Stability analysis of a pile completely embedded into elastic foundation

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Abstract. A building mechanics problem on pile stability is solved. A pile is completely embedded into an elastic foundation and experiences an axial load. Both critical forces and critical modes are calculated. Some of critical states prove to be ones of bifurcation and each bifurcation state corresponds to two neighbor critical modes. Various fixing conditions at the endpoints are considered.

1. Introduction
Pile foundations are widely used as bearing systems and are subjected to static and dynamic loads. In practice, each building process starts from the pile fields setting up and pile stability is an important factor when designing a construction or erection. Through the literature, various models are proposed for interaction between a foundation (soil) and a pile. The Vinkler models appear to be the simplest ones in calculation aspect and these models are sufficiently effective [1-11]. In this case, a pile is modeled by the Euler-Bernoulli shaft [1, 3, 5, 9, 10, 12-15], the Timoshenko shaft [4, 16], the stepwise shaft [17], and shafts of variable bending stiffness [18, 19]. Besides the Vinkler models, various models of elastic foundation are applied: non-linear and heterogeneous [20-22], lamellar [23, 24], and two-parametrical ones [10, 15, 19]. In [25-28], frictional forces are taken into account among all the forces that determine pile’s stress-strain state. The influence of boundary conditions on critical forces and on proper modes is studied in some works [3, 27, 29, 30]. The effect of depth of embedding into a foundation is also studied in [6, 11, 13, 15, 31]. Besides analytical methods, numerical ones are also used in stability analysis: finite elements [32], finite difference [33], iteration [12, 18], and also the optimization problem for the system “pile-foundation” is considered [34].

In this work, a new matrix form of representation the resolving system of equations for a partially embedded pile is employed. This is achieved by application of initial parameters and a vector representation for a stress-strain state in each cross section of both a pile and a foundation. It is proposed that a certain reference horizon exists and it allows neglecting the frictional forces of pile’s lateral surface and their contribution into the bearing capability. A foundation is proposed to be solid, uniform, and linearly elastic and the latitude reaction modular does not change depending on depth. This approach is especially effective when a number of sites of compound pile and a number of foundation layers are large.
2. Problem statement
Consider an elastic pile of length \( L \) that is completely embedded into an elastic foundation. The equation of bending pile axes (figure 1) is of the form [1-5]

\[
EI \frac{d^4 y}{dx^4} + N \frac{d^2 y}{dx^2} + ky = 0
\]  

(1)

where \( y = y(x) \) – deflection, \( E \) – elasticity module, \( k \) – stiffness coefficient of foundation, \( I \) – inertia moment of pile’s cross section, \( N \) – pressing force. By introduction of some dimensionless variables and parameters

\[
\xi = x/L, \quad w = y/L, \quad \overline{N}^2 = NL^2/(4EI), \quad \alpha^4 = kL^4/(4EI)
\]

the other form of (1) is obtained:

\[
d^4w/d\xi^4 + 4\overline{N}^2 d^2w/d\xi^2 + 4\alpha^4 w = 0.
\]  

(2)

We search the solution of equation (2) by the Euler substitution

\[
w = Ae^{n\xi}
\]  

(3)

where \( A \) and \( n \) – constants to be determined. The characteristic equation for (2) is of the form

\[
n^4 + 4\overline{N}^2 n^2 + 4\alpha^4 = 0.
\]  

(4)

By substitution \( n^2 = m \), the biquadratic equation (4) takes the form of the quadratic one

\[
m^2 + 4\overline{N}^2 m + 4\alpha^4 = 0.
\]  

(5)

Its roots are expressed as (depending on the relation between \( \overline{N} \) and \( \alpha \)):

\[
m_{1,2} = -2\overline{N}^2 \pm 2\sqrt{\overline{N}^4 - \alpha^4} \quad \text{if} \quad \overline{N} > \alpha
\]  

(6)

\[
m_{1,2} = -2\overline{N}^2 \pm 2i\sqrt{\alpha^4 - \overline{N}^4} \quad \text{if} \quad \overline{N} < \alpha.
\]  

(7)

Notice that if to introduce a new parameter \( N_0 = 2\sqrt{kEI} \) of dimensional “force” and therefore called “conditional force” then the relations \( \overline{N} > \alpha, \overline{N} = \alpha, \) and \( \overline{N} < \alpha \) can be replaced by the relation between dimensional (physical) variables \( N \) and \( N_0 \):

\[
N > N_0, \quad N = N_0, \quad \text{and} \quad N < N_0
\]

because the following equalities hold:

\[
\overline{N} = \sqrt{NL^2/(4EI)} \quad \text{and} \quad \alpha = \sqrt{N_0L^2/(4EI)}.
\]

