Statistical anisotropy in the inflationary universe

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During cosmological inflation, quasiclassical perturbations are permanently generated on super-Hubble spatial scales, their power spectrum being determined by the fundamental principles of quantum field theory. By the end of inflation, they serve as primeval seeds for structure formation in the universe. At early stages of inflation, such perturbations break homogeneity and isotropy of the inflationary background. In the present paper, we perturbatively take into account this quasiclassical background inhomogeneity of the inflationary universe while considering the evolution of small-scale (sub-Hubble) quantum modes. As a result, the power spectrum of primordial perturbations develops \textit{statistical anisotropy}, which can subsequently manifest itself in the large-scale structure and cosmic microwave background. The statistically anisotropic contribution to the primordial power spectrum is predicted to have almost scale-invariant form dominated by a quadrupole. Theoretical expectation of the magnitude of this anisotropy depends on the assumptions about the physics in the trans-Planckian region of wave numbers.

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I. INTRODUCTION

The scenario of cosmological inflation [1] was designed to address several puzzles of the standard cosmology, most of them connected with the property of large-scale homogeneity, isotropy and flatness of the observable universe. It was also realized that an inflationary epoch can explain the origin of primordial density perturbations [2]. Specifically, quantum fluctuations of the inflaton and metric fields, being blown up by inflation, acquire quasiclassical features on large (super-Hubble) spatial scales. The predictions of the inflationary theory — most notably, the spatial flatness of the universe and adiabatic primordial perturbations with almost scale-invariant spectrum — are surprisingly well verified by modern observations of the large-scale structure (LSS) and cosmic microwave background (CMB).

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Primordial perturbations which serve as primeval seeds for the formation of the observable structure are generated at the late stages of inflation (the last 60 or so e-foldings). However, depending on a concrete realization of the inflationary scenario, the inflationary stage can be rather long, in principle, starting right from the largest classically allowable energy density, the Planck density. The total number of inflationary e-foldings can be enormous, with typical values $\sim 10^{12}$ in the scenario of chaotic inflation. It is clear that generation of super-Hubble scalar and tensor perturbations with quasiclassical features cannot be considered as specific to the last stage of inflation, but that this process should occur all along the inflationary history. This observation is the basis for the theory of the self-regenerating eternal inflationary universe [1, 3].

Thus one is lead to the idea that the inflationary universe at all times should be inhomogeneous on super-Hubble spatial scales. It then makes sense to take this long-wave (super-Hubble) quasiclassical inhomogeneity into account when considering the evolution of small-scale (sub-Hubble) quantum fluctuations on the background of an inflationary universe. Propagating with speed of light, the small-scale quantum fluctuations travel about a Hubble distance in a Hubble time, thus passing through inhomogeneities on a Hubble spatial scale during inflation. The background inhomogeneity affects the evolution of the small-scale modes determining the vacuum state and, as was shown in our previous publications [4], the resulting power spectrum of primordial perturbations acquires statistical anisotropy of quite specific form.\(^1\) Namely, the correlation function, say, of the relativistic potential $\Phi$ at the end of inflation is expressed as

$$\langle \Phi(x)\Phi(y) \rangle = \int \frac{d^3k}{(2\pi)^3} P_k (1 + \nu_k) e^{i(k(x-y))},$$

where $P_k$ describes the standard isotropic power spectrum, and $\nu_k$ is a new anisotropic correction, which turns out to be almost scale-invariant. Specifically, its multipole expansion is given by

$$\nu_k = (n_S - 1) \left( \Lambda_{ij} \frac{k^i k^j}{k^2} + \Lambda_{ijkl} \frac{k^i k^j k^k k^l}{k^4} + \ldots \right),$$

where $n_S(k)$ is the spectral index of scalar perturbations (which is known to be close to unity), and the quantities $\Lambda_{\ldots}(k)$ depend on $k$ very weakly. Moreover, the expression (2) is essentially dominated by the first term, with quadrupole dependence on the unit vector $n = k/k$.

In deriving the result (1), only scalar quasiclassical background inhomogeneities during inflation were taken into account in [4]. The magnitude of such inhomogeneities in terms of

\(^1\) As noted in [4], this bears some similarity with the Sachs–Wolfe effect [5], in which the large-scale inhomogeneity of the universe at a later cosmological epoch is responsible for the locally observed anisotropy in the temperature of the cosmic microwave background radiation.
the relativistic gravitational potential is estimated to be \( \Phi_c^2 \sim G |\dot{H}| \) (here, \( G \) is the Newton’s gravitational constant, and \( H \) is the Hubble parameter). However, inflation also produces tensor perturbations [6, 7] with magnitude \( h_c^2 \sim GH^2 \) on super-Hubble scales, which is higher than \( \Phi_c^2 \) especially at the early stages of inflation, where \( |\dot{H}| \ll H^2 \). Therefore, tensor inhomogeneities of the background of the inflationary universe also necessarily must be taken into account, and this will be the subject of the present paper. We will show that the results (1) and (2) for the spectrum of primordial scalar perturbations remain to be valid; however, in most interesting cases, a major contribution to the quantity \( \nu_k \) comes from the tensor inhomogeneities of the background of the inflationary universe.

We will see that the expected magnitude of the anisotropic part (2) crucially depends on the moment of time where one sets the initial (vacuum) conditions for the quantum fluctuations of interest at the inflationary state. The earlier this moment is, the larger is the number of inhomogeneous quasiclassical modes that have affected the evolution of the small-scale quantum fluctuations, hence, the larger is the expected statistical anisotropy. This gives rise to an important issue that was first discussed in [4]. It was observed that the physical wave number or frequency of a particular quantum mode in the cosmological frame exceeds the Planck value \( M_P \simeq G^{-1/2} \) during most part of the inflationary universe. A natural question arises, whether it is legitimate to use the standard field theory in cosmology on spatial scales below the Planck length. Recently, this trans-Planckian issue has become a subject of discussion in connection with the possible modification of the form of (statistically isotropic) primordial power spectrum [8] and cosmological particle creation [9]. We will present several arguments from the literature in favor of the possibility of dealing with the trans-Planckian region in cosmology. If the trans-Planckian issue can be safely ignored, then, as we will see, the statistical anisotropy can be significant, with the dominant term \( \Lambda_{ij} \) being up to the order of unity. If, however, one is allowed to specify locally homogeneous and isotropic initial conditions for quantum modes only after the Planck-radius crossing, then the resulting statistical anisotropy is rather small, \( \nu_k \sim 10^{-7} \).

Statistical anisotropy of the power spectrum of primordial scalar perturbations will be manifest in the LSS and CMB. The first effect was already discussed in our previous paper [4], while the statistical anisotropy of the CMB was a subject of a number of recent studies [10–16]. We will briefly describe the arising effects in the present paper.

Our work represents a theoretical contribution to the issue of the possible origin of statistical anisotropy in the universe (see [16, 17] for other ideas on this subject). The predictions made in this paper have an advantage of being based on the usual theory of chaotic inflation.

\(^2\) Throughout this paper, we use the system of units in which \( \hbar = c = 1 \).
without any additional physical ingredients. Observation of statistical anisotropy in the scale-invariant form (2) may thus be regarded as a nontrivial confirmation of the chaotic inflation scenario; this is the only possible test of chaotic inflation that we are aware of. On the other hand, since the magnitude of the statistical anisotropy depends on both the details of the inflationary scenario and the physical assumptions about the trans-Planckian region, its nondetection by itself will not invalidate any of the existing models of inflation.

The paper is organized as follows. In Sec. II, we describe the standard theory of quantum scalar and tensor perturbations during inflation and introduce our basic notation. In Sec. III, we discuss the quasiclassical status of perturbations on super-Hubble spatial scales during inflation. In Sec. IV, we give a detailed treatment of the evolution of scalar quantum fluctuations on small (sub-Hubble) spatial scales on the background of an inflationary universe taking into account the effect of scalar and tensor quasiclassical inhomogeneities on large (super-Hubble) spatial scales. In Sec. V, we calculate the resulting spectrum of primordial scalar perturbations which are seeds for the observable large-scale structure. The trans-Planckian issue is discussed in Sec. VI. The observable effects of statistical anisotropy of the large-scale structure and cosmic microwave background are described in Sec. VII. The results are summarized in Sec VIII. Some calculations are moved to the appendix.

II. PERTURBATIONS DURING INFLATION

Throughout this paper, we consider a simple model of inflation based on a single scalar field $\varphi$ with canonical Lagrangian and potential $V(\varphi)$. As a concrete example which is compatible with current observations, we will use the quadratic potential

$$V(\varphi) = \frac{1}{2} m_\varphi^2 \varphi^2$$

with constant mass $m_\varphi$.

The expanding spatially flat homogeneous and isotropic universe filled with a scalar field is described by the set of classical equations

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV(\varphi)}{d\varphi} = 0,$$  

$$H^2 = \frac{8\pi G}{3} \left[ \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right],$$

$$\dot{H} = -4\pi G \dot{\varphi}^2,$$

where the dot denotes differentiation with respect to the cosmological time $t$. The so-called slow-roll regime of the scalar field is realized when the “friction” term $3H\dot{\varphi}$ dominates over
the term with the second derivative in (4), so that we can write

$$
\dot{\phi} \approx -\frac{1}{3H} \frac{dV}{d\varphi}.
$$

(7)

In view of (5) and (6), the condition for inflation

$$
|\dot{H}| \ll H^2
$$

(8)

is equivalent to

$$
\dot{\phi}^2 \ll V(\varphi),
$$

(9)

which, for the scalar-field potential (3) during the slow-roll regime (7), reduces to

$$
\varphi^2 \gg \frac{1}{6\pi G}.
$$

(10)

A. Scalar perturbations

In the longitudinal gauge, metric perturbations of scalar type are described by the so-called relativistic potentials $\Phi$ and $\Psi$, which enter the metric as follows:

$$
ds^2 = a^2(\tau) \left[ (1 + 2\Phi) d\tau^2 - (1 - 2\Psi) dx^2 \right],
$$

(11)

where $\tau$ is the conformal time.

Perturbations of the metric are accompanied by those of the inflaton field $\chi \equiv \delta \varphi$. For perturbations in the linear approximation, one obtains the following system of equations in the spatially flat case under consideration:

$$
\begin{align*}
\Psi &= \Phi, \\
4\pi G \dot{\phi} \chi &= \ddot{\Psi} + H \Phi, \\
\dot{H} \frac{\partial}{\partial t} \left[ \frac{H^2}{aH} \frac{\partial}{\partial t} \left( \frac{a\Phi}{H} \right) \right] - \nabla^2 \Phi a^2 &= 0.
\end{align*}

(12) \quad (13) \quad (14)

This system implies a useful relation that expresses $\Phi$ in terms of $\chi$ and $\chi'$:

$$
(\nabla^2 + 4\pi G \phi^2) \Phi = \frac{4\pi G \phi^2}{a} \left( \frac{a\chi}{\varphi'} \right)'.
$$

(15)

Here and below, the prime in the time-dependent quantities denotes the derivative with respect to the conformal time $\tau$, and the symbol $\nabla^2$ is the usual Laplace operator in the Euclidean (comoving) three-space.

Because of the constraint equations (12) and (13), perturbations of scalar type are described by a single scalar degree of freedom. Hence, proceeding to quantization, we note
that there exists a single set of creation and annihilation operators realizing the field decomposition

\[ \chi = \int \frac{d^3 k}{(2\pi)^3} \left[ \chi_k e^{ikx} a_k + \chi_k^* e^{-ikx} a_k^\dagger \right], \quad (16) \]

\[ \Phi = \int \frac{d^3 k}{(2\pi)^3} \left[ \Phi_k e^{ikx} a_k + \Phi_k^* e^{-ikx} a_k^\dagger \right], \quad (17) \]

with the commutation relations

\[ \left[ a_{k_1}, a_{k_2}^\dagger \right] = (2\pi)^3 \delta (k_1 - k_2). \quad (18) \]

The spatial dependence of the mode functions is given by the standard exponent \( e^{ikx} \), and their temporal behavior, described by the functions \( \chi_k(\tau) \) and \( \Phi_k(\tau) \), depends only on the absolute value of \( k \), which implies the spatial homogeneity and isotropy of the vacuum state defined by the annihilation operators.

