A GEOMETRIC CONSTRUCTION FOR PERMUTATION EQUIVARIANT CATEGORIES FROM MODULAR FUNCTORS

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Abstract

Let $G$ be a finite group. Given a finite $G$-set $\mathcal{X}$ and a modular tensor category $\mathcal{C}$, we construct a weak $G$-equivariant fusion category $\mathcal{C}^{\mathcal{X}}$, called the permutation equivariant tensor category. The construction is geometric and uses the formalism of modular functors. As an application, we concretely work out a complete set of structure morphisms for $\mathbb{Z}/2$-permutation equivariant categories, finishing thereby a program we initiated in [BFRS10].

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1 Introduction

The correspondence between commutative Frobenius algebras over a field \( k \) and two-dimensional topological field theories with values in \( k \)-vector spaces is by now a classic result of mathematical physics [Ati88, Dij89]. Its most precise form is the assertion [Koc04] that the tensor categories of two-dimensional topological field theories and of commutative Frobenius algebras are equivalent as symmetric tensor categories.

Various generalizations of this theorem have been addressed. On the one hand side, given any (finite) group \( G \), \( G \)-equivariant two-dimensional topological field theories have lead to the notion of a \( G \)-Frobenius algebra [MS06, Tur99]. For three-dimensional topological field theories on the other hand, one is lead to the algebraic structure of a modular tensor category. A structure related to three-dimensional topological field theories that is more appropriate for our purposes is given by the notion of a modular functor. For any abelian category \( C \) satisfying suitable finiteness conditions, the following correspondences have been established [BK01]:

- \( C \)-extended genus 0 modular functors correspond to structures of (weakly) ribbon categories on \( C \).
- Higher genus modular functors correspond to structures of a modular category on \( C \).

We now fix a finite group \( G \). Equivariant versions of three-dimensional topological field theories have been constructed from crossed group categories in [Tur00]; for the related notion of a \( G \)-modular functor see [KP08]. A partial generalization of the preceding statements asserts [KP08] that the structure of a (weakly) \( G \)-equivariant fusion category on a given \( G \)-equivariant abelian category \( C^G \) is equivalent to a \( C^G \)-extended \( G \)-equivariant genus 0 modular functor.

The present paper is devoted to the construction of a \( G \)-equivariant fusion category \( C^X \) from a finite \( G \)-set \( X \) and a modular tensor category \( C \). We address the problem by constructing from these data a \( G \)-modular functor.

Let us pause to explain the importance of this construction: a general construction [Kir04] allows to associate to any \( G \)-equivariant modular category \( C^G \) a modular category \( C^G/\!/G \), the orbifold category. In the special case of the permutation group \( G = S_N \) acting on the set \( X = N := \{1, 2, \ldots, N\} \) of \( N \) elements, one obtains from permutation equivariant categories permutation orbifolds. These categories [FRS03] enter in the construction of boundary conditions for tensor product theories that break permutation symmetries, so-called permutation branes [Rec03]. Moreover, they conveniently encode refined aspects of the family of representations of mapping class groups that is associated to the modular tensor category \( C \). This explains the role of permutation orbifolds in Bantay’s approach to the congruence subgroup conjecture ([Ban02, Ban03]). (For a different proof of the congruence subgroup conjecture that is based on generalized Frobenius-Schur indicators, see [NS07].) We expect that the module categories over \( C^{G_N} \) constructed in this work describe permutation modular invariants on \( C^{G_N} \); our construction can thus also be seen as a first step towards showing that these modular invariants are physical.

We now summarize the content of this paper: given a finite \( G \)-set \( X \), we construct a symmetric monoidal functor \( F_X \) from the category \( G_{\text{cob}}(d) \) of \( G \)-cobordisms to the category \( \text{cob}(d) \) of
cobordisms. This functor assigns to a principal $G$-cover $(P \to M)$ the total space of the associated bundle

$$F_X(P \to M) := X \times_G P = X \times P/((g^{-1}x, p) \sim (x, gp)) \quad (1)$$

Pulling back topological field theories along this functor $F_X$, we find $G$-equivariant theories. This functor is introduced in subsection 2.1 and used in subsections 2.2 and 2.3 to study geometric and algebraic properties of two-dimensional $G$-equivariant field theories.

These sections also contain necessary preparation for the construction of a $C^G$-extended modular functor for which we need to know the corresponding category $C^G$ as a $G$-equivariant abelian category. We gain the necessary insight in the structure for the correct ansatz by first representing a given two-dimensional TFT by the corresponding commutative Frobenius algebra $(R, m, \eta, \Delta, \epsilon)$ and then reading off the full $G$-Frobenius algebra $A$ of the induced $G$-TFT.

In section 3 we present theorem 26, one of our main results: we use the cover functor $F_X$ to obtain a $G$-equivariant modular functor for every $G$-set $X$ and modular tensor category $C$.

We illustrate the situation by the following diagram:

$$\begin{array}{ccc}
\text{modular category } C & \longrightarrow & \text{C-extended modular functor } \tau \\
\downarrow F_X\text{-construction} & & \\
G\text{-modular category } C^X & \longrightarrow & C^X\text{-extended modular functor } \tau^G
\end{array}$$

The upper arrow that points to the left is dashed, since a $C$-extended modular functor endows an abelian category $C$ only with a weak duality [BK01] while on a modular tensor category one has a strong duality. The lower arrow pointing to the left is dashed not only for this reason; moreover the algebraic structure corresponding to higher genus $G$-equivariant modular functors has, so far, not yet fully been worked out.

There is no genus zero version of our results and the the requirement on $C$ to be modular cannot be weakened: the modular functor corresponding to $C$ has to be applied to total spaces of covers of manifolds which can have higher genus, even if the base manifold is of genus 0.

In section 4 we use these results to derive in full detail the structure of a $\mathbb{Z}/2$-equivariant fusion category obtained from the permutation action of the group $\mathbb{Z}/2$ on the set of two elements, completing thus the program initiated in [BFRS10]. In contrast to the ad hoc ansatz used in [BFRS10], the geometric structure unraveled in this paper provides clear guiding principles to write down a consistent set of constraint morphisms.

We fix the following conventions for this paper: $G$ is a finite group and $X$ is a finite $G$-set, $k$ is an algebraically closed field of characteristic zero. All manifolds are smooth oriented manifolds and all maps are smooth and orientation preserving. We freely use the graphical calculus for morphisms in ribbon categories for which we refer to [JS91, FRS02].
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2 Cover functors and two-dimensional topological field theories

2.1 G-cobordisms and cover functors

In this subsection we recall the notion of a two-dimensional $G$-equivariant topological field theory. We first define a category $Gcob(d)$ of cobordisms with $G$-covers. Throughout this paper, cobordism categories will be true categories rather than higher categories; correspondingly, topological field theories will not be extended or $n$-tier topological field theories.

An object $(P \to \Sigma, e)$ of $Gcob(d)$ consists of a $(d-1)$-dimensional closed oriented manifold $\Sigma$, together with a principal left $G$-bundle $P$ on $\Sigma$. For technical reasons, we also fix a marked point $e$ on each connected component of $P$. For any oriented manifold $M$, the manifold with the opposite orientation will be denoted by $\overline{M}$. It is sufficient to choose orientations of the base manifolds:

Lemma 1. Let $\pi : P \to M$ be a discrete cover of an oriented manifold $M$. Then the orientation of $M$ induces a canonical orientation on $P$.

Proof. The global orientation of $M$ can be represented by global section of the orientation bundle on $M$, which can be pulled back along the local diffeomorphism $\pi$ to a global section of the orientation bundle of $P$. □

The morphisms in $Gcob(d)$ from $(P_1 \to \Sigma_1, e_1)$ to $(P_2 \to \Sigma_2, e_2)$ are diffeomorphism classes of cobordisms of the base and the total space of the principal bundles. More precisely, consider pairs $(M, E)$, consisting of a left principal $G$-bundle $E$ over a $d$-dimensional oriented manifold $M$ and an orientation preserving diffeomorphism of $G$-bundles from the restriction $\partial E \to \partial M$ to the $G$-bundles on the boundary, i.e. diffeomorphisms

$$\partial M \xrightarrow{\cong} \Sigma_1 \sqcup \Sigma_2, \quad E|_{\Sigma_1} \xrightarrow{\cong} P_1 \quad \text{and} \quad E|_{\Sigma_2} \xrightarrow{\cong} \overline{P_2}.$$  

The morphisms in $Gcob(d)$ are now obtained by modding out diffeomorphisms of $E$. We write $(E \to M)$ for a representative.

The composition of morphisms is gluing along the boundary of both the base and the total space: Let $(P_i \to \Sigma_i, e_i)$ for $i = 1, 2, 3$ be objects in $Gcob(d)$ and $(E \to M) : (P_1 \to \Sigma_1, e_1) \to (P_2 \to \Sigma_2, e_2)$ and $(E' \to M') : (P_2 \to \Sigma_2, e_2) \to (P_3 \to \Sigma_3, e_3)$ be cobordisms. Then the manifold $M \sqcup_{\Sigma_2} M'$ obtained by gluing the base spaces is naturally equipped with the $G$-principal bundle $E \sqcup_{P_2} E'$. The identity on $(P \to \Sigma, e)$ is the diffeomorphism class of the cylinder over $\Sigma$ with trivial $G$-cover. The category $Gcob(d)$ has a natural structure of a symmetric monoidal category,
with tensor product given by disjoint union of manifolds and bundles. The empty set with empty $G$-bundle is the tensor unit.

We comment on the role of the marked point $e$ on the $G$-cover $P$ of an object $(P \to \Sigma, e)$ which we have chosen as an auxiliary datum. Its projection on $\Sigma$ determines a base point $x \in \Sigma$. Moreover, it determines an identification of the fibre $P_x$ over $x$ with $G$. We did not choose marked points on the covers of morphisms. Thus different choices for $e \in P$ give objects that are isomorphic in $\text{Gcob}(d)$ with the isomorphism being the cylinder with the trivial $G$-cover. In the case of the trivial group $G = 1$, one can forget the marked point and obtains an equivalence of categories of $\text{Gcob}(d)$ to the usual category $\text{cob}(d)$ of manifolds and cobordisms.

**Definition 2.** The category of $d$-dimensional $G$-equivariant topological field theories (or $G$-TFTs for short) is the category of symmetric monoidal functors

$$\text{tft}^G : \text{Gcob}(d) \to \text{Vect}_k.$$  \hspace{1cm} (3)

with monoidal natural transformations as morphisms.

One ingredient in our construction of $G$-TFTs is a finite left $G$-set $X$. In the construction we will need to make choices for all partitions of $X$. To this end, we fix an order on $X$ as an auxiliary datum.

For any object $(P \to \Sigma, e)$ of $\text{Gcob}(d)$, we consider the smooth $(d-1)$-manifold $X \times_G P$, where $X \times_G P = (X \times P)/((g^{-1}x, p) \sim (x, gp))$. Similarly, we obtain smooth $d$-manifolds for morphisms of $\text{Gcob}(d)$. For any $G$-bundle $P \to M$, the manifold $X \times_G P$ is the total space of an $|X|$-fold cover of $M$, we call this cover the associated $G$-cover. We agree to write $[x, p] \in X \times_G M$ for the equivalence class of $(x, p) \in X \times M$ for any $G$-manifold $M$.

**Proposition 3.** Let $X$ be a finite ordered left $G$-set. The assignment $(P \to \Sigma, e) \mapsto X \times_G P$ defines a symmetric monoidal functor

$$\mathcal{F}_X : \text{Gcob}(d) \to \text{cob}(d)$$  \hspace{1cm} (4)

**Proof.** The proof is straightforward, including its most intricate aspect, the fact that gluing of cobordisms is respected. \hfill $\square$

**Definition 4.** The functor $\mathcal{F}_X$ in proposition 3 is called the $d$-dimensional cover functor for the $G$-set $X$.

**Corollary 5.** Let $\text{tft} : \text{cob}(d) \to \text{Vect}_k$ be a topological field theory and let $X$ be a $G$-set. Then the composite functor

$$\text{tft}^X : \text{Gcob}(d) \xrightarrow{\mathcal{F}_X} \text{cob}(d) \xrightarrow{\text{tft}} \text{Vect}_k$$  \hspace{1cm} (5)

is a $d$-dimensional $G$-equivariant topological field theory in the sense of definition 2.
2.2 Covers of two-dimensional cobordisms

From now on, we specialize to dimension $d = 2$. We recall the following definition [BK00, BK01, KP08]:

**Definition 6.**

1. An extended surface is a compact oriented smooth two-dimensional manifold $M$, possibly with boundary, together with a choice of a marked point on each connected component of the boundary $\partial M$. The set of boundary components of $M$ is denoted by $A(M)$ and we write extended surfaces as $(M, \{e_a\}_{a \in A(M)})$. A morphism of extended surfaces is a smooth map that preserves marked points.

2. A $G$-cover of an extended surface $(M, \{e_a\}_{a \in A(M)})$ is a pair $(P \to M, \{p_a\}_{a \in A(M)})$, where $P \to M$ is a principal $G$-cover of $M$ and the $p_a$ are marked points in the fibre over $e_a$. A morphism of $G$-covers of extended surfaces is a smooth bundle map that preserves marked points.

The morphisms in $\text{Gcob}(2)$ are thus diffeomorphism classes of $G$-covers of extended surfaces. We also consider a category $\text{Ext}$ whose objects are extended surfaces and whose morphisms are orientation preserving diffeomorphisms of extended surfaces. Similarly, we have a $G$-equivariant version, a category $\text{GExt}$ with $G$-covers of extended surfaces as objects and orientation preserving diffeomorphisms of $G$-covers of extended surfaces as morphisms.

The cover functor $\mathcal{F}_X$ induces a functor $\text{GExt} \to \text{Ext}$ which we will also denote by $\mathcal{F}_X$. To see this, let $(E \to M, \{p_a\}_{a \in A(M)})$ be an object in $\text{GExt}$, i.e. a $G$-cover of an extended surface. By definition, $E$ is endowed with a marked point $p_a$ for each connected component of the boundary of $M$. This yields on the boundary of $\mathcal{F}_X(E \to M)$ the marked points $[x, p_a]$ for every element $x \in \mathcal{X}$ of the $G$-set and every boundary component $a$ of $M$. To turn $\mathcal{F}_X(E \to M)$ into an extended surface, we need to choose just a single point for every connected component of the boundary of $\mathcal{F}_X(E \to M)$. At this point, we use the auxiliary structure of an ordering on the $G$-set $\mathcal{X}$ to choose the point $[x, p_a]$ with the smallest value of $x \in \mathcal{X}$ in that boundary component.

We will now analyze covers of the following basic manifolds:

1. The vector spaces relevant for a $G$-equivariant topological field theory are given by evaluations of the TFT functor on $G$-covers of the circle $S^1$.

2. The multiplicative structure on the vector spaces underlying a topological field theory comes from the 3-punctured sphere, the so-called pair-of-pants. To set the stage for the discussion in section 4, we consider covers of the $n$-punctured sphere.

2.2.1 Covers of the circle

For any element $g \in G$, we introduce the principal $G$-bundle $P_g$ of $S^1$ with total space $P_g := \mathbb{R} \times G/(t + 2\pi, h) \sim (t, hg)$ and distinguished point $[0, 1_G]$. With respect to this point, the monodromy of the bundle is given by $g$. One easily checks that every principal $G$-bundle over $S^1$ is isomorphic to $P_g$ for some $g \in G$.

Given a finite $G$-set $\mathcal{X}$, we define for every $g \in G$ a $|\mathcal{X}|$-fold cover of $S^1$ with total space $E_g := \mathbb{R} \times \mathcal{X}/(t + 2\pi, x) \sim (t, g^{-1}x)$. The following lemma is straightforward:
Lemma 7. For any $g \in G$, the covers $E_g$ and $X \times_G P_g$ over $S^1$ are isomorphic.

As a closed one-dimensional manifold, the total space $E_g$ is a disjoint union of circles. We describe the connected components of $E_g$:

Lemma 8. For any element $g \in G$ there is a one-to-one correspondence between the connected components of the manifold $E_g$ and the orbits of $X$ under the action of the cyclic group $\langle g \rangle$.

We denote the set of orbits of $X$ under the action of the cyclic group $\langle g \rangle \subset G$ by $O_g$; let $b_g := |O_g|$ be the number of orbits.

The following lemma follows from an easy calculation, as well.

