Random matrix products when the top Lyapunov exponent is simple

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Abstract

In the present paper, we treat random matrix products on the general linear group GL(V), where V is a vector space defined on any local field, when the top Lyapunov exponent is simple, without irreducibility assumption. In particular, we show the existence and uniqueness of the stationary measure ν on $P(V)$ that is relative to the top Lyapunov exponent and we describe the projective subspace generated by its support. We observe that the dynamics takes place in an open set of $P(V)$ which has the structure of a skew product space. Then, we relate this support to the limit set of the semi-group $T_\mu$ of GL(V) generated by the random walk. Moreover, we show that ν has Hölder regularity and give some limit theorems concerning the behavior of the random walk and the probability of hitting a hyperplane. These results generalize known ones when $T_\mu$ acts strongly irreducibly and proximally (i-p to abbreviate) on V. In particular, when applied to the affine group in the so-called contracting case or more generally when the Zariski closure of $T_\mu$ is not necessarily reductive, the Hölder regularity of the stationary measure together with the description of the limit set are new. We mention that we don’t use results from the i-p setting; rather we see it as a particular case.

Keywords: Random matrix products, Stationary measures, Lyapunov exponents, Limit sets, Large deviations

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1 Introduction

Let \( V \) be a finite dimensional vector space over a local field \( k \) and \( \mu \) a probability measure on the general linear group \( \text{GL}(V) \). Random Matrix Products Theory studies the behavior of a random walk on \( \text{GL}(V) \) whose increments are taken independently with respect to \( \mu \). This theory is well-developed when the sub-semigroup \( T_\mu \) generated by the support of \( \mu \) is strongly irreducible (algebraic assumption) and contains a proximal element (dynamical assumption) \([\text{Fur63}], [\text{BL85}], [\text{GR85}], [\text{BQ16b}]\). The latter framework, which will be abbreviated from now by i-p, had shown to be a powerful tool for understanding the actions of reductive algebraic groups \([\text{Gui90}], [\text{BQ11}], [\text{Aou11}], [\text{Bre14}]\)... One reason is that a great information on the structure of a reductive algebraic group is encoded in its irreducible and proximal representations. This setting had also proved its efficiency in the solution to some fundamental problems involving stochastic recursions \([\text{Kes73}], [\text{GLP16}]\).

In this article, we extend this theory from the i-p setting to a more general and natural framework. More precisely, we consider a probability measure \( \mu \) on \( \text{GL}(V) \) and assume only that its first Lyapunov exponent is simple; in some sense we keep the dynamical condition and assume no algebraic condition on the support of \( \mu \). Recall that by a fundamental theorem of Guivarc’h-Raugi \([\text{GR85}]\), our setting includes the i-p setting. But it also includes new settings as random walks on the affine group in the called contracting case or more generally any probability measure on a subgroup \( G \) of \( \text{GL}(V) \) that may fix some proper subspace \( L \) of \( V \) provided the action on \( L \) is less expanding than that on the quotient \( V/L \).

Our goal is then to obtain limit theorems concerning the random walk and the existence, uniqueness and regularity of stationary probability measure on the projective space of \( V \). Our results give also new information about the limit sets of some non irreducible linear groups. In our proofs we don’t use results from the i-p setting but rather see it as a particular case where our assumption concerning the Lyapunov exponent is satisfied. When applied to a probability measure on the affine group in the contracting case, the regularity of the stationary probability measure as well as the description of its support using the limit set of \( T_\mu \) are new. More generally, we show that the dynamics takes place on an open subset of \( P(V) \) which has essentially the structure of a skew product space with basis a projective space and fiber an affine space. We believe that this generalization can be useful to treat random walks on non necessarily reductive algebraic groups just as the i-p setting has proved its efficiency.

Here is the structure of the article.

• In Section 2 we state formally our results. We note that Section 2.2 shows the geometry behind our results and gives main examples that can be guiding ones through our paper.

• Section 3 consists of some preliminary results concerning orthogonality in non-Archimedean local fields and some results on Lyapunov exponents.

• In section 4, we show the existence and uniqueness of the stationary measure on the projective space whose cocycle average is the top Lyapunov exponent (Theorem 2.4 stated in Section 2). In addition, we describe the projective subspace generated by its support and show that it is not degenerate on it.

The existence appeals to Oseledets theorem. The uniqueness is explicit: we show in Proposition 4.6 that when \( \lambda_1 > \lambda_2 \), every limit point of the normalized random walk \( M_n/\|M_n\| \) is almost surely of rank one, and the projection of its image in \( P(V) \) is a random variable of law \( \nu \). A convergence in the KAK decomposition is also stated.
In Section 5, we make more precise the results of Section 4 by relating the support of our unique stationary measure to the limit set of $T_\mu$ (Theorem 2.6 stated in Section 2).

In Section 6, we show the Hölder regularity of the stationary measure (stated in Theorem 2.9). Moreover, we describe an important related large deviation estimate for the hitting probability of a hyperplane (Proposition 2.12).

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2 Statement of the results

2.1 Uniqueness of the Stationary Measure

From now on, $k$ is a local field of any characteristic, $V$ a finite dimensional vector space defined over $k$. Denote by $P(V)$ the projective space of $V$. We consider a probability measure $\mu$ on the general linear group $GL(V)$ and denote by $T_\mu$ (resp. $G_\mu$) the semi-group (resp. subgroup) of $GL(V)$ generated by the support of $\mu$. We define on the same probabilistic space $(\Omega, F, P)$ a sequence $(X_i)_{i \in \mathbb{N}^*}$ of independent identically distributed random variables of law $\mu$. The right (resp. left) random walk a time $n$ is by definition the random variable $M_n = X_1 \cdots X_n$ (resp. $S_n = X_n \cdots X_1$). Endow $V$ with any norm $\| \cdot \|$ and keep for simplicity the same symbol for the operator norm on $\text{End}(V)$. We will always assume that $\mu$ has a moment of order one, i.e. $\mathbb{E}(\log + \|X_1\|) < +\infty$ and denote by $\lambda_1 \geq \cdots \geq \lambda_d$ the Lyapunov exponents of $\mu$ defined recursively by:

$$
\lambda_1 + \cdots + \lambda_i = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E}(\log \| \bigwedge_i S_n \|) = \lim_{n \to +\infty} \frac{1}{n} \log \| \bigwedge_i S_n \|,
$$

the last equality is an almost sure equality and is guaranteed by the subadditive ergodic theorem of Kingman [Kin73].

