Towards Optimal Schemes for the Half-Duplex Two-Way Relay Channel

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Abstract—A restricted two-way communication problem in a small network is investigated. The network consists of three nodes, all having access to a common channel with half-duplex constraint. Two nodes want to establish a dialog while the third node can assist in the bi-directional transmission process. All nodes have agreed on a transmission protocol a priori and the problem is restricted to the dialog encoders not being allowed to establish a cooperation by the use of previous receive signals. The channel is referred to as the restricted half-duplex two-way relay channel. Here the channel is defined and an outer bound on the achievable rates is derived by the application of the Cut-set Theorem. This shows that the problem consists of six parts. By the use of random codes and suboptimal decoders an inner bound on the achievable rates with decoding at the relay is established. Restricting to this relaying strategy and fixed input distributions makes it possible to determine optimal transmission schemes with respect to maximizing a rate objective or to minimizing a cost objective by solving a small-scale linear program. Simulations for an AWGN-channel model then show that it is possible to simultaneously increase the communication rate of both dialog messages compared to two-way communication without relay.

Index Terms—networks, cooperation, relay, half-duplex constraint, two-way channel, channel coding, resource allocation.

I. INTRODUCTION

In the last years it has been recognized that operating wireless systems in a point-to-point fashion might not be optimal. Other users act as interferers in the transmission process and are therefore equivalent to noise. It is believed that cooperation can outperform this competitive approach. One such cooperative concept, that has received increasing attention lately, is known as relaying. Source and destination connect over one or many intermediate nodes if isolated from each other or facing bad channel conditions for direct communication. The potential to further extend the efficient use of resources when interchanging data between two nodes in a bi-directional way over a relay [1] has led to a lot of recent works. Although, some years ago, it has been shown that with a careful design of transmission protocols [2], relaying can also be used to increase the communication rate in the presence of a direct path, two-way relaying is still mostly considered in the context of a connectivity problem, e.g., [1] [3] [4]. There two nodes cannot establish a direct communication and convey messages over a third node, the relay. Here a more general approach to half-duplex two-way relaying is taken. Two nodes that have a direct connection want to perform a dialog. It is to decide if a third node should join the communication process in order to facilitate the exchange of messages. The motivation to do so could be to maximize the communication rates for given resources or to minimize resources for given rates. The ultimate solution to both problems is to use the channel in the most efficient way by organizing the communication process in the network.

A. Wireline Example

Consider a fully-connected wireline network with three nodes [5, Section 9.1]. Each node faces a half-duplex constraint, i.e., it can not receive and transmit simultaneously. For simplicity all links support one reliable bit per channel use. Nodes 1 and 3 want to exchange messages (bits) while node 2 can assist the communication process as a relay. The simplest way to communicate is to let the dialog nodes send sequentially to each other while node 2 stays turned off (see Fig. 1). This communication strategy, referred to as two-way channel (TWC), allows to transmit two bits within two steps. The communication rate is \( R = 1.0 \) bits per step (bps). Asking for the cost \( C_T \) of the communication protocol we conclude that two links have been used, \( C_T = 1.0 \) links per bit (lpb) or two nodes have been activated, \( C_T = 1.0 \) nodes per bit (npb). A popular two-way relaying scheme [1] is provided by the two steps of Fig. 2. For this scheme the rate stays at \( R = 1.0 \) bps while the cost increases to \( C_T = 2.0 \) lpb or \( C_T = 1.5 \) npb. Note that this scheme also ignores the possibility to use the direct path between nodes 1 and 3. An improvement in transmission rate is provided by the three-step scheme sketched in Fig. 3. The rate attains \( R = 1.33 \) bps while the cost is \( C_T = 1.5 \) lpb or \( C_T = 1.5 \) npb. Alternatively, one could think of a different three-step scheme [6] (see Fig. 4) in order to activate less nodes in the communication process. For this protocol the rate stays at \( R = 1.33 \) bps while the cost is \( C_T = 1.5 \) lpb or \( C_T = 0.75 \) npb. Note that interestingly

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Fig. 1: Two-way channel (1.0 bps / 1.0 lpb / 1.0 npb)

Fig. 2: Two-step scheme (1.0 bps / 2.0 lpb / 1.5 npb)

Fig. 3: Three-step scheme (1.33 bps / 1.5 lpb / 1.5 npb)

Fig. 4: Different three-step scheme (1.33 bps / 0.75 lpb / 1.5 npb)
with this scheme the rate is higher than in the simple two-way communication without relay at a lower node-based cost. Finally, consider a four-step scheme like depicted in Fig. 5.

The rate has reached $R = 1.5$ bps at a cost of $C_T = 1.33$ lpb or $C_T = 1.0$ npb.

