On the structure of even $K$-groups of rings of algebraic integers

by

MENG FAI LIM (Wuhan)

1. Introduction. Throughout, $p$ will denote a fixed odd prime. Let $F$ be a number field with ring of integers $\mathcal{O}_F$. Inspired by the work of Tate \[30\], Browkin, Keune and Kolster made extensive studies in comparing the $K_2$-group $K_2(\mathcal{O}_F)/p^n$ with the $S$-class group $\text{Cl}_S(F(\mu_{p^n}))$ of $F(\mu_{p^n})$ in a series of papers \[4\,12\,14\,15\]. We should perhaps mention that Coates \[6\] also performed a study of this sort but instead over an (infinite) cyclotomic $\mathbb{Z}_p$-extension (see also the related work of Gras \[9\] and Jaulent \[11\]).

In this paper, we are interested in extending the work of Browkin, Keune and Kolster to the higher even $K$-groups of $\mathcal{O}_F$. In slightly more detail, let $i$ be a positive integer. Write $a := a(F)$ for the largest integer such that $F(\mu_p) = F(\mu_{p^a})$, and set $b = b(i)$ to be the largest integer such that $p^b$ divides $i$. Our main theorems compare the group $K_{2i}(\mathcal{O}_F)/p^n$ with certain eigenspaces of the $S$-class group of an appropriate cyclotomic extension of $F$. In particular, if $n \leq a + b$, we compare $K_{2i}(\mathcal{O}_F)/p^n$ with a certain eigenspace of the $S$-class group of $F(\mu_p)$ (see Theorem 4.2 for a precise statement). When $n > a + b$, the comparison is done over the field $F(\mu_{p^n-b})$ (see Theorem 4.5). Note that in the case of $K_2$, the quantity $b$ is always zero and so the comparison for large $n$ is always done over $F(\mu_{p^n})$. However, for higher even $K$-groups, the presence of $b$ requires extra care.

As illustrations of our results, we specialize them to the context of a quadratic field, where the results become slightly more explicit. In the context of $K_2$, such $p^n$-rank results were also worked out by Browkin \[4\] and Zhou \[36\]. Our discussion here generalizes those results. We note that many authors have applied the results of Browkin and Keune to obtain explicit results for $K_2$ in various specific number fields (for instance, see \[5\,17\,25\].

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It would definitely be an interesting project to apply the results of this paper to perform analogous studies for higher even $K$-groups for the above-mentioned number fields. In this note, we content ourselves with the case of a quadratic field (although we hope to return to this subject in future work).

We next turn our attention to the case of a totally real field, where we give an algebraic-$K$-theoretical formulation of Kummer’s criterion in the sense of Greenberg [10] and Kida [13] (see Theorem 6.1 and Remark 6.2 for the details). We then prove it via the algebraic-$K$-theoretical results developed in this paper. In particular, our proof does not make use of $p$-adic $L$-functions.

We end this introductory section by giving an outline of the paper. In Section 2, we collect certain basic algebraic notions and notation required for our discussion. Section 3 is where we recall the relation between Sylow $p$-subgroups of even $K$-groups and various Galois cohomology groups. This paves the way for the proof of our main results in Section 4. Section 5 is where we specialize our main results to the context of a quadratic field. Finally, in Section 6, we give the algebraic-$K$-theoretical formulation of Kummer’s criterion for a totally real field and its proof.

2. Algebraic preliminaries. To begin, we introduce some terminology and notation that will be used throughout. Let $p$ be a prime. For a positive integer $t$, write $v_p(t)$ for the $p$-adic valuation of $t$. In particular, if $t$ is a $p$-power, we have $t = p^{v_p(t)}$. For a finite abelian group $N$, denote by $N[p^n]$ the subgroup of $N$ consisting of elements annihilated by $p^n$. Plainly, $N[p^n-1] \subseteq N[p^n]$ for $n \geq 1$, where $N[p^0]$ is understood to be the trivial group. The quotient module $N[p^n]/N[p^n-1]$ can be viewed as an $\mathbb{F}_p$-vector space, where $\mathbb{F}_p$ is the finite field with $p$ elements. Therefore, it makes sense to define the $p^n$-rank $r_{p^n}(N) = \dim_{\mathbb{F}_p}(N[p^n]/N[p^n-1])$.

Throughout the paper, we also write $N/p^n$ for $N/p^nN$. The following elementary lemma is left to the reader as an exercise.

**Lemma 2.1.** Let $N$ be a finite abelian group. Then

$$r_{p^n}(N) = r_{p^n}(N/p^n).$$

In this paper, we are interested in the quantity $r_{p^n}(K_{2i}(\mathcal{O}_F))$ which, by the above lemma, leads us to analyse $r_{p^n}(K_{2i}(\mathcal{O}_F)/p^n)$. We shall frequently make use of this viewpoint without further mention.

Finally, for a given profinite group $G$ and a $G$-module $M$, we let $M^G$ be the subgroup of $M$ consisting of elements fixed by $G$, and we let $M_G$ be the largest quotient of $M$ on which $G$ acts trivially. If $M$ is a discrete $G$-module, we write $H^k(G, M)$ for the $k$th cohomology group of $G$ with coefficients in $M$. 
3. Arithmetic preliminaries. For the discussion of our main results, we need to recall the relation between algebraic $K$-groups and étale/Galois cohomology. This relation is essentially well-known (see for instance [24, 31, 33]), but for the convenience of the readers, we collect some aspects of that relation in this section. Again, we emphasize that we always work under the assumption that the prime $p$ is odd.

3.1. $i$-fold tensor product. To begin, fix once and for all an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. Therefore, an algebraic (possibly infinite) extension of $\mathbb{Q}$ will mean a subfield of $\bar{\mathbb{Q}}$. In particular, a number field is understood to be a finite extension of $\mathbb{Q}$ contained in $\bar{\mathbb{Q}}$.

