Role of electronic nematicity in the interplay between s- and d-wave broken-symmetry states

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To understand the role of electronic nematic order in the interplay between s- and d-wave particle-particle or particle-hole condensate states, relations between various s- and d-wave order parameters are studied. We find that the nematic operator transforms two independent six-dimensional vectors. The d-wave superconducting, d-density wave, and antiferromagnetic orders are organized into one vector, and the s-wave superconducting, charge density wave, and spin-triplet d-density wave orders into the other vector. Each vector acts as a superspin and transforms under the action of SO(6) where charge, spin, η- and π-pairing, spin-triplet nematic operators satisfy the SO(6) Lie algebra. Electronic nematic order is not a part of the SO(6) group. It commutes with all 15 generators. Our findings imply that nematic order does not affect the competition among the order parameters within the same superspin, while it strongly interferes the interplay between two order parameters that belong to different superspins. For example, nematicity allows a linear coupling between d- and s-wave superconducting order parameters which modifies the superconducting transition temperature. A generalized Ginzburg-Landau theory and further physical implications are discussed.

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INTRODUCTION

A minimal model for strongly correlated materials including the most complex systems such as the high temperature cuprates is the Hubbard or t-J model. However, even within the simplified Hubbard model away from the half-filling, there has been no consensus about the ground states, and various theoretical proposals have been made for the phase diagram of the high temperature cuprates.

One of such states, the staggered flux phase (d-density wave), was discussed in exitonic condensation[1], t-J model, and Hubbard model[2–5], and further suggested as a pseudogap phase of the cuprates[6]. The spin-triplet version of the staggered flux phase (spin-triplet d-density wave) was also discussed in the context of high temperature cuprates.[7–9].

Another proposed particle-hole condensate state of the angular momentum $l=2$ channel with broken rotational symmetry is the electronic nematic state.[10–18]. The spontaneous formation of nematicity has been invoked to explain the anisotropic transport observed in a two-dimensional electron gas in a high magnetic field[19, 20] and in Ru-based oxides[21]. Its relevance to high temperature cuprates was evidenced by a neutron scattering measurement where anisotropic scattering patterns have been observed in YBa$_2$Cu$_3$O$_{6.5}$.[22] From a weak-coupling point of view, it is sometimes also called Pomeranchuk instability[23].

Given various proposed order parameters, the interplay between s- and d-wave order parameters has been of intensive theoretical study. Examples for s- and d-wave order parameters are s- and d-wave superconductors, charge density wave, spin density wave, d-density wave, spin-triplet d-density wave, and nematic states.

Among them, it was reported that nematicity plays an important role in the interplay between s- and d-wave superconductors[24], and the spin density wave and spin-triplet d-density wave states[9]. However, a full set of the relations between them is still lacking.

In this paper, we offer a complete theory on how s- and d-wave orders transform via the nematic order, and relations between them. We found that the order parameters listed above can be organized in two independent six-dimensional vectors. One vector is composed of d-wave superconducting, d-density wave, and spin density wave order parameters, while the other vector contains s-wave superconducting, charge density wave, and spin-triplet d-density wave order parameters. Each vector transforms under the action of SO(6). Charge, spin, η-pairing[25], π-pairing[26], and spin-triplet nematic[27] operators together satisfy the SO(6) Lie algebra.

The nematic order parameter commutes with all generators, and hence is not a part of the SO(6) group. However, it transforms the two independent vectors connecting s- and d-wave order parameters. Our findings imply that nematic order does not interfere the competition between the order parameters within the same vector, but strongly affects the interplay between two order parameters belonging to different vectors. For example, it allows a linear coupling between s- and d-wave order parameters in the Ginzburg-Landau (GL) free energy, which modifies the physical properties of both phases. We found that the conditions for non-zero linear coupling differ for particle-particle and particle-hole condensate states.

Below we will review an SO(6) group theory and present the relations between the nematic order, the generators, and superspins. We will show the role of nematic order for particle-particle condensate states, and elabo-
rate a similar process for particle-hole cases. We will also discuss the implications of such relations in the context of GL free energy, and superconducting transition temperature related to high temperature cuprates.

