The on-shell self-energy of the uniform electron gas in its weak-correlation limit

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PACS 71.10.Ca, 05.30.Fk

The ring-diagram partial summation (or RPA) for the ground-state energy of the uniform electron gas (with the density parameter $r_s$) in its weak-correlation limit $r_s \to 0$ is revisited. It is studied, which treatment of the self-energy $\Sigma(k,\omega)$ is in agreement with the Hugenholtz-van Hove (Luttinger-Ward) theorem $\mu - \mu_0 = \Sigma(k_F,\mu)$ and which is not. The correlation part of the lhs has the RPA asymptotics $a \ln r_s + a' + O(r_s)$ [in atomic units]. The use of renormalized RPA diagrams for the rhs yields the similar expression $a \ln r_s + a'' + O(r_s)$ with the sum rule $a' = a''$ resulting from three sum rules for the components of $a'$ and $a''$. This includes in the second order of exchange the sum rule $\mu_{2x} = \Sigma_{2x}$ [P. Ziesche, Ann. Phys. (Leipzig), 2006].
I. INTRODUCTION

Although not present in the Periodic Table the uniform or homogeneous electron gas (HEG) is still an important model system for electronic structure theory, cf. e.g. [1]. In its spin-unpolarized version, the HEG ground state is characterized by only one parameter $r_s$, such that a sphere with the radius $r_s$ contains on average one electron [2]. It determines the Fermi wave number as $k_F = 1/(\alpha r_s)$ in a.u. with $\alpha = [4/(9\pi)]^{1/3} \approx 0.521062$ and it measures simultaneously both the interaction strength and the density such that high density corresponds to weak interaction and hence weak correlation [3]. For recent papers on this limit cf. [4, 5, 6, 7]. Usually the total ground-state energy per particle is written as (here and in the following are wave numbers measured in units of $k_F$ and energies in $k_F^2$)

$$ e = e_0 + e_x + e_c, \quad e_0 = \frac{3}{5} \frac{1}{2}, \quad e_x = -\frac{3}{4} \frac{\alpha r_s}{\pi}, \quad e_c = (\alpha r_s)^2[\alpha \ln r_s + b + b_{2x} + O(r_s)], \quad (1.1) $$

where $e_0$ is the energy of the ideal Fermi gas, $e_x$ is the exchange energy in lowest (1st) order, and $e_c$ is referred to as correlation energy given here in its weak-correlation limit with $a = (1 - \ln 2)/\pi^2 \approx 0.031091$ after Macke [9] and $b \approx -0.0711$ after Gell-Mann and Brueckner [10]. $e_c$ contains also the 2nd-order of exchange with $b_{2x} \approx +0.02418$ after Onsager, Mittag, and Stephen [11]. Notice that $\tilde{e} = k_F^2 e = e/(\alpha r_s)^2$ gives the energy in a.u., e.g. the energy in zeroth order and the lowest-order exchange energy are $\tilde{e}_0 = 3/(10 \alpha^2 r_s^2)$ and $\tilde{e}_x = -3/(4\pi \alpha r_s)$, respectively.

Revisiting how Macke [9], Gell-Mann/Brueckner [10], and Onsager/Mittag/Stephen [11, 12] derived $e_c$ in its weak-correlation limit, it is shown here, that and how an analogous procedure - also called RPA (= random phase approximation) - applies to the self-energy $\Sigma(k, \omega)$. This latter quantity determines (i) the one-body Green’s function $G(k, \omega)$, from which follow the quasi-particle dispersion and damping and the momentum distribution $n(k)$ [13]. It furthermore appears (ii) in the Galitskii-Migdal formula for the potential energy [14] ($C_+$ means the closing of the contour in the upper complex $\omega$-plane),

$$ v = \frac{1}{2} \int d(k^3) \int \frac{d\omega}{2\pi i} \lim_{\delta \to 0} G(k, \omega) \Sigma(k, \omega), \quad (1.2) $$

which is related to the total energy $e$ through the virial theorem [15]

$$ v = r_s \frac{d}{dr_s} e. \quad (1.3) $$
(iii) Besides, $\Sigma(k, \omega)$ appears in the Luttinger theorem $\text{Im } \Sigma(1, \mu) = 0$ [16], in the Hugenholtz-van Hove theorem $\mu - \mu_0 = \Sigma(1, \mu)$ [17], and in the Luttinger-Ward formula for the quasi-particle weight $z_F$ [18]:

$$z_F = \frac{1}{1 - \Sigma'}, \quad \Sigma' = \text{Re } \frac{\partial \Sigma(1, \omega)}{\partial \omega} \bigg|_{\omega = \mu}. \quad (1.4)$$

So $\Sigma(1, \omega)$ is related to the chemical potential $\mu$, which can be calculated from $e$ according to the Seitz theorem [19]

$$\mu = \left( \frac{5}{3} - \frac{1}{3} r_s \frac{d}{dr_s} \right) e, \quad (1.5)$$

supposed $e$ is known as a function of $r_s$. Thus from Eq. (1.1) it follows for $\mu$:

$$\mu_0 = \frac{1}{2}, \quad \mu_x = -\frac{\alpha r_s}{\pi}, \quad \mu_c = (\alpha r_s)^2 \left[ a \ln r_s + \left( -\frac{1}{3}a + b + b_{2x} \right) + O(r_s) \right]. \quad (1.6)$$

Similarly as in Eqs. (1.1) and (1.6) it is

$$\Sigma(k, \omega) = \Sigma_x(k) + \Sigma_c(k, \omega), \quad \Sigma_x(k) = -\left( 1 + \frac{1 - k^2}{2k} \ln \left|\frac{k + 1}{k - 1}\right| \right) \frac{\alpha r_s}{\pi}. \quad (1.7)$$

Notice that $\Sigma(k, \omega)$ in lowest order (of exchange) does not depend on $\omega$. In particular, it is $\Sigma_x(1) = -\alpha r_s / \pi$, thus $\mu_x = \Sigma_x(1)$. Similarly, in 2nd order of exchange the sum rule $\mu_{2x} = \Sigma_{2x}(1, \frac{1}{2})$ holds [20]. With $\Sigma_{2x}(1, \frac{1}{2}) = (\alpha r_s)^2 c_{2x}$ it takes the form $b_{2x} = c_{2x}$. The asymptotic behavior $\Sigma_c(1, \mu) = (\alpha r_s)^2[a \ln r_s + c + c_{2x} + O(r_s)]$ and the sum rule

$$-\frac{1}{3}a + b = c \quad (1.8)$$

are a must as a consequence of the Hugenholtz-van Hove theorem. But the question is: which partial summation of Feynman diagrams has to be used for $\Sigma_c$ and what has to be used for $\mu$? The obvious answer to the first question seems to be $\Sigma_c = \Sigma_t + \cdots$ (the subscript ”r” means ”ring diagram”). Symbolically written it is defined as $\Sigma_t = G_0 \cdot (v_t - v_0)$ in terms of the Feynman-diagram building elements $G_0$, $v_0$, and $Q(k, \omega)$ [ = polarization propagator in RPA], cf. Eqs. (A.2), (A.4), and Fig. 1. $v_t$ is the effectively screened Coulomb repulsion following from $v_t = v_0 + v_0 Q v_t$, see Fig. 2. The Feynman diagrams of $\Sigma_t$ are shown in Fig. 3. To what extent this (naive) ansatz has to be changed in a particular way (to answer also the second question) will be discussed at the end of Sec. III.

