Discretized Light-Cone Quantization: Solving a 1+1 dimensional field theory (QED)

Stephan Elser, Hans-Christian Pauli and Alex C. Kalloniatis
Max-Planck-Institut für Kernphysik
Postfach 10 39 80
D-69029 Heidelberg, Germany
March 28, 2022
Preprint MPIH-V17-1995

Abstract

We describe the programming method for generating the spectrum of bound states for relativistic quantum field theories using the nonperturbative Hamiltonian approach of Discretized Light-Cone Quantization. The method is intended for eventual application to quantum chromodynamics in 3+1 dimensions. Here the fundamental principles are illustrated concretely by treatment of QED in two dimensions. The code is intended as a basis for extensions to include more complicated gauge symmetry groups, such as SU(3) color, and other quantum numbers. The code was written in Fortran 77 and implemented on a DEC 5000-260 workstation with a typical runtime of 0.2s at total momentum 10.

Program Summary

Title of program: ms_main.f
Catalogue number: ???.
Program obtainable from: MPI für Kernphysik, Heidelberg, FRG
Licensing provisions: none
Computer: workstation DEC 5000-260; Installation: MPI für Kernphysik, Heidelberg, FRG
Operating System: ULTRIX version 4.2
Programming language used: Fortran 77
Program storage required: 255kbyte

*Present Address: Institut für Physik/COM, Humboldt-Universität, Invalidenstrasse 110, D-10115 Berlin, Germany
Data storage required: 80kbyte

Peripherals used: terminal

No. of bits per word: 32

Number of lines in program: 1894

Keywords: particle physics, field theory, Hamiltonian method, QED, Schwinger model, Discretized Light-Cone Quantization

Nature of the physical problem
Gauge field theories are notoriously difficult to solve non-perturbatively, in particular non-Abelian gauge theory which underlies quantum chromodynamics. We describe the relatively new approach of [1] to this problem, Discretized Light-Cone Quantization (DLCQ), which in many respects is complementary to Lattice Gauge Theory. We detail the method in some generality but present the programming application to Quantum Electrodynamics in 1+1 dimensions [2]. Mathematically, in DLCQ one formulates the theory in lightcone coordinates on a finite space-interval or volume with periodic (and/or antiperiodic) boundary conditions on the quantum fields. Correspondingly the plane wave Fock basis is discretized. A particle number and momentum cut-off lead to a completely specified numerical problem. One can thus solve the Schrödinger equation for bound states. The continuum limit must be obtained by numerical extrapolation of, for example, the invariant masses. The generalization to other theories is evident. The program is intended as a template for further calculations in theories approaching the complexity of QCD(3+1), and has been already used as such.

Method of solution
The algorithm can be briefly described as follows:

**Step 1:** Choose a harmonic resolution and particle number cut-off.

**Step 2:** Construct the Hilbert space (Fock states) respecting statistics and symmetries.

**Step 3:** Evaluate for each matrix element the analytic expression, making sure all conservation laws are obeyed.

**Step 4:** Add all allowed terms to obtain the Hamiltonian matrix.

**Step 5:** Diagonalize, yielding the complete mass spectrum and wave functions.

**Step 6:** Repeat steps 1-5 for higher resolutions to check convergence.

Restrictions: The resolution is restricted by the available memory, as the Hamiltonian matrix is stored. The run time is negligibly restrictive for matrices of dimension 1500, but grows with matrix dimension $d^3$.

Typical run time: 0.2s for the test case of resolution 10.

References:
[1] H.C. Pauli, S.J. Brodsky, Phys. Rev. D32 (1985) 1993, 2001
[2] J. Schwinger, Phys. Rev. 128 (1962) 2425
Long Write-Up

1 Introduction

The major physical theory calling for the development of nonperturbative techniques for solving quantum field theories is quantum chromodynamics (QCD). It is arguably the theory of strong interactions describing hadrons. However to extract the physics of mesons and baryons within a picture of relativistic bound states of quarks and gluons is extremely difficult. Lattice Gauge Theory [1] is one nonperturbative method which is presently computing ratios of hadron masses with ever improving success [2]. However it offers little insight into how the above constituent or partonic picture emerges out of QCD. Discretized light-cone quantization (DLCQ) exploits the computational elegance and simplicity of the lattice method – by working with a momentum space lattice. However it also employs the physically intuitive power of Dirac’s ‘front form’ [3] in which the vacuum is essentially trivial and one of the momentum operators is independent of the interaction and positive definite. Thus the low energy part of the QCD hadron spectrum in such a framework would be well described by a low number of light-cone Fock space quanta built from the simple light-cone vacuum. DLCQ obtains both the spectrum and the wavefunctions of relativistic bound states of the theory by direct diagonalization on the computer of a finite dimensional Hamiltonian matrix. How a Lagrangian field theory is reduced to such a problem is the whole art of DLCQ.