B. Potentials, Critics and Outline

The simple example given indicates that two-way relaying schemes on half-duplex communication systems might offer possibilities to enhance the performance of bi-directional communication even in the presence of a direct connection. Obviously, such wireline examples neglect the properties of wireless channels, i.e., links supporting asymmetric rates, statistical dependence of links resulting in broadcasting and superposition. But note that there are wireless channels that might allow to enforce situations similar to wireline through pre-coding techniques, e.g., MIMO wireless channels [7]. The example also suggests that treating the fully-connected problem like a separated two-way relay channel [4] and ignoring the direct path might be not optimal. This directly raises the following question: Which scheme is the optimal one for the problem sketched here and how can it be found? All the following is a first step towards an answer to this particular question. After summarizing related works, a definition of the communication problem is given. The Cut-set Theorem [8, Theorem 15.10.1] is applied in order to derive an outer bound on the achievable rate pairs. Then an inner bound with decoding at the relay is given. We comment on the problem of finding optimal schemes with respect to various objectives for the given bounds. Subsequently, the focus lies on a scalar AWGN-channel model. For simulations we determine the scheme that maximizes the symmetric rate of two-way communication with decoding at the relay and visualize the possible performance gain.

II. RELATED WORK

The relay channel was introduced in [9]. The seminal work [2] derives an upper bound on the capacity of the relay channel. Moreover, different relaying strategies are presented, among them the decode-and-forward strategy which is shown to be capacity achieving for the degraded relay channel [2, Theorem 1]. In [5, Section 9] relaying strategies are comprehensively revised. The half-duplex relay channel is explored in [10]. Two-way communication channels were introduced in [11]. The work [1] establishes the idea of exchanging messages in a bi-directional way over a relay. In [3] the broadcast phase of the two-phase two-way relay channel (see Fig. 2) is considered and the capacity achieving coding scheme is derived if the relay has available both dialog messages. Using a compress-and-forward strategy for this relay channel is analyzed in [12] while [13] and [14] show how to outperform this by using structured codes. [6] proposes a three-phase scheme (see Fig. 4) and obtains the achievable rates with network coding. In [15] this scheme is also examined and additionally a four-phase scheme is put forward. [16] studies the full-duplex two-way relay channel while [4] focuses on a separated full-duplex model. [17] comes up with a deterministic approach to approximate the capacity of the two-way relay channel.

III. COMMUNICATION PROBLEM

The studied network consists of three nodes labeled by $i = 1, 2, 3$. Each node is only allowed to operate in half-duplex mode. The message $W_{13}$ is to be communicated from node 1 to node 3 and the message $W_{31}$ from node 3 to node 1. Node 2 has no own message. The two messages are considered to be independent and drawn from uniform distributions. Each node $i$ is equipped with an input $X_i$ to and an output $Y_i$ from a common channel (see Fig. 6). The channel is time-invariant, discrete and memoryless. In contrast to the full-duplex model [16], a network state variable $S$ determines the receive-transmit configuration of the network nodes. Therefore, the channel can be defined as

$$\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3, P(y_1y_2y_3|x_1x_2x_3s), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3, S$$ (1)

where $\mathcal{X}_i, \mathcal{Y}_i$ are finite input and output alphabets. The network state variable $S := [S_1 \ S_2 \ S_3]$ with $s_i \in \{0, 1\}$ imposes the following restrictions on the output values and the input distributions

$$P_{X_i}(\emptyset) = 1 \quad \text{if} \quad s_i = 0$$
$$y_i = \emptyset \quad \text{if} \quad s_i = 1 \quad i = 1, 2, 3$$ (2)

Fig. 6: Half-Duplex Two-Way Relay Channel
with "0" being understood as a deactivation symbol. As the network state is determined by three binary variables the alphabet $S$ can contain at the utmost eight symbols. In the following, fixed input distributions for each individual network state are assumed. This excludes the use of time-sharing techniques on the input distributions. According to standard assumptions, all nodes have available the codebooks used in the network and the conditional distribution characterizing the channel. The communication process is neither limited by delay nor complexity.

### A. Restricted Codes

Consider $n$ channel uses, a scheme $s^n$ and a choice of $2 \leq L \leq 8$ used network states. $s_k$ takes values in $S = \{1, \ldots, L\}$ and determines the individual network state $s$ for the $k$-th of $n$ channel uses. A code of length $n$ and rate $R = [R_{13} \quad R_{31}]^T$ consists of two message sets

$$W_{13} = \{1, \ldots, 2^n R_{13}\}, \quad W_{31} = \{1, \ldots, 2^n R_{31}\}, \quad (3)$$

two encoding functions

$$f_1 : W_{13} \times S^n \rightarrow X_1^n, \quad f_3 : W_{31} \times S^n \rightarrow X_3^n, \quad (4)$$

a set of relaying functions

$$\{f_{2,k}\}_{k=1}^n \quad \text{s.t.} \quad x_{2,k} = f_{2,k}(s_k, Y_{2,1}, \ldots, Y_{2,k-1}) \quad (5)$$

and two decoding functions

$$g_1 : Y_1^n \times S^n \times W_{13} \rightarrow W_{31}$$
$$g_3 : Y_3^n \times S^n \times W_{31} \rightarrow W_{13}, \quad (6)$$

The code is restricted as the encoding functions are independent of previous receive signals. In the following $n_l$ denotes the number of occurrences of the network state $l$ in $n$ channel uses and $\tau_l$ is defined as $\tau_l = n_l/n$.

