Ternary Codes Associated with Symplectic Groups and Power Moments of Kloosterman Sums with Square Arguments

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Abstract – In this paper, we construct two ternary linear codes associated with the symplectic groups $Sp(2, q)$ and $Sp(4, q)$. Here $q$ is a power of three. Then we obtain recursive formulas for the power moments of Kloosterman sums with square arguments and for the even power moments of those in terms of the frequencies of weights in the codes. This is done via Pless power moment identity and by utilizing the explicit expressions of “Gauss sums” for the symplectic groups $Sp(2n, q)$.

Key words – ternary linear code, power moment, Kloosterman sum, square argument, Pless power moment identity, Gauss sum, symplectic group, weight distribution.

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1 Introduction

Let $\psi$ be a nontrivial additive character of the finite field $\mathbb{F}_q$ with $q = p^r$ elements ($p$ a prime). Then the Kloosterman sum $K(\psi; a)(\mathbb{F}_q)$ is defined by

$$K(\psi; a) = \sum_{a \in \mathbb{F}_q^*} \psi(a + a^{-1})(a \in \mathbb{F}_q^*).$$

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The Kloosterman sum was introduced in 1926 ([7]) to give an estimate for the Fourier coefficients of modular forms. For each nonnegative integer \( h \), we denote by \( MK(\psi)^h \) the \( h \)-th moment of the Kloosterman sum \( K(\psi; a) \), i.e.,

\[
MK(\psi)^h = \sum_{a \in \mathbb{F}_q^*} K(\psi; a)^h.
\]

If \( \psi = \lambda \) is the canonical additive character of \( \mathbb{F}_q \), then \( MK(\lambda)^h \) will be simply denoted by \( MK^h \).

Also, we introduce an incomplete power moments of Kloosterman sums. Namely, for every nonnegative integer \( h \), and \( \psi \) as before, we define

\[
SK(\psi)^h = \sum_{a \in \mathbb{F}_q^*; \ a \ \text{square}} K(\psi; a)^h,
\]

which is called the \( h \)-th moment of Kloosterman sums with “square arguments.” If \( \psi = \lambda \) is the canonical additive character of \( \mathbb{F}_q \), then \( SK(\lambda)^h \) will be denoted by \( SK^h \), for brevity.

Explicit computations on power moments of Kloosterman sums were initiated in the paper [13] of Salić in 1931, where it is shown that for any odd prime \( q \),

\[
MK^h = q^2M_{h-1} - (q - 1)^{h-1} + 2(-1)^{h-1}(h \geq 1).
\]

Here \( M_0 = 0 \), and for \( h \in \mathbb{Z}_{>0} \),

\[
M_h = | \{ (\alpha_1, \ldots, \alpha_h) \in (\mathbb{F}_q^*)^h \mid \sum_{j=1}^h \alpha_j = 1 = \sum_{j=1}^h \alpha_j^{-1} \} | .
\]

For \( q = p \) odd prime, Salić obtained \( MK^1, MK^2, MK^3, MK^4 \) in [13] by determining \( M_1, M_2, M_3 \). \( MK^5 \) can be expressed in terms of the \( p \)-th eigenvalue for a weight 3 newform on \( \Gamma_0(15) \) (cf. [9], [12]). \( MK^6 \) can be expressed in terms of the \( p \)-th eigenvalue for a weight 4 newform on \( \Gamma_0(6) \) (cf. [3]). Also, based on numerical evidence, in [11] Evans was led to propose a conjecture which expresses \( MK^7 \) in terms of Hecke eigenvalues for a weight 3 newform on \( \Gamma_0(525) \) with quartic nebentypus of conductor 105.

Assume now that \( q = 3^r \). Recently, Moisio was able to find explicit expressions of \( MK^h \), for \( h \leq 10 \) (cf. [11]). This was done, via Pless power moment identity, by connecting moments of Kloosterman sums and the frequencies of weights in the ternary Melas code of length \( q - 1 \), which were known by the work of Geer, Schoof and Vlugt in [2]. In [6], two infinite families of ternary linear codes associated with double cosets in the symplectic group \( Sp(2n, q) \)
were constructed in order to generate infinite families of recursive formulas
for the power moments of Kloosterman sums with square arguments and for
the even power moments of those in terms of the frequencies of weights in
those codes.

