Abstract. Perfectly rational decision-makers maximize expected utility, but crucially ignore the resource costs incurred when determining optimal actions. Here we propose an axiomatic framework for bounded rational decision-making based on a thermodynamic interpretation of resource costs as information costs. We show that this axiomatic framework enforces a unique conversion law between utility and information, which can be characterized by a variational “free utility” principle akin to thermodynamical free energy. This variational principle constitutes a normative criterion that trades off utility and information costs, the latter measured by the Kullback-Leibler deviation between a distribution representing a desired policy and a reference distribution representing an initial default policy. We show that bounded optimal control solutions can be derived from this variational principle, which leads in general to stochastic policies. Furthermore, we show that risk-sensitive and robust (minimax) control schemes fall out naturally from this framework if the environment is considered as an adversarial opponent. When resource costs are ignored, the maximum expected utility principle is recovered.

1 Introduction

Rational decision-making is usually based on the principle of maximum expected utility (MEU) [17]. According to MEU, a rational agent chooses its action $a$ so as to maximize its expected utility $E[U|a] = \sum_s P(s|a)U(s)$ given the probability $P(s|a)$ that action $a \in A$ will lead to outcome $s \in S$ and given that the desirability of the outcome $s$ is measured by the utility $U(s) \in \mathbb{R}$. Thus, expected utilities express betting preferences over lotteries with uncertain outcomes. The optimal action $a^* \in A$ is defined as the one that maximizes the expected utility, that is $a^* := \arg \max_a E[U|a]$. However, computing such optimal actions is often very difficult in practice due to prohibitive resource costs that are associated with the process of finding the optimal action. Such resource costs are ignored by MEU.

In contrast, a bounded rational decision-maker has only limited resources and cannot afford an unlimited search for the optimal action [11]. Therefore, such decision-makers have to trade off the utility that an action achieves against the resource cost of finding the action. Imagine, for example, you want to invest

* A shortened version of this paper has been published in Lecture Notes on Artificial Intelligence 6830, pp. 269–274.
some of your savings and you start reading up on several options, asking your local bank, etc. However, as a bounded agent in the real world you cannot extend this search forever, as you will loose out in the meanwhile. Therefore, you have to trade off somehow the time invested in this search and a satisfactory return from some investment option.

In this paper we propose an axiomatic formalization of bounded rationality that leads to such a trade-off based on a thermodynamic interpretation of resource costs [4]. The intuition behind this interpretation is that ultimately any real decision-maker has to be incarnated in a physical system, since any process of information processing must always be accompanied by a pertinent physical process [15]. Thermodynamics provides the tools to study these general physical systems. In Section 2 we discuss the thermodynamical notion of resource costs in information processing systems. In Section 3 we show how a set of simple choice axioms leads to a variational principle that allows computing bounded optimal policies in systems with resource costs. In Section 4 we apply this framework and show how to derive bounded optimal solutions for decision-making under resource costs in different environments. We also show how to obtain classic maximum expected utility solutions in the limit of negligible resource costs.

2 Resource Costs

In the following we conceive of information processing as changes in information states, i.e. ultimately changes of probability distributions that are represented in physical systems. Changing an information state therefore implies changes in physical states, such as flipping gates in a transistor, changing voltage on a microchip, or even changing location of a gas particle. Changing such states is costly and requires thermodynamical work [4]. Imagine, for example, that we use an ideal gas particle in a box with volume $V_i$ as an information processing system to represent a uniform probability density over a random variable with $p_i = \frac{1}{V}$. If we now want to update this probability to $p_f$, because we gained information $-\log p = -\log \frac{p_i}{p_f} > 0$, we have to reduce the original volume to $V_f = pV_i$. However, this decrease in volume requires the work $W = -\int_{V_i}^{V_f} \frac{NkT}{V} dV = NkT \ln \frac{V_i}{V_f}$, where $N$ is the number of gas molecules, $k$ is the Boltzmann constant, and $T$ is temperature. Thus, in this simple example we can compute the relation between the change in information state and the required work, that is $W = -\alpha \log p$, with $\alpha = \frac{NkT}{\log e} > 0$ being the conversion factor between information and energy. The conversion factor $\alpha$ depends on the underlying properties of the physical system and determines how expensive it is to process information. In the next two sections, we derive a general expression of information costs for physical systems that represent bounded rational decision-makers. Since such decision-makers need to trade off utility and information costs, we will first investigate the relation between information and utility [10] and then show how information costs appear as an additional term in the utility in physically implemented decision-makers.
3 Conversion between utility and information

