Exact Non-Stationary Probabilities in the Asymmetric Exclusion Process on a Ring

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By a geometrical treatment of the Bethe ansatz, we obtain an exact solution for the totally asymmetric exclusion process on a ring. We derive an explicit determinant expression for the non-stationary conditional probability $\text{Prob}(x_1, \ldots, x_P; t|x_1^0, \ldots, x_P^0; 0)$ of finding $P$ particles on sites $x_1, \ldots, x_P$ at time $t$ provided they are on sites $x_1^0, \ldots, x_P^0$ at time $t=0$.

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The one-dimensional asymmetric exclusion process (ASEP) has been intensively studied as one of the simplest examples of a system with stochastic dynamics and exclusion interaction $[1,2,3,4]$. In the steady state, many properties of the ASEP have been calculated exactly (see, e.g., $[5,6,7,8]$ and references therein). The structure of states out of stationarity is more complicated and description of dynamics is harder to obtain $[9,10,11]$. In general, evaluation of the conditional probability of finding $P$ particles on lattice sites $x_1, \ldots, x_P$ at time $t$ with initial occupation $x_1^0, \ldots, x_P^0$ at time $t=0$, needs solving a non-trivial master equation. The solution of the master equation for the ASEP with a finite number of particles on an infinite lattice was obtained in $[11]$ where an explicit determinant expression for the probability $\text{Prob}(x_1, \ldots, x_P; t|x_1^0, \ldots, x_P^0; 0)$ was derived. At the same time, a non-stationary solution for the finite density of particles is desirable for considering the evolution of a system from arbitrary initial conditions to the steady state.

It is the aim of this Letter to present an exact expression for the conditional probability of finding $P$ particles on an infinite ring of $L$ sites after the evolution with the ASEP dynamics during finite time $t$. In the theory of exactly soluble models, the determinant solutions on finite two-dimensional domains permitting a dynamical interpretation in space-time coordinates are known either for so-called free-fermion models $[1,12,13]$, or for particular domain-wall boundary conditions $[12,13]$. The ASEP does not belong to the free-fermion class and cannot be restricted to a domain wall geometry. However, we will see that a specific property of the pair interaction in the ASEP allows one to consider the model on the finite ring and express the solution in a form of determinant of a $P \times P$ matrix.

Instead of explicitly solving the master equation on the ring, we use here a geometrical treatment of the Bethe ansatz which allows us to analyse a tangled system of allowed and forbidden trajectories of particles. To make the presentation more transparent, we consider a discrete space-time version of the totally ASEP. The continuous time limit then follows from the final expressions by the straightforward substitution of the Bernoulli distribution by its Poisson analogue.

Consider the triangle lattice $\Lambda$ obtained from the square lattice by adding a diagonal between the upper left corner and the lower right corner of each elementary square. The lattice is periodic in the horizontal direction with the period $L$. Let $(x, t)$ be integer coordinates of a particle on $\Lambda$ where the vertical time axis is directed down and the horizontal space axis is directed right. A trajectory of the particle is a sequence of connected vertical and diagonal bonds of $\Lambda$. Each diagonal bond corresponds to one jump of the particle to its right for a unit of time and has a statistical weight $z$. The vertical bond corresponds to a stay at given site for the unit time and has a statistical weight $y$. The generating function of all possible free trajectories of one particle for time $t$ is $(z+y)^t$. For trajectories starting at an initial position $(x^0,0)$ and ending at $(x,t)$, we define

$$B(N,t) = \left( \frac{t}{N} \right) z^N y^{t-N}$$

(1)

where $N = x - x^0$ is the travelled distance. To provide $B(N,t)$ with probabilistic meaning, we put $z < 1$, $z+y = 1$. Then, $B(N,t)$ is the probability to reach $(x,t)$ from $(x^0,0)$ for $t$ time steps provided $z$ is the probability of one step in the right direction. In the continuum limit, $z \to 0$, $t \to \infty$, we have

$$B(N,t) = \frac{e^{-t} t^N}{N!}$$

(2)

where $t$ is rescaled continuous time $tz \to t$. For $N < 0$, we put $B(N,t) = 0$.

