( Q_2(\lambda, \partial) = (2\lambda + \partial)(11(\lambda^2 + \lambda\partial)^4 + 2(\lambda^2 + \lambda\partial)^3\partial^2) \) satisfies Equation (2.20).

Proof. This follows from straightforward calculations. Indeed, by the property of skew-symmetry and the assumption that \( \deg\partial Q_3(\lambda, \partial) = -c \), we can give a general form for \( Q_3(\lambda, \partial) \). Then we just need to check Equation (2.20) for \( Q_3(\lambda, \partial) \).

2.6. The classification.

Theorem 2.21. Let \( R \) be a non-semisimple rank two Lie conformal algebra. Then, up to isomorphism, \( R \) is one of the following types,

1. If \( R \) is nilpotent, then \( R \cong R_{\text{nil}}(Q(\lambda, \partial)) \) and has a basis \( \{ A, B \} \) satisfying \([A, A] = [B, A] = 0 \) and \([B, B] = Q(\lambda, \partial)A \) for some skew-symmetric polynomial \( Q(\lambda, \partial) \). Moreover, \( R_{\text{nil}}(Q(\lambda, \partial)) \cong R_{\text{nil}}(Q'(\lambda, \partial)) \) if and only if \( Q(\lambda, \partial) = kQ'(\lambda, \partial) \) for some \( k \in \mathbb{C}^\times \).

2. If \( R \) is solvable but not nilpotent, then \( R \cong R_{\text{sol}}(a(\lambda)) \) and has a basis \( \{ A, B \} \) satisfying \([A, A] = [B, A] = 0 \) and \([B, A] = a(\lambda)A \) for some non-zero polynomial \( a(\lambda) \in \mathbb{C}[\lambda] \). Moreover, \( R_{\text{sol}}(a(\lambda)) \cong R_{\text{sol}}(a'(\lambda)) \) if and only if \( a(\lambda) = ka'(\lambda) \) for some \( k \in \mathbb{C}^\times \).

3. If \( R \) is not solvable, then we have two classes.

3i) \( R \) is the direct sum of a rank one commutative Lie conformal algebra and the Virasoro Lie conformal algebra.

3ii) \( R \) has a basis \( \{ A, B \} \), such that \([A, A] = 0, [B, A] = (c\lambda + d + \partial)A \) and \([B, B] = Q_c(\lambda, \partial)A + (2\lambda + \partial)B \) for some constants \( c, d \in \mathbb{C} \) and some skew-symmetric polynomial \( Q_c(\lambda, \partial) \). We denote such \( R \) in this class by \( R(c, d, Q_c(\lambda, \partial)) \). Moreover, \( Q_c(\lambda, \partial) \neq 0 \) only when \( d = 0 \) and \( c \in \{ 1, 0, -1, -4, -6 \} \), in which case we document the explicit formulae for \( Q_c(\lambda, \partial) \) in the following table.

| \( c \) | \( Q_c(\lambda, \partial) \), \( \beta, \gamma \in \mathbb{C} \) |
|---|---|
| 1 | \( \beta(2\lambda + \partial) \) |
| 0 | \( \beta(2\lambda + \partial)(\lambda^2 + \lambda\partial) + \gamma(2\lambda + \partial)\partial \) |
| -1 | \( \beta(2\lambda + \partial)\partial^2 + \gamma(2\lambda + \partial)(\lambda^2 + \lambda\partial)\partial \) |
| -4 | \( \beta(2\lambda + \partial)(\lambda^2 + \lambda\partial)^3 \) |
| -6 | \( \beta(2\lambda + \partial)(11(\lambda^2 + \lambda\partial)^4 + 2(\lambda^2 + \lambda\partial)^3\partial^2) \) |

Moreover, \( R(c, d, Q_c(\lambda, \partial)) \cong R(c', d', Q'_{c'}(\lambda, \partial)) \) if and only if \( c = c', d = d' \) and \( Q_c(\lambda, \partial) = kQ'_{c'}(\lambda, \partial) \) for some \( k \in \mathbb{C}^\times \).
Proof. It is clear that Lie conformal algebras of different types in (1)-(3) are non-isomorphic. The non-semisimple rank two Lie conformal algebras are divided in three cases by Lemma 2.12. The solvable case (Case 1) is done in Proposition 2.13 and Lemma 2.14 which gives us the types (1)-(2) in the list. For the non-solvable Lie conformal algebras, we have Case 2 and Case 3. Case 2 is done in Proposition 2.24 and it gives us the class (3i). We have divided Case 3 into two subcases, Case 3a and Case 3b. Case 3a is done in Proposition 2.24 and it gives us the $d = 0$ and $Q_c(\lambda, \partial) = 0$ part of the class (3ii).

