Quantification of entanglement by means of convergent iterations

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An iterative procedure is proposed for the calculation of the relative entropy of entanglement of a given bipartite quantum state. When this state turns out to be non-separable the algorithm provides the corresponding optimal entanglement witness measurement.

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Entanglement is an important resource of quantum information processing. Although there are quantum protocols based on other features of quantum mechanics such as quantum superposition principle in celebrated Deutsch’s searching algorithm [1], or protocols where the use of entanglement can be advantageous but is not essential, such as signaling through depolarizing channels with memory [2], most quantum protocols rely on the existence of non-separable states. For practical purposes it is very important to quantify entanglement generated by realistic laboratory sources and thus evaluate the potential usefulness of a given realistic source for the quantum processing/communication purposes.

One of the measures of entanglement thoroughly studied over the past decade is the relative entropy of entanglement defined as

\[ E(\sigma) = \inf_{\rho_{\text{sep}}} S(\sigma \| \rho), \]

where \( S \) is the quantum relative entropy,

\[ S(\sigma \| \rho) = \text{Tr}(\sigma \ln \sigma - \sigma \ln \rho), \]

between states \( \sigma \) and \( \rho \); the infimum in Eq. (1) is taken over the set of separable states. This functional is one possible generalization of the classical relative entropy between two probability distributions [3] to quantum theory. Let us mention that unlike in the case of entropy this generalization is by no means unique. This quantity can be given a geometrical interpretation as a quasi-distance between the state whose entanglement we are interested in and the convex set of separable states. \( E \) fulfills most of the requirements usually imposed on a good entanglement measure and has other good properties. Most notably, on the set of pure states it coincides with the Von Neumann reduced entropy [4, 5], and is closely related to some other measures of entanglement [6]. Relative entropy makes also a good entanglement measure for multipartite [7] and infinite-dimensional [8] quantum systems.

The analytical form of \( E \) is known only for some special sets of states of high symmetry [9, 10, 11]. Generally one has to resort to numerical calculation. In a sense the problem resembles the reconstruction of quantum states using the maximum likelihood principle [12, 13]. Here the given input state \( \sigma \) plays the role of experimental data; once this state is known, the statistics of any possible measurement performed on it is available. The solution can be obtained by means of several numerical methods. The formulation given in [13] is just an example corresponding to an implementation of the downhill simplex method. Its efficiency strongly depends on the dimensionality of the problem.

In the procedure proposed here more analytical approach will be adopted. We will derive a set of extremal equations for \( E \) and will show how to solve them by means of repeated convergent iterations. But this is not the only goal. The extremal equations indicate that there is a structure of quantum measurement associated with the extremal solution. The separable measurement obtained in this way specifies the extremal separable state and, significantly, it provides the optimal entanglement witness operator revealing the possible entanglement of the input state \( \sigma \).

Let us denote \( \rho^* \) the separable state having the smallest quantum relative entropy with respect to \( \sigma \). Let

\[ f(x, \rho^*, \rho) = S(\sigma \| (1 - x)\rho^* + x\rho) \]

be the relative entropy of a state obtained by moving from \( \rho^* \) towards some \( \rho \). We are looking for the global maximum of a convex functional on the convex set of separable states. Two cases may arise. When \( \sigma \) is separable the necessary and sufficient condition for the maximum of \( S \) is that its variations along the paths lying in the set of separable states vanish,

\[ \frac{\partial f}{\partial x}(0, \rho^*, \rho) = 0, \quad \forall \rho \quad \text{separable}. \]

When \( \sigma \) is entangled \( S \) attains its true maximum outside the set of separable states and we must carry on the maximization on the boundary. In that case Eq. (3) holds only for variations along the boundary. It is well known that any separable state from the Hilbert space of dimension \( p = d \otimes d \) can be expressed as a convex sum of (at most) \( p^2 \) projectors on disentangled pure states (Caratheodory’s theorem, see also [14]),

\[ \rho = \sum_{k=1}^{p^2} |\varphi_k^1\rangle \langle \varphi_k^1| \otimes |\varphi_k^2\rangle \langle \varphi_k^2|, \]