**NEMATIC ORDER PARAMETER AND SO(6) GROUP**

It was first pointed out by Yang[25] that the $\eta$-pairing state is an eigenstate of the Hubbard model. The $\eta$-pairing operator is defined as $\hat{n}^\pm = -i \sum_{k,\sigma,\sigma'} c^\dagger_{k,\sigma,\sigma'} c_{k,\sigma} - c^\dagger_{k,\sigma} c_{k,\sigma'}$, and $\eta^\pm = (\hat{n}^\pm)^\dagger$, where $Q = (\pi, \pi)$ and $\sigma^y$ is a Pauli matrix. The $\eta$ operator carries charge 2e and spin 0, and commutes with the Hubbard Hamiltonian at half filling $\mu = U/2$, where $\mu$ is the chemical potential and $U$ is the on-site Hubbard interaction. It is also an eigenstate of the momentum operator with the eigenvalue $Q$.

Later it was found that the $\eta$-pairing operators combined with the charge operator satisfy an SU(2) algebra (named pseudospin)[28], and further recognized that the Hubbard model has two sets of commuting SU(2) symmetries. One set is characterized by the pseudospin of $\eta$-pairing and charge operators, and the other is conventional spin operator.[29] The three-dimensional vector transforming under the action of pseudospin SU(2) forms a superspin, where its three components are s-wave superconductor, and charge density wave with the ordering wave vector $Q$. It was also reported that the pseudospin SU(2) rotates another superspin composed of d-wave superconducting and d-density wave order parameters.[30]

On the other hand, the vector transforming under the spin SU(2) is the spin density wave with the ordering wave vector $Q$. Under a particle-hole transformation for one spin species, $c^\dagger_{1,i} \rightarrow (-1)^i c^\dagger_{i,1}$, the role of the two sets of SU(2) generators is interchanged. The same particle-hole transformation maps the positive Hubbard model to the negative Hubbard model, and it also maps the $\eta$-pairing to the Nagaoka ferromagnetic state.[31] A further generalization of the concept of the exact SO(4) symmetry of the Hubbard model to a unified theory of antiferromagnetism and d-wave superconductivity based on SO(5) symmetry was later proposed to understand the physics of the high temperature cuprates.[26, 32]

Here we present a full list of relations between $s$- and d-wave order parameters including those mentioned above. The ground state order parameters transform as a six-dimensional vector under the action of SO(6). They can be organized into a vector $\hat{n}_a (a = 1...6)$ which should satisfy

$$[\hat{L}_{ab}, \hat{n}_c] = -i (\delta_{bc} \hat{n}_a - \delta_{ac} \hat{n}_b).$$

where $\hat{L}_{ab}$ are generators of SO(6) based on the following operators:

$$\hat{Q} = -\frac{1}{2} \sum_{k\sigma} \left( c^\dagger_{k,\sigma} c_{k,\sigma} + c^\dagger_{k+Q,\sigma} c_{k+Q,\sigma} - 1 \right),$$

$$\hat{S}_a = \frac{1}{2} \sum_{k\sigma\sigma'} \left( c^\dagger_{k,\sigma} \sigma^\alpha_{\sigma\sigma'} c_{k,\sigma'} + c^\dagger_{k+Q,\sigma} \sigma^\alpha_{\sigma\sigma'} c_{k+Q,\sigma'} \right),$$

$$\hat{R}_a = \frac{1}{2} \sum_{k\sigma} d(k) \left( c^\dagger_{k,\sigma} \sigma^\alpha_{\sigma\sigma'} c_{k+Q,\sigma'} - c^\dagger_{k+Q,\sigma} \sigma^\alpha_{\sigma\sigma'} c_{k,\sigma'} \right),$$

$$\hat{\Pi}^+_a = \sum_{k\sigma} d(k) c^\dagger_{k,\sigma} (\sigma^\alpha \sigma^\beta)_{\sigma\sigma'} c_{k+Q,\sigma'}, \quad \hat{\Pi}^- = (\hat{\Pi}^+)\dagger,$$

$$\hat{\eta}^+ = -i \sum_{k\sigma\sigma'} c^\dagger_{k,\sigma} \sigma^\alpha_{\sigma\sigma'} c_{k+Q,\sigma'} \quad \hat{\eta}^- = (\hat{\eta}^+)\dagger,$$

where $k$ runs over the reduced Brillouin zone, $d(k) = \cos k_x - \cos k_y$, $\alpha$ takes the values $x, y, z$, and $\sigma^\alpha$ are the Pauli matrices.