The roots (6) are used when \( \overline{N} > \alpha \) (in physical form, \( N > N_0 \)). Then, the roots of (4) are imaginary:

\[
n_{1,2} = \pm ic, \quad n_{3,4} = \pm id, \quad \text{where} \quad c = \sqrt{\overline{N}^2 - \alpha^2} + \sqrt{\overline{N}^2 + \alpha^2}, \quad d = \sqrt{\overline{N}^2 - \alpha^2} - \sqrt{\overline{N}^2 + \alpha^2}.
\]  

(8)

The deflection function (3) takes in such case the form

\[
w = A_1 \cos c \xi + A_2 \sin c \xi + A_3 \cos d \xi + A_4 \sin d \xi.
\]  

(9)
If $\alpha < N$ then the roots $m_{1,2}$ of the form (7) are used. In such case, the roots of (4) become complex:

$$n_{1,2,3,4} = \pm a \pm ib,$$

where $a = \sqrt{\alpha^2 - N^2}$, $b = \sqrt{\alpha^2 + N^2}$. \hfill (10)

The proper deflection function is of the form

$$w = A_1 \cosh \alpha \cos \xi + A_2 \sinh \alpha \cos \xi + A_3 \cosh \alpha \sin \xi + A_4 \sinh \alpha \sin \xi.$$

If $\alpha = N$ ($N = N_0$), one obtains from (6) (or (7)) multiple roots $m_{1,2} = -2N^2$ and correspondingly double roots of (4):

$$n_{1,2} = \pm ib, n_{3,4} = \mp ib$$

and the deflection function is of the form

$$w = (A_i + A_2 \xi) \cos \alpha \xi + (A_i + A_4 \xi) \sin \alpha \xi.$$

The integration constants $A_i$ ($1 \leq i \leq 4$) in the solutions of equation (2) are expressed by the initial parameters:

$$w_0 = w(0), w'_0 = w'(0), w''_0 = w''(0), w'''_0 = w'''(0).$$

Besides that the constants $A_i$ ($1 \leq i \leq 4$) gain a physical meaning, such form of solution is convenient when a partially embedded pile is studied; when a pile is embedded into a lamellae foundation; when a pile is of a stepwise construction, etc. The transformation procedure is demonstrated by an example using the function (9). By differentiation this function in $\xi$, we obtain dimensionless functions of rotation angle $\omega'$ of a cross section, of bending moment $\omega''$, and of transversal force $\omega'''$.

$$\omega' = c(-A_1 \sin \alpha \xi + A_2 \cos \alpha \xi) + d(-A_3 \sin \alpha \xi + A_4 \cos \alpha \xi)$$

$$\omega'' = c^2(-A_1 \cos \alpha \xi - A_2 \sin \alpha \xi) + d^2(-A_3 \cos \alpha \xi - A_4 \sin \alpha \xi)$$

$$\omega''' = c^3(\alpha_1 \sin \alpha \xi - \alpha_2 \cos \alpha \xi) + d^3(\alpha_3 \sin \alpha \xi - \alpha_4 \cos \alpha \xi).$$

Introduce a state vector $\vec{w}$ for an arbitrary cross section $\xi$:

$$\vec{w} = [\omega, \omega', \omega'', \omega''']^T,$$

vector of integration constants

$$\vec{A} = [A_1, A_2, A_3, A_4]^T,$$

and a matrix-function which is obtained by differentiating the function (9):

$$M_{4 \times 4} = \begin{pmatrix}
\cos \alpha \xi & \sin \alpha \xi & \cos \alpha \xi & \sin \alpha \xi \\
-c \sin \alpha \xi & c \cos \alpha \xi & -d \sin \alpha \xi & d \cos \alpha \xi \\
-c^2 \cos \alpha \xi & -c^2 \sin \alpha \xi & -d^2 \cos \alpha \xi & -d^2 \sin \alpha \xi \\
c^3 \sin \alpha \xi & -c^3 \cos \alpha \xi & d^3 \sin \alpha \xi & -d^3 \cos \alpha \xi
\end{pmatrix}.$$

Now, the system of equations (9), (14) can be represented in a simple matrix form:

$$\vec{w} = M\vec{A}.$$ \hfill (15)

Evaluating the functions (9) and (14) at the coordinate system origin $\xi = 0$, the following matrix equation is obtained that links the initial parameters with the integration constants $A_i$ ($i = 1, 4$):

$$\vec{w}_0 = L\vec{A} \text{ or equivalently } \vec{A} = L^{-1}\vec{w}_0$$

where vector $\vec{w}_0$, matrix $L$, and reverse matrix $L^{-1}$ are of the following form:

$$\vec{w}_0 = [w_0, w'_0, w''_0, w'''_0]^T.$$
Substituting vector $\vec{A}$ from (16) into (15), we obtain the matrix equation

$$\vec{w} = \Phi(\xi)\vec{w}_0,$$

where matrix $\Phi = ML^{-1}$ describes the influence of the initial parameters on the state of arbitrary pile’s cross section. The matrix components $\Phi = \{\Phi_{ij}\}$ $(i, j = 1, 2, 3, 4)$ takes the form

