Cell algebra structure on generalized Schur algebras

Robert May
Department of Mathematics and Computer Science
Longwood University
201 High Street, Farmville, VA 23909
rmay@longwood.edu
January 18, 2016

1 Introduction

A family of “generalized Schur algebras” were first introduced in [6] and [2]. In [3] the left and right generalized Schur algebras were shown to be “double coset algebras”. In [2] and [5] a stratification of these algebras was given leading to a parameterization, in most cases, of their irreducible representations. In this paper we obtain cell algebra structures for these algebras in the sense of [4]. (The Cell algebras of [4] coincide with the standardly based algebras previously introduced by Du and Rui in [8].) The properties of cell algebras combined with the parameterization of the irreducible representations leads to a more concrete description of all these irreducibles. In certain cases these algebras are shown to be quasi-hereditary.

In section 2 we review the definition and properties of cell algebras as presented in [4]. In section 3 we describe the cell bases found in [4] for a family of semigroups including the full transformation semigroups $T_r$ and the rook semigroups $R_r$. In section 4 we give cell bases for the left and right generalized Schur algebras corresponding to these semigroups. Finally, in section 5 we use the cell algebra structure to describe the irreducible representations of these algebras and to determine when they are quasi-hereditary.

2 Cell algebra structures

In this section we review, without proofs, the definition and properties of cell algebras as presented in [4]. (These algebras were previously studied as “standardly based algebras” by Du and Rui in [8].) Let $R$ be a commutative integral domain with unit 1 and let $A$ be an associative, unital $R$-algebra. Let $\Lambda$ be a finite set with a partial order $\leq$ and for each $\lambda \in \Lambda$ let $L(\lambda), R(\lambda)$
be finite sets of “left indices” and “right indices”. Assume that for each $\lambda \in \Lambda$, $s \in L(\lambda)$, and $t \in R(\lambda)$ there is an element $sC_t^\lambda \in A$ such that the map $(\lambda, s, t) \mapsto sC_t^\lambda$ is injective and

$$C = \{sC_t^\lambda : \lambda \in \Lambda, s \in L(\lambda), t \in R(\lambda)\}$$

is a free $R$-basis for $A$. Define $R$-submodules of $A$ by

$$A^\lambda = R \cdot \text{span of } \{sC_t^\mu : \mu \in \Lambda, \mu \geq \lambda, s \in L(\mu), t \in R(\mu)\}$$

and

$$\hat{A}^\lambda = R \cdot \text{span of } \{sC_t^\mu : \mu \in \Lambda, \mu > \lambda, s \in L(\mu), t \in R(\mu)\}.$$

**Definition 2.1.** Given $A, \Lambda, C$, $A$ is a cell algebra with poset $\Lambda$ and cell basis $C$ if

1. For any $a \in A, \lambda \in \Lambda$, and $s, s' \in L(\lambda)$, there exists $r_L = r_L(a, \lambda, s, s') \in R$ such that, for any $t \in R(\lambda)$, $a \cdot sC_t^\lambda = \sum_{s' \in L(\lambda)} r_L \cdot sC_t^\lambda \mod \hat{A}^\lambda$, and

2. For any $a \in A, \lambda \in \Lambda$, and $t, t' \in R(\lambda)$, there exists $r_R = r_R(a, \lambda, t, t') \in R$ such that, for any $s \in L(\lambda)$, $sC_t^\lambda \cdot a = \sum_{t' \in R(\lambda)} r_R \cdot sC_{t'}^\lambda \mod \hat{A}^\lambda$.

Consider a fixed cell algebra $A$ with poset $\Lambda$ and cell basis $C$.

**Lemma 2.1.**

(a) $A^\lambda$ and $\hat{A}^\lambda$ are two sided ideals in $A$ for any $\lambda \in \Lambda$.

(b) For $\lambda \in \Lambda, t, t' \in R(\lambda), s, s' \in L(\lambda)$, $r_L(s, s'C_t^\lambda, \lambda, s, s') = r_R(s, s'C_{t'}^\lambda, \lambda, t, t')$.

(c) Given $\lambda \in \Lambda, t \in R(\lambda), s \in L(\lambda)$, there exists $r_{st} \in R$ such that for any $s' \in L(\lambda), t' \in R(\lambda)$ we have $sC_t^\lambda \cdot s'C_{t'}^\lambda = r_{st} \cdot s'C_{t'}^\lambda \mod \hat{A}^\lambda$. In fact $r_{st} = r_L(s, s'C_t^\lambda, \lambda, s, s') = r_R(s, s'C_{t'}^\lambda, \lambda, t, t')$.

By lemma 2.1 part (a), $A/\hat{A}^\lambda$ is a unital $R$-algebra and $A^\lambda/\hat{A}^\lambda$ is a two sided ideal in $A/\hat{A}^\lambda$. Observe that as an $R$-module $A^\lambda/\hat{A}^\lambda$ is free with a basis $\{sC_t^\lambda + \hat{A}^\lambda : s \in L(\lambda), t \in R(\lambda)\}$.

For a fixed $t \in R(\lambda)$, define $LC_t^\lambda$ as the free $R$-submodule of $A^\lambda/\hat{A}^\lambda$ with basis $\{sC_t^\lambda + \hat{A}^\lambda : s \in L(\lambda)\}$. By property (i), $LC_t^\lambda$ is a left $A$-module and $LC_t^\lambda \cong LC_{t'}^\lambda$ as left $A$-modules for any $t, t' \in R(\lambda)$. Evidently, as left $A$-modules we have $A^\lambda/\hat{A}^\lambda \cong \bigoplus_{t \in R(\lambda)} LC_t^\lambda$.

**Definition 2.2.** The left cell module for $\lambda$ is the left $A$-module $LC_t^\lambda$ defined as follows: Take the free $R$-module with a basis $\{sC_t^\lambda : s \in L(\lambda)\}$ and define the left action of $A$ by $a \cdot sC_t^\lambda = \sum_{s' \in L(\lambda)} r_L(a, \lambda, s, s') s'C_t^\lambda$ for $a \in A$. 2
For any \( t \in R(\lambda) \), \( sC^\lambda \mapsto sC^\lambda_t + \hat{A}^\lambda \) gives a left \( A \)-module isomorphism \( \phi_t : \_L^C^\lambda \rightarrow \_L^C^\lambda_t \). Then \( \_A^\lambda/\_A^\hat{A}^\lambda \cong \bigoplus_{t \in R(\lambda)} \_L^C^\lambda_t \) is isomorphic to the direct sum of \( |R(\lambda)| \) copies of \( \_L^C^\lambda \).

In a parallel way, for a fixed \( s \in L(\lambda) \), define \( sC_R^\lambda \) as the free \( R \)-module with basis \( \{ sC^\lambda_t + \hat{A}^\lambda : t \in R(\lambda) \} \), an \( R \)-submodule of \( \_A^\lambda/\_A^\hat{A}^\lambda \). By property (ii), \( sC_R^\lambda \) is a right \( A \)-module and \( sC_R^\lambda \cong sC^\lambda,R \) as right \( A \)-modules for any \( s, s' \in L(\lambda) \). As right \( A \)-modules we have \( \_A^\lambda/\_A^\hat{A}^\lambda \cong \bigoplus_{s \in L(\lambda)} sC^\lambda,R \).

**Definition 2.3.** The right cell module for \( \lambda \) is the right \( A \)-module \( C_R^\lambda \) defined as follows: Take the free \( R \)-module with a basis \( \{ C^\lambda_t : t \in R(\lambda) \} \) and define the right action of \( A \) by \( C^\lambda_t : a = \sum_{t' \in R(\lambda)} r_R(a, \lambda, t, t') C^\lambda_{t'} \) for \( a \in A \).

For any \( s \in L(\lambda) \), \( C^\lambda_t \mapsto sC^\lambda_t + \hat{A}^\lambda \) gives a right \( A \)-module isomorphism \( \phi : C_R^\lambda \rightarrow sC^\lambda,R \). Then \( \_A^\lambda/\_A^\hat{A}^\lambda \cong \bigoplus_{s \in L(\lambda)} sC^\lambda,R \) is isomorphic to the direct sum of \( |L(\lambda)| \) copies of \( C^\lambda,R \).

For each \( \lambda \in \Lambda \) there is an \( R \)-bilinear map \( \langle , \rangle : C^\lambda_R \times \_L^C^\lambda \rightarrow R \) defined on basis elements by \( \langle C^\lambda_t, sC^\lambda \rangle = r_{st} \), where \( r_{st} \in R \) is as given in lemma 2.1.

**Definition 2.4.** The right \( C_R^\lambda \) radical is

\[
\text{rad} \left( C_R^\lambda \right) = \{ x \in C_R^\lambda : \langle x, y \rangle = 0 \text{ for all } y \in \_L^C^\lambda \}.
\]

The left \( \_L^C^\lambda \) radical is

\[
\text{rad} \left( \_L^C^\lambda \right) = \{ y \in \_L^C^\lambda : \langle x, y \rangle = 0 \text{ for all } x \in C_R^\lambda \}.
\]

The radical \( \text{rad} \left( C_R^\lambda \right) \) is a right \( A \)-submodule of \( C_R^\lambda \) and \( \text{rad} \left( \_L^C^\lambda \right) \) is a left \( A \)-submodule of \( \_L^C^\lambda \).

**Definition 2.5.** \( D_R^\lambda = \frac{C_R^\lambda}{\text{rad} \left( C_R^\lambda \right)} \) and \( D_C^\lambda = \frac{\_L^C^\lambda}{\text{rad} \left( \_L^C^\lambda \right)} \).

Then \( D_R^\lambda \) is a right \( A \)-module and \( \_L^C^\lambda \) is a left \( A \)-module. The following lemma follows at once from the definitions.

**Lemma 2.2.** The following conditions are equivalent:

(i) \( D_R^\lambda = 0 \);

(ii) \( \text{rad} \left( C_R^\lambda \right) = C_R^\lambda \);

(iii) \( \langle x, y \rangle = 0 \) for all \( x \in C_R^\lambda, y \in \_L^C^\lambda \);

(iv) \( \text{rad} \left( \_L^C^\lambda \right) = \_L^C^\lambda \); and

(v) \( \_L^C^\lambda = 0 \).

**Definition 2.6.** \( \Lambda_0 = \{ \lambda \in \Lambda : \langle x, y \rangle \neq 0 \text{ for some } x \in C_R^\lambda, y \in \_L^C^\lambda \} \).

Evidently, \( \Lambda_0 \supseteq \Lambda_0 \iff D_R^\lambda \neq 0 \iff \_L^C^\lambda \neq 0 \).

When \( R = k \) is a field, one can characterize the irreducible modules in a cell algebra in terms of the set \( \Lambda_0 \).

**Proposition 2.1.** Let \( R = k \) be a field and take \( \lambda \in \Lambda_0 \). Then
(a) \( D^\lambda_R \) is an irreducible right \( A \)-module.
(b) \( \text{rad} \left( C^\lambda_R \right) \) is the unique maximal right submodule in \( C^\lambda_R \).
(c) \( L^\lambda \) is an irreducible left \( A \)-module.
(d) \( \text{rad} \left( L^C^\lambda \right) \) is the unique maximal left submodule in \( L^C^\lambda \).

The modules \( \{ D^\lambda_R : \lambda \in \Lambda \}_0 \) are shown to be absolutely irreducible and pairwise inequivalent and similarly for \( \{ L^\lambda : \lambda \in \Lambda \}_0 \).

A major result of [4] is the following:

**Theorem 2.1.** Assume \( R = k \) is a field. Then \( \{ D^\mu_R : \mu \in \Lambda \}_0 \) is a complete set of pairwise inequivalent irreducible right \( A \)-modules and \( \{ L^\mu : \mu \in \Lambda \}_0 \) is a complete set of pairwise inequivalent irreducible left \( A \)-modules.

In [4] it the following result is also obtained:

**Corollary 2.1.** If \( \Lambda = \Lambda \_0 \), then \( A \) is quasi-hereditary.

### 3 Cell bases for certain monoid algebras

In this section we review (again omitting most of the proofs) the cellbases given in [4] for the monoid algebras \( R[M] \) corresponding to a class of monoids \( M \) containing the full transformation semigroups \( T_r \) and the rook monoids \( R_r \).

Some of the notation and results will be needed in the next section on generalized Schur algebras.

Let \( \bar{\tau}_r = \{ 1, 2, \ldots, r \} \) and let \( \tau_r \) be the monoid of all maps \( \alpha : \bar{\tau}_r \cup \{ 0 \} \to \bar{\tau}_r \cup \{ 0 \} \) such that \( \alpha (0) = 0 \). Note that \( \tau_r \) can be identified with the partial transformation semigroup \( \mathcal{PT}_r \) of all “partial maps” of \( \bar{\tau}_r \) to itself. The full transformation semigroup \( \mathcal{T}_r \) of all maps \( \bar{\tau}_r \to \bar{\tau}_r \) can be identified with the submonoid of \( \tau_r \) consisting of maps with \( \alpha^{-1} (0) = 0 \). The rook monoid \( R_r \) can be identified with the submonoid of \( \tau_r \) consisting of maps such that \( \alpha^{-1} (i) \) has at most one element for each \( i \in \bar{\tau}_r \). With these identifications, the symmetric group \( S_r \) is the intersection \( \mathcal{T}_r \cap R_r \).

Let \( M \) be any monoid contained in \( \tau_r \) and containing \( S_r \). Let \( R \) be a commutative domain with unit \( 1 \) and let \( R[M] \) be the monoid algebra over \( R \). We will describe a cell basis for \( R[M] \).

For \( \alpha \in M \), the index of \( \alpha \) is the number of nonzero elements in the image of \( \alpha \), index \( (\alpha) = |\text{image}(\alpha) - \{0\}| \). Let \( I (M) \subseteq \bar{\tau}_r \cup \{ 0 \} \) be the set of indices of elements in \( M \), that is,

\[
I (M) = \{ i : \exists \alpha \in M \text{ with index} (\alpha) = i \}.
\]

For \( i \in \bar{\tau}_r \), let \( \Lambda (i) \) be the set of all (integer) partitions of \( i \). Let \( \Lambda (0) \) be a set with one element \( \lambda_0 \). Then define \( \Lambda = \bigcup_{i \in I (M)} \Lambda (i) \). For \( \lambda \in \Lambda \) define the index \( i (\lambda) \) to be the integer such that \( \lambda \in \Lambda (i (\lambda)) \). Finally, define a partial order on \( \Lambda \) by

\[
\lambda \succeq \mu \iff i (\lambda) < i (\mu) \text{ or } i (\lambda) = i (\mu) \text{ and } \lambda \succeq \mu
\]
where $\succeq$ is the usual dominance relation on partitions.