Here \( |\varphi_k^1\rangle \) and \( |\varphi_k^2\rangle \) are pure states (not normalized) of the systems 1 and 2, respectively. Now by taking squares of the projectors, \( |\varphi_k^{1,2}\rangle \langle \varphi_k^{1,2}| \rightarrow (|\varphi_k^{1,2}\rangle \langle \varphi_k^{1,2}|)^2 \), one can remove the boundary and make condition (3) universal. The derivation in Eq. (3) can easily be calculated using...
an integral representation of the logarithm of a positive operator $\rho$. It reads
\[
\frac{\partial f}{\partial x}(0, \rho^*, \rho) = \int_0^\infty \text{Tr}((\rho^* + t)^{-1}\sigma(\rho^* + t)^{-1} \delta\rho) dt
\]  
\[
= \text{Tr}A\delta\rho,
\]
where we denoted $(1 - x)\rho^* + x\rho = \rho^* + \delta\rho$, and operator $A$ has the following matrix elements in the eigenbasis $\{|\lambda_n\rangle\}$ of $\rho^*$
\[
|\lambda_m\rangle|A|\lambda_n\rangle = \log \frac{\lambda_n - \log \lambda_m}{\lambda_n - \lambda_m} |\lambda_m\rangle |\lambda_n\rangle.
\]
Its meaning will be discussed later. The right hand side
\[
\text{Eq. (7)}\text{.}
\]
\[
\text{that the length of the step can be made smaller by mixing the operators $R_k^{1,2}$ with the unity operator:}
\]
\[
R_k^{2,1} \rightarrow (\mathbb{1} + \frac{1}{2}\alpha R_k^{1,2})/\left(1 + \frac{1}{2}\alpha\right).
\]
Indeed, when $\alpha$ is sufficiently small the algorithm converges monotonically. This can be seen by considering an infinitesimal step with $\alpha < 1$. It is convenient to split one iteration of Eqs. (7) into two subsequent steps corresponding to the two rows of Eqs. (5) (projectors of only one of the subsystems are updated at a time). The two steps are completely symmetrical, so we will consider an infinitesimal iteration on, say, the projectors $|\phi_k^1\rangle\langle\phi_k^1|$ of the first system obtained after the $i$-th iteration. We want to show that after one such step the quantum relative entropy is never increased, $S(\sigma||\rho^{i+1}) \leq S(\sigma||\rho^i)$. Using Eq. (10) in Eqs. (9) we get to the first order in $\alpha$,
\[
\rho^{i+1} = (1 - \alpha)\rho^i + \alpha\tilde{\rho},
\]
where $\tilde{\rho} = \frac{1}{2} \sum_k (R_k^{11}|\phi_k^1\rangle\langle\phi_k^1| + |\phi_k^1\rangle\langle\phi_k^1|R_k^{21}) |\phi_k^2\rangle\langle\phi_k^2|$, and thus
\[
S(\sigma||\rho^{i+1}) - S(\sigma||\rho^i) \propto \frac{\partial f(0, \rho^i, \tilde{\rho})}{\partial \alpha} = 1 - \text{Tr}A^i\tilde{\rho}.
\]
It remains to show that $\text{Tr}A^i\tilde{\rho} \geq 1$. Let us denote $\lambda_k^i = |\langle\chi_k^i|\rho^i||\phi_k^1\rangle|\langle\chi_k^i|\rho^i||\phi_k^1|$. Notice that $\sum_k \lambda_k^i = 1$ by the normalization of $\rho^i$. Then by using the Swartz inequality and the concavity of the square function we obtain
\[
\text{Tr}A^i\tilde{\rho} = \sum_k \lambda_k^i \text{Tr}(R_k^{11}|\phi_k^1\rangle\langle\phi_k^1|) \geq \sum_k \lambda_k^i \left[\text{Tr}(R_k^{11}|\phi_k^1\rangle\langle\phi_k^1|)\right]^2 \geq \sum_k \lambda_k^i \left[\text{Tr}(R_k^{11}|\phi_k^1\rangle\langle\phi_k^1|)\right]^2 \geq (\text{Tr}A^i\tilde{\rho})^2 = 1,
\]
which completes our proof. This means that with a sufficient amount of regularization (10) the algorithm (7) converges monotonically. In practice, the parameter $\alpha$ need not be very small. In 2 $\otimes$ 2 and 4 $\otimes$ 4 dimensional problems we tried monotonically convergent was observed even with $\alpha$ of the order of unity.

