Superintegrable systems with spin invariant with respect to the rotation group

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Abstract
Quantum non-relativistic systems with $2 \times 2$ matrix potentials are investigated. Physically, they simulate charged or neutral fermions with non-trivial dipole moments, interacting with an external electric field. Assuming rotational invariance of the Hamiltonian, all such systems allowing second order integrals of motion are identified. It is shown that the integrals of motion can be effectively used to separate variables and to reduce the systems to decoupled ordinary differential equations. Solutions for two of the problems discussed are presented explicitly.

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1. Introduction

Exactly solvable systems of quantum mechanics are favored subjects for many physicists and mathematicians. The beauty of such systems (like the hydrogen atom or the harmonic oscillator) is that they are simple enough to be solved in a way free of the uncertainties and inconveniences of various approximate approaches. Meanwhile, they are sufficiently complicated to model the physical reality. In addition, the complete sets of their exact solutions supply us with convenient bases for the expansions of solutions of other problems. Many exactly solvable systems admit nice hidden symmetries, which are very interesting on their own account.

The exact solvability of quantum mechanical systems is usually caused by a specific property of theirs, called superintegrability. The system with $n$ degrees of freedom is integrable if it admits $n - 1$ commuting integrals of motion in addition to its Hamiltonian. The system is superintegrable if it is integrable and admits at least one more integral of motion.

The systematic search for superintegrable systems began with paper [1]. We will not recount the details of the rather inspiring history of this search or discuss all obtained fundamental results related to quantum mechanical and classical scalar systems. On the contrary, we restrict ourselves to a discussion of superintegrable systems with spin.
Such systems have been studied methodically in recent papers [2–4]. The problem of classifying superintegrable systems with spin was stated in [2] in which 2D systems with spin–orbit interaction and first order integrals of motion were presented. These results were then generalized to the cases of three-dimensional (3D) Euclidean space [3] and second order integrals of motion [4]. The results presented in [4] were restricted to rotationally invariant systems and to integrals of motion which are rotational vectors or scalars.

However, spin–orbit coupling is not the only spin effect which can be present in quantum mechanical systems. Another coupling which is very important and has a well-observable effect is the dipole, or Pauli interaction. This interaction is represented by the Stern–Gerlach term $\sim S \cdot B$ (where $S$ and $B$ are the spin and magnetic field strength vectors) or, more generally, by a matrix term linear in an external field. Moreover, the Pauli interaction affects even neutral particles, provided that, as in the case of the neutron, they have non-trivial dipole moments.

An important example of a 2D superintegrable system with dipole–spin interaction was presented a long time ago by Pron’ko and Stroganov [5]. However, a systematical search for such systems is only just beginning. The classification of 2D systems with first order integrals of motion was presented in [6], while the 3D systems with Fock type dynamical symmetry were derived in [7]. Generalizations of the Pron’ko–Stroganov system to the case of arbitrary spin were discussed in [8] and [9]. However, a classification of integrable and superintegrable systems with the dipole interaction, admitting second and higher order integrals of motion, was not carried out until now.

In the present paper, 3D superintegrable systems with dipole interaction are classified. We restrict ourselves to systems which are invariant with respect to the rotation group and admit second order integrals of motion. The list of such systems is not too long, but two of them are defined up to arbitrary functions, and so the number of inequivalent systems is infinite. In particular, it includes a system with Fock symmetry, discussed recently in [7], and a supersymmetric system with a matrix potential. For a classification of supersymmetric matrix potentials see papers [10–12].

There are close relations between superintegrability and exact solvability; see, e.g., [13–16]. In particular, all the spinless 2D superintegrable systems whose integrals are given by second order differential operators are exactly solvable as well, and it was conjectured that this property is also valid for higher dimensional superintegrable systems [15].

For systems with spin the situation is more complicated due to the presence of the additional dichotomous variable. As was noticed in [7], such systems with three spatial variables are not necessarily exactly solvable even if they admit more than three second order integrals of motion.

Thus the solvability of the discussed systems should be examined separately. Of course, it is impossible to solve all systems with arbitrary functions presented in the current paper. We restrict ourselves to two particular cases which seem to be physically interesting since they include the field of point charge. The corresponding exact solutions are presented in section 6, in which some other exactly solvable problems are also indicated. In addition, we apply integrals of motion to reduce generic (i.e., including arbitrary functions) eigenvalue problems to decoupled systems of ordinary differential equations for radial wave functions. However, in general these equations are not exactly solvable.

2. Schrödinger–Pauli equations for neutral particles

To describe spin effects in non-relativistic quantum mechanics the Schrödinger equation should be generalized by the Pauli term proportional to the scalar product of spin with the vector of the magnetic field strength. For neutral particles with non-trivial dipole moments (e.g., neutrons)
this term becomes dominant since in this case the minimal interaction is absent. The Pauli-like term is also requested for describing the interaction of charged particles having non-trivial electric moments with an external electric field.

In other words, there are several reasons to study the Schrödinger–Pauli equations of the following generic form:

\[ H \psi = \left( \frac{p^2}{2m} + \frac{\lambda}{2m} \sigma \cdot \mathbf{K} + \omega V(x) \right) \psi = E \psi \]  

(1)

where \( \sigma \) is the matrix vector whose components are Pauli matrices, and \( \mathbf{K} \) and \( V \) are vector and scalar external fields. Moreover, \( \lambda \) and \( \omega \) are coupling constants and \( E \) denotes an eigenvalue of Hamiltonian \( H \).

To obtain more compact formulae in the following calculations let us rescale the variables and reduce the Hamiltonian to the following simplified form:

\[ H = -\nabla^2 + \tilde{V}(x) \equiv -\nabla^2 + \sigma \cdot \mathbf{F} + F^0 \]  

(2)

where \( \nabla^2 \) is the Laplace operator and \( \tilde{V}(x) \) is a matrix potential. For this purpose we change in (1)

\[ E \rightarrow \tilde{E} = 2mE, \quad \mathbf{K} \rightarrow \mathbf{F} = \lambda \mathbf{K}, \quad V \rightarrow F^0 = 2m \omega V. \]  

(3)

Only Hamiltonians (2), including the generic \( 2 \times 2 \) matrix potential \( \tilde{V} \), will be the subject of our classification. Our goal is to find all possible external fields \( \mathbf{F} = (F^1, F^2, F^3) \) and \( F^0 \) such that the systems with such Hamiltonians are superintegrable.