Let $F$ be a number field, and $\mathcal{O}_F$ its ring of integers. Throughout, $S$ will denote the finite set consisting of all primes of $F$ above $p$ and of all infinite primes of $F$. We then write $\mathcal{O}_{F,S}$ for the ring of $S$-integers. Let $F_S$ be the maximal algebraic extension of $F$ unramified outside $S$ and denote by $G_S(F)$ the Galois group $\text{Gal}(F_S/F)$. For a given finite extension $L$ of $F$ contained in $F_S$, we write $S(L)$ for the set of primes of $L$ above $S$. Equivalently, $S(L)$ consists of the primes of $L$ above $p$ and the infinite primes of $L$. Since $L$ is contained in $F_S$, it is straightforward to verify that $F_S = L_{S(L)}$. We then write $G_S(L)$ for the Galois group $\text{Gal}(F_S/L)$.

Now, a $p$-primary discrete $G_S(F)$-module $M$ may also be viewed as a discrete $G_S(L)$-module by restricting the action. From this, we have the restriction map

$$H^k(G_S(F), M) \to H^k(G_S(L), M)$$

and the corestriction map

$$H^k(G_S(L), M) \to H^k(G_S(F), M)$$

on cohomology. For the subsequent discussion, we require the following standard facts.

**Lemma 3.1.** Suppose that $M$ is a $p$-primary discrete $G_S(F)$-module. Let $L$ be a Galois extension of $F$ contained in $F_S$.

(i) The corestriction map induces an isomorphism

$$H^2(G_S(L), M)_{\text{Gal}(L/F)} \cong H^2(G_S(F), M).$$

(ii) If the group $\text{Gal}(L/F)$ is finite of order coprime to $p$, then the restriction map induces an isomorphism

$$H^2(G_S(F), M) \cong H^2(G_S(L), M)^{\text{Gal}(L/F)}.$$  

**Proof.** The first assertion follows from reading off the initial term of the Tate spectral sequence (cf. [21, Theorem 2.5.3]). On the other hand, in view of $\text{Gal}(L/F)$ being finite of order coprime to $p$, the Hochschild–Serre spectral
sequence
\[ H^r(\text{Gal}(L/F), H^s(G_S(L), M)) \implies H^{r+s}(G_S(F), M) \]
is concentrated on the line \( r = 0 \), yielding
\[ H^s(G_S(F), M) \cong H^s(G_S(L), M)^{\text{Gal}(L/F)} \]
for every \( s \geq 0 \). In particular, taking \( s = 2 \), we have the second assertion. ■

We now introduce the discrete modules that will be frequently considered. Denoting by \( \mu_p^n \) the cyclic group generated by a primitive \( p^n \)th root of unity, we write \( \mu_p^\infty \) for the direct limit of the groups \( \mu_p^n \). These have natural \( G_S(F) \)-module structures. Furthermore, for an integer \( i \geq 2 \), the \( i \)-fold tensor products \( \mu_p^{\otimes i} \) and \( \mu_p^\infty \) can be endowed with \( G_S(F) \)-module structures via the diagonal action. Therefore, we may speak of the Galois cohomology groups
\[ H^k(G_S(F), \mu_p^{\otimes i}) \text{ and } H^k(G_S(F), \mu_p^\infty), \]
noting that
\[ H^k(G_S(F), \mu_p^{\otimes i}) \cong \lim_{n \to \infty} H^k(G_S(F), \mu_p^{\otimes i}). \]

In particular, for \( 1 \leq m \leq \infty \), we write \( \mu_p^{\otimes i}(L) \) for \( H^0(G_S(L), \mu_p^{\otimes i}) = (\mu_p^{\otimes i})^{G_S(L)} \).

Let \( S_p \) be the set of primes of \( F \) above \( p \). For a finite extension \( L \) of \( F \) contained in \( F_S \), we will write \( S_p(L) \) for the set of primes of \( L \) above \( S_p \). If \( w \) is a prime in \( S_p(L) \), we write \( L_w \) for the completion of \( L \) at \( w \). One has a natural group homomorphism
\[ \text{Gal}(\overline{L}/L) \to \text{Gal}(\overline{L}/L) \to \text{Gal}(F_S/L). \]

Via this homomorphism, \( \mu_p^{\otimes i} \) may be viewed as a \( \text{Gal}(\overline{L}/L) \)-module for \( 1 \leq m \leq \infty \), and we set \( \mu_p^{\otimes i}(L_w) := H^0(\text{Gal}(\overline{L}/L_w), \mu_p^{\otimes i}). \)

The next lemma gives a precise description of these groups.

**Lemma 3.2.** Suppose that \( i \geq 1 \). Let \( L \) be a finite extension of \( F \) contained in \( F_S \). Let \( \mathcal{L} \) denote either \( L \) or \( L_w \) for some \( w \in S_p(L) \). Set \( a(\mathcal{L}) \) to be the largest integer such that \( \mathcal{L}(\mu_p) \) contains a primitive \( p^n(\mathcal{L}) \)th root of unity, and set \( b(i) = v_p(i) \). Then
\[ \mu_p^{\otimes i}(\mathcal{L}) = \begin{cases} 
\mu_p^{\otimes i}[\mathcal{L}] & \text{if } i \equiv 0 \mod |\mathcal{L}(\mu_p) : \mathcal{L}|, \\
1 & \text{if } i \not\equiv 0 \mod |\mathcal{L}(\mu_p) : \mathcal{L}|.
\end{cases} \]

In particular,
\[ \mu_p^{\otimes i}(\mathcal{L}) = \begin{cases} 
\mu_p^{\otimes i} & \text{if } i \equiv 0 \mod |\mathcal{L}(\mu_p) : \mathcal{L}| \text{ and } n \leq a(\mathcal{L}) + b(i), \\
\mu_p^{\otimes i}[\mathcal{L}] & \text{if } i \equiv 0 \mod |\mathcal{L}(\mu_p) : \mathcal{L}| \text{ and } n > a(\mathcal{L}) + b(i), \\
1 & \text{if } i \not\equiv 0 \mod |\mathcal{L}(\mu_p) : \mathcal{L}|.
\end{cases} \]

**Proof.** See [33, Chap. VI, Proposition 2.2]. ■
Remark 3.3. Note that the final equality also says that whenever \( i \equiv 0 \mod \lfloor L(\mu_p) : L \rfloor \) and \( n \leq a(L) + b(i) \), \( \mu_{p^n}^{\otimes i} \) is a trivial \( G \)-module, where \( G \) is \( G_S(L) \) or \( \text{Gal}(\mathbb{L}_w/L_w) \) according as \( L \) is \( L \) or \( L_w \).

