CAUSALITY IN STRONG SHEAR FLOWS

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ABSTRACT

It is well known that the standard transport equations violate causality when gradients are large or when temporal variations are rapid. We derive a modified set of transport equations that satisfy causality. These equations are obtained from the underlying Boltzmann equation. We use a simple model for particle collisions which enables us to derive moment equations non-perturbatively, i.e. without making the usual assumption that the distribution function deviates only slightly from its equilibrium value. We also retain time derivatives of various moments and choose closure relations so that the final set of equations are causal. We apply the model to two problems: particle diffusion and viscous transport. In both cases we show that signals propagate at a finite speed and therefore that the formalism obeys causality. When spatial gradients or temporal variations are small, our theory for particle diffusion and viscous flows reduces to the usual diffusion and Navier-Stokes equations respectively. However, in the opposite limit of strong gradients the theory produces causal results with finite transport fluxes, whereas the standard theory gives results that are physically unacceptable. We find that when the velocity gradient is large on the scale of a mean free path, the viscous shear stress is suppressed relative to the prediction of the standard diffusion approximation. The shear stress reaches a maximum at a finite value of the shear amplitude and then decreases as the velocity gradient increases. The decrease of the stress in the limit of large shear is qualitatively different from the case of scalar particle diffusion where the diffusive flux asymptotes to a constant value in the limit of large density gradient. In the case of a steady Keplerian accretion disk with hydrodynamic turbulent viscosity, the stress-limit translates to an upper bound on the Shakura-Sunyaev $\alpha$-parameter, namely $\alpha < 0.07$. The limit on $\alpha$ is much stronger in narrow boundary layers where the velocity shear is larger than in a Keplerian disk.

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1. INTRODUCTION

According to the standard theory of diffusion the particle flux increases linearly with the gradient of the particle density. Thus, the theory predicts that a sharp gradient will result in a divergent flux. This violates causality, since the particle flux is obviously limited by the finite particle speed. The standard theory runs into a similar problem when the particle density changes on a very short time scale locally (e.g. Morse & Feshbach 1953, Narayan 1992).

A similar limitation is well-known in viscous interactions. In this case the viscous stress tensor is linearly proportional to the velocity gradient in the diffusion approximation. When this relation is used in the Navier-Stokes equation one finds an instantaneous propagation of viscous signals, in violation of causality. Since the particle speed does not enter explicitly in the underlying theory, the standard equations cannot be modified in a straightforward way to limit the signal propagation speed.

The diffusion equation and the Navier-Stokes equation are valid only when particles have suffered many collisions and their distribution has relaxed to have weak spatial gradients and slow temporal variations. However, there are physical situations both in the laboratory and in astrophysical gases where gradients are large on the scale of a collisional mean free path or temporal changes are rapid relative to the mean collision time. Examples include radiation hydrodynamics in optically thin media (Levermore and Pomraning 1981), electron heat transport in laser produced plasma (Max 1981), and viscous angular momentum transport in boundary layers of accretion disks (Popham & Narayan 1992). One needs to go beyond the standard fluid equations to model such conditions.

Specific prescriptions have been proposed to incorporate causality in individual problems. In radiation hydrodynamics, Levermore and Pomraning (1981) introduced a flux-limiter such that, regardless of the gradient, the radiative flux is never permitted to exceed the product of the radiation energy density and the speed of light. In the accretion disk problem, Narayan (1992) proposed a modifying factor for the viscosity coefficient, such that in steady state flows the viscosity vanishes when the flow speed exceeds the maximum random speed of the particles.

While the above prescriptions have worked well in their individual applications, it would be useful to derive a general formalism that automatically yields a causal limit to different transport phenomena. We present such a formalism in this paper. We base our work on the Boltzmann equation which is strictly causal. We make a simple non-perturbative approximation to the scattering term in the Boltzmann equation, take successive moments, and use appropriate closure relations. The standard diffusion approximation is then recovered if we neglect certain terms involving time derivatives. This approximation is valid
if temporal variations are slow compared to the collision time of particles, and spatial
gradients are small, but it breaks down whenever there are rapid variations and the flux
is limited by causality. When we retain all terms in the moment equations, we obtain a
causal set of equations.

The technical discussion in the paper is divided into two main sections. In §2 we
discuss the diffusion of particles in a fixed background, while in §3 we consider the stress
tensor.

The diffusion problem provides a simple test-bed for our approximations since it in-
volves both temporal and spatial variations of the gas properties. We introduce the basic
features of our approach in §2.1 where we discuss particle diffusion in one dimension. There
are two main simplifications which permit us to obtain a set of closed causal equations
in this case. First, we write the scattering term of the Boltzmann equation in a simple
non-perturbative form. This explicit form allows us to take moments of the Boltzmann
equation and to obtain equations that are valid for arbitrarily strong gradients in the par-
ticle density. This particular approximation of the scattering term is used throughout the
paper and is a key ingredient of our approach. Second, we close the equations by assuming
that the second velocity moment of the distribution function is constant. In §2.2 we explore
the properties of our one-dimensional diffusion equation and verify the validity of these
assumptions by comparing results with numerical tests. We then generalize the theory to
three-dimensional diffusion in §§2.3 and 2.4.

Section 3 describes the viscous transport of momentum in the presence of strong
velocity gradients and presents the main results of this paper. In §3.1 we derive a causal
equation for the evolution of the stress tensor. We use the same approximation to the
scattering term that was employed in §2. However, since the elements of the stress tensor
are themselves second moments of the distribution function, we close the equations at
the level of the third moments rather than the second moments. We use the simplest
approximation allowed, assuming that all third moments vanish. The causal equations we
thus derive reduce to the Navier-Stokes equation whenever the velocity gradients are weak,
but can also be used when the gradients are large. We apply the new equations to a number
of problems involving a steady state shear (§3.2–§3.5), and show that in the presence of
a large velocity gradient the viscous stress is suppressed compared to the prediction of
the standard diffusion approximation. We consider the effect of steady advection on the
viscous shear stress in §3.4 and discuss bulk viscosity in §3.5. In §3.6 we extend our analysis
to rotating flows. The presence of a Coriolis force introduces the epicyclic frequency into
the problem, and this leads to a generalization of our formula for the modified shear stress.
In §3.7 we discuss the implications of this formula for accretion disks and show that the \( \alpha \)
parameter introduced by Shakura and Sunyaev (1973) has a strict upper limit whenever the viscosity is mediated by hydrodynamic interactions. Finally, we summarize the main conclusions in §4.

Parts of this paper overlap previous work in the subject. The causal particle diffusion equations which we derive in equations (2.1.11) and (2.3.6) have appeared several times in the literature (Israel and Stewart 1980 and references therein, Schweizer 1984). Many of the previous discussions have been somewhat phenomenological whereas we derive the equations through a systematic procedure which reveals clearly the specific assumptions which we make. In particular, we discuss the limits of the diffusion theory and identify exactly which phenomena are described well by the theory and which are not. Our discussion of the shear problem, especially the effects of strong shear and advection, appears to be largely new. Some of the results on rotating flows have been derived independently by Kato and Inagaki (1993) whose preprint we received at a late stage of our work.
2. PARTICLE DIFFUSION

2.1 The One-Dimensional Problem

We introduce our notation and explain our basic ideas and approximations by discussing first a one-dimensional problem. Consider a gas of light particles diffusing in a fixed background of much heavier particles. Let the light particles be described by a distribution function $f(t, x, v)$ where $t$ is time, $x$ is particle position, and $v$ is particle velocity. We define various moments of $f$ in the usual way:

$$n = \int fdv, \quad \overline{v} = \frac{1}{n} \int vf dv, \quad \overline{v^2} = \frac{1}{n} \int v^2 f dv.$$  \hfill (2.1.1)

We assume that the particles experience no accelerations in between scatterings.

The distribution function $f$ satisfies a Boltzmann equation of the form

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(vf) = \Gamma_{\text{scatt}} + \dot{f}_s,$$  \hfill (2.1.2)

where $\Gamma_{\text{scatt}}$ describes the effect of scattering and $\dot{f}_s(t, x, v)$ is the rate at which particles are introduced or removed from a phase space element. Let us write $\Gamma_{\text{scatt}}$ as

$$\Gamma_{\text{scatt}} \equiv \Gamma^+ - \Gamma^- = \frac{1}{\tau}(f_0 - f).$$  \hfill (2.1.3)

The term $\Gamma^- = -f/\tau$ represents the rate at which particles are removed from a phase space element, where $\tau(v)$ is the mean free time which is in general a function of velocity. $\Gamma^+ = f_0/\tau$ describes the phase space distribution of the particles after they are scattered. Provided we make $f_0$ a function of $f$ and normalize $f_0$ so as to conserve particles, equation (2.1.3) will always be true. Note that this equation does not imply that $f$ is close to $f_0$ in any way. Indeed many of the cases we consider in this paper involve highly non-linear situations where $f$ is very different from $f_0$. In this respect we differ from the usual approach to transport theory where one expands $f$ around an equilibrium $f_0$ and assumes that the deviations are small (see e.g. Krook’s equation [Shu 1992]).

In general, the scattering is a complicated function of velocity which makes the Boltzmann equation very difficult to handle. A major simplification is achieved if we make two approximations (see Appendix B of Grossman, Narayan, and Arnett 1993, also Kato and Inagaki 1993). First, we assume that $\tau$ is a constant, independent of velocity. Second, we make some simplifying assumptions regarding the post-scattering distribution function $f_0$, viz. we assume that each scattering is elastic in the frame of the fixed background, that
it conserves particles, and that it leads to a total randomization of the initial velocities. This allows us to write simple relations for the moments of $f_0$:

$$\int f_0 dv = \int f dv = n, \quad \int v f_0 dv = 0, \quad \int v^2 f_0 dv = n\overline{v^2}. \quad (2.1.4)$$

In the same spirit, we assume that the source function $\dot{f}_s$ has zero mean velocity, and write

$$\int \dot{f}_s dv = s(t, x), \quad \frac{1}{s} \int v \dot{f}_s dv = 0, \quad \frac{1}{s} \int v^2 \dot{f}_s dv = \overline{v^2}. \quad (2.1.5)$$

With the above assumptions, we now take the first two moments of equation (2.1.2). Integrating equation (2.1.2) over $v$, we obtain

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\overline{v}) = s. \quad (2.1.6)$$

Multiplying equation (2.1.2) by $v$ and integrating over $v$, we obtain

$$\frac{\partial}{\partial t}(n\overline{v}) + \frac{\partial}{\partial x}(n\overline{v^2}) = -\frac{1}{\tau} n\overline{v}. \quad (2.1.7)$$

Equations (2.1.6) and (2.1.7) are two coupled equations describing the evolution of the particle moments. Unfortunately, these equations involve three different moments, $n$, $\overline{v}$ and $\overline{v^2}$. Therefore, we need a closure relation. We make the simplest assumption possible, namely that the velocity dispersion of the particles, $\overline{v^2}$, is a constant, independent of $x$ and $t$:

$$\overline{v^2} \equiv \sigma^2 = \text{constant}. \quad (2.1.8)$$

This condition is exactly valid if all particles, including those created by the source function $\dot{f}_s$, have a single speed, $v = \pm \sigma$, and provided the scattering is elastic. In a more general situation, this condition may not be satisfied but one might hope that it will perhaps still capture the essential features of particle transport phenomena. We discuss later the degree to which the approximation (2.1.8) is valid and where it breaks down. Equation (2.1.7) now becomes

$$\frac{\partial}{\partial t}(n\overline{v}) + \sigma^2 \frac{\partial n}{\partial x} = \frac{1}{\tau} n\overline{v}. \quad (2.1.9)$$

Equations (2.1.6) and (2.1.9) provide a closed pair of equations for the two moments $n$ and $\overline{v}$.

If we ignore the time derivative in equation (2.1.9) we get the usual formulation of diffusion, where the diffusive flux $n\overline{v}$ is given by $-D \partial n / \partial x$, with a diffusion constant $D = \sigma^2 \tau$. In this approximation $n$ satisfies the standard diffusion equation

$$\frac{\partial n}{\partial t} - D \frac{\partial^2 n}{\partial x^2} = s, \quad D = \sigma^2 \tau. \quad (2.1.10)$$
As is well-known, this equation violates causality (e.g. Morse & Feshbach 1953, Narayan 1992).

If, on the other hand, we retain the time derivative in (2.1.9) we obtain a modified diffusion equation which does preserve causality. Differentiating equation (2.1.9) with respect to \( x \) and combining with equation (2.1.6) we obtain

\[
\frac{\partial n}{\partial t} - \sigma^2 \tau \left( \frac{\partial^2 n}{\partial x^2} - \frac{1}{\sigma^2} \frac{\partial^2 n}{\partial t^2} \right) = s + \tau \frac{\partial s}{\partial t}.
\] (2.1.11)

This equation has been derived previously in the literature using the theory of “transient thermodynamics” (Israel and Stewart 1980, Schweizer 1984). The most interesting feature of equation (2.1.11) is the presence of the wave operator on the left hand side, which ensures that signals cannot propagate faster than the r.m.s. particle speed \( \sigma \).

The Green’s function \( G_1(t, x) \) of the one-dimensional diffusion equation (2.1.11) is obtained by solving the equation

\[
\frac{\partial G_1}{\partial t} - \sigma^2 \tau \left( \frac{\partial^2 G_1}{\partial x^2} - \frac{1}{\sigma^2} \frac{\partial^2 G_1}{\partial t^2} \right) = \left[ \delta(t) + \tau \frac{d}{dt} \delta(t) \right] \delta(x).
\] (2.1.12)

Morse and Feshbach (1953) have given the solution when the right hand side consists only of the first term in the square brackets. Modifying their solution for our case (Schweizer 1984, Nagel and Mészáros 1985), we obtain

\[
G_1(t, x) = \left[ \frac{1}{4\sigma \tau} \left\{ I_0(u) + \frac{t}{2\tau} \frac{I_1(u)}{u} \right\} + \frac{1}{2} \left\{ \delta(x - \sigma t) + \delta(x + \sigma t) \right\} \right] e^{-t/2\tau}, \ |x| \leq \sigma t,
\]

\[
= 0 \quad \text{, } \ |x| > \sigma t,
\] (2.1.13)

where \( I_0 \) and \( I_1 \) are modified Bessel functions and

\[
u = \frac{1}{2\tau} \left( t^2 - \frac{x^2}{\sigma^2} \right)^{1/2}.
\] (2.1.14)

Note that the Green’s function cuts off for \( |x| > \sigma t \). This demonstrates explicitly that equation (2.1.11) satisfies causality with a maximum propagation speed of \( \sigma \).

