Axial Torsion-Dirac spin Effect in Rotating Frame with Relativistic Factor

C. M. Zhang

Research Center for Theoretical Astrophysics

School of Physics, University of Sydney, NSW 2006, Australia

zhangcm@physics.usyd.edu.au

Abstract

In the framework of spacetime with torsion and without curvature, the Dirac particle spin precession in the rotational system is studied. We write out the equivalent tetrad of rotating frame, in the polar coordinate system, through considering the relativistic factor, and the resultant equivalent metric is a flat Minkowski one. The obtained rotation-spin coupling formula can be applied to the high speed rotating case, which is consistent with the expectation.

04.25.Nx, 04.80.Cc, 04.50.+h, 04.20.Jb
I. INTRODUCTION

The tetrad theory of gravitation has been pursued by a number of authors [1–7], where the spacetime is characterized by the torsion tensor and the vanishing curvature, the relevant spacetime is the Weitzenböck spacetime [1], which is a special case of the Riemann-Cartan spacetime with the constructed metric-affine theory of gravitation [8–10]. The tetrad theory of gravitation will be equivalent to general relativity when the convenient choice of the parameters of the Lagrangian.

We will use the greek alphabet (μ, ν, ρ, ⋯ = 1, 2, 3, 4) to denote tensor indices, that is, indices related to spacetime. The latin alphabet (a, b, c, ⋯ = 1, 2, 3, 4) will be used to denote local Lorentz (or tangent space) indices. Of course, being of the same kind, tensor and local Lorentz indices can be changed into each other with the use of the tetrad $e^a_\mu$ which satisfy

$$e^a_\mu e_a^\nu = \delta_\mu^\nu ; \quad e^a_\mu e_b^\mu = \delta^a_b .$$

A nontrivial tetrad field can be used to define the linear Cartan connection [1,5]

$$\Gamma^a_{\mu\nu} = e^a_\sigma \partial_\nu e^{\sigma}_\mu ,$$

with respect to which the tetrad is parallel:

$$\nabla_\nu e^a_\mu \equiv \partial_\nu e^a_\mu - \Gamma^a_{\rho\mu} e^\rho_\mu = 0 .$$

The Cartan connection can be decomposed according to

$$\Gamma^a_{\mu\nu} = \tilde{\Gamma}^a_{\mu\nu} + K^a_{\mu\nu} ,$$

where
\[ \Gamma^\sigma_{\mu \nu} = \frac{1}{2} g^{\sigma \rho} \left[ \partial_\mu g_{\rho \nu} + \partial_\nu g_{\rho \mu} - \partial_\rho g_{\mu \nu} \right] \]  \hspace{1cm} (5)

is the Levi–Civita connection of the metric

\[ g_{\mu \nu} = \eta_{ab} \epsilon^a_{\mu} e^b_\nu, \]  \hspace{1cm} (6)

where \( \eta^{ab} \) is the metric in flat space with the line element

\[ d\tau^2 = g_{\mu \nu} dx^\mu dx^\nu, \]  \hspace{1cm} (7)

and

\[ K^\sigma_{\mu \nu} = \frac{1}{2} \left[ T^\sigma_{\mu \nu} + T^\sigma_{\nu \mu} - T^\sigma_{\mu \nu} \right] \]  \hspace{1cm} (8)

is the contorsion tensor, with

\[ T^\sigma_{\mu \nu} = \Gamma^\sigma_{\mu \nu} - \Gamma^\sigma_{\nu \mu} \]  \hspace{1cm} (9)

the torsion of the Cartan connection \([1,5]\). The irreducible torsion vectors, i.e., the torsion vector and the torsion axial-vector, can then be constructed as \([1,5]\)

\[ V_\mu = T^\nu_{\nu \mu}, \]  \hspace{1cm} (10)

\[ A_\mu = \frac{1}{6} \epsilon_{\mu \nu \rho \sigma} T^\nu_{\nu \rho \sigma}, \]  \hspace{1cm} (11)

with \( \epsilon_{\mu \nu \rho \sigma} \) being the completely antisymmetric tensor normalized as \( \epsilon_{0123} = \sqrt{-g} \) and \( \epsilon^{0123} = \frac{1}{\sqrt{-g}} \).

The spacetime dynamic effects on the spin is incorporated into Dirac equation through the “spin connection” appearing in the Dirac equation in gravitation \([1]\). In Weitzenböck spacetime, as well as the general version of torsion gravity, it has been shown by many authors \([1,2,11–17]\) that the spin precession of a Dirac particle is intimately related to the torsion axial-vector, and it is interesting to note that the torsion axial-vector represents the deviation of the axial symmetry from the spherical symmetry \([2]\).
\[
\frac{dS}{dt} = -\frac{3}{2} A \times S, \tag{12}
\]

where \( S \) is the semiclassical spin vector of a Dirac particle, and \( A \) is the spacelike part of the torsion axial-vector. Therefore, the corresponding extra Hamiltonian energy is of the form,

\[
\delta H = -\frac{3}{2} A \cdot S. \tag{13}
\]

Throughout this paper we use the relativistic unit, \( c = 1 \).