Lemma 9. The map

$$
E_g \rightarrow E_{hgh^{-1}}
$$

$$
[t, x] \mapsto [t, hx]
$$

(6)

is an isomorphism of covers of $S^1$. The induced map on the sets of connected components is given by the map

$$
O_g \rightarrow O_{hgh^{-1}}
$$

$$
o \mapsto ho.
$$

(7)

between the sets of orbits of cyclic groups.

2.2.2 Covers of the $n$-punctured sphere

Next, we investigate covers of the $n$-punctured sphere; to this end, we fix a standard model [BK00, BK01] of this manifold:

Definition 10. For every $n \in \mathbb{N}$ the standard sphere $S_n$ is the complex sphere $\mathbb{C}$ with standard orientation and with discs of radius $\frac{1}{3}$ centered around the first $n$ positive integers removed. As marked points on the boundary components of $S_n$, we choose $k - \frac{i}{3}$ for $k = 1, \ldots, n$.

We need $G$-covers of the standard sphere $S_n$; as standard models for these covers, we use the so-called standard blocks [Pri07]. To construct the standard blocks, we remove from $S_n$ the straight lines connecting the points $k + \frac{i}{3}$ to the point $\infty$. The resulting manifold $S_n \setminus \text{cuts}$ is contractible, hence it only has the trivial $G$-cover $((S_n \setminus \text{cuts}) \times G \rightarrow S_n)$. For any $n$-tuple $g_1, \ldots, g_n$ of elements in $G$ whose product is the neutral element, we obtain a $G$-cover of $S_n$ by gluing the $j$-th cut in $(S_n \setminus \text{cuts}) \times G$ with the action of $g_j \in G$. The following picture shows the situation with a view in the direction of the negative imaginary axis:
We finally have to specify marked points on the boundary components of \(((S_n \setminus \text{cuts}) \times G) / \text{gluing} \). To this end, we choose another \(n\)-tuple \(h_1, \ldots, h_n\) of elements in \(G\) and take as marked points \([k - \frac{1}{3}, h_k]\) for \(k = 1, \ldots, n\). We write \(S_n(g_1, \ldots, g_n; h_1, \ldots, h_n)\) for these marked \(G\)-covers over \(S_n\). In fact, any \(G\)-cover over an \(n\)-punctured sphere is diffeomorphic to one \(G\)-cover of the form \((S_n(g_1, \ldots, g_n; h_1, \ldots, h_n) \to S_n)\). These covers have monodromies \(h_i g_i^{-1} h_i^{-1}\) around the \(i\)-th boundary circle of \(S_n\).

In the definition of standard blocks, the orientation of a boundary component depends on whether we consider the component as ingoing or outgoing. For example, for the pair-of-pants with one outgoing circle the third circle is given a clockwise orientation. The cover of \(S_3\) that is most important in the following discussion is \(S_3(g_1, g_2, (g_1 g_2)^{-1}; 1, 1, 1)\). When orienting the third boundary component as outgoing, this cover has monodromies \(g_1^{-1}\) and \(g_2^{-1}\) at the ingoing components and \((g_1 g_2)^{-1}\) at the outgoing component. We sometimes abbreviate \(E_{g_1; g_2} := \mathcal{F}_X(S_3(g_1, g_2, (g_1 g_2)^{-1}; 1, 1, 1) \to S_3)\).

To analyze how the structure of \(\mathcal{F}_X(S_3(g_1, g_2, (g_1 g_2)^{-1}; 1, 1, 1) \to S_3)\) depends on the group elements \(g_1\) and \(g_2\), we introduce the following paths in the base manifold, the pair-of-pants \(S_3\):

- \(\gamma_{g_1}\), the path with winding number one around the ingoing boundary circle which has monodromy \(g_1^{-1}\).
- \(\gamma_{g_2}\) the path winding once around the ingoing boundary circle which has monodromy \(g_2^{-1}\).
- \(\gamma_{g_1 g_2}\) the path winding once around the outgoing boundary circle which has monodromy \((g_1 g_2)^{-1}\).
- \(\alpha, \beta\) open paths connecting the base points of the ingoing circles with the base point of the outgoing circle.
The following lemma describes the connected components of $E_{g_1;g_2}$:

**Lemma 11.**

(i) There is a natural bijection between the connected components of

$$\mathcal{F}_X(S_3(g_1, g_2, (g_1 g_2)^{-1}; 1, 1, 1) \to S_3) = E_{g_1;g_2}$$

and orbits of the $G$-set $X$ under the action of the subgroup $\langle g_1, g_2 \rangle \subset G$ of $G$ generated by the elements $g_1$ and $g_2$.

(ii) By lemma [2] the restriction of $E_{g_1;g_2}$ to the boundary with monodromy $g_1^{-1}$ is diffeomorphic to $E_{g_1}$, and similarly for the other boundaries. Let $o$ be a $\langle g_1, g_2 \rangle$-orbit of $X$ and write $E^o_{g_1;g_2}$ for the connected component of $E_{g_1;g_2}$ corresponding to the orbit $o$. The boundary components of $E^o_{g_1;g_2}$ correspond to precisely those orbits of the cyclic subgroups $\langle g_1 \rangle$, $\langle g_2 \rangle$ and $\langle g_1 g_2 \rangle$ that are contained in the orbit $o$ of the group $\langle g_1, g_2 \rangle$.

(iii) In particular, the number of sheets of the cover $E^o_{g_1;g_2} \to S_3$ is $|o|$.

This is seen by choosing appropriate lifts of the paths $\gamma_{g_1}, \gamma_{g_2}, \gamma_{g_1 g_2}, \alpha$ and $\beta$. We leave the details to the reader as an exercise. We write $b^o_{g_1}$ for the number of $\langle g_1 \rangle$-orbits that are contained in $o$ and similarly for $g_2$ and $g_1 g_2$. We can now describe the topology of the connected components of the cover:

**Lemma 12.** Let $o$ be a $\langle g_1, g_2 \rangle$-orbit on $X$. Then the component $E^o_{g_1;g_2}$ of

$$\mathcal{F}_X(S_3(g_1, g_2, (g_1 g_2)^{-1}; 1, 1, 1) \to S_3)$$

is a surface of genus

$$\frac{2 - b^o_{g_1} - b^o_{g_2} - b^o_{g_1 g_2} + |o|}{2}$$

(10)

**Proof.** We have an unramified $|o|$-fold cover of the pair-of-pants $S_3$ with Euler characteristic $\chi(S_3) = 2 - 3 = -1$. Hence, by the theorem of Riemann Hurwitz, $\chi(E^o_{g_1;g_2}) = -|o|$. As described in lemma [11] the number of boundary components of $E^o_{g_1;g_2}$ is $b^o_{g_1} + b^o_{g_2} + b^o_{g_1 g_2}$. This implies formula (10) for the genus of $E^o_{g_1;g_2}$. □
Finally, we consider the special case of the cylinder \( S_2 \) with principal bundle \( S_2(g, g^{-1}; 1, h) \), where the first boundary is oriented as ingoing and the second boundary as outgoing. Then \( S_2(g, g^{-1}; 1, h) \) has monodromies \( g^{-1} \) around the ingoing boundary and \( hg^{-1}h^{-1} \) around the outgoing boundary. We identify the ingoing boundary of \( (S_2(g, g^{-1}; 1, h) \to S_2) \) with \( (P_g \to S^1_g, [0, 1_G]) \) and the outgoing boundary with \( (P_{g^{-1}} \to S^1, [0, h]) \). The latter is isomorphic to \( (P_{hg^{-1}h^{-1}} \to S^1, [0, 1_G]) \) under the map \([t, k] \mapsto [t, kh^{-1}] \) of bundles. Hence the boundaries of \( F_X(S_2(g, g^{-1}; 1, h) \to S_2) \) are isomorphic to \( E_{g^{-1}} \) and \( E_{hg^{-1}h^{-1}} \) respectively.

**Lemma 13.** The manifold \( F_X(S_2(g, g^{-1}; 1, h) \to S_2) \) is a disjoint union of cylinders. These cylinders interpolate between the connected component of \( E_{g^{-1}} \) corresponding to the \( (g) \)-orbit \( o \) of \( X \) and the connected component of \( E_{hg^{-1}h^{-1}} \) that corresponds to the \( (hg^{-1}h^{-1}) \)-orbit \( ho \).

**Proof.** Consider the following paths in the base cylinder \( S_2 \):

- \( \gamma_g \) the path winding once around the ingoing boundary circle which has monodromy \( g^{-1} \).
- \( \gamma_{hg^{-1}h^{-1}} \) the path winding once around the outgoing boundary circle which has monodromy \( hg^{-1}h^{-1} \).
- \( \alpha \) a path connecting the base points of both boundary circles.

Let \( x \in o \) be an element of the \( (g) \)-orbit \( o \). The point \([0, x] \) in \( E_{g^{-1}} \) is connected to the points \([0, g^ix] \) of \( E_{g^{-1}} \) by repeated lifts of \( \gamma_g \), similar to the proof of lemma 11. By lifting \( \alpha \) to a path \( \hat{\alpha}_{g^ix} \) in \( F_X(S_2(g, g^{-1}; 1, h) \to S_2) \) with initial point \( \hat{\alpha}_{g^ix}(0) = [0, g^ix] \in E_{g^{-1}} \), these points are connected to the points \([0, hg^ix] \in E_{hg^{-1}h^{-1}} \), where the map from lemma 9 is used to identify the outgoing boundary of \( F_X(S_2(g, g^{-1}; 1, h) \to S_2) \) with \( E_{hg^{-1}h^{-1}} \). These again are connected by lifts of \( \gamma_{hg^{-1}h^{-1}} \) only connected to points of the same form. By lemma 8, the connected component of \( E_{hg^{-1}h^{-1}} \) containing these points, corresponds to the \( (hg^{-1}h^{-1}) \)-orbit \( ho \).

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**2.3 Equivariant Frobenius algebras from equivariant topological field theories**

In this subsection, we review the equivariant generalization of the correspondence \([\text{Koc04}]\) between two-dimensional topological field theories and commutative Frobenius algebras to set the stage for the discussion of \( G \)-equivariant modular functors. We then present a decategorified version of the main construction of this paper.

**2.3.1 From \( G \)-equivariant TFTs to \( G \)-Frobenius algebras**

We start by recalling definitions from \([\text{MS06}, \text{Tur99}])

**Definition 14.** A \( G \)-Frobenius algebra (or crossed \( G \) Frobenius algebra or Turaev algebra) is a \( G \)-graded associative unital algebra \( A = \bigoplus_{g \in G} A_g \) together with a group homomorphism \( \alpha : G \to \text{Aut}(A) \) such that
1. The $G$ action is compatible with the $G$-grading via the adjoint action of $G$ on itself, $\alpha_h : A_g \to A_{gh^{-1}}$.

2. The restriction of $\alpha_h$ to $A_h$ is the identity.

3. $A$ is twisted commutative: For all $a \in A_g, b \in A_h$ we have $\alpha_h(a)b = ba$.

4. There is a $G$-invariant trace $\epsilon : A_1 \to k$ such that the induced pairing $A_g \otimes A_{g^{-1}} \to A_1 \to k$ is non-degenerate.

5. For all $g,h \in G$ we have
   \[ \sum \alpha_h(\xi_i)\xi^i = \sum \eta_i\alpha_g(\eta^i) \in A_{gh^{-1}g^{-1}} \] (11)
   where $(\xi_i, \xi^i)$ and $(\eta_i, \eta^i)$ are pairs of dual bases of $A_g, A_{g^{-1}}$ and $A_h, A_{h^{-1}}$ respectively.

We call $A_g$ the $g$-graded component and $A_1$ the neutral component. A morphism of $G$-Frobenius algebras is a morphism of unital algebras that respects the trace, the $G$-action and the grading. One verifies that all morphisms of $G$-Frobenius algebras are isomorphisms.

The following theorem [MS06, Tur99] holds:

**Theorem 15.** The symmetric tensor categories of $G$-TFTs and $G$-Frobenius algebras are equivalent.

Instead of reviewing the complete proof (see [Tur99] or [MS06] with slightly different conventions), we recall how to extract the data of a $G$-Frobenius algebra from a $G$-TFT $\text{tft}^G$.

- For $g \in G$, the $g$-graded component is defined as
  \[ A_g := \text{tft}^G(P_{g^{-1}} \to S^1, [0, 1_G]) \]
  where $P_g$ is the principal $G$-bundle on $S^1$ introduced subsection 2.2.1

- For any pair of group elements $g, h \in G$ consider the standard three-point block
  \( (S_3(g,h,(gh)^{-1}; 1,1,1) \to S_3) \)
  as a cobordism
  \[ (P_{g^{-1}} \to S^1, [0, 1_G]) \sqcup (P_{h^{-1}} \to S^1, [0, 1_G]) \to (P_{(gh)^{-1}} \to S^1, [0, 1_G]) \] (12)
  Its image under the functor $\text{tft}^G$ is a morphism $m_{g,h} : A_g \otimes A_h \to A_{gh}$ which yields an associative product $\sum_{g,h \in G} m_{g,h}$ on the $G$-graded vector space $A = \bigoplus_{g \in G} A_g$.

- Unit $\eta$ and counit $\epsilon$ of $A$ are obtained from cobordisms with the topology of a disc $D$. Since the disc is contractible, it only admits the trivial cover $D \times G \to D$ which restricts unit and counit to be trivial on $A_g$ for $g \neq 1$.

The unit $\eta$ of $A$ is obtained as $\text{tft}^G(D \times G \to D)$ with the disc viewed as a cobordism from the empty set to $(P_1 \to S^1, [0, 1_G])$. Hence $\eta : k \to A_1$. Similarly we define the counit as $\text{tft}^G(D \times G \to D)$, where this time the disc is seen as a cobordism from $(P_1 \to S^1, [0, 1_G])$ to the empty set.
We finally obtain the action of $h \in G$ on $A_g$ from the cover

$$S_2(g, g^{-1}; 1, h) \rightarrow S_2$$

of the cylinder where the marked point over the outgoing boundary has been shifted by $h \in G$. The discussion in subsection 2.2.2 shows that the boundaries are isomorphic to the bundles $(P_{g^{-1}} \rightarrow S^1, [0, 1_G])$ and $(P_{hg^{-1}h^{-1}} \rightarrow S^1, [0, 1_G])$ respectively, hence $\alpha_h$ has the correct domain and target.

We refer to [MS06] for a detailed proof that this endows $A = \bigoplus_{g \in G} A_g$ with the structure of a $G$-Frobenius algebra.

2.3.2 Permutation equivariant Frobenius algebras

We now describe a construction that can be seen as the decategorified version of the main construction of this article. Suppose we are given a finite ordered $G$-set $X$ and a commutative Frobenius algebra $(R, \eta, m, \epsilon, \Delta)$, playing the role of a decategorification of a modular tensor category. A Frobenius algebra structure on an associative unital algebra $(R, m, \eta)$ can be equivalently described by a linear form $\epsilon$ such that the induced bilinear pairing on $R$ is non-degenerate or by a coalgebra structure $(R, \Delta, \epsilon)$ such that $\Delta$ is a morphism of $R$-bimodules. Here, we prefer the latter description. We want to to construct a $G$-Frobenius algebra $A$ with neutral component $A_1 = \otimes_X R$, where the $G$-action on $A_1$ is induced by the $G$-action on $X$.

We start by constructing the underlying $G$-graded vector space:

- Composing the 2-dimensional topological field theory associated to $R$

$$tft_R : \text{cob}(2) \rightarrow \text{Vect}_k. \tag{13}$$

with the tensor functor $F_X$, we obtain by corollary 5 a 2-dimensional $G$-equivariant TFT,

$$tft^X_R := tft_R \circ F_X.$$

- To describe the $G$-Frobenius algebra $A = \bigoplus_{g \in G} A_g$ that corresponds to this $G$-TFT, we first describe the vector spaces $A_g$ for $g \in G$. By lemma 7 we have $F_X(P_{g^{-1}} \rightarrow S^1, [0, 1_G]) \cong E_{g^{-1}}$; since the functor $tft_R$ is monoidal, we only need to know the number of connected components of $E_{g^{-1}}$ which by lemma 8 is the number $b_g$ of orbits of the cyclic group $\langle g \rangle$ on $X$. Thus, as a vector space,

$$A_g \cong tft_R(E_{g^{-1}}) \cong tft_R(\sqcup_{o \in O_g} S^1) \cong R^{\otimes b_g} \cong \bigotimes_{o \in O_g} R_o, \tag{14}$$

with $R_o \cong R$ as a vector space. Hence an element of $A_g$ is a linear combination of elements of the form $r_{o_1} \otimes \cdots \otimes r_{o_{b_g}}$ with $r_{o_i} \in R$.