Naively one should expect that in the weak-correlation limit the Coulomb repulsion $\epsilon^2/r$ [3] can be treated as perturbation. But in the early theory of the HEG, Heisenberg [8] has
shown, that ordinary perturbation theory with $e_c = e_2 + e_3 + \cdots$ and $e_n \sim (\alpha r_s)^n$ does not apply. Namely, in 2nd order, there is a direct term $e_{2d}$ and an exchange term $e_{2x}$, so that $e_2 = e_{2d} + e_{2x}$. Whereas $e_{2x}/(\alpha r_s)^2$, cf. Fig. 4, is a pure finite number $b_{2x}$ (not depending on $r_s$), the direct term $e_{2d}$ logarithmically diverges along the Fermi surface (i.e. for vanishing transition momenta $q$): $e_{2d} \to \ln q$ for $q \to 0$. This failure of perturbation theory has been repaired by Macke [9] with an appropriate partial summation of higher-order terms $e_{3r}, e_{4r}, \cdots$ (the subscript "r" means "ring diagram") up to infinite order. This procedure replaces the logarithmic divergence for $q \to 0$ by another logarithmic divergence, namely for $r_s \to 0$, cf. Eq. (1.1). This simultaneously "explains", why perturbation theory fails. The coefficient $a$, first found by Macke [9], has been confirmed later by Gell-Mann and Brueckner [10], who in addition to the logarithmic term numerically calculated two contributions to the next term $b$, namely $b_r$ and $b_{2d}$. More precisely, instead of $e_r = e_{2d} + e_{3r} + \cdots$ (notice that $e_{2r} = e_{2d}$) they calculated a more easily doable approximation $e_r^0 = e_{2d}^0 + e_{3r}^0 + \cdots$ (which is sufficient in the weak-correlation limit) with the result $e_r^0 = (\alpha r_s)^2 [a \ln r_s + b_r + O(r_s)]$, so that $e_r = e_r^0 + \Delta e_{2d} + O(r_s^3)$ with $\Delta e_{2d} = e_{2d} - e_{2d}^0 = (\alpha r_s)^2 b_{2d} + O(r_s^3)$. In summary,

$$e_c = e_r + e_{2x} + O(r_s^3)$$

$$= (e_{2d} + e_{3r} + \cdots) + e_{2x} + O(r_s^3)$$

$$= (e_{2d}^0 + e_{3r}^0 + \cdots) + (e_{2d} - e_{2d}^0) + e_{2x} + O(r_s^3)$$

$$= e_r^0 + \Delta e_{2d} + e_{2x} + O(r_s^3)$$

$$= (\alpha r_s)^2 \{[a \ln r_s + b_r + O(r_s)] + b_{2d} + b_{2x} + O(r_s)\} \ .$$  \ (1.9)

The result is Eq. (1.1) with $b = b_r + b_{2d}$. This procedure is revisited in Sec. 2 and then in Sec. 3 applied *mutatis mutandis* to the on-the-chemical-potential-shell self-energy $\Sigma(1, \mu)$, the rhs of the Hugenholtz-van Hove theorem. This is a contribution to the mathematics of the weakly-correlated (high-density) HEG. It concerns the HEG self-energy in RPA, extending and completing the paper [6].
II. THE TOTAL ENERGY

The Heisenberg-Macke story starts with the 2nd-order perturbation theory, \( e_2 = e_{2d} + e_{2x} \).

Its components are the direct (d) term \( e_{2d} \) (with \( q_0 \geq 0 \)) and the exchange (x) term \( e_{2x} \):

\[
e_{2d} = - (\alpha r_s)^2 \frac{2 \cdot 3}{(2\pi)^5} \int_{q > q_0} \frac{d^3q}{q^4} \frac{d^3k_1}{q^4} \frac{d^3k_2}{q^4} \frac{P}{q \cdot (k_1 + k_2 + q)}, \quad k_{1,2} < 1, \quad |k_{1,2} + q| > 1, \quad (2.1)
\]

\[
e_{2x} = + (\alpha r_s)^2 \frac{3}{(2\pi)^5} \int q^2 (k_1 + k_2 + q)^2 \frac{P}{q \cdot (k_1 + k_2 + q)}, \quad k_{1,2} < 1, \quad |k_{1,2} + q| > 1. \quad (2.2)
\]

\( P \) means the Cauchy principle value. (Notice the prefactor \(-1/2\) and the replacement \( q^4 \to q^2 (k_1 + k_2 + q)^2 \), when going from \( e_{2d} \) to \( e_{2x} \), and note that the 2nd-order vacuum diagram of Fig. 3 does not contribute.) As already mentioned, the integral \( 2.2 \) has been ingeniously calculated by Onsager et al. \[11\] with the result \( e_{2x} = (\alpha r_s)^2 b_{2x}, b_{2x} = \frac{1}{6} \ln 2 - \frac{3 \zeta(3)}{4 \pi^2} \approx +0.0242 \). Unlike \( e_{2x} \), the direct term \( e_{2d} \) logarithmically diverges for \( q_0 \to 0 \), i.e. along the Fermi surface. This is seen from

\[
e_{2d} = - (\alpha r_s)^2 \frac{2 \cdot 3}{(2\pi)^5} \int_{q > q_0} \frac{d^3q}{q^4} I(q), \tag{2.3}
\]

where the Pauli principle makes the function

\[
I(q) = \int \frac{d^3k_1}{q \cdot (k_1 + k_2 + q)} \frac{d^3k_2}{q^4} \frac{P}{P}, \quad k_{1,2} < 1, \quad |k_{1,2} + q| > 1 \quad (2.4)
\]

