On Dart-Zobel Algorithm for Testing Regular Type Inclusion

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Abstract. This paper answers open questions about the correctness and the completeness of Dart-Zobel algorithm for testing the inclusion relation between two regular types. We show that the algorithm is incorrect for regular types. We also prove that the algorithm is complete for regular types as well as correct for tuple distributive regular types. Also presented is a simplified version of Dart-Zobel algorithm for tuple distributive regular types.

Keywords: type, regular term language, regular term grammar, tuple distributivity

1 Introduction

Types are ubiquitous in programming languages [4]. They make programs easier to understand and help detect errors since a large number of errors are type errors. Types have been introduced into logic programming in the forms of type checking and inference [3, 7, 11, 23, 25] or type analysis [22, 29, 15, 17, 12, 20, 8, 21] or typed languages [14, 19, 23, 27]. Recent logic programming systems allow the programmer declare types for predicates and type errors are then detected either at compile time or at run time. Even in early logic programming systems, built-in predicates are usually typed and type checking for these predicates are performed at run time. The reader is referred to [24] for more details on type in logic programming.

A type is a possibly infinite set of ground terms with a finite representation. An integral part of any type system is its type language that specifies which sets of ground terms are types. To be useful, types should be closed under intersection, union and complement operations. The decision problems such as the emptiness of a type, inclusion of a type in another, equivalence of two types should be decidable. Regular term languages [13, 8], called regular
types, satisfy these constraints and has been used widely used as types \[20, 22, 29, 13, 19, 23, 27, 11, 28, 17, 12, 20, 5, 21\].

Most type systems use *tuple distributive regular types* which are strictly less powerful than regular types \[20, 22, 29, 15, 19, 25, 27, 11, 28, 17, 12, 20, 5, 21\]. Tuple distributive regular types are regular types closed under tuple distributive closure. Intuitively, the tuple distributive closure of a set of terms is the set of all terms constructed recursively by permuting each argument position among all terms that have the same function symbol \[28\]. Tuple distributive regular types are discussed in section 5.

To our knowledge, Dart and Zobel’s work \[8\] is the only one to present, among others, an inclusion algorithm for regular types with respect to a given set of type definitions without the tuple distributive restriction. Set-based analysis can also be used to deriving types based on set constraint solving \[2, 1, 18, 16, 10\]. However, set constraint solving methods are intended to infer descriptive types \[25\] rather than for testing inclusion of a prescriptive type \[25\] in another. Therefore, they are useful in different settings from Dart-Zobel algorithm. Dart-Zobel algorithm has been used in type or type related analyses \[4, 9\]. However, the completeness and the correctness of the algorithm are left open. This paper provides answers to these open questions. We show that the algorithm is incorrect for regular types. We also prove that the algorithm is complete for regular types in general as well as correct for tuple distributive regular types. These results lead to a simplified version of Dart-Zobel algorithm that is complete and correct for tuple distributive regular types.

The remainder of this paper is organised as follows. Section 2 defines regular types by regular term grammars. Section 3 recalls Dart-Zobel algorithm for testing if a regular type is a subset of another regular type. Section 4 addresses the completeness and the correctness of their algorithm that have been left open. In section 5 we show that their algorithm is both complete and correct for tuple distributive regular types and provides a simplified version of their algorithm for tuple distributive regular types.
2 Regular types

Several equivalent formalisms such as tree automata [13, 6], regular term grammars [13, 6], regular unary logic programs [28] have been used to describe regular types. In [8], Dart and Zobel use regular term grammars to describe regular types that are sets of ground terms over a ranked alphabet Σ.

A regular term grammar is a tuple $G = \langle \Pi, \Sigma, \Delta \rangle$ where

- $\Sigma$ is a fixed ranked alphabet. Each symbol in $\Sigma$ is called a function symbol and has a fixed arity. It is assumed that $\Sigma$ contains at least one constant that is a function symbol of arity 0.
- $\Pi$ is a set of symbols called nonterminals. These terminals will be called type symbols as they represent types. Type symbols are of arity 0. It is assumed that $\Pi \cap \Sigma = \emptyset$.
- $\Delta$ is a set of production rules of the form $\alpha \rightarrow \tau$ with $\alpha \in \Pi$ and $\tau \in T(\Sigma \cup \Pi)$ where $T(\Sigma \cup \Pi)$ is the set of all terms over $\Sigma \cup \Pi$. Terms in $T(\Sigma \cup \Pi)$ will be called pure type terms.

Example 1. Let $\Sigma = \{0, s(), nil, cons(), \}$ and $\Pi = \{Nat, NatList\}$. $G = \langle \Pi, \Sigma, \Delta \rangle$ defines natural numbers and lists of natural numbers where

$$\Delta = \begin{cases} Nat \rightarrow 0, \\
Nat \rightarrow s(Nat), \\
Natlist \rightarrow nil, \\
Natlist \rightarrow cons(Nat, Natlist) \end{cases}$$

The above presentation is slightly different from [8] where production rules with the same type symbol on their lefthand sides are grouped together and called a type rule. For instance, production rules in the above examples are grouped into two type rules $Nat \rightarrow \{Nat, NatList\}$ and $Natlist \rightarrow \{nil, cons(Nat, NatList)\}$.