### IV. Outer Rate Bound

The Cut-set Theorem is applied to outer bound the achievable rates. The proof is given in the appendix.

**Theorem 1:** All rate pairs of the discrete memoryless restricted half-duplex two-way relay channel that are achievable for some joint probability distributions

$$P(x_1^{(1)} y_2^{(1)} y_3^{(1)}) = P(x_1^{(1)}) P(y_2^{(1)}|y_3^{(1)})$$
$$P(x_2^{(2)} y_1^{(2)}) = P(x_2^{(2)}) P(y_1^{(2)}|x_2^{(2)})$$
$$P(x_3^{(3)} y_1^{(3)} y_2^{(3)}) = P(x_3^{(3)}) P(y_2^{(3)}|x_3^{(3)})$$
$$P(x_2^{(4)} y_1^{(4)} y_3^{(4)}) = P(x_2^{(4)}) P(y_1^{(4)}|y_3^{(4)})$$
$$P(x_2^{(5)} y_1^{(5)} y_3^{(5)}) = P(x_2^{(5)}) P(y_1^{(5)}|y_3^{(5)})$$
$$P(x_1^{(6)} y_2^{(6)} y_3^{(6)}) = P(x_1^{(6)}) P(y_2^{(6)}|y_3^{(6)})$$

must satisfy

$$R_{13} \leq \tau_1 I(X_1^{(1)}; Y_2^{(1)}) + \tau_3 I(X_3^{(3)}; Y_2^{(3)}|X_3^{(3)}) + \tau_6 I(X_1^{(6)}; Y_3^{(6)}|X_2^{(6)})$$
$$R_{31} \leq \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_4 I(X_2^{(4)}; Y_4^{(4)}) + \tau_6 I(X_1^{(6)}; Y_3^{(6)}|X_2^{(6)})$$
$$R_{31} \leq \tau_2 I(X_3^{(2)}; Y_1^{(2)}) + \tau_4 I(X_2^{(4)}; Y_4^{(4)}) + \tau_6 I(X_3^{(5)}; Y_1^{(5)}|X_2^{(5)})$$
$$R_{31} \leq \tau_2 I(X_3^{(2)}; Y_1^{(2)}) + \tau_4 I(X_2^{(4)}; Y_4^{(4)}) + \tau_6 I(X_3^{(5)}; Y_1^{(5)}|X_2^{(5)})$$

where $0 \leq \tau_l$ and $\sum_l \tau_l \leq 1$.

Here it shows up that the problem contains six separate parts without specific order. Three parts where one node sends to two other nodes and three parts where two nodes send to one node. The network configurations where all nodes transmit or receive do not offer a positive information flow in the network. Note that for any scheme which uses less than the six relevant network states an individual performance outer bound can be derived without further proof by setting the probability of unused network states to zero, i.e., $P_S(l) = \tau_l = 0$.

### V. Inner Rate Bound

Now the achievable rates with a scheme that uses all six relevant network states are derived. The coding proof is outlined in the appendix.

**Theorem 2:** All rate pairs of the discrete memoryless restricted half-duplex two-way relay channel that satisfy

$$R_{13} \leq \tau_1 I(X_1^{(1)}; Y_2^{(1)}) + \tau_3 I(X_3^{(3)}; Y_2^{(3)}|X_3^{(3)}) + \tau_6 I(X_1^{(6)}; Y_3^{(6)}|X_2^{(6)})$$
$$R_{31} \leq \tau_1 I(X_1^{(1)}; Y_3^{(1)}) + \tau_4 I(X_2^{(4)}; Y_4^{(4)}) + \tau_6 I(X_1^{(6)}; Y_3^{(6)}|X_2^{(6)})$$
$$R_{31} \leq \tau_2 I(X_3^{(2)}; Y_1^{(2)}) + \tau_4 I(X_2^{(4)}; Y_4^{(4)}) + \tau_6 I(X_3^{(5)}; Y_1^{(5)}|X_2^{(5)})$$
$$R_{31} \leq \tau_2 I(X_3^{(2)}; Y_1^{(2)}) + \tau_4 I(X_2^{(4)}; Y_4^{(4)}) + \tau_6 I(X_3^{(5)}; Y_1^{(5)}|X_2^{(5)})$$
subject to 
\[
R_{13} + R_{31} \leq \tau_l I(X^{(1)}; Y^{(1)}) + \tau_2 I(X^{(2)}; Y^{(2)}) + \\
\tau_3 I(X^{(3)}; Y^{(3)}; Y^{(3)}) + \tau_6 I(X^{(6)}; Y^{(6)})|X^{(5)} + \\
\tau_6 I(X^{(6)}; Y^{(6)})|X^{(6)}
\]

with \(0 < \tau_l \) and \(\sum_{l=1}^{6} \tau_l \leq 1\), for some joint probability distributions

\[
P(x^{(1)}_1, y^{(1)}_2) = P(x^{(2)}_3, y^{(2)}_2, x^{(2)}_3) = P(x^{(3)}_1, x^{(3)}_2, y^{(3)}_3) = P(x^{(3)}_1, y^{(3)}_3|x^{(3)}_3) = \\
P(x^{(4)}_1, y^{(4)}_3) = P(x^{(5)}_2, x^{(5)}_3, y^{(5)}_1) = P(x^{(5)}_2, x^{(5)}_3, y^{(5)}_1, x^{(5)}_3) = \\
P(x^{(6)}_1, x^{(6)}_2, y^{(6)}_3) = P(x^{(6)}_1, x^{(6)}_2, y^{(6)}_3, x^{(6)}_2)
\]

are achievable with a decode-and-forward (DF) strategy.

