QCD Finite Energy Sum Rules
and the Isoscalar Scalar Mesons

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Abstract

We apply QCD Finite Energy Sum Rules to the scalar-isoscalar current to determine the lightest $u\bar{u}+d\bar{d}$ meson in this channel. We use ‘pinch-weights’ to improve the reliability of the QCD predictions and reduce the sensitivity to the cut-off $s_0$. A decaying exponential is included in the weight function to allow us to focus on the contribution from low mass states to the phenomenological integral. On the theoretical side we include OPE contributions up to dimension six and a contribution due to instantons taken from the Instanton Liquid Model. Phenomenologically, we incorporate experimental data by using a coupling scheme for the scalar current which links the vacuum polarisation to the $\pi\pi$ scattering amplitude via the scalar form factor. We find that the sum rules are well saturated for certain instanton parameters. We conclude that the $f_0(400-1200)$ definitely contains a large $u\bar{u}+d\bar{d}$ component, whereas the $f_0(980)$ most likely does not. We are able to estimate the average light quark mass and find $m_q(1\text{ GeV}^2) = 5.2 \pm 0.6$ MeV.
1 Introduction

Currently, experiment suggests that there are more scalar mesons than can be accommodated in a single quark-model nonet. Many QCD motivated models predict the existence of non-$q\bar{q}$ mesons such as glueballs [1], hybrids [2], $qq\bar{q}\bar{q}$ [3] and $K\bar{K}$ [4] states. It is tempting to say that these ‘spare’ experimental states are the unconventional mesons allowed by QCD but, with scalar glueball and multiquark states expected to have masses comparable to the conventional $q\bar{q}$ states, which are the extra states? In the work that follows we hope to shed some light on this question by attempting to determine which of the 5 scalar-isoscalar mesons currently listed in the Particle Data Group tables [5] is the lightest $u\bar{u} + d\bar{d}$ meson in this channel. To address this question we will use the QCD sum rule technique. The answer has importance for chiral symmetry breaking in QCD, see for instance [6, 7].

QCD sum rules are integral expressions that relate the hadronic and partonic regimes, i.e. the low energy world of resonances with the high energy world of (tractable) QCD. Since their conception over twenty years ago [8], QCD sum rules have become an established technique both for calculating hadronic properties in channels, where the QCD expressions are under control, and, conversely, estimating QCD parameters (such as the masses of the quarks) in channels where there is good experimental information.

Using QCD sum rules in the scalar channel is not a new idea. They were first applied to the scalar mesons in [9] where Laplace sum rules were used (for a recent review of Laplace Sum Rules see [10]). Here the phenomenological side was represented using the, now standard, resonance + continuum ansatz, with resonances being represented as $\delta$-functions and the continuum being calculated entirely within perturbative QCD. As only the Operator Product Expansion (OPE) terms, discussed in Sect. 3, were taken into account on the theory side, exact mass degeneracy between the lightest isoscalar and isovector was found, with $m_{f_0} = m_{a_0} = 1.00 \pm 0.03$ GeV. The $s\bar{s}$ state was predicted to have a mass of around 1.35 GeV. These findings were supported by Bramon and Narison [11], who used QCD sum rules to calculate the couplings of the $f_0(980)$ and $a_0(980)$ to two photons. Again modelling the resonances with $\delta$-functions the authors concluded that a $q\bar{q}$ interpretation of these two mesons could not be ruled out with the data then available.

The effects of going beyond the OPE were studied in [12], where instantons (see Sect. 4) were included on the theoretical side, but not in the QCD continuum on the phenomenological side. Once again the resonances were treated as $\delta$-functions and only the isospin-1 channel was considered. It was concluded that the lightest $q\bar{q}$ state in this channel had a mass $\lesssim 1$ GeV.
For the light scalar mesons the zero-width approximation is not a good one and the first attempt to go beyond it in a sum rule investigation was by Elias et al. Here resonances were represented by Breit-Wigner formulae, which, whilst better than a δ-function, still do not adequately describe the complex structure in the scalar sector. A further abstraction was introduced by replacing these Breit-Wigner shapes with a Riemann sum of rectangular pulses. Laplace sum rules were used, but in an integral form which requires the calculation of perturbative expressions at rather low energy scales. Again instantons were included, but not in the continuum. In the isovector channel, the \( a_0(980) \) was found to decouple from the sum rule and the mass of the lightest quarkonium isovector was consistent with the \( a_0(1450) \). In the isoscalar channel the conclusions were not so clear cut, it was predicted that the lightest quarkonium here should have a width less than half of its mass, and although the \( f_0(980) \) could not be ruled out as the main contributor a lighter state was preferred. Similar conclusions were reached in [14]. This study used both Laplace and Finite Energy sum rules, again requiring perturbative expressions to be evaluated at low energies. The authors noted, that for a consistent treatment of the Borel sum rules, the instanton effects should also be included in the QCD continuum contribution. The sum rules were dominated by an isoscalar with mass around 1 GeV and an isovector with mass around 1.5 GeV, thus suggesting the \( f_0(980) \) and \( a_0(1450) \) as the lightest \( q\bar{q} \) states in their respective channels. However, in the same work, a comparison to a more realistic resonance shape predicted resonance parameters for the lightest quarkonium of \( m \sim 860 \text{ MeV} \) and \( \Gamma \sim 340 \text{ MeV} \). These parameters are not consistent with the \( f_0(980) \), but could describe a Breit-Wigner fit to the \( f_0(400-1200) \).

In [15], Maltman used pinched weight finite energy sum rules to calculate the decay constants of the \( a_0(980) \) and \( a_0(1450) \), which were then compared with the decay constant of the, presumably \( q\bar{q}, K_0^* (1430) \). Instantons were included and all QCD integrals were carried out in the complex plane, which improves the convergence of perturbative expressions. The isovector decay constants were found to be of comparable size, suggesting a similar structure for both. The author concluded that this favoured a Unitarised Quark Model [16] scenario, where two physical hadrons can arise from one ‘seed’ state.