Equation (2.1.11) and the Green’s function (2.1.13), which are derived under several approximations, represent the physical situation exactly in the particular case when all particles have the same speed \( \sigma \) and the same mean free time \( \tau \). The \( \delta \) functions and the causal fronts at \( x = \pm \sigma \tau \) in equation (2.1.13) are of course a consequence of the mono-speed assumption. The generalization of the Green’s function for an arbitrary distribution of
speeds, $f(|v|)$, and an arbitrary dependence of the mean free time, $\tau(|v|)$, is straightforward and requires merely integrating equation (2.1.13) over the particular distributions*.

One effect that is not described by equations (2.1.11) and (2.1.13) is the smoothing of density inhomogeneities by phase mixing (see e.g. Binney & Tremaine 1987). Particles with different velocities travel different distances within a collision period and therefore wash out inhomogeneities. The damping of inhomogeneities by free streaming of particles is not described by equation (2.1.11) since it corresponds to particles with a single speed. However, for elastic collisions this problem can be easily fixed by integrating the Green’s function (2.1.13) over the velocity distribution of particles.

Another way to improve the description of particle diffusion is to drop the assumption of a constant $v^2$ (which crudely speaking corresponds to an isothermal system) in the moment equations. We can treat $v^2$ as an independent moment and solve for it by considering higher moments of the Boltzmann equation. This extension of the theory describes both particle and heat diffusion. We consider up to third moments of the Boltzmann equation, and use a closure on the fourth moments of the form $\overline{v^4} = \zeta (\overline{v^2})^2$ with $\zeta$ equal to some constant (e.g. $\zeta = 3$ for a Gaussian distribution). Let us generalize equations (2.1.4) and (2.1.5) suitably for the higher moments,

$$\int v^3 f_0 dv = 0, \quad \int v^4 f dv = \zeta n(\overline{v^2})^2,$$

$$\frac{1}{s} \int v^3 f_s dv = 0.$$

We then obtain

$$\frac{\partial}{\partial t} (n \overline{v^2}) + \frac{\partial}{\partial x} (n \overline{v^3}) = s \overline{v^2}_s,$$  \hspace{1cm} (2.1.17)

$$\frac{\partial}{\partial t} (n \overline{v^3}) + \zeta \frac{\partial}{\partial x} [n(\overline{v^2})^2] = -\frac{1}{\tau} n \overline{v^3}.$$  \hspace{1cm} (2.1.18)

By taking the $x-$derivative of equation (2.1.18) and substituting equation (2.1.17) one gets a diffusion equation for $n \overline{v^2}$,

$$\frac{\partial (n \overline{v^2})}{\partial t} - \tau \left( \zeta \frac{\partial^2 (n \overline{v^2})}{\partial x^2} - \frac{\partial^2}{\partial t^2} (n \overline{v^2}) \right) = \tau \frac{\partial (sv^2_s)}{\partial t} + (sv^2_s),$$

which is coupled to the diffusion equation for $n$,

$$\frac{\partial n}{\partial t} - \tau \left( \frac{\partial^2 [n \overline{v^2}]}{\partial x^2} - \frac{\partial^2 n}{\partial t^2} \right) = \tau \frac{\partial s}{\partial t} + s.$$  \hspace{1cm} (2.1.19)

* The conditions, however, are different in more complex gases, like plasmas. Collisionless Landau damping may dominate over collisional dissipation whenever the plasma properties undergo strong temporal or spatial variations.
Note that the limiting speed for the propagation of a pressure perturbation is greater by a factor $\sim \sqrt{\zeta}$ than that of a density perturbation.

Equations (2.1.19) and (2.1.20) provide a more accurate representation of diffusion than the simpler version of the theory presented earlier. Furthermore, these equations are causal as is evident from the wave operator on the left side. In fact, these equations also describe thermal diffusion since the quantity $n\overrightarrow{v}$ in equations (2.1.17) and (2.1.18) represents the flux of heat. We do not discuss these extended equations further in this paper but concentrate instead on the properties of the simpler equation (2.1.11).

2.2 Evaluating the Accuracy of the Approximate Theory

Figure 1 shows the Green’s function $G_1$ at two different times, $t = 0.1\tau$, and $10\tau$. For comparison, two other Green’s functions are also shown for each case. One is the Green’s function $G_s$ of the standard diffusion equation (2.1.10). The other is the Green’s function $G_M$ for a Maxwellian distribution of particles with an r.m.s. one-dimensional velocity $\sigma$. This is calculated by integrating equation (2.1.13) over a Maxwellian in $\sigma$, and assuming $\tau$ to be independent of $v$. At the detailed level of the shape of the Green’s function, neither equation (2.1.10) nor equation (2.1.11) does a particularly good job of fitting the Maxwellian Green’s function, particularly at early time. This is not surprising since the two theories have only one parameter ($D$) and two parameters ($\tau, \sigma^2$) respectively, and therefore they cannot describe the exact spatial distribution of particles with a continuous velocity distribution.

Although the shape of the Green’s function $G_1$ differs significantly from $G_M$, the spatial extent of $G_1$ is close to that of the Maxwellian Green’s function $G_M$ at all times (see figure 1). This is in contrast to the Green’s function $G_s$ of equation (2.1.10) which is much too wide at early times (the usual signature of acausal behavior). We now show that the mean square width $\langle x^2 \rangle$ of $G_1$ is, in fact, exactly equal to $\langle x^2 \rangle$ of $G_M$.

Let us define $\langle x^2 \rangle$ for $G_1$ as follows:

$$\langle x^2 \rangle = \int dx \, x^2 G_1(t,x), \quad \text{where } \int dx \, G_1(t,x) \equiv 1. \quad (2.2.1)$$

Multiplying equation (2.1.12) by $x^2$ and integrating over $dx$ we find

$$\frac{\partial \langle x^2 \rangle}{\partial t} + \tau \frac{\partial^2 \langle x^2 \rangle}{\partial t^2} = 2\sigma^2 \tau, \quad (2.2.2)$$

whose solution with the appropriate boundary conditions is

$$\langle x^2 \rangle = 2\sigma^2 \tau t + 2\sigma^2 \tau^2 [\exp(-t/\tau) - 1]. \quad (2.2.3)$$
At early times, i.e. for $t \ll \tau$, we see that $\langle x^2 \rangle = \sigma^2 t^2$ which corresponds to particles streaming freely with a speed $\sigma$. At late times, however, we have $\langle x^2 \rangle \rightarrow 2\sigma^2 \tau t$ which corresponds to the usual diffusion limit.

Since the Green’s function $G_1$ corresponds exactly to the case of particles with a single speed $\sigma$, equation (2.2.3) is the exact result for the evolution of $\langle x^2 \rangle$ for a mono-speed population of particles. The interesting point is that the expression for $\langle x^2 \rangle$ is directly proportional to $\sigma^2$. Therefore, any distribution of velocities which has a mean square velocity equal to $\sigma^2$ will evolve exactly according to equation (2.2.3). This equation is therefore valid more generally than for just mono-speed particles — all it requires is that all the particles should have the same mean free time.

If particles with different speeds do not have the same mean free time, then obviously we do not have a perfect correspondence between (2.2.3) and the exact result for $\langle x^2 \rangle$. However, even in this case we find that the theory performs quite well. As a demonstration of this result, suppose we have particles with a Maxwellian distribution of one-dimensional speeds,

$$f(v)dv = \sqrt{\frac{2}{\pi \sigma^2}} \exp \left( -\frac{v^2}{2\sigma^2} \right) dv, \quad v \geq 0,$$

and let us assume that all the particles have the same mean free path $l$. The mean free time is then a function of $v$,

$$\tau(v) = \frac{l}{v}.$$  \hspace{1cm} (2.2.5)

To obtain the evolution of $\langle x^2 \rangle$ for this problem, we replace $\sigma^2$ by $v^2$ in equation (2.2.3) and integrate over the Maxwellian $f(v)dv$. This gives

$$\langle x^2 \rangle = 2\sqrt{\frac{2}{\pi}} \sigma l t + 2l^2 \left[ \exp \left( \frac{\sigma^2 l^2}{2t^2} \right) \text{erfc} \left( \frac{\sigma t}{\sqrt{2l}} \right) - 1 \right].$$ \hspace{1cm} (2.2.6)

At early and late times this has the following asymptotic dependences: $\langle x^2 \rangle \rightarrow \sigma^2 t^2, \; t \ll \tau$; $\langle x^2 \rangle \rightarrow 2(2/\pi)^{1/2} \sigma lt, \; t \gg \tau$. We can fit these dependences with the single-speed result (2.2.3) provided we choose the mean free time $\tau$ to be

$$\tau = \sqrt{\frac{2}{\pi}} \frac{l}{\sigma}.$$ \hspace{1cm} (2.2.7)

Figure 2 shows a comparison between the exact result (2.2.6) and the approximate result (2.2.3) for this particular choice of $\tau$. The agreement is very good. In contrast, the standard diffusion equation (2.2.10), which gives $\langle x^2 \rangle = 2\sigma^2 \tau t$ for all $t$, clearly makes a large error for $t < \tau$. We expect our theory to give similar good agreement with exact results for
other distribution functions \( f(v) \) or prescriptions for the mean free time \( \tau(v) \), provided we define the mean free time \( \tau \) appropriately.

We thus conclude that the causal diffusion theory developed here provides a good representation of particle diffusion, and in particular it predicts the mean square distance traveled by particles very well at all times.

### 2.3 Particle Diffusion in Three Dimensions

We work in a Cartesian representation with position indicated by the coordinates \( \vec{r} = (x_1, x_2, x_3) \) and velocity by \( \vec{v} = (v_1, v_2, v_3) \). In the absence of any forces, but with scattering, the Boltzmann equation gives

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}(v_i f) = \Gamma_{\text{scatt}} + \dot{f}_s = \frac{1}{\tau} (f_0 - f) + \dot{f}_s. \tag{2.3.1}
\]

We use the summation convention on repeated indices. Following the one dimensional example of §2.1, we have written the scattering term simply in terms of a mean free time \( \tau \). We also make similar assumptions as in equations (2.1.4) and (2.1.5), viz.

\[
\begin{align*}
\int f_0 dv &= n, \\
\int v_i f_0 dv &= 0, \\
\int \dot{f}_s dv &= s, \\
\frac{1}{s} \int v_i \dot{f}_s dv &= 0.
\end{align*}
\]

Taking the zeroth and \( v_j \)th moments of (2.3.1) we obtain

\[
\begin{align*}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i} n \bar{v}_i &= s, \tag{2.3.2} \\
\frac{\partial}{\partial t} (n \bar{v}_j) + \frac{\partial}{\partial x_i} (n \bar{v}_i \bar{v}_j) &= - \frac{1}{\tau} n \bar{v}_j. \tag{2.3.3}
\end{align*}
\]

These are not a closed set of equations because they involve the six second moments \( \bar{v}_i \bar{v}_j \). To close the equations, we simplify the second moments by assuming that the velocities are isotropically distributed and that the mean square velocity is independent of \( t \) and \( \vec{r} \):

\[
\bar{v}_i \bar{v}_j = \frac{1}{3} \sigma^2 \delta_{ij}, \quad \sigma^2 = \text{constant}. \tag{2.3.4}
\]

This approximation is technically inconsistent, particularly at early times. While the mean square velocity itself can be constant, for instance in the case of a mono-speed population of particles, the assumption of isotropy is rather drastic and we may expect some pathological behavior in the solutions in certain limits. Nevertheless, the resulting theory is simple and causal, and therefore we explore it further. Substituting equation (2.3.4) into equation (2.3.3) we find

\[
\begin{align*}
\frac{\partial}{\partial t} (n \bar{v}_j) + \frac{\sigma^2}{3} \frac{\partial n}{\partial x_j} &= - \frac{1}{\tau} n \bar{v}_j. \tag{2.3.5}
\end{align*}
\]
Combining equations (2.3.2) and (2.3.5) we then obtain

\[
\frac{\partial n}{\partial t} - \frac{1}{3} \sigma^2 \tau \left( \nabla^2 n - \frac{3}{\sigma^2} \frac{\partial^2 n}{\partial t^2} \right) = s + \tau \frac{\partial s}{\partial t}.
\]  

(2.3.6)

As before, we find a wave operator appearing in the equation which enforces causality. Rather surprisingly, the relevant wave speed appears to be \(\sigma/\sqrt{3}\). This is however somewhat deceptive and the true speed turns out to be \(\sigma\) in certain situations as we show in §2.4. For reference we note that the standard diffusion equation in three dimensions is

\[
\frac{\partial n}{\partial t} - D \nabla^2 n = s, \quad D = \frac{1}{3} \sigma^2 \tau.
\]  

(2.3.7)

Morse and Feshbach (1953) show that the three-dimensional Green’s function \(G_3(t, r)\) of the operator in equation (2.3.6) can be derived from \(G_1(t, x)\) discussed in §2.1. The relation is

\[
G_3(t, r) = -\frac{1}{2\pi r} \frac{d}{dr} G_1(t, r),
\]  

(2.3.8)

with \(\sigma\) in equation (2.1.13) replaced by \(\sigma/\sqrt{3}\) because of the modified wave operator in equation (2.3.6). The appearance of derivatives of the \(\delta\)-function in \(G_3\) means that this Green’s function can give unphysical negative particle densities under certain circumstances. This may be somewhat surprising considering the fact that the one dimensional Green’s function was not unphysical in any way. There is, however, a significant physical difference between the one-dimensional and three-dimensional problems. In one dimension, we showed that equation (2.1.11) corresponds to a particular real physical situation, namely the case of mono-speed particles. This is unfortunately no longer true in three dimensions. There is no physical three dimensional system which behaves exactly as described by equation (2.3.6). In particular, a mono-speed population of particles does not obey this equation. The reason is associated with the closure assumption that the second moment tensor of the velocity is isotropic (eq. 2.3.4). This assumption is obviously violated close to an impulsive point source before collisions take place.

