HARDY’S THEOREM FOR COMPACT LIE GROUPS

S. THANGAVELU

Abstract. We show that Hardy’s uncertainty principle can be reformulated in such a way that it has an analogue even for compact Lie groups and symmetric spaces of compact type.

1. Introduction

By Hardy’s theorem we refer to the following result proved by Hardy [2] in 1933 on Fourier transform pairs: if a nontrivial function $f$ and its Fourier transform $\hat{f}$ on $\mathbb{R}$ satisfies the conditions

$$|f(x)| \leq Ce^{-ax^2}, \quad |\hat{f}(y)| \leq Ce^{-by^2}$$

for $a, b > 0$ then necessarily $ab \leq 1/4$. In other words, if the above estimates are valid for any pair with $ab > 1/4$ then $f$ has to be identically zero. An analogue of this result is true on $\mathbb{R}^n$ also and it is informative to state the result in terms of the heat kernel

$$p_t(x) = (2\pi t)^{-n/2}e^{-\frac{1}{4t}|x|^2}, \quad x \in \mathbb{R}^n$$

associated to the Laplacian $\Delta$ on $\mathbb{R}^n$. If

$$\hat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$$

then it is well known that $\hat{p}_t(\xi) = e^{-t|\xi|^2}$. Hardy’s theorem on $\mathbb{R}^n$ takes the following form: if

$$|f(x)| \leq Cp_s(x), \quad |\hat{f}(\xi)| \leq C\hat{p}_t(\xi)$$

where $0 < s < t$ then $f = 0$; when $s = t$, $f = Cp_t$.

The heat kernel version of Hardy’s theorem has been extended to Fourier transforms on Lie groups such as non-compact semisimple Lie groups, Heisenberg groups and also for the Helgason Fourier transform on non-compact Riemannian symmetric spaces. In all these cases if

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is the heat kernel associated to the suitable Laplacian (or sublaplacian) on the group then estimates of the form (1.1) on \( f \) and its Fourier transform with \( s < t \) always lead to the conclusion \( f = 0 \). See \[12\] and the references there for the history of Hardy’s theorem in various setups. The equality case \( s = t \) remains open in many situations and fails to be true (e.g. \( G = SL(2, \mathbb{R}) \)) in some cases.

But what about Hardy’s theorem on compact Lie groups? Well, when \( K \) is a such a group with \( p_t \) the heat kernel associated to the Laplacian \( \Delta \), the condition \( |f(k)| \leq Cp_s(k) \) does not say much as any bounded function satisfies the above condition for any \( s > 0 \) since \( K \) is compact and \( p_s(k) > 0 \). Eventhough the condition ‘\( |\hat{f}(\xi)| \leq C\hat{p}_t(\xi) \)’ has a natural analogue, due to the lack of information from \( |f(k)| \leq Cp_s(k) \), it looks as though Hardy’s theorem has no analogue on compact Lie groups.

Nevertheless, let us turn around and have a fresh look at Hardy’s theorem- or rather the proof of the theorem on \( \mathbb{R} \). The second condition, namely the one on \( \hat{f} \) allows us to extend \( f \) as an entire function to the whole of \( \mathbb{C} \) and we also get the estimate

\[
|f(x + iy)| \leq Ce^{\frac{1}{4}y^2}, \quad x, y \in \mathbb{R}.
\]

The entire function \( f(z) \) when restricted to \( \mathbb{R} \) satisfies the Gaussian decay \( |f(x)| \leq Ce^{-ax^2} \). By considering the equality case \( ab = 1/4 \) Hardy concluded by a clever application of Phragmen-Lindelof theorem that \( f(z) = Ce^{-az^2} \) and the case \( ab > 1/4 \) follows from this immediately. Thus we see that the growth of the entire function \( f \) and the decay of the Fourier transform of its restriction to \( \mathbb{R} \) are related.

Viewing Hardy’s theorem in the light of the above observation allows us to formulate a version for compact Lie groups.

2. Hardy’s theorem for compact Lie groups

Let \( K \) be any compact connected Lie group with Lie algebra \( k \). Any inner product on \( k \) which is invariant under the adjoint action of \( K \) determines a biinvariant Riemannian metric on \( K \). Let \( \Delta \) be the Laplace-Beltrami operator associated with this Riemannian metric and let \( p_t \) stand for the kernel of the semigroup \( e^{-t\Delta} \) generated by \( \Delta \). We fix a Haar measure \( dk \) on \( K \). Let \( \hat{K} \) stand for the unitary dual of \( K \). Then for any \( \pi \in \hat{K} \) the matrix coefficients \( \pi_{ij}(k) \) of \( \pi \) are eigenfunctions of
In particular, the character $\chi_{\pi}$ is an eigenfunction of $\Delta$. Let us denote the corresponding eigenvalues by $\lambda_{\pi}^2$ and let $d_{\pi}$ stand for the dimension of the representation $\pi$. The heat kernel $p_t$ has the following expansion:

$$p_t(k) = \sum_{\pi \in \hat{K}} d_{\pi} e^{-\lambda_{\pi}^2 t} \chi_\pi(k).$$

Note that for $f \in L^2(K)$ the function $u(k, t) = f \ast p_t(k)$ solves the heat equation $\partial_t u(k, t) = -\Delta u(k, t)$ with initial condition $u(k, 0) = f(k)$.