In the work that follows we will apply QCD Finite Energy Sum Rules introduced in Sect. 2 to the scalar-isoscalar channel. On the theoretical side we will include in Sects. 3 and 4 instanton effects as well as the OPE terms up to dimension six and use the contour improvement prescription to increase the convergence of perturbative expressions. On the phenomenological side in Sect. 5 we will incorporate experimental data directly by relating the hadronic vacuum polarisation to the \( \pi\pi \) scattering amplitude via the pion’s scalar form-factor. The sum rules are evaluated in Sect. 6 and we present our conclusions in Sect. 7.
2 The Sum Rules

The connection between quarks and hadrons is through the vacuum polarisation, $\Pi(s)$. This is defined by the correlation function of two currents, \( i.e. \)

\[
\Pi(s) = i \int d^4x \ e^{iqx} \langle 0| T\{J(x)J(0)\}|0 \rangle ,
\]

(1)

where \( s = q^2 = -Q^2 \) and the currents are chosen so as to select the desired channel. In this work we are considering the scalar-isoscalar mesons and so we choose the Renormalisation Group invariant current

\[
J(x) = \frac{m_q}{2} \{ \bar{u}(x)u(x) + \bar{d}(x)d(x) \} = \frac{m_q}{\sqrt{2}} \pi(x)n(x) ,
\]

(2)

where \( m_q \equiv \frac{1}{2}(m_u + m_d) \) and we use the symbol \( n \) to denote an effective light-quark. The second equality follows because we ignore the two-gluon intermediate state, whose contribution is \( O(m_q^2) \) and so can be safely neglected.

\[ \begin{align*}
\text{Figure 1: The complex } s-\text{plane showing the so called \textquote{Pacman} contour which is used to derive the FESRs.}
\end{align*} \]

$\Pi(s)$ is analytic everywhere in the complex $s-$plane except along the time-like axis, so for any weight function, $w(s)$, which is real for all real $s$ and analytic within the ‘Pacman’ contour shown in Fig. 1 we can write

\[
\frac{1}{\pi} \int_{0}^{s_0} w(s) \text{Im}\Pi(s) \, ds = -\frac{1}{2\pi i} \oint_{|s|=s_0} w(s) \Pi(s) \, ds .
\]

(3)

On the left-hand side of Eq. (3) we have an integral in the physical region, and here we choose to describe the integrand using hadronic physics (either with experimental data or
a phenomenological parameterisation). On the right-hand side we have an integral in the non-physical region (complex $s$), and here we use field theory (i.e. QCD) to calculate the integrand. The assumption that Eq. (3) holds for a range of values of $s_0$ is a statement of global quark-hadron duality, i.e. the integral of an hadronic expression is equal to the integral of an expression written in terms of partonic degrees of freedom.

Early sum rule work used positive integer powers of $s$ as weight functions. In this work we wish to investigate the lightest meson in the $I = J = 0$ channel, and so we choose a decaying exponential, $w(s) = \exp(-s/M^2)$. On the phenomenological side, this weight function will suppress the region of the integral $s > M^2$, and so by varying $M^2$ we can control which regions of the integrand contribute most significantly to the sum rule. If we also include integer powers of $s$ in the weight function then we can build up a family of sum rules.

Almost by definition, at energies where resonances dominate the spectral density, the implications of QCD are not straightforwardly calculable. If QCD were tractable then the technology of QCD sum rules would be redundant. This means that, for moderate values of $s_0$, our straightforward QCD description of $\Pi(s)$ must fail, at the very least, in the region of the positive real axis (see Fig. [1]). However, it is believed [17] that it is only in the region of the positive, real axis that the straightforward QCD description is particularly unreliable and that in the rest of the $s$-plane calculable QCD provides a good description of the correlator. Consequently we introduce a zero into the weight function at the point where the circle joins the real axis, i.e. $s_0$. Thus our sum rules have the weight function $w(s) = s^k (1 - s/s_0) e^{-s/M^2}$ and we denote them by the symbol $T_k$. Elsewhere in the literature they are often referred to as 'pinched weight' sum rules. Then the left hand side of Eq. (3) becomes

$$T_k(s_0, M^2) \equiv \frac{1}{\pi} \int_0^{s_0} s^k \left(1 - \frac{s}{s_0}\right) \exp \left(-\frac{s}{M^2}\right) \text{Im}\Pi(s) \, ds .$$

Introducing this zero into the weight function has a number of advantages. Firstly, on the theoretical side we avoid the problem, just described, of the failure of QCD on the part of the circle near the positive real axis. Secondly, on the phenomenological side, we have introduced a second factor with a tendency to suppress the higher energy portion of the integral and consequently reduce the dependence on the exact value of $s_0$ [18, 19].

The parameter $s_0$ is not entirely free: it must be large enough that the QCD expressions we are able to calculate are expected to be a good approximation to the full correlator, yet $s_0$ should be small enough that we have experimental information. The art of sum-ruling is to find (sensible) values of $s_0$ for which $[T_k(s_0, M^2)]_{\text{had}} = [T_k(s_0, M^2)]_{\text{QCD}}$ over a wide range of the unphysical parameter $M^2$. When this occurs we say that the sum rules are saturated.
Although the current Eq. (2) is RG-invariant, the correlator it gives rise to is not. As well as making the calculation of the correlator within QCD more difficult, this would also lead to an unwanted scale dependence in our final results. The second derivative of this correlator, $\Pi''(s)$, is RG-invariant. As $\Pi''(s)$ is also analytic within the contour shown in Fig. [1], we could choose to write the sum rules completely in terms of this quantity, but $\text{Im} \, \Pi''(s)$ is not directly related to any physical quantity and so we would lose our link to experiment. Instead we use partial integration to re-express the theoretical side of the sum rules in terms of $\Pi''(s)$. To do this we write the integral on the theoretical side as

$$i_1 = \oint v''(s) \Pi(s) \, ds$$

(5)

We now integrate this expression by parts twice to give,

$$i_1 = v'(s) \Pi(s) - v(s) \Pi'(s) + \oint v(s) \Pi''(s) \, ds$$

(6)

and arrange the constants of integration which appear in $v(s)$ and $v'(s)$ so that these functions disappear at $s = s_0$. Then our sum rule is

$$\frac{1}{\pi} \int_{0}^{s_0} w(s) \left[ \text{Im} \Pi(s) \right]_{\text{had}} \, ds = -\frac{1}{2\pi i} \oint_{|s|=s_0} v(s) \left[ \Pi''(s) \right]_{\text{QCD}} \, ds$$

