EXTENSIONS OF YANG–BAXTER SETS

V. G. Bardakov*†‡ and D. V. Talalaev§¶

This paper is a first step in constructing the category of braided sets and its closest relative, the category of Yang–Baxter sets. Our main emphasis is on the construction of morphisms and extensions of Yang–Baxter sets. This problem is important for the possible constructions of new solutions of the Yang–Baxter equation and the braid equation. Our main result is the description of a family of solutions of the Yang–Baxter equation on $B \otimes C$ and on $B \times C$, given two linear (set-theoretic) solutions $(B, R_B)$ and $(C, R_C)$ of the Yang–Baxter equation.

Keywords: Yang–Baxter equation, set-theoretic solution, quandle, Hopf algebra, extension of Yang–Baxter sets, product of Yang–Baxter sets, Drinfeld twist

DOI: 10.1134/S0040577923050021

1. Introduction

A solution of the quantum Yang–Baxter equation (YBE) is a linear map $R: V \otimes V \to V \otimes V$ satisfying

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $V$ is a vector space over a field $K$ and $R_{ij}: V \otimes V \otimes V \to V \otimes V \otimes V$ acts as $R$ on the $(i, j)$ tensor factor and as the identity on the remaining factor. Buchstaber [1] called the map $R$ that satisfies the YBE by a Yang–Baxter map. The pair $(V, R)$ is said to be a solution of the YBE or simply a solution.

The Yang–Baxter equation, or the 2-simplex equation, or the triangle equation, is one of the basic equations in mathematical physics and in low-dimension topology. It lies in the foundation of the theory of quantum groups, solvable models of statistical mechanics, knot theory, and braid theory. The YBE first appeared in the paper of Yang [2] in studying the many-body problem. Later, in studying solvable vertex models in statistical mechanics, Baxter [3] introduced this equation as the commutativity condition for transfer matrices. Another derivation of the YBE follows from the factorization of the $S$-matrix in 1 + 1 dimensional quantum field theory (see [4], [5]). The YBE is also essential in the quantum inverse scattering method for integrable systems [6], [7].
Drinfeld [8] suggested focusing on a specific class of solutions: the set-theoretic ones, i.e., solutions for which the vector space $V$ is spanned by a set $X$, and $R$ is the linear operator induced by a map $R: X \times X \to X \times X$. In this case, we say that $(X, R)$ is a set-theoretic solution of the Yang–Baxter equation or simply a solution of the YBE. We also call the pair $(X, R)$ the Yang–Baxter set. Set-theoretic solutions have connections, for example, with groups of I-type, Bieberbach groups, bijective 1-cocycles, Garside theory, and a wide class of integrable discrete dynamical systems [9]–[11].

It is easy to see that for any $X$, the map $P(x, y) = (y, x)$ gives a set-theoretic solution of the YBE. On the other hand, if $R$ is a solution of the YBE, then the map $S = PR$ satisfies the braid relation

$$(S \times \text{id})(\text{id} \times S)(S \times \text{id}) = (\text{id} \times S)(S \times \text{id})(\text{id} \times S),$$

which is the defining relation in the braid group $B_n$. Topologically, the braid relation is simply the third Reidemeister move of planar diagrams of links. In the 1980s, Joyce [12] and Matveev [13] introduced quandles as invariants of knots and links and proved that any quandle gives an elementary set-theoretic solution of the braid equation.

In this paper, we develop such a point of view on solutions of the Yang–Baxter equation or on braidings as a representative of a PROP structure with biarity $(2, 2)$ morphisms [14]. We recall that PROP is a generalization of the concept of an operad with operations of the highest valency, in particular, an operation that brings a pair of values given two arguments. These structures are of great importance in the study of multivalued groups [15]. Algebras over PROPs with biarity $(2, 2)$ morphisms interpolate between algebras and coalgebras. Apparently, this is why the Yang–Baxter equation plays such a significant role in the theory of Lie bialgebras and Hopf algebras. In this context, the following questions are of great importance: the functors and equivalences between such PROPs, the extensions of such categories and possible classifications. It turns out that in contrast to the notion of extension that is natural in the category of groups, the notion close to the bicrossed product of groups is here more general and meaningful. This relates to the main result in this paper. We develop the formalism of extensions in the vector space category in the case of extensions of quasitriangular Hopf algebras and in the set-theoretical case.

This paper is organized as follows. In Sec. 2, we recall the known facts about Hopf algebras and extensions of braided sets related to group structures. In Secs. 3 and 4, we elaborate on the extension procedure in the Hopf algebra and the set-theoretic case respectively.

2. Preliminaries

2.1. Hopf algebra. We recall the notation from the theory of Hopf algebras (see, e.g., [16]). A Hopf algebra $(H, m, \Delta, \varepsilon, S)$ over a field $K$ is an associative algebra with multiplication $m$, a coassociative comultiplication

$$\Delta: H \to H \otimes H,$$

which is a homomorphism of algebras, a counit $\varepsilon: H \to K$ such that

$$\sum h_i^{(1)} \varepsilon(h_i^{(2)}) = \sum \varepsilon(h_i^{(1)}) h_i^{(2)}, \quad \text{with} \quad \Delta(h) = \sum h_i^{(1)} \otimes h_i^{(2)},$$

and an antipode, which is an antihomomorphism $S: H \to H$ such that

$$\sum h_i^{(1)} \cdot S(h_i^{(2)}) = \sum S(h_i^{(1)}) \cdot h_i^{(2)} = \varepsilon(h) \cdot 1.$$