The product morphisms $m_{g_1, g_2} : A_{g_1} \otimes A_{g_2} \rightarrow A_{g_1g_2}$ are induced by the covers

$$(S_3(g_1, g_2, (g_1g_2)^{-1}; 1, 1, 1) \rightarrow S_3)$$
of the three-punctured sphere. On these covers the cover functor $\mathcal{F}_X$ gives the surfaces $E_{g_1:g_2}$.

By lemma 11 each $\langle g_1, g_2 \rangle$-orbit $o$ on $X$ gives a connected component $E_{g_1:g_2}^o$ of $E_{g_1:g_2}$ and thus a contribution to the product morphism. We describe these contributions separately and write the elements in $A_{g_1}$ as products of elements $r_o' \in R$ and the elements in $A_{g_2}$ as products of elements $s_o'' \in R$.

- First multiply all elements $r_o'$ and $s_o''$ for all $\langle g_1 \rangle$-orbits $o'$ and the $\langle g_2 \rangle$-orbits $o''$ that are contained in the $\langle g_1, g_2 \rangle$-orbit $o$. No choices are involved, because the Frobenius algebra $R$ is commutative.

- By lemma 12, the genus of $E_{g_1:g_2}^o$ equals $p := \frac{2 - b_{g_1}^o - b_{g_2}^o - b_{g_1:g_2}^o + |o|}{2}$. In a second step, apply the endomorphism $(m \circ \Delta)^p$ of $R$ to the product of the previous step.

- Let $o_1, \ldots, o_k$ be those orbits of the cyclic group $\langle g_1 \cdot g_2 \rangle$ that are contained in the $\langle g_1, g_2 \rangle$-orbit $o$. To the element of $R$ obtained in the previous step, apply the $k$-fold coproduct of $R$, so we get an element in $R_{o_1} \otimes \cdots \otimes R_{o_k}$. This element is well defined by coassociativity of $R$.

- Map the factors $R_{o_i}$ of the previous step to the corresponding factors of $A_{g_1:g_2}$. No choices are involved, since the coproduct on the Frobenius algebra $R$ is cocommutative.

This provides the prescription for the product on the $G$-Frobenius algebra. As explained in subsection 2.3.1, the unit of $A$ is obtained as the evaluation of the functor $tft^X_R$ on the disc with the trivial $G$-cover. The cover functor maps $(D \times G \to D)$ to the disjoint union $\bigsqcup_{x \in X} D$ of $|X|$-many discs, hence the unit of the $G$-Frobenius algebra $A$ is just the tensor product of the units $\bigotimes_{x \in X} \eta : k \to A_1 \cong \bigotimes_{x \in X} R$ of $R$. Similarly we find that the counit of $A$ is given by tensor product of the counits of $R$.

From lemma 13 we deduce that the $G$-action $\alpha_h : A_g \to A_{hgh^{-1}}$ is given by the permutation of factors: The connected components of the cover $\mathcal{F}_X(S_2(g, g^{-1}; 1, h))$ of the cylinder are again cylinders. For any $\langle g \rangle$-orbit $o$ of $X$ the factor $R_o$ of $A_g$ is thus mapped to the factor $R_{ho}$ of $A_{hgh^{-1}}$.

### 3 G-modular functors and G-equivariant ribbon categories

Let $\mathcal{C}$ be a modular category over a field $k$; we assume $\mathcal{C}$ to be strict. Let $X$ be a finite ordered $G$-set. The goal of this section is to construct for any pair $(\mathcal{C}, X)$ a $G$-equivariant modular functor. Our construction is based on the decategorified version of the construction in section 2.3.

#### 3.1 Definitions and notation

We start by recalling some definitions [KP08]:

**Definition 16.** A $G$-equivariant category is an abelian category $C^G$ with the following structure:

- A decomposition $C^G \cong \bigoplus_{g \in G} C_g^G$ into full abelian subcategories.
\begin{itemize}
  \item A $G$-action covering the adjoint action of $G$ on itself.

In more detail, we have for any group element $g \in G$ a functor $R_g : \mathcal{C}^G \to \mathcal{C}^G$ and for any pair $g, h \in G$ of group elements isomorphisms $\alpha_{g,h} : R_g \circ R_h \Rightarrow R_{gh}$ such that $R_1 = \text{Id}_{\mathcal{C}^G}$, $R_g(\mathcal{C}^G) \subset \mathcal{C}^G_{\text{gh}^{-1}}$. The isomorphisms $\alpha_{g,h}$ are required to satisfy an associativity condition.

As a shorthand, we introduce the notation $gV \equiv R_g(V)$ for $g \in G$ and $V$ in $\mathcal{C}^G$.

\textbf{Definition 17.} Let $\mathcal{C}^G$ be a $G$-equivariant category. We assume from now on that $\mathcal{C}^G$ is enriched over the category of finite-dimensional $k$-vector spaces. We denote by $\boxtimes$ the Deligne tensor product of $k$-linear categories. An object $\mathcal{R} \in \mathcal{C}^G \boxtimes \mathcal{C}^G$ is called a \textit{gluing object} if

\begin{itemize}
  \item $\mathcal{R}$ is of vanishing total degree, $\mathcal{R} \in \bigoplus_h \mathcal{C}^G_h \boxtimes \mathcal{C}^G_{\text{h}^{-1}}$.
  \item $\mathcal{R}$ is symmetric, i.e. $\mathcal{R} \cong \mathcal{R}^{op}$. Here $\mathcal{R}^{op}$ is obtained by the permutation action on the two factors.
  \item $\mathcal{R}$ is $G$-invariant: For every group element $g \in G$ there is an isomorphism $(R_g \boxtimes R_g)(\mathcal{R}) \cong \mathcal{R}$.
  \item These isomorphisms are compatible with each other.
\end{itemize}

We write $\mathcal{R}_h$ for the component of $\mathcal{R}$ in $\mathcal{C}^G_h \boxtimes \mathcal{C}^G_{\text{h}^{-1}}$; sometimes we use the Sweedler-like notation $\mathcal{R} = \mathcal{R}^{(1)} \boxtimes \mathcal{R}^{(2)}$.

We are now in a position to give the definition \cite{KP08} of a $G$-modular functor. To make the notation less cumbersome, we sometimes use the abbreviation $P \equiv (P \to M, \{p_a\}_{a \in A(M)})$ for $G$-covers.

\textbf{Definition 18.} Let $\mathcal{C}^G$ a $G$-equivariant category enriched over the category of finite-dimensional $k$-vector spaces and $\mathcal{R}$ be a gluing object for $\mathcal{C}^G$. A $\mathcal{C}^G$-\textit{extended} $G$-equivariant modular functor consists of the following data:

\begin{enumerate}
  \item Functors for $G$-covers:
    For every $G$-cover $(P \to M, \{p_a\}_{a \in A(M)})$ of an extended surface a functor
    \begin{equation}
    \tau^G(P \to M, \{p_a\}_{a \in A(M)}) : \bigotimes_{a \in A(M)} \mathcal{C}^G_{m_a^{-1}(M)} \to \text{Vect},
    \end{equation}
    where $m_a$ is the monodromy of $P$ around the $a$-th boundary component of $M$. We will often write $\tau^G(P; \{V_a\})$ for the value of the functor on a family $\{V_a\}$ of suitable objects.

  \item Functorial isomorphisms for morphisms of $G$-covers:
    For every isomorphism $f : (P \to M, \{p_a\}_{a \in A(M)}) \to (P' \to M', \{p'_a\}_{a \in A(M')})$ of extended surfaces a functorial isomorphism
    \begin{equation}
    f_* : \tau^G(P \to M, \{p_a\}_{a \in A(M)}) \to \tau^G(P' \to M', \{p'_a\}_{a \in A(M')})
    \end{equation}
    that depends only on the isotopy class of $f$.

  \item Isomorphisms $\tau^G(\emptyset) \cong k$ and $\tau^G(P \sqcup P') \cong \tau^G(P) \otimes_k \tau^G(P')$.
\end{enumerate}
4. Functorial gluing isomorphisms:

Let \((P \to M, \{p_a\}_{a \in A(M)})\) be a \(G\)-cover of an extended surface and let \(\alpha, \beta \in A(M), \alpha \neq \beta\) such that the monodromies are inverse, \(m_\alpha = m_\beta^{-1}\). Then gluing of \(P\) along the boundary components over \(\alpha\) and \(\beta\) is well defined. We have functorial gluing isomorphisms

\[
G_{\alpha, \beta} : \tau^G(P; \{V_a\}, R_{\alpha, \beta}) \sim \tau^G(\sqcup_{\alpha, \beta} P; \{V_a\})
\]

where \(R_{\alpha, \beta}\) indicates that the summand \(R_{m_\alpha}\) of \(R\) is assigned to the boundary components \(\alpha\) and \(\beta\) respectively. This is well defined by symmetry of \(R\). Here \(\sqcup_{\alpha, \beta} M\) is the surface with the boundary components \(\alpha\) and \(\beta\) glued. \(\sqcup_{\alpha, \beta} P\) denotes the \(G\)-cover of \(\sqcup_{\alpha, \beta} M\) that is obtained by gluing the corresponding boundary components over \(\alpha\) and \(\beta\).

5. Equivariance under the \(G\)-action:

For any \(G\)-cover \((P \to M, \{p_a\}_{a \in A(M)})\) and any tuple of group elements \(g = (g_a)_{a \in A(M)} \in G^{A(M)}\) we have functorial isomorphisms

\[
T_g : \tau^G(P \to M, \{p_a\}_{a \in A(M)}; \{V_a\}) \sim \tau^G(P \to M, \{g_a p_a\}_{a \in A(M)}; \{g_a V_a\})
\]

These data are subject to the following conditions:

- \((f \circ g)_* = f_* \circ g_*\) and \(id_* = id\).
- All morphisms are functorial in \((P \to M, \{p_a\}_{a \in A(M)})\) and compatible with each other.
- When identifying \(\mathcal{R} \cong \mathcal{R}^{op}\) we have \(G_{\alpha, \beta} = G_{\beta, \alpha}\).
- Normalization: \(\tau^G(S^2 \times G \to S^2) \cong \mathbb{k}\)

Remark 19. Specializing to the trivial group, \(G = \{1\}\) and suppressing the morphisms \(T_g\) from \([17]\) implementing equivariance, we recover the usual definition of a modular functor, see e.g. \([\text{BK01}]\).

For our purposes, the notion of a \(G\)-equivariant monoidal structure \([\text{Tur00} \; \text{Kir04} \; \text{KP08}]\) will be important.

Definition 20.

1. A \(G\)-equivariant monoidal category is a semisimple \(G\)-equivariant category \(\mathcal{C}^G\) with a monoidal structure that is compatible with the grading, i.e. \(X \otimes Y \in \mathcal{C}^G_{gh}\) for \(X \in \mathcal{C}^G_g, Y \in \mathcal{C}^G_h\) and for which the functors \(R_g\) implementing equivariance are endowed with the structure of tensor functors.

2. A \(G\)-equivariant monoidal category is called braided, if for any pair of objects \(X \in \mathcal{C}^G_g, Y \in \mathcal{C}^G_h\) there are isomorphisms

\[
C_{X, Y} : X \otimes Y \to g Y \otimes X
\]

that satisfy two \(G\)-equivariant hexagon axioms.
3. An object $V$ in a $G$-equivariant monoidal category $\mathcal{C}^G$ has a \textit{weak dual} if there is an object $V^* \in \mathcal{C}^G$ representing the functor $\text{Hom}_{\mathcal{C}^G}(1, V \otimes ?)$. This amounts to the existence of functorial isomorphisms $\text{Hom}_{\mathcal{C}^G}(1, V \otimes T) \cong \text{Hom}_{\mathcal{C}^G}(V^*, T)$ for all $T \in \mathcal{C}^G$.

4. A $G$-equivariant monoidal category is called \textit{weakly rigid} if every object has a weak dual. It is called \textit{rigid}, if there are compatible duality morphisms.

5. A $G$-equivariant monoidal category is called \textit{weakly ribbon} if it is weakly rigid, braided and for every object $V \in \mathcal{C}_g^G$ there is a functorial isomorphism $\Theta_V: V \to gV$, satisfying certain coherence conditions spelled out in [Kir04, Section 2]. A weakly ribbon category is called a \textit{ribbon category}, if it is rigid rather than only weakly rigid.

The discussion in the following subsection 3.2 strongly uses the following theorem from [KP08]:

\textbf{Theorem 21.} A genus 0 $\mathcal{C}^G$-extended $G$-modular functor is equivalent to the structure of a $G$-equivariant weakly ribbon category on $\mathcal{C}^G$.

In [KP08] no explicit prescription is given how to obtain from a $\mathcal{C}^G$-extended $G$-modular functor $\tau^G$ the structure morphisms endowing the equivariance functors $R_g: \mathcal{C}^G \to \mathcal{C}^G$ with the structure of tensor functors. We will need the explicit form of these structure morphisms and therefore explain them in some detail.

In the definition in subsection 2.2.2 of standard blocks on the $n$-punctured sphere $S_n$ as a quotient of $S_n \setminus \text{cuts} \times G$ a point of the total space $S_n(g_1, \ldots, g_n; h_1, \ldots, h_n)$ is an equivalence class $[z, x]$ with $z \in S_n \setminus \text{cuts}$ and $x \in G$. For every group element $k \in G$ the map $[z, x] \mapsto [z, xk]$ induces an isomorphism of $G$-covers

$$\tilde{k}: S_n(g_1, \ldots, g_n; h_1, \ldots, h_n) \to S_n(k^{-1}g_1k, \ldots, k^{-1}g_nk; h_1k, \ldots, h_nk)$$

(18)

The corresponding natural transformations enter in the construction of the tensoriality constraints. To construct these morphisms, let $h \in G$ and $A \in \mathcal{C}^G_{g_1}$ and $B \in \mathcal{C}^G_{g_2}$ be objects of $\mathcal{C}^G$. The main step is to construct a natural isomorphism between the functors $\mathcal{C} \to \text{Vect}_k$ given by

$$X \mapsto \tau^G(S_2(hg_2^{-1}g_1^{-1}h^{-1}, hg_1g_2h^{-1}; 1, 1); X, h(A \otimes B))$$

(19)

and

$$X \mapsto \tau^G(S_2(hg_2^{-1}g_1^{-1}h^{-1}, hg_1g_2h^{-1}; 1, 1); X, hA \otimes hB)$$

(20)

Since our categories are, by assumption, semi-simple with finitely many isomorphism classes of simple objects, every functor is representable. In [KP08] the objects $h(A \otimes B)$ and $hA \otimes hB$ have been introduced as the objects representing the functors (19) and (20) respectively. Thus, by the Yoneda lemma the image of the identity for $X = h(A \otimes B)$ under the natural isomorphism gives an isomorphism

$$\varphi^h_{A,B}: h(A \otimes B) \to hA \otimes hB.$$
The natural transformation we use is given by

\[
\begin{align*}
T_{(1,h^{-1})} & \mapsto \tau^G(S_2(hg_2^{-1}g_1^{-1}h^{-1}, hg_1g_2h^{-1}; 1, 1); X, h(A \otimes B)) \\
(\hat{h}^{-1}) & \mapsto \tau^G(S_2(g_2^{-1}g_1^{-1}, g_1g_2; h, 1); X, A \otimes B) \\
G^{-1} & \mapsto \tau^G(S_2(g_2^{-1}g_1^{-1}, g_1g_2; h, 1); X, \mathcal{R}^{(1)}(1) \otimes \kappa \tau^G(S_2(g_2^{-1}g_1^{-1}, g_1g_2; 1, 1); \mathcal{R}^{(2)}(1); A \otimes B) \\
def & \mapsto \tau^G(S_2(g_2^{-1}g_1^{-1}, g_1g_2; h, 1); X, \mathcal{R}^{(1)}(1) \otimes \kappa \tau^G(S_3(g_2^{-1}g_1^{-1}, g_1, g_2; 1, 1); \mathcal{R}^{(2)}(1), A, B) \\
G \mapsto \tau^G(S_3(g_2^{-1}g_1^{-1}, g_1, g_2; h, 1, 1); X, A, B) \\
(\hat{h}^{-1}) & \mapsto \tau^G(S_3(hg_2^{-1}g_1^{-1}h^{-1}, hg_1h^{-1}, hg_2h^{-1}; 1, 1, h^{-1}h^{-1}); X, A, B) \\
def & \mapsto \tau^G(S_3(hg_2^{-1}g_1^{-1}h^{-1}, hg_1h^{-1}, hg_2h^{-1}; 1, 1, 1); X, hA, hB) \\
def & \mapsto \tau^G(S_3(hg_2^{-1}g_1^{-1}h^{-1}, hg_1h^{-1}, hg_2h^{-1}; 1, 1); X, hA \otimes hB)
\end{align*}
\]

for any \( X \) in \( \mathcal{C}^G_{hg_2^{-1}g_1^{-1}h^{-1}} \). The idea in the definition is to use the equivariance isomorphisms defined in equation \([17]\) to shift the \( G \)-action from objects in the category to geometric quantities. Then a factorization is applied to be able to use the definition of the tensor product and finally, the \( G \)-action is shifted again to objects.