Let $G = K_\mathbb{C}$ be the universal complexification of $K$ with Lie algebra $k + i k$. Then $K$ is a maximal compact subgroup of $G$ and $X = G/K$ is a non-compact Riemannian symmetric space. Let $\Delta_X$ be the Laplace-Beltrami operator on $G/K$ with associated heat kernel $q_t(g)$. It is a $K -$biinvariant function on $G$. By the polar decomposition, every element of $G$ can be written as $g = ke^{iY}$ where $Y \in k$. By $|Y|$ we denote the norm of $Y$ defined by the $Ad$-$K$ invariant inner product on $k$. We are now ready state a version of Hardy’s theorem for $K$.

**Theorem 2.1.** For a function $f \in L^2(K)$ let us set

$$b = \sup\{t \geq 0 : \|\pi(f)\|_{HS} \leq Ce^{-t\lambda_{\pi}^2}, \forall \pi \in \hat{K}\}.$$ 

Suppose $f$ has a holomorphic extension to $G$ and satisfies the estimate

$$|f(ke^{iY})|^2 \leq Cp_{2a}(e^{2iY})$$

for all $k \in K$ and $Y \in k$. Then $f = 0$ whenever $a > b$.

**Proof.** Let $dg$ is the Haar measure on the complex group $G$. We first show that

$$\int_G |f(g)|^2 q_{t/2}(g) dg < \infty$$

for every $0 < t < a$. To prove this we make use of several results: an explicit formula for the heat kernel $q_t$ due to Gangolli [3], a theorem of Hall on Segal-Bargmann transforms [4] and a Gutzmer’s formula due to Lassalle [7], [8]. In order state Gangolli’s formula we need to recall some facts about $G$. Let $T$ be a maximal torus in $K$ with Lie algebra $t$. We identify $t^*$ with $k$ via the inner product on $k$ restricted to $t$. Let $R$ be the set of all real roots, $R^+$ the set of all positive roots and $\rho$ be the half sum of positive roots.
The polar decomposition of $G$ reads as $g = k_1 e^{iH} k_2$ where $k_1, k_2 \in K$ and $H \in \mathfrak{t}$. Then for $g = k_1 e^{iH} k_2$ we have

$$q_t(g) = (4\pi t)^{-n/2} e^{-4t|\rho|^2} e^{-\frac{|H|^2}{4t}} \Pi_{\alpha \in R^+} \frac{(\alpha, H)}{\sinh(\alpha, H)}.$$ 

If we let $\Phi$ stand for the unique $Ad$-$K$ invariant function on $k$ which coincides with $\Pi_{\alpha \in R^+} \frac{(\alpha, H)}{\sinh(\alpha, H)}$ on $t$ then we can write the formula as

$$q_t(k e^{iY}) = (4\pi t)^{-n/2} e^{-4t|\rho|^2} e^{-\frac{|H|^2}{4t}} \Phi(Y), \quad k \in K, Y \in \mathfrak{k}.$$ 

Since $0 < t < a$, $p_a(g) = p_{a-t} * p_t(g)$ and by the theorem of Hall we have

$$\int_G |p_a(g)|^2 q_{t/2}(g) dg = c \int_K |p_{a-t}(k)|^2 dk.$$ 

The expression for the Haar measure on $G$ in polar coordinates is given by

$$dg = c_n \Phi(Y)\,dY \,dk$$

where $dY$ is the Lebesgue measure on the Lie algebra $\mathfrak{k}$. Integrating in polar coordinates, the above leads to

$$\int_K \int_{\mathfrak{k}} |p_a(k e^{iY})|^2 q_{t/2}(k e^{iY}) \Phi(Y)^{-2} dY \,dk < \infty.$$ 

Since the heat kernel $q_t$ is $K$-biinvariant, we can rewrite the above as

$$\int_K \int_{\mathfrak{k}} \left(\int_{K \times K} |p_a(uke^{iY}v)|^2 dudv\right) q_{t/2}(k e^{iY}) \Phi(Y)^{-2} dY \,dk.$$ 