to linearly behave as \( I(q \to 0) = \frac{8\pi^4}{3} a q + O(q^3) \), see App. C, Eq. \( (C.2) \). Thus \( e_{2d} = (\alpha r_s)^2 [a \ln q_0^2 + \ldots] \), what agrees with \( 1.1 \) for \( q_0^2 \sim r_s \). The ring-diagram (or RPA) partial summation of Macke \[9\] and Gell-Mann/Brueckner \[10\] replaces the artificial (by hand) cut-off \( q_0 \) by a natural cut-off \( q_c \sim \sqrt{r_s} \). This is made replacing the divergent direct term \( e_{2d} \) by the non-divergent ring-diagram sum

\[
e_r = - \frac{3}{16\pi} \int d^3q \int \frac{dn}{2\pi i} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \left[ \left( \frac{q_c}{q} \right)^2 Q(q, \eta) \right]^n, \quad q_c = \sqrt{\frac{4\alpha r_s}{\pi}}. \quad (2.5)
\]

For \( Q(q, \eta) \), the polarization function in lowest order, is given in Eq. \( A.1 \). With \( \eta = iqu \) the contour integration along the real axis is turned to the imaginary axis:

\[
e_e = - \frac{3}{8\pi} \int du \int_0^{\infty} d(q^2) \left[ q^2 \ln \left( 1 + \frac{q_c^2}{q^2} \right) R(q, u) - q_c^2 R(q, u) \right]. \tag{2.6}
\]
This has the advantage, that \( R(q, u) = Q(q, iqu) \) is a real function, being symmetric in \( u \), cf. Eq. (A.2). Let us control Eq. (2.6): The small-\( r_s \) expansion of the \( u \)-integrand starts with \((-1/2)(q_c/q)^4 R^2(q, u)\), which just reproduces the 2nd-order direct term \( e_{2d} \) with the help of the integral identity (C.7). For \( r_s \to 0 \), a direct numerical investigation of Eq. (2.6) yields \( e_r \to (\alpha r_s)^2(0.031091 \ln r_s - 0.0711 + \cdots) \). This result is analytically rederived in the following.

Namely, in the weak-correlation limit \( r_s \to 0 \) one can approximate \( R(q, u) \approx \Theta(q_1 - q)R_0(u) + \cdots \) with \( R_0(u) = 1 - u \arctan 1/u \), so that \( e_r = e_r^0 + O(r_s^3) \), where \( e_r^0 \) contains only the \( q \)-independent \( R_0(u) \) and its \( q \)-integration is restricted to \( 0 \leq q \leq q_1 \):

\[
e_r^0 = \frac{3}{8\pi} \int_0^{q_1} dq^2 \left[ q^2 \ln[q^2 + q_c^2 R_0(u)] - q^2 \ln q^2 - q_c^2 R_0(u) \right]
\]

\[
= \frac{3}{8\pi} \int_0^{q_1} du \left[ \frac{1}{2} q^2 R_0^2(u) \left[ \ln \left( \frac{q^2}{q_c^2} R_0(u) \right) - \frac{1}{2} \right] + O(r_s^3) \right].
\]

(For a discussion of the divergent/convergent behavior of the \( q \)-series cf. 10, text after their Eq. (23).) With \( q_c^2 = 4\alpha r_s/\pi \) it is

\[
e_r^0 = (\alpha r_s)^2 \frac{3}{\pi^3} \int_0^{q_1} du R_0^2(u) \left[ \ln r_s + \ln \frac{4\alpha}{\pi} - \frac{1}{2} + \ln R_0(u) - 2 \ln q_1 \right] + O(r_s^3). \tag{2.7}
\]

So it results \( e_r^0 = (\alpha r_s)^2 \left[ a \ln r_s + b_r - 2a \ln q_1 + O(r_s^3) \right] \) with

\[
\frac{3}{\pi^3} \int_0^{q_1} du R_0^2(u) = a \approx 0.031091,
\]

\[
b_r = a \left( \ln \frac{4\alpha}{\pi} - \frac{1}{2} \right) + \frac{3}{\pi^3} \int_0^{q_1} du R_0^2(u) \ln R_0(u) \approx -0.045423. \tag{2.9}
\]

For the integrals cf. Eq. (B.3).

As it has been explained before and in Eq. (1.9), the difference between the correct 2nd-order term of Eq. (2.5) and the first term in the expansion of \( e_r^0 \), namely

\[
e_{2d}^0 = -(\alpha r_s)^2 \frac{2 \cdot 3}{\pi^3} \int_{q_0}^{q_1} dq \int_0^{q} du R_0^2(u) + O(r_s^3) = -(\alpha r_s)^2 \frac{2a}{q_0} \int_{q_0}^{q_1} dq + O(r_s^3), \tag{2.10}
\]

gives

\[
\Delta e_{2d} = e_{2d} - e_{2d}^0 = -(\alpha r_s)^2 \frac{2 \cdot 3}{\pi^3} \left[ \int_{q_0}^{q} dq I(q) \frac{q}{8\pi q} - \left( \frac{1}{q_0} + \frac{q_1}{q} \right) \int_{q_0}^{q_1} \frac{dq}{q} \frac{\pi^3}{3} \right] + O(r_s^3). \tag{2.11}
\]
\[
\Delta e_{2d} = (\alpha r_s)^2 [b_{2d} + 2a \ln q_1 + O(r_s)] \text{ shows, that the sum } e_0 + \Delta e_{2d} \text{ does not depend on } q_1 \text{ for } r_s \to 0. \text{ Besides, the first term of } b_{2d} \text{ is no longer divergent with } q_0 \to 0, \text{ therefore it can be set } q_0 = 0:
\]

\[
b_{2d} = \frac{3}{4\pi^4} \int_0^1 \frac{dq}{q} \left[ I(q) - \frac{8\pi^4}{3} a \Theta(1 - q) \right] = \frac{1}{4} + \frac{1}{\pi^2} \left[ -\frac{11}{6} - \frac{8}{3} \ln 2 + 2(\ln 2)^2 \right] \approx -0.025677. \tag{2.12}
\]

Together it is \( b = b_t + b_{2d} \approx -0.0711 \), what agrees with the above mentioned numerical evaluation of Eq. (2.6).

**III. THE SELF-ENERGY**

Here - after the training of Sec. II - , it is aimed to calculate \( \Sigma_c(1, \mu) \) in the weak-correlation limit, where there is a scheme for \( \Sigma_c(1, \mu) \) analog to Eq. (1.9) for \( e_c \) with one difference. Namely, whereas the chemical-potential shift \( \mu \) results from vacuum diagrams, the self-energy \( \Sigma(k, \omega) \) results from non-vacuum diagrams, which are functions of \( k \) and \( \omega \), see the discussion at the end of this Section.

