Self-localization of directed polymers and oppressive population control

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We construct a phenomenological theory of self-localization of directed polymers in $d+1$ dimensions. In $d=1$ we show that the polymer is always self-localized, whereas in $d=2$ there is a phase transition between localized and free states. We also map this system to a model of population dynamics with fixed total population. Our previous results translate to static and expanding population clusters, depending on the birth and death rates. A novel “pseudo-travelling wave” is observed in some sectors of parameter space.

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Directed polymers are important topological objects in many condensed matter systems. Examples are flux lines in superconductors, domain walls in ferromagnets, and atomic steps on vicinal surfaces. In the case of an isolated directed polymer, theoretical studies have concentrated on the effects of quenched disorder, which may be either point-like or columnar. Using disorder to suppress the wandering of flux lines is a central concept in designing useful high-temperature superconductors.

Another mechanism by which the wandering of directed polymers may be suppressed is self-localization, mediated by elastic distortions of the medium in which the polymer is embedded. This has been studied previously via a variational free energy functional, which is the analogue of the quantum action for the polaron problem. In the first part of this Letter we study the phenomenon from an alternative viewpoint, using the statistical mechanics of directed polymers. Such an approach leads us to a simple phenomenological equation for the probability density $P$ of the directed polymer. We show that in $d=1$ the directed polymer is always self-localized, while in $d=2$ the polymer undergoes a transition to self-localization above a critical value of the coupling constant $\lambda$ (which measures the strength of interaction between the polymer and the embedding environment). The details of this transition are investigated by numerical integration.

In the latter half of this Letter we show that the phenomenological equation for $P$ may be re-interpreted in the language of population dynamics. In this case $P$ corresponds to the local population density. The precise form of the dynamics is unusual, since along with the birth and death terms, we have a global constraint that fixes the total population. [Similar constraints have been studied recently in the context of plant population models, and a link has been made to self-organized criticality. Also, interesting links have been made between “non-Hermitian quantum mechanics” and population biology.] The issue in the present work is not whether the population thrives or becomes extinct, but rather how the individuals spatially distribute themselves. Different phases are observed corresponding to self-localized and expanding populations. We also test the predictions of the theory by simulations of a discrete reaction-diffusion model with a global constraint. Finally, within the context of population dynamics we are at liberty to reverse the sign of the coupling constant $\lambda$. In this case, we find that in $d=1$ the density is delocalized via a pseudo-travelling wave – an unusual scenario which interpolates between dynamical scaling and travelling waves. We end the Letter by noting the general mapping between directed polymers and constrained population dynamics.

We consider a directed polymer with elastic constant $\kappa$ in a space with $d$ transverse dimensions $r$, and one longitudinal dimension $t$. The polymer experiences thermal fluctuations from a reservoir at temperature $T$, and a potential $V(r,t)$. One can write the partition function $Z(r,t)$ for such a polymer with ends fixed at $(0,0)$ and $(r,t)$ in the form of a path integral. Following standard methods, this path integral may be rewritten as a partial differential equation of the form

$$\partial_t Z = \nu \nabla^2 Z - V Z,$$

where $\nu = T/2\kappa$ and we have absorbed a factor of $T$ into the potential. The probability density of the polymer is constructed by normalizing $Z$: $P(r,t) = Z(r,t)/\int d^d r' Z(r',t)$. Inserting this definition into the equation for $Z$ yields a closed equation for $P$:

$$\partial_t P = \nu \nabla^2 P - V P + \left[ \int d^d r' V(r',t) P(r',t) \right] P.$$

To our knowledge, this description of the directed polymer has not been studied previously. Note this equation does not have the form of a Fokker-Planck equation – conservation of probability is not achieved via an equation of continuity. The physics of directed polymers is intrinsically non-local.