For every finite dimensional representation $(\rho, V)$ of $G_\mu$, we denote by $\lambda(\rho, V)$ the top Lyapunov exponent relative to the pushforward probability measure $\rho(\mu)$ of $\mu$ by the map $\rho$, when the latter has a moment of order one. When there is no confusion on the action of $G_\mu$ on $V$, we will simply denote this exponent by $\lambda(V)$. To simplify, we will refer to it as the Lyapunov of $V$. By convention, if $(\rho, V)$ is the null representation, then $\lambda(\rho, V) = -\infty$.

Finally recall that if $T$ is a topological semi-group acting continuously on a topological space $X$ and $\mu$ is a Borel probability measure on $T$, then a Borel probability measure $\nu$ on $X$ is said to be $\mu$-stationary, or $\mu$-invariant, if for every continuous real function $f$ defined on $X$, the following equality holds:

$$
\int_{G \times X} f(g \cdot x) d\mu(g) d\nu(x) = \int_X f(x) d\nu(x).
$$

Proposition/Definition 2.1. Let $\mathcal{F}$ be the set of all $G_\mu$-stable vector subspaces of $V$ ordered by inclusion. Let

$$
L_\mu := \sum_{S \in \mathcal{F}, \lambda(S) < \lambda_1} S.
$$

Then $L_\mu$ is a proper $G_\mu$-stable subspace of $V$ whose Lyapunov exponent is less that $\lambda_1$, and is the greatest element of $\mathcal{F}$ with these properties.
Let \( \lambda_1 = \beta^0(\mu) > \beta^1(\mu) > \beta^2(\mu) > \cdots > \beta^r(\mu) \) such that if \( x \in L_i \setminus L_{i+1} \), then almost surely,
\[
\lim_{n \to +\infty} \frac{1}{n} \log \|S_n x\| = \beta^i(\mu).
\]

Remark 2.3. 1. It follows immediately from Proposition/Definition 2.1 and the theorem above that \( L_\mu \) coincide with \( L_1 \). Hence we will be using in the rest of article, the following useful equivalence:
\[
x \notin L_\mu \iff \text{a.s. } \lim_{n \to +\infty} \frac{1}{n} \log \|S_n x\| = \lambda_1.
\]

2. Furstenberg and Kifer gave actually an expression of \( \lambda_1 \) in terms of the “cocycle average” of stationary measures. More precisely, let \( N \) be the set of all \( \mu \)-stationary measures on \( P(V) \). For every \( \nu \in N \), let
\[
\alpha(\nu) = \int_{GL(V) \times P(V)} \frac{\log \|g v\|}{\|v\|} d\mu(g) d\nu([v])
\]
Then, they showed that
\( (a) \) \( \lambda_1 = \sup \{\alpha(\nu) ; \nu \in N\} \).
\( (b) \) \( L_\mu = \{0\} \), if and only if, \( \alpha(\nu) \) is the same for all \( \nu \in N \) (and hence equal to \( \lambda_1 \)).

3. Note that the filtration given by Furstenberg and Kifer is deterministic, unlike the one given by Oseledets theorem.

For every \( x \in V \setminus \{0\} \), we denote by \([x]\) the projection of \( x \) on \( P(V) \). For every subspace \( E \) of \( V \), let \([E] := \{ [x] ; x \in E \setminus \{0\}\} \). Let \( \delta \) the Fubini-Study distance on \( P(V) \) defined for every \([x],[y] \in P(V)\) by:
\[
\delta([x],[y]) = \frac{\|x \wedge y\|}{\|x\| \|y\|}.
\]
See Section 3 below for more details on \( \delta \) (and on the chosen norms on \( V \) and \( \Lambda^2 V \)).

One first result describes the stationary measures on \( P(V) \):

**Theorem 2.4.** Let \( \mu \) be a probability measure on \( GL(V) \) such that \( \lambda_1 > \lambda_2 \). Then,
\( a) \) There exists a unique \( \mu \)-stationary probability measure \( \nu \) on \( P(V) \) which satisfies \( \nu([L_\mu]) = 0 \).
\( b) \) The projective subspace of \( P(V) \) generated by the support of \( \nu \) is \([U_\mu]\), where
\[
U_\mu := \bigcap_{\lambda(\beta) = \lambda_1} S_
\]
Moreover, \( \nu \) is proper on \([U_\mu]\) (i.e. \( \nu \) gives zero mass to every proper projective subspace of \([U_\mu]\)).
\( c) \) \( (P(V),\nu) \) is a \( \mu \)-boundary in the sense of Furstenberg ([Fur72]), i.e. there exists a random variable \( \omega \to Z(\omega) \in P(V) \) such that, for \( \Omega := \mu^{\otimes \mathbb{N}} \)-almost every \( \omega := (g_n)_{n \in \mathbb{N}} \in GL(V)^\mathbb{N} \), such that \( g_1 \cdots g_n \cdot \nu \) converges weakly to \( \delta_{Z(\omega)} \).
Remark 2.5. After finishing this paper, it came to our knowledge that Benoist and Bruère have studied recently and independently the existence and uniqueness of stationary measures on projective spaces over $\mathbb{R}$ in a non irreducible context, in order to study recurrence on affine grassmannians. We will state one of the main results of the authors, namely [BB17, Theorem 1.6], then discuss the similarities and differences with Theorem 2.4 stated above.