$$\Phi_{11} = \frac{d^2 \cos c \xi - c^2 \cos d \xi}{d^2 c^2}, \quad \Phi_{12} = \frac{d^3 \sin c \xi - c^3 \sin d \xi}{d^3 c^3}, \quad \Phi_{13} = \frac{\cos c \xi - \cos d \xi}{c^2 d^2 - c^4 d^2}, \quad \Phi_{14} = \frac{d \sin c \xi - c \sin d \xi}{d^2 c^2 - c^4 d^2};$$

$$\Phi_{21} = -(cd)^2 \Phi_{11}; \quad \Phi_{22} = \Phi_{11}; \quad \Phi_{23} = \Phi_{13}; \quad \Phi_{24} = \Phi_{13};$$

$$\Phi_{31} = -(cd)^2 \Phi_{13}; \quad \Phi_{32} = \Phi_{22}; \quad \Phi_{33} = \Phi_{23}; \quad \Phi_{34} = \Phi_{23};$$

$$\Phi_{41} = -(cd) \Phi_{23}; \quad \Phi_{42} = \Phi_{32}; \quad \Phi_{43} = \Phi_{33}; \quad \Phi_{44} = \Phi_{33}.$$

By application the previous procedure (16), we obtain the matrix equations

$$\vec{w} = \chi(\xi)\vec{w}_0 \quad \text{if} \quad N < \alpha \quad (N < N_o) \quad (18)$$

$$\vec{w} = \psi(\xi)\vec{w}_0 \quad \text{if} \quad N = \alpha \quad (N = N_o). \quad (19)$$

The matrix elements $\chi = \{\chi_{ij}\}$ are the functions

$$\chi_{11} = cha_2 \cos b \xi \sin h \xi - \frac{a^2 - b^2}{2ab}sha_2 \sin b \xi; \quad \chi_{21} = -(a^2 + b^2) \chi_{11};$$

$$\chi_{12} = \frac{3a^2 - b^2}{2a(a^2 + b^2)}sha_4 \cos b \xi - \frac{a^2 - 3b^2}{2b(a^2 + b^2)}cha_2 \sin b \xi; \quad \chi_{22} = \chi_{11};$$

$$\chi_{13} = \frac{1}{2ab}sha_2 \sin h \xi; \quad \chi_{23} = \frac{1}{2ab} (ach^2 \sin h \xi + bsha_2 \cos b \xi);$$

$$\chi_{14} = \frac{1}{2(a^2 + b^2)} (\frac{1}{b} cha_2 \sin h \xi - \frac{1}{a} sha_2 \cos b \xi); \quad \chi_{24} = \chi_{13};$$

$$\chi_{31} = -(a^2 + b^2) \chi_{11}; \quad \chi_{41} = -(a^2 + b^2) \chi_{11};$$

$$\chi_{32} = cha_2 \cos b \xi + \frac{a^2 - b^2}{2ab}sha_2 \sin b \xi; \quad \chi_{42} = \chi_{13};$$

$$\chi_{33} = cha_2 \cos b \xi + \frac{a^2 - b^2}{2ab}sha_2 \sin b \xi; \quad \chi_{43} = \frac{3a^2 - b^2}{2a}sha_2 \cos b \xi + \frac{a^2 - 3b^2}{2b}cha_2 \sin b \xi;$$

$$\chi_{34} = \chi_{23}; \quad \chi_{44} = \chi_{33};$$

The matrix elements $\psi = \{\psi_{ij}\}$ are the functions

$$\psi_{11} = \cos b \xi + \frac{h \xi}{2} \sin h \xi; \quad \psi_{12} = \frac{1}{2b} (3 \sin h \xi - b \xi \cos b \xi); \quad \psi_{13} = \frac{\xi \sin h \xi}{2b}; \quad \psi_{14} = \frac{\sin h \xi - b \xi \cos b \xi}{2b^2};$$

$$\psi_{21} = b \psi_{11}; \quad \psi_{22} = \psi_{11}; \quad \psi_{23} = \psi_{13}; \quad \psi_{24} = \psi_{13};$$

$$\psi_{31} = -b \psi_{13}; \quad \psi_{32} = \psi_{13}; \quad \psi_{33} = \psi_{13}; \quad \psi_{34} = \psi_{13};$$

$$\psi_{41} = -b \psi_{11}; \quad \psi_{42} = \psi_{11}; \quad \psi_{43} = \psi_{13}; \quad \psi_{44} = \psi_{13}.$$

Thus, pile’s stress-strain state is described by one of the relations (17)-(19). Further analysis of the obtained solutions (17)-(19) is possible only after concretization of boundary values.
3. Results and discussion

3.1. Critical forces of a pile

In this Subsection, all possible combinations of critical force \( N \) and “conditional force” \( N_0 \) (this force characterizes the stiffness of the system “pile-foundation” under condition of symmetrical fixing of pile’s endpoints: hinges, clamping, and free endpoints) are studied in order to determine both critical forces and critical modes.