In analogy to Eqs. (2.1) and (2.2), the self-energy in 2nd order is \( \Sigma_{2d}(k, \omega) = \Sigma_{2d}(k, \omega) + \Sigma_{2x}(k, \omega) \), the 2nd-order self-energy diagram of Fig. 5 vanishes. From (A.6) it follows for the direct term

\[
\Sigma_{2d}(1, \omega) = \frac{(\alpha r_s)^2}{2\pi^4} \int_{q > q_0} d^3q d^3k_2 \left[ \frac{\Theta(\lvert e_1 + q \rvert - 1)}{\omega - \frac{1}{2} - q \cdot (e_1 + k_2 + q) + i\delta} + \frac{\Theta(1 - \lvert e_1 + q \rvert)}{\omega - \frac{1}{2} - q \cdot (e_1 - k_2) - i\delta} \right] \Theta(1 - k_2) \Theta(\lvert k_2 + q \rvert - 1). \tag{3.1}
\]

For the corresponding exchange term \( \Sigma_{2x}(1, \omega) \) cf. Fig. 4 and ref. [20], where it has been shown that \( \Sigma_{2x} = \text{Re} \Sigma_{2x}(1, \frac{1}{2}) = (\alpha r_s)^2 c_{2x} \) with the sum rule \( c_{2x} = b_{2x} \approx +0.0242 \). On the other hand, the direct term \( \Sigma_{2d} \) diverges logarithmically for \( q_0 \to 0 \). This is seen from

\[
\Sigma_{2d} = \text{Re} \Sigma_{2d} \left( 1, \frac{1}{2} \right) = -\frac{(\alpha r_s)^2}{2\pi^4} \int_{q > q_0} \frac{d^3q}{q^4} J(q), \tag{3.2}
\]
where the Pauli principle makes the function

\[
J(q) = \int \frac{d^3q}{4\pi^3} \frac{e^1}{d^3k_2} \left[ \Theta(|e_1 + q| - 1)P \frac{q \cdot (e_1 + k_2 + q)}{q \cdot (e_1 - k_2)} + \Theta(1 - |e_1 + q|)P \right] \Theta(1 - k_2) \Theta(|k_2 + q| - 1),
\]

(3.3)

to linearly behave as \( J(q \to 0) = \pi^3a q + O(q^3) \), see App. D, Eq. (D.2). Thus \( \Sigma_{2d} = (\alpha r_s)^2(a \ln q_0^2 + \cdots) \), what is for \( q_0^2 \sim r_s \) in full agreement with the Hugenholtz-van Hove theorem (1.4) and the perturbation expansion of \( \mu \), which - because of (1.5) - gives \( \mu_{2d} = e_{2d} = (\alpha r_s)^2(a \ln r_s + \cdots) \). In the ring-diagram partial summation the divergent direct term \( \Sigma_{2d}(k, \omega) \) is replaced by the non-divergent sum (its Feynman diagrams are shown in Fig. 3)

\[
\Sigma(k, \omega) = (\alpha r_s)^2 \frac{2}{\pi^3} \int \frac{d^3q}{q^2} \int \frac{d\eta}{2\pi^1} \frac{Q(q, \eta)}{q^2 + q^2 Q(q, \eta)} \times \\
\times \left[ \frac{\Theta(|k + q| - 1)}{\omega + \eta - \frac{1}{2}k^2 - q \cdot (k + \frac{q}{\eta}q) + i\delta} + \frac{\Theta(1 - |k + q|)}{\omega + \eta - \frac{1}{2}k^2 - q \cdot (k + \frac{q}{\eta}q) - i\delta} \right].
\]

(3.4)

Next, this expression is carefully controlled:

(i) If the term \( q^2 Q(q, \eta) \), which describes the RPA screening of the bare Coulomb interaction \( 1/q^2 \), is deleted, then \( \Sigma(k, \omega) \) changes to \( \Sigma_{2d}(k, \omega) \), as it is seen from Eq. (A.5).

(ii) Use of Eq. (3.4) in the Galitskii-Migdal formula (1.2) yields the ring-diagram summation for the potential energy, \( v_r \), which follows from \( e_r \) through the virial theorem (1.3).

(iii) The expression (3.4) allows to calculate the derivative \( \Sigma'_r(k, \omega) = \partial \Sigma_r(k, \omega)/\partial \omega \). From \( \Sigma'_r = \text{Re} \Sigma'_r(1, \frac{k}{2}) \) one obtains \( z_F \) in RPA by means of the Luttinger-Ward formula (1.4) as \( z_F = 1 + \Sigma'_r + \cdots \) with

\[
\Sigma'_r = \frac{\alpha r_s}{\pi^2} \int_0^\infty du R_0(u) \frac{R_0(u)}{R_0(u)} \arctan \frac{1}{u} + \cdots = -0.177038 \ r_s + \cdots.
\]

(3.5)

This is just the well-known RPA result for \( z_F \). For the integral see Eq. (3.4).

After this controlling and training, \( \Sigma_r = \text{Re} \Sigma_r(1, \frac{k}{2}) \) is derived from Eq. (3.4) in a similar way as \( e_r \) in Eqs. (2.17) - (2.12). The next steps again are substitution \( \eta = iqu \) and contour deformation from the real to the imaginary axis with \( x = e \cdot e_q \) and \( |e + q| \geq 1 \pm \delta \):

\[
\Sigma_r = -\frac{(\alpha r_s)^2}{\pi^4} \int \frac{d^3q}{q^2} \int \frac{R(q, u)}{q^2 + q^2 R(q, u)} \cdot \frac{1}{(x + \frac{q}{u}) - iu} \\
= -\frac{(\alpha r_s)^2}{\pi^4} \int_0^\infty du \int \frac{d^3q}{q^2} \frac{R(q, u)}{q^2 + q^2 R(q, u)} \cdot \frac{2(x + \frac{q}{u})}{(x + \frac{q}{u})^2 + u^2}.
\]