Types denoted by a pure type term is given by a rewrite rule $\Rightarrow_G$ associated with $G$. $t \Rightarrow_G s$ if $\Delta$ contains a rule $\alpha \rightarrow \tau$, $\alpha$ occurs in $t$ and $s$ results from replacing an occurrence of $\alpha$ in $t$ by $\tau$. Let $\Rightarrow_G^*$

1 A start symbol is not needed in our setting.
be the reflexive and transitive closure of $\Rightarrow_G$. The type denoted by a pure type term $\tau$ is defined as follows.

$$\left[\tau\right]_G \overset{\text{def}}{=} \{ t \in \mathcal{T}(\Sigma) \mid \tau \Rightarrow^* t \}$$

$[\tau]_G$ is the set of terms over $\Sigma$ that can be derived from $\tau$ by repeatedly replacing the lefthand side of a rule in $\Delta$ with its righthand side.

**Example 2.** Let $G$ be the regular term grammar in example 1. We have

$$\text{Natlist} \Rightarrow_G \text{cons} (\text{Nat}, \text{Natlist})$$

$$\Rightarrow_G \text{cons} (s(\text{Nat}), \text{Natlist})$$

$$\Rightarrow_G \text{cons} (s(0), \text{Natlist})$$

$$\Rightarrow_G \text{cons} (s(0), \text{nil})$$

Thus, $[\text{Natlist}]_G$ contains $\text{cons}(s(0), \text{nil})$.

The type represented by a sequence $\psi$ of pure type terms and a set $\Psi$ of sequences of pure type terms are defined as follows.

$$\left[\epsilon\right]_G \overset{\text{def}}{=} \{ \epsilon \}$$

$$\left[(\tau) + \psi\right]_G \overset{\text{def}}{=} [\tau]_G \times [\psi]_G$$

$$[\Psi]_G \overset{\text{def}}{=} \bigcup_{\psi \in \Psi} [\psi]_G$$

where $\epsilon$ is the empty sequence, $+$ is the infix sequence concatenation operator, $(\tau)$ is the sequence consisting of the pure type term $\tau$ and $\times$ is the Cartesian product operator.

The set $\Pi$ of nonterminals in Dart and Zobel’s type language also contains constant type symbols. Constant type symbols are not defined by production rules and they denote constant types. In particular, $\Pi$ contains $\mu$ denoting the set of all terms over $\Sigma$ and $\phi$ denoting the empty set of terms. We will leave out constant type symbols in this paper in order to simplify presentation. Re-introducing constant type symbols will not affect the results of the paper.

Dart-Zobel algorithm works with simplified regular term grammars. A regular term grammar $G = \langle \Pi, \Sigma, \Delta \rangle$ is simplified if $[\alpha]_G \neq \emptyset$ for each $\alpha \in \Pi$ and $\tau \notin \Pi$ for each $(\alpha \rightarrow \tau) \in \Delta$. Every regular grammar can be simplified.
3 Dart-Zobel Inclusion Algorithm

This section recalls Dart and Zobel's inclusion algorithm for regular types. As indicated in section 2, we shall disregard constant type symbols and simplify their algorithm accordingly. We note that without constant type symbols, many functions in their algorithm can be greatly simplified. In place of a type rule, we use the corresponding set of production rules. These superficial changes don’t change the essence of the algorithm but facilitate the presentation. We shall assume that \( G \) is a simplified regular term grammar and omit references to \( G \) where there is no confusion.

We first describe the ancillary functions used in their algorithm. Let \( \psi = \tau_1 \tau_2 \cdots \tau_n \) be a non-empty sequence of pure type terms and \( \Psi \) be a set of non-empty sequences of pure type terms. \( \text{head}(\psi) \overset{\text{def}}{=} \tau_1 \) and \( \text{tail}(\psi) \overset{\text{def}}{=} \tau_2 \cdots \tau_n \). \( \text{heads} \) and \( \text{tails} \) are defined as:

\[
\text{heads}(\psi) \overset{\text{def}}{=} \{ \text{head}(\psi) \mid \psi \in \Psi \}
\]

\[
\text{tails}(\psi) \overset{\text{def}}{=} \{ \text{tail}(\psi) \mid \psi \in \Psi \}
\]

The function \( \text{expand} \) rewrites a non-empty sequence into a set of sequences when necessary.

\[
\text{expand}(\psi) \overset{\text{def}}{=} \left\{ \begin{array}{ll}
\{ \psi \} & \text{if head}(\psi) \not\in \Pi \\
\{ (\tau) + \text{tail}(\psi) \mid (\text{head}(\psi) \rightarrow \tau) \in \Delta \} & \text{if head}(\psi) \in \Pi
\end{array} \right.
\]

\( \text{expands}(\Psi) \overset{\text{def}}{=} \bigcup_{\psi \in \Psi} \text{expand}(\psi) \).

The function \( \text{selects}(\tau, \Psi) \) defined below applies when \( \tau \) is pure type term and \( \tau \not\in \Pi \) and \( \Psi \) is a set of non-empty sequences with \( \text{heads}(\Psi) \cap \Pi = \emptyset \). The output of \( \text{selects}(\tau, \Psi) \) is the set of the sequences in \( \Psi \) that have the same principal function symbol as \( \tau \).

\[
\text{selects}(f(\tau_1, \cdots, \tau_n), \Psi) \overset{\text{def}}{=} \{ \psi \in \Psi \mid \text{head}(\psi) = f(\omega_1, \cdots, \omega_n) \}
\]

Note that \( f(\tau_1, \cdots, \tau_n) \) is a constant when \( n = 0 \).

