A comparative study on stress and compliance based structural topology optimization

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Abstract. Most of structural topology optimization problems have been formulated and solved to either minimize compliance or weight of a structure under volume or stress constraints, respectively. Even if, a lot of researches are conducted on these two formulation techniques separately, there is no clear comparative study between the two approaches. This paper intends to compare these formulation techniques, so that an end user or designer can choose the best one based on the problems they have. Benchmark problems under the same boundary and loading conditions are defined, solved and results are compared based on these formulations. Simulation results show that the two formulation techniques are dependent on the type of loading and boundary conditions defined. Maximum stress induced in the design domain is higher when the design domains are formulated using compliance based formulations. Optimal layouts from compliance minimization formulation have complex layout than stress based ones which may lead the manufacturing of the optimal layouts to be challenging. Optimal layouts from compliance based formulations are dependent on the material to be distributed. On the other hand, optimal layouts from stress based formulation are dependent on the type of material used to define the design domain. High computational time for stress based topology optimization is still a challenge because of the definition of stress constraints at element level. Results also shows that adjustment of convergence criterions can be an alternative solution to minimize the maximum stress developed in optimal layouts. Therefore, a designer or end user should choose a method of formulation based on the design domain defined and boundary conditions considered.

1. Introduction

Structural optimization, which deals about assemblage of material to sustain loads in the best way, is one type of engineering optimization tools. It has been used in different areas of applications[1-3]. Problems of structural optimization have been modeled and solved using size, shape or topology optimization. In size and shape optimization, some predefined size and shape of solid elements are included in addition to the loading and boundary conditions before the optimization starts[4, 5]. Unlike size and shape optimization, in topology optimization there is no predefined size and shape of elements except the definition of boundary and loading conditions, which makes optimal layout to be independent of the knowledge and experience of the end user. Topology optimization is a mathematical approach which seeks optimal material layout within a given design domain for a given set of boundary and loading conditions. It includes determination of connectivity, geometries of cavities and location of voids in the design domain[6].

Problems in structural topology optimization have been, formulated and solved to minimize compliance of the structure under volume constraint[7, 8]. This type of formulation, which is the most common way of formulating topology optimization problems, has some drawbacks such as variation of results with the amount of material distributed and unable to consider stress and displacement[6, 9]. These limitations may prevent optimal layouts to be applicable in the real-world scenario unless some post processing action such as sizing and shape optimization, is involved to accommodate stress concentrations and major deflections. In this type of formulation, the optimal material distribution is highly affected by the amount
of materials to be distributed, which will let the optimal layout be different for different values of the constraint.

For those designs areas and applications where stress is main design variable, it will be difficult to directly take optimal layouts from classical compliance minimization formulations. Researches have been conducting to include stress constraints in topology optimization process[10, 11]. In this type of formulation and solution of topology optimization problems, which considers stress constraints, the optimal lay out always favors the failure criterion other than the amount of material to be distributed. This will make the optimal material layout to be independent of amount of material distributed, safe and secured from failures associated with the loading conditions. Though, this approach is accepted and more realistic from engineering point of view[9], especially for those designs where stress is main concern, it has three challenges associated with the design variables namely singularity phenomenon, local nature of stress constraint and nonlinear behavior of stress constraint. So far, structural topology optimization problems have been formulated based on the need and experience of the designer and end users. This paper aims to show how the formulation technique used can alter the optimal layout of a given design domain and propose a guideline how to choose an appropriate formulation technique. All topology optimization problems defined in Section 2 are formulated using SIMP method and tested using bench mark problems.

2. Problem formulation

In this paper, weight minimization under stress constraints and classical compliance minimization under volume constraint are considered.

2.1. Weight Minimization Problem. Eq.1 shows a stress based topology optimization (STO) problem formulated to minimize weight of a given design domain under stress constraints. Stress constraints are defined based von Mises stress failure theory, which states, “A material will fail when the von Mises stress induced in the material exceeds yield strength”.

\[
\begin{align*}
\min_{x} & \quad V = \sum_{e=1}^{N} (x_e)^{P_e} v_e \\
\text{Subjected to:} & \quad g(x_e) = \frac{\sigma_{\text{vm}}}{\sigma_{\text{yield}}} < 1 \\
& \quad KU = F \\
& \quad 0 < x_{\min} \leq x_e \leq 1
\end{align*}
\]

Where, \( V \) is the volume (objective function), \( N \) total number of elements which defines the design domain, \( e \) elements within the design domain, \( v_e \) volume of each element in the design domain, \( \sigma_{\text{yield}} \) Von mises stress is maximum (yield stress), \( K \) is global stiffness matrix, \( U \) is global displacement vector, \( F \) global force vector, \( x_e \) is relative density/design variable, \( x_{\min} \) is the minimum relative density to control the singularity phenomenon associated with the design variable.