To define the sets $L(\lambda)$ and $R(\lambda)$ we need some preliminaries. First, for $i \in \bar{\tau} \cup \{0\}$ let $C(i, r)$ be the collection of all $i$-sets of elements in $\bar{\tau}$, that is, $C(i, r) = \{C = \{c_1, c_2, \ldots, c_i\} : 1 \leq c_1 < c_2 < \cdots < c_i \leq r\}$. $(C(0, r)$ contains one element, the empty set.) For any $C \in C(i, r)$, define a map $\phi_C : i \cup \{0\} \rightarrow \bar{\tau} \cup \{0\}$ by $\phi_C(j) = c_j$ for $j \in i$, $\phi_C(0) = 0$.

Next, choose any ordering of the $2^n$ subsets of $\bar{\tau}$ and label these subsets $d_j$ so that $d_1 < d_2 < \cdots < d_{2^n}$. Let $D(i, r)$ be the collection of sets of $i$ nonempty, pairwise disjoint subsets of $\bar{\tau}$, that is,

$$D(i, r) = \{\{d_{a_1}, d_{a_2}, \cdots, d_{a_k}\} : d_{a_j} \neq \emptyset, d_{a_j} \cap d_{a_k} = \emptyset \text{ and } d_{a_j} < d_{a_k} \text{ for } j < k\}.$$ 

$(D(0, r)$ also contains one element, the empty set.) For any $D \in D(i, r)$ define a map $\psi_D : \bar{\tau} \cup \{0\} \rightarrow \bar{i} \cup \{0\}$ by $\psi_D(x) = j$ for $x \in d_{a_j}$, $\psi_D(x) = 0$ when $x \notin d_{a_j}$ for any $j$.

Regard an element $\sigma$ in the symmetric group $S_i$ as a mapping $\sigma : i \cup \{0\} \rightarrow i \cup \{0\}$ such that $\sigma(0) = 0$. Then for any $\sigma \in S_i$, $C \in C(i, r)$, $D \in D(i, r)$, define an element $\alpha = \alpha(\sigma, C, D) \in \bar{\tau}$ by $\alpha = \phi_C \circ \sigma \circ \phi_D$. Then $\alpha(\sigma, C, D)$ has index $i$. Note that $\alpha(x) = \{c_{\sigma(j)}(x) \in D_{a_j} \text{ if } x \in D_{a_j}, 0 \text{ if } x \notin D_{a_j} \text{ for any } j\}$.

**Lemma 3.1.** For any $\alpha \in \bar{\tau}$, of index $i$, there exist unique $\sigma_\alpha \in S_i$, $C_\alpha \in C(i, r)$, $D_\alpha \in D(i, r)$ such that $\alpha = \phi_{C_\alpha} \circ \sigma_\alpha \circ \phi_{D_\alpha}$.

Notice that $S_\tau$ acts on the left on $C(i, r)$: for $C = \{c_1, c_2, \cdots, c_i\} \in C(i, r)$ and $\sigma \in S_\tau$, let $\sigma C = \{\sigma c_1, \sigma c_2, \cdots, \sigma c_i\}$.

**Lemma 3.2.** Given $C, C' \in C(i, r)$ and $\pi \in S_i$, there exists a $\sigma \in S_\tau$ such that $C' = \sigma C$ and $\sigma \circ \phi_C = \phi_{C'} \circ \pi$.

Given $C \in C(i, r)$, $D \in D(i, r)$, let $A(C, D)$ be the free $R$-submodule of $R[\bar{\tau}]$ with basis $\{\alpha \in \tau : C_\alpha = C, D_\alpha = D\}$. Note that if $i = 0$, then $C(0, r) = D(0, r) = \{0\}$, a set with one element. $A(0, 0)$ is then one dimensional with basis $z$, where $z$ is the zero map such that $z(j) = 0$ for all $j \in \bar{\tau} \cup \{0\}$. Evidently, as an $R$-module

$$R[\bar{\tau}] = \bigoplus_{i \in \tau \cup \{0\}} \bigoplus_{C \in C(i, r), D \in D(i, r)} A(C, D).$$

**Lemma 3.3.** Suppose that for $D \in D(i, r)$ there exists an $\alpha \in M$ with $D_\alpha = D$. Then $A(C, D) \cap R[M] = A(C, D)$ for every $C \in C(i, r)$.

Define $D(M, i, r) = \{D \in D(i, r) : \exists \alpha \in M$ with $D_\alpha = D\}$. Then as an $R$-module, $R[M] = \bigoplus_{i \in \tau \cup \{0\}} \bigoplus_{C \in C(i, r), D \in D(M, i, r)} A(C, D)$ by lemma 3.3. So choosing a basis for each free $R$-module $A(C, D)$, $C \in C(i, r)$, $D \in D(M, i, r)$ will give a basis for $R[M]$.

**Definition 3.1.** For $C \in C(i, r)$, $D \in D(i, r)$, $i > 0$, define a map of $R$-modules $H_{C,D} : R[S_i] \rightarrow A(C, D)$ by $H_{C,D}(\sigma) = \phi_C \circ \sigma \circ \psi_D$. 

5
By lemma 3.1, \( H_{C,D} \) is well-defined and is a bijection between free \( R \)-modules. So any basis for \( R [\mathfrak{S}_i] \) transfers to a basis for \( A (C, D) \). Let \( B_i = \{ s C^\lambda_i : \lambda \in \Lambda (i) , s , t \ \text{standard} \ \lambda \ \text{tableaux} \} \) be the standard Murphy cellular basis for the cellular algebra \( R [\mathfrak{S}_i] \) \( \text{(See e.g. [11] or [7]).} \) Then \( \{ H_{C,D} (s C^\lambda_i) : s C^\lambda_i \in B_i \} \) is a basis for \( A (C, D) \).

We can now finally define our index sets \( L (\lambda) \) and \( R (\lambda) \). Given \( \lambda \in \Lambda (i) , i \in I (M) , i > 0 \), define

\[
L (\lambda) = \{ (C, s) : C \in C (i, r) , s \ \text{a standard} \ \lambda \ \text{tableau} \}
\]

and

\[
R (\lambda) = \{ (D, t) : D \in D (M, i, r) , t \ \text{a standard} \ \lambda \ \text{tableau} \}.
\]

Then for any \( \lambda \in \Lambda , (C, s) \in L (\lambda) , (D, t) \in R (\lambda) \) define

\[
(C, s) C^\lambda_{(D, t)} = H_{C,D} (s C^\lambda_i) \in A (C, D) \subseteq R [M].
\]

If \( 0 \in I (M) \), that is, if the zero map \( z \) such that \( z (j) = 0 \) for all \( j \in \overline{r} \cup \{ 0 \} \) is in \( M \), we define \( \Lambda (0) \) to have a single element \( \lambda_0 \) and define \( L (\lambda_0) = R (\lambda_0) = \{ 0 \} \) each to be sets containing one element, \( \emptyset \). Then define \( \emptyset C^\lambda_0 = z \). The set

\[
\{ (C, s) C^\lambda_{(D, t)} : \lambda \in \Lambda , (C, s) \in L (\lambda) , (D, t) \in R (\lambda) \}
\]

is a union of the bases for the various direct summands \( A (C, D) \) and is therefore a basis for the free \( R \)-module \( R [M] \). We will show that it is a cell-basis for \( R [M] \).

Write \( A_i \) for the cellular algebra \( R [\mathfrak{S}_i] \) and \( \hat{A}_i^\lambda \) for the two sided ideal in \( R [\mathfrak{S}_i] \) spanned by \( \{ s C^\mu_i : \mu > \lambda \} \). The following observation will be useful. Recall that \( \hat{A}^\lambda \) is the \( R \)-submodule of \( R [M] \) spanned by \( \{ (C, s) C^\mu_{(D, t)} : \mu > \lambda \} \).

**Lemma 3.4.** For any \( C \in C (i, r) , D \in D (i, r) \) and \( \lambda \in \Lambda (i) \), \( H_{C,D} (\hat{A}_i^\lambda) \subseteq \hat{A}^\lambda \).

**Lemma 3.5.** For \( \alpha \in M \), \( C \in C (i, r) \), suppose \( C' = \alpha (C) \in C (i, r) \). Then there exists \( \rho \in \mathfrak{S}_i \) such that \( \alpha \circ \phi_C = \phi_{C'} \circ \rho \).

**Lemma 3.6.** For \( \alpha \in M \), \( D = \{ d_j : j \in \overline{i} \} \in D (i, r) \), suppose that \( \alpha^{-1} (d_j) \neq \emptyset \) for all \( j \), so that \( D' = \{ \alpha^{-1} (d_j) : j \in \overline{i} \} \in D (i, r) \). Then there exists \( \rho \in \mathfrak{S}_i \) such that \( \psi_D \circ \alpha = \rho \circ \psi_{D'} \). Furthermore, if \( D \in D (M, i, r) \), then \( D' \in D (M, i, r) \).

**Proposition 3.1.** \( C = \{ (C, s) C^\lambda_{(D, t)} : \lambda \in \Lambda , (C, s) \in L (\lambda) , (D, t) \in R (\lambda) \} \) is a cell basis for \( A = R [M] \).

For \( \lambda \in \Lambda \), the right cell module \( C^\lambda_R \) is a right \( A \)-module and a free \( R \)-module with basis \( \{ C^\lambda_{(D, t)} : (D, t) \in R (\lambda) \} \), while the left cell module \( L C^\lambda \) is a left \( A \)-module and a free \( R \)-module with basis \( \{ (C, s) C^\lambda : (C, s) \in L (\lambda) \} \). The
bracket for $A$ is an $R$-bilinear map $\langle -, - \rangle : C^A_R \times L^A \to R$ defined on basis elements by $\langle C^A_{(D,t)}, (c,s)C^A \rangle = r_{(c,s),(D,t)} \in R$ where $(c',s')(C^A_{(D,t)})^*\cdot (c,s)C^A_{(D',t')} = r_{(c,s),(D,t)} \cdot (c',s')C^A_{(D',t')}$ mod $\bar{A}^A$.

**Lemma 3.7.** Assume $C, C' \in C(i, r), D, D' \in D(i, r)$ and $x, y \in R(\mathfrak{S}_i)$. Then

(a) If $\rho = \psi_D \circ \phi_C : \bar{i} \to \bar{i}$ is not bijective, then $H_{C',D}(x) \cdot H_{C,D'}(y) \in J_{i-1}$.

(b) If $\rho = \psi_D \circ \phi_C : \bar{i} \to \bar{i}$ is bijective, then for any $\pi \in \mathfrak{S}_i$, $H_{C',D}(x) \cdot H_{C,D'}(\pi y) = H_{C',D'}(x \rho \pi y)$.

**Lemma 3.8.** Let $i \in J(M)$, $\lambda \in \Lambda (i)$, $(C, s) \in L(\lambda)$, and $(D, t) \in R(\lambda)$. Then

(a) If $\rho = \psi_D \circ \phi_C : \bar{i} \to \bar{i}$ is not bijective, $\langle C^A_{(D,t)}, (c,s)C^A \rangle = 0$.

(b) If $\rho = \psi_D \circ \phi_C : \bar{i} \to \bar{i}$ is bijective, then for any $\pi \in \mathfrak{S}_i$,

$$\langle C^A_{(D,t)}, \pi' \cdot (c,s)C^A \rangle = \langle C^A_i, \rho \pi \cdot sC^A \rangle_i.$$

Here $\langle - , - \rangle$ is the bracket in the cellular algebra $R[\mathfrak{S}_i]$ and $\pi' = H_{C,D'}(\pi)$ for any $D' \in D(M, i, r)$ such that $\psi_{D'} \circ \phi_C = \text{id} : \bar{i} \to i$.

Recall that the radical, $\text{rad}(C^A_R)$, of a right cell module is the right $A$-module given by $\text{rad}(C^A_R) = \{ x \in C^A_R : \langle x, y \rangle = 0 \text{ for all } y \in L^A \}$.

**Proposition 3.2.** Let $i \in I(M)$, $\lambda \in \Lambda (i)$. Then $\text{rad}(C^A_R) = C^A_R$ in $A$ if and only if $\text{rad}(C^A) = C^A$ in $A_i$.

**Proof.** Assume first that $\text{rad}(C^A) = C^A$ in $A_i$, so $\langle x, y \rangle_i = 0$ for all $x, y$. To show $\text{rad}(C^A_R) = C^A_R$ it suffices to show that $\langle C^A_{(D,t)}, (c,s)C^A \rangle = 0$ for any $(D, t) \in R(\lambda), (C, s) \in L(\lambda)$. If $\rho = \psi_D \circ \phi_C : \bar{i} \to \bar{i}$ is not bijective, lemma 3.8(a) gives $\langle C^A_{(D,t)}, (c,s)C^A \rangle = 0$. If $\rho = \psi_D \circ \phi_C : \bar{i} \to \bar{i}$ is bijective, take $\pi$ in lemma 3.8(b) to be the identity so that $\pi'(c,s)C^A = (c,s)C^A$. Then by lemma 3.8(b), $\langle C^A_{(D,t)}, (c,s)C^A \rangle = \langle C^A_{(D,t)}, \pi' \cdot (c,s)C^A \rangle = \langle C_i^A, \rho \pi \cdot sC^A \rangle_i = 0$.

Now assume $\text{rad}(C^A_R) = C^A_R$ in $A$, so $\langle x, y \rangle = 0$ for any $x \in C^A_R, y \in L^A$. To show $\text{rad}(C^A) = C^A$ in $A_i$, it suffices to show that $\langle C_i^A, sC^A \rangle_i = 0$ for any $i, s$. Take any $D \in D(M, i, r)$ and choose $C \in C(i, r)$ such that $\rho = \psi_D \circ \phi_C$ is bijective. Then apply lemma 3.8(b) with $\pi = \rho^{-1}$ to get $\langle C_i^A, sC^A \rangle_i = \langle C^A_{(D,t)}, \pi' \cdot (c,s)C^A \rangle = 0$. 