As a final comment we note that if we consider a particular case when all the gradients are restricted to one direction, say the \(x_1\) axis, then the three-dimensional equation (2.3.6) reduces to

\[
\frac{\partial n}{\partial t} - \frac{1}{3} \sigma^2 \tau \left( \frac{\partial^2 n}{\partial x_1^2} - \frac{3}{\sigma^2} \frac{\partial^2 n}{\partial t^2} \right) = s + \tau \frac{\partial s}{\partial t}.
\]  

(2.3.11)

This is identical to the one-dimensional equation (2.1.11) except for the identification \(\sigma^2 \rightarrow \sigma^2/3\). As we have seen in §2.1 and §2.2, the one-dimensional problem behaves quite reasonably (it has no negative densities), and therefore the three-dimensional version derived here should be well behaved when gradients are nearly uni-directional.
In §3, we derive a causal equation of evolution for the shear stress and consider a number of flows with gradients in the velocities. By analogy with the particle diffusion problem we expect well behaved results in the shear flow problem when gradients are in the same direction everywhere. All the situations we consider in §3 do have gradients limited to a single direction.

2.4 Accuracy of the Three-Dimensional Diffusion Model

We have seen that the Green’s function $G_3$ contains the derivative of a $\delta$-function which yields an unphysical density at early times. Thus, the solutions to equation (2.3.6) cannot be accurate in detail. Following the discussion in §2.2, we can however ask whether equation (2.3.6) is accurate in some more limited sense. For instance, does it fit some spatial moment of the particle distribution? The answer is that equation (2.3.6) does indeed do an excellent job of predicting the evolution of the mean square distance $\langle r^2 \rangle$ traveled by particles.

To show this we set $s = \delta(t)\delta^3(\vec{r})$ in equation (2.3.6), multiply the equation by $r^2$, and integrate over $4\pi r^2 dr$. We find that

$$\frac{\partial \langle r^2 \rangle}{\partial t} + \tau \frac{\partial^2 \langle r^2 \rangle}{\partial t^2} = 2\sigma^2 \tau,$$

whose solution for the given boundary conditions is (cf. eq. 2.2.3)

$$\langle r^2 \rangle = 2\sigma^2 \tau t + 2\sigma^2 \tau^2 [\exp(-t/\tau) - 1].$$

This result shows that $\langle r^2 \rangle$ varies as $\sigma^2 t^2$ at early time ($t \ll \tau$), corresponding to free streaming at a speed $\sigma$. (This is a little surprising since equation (2.3.6) has an apparent speed limit of $\sigma/\sqrt{3}$ in the wave operator. The Green’s function $G_3(t, r)$ too cuts off for $r > \sigma t/\sqrt{3}$. Mathematically, it is the derivative of the $\delta$-function in $G_3$ which leads to an r.m.s. particle distance of $\sigma t$ even though the Green’s function cuts off at a smaller radius.) At late time, $\langle r^2 \rangle$ has the standard dependence due to diffusion, viz. $\langle r^2 \rangle = 2\sigma^2 \tau t$.

Motivated by the early time behavior of $\langle r^2 \rangle$ in equation (2.4.2), let us find the ensemble averaged evolution of $\langle r^2 \rangle$ for a set of particles with a mean square velocity $\sigma^2$ which are injected at the origin at a time $t = 0$. The Brownian motion of each particle is described by Langevin’s equation (Reif 1965),

$$\frac{d\vec{v}}{dt} = -\frac{\vec{v}}{\tau} + \vec{F}(t),$$

The first term on the right hand side represents a frictional force, where $\tau$ here is the same as the mean free time in our theory and is taken to be a constant as per our assumptions.
The second term, $\vec{F}(t)$, is a rapidly fluctuating force with zero mean. We assume that this force is arranged so that the mean square velocity $\langle v^2 \rangle \equiv \sigma^2$ is independent of time (again as in our model of the scattering). Multiplying equation (2.4.3) by $\vec{r}$, averaging over the ensemble of particles, and noting that $\langle \vec{r} \cdot \vec{F} \rangle = 0$, we get

$$\langle \frac{d}{dt}(\vec{r} \cdot \vec{v}) \rangle = \sigma^2 - \frac{1}{\tau} \langle \vec{r} \cdot \vec{v} \rangle,$$  

(2.4.4)

With the identity $\langle \vec{r} \cdot \vec{v} \rangle = \frac{1}{2} \langle d \langle \vec{r}^2 \rangle \rangle$, equation (2.4.4) admits the solution,

$$\langle \vec{r} \cdot \vec{v} \rangle = 2 \sigma^2 \tau + 2 \sigma^2 \tau^2 [\exp(-t/\tau) - 1].$$  

(2.4.5)

which is identical to equation (2.4.2). This means that equation (2.3.6) predicts the evolution of the mean square distance $\langle \vec{r} \cdot \vec{v} \rangle$ exactly for any population of particles with mean square velocity $\sigma^2$ and constant mean free time $\tau$. This result clarifies in what sense equation (2.3.6) is a good representation of diffusion in three dimensions — it is not very good at predicting the detailed shape of the density distribution but it does provide an exact fit of the mean square distance diffused by particles.

As an extension, we ask what would happen if the mean free times of the particles are not all equal. As before we consider the particular case when all particles have the same mean free path $l$, i.e.

$$\tau(v) = \frac{l}{v},$$  

(2.4.6)

and where the distribution of speeds is Maxwellian in three dimensions:

$$f(v)dv = \frac{4\pi}{\sigma_M^3} \left( \frac{3}{2\pi} \right)^{3/2} v^2 \exp \left( -\frac{3v^2}{2\sigma_M^2} \right) dv, \quad v \geq 0.$$  

(2.4.7)

After replacing $\sigma$ by $v$, one can integrate equation (2.4.2) over this $f(v)$ to find

$$\langle r^2 \rangle = 2 \sqrt{\frac{8}{3\pi}} \sigma_M l t - 4l^2 \frac{d}{da} \left[ \frac{1}{\sqrt{a}} \exp(w^2) \text{erfc}(w) \right]_{a \rightarrow 1} - 2l^2, \quad w = \frac{\sigma_M t}{l \sqrt{6a}}.$$  

(2.4.8)

This result has the following asymptotic behaviors: $\langle r^2 \rangle \rightarrow \sigma_M^2 t^2$ for $t \ll \tau$; and $\langle r^2 \rangle \rightarrow 2(8/3\pi)^{1/2} \sigma_M l t$ for $t \gg \tau$. Comparing with equation (2.4.2) we see that we can obtain agreement in the behavior of $\langle r^2 \rangle$ in the two limits if we choose the mean free time $\tau$ in equation (2.4.2) to be

$$\tau = \sqrt{\frac{8}{3\pi}} \frac{l}{\sigma_M^2}.$$  

(2.4.9)

Figure 3 shows a comparison between the exact result (2.4.8) and the approximate result (2.4.2) when $\tau$ is defined as in (2.4.9). The agreement is excellent, showing that our diffusion theory predicts $\langle r^2 \rangle$ very well even when the mean free time is not constant. Also shown in figure 3 is the result from the standard diffusion equation, viz. $\langle r^2 \rangle = 2(8/3\pi)^{1/2} \sigma_M l t$, which is very inaccurate at early time.
3. STRESS TENSOR

3.1 Evolution Equation for the Stress Tensor

Having introduced our approach through the relatively simple problem of particles diffusing through a fixed background, we now consider the case when particles scatter off one another. We decompose the velocity of a particle $v_i$ into a mean velocity $\overline{v}_i$ plus a relative velocity $u_i$:

$$v_i = \overline{v}_i + u_i.$$  \hspace{1cm} (3.1.1)

Particles experience an acceleration $\vec{a} = (a_1, a_2, a_3)$, where the $a_i$ may be functions of time, position, and in general velocity as well. We ignore the possibility of particles being added or removed from the system. We thus have the Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}(v_i f) + \frac{\partial}{\partial v_i}(a_i f) = \Gamma_{\text{scatt}} = \frac{1}{\tau}(f_0 - f). \hspace{1cm} (3.1.2)$$

We depart from the discussion of §2 in the properties we assume for the scattering. Because the particles scatter off one another, the relevant frame in which we should specify the properties of $\Gamma_{\text{scatt}}$ is not some fixed background frame but rather the local rest frame of the gas of particles. Since each scattering event conserves momentum, the mean velocity of the particles must be preserved. Thus

$$\left(\overline{v}_i\right)_0 \equiv \frac{1}{n} \int v_i f_0 d^3v = \overline{v}_i, \quad \left(\overline{u}_i\right)_0 \equiv \frac{1}{n} \int u_i f_0 d^3v = 0. \hspace{1cm} (3.1.3)$$

Furthermore, we specify the second moments of the post-scattering distribution function $f_0$ to be

$$\left(u_i u_j\right)_0 \equiv \frac{1}{n} \int u_i u_j f_0 d^3v = (1 - \xi) \frac{\sigma^2}{3} \delta_{ij}, \quad \sigma^2 = u_i u_i = u_1^2 + u_2^2 + u_3^2. \hspace{1cm} (3.1.4)$$

In essence we assume that the scattering completely isotropizes the particle velocities so that immediately after scattering, (i) there are no off-diagonal velocity correlations such as $(u_1 u_2)_0$, and (ii) the mean square velocity is independent of direction, i.e. $(u_1^2)_0 = (u_2^2)_0 = (u_3^2)_0$. A new feature in equation (3.1.4) is the introduction of the parameter $\xi$, which allows for the possibility of inelastic scattering ($\xi = 0$ corresponds to elastic scattering). The point is that particles heat up when there are stresses present, and in real gases some of this heat is lost from the system by, for example, radiative processes. We introduce $\xi$ to model this kind of “cooling” which has to be modeled if we are interested in considering steady state situations.
Let us take the zeroth and first moments of the Boltzmann equation (3.1.2). These give the standard continuity and momentum equations:

\[
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i}(n v_i) = 0, \tag{3.1.5}
\]

\[
\frac{\partial \bar{v}_j}{\partial t} + \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} = \bar{a}_j - \frac{1}{n} \frac{\partial}{\partial x_i}(n \bar{u}_i \bar{u}_j). \tag{3.1.6}
\]

In equation (3.1.6), there are two kinds of acceleration, a mean acceleration \( \bar{a}_j \) due to the external forces and a contribution from the gradient of the stress tensor \( n \bar{u}_i \bar{u}_j \). In order to be general and to permit velocity-dependent external forces, we Taylor expand the acceleration \( a_j \) in the form

\[
a_j = \bar{a}_j + \frac{\partial a_j}{\partial v_i} u_i, \tag{3.1.7}
\]

where \( \bar{a}_j \) is the mean force and the second term is the excess force on a particle due to its relative velocity. Equation (3.1.7) is quite general and the only approximation made is that the expansion in velocity has been truncated at the linear term.

We now derive equations for the evolution of the stress tensor. To do this it is convenient to switch from the distribution function \( f(t, \{x_i\}, \{v_i\}) \) to another “relative” distribution function \( f_R(t, \{x_i\}, \{u_i\}) \) (cf. Kato 1970, Grossman et al. 1993) which is a function of the relative velocities \( u_i \). This distribution function satisfies a Boltzmann equation

\[
\frac{\partial f_R}{\partial t} + \frac{\partial}{\partial x_i}[(\bar{v}_i + u_i)f_R] + \frac{\partial}{\partial u_i} (\dot{u}_i f_R) = \frac{1}{\tau} (f_0 - f_R), \tag{3.1.8}
\]

where \( \dot{u}_i \) is the rate of change of the relative velocity of a particle as we follow it along its trajectory:

\[
\dot{u}_i = \ddot{v}_i - \frac{\partial \bar{v}_i}{\partial t} = \ddot{v}_i + \frac{\partial a_i}{\partial v_j} u_j - \frac{\partial \bar{v}_i}{\partial t} - (\bar{v}_j + u_j) \frac{\partial \bar{v}_i}{\partial x_j}.
\]

\[
= \frac{\partial a_i}{\partial v_j} u_j - \frac{\partial \bar{v}_i}{\partial x_j} u_j + \frac{1}{n} \frac{\partial}{\partial x_j}(n \bar{u}_i \bar{u}_j). \tag{3.1.9}
\]

We now substitute equation (3.1.9) into equation (3.1.8) and take the \( u_j u_k \) moment to obtain the equations of evolution of the stress components,

\[
\frac{\partial}{\partial t}(n \bar{u}_j \bar{u}_k) + \frac{\partial}{\partial x_i}(n \bar{u}_i \bar{u}_j \bar{u}_k) + \frac{\partial}{\partial x_i}(n \bar{u}_i \bar{u}_j \bar{u}_k) = - \left( \frac{\partial \bar{v}_j}{\partial x_i} - \frac{\partial a_j}{\partial v_i} \right) n \bar{u}_k \bar{u}_i
\]

\[
- \left( \frac{\partial \bar{v}_k}{\partial x_i} - \frac{\partial a_k}{\partial v_i} \right) n \bar{u}_j \bar{u}_i + \frac{(1 - \xi)}{3\tau} n \sigma^2 \delta_{jk} - \frac{1}{\tau} n \bar{u}_j \bar{u}_k. \tag{3.1.10}
\]
Equivalently the equations for second moments are obtained by combining (3.1.5) and (3.1.10),

$$
\begin{align*}
\frac{\partial}{\partial t}(u_j u_k) + u_i \frac{\partial}{\partial x_i}(u_j u_k) &= -\frac{1}{n} \frac{\partial}{\partial x_i}(n u_i u_j u_k) - \left( \frac{\partial v_j}{\partial x_i} - \frac{\partial a_j}{\partial v_i} \right) u_k u_i \\
&\quad - \left( \frac{\partial v_k}{\partial x_i} - \frac{\partial a_k}{\partial v_i} \right) u_j u_i + \frac{(1 - \xi)}{3\tau} \sigma^2 \delta_{jk} - \frac{1}{\tau} u_j u_k.
\end{align*}
$$

(3.1.11)

As usual, the above equations involve higher order moments (in this case the third moments $u_j u_k u_i$) and we need additional relations to close the set of equations. We faced such requirements for closures in the discussion on diffusion in §2. There we needed closures for the second moments and we made the simplest assumption possible, viz. that the second moments are constant and isotropic. Since the third moments are odd in velocity, the simplest assumption here is to set all the third moments to zero:

$$
u_i u_j u_k = 0.
$$

(3.1.12)

This is a reasonable approximation whenever the gradients of the second moments are small. More precisely, we expect odd moments in the velocity to be negligible whenever there is sufficient spatial symmetry in the neighborhood of the fluid element under consideration. Note that equation (3.1.12) does not require velocity gradients to be small. Indeed much of what we discuss later in the paper deals with large velocity gradients. We do however make sure that there is sufficient “right-left” spatial symmetry in the model problems we solve so that equation (3.1.12) will be valid. In the Appendix we derive a conditions on the second and third derivatives of the mean velocity which are necessary to make relation (3.2.11) viable for a steady shear flow. We should mention that there are other possible closures for the third moments. For instance, one could use a “diffusion approximation” to relate the third moments to gradients of the second moments, e.g. $u_1^3 \propto u_1^2 \tau \partial u_1^2 / \partial x_1$, and so on. Relations like this are reasonable, but they ultimately correspond to setting time derivatives to zero in the third moment evolution equations and using a Gaussian-like closure on fourth moments (see equation [2.1.18]). The problem with the neglect of time derivatives is that it may lead to a violation of causality. As an analogy, note that in the particle diffusion problem, if we neglect the time derivative in (2.1.9) then we obtain the usual diffusion approximation which gives the acausal equation (2.1.10). In the same way we expect a diffusion-like closure relation on third moments to lead to a violation of causality.