3.1.1 Hinged support of pile’s endpoints

In this case, \( \xi = 0 \) and \( \xi = 1 \). The boundary conditions are of the form

\[
\begin{align*}
(w_0)' &= w_0'' = 0, \\
(w_1)' &= w_1'' = 0.
\end{align*}
\]

We consider that foundation stiffness \( k \) (if pile’s bending stiffness \( EI \) is fixed) corresponds to conditional force \( N_0 \). Initially, we adopt that pile’s critical force is equal to some “conditional force” \( N_0 \). It follows the variant (12): \( N = \alpha \) (or \( N = N_0 \)). The deflection field is determined by the function

\[
w = w_0 \psi_{12}(\xi) + w_0' \psi_{14}(\xi) + w_0'' \psi_{13}(\xi) + w_0''' \psi_{14}(\xi)
\]

which is obtained from the matrix equation (19) by multiplying the first row of matrix \( w_0 \). Taking into account the first pair of boundary conditions (20), the deflection field \( w(\xi) \) takes the form

\[
w = w_0 \psi_{12}(\xi) + w_0'' \psi_{14}(\xi).
\]

Meeting the requirements of second pare in conditions (20), obtain a system of algebraic equations relative to unknown initial parameters \( w_0' \) and \( w_0'' \).

\[
\begin{align*}
\psi_{12}(1) w_0' + \psi_{14}(1) w_0'' &= 0, \\
\psi_{32}(1) w_0' + \psi_{34}(1) w_0'' &= 0.
\end{align*}
\]

The condition of existence of non-zero solutions of a given heterogeneous system of linear equations is that a proper determinant equals zero. That is,

\[
\begin{vmatrix}
\psi_{12}(1) & \psi_{14}(1) \\
\psi_{32}(1) & \psi_{34}(1)
\end{vmatrix} = \psi_{12}(1) \psi_{34}(1) - \psi_{14}(1) \psi_{32}(1) = 0.
\]

Substituting elements \( \psi_{ij}(1) \) of matrix \( \psi(\xi) \) into equation (23), obtain

\[
sin b = 0
\]

in what follows that

\[
b_n = n\pi, \quad \text{where } n \text{ is an integer number.}
\]

By using the denotation \( b = \sqrt{\alpha^2 + \bar{N}^2} \) and taking into account \( \bar{N} = \alpha \) \((N = N_0)\), it follows from (24) that

\[
\bar{N}_n = b_n / \sqrt{2} = n\pi / \sqrt{2}
\]

where \( \bar{N}_n \) – sequence of dimensionless force values and each value corresponds with a certain bending critical mode. Considering a critical force as a minimum force under which the pile stability is violated, we accept critical mode’s number \( n = 1 \). Then,

\[
\bar{N}_1 = \bar{N}_n = \pi / \sqrt{2}
\]
Returning to dimensional compressing force $N$, obtain $$\sqrt{\frac{N_e L^2}{4EI}} = \frac{\pi}{\sqrt{2}}.$$ It follows that $N_e = 2N_{e0}$, where $N_{e0} = \frac{\pi^2 EI}{L^2}$ – critical Euler’s force of a free (i.e., not supported by foundation) rod with the same fixing conditions as ones for a pile. It follows from $N = N_0$ that

$$N_e = N_0 = 2N_{e0}.$$ (25)

The result (25) should be treated in the following way: if a combination the mechanical parameters of a pile and a foundation leads to $N_0 = 2N_{e0}$ then the critical force will be equal to $N_e = N_0$. If a combination of the mechanical parameters yields $N_0 \neq 2N_{e0}$ then the critical force calculation should be carried out according to either variant (8)-(9) or variant (10)-(11) depending on the sign of the difference $N_0 - 2N_{e0}$. Substituting the functions $\psi_{12}(\xi)$ and $\psi_{14}(\xi)$ into (21) along with the relation $w_0^* / w_0^* = -b^2$, which may be obtained from any equation of system (22), the equation of bending pile’s axis takes the form

$$w_0 = f \sin n\pi \xi$$

where $f$ – bending deflection. This sinusoid has $n$ half-waves. If $n = 1$ then the critical mode has one half-wave:

$$w = f \sin \pi \xi.$$ By taking sequentially $n = 2, 3, ..., $ we obtain all possible critical modes. As for the critical force, it will be equal to $N_2 = 8N_{e0}, \ N_3 = 18N_{e0}, ... .$