In this paper, we will be able to produce two recursive formulas generat-
ing power moments of Kloosterman sums with square arguments over finite
fields of characteristic three. To do that, we will construct two ternary linear
codes $C(\text{Sp}(2,q))$ and $C(\text{Sp}(4,q))$, respectively associated with the symplec-
tic groups $\text{Sp}(2,q)$ and $\text{Sp}(4,q)$, and express those power moments in terms
of the frequencies of weights in each code. Then, thanks to our previous
results on the explicit expressions of “Gauss sums” for the symplectic groups
$\text{Sp}(2n,q)$ [4], we can express the weight of each codeword in the duals of
the codes in terms of Kloosterman sums with square arguments. Then our
formulas will follow immediately from the Pless power moment identity (cf.
(29)).

Theorem 1 in the following(cf. (4), (5), (7), (8)) is the main result of this
paper. Henceforth, we agree that, for nonnegative integers $a, b, c,$
\[
\binom{c}{a,b} = \frac{c!}{a!b!(c-a-b)!}, \quad \text{if} \quad a + b \leq c, \tag{2}
\]
and
\[
\binom{c}{a,b} = 0, \quad \text{if} \quad a + b > c. \tag{3}
\]

**Theorem 1.** Let $q = 3^r$. Then we have the following.

(a) For $h = 1, 2, \ldots,$
\[
SK^h = \sum_{j=0}^{h-1} (-1)^{h+j+1}\binom{h}{j}(q^2 - 1)^{h-j}SK^j + q^{1-h} \sum_{j=0}^{\min\{N_1, h\}} (-1)^{h+j}C_{1,j} \sum_{t=j}^{h} t!S(h, t)3^{h-t}2^{t-h-j-1}\binom{N_1 - j}{N_1 - t}, \tag{4}
\]

where $N_1 = |\text{Sp}(2,q)| = q(q^2 - 1)$, and $\{C_{1,j}\}_{j=0}^{N_1}$ is the weight distribution of
the ternary linear code $C(\text{Sp}(2,q))$ given by
\[
C_{1,j} = \sum \binom{q^2}{\nu_1, \mu_1} \binom{q^2}{\nu_{-1}, \mu_{-1}} \prod_{\beta^2-1 \neq 0 \text{ square}} \binom{q^2 + q}{\nu_{\beta}, \mu_{\beta}} \prod_{\beta^2-1 \text{ nonsquare}} \binom{q^2 - q}{\nu_{\beta}, \mu_{\beta}} (j = 0, \ldots, N_1). \tag{5}
\]
Here the sum is over all the sets of nonnegative integers \( \{ \nu_{\beta} \}_{\beta \in \mathbb{F}_q} \) and \( \{ \mu_{\beta} \}_{\beta \in \mathbb{F}_q} \) satisfying \( \sum_{\beta \in \mathbb{F}_q} \nu_{\beta} + \sum_{\beta \in \mathbb{F}_q} \mu_{\beta} = j \) and \( \sum_{\beta \in \mathbb{F}_q} \nu_{\beta} \beta = \sum_{\beta \in \mathbb{F}_q} \mu_{\beta} \beta \). In addition, \( S(h, t) \) is the Stirling number of the second kind defined by

\[
S(h, t) = \frac{1}{t!} \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} j^h.
\]  

(6)

(b) For \( h = 1, 2, \ldots \),

\[
SK^{2h} = \sum_{j=0}^{h-1} (-1)^{h+j+1} \binom{h}{j} (q^6 - q^4 - q^3 - q^2 + q + 1)^{h-j} SK^{2j} + q^{1-4h} \sum_{j=0}^{\min\{N_2, h\}} (-1)^{h+j} C_{2j} \sum_{t=1}^{h} t! S(h, t) 3^{h-t} 2^{t-h-j-1} \left( \frac{N_2 - j}{N_2 - t} \right).
\]  

