ENERGY, TEMPERATURE, AND ENTROPY OF BLACK HOLES DRESSED WITH QUANTUM FIELDS

James W. York, Jr.  
Institute of Field Physics and  
Theoretical Astrophysics and Relativity Group  
Department of Physics and Astronomy  
The University of North Carolina  
Chapel Hill, North Carolina 27599-3255, USA

1. Introduction

A deeper understanding of the thermal properties of black holes than we presently have depends to a large degree on obtaining a firmer grasp of the properties of the entropy. For such an understanding we must at least know the basic relations among entropy, energy, and temperature of a black hole in thermal equilibrium with quantized matter fields. Limiting attention to spherical, uncharged ("Schwarzschild") holes, we will find that the basic Bekenstein-Hawking relations have to be generalized when the hole is dressed by quantum fields. Though this fact is not surprising, the corrections contain surprises and are very instructive. My purpose here is to discuss several aspects of this problem and to display some concrete results within the framework of the semi-classical theory of quantum fields in curved spacetime. Two key ideas that emerge are the following: (1) The calculated thermodynamical entropy $\Delta S$ by which the quantum fields we consider augment the usual Bekenstein-Hawking entropy is positive and monotonically increasing in a suitable sense if and only if the back-reaction of the quantum fields on the spacetime geometry is taken into account. (2) The total thermodynamical entropy within a sphere of surface area $4\pi r^2$, enclosing the black hole and fields, can be calculated from the temperature and the quasi-local energy determined at radius $r$ without any reference to asymptotic values of temperature or energy.

The quasi-local energy also forms a key part of a new fundamental approach to the statistical mechanics of self-gravitating systems. Such a statistical mechanics, that when sufficiently developed should further illuminate black hole thermodynamics, has been introduced recently by means of a microcanonical functional integral. The later takes advantage of the property of gravity by which the total energy of a system may be known by the behavior of the gravitational field at its boundary. This behavior defines the quasi-local energy. In this report, however, the microcanonical functional integral itself will not be used.

Progress in black hole thermodynamics depends upon accounting for those features of gravity, such as its long-range unscreened nature, that partially contradict the usual thermostatistical assumptions. It is clear that the gravitational

---

*Based on a talk given at the Fifth Canadian Conference on General Relativity and Relativistic Astrophysics, Waterloo, May, 1993.
effects of the radiation that would equilibrate a black hole (or a star) prevent the existence of the usual “thermodynamical limit”. Therefore we should contemplate finite systems even when back-reaction is being ignored, in order properly to set the stage for further work. Thermodynamics will thus not look exactly as it does in textbooks: it becomes more interesting. The generality and simplifying power that make thermodynamics a powerful tool remain.

2. Temperature and Energy

Consider a static spacetime $M$. (Stationary spacetimes are treated similarly but involve further subtleties, not discussed here, that have been treated in Refs. 7 and 8.) We learned from Tolman that quantities like temperature and chemical potential, usually regarded as intensive, in the presence of gravity are no longer purely so. For example, consider the temperature $T$. Then we have as a consequence of the “red shift” law that

$$N(\vec{x}_1) T(\vec{x}_1) = N(\vec{x}_2) T(\vec{x}_2)$$

(1)

for any points $\vec{x}_1$ and $\vec{x}_2$ on a static time slice $t = \text{constant}$. Here $T(\vec{x})$ denotes the temperature measured locally by a static observer at $\vec{x}$ and

$$N(\vec{x}) = [-g^{tt}(\vec{x})]^{-1/2}$$

(2)

is the lapse function. If there is an “asymptotically flat region at spatial infinity” where $N \to 1$ then there is a corresponding “temperature at infinity” $T_\infty$. This is the temperature usually meant, for example, when one speaks of “the Hawking temperature of a black hole”. However, as we shall see, one can just as well, indeed better, use a finite sphere rather than the asymptotic “sphere at infinity” as a reference locus for the temperature of a star or a black hole. (In the absence of spherical symmetry, other generally finite reference surfaces may prove simpler to use. In principle, any two-surface enclosing the system will suffice.)