Equations (3.1.5), (3.1.6) and (3.1.11) with third moments set to zero provide 10 equations (one continuity equation, three momentum equations, and six stress equations) that
describe the evolution of the 10 moments: \( n, \overline{v_t}, \) and \( \overline{u_iu_j}. \) The Navier-Stokes equation is a special case of these equations. When the spatial and temporal derivatives are small on a collision scale (3.1.11) gives the standard linear relation between the stress and the velocity gradient. This result can in turn be substituted into equation (3.1.6) to get the Navier-Stokes equation. While the Navier-Stokes equation does not satisfy causality, our full equations can handle strong velocity gradients and satisfy causality. The combined equations (3.1.6) and (3.1.11) can potentially replace the Navier-Stokes equation in situations where the gradients or the temporal variations are arbitrarily large. In the following sections we explore the causal properties of these equations especially for the particular case of steady flows with a one-dimensional velocity gradient. More work, however, is needed in order to examine the full regime of applicability of equations (3.1.5), (3.1.6) and (3.1.11) under general conditions. Note that, at the level of our approximation there is no independent equation of state. Our system of particles behaves essentially like an ideal gas, and whatever prescription we may use for the cooling will determine the degree to which the gas is non-adiabatic. Deviations from an ideal gas behavior could be introduced by allowing for additional degrees of freedom in the particles but we do not explore this possibility here.

It is useful to write the stress tensor explicitly as the sum of the diagonal terms and the off-diagonal terms. The diagonal terms can be decomposed into two parts:

\[
nu u_i^2 = p + n \left( \overline{u_i^2} - \frac{1}{3} \sigma^2 \right), \quad p \equiv \frac{n \sigma^2}{3}, \tag{3.1.13}
\]

where the first term is the isotropic pressure. The second term describes the degree of anisotropy in the diagonal elements and gives rise to bulk viscosity. The off-diagonal components of the stress tensor constitute the shear stress. We discuss the physics of the shear and bulk viscous stresses within our formalism in the following sections by considering various limiting cases of equation (3.1.11).

Before we proceed to discuss steady state conditions let us show that the stress equation (3.1.11) satisfies causality. For simplicity, consider a flow, in the absence of external forces, where the mean velocity at each point is oriented along the \( x_2 \) axis and its magnitude \( \overline{v_2} \) is a function of \( x_1 \). Since this flow is divergence-free we take the particle density \( n \) to be constant. In addition we take \( \overline{u_1^2} \approx \sigma^2/3 \) to be constant, which is a valid assumption for a sufficiently weak shear (\( \overline{u_1u_2} \ll \sigma^2 \)). Under these conditions the (1,2) component of equation (3.1.11) together with the \( x_2 \)-component of the momentum equation (3.1.6) yield,

\[
\frac{\partial \overline{v_2}}{\partial t} - \nu_0 \left( \frac{\partial^2 \overline{v_2}}{\partial x_1^2} - \frac{3}{\sigma^2} \frac{\partial^2 \overline{v_2}}{\partial t^2} \right) = 0, \tag{3.1.14}
\]
where \( \nu_0 = \sigma^2 \tau / 3 \) is the viscosity coefficient. Equation (3.1.14) has the same structure and shares the same causal properties as the diffusion equation (2.3.11). Thus, based on the discussion in \( \text{s}2 \) we expect our shear flow theory to be causal and well behaved in one dimension.

### 3.2 Steady State Linear Shear

We begin our discussion of steady state shear flows by considering a flow where there are no external accelerations, i.e. \( a_i = 0 \), and where the mean velocity at each point is oriented along the \( x_2 \)-axis. We assume that the magnitude of the velocity is independent of \( x_2 \) and \( x_3 \), and varies linearly with \( x_1 \), i.e.

\[
\overline{v_1} = 0, \quad \overline{v_2} = 2Ax_1, \quad \overline{v_3} = 0. \tag{3.2.1}
\]

We assume that the system is in a steady state, and take \( n \) to be a constant, as is consistent with the divergence-free nature of the flow. Under these conditions, the momentum equation (3.1.6) shows that the various second moments have no spatial gradients. Furthermore, by the symmetries of the problem, the only non-vanishing second moments are \( u_1^2, u_2^2, v_3^2, u_1 v_2 \). Thus, the four non-vanishing components of the second moment equation (3.1.11) are given by

\[
(1 - \xi) \sigma^2 - 3u_1^2 = 0, \tag{3.2.2}
\]
\[
(1 - \xi) \sigma^2 - 3u_2^2 - 12A\tau u_1 u_2 = 0, \tag{3.2.3}
\]
\[
(1 - \xi) \sigma^2 - 3u_3^2 = 0, \tag{3.2.4}
\]
\[
2A\tau u_1^2 + u_1 u_2 = 0. \tag{3.2.5}
\]

We add to these four equations the definition of \( \sigma^2 \) given in equation (3.1.4).

If we sum equations (3.2.2)—(3.2.4) we obtain the following relation,

\[
-\overline{u_1 u_2} \cdot 2A = \frac{\xi}{\tau} \cdot \frac{1}{2} \sigma^2. \tag{3.2.6}
\]

The left-hand side of this equation is the work done (stress\times shear) per particle per unit time by the shear stress. The right hand side represents the rate at which kinetic energy is lost per particle as a result of cooling. These two quantities should obviously be equal in a steady state and equation (3.2.6) shows that our system of equations is physically consistent. More importantly, it reveals the need for the cooling parameter \( \xi \), since without its presence there are no steady state solutions to the equations (except for the trivial case of a zero shear).
Going back to equations (3.2.2)–(3.2.5), we can solve for $\xi$ to obtain

$$\xi = \frac{8A^2\tau^2}{3 + 8A^2\tau^2}, \quad (3.2.7)$$

and from this we solve for all the second moments in terms of $\sigma^2$:

$$\overline{u_1^2} = \overline{u_3^2} = \frac{1}{3 + 8A^2\tau^2}\sigma^2, \quad (3.2.8)$$

$$\overline{u_2^2} = \frac{1 + 8A^2\tau^2}{3 + 8A^2\tau^2}\sigma^2, \quad (3.2.9)$$

$$\overline{u_1u_2} = -\frac{2A\tau}{3 + 8A^2\tau^2}\sigma^2. \quad (3.2.10)$$

Note that in deriving these results we did not restrict the linear shear parameter $2A$ in any way and therefore these formulae are valid even for a large linear shear as long as the spatial derivatives of the shear are sufficiently small (see the Appendix). However, we did assume a steady state. Therefore, the results may not be applicable when the flow varies on a collision time. Also, we assumed that all particles have the same mean free time. Even when this is not the case, we expect that we can choose $\tau$ appropriately, so that our equations will describe the behavior of the system well. This point was demonstrated for the particle diffusion problem in §2.2 and §2.4, and will be discussed further in the next subsection. Similarly, there is no problem in principle with spatial variations of $\tau$.

Finally, we note that the stress tensor depends on the form of the cooling function assumed. However, the main qualitative features of the results, such as the decrease in the stress $\overline{u_1u_2}$ at large shear amplitude, appear to be universal.

Equations (3.2.7)–(3.2.10) show that the nature of the steady state stress tensor depends on the magnitude of the dimensionless parameter $A\tau$. In the limit of a weak shear, $A\tau \ll 1$, we find that the velocity distribution of the particles is essentially isotropic, i.e. $\overline{u_1^2} \approx \overline{u_2^2} \approx \overline{u_3^2} \approx \sigma^2/3$, and the shear stress is given by

$$n\overline{u_1u_2} = -n\frac{\sigma^2\tau}{3} \cdot 2A = -n\nu_0 \cdot \frac{\partial v_2}{\partial x_1}. \quad (3.2.11)$$

This is the standard linear relation between the viscous stress and the velocity shear. The kinematic coefficient of viscosity is

$$\nu_0 = \frac{\sigma^2\tau}{3}. \quad (3.2.12)$$

The Navier-Stokes equation is based on a relation of the form (3.2.11) for the shear stress. This relation, plus an equivalent one for the bulk viscosity which we discuss in §3.5, are
introduced into the momentum equation to give a closed set of dynamical equations for the flow. We see that this approximation is valid only when the shear is weak, i.e. only when the shear frequency $2A$ is small compared to the scattering frequency $1/\tau$. For larger velocity gradients we have to use the more complete set of equations presented here.

When the shear is strong, i.e. $A\tau \gg 1$, we find a completely different behavior than the one expressed by equation (3.2.11). As equations (3.2.8)–(3.2.10) show the velocity distribution becomes highly elongated along the $u_2$ axis, and the shear stress now is inversely proportional to the velocity shear,

$$\overline{u_1 u_2} \approx -\frac{1}{4A\tau} \sigma^2.$$ \hspace{1cm} (3.2.13)

The concept of a viscosity coefficient is not useful in this regime, but if we were to formally define $\nu$ as in equation (3.2.11) we would find that $\nu$ varies inversely as the square of the shear.

The turnover and reduction of the shear stress when the shear is large is a manifestation of causality in the theory. From the expression given in (3.2.10) we find that the maximum value of $|\overline{u_1 u_2}|$ is

$$|\overline{u_1 u_2}|_{\text{max}} = \frac{1}{2\sqrt{6}} \sigma^2,$$ \hspace{1cm} (3.2.14)

which shows that the shear stress $n\overline{u_1 u_2}$ is always limited to be smaller than the pressure ($p = n\sigma^2/3$), regardless of the magnitude of the shear. This automatic stress limiter is a very natural and welcome feature of the theory and is an improvement over the simple relation (3.2.11). The reason for the reduction of $\overline{u_1 u_2}$ at a large shear is not difficult to understand. For large values of shear, particles coming from a mean free distance, $u_1\tau$, along the $x_1$ axis, develop a transverse velocity difference $u_2 \sim A(u_1\tau)$. Thus for $A\tau \gg 1$, $\overline{u_1^2} \approx \overline{u_2^2}/(A\tau)^2 \sim \sigma^2/(A\tau)^2$. Therefore the stress, $|\overline{u_1 u_2}| \sim \overline{u_1^2}A\tau \sim \sigma^2/A\tau$, decreases for a large shear.

The asymptotic behavior of the shear stress in the limit of large shear is qualitatively very different from that exhibited by the diffusive flux in particle diffusion. As we have shown above, the shear stress actually decreases with increasing shear and goes to zero in the limit $2A \to \infty$. In particle diffusion on the other hand, as the density gradient increases the diffusive flux asymptotes to a constant value which is equal to the product of the particle density and the r.m.s. particle speed (Levermore and Pomraning 1981, Narayan 1992).

### 3.3 Shear Stress from Numerical Simulations
The solution written down in equations (3.2.7)–(3.2.10) is valid even for highly non-linear steady state shears, where by non-linear we mean that \( f \) deviates considerably from \( f_0 \). We have tested this solution using numerical simulations and describe some of our results here.

We set up a shearing cell extending from \( x_1 = -0.5 \) to \( +0.5 \) with a large number \( N \) of particles and with boundary conditions imposed so as to enforce the mean velocity to follow equation (3.2.1). The particles scatter with a specified mean free time \( \tau \) and are re-injected with the appropriate cooling \( \xi \) and an isotropic Maxwellian velocity distribution in the local rest frame. We let the system evolve until it achieves steady state and then compare the results with the theoretical predictions. Figure 4 shows a case with \( N = 10^5 \), \( 2A = 1 \), \( \tau = 1 \), \( (1 - \xi)\sigma^2 = 1 \). We display the variation of the moments with time. Note that the system achieves a steady state in less than 10 mean free times.

Figure 5 shows the distribution of the velocity components \( u_1 \) and \( u_2 \) at the end of the run for the three cases that we have simulated. The steady state distribution \( f \) is only slightly deformed from the Maxwellian post-scattering distribution \( f_0 \) when \( 2A\tau = 0.1 \). This is a case of weak shear and it is understandable that \( f \) is only slightly perturbed from \( f_0 \). Indeed perturbation theory is valid in this case, and so is the standard weak shear formula equation (3.2.11) for the shear stress. In the other two cases that we have presented, viz. \( 2A\tau = 1, 10 \), we find that \( f \) is strongly distorted by the shear. We are therefore definitely into the regime of non-linear shear. Nevertheless, we have verified that all the second moments calculated analytically (eqs. [3.2.8]–[3.2.10]) are in excellent agreement with the numerical simulation.

We next consider a different situation where the particles have a constant mean free path \( l \) rather than a constant \( \tau \). Figure 6 shows \( \frac{\mathbf{u}_1 \cdot \mathbf{u}_2}{\sigma^2} \) as a function of the shear amplitude \( 2Al/\sigma \) in this case. The points are results from a simulation of \( N = 10^3 \) particles, and the curves were obtained from equation (3.2.10) when we set \( \tau = l/\sigma \) (dashed line) or \( \tau = 0.55l/\sigma \) (solid line). Note the interesting result that equation (3.2.10) predicts the maximum value of stress accurately for particles with a constant \( l \), although it was derived for the constant \( \tau \) case.