The generators of SO(6) can be represented by an antisymmetric 6x6 matrix, $\hat{L}_{ab} = -\hat{L}_{ba}$.

$$\hat{L}_{ab} = \begin{pmatrix} 0 & \hat{Q} & \hat{\Pi}_x & \hat{\Pi}_y & \hat{\Pi}_z & \hat{\eta} \\ \hat{\eta} & 0 & \hat{\Pi}_x & \hat{\Pi}_y & \hat{\Pi}_z & 0 \\ \hat{\Pi}_x & \hat{\Pi}_y & 0 & \hat{\Pi}_x & \hat{\Pi}_y & \hat{\Pi}_z \\ \hat{\Pi}_y & \hat{\Pi}_z & \hat{\Pi}_x & 0 & \hat{\Pi}_x & \hat{\Pi}_y \\ \hat{\Pi}_z & 0 & \hat{\Pi}_y & \hat{\Pi}_x & 0 & \hat{\Pi}_x \\ \hat{\eta} & \hat{\Pi}_z & \hat{\Pi}_y & \hat{\Pi}_x & \hat{\Pi}_y & 0 \end{pmatrix},$$

where $\Re \hat{O} \equiv \frac{1}{2}(\hat{O}^- + \hat{O}^+)$ and $\Im \hat{O} \equiv \frac{1}{2}(\hat{O}^- - \hat{O}^+)$. It satisfies the correct SO(6) Lie algebra,[33]

$$[\hat{L}_{ab}, \hat{L}_{cd}] = -i \left( \delta_{ad} \hat{L}_{bc} + \delta_{bc} \hat{L}_{ad} - \delta_{bd} \hat{L}_{ac} - \delta_{ac} \hat{L}_{bd} \right).$$

Here $L_{12}$ is the charge operator, and $L_{34}, L_{35},$ and $L_{45}$ are the three components of the spin operator. $L_{16}$ and $L_{26}$ represent real and imaginary part of the $\eta$-pairing,[25] $L_{13}, L_{14},$ and $L_{15}$ ($L_{23}, L_{24}$ and $L_{25}$) denote $x, y$ and $z$ component of the real (imaginary) part of the $\eta$-pairing which carries charge 2e and spin 1 and represents a broken translational symmetry.[26] $L_{36}, L_{46},$ and $L_{56}$ correspond to the spin-triplet nematic order parameter carrying spin 1 and representing a broken $x$-$y$ symmetry on a square lattice.[27]

There exist two independent vectors. Each vector acts as a superspin, and transforms under the SO(6). This observation was first reported in Ref. [34], and a similar SO(6) symmetry was found in Fe-pnictide superconductors.[35] One superspin (superspin-1) consists of spin-density wave ($\Delta_{sdw}$), d-density wave ($\Delta_{ddw}$), and d-wave superconducting ($\Delta_{dsc}$) order parameters:

$$\Delta_{sdw}^\alpha = \frac{1}{2} \sum_{k\sigma\sigma'} \left( c^\dagger_{k,\sigma} \sigma^\alpha_{\sigma\sigma'} c_{k+Q,\sigma} + c^\dagger_{k+Q,\sigma} \sigma^\alpha_{\sigma\sigma'} c_{k,\sigma} \right),$$

$$\Delta_{dsc}^+ = \sum_{k} d(k) \left( c^\dagger_{k,\sigma} c^\dagger_{k+Q,\sigma} \sigma^\alpha_{\sigma\sigma'} c_{k,\sigma'} - c^\dagger_{k+Q,\sigma} c^\dagger_{k,\sigma} \sigma^\alpha_{\sigma\sigma'} c_{k+Q,\sigma'} \right),$$

$$\Delta_{dsc}^- = (\Delta_{dsc}^+)^\dagger,$$

$$\Delta_{ddw}^- = \frac{i}{2} \sum_{k\sigma} d(k) \left( c^\dagger_{k,\sigma} c_{k+Q,\sigma} - c^\dagger_{k+Q,\sigma} c_{k,\sigma} \right).$$
where $\hat{n}_1 = \Re \hat{\Delta}_{dsc}$, $\hat{n}_2 = \Im \hat{\Delta}_{dsc}$, $\hat{n}_3 = \hat{\Delta}_{tsf}$, $\hat{n}_4 = \hat{\Delta}_{cdw}^x$, $\hat{n}_5 = \hat{\Delta}_{cdw}^y$, and $\hat{n}_6 = \hat{\Delta}_{cdw}$.

