Charge asymmetry in the differential cross section of high-energy bremsstrahlung in the field of a heavy atom

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Abstract

The distinction between the charged particle and antiparticle differential cross sections of high-energy bremsstrahlung in the electric field of a heavy atom is investigated. The consideration is based on the quasiclassical approximation to the wave functions in the external field. The charge asymmetry (the ratio of the antisymmetric and symmetric parts of the differential cross section) arises due to the account for the first quasiclassical correction to the differential cross section. All evaluations are performed with the exact account of the atomic field. We consider in detail the charge asymmetry for electrons and muons. For electrons, the nuclear size effect is not important while for muons this effect should be taken into account. For the longitudinal polarization of the initial charged particle, the account for the first quasiclassical correction to the differential cross section leads to the asymmetry in the cross section with respect to the replacement $\varphi \rightarrow -\varphi$, where $\varphi$ is the azimuth angle between the photon momentum and the momentum of the final charged particle.

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I. INTRODUCTION

The theoretical investigation of high-energy bremsstrahlung and high-energy particle-antiparticle photoproduction in the electric field of a heavy nucleus or atom has a long history because of importance of these processes for various applications; for the latter process see reviews in Refs. [1, 2]. These processes should be taken into account when considering electromagnetic showers in detectors, they also give the significant part of the radiative corrections in many cases. Therefore, it is necessary to know the cross sections of these processes with high accuracy. In the Born approximation, the cross sections of both processes have been obtained for arbitrary energies of particles and for arbitrary atomic form factors [3, 4] (see also Ref. [5]). The Coulomb corrections to the cross section, which are the difference between the exact in the parameter $\eta = Z\alpha$ cross section and the Born cross section, are very important (here $Z$ is the atomic charge number, $\alpha$ is the fine-structure constant, $\hbar = c = 1$). There are formal expressions for the Coulomb corrections to the cross sections exact in $\eta$ and energies of particles [6]. However, the numerical computations based on these expressions become more and more difficult when energies are increasing, and, for instance, the numerical results for $e^+e^-$ photoproduction have been obtained so far only for the photon energy $\omega < 12.5$ MeV [7].

At high energies of initial particles, the final particle momenta usually have small angles with respect to the incident direction. In this case typical angular momenta, which provide the main contribution to the cross section, are large ($l \sim E/\Delta \gg 1$, where $E$ is energy and $\Delta$ is the momentum transfer). This is why the quasiclassical approximation, based on the account of large angular momenta contributions, becomes applicable. In this approximation, the wave functions and the Green’s functions of the Dirac equation in the external field have very simple forms which drastically simplify their use in specific calculations. The wave functions, obtained in the leading quasiclassical approximation for the Coulomb field, are the famous Furry-Sommerfeld-Maue wave functions [8, 9] (see also Ref. [5]). The quasiclassical Green’s function have been derived in Ref. [10] for the case of a pure Coulomb field, in Ref. [11] for an arbitrary spherically symmetric field, in Ref. [12] for a localized field which generally possesses no spherical symmetry, and in Ref. [13] for combined strong laser and atomic fields.

In the leading quasiclassical approximation, the cross sections for pair photoproduction
and bremsstrahlung have been obtained in [14–18]. The first quasiclassical corrections to the spectra of both processes, as well as to the total cross section of pair photoproduction, have been obtained in Refs. [19–22]. Recently, the first quasiclassical correction to the fully differential cross section was obtained in Ref. [23] for $e^+e^-$ pair photoproduction and in Ref. [24] for $\mu^+\mu^-$ pair photoproduction. As a result, the charge asymmetry in these processes (the asymmetry of the cross section with respect to permutation of particle and antiparticle momenta) was predicted. This asymmetry is absent in the cross section calculated in the Born approximation and also in the cross section exact in the parameter $\eta$ but calculated in the leading quasiclassical approximation. Thus, the charge asymmetry appears solely due to the quasiclassical corrections to the Coulomb corrections. The difference between the atomic field and the Coulomb field of a nucleus results in the modification of the cross sections (effect of screening). The influence of screening on the Coulomb corrections to $e^+e^-$ pair photoproduction is small for the differential cross section and for the total cross section [15]. However, screening is important for the Born term. The quantitative investigation of the effect of screening on the Coulomb corrections to the photoproduction cross section is performed in Ref. [19].

The influence of screening on the bremsstrahlung cross section in a nuclear field is more complicated. It is shown in Refs. [16, 20] that the Coulomb corrections to the differential cross section are very susceptible to screening. However, the Coulomb corrections to the cross section integrated over the momentum of final charged particle (electron or muon) are independent of screening in the leading approximation over a small parameter $1/m_em_{scr}r_{scr}$, where $r_{scr} \sim Z^{-1/3}(m_e\alpha)^{-1}$ is a screening radius and $m_e$ is the electron mass. The quantitative investigation of the effect of screening on the Coulomb corrections to the spectrum of bremsstrahlung is performed in Ref. [20]. The differential cross section of bremsstrahlung, calculated in the leading quasiclassical approximation, is the same for $e^+$ and $e^-$ (for $\mu^+$ and $\mu^-$). Therefore, to predict the charge asymmetry (the difference between the bremsstrahlung differential cross section for particles and antiparticles in the atomic field), one should perform calculations in the next-to-leading quasiclassical approximation. This is the main goal of our paper. The result is obtained exactly in the parameter $\eta$. Besides, for the case of muons the nuclear size effect is taken into account.

The bremsstrahlung differential cross section from high-energy charged particle in an
atomic field, $d\sigma(p, q, k, \eta)$, can be written as

$$
\begin{align*}
    d\sigma(p, q, k, \eta) &= d\sigma_s(p, q, k, \eta) + d\sigma_a(p, q, k, \eta), \\
    d\sigma_s(p, q, k, \eta) &= \frac{d\sigma(p, q, k, \eta) + d\sigma(p, q, -\eta)}{2}, \\
    d\sigma_a(p, q, k, \eta) &= \frac{d\sigma(p, q, k, \eta) - d\sigma(p, q, -\eta)}{2},
\end{align*}
$$

(1)

where $k$ is the photon momentum, $p$ and $q$ are the initial and final charged particle momenta, respectively. Evidently, the bremsstrahlung differential cross section from high-energy antiparticle can be obtained from $d\sigma(p, q, k, \eta)$ by the replacement $\eta \rightarrow -\eta$, so that it is equal to $d\sigma_s(p, q, k, \eta) - d\sigma_a(p, q, k, \eta)$. We show that the antisymmetric part of the differential cross section, $d\sigma_a(p, q, k, \eta)$, is independent of screening in the kinematical region which provides the main contribution to the antisymmetric part of the spectrum.

The paper is organized as follows. In Sec. II we derive the general expression for the quasiclassical matrix element of the process. In Sec. III we find in the quasiclassical approximation all structures of the Green’s function of the squared Dirac equation for a charged particle in arbitrary localized potential and the corresponding structures of the wave functions. We obtain the leading terms and the first quasiclassical corrections as well. Using these wave functions, we derive in Sec. IV the matrix element of the process and the corresponding differential cross section for arbitrary localized potential and in the particular case of the pure Coulomb field. In Sec. V we investigate in detail the charge asymmetry in high-energy bremsstrahlung from electrons. In this case the nuclear size effect is not important. In Sec. VI we investigate the charge asymmetry in high-energy bremsstrahlung from muons which is sensitive to the deviation at small distances of the nuclear atomic field from the pure Coulomb field. Finally, in Sec. VII the main conclusions of the paper are presented.