3.1.2 Hinged support: critical force $N$ is greater than conditional force $N_0$

Selecting the deflection function

$$w = w_0 \phi_1(\xi) + w_0^* \phi_2(\xi) + w_0^* \phi_3(\xi) + w_0^* \phi_4(\xi)$$ (26)

from the matrix equation (17) and using the boundary conditions (20), we obtain

$$\sin c \sin d = 0$$

with the solutions

$$C_n = n\pi \quad \text{and} \quad d_n = n\pi.$$ (27)

Introduce new denotations $\tilde{N} = N / N_{e0}$ and $\tilde{N}_0 = N_0 / N_{e0}$. Then, it follows from (27):

$$\tilde{N} = n^2 + \frac{\tilde{N}_0^2}{4n^2}.$$ (28)

and by letting $n = 1, 2, 3, ..., \$ a sequence of force $\tilde{N}$ values is obtained. In contrast to a free bending rod, the minimum number of half-waves in a critical mode is not always equal to 1. This number has to be determined by differentiating (28) in $n$ aiming to find the point of minimum [35]. It follows that

$$n = \sqrt{\tilde{N}_0 / 2}.$$ (29)

Function $\tilde{N}(n)$ is a linear one relative to the parameter $\tilde{N}_0^2 / 4$. By letting $n = 1, 2, 3, ...$, obtain a set of straight lines of the form depicting in figure 2:

$$\tilde{N} = A + Bx$$ where $A = n^2, \ B = \frac{1}{n^2}, \ x = \frac{\tilde{N}_0^2}{4}$. 

6
The transition from \( n \)-mode to \((n+1)\)-mode corresponds to the equation

\[
n^2 + \frac{N_0^2}{4n^2} = (n + 1)^2 + \frac{N_0^2}{4(n + 1)^2}.
\]

in what follows that

\[
\frac{N_0}{2} = n^2(n + 1)^2.
\]

That is, if \( n = (N_0/2)^2 \leq 4 \) then one half-wave of length \( L \) is formed during the equilibrium loss. The critical force is calculated by

\[
\tilde{N}_{cr} = 1 + (N_0/2)^2.
\]

If the inequality \( 4 \leq (N_0/2)^2 \leq 36 \) holds then two half-waves of length \( L/2 \) form during the stability loss. The critical force in such case is calculated by

\[
\tilde{N}_{cr} = 4 + (N_0/2)^2 / 4.
\]

The dependence (28) shows that \( N > N_0 \) for every \( N_0 \neq 2N_{cr0} \). It means that the variant \( N < N_0 \) is not realized. Figure 3 shows the increase in critical force. The critical modes are described by (21) if \( \tilde{N}_{cr} = 2, 8, 18, ... \) and by (26) in all other events taking into account the boundary conditions \( w_0 = w_0^* = 0 \).

**Figure 2.** Modes (shapes) of stability loss

**Figure 3.** Critical force versus conditional force under hinged fixing
3.2. Clamped pile’s endpoints.

The boundary conditions take the form

\[ w_0 = w_0' = 0 \]
\[ w(1) = w'(1) = 0. \]  

(30)

By analogy with Subsection 3.1.1, suppose that \( N = N_0 \) or (in the dimensionless form) \( \tilde{N} = \alpha \). The deflection function \( w(\tilde{\xi}) \), from the matrix equation (19), together with the first pair of boundary conditions (30) takes the form

\[ w = w_0^* \psi_{13}(\tilde{\xi}) + w_0^\alpha \psi_{14}(\tilde{\xi}). \]

Taking into account the second pair of equations (30), we deduce a system of algebraic equations relative to unknown initial parameters \( w_0^* \) and \( w_0^\alpha \), which is analogical to the system (22). Letting the determinant of the new system be equal to zero, we obtain the equation for critical force

\[ \psi_{13}(1) \psi_{24}(1) - \psi_{14}(1) \psi_{23}(1) = 0. \]  

(31)

The substitution here the functions entering the equation (31) yields

\[ b^2 = \sin^2 b, \]

the only root of which is

\[ b = 0. \]

Taking into account \( \tilde{N} = b / \sqrt{2} \), obtain \( N_{cr} = N_0 = 0 \). It means that the supposition \( N = N_0 \) leads to the contradiction and this case proves to be non-realistic.

Now, suppose that \( \tilde{N} < N_0 \) or \( \tilde{N} < \alpha \). By analogy to the previous sections, deduce an equation for critical force by writing the matrix equation (18) along with the boundary conditions (30). Require that the determinant of the system deduced be equal to zero:

\[ \chi_{13}(1) \chi_{24}(1) - \chi_{14}(1) \chi_{23}(1) = 0. \]  

(32)

Substituting proper elements of matrix \( \chi(\tilde{\xi}) = \{ \chi_0 \} \), into (32) obtain the equation

\[ 2\alpha^2 (\text{sha} \sin b)^2 - a^2 (\text{cha} \sin b)^2 + b^2 (\text{sha} \cos b)^2 = 0. \]  

(33)

Taking into account \( a \neq 0 \), because of \( \alpha \neq \tilde{N} (N_0 \neq N) \), the minimum root of (33) equals to

\[ b = \sqrt{\alpha^2 + \tilde{N}^2} = 0. \]

It means that

\[ \alpha = \tilde{N} = 0 \]

that is, the supposition \( N < N_0 \) leads to a contradiction.