This is not the place to detail the history or subtleties of DLCQ for which the reader is referred to [4]. It suffices to say that QCD in (3+1) dimensions has not yet been attacked with the full power of the method. Rather it is being worked towards via progressively more complicated models by ourselves [5] and others [6, 7, 8, 9]. This is therefore an opportune point in the development of the method to present the programming details at their simplest but with enough richness to display the power of the method. We turn to quantum electrodynamics in one space and one time dimension, QED(1+1), originally treated by Eller [10]. This model shares with QCD(3+1) the feature of being a relativistic theory of gauge vector bosons coupled to fermions. It is, however, unencumbered by the technicalities of color and transverse dimensions. These demand a richer Fock space and additional Hamiltonian structures the programming of which can nevertheless be grafted onto the basic procedures we outline. Moreover the simplicity of the theory here means Hamiltonian matrix elements can be analytically calculated by hand. This may be technically inefficient in more elaborate theories in which case part of this task could also be programmed. Two aspects, however, could change the underlying method of programming: nontrivial renormalization and dynamical zero modes of gauge boson fields. These problems certainly become acute in higher dimensions and remain at the frontier of present research into the method [11, 12] but we will not say anything more about them in the present work.
In the following section we outline briefly how one builds from the Lagrangian defining the field theory to the matrix problem via DLCQ. In Section 3 there is a description of the various components of the program and how they can be implemented in a test run. The final section gives a statement of conclusions.

2 Light-Cone Hamiltonian Matrix

This section is intended to explain notation and concepts in the formulae and the corresponding computer codes. To show the generality of the method we describe the procedure for QCD(3+1), and later show how this concretely is worked out for the model case QED(1+1).

Derivation. For a given quantum field theory in an arbitrary number of space-dimensions, DLCQ sets out to solve the eigenvalue equation

\[ H_{LC} |\Psi\rangle = m^2 |\Psi\rangle. \]  

Here \( H_{LC} \equiv P^\mu P_\mu = 2P^- P^+ - P^2_\perp \) is the so-called ‘light-cone Hamiltonian’. In two dimensions \( P_\perp = 0 \). Evidently Eq.1 is a realization of the relativistic energy-momentum relationship on physical eigenstates of the given theory. The (four) momentum operator \( P^\mu \) is built from the energy-momentum tensor, a conserved quantity under evolution in light-cone time \( x^+ \equiv \frac{1}{\sqrt{2}}(x^0 + x^3) \). Our conventions follow Kogut and Soper [13]. Initial data for the independent Lagrangian fields are specified on the null-plane surface \( x^+ = 0 \). This will be done by expanding the independent fields at \( x^+ = 0 \) in momentum modes. A momentum space ‘lattice’ is obtained by restricting the one dimensional light-cone space variable \( x^- \equiv \frac{1}{\sqrt{2}}(x^0 - x^3) \) to a finite interval from \(-L\) to \(+L\) and invoking boundary conditions on the fields. A similar demand can be made on \( x_\perp \) in higher dimensions giving discretized \( k_\perp \). Periodic boundary conditions for bosons is the only choice that is consistent with the standard Euler-Lagrange equations. As fermions appear bilinearly in the Lagrangian of any of our candidate gauge theories, either periodic or antiperiodic boundary conditions can be used. The program we present in the sequel allows the user to choose. Thus the corresponding momenta \( k^+ \) are discretized in, respectively, integer or half-integer units of \( \pi/L \). Due to light-cone kinematics, these integers are either positive or zero. Ignoring zero modes of gauge bosons then, we can choose the light-cone gauge \( A^+ = 0 \). The Maxwell and Dirac equations give constraints allowing one to eliminate the gauge potential \( A^- \) and the lower spinor component \( \Psi_L \). Thus, one is brought to the independent degrees of freedom, which in physical dimensions are the transverse gluons \( A_i^i, i = 1, 2 \) and the upper spinor component \( \Psi_R \). These are subject to quantization. Quantization is achieved by imposing equal \( x^+ \) (anti)commutation relations on the dynamical fields. These translate into (anti)commutators on the Fourier coefficients of the aforementioned momentum mode expansions. The coefficients become Fock operators of boson, fermion and antifermion states. The above eigenvalue problem is then expressed in terms of these operators: the
light-cone Hamiltonian is written in second quantized form, and the wavefunctions are built from the Fock operators as an expansion in the number of fundamental partons with amplitude coefficients which are obtained by solving Eq.(1). The eigenvalue problem for the bound states is thus completely specified numerically.

Now we specify using the model theory QED(1+1). First, in 2 dimensions there is no spin and, neglecting zero modes, there are no dynamical photons. Thus \( \Psi_R \) is the only independent quantum field leading to fermion and antifermion Fock operators in terms of which the eigenvalue problem must be specified. In the light-cone Hamiltonian one can separate out the dimensionful Lagrangian parameters of fermion mass \( m_f \) and coupling constant \( g \) by invoking the dimensionless operator functions in \( K, T, C, F \) and \( S \)

\[
H_{LC} = 2K(m_f^2 + \frac{g^2}{\pi})\left[(1 - \frac{1}{1 + \pi(\frac{m_f}{g})^2})T + \frac{1}{1 + \pi(\frac{m_f}{g})^2}(C + F + S)\right].
\]  

(2)

These operators are built from combinations of the fermion and antifermion Fock operators and analytic expressions for their matrix elements will be given in the sequel. Here \( K \) represents the dimensionless momentum called the harmonic resolution and \( T \) is the kinetic energy. In the literature, the potential energy terms \( C, F \) and \( S \) are called respectively ‘contractions’, ‘forks’ and ‘seagulls’ because of a diagrammatic representation of their operator structure \[14, 15\]. The simplicity of QED(1+1) allows application of the Wick expansion leading to analytic expressions for the matrix elements of \( H_{LC} \) between arbitrary Fock states. These expressions are given below. The matrix is normalised by a factor \((m_f^2 + \frac{g^2}{\pi})^{-1}\).