The Case 3b is the core of our work. Recall that in this subcase, we assume that $R$ has a basis $\{A, B\}$ satisfying $[A, A] = 0, [B, A] = (c\lambda + \partial)A$ and $[B, B] = Q(\lambda, \partial)A + (2\lambda + \partial)B$. Let us write $Q(\lambda, \partial) = \sum_{n \geq 1} Q_n(\lambda, \partial)$, where $Q_n(\lambda, \partial)$ is the homogeneous component of degree $n$ of $Q(\lambda, \partial)$ as a polynomial in $\lambda$ and $\partial$.

If $c \notin \{1, 0, -1, -4, -6\}$, then by Lemma 2.12 we can kill $Q_n(\lambda, \partial)$ for $n \leq 4$. By Corollary 2.15, Corollary 2.18 and Lemma 2.19 we can kill $Q_n(\lambda, \partial)$ for $n \geq 5$. Thus by a suitable change of basis, we can set $Q(\lambda, \partial) = 0$.

If $c \in \{1, 0, -1\}$, then by Corollary 2.15 we can kill $Q_n(\lambda, \partial)$ for $n \geq 5$. For $n \leq 4$, Lemma 2.12 determines the parts of $Q_n(\lambda, \partial)$ which can be killed.

If $c \in \{-4, -6\}$, then by Lemma 2.12 we can kill $Q_n(\lambda, \partial)$ for $n \leq 4$. By Corollary 2.15, we can kill $Q_n(\lambda, \partial)$ for $n \geq 5$ and $n \neq -3 - c$. For $n = -3 - c$, by Lemma 2.16 we can assume that $\deg Q_{3+c}(\lambda, \partial) = -c$ if it is not zero, and is unique up to scalar multiples by Remark 2.17. Then by Lemma 2.20 we get the formulae for $Q_{3+c}(\lambda, \partial)$ as listed in the table.

We prove the isomorphism between different types of Lie conformal algebras in the class (3ii) in Proposition 2.24. □

Remark 2.22. The dimension of the solution space of the polynomials $Q_c(\lambda, \partial)$ in the table of Theorem 2.24 were determined by Theorem 7.2 in [2], where the cohomology of the Virasoro Lie conformal algebra with coefficients in rank one modules was calculated. We call the $\lambda$-brackets in Theorem 2.24 the normalized $\lambda$-brackets.

Lemma 2.23. If $Q_c(\lambda, \partial) \neq 0$ is one of the polynomials appearing in the table of Theorem 2.24, then for any $f(\partial) \in \mathbb{C}[\partial]$,

$$Q_c(\lambda, \partial) \neq f(\lambda + \partial)(c\lambda + \partial) + f(-\lambda)(c\lambda + c\partial - \partial) - (2\lambda + \partial)f(\partial).$$

Proof. Denote by $S_{f(\partial)}^c(\lambda) := f(\lambda + \partial)(c\lambda + \partial) + f(-\lambda)(c\lambda + c\partial - \partial) - (2\lambda + \partial)f(\partial)$ for a polynomial $f(\partial)$. It is enough to prove the lemma for the homogeneous components of $Q_c(\lambda, \partial)$ as a polynomial in $\lambda$ and $\partial$. Let $h(\lambda, \partial)$ be a homogeneous component of $Q_c(\lambda, \partial)$ of degree $m + 1$. If $h(\lambda, \partial) = S_{f(\partial)}^c(\lambda)$ for some polynomial $f(\partial)$, then $f(\partial)$ must have a degree $m$ monomial $k\partial^m$ and $h(\lambda, \partial) = S_{k\partial^m}^c(\lambda)$.

