Abstract

In this paper which is an extension of the work [1], we study the conditions required for validity of the generalized second law in phantom dominated universe in the presence of Schwarzschild black hole. Our study is independent of the origin of the phantom like behavior of the considered universe. We also discuss the generalized second law in the neighborhood of transition (from quintessence to phantom regime) time. We show that even for a constant equation of state parameter, the generalized second law may be satisfied provided that the temperature is not taken as de Sitter temperature. It is shown that in models with (only) a transition from quintessence to phantom regime the generalized second law does not hold in the transition epoch.

1 Introduction

Astrophysical data show that the universe is accelerating [2]. Based on some data, it is possible to consider an evolving equation of state parameter, \( \omega \), less than \(-1 \) at present time from \( \omega > -1 \) in the near past [3]. In this view we may assume that the universe is filled with a perfect fluid with a negative pressure and \( \omega < -1 \), dubbed as phantom dark energy [4]. A candidate for phantom dark energy is a phantom scalar field with wrong sign for kinetic energy term [5]. Another method to study the present inflation is to use a running cosmological constant based on principles of quantum field theory (specially on the renormalization group) which can mimic the phantom like behavior of the universe [6].

This description of the universe may contain finite time future singularity accompanied with dark energy density singularity called big rip. The big rip may be avoided by the effect of gravitational backreactions which can
end the phantom dominated regime \cite{7}. We can consider horizons for the accelerating universe and associate entropy (as a measure of our ignorance about what is going behind it) and temperature to them \cite{8,9,10,11,12,13,14}. In this way one is able to study the thermodynamics of a system consisting of dark energy perfect fluid and the horizon.

In phantom dominated universe black holes lose their masses by accreting phantom fluid \cite{16}. Therefore their areas and consequently their entropies will decrease. So it may be of interest to know that if the generalized second law of thermodynamics (GSL) is satisfied in this situation. Indeed if the thermodynamics parameters assigned to the universe are the same as the ordinary thermodynamics parameters known in physical systems, then one expects that thermodynamics laws be also satisfied for the universe.

Thermodynamics of an accelerating universe has been studied in several papers \cite{17}. In \cite{8} and \cite{9}, the generalized second law for cosmological models that depart slightly from de Sitter space and also when the horizon shrinks, was studied respectively. The thermodynamics of super-accelerated universe in a de Sitter and quasi de Sitter space-time was the subject of the paper \cite{10}.

In \cite{11}, it was shown that for a phantom dominated universe with constant $\omega$ the total entropy is a constant and for time dependent $\omega$, via two specific examples, the validity of GSL was verified. In \cite{12} the conditions of validity of GSL in more general cases, including the transition epoch (from quintessence to phantom), and for temperatures proportional to de Sitter temperature were studied independently of the origin of dark energy.

In a recent paper the author of \cite{1}, using phantom scalar field model, showed that GSL is violated in the presence of a Schwarzschild black hole in the cases studied in \cite{11} and in phantom dominated era. In this paper we try to study the same problem but by considering a temperature other than de Sitter temperature. Our study is independent of the origin of phantom like behavior of the universe. We also consider the possibility of transition from quintessence to phantom regime and discuss the validity of GSL in the neighborhood of transition time in the presence of the black hole.

We use the units $\hbar = c = G = k_B = 1$.

## 2 GSL in the phantom dominated FRW universe in the presence of a Schwarzschild black hole

We consider spatially flat Friedman Robertson Walker (FRW) metric with scale factor $a(t)$:

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2).$$  \hspace{1cm} (1)