Suppose we know the entropy $S$ as a function of energy $E$, charge $Q$, and other conserved quantities. For simplicity let us assume spherical symmetry in what follows. Then, by definition,

$$\beta(r) = \left(\frac{\partial S}{\partial E}\right)_{r,Q,...}$$

(3)

What energy is “conjugate” to $\beta(r)$? If $r = \infty$ is an asymptotically flat region, then $E$ should be the ADM mass $M$ of the system. More generally, however, $E$ is the quasi-local energy obtained by a geometrical analysis of the surface terms belonging to the Hamiltonian generating the motion of the system bounded spatially by the two-surface $B$, in this case $r = \text{constant}$. As such, the expression for $E$ is not offered as a proposal for the quasi-local energy; it is dictated by the action integral of the theory. One finds

$$E(B) = \frac{1}{8\pi} \int_B (k - k_0) \sqrt{\sigma} d^2x,$$

(4)

where $\sigma_{ij}$ is the two-metric induced on $B$ and $k = \sigma^{ij} k_{ij}$ is the trace of the extrinsic curvature $k_{ij}$ of $B$ as embedded in the time slice $t = \text{constant}$. (Units are chosen such that $G = c = k_B = 1$, but $\hbar \neq 1$.) The term $k_0$ serves to define the zero of energy, which is no more intrinsically fixed in general relativity than it is in any
other theory. (The usual convention is that Minkowski spacetime has zero energy.) A convenient definition of $k_0$, when it can be implemented, is that it is the trace of the second fundamental tensor of the two-surface $B$ when the latter is isometrically embedded in flat three-space. This assures that $E$ becomes asymptotically the ADM mass when the latter is defined. In any event, no physical result should depend in an essential way on the zero of energy, and that is true of the results reviewed in this paper.

Consider a static spherically symmetric spacetime in the usual coordinates, such that $r$ is the “areal” radius. Temporarily ignore quantum fields and back-reaction but let $r < \infty$. Then for the Schwarzschild black hole, using Eq. (4), one finds

$$E(r) = r - r \left( 1 - \frac{2M}{r} \right)^{1/2},$$

and, more generally, for the Reissner-Nordström black hole

$$E(r) = r - r \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{1/2}.$$  
\hspace{1cm} (6)

Note that Eq. (6) implies

$$M = E - E^2 + \frac{E^2}{2r}.$$  
\hspace{1cm} (7)

The second and third terms on the right have the expected signs for gravitational and electrostatic binding energies associated with constructing a shell of radius $r$, energy $E$, and charge $Q$. But Eq. (7) just helps us build a “word picture” to make us feel more comfortable with the energy $E$. Ref. 4 contains further heuristic discussions along these lines. However, the important expression is Eq. (6), which can be expressed without using the ADM mass $M$. In fact, $M$ has no essential role as mass in this problem. It is rather just a parameter perhaps better expressed in terms of the gravitational radius of the hole,

$$r_+ = M + M \left( 1 - \frac{Q^2}{M^2} \right)^{1/2}.$$  
\hspace{1cm} (8)

In place of Eq. (3) we find

$$E(r) = r - r \left[ \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{Q^2}{r_+} \right) \right]^{1/2}.$$  
\hspace{1cm} (9)

Given that the Bekenstein-Hawking entropy is $S_{BH} = \pi r_+^2 \hbar^{-1}$, we find

$$\left( \frac{\partial S_{BH}}{\partial E} \right)_{r,Q} = \beta(r) = \left[ \frac{4\pi r_+}{\hbar} \left( 1 - \frac{Q^2}{r_+^2} \right)^{-1/2} \right] \left[ \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{Q^2}{r_+} \right) \right]^{1/2}.$$  
\hspace{1cm} (10)

This is the inverse Hawking temperature, blue-shifted from infinity to $r$. Note that the result is independent of the choice of the zero-point for $E$ (represented by the first term on the right in Eq. (10)). The inverse of the first factor in the rectangular brackets on the right of Eq. (10) is just $\kappa \hbar/2\pi$, where $\kappa$ is the surface gravity of the event horizon. The second factor in the rectangular brackets on the right of Eq. (10) is just $\kappa \hbar/2\pi$, where $\kappa$ is the surface gravity of the event horizon.
is the lapse function $N(r)$. The result for $\beta(r)$ can be expressed in terms of $r$, $E$, and $Q$ by using Eq. (7) and Eq. (8) in Eq. (10). The conclusion is that the energy $E$ as defined by the surface term Eq. (3) from the behavior of the gravitational field on the two-boundary $B$ is both the energy defined by the total Hamiltonian of the system and the total internal energy of the system in the sense of thermodynamics. Identification of these quantities is a significant step in the unification of gravitation and thermodynamics that was anticipated when the Hawking effect was discovered.