II. GENERAL DISCUSSION

The differential cross section of bremsstrahlung in the electric field of a heavy atom reads

$$
    d\sigma = \frac{\alpha \omega q \varepsilon_q}{(2\pi)^4} d\Omega_k d\Omega_q d\omega |M|^2,
$$

(2)

where $d\Omega_k$ and $d\Omega_q$ are the solid angles corresponding to the photon momentum $k$ and the final charged particle momentum $q$, $\omega = \varepsilon_p - \varepsilon_q$ is the photon energy, $\varepsilon_p = \sqrt{p^2 + m^2}$,
The wave functions in the atomic field can be found from the Green's function following relations hold

\[ M = \int dr \, \bar{u}_q^{(-)}(r) \gamma \cdot e^* u_p^{(+)}(r) \exp(-ik \cdot r), \]  \hspace{1cm} (3)

where \( \gamma^\mu \) are the Dirac matrices, \( u_p^{(+)}(r) \) and \( u_q^{(-)}(r) \) are the solutions of the Dirac equation in the external field, \( e \) is the photon polarization vector. The superscripts \((-)\) and \((+)\) remind us that the asymptotic forms of \( u_q^{(-)}(r) \) and \( u_p^{(+)}(r) \) at large \( r \) contain, in addition to the plane wave, the spherical convergent and divergent waves, respectively. The wave functions \( u_p^{(+)}(r) \) and \( u_q^{(-)}(r) \) have the form \[24\]

\[ u_p^{(+)}(r) = \bar{u}_q[f_0(r, q) - \alpha \cdot f_1(r, q) - \Sigma \cdot f_2(r, q)], \]

\[ u_q^{(-)} = \frac{\phi}{\sqrt{\phi \cdot \phi}}, \quad u_p = \frac{\chi}{\sqrt{\chi \cdot \chi}}, \]  \hspace{1cm} (4)

where \( \phi \) and \( \chi \) are spinors, \( \alpha = \gamma^0 \gamma, \Sigma = \gamma^0 \gamma^5 \gamma, \) and \( \sigma \) are the Pauli matrices. The following relations hold

\[ g_0(r, q) = f_0(r, -q), \quad g_1(r, q) = f_1(r, -q), \quad g_2(r, q) = -f_2(r, -q). \]  \hspace{1cm} (5)

The wave functions in the atomic field can be found from the Green’s function \( D(r_2, r_1|\varepsilon) \) of the “squared” Dirac equation in this field using the relations \[24\]

\[ \frac{\exp iqr_2}{4\pi r_2} u_q^{(-)}(r_1) = - \lim_{r_2 \to \infty} u_q D(r_2, r_1|\varepsilon), \quad q = qn_2, \]

\[ \frac{\exp ipr_1}{4\pi r_1} u_p^{(+)}(r_2) = - \lim_{r_1 \to \infty} D(r_2, r_1|\varepsilon_p) u_p, \quad p = -pn_1, \]  \hspace{1cm} (6)

where \( n_1 = r_1/r_1, n_2 = r_2/r_2, \) and

\[ D(r_2, r_1|\varepsilon) = \langle r_2 | \frac{1}{p^2 - m^2 + i0} | r_1 \rangle \]

\[ = \langle r_2 | \left[ (\varepsilon - V(r))^2 + \nabla^2 - m^2 + i\alpha \cdot \nabla V(r) + i0 \right]^{-1} | r_1 \rangle. \]  \hspace{1cm} (7)

Here \( \hat{P} = \gamma^\mu P_\mu, \ P_\mu = (\varepsilon - V(r), i\nabla), \) and \( V(r) \) is the atomic potential. It follows from Eq. \[7\] that the Green’s function \( D(r_2, r_1|\varepsilon) \) can be written as

\[ D(r_2, r_1|\varepsilon) = d_0(r_2, r_1) + \alpha \cdot d_1(r_2, r_1) + \Sigma \cdot d_2(r_2, r_1). \]  \hspace{1cm} (8)
It is convenient to calculate the matrix element for definite helicities of the particles. Let \( \mu_p \), \( \mu_q \), and \( \lambda \) be the signs of the helicities of initial charged particle, final charged particle, and photon, respectively. We fix the coordinate system so that \( \nu = k/\omega \) is directed along \( z \)-axis and \( q \) lies in the \( xz \) plane with \( q_z > 0 \). Denoting helicities by the subscripts, we have

\[
\phi_{\mu_p} = \frac{1 + \mu_p \sigma \cdot n_p}{4 \cos(\theta_p/2)} \left( 1 + \mu_p \right) \approx \frac{1}{4} \left( 1 + \frac{\theta_p^2}{8} \right) \left( 1 + \mu_p \sigma \cdot n_p \right) \left( 1 + \mu_p \right),
\]

\[
\chi_{\mu_q} = \frac{1 + \mu_q \sigma \cdot n_q}{4 \cos(\theta_q/2)} \left( 1 + \mu_q \right) \approx \frac{1}{4} \left( 1 + \frac{\theta_q^2}{8} \right) \left( 1 + \mu_q \sigma \cdot n_q \right) \left( 1 + \mu_q \right),
\]

\[
e_\lambda = \frac{1}{\sqrt{2}} (e_x + i \lambda e_y),
\]

where \( \theta_p \) and \( \theta_q \) are the polar angles of the vectors \( p \) and \( q \), respectively. The unit vectors \( e_x \) and \( e_y \) are directed along \( q_\perp \) and \( k \times q \), respectively, where the notation \( X_\perp = X - (X \cdot \nu) \nu \) for any vector \( X \) is used. We also introduce the vectors \( \theta_p = p_\perp/p, \theta_q = q_\perp/q \), and the matrix \( F = u_{p_\mu_p} \bar{u}_{q_\mu_q} \), which can be written as

\[
F = \frac{1}{8} (a_{\mu_p\mu_q} + \Sigma \cdot b_{\mu_p\mu_q}) [\gamma^0(1 + PQ) + \gamma^0\gamma^5(P + Q) + (1 - PQ) - \gamma^5(P - Q)],
\]

\[
P = \frac{\mu_p p}{\epsilon_p + m}, \quad Q = \frac{\mu_q q}{\epsilon_q + m},
\]

where \( a_{\mu_p\mu_q} \) and \( b_{\mu_p\mu_q} \) are defined from

\[
\phi_{\mu_p} \chi_{\mu_q}^\dagger = \frac{1}{2} (a_{\mu_p\mu_q} + \sigma \cdot b_{\mu_p\mu_q}).
\]

We obtain from Eq. (9)

\[
a_{\mu \mu} = 1 - \frac{\theta_p^2}{8} - \frac{i \mu}{4} \nu \cdot [\theta_p \times \theta_q],
\]

\[
a_{\mu \bar{\mu}} = \frac{\mu}{\sqrt{2}} e_\mu \cdot \theta_{pq},
\]

\[
b_{\mu \mu} = \left\{ \mu \left[ 1 - \frac{1}{8} (\theta_p + \theta_q)^2 \right] + \frac{i}{4} \nu \cdot [\theta_p \times \theta_q] \right\} \nu
\]

\[
+ \frac{\mu}{2} (\theta_p + \theta_q) + \frac{i}{2} [\theta_{pq} \times \nu],
\]

\[
b_{\mu \bar{\mu}} = \sqrt{2} e_\mu - \frac{1}{\sqrt{2}} (e_\mu, \theta_p + \theta_q) \nu,
\]

where \( \theta_{pq} = \theta_p - \theta_q \) and \( \bar{\mu} = -\mu \). The matrix element \( M \), Eq. (3), can be written as follows

\[
M = \int d\tau e^{-i k \cdot r} \text{Sp}\{(f_0 - \alpha \cdot f_1 - \Sigma \cdot f_2) \gamma \cdot e_\lambda^r (g_0 - \alpha \cdot g_1 - \Sigma \cdot g_2) F\}.
\]
Note that only the terms with \((P+Q)\) and \((1+PQ)\) in \(F\), Eq. (10), contribute to the matrix element \((13)\) because it contains the odd number of the gamma-matrices.