**Example 2.1.** Let $G$ be a group and $K[G]$ be the group algebra. We define the comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$ on elements of $G$ as

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}$$

and extend to $K[G]$ by linearity. We then obtain a cocommutative Hopf algebra.
Example 2.2. Let $G$ be a finite group. The Hopf algebra $K[G]^*$ has a basis $P_g$, $g \in G$, on which comultiplication and multiplication are defined by

$$\Delta(P_g) = \sum_h P_h \otimes P_{h^{-1}g}, \quad P_g P_h = \delta_{g,h} P_g.$$ 

This means that $\{P_g \mid g \in G\}$ is a set of pairwise orthogonal idempotents whose sum is equal to unity. The counit is defined by

$$\varepsilon(P_1) = 1, \quad \varepsilon(P_g) = 0, \quad g \in G, \quad g \neq 1,$$

and the antipode is defined by the equality $S(P_g) = P_{g^{-1}}$.

2.2. Extensions of Yang–Baxter sets. Let $X$ be a nonempty set and $R : X \times X \rightarrow X \times X$ be a solution of the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$ 

We let $R(x, y) = (\sigma_y(x), \tau_x(y))$, with $x, y \in X$, denote the components of $R$. If $(X, R^X)$ and $(Y, R^Y)$ are two Yang–Baxter sets, then a map $f : X \rightarrow Y$ is said to be a morphism if the diagram

$$\begin{array}{ccc}
X \times X & \xrightarrow{f \times f} & Y \times Y \\
R^X & \downarrow & R^Y \\
X \times X & \xrightarrow{f \times f} & Y \times Y
\end{array}$$

is commutative, i.e., for any $x, x' \in X$, we have $R^Y(f \times f)(x, x') = (f \times f)R^X(x, x')$. For any $y \in Y$, we can define its preimage

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}.$$ 

We say that $f$ is homogeneous if the cardinalities of all preimages $f^{-1}(y)$ are equal. In this case, we can find a set of different elements $y_i$, $i \in I$, such that $X$ is the disjoint union

$$X = \coprod_{i \in I} f^{-1}(y_i).$$ 

If the inclusion

$$R^X(f^{-1}(y_k) \times f^{-1}(y_k)) \subseteq f^{-1}(y_k) \times f^{-1}(y_k)$$

holds for some $k \in I$, then we say that there is a homomorphism of the solution $(X, R^X)$ to the solution $((Y, y_k), R^Y)$ with the kernel $(f^{-1}(y_k), R_{f^{-1}(y_k)})$.

In some problems, different equivalence relations arise between solutions of the Yang–Baxter equation. In the case of a braided set, the so-called guitar map has been found that transforms a solution of the YBE on $X$ into a solution of a special kind,

$$R'(x, y) = (\sigma_y(\sigma_x^{-1}(y)) \sigma_x(x), y).$$

This transformation was introduced by Soloviev [17] and developed by Lebed and Vendramin [18].
2.3. Extension of braided sets induced by a group structure. We recall some ideas and results from [19]. A solution $S$ of the braid equation

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23}$$

(2.1)

can be associated with a solution of the Yang–Baxter equation of the form $R(x, y) = PS(x, y)$. We recall that $X$ is called a braided set if $X$ is equipped with $S: X^2 \to X^2$, which is a solution of braid equation (2.1).

**Definition 2.1.** We call a set $X$ with a binary algebraic operation $*$ self-distributive if $*$ satisfies

$$(x * y) * z = (x * z) * (y * z).$$

**Proposition 2.1.** The set $(X, *)$ is self-distributive if and only if the map $S_*(x, y) \overset{\text{def}}{=} (y, x * y)$ defines a braided set on $X$.

**Example 2.3.** Any group $G$ with the conjugation operation $x * y = y^{-1}xy$ is a self-distributive set. We call such self-distribute sets grouplike.

These observations allow connecting groups with braided sets. In particular, they allow describing extensions of grouplike braided sets. This is principally due to the well-developed theory of group extensions defined by group cohomology [20]. This idea is exploited in [19] (also see [21]) to describe solutions of the parametric Yang–Baxter equation.

3. Extension of quasitriangular Hopf algebras

We recall that a quasitriangular Hopf algebra is a Hopf algebra $A$ together with an invertible element $R \in A \otimes A$ (the quantum $R$-matrix) satisfying the “exchange” condition

$$\Delta^\text{op}(a) = R\Delta(a)R^{-1}, \quad a \in A,$$

and the compatibility condition

$$(\Delta \otimes \text{id})R = R_{23}R_{13}, \quad (\text{id} \otimes \Delta)R = R_{12}R_{13}.$$  

These conditions imply that $R$ satisfies the quantum Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$  

The “classical analogue” of a quasitriangular Hopf algebra is realized in the context of Lie algebras. A Lie bialgebra is a dual pair $(g, g^*)$ of Lie algebras for which the dual of commutator on $g^*$, $c: g \to g \wedge g$, is a cocycle with respect to the adjoint representation. The Lie bialgebra is called quasitriangular if it is equipped with an element $r \in g \otimes g$ (the classical $r$-matrix) satisfying conditions that are “infinitesimal versions” of those for $R$, the most important of them being the classical Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{12}, r_{23}] = 0.$$  

We regard $g$ as embedded in $U(g)$.
3.1. Product of Hopf algebras. We recall some definitions and constructions that can be found in Chapter 4 in [22].