In [BB17, Theorem 1.6 b) ], the authors consider a real vector space $V$, $G$ Zariski connected algebraic group subgroup $G$ of $GL(V)$, $W$ a $G$-invariant subspace of $V$ such that $W$ has no complementary $G$-stable subspace, the action of $G$ on $W$ and the quotient $V/W$ is i-p and such that the representations of $G$ in $W$ and $V/W$ are not equivalent. Then for every probability measure $\mu$ such that $\lambda(V/W) > \lambda(W)$ and whose support is compact and generates a Zariski dense subgroup of $G$, the authors show that there exists a unique $\mu$-invariant probability measure on the open set $P(V) \setminus [W]$ and that the Cesaro mean $\frac{1}{n} \sum_{j=1}^{n} \mu^j \ast \delta_x$ converges weakly to $\nu$.

Theorem 2.4 recovers the aforementioned result. Indeed, $\mu$ has a moment of order one since its support is assumed to be compact. The conditions on the Lyapunov exponents imply that $L_\mu = W$ and $\lambda_1 > \lambda_2$. Moreover, since $W$ has no complementary $G$-stable subspace, then $U_\mu = V$.

Theorem 2.4 permits actually to relax the i-p assumption on the action on $W$ in the previous statement; only the condition i-p on the quotient and $\lambda(V/W) > \lambda(W)$ is enough. Moreover, there is no need for the compactness of the support of $\mu$; a moment of order one is enough. Furthermore, $\mu^j \ast \delta_x$ converges weakly to $\nu$ (see Remark 6.3), not only in average. In addition, the vector space $V$ can be defined on any local field $k$.

We note that, in the rest of the present paper, we will be interested in understanding further properties of this stationary measure. Namely in Theorem 2.6 (Section 2.3) below, we describe more precisely the support of $\nu$ in terms of the the limit set of $T_\mu$ and we prove its Hölder regularity in Theorem 2.9 (Section 2.4).

It is worth-mentioning that in [BB17, Theorem 1.6 a) ], the authors show that when $\lambda(W) \geq \lambda(V/W)$, there is no $\mu$-stationary probability measure on $P(V) \setminus [W]$ and that the above Cesaro mean converges weakly to zero. This says somehow that $G$-stable subspaces with top Lyapunov exponent guide the dynamics. This information is not disjoint from the one given by Part b) of Theorem 2.4 saying that the projective subspace generated by the support of $\nu$ is $[U_\mu]$.

The techniques used in the two papers are highly different. In the present paper we obtain the existence of such a stationary measure via Oseledects theorem while Benoist and Bruère use Banach-Alaoglu theorem and a method developed in [EM01] for the situation of locally symmetric spaces. Concerning the uniqueness of the stationary measure, Benoist and Bruère’s proof is by contradiction via a beautiful argument of joining measure and previous results on stationary measures on the projective space by Benoist-Quint [BQ14]. Here we use methods of [Fur73] and [GR85] based on the $\mu$-boundary property. Our method is more explicit as it was described in the introduction (see Propositions 4.6 and Proposition 4.7).

2.2 The geometry behind Theorem 2.4 and guiding Examples

2.2.1 The geometry behind Theorem 2.4

Theorem 2.4 shows that an attractor for the dynamics is the open subset $F := P(V) \setminus [L_\mu]$ of $P(V)$ (actually the real attractor is $[U_\mu] \setminus [U_\mu \cap L_\mu]$). Let us assume for simplicity that $U_\mu = V$ and write simply $L$ for $L_\mu$. In that case, since we assumed that $\lambda_1 > \lambda_2$, this forces the action on $V/L$ to be strongly irreducible and to contain a proximal element (see Lemma 3.12). Fix a supplementary $L^\perp$ of $L$ in $V$. From now on, we identify the quotient vector space $V/L$ with $L^\perp$ in the natural way. For every $[x] \in P(V) \setminus [L]$, one can choose $t \in L$ and $s \in L^\perp$ such that $||s|| = 1$ and $x = t + s$. Let $S(L^\perp)$ be the unit sphere of $L^\perp$ and $U_k$ be the group of units of $k$. Two couples $(t, s)$ and $(t', s')$ in $L \times S(L^\perp)$ yield the same element $[x] \in P(V) \setminus [L]$ if and only if there exists a unit $u \in U_k$ such that $t' = ut$ and $s' = us$. This gives a bijection $\phi$ between
We observe that the fiberwise action is given, for \( \eta \in \mathcal{U}_k \), by

\[ X := L \times S(V/L) \]

and \( \mathcal{U}_k \) acts on \( X \) in the natural way. When we endow \( X/\mathcal{U}_k \) with the quotient topology, it is immediate that \( \phi \) is a homeomorphism.

Using the bijection \( \phi \), the space \( X/\mathcal{U}_k \) is endowed with a natural structure of \( G_\mu \)-space. Part a) of Theorem 2.4 is then equivalent to saying that \( X/\mathcal{U}_k \) has a unique \( \mu \)-stationary probability measure \( \nu \). It is easy to verify that the pushforward measure of \( \nu \) by the natural map \( X/\mathcal{U}_k \to P(V/L) \) is the unique \( \mu \)-invariant probability measure on \( P(V/L) \). We note that the existence and uniqueness of the latter is due to Guivarc’h-Raugi [GR85] and is based on techniques developed by Furstenberg [Fur73], but is also a particular case of Theorem 2.4 as we will see in Example 1 of Section 2.2.2 below. The action of \( G_\mu \) on \( X/\mathcal{U}_k \) lifts to an action of \( G_\mu \) on \( X \) which commutes with the natural action of \( \mathcal{U}_k \) on \( X \). Moreover, the probability measure \( \nu \) lifts to a probability measure on \( X \) which is \( \mathcal{U}_k \)-invariant, \( \mu \)-stationary and unique for these properties.