The following observation follows from the compatibility of all occurring morphisms and the definition of the associativity constraints in \( \mathcal{C} \).

**Lemma 22.**

The morphisms \( \varphi^h_{A,B} \) endow the functor \( R_h \) with the structure of a monoidal functor.

Next, we adapt the discussion of [BK01, prop. 5.3.13] of conditions ensuring that a weakly ribbon category is a ribbon category to the \( G \)-equivariant case. Suppose \( \mathcal{C}^G \) is a weakly rigid \( G \)-equivariant category in which biduals of objects can be functorially identified with the objects, \((V^*)^* \cong V\) for all \( V \) in \( \mathcal{C}^G \). This condition is fulfilled for all categories with tensor structure obtained from a modular functor. The image of the identity on \( V^* \) under the functorial isomorphisms of definition \([20, 3]\) provides us with a morphism \( i_V : 1 \to V \otimes V^* \) for any object \( V \in \mathcal{C}^G \). Consider for a simple object \( V \) of \( \mathcal{C}^G \) the morphism

\[
\alpha^{-1}_{V^*,V,V^*} \circ (\text{id}_{V^*} \otimes i_V) : V^* \to (V^* \otimes V) \otimes V^*
\]

constructed from \( i_V \) and the associativity constraint \( \alpha \) of \( \mathcal{C}^G \). Since \( \mathcal{C}^G \) is semisimple, we can decompose \( V^* \otimes V \cong \bigoplus_j V_j \) into a direct sum of simple objects \( V_j \). Multiplicities can occur, but since \( \dim_x \text{Hom}_{\mathcal{C}^G}(1, V^* \otimes V) = 1 \) we can decompose the morphism

\[
\alpha^{-1}_{V^*,V,V^*} \circ (\text{id}_{V^*} \otimes i_V) = a_V \otimes \text{id}_{V^*} + \sum_j \psi_j
\]

into a morphism \( a_V : 1 \to V^* \otimes V \) in the one-dimensional vector space \( \text{Hom}(1, V^* \otimes V) \) and certain morphisms \( \psi_j \). By the same arguments as in [BK01], we have

**Proposition 23.** The category \( \mathcal{C}^G \) is rigid, if and only if \( a_V \neq 0 \) for all simple objects \( V \).


3.2 \textit{G}-equivariant modular functors from the cover functor

We fix a finite ordered \( G \)-set \( \mathcal{X} \) and wish to use the 2-dimensional cover functor

\[
\mathcal{F}_X : \text{GExt} \rightarrow \text{Ext}
\]

on extended surfaces to associate to every modular tensor category \( \mathcal{C} \) a \( \text{G} \)-equivariant modular functor.

We fix a modular tensor category \( \mathcal{C} \) and choose representatives \((U_i)_{i \in \mathcal{I}}\) for the isomorphism classes of simple objects of \( \mathcal{C} \) and denote by \( \theta_i \in \mathbb{k} \) the eigenvalue of the twist on \( U_i \) and by \( d_i \) the dimension of \( U_i \). As usual, we introduce the scalars \( p^\pm := \sum_{i \in \mathcal{I}} \theta_i^\pm d_i^2 \) and assume that \( p^+ = p^- \).

Finally, we introduce

\[
D = \sqrt{p^+ p^-} = \sqrt{\sum_i (d_i^2)} .
\]

With this assumption, the modular category \( \mathcal{C} \) defines (see e.g. [Tur94, BK01]) a \( \mathcal{C} \)-extended modular functor \( \tau \) which acts on objects as

\[
\tau(S_n; V_1, \ldots, V_n) := \text{Hom}_\mathcal{C}(1, V_1 \otimes \cdots \otimes V_n) .
\]

To define a \( \text{G} \)-modular functor, we need a \( \text{G} \)-equivariant category as input. Given \((\mathcal{C}, \mathcal{X})\), we define a \( \text{G} \)-equivariant category by transferring the results of subsection 2.1 to categorical structures.

The first part of data required to define a \( \text{G} \)-modular functor is a \( \text{G} \)-equivariant category.

**Definition 24.** Let \( \text{G} \) be a finite group, \( \mathcal{X} \) a finite \( \text{G} \)-set and \( \mathcal{C} \) a modular tensor category. Then the following data define a \( \text{G} \)-equivariant category \( \mathcal{C}^\mathcal{X} \):

1. For \( g \in \text{G} \) set

\[
\mathcal{C}_g^\mathcal{X} := \mathcal{C}^{b_g} \equiv \mathcal{C}_{o_1} \boxtimes \cdots \boxtimes \mathcal{C}_{o_{b_g}} ,
\]

where \( b_g \) is the number of \( \langle g \rangle \)-orbits \( o_i \) of \( \mathcal{X} \). Moreover, \( \mathcal{C}_{o_i} \cong \mathcal{C} \) as abelian categories.

2. For \( g, h \in \text{G} \) define a functor \( R_h : \mathcal{C}_g^\mathcal{X} \rightarrow \mathcal{C}_{gh^{-1}}^\mathcal{X} \) by permutation of factors mapping the factor \( \mathcal{C}_o \) of \( \mathcal{C}_g^\mathcal{X} \) to the factor \( \mathcal{C}_{ho} \) of \( \mathcal{C}_{gh^{-1}}^\mathcal{X} \).

The equivariance functors obey the strict identities \( R_h \circ R_k = R_{hk} \) for all \( h, k \in \text{G} \) which allows us to choose trivial composition constraints, \( \alpha_{h,k} = \text{id} \) for all \( h, k \in \text{G} \).

The \( \text{G} \)-equivariant category \( \mathcal{C}^\mathcal{X} \) is semisimple and has only finitely many isomorphism classes of simple objects. Representatives of the isomorphism classes of simple objects in \( \mathcal{C}_g^\mathcal{X} \) are \( U_{i_1} \times \cdots \times U_{i_{b_g}} \), where the \( U_i \) are representatives of the isomorphism classes of simple objects of \( \mathcal{C} \).

This ansatz agrees with the description of simple objects in \( \mathbb{Z}/2 \)-permutation orbifolds found in [BHS98] and the analysis of more general permutation orbifolds in [Ban98]. We briefly explain the simplest case of \( \mathbb{Z}/2 \)-permutation orbifolds: in the orbifold theory, simple objects in the twisted sector come in pairs which are in correspondence to simple objects in \( \mathcal{C} \). The operation inverse to orbifolding is, in the situation at hand, a simple current extension with a simple current of
order two. In the twisted sector, simple current “fixed points” do not occur; hence the pair of simple objects gives rise to a single simple object in the equivariant theory. Indeed, in this specific situation, the generating element \( g \) of \( \mathbb{Z}/2 \) has a single orbit and thus our ansatz for the twisted sector \( \mathcal{O}_g \) as an abelian category is \( \mathcal{C} \). The analysis of the untwisted sector is similar, and the whole analysis can be extended to general permutation orbifolds.

We next define the gluing object

\[
\mathcal{R} \in \bigoplus_{g \in G} \mathcal{C}_g^X \boxtimes \mathcal{C}_g^{X-1};
\]

its component \( \mathcal{R}_g \) in \( \mathcal{C}_g^X \boxtimes \mathcal{C}_g^{X-1} \) is

\[
\mathcal{R}_g := \bigoplus_{i_1,i_2,...,i_{k_g}} (U_{i_{o_1}} \times \cdots \times U_{i_{o_k}}) \times (U_{i_{o_1}'} \times \cdots \times U_{i_{o_k}'});
\]

where the direct sum is taken over all isomorphism classes of simple objects of \( \mathcal{C} \). Thus the direct sum in (28) is taken over representatives of all simple objects of \( \mathcal{C}_{\text{sym}} \). With this definition, \( \mathcal{R} \) is clearly symmetric and \( G \)-invariant.

Now we are able to define a \( \mathcal{C}^X \)-extended \( G \)-equivariant modular functor \( \tau^X \). Let \((P \to M, \{p_a\}_{a \in A(M)})\) be a \( G \)-cover of an extended surface \( M \). Consider the surface

\[
\mathcal{F}_X(P \to M, \{p_a\}_{a \in A(M)}) = \mathcal{X} \times_G P.
\]

It can be viewed as the total space of an \( |\mathcal{X}| \)-fold cover of \( M \) and has, by the discussion in subsection 2.2 the structure of an extended surface. Let \( a \in A(M) \) be a boundary component of \( M \). By lemmas 7 and 8 the restriction of \( \mathcal{X} \times_G P \) to \( a \) has a connected component for every \( \langle m_a \rangle \)-orbit of \( \mathcal{X} \), where \( m_a \) is the monodromy of \( P \) around \( a \).

**Definition 25.** Let \((P \to M, \{p_a\}_{a \in A(M)})\) be a \( G \)-cover of an extended surface \( M \). Define a functor

\[
\tau^X(P \to M, \{p_a\}_{a \in A(M)} : \bigotimes_{a \in A(M)} \mathcal{C}_m^X \to \mathcal{C} \text{Vect}_k
\]

by

\[
\tau^X(P \to M, \{p_a\}_{a \in A(M)}) := \tau(\mathcal{F}_X(P \to M, \{p_a\}_{a \in A(M)})) = \tau(\mathcal{X} \times_G P).
\]

This is indeed well-defined: from \( \bigotimes_{a \in A(M)} \mathcal{C}_m^X = \bigotimes_{a \in A(\mathcal{X} \times_G P)} \mathcal{C} \) we conclude that

\[
\bigotimes_{a \in A(M)} \mathcal{C}_m^{X-1} = \bigotimes_{a \in A(\mathcal{X} \times_G P)} \mathcal{C}.
\]

Note that the definition of \( \tau^X \) also depends on the choice of marked points in \( \mathcal{X} \times_G P \).

We will now describe all the additional data that is needed to turn \( \tau^X \) into a \( \mathcal{C}^X \)-extended \( G \)-equivariant modular functor and prove that all axioms are satisfied.

- Let \( f : (P \to M, \{p_a\}_{a \in A(M)}) \xrightarrow{\sim} (P' \to M', \{p'_a\}_{a \in A(M')}) \) be an isomorphism in \( \text{GExt} \). This gives a morphism \( \tilde{f} := \mathcal{F}_X(f) : \mathcal{X} \times_G P \xrightarrow{\sim} \mathcal{X} \times_G P' \) in \( \text{Ext} \). By definition, the \( \mathcal{C} \)-extended modular functor \( \tau \) gives an isomorphism \( \tilde{f}_* : \tau(\mathcal{X} \times_G P) \xrightarrow{\sim} \tau(\mathcal{X} \times_G P') \) of functors, hence an isomorphism \( \tilde{f}_* : \tau^X(P \to M, \{p_a\}_{a \in A(M)}) \xrightarrow{\sim} \tau^X(P' \to M', \{p'_a\}_{a \in A(M')}) \). This isomorphism only depends on the isotopy class of \( f \) and it obeys \( (f \circ \tilde{g})_* = \tilde{f}_* \circ \tilde{g}_* \).
• The \(C\)-extended modular functor has enough structure to provide an isomorphism of functors

\[
\tau^X(\emptyset) = \tau(\mathcal{X} \times_G \emptyset) = \tau(\emptyset) \cong k
\]  

(31)

and for \(G\)-covers \((P \to M, \{p_a\}_{a \in A(M)})\) and \((P' \to M', \{p'_a\}_{a \in A(M')})\)

\[
\tau^X(P \sqcup P') = \tau(\mathcal{X} \times_G (P \sqcup P')) \cong \tau((\mathcal{X} \times_G P) \sqcup (\mathcal{X} \times_G P'))
\]

\[
\Rightarrow \tau(\mathcal{X} \times_G P) \otimes_k \tau(\mathcal{X} \times_G P') = \tau(X) \otimes_k \tau(X'),
\]

(32)

• Next we describe the gluing isomorphisms. Let \((P \to M, \{p_a\}_{a \in A(M)})\) be a \(G\)-cover of \(M\) and let \(\alpha, \beta \in A(M)\) be two different boundary components of \(M\) with inverse monodromies, \(m_\alpha = m_\beta^{-1}\). This condition ensures that we can glue \(P\) along \(\alpha\) and \(\beta\). For all \(a \in A(M) \setminus \{\alpha, \beta\}\), let \(W_a\) be an object of \(\mathcal{C}^X_{m_\alpha}\). Then

\[
\tau^X(P, \{W_a\}, \mathcal{R}_{\alpha, \beta}) \cong \bigoplus \tau(\mathcal{X} \times_G P, \{W_a\}, U_{a_1} \times \cdots \times U_{a_k}, U^\vee_{a_1} \times \cdots \times U^\vee_{a_k}).
\]  

(33)

Here the \(o_i\) are the \(m_\alpha^{-1}\)-orbits of \(\mathcal{X}\). Since the monodromies are inverses, \(m_\beta = m_\alpha^{-1}\), these orbits are equal to the \(m_\beta\)-orbits so that the assignment of the simple objects in the first component of \(\mathcal{R}_{m_\alpha}\) to the boundary components of \(\mathcal{X} \times_G P\) over \(\alpha\) and the simple objects in the second component of \(\mathcal{R}_{m_\alpha^{-1}}\) to the boundary components over \(\beta\) is compatible. The direct sum is taken over all simple objects as in definition \(28\) where \(\mathcal{R}\) has been introduced.

Since \(\tau\) is a modular functor, we get gluing isomorphisms for all boundary components of the cover \(\mathcal{X} \times_G P\) over \(\alpha\) and \(\beta\) by gluing \(\mathcal{X} \times_G P\) along each boundary component over \(\alpha\) and \(\beta\) separately. The gluing isomorphisms of \(\tau\) satisfy an associativity condition; thus the procedure does not depend on the order in which the boundary components are glued. Hence we get gluing isomorphisms

\[
G_{\alpha, \beta} : \tau^X(P, \{W_a\}, \mathcal{R}_{\alpha, \beta}) \xrightarrow{\cong} \tau^X(\sqcup_{\alpha, \beta} P, \{W_a\})
\]

(34)

that are functorial in the objects \(W_a\). The procedure in the definition of \(G_{\alpha, \beta}\) relies on the fact that the cover functor \(\mathcal{F}_{\mathcal{X}}\) respects gluing of covers.

• Next we implement \(G\)-equivariance. Let \((P \to M, \{p_a\}_{a \in A(M)})\) be a \(G\)-cover of an extended surface \(M\) with marked points \(\{p_a\}\) and let \(g = (g_a)_{a \in A(M)} \in G^{A(M)}\) be a tuple of elements of \(G\) for each boundary component of \(M\). For all \(a \in A(M)\) let \(W_a\) be an object in \(\mathcal{C}^G_{m_\alpha}\). We need to give functorial isomorphisms

\[
T_g : \tau^X(P \to M, \{p_a\}, \{W_a\}) \xrightarrow{\cong} \tau^X(P \to M, \{g_a p_a\}, \{g_a W_a\}).
\]  

(35)

implementing the action of \(g\) on the boundary components.

The boundary component of \((P \to M, \{p_a\})\) corresponding to the \(a\)-th boundary component of \(M\) is isomorphic to the cover \((P_{m_a} \to S^1, [0, 1_G])\). Now we analyse the situation on the right hand side of \(35\). The boundary of \((P \to M, \{g_a p_a\})\) is isomorphic to \((P_{m_a} \to S^1, [0, g_a]) \cong (P_{g_a m_a g_a^{-1}} \to S^1, [0, 1_G])\). This induces precisely the map \(E_{m_a} \to E_{g_a m_a g_a^{-1}}\) in lemma \([9]\). After
identifying boundary components and orbits, a \( \langle m_a \rangle \)-orbit \( o \) is mapped under this map to the \( \langle g_a m_a g_a^{-1} \rangle \)-orbit \( g_o o \). On the other hand, the action of \( R_{g_a} \) on \( C^X_{m_a^{-1}} \) permutes the objects in \( C^X_{m_a^{-1}} \) in exactly the same way. Hence we can choose the equivariance isomorphisms \( T_g \) to be identity morphisms.