A Hopf algebra $A$ over a commutative ring $k$ is said to be almost cocommutative if there is an invertible element $R \in A \otimes A$ such that $\Delta^\text{op}(a) = R \Delta(a) R^{-1}$ for all $a \in A$.

The known construction for the product of Hopf algebras is as follows. Let $B(\Delta^B, \varepsilon^B, S^B, R^B)$ and $C(\Delta^C, \varepsilon^C, S^C, R^C)$ be Hopf algebras over a commutative ring $K$, and let $R \in C \otimes B$ be an invertible element such that
\[
(\Delta^C \otimes \text{id})R = R_{23}R_{13}, \quad (\text{id} \otimes \Delta^B)R = R_{12}R_{13}, \quad (\text{id} \otimes S^B)R = R^{-1}, \quad (S^C \otimes \text{id})R = R^{-1}.
\] (3.1)

Then the tensor product $B \otimes C$ can be endowed with a Hopf algebra structure with the tensor product multiplication, the comultiplication given by
\[
\Delta(b \otimes c) = R_{23}\Delta^B_{13}(b)\Delta^C_{24}(c)R_{23}^{-1},
\]
the antipode
\[
S(b \otimes c) = R_{21}^{-1}(S^B(b) \otimes S^C(c))R_{21}
\]
and the counit
\[
\varepsilon(b \otimes c) = \varepsilon^B(b)\varepsilon^C(c).
\]
This algebra is denoted by $B \otimes_C R$.

**Theorem 3.1.** Under the conditions of the above construction, let the Hopf algebras $B$, $C$ be quasi-triangular, i.e., such that the comultiplication is quasico-commutative,
\[
(\Delta^B)^\text{op} = R^B \Delta^B(R^B)^{-1}, \quad (\Delta^C)^\text{op} = R^C \Delta^C(R^C)^{-1},
\]
and moreover the following conditions hold:
\[
(\Delta^B \otimes \text{id})R^B = R^B_{13}R^B_{23}, \quad (\text{id} \otimes \Delta^B)R^B = R^B_{13}R^B_{12}, \quad (\Delta^C \otimes \text{id})R^C = R^C_{13}R^C_{23}, \quad (\text{id} \otimes \Delta^C)R^C = R^C_{13}R^C_{12}.
\]

Then the Hopf algebra $B \otimes C$ is also quasitriangular with the structure element (the quantum $R$-matrix)
\[
R = R_{41}R^B_{13}R^C_{24}R_{23}^{-1}.
\]
A part of this theorem can be found in [23].

**Proof.** We represent the opposite comultiplication on $B \otimes C$ in the form
\[
\Delta^\text{op}(b \otimes c) = P_{13}P_{24}\Delta(b \otimes c) = R_{41}(\Delta^B_{13})^\text{op}(b)(\Delta^C_{24})^\text{op}(c)R_{41}^{-1} =
\]
\[
= R_{41}R^B_{13}\Delta^B_{13}(b)(R^B_{13})^{-1}R^C_{24}\Delta^C_{24}(c)(R^C_{24})^{-1}R_{41}^{-1} =
\]
\[
= R_{41}R^B_{13}R^C_{24}R_{23}^{-1}\Delta(b \otimes c)R_{23}(R^B_{13})^{-1}(R^C_{24})^{-1}R_{41}^{-1}.
\]

Then
\[
\Delta^\text{op}(b \otimes c) = R\Delta(b \otimes c)R^{-1}.
\]
We now prove that this Hopf algebra is quasitriangular. We derive several consequences of conditions (3.1) on $R$:

$$R_{12}^C R_{23} R_{13} = R_{13} R_{23} R_{12}^C, \quad R_{23}^B R_{12} R_{13} = R_{13} R_{12} R_{23}^B. \quad (3.2)$$

We use multiindices to denote internal tensor components. For example, $\mathcal{R} = R_{41}^B R_{24}^B R_{23}^{-1}$ is indexed as $\mathcal{R}_{(12)(34)}$. This emphasizes that $\mathcal{R}$ is an element of the space $(B \otimes C) \otimes (B \otimes C)$. We now prove that

$$\Delta_{(12)} \otimes \text{id}) \mathcal{R} = \mathcal{R}_{(12)(56)} \mathcal{R}_{(34)(56)}. \quad (3.3)$$

The right-hand side of this expression takes the form

$$R_{61}^C R_{15}^B R_{26} R_{25}^{-1} R_{63} R_{35}^B R_{46} R_{45}^{-1}. \quad (3.4)$$

We calculate the left-hand side sequentially. First, we apply the operation $P_{23}(\Delta_B \otimes \Delta_C \otimes \text{id} \otimes \text{id})$ to $\mathcal{R}_{(12)(34)}$, which gives $R_{61}^C R_{15}^B R_{33}^B R_{26}^B R_{46} C \otimes R_{45}^{-1} R_{45}^{-1}$. Then we conjugate the expression with $R_{23}$:

$$R_{23} R_{61} R_{15} R_{15}^B R_{35}^B R_{26}^C R_{46} R_{45}^{-1} R_{23}^{-1} R_{23} = R_{61} R_{23} R_{63} R_{35}^B R_{26}^B R_{46} R_{45}^{-1} R_{23}^{-1},$$

$$R_{23} R_{61} R_{15}^B R_{26} R_{35}^B R_{26}^C R_{46} R_{45}^{-1} R_{23}^{-1} = R_{61} R_{15} R_{26} R_{35} R_{26}^B R_{46} R_{45}^{-1} R_{23}^{-1},$$

$$R_{23} R_{61} R_{15}^B R_{26} R_{35} R_{26}^B R_{46} R_{45}^{-1} R_{23}^{-1} = R_{61} R_{15} R_{26} R_{35} R_{26}^B R_{46} R_{45}^{-1} R_{23}^{-1}.$$