The function \( \text{open}(\psi') \) defined below applies when \( \psi' \) is a non-empty sequence with \( \text{head}(\psi') \not\in \Pi \). \( \text{open}(\psi') \) replaces the head of \( \psi' \) with its arguments.

\[
\text{open}(f(\tau_1, \cdots, \tau_n) + \psi) \overset{\text{def}}{=} \tau_1 \tau_2 \cdots \tau_n + \psi
\]

When \( n = 0 \), \( \text{open}(f(\tau_1, \cdots, \tau_n) + \psi) = \psi \). Without constant type symbols, \( \text{open} \) doesn’t need an extra argument as in \[8\] that is used to test membership of a term in a constant type and to indicate the
required number of arguments when the constant type symbol is \( \mu \).

\[ \text{opens}(\Psi) \overset{\text{def}}{=} \{ \text{open}(\psi) \mid \psi \in \Psi \} \]

The inclusion algorithm \( \text{subset}(\tau_1, \tau_2) \) takes two pure type terms \( \tau_1 \) and \( \tau_2 \) and is intended to decide if \( [\tau_1]_G \subseteq [\tau_2]_G \) is true or false. The core part \( \text{subsetv} \) of the inclusion algorithm takes a sequence \( \psi \) of pure type terms and a set \( \Psi \) of sequences of pure type terms that are of the same length as \( \psi \) and is intended to decide if \( [\psi]_G \subseteq \Psi G \).

\( \text{subsetv} \) takes a third argument \( C \) to ensure termination. \( C \) is a set of pairs \( \langle \beta, \Upsilon \rangle \) where \( \beta \in \Pi \) is a type symbol and \( \Upsilon \subseteq T(\Sigma \cup \Pi) \) is a set of pure type terms. A pair \( \langle \beta, \Upsilon \rangle \) in \( C \) can be read as \( [\beta]_G \subseteq [\Upsilon]_G \).

The functions \( \text{subset} \) and \( \text{subsetv} \) are defined in the following. Where several alternative definitions of \( \text{subsetv} \) apply, the first is used.

\[
\text{subset}(\tau_1, \tau_2) \overset{\text{def}}{=} \text{subsetv}(\langle \tau_1 \rangle, \{\langle \tau_2 \rangle\}, \emptyset)
\]

\[
\text{subsetv}(\psi, \Psi, C) \overset{\text{def}}{=} \begin{cases} 
\text{false} & \text{if } \Psi = \emptyset \\
\text{true} & \text{if } \psi = \epsilon \\
\text{subsetv}(\text{tail}(\psi), \text{tails}(\Psi), C) & \text{if } \langle \text{head}(\psi), \Upsilon \rangle \in C \text{ and } \text{heads}(\Psi) \supseteq \Upsilon \\
\forall \psi' \in \text{expand}(\psi).\text{subsetv}(\psi', \Psi, C \cup \{\langle \text{head}(\psi), \text{heads}(\Psi) \rangle\}) & \text{if } \text{head}(\psi) \in \Pi \\
\text{subsetv}(\text{open}(\psi), \text{opens}(\text{selects}(\text{head}(\psi), \text{expands}(\Psi))), C) & \text{if } \text{head}(\psi) = f(\tau_1, \ldots, \tau_n) \\
\end{cases}
\]

The second condition \( \text{heads}(\Psi) \supseteq \Upsilon \) for the third alternative is obviously mistaken to be \( \text{heads}(\Psi) \subseteq \Upsilon \) in [8]. The first two alternatives deal with two trivial cases. The third alternative uses pairs in \( C \) to force termination. As we shall see later, this is fine for tuple distributive regular types but is problematic for regular types in general. The fourth alternative expands \( \psi \) into a set of sequences \( \psi' \) and compares each of them with \( \Psi \). The fifth alternative applies when \( \psi = f(\tau_1, \ldots, \tau_n) + \psi' \). Sequences in \( \Psi \) are expanded and the expanded sequences of the form \( f(\sigma_1, \ldots, \sigma_n) + \omega' \) are selected. \( \psi \) and the set of the selected sequences are then compared after replacing \( f(\tau_1, \ldots, \tau_n) \) with \( \tau_1 \cdots \tau_n \) in \( \psi \) and replacing \( f(\sigma_1, \ldots, \sigma_n) \) with \( \sigma_1 \cdots \sigma_n \) in each \( f(\sigma_1, \ldots, \sigma_n) + \omega' \).
4 Correctness and Completeness

We now address the correctness and the completeness of Dart-Zobel algorithm that were left open. We first show that the algorithm is incorrect for regular types by means of a counterexample. We then prove that the algorithm is complete for regular types. Thus, the algorithm provides an approximate solution to the inclusion problem of regular types in that it returns true if inclusion relation holds between its two arguments while the reverse is not necessarily true.

4.1 Correctness - a counterexample

The following example shows that Dart-Zobel algorithm is incorrect for regular types.