In the quasiclassical approximation the relative magnitude of the functions \(f_0, f_1, g_0, \) and \(g_1, 2\) is different, so that

\[
f_0 \sim l_c f_1 \sim l_c^2 f_2, \quad g_0 \sim l_c g_1 \sim l_c^2 g_2, \quad d_0 \sim l_c d_1 \sim l_c^2 d_2,
\]

(14)

where \(l_c \sim \varepsilon/\Delta \gg 1\) is the characteristic value of the angular momentum in the process, \(\Delta = q+k-p\) is the momentum transfer. To find the distinction between the differential cross section of bremsstrahlung from particles and antiparticles, it is necessary to take into account the first quasiclassical corrections to the functions \(f_0, g_0, f_1, \) and \(g_1\), while the functions \(f_2\) and \(g_2\) can be taken in the leading quasiclassical approximation. Let us introduce the quantities

\[
(A_{00}, A_{01}, A_{10}, A_{02}, A_{20}) = \int dr \exp (-ik \cdot r)(f_0 g_0, f_0 g_1, f_1 g_0, f_0 g_2, f_2 g_0).
\]

(15)

In terms of these quantities, the matrix element \(M\) has the form

\[
M = \delta_{\mu \nu q} \left[ \delta_{\lambda \mu \rho} (e_\lambda^*, -\theta_q A_{00} - 2A_{10} + 2\mu_p A_{20}) + \delta_{\lambda \mu \rho} (e_\lambda^*, -\theta_p A_{00} + 2A_{01} + 2\mu_p A_{02}) \right] - \frac{m \mu_p (p-q)}{\sqrt{2pq}} \delta_{\mu \rho \lambda \mu} A_{00}.
\]

(16)

Below we calculate all quantities in (16) for arbitrary atomic potential \(V(r)\) which includes the effect of screening and the nuclear size effect as well.

### III. GREEN’S FUNCTIONS AND WAVE FUNCTIONS

Let us consider the case of arbitrary central localized potential \(V(r)\). We expand the Green’s function \(D(r_2, r_1|\varepsilon)\), Eq. (7), up to the second order with respect to the correction \(\alpha \cdot \nabla V(r)\):

\[
D(r_2, r_1|\varepsilon) = \langle r_2 | \frac{1}{\mathcal{H}} - \frac{1}{\mathcal{H}} i\alpha \cdot \nabla V(r) \frac{1}{\mathcal{H}} + \frac{1}{\mathcal{H}} i\alpha \cdot \nabla V(r) \frac{1}{\mathcal{H}} i\alpha \cdot \nabla V(r) \frac{1}{\mathcal{H}} | r_1 \rangle,
\]

\[
\mathcal{H} = \varepsilon^2 - m^2 - 2\varepsilon \varphi(r) + \nabla^2 + i0, \quad \varphi(r) = V(r) - \frac{V^2(r)}{2\varepsilon}.
\]

(17)

The function \(D^{(0)}(r_2, r_1|\varepsilon) = \langle r_2 | \mathcal{H}^{-1} | r_1 \rangle\) is the Green’s function of the Klein-Gordon equation. This function was found in the quasiclassical approximation with the first correction.
taken into account \[12\]:

\[
D^{(0)}(r_2, r_1 | \varepsilon) = \frac{ie^{ikr}}{4\pi^2r} \int dQ \exp \left[ iQ^2 - ir \int_0^1 dx V(R_x) \right] 
\]

\[
\times \left\{ 1 + \frac{i\varepsilon}{2\varepsilon} \int_0^1 dx \int dy(x - y) \nabla_\perp V(R_x) \cdot \nabla_\perp V(R_y) \right\} ,
\]

\[
r = r_2 - r_1 , \quad R_x = r_1 + xr + Q \sqrt{\frac{2r_1r_2}{kr}} ,
\]

(18)

where \( Q \) is a two-dimensional vector perpendicular to \( r \) and \( \nabla_\perp \) is the component of the gradient perpendicular to \( r \). Within the same accuracy, \( D^{(0)}(r_2, r_1 | \varepsilon) \) coincides with the contribution \( d(r_2, r_1) \) to the Green’s function \( D(r_2, r_1 | \varepsilon) \), Eq. [5].

Using this formula and Eqs. (4), (6), and (8), we obtain the function \( f_0(r, q) \),

\[
f_0(r, q) = -\frac{i}{\pi} e^{-iqr} \int dQ \exp \left[ iQ^2 - i \int_0^\infty dx V(r_x) \right] 
\]

\[
\times \left\{ 1 + \frac{i}{2\varepsilon} \int_0^\infty dx \int dy(x - y) \nabla_\perp V(r_x) \cdot \nabla_\perp V(r_y) \right\} ,
\]

\[
r_x = r + xn_q + Q \sqrt{\frac{2r}{\varepsilon_q}} , \quad Q \cdot n_q = 0 ,
\]

(19)

where \( \nabla_\perp \) is the component of the gradient perpendicular to \( n_q = q/q \). Then we use the relation

\[
i \nabla V(r) = \frac{1}{2\varepsilon} [p, \mathcal{H}] + \frac{i}{2\varepsilon} \nabla V^2(r) ,
\]

(20)

and write the linear in \( \nabla V(r) \) term in Eq. (17) as \( \alpha \cdot d_1(r_2, r_1) \), where

\[
d_1(r_2, r_1) = -\frac{i}{2\varepsilon} (\nabla_1 + \nabla_2) D^{(0)}(r_2, r_1 | \varepsilon) + \delta d_1(r_2, r_1) ,
\]

\[
\delta d_1(r_2, r_1) = -\langle r_2 | \frac{1}{\mathcal{H} 2\varepsilon} \nabla V^2(r) \frac{1}{\mathcal{H}} | r_1 \rangle .
\]

(21)

If we replace \( V(r) \) by \( V(r) + \delta V(r) \) in the operator \( \mathcal{H} \), where \( \delta V(r) = -i\alpha \cdot \nabla V^2(r)/(2\varepsilon)^2 \), then we obtain from Eq. (18)

\[
\delta d_1(r_2, r_1) = -\frac{ie^{ikr}}{16\pi^2\varepsilon^2} \int dQ \exp \left[ iQ^2 - ir \int_0^1 dx V(R_x) \right] \int_0^1 dx \nabla V^2(R_x) ,
\]

(22)

where \( R_x \) is given in (18). Using Eqs. (4), (6), and (8), we find the function \( f_1(r, q) \),

\[
f_1(r, q) = \frac{1}{2\varepsilon} (i \nabla - q)f_0(r, q) + \delta f_1(r, q) ,
\]

8
\[ \delta f_1(r, q) = -\frac{i}{4\pi \varepsilon^2} e^{-iqr} \int dQ \exp \left[ iQ^2 - i \int_0^\infty dx \nabla V(r_x) \right] \int_0^\infty dx \nabla V^2(r_x). \] (23)

where \( r_x \) is given in Eq. (19).