The Hubble parameter is given by $H = \dot{a}/a$. The over dot shows derivative with respect to the comoving time $t$. The equation of state of the universe
which is assumed to behave as a perfect fluid at large scale is given by

\[ p = \omega \rho, \]

where \( \omega \) is the equation of state parameter. For an accelerating universe, i.e. \( \ddot{a} > 0 \), we have \( \omega < -1/3 \). The future event horizon, \( R_h \), is given by

\[ R_h(t) = a(t) \int_t^{\infty} \frac{dt'}{a(t')} \]

where \( \lim_{t \to \infty} a(t) = \infty \) and \( \int_t^{\infty} dt'/a(t') < 0 \). In the presence of big rip singularity at \( t_s \), we must replace \( \infty \) by \( t_s \) in the integration. For a de Sitter space-time \( a(t) \propto \exp(\mathcal{H}t) \), and the future event horizon reduces to de Sitter horizon: \( R_h = 1/\mathcal{H} \). In this space time the equation of state parameter is \( \omega = -1 - 2\dot{\mathcal{H}}/(3\mathcal{H}^2) = -1 \).

If the system remains in quintessence phase, defined by \(-1 < \omega < -1/3 \) (or \( \dot{\mathcal{H}} < 0 \)), the future event horizon satisfies \( \dot{R}_h \geq 0 \). Instead, for an universe which will remain in phantom dominated era, defined by \( \omega < -1 \) (or \( \dot{\mathcal{H}} > 0 \)), we have \( \dot{R}_h \leq 0 \). It is worth to note that these behaviors of the future event horizon depend on the entire future circumstances, e.g., if the phantom ends to quintessence phase, we may have \( \dot{R}_h \geq 0 \) even in the phantom dominated era.

One can consider an entropy for the future event horizon as a measure of information hidden behind it:

\[ S_h = \pi R_h^2. \]  

(4)

By adopting this point of view, we obtain the total entropy of the universe, \( S \), as the sum of the entropy inside the horizon, \( S_{in} \), and \( S_h \):

\[ S = S_{in} + S_h \]

(5)

The perfect fluid is supposed to be in thermal equilibrium with the future event horizon (note that FRW model requires thermal equilibrium). When the space-time \( \text{is de Sitter} \), i.e. when the future event horizon is the same as de Sitter horizon, we can consider the temperature as \( T = \mathcal{H}/(2\pi) \). Note that this has been only verified for de Sitter horizons \[15\]. For a non-de Sitter space-time, i.e. when \( R_h \neq 1/\mathcal{H} \), we assume that the future event horizon temperature is proportional to de Sitter temperature (which is the only temperature scale we have at our disposal) \[8\]

\[ T = \frac{bH}{2\pi}, \]

(6)

where \( b \) is a real constant.

Besides the dark energy and dark matter, we introduce a Schwarzschild black hole inside the future event horizon. The mass of the black hole, \( M \),
is assumed to be enough small so that the metric (11) remains unchanged. Using \( \rho = 3H^2/(8\pi) \), where \( \rho \) is the energy density inside the future event horizon, this condition reduces to

\[
MH \ll \frac{R_h^3 H^3}{2}
\]  

(7)

\( S_{in} \) may be divided into two parts: entropy of the black hole, denoted by \( S_{bd} \) and the entropy of perfect fluids denoted by \( S_d \)

\[
S_{in} = S_{bd} + S_d.
\]  

(8)

In a fluid with the energy density \( \rho \) and the pressure \( P \), the change rate of the black hole mass is \[16\]

\[
\dot{M} = 4\pi Ar_h^2 (P + \rho) = -4AM^2 \dot{H},
\]  

(9)

where \( r_h \) is the radius of the black hole horizon and \( A \) is a positive numerical constant. So, in terms of the Hubble parameter, the black hole mass may be obtained as

\[
M = \frac{1}{C + 4AH}
\]  

(10)

where \( C \) is a numerical constant.