For $c \in \{0, 1, -1\}$, the total degree of $Q_c(\lambda, \partial)$ is less than or equal to 4. Hence if $Q_c(\lambda, \partial) = S_{f(\partial)}^c(\lambda)$ for some polynomial $f(\partial)$, then we can assume that $\deg f(\partial) \leq 3$. For $f(\partial) = \sum_{i=0}^3 k_i \partial^i$, we have

$$S_{f(\partial)}^c(\lambda) = (2\lambda + \partial)[k_0(c - 1) + k_2c(\lambda^2 + \lambda\partial) + k_3(c + 1)(\lambda^2 + \lambda\partial)\partial].$$
Now it is clear that $Q_c(\lambda, \partial) \neq S^c_{f(\partial)}(\lambda)$ for any $f(\partial)$.

For $c \in \{-4, -6\}$ and $Q_c(\lambda, \partial) \neq 0$, the total degree of $Q_c(\lambda, \partial)$ is $3 - c$ and $\deg_\partial Q_c(\lambda, \partial) = -c$. Thus we can assume $f(\partial) = k\partial^{2-c}$ if $Q_c(\lambda, \partial) = S^c_{f(\partial)}(\lambda)$. But by straightforward calculations, we have $\deg_\partial S^c_{k\partial^{2-c}}(\lambda) = 1 - c$ if $k \neq 0$. 

\[\square\]

**Proposition 2.24.** The rank two Lie conformal algebras $R(c, d, Q_c(\lambda, \partial))$ and $R(c', d', Q'_{c'}(\lambda, \partial))$ are isomorphic if and only if $c = c', d = d'$ and $Q_c(\lambda, \partial) = kQ'_{c'}(\lambda, \partial)$ for some $k \in \mathbb{C}^\times$.

**Proof.** Let $\{A, B\}$ and $\{A', B'\}$ be bases of $R(c, d, Q_c(\lambda, \partial))$ and $R(c', d', Q'_{c'}(\lambda, \partial))$, respectively, satisfying the normalized $\lambda$-brackets as in Theorem 2.21. Assume that $\varphi$ is an isomorphism between $R(c, d, Q_c(\lambda, \partial))$ and $R(c', d', Q'_{c'}(\lambda, \partial))$, such that $\varphi(A) = f(\partial)A' + g(\partial)B'$ and $\varphi(B) = p(\partial)A' + q(\partial)B'$ for some polynomials $f(\partial), g(\partial), p(\partial)$ and $q(\partial) \in \mathbb{C}[\partial]$. Then the equation

$$\varphi([A, A]) = 0 = [\varphi(A), \varphi(A)]$$

implies that $g(\partial) = 0$. Since $\{\varphi(A), \varphi(B)\}$ forms a basis, we have $f(\partial) = s$ and $q(\partial) = t$ for some $s, t \in \mathbb{C}^\times$. Moreover, $\varphi([B, A]) = [\varphi(B), \varphi(A)]$ implies that $t = 1$ and $c = c', d = d'$.

When $d = d' \neq 0$, there is nothing to say, since $Q_c(\lambda, \partial) = Q'_{c'}(\lambda, \partial) = 0$.

When $d = d' = 0$, from $\varphi([B, A]) = [\varphi(B), \varphi(A)]$, we get

$$Q_c(\lambda, \partial) - sQ'_{c'}(\lambda, \partial) = p(\partial + \partial)(c\lambda + \partial) + p(-\lambda)(c\lambda + c\partial - \partial) - (2\lambda + \partial)p(\partial).$$

Since $c = c', d = d'$, $Q_c(\lambda, \partial) - sQ'_{c'}(\lambda, \partial)$ is also a polynomial of the form as listed in the table of Theorem 2.21. Thus by Lemma 2.23, we have $Q_c(\lambda, \partial) = sQ'_{c'}(\lambda, \partial)$.