Example 3. Let \( G = \langle \Pi, \Sigma, \Delta \rangle \) with \( \Pi = \{ \alpha, \beta, \theta, \sigma, \omega \} \), \( \Sigma = \{ a, b, g(), h() \} \) and

\[
\Delta = \begin{cases}
\alpha \rightarrow g(\omega) \\
\beta \rightarrow g(\theta) \mid g(\sigma) \\
\theta \rightarrow a \mid h(\theta, a) \\
\sigma \rightarrow b \mid h(\sigma, b) \\
\omega \rightarrow a \mid b \mid h(\omega, a) \mid h(\omega, b)
\end{cases}
\]

where, for instance, \( \theta \rightarrow a \mid h(\theta, a) \) is an abbreviation of two rules \( \theta \rightarrow a \) and \( \theta \rightarrow h(\theta, a) \). Let \( \Sigma_h = \Sigma \setminus \{ h \} \). We have

\[
[\theta]_G = \{ t \in T(\Sigma_h) \mid t \text{ is left-skewed and leaves of } t \text{ are a’s} \}
\]

\[
[\sigma]_G = \{ t \in T(\Sigma_h) \mid t \text{ is left-skewed and leaves of } t \text{ are b’s} \}
\]

\[
[\omega]_G = \{ t \in T(\Sigma_h) \mid t \text{ is left-skewed} \}
\]

\[
[\alpha]_G = \{ g(t) \mid t \in [\omega]_G \}
\]

\[
[\beta]_G = \{ g(t) \mid t \in [\theta]_G \cup [\sigma]_G \}
\]

Let \( t = g(h(h(a, b), a)) \). \( t \in [\alpha]_G \) and \( t \not\in [\beta]_G \). Therefore, \( [\alpha]_G \not\subseteq [\beta]_G \). The incorrectness of Dart-Zobel algorithm is illustrated by showing \( \text{subset}(\alpha, \beta) = \text{true} \) as follows. Let \( C_0 = \{ \langle \alpha, \{ \beta \} \rangle \} \). We have

\[
\text{subset}(\alpha, \beta) = \text{subsetv}(\langle \alpha \rangle, \{ \{ \beta \} \}, \emptyset) \quad \text{by def. of subset}
\]

\[
= \text{subsetv}(\langle g(\omega) \rangle, \{ \{ \beta \} \}, C_0) \quad \text{by 4th def. of subsetv}
\]

\[
= \text{subsetv}(\langle \omega \rangle, \{ \langle \theta \rangle, \{ \sigma \} \}, C_0) \quad \text{by 5th def. of subsetv}
\]
Let \( C_1 = C_0 \cup \{ \langle \omega, \{ \theta, \sigma \} \rangle \} \). By the fourth definition of \( \text{subset}_v \) and the above equation,

\[
\text{subset}(\alpha, \beta) = \begin{cases} 
\text{subset}(\langle a \rangle, \{ \langle \theta \rangle, \langle \sigma \rangle \}, C_1) \\
\land 
\text{subset}(\langle b \rangle, \{ \langle \theta \rangle, \langle \sigma \rangle \}, C_1) \\
\land \text{subset}(\langle h(\omega, a) \rangle, \{ \langle \theta \rangle, \langle \sigma \rangle \}, C_1) \\
\land \text{subset}(\langle h(\omega, b) \rangle, \{ \langle \theta \rangle, \langle \sigma \rangle \}, C_1)
\end{cases}
\]

(1)

By applying the fifth and then the second definitions of \( \text{subset}_v \),

\[
\text{subset}(\langle a \rangle, \{ \langle \theta \rangle, \langle \sigma \rangle \}, C_1) = \text{subset}(\epsilon, \{ \epsilon \}, C_1) = \text{true}
\]

In the same way, we obtain \( \text{subset}(\langle a \rangle, \{ \langle \theta \rangle, \langle \sigma \rangle \}, C_1) = \text{true} \).

We can show \( \text{subset}(\langle h(\omega, a) \rangle, \{ \langle \theta \rangle, \langle \sigma \rangle \}, C_1) = \text{true} \) in the same way as above. Therefore, by equation (1), \( \text{subset}(\alpha, \beta) = \text{true} \) and \( \text{subset} \) is incorrect for regular types.

\( \square \)

The problem with the algorithm stems from the way the set \( C \) is used in the third definition of \( \text{subset}_v \). As the above example indicates, the third definition of \( \text{subset}_v \) severs the dependency between the terms in a tuple, i.e., subterms of a term.

In [8], Dart and Zobel show by an example that their algorithm works for some regular types which are not tuple distributive. We don’t know what is the largest subclass of the class of regular types for which the algorithm is correct.

### 4.2 Completeness

We now prove that Dart-Zobel algorithm is complete for regular types in the sense that \( \text{subset}(\tau_1, \tau_2) = \text{true} \) whenever \([\tau_1]_G \subseteq [\tau_2]_G\). Let \( C \) be a set of pairs \( \langle \beta, \gamma \rangle \) with \( \beta \in \Pi \) and \( \gamma \subseteq T(\Sigma \cup \Pi) \). A pair \( \langle \beta, \gamma \rangle \) in \( C \) states that the denotation of \( \beta \) is included in that of \( \gamma \), i.e., \([\beta]_G \subseteq [\gamma]_G\) for regular types. Define

\[
\Gamma_{C, G} \overset{\text{def}}{=} \land_{\langle \beta, \gamma \rangle \in C} [\beta]_G \subseteq [\gamma]_G
\]
The completeness of \textit{subset} follows from the following theorem which asserts the completeness of \textit{subsetv}.