To transform the third term in (17), we replace \( i \nabla V(r) \) by \( \frac{1}{2\varepsilon}[p, \mathcal{H}] \). Then it follows from Eqs. (4), (6), and (8) that the function \( d_2(r_2, r_1) \) is

\[ d_2(r_2, r_1) = -\frac{i}{(2\varepsilon)^2} [\nabla_2 \times \nabla_1] D(0)(r_2, r_1 | \varepsilon) + \delta d_2(r_2, r_1), \]

\[ \delta d_2(r_2, r_1) = l_{21} \langle r_2 \rangle \frac{V'(r)}{\mathcal{H}} \frac{1}{2\varepsilon r} \frac{\nabla}{\mathcal{H}} r_1, \] (24)

where \( l_{21} = -(i/2)(r_2 \times \nabla_2 - r_1 \times \nabla_1) \) and \( V'(r) = \partial V(r)/\partial r \). In (24) we use the relation \([l, \mathcal{H}] = 0\). If we replace \( V(r) \) by \( V(r) + \delta V(r) \) in the operator \( \mathcal{H} \), where \( \delta V(r) = r^{-1}V'(r)/(2\varepsilon)^2 \), then we obtain from Eq. (18)

\[ \delta d_2(r_2, r_1) = l_{21} \frac{e^{iqr}}{16\pi^2 \varepsilon^2} \int dQ \exp \left[ iQ^2 - i r \int_0^1 dx \nabla V(R_x) \right] \int_0^1 dx \frac{V'(R_x)}{R_x}. \] (25)

Substituting this expression in (24), we finally find \( d_2(r_2, r_1) \),

\[ d_2(r_2, r_1) = -\frac{re^{iqr}}{16\pi^2 \varepsilon^2} \int dQ \exp \left[ iQ^2 - i r \int_0^1 dx \nabla V(R_x) \right] \]

\[ \times \int_0^1 dx \int_0^x dy \left[ \nabla V(R_x) \times \nabla V(R_y) \right]. \] (26)

The corresponding function \( f_2(r, q) \) is

\[ f_2(r, q) = -\frac{e^{-iqr}}{4\pi \varepsilon^2} \int dQ \exp \left[ iQ^2 - i r \int_0^1 dx \nabla V(r_x) \right] \]

\[ \times \int_0^\infty dx \int_0^x dy \left[ \nabla V(r_x) \times \nabla V(r_y) \right]. \] (27)

For the Coulomb field \( V_c(r) = -\eta/r \), we find from (19), (23), and (27)

\[ f_0(r, q) = F_A + (1 + n_q \cdot n) F_C, \]

\[ f_1(r, q) = (n_q + n) \eta F_B, \]

\[ f_2(r, q) = -i \Sigma \cdot [n_q \times n] F_C, \] (28)

where

\[ F_A(r, q, \eta) = \exp \left( \frac{\pi \eta}{2} - i q \cdot r \right) \right[ (1 - i\eta) F(i\eta, 1, iz) \]
\[
\frac{\pi \eta^2 e^{i \pi \eta}}{2 \sqrt{2qr}} \Gamma(1/2 - i \eta) F(1/2 + i \eta, 1, iz) + \frac{\pi \eta^2 e^{i \pi \eta}}{2 \sqrt{2qr}} \Gamma(1/2 - i \eta) F(3/2 + i \eta, 2, iz) \],
\]
\[
F_B(r, q, \eta) = -\frac{i}{2} \exp \left( \frac{\pi \eta}{2} - i q \cdot r \right) \left[ \Gamma(1 - i \eta) F(1 + i \eta, 2, iz) \right] + \frac{\pi \eta^2 e^{i \pi \eta}}{8 \sqrt{2qr}} \Gamma(1/2 - i \eta) F(3/2 + i \eta, 2, iz),
\]
\[
z = (1 + \mathbf{n} \cdot \mathbf{n_q})qr, \quad \mathbf{n} = \frac{r}{r}, \quad (29)
\]

Here \( \Gamma(x) \) is the Euler Gamma function and \( F(\alpha, \beta, x) \) is the confluent hypergeometric function. The results (28) and (29) are in agreement with that obtained in [23].

### IV. Calculation of the Matrix Element

The calculation of the quantities \( A_{00}, A_{01}, A_{10}, A_{02}, \) and \( A_{20} \) is performed in the same way as in Ref.[20]. We present details of this very tricky calculation in Appendix. We obtain

\[
A_{00} = \frac{1}{\omega m^4} \int dr \exp \left[ -i \mathbf{\Delta} \cdot r - i \chi(\rho) \right] \left[ i 2 \varepsilon_p \varepsilon_q \xi_q \xi_q (p_\perp + q_\perp) + m^2 (\varepsilon_p \xi_p - \varepsilon_q \xi_q) \int_0^\infty dx x \nabla_\perp V(r - x \nu) \cdot \nabla_\perp V(r) \right],
\]
\[
A_{01} = \frac{\varepsilon_q \xi_q}{\omega m^2} \int dr \exp \left[ -i \mathbf{\Delta} \cdot r - i \chi(\rho) \right] \nabla_\perp V(r) + \frac{\mathbf{\Delta}}{2 \varepsilon_p} \int_0^\infty dx x \nabla_\perp V(r - x \nu) \cdot \nabla_\perp V(r) + \frac{i}{2 \varepsilon_p} \nabla_\perp V^2(r),
\]
\[
A_{02} = -\frac{\varepsilon_q \xi_q}{2 \omega \varepsilon_p m^2} \int dr \exp \left[ -i \mathbf{\Delta} \cdot r - i \chi(\rho) \right] \nabla V(r - x \nu) \times \nabla V(r),
\]
\[
A_{10} = -A_{01}(\varepsilon_q \leftrightarrow \varepsilon_p, \xi_q \leftrightarrow \xi_p), \quad A_{20} = -A_{02}(\varepsilon_q \leftrightarrow \varepsilon_p, \xi_q \leftrightarrow \xi_p),
\]
\[
\chi(\rho) = \int_{-\infty}^{\infty} V(z, \rho) dz, \quad \xi_p = \frac{m^2}{m^2 + p_\perp^2}, \quad \xi_q = \frac{m^2}{m^2 + q_\perp^2}. \quad (30)
\]

Substituting Eq. (30) in Eq. (16), we find the matrix element \( M \),

\[
M = -\delta_{\mu \nu \rho \eta} (\varepsilon_p \delta_{\lambda \mu p} + \varepsilon_q \delta_{\lambda \rho q}) [N_0(e^*_\lambda, \xi_p p_\perp - \xi_q q_\perp) + N_1(e^*_\lambda, \varepsilon_p \xi_p p_\perp - \varepsilon_q \xi_q q_\perp)]]
\]
\[-\frac{1}{\sqrt{2}}m\mu_p\delta_{\mu_p\mu_q}\delta_{\lambda_p\lambda_q}(\varepsilon_p - \varepsilon_q)[N_0(\xi_p - \xi_q) + N_1(\varepsilon_p\xi_p - \varepsilon_q\xi_q)],\]

\[N_0 = \frac{2i}{\omega m^2\Delta^2} \int dr \exp[-i\Delta \cdot r - i\chi(p)] \Delta \cdot \nabla V(r),\]

\[N_1 = \frac{1}{\omega m^2\varepsilon_p\varepsilon_q} \int dr \exp[-i\Delta \cdot r - i\chi(p)] \int_0^\infty dx x \nabla V(r - x\nu) \cdot \nabla V(r). \quad (31)\]

Note that in Eq. (10) the contributions of \(A_{02}\) and \(A_{20}\) cancel out the contributions of the terms with \(\nabla \cdot V^2(r)\) in \(A_{01}\) and \(A_{10}\) (31). The amplitude \(M\) is exact in the potential \(V(r)\). It contains the leading quasiclassical contribution and the first quasiclassical correction as well. For high-energy bremsstrahlung from electrons in the field of a heavy atom, it is necessary to take into account the effect of screening. For high-energy bremsstrahlung from muons it is necessary to take also into account the finite nuclear radius \(R\) (nuclear size effect), because the muon Compton wavelength, \(\lambda_\mu = 1/m_\mu = 1.87\,\text{fm}\), is smaller than \(R\), \(R = 7.3\,\text{fm}\) for gold and \(R = 7.2\,\text{fm}\) for lead, \(m_\mu\) is the muon mass.