VI. LINEAR RESOURCE ALLOCATION PROBLEMS

Assuming fixed input distributions the mutual informations of the outer and inner bound are constant and only the right time-allocation \(\tau_l\) for \(l = 1, \ldots, L\) has to be found. Fortunately, for various objectives this problem can be stated as a small-scale linear program of the form

\[
\text{max or min } c^T x \\
\text{s.t. } Ax \leq b, 0 \leq x, \text{const}(x)
\]

where \(\text{const}(x)\) denotes that additional constraints on \(x\) are satisfied, e.g., discrete time-lengths. Possible rate objectives are for example

\[
R_{SRMAX} = \max_{\tau} \{R_{13} + R_{31}\} \\
R_{WSRMAX} = \max_{\tau} \{\lambda R_{13} + (1 - \lambda) R_{31}\} \quad \lambda \in [0, 1] \\
R_{MCBMIN} = \max_{\tau} \{\min\{R_{13}, R_{31}\}\}
\]

If one is interested in minimizing resources or cost for certain rates \(R = [R_{13} \ R_{31}]^T\) one solves

\[
C_{TMCB} = \min_{\tau} C_T(R)
\]

by a linear program after having associated a cost \(c_t\) with each phase \(l\) of unit duration. Solving the time-allocation problem can result in some phases to be turned off and therefore gives the optimal scheme for the considered relaying strategy. Note that changing the order of the six phases or splitting them up and permuting the individual parts in an arbitrary way will not result in a better performance as is the problem is restricted and all nodes have agreed on the scheme a priori.

A. Resource Allocation for Wireless Systems

For time-division duplex (TDD) systems the allocation of time-lengths to the individual phases might have relevance in order to maximize one of the mentioned rate objectives while the channel is used all the time. An example for a cost problem might be to minimize the overall duration of the two-way communication for certain rates in order to allow the nodes to perform other communication tasks for the rest of the time. The two-way relaying problem has been stated under the assumption that the nodes are not allowed to transmit and receive at the same time. Although a rigorous proof would require block-Markov arguments as used in [2], the achievable rate expressions hold also for frequency-division duplex (FDD) systems where the different communication parts are carried out in parallel on orthogonal frequencies [10]. For band-limited Gaussian channels [8, Section 9.3], \(\tau_l\) can be reinterpreted as the bandwidth \(\omega_l\) associated with the \(l\)-th network state. Therefore, one might want to maximize a rate objective by allocating the optimal bandwidth to each of the communication parts. A reasonable cost problem might be to minimize the overall used bandwidth for given rates.

VII. GAUSSIAN TWO-WAY RELAY CHANNEL

Now we consider a scalar AWGN-channel model where an active output at node \(j\) in phase \(l\) has the form

\[
Y^{(l)}_j = \sum_{i=1}^{3} h_{ij} x^{(l)}_i + Z^{(l)}_j
\]

with \(x^{(l)}_i, y^{(l)}_j, z^{(l)}_j, h_{ij} \in \mathbb{C}\) and the additive Gaussian noise \(Z^{(l)}_j\) being independent, zero mean and of unit variance, i.e., \(Z^{(l)}_j \sim \mathcal{N}(0, 1)\). The active input distributions are limited by a per-symbol power constraint

\[
E \left[ |x^{(l)}_i|^2 \right] \leq P_i \quad \forall i = 1, 2, 3
\]

and the input variables have the form

\[
\begin{align*}
X^{(1)}_1 &= \sqrt{P_1} f^{(1)}_1(W_{13}) \\
X^{(2)}_2 &= \sqrt{P_2} f^{(2)}_2(W_{31}) \\
X^{(3)}_1 &= \sqrt{P_1} f^{(3)}_1(W_{13}) \\
X^{(3)}_3 &= \sqrt{P_3} f^{(3)}_3(W_{31}) \\
X^{(4)}_3 &= \sqrt{P_2} f^{(4)}_2(W_{31}, W_{13}) \\
X^{(5)}_2 &= \sqrt{P_2} f^{(5)}_2(W_{31}) \\
X^{(5)}_3 &= \sqrt{P_3} f^{(5)}_3(W_{31}) + \sqrt{(1 - \beta) P_3} f^{(5)}_3(W_{31}) \\
X^{(6)}_2 &= \sqrt{P_2} f^{(6)}_2(W_{13}) \\
X^{(6)}_3 &= \sqrt{\gamma P_2} f^{(6)}_2(W_{13}) + \sqrt{(1 - \gamma) P_1} f^{(6)}_1(W_{13})
\end{align*}
\]