We want to consider the potential $V$ to have been caused purely from elastic distortions of the embedding environment, due to the presence of the directed polymer itself. In principle we could formulate a detailed microscopic model which includes these elastic degrees of freedom explicitly, and then attempt to integrate them out to find the effective potential. However, we are content to take a phenomenological viewpoint: the elastic distortions caused by the polymer will provide an attractive
potential which is strongest in the immediate vicinity of the polymer, and negligible far from the polymer. If we try to construct such a potential from the probability density \( P \), we come at once upon the particularly simple form \( V(\mathbf{r}, t) = -\lambda P(\mathbf{r}, t) \), where \( \lambda \) is a non-negative coupling constant. Inserting this relation into Eq.\( \text{(3)} \) we have a closed equation for \( P \) of the form

\[
\partial_t P = \nu \nabla^2 P + \lambda P^2 - \lambda \sigma(t)P , \quad (3)
\]

where \( \sigma(t) \equiv \int d^d \mathbf{r} \, P(\mathbf{r}, t)^2 \).

Given we have a non-linear integro-differential equation for \( P \) there is little hope of a complete analytic solution. However we will uncover much of the physics from scaling arguments, and numerical integration. First, let us consider the possibility of simple scaling, and make a similarity Ansatz, which, based on the radial symmetry of the problem, we take as \( P(\mathbf{r}, t) = \xi(t)^{-d} F(z) \), with \( z = r/\xi \). Note, the prefactor is fixed by normalization of \( P \). Inserting this Ansatz into Eq.\( \text{(3)} \) yields

\[
\dot{\xi}(dF + zF') + (\nu/\xi)(F'' + (d-1)F'/z) + \lambda/\xi^{d-1}(F - \sigma_0)F = 0 , \quad (4)
\]

where \( \sigma_0 = S_d \int dz z^{d-1} F(z)^2 \), and \( S_d \) is the area of the unit hypersphere. We need to balance the terms arising from the derivative with respect to \( t \) and the Laplacian. This implies \( \dot{\xi} = (\nu/\xi) \), which in turn gives \( \xi = (2\nu t)^{1/2} \).

So for large \( t \) the correlation length \( \xi \) is large, and we see on comparing the Laplacian terms to the interaction terms (i.e. those proportional to \( \lambda \)) that for \( d > 2 \) the interaction terms are negligible. In other words, for \( d > 2 \) the behaviour of \( P \) corresponds to thermal diffusion (at least for modest values of \( \lambda \)).

With the assumption of a \( t \)-dependent correlation length, we see that the interaction terms dominate for \( d < 2 \), and there is no self-consistent solution. We therefore demand that \( \xi \) is \( t \)-independent, and denote it as \( \xi_0 \). Focusing on \( d = 1 \), we find

\[
F'' + (\lambda \xi_0/\nu)(F - \sigma_0)F = 0 . \quad (5)
\]

This equation is simple enough to yield an exact solution. Expressed in terms of the original spatial variable \( x \), we find the \( t \)-independent solution

\[
P(x) = (1/4\xi_0) \text{sech}^2(x/2\xi_0) , \quad (6)
\]

where the localization length is self-consistently found to be \( \xi_0 = 6\nu/\lambda \). We have confirmed the stability of this solution by direct numerical integration of Eq.\( \text{(3)} \) using a variety of initial conditions. In all cases we find rapid convergence to the form given above. It is remarkable that our solution coincides exactly with the well-known form for the polaron probability density, first found by Rashba\( \text{(3)} \). This agreement is satisfying from a physical viewpoint, but by no means obvious given the mathematical difference between Eq.\( \text{(3)} \) and the non-linear Schrödinger equation governing polaron physics.