The other superspin (superspin-2) rotated by the same 15 generators $\hat{L}_{ab}$ is composed of spin-triplet d-density wave ($\Delta_{cdw}$), charge density wave ($\Delta_{cdw}$), and s-wave superconducting ($\Delta_{ssc}$) order parameters:

$$\hat{N}^a_{fs} = \frac{i}{2} \sum_{k\sigma \sigma'} d(k) \left( c_{k,\sigma}^\dagger \sigma^a c_{k,\sigma'} + c_{k,\sigma'}^\dagger \sigma^a c_{k,\sigma} \right),$$

$$\hat{N}^+_ss = \sum_k \left( c_{k,\uparrow}^\dagger c_{k,\downarrow} + c_{k,\downarrow}^\dagger c_{k,\uparrow} \right),$$

$$\hat{N}^-_{ssc} = \left( \hat{N}^+_{ssc} \right)^\dagger,$n

$$\hat{N}_{cdw} = \frac{1}{2} \sum_{k\sigma} \left( c_{k,\sigma}^\dagger c_{k,\sigma} + c_{k,\sigma}^\dagger c_{k,\sigma} \right),$$

where the superspin is arranged as $\hat{N}_1 = \Re \hat{\Delta}_{ssc}$, $\hat{n}_2 = \Im \hat{\Delta}_{ssc}$, $\hat{n}_3 = \hat{\Delta}_{tsf}$, $\hat{n}_4 = \hat{\Delta}_{tsf}$, $\hat{n}_5 = \hat{\Delta}_{tsf}$, and $\hat{n}_6 = \hat{\Delta}_{tsf}$.

What is the role of the nematic order parameter within the SO(6) group? The nematic order parameter is given by

$$\hat{N} = \sum_{k\sigma} d(k) \left( c_{k,\sigma}^\dagger c_{k,\sigma} - c_{k,\sigma}^\dagger c_{k,\sigma} \right).$$

When $\langle \hat{N} \rangle \equiv N_0 \neq 0$, the phase is characterized by a broken x-y symmetry of the square lattice, and it is trivial to generalize to a broken point group symmetry in other lattices. Note that the nematic order parameter commutes with all 15 generators:

$$[\hat{N}, \hat{L}_{ab}] = 0.$$

This means that nematicity is not an SO(6) symmetry breaking field, and does not interfere with the competition between the order parameters within the superspin. For example, the phase diagram between antiferromagnetism and d-wave superconductivity (both belong to superspin-1) studied in the t-J model based on SO(5) symmetry[26] (a subset of the SO(6) in this study) is not modified by the presence of nematicity.

However, the nematic operator does not commute with the following conventional quantum rotor model.

$$H_{QR} = \frac{1}{2\chi} \sum_{i,a<b} \hat{L}_{i,a}^2 + \sum_{i<j,a} \hat{r}_a \hat{n}_i^a \hat{n}_j^a,$$ (8)

where the first term is the kinetic term and $\chi$ is the moment of inertia, and the second term is the potential term. The Hamiltonian has SO(6) symmetry when $r_a$ is identical to all $a$. Note that the nematic operator commutes with the first term, but not the second term. On the other hand, the competition between the order parameters in the same superspin $n^a$ is determined by difference in $r_a$.