To determine the temporal dependence of the mode functions, hence, the vacuum state, one can introduce the variable

\[ v = a \left( \chi + \dot{\Phi} \frac{\dot{\Psi}}{H} \right) \quad (19) \]

that brings the quadratic action for scalar perturbations into a canonical form\(^3\) (after the constraints are taken into account)

\[ S_v = \frac{1}{2} \int \left( v'^2 - \partial_i v \partial^i v + \frac{z''}{z} v^2 \right) d\tau d^3 x, \quad (20) \]

where \( z = a\dot{\varphi}/H \). In the high-frequency regime, where \( k \gg aH \), one deals with the quantum theory of a free massless scalar field in Minkowski space with a well-defined quantum state determined by positive-frequency solutions.

For our purposes, however, it will be more convenient to use the variable \( \chi \) in the high-frequency region. Using constraint (15), one can see from (19) that \( v \approx a\chi \) in this region, so that the two variables are simply related. One can describe this fact as insignificance of the self-gravity of the inflaton field in the high-frequency region. In this case, the appropriately normalized positive-frequency modes for the inflaton field have the form

\[ \chi_k \approx \frac{1}{a\sqrt{2k}} e^{-ik\tau}, \quad k \gg aH. \quad (21) \]

The expression for the modes of the relativistic potential \( \Phi_k \) in the high-frequency region are then obtained using equation (15):

\[ \Phi_k \approx i\pi G\dot{\varphi} \left( \frac{2}{k} \right)^{3/2} e^{-ik\tau}, \quad k \gg aH. \quad (22) \]

\( ^3 \) Here and below, the spatial indices are raised and lowered with the use of the Euclidean metric \( \delta_{ij} \).
Here we used the fact that the value of $\dot{\phi}$ changes slowly with conformal time relative to the frequency $k \gg aH$ (during the slow-roll regime, it varies slowly with time even relative to $H$).

A more careful analysis of equation (14) shows that the approximate solution (22) is, in fact, valid in a somewhat broader range of wave numbers, namely, for $k \gg a|\dot{H}|^{1/2}$ (we remember that $|\dot{H}| \ll H^2$ during inflation). Indeed, after introducing a new variable

$$\phi = |\dot{H}|^{-1/2} \Phi,$$

we can write equation (14) in terms of the conformal time as

$$\phi'' + (F - \nabla^2) \phi = 0,$$

in which the time-dependent term

$$F = a^2 \left[ \frac{\ddot{H}}{2H} - \frac{H\dddot{H}}{2H} - \frac{3}{4} \left( \frac{\dot{H}}{H} \right)^2 + 2\dot{H} \right],$$

is of the order of $a^2|\dot{H}|$ during slow-roll inflation. Taking into account equation (6), one can see that (22) is a good approximate solution of (23), (24) in the domain $k \gg a|\dot{H}|^{1/2}$. Then, using (13) and the slow-roll condition, we can write the approximate solutions for the modes valid in a broader frequency range as follows:

$$\Phi_k \approx i\pi G \dot{\phi} \left( \frac{2}{k} \right)^{3/2} e^{-ik\tau}, \quad k \gg a|\dot{H}|^{1/2},$$

$$\chi_k \approx \frac{1}{a\sqrt{2k}} \left( 1 + i\frac{aH}{k} \right) e^{-ik\tau}, \quad k \gg a|\dot{H}|^{1/2}. \quad (26)$$

The Fourier amplitude of the solution of equation (14) can also conveniently be written as

$$\Phi_k = A_k \frac{H}{a} \int \frac{a\dot{H}}{H^2} dt + B_k \frac{H}{a},$$

where $A_k$ and $B_k$ are some time-dependent functions. As the physical wavelength $\lambda = a/k$ of the perturbation increases, the functions $A_k$ and $B_k$ approach constant values. This is clear from equation (14), in which one can neglect the Laplacian for $k \ll aH$. In this limit, solution (28) with $A_k$ and $B_k$ which are constant in time becomes exact.

To write an approximate solution in the region $k \ll aH$, we note that the second term in (28) rapidly decreases during inflation and becomes insignificant. The whole expression (28) can then be approximated as

$$\Phi_k \approx A_k \frac{\dot{H}}{H^2}, \quad k \ll aH.$$
Then equations (12) and (13) imply

\[ \chi_k \approx -A_k \frac{\dot{\phi}}{H}, \quad k \ll aH. \tag{30} \]

One can determine the amplitude \( A_k \) by noting that the frequency domains in (26), (27), (29), and (30) overlap. One obtains

\[ A_k \approx -\frac{i}{\sqrt{2}} k^{3/2} \frac{H^2}{\dot{\phi}} e^{-ikr}, \quad |\dot{H}|^{1/2} \ll \frac{k}{a} \ll H. \tag{31} \]

During the future evolution, the value of \( A_k \) should tend to a constant, which is then estimated by the value of the ratio \( H^2/\dot{\phi} \) at the moment of time where \( a|\dot{H}|^{1/2} = k \). We will supply the homogeneous fields evaluated at this moment of time by the index “\( k \)” so that, for instance, \( \dot{\phi}_k = \dot{\phi} \mid_{a|\dot{H}|^{1/2}=k} \). Thus, we have

\[ P_k \equiv k^3 |A_k|^2 \approx \frac{H_k^4}{2\dot{\phi}_k^2}, \quad a|\dot{H}|^{1/2} > k. \tag{32} \]

In view of definition (29), the last expression gives the primordial power spectrum of perturbations for the variable \( \Phi \) at the end of inflation, which is conventionally defined as the moment of time where \( |\dot{H}| = H^2 \). The quantity \( P_k \) typically is a logarithmic function of the wave number \( k \). In the slow-roll approximation with potential (3), we have\(^5\)

\[ \varphi^2 \approx \frac{m^2_{\varphi}}{12\pi G} = \text{const}, \quad H^2 \approx \frac{4\pi G}{3} m^2_{\varphi} \varphi^2, \tag{33} \]

so that

\[ a \approx a_f e^{-2\pi G (\varphi^2 - \varphi_f^2)}, \tag{34} \]

where the index “\( f \)” refers to the moment of the end of inflation, where \( |\dot{H}| = H^2 \). Combining these expressions, we eventually get

\[ P_k \approx \frac{8\pi}{3} G m^2_{\varphi} \left( \frac{1}{2} + \ln \frac{a_f H_f}{k} \right)^2, \tag{35} \]

in which

\[ \ln \frac{a_f H_f}{k} \approx 50 \tag{36} \]

\(^4\) Since the universe expands exponentially rapidly during inflation, the value of the slowly evolving quantity \( H^2/\dot{\phi} \) does not change much in the time interval during which \( |\dot{H}|^{1/2} < k/a < H \). For this reason, the amplitude of perturbations is frequently estimated at the moment of the Hubble-radius crossing during inflation, i.e., at \( aH = k \).

\(^5\) To our knowledge, relations (33) and (34) were first obtained in [18], where the dynamics of a universe dominated by a massive scalar field was under consideration.
for the typical wavelengths corresponding to the large-scale structure. Since $P_k/2\pi^2 \approx 2.4 \times 10^{-9}$ according to the observations of the angular power spectrum of the CMB temperature anisotropy [19], we get the following estimate for the mass of the inflaton field:

$$Gm_\phi^2 \approx 2 \times 10^{-12}. \quad (37)$$

The running spectral index $n_S$ of scalar perturbations is conventionally defined as

$$n_S = 1 + \frac{d\ln P_k}{d\ln k}, \quad (38)$$

so that a scale-invariant power spectrum corresponds to $n_S = 1$. For spectrum (35) and for (36), we calculate

$$n_S \approx 1 - 2 \left( \ln \frac{a_f H_f}{k} \right)^{-1} \approx 0.96. \quad (39)$$

**B. Tensor perturbations**

The idea of parametric amplification of quantum tensor fluctuations (gravitational waves) in a homogeneous and isotropic universe was proposed in [20]. The spectrum of gravitational waves produced during the de Sitter stage in the early universe was first calculated in [6] and later analyzed in [7] in the context of the theory of cosmological inflation. Here we provide the relevant relations that will be used in the following sections.

Tensor perturbations can be presented in the form

$$g_{\mu\nu} = a^2 [\eta_{\mu\nu} + h_{\mu\nu}], \quad (40)$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and $h_{\mu\nu}$ denotes the small perturbation. One can work in the transverse traceless gauge

$$h_{00} = h_{0i} = 0, \quad \delta^j h_{ij} = \partial^i h_{ij} = 0. \quad (41)$$

For a plane wave with wave vector $k$, there exist two independent polarizations, which are usually denoted by $\sigma = +, \times$.

The classical dynamics of tensor perturbations is determined by the expansion of the Hilbert–Einstein action to quadratic order in $h_{ij}$:

$$S_g = \frac{1}{64\pi G} \int a^2(\tau) \eta^{\mu\nu} \partial_\mu h^{ij} \partial_\nu h_{ij} d\tau d^3x. \quad (42)$$

One can proceed to the Fourier representation

$$h_{ij}(x, \tau) = \int \frac{d^3k}{(2\pi)^3} \sum_{\sigma = +, \times} h_{k,\sigma}(\tau) e_{ij}(k, \sigma) e^{ikx}, \quad (43)$$
where \( e_{ij}(k, \sigma) \) are symmetric traceless polarization tensors with real components satisfying the conditions
\[
e_{ij}(k, \sigma) = e_{ij}(-k, \sigma), \quad k^i e_{ij}(k, \sigma) = 0, \quad e^{ij}(k, \sigma) e_{ij}(k, \sigma') = \delta_{\sigma\sigma'}.
\] (44)

The action then takes the form
\[
S_g = \frac{1}{64\pi G} \int \frac{d^3k}{(2\pi)^3} \sum_{\sigma=\pm} \left( h_{k,\sigma} h_{k,\sigma}^* - k^2 h_{k,\sigma} h_{k,\sigma}^* \right) a^2 d\tau.
\] (45)

Quantization is performed by interpreting \( h_{k,\sigma}(\tau) \) as operators with the decomposition
\[
h_{k,\sigma}(\tau) = h_{k,\sigma}(\tau) a_{k,\sigma} + h_{k,\sigma}^*(\tau) a_{k,\sigma}^\dagger
\] (46)
and commutation relations
\[
\left[ a_{k_1,\sigma_1}, a_{k_2,\sigma_2}^\dagger \right] = (2\pi)^3 \delta_{\sigma_1\sigma_2} \delta(k_1 - k_2).
\] (47)

A new, canonically normalized, variable
\[
v_k = \sqrt{\frac{1}{32\pi G}} a h_k,
\] (48)
satisfies the equation
\[
v''_k + \omega^2_k(\tau)v_k = 0,
\] (49)
where
\[
\omega^2_k(\tau) = k^2 - \frac{\alpha''}{\alpha} = k^2 - a^2 \left( \dot{H} + 2H^2 \right).
\] (50)
The initial conditions for the normalized solution with positive frequency at the moment of time \( \tau_i \) are
\[
v_k(\tau_i) = \frac{1}{\sqrt{2\omega_k}}, \quad v'_k(\tau_i) = -i\sqrt{\frac{\omega_k}{2}}.
\] (51)
Note that these initial conditions make sense only for sufficiently small wavelengths, for which \( \omega^2_k > 0 \), or \( k^2 > a^2 \left( \dot{H} + 2H^2 \right) \), at the initial moment of time.