Let us describe the above paragraph with equations. Let \( l \) (resp. \( d \)) be the dimension of \( L \) (resp. \( V \)). Every element \( h \in T_\mu \) can be written in a basis compatible with the decomposition \( V = L \oplus L^\perp \) in the form \( h = \begin{pmatrix} A & B \\ 0 & g \end{pmatrix} \) with \( A \) is a \( l \times l \) matrix representing the action of \( h \) on \( L \), \( g \) is a \((d - l) \times (d - l)\) matrix representing the action on the quotient \( V/L \) and \( b \) is a \( l \times (d - l) \) rectangular matrix. The action of \( G \) on \( X = L \times S(V/L) \) described above is explicitly:

\[ h(t,s) = \begin{pmatrix} A & B \\ 0 & g \end{pmatrix} \cdot \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} A t + B s \\ g s \end{pmatrix}, \]

where \( Bs \) is a multiplication of a \( l \times (d - l) \) rectangular matrix by a \((d - l) \times 1\) vector.

We observe that the \( G \)-space \( X \) has a skew product structure given by the above formula. We note that the fiberwise action is given, for \( g \) and \( s \) fixed, by affine maps. The \( \mu \)-random walk on \( X \) (or \( P(V) \setminus [L] \)) is given by the following recursive stochastic equation:

\[ t_n = \frac{A_n t_{n-1} + B_n s_{n-1}}{||g_n s_{n-1}||}, \quad s_n = \frac{g_n s_{n-1}}{||g_n s_{n-1}||} \]

where \( \left\{ \begin{pmatrix} A_n & B_n \\ 0 & g_n \end{pmatrix} ; n \in \mathbb{N} \right\} \) is a sequence of independent random variable on \( GL(V) \) of same law \( \mu \).

Such stochastic recursions appeared recently in [GLP16, Section 5] with \( \dim(L_\mu) = 1 \) in order to prove the homogeneity at infinity of the measure \( \nu \), in the affine situation.

### 2.2.2 Guiding Examples

The guiding examples through this article are the following. The first two (i-p setting and the affine one) are standard and we just check that our general framework include them. The third example is an interesting new one that mixes somehow the first two. Together with the simulations of Section 5.2, they illustrate our new geometric setting and the dynamic on it.

1. **The irreducible linear groups.**
   If \( T \) is a sub-semi-group of \( GL(V) \) that acts irreducibly on \( V \), then for every probability measure \( \mu \) such that \( T_\mu = T \), we have by irreducibility \( L_\mu = \{0\} \) and \( U_\mu = V \). By a theorem of Guivarc’h-Raugi [GR85], the condition \( \lambda_1 > \lambda_2 \) is equivalent to saying that \( T \) is i-p (strongly irreducible and contains a proximal element). The results given by Theorem 2.4 are known in this case and are due also to Guivarc’h and Raugi in the same paper.

2. **The affine group.**
   Let \( d \geq 2 \) and \( T \) a sub-semi-group of affinities of the affine space \( k^{d-1} \).
We can let $T$ act linearly on the vector space $V = k^d$, in the usual way, and represent then $T$ as a sub-semigroup of $GL(V)$ whose matrices are all of the form \(egin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}\). Denote by $H$ be the hyperplane of $V$ spanned by the first $d-1$ vectors of the canonical basis of $k^d$.

Let $\mu$ be a probability measure on $GL(V)$ such that $T_\mu = T$. We denote by $a_1$ (resp. $a_2$) be the top (resp. second) Lyapunov exponent of the probability measure $A(\mu)$, relative to the linear part of $\mu$. Then by Lemma 3.8 and Corollary 3.9 below, the following equalities hold

\[
\lambda_1(\mu) = \max\{a_1, 0\}, \quad \lambda_2(\mu) = \min\{a_1, \max\{a_2, 0\}\}
\]

The subspaces $L_\mu$ and $U_\mu$ of $V$ depend on the measure $\mu$, unlike the previous example. More precisely,

(a) Contracting case ($a_1 < 0$). In this case, $0 = \lambda_1 > \lambda_2 = a_1$ and $L_\mu = H$. Note that the open set $P(V) \setminus [L_\mu]$ is homeomorphic to the affine space $k^{d-1}$, who compactifies in $P(V)$ with the hyperplane at infinity. If we assume moreover that $T$ does not fix any proper affine subspaces of $k^{d-1}$, then this translates to the linear action by saying that every $T$-stable vector space of $k^d$ is included in $H$. In particular we have $U_\mu = k^d$.

We can then apply Theorem 2.4. Its content translates back to the affine action by saying that there is a unique $\mu$-invariant probability measure on $k^{d-1}$ and that this measure gives zero mass to any affine hyperplane. This result is well known (see for instance [Kes73], [BP92]).

(b) Expansive case ($a_1 > 0$): In this case, $\lambda_1 = a_1$ and $\lambda_2 = \sup\{a_2, 0\}$. Assume for simplicity that the sub-semigroup $A_T$ generated by $A(\mu)$ acts irreducibly on $H$. Hence the condition $\lambda_1 > \lambda_2$ is equivalent to saying that $A_T$ is i-p. With these assumptions, we have $U_\mu = H$ and $L_\mu = \{0\}$ unless $T$ fixes a point in the affine space $k^{d-1}$.