The ASEP in discrete space-time can be defined as follows. Consider trajectories of $P$ particles on the lattice $\Lambda$ starting at points $(x_1^0,0), \ldots, (x_P^0,0)$, $0 \leq x_1^0 < x_2^0 < \ldots < x_P^0 < L$ and ending at points $(x_1, t), \ldots, (x_P, t)$, $0 \leq x_1 < x_2 < \ldots < x_P < L$ after an arbitrary number of rotations around the ring. The exclusion rules read:
(a) trajectories of particles do not intersect; (b) for an arbitrary elementary square of \( \Lambda \), if two vertical bonds are occupied by adjacent trajectories, the weight of the left bond is changed from \( y \) to \( 1 \). The rule (a) is the usual condition of occupation of every site by at most one particle. The rule (b) implies that the moving particle stays at a given site with probability 1 if the target site is occupied by a standing particle.

The generating function \( G_P(x_1, \ldots, x_P; t|x_1^0, \ldots, x_P^0; 0) \) of the discrete ASEP is the sum over all trajectories of \( P \) particles allowed by the exclusion rule (a) and weighted according to the rule (b). Due to the exclusion rules, \( G_P \) is 

\[
G_P = \text{Prob}(x_1, \ldots, x_P; t|x_1^0, \ldots, x_P^0; 0).
\]

To formulate the main result of this Letter, consider the function

\[
F_m(x_0^0, x_j^0|t) = \sum_{k=0}^{m} \binom{k+m-1}{m-1} F_0(x_{i-1}^0 - k, x_j^0|t) \quad (3)
\]

if integer \( m > 0 \), and

\[
F_m(x_0^0, x_j^0|t) = -m \sum_{k=0}^{-m} (-1)^k \binom{-m}{k} F_0(x_{i-t}^0 - k, x_j^0|t) \quad (4)
\]

if integer \( m < 0 \). For \( m = 0 \),

\[
F_0(x_0^0, x_j^0|t) = B(x_j - x_0^0, t) \quad (5)
\]

where \( B(N,t) \) is given by Eq.(1). Then, for integer \( L > 1 \) and \( 0 < P < L \) the generating function \( G_P(x_1, \ldots, x_P; t|x_1^0, \ldots, x_P^0; 0) \) is

\[
G_P = \sum_{n_1=-\infty}^{\infty} \ldots \sum_{n_P=-\infty}^{\infty} (-1)^{\sum_i n_i - \sum_j n_j} \det M \quad (6)
\]

Elements of the \( P \times P \) matrix \( M \) are

\[
M_{ij} = F_{s_{ij}}(x_0^0, x_j + n_j L|t) \quad (7)
\]

where

\[
s_{ij} = (P-1)n_j - \sum_{k \neq j} n_k + j - i \quad (8)
\]

In the continuous limit, \( B(N,t) \) is given by Eq.(1) and \( G_P \) coincides with the probability \( \text{Prob}(x_1, \ldots, x_P; t|x_1^0, \ldots, x_P^0; 0) \) of the totally ASEP in its standard formulation.

The derivation of this result is based on a common property of integrable models admitting of a two-dimensional graphic representation: interchanging of end points of two trajectories leads to their crossing. The idea of the Bethe-ansatz is to represent trajectories of interacting particles by a set of free trajectories given by Eq.(1) or Eq.(2). Then, using the one-to one correspondence between intersections and permutations, one can reduce enumerating all interacting trajectories to a proper choice of signs of permutations.

We start with the case of two particles \( P = 2 \). According to the Bethe ansatz, we try to represent the motion of particles by free trajectories from \( (x_0^0, 0) \) to \( (x_1, t), i = 1, 2 \). Consider an elementary square of \( \Lambda \) with space coordinates \( x \) of the left side and \( x+1 \) of the right side. Assume that particles come for the first time to neighboring sites at a moment \( t' \) when one trajectory reaches the site \( (x, t') \) from \( (x_0^0, 0) \) and another reaches the site \( (x+1, t') \) from \( (x_0^0, 0) \). To ensure correct weights of the next steps of interacting particles after moment \( t' \), we have to exclude two possibilities from all continuations of trajectories (Fig.1a):

(i) for the first particle, the step from \( (x, t') \) to \( (x+1, t' + 1) \) with weight \( z \) and then from \( (x+1, t' + 1) \) to \( (x_1, t) \); for the second particle, the step from \( (x+1, t) \) to \( (x+1, t' + 1) \) with weight \( y \) and then from \( (x+1, t' + 1) \) to \( (x_2, t) \).