with encoding functions \(f^{(l)}_i \sim \mathcal{N}(0, 1)\). Parameters \(\beta, \gamma \in [0, 1]\) control coherent signaling of nodes 3 and 1 with node
2. The achievable rates are
\[ R_{13} \leq \tau_1 \log\left(1 + |h_{12}|^2 P_1\right) + \tau_3 \log\left(1 + |h_{13}|^2 P_1\right) + \tau_6 \log\left(1 + |h_{13}|^2 (1-\gamma)P_1\right) \]
\[ R_{13} \leq \tau_1 \log\left(1 + |h_{13}|^2 P_1\right) + \tau_4 \log\left(1 + |h_{23}|^2 P_2\right) + \tau_6 \log\left(1 + |h_{13}|^2 P_1 + |h_{23}|^2 P_2 + 2|h_{13}h_{23}|\sqrt{\gamma P_1 P_2}\right) \]
\[ R_{31} \leq \tau_2 \log\left(1 + |h_{32}|^2 P_3\right) + \tau_4 \log\left(1 + |h_{32}|^2 P_3\right) + \tau_5 \log\left(1 + |h_{31}|^2 (1-\beta)P_3\right) \]
\[ R_{31} \leq \tau_2 \log\left(1 + |h_{31}|^2 P_3\right) + \tau_4 \log\left(1 + |h_{31}|^2 P_3\right) + \tau_5 \log\left(1 + |h_{31}|^2 P_3 + |h_{21}|^2 P_2 + 2|h_{31}h_{21}|\sqrt{\beta P_2 P_3}\right) \]
\[ R_{13} + R_{31} \leq \tau_1 \log\left(1 + |h_{12}|^2 P_1\right) + \tau_2 \log\left(1 + |h_{32}|^2 P_3\right) + \tau_3 \log\left(1 + |h_{12}|^2 P_1 + |h_{32}|^2 P_3\right) + \tau_5 \log\left(1 + |h_{31}|^2 (1-\beta)P_3\right) + \tau_6 \log\left(1 + |h_{13}|^2 (1-\gamma)P_1\right). \tag{13} \]

VIII. Simulations

For simulations a plane-network model [18] is used where the positions \( p_i \) of node 1 and 3 are fixed to \( p_1 = [0 \ 0]^T \) and node 2 can be placed at position \( p_2 = [x \ y]^T \) with \( x, y \in (-\infty, \infty) \). The channel coefficient
\[ h_{ij} = 1/d_{ij}^\alpha \tag{14} \]
is determined by the path-loss exponent \( \alpha \) and the distance \( d_{ij} \) between nodes \( i \) and \( j \). For a scenario with \( \alpha = 3 \) and a power constraint \( P_1 = 10 \) like used in [16], we move the relay to different positions, sample over the parameters \( \beta, \gamma \) and solve time allocation with the objective of maximizing the symmetric two-way rate \( R_{\text{MAXMIN}} \). After sampling we pick \( \beta^*, \gamma^* \) in conjunction with the time allocation solution \( \tau^* \) that yield the highest symmetric rate. Note that if, due to system constraints, coherent signaling of two nodes is not possible one sets \( \beta = \gamma = 0 \) and solves the problem at each position by one linear program. Fig. 8 shows the achievable rate gain compared to two-way communication without relay. It shows up that in the area between the nodes an increase in symmetric rate of 25 to 50 percent is possible. Fig. 9 compares the achievable symmetric rates with their upper bound. This shows that with the relay being located in the middle between the two dialog nodes still an improvement of 20 percent might be possible with other relaying strategies. For the other regions DF performs within 10 percent distance to the upper bound.

IX. Conclusion

The problem of exchanging messages between two nodes in a fully-connected half-duplex three-node network has been defined and an outer bound on the achievable rates has been derived. This bound alludes to a scheme that takes into account all possible network states that are relevant. For such a scheme a coding procedure with decoding at the relay has been proposed. For fixed input distributions the optimal time-allocation to the individual network states can be found at very low complexity. The relevance of this property for wireless systems has been outlined. Simulations for the established communication protocol show a significant increase in symmetric rate compared to two-way communication without relay. This provides the inside that two-way relaying can also be used to improve the system performance in half-duplex wireless scenarios without connectivity problem.

APPENDIX A

Outer Bound on the Achievable Rates

Consider a full-duplex three-node network and the two possible cut-set partitions \( \Omega_1, \Omega_3 \) separating nodes 1 and 3. Under the assumption of zero-error codes it holds with the Cutset Theorem [8, Theorem 15.10.1] that the achievable rates are
for some joint input distribution $P(x_1, x_2, x_3)$. As here the encoders are not allowed to cooperate, the marginal $P(x_1, x_3)$ is restricted to distributions $P(x_1, x_3) = P(x_1)P(x_3)$. Introducing a state variable $S$ known by all nodes, taking values in $S : \{1, \ldots, L\}$ and being distributed according to $P_S(l) = \frac{\alpha}{\alpha + \beta} = \tau_1$ the rate constraints can be written as

\[
R_{13} \leq \sum_{l=1}^{L} P(l) I(X_1^{(l)}; Y_2^{(l)} | X_2^{(l)}X_3^{(l)} S = l) \\
R_{31} \leq \sum_{l=1}^{L} P(l) I(X_3^{(l)}; Y_1^{(l)} | X_1^{(l)}X_2^{(l)} S = l)
\]