Note: In the special case when $0 \in I(M)$, so $\Lambda (0) = \{ \lambda_0 \} \subseteq \Lambda$, the cell modules $C^A_{R'}, LC^A_{\lambda_0}$ are one dimensional with generators $C^A_{\emptyset, 0} \cdot gC^A_{\lambda_0}$ and $\langle C^A_{\emptyset, 0} \cdot gC^A_{\lambda_0} \rangle = 1$ (since $z \cdot z = z$ where $z = gC^A_{\emptyset, 0} : j \mapsto 0$ for all $j \in i \cup \{0\}$). Then $\text{rad}(C^A_{R'}) = 0$. 

7
4 Generalized Schur algebras

For a monoid $M$ as in section 3 and a domain $R$, a “generalized Schur algebra”, $S(M, R)$, was defined in [3] and [2]. This algebra is isomorphic to $R \otimes A^Z$ where $A^Z$ is a certain “$Z$-form”. As shown in [3], there are actually two relevant $Z$-forms, the left and right generalized Schur algebras $LGS(M, G) = A^Z_L$ and $RGS(M, G) = A^Z_R$, corresponding to the monoid $M$ and the family of subgroups $G = \{G_{\mu} : \mu \in \Lambda(r,n)\}$. Here $\Lambda(r,n)$ is the set of all compositions of $r$ with $n$ parts, $G_{\mu}$ is the “Young subgroup” corresponding to $\mu \in \Lambda(r,n)$, and we will always assume $n \geq r$. We sketch the description of these two algebras; for details see [3].

For compositions $\mu, \nu \in \Lambda(r,n)$, let $^{\mu}A^{\nu}$ be the $\mathbb{Z}$-submodule of $A = \mathbb{Z}[M]$ which is invariant under the action of $G_{\mu}$ on the left and $G_{\nu}$ on the right. Let $\mu M_\nu$ be the set of double cosets of the form $G_{\mu}mG_{\nu}$, $m \in M$. For $D \in \mu M_\nu$, define $X(D) = \sum m \in \mathbb{Z}[M]$. Then $^{\mu}A^{\nu}$ is a free $\mathbb{Z}$-module with basis $\{X(D) : D \in \mu M_\nu\}$. Let $\tilde{A} = \bigoplus_{\mu, \nu \in \Lambda(r,n)}^{\mu}A^{\nu}$, the direct sum of disjoint copies of submodules of $A$. Then $\tilde{A}$ is a free $\mathbb{Z}$-module with basis $\{b_D = X(D) : D \in \mu M_\mu, \mu, \nu \in \Lambda(r,n)\}$. Notice that if $D_1 \in \mu M_\mu$, $D_2 \in \nu M_\nu$, then the product $X(D_1)X(D_2)$ (defined in $A$) is invariant under multiplication by $G_{\mu}$ on the left and by $G_{\nu}$ on the right, i.e., $X(D_1)X(D_2) \in ^{\mu}A^{\nu}$. It is therefore a $\mathbb{Z}$-linear combination of $\{X(D) : D \in \mu M_\mu\}$: $X(D_1)X(D_2) = \sum_{D \in \mu M_\mu} a(D_1, D_2, D)X(D)$, with coefficients $a(D_1, D_2, D) \in \mathbb{Z}$. Then an associative, bilinear product on $\tilde{A}$ is defined on the basis elements $b_{D_i} = X(D_i)$ corresponding to $D_1 \in \mu M_\mu$, $D_2 \in \nu M_\nu$ by

$$b_{D_1}b_{D_2} = \begin{cases} \sum_{D \in \nu M_\nu} a(D_1, D_2, D)b_D & \text{if } \nu = \rho \\ 0 & \text{if } \nu \neq \rho \end{cases}.$$

$\tilde{A}$ with this multiplication fails (in general) to have an identity. To obtain the $\mathbb{Z}$-forms $A^Z_L$ and $A^Z_R$, which are $\mathbb{Z}$-algebras with identity, we define new “left” and “right” products, $\ast_L$ and $\ast_R$ on $\tilde{A}$.

For $D \in \mu M_\mu$, let $n_L(D)$ be the number of elements in any left $G_{\mu}$-coset $C \subseteq D$. Then the product $\ast_L$ is defined on the basis elements $b_{D_i} = X(D_i)$ corresponding to $D_1 \in \mu M_\mu$, $D_2 \in \nu M_\nu$ by

$$b_{D_1} \ast_L b_{D_2} = \begin{cases} \sum_{D \in \nu M_\nu} \frac{n_L(D)}{n_L(D_1) n_L(D_2)} a(D_1, D_2, D)b_D & \text{if } \nu = \rho \\ 0 & \text{if } \nu \neq \rho \end{cases}.$$

Similarly, for $D \in \mu M_\nu$, let $n_R(D)$ be the number of elements in any right $G_{\nu}$-coset $C \subseteq D$. Then the product $\ast_R$ is defined on the basis elements $b_{D_i} =
for the individual submodules $O$ be the corresponding Young subgroup.

We will obtain a cell basis for $\bar{\Lambda}(\pi)$ for orbits $O\subseteq A^R_{\mu}$ in $A^R_{\mu}$ that both of these algebras, and the associated generalized Schur algebras $R \otimes A^L_{\mu}$ and $R \otimes A^R_{\mu}$ for any domain $R$, have cell bases and are cell algebras.

We use the notation and definitions of section 3. For a composition $\mu \in \Lambda(r,n)$, $\mathfrak{S}_\mu$ acts on the left on $C(i,r)$ : if $C = \{c_1, c_2, \cdots, c_i\} \in C(i,r)$ and $\sigma \in \mathfrak{S}_\mu$ define $\sigma C = \{\sigma c_1, \cdots, \sigma c_i\}$. Then write $O(\mu, C) = \{\rho C : \rho \in \mathfrak{S}_\mu\}$ for the orbit of $C \subseteq C(i,r)$ under $\mathfrak{S}_\mu$. Similarly, for a composition $\nu \in \Lambda(r,n)$, $\mathfrak{S}_\nu$ acts on the right on $D(M,i,r)$ for $D = \{d_1, d_2, \cdots, d_i\} \in D(M,i,r)$ and $\pi \in \mathfrak{S}_\nu$.

Then write $O(\nu, D) = \{D \pi : \pi \in \mathfrak{S}_\nu\}$ for the orbit of $D \subseteq D(M,i,r)$ under $\mathfrak{S}_\nu$.

Any double coset $D \in \mathfrak{S}_\mu \alpha \mathfrak{S}_\nu = \mu M_{\nu}$ has a well defined index $i$ since any $\beta \in D$ has the same index as $\alpha$. We also have $O(\mu, C_\beta) = O(\mu, C_{\alpha})$ for any $\beta \in D$, so $D$ has a well defined orbit $O(\mu, D) = O(\mu, C_{\alpha})$ for the action of $\mathfrak{S}_\mu$ on $C(i,r)$. Similarly, $D$ has a well defined orbit $O(\nu, D) = O(\nu, D_{\alpha})$ for the action of $\mathfrak{S}_\nu$ on $D(M,i,r)$.

Let $O_{\mu, C(i,r)}$ be the set of orbits for the action of $\mathfrak{S}_\mu$ on $C(i,r)$ so $C(i,r) = \bigcup_{O \in O_{\mu, C(i,r)}} O$. Similarly, let $O_{\nu, D(M,i,r)}$ be the set of orbits for the action of $\mathfrak{S}_\nu$ on $D(M,i,r)$ so $D(M,i,r) = \bigcup_{O \in O_{\nu, D(M,i,r)}} O$.

Finally, for $(O_{\mu, O_{\nu}}) \in O(\mu, \nu, M, i)$ define a collection of double cosets

$$M(O_{\mu, O_{\nu}}) = \{D \in \mu M_{\nu} : \text{index}(D) = i, O(\mu, D) = O_{\mu}, O(\nu, D) = O_{\nu}\}.$$ Then for orbits $O_{\mu} \in O_{\mu, C(i,r)}, O_{\nu} \in O_{\nu, D(M,i,r)}$ define $O_{\nu} A^{O_{\nu}}$ to be the free module with basis $\{X(D) : D \in M(O_{\mu, O_{\nu}})\}$. Evidently

$$O_{\nu} A^{O_{\nu}} = \bigoplus_{i \in I(M)} \bigoplus_{(O_{\mu}, O_{\nu}) \in O(\mu, \nu, M, i)} O_{\nu} A^{O_{\nu}}.$$ We will obtain a cell basis for $\bar{A} = \bigoplus_{\mu, \nu \in \Lambda(r,n)} \mu^* A^{\nu}$ by taking the union of bases for the individual submodules $O_{\mu} A^{O_{\nu}}$.

For $\mu \in \Lambda(r,n)$ and $C \subseteq C(i,r)$ let $\mu(C) \in \Lambda(i,n)$ be the composition of $i$ obtained as follows: if $b^\mu_j$, $j \in \bar{n}$, is the $j$th block of $\mu$, then the $j$th block of $\mu(C)$ is given by $b^{\mu_j}_{C}(\mu) = \phi^{-1}_{C}(b^\mu_j)$, $j \in \bar{n}$. Notice that $\mu(C)_{j} = |b^\mu_j \cap C|$, the number of the $i$ elements in $C$ which lie in the $j$th block of $\mu$. Since elements of $\mathfrak{S}_\mu$ preserve the blocks of $\mu$ (by definition), the composition $\mu(C)$ depends only on the orbit $O(\mu, C)$, that is, $\mu(\rho C) = \mu(C)$ for any $\rho \in \mathfrak{S}_\mu$. Let $\mathfrak{S}_{\mu}(C) \subseteq \mathfrak{S}_i$ be the corresponding Young subgroup.
Lemma 4.1. Given $C \in C(i, r)$ and $\rho \in \mathfrak{S}_\mu$, let $\rho C \in O(\mu, C)$ be the image of $C$ under $\rho$. Then there exists a unique $\rho C \in \mathfrak{S}_\mu(\nu)$ such that $\rho \cdot \phi_C = \phi_{\rho C} \cdot \rho C$. Conversely, given any $\rho C \in \mathfrak{S}_\mu(\nu)$ and any $\rho C \in O(\mu, C)$, there exists a $\rho' \in \mathfrak{S}_\mu$ such that $\rho' \cdot \phi_C = \phi_{\rho C} \cdot \rho_C$ and $\rho' C = \rho C \in O(\mu, C)$.

Proof. $\rho \cdot \phi_C$ and $\phi_{\rho C}$ both map $\overline{1}$ one to one onto $\rho C$. So define $\rho C \in \mathfrak{S}_i$ by letting $\rho_C(k)$ be the unique element in $\phi_{\rho C}^{-1}[\rho \cdot \phi_C(k)]$. Then $\rho C$ is the unique element in $\mathfrak{S}_i$ such that $\rho \cdot \phi_C = \phi_{\rho C} \cdot \rho_C$, and we need only prove that $\rho C \in \mathfrak{S}_\mu(\nu)$. So suppose $j$ is in the $k$th $\mathfrak{S}_\mu(\nu)$ block $b_k^{(\mu)}$. We must show that $\rho C(j) \in b_k^{(\mu)}(\nu)$. But $j \in b_k^{(\mu)}(\nu) \Rightarrow \phi_C(j) \in b_k^{(\mu)} \Rightarrow \rho \cdot \phi_C(j) \in b_k^{(\mu)}$ (since $\rho \in \mathfrak{S}_\mu$). Then $\phi_{\rho C} \cdot \rho C(j) \in b_k^{(\mu)}$ which implies $\rho C(j) \in b_k^{(\mu)}$. But $\mu(\rho C) = \mu(C)$, so $\rho C(j) \in b_k^{(\mu)}$ as desired.

Now consider an element $\nu \in \Lambda(r, n)$, $\mathfrak{S}_\nu$ acts on the right on $P_r$: if $S \in P_r$ and $\pi \in \mathfrak{S}_\nu$, then $S \pi = \pi^{-1}(S) = \{\pi^{-1} \{s\} : s \in S\} \in P_r$. Choose a total order on the orbits of $\mathfrak{S}_\nu$ acting on $P_r$ and then label the orbits $O_i$ so that $O_1 < O_2 < \cdots < O_{N_\nu}$, where $N_\nu$ is the number of orbits. Then choose a total order of all the subsets in $P_r$ which is compatible with the ordering of the $\mathfrak{S}_\nu$ orbits: $S_{a_1} \in O_{a_1}$ and $O_{a_1} < O_{a_2} \Rightarrow S_{a_1} < S_{a_2}$. We label the subsets $S_i$ so that $S_1 < S_2 < \cdots < S_{2r}$.

Now consider an element $D \in D(M, i, r)$. $D$ consists of $i$ sets in $P_r$, so $D = \{\{S_{a_1} < S_{a_2} < \cdots < S_{a_i}\} \}$. Define a composition $\nu(D) \in \Lambda(i, N_\nu)$ by $\nu(D)_j = |D \cap O_j|$, $j = 1, 2, \cdots, N_\nu$. Then $S_{a_k}, S_{a_l}$ are in the same orbit $O_j$ if and only if $k, l$ are in the same block of the composition $\nu(D)$, $k, l \in b_j^{(\nu)}$. $\mathfrak{S}_\nu$ acts on the right on $D(M, i, r)$: For $\pi \in \mathfrak{S}_\nu, D \pi = \{S_{a_j} \pi : j \in i\} = \{\pi^{-1}[S_{a_j}] : j \in i\} \in D(M, i, r)$. Since by definition $\mathfrak{S}_\nu$ preserves orbits, $|D \pi \cap O_j| = |D \pi \cap O_j|$ for any $\pi \in \mathfrak{S}_\nu$. So the composition $\nu(D \pi) = \nu(D)$ depends only on the orbit $O(\nu, D)$ of $D$ under the action of $\mathfrak{S}_\nu$. Let $\mathfrak{S}_{\nu(D)} \subseteq \mathfrak{S}_i$ be the corresponding Young subgroup.

Lemma 4.2. Given $D \in D(M, i, r)$ and $\pi \in \mathfrak{S}_\nu$, let $D \pi \in O(\nu, D)$ be the image of $D$ under $\pi$. Then there exists a unique $\pi D \in \mathfrak{S}_{\nu(D)}$ such that $\psi_D \cdot \pi = \pi D \cdot \psi_{\pi D}$. Conversely, given any $\pi D \in \mathfrak{S}_{\nu(D)}$ and any $D \pi \in O(\nu, D)$, there exists a $\pi' \in \mathfrak{S}_\nu$ such that $\psi_D \cdot \pi' = \pi_D \cdot \psi_{\pi D}$ and $D \pi' = D \pi \in O(\nu, D)$.