We can now evaluate the inner integral using Lassalle’s formula. Recall that we can write $Y = Ad(k_1)H, H \in \mathfrak{t}$ for some $k_1 \in K$ and hence the inner integral above is given by

$$\int_{K \times K} |p_a(uke^{iY}v)|^2 dudv = \int_{K \times K} |p_a(ue^{iH}v)|^2 dudv$$

which by Lassalle’s formula (Theorem 1 in Section 9 of [8]) is equal to

$$\sum_{\pi \in \hat{K}} e^{-2\alpha^2} \chi_{\pi}(e^{2iH}) = p_{2a}(e^{2iY})$$

where we have used the fact that $p_a$ is a class function. The finiteness of the integral in (2.2) shows that

$$\int_K \int_{\mathfrak{k}} p_{2a}(e^{2iY}) q_{t/2}(k e^{iY}) \Phi(Y)^{-2} dk dY < \infty.$$ 

The hypothesis on $f$ now shows that the integral in (2.1) is finite proving our claim. Once again we appeal to the theorem of Hall to conclude that $f = h * p_t$ for some $h \in L^2(K)$. But this means that

$$\pi(f) = e^{-t\lambda^2} \pi(h)$$

and hence by the definition of $b$ we conclude that
\( t \leq b \). As this is true for any \( t < a \) we get \( a \leq b \) which proves the theorem.

\[ \square \]

By using known estimates on the heat kernel \( p_t(g) \) we can restate the above theorem in a more familiar form.

**Theorem 2.2.** For a function \( f \in L^2(K) \) let us set
\[
 b = \sup \{ t \geq 0 : \| \pi(f) \|_{HS} \leq C e^{-t\lambda^2}, \forall \pi \in \hat{K} \}. 
\]
Suppose \( f \) has a holomorphic extension to \( G \) and satisfies the estimate
\[
 |f(ke^{iY})| \leq C e^{a|Y|^2} 
\]
for all \( k \in K \) and \( Y \in k \). Then \( f = 0 \) whenever \( ab < 1/4 \).

**Corollary 2.3.** Suppose \( F \) is a holomorphic function on \( G = K_C \) which is of exponential type, i.e., it satisfies
\[
 |F(ke^{iY})| \leq C e^{a|Y|}, \ k \in K, Y \in k. 
\]
Let \( f \) be the restriction of \( F \) to \( K \). Then \( \| \pi(f) \|_{HS} \leq C e^{-t\lambda^2} \), \( \forall \pi \in \hat{K} \) for all \( t > 0 \).

**Remark 2.4.** In the equality case of Theorem 2.1, that is when \( a = b \), we cannot conclude that \( f = c p_a \). To see this, consider the function \( f \) on \( K = S^1 \) whose Fourier expansion is given by
\[
 f(x) = \sum_{k=-\infty}^{\infty} e^{-c|k|-ak^2} e^{ikx}. 
\]
Then it can be easily checked that the holomorphic extension of \( f \) to \( \mathbb{C} \) satisfies
\[
 |f(x + iy)|^2 \leq C p_{2a}(2iy). 
\]
However, from the proof of the above theorem we see that when \( a = b \) we can write \( f \) as \( h \ast p_t \) for any \( t < a \).

**Remark 2.5.** An analogue of Theorem 2.1 is true for any Riemannian symmetric space of compact type. In [11] Stenzel has studied the Segal-Bargmann transform on compact symmetric spaces and proved an analogue of Hall’s theorem. If \( p_t \) is the heat kernel associated to the Laplace-Beltrami operator on a compact symmetric spaces \( X \) then the image of \( L^2(X) \) under the Segal-Bargmann transform is a weighted Bergman space. The weight function is given by \( q_{2t}(e^{2iH}) \) where \( q_t \) is the heat kernel associated to the Laplace-Beltrami operator on the noncompact dual of \( X \), see Faraut [6] and also [13]. We leave the formulation and proof of a Hardy’s theorem on \( X \) to the interested reader.
3. HARDY’S THEOREM ON EUCLIDEAN SPACES REVISITED

For functions on $\mathbb{R}^n$ the assumption $|\hat{f}(\xi)| \leq Ce^{-b|\xi|^2}$ leads to the estimate $|f(x + iy)| \leq Cp_a(iy)$ where $p_t$ is the Euclidean heat kernel. However, due to the noncompactness of $\mathbb{R}^n$ these two estimates cannot be used as in the case of compact Lie groups to prove a version of Hardy’s theorem. On the other hand, when $g$ is a Schwartz function the holomorphic extension of $f = g * p_a$ satisfies the estimates

$$|f(x + iy)|^2 \leq C_m (1 + |x|^2 + |y|^2)^{-m}p_{2a}(2iy)$$

for every non-negative integer $m$ as shown by Bargmann in [1]. This allows us to prove the following version of Hardy’s theorem.