In this case, Theorem 2.4 says that there exists a unique $\mu$-invariant probability measure on the (compactified) affine space $k^{d-1}$ and that it is concentrated on the hyperplane at infinity. This probability measure corresponds to the unique $A(\mu)$-stationary probability measure on the projective space $P(H)$ of $H$ (we are back to Example 1).

We note that our results are not applied to the interesting case $a_1 = 0$, called the critical case.

3. The Automorphism group of the Heisenberg group. Let $G$ be the following subgroup of $GL_3(\mathbb{R})$:

\[
G = \left\{ h = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}; a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}^2; g \in GL_3(\mathbb{R}) \right\} \subset GL_3(\mathbb{R}).
\]

The group $G$ can be though as a dual of the affine group on $\mathbb{R}^2$. In this context, random walks on $G$ appeared naturally in [GLP16, Section 5] as we have mentioned in the previous section. Also, if one imposes the condition $|a| = \det(g)$ in the definition of $G$, then by letting act the continuous Heisenberg group $\mathcal{H}_3$ on its Lie algebra, it can be proved (see [Fol89]) that $G$ is isomorphic to the automorphism group of $\mathcal{H}_3$; the one dimensional fixed subspace of $\mathbb{R}^3$ being the center of $\mathcal{H}_3$.

Let $L = \mathbb{R}(1,0,0)$ and $V = \mathbb{R}^3$. The open subset $F := P(V) \setminus [L]$ of $P(V)$ is a quotient of the cylinder $L \times S(V/L) \simeq \mathbb{R} \times S^1$ (see Section 2.2). The latter compactifies at infinity in $P(V)$ with the point $[L]$. Now let $\mu$ be a probability measure on $G$. Assume that:

(a) the action of $T_\mu$ on $V/L$ is i-p

(b) $\int_C \log |a(h)| \, d\mu(h) < \lambda(V/L)$.

In this case, $\lambda_1 > \lambda_2$ and $L_\mu = L$. If at least there is one $h \in T_\mu$ with $b \neq 0$, one has $U_\mu = V$. With these conditions, the content of Theorem 2.4 is new. It says that there exists a unique $\mu$-invariant probability measure $\nu$ on the open subset $F$ of $P(V)$. The latter has a structure of skew-product space whose base is a projective line and fibers the line $L$.

The projection of $\nu$ on the base is the unique $\mu$-invariant probability measure on the i-p semigroup of $GL_2(\mathbb{R})$, projection of $T_\mu$ on $V/L$. We refer to the simulations of Section 5.2.
2.3 The support of the stationary measure and Limit Sets

Our next goal will be to relate the support of the stationary measure \( \nu \) obtained above with the limit set of \( T_\mu \). We refer to [GG96] and [Gui90] when such a study is conducted in the strong irreducible and proximal case. We begin by some notations for a general semigroup \( T \subset \text{GL}(V) \) and two \( T \)-invariant subspaces \( L \) and \( U \) of \( V \) such that \( U \not\subset L \). The first one plays the role of a repeller and the second that of an attractor. Denote by \( PT \subset \text{PGL}(V) \) the projection of \( T \) on \( \text{PGL}(V) \).

We will need the notion and some properties of quasi-projective transformation introduced by [Fur73] and developed in [GM89]. Recall that a quasi-projective transformation is a map from \( \text{P}(V) \) to itself obtained by a pointwise limit of a sequence projective transformations. Denote by \( Q \) the set of quasi-projective maps.

- We denote by \( \hat{T} \subset Q \) the set of quasi-projective transformations \( \delta : \text{P}(V) \rightarrow \text{P}(V) \), pointwise limits of projective maps \( g_\delta \in PT \) with the following property: there exists a proper projective subspace \( S \) of \( V \) such that \( [U] \not\subset S \) and for every \( x \not\in S \), \( \delta x \) is pointwise limit of a sequence projective transformations. Denote by \( U \) the set of quasi-projective maps.

\[
\Lambda(T) = \{ p(\delta); \delta \in \hat{T} \} \subset [U].
\]

We will check in Lemma 5.1 that this is a closed \( T \)-invariant subset of \( \text{P}(V) \). We will call it the limit set of \( T \) (note that it depends on the subspaces \( L \) and \( U \)).

- We consider the \( T \)-space \( F = \text{P}(V) \setminus [L] \) and we endow it with the topology induced from that of \( \text{P}(V) \). If \( X \subset F \), we denote by \( \overline{X} \) its closure in \( \text{P}(V) \) and by \( \overline{X}_F \) its closure in \( F \). Let \( \Lambda^a(T) = \Lambda(T) \cap F \) so that \( \Lambda^a(T) \) is a closed \( T \)-invariant subset of \( F \).

- Let \( T_0 \) (resp. \( T_0^\mu \)) the subset of \( T \) which consists of elements \( g \) with a simple and unique dominant eigenvalue corresponding to a direction \( p^+(g) \in [U] \) (resp. \( p^+(g) \in [U \setminus L] \)).

**Theorem 2.6.** Let \( T \) be a semi-group of \( \text{GL}(V) \), \( L \) and \( U \) be \( T \)-invariant subspaces such that \( U \not\subset L \). Let \( \mu \) be a probability measure on \( \text{GL}(V) \) such that \( \lambda_1 > \lambda_2 \), \( T_\mu = T \), \( L_\mu = L \) and \( U_\mu = U \). Let \( \nu \) be the unique stationary measure on \( \text{P}(V) \setminus [L] \). Then,

1. \( T_0^\mu \neq \emptyset \) and \( \text{Supp}(\nu) = \overline{p^+(T_0^\mu)} \).
2. \( \text{Supp}(\nu) = \Lambda(T) \).
3. For any \( x \in \text{P}(V) \setminus [L] \), we have \( \Lambda(T) \subset \overline{T^+x} \). In particular, \( \Lambda^a(T) \) is the unique \( T \)-minimal subset of \( \text{P}(V) \setminus [L] \).