Proof. Let $D = \{S_{a_1} < S_{a_2} < \cdots < S_{a_i}\}$, so $D \pi = \{\pi^{-1}(S_{a_j}) : j \in i\}$. Arrange the sets in $D \pi$ in order and define $k(j) \in i$ such that $\pi^{-1}(S_{a_j})$ is the $k(j)$th set in the sequence. Then $\psi_D \cdot \pi$ maps elements in $\pi^{-1}(S_{a_j})$ to $j$, while $\psi_{\pi D}$ maps elements in $\pi^{-1}(S_{a_j})$ to $k(j)$. Define $\pi D \in \mathfrak{S}_i$ by $\pi_D(k(j)) = j, j \in i$. Then $\pi_D$ is the unique element in $\mathfrak{S}_i$ such that $\psi_D \cdot \pi = \pi_D \cdot \psi_{\pi D}$, and it remains to show that $\pi_D \in \mathfrak{S}_{\nu(D)}$. For this, we must show that $j$ and $k(j)$ are always in the same block of the composition $\nu(D)$. But $D \pi$ and $D$ contain the same
number of subsets in each orbit of \( \mathcal{S}_\nu \). So if \( S_{a_j} \) and \( \pi^{-1}(S_{a_j}) \) are both in the \( l \)th orbit of \( \mathcal{S}_\nu \), then their indices \( j \) and \( k(j) \) are both in the \( l \)th block of \( \nu(D) \).

Now take any \( \pi_D \in \mathcal{S}_\nu(D) \) and any \( D\pi \in O(\nu, D) \). Recall that \( \bar{r} \supseteq \bigcup_{k \in \bar{r}} \pi^{-1}(S_{a_k}) \) and that the sets \( \pi^{-1}(S_{a_k}) \) are pairwise disjoint, so we can define a \( \pi' \in \mathcal{S}_r \) by defining \( \pi' \big| \pi^{-1}(S_{a_k}) \) for each \( k \). Suppose \( k \) is in the \( j \)th block of \( \nu(D) \). Then both \( (\psi_D)^{-1}(k) = S_{a_k} \) and \( (\pi_D \cdot \psi_D)^{-1}(k) = \pi^{-1}(S_{a_m(k)}) \) (for some \( m(k) \)) are in the \( j \)th orbit of \( \mathcal{S}_\nu \). Define \( \pi' \in \mathcal{S}_r \) by \( \pi' \big| (\pi^{-1}(S_{a_m(k)})) = \sigma_k \big| (\pi^{-1}(S_{a_m(k)})) \) where \( \sigma_k \in \mathcal{S}_\nu \) maps \( \pi^{-1}(S_{a_m(k)}) \) one to one onto \( S_{a_k} \). Then \( (\psi_D \cdot \pi')^{-1}(k) = (\pi')^{-1}(\psi_D^{-1}(k)) = (\pi')^{-1}(S_{a_k}) = \pi^{-1}(S_{a_m(k)}) = (\pi_D \cdot \psi_D)^{-1}(k) \) for all \( k \). So \( \psi_D \cdot \pi' = \pi_D \cdot \psi_D \pi \) and it remains to show that \( \pi' \in \mathcal{S}_\nu \). It suffices to show that if \( l \) is in the \( j \)th block of \( \nu \) then \( \pi'(l) \) is also in the \( j \)th block. But any \( l \) is in a unique \( \pi^{-1}(S_{a_k}) \), so \( \pi'(l) = \sigma_k(l) \) is in fact in the \( j \)th block since \( \sigma_k \in \mathcal{S}_\nu \).

Consider compositions \( \mu, \nu \in \Lambda(r, n) \) and an element \( \alpha \in M \) of index \( i \). There exist unique \( \sigma_\alpha \in \mathcal{S}_r \), \( C_\alpha \in C(i, r) \), \( D_\alpha \in D(M, i, r) \) such that \( \alpha = \phi_{C_\alpha} \circ \sigma_\alpha \circ \psi_{D_\alpha} \). For \( C = \rho C_\alpha \in O(\mu, C_\alpha) \), define

\[
\phi_C \cdot \mathcal{S}_{\nu(C)} \cdot \sigma_\alpha \circ \psi_{D_\alpha} = \{ \phi_C \cdot \kappa \cdot \sigma_\alpha \circ \psi_{D_\alpha} : \kappa \in \mathcal{S}_{\nu(C)} \} \subseteq M.
\]

Similarly, for \( D = D_\alpha \pi \in O(\nu, D_\alpha) \), define

\[
\phi_{C_\alpha} \circ \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \cdot \psi_D = \{ \phi_{C_\alpha} \circ \sigma_\alpha \cdot \gamma \cdot \psi_D : \gamma \in \mathcal{S}_{\nu(D)} \}.
\]

Finally, for \( C \in O(\mu, C_\alpha) \), \( D \in O(\nu, D_\alpha) \), define

\[
\phi_C \cdot \mathcal{S}_{\nu(C)} \cdot \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \cdot \psi_D = \{ \phi_C \cdot \kappa \cdot \sigma_\alpha \cdot \gamma \cdot \psi_D : \kappa \in \mathcal{S}_{\nu(C)}, \gamma \in \mathcal{S}_{\nu(D)} \}.
\]

**Proposition 4.1.** For compositions \( \mu, \nu \in \Lambda(r, n) \) and an element \( \alpha = \phi_{C_\alpha} \circ \sigma_\alpha \circ \psi_{D_\alpha} \in M \) of index \( i \),

(a) For \( C_1, C_2, C \in O(\mu, C_\alpha), D_1, D_2, D \in O(\nu, D_\alpha) \),

\[
(\phi_{C_1} \cdot \mathcal{S}_{\nu(C)} \cdot \sigma_\alpha \circ \psi_{D_1}) \cap (\phi_{C_2} \cdot \mathcal{S}_{\nu(C)} \cdot \sigma_\alpha \circ \psi_{D_1}) = \emptyset
\]

unless \( C_1 = C_2 \) and \( D_1 = D_2 \). The double coset \( \mathcal{S}_{\mu, \alpha} \mathcal{S}_{\nu} \) is a disjoint union

\[
\mathcal{S}_{\mu, \alpha} \mathcal{S}_{\nu} = \bigcup_{C \in O(\mu, C_\alpha), D \in O(\nu, D_\alpha)} \phi_C \cdot \mathcal{S}_{\nu(C)} \cdot \sigma_\alpha \circ \psi_{D_\alpha}.
\]

(b) For \( C_1, C_2 \in O(\mu, C_\alpha) \),

\[
(\phi_{C_1} \cdot \mathcal{S}_{\nu(C)} \cdot \sigma_\alpha \circ \psi_{D_1}) \cap (\phi_{C_2} \cdot \mathcal{S}_{\nu(C)} \cdot \sigma_\alpha \circ \psi_{D_1}) = \emptyset
\]

unless \( C_1 = C_2 \).

\[
\mathcal{S}_{\mu, \alpha} = \bigcup_{C \in O(\mu, C_\alpha)} \phi_C \cdot \mathcal{S}_{\nu(C)} \cdot \sigma_\alpha \circ \psi_{D_\alpha},
\]

a disjoint union.
(c) For \( D_1, D_2 \in O(\nu, D_\alpha) \),
\[
(\phi_{C_\alpha} \cdot \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \cdot \psi_{D_1}) \cap (\phi_{C_\alpha} \cdot \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \cdot \psi_{D_2}) = \emptyset
\]
unless \( D_1 = D_2 \).

\[
\alpha \mathcal{S}_\nu = \bigcup_{D \in O(\nu, D_\alpha)} \phi_{C_\alpha} \cdot \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \circ \psi_D,
\]
a disjoint union.

Proof. For part (a), if
\[
\beta \in (\phi_{C_1} \cdot \mathcal{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \cdot \psi_{D_1}) \cap (\phi_{C_2} \cdot \mathcal{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \cdot \psi_{D_1})
\]
then \( C_1 = C_\beta = C_2 \) and \( D_1 = D_\beta = D_2 \), so
\[
(\phi_{C_1} \cdot \mathcal{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \cdot \psi_{D_1}) \cap (\phi_{C_2} \cdot \mathcal{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \cdot \psi_{D_1}) = \emptyset
\]
unless \( C_1 = C_2 \) and \( D_1 = D_2 \).

If \( \beta \in \mathcal{S}_\mu \alpha \mathcal{S}_\nu \), then \( \beta = \rho \circ \phi_{C_\alpha} \circ \sigma_\alpha \circ \psi_{D_\alpha} \circ \pi \) for some \( \rho \in \mathcal{S}_\mu \), \( \pi \in \mathcal{S}_\nu \).

By lemmas 4.1 and 4.2 there exist \( \rho C_\alpha \in \mathcal{S}_{\mu(C)} \), \( \pi D \in \mathcal{S}_{\nu(D)} \) such that
\[
\beta = \phi_{\rho C_\alpha} \circ \rho C_\alpha \circ \pi D \circ \psi_{D_\alpha} \circ \pi \in \phi_{\rho C_\alpha} \cdot \mathcal{S}_{\mu(C)} \circ \sigma_\alpha \circ \mathcal{S}_{\nu(D)} \circ \psi_{D_\alpha} \circ \pi .
\]
So \( \mathcal{S}_\mu \alpha \mathcal{S}_\nu \subseteq \bigcup_{C \in O(\mu, C_\alpha), D \in O(\nu, D_\alpha)} \phi_{C} \cdot \mathcal{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \cdot \psi_D \). On the other hand, if
\[
\beta = \phi_{\rho C_\alpha} \circ \rho C_\alpha \circ \pi D \circ \psi_{D_\alpha} \circ \pi \in \phi_{\rho C_\alpha} \cdot \mathcal{S}_{\mu(C)} \circ \sigma_\alpha \circ \mathcal{S}_{\nu(D)} \circ \psi_{D_\alpha} \circ \pi ,
\]
then by lemmas 4.1 and 4.2 there exist \( \rho' \in \mathcal{S}_\mu \), \( \pi' \in \mathcal{S}_\nu \) such that
\[
\beta = \rho' \circ \phi_{C_\alpha} \circ \sigma_\alpha \circ \psi_{D_\alpha} \circ \pi' = \rho' \alpha \pi' \in \mathcal{S}_\mu \alpha \mathcal{S}_\nu .
\]
So
\[
\bigcup_{C \in O(\mu, C_\alpha), D \in O(\nu, D_\alpha)} \phi_{C} \cdot \mathcal{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathcal{S}_{\nu(D)} \cdot \psi_D \subseteq \mathcal{S}_\mu \alpha \mathcal{S}_\nu ,
\]
completing the proof of part (a).

(b) follows from (a) by taking \( D_1 = D_2 = D_\alpha \) and \( \nu \) to be the composition \( \nu_i = 1 \), \( \forall i \), so \( \mathcal{S}_\nu = \mathcal{S}_{\nu(D)} = \{1\} \). Similarly, (c) follows from (a) by taking \( C_1 = C_2 = C_\alpha \) and \( \mu \) to be the composition \( \mu \alpha = 1 \), \( \forall i \), so \( \mathcal{S}_\mu = \mathcal{S}_{\mu(C)} = \{1\} \).

Any double coset \( D = \mathcal{S}_\mu \alpha \mathcal{S}_\nu \in \mu M_\nu \) has a well defined index \( i \) since any \( \beta \in D \) has the same index as \( \alpha \). We also have \( O(\mu, C_\beta) = O(\mu, C_\alpha) \) for any \( \beta \in D \), so \( D \) has a well defined orbit \( O(\mu, D) = O(\mu, D_\alpha) \) for the action of \( \mathcal{S}_\mu \) on \( C \). There is also a well defined composition \( \mu(D) \) and a Young subgroup \( \mathcal{S}_{\mu(D)} \) where \( \mu(D) = \mu(C) \) for any \( C \in O(\mu, D) \). Both \( \mu(D) \) and \( \mathcal{S}_{\mu(D)} \) depend only on the orbit \( O(\mu, D) \). For a double coset \( D = \mathcal{S}_\mu \alpha \mathcal{S}_\nu \in \mu M_\nu \), a left coset
\[
C \subseteq \mathcal{S}_\mu \alpha \mathcal{S}_\nu \text{ has the form } C = \mathcal{S}_\mu \alpha \cdot \pi = \bigcup_{C \in O(\mu, C_\alpha)} \phi_{C} \cdot \mathcal{S}_{\mu(C_\alpha)} \cdot \sigma_\alpha \circ \psi_{D_\alpha} \circ \pi
\]
for some $\pi \in S_\nu$, where the union is disjoint by proposition 4.1. Then $n_L(D) = |C| = |O(\mu, C_\alpha)| \cdot |SSt(\mu(C_\alpha))| = |O(\mu, D)| \cdot |SSt(\mu(D))|$, which depends only on the orbit $O_\mu = O(\mu, D) = O(\mu, C_\alpha)$. So we can define $n_L(O_\mu) = n_L(D)$ for any $D$ with $O_\mu = O(\mu, D)$.

Similarly, $D$ has a well defined orbit $O(\nu, D) = O(\nu, D_\nu)$ for the action of $SSt(\nu)$ on $D(M, i, \nu)$ and there are a composition $\nu(D)$ and a Young subgroup $SSt(\nu(D))$ where $\nu(D) = \nu(D)$ for any $D \in O(\nu(D))$. Then $\nu(D), SSt(\nu(D))$, and $n_R(D) = |O(\nu, D)| \cdot |SSt(\nu(D))|$ depend only on the orbit $O(\nu, D)$ and we can define $n_R(O_\nu) = n_R(D)$ for any $D$ with $O_\nu = O(\nu, D)$.