So, the only opportunity \( \tilde{N} > N_0 \) can be realized when both pile’s endpoints are clamped. In this event, the critical mode is described by (25) along with \( w_0 = w_0' = 0 \). We have:

\[ w = w_0^* \phi_{13}(\tilde{\xi}) + w_0^\alpha \phi_{14}(\tilde{\xi}). \]

By using the second pare of boundary conditions (30), obtain the equation for the critical force:

\[ \phi_{13}(1) \phi_{24}(1) - \phi_{14}(1) \phi_{23}(1) = 0. \]  

(34)

Substituting here proper elements of matrix \( \phi(\tilde{\xi}) = \{ \phi_0(\tilde{\xi}) \} \) we obtain the equation

\[ \frac{\cos c - \cos d}{d^2 - c^2} + \frac{(c \sin c - d \sin d)(d \sin c - c \sin d)}{cd(d^2 - c^2)} = 0. \]  

(35)

In order to find a general solution of equation (35), it is convenient to move to forces \( N \) and \( N_0 \) by means of transformations
\[ c - d = 2\sqrt{N^2 + \alpha^2}; \quad c + d = 2\sqrt{N^2 - \alpha^2}; \]
\[ c^2 - d^2 = 4\sqrt{N^2 - \alpha^2}; \quad c^2 + d^2 = 4N^2; \quad cd = -2\alpha^2; \]
\[ \tilde{N}^2 = \pi^2 \frac{N}{N_{cr0}}; \quad \alpha^2 = \pi^2 \frac{N_0}{N_{cr0}}. \]

Here, \( N_{cr0} = 4\pi^2 EI / L^2 \) – critical force of a free rod clamped at its endpoints. After some algebraic transformations and trigonometric substitutions, deduce the equation for critical force:
\[ \left(\frac{x}{y} + 1\right)\cos 2\pi \sqrt{y-x} + \left(\frac{x}{y} - 1\right)\cos 2\pi \sqrt{y+x} = 2 \frac{x}{y}. \] (36)
Here, \( x = N_0 / N_{cr0} \) and \( y = N / N_{cr0} \).

**Figure 4.** Critical force depending on conditional force (endpoint are clamped)

In figure 4, the dependence of critical force \( \tilde{N} \) on the stiffness parameter “pile-foundation” \( N_0 \) is shown. The salient points (\( x = 1.5, 4.5, 6.2 \) and so on) are points of changing of the first mode of stability lost under increasing in stiffness of the “pile-foundation” system.

### 3.3 Free pile’s endpoints

When considering one more case of symmetric boundary conditions – free ends, one can show that in this case also the only variant \( N > N_0 \) (or \( \tilde{N} > \alpha \)) is realized. Then, the boundary conditions are of the form
\[ w_0'' = 0, \quad w_0'' + 4\tilde{N}^2 w_0' = 0, \]
\[ w''(1) = 0, \quad w''(1) + 4\tilde{N}^2 w'(1) = 0. \]

Given the condition \( N_{cr} = N_0 \) (\( \tilde{N} = \alpha \)) and making calculations by analogy with the previous Subsection 3.1 and Subsection 3.2, the following equation for critical force is obtained:
\[ \psi_{31}(1)\psi_{21}(1) - \psi_{31}(1)\psi_{21}(1) + 4\tilde{N}^2 \left[ \psi_{31}(1)\psi_{23}(1) - \psi_{33}(1)\psi_{21}(1) \right] + 16\tilde{N}^4 \left[ \psi_{33}(1)\psi_{23}(1) - \psi_{31}(1)\psi_{23}(1) \right] = 0. \] (37)
Substituting the values \( \psi_{31}(1) \), into (37), obtain the equation
\[ b^2 - 9\sin^2 b = 0 \] (38)
with the minimum root $b = 0$. That is, the proposition $N_{cr} = N_0$ ($\bar{N} = \alpha$) is not valid and contradicts to the problem conditions.

Given the condition $N < N_0$ ($\bar{N} \alpha$), the analysis of equation

$$
\chi_{31}(1)\chi_{32}(1) - \chi_{31}(1)\chi_{41}(1) + 4\bar{N}^2\left[\chi_{31}(1)(\chi_{32}(1) - \chi_{41}(1)) + \chi_{34}(1)(\chi_{31}(1) - \chi_{42}(1))\chi_{32}(1)\chi_{32}(1)\right] +
+16\bar{N}^4\left[\chi_{31}(1)\chi_{31}(1) - \chi_{32}(1)\chi_{34}(1)\right] = 0
$$

shows that the condition $N < N_0$ ($\bar{N} \alpha$), is also invalid.