We easily deduce the following characterization of the compactness of \( \text{Supp}(\nu) \) in the open subset \( F = \text{P}(V) \setminus [L] \) of \( \text{P}(V) \).

**Corollary 2.7.** In this corollary only, \( \nu \) is seen as a probability measure on the open subset \( F = \text{P}(V) \setminus [L] \) of \( \text{P}(V) \) and its support \( \nu \) is understood in \( F \). The following are equivalent:

1. \( \text{Supp}(\nu) \) is compact in \( F \)
2. There exists \( x \in F \) such that \( \overline{T^+x} \) is compact in \( F \)
3. \( \delta (p^+(T_0^\mu), [L]) > 0 \), where \( \delta \) is the Fubini-Study metric on \( \text{P}(V) \).

**Remark 2.8.**

1. The support of \( \nu \) is not \( T \)-minimal, unless it is compact in \( F \) (in particular when \( L = \{0\} \)). Note also that even when it is minimal, it is not unique unless \( L = \{0\} \). Hence the superscript “a” in item 3 of Theorem 2.6 is required.
2. It follows from item 1 of Theorem 2.6 that a sufficient condition for the non compactness of \( \text{Supp}(\nu) \) in \( F \) is the existence of at least one proximal element \( g \in T \) with an attracting direction \( p^+(g) \in [L] \). For the situation where \( T \) is non degenerated semi-group of the affine transformations of the real line in the contracting case, this says that the support of the unique stationary measure \( \nu \) on the affine line is non compact when there exists at least one transformation \( x \mapsto ax + b \) with \( |a| > 1 \). This result is well-known and actually a converse holds, provided some boundness condition on the translation part \( b \) is imposed (see for instance [Kes73], [GLP15] for more on the support of \( \nu \)). In our general setting, such a converse fails; we refer to Example 5.2.
3. Assume now that \( \text{Supp}(\nu) \) is not compact in \( F \). Note that \( F \cup \{L\} (= P(V)) \) is a natural compactification of \( F \) in \( P(V) \). Let \( \Lambda^\infty(T) = \Lambda(T) \cap \{L\} \). We observe that property 1 of Theorem 2.6 implies that the closure of \( p^\ast(T^n) \) in \( F \) is \( \Lambda^\infty(T) \), while its closure in \( P(V) \) is \( \Lambda(T) = \Lambda^\infty(T) \cup \Lambda^\infty(T) \). In particular, if \( T = T_\mu \) is a sub-semigroup of the affine group in the contracting case (see Section 2.2, Example 2) and if the linear part of \( T_\mu \) is i-p, then the limit set \( \Lambda(T) \) of \( T \) is the union of \( \Lambda^\infty(T) \subseteq F \) and of a subset of the hyperplane at infinity which corresponds to the limit set of the semi-group generated by the support of \( \mathbb{A}(\mu) \), \( \mathbb{A}(\mu) \) being the linear part of \( \mu \) (see Section 2.2, Example 2). The set of points at infinity of \( \Lambda^\infty(T) \) is \( \Lambda^\infty(T) \).

### 2.4 Regularity of the stationary measure

The following result shows that the unique stationary measure \( \nu \) given by Theorem 2.4 has Hölder regularity when \( \mu \) has an exponential moment, i.e. when \( \int_{GL(V)} ||g^\pm 1||^\tau \ d\mu(g) < +\infty \) for some \( \tau > 0 \). We recall that the projective subspace of \( P(V) \) generated by \( \nu \) is \( [\mu] \) and that \( \nu \) is proper on it. Hence, the following result gives a precision of that fact.

**Theorem 2.9.** Let \( \mu \) be a probability measure on \( GL(V) \) such that \( \lambda_1 > \lambda_2 \). If \( \mu \) has an exponential moment, then there exists \( c > 0 \) such that

\[
\sup_{H \text{ hyperplane of } U_{\mu}} \int \delta^{-\epsilon}([x],[H]) \ d\nu([x]) < +\infty.
\]

We note that we will give a slightly more general statement in Theorem 6.1 involving the distance to any hyperplane of \( V \).

In the i-p case, the previous result is known and is due to Guivarc’h [Gui90]. When applied to the affine group it is new. More precisely,

**Corollary 2.10.** Let \( \mu \) be a probability measure on the group of affinities of an affine space \( W \) whose support does not fix any proper affine subspace. Assume that the Lyapunov exponent of the linear part of \( \mu \) is negative (contracting case). Then the unique \( \mu \)-stationary probability measure \( \nu \) on \( W \) is Hölderian and has therefore a positive Hausdorff dimension.

**Remark 2.11.** We note that the problem of estimation of the Hausdorff dimension of \( \nu \) was initially considered by Erdős (see for instance [PSS00]) if \( T \subseteq \text{Aff}(\mathbb{R}) \) preserves an interval of the line. It led recently to deep results in similar situations (see [Hoc14], [BV16] for example).

In the more general situation of this paper, we get only qualitative results on the dimension of \( \nu \).

One of the important estimates in random matrix products theory is the probability of return of the random walk to hyperplanes. It is well studied in the i-p case and leads to fundamental spectral gap results [BG08], [BG10], [BdS16], [Bre].... The general setting studied in this paper leads to new estimates in this direction.