Given an orbit pair $(O_\mu, O_\nu) \in O(\mu, \nu, M, i)$, define compositions $\mu(O_\mu)$ and $\nu(O_\nu)$ of $i$ and corresponding Young subgroups $SSt(\mu(O_\mu)), SSt(\nu(O_\nu))$, where $\mu(O_\mu) = \mu(C)$ for any $C \in O_\mu$ and $\nu(O_\nu) = \nu(D)$ for any $D \in O_\nu$. If $\mu(O_\mu)B^{\nu(O_\nu)} \subseteq Z[SSt(\nu)]$ is the $Z$-submodule of $B = Z[SSt(\nu)]$ which is invariant under the action of $SSt(\mu(O_\mu))$ on the left and $SSt(\nu(O_\nu))$ on the right, then $\mu(O_\mu)B^{\nu(O_\nu)}$ is a free $Z$-module with basis $\{X(\nu(O_\nu)| \tau SSt(\nu(O_\nu)) : \tau \in SSt(\nu)\}$. Define a $Z$-linear map $\Phi(O_\mu, O_\nu): \mu(O_\mu)B^{\nu(O_\nu)} \to O_\nu A^{O_\nu}$ by

$$\Phi(O_\mu, O_\nu)(x) = \sum_{C \in O_\mu} \sum_{D \in O_\nu} \phi_C \circ x \circ \psi_D.$$ 

By proposition 4.1, $\Phi(O_\mu, O_\nu)$ is an isomorphism of free $Z$-modules taking the basis elements $X(\nu(O_\nu)| \tau SSt(\nu(O_\nu))$ for $\mu(O_\mu)B^{\nu(O_\nu)}$ one to one onto the basis elements $X(\nu(O_\nu)| \tau SSt(\nu(O_\nu))$ for $O_\nu A^{O_\nu}$ where $\alpha = \phi_C \circ \sigma \circ \psi_D$ for any $C \in O_\mu, D \in O_\nu$.

Now as a $Z$-module, $\mu(O_\mu)B^{\nu(O_\nu)}$ can be identified with a direct summand of the standard Schur algebra $S^Z(\nu, M)$. The standard cellular basis for the Schur algebra $S^Z(\nu, M)$ then yields a basis $\{s C \lambda \}$ for $\mu(O_\mu)B^{\nu(O_\nu)}$, where $\lambda$ is a partition of $\nu$, $S$ is a semistandard $\lambda$ tableau of type $\mu(O_\mu)$ and $T$ is a semistandard $\lambda$ tableau of type $\nu(O_\nu)$. Then $\{\Phi(O_\mu, O_\nu)(s C \lambda)\}$ gives a basis $B(O_\mu, O_\nu)$ for $O_\nu A^{O_\nu}$. These piece together to give a basis for $A$ which turns out to be a cell basis for $A_L$ or $A_R$.

Let $\Lambda = \bigcup_{i \in I(M)} \Lambda(i)$ with the same partial order as for the cell algebra $Z[M]$ of section 7. For $\nu \in \Lambda(i) \subseteq \Lambda$ define

$$L(\lambda) = \{O_\mu, S : \mu \in \Lambda(i, \nu), O_\mu \in O(\mu, C(i, r)), S \in SSt(\lambda, \mu(O_\mu))\}$$

where $SSt(\lambda, \mu(O_\mu))$ is the set of semistandard $\lambda$ tableaux of type $\mu(O_\mu)$ and define

$$R(\lambda) = \{O_\nu, T : \nu \in \Lambda(i, \nu), O_\nu \in O(\nu, D(M, i, r)), T \in SSt(\lambda, \nu(O_\nu))\}$$

where $SSt(\lambda, \nu(O_\nu))$ is the set of semistandard $\lambda$ tableaux of type $\nu(O_\nu)$. Then for $\lambda \in \Lambda, O_\nu, S \in L(\lambda), (O_\nu, T) \in R(\lambda)$, define $(O_\nu, S)C^\lambda_{\nu(T)} = \Phi(O_\mu, O_\nu)(s C^\lambda)$. As just mentioned, elements of this type provide a $Z$-basis for each direct summand $O_\nu A^{O_\nu}$ and hence for all of $A$.

Note that if $M$ contains the zero map $z$ where $z(j) = 0$ for all $j$, then $\Lambda$ contains the empty partition $\lambda^0$ of index 0. For any partitions $\mu, \nu$ the double
Lemma 4.4. Let $\mathbf{S}_\mu \mathbf{S}_\nu = \{ z \} \subseteq \mu A^{\nu}$. $C(0,r) = D(M,0,r) = \emptyset$, so for each partition there is one orbit $O_\mu$ or $O_\nu$. Then $L(\lambda^0)$ contains one element $(O_\mu, \emptyset)$ for each partition $\mu$, and similarly for $R(\lambda^0)$. Then for each pair of partitions $\mu, \nu$ our basis contains an element $(O_\mu, \emptyset) C^\lambda_{(O_\mu, \emptyset)} = z \in \mu A^{\nu}$.

We now show that the basis just described is a cell basis. We first check that these basis elements have the left and right cell algebra properties (i) and (ii) for the “ordinary” product in $\bar{A}$. We then check that the properties also hold for the products $*_L$ and $*_R$ in $A^2$ and $A^R$.

Notice that each basis element $(O_\mu, S) C^\lambda_{(O_\mu, T)}$ for $\bar{A}$ is a sum of basis elements in the cell algebra $A = \mathbb{Z}[M]$ of section 7 of the form $C_{s,t} C^\lambda_{t,D}$ for the same $\lambda$ (where $s, t$ are standard tableaux of type $S, T$, $C \in O_\mu$, $D \in O_\nu$). As in section 7, let $A^1, \bar{A}^1$ be the ideals in the cell algebra $A = \mathbb{Z}[M]$ and let $\bar{A}^1, \bar{A}^1$ be the corresponding submodules of $\bar{A}$ (spanned by basis elements $(O_\mu, S) C^\lambda_{(O_\mu, T)}$ with $\kappa \geq \lambda$ or $\kappa > \lambda$ respectively). Then $\bar{A}^1 \cap \mu A^{\nu} = \bar{A}^1 \cap \mu A^{\nu}$ for any $\lambda, \mu, \nu$.

**Lemma 4.3.** Let $(O_{\mu_1}, S_1) C^\lambda_{(O_{\mu_1}, T_1)} \in O_{\mu_1} A^{O_{\nu_1}}$ for $i = 1, 2$. Then in $\bar{A}, (O_{\mu_1}, S_1) C^\lambda_{(O_{\mu_1}, T_1)}$ \((O_{\mu_2}, s_2) C^\lambda_{(O_{\mu_2}, T_2)} \sum_{\mu', S'} r \cdot (O_{\nu'}, S') C^\lambda_{(O_{\nu'}, T')} \mod \bar{A}^2 \) where the coefficients $r \in \mathbb{Z}$ are independent of $O_{\nu_2}$ and $T_2$.

**Proof.** Write $(O_{\mu_2}, s_2) C^\lambda_{(O_{\mu_2}, T_2)}$ as a sum of terms $C_{s,t} C^\lambda_{t,D}$ where $t$ is a standard tableau of type $T$ and $D \in O_{\nu_2}$. Then using property (i) for the cell algebra $A$, we have $(O_{\mu_1}, S_1) C^\lambda_{(O_{\mu_1}, T_1)} \cdot (O_{\mu_2}, s_2) C^\lambda_{(O_{\mu_2}, T_2)} = \sum_{\mu', S'} r \cdot C_{s,t} C^\lambda_{t,D} \mod \bar{A}^2$ where the coefficients $r$ are independent of $D$ and $t$ and therefore of $O_{\nu_2}$ and $T_2$. Also $(O_{\mu_1}, S_1) C^\lambda_{(O_{\mu_1}, T_1)} \cdot (O_{\mu_2}, s_2) C^\lambda_{(O_{\mu_2}, T_2)} \in \mu_1 A^{\nu_2}$. So the terms $C_{s,t} C^\lambda_{t,D}$ must regroup into a linear combination of terms of the form $(O_{\nu'}, S') C^\lambda_{(O_{\nu'}, T')}$. Then using $\bar{A}^1 \cap \mu_1 A^{\nu_2} = \bar{A}^1 \cap \mu_1 A^{\nu_2}$ gives the result. \[\square\]

Since the $(O_\mu, S) C^\lambda_{(O_\mu, T)}$ form a basis for $\bar{A}$, linearity gives the following corollary.

**Corollary 4.1.** For any $x \in \bar{A}$,

$$x \cdot (O_\mu, S) C^\lambda_{(O_\mu, T)} = \sum_{\mu', S'} r \cdot (O_{\nu'}, S') C^\lambda_{(O_\nu', T')} \mod \bar{A}^1$$

where $r = r(x, \mu, S, \mu', S')$ is independent of $O_{\nu}, T$.

Similar arguments give the following results.

**Lemma 4.4.** Let $(O_{\mu_1}, S_1) C^\lambda_{(O_{\mu_1}, T_1)} \in O_{\mu_1} A^{O_{\nu_1}}$ for $i = 1, 2$. Then in $\bar{A}, (O_{\mu_1}, S_1) C^\lambda_{(O_{\mu_1}, T_1)}$ \((O_{\mu_2}, s_2) C^\lambda_{(O_{\mu_2}, T_2)} \sum_{\nu', T'} r \cdot (O_{\nu'}, S_{\nu'}) C^\lambda_{(O_{\nu'}, T')} \mod \bar{A}^1 \) where the coefficients $r \in \mathbb{Z}$ are independent of $O_{\mu_1}$ and $S_1$.
Corollary 4.2. For any $x \in \hat{A}$,

$$(O_{\mu}, S)^C_{\lambda} (O_{\nu}, T) \cdot x = \sum_{\nu', T'} r \cdot (O_{\mu}, S)^C_{\lambda} (O_{\nu'}, T') \mod \hat{A}^\lambda$$

where $r = r(x, \nu, T, \nu', T')$ is independent of $O_{\mu}, S$.

We now transfer our results to $A^2_L$ and $A^2_R$. We need the following lemma.

Lemma 4.5. Let $B = \cup B(O_{\mu}, O_{\nu})$ be a basis for $\hat{A}$ where each $B(O_{\mu}, O_{\nu})$ is a basis for the direct summand $O_{\nu}A^{O_{\nu}}$. For $b \in B(O_{\mu}, O_{\nu})$ define $n_L(b) = n_L(O_{\mu})$ and $n_R(b) = n_R(O_{\nu})$. Assume that $b_1 b_2 = \sum_{b \in B} c(b_1, b_2, b) b$ with structure constants $c(b_1, b_2, b) \in \mathbb{Z}$ (using the “ordinary” product in $\hat{A}$). Then $b_1 *_L b_2 = \sum_{b \in B} n_L(b_1)n_L(b_2) c(b_1, b_2, b)$ and $b_1 *_R b_2 = \sum_{b \in B} n_R(b_1)n_R(b_2) c(b_1, b_2, b)$.

Proof. The result is true by definition for the standard basis $\{b_D = X(D)\}$. Each $b \in B(O_{\mu}, O_{\nu})$ is a linear combination of standard basis vectors $b_D \in B(O_{\mu}, O_{\nu})$ and each $b_D \in O_{\nu}A^{O_{\nu}}$ is a linear combination of the new $b \in B(O_{\mu}, O_{\nu})$. Then since the values $n_L(b), n_R(b)$ depend only on the orbits $O_{\mu}, O_{\nu}$, the result for the new basis follows by linearity. \(\square\)

Lemma 4.6. Let $(O_{\mu_i}, S_i)^C_{\lambda} (O_{\nu_i}, T_i) \in O_{\nu_i}A^{O_{\nu_i}}$ for $i = 1, 2$. Assume $\nu_1 = \mu_2$.

(a) In $A^2_L$,

$$(O_{\mu_1}, S_1)^C_{\lambda_1} (O_{\nu_2}, T_1) *_L (O_{\mu_2}, S_2)^C_{\lambda_2} (O_{\nu_2}, T_2) = \sum_{\mu', S'} r \cdot (O_{\mu'}, S')^C_{\lambda_2} (O_{\nu_2}, T_2) \mod \hat{A}^{\lambda_2}$$

where the coefficients $r \in \mathbb{Z}$ are independent of $O_{\nu_2}$ and $T_2$.

(b) In $A^2_L$,

$$(O_{\mu_1}, S_1)^C_{\lambda_1} (O_{\nu_2}, T_1) *_L (O_{\mu_2}, S_2)^C_{\lambda_2} (O_{\nu_2}, T_2) = \sum_{\mu', S'} r \cdot (O_{\mu_1}, S_1)^C_{\lambda_2} (O_{\nu_2}, T_2) \mod \hat{A}^{\lambda_2}$$

where the coefficients $r \in \mathbb{Z}$ are independent of $O_{\mu_1}$ and $S_1$.

(c) In $A^2_R$,

$$(O_{\mu_1}, S_1)^C_{\lambda_1} (O_{\nu_2}, T_1) *_R (O_{\mu_2}, S_2)^C_{\lambda_2} (O_{\nu_2}, T_2) = \sum_{\mu', S'} r \cdot (O_{\mu_1}, S_1)^C_{\lambda_2} (O_{\nu_2}, T_2) \mod \hat{A}^{\lambda_2}$$

where the coefficients $r \in \mathbb{Z}$ are independent of $O_{\nu_2}$ and $T_2$.

(d) In $A^2_R$,

$$(O_{\mu_1}, S_1)^C_{\lambda_1} (O_{\nu_2}, T_1) *_R (O_{\mu_2}, S_2)^C_{\lambda_2} (O_{\nu_2}, T_2) = \sum_{\mu', S'} r \cdot (O_{\mu_1}, S_1)^C_{\lambda_2} (O_{\nu_2}, T_2) \mod \hat{A}^{\lambda_1}$$

where the coefficients $r \in \mathbb{Z}$ are independent of $O_{\mu_1}$ and $S_1$. 15
Proof. Notice that a basis element \( b = (O_{\nu}, S)C_{\lambda}^{\mu}(O_{\nu}, T) \) is in \( O_{\nu}A_{\nu} \), so in the notation of lemma 4.6, we have \( n_L(b) = n_L(O_{\mu}) \), \( n_R(b) = n_R(O_{\nu}) \).