3. Quantum Stress-energy Tensors and Back-reaction

A black hole can exist in thermodynamical equilibrium provided that it is surrounded by radiation with a suitable distribution of stress-energy. In the semi-classical approach, such radiation is characterized by the expectation value of a stress-energy tensor obtained by renormalization of a quantum field on the classical spacetime geometry of a black hole. One can use such a stress-energy tensor as a source in the semi-classical Einstein equation,

$$G^\mu_\nu = 8\pi \langle T^\mu_\nu \rangle_{\text{renormalized}},$$

(11)

to calculate the change effected by the stress-energy tensor in the black hole’s spacetime metric. This is the “back-reaction” problem associated with the spacetime geometry of a black hole in equilibrium.

We shall see, from the properties of the renormalized stress-energy tensors we employ and of the semi-classical Einstein equation, that we can obtain accurate fractional corrections to the metric only in $O(\epsilon)$, where $\epsilon = \hbar M^{-2}$, $M_{Pl} = h^{1/2}$ is the Planck mass and $M$ is the mass of the black hole. Because the usual black hole entropy $S_{BH} = (4\pi M^2)\hbar^{-1} = O(\epsilon^{-1})$, corrections to $S_{BH}$ can be obtained in $O(\epsilon^0) = O(1)$ from the fractional corrections of $O(\epsilon)$ in the metric. It turns out that these corrections are of the same order as the naive flat space radiation entropy $(4/3)aT_H^3 V$, where $a = (\pi^2/15\hbar^3)$, $T_H = \hbar (8\pi M)^{-1}$ is the uncorrected Hawking temperature of a Schwarzschild black hole, and $V$ is the flat space volume. From this fact alone it follows that the back-reaction cannot be ignored.

Stress-energy tensors renormalized on a Schwarzschild background have been obtained in exact form for conformal scalar fields and for $U(1)$ gauge fields, respectively, by Howard and by Jensen and Ottewill. Both results can be written in the form

$$\langle T^\mu_\nu \rangle_{\text{renormalized}} = \langle T^\mu_\nu \rangle_{\text{analytic}} + \left(\frac{\hbar}{\pi^2 (4M)^4}\right) \Delta^\mu_\nu,$$

(12)

where the analytic piece, in the case of the conformal scalar field, was given by Page. The term $\Delta^\mu_\nu$ is obtained from a numerical evaluation of a mode sum. The numerical piece is small compared to the analytic piece, and we do not include it in the calculations in this paper. This does not change any of the results qualitatively because both pieces separately obey the required regularity and consistency conditions. The analytic piece has the exact trace anomaly in both cases. The stress-energy tensors have vanishing covariant divergence on the Schwarzschild background. They represent the stress-energy distribution required to equilibrate the black hole with its own Hawking radiation. Each satisfies $\langle T^r_r \rangle = \langle T^\theta_\theta \rangle$ at the horizon $r = 2M$, which is required for regularity of the spacetime geometry. Each has the asymptotic form of a flat spacetime radiation stress-energy tensor at the uncorrected Hawking temperature at infinity of an ordinary Schwarzschild black hole.
denoted by $T_H = h(8\pi M)^{-1}$. The cases worked out so far include the conformal scalar field, the massless spin $1/2$ field, and the $U(1)$ vector field. Only the latter will be displayed here.

Dropping the angular brackets and displaying the analytic piece, one has for the $U(1)$ vector field, with $w = 2M/r$,

\begin{align}
T_t^t &= -\frac{1}{3} aT_H^4 (3 + 6w + 9w^2 + 12w^3 - 315w^4 + 78w^5 - 249w^6), \quad (13) \\
T_r^r &= \frac{1}{3} aT_H^4 (1 + 2w + 3w^2 - 76w^3 + 295w^4 - 54w^5 + 285w^6), \quad (14) \\
T_\theta^\theta &= T_\phi^\phi = \frac{1}{3} aT_H^4 (1 + 2w + 3w^2 + 44w^3 - 305w^4 + 66w^5 - 579w^6). \quad (15)
\end{align}

Note that $T_r^r > 0$ and that the energy density $-T_t^t$ is negative in the vicinity of the event horizon, thus violating the weak energy condition. The energy density is negative from $r = 2M$ to $r \approx 5.14M$. The dominant energy condition is also violated in a region surrounding and bordering the horizon. It is convenient in what follows to write

$$
\frac{1}{3} aT_H^4 = \frac{\epsilon}{48\pi KM}, \quad (16)
$$

where $K = 3840\pi$.