In the inflationary regime, where \( |\dot{H}| \ll H^2 \), the normalized solution which has positive frequency in the asymptotic past is given by the approximate expression [compare with (27)]
\[
v_k \approx \frac{1}{\sqrt{2k}} \left( 1 + \frac{iaH}{k} \right) e^{-ik\tau}, \quad k \gg a|\dot{H}|^{1/2}.
\] (52)
It is exact in the de Sitter space, where \( H \equiv \text{const.} \) On small spatial scales, the nondecaying solution of (49), (50) is \( v_k \propto a \), so that
\[
v_k \approx \frac{iaH_k}{\sqrt{2k^{3/2}}} e^{-ik\tau}, \quad k \lesssim a|\dot{H}|^{1/2}.
\] (53)
III. QUANTUM AND CLASSICAL PERTURBATIONS

Observing expressions (26), (27) and (52), one can notice a qualitative difference between the behavior of the vacuum modes on sub-Hubble spatial scales \(k \gg aH\) and their behavior on super-Hubble spatial scales \(k \ll aH\). On sub-Hubble spatial scales, they are quite similar to the corresponding vacuum modes in the Minkowski space: they oscillate with positive frequency in time, and the scalar-field amplitude has the characteristic behavior \(\propto k^{-1/2}\). The field’s quantum state in this domain of wave numbers features the usual vacuum in a flat space. On super-Hubble spatial scales, the temporal evolution is “frozen” and the scalar-field amplitude contains an extra factor \(k^{-1}\). The property of the quantum state allows one to speak about the generation of a quasiclassical field condensate on large spatial scales (see [1]). Indeed, the equal-time correlation function, say, for the perturbations of the inflaton field, is given by

\[
\langle \chi(x)\chi(y) \rangle = \int \frac{d^3k}{(2\pi)^3} |\chi_k|^2 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} = \int \frac{d^3k_{\text{ph}}}{(2\pi)^3} \frac{1}{k_{\text{ph}}} \left( \frac{1}{2} + \frac{H^2}{2k_{\text{ph}}^2} \right) e^{i\mathbf{k}_{\text{ph}}\cdot(\mathbf{x}_{\text{ph}}-\mathbf{y}_{\text{ph}})},
\]

(54)

where

\[
k_{\text{ph}} = \frac{k}{a}, \quad \mathbf{x}_{\text{ph}} = a\mathbf{x}
\]

are the physical wave number and physical distance, respectively. The first term in the brackets of (54) corresponds to the usual quantum fluctuations in the flat space. The second term, from the viewpoint of flat space, can be interpreted as the presence of particles with occupation numbers

\[
N_k = \frac{H^2}{2k_{\text{ph}}^2},
\]

(56)

leading to nonzero correlation in (54) for exponentially large physical distances \(H \ll |\mathbf{x}_{\text{ph}}-\mathbf{y}_{\text{ph}}| \lesssim Ha/a_i\), where \(a_i\) is the scale factor at the beginning of inflation.

The behavior of the quasiclassical part of the inflaton \(\varphi\) in the inflationary universe can be described in terms of Brownian motion induced by conversion of quantum fluctuations of the field \(\varphi\) into quasiclassical inhomogeneities. The probability distribution of the local value of the inflaton field obeys an effective diffusion equation (see [1, 3, 4, 21] for the development of this theory).

At the end of inflation, perturbations on super-Hubble spatial scales constitute the primordial density perturbations. It is usually assumed that they represent a realization of a classical Gaussian random field described by the correlation function (54) and accompanied by the corresponding classical perturbations in the relativistic potential \(\Phi\). The mechanism of this transition — from the quantum state that does not break the symmetries of the background space to a classical state that obviously breaks these symmetries — remains
to be an open issue. It is presumably related to the problem of collapse of wave function, hence, to the deep interpretational issues of quantum mechanics (see [22]).

IV. SCALAR PERTURBATIONS ON THE INHOMOGENEOUS INFLATIONARY BACKGROUND

A. Self-regenerating inflation

It is clear that the generation of scalar and tensor perturbations on super-Hubble spatial scales that subsequently behave quasiclassically is not restricted specifically to the last stage of inflation, where it is supposed to produce the primordial seeds for the subsequent formation of large-scale structure and primordial gravity waves, but that this process occurs all along the inflationary history. Therefore, the quasiclassical parts of $\Phi$ and $h_{ij}$ are permanently generated during inflation constituting a (random) inhomogeneity and anisotropy of the inflationary universe. It thus makes sense to make the next step in the perturbation theory and consider the evolution of short-wave ($k \gg aH$) perturbations (to become the observable large-scale structure) on the background of the long-wave ($k \lesssim aH$) quasiclassical inhomogeneities permanently generated during inflation. In order to set up a valid theoretical background for this procedure, we need to address the issue of the evolution of the inhomogeneous inflationary universe. This is the subject of the present subsection.

Because of the permanent generation of classical long-wave perturbations and their subsequent evolution, the inflationary universe considered as a whole may be very inhomogeneous, its different spatially remote parts having quite different histories. Indeed, the dispersion of the inhomogeneity of the classical inflaton field generated during one Hubble time is equal to $\langle \chi^2 \rangle \approx H^2/2\pi$ according to (54). At early stages of inflation, a typical random change of the inflaton can be larger than its regular change $\Delta \varphi = \dot{\varphi}H^{-1}$ caused by slow rolling; a criterion for this property is $\Delta \varphi < \sqrt{\langle \chi^2 \rangle}$, or

$$\left| \frac{dV(\varphi)}{d\varphi} \right| < \frac{3H^3}{2\pi}. \tag{57}$$

For the scalar-field potential (3), this condition becomes

$$\phi^2 > \phi_*^2 = \frac{\sqrt{3}}{2\sqrt{2\pi} C^{3/2} m_\varphi}, \tag{58}$$

which is considerably higher than the inflationary boundary (10) but still lower than the boundary of the Planckian energy density $\rho_p \simeq G^{-2}$ in view of (37):

$$V(\varphi_*) \simeq m_\varphi G^{-3/2} \ll G^{-2}. \tag{59}$$
If the inflaton field is above the threshold given by (57) or (58), then its value has a significant chance to increase with time due to the mechanism of quantum fluctuations described above. This random behavior of the classical part of the inflaton field leads to a picture of eternal self-regenerating inflation [1, 3]. One of the consequences of this picture is that the inflationary universe as a whole cannot be split into a homogeneous and isotropic background and a small classical perturbation because very soon one has to deal with perturbations with relative amplitudes larger than unity, which are impossible to treat in a perturbative way.

To overcome this difficulty, we will follow the history of a spatially bounded domain of the inflationary universe choosing its variable comoving size to be conveniently larger than the size of the horizon at every moment of time and, on the other hand, not very large, so that inhomogeneity within this domain can still be treated as small. This approach was also adopted in [4]. Specifically, in terms of the original coordinates \((\tau, x)\), at every moment of time, we deal with a domain of comoving size \(\Delta x = (\zeta aH)^{-1}\), where the constant \(\zeta \ll 1\), i.e., a domain well exceeding the Hubble size. Within this domain, at every moment of time, we can split the quasiclassical perturbations generated during inflation into the spatially homogeneous parts \(\Phi\) and \(\tilde{h}_{ij}\) and the remaining parts with zero spatial average, which we denote by \(\Phi_c\) and \(h^c_{ij}\) (endowing them with the subscript or superscript “c”). After that, we eliminate the homogeneous parts \(\Phi\) and \(\tilde{h}_{ij}\), depending only on time, by making the transformation of coordinates and of the scale factor within the domain under consideration as follows.

To eliminate \(\Phi\), it suffices to proceed to a new time coordinate: \(d\tau \rightarrow (1 - 2\Phi) d\tau\). This will redefine the scale factor as \(a \rightarrow a(1 - \Phi)\). After this transformation, the homogeneous part of the metric takes the form

\[
\text{ds}^2 = a^2 \left[ d\tau^2 - (\delta_{ij} - \tilde{h}_{ij}) dx^i dx^j \right],
\]

where \(\tilde{h}_{ij}\) depend only on \(\tau\).

To get rid of the global anisotropy in the spatial part of the metric (60), we make the coordinate transformation

\[
x^i \rightarrow x^i + \frac{1}{2} \tilde{h}^i_j x^j, \quad \tau \rightarrow \tau - \frac{1}{4} \tilde{h}^i_j x^i x^j,
\]

where we set the origin of coordinates \(x^i\) to be at the center of the domain under consideration. This transformation brings the metric (60) into the form

\[
\text{ds}^2 = a^2 \left[ (1 + 2\Phi_h) d\tau^2 - (1 - 2\Psi_h) dx^2 \right],
\]

where

\[
\Phi_h = -\frac{1}{4} \left( \tilde{h}''_{ij} + \alpha \tilde{h}'_{ij} \right) x^i x^j, \quad \Psi_h = \frac{1}{4} \alpha \tilde{h}'_{ij} x^i x^j,
\]

(63)
and $\alpha \equiv a'/a$.

The complete classical metric within the domain under consideration is then given by

$$ds^2 = a^2 \left[ (1 + 2\Phi_c) d\tau^2 - (1 - 2\Psi_{hc} - 2\Phi_c) dx^2 + h_{ij} dx^i dx^j \right].$$

(64)

Since the quantities $\Phi_c$ and $h_{ij}^c$ describe inhomogeneities with zero spatial average over the specified domain, they can be approximated by the following expressions with conventionally chosen time-dependent exponential momentum cutoff

$$\Phi_c = \int \frac{d^3k}{(2\pi)^3} \sum_{\sigma = +, \times} \Phi_k(\tau) \left( e^{ikx} - e^{-k/\zeta aH} \right) a_k^\dagger e^{-k/\varepsilon aH},$$

(65)

$$h_{ij}^c = \int \frac{d^3k}{(2\pi)^3} \sum_{\sigma = +, \times} e_{ij}(k, \sigma) \left[ h_k(\tau) \left( e^{ikx} - e^{-k/\zeta aH} \right) a_k^\dagger + h_k^*(\tau) \left( e^{-ikx} - e^{-k/\zeta aH} \right) a_k^\dagger \right] e^{-k/\varepsilon aH},$$

(66)

where the exponent with the constant $\varepsilon \lesssim 1$ determines the momentum integration region in (65) and (66) ensuring that we are well in the quasiclassical domain, and the exponent with the constant $\zeta \ll 1$ ensures the property of negligible spatial average over our monitored spatial domain with comoving size $(\zeta aH)^{-1}$. The quantity $\bar{h}_{ij}$ entering (63) is the average of the classical part of the corresponding field over the spatial domain under consideration and, therefore, is given by the expression

$$\bar{h}_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_{\sigma = +, \times} c_{ij}(k, \sigma) \left[ h_k(\tau) a_{k,\sigma} + h_k^*(\tau) a_{k,\sigma}^\dagger \right] e^{-k/\zeta aH - k/\varepsilon aH},$$

(67)

in which the effective momentum cutoff parameter in the exponent is $\zeta_{\text{eff}}^{-1} = \zeta^{-1} + \varepsilon^{-1} \approx \zeta^{-1}$ in view of the condition $\zeta \ll \varepsilon \lesssim 1$.

### B. Small-scale perturbations on the inhomogeneous inflationary background

Now we are going to investigate the propagation of small-scale scalar modes on the background of the metric (64). In our treatment [4], this was done by taking into consideration only scalar classical inhomogeneities $\Phi_c$ present in (64). In this paper, we are going to take into account both scalar and tensor classical background inhomogeneities.

For an inhomogeneous background, the convenient classification of perturbations into scalar, vector, and tensor types is no longer exact. In particular, if one wishes to use it, then one has to deal with mixing and interaction between these types of modes mediated by the background inhomogeneity. However, for the modes with extremely short wavelengths, much below the Hubble scale, this mixing can be neglected. Indeed, the self-gravity of the scalar inflaton field is negligible at such short wavelengths, and perturbations in a scalar
field are described only by scalar quantities. We, therefore, use the same method as in [4]. Specifically, for the scalar modes in the high-frequency regime (with physical wavelengths $\lambda \ll H^{-1}$), we neglect the effects of self-gravity and consider propagation of the inflaton perturbations on the background of the metric $g^{c}_{\mu\nu}$ defined in (64). The corresponding small-scale perturbations $\chi \equiv \delta \varphi$ are then described by the equation

$$\frac{1}{\sqrt{-g^c}} \partial_{\mu} \left( \sqrt{-g^c} g^c_{\mu\nu} \partial_{\nu} \chi \right) + m^2_{\varphi} \chi = 0,$$

where

$$m^2_{\varphi}(\varphi_c) = \frac{d^2V(\varphi_c)}{d\varphi^2}$$

is the scalar-field effective mass, and $\varphi_c$ denotes the quasiclassical part of the scalar field, defined similarly to (65). For the free scalar field, $m_{\varphi}(\varphi_c)$ will be just the constant inflaton mass $m_{\varphi}$. In what follows, however, we will see that the mass term can be neglected altogether for the high-frequency modes described by (68).