**Proposition 2.12.** Let \( V \) a finite dimensional vector space and \( \mu \) be a probability measure on \( GL(V) \) with an exponential moment such that \( \lambda_1 > \lambda_2 \). Then, for every \( \epsilon > 0 \), there exist \( C(\epsilon) > 0 \), \( n_0(\epsilon) \in \mathbb{N} \) such that for every \( n \geq n_0(\epsilon) \) and every compact subset \( K \) (resp. \( K' \)) respectively of \( P(V) \setminus [L_{\mu}] \) (resp. of \( P(V^\ast) \setminus [L_{\bar{\mu}}] \)), one has:

\[
\sup_{x \in K} \sup_{f \in K'} \mathbb{P} [\delta (S_n[x],[Ker(f)]) \leq \exp(-\epsilon n)] \leq C(K,K') \exp(-nC(\epsilon)).
\]

In this statement \( V^\ast \) is the dual space of \( V \) and \( \bar{\mu} \) is the probability measure \( f(\mu) \) where \( f : GL(V) \rightarrow GL(V^\ast), g \mapsto g^\ast \) maps any \( g \in GL(V) \) to its transpose linear map \( g^\ast \) on \( V^\ast \). Moreover,

\[
C(K,K') := \max \left\{ \sup_{x \in K} \frac{1}{\delta([x],[L_{\mu}])}, \sup_{f \in K'} \frac{1}{\delta ([f],[L_{\bar{\mu}}])} \right\}.
\]
3 Preliminaries

3.1 Linear algebra preliminaries

Our proofs rely on suitable choice of norms on our vector spaces and on the expression of the distance between a point and a projective subspace of $P(V)$ (Lemma 3.7 below). For the convenience of the reader, we recall in Section 3.1.1 basic facts about orthogonality in non-Archimedean vector spaces. The reader interested only in vector spaces over Archimedean fields can check directly Section 3.1.2.

3.1.1 Non-Archimedean orthogonality

Let $(k, |·|)$ be a non-Archimedean local field. We denote by $\mathcal{O}_k = \{x \in k; |x| \leq 1\}$ its ring of integers. We recall that $\mathcal{O}_k$ is a Principal Integral Domain (PID) and its group of units of $\mathcal{O}_k$ is $\mathcal{O}_k^\times = \{x \in k; |x| = 1\}$. We will use standard notions of orthogonality in non-Archimedean vector spaces (c.f. [MS65]).

**Definition 3.1.** Let $V$ a vector space over $k$ endowed with a norm $||·||$.

1. We say that $(V, ||·||)$ is a non-Archimedean vector space if $||v + w|| \leq \max\{||v||, ||w||\}$ for every $v, w \in V$.

   From now, we assume that $(V, ||·||)$ is non-Archimedean. We say that

2. two subspaces $E$ and $F$ of $V$ are orthogonal when $||v + w|| = \max\{||v||, ||w||\}$ for every $v \in E$ and $w \in F$.

3. two vectors $x, y \in V$ are orthogonal when the subspaces $\langle x \rangle$ and $\langle y \rangle$ generated respectively by $x$ and $y$ are orthogonal.

4. a family of vectors $(v_1, \ldots, v_r)$ in $V$ is orthogonal if for every $\lambda_1, \ldots, \lambda_r \in K$,

   $||\lambda_1 v_1 + \cdots + \lambda_r v_r|| = \max\{||\lambda_1|| ||v_1||, \ldots, ||\lambda_r|| ||v_r||\}$.

   It is said to be orthonormal if it is orthogonal and moreover $||v_i|| = 1$ for every $i = 1, \ldots, r$.

5. a subspace $E$ in $V$ is orthocomplemented if there exists a subspace $F$ of $V$ such that $V = E \oplus F$ and $E$ and $F$ are orthogonal. In that case, we say that $F$ is an orthogonal complement of $E$ and we denote it by $E^\perp$ (although it may not be unique, see Remark 3.4).

6. that $(V, ||·||)$ is orthogonalizable if $V$ admits an orthogonal basis.

**Remark 3.2.** Let $d \in \mathbb{N}^*$ and $V = k^d$. Then the norm $||·||_0$ defined on $V$ by

$||(x_1, \ldots, x_n)||_0 = \max\{|x_i|; i = 1, \ldots, d\}$

is Archimedean and the canonical basis is orthonormal, so that $(k^d, ||·||_0)$ is orthogonalizable. Conversely it is easy to see that every orthogonalizable space $(V, ||·||)$ of dimension $d$ is isomorphic algebraically and topologically to $(k^d, ||·||_0)$.

We recall that if $d \in \mathbb{N}^*$, then $GL_d(\mathcal{O}_k)$ is the subgroup of the general linear group $GL_d(k)$ formed by the matrices $g$ such that the coefficients of $g$ and $g^{-1}$ have coefficients in $\mathcal{O}_k$; which is equivalent to impose that $g$ has coefficients in $\mathcal{O}_k$ and that $\det(g) \in \mathcal{O}_k^\times$. One can show that $GL_d(\mathcal{O}_k)$ is a maximal compact subgroup of $GL_d(k)$. Hence, it is analogous to the orthogonal (resp. unitary) group $O_d(\mathbb{R})$ (resp. $U_d(\mathbb{C})$) when $k = \mathbb{R}$ (resp. $k = \mathbb{C}$). The following lemma gives crucial results of orthogonality in non-Archimedean vector spaces which are classic in the Archimedean setting, i.e. when $(V, ||·||)$ is an inner vector space over the field of real or complex numbers and when $\mathcal{O}_k$ is replaced by the classical groups stated above.

**Lemma 3.3.** Let $(V, ||·||)$ be an orthogonalizable Archimedean vector space of dimension $d \in \mathbb{N}^*$ and $B_0 = (e_1, \ldots, e_d)$ an orthonormal basis. Hence,

1. For every basis $B = (v_1, \ldots, v_d)$ of $V$, the following assertions are equivalent:
   i. $B$ is orthonormal
   ii. The transition matrix from $B_0$ to $B$ belongs to $GL_d(\mathcal{O}_k)$
   iii. $B$ is a basis of the $\mathcal{O}_k$-module $\mathcal{O}_k e_1 \oplus \cdots \oplus \mathcal{O}_k e_d \simeq \mathcal{O}_k^d$. 

