Statistics of $2^+$ Levels in Even–Even Nuclei

A. Y. Abul–Magd,¹ H.L. Harney,² M.H. Simbel,¹ and H.A. Weidenmüller²

¹Faculty of Science, Zagazig University, Zagazig, Egypt
²Max-Planck-Institut für Kernphysik, Heidelberg

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Abstract

Using all the available empirical information, we analyze the spacing distributions of low–lying $2^+$ levels of even–even nuclei. To obtain statistically relevant samples, the nuclei are grouped into classes defined by the ratio $R_{4/2}$ of the excitation energies of the first $4^+$ and $2^+$ levels. This ratio serves as a measure of collectivity in nuclei. With the help of Bayesian inference, we determine the chaoticity parameter for each class. This parameter is found to vary strongly with $R_{4/2}$ and takes particularly small values in nuclei that have one of the dynamical symmetries of the interacting Boson model.

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I. MOTIVATION AND PURPOSE

During the past decades, a vast amount of nuclear spectroscopic data has been accumulated. Level schemes involving tens and sometimes hundreds of levels with reliably known values of spin and parity are now available for hundreds of nuclei (see Ref. [1]). The wealth of published spectroscopic data allows for an extensive study of the level statistics of nuclei at low excitation energies. In this paper we report on the statistical analysis of low-lying states with spin and parity $2^+$. The interest in such a study derives from the success of random-matrix theory (RMT) in describing the spectral properties of nuclear levels (actually: resonances) near neutron threshold and proton threshold [2, 3]. Careful analysis has shown that the spectral fluctuation properties of these resonances are in very good agreement with the predictions of the Gaussian orthogonal ensemble (GOE) of random matrices. This statement applies, in particular, to the nearest-neighbor spacing (NNS) distribution which is well approximated by Wigner’s surmise

$$p_W(s) = \frac{\pi}{2} s \exp \left(-\frac{\pi s^2}{4}\right).$$  \hspace{1cm} \text{(1)}$$

Here, $s$ is the NNS in units of the mean spacing. In view of the conjecture by Bohigas, Giannoni and Schmit [5], the agreement between the spectral fluctuation properties of the resonances and the GOE predictions was taken as an indication of chaotic motion in medium-weight and heavy nuclei near neutron threshold. Interest then turned to the ground-state domain. Here, integrable models often successfully describe the spectroscopic data, and one would, therefore, expect the spectral fluctuation properties to be close to those predicted for regular systems. For such systems, the NNS distribution is generically given by the Poisson distribution,

$$p_P(s) = \exp (-s).$$  \hspace{1cm} \text{(2)}$$

A statistical analysis requires complete (few or no missing levels) and pure (few or no unknown spin-parities) level schemes. Some 15 years ago, complete and pure level schemes were available for only a limited number of nuclei (see, e.g., Refs. [6, 7]). The work of Ref. [8] then suggested that the NNS distribution of low-lying nuclear levels lies between the Wigner and the Poisson distributions. The evidence presented in Ref. [8] has since become an established fact through the work in Refs. [9, 10, 11, 12, 13, 14, 15].
The wealth of spectroscopic data now available in the Nuclear Data tables has motivated us to investigate once again the nuclear ground–state domain. We are able to make more definitive and precise statements about regularity versus chaos in this domain than has been possible so far. As in Ref. [8], we focus attention on $2^+$ states of select even–even nuclei. These nuclei are grouped into classes. The classes are defined in terms of the ratio $R_{4/2}$, i.e., the ratio of the excitation energies of the first $4^+$ and the first $2^+$ level in each nucleus. We argue below that the classes define a grouping of nuclei that have common collective behavior. The sequences of $2^+$ states are unfolded and analyzed with the help of Bayesian inference. The chaoticity parameter $f$ defined below is determined for each class.