- It follows from functoriality of the cover functor \( F_X \) and of \( \tau \) that all isomorphisms constructed above are functorial in \( (P \to M, \{p_a\}) \). Similarly, one concludes that all isomorphisms are compatible.

- The \( G \)-equivariant modular functor \( \tau^X \) is normalized:

\[
\tau^X(S^2 \times G \to S^2) = \tau(X \times_G (S^2 \times G)) \cong \tau(X \times S^2) \cong k \otimes \cdots \otimes k \cong k \tag{36}
\]

We summarize these findings in the following theorem, which is one of the main results of this paper:

**Theorem 26.** Let \( G \) be a finite group, \( X \) a finite ordered \( G \)-set and \( C \) a \( k \)-linear modular category. Denote by \( C^X \) the \( G \)-equivariant category introduced in definition 24. Then the functor \( \tau^X \) defined by

\[
\tau^X(P \to M, \{p_a\}_{a \in A(M)}) := \tau(F_X(P \to M, \{p_a\}_{a \in A(M)}) \tag{37}
\]

is a \( C^X \)-extended \( G \)-equivariant modular functor.

This implies in particular:

**Corollary 27.**

1. The \( G \)-equivariant modular functor \( \tau^X \) induces a \( G \)-equivariant monoidal structure on the \( G \)-equivariant category \( C^X \).

2. The equivariant modular functor \( \tau^X \) induces on the \( G \)-equivariant monoidal category \( C^X \) the structure of a weakly fusion category. Since the \( G \)-modular functor in theorem 26 is defined for arbitrary genus, we expect this structure to be \( G \)-modular, so that orbifold theories exist. Unfortunately, \( G \)-equivariant modular functors for higher genus are not yet well understood.

3. By restriction, the \( G \)-equivariant functor \( \tau^X \) endows for any group element \( g \in G \) the abelian category \( C^g \)-th \( C \)-with the structure of a module category over the tensor category \( C^{\otimes X} \).

In the next section, we make this structure explicit in the case of \( G = \mathbb{Z}/2 \) acting non-trivially on a two-element set \( X \). In this case, we can show that the equivariant category is rigid rather than only weakly rigid. It is also known in this case \[BFRS10\] that the module category structure on \( C \) over \( C \otimes C \) describes the permutation modular invariant.

For our construction, it is indispensable to require \( C \) to be modular. A genus 0 \( C^X \)-extended \( G \)-equivariant modular functor has to be defined on \( G \)-covers of extended surfaces of genus 0. According to lemma 12 the total space of the associated \( |X| \)-fold cover, however, can have a non-zero genus. Hence the \( C \)-extended modular functor \( \tau \) has to be defined for any genus, and thus we can define, as in theorem 26, the equivariant modular functor \( \tau^X \) for any genus as well.
4 \( \mathbb{Z}/2 \)-permutation equivariant fusion categories

4.1 Notation and conventions

From now on, we restrict ourselves on the case where the cyclic group \( G = \mathbb{Z}/2 \) acts on the ordered two-element set \( X = \{1, 2\} \) by permutation of elements. We denote the generator of \( \mathbb{Z}/2 \) by \( g \) and hence write \( \mathbb{Z}/2 = \{1, g\} \) with \( g^2 = 1 \).

We note that in this special case, the \( \mathbb{Z}/2 \)-cover \( P \to M \) and the cover \( X \times \mathbb{Z}/2 P \to M \) are isomorphic as covers over the manifold \( M \). However, whereas \( P \) has only one marked point for every connected component of \( \partial M \), the cover \( X \times \mathbb{Z}/2 P \to M \) has one marked point in every connected component of the boundary of the cover. We assume that one distinguished point has been chosen over each connected component of \( \partial P \) over \( \partial M \).

Our construction will involve choices: we define the value of functors like the duality functor or the tensor product functor by objects representing functors constructed from the equivariant modular functor \( \tau^X \). This is possible, since by [BK01, lemma 5.3.1] any additive functor \( F : \mathcal{D} \to \text{Vect}_k \) from a semisimple abelian category \( \mathcal{D} \) with finitely many objects is representable. By definition of the dual object this shows that any object \( V \) is determined by the functor \( \tau^X(S_2, ?, V) \).

The representing objects from the Yoneda lemma will only be unique up to canonical isomorphism. Different choices of representing objects lead to different structures of ribbon categories on \( C^X \), which are equivalent via a tensor functor which is the identity functor on \( C^X \) and whose structure morphisms are provided the Yoneda lemma. We will indicate such choices in our discussion.

For the value of the equivariant modular functor \( \tau^X \) on the standard blocks of subsection \( \text{2.1} \) we introduce the shorthand notation

\[
\langle V_1, \ldots, V_n \rangle_X := \tau^X(S_n(g_1, \ldots, g_n; 1, \ldots, 1); V_1, \ldots, V_n).
\]

(38)

with objects \( V_i \in C^X_{g_i} \). We introduce a similar shorthand for the \( C \)-extended modular functor \( \tau \) as well,

\[
\langle W_1, \ldots, W_n \rangle := \tau(S_n; W_1, \ldots, W_n),
\]

(39)

where \( W_1, \ldots, W_n \) are objects in \( C \).

As an abelian category, the \( \mathbb{Z}/2 \)-equivariant category \( C^X \) is of the form \( (C \boxtimes C) \oplus (C) \), where \( C \boxtimes C \) is the neutral and \( C \) the twisted component or sector of \( C^X \). We denote objects in the neutral component by \( A_1 \times A_2, B_1 \times B_2, \ldots \) and objects in the twisted component by \( M, N, \ldots \).

The dual of an object \( V \) in \( C \) will be denoted by \( V^\vee \). We agree to drop the tensor product symbol for objects of \( C \) when using the monoidal structure of \( C \), so we write \( AB \equiv A \otimes_C B \). The braiding of two objects in \( C \) will be denoted by \( c_{A,B} \) and the equivariant braiding in \( C^X \) by \( C_{A,B} \) in capital letters. Similarly, \( \theta_U \) denotes the twist, \( b_U \) the coevaluation and \( d_U \) the evaluation in \( C \), while for the corresponding morphisms \( \Theta_A, B_A \) and \( D_A \) of \( C^X \) capital letters are used.

When dealing with modular functors we use a graphical notation. First note that \( \mathbb{Z}/2 \) covers over \( S_n \) can have the following two types of boundary components:
The first is a boundary component of non-trivial monodromy $g$, the second with trivial monodromy $1 \in \mathbb{Z}/2$.

A decorated surface $\Sigma$ also represents the corresponding vector space $\tau(\Sigma; V_1, \ldots, V_n)$: we then write objects of the appropriate categories to the boundary components:

When we write an object $A_1 \times A_2$ next to a boundary embedded in $\mathbb{R}^3$, our convention is such that the first object is assigned to the outer circle and the second object to the inner circle.

To keep track of diffeomorphisms of a surface $\Sigma$, we use techniques from the Lego-Teichmüller-Game ([BK00]), or LTG for short and find a graphical representation of these diffeomorphisms. For the definition of the LTG, we refer to [BK00]; we use the notations of this paper. The LTG requires to draw marking graphs with a base point on the surfaces under consideration:

We fix a standard marking on the standard sphere $S_n$ by taking the $n$ straight lines that relate the marked point $k - \frac{i}{3}$ to the point $-2i$ for $k = 1, 2 \ldots n$.

Diffeomorphisms map marking graphs to marking graphs. Different marking graphs are connected by finite sequences of LTG moves. This sequence is unique [BK00 Theorem 4.24] up to a known set of relations. The LTG gives rules to translate these moves into natural isomorphisms between the corresponding vector spaces. In most cases, we will suppress the LTG-move $Z$ (the
rotation move) in manipulations of the marking graphs; when translating the LTG-moves into morphisms in $C$, we will point out at what point one has to insert $Z$-moves.

Similarly, if a surface has been obtained by gluing two boundary circles, the corresponding circle will be drawn on the surface; such circles are called cuts. A single base point of the sewn surface is obtained by contracting the link crossing the cut.

### 4.2 Dual objects

We start by finding for every object $V \in C^X$ a candidate for the dual object. This will be the object $V^*$ representing the functor

$$C^X \to \text{Vect}_k, \quad X \mapsto \langle V, X \rangle_X.$$  \hspace{1cm} (43)

We consider two cases separately.

- For $V$ in the neutral component, $V = A_1 \times A_2 \in C \otimes C$, we consider the standard block $S_2(1,1;1,1)$ whose cover spaces is the disjoint union of two copies of $S_2$ and find the object representing the functor

$$C \otimes C \to \text{Vect}_k, \quad X_1 \times X_2 \mapsto \langle A_1 \times A_2, X_1 \times X_2 \rangle_X.$$  \hspace{1cm} (44)

Here, we restricted our attention to the value of the functor on the neutral component, since the grading implies that the functor is zero on the twisted component. We compute

$$\tau^X(S_2(1,1;1,1); V, X) = \tau(S_2 \sqcup S_2; A_1, A_2, X_1, X_2)$$

$$\cong \tau(S_2; A_1, X_1) \otimes_k \tau(S_2; A_2, X_2)$$

$$\overset{\text{def}}{=} \text{Hom}_C(1, A_1 X_1) \otimes_k \text{Hom}_C(1, A_2 X_2)$$

$$\cong \text{Hom}_C(A_1^\vee, X_1) \otimes_k \text{Hom}_C(A_2^\vee, X_2)$$

$$\overset{\text{def}}{=} \text{Hom}_{C^X}(A_1^\vee \times A_2^\vee, X_1 \times X_2)$$

$$\overset{\text{def}}{=} \text{Hom}_{C^X}(A_1^\vee \times A_2^\vee, X_1 \times X_2).$$ \hspace{1cm} (45)

Let us explain in this example the various steps in detail; in subsequent calculations, we will only explain additional new features. The first equality is by definition of the equivariant modular functor $\tau^X$ via the cover functor $F^X$; the second isomorphy is the tensoriality of $\tau$. The third equality follows from the definition of the modular functor $\tau$ from the modular category $C$. The next isomorphism is a consequence of the duality in the category $C$, while the last identities follow from the definition of the Deligne product $C \otimes C$ and the definition of $C^X$.

Hence we find that $(A_1 \times A_2)^* \cong A_1^\vee \times A_2^\vee$. A choice of the representing object and of a diffeomorphism is involved in equation (43); different choices ultimately lead to equivalent dualities on $C^X$. 

25
Let $M$ be an object in the twisted component of $C^X$, hence $M \in \mathcal{C}$. The total space of the cover $(S_2(g, g; 1, 1) \to S_2)$ has only one connected component and is in fact diffeomorphic to $S_2$ as a smooth manifold, so we find

$$\tau^X(S_2(g, g; 1, 1); M, X) = \tau(S_2; M, X)$$

$$\overset{\text{def}}{=} \text{Hom}_C(1, MX)$$

$$\cong \text{Hom}_C(M^\vee, X)$$

We thus find that $M^* \cong M^\vee$. Again, a choice of a representing object and of a diffeomorphism $S_2(g, g; 1, 1) \xrightarrow{\cong} S_2$ are involved.

So far we only found dual objects, for the further discussion on duality in $C^X$ we need a tensor product on $C^X$. For this reason, we return to more aspects of a ribbon structure in section 4.7.

### 4.3 The tensor product

The tensor product of two objects $V, W \in C^X$ is defined as the object representing the functor $\langle ?, V, W \rangle_X$, i.e. $\langle T, V \otimes W \rangle_X = \langle T, V, W \rangle_X$. This again involves a choice of the representing object. The total space of the standard cover $S_3((g_1g_2)^{-1}, g_1, g_2; 1, 1, 1)$ is isomorphic to the $n$-punctured sphere $S_n$ for a value of $n$ that depends on the group elements $g_1$ and $g_2$. We will have to choose such a diffeomorphism as well. Different choices of this diffeomorphism lead to isomorphic tensor products, but the choice will enter in the associativity constraints; hence we have to keep track of this choice. This is done by considering the image of the standard marking graph on $S_n$ that has been introduced at the end of section 4.1. We call this image the standard marking graph on $S_3((g_1g_2)^{-1}, g_1, g_2; 1, 1, 1)$.

We first consider two objects $V \equiv A_1 \times A_2$ and $W \equiv B_1 \times B_2$ in the neutral component of $C^X$. Since total space of the cover $S_3(1, 1, 1; 1, 1, 1)$ is just the disjoint union of two three-holed spheres, we find

$$\langle T, V \otimes W \rangle_X \overset{\text{def}}{=} \langle T_1 \times T_2, A_1 \times A_2, B_1 \times B_2 \rangle_X$$

$$\overset{\text{def}}{=} \tau^X(S_3(1, 1, 1; 1, 1, 1); T_1 \times T_2, A_1 \times A_2, B_1 \times B_2)$$

$$\overset{\text{def}}{=} \tau(S_3 \sqcup S_3; T_1, T_2, A_1, A_2, B_1, B_2)$$

$$\cong \tau(S_3; T_1, A_1, B_1) \otimes_k \tau(S_3; T_2, A_2, B_2)$$

$$\overset{\text{def}}{=} \text{Hom}_C(1, T_1A_1B_1) \otimes_k \text{Hom}_C(1, T_2A_2B_2)$$

$$\cong \text{Hom}_C(T_1^\vee, A_1B_1) \otimes_k \text{Hom}_C(T_2^\vee, A_2B_2)$$

$$\overset{\text{def}}{=} \text{Hom}_{C \times C}(T_1^\vee \times T_2^\vee, A_1B_1 \times A_2B_2)$$

Our definition of the tensor product on objects of the untwisted component thus yields $(A_1 \times A_2) \otimes (B_1 \times B_2) := A_1B_1 \times A_2B_2$, i.e. the usual tensor product on $C \otimes C$. The standard marking graph on the cover $S_3(1, 1, 1; 1, 1, 1)$ is then
• Next we consider the tensor product of an object in the untwisted component $A_1 \times A_2$ with an object $M$ in the twisted component. The total space $S_3(g, 1, g; 1, 1, 1)$ of the relevant cover is diffeomorphic to $S_4$ as can be seen by the following chain of diffeomorphisms:

In the first and second step we move the inner hole labelled by $A_2$ around the component of the boundary with non-trivial monodromy, labelled by $M$. The last step is also an isomorphism of manifolds; the reader should not be confused by the fact that we have to draw two-dimensional manifolds immersed into the three-dimensional space $\mathbb{R}^3$. Hence we
find that
\[
\langle T, (A_1 \times A_2) \otimes M \rangle_X = \langle T, A_1 \times A_2, M \rangle_X \\
\overset{\text{def}}{=} \tau^X(S_3(g, 1, 1, 1); T, A_1 \times A_2, M) \\
\overset{=}{=} \tau(S_3(g, 1, 1, 1); \tau(T, A_1, A_2, M)) \\
\overset{\text{def}}{=} \tau(S_4; T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_3(g, 1, 1, 1); T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_4; T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_3(g, 1, 1, 1); T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_4; T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_3(g, 1, 1, 1); T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_4; T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_3(g, 1, 1, 1); T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_4; T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_3(g, 1, 1, 1); T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_4; T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_3(g, 1, 1, 1); T, A_1, A_2, M) \\
\overset{\text{def}}{=} \tau(S_4; T, A_1, A_2, M)
\]

We are thus lead to the definition \((A_1 \times A_2) \otimes M :\) The image of the standard marking graph on \(S_4\) under the diffeomorphism described in equation \((49)\) gives the following marking on \(S_3(g, 1, 1, 1):\)

- The discussion of the tensor product \(M \otimes (A_1 \times A_2)\) closely parallels the preceding discussion.
- Again the manifold \(S_3(g, g, 1; 1, 1, 1)\) is diffeomorphic to the four-punctured sphere \(S_4\). We introduce a diffeomorphism similar to the one defined in equation \((49)\):
We find:

\[
\langle T, M \otimes (A_1 \times A_2) \rangle_x \overset{\text{def}}{=} \langle T, M, A_1 \times A_2 \rangle_x \\
\overset{\text{def}}{=} \tau^x(S_3(g, g, 1; 1, 1, 1); T, M, A_1 \times A_2) \\
\overset{\text{def}}{=} \tau(S_3(g, g, 1; 1, 1, 1); T, M, A_1, A_2) \\
\overset{\cong}{=} \tau(S_4; T, M, A_1, A_2) \\
\overset{\text{def}}{=} \text{Hom}_C(\mathbf{1}, TMA_1A_2) \\
\overset{\cong}{=} \text{Hom}_C(T', MA_1A_2)
\]

so that we are lead to define \( M \otimes (A_1 \times A_2) \):= \( MA_1A_2 \) and have the standard marking graph on \( S_3(g, g, 1; 1, 1, 1) \)

These definitions coincide with the ad hoc definitions made in [BFRS10].
• We now turn to a new result about the $\mathbb{Z}/2$-equivariant tensor product: the tensor product of two objects in the twisted component of $C^X$. To this end, we first note that the tensor product functor

$$C \boxtimes C \to C,$$

$$\bigoplus_l V_l \times W_l \mapsto \bigoplus_l V_l W_l$$

(55)

has the right adjoint [KR09, Thm. 2.21]

$$R : C \to C \boxtimes C,$$

$$V \mapsto \bigoplus_{i \in I} V U_i^\vee \times U_i,$$

(56)

where the sum is over representatives of isomorphism classes of simple objects of $C$.