**3. Determining equations**

Let us search for the first and second order integrals of motion \( Q \) for systems described by equation (1). By definition, such integrals are first and second order differential operators commuting with Hamiltonian \( H \):

\[ [H, Q] = HQ - QH = 0. \]  

(4)

We suppose these operators to be formally self-adjoint. Then, without loss of generality, they can be written in the following form:

\[ Q = \frac{1}{2} \sigma^a \{ \Phi^{\mu ab}, \nabla_a \}, \nabla_b \} + i \sigma^a \{ \Lambda^{\mu a}, \nabla_a \} + \sigma^a \Omega^a \]  

(5)

where \( \Phi^{\mu ab}, \Lambda^{\mu a} \) and \( \Omega^a \) are (unknown) real functions of \( x \) and \( \{ \Phi^{\mu ab}, \nabla_a \} = \Phi^{\mu ab} \nabla_a + \nabla_a \Phi^{\mu ab}, \nabla_a = \frac{\partial}{\partial x^a}, \sigma^a \) are Pauli matrices with \( \sigma^0 \) being the \( 2 \times 2 \) unit matrix. In addition, here and in the following the summation is imposed over the repeated indices. Moreover, in all equations the Latin and Greek indices take the values 1, 2, 3 and 0, 1, 2, 3 correspondingly.

Substituting (2) and (5) into (4), using the relations

\[ \sigma^a \sigma^b = \delta^{ab} + i \epsilon^{abc} \sigma^c \]

where \( \epsilon^{abc} \) is the Levi-Civita symbol, and equating coefficients for linearly independent matrices and differential operators, we obtain the following system of determining equations for functions \( \Phi^{\mu ab}, \Lambda^{\mu a}, \Omega^a \) and \( F^\mu \):

\[ \Phi^{\mu ab} + \Phi^{\mu cb} + \Phi^{\mu ac} = 0, \]  

(6)

\[ \Lambda^a_b + \Lambda^b_a = 0, \]  

(7)
\[ \Lambda^m_a + \Lambda^m_b + \varepsilon^{mab} B^k \Phi^{\mu ab} = 0, \quad (8) \]

\[ \Phi^\mu_{ab} F^b_k + \Phi^\mu_{ba} F^b_k - 2 \varepsilon^{bck} \Lambda^{ca} F^k + \Omega^\mu_a = 0, \quad (9) \]

\[ \Lambda^{ak} F^a_k + \Lambda^{bk} F^b_k + \varepsilon^{abc} \Omega^\nu c F^k + \varepsilon^{mkd} \Phi^{\nu cd} F^k = 0, \quad (10) \]

\[ \Phi^\mu_{ak} F^a_k + \Omega^\mu_0 = 0, \quad (11) \]

\[ \Lambda^{\mu k} F^k_\mu = 0. \quad (12) \]

Here the subindices denote derivatives with respect to the independent variables, i.e.,
\[ F^\mu_a = \frac{\partial F^\mu}{\partial x_a}, \text{ etc.} \]

Thus to classify Hamiltonians (2) which admit first and second order integrals of motion we are supposed to solve the system of determining equations (6)–(12). Moreover, when searching for the first order integrals of motion we should \textit{a priori} set \( \Phi^{\mu ab} = 0 \).

4. Decoupling of the determining equations

The system (6)–(12) is rather complicated and includes 77 coupled nonlinear partial differential equations for 44 variables. However, it is rather symmetric, and some of its constituents are easily integrable. The general solution of this system with two independent variables for the case \( \Phi^{\mu ab} = 0 \) was found in [7].

In this paper we find solutions of equations (6)–(12) compatible with the supposition that Hamiltonian (2) is invariant with respect to the rotation group \( O(3) \). In other words we suppose that \( H \) commutes with generators of this group which are nothing but components of the total orbital momentum vector \( J \):

\[ J = x \times p + \frac{1}{2} \sigma. \quad (13) \]

This condition reduces the general form of the external fields to:

\[ F^0 = \phi(x), \quad F^a = x^a \varphi(x) \]

where \( \phi \) and \( \varphi \) are functions of \( x = \sqrt{x_1^2 + x_2^2 + x_3^2} \).

Let us start with the first order integrals of motion. The corresponding functions \( \Phi^{\mu ab} \) are equal to zero, and equations (6)–(8) are reduced to the following:

\[ \Lambda^m_a + \Lambda^m_b = 0. \quad (15) \]

These equations are easily integrated:

\[ \Lambda^m_a = \varepsilon^{abc} x_c \alpha^b + v^a, \quad \Lambda^m_a = \sum_i \Lambda^m_{ai} \quad (16) \]

where

\[ \Lambda^m_{1a} = \nu \delta^m_a, \quad \Lambda^m_{2a} = \mu \varepsilon^{mac} x^c, \quad (17) \]

\[ \Lambda^m_{3a} = \varepsilon^{mac} \mu c, \quad \Lambda^m_{4a} = \delta^m_a x^c c - x^m i^a, \quad (18) \]

\[ \Lambda^m_{5a} = \varepsilon^{abc} x^b \mu^c, \quad \Lambda^m_{6a} = v^m a. \quad (19) \]

Here \( \alpha^a, v^a, \nu, \mu, \mu^c, v^ma \) and \( \mu^ma \) are integration constants. Moreover, without loss of generality we can set \( \alpha^b = 0 \), since non-trivial \( \alpha^b \) correspond to the already declared integrals of motion (13).
Thus to find the first order integrals of motion it is sufficient to solve equations (9)–(12) with the given coefficients (16) and trivial $\Phi_{iab}$. Moreover, all cases enumerated in (17)–(19) (which correspond to scalar, vector and tensor integrals of motion) should be considered separately.