From (31) we have:

\[\sum_{\lambda\mu_\eta} |M|^2 = S_0 + S_1 + S_2,\]

\[S_0 = m^2|N_0|^2 \left[\frac{\Delta^2}{m^2}(\varepsilon_p^2 + \varepsilon_q^2)(\xi_p\xi_q - 2\varepsilon_p\varepsilon_q(\xi_p - \xi_q))^2\right],\]

\[S_1 = \frac{m^2\text{Re}N_0N_1^*}{2} \left\{\frac{\Delta^2}{m^2}(\varepsilon_p^2 + \varepsilon_q^2)(\varepsilon_p + \varepsilon_q)\xi_p\xi_q + \left[(\varepsilon_p^2 + \varepsilon_q^2)(\varepsilon_p - \varepsilon_q) - 4\varepsilon_p\varepsilon_q(\varepsilon_p\xi_p - \varepsilon_q\xi_q)\right](\xi_p - \xi_q)\right\},\]

\[S_2 = -\mu_p\text{Im}N_0N_1^* \omega^2(\varepsilon_p + \varepsilon_q)\xi_p\xi_q [\mathbf{p}_\perp \times \mathbf{q}_\perp] \cdot \nu. \quad (32)\]

The quantity \(S_0\) is the even function of \(\eta\), it contributes to the symmetric term \(d\sigma_s(p, q, k, \eta)\) of the cross section (1). The quantity \(S_1\) is the odd function of \(\eta\), it contributes to the antisymmetric term \(d\sigma_a(p, q, k, \eta)\) of the cross section (1). The quantity \(S_2\) is the even function of \(\eta\), it contributes to the symmetric term \(d\sigma_s(p, q, k, \eta)\) of the cross section (1) which vanishes after averaging over the helicity \(\mu_p\) of the initial electron. Note that the contribution of \(S_2\) to the cross section is responsible for the effect of asymmetry with respect to the replacement \(\varphi_i \to -\varphi_i\), where \(\varphi_i\) are the azimuth angles of the final particles in the frame where the \(z\) axis is directed along \(p\). Such asymmetry is absent in the cross section calculated in the leading quasiclassical approximation. We emphasize that the contributions
S₁ and S₂ are nonzero due to accounting for the next-to-leading quasiclassical terms.

The coefficients N₀ and N₁ depend on the momenta p, q, and k via the momentum transfer Δ. Therefore, it is easy to find from (2) and (32) the cross section dσ/dωdΔ⊥. A simple integration gives

\[
\frac{d\sigma_s}{d\omega d\Delta_{\perp}} = \frac{\alpha \varepsilon_p m^4}{(2\pi)^3 \varepsilon_p} |N_0|^2 \Phi(\zeta), \quad \frac{d\sigma_a}{d\omega d\Delta_{\perp}} = \frac{\alpha \varepsilon_p (\varepsilon_p + \varepsilon_q) m^4}{(2\pi)^3 \varepsilon_p} \text{Re} N_0 N_1^* \Phi,
\]

\[\Phi = \frac{\ln(\zeta + \sqrt{1 + \zeta^2})}{\zeta \sqrt{1 + \zeta^2}} \left( \frac{\varepsilon_p^2 + \varepsilon_q^2}{\varepsilon_p \varepsilon_q} + 1 \right) - 1, \quad \zeta = \frac{\Delta_{\perp}}{2m}. \quad (33)\]

The function Φ has the following asymptotic forms:

\[\Phi = \left( \frac{\varepsilon_p^2 + \varepsilon_q^2}{\varepsilon_p \varepsilon_q} - \frac{2}{3} \right) \zeta^2 \quad \text{at } \zeta \ll 1, \]

\[\Phi = \frac{\varepsilon_p^2 + \varepsilon_q^2}{\varepsilon_p \varepsilon_q} \ln(2\zeta) - 1 \quad \text{at } \zeta \gg 1. \quad (34)\]

V. CHARGE ASYMMETRY IN HIGH-ENERGY BREMSSTRAHLUNG FROM ELECTRONS

As is known, the main contribution to the Coulomb corrections to the symmetric part of the differential cross section of bremsstrahlung is given by the region Δ ∼ max(r⁻¹, Δmin) [16, 20], where Δmin = p − q − ω ≈ m²ω/2є_pє_q. However, the main contribution to the charge asymmetry is given by the region Δ ≫ max(r⁻¹, Δmin). In this region we can neglect the effect of screening, replace V(r) by the Coulomb potential V_c(r) = −η/r, and neglect also Δ∥ in comparison with Δ⊥. A simple calculation gives for the coefficient N₀ and N₁ in (31):

\[N_0 = \frac{8\pi \eta (L \Delta)^{2i\eta}}{\omega m^2 \Delta^2} \frac{\Gamma(1 - i\eta)}{\Gamma(1 + i\eta)}, \quad N_1 = \frac{2\pi^2 \eta^2 (L \Delta)^{2i\eta}}{\omega m^2 \varepsilon_p \varepsilon_q \Delta} \frac{\Gamma(1/2 - i\eta)}{\Gamma(1/2 + i\eta)}, \quad (35)\]

where L ∼ min(є_p/m², r_scr). Note that the factor (LΔ)²πη is irrelevant because it disappears in |M|². Then we obtain for the coefficients in S₀, S₁, and S₂ (32):

\[|N_0|^2 = \left( \frac{8\pi \eta}{\omega m^2 \Delta^2} \right)^2, \quad \text{Re} N_0 N_1^* = \frac{\pi \text{Re} g(\eta) \Delta}{4\varepsilon_p \varepsilon_q} |N_0|^2, \]

\[\text{Im} N_0 N_1^* = \frac{\pi \text{Im} g(\eta) \Delta}{4\varepsilon_p \varepsilon_q} |N_0|^2, \quad g(\eta) = \frac{\eta (1 - i\eta) \Gamma(1/2 + i\eta)}{\Gamma(1 + i\eta) \Gamma(1/2 - i\eta)}. \quad (36)\]

As it should, in the region Δ ∼ m there are no Coulomb corrections to dσ_s(p, q, k, η) calculated in the leading quasiclassical approximation. The Coulomb corrections to dσ_a(p, q, k, η)
(the term $S_1$) and $d\sigma_s(p, q, k, \eta)$ (the term $S_2$) are accumulated in the functions $\text{Re}g(\eta)$ and $\text{Im}g(\eta)$, respectively. These functions are shown in Fig.1. At $\eta \ll 1$ we have $\text{Re}g(\eta) \approx \eta$ and $\text{Im}g(\eta) \approx -(4 \ln 2) \eta^2$. It is seen from Fig.1 that the functions $\text{Re}g(\eta)$ and $\text{Im}g(\eta)$ differ significantly from their small-$\eta$ asymptotic forms already at very small $\eta$.

At $\omega \ll \varepsilon_p$, the ratio of the antisymmetric part of the cross section to the symmetric one,

$$\frac{S_1}{S_0} = \frac{\pi \Delta \text{Re}g(\eta)}{2 \varepsilon_p}, \quad \text{(37)}$$

increases with $\Delta/\varepsilon_p$ and can be more than ten percent. The ratio $S_2/S_0$ is small at $\omega \ll \varepsilon_p$ because it is suppressed by the factor $(\omega/\varepsilon_p)^2$.

If $p_\perp \gg m$ and $q_\perp \gg m$, then

$$\frac{S_1}{S_0} = \frac{\pi \text{Re}g(\eta)}{2\Delta} \Delta \cdot \theta_{qp}, \quad \frac{S_2}{S_0} = \mu_p \frac{\pi \omega (\varepsilon_p + \varepsilon_q) \text{Im}g(\eta)}{2(\varepsilon_p^2 + \varepsilon_q^2) \Delta} [\Delta \times \theta_{qp}] \cdot \nu, \quad \text{(38)}$$

where $\theta_{qp} = p_\perp/p - q_\perp/q$. Thus, the azimuth asymmetry increases with $\omega$ and may be important.