We obtain fractional corrections $h^\alpha_\nu$ to the metric by setting

\begin{equation}
g_{\mu\nu} = \hat{g}_{\mu\alpha} (\delta^\alpha_\nu + \epsilon h^\alpha_\nu)
\end{equation}

in the semi-classical Einstein equation Eq. (11), where $\hat{g}_{\mu\nu}$ is the uncorrected Schwarzschild metric. We work in linear order in $\epsilon$ as required by $\hat{\nabla}_\mu T_\nu^\mu = 0$ and $\hat{\nabla}_\mu (\delta G_\nu^\mu) = 0$, where $\delta G_\nu^\mu$ is the Einstein operator linearized on a background satisfying $G_\nu^\mu = 0$. The corrected geometry will be taken to be static and spherically symmetric. Working out the equations as in Refs. 16 and 17, we find the corrected metric can be written as

\begin{equation}
ds^2 = -\left(1 - \frac{2m(r)}{r}\right) (1 + 2\bar{\rho}(r)) dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\omega^2, \quad (18)
\end{equation}

where $d\omega^2$ is the standard metric of a normal round unit sphere. To obtain $m(r)$ and $\bar{\rho}(r)$ requires only simple radial integrals involving $T_t^t$ and $T_r^r$. The angular components enter linearized Einstein equations that hold automatically by virtue of $\hat{\nabla}_\mu T_\nu^\mu = 0$ in a static spherical geometry.

The mass function $m(r)$ has the form

\begin{equation}
m(r) = M \left(1 + \epsilon \mu(r) + \epsilon CK^{-1}\right), \quad (19)
\end{equation}

with

\begin{equation}
\mu(r) = \frac{1}{\epsilon M} \int_{2M}^r (-T_t^t) 4\pi \bar{\rho}^2 \, d\bar{r}, \quad (20)
\end{equation}

so $\mu(r)$ vanishes at the horizon. In Eq. (19), $C$ is an undetermined integration constant that inspection of Eq. (18) shows is to be absorbed into $M$ to obtain a renormalized mass for the black hole. Thus, setting $g^{rr} = 0$ shows that $r = 2m = 2M(1 + \epsilon CK^{-1}) = 2M_{\text{renormalized}}$ locates the event horizon. Note that, to the
order we are working, we can write \( m(r) = M(1 + \epsilon CK^{-1})(1 + \epsilon \mu(r)) \equiv M_{\text{ren}}(1 + \epsilon \mu(r)) \). The renormalized mass will not be distinguished notationally from the original Schwarzschild mass \( M \) in what follows, as the bare Schwarzschild mass has no physical meaning in the back-reaction problem. Therefore, we write

\[
m(r) = M (1 + \epsilon \mu(r)) \equiv M + M_{\text{rad}}(r)
\]  

where, using Eq. (20), we see that \( M_{\text{rad}} = \epsilon M \mu \) is the usual expression for the effective mass of a spherical source. One finds

\[
K \mu = \frac{2}{3} w^{-3} + 2w^{-2} + 6w^{-1} - 8\ln(w) + 210w - 26w^2 + \frac{166}{3} w^3 - 248.
\]  

(22)

In Eq. (22), we note that the first term on the right, multiplied by \( \epsilon MK^{-1} \), gives the naive flat-space value \( aT^A_T \) for radiation energy.

The metric is completed by a determination of \( \bar{\rho} \) which, like \( \mu \), can be found from an elementary integration. Defining

\[
K \bar{\rho} \equiv K \rho + k,
\]  

(23)

where \( k \) is constant of integration (not the same as the \( k \) in Eq. (4)), we have

\[
\rho = \frac{1}{\epsilon} \int_{2M}^{r} (T^r_r - T^t_t)(\tilde{r} - 2M)^{-1} 4\pi \tilde{r}^2 d\tilde{r}.
\]  

(24)

We find

\[
K \rho = \frac{2}{3} w^{-2} + 4w^{-1} - 8\ln(w) + \frac{40}{3} w + 10w^2 + 4w^3 - 32,
\]  

(25)

with \( \rho(1) = 0 \) at \( w = 1 \).