Consider the second order integrals of motion. In accordance with (6) functions $\Phi_{iab}$ should satisfy the equations for Killing tensors of rank 2. Thus they are second order polynomials in $x^a$ which can be represented in the following form [20]

$$\Phi_{iab} = \Phi_1^{iab} + \Phi_2^{iab} + \Phi_3^{iab} + \Phi_4^{iab}$$

(20)

where

$$\Phi_1^{iab} = \lambda_1 \delta^{ab} + \lambda_2 (\delta^{ab} - x^a x^b),$$  

(21)

$$\Phi_2^{iab} = \lambda_1 \delta^{ab} + \lambda_2 \delta^{ab} - 2 \delta^{ab} x^c x^d,$$  

(22)

$$\Phi_3^{iab} = \lambda_1 \delta^{ab} - \lambda_2 x^a x^b - \lambda_3 x^a x^b + \delta^{ab} x^c x^d,$$  

(23)

$$\Phi_4^{iab} = \lambda_1 \delta^{ab} x^c x^d + \lambda_3 \delta^{ab} x^c x^d.$$  

(24)

Here $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are (real) integration constants. Moreover, $\lambda_1^{ab}, \lambda_2^{ab}$ and $\lambda_3^{ab}$ are symmetric and traceless tensors. In the right brackets the covariant properties of the integration constants and parities of $\Phi_{iab}^{ab}$ as functions of $x^a$ are indicated.

Functions $\Phi_{iab}$ with a fixed value of $m \neq 0$ also are Killing tensors of rank 2 with respect to indices $a$ and $b$. Their general form is analogous to (20) but more complicated thanks to the additional free index $m$:

$$\Phi_{iab} = \sum_{i=1}^{7} \Phi_i^{iab}$$

(25)

where

$$\Phi_1^{iab} = \lambda (2x^m \delta^{ab} - x^a x^b - \delta^{ams} x^m),$$  

(26)

$$\Phi_2^{iab} = \lambda_1 \delta^{ab} + \delta^{ab} \lambda_2 + \delta^{ab} \lambda_3 + \delta^{ab} (\delta^{ab} - x^a x^b)$$

$$+ \delta^{ab} (\lambda_1 \delta^{ab} x^c - \lambda_2 \delta^{ab} x^c - \lambda_3 \delta^{ab} x^c),$$  

(27)

$$\Phi_3^{iab} = (\epsilon_{mac} \lambda_2^{ab} + \epsilon_{mcba} \lambda_3^{ab}) x^c + \lambda_4^{ab} (\delta^{ab} x^c + \delta^{ab} x^c),$$.  

(28)

$$\Phi_4^{iab} = \epsilon \lambda_3 \delta^{ab} x^c + \epsilon \lambda_5 \delta^{ab} x^c + \delta^{ab} \lambda_6 \delta^{ab} x^c,$$  

(29)

$$\Phi_5^{iab} = \lambda_1 \delta^{ab} x^c + \lambda_2 \delta^{ab} x^c - 2 \delta^{ab} \lambda_3 x^c + \delta^{ab} \lambda_4 x^c + \delta^{ab} \lambda_5 x^c + \delta^{ab} \lambda_6 x^c,$$  

(30)

$$\Phi_6^{iab} = \lambda_2 \delta^{ab} x^c + \lambda_3 \delta^{ab} x^c + \lambda_4 \delta^{ab} x^c.$$  

(31)

$$\Phi_7^{iab} = \lambda_3 \delta^{ab} x^c + \lambda_4 \delta^{ab} x^c + \lambda_5 \delta^{ab} x^c + \delta^{ab} \lambda_6 x^c + \delta^{ab} \lambda_7 x^c.$$  

(32)
Formulae (20) and (25) give the general solution of equations (6). Another subsystem of the determining equations which can be easily integrated is given by formula (7). In this case we deal with the equation for Killing vectors, whose general solution is given by equation (16).

Solving the remaining equations (9)–(12) is a much more complicated problem which, however, can be effectively separated into relatively simple subproblems using the following reasoning.

- By definition, Hamiltonian (2) admits integrals of motion (13) and is an integral of motion in itself. Thus without loss of generality we can set $\alpha^b = 0$, $\Phi_{iab}^0 = 0$ and $\lambda_{ab}^b$ in (16), (20) and (23) respectively, since they correspond to higher order terms of operators $H$, $J_a$ and $J_a J_b$.
- Functions (26), (22), (27) and (28) correspond to scalar and vector integrals of motion while the remaining solutions generate tensor operators (5). Since scalars, vectors and tensors transform differently under rotation transformations which keep the Hamiltonian invariant, all of them should satisfy the commutativity condition (4) independently. In other words, the determining equations (6)–(12) should be solved separately for scalar, vector and tensor operators.
- Integrals of motion which are tensors of rank 3 are forbidden since the number of their components exceeds the maximal admissible number of integrals of motion. Thus it is possible a priori to set in (25) $\Phi_{iab}^m = \Phi_{ijab}^m = 0$.
- In accordance with (14) the external field $F^a$ and potential $F^0$ are odd and even functions correspondingly. Parities of solutions (22)–(24) and (26)–(32) are transparent also. Analyzing the properties of equations (8)–(12) under the space inversion we conclude that $\Phi_{iab}^m$, $\Lambda^{ma}$ and $\Omega^m$ should have the same parity, which is opposite to that of functions $\Phi_{iab}^0$, $\Lambda^0$ and $\Omega^0$. Thus the system (6)–(12) should be solved separately for the cases in which $(\Phi_{iab}^m, \Lambda^{0a}, \Omega^0)$ are even and odd, with $(\Phi_{iab}^m, \Lambda^{ma}, \Omega^m)$ being odd and even, respectively.