### 3.4 Steady State Shear with Advection

The stress-limiter discussed in §3.2 provides a strong indication that our theory satisfies causality. To demonstrate causality in a different and possibly more transparent way we consider here a generalization of the previous case where, in addition to the shear in \( \mathbf{v}_2 \), we now also allow motion along the \( x_1 \) axis (\( \mathbf{v}_1 \)). We refer to this additional motion as *advection*. We do not restrict \( \mathbf{v}_1 \) or \( \mathbf{v}_2 \) in any way except to assume that the gradients
of both velocities are in the $x_1$ direction and that there are no external forces. Also, we retain the assumption of a steady state. The reader may notice a close analogy between the flow considered here and that in a steady state accretion disk where $x_1$ and $x_2$ would refer to the radial ($R$) and azimuthal ($\phi$) directions respectively. We do not, however, discuss rotation in this section (but see § 3.6 & 3.7).

In a steady state, the continuity equation (3.1.5), the $x_2$ component of the momentum equation (3.1.6), and the $u_1 u_2$ component of the second moment equation (3.1.11) give

$$\frac{1}{n} \frac{\partial n}{\partial x_1} = -\frac{1}{v_1} \frac{\partial v_1}{\partial x_1},$$

$$\frac{v_1}{n} \frac{\partial v_2}{\partial x_1} = -\frac{\partial u_1 u_2}{\partial x_1} - \frac{u_1 u_2}{n} \frac{\partial n}{\partial x_1} = -\frac{\partial u_1 u_2}{\partial x_1} + \frac{u_1 u_2}{v_1} \frac{\partial v_1}{\partial x_1},$$

$$\frac{v_1}{n} \frac{\partial u_1 u_2}{\partial x_1} = -\frac{u_1^2}{v_1} \frac{\partial v_2}{\partial x_1} - \frac{1}{\tau} u_1 u_2 - \frac{u_1 u_2}{v_1} \frac{\partial v_1}{\partial x_1}.$$

Substituting (3.4.2) into (3.4.3) and rearranging terms we find the following expression for the steady state shear stress $\overline{u_1 u_2}$:

$$\overline{u_1 u_2} = -\left(\frac{1}{1 + 2\tau \frac{\partial v_1}{\partial x_1}}\right) \left(1 - \frac{v_1^2}{u_1^2}\right) \overline{u_1^2} \frac{\partial v_2}{\partial x_1}.$$  

(3.4.4)

This relation has several interesting features.

First notice that if $\overline{v_1} = 0$ the viscosity coefficient that relates $\overline{u_1 u_2}$ to the shear $\partial v_2/\partial x_1$ is just $\overline{u_1^2}/\tau$. This agrees exactly with the results of the previous analysis (eqs. [3.2.10] and [3.2.8]). When there is advection, the viscosity coefficient is modified by two factors. Consider first the factor $(1 - \overline{v_1^2}/\overline{u_1^2})$, which reduces the viscosity whenever there is a steady flow, and is very similar to results previously obtained by Narayan (1992). To interpret this result, we need first to ask under what conditions we can have a shear in a steady flow such as we are considering. Since we have ignored external forces, there is no outside mechanism that can induce shear in the flow; the only reason for there to be a shear is a downstream boundary condition. For instance, let us suppose that the flow comes in from negative $x_1$ with a positive $\overline{v_1}$ and with $\overline{v_2} = 0$. Let us assume that at some large positive $x_1$ the flow meets a sideward moving interface of some sort — a “conveyor belt” in the language of Syer and Narayan (1993). The downstream boundary condition can be satisfied only if the flow sets up a shear $\partial v_2/\partial x_1$. Now, information about this downstream requirement has to be transported upstream by the particles themselves. As the flow velocity (as measured in the frame in which $\partial/\partial t = 0$) increases, the net flux of upstream moving particles reduces because only a fraction of the particles has large
enough negative velocities to overcome the advection. Therefore, the number of particles that participate in the shear stress reduces and this is reflected by a reduction in the viscosity coefficient.

Equation (3.4.4) shows that the viscosity coefficient goes to zero when \( \overline{\nu_1} = (\overline{\nu_1^2})^{1/2} \). In particular, when the shear is weak, we know that \( \overline{u_1^2} \approx \sigma^2 / 3 \), and the cutoff occurs at \( \overline{\nu_1} = \sigma / \sqrt{3} \). The quantity \( (\overline{u_1^2})^{1/2} \) is the limiting speed of particles in the \( x_1 \) direction in our theory. When \( \overline{\nu_1} \) is equal to this speed, no particles are able to move fast enough to beat the advection and as a result no viscous stress is transmitted upstream. This is the reason for the cut-off of the viscosity coefficient. In real gases the cut-off may be more gradual due to the existence of a high-velocity tail in the particle distribution function.

We note an apparent paradox in the fact that the viscosity coefficient in equation (3.4.4) actually changes sign when \( \overline{\nu_1^2} > \overline{u_1^2} \), which means that the shear stress begins to point in the same direction as the velocity shear. It implies that the work done by the shear stress, \( -\overline{u_1 u_2} \partial \overline{\nu_1} / \partial x_1 \) (see equation 3.2.6), is negative, i.e. the shear stress extracts heat energy from the flow and converts it into mechanical energy. Since this is not reasonable, we interpret the change of sign in (3.4.4) to mean that, when \( \overline{\nu_1^2} > \overline{u_1^2} \) there can be no steady state shear in the flow.

We now come to the second factor \( (1 + 2\tau \partial \overline{\nu_1} / \partial x_1)^{-1} \) in equation (3.4.4). It shows that any divergence in the advection velocity modifies the viscosity coefficient. This is fairly straightforward to interpret. Consider first the case when \( \partial \overline{\nu_1} / \partial x_1 > 0 \), which corresponds to an expansion of the flow. In this case, as a particle moves from one region of the flow to another it finds that its velocity becomes closer and closer to the local bulk velocity \( \overline{\nu_1} \). Therefore, the distance that a particle is able to move in “comoving fluid coordinates” before it is scattered is reduced. As a result, the number of particles that can reach any point in the flow from downstream is reduced, and this causes a reduction in the shear stress. The case of compression, when \( \partial \overline{\nu_1} / \partial x_1 < 0 \), is similar up to a point. In the presence of compression, downstream particles find it easier to move upstream and this enhances the shear stress. The interesting point is that the shear stress diverges when \( 2\tau \partial \overline{\nu_1} / \partial x_1 = -1 \). What this means is that a flow with such a large compression is unphysical. When \( 2\partial \overline{\nu_1} / \partial x_1 < -1 / \tau \), the flow compresses down to a zero volume in less than one scattering time. Obviously such a situation can occur only in transient flows and not in a steady state.

Equation (3.4.4) is a general result for steady flows which clarifies several physical issues. However, because the expression for the stress is given in terms of \( \overline{u_1^2} \), whose magnitude relative to \( \sigma^2 \) depends on the strength of the shear, it does not provide a useful formula for the viscosity coefficient. To find a more practical formula we restrict ourselves
to a steady flow where \( \overline{v_1}, n \) and \( \sigma^2 \) are constant. In this situation the momentum equation gives

\[
\frac{\partial u_1 u_2}{\partial x_1} = -\overline{v_1} \frac{\partial \overline{v_2}}{\partial x_1}.
\] (3.4.5)

and the four relevant components of the second moment equation (3.1.11) are

\[
\frac{v_1}{3\tau} \frac{\partial u_1^2}{\partial x_1} + \frac{u_1^2}{\tau} = \frac{(1 - \xi)}{3\tau} \sigma^2 = 0,
\] (3.4.6)

\[
-\frac{(1 - \xi)}{3\tau} \sigma^2 + v_1 \frac{\partial u_2^2}{\partial x_1} + \frac{u_2^2}{\tau} + 2u_1 u_2 \frac{\partial \overline{v_2}}{\partial x_1} = 0,
\] (3.4.7)

\[
-\frac{(1 - \xi)}{3\tau} \sigma^2 + \frac{u_2^2}{\tau} = 0,
\] (3.4.8)

\[
\frac{u_1}{3\tau} \frac{\partial v_2}{\partial x_1} + \frac{u_1 u_2}{\tau} = 0.
\] (3.4.9)

From the momentum equation (3.1.6) we find that \( \frac{\partial u_1^2}{\partial x_1} = 0 \), and since \( \sigma^2 \) is independent of position, it follows that \( \frac{\partial u_2^2}{\partial x_1} = -\frac{\partial u_3^2}{\partial x_1} \). We now solve equations (3.4.5)–(3.4.9) making use of these relations, and find

\[
u_1 u_2 = -\frac{\sigma^2 \tau}{3 + 8A^2 \tau^2} \left( 1 - \frac{3v_1^2}{\sigma^2} \right) \frac{\partial \overline{v_2}}{\partial x_1},
\] (3.4.11)

which is just equation (3.2.10) modified by the causality factor \((1 - 3\overline{v_1}^2/\sigma^2)\). This provides a useful special case of equation (3.4.4) for flows with uniform advection.

### 3.5 Bulk Viscosity

We now explore the nature of bulk viscosity in our theory. We consider a flow where the mean velocity at each point is oriented along the \( x_1 \)-axis and where all gradients are also along \( x_1 \). Let us define the total time derivative \( d/dt \) by

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \overline{v_1} \frac{\partial}{\partial x_1}.
\] (3.5.1)

The continuity and momentum equations (ignoring external accelerations) give

\[
\frac{dn}{dt} + n \frac{\partial \overline{v_1}}{\partial x_1} = 0,
\] (3.5.2)

\[
n \frac{d\overline{v_1}}{dt} + \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_1} \left[ n \left( \frac{\overline{v_1}^2 - \sigma^2}{3} \right) \right] = 0,
\] (3.5.3)
where in (3.5.3) we have split the stress tensor into the sum of the pressure \( p \) and an anisotropic term which will be associated with bulk viscosity (cf. eq. [3.1.13]).

By the symmetry of this problem, there are only three non-vanishing second moments, namely \( \overline{u_1^2} \), \( \overline{u_2^2} \), \( \overline{u_3^2} \). Ignoring the acceleration terms, these components evolve according to

\[
\frac{du_1^2}{dt} + \left( 2 \frac{\partial \overline{v_1}}{\partial x_1} + \frac{1}{\tau} \right) \overline{u_1^2} - \frac{1 - \xi}{3\tau} \sigma^2 = 0, \tag{3.5.4}
\]

\[
\frac{du_2^2}{dt} + \frac{\overline{u_2^2}}{\tau} - \frac{1 - \xi}{3\tau} \sigma^2 = 0, \tag{3.5.5}
\]

\[
\frac{du_3^2}{dt} + \frac{\overline{u_3^2}}{\tau} - \frac{1 - \xi}{3\tau} \sigma^2 = 0. \tag{3.5.6}
\]

Adding equations (3.5.4)–(3.5.6) we find

\[
- \overline{u_1^2} \frac{\partial \overline{v_1}}{\partial x_1} = \left( \frac{d}{dt} + \frac{\xi}{\tau} \right) \cdot \frac{1}{2} \sigma^2. \tag{3.5.7}
\]

This is analogous to equation (3.2.6). The left-hand-side gives the work done per particle per unit time. Note that the stress \( n\overline{u_1^2} \) is the sum of the pressure and the bulk viscous contribution. The right hand side of (3.5.7) is the rate at which particles acquire kinetic energy, some of which is lost through cooling. Equation (3.5.7) thus expresses energy conservation.

Let us first consider the case when the velocity gradient \( \partial \overline{v_1}/\partial x_1 \) is small compared to \( 1/\tau \). In this case we expect \( \overline{u_1^2} \approx \overline{u_2^2} \approx \overline{u_3^2} \approx \sigma^2/3 \). Equation (3.5.4) gives

\[
\frac{1}{\tau} \left( \overline{u_1^2} - \frac{\sigma^2}{3} \right) = -2\overline{u_1} \frac{\partial \overline{v_1}}{\partial x_1} - \frac{du_1^2}{dt} - \frac{\xi}{3\tau} \sigma^2. \tag{3.5.8}
\]

Substituting \( \overline{u_1^2} \approx \sigma^2/3 \) into the right hand side and using equation (3.5.7) we find

\[
\frac{n}{3} \left( \overline{u_1^2} - \frac{\sigma^2}{3} \right) \approx - \frac{4}{3} \nu_0 n \frac{\partial \overline{v_1}}{\partial x_1}, \tag{3.5.9}
\]

where \( \nu_0 \) is the kinematic coefficient of shear viscosity introduced in equation (3.2.12). We thus recover the standard result, viz. that in the limit of weak velocity gradients, the shear and bulk viscosity coefficients are related by a factor of \( 4/3 \). If this relation and the analogous equation (3.2.11) are substituted in the momentum equation we obtain the Navier-Stokes equation for the evolution of a viscous fluid. These relations are, however, valid only for weak velocity gradients and for slow variations of the flow parameters. When these conditions are violated we need the more complete theory presented in this paper.
Let us now proceed to the case when the velocity gradient is not necessarily small. For simplicity we assume that the velocity gradient is independent of $x_1$:

$$\frac{\partial v_1}{\partial x_1} = 2C = \text{constant}, \quad (3.5.10)$$

Let us also assume that the temperature, which is proportional to $\sigma^2$, is the same everywhere and does not change with time. It follows from equation (3.5.7) that $\overline{u_1^2}$ must also be independent of position and time. It is now straightforward to solve equations (3.5.5)–(3.5.7), which yields

$$\xi = -\frac{4C\tau}{3 + 8C\tau}, \quad (3.5.11)$$

$$\overline{u_1^2} = \frac{1}{1 + 8C\tau/3} \sigma^2. \quad (3.5.12)$$

Note that $\xi$ is negative for positive $C$. This is because particles cool when there is an expansion of the flow and we need a source of heating (rather than cooling) to maintain the moments at a constant value as we have assumed.