The entropy of the black hole is \( S_{bd} = 4\pi M^2 \) \[18\], therefore

\[
\dot{S}_{bd} = -32\pi AM^3 \dot{H}.
\]  

(11)

The entropy of the phantom fluid inside the cosmological horizon is related to the energy and the pressure via the first law of thermodynamics

\[
TdS_d = dE + PdV = (P + \rho)dV + Vd\rho,
\]  

(12)

where \( V = (4/3)\pi R_h^3 \) is the volume inside the future event horizon. Using \[12\] we obtain \[12\]

\[
T\dot{S}_d = \dot{H}R_h^2.
\]  

(13)

Note that if \( T > 0 \) then \( \dot{S}_d > 0 \). The generalized second law asserts that the sum of the ordinary entropy, the future event horizon entropy and the black hole entropy cannot decrease with time: \( \dot{S}_d + \dot{S}_{bd} + \dot{S}_h \geq 0 \). This leads to

\[
\dot{H}(\frac{R_h^2}{T} - 32\pi AM^3) + 2\pi R_h \dot{R}_h \geq 0.
\]  

(14)

Note that for a de Sitter space-time, GSL is satisfied: \( \dot{H} = 0 \) and \( \dot{S}_d + \dot{S}_{bd} + \dot{S}_h = 0 \).

In phantom era \( \dot{H} > 0 \), therefore for a system remaining in phantom phase, \( T > 0 \) is a necessary condition for validity of GSL (Note that in such
system we have $\dot{R}_h \leq 0$). Also in the presence of the black hole, GSL is violated in phantom models with negative temperature. Using (14) we find

$$\dot{H} \left( \frac{R_h^2}{H} - 16bAM^3 \right) + bR_h \dot{R}_h \geq 0. \tag{15}$$

This results in that in order that GSL holds at $t_0$, where $\dot{H}(t_0) = 0$, we must have $\dot{R}_h(t_0) \geq 0$. In the phantom regime $\dot{H} > 0$, hence $H$ is an increasing function of time, so that we may write (15) as

$$\frac{b}{2} \frac{dR_h^2}{dH} + \frac{R_h^2}{H} \geq 16bAM^3 \tag{16}$$

To go further let us study the validity of GSL in some special cases which are of interest: For example consider a phantom dominated universe with a constant equation of state parameter, $\omega(t) = \omega_0 \neq -1$, with a big rip at $t = t_s$. The Hubble parameter is then

$$H = \frac{2}{3(1 + \omega_0)(t - t_s)}. \tag{17}$$

Using

$$\dot{R}_h = H R_h - 1, \tag{18}$$

we obtain

$$R_h = 3 \frac{1 + \omega_0}{1 + 3\omega_0} (t_s - t) \tag{19}$$

which leads to

$$HR_h = \beta, \tag{20}$$

where $\beta$ is a constant, $\beta = -2/(3\omega_0 + 1) < 1$, in agreement with the expected decreasing behavior of the future event horizon. Note that even for $\omega_0 = -1$, which describes a de Sitter space, (20) is still valid. The solution of (9) is

$$M = \frac{t_s - t}{-\frac{8A}{3(1+\omega_0)} + C(t_s - t)}. \tag{21}$$

$C$ is given by:

$$C = \frac{1}{M(t_i)} + \frac{8A}{3(1+\omega_0)} \frac{1}{t_s - t_i}, \tag{22}$$

where $t_i$ is an arbitrary time in phantom dominated era. Combining (21) and (17) we arrive at

$$MH = \frac{1}{4A - \frac{8A}{2}(1 + \omega_0)(t_s - t)}. \tag{23}$$

Using (20) we can write (16) in the form

$$M^3H^3 \leq \beta^2 \frac{1 - \frac{b}{16bA}}{rac{1}{16bA}}. \tag{24}$$
which, for \( b = 1 \), does not hold and the generalized second law is not respected, in agreement with the claim of [1]. But it seems that for \( b < 1 \), GSL may be respected for suitably chosen parameters, at least in the domain of validity of the approximation (7). To see this, we proceed as follows: For \( b < 1 \) and \( \dot{S} > 0 \), in order to satisfy the GSL, we must have

\[
\frac{1}{4A - \frac{3C}{2}(1 + \omega_0)(t_s - t)} < (\beta^2 \frac{1-b}{16bA})^3 \tag{25}
\]

in addition, for validity of our approximation (7), we require

\[
\frac{1}{4A - \frac{3C}{2}(1 + \omega_0)(t_s - t)} \ll \frac{1}{2}\beta^3. \tag{26}
\]

