Unlinking Theorem for Symmetric Quasi-convex Polynomials

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Abstract  Let $\mu_n$ be the standard Gaussian measure on $\mathbb{R}^n$ and $X$ be a random vector on $\mathbb{R}^n$ with the law $\mu_n$. U-conjecture states that if $f$ and $g$ are two polynomials on $\mathbb{R}^n$ such that $f(X)$ and $g(X)$ are independent, then there exist an orthogonal transformation $L$ on $\mathbb{R}^n$ and an integer $k$ such that $f \circ L$ is a function of $(x_1, \ldots, x_k)$ and $g \circ L$ is a function of $(x_{k+1}, \ldots, x_n)$. In this case, $f$ and $g$ are said to be unlinked. In this note, we prove that two symmetric, quasi-convex polynomials $f$ and $g$ are unlinked if $f(X)$ and $g(X)$ are independent.

Keywords  U-conjecture; quasi-convex polynomial; Gaussian correlation conjecture

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1 Introduction and main result

Let $\mu_n$ be the standard Gaussian measure on $\mathbb{R}^n(n \geq 2)$ and $X$ be a random vector on $\mathbb{R}^n$ with the law $\mu_n$. In 1973, Kagan, Linnik and Rao [6] considered the following problem: if $f$ and $g$ are two polynomials on $\mathbb{R}^n$ such that $f(X)$ and $g(X)$ are independent, then is it possible to find an orthogonal transformation $L$ on $\mathbb{R}^n$ and an integer $k$ such that $f \circ L$ is a function of $(x_1, \ldots, x_k)$ and $g \circ L$ is a function of $(x_{k+1}, \ldots, x_n)$? If the answer is positive, then $f$ and $g$ are said to be unlinked. This problem is called U-conjecture and is still open.

The U-conjecture is true for the case $n = 2$, and some special cases have been proved for larger
number of variables (see Sections 11.4-11.6 of [6]). In 1994, Bhandari and DasGupta [2] proved that the U-conjecture holds for two symmetric convex functions \( f \) and \( g \) under an additional condition. The additional condition can be canceled since the Gaussian correlation conjecture
has been proved (see Royen [10] or Latalska and Matlak [7]).

Bhandari and Basu [11] proved that the U-conjecture holds for two nonnegative convex polynomials \( f \) and \( g \) with \( f(0) = 0 \). Hargé [4] proved that if \( f, g : \mathbb{R}^n \to \mathbb{R} \) are two convex functions in \( L^2(\mu_n) \), and \( f \) is a real analytic function satisfying \( f(x) \geq f(0), \forall x \in \mathbb{R}^n \), and \( f \) and \( g \) are independent with respect to \( \mu_n \), then they are unlinked.

Malicet et al. [8] proved that the U-conjecture is true when \( f, g \) belong to a class of polynomials, which is defined based on the infinitesimal generator of Ornstein-Uhlenbeck semigroup.

In Remark 2 of [1], the authors wish that their result could be extended to symmetric, quasi-convex polynomials. In this note, we will give an affirmative answer based on the first author’s master thesis [5] and prove the following result.

**Theorem 1.1** Two symmetric, quasi-convex polynomials \( f \) and \( g \) are unlinked if \( f \) and \( g \) are independent with respect to \( \mu_n \).

### 2 Proof of Theorem 1.1

Before giving the proof of Theorem 1.1, we present some preliminaries.

A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called quasi-convex if for any \( \alpha \in [0, 1] \) and any \( x, y \in \mathbb{R}^n \),

\[
f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}.
\]

It’s easy to know that a convex function is quasi-convex. About the properties of quasi-convex functions, and the relations between convex and quasi-convex functions, refer to a survey paper Greenberg and Pierskalla [3].

**Lemma 2.1** Suppose that \( g : \mathbb{R} \to \mathbb{R} \) is a quasi-convex polynomial and there exist \( \lambda_1, \lambda_2 \in \mathbb{R} \) such that \( g(\lambda_1) \neq g(\lambda_2) \). Then one of the following two claims holds.