Let \( L \) be a finite extension of \( F \). To ease notation slightly, we shall write \( \text{Cl}_S(\mathcal{O}_L) \) for the \( S(L) \)-class group \( \text{Cl}_{S(L)}(\mathcal{O}_L) \). We then write \( A_L^{S} \) (resp. \( A_L^S \)) for the Sylow \( p \)-subgroup of \( \text{Cl}(\mathcal{O}_L) \) (resp., Sylow \( p \)-subgroup of \( \text{Cl}_S(\mathcal{O}_L) \)). The following exact sequence relates this class group to the cohomology group \( H^2(G_S(L), \mu_{p^n}) \).

**Proposition 3.4.** Let \( L \) be a finite extension of \( F \) contained in \( F_S \). Then for every \( n \geq 1 \), one has an exact sequence

\[
0 \to A_L^S/p^n \to H^2(G_S(L), \mu_{p^n}) \to \bigoplus_{w \in S(p)(L)} \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0.
\]

**Proof.** See [23, Satz 4] or [21, Proposition 8.3.11]. \( \square \)

3.2. A brief interlude on étale cohomology. On the other hand, we can view \( \mu_{p^n}^{\otimes i} \) as an étale sheaf over the scheme \( \text{Spec}(\mathcal{O}_{F,S}) \) in the sense of [19, Chap. II], and consider the étale cohomology groups \( H^k_{\text{ét}}(\text{Spec}(\mathcal{O}_{F,S}), \mu_{p^n}^{\otimes i}) \).

By [19, Chap. II, Proposition 2.9], the latter is isomorphic to the Galois cohomology group \( H^k(G_S(F), \mu_{p^n}^{\otimes i}) \). Writing \( \mathbb{Z}_p(i) = \varprojlim_n \mu_{p^n}^{\otimes i} \) and taking the inverse limit, we obtain

\[
\varprojlim_n H^k_{\text{ét}}(\text{Spec}(\mathcal{O}_{F,S}), \mu_{p^n}^{\otimes i}) \cong \varprojlim_n H^k(G_S(F), \mu_{p^n}^{\otimes i}) \cong H^k_{\text{cts}}(G_S(F), \mathbb{Z}_p(i)),
\]

where \( H^k_{\text{cts}}(\cdot, \cdot) \) is the continuous cohomology group of Tate (see [21, Chap. 2, §7]), and where the second isomorphism is a consequence of [21, Corollary 2.7.6 and Theorem 8.3.20(i)]. To simplify notation, we shall denote the left-hand side above by \( H^k_{\text{ét}}(\mathcal{O}_{F,S}, \mathbb{Z}_p(i)) \).

3.3. Algebraic \( K \)-theory. We come to the algebraic-\( K \)-theoretical aspects. As before, \( p \) will always denote an odd prime. For a ring \( R \) with identity, \( K_n(R) \) will always denote the algebraic \( K \)-groups of \( R \) in the sense of Quillen [27, 28] (see also [16, 33]). In particular, we are interested in the groups \( K_{2i}(\mathcal{O}_F) \) for \( i \geq 1 \), which are well-known to be finite by the results of Quillen (see [28, 31, 38]).

In [24], Soulé connected higher \( K \)-groups with étale cohomology groups via the \( p \)-adic Chern class maps

\[
\text{ch}_{i,k}^{(p)} : K_{2i+2-k}(\mathcal{O}_F) \otimes \mathbb{Z}_p \to H^k_{\text{ét}}(\mathcal{O}_{F,S}, \mathbb{Z}_p(i+1))
\]

for \( i \geq 1 \) and \( k = 1, 2 \). (For the precise definition of these maps, see [24]) The famed Quillen–Lichtenbaum Conjecture asserts that these maps are isomorphisms (see [16, 33] for the history). Thanks to the efforts of many, this prediction is now known to be true.
**Theorem 3.5.** For an odd prime $p$, the $p$-adic Chern class maps are isomorphisms for $i \geq 1$ and $k = 1, 2$.

*Proof.* Soulé first proved that these maps are surjective (see [24, Théorème 6(iii)]; see also the work of Dwyer and Friedlander [7, Theorem 8.7]). It is folklore (for instance, see [16, Theorem 2.7]) that the asserted bijectivity is a consequence of the norm residue isomorphism theorem (previously also known as the Bloch–Kato(–Milnor) conjecture; see [2, 20] for details). The said norm residue isomorphism theorem was proved by the deep work of Rost–Voevodsky [31] (see also [22]) with the aid of a patch from Weibel [32].

**Corollary 3.6.** Let $p$ be an odd prime. For $i \geq 1$, we have

$$K_{2i}(O_F)[p^\infty] \cong H^2_{cts}(\mathbb{G}_S(F), \mathbb{Z}_p(i + 1)).$$

Furthermore, for each $n \geq 1$,

$$K_{2i}(O_F)/p^n \cong H^2(\mathbb{G}_S(F), \mu_p^{\otimes i+1}).$$

*Proof.* Since the group $K_{2i}(O_F)$ is finite, it follows that $K_{2i}(O_F)[p^\infty] \cong K_{2i}(O_F) \otimes \mathbb{Z}_p$. By the preceding theorem, the latter is isomorphic to $H^2_{\text{ct}}(O_F, \mathbb{Z}_p(i + 1))$ which identifies with the corresponding continuous Tate cohomology group as noted in Subsection 3.2. The final isomorphism now follows by considering the long exact $\mathbb{G}_S(F)$-cohomology sequence of the short exact sequence

$$0 \rightarrow \mathbb{Z}_p(i + 1) \xrightarrow{p^n} \mathbb{Z}_p(i + 1) \rightarrow \mu_p^{\otimes i+1} \rightarrow 0$$

and noting that $H^3_{\text{ct}}(\mathbb{G}_S(F), -) = 0$ (cf. [21, Proposition 10.11.3]).

**Remark 3.7.** We should mention that the conclusions of Theorem 3.5 and Corollary 3.6 are not true in general when $p = 2$. As the current paper is concerned with the case of $p$ odd, we shall say nothing more on this but refer the interested readers to [22] for details.