\textbf{Notation.} In the absence of spin and color, the only quantum number needed to specify states is longitudinal momentum. In the following, we use \( k \) to represent this momentum for single electron states and \( p \) for positron states. The quantity \( N \) will represent the particle number. As we will construct multiparticle Fock states, we attach a label to the momenta, \( k_i \), to indicate the position of the corresponding Fock operator in what will be \( momentum \ ordered \) states. Thus, for example, \( k_1 \) will always represent the electron with lowest momentum in the state. The summation indices \( i, j \ldots \) of the sums thus run over the positions \( 1, 2 \ldots N \). Incoming momenta \( k_i, p_i \) carry no prime while outgoing momenta, \( k'_i, p'_i \), are primed. As usual, ‘incoming’ means those states on the right-hand side of expressions. We denote incoming and outgoing states by \( |i\rangle \) and \( \langle f| \) respectively. For convenience we suppress the forbidden zero momentum transfer \( 1 \) terms in the formulae.

Because of contractions between states of like-momentum we can use the ‘spectator/participant’ picture for the interaction. Particles not involved in the interaction of up to four participants have to fulfill a pairwise momentum conservation. This will be represented by the symbol \( \Delta_s \)

\[
\Delta_s = \prod_i \delta_{k_i,k'_i} \prod_j \delta_{p_j,p'_j},
\]

with indices \( i, j \) running only over spectator positions in the Fock space vectors.
The relative sign $\Sigma_r$ of graphs can be calculated using reference graphs and the number of transpositions of particles necessary to get to these. This leads to terms $\Sigma_r = (-1)^{(a+b+c+d)}$ and $c_{a,b} = \begin{cases} 1 & \text{if } b > a; \ 0 & \text{else}. \end{cases}$

**Matrix elements.** We thus obtain the following c-number expressions for the matrix elements of the aforementioned operators.

\[
\langle f|K|i \rangle = \delta_{NN'} \prod_{n=1}^{N} \delta_{k_n k'_n} \prod_{n=1}^{N} \delta_{p_n p'_n} \sum_{n=1}^{N} (k_n + p_n),
\]

\[
\langle f|T|i \rangle = \delta_{NN'} \prod_{n=1}^{N} \delta_{k_n k'_n} \prod_{n=1}^{N} \delta_{p_n p'_n} \sum_{n=1}^{N} \frac{1}{2} \left( \frac{1}{k_n} + \frac{1}{p_n} \right),
\]

\[
\langle f|C|i \rangle = \delta_{NN'} \prod_{n=1}^{N} \delta_{k_n k'_n} \prod_{n=1}^{N} \delta_{p_n p'_n} \times
\sum_{n=1}^{N} \sum_{q=1}^{N} \frac{1}{2} \left( \frac{1}{(k_n - q)^2} + \frac{1}{(p_n - q)^2} - \frac{1}{(k_n + q)^2} - \frac{1}{(p_n + q)^2} \right),
\]

\[
\langle f|S|i \rangle = \delta_{NN'} \sum_{n,m,s,t=1}^{N} \Delta_s \times
\left( \delta_{k_n + p_m - k'_s - p'_t, 0} (-1)^{(n+m+s+t)} \left( \frac{1}{(k_n + p_m)^2} - \frac{1}{(k_n - k'_s)^2} \right) \right.
\]
\[
+ \frac{1}{2} \delta_{k_n + k_m - k'_s - k'_t, 0} (-1)^{(n+m+s+t+c_{n,m}+c_{s,t})} \frac{1}{(k_n - k'_s)^2}
\]
\[
+ \frac{1}{2} \delta_{p_n + p_m - p'_s - p'_t, 0} (-1)^{(n+m+s+t+c_{n,m}+c_{s,t})} \frac{1}{(p_n - p'_s)^2},
\]

\[
\langle f|F|i \rangle = \Delta_s \sum_{n=1}^{N} \delta_{N+2,N'} \times
\left( \delta_{k_n - k'_m - k'_s - p'_t, 0} (-1)^{(n+m+s+t+c_{n,s}+N)} \left( \frac{1}{(k_n - k'_m)^2} \right) \right.
\]
\[
+ \delta_{p_n - p'_m - p'_s - p'_t, 0} (-1)^{(n+m+s+t+c_{n,m}+N)} \left( \frac{1}{(p_n - p'_m)^2} \right)
\]
\[
+ \Delta_s \sum_{n,m,s=1}^{N} \delta_{N,N'+2} \times
\left( \delta_{k_n + k_m + p_s - k'_t, 0} (-1)^{(n+m+s+t+c_{n,m}+N')} \left( \frac{1}{(k_n - k'_t)^2} \right) \right.
\]
\[
+ \delta_{p_n + p_m + p_s - p'_t, 0} (-1)^{(n+m+s+t+c_{n,m}+N')} \left( \frac{1}{(p_n - p'_t)^2} \right) \right].
\]

These analytic expressions were implemented in the computer code we discuss in the following section.