3.1 Choice axioms

Consider a decision-maker whose behavior is represented by a probability space 
\((\Omega, \mathcal{F}, P)\) with sample set \(\Omega\) and \(\sigma\)-algebra \(\mathcal{F}\) of measurable events between 
which the decision-maker can choose. We assume that the decision-maker can 
choose freely any probability measure \(P\) representing his choice behavior. Thus, if 
\(P(A) > P(B)\), then the propensity of choosing \(A\) is higher than that of choosing 
\(B\). This difference in probability can be given a utilitarian interpretation: \(A\) 
is chosen with higher probability than \(B\) because \(A\) is more desirable than \(B\).

The measure that quantifies such differences in desirability is commonly called 
a utility function. If there is such a measure, then it is reasonable to demand 
the following properties:

i. Utilities should be mappings from events into real numbers.

ii. Absolute values of utility are irrelevant, only relative differences in utility 
should matter ("utility gains").

iii. Utility gains should be additive.

iv. A decision-maker should assign more probability mass to events with high 
utility and less probability mass to events with low utility.

v. An adversarial agent should make the reverse assignment of probability mass.

These postulates are summarized in the following definition.

**Definition 1 (Axioms of Choice).** Let \((\Omega, \mathcal{F}, P)\) be a probability space. A set 
function \(U: \mathcal{F} \to \mathbb{R}\) is a utility function for a decision-maker with probability 
measure \(P\) iff its utility gain function \(u(A|B) := U(A \cap B) - U(B)\) has the 
following three properties for all events \(A, B, C, D \in \mathcal{F}\):

\[
\begin{align*}
A1 &. \exists f, u(A|B) = f(P(A|B)) \in \mathbb{R}, \quad \text{(real-valued)} \\
A2 &. u(A \cap B|C) = u(A|C) + u(B|A \cap C), \quad \text{(additive)} \\
A3 &. P(A|B) > P(C|D) \iff u(A|B) > u(C|D). \quad \text{(monotonic increasing)} \\
A4 &. P(A|B) > P(C|D) \iff u(A|B) < u(C|D). \quad \text{(monotonic decreasing)}
\end{align*}
\]

If the decision-maker is an adversarial opponent, the inequality of \(A3\) is reversed

\[A4. \quad P(A|B) > P(C|D) \iff u(A|B) < u(C|D). \quad \text{(monotonic decreasing)}\]

Furthermore, we use the abbreviation \(u(A) := u(A|\Omega)\).

The following theorem shows that these three properties enforce a strict 
mapping between probabilities and utility gains.

**Theorem 1 (Utility Gain ↔ Probability).** If \(f\) is such that \(u(A|B) = f(P(A|B))\) for any probability space \((\Omega, \mathcal{F}, P)\), then \(f\) is of the form 
\[
f(\cdot) = \alpha \log(\cdot),
\]

where \(\alpha\) is an arbitrary strictly positive constant in case of \(A3\) or an arbitrary 
strictly negative constant in case of \(A4\).
4 Bounded Rationality

The proof is provided elsewhere [10, 8]. If one is willing to accept Definition 1, then one obtains the relations

\[ U(A \cap B) - U(B) = \alpha \log P(A|B). \] (1)

In this relation, \( \alpha \) plays the role of a conversion factor between utilities and information. A bounded rational decision-maker is characterized by \( \alpha > 0 \), whereas an adversarial opponent can be described by \( \alpha < 0 \). Unless otherwise stated, we will assume \( \alpha > 0 \) in the following. If a probability measure \( P \) and a utility function \( U \) satisfy the relation (1), then we say that they are conjugate. Given that this transformation between utility gains and probabilities is a bijection, one can rewrite any probability \( P(A|B) \) as a Gibbs measure:

\[ P(A|B) = \frac{\sum_{\omega \in A \cap B} \exp \frac{1}{\alpha} U(\omega)}{\sum_{\omega \in B} \exp \frac{1}{\alpha} U(\omega)}. \] (2)

where we have used the abbreviation \( U(\omega) := U(\{\omega\}) \). This transformation implies that the probability measure \( P \) is the Gibbs measure with temperature \( \alpha \) and energy levels \( e(\omega) := -U(\{\omega\}) \). As the conversion factor \( \alpha \) approaches zero, the probability measure \( P(\omega) \) approaches a delta function \( \delta_{\omega^*} \) with \( \omega^* = \text{arg max}_\omega U(\omega) \), or in case of several maxima the uniform distribution over the maximal set \( \Omega_{\text{max}} := \{ \omega^* \in \Omega | \omega^* = \text{arg max}_\omega U(\omega) \} \). Similarly, as \( \alpha \to \infty \), \( P(\omega) \to \frac{1}{|\Omega|} \), i.e. the uniform distribution over the whole outcome set \( \Omega \).