We can achieve a perfect balance of all the terms in Eq.\( \text{(3)} \) in exactly two dimensions. In this case, the correlation length drops out of the equation and we have an ordinary differential equation for \( F(z) \) of the form

\[
F'' + (z + 1/z)F' + 2F + \lambda(F - \sigma_0)F = 0 , \quad (7)
\]

where \( \lambda = \lambda/\nu \) is a dimensionless coupling constant. We have been unable to derive an analytic solution to this equation. We have studied its properties using a straightforward numerical integration scheme, starting with a localized initial condition, generally taken to be Gaussian. For modest values of \( \lambda \) the probability density diffuses outwards, and we find good data collapse in accordance with the similarity Ansatz defined in terms of \( z = r/\sqrt{\tau} \).

As we increase \( \lambda \) we find a sudden transition at \( \lambda_c \approx 30.5 \), above which the probability density shrinks to a microscopically collapsed state. Details of this transition will be given in a longer paper\( \text{(4)} \).

We now move away from the physics of directed polymers, and re-interpret our central equation\( \text{(3)} \) within a very different context, namely reaction kinetics or population dynamics. Consider a species \( A \) (either chemical, biological or homo sapiens) which undergoes the following birth and death processes: \( A \to 0 \) and \( 2A \to 3A \), with rates \( k \) and \( k' \) respectively, which are in general time dependent. We denote the relative density of \( A \) by \( \rho(x,t) \), and stress that the symbol \( t \) now represents real time. We may write a reaction-diffusion equation to describe these processes:

\[
\partial_t \rho = D \nabla^2 \rho + k'(t)\rho^2 - k(t)\rho , \quad (8)
\]

where \( D \) is the effective diffusion constant of individual \( A \) ‘particles’. The above equation is only an approximate description of this process, and \( a \, \text{priori} \) is only expected to be valid above some critical dimension \( d_c \). For lower dimensions, it is often necessary to explicitly include the appropriate (microscopically derived) noise term in the above equation\( \text{(4)} \). For a qualitative understanding of the model (which is further refined below), such a noise term is not required.

At this point, we shall introduce a somewhat unusual feature – a global constraint on the population. In other words we fix the normalization of the relative density to unity: \( \int d^d \mathbf{r} \rho = 1 \). From Eq.\( \text{(3)} \) we see this imposes \( k'(t)/k(t) = \int d^d x \rho^2 \). We can imagine two special cases of the above. First, we can fix the death rate \( k \) to be constant, which implies that the total population is controlled by tuning the birth rate \( k' \) (which is not unknown). Alternatively, we have the chilling mechanism of total population control in which the birth rate is fixed, equal to \( k_0 \) say, and the death rate is adjusted according to \( k(t) = k_0 \int d^d x \rho^2 \). At the level of our phenomenological description, these two choices lead to similar behaviour, so we shall concentrate on the second.
we find that this profile spreads outwards, with a flat central region of height \( A/\xi_0 \), where \( \xi_0 \) is the lateral width. The solution for Eq. (11) is compared with simulations in Fig. 1, with \( \xi_0 \) fitted to the mean cluster size.

In the directed polymer context there is no compelling reason to study the system for \( \lambda < 0 \). However, within the context of population dynamics, this is a perfectly acceptable sector of parameter space to explore. In order to motivate such a model, we shall consider a different birth/death process described above. Our model consists of \( N \) random walkers on a \( d \)-dimensional hypercubic lattice. The walkers obey exclusion statistics. If two walkers are at adjacent lattice sites, there is a probability \( q \) for them to spawn a new walker at a neighbouring unoccupied site. If this occurs, a walker is randomly selected from the entire population and is killed - thus conserving the overall population. We have studied this model in various dimensions. In \( d = 1 \), and starting from a compact cluster of 100 particles, we find that the cluster spreads out somewhat, but that the mean cluster size asymptotes to a constant. On averaging over realizations we find that the density profile of the cluster (in the center of mass frame) is in approximate agreement with the solution of the continuum model. We refer the reader to Fig. 1 where a direct comparison is made.