For a small velocity gradient, $|C\tau| \ll 1$, equation (3.5.12) recovers the usual result

$$\overline{u_1^2} - \frac{\sigma^2}{3} \approx -\frac{4}{3} \cdot \frac{\sigma^2 \tau}{3} \cdot 2C, \quad (3.5.13)$$

which is identical to equation (3.5.9). When $C\tau$ is large and positive we find $\xi \to -1/2$ and $\overline{u_1^2}/\sigma^2 \to 0$. For an extremely rapid expansion the particles are very cold in the $x_1$ direction and their velocity distribution is restricted nearly to the $u_2-u_3$ plane. This is reasonable. When $C\tau$ is large and negative, equations (3.5.11), (3.5.12) appear to reveal a problem. If $C\tau < -1/4$, then we find $\overline{u_1^2} > \sigma^2$ which is physically inconsistent. There is, however, a simple explanation for this result. Precisely when $C\tau = -1/4$ we notice that $\xi = 1$. Going back to the original equation (3.1.4) where $\xi$ was defined, we see that $\xi = 1$ is a physical upper limit to the cooling, representing the case when all the energy in a particle is removed entirely in a collision. However, $C\tau = -1/4$ corresponds to an infinite compression of the gas in one collision time, which generates an infinite amount of heat. Clearly our cooling term is unable to remove this heat, and thus the assumption of a steady state breaks down. The present limit on $C\tau$ is exactly equivalent to the limit on $\partial \overline{u_1^2}/\partial x_1$ discussed in §3.4, and these have the same underlying physics.

### 3.6 Shear Viscosity in Differential Rotation

We consider finally a shearing rotating flow as an example of an application with velocity-dependent accelerations. This is also of particular importance in astrophysics.
because of applications in accretion disks. Kato and Inagaki (1993) discuss rotating flows in greater detail than we do, but they have limited themselves to a small shear, whereas the following discussion is applicable even for a large shear. (See also Goldreich & Tremaine 1978, who have derived an expression for the shear stress in a rotating disk of particles in Saturn’s ring using a different approach.)

We consider a rotating flow in a local Cartesian approximation, called the shearing sheet approximation (Goldreich & Lynden-Bell 1965, Narayan, Goldreich & Goodman 1987). Let the flow take place in a central potential which produces an inward radial acceleration $g(R)$. Consider a reference radius $R_0$ and let $\Omega = \left[\frac{g(R_0)}{R_0}\right]^{1/2}$ be the equilibrium angular velocity such that centrifugal acceleration just cancels $g$ at $R = R_0$.

We work in a frame which rotates with angular velocity $\Omega$ and is centered on $R_0$. We take our local Cartesian grid to have $x_1 = R - R_0$, $x_2$ along the azimuthal direction, and $x_3$ parallel to $\vec{\Omega}$. We have arranged so that a particle at rest at $x_1 = x_2 = x_3 = 0$ is in equilibrium. However, particles at non-zero $x_1$ experience an additional acceleration of the form $g’x_1$, arising from a combination of the central potential and centrifugal force; we have $g’ = \frac{dg}{dR} + \Omega^2$. In addition, moving particles in the rotating frame experience a Coriolis force. Therefore, the three components of the acceleration are given by

$$a_1 = g’x_1 + 2\Omega v_2, \quad a_2 = -2\Omega v_1, \quad a_3 = 0. \quad (3.6.1)$$

For a truly spherical central potential, $a_3 \neq 0$, but we have simplified matters by assuming a cylindrically central potential (as appropriate for a thin disk).

Let us consider a steady state shear flow in this rotating frame, and let us assume that $n$ is constant and that (cf. §3.2)

$$\overline{v_1} = 0, \quad \overline{v_2} = 2Ax_1, \quad \overline{v_3} = 0. \quad (3.6.2)$$

The four non-vanishing second moments, $\overline{u_1^2}$, $\overline{u_2^2}$, $\overline{u_3^2}$, and $\overline{u_1 u_2}$, satisfy the following equations (compare to equations 3.2.2—3.2.5):

$$3\overline{u_1^2} - (1 - \xi)\sigma^2 - 12\Omega\tau \overline{u_1 u_2} = 0, \quad (3.6.3)$$

$$3\overline{u_2^2} - (1 - \xi)\sigma^2 + 12B\tau \overline{u_1 u_2} = 0, \quad (3.6.4)$$

$$3\overline{u_3^2} - (1 - \xi)\sigma^2 = 0, \quad (3.6.5)$$

$$2B\tau \overline{u_1^2} - 2\Omega\tau \overline{u_2^2} + \overline{u_1 u_2} = 0, \quad (3.6.6)$$

where we have defined the vorticity $2B$ by

$$2B = 2A + 2\Omega. \quad (3.6.7)$$
As before, summing equations (3.6.3), (3.6.4) and (3.6.5) we obtain the usual energy conservation relation

\[-u_1u_2 \cdot 2A = \frac{\xi}{\tau} \cdot \frac{1}{2} \sigma \cdot 2A.\]  

(3.6.8)

Solving the equations for the cooling constant \(\xi\) and the various second moments we find

\[\xi = \frac{8A^2 \tau^2}{3 + 8A^2 \tau^2 + 12\kappa^2 \tau^2},\]  

(3.6.9)

\[\overline{u_1^2} = 1 + 8\Omega^2 \tau^2 + 2\kappa^2 \tau^2 \overline{\sigma^2},\]  

(3.6.10)

\[\overline{u_2^2} = 1 + 8B^2 \tau^2 + 2\kappa^2 \tau^2 \overline{\sigma^2},\]  

(3.6.11)

\[\overline{u_3^2} = \frac{1 + 4\kappa^2 \tau^2}{3 + 8A^2 \tau^2 + 12\kappa^2 \tau^2 \sigma^2},\]  

(3.6.12)

\[\overline{u_1u_2} = -\frac{2A\tau}{3 + 8A^2 \tau^2 + 12\kappa^2 \tau^2 \sigma^2},\]  

(3.6.13)

where \(\kappa^2\) is the epicyclic frequency,

\[\kappa^2 = 4\Omega B.\]  

(3.6.14)

These are generalizations of the results of §3.2 when there is rotation. We notice that the results now depend on two dimensionless parameters, \(A\tau\) and \(\kappa\tau\), rather than one, because the flow now has two frequencies associated with it, the shear \(2A\) and the epicyclic frequency \(\kappa\).

We may consider various limits of equation (3.6.13). If \(A\tau, \kappa\tau \ll 1\), which corresponds to very rapid collisions, we recover the usual result

\[\overline{u_1u_2} = -\frac{2A\tau}{3 + 8A^2 \tau^2 + 12\kappa^2 \tau^2 \sigma^2},\]  

(3.6.15)

which is the standard formula for shear viscosity. In this limit, rotation plays no role. Similarly, if \(\kappa\tau \ll 1\) and if \(A\tau \gg 1\), i.e. if we have strong shear in the presence of weak rotation, we again recover equation (3.2.10). On the other hand, if \(\kappa\tau \gg 1\), then even if the shear is weak (i.e. \(A\tau \ll 1\)), the shear stress is suppressed relative to the non-rotating case (see also Kato and Inagaki 1993),

\[\overline{u_1u_2} \simeq -\frac{1}{1 + 4\kappa^2 \tau^2} \cdot \frac{\sigma^2 \tau}{3} \cdot 2A.\]  

(3.6.16)

The reason for this is straightforward. When \(\kappa \gg 1/\tau\) a particle undergoes many epicycles within a mean free time and travels a distance equal only to the radius of an epicycle in
the $x_1$ direction. Therefore, the effective mean free path is much shorter than that in the non-rotating case and this suppresses the viscosity. Finally, if $A\tau, \kappa\tau$ are both $\gg 1$, then the viscous stress is reduced by an even larger factor.

In the derivation of the basic equations we had set the third moment of the velocity to zero. Physically this condition is satisfied when the velocity distribution of particles has a reflection symmetry, or in particular when the distribution is constant on ellipses. This turns out to be a very good approximation for shear flows in rotating systems, where the epicyclic motion of particles results in a quite symmetric velocity distributions (see figure 8).

Obviously, the shear stress $\eta \overline{u_1u_2}$ in the presence of rotation is limited by the two parameters discussed above, instead of just one as in the non-rotating case (eq. [3.2.14]). The maximum stress is thus smaller than in non-rotating flows. We discuss the stress limiter further in §3.7, where we specialize some of these results to the case of a Keplerian disk.

To demonstrate causality more explicitly, we restore the temporal and spatial derivatives in the $\overline{v_2}$ component of the momentum equation and the equation for $\overline{u_1u_2}$:

$$\frac{\partial \overline{v_2}}{\partial t} = -\frac{\partial}{\partial x_1}(\overline{u_1u_2}), \quad (3.6.17)$$

$$\frac{\partial \overline{u_1u_2}}{\partial t} + \overline{u_1} \frac{\partial \overline{v_2}}{\partial x_1} + (\overline{u_1^2} - \overline{u_2^2})2\Omega = -\frac{1}{\tau} \overline{u_1u_2}. \quad (3.6.18)$$

We continue to assume that $n$ is constant and that $\overline{v_1} = 0$. This assumption is technically inconsistent since variations in $\overline{v_2}$ will cause fluctuations in $\overline{v_1}$ (through the Coriolis acceleration) which will lead to compressive motions and changes in $n$. However, compression in $n$ will cause sound waves which will complicate the analysis. We filter out the sound waves by ignoring variations in $\overline{v_1}$ and concentrating only on $\overline{v_2}$ and $\overline{u_1u_2}$. Let us further assume that the velocity shear $\partial \overline{v_2}/\partial x_1$ is small. In this limit, we have $2A\tau \ll 1$, $B \approx \Omega$, $\kappa \approx 2\Omega$. Equations (3.6.10) and (3.6.11) then indicate that $\overline{u_1^2} \approx \overline{u_2^2} \approx \sigma^2/3$. Substituting these relations into (3.6.18), we find

$$\frac{\partial \overline{u_1u_2}}{\partial t} + \frac{\sigma^2}{3} \frac{\partial \overline{v_2}}{\partial x_1} = -\frac{1}{\tau} \overline{u_1u_2}. \quad (3.6.19)$$

We now consider small perturbations in $\overline{v_2}$ and $\overline{u_1u_2}$:

$$\overline{v_2} \rightarrow \overline{v_2} + \delta \overline{v_2}, \quad \overline{u_1u_2} \rightarrow \overline{u_1u_2} + \delta \overline{u_1u_2}. \quad (3.6.20)$$

Equations (3.6.17) and (3.6.19) then combine to give

$$\frac{\partial \delta \overline{v_2}}{\partial t} - \frac{\sigma^2\tau}{3} \left[ \frac{\partial^2 \delta \overline{v_2}}{\partial x^2} - \frac{3}{\sigma^2} \frac{\partial^2 \delta \overline{v_2}}{\partial t^2} \right] = 0 \quad (3.6.19)$$
This is analogous to equation (2.3.11) for particle diffusion and has the familiar wave operator which enforces causality. Thus, we have proved that even in the presence of strong rotation, viscous signals travel at a finite speed and satisfy causality.

3.7 Limits on $\alpha$–Viscosity

Shakura and Sunyaev (1973) used simple scaling arguments to write the kinematic viscosity coefficient in a turbulent thin accretion disk in the form

$$\nu = \alpha \frac{c_s^2}{\Omega_K}, \quad \alpha \leq 1,$$

(3.7.1)

where $\Omega_K$ is the Keplerian angular velocity. Their argument was that viscosity depends on two quantities: (i) the speed $\sigma$ of the turbulent eddies, which is likely to be limited by the sound speed $c_s$, and (ii) the mean free path $l$ which is likely to be no larger than the vertical scale height of the disk, the latter quantity being $\sim c_s/\Omega_K$. Since $\nu \sim \sigma l$, they thus obtained equation (3.7.1) with $\alpha$ limited to be $\lesssim 1$.

In §3.6 we have developed a fairly complete treatment of viscous interactions between particles in a differentially rotating system and we can therefore quantify equation (3.7.1) more carefully. Our theory has two parameters. One of these is the $r.m.s.$ particle velocity $\sigma$. In the context of disk viscosity, our “particles” are turbulent eddies or blobs, and in a purely hydrodynamic situation we expect the blob velocity $\sigma$ to be smaller than the local sound speed, otherwise it would be dissipated by shocks. Let us therefore write

$$\sigma = \alpha_1 c_s, \quad \alpha_1 \leq 1.$$ 

(3.7.2)

Our second parameter is the mean free time $\tau$ between collisions. Since we cannot a priori assign any limits to $\tau$, let us simply express it in terms of the rotation frequency $\Omega$, 

$$\tau = \frac{\alpha_2}{\Omega},$$

(3.7.3)

where we do not restrict $\alpha_2$ to any particular range. We showed in §3.6 that the viscosity depends on two frequencies, $A$ and $\kappa$, which are given by

$$\frac{2A}{\Omega} = \frac{R}{\Omega} \frac{d\Omega}{dR} = \frac{d\ln \Omega}{d\ln R},$$

(3.7.4)

$$\frac{\kappa^2}{\Omega^2} = \frac{2}{\Omega R} \frac{d}{dR} (\Omega R^2) = 2 \frac{d\ln (\Omega R^2)}{d\ln R}.$$ 

(3.7.5)
Substituting these relations into equation (3.6.13) we find the following result for the coefficient of viscosity

$$\nu = \frac{\alpha_1^2 \alpha_2}{3 + 2\alpha_2^2 (d \ln \Omega / d \ln R)^2 + 24 \alpha_2^2 (d \ln \Omega R^2 / d \ln R)} \frac{c_s^2}{\Omega} \equiv \alpha \frac{c_s^2}{\Omega},$$

(3.7.6)

which gives an expression for the Shakura-Sunyaev $\alpha$ parameter in terms of $\alpha_1$ and $\alpha_2$. In the particular case of a Keplerian disk, where $\Omega(R) \propto R^{-3/2}$, we have $2A/\Omega = -3/2$ and $\kappa^2/\Omega^2 = 1$, and (3.7.6) becomes

$$\nu_K = \frac{\alpha_1^2 \alpha_2}{3 + 33 \alpha_2^2 / 2} \frac{c_s^2}{\Omega} \equiv \alpha_K \frac{c_s^2}{\Omega}.$$

(3.7.7)

The most interesting feature of these relations is the fact that $\nu$ and $\nu_K$ each has an absolute maximum value regardless of the value of $\alpha_2$. Equation (3.7.7) for instance reaches its maximum value when $\Omega \tau \equiv \alpha_2 = \sqrt{2/11}$ at which point we have

$$(\alpha_K)_{max} = \frac{\alpha_1^2}{\sqrt{198}} = 0.071 \alpha_1^2.$$

(3.7.8)

This is an extremely small limit for $\alpha_K$ considering that $\alpha_1$ is itself limited to lie below unity. Note that if $\alpha$ is defined alternatively through the stress tensor relation $u_1 u_2 = -\alpha c_s^2$, then the above bound should be multiplied by 3/2.