\textbf{Theorem 1.} Let \( \psi \) be a sequence of pure type terms and \( \Psi \) a set of sequences of pure type terms of the same length as \( \psi \), \( C \) a set of pairs \( \langle \beta, \gamma \rangle \) with \( \beta \in \Pi \) and \( \gamma \subseteq T(\Sigma \cup \Pi) \). If \( \Gamma_{C,G} \models [\psi]_G \subseteq [\Psi]_G \) then \textit{subsetv}(\( \psi, \Psi, C \)) = true.

\textbf{Proof.} Assume \textit{subsetv}(\( \psi, \Psi, C \)) = false. The proof is done by showing \( \Gamma_{C,G} \not\models [\psi]_G \subseteq [\Psi]_G \). This is accomplished by induction on \( \langle dp(\psi, \Psi, C), lg(\psi) \rangle \) where \( lg(\psi) \) is the length of \( \psi \) and \( dp(\psi, \Psi, C) \) is the depth of the computation tree for \textit{subsetv}(\( \psi, \Psi, C \)). Define \( \langle k, l \rangle \) def \( (k < k') \lor (k = k') \land (l < l') \).

\textbf{Basis.} \( dp(\psi, \Psi, C) = 0 \) and \( lg(\psi) = 0 \). \( \psi = \epsilon \) and \( \Psi = \emptyset \) since \textit{subsetv}(\( \psi, \Psi, C \)) = false. Let \( t = \epsilon \). \( t \in [\psi]_G \) and \( t \notin [\Psi]_G \). So, \( \Gamma_{C,G} \not\models [\psi]_G \subseteq [\Psi]_G \).

\textbf{Induction.} \( dp(\psi, \Psi, C) \neq 0 \) or \( lg(\psi) \neq 0 \). By the definition of \textit{subsetv},

\begin{enumerate}
\item \( \Psi = \emptyset \); or
\item \( \textit{subsetv}(\text{tail}(\psi), \text{tails}(\Psi), C) = false \) and there is \( \gamma \subseteq T(\Sigma \cup \Pi) \) such that \( \langle \text{head}(\psi), \gamma \rangle \subseteq C \land (\text{heads}(\Psi) \supseteq \gamma) \); or
\item \( \text{head}(\psi) \in \Pi \) and \( \exists \psi' \in \text{expand}(\psi). \text{subsetv}(\psi', \Psi, C') = false \) where \( C' = C \cup \{ \langle \text{head}(\psi), \text{heads}(\Psi) \rangle \} \); or
\item \( \text{head}(\psi) = f(\tau_1, \cdots, \tau_n) \) and \( \text{subsetv}(\psi', \Psi', C) = false \) where \( \psi' = \text{open}(\psi) \) and \( \Psi' = \text{opens}(\text{selects}(\text{head}(\psi), \text{expand}(\Psi))) \).
\end{enumerate}

It remains to prove that \( \Gamma_{C,G} \not\models [\psi]_G \subseteq [\Psi]_G \) in each of the cases (a)-(d). The case (a) is trivial as \( G \) is simplified and hence \( [\psi]_G \neq \emptyset \).

In the case (b), we have \( dp(\text{tail}(\psi)), \text{tails}(\Psi), C \leq dp(\psi, \Psi, C) \) and \( lg(\text{tail}(\psi)) < lg(\psi) \). By the induction hypothesis, \( \Gamma_{C,G} \not\models [\text{tail}(\psi)]_G \subseteq [\text{tails}(\Psi)]_G \). Thus, \( \Gamma_{C,G} \models \exists t'. (t' \in [\text{tail}(\psi)]_G \land t' \notin [\text{tails}(\Psi)]_G) \). Let \( t \in [\text{head}(\psi)]_G \) and \( t = t + t' \). Note that \( t \) exists as \( G \) is simplified. We have \( \Gamma_{C,G} \models t \in [\psi]_G \land t \notin [\Psi]_G \). So, \( \Gamma_{C,G} \not\models [\psi]_G \subseteq [\Psi]_G \).

In the case (c), \( dp(\psi, \Psi, C') < dp(\psi, \Psi, C) \). By the induction hypothesis, \( \Gamma_{C',G} \not\models [\psi']_G \subseteq [\Psi']_G \). Note that \( \Gamma_{C',G} = \Gamma_{C,G} \land [\text{head}(\psi)]_G \subseteq [\text{heads}(\Psi)]_G \). So, we have \( \Gamma_{C,G} \models [\psi]_G \subseteq [\Psi]_G \lor [\text{head}(\psi)]_G \subseteq [\text{heads}(\Psi)]_G \). Assume \( \Gamma_{C,G} = true \). Either (i) \( [\psi]_G \subseteq [\Psi]_G \lor [\text{head}(\psi)]_G \subseteq [\text{heads}(\Psi)]_G \) or (ii) \( [\text{head}(\psi)]_G \subseteq [\text{heads}(\Psi)]_G \). In the case (i), \( \exists t'. (t' \in [\psi']_G \land t' \notin [\Psi']_G) \). By proposition 5.26 in [3], we have \( \Gamma_{C,G} \not\models [\psi]_G \subseteq [\Psi]_G \). In the case (ii), \( \exists t. (t \in [\psi]_G \land t \notin [\Psi]_G) \).
Thus, \( \Gamma \subset \tau \). We have \( \tau \subseteq \{ \psi \} \) and \( \tau \not\subseteq \{ \psi \} \).