We have the right to treat equation (68) only in linear approximation in the quasiclassical inhomogeneities $\Phi_h$, $\Psi_h$, $\Phi_c$ and $h^c_{ij}$ since our quasiclassical metric (64) is valid only in linear approximation in these fields. Equation (68), written explicitly to linear order in terms of the background classical inhomogeneities, reads

$$\chi'' - \nabla^2 \chi + 2a \chi' + a^2 m^2_{\varphi} \chi = (4\Phi_c' + \Phi'_h + 3\Psi'_h) \chi' + (4\Phi_c + 2\Phi_h + 2\Psi_h) \nabla^2 \chi + \sum_{ij} [\delta_{ij} \partial_i (\Phi_h - \Psi_h) \partial_j \chi + h^c_{ij} \partial_i \partial_j \chi] - 2a^2 m^2_{\varphi} \Phi_c \chi,$$

where $\alpha \equiv a'/a$, and we remind the reader that the spatial indices are raised and lowered using the Euclidean metric $\delta_{ij}$. The mass function (69) for an arbitrary inflaton potential is also supposed to be expanded in (70) in the linear perturbation of the inflaton. However, as we already mentioned, the corresponding terms in (70) will be insignificant.

The quantum field has the decomposition quite similar to (16):

$$\chi = \int d^3k \left( \chi_k e^{i k \cdot x} a_k + \chi_k^* e^{-i k \cdot x} a^\dagger_k \right),$$

with the creation and annihilation operators satisfying the commutation relations (18), and the modes $\chi_k(\tau, x)$, which now depend on the whole vector $k$ as well as on $x$, satisfying the usual normalization conditions

$$\langle \chi_{k_1} \chi_{k_2} \rangle = \int_{\tau = \text{const}} i \left( \chi_{k_1}^* \chi_{k_2} - \chi_{k_1} \chi_{k_2}^* \right) e^{i(k_1 - k_2)x} \sqrt{-g^c} g^{c\tau\tau} d^3x = (2\pi)^3 \delta(k_1 - k_2).$$

Following [4], we look for the solution of (70) in the form

$$\chi_k = \chi_k e^{-i S_k - i T_k},$$

(73)
where \( \chi_k(\tau) \) are solutions \((21)\) for the unperturbed modes, and the (complex) phases \( S_k(\tau, x) \) and \( T_k(\tau, x) \) appear due to the influence of the scalar and tensor quasiclassical inhomogeneities, respectively. The dependence of \( S_k \) on \( \Phi_c \), and of \( T_k \) on \( h_c^{ij} \), will be considered in linear approximation. Furthermore, since we are working in the domain of high frequencies, we will use the following formal expansion in powers of \( k^{-1} \):

\[
S_k = kS_0 + iS_1 + \mathcal{O}(k^{-1}) \\
T_k = kT_0 + iT_1 + \mathcal{O}(k^{-1}) ,
\]

where we have omitted the implicit index \( k \) in the functions \( S_0, S_1, T_0, \) and \( T_1 \). In fact, this is the expansion in powers of the dimensionless parameter \((k\ell)^{-1} \ll 1\), where \( \ell \) is the comoving size of the quasiclassical background inhomogeneities. With this notation, all the functions \( S_0, S_1, T_0, \) and \( T_1 \) turn to be real, as we will see soon.

Substituting \((73)\) and \((74)\) into \((70)\), we get the following system of equations for the phases:

\[
\begin{align*}
(\partial_\tau + n \cdot \nabla) S_0 &= 2\Phi_c , \\
(\partial_\tau + n \cdot \nabla) S_1 &= 2\Phi_c' - \frac{1}{2} \left( S''_0 - \nabla^2 S_0 \right) , \\
(\partial_\tau + n \cdot \nabla) T_0 &= \Phi_h + \Psi_h + Q , \\
(\partial_\tau + n \cdot \nabla) T_1 &= \frac{1}{2} \left( \Phi_h' + 3\Psi_h' \right) - \frac{1}{2} n \cdot \nabla (\Phi_h - \Psi_h) - \frac{1}{2} \left( T''_0 - \nabla^2 T_0 \right) ,
\end{align*}
\]

where \( n \equiv k/k \) is the unit vector with components \( n_i \), and

\[
Q \equiv \frac{1}{2} h_c^{ij} n_i n_j .
\]

The solution for the components of the phases \( S_k[\Phi_c] \) and \( T_k[h_c^{ij}] \) is straightforward:

\[
\begin{align*}
S_0(\tau, x) &= \int_{\tau_i}^\tau 2\Phi_c [\tilde{\tau}, x - n (\tau - \tilde{\tau})] d\tilde{\tau} , \\
S_1(\tau, x) &= \Phi_c(\tau, x) + \left[ \nabla^2 - (n \cdot \nabla)^2 \right] \int_{\tau_i}^\tau \Phi_c [\tilde{\tau}, x - n (\tau - \tilde{\tau})] (\tau - \tilde{\tau}) d\tilde{\tau} , \\
T_0(\tau, x) &= \int_{\tau_i}^\tau (\Phi_h + \Psi_h + Q) [\tilde{\tau}, x - n (\tau - \tilde{\tau})] d\tilde{\tau} , \\
T_1(\tau, x) &= n \cdot \nabla \int_{\tau_i}^\tau Q [\tilde{\tau}, x - n (\tau - \tilde{\tau})] d\tilde{\tau} + \frac{1}{2} Q(\tau, x) + \Psi_h(\tau, x) \\
&\quad + \frac{1}{2} \left[ \nabla^2 - (n \cdot \nabla)^2 \right] \int_{\tau_i}^\tau (\Phi_h + \Psi_h + Q) [\tilde{\tau}, x - n (\tau - \tilde{\tau})] (\tau - \tilde{\tau}) d\tilde{\tau} .
\end{align*}
\]

The moment of time \( \tau_i \) in \((80)-(83)\) is taken to be the time at which the initial conditions for perturbations are set. It may or may not coincide with the moment of the beginning of
inflation (we will discuss this issue in Sec. VI). For simplicity, we can assume the background quasiclassical inhomogeneities to be absent at this moment of time, i.e., we can set $\Phi_c(\tau, x) \equiv 0$ and $h^{ij}_c(\tau, x) \equiv 0$. This enables one to speak about the homogeneous and isotropic initial conditions for quantum fluctuations in a homogeneous and isotropic initial universe, hence, to ensure that the inhomogeneity and anisotropy are generated intrinsically in a self-consistent way. Alternatively, one can consider a region of the initial universe with size much smaller than the local Hubble radius and much larger than the wavelength of the small-scale modes under consideration, and by making the local change of time and space coordinates in that region in the neighborhood of the initial moment of time, one can make sure that solutions (80)–(83) with the initial conditions $S_k(\tau, x) = T_k(\tau, x) \equiv 0$ correspond to the locally homogeneous and isotropic vacuum. (See [4] for more details in the case of scalar background inhomogeneity.)

We trace the evolution of the mode $\chi_k$ in the form (73) until some later moment of time which is close (say, a dozen $e$-foldings) to the moment of its Hubble-radius crossing. At this time, the mode is still in the high-frequency regime, but the classical inhomogeneities $\Phi_c$ and $h^{ij}_c$ that substantially influenced its evolution during the preceding inflationary period are already stretched well beyond the Hubble radius because of inflation. Therefore, one can safely neglect the influence of $\Phi_c$ and $h^{ij}_c$ at this stage and return to the purely linear theory. We also need to take into account the self-gravity of the inflaton perturbations since it is going to become important after the Hubble-radius crossing. Thus, we need to determine the behavior of the mode $\Phi_k$ at this period of time, well before the Hubble-radius crossing.

The modes $\Phi_k$ enter the field decomposition similarly to (17):

$$\Phi = \int \frac{d^3k}{(2\pi)^3} \left[ \Phi_k e^{ikx} \partial_k + \Phi^*_k e^{-ikx} \partial_k^* \right].$$  

We look for the solution of $\Phi_k$ in the form similar to (73):

$$\Phi_k = \Phi_k e^{-i\tilde{S}_k - i\tilde{T}_k},$$

where $\Phi_k(\tau)$ is the unperturbed solution (22), and $\tilde{S}_k(\tau, x)$ and $\tilde{T}_k(\tau, x)$ are the phases corresponding to $S_k$ and $T_k$ in (73), describing the influence of the scalar and tensor quasiclassical inhomogeneities, respectively. We can expand them in powers of $k^{-1}$ similarly to (74):

$$\tilde{S}_k = k\tilde{S}_0 + i\tilde{S}_1 + \mathcal{O}(k^{-1}) , \quad \tilde{T}_k = k\tilde{T}_0 + i\tilde{T}_1 + \mathcal{O}(k^{-1}).$$

The equation of the linear perturbation theory that connects $\Phi_k$ and $\chi_k$ is (15). We substitute (73), (74) and (85), (86) into this equation and, by collecting equal powers of $k^{-1}$, obtain the following expressions for the new phases:

$$\tilde{S}_0 = S_0 , \quad \tilde{S}_1 = S_1 + S'_0 + 2n \cdot \nabla S_0 = S_1 + n \cdot \nabla S_0 + 2\Phi_c ,$$

$$\tilde{T}_0 = T_0 , \quad \tilde{T}_1 = T_1 + T'_0 + 2n \cdot \nabla T_0 = T_1 + n \cdot \nabla T_0 + \Phi_h + \Psi_h + Q.$$
On the next step, we should link the solution (85) for the mode well before the Hubble-radius crossing to the solution well after the Hubble-radius crossing. The solution in this last regime can be expressed similarly to (29):

\[ \Phi_k \approx A_k(x) \frac{\dot{H}}{H^2} , \]  

with the quantity \( A_k(x) \) depending on \( x \) as well as on \( k \). Comparing this equation to (85), one can find a nontrivial phase factor in \( A_k \):

\[ A_k(x) = A_k e^{-i\tilde{S}_k(\tau_k,x) - i\tilde{T}_k(\tau_k,x)} , \]  

which we can evaluate at the moment \( \tau_k \), defined by the condition \( a|\dot{H}|^{1/2} = k \), because, as we have seen in Sec. II A, the high-frequency approximation for \( \Phi_k \) can be extended to the domain \( k \gtrsim a|\dot{H}|^{1/2} \) [see (26)].\(^6\) The isotropic quantity \( A_k \) was also defined in Sec. II A.

Therefore, at the end of inflation, which we defined by the condition \( |\dot{H}| = H^2 \), we have

\[ \Phi_k(\tau_f, x) \approx -A_k e^{-i\tilde{S}_k(\tau_k,x) - i\tilde{T}_k(\tau_k,x)} . \]  

This result enables us to calculate the power spectrum of the primordial perturbations.

V. POWER SPECTRUM OF THE PRIMORDIAL PERTURBATIONS

A. General expression

The correlation function of the primordial scalar perturbations \( \langle \Phi(x)\Phi(y) \rangle \) is associated with the symmetrized quantum-mechanical average of the corresponding field operator. However, in the theory under consideration, the quantum operators \( a_k \) and \( a_k^\dagger \) enter the field \( \Phi \) not only explicitly in the decomposition (84) but also implicitly in the phases \( \tilde{S}_k \) and \( \tilde{T}_k \) in (85). These phases depend on the field inhomogeneities with wavelengths greatly exceeding the size of the observable universe; therefore, they cannot be included in the quantum averaging procedure. The uncertainty in the correlation function due to the presence of these modes is to be regarded as an instance of cosmic variance.

Using the new expression (91) for the modes, we can write the correlation function at the end of inflation

\[ \langle \Phi(x)\Phi(y) \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} |A_k|^2 e^{ik(x-y)} \times \left[ e^{-i\tilde{S}_k(\tau_k,x) - i\tilde{T}_k(\tau_k,x)} + e^{-i\tilde{S}_{-k}(\tau_k,y) + i\tilde{T}_{-k}(\tau_k,y)} + e^{-i\tilde{S}_k(\tau_k,y) + i\tilde{T}_k(\tau_k,y)} + e^{-i\tilde{S}_{-k}(\tau_k,x) + i\tilde{T}_{-k}(\tau_k,x)} \right] , \]  

\(^6\) The dependence of the phases in (90) on \( \tau_k \) in the neighborhood of the Hubble-radius crossing is very weak, as will become clear later, so the precise choice of the moment of time \( \tau_k \) is not very important.
which we symmetrized with respect to the $x \leftrightarrow y$ interchange.