Let us discuss the cross section $d\sigma/d\omega d\Delta_\perp$ at $\Delta \gg \max(r^{-1}_{\text{cor}}, \Delta_{\text{min}})$, Eq. (33):

$$\frac{d\sigma_s}{d\omega d\Delta_\perp} = \frac{8 \alpha \eta^2 \varepsilon_q \Phi}{\pi \omega \varepsilon_p \Delta_\perp^2}, \quad \frac{d\sigma_a}{d\omega d\Delta_\perp} = \frac{\pi \text{Re}g(\eta)(\varepsilon_p + \varepsilon_q) \Delta}{4 \varepsilon_p \varepsilon_q} \frac{d\sigma_s}{d\omega d\Delta_\perp}. \quad \text{(39)}$$

In Fig.2 we show the dependence of $A = \sigma_0^{-1} d\sigma_a/d\omega d\Delta_\perp$ on $\zeta = \Delta_\perp/2m$ for a few values of $t = \varepsilon_q/\varepsilon_p$; $\sigma_0 = \alpha \eta^2 \text{Re}g(\eta)/(2m^2 \omega \varepsilon_p \Delta_\perp)$. This figure confirms our statement that the
main contribution to the antisymmetric part of the cross section is given by the region \( \Delta \sim m \). Performing integration over \( \Delta_\perp \) in Eq. (33), we obtain the antisymmetric correction to the spectrum

\[
\frac{d\sigma}{d\omega} = \frac{\alpha \pi^2 \eta^2 \text{Reg}(\eta)}{4m \omega \varepsilon_p^2} \left[ 2 \frac{\varepsilon_p^2 + \varepsilon_q^2}{\varepsilon_p \varepsilon_q} - 1 \right] (\varepsilon_p + \varepsilon_q). \tag{40}
\]

This result coincides with the corresponding result of Ref. [19, 20].

In Fig. 3 we show the dependence of \( A_1 = \sigma_1^{-1} d\sigma_a/d\omega dk_\perp \) on \( \zeta_1 = k_\perp / m \) for a few values of \( t = \varepsilon_q/\varepsilon_p; \sigma_1 = \alpha \eta^2 \text{Reg}(\eta)/(2m^3 \omega \varepsilon_p) \). Here \( k_\perp \) is the component of \( k \) perpendicular to the vector \( p \), \( k_\perp = -\omega p_\perp / p \). The result is obtained by numerical integration of the differential cross section \( d\sigma_a(p, q, k, \eta) \) over \( dq_\perp \). As is known (see, e.g., Ref. [5]), the cross section \( d\sigma_{\text{brem}}(\omega, \varepsilon_p, k_\perp)/d\omega dk_\perp \) of bremsstrahlung (\( \varepsilon_p \) is the initial electron energy) can be obtained from the cross section \( d\sigma_{\text{photo}}(\omega, \varepsilon_p, p_\perp)/d\varepsilon_p dp_\perp \) of photoproduction (\( \varepsilon_p \) is the positron energy) by the relation

\[
\frac{d\sigma_{\text{brem}}(\omega, \varepsilon_p, k_\perp)}{d\omega dk_\perp} = \frac{\varepsilon_p^2 d\sigma_{\text{photo}}(-\omega, -\varepsilon_p, p_\perp)}{\omega^2 d\varepsilon_p dp_\perp}, \tag{41}
\]

where \( p_\perp = pk_\perp / \omega \). The antisymmetric part of \( d\sigma_{\text{photo}}(\omega, \varepsilon_p, p_\perp)/d\varepsilon_p dp_\perp \) was obtained in Ref. [23]. Using Fig. 2 in that paper and Eq. (41), we find that our result shown in Fig. 3 is in agreement with the corresponding result in Ref. [23].
FIG. 3: Dependence of $A_1 = \sigma_1^{-1}d\sigma_a/d\omega d\mathbf{k}_\perp$ on $\zeta_1 = k_\perp/m$, for a few values of $t = \varepsilon_q/\varepsilon_p$; $\sigma_1 = \alpha\eta^2\text{Reg}(\eta)/(2m^3\omega\varepsilon_p)$: $t = 0.25$ (solid curve), $t = 0.5$ (dashed curve), and $t = 0.75$ (dotted curve).

VI. CHARGE ASYMMETRY IN HIGH-ENERGY BREMSSTRAHLUNG FROM MUONS

The charge asymmetry in the differential cross section of high-energy $\mu^+\mu^-$ photoproduction in the electric field of a heavy atom was investigated in detail in Ref. [24]. The deviation of the nuclear electric field from the Coulomb field at small distances due to the finite nuclear radius $R$ (nuclear size effect) is crucially important for the charge asymmetry. Though the Coulomb corrections to the total cross section are negligibly small, it was shown in Ref. [24] that the charge asymmetry is not negligible for selected final states of $\mu^+$ and $\mu^-$. In this section we study the charge asymmetry in the differential cross section of high-energy bremsstrahlung from muons in the field of a heavy atom.

We write the Fourier transformation $V_F(\Delta^2)$ of the potential $V(r)$ as

$$V_F(\Delta^2) = -\frac{4\pi\eta F(\Delta^2)}{\Delta^2},$$

where $F(\Delta^2)$ is the form factor which differs essentially from unity at $\Delta \gtrsim 1/R$ and $Q \lesssim 1/r_{scr}$. Let us first discuss the Coulomb corrections to the symmetric part of the cross section calculated in the leading quasiclassical approximation. In this case the cross section
$d\sigma_s$ depends on the parameters of the field via the factor $N_0\,^{(31)}$. In the Born approximation

$$N_{0B} = \frac{2i}{\omega m^2 \Delta^2} \int d\mathbf{r} \exp(-i\Delta \cdot \mathbf{r}) \Delta \cdot \nabla \perp V(\mathbf{r}) = -\frac{2V_F(\Delta^2)}{\omega m^2}. \quad (43)$$

We define the Coulomb corrections $N_0$ to the quantity $|N_0|^2$ as

$$N_0 = |N_0|^2 - |N_{0B}|^2. \quad (44)$$

The quantity $N_0$ vanishes at $r_{\text{scr}}^{-1} \ll \Delta \ll R^{-1}$ and has two peaks: at $\Delta \sim r_{\text{scr}}^{-1}$ and at $\Delta \sim R^{-1}$. The contributions of these peaks to the integral $\int \Delta^2 N_0 d\Delta_\perp$ are $^{[20]}$

$$\int \Delta^2 N_0 d\Delta_\perp = \mp \frac{128\pi^3 \eta^2 f(\eta)}{\omega^2 m^4}, \quad f(\eta) = \text{Re}\psi(1 + i\eta) - \psi(1), \quad (45)$$

where $\psi(x) = d\ln \Gamma(x)/dx$, the negative contribution corresponds to the peak at $\Delta \sim r_{\text{scr}}^{-1}$, and the positive contribution corresponds to the peak at $\Delta \sim R^{-1}$. These two contributions are the universal functions of $\eta$ independent of the form of the potential in the regions $r \sim r_{\text{scr}}$ and $r \sim R$. While the function $N_0$ is very sensitive to the form of the potential in these regions $^{[16, 20]}$. Since $m \ll R^{-1}$ for electrons, only the region $r \sim r_{\text{scr}}$ gives the Coulomb corrections to $d\sigma_s/d\omega$ $^{[16]}$, 

$$\frac{d\sigma_C}{d\omega} = -\frac{4\alpha \eta^2 f(\eta)}{m^2 \omega} \left(t^2 - \frac{2}{3}t + 1\right), \quad t = \frac{\varepsilon_q}{\varepsilon_p}. \quad (46)$$