(3.6)
In the last step the \( u \)- and \( q \)-integrations are exchanged and it is used that \( R(q, u) \) is even in \( u \), cf. Eq. (B.1); so the imaginary part again vanishes. Next the \( q \)-integration is specified as

\[
\Sigma_r = -\frac{(\alpha r_s)^2}{\pi^3} \int_0^\infty du \int_0^\infty \frac{R(q, u)}{q^2 + q_s^2 R(q, u)} \cdot \left[ \frac{2(x + \frac{q}{2})}{(x + \frac{q}{2})^2 + u^2} \right] dx
\]

\[
= -\frac{(\alpha r_s)^2}{\pi^3} \int_0^\infty du \int_0^\infty dq \frac{R(q, u)}{q^2 + q_s^2 R(q, u)} \cdot \ln \left( \frac{q}{2} + 1 \right)^2 + u^2 . \tag{3.7}
\]

Let us control Eq. (3.7): The small-\( r_s \) expansion of the \( u \)-integrand starts with \( R(q, u)/q^2 \), which just reproduces the 2nd-order direct term (3.3) with the help of the integral identity (B.7). In the limit \( r_s \to 0 \), Eq. (3.7) numerically gives \( \Sigma_r \approx (\alpha r_s)^2(0.031091 \ln r_s - 0.081463 + \cdots) \). This result is analytically confirmed by the following. The asymptotic behavior for \( r_s \to 0 \) is determined by the lower integration limit \( q \to 0 \), therefore \( R(q, u) \) and \( \ln \cdots \) can be approximated by \( R_0(u) = 1 - u \arctan 1/u \) and \( L_0(u) = 2q/(1 + u^2) \), respectively:

\[
\Sigma_r = -\frac{(\alpha r_s)^2}{\pi^3} \int_0^\infty du \int_0^\infty d(q^2) \frac{[R_0(u) + R_1(u)q^2 + \cdots][L_0(u) + L_1(u)q^2 + \cdots]}{q^2 + q_s^2 R_0(u)} = \Sigma_r^0 + O(r_s^3) . \tag{3.8}
\]

Finally the \( q \)-integration yields

\[
\Sigma_r^0 = -\frac{(\alpha r_s)^2}{\pi^3} \int_0^\infty du \frac{R_0(u)}{1 + u^2} \int_0^{q_s^2} dq \frac{dq}{q^2 + q_s^2 R_0(u)}
\]

\[
= -\frac{(\alpha r_s)^2}{\pi^3} \int_0^\infty du \frac{R_0(u)}{1 + u^2} \ln[q^2 + q_s^2 R_0(u)]_0^{q_s^2}
\]

\[
= -\frac{(\alpha r_s)^2}{\pi^3} \int_0^\infty du \frac{R_0(u)}{1 + u^2} [2 \ln q_s - \ln q_s^2 R_0(u)] . \tag{3.9}
\]

With Eq. (B.3) it turns out \( \Sigma_r^0 = (\alpha r_s)^2[a \ln r_s + c_t - 2a \ln q_s + O(r_s)] \),

\[
c_t = a \ln \frac{4\alpha}{\pi} + \frac{2}{\pi^3} \int_0^\infty du \frac{R_0(u) \ln R_0(u)}{1 + u^2} \approx -0.035059 . \tag{3.10}
\]

The difference between the exact 2nd-order term of Eq. (3.2) and the first term in the \( q \)-expansion of \( \Sigma_r^0 \), namely

\[
\Sigma_{2q} = -\frac{(\alpha r_s)^2}{\pi^3} \int_0^{q_s^2} dq \int_0^\infty du \frac{R_0(u)}{1 + u^2} = -\frac{(\alpha r_s)^2}{\pi^3} \int_0^{q_s^2} dq \frac{2q}{q}(1 - \ln 2) , \tag{3.11}
\]
yields
\[
\Delta \Sigma_{2d} = \Sigma_{2d} - \Sigma^0_{2d} = - (\alpha r_s)^2 \frac{2}{\pi^3} \left[ \int_{q_0}^{\infty} \frac{dq}{q} J(q) - \int_{q_0}^{2d} \frac{dq}{q} \pi(1 - \ln 2) \right] \quad (3.12)
\]
\[
= (\alpha r_s)^2 \left\{ - \frac{2}{\pi^3} \left[ \int_{q_0}^{1} \frac{dq}{q} \left( \frac{J(q)}{q} - \pi(1 - \ln 2) \right) \right] + \int_{1}^{\infty} \frac{dq}{q} \frac{J(q)}{q} + 2a \right\}.
\]
\[
\Delta \Sigma_{2d} = (\alpha r_s)^2 [c_{2d} + 2a \ln q_2 + \cdots] \text{ shows, that the sum } \Sigma^0_r + \Delta \Sigma_{2d} \text{ does not depend on } q_2 \text{ for } r_s \to 0. \text{ Besides the first term of } c_{2d} \text{ is no longer divergent with } q_0 \to 0, \text{ therefore it can be set } q_0 = 0:
\]
\[
c_{2d} = - \frac{2}{\pi^3} \int_{0}^{\infty} \frac{dq}{q} \left[ \frac{J(q)}{q} - \pi(1 - \ln 2) \Theta(1 - q) \right] \approx -0.046404. \quad (3.13)
\]
For the } J(q)-\text{integral cf. Eq. (D.4). Together it is } c = c_r + c_{2d} \approx -0.08146, \text{ to be compared with } -\frac{1}{3}a + b = -0.08146. \text{ This is just the expected sum rule (1.8). But the difference } \Sigma_r(1, \frac{1}{2} + \mu_x \cdots) - \Sigma_r(1, \frac{1}{2}) \neq 0 \text{ seems to disturb this sum rule } c = -\frac{1}{3}a + b. \text{ This 'misfit' is removed by an additional partial summation replacing Fig. 3 by Fig. 4 i.e. replacing } \Sigma_r = G_0 \cdot (v_r - v_0) \text{ by } \Sigma^x_r = G_x \cdot (v_r - v_0) \text{ with the renormalized one-body Green's function}
\]
\[
G_x(k, \omega) = \frac{1}{\omega - \frac{1}{2}k^2 - \Sigma^x(k) \pm i\delta}, \quad (3.14)
\]
see Fig. 7. For } k = 1 \text{ it is } G_x(1, \omega) = 1/[\omega - \frac{1}{2} - \mu_x \pm i\delta]. \text{ Thus } \Sigma^x_r(1, \frac{1}{2} + \mu_x + \cdots) = \Sigma_r(1, \frac{1}{2}) + \cdots \text{ in the limit } r_s \to 0. \text{ So the conjectured relation (1.8) holds. This can be seen still in another way.}

Namely, note the similarities of Eqs. (3.10) and (2.9) as well as Eqs. (3.13) and (2.12). Their differences are
\[
c_r - b_r = \frac{1}{3}a \quad \text{and} \quad c_{2d} - b_{2d} = -\frac{2}{3}a \quad (3.15)
\]
using the identities (D.5) and (D.8), respectively. Thus, with } b = b_r + b_{2d} \text{ and } c = c_r + c_{2d}, \text{ the sum rule (1.8) is proven once more.