2.2. Classical Compliance Minimization. Eq.2 shows a classical topology optimization problem to minimize compliance of a given structure under volume constraint[12].

\[
\begin{align*}
\min_{x} & \quad C(x) = U^T K U = \sum_{e=1}^{N} (x_e)^{p} u_e^T K_e u_e \\
& \quad \frac{V(x)}{V} = \text{volfrac} \\
& \quad KU = F \\
& \quad 0 < x_{\min} \leq x_e \leq 1
\end{align*}
\]

Where, \( C \) is compliance, \( U \) and \( K \) are global displacement vector and global stiffness matrix, respectively, \( u_e \) and \( K_e \) elemental displacement vector and elemental stiffness matrix, respectively, \( p \) is the penalization factor, \( V(x) \) and \( V \) material and design domain volume, respectively, \( \text{volfrac} \) is volume fraction, \( F \) global force vector, \( x_{\min} \) is the minimum value for the design variable to avoid singularity phenomenon associated with
removal of material from the design domain and $x_e$ is a vector of design variable.

3. Methodology

All the optimization problems formulated are solved using an optimality criteria method, by which updating scheme for design variables can be expressed as shown in Eq.3[13].

$$
\begin{align*}
\text{if } x_e \beta e &\leq \max(x_{\text{min}}, x_e - m) \\
x_e^{\text{new}} &= \max(x_{\text{min}}, x_e - m) \\
\text{if } \max(x_{\text{min}}, x_e - m) < x_e \beta e &< \min(1, x_e + m) \\
x_e^{\text{new}} &= x_e \beta e \\
\text{if } \min(1, x_e + m) < x_e \beta e \\
x_e^{\text{new}} &= \min(1, x_e + m)
\end{align*}
$$

Where $m$ is a positive limit, which usually takes a value of 0.2, $\eta$ is a numerical damping coefficient with a value of 0.5[13], $\beta$ which will be dependent on the type of problems defined in Eq. 1 and Eq.2 as shown in Eq. 4 and Eq.5, respectively.

$$
\begin{align*}
\beta_e &= \frac{\partial v}{\partial x} \\
\beta_e &= \frac{\partial g}{\partial x}
\end{align*}
$$

$$
\begin{align*}
\beta_e &= \frac{\partial c}{\partial v} \\
\beta_e &= \frac{\partial c}{\partial x}
\end{align*}
$$

Where $\lambda$ is a Lagrangian multiplier, where the value can be found using bisection method. The sensitivity analysis for problems defined can be found as shown in Eq.6 for stress based topology optimization and Eq.7 for classical compliance minimization problem.

$$
\begin{align*}
\frac{\partial v}{\partial x} &= px^{p-1}v_e \\
\frac{\partial g}{\partial x} &= (p-q)x^{p-q-1}D_0 B_e u_e - x^{p-q}D_0 B_e \frac{\partial k}{\partial x} k \\
\frac{\partial C}{\partial x} &= px^{p-1}U_e^T K_0 U_e
\end{align*}
$$

4. Numerical Result

To validate formulated problems simply supported beam and cantilever beams under different loading and boundary conditions were taken[13, 14]. All the design domains are discretized by square rectangular finite elements. The material considered for all cases described in this paper has a Young’s modulus of $E = 1.0$ MPa, a Poisson’s ratio of $\nu = 0.3$ and a von Mises stress of 1.2Mpa subjected to a unit load[15].

The maximum stresses induced for compliance based formulations is calculated for each iteration and compared to the maximum stress of design domains under stress based formulation. Compliance of the design domains under stress based formulation is calculated for each case study and compared with the respective compliance based formulations.

The design domains and loading conditions for different case studies used in this study are defined followed by their respective optimal material distribution layout, stress distribution plots and variation of maximum
stress and compliance. To have a full control on the stress measure in each element all the stress constraints are defined at element level. A Matlab simulation code is used for analysis on Dell –i5- 4GB RAM (3.41)-3.2GHz computer.