3 Features of the Program
3.1 General Run Features

The program ‘ms\_main.f’ was written in Fortran77 and was compiled and run on a Unix-system DEC 5000-260 workstation, using routines of the Numerical Algorithm Group library NAGLIB Mark 15. It observes the standard Fortran 77 conventions, but uses the comment sign ‘!’ and longer variable names which can be easily shortened if necessary.

The ‘physical’ input parameters are: choice of fermion boundary conditions as a logical constant, the harmonic resolution as a natural number, the fermion mass and the coupling constant as positive real numbers. The desired particle number truncation is given via two switches, fixed particle number and maximal particle number. The value ‘Zero’ signifies no constraint, so that obviously at most one of these can larger than zero, both zero standing for no constraint at all. Other options control the amount of output produced.

There are default input values given specifying QED(1+1) with massive fermions and a strong coupling \( g = 2.38m_f \). If the inputs are internally consistent they are exchanged against the defaults (check this, if there are problems). A convenient deviation made in the programming code is that the momenta and correspondingly the harmonic resolution are stored and assigned twice the value as in the formulae to make them integers instead of half-integers. For example, a harmonic resolution 10 is in fact \( K = 20 \) in the code and input files.

A standard test run should take about 0.2s CPU time. Matrices of dimension of about 1500 take no more than 2000s, but use about 10 Mbyte RAM. This is because the whole matrix is handled as an array, the simplest possibility for the small matrices we are dealing with. The CPU time needed to generate the Fock space is generally negligible (Test Run: 0.008s) in comparison to the time needed to calculate the LC Hamiltonian matrix elements (0.098s) and diagonalize the matrix (0.062s). The diagonalization time is cubic in the matrix dimension and clearly dominates in the consumption of CPU time.

3.2 Program structure

The main program is ‘ms\_main.f’. It

- sets the maximum value for the matrix dimension (= number of basis Fock states), number of partons, resolution and the number of lines on one output page by including ‘ms\_parameters.f’. Default values are dimension 700, particle number 10, resolution 600 and pagelength 60 lines.

- sets input data defaults. The values correspond to QED(1+1) with Periodic Boundary Conditions (BC = true), fermion mass \( m_f = 1.0 \), a rather strong coupling \( g = 2.38 \) and a small resolution \( K = 20 \). Using the switch ‘fixed particle number =0’ and ‘maximal particle number =0’ we include the complete Fock space, e.g. all states with particle number up to the highest theoretically possible one. The output is limited to one page for each of the Fock state basis elements, mass eigenvalues and wave functions.
reads the input data in subroutine ‘input’ (viz. screen output), tests it and exchanges it against the defaults if consistent.

- calls the master subroutine ‘master’.

- writes out to the file ‘masses.out’, first a leading page with the input data (and the defaults if these were taken for the calculation), then the requested number of pages of Fock state basis, mass eigenvalues and wave functions with a header repeating the most important input data at the top of each page.

- as a check writes out the actual input values to standard i/o.

In the master subroutine ‘master’

- the Fock space basis is constructed using the subroutine ‘con’.

- the mass squared matrix parts $2KT$, $2KC$, $2KS$ and $2KF$ are calculated using the subroutines ‘kinetic’, ‘contractions’, ‘seagulls’ and ‘forks’.

- these parts are added to the correct mass squared matrix, and this is diagonalized using a NAGLIB routine.

The important subroutines within ‘con’, ‘kinetic’, ‘contractions’, ‘seagulls’ and ‘forks’ are:

- ‘fermion’ – constructs the charge singlet Fock states built from electrons and positrons. This essentially is the heart of the program and so we take the space here to explain it in some detail. This subroutine is called from the routine ‘con’ and works as follows. For each fixed particle number allowed, the routine begins by building a reference state which consists of the lowest momentum electrons, namely a sequence of integers one after the other such that their sum does not exceed the total allowed momentum. This is stored as the first possible electron sub-state. From this state further states are now generated. The highest integer in the reference sequence is incremented. The total momentum of the new set is checked for total momentum saturation. If not saturated, the state is allowed and added to the electron sub-state array together with the sum of the particle momenta and the particle number. The procedure continues with incrementation of the highest electron momentum until total momentum is exhausted. The next-to-last momentum is then incremented, all higher momenta are reset to their (lowest) starting positions, and the routine proceeds with the highest momentum electron in an identical way to generate more states, always checking against total momentum saturation. Thus the program works its way down to the lowest momentum number. In this way an overly-large electron Fock space is built, but with all states satisfying the Pauli principle and momentum ordered. A positron space is built by copying the electron states. The two spaces are then joined by combining equal number electron and positron states. These combined states are then only checked against total momentum conservation as charge
conservation is already implicit. The stored particle numbers and total momenta of the sub-states allows for this to be very fast, so that further improvements were not implemented.