So, \( \Gamma \not\models [\psi] \subseteq [\psi] \) in the case (c).

In the case (d), we have \( \psi = \tau_1 \cdots \tau_n + \text{tail}(\psi) \) and \( dp(\psi', \psi, C) < dp(\psi, \psi, C) \). By the induction hypothesis, \( \Gamma \not\models [\psi'] \subseteq [\psi] \). Thus, \( \Gamma \not\models \exists \tau_1.\exists \tau_2. (\text{length}(\tau_1) = n) \land ((\tau_1 + \tau_2) \not\subseteq \{ \psi \}) \land ((\tau_1 + \tau_2) \not\subseteq \{ \psi \}) \), which implies \( \Gamma \not\models \exists \tau_1.\exists \tau_2. (\langle f(\tau_1) \rangle + \tau_2) \not\subseteq \{ \psi \}) \land ((\langle f(\tau_1) \rangle + \tau_2) \not\subseteq \{ \psi \}) \). So, \( \Gamma \not\models [\psi] \subseteq [\psi] \).

\( \square \)

The completeness of \( \text{subset} \) is a corollary of the above theorem.

Corollary 2. Let \( \tau_1 \) and \( \tau_2 \) be pure type terms. If \( [\tau_1] \subseteq [\tau_2] \) then \( \text{subset}(\tau_1, \tau_2) = \text{true} \).

Proof. \( \text{subset}(\tau_1, \tau_2) = \text{subset}(\{ \langle \tau_1 \rangle \}, \{ \langle \tau_2 \rangle \}, \emptyset) \) by the definition of \( \text{subset} \). We have \( \Gamma \models [\langle \tau_1 \rangle] \subseteq [\langle \tau_2 \rangle] \) since \( [\tau_1] \subseteq [\tau_2] \). The corollary now follows from the above theorem as \( \Gamma \models \text{true} \).

\( \square \)

5 Tuple Distributive Regular Types

Most type languages in logic programming use tuple distributive closures of regular term languages as types \( [20, 22, 23, 15, 19, 25, 24, 11, 21] \). The notion of tuple distributivity is due to Mishra [22]. The following definition of tuple distributivity is due to Heintze and Jaffar [15]. Each function symbol of arity \( n \) is associated with \( n \) projection operators \( f_1^{-1}, f_2^{-1}, \ldots, f_n^{-1} \). Let \( S \) be a set of ground terms in \( \mathcal{T}(\Sigma) \). \( f_1^{-1} \) is defined as follows.

\[
\forall (i) \quad f_1^{-1}(S) \triangleq \{ t_i \mid f(t_1, \ldots, t_i, \ldots, t_n) \in S \}
\]

The tuple distributive closure of \( S \) is

\[
S^* \triangleq \{ c \mid c \in S \land \forall c \in \Sigma_0 \cup \{ f(t_1, \ldots, t_n) \mid t_i \in (f_1^{-1}(S))^* \}
\]

where \( \Sigma_0 \) is the set of constants in \( \Sigma \).

The following proposition results from the fact that \((.)*\) is a closure operator and preserves set inclusion, i.e., \( S_1 \subseteq S_2 \) implies \( S_1^* \subseteq S_2^* \).
Proposition 3. Let $S_1, S_2 \subseteq T(\Sigma)$. $(S_1 \cup S_2)^* = (S_1^* \cup S_2^*)^*$.

The tuple distributive regular type $\langle \tau \rangle_G$ associated with a pure type term $\tau$ is the tuple distributive closure of the regular type $[\tau]_G$ associated with $\tau$. 

\[ \langle \tau \rangle_G \overset{\text{def}}{=} [\tau]_G \]

Let $\psi$ be a sequence of pure type terms, $\Psi$ be a set of sequences of pure type terms of the same length.

\[ \langle \psi \rangle_G \overset{\text{def}}{=} \{ (t) + t \mid t \in \{ \text{head}(\psi) \}_G \land t \in \{ \text{tail}(\psi) \}_G \} \]

\[ \langle \Psi \rangle_G \overset{\text{def}}{=} \left( \bigcup_{\psi \in \Psi} \{ \text{head}(\psi) \}_G^* \times \{ \text{tails}(\psi) \}_G \right) \]

The correctness of $\langle \Psi \rangle_G$ makes use of tuple distributivity and hence severs the inter-dependency between components of a sequences of terms.

5.1 Correctness

We now prove that Dart-Zobel algorithm is correct for tuple distributive regular types in the sense that if $\text{subset}(\tau_1, \tau_2) = \text{true}$ then $\langle \tau_1 \rangle_G \subseteq \langle \tau_2 \rangle_G$. Let $C$ be a set of pairs $\langle \beta, \Upsilon \rangle$ with $\beta \in \Pi$ and $\Upsilon \subseteq T(\Sigma \cup \Pi)$. A pair $\langle \beta, \Upsilon \rangle$ in $C$ represents $\langle \beta \rangle_G \subseteq \langle \Upsilon \rangle_G$ for tuple distributive regular types. Define

\[ \Phi_{C,G} \overset{\text{def}}{=} \land_{\langle \beta, \Upsilon \rangle \in C} \langle \beta \rangle_G \subseteq \langle \Upsilon \rangle_G \]

The correctness of $\text{subset}$ follows from the following theorem which asserts the correctness of $\text{subsetv}$ for tuple distributive regular types.