4. Main results

4.1. **Twist.** As before, $p$ is an odd prime. Let $m$ be a positive integer. Once and for all, we fix a primitive $p^m$th root of unity $\zeta$ in $\overline{\mathbb{Q}}$. With such a choice, denote by $\sigma_c$ the element of $\text{Gal}(\mathbb{Q}(\mu_{p^m})/\mathbb{Q})$ defined by $\sigma_c(\zeta) = \zeta^c$. This assignment induces a group isomorphism $\text{Gal}(\mathbb{Q}(\mu_{p^m})/\mathbb{Q}) \cong (\mathbb{Z}/p^m\mathbb{Z})^\times$.

For a number field $F$, we let $\Delta$ denote the maximal subgroup of the Galois group $\text{Gal}(F(\mu_{p^m})/F)$ with order coprime to $p$. In other words, $\Delta \cong \text{Gal}(F(\mu_p)/F)$. (Of course, $\Delta$ may be trivial, for instance if $F$ contains $\mu_p$.) We shall write $d = |\Delta|$. The Galois group $\text{Gal}(F(\mu_{p^m})/F)$ may be identified with a subgroup of $\text{Gal}(\mathbb{Q}(\mu_{p^m})/\mathbb{Q})$, and under this identification, $\Delta$ can be identified with a subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$. 


For our subsequent discussion, we require an explicit description of this embedding of $\Delta$ into $(\mathbb{Z}/p\mathbb{Z})^\times$. Fix a generator $g$ of $(\mathbb{Z}/p^n\mathbb{Z})^\times$, and write $\sigma := (\sigma g)^{p-1}$. In this setting, the elements of $\Delta$ may be identified with elements of the form $\sigma^k(p-1)/d$, where $k = 0, 1, \ldots, d - 1$. The surjection $(\mathbb{Z}/p^n\mathbb{Z})^\times \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^\times$ maps $g$ to a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$ which by abuse of notation is still denoted as $g$. Therefore, correspondingly, $g^{(p-1)/d}$ is a generator of the subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$ which is the image of $\Delta$ under the embedding of $\Delta$ into $(\mathbb{Z}/p\mathbb{Z})^\times$.

Let $\omega$ be the Teichmüller character of the group $(\mathbb{Z}/p\mathbb{Z})^\times$. For a given integer $j$, we define

$$\varepsilon_j := \varepsilon_j(F) := \frac{1}{d} \sum_{k=0}^{d-1} \omega(g)^{j(p-1)/d} \sigma^{-k(p-1)/d},$$

which lives in $\mathbb{Z}_p[\Delta] \subseteq \mathbb{Z}_p[\text{Gal}(F(\mu_{p^m})/F)]$. One can check easily that $\varepsilon_j = \varepsilon_{j'}$ whenever $j \equiv j' \mod |\Delta|$, and $\varepsilon_0, \ldots, \varepsilon_{d-1}$ form a collection of primitive idempotents of the group ring $\mathbb{Z}_p[\Delta]$.

**Lemma 4.1.** Let $M$ be a $\mathbb{Z}_p[\Delta]$-module. Then the following hold:

(i) $M^\Delta = \varepsilon_0 M$.

(ii) For every $i \geq 1$, we have $(\mu_{p^m} \otimes M)^\Delta = \mu_{p^m} \otimes \varepsilon_{-i} M$.

**Proof.** The first assertion is immediate from the definition of $\varepsilon_0$. For the second, it suffices to show that

$$\varepsilon_{j+1}(\mu_{p^m} \otimes M) = \mu_{p^m} \otimes \varepsilon_j M$$

for every $j$. Indeed, the conclusion of the lemma follows from a recursive application of this equality. Via the natural surjection $(\mathbb{Z}/p^n\mathbb{Z})^\times \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^\times$, we may also view $\omega$ as a character of $(\mathbb{Z}/p^n\mathbb{Z})^\times$. It then follows that $\omega(g)^{p-1} = 1$ and $\omega(g) = g \mod p$, where the latter implies $\omega(g) \equiv \omega(g)^{p-1} \equiv g^{p-1} \mod p^{m-1}$. Consequently,

$$\sigma(\zeta) = \zeta^{g^{p-1}} = \zeta^{\omega(g)}.$$

Let $x \in M$. Then

$$\varepsilon_{j+1}(\zeta \otimes x) = \frac{1}{d} \sum_{k=0}^{d-1} \omega(g)^{j(p-1)/d} \sigma^{-k(p-1)/d} (\zeta \otimes x)$$

$$= \frac{1}{d} \sum_{k=0}^{d-1} \omega(g)^{j(p-1)/d} \sigma^{-k(p-1)/d} (\zeta^{\omega(g)^{k(p-1)/d}} \otimes x)$$

$$= \frac{1}{d} \sum_{k=0}^{d-1} \omega(g)^{j(p-1)/d} (\zeta \otimes \sigma^{-k(p-1)/d} x) = \zeta \otimes \varepsilon_j x.$$

This establishes our claim, and the proof of the lemma is thus complete. ■
In what follows, we will study the $p^n$-rank of $K_{2i}(\mathcal{O}_F)$ for a given number field $F$ and integer $i \geq 1$. Let $a := a(F)$ be the largest integer such that $F(\mu_p) = F(\mu_{p^n})$. Set $b = v_p(i)$. In other words, $b$ is the largest integer such that $p^b$ divides $i$. We first consider the case when $n \leq a + b$.

4.2. $p^n$-rank for $n \leq a + b$. For a given integer $i \geq 1$, let $S_p^{(i)}$ be the set of primes $v$ in $S_p$ such that $i$ is divisible by $|\Delta_v|$, where $\Delta_v$ is the decomposition group of $\Delta$ at $v$. In particular, $S_p^{(1)}$ is the set of primes $v$ in $S_p$ which split completely in $F(\mu_p)/F$. We can now prove the following.