In accordance with the above the system of determining equations (8)–(12) is decoupled to five subsystems corresponding to scalar, vector and tensor functions $\Phi_{iab}^0$, $\Phi_{iab}^m$ and $\Lambda^{0a}$, namely:

\[
\begin{align*}
\Phi_{iab}^0 &= 0, & \Phi_{iab}^m &= \Phi_{i1ab}^m, \\
\Phi_{iab}^0 &= \Phi_{i2ab}^0, & \Phi_{iab}^m &= \Phi_{i2ab}^m, \\
\Phi_{iab}^0 &= 0, & \Phi_{iab}^m &= \Phi_{i3ab}^m, & \Lambda^{0a} &= \lambda^a, \\
\Phi_{iab}^0 &= \Phi_{i4ab}^0, & \Phi_{iab}^m &= \Phi_{i5ab}^m, \\
\Phi_{iab}^0 &= \Phi_{i4ab}^0, & \Phi_{iab}^m &= \Phi_{i4ab}^m
\end{align*}
\]

where functions $\Phi_{iab}^m$ are defined by equations (22)–(24) and (26)–(30). Except in the case (35) the tensor $\Lambda^{0a}$ should be trivial.

5. Classification results

Thus to identify Hamiltonians (2) admitting first and second order integrals of motion it is sufficient to solve equations (10)–(12) with given functions (17), (18) or (19) and trivial $\Phi_{iab}^0$, and equations (9)–(12) with given functions (33), (34), (35), (36) or (37). As a result we
obtain the following list of Hamiltonians (2) together with the admissible constants of motion additional to (13)\(^1\):

\[
H = H_1 = -\nabla^2 + \frac{\lambda}{x} \sigma \cdot \mathbf{x} + \varphi(x),
Q_1 = \sigma \cdot \mathbf{L} + 1 + \lambda \sigma \cdot \mathbf{n};
\]

\[
H = H_2 = -\nabla^2 + \sigma \cdot \mathbf{n} f' + f^2 - \frac{\alpha}{x},
Q_2 = (i\sigma \cdot \mathbf{p} + f)(\sigma \cdot \mathbf{L} + 1) + \frac{\alpha}{2} \sigma \cdot \mathbf{n};
\]

\[
H = H_3 = -\nabla^2 + \frac{\lambda}{x} \sigma \cdot \mathbf{n} + \frac{\lambda^2}{x^2} - \frac{\alpha}{x},
Q_3 = Q_2 \big|_{f = \frac{\lambda}{x}};
\]

\[
H = H_4 = -\nabla^2 + \frac{\lambda}{x} \sigma \cdot \mathbf{x}
R = \frac{1}{2}(\mathbf{p} \times \mathbf{J} - \mathbf{J} \times \mathbf{p}) + \frac{\lambda \mathbf{x} \sigma \cdot \mathbf{x}}{x^2}.
\]

Here \(\mathbf{L} = \mathbf{x} \times \mathbf{p}\) is the orbital momentum, \(\mathbf{n} = \frac{\mathbf{x}}{x}\) and \(\varphi(x)\) and \(f\) are arbitrary functions of \(x\), \(f' = \frac{\partial f}{\partial x}\).

Hamiltonian (40) is a particular case of operators presented in (38) and (39), which corresponds to a more extensive number of integrals of motion.

Hamiltonian (41) admits three integrals of motion, additional to (13). They are components of vector \(\mathbf{R}\), which generalizes the Laplace–Runge–Lenz vector to the case of a system with spin. The integrals \(\mathbf{J}\) and \(\mathbf{R}\) generate the dynamical symmetry with respect to group \(O(4)\). This system was discussed in paper [7], in which its shape invariance was proven and exact solutions of the corresponding eigenvalue problem (1) were found.

Notice that all Hamiltonians (38)–(41) admit one more (discrete) symmetry. Namely, they commute with the following operator

\[
Q_4 = (\sigma \cdot \mathbf{L} + 1)\sigma \cdot \mathbf{p}
\]

where \(\sigma \cdot \mathbf{p}\) is the space inversion operator which changes the sign of independent variables, i.e., it acts on the wave function as \(p\psi(x) = \psi(-x)\).

Another symmetry which includes the space reflection is valid for Hamiltonians \(H_2\). Namely, these Hamiltonians commute with the following operator:

\[
Q_5 = (\sigma \cdot \mathbf{p} + f)\sigma \cdot \mathbf{p} + \alpha \sigma \cdot \mathbf{n}.
\]

Of course, this symmetry with \(f = \frac{\lambda}{x}\) is valid for Hamiltonian \(H_3\).

6. Algebraic properties of integrals of motion

Let us discuss some general properties of the constants of motion presented in the previous section.

By construction, operators \(Q_1\), \(Q_2\), \(Q_3\) and \(\mathbf{R}\) commute with the corresponding Hamiltonians, i.e., satisfy conditions (4).

Operators \(Q_1\), \(Q_2\), and \(Q_3\) are rotational scalars and so commute with the total orbital momentum (13):

\[
[Q_a, \mathbf{J}] = 0, \quad a = 1, 2, 3.
\]

\(^1\) We do not present the calculation details which can be found in [21].
In addition, these operators satisfy the following algebraic relations:

\[ Q_1^2 = J^2 + \lambda^2 + \frac{1}{4}, \quad (44) \]

\[ Q_2^2 = \left( J^2 + \frac{1}{4} \right) H_2 + \frac{\alpha^2}{4}, \quad (45) \]

\[ Q_3^2 = \left( J^2 + \frac{1}{4} \right) H_3 + \frac{\alpha^2}{4}, \quad (46) \]

\[ Q_1 Q_3 + Q_3 Q_1 = \alpha \lambda, \quad [Q_1^2, Q_3] = 0, \quad [Q_2^2, Q_3] = 0. \quad (47) \]

We see that the integrals of motion form rather non-trivial superalgebraic structures. Relations (46) and (47) will be used to explain the degeneration of the spectrum of Hamiltonian (40).

In the particular case \( \varphi = \frac{\alpha}{x^2} \) integrals of motion (13) and (38), together with operators \( D = x^1 p_1 + x^2 p_2 + x^3 p_3 \) and \( K = x^2 / 2 \) form a basis for the seven-dimensional Lie algebra since the following commutation relations are satisfied:

\[ [H_1, D] = -2iH, \quad [K, H_1] = iD, \]

\[ [J^\mu, J^\nu] = i\epsilon^{abc} J^c, \quad [K, D] = 2iK \quad (48) \]

while all the other commutators are trivial. In other words, there is a direct sum of the conformal algebra \( \text{so}(1,2) \cong \langle H_1, D, K \rangle \), the Lie algebra of the rotation group \( \text{so}(3) \cong \langle J^1, J^2, J^3 \rangle \) and the one-dimension algebra spanned on \( Q_1 \). Thanks to the conformal symmetry the discussed system can be interpreted as a model of conformal quantum mechanics; see [19] for definitions.