Since $\tilde{S}_k$ and $\tilde{T}_k$ depend on the modes with wavelengths much exceeding the size of the observable universe, we can expand these quantities in power series in the coordinate distance $x$ in the neighborhood of the observer. Expansions (86), as we said, are, in fact, expansions in powers of $(k\ell)^{-1}$, where $\ell$ is the comoving scale of inhomogeneity of the phases $\tilde{S}_k$ and $\tilde{T}_k$; therefore, we can restrict ourselves to terms at most linear in $x$ in the expansion of $\tilde{S}_0$ and $\tilde{T}_0$ and retain only the zero-order terms in the expansion of $\tilde{S}_1$ and $\tilde{T}_1$ (i.e., evaluate them at $x = 0$, which we take to be the center of the region of observation). This procedure will produce terms linear in the difference $x - y$ in the exponents of (92), so that the coordinate-dependent parts of the phases will take the form $(k - \delta k)(x - y)$. It is then convenient to change the integration variables from $k$ to $k - \delta k$ in (92). After this cumbersome but straightforward procedure (see Appendix A for details), we arrive at the following expression for the correlation function, which we now write in linear approximation in $\Phi_c$ and $h_{ij}^c$:

$$\langle \Phi(x)\Phi(y) \rangle = \int \frac{d^3k}{(2\pi)^3} |A_k|^2 (1 + \nu^S_k + \nu^T_k) e^{ik(x-y)}.$$  

The quantities $\nu^S_k$ and $\nu^T_k$ are the resulting effects of $\tilde{S}_k$ and $\tilde{T}_k$ in (92), i.e., of the scalar and tensor quasiclassical background inhomogeneities, respectively. They are given by the expressions (see Appendix A):

$$\nu^S_k = \frac{1}{2} (n_S - 1) n \cdot \nabla \left[ (S_0)_k - (S_0)_{-k} \right],$$

$$\nu^T_k = \frac{1}{2} (n_S - 1) n \cdot \nabla \left[ (T_0)_k - (T_0)_{-k} \right].$$

Here, we have restored the index $k$ in the components $\tilde{S}_0 \equiv S_0$ and $\tilde{T}_0 \equiv T_0$ and used the definition (38) of the running spectral index. Expressions (94) and (95) are to be evaluated at $\tau = \tau_k$ and $x = 0$.

**B. The form and magnitude of statistical anisotropy**

1. **The scalar part, $\nu^S_k$**

The contribution of the scalar inhomogeneity to the statistical anisotropy, $\nu^S_k$, was described and studied in the previous papers [4]. Here, this analysis will be carried out in more detail. Using (65), (80) and (94), we have

$$\nu^S_k = (n_S - 1) I^S_k,$$

$$I^S_k = 2 \int_{\tau_i}^{\tau_k} d\tau \int \frac{d^3p}{(2\pi)^3} \Phi_p(\tau)(\mathbf{p}) \sin [\mathbf{p}n(\tau_k - \tau)] e^{-p/\epsilon a H} a_{\mathbf{p}} + \text{H.c.},$$
where \( n = k/k \), the scalar product of \( p \) and \( n \) is denoted by \( pn \), and “H.c.” denotes the Hermitian conjugate of the preceding expression.

The variance of \( I_k^S \) is estimated in Appendix B, with the result

\[
\left\langle \left( I_k^S \right)^2 \right\rangle \simeq \frac{\varepsilon^4}{10\pi} G \left( H_i^2 - H_k^2 \right). \tag{98}
\]

Hence,

\[
\left\langle \left( \nu_k^S \right)^2 \right\rangle \simeq \frac{\varepsilon^4}{10\pi} (n_S - 1)^2 G \left( H_i^2 - H_k^2 \right). \tag{99}
\]

The numerical value of this estimate is obtained by using (39) and setting \( \varepsilon \simeq 1 \):

\[
\left\langle \left( \nu_k^S \right)^2 \right\rangle \simeq 5 \times 10^{-5} G \left( H_i^2 - H_k^2 \right) \simeq 5 \times 10^{-5} G H_i^2, \tag{100}
\]

where the last equality assumes \( H_i \gg H_k \). Thus, if we set our initial conditions at the largest possible energy density — the Planck energy density (assuming that inflation starts from this energy density), — we will have \( G H_i^2 \sim \) a few and \( \nu_k^S \sim 10^{-2} \). The results (99) and (100) were first obtained in [4].

Concerning the form of the anisotropic part \( \nu_k^S \), one can see from (96), (97) that it strongly depends on the direction \( n = k/k \) but very weakly on the absolute value of \( k \) for given \( n \). Indeed, the spectral index \( n_S \) depends on \( k \) in a logarithmic way, as can be seen from (39).

The derivative of integral (97) with respect to \( \ln k \) for fixed \( n = k/k \) is given by

\[
\frac{\partial I_k^S_n}{\partial \ln k} = 2 \frac{d\tau_k}{d\ln k} \int_{\tau_i}^{\tau_k} d\tau \int \frac{d^3p}{(2\pi)^3} \Phi_p(\tau) (pn)^2 \cos [pn(\tau_k - \tau)] e^{-p/\varepsilon a H} a_p + \text{H.c.} \tag{101}
\]

The variance of this quantity can be estimated quite similarly to the procedure (B3) of Appendix B using the relation

\[
\frac{d\tau_k}{d\ln k} \simeq \frac{|\dot{H}_k|^{1/2}}{k H_k} \tag{102}
\]

(which one can easily verify), and we obtain the following result:

\[
\left\langle \left( \frac{\partial I_k^S_n}{\partial \ln k} \right)^2 \right\rangle \simeq \frac{\varepsilon^4}{15\pi} G |\dot{H}_k| . \tag{103}
\]

The ratio of the variances is given by

\[
\frac{\left\langle \left( \frac{\partial I_k^S_n}{\partial \ln k} \right)^2 \right\rangle}{\left\langle \left( I_k^S \right)^2 \right\rangle} \simeq \frac{2|\dot{H}_k|}{3 (H_i^2 - H_k^2)} \ll 1 . \tag{104}
\]

This establishes very weak dependence of \( \nu_k^S \) on the absolute value of \( k \) for fixed \( n = k/k \).
Let us now consider the dependence of the anisotropic part \( \nu_k^S \) on the direction \( n = k/k \). To this end, we can expand the sine under the integral in (97) in powers of its argument, thus obtaining the formal series

\[
I_k^S = \sum_{m=0}^{\infty} I_m^S(k, n), \tag{105}
\]

where

\[
I_m^S(k, n) = \frac{2(-1)^m}{(2m+1)!} \int_{\tau_i}^{\tau_k} d\tau (\tau_k - \tau)^{2m+1} \int \frac{d^3p}{(2\pi)^3} \Phi_p(\tau)(pn)^{2(m+1)}e^{-p/\varepsilon aH a_p + H.c.} \tag{106}
\]

This is nothing but a multipole expansion in powers of \( n \).

Proceeding exactly like in the derivation of (B3), we can estimate the variance of every partial term in (105):

\[
\langle (I_m^S)^2 \rangle \simeq \frac{5 \varepsilon^{4m} \langle (I_0^S)^2 \rangle}{(4m+5)(m+1)^2 [(2m+1)!]^2} \tag{107}
\]

One can see that the quantity \( I_k^S \) is dominated by the zeroth term \( I_0^S \), since the next term already has much smaller variance:

\[
\frac{\langle (I_S^S)^2 \rangle}{\langle (I_0^S)^2 \rangle} \simeq \frac{5 \varepsilon^4}{6^4} \approx 0.0039 \varepsilon^4, \tag{108}
\]

so that \( I_1^S \approx 0.06 \varepsilon^2 I_0^S \ll I_0^S \).

2. The tensor part, \( \nu_k^T \)

The contribution of the tensor inhomogeneity to the statistical anisotropy, \( \nu_k^T \), is a new result of the present paper. The relevant expression is given by formula (95) with \( T_0 \) given by (82). One can write

\[
\nu_k^T = (n_S - 1) (I_k^T + J_k^T), \tag{109}
\]

\[
I_k^T = \frac{1}{2} \int_{\tau_i}^{\tau_k} n \cdot \nabla Q [\tau, -n (\tau_k - \tau)] d\tau + \left\{ n \rightarrow -n \right\}, \tag{110}
\]

\[
J_k^T = \frac{1}{2} \int_{\tau_i}^{\tau_k} n \cdot \nabla (\Phi_h + \Psi_h) [\tau, -n (\tau_k - \tau)] d\tau + \left\{ n \rightarrow -n \right\}. \tag{111}
\]

The quantity \( J_k^T \) is calculated by recalling definitions (63) with the result

\[
J_k^T = \frac{1}{2} n^i n^j \int_{\tau_i}^{\tau_k} h_{ij}''(\tau)(\tau_k - \tau) d\tau. \tag{112}
\]

It has only quadrupole dependence on the unit vector \( n \).
The quantity $I_T^k$ is given by

$$
I_T^k = \frac{1}{2} \int_{\tau_i}^{\tau_k} d\tau \int \frac{d^3p}{(2\pi)^3} h_p(\tau) \sum_{\sigma=+,-} e_{ij}(p, \sigma) n^i n^j(p\nu) \sin[p(n_\tau - \tau)] e^{-p/\epsilon a H} a_{p,\sigma} + H.c.
$$

(113)

It can further be expanded in polynomials in the unit vector $n$ in a way similar to (105), (106):

$$
I_T^k = \sum_{m=0}^{\infty} I_T^m(k, n),
$$

(114)

where

$$
I_T^m(k, n) = \frac{(-1)^m}{2(2m+1)!} \int_{\tau_i}^{\tau_k} d\tau (\tau_k - \tau)^{2m+1} \times \int \frac{d^3p}{(2\pi)^3} h_p(\tau) \sum_{\sigma=+,-} e_{ij}(p, \sigma) n^i n^j(p\nu) (2m+1) e^{-p/\epsilon a H} a_{p,\sigma} + H.c.
$$

(115)

The variance of the quadrupole term (112) can be estimated using (48), (53) and (67) as follows:

$$
\langle (J_T^k)^2 \rangle \simeq \frac{16G}{15\pi} \int_{\tau_i}^{\tau_k} d\tau_1 (\tau_k - \tau_1) \int_{\tau_i}^{\tau_1} d\tau_2 (\tau_k - \tau_2) \int_{0}^{\zeta_{eff} a_2 H_2} dp p^3 \frac{H^2_p}{\zeta_{eff}} a_{k} \simeq \frac{2G}{15\pi} \int_{a_k}^{a_k} H^2 \frac{da}{a}.
$$

(116)

This variance is independent of the effective cutoff parameter $\zeta_{eff}$ in Eq. (67) due to the scale invariance of the spectrum of tensor perturbations. For our model (3), using the formulas of Sec. II A, we have

$$
\langle (J_T^k)^2 \rangle \simeq \frac{2G}{15\pi} \int_{a_k}^{a_k} H^2 \frac{da}{a} \simeq \frac{G (H^4_i - H^4_k)}{10\pi m^2_{\phi}} \simeq \frac{GH^4_i}{10\pi m^2_{\phi}},
$$

(117)

where the last estimate takes into account the relation $H^4_i - H^4_k \simeq H^4_i$.

In the same way, one can estimate the variances of the multipole terms (115):

$$
\langle [I_m^T(k, n)]^2 \rangle \simeq \frac{2 \varepsilon^4 (m+1) G}{\pi (4m+5)(4m+7)(4m+9)(m+1)^2 [(2m+1)!]^2} \int_{a_k}^{a_k} H^2 \frac{da}{a} \simeq \frac{3 \varepsilon^4 (m+1) G}{2\pi (4m+5)(4m+7)(4m+9)(m+1)^2 [(2m+1)!]^2} \frac{H^4_i - H^4_k}{m^2_{\phi}}.
$$

(118)

Again we see that the term with $m = 0$ is dominating; for example,

$$
\frac{\langle (I_1^T)^2 \rangle}{\langle (I_0^T)^2 \rangle} \approx 0.0017 \varepsilon^4,
$$

(119)
so that $I^T_T \simeq 0.04 \varepsilon^2 I^T_0 \ll I^T_0$.