(7)

### 3 The Operator Product Expansion

In order to calculate the vacuum polarisation within QCD we make use of the Operator Product Expansion (OPE) \([20]\). This replaces the matrix element of the product of currents in Eq. (1) by a linear combination of local operators ordered in terms of increasing dimensionality,

$$\Pi(s = -Q^2) = \sum_n C_n(Q^2) \langle \mathcal{O}_n \rangle$$

(8)

The first term in this expansion is the purely perturbative contribution, with higher dimension terms giving the contribution due to various condensates. We will consider terms up to dimension six, higher order terms are expected to be negligible. Expressions are available in the literature for the dimension zero term to four-loop order, the dimension four term to two loops and the dimension six term at leading order. We quote these expressions with references to the original works for the full information. All expressions are given in the $\overline{\text{MS}}$ scheme.
The dimension zero expression can be found in \[21\] and is given by
\[
\Pi''_0(Q^2) = \frac{6 m_q^2}{2(4\pi)^2 Q^2} \left\{ 1 + a_s \left[ \frac{11}{3} - 2 \log \frac{Q^2}{\mu^2} \right] \\
+ a_s^2 \left[ \frac{5071}{144} - \frac{35}{2} \zeta(3) \right] - \frac{139}{6} \log \frac{Q^2}{\mu^2} + \frac{17}{4} \log^2 \frac{Q^2}{\mu^2} \right\} \\
+ a_s^3 \left[ -\frac{4781}{9} + \frac{b_1}{6} + \frac{475}{4} \zeta(3) \right] + \left( -\frac{2720}{9} + \frac{475}{4} \zeta(3) \right) \log \frac{Q^2}{\mu^2} \right\} ,
\]
where
\[
b_1 = \frac{4748953}{864} - \frac{91519}{36} \zeta(3) - 15 \zeta(4) + \frac{715}{2} \zeta(5) ,
\]
and \(\zeta(n)\) is the Riemann Zeta function. As \(\Pi''(s)\) is renormalisation group invariant, we are free to make any convenient choice of the renormalisation scale. We use the ‘contour improvement’ prescription, whereby the logs in Eq. (9) are ‘re-summed’ by making the choice \(\mu^2 = Q^2\). This improves the convergence of the perturbative series by removing the dependence on large powers of possibly large logs which would appear in the (unknown) higher order terms. The contour improved perturbative contribution to our correlator is then,
\[
\Pi''_0(Q^2) = \frac{6 m_q^2}{2(4\pi)^2 Q^2} \left\{ 1 + \frac{11 a_s(Q^2)}{3} + a_s^2(Q^2) \left[ \frac{5071}{144} - \frac{35}{2} \zeta(3) \right] \\
+ a_s^3(Q^2) \left[ -\frac{4781}{9} + \frac{b_1}{6} + \frac{475}{4} \zeta(3) \right] \right\} .
\]

Naïvely, the only gauge-invariant operator of dimension two that can be constructed within QCD is the so-called mass insertion term, \(m_q^2\). For the light quarks that we are considering this term can safely be ignored. Some recent works have suggested that there may be mechanisms outside of the normal OPE, which could lead to dimension two contributions to correlation functions. These mechanisms include renormalons \[22\] and a tachyonic (i.e. imaginary) gluon mass \[23\]. It has been argued that, if they do exist, the effects of dimension two operators would not be significant in most sum rule analyses and that even in calculations where they would be expected to be most noticeable, these effects are consistent with zero \[24\]. Hence, in this work, we choose to ignore them.

The contour-improved contribution at dimension four can be inferred from \[25, 26, 27\] and is given by
\[
\Pi''_4(Q^2) = \frac{m_q^2(Q^2)}{2Q^6} \left\{ 6 \langle m_q \bar{q} q \rangle \left[ 1 + \frac{22 a_s(Q^2)}{3} \right] - \frac{I_g}{9} \left[ 1 + \frac{121 a_s(Q^2)}{18} \right] \right\} \\
+ \frac{I_s}{9} \left[ 1 + \frac{3 m_q^4(Q^2)}{28 \pi^2} \right] ,
\]
where
\[ I_G = -\frac{9\langle a_s G^2 \rangle}{4} \left( 1 + \frac{16a_s}{9} \right) + 4a_s\langle m_s \bar{s}s \rangle \left( 1 + \frac{91a_s}{24} \right) + \frac{3m_s^4}{4\pi^2} \left( 1 + \frac{4a_s}{3} \right), \] (12)
and
\[ I_s = \langle m_s \bar{s}s \rangle + \frac{3m_s^4}{7\pi^2} \left( \frac{1}{a_s} - \frac{53}{24} \right). \] (13)

The leading order dimension six contribution can be taken from [25] and we follow this reference in taking this term at the fixed scale of 1 GeV. The dimension six contribution is then
\[ \Pi_6''(Q^2) = \frac{3m_q^2(1 \text{ GeV}^2)}{Q^8} \left( m_q \langle g\bar{q}\sigma_{\mu\nu}G^{\mu\nu}a^a q \rangle \right. \\
+ \pi^2 a_s \left[ \langle \bar{q}\sigma_{\mu\nu}a^a q\bar{q}\sigma^{\mu\nu} a^a q \rangle + \frac{2}{3} \langle \bar{q}\gamma_{\mu} a^a q\bar{q}\gamma^{\mu} a^a q \rangle \right] \right), \] (14)
where \( \gamma_{\mu} \) are the Dirac matrices, \( \sigma_{\mu\nu} = \frac{1}{2} \{ \gamma_{\mu}, \gamma_{\nu} \} \) and \( \lambda^a \) are the usual Gell-Mann matrices normalised such that \( \text{Tr}[\lambda^a\lambda^b] = 2\delta^{ab} \). Currently there is no reliable method to calculate the vacuum expectation values appearing in Eq. (14), so we follow standard practice and relate these dimension six condensates to the (RG variant) dimension three light quark condensate \( \langle \bar{q}q \rangle \). For the mixed condensate, \( \langle g\bar{q}\sigma_{\mu\nu}G^{\mu\nu}a^a q \rangle \), this is done via the parameterisation
\[ \langle g\bar{q}\sigma_{\mu\nu}G^{\mu\nu}a^a q \rangle = m_0^2 \langle \bar{q}q \rangle, \] (15)
and for the four-quark condensates, \( \langle \bar{q}\sigma_{\mu\nu}a^a q\bar{q}\sigma^{\mu\nu} a^a q \rangle \) and \( \langle \bar{q}\gamma_{\mu} a^a q\bar{q}\gamma^{\mu} a^a q \rangle \), via the vacuum saturation hypothesis [8]. To take into account the possible violation of the vacuum saturation hypothesis we include a multiplicative factor \( V_{qs} \). Thus our final expression for the dimension six contribution is
\[ \Pi_6''(Q^2) = \frac{6m_q^2}{2Q^8} \left( m_q^2 \langle q\bar{q} \rangle - \frac{176\pi^2 a_s V_{qs}}{27} \langle \bar{q}q \rangle^2 \right). \] (16)