This coincides with (3.4). Here, we used (3.2), with the corresponding factors shown with an underbrace. The factors that can be transposed freely are underlined. We similarly prove the identity

$$(\text{id} \otimes \Delta_{(34)}) \mathcal{R} = \mathcal{R}_{(12)(56)} \mathcal{R}_{(12)(34)}.$$

**Remark.** The theorem, in particular, allows constructing new solutions of the Yang–Baxter equation in the case $C = B$ as follows. We consider $R^C = R^B$ and take $R = R_{21}^B$. Conditions (3.1) are then satisfied. We verify the second one. We can write

$$(\text{id} \otimes \Delta^B) R = (\text{id} \otimes \Delta^B) P_{12} R^B P_{12} = P_{12} P_{23} (\Delta^B \otimes \text{id}) R^B P_{23} P_{12} = P_{12} P_{23} R_{13}^B R_{23}^B P_{23} P_{12} = R_{21}^B R_{31}^B = R_{13} R_{13}.$$

The expression for $\mathcal{R}$ in this case takes the form $\mathcal{R} = R_{14}^B R_{13}^B R_{24}^B R_{23}^{-1}$. By our theorem, this is a solution of the Yang Baxter equation, different from the known $\overline{\mathcal{R}} = R_{14}^B R_{13}^B R_{24}^B R_{23}^2$. They coincide only in the involutive case.

**Example 3.1.** Let $(X, *)$ be a rack, i.e., a self-distributive groupoid with an operation $\bar{*}: X \times X \to X$ such that

$$(x * y) \bar{*} y = x = (x \bar{*} y) * y, \quad x, y \in X.$$

Then the map

$$R: X \times X \to X \times X, \quad R(x, y) = (x, y \bar{*} x), \quad x, y \in X,$$

gives an elementary (one component is fixed) invertible solution of the YBE. Its inverse $R^{-1}$ is defined by the rule $R^{-1}(x, y) = (x, y \bar{*} x)$. Also, $R_{21} = P_{12} R P_{12}$ is defined by the rule $R_{21}(x, y) = (x * y, y)$. We also have

$$R_{21}^{-1}(x, y) = (x \bar{*} y, y), \quad R_{12} R_{21}(x, y) = (x * y, y * (x * y)).$$
We find the action of \( R = R_{14} R_{13} R_{24} R_{32}^{-1} \) on \( X^4 \). We have
\[
R(x_1, y_1, x_2, y_2) = R_{14} R_{13} R_{24} R_{32}^{-1}(x_1, y_1, x_2, y_2) = R_{14} R_{13} R_{24}(x_1, y_1) \ast x_2, x_2, y_2 =
= R_{14} R_{13}(x_1, y_1) \ast x_2, x_2, y_2 \ast (y_1 \ast x_2) =
= R_{14}(x_1, y_1) \ast x_2, x_2 \ast x_1, y_2 \ast (y_1 \ast x_2) =
= (x_1, y_1) \ast x_2, x_2 \ast x_1, (y_2 \ast (y_1 \ast x_2)) \ast x_1).
\]

**Remark.** There is another possibility to define a solution on the square \( X^2 \) of a rack \( X \). We can define the rack operation on \( X^2 \) and define the rack solution on \( X^2 \) as in the beginning of the preceding example. But in this case we obtain an elementary solution.

### 3.2. Drinfeld twist.
Let \( T \in GL(V \otimes V) \) satisfy the braid relation
\[
T_{12} T_{23} T_{12} = T_{23} T_{12} T_{23}.
\]
We consider \( F \in GL(V \otimes V) \) and \( \Psi, \Phi \in GL(V \otimes V \otimes V) \) such that
\[
F_{12} \Psi = F_{23} \Phi, \quad \Phi T_{23} = T_{23} \Phi, \quad \Psi T_{12} = T_{12} \Psi.
\]
Then \( \hat{T} = F T F^{-1} \) also satisfies the braid relation
\[
\hat{T}_{12} \hat{T}_{23} \hat{T}_{12} = \hat{T}_{23} \hat{T}_{12} \hat{T}_{23}.
\]
Such a transformation is called the Drinfeld twist [24]. This construction was originally proposed in the context of deformations of quasitriangular Hopf algebras. In fact, the construction in Theorem 3.1 can be considered a version of the Drinfeld twist. Indeed, we first pass to the braid notation with the help of respective transpositions \( P^B \) and \( P^C \) in \( B \otimes B \) and \( C \otimes C \):
\[
T^B = P^B R^B, \quad T^C = P^C R^C, \quad \hat{T}^{(12)(34)} = P^{13} P^{24} R^{(12)(34)}.
\]
Then \( \hat{T}^{(12)(34)} T = R_{23} T^{13} T^{24} R_{23}^{-1} \) can be viewed as a conjugation of the obvious solution \( T^{(12)(34)} = T^{13} T^{24} \) for the braid relation on the tensor product \( B \otimes C \) by an element \( R_{23} \). We compare this transformation with the Drinfeld twist. In terms of \( T^B \) and \( T^C \), Eqs. (3.2) take the form
\[
T^{12}_2 R_{23} R_{13} = R_{23} R_{13} T^{12}_2, \quad T^{B}_{23} R_{12} R_{13} = R_{12} R_{13} T^{B}_{23}. \tag{3.5}
\]
To exactly obtain the Drinfeld twist, we have to find elements \( \Psi \) and \( \Phi \) in \( (B \otimes C)^{\otimes 3} \) such that
\[
F^{(12)(34)} \Psi^{(12)(34)(56)} = F^{(34)(56)} \Phi^{(12)(34)(56)},
\Phi^{(12)(34)(56)} T^{13}_3 T^{16}_3 = T^{13}_3 T^{16}_3 \Phi^{(12)(34)(56)},
\Psi^{(12)(34)(56)} T^{B}_{24} = T^{B}_{24} \Psi^{(12)(34)(56)}.
\tag{3.6}
\]
Due to (3.5), the choice \( \Phi = R_{23} R_{25} \) and \( \Psi = R_{45} R_{25} \) guarantees that Eq. (3.6) is satisfied. This choice is suggested by [25].