- ‘ep_graph’ – calculates the matrix element contributions from electron-positron annihilation and electron-positron scattering graphs. It is called from the subroutine ‘seagulls’. It creates all possible matrix elements for these interaction operators by using the above-created space of states. In a second step, matrix elements are weeded out by checking first for particle number conservation. Then all possible participant combinations are tested for momentum conservation. Using the subroutine ‘spec’, the spectators are then checked for correct pairwise momentum conservation. As zero momentum transfers are unphysical in our theory, these are also excluded in a final step. The subroutines ‘ee_graph’ and ‘pp_graph’ are modelled on ‘ep_graph’ and all three taken together generate the ‘seagull’ matrix elements.

- ‘n_2_sec.e’ – calculates the matrix element contributions from the particle number (n-2,n) sector graphs of an incoming electron undergoing pair production. It is called from the routine ‘forks’. Again, as in ‘ep_graph’, all possible matrix elements are created and then non-allowed elements removed by checking that the number of ‘ingoing’ and ‘outgoing’ particles differ by two and that momentum is conserved in the appropriate way. A further feature in this subroutine is the explicit inclusion of a relative minus sign between states with two indistinguishable electrons and/or positrons which arises directly from the Wick expansion. This subroutine is the template for the routines ‘n_2_sec_p’, ‘n_sec_eep’ and ‘n_sec_epp’ and altogether give the ‘fork’ matrix elements of the light-cone Hamiltonian.

- ‘spec’ – weeds out matrix elements for which spectators do not satisfy momentum conservation. The routine is given the participant configuration via the general participant vector ‘s’ holding two ingoing and outcoming electron and positron momenta respectively. From that it is able to identify the remaining particles in a pair of states as spectators. For these pairwise momentum conservation is checked and when the check fails, the states are eliminated.

- ‘diag’ – handles the calling of the general black box diagonalisation routine of the NAGLIB ‘F02ABF’. We choose that routine because the QED(1+1) mass squared operator is real and symmetric (being hermitian). It reduces the matrix to real symmetric tridiagonal form by Householder’s method and then uses the QL-Algorithm to extract eigenvalues and eigenvectors. A way of checking the results is to change this to routines ‘F02AGF’ and ‘F02AXF’, which deal with general nonsymmetric and hermitian matrixes, respectively. The matrix is stored as an array, which causes this method to use rather huge amounts of storage. However it is reliable, conceptionally simple and obtains the full spectra and wavefunctions. Note that the eigenvectors
are stored in columns in an array called ‘vector’.

3.3 Program package components

In the package there are the following Fortran files:
1. ms\_main.f
2. ms\_parameters.f
3. ms\_master\_sub.f
4. ms\_construct\_sub.f
5. ms\_kinetic\_sub.f
6. ms\_contractions\_sub.f
7. ms\_seagulls\_sub.f
8. ms\_forks\_sub.f
9. ms\_spectator\_sub.f
10. ms\_diagonalization\_sub.f
11. ms\_in-out\_sub.f
12. ms\_input\_data

File 1 contains the main program, file 2 the parameter definitions used by all routines and the files 3-11 the subroutines with self-explanatory file names. In file 12 the input of the test run is given in a form already prepared for automatized use.

3.4 Reliability and Uses

The routines shown here have already been applied to obtain mass spectra for various coupling constants and with different mass normalizations. The confirmation of their reliability comes from the very good comparison of the results with spectra from alternate analytic and numerical methods. In particular, these are [17, 18, 19]. Moreover we can directly compare against the $m_f = 0$ case of the exact Schwinger model [20], where the lowest mass eigenvalues should be 1 and 2. Even a resolution of 10 ($K = 20$) already yields values of $m_1 = 0.99875$ and $m_2 = 1.9994$. The wavefunctions obtained with this method can be used to extract structure functions [15]. A novel example of uses for the so-obtained spectra is the computation of thermodynamic quantities for the given theory at finite temperature [21].

A generalization of this Abelian gauge theory model would be to include extra flavours. Moreover these routines are already the basis for application to other field theories in 1+1 dimensions such as QED with massive photons [22].

3.5 Test Run

In Appendix [A] we give a sample screen output with the sequence of data inputting, checking and confirmation. The input values used here are the default values. The data file output of a test corresponding to this default input is shown in Appendix [B]. This
should be the basis of checking a test run of the program package. As a second option, in Appendix (C) we give a suggested input file which can be piped into the program while running our routines in batch mode.