For part (a), lemma 4.3 gives

\[
(O_{\nu_1}, S_1)C_{\lambda_1}^{\mu_1}(O_{\nu_1}, T_1) \cdot (O_{\nu_2}, S_2)C_{\lambda_2}^{\mu_2}(O_{\nu_2}, T_2) = \sum_{\mu', S'} r' \cdot (O_{\nu'}, S')C_{\lambda_2}^{\mu_2}(O_{\nu_2}, T_2) \mod \hat{A}^{\lambda_2}
\]

where the coefficients \( r' \in \mathbb{Z} \) are independent of \( O_{\nu_2} \) and \( T_2 \). Then lemma 4.5 gives

\[
\sum_{\mu', S'} \frac{n_L(O_{\mu})}{n_L(O_{\mu_1})} \cdot \frac{n_L(O_{\mu_2})}{n_L(O_{\mu_2})} \cdot r' \cdot (O_{\nu_1}, S_1)C_{\lambda_1}^{\mu_1}(O_{\nu_1}, T_1) \mod \hat{A}^{\lambda_1}
\]

where the coefficients \( r' \in \mathbb{Z} \) are independent of \( O_{\mu_1} \) and \( S_1 \). Then lemma 4.5 gives

\[
\sum_{\nu', T'} \frac{n_L(O_{\nu_1})}{n_L(O_{\nu_2})} \cdot \frac{n_L(O_{\nu_2})}{n_L(O_{\nu_2})} \cdot r' \cdot (O_{\nu_1}, S_1)C_{\lambda_1}^{\mu_1}(O_{\nu_1}, T_1) \mod \hat{A}^{\lambda_1}
\]

Then \( r = \frac{n_L(O_{\nu_1})}{n_L(O_{\nu_2})} \cdot \frac{1}{n_L(O_{\nu_2})} \cdot r' \) is independent of \( O_{\mu_1} \) and \( S_1 \) and the result (b) follows.

Parts (c) and (d) are proved similarly.

Proposition 4.2. \( \left\{ (O_{\nu}, S)C_{\lambda}^{\mu}(O_{\nu}, T) \right\} \) is a cell basis for both \( A_{\mathbb{Z}}^{\mathbb{Z}} \) and \( A_{R}^{\mathbb{Z}} \), which are therefore cell algebras.

Proof. We have shown the \( \left\{ (O_{\nu}, S)C_{\lambda}^{\mu}(O_{\nu}, T) \right\} \) form a basis and the multiplication rules (i) and (ii) for a cell algebra then follow at once by linearity from lemma 4.6.

Corollary 4.3. For any domain \( R \), the left and right generalized Schur algebras \( LGS_R(M, G) = R \otimes \mathbb{Z} A_{\mathbb{Z}}^{\mathbb{Z}} \) and \( RGS_R(M, G) = R \otimes \mathbb{Z} A_{R}^{\mathbb{Z}} \) are cell algebras with a cell basis \( \left\{ (O_{\nu}, S)C_{\lambda}^{\mu}(O_{\nu}, T) \right\} \).
5 Irreducible modules for generalized Schur algebras

The cell basis \( \{ (O_{\nu}, S) C_{(O_{\nu}, T)}^\lambda \} \) for the cell algebra \( A_L^\mathbb{Z} \) or \( A_R^\mathbb{Z} \) found above depends on the choice of an ordering of the orbits of \( \mathfrak{S}_\nu \) acting on \( P_r \), and of an ordering of the subsets in \( P_r \) compatible with the ordering of the orbits. We now choose orderings which will simplify the calculations of the brackets in these cell algebras.

For \( d \in P_r \), \( |d| = i \), define an increasing string of \( i \) integers, \( s(d) \), to be the \( i \) elements of \( d \) arranged in ascending order. Then define a non-decreasing string of \( i \) of positive integers, \( s(\nu, d) \), by replacing each \( x \in s(d) \) by \( b(x) \), where \( x \) is in the \( b(x)^{th} \) block \( b_{\nu}(x) \) of the composition \( \mathfrak{S}_\nu \). Finally, define a string of \( r \) non-negative integers, \( \bar{s}(\nu, d) \), by adding \( r - i \) zeroes to the end of \( s(\nu, d) \). Note that \( \bar{s}(\nu, d) \) depends only on the \( \mathfrak{S}_\nu \)-orbit of \( d \). In fact, \( \bar{s}(\nu, d) = \bar{s}(\nu, d') \iff d, d' \) are in the same \( \mathfrak{S}_\nu \)-orbit. We then get a total ordering of the \( \mathfrak{S}_\nu \)-orbits by ordering the corresponding strings \( \bar{s}(\nu, d) \) lexicographically: if \( \bar{s}(\nu, d) \) represents the \( j \)th element of the string, then \( \bar{s}(\nu, d)_{j} < \bar{s}(\nu, d')_{j} \) for some \( J \) between 1 and \( r \) we have \( \bar{s}(\nu, d)_{j} = \bar{s}(\nu, d')_{j} \) for all \( j < J \), while \( \bar{s}(\nu, d)_{J} < \bar{s}(\nu, d')_{J} \).

We then define our order on \( P_r \) by: \( d < d' \) if (1) \( \bar{s}(\nu, d) < \bar{s}(\nu, d') \) or (2) \( \bar{s}(\nu, d) = \bar{s}(\nu, d') \) (so \( d, d' \) are in the same \( \mathfrak{S}_\nu \)-orbit) and \( s(d) < s(d') \) in lexicographical order.

Note the following special cases of our ordering:

The smallest set in \( P_r \) is the empty set \( \emptyset \).

If \( \{a\}, \{b\} \) are one element sets in \( P_r \), then \( \{a\} < \{b\} \iff a < b \).

If \( \{a\}, d \in P_r \) and \( d \) has more than one element with smallest element \( b \), then \( \{a\} < d \) if the \( \nu \)-block containing \( a \) is less than the \( \nu \)-block containing \( b \), while \( \{a\} > d \) if the \( \nu \)-block containing \( a \) is greater than the \( \nu \)-block containing \( b \).

We will assume our cell bases are chosen with respect to these orderings.

In this section we assume that \( R = k \) is a field. We will write \( S_L(M, k) \) for the left generalized Schur algebra \( LGS_k(M, \mathbb{G}) = k \otimes A_L^\mathbb{Z} \) and \( S_R(M, k) \) for the right generalized Schur algebra \( RGS_k(M, \mathbb{G}) = k \otimes A_R^\mathbb{Z} \). For either of these algebras, if \( \lambda \in \Lambda \) then \( \lambda \in \Lambda (i, n) \) for some \( i \) with \( i \leq r \leq n \). Recall that in these cell algebras \( \Lambda_0 \) is the subset of \( \Lambda \) consisting of \( \lambda \) for which the bracket \( \langle O_{\nu}, S^\lambda, C_{(O_{\nu}, T)}^\lambda \rangle \) is not identically zero. By corollary 2.1, there is one isomorphism class of irreducible modules for each \( \lambda \in \Lambda_0 \). We will determine \( \Lambda_0 \) when \( M = T_r \) or when \( M \) contains the rook monoid \( \mathfrak{R}_r \).

**Theorem 5.1.** Let \( k \) be a field of characteristic 0 and let \( M = T_r \). Then \( \Lambda_0 = \Lambda \) for both \( S_L(\tau_r, k) \) and \( S_R(\tau_r, k) \). Both \( S_L(\tau_r, k) \) and \( S_R(\tau_r, k) \) are quasi-hereditary algebras.

**Proof.** Take any \( \lambda \in \Lambda \) with index(\( \lambda \)) = \( i > 0 \). Let \( k \) be the largest index such that \( \lambda_k > 0 \), so \( \lambda_j = 0 \), \( j > k \). Let \( \mu \) be the partition of \( r \) where

\[
\mu_j = \begin{cases} 
\lambda_j & \text{if } j \leq k \\
1 & \text{if } j = k + 1, k + 2, \ldots, k + (r - i) 
\end{cases}
\]
Let $C = \{1, 2, \cdots, i\}$. Then $\mu(C) = \lambda \in \Lambda(i, n)$. Let $S$ be the semistandard $\lambda$-tableau of type $\mu(C)$ where $S_{j,l} = j$, for $1 \leq l \leq \lambda_j$, $j = 1, 2, \cdots, k$. There is only one standard $\lambda$-tableau of type $S$, namely $s = id$, the identity in $\mathcal{S}_i$, 

The orbit $O(C, \mu) = \{C\}$, so $\#O(C, \mu) = 1$. Also, $\phi_C : l \rightarrow \bar{l}$ is the identity $\phi_C(j) = j$, $j = 1, 2, \cdots, i$.

Next let $D = \{\{1\}, \{2\}, \cdots, \{i - 1\}, \{i, i + 1, i + 2, \cdots, r\}\}$. Then $\mu(D) \in \Lambda(i, n)$ is given by

$$
\mu(D)_j = \begin{cases}
\lambda_j & \text{if } j < k \\
\lambda_k - 1 & \text{if } j = k \\
1 & \text{if } j = k + 1 \\
0 & \text{if } j > k + 1.
\end{cases}
$$

Let $T$ be the semistandard $\lambda$-tableau of type $\mu(D)$ where

$$
T_{j,l} = \begin{cases}
j & \text{for } 1 \leq l \leq \lambda_j, j = 1, 2, \cdots, k - 1 \\
k & \text{for } 1 \leq l \leq \lambda_k - 1, j = k \\
k + 1 & \text{for } l = \lambda_k, j = k.
\end{cases}
$$

There is only one standard $\lambda$-tableau of type $T$, namely $t = id$, the identity in $\mathcal{S}_i$. Let $b^k_\mu$ be the $k$th block in the partition $\mu$. For $a \in b^k_\mu$, define $D_a = \{\{1\}, \{2\}, \cdots, \{i - 1\}, \{i\}, \{a, i + 1, i + 2, \cdots, r\}\} - \{\{a\}\}$. Then the orbit $O(D, \mu) = \{D_a : a \in b^k_\mu\}$ and $\#O(D, \mu) = |b^k_\mu| = \mu_k = \lambda_k$. Also $\psi_{D_a} : \bar{l} \rightarrow \bar{i}$ is given by

$$
\psi_{D_a}(j) = \begin{cases}
j & \text{if } j < a \\
j - 1 & \text{if } a < j \leq i \\
i & \text{if } j = a \text{ or } j > i.
\end{cases}
$$

Then $\psi_{D_a} \circ \phi_C : l \rightarrow \bar{l}$ is a cyclic permutation $\sigma_a = (a, i, i - 1, i - 2, \cdots, a + 1) \in \mathcal{S}_\lambda$.

We have $SC^\lambda_T = SC^\lambda_T = id \cdot r_\lambda \cdot id = r_\lambda$, so

$$
SC^\lambda_T \circ \psi_{D_a} \circ \phi_C \circ SC^\lambda_T = r_\lambda \cdot \sigma_a \cdot r_\lambda = r_\lambda \cdot r_\lambda = o(\mathcal{S}_\lambda) r_\lambda.
$$

Then writing $b =_{O(C, \mu), S} C^\lambda_{O(D, \mu), T} = \phi_C \cdot SC^\lambda_T \cdot \sum_{a \in b^k_\mu} \psi_{D_a}$, compute

$$
b \cdot b = \phi_C \cdot SC^\lambda_T \cdot \sum_{a \in b^k_\mu} \psi_{D_a} \cdot \phi_C \cdot SC^\lambda_T \cdot \sum_{a \in b^k_\mu} \psi_{D_a}
$$

$$
= \lambda_k \cdot \phi_C \cdot o(\mathcal{S}_\lambda) \cdot r_\lambda \cdot \sum_{a \in b^k_\mu} \psi_{D_a}
$$

$$
= \lambda_k \cdot o(\mathcal{S}_\lambda) \cdot b.
$$

Then

$$
b *_L b = \lambda_k \cdot o(\mathcal{S}_\lambda) \cdot \frac{n_L(b)}{n_L(b) n_L(b)} \cdot b = \lambda_k \cdot o(\mathcal{S}_\lambda) \cdot \frac{1}{n_L(b)} \cdot b = \lambda_k \cdot b.
$$
where  
\[ n_L(b) = n_L(O(C, \mu)) = \#O(C, \mu) \cdot o(\mathfrak{g}_{\mu(C)}) = 1 \cdot o(\mathfrak{g}_{\lambda}). \]

Similarly,  
\[ b \ast_R b = \lambda_k \cdot o(\mathfrak{g}_{\lambda}) \cdot \frac{n_R(b)}{n_R(b)} \cdot b = \lambda_k \cdot o(\mathfrak{g}_{\lambda}) \cdot \frac{1}{n_R(b)} \cdot b = \lambda_k \cdot b \]

where  
\[ n_R(b) = n_R(O(D, \mu)) = \#O(D, \mu) \cdot o(\mathfrak{g}_{\mu(D)}) = \lambda_k \cdot o(\mathfrak{g}_{\lambda})/\lambda_k. \]

Then computing the bracket (in either \( S_L(\tau, k) \) or \( S_R(\tau, k) \)) we find that  
\[ \left\langle C^\lambda_{O(D, \mu)} \cdot T \cdot o(C, \mu), S^\lambda \right\rangle = \lambda_k \neq 0 \]  
(since characteristic of \( k \) is 0). So \( \lambda \in \Lambda_0 \).

By corollary \( 2.1 \) a cell algebra with \( \Lambda_0 = \Lambda \) is quasi-hereditary, so the proof is complete.

Now assume \( k \) is a field of characteristic \( p > 0 \). For a partition \( \lambda \in \Lambda(i), \; 1 \leq i \leq r \), define an integer \( k(p, \lambda) \geq 0 \) to be the highest power of \( p \) which divides \( \lambda_j \) for every \( j \). (So for all \( j, \; p^{k(p, \lambda)}(p \lambda_j) \) while for at least one \( j, \; p^{k(p, \lambda)+1} \) does not divide \( \lambda_j \).) Then define \( \Lambda_p = \{ \lambda \in \Lambda : p^{k(p, \lambda)} \) divides \( r - i \) for \( i = \text{index}(\lambda) \} \).

**Lemma 5.1.** For a field \( k \) of characteristic \( p \) and the cell algebra \( S_R(\mathcal{T}_r, k) \), \( \Lambda_p \subseteq \Lambda_0 \).