Finally, let us consider the superintegrable system (41) admitting vector integrals of motion. Components \( R_a \) of vector operator \( R \) satisfy the following commutation relations:

\[ [J_a, J_b] = i\epsilon_{abc} J_c, \quad [R_a, J_b] = i\epsilon_{abc} R_c, \]

\[ [R_a, R_b] = -2i\epsilon_{abc} J_c H_4. \quad (49) \]

Being considered on eigenvectors of Hamiltonian \( H_4 \) corresponding to coupled states, algebra (49) is isomorphic to the Lie algebra of group \( \text{O}(4) \). In other words, the system with Hamiltonian \( H_4 \) admits the same dynamical symmetry as the hydrogen atom. Detailed analysis of this system is presented in [7].

7. Exact solutions

Since Hamiltonians \( H_1 \) and \( H_2 \) are defined up to arbitrary functions, they represent an infinite set of superintegrable models. In this section we consider two important particular cases of such models which involve the Coulomb potential, and present a constructive way of finding solutions for the systems with arbitrary potentials.

7.1. Charged particle with the electric dipole moment interacting with the field of point charge

Let us start with Hamiltonian \( H_1 \). It includes an arbitrary scalar potential \( \varphi \) and a dipole interaction term \( \sim \sigma \cdot F \) with coupling constant \( \lambda \) and external field \( F = \frac{\sigma}{x^2} \). This field can be interpreted as an electric field generated by a point charge. Thus it is natural to choose \( \varphi = \frac{\alpha}{x^2} \), then the corresponding operator \( H_1 \) can be interpreted as a Hamiltonian of a charged particle with spin 1/2 and a non-trivial dipole electric moment.
Consider the eigenvalue problem for a specified Hamiltonian $H_1$:

$$\left( -\Delta + \lambda \frac{\sigma \cdot x}{x^2} - \frac{\alpha}{x} \right) \psi = \hat{E} \psi. \quad (50)$$

Here $\psi = \psi(x)$ is a two-component function which is supposed to be normalizable and vanishes at $x = 0$. In addition, to obtain a system with coupled states we suppose that $\alpha > 0$.

Equation (50) admits three constants of motion $J_3, J_2$ and $Q_1$ which commute each other. Thus we can expand $\psi = \psi(x)$ via eigenvectors $\hat{\Omega}_{j,k,v}(\varphi, \theta)$ of these operators:

$$\psi = \sum_{j,k,v} \psi_{jkv}(x) \hat{\Omega}_{j,k,v}(\varphi, \theta). \quad (51)$$

Here $x, \varphi$ and $\theta$ are spherical coordinates, $j = \frac{1}{2}, \frac{3}{2}, \ldots, \kappa = -j, -j + 1, \ldots, j$, and $\nu = \epsilon \mu$ (where $\mu = \sqrt{j(j+1) + \lambda^2 + \frac{1}{4}}$ and $\epsilon = \pm 1$) are quantum numbers which label the eigenvalues:

$$J_2 \hat{\Omega}_{j,k,v} = j(j+1) \hat{\Omega}_{j,k,v},$$
$$J_3 \hat{\Omega}_{j,k,v} = \kappa \hat{\Omega}_{j,k,v}, \quad (52)$$
$$Q_1 \hat{\Omega}_{j,k,v} = \nu \hat{\Omega}_{j,k,v}. \quad (53)$$

The explicit form of $\hat{\Omega}_{j,k,v}$ will be specified later. Notice that the eigenvalues of $Q_1$ can be found algebraically starting with relation (44) and the first of relations (52).

Substituting (51) into (50) we obtain the following ordinary differential equations for radial functions $\psi_{jkv}$:

$$\left( -\frac{d^2}{dx^2} + \frac{\nu(\nu+1) - \lambda^2}{x^2} - \frac{\alpha}{x} \right) \psi_{jkv} = \hat{E} \psi_{jkv}. \quad (54)$$

This equation is solved by the following functions:

$$\psi_{jkv} = C_{jkv} e^{k_\nu + \frac{1}{2}} \exp(\sqrt{-\hat{E}x}) F(a_\nu, 2k_\nu + 1, 2\sqrt{-\hat{E}x}) \quad (55)$$

where $C_{jkv}$ is an integration constant, $F$ is the confluent hypergeometric function, and

$$k_\nu = \sqrt{\nu(\nu+1) - \lambda^2}, \quad a_\nu = k_\nu + \frac{1}{2} - \frac{1}{\sqrt{-\hat{E}}}, \quad \nu = \pm \sqrt{j(j+1) + \lambda^2 + \frac{1}{4}}. \quad (56)$$

In order for function (55) to be bounded at infinity, the argument $a_\nu$ has to be a negative integer or zero, i.e.,

$$k_\nu + \frac{1}{2} - \frac{1}{\sqrt{-\hat{E}}} = -n, \quad n = 1, 2, \ldots \quad (57)$$

The corresponding eigenvalue $\hat{E}$ in (50) and the energy value $E$ from (3) are given by the following equations:

$$\hat{E} = -\frac{\alpha^2}{4N^2} \quad \text{and} \quad E = -\frac{ma^2}{2N^2}. \quad (58)$$

where

$$N = \sqrt{\nu(\nu+1) - \lambda^2 + n + \frac{1}{2}}, \quad n = 0, 1, 2, \ldots \quad (59)$$