IV. SUMMARY

Summarizing, the Hugenholtz-van Hove theorem } \mu - \mu_0 = \Sigma(1, \mu) \text{ takes for the HEG ground state in its weak-correlation limit } r_s \to 0 \text{ the asymptotic form}
\[
- \frac{\alpha r_s}{\pi} + (\alpha r_s)^2 [a \ln r_s + \left( -\frac{1}{3}a + b_r + b_{2d} \right) + b_{2x} + O(r_s)] =
\]
\[-\frac{\alpha r_s}{\pi} + (\alpha r_s)^2 [a \ln r_s + (c_r + c_{2d}) + c_{2x} + O(r_s)] . \]  

(4.1)

So the sum rules [with \(a = \frac{1}{\pi^2} (1 - \ln 2)\) and \(b_r, b_{2d}, c_r, c_{2d}\) given in Eqs. [2.9], [2.12], [3.10], [3.13]]

\[ \frac{1}{3} a + b_r = c_r, \quad -\frac{2}{3} a + b_{2d} = c_{2d}, \quad b_{2x} = c_{2x} \]  

(4.2)

hold. The last relation or \(\mu_{2x} = \Sigma_{2x}\) has been shown in [20]. The sum rules [4.2] are relations between the Macke number \(a\), the Gell-Mann/Brueckner numbers \(b_r, b_{2d}\), and the Onsager/Mittag/Stephen number \(b_{2x}\) (which altogether describe the \(r_s \to 0\) asymptotics of the correlation energy \(e_c\)) on the one hand and corresponding numbers \(c_r, c_{2d}, c_{2x}\) of the on-shell self-energy \(\Sigma(1, \mu)\) on the other hand. Eqs. (4.1) and (4.2) answer the question which partial summation of Feynman diagrams has to be used in the weak-correlation limit for the self-energy \(\Sigma\) on the rhs of the Hugenholtz-van Hove theorem. They result from the renormalized ring-diagram (or RPA) partial summation (symbolically written as) \(\Sigma \approx \Sigma^x = G_x \cdot v_r\) with \(v_r = v_0/(1 - Q v_0)\) and \(G_x(k, \omega) = \) renormalized one-body Green’s function and \(Q(q, \eta) = \) polarization propagator in RPA, see Eqs. (3.14) and (A.4), respectively, and Figs. 1, 2, 6, 7. (They not result from \(\Sigma \approx \Sigma^{HF} = G \cdot v_0\), see Fig. 8 as an alternative ansatz with \(G(k, \omega) = \) full one-body Green’s function of the interacting system [21].) Byproducts are the analytical representation of \(b_{2d}\), a detailed description of the momentum-transfer or Macke function \(I(q)\) for the RPA vacuum diagrams (App. C), the introduction and discussion of an analog function \(J(q)\) for the RPA self-energy diagrams (App. D), and the proof of integral identities, which relate \(I(q)\) and \(J(q)\) to the polarization-propagator function \(R(q, u)\), cf. Apps. C and D.

**Acknowledgments**

The author thanks G. Diener and E. Runge for valuable hints and F. Tasnádi for technical help and acknowledges P. Fulde for supporting this work.

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APPENDIX A: ONE-BODY GREEN’S FUNCTION, PARTICLE-HOLE PROPAGATOR, AND 2ND-ORDER SELF-ENERGY

In the following the identities (with $z = x + iy$)

$$
\int_{C_\pm} \frac{dz}{2\pi i (z - z_1)(z - z_2)} = \begin{cases}
0 & \text{for sign } y_1 = \text{sign } y_2 \\
\frac{1}{z_1 - z_2} & \text{for } y_1 > 0 \text{ and } y_2 < 0 \\
\frac{1}{z_2 - z_1} & \text{for } y_1 < 0 \text{ and } y_2 > 0
\end{cases}
$$

(A.1)

for contour integrations in the complex $z$-plane are used ($z = x + iy$, $C_\pm =$ contour along the real axis, closed above or below with a half circle). The building elements of the Feynman diagrams are

$$
G_0(k, \omega) = \frac{\Theta(k - 1)}{\omega - \frac{1}{2}k^2 + i\delta} + \frac{\Theta(1 - k)}{\omega - \frac{1}{2}k^2 - i\delta}, \quad \delta \to 0 \quad \text{and} \quad v_0(q) = \frac{4\pi\alpha r_s}{q^2}.
$$

(A.2)

From $G_0$ follows the particle-hole propagator $Q$ in RPA according to

$$
Q(q, \eta) = -\int \frac{d^3k}{4\pi} \int \frac{d\omega}{2\pi i} G_0(k, \omega)G_0(|k + q|, \omega + \eta)
$$

(A.3)

with the result

$$
Q(q, \eta) = \int \frac{d^3k}{4\pi} \left[ \frac{1}{q(k + \frac{1}{2}q) - \eta - i\delta} + \frac{1}{q(k + \frac{1}{2}q) + \eta - i\delta} \right] \Theta(1 - k)\Theta(|k + q| - 1).
$$

(A.4)
and \((A.4)\) used in the direct term of the 2nd-order off-shell self-energy
\[
\Sigma_{2d}(k, \omega) = (\alpha r_s)^2 \frac{2}{\pi^3} \int_{q > q_0} \frac{d^3q}{q^4} \int \frac{d\eta}{2\pi i} \, Q(q, \eta) G_0(|k + q|, \omega + \eta)
\]
\[(A.5)\]
yields
\[
\Sigma_{2d}(k, \omega) = (\alpha r_s)^2 \frac{2}{\pi^3} \int_{q > q_0} \frac{d^3q}{q^4} \int d^3k' \left[ \frac{\Theta(|k + q| - 1)}{\omega - \frac{1}{2} k^2 - q \cdot (k + k' + q) + i\delta} \right.
\]
\[
+ \frac{\Theta(1 - |k + q|)}{\omega - \frac{1}{2} k^2 - q \cdot (k - k') - i\delta} \right] \Theta(1 - k') \Theta(|k' + q| - 1).
\]
\[(A.6)\]
This expression used in \((1.2)\) yields \(v_{2d} = 2e_{2d}\) in agreement with the virial theorem \((1.3)\).