The converse is clear by defining the map $\varphi(A) = k^{-1}A'$ and $\varphi(B) = B'$. 

\[\square\]

### 3. Automorphism groups of rank two Lie conformal algebras

We devote this final section to the calculation of the automorphism groups of rank two Lie conformal algebras. We denote by $R_c$ and $R_{sa}$ the commutative and semisimple rank two Lie conformal algebra, respectively, and by $R_{cs}$ the direct sum of the rank one commutative Lie conformal algebra and the Virasoro Lie conformal algebra. It is clear that $\text{Aut} R_c \cong GL(2, \mathbb{C}[\partial])$, since an automorphism of $R_c$ as a Lie conformal algebra is same as an automorphism as a $\mathbb{C}[\partial]$-module.

Let $\{L^1, L^2\}$ be a basis of $R_{sa}$ satisfying $[L^1, L^2] = \delta_{i,j}(2\lambda + \partial)L^i$. Assume that $\varphi(L^1) = f_1(\partial)L^1 + g_1(\partial)L^2$ is an automorphism of $R_{sa}$, where $f_1(\partial), g_1(\partial) \in \mathbb{C}[\partial]$. From the equation

$$\varphi([L^1, L^2]) = \delta_{i,j}(2\lambda + \partial)\varphi(L^i) = [\varphi(L^i), \varphi(L^j)]$$

we conclude that

$$\delta_{i,j}f_1(\partial) = f_1(-\lambda)f_j(\lambda + \partial) \quad \text{and} \quad \delta_{i,j}g_1(\partial) = g_1(-\lambda)g_j(\lambda + \partial) \quad (3.1)$$
for $i,j \in \{1,2\}$. Setting $i = j = 1$ and $i = j = 2$ in Equation (3.1), we get $f_i(\partial), g_i(\partial) \in \{0,1\}$ for $i \in \{1,2\}$. Setting $i = 1, j = 2$ in Equation (3.1), we get $f_1(-\lambda)f_2(\lambda + \partial) = g_1(-\lambda)g_2(\lambda + \partial) = 0$. So

\[
\begin{pmatrix}
  f_1(\partial) & g_1(\partial) \\
  f_2(\partial) & g_3(\partial)
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix},
\]

i.e., $\text{Aut } R_{cs} \cong \mathbb{Z}_2$.

Let $\{K, L\}$ be a basis of $R_{cs}$ such that $[K, K] = [K, L] = 0, [L, L] = (2\lambda + \partial)L$. Let $\phi \in \text{Aut } R_{cs}$ be an automorphism. Then $[\phi(K), \phi(K)] = 0$ implies that $\phi(K) = f(\partial)K$ and $[\phi(L), \phi(L)] = (2\lambda + \partial)\phi(L)$ implies that $\phi(L) = L$ or $\phi(L) = g(\partial)K$, where $f(\partial), g(\partial) \in \mathbb{C}[\partial]$. But $\{\phi(K), \phi(L)\}$ forms a basis of $R_{cs}$, so $f(\partial) = k \in \mathbb{C}^\times$ and $\phi(L) = L$, i.e., $\text{Aut } R_{cs} \cong \mathbb{C}^\times$.

**Lemma 3.1.** Let $R$ be a non-commutative rank two Lie conformal algebra of type $R_{nil}(Q(\lambda, \partial)), R_{sol}(a(\lambda))$ or $R(c, d, Q_c(\lambda, \partial))$ with basis $\{A, B\}$ satisfying the normalized $\lambda$-brackets in Theorem 2.1. Let $\varphi \in \text{Aut } R$ be an automorphism. Then $\varphi(A) = k_1 A + p(\partial)A$ and $\varphi(B) = k_2 B + p(\partial)A$ for some $k_1, k_2 \in \mathbb{C}^\times$ and some $p(\partial) \in \mathbb{C}[\partial]$.

**Proof.** Let us assume that $\varphi(A) = f(\partial)A$ and $\varphi(B) = g(\partial)B$ where $f(\partial), g(\partial) \in \mathbb{C}[\partial]$. We only need to show that $g(\partial) = 0$, because $\{\varphi(A), \varphi(B)\}$ forms a basis of $R$ which implies that $f(\partial)$ and $q(\partial)$ are some nonzero constants.