Hence GSL is respected in times: \( t \), satisfying (25) and (26). Near \( t = t_s \), the approximation (26) is not satisfied for \( A \sim O(1) \). If \( C > 0 \) and if GSL holds for a specific \( t = t_i \), it will hold for \( t < t_i \).

For \( b < 1 \) and \( \dot{S} = 0 \) (corresponding to reversible adiabatic expansion), we obtain

\[
\gamma = \beta^2 \frac{1 - b}{16bA} \tag{27}
\]

where \( \gamma = \beta^2 (1 - b)/(16bA) \). Now from (27) and (23) we can determine \( \gamma \) and \( C \): \( \gamma = 1/(4A) \), \( C = 0 \). On the other hand the validity of the approximation (7) requires: \( \gamma \ll \beta^3/2 \), which is only valid for large \( A \).

As another example consider time depending \( \omega(t) \) and \( \dot{S} = 0 \). In this case one can determine \( R_h \) as a function of time. Applying \( \dot{S} = 0 \) in (16) gives

\[
R_h^2(H) = H^{-\frac{3}{2}} \left( C_1 + 32A \int M^3(H)H^{\frac{3}{2}}dH \right). \tag{28}
\]

\( C_1 \) is a numerical constant. Inserting (11) into the above integral yields

\[
R_h^2 = C_1 H^{-\frac{3}{2}} + \frac{4}{(AH + C)^2} [2 + \frac{8HA}{C} - b] + \frac{8}{C^2b} (1 - \frac{2}{b}) \Phi(-\frac{4HA}{C}, 1, \frac{2}{b}) \tag{29}
\]

where \( \Phi \) is the Lerchphi function. But following the approximation (7), the solution (29) is only valid when \( 4A + C/H \gg 1 \). For \( A \sim O(1) \) and \( C/H \gg 1 \), by considering the series representation of Lerchphi function, we obtain

\[
R_h^2 H^2 = C_1 H^{2-\frac{3}{2}} + \frac{32Ab}{b+2} \frac{H}{C^3} + O((\frac{H}{C})^4). \tag{30}
\]

This equation with (18) determine \( R_h \). Up to the order \( O((H/c)^3) \), by inserting (30) into (18) we find

\[
\dot{R}_h - C_1^2 R_h^{1-b} + 1 = 0. \tag{31}
\]
For $b = 1$, the problem reduces to $\omega = \omega_0 = C_{1}^{1/2}$, discussed in the previous part. For $b \neq 1$, solution of (31) satisfies

$$R_h \Phi(C_{b}^{0} R_h^{1-b} : 1, \frac{1}{1-b}) = d - t.$$  \hspace{1cm} (32)

At $t = d$, we have $R_h = 0$. Note that, in this approximation $R_h^b H = C_{1}^b$. Comparing of this result with that obtained in [12] indicates that the presence of the black hole in the domain of validity of GSL and the approximation \[], up to the order $M^3$, does not change the behavior of $R_h$.