In order to evaluate all these expressions we need to calculate the running coupling and quark masses at complex values. This is done by solving the four-loop Renormalisation Group Equations that define the QCD \( \beta \) and \( \gamma \) functions (our conventions for these functions are given in Appendix [3]).

For the coupling, we choose to solve the implicit equation
\[ \int_{a_s(m_0^2)}^{a_s(\mu^2)} \frac{da'}{\beta(a')} - \log \left( \frac{\mu^2}{m_0^2} \right) = 0, \] (17)
iteratively using Newton’s method. As input, we take the experimental value \( \alpha_s(m_T^2) = 0.334 \pm 0.022 \), as measured by the ALEPH collaboration [28]. For reference, had we used the standard expansion for \( \alpha_s \), then at four-loops and with three active flavours, this central input value would correspond to \( \Lambda_{QCD} \approx 365 \text{ MeV} \). For higher values of \( \alpha_s(m_T^2) \), or correspondingly \( \Lambda_{QCD} \), the radiative corrections to Eq. (10) become more important and start to swamp the higher dimension terms, *i.e.* it becomes more important to include higher loop corrections to the perturbative contribution than to include the condensates. It has been argued [29] that the sum rule methodology could become invalid for \( \Lambda_{QCD} \gtrsim 330 \text{ MeV} \). This is especially problematic for attempts to determine condensate values via the sum rule technique. As this is not our aim and considering the success of previous sum rule studies with similar and higher values of \( \alpha_s(m_T^2) \) (and \( \Lambda_{QCD} \)), we feel justified in applying sum rules to this problem.

The momentum dependence of the quark mass is given by

\[
m_q(\mu^2) = m_q(1 \text{ GeV}^2) \exp \left[ \int_{a_s(1 \text{ GeV}^2)}^{a_s(\mu^2)} \frac{\gamma(a')}{\beta(a')} \text{d}a' \right].
\]  

(18)

### 4 Instantons

The OPEs for the scalar-isoscalar and scalar-isovector channels are identical. This was originally interpreted as an explanation for the almost exact mass degeneracy of the \( f_0(980) \) and the \( a_0(980) \) [9]. However, we know that the OPE is not enough to describe the QCD vacuum completely. Other effects are important, particularly in the scalar and pseudoscalar channels [30].

The complex nature of the QCD vacuum means that it is not currently possible to solve the equations of motion of the full theory directly. We must make use of approximations, which in this work we choose to be the Instanton Liquid Model [31, 32, 33]. Here the QCD vacuum is modelled as a four-dimensional ‘liquid’ of instantons, which is assumed to be dilute enough to make the idea of individual instantons meaningful. The liquid is characterised by the average instanton size \( \rho_c \) and four-volume density \( n_c \) (or alternatively the average instanton separation, \( r_c = n_c^{-1/4} \)). The Instanton Liquid Model is then thought to be a good approximation, if the ratio \( \rho_c/r_c \) is small. Within this model the instanton contribution to the scalar-isoscalar light-quark correlator was found in [33, 34] to be

\[
\Pi_i(s) = \frac{3 Q^2 m_q^2}{2 \pi^2} \left[ K_1(\rho_c \sqrt{Q^2}) \right]^2,
\]  

(19)

where \( K_1(x) \) is a modified Bessel function of the second kind (or MacDonald function).
Taking two derivatives we obtain

$$\Pi''(s) = \frac{3 \rho_c^2 m_q^2}{4 \pi^2} \left[ K_0^2(\rho_c \sqrt{Q^2}) + K_1^2(\rho_c \sqrt{Q^2}) \right]. \tag{20}$$

As stated earlier $\Pi''(s)$ is an RG-invariant quantity. The only quantity in Eq. (20) that could have a renormalisation scale dependence is the quark mass. This renormalisation scale dependence cannot be cancelled out anywhere else, so we conclude that the quark mass appearing in Eq. (20) must be the quark mass at some fixed scale, $\nu^2$. A priori we do not know what this scale is. It is reasonable to expect that it is related to the instanton scale given by $1/\rho_c$ or the hadronic scale of $\sim 1$ GeV, but we have no physical reason for choosing $1/\rho_c$ rather than e.g. $2/\rho_c$. Thus we take this scale as an input parameter in the calculation. If this scale is found to be considerably different from the hadronic and instanton scales, then it would call into doubt our treatment of the instanton contribution.

5 The Phenomenological Side

5.1 The Coupling Schemes

At low energies the only hadronic process possible in the scalar-isoscalar channel is $\pi \pi$ scattering. As pions are made of non-strange quarks, it is reasonable to expect that, in the elastic region, the spectral density for the correlator is related to the absorptive part of the $\pi \pi \to \pi \pi$ scattering amplitude, $\mathcal{T}(s)$. This link can be made via the scalar form-factor of the pion, $d(s)$, which is defined by

$$d(s) = \langle 0 | m_q q q | \pi \pi \rangle. \tag{21}$$

Inserting a complete set of hadronic states into the correlator Eq. (1) and keeping only the lowest state, i.e. two pions, gives the spectral density in the elastic region,

$$\frac{\text{Im}\Pi(s)}{\pi} = \frac{3 \beta(s)}{32 \pi^2} |d(s)|^2, \tag{22}$$

where $\beta(s) = \sqrt{1 - 4m_q^2/s}$.