Agreeing to use $L = 8$ network states, here defined as

\[l = 1 : s = [1 \ 0 \ 0], \quad l = 5 : s = [0 \ 1 \ 1]\\
\]

\[l = 2 : s = [0 \ 0 \ 1], \quad l = 6 : s = [1 \ 1 \ 0]\\
\]

\[l = 3 : s = [1 \ 0 \ 1], \quad l = 7 : s = [1 \ 1 \ 1]\\
\]

\[l = 4 : s = [0 \ 1 \ 0], \quad l = 8 : s = [0 \ 0 \ 0],
\]

establishes the theorem. ■

**Appendix B**

**Achievable Rates with Decode-and-Forward**

For the proof it is assumed that the transmission is performed with $n \geq 6$ channel uses and six subsequent phases, each with an individual network state, i.e., $L = 6$. Phase $l$ features $n_l \geq 1$ transmission slots and $\sum_{l=1}^{L} n_l = n$. If $n$ grows each $n_l$ is assumed to grow at the same speed. For large $n$, $\frac{n}{m} \rightarrow \tau_1 > 0$. The codebooks are labeled by $M = 12$

\[w_m = 1, \ldots, 2^{nR_m}, \quad m = 1, \ldots, M
\]

and $w_m(i)$ denotes the estimate of the $m$-th index at node $i$.

**1) Codebook Construction:** Generate $2^{n(R_3 + R_2 + R_1)}$ $n_1$-sequences $x_1^{(1)}(w_1, w_2, w_3)$ by choosing each $x_1^{(1)}$ independently according to $P_{X_1^{(1)}}(\cdot)$. Generate $2^{n(R_2 + R_3 + R_0)}$ $n_2$-sequences $x_2^{(2)}(w_2, w_7, w_8, w_9)$ by choosing each $x_2^{(2)}$ independently according to $P_{X_2^{(2)}}(\cdot)$. Generate $2^{n(R_1 + R_3 + R_0 + R_1)}$ $n_3$-sequences $x_3^{(3)}(w_10, w_11)$ by choosing each $x_3^{(3)}$ independently according to $P_{X_3^{(3)}}(\cdot)$. Generate $2^{n(R_1 + R_2 + R_0 + R_1)}$ $n_4$-sequences $x_3^{(4)}(w_1, w_4, w_7, w_10)$ by choosing each $x_3^{(4)}$ independently according to $P_{X_3^{(4)}}(\cdot)$. Generate $2^{n(R_2 + R_3 + R_0 + R_1)}$ $n_5$-sequences $x_2^{(5)}(w_8, w_{11})$ by choosing each $x_2^{(5)}$ independently according to $P_{X_2^{(5)}}(\cdot)$.

**2) Input at node 1:** The message $w_{13}$ is reindexed by $(w_1, w_2, w_3, w_4, w_5)$. Node 1 transmits $x_1^{(1)}(w_1, w_2, w_3)$ and $x_1^{(3)}(w_4, w_5)$.

**3) Input at node 3:** The message $w_{31}$ is reindexed by $(w_7, w_8, w_9, w_{10}, w_{11}, w_{12})$. Node 3 transmits $x_2^{(2)}(w_7, w_8, w_9)$, $x_3^{(3)}(w_{10}, w_{11})$ and $x_3^{(4)}(w_8, w_{11}, w_{12})$.

**4) Output at node 2:** The sequences $y_1^{(2)}, y_2^{(2)}$ and $y_3^{(2)}$ are observed. After that node 2 tries to find a $(\hat{w}_1, \hat{w}_2, \hat{w}_3)$ such that

\[
(x_1^{(1)}(\hat{w}_1, \hat{w}_2, \hat{w}_3), y_2^{(1)}) \in T_{\epsilon}^{-1}(P_{X_2^{(1)}X_3^{(1)}}). \tag{19}
\]

If there is none or more than one such $(\hat{w}_1, \hat{w}_2, \hat{w}_3)$ an error is declared. Otherwise, the found $(\hat{w}_1, \hat{w}_2, \hat{w}_3)$ is the estimate $(\hat{w}_1(2), \hat{w}_2(2), \hat{w}_3(2))$. Then node 2 tries to find a $(\hat{w}_7, \hat{w}_8, \hat{w}_9)$ such that

\[
(x_3^{(2)}(\hat{w}_7, \hat{w}_8, \hat{w}_9), y_2^{(2)}) \in T_{\epsilon}^{-1}(P_{X_3^{(2)}Y_2^{(2)}}). \tag{20}
\]