(7)

where \( N_2 = |Sp(4, q)| = q^4 (q^2 - 1)(q^4 - 1) \), and \( \{ C_{2j} \}_{j=0}^{N_2} \) is the weight distribution of the ternary linear code \( C(\text{Sp}(4, q)) \) given by

\[
C_{2j} = \sum_{\nu_0, \mu_0} \left( q^4 (\delta(2, q; 0) + q^5 - q^2 - 3q + 3) \right)_{\nu_0, \mu_0} \times \prod_{\beta \in \mathbb{F}_q^*} \left( q^4 (\delta(2, q; \beta) + q^5 - q^2 - 3q + 3) \right)_{\nu_\beta, \mu_\beta} (j = 0, \ldots, N_2).
\]  

(8)

Here the sum is over all the sets of nonnegative integers \( \{ \nu_{\beta} \}_{\beta \in \mathbb{F}_q} \) and \( \{ \mu_{\beta} \}_{\beta \in \mathbb{F}_q} \) satisfying \( \sum_{\beta \in \mathbb{F}_q} \nu_{\beta} + \sum_{\beta \in \mathbb{F}_q} \mu_{\beta} = j \) and \( \sum_{\beta \in \mathbb{F}_q} \nu_{\beta} \beta = \sum_{\beta \in \mathbb{F}_q} \mu_{\beta} \beta \), and, for every \( \beta \in \mathbb{F}_q \),

\[
\delta(2, q; \beta) = \left| \{(\alpha_1, \alpha_2) \in (\mathbb{F}_q^*)^2 \mid \alpha_1 + \alpha_1^{-1} + \alpha_2 + \alpha_2^{-1} = \beta \} \right|.
\]

2 \( Sp(2n, q) \)

For more details about this section, one is referred to the paper [4]. Throughout this paper, the following notations will be used:

- \( q = 3^r \ (r \in \mathbb{Z}_{\geq 0}) \),
- \( \mathbb{F}_q \) = the finite field with \( q \) elements,
- \( TrA \) = the trace of \( A \) for a square matrix \( A \),

- \( 4 \)
$tB$ = the transpose of $B$ for any matrix $B$.

The symplectic group $Sp(2n, q)$ over the field $\mathbb{F}_q$ is defined as:

$$Sp(2n, q) = \{w \in GL(2n, q) \mid t^wJw = J\},$$

with

$$J = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}.$$

Let $P = P(2n, q)$ be the maximal parabolic subgroup of $Sp(2n, q)$ defined by:

$$P(2n, q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & tA^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \mid A \in GL(n, q), t^B = B \right\}.$$

Then, with respect to $P = P(2n, q)$, the Bruhat decomposition of $Sp(2n, q)$ is given by

$$Sp(2n, q) = \biguplus_{r=0}^{n} P\sigma_r P,$$  \hspace{1cm} (9)

where

$$\sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ -1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix} \in Sp(2n, q).$$

Put, for each $r$ with $0 \leq r \leq n$,

$$A_r = \{w \in P(2n, q) \mid \sigma_r w \sigma_r^{-1} \in P(2n, q)\}.$$

Expressing $Sp(2n, q)$ as a disjoint union of right cosets of $P = P(2n, q)$, the Bruhat decomposition in (9) can be written as

$$Sp(2n, q) = \biguplus_{r=0}^{n} P\sigma_r (A_r \setminus P).$$  \hspace{1cm} (10)

The order of the general linear group $GL(n, q)$ is given by

$$g_n = \prod_{j=0}^{n-1} (q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^{n} (q^j - 1).$$

