Reduced phase-space quantization of constrained systems

February 5, 2020

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Abstract

The Hamilton - Jacobi method of constrained systems is discussed. The equations of motion for three singular systems are obtained as total differential equations in many variables. The integrability conditions for these systems lead us to obtain the canonical reduced phase space coordinates without using any gauge fixing condition. The operator and the path integral quantization of these systems are discussed.
1 Introduction

Recently, the canonical method [1-4] has been developed to investigate constrained systems. The equations of motion are obtained as total differential equations in many variables which require the investigation of integrability conditions. If the system is integrable, one can solve the equations of motion without using any gauge fixing conditions.

Now we would like to give a brief discussion of the canonical method. This method gives the set of Hamilton - Jacobi partial differential equations [HJPDE] as

\[ H'_\alpha(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_a}) = 0, \]
\[ \alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r, \] (1)

where

\[ H'_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_a, \] (2)

and \( H_0 \) is defined as

\[ H_0 = p_a w_a + p_\mu \dot{q}_\mu |_{\nu = -H_\nu} - L(t, q_i, \dot{q}_\nu, \dot{q}_a = w_a), \]
\[ \mu, \nu = n - r + 1, ..., n. \] (3)

The equations of motion are obtained as total differential equations in many variables as follows:

\[ dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad dp_\beta = -\frac{\partial H'_\alpha}{\partial t_\beta} dt_\alpha. \] (4)

\[ dz = (-H_\alpha + p_\beta \frac{\partial H'_\alpha}{\partial q_a}) dt_\alpha; \] (5)
\[ \alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r \]

where \( z = S(t_\alpha; q_a) \). The set of equations (4,5) is integrable [3,4] if

\[ dH'_0 = 0, \quad dH'_\mu = 0, \mu = n - p + 1, ..., n. \] (6)
If condition (6) are not satisfied identically, one considers them as new constraints and again tests the consistency conditions. Hence, the canonical formulation leads to obtain the set of canonical phase space coordinates $q_a$ and $p_a$ as functions of $t_\alpha$, besides the canonical action integral is obtained in terms of the canonical coordinates. The Hamiltonians $H'_\alpha$ are considered as the infinitesimal generators of canonical transformations given by parameters $t_\alpha$ respectively.

2 Quantization of constrained systems

For the quantization of constrained systems we can use the Dirac’s method of quantization [5,6], or the path integral quantization method [7,8]. Now will shall give a brief information about these two methods.

2.1 Operator quantization

For the Dirac’s quantization method we have

$$H'_\alpha \Psi = 0, \quad \alpha = 0, n - r + 1, \ldots, n,$$

(7)

where $\Psi$ is the wave function. The consistency conditions are

$$[H'_\mu, H'_\nu] \Psi = 0, \quad \mu, \nu = 1, \ldots, r,$$

(8)

where $[,]$ is the commutator. The constraints $H'_\alpha$ are called first-class constraints if they satisfy

$$[H'_\mu, H'_\nu] = C^\gamma_{\mu\nu} H'_\gamma.$$

(9)

In the case when the Hamiltonians $H'_\mu$ satisfy

$$[H'_\mu, H'_\nu] = C_{\mu\nu},$$

(10)

with $C_{\mu\nu}$ do not depend on $q_i$ and $p_i$, then from (8) there arise naturally Dirac’ brackets and the canonical quantization will be performed taking Dirac’s brackets into commutators.
2.2 Path integral quantization method

The path integral quantization is an alternative method to perform the quantization of constrained systems.

Now we shall give a brief review of the canonical path integral formulation of constrained systems [7,8].

If the set of equations (4) is integrable then one can solve them to obtain the canonical phase-space coordinates as

\[ q_a \equiv q_a(t, t_\mu), \quad p_a \equiv p_a(t, t_\mu), \quad \mu = 1, \ldots, r, \quad (11) \]

In this case, the path integral representation may be written as [7,8]

\[ \langle \text{Out} | S | \text{In} \rangle = \int \prod_{a=1}^{n-r} dq^a \, dp^a \exp \left\{ i \left[ \int_{t_{\alpha}}^{t_{\alpha}'} \left( -H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_{\alpha} \right] \right\}, \]

\[ a = 1, \ldots, n - r, \quad \alpha = 0, n - r + 1, \ldots, n. \quad (12) \]

One should notice that the integral (12) is an integration over the canonical phase-space coordinates \((q_a, p_a)\).