Quite similarly we can calculate the variances of the derivatives of the relevant quantities with respect to $\ln k$, which turn out to be relatively very small:

\[
\left\langle \left( \frac{\partial J^T_k}{\partial \ln k} \right)^2 \right\rangle \simeq \frac{4}{15 \pi} G H_k^2 \ll \left\langle (J^T_k)^2 \right\rangle, \tag{120}
\]

\[
\left\langle \left( \frac{\partial I^T_k}{\partial \ln k} \right)^2 \right\rangle \simeq \frac{2 \varepsilon^4}{945 \pi} G H_k^2 \ll \left\langle (I^T_k)^2 \right\rangle. \tag{121}
\]

This establishes very weak dependence of $\nu_k^T$ on the absolute value of $k$ for fixed $n = k/k$.

C. Summary of the main results of this section

In this section, we have found that the large-scale inhomogeneity (and anisotropy) of the inflationary universe due to the generation of quasiclassical scalar and tensor perturbations on super-Hubble spatial scales leads to statistically anisotropic corrections to the power spectrum of the post-inflationary primordial scalar perturbations. The correlation function of the relativistic potential $\Phi$ has the form

\[
\langle \Phi(x) \Phi(y) \rangle = \int \frac{d^3k}{(2\pi)^3} |A_k|^2 (1 + \nu_k) e^{ik(x-y)}, \tag{122}
\]

where $\nu_k = \nu_k^S + \nu_k^T$, and the quantities $\nu_k^S$ and $\nu_k^T$ describe the influence of scalar and tensor quasiclassical inhomogeneities, respectively. The leading contributions to these quantities are given by

\[
\nu_k^S = (n_S - 1) \left( \Lambda^{S}_{ij} n^i n^j + \Lambda^{S}_{ijkl} n^i n^j n^k n^l \right), \tag{123}
\]

\[
\nu_k^T = (n_S - 1) \left( \Lambda^{T}_{ij} n^i n^j + \Lambda^{T}_{ijkl} n^i n^j n^k n^l \right), \tag{124}
\]

in which the quantities $\Lambda^{S,T}_{..}$ very weakly depend on the wave number $k$. One can introduce the combined quantities

\[
\Lambda_{ij} = \Lambda^{S}_{ij} + \Lambda^{T}_{ij}, \quad \Lambda_{ijkl} = \Lambda^{S}_{ijkl} + \Lambda^{T}_{ijkl}, \tag{125}
\]

so that

\[
\nu_k = (n_S - 1) \left( \Lambda_{ij} n^i n^j + \Lambda_{ijkl} n^i n^j n^k n^l \right). \tag{126}
\]

Results (123), (124) or (126) are proportional to $(n_S - 1)$, the deviation of the scalar spectral index from its value for the exactly scale-invariant form.
The quantities $\Lambda^{S,T}_{ij}$ are given by the following expressions:

\[
\Lambda^S_{ij} = 2 \int_{\tau_i}^{\tau_k} d\tau (\tau_k - \tau) \int \frac{d^3P}{(2\pi)^3} \Phi_p(\tau)p_ip_je^{-p/\xi aH}a_p + \text{H.c.} , \tag{127}
\]

\[
\Lambda^S_{ijkl} = -\frac{1}{3} \int_{\tau_i}^{\tau_k} d\tau (\tau_k - \tau)^{3} \int \frac{d^3P}{(2\pi)^3} \Phi_p(\tau)p_ip_kpe^{-p/\xi aH}a_p + \text{H.c.} , \tag{128}
\]

\[
\Lambda^T_{ij} = \frac{1}{2} \int_{\tau_i}^{\tau_k} h''_{ij} (\tau_k - \tau) d\tau , \tag{129}
\]

\[
\Lambda^T_{ijkl} = \frac{1}{2} \int_{\tau_i}^{\tau_k} d\tau (\tau_k - \tau) \int \frac{d^3P}{(2\pi)^3} h_p(\tau) \sum_{\sigma = +, x} e_{ij}(p, \sigma)p_kpe^{-p/\xi aH}a_{p, \sigma} + \text{H.c.} , \tag{130}
\]

where, according to (67), the quantity $h''_{ij}$ is given by

\[
h''_{ij} = \frac{1}{\xi^2} \int \frac{d^3P}{(2\pi)^3} \tau^2h_p(\tau) \sum_{\sigma = +, x} e_{ij}(p, \sigma)e^{-p/\xi aH}a_{p, \sigma} + \text{H.c.} \tag{131}
\]

The variances of the anisotropic parts of the power spectrum are estimated as

\[
\langle (\Lambda^S_{ij} n^i n^j)^2 \rangle = \langle (\Lambda^T_{ij} n^i n^j)^2 \rangle \approx \frac{\xi^4}{10\pi} G(H_i^2 - H_k^2) , \tag{132}
\]

\[
\langle (\Lambda^S_{ijkl} n^i n^j n^k n^l)^2 \rangle = \langle (\Lambda^T_{ijkl} n^i n^j n^k n^l)^2 \rangle \approx \frac{\xi^8}{2592\pi} G(H_i^4 - H_k^4) , \tag{133}
\]

\[
\langle (\Lambda^T_{ij} n^i n^j)^2 \rangle = \langle (\Lambda^T_{ij} n^i n^j)^2 \rangle \approx \frac{G(H_i^4 - H_k^4)}{10\pi m_\phi^2} , \tag{134}
\]

\[
\langle (\Lambda^T_{ijkl} n^i n^j n^k n^l)^2 \rangle = \langle (\Lambda^T_{ijkl} n^i n^j n^k n^l)^2 \rangle \approx \frac{\xi^4 G(H_i^4 - H_k^4)}{210\pi m_\phi^2} . \tag{135}
\]

The total anisotropy is dominated by the tensor contribution, which is clear from the ratio of their variances:

\[
\frac{\langle (\Lambda^S_{ij} n^i n^j)^2 \rangle}{\langle (\Lambda^T_{ij} n^i n^j)^2 \rangle} \approx \frac{\xi^4 m_\phi^2}{H_i^2} \ll 1 , \quad \frac{\langle (\Lambda^S_{ijkl} n^i n^j n^k n^l)^2 \rangle}{\langle (\Lambda^T_{ijkl} n^i n^j n^k n^l)^2 \rangle} \approx 10^{-1} \frac{\xi^4 m_\phi^2}{H_i^2} \ll 1 . \tag{136}
\]

Within the tensor (as well as within the scalar) contributions, the quadrupole component dominates:

\[
\frac{\langle (\Lambda^T_{ijkl} n^i n^j n^k n^l)^2 \rangle}{\langle (\Lambda^T_{ij} n^i n^j)^2 \rangle} \approx \frac{\xi^4}{21} , \quad \frac{\langle (\Lambda^S_{ij} n^i n^j)^2 \rangle}{\langle (\Lambda^S_{ij} n^i n^j)^2 \rangle} \approx \frac{\xi^4}{250} . \tag{137}
\]

The full traces of the symmetric tensors $\Lambda^T_{ij}$ and $\Lambda^T_{ijkl}$ are equal to zero, so that the quantity $\nu^T_k$ being averaged over the directions of $\mathbf{n}$ gives zero. On the contrary, the quantity $\nu^S_k$ gives a nonzero contribution to the isotropic part of the power spectrum:

\[
\nu^S_k \equiv \frac{1}{4\pi} \int \nu^S_k d\Omega_n = (n_\perp - 1) \left( \frac{1}{3} \Lambda^S_i + \frac{1}{5} \Lambda^S_{ij} \right) , \tag{138}
\]

which, however, is rather small in view of the estimate (100).
VI. NUMERICAL ESTIMATES AND THE TRANS-PLANCKIAN ISSUE

Observing the variances (132)–(135), one can see that they depend crucially on the value $H_i$ of the Hubble parameter at the time $\tau_i$, which is the moment of time where the initial conditions for quantum fluctuations are specified as $S_k(\tau_i, x) = T_k(\tau_i, x) \equiv 0$ (see Sec. IV B). The evolution of the scalar modes is traced beginning from this moment of time roughly until the moment $\tau_k$, which, we remember, is defined by the condition $a_k|\dot{H}_k|^{1/2} = k$; this is roughly the moment of the Hubble-radius crossing at the inflationary stage.

The question is: at what time $\tau_i$ can one set the initial conditions? First of all, we remember that our calculation is based on perturbation theory in terms of the quasiclassical inhomogeneities; this implies that we can use our expressions (132)–(135) without significant corrections as long as they are smaller than unity. The largest of them, (134), then determines the highest initial value for the Hubble parameter:

$$\frac{GH_i^4}{10\pi m_{\varphi}^2} \lesssim 1 \quad \Rightarrow \quad \varphi_i^2 \lesssim \frac{3\sqrt{10}}{4\sqrt{\pi G^{3/2} m_{\varphi}}} \simeq \varphi_\ast^2,$$

where $\varphi_\ast$ is the typical value of the inflaton field that determines the boundary of the self-regenerating inflation, as described in Sec. IV A [see Eq. (58)]. As noted in that section, this is still much lower than the boundary of the Planckian energy density [see (59)]. Thus, we might expect the variance (134) of the quadrupole part of the statistical anisotropy to be of any value between zero and unity, while that of the next leading term (135) to be an order of magnitude smaller. For $\varphi_i \sim \varphi_\ast$, variances (132) and (133) of the contributions from scalar inhomogeneities are small and can be neglected:

$$\left\langle \left( \Lambda_{ij}^i n^j \right)^2 \right\rangle \simeq \frac{\varepsilon^4}{10\pi} G \left\langle n^i \right\rangle^2 \simeq \frac{\varepsilon^4}{\sqrt{10\pi}} G^{1/2} m_{\varphi} \lesssim 10^{-7}. \quad (140)$$

The dispersion of the anisotropic part in this case is due to the contribution from tensor inhomogeneity and is estimated as $\sigma_{\nu_k} \sim (n_S - 1) \simeq 0.04$. We emphasize, however, that this is only the value at which our perturbative approach will possibly require corrections; in reality, if $\phi_i \gg \phi_\ast$, the anisotropy may be larger.

There is another important issue that affects the resulting statistical anisotropy. The point is that the physical wavelength $\lambda = a/k$ of a particular mode is much smaller than the Planck length $\ell_P \simeq G^{1/2}$ during most part of the inflationary universe. This situation is illustrated in Fig. 1. A natural question arises whether it is legitimate to use field theory in

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7 The scalar and tensor contributions to the anisotropy become equal by the order of magnitude if the initial value of the Hubble parameter is as low as $H_i \simeq m_{\varphi}$, which is very close to the end of inflation. Thus, in all interesting cases, the dominating contribution to the statistical anisotropy comes from tensor inhomogeneities.
cosmology on spatial scales below the Planck length. This is the so-called trans-Planckian issue. In the context of the origin of statistical anisotropy, it was pointed out in our papers [4], and recently it was a subject of discussion in several works in connection with the possible modification of the (statistically isotropic) primordial power spectrum [8] and cosmological particle creation [9].