3.2 Variational principle

It is well known in statistical physics that the Gibbs measure satisfies a variational problem in the free energy [3]. Since utilities correspond to negative energies, we can formulate a free utility principle that is maximized by a decision-maker that acts according to (2).

**Theorem 2.** Let \( X \) be a random variable with values in \( \mathcal{X} \). Let \( P \) and \( U \) be a conjugate pair of probability measure and utility function over \( X \). Define the free utility functional as

\[ J(\mathbb{P}; U) := \sum_{x \in \mathcal{X}} \mathbb{P}(x) U(x) - \alpha \sum_{x \in \mathcal{X}} \mathbb{P}(x) \log \mathbb{P}(x), \]

where \( \mathbb{P} \) is an arbitrary probability measure over \( X \). Then,

\[ J(\mathbb{P}; U) \leq J(P; U) = U(\Omega). \]

A proof can be found in [7]. The free utility is a combined measure of a system’s expected utility and its uncertainty. The variational principle implies that the Gibbs measure \( P \) maximizes the free utility for a given utility function \( U \), as \( P = \text{arg max}_\mathbb{P} J(\mathbb{P}; U) \).

The variational principle of the free utility also allows measuring the cost of transforming the state of a stochastic system required for information processing.
Consider an initial system having probability measure $P_i$ and utility function $U_i$. This system satisfies the equation

$$J_i := \sum_{x \in X} P_i(x) U_i(x) - \alpha \sum_{x \in X} P_i(x) \log P_i(x) = U_i(\Omega).$$

If we add new constraints represented by the utility function $U_*$ then the resulting utility function $U_f$ is given by the sum

$$U_f = U_i + U_*,$$

and the resulting probability measure $P_f$ maximizes

$$J(P_f, U_f) = \sum_{x \in X} P_f(x) U_f(x) - \alpha \sum_{x \in X} P_f(x) \log P_f(x)$$

$$= \sum_{x \in X} P_f(x)(U_i(x) + U_*(x)) - \alpha \sum_{x \in X} P_f(x) \log P_f(x)$$

$$= \sum_{x \in X} P_f(x) U_*(x) - \alpha \sum_{x \in X} P_f(x) \log \frac{P_f(x)}{P_i(x)} + U_i(\Omega).$$

Let $J_f := J(P_f, U_f)$. The difference in free utility is

$$J_f - J_i = \sum_{x \in X} P_f(x) U_*(x) - \alpha \sum_{x \in X} P_f(x) \log \frac{P_f(x)}{P_i(x)}.$$

(3)

In physical systems with constant $\alpha$, this difference measures the amount of work necessary to change the state of the system from state $i$ to state $f$. The first term of the equation measures the expected utility difference $U_*(x)$, while the second term measures the information cost of transforming the probability distribution from state $i$ to state $f$. These two terms can be interpreted as determinants of bounded rational decision-making in that they formalize a trade-off between an expected utility $U_*$ (first term) and the information cost of transforming $P_i$ into $P_f$ (second term). In this interpretation $P_f$ represents an initial probability or policy, which includes the special case of the uniform distribution where the decision-maker has initially no preferences. Deviations from this initial probability incur an information cost measured by the KL divergence. If this deviation is bounded by a non-zero value, we have a bounded rational agent.

In thermodynamics there are two dominant formulations of the second law that allow determining the equilibrium distribution: the first and maybe more familiar formulation is the principle of maximum entropy, and the second principle is the principle of minimum energy [3]. The corresponding variational problems are typically formulated such that in the case of maximum entropy we hold the mean energy fixed (i.e. in our case the expected utility), and in the case of minimum energy (i.e. in our case maximum utility) we hold the entropy fixed. Mathematically, the constraints of fixed entropy and fixed utility are added by Lagrange multipliers. In our context with respect to equation (3), this leads to two different variational principles:
1. **Control.** The minimum energy principle translates into a bounded maximum utility principle. Given an initial policy represented by the probability measure $P_i$ and the constraint utilities $U_*$, we are looking for the final system $P_f$ that optimizes the trade-off between utility and resource costs. That is,

$$P_f = \arg \max_{P_f} \sum_{x \in X} \Pr(x)U_*(x) - \alpha \sum_{x \in X} \Pr(x) \log \frac{\Pr(x)}{P_i(x)}. \quad (4)$$