Because the radiation stress-energy tensors are asymptotically constant, it is clear that the system composed of black hole plus equilibrium radiation must be put in a finite “box”. Otherwise, the fractional corrections \( \epsilon h^a_\alpha \) to the metric would not remain small for sufficiently large radius. Physically this means that the radiation in a box that is too large would collapse onto the black hole, producing a larger one. Hence, we must choose the radius \( r_\circ \) of the box such that it is less than the second positive root \( r_* \) for \( r \) in \( g^{rr} = 0 \) (the first zero corresponds to the horizon \( r = 2M \)). We shall also assume that the box radius \( r_\circ \) is sufficiently large that the stress-energy tensors we employ, which were constructed for infinite asymptotically flat spacetime, are a good approximation. Clearly, a finite radius would cut out some of the radial modes that were used in these calculations. However, if \( r_\circ \) is somewhat greater than the longest wavelength characteristic of Hawking radiation, which in turn is associated with the least-damped quasi-normal mode of lowest angular momentum for the field in question, then this effect should be negligible. This wavelength \( \lambda_* \) is about 42M for the conformal scalar field and is smaller for the higher-spin massless fields. Also, if \( r_\circ > \lambda_* \), then the explicit nature of the walls of the box (e.g., adiabatic versus diathermic) should not be important. For these reasons we shall assume throughout the remainder of this work that \( \lambda_* < r_\circ < r_* \). (Of course, one must also assume that \( M \gtrsim M_{\text{Pl}} \), in any treatment based on Eq. (11).)

One convenient way to fix the constant \( k \) is to impose a microcanonical boundary condition.\(^{16}\) We fix \( r_\circ \) and imagine placing there an ideal massless perfectly reflecting wall. Outside \( r_\circ \), we then have an ordinary Schwarzschild spacetime.
\[ ds^2 = -\left(1 - \frac{2m(r_o)}{r}\right) dt^2 + \left(1 - \frac{2m(r_o)}{r}\right)^{-1} dr^2 + r^2 d\omega^2, \]  
\eqno{(26)}

for \( r \geq r_o \). Continuity of the three-metric induced by the metrics Eq. (18) and Eq. (26) on the world tube \( r = r_o \) fixes the constant \( k \) in \( \bar{\rho} \) by the relation
\[ k = -K\rho(r_o). \]  
\eqno{(27)}

There are finite discontinuities in the extrinsic curvature of the world tube \( r = r_o \),
but these, and other properties of the box wall, are of no interest in the present analysis, as we argued above. The space-time geometry, including back-reaction, is now completely determined by Eq. (26) for \( r \geq r_o \), and for \( r \leq r_o \) by Eq. (18) and Eq. (27).

4. Temperature and Entropy

If we release a small packet of energy from a closed box containing a black hole through a long thin radial tube, it will undergo a red-shift and approach the asymptotic temperature
\[ T_\infty = \frac{\kappa_H \hbar}{2\pi}, \]  
\eqno{(28)}

where \( \kappa_H \) is the surface gravity of the event horizon. For an ordinary Schwarzschild black hole (ignoring radiation), one finds \( \kappa_H = (4M)^{-1} \) and \( T_\infty = T_H = \hbar (8\pi M)^{-1} \) . However, the stress-energy of the radiation changes the surface gravity of the horizon to
\[ \kappa_H = \left. \frac{1}{4M} \left[ 1 + \epsilon(\bar{\rho} - \mu) + 8\pi r^2 T_t^t \right] \right|_{r = 2M}, \]  
\eqno{(29)}

as a straightforward calculation shows. With the microcanonical boundary conditions, we can use Eq. (27) to obtain from Eq. (28) and Eq. (29)
\[ T_\infty = \frac{\hbar}{8\pi M} \left[ 1 - \epsilon \bar{\rho}(r_o) + \epsilon nK^{-1} \right], \]  
\eqno{(30)}

where \( n \) takes the value 304 for the vector field. (It has other values for other fields.) The local temperature at the boundary of the box is obtained by blue-shifting Eq. (30) from infinity back to \( r_o \). We find from
\[ T_{loc} = T_\infty \left[ -g_{tt}(r_o) \right]^{-1/2}, \]  
\eqno{(31)}

that
\[ T_{loc}(r_o) = \frac{\hbar}{8\pi M} \left[ 1 - \epsilon \bar{\rho}(r_o) + \epsilon nK^{-1} \right] \left[ 1 - \frac{2m(r_o)}{r_o} \right]^{-1/2}. \]  
\eqno{(32)}

The temperature \( T_{loc} \), unlike \( T_\infty \), is actually independent of the boundary condition that determines the constant \( k \), as explained in detail in Ref. 16. Indeed, it can be readily verified that \( k \) cancels out in \( O(\epsilon) \) in the expression Eq. (31) for \( T_{loc} \). Either measure of temperature, \( T_\infty \) or \( T_{loc} \), can be used to calculate the same entropy in conjunction with an appropriate measure of energy. This is quite important: it
means that the specific boundary condition chosen does not affect the calculated entropy, as we shall see below.