2.1 GSL near the transition time

Based on astrophysical data, which seem to favor an evolving dark energy with $\omega$ less than $-1$ at present epoch from $\omega > -1$ in the near past [3], it may be interesting to study the validity of GSL near the transition time(time of $\omega = -1$ crossing). In the phantom regime $\dot{H} > 0$ and in the quintessence regime we have $\dot{H} < 0$, therefore if the Hubble parameter has a Taylor series at transition time, which is taken to be at $t = 0$, $\dot{H}(0) = 0$ and we can write

$$H = h_0 + h_1 t^a,$$  \hspace{1cm} (33)

where $h_0 = H(t = 0)$ and $a$, a positive even integer number, is the order of the first nonzero derivative of $H$ at $t = 0$. $h_1 = H^{(a)}/(a!)$ and $H^{(a)} = d^a H/dt^a$. In the case of transition from quintessence to phantom phase we must have $h_1 > 0$. Using (18) it can be shown that $R(t)$ has the following expansions:

$$R_h(t) = R_h(0) + (h_0 R_h(0) - 1) t + O(t^2),$$  \hspace{1cm} (34)

for $\dot{R}_h(0) \neq 0$, and

$$R_h(t) = R_h(0)(1 + \frac{h_1}{a + 1} t^{a+1}) + O(t^{a+2}),$$  \hspace{1cm} (35)

for $\dot{R}_h(0) = 0$, at $t = 0$. Near the transition time (7) reduces to $h_0^2 R_h^3(0) \gg 2M(0)$.

The condition of validity of GSL near the transition time, $t = 0$, for $\dot{R}_h(0) = 0$, can be investigated by inserting $H = h_0 + h_1 t^a$ and (33) into (15):

$$ah_1 \left( \frac{R_h(0)^2}{h_0} - 16b AM(0)^3 \right) t^{a-1} + O(t^{a}) \geq 0.$$  \hspace{1cm} (36)

Note that $(a - 1)$ is an odd integer, therefore if $R_h(0)^2/h_0 - 16b AM(0)^3 \geq (\leq) 0$, GSL is not respected in quintessence (phantom) phase before (after) the transition. Indeed the black hole mass $M(0)$, gives the possibility that GSL becomes respected in the quintessence era before the transition.
In the same way, for $\dot{R}_h(0) \neq 0$ we obtain

$$bR_h(0)(h_0R_h(0) - 1) + O(t) \geq 0.$$  \hspace{1cm} (37)

Therefore the generalized second law is respected at least in both sides of the transition time provided that $\dot{R}_h(0) > 0$, in agreement with the discussion after eq.(15). Then the continuity of $\dot{R}_h$, for $t$’s belonging to an open set including $t = 0$, results in $\dot{R}_h(t) > 0$.

In [9], it was shown that the future event horizon in the quintessence model is a nondecreasing function of time. Using the same method, in [12] it was proved that the future event horizon is non increasing in phantom dominated era. In the first view, combining these results leads to $\dot{R}_h(t = 0) = 0$ which prompts us to choose (35). But this causes a conflict: near the transition time, (35) results in $\dot{R}_h(t) = R_h(0)h_1t^a$, which is positive because $h_1 > 0$ and $a$ is even, and this is in contradiction with the assumption $\dot{R}_h(t > 0) \leq 0$ proved in [12]. On the other hand if we adopt $\dot{R}_h(t = 0) \neq 0$, due to continuity of $\dot{R}_h$ (see (18)), there is an open set containing the transition time in which the sign of $\dot{R}_h(t)$ is the same as the sign of $\dot{R}_h(t = 0)$, i.e. we have either $\dot{R}_h(t) < 0$ in the quintessence phase before the transition or $\dot{R}_h(t) > 0$ in the phantom phase after the transition. This conflict can be solved by noting that the verifications of nondecreasing (non increasing) behavior of $R_h$ in [9] (12), were based on the fact that the system remains in quintessence (phantom) phase for all future time. So in the presence of quintessence(phantom) to phantom (quintessence) phase transition, it may be in general possible to have $\dot{R}_h(t) < 0 (> 0)$ for some $t$’s in quintessence (phantom) era.

Following the above discussion we conclude in an universe which remains in phantom phase after the transition, GSL is not respected in the neighborhood of transition time, indeed for this universe $\dot{R}_h \neq 0$ and in the vicinity of transition time we have $\dot{R}_h < 0$. To find an example of this situation see [12].

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