What are relations between nematic order and other order parameters? The nematic operator transforms the components of the two independent superspins as follows:

$$\hat{\Delta}^+_{dsc}, \hat{\Delta}^-_{ssc} = 2\hat{N}, \quad \hat{\Delta}^-_{dsc}, \hat{\Delta}^+_{ssc} = 2\hat{N},$$

$$\hat{\Delta}_{dsc}, \hat{\Delta}_{cdw} = \hat{N}, \quad \hat{\Delta}_{tsf}, \hat{\Delta}_{cdw} = \hat{N},$$

$$\hat{\Delta}^+_{tsf}, \hat{\Delta}^-_{cdw} = \frac{i}{2}\hat{N}, \quad \hat{\Delta}^-_{tsf}, \hat{\Delta}^+_{cdw} = \frac{i}{2}\hat{N}.$$ (9)

The nematic operator transforms s - to d-wave order parameters which belong to different superspins.

The above results are summarized in the table below.

| SO(6) generators | $Q$, $\hat{S}$, $\hat{\Phi}$, $\hat{\Psi}$, spin-pairing, spin operators | nematic operator | commutes with generators & transforms superspin-1 and -2 |
|----------------|-------------------------------------------------|----------------|-------------------------------------------------|
| superspin-1    | dSC, dDW, SDW                                   | nematic operator | commutes with generators & transforms superspin-1 and -2 |
| superspin-2    | sSC, CDW, spin-triplet dDW                     | nematic operator | commutes with generators & transforms superspin-1 and -2 |

TABLE I: A summary of the SO(6) group and the relations to nematic order.

In the following section, we discuss the physical implications of the commutation relations using a GL free energy theory assuming that nematic order is present.[36]

**GINZBURG LANDAU THEORY**

The commutation relations in Eq. 9, $[A, B] = \hat{N}$, indicate that if $\langle \hat{N} \rangle \equiv N_0$ is finite, a linear coupling between A and B phases, such as $\gamma \hat{\Psi} \Phi$ with $\Phi = \langle A \rangle$ and $\Psi = \langle B \rangle$, may be present in the GL free energy. The GL free energy is then given by

$$\mathcal{F} = \frac{a}{2} \hat{\Psi}^2 + \frac{b}{2} \hat{\Phi}^2 + \gamma \hat{\Psi} \hat{\Phi} + a \hat{\Psi}^4 + b \hat{\Phi}^4 + ...$$ (10)

Assuming that $a > 0$, $b > 0$, and $ab > \gamma^2$ (none of the phases represented by $\Phi$ and $\Psi$ is ordered), the solutions of the two coupled equations for $\Phi$ and $\Psi$ leads to the following dispersion of modes:[37]

$$\chi \omega^2(k) = \frac{\epsilon_1(k) + \epsilon_2(k)}{2} \pm \frac{1}{2} \sqrt{(\epsilon_1(k) - \epsilon_2(k))^2 + 4\gamma^2},$$ (11)

where

$$\epsilon_1(k) = a + \rho \{(1 + N_0)k_x^2 + (1 - N_0)k_y^2\},$$

$$\epsilon_2(k) = b + \rho \{(1 + N_0)k_x^2 + (1 - N_0)k_y^2\}.$$ (12)
deviations from an ordering wave-vector which is either 0 or $Q$ depending on the nature of $\Psi$ (or $\Phi$).

One of the solutions becomes 0 when $\gamma = \sqrt{ab}$, leading to an ordered state. The condensed state is a linear combination of $\Psi$ and $\Phi$, and the dominant contribution depends on $a$ and $b$. Also, if one of them, say $\Psi$, is finite (when $a < 0$), the other, $\Phi$, is always induced as long as $\gamma$ is finite.

Is $\gamma$ always finite if nematic order exists? For example, consider a system in the nematic state with SO(6) symmetry at high temperatures. At low energy, the system spontaneously breaks the SO(6) symmetry, and one of the phases represented by $\Psi$ is stabilized. If $\Psi$ represents the d-wave superconducting state, $\Phi$ is the s-wave component. Similarly if $\Psi$ is the spin density wave, $\Phi$ should be the spin triplet d-density wave.[9] Does nematicity always lead to an induced order parameter of $\Phi$ without any extra condition? Below we show that it requires another condition for a non-zero linear coupling (in addition to the nematic order), and that the condition for a finite $\gamma$ differs for particle-particle and particle-hole condensates.