For integers $n, r$ with $0 \leq r \leq n$, the $q$-binomial coefficients are defined as:

$$\left[ \begin{array}{c} n \\ r \end{array} \right]_q = \prod_{j=0}^{r-1} (q^{n-j} - 1)/(q^{r-j} - 1).$$
Then, for integers $n, r$ with $0 \leq r \leq n$, we have
\[ \frac{g_n}{g_{n-r}g_r} = q^{r(n-r)} [n]_{r,q}. \] (11)

In [4], it is shown that
\[ |A_r| = g_r g_{n-r} q^{\frac{n+1}{2} r^2 q^{r(2n-3r-1)}/2}. \] (12)

Also, it is immediate to see that
\[ |P(2n, q)| = q^{\frac{n+1}{2}} g_n. \] (13)

So, from (11)-(13), we get
\[ |A_r \setminus P(2n, q)| = q^{\frac{n+1}{2}} [n]_{r,q}. \] (14)

and
\[ |P(2n, q)|^2 |A_r|^{-1} = q^{n^2} [n]_{q, q^{1/2}} q^r \prod_{j=1}^{n} (q^j - 1). \] (15)

Also, from (10) and (15), we have
\[ |Sp(2n, q)| = \sum_{r=0}^{n} |P(2n, q)|^2 |A_r|^{-1} = q^{n^2} \prod_{j=1}^{n} (q^{2j} - 1), \] (16)

where one can apply the following $q$-binomial theorem with $x = -q$:
\[ \sum_{r=0}^{n} \left[ \begin{array}{c} n \\ r \end{array} \right]_q (-1)^r q^{r(\frac{r}{2})} x^r = (x; q)_n, \]
with $(x; q)_n = (1 - x)(1 - qx) \cdots (1 - q^{n-1}x)$.

3 **Gauss sums for** $Sp(2n, q)$

The following notations will be employed throughout this paper.
\[ tr(x) = x + x^3 + \cdots + x^{3r-1} \] the trace function $\mathbb{F}_q \to \mathbb{F}_3$,
\[ \lambda_0(x) = e^{2\pi ix/3} \] the canonical additive character of $\mathbb{F}_3$,
\[ \lambda(x) = e^{2\pi i tr(x)/3} \] the canonical additive character of $\mathbb{F}_q$. 
Then any nontrivial additive character $\psi$ of $\mathbb{F}_q$ is given by $\psi(x) = \lambda(ax)$, for a unique $a \in \mathbb{F}_q^*$. 

For any nontrivial additive character $\psi$ of $\mathbb{F}_q$ and $a \in \mathbb{F}_q^*$, the Kloosterman sum $K_{GL(t,q)}(\psi; a)$ for $GL(t, q)$ is defined as 

$$K_{GL(t,q)}(\psi; a) = \sum_{w \in GL(t,q)} \psi(Tw + aTw^{-1}).$$

Notice that, for $t = 1$, $K_{GL(1,q)}(\psi; a)$ denotes the Kloosterman sum $K(\psi; a)$.

In [4], it is shown that $K_{GL(t,q)}(\psi; a)$ satisfies the following recursive relation: for integers $t \geq 2$, $a \in \mathbb{F}_q^*$,

$$K_{GL(t,q)}(\psi; a) = q^{t-1}K_{GL(t-1,q)}(\psi; a)K(\psi; a) + q^{2t-2}(q^{t-1} - 1)K_{GL(t-2,q)}(\psi; a),$$

where we understand that $K_{GL(0,q)}(\psi; a) = 1$. From (17), in [4] an explicit expression of the Kloosterman sum for $GL(t, q)$ was derived.

**Theorem 2 ([4]).** For integers $t \geq 1$, and $a \in \mathbb{F}_q^*$, the Kloosterman sum $K_{GL(t,q)}(\psi; a)$ is given by

$$K_{GL(t,q)}(\psi; a) = q^{(t-2)(t+1)/2} \sum_{l=1}^{\lfloor (t+2)/2 \rfloor} q^l K(\psi; a)^{t+2-2l} \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1),$$

where $K(\psi; a)$ is the Kloosterman sum and the inner sum is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l - 1 \leq j_{l-1} \leq j_{l-2} \leq \cdots \leq j_1 \leq t + 1$. Here we agree that the inner sum is 1 for $l = 1$.