**Theorem 4.2.** Suppose that $n \leq a + b$.

(i) If $i \equiv 0 \mod |F(\mu_p) : F|$, we have an exact sequence

$$0 \to \mu_p^\otimes i \otimes A_F^S \to K_{2i}(\mathcal{O}_F)/p^n \to \bigoplus_{v \in S_p} \mu_p^\otimes i \to 0.$$

(ii) If $i \not\equiv 0 \mod |F(\mu_p) : F|$, we have an exact sequence

$$0 \to \mu_p^\otimes i \otimes \varepsilon_i A_F^S(\mu_p) \to K_{2i}(\mathcal{O}_F)/p^n \to \bigoplus_{v \in S_p^{(i)}} \mu_p^\otimes i \to 0.$$

**Proof.** We first consider the situation when $i \equiv 0 \mod |F(\mu_p) : F|$. From Proposition 3.4, we have an exact sequence

$$0 \to A_F^S/p^n \to H^2(G_S(F), \mu_{p^n}) \to \bigoplus_{v \in S_p} \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0.$$

Applying $\mu_p^\otimes i \otimes -$, we obtain

$$0 \to \mu_p^\otimes i \otimes A_F^S \to \mu_p^\otimes i \otimes H^2(G_S(F), \mu_{p^n}) \to \bigoplus_{v \in S_p} \mu_p^\otimes i \to \mu_p^\otimes i \to 0.$$

Since $i \equiv 0 \mod |F(\mu_p) : F|$, the group $G_S(F)$ acts trivially on $\mu_p^\otimes i$, and so

$$\mu_p^\otimes i \otimes H^2(G_S(F), \mu_{p^n}) \cong H^2(G_S(F), \mu_p^\otimes i + 1).$$

By Corollary 3.6, the latter is $K_{2i}(\mathcal{O}_F)/p^n$. This proves (i).

Now suppose that $i \not\equiv 0 \mod |F(\mu_p) : F|$. In this case, $G_S(F(\mu_p))$ acts trivially on $\mu_p^\otimes i$ by Lemma 3.1, since $a(F(\mu_p)) = a$. By a similar argument to the above, we obtain an exact sequence

$$0 \to \mu_p^\otimes i \otimes A_F^S(\mu_p) \to H^2(G_S(F(\mu_p)), \mu_p^\otimes i + 1)$$

$$\to \bigoplus_{w \in S_p(F(\mu_p))} \mu_p^\otimes i \to \mu_p^\otimes i \to 0.$$
Taking the $\Delta$-invariant, we have

$$0 \to (\mu_{p^n}^{\otimes i} \otimes A_F^{S(\mu_p)}))^\Delta \to H^2(G_S(F(\mu_p)), \mu_{p^n}^{\otimes i+1})^\Delta \to \bigoplus_{w \in S_p(F(\mu_p))} (\mu_{p^n}^{\otimes i})^\Delta \to (\mu_{p^n}^{\otimes i})^\Delta \to 0.$$ 

Note that the above sequence is still exact, as $|\Delta|$ is coprime to $p$. In view of this property, Lemma 3.1 also implies that $H^2(G_S(F(\mu_p)), \mu_{p^n}^{\otimes i+1})^\Delta \cong H^2(G_S(F), \mu_{p^n}^{\otimes i+1})$, where the latter is $K_{2i}(\mathcal{O}_F)/p^n$ by Corollary 3.6. The first term in the exact sequence is $\mu_{p^n}^{\otimes i} \otimes \varepsilon_{-i} A_F^{S(\mu_p)}$ by Lemma 4.1. On the other hand, as $i \not\equiv 0 \mod |F(\mu_p) : F|$, one has $(\mu_{p^n}^{\otimes i})^\Delta = 0$ by Lemma 3.1. For the remaining local terms, we note that

$$\left( \bigoplus_{v \in S_p} (\mu_{p^n}^{\otimes i})^\Delta \right) \cong \bigoplus_{v \in S_p} \left( \bigoplus_{w|v} (\mu_{p^n}^{\otimes i})^\Delta_v \right),$$

where the second isomorphism follows from Shapiro’s lemma. By another application of Lemma 3.1, one then has

$$(\mu_{p^n}^{\otimes i})^\Delta_v = \begin{cases} \mu_{p^n}^{\otimes i} & \text{if } i \text{ is divisible by } |\Delta_v|, \\ 0 & \text{otherwise.} \end{cases}$$

Combining all the above observations, we obtain the conclusion of (ii). 

We discuss some consequences of the preceding theorem.

**Corollary 4.3.** Suppose that $n \leq a + b$. Then

$$r_{p^n}(K_{2i}(\mathcal{O}_F)) = \begin{cases} r_{p^n}(A_F^S) + |S_p| - 1 & \text{if } i \equiv 0 \mod |F(\mu_p) : F|, \\ r_{p^n}(\varepsilon_{-i} A_F^{S(\mu_p)}) + |S_p^{(i)}| & \text{if } i \not\equiv 0 \mod |F(\mu_p) : F|. \end{cases}$$

**Proof.** Suppose that $i = 0 \mod |F(\mu_p) : F|$. Set $N$ to be the kernel of the map $\bigoplus_{v \in S_p} \mu_{p^n}^{\otimes i} \to \mu_{p^n}^{\otimes i}$. Then Theorem 4.2(i) yields a short exact sequence

$$0 \to \mu_{p^n}^{\otimes i} \otimes A_F^S \to K_{2i}(\mathcal{O}_F)/p^n \to N \to 0$$

of $\mathbb{Z}/p^n\mathbb{Z}$-modules. Plainly, $N$ is a free $\mathbb{Z}/p^n\mathbb{Z}$-module of rank $|S_p| - 1$, and so the above short exact sequence splits, which in turn supplies an isomorphism

$$K_{2i}(\mathcal{O}_F)/p^n \cong (\mu_{p^n}^{\otimes i} \otimes A_F^S) \oplus N$$

of $\mathbb{Z}/p^n\mathbb{Z}$-modules. The asserted $p^n$-rank of $K_{2i}(\mathcal{O}_F)$ is now an immediate consequence.
Suppose that \( i \not\equiv 0 \mod |\Delta| \). By Theorem 4.2, we have a short exact sequence

\[
0 \to \mu_p^\otimes i \otimes \varepsilon_{-i} A_F^S(\mu_p) \to K_{2i}(\mathcal{O}_F)/p^n \to \bigoplus_{v \in S_p^{(i)}} \mu_p^\otimes i \to 0,
\]

where \( \bigoplus_{v \in S_p^{(i)}} \mu_p^\otimes i \) is a free \( \mathbb{Z}/p^n\mathbb{Z} \)-module of rank \( |S_p^{(i)}| \). Therefore, the sequence splits, proving the asserted \( p^n \)-rank of \( K_{2i}(\mathcal{O}_F) \).