We have to consider the manifold $S_3(1, g, g; 1, 1, 1)$ together with the following diffeomorphisms

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\cdots
\end{array}
\end{array}
\end{align*}
\]
We thus find
\[
\langle T_1 \times T_2, M \otimes N \rangle_X \overset{\text{def}}{=} \langle T_1 \times T_2, M, N \rangle_X
\]
\[
\overset{\text{def}}{=} \tau_X^X(S_3(1, g, g; 1, 1, 1); T_1 \times T_2, M, N)
\]
\[
\overset{\text{def}}{=} \tau(S_4; T_2, T_1, M, N)
\]
\[
\overset{\text{def}}{=} \tau(S_3(1, g, g; 1, 1, 1); T_1, T_2, M, N)
\]
\[
\overset{\text{def}}{=} \tau(S_3; T_2, T_1, M, N)
\]
\[
\overset{\text{def}}{=} \tau(S_4; T_1 \times T_2, M, N)
\]
\[
\overset{\text{def}}{=} \tau(S_3(1, g, g; 1, 1, 1); T_1, T_2, M, N)
\]
\[
\overset{\text{def}}{=} \tau(S_4; T_2, T_1, M, N)
\]
\[
\overset{\text{def}}{=} \tau(S_3; T_2, T_1, M, N)
\]
\[
\overset{\text{def}}{=} \tau(S_4; T_1 \times T_2, M, N)
\]
where the last isomorphism is given by the adjunction between \( R \) and the tensor product functor. Hence we set
\[
M \otimes N = R(MN) = \bigoplus_{i \in I} MNU_i^\vee \times U_i.
\]
We note that the adjunction morphism
\[
\text{Hom}_C(T_1^\vee T_2^\vee, MN) \overset{\text{def}}{=} \bigoplus_{i \in I} \text{Hom}_C(T_1^\vee T_2^\vee, MNU_i^\vee \times U_i)
\]
\[
\overset{\text{def}}{=} \bigoplus_{i \in I} \text{Hom}_C(T_1^\vee, MNU_i^\vee) \otimes_k \text{Hom}_C(T_2^\vee, U_i)
\]
coincides with the gluing isomorphism
\[
\overset{\text{def}}{=} \bigoplus_{i \in I} \text{Hom}_C(T_1^\vee, MNU_i^\vee) \otimes_k \text{Hom}_C(T_2^\vee, U_i)
\]
so that on the total space \( S_3(1, g, g; 1, 1, 1) \) of the cover we have the standard marking graph
which has a cut drawn on the manifold close to the insertion of $T_2$. The second arrow in the marking on $S_3(1,g,g;1,1,1)$ comes from the marking on $S_2(U_i,T_2)$ and fixes the order of the objects in $\text{Hom}_C(1,U;T_2)$.

Formulae for the dimensions of spaces of conformal blocks in permutation orbifolds have been given in [BHS98, Ban98]. It is easy to see that these formulae imply that dimension of the spaces of conformal three-point blocks of the equivariant theory on the sphere involving two objects in the twisted sector are given by the dimension of the spaces of four-point blocks on the sphere for $C$. This nicely follows from the geometry of the cover functor: the total space of the relevant cover is isomorphic to the four-punctured sphere. The analysis can be extended to other conformal blocks as well.

We finally determine the tensor unit $1_{\mathbb{Z}/2}$ of $C^X$ that is defined by

$$\langle 1_{\mathbb{Z}/2}, U \rangle_X := \langle U \rangle_X$$

for all $U$ in $C^X$. The classification of covers of the one-holed sphere implies that $\langle U \rangle_X$ is only defined when $U$ is in the neutral component, $U \in \mathcal{C}_1^X$. With $U = A_1 \times A_2 \in C \boxtimes C$, we find $\langle A_1 \times A_2 \rangle_X \cong \langle A_1 \rangle \otimes_k \langle A_2 \rangle$ and hence

$$1_{\mathbb{Z}/2} = 1 \times 1.$$  

Since the modular tensor category $C$ was supposed to be strict, the unit constraints of $C^X$ are the identity morphisms.

### 4.4 The associativity constraints

The next step is to derive the associativity constraints for the tensor product given in the previous section. As special cases, we will find the mixed associativity constraints for which ansätze were proposed in [BFRS10].

For any choice of three elements $p, q, r \in \mathbb{Z}/2$ we need to consider the $\mathbb{Z}/2$-cover

$$S_4((pqr)^{-1}, p, q, r; 1, 1, 1, 1)$$

of the four-holed sphere $S_4$. This cover will be cut in two different ways, representing the tensor product $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$, respectively. The two ways of cutting give two different
markings on $S_4((pqr)^{-1}, p, q, r; 1, 1, 1, 1)$ which in turn represent two different diffeomorphisms from the total space of the cover

$$f, g : S_4((pqr)^{-1}, p, q, r; 1, 1, 1, 1) \overset{\cong}{\rightarrow} S$$

(65)

to the appropriate standard block $S$, which is either a punctured sphere, a disjoint union of punctured spheres or a surface of genus one. We then consider for all objects $T \in \mathcal{C}^X$ the following chain of natural transformations:

$$\begin{align*}
\text{Hom}_{\mathcal{C}^X}(T^*, (A \otimes B) \otimes C) & \overset{\cong}{\rightarrow} \tau(S; T, A, B, C) \\
\overset{f^{-1}}{\rightarrow} \tau(S_4((pqr)^{-1}, p, q, r; 1, 1, 1, 1); T, A, B, C) & \overset{\cong}{\rightarrow} \tau(S; T, A, B, C) \\
\overset{g^{-1}}{\rightarrow} \text{Hom}_{\mathcal{C}^X}(T^*, A \otimes (B \otimes C)).
\end{align*}$$

The first and the last isomorphism in (66) are determined by the definition of the tensor products $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ as objects in $\mathcal{C}$ or $\mathcal{C} \otimes \mathcal{C}$ respectively. The Yoneda lemma implies that this natural transformation comes from a natural isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \overset{\cong}{\rightarrow} A \otimes (B \otimes C)$. The arguments of [KP08, BK01] then imply that these isomorphisms satisfy the pentagon axiom.

We summarize our strategy:

- Determine the two marking graphs on $S_4((pqr)^{-1}, p, q, r; 1, 1, 1, 1)$ induced on the cover by cutting $S_4$ in the two ways determined by associativity and by our definition of the tensor product. Denote the marking graph representing $(A \otimes B) \otimes C$ by $M_1$ and the marking graph representing $A \otimes (B \otimes C)$ by $M_2$.

- Determine the standard manifold $S$ such that $S \cong S_4((pqr)^{-1}, p, q, r; 1, 1, 1, 1)$. Transform the surface $S_4((pqr)^{-1}, p, q, r; 1, 1, 1, 1)$ with the marking graph $M_1$ to the standard manifold $S$ in the way prescribed by the marking $M_2$.

- This yields $S$, together with a marking graph. Determine the LTG-moves that transform this graph into the standard marking graph on $S$.

- Translate these LTG-moves into morphisms in $\mathcal{C}$ or $\mathcal{C} \otimes \mathcal{C}$.

Since we need to do this for any choice of $p, q, r \in \mathbb{Z}/2$, we will get eight different associativity constraints $\alpha_{A,B,C}$, depending on the sector of the objects $A, B, C$.

### 4.4.1 Three objects from the untwisted component

To illustrate the method, we start with the easiest case, three objects $A_1 \times A_2$, $B_1 \times B_2$ and $C_1 \times C_2$ in the untwisted component. The total space $S_4(1, 1, 1, 1; 1, 1, 1, 1)$ is just a disjoint union of two four-holed spheres. Both cutting procedures yield the same marking graph on $S_4 \sqcup S_4$:
Hence we arrive at the trivial LTG-move. Taking into account that $C$ is supposed to be strict, we find that $\alpha_{A_1 \times A_2, B_1 \times B_2, C_1 \times C_2}$ is the identity on $A_1 B_1 C_1 \times A_2 B_2 C_2$.

### 4.4.2 One object from the twisted component

Next we derive the associativity isomorphisms involving one object in the twisted component and two objects in the untwisted component. To explain our prescription in detail, we first discuss the associativity constraint $\alpha_{A_1 \times A_2, B_1 \times B_2, M}$ in greater detail.

The first cutting amounts on the total space of $S_4(g, 1, 1, g; 1, 1, 1)$ to the gluing isomorphism in $C \boxtimes C$:

$$\bigoplus_{i,j \in I} \langle T, U_i^\vee \times U_j^\vee, M \rangle_{\mathcal{X}} \otimes_k \langle U_i \times U_j, A_1 \times A_2, B_1 \times B_2 \rangle_{\mathcal{X}} \xrightarrow{\sim} \langle T, A_1 \times A_2, B_1 \times B_2, M \rangle_{\mathcal{X}}$$

With the standard markings on $S_3$-covers introduced in section 4.1 we arrive at the following picture:
By contracting the marking along the factorizing link, we get on $S_4(g, 1, 1, g; 1, 1, 1, 1)$ the marking

$$T \quad A_1 \times A_2$$

$$M \quad B_1 \times B_2$$

(70)

The second gluing procedure is the isomorphism

$$\bigoplus_{i \in I} (T, A_1 \times A_2, U^\vee_i \times_k (U_i, B_1 \times B_2, M)) \quad (71)$$

with the pictorial description

$$A_1 \times A_2 \quad B_1 \times B_2$$

$$U^\vee_i \quad U_i$$

$$T \quad M$$

(72)

Again contracting the marking along the factorizing link, we get on $S_4(g, 1, 1, g; 1, 1, 1, 1)$ the second marking

$$T \quad A_1 \times A_2$$

$$M \quad B_1 \times B_2$$

(73)

The surface $S_4(g, 1, 1, g; 1, 1, 1, 1)$ is isomorphic to a sphere $S_6$ with 6 punctures. We draw the graph (70) obtained from the first gluing procedure on $S_4(g, 1, 1, g; 1, 1, 1, 1)$ and use the isomorphism to $S_6$ encoded in the graph (73) obtained from the second gluing procedure:
The LTG-moves that transform this marking into the standard marking on $S_6$ are easily read off:

The LTG-move $B_{B_1, A_2}$ corresponds to the natural transformation on

(75)
Hom$_C(1, TA_1 B_1 A_2 B_2 M)$ that is induced by the braiding in $C$, pictorially,

\[
\begin{array}{c}
T A_1 B_1 A_2 B_2 M \\
\end{array} \xrightarrow{B_{B_2}} \begin{array}{c}
T A_1 A_2 B_1 B_2 M \\
\end{array}
\]

We now apply the isomorphism $\text{Hom}_C(1, TA_1 B_1 A_2 B_2 M) \cong \text{Hom}_C(T^\vee, A_1 B_1 A_2 B_2 M)$ induced by the ribbon structure on $C$. Setting $T^\vee := A_1 B_1 A_2 B_2 M$ and evaluating the natural transformation (76) on the identity, we obtain the morphism for the associativity constraint as $\alpha_{A_1 \times A_2, B_1 \times B_2, M} = \text{id}_{A_1} \otimes c_{B_1, A_2} \otimes \text{id}_{B_2 M}$.

These are precisely the mixed associativity constraints proposed in [BFRS10]; in that paper, a whole family of mixed associativity constraints was given. Here we read off the morphisms from marking graphs on surfaces. These graphs depend on the choice of diffeomorphisms made in subsection 4.1. Different choices of diffeomorphisms lead to different associativity constraints, of which some were already proposed in [BFRS10]. Different choices of diffeomorphisms are related by elements of the mapping class group of the relevant surface. These group elements can be translated into morphisms in $C$ which can be used to endow the identity functor on $C^X$ with the structure of a monoidal equivalence of monoidal categories.

For the associativity constraint $\alpha_{M, A_1 \times A_2, B_1 \times B_2}$, we get a similar picture. The first gluing procedure on $S_4(g, g, 1, 1; 1, 1, 1, 1)$ gives the marking

\[
\begin{array}{c}
T M \\
B_1 \times B_2 A_1 \times A_2 \\
\end{array}
\]

while the second procedure gives the marking

\[
\begin{array}{c}
T M \\
B_1 \times B_2 A_1 \times A_2 \\
\end{array}
\]
On the sphere $S_6$ with 6 punctures, this gives the LTG-move

\[
\begin{array}{c}
B_2 \quad T \quad M \quad A_1 \quad B_1 \\
A_2 \quad B_1 \quad A_2
\end{array}
\xrightarrow{B_{B_1,A_2}^{-1}}
\begin{array}{c}
T \quad M \quad B_2 \quad A_1 \\
A_2 \quad B_1
\end{array}
\]

(79)

By the same reasoning as after equation (76), we conclude that $\alpha_{M,A_1 \times A_2,B_1 \times B_2} = \text{id}_{M,A_1} \otimes c_{B_1,A_2}^{-1} \otimes \text{id}_{B_2}$. This is one of the constraints in [BFRS10, Corollary 3].

For the associativity constraint $\alpha_{A_1 \times A_2,M,B_1 \times B_2}$ we consider $S_4(g,1,g,1;1,1,1,1) \cong S_6$. The two gluing procedures give the two markings

\[
\begin{array}{c}
T \quad A_1 \times A_2 \\
B_1 \times B_2 \quad M
\end{array}
\]

\[
\begin{array}{c}
T \quad A_1 \times A_2 \\
B_1 \times B_2 \quad M
\end{array}
\]

(80)

Our general prescription gives the following marking graph on $S_6$:

\[
\begin{array}{c}
T \quad A_1 \\
B_2 \quad B_1 \quad M \quad A_2
\end{array}
\]

(81)

The following sequence of four LTG-moves transforms this marking graph into the standard
marking graph on $S_6$:

\[
\begin{align*}
T & \quad A_1 \\
B_2 & \Rightarrow A_2 \\
B_1 & \quad T \\
M & \\
\end{align*}
\]

\[
\begin{align*}
T & \quad B_2 \\
A_1 & \Rightarrow B_1 \\
B_2 & \quad T \\
M & \\
\end{align*}
\]

\[
\begin{align*}
T & \quad A_1 \\
B_2 & \Rightarrow A_2 \\
B_1 & \quad T \\
M & \\
\end{align*}
\]

\[
\begin{align*}
T & \quad B_2 \\
A_1 & \Rightarrow B_1 \\
B_2 & \quad T \\
M & \\
\end{align*}
\]

\[
\begin{align*}
T & \quad A_1 \\
B_2 & \Rightarrow A_2 \\
B_1 & \quad T \\
M & \\
\end{align*}
\]

The sequence

\[
B_{TA_1A_2,B_2} \circ B_{B_2MB_1,A_2} \circ B_{M_1B_1,A_2}^{-1} \circ B_{TA_1,B_2}^{-1}
\]

of LTG-moves is translated into the natural transformation

\[
\text{Hom}_C(1, TA_1A_2MB_1B_2) \xrightarrow{\cong} \text{Hom}_C(1, TA_1A_2MB_1B_2),
\]
where we will need to insert the appropriate $Z$-moves implementing cyclicity:

Using again the isomorphy $\text{Hom}_C(\mathbf{1}, TA_1A_2MB_1B_2) \cong \text{Hom}_C(T^\vee, A_1A_2MB_1B_2)$ implied by duality and evaluating on the identity morphism gives the associativity constraint

\[ \alpha_{A_1 \times A_2, M, B_1 \times B_2} = \]

(86)
4.4.3 Two objects in the twisted component

We next discuss associativity constraints involving two objects in the twisted and one in the untwisted component of $C^x$. The tensor product of three such objects lies in the untwisted component. Again the relevant marking graphs on covers of the four-punctured sphere contain cuts.