The solution is given by

$$P_f(x) \propto P_i(x) \exp \left( \frac{1}{\alpha} U_*(x) \right).$$

In particular, at very low temperature $\alpha \approx 0$, (3) becomes

$$J_f - J_i \approx \sum_{x \in X} P_f(x)U_*(x),$$

and hence resource costs are ignored in the choice of $P_f$, leading to $P_f \approx \delta_{x^*}(x)$, where $x^* = \max_x U_*(x)$. Similarly, at a high temperature, the difference is

$$J_f - J_i \approx -\alpha \sum_{x \in X} P_f(x) \log \frac{P_f(x)}{P_i(x)},$$

and hence only resource costs matter, leading to $P_f \approx P_i$.

2. **Estimation.** The maximum entropy principle translates into a minimum relative entropy principle for estimation. Given a final probability measure $P_f$ that represents the environment and the constraint utilities $U_*$, we are looking for the initial system $P_i$ that satisfies

$$P_i = \arg \max_{P_i} \sum_{x \in X} P_f(x)U_*(x) - \alpha \sum_{x \in X} P_f(x) \log \frac{P_f(x)}{P_i(x)} \quad (5)$$

$$= \arg \min_{P_i} \sum_{x \in X} P_f(x) \log \frac{P_f(x)}{P_i(x)},$$

and thus we have recovered the minimum relative entropy principle for estimation, having the solution

$$P_i = P_f.$$

The minimum relative entropy principle for estimation is well-known in the literature as it underlies Bayesian inference [5], but the same principle can also be applied to problems of adaptive control [9]. In the following we focus on applications of the first principle on bounded optimal control.

### 4 Applications

Consider a system that first emits an action symbol $x_1$ with probability $P_0(x_1)$ and then expects a subsequent input signal $x_2$ with probability $P_0(x_2|x_1)$. Now
we impose a utility on this decision-maker that is given by $U(x_1)$ for the first symbol and $U(x_2|x_1)$ for the second symbol. How should this system adjust its action probability $P(x_1)$ and expectation $P(x_2|x_1)$? Given the boundedness constraints $c_1$ and $c_2$ on the relative entropies, the variational problem is given by

$$\max_{p(x_1)p(x_2|x_1)} \sum_{x_1} p(x_1)U(x_1) - \alpha \left( \sum_{x_1} p(x_1) \log \frac{p(x_1)}{p_0(x_1)} - c_1 \right) + \sum_{x_1,x_2} p(x_1)p(x_2|x_1)U(x_2|x_1)$$

$$- \beta \left( \sum_{x_1,x_2} p(x_1)p(x_2|x_1) \log \frac{p(x_2|x_1)}{p_0(x_2|x_1)} - c_2 \right),$$

with $\alpha$ and $\beta$ as Lagrange multipliers. We can rewrite this sum as a nested expression and drop all constants

$$\max_{p(x_1)p(x_2|x_1)} \sum_{x_1} p(x_1) \left[ U(x_1) - \alpha \log \frac{p(x_1)}{p_0(x_1)} + \sum_{x_2} p(x_2|x_1) \left[ U(x_2|x_1) - \beta \log \frac{p(x_2|x_1)}{p_0(x_2|x_1)} \right] \right].$$

We have then an inner variational problem:

$$\max_{p(x_2|x_1)} \sum_{x_2} p(x_2|x_1) \left[ -\beta \log \frac{p(x_2|x_1)}{p_0(x_2|x_1)} + U(x_2|x_1) \right]$$

with the solution

$$p(x_2|x_1) = \frac{1}{Z_2} p_0(x_2|x_1) \exp \left( \frac{1}{\beta} U(x_2|x_1) \right)$$

and the $x_1$-dependent normalization constant

$$Z_2 = \sum_{x_2} p_0(x_2|x_1) \exp \left( \frac{1}{\beta} U(x_2|x_1) \right)$$

and an outer variational problem

$$\max_{p(x_1)} \sum_{x_1} p(x_1) \left[ -\alpha \log \frac{p(x_1)}{p_0(x_1)} + U(x_1) + \beta \log Z_2 \right]$$

with the solution

$$p(x_1) = \frac{1}{Z_1} p_0(x_1) \exp \left( \frac{1}{\alpha} (U(x_1) + \beta \log Z_2) \right)$$