Given the condition $N > N_0$, the critical mode is described by the function

$$
w(\xi) = w_0\varphi_{32}(\xi) + w_0'\left[\varphi_{32}(\xi) - 4\bar{N}^2\varphi_{34}(\xi)\right].
$$

In order to calculate the critical force, employ an equation that is obtained from (37) if to replace $\Psi_j(\xi)$ by $\varphi_{ij}(\xi)$:

$$
\varphi_{31}(1)\varphi_{32}(1) - \varphi_{32}(1)\varphi_{41}(1) + 4\bar{N}^2\left[\varphi_{31}(1)(\varphi_{32}(1) - \varphi_{41}(1)) + \varphi_{34}(1)(\varphi_{31}(1) - \varphi_{32}(1))\right] +
+16\bar{N}^4\left[\varphi_{24}(1)\varphi_{31}(1) - \varphi_{24}(1)\varphi_{34}(1)\right] = 0.
$$

### 3.4 Cantilevered fixing

Consider non-symmetrical fixing of pile’s endpoints. Let the lower endpoint be clamped while the upper one be free – see figure 5.

![Figure 5. Cantilevered fixing of pile](image)

The boundary conditions are of the form

$$
w_0 = w'_0 = 0 \quad (40)
$$

$$
w''(1) = 0, \quad w''(1) + 4\bar{N}^2w'(1) = 0.
$$

By analogy with the previous, consider the case $N_{cr} = N_0$ ($\bar{N} = \alpha$). Write the matrix equation (19) in a detailed form:

$$
\begin{pmatrix}
w(1) \\
w'(1) \\
w''(1)
\end{pmatrix} =
\begin{pmatrix}
\psi_{11}(1) & \psi_{12}(1) & \psi_{13}(1) & \psi_{14}(1) \\
\psi_{21}(1) & \psi_{22}(1) & \psi_{23}(1) & \psi_{24}(1) \\
\psi_{31}(1) & \psi_{32}(1) & \psi_{33}(1) & \psi_{34}(1) \\
\psi_{41}(1) & \psi_{42}(1) & \psi_{43}(1) & \psi_{44}(1)
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
w_0'' \\
w_0''
\end{pmatrix},
$$

It follows (along with the second condition of (40)) the system of algebraic equations relative to unknown parameters $w_0''$ and $w_0''$:

$$
\begin{align*}
\psi_{33}(1)w_0'' + \psi_{34}(1)w_0'' &= 0 \\
(\psi_{43}(1) + 4\bar{N}^2\psi_{23}(1)w_0'' + (\psi_{44}(1) + 4\bar{N}^2\psi_{24}(1))w_0'' &= 0
\end{align*}
$$

where the determinant value must equal to zero:
Expanding this determinant, obtain the equation

\[ \psi_{33}(1)\psi_{44}(1) - \psi_{34}(1)\psi_{43}(1) + 4\overline{N}^2(\psi_{24}(1)\psi_{33}(1) - \psi_{23}(1)\psi_{34}(1)) = 0. \quad (41) \]

Substituting functions \( \psi_{ij} \) into (41) and taking into account that \( \overline{N} = b / \sqrt{2} \) under the condition \( \overline{N} = \alpha \), obtain the transcendent equation

\[ \cos^2 b = \frac{1}{3}(b^2 - 1), \quad (42) \]

the minimal root of which \( b = 1.1896 \) allows to determine the critical force \( N \) from the equality

\[ \sqrt{\frac{NL}{4EI}} = \frac{b}{\sqrt{2}}. \]

The critical force for a proper free rod is

\[ N_{cr0} = \frac{\pi^2 EI}{4L}. \]

Then, the sought-for critical force (equal to the conditional force) is determined from the expression

\[ N_{cr} = N_0 = \left(\frac{2\sqrt{2}}{\pi}b\right)^2 N_{cr0} \approx 1.15 N_{cr0} \quad (43) \]

Thus, if a combination of cantilevered pile and foundation parameters leads to the equality \( N_0 = 1.15 N_{cr0} \) then the critical force is also equal to \( 1.15 N_{cr0} \). If \( N_0 \neq 1.15 N_{cr0} \) then calculation of critical force is carried out according to (8-9) if \( N > N_0 \) or according to (10-11) if \( N < N_0 \).