**II. THE ROTATION-SPIN EFFECT**

Now we discuss the Dirac equation in the rotational coordinate system with the polar coordinates \((t, r, \phi, z)\), and the system is rotating with the angular velocity \( \Omega \), and the rotation axis is set in \( z \)-direction.

In the case of considering the relativistic factor \( \gamma = 1/\sqrt{1 - (\Omega r)^2} \), the tetrad can be expressed by the dual basis of the differential one-form [18] through choosing a coframe of the rotational coordinate system, so we define,

\[
d\vartheta^0 = \gamma [dt - \Omega r (rd\phi)], \tag{14}
\]
\[
d\vartheta^1 = dr, \tag{15}
\]
\[
d\vartheta^2 = \gamma [(rd\phi) - \Omega r dt], \tag{16}
\]
\[
d\vartheta^3 = dz, \tag{17}
\]

however our result here is added by the relativistic factor \( \gamma \) because the high speed rotation is taken into account. If \( \Omega r \) is much less than the speed of light, then we have \( \gamma = 1 \) and the classical coframe expression is recovered, which is same as those applied in the existed references [18,19]. Therefore Eq.(15) and Eq.(17) is a generalised coframe expression for any rotation velocity.

The tetrad can be obtained with the subscript \( \mu \) denoting the column index (c.f. [18,19]),
\[ e^a_{\mu} = \begin{pmatrix} \gamma & 0 & -\gamma \Omega r^2 & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma \Omega r & 0 & \gamma r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18} \]

with the inverse \( e^a_{\mu} = g^{\mu\nu} e^b_{\nu} \eta_{ab}, \) and so

\[ e^a_{\mu} = \begin{pmatrix} \gamma & 0 & \gamma \Omega & 0 \\ 0 & 1 & 0 & 0 \\ \gamma \Omega r & 0 & \gamma / r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{19} \]

We can inspect that Eqs. (18) and (19) satisfy the conditions in Eqs. (1) and (6).

The obtained metric is obtained as

\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{20} \]

\[ g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{21} \]

and with the determinant of the metric

\[ g = det|g_{\mu\nu}| = -r^2. \tag{22} \]

From the tetrad and metric given above, we have the line element,

\[ d\tau^2 = \eta_{ab} d\phi^a \otimes d\phi^b = g_{\mu\nu} dx^\mu dx^\nu \]

\[ = dt^2 - (dr^2 + r^2 d\phi^2 + dz^2), \tag{23} \]

and we find that the metric in Eq. (23) is that of flat spacetime, which will result in the null curvature. Although the curvature vanishes, the torsion (field) may have nonzero components determined by tetrads and not by metrics. In other words, the basic element in torsion
gravity without curvature is tetrad and the metric is just a by-product [1]. From Eqs.
(18) and (19), we can now construct the Cartan connection, whose nonvanishing
components are:

\[ \Gamma^0_{01} = \gamma^2 \Omega^2 r, \quad \Gamma^2_{01} = -\gamma^2 \Omega / r, \quad \Gamma^0_{21} = -\gamma^2 \Omega r, \quad \Gamma^2_{21} = \gamma^2 / r . \]  

(24)

The corresponding nonvanishing torsion components contributed to the axial torsion-vectors are:

\[ T^2_{01} = -\gamma^2 \Omega / r, \quad T^0_{21} = -\gamma^2 \Omega r , \]  

(25)

The nonvanishing axial torsion-vectors are consequently

\[ A_3 = \frac{2}{3} \gamma^2 \Omega , \quad A_k = 0, k = 0, 1, 2 . \]  

(26)

As shown, \( A_1 = A_2 = 0 \) is on account of the Z-axis symmetry which results in the
cancelling of the \( r \) and \( \phi \) components, and then generally we can write \( \mathbf{A} = \frac{2}{3} \gamma^2 \Omega \). From the
spacetime geometry view, the torsion axial-vector represents the deviation from the spherical
symmetry [2], i.e., which will disappear in the spherical case (Schwarzschild spacetime for
instance) and occurs in the axisymmetry case (Kerr spacetime for instance). Therefore the
torsion axial-vector corresponds to an inertia field with respect to Dirac particle, which is
now explicitly expressed by Eq.(12),

\[ \frac{dS}{dt} = -\gamma^2 \Omega \times S . \]  

(27)