For muons $m_\mu \gg R^{-1}$, so that the sum of the contributions from both peaks, $\Delta \gtrsim 1/R$ and $Q \lesssim 1/r_{\text{scr}}$, gives the total Coulomb corrections to $d\sigma_s/d\omega$. As a result, the total Coulomb corrections vanish, see Eq.$^{(45)}$. However, the Coulomb corrections to the differential cross section at $\Delta \sim R^{-1}$ are large. To illustrate this statement, we consider the form factor $F(\Delta^2)$ in the form

$$F(\Delta^2) = \frac{\Lambda^2}{\Delta^2 + \Lambda^2}, \quad (47)$$

where $\Lambda \sim 60$ MeV for heavy nuclei. This form is valid for $\Delta \gg r_{\text{scr}}^{-1}$, where the factor $N_0$ is given by

$$N_0 = \frac{8\pi \eta}{\omega m^2 \Delta^2} \int_0^\infty d\rho J_1(\rho) \left[1 - \frac{\rho}{\beta} K_1\left(\frac{\rho}{\beta}\right)\right] \exp \left\{-2i\eta \left[\ln \frac{\rho}{\beta} + K_0\left(\frac{\rho}{\beta}\right)\right]\right\},$$

$$N_{0B} = \frac{8\pi \eta}{\omega m^2 \Delta^2 (1 + \beta^2)}, \quad \beta = \frac{\Delta}{\Lambda}. \quad (48)$$

Here $J_n(x)$ is the Bessel function and $K_n(x)$ is the modified Bessel function of the second kind. In Fig. 4 we show the dependence of $G_0 = |N_0|^2/|N_{0B}|^2 - 1$ on $\beta = \Delta/\Lambda$ for a few
FIG. 4: Dependence of \( G_0 = |N_0|^2/|N_{0B}|^2 - 1 \) on \( \beta = \Delta/\Lambda \) at \( \beta \gg 1/(r_{sc} \Lambda) \) and a few values of \( \eta \); \( \eta = 0.34 \) (Ag, solid curve), \( \eta = 0.6 \) (Pb, dashed curve), and \( \eta = 0.67 \) (U, dotted curve).

values of \( \eta \). Note that very narrow peak at \( \Delta \sim r_{sc}^{-1} \) \( (\delta \beta \sim 1/(r_{sc} \Lambda) \ll 1) \) is not shown in this figure. The dependence of the peak on the shape of the atomic potential at \( \Delta \sim r_{sc}^{-1} \) was investigated in detail in Ref. [20]. It is seen from Fig. 4 that the Coulomb corrections to \( |N_0|^2 \) are significant in the region \( \Delta/\Lambda \sim 1 \).

Let us consider the factor \( N_1 \) (31). In the lowest in \( \eta \) approximation it reads

\[
N_{1B} = \frac{\mathcal{J}(\Delta)}{\omega m^2 \varepsilon_p \varepsilon_q},
\]

\[
\mathcal{J}(\Delta) = \int \frac{ds}{(2\pi)^3} [V_F(Q_+)V_F'(Q_-) + (\Delta^2 - 4s^2_s) V_F(Q_+) V_F'(Q_-)],
\]

\[
Q_{\pm} = (s \pm \Delta/2)^2, \quad s_s = s \cdot \Delta/\Delta.
\]

(49)

where \( V'_F(Q) = \partial V_F(Q) / \partial Q \), see Ref. [24]. For the form factor (47), the explicit form of \( \mathcal{J}(\Delta) \) is given in Ref. [24]:

\[
\mathcal{J}(\Delta) = \frac{2\pi^2 \eta^2}{\Delta} \mathcal{F}(\beta), \quad \beta = \frac{\Delta}{\Lambda},
\]

\[
\mathcal{F}(\beta) = 1 + \frac{2}{\pi} \arcsin \frac{\beta}{\sqrt{\beta^2 + 4}} - \frac{4}{\pi} \arcsin \frac{\beta}{\sqrt{\beta^2 + 1}} - \frac{12\beta}{\pi(\beta^2 + 4)(\beta^2 + 1)}. \quad (50)
\]

The exact in \( \eta \) factor \( N_1 \) at \( \Delta \gg r_{sc}^{-1} \) is given by

\[
N_1 = \frac{2\pi^2 \eta^2}{\omega m^2 \varepsilon_p \varepsilon_q \Delta} \int_0^{\infty} \int_0^{\infty} dx d\rho \mathcal{F}(\beta x/\rho) J_0(\rho) J_0(x)
\]

\[
\times \exp \left\{ -2i\eta \left[ \ln \frac{\rho}{2} + K_0 \left( \frac{\rho}{\beta} \right) \right] \right\}. \quad (51)
\]
To demonstrate the influence of the nuclear size effect on the ratios $S_1/S_0$ and $S_2/S_0$ (32), we plot in Figs. 5 and 6 the quantities $G_1$ and $G_2$,

$$G_1 = \frac{\text{Re} N_0 N_1^*}{|N_0|^2 \Sigma_R}, \quad \Sigma_R = \frac{\pi \text{Re} g(\eta) \Delta}{4 \varepsilon_p \varepsilon_q},$$

$$G_2 = \frac{\text{Im} N_0 N_1^*}{|N_0|^2 \Sigma_I}, \quad \Sigma_I = \frac{\pi \text{Im} g(\eta) \Delta}{4 \varepsilon_p \varepsilon_q},$$

(52)
as a function of $\beta = \Delta/\Lambda$. For a pure Coulomb field $G_1 = G_2 = 1$ (36).

FIG. 5: Dependence of $G_1 = \Sigma_R^{-1} \text{Re} N_0 N_1^* / |N_0|^2$ on $\beta = \Delta/\Lambda$ (52), for a few values of $\eta$; $\eta = 0.34$ (Ag, solid curve), $\eta = 0.6$ (Pb, dashed curve), and $\eta = 0.67$ (U, dotted curve).

FIG. 6: Dependence of $G_2 = \Sigma_I^{-1} \text{Im} N_0 N_1^* / |N_0|^2$ on $\beta = \Delta/\Lambda$ (52), for a few values of $\eta$; $\eta = 0.34$ (Ag, solid curve), $\eta = 0.6$ (Pb, dashed curve), and $\eta = 0.67$ (U, dotted curve).
It is seen that the quantities $G_1$ and $G_2$ decrease rapidly with increasing $\beta$ at $\beta \lesssim 1$.

VII. CONCLUSION

We have investigated in detail the charge asymmetry in the process of high-energy bremsstrahlung in the field of a heavy atom. The charge asymmetry arises due to the account for the first quasiclassical correction to the differential cross section of the process. The results are exact in the parameters of the atomic field and are valid even for $\eta \sim 1$, they take into account the effect of screening and the nuclear size effect. The latter is important for high-energy bremsstrahlung from muons where the charge asymmetry is very sensitive to the shape of the nuclear form factor. It is shown that the Coulomb corrections essentially modify the charge asymmetry as compared with the leading in $\eta$ result already for the relatively small $\eta$. In the experimental region of interest, where $\varepsilon_p \gg p_\perp \gg m$ and $\varepsilon_q \gg q_\perp \gg m$ but $\Delta \lesssim 1/R$, the asymmetry can be as large as a few tens of percent. For the longitudinal polarization of the initial charged particle, due to the account for the first quasiclassical correction, the differential cross section reveals the asymmetry with respect to the replacement $\varphi \to -\varphi$, where $\varphi$ is the azimuth angle between the photon momentum $\mathbf{k}$ and the momentum $\mathbf{q}$ of the final charged particle in the frame where the $z$ axis is directed along $\mathbf{p}$. Due to account for the first quasiclassical correction, our results for the differential cross section of high-energy bremsstrahlung have essentially higher precision than the famous results in Ref. [16] and should be taken into account in precision experiments and at data analysis in detectors.