**Proof.** Take any \( \lambda \in \Lambda_p \) of index \( i \). Let \( m \) be the lowest nonzero row of \( \lambda \), that is, assume \( \lambda_m > 0 \), \( \lambda_j = 0 \) for \( j > m \). Put \( k = k(p, \lambda) \). Let \( a \) be the lowest row (i.e., largest integer) such that \( p^{k+1} \) does not divide \( \lambda_a \). Since \( \lambda \in \Lambda_p \), \( p^k \) divides \( r - i \), so \( q = (r - i)/p^k \) is an integer. Define a composition \( \mu \) of \( r \) by splitting off the last \( p^k \) elements of row \( a \) of \( \lambda \) and then adding \( q \) additional rows of size \( p^k \):

\[
\mu_j = \begin{cases} 
\lambda_j & \text{if } j < a \\
\lambda_a - p^k & \text{if } j = a \\
p^k & \text{if } j = a + 1 \\
\lambda_{a+l} & \text{if } j = a + l + 1 \text{ for } l = 1, 2, \cdots, m - a \\
p^k & \text{if } j = m + 1 + l \text{ for } l = 1, 2, \cdots, q
\end{cases}
\]

Let \( C = \{1, 2, \cdots, i\} \). Then a composition of \( i \) is given by \( \mu(C)_j = \mu_j, \; 1 \leq j \leq m + 1 \). Let \( S \) be the semistandard \( \lambda \)-tableau of type \( \mu(C) \) where

\[
S_{j,l} = \begin{cases} 
j & \text{for } 1 \leq l \leq \lambda_j, \; j < a \\
a & \text{for } 1 \leq l \leq \lambda_a - p^k, \; j = a \\
a + 1 & \text{for } \lambda_a - p^k < l \leq \lambda_a, \; j = a \\
j + 1 & \text{for } 1 \leq l \leq \lambda_j, \; a < j \leq m .
\end{cases}
\]

There is only one standard \( \lambda \)-tableau of type \( S \), namely \( s = \text{id} \), the identity in \( \mathfrak{S}_i \). The orbit \( O(C, \mu) = \{C\} \), so \( \#O(C, \mu) = 1 \). Also, \( \phi_C : i \rightarrow \bar{r} \) is the identity \( \phi_C(j) = j, \; j = 1, 2, \cdots, i \).
Now define $D \in D(i, \tau, r)$ as follows: $D$ contains $i - p^k$ single elements sets, one set $\{l\}$ for each entry $l$ in rows $1$ through $a$ or rows $a + 2$ through $m + 1$ of $\mu$. $D$ also contains $p^k$ sets with $q + 1$ entries: for $1 \leq j \leq p^k$, the $j$ set $D_j$ contains the $j$th entry in row $a + 1$ and in each of the last $q$ rows of $\mu$. As a composition of $i$, $\mu(D) = \mu(C)$, so we can take $T = S$ as a semistandard $\lambda$-tableau of type $\mu(D)$. Then again there is only one standard $\lambda$-tableau of type $T$, namely $t = \text{id}$, the identity in $S$. To define the orbit space $O(D, \mu)$, let $\mathfrak{S}_{\mu_{m+1+j}} \subseteq \mathfrak{S}_\mu$ be the group of permutations of row $m + 1 + j$ of $\mu$. For each $1 \leq j \leq q$, $\mathfrak{S}_{\mu_{m+1+j}} \cong \mathfrak{S}_{p^k}$. Let $G = \prod_{j=1}^{\mu} \mathfrak{S}_{\mu_{m+1+j}} \subseteq \mathfrak{S}_\mu$ and for $\sigma \in G$ let $D_\sigma = D \sigma \in D(i, \tau, r)$. Then $O(D, \mu) = \{D_\sigma : \sigma \in G\}$. Then $\#O(D, \mu) = o(G) = o\left(\mathfrak{S}_{p^k}\right)^{q} = (p^k)^{q}.$

Notice that with our choice of an ordering of the subsets of $r$, we get $\psi_{D_\sigma}(j) = j, 1 \leq j \leq i$, for any $\sigma \in G$. Then $\psi_{D_\sigma} \circ \phi_C = \text{id}$, the identity mapping $i \rightarrow i$. We have $S C^\lambda_D \cdot S C^\lambda_D = \text{id} \cdot r_\lambda \cdot \text{id} = r_\lambda$, so

$$SC^\lambda_D \circ \psi_{D_\sigma} \circ \phi_C \circ SC^\lambda_D = r_\lambda \cdot \text{id} \cdot r_\lambda = r_\lambda \cdot r_\lambda = o(\mathfrak{S}_\lambda) \cdot r_\lambda.$$

Then writing $b = O(C, \mu), S \cdot C^\lambda_{O(D, \mu), T} = \phi_C \circ S C^\lambda_D \cdot \sum_{\sigma \in G} \psi_{D_\sigma}$, compute

$$b \cdot b = \phi_C \cdot S C^\lambda_D \cdot \sum_{\sigma \in G} \psi_{D_\sigma} \cdot \phi_C \cdot S C^\lambda_D \cdot \sum_{\sigma \in G} \psi_{D_\sigma}
= o(G) \cdot \phi_C \cdot o(\mathfrak{S}_\lambda) \cdot r_\lambda \cdot \sum_{\sigma \in G} \psi_{D_\sigma}
= o(G) \cdot o(\mathfrak{S}_\lambda) \cdot b.$$

Then

$$b * R b = o(G) \cdot o(\mathfrak{S}_\lambda) \cdot \frac{n_R(b)}{n_R(b) n_R(b)} \cdot b
= o(G) \cdot o(\mathfrak{S}_\lambda) \cdot \frac{1}{n_R(b)} \cdot b
= \frac{o(G) \cdot o(\mathfrak{S}_\lambda)}{\#O(D, \mu) \cdot o(\mathfrak{S}_{\mu(D)})} \cdot b$$

where $n_R(b) = n_R(O(D, \mu)) = \#O(D, \mu) \cdot o(\mathfrak{S}_{\mu(D)})$. Since $\#O(D, \mu) = o(G)$ and $\frac{o(\mathfrak{S}_{\mu(D)})}{o(\mathfrak{S}_{\mu(D)})} = \frac{\lambda_a}{(\lambda_a - p^k)p^k} = \left(\begin{array}{c}
\lambda_a \\
p^k
\end{array}\right)$, we get $b * R b = \left(\begin{array}{c}
\lambda_a \\
p^k
\end{array}\right) \cdot b$.

Computing the bracket gives $\left(S^\lambda_{O(D, \mu), T} \cdot O(C, \mu), S C^\lambda_D\right) = \left(\begin{array}{c}
\lambda_a \\
p^k
\end{array}\right)$. Since $p^k$ divides $\lambda_a$ but $p^{k+1}$ does not, it is easily checked that $\left(\begin{array}{c}
\lambda_a \\
p^k
\end{array}\right)$ is not congruent to $0$ mod $p$. So the bracket is not identically zero in $S_R(\tau_\tau, k)$ and $\lambda \in \Lambda_0$ as desired.

We claim that in fact $\Lambda_p = \Lambda_0$, that is, that every irreducible representation of $S_R(\tau_\tau, k)$ corresponds to some $\lambda \in \Lambda_p$. In [2] or [4], a parameterization
of the isomorphism classes of irreducible representations of $S_R(T_r,k)$ is given. There is one isomorphism class corresponding to the following set of data: i) a set of nonnegative integers $s_m, s_{m+1}, \ldots, s_M$ with $s_m > 0, m \geq 0$ such that $r = s_mp^m + s_{m+1}p^{m+1} + \cdots + s_Mp^M$, ii) a $p$-restricted partition of $s_i$ for each $m < i \leq M$, and iii) a $p$-restricted partition of $i$ for some integer $1 \leq i \leq s_m$. (The index of the corresponding irreducible is $r - (s_m - i)p^m$.) We will show that each such set of data corresponds with a unique element $\lambda \in \Lambda_p$, so that the number of isomorphism classes is less than or equal to $\#\Lambda_p$. But we know that the number of isomorphism classes is $\#\Lambda_0$ and by the lemma $\Lambda_p \subseteq \Lambda_0$. So we must have $\Lambda_p = \Lambda_0$.

We will need a certain “decomposition” operation on partitions. For any integer $n > 0$, let $\lambda$ be a partition of $n$ with $R$ non-zero parts, $\lambda_1 + \lambda_2 + \cdots + \lambda_R = n$. For $1 \leq i \leq R$ define the row length differences $\Delta_i = \lambda_i - \lambda_{i+1}$. Then define an integer $k(\lambda) \geq 0$ to be the highest power of $p$ which is less than or equal to at least one $\Delta_i$. Then we can find nonnegative integers $q_i, r_i$ such that $\Delta_i = q_ip^{k(\lambda)} + r_i$ where each $r_i < p^{k(\lambda)}$, each $q_i < p$, and at least one $q_i > 0$. Define $s(\lambda) = \sum_{i=1}^{R} i \cdot q_i$. We will construct a $p$-restricted partition of $s(\lambda)$ and a partition $\lambda$ of $n - s(\lambda)p^{k(\lambda)}$ with $k(\lambda) < k(\lambda')$. Notice that there are $\Delta_i = q_ip^{k(\lambda)} + r_i$ columns of height $i$ in $\lambda$. We break $\lambda$ into two partitions $\lambda_q, \lambda_r$ by placing $q_ip^{k(\lambda)}$ columns of height $i$ in the first partition and $r_i$ columns of height $i$ in the second. Then $\lambda_q$ is a partition of $i \cdot q_ip^{k(\lambda)} = s(\lambda)p^{k(\lambda)}$ with row differences $q_ip^{k(\lambda)}$. By replacing each set of $p^{k(\lambda)}$ consecutive boxes in a row of $\lambda_q$ by a single box, we obtain a partition of $s(\lambda)$ with row differences $q_i < p$, i.e., a $p$-restricted partition of $s(\lambda)$. The second partition $\lambda_r$ is a partition of $n - s(\lambda)p^{k(\lambda)}$ with row differences $r_i < p^{k(\lambda)}$. Then $k(\lambda_r) < k(\lambda)$ and we take $\lambda = \lambda_r$.

We can now replace $\lambda$ with $\lambda_r$ and iterate the construction until we reach a case when all $r_i$ are 0. The result is a sequence of nonnegative integers $s_m, s_{m+1}, \ldots, s_M$ with $s_m > 0, m \geq 0$ such that $n = s_mp^m + s_{m+1}p^{m+1} + \cdots + s_Mp^M$ and a $p$-restricted partition of $s_i$ for each $m \leq i \leq M$. By replacing each box in the partition of $s_i$ by a row of $p^i$ boxes and then joining the resulting partitions (taking the union of the boxes in each row of each partition) we recover uniquely the original partition $\lambda$ of $n$. Notice that $k(p, \lambda) = m$.

Now take any isomorphism class of irreducible $S_R(T_r,k)$ modules and consider the unique corresponding data i) nonnegative integers $s_m, s_{m+1}, \ldots, s_M$ with $s_m > 0$ such that $r = s_mp^m + s_{m+1}p^{m+1} + \cdots + s_Mp^M$, ii) a $p$-restricted partition of $s_i$ for each $m < i \leq M$, and iii) a $p$-restricted partition of $s'_m$ for some integer $1 \leq s'_m \leq s_m$.

Then $s'_mp^m + s_{m+1}p^{m+1} + \cdots + s_Mp^M = r - (s_m - s'_m)p^m$, so our construction gives a unique partition $\lambda$ of $r - (s_m - s'_m)p^m$ with $k(p, \lambda) = m$. Then $p^{k(p, \lambda)} = p^m$ divides $r - \text{index}(\lambda) = r - (r - (s_m - s'_m)p^m) = (s_m - s'_m)p^m$, so $\lambda \in \Lambda_p$.

So the number of isomorphism classes is $\leq \#\Lambda_p$ as desired.

As remarked above, this proves the following result.

**Theorem 5.2.** If $k$ is a field of characteristic $p$, then $\Lambda_p = \Lambda_0$ in $S_R(T_r,k)$. 21
Corollary 5.1. If $k$ is a field of characteristic $p$ and $r = ap^j$ for $1 \leq a < p$ and some $l = 0, 1, 2, \cdots$, then $S_R(T_r, k)$ is quasi-hereditary.

Proof. By corollary 2.1 we must show that $\Lambda = \Lambda_0$, that is, that any $\lambda \in \Lambda$ is actually in $\Lambda_p = \Lambda_0$. So suppose $\lambda$ is a partition of $i$ for some $0 < i \leq r$. Put $k = k(p, \lambda)$. Since $p^k$ divides $\lambda_j$ for every $j$, $p^k$ divides $i = \sum \lambda_j$, say $i = bp^k$ for some $b > 0$. Now $bp^k = i \leq r = ap^j < p^{j+1}$ (since $a < p$), so $k \leq l$. Then $r - i = ap^j - bp^k = (ap^{j-k} - b)p^k$, so $p^k$ divides $r - i$ and $\lambda \in \Lambda_p$ as desired. □

Now consider $S_R(T_r, k)$ for characteristic $p$. Define

$$\Lambda_{L,p} = \{ \lambda \in \Lambda : p \text{ does not divide } \lambda_j \text{ for at least one } j \} \cup \Lambda(r).$$

Lemma 5.2. For a field $k$ of characteristic $p$ and the cell algebra $S_L(T_r, k)$, $\Lambda_{L,p} \subseteq \Lambda_0$.

Proof. First suppose $\lambda \in \Lambda(r)$, a partition of maximal index $r$. Then take $\mu = \lambda$ as a composition of $r$ and let $C = \{1, \cdots, r\}$. Then $\mu(C) = \mu = \lambda$ and a semistandard $\lambda$-tableau $S$ of type $\mu(C)$ is given by $S_{j,l} = j$, $1 \leq l \leq j$. There is only one standard $\lambda$-tableau of type $S$, namely $s = id$, the identity in $S$. The orbit $O(C, \mu) = \{C\}$, so $\#O(C, \mu) = 1$. Also, $\phi_C : \tilde{r} \rightarrow \tilde{r}$ is the identity $\phi_C(j) = j, j = 1, 2, \cdots, r$.

Define $D \in D(r, r, r)$ by $D = \{\{1\}, \{2\}, \cdots, \{r\}\}$. As a composition of $r$, $\mu(D) = \mu(C) = \lambda$, so we can take $T = S$ as a semistandard $\lambda$-tableau of type $\mu(D)$. Then again there is only one standard $\lambda$-tableau of type $T$, namely $t = id$, the identity in $S$.