(a) There exists \( \lambda_0 \) such that \( g(u) < g(v) \) for any \( \lambda_0 \leq u < v \) and \( \lim_{\lambda \to \infty} g(\lambda) = \infty \).

(b) There exists \( \lambda_0 \) such that \( g(u) < g(v) \) for any \( v < u \leq \lambda_0 \) and \( \lim_{\lambda \to -\infty} g(\lambda) = \infty \).

**Proof.** Since \( g \) is a polynomial on \( \mathbb{R} \), we can write it as

\[
g(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.
\]

By the assumption, \( g \) is not a constant, so \( n \geq 1 \) and \( a_n \neq 0 \). By (2.1), we obtain

\[
g'(\lambda) = na_n\lambda^{n-1} + (n-1)a_{n-1}\lambda^{n-2} + \cdots + a_1.
\]

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Since \( h \) is quasi-convex, we know that

\[
\text{By (2.8) and (2.1), we know that one of the following two claims must hold:}
\]

\[
\text{By (2.10), there exists } \lambda \text{ such that for any } \lambda > \lambda_2, \ h(\lambda) \leq h(\lambda), \ i.e. \]

\[
g(\lambda) \leq g(\lambda), \ \forall \lambda > \lambda_2.
\]

By (2.1) and (2.4), we get that \( a_n > 0 \), and thus

\[
\lim_{\lambda \to \infty} g(\lambda) = \lim_{\lambda \to \infty} \left( a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 \right) = \infty. \tag{2.5}
\]

If \( n = 1 \), then \( g(\lambda) = a_1 \lambda + a_0 \) with \( a_1 > 0 \), and thus (a) holds in this case. If \( n \geq 2 \), then

\[
\lim_{\lambda \to \infty} g'(\lambda) = \lim_{\lambda \to \infty} \left( na_n \lambda^{n-1} + (n-1) a_{n-1} \lambda^{n-2} + \cdots + a_1 \right) = \infty. \tag{2.6}
\]

By (2.6), there exists \( \lambda_0 \) such that for any \( \lambda > \lambda_0 \), \( g'(\lambda) > 0 \), which together with (2.3) implies that (a) holds in this case.

**Case 2:** \( g(\lambda_1) > g(\lambda_2) \). Define \( \tilde{h}(\lambda) := g(\lambda) - g(\lambda_2) \). Then \( \tilde{h}(\lambda_2) = 0 \), and as in Case 1, \( \tilde{h} \) is a quasi-convex function and for any \( \lambda < \lambda_1 \), we have

\[
\tilde{h}(\lambda) = \tilde{h} \left( \frac{\lambda_1 - \lambda}{\lambda_2 - \lambda} \right) \leq \max\{\tilde{h}(\lambda_2), \tilde{h}(\lambda)\} = \max\{0, \tilde{h}(\lambda)\}. \tag{2.7}
\]

Since \( \tilde{h}(\lambda_1) = g(\lambda_1) - g(\lambda_2) > 0 \), by (2.7), we obtain that for any \( \lambda < \lambda_1 \), \( \tilde{h}(\lambda) \leq \tilde{h}(\lambda_1) \), i.e.

\[
g(\lambda_1) \leq g(\lambda), \ \forall \lambda < \lambda_1. \tag{2.8}
\]

By (2.8) and (2.1), we know that one of the following two claims must hold:

(i) \( n \) is even and \( a_n > 0 \); (ii) \( n \) is odd and \( a_n < 0 \).

If (i) holds, then by the proof of Case 1 above, we know that (a) is true.