3 Examples

As a first example we shall treat the relativistic particle as a constrained system and demonstrate the fact that the gauge fixing problem is solved naturally if the canonical path integral method is used.

Let us consider the action of the a relativistic particle as

\[ S = \frac{1}{2} \int (\frac{\dot{x}^\mu \dot{x}_\mu}{e} - em^2) d\tau, \quad (13) \]

where \(x^\mu(\tau)\) and \(e(\tau)\) are even variables. The canonical momenta are defined as

\[ p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{e}(\dot{x}_\mu), \quad \pi_e = \frac{\partial L}{\partial \dot{e}} = 0 = -H_1. \quad (14) \]

The canonical Hamiltonian \(H_0\) can be obtained as
\[ H_0 = p_{\mu} \dot{x}^{\mu} - e H_1 - L = + \frac{e}{2} (p^2 + m^2). \]  

(15)

Making use of equations (1,2) and (14, 15), the set of Hamilton- Jacobi partial differential equations reads

\[
H'_0 = p^{(\tau)} + H_0 = 0; \quad p^{(\tau)} = \frac{\partial S}{\partial \tau},
\]

(16)

\[
H'_1 = \pi_e = 0; \quad \pi_e = \frac{\partial S}{\partial \epsilon},
\]

(17)

This set leads to the total differential equations as

\[
dx_{\mu} = (cp_{\mu})d\tau, \quad dp_{\mu} = 0, \quad d\pi_e = -\frac{1}{2}(p^2 + m^2)d\tau = 0, \quad dp^{(\tau)} = 0. \]

(19)

To check whether the set of equations (19) is integrable or not let us consider the total variations of \(H'_0\) and \(H'_1\). In fact

\[
dH'_1 = -\frac{1}{2}(p^2 + m^2)d\tau = 0 = H'_2d\tau.
\]

(20)

The total differentials of \(H'_0\) and \(H'_2\) vanish identically, the equations of motion are integrable and the canonical phase space coordinates \((q_{\mu}, p_{\mu})\) are obtained in terms of parameters \((\tau, \epsilon)\).

To obtain the operator quantization of this system one can follow the procedure discussed in section (2.1). In this case one takes the constraint equation as an operator whose action on the allowed Hilbert space vectors is constrained to zero, i.e., \(H'_2\Psi = 0\), we obtain

\[
[p^{\mu} \hat{p}_{\mu} + m^2] \Psi = 0,
\]

(21)

Now to obtain the path integral quantization of this system, we can use equation (5) to obtain the canonical action as

\[
S = \int \frac{e}{2} (p^2 - m^2)d\tau.
\]

(22)
Making use of (22) and (12) the path integral for the system (13) is obtained as
\[
\langle q_\mu, e, \tau; q'_\mu, e', \tau' \rangle = \int_{q_\mu}^{q'_\mu} \prod_\mu dq^\mu \, dp^\mu \exp\{i\{\int_\tau^{\tau'} \frac{e}{2}(p^2 - m^2) \, d\tau}\}. \tag{23}
\]

This path integral representation is an integration over the canonical phase space coordinates \(q_\mu\) and \(p^\mu\).

Now for a system with \(n\) degrees of freedom and \(r\) first class constraints \(\phi^\alpha\), the matrix element of the \(S\) - matrix is given by Faddeev and Popov [9,10] as
\[
\langle \text{Out} | S | \text{In} \rangle = \int \prod_t d\mu(q_j, p_j) \exp[i\{\int_{-\infty}^{\infty} dt(p_j \dot{q}_j - H_0)\}], \quad j = 1, \ldots, n, \tag{24}
\]
where the measure of integration is given as
\[
d\mu = \det|\{\phi^\alpha, \chi^\beta\}| \prod_{\alpha=1}^r \delta(\chi^\alpha) \delta(\phi^\alpha) \prod_{j=1}^n dq_j dp_j, \tag{25}\]
and \(\chi^a\) are \(r\)-gauge constraints.