DIFERENCE BETWEEN PARTICLE-PARTICLE AND PARTICLE-HOLE PAIRS

Let us compute $\gamma$ for particle-particle condensate states. To check the condition for a non-zero linear coupling coefficient $\gamma$ between d-wave and s-wave superconducting cases ($\Psi = Re(\Delta_{dsc})$ and $\Phi = Re(\Delta_{ssc})$), we introduce $\psi^\dagger_k = (\xi^{\dagger}_{k_0}, e^{\dagger}_{k-k_0})$. Then the order parameter is written as $\Delta_{dsc}^\dagger = \sum_k \psi^\dagger_k \tau_1 \psi_k$. Inside the nematic state, the quasiparticle Green’s function is written as

$$ G^{-1}(\mathbf{k}, i\omega_n) = -i\omega_n + \epsilon_k - \mu, $$

where

$$ \epsilon_k = -2t(cos k_x + cos k_y) + 2td(\mathbf{k})N_0 - 4t' cos k_x cos k_y, $$

and $\mu$ is the chemical potential. $t$ and $t'$ represent the nearest neighbor and second nearest neighbor hoppings, respectively. Assuming that d- and s-wave superconducting fluctuations couple to fermions with interactions of $g_1$ and $g_2$, the $\gamma$ coefficient becomes

$$ \gamma_{dsc-ssc} = g_1 g_2 \sum_{\mathbf{k}} d(\mathbf{k}) \sum_{i\omega_n} Tr (G(\mathbf{k}, i\omega_n) \tau_1 G(\mathbf{k}, i\omega_n) \tau_1) $$

$$ = g_1 g_2 \sum_{\mathbf{k}} d(\mathbf{k}) \frac{n_F(\epsilon_k - \mu)}{2\xi_k}, $$

where $\xi_k = \epsilon_k - \mu$. $\gamma$ is always finite as long as $\mu$ and/or $t'$ is finite. In other words, when the particle-hole symmetry is broken and nematic order is present, the linear coupling term induces d- or s-wave superconducting order as we discussed in Eq. 10.

However, the above result is not true for particle-hole pairs. $\gamma$ then is zero independent of particle-hole symmetry. To examine the condition for particle-hole cases, let us introduce $\psi^\dagger_{k_0} = (\xi^{\dagger}_{k_0}, e^{\dagger}_{k+Q_0})$. In this basis, the Green’s function becomes

$$ G^{-1}(\mathbf{k}, i\omega_n) = -i\omega_n I + \epsilon_k \tau_3 - \mu_k I, $$

where $\epsilon_k = -2t(cos k_x + cos k_y) + 2td(\mathbf{k})N_0 - 4t' cos k_x cos k_y + \mu = \mu_k + Q_0$. The $\gamma$ coefficient for example between the charge density wave and the d-density wave order is then obtained as

$$ \gamma_{cdw-dsc} \propto T \sum_{\mathbf{k}} d(\mathbf{k}) \sum_{i\omega_n} Tr (G(\mathbf{k}, i\omega_n) \tau_1 G(\mathbf{k}, i\omega_n) \tau_2) $$

$$ = 0. $$

This is similar to the coupling between $Re\Delta_{dsc}$ and $Im\Delta_{ssc}$. This linear coupling is not allowed in the free energy due to symmetry.

Let us consider the coupling between different directions of spin density wave and spin-triplet d-density wave. For example, the coupling between antiferromagnetic fluctuations along the $x$-direction and spin-triplet d-density wave fluctuations along the $y$-direction are given by $\delta \Delta_{dsc}^x \propto \psi^\dagger_k \tau_1 \psi_k$ and $\delta \Delta_{tsf}^y \propto \psi^\dagger_k \tau_1 \psi_k$ where $\psi^\dagger_k = (\xi^{\dagger}_{k, 1}, e^{\dagger}_{k+Q, 1})$. Then the coefficient $\gamma$ is obtained as

$$ \gamma_{tsf-dsc} \propto \sum_{\mathbf{k} i\omega_n} d(\mathbf{k}) \sum_{\mathbf{k} i\omega_n} Tr (\tau_1 G(\mathbf{k} i\omega_n) \tau_1 G(\mathbf{k} i\omega_n)) $$

$$ = \sum_{\mathbf{k}} \frac{d(\mathbf{k})}{2\epsilon_k} (n_F(\epsilon_k - \mu) - n_F(\epsilon_k - \mu)) $$

$$ = 0. $$

Note that the coupling is also 0, because both $d(\mathbf{k})$ and $\epsilon_k$ change sign under $\mathbf{k} \rightarrow \mathbf{k} + Q$, while $\mu_k$ does not. The physical reason is that the spin density wave state breaks time reversal symmetry, while the triplet staggered flux does not. This fact is reflected in the commutation relations, where Eq. 9 has the imaginary factor $i$. Therefore, a linear coupling is not allowed between s- and d-wave particle-hole condensate states.