Watson’s Final State Interaction Theorem [35] implies that the form factor and the scattering amplitude have the same phase. Thus, in the region where elastic unitarity holds, we can write

$$\mathcal{T}(s) = \alpha(s) d(s), \tag{23}$$
where the coupling function, $\alpha(s)$, is real for real $s > 4m^2$. Substituting Eq. (23) into Eq. (22) and making use of the elastic unitarity relationship $\text{Im}T = \beta |T|^2$, we obtain
\[
\frac{\text{Im}\Pi(s)}{\pi} = \frac{3}{32\pi^2} \frac{\text{Im}T}{\alpha^2(s)}.
\] (24)

It is well known that $\pi\pi$ scattering contains an Adler zero [36], and that this zero does not appear in the scalar pion form factor. We now make the simplifying assumption that this is the only $s$-dependence appearing in $\alpha(s)$, i.e.
\[
\alpha(s) = \alpha_0(s - s_A)
\] (25)
where $s_A$ is the position of the Adler zero and $\alpha_0$ is a constant. Substituting Eq. (25) into Eq. (24) gives our first ansatz
\[
\frac{\text{Im}\Pi(s)}{\pi} = f \frac{\text{Im}T(s)}{(s - s_A)^2},
\] (26)
where $f$ is an unknown constant of proportionality.

Although the $4\pi$ threshold is at 560 MeV, it is well known that multi-pion channels are not significant below around 1400 MeV, when quasi two-body channels, e.g. $\sigma\sigma$, $\rho\rho$, become important [37, 38]. The first important inelastic channel is $\pi\pi \rightarrow K\bar{K}$. Even after this has opened the $\pi\pi$ final state will continue to pick out the non-strange contribution to the sum over states. Thus we might expect Eq. (26) to give a reasonable approximation to the true spectral density even above 1 GeV. We call this Coupling Scheme I.

The multi-pion final states mentioned above will increase the $n\bar{n}$ spectral density, causing Eq. (24) to be an underestimate at higher energies. Phenomenologically, we might expect the true spectral density to be enhanced by the ratio of the total to elastic $\pi\pi$ cross-sections, thus giving
\[
\frac{\text{Im}\Pi(s)}{\pi} = f \frac{(\text{Im}T(s))^2}{\beta(s)(s - s_A)^2 |T(s)|^2},
\] (27)
which we call Coupling Scheme II. In the elastic region, the two coupling schemes are, of course, identical. Above the $K\bar{K}$ threshold, whereas Coupling Scheme I ignores all inelastic channels and so will underestimate the spectral density, Coupling Scheme II takes into account all inelastic channels, including those with hidden strangeness, and so may be an overestimate.

As input we take the parameterisation of the $I = J = 0$ $\pi\pi$ scattering data carried out by Bugg et al [39]. However, when continued down to threshold this fails to reproduce the scattering length predicted by Chiral Perturbation Theory ($\chi$PT) and so in the region
Figure 2: A sketch showing the absorptive part of the $I = J = 0$ $\pi\pi$ scattering amplitude as calculated from the data set described in the text (line). Superimposed are experimental points from [12] (diamonds), [13] (boxes), [14] (triangles) and [15] (circles).

below 450 MeV we use the Roy equation extrapolation of the classic Ochs-Wagner phase-shifts [10], as carried out by Pennington and used in the discussion of $\gamma\gamma \rightarrow \pi\pi$ [11].

Fig. 2 shows a sketch of the absorptive part of the isospin zero, $S$-wave amplitude for $\pi\pi$ scattering as calculated from our data set. From this we can clearly see that the $f_0(400 - 1200)$ and the $f_0(1370)$ appear as peaks, while the $f_0(980)$ and the $f_0(1500)$ show up as sharp dips. Thus these latter two states, and in particular the $f_0(980)$, will largely decouple from the spectral density and if saturation of the sum rules is achieved with either of our two Coupling Schemes, it will not be due in any great part to the $f_0(980)$ or $f_0(1500)$. By varying both $M^2$ and $s_0$ it may be possible to determine which of the $f_0(400 - 1200)$ and the $f_0(1370)$ plays the dominant role in saturation, but for the evaluation of the Wilson coefficients in the OPE to be trustworthy, we would expect $s_0$ to be too high to exclude the $f_0(1370)$. Nevertheless, as the decaying exponential in the weight functions of our sum rules suppresses the integrand for $s > M^2$, it is still possible to determine which resonance is most important for saturation.
5.2 Normalisation

A value for the normalisation factor $f$, can be determined by making a comparison with the spectral density obtained using an Omnès representation \[10\] of the form factor. This representation is given by

$$d(s) = d(0) \exp \left[ \frac{s}{\pi} \int_{4m^2_\pi}^{\infty} \frac{\phi(s') \, ds'}{s'(s' - s)} \right], \quad (28)$$

where $\phi(s)$ is the phase of $I = J = 0 \, \pi\pi$ scattering. The form factor at zero momentum transfer, which normalises the Omnès function, can be determined from the Feynman-Hellman Theorem \[17\] and is given by

$$d(0) = m_q \frac{\partial m^2_\pi}{\partial m_q}. \quad (29)$$

To leading order in $\chi$PT, the physical pion mass is given by the Gell-Mann-Oakes-Renner relation, $4\langle m_q \overline{q} q \rangle = -m^2_\pi f^2_\pi \rangle \[18\]$, and thus we see that

$$d(0) = m^2_\pi. \quad (30)$$

Going to next-to-leading order in $\chi$PT would introduce a correction to Eq. (30) of the order of 1% (see for example \[19\]), which is safe to neglect.