(ii) for the first particle, the step from \( (x, t') \) to \( (x+1, t) \) with weight \( y-1 = -z \) and then from \( (x+1, t + 1) \) to \( (x_2, t) \);

for the second particle, the step from \( (x+1, t) \) to \( (x+1, t' + 1) \) with weight \( y \) and then from \( (x+1, t' + 1) \) to \( (x_2, t) \).

\[
W_1 = W(x_0^0, x|z|x + 1, x_1)W(x_0^0, x + 1|y|x + 1, x_2) \quad (9)
\]

The contribution from the diagram (ii) is

\[
W_2 = -W(x_0^0, x|z|x, x_1)W(x_0^0, x + 1|y|x + 1, x_2) \quad (10)
\]

Consider now the trajectories where the end points are interchanged (Fig 1b). The contribution from these diagrams is

\[
W(x_0^0, x|z|x + 1, x_1)W(x_0^0, x + 1|y|x + 1, x_2) - W(x_0^0, x|z|x + 1, x_1)W(x_0^0, x + 1|y|x + 1, x_2) \quad (11)
\]

FIG. 1: The interaction between two trajectories (see text).

Case (i) is the forbidden step of the first particle toward the standing second particle. Case (ii) is a correction of the weight of the vertical step of the first particle which must be 1 instead of \( y \) according to the ASEP rule (b). The generating function of paths of the first particle in case (i) is a product of three factors \( B(x - x_0^0, t')zB(x_1 - x - 1, t' - 1) \). Using symbolic notations \( W(a, x|z)x + 1, b) \) for the generating function of trajectories passing points \( a, x, x + 1, b \) at moments \( 0, t', t + 1, t \) and making a step with weight \( z \) between \( t' \) and \( t' + 1 \), we can write the contribution from diagram (i) in the form

\[
W_1 = W(x_0^0, x|z|x + 1, x_1)W(x_0^0, x + 1|y|x + 1, x_2) \quad (9)
\]

The contribution from the diagram (ii) is

\[
W_2 = -W(x_0^0, x|z|x, x_1)W(x_0^0, x + 1|y|x + 1, x_2) \quad (10)
\]
\(W(x_1^0, x | x, x_2)W(x_2^0, x + 1 | y | x + 1, x_1)\). We are going to take the diagrams in Fig.1b with opposite signs to cancel \(W_1 + W_2\). The left diagrams in Fig.1a and Fig.1b are equivalent, however the right ones are different. To cancel all unwanted diagrams, we add to the diagrams in Fig.1b a set of auxiliary trajectories. Namely, add to trajectories of the second particle those starting in point \(x_2^0 - 1\) and taken with minus sign. Also, we add to trajectories of the first particle a set of trajectories starting at the points shifted along the ring in negative directions: \(x_1^0 - 1, x_1^0 - 2, x_1^0 - 3, \ldots\). If a shift exceeds \(kL\), where \(k > 0\) is integer, the trajectory wraps the cylinder \(\Lambda_k\) \(\times \) trajectories of the second particle those starting in point \(x_1^0 - 1, x_1^0 - 2, x_1^0 - 3, \ldots\). If a shift exceeds \(kL\), where \(k > 0\) is integer, the trajectory wraps the cylinder \(\Lambda_k\) times. Then, the contribution from diagram (i) in Fig.1b will be

\[
\tilde{W}_1 = W_1^+ W_1^- \quad (11)
\]

where

\[
W_1^+ = \sum_{k=0}^{\infty} W(x_1^0 - k, x - k | z | x + 1 - k, x_2) \quad (12)
\]

and

\[
W_1^- = W(x_2^0, x + 1 | y | x + 1, x_1) - W(x_2^0 - 1, x | y | x, x_1) \quad (13)
\]