If there is none or more than one such $(\hat{w}_7, \hat{w}_8, \hat{w}_9)$ an error is declared. Otherwise, the found $(\hat{w}_7, \hat{w}_8, \hat{w}_9)$ is the estimate $(\hat{w}_7(2), \hat{w}_8(2), \hat{w}_9(2))$. Then node 2 tries to find a $(\hat{w}_4, \hat{w}_5, \hat{w}_{10}, \hat{w}_{11})$ such that

\[
(x_1^{(3)}(\hat{w}_4, \hat{w}_5), x_3^{(3)}(\hat{w}_{10}, \hat{w}_{11}), y_2^{(3)}) \in T_{\epsilon}^{-1}(P_{X_1^{(3)}X_3^{(3)}Y_2^{(3)}}). \tag{21}
\]

If there is none or more than one such $(\hat{w}_4, \hat{w}_5, \hat{w}_{10}, \hat{w}_{11})$ an error is declared. Otherwise, the found $(\hat{w}_4, \hat{w}_5, \hat{w}_{10}, \hat{w}_{11})$ is the estimate $(\hat{w}_4(2), \hat{w}_5(2), \hat{w}_{10}(2), \hat{w}_{11}(2))$.

**5) Input at node 2:** Node 2 sends the sequences $x_2^{(2)}(\hat{w}_1(2), \hat{w}_4(2), \hat{w}_7(2), \hat{w}_{10}(2)), x_3^{(2)}(\hat{w}_8(2), \hat{w}_{11}(2))$ and $x_3^{(3)}(\hat{w}_4(2), \hat{w}_5(2))$.

**6) Output at node 1:** The sequences $y_1^{(3)}, y_1^{(4)}$ and $y_1^{(5)}$ are observed. After that node 1 tries to find a $(\hat{w}_7, \hat{w}_10)$ such that

\[
(x_2^{(4)}(\hat{w}_1, w_4, w_7, w_{10}), y_1^{(4)}) \in T_{\epsilon}^{-1}(P_{X_2^{(4)}Y_2^{(4)}}). \tag{22}
\]

If there is none or more than one such $(\hat{w}_7, \hat{w}_{10})$ an error is declared. Otherwise, the found $(\hat{w}_7, \hat{w}_{10})$ is the estimate $(\hat{w}_7(1), \hat{w}_{10}(1))$. Then node 1 tries to find a $(\hat{w}_8, \hat{w}_{11})$ such that

\[
(x_2^{(5)}(\hat{w}_8, \hat{w}_{11}), y_1^{(5)}) \in T_{\epsilon}^{-1}(P_{X_2^{(5)}Y_2^{(5)}}). \tag{23}
\]
If there is none or more than one such \((\hat{w}_8, \hat{w}_{11})\) an error is declared. Otherwise, the found \((\hat{w}_8, \hat{w}_{11})\) is the estimate \((\hat{w}_8(1), \hat{w}_{11}(1))\). Then node 1 tries to find a \(\hat{w}_{12}\) such that
\[
\begin{align*}
(x^n_3(\hat{w}_8(1), \hat{w}_{11}(1), \hat{w}_{12}), x^n_2(\hat{w}_8(1), \hat{w}_{11}(1), y^n_7) & \in T^n_e(P_{X_3^n}X_2^nY_1^n). \quad (24)
\end{align*}
\]
If there is none or more than one such \(\hat{w}_{12}\) an error is declared. Otherwise, the found \(\hat{w}_{12}\) is the estimate \(\hat{w}_{12}(1)\). Finally node 1 tries to find a \(\hat{w}_9\) such that
\[
\begin{align*}
(x^n_2(\hat{w}_7(1), \hat{w}_8(1), \hat{w}_9), y^n_2) & \in T^n_e(P_{X_2^n}Y_1^n). \quad (25)
\end{align*}
\]
If there is none or more than one such \(\hat{w}_9\) an error is declared. Otherwise, the found \(\hat{w}_9\) is the estimate \(\hat{w}_9(1)\). The message estimate \(\hat{w}_{31}(1)\) is found by reindexing \((\hat{w}_7(1), \hat{w}_8(1), \hat{w}_9(1), \hat{w}_{10}(1), \hat{w}_{11}(1), \hat{w}_{12}(1))\).