To see how this consideration affects our results, let us calculate the values of the inflaton field \( \varphi_k \) (at the moment of Hubble-radius crossing, where, according to our definition, \( k = a|\dot{H}|^{1/2} \)) and \( \varphi_{Pk} \) (at the moment where the physical wave number crosses the Planck boundary, \( k \approx aG^{-1/2} \)) for our quadratic inflaton potential. Using the formulas of Sec. II A, we have

\[
4\pi G \varphi_k^2 \approx 2 \ln \frac{a_f H_f}{k} \approx 100, \quad 4\pi G \varphi_{Pk}^2 = 4\pi G \varphi_k^2 + \ln \frac{3}{Gm_\varphi^2} \approx 128. \tag{141}
\]

Together with (34), these estimates imply that the Planck-radius crossing occurs only about 2\(\pi G (\varphi_{Pk}^2 - \varphi_k^2) \approx 14 \) e-foldings prior to the Hubble-radius crossing during inflation. The difference between \( \varphi_{Pk} \) and \( \varphi_k \) is rather small, and, if we set the isotropic initial conditions for the modes at the Planck-radius crossing, so that \( \varphi_i = \varphi_{Pk} \), then, for variance (134) of the leading contribution to the anisotropy, we will have

\[
\left\langle \left( \Lambda_{ij} n^i n^j \right)^2 \right\rangle \approx \frac{G (H_{Pk}^4 - H_k^4)}{10\pi m_\varphi^2} \approx 10 Gm_\varphi^2 \approx 10^{-11}, \tag{142}
\]
which is extremely small. The statistical anisotropy in this case is negligible: $\langle \nu_k^2 \rangle \sim 10^{-14}$.

There are several arguments in favor of the possibility of working in the trans-Planckian frequency region. First of all, one can note that the wavelength and frequency of a particular mode are not Lorentz-invariant quantities and can locally be changed to any value by the transformation of coordinates. Thus, the authors of [9] argue that nothing dangerous results from this procedure in a locally Lorentz-invariant theory as long as the invariant $k^\mu k_\mu \ll M_P^2$. It is also pointed out there that a similar trans-Planckian problem arises formally in the calculation of electron-positron pair creation in a constant electric field $E_0$ if the gauge $A = -E_0 t$ is used [23]; in this case, in order to obtain correct results for the effective action, it is even obligatory to consider field modes with arbitrarily high momenta. Similar arguments in cosmological context are given in [24], and the authors of [25] consider the trans-Planckian effects in quantum gravity and argue in favor of the asymptotic safety in this region.

The trans-Planckian issue can be put in a slightly different way. Given that there exists a formal moment of Planck-radius crossing for every mode (see Fig. 1), one can ask whether the quantum state in which a particular mode finds itself at this moment of time will be sensitive to the past history of the inflationary expansion. The unknown physics taking place in the trans-Planckian region might have lead to an effect similar to that of our field-theory consideration, producing the modes $\chi_k$ and $\Phi_k$ at the Planck-radius crossing similar to those calculated in this paper. This issue will hopefully be clarified in a reliable quantum theory capable of dealing with sub-Planckian spatial scales.

One can note that quite a similar trans-Planckian problem arises in the theory of quantum radiation from a black hole which is formed as a result of gravitational collapse: since all outgoing modes originate as incoming modes that arrive at the black hole before the formation of the event horizon, the frequency of the incoming modes diverges as the time of the corresponding outgoing modes goes to infinity (see [26] for a review). While the prediction of quantum thermal radiation of black holes hardly can be tested experimentally because of its extremely small value for black holes with realistic masses, we have a different situation in the case of the cosmological prediction made in this paper, which, in principle, can be verified by observations. We discuss such testable predictions in the next section.

VII. OBSERVABLE EFFECTS

Statistical anisotropy of the primordial power spectrum will broadly be manifest in the large-scale structure and cosmic microwave background. The first effect was discussed in our previous paper [4], while the statistical anisotropy of the CMB was a subject of a number of recent studies [10–16].
A. Statistical anisotropy of the LSS

Since the leading contribution to the statistical anisotropy is of quadrupole form [the corresponding variance is given by (134)], the formulas of [4] are directly applicable to the present case. There is a linear homogeneous and isotropic connection between perturbations in the relativistic potential $\Phi$ and density perturbations $\delta = \delta \rho / \rho$ on sub-Hubble spatial scales. Therefore, the anisotropic factor $\nu_k$ is inherited in the power spectrum of the density contrast. The corresponding correlation function is given by

$$\xi(x) = \xi_0(x) + \xi_1(x), \quad (143)$$

where

$$\xi_0(x) = \int \frac{d^3k}{(2\pi)^3} \delta_k^2 e^{i k x} = \frac{1}{2\pi^2} \int d k k^2 \delta_k^2 \frac{\sin k x}{k x} \quad (144)$$

and

$$\xi_1(x) = \int \frac{d^3k}{(2\pi)^3} \delta_k^2 \nu_k e^{i k x} \quad (145)$$

are the isotropic and anisotropic parts, respectively. The anisotropic quantity $\nu_k$ is dominated by the first term in the brackets of (126) in which $\Lambda_{ij} \approx \Lambda_{ij}^T$. In view of the fact that the product $(n_S - 1) \Lambda_{ij}$ depends very weakly on the wave number $k$, one can treat it as a constant and take out of the integral in (145). Since the matrix $\Lambda_{ij}^T$ is traceless, one gets the following expression for the anisotropic part of the correlation function:

$$\xi_1(x) \approx (n_S - 1) \Lambda_{ij} f(x) \hat{x}^i \hat{x}^j, \quad (146)$$

where $\hat{x} = x / x$ and

$$f(x) = \xi_0(x) + \frac{3}{2\pi^2} \int d k k^2 \delta_k^2 \frac{k x \cos k x - \sin k x}{(k x)^3}. \quad (147)$$

The anisotropic part of the correlation function has the specific form (146), (147) which, in principle, can be tested by observations. Measurement of the quadrupole of the power spectrum can simply be done by comparing the amplitudes of Fourier modes in different directions. The standard error to the quadrupole coefficients $q_{ij} = (n_S - 1) \Lambda_{ij}$ using the SDSS data is estimated in [12] to be $\sigma_q \sim 10^{-2}$.

The velocities on large spatial scales are connected with the relativistic potential as $v \propto \nabla \Phi$. Therefore, one can also test the statistical anisotropy by using the velocity correlation function $\sigma_{ij}(x) = \langle v_i(x_0 - \hat{x}) v_j(x_0 + \hat{x}) \rangle$. Of a particularly simple form is the dispersion of the velocity components at one point:

$$\sigma_{ij}(0) = \langle v_i(x_0) v_j(x_0) \rangle = \frac{1}{3} \langle v^2 \rangle \left( \delta_{ij} + \frac{2}{3} \Lambda_{ij} \right). \quad (148)$$
This relation is valid on large scales, where the evolution of density perturbations is still in the linear regime, and, in principle, can be used to determine the matrix $\Lambda_{ij}$.

The effect of nonlinear evolution of initially anisotropic spectrum was recently under investigation in [27]; the authors reported the suppression of primordial anisotropy by $\sim 7\%$ in quasilinear regime on small spatial scales, $k \sim 0.1\text{Mpc}^{-1}$.

### B. Statistical anisotropy of the CMB

Statistical anisotropy of the primordial power spectrum in the form (126) falls into the category that was recently studied in detail in [12]. It is given by the general formula

$$P_k = P_k \left[ 1 + \sum_{LM} g_{LM}(k) Y_{LM}(\mathbf{n}) \right],$$

(149)

where $P_k$ is the isotropic part of the anisotropic power spectrum $P_k$ of the density profile $\delta = \delta \rho / \rho$, and $g_{LM}(k)$ are the coefficients of the expansion of the statistical anisotropy into spherical harmonics $Y_{LM}(\mathbf{n})$, where, as usual, $\mathbf{n} = \mathbf{k} / k$. In our case, the sum in (149) is dominated by harmonics with $L = 2$, with coefficients $g_{2M} \sim (nS - 1) \Lambda_{ij}$ that depend on $k$ very weakly.

The primordial inhomogeneity described by the power spectrum (149) leads to the anisotropic distribution of the CMB temperature $T(\hat{n})$ on the sky; the relevant relation can be expressed as follows:

$$T(\hat{n}) = T_0 \int \frac{d^3k}{(2\pi)^3} \sum_l (-i)^l (2l + 1) P_l(\mathbf{k} \cdot \hat{n}) \Theta_l(k) \delta_k,$$

(150)

where $\hat{n}$ denotes a direction on the sky, and $\Theta_l(k)$ is the kernel of the linear relation between $T(\hat{n})$ and $\delta_k$ describing the contribution to the $l$th temperature moment from the wave vector $\mathbf{k}$ [with these conventions, $\Theta_l(k)$ is real].

After expansion of the temperature map $T(\hat{n})$ into spherical harmonics

$$T(\hat{n}) = T_0 \sum_{lm} a_{lm} Y_{lm}(\hat{n}),$$

(151)

one can calculate the expectation value (covariance matrix)

$$\langle a_{lm} a_{l'm'}^* \rangle = \delta_{ll'} \delta_{mm'} C_l + \sum_{LM} \xi_{LM}^{l'm'} D_{ll'}^{LM},$$

(152)

with

$$C_l = \frac{2}{\pi} \int_0^\infty dk k^2 P_k \Theta_l^2(k),$$

(153)
\[ D_{l'l'}^{LM} = \frac{2(-i)^{l-l'}}{\pi} \int_0^\infty dk k^2 P_k g_{LM}(k) \Theta_l(k) \Theta_{l'}(k), \]  
(154)

\[ \xi_{lm'm'}^{LM} = \int d\hat{n} Y_{lm}^*(\hat{n}) Y_{l'm'}(\hat{n}) Y_{LM}^*(\hat{n}). \]  
(155)

The quantities \( D_{l'l'}^{LM} \) are the generalization of the coefficients \( C_l \) to the case of statistical anisotropy. One can note that, in the case where the dependence of \( g_{LM}(k) \) on \( k \) can be neglected, there arises the relation \( D_{l'l'}^{LM} = g_{LM} W_{l'l'} \) with \( W_{l'l'} \) depending only on the isotropic cosmology (in particular, \( W_{l'l'} = C_l \)). Thus, the combinations \( D_{l'l'}^{LM} / W_{l'l'} \) can be regarded as convenient estimators for the coefficients \( g_{LM} \) in this case. The estimators for \( D_{l'l'}^{LM} \) were constructed in [12] and essentially coincide with the bipolar-spherical-harmonic coefficients of [10]. For the case \( L = 2 \) of interest for our work, it is shown in [12] that the variances with which the quantities \( g_{2M} \) can be measured using the temperature maps of WMAP and Planck are \( \sigma_g \sim 1.2 \times 10^{-2} \) and \( 3.8 \times 10^{-3} \), respectively. This is comparable to the highest theoretical estimates that we get within the limits of our approximation, linear in the background inhomogeneity of the inflationary universe. In a complete nonlinear calculation, therefore, the resulting statistical anisotropy may well exceed the current threshold of its detection.

Our theory does not predict any significant dipole statistical asymmetry of the CMB or LSS, which is a simple consequence of the fact that the correlator \( \langle \Phi(x) \Phi(y) \rangle \) in our approximation is a symmetric function of the difference \( x - y \). In fact, statistical inhomogeneity in our theory would be obtained in the next order of expansion of the phases in (92) in powers of \( x \) and \( y \), and one can show that this effect is very small. Specifically, the spatial variance of the quantity \( \nu_k \) can be estimated as \( \langle (x \cdot \nabla \nu_k)^2 \rangle \sim (n_S - 1)^2 (k x)^2 G m_\phi^2 \sim 10^{-15} (k x)^2 \). Even if one takes the comoving length \( x \) as large as the size of the observable universe, \( x \sim 3000 \) Mpc, and comoving scale \( k \) as large as \( k \sim 1 \) Mpc\(^{-1} \), one gets the estimate \( \langle (x \cdot \nabla \nu_k)^2 \rangle \sim 10^{-8} \). In particular, this implies that our theory by itself does not predict any significant dipole modulation of the CMB power spectrum, which is suggested to explain the observational evidence of the hemispherical power asymmetry in the WMAP data [13] (see [14] for a recent discussion of this issue). However, the quadrupole modulation of the CMB sky, which is present in our theory, together with anisotropy in CMB data (caused, e.g., by the cut-sky masks used to exclude foregrounds, see [15]), might be responsible for this effect. Indeed, in this case, effectively nonantipodal regions of the sky may be involved in the comparison of the power spectra of two antipodal hemispheres, and then the quadrupole asymmetry may produce the effect of antipodal asymmetry. In our case, the effect is described by an almost scale-independent traceless matrix \( \Lambda_{ij} \), hence, it is expected to exhibit universal axes of asymmetry for statistical anisotropy on all angular scales. This question remains to be investigated in all detail.
In this paper, we considered the evolution of small-scale quantum fluctuations on the background of an inhomogeneous inflationary universe and calculated the resulting primordial power spectrum. The background metric inhomogeneities during inflation are generated in a self-consistent way on super-Hubble spatial scales.