Theorem 4. Let $\psi$ be a sequence of pure type terms and $\Psi$ a set of sequences of pure type terms of the same length as $\psi$, $C$ a set of pairs $\langle \beta, \Upsilon \rangle$ with $\beta \in \Pi$ and $\Upsilon \subseteq T(\Sigma \cup \Pi)$. If $\text{subsetv}(\psi, \Psi, C) = \text{true}$ then $\Phi_{C,G} \models \langle \psi \rangle_G \subseteq \langle \Psi \rangle_G$.

Proof. Assume $\text{subsetv}(\psi, \Psi, C) = \text{true}$. The proof is done by induction on $\langle dp(\psi, \Psi, C), lg(\psi) \rangle$. 

Basis. \( dp(\psi, \Psi, C) = 0 \) and \( lg(\psi) = 0 \). \( \psi = \epsilon \) and \( \Psi \neq \emptyset \) by the second definition of \( \text{subsetv} \). So, \( \Psi = \{\epsilon\} \) and \( \Phi_{C,G} \models \langle \psi \rangle_G \subseteq \langle \Psi \rangle_G \).

Induction. \( dp(\psi, \Psi, C) \neq 0 \) or \( lg(\psi) \neq 0 \) By the definition of \( \text{subsetv} \),

(a) \( \text{subsetv}(tail(\psi), tails(\Psi), C) = \text{true} \) and there is \( \Upsilon \subseteq T(\Sigma \cup \Pi) \) such that \( (\langle head(\psi), \Upsilon \rangle \in C) \land (heads(\Psi) \supseteq \Upsilon) \); or

(b) \( head(\psi) \in \Pi \) and \( \forall \psi'. \in \text{expand}(\psi). \text{subsetv}(\psi', \Psi, C') = \text{true} \) where \( C' = C \cup \{\langle head(\psi), heads(\Psi)\rangle\} \); or

(c) \( head(\psi) = f(\tau_1, \ldots, \tau_n) \) and \( \text{subsetv}(\psi', \Psi', C') = \text{true} \) where \( \psi' = \text{open}(\psi) \) and \( \Psi' = \text{opens}(\text{selects}(head(\psi), \text{expands}(\Psi))) \).

It remains to prove that \( \Phi_{C,G} \models \langle \psi \rangle_G \subseteq \langle \Psi \rangle_G \) in each of the cases (a)-(c).

In the case (a), we have \( dp(tail(\psi), tails(\Psi), C) \leq dp(\psi, \Psi, C) \) and \( lg(tail(\psi)) < lg(\psi) \). By the induction hypothesis, \( \Phi_{C,G} \models \langle \text{tail}(\psi) \rangle_G \subseteq \langle \text{tails}(\Psi) \rangle_G \). \( (\langle head(\psi), \Upsilon \rangle \in C) \land (heads(\Psi) \supseteq \Upsilon) \) imply \( \Phi_{C,G} \models \langle \text{head}(\psi) \rangle_G \subseteq \langle \text{heads}(\Psi) \rangle_G \). Thus, \( \Phi_{C,G} \models \langle \psi \rangle_G \subseteq \langle \Psi \rangle_G \) by the definitions of \( \langle \psi \rangle_G \) and \( \langle \Psi \rangle_G \). Note that tuple distributivity is used in the definition of \( \langle \Psi \rangle_G \).

In the case (b), \( dp(\psi', \Psi, C') < dp(\psi, \Psi, C) \). By the induction hypothesis, \( \Phi_{C',G} \models \langle \psi' \rangle_G \subseteq \langle \Psi \rangle_G \). Note that \( \Phi_{C',G} = \Phi_{C,G} \land (\langle head(\psi) \rangle_G \subseteq \langle heads(\Psi) \rangle_G) \).

So,

\[ \Phi_{C,G} \models \langle \psi \rangle_G \subseteq \langle \Psi \rangle_G \lor \langle head(\psi) \rangle_G \subseteq \langle heads(\Psi) \rangle_G \]

\( \text{subsetv}(head(\psi), heads(\Psi), C) = \text{true} \) since \( \text{subsetv}(\psi, \Psi, C) = \text{true} \).

We have \( \Phi_{C,G} \models \langle head(\psi) \rangle_G \subseteq \langle heads(\Psi) \rangle_G \) by the induction hypothesis since \( dp(head(\psi), heads(\Psi), C) < dp(\psi, \Psi, C) \). So, \( \Phi_{C,G} \models \langle \psi' \rangle_G \subseteq \langle \Psi \rangle_G \). \( \Phi_{C,G} \models \langle \{\psi' \mid \psi' \in \text{expand}(\psi)\} \rangle_G \subseteq \langle \Psi \rangle_G \) since \((.)^*\) is a closure operator and hence \( \Phi_{C,G} \models \langle \psi \rangle_G \subseteq \langle \Psi \rangle_G \).

In the case (c), we have \( \psi' = \tau_1 \cdots \tau_n + tail(\psi) \) and \( dp(\psi', \Psi', C) < dp(\psi, \Psi, C) \). By the induction hypothesis, \( \Phi_{C,G} \models \langle \psi' \rangle_G \subseteq \langle \Psi' \rangle_G \). By proposition 5.29 in [1], \( \Phi_{C,G} \models \langle \psi \rangle_G \subseteq \langle \Psi \rangle_G \). This completes the proof of the theorem.