We start with the associativity constraint $\alpha_{M,N,A_1 \times A_2}$. The two gluing procedures over $S_4$ yield the following two markings on $S_4(1, g, g, 1; 1, 1, 1, 1)$, where we already removed the cuts and contracted the factorizing links:

\begin{equation}
\begin{align*}
A_1 \times A_2 & \quad T_1 \times T_2 & A_1 \times A_2 & \quad T_1 \times T_2 \\
M & & N & & M \\
\end{align*}
\end{equation}

This gives the following marking on the six-punctured sphere $S_6$:

\begin{equation}
\begin{align*}
A_2 & \quad T_2 & A_1 & \quad T_1 \\
M & & N \\
\end{align*}
\end{equation}

It is transformed into the standard marking on $S_6$ by the LTG-moves $B_{A_1 \cdot A_2}^{-1} \circ B_{A_1 \cdot T_2}^{-1}$. Now we need to translate this into a natural isomorphism

\begin{equation}
\begin{align*}
\bigoplus_{i \in I} \Hom_C(1, T_1 MNU_i^\vee A_1) \otimes_k \Hom_C(1, T_2 U_i A_2) \\ \cong \bigoplus_{j \in I} \Hom_C(1, T_1 MA_1 A_2 U_j^\vee) \otimes_k \Hom_C(1, T_2 U_j) \\
\end{align*}
\end{equation}

Removing the cuts in the markings on $S_4(1, g, g, 1; 1, 1, 1, 1)$ amounts to the isomorphism

\begin{equation}
\begin{align*}
\bigoplus_{i \in I} \Hom_C(1, T_1 MNU_i^\vee A_1) \otimes_k \Hom_C(1, T_2 U_i A_2) & \rightarrow \Hom_C(1, T_1 MNA_2 T_2 A_1) \\
\end{align*}
\end{equation}
which is given by the generalized $F$-move:

\[
\bigoplus_{i \in I} M U_i \nabla T_1 T_2 U_i A_1 \rightarrow \bigoplus_{i \in I} \otimes_k M U_i \nabla T_1 T_2 U_i A_1
\]

Then applying the LTG-moves $B_{A_1, A_2}^{-1} \circ B_{A_1, T_2}^{-1}$ gives

\[
\bigoplus_{i \in I} M U_i \nabla T_1 T_2 U_i A_1 \nabla T_2
\]

to which we now apply the adjunction (60). After applying duality morphisms, we obtain the following ribbon graph for the morphism on the right hand side of (89):

\[
\bigoplus_{i, j \in I} \sum_{\alpha} M N A_1 A_2 \nabla T_1 T_2 U_i U_j \nabla \alpha \nabla T_2
\]

The sum is taken over a basis of $\text{Hom}_C(U_j, T_2')$ and the corresponding dual basis of $\text{Hom}_C(T_2', U_j)$. It is obvious that the morphism does not depend on the choice of basis. To use again the Yoneda lemma, we evaluate the corresponding natural transformation on $T_1' = MNU_i' A_1$ and $T_2' = U_i A_2$ for the identity. We get the following explicit formula for the associativity constraint:
\[
\alpha_{M,N,A_1 \times A_2} = \bigoplus_{i,j \in I} \sum_{\alpha} \alpha
\]

For the associativity constraint \(\alpha_{M,A_1 \times A_2,N}\) gluing \(S_4(1, g, 1, g; 1, 1, 1, 1)\) over the four-punctured sphere \(S_4\) gives, after removing the cuts, the graphs

This results in the following marking on the 6-punctured sphere \(S_6\):

This marking is transformed into the standard graph on \(S_6\) by the following chain of LTG-moves:

\[
B_{T_2T_1,MA_1,A_2}^{-1} \circ B_{N,A_2,T_2}^{-1} \circ B_{N,T_2} \circ B_{T_1,MA_1,A_2}
\]

(97)
They eventually lead to the associativity isomorphism

$$\alpha_{M,A_1 \times A_2,N} = \bigoplus_{i \in I} \bigotimes_k$$

(98)

For the associativity constraints $\alpha_{A_1 \times A_2, M, N}$ gluing over $S_4$ gives the graphs

$$T_1 \times T_2 \quad A_1 \times A_2 \quad T_1 \times T_2 \quad A_1 \times A_2$$

(99)

Transforming $S_4(1, 1, g, g; 1, 1, 1, 1)$ to $S_6$ gives the following marking on $S_6$:

$$T_1 \quad A_1 \quad T_2$$

(100)

The transformation into the standard graph on $S_6$ is just given by the LTG-move $B_{T_1 A_1, A_2}$. The procedure outlined before then gives the constraint

$$\alpha_{A_1 \times A_2, M, N} = \bigoplus_{i \in I} \sum_{\alpha}$$

(101)
where the $\alpha$-summation is over a basis of $\text{Hom}_C(U_j, A^\vee_2 U_i)$ and the corresponding dual basis of $\text{Hom}_C(A^\vee_2 U_i, U_j)$.

### 4.4.4 Three objects in the twisted component

The last associativity isomorphism $\alpha_{M,N,O}$ for three objects $M, N, O$ in the twisted component of $\mathcal{C}^\vee$ is more involved: the total space $S_4(g, g, g, g; 1, 1, 1, 1)$ of the relevant cover is a surface of genus one. The two gluing procedures over $S_4$ give the following markings:

![Diagram](image)

Since the cover is not a genus zero surface any longer, the rules of the LTG do not allow us to remove the cuts and to contract lines of the marking. Hence we keep lines indicating the cuts which are drawn with dotted lines. Transforming $S_4(g, g, g, g; 1, 1, 1, 1)$ into a standard block along the second marking gives the following surface of genus one with four boundary components:
The transformation

\[
\begin{array}{c}
\text{(103)}
\end{array}
\]

of the resulting marking into the standard marking on the four-holed torus is given by the following sequence of LTG-moves:

\[
S^{-1} \circ B_{TMN,R(1)} \circ B_{O,R(2)}^{-1}
\]  

(105)
Translating these moves into a morphism in the modular tensor category $\mathcal{C}$ gives

$$\alpha_{M,N,O} = \bigoplus_{i,j \in I} d_i^j \mathcal{D}$$

with $\mathcal{D} = \sqrt{p^+ p^-}$ and $p^\pm$ defined as in section 3.2. (Note that the morphism corresponding to the $S$-move in [BK01] is defined by a direct sum of morphisms $U_i U_j^\vee \to U_k U_k^\vee$; we have inserted two pairs of isomorphisms identifying for a simple object $U_k \cong U_k^\vee$ which cancel pairwise.)

These associativity constraints have to satisfy mixed pentagon axioms for any choice of four objects in the two sectors of $\mathcal{C}^X$, yielding in total 16 different types of pentagon diagrams. Theorem 26 asserts that our construction yields a $G$-equivariant functor; the general results of [KP08, Section 7.2] then ensure that all associativity constraints obtained in this section satisfy the pentagon axiom. We have checked this by hand as well; only the pentagon with four objects in the twisted component is more involved.

4.5 Tensoriality of the $\mathbb{Z}/2$-action

We next derive the isomorphisms $\varphi_{A,B} : h(A \otimes B) \to hA \otimes hB$ that turn the equivariance functor $R_h$ into a tensor functor. They have been described in general before lemma 22. The functor $R_1$ is the identity functor and the tensoriality constraints $\varphi^1_{A,B}$ are identity morphisms. The non-trivial element $g \in \mathbb{Z}/2$ acts by permutation of factors on the untwisted component $\mathcal{C} \boxtimes \mathcal{C}$ and as the identity on the twisted component $\mathcal{C}$. Hence we only compute the tensoriality constraint $\varphi^g_{A,B}$.

Before we proceed, we will have a look at the $S_2$ cover of the form $(S_2(g,g;1,g) \to S_2)$, together with its marking. As a smooth manifold the total space $S_2(g,g;1,1)$ is diffeomorphic to $S_2(g,g;1,1)$ by a half turn around the second hole. However this is not a map of marked covers over $S_2$. We choose a half turn diffeomorphism (say by rotating the second circle clockwise by $\pi$) to identify the total spaces $S_2(g,g;1,g)$ and $S_2(g,g;1,1)$. Both spaces are diffeomorphic to the two-punctured sphere $S_2$ and the corresponding diffeomorphism of $S_2$ induces a natural isomorphism $\text{Hom}_C(1,UV) \cong \text{Hom}_C(1,U^gV)$. Using the duality, we get an isomorphism $\sigma_V : V \to U^gV = V$. We use this isomorphism to identify the respective hom-spaces. On marking graphs this introduces an additional move which we call the $\sigma$-move,

$$A \xrightarrow{\sigma_B} B$$

(107)
Figure 1: The marking graphs on $S_3$-covers that represent the non-trivial tensor products. We show three Riemann spheres $\mathbb{C}$ from above. The arrow points to the disc where the test object $T$ is inserted. All covers are twofold; their total space has the topology of a four-holed sphere. The dashed line is a branch cut linking the two insertions with non-trivial monodromy and indicates a self-intersection in the immersion of the total space of the cover into three-dimensional space.

on the two-punctured sphere $S_2$ and

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\end{array}
\end{array}

(108)

on the total space $S_2(g, g; 1, g)$ of the cover. Applying the half-turn diffeomorphism twice is the Dehn-twist, hence we have the identity $\sigma_V^2 = \theta_V \in \text{Hom}_C(V, V)$.

From now on we will draw the standard spheres $S_n$ as the one-point compactification $\mathbb{C}$ of the complex plane with $n$ discs of radius $1/3$ centered at $0, 1, 2, \ldots n - 1$ removed. As in subsection 2.2.1 the standard blocks are obtained by identifying the trivial cover of the cut sphere along the cuts, i.e. as $(S_n \setminus \text{cuts} \times G)/\sim$. For instance the marking graphs on covers of $S_3$ that represent the tensor products are shown in figure 1.

We are now ready to compute the tensoriality constraint $\varphi^g_{A, B}$. We apply the $\sigma$-move introduced in picture (108) to the relevant cover of $S_2$ obtain the marking graph
on $S_2(g, g; 1, g)$. Then we apply the sequence of transformations in \[21\] on $S_2(g, g; 1, g)$ and translate them into morphisms, which have to be applied after the morphism $\sigma$.

We outline our procedure to determine the tensoriality constraint $\varphi^g_{A,B}$:

1. Determine the cover of the two-punctured sphere that is appropriate for the pair of objects $(A, B)$.
2. If the tensor product takes its value in the twisted component, $A \otimes B \in \mathcal{C}_g^X$, use the half-twist $\sigma$ to identify the covers $S_2(g, g; 1, 1)$ and $S_2(g, g; 1, g)$.
3. Apply the diffeomorphism $\tilde{g}$ from equation \[18\] to the cover.
4. Now we can glue in the cover of the three-punctured sphere $S_3$ that implements the tensor product $A \otimes B$.
5. Apply the diffeomorphism $\tilde{g}^{-1} = \tilde{g}$ again.
6. Transform the resulting cover of $S_3$ back to a standard block using the marking from the tensor product $^gA \otimes ^gB$.
7. Read off the LTG-moves that transform the resulting marking into the standard marking.
   If $A \otimes B$ is in the untwisted component, this is the tensoriality constraint. Otherwise, if $A \otimes B \in \mathcal{C}_g^X$, the tensoriality constraint is obtained by first applying the morphism $\sigma_{A \otimes B}$ to $^g(A \otimes B)$ to take into account step 2, and then the LTG-moves.

The choice of the diffeomorphism $\sigma$ to identify the total spaces $S_2(g, g; 1, 1)$ and $S_2(g, g; 1, g)$ is non-canonical. Since the mapping class group of the cylinder is generated by the Dehn-twist, different choices of identifications differ by powers of the Dehn-twist, which in our conventions is the square of $\sigma$. A different choice of the identification of $S_2(g, g; 1, 1)$ and $S_2(g, g; 1, g)$ gives different markings on the cover of $S_3$ in step 6 of our procedure. As a consequence, the LTG-moves will differ by powers of the twist on $^g(A \otimes B)$. The difference then cancels in step 7, so that the tensoriality constraint is independent of the choice of identifications of $S_2(g, g; 1, 1)$ and $S_2(g, g; 1, g)$. 

\[109\]
We will start by deriving the tensoriality constraint \( \varphi^g_{A_1 \times A_2, B_1 \times B_2} \) for \( g \) on the tensor product of two objects in the untwisted component. Our procedure gives the standard marking

\[
A_1 \times A_2 \quad B_1 \times B_2 \quad C_1 \times C_2
\]

hence we deduce that the constraint is the identity, \( \varphi^g_{A_1 \times A_2, B_1 \times B_2} = \text{id}_{A_2 B_2 \times A_1 B_1} \).

For the constraint \( \varphi^g_{\mathcal{M}, A_1 \times A_2} \), we first have to use \( \sigma \) to identify \( S_2(g, g; 1, 1) \) and \( S_2(g, g; 1, g) \); then we apply the sequence of operations described in [21]:

\[
(110)
\]
In the first step we apply the half-twist $\sigma_{MA_1A_2}$, in the second step the application of $\tilde{g}$ as defined in (18) exchanges the two sheets of the cover. In the third step we glue in the three-puncture sphere $S_3(g, g, 1; 1, 1, 1)$ with the marking representing the tensor product $M \otimes (A_1 \times A_2)$, see figure 1. In the fourth step we perform the gluing on the marking graph. In the fifth step we apply $\tilde{g}$ again, which in particular exchanges the holes labelled by $A_1$ and $A_2$. The last step is merely a simplification of the graph: we move the lines around the back side of the sphere.

We use the marking on $S_3(g, g, 1; 1, 1, 1)$ in figure 1 that represents the tensor product $M \otimes (A_2 \times A_1)$ to get an isomorphism to $S_4$. This marking instructs us to move the disc labelled by
$A_1$ around the disc labelled by $M$. This yields the left figure in the following line:

In the second picture we redraw the marking graph in a more convenient shape. The last picture is obtained by a diffeomorphism of non-embedded manifolds.

The marking graph in the third picture of (112) is transformed into the standard marking graph on $S_4$ by the following sequence of LTG-moves:

$$\theta_{A_1}^{-1} \circ \sigma_M \circ B_{M A_2, A_1}^{-1} \circ B_{A_2, M} \circ B_{T, M}$$

where $\theta$ is an abbreviation for the Dehn twist move $Z \circ B^{-1}$. When we translate the LTG moves into morphisms, several Dehn twists occur in manipulations of the ribbon graphs. They have to be combined with the morphism $\sigma$ which squares to the twist. This leads to powers of $\sigma$ that differ from the naive expectations, and we arrive at the tensoriality constraint

$$\varphi_{M, A_1 \times A_2}^{g, M, A_1} = \varphi_{M, A_1 \times A_2, M, A_1}$$

where we have included $\sigma_{M A_1 A_2}$ according to our general prescription, step 7.

For tensoriality constraint $\varphi_{A_1 \times A_2, M}^g$, we proceed in a similar way. Again we identify the total spaces of the covers $S_2(g, g; 1, 1)$ and $S_2(g, g; 1, g)$ with the diffeomorphism $\sigma$ and apply (21). This gives:
The first picture is the result of (21). The second picture is the result of transforming $S_3(g, 1, g; 1, 1, 1)$ to $S_4$ using the marking on $S_3(g, 1, g; 1, 1, 1)$ that represents the tensor product $^g(A_1 \times A_2) \otimes ^g M = (A_2 \times A_1) \otimes M$.