In Section 5 of [4], it is shown that the Gauss sum for $Sp(2n, q)$ is given by:

$$\sum_{w \in Sp(2n,q)} \psi(Tw) = \sum_{r=0}^{n} |A_r \setminus P| \sum_{w \in P} \psi(Tw\sigma_r) = q^{(n+1)/2} \sum_{r=0}^{n} |A_r \setminus P| q^{r(n-r)} a_r K_{GL(n-r,q)}(\psi; 1).$$

Here $\psi$ is any nontrivial additive character of $\mathbb{F}_q$, $a_0 = 1$, and, for $r \in \mathbb{Z}_{>0}$, $a_r$ denotes the number of all $r \times r$ nonsingular alternating matrices over $\mathbb{F}_q$, which is given by 

$$a_r = \begin{cases} 
0, & \text{if } r \text{ is odd}, \\
q^{r(r-1)/2} \prod_{j=1}^{r/2} (q^{2j-1} - 1), & \text{if } r \text{ is even}.
\end{cases}$$
(cf. [4], Proposition 5.1). So

\[
\sum_{w \in Sp(2n,q)} \psi(Trw) = q^{n+1 \over 2} \sum_{0 \leq r \leq n, \ r \ even} q^{n-r^2} \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{GL(n-r,q)}(\psi; 1).
\]

For our purposes, we only need the following two expressions of the Gauss sums for \( Sp(2,q) \) and \( Sp(4,q) \). So we state them separately as a theorem. Also, for the ease of notations, we introduce

\[
G_1(q) = Sp(2,q), \quad G_2(q) = Sp(4,q).
\]

**Theorem 3.** Let \( \psi \) be any nontrivial additive character of \( \mathbb{F}_q \). Then we have

\[
\sum_{w \in G_1(q)} \psi(Trw) = qK(\psi; 1),
\]

\[
\sum_{w \in G_2(q)} \psi(Trw) = q^4(K(\psi; 1)^2 + q^3 - q).
\]

The next corollary follows from Theorems 4 and simple change of variables.

**Corollary 4.** Let \( \lambda \) be the canonical additive character of \( \mathbb{F}_q \), and let \( a \in \mathbb{F}_q^* \). Then we have

\[
\sum_{w \in G_1(q)} \lambda(a Trw) = qK(\lambda; a^2), \quad (18)
\]

\[
\sum_{w \in G_2(q)} \lambda(a Trw) = q^4(K(\lambda; a^2)^2 + q^3 - q). \quad (19)
\]

**Proposition 5. ([5, (5.3–5)])** Let \( \lambda \) be the canonical additive character of \( \mathbb{F}_q \), \( m \in \mathbb{Z}_{\geq 0} \), \( \beta \in \mathbb{F}_q \). Then

\[
\sum_{a \in \mathbb{F}_q^*} \lambda(-a\beta) K(\lambda; a^2)^m = q\delta(m, q; \beta) - (q - 1)^m, \quad (20)
\]

where, for \( m \geq 1 \),

\[
\delta(m, q; \beta) = | \{ (\alpha_1, \ldots, \alpha_m) \in (\mathbb{F}_q^*)^m \mid \alpha_1 + \alpha_1^{-1} + \cdots + \alpha_m + \alpha_m^{-1} = \beta \} |, \quad (21)
\]

and

\[
\delta(0, q; \beta) = \begin{cases} 1, & \text{if } \beta = 0, \\ 0, & \text{otherwise.} \end{cases}
\]
Let $G(q)$ be one of finite classical groups over $\mathbb{F}_q$. Then we put, for each $\beta \in \mathbb{F}_q$,

$$N_{G(q)}(\beta) = | \{ w \in G(q) \mid Tr(w) = \beta \} | .$$

Then it is easy to see that

$$qN_{G(q)}(\beta) = |G(q)| + \sum_{a \in \mathbb{F}_q^*} \lambda(-a\beta) \sum_{w \in G(q)} \lambda(aTrw). \tag{22}$$