4 Concluding Remarks

In this paper we have described a program to solve gauge theories for the bound state spectra and wavefunctions via the formal method of Discretized Light-Cone Quantization. The concrete test case for which we gave details for was quantum electrodynamics in one space and one time dimension. This model theory shares many features with the real problem to which DLCQ is eventually intended to be applied, namely quantum chromodynamics. These common features are the presence of Lorentz and gauge symmetry, fermions and a linear confining potential. However, the absence of spin, color and transverse momentum quantum numbers allow a simple exposition of the essentials of the programming procedure. We thus descried routines which constructed a Fock space of electrons and positrons, and which then computed matrix elements of the Hamiltonian for charge singlet combinations of the Fock states. Diagonalization of this Hamiltonian has been used to generate, for example, spectra of bound states of electron-positron pairs in two-dimensions.

Extension to non-Abelian gauge theories, namely QCD in two dimensions, is technically more complicated in that color statistics must be introduced into the Fock space construction and more complicated interactions built into the code. However these represent a ‘grafting’ onto the type of code we explicitly discussed here. Further extension to include more space-dimensions is also feasible in the same spirit pursued here, where color is compounded by the presence of genuinely dynamical gluons in the Fock space construction. In both cases, the methods used here have been the basis for codes developed in, for example, [5, 23]. These two examples far from exhaust the variations of simplifications on full QCD which could be constructed in order to build toward the full problem and for which DLCQ, as a general method, could be insightful. As a further example we mention the novel application to two-dimensional Yang-Mills theory coupled to scalar adjoint matter by [8]. For a treatment of full QCD, the major problems which could change some aspects of the programming method we describe are the renormalization and zero mode problems. These difficult formal problems are under active consideration.
A  Screen output

| 1 | The Output of 'ms' on your terminal screen |
| 2 | |
| 3 | please give input data: |
| 4 | Next input data set? (1=yes) |
| 5 | boundary conditions |
| 6 | (antiperiodic = T, periodic = F) |
| 7 | boundary conditions are T |
| 8 | harmonic resolution |
| 9 | harmonic resolution is 20 |
| 10 | fermion mass (with decimal point) |
| 11 | fermion mass is 1.00 |
| 12 | coupling constant (with decimal point) |
| 13 | coupling constant is 2.38 |
| 14 | particle number limits |
| 15 | (one of these must be zero; both zero is a full fock space request) |
| 16 | fixed particle number |
| 17 | fixed particle number is 0 |
| 18 | maximal particle number |
| 19 | maximal particle number is 0 |
| 20 | front page (1=yes) |
| 21 | front page printed |
| 22 | maximal number of Fock space pages |
| 23 | maximal number of Fock space pages is 1 |
| 24 | maximal number of eigenvalue pages |
| 25 | maximal number of eigenvalue pages is 1 |
| 26 | maximal number of wavefunction pages |
| 27 | maximal number of wavefunction pages is 1 |
check output - you calculated with data:

BC - boundary conditions: True (antiperiodic)

res - harmonic resolution: 20

m_f - fermion mass: 1.00

c_c - coupling constant: 2.38

Output is: 1 of max 2 page(s) of Fock space

CPU-time for the DLCQ matrix. Dimension is 42.
total time: 0.168 s fock space: 0.008 s matrix: 0.098 s diagonalisation: 0.062 s

The Output of 'ms' on your terminal screen

please give input data:

Next input data set? (1=yes)

program 'ms' calculated the mass spectrum of QED 1+1 with the accepted input data:

BC - boundary conditions: True
(antiperiodic)  |
14  |
15  | res - harmonic resolution: 20
16  |
17  | m_f - fermion mass: 1.00
18  |
19  | c_c - coupling constant: 2.38
20  |
21  | fixed particle number: 0
22  |
23  | maximal particle number: 0
24  | full Fock space - max. part. number: 6
25  |
26  | Output is: 1 of max 2 page(s) of Fock space
27  | 1 of max 1 page(s) of mass values
28  | 1 of max 12 page(s) of eigenvectors
29  |
30  | CPU-time for the DLCQ matrix. Dimension is 42 .
31  | total time: 0.168 s fock space: 0.008 s
32  | matrix: 0.098 s diagonalisation: 0.062 s
33  |
34  |
+-------------------------------------------------------------------------+
::: page 2 :::::::::
05-Jul-94
BC: True m_f= 1.00 c_c= 2.38 Nmax= 6
res= 20
+-------------------------------------------------------------------------+
Fock space basis used by 'ms' 1 - 42 of 42
+-------------------------------------------------------------------------+
state mass^2 part  lowest 5 electron momenta  lowest 5 positron momenta
10  1 21.05 2 1
19  2 7.84 2 3
17  3 5.33 2 5
15  4 4.40 2 7
13  5 4.04 2 9
11  6 4.04 2 11
9  7 4.40 2 13
7  8 5.33 2 15
5  9 7.84 2 17
3  10 21.05 2
1 11 48.00 4 3 1 15
1 12 34.87 4 3 1 13
3 13 32.48 4 3 1 11
5 14 31.75 4 3 1 9
7 15 45.54 4 5 1 13
1 16 32.48 4 5 1 11
3 17 30.22 4 5 1 9
5 18 44.68 4 7 1 11
7 19 31.75 4 7 1 9
3 20 29.71 4 7 1 7
| Mass^2 eigenvalues calculated by 'ms' 1 - 42 of 42 |
|-----------------------------------------------|
| state 1 2 3 4 5 6 7       | 8 |
|-----------------------------------------------|
| mass^2 3.22 7.07 11.21 11.72 12.38 14.43 16.09 |