To demonstrate how small $\alpha_K$ is in a Keplerian disk we show in figure 9 the variation of $\alpha_K$ as a function of the parameter $\alpha_2 \equiv \Omega \tau$. We see that, if $\alpha_2$ lies more than a factor of few outside the optimum value of $\sqrt{2/11}$, then $\alpha_K$ falls even further. Also, $\alpha_1$ itself is presumably somewhat less than unity since turbulent blobs are likely to have a range of speeds with the fastest blobs limited to speeds less than $c_s$. Thus, we conclude that turbulent viscosity in accretion disks, when expressed in the Shakura-Sunyaev form (3.7.1), is likely to have values of $\alpha_K$ which are somewhat smaller than 0.07. Lower values of $\alpha_K$ imply disks with larger surface densities and spectra that are more nearly thermal.

In regions away from the Keplerian region of a disk, for example in the inner boundary layer, we need to use the full expression given in equation (3.7.6). We see that in regions of large shear the effective $\alpha$ will be much smaller than in the Keplerian case. Figure 9 shows a case where $d \ln \Omega / d \ln R = 5$, a modest value by the standards of boundary layers (e.g. Narayan and Popham 1993, Popham et al. 1993). Note how small the viscosity coefficient becomes in this case. When we add to this the fact that the boundary layer region has radial flows which also reduce $\nu$ through the causality factor discussed in §3.4, we see that accretion disk boundary layers will in general have very weak viscosity. The effect that
this will have on the structure of the boundary layer is yet to be investigated. The role of
third moments, which may not be negligible in boundary layers, is also unclear.

The formulae we have developed are based on the assumption of a constant mean free
time, and one may wonder whether the results would change significantly with other scat-
tering laws. However, to leading order, different scattering laws should all be describable
through an effective mean free time, and since we did not restrict $\Omega\tau$ in any way our limit
on $\alpha$ would appear to be robust. The point is that, ultimately, the limit on $\alpha$ comes from
the stress-limiter, which inhibits the viscous stress from exceeding the pressure, coupled
with geometrical factors which further limit the shear stress because of the epicyclic mo-
tion. These are sufficiently general principles that we cannot envisage violating the limits
shown in figure 9 by any significant factor. Obviously, if the shear stress is produced by
non-hydrodynamic effects, e.g. magnetic fields, then our limit could be exceeded.
4. SUMMARY AND DISCUSSION

In this work we have attempted to incorporate causality into the theory of transport phenomena. We have developed equations for particle diffusion in one and three dimensions which are an improvement over the standard diffusion equation whenever gradients are large and causality limits the flux. We have also generalized the equations of viscous hydrodynamics to include situations where velocity gradients are large or time variations are rapid. In such cases the standard Navier-Stokes equation breaks down.

Our approach is based on taking moments of the Boltzmann equation and using an approximate form for the collision term (equation 2.1.3). We assume that particles collide with a mean free time $\tau$ which is independent of velocity. Furthermore, we assume that each scattering completely isotropizes the velocity distribution of the particles. These assumptions permit us to write simple expressions for the velocity moments of the post-scattering distribution function $f_0$ in terms of the moments of the pre-scattering function $f$. The theory is constructed to handle situations where $f$ differs appreciably from $f_0$. This is an important departure from the usual perturbative method of analyzing transport phenomena where the quantity $(f - f_0)$ in equation (2.1.3) is assumed to be small compared to the equilibrium distribution $f_0$. The non-perturbative nature of our theory allows us to write down moments of the Boltzmann equation which are valid even in situations that involve large departures from equilibrium.

The first example we have presented is the case of light particles diffusing in one dimension in a fixed background (§2.1, §2.2). We take the zeroth and first moments of the Boltzmann equation and close the equations by setting the second moment of the velocity, $v^2$, to a constant. This leads to a modified diffusion equation (2.1.11) which differs from the usual diffusion equation in having a wave operator instead of the standard Laplacian operator. The presence of the second time derivative in the wave operator enforces causality (Israel and Stewart 1980, Schweizer 1984). This term arises from a time derivative term in the first moment equation, which is neglected in deriving the standard diffusion equation. We therefore deduce that, if we wish to preserve causality, it is important to retain consistently all time derivatives of higher moments. We follow this prescription in the various problems we have analyzed and we find that we obtain equations that are causal in all cases.

The Green’s function of the one dimensional diffusion problem reveals several interesting features. First, it vanishes outside the particle “velocity cone,” $|x| > \sigma t$, where $x$ is distance from the point of injection of the particles and $\sigma$ is the r.m.s. velocity of the particles. This explicitly demonstrates the causal nature of the equations. Secondly,
the Green’s function has $\delta$-functions exactly at the two edges of the velocity cone. This shows that the theory effectively replaces the full velocity space distribution $f(v)$ by two $\delta$-functions at $v = \pm \sigma$. This simplification means that the theory cannot treat phenomena like phase mixing which arise from a continuous distribution of velocities. If such effects are important in any application then the theory we have presented is inadequate. One should either integrate the single velocity Green’s function over the particle distribution function or use higher order moments (see equations [2.1.19] & [2.1.20]). The theory also ignores the tail of high velocity particles which are normally found in a real distribution function $f(v)$. Real gases do not diffuse with an exact cutoff in space. While the bulk of the signal is limited to a speed equal to the r.m.s. speed $\sigma$ of the particles, there is always an exponentially small number of particles at large velocities propagating faster than $\sigma$. By eliminating the high velocity tail altogether our theory ignores this weak precursor signal and therefore has an exact cutoff of signals at a maximum propagation speed $\sigma$.

Leaving aside the issues of phase mixing and the exponential tail, we show that the theory does well in describing the behavior of the bulk of the particles. In fact, when all particles have a constant mean free time $\tau$ the theory gives the mean square distance $\langle x^2 \rangle$ traversed by particles exactly, regardless of their velocity distribution function. The agreement is perfect both at early times, when the particles are streaming freely, and at late times, when they have undergone many scatterings. Furthermore, even for a more general scattering law we can usually find an equivalent $\tau$ such that our theory provides a satisfactory description of the true evolution of $\langle x^2 \rangle$ (see figure 2).

It is a straightforward matter to generalize the theory to diffusion in three dimensions. The only complication originates from the fact that the minimal extension of the one-dimensional closure relation requires a somewhat more extreme assumption in three dimensions, namely that the velocity second moment tensor is not only constant but also isotropic. This assumption leads to a causal but somewhat unphysical Green’s function with negative densities. Despite this unsatisfactory feature, the theory provides an excellent description of the evolution of the mean square distance $\langle r^2 \rangle$ travelled by the particles.

An interesting special case of the three dimensional problem is when the density gradient is limited to a single direction, say the $x_1$ axis. In this case, we find that our three-dimensional diffusion equation reduces to a form that is identical to the one dimensional equation except that the speed of propagation of signals is no longer the r.m.s. particle speed $\sigma$ but rather $\sigma/\sqrt{3}$. In this particular case, it is clear that we effectively replace the full distribution function $f(v_1, v_2, v_3)$ by one that is restricted to two planes in velocity space, i.e. $v_1 = \pm \sigma/\sqrt{3}$. We have already seen the limitations associated with such an approximation, namely the inability to model phase mixing and signal propagation by the
exponential tail of fast particles. But apart from this, we expect the one-dimensional projection of the three-dimensional equation to behave physically and to predict the behavior of the majority of particles in a satisfactory way. In particular, the Green’s function for this problem does not have unphysical negative densities.

In §3 we have used our method to derive a general causal equation (3.1.11) for the evolution of the stress tensor. Since the stress tensor is of second order in the velocities, we take the zeroth, first and second moments of the Boltzmann equation, assuming as in the diffusion problem a constant mean free time $\tau$. We also set the third moments of the velocity to zero, which allows us to close the moment sequence and to derive a self-contained set of equations. The main additional feature in our model of the scattering is the introduction of a “cooling” factor $\xi$ (see equation 3.1.4) whereby scattering events are allowed to be inelastic. This modification allows us to model cooling processes where some of the kinetic energy of colliding particles is converted into another form of energy, e.g. radiation, and removed from the system. By including cooling, we are able to investigate steady state problems even when there is heating due to stresses.

Before proceeding to describe our results on the stress tensor, we discuss the effect of the various approximations we have made. The physical systems we have investigated involve velocity gradients along a single direction, a case where we can expect the theory to be well-behaved since we know that the particle diffusion problem admits a physical solution. The theory assumes a constant mean free time for all particles and therefore involves a simplification of the real collision integral. However, as we have shown in detail in the diffusion problem, and with some limited numerical experiments in the stress problem as well (figure 6), general scattering laws can usually be reduced to an effective constant $\tau$ with very little qualitative differences. Therefore, the assumption of a constant $\tau$ is unlikely to be a serious limitation.

The most serious simplification of our theory is the neglect of third moments. This approximation is reasonable when the velocity distribution has a reflection symmetry, or when the mean velocity profile is nearly linear in the vicinity of the region of interest. As we show in the Appendix, the third moment terms will be small provided the dimensionless second velocity derivative, $\sigma \tau^2 \partial^2 \bar{v} / \partial x^2$, is small compared to either unity or the square of the dimensionless shear (equation A10). There is another similar condition on the third derivative of the velocity (equation A11). These conditions may not be satisfied in highly transient flows or in the vicinity of shocks, but should be valid in smooth steady state flows and especially in rotating systems where the distortion of the particle velocity distribution tends to be limited (see figure 8). It is important to emphasize that the neglect of the third moments does not require the velocity gradient to be small; the theory we have developed
is valid for arbitrarily large gradients as long as the gradient itself is spatially constant, i.e. the second derivative of the velocity is sufficiently small.

Subject to the above caveats, equations (3.1.6) & (3.1.11) may serve as potential candidates to replace the Navier-Stokes equation whenever spatial or temporal derivatives are large. We have not tested these equations in their full generality. Rather, we have concentrated on special cases involving steady state shear flows to demonstrate several physical effects involving causality in viscous transport processes.

In §3.2–3.4, we derive some general results for the shear stress in a steady shear flow. According to the standard theory, a velocity shear $\partial \overline{v_2}/\partial x_1 \equiv 2A$ gives rise to a shear stress $n\nu(2A)$, where $n$ is the density and $\nu$ is the kinematic coefficient of viscosity. Our causal theory reproduces this result in limit that $A$ is small, with $\nu = \sigma^2\tau/3$, but more generally reveals three distinct new effects described below:

1. In §3.2 we consider steady shear of arbitrary magnitude $2A$ without advection and show that the shear stress is given by $n\nu(2A)(1 + 2[2A\tau]^2/3)^{-1}$ (equation 3.2.10). This expression gives a significantly smaller stress than the standard result mentioned above when the dimensionless shear $2A\tau$ is large. Whereas the standard relation for the shear stress diverges in the limit of large shear, the modified expression reaches a maximum value, which is less than the pressure, at a finite value of the dimensionless shear $2A\tau$ and reduces for larger values of the shear. This behavior is a consequence of causality. The particles at any given point typically have originated a mean free path away. Therefore, for a large shear, they have a large transverse velocity spread $\overline{u_2^2} \sim \sigma^2 \gg \overline{u_1^2}$. This translates to a stress limit $|\overline{u_1u_2}| < (\overline{u_1^2} \overline{u_2^2})^{1/2} \ll \sigma^2$. We have tested our result for the shear stress through numerical particle simulations spanning a range of values of the dimensionless shear $2A\tau$ (§3.3). The agreement is essentially perfect (to within numerical precision) when all particles have the same mean free time. Even when we instead consider the mean free path to be constant, our theory agrees well with the numerical simulations (figure 6). In particular, the theory predicts the maximum value of the shear stress accurately. Note that in the case of the shear stress, not only does causality set a limit to the maximum stress, it actually causes a reduction of the stress in the limit of large shear. This asymptotic behavior is very different from the behavior of scalar particle diffusion (e.g. Levermore and Pomraning 1981, Narayan 1992), where the flux asymptotically approaches to a constant value in the limit of a large density gradient.

2. When a constant advection velocity, $\overline{v_1}$, is added to the steady shear flow, we find that the shear stress is reduced further by a second suppression factor $(1 - 3\overline{v_1^2}/\overline{u_1^2})$ (cf. eq. [3.4.4]). When there is advection, the shear stress has to be communicated
by particles that move upstream faster than the advection speed $\overline{v}_1$. The flux of such upstream moving particles reduces as the advection speed increases and cuts off when $\overline{v}_1$ exceeds the r.m.s. velocity $\langle u_1^2 \rangle^{1/2}$ of particles in the direction of the flow. A similar qualitative result was obtained by Narayan (1992) who showed that causality leads to a reduction and ultimate cutoff of viscosity when there is advection. Narayan considered somewhat general velocity distribution functions whereas our discussion corresponds to a very simple velocity distribution consisting of two $\delta-$functions at $u_1 = \pm \langle u_1^2 \rangle^{1/2}$. On the other hand, Narayan assumed that the shear is small, whereas we allow arbitrarily large velocity gradients. One other difference is that Narayan’s expression for the viscosity coefficient left undetermined the critical advection velocity at which the viscosity will vanish. The present theory shows that, at this level of approximation, the cutoff occurs exactly at the sound speed $\sigma/\sqrt{3}$. We caution, however, that in real gases the advection factor may not involve a sudden truncation of the viscosity coefficient but rather an exponential cut off as $\overline{v}_1$ exceeds $\langle u_1^2 \rangle^{1/2}$. This is because of the high velocity tail in the particle distribution function which may be able to transmit a weak signal upstream even at large advection speeds.