For brevity, we write

$$n_1(\beta) = N_{G_1(q)}(\beta), n_2(\beta) = N_{G_2(q)}(\beta). \tag{23}$$

Using (16), (18)-(20), and (22), one derives the following proposition. Here one notes that

$$\delta(1, q; \beta) = | \{ x \in \mathbb{F}_q \mid x^2 - \beta x + 1 = 0 \} |$$

$$= \begin{cases} 2, & \text{if } \beta^2 - 1 \neq 0 \text{ is a square}, \\ 1, & \text{if } \beta = \pm 1, \\ 0, & \text{if } \beta^2 - 1 \text{ is a nonsquare}. \end{cases}$$

**Proposition 6.** With $n_1(\beta), n_2(\beta)$ as in (23) and $\delta(m, q; \beta)$ as in (21), we have:

$$n_1(\beta) = q\delta(1, q; \beta) + q^2 - q = \begin{cases} q^2 + q, & \text{if } \beta^2 - 1 \neq 0 \text{ is a square}, \\ q^2, & \text{if } \beta = \pm 1, \\ q^2 - q, & \text{if } \beta^2 - 1 \text{ is a nonsquare}. \end{cases} \tag{24}$$

$$n_2(\beta) = \begin{cases} q^4(\delta(2, q; 0) + q^5 - q^2 - 3q + 3), & \text{if } \beta = 0, \\ q^4(\delta(2, q; \beta) + q^5 - q^2 - q^2 - 2q + 3), & \text{if } \beta \neq 0. \end{cases} \tag{25}$$

**Proposition 7.** $Tr : Sp(2n, q) \to \mathbb{F}_q$ is surjective.

*Proof.* Under the map $Tr$, for any $\alpha \in \mathbb{F}_q$,

$$Sp(2n, q) \ni \begin{bmatrix} \alpha & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-1} \\ -1 & 0 & 0 & 0 \\ 0 & -1_{n-1} & 0 & 0 \end{bmatrix} \mapsto \alpha.$$
4 Construction of codes

Let

\[ N_1 = |G_1(q)| = q(q^2 - 1), \quad N_2 = |G_2(q)| = q^4(q^2 - 1)(q^4 - 1). \quad (26) \]

Here we will construct two ternary linear codes \( C(G_1(q)) \) of length \( N_1 \) and \( C(G_2(q)) \) of length \( N_2 \), respectively associated with the symplectic groups \( G_1(q) \) and \( G_2(q) \).

By abuse of notations, for \( i = 1, 2 \), let \( g_1, g_2, \ldots, g_{N_i} \) be a fixed ordering of the elements in the group \( G_i(q) \). Also, for \( i = 1, 2 \), we put

\[ v_i = (Tr g_1, Tr g_2, \ldots, Tr g_{N_i}) \in \mathbb{F}_q^{N_i}. \]

Then, for \( i = 1, 2 \), the ternary linear code \( C(G_i(q)) \) is defined as

\[ C(G_i(q)) = \{ u \in \mathbb{F}_3^{N_i} \mid u \cdot v_i = 0 \}, \quad (27) \]

where the dot denotes the usual inner product in \( \mathbb{F}_q^{N_i} \).

The following Delsarte’s theorem is well-known.

**Theorem 8** ([10]). Let \( B \) be a linear code over \( \mathbb{F}_q \). Then

\[ (B|_{\mathbb{F}_3})^\perp = \text{tr}(B^\perp). \]

In view of this theorem, the dual \( C(G_i(q))^\perp \) \( (i = 1, 2) \) is given by

\[ C(G_i(q))^\perp = \{ c_i(a) = (tr(aTr g_1), \ldots, tr(aTr g_{N_i})) \mid a \in \mathbb{F}_q \}. \quad (28) \]

**Proposition 9.** For every \( q = 3^r \), and \( i = 1, 2 \), the map \( \mathbb{F}_q \to C(G_i(q))^\perp \) \((a \mapsto c_i(a))\) is an \( \mathbb{F}_3 \)-linear isomorphism.