Now let us go back to the extremal equations (11). The left hand sides are generated by the operator $A$, which depends on $\sigma$ through Eq. (11). Let us assume for now that $\sigma$ is an entangled state. Then $A$ represents the gradient of the quantum relative entropy $S(\sigma||\rho)$ at $\rho^i$—the separable state closest to $\sigma$. Loosely speaking, the states giving the same expectation, $\text{Tr}A(\sigma)\rho = \text{const.}$, form hyperplanes that are perpendicular to the line connecting $\sigma$ and $\rho^i$. Since $\text{Tr}A(\sigma)\rho = 1$ and $\rho^i$ lies at the boundary of the set of separable states, the conjecture is that the operator $A$ is up to a shift of its spectrum a witness operator (14) detecting the entanglement of $\sigma$. In the following we will show that this is indeed the case, and that the operator
\[
W(\sigma) = I - A(\sigma)
\]
is indeed the optimal witness of the entanglement of $\sigma$. The mutual relationship of $\sigma$, $\rho^i$, and the states detected by $W$ is shown in Fig. 1.
First we will show that $\text{Tr} A \rho \leq 1$ if $\rho$ is separable. To this end let us note that

$$1 - \text{Tr} A \rho = \frac{\partial f}{\partial x}(0, \rho^*, \rho) = \lim_{x \to 0} \frac{S(\sigma || (1-x)\rho^* + x\rho) - S(\sigma || \rho^*)}{x}.$$  \hspace{1cm} (15)

Now, since both $\rho$ and $\rho^*$ are separable states, so is their convex combination $(1-x)\rho^* + x\rho$. But $\rho^*$ minimizes $S(\sigma || \rho)$ over the set of separable states. Therefore, $S(\sigma || (1-x)\rho^* + x\rho) - S(\sigma || \rho^*) \geq 1$. This holds for all $x$ so we have,

$$\text{Tr} W(\sigma) \rho = 1 - \text{Tr} A(\sigma) \rho \geq 0, \quad \forall \rho \text{ separable} \quad (16)$$

This already means that $W$ is an entanglement witness operator. To show that $W$ detects $\sigma$ we will again make use of Eq. (15) with $\rho$ now being substituted by the entangled state $\sigma$. Now, because of convexity of $S$,

$$\frac{S(\sigma || (1-x)\rho^* + x\sigma) - S(\sigma || \rho^*)}{x} \leq -S(\sigma || \rho^*) < 0.$$  \hspace{1cm} (17)

The last inequality follows from the assumed non-separability of $\sigma$. Eq. (17) also holds for any $x$ so we obtain

$$\text{Tr} W(\sigma) \sigma = 1 - \text{Tr} A(\sigma) \sigma < 0, \quad \forall \sigma \text{ entangled} \quad (18)$$

which we set out to prove.

Possible applications of our algorithm are twofold: First, it can be used for checking whether a given state is separable or not. Second, it can be used for quantifying the amount of entanglement the state contains. As a test of separability we tested the algorithm on many randomly generated separable and NPT states of dimensions $2 \otimes 2$ and $4 \otimes 4$; typical results are summarized in Fig. 4.