We record the following special case of the preceding corollary which generalizes a previous result of Kolster (see [14, Corollary 1.8] and [15, Corollary 3.4]), where a formula for the \( p^n \)-rank of \( K_{2i} \) is obtained.

**Corollary 4.4.** Suppose \( n \leq a + b \), \( |F(\mu_p) : F| = 2 \) and \( i \) is an odd integer. Then

\[
\text{r}_p^n(K_{2i}(\mathcal{O}_F)) = \text{r}_p^n(\varepsilon_1 A_F^S(\mu_p)) + |S_p^{(1)}|.
\]

4.3. \( p^n \)-rank for \( n > a + b \). We now consider the situation when \( n \) is greater than \( a + b \).

**Theorem 4.5.** If \( n > a + b \), we have an exact sequence

\[
0 \to (\mu_p^\otimes i \otimes A_E^S)G \to K_{2i}(\mathcal{O}_F)/p^n \to \bigoplus_{v \in S_p} \mu_p^\otimes i(F_v) \to \mu_p^\otimes i(F) \to 0
\]

where \( E := E_n := F(\mu_{p^{-b}}) \) and \( G = \text{Gal}(E/F) \).

In preparation for the proof of the theorem, we first establish the following lemma which generalizes an observation made by Keune [12, Lemma 6.5]. We note that our approach in the proof differs from Keune’s.

**Lemma 4.6.** Retain the setting of Theorem 4.5. Then \( \mu_p^\otimes i \) is a cohomologically trivial \( H \)-module for every subgroup \( H \) of \( G \).

**Proof.** By replacing \( F \) by \( E^H \), it suffices to prove that \( \mu_p^\otimes i \) is a cohomologically trivial \( G \)-module. Set \( \Gamma = \text{Gal}(F(\mu_{p^\infty})/F) \) and \( \Gamma_n = \text{Gal}(F(\mu_{p^\infty})/E) \).

Consider the spectral sequence

\[
H^r(G, H^s(\Gamma_n, \mu_p^\otimes i)) \Rightarrow H^{r+s}(\Gamma, \mu_p^\otimes i).
\]

Since the groups \( \Gamma_n \) and \( \Gamma \) have \( p \)-cohomological dimension 1, one has

\[
H^t(\Gamma_n, \mu_p^\otimes i) = H^t(\Gamma, \mu_p^\otimes i) = 0
\]

for \( t \geq 2 \). On the other hand, Tate’s lemma [29] tells us that

\[
H^1(\Gamma_n, \mu_p^\otimes i) = H^1(\Gamma, \mu_p^\otimes i) = 0.
\]

Hence the spectral sequence degenerates, yielding

\[
H^r(G, H^0(\Gamma_n, \mu_p^\otimes i)) = 0
\]

for every \( r \geq 1 \). But also \( H^0(\Gamma_n, \mu_p^\otimes i) = \mu_p^\otimes i \) by Lemma 3.1. This completes the proof of the lemma. ■
We can now give the proof of Theorem 4.5.

Proof of Theorem 4.5. Since $G_S(E)$ acts trivially on $\mu_{p^n}^i$, we may apply a similar argument to the beginning of the proof of Theorem 4.2 to obtain an exact sequence
\[ 0 \to \mu_{p^n}^i \otimes A_E^S \to H^2(G_S(E), \mu_{p^n}^{i+1}) \to \bigoplus_{w \in S_p(E)} \mu_{p^n}^i \to \mu_{p^n}^i \to 0. \]

By Lemma 4.6, $\mu_{p^n}^i$ is a cohomologically trivial $G$-module. We now proceed to show that so is $\bigoplus_{w \in S_p(E)} \mu_{p^n}^i$. Observe that
\[ H^r\left( G, \bigoplus_{w \in S_p(E)} \mu_{p^n}^i \right) \cong \bigoplus_{v \in S_p} H^r\left( G, \bigoplus_{w | v} \mu_{p^n}^i \right) \cong \bigoplus_{v \in S_p} H^r(G_v, \mu_{p^n}^i), \]
where the second isomorphism is a consequence of Shapiro’s lemma. Since $G_v$ may be identified with a subgroup of $G$, we may apply Lemma 4.6 to conclude that $H^r(G_v, \mu_{p^n}^i) = 0$ for $r \geq 1$. Hence $\bigoplus_{w \in S_p(E)} \mu_{p^n}^i$ is a cohomologically trivial $G$-module. Set
\[ Y := \ker\left( \bigoplus_{w \in S_p(E)} \mu_{p^n}^i \to \mu_{p^n}^i \right). \]

Straightforward cohomological considerations show that $Y$ is also a cohomologically trivial $G$-module. Therefore, after taking the $-G$ functor, we get the short exact sequences
\[ 0 \to (\mu_{p^n}^i \otimes A_E^S)_G \to \left( H^2(G_S(E), \mu_{p^n}^{i+1}) \right)_G \to Y_G \to 0, \]
\[ 0 \to Y_G \to \left( \bigoplus_{w \in S_p(E)} \mu_{p^n}^i \right)_G \to (\mu_{p^n}^i)_G \to 0. \]
Splicing these, we obtain
\[ 0 \to (\mu_{p^n}^i \otimes A_E^S)_G \to \left( H^2(G_S(E), \mu_{p^n}^{i+1}) \right)_G \to \left( \bigoplus_{w \in S_p(E)} \mu_{p^n}^i \right)_G \to (\mu_{p^n}^i)_G \to 0. \]
By Lemma 3.1, $(H^2(G_S(E), \mu_{p^n}^{i+1}))_G \cong H^2(G_S(F), \mu_{p^n}^{i+1})$, which is precisely $K_2(G_F/p^n)$ by Corollary 3.6. Finally, for every cohomologically trivial $G$-module $N$, one has a natural isomorphism $N_G \cong N^G$ induced by the norm map. Therefore, for the rightmost two terms, we may switch $-G$ to $-G$. Putting all these observations together, we obtain the conclusion of our theorem.

5. Quadratic fields. We now specialize to quadratic fields. Let $m$ be a squarefree integer $\neq \pm 1, (-1)^{(p-1)/2} p$. Set $F = \mathbb{Q}(\sqrt{m})$. Note that in this situation, one has $a = a(F) = 1$. As before, we write $b = v_p(i)$. 
We begin with the case of small $n$.