To evaluate the integral in Eq. (28) exactly requires knowledge of the scattering phase in the high energy region where it is unknown. However, the integrand is dominated by the region around $s$. So if we restrict our comparison to the low energy region then the behaviour of the phase above some point $s_1$, where $s_1 \gg s$, will not greatly affect the value of the integral. We take $s_1$ to correspond to the last data point and approximate the phase above this point as a constant, $\phi_1$. Thus, the spectral density, as calculated from the Omnès representation, is written

$$\frac{\text{Im} \Pi(s)}{\pi} = \frac{3m^4_\pi}{32\pi^2} \beta(s) \left( \frac{s_1}{s_1 - s} \right)^{2\phi_1/\pi} \exp \left[ \frac{2s}{\pi} \int_{4m^2_\pi}^{s_1} \frac{\phi(s') \, ds'}{s'(s' - s)} \right]. \quad (31)$$

where the bar on the integral sign denotes the Cauchy Principal Value.

The value of the constant, $f$, can then be determined by fitting Eq. (28) to Eq. (31). The position of the Adler zero for isospin-0, S-wave $\pi\pi$ scattering is fixed to be close to $m^2_\pi/2$, which is its position at lowest order in $\chi$PT and so we essentially only have one parameter, $f$, to determine. The fit, shown in Fig. 3, was carried out up to a maximum energy of 400 MeV. Over this energy range it does not matter which Coupling Scheme is used. Each gives the value $f = 6.9 \times 10^{-7}$ GeV. However, this value is quite sensitive to the maximum energy of the fit and should not be trusted to an accuracy better than 20%. 

13
Figure 3: The spectral density below 400 MeV as calculated from our Coupling Scheme (points) and the single channel Omnès representation (solid line).

6 Results

In any sum rule calculation, we hope to find regions of the parameter space where the theoretical and phenomenological sides are equal. We measure this saturation of the sum rules by introducing the double ratios

$$D_k(s_0, M^2) = \frac{[T_{k+1}/T_k]_{\text{had}}}{[T_{k+1}/T_k]_{\text{QCD}}}.$$  \hspace{1cm} (32)

This ratio is independent of the overall normalising constants on either side, i.e. $f$ and $m_q^2(1 \text{ GeV}^2)$, and obviously is equal to one when the sum rules are saturated. As sum rule analyses are generally accurate to around 10-20\% we set an agreement region of $(1 \pm 0.1)$. In practice we shall use only the lowest such double ratio for each type of sum rule, i.e. $D_0$, as the integer powers of $s$ in the higher sum rules increase the importance of the upper end of the integrand on the phenomenological side.

Our sum rules contain a number of input parameters which we list in Table 1 along with the values used. Any deviations from these values will be discussed in Sects. 6.1, 6.2 and 6.3. Where appropriate we also include a brief description of where these values come from.
Table 1: Standard values of all parameters used to evaluate the sum rules.

| Parameter | Value                  | Source                          |
|-----------|------------------------|---------------------------------|
| $\alpha_s(m_T^2)$ | 0.334                 | From Experiment [28]             |
| $\langle m_q q \rangle$ | $-(95\text{MeV})^4$ | From GMOR relation [48]         |
| $a_s G^2$         | (381 MeV)$^4$          |                                 |
| $\langle m_s s \rangle$ | $-(195 \text{MeV})^4$ |                                 |
| $m_s(1 \text{GeV}^2)$ | 159 MeV              |                                 |
| $\langle q q \rangle$ | $-(225 \text{MeV})^3$ |                                 |
| $m_0^2$           | 0.8 GeV$^2$            |                                 |
| $V_{vs}$           | 1                     |                                 |
| $\rho_c$          | $(600 \text{MeV})^{-1}$ |                                 |
| $\nu$             | $\sim 1 \text{GeV}$   |                                 |
| $s_0$             | 3 GeV$^2$              |                                 |

6.1 The Instanton Parameters

The first task is to determine $\nu^2$, the scale of the quark mass in the instanton contribution. To evaluate Eq. (20), we introduce a new parameter, $\lambda$, which is equal to the ratio of $m_q^2(\nu^2)$ to $m_q^2(1 \text{GeV}^2)$, i.e.,

$$
\lambda = \left[ \frac{m_q(\nu^2)}{m_q(1 \text{GeV}^2)} \right]^2 = \exp \left[ 2 \int_{a_s(1 \text{GeV}^2)}^{a_s(\nu^2)} \frac{\gamma(a')}{\beta(a')} \, da' \right],
$$

where the second equality follows from Eq. (18). In Fig. 4 we show the stability curves obtained for various values of this parameter using Coupling Scheme II and $s_0 = 3.7 \text{GeV}^2$. We have found that the saturation of the sum rules is practically independent of $s_0$ in the region $3 \text{GeV}^2 \lesssim s_0 \lesssim 4 \text{GeV}^2$ and so no significance should be attached to this choice of $s_0$. However, it was found that saturation could not be achieved with the standard value of $\rho_c = (600 \text{MeV})^{-1}$, and in Fig. 4 we pre-empt the results of Figs. 5 and 6 by using the non-standard value $\rho_c = (480 \text{MeV})^{-1}$.

It is immediately clear that the sum rules are far from satisfied when $\lambda = 0$, i.e. when the instanton contribution is removed. However, the sum rules are well saturated for $\lambda \approx 1.2$ and we have found that the best value is $\lambda = 1.25$, which we will use from now on. This value of $\lambda$ corresponds to a scale $\nu = 0.96 \text{GeV}$.

As stated earlier saturation can only be achieved by use of a non standard value of $\rho_c$. In Figs. 5 and 6 we show the effect that varying this parameter has on the saturation curves. We see that good saturation is achieved for $1/\rho_c = 470 - 490 \text{MeV}$, with the ‘best’ value being $\rho_c = (480 \text{MeV})^{-1} \approx 0.42 \text{fm}$. This value is slightly larger (25%) than the estimate [32] based on the size of the gluon condensate which has become the standard value employed in the Instanton Liquid Model. However, it is entirely consistent with current lattice determinations which tend to find sizes in the range $0.32 - 0.43 \text{fm}$ and, unless stated otherwise, we will use this ‘best’ value of $\rho_c$ from now on.
Figure 4: The saturation curves obtained for various values of $\lambda$, with $\rho_c = (480 \text{ MeV})^{-1}$ and all other parameters as listed in Table [1].