**Proof.** The map is clearly \( \mathbb{F}_3 \)-linear and surjective. Let \( a \) be in the kernel of the map. Then, in view of Proposition 7, \( tr(a \beta) = 0 \), for all \( \beta \in \mathbb{F}_q \). Since the trace function \( \mathbb{F}_q \to \mathbb{F}_3 \) is surjective, \( a = 0 \).

5 Recursive formulas for power moments of Kloosterman sums with square arguments

In this section, we will be able to find, via Pless power moment identity, recursive formulas for the power moments of Kloosterman sums with square arguments and even power moments of those with square arguments in terms of the frequencies of weights in \( C(G_i(q)) \), for each \( i = 1, 2 \).
Theorem 10 (Pless power moment identity). Let $B$ be an $q$-ary $[n,k]$ code, and let $B_i$ (resp. $B_i^\perp$) denote the number of codewords of weight $i$ in $B$ (resp. in $B^\perp$). Then, for $h = 0, 1, 2, \ldots$,

$$
\sum_{j=0}^{n} j^h B_j = \min\{n,h\} \sum_{j=0}^{h} t! S(h,t) q^{k-t} (q-1)^{t-j} \binom{n-j}{n-t},
$$

(29)

where $S(h,t)$ is the Stirling number of the second kind defined in (6).

Lemma 11. Let $c_i(a) = (tr(a Tr g_1), \ldots, tr(a Tr g_{N_i})) \in C(G_i(q))^\perp$, for $a \in \mathbb{F}_q^*$, and $i = 1, 2$. Then the Hamming weight $w(c_i(a))$ can be expressed as follows:

(a) $w(c_1(a)) = \frac{2}{3} q(q^2 - 1 - K(\lambda;a^2))$,

(b) $w(c_2(a)) = \frac{2}{3} q^4 \{(q^2 - 1)(q^4 - 1) - (K(\lambda;a^2)^2 + q^3 - q)\}$.

(30) 

(31)

Proof. For $i = 1, 2$,

$$
w(c_i(a)) = \sum_{j=1}^{N_i} (1 - \frac{1}{3} \sum_{\alpha \in \mathbb{F}_3} \lambda_0(\alpha tr(a Tr g_j)))
$$

$$
= N_i - \frac{1}{3} \sum_{\alpha \in \mathbb{F}_3} \sum_{w \in G_i(q)} \lambda(\alpha Tr w)
$$

$$
= \frac{2}{3} N_i - \frac{1}{3} \sum_{\alpha \in \mathbb{F}_3} \sum_{w \in G_i(q)} \lambda(\alpha Tr w).
$$

Our results now follow from (18), (19), and (26).

Fix $i(i = 1, 2)$, and let $u = (u_1, \ldots, u_{N_i}) \in \mathbb{F}_3^{N_i}$, with $\nu_\beta$ 1’s and $\mu_\beta$ 2’s in the coordinate places where $Tr(g_j) = \beta$, for each $\beta \in \mathbb{F}_q$. Then we see from the definition of the code $C(G_i(q))$ (cf. (27)) that $u$ is a codeword with weight $j$ if and only if

$$
\sum_{\beta \in \mathbb{F}_q} \nu_\beta + \sum_{\beta \in \mathbb{F}_q} \mu_\beta = j \quad \text{and} \quad \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = \sum_{\beta \in \mathbb{F}_q} \mu_\beta \beta \quad \text{an identity in} \ \mathbb{F}_q.
$$