For $R = R(c, d, Q_c(\lambda, \partial))$, we have $g(\partial) = 0$ because

\[
0 = [\varphi(A) - g(\partial)] = g(\lambda + \partial)(2\lambda + \partial)B \mod \mathbb{C}[\partial]A.
\]

For $R = R_{nil}(Q(\lambda, \partial))$ and $Q(\lambda, \partial) \neq 0$, we have $g(\partial) = 0$ because

\[
0 = [\varphi(A) - g(\partial)] = g(\lambda + \partial)Q(\lambda, \partial)A.
\]

For $R = R_{sol}(a(\lambda))$ and $a(\lambda) \neq 0$, we have $g(\partial) = 0$ because

\[
a(\lambda)[\varphi(A) - g(\partial)] = 0 \mod \mathbb{C}[\partial]A.
\]

\[\square\]

**Lemma 3.2.** If $c, d \in \mathbb{C}, f(\partial) \in \mathbb{C}[\partial]$ satisfying

\[
f(\lambda + \partial)(c\lambda + d + \partial) + f(-\lambda)(c\lambda + c\partial - d - \partial) - (2\lambda + \partial)f(\partial) = 0,
\]

then

\[
f(\partial) =
\begin{cases}
  k \left(1 + \frac{1-c}{d}\right) & \text{if } d \neq 0, \\
  \delta_{c,1} a_0 + a_1 \partial + \delta_{c,0} a_2 \partial^2 + \delta_{c,-1} a_3 \partial^3 & \text{if } d = 0,
\end{cases}
\]

where $a_i, k \in \mathbb{C}$.

**Proof.** For $d \neq 0$, setting $\lambda = 0$ in Equation (3.2), we get

\[
f(\partial)d + f(0)(c - 1)\partial - d) = 0,
\]
i.e., \( f(\partial) = k \left( 1 + \frac{1 - c}{d} \partial \right) \), where \( f(0) = k \in \mathbb{C} \). By straightforward calculations, we can see that such a polynomial always satisfies Equation (3.2).

For \( d = 0 \), let us assume that \( f(\partial) = \sum_i a_i \partial^i \). Note that Equation (3.2) is equivalent to
\[
a_i \left[ (\lambda + \partial)^i (c\lambda + \partial) + (-\lambda)^i (c\lambda + c\partial - \partial) - (2\lambda + \partial)\partial^i \right] = 0
\]
for all \( i \). By straightforward calculations, we have
\[
(\lambda + \partial)^i (c\lambda + \partial) + (-\lambda)^i (c\lambda + c\partial - \partial) - (2\lambda + \partial)\partial^i \neq 0 \text{ for } i \geq 4.
\]
Thus \( a_i = 0 \) for \( i \geq 4 \), and \( f(\partial) = a_0 + a_1 \partial + a_2 \partial^2 + a_3 \partial^3 \). By Equation (3.2), we get
\[
a_0(c - 1)(2\lambda + \partial) + a_2 c(2\lambda + \partial)(\lambda^2 + \lambda\partial) + a_3(c + 1)(2\lambda + \partial)(\lambda^2 + \lambda\partial)\partial = 0,
\]
i.e., \( a_1 \in \mathbb{C} \) and \( a_0(c - 1) = a_2 c = a_3(c + 1) = 0. \)

**Theorem 3.3.** The automorphism groups of rank two Lie conformal algebras are as follows:

1. \( \text{Aut } R_c \cong GL(2, \mathbb{C}[\partial]), \text{Aut } R_{ss} \cong \mathbb{Z}_2, \text{ and } \text{Aut } R_{cs} \cong \mathbb{C}^\times. \)
2. \( \text{Aut } R_{nil}(Q(\lambda, \partial)) \cong \mathbb{C}^\times \ltimes \mathbb{C}[\partial]. \)
3. \( \text{Aut } R_{sol}(a(\lambda)) \cong \mathbb{C}^\times \ltimes \mathbb{C}. \)
4. \( \text{Aut } R(c, d, Q_c(\lambda, \partial)) \cong \begin{cases} \mathbb{C}^\times \ltimes \mathbb{C} & \text{if } d \neq 0, \\ \mathbb{C} & \text{if } d = 0, c \notin \{1, 0, -1\}, Q_c(\lambda, \partial) \neq 0, \\ \mathbb{C}^\times \times \mathbb{C} & \text{if } d = 0, c \notin \{1, 0, -1\}, Q_c(\lambda, \partial) = 0, \\ \mathbb{C}^2 & \text{if } d = 0, c \in \{1, 0, -1\}, Q_c(\lambda, \partial) \neq 0, \\ \mathbb{C}^\times \times \mathbb{C}^2 & \text{if } d = 0, c \in \{1, 0, -1\}, Q_c(\lambda, \partial) = 0. \end{cases} \)