$$= \frac{1}{Z_1} p_0(x_1) \exp \left( \frac{1}{\alpha} U(x_1) + \beta \log \sum_{x_2} p_0(x_2|x_1) \exp \left( \frac{1}{\beta} U(x_2|x_1) \right) \right)$$

and the normalization constant

$$Z_1 = \sum_{x_1} p_0(x_1) \exp \left( \frac{1}{\alpha} (U(x_1) + \beta \log Z_2) \right)$$

$$= \sum_{x_1} p_0(x_1) \exp \left( \frac{1}{\alpha} U(x_1) + \beta \log \sum_{x_2} p_0(x_2|x_1) \exp \left( \frac{1}{\beta} U(x_2|x_1) \right) \right).$$
For notational convenience we introduce $\lambda = \frac{1}{\alpha}$ and $\mu = \frac{1}{\beta}$. Depending on the values of $\lambda$ and $\mu$ we can discern the following cases:

1. **Risk-seeking bounded rational agent:** $\lambda > 0$ and $\mu > 0$
   When $\lambda > 0$ the agent is bounded and acts in general stochastically. When $\mu > 0$ the agent considers the move of the environment as if it was his own move (hence “risk-seeking” due to the overtly optimistic view). This follows immediately from the choice axioms presented in section 3.1. We can also see this from the relationship between $Z_1$ and $Z_2$ in (9), if we assume $\mu = \lambda$ and introduce the value function $V_t = \frac{1}{\lambda} \log Z_t$, which results in the recursion
   \[ V_{t-1} = \frac{1}{\lambda} \log \sum_{x_{t-1}} p_0(x_{t-1}) \exp (\lambda (U(x_{t-1}) + V_t)) \, . \]

   Similar recursions based on the log-transform have been previously exploited for efficient approximations of optimal control solutions both in the discrete and the continuous domain [2,6,14]. In the perfectly rational limit $\lambda \to +\infty$, this recursion becomes the well-known Bellman recursion
   \[ V^*_t := \max_{x_{t-1}} (U(x_{t-1}) + V^*_t) \]
   with $V_t^* = \lim_{\lambda \to +\infty} V_t$.

2. **Risk-neutral perfectly rational agent:** $\lambda \to +\infty$ and $\mu \to 0$
   This is the limit for the standard optimal controller. We can see this from (9) by noting that
   \[ \lim_{\mu \to 0} \frac{1}{\mu} \log \sum_{x_2} p_0(x_2|x_1) \exp (\mu U(x_2|x_1)) = \sum_{x_2} p_0(x_2|x_1) U(x_2|x_1), \]
   which is simply the expected utility. By setting $U(x_1) \equiv 0$, and taking the limit $\lambda \to +\infty$ in (9), we therefore obtain an expected utility maximizer
   \[ p(x_1) = \delta(x_1 - x_1^*) \]
   with
   \[ x_1^* = \arg \max_{x_1} \sum_{x_2} p_0(x_2|x_1) U(x_2|x_1). \]
   As discussed previously, action selection becomes deterministic in the perfectly rational limit.

3. **Risk-averse perfectly rational agent:** $\lambda \to +\infty$ and $\mu < 0$
   When $\mu < 0$ the decision-maker assumes a pessimistic view with respect to the environment, as if the environment was an adversarial or malevolent agent. This attitude is sometimes called risk-aversion, because such agents act particularly cautiously to avoid high uncertainty. We can see this from (9) by writing a Taylor series expansion for small $\mu$
   \[ \frac{1}{\mu} \log \sum_{x_2} p_0(x_2|x_1) \exp (\mu U(x_2|x_1)) \approx E[U] - \frac{\mu}{2} \text{VAR}[U], \]
where higher than second order cumulants have been neglected. The name risk-sensitivity then stems from the fact that variability or uncertainty in the utility of the Taylor series is subtracted from the expected utility. This utility function is typically assumed in risk-sensitive control schemes in the literature \[18\], whereas here it falls out naturally. The perfectly rational actor with risk-sensitivity $\mu$ picks the action

$$p(x_1) = \delta(x_1 - x_1^*)$$

with

$$x_1^* = \arg \max_{x_1} \frac{1}{\mu} \log \sum_{x_2} p_0(x_2|x_1) \exp (\mu U(x_2|x_1)),$$

which can be derived from (9) by setting $U(x_1) \equiv 0$ and by taking the limit $\lambda \to +\infty$. Within the framework proposed in this paper we might also interpret the equations such that the decision-maker considers the environment as an adversarial opponent with bounded rationality $\mu$.