One way to calculate the entropy is as follows. Fix the radius $r_\circ$ of a closed box. The measure of energy in the box conjugate to the asymptotic inverse temperature $\beta_\infty \equiv T_\infty^{-1}$ is then the ADM mass $m(r_\circ)$ determined at spatial infinity. The first law of thermodynamics for slightly differing equilibrium configurations with the same areal radius tells us that

$$dS = \beta_\infty dm \quad (dr_\circ = 0),$$  \hspace{1cm} (33)

where $S(r_\circ)$ is the total entropy in the box. By this method we seem to obtain only the total entropy $S(r_\circ)$ rather than the distribution of entropy in the given box, $S(r)$, for $r \leq r_\circ$, where $S(r)$ denotes the total entropy inside the radius $r$. However, the latter can be obtained by using the quasi-local energy $E$,\footnote{which for static spherical metrics like those treated here is given by}

$$E(r) = r - r [g^{rr}(r)]^{1/2},$$  \hspace{1cm} (34)

with $g^{rr}(r)$ determined by the metric for $r \leq r_\circ$. This energy, unlike $m$, does not depend on asymptotic flatness in its definition, nor even on the existence of an asymptotically flat region.\footnote{Furthermore, even the “normalization” of the zero of energy that is incorporated in $E$ as given in Eq. (34) does not affect the calculated entropy, as it certainly should not. (Recall that this “normalization” is intended to make $E$ approach the ADM mass in an asymptotically flat region, if such a region exists.) Similarly, the inverse local temperature $\beta(r) \equiv T_{\text{loc}}^{-1}(r), r \leq r_\circ$, is independent of the boundary condition as mentioned above. Hence, the value of the entropy depends neither on the zero of energy nor the existence of an asymptotic region.}

Therefore, to obtain $S(r)$, in place of Eq. (31) we can write

$$dS = \beta dE \quad (dr = 0, \, r \leq r_\circ).$$  \hspace{1cm} (35)

Choosing $M$ and $r$ as independent variables, and fixing $r$, we can readily integrate Eq. (33) to obtain $S$ up to a function of $r$ and a constant. From Eq. (32) we have

$$\beta(r) = \frac{8\pi M}{\hbar} \left[ 1 + \epsilon \rho(r) - \epsilon nK^{-1} \right] \left( 1 - \frac{2m(r)}{r} \right)^{1/2},$$  \hspace{1cm} (36)

and from Eq. (21) and Eq. (34), holding $r$ fixed,

$$dE = \left[ 1 - \epsilon \mu + \epsilon M \frac{\partial \mu}{\partial M} \right] \left( 1 - \frac{2m(r)}{r} \right)^{-1/2} dM.$$  \hspace{1cm} (37)

One can see directly for any $r \leq r_\circ$ that $\beta_\infty dm = \beta dE$ where, of course, one replaces $r_\circ$ by $r$ in the formulas for $\beta_\infty$ and $m$ to establish this result. This equality means that we can calculate $S(r)$ for any $r \leq r_\circ$. The key point of this discussion is that Eq. (33) is independent of the boundary conditions (does not depend on $k$). This means that the $\Delta S(r)$ calculated does not depend on whether there is empty space just outside $r$ or more radiation. The result will hold for any $r$ such that $2M \leq r < r_\circ$.

Observe that from the fractional changes of $O(\epsilon)$ in the metric, which affect the surface gravity and temperature in this order, we are able to calculate from Eq. (33) departures of $O(\epsilon^0) = O(1)$ from the usual black hole entropy $S_{\text{BH}} = (4\pi M^2)\hbar^{-1} = 4\pi \epsilon^{-1}$. But in fact all of the corrections to the entropy are of the same order as the naive flat-space entropy itself:
\[
\frac{4}{3}aT^3_HV = \frac{4}{3} \left( \frac{\pi^2}{15\hbar^3} \right) \left( \frac{\hbar}{8\pi M} \right)^3 \left( \frac{4}{3} \pi r^3 \right) = \frac{8\pi}{K} \left( \frac{8}{9} w^{-3} \right) = O(1) \times w^{-3}. \tag{38}
\]