**Proof.** Part (1) is done in the beginning of this section. For (2)–(4), we denote by \( \{A, B\} \) the basis satisfying the normalized \( \lambda \)-brackets in Theorem 2.21. So for \( R_{nil}(Q(\lambda, \partial)) \), we assume that \([A_\lambda A] = [B_\lambda A] = 0\) and \([B_\lambda B] = Q(\lambda, \partial) A\). For \( R_{sol}(a(\lambda)) \), we assume that \([A_\lambda A] = [B_\lambda B] = 0\) and \([B_\lambda A] = a(\lambda) A\). For \( R(c, d, Q_c(\lambda, \partial)) \), we assume that \([A_\lambda A] = 0, [B_\lambda A] = (c\lambda + d\partial) A\) and \([B_\lambda B] = Q_c(\lambda, \partial) A + (2\lambda + \partial) B\).

Let \( \varphi \) be an automorphism of \( R \). Then Lemma 3.1 implies that \( \varphi(A) = k_1 A \) and \( \varphi(B) = k_2 B + f(\partial) A \) for some \( k_1, k_2 \in \mathbb{C}^\times \) and \( f(\partial) \in \mathbb{C}[\partial] \). We use the matrix
\[
\begin{pmatrix} k_1 & f(\partial) \\ 0 & k_2 \end{pmatrix}
\]
to represent \( \varphi \). Note that \( \varphi([A_\lambda A]) = [\varphi(A) A_\lambda \varphi(A)] \) is already satisfied when \( \varphi \) is of the above form. So we only need to consider \( \varphi([B_\lambda A]) = [\varphi(B) A_\lambda \varphi(A)] \) and \( \varphi([B_\lambda B]) = [\varphi(B) B_\lambda \varphi(B)]. \)

For \( R = R_{nil}(Q(\lambda, \partial)) \), \( \varphi([B_\lambda A]) = [\varphi(B) A_\lambda \varphi(A)] \) is already satisfied and \( \varphi([B_\lambda B]) = [\varphi(B) B_\lambda \varphi(B)] \) implies that \( k_1 = k_2^2 \). Thus
\[
\text{Aut } R_{nil}(Q(\lambda, \partial)) \cong \left\{ \begin{pmatrix} k_2^2 & f(\partial) \\ 0 & k_2 \end{pmatrix} \mid k_2 \in \mathbb{C}^\times, f(\partial) \in \mathbb{C}[\partial] \right\} \cong \mathbb{C}^\times \ltimes \mathbb{C}[\partial].
\]
For $R = R_{sol}(a(\lambda))$, $\varphi([B_\lambda A]) = [\varphi(B)_\lambda \varphi(A)]$ implies that $k_2 = 1$ and $\varphi([B_\lambda B]) = [\varphi(B)_\lambda \varphi(B)] = 0$ implies that $f(\lambda + \delta)a(\lambda) = f(-\lambda)a(-\lambda - \delta)$. Since $a(\lambda) \neq 0$, we get $f(\delta) = ka(\delta)$ for some $k \in \mathbb{C}$. Thus

$$\text{Aut } R_{sol}(a(\lambda)) \cong \left\{ \begin{pmatrix} k_1 & ka(\delta) \\ 0 & 1 \end{pmatrix} \mid k_1 \in \mathbb{C}^\times, k \in \mathbb{C} \right\} \cong \mathbb{C}^\times \ltimes \mathbb{C}.$$

For $R = R(c, d, Q_c(\lambda, \delta))$, $\varphi([B_\lambda B]) = [\varphi(B)_\lambda \varphi(B)]$ implies that $k_2 = 1$ and $(k_1 - 1)Q_c(\lambda + \delta) = f(\lambda + \delta)(c\lambda + d + \delta) + f(-\lambda)(c\lambda + c\delta - d - \delta) - (2\lambda + \delta)f(\delta)$. (3.3)

Note that when $k_2 = 1$, $\varphi([B_\lambda A]) = [\varphi(B)_\lambda \varphi(A)]$ is also satisfied.