4. **Robust perfectly rational agent**: $\lambda \to +\infty$ and $\mu \to -\infty$

When $\mu \to -\infty$ the decision-maker makes a worst case assumption about the adversarial environment, namely that it is also perfectly rational. This leads to the well-known game-theoretic minimax problem with the solution

$$x_1^* = \arg \max_{x_1} \arg \min_{x_2} U(x_2|x_1),$$

which can be derived from (9) by setting $U(x_1) \equiv 0$, taking the limits $\lambda \to +\infty$ and $\mu \to -\infty$ and by noting that $p(x_1) = \delta(x_1 - x_1^*)$. Minimax problems have been used to reformulate robust control problems that allow controllers to cope with model uncertainties [1]. Robust control problems are also known to be related to risk-sensitive control [1]. Here we derived both control types from the same variational principle.

5. **Conclusion**

In this paper we have proposed a thermodynamic interpretation of bounded rationality based on a free utility principle. Accordingly, bounded rational agents trade off utility maximization against resource costs measured by the KL divergence with respect to an initial policy. The use of the KL divergence as a cost function for control has been previously proposed to measure deviations from passive dynamics in Markov systems [13,14]. Other methods of statistical physics have been previously proposed as an information-theoretic approach to interactive learning [12] and to game theory with bounded rational players [19]. The contribution of our study is to devise a single axiomatic framework that allows for the treatment of control problems, game-theoretic problems and estimation and learning problems for perfectly rational and bounded rational agents. In the future it will be interesting to relate the thermodynamic resource costs of bounded rational agents to more traditional notions of resource costs in computer science like space and time requirements when computing optimal actions [10].
10 Bounded Rationality

References

1. T. Basar and P. Bernhard. *H-Infinity Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*. Birkhauser Boston, 1995.
2. D.A. Braun, P.A. Ortega, E. Theodorou, and S. Schaal. Path integral control and bounded rationality. In *IEEE symposium on adaptive dynamic programming and reinforcement learning*, 2011.
3. H.B. Callen. *Thermodynamics and an Introduction to Thermostatistics*. John Wiley & Sons, 2nd edition, 1985.
4. R. P. Feynman. *The Feynman Lectures on Computation*. Addison-Wesley, 1996.
5. D. Haussler and M. Opper. Mutual information, metric entropy and cumulative relative entropy risk. *The Annals of Statistics*, 25:2451–2492, 1997.
6. B. Kappen. A linear theory for control of non-linear stochastic systems. *Physical Review Letters*, 95:200201, 2005.
7. G. Keller. *Equilibrium States in Ergodic Theory*. London Mathematical Society Student Texts. Cambridge University Press, 1998.
8. P. Ortega. A unified framework for resource-bounded autonomous agents interacting with unknown environments. PhD thesis, 2011.
9. P.A. Ortega and D.A. Braun. A bayesian rule for adaptive control based on causal interventions. In *The third conference on artificial general intelligence*, pages 121–126, Paris, 2010. Atlantis Press.
10. P.A. Ortega and D.A. Braun. A conversion between utility and information. In *The third conference on artificial general intelligence*, pages 115–120, Paris, 2010. Atlantis Press.
11. H Simon. *Models of Bounded Rationality*. MIT Press, 1982.
12. S. Still. An information-theoretic approach to interactive learning. *Europhysics Letters*, 85:28005, 2009.
13. E. Todorov. Linearly solvable markov decision problems. In *Advances in Neural Information Processing Systems*, volume 19, pages 1369–1376, 2006.
14. E. Todorov. Efficient computation of optimal actions. *Proceedings of the National Academy of Sciences U.S.A.*, 106:11478–11483, 2009.
15. M. Tribus and E.C. McIrvine. Energy and information. *Scientific American*, 225:179–188, 1971.
16. P.M.B. Vitanyi. Time, space, and energy in reversible computing. In *Proceedings of the 2nd ACM conference on Computing frontiers*, page 435444, 2005.
17. J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.
18. P. Whittle. *Risk-sensitive optimal control*. John Wiley and Sons, 1990.
19. D.H. Wolpert. Information theory - the bridge connecting bounded rational game theory and statistical physics. In: *Complex Engineering Systems*. Braha, D. and Bar-Yam, Y. (Eds.). Perseus Books, 2004.