3. We have also investigated the effect of a non-uniform advection and find that the shear stress is modified by a third factor, $(1 + 2\tau \partial \overline{v}_1 / \partial x_1)^{-1}$ (cf. eq. [3.4.4]). This shows that the shear stress is suppressed when the flow expands, i.e. when $\partial \overline{v}_1 / \partial x_1 > 0$, but is enhanced in the presence of compression. Curiously, the stress diverges when $2\tau \partial \overline{v}_1 / \partial x_1 = -1$, but this is not a serious problem since it refers to an unphysical limit where the gas is compressed to an infinite density in a collision time. The same limiting compression appears also in the context of our analysis of bulk viscosity in §3.5. Kato and Inagaki (1993) have independently noted that expansion or contraction of the advecting flow can modify the shear stress.

In §3.6, we extend the analysis to rotating flows and find that the above results for the shear stress are further modified. We find that the shear stress in a rotating flow in the absence of advection is given by $nu(2A)(1 + 2[2A\tau]^2/3 + 4[\kappa \tau]^2)^{-1}$, where $\Omega$ is the angular velocity and $\kappa^2 = 4\Omega(A+\Omega)$ is the square of the epicyclic frequency. Thus, in the presence of rotation, the shear stress is reduced compared to the classical diffusion formula $nu(2A)$ when either the shear or the epicyclic frequency is large compared to the collision frequency $1/\tau$. The suppression effect due to $\kappa$ has been noted in previous studies by Goldreich and Tremaine (1978) and Kato and Inagaki (1993). Kato and Inagaki’s approach is very similar to ours, whereas Goldreich and Tremaine treat two particle inelastic collisions in much greater detail, but they close the moment equations by assuming that the distribution function in velocity space has a triaxial Gaussian shape. When $\kappa \tau \gg 1$, a typical particle
undergoes many epicycles within a collision time and consequently moves radially only across the limited scale of an epicycle. The effective mean free path is therefore smaller than in the non-rotating case by a factor $\sim \kappa \tau$, and the stress is reduced by the square of this factor. For the same reason the distribution of particle velocities in phase space does not stretch indefinitely as in the non-rotating case (figures 5 and 7), but rather reaches a limiting oval shape as $\Omega$ increases beyond $1/\tau$ (figure 8). The new feature that comes out of our theory is that the viscosity coefficient is suppressed not just by the dimensionless epicyclic frequency $\kappa \tau$ but by a combination of the dimensionless shear $2A\tau$ and $\kappa \tau$. This result is of potential importance in astrophysics. For instance, in Keplerian disks, the two frequencies $2A$ and $\kappa$ are of comparable magnitudes.

Finally, in §3.7, we have applied our results to thin accretion disks, where we find that the causal limit on the shear stress leads to a constraint on the value of the disk viscosity coefficient $\nu$. This limit can be translated to an upper bound on the dimensionless $\alpha$-parameter defined by $\nu = \alpha c_s^2 / \Omega$, where $c_s$ is the sound speed in the disk. For hydrodynamic turbulence in a Keplerian disk we find that $\alpha < \alpha_1^2 / \sqrt{198} < 0.07$ for any value of $\Omega \tau$ (see figure 9), where $\alpha_1 < 1$ is the ratio between the r.m.s. turbulent blob speed and the sound speed in the disk. This bound can be exceeded only if the turbulence is non-hydrodynamic (e.g. magnetic) and $\alpha_1 > 1$. Note that our upper bound should be scaled up by a factor of $3/2$ if $\alpha$ is instead defined through the stress tensor relation $\overline{u_1 u_2} = -\alpha c_s^2$. The upper limit on $\alpha$ is much stronger in boundary layers where the velocity shear is larger than Keplerian.

To conclude, we note three issues related to viscous flows that we have not addressed in this paper, namely (i) the effect of advection in rotating flows (where the physics appears to be a little different than in the non-rotating case because of the Coriolis acceleration), (ii) the structure of viscous shocks, and (iii) time-dependent effects. However, the basic equations of our theory, especially the stress-evolution equation (2.1.11), appear to be capable of dealing with these issues. Work along these directions may provide further insight into the role of causality in astrophysical accretion disks.

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Appendix

In the following we rederive the results given in §3.2 for the components of the stress tensor in the presence of an arbitrary velocity shear, \( \partial v_2 / \partial x_1 = 2A \). We employ here a different, and possibly more transparent, approach in which we follow the trajectories of individual particles. We then extend this approach to investigate under which conditions our assumption of vanishing third moments is valid. This helps us identify the limit of validity of the theory developed in the paper.

Consider the particles at a particular plane, \( x_1 = 0 \), at a given time and label the particles by the coordinate \( x_1 \) at which they last suffered a scattering. Let the relative velocity component of a particle immediately after the last scattering be given by \( (u_1)_0, (u_2)_0, (u_3)_0 \). The relative velocity components at position \( x_1 = 0 \) are then

\[
  u_1 = (u_1)_0, \quad u_2 = (u_2)_0 + 2Ax_1 + \frac{1}{2} \frac{\partial^2 v_2}{\partial x_1^2} x_1^2 \cdots, \quad u_3 = (u_3)_0,
\]

where we have Taylor-expanded the mean velocity \( v_2 \) as a function of \( x_1 \). By the assumption that the scattering is isotropic, we have

\[
  (u_1)_0 = (u_2)_0 = (u_3)_0. \quad \text{(A2)}
\]

Furthermore, since \( u_1 \) and \( u_3 \) of a particle do not vary in between scatterings, we have

\[
  \overline{u_1^2} = (u_1^2)_0, \quad \overline{u_3^2} = (u_3^2)_0. \quad \text{(A3)}
\]

Consider a particle at \( x_1 = 0 \) with a particular value of \( u_1 \). Given our assumptions of constant density, constant second moments, and constant mean free time, the probability that this particle had its last scattering between \( x_1 \) and \( x_1 + dx_1 \) is given by

\[
  p(x_1)dx_1 = \frac{1}{|u_1|\tau} \exp(x_1/u_1\tau)dx_1, \quad x_1u_1 < 0. \quad \text{(A4)}
\]

We then see that, averaged over all particles at \( x_1 = 0 \), odd moments of \( x_1 \) such as \( \langle x_1 \rangle \) and \( \langle x_1^3 \rangle \) vanish, while even moments become

\[
  \langle x_1^2 \rangle = 2\overline{u_1^2}\tau^2, \quad \langle x_1^4 \rangle = 24\overline{u_1^4}\tau^4, \quad \cdots \quad \text{(A5)}
\]

Let us consider now the case discussed in §3.2, where the velocity profile is exactly linear with a slope \( 2A \) and where the second derivative of the velocity is zero. From equation (A1), we find the second moment of \( u_2 \) to be given by

\[
  \overline{u_2^2} = (u_2^2)_0 + 4A^2\overline{x_1^2} = (1 + 8A^2\tau^2)\overline{u_1^2}. \quad \text{(A6)}
\]
Using now the relation, \( \sigma^2 = u_1^2 + u_2^2 + u_3^2 \), we find that
\[
\overline{u_1^2} = \overline{u_3^2} = \frac{\sigma^2}{3 + 8A^2\tau^2}, \quad \overline{u_2^2} = \frac{1 + 8A^2\tau^2}{3 + 8A^2\tau^2} \sigma^2.
\] (A6)

Similarly, we find the velocity moment \( \overline{u_1 u_2} \) to be given by
\[
\overline{u_1 u_2} = (u_1 u_2)_0 + (2A x_1 (u_1))_0 = -2A\tau \overline{u_1^2} = -\frac{2A\tau}{3 + 8A^2\tau^2} \sigma^2.
\] (A7)

These relations are identical to the results derived in §3.2.

Let us proceed next to a case where the second velocity derivative \( \partial^2 \overline{v_2}/\partial x_1^2 \) and higher derivatives are not zero but small and let us estimate the contribution that they make to the shear-stress. This is best accomplished by considering the \( u_1-u_2 \) component of equation (3.1.11) which is given below:
\[
\overline{u_1 u_2} = -2\tau A\overline{u_1^2} - \tau \frac{\partial \overline{u_1^2 u_2}}{\partial x_1}.
\] (A8)

We can estimate \( \overline{u_1^2 u_2} \) using equations (A1) and (A4). Substituting this into equation (A8) we find that the modification to the shear stress is small if the following condition on the derivatives of \( \overline{v_2} \) is satisfied:
\[
\left| \frac{\partial}{\partial x_1} \left[ \tau^2 u_1^4 \frac{\partial^2 \overline{v_2}}{\partial x_1^2} \right] \right| \ll 2A\overline{u_1^2},
\] (A9)

Using the Gaussian closure relation, \( \overline{u_1^4} = 3(\overline{u_1^2})^2 \), this means that we require
\[
\sigma^2 \tau \left| \frac{\partial^3 \overline{v_2}}{\partial x_1^3} \right| \ll \frac{(3 + 8A^2\tau^2)}{\sqrt{24}},
\] (A10)
and
\[
\sigma^2 \tau^3 \left| \frac{\partial^3 \overline{v_2}}{\partial x_1^3} \right| \ll \frac{2A\tau(3 + 8A^2\tau^2)}{3}.
\] (A11)

We thus see that, when the dimensionless shear \( 2A\tau \ll 1 \), the condition for the third moments to have a negligible effect is that (i) the dimensionless second velocity derivative in the left-hand side of equation (A10) must be small compared to unity, and (ii) the dimensionless third derivative in (A11) must be small compared to \( 2A\tau \). When \( 2A\tau \gg 1 \), both of these criteria are weakened by a factor of \( A^2\tau^2 \).

For a rotating system the value of \( x_1 \) is limited by the radial extent of an epicycle \( \sim u_1/\kappa \). In the case of a thin disk with a scale-height \( h \ll r \) and a Keplerian velocity profile, the conditions (A10) and (A11) will be automatically satisfied since the left hand side is always of order \( (h/r) \ll 1 \). Thus the limit on \( \alpha \) which was derived in §3.7 under the assumption that the third velocity moments vanish is self-consistent.
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FIGURE CAPTIONS

**Fig. 1:** Comparison between the Green’s function for the one-dimensional diffusion of a Maxwellian distribution function (dot-dashed line), the Green’s function of the standard diffusion equation (dotted), and the Green’s function (2.1.13) of the modified diffusion equation (2.1.11) (solid). The curves are normalized to a unit area and are shown at two different times: \( t = 0.1 \tau \) (upper panel) and \( t = 10 \tau \) (lower panel), where \( \tau = \text{const} \) is the collision mean free time. In the lower panel, the two \( \delta \)-functions which propagate at \( v = \pm \sigma \) have not been plotted since they are suppressed by \( e^{-t/2\tau} = e^{-5} \).

**Fig. 2:** The evolution of the r.m.s. particle distance \( \langle x^2 \rangle^{1/2} \) for one-dimensional diffusion away from an instantaneous point source. Length and time are normalized by the mean free path \( l \) and the mean free time \( l/\sigma \), respectively. The different curves compare the exact result for a one-dimensional Maxwellian distribution of particles with a constant mean free path (solid line) to our one-dimensional causal model that assumes a constant mean free time \( \tau = (2/\pi)^{1/2}l/\sigma \) (short dashed) and the standard diffusion approximation result (long dashed). The causal model fits the true behavior very well. Note that our model describes the evolution of \( \langle x^2 \rangle \) precisely for any system of particles that has a constant mean free time.

**Fig. 3:** The same as in figure 2 but for the r.m.s. particle distance \( \langle r^2 \rangle \) in three-dimensional diffusion.

**Fig. 4:** Evolution of the second velocity moments with time. The results were obtained from a numerical simulation of a linear shear flow with \( 10^5 \) particles, corresponding to \( 2A = 1, (1-\xi)\sigma^2 = 1, \) and \( \tau = 1 \). Note that the moments achieve their steady-state values \( \overline{u_1^2} = \overline{u_3^2} = 1/3, \overline{u_2^2} = 1 \) and \( \overline{u_1u_2} = -1/3 \) in less than 10 mean free times.

**Fig. 5:** Phase-space distribution of particles in the \( u_1-u_2 \) plane for steady-state flow with a linear shear. Results are shown for three different shear amplitudes \( 2A\tau \), namely: (a) 0.1, (b) 1.0, and (c) 10, where \( \tau \) is the mean free time, taken to be independent of particle velocity. Note the strong distortion of the velocity distribution at large shear.

**Fig. 6:** The stress in a steady-state shear flow of particles with a constant mean free path \( l \). The points show \( \overline{u_1u_2}/\sigma^2 \) as a function of the shear amplitude \( 2Al/\sigma \), according to numerical simulations with \( 10^3 \) particles each. The curves show the prediction of equation (3.2.10) when we set \( \tau = l/\sigma \) (dashed line) or \( \tau = 0.55l/\sigma \) (solid line). The maximum
stress is predicted accurately by our model even though it assumes a constant mean free time \( \tau \) instead of a constant \( l \).

**Fig. 7:** Phase-space distribution of particles in the \( u_1-u_2 \) plane for steady-state flow of particles with a linear shear \( 2A \) and a constant mean free path \( l \). The different panels correspond to different shear amplitudes \( 2Al/\sigma \), namely: (a) 0.1, (b) 1.0, and (c) 10.

**Fig. 8:** Phase space distribution of particles in the \( u_1-u_2 \) plane for a steady shear in a rotating Keplerian disk. Results are shown for four values of \( \Omega \tau \), namely: (a) 0.1, (b) 0.4264, (c) 2.0, and (d) 10. Case (b) yields the maximum value of \( u_1u_2 \). Note the limited distortion of the velocity distribution for large shear, in contrast to the cases shown in figures (5) and (7). This is because the asymptotic form of the velocity distribution in a rotating flow under a large shear is moderated by the epicyclic motion (see equations [3.6.10]-[3.6.12]).

**Fig. 9:** The value of the \( \alpha \)-viscosity parameter for hydrodynamic turbulence in thin disks (cf. eq. [3.7.6]) as a function of \( \alpha_2 \equiv \Omega \tau \), where \( \Omega \) is the disk angular velocity and \( \tau \) is the mean time between collisions of turbulent blobs. The value of \( \alpha \) is divided by \( \alpha_1^2 \), where \( \alpha_1 \equiv \sigma/c_s < 1 \) is the ratio between the r.m.s. blob velocity and the sound speed in the disk. Results are presented for two rotation profiles: a Keplerian disk (solid line) and a “boundary layer” profile with \( 2A = 5\Omega \) (dashed).