We show that $(k_1 - 1)Q_c(\lambda, \delta) = 0$. For $d \neq 0$, we already have $Q_c(\lambda, \delta) = 0$, so we are done. For $d = 0$, if $(k_1 - 1)Q_c(\lambda, \delta) \neq 0$, then $Q_c(\lambda, \delta)$ is of the form $\mathcal{S}_f(\lambda)$ as in Lemma 2.23 which is impossible. Hence both sides of Equation (3.3) must be zero. Thus $k_1 = 1$ if $Q_c(\lambda, \delta) \neq 0$ and $k_1 \in \mathbb{C}^\times$ if $Q_c(\lambda, \delta) = 0$. The polynomial $f(\delta)$ is given in Lemma 3.2.

For $d \neq 0$, we have $k_1 \in \mathbb{C}^\times$ since $Q_c(\lambda, \delta) = 0$. By Lemma 3.2 we have

$$f(\delta) = k \left( 1 - \frac{c - 1}{d} \delta \right)$$

for some $k \in \mathbb{C}$, hence

$$\text{Aut } R(c, d, 0)_{d \neq 0} \cong \left\{ \begin{pmatrix} k_1 & k \left( 1 - \frac{c - 1}{d} \delta \right) \\ 0 & 1 \end{pmatrix} \mid k_1 \in \mathbb{C}^\times, k \in \mathbb{C} \right\} \cong \mathbb{C}^\times \ltimes \mathbb{C}.$$

Using similar arguments, we have the following list of automorphism groups of $R(c, d, Q_c(\lambda, \delta))$ for $d = 0$.

(i) If $c \notin \{1, 0, -1\}$ and $Q_c(\lambda, \delta) \neq 0$, then $k_1 = 1$ and $f(\delta) = k\delta$, hence

$$\text{Aut } R(c, 0, Q_c(\lambda, \delta)) \cong \left\{ \begin{pmatrix} 1 & k\delta \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{C} \right\} \cong \mathbb{C}.$$

(ii) If $c \notin \{1, 0, -1\}$ and $Q_c(\lambda, \delta) = 0$, then $k_1 \in \mathbb{C}^\times$ and $f(\delta) = k\delta$, hence

$$\text{Aut } R(c, 0, Q_c(\lambda, \delta)) \cong \left\{ \begin{pmatrix} k_1 & k\delta \\ 0 & 1 \end{pmatrix} \mid k_1 \in \mathbb{C}^\times, k \in \mathbb{C} \right\} \cong \mathbb{C}^\times \ltimes \mathbb{C}.$$

(iii) If $c \in \{1, 0, -1\}$ and $Q_c(\lambda, \delta) \neq 0$, then $k_1 = 1$ and $f(\delta) = a_1 \delta + \delta_{c, 1}a_0 + \delta_{c, 0}a_2\delta^2 + \delta_{c, -1}a_3\delta^3$ where $a_i \in \mathbb{C}$, hence

$$\text{Aut } R(c, 0, Q_c(\lambda, \delta)) \cong \left\{ \begin{pmatrix} f(\delta) \\ 1 \end{pmatrix} \right\} \cong \mathbb{C}^2.$$

(iv) If $c \in \{1, 0, -1\}$ and $Q_c(\lambda, \delta) = 0$, then $k_1 \in \mathbb{C}^\times$ and $f(\delta) = a_1 \delta + \delta_{c, 1}a_0 + \delta_{c, 0}a_2\delta^2 + \delta_{c, -1}a_3\delta^3$ where $a_i \in \mathbb{C}$, hence

$$\text{Aut } R(c, 0, Q_c(\lambda, \delta)) \cong \left\{ \begin{pmatrix} k_1 & f(\delta) \\ 0 & 1 \end{pmatrix} \mid k_1 \in \mathbb{C}^\times \right\} \cong \mathbb{C}^\times \ltimes \mathbb{C}^2.$$

□
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