615
3.3. Infinitesimal version in the tensor case. Here, we deduce a classical limit of Theorem 3.1.

**Theorem 3.2.** Let $r^B$ and $r^C$ be solutions of the classical Yang–Baxter equation (CYBE) on respective Lie algebras $B$ and $C$ and let $r$ satisfy the equations

\[
[r^C_{12}, r_{13} + r_{23}] = [r_{23}, r_{13}], \quad (3.7) \\
[r^B_{12}, r_{12} + r_{13}] = [r_{12}, r_{13}], \quad (3.8)
\]

Then

\[
\tilde{r}_{(12)(34)} = r^B_{13} + r^C_{23} + r_{41} - r_{23}
\]
solves the CYBE on $B \otimes C$.

**Proof.** The CYBE in this case takes the form

\[
\left[\tilde{r}_{(12)(34)}, r_{(12)(56)} + \tilde{r}_{(34)(56)}\right] + \left[\tilde{r}_{(12)(56)}, \tilde{r}_{(34)(56)}\right] = 0.
\]

This can be expressed as

\[
\left[r^B_{13} + r^C_{24} + r_{41} - r_{23}, r^B_{15} + r^C_{26} + r_{61} - r_{25} + r^B_{35} + r^C_{46} + r_{63} - r_{45}\right] + \\
\left[r^B_{15} + r^C_{26} + r_{61} - r_{25}, r^B_{35} + r^C_{46} + r_{63} - r_{45}\right] = 0.
\]

All commutators of $r^B_{ij}$ and $r^B_{kl}$ yield the CYBE for $r^B$,

\[
[r^B_{13}, r^B_{15}] + [r^B_{13}, r^B_{35}] + [r^B_{15}, r^B_{35}] = 0.
\]

The same follows for the commutators of $r^C$. We observe that the commutators between $r^B$ and $r^C$ are all trivial due to their localization in different tensor components. We consider the commutators of $r^B_{13}$ with different $r$ in more detail:

\[
[r^B_{13}, r_{61} + r_{63}] + [r_{61}, r_{63}] = 0.
\]

This is zero due to (3.8). Similar formulas follow from the with other terms.

**Remark.** Conditions (3.7) and (3.8) on $r$, which are necessary for the extension in the infinitesimal case, can be expressed as the Maurer–Cartan condition in a suitable differential graded Lie algebra. We suppose that this is a relevant version of the cohomological characterization of extensions in this case. We plan to formulate this approach in greater detail elsewhere.

4. Extensions in the set-theoretic case

4.1. Product of Yang–Baxter sets. In this section, we address the following question. Let $B$ and $C$ be two nonempty sets and $R^B: B \times B \rightarrow B \times B$ and $R^C: C \times C \rightarrow C \times C$ be Yang–Baxter maps (YBMs) on them. It means that these maps satisfy the equalities

\[
R^B_{12}R^B_{13}R^B_{23} = R^B_{23}R^B_{13}R^B_{12}, \quad R^C_{12}R^C_{13}R^C_{23} = R^C_{23}R^C_{13}R^C_{12}.
\]

What YBMs can be defined on the direct product $B \times C$?

An obvious way is to take the direct product $R^B \times R^C$, which acts by the rule

\[
(R^B \times R^C)((b_1, c_1), (b_2, c_2)) = ((\sigma^B_{b_2}(b_1), \sigma^C_{c_2}(c_1)), (\tau^B_{b_1}(b_2), \tau^C_{c_1}(c_2))),
\]

where

\[
R^B(b_1, b_2) = (\sigma^B_{b_2}(b_1), \tau^B_{b_1}(b_2)), \quad R^C(c_1, c_2) = (\sigma^C_{c_2}(c_1), \tau^C_{c_1}(c_2)).
\]
Remark. If $B$ and $C$ are not only sets but some algebraic systems (groups, racks, bi-racks, or skew braces), we can use extensions of these systems. The group case is expounded in Sec. 2.3. Under these constructions, the roles of the sets $B$ and $C$ are typically nonsymmetric: one of them is the image and the other is the kernel of a homomorphism.

We define a map $R: C \times B \to C \times B$, $R(c, b) = (\mu(c, b), \nu(c, b))$ such that

$$R_{23}^B R_{12} R_{13}^B = R_{13} R_{12} R_{23}^B, \quad R_{12}^C R_{23} R_{13}^C = R_{13} R_{23} R_{12}^C.$$  

Both sides of the first equality are maps on $C \times B \times B \to C \times B \times B$, and those of the second one, maps on $C \times C \times B \to C \times C \times B$.