| Mass^2 wavefunction (times 10) page 1 of 4 |
|-------------------------------------------|
| state 1 2 3 4 5 6 7       | 8 |
|-------------------------------------------|
| mass^2 3.22 7.07 11.21 11.72 12.38 14.43 16.09 |
| 197 | 3.6375 | 2.8099 | 1.6692 | 1.1390 | -2.4319 | 1.4712 | 2.6845 |
| 198 | 3.2898 | 3.8657 | -1.6452 | -0.1565 | -0.0947 | 2.2263 | -0.6816 |
| 199 | 2.7551 | 3.9533 | -3.6808 | -0.9977 | 1.5350 | -1.8788 | -1.4141 |
| 200 | 1.9568 | 2.9886 | -3.3115 | -0.9786 | 1.6094 | -2.6508 | -2.7312 |
| 201 | 0.0508 | 0.1227 | 0.5743 | -0.0947 | 0.4703 | -1.4469 | 1.2120 |
| 202 | 0.0107 | 0.1923 | 0.4471 | -0.3046 | 0.7164 | 2.2283 | 0.6816 |
| 203 | 0.0112 | 0.1634 | 0.1320 | -0.1184 | -2.1230 | -1.0377 | -0.9995 |
| 204 | -0.0002 | 0.0651 | -0.0076 | 2.4650 | 1.2108 | -0.1620 | 1.0982 |
| 205 | 0.0484 | 0.3630 | 0.9904 | -0.5019 | 1.0885 | 0.9985 | -0.1918 |
| 206 | 0.0083 | 0.3870 | 0.4591 | -0.4231 | -1.6197 | 1.4899 | -2.0400 |
| 207 | -0.0004 | 0.2494 | 0.0161 | 2.6933 | -1.0611 | -1.3940 | 0.5794 |
| 208 | 0.0275 | 0.5813 | 0.7688 | -0.4837 | -1.2287 | 0.3499 | -2.1919 |
| 209 | -0.0003 | 0.4728 | 0.0473 | 2.4318 | -0.3335 | 1.4770 | 0.1299 |
| 210 | -0.0021 | 0.1830 | -0.1102 | -0.1730 | -2.6526 | -1.3541 | 0.8304 |
| 211 | 0.0000 | 0.6695 | 0.0000 | 2.0339 | -0.1441 | 0.3881 | 0.0000 |
| 212 | -0.0092 | 0.3973 | -0.3850 | -0.4348 | -1.7352 | 1.6020 | 1.7598 |
| 213 | -0.0275 | 0.5813 | -0.7688 | -0.4837 | -1.2287 | 0.3499 | 2.1919 |
| 214 | -1.5719 | 20 | 1.0543 | -0.0001 | 0.0895 | -0.0056 | 3.8877 | 1.9939 | -0.2040 | 0.5719 |
| 215 | -1.5882 | 21 | 0.0092 | 0.3973 | 0.3850 | -0.4348 | -1.7352 | 1.6020 | -1.7598 |
| 216 | -0.0508 | 0.1227 | -0.5743 | -0.2382 | 0.4703 | -1.4469 | -1.1210 |
| 217 | -0.1230 | 0.2058 | -0.4299 | -0.4354 | 0.8182 | 2.7150 | 0.6096 |
| 218 | 0.0021 | 0.2121 | 0.1534 | -0.1939 | -3.1555 | -1.6260 | -1.3350 |
| 219 | -0.0001 | 0.0895 | -0.0056 | 3.8877 | 1.9939 | -0.2040 | 0.5719 |
| 220 | -1.7284 | 22 | 0.0092 | 0.3973 | 0.3850 | -0.4348 | -1.7352 | 1.6020 | -1.7598 |
| 221 | 0.0000 | 0.3037 | 0.0000 | 3.4922 | -1.4084 | -1.8976 | 0.0000 |
| 222 | 0.0003 | 0.4728 | -0.0473 | 2.4318 | -0.3335 | 1.4770 | -0.1299 |
| 223 | -0.0021 | 0.2121 | -0.1534 | -0.1939 | -3.1555 | -1.6260 | 1.3350 |
| 224 | 0.0000 | 0.3037 | 0.0000 | 3.4922 | -1.4084 | -1.8976 | 0.0000 |
| 225 | -0.0107 | 0.1923 | -0.4471 | -0.3046 | 0.7164 | 2.2283 | 0.6816 |
| 226 | -1.8602 | 23 | 0.0001 | 0.0895 | 0.0056 | 3.8877 | 1.9939 | -0.2040 | -0.5719 |
| 227 | 0.0004 | 0.2494 | -0.0161 | 2.6933 | -1.0611 | -1.3940 | -0.9794 |
| 228 | -0.0012 | 0.1634 | -0.1320 | -0.1184 | -2.1230 | -1.0377 | 0.9995 |
| 229 | 2.6779 | 24 | 0.0000 | 0.0651 | 0.0075 | 2.4650 | 1.2108 | -0.1620 | -1.0982 |
| 230 | -3.8333 | 25 | 0.0000 | -0.0001 | 0.0045 | 0.0106 | 0.0180 | 0.0109 | -0.0823 |
| 231 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