If the physics measurement is performed in the rotating frame, the time \( dt \) is taken as the
proper time through setting the null space difference. Then we have \( dt = d\theta^0 / \gamma \), and so

\[ \frac{dS}{d\theta^0} = -\gamma \Omega \times S , \]  

(28)

and the additive Hamiltonian energy measured in the rest frame is,

\[ \delta H = -\gamma \Omega \cdot S , \]  

(29)

which is same as that expected by Mashhoon (c.f. Ref. [20,21]).
III. DISCUSSIONS AND CONCLUSIONS

The rotation Dirac spin coupling for the high speed rotation in the framework of the torsion spacetime without curvature has been derived. This effect was first proposed by Mashhoon [20,21], and the straightforward theoretical derivation was performed by Hehl and Ni [18]. However, the relativistic factor has not been considered in the previous theoretical work on rotation-spin [18,19]. So, in this paper, we follow the axial torsion spin treatment [19], and extend that method to the high speed case, which successfully presents the relativistic $\gamma$ factor into the rotation-spin coupling term. One fact seems to be interested to be paid attention that the choice of the tetrad results in the flat metric, which produces the null curvature, i.e., Riemannian curvature and Cartan curvature. It is remarked that the flat metric will arise the Minkowski spacetime, not Riemannian spacetime, but the axial torsion has nothing to do with the metric and is just related to the tetrad. Technically, the diagonal tetrad will produce the diagonal metric, but the inverse is not true. So our non-diagonal tetrad, similar to Lorentz transformation, arises from the rotation velocity and relativistic factor, results in the diagonal metric. In other words, the metric determines the curvature of spacetime, which reflects the gravitation and define the geodesic of free particle, then tetrad or torsion, determining the metric as well, defines the rotation-spin motion. Although our final conclusion is same as that by Hehl and Ni [18], they obtained the non-diagonal metric, which was derived from the Fermi-Walker transport, i.e., non-Minkowski spacetime. The relation between the geometrical meaning and physical meaning has not yet been clearly and needs the further investigation. Nonetheless, the axial torsion-spin coupling is also successfully applied to the Kerr spacetime [6], and the gravitomagnetic effect on Dirac particle [21] has been obtained, where the exact Kerr tetrad has been exploited. This means that the axial torsion-spin method is applicable to the axisymmetric interaction effect on the Dirac particle.
ACKNOWLEDGMENTS

Discussions and suggestions, as well as critic reading, from G. Lambiase are highly appreciated.
REFERENCES

[1] Hayashi, K. and Shirafuji, T. 1979 *Phys. Rev. D* **19** 3524

[2] Nitsch, J. and Hehl, F.W. 1980 *Phys. Lett. B* **90** 98

[3] de Andrade V.C. and Pereira J.G. 1998 *Phys. Rev. D* **56** 4689

[4] de Andrade, V.C. and Pereira, J.G. 1997 *Gen. Rel. Grav.* **30** 263

[5] Aldrovandi, R. and Pereira, J. G. 1995 *An Introduction to Geometrical Physics*, World Scientific, Singapore.

[6] Pereira, J.G., T. Vargas, and Zhang, C.M. 2001 *Class. Quant. Grav.* **18** 833, gr-qc/0102070

[7] Maluf, J.W. 1999 *Gen. Rel. Grav.* **31** 173

[8] Hehl, F.W., von der Heyde, P., Kerlick, G.D., and Nester, J.M., 1976 *Rev. Mod. Phys.* **48** 393

[9] Hehl, F.W., McCrea, J.D., Mielke, E., and Ne’eman, Y. 1995 *Phys. Rep.* **258** 1

[10] Hehl, F.W., and Macias, A. 1999, Int. J. Mod. Phys. **D8**, 399, gr-qc/9902076

[11] Hammond, R.T. 1994 *Gen. Rel. Grav.* **26**, 247

[12] Hammond, R. T. 1995 *Cont. Phys.* **36** 103

[13] Hehl, F. W. 1971 *Phys. Lett. A* **36** 225

[14] Trautman, A. 1972 *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.* **20**, 895

[15] Rumpf, H. 1980 in *Cosmology and Gravitation*, eds. Bergmann, P. G. and de Sabbata, V. (New York: Plenum)

[16] Yasskin, P. B. and Stoeger W R 1980 *Phys. Rev. D* **21** 2081

[17] Audretsch, J. 1981 *Phys. Rev. D* **24** 1470
[18] Hehl, F. W., and Ni, W. T. 1990 *Phys. Rev. D* 42, 2045

[19] Zhang, C.M., and Beesham, A. 2002 *Gen. Rel. Grav.* 34, 679

[20] Mashhoon, B. 1988 *Phys. Rev. Lett.* 61 2639.

[21] Mashhoon, B. 2000 *Class. Quantum Grav.* 17, 2399