If we perform now the usual path integral quantization [9,10] using (24) for system (13), one must choose two gauge fixing conditions to obtain the path integral quantization over the canonical phase-space coordinates.

As a second example, we consider the Lagrangian
\[
L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{\dot{q}_1^2}{4q_2} - q_2(q_1^2 + \frac{q_2^2}{3} - R^2), \tag{26}
\]
excluding the line \(q_2 = 0\) on the configuration space. The canonically conjugated momenta are obtained as
\[
p_1 = \frac{\dot{q}_1}{2q_2}, \tag{27}
\]
\[
p_2 = 0. \tag{28}
\]
The canonical Hamiltonian can be obtained as
\[
H_0 = q_2p_1^2 + q_2(q_1^2 + \frac{q_2^2}{3} - R^2). \tag{29}
\]
Making use of eqns. (1,2) and (28,29), the set of HJPDE reads

\begin{align}
H'_{0} &= p_{0} + H_{0} = 0, \\
H'_{1} &= p_{2} = 0,
\end{align}

where \(p_{0} = \frac{\partial S}{\partial t}\), \(p_{2} = \frac{\partial S}{\partial q_{2}}\), here \(S = S(q_{1}, q_{2}, t)\) represents the action.

This set leads to the following total differential equations

\begin{align}
\text{dq}_{1} &= 2p_{1}q_{2} dt, \\
\text{dp}_{1} &= -2q_{1}q_{2} dt, \\
\text{dp}_{2} &= -(q_{1}^{2} + q_{2}^{2} + p_{1}^{2} - R^{2}) dt = 0.
\end{align}

According to eqns. (6) and (31,34) the vanishing of the total differential of \(H'_{1}\) leads to the constraint

\[H'_{2} = q_{1}^{2} + q_{2}^{2} + p_{1}^{2} - R^{2}.\]

Since \(H'_{2}\) is not identically zero, we consider it as new constraint. Thus for a valid theory, total variation of \(H'_{2}\) should be zero. Thus one gets

\[dq_{2} = 0,\]

which has the following solution

\[q_{2} = c = \pm \sqrt{R^{2} - p_{1}^{2} - q_{1}^{2}},\]

where \(c\) is an arbitrary constant.

Now the set of equations (32-34) is integrable and the canonical phase space coordinates \((q_{1}, p_{1})\) are obtained as functions of \((t, q_{2} = c)\) and in this case it is disc of radius \(R\) without the boundary. Choosing \(q_{2}\) to be positive, the canonical Hamiltonian can be written as

\[H_{0} = \frac{2}{3}(R^{2} - p_{1}^{2} - q_{1}^{2})^{3/2}.\]

To proceed the canonical quantization of this system one can follow the procedure discussed in section (2.1). In this case one takes the constraint
equation as an operator whose action on the allowed Hilbert space vectors
is constrained to \( \text{zero}, \) i.e., \( H_0' \Psi(q_1) = 0, \) we obtain
\[
[H_0' = \hat{p}_0 + \frac{2}{3}(R^2 - \hat{p}_1^2 - \hat{q}_1^2)^{3/2}] \Psi(q_1) = 0.
\] (39)
in his case, the Hilbert space \( \mathcal{H} \) consists of square integrable functions in the interval \(-R < q_1 < R\).