The \( \hbar \)'s in Eq. (38) cancel out, leaving only a function of \( w = 2Mr^{-1} \). Combining Eq. (36) and Eq. (37) yields

\[
dS = \frac{8\pi M^2}{\hbar} dM + 8\pi \left[ w^{-1}(\rho - \mu) + \frac{\partial\mu}{\partial w} - nK^{-1}w^{-1} \right] dw, \tag{39}
\]

with \( dr = 0 \). Integration of Eq. (39) gives an expression of the form

\[
S = \frac{4\pi M^2}{\hbar} + \Delta S(w) + f \left( \frac{r}{\hbar^{1/2}} \right), \quad (1 \leq w \leq w_o = 2M/r_o) \tag{40}
\]

where the first term is the usual Bekenstein-Hawking expression \( S_{BH} \) for the black hole entropy, the second term is a function of \( w \) determined up to an additive integration constant by the second term on the right of Eq. (39), and \( f \) is a dimensionless function of \( r \) that does not depend on \( M \). The appearance of the function \( f \) in Eq. (40) can be understood as follows. Since our problem involves the three mass or length scales \( M_{Planck} = \hbar^{1/2} \), the mass of the black hole, \( M \), and the radius \( r \leq r_o \), there are, for a given \( r \), three relevant dimensionless parameters one can define, namely, \( \epsilon = \hbar M^{-2} \), \( w = 2M/r \) and \( r/\hbar^{1/2} \). However, the first two terms on the right of Eq. (40) depend only on \( \epsilon \) and \( w \), respectively. Thus, if the entropy \( S \) depends on \( r/\hbar^{1/2} \), it can only do so through a separate function of this parameter.

Let us first dispose of the dimensionless function \( f \), which clearly can depend only on \( (r/\hbar^{1/2}) \), where \( \hbar^{1/2} \) is the Planck length in our units. It seems that such a term could only arise in a theory taking quantum gravity into explicit account because the semi-classical theory has incorporated the dimensionless terms involving \( \hbar/M^2 \) and \( 2M/r \). (Of course, quantum gravity could modify terms of these latter two types quantitatively.) On dimensional grounds, therefore, we take \( f = 0 \) in the semi-classical theory. The possibility of an additive constant will be discussed when we treat \( \Delta S \) below.

In considering \( \Delta S \), which will be given explicitly below, we first note the significant property that

\[
\frac{\partial(\Delta S)}{\partial w} = 8\pi \left[ w^{-1}(\rho - \mu) + \frac{\partial\mu}{\partial w} - nK^{-1}w^{-1} \right] \tag{41}
\]

vanishes at the horizon \( w = 1 \). Therefore, for a fixed black hole mass \( M \), the derivative with respect to \( r \) of \( \Delta S \) vanishes at the horizon. Thus \( \Delta S \) has a local extremum with respect to \( r \) at the horizon. This result follows from several general features that will be enjoyed by all regular renormalized stress-energy tensors on the Schwarzschild background and the back-reactions they induce, not just the presently known cases. First, \( \mu \) vanishes at the horizon by virtue of the black hole’s mass having been suitably renormalized. Second, \( \rho \) vanishes at the horizon, as follows from Eq. (24) and the regularity condition \( T^t_t = T^r_r \) at the horizon. More precisely, we have that

\[
\lim_{w \to 1^+} \left( \frac{T^t_t - T^r_r}{1 - w} \right) \tag{42}
\]

exists. Third, the last two terms on the right side of Eq. (39) add to zero at the horizon because there the Hamiltonian constraint \( \left( G_t^t - 8\pi T^t_t = 0 \right) \) holds. Furthermore,
note that if the fractional effects of $O(\varepsilon)$ in the temperature induced by the back reaction were neglected, the derivative of Eq. (41) would not vanish at the horizon.

Is the local extremum of $\Delta S$ at the horizon a local minimum? To answer this we calculate

\[
\frac{\partial^2 (\Delta S)}{\partial w^2} = 8\pi \left[ -w^{-2}(\rho - \mu) + w^{-1} \left( \frac{\partial \rho}{\partial w} - \frac{\partial \mu}{\partial w} \right) + \frac{\partial^2 \mu}{\partial w^2} + nK^{-1}w^{-2} \right],
\]

which becomes, at the horizon $w = 1$,

\[
\frac{\partial^2 (\Delta S)}{\partial w^2} \bigg|_{w=1} = 8\pi \left( \frac{\partial \rho}{\partial w} + \frac{\partial^2 \mu}{\partial w^2} \right) \bigg|_{w=1}
\]

or, equivalently, with $M$ fixed,

\[
\frac{\partial^2 (\Delta S)}{\partial r^2} \bigg|_{r=2M} = \frac{32\pi^2 M^2}{\hbar} \left[ 4M \frac{\partial (-T^r_r)}{\partial r} - 8T^r_r - \left( \frac{T^r_r - T^t_t}{1 - 2M/r} \right) \right] \bigg|_{r=2M}.
\]

Hence we need only examine the stress-tensors. In all the cases treated so far\textsuperscript{14} (conformal scalar, vector, massless fermion), Eq. (44) and Eq. (45) are positive so that $\Delta S$ takes a local minimum with respect to the radius at the horizon. This suggests, but does not prove, that $\Delta S$ is non-negative.