Correspondingly, for the diagram (iii) in Fig.1b, we have

\[
\tilde{W}_2 = W_2^+ W_2^- \quad (14)
\]

where

\[
W_2^+ = - \sum_{k=0}^{\infty} W(x_1^0 - k, x - k | z | x - k, x_2) \quad (15)
\]

and \(W_2^- = W_1^-\). Taking into account that generating functions of trajectories from \((x_1^0 - k, 0)\), \(i = 1, 2\), to \((x - k, t')\) are equal for all \(k\) due to translation invariance, one can check the identity

\[
W_1 + W_2 - \tilde{W}_1 - \tilde{W}_2 = 0 \quad (16)
\]

comparing all positive and negative terms.

Consider first the evaluation of generating function \(G_2(x_1, x_2; t | x_1^0, x_2^0; 0)\) in the case \(L \gg t\) and \(L \gg x_2^0 \gg x_1^0 \gg 0\) which is equivalent to the ASEP on an infinite lattice solved in \([10]\). In this case,

\[
G_2 = B(x_1 - x_1^0, t)B(x_2 - x_2^0, t) - B(x_1 - x_2^0 + 1, t) \sum_{k=0}^{\infty} B(x_2 - x_1^0 + k, t) \quad (17)
\]

Indeed, the first term in Eq.(17) generates all possible free trajectories from initial to end points. When one particle approaches another, the second term produces trajectories cancelling unwanted terms. On the other hand, the order of starting and ending points in the second term is interchanged. Therefore, each trajectory from the second term starting at \(x_1^0 - k\) or \(x_1^0 - k\) approaches at least once the point \((x - k, t')\) or \((x + 1 - k, t')\) where it participates in the cancellation procedure.

Each free trajectory from \(a\) to \(b\) making the vertical step at the collision site \(x\) can be decomposed into two parts \(W(a, x | 1 | x) + W(a, x | -1 | x)\). The second part is unwanted and is cancelled, the first one corresponds to trajectories which continue with true weights up to the next collision. As the second term in Eq.\((17)\) contains intersecting trajectories only, all of them will be cancelled eventually and only true allowed trajectories from the first term survive.

The ASEP on the ring has several peculiarities. To fix them, let us note that two intersecting trajectories are non-equivalent: one of them belongs to the overtaking particle and we may call it ”active”. On the contrary, the second particle can be called ”passive”. In the case of infinite lattice, the active and passive trajectories are ordered: for each pair \(i, i + 1\), the trajectory of \(i\)-th particle with respect to \(i + 1\) particle is always active. On the ring, each of two trajectories can be active or passive independently on initial conditions. Moreover, one trajectory can intersect another \(m\) times if the numbers of rotations differ by \(m\) for two particles.

Assume, that the trajectory of given particle has \(m\) active intersections. It means that it participates \(m\) times in the cancellation procedure and its starting point is shifted \(m\) times to arbitrary distances in the negative direction of the ring. As a result, the auxiliary set associated with the free trajectory between points \(x_i^0\) and \(x_j\) becomes

\[
B(x_j - x_i^0, t) \rightarrow \sum_{k=0}^{\infty} \left( \frac{k + m - 1}{m - 1} \right) B(x_j - x_i^0 + k, t) \quad (18)
\]

because the shift by \(k\) positions for \(m\) attempts can be done in \((k + m - 1)! / (m - 1)!\)! ways. The above can be expressed in an operator form

\[
B(x_j - x_i^0, t) \rightarrow \frac{1}{(1 - \tilde{\alpha})^m} B(x_j - x_i^0, t) \quad (19)
\]

where the operator \(\tilde{\alpha}\) shifts \(x_i^0\) by one step in the negative direction. Similarly, for trajectories having \(m\) passive intersections we get

\[
B(x_j - x_i^0, t) \rightarrow \sum_{k=0}^{\infty} (-1)^k \left( \frac{m}{k} \right) B(x_j - x_i^0 + k, t) \quad (20)
\]

because the right-hand side is the result of action of the operator \((1 - \tilde{\alpha})^m\). Note that Eq.\((17)\) coincides with Eq.\((18)\) and Eq.\((19)\) with Eq.\((20)\) where the index \(m\) can be called ”activity”.