II. DATA SET

The data on low–lying $2^+$ levels of even–even nuclei are taken from the compilation by Tilley et al. [16] for mass numbers $16 \leq A \leq 20$, from that of Endt [17] for $20 \leq A \leq 44$, and from the Nuclear Data Sheets for heavier nuclei. We considered nuclei for which the spin–parity $J^\pi$ assignments of at least five consecutive $2^+$-levels are unambiguous. In cases, where the spin-parity assignments were uncertain and where the most probable value appeared in brackets, we accepted this value. We terminated the sequence when we arrived at a level with unassigned $J^\pi$, or when an ambiguous assignment involved a $2^+$ spin-parity among several possibilities, as e.g. $J^\pi = (2^+, 4^+)$. We made an exception when only one such level occurred and was followed by several unambiguously assigned levels containing at least two $2^+$ levels, provided that the ambiguous $2^+$ level is found in a similar position in the spectrum of a neighboring nucleus. However, this situation occurred for less than 5% of the levels considered. In this way, we obtained 1306 levels of spin–parity $2^+$ belonging to 169 nuclei. The composition of this ensemble is as follows: 5 levels from each of 47 nuclei, 6 levels from each of 32 nuclei, 7 levels from each of 22 nuclei, 8 levels from each of 22 nuclei, 9 levels from each of 16 nuclei, 10 levels from each of 14 nuclei, 11 levels from each of 5 nuclei, 12 levels from each of 2 nuclei, and sequences of 13, 14, 15, 17, 20, 21, 24, 30, and 32 levels, each belonging to a single nucleus.
III. CLASSIFICATION OF NUCLEI

A class is defined by choosing an interval within which the ratio

\[ R_{4/2} = \frac{E(4^+_1)}{E(2^+_1)} \]

of excitation energies of the first \(4^+\) and the first \(2^+\) excited states, must lie. The width of the intervals was taken to be 0.1 when the total number of spacings falling into the corresponding class was about 100 or more. Otherwise, the width of the interval was increased. The use of the parameter (3) as an indicator of collective dynamics is justified both empirically and by theoretical arguments. We recall the arguments in turn.

(i) Casten et al. [18] plotted \(E(4^+_1)\) versus \(E(2^+_1)\) for all nuclei with \(38 \leq Z \leq 82\) and with \(2.05 \leq R_{4/2} \leq 3.15\). The authors found that the data fall on a straight line. This suggests that nuclei in this wide range of \(Z\)-values behave like anharmonic vibrators with nearly constant anharmonicity. As the ratio \(R_{4/2}\) approaches the rotor limit \(R_{4/2} = 3.33\), the slope of the curve showing \(E(4^+_1)\) versus \(E(2^+_1)\) decreases within a narrow range of \(E(2^+_1)\)–values, asymptotically merging the rotor line of slope 3.33. In a subsequent paper [19] it was found that a linear relation between \(E(4^+_1)\) and \(E(2^+_1)\) holds for pre–collective nuclei with \(R_{4/2} < 2\). Thus, from an empirical perspective, the dynamical structure of medium–weight and heavy nuclei can be quantified in terms of \(R_{4/2}\).

(ii) Theoretical calculations based on the IBM-1 model [20] support the conclusion that \(R_{4/2}\) is an appropriate measure for collectivity in nuclei. The model has three dynamical symmetries, obtained by constructing the chains of subgroups of the \(U(6)\) group that end with the angular momentum group \(SO(3)\). The symmetries are labeled by the first subgroup appearing in the chain which are \(U(5)\), \(SU(3)\), and \(O(6)\) corresponding, respectively, to vibrational, rotational and \(\gamma\)–unstable nuclei. Extensive numerical calculations for the classical as well as the quantum–mechanical IBM Hamiltonian by Alhassid et al. [21] indeed showed a considerable reduction of the standard measures of chaoticity when the parameters of the IBM model approach one of the three cases just mentioned. The IBM calculation of energy levels yields values of \(R_{4/2} = 2.00, 3.33,\) and 2.50 for the dynamical symmetries \(U(5)\), \(SU(3)\), and \(O(6)\), respectively. Thus, we may expect increased regularity of nuclei having one of these values of \(R_{4/2}\).

One might expect that the chaoticity parameter also assumes small values for nuclei
near magic numbers, where $R_{4/2} \approx 1$. For mass numbers in this domain, our data set is unfortunately too small to allow us to draw definitive conclusions, see Fig. 1.

**FIG. 1:** Mean value $\bar{f}$ (solid lines) and $\sigma$ (error bars) (see Eqs. (15)) of the chaoticity parameter for nuclei in several classes defined in terms of $R_{4/2}$, obtained by Bayesian inference.