**APPENDIX B: THE FUNCTION \(R(q, u)\)**

\(Q(q, \eta)\) becomes real for imaginary \(\eta\):
\[
R(q, u) = Q(q, iqu) = \frac{1}{2} \left[ 1 + \frac{1 + u^2 - q^2}{2q} \ln \left( \frac{q^2 + 1 + u^2}{(q^2 - 1)^2 + u^2} \right) \right.
\]
\[
- u \left( \frac{\arctan \left( \frac{1 + q}{u} \right) + \arctan \left( \frac{1 - q}{u} \right)}{2} \right) \right].
\]
\[(B.1)\]
The function \(R(q, u)\) has the \(q\)-expansion \(R(q, u) = R_0(u) + q^2 R_1(u) + \cdots\) with
\[
R_0(u) = 1 - u \arctan \frac{1}{u}, \quad R_1(u) = - \frac{1}{12(1 + u^2)^2}, \quad R_2(u) = - \frac{1 - 5u^2}{240(1 + u^2)^4}.
\]
\[(B.2)\]
Here is a list of integrals:
\[
\int_0^\infty du \frac{R_0^2(u)}{1 + u^2} = \frac{\pi^3}{2} a \approx 0.321336, \quad \int_0^\infty du \frac{R_0^2(u)}{1 + u^2} \ln R_0(u) \approx -0.176945,
\]
\[(B.3)\]
\[
\int_0^\infty du \frac{R_0(u)}{1 + u^2} = \frac{\pi^3}{2} a \approx 0.482003, \quad \int_0^\infty du \frac{R_0(u) \ln R_0(u)}{1 + u^2} \approx -0.345751
\]
\[(B.4)\]
The last but one integral appears in the weak-correlation limit of the quasi-particle weight \(z_F\) \[24\]. The identity
\[
\frac{2}{\pi^3} \int_0^\infty du \frac{R_0(u)}{1 + u^2} \ln R_0(u) \left[ \frac{1}{1 + u^2} - \frac{3}{2} R_0(u) \right] = -\frac{1}{6} a
\]
\[(B.5)\]
leads to the sum rule \((3.15)\).
APPENDIX C: THE FUNCTION $I(q)$

Using cylindrical coordinates and the centre of the vector $q$ as origin, Macke succeeded to calculate $I(q)$ explicitly as

\[
I(q \leq 2) = \pi^2 \left[ \left( \frac{29}{15} - \frac{8}{3} \ln 2 \right) q - \frac{q^3}{20} + \frac{1}{q} \left( \frac{16}{15} + q - \frac{q^3}{6} + \frac{q^5}{80} \right) \ln \left( 1 + \frac{q}{2} \right) + \frac{1}{q} \left( \frac{16}{15} - q + \frac{q^3}{6} - \frac{q^5}{80} \right) \ln \left( 1 - \frac{q}{2} \right) \right].
\]

\[
I(q \geq 2) = \pi^2 \left[ \frac{1}{30} \left[ 4 (22 + q^2) + \frac{1}{q} (q + 2)^3 (4 - 6q + q^2) \ln \left( 1 + \frac{2}{q} \right) + \frac{1}{q} (q - 2)^3 (4 + 6q + q^2) \ln \left( 1 - \frac{2}{q} \right) \right] \right].
\] (C.1)

Therefore $I(q)$ is referred to as Macke function, cf. also [23]. (The last two lines of (C.1) correct errors in [4], Eq. (A.1).) $I(q)$ has the properties

\[
I(q \to 0) = \frac{8\pi^2}{3} \left( 1 - \ln 2 \right) q - \frac{\pi^2}{6} q^3 + \cdots, \quad I(q \to \infty) = \left( \frac{4\pi}{3} \right)^2 \left( \frac{1}{q^2} + \frac{2}{5} \frac{1}{q^4} + \cdots \right),
\]

\[
I(2) = \frac{4\pi^2}{15} (13 - 16 \ln 2) \approx 5.02598, \quad I'(2) = -\frac{8\pi^2}{5} (-1 + 2 \ln 2) \approx -6.10012,
\]

\[
I''(2^+) = \frac{8\pi^2}{15} (2 + \ln 2) \approx 14.1762, \quad I''(2^-) = -\frac{2\pi^2}{15} (7 - 4 \ln 2) \approx -5.56305.
\] (C.2)

$I(q)$ has a maximum of 7.12 at $q = 1.36$. $I(q)$ and $I'(q)$ are continuous at $q = 2$, but $I''(q)$ has there a jump discontinuity of $2\pi^2$. This is because the topology changes from overlapping to non-overlapping Fermi spheres, when passing $q = 2$ from below. Its normalization is

\[
\int_0^\infty dq \, I(q) = \frac{8\pi^2}{45} (-3 + \pi^2 + 6 \ln 2) \approx 19.3505.
\] (C.3)

$I(q)$ is shown in Fig. 9. Multiplying the integral

\[
\int_0^\infty \frac{dq}{q^2} \left[ I(q) - \frac{8\pi^2}{3} (1 - \ln 2) q \Theta(1 - q) \right] = \frac{\pi^2}{9} \left[ 22 - 3\pi^2 + 32 \ln 2 - 24 (\ln 2)^2 \right] \approx 3.343456
\]

with $-3/(4\pi^4)$ yields Eq. (2.12).

The original expression for $I(q)$ arises from the diagram rules for $e_{2d}$ with Eq. (3.3) as

\[
I(q) = \frac{1}{2} (4\pi)^2 \text{Re} \int \frac{dn}{2\pi i} Q^2(q, \eta).
\] (C.5)
One way is to insert (A.4) into (C.5). It results (with $x_i = q \cdot (k_i + \frac{1}{2}q)$, $i = 1, 2$)

$$I(q) = \text{Re} \int \frac{d^3k_1d^3k_2}{x_1 + x_2 - i\delta} = \int \frac{d^3k_1d^3k_2}{q \cdot (k_1 + k_2 + q)}$$

(C.6)

in agreement with Eq. (2.4). Another way is the analytical continuation and the deformation of the integration contour from the real to the imaginary axis with the advantage that $R(q, u) = Q(q, iqu)$ is a real function. This yields

$$I(q) = 2 \cdot 4\pi q \int_0^\infty du \ R^2(q, u)$$

(C.7)

as an integral identity.