The energy levels (58) are degenerated with respect to quantum number $\kappa$, which are eigenvalues of the third component of the total orbital momentum. There are no other degenerations.
Let us present the basic functions \(\hat{\Omega}_{ji,\kappa,\nu} = \hat{\Omega}_{ji,\kappa,\nu}(\phi, \theta)\), which are used in formula (51):

\[
\hat{\Omega}_{ji,\kappa,\nu} = \frac{1}{2\sqrt{\mu}} \left( \sqrt{\frac{j + j + \frac{1}{2}(j + \kappa)}{j}} Y_{j - \frac{1}{2},\kappa + \frac{1}{2}} + \frac{\lambda}{j(\mu + j + \frac{1}{2})} \frac{(j + j + \frac{1}{2})(\mu + j + \frac{1}{2})}{(j + 1)} Y_{j + \frac{1}{2},\kappa + \frac{1}{2}} \right)
\]  
(60)

if \(\nu = \mu > 0\), and

\[
\hat{\Omega}_{ji,\kappa,\nu} = \frac{1}{2\sqrt{\mu}} \left( \frac{\lambda}{j(\mu + j + \frac{1}{2})} Y_{j - \frac{1}{2},\kappa + \frac{1}{2}} + \frac{(j + j + \frac{1}{2})(\mu + j + \frac{1}{2})}{(j + 1)} Y_{j + \frac{1}{2},\kappa + \frac{1}{2}} \right)
\]  
(61)

if \(\nu = -\mu < 0\). Here \(Y_{j \pm \frac{1}{2}, k \pm \frac{1}{2}}\) are spherical functions.

7.2. Supersymmetric system

Let us discuss the maximally superintegrable system whose Hamiltonian \(H_3\) is defined by equation (40), and consider the related eigenvalue problem:

\[
H_3 \psi \equiv \left( -\Delta + \frac{\lambda}{x} \sigma \cdot n + \frac{\lambda^2}{x^2} - \frac{a}{x} \right) \psi = \hat{E} \psi.
\]  
(62)

Equation (62) admits a number of symmetry operators given by relations (13) and (40). Among them there are two commuting integrals of motion \(J_3\), \(J_2\) and two anticommuting constants of motion \(Q_1\) and \(Q_3\). Symmetries \(J_3\) and \(J_2\) make it possible to separate the variables. The constants of motion \(Q_1\) and \(Q_3\) enable us to decouple the system of equations in radial variables. In addition, it will be shown that these constants of motion cause a specific degeneration of the energy spectrum.

Like (50), equation (62) is exactly solvable. Expanding its solutions via basis vectors (60) and (61), i.e., using representation (51), we come to the following equations for radial functions:

\[
H_{ij} \psi_{j\kappa\nu} \equiv \left( -\frac{\partial^2}{\partial x^2} + \nu(\nu + 1) + \frac{\alpha^2}{x^2} - \frac{a}{x} \right) \psi_{j\kappa\nu} = \hat{E} \psi_{j\kappa\nu}.
\]  
(63)

where \(\nu\) are parameters defined in (56).

Hamiltonian \(H_\nu\) is shape invariant. In other words, it can be factorized:

\[
H_\nu = a_\nu^+ a_\nu + c_\nu
\]  
(64)

where

\[
a_\nu = \frac{\partial}{\partial x} + W_\nu, \quad a_\nu^+ = -\frac{\partial}{\partial x} + W_\nu, \quad c_\nu = -\frac{\alpha^2}{(|\nu| + \frac{\epsilon}{2})}, \quad \epsilon = \text{sign} \nu,
\]  
(65)

and \(W_\nu\) is a superpotential:

\[
W_\nu = \frac{2\alpha}{2|\nu| + \epsilon + 1} - \frac{2|\nu| + \epsilon + 1}{4x}.
\]

Moreover, operators \(\hat{H}_\nu = H_\nu - c_\nu\) satisfy the intertwining relations

\[
\hat{H}_\nu a_\nu^+ = a_\nu^+ \hat{H}_{\nu+1}.
\]  
(66)

This means that eigenvalues of Hamiltonian \(H_\nu\) (63) can be found algebraically using tools of supersymmetric quantum mechanics. As a result we obtain these eigenvalues in the form (58) where

\[
N = n + |\nu| + \frac{\epsilon + 1}{2}, \quad n = 0, 1, 2, \ldots
\]  
(67)
As for the hydrogen atom, energy values (58), (67) are proportional to the inverse square of the main quantum number \( N \). However, in contrast with the hydrogen atom, \( N \) is not a linear combination of two independent non-negative integers, and so there is no \( N \)-fold degeneration. However, since \( \nu \) can take both positive and negative values, there is a two-fold degeneration typical for supersymmetric systems. In addition, since the quantum number \( \kappa \) does not affect the energy values, there is the additional \( (2j + 1) \)-fold degeneration with respect to \( \kappa \).

Let us show that the supersymmetric nature of spectrum (58), (67) is caused by integrals of motion \( Q_1 \) and \( Q_3 \).

It follows from (47) that operator \( Q_4 = \frac{1}{2}[Q_1, Q_3] \) anticommutes with \( Q_1 \) and \( Q_3 \). On the set of eigenfunctions of operators \( \mathbf{J}^2 \) and \( Q_1 \) it is possible to define the rescaled operators

\[
\hat{Q}_3 = \frac{Q_3}{j + \frac{1}{2}}, \quad \hat{Q}_4 = \frac{Q_4}{\nu} + \frac{\alpha Q_1}{(2j + 1)\nu}
\]

which satisfy the following anticommutation relations

\[
\hat{Q}_a \hat{Q}_b + \hat{Q}_b \hat{Q}_a = 2\delta_{ab}\hat{H}.
\]

Here any of the subindices \( a \) and \( b \) independently take the values 3 and 4, and \( \hat{H} = H_3 + \frac{\alpha^2}{(2j + 1)\nu^2} \).

By construction, both \( \hat{Q}_3 \) and \( \hat{Q}_4 \) commute with \( \hat{H} \), but this is also a consequence of (69).

Thus the rescaled integrals of motion \( \hat{Q}_3, \hat{Q}_4 \) and Hamiltonian \( \hat{H} \) form a basis of the superalgebra of supersymmetric quantum mechanics. The two-fold degeneration of the spectrum of Hamiltonian \( \hat{H} \) (and therefore of Hamiltonian \( H_3 \)) is a direct consequence of algebraic relations (69).