We have seen that the resulting power spectrum of scalar primordial perturbations acquires the statistically anisotropic part and can be put in the form (1), (2). The quantities $\Lambda_{ij}$ and $\Lambda_{ijkl}$ in (2) very weakly depend on the wave number $k$. Being a result of metric inhomogeneities generated during inflation, they represent Gaussian random variables with zero average; in our universe, we observe only one particular realization of these quantities, hence, we can only speak about their statistical properties.

The variance (which characterizes the expected magnitude) of the statistical anisotropy crucially depends on the moment of time at which one specifies the initial conditions for the modes describing the vacuum state. It is here that we encounter the trans-Planckian issue, which we discussed in Sec. VI. If one specifies locally homogeneous and isotropic initial conditions for quantum modes at the moment of Planck-radius crossing, then the statistical anisotropy which we are talking about turns out to be very small, with variance $\langle \nu_k^2 \rangle \sim 10^{-14}$. However, one can argue in favor of the possibility of working in the trans-Planckian domain of wave numbers (assuming exact local Lorentz invariance) and specify the initial conditions much earlier along the inflationary epoch. In this case, the expected value of the statistical anisotropy can be appreciable; our perturbative approach breaks down at the values $\langle \nu_k^2 \rangle \sim 10^{-3}$; in reality, the anisotropy might be considerably higher and might be detectable.

Where the statistical anisotropy is sufficiently high, it is mainly caused by the influence from the background metric inhomogeneities of tensor type. In this case, the full traces of the tensors $\Lambda_{ij}$ and $\Lambda_{ijkl}$ in (2) are equal to zero, and the ratio of the corresponding variances is estimated in (137) to be

$$\frac{\langle \Lambda_{ijkl} n^i n^j n^k n^l \rangle^2}{\langle \Lambda_{ij} n^i n^j \rangle^2} \lesssim \frac{1}{21}.$$  

(156)

The specific scale-invariant form (2), dominated by the quadrupole term, might facilitate the search for the predicted statistical anisotropy in the large-scale structure and cosmic microwave background. Its detection may be regarded as another nontrivial confirmation of the chaotic inflation scenario and elucidate some aspects of the trans-Planckian issue.
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APPENDIX A: DERIVATION OF EQ. (93)

Since the contributions from $\tilde{S}_k$ and $\tilde{T}_k$ add up in the linear approximation, we illustrate the derivation of (93) from (92) by considering only one of these quantities. Consider then the integral

$$I = \int \frac{d^3k}{(2\pi)^3} |A_k|^2 e^{i k(x-y) - i\tilde{T}_k(\tau_k, x) + i\tilde{T}_k^*(\tau_k, y)}.$$  \hspace{1cm} (A1)

According to the procedure described in Sec. VA, we expand the exponent into a series in the spatial coordinates retaining the leading terms:

$$\tilde{T}_k(\tau_k, x) \approx k\tilde{T}_0^0(\tau_k, 0) + k x \cdot \nabla \tilde{T}_0^0(\tau_k, 0) + i\tilde{T}_1^0(\tau_k, 0).$$  \hspace{1cm} (A2)

The exponent of (A1) becomes

$$i k(x-y) - i\tilde{T}_k(\tau_k, x) + i\tilde{T}_k^*(\tau_k, y) \approx i(k - \delta k)(x-y) + 2T_1(\tau_k, 0) + 2n \cdot \nabla T_0(\tau_k, 0) + 2Q(\tau_k, 0),$$  \hspace{1cm} (A3)

where

$$\delta k = k\nabla T_0(\tau_k, 0),$$  \hspace{1cm} (A4)

and we have used (88). In what follows, we omit the argument $(\tau_k, 0)$ which is going to be present in all functions. The integral (A1) then becomes, in the linear approximation in the quasiclassical field $h^c_{ij}$,

$$I = \int \frac{d^3k}{(2\pi)^3} |A_k|^2 (1 + 2T_1 + 2n \cdot \nabla T_0 + 2Q) e^{i(k - \delta k)(x-y)}.$$  \hspace{1cm} (A5)

Now we change the integration variables from $k$ to $k - \delta k$. We have to take into account the transformation of the amplitude $|A_k|$ and the Jacobian of transformation to new variables, all in linear approximation with respect to $\delta k$ defined in (A4). The first part gives

$$|A_{k+\delta k}|^2 = |A_k|^2 + \frac{d|A_k|^2}{dk} \delta k = |A_k|^2 \left( 1 + \frac{d\ln |A_k|^2}{d\ln k} n \cdot \nabla T_0 \right).$$  \hspace{1cm} (A6)

The Jacobian in linear approximation is calculated using (79), (82) and (83) as follows:

$$J = 1 + \text{tr} \frac{\partial \delta k}{\partial k} = 1 + n \cdot \nabla T_0 - 2T_1 - Q + [n \cdot \nabla Q - (n \cdot \nabla)^2 T_0] \frac{d\tau_k}{d\ln k}.$$  \hspace{1cm} (A7)
Combining (A5), (A6) and (A7), we obtain

\[ I = \int \frac{d^3k}{(2\pi)^3} |A_k|^2 \left[ 1 + \mu_k^T \right] e^{ik(x-y)}, \tag{A8} \]

where

\[ \mu_k^T = \frac{d\ln (k^3|A_k|^2)}{d\ln k} n \cdot \nabla T_0 + Q + \left[ n \cdot \nabla Q - (n \cdot \nabla)^2 T_0 \right] \frac{d\tau_k}{d\ln k}. \tag{A9} \]

Considering the second integral in (92), we arrive at an expression similar to (A8) but with \( \mu_{k-k}^T \) replacing \( \mu_k^T \). By adding these two integrals with the factors \( 1/2 \), we obtain

\[ I_T = \int \frac{d^3k}{(2\pi)^3} |A_k|^2 (1 + \nu_k^T) e^{ik(x-y)}, \tag{A10} \]

where

\[ \nu_k^T = \frac{1}{2} \left( \mu_k^T + \mu_{k-k}^T \right) = Q + \frac{1}{2} (n_S - 1) n \cdot \nabla \left[ (T_0)_k - (T_0)_{-k} \right] - \frac{1}{2} (n \cdot \nabla)^2 \left[ (T_0)_k + (T_0)_{-k} \right] \frac{d\tau_k}{d\ln k}. \tag{A11} \]

The variance of the local quantity \( Q \) can be estimated using (66) and (53), and we get

\[ \langle Q^2 \rangle \simeq \int \frac{d^3p}{(2\pi)^3} |h_p(\tau_k)|^2 \simeq GH_k^2 \ln \frac{\xi}{\zeta} \simeq G^2m_{\phi k}^2 \rho_k \simeq 10^{-12}. \tag{A12} \]

Because of its smallness, the local term \( Q \) can be neglected in (A11).

The last term in (A11), containing the derivative \( d\tau_k/d\ln k \simeq |\dot{H}_k|^{1/2}/kH_k \), arises from the Jacobian (A7) as a result of the dependence of \( T_0 \) on the time \( \tau_k \). This term is also much smaller than the second, leading term in (A11), which should be clear from the fact that it is of higher order in the expansion parameter \( (k\ell)^{-1} \). One can calculate the variance of this term to find that it is of the order \( Gm_{\phi k}^2 \sim 10^{-12} \). Thus, we should drop this term as well from the correction to the power spectrum. Eventually, we get our result (95).

Quite similarly, one obtains contribution (94) from the phase \( \tilde{S}_k \). The local quantity \( \Phi_c \), which is analogous to \( Q \) in (A11), as well as the expression similar to the last term in (A11) are both much smaller than the leading term (94).

In principle, expression (93) for the correlation function should be revised in view of the fact that the spatial coordinates \( x \) entering the metric at the end of inflation are not exactly Euclidean because of the presence of the large-scale quasiclassical inhomogeneities in metric (64). However, proceeding to the Euclidean coordinates will generate additional local terms of the form and order of \( \Phi_c \) and \( Q \) in the anisotropic part of the correlation function, which thus also can be neglected.
APPENDIX B: ESTIMATES OF THE VARIANCES

The variance of integral (97) is given by

\[ \left\langle (I_k^S)^2 \right\rangle = 4 \int_{\tau_i}^{\tau_k} d\tau_1 \int_{\tau_i}^{\tau_k} d\tau_2 \times \int \frac{d^3 p}{(2\pi)^3} \Phi_p(\tau_i) \Phi_p^*(\tau_2) (\mathbf{p} n) \sin |\mathbf{p} n(\tau_k - \tau_1)| \sin |\mathbf{p} n(\tau_k - \tau_2)| e^{-p \left( \frac{1}{a_1 H_1} + \frac{1}{a_2 H_2} \right)}, \quad (B1) \]

where \( a_1 = a(\tau_1) \), \( H_1 = H(\tau_1) \) etc. The cutoff value of momentum integration is sufficiently low, so that the argument of the sine in the effective integration region satisfies

\[ |\mathbf{p} n(\tau_k - \tau)| < \varepsilon a H(\tau_k - \tau) \lesssim 1. \quad (B2) \]

Taking into account (29) and (31), we can estimate variance (B1) as follows:

\[ \left\langle (I_k^S)^2 \right\rangle \approx 4 \int_{\tau_i}^{\tau_k} d\tau_1 \frac{H_1}{H_1^2} (\tau_k - \tau_1) \int_{\tau_i}^{\tau_k} d\tau_2 \frac{H_2}{H_2^2} (\tau_k - \tau_2) \int \frac{d^3 p}{(2\pi)^3} |A_p|^2 (\mathbf{p} n)^4 e^{-p \left( \frac{1}{a_1 H_1} + \frac{1}{a_2 H_2} \right)} \]

\[ = \frac{2}{5\pi^2} \int_{\tau_i}^{\tau_k} d\tau_1 \frac{H_1}{H_1^2} (\tau_k - \tau_1) \int_{\tau_i}^{\tau_k} d\tau_2 \frac{H_2}{H_2^2} (\tau_k - \tau_2) \int_0^{\infty} p^2 dp |A_p|^2 p^4 e^{-p \left( \frac{1}{a_1 H_1} + \frac{1}{a_2 H_2} \right)} \]

\[ \approx \frac{1}{10\pi^2} \int_{\tau_i}^{\tau_k} d\tau_1 \frac{H_1}{H_1^2} (\tau_k - \tau_1) \int_{\tau_i}^{\tau_k} d\tau_2 \frac{H_2}{H_2^2} (\tau_k - \tau_2) (\varepsilon a_2 H_2)^4 \frac{H_2^4}{\dot{\varphi}^4} \]

\[ = \frac{8\varepsilon^4 G^2}{5} \int_{\tau_i}^{\tau_k} d\tau_1 \frac{\dot{\varphi}^4}{H_1^2} (\tau_k - \tau_1) \int_{\tau_i}^{\tau_k} d\tau_2 (\tau_k - \tau_2) a_2^4 H_2^6, \quad (B3) \]

where, in the last equality, we have used equations (6) and (7). Now using the approximate relation

\[ \tau_k - \tau \approx \frac{1}{a H} - \frac{1}{a_k H_k}, \quad (B4) \]

and taking into account the slow variation of the Hubble parameter as a function of the scale factor \( a \) during inflation, we get the estimate

\[ \left\langle (I_k^S)^2 \right\rangle \approx \frac{4\varepsilon^4 G^2}{5} \int_{a_i}^{a_k} \dot{\varphi}^2 \frac{da}{a} \approx \frac{4\varepsilon^4 G^2}{5} \int_{\varphi_i}^{\varphi_k} \dot{\varphi}^2 H dt \approx \frac{4\varepsilon^4 G^2}{15} \int_{\varphi_i}^{\varphi_k} \frac{dV(\varphi)}{d\varphi} d\varphi \]

\[ = \frac{4\varepsilon^4 G^2}{15} (V_i - V_k) \approx \frac{\varepsilon^4}{10\pi} G \left( H_i^2 - H_k^2 \right). \quad (B5) \]

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