**APPENDIX D: THE FUNCTION $J(q)$**

Using again the method of Macke yields

$$J(q \leq 2) = \frac{\pi}{4} q \left[ \frac{8}{3} - 4 \ln 2 + \frac{1}{3} \left( 2 - \frac{q}{2} \right) \left( 1 + \frac{2}{q} \right)^2 \ln \left( 1 + \frac{q}{2} \right) \right.$$

$$+ \frac{1}{3} \left( 2 + \frac{q}{2} \right) \left( 1 - \frac{2}{q} \right)^2 \ln \left( 1 - \frac{q}{2} \right) \left. \right] ,$$

$$J(q \geq 2) = \frac{4\pi}{3} q \left[ 1 + \frac{1}{8q} (1 - q)(2 + q)^2 \ln \left( 1 + \frac{2}{q} \right) 

- \frac{1}{8q} (1 + q)(2 - q)^2 \ln \left( 1 - \frac{2}{q} \right) \right] .$$

(D.1)

$J(q)$ has the properties

$$J(q \to 0) = \pi (1 - \ln 2) q - \frac{\pi}{48} q^3 + \cdots , \quad J(q \to \infty) = \frac{4\pi}{3} \left( \frac{1}{q^2} + \frac{8}{15} \frac{1}{q^4} + \cdots \right) ,$$

$$J(2) = \frac{4\pi}{3} (1 - \ln 2), \quad J'(2^+) = -\frac{\pi}{3} (-1 + 4 \ln 2), \quad J'(2^-) = -\frac{\pi}{6} (-5 + 8 \ln 2) .$$

(D.2)

(Notice $I(q \to 0) = \frac{8\pi}{3} J(q \to 0)$.) $J(q)$ has a maximum of 1.3 at $q = 1.9$. $J(q)$ is continuous at $q = 2$, but $J'(q)$ has there a jump discontinuity. Its normalization is

$$\int_0^\infty dq \ J(q) = \frac{\pi}{9} (-3 + \pi^2 + 6 \ln 2) .$$

(D.3)

$J(q)$ is shown in Fig. 10. Multiplying the integral

$$\int_0^\infty \frac{dq}{q^2} [J(q) - \pi (1 - \ln 2) q \Theta(1 - q)] = \frac{\pi}{8} [10 - \pi^2 + 8(1 - \ln 2) \ln 2] \approx 0.719405$$

(D.4)
with $-2/\pi^3$ yields (3.13).

The original expression for $J(q)$ arises from the diagram rules for $\Sigma_{2d}$ with Eq. (3.3) as

$$J(q) = \text{Re} \int d^2c \int \frac{d\eta}{2\pi i} Q(q, \eta) \left[ \frac{\Theta(|e + q| - 1)}{q(e + \frac{1}{2}q) - \eta - i\delta} + \frac{\Theta(1 - |e + q|)}{q(e + \frac{1}{2}q) - \eta + i\delta} \right]$$  \hspace{1cm} (D.5)

One way is to insert (A.4). It results with $x_1 = q(e_1 + \frac{1}{2}q)$, $x_2 = q(k_2 + \frac{1}{2}q)$

$$J(q) = \int \frac{d^2c_1 d^3k_2}{4\pi} \left[ \frac{\Theta(|e_1 + q - 1| P}{x_1 + x_2} + \frac{\Theta(1 - |e_1 + q|)}{x_1 - x_2} \right] \Theta(1 - k_2) \Theta(|k_2 + q| - 1)$$  \hspace{1cm} (D.6)

in agreement with Eq. (3.4). Another way is the deformation of the integration contour from the real to the imaginary axis with $\eta = iqu$. It yields

$$J(q) = \int_0^\infty du \left[ \ln \frac{u^2 + (1 + \frac{q^2}{2})^2}{u^2 + (1 - \frac{q^2}{2})^2} \right] R(q, u)$$  \hspace{1cm} (D.7)

as an integral identity.

Comparing $J(q)$ with $I(q)$:

$$\int_0^\infty dq \left[ \frac{3}{8\pi} I(q) - J(q) \right] = \frac{\pi^3}{3} a.$$ \hspace{1cm} (D.8)

Note $\frac{3}{8\pi} I(q \to 0) = J(q \to 0) = \pi^3 a$. The identity (D.8) leads to the sum rule (3.15).

**Figures**

![Feynman diagrams](image)

FIG. 1: Feynman diagrams for the one-body Green’s function of the ideal Fermi gas $G_0(k, \omega)$, the bare Coulomb repulsion $v_0(q)$, and the RPA polarization propagator $Q(q, \eta)$ as defined in Eqs. (A.2)-(A.4).
FIG. 2: Feynman diagrams for \( v_r = v_0 + v_0 Q v_r \), the screened Coulomb repulsion in RPA.

\[ \text{Diagram} = \text{Diagram} + \text{Diagram} + \ldots \]

FIG. 3: Feynman diagrams for the ring-diagram-summed self-energy \( \Sigma_r = G_0 \cdot (v_r - v_0) \) as defined in Eq. (3.4).

\[ \text{Diagram} = \text{Diagram} + \text{Diagram} + \ldots \]

FIG. 4: Feynman diagrams for \( e_{2x} \) and \( \Sigma_{2x} \).

\[ \text{Diagram} = \text{Diagram} + \text{Diagram} + \ldots \]

FIG. 5: Feynman diagrams, which do not contribute to \( e_2 \) and \( \Sigma_2 \), respectively.

\[ \text{Diagram} = \text{Diagram} + \text{Diagram} + \ldots \]

FIG. 6: Feynman diagrams for \( \Sigma^x_r = G_x \cdot (v_r - v_0) \). For \( G_x \) see Fig. 7 and Eq. (3.14).
FIG. 7: Feynman diagrams for the renormalized one-body Green’s function $G_x = G_0 + G_0 \Sigma_x G_x$, see Eq. \ref{eq:renormalized_green_function}.

\[ = \quad + \]

FIG. 8: Feynman diagrams for $\Sigma_{HF} = G v_0$ with $G = \text{full one-body Green’s function of the interacting system}$, $G = G_0 + G_0 \Sigma G$, $\Sigma = \text{full self-energy}$. The lowest-order term is $\Sigma_x = G_0 v_0$, therefore the correlation part is $\Sigma_{c HF} = (G - G_0) v_0$.

\[ = \quad + \quad + \ldots \]

FIG. 9: The Macke function $I(q)$ according to Eq. \ref{eq:macke_function}, $I(q \to 0) = \frac{8\pi^4}{3} a q$. 

\[ I(q) \]

\[ q \]

$7$ $6$ $5$ $4$ $3$ $2$ $1$ $0$ $1$ $2$ $3$ $4$ $5$
FIG. 10: The function $J(q)$ according to Eq. \((D.1)\), $J(q \to 0) = \pi^3 q$. 