3.4.1 Critical force exceeds the conditional one

In this case \( N > N_0 \), the equation for critical force is obtained after replacing \( \psi_{ij}(1) \) in equation (41) by \( \varphi_{ij}(1) \). These results in

\[ \varphi_{33}(1)\varphi_{44}(1) - \varphi_{34}(1)\varphi_{43}(1) + 4\overline{N}^2(\varphi_{24}(1)\varphi_{33}(1) - \varphi_{23}(1)\varphi_{34}(1)) = 0. \quad (44) \]

Substitution the functions \( \varphi_{ij}(1) \) into (44) yields

\[ \left(2\left(\frac{\alpha}{\overline{N}}\right)^4\right)\cos c \cos d - \left(\frac{\alpha}{\overline{N}}\right)^2 \sin c \sin d - \left(\frac{\alpha}{\overline{N}}\right)^4 = 0 \]

or, taking into account \( \alpha / \overline{N} = \sqrt{N_0 / N} \), reduce the latest equation to

\[ 2 - \left(\frac{x}{y}\right)^2 \cos c \cos d - \frac{x}{y} \sin c \sin d - \left(\frac{x}{y}\right)^2 = 0 \quad (45) \]

where \( x = N_0 / N_{cr} \), \( y = N / N_{cr0} \), and \( N_{cr0} = \pi^2 EI / (4L^2) \). Relative critical force versus relative conditional force is shown in figure 6.

3.4.2 Conditional force exceeds the critical one

In case \( N < N_0 \) (\( \overline{N} < \alpha \)), the equation for critical force is of the same form as (41) and (44). Substituting the function values \( \varphi_{ij}(1) \), into this equation, obtain

\[ \varphi_{33}(1)\varphi_{44}(1) - \varphi_{34}(1)\varphi_{43}(1) + 4\overline{N}^2(\varphi_{24}(1)\varphi_{33}(1) - \varphi_{23}(1)\varphi_{34}(1)) = 0. \quad (46) \]

After some transformations, equation (46) is reduced to

\[ (b^2 - 3a^2)b^2ch^2a - (3b^2 - a^2)a^2 \cos^2 b = (a^2 - b^2)^2. \quad (47) \]
Replacing $a$ and $b$ by $\bar{N}$ and $\alpha$ by means of the relations

$$
\begin{align*}
a^2 &= \alpha^2 - \bar{N}^2, & b^2 &= \alpha^2 + \bar{N}^2, & a^2 + b^2 &= 2\alpha^2, & a^2 - b^2 &= -2\bar{N}^2
\end{align*}
$$

transform (47) to the equation

$$
2(\bar{N}/\alpha)^4 + (\bar{N}/\alpha)^2 - 1 = c h^2 \alpha - 2(\bar{N}/\alpha)^4 + (\bar{N}/\alpha)^2 + 1) \cos^2 b + 2(\bar{N}/\alpha)^4
$$

or, factorizing the coefficients at $ch^2 \alpha$ and $\cos^2 b$, obtain

$$
2(\frac{N}{N_0} - 1) (1 + \frac{N}{N_0}) ch^\alpha \sqrt{1 - \frac{N}{N_0}} = 2(\frac{N}{N_0} + 1) (1 - \frac{N}{N_0}) \cos^2 \alpha \sqrt{1 + \frac{N}{N_0}} + 2(\frac{N}{N_0})^2.
$$

Note that $\bar{N} < \alpha$ and consequently $N/N_0 < 1$. It means that the right side of (48) is positive. In order that both the left and the right sides be positive, it is necessary that the relative force meets the double inequality $0.5 < N/N_0 < 1$. If to consider both critical and conditional forces be related to $N_{cr0}$, then this inequality along with equation (48) take the form

$$
\frac{y}{x} - 1 + 2\left(\frac{y}{x}\right)^2 ch^2 \frac{\pi}{4} \sqrt{x-y} = \frac{y}{x} + 1 - 2\left(\frac{y}{x}\right)^2 \cos^2 \frac{\pi}{4} \sqrt{x+y} + 2\left(\frac{y}{x}\right)^2.
$$

In figure 7, the dependence of critical force versus conditional one is depicted.

The results obtained coincide with [29]. Note that the results [29] were obtained without the use of state vector, influence matrix, and initial parameters matrix. This restricts the application [29] to the stability analysis of piles embedded partially into a foundation, ladder-shaped piles, and lamellar foundations.

**Figure 6.** Critical force versus conditional one for cantilevered pile
4. Conclusion

Analytical relationships for critical force determination during stability loss of a compressed rod have been constructed. Here, the compressed rod simulates a standing pile embedded completely into an elastic foundation. These relationships were obtained in the framework of the initial parameters method using state vector of rod’s cross section and influence matrices (here, the influence of initial parameters on the cross section state is considered). Functional dependencies of critical force on a generalized parameter of stiffness (conditional force) for the system “rod-foundation” under various fixing conditions for pile’s endpoints, both symmetric (hinge-hinge, clamping, and free endpoints) and non-symmetric (cantilever), were studied. Both critical force and conditional force are expressed via proper critical force of a free rod.

The usage of initial parameters allows applying the developed methodology to the stability analysis of ladder-shaped piles, built-up piles, and piles embedded partially or completely into lamellar foundations.

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