Note that there are \( \prod_{\beta \in \mathbb{F}_q} \binom{n_i(\beta)}{\nu_\beta, \mu_\beta} \) (cf. (2), (3)) many such codewords with weight $j$. Now, we get the following formulas in (32) and (33), by using the explicit values of $n_i(\beta)$ in (24) and (25).
Theorem 12. Let \( \{C_{i,j}\}_{j=0}^{N_i} \) be the weight distribution of \( C(G_i(q)) \), for \( i = 1, 2 \). Then

\[
(a) \quad C_{1,j} = \sum \left( \frac{q^2}{\nu_1, \mu_1} \right) \left( \frac{q^2}{\nu_{-1}, \mu_{-1}} \right) \prod_{\beta^2 - 1 \neq 0 \text{ square}} \left( \frac{q^2 + q}{\nu_\beta, \mu_\beta} \right) \prod_{\beta^2 - 1 \text{ nonsquare}} \left( \frac{q^2 - q}{\nu_\beta, \mu_\beta} \right) (j = 0, \ldots, N_1),
\]

where the sum is over all the sets of nonnegative integers \( \{\nu_\beta\}_{\beta \in \mathbb{F}_q} \) and \( \{\mu_\beta\}_{\beta \in \mathbb{F}_q} \) satisfying

\[
\sum_{\beta \in \mathbb{F}_q} \nu_\beta + \sum_{\beta \in \mathbb{F}_q} \mu_\beta = j \quad \text{and} \quad \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = \sum_{\beta \in \mathbb{F}_q} \mu_\beta \beta.
\]

\[
(b) \quad C_{2,j} = \sum \left( \frac{q^4(\delta(2, q; 0) + q^5 - q^2 - 3q + 3)}{\nu_0, \mu_0} \right) \prod_{\beta \in \mathbb{F}_q^*} \left( \frac{q^4(\delta(2, q; \beta) + q^5 - q^3 - q^2 - 2q + 3)}{\nu_\beta, \mu_\beta} \right) (j = 0, \ldots, N_2),
\]

where the sum is over all the sets of nonnegative integers \( \{\nu_\beta\}_{\beta \in \mathbb{F}_q} \) and \( \{\mu_\beta\}_{\beta \in \mathbb{F}_q} \) satisfying

\[
\sum_{\beta \in \mathbb{F}_q} \nu_\beta + \sum_{\beta \in \mathbb{F}_q} \mu_\beta = j \quad \text{and} \quad \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = \sum_{\beta \in \mathbb{F}_q} \mu_\beta \beta,
\]

and, for every \( \beta \in \mathbb{F}_q \),

\[
\delta(2, q; \beta) = | \{(\alpha_1, \alpha_2) \in (\mathbb{F}_q^*)^2 | \alpha_1 + \alpha_1^{-1} + \alpha_2 + \alpha_2^{-1} = \beta\} |.
\]

We now apply the Pless power moment identity in (29) to each \( C(G_i(q)) \), for \( i = 1, 2 \), in order to obtain the results in Theorem 1(cf. (4), (7)) about recursive formulas.

Then the left hand side of the identity in (29) is equal to

\[
\sum_{a \in \mathbb{F}_q^*} w(c_i(a))^h,
\]

with the \( w(c_i(a)) \) in each case given by (30) and (31).
For $i = 1$, (34) is

$$(\frac{2q}{3})^h \sum_{a \in \mathbb{F}_q^*} (q^2 - 1 - K(\lambda; a^2))^h$$

$$= (\frac{2q}{3})^h \sum_{a \in \mathbb{F}_q^*} \sum_{j=0}^{h-1} (-1)^j \binom{h}{j} (q^2 - 1)^{h-j} K(\lambda; a^2)^j$$

$$= 2(\frac{2q}{3})^h \sum_{j=0}^{h-1} (-1)^j \binom{h}{j} (q^2 - 1)^{h-j} SK^j.$$  \hspace{1cm} (35)

Similarly, for $i = 2$, (34) equals

$$2(\frac{2q^4}{3})^h \sum_{j=0}^{h-1} (-1)^j \binom{h}{j} (q^6 - q^4 - q^3 - q^2 + 1)^{h-j} SK^{2j}.$$  \hspace{1cm} (36)

Here one has to separate the term corresponding to $j = h$ in (35) and (36), and note $\dim_{\mathbb{F}_q} C(G_i(q))^\perp = r$.

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