**Proposition 5.1.** Suppose that $n \leq 1 + b$.

(i) If $i \equiv 0 \mod p - 1$, then

$$r_p^n(K_{2i}(O_F)) = \begin{cases} r_p^n(A_F^S) + 1 & \text{if } p \text{ splits in } F/\mathbb{Q}, \\ r_p^n(A_F^S) & \text{otherwise}. \end{cases}$$

(ii) If $i \not\equiv 0 \mod p - 1$, then

$$r_p^n(K_{2i}(O_F)) = \begin{cases} r_p^n(\varepsilon_i A_{F(\mu_p)}^S) + 1 & \text{if } i \equiv 0 \mod (p - 1) / 2 \text{ and } m = (-1)^{(p-1)/2} pm_1 \\ r_p^n(\varepsilon_i A_{F(\mu_p)}^S) & \text{otherwise}. \end{cases}$$

**Proof.** (i) Since $|S_p| = 2$ or 1 according as the prime $p$ splits in $F/\mathbb{Q}$ or not, the asserted $p^n$-rank formula is a consequence of Corollary 4.3.

(ii) From Corollary 4.3, one has

$$r_p^n(K_{2i}(O_F)) = r_p^n(\varepsilon_i A_{F(\mu_p)}^S) + |S^{(i)}|.$$ 

From the discussion in 

From the discussion in 

Section 3], we see that for a prime $v \in S_p$, $|F_v(\mu_p) : F_v| = (p - 1)/2$ precisely when $p$ is both ramified in $F/\mathbb{Q}$ and split in $\mathbb{Q}(\sqrt{m_1})/\mathbb{Q}$, where $m = (-1)^{(p-1)/2} pm_1$. The discussion in also tells us that in all other cases, one has $|F_v(\mu_p) : F_v| = p - 1$. Hence $|S_p^{(i)}| = 1$ when $i$ is divisible by $(p - 1)/2$, and $m = (-1)^{(p-1)/2} pm_1$ with $p$ splitting in $\mathbb{Q}(\sqrt{m_1})/\mathbb{Q}$, and $|S_p^{(i)}| = 0$ otherwise. This yields the asserted conclusion.

For large $n$, we may apply Theorem 4.5 to obtain the following. Since the argument is quite similar to that of the preceding proof, we omit it.

**Proposition 5.2.** Suppose that $n > 1 + b$. Set $E = F(\mu_{p^n - b})$ and $G = \text{Gal}(E/F)$.

(i) Suppose that $i \equiv 0 \mod p - 1$.

(a) If $p$ is non-split in $F/\mathbb{Q}$, then

$$(\mu_{p^n} \otimes A_E^S)_G \cong K_{2i}(O_F)/p^n$$

and therefore

$$r_p^n(K_{2i}(O_F)) = r_p^n((\mu_{p^n} \otimes A_E^S)_G).$$

(b) If $p$ splits in $F/\mathbb{Q}$, then we have an exact sequence

$$0 \to (\mu_{p^n} \otimes A_E^S)_G \to K_{2i}(O_F)/p^n \to \mu_{p^{1+b}} \to 0.$$
(ii) Suppose that \( i \not\equiv 0 \mod p - 1 \). Then
\[
K_{2i}(\mathcal{O}_F)/p^n \cong (\mu_{p^n}^\otimes i \otimes A_E^S)_G
\]
except when \( i \equiv 0 \mod (p - 1)/2 \) and \( m = (-1)^{(p-1)/2}p^m \) with \( p \) being split in \( \mathbb{Q}(\sqrt{m_1})/\mathbb{Q} \). In this exceptional case, one has instead a short exact sequence
\[
0 \to (\mu_{p^n}^\otimes i \otimes A_E^S)_G \to K_{2i}(\mathcal{O}_F)/p^n \to \mu_{p^{i+b}}^\otimes \to 0.
\]

6. On Kummer’s criterion. In this final section, \( F \) always denotes a totally real field. We let \( \zeta_p \) denote a fixed primitive \( p \)-th root of unity, and set \( d = |F(\zeta_p) : F| \). The following theorem is the main result of this section.

**Theorem 6.1.** Suppose that \( F \) is a totally real field such that no prime in \( S_p(F(\zeta_p+\zeta_p^{-1})) \) splits in \( F(\zeta_p) \). Then the following statements are equivalent:

1. The class number of \( F(\zeta_p) \) is divisible by \( p \).
2. \( p \) divides the order of \( K_{2i}(\mathcal{O}_F) \) for some odd \( i \) with \( 1 \leq i \leq d - 1 \).

**Remark 6.2.** Before giving the proof, we explain why the theorem is equivalent to Kummer’s criterion in the sense of [10, Theorem 1] and [13, Theorem 1], and thus can be thought of as an algebraic-\( K \)-theoretical formulation of it. Write \( | \ ) \) for the \( p \)-adic norm which is normalized by \( |p|_p = 1/p \). Denote by \( \zeta_F(s) \) the Dedekind zeta function of \( F \). For \( a, b \in \mathbb{Q} - \{0\} \), we write \( a \sim_p b \) if \( a/b \) is a \( p \)-unit. For each odd integer \( i \) with \( 1 \leq i \leq d - 1 \), it follows from the main conjecture of Iwasawa as proven by Wiles [34] that
\[
\zeta_F(1 - (i + 1)) \sim_p \frac{|K_{2i}(\mathcal{O}_F)|}{|K_{2i+1}(\mathcal{O}_F)|}.
\]

(Strictly speaking, the theorem of Wiles only yields
\[
\zeta_F(1 - (i + 1)) \sim_p \frac{|H^2(G_S(F), \mathbb{Z}_p(i + 1))|}{|H^1(G_S(F), \mathbb{Z}_p(i + 1))|},
\]
see the paper of Báyer and Neukirch [1]. But we now know that this cohomological version is equivalent to the \( K \)-theoretical version as stated above, in view of the work of Rost–Voevodsky [31].)