Notice that the ground state with energy \( \hat{E} = \frac{\alpha^2}{(2j + 1)\nu^2} \) (see equations (58), (67) for \( n = 0 \) and \( \epsilon = -1 \)) is not degenerated. Thus the supersymmetry of system (62) is exact.

Let us also present the state vectors corresponding to eigenvalues (58), (59):

\[
\psi_{j\kappa\nu}^{(n)} = C_n x^{|\nu|+\frac{1}{2}} \exp\left(-\sqrt{-\hat{E} x}\right) F(-n, 2|\nu| + 1, \sqrt{-\hat{E} x}).
\]

Like in (55), \( F(-n, 2|\nu| + 1, x) \) is the confluent hypergeometric function. However, its arguments differ from the arguments of function (55).

### 7.3. Equations for radial wave functions for arbitrary potentials

The eigenvalue problem for Hamiltonians \( H_1 \) and \( H_2 \) can be effectively decoupled for the case of arbitrary function \( \varphi \) and \( f \) present in their definitions (38) and (39). Here we deduce the decoupled equations for radial wave functions.

For Hamiltonian \( H_1 \) it can be done in complete analogy with section 7.1. Considering the eigenvalue problem for this Hamiltonian with arbitrary function \( \varphi \) and repeating all steps presented in this section before equation (54), we obtain the following equation:

\[
H_1 \psi_{j\kappa\nu} = \left(-\frac{d^2}{dx^2} + \frac{\nu(\nu + 1)}{x^2} - \frac{\lambda^2}{x^2} + \varphi(x)\right)\psi_{j\kappa\nu} = \hat{E} \psi_{j\kappa\nu}
\]

where \( \nu \) is the parameter defined in (56).

Thus expanding solutions via eigenvectors of the commuting integrals of motion, satisfying (52) and (53), it is possible to reduce the eigenvalue problem for Hamiltonian \( H_1 \) (38), which is a 3D system of two coupled equations, to the infinite set of decoupled ordinary differential equations (71).

In order for radial functions \( \psi_{j\kappa\nu} \) to have a good behavior at \( x = 0 \), the potential \( \varphi(x) \) should increase no faster than \( \frac{1}{x^2} \) when \( x \to 0 \). If for small \( x \) this potential increases as \( \frac{1}{x^2} \), then parameter \( \alpha \) should satisfy the condition \( \alpha > \sqrt{1 + \lambda^2} - \frac{1}{4} \).
For some particular potentials \( \varphi(x) \) equations (71) can be solved explicitly. Examples of such potentials are:

\[
\varphi(x) = -\frac{\alpha}{x}, \quad (72)
\]

\[
\varphi(x) = \frac{\alpha}{x} + \frac{\beta}{x^2}, \quad \beta \neq 0, \quad (73)
\]

\[
\varphi(x) = \alpha^2 x^2. \quad (74)
\]

In the present paper only solutions corresponding to the Coulomb potential (72) are discussed; see section 6.1.

Consider now the eigenvalue problem for Hamiltonian \( H_2 \) with arbitrary potential:

\[
H_2 \psi = \left( -\nabla^2 + \sigma \cdot n f' + f^2 - \frac{\alpha}{x} \right) \psi = \hat{E} \psi. \quad (75)
\]

In addition to (75), we impose on wave function \( \psi \) the following condition:

\[
Q_2 \psi = \left( (i\sigma \cdot p + f)(\sigma \cdot L + 1) + \frac{\alpha}{2} \sigma \cdot n \right) \psi = q \psi \quad (76)
\]

where eigenvalues \( \hat{E} \) and \( q \) are connected by the following algebraic relation:

\[
4q^2 = (2j + 1)^2 \hat{E} + \alpha^2. \quad (77)
\]

Equations (75) and (76) are compatible since operators \( H_2 \) and \( Q_2 \) commute each other and satisfy relations (39).

To separate the variables, let us expand the solutions of equations (75) and (76) via the complete set of eigenfunctions of the commuting operators \( J_3^2, J_3, \) and \( \sigma \cdot L + 1 \):

\[
\psi = \frac{1}{r} \sum_{j,\kappa,\epsilon\mu} \psi_{j,\kappa,\epsilon\mu}(r) \Omega_{j,\kappa,\epsilon\mu}(\varphi, \theta). \quad (78)
\]

Here \( \Omega_{j,\kappa,\epsilon\mu}(\varphi, \theta) \) are spherical spinors, \( j, \kappa, \epsilon = \pm 1 \) and \( \mu = j + \frac{1}{2} \) are quantum numbers labeling the eigenvalues of the above mentioned operators:

\[
J_3^2 \Omega_{j,\kappa,\epsilon\mu} = j(j+1) \Omega_{j,\kappa,\epsilon\mu},
\]

\[
J_3 \Omega_{j,\kappa,\epsilon\mu} = \kappa \Omega_{j,\kappa,\epsilon\mu},
\]

\[
(\sigma \cdot L + 1) \Omega_{j,\kappa,\epsilon\mu} = \epsilon \mu \Omega_{j,\kappa,\epsilon\mu}.
\]

As a result equation (75) is reduced to the following coupled system of ordinary differential equations for the radial wave function:

\[
\left( -\frac{\hat{d}^2}{dx^2} + \frac{\mu(\mu + 1)}{x^2} + f^2 - \frac{\alpha}{x} \right) \psi_{j,\kappa,\epsilon\mu}(x) + f' \psi_{j,\kappa,-\epsilon\mu}(x) = \hat{E} \psi_{j,\kappa,\epsilon\mu}(x). \quad (79)
\]

\[
\left( -\frac{\hat{d}^2}{dx^2} + \frac{\mu(\mu - 1)}{x^2} + f^2 - \frac{\alpha}{x} \right) \psi_{j,\kappa,-\epsilon\mu}(x) + f' \psi_{j,\kappa,\epsilon\mu}(x) = \hat{E} \psi_{j,\kappa,-\epsilon\mu}(x). \quad (80)
\]