If (ii) holds, then

\[
\lim_{\lambda \to -\infty} g(\lambda) = \lim_{\lambda \to -\infty} \left( a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 \right) = \infty. \tag{2.9}
\]

If \( n = 1 \), then \( g(\lambda) = a_1 \lambda + a_0 \) with \( a_1 < 0 \), and thus (b) holds in this case. If \( n \geq 3 \), then

\[
\lim_{\lambda \to -\infty} g'(\lambda) = \lim_{\lambda \to -\infty} \left( na_n \lambda^{n-1} + (n-1) a_{n-1} \lambda^{n-2} + \cdots + a_1 \right) = -\infty. \tag{2.10}
\]

By (2.10), there exists \( \lambda_0 \) such that for any \( \lambda < \lambda_0 \), \( g'(\lambda) < 0 \), which together with (2.9) implies that (b) holds in this case. \( \Box \)
**Corollary 2.2** Let $g : \mathbb{R} \to \mathbb{R}$ be a quasi-convex polynomial. If $g$ has an upper bound, then $g$ is a constant function.

**Corollary 2.3** Let $U : \mathbb{R}^n \to \mathbb{R}$ be a quasi-convex polynomial. Suppose that for two fixed vectors $\beta_1, \beta_2 \in \mathbb{R}^n$, $U(\beta_1 + \lambda \beta_2)$ is a constant function of $\lambda \in \mathbb{R}$. Then for any fixed vector $b \in \mathbb{R}^n$, $U(b + \lambda \beta_2)$ is a constant function of $\lambda$.

**Proof.** For any fixed vector $b \in \mathbb{R}^n$, define $g(\lambda) = U(b + \lambda \beta_2), \lambda \in \mathbb{R}$. Then $g(\lambda)$ is a polynomial of $\lambda$. By the quasi-convexity of $U$, we know that for any $\alpha \in [0, 1]$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$
g(\alpha \lambda_1 + (1 - \alpha) \lambda_2) = U(b + (\alpha \lambda_1 + (1 - \alpha) \lambda_2) \beta_2) = U(\alpha (b + \lambda_1 \beta_2) + (1 - \alpha)(b + \lambda_2 \beta_2)) \leq \max\{U(b + \lambda_1 \beta_2), U(b + \lambda_2 \beta_2)\} = \max\{g(\lambda_1), g(\lambda_2)\}.
$$

Thus $g(\lambda)$ is a quasi-convex polynomial. By the quasi-convexity of $U$,

$$
g(\lambda) = U(b + \lambda \beta_2) = U\left(\frac{1}{2}(2b - \beta_1) + \frac{1}{2}(\beta_1 + 2\lambda \beta_2)\right) \leq \max\{U(2b - \beta_1), U(\beta_1 + 2\lambda \beta_2)\}. \tag{2.11}
$$

By (2.11) and the assumption that $U(\beta_1 + \lambda \beta_2)$ is a constant function of $\lambda$, we get that the quasi-convex polynomial $g(\lambda)$ has an upper bound. Hence by Corollary 2.2 we know that $U(b + \lambda \beta_2)$ is a constant function of $\lambda$. \qed

**Corollary 2.4** Let $U : \mathbb{R}^n \to \mathbb{R}$ be a quasi-convex polynomial with $U(0) = 0$. Define

$$
S_U := \{\alpha : U(\lambda \alpha) = 0, \forall \lambda \in \mathbb{R}\}. \tag{2.12}
$$

Then $S_U$ is a vector subspace of $\mathbb{R}^n$.

**Proof.** Let $\alpha_1, \alpha_2 \in S_U$. For any $c_1, c_2, \lambda \in \mathbb{R}$, by Corollary 2.3 we get that

$$
U(\lambda (c_1 \alpha_1 + c_2 \alpha_2)) = U(\lambda c_1 \alpha_1 + \lambda c_2 \alpha_2) = U(\lambda c_1 \alpha_1) = 0.
$$

Hence $c_1 \alpha_1 + c_2 \alpha_2 \in S_U$, and thus $S_U$ is a vector subspace of $\mathbb{R}^n$. \qed

Now suppose that $U$ and $V$ are two quasi-convex polynomials from $\mathbb{R}^n$ into $\mathbb{R}$ satisfying that $U(0) = V(0) = 0$. Define $S_U$ by (2.12). Similarly, define $S_V$.