However, in the presence of a magnetic field $h$, the result alters. Note that the particle-particle and particle-hole order parameters are related by the particle-hole transformation, which also maps the chemical potential to the magnetic field to be discussed in detail below. In the presence of an external magnetic field, $\gamma$ changes to

$$ \gamma_{tsf-dsd} (h \neq 0) \propto \sum_{\mathbf{k}} \frac{d(\mathbf{k})}{2(\epsilon_k + h)} (n_F(\epsilon_k + h - \mu) - n_F(\epsilon_k + h - \mu)). $$
\( \gamma \) between the spin-triplet d-density wave and spin density wave is finite when \( N_0 \) and \( h \) are finite. The leading contribution of \( h \) and \( N_0 \) to \( \gamma(N_0, h) \) can be written as
\[
\gamma(N_0, h) = \gamma_0 N_0 h, \quad \gamma_0 \text{ depends on the interactions between fermions and the fluctuations of the order parameters.} \ [9]
\]

To understand the difference between particle-particle and particle-hole condensates, let us consider the particle-hole transformation. The particle-hole transformation mapping the positive to the negative Hubbard model discussed above maps each component of the superspins as follows:
\[
\hat{\Delta}^{\alpha}_{sdw} \rightarrow \left( \hat{\Delta}^{x}_{ssc}, \hat{\Delta}^{y}_{cdw} \right), \quad \hat{\Delta}^{\alpha}_{tsf} \rightarrow \left( \hat{\Delta}^{x}_{dsc}, \hat{\Delta}^{y}_{ddw} \right), \quad (19)
\]
where \( \alpha = x, y, z \).

In addition to the known result that the antiferromagnetic order transforms to the s-wave superconducting and charge density wave orders, we found that the spin-triplet d-density wave phase transforms to the d-wave superconducting and d-density wave orders, while nematic order is invariant. Since the chemical potential maps to the magnetic field under the particle-hole transformation, the conditions for a finite \( \gamma \) between particle-particle and particle-hole condensates are also related by the particle-hole transformation \( - \gamma_0 \sqrt{h} \langle \mathcal{R} \Delta_{dsc} \rangle / \langle \mathcal{R} \Delta_{ssc} \rangle \) maps to \( \gamma_0 \sqrt{h} \langle \mathcal{R} \Delta_{tsf} \rangle / \langle \mathcal{R} \Delta_{ssc} \rangle \) under the particle-hole transformation. Therefore, the linear coupling between d- and s-wave superconducting order parameters requires a finite chemical potential, while the coupling between spin-triplet d-density wave and spin density wave requires a magnetic field. Note that both breaks SO(6) symmetry, as the chemical potential and magnetic field appear as \( \mu L_{12} \) and \( hL_{34} \) in Hamiltonian, respectively.

**EFFECT ON SUPERCONDUCTING TRANSITION TEMPERATURE**

Let us reexamine the GL free energy, Eq. 10, to see if the superconducting transition temperature is modified by the coupling between the d- and s-wave superconducting order parameters. We consider \( \Psi = \langle \mathcal{R} \Delta_{dsc} \rangle \) and \( \Phi = \langle \mathcal{R} \Delta_{ssc} \rangle \), and \( \gamma \) is finite and proportional to the nematic strength \( N_0 \), and particle-hole symmetry is assumed to be broken.

Since the chemical potential couples to the charge operator \( L_{12} \), it favors the d-wave superconducting state over the antiferromagnetic and d-density wave states. The competition between antiferromagnetism, d-wave superconductor, and d-density wave is determined by SO(6) symmetry breaking terms, where nematicity does not affect the interplay between them.