We have $O(D, \mu) = \{D\}$, $\#O(D, \mu) = 1$, $\psi_D(j) = j, 1 \leq j \leq r$. Then $\psi_D \circ \phi_C = id : \tilde{r} \rightarrow \tilde{r}$. We have $sC^\lambda_T = sC^\lambda_T = id \cdot r_\lambda \cdot id = r_\lambda$, so $sC^\lambda_T \circ \psi_D \circ \phi_C = sC^\lambda_T = r_\lambda \cdot id = r_\lambda \cdot r_\lambda = o(\tilde{S}) \cdot r_\lambda$. Then writing $b = o(C, \mu) \cdot sC^\lambda_T \cdot \psi_D$, compute $b \cdot b = \phi_C \cdot sC^\lambda_T \cdot \psi_D \cdot \phi_C \cdot sC^\lambda_T \cdot \psi_D = \phi_C \cdot o(\tilde{S}) \cdot r_\lambda \cdot r_\lambda = o(\tilde{S}) \cdot b$. Then $b \ast_L b = o(\tilde{S}) \cdot \frac{n_L(b)}{n_L(b)_{\mu(L)}} \cdot b = o(\tilde{S}) \cdot \frac{1}{n_L(b)} \cdot b = \#O(C, \mu) \cdot o(\tilde{S}) \cdot b$ where $n_L(b) = n_L(O(C, \mu)) = \#O(C, \mu) \cdot o(\tilde{S}) \cdot b$. Since $\#O(C, \mu) = 1$ and $o(\tilde{S}) = o(\tilde{S})$, we get $b \ast_L b = b$. Computing the bracket gives $\langle C^\lambda_{O(D, \mu)} \cdot O(C, \mu) \cdot sC^\lambda_T \rangle = 1 \neq 0$. So the bracket is not identically zero in $S_L(S, k)$ and $\lambda \in \Lambda_0$ as desired.

Now take any $\lambda \in \Lambda_{L,p}$ of index $i < r$. Let $m$ be the lowest nonzero row of $\lambda$, that is, assume $\lambda_m > 0, \lambda_j = 0$ for $j > m$. Let $a$ be the largest integer such that $p$ does not divide $\lambda_a$. Define a composition $\mu$ of $r$ by splitting off the last element of row $a$ of $\lambda$ and also adding $r - i$ additional rows of length 1:

$$\mu_j = \begin{cases} 
\lambda_j & \text{if } j < a \\
\lambda_a - 1 & \text{if } j = a \\
1 & \text{if } j = a + 1 \\
\lambda_{j-1} & \text{if } a + 2 \leq j \leq m + 1 \\
1 & \text{if } m + 2 \leq j \leq (m + 1) + (r - i) .
\end{cases}$$
Let $C = \{1, 2, \ldots, i\}$. Then a composition of $i$ is given by $\mu(C)_j = \mu_j$, $1 \leq j \leq m + 1$. Let $S$ be the semistandard $\lambda$-tableau of type $\mu(C)$ where

$$S_{j,l} = \begin{cases} 
  j & \text{for } 1 \leq l \leq \lambda_j, \ j < a \\
  a & \text{for } 1 \leq l \leq \lambda_a - 1, \ j = a \\
  a + 1 & \text{for } l = \lambda_a, \ j = a \\
  j + 1 & \text{for } 1 \leq l \leq \lambda_j, \ a < j \leq m .
\end{cases}$$

There is only one standard $\lambda$-tableau of type $S$, namely $s = \text{id}$, the identity in $S_t$. The orbit $O(C, \mu) = \{C\}$, so $\# O(C, \mu) = 1$. Also, $\phi_C : i \rightarrow \tilde{i}$ is the identity in $S_t$.

Now define $D \in D(i, \tau, r)$ as follows: Let $x$ be the last element in row $\lambda_a$, that is, $x = \lambda_1 + \lambda_2 + \cdots + \lambda_a$. $D$ contains $i - 1$ single elements sets, one set $\{l\}$ for every $1 \leq l \leq i$ except $l = x$. $D$ also contains one set with $r - i + 1$ elements: $\{x, i + 1, i + 2, \ldots, r\}$. As a composition of $i$, $\mu(D) = \mu(C)$, so we can take $T = S$ as a semistandard $\lambda$-tableau of type $\mu(D)$. Then again there is only one standard $\lambda$-tableau of type $T$, namely $t = id$, the identity in $S_t$.

We have $O(D, \mu) = \{D\}$, $\# O(D, \mu) = 1$, $\psi_D(j) = \begin{cases} 
  j & \text{if } 1 \leq j \leq i \\
  x & \text{if } j > i .
\end{cases}$

Then $\psi_D \circ \phi_C = \text{id} : i \rightarrow \tilde{i}$. We have $sC_\lambda^\lambda = sC_\lambda^\lambda = \text{id} \cdot r_\lambda \cdot \text{id} = r_\lambda$, $sC_\lambda^\lambda \circ \psi_D \circ \phi_C \circ sC_\lambda^\lambda = r_\lambda \cdot \text{id} \cdot r_\lambda = r_\lambda \cdot r_\lambda = o(\bar{S}_\lambda) r_\lambda$. Then writing $b = O(C, \mu), sC_\lambda^\lambda = o(\bar{S}_\lambda) r_\lambda \cdot \psi_D$, compute

$$b \cdot b = \phi_C \cdot sC_\lambda^\lambda \cdot \psi_D \cdot \phi_C \cdot sC_\lambda^\lambda \cdot \psi_D$$

$$= \phi_C \cdot o(\bar{S}_\lambda) r_\lambda \cdot \psi_D$$

$$= o(\bar{S}_\lambda) \cdot b .$$

Then

$$b \ast_L b = o(\bar{S}_\lambda) \cdot \frac{n_L(b)}{n_L(b) n_L(b)} \cdot b = o(\bar{S}_\lambda) \cdot \frac{1}{n_L(b)} .$$

$$= \frac{o(\bar{S}_\lambda)}{\# O(C, \mu) o(\bar{S}_\mu(C))} .$$

where $n_L(b) = n_L(O(C, \mu)) = \# O(C, \mu) \cdot o(\bar{S}_\mu(C))$. Since $\# O(C, \mu) = 1$ and $\frac{o(\bar{S}_\lambda)}{o(\bar{S}_\mu(C))} = \frac{\lambda_a}{\mu_a}$, we get $b \ast_L b = \lambda_a \cdot b$.

Computing the bracket gives $\langle C_\lambda^\lambda o(C, \mu), O(C, \mu), sC_\lambda^\lambda \rangle = \lambda_a$. Since $p$ does not divide $\lambda_a$ (by definition), $\lambda_a \neq 0$ in $k$. So the bracket is not identically zero in $S_L(T_r, k)$ and $\lambda \in \Lambda_0$ as desired.

We claim that $\Lambda_{L,p} = \Lambda_0$, that is, that every irreducible representation of $S_L(T_r, k)$ corresponds to some $\lambda \in \Lambda_{L,p}$. [4] gives the following parameterization of the isomorphism classes of irreducible representations of $S_L(T_r, k)$. There is one isomorphism class corresponding to the following set of data: i) a set of nonnegative integers $s_m, s_{m+1}, \ldots, s_M$ with $m \geq 0$, $s_m > 0$ such that $r = s_m p^m + s_{m+1} p^{m+1} + \cdots + s_M p^M$, ii) a $p$-restricted partition of $s_i$ for each

23
Consider first a set of data for the case $k > p$. Then we have $r = s_m p + s_{m+1} p^{m+1} + \cdots + s_M p^M$ and a $p$-restricted partition of $s_m$ for all $m$. So by the construction preceding theorem 5.2 there is a unique partition $\lambda$ of index $r$ corresponding to the data, and since the index of $\lambda$ is $r$ we have $\lambda \in \Lambda_{L,p}$. Next consider a set of data for the case $m = 0$. Putting $s'_0 = i$ we have $s_0' + s_1 p^1 + \cdots + s_M p^M = r - (s_0 - s'_0)$ with $p$-restricted partitions of $s'_0 = i$ and all $s_i, i > 0$. The result is a unique partition $\lambda$ of $r - (s_0 - i)$ with $k(p, \lambda) = m = 0$. But $k(p, \lambda) = 0$ means that at least one row length $\lambda_j$ of $\lambda$ is not divisible by $p^{(p-\lambda)+1} = p$, that is, that $\lambda \in \Lambda_{L,p}$. So corresponding to each set of data there is a unique element $\lambda \in \Lambda_{L,p}$ as desired. As remarked above, this proves the following theorem.

**Theorem 5.3.** If $k$ is a field of characteristic $p$, then $\Lambda_{L,p} = \Lambda_0$ in $S_L(T_r, k)$.

Notice that if $p \geq r$ then $\Lambda_0 = \Lambda_{L,p} = \Lambda$ and $S_L(T_r, k)$ is quasi-hereditary. However, when $r > p$ we have $\Lambda_0 = \Lambda_{L,p} \neq \Lambda$ and $S_L(T_r, k)$ is not quasi-hereditary for the given poset structure $\Lambda$.

Now consider the case when $M$ contains the rook monoid $\mathcal{R}_r$.

**Theorem 5.4.** Assume $M$ contains the rook monoid $\mathcal{R}_r$. Then for any field $k$, $\Lambda_0 = \Lambda$ for both cell algebras $S_L(M, k)$ and $S_R(M, k)$. Both $S_L(M, k)$ and $S_R(M, k)$ are quasi-hereditary.

Remark: If $M$ is just the rook monoid, $M = \mathcal{R}_r$, then $S_L(M, k)$ and $S_R(M, k)$ are both actually cellular algebras and are anti-isomorphic as algebras.

**Proof.** Take any partition $\lambda$ of $i$, $0 \leq i \leq r$. We must show that $\lambda \in \Lambda_0$. Let $m \geq 0$ be the smallest integer such that $\lambda_j = 0$ for $j > m$. Define a composition $\mu$ of $r$ by adding $r - i$ rows of length 1 to $\lambda$. So

$$
\mu_j = \begin{cases} 
\lambda_j & \text{if } j \leq m \\
1 & \text{if } m + 1 \leq j \leq m + r - i \\
0 & \text{if } j > m + r - i.
\end{cases}
$$

Let $C = \{1, 2, \ldots, i\}$. Then $\mu(C) = \lambda$. Let $S$ be the semistandard $\lambda$-tableau of type $\mu(C)$ where $S_{j,l} = j, 1 \leq l \leq \lambda_j, 1 \leq j \leq m$. There is only one standard $\lambda$-tableau of type $S$, namely $s = id$, the identity in $S_i$. The orbit $O(C, \mu) = \{C\}$, so $\# O(C, \mu) = 1$. Also, $\phi_C : i \mapsto r$ is the identity $\phi_C(j) = j, j = 1, 2, \ldots, i$.

Define $D \in D(i, r)$ by $D = \{\{j\} : 1 \leq j \leq i\}$. $M$ contains the rook monoid $\mathcal{R}_r$, so it contains the map $\alpha : r \cup 0 \to r \cup 0$ given by $\alpha(j) = \begin{cases} j & \text{if } 1 \leq j \leq i \\
0 & \text{if } j > i
\end{cases}$. 

24
Then \( D = D_\alpha \in D(i, M, r) \). As a composition of \( i, \mu(D) = \mu(C) = \lambda \), so we can take \( T = S \) as a standard \( \lambda \)-tableau of type \( \mu(D) \). Then again there is only one standard \( \lambda \)-tableau of type \( T \), namely \( t = id \), the identity in \( S_\lambda \).

We have \( O(D, \mu) = \{ \{ D \} \} \), \( \#O(D, \mu) = 1 \), \( \psi_D(j) = \begin{cases} j & \text{if } 1 \leq j \leq i \\ 0 & \text{if } j > i \end{cases} \). Then \( \psi_D \circ \phi_C = id : \tilde{i} \to \tilde{i} \).

We have \( sC_T^\lambda = sC_t^\lambda = id \cdot r_\lambda \cdot id = r_\lambda \), so \( sC_T^\lambda \circ \psi_D \circ \phi_C \circ sC_T^\lambda = r_\lambda \cdot id \cdot r_\lambda = r_\lambda \cdot r_\lambda = o(S_\lambda) \cdot r_\lambda \). Then writing \( b = \phi_{C(\mu)} \cdot sC_T^\lambda \circ \psi_D \circ \phi_C \cdot sC_T^\lambda = \phi_C \cdot sC_T^\lambda \cdot \psi_D = \phi_C \cdot o(S_\lambda) \cdot r_\lambda \cdot \psi_D = o(S_\lambda) \cdot b \).

Then \( b \ast_L b = o(S_\lambda) \cdot \frac{1}{\#O(C, \mu) \cdot \#O(D, \mu)} \cdot b \) where \( n_L(b) = n_L(O(C, \mu)) = \#O(C, \mu) \cdot o(S_{\mu(C)}) \). Since \( \#O(C, \mu) = 1 \) and \( S_\lambda = S_{\mu(C)} \), we get \( b \ast_L b = b \).

Computing the bracket in \( S_L(M, k) \) gives \( \left\langle C_T^\lambda(O(D, \mu), T) : O(C, \mu), sC_T^\lambda \right\rangle \neq 0 \). So the bracket is not identically zero in \( S_L(M, k) \) and \( \lambda \in \Lambda_0 \) as desired.

Similarly, \( b \ast_R b = o(S_\lambda) \cdot \frac{1}{\#O(D, \mu) \cdot \#O(C, \mu)} \cdot b \) where \( n_R(b) = n_R(O(D, \mu)) = \#O(D, \mu) \cdot o(S_{\mu(D)}) \). Since \( \#O(D, \mu) = 1 \) and \( S_\lambda = S_{\mu(D)} \), we get \( b \ast_R b = b \).

Computing the bracket in \( S_R(M, k) \) again gives \( \left\langle C_T^\lambda(O(D, \mu), T) : O(C, \mu), sC_T^\lambda \right\rangle = 1 \neq 0 \). So the bracket is not identically zero in \( S_R(M, k) \) and \( \lambda \in \Lambda_0 \) as desired.

By corollary 2.1 cell algebras with \( \Lambda_0 = \Lambda \) are quasi-hereditary. \( \square \)

References

[1] Graham, J.J. and Lehrer, G.I., Cellular algebras, Invent. Math. 123(1996) 1-34.

[2] May, Robert, Representations of certain generalized Schur algebras, J. Algebra 333(2011)180-201.

[3] May, Robert, Double coset algebras, J. Pure Appl. Algebra 218(2014)2081-2095.

[4] May, Robert, Cell algebras, J. Algebra 425(2015)107-132

[5] May, Robert, Generalized Schur algebras, [arXiv:1601.01711v2].

[6] May, Robert and Abrams, William, A generalization of the Schur algebra to \( k[\tau_r] \). J. Algebra 295(2006) 524 - 542.

[7] Mathas, Andrew, Iwahori-Hecke Algebras and Schur Algebras of the Symmetric Group, American Mathematical Society, Providence, 1999.

[8] Du, J. and Rui, H., Based algebras and standard bases for quasi-hereditary algebras, Trans. Amer. Math. Soc. 350(8) (1998) 3207-3235.

25