**Definition 2.5** $U$ and $V$ are said to be concordant of order $r$, if

$$
\dim(S_U^\perp) - \dim(S_U^\perp \cap S_V) = r. \tag{2.13}
$$
Note that this definition is symmetric in $U$ and $V$, i.e. if $2.13$ holds, then (see [2])

$$\dim(S_U^+) - \dim(S_V^+ \cap S_U) = r.$$ 

**Theorem 2.6** Let $X$ be an $n \times 1$ random vector distributed as $N(0, I_n)$. Let $U$ and $V$ be two symmetric (i.e. $U(x) = U(-x)$, $V(x) = V(-x)$) quasi-convex polynomials on $\mathbb{R}^n$ satisfying $\text{Cov}(U(X), V(X)) = 0$. Furthermore, assume that $U(0) = V(0) = 0$, and $U$ and $V$ are concordant of order $r$. Then there exists an orthogonal transformation $Y = LX$ such that $U$ and $V$ can be expressed as functions of two different sets of components of $Y$, i.e. $U$ and $V$ are unlinked.

**Proof.** Based on the lemmas and corollaries established above, the proof of this theorem is similar to the one of [2]. For the reader's convenience, we spell out the details in the following.

Let $\{\alpha_1, \ldots, \alpha_{r+t}\}$, $\{\alpha_{r+1}, \ldots, \alpha_{r+t}\}$, $\{\alpha_1, \ldots, \alpha_{r+t+m}\}$ and $\{\alpha_1, \ldots, \alpha_n\}$ be orthonormal bases of $S_U^+$, $S_V^+ \cap S_U$, $S_U^+ + S_V^+$, and $\mathbb{R}^n$, respectively. We will show that if $r > 0$ then $\text{Cov}(U(X), V(X)) > 0$, which contradicts the condition given in the theorem, and so we get $r = 0$, and thus $U$ and $V$ are unlinked.

Define $Y_1, Y_2, \ldots, Y_n$ by $X = \sum_{i=1}^n Y_i \alpha_i$, i.e. $Y_i$ is the $i$-th component of $X$. Then $Y_1, Y_2, \ldots, Y_n$ are i.i.d. as $N(0, 1)$. By Corollary 2.3

$$U(X) = U\left(\sum_{i=1}^n Y_i \alpha_i\right) = U\left(\sum_{i=1}^r Y_i \alpha_i + \sum_{i=r+1}^{r+t} Y_i \alpha_i\right),$$
$$V(X) = V\left(\sum_{i=1}^n Y_i \alpha_i\right) = U\left(\sum_{i=1}^r Y_i \alpha_i + \sum_{i=r+t+1}^{r+t+m} Y_i \alpha_i\right).$$

Assume that $r > 0$. Let $y^* = (y_1, \ldots, y_r)'$ be a nonzero vector in $\mathbb{R}^r$. Define

$$U^*(y^*) := E\left[U\left(\sum_{i=1}^r y_i \alpha_i + \sum_{i=r+1}^{r+t} Y_i \alpha_i\right)\right],$$
$$V^*(y^*) := E\left[V\left(\sum_{i=1}^r y_i \alpha_i + \sum_{i=r+t+1}^{r+t+m} Y_i \alpha_i\right)\right].$$

Then $U^*$ and $V^*$ are two symmetric quasi-convex polynomials of $y^*$.

By the choice of the bases, $U(\lambda \sum_{i=1}^r y_i \alpha_i)$ is not a zero function of $\lambda$. By Corollary 2.3 and the condition $U(0) = 0$, we know that $U(\lambda \sum_{i=1}^r y_i \alpha_i + \sum_{i=r+1}^{r+t} y_i \alpha_i)$ is not a constant of $\lambda$. In addition, by the symmetry and quasi-convexity of $U$, $U(x) \geq U(0), \forall x \in \mathbb{R}^n$. Hence by Lemma 2.1 we get that when $\lambda \to \infty$,