6.2 Saturation

As shown in Fig. [3], once the instanton parameters have been determined the quality of the saturation of our sum rules is very good. This means that the $f_0(980)$, being largely decoupled from our Coupling Schemes, does not play an important role in this saturation. On the other hand, the saturation curves are seen to be flat from quite low values of $M^2$. With our weight function suppressing the portion of the integrand $s > M^2$ relative to the low energy region, the shape of the curves implies that saturation is achieved long before the $f_0(1370)$ contributes significantly to the phenomenological integral. This suggests that it is the $f_0(400 - 1200)$ that plays the most important role in saturating our sum rules.

As the values of the condensates have not been fixed definitively, we have performed a number of tests to determine how sensitive the saturation of our sum rules is to variations of these input values. To test the dependence on the size of the gluon condensate, we write

$$\langle a_s G^2 \rangle = c \langle a_s G^2 \rangle_0$$

where $\langle a_s G^2 \rangle_0$ is the standard value listed in Table [1]. In Fig. [4] we show the saturation curves for various values of this factor, $c$. We see that the saturation is not strongly dependent on the size of the gluon condensate, but the best results are obtained in the
Figure 5: The saturation curves obtained for various values of $\rho_c$, with $\lambda = 1.25$ and all other parameters as listed in Table [1].

Figure 6: As Fig. 5 but for finer steps in $\rho_c$. 
Figure 7: The saturation curves, $D_0$, for various values of the gluon condensate.

Figure 8: The saturation curves, $D_0$, for various values of the vacuum saturation violation parameter, $V_{vs}$.
range $0 \leq c \leq 2$. In Fig. 8 we see how deviations from vacuum saturation affect our results. We find reasonable agreement for $0 \leq V_{vs} \leq 2$, suggesting that vacuum saturation may be violated by 100% with little effect. For the remaining OPE parameters, varying their values from 0 to 200% was found to have no noticeable effect. However, the best results are obtained with the strong coupling taken to be close to the central value quoted in Table I.

6.3 The Average Light Quark Mass

Another way to measure whether the sum rules are saturated is by making a prediction for a physical quantity. As $M^2$ is an unphysical parameter, for ‘sensible’ values of $s_0$ this prediction should be effectively independent of $M^2$ over a wide range. From Eq. (18), we see that the theoretical side of our sum rules contains an overall factor of $m^2_q(1 \text{ GeV}^2)$, we can thus make an estimate for the average light quark mass from

$$m_q(1 \text{ GeV}^2) = \sqrt{\frac{[T_k]_{\text{had}}}{[T_k]_{\text{calc}}}}. \quad (35)$$

If the graph of this quantity against $M^2$ is found to be flat then we can say that the sum

Figure 9: The quark mass curves for our two Coupling Schemes. The dashed line shows Coupling Scheme I and the solid line corresponds to Coupling Scheme II.
rules are saturated. Although the actual value of the quark mass extracted will depend on the constant $f$, its $M^2$ dependence will not, thus as a test of saturation Eq. (35) is independent of any uncertainties in the normalisation of the hadronic side. Once again we will use only the lowest sum rule, \textit{i.e.} $T_0$, to estimate the quark mass.

![Graphs showing the effect of various parameters on the quark-mass curves.](image)

(a) Average Instanton Size  
(b) The Gluon Condensate  
(c) Vacuum Saturation Violation Parameter

Figure 10: The effect of various parameters on the quark-mass curves.

In Fig. 9 we show the curve obtained from Eq. (35) using the instanton parameters determined in Sect. 6.1 for our two Coupling Schemes. We notice immediately that the curve is extremely flat, in particular for Coupling Scheme II. In both cases the variation is less than 1% in the range $0.6 \text{ GeV}^2 \leq M^2 < s_0$. This confirms that our sum rules are well satisfied even for quite low values of $M^2$. The difference between the results of the two Coupling Schemes is $\approx 1\%$. Thus we estimate $m_q(1 \text{ GeV}^2) = 5.2$ MeV.
We have performed a number of tests to determine how stable this estimate is to the variation of input parameters within the ranges found to give good saturation curves. The parameters $\rho_c$, $\langle a_s G^2 \rangle$ and $V_{vs}$ each yield an uncertainty in the quark mass of 0.1 MeV, see Fig. 10. Varying the remaining condensate values from zero to twice the value listed in Table 1 produces no noticeable effect on the quark mass obtained.

The largest source of uncertainty in our calculation is the value of the phenomenological constant of proportionality, $f$, defined in Eqs. (27,28). This could only be determined to an accuracy of 20%, and so gives the dominant uncertainty in the final result of 0.5 MeV. Our final estimate is then $m_q(1 \text{ GeV}^2) = 5.2 \pm 0.6 \text{ MeV}$. In Table 2 we list some other recent determinations of this quantity. We see that our estimate is consistent, though lower, than earlier sum rule determinations. Though not agreeing with the results of quenched lattice QCD, our value is reassuringly consistent with the early results of unquenched lattice calculations which are just beginning to appear in the literature.

### Table 2: Recent determinations of $m_q(1 \text{ GeV}^2)$. Note that Lattice QCD results are generally quoted as $m_q(4 \text{ GeV}^2)$ and have here been scaled up by a factor of 1.42 as required by Eq. (18) with $\alpha_s(m_t^2) = 0.334$. |
| Technique | $m_q(1 \text{ GeV}^2)$ (MeV) |
|-----------------|-------------------------------|
| Chetyrkin et al. [23] | Pseudoscalar Sum Rule | 5.7 ± 1.2 |
| Prades [51] | | 6.4 ± 2.3 |
| Maltman & Kambor [52] | | 5.6 ± 0.8 |
| Dosch [53] | SR for $\langle \bar{q}q \rangle$ + GMOR | 4.7 − 7.9 |
| APE 98 [54] | Quenched Lattice QCD | 6.8 ± 0.6 |
| JLQCD 99 [55] | | 6.0 ± 0.4 |
| QCDSF 99 [56] | | 6.3 ± 0.3 |
| APE 99 [57] | | 6.8 ± 0.7 |
| CP-PACS 00 [58] | | 6.2 ± 0.2 |
| SESAM 98 [59] | Unquenched Lattice QCD | 3.9 ± 0.2 |
| CP-PACS 00 [58] | | 4.9 ± 0.3 |
| QCDSF - UKQCD 01 [60] | | 5.0 ± 0.3 |