\( \square \)

The correctness of \( \text{subset} \) is a corollary of the above theorem.

**Corollary 5.** Let \( \tau_1 \) and \( \tau_2 \) be pure type terms. If \( \text{subset}(\tau_1, \tau_2) = \text{true} \) then \( \langle \tau_1 \rangle_G \subseteq \langle \tau_2 \rangle_G \).
Proof. Let \( \text{subset}(\tau_1, \tau_2) = \text{true} \). \( \text{subsetv}(\langle \tau_1 \rangle, \{\langle \tau_2 \rangle\}, \emptyset) = \text{true} \) by the definition of \( \text{subset} \). Thus, \( \Phi_{0,G} \models \langle \langle \tau_1 \rangle \rangle_G \subseteq \{\langle \tau_2 \rangle\}\}_G \) according to the above theorem. So, \( \langle \tau_1 \rangle_G \subseteq \langle \tau_2 \rangle_G \) as \( \Phi_{0,G} = \text{true} \). □

5.2 Completeness

This section presents the completeness of Dart-Zobel algorithm for tuple distributive regular types. The following theorem is the counterpart of theorem \( \ref{thm:subset} \).

**Theorem 6.** Let \( \psi \) be a sequence of pure type terms and \( \Psi \) a set of sequences of pure type terms of the same length as \( \psi \), \( C \) a set of pairs \( \langle \beta, \Upsilon \rangle \) with \( \beta \in \Pi \) and \( \Upsilon \subseteq T(\Sigma \cup \Pi) \). If \( \Phi_{C,G} \models \langle \psi \rangle_G \subseteq \{ \Psi \}_G \) then \( \text{subsetv}(\psi, \Psi, C) = \text{true} \).

**Proof.** The proof can be obtained from that for theorem \( \ref{thm:subset} \) by simply replacing \( \Gamma_{\cdot, \cdot} \) with \( \Phi_{\cdot, \cdot} \) and \( \cdot \}_G \) with \( \cdot \}_G \). □

The following completeness result of Dart-Zobel algorithm for tuple distributive regular types follows from the above theorem.

**Corollary 7.** Let \( \tau_1 \) and \( \tau_2 \) be pure type terms. If \( \langle \tau_1 \rangle_G \subseteq \langle \tau_2 \rangle_G \) then \( \text{subset}(\tau_1, \tau_2) = \text{true} \).

**Proof.** The proof can be obtained from that for corollary \( \ref{cor:subset} \) by simply replacing \( \Gamma_{\cdot, \cdot} \) with \( \Phi_{\cdot, \cdot} \), \( \cdot \}_G \) with \( \cdot \}_G \) and theorem \( \ref{thm:subset} \) with theorem \( \ref{thm:subsetv} \). □

5.3 A Simplified Algorithm

Now that Dart-Zobel algorithm is complete and correct for tuple distributive regular types but not correct for general regular types, it is desirable to specialise Dart-Zobel algorithm for tuple distributive regular types which was originally proposed for general regular types. The following is a simplified version of the algorithm for tuple distributive regular types.
subset′(τ₁, τ₂) def = subset′(τ₁, {τ₂}, ∅)
subset′(τ, Υ, C) def =
\[ \left\{ \begin{array}{ll}
  \text{false} & \text{if } Υ = ∅ \\
  \text{true} & \text{if } (⟨τ, τ′⟩ ∈ C) ∧ (Υ ⊇ τ′) \\
  \forall τ′ ∈ \text{expand′}(τ).\text{subset′}(τ′, Υ, C ∪ \{⟨τ, τ⟩⟩) & \text{if } τ ∈ \Pi \\
  \text{subset′}(τ₁ \cdots τₙ, \{σ₁ \cdots σₙ | f(σ₁, \cdots, σₙ) ∈ \text{expands′}(Υ), C) & \text{if } τ = f(τ₁, \cdots, τₙ)
\end{array} \right. \]
subset′(ε, {ε}, C) def = true
subset′(ψ, Ψ, C) def =
 subset′(head(ψ), heads(Ψ), C) ∧ subset′(tail(ψ), tails(Ψ), C)
expand′(τ) def = \{ {τ} \} if τ ∉ Π
\{ {σ | τ → σ} ∈ Δ} if τ ∈ Π
expands′(Υ) def = \bigcup_{τ ∈ Υ} \text{expand′}(τ)

While Dart-Zobel algorithm mainly deals with sequences of pure type terms, the simplified algorithm primarily deals with pure type terms by breaking a sequence of pure type terms into its component pure type terms. This is allowed because tuple distributive regular types abstract away inter-dependency between component terms in a sequence of ground terms. We forgo presenting the correctness and the completeness of the simplified algorithm because they can be proved by emulating proofs for theorems 1 and 4.

6 Conclusion

We have provided answers to open questions about the correctness and the completeness of Dart-Zobel algorithm for testing inclusion of one regular type in another. The algorithm is complete but incorrect for general regular types. It is both complete and correct for tuple distributive regular types. It is our hope that the results presented in this paper will help identify the applicability of Dart-Zobel algorithm. We have also provided a simplified version of Dart-Zobel algorithm for tuple distributive regular types.
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