The following lemma is evident.

Lemma 4.1. The following relations hold:

\begin{align*}
R_{15}^B R_{41} R_{45} &= R_{45} R_{41} R_{15}^B, \quad (4.1a) \\
R_{45}^{-1} R_{15}^B R_{41} &= R_{41} R_{15} R_{45}^{-1}, \quad (4.1b) \\
R_{35}^B R_{23} R_{25} &= R_{23} R_{25} R_{35}^B, \quad (4.1c) \\
R_{23}^{-1} R_{25} R_{35}^B &= R_{35} R_{25}^{-1} R_{23}^{-1}, \quad (4.1d) \\
R_{13}^B R_{61} R_{63} &= R_{63} R_{61} R_{13}^B, \quad (4.1e) \\
R_{26}^C R_{63} R_{23} &= R_{23} R_{63} R_{26}^C, \quad (4.1f) \\
R_{23}^{-1} R_{26} R_{63} &= R_{63} R_{26} R_{23}^{-1}, \quad (4.1g) \\
R_{24}^C R_{45} R_{25} &= R_{25} R_{45} R_{24}^C, \quad (4.1h) \\
R_{45}^{-1} R_{25} R_{24} &= R_{45} R_{25}^{-1} R_{24}^{-1}, \quad (4.1i) \\
R_{36}^C R_{61} R_{41} &= R_{41} R_{61} R_{36}^C. \quad (4.1j)
\end{align*}

We now prove a set-theoretic analogue of Theorem 3.1.

Theorem 4.1. The pair $(B \times C, R)$, where $R = R_{41} R_{13}^B R_{24} R_{23}^{-1}$, is a YB set.

Proof. We have to verify the equality

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}.$$  

Its right-hand side is

$$\mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} = R_{63} R_{35} R_{46} (R_{45}^{-1} \cdot R_{61}) R_{15}^B (R_{26}^C R_{25}^{-1} \cdot R_{41}) R_{13}^B R_{24} R_{23}^{-1}.$$  

Using the commutativity relations and Lemma 4.1, we move $R_{41}$ to the left. Because $R_{41}$ commutes with $R_{25}^{-1}$ and $R_{26}^C$, and $R_{45}^{-1}$ commutes with $R_{61}$, we obtain

$$\mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} = R_{63} R_{35} R_{46} R_{61} (R_{45}^{-1} R_{15}^B R_{41}) R_{26}^C R_{25}^{-1} \cdot R_{13}^B R_{24} R_{23}^{-1}.$$  

By (4.1b),

$$\mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} = R_{63} R_{35} R_{46} R_{61} R_{41} R_{13}^B R_{45}^{-1} R_{26}^C R_{25}^{-1} \cdot R_{13}^B R_{24} R_{23}^{-1}.$$  

617
By (4.1j),
\[ R_{23} R_{13} R_{12} = R_{63} R_{35} B C R_{46} R_{61} R_{41} R_{13} R_{15} R_{26} R_{25} R_{24} R_{23} . \]

Hence,
\[ R_{23} R_{13} R_{12} = R_{63} R_{41} R_{35} R_{61} R_{46} R_{13} R_{15} R_{26} R_{25} R_{24} R_{23} \]
and
\[ R_{23} R_{13} R_{12} = R_{63} R_{41} R_{35} R_{61} R_{46} R_{13} R_{15} R_{26} R_{25} R_{24} R_{23} , \]
which is equal to
\[ R_{23} R_{13} R_{12} = R_{41} R_{63} R_{61} R_{35} R_{46} R_{15} R_{26} R_{25} R_{24} R_{23} . \]

For the left-hand side, we have
\[ R_{12} R_{13} R_{23} = R_{41} R_{13} R_{24} (R_{23} R_{15} R_{63} R_{35} R_{46} R_{13} R_{15} R_{26} R_{25} R_{24} R_{23} . \]

Using the commutativity relations and Lemma 4.1 we move \( R_{63} \) to the left. Because \( R_{23} R_{24} R_{23} \) commutes with \( R_{61} R_{15} \) and \( R_{63} \) commutes with \( R_{25} R_{24} \), we obtain
\[ R_{12} R_{13} R_{23} = R_{41} R_{13} R_{24} R_{61} R_{15} R_{26} R_{25} R_{24} R_{23} . \]

By (4.1g), \( R_{23} R_{24} R_{23} = R_{63} R_{26} R_{24} R_{23} \). Hence
\[ R_{12} R_{13} R_{23} = R_{41} R_{13} R_{24} R_{61} R_{15} R_{26} R_{25} R_{24} R_{23} . \]

By the commutativity condition, we have
\[ R_{12} R_{13} R_{23} = R_{41} R_{13} R_{24} R_{61} R_{15} R_{26} R_{25} R_{24} R_{23} . \]

Because \( R_{24} R_{23} \) commutes with the product \( R_{61} R_{63} \), we obtain
\[ R_{12} R_{13} R_{23} = R_{41} R_{13} R_{24} R_{61} R_{15} R_{26} R_{25} R_{24} R_{23} . \]

By (4.1e),
\[ R_{12} R_{13} R_{23} = R_{41} R_{13} R_{24} R_{61} R_{15} R_{26} R_{25} R_{24} R_{23} . \]