**Theorem 3.1.** Let $b = \sup\{t \geq 0 : |\hat{f}(\xi)| \leq Ce^{-t|\xi|^2}\}$ for a Schwartz class function $f$. Assume that $f$ has a holomorphic extension to $\mathbb{C}^n$ and satisfies the estimates

$$|f(x + iy)|^2 \leq C_m (1 + |x|^2 + |y|^2)^{-m}p_{2a}(2iy)$$

for every non-negative integer $m$. Then for $f = 0$ whenever $a > b$.

**Proof.** The proof depends on Bargmann’s theorem which states that an entire function $f$ satisfies the hypothesis of the theorem if and only if it is of the form $g * p_a$ for a Schwartz function $g$. But then the hypothesis on $\hat{f}$ forces $a$ to be at most $b$. $\square$

The following $L^2$ version of the above result can be considered as a reformulation of Cowling-Price theorem (see [12]).

**Theorem 3.2.** Let $b = \sup\{t \geq 0 : |\hat{f}(\xi)| = g_t(\xi)e^{-t|\xi|^2}, g_t \in L^2(\mathbb{R}^n)\}$ for a function $f \in L^2(\mathbb{R}^n)$. Assume that $f$ has a holomorphic extension to $\mathbb{C}^n$ and satisfies the estimate

$$\int_{\mathbb{R}^n} |f(x + iy)|^2 dx \leq Cp_{2a}(2iy).$$

Then for $f = 0$ whenever $a > b$.

**Proof.** We only need to observe that for any $t < a$, $f(x + iy)$ is square integrable with respect to $p_{2a}(2iy)$ and hence by the theorem of Segal and Bargmann we have $f = h_t * p_t$ for some $h_t \in L^2(\mathbb{R}^n)$. $\square$

We also have the following theorem which can be considered as a reformulation of Hardy’s theorem for Hermite expansions. By rescaling, the decay rate $b$ of the Fourier transform of the function $f$ can be assumed to be at most $1/2$. With this modification we have the following
result. We let $H$ stand for the Hermite operator $-\Delta + |x|^2$ and write its spectral projection as $H = \sum_{k=0}^{\infty} (2k + n) P_k$, see [14].

**Theorem 3.3.** Let $b = \sup \{t > 0 : |\hat{f}(\xi)| \leq C e^{-\frac{1}{2}(\tanh 2t)|\xi|^2} \}$ for a function $f \in L^2(\mathbb{R}^n)$. Assume that $f$ has a holomorphic extension to $\mathbb{C}^n$ and satisfies the estimates

$$ |f(x + iy)| \leq C e^{-\frac{1}{2}(\tanh 2a)|x|^2 + \frac{1}{2}(\coth 2a)|y|^2}. $$

Then for $f$ to be nontrivial it is necessary that $a \leq b$. Moreover, for every $t < a$ the Hermite projections of $f$ have the decay $\|P_k f\|_2 \leq C e^{-(2k+n)t}$.

**Proof.** For the proof we need the following characterisation of the image of $L^2(\mathbb{R}^n)$ under the Hermite semigroup. If $f \in L^2(\mathbb{R}^n)$ then $e^{-tH} f$ extends to $\mathbb{C}^n$ as a holomorphic function and we have the identity

$$ \int_{\mathbb{C}^n} |e^{-tH} f(x + iy)|^2 U_t(x, y) dxdy = \int_{\mathbb{R}^n} |f(x)|^2 dx $$

where the weight function $U_t$ is given explicitly by

$$ U_t(x, y) = c_n (\sinh(2t))^{-n/2} e^{\frac{1}{2}(\tanh 2t)|x|^2 - \frac{1}{2}(\coth 2t)|y|^2}. $$

And the converse is also true, see [14]. Under our assumption on $f$ we see that for any $t < a$ the holomorphic function $f(x + iy)$ is square integrable with respect to $U_t(x, y) dxdy$ and hence $f(x) = e^{-tH} g(x)$ for some $g \in L^2(\mathbb{R}^n)$. As $e^{-tH}$ commutes with the Fourier transform we have $\hat{f} = e^{-tH} \hat{g}$. In [9] the authors have shown that the holomorphic function $\hat{f}(x + iy) = e^{-tH} \hat{g}(x + iy)$ satisfies the estimate

$$ |\hat{f}(x + iy)| \leq C e^{-\frac{1}{2}(\tanh 2s)|x|^2 + \frac{1}{2}(\coth 2s)|y|^2} $$

for any $s < t$. Consequently, we get $s < b$ and as this is true for any $s < t < a$ we conclude that $a \leq b$ as claimed. \[\square\]

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department of mathematics, Indian Institute of Science, Bangalore - 560 012, India

E-mail address: veluma@math.iisc.ernet.in