The local minimum of $\Delta S$ at the horizon and the fact the $S_{BH}$ in the expression Eq. (40) for the total entropy $S$ contains the renormalized mass $M$ of the hole motivate the choice of the remaining additive constant in $\Delta S$, which can only be a pure number, to be such that $\Delta S = 0$ at $w = 1$. For $w = 1$, with no “room” for the fields to contribute anything further, one then obtains only the Bekenstein-Hawking entropy $(1/4)A_H\hbar^{-1}$, as would be expected. With the choice $\Delta S(w = 1) = 0$, we obtain for the vector field

\[
\Delta S = \frac{8\pi}{K} \left( \frac{8}{9}w^{-3} + \frac{8}{3}w^{-2} + 8w^{-1} - 96\ln(w) + \frac{40}{3}w - 8w^2 + \frac{344}{9}w^3 - \frac{496}{9} \right).
\]

(Positive entropy $\Delta S$ for the conformal scalar field was first obtained in Refs. 18 and 19.) In this expression, the naive flat-space radiation entropy term Eq. (38) appears as the first term on the right. $\Delta S$ is positive for $1 \geq w \geq w_o \geq w_* = 2M/t_s^{-1}$ and vanishes at $w = 1$. Hence, in that it is positive, it is amenable to arguments relating thermodynamical and statistical entropy. It is not heretofore been evident that this desirable feature would be present in the semi-classical theory. The reader can verify, by omitting the back-reaction terms in the inverse temperature Eq. (30), that not only is the vanishing slope of $\Delta S$ at $w = 1$ lost, but also that the value of the resulting “$\Delta S$” normalized as above, is no longer positive for the range $1 \geq w \geq w_o$ . In this fundamental sense, we conclude that the back reaction, however small quantitatively in its effects on the metric near a black hole, can never be regarded as negligible.

5. Acknowledgements

I thank G. L. Comer, D. Hochberg, and T. W. Kephart for collaboration in part of this work and for helpful discussions. This research was supported by
National Science Foundation grants PHY-8407492 and PHY-8908741.

6. References

1. N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
2. J. D. Bekenstein, *Phys. Rev.* D7 (1973) 2333.
3. S. W. Hawking, *Commun. Math. Phys.* 43 (1975) 199.
4. J. D. Brown and J. W. York, *Phys. Rev.* D47 (1993) 1407.
5. J. D. Brown and J. W. York, *Phys. Rev.* D47 (1993) 1420.
6. J. W. York, *Phys. Rev.* D33 (1986) 2092.
7. J. D. Brown, E. A. Martinez, and J. W. York, *Ann. N. Y. Acad. Sci.* 631 (1991) 225.
8. J. D. Brown, E. A. Martinez, and J. W. York, *Phys. Rev. Lett.* 66 (1991) 2281.
9. R. C. Tolman, *Phys. Rev.* 35 (1930) 904.
10. H. W. Braden, J. D. Brown, B. F. Whiting, and J. W. York, *Phys. Rev.* D42 (1990) 3376.
11. K. W. Howard, *Phys. Rev.* D30 (1984) 2532.
12. B. P. Jensen and A. C. Ottewill, *Phys. Rev.* D39 (1989) 1130.
13. D. N. Page, *Phys. Rev.* D25 (1982) 1499.
14. D. Hochberg, T. W. Kephart, and J. W. York, *Phys. Rev.* D48 (1993) 479.
15. M. R. Brown, A. C. Ottewill, and D. N. Page, *Phys. Rev.* D33 (1986) 2840.
16. J. W. York, *Phys. Rev.* D31 (1985) 775.
17. D. Hochberg and T. W. Kephart, *Phys Rev.* D47 (1993) 1465.
18. J. W. York, “Entropy of a Conformal Scalar Field and a Black Hole” (1985) unpublished.
19. G. L. Comer, “The Thermodynamic Stability of Systems Containing Black Holes”, University of North Carolina doctoral thesis (1990) unpublished.