7 Discussion

The application of QCD sum rules to the isoscalar-scalar current requires a realistic representation of the contributing resonances. None in this channel remotely approximates a $\delta-$function, Fig. 2. Indeed, states like the $\sigma$ (or $f_0(400 - 1200)$) are far too broad to be sensibly described by simple Breit-Wigner forms. Moreover they overlap with other resonances like the $f_0(980)$ and strongly coupled thresholds, like $K \bar{K}$. It is quite natural
that the $I = J = 0$ correlator is expressly related to the $\pi\pi$ scattering amplitude and so the vacuum polarization can be described directly in terms of experimental information without the need for further (Breit-Wigner-like) approximations. We have shown that such Coupling Schemes, defined in Sect. 5.1, are perfectly capable of saturating QCD finite energy sum rules. In $\pi\pi$ scattering the $f_0(980)$ appears as a sharp dip. Consequently, in our coupling schemes this state makes relatively little contribution to the correlator, in total contrast to the conclusions of Shakin and Wang [61]. That good saturation is possible strongly suggests then that the $f_0(980)$ does indeed make only a small contribution to the scalar-isoscalar sum rules, which in turn implies that it does not contain a large $\bar{u}u$ component in its wave function. This result is also at odds with the conclusion reached in [13, 14]. However, as explained above, when the same authors [14] considered a more realistic approximation for the resonance shape, the parameters found were more consistent with the $f_0(400 - 1200)$ than the $f_0(980)$. From the present analysis, beyond saying that it is not strongly $\bar{u}u$, we can make no comment on the structure of the $f_0(980)$.

The saturation curves we obtain are remarkably flat and close to one from low values of $M^2$, see Figs. 4-8. As the exponential in the weight function tends to suppress the integrand in the region $s > M^2$, saturation at these values of $M^2$ means it is the lower energy portion of the experimental data that is most important in saturating the sum rules. Furthermore, the $f_0(1370)$ (with mass $m$ and width $\Gamma$) will not contribute until $M^2 \gtrsim (m - \Gamma/2)^2 \approx 1.4 \text{ GeV}^2$. That the sum rules are fulfilled from $M^2 \approx 1 \text{ GeV}^2$ implies it is the $f_0(400 - 1200)$ that plays the most important role in saturating the $\bar{u}u$ scalar-isoscalar sum rules.

Our results again show that the Operator Product Expansion is not enough to describe the physics of the scalar mesons completely. We find saturation only when instantons are included. One potentially worrying point of this analysis is the need to tune the average instanton size away from its accepted value of $\rho_c \approx (600 \text{ MeV})^{-1} \approx 0.33 \text{ fm}$ to $\rho_c \approx (470 - 490 \text{ MeV})^{-1}$ or $0.40 - 0.42 \text{ fm}$, which is about 25% larger. The validity of the Instanton Liquid Model relies on the mean separation of instantons being significantly larger than their average size. If this is not true then neighbouring instantons will overlap significantly and it becomes meaningless to talk about individual instantons. The assumed average instanton separation of 1 fm is only about 2.5 times the value of $\rho_c$ required by our sum rule, and so we are probably approaching this limit. Reassuringly though, lattice simulations of the topological features of the QCD vacuum seem to find average instanton sizes similar to the one we are forced to use (for a summary of lattice determinations of instanton parameters see [62]).
The above results follow from ratios of the finite energy sum rules defined in Eq. (33) and displayed in Figs. 4-8. Moreover, the absolute value of each is determined by the quark mass curves of Sect. 6.3. These are also found to be flat and this constancy sets in already below $M^2 = 1 \text{ GeV}^2$, Fig. 9. This is again evidence that our Coupling Schemes are capable of satisfying the sum rules and that the state most important for this is the $f_0(400-1200)$. The value we find for the average $u$ and $d$ quark mass is

$$m_q(1 \text{ GeV}^2) = 5.2 \pm 0.6 \text{ MeV}. \quad (36)$$

The most conservative way to interpret this result is merely as a consistency check. A number of assumptions are made in this analysis, e.g. the validity of the Coupling Schemes, the truncation of the OPE at dimension-6 and the perturbative contribution at $O(a_s^3)$, the form of the instanton contribution etc. That our final result is of the correct size reassures us that none of our assumptions are too far off the mark. Going further and taking it foregranted that the isoscalar-scalar correlator is indeed directly related to $\pi\pi$ scattering, then the sum rules provide an independent determination of the average light quark mass, as given in Eq. (37). Our result is quite compatible with sum rule determinations in the pseudoscalar sector and with the early unquenched lattice estimates that are starting to appear in the literature. We can then be certain that the $f_0(400-1200)$ is the dominant contribution to the $u\bar{u} + d\bar{d}$ current.

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A Renormalisation Group Conventions

The QCD $\beta$ and $\gamma$-functions are defined by the the Renormalisation Group Equations

$$\mu^2 \frac{d a_s}{d\mu^2} = -\beta(a_s), \quad \beta(a_s) = a_s^2 \sum_{i \geq 0} \beta_i a_s^i,$$

and

$$\mu^2 \frac{d m_q}{d\mu^2} = -m_q \gamma(a_s), \quad \gamma(a_s) = a_s \sum_{i \geq 0} \gamma_i a_s^i,$$

where $a_s = \alpha_s / \pi$.

Currently, the QCD $\beta$ and $\gamma$-functions are known to four-loop order. For three active flavours, the coefficients of the $\beta$ function within the $\overline{\text{MS}}$ scheme are $[63]$

$$\beta_0 = \frac{9}{4}, \quad \beta_1 = 4, \quad \beta_2 = \frac{3863}{384}, \quad \beta_3 = \frac{140599}{4608} + \frac{445}{32} \zeta(3).$$

The coefficients of the $\gamma$-function, also within the $\overline{\text{MS}}$ scheme, are $[64, 65]$

$$\gamma_0 = 1, \quad \gamma_1 = \frac{91}{24}, \quad \gamma_2 = \frac{8885}{576} - \frac{5}{2} \zeta(3),$$

$$\gamma_3 = \frac{2977517}{41472} - \frac{9295}{432} \zeta(3) + \frac{135}{16} \zeta(4) - \frac{125}{12} \zeta(5).$$

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