In \( d = 2 \) we find a phase transition on varying the spawning probability \( q \). For \( q < q_c \) the cluster asymptotically diffuses outwards, with a cluster size growing as \( \sqrt{t} \). For \( q > q_c \) the cluster remains compact. The value of \( q_c \) is generally very small. For example, for \( N = 400 \), we find \( q_c \approx 0.004 \). The precise \( N \) dependence of \( q_c \), along with details of the simulations will be given in a longer work [14]. At a qualitative level, we see that the continuum formulation [3] provides a good description of the underlying microscopic model.
where $F(y)$ is simply related to the initial profile. Since this solution does not have a canonical dynamical scaling form, there may be other relevant scales; and indeed we find that there is a second important length scale, which is the width $W(t)$ of the interface separating the region of "flat" nonzero density and the region of vanishing density. This quantity changes in time according to $W(t) \sim \xi(t)\xi_0/\xi(t)^3$. Therefore the interface width shrinks (grows) with time for $\beta > 1$ ($\beta < 1$). The above solution is rather unusual as it, in some way, interpolates between a travelling wave solution $F(x - vt)$ and a scaling (or similarity) solution $F(x/\xi(t))$ [12]. We term it a "pseudo-travelling wave". [We note in passing that an initial condition with tails decaying with a power $q$ evolve under normal dynamical scaling with $\rho \sim (1/t)F(|x|/t)$ and $F(z) \sim z^{-q}$ for $z \gg 1$.]

One may now ask how the above picture is modified when one re-instates the local coupling $D$. We have several arguments which all agree that, so long as the initial profile has a finite transverse scale, then for late times the density will have the form (11), but with a selected value of $\beta = 1/2$. This implies that the profile grows outwards with a transverse scale $\xi(t) \sim t^{2/3}$ and interfacial width $W(t) \sim t^{1/3}$. The simplest (albeit non-rigorous) way to see why this value of $\beta$ is selected is to assume that for non-zero $D$ the density profile has a form approximately given by (11), and to then ask how the Laplacian term in (10) can balance with the reaction terms. Since the profile is flat for $|x| \ll \xi(t)$ and vanishing for $|x| \gg \xi(t)$ it is clear that the size of the Laplacian is set by the width of the interfacial region. We therefore estimate $\partial_x^2 \rho \sim 1/W^2 \xi$. Similarly the reaction terms are of order $\rho^\beta \sim 1/\xi^2$. Balancing the two leads us to $W(t) \sim \xi(t)^{1/2}$, which on comparing with our previous result $W(t) \sim \xi(t)^{1-\beta}$ gives $\beta = 1/2$. We have performed numerical integration of Eq.(10) in $d = 1$ for a variety of initial conditions, and with zero or non-zero $D$. The exact results for $D = 0$ have been numerically confirmed, however the situation for $D > 0$ is less clear. In Fig.2 we show an example of the evolving density grown from an initial Gaussian profile, with $D = 1$ and $r_0 = 5$. In the inset we plot the profile size and interfacial width. These quantities exhibit very strong corrections to simple power law scaling, but a closer analysis (studying the ‘running exponents’) indicates that the asymptotic power laws are approximately 0.65(1) and 0.30(1) respectively, which are close to the values obtained above.

In conclusion we have considered two very different physical scenarios which share a common mathematical description. These are self-localization of a directed polymer, and birth/death models with fixed population number. The physics of these processes is non-local which leads to a number of interesting results, including phase transitions from localized to non-localized states, and pseudo-travelling waves.

Finally, it is noteworthy that the fundamental directed polymer equation [4] with a general external potential $V$ can be recast in terms of constrained population dynamics. The case of a random potential is especially interesting, and corresponds to local, random birth/death processes, along with a global feedback to control the population. We are currently using this mapping to gain new insights into these two fascinating problems.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Density profiles for three different times, from numerical integration of (10) with $D = 1$ and $r_0 = 5$. The inset shows a log-log plot of the time evolution of the cluster size $\xi$ and the interface width $W$. The straight lines (which are a guide to the eye) have slopes of 2/3 and 1/3 respectively.
}
\end{figure}

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