The resulting marking graph is transformed into the standard graph by the sequence

$$B^{-1}_{A_2,A_1} \circ B^{-1}_{A_2,M} \circ \theta^{-1}_{A_2} \circ B^{-1}_{M,A_2} \circ \sigma^{-1}_M$$

of LTG-moves. This gives the morphism

$$\varphi^{g}_{A_1 \times A_2,M} =$$

(117)

The last tensoriality constraint to be determined is $\varphi^{g}_{M,N}$. We apply (21) to $S_2(1, 1; 1, 1)$ and get

$$T_1 \times T_2 M$$

(118)
The first picture is again the result of (21), the second picture is the marking graph we obtain on $S_4$. Now this marking is transformed into the standard marking by the LTG-moves

$$B_{NT_2,M} \circ B_{N,T_2}^{-1} \circ B_{N,M}^{-1} \circ \sigma_N^{-1} \circ \sigma_M \quad (119)$$

Recall that $g(M \otimes N) = \bigoplus U_i \times MNU_i^\vee$. Hence, we have to apply the corresponding transformations to $\text{Hom}_C(1, T_i U_i) \otimes_k \text{Hom}_C(1, T_2 MNU_i^\vee) \cong \text{Hom}_C(1, T_1 T_2 M N)$ and finally use the adjunction (60) to obtain

$$\varphi_{M,N}^g = \bigoplus_{i,j \in I} \sum_{\alpha} \alpha \otimes_k M U_i U_j^\vee \varphi_{M,N}(120)$$

where the summation over $\alpha$ runs over a basis of $\text{Hom}_C(U_j, MNU_i^\vee)$ and the corresponding dual basis. Again, we had to take into account Dehn twists which shifted the powers of $\sigma$.

We have checked directly that all morphisms derived indeed satisfy all identities needed to endow the functor $R_g$ with the structure of a tensor functor.

### 4.6 The braiding

We finally derive the braiding on the $\mathbb{Z}/2$-equivariant category $C^X$ which consists of isomorphisms $C_{U,V} : U \otimes V \xrightarrow{\cong} pV \otimes U$ with $U \in C^X_p$ and $V \in C^X_q$. To this end, we lift the braiding diffeomorphism

$$\varphi_B : S_3((pq)^{-1}, p, q; 1, 1, 1) \xrightarrow{\cong} S_3((pq)^{-1}, pqp^{-1}, p; 1, p^{-1}, 1) \quad (121)$$

of the three-holed sphere $S_3$ to appropriate covers and obtain a diffeomorphism

$$\varphi_B : S_3((pq)^{-1}, p, q; 1, 1, 1) \xrightarrow{\cong} S_3((pq)^{-1}, pqp^{-1}, p; 1, p^{-1}, 1) \quad (122)$$

It induces for any object $T \in C^X$ a natural isomorphism

$$\langle T, U \otimes V \rangle_X \xrightarrow{\text{def}} \langle T, U, V \rangle_X \xrightarrow{\text{def}} \tau^X(S_3((pq)^{-1}, p, q; 1, 1, 1); T, U, V) \xrightarrow{(\varphi_B)_*} \tau^X(S_3((pq)^{-1}, pqp^{-1}, p; 1, p^{-1}, 1); T, V, U) = \tau^X(S_3((pq)^{-1}, pqp^{-1}, p; 1, 1, 1); T, pV, U) \xrightarrow{\text{def}} \langle T, pV, U \rangle_X \xrightarrow{\text{def}} \langle T, pV \otimes U \rangle_X. \quad (123)$$
Thus the procedure to determine braidings is analogous to the one to determine the associativity constraints:

1. Start with the standard marking on the cover $S_3((pq)^{-1}, p, q; 1, 1, 1)$ that represents the tensor product $U \otimes V$.

2. Apply the diffeomorphism $\tilde{\varphi}_B$ defined in (122).

3. The result is the cover $S_3((pq)^{-1}, pq^{-1}, p; 1, p^{-1}, 1)$ and has to be transformed into a standard block, using the marking representing the tensor product $pV \otimes U$.

4. Next use the LTG.moves to transform the resulting marking graph on the standard block into the standard marking graph.

5. Finally translate the LTG-moves into morphisms in $C$ or $C \boxtimes C$, respectively.

Not surprisingly, the braiding of two objects $A_1 \times A_2$ and $B_1 \times B_2$ in the neutral component of $C^X$ turns out to be the braiding on $C \boxtimes C$. To see this, we lift the braiding diffeomorphism $\varphi_B$ to $S_3(1, 1, 1; 1, 1, 1)$. As a smooth manifold, this is just isomorphic to the disjoint union $S_3 \sqcup S_3$ of two three-holed spheres. The lift $\tilde{\varphi}_B$ of the braiding isomorphism is just $\varphi_B$ applied to both components. Hence

$$C_{A_1 \times A_2, B_1 \times B_2} = c_{A_1, B_1} \otimes c_{A_2, B_2}. \quad (124)$$

We now discuss the more complicated situations. For the braiding $C_{A_1 \times A_2, M} : (A_1 \times A_2) \otimes M \to M \otimes (A_1 \times A_2)$, we have to consider the cover $S_3(g, 1, g; 1, 1, 1)$ of $S_3$ with its standard marking graph. Lifting the braiding $\varphi_B$ gives the marking

![Diagram](image)

on $S_3(g, g, 1; 1, 1, 1)$. Applying the diffeomorphism to $S_4$ that represents the tensor product $^1M \otimes (A_1 \times A_2) = M \otimes (A_1 \times A_2)$ gives the marking

![Diagram](image)
on $S_4$. It is connected to the standard marking of the four-punctured sphere $S_4$ by the following sequence of LTG-moves:

$$\mathbf{B}_{A_1,M} \circ \mathbf{B}_{T,A_2} \circ \mathbf{B}_{A_1,A_2}$$

(127)

Applied to $\text{Hom}_C(1, TA_1A_2M)$, this induces the following braiding isomorphism on $C^x$

$$C_{A_1 \times A_2, M} =$$

(128)

For the braiding $C_{M,A_1 \times A_2} : M \otimes (A_1 \times A_2) \rightarrow \theta^9(A_1 \times A_2) \otimes M = (A_2 \times A_1) \otimes M$ we consider the cover $S_3(g, g, 1; 1, 1, 1)$ of $S_3$ with its standard marking graph. We lift the braiding $\varphi_B$ and get the marking

(129)

on $S_3(g, 1, g; 1, g, 1)$. Note that the insertions of $A_1$ and $A_2$ have exchanged the sheet, when they were moved pass the self-intersection. We apply the diffeomorphism to $S_4$ given by the marking
on $S_3(g, 1, g; 1, g, 1)$ that represents the tensor product $g(A_1 \times A_2) \otimes M = (A_2 \times A_1) \otimes M$. The result is the following marking on $S_4$:

This marking on $S_4$ is transformed into the standard marking on $S_4$ by the LTG-moves

$$B_{A_1M, A_2} \circ B_{M, A_1}.$$  

We get the braiding isomorphism on $C^X$:

Finally we describe the braiding isomorphism $C_{M,N} : M \otimes N \to gN \otimes M = N \otimes M$. We lift the braiding $\varphi_B$ to $S_3(1, g, g; 1, 1, 1)$. The standard marking on $S_3(1, g, g; 1, 1, 1)$ involves a cut: we first remove this cut and then apply $\tilde{\varphi}_B$.

$$\varphi_B$$  

$$\tilde{\varphi}_B$$  

$$\varphi_B$$
We transform the resulting manifold into $S_4$ and arrive at

\[ \text{(134)} \]

This marking is transformed into the standard marking on $S_4$ by the following sequence of moves:

\[ \mathbf{B}_{T_1,N}^{-1} \circ \mathbf{B}_{M,T_2}^{-1} \circ \mathbf{B}_{T_1,T_2}^{-1} \circ \sigma_N^{-1} \]  

\[ \text{(135)} \]

When we apply this to $\text{Hom}_C(1, T_1 MNT_2)$ and perform the gluing process in the adjunction (56), we arrive at

\[ C_{M,N} = \bigoplus_{i \in I} \otimes_k U_i \]

\[ \text{(136)} \]

Again powers of $\sigma$ are changed by taking into account Dehn twists.

By theorem [26] our construction yields a $G$-equivariant modular functor, and by the general results of [KP08], braiding morphisms obtained from a $G$-equivariant modular functor satisfy $\mathbb{Z}/2$-equivariant generalizations of the hexagon axioms. We have also directly verified that the morphisms presented in this subsection satisfy the hexagon axioms.

### 4.7 Equivariant ribbon structure

We now return to study the existence of a $\mathbb{Z}/2$-equivariant ribbon structure on $C^X$. The general results of [KP08] ensure that $C^X$ has a twist but do not guarantee the existence of duality morphisms like the evaluation $D_U : U \otimes U^* \to 1$. However, in the case of $\mathbb{Z}/2$-permutation equivariant categories, there are indeed compatible duality morphisms that endow $C^X$ with a ribbon structure.

We obtain [KP08, Section 7.2] the twist morphism $\Theta_U : U \to \nu U$ for $U \in \mathcal{C}^X_p$ by the Yoneda
lemma from a natural transformation of functors
\[
\tau_X(S_2(p, p^{-1}; 1, 1); U, T) \xrightarrow{(\tilde{\varphi}_B^{-1})^*} \tau_X(S_2(p^{-1}, p; 1, p^{-1}); T, U)
\]
\[
T_{(1,p)} \tau_X(S_2(p^{-1}, p; 1, 1); T, pU)
\]
\[
(\tilde{\varphi}_Z^*)^* \tau_X(S_2(p, p^{-1}; 1, 1); pU, T)
\]
(137)

Here \(\varphi_B\) is the braiding of the two holes of \(S_2\) and \(\tilde{\varphi}_B\) its lift to the cover \(S_2(p^{-1}, p; 1, p^{-1})\). The equivariance morphism \(T_{(1,p)}\) is, in our case, the identity. Finally, \(\varphi_Z\) is the diffeomorphism of the standard sphere inducing a cyclic move of the distinguished edge, and \(\tilde{\varphi}_Z\) is its lift.

For an object \(U = A_1 \times A_2\) in the untwisted component of \(C^X\), the total space \(S_2(1, 1; 1, 1)\) of the cover is again just a disjoint union of two copies of \(S_2\) with the lifts of \(\varphi_B^{-1}\) and \(\varphi_Z\) being the application of the diffeomorphisms to both components separately. Hence the twist on \(C_1^X\) is just the usual twist on the category \(C \boxtimes C\):

\[
\Theta_{A_1 \times A_2} = \theta_{A_1} \otimes_k \theta_{A_2}
\]
(138)

To calculate the twist morphism for an object \(U = M \in C^X_g\) in the twisted component, our moves amount to

\[
(139)
\]

Transforming the total space of this cover into the standard sphere \(S_2\) gives the following marking:

\[
(140)
\]
Here, we redrew the figure by pulling the line connecting to $M$ along the backside of the sphere. The last figure is transformed into the standard marking on $S_2$ by applying the half turn move $\sigma_M$. Hence we find

$$\Theta_M = \sigma_M.$$ \hfill (141)

Permutation orbifolds of rational conformal field theories have been analyzed with representation theoretic tools in [BHS98]. The formula (141) for the twist in the twisted component is in full agreement with formula [BHS98, (4.21)] for the conformal weights.

Up to this point, we have derived all structure on $C^X$ from the approach of $\mathbb{Z}/2$-equivariant modular functors [KP08]. In fact, we have fully exploited this ansatz and obtained the structure of a weakly rigid $\mathbb{Z}/2$-equivariant monoidal category. We next check that the condition in proposition 23 that ensures that the tensor category $C^X$ is even rigid is satisfied. The criterion is easy to check for simple objects $U_i \times U_j$ in the untwisted component of $C^X$. In this case the morphism

$$i_{U_i \times U_j} : 1 \times 1 \to U_i U_i^\vee \times U_j U_j^\vee$$ \hfill (142)

is just given by the tensor product of two coevaluations,

$$i_{U_i \times U_j} = b_{U_i} \otimes_k b_{U_j}.$$ \hfill (143)

The morphisms in proposition 23 $a_{i \times j}$ are non-zero by rigidity of $C \boxtimes C$. The ribbon structure on the neutral component $C^X_1$ is then just the usual ribbon structure on $C \boxtimes C$.

Now we look at a simple object $U_i$ in $C^X_g$. As we have seen before, the dual object in $C^X$ of $U_i$ is $U_i^\vee$. The adjunction (60) with $T_1 = T_2 = 1$ and $N = U_i^\vee$ gives isomorphisms

$$\text{Hom}_C(U_i^\vee, U_i^\vee) \cong \text{Hom}_C(1, U_i U_i^\vee) \cong \text{Hom}_{C^{\boxtimes 2}}(1 \times 1, R(U_i U_i^\vee))$$ \hfill (144)

Evaluating the isomorphism on $1 \otimes U_i^\vee$ gives a morphism in $\bigoplus_{j \in I} \text{Hom}_{C^{\boxtimes 2}}(1 \times 1, U_i U_i^\vee U_j^\vee \times U_j)$ whose only non-vanishing component appears for $j = 0$. It reads

$$i_{U_i} := \begin{array}{ccc}
U_i & U_i^\vee \\
\otimes_k & 1 \\
1
\end{array}$$ \hfill (145)

Now compute $\alpha_{U_i, U_i, U_i}^{-1} \circ (\text{id}_{U_i^\vee} \otimes i_{U_i})$ using equation (106) for the associativity constraint:

$$\alpha_{U_i, U_i, U_i}^{-1} \circ (\text{id}_{U_i^\vee} \otimes i_{U_i}) = \bigoplus_{k \in I} d_k \frac{d_k}{D}$$ \hfill (146)

60
Here $d_k$ is the dimension of the simple object $U_k$ and $D$ is the dimension of the category $C$ introduced in (25). The morphism

$$a_i : 1 \times 1 \to \bigoplus_k U_i^\vee U_i^\vee \times U_k$$

introduced in proposition 23 can only have a non-vanishing component for $k = 0$,

$$a_i^{(0)} : 1 \times 1 \to U_i^\vee U_i \times 1.$$

Since the tensor unit $1$ is absolutely simple, i.e. $\text{End}(1) = k\text{id}_1$, this component $a_i^{(0)}$ is of the form

$$a_i^{(0)} = \pi_i \otimes_k \text{id}_1$$

with $\pi_i \in \text{Hom}_C(1, U_i^\vee U_i)$. We then have

$$a_{U_i^\vee, U_i, U_i^\vee}^{-1} \circ (\text{id}_{U_i^\vee} \otimes i_{U_i}) = \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array}
+ \sum_{k \neq 0} \cdots
$$

(147)

Hence

$$\begin{array}{c}
\begin{array}{c}
1 \\
D
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array}
$$

(148)

To determine $\pi_i$, we take a partial trace on both sides an arrive at

$$\begin{array}{c}
\begin{array}{c}
1 \\
D
\end{array} \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array} = d_i \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
U_i^\vee \\
U_i
\end{array}
\end{array}
$$

(149)
so that

\[
\begin{array}{c}
\begin{array}{c}
\pi \\
\otimes_k \\
\end{array}
\end{array}
\begin{array}{c}
U_i \otimes U_i \\
U_i \otimes U_i \otimes_k \\
\end{array}
\begin{array}{c}
= \frac{1}{Dd_i} \\
\end{array}
\begin{array}{c}
\cup \\
\end{array}
\begin{array}{c}
U_i \otimes U_i \\
\end{array}
\end{array}
\]  

(150)

and hence

\[
\begin{array}{c}
\begin{array}{c}
\cup \\
\otimes_k \\
\end{array}
\end{array}
\begin{array}{c}
U_i \otimes U_i \\
U_i \otimes U_i \otimes_k \\
\end{array}
\begin{array}{c}
a_i = \frac{1}{Dd_i} \\
\end{array}
\begin{array}{c}
1 \\
\end{array}
\]  

(151)

This morphism is non-zero since the coevaluation of \( C \) is non-zero. Hence the \( \mathbb{Z}/2 \)-permutation equivariant tensor category \( \mathcal{C}^X \) is rigid.

The evaluation morphisms \( D_{U_i} : U_i^\vee \otimes U_i \to 1 \times 1 \) are fixed by the condition \( D_{U_i} \circ a_i = \text{id}_{1 \times 1} \). It is easy to see that the right evaluation reads

\[
\begin{array}{c}
\begin{array}{c}
\cup \\
\otimes_k \\
\end{array}
\end{array}
\begin{array}{c}
U_i \otimes U_i \\
U_i \otimes U_i \otimes_k \\
\end{array}
\begin{array}{c}
D_{U_i} = D \\
\end{array}
\begin{array}{c}
1 \\
\end{array}
\]  

(152)

Similarly we find for the left evaluation

\[
\begin{array}{c}
\begin{array}{c}
\cup \\
\otimes_k \\
\end{array}
\end{array}
\begin{array}{c}
U_i \otimes U_i \\
U_i \otimes U_i \otimes_k \\
\end{array}
\begin{array}{c}
\tilde{D}_{U_i} = \tilde{D} \\
\end{array}
\begin{array}{c}
1 \\
\end{array}
\]  

(153)
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