C Batch job input data

1 1 !---------- input file for 'ms' -----------
2 .true. !BC boundary condition
3 20 !res harmonic resolution
4 0.0 !m_f fermion mass
5 2.38 !g coupling constant
6 0 !N_fix fixed parton number
7 0 !N_max maximal parton number
8 1 !pages(1) front page print option
9 1 !pages(2) Fock space print option
10 1 !pages(3) eigenvalue print option
11 1 !pages(4) wavefunction print option
12 -1 !---------- input file for 'ms' -----------
References

[1] J.B. Kogut, Rev.Mod.Phys. 51 (1979) 659.

[2] F. Butler, H. Chen, J. Sexton, A. Vaccarino, D. Weingarten, Phys.Rev.Lett. 70 (1993) 2849.

[3] P.A.M. Dirac, Rev.Mod.Phys. 21 (1949) 392.

[4] S.J. Brodsky, H.C. Pauli, in Recent Aspects of Quantum Fields, eds. H. Mitter, H. Gausterer (Lecture Notes in Physics, Vol. 396, Springer-Verlag, Heidelberg 1991); S.J. Brodsky, G. McCartor, H.C. Pauli, S.S. Pinsky, Particle World 3 (1993) 109.

[5] F. Wölz, On the Spectrum of Normal modes in Quantum Chromodynamics and the Theory of the Effective Interaction from the Tamm-Dancoff Method, PhD thesis, Ruprecht-Karls-Universität, Heidelberg, 1995.

[6] J.J. Wivoda, J.R. Hiller, Phys.Rev. D48 (1993) 4647.

[7] S.S. Pinsky, B. van de Sande, Phys.Rev. D49 (1994) 2001;
S.S. Pinsky, B. van de Sande, J.R. Hiller, Phys.Rev. D51 (1995) 726.

[8] K. Demeterfi, I.R. Klebanov, G. Bhanot, Nucl.Phys. B418 (1994) 15.

[9] A. Tam, C.J. Hamer, C.M. Yung, Light-Cone Quantization Approach to Quantum Electrodynamics in (2+1) Dimensions, New South Wales University Preprint PRINT-94-0182 (1994).

[10] Th. Eller, H.C. Pauli, S.J. Brodsky, Phys.Rev. D35 (1987) 1493;
Th. Eller, H.C. Pauli, Z.Phys. C42 (1989) 59.

[11] R.J. Perry, K.G. Wilson, Nucl.Phys. B 403 (1993) 587.

[12] A.C. Kalloniatis, H.C. Pauli, Z.Phys. C 63 (1994) 161
A.C. Kalloniatis, D.G. Robertson, Phys.Rev. D 50 (1994) 5262;
A.C. Kalloniatis, H.-C. Pauli, S.S. Pinsky, Phys.Rev. D 50 (1994) 6633;
H.-C. Pauli, A.C. Kalloniatis, S.S. Pinsky, Towards Solving QCD – the Transverse Zero Modes in Light-Cone Quantization, MPI für Kernphysik Heidelberg preprint MPIH-V3-1995 (MPIH-V23-1994) (Revised). To appear in Phys.Rev. D.

[13] J. Kogut, D. Soper, Phys.Rev. D1 (1970) 2901.

[14] H.C. Pauli, S.J. Brodsky, Phys.Rev. D32 (1985) 1993 and 2001.

[15] S. Elser, The Spectrum of QED\textsubscript{1+1} in the framework of the DLCQ method, diploma thesis, Ruprecht-Karls-Universität Heidelberg, 1994; The Spectrum of QED\textsubscript{1+1} in the framework of the DLCQ method, in Hadron Structure ’94 (Proceedings), eds. J. Urbán and J. Vrláková, Kosice, 1994.
[16] C. Lanczos, J.Res.Nat.Bur.Stand. **45** (1950) 255

[17] D.P. Crewther, C.J. Hamer, Nucl.Phys. **B170** (1980) 353.

[18] S. Coleman, R. Jackiw, L. Susskind, Ann.Phys.(N.Y.) **93** (1975) 267;  
S. Coleman, Ann.Phys.(N.Y.) **101** (1976) 239.

[19] H. Bergknoff, Nucl.Phys. **B122** (1977) 215.

[20] J. Schwinger, Phys.Rev. **128** (1962) 2425.

[21] S. Elser, A.C. Kalloniatis, in preparation.

[22] L. Martinovic, S. Elser, work in progress.

[23] M. Heyssler, *Numerical Methods for Solving QCD$_{1+1}$ in the context of the DLCQ Method*, diploma thesis, Ruprecht-Karls-Universität Heidelberg, 1994; *Constituent Quark Picture out of QCD in two-dimensions — on the Light-Cone*, MPI für Kernphysik Heidelberg preprint MPIH-V25-1994 (Revised). Submitted to Phys.Lett. B.