Assuming that the superconducting state is stabilized in a finite window of phase space, and the transition temperature is set by \( T_c^0 \) (\( a < 0 \) below \( T_c^0 \) and assume \( b > 0 \)), we are interested in the effect of nematicity on the superconducting transition temperature. It is straightforward to check that the effective mass term \( a_{eff} \) is modified by \( a - \frac{\gamma^2}{2} \) after integrating out the \( \Phi \) field. Note that \( a \propto (T - T_c^0) \) and \( a_{eff} \propto (T - T_c) \) where \( T_c \) is the transition temperature modified by the coupling \( \gamma \). Since the effective mass gets smaller due to the coupling to the s-wave component, the transition temperature \( T_c \) is higher than \( T_c^0 \). However, one should note that the current description is based on a classical theory, and quantum fluctuations beyond the present study should be taken into account to see if the result may qualitatively change.

**DISCUSSION AND SUMMARY**

We have studied the role of the nematic order parameter in the interplay between s- and d-wave particle-particle or particle-hole condensate states. These condensate states include d- and s-wave superconductors, d-density wave, spin-triplet d-density wave, spin density wave, and charge density wave phases. We found that the nematic operator transforms d- to s-wave superconductors, spin-triplet d-density wave to (s-wave) spin-density wave, and d-density wave to (s-wave) charge-density wave operators. This can be summarized as a transformation between two different six-dimensional vectors. One vector is composed of d-wave superconductor, d-density wave, and spin-density wave order parameters, while the other vector consists of s-wave superconductor, charge-density wave, and spin-triplet d-density wave order parameters. Each vector acts as a superspin and transforms under the action of SO(6). There exist 15 generators, which correspond to charge, spin, spin-triplet nematic, \( \eta \)- and \( \pi \)-pairing operators, which form the SO(6) group.

The transformation between the two superspins via nematicity implies that a linear coupling between two order parameters that belong to two different vectors can be present in the GL free energy. Such a linear coupling allows induced ordering when one of them is condensed. However, we found that there is an additional condition for a non-zero linear coupling, which differs for particle-particle and particle-hole condensates. For example, when d-wave superconductor (particle-particle condensate) and nematic order coexist, s-wave superconducting order is induced, only when the particle-hole symmetry is broken. On the other hand, when spin-density wave (particle-hole condensate) and nematic order coexist, a similar transformation allows an induced spin-triplet d-density wave, only when time-reversal symmetry is broken. These results are consistent with symmetry considerations. Since the spin-triplet d-density wave does not break time reversal symmetry, while spin-density wave does, a linear coupling between the two order parameters is allowed when time reversal symmetry is broken by
an external magnetic field.

It is also interesting to notice that the nematic operator commutes with the generators. When the Hamiltonian contains a term $-g \sum_{ij} \hat{N}_i \hat{N}_j$ which favors nematic ordering, it does not act as an SO(6) symmetry breaking field. It means that the nematic order can exist without interfering the competition among the six different order parameters within a superspin. It is merely a spectator. However, it affects the interplay between order parameters which belong to two different superspins. Nematicity allows a linear coupling between the two order parameters, and affects the physical properties of both phases. As an example, we showed that the d-wave superconducting transition temperature is modified by the coupling to the s-wave superconducting order parameters which happens when nematicity is present and particle-hole symmetry is broken.

The nematic order parameter has been widely discussed in the context of strongly correlated materials. In particular, the phase diagram of the high temperature cuprates is complex and its complete understanding requires further experimental and theoretical investigation. Our results indicate that the proposed nematic phase affects phenomena in the superconducting phase such as an anisotropy in the spin susceptibility and an increase in superconducting transition temperature. It also affects antiferromagnetism via the coupling to the spin-triplet d-density wave when a magnetic field is applied. We do not attempt to find a microscopic Hamiltonian with SO(6) symmetry which is beyond the scope of the current study. However, we emphasize that Eq. 7 and 9 are exact independent of symmetry of Hamiltonian, and SO(6) symmetry which is useful to identify the compact relations between the nematic and other order parameters suggested in the context of high temperature cuprates. The GL free energy analysis hints the importance of nematicity for the phase diagram of antiferromagnetism and d-wave superconducting phase.

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