Comparing the right-hand side of the last equality with the corresponding equality for \( R_{23} R_{13} R_{12} \), we see that we can reduce both sides from the left by \( R_{41} R_{63} R_{61} \). Hence, the equality
\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \]

takes the form
\[ R_{13} R_{24} R_{61} R_{15} R_{26} R_{25} R_{24} R_{23} = R_{35} R_{46} R_{13} R_{15} R_{26} R_{25} R_{24} R_{23} . \]

By (4.1d) and the commutativity relations, we obtain
\[ R_{13} R_{24} R_{61} R_{15} R_{26} R_{25} R_{24} R_{23} = R_{35} R_{46} R_{13} R_{15} R_{26} R_{25} R_{24} R_{23} . \]
Similarly, we deduce that
\[ R_{13} B C R_{13} B C R_{26} R_{35} R_{26} R_{35} R_{25} R_{23} R_{45}^{-1} R_{45} = R_{35} R_{46} R_{13} R_{26} R_{20} R_{13} (R_{45} R_{25} R_{24}) R_{23}^{-1}. \]

Equation (4.1i) gives
\[ R_{13} B C R_{15} R_{26} R_{35} R_{46} R_{25} R_{23} R_{45}^{-1} R_{45} = R_{35} R_{46} R_{15} R_{26} R_{13} R_{24} R_{25} R_{45}^{-1} R_{23}. \]

Because \( R_{45}^{-1} \) commutes with \( R_{23}^{-1} \), we can reduce both sides of the last equality by \( R_{25}^{-1} R_{23}^{-1} R_{45}^{-1} \) from the right,
\[ R_{13} R_{15} R_{35} R_{24} R_{25} R_{26} R_{46} = R_{35} R_{15} R_{13} R_{26} R_{24} R_{26} R_{24}, \]

The last equality is equivalent to
\[ R_{13} R_{15} R_{35} R_{24} R_{25} R_{26} R_{46} = R_{35} R_{15} R_{13} R_{26} R_{24} R_{26} R_{24}, \]

which is obviously true.

**4.2. Set-theoretic Drinfeld twist.** Let \((X, T)\) be a set-theoretic solution of the braid equation and let there exist \( F \in \text{Sym}(X \times X) \) and \( \Phi, \Psi \in \text{Sym}(X \times X \times X) \) such that
\[ F_{12} \Psi = F_{23} \Phi, \quad \Phi T_{23} = T_{23} \Phi, \quad \Psi T_{12} = T_{12} \Psi. \quad (4.2) \]

Then \( \hat{T} = F T F^{-1} \) also satisfies the braid equation
\[ \hat{T}_{12} \hat{T}_{23} \hat{T}_{12} = \hat{T}_{23} \hat{T}_{12} \hat{T}_{23}. \]

Using the notation in the preceding section, we put \( S^B = P R^B \) and \( S^C = P R^C \). Then the relations
\[ R_{12} R_{13} R_{23} R_{23} = R_{23} R_{13} R_{12} R_{23}, \quad R_{12} R_{13} R_{23} R_{23} = R_{23} R_{13} R_{12} R_{23} \]

imply the braid relations
\[ S_{12}^B S_{23}^B S_{12}^B = S_{23}^B S_{12}^B S_{23}^B, \quad S_{12}^C S_{23}^C S_{12}^C = S_{23}^C S_{12}^C S_{23}^C. \]

The relations
\[ R_{12} R_{23} R_{13} = R_{13} R_{23} R_{12}, \quad R_{23} R_{12} R_{13} = R_{13} R_{12} R_{23} \]
yield
\[ S_{12}^C R_{23} R_{13} = R_{23} R_{13} S_{12}^C, \quad S_{23}^B R_{12} R_{13} = R_{12} R_{13} S_{23}^B. \]

As a result, we obtain the \( R \)-matrix
\[ \mathcal{R} = R_{41} R_{13} R_{24} R_{23}^{-1} = P_{12} P_{24} (R_{23} S_{13} S_{24} R_{23}^{-1}) \]

and the elements
\[ \mathcal{R}_{12} = P_{12} P_{24} (R_{23} S_{13} S_{24} R_{23}^{-1}), \quad \mathcal{R}_{13} = P_{15} P_{26} (R_{25} S_{15} S_{26} R_{25}^{-1}), \quad \mathcal{R}_{23} = P_{35} P_{46} (R_{45} S_{35} S_{46} R_{45}^{-1}). \]

The following lemma is not difficult to prove.
Lemma 4.2. The relation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ implies the equality

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23},$$

where $S_{12} = R_{23}S_{13}S_{24}R_{23}^{-1}$ and $S_{23} = R_{45}S_{35}S_{46}R_{45}^{-1}$.

Hence, we have shown that if we take $T = T_{13}^{-1}S_{24}^B$, which evidently satisfies the braid equation, then $S = T = T_{13}^{-1}$ also satisfies the braid equation, where $F = F_{12} = R_{23}$ and $F_{23} = R_{45}$. We wish to interpret $S$ as a particular case of the Drinfeld twist of $T = T_{13}^{-1}S_{24}^B$.

We take $\Phi = R_{23}R_{25}$, $\Psi = R_{45}R_{25}$. Then the first equality of system (4.2) becomes

$$R_{23} \cdot (R_{45}R_{25}) = R_{45} \cdot (R_{23}R_{25}).$$

This relation holds due to the commutativity of $R_{23}$ and $R_{45}$. Lemma 4.1 implies the following lemma.