To find the generating function of two particles on the ring, we map the trajectories wrapping the cylinder \(\Lambda\) on an infinite plane introducing coordinates \(x + nL\), \(n\) integer, for equivalent points. We call trajectories of two particles compatible if there is at least one possibility to draw
them without intersections. Given starting points $x_1^0$ and $x_2^0$ at $t = 0$, the pairs of compatible trajectories correspond to end points $x_1, x_2; x_1 + L; \ldots; x_1 + nL, x_2 + nL; x_2 + nL, x_1 + (n + 1)L; \ldots$ at time $t$. The index of activity of compatible trajectories is 0. A trajectory ending at $x_1 + nL$ may interact with the trajectories ending at $x_2 + (n - 1)L$ or $x_2 + nL$. Following the Bethe-ansatz prescription, we should add to the set of compatible trajectories two sets of intersecting trajectories taken with minus sign: the set which is obtained by permuting end points $x_2 + (n - 1)L$ and $x_1 + nL$ and the second one obtained by permuting $x_1 + nL$ and $x_2 + nL$. Each new trajectory, in its turn, interacts with neighbouring trajectories and we should permute their end points again. Continuing this procedure, we obtain all possible pairs of trajectories of the first particle with end point $x_1 + nL$, the second one with end point $x_2 + nL$, or vice versa, for arbitrary integer $n_1$ and $n_2$. The permutations of end points can be expressed by the determinant as can be seen in Eq. (8). The sign of a pair is defined by the number of permutations needed to obtain given pair from a compatible one. The index of activity of each trajectory is defined by the number of overtakes to the moment $t$. Evaluation of the number of permutations gives the prefactor in Eq. (8). The number of overtakes is given by Eq. (8).

If the number of particles $P \geq 3$, the elementary squares shown in Fig.1 may occur several times in one horizontal strip of $\Lambda$. If squares filled by interacting trajectories are separated one from another by a gap of empty sites, the above arguments can be applied to each pair of interacting trajectories separately. The crucial case for the Bethe ansatz is a situation when the elementary squares are nearest neighbors. The specific property of the totally $\Lambda$SEP is that, in each pair of interacting trajectories, the right trajectory remains free and interacts with the next trajectory independently on its left neighbors. Therefore, we can analyse the interaction between particles considering successively elementary squares in each row from left to right starting from an arbitrary empty square and then from the top to bottom of the lattice until all unwanted trajectories on $\Lambda$ will be removed. As above, all trajectories ending at points $x_i + n_i L, i = 1, \ldots, P$ must be involved in consideration.

Then, Eq. (10) is a straightforward generalisation of the case $P = 2$, and Eq. (8) gives the index of activity for an arbitrary number of intersecting trajectories. A new element in the many-particle case is that each trajectory may get $m$ active intersections and $n$ passive ones. The operator form of Eq. (19) and Eq. (20) shows that the resulting activity in this case is $m - n$.

Because of conditions $B(N, t) = 0$ for $N < 0$, the infinite summations in Eq. (3) and Eq. (6) are actually finite for finite $t$. An advantage of the discrete formulation of the $\Lambda$SEP is a possibility to illustrate each step of derivation by simple examples. For instance, in the case $P = 2, L = 3, t = 6, x_1^0 = x_1 = 1, x_2^0 = x_2 = 2$, the generating function is $G_2 = z^3 + 3z^2y^2 + 3z^3 + z^6 + 90z^4y^2 + 20z^6y^4 + 60z^8y^6 + 30z^9y^8 + 5y^6 + 30z^6y^5 + 90z^5y^4 + 20z^3y^3 + 20z^3y^6 + 30z^3y^5 + y^6$. All 243 allowed configurations enter this expression with proper weights. In the general case, the obtained expressions open a prospect for detailed investigations of non-stationary solutions for various initial conditions.

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