Using (76) this system can be decoupled. Substituting (78) into (76) and taking into account the relations

\[
\sigma \cdot p = \sigma \cdot n \sigma \cdot p = \left( x \cdot p - \frac{1}{x} \right) \sigma \cdot n + \frac{i}{x} \sigma \cdot n (\sigma \cdot L + 1),
\]

\[
\sigma \cdot n (\sigma \cdot L + 1) = -(\sigma \cdot L + 1) \sigma \cdot n,
\]

\[
(\sigma \cdot L + 1)^2 = J_3^2 + \frac{1}{4}, \quad (\sigma \cdot n)^2 = 1,
\]

\[
\sigma \cdot n \Omega_{j,\kappa,\epsilon\mu} = \Omega_{j,\kappa,-\epsilon\mu}, \quad x \cdot p \Omega_{j,\kappa,\epsilon\mu} = 0, \quad x \cdot p \psi_{j,\kappa,\epsilon\mu} = -i \frac{\hat{d}}{dx} \psi_{j,\kappa,\epsilon\mu}
\]
we obtain the following system of first order equations:
\[
(f - \tilde{q})\psi_{j,k,\mu} + a_\mu^\dagger \psi_{j,k,-\mu} = 0,
\]
\[
(f + \tilde{q})\psi_{j,k,-\mu} + a_j \psi_{j,k,\mu} = 0
\]
where
\[
a_\mu = \left( \frac{\partial}{\partial x} + \frac{\mu}{x} - \frac{\alpha}{2\mu} \right), \quad a_\mu^\dagger = \left( - \frac{\partial}{\partial x} + \frac{\mu}{x} - \frac{\alpha}{2\mu} \right)
\]
and
\[
\tilde{q} = \frac{2q}{2j+1} = \pm \left( \hat{E} + \frac{\alpha^2}{(2j+1)^2} \right)^{\frac{1}{2}}.
\]

Solving equation (82) for \(\psi_{j,k,-\mu}\) and substituting the derived expression into (81) or (79) we obtain a decoupled system of second order equations:
\[
\left( a_\mu^\dagger a_\mu + \frac{f'}{f + \tilde{q}} a_\mu + f^2 - \frac{\alpha^2}{4\mu^2} \right) \psi_{j,k,\mu} = \hat{E} \psi_{j,k,\mu}.
\]  
(83)

Thus equation (75) with arbitrary potential function \(f = f(x)\) can be reduced to the decoupled system of ordinary differential equations (83) for radial functions. However, it is doubtful whether the latter equations can be solved exactly for a fixed function \(f = f(x)\) if \(\alpha f' \neq 0\).

To end this section, let us write system (79), (80) as a single equation with a matrix potential:
\[
\mathcal{H} \psi_{j,k,\mu} \equiv \left( - \frac{\partial^2}{\partial x^2} + U \right) \psi_{j,k,\mu} = \left( \hat{E} + \frac{\alpha^2}{(2j+1)^2} \right) \psi_{j,k,\mu}
\]
where \(\psi_{j,k,\mu} = \text{column}(\psi_{j,k,\mu}, \psi_{j,k,-\mu})\), and
\[
U = W^2 - W', \quad W = \left( \frac{\mu}{x} - \frac{\alpha}{2\mu} \right) \sigma_3 + f \sigma_1.
\]

In other words, the effective potential \(U\) can be expressed via superpotential \(W\). This circumstance makes it possible to find the ground states \(\psi_{j,k,\mu}^{(0)}\) of Hamiltonians \(\mathcal{H}\) which are solutions of the equation \(\frac{\partial}{\partial x} + W) \psi_{j,k,\mu}^{(0)} = 0\). The latter equation is nothing but the system (81), (82) with \(\tilde{q} = 0\).

8. Discussion

We present the complete list of 3D Hamiltonians (2) with matrix potentials, which are invariant with respect to the rotation group and admit first and second order integrals of motion. This list is given by equations (38)–(41) and includes four representatives, two of which are defined up to arbitrary functions.

The presented systems have a clear physical interpretation and describe particles with spin 1/2 and non-trivial dipole moment. The system (41) was discussed in paper [7]. As for the hydrogen atom, it admits six integrals of motion satisfying algebra o(4). However, in contrast with symmetries of the hydrogen atom, the Laplace–Runge–Lenz vector \(R\) in (41) is dependent on spin. In the present paper we prove that this system is unique, and there are no other \(2 \times 2\) matrix potentials compatible with this symmetry.

It is possible to verify by direct calculations that Hamiltonians \(H_1 \ldots H_4\) do commute with the presented integrals of motion and do not admit other integrals of motion independent from those given by relations (38)–(41). A much more difficult problem is to prove that the
list of superintegrable systems (38)–(41) is complete. To do this, it is necessary to find all non-
equivalent solutions of the determining equations (6)–(12). To save room we do not present
here the related detailed calculations, which can be found in an extended version of this paper
published as a preprint [21].

Solving the determining equations, we did not restrict ourselves to scalar and vector
integrals of motion. On the contrary, we considered also the tensor integrals of motion and
proved that for equation (2) they do not exist.

The eigenvalue problems for Hamiltonians (2) are systems of coupled partial differential
equations of second order, which are rather complicated. However, these Hamiltonians admit
at least three commuting integrals of motion. Thank to this, the eigenvalue problems can be
effectively decoupled and reduced to ordinary differential equations in radial variables.
Solutions for two of the eigenvalue problems which include the Coulomb potential are
presented in section 6, while the eigenvalues and eigenvectors of Hamiltonian (41) have
been found in paper [7].

In contrast to the case of scalar superintegrable systems, the systems with spin have been
classified only partially. Thus they belong to a prospective field of research, and it would be
interesting to extend our knowledge of these interesting and important subjects. In particular,
it would be desirable to classify second order superintegrable systems with dipole (and spin–
orbit) interactions, which are not a priori rotationally invariant. This problem is much more
complicated than in the case of scalar systems, and it would be a challenge to solve it.

Another interesting task would be to search for relativistic counterparts of non-relativistic
superintegrable systems. The relativistic analogues of the Pron’ko–Stroganov system [5] and
the system with Hamiltonian (41) are discussed in papers [22] and [7], and it would not be too
difficult to extend this discussion to the cases presented in (38)–(40).

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