$$U\left(\lambda \sum_{i=1}^r y_i \alpha_i + \sum_{i=r+1}^{r+t} Y_i \alpha_i\right) + U\left(-\lambda \sum_{i=1}^r y_i \alpha_i + \sum_{i=r+1}^{r+t} Y_i \alpha_i\right) \to \infty.$$ (2.14)
Similarly, and simple calculations, we have \( e.g. [9, \text{Theorem 21.3}] \) or \([11, \text{Remark 2.3.6(1)}]\), we obtain

\[
U^*(\lambda y^*) \to \infty \quad \text{as} \quad \lambda \to \infty. \tag{2.15}
\]

Taking the expectation of (2.14) with respect to \( Y \)

\[
\text{Symmetric convex sets (see [3, Table II]). By the Gaussian correlation inequality (see [10] or [7]),}
\]

\[
A_k \quad \text{When} \quad M \quad \text{By (2.15), (2.16), and Lemma 2.1, the Lebesgue measure of } \{M \in \mathbb{R}^n : y^* \leq k\} \quad \text{is positive. Hence by (2.17),}
\]

\[
A_{k_1} = \{y^* : U^*(y^*) \leq k_1\}, \quad B_{k_2} = \{y^* : V^*(y^*) \leq k_2\}.
\]

Define \( Y^* = (Y_1, \ldots, Y_r)' \). By the independence of components of \( X = (Y_1, \ldots, Y_r, Y_{r+1}, \ldots, Y_n)' \) and simple calculations, we have

\[
\text{Cov}(U(X), V(X)) = E[U(X)V(X)] - E[U(X)]E[V(X)]
\]

\[
= E[U^*(Y^*)V^*(Y^*)] - E[U^*(Y^*)]E[V^*(Y^*)]
\]

\[
= \int_0^{\infty} \int_0^{\infty} \left[ P(Y^* \in A_{k_1}^c \cap B_{k_2}^c) - P(Y^* \in A_{k_1}) P(Y^* \in B_{k_2}) \right] \, dk_1 \, dk_2
\]

where

\[
P(Y^* \in A_{k_1} \cap B_{k_2}) - P(Y^* \in A_{k_1}) P(Y^* \in B_{k_2}) \geq 0. \tag{2.18}
\]

Define a set

\[
M = \{(k_1, k_2) \in (0, \infty) \times (0, \infty) | A_{k_1} \subset B_{k_2}, P(Y^* \in B_{k_2}^c) > 0, P(Y^* \in A_{k_1}) > 0\}
\]

When \( A_{k_1} \subset B_{k_2} \), we have

\[
P(Y^* \in A_{k_1} \cap B_{k_2}) - P(Y^* \in A_{k_1}) P(Y^* \in B_{k_2})
\]

\[
= P(Y^* \in A_{k_1})(1 - P(Y^* \in B_{k_2}))
\]

\[
= P(Y^* \in A_{k_1}) P(Y^* \in B_{k_2}^c).
\]

Hence we obtain

\[
M \subset \{(k_1, k_2) \in (0, \infty) \times (0, \infty) | P(Y^* \in A_{k_1} \cap B_{k_2}) - P(Y^* \in A_{k_1}) P(Y^* \in B_{k_2}) > 0\}. \tag{2.19}
\]

By (2.15), (2.16), and Lemma 2.1, the Lebesgue measure of \( M \) is positive. Hence by (2.17), (2.18) and (2.19), we obtain

\[
\text{Cov}(U(X), V(X)) > 0,
\]

which contradicts the assumption, and so \( r = 0 \). \( \square \)
Remark 2.7  The assumption that \( U(0) = V(0) = 0 \) in Theorem 2.6 can be taken out since by the symmetry and quasi-convexity of \( U \) and \( V \), we have \( U(x) \geq U(0), V(x) \geq V(0) \) for all \( x \in \mathbb{R}^n \), and we can consider the polynomials \( U(x) - U(0) \) and \( V(x) - V(0) \), which satisfy the conditions in Theorem 2.6.

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