Lemma 4.3. The following relations hold:

$$S_{15}R_{41}R_{45} = R_{41}R_{45}S_{15}^B,$$  (4.3a)
$$S_{35}R_{23}R_{25} = R_{23}R_{25}S_{35}^B,$$  (4.3b)
$$S_{13}R_{61}R_{63} = R_{61}R_{63}S_{13}^B,$$  (4.3c)
$$S_{26}R_{63}R_{23} = R_{63}R_{23}S_{26}^C,$$  (4.3d)
$$S_{24}R_{45}R_{25} = R_{45}R_{25}S_{24}^C,$$  (4.3e)
$$S_{46}R_{61}R_{41} = R_{61}R_{41}S_{46}^C.$$  (4.3f)

The second equality of system (4.2) has the form

$$(R_{23}R_{25})(S_{35}^B S_{46}^C) = (S_{35}^B S_{46}^C)(R_{23}R_{25}).$$

Using (4.3b) and the commutativity, we rewrite the left-hand side as

$$(R_{23}R_{25}S_{35}^B)S_{46}^C = S_{35}^B R_{23}R_{25}S_{46}^C = S_{35}^B S_{46}^C R_{23}R_{25}.$$

The third equality of system (4.2) has the form

$$(R_{45}R_{25})(S_{13}^B S_{24}^C) = (S_{13}^B S_{24}^C)(R_{45}R_{25}).$$

We verify this equality using the same arguments as in the foregoing.

Conflicts of interest. The authors declare no conflicts of interest.

REFERENCES

1. V. M. Buchstaber, “The Yang–Baxter transformation,” Russian Math. Surveys, 53, 1343–1345 (1998).
2. C. N. Yang, “Some exact results for the many-body problem in one dimension with repulsive delta-function interaction,” Phys. Rev. Lett., 19, 1312–1315 (1967).
3. R. J. Baxter, “Partition function of the eight-vertex lattice model,” Ann. Phys., 70, 193–228 (1972).
4. A. B. Zamolodchikov, “Tetrahedra equations and integrable systems in three-dimensional space,” Soviet Phys. JETP, 52, 325–336 (1980).
5. A. B. Zamolodchikov, “Tetrahedron equations and the relativistic S-matrix of straight-strings in 2 + 1-dimensions,” Commun. Math. Phys., 79, 489–505 (1981).
6. E. K. Sklyanin, L. A. Takhtadzhyan, and L. D. Faddeev, “Quantum inverse problem method. I,” Theoret. and Math. Phys., 40, 688–706 (1979).
7. L. A. Takhtadzhyan and L. D. Faddeev, “The quantum method of the inverse problem and the Heisenberg XYZ model,” Russian Math. Surveys, 34, 11–68 (1979).
8. V. G. Drinfel’d, “On some unsolved problems in quantum group theory,” in: Quantum Groups, Lecture Notes in Mathematics, Vol. 1510 (P. P. Kulish, ed.), Springer, Berlin, Heidelberg (1992), pp. 1–8.
9. A. P. Veselov, “Integrable maps,” Russian Math. Surveys, 46, 1–51 (1991).
10. A. P. Veselov, “Yang–Baxter map and integrable dynamics,” Phys. Lett. A, 314, 214–221 (2003).
11. V. V. Bazhanov and S. M. Sergeev, “Yang–Baxter maps, discrete integrable equations and quantum groups,” Nucl. Phys. B, 926, 509–543 (2018).
12. D. Joyce, “A classifying invariant of knots: the knot quandle,” J. Pure Appl. Algebra, 23, 37–65 (1982).
13. S. V. Matveev, “Distributive groupoids in knot theory,” Math. USSR-Sb., 47, 73–83 (1984).
14. M. Markl, Handbook of Algebra, Vol. 5, Elsevier, North-Holland, Amsterdam (2008).
15. V. M. Buchstaber and E. G. Rees, “Multivalued groups and Hopf n-algebras,” Russian Math. Surveys, 51, 727–729 (1996).
16. C. Kassel, Quantum Groups, Graduate Texts in Mathematics, Vol. 155, Springer, New York (1995).
17. A. Soloviev, “Non-unitary set-theoretical solutions to the quantum Yang–Baxter equation,” Math. Res. Lett., 7, 577–596 (2000).
18. V. Lebed and A. Vendramin, “Homology of left non-degenerate set-theoretic solutions to the Yang–Baxter equation,” Adv. Math., 304, 1219–1261 (2017).
19. M. M. Preobrazhenskaya and D. V. Talalaev, “Group extensions, fiber bundles, and a parametric Yang–Baxter equation,” Theoret. and Math. Phys., 207, 670–677 (2021).
20. K. S. Brown, Cohomology of Groups, Graduate Texts in Mathematics, Vol. 87, Springer, New York (1982).
21. V. Bardakov, B. Chuzinov, I. Emel’yanenkov, M. Ivanov, T. Kozlovskaya, and V. Leshkov, “Set-theoretical solutions of simplex equations,” arXiv: 2206.08906.
22. V. Chari and A. Pressley, A Guide to Quantum Groups, Cambridge Univ. Press, Cambridge (1995).
23. N. Yu. Reshetikhin and M. A. Semenov-Tian-Shansky, “Quantum R-matrices and factorization problem,” J. Geom. Phys., 5, 533–550 (1988).
24. V. G. Drinfeld, “Quasi-Hopf algebras,” Leningrad Math. J., 1, 1419–1457 (1990).
25. P. P. Kulish and A. I. Mudrov, “On twisting solutions to the Yang–Baxter equation,” Czech. J. Phys., 50, 115–122 (2000).