4.1. Simply Supported Beam. The first case corresponds to a simply supported beam, as shown in Fig.1. The design domain is discretized into 120 X 48 elements. Fig.2a and Fig.2b shows optimal material distribution under stress constrained topology optimization problem and classical compliance minimization problem, respectively.

![Figure 1 Design domain definition](image)

As it can be seen from the optimal layout plots, the optimal layout from classical compliance minimization problem is more complex than from that of stress based optimization problem. This complex layout will make the manufacturing process to be somewhat difficult and the need for post processing to be higher as shown in Fig.2b.

![Material distribution for stress based formulation](image)  ![Material distribution for compliance based formulation](image)

**Figure 2** Material distribution for a) stress based formulation b) Compliance based formulation

Fig. 3a shows a plot of stress vs number of iterations during the optimization process. From the plot it can be seen that maximum stress induced in classical compliance minimization problem is greater than that of stress based topology optimization problem.

![Maximum stress Vs Iteration](image)  ![Compliance Vs Iteration](image)

**Figure 3** variation of stress and compliance with iteration

4.2. Cantilever Beam: Under this case study different loading and boundary conditions of a cantilever beam are considered as shown in Fig.4. All the design domains are defined and discretized into 120X60.

![Classical cantilever beam](image)  ![Cantilever beam with predefined shape](image)  ![Cantilever beam with multiple loading](image)

**Figure 4** Design domain definition
Fig. 5a, Fig. 5b and Fig. 5c show optimal materials distribution of a cantilever beam using stress based formulation under the loading and boundary conditions defined in their respective design domains. Fig. 5d, Fig. 5e and Fig. 5f show optimal materials distribution of a cantilever beam under compliance for the respective design domains. From the optimal layouts of the cantilever beam under stress and compliance based formulations, the optimal layouts from stress based formulation are simple and clearer than that of optimal layouts from compliance based formulation. This makes the need for post processing for layouts form stress based topology optimization to be less than that of layouts from compliance based formulations.

Figure 5 Optimal material distribution of cantilever beam under stress based formulation (a, b and c) and compliance based formulation (d, e, and f)

The maximum stress induced in the design domains defined is higher when the problems are solved using compliance based formulations except the multiple loading case as shown in Fig.6a, Fig.6c and Fig.6e. This shows, the solutions and formulations of a problem should be based on the type of design domain defined, boundary and loading conditions.

Fig. 6b, Fig. 6d and Fig. 6f shows variation of compliance for a cantilever beam under stress and compliance based formulations considering the loading and boundary conditions defined in Fig 4a, Fig4b and Fig.4c, respectively. For the first two case studies considered in this section, the design domains have higher compliance when the problems are formulated using stress based formulation. But for the third case study considered the change in the compliance of the design domain under the two types of formulation decreases at the problem converges to its optimal solution. From which we can conclude that the selection problem formulation technique must be selected carefully ahead of generating optimal solutions.

(a) Maximum stress vs iteration for classical Cantilever beam
(b) Compliance vs iteration for a classical cantilever beam
(c) Maximum stress vs iteration for Cantilever beam with pre-defined shape
(d) Compliance vs iteration for a cantilever beam with predefined shape
From the simulation results of different cases considered in this paper, optimal material distribution of design domains from compliance based formulation have complex optimal layout than stress based formulations. The variation of compliance and maximum stress induced in the design domain are dependent on the type of design domain and loading conditions defined. For designs where stress is main design constraint, most of design using ductile material, optimal layouts from weight minimization results will be preferable.

5. Conclusion

Based on the benchmark case studies considered and simulation results, both the formulations can generate safe optimal material distribution. For all case studies considered, the maximum stress induced and compliance are dependent on type of problem considered. In stress based topology optimization, material within the design domain is distributed in such a way that it can sustain the applied loads under defined boundary and loading conditions in a best way. In classical compliance minimization problem, the optimal volume fraction, which is defined by the end user, is one of the key input variables. This makes the optimal material distribution and induced stress within the material to be highly dependent on the knowledge and experience of the end-user.

Adjustment of convergence criterions can be an alternative solution to minimize the maximum stress developed in optimal layouts. The effect of convergence criterion on other physical and mechanical properties and advantage of adjusting the convergence criterion over post processing can be studied as a future work. In addition, the effect of algorithms on the optimal layouts, stress distribution, compliance of design domain and maximum stress induced in the design domain, and alternatively a multi objective optimization which includes both objective functions under their respective design constraints can be studied further.

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