7) Output at node 3: The sequences \(y^n_1, y^n_3, d^n_4\) and \(y^n_6\) are observed. After that node 3 tries to find a \((\hat{w}_1, \hat{w}_4)\) such that
\[
\begin{align*}
(x^n_2(\hat{w}_1, \hat{w}_4, \hat{w}_7, \hat{w}_{10}), y^n_3) & \in T^n_e(P_{X_2^n}Y_3^n). \quad (26)
\end{align*}
\]
If there is none or more than one such \((\hat{w}_1, \hat{w}_4)\) an error is declared. Otherwise, the found \((\hat{w}_1, \hat{w}_4)\) is the estimate \((\hat{w}_1(3), \hat{w}_4(3))\). Then node 3 tries to find a \((\hat{w}_2, \hat{w}_5)\) such that
\[
\begin{align*}
(x^n_2(\hat{w}_2, \hat{w}_5), y^n_3) & \in T^n_e(P_{X_2^n}Y_3^n). \quad (27)
\end{align*}
\]
If there is none or more than one such \((\hat{w}_2, \hat{w}_5)\) an error is declared. Otherwise, the found \((\hat{w}_2, \hat{w}_5)\) is the estimate \((\hat{w}_2(3), \hat{w}_5(3))\). Then node 3 tries to find a \(\hat{w}_6\) such that
\[
\begin{align*}
(x^n_2(\hat{w}_2(3), \hat{w}_5(3), \hat{w}_6), x^n_3(\hat{w}_2(3), \hat{w}_5(3)), y^n_3) & \in T^n_e(P_{X_3^n}X_2^nY_3^n). \quad (28)
\end{align*}
\]
If there is none or more than one such \(\hat{w}_6\) an error is declared. Otherwise, the found \(\hat{w}_6\) is the estimate \(\hat{w}_6(3)\). Finally node 3 tries to find a \(\hat{w}_3\) such that
\[
\begin{align*}
(x^n_2(\hat{w}_1(3), \hat{w}_2(3), \hat{w}_3), y^n_3) & \in T^n_e(P_{X_1^n}Y_2^n). \quad (29)
\end{align*}
\]
If there is none or more than one such \(\hat{w}_3\) an error is declared. Otherwise, the found \(\hat{w}_3\) is the estimate \(\hat{w}_3(3)\). The message estimate \(\hat{w}_{13}(3)\) is found by reindexing \((\hat{w}_1(3), \hat{w}_2(3), \hat{w}_3(3), \hat{w}_4(3), \hat{w}_5(3), \hat{w}_6(3))\).

8) Error Analysis: The average error probability \(\hat{P}_e\) can be upper bounded by the sum of the individual decoding error probabilities each calculated under the assumption that all prior decoding steps in the network have been performed error-free. According to the properties of \(\epsilon\)-letter typical sequences [5], the probabilities of the individual errors can be made arbitrarily small by choosing \(n\) sufficiently large, \(\epsilon > 0\) but small, while the rate constraints
\[
\begin{align*}
R_1 + R_2 + R_3 & < \tau_1 I(X_1^3; Y_2^3) \\
R_7 + R_8 + R_9 & < \tau_2 I(X_3^2; Y_2^2) \\
R_4 + R_5 & < \tau_3 I(X_1^3; Y_3^2 | X_3^3) \\
R_{10} + R_{11} & < \tau_3 I(X_3^3; Y_3^2 | X_3^3) \\
R_4 + R_5 + R_{10} + R_{11} & < \tau_3 I(X_1^3; X_3^3; Y_2^3) \quad (30)
\end{align*}
\]
at node 2,
\[
\begin{align*}
R_7 + R_{10} & < \tau_4 I(X_4^4; Y_3^4) \\
R_8 + R_{11} & < \tau_5 I(X_5^5; Y_4^4) \\
R_{12} & < \tau_3 I(X_3^5; Y_1^1 | X_2^5) \\
R_9 & < \tau_2 I(X_3^2; Y_1^1) \quad (31)
\end{align*}
\]
at node 1 and
\[
\begin{align*}
R_1 + R_4 & < \tau_4 I(X_4^4; Y_3^4) \\
R_2 + R_5 & < \tau_6 I(X_2^6; Y_4^4) \\
R_6 & < \tau_6 I(X_1^6; Y_4^4 | X_2^6) \\
R_3 & < \tau_1 I(X_1^1; Y_1^1) \quad (32)
\end{align*}
\]
at node 3 are satisfied. Using the fact that
\[
\begin{align*}
R_{13} & = R_1 + R_2 + R_3 + R_4 + R_5 + R_6 \\
R_{31} & = R_7 + R_8 + R_9 + R_{10} + R_{11} + R_{12} \quad (33)
\end{align*}
\]
shows that
\[
\begin{align*}
R_{13} & < \tau_1 I(X_1^1; Y_2^1) + \tau_3 I(X_1^3; Y_3^2 | X_3^3) + \\
& + \tau_6 I(X_1^6; Y_3^4 | X_2^6) \\
R_{13} & < \tau_1 I(X_1^1; Y_3^1) + \tau_4 I(X_2^4; Y_3^4) + \tau_6 I(X_2^6; Y_3^6) + \\
& + \tau_6 I(X_1^6; Y_3^4 | X_2^6) \\
R_{31} & < \tau_2 I(X_4^2; Y_2^2) + \tau_3 I(X_3^2; Y_3^3 | X_3^3) + \\
& + \tau_5 I(X_3^5; Y_1^1 | X_2^5) \\
R_{31} & < \tau_2 I(X_4^2; Y_2^2) + \tau_3 I(X_2^4; Y_1^1) + \tau_5 I(X_2^5; Y_1^5) + \\
& + \tau_5 I(X_3^5; Y_1^5 | X_2^5) \quad (34)
\end{align*}
\]
subject to
\[
\begin{align*}
R_{13} + R_{31} & < \tau_1 I(X_1^1; Y_2^1) + \tau_2 I(X_3^3; Y_2^3) + \\
& + \tau_3 I(X_1^3; X_3^3; Y_2^3) + \tau_5 I(X_5^5; Y_1^5) | X_2^5) + \\
& + \tau_6 I(X_1^6; Y_3^4 | X_2^6). \quad (35)
\end{align*}
\]
must be satisfied in order to allow reliable communication over the two-way relay channel.

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