From [33, Chap. VI, Theorem 9.5], we see that \(|K_{2i+1}(\mathcal{O}_F)| \sim_p w_{i+1}^{(p)}(F)\), where \( w_j^{(p)}(F) \) is the order of \( \mu_{p^{\infty}}^\otimes j(F) = (\mu_{p^{\infty}}^\otimes)^{\operatorname{Gal}(\bar{F}/F)} \). Lemma 3.2 then tells us that
\[
w_{i+1}^{(p)}(F) = \begin{cases} 
1, & 1 \leq i < d - 1, \text{ i odd}, \\
p^{a(F)}, & i = d - 1.
\end{cases}
\]

Upon combining these observations, we obtain
\[
|K_{2i}(\mathcal{O}_F)| \sim_p \begin{cases} 
\zeta_F(1 - (i + 1)), & 1 \leq i < d - 1, \text{ i odd}, \\
p^{a(F)}\zeta_F(1 - d), & i = d - 1.
\end{cases}
\]
Therefore, statement (2) of Theorem 6.1 is equivalent to saying that $p$ divides one of the numerators of the rational numbers

$$\zeta_F(1-(i+1)) \ (1 \leq i < d-1, \ i \ odd), \ \ p^{a(F)}\zeta_F(1-d).$$

Thus our theorem is equivalent to [10, Theorem 1] and [13, Theorem 1].

We proceed with the proof of Theorem 6.1; we emphasize that our proof does not make use of $p$-adic $L$-functions. We should however make the following remark.

**Remark 6.3.** Although the proof of Theorem 6.1 does not make use of $p$-adic $L$-functions, in order to see that the theorem is equivalent to Kummer’s criterion, one requires to be able to relate the special values of the Dedekind zeta function and the size of the $K$-groups, and a such relation is predicted by a conjecture of Lichtenbaum [18]. At our current knowledge, it would seem that the only way to study this relation is via the Iwasawa main conjecture, whose formulation requires $p$-adic $L$-functions.

For the proof of Theorem 6.1, we require the following lemma.

**Lemma 6.4.** Retain the setting of Theorem 6.1. For every odd integer $i$ such that $1 \leq i \leq d-1$, we have

$$r_p(K_2i(O_F)) = r_p(\varepsilon_{-i}A_{F(\zeta_p)}).$$

**Proof.** Set $E = F(\zeta_p)$. Recall that by Corollary 4.3

$$r_p(K_2i(O_F)) = r_p(\varepsilon_{-i}A_E^S) + |S_p^{(i)}|.$$

From the assumption that the primes in $S_p(F(\zeta_p + \zeta_p^{-1}))$ do not split in $E$, we see that $|\Delta_v|$ is even for every $v \in S_p$. But since $i$ is odd, this in turn implies that the set $S_p^{(i)}$ is empty. It therefore remains to show that

$$\varepsilon_{-i}A_E^S = \varepsilon_{-i}A_E$$

for every odd $i$. Since $d$ is even, this is equivalent to showing that $\varepsilon_iA_E^S = \varepsilon_iA_E$ for every odd $i$. Let $\lambda$ be the natural surjection $\text{Cl}(O_E) \twoheadrightarrow \text{Cl}_S(O_E)$. Then $\ker \lambda$ is generated by the class of primes in $S_p(E)$. Now observe that for any odd $i$, we have

$$\varepsilon_i = \frac{1}{d} \sum_{k=0}^{d-1} \omega(g)^{-ik(p-1)/d} \sigma^{-k(p-1)/d}$$

$$= \frac{1}{d} \sum_{k=0}^{d/2-1} \left( \omega(g)^{ik(p-1)/d} \sigma^{-k(p-1)/d} \right.$$  

$$+ \omega(g)^{i(k+d/2)(p-1)/d} \sigma^{-(k+d/2)(p-1)/d})$$
\[
= \frac{1}{d} \sum_{k=0}^{d/2-1} \omega(g)^{ik(p-1)/d} \sigma^{-k(p-1)/d} (1 + (-1)^i \sigma^{-d/2(p-1)/d})
\]
\[
= \frac{1}{d} \sum_{k=0}^{d/2-1} \omega(g)^{ik(p-1)/d} \sigma^{-k(p-1)/d} (1 - \sigma^{p-1}/2),
\]

where we note that \(\sigma^{-(p-1)/2} = \sigma^{(p-1)/2}\) is the complex conjugation and hence a generator of the group \(\text{Gal}(E/F(\zeta_p + \zeta_p^{-1}))\). Since the primes in \(S_p(F(\zeta_p + \zeta_p^{-1}))\) do not split in \(F(\zeta_p)\), they are invariant under the complex conjugation \(\sigma^{(p-1)/2}\) and so \(\ker \lambda\) is annihilated by \(1 - \sigma^{(p-1)/2}\). This implies that \((\ker \lambda)[p^\infty]\) is annihilated by \(\varepsilon_i\) for every odd \(i\), which in turn yields \(\varepsilon_i A^S_E = \varepsilon_i A_E\) for every odd \(i\). The proof of the lemma is now complete. \(\blacksquare\)

**Proof of Theorem 6.1.** Let \(A\) denote the Sylow \(p\)-subgroup of the class group of \(E := F(\zeta_p)\). Then there is a decomposition
\[
A = \bigoplus_{i=0}^{d-1} \varepsilon_i A,
\]
where one notes that
\[
\bigoplus_{0 \leq i \leq d-1, \text{even}} \varepsilon_i A
\]
is the Sylow \(p\)-subgroup of the class group of \(F(\zeta_p + \zeta_p^{-1})\). By [10, Section 4], the class number of \(E\) is divisible by \(p\) if and only if the relative class number of \(E/F(\zeta_p + \zeta_p^{-1})\) is divisible by \(p\). In view of the above decomposition, the latter divisibility is equivalent to
\[
r_p\left( \bigoplus_{1 \leq i \leq d-1, \text{odd}} \varepsilon_i A \right) \geq 1.
\]
By Lemma 6.4, this is in turn equivalent to \(r_p(K_{2i}(O_F)) \geq 1\) for some odd \(i\) with \(1 \leq i \leq d - 1\). The proof of the theorem is therefore complete. \(\blacksquare\)

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Meng Fai Lim
School of Mathematics and Statistics &
Hubei Key Laboratory of Mathematical Sciences
Central China Normal University
Wuhan, 430079, P.R. China
E-mail: limmf@ccnu.edu.cn