### IV. UNFOLDING

Every sequence has to be "unfolded", see Ref. [22], to obtain a new sequence with unit mean level spacing. This is done by fitting a theoretical expression to the number $N(E)$ of levels below excitation energy $E$. The expression used here is the constant–temperature formula [6],

$$N(E) = N_0 + \exp\left(\frac{E - E_0}{T}\right).$$

We deal with many short sequences of levels. In this case, the unfolding procedure introduces a bias towards the GOE. This is shown and discussed in Ref. [23], and will have to be taken into account when we discuss our results. The three parameters $N_0$, $E_0$ and $T$ obtained for each nucleus vary considerably with mass number. Nevertheless, all three show a clear tendency to decrease with increasing mass number. For the effective temperature, for example, we find, assuming a power–law dependence, the result $T = (15 \pm 4)A^{-(0.62 \pm 0.05)}$ MeV. This
result is consistent with an analysis of the level density of nuclei in the same range of excitation energy carried out by von Egidy et al. These authors find $T = (19 \pm 2)A^{-0.68\pm0.02}$ MeV.

V. METHOD OF ANALYSIS

A detailed account of our method has been given in Ref. [23]. Here we confine ourselves to the central aspects. We are guided by the idea that the intermediate behavior of the NNS distribution of low–lying nuclear levels does not necessarily imply that nuclei in the vicinity of the ground state have mixed regular–chaotic dynamics. The key ingredient of our analysis is the assumption that the deviation of the NNS distribution of low–lying nuclear levels from the GOE statistics is caused by the neglect of possibly existing conserved quantum numbers other than energy, spin, and parity. A given sequence $S$ of levels can then be represented as a superposition of $m$ independent sequences $S_j$ each having fractional level density $f_j$, with $j = 1, ..., m$, and with $0 < f_j \leq 1$ and $\sum_{j=1}^{m} f_j = 1$. We assume that the NNS distribution $p_j(s)$ of $S_j$ obeys GOE statistics. The exact NNS distribution $p(s)$ has been given in Ref. [24]. It depends on the $(m - 1)$ parameters $f_j$, $j = 1, \ldots, m - 1$. In [25], this expression has been simplified by observing that $p(s)$ is mainly determined by short-range level correlations. This reduces the number of parameters to unity and the proposed NNS distribution of the spectrum is

$$p(s, f) = [1 - f + Q(f)\frac{\pi s}{2}] \exp \left[ -(1 - f)s - Q(f)\frac{\pi s^2}{4} \right]. \tag{5}$$

Here, $f = \sum_{j=1}^{n} f_j^2$ is the mean fractional level density for the superimposed sequences; it is the single parameter characterizing the distribution. We determine the function $Q(f)$ from the requirement that the expectation value of $s$ is unity, $\int ds \, sp(s, f) = 1$. This relates $Q$ to the error function. We have numerically approximated it and obtain for $f$ in the interval of $0.1 \leq f \leq 0.9$ the parabolic relation

$$Q(f) = f \left( 0.7 + 0.3 f \right). \tag{6}$$

For a superposition of a large number $m$ of sequences, $f$ is of order $1/m$. In the limit of $m \to \infty$, $p(s, f) \to p(s, 0) = p_p(s)$ as given by Eq. [2]. This expresses the well–known fact that the superposition of very many GOE sequences produces a Poisson distribution.
On the other hand, for \( f \to 1 \), \( p(s, f) \) approaches the Wigner distribution (1) expected for a single GOE. We therefore refer to \( f \) as to the chaoticity parameter. Our parameterization (5) is not restricted to statistically independent sequences \( S_j \). A system with partially broken symmetries can also be approximately represented by a superposition of independent sequences [26]. In this case, the distribution (5) which differs from zero at \( s = 0 \), is not accurate for a domain of very small spacings. The magnitude of this domain depends on the ratio of the strength of the symmetry-breaking interaction and the mean level spacing.

We determine the parameter \( f \) by the method of Bayesian inference [26]. Given a sequence of spacings \( s = (s_1, s_2, ..., s_N) \), the joint probability distribution \( p(s|f) \) of these spacings, conditioned by the parameter \( f \), is given by

\[
p(s|f) = \prod_{i=1}^{N} p(s_i, f).
\]

Eq. (7) holds if the experimental \( s_i \) are taken to be statistically independent. This assumption is justified as long as we confine ourselves to the investigation of the NNS distribution. Bayes’ theorem then provides the posterior distribution

\[
P(f|s) = \frac{p(s|f)\mu(f)}{M(s)}
\]

of the parameter \( f \) given the events \( s \). Here, \( \mu(f) \) is the prior distribution and \( M(s) = \int_0^1 p(s|f)\mu(f)df \) is the normalization. The prior distribution is found from Jeffreys’ rule [29, 30]

\[
\mu(f) \propto \left| \int p(s|f) \left[ \partial \ln p(s|f)/\partial f \right]^2 ds \right|^{1/2}.
\]