Recently, another numerical test of separability has been proposed consisting of a hierarchy of gradually more and more complex separability criteria that can be formulated as separate problems of the linear optimization theory. The algorithm we propose is much more simple. There is just one set of equations to be solved by repeated iterations and after that one finds not only whether the input state is entangled but also how much.

Unfortunately, explicit formulas for the quantum relative entropy are known only in very few cases. One such exception is the family of Werner states defined as
follows,
\[ \rho_w = \frac{d - f}{d(d^2 - 1)} \mathbb{I} + \frac{fd - 1}{d(d^2 - 1)} F, \]
where $F$ is the flip operator $F(\psi_1 \otimes \psi_2) = \psi_1 \otimes \psi_2$ and $f = \text{Tr} \rho_w F \in [-1, 1]$ is a parameter. Eq. (19) shows the performance of our algorithm for several entangled Werner states of dimension $4 \otimes 4$. It is worth mentioning that the optimal entanglement witness $W$ for the detection of Werner states generated by the operator $A$ Eq. (4) is simply $W = 1 - 2|\Psi_-\rangle\langle\Psi_-|$, where $\Psi_-$ is the singlet state. The expectation value of $W$ is then a renormalized singlet fraction.

The curves appearing in Figs. 2 and 3 suggest that the convergence of the proposed algorithm is faster than polynomial but slower than exponential. In some cases such as that shown in Fig. 3 the convergence is nearly exponentially fast. This is just a qualitative statement since we did not attempt to do any optimization of the length of the iteration step. In all probability such optimization would result in further speedup compared to our examples given in Figs. 2 and 3.

In conclusion, we derived a convergent iterative algorithm for the calculation of the relative entropy of entanglement. It can be used for checking whether a given input state is entangled. If it is, the algorithm calculates its relative entropy of entanglement, finds its closest separable state, and provides the optimal entanglement witness measurement.

Acknowledgments

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[1] D. Deutsch and R. Jozsa, Proc. R. Soc. Lond. A 439, 553 (1992).
[2] C. Macchiavello and G. Palma, quant-ph/0107052.
[3] V. Vedral, M. Plenio, K. Jacobs, and P. Knight, Phys. Rev. A 56, 4452 (1997).
[4] S. Braunstein, Phys. Lett. A 219, 169 (1996).
[5] V. Vedral and M. Plenio, Phys. Rev. A 57, 1619 (1998).
[6] M. Plenio, S. Virmani, and P. Papadopoulos, J. Phys. A: Math. Gen. 33, 193 (2000).
[7] L. Henderson and V. Vedral, Phys. Rev. Lett. 84, 2263 (2000).
[8] M. Plenio and V. Vedral, J. Phys. A: Math. Gen. 34, 6997 (2001).
[9] J. Eisert, C. Simon, and M. Plenio, quant-ph/0112064.
[10] V. Vedral, M. Plenio, M. Rippin, and P. Knight, Phys. Rev. Lett. 78, 2275 (1997).
[11] K. Audenaert, J. Eisert, E. Jané, M. Plenio, S. Virmani, and B. D. Moor, Phys. Rev. Lett. 87, 217902 (2001).
[12] K. Vollbrecht and R. Werner, Phys. Rev. A 64, 062307 (2001).
[13] Z. Hradil, Phys. Rev. A 55, 1561(R) (1997).
[14] J. Reháček, Z. Hradil, and M. Ježek, Phys. Rev. A 63, 040303(R) (2001).
[15] C. Byrne, IEEE Trans. Image Proc. 2, 96 (1993).
[16] R. Werner, Phys. Rev. A 40, 4277 (1989).
[17] B. Terhal, Phys. Lett. A 271, 2000 (2000).
[18] A. Doherty, P. Parrilo, and F. Spedalieri, quant-ph/0112007.
[19] When $m = n$ or $\lambda_n = \lambda_m$ the corresponding coefficient should be replaced the limit value of $\lambda_m^{-1}$.