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Appendix

In this appendix, following the method of [20], we derive the expression (30) for the quantity $A_{00}$,

$$A_{00} = \int dr \exp(-i k \cdot r) f_0 g_0,$$

where the function $f_0(r, q)$ is given in (19) and $g_0(r, p) = f_0(r, -p)$. Other quantities in (30) are calculated in the same way. We split the integration region into two, $z > 0$ and $z < 0$, and denote the corresponding contributions to $A_{00}$ as $A_{00}^+$ and $A_{00}^-$. For $z > 0$, the function $f_0$ has a simple eikonal form

$$f_0(r, q) = e^{-i q \cdot r} \exp \left[ - i \int_0^\infty dx \mathcal{V}(r + x n_q) \right],$$

so that

$$A_{00}^+ = \int_{z > 0} dr \frac{dQ}{i \pi} \exp \left\{ i Q^2 - i \Delta \cdot r - i \int_0^\infty dx \mathcal{V}(r_x) + \mathcal{V}(r + x n_q) \right\} \times \left[ 1 + \frac{i}{2 \varepsilon_p} \int_0^\infty dx \int_0^x dy (x - y) \nabla \perp \mathcal{V}(r_x) \cdot \nabla \perp \mathcal{V}(r_y) \right],$$

where $r_x = r - x n_p + Q \sqrt{2r/p}$. Within our accuracy we can replace the quantity $\mathcal{V}(r + x n_q)$ in (55) by $\mathcal{V}(r n_q + Q \sqrt{2r/p})$, shift $\rho \rightarrow \rho - Q \sqrt{2r/p}$, and take the integral over $Q$. We obtain

$$A_{00}^+ = \int_{z > 0} dr \exp \left\{ - i \frac{z}{2 \varepsilon_p} \Delta^2 \perp + i \Delta \cdot r - i \int_0^\infty dx \mathcal{V}(r - x n_p) + \mathcal{V}(r + x n_q) \right\} \times \left[ 1 + \frac{i}{2 \varepsilon_q} \int_0^\infty dx \int_0^x dy (x - y) \nabla \perp \mathcal{V}(r - x n_p) \cdot \nabla \perp \mathcal{V}(r - y n_p) \right].$$

In the same way, we obtain

$$A_{00}^- = \int_{z < 0} dr \exp \left\{ i \frac{z}{2 \varepsilon_q} \Delta^2 \perp - i \Delta \cdot r - i \int_0^\infty dx \mathcal{V}(r - x n_p) + \mathcal{V}(r + x n_q) \right\} \times \left[ 1 + \frac{i}{2 \varepsilon_q} \int_0^\infty dx \int_0^x dy (x - y) \nabla \perp \mathcal{V}(r + x n_q) \cdot \nabla \perp \mathcal{V}(r + y n_q) \right].$$

There are two overlapping regions of the momentum transfer $\Delta$:

I. $\Delta \ll \frac{m \omega}{\varepsilon_p}$

II. $\Delta \gg \Delta_{\text{min}} = \frac{m^2 \omega}{2 \varepsilon_p \varepsilon_q}$.
In the first region, one can neglect the term proportional to $\Delta_±^2$ in the exponents in (56) and (57). Then the sum $A_{00} = A_{00}^+ + A_{00}^-$ reads

$$A_{00} = \int dr \exp \left\{ -i\Delta \cdot r - i \int_0^\infty dx [V(r - xn_p) + V(r + xn_q)] \right\}$$

\[ \times \left[ 1 + \frac{i}{2\varepsilon_p} \int_0^\infty dx \int_0^x dy (x - y) \nabla_\perp V(r - xn_p) \cdot \nabla_\perp V(r - yn_p) \right. \]

\[ + \frac{i}{2\varepsilon_q} \int_0^\infty dx \int_x^\infty dy (x - y) \nabla_\perp V(r + xn_q) \cdot \nabla_\perp V(r + yn_q) \].

(59)

In the prefactor we make the replacement $n_p, n_q \to \nu$, and in the exponent we take into account the linear term of expansion in $n_p - \nu$ and $n_q - \nu$ of the integral. Besides, in the arguments of the functions $V(r + y\nu)$ and $V(r - y\nu)$ we make the substitutions $z \to z - y$ and $z \to z + y$, respectively. After that we take the integral over $y$ and obtain the contribution of the first region

$$A_{00} = \int dr \exp\left[-i\Delta \cdot r - i\chi(\rho)\right] \nabla_\perp V(r)$$

\[ \cdot \left[ \frac{\partial_{qp}}{\Delta_±^2} - \frac{\omega}{2\Delta_±\varepsilon_p\varepsilon_q} \int_0^\infty dxx \nabla_\perp V(r - x\nu) \right] , \]

$$\chi(\rho) = \int_{-\infty}^{\infty} V(z, \rho) dz ,$$

(60)

where $\Delta_± = \nu \cdot \Delta$ and $\theta_{qp} = q_\perp / q - p_\perp / p$.

Now we pass to the calculation of $A_{00}$ in the second region (58). In Eq. (56) for $A_{00}^+$, we make the replacement $n_q \to n_p$ and $z\Delta_±^2/2p \to (n_q \cdot n)\Delta_±^2/2p$. The polar angle of $n$ is small, and we can integrate in (56) over the region $n_q \cdot n > 0$. After the integration over $z$, we obtain

$$A_{00}^+ = \frac{1}{\Delta \cdot n_p + \Delta_±^2/2p} \int d\rho \exp\left[-i\Delta_\perp \cdot \rho - i\chi(\rho)\right]$$

\[ \times \left[ -i + \int_{-\infty}^{\infty} dz \int_0^\infty dxx \nabla_\perp V(r) \cdot \nabla_\perp V(r - xn_p) \right] . \]

(61)

The calculation of $A_{00}^-$ is performed quite similarly. As a result we have

$$A_{00}^- = \frac{1}{-\Delta \cdot n_q + \Delta_±^2/2q} \int d\rho \exp\left[-i\Delta_\perp \cdot \rho - i\chi(\rho)\right]$$

\[ \times \left[ -i + \int_{-\infty}^{\infty} dz \int_0^\infty dxx \nabla_\perp V(r) \cdot \nabla_\perp V(r - xn_q) \right] . \]

(62)
Taking into account that
\[
\Delta \cdot n_p + \Delta_\perp^2/2p = -\frac{m^2\omega}{2\varepsilon_p \varepsilon_q \xi_p}, \quad -\Delta \cdot n_q + \Delta_\perp^2/2q = \frac{m^2\omega}{2\varepsilon_p \varepsilon_q \xi_p},
\]
\[
\xi_p = \frac{m^2}{m^2 + p_\perp^2}, \quad \xi_q = \frac{m^2}{m^2 + q_\perp^2},
\]
we obtain for \( A_{00} = A_{00}^+ + A_{00}^- \) in the second region
\[
A_{00} = \frac{1}{m^4\omega} \int d\rho \exp\left[-i\Delta \cdot \rho - i\chi(\rho)\right] \left[2i\varepsilon_p \varepsilon_q \xi_p \xi_q (p_\perp + q_\perp) \cdot \nabla_\perp \chi(\rho) \right. \\
\left. + m^2(\varepsilon_p \xi_p - \varepsilon_q \xi_q) \int_{-\infty}^\infty dz \int_0^\infty dx \nabla_\perp V(r - x_\nu) \cdot \nabla_\perp V(r) \right].
\]
(63)

Now we can compare (63) and (60) and write the expression for \( A_{00} \) which is valid in all region of \( \Delta \). We finally arrive at the following result
\[
A_{00} = \frac{1}{m^4\omega} \int d\rho \exp\left[-i\Delta \cdot \rho - i\chi(\rho)\right] \left[2i\varepsilon_p \varepsilon_q \xi_p \xi_q (p_\perp + q_\perp) \right. \\
\left. + m^2(\varepsilon_p \xi_p - \varepsilon_q \xi_q) \int_0^\infty dx \nabla_\perp V(r - x_\nu) \right] \cdot \nabla_\perp V(r).
\]
(64)

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