We substitute Eq. (8) into formula (9), evaluate the integral numerically and approximate the result by the polynomial

\[
\mu(f) = 1.975 - 10.07f + 48.96f^2 - 135.6f^3 + 205.6f^4 - 158.6f^5 + 48.63f^6.
\]

Even for only moderately large \( N \), it is useful to write \( p(s|f) \) in the form

\[
p(s|f) = e^{-N\phi(f)},
\]

where

\[
\phi(f) = (1-f)s + \frac{\pi}{4}f(0.7+0.3f)s^2 - \ln[1-f + \frac{\pi}{2}f(0.7+0.3f)s] + \ln[1-f + \frac{\pi}{4}f(0.7+0.3f)s].
\]
Here the notation $\langle x \rangle = (1/N) \sum_{i=1}^{N} x_i$ has been used. By calculating the mean values $\langle \cdots \rangle$ in Eq. (12) for various spectra, one finds that the function $\phi(f)$ has a deep minimum, say at $f = f_0$. One can therefore represent the numerical results in analytical form by parametrizing $\phi$ as

$$\phi(f) = A + B(f - f_0)^2 + C(f - f_0)^3. \quad (13)$$

We then obtain

$$P(f|s) = c \mu(f) \exp(-N[B(f - f_0)^2 + C(f - f_0)^3]) , \quad (14)$$

where $c = e^{-NA}/M(s)$ is a normalization constant. The error interval $\overline{f} \pm \sigma^{1/2}$ of the chaoticity parameter is defined by the mean value $\overline{f}$ and the variance $\sigma^2$, with

$$\overline{f} = \int_0^1 f P(f|s) \, df \quad \text{and} \quad \sigma^2 = \int_0^1 (f - \overline{f})^2 P(f|s) \, df . \quad (15)$$

VI. CHAOTICITY PARAMETER

The results obtained for $\overline{f}$ and $\sigma$ are given in Fig. 1. Figure 2 shows a comparison of the spacing distributions conditioned by $\overline{f}$ and the histograms for each class of nuclei. In view of the small number of spacings within each class, the agreement seems satisfactory.

We recall that the analysis of many short sequences of levels tends to overestimate $\overline{f}$. Therefore, we focus attention not on the absolute values of $\overline{f}$ but on the way $\overline{f}$ changes with $R_{4/2}$. The graph of $\overline{f}$ against $R_{4/2}$ in Figure 1 has deep minima at $R_{4/2} = 2.0, 2.5$, and $3.3$. These values of $R_{4/2}$ are associated with the dynamical symmetries of the IBM mentioned above. Another minimum of statistical significance occurs for $2.25 \leq R_{4/2} \leq 2.35$. This minimum may indicate that nuclei which lie between the limiting cases of the $U(5)$ and $O(6)$ dynamical symmetries, are relatively regular. One may associate this region with the critical point of the $U(5)$–$O(6)$ shape transition in nuclei. Iachello [27] has recently shown that this transition is approximately governed by the "critical" $E(5)$ dynamical symmetry. Nuclei with $E(5)$ dynamical symmetry have $R_{4/2} = 2.2$. Experimental examples of this critical symmetry have been found by Casten and Zamfir [28].
FIG. 2: Comparison of the spacing distributions calculated from Eq. (5) using the values of $f$ given in Fig. 1 with the histograms for the empirical NNS distributions for nuclei in several classes defined in terms of $R_{4/2}$.

VII. SUMMARY

With the help of a systematic analysis of the NNS distributions for $2^+$ levels of even–even nuclei, we have determined the chaoticity parameter $f$ for nuclei at low excitation energy. While in a single nucleus the number of states with reliable spin–parity assignments is not sufficient for a meaningful statistical analysis, a combination of sequences of levels taken from similar nuclei provides a sufficiently large ensemble. As the measure of similarity we have taken the ratio $R_{4/2}$ of the excitation energies of the lowest $4^+$ and $2^+$ levels in each nucleus. As seen in Figure 1, the chaoticity parameter $f$ is indeed dependent on $R_{4/2}$. It has deep minima at $R_{4/2} = 2.0$, $2.5$, and $3.3$. These minima correspond, respectively, to the $U(5)$, $SO(6)$, and $SU(3)$ dynamical symmetries of the IBM. A further minimum may relate to the critical $E(5)$ symmetry.

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