Now to obtain the path integral quantization of this system, we can use equation (5) to obtain the canonical action as
\[
S = \int [2p_1^2 \sqrt{R^2 - p_1^2 - q_1^2} - \frac{2}{3}(R^2 - p_1^2 - q_1^2)^{3/2}] dt.
\] (40)
Making use of (40) and (12) the path integral for the system (26) is obtained as
\[
\langle q_1, t; q'_1, t' \rangle = \int q'_1 \prod dq_1 dp_1 \exp \{i \left[ \int_t^{t'} (2p_1^2 \sqrt{R^2 - p_1^2 - q_1^2} - \frac{2}{3}(R^2 - p_1^2 - q_1^2)^{3/2}) dt \right] \}.
\] (41)
Consider next, the third Lagrangian
\[
L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{\dot{q}_1^2}{4q_2} - q_2(\dot{q}_1^2 - \frac{q_2^2}{3} - R^2),
\] (42)
The canonical Hamiltonian can be obtained as
\[
H_0 = q_2p_1^2 + q_2(\dot{q}_1^2 - \frac{q_2^2}{3} - R^2).
\] (43)
The set of HJPDE reads
\[
H_0' = p_0 + H_0 = 0,
\] (44)
\[
H_1' = p_2 = 0.
\] (45)
As for previous system, the equations of motion are integrable and the canonical phase space coordinates \((q_1, p_1)\) are obtained as functions of \((t, q_2 = c)\). Here \(q_2\) has to branches as
\[
q_2 = c = \pm \sqrt{p_1^2 + q_1^2 - R^2},
\] (46)
where \(c\) is an arbitrary constant. For each choice the canonical phase space is 2-dimensional infinite plane with a hole of radius \(R\) at the center, where we restricted \(q_2\) to be positive.

This system can be quantized using the method of section (2.1). The Hilbert space consists of square-integrable functions on real line \(\mathcal{R}^1\) excluding the interval \([-R, R]\). Hence, we obtain

\[
[\hat{p}_0 + \frac{2}{3}(\hat{p}_1^2 + \hat{q}_1^2 - R^2)^{3/2}]\Psi(q_1) = 0. \tag{47}
\]

The the path integral for the system (42) is obtained as

\[
\langle q_1, t; q_1', t' \rangle = \int_{q_1}^{q_1'} \prod dq_1 \, dp_1 \exp[i\{\int_t^{t'} (2p_1^2 \sqrt{p_1^2 + q_1^2 - R^2} - R^2 - \frac{2}{3}(p_1^2 + q_1^2 - R^2)^{3/2})dt\}]. \tag{48}
\]

One should notice that this path integral is an integration over the canonical phase space coordinates \(q_1\) and \(p_1\).

\section{Conclusion}

In this work we have obtained the quantization for three singular systems. In the relativistic particle problem, (13). The integrability conditions \(dH'_0\) and \(dH'_1\) are satisfied, the system is integrable. Hence the canonical phase space coordinates \(q_\mu\) and \(p_\mu\) are obtained in terms of parameters \(\tau\) and \(e\). Although \(e\) is introduced as a coordinate in the Lagrangian the presence of the constraints and the integrability conditions forces as to treat it as a parameter like \(\tau\). In this case the Hamiltonians \(H'_0\) and \(H'_1\) are considered as infinitesimal generators of canonical transformations given by parameters \(\tau\) and \(e\) respectively and the path integral is obtained directly as an integration over the canonical phase space coordinates \(q_\mu\) and \(p_\mu\) without using any gauge fixing conditions.

When applying the Faddeev and Popov method to this model one has to choose two gauge fixing of the form and to integrate over the extended phase space coordinates and after integration over the redundant variables one can arrive at the result (23).
For the second and the third Lagrangians, these systems are integrable and leads to obtain \( dq_2 = 0 \). Hence, the canonical phase space coordinates \((q_1, p_1)\) are obtained as functions of \((t, q_2 = \text{constant})\). In this case the path integral follows directly as an integration over the canonical phase-space coordinates \((q_1, p_1)\). In the usual formulation \([11]\), one has to integrate over the extended phase space coordinates \((q_1, p_1, q_2, p_2)\) and one can get rid of redundant variables \((q_2, p_2)\) by using delta functions \(\delta(p_2), \delta(q_2^2 + q_1^2 + p_1^2 - R^2)\) for the second system and the delta functions \(\delta(p_2), \delta(q_2^2 + R^2 - q_1^2 - p_1^2)\) for the third system.

Unlike conventional methods one can perform the path integral quantization of this system using the canonical path integral method to obtain the action directly without considering any Lagrange multipliers and without using delta functions in the measure.

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