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Every subspace $E$ of $V$ has an orthonormal basis and admits an orthogonal complement $E^\perp$.

Proof. By Remark 3.2, we can without loss of generality assume that $V = k^d$, $B_0$ the canonical basis and $\|\langle x, \cdots, x_n \rangle \| = \max \{|x|; i = 1, \cdots, d\}$ for every $(x_1, \cdots, x_d) \in k^d$. One can easily show that $GL_d(O_k)$ is the isometry group of $(V, \| \cdot \|)$.

1. The equivalence between items i., ii. and iii. is an easy consequence of the fact that $GL_d(O_k)$ acts by isometries on $V$.

2. Let $r$ be the dimension of $E$ as a $k$-vector space, $M = O_k^2$ and $E' = E \cap M$. Then $M$ is a free $O_k$-module of rank $d$ and $E'$ is a submodule. Since $k$ is a local field, then $O_k$ is a Principal Integral Domain (PID). Then the structure theorem of modules over PID's assures that there exists a basis $B = \{v_1, \cdots, v_n\}$ of $M$, $r \in \mathbb{N}^*$ and scalars $d_1, \cdots, d_r \in O_k$ such that $(d_1v_1, \cdots, dv_r)$ is a basis of $E'$ as a $O_k$-module. The set $B$ is clearly also a basis of the $k$-vector space $V$, $r$ the dimension of $E$ as $k$-vector space and $(v_1, \cdots, v_r)$ a basis of the subspace $E$ of $V$. By the equivalence between 1.i. and 1.iii., $B$ is orthonormal. Hence items 2 and 3 follow immediately.

Remark 3.4. Unlike the Archimedean case, a subspace $E$ of a $k$-vector space $V$ may have more than one orthogonal complement. Here is an example. Consider $k = \mathbb{Q}_2$, $V = k^2$ and $E$ the line generated by the vector $u = (3, 2)$. We denote by $E_1$ and $E_2$ the one dimensional subspaces of $V$ generated respectively by the vectors $v_1 = (1, 1)$ and $v_2 = (2, 3)$. Then $E_1$ and $E_2$ are two distinct orthogonal complements of $E$ because the matrices $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ belong to $SL_2(\mathbb{Z}_2)$. One can also check this fact by applying the initial Definition 3.1.

3.1.2 The Fubini-Study metric

Now $(k, \langle \cdot, \cdot \rangle)$ is a local field and $V$ a vector space over $k$ of dimension $d \geq 2$. One can endow $V$ with a norm $\| \cdot \|$ such that $(V, \| \cdot \|)$ is a inner product space when $k$ is Archimedean and $(V, \| \cdot \|)$ is an orthogonalizable space when $k$ is non-Archimedean. In other terms $(V, \| \cdot \|)$ is algebraically and topologically isomorphic to $k^d$ endowed with standard norm; the Euclidean one when $k$ is Archimedean and the norm $\| \cdot \|_0$ defined in Remark 3.2 otherwise. Before we define the Fubini-Study metric and give the desired properties we will be using later, we need some preliminaries on the relation between orthogonality in $V$ and that on the wedge product $\Lambda^2 V$.

Lemma 3.5. Let $(V, \| \cdot \|)$ as above and $(e_1, \cdots, e_d)$ an orthonormal basis of $V$. We consider the norm on $\Lambda^2 V$, which will be designed also by $\| \cdot \|$, such that $(e_i \wedge e_j)_{1 \leq i < j \leq d}$ is an orthonormal basis of $\Lambda^2 V$.

1. If $(v_i)_{i=1}^d$ is an orthonormal basis of $V$, then $(v_i \wedge v_j)_{1 \leq i < j \leq d}$ is an orthonormal basis of $\Lambda^2 V$.

2. For every $x, y \in V$, $\| x \wedge y \| \leq \| x \| \| y \|$.

3. If moreover $x$ and $y$ are orthogonal in $V$, then $\| x \wedge y \| = \| x \| \| y \|$.

Proof. We will treat only the case where $k$ is non-Archimedean, the Archimedean case being more classic. Without loss of generality, we can assume that $V = k^d$ and $\| \cdot \|$ is the norm max (See Remark 3.2).

1. By Lemma 3.3, there exists $P \in GL_d(O_k)$ such that $v_i = P e_i$ for every $i = 1, \cdots, d$. Since $(e_i \wedge e_j)_{1 \leq i < j \leq d}$ is an orthonormal basis of $\Lambda^2 V$ and $GL_d(O_k)$ acts by isometries on $\Lambda^2 V$, it follows that $(v_i \wedge v_j)_{1 \leq i < j \leq d}$ is also orthonormal.

2. Easy to check using the Archimedean triangle inequality in $k$.

3. Without loss of generality, we can assume that $x$ and $y$ are of norm one. By item iii. of Lemma 3.3, there exists an orthonormal basis $(v_1, v_2, \cdots, v_d)$ of $V$ such $(v_1, v_2)$ is a basis of $E := \text{Span}(x, y)$. By item i. of the same lemma, there exists $g \in GL_2(O_k)$ such that $x = gv_1$ and $y = gv_2$ so that $x \wedge y = |\det(g)| v_1 \wedge v_2 = v_1 \wedge v_2$. But the family $(v_1 \wedge v_j)_{1 \leq i < j \leq d}$ is an orthonormal basis of $\Lambda^2 V$, so that $\| x \wedge y \| = \| v_1 \wedge v_2 \| = 1$. 

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Hence, for every \( y \) belongs to subspace of \( \text{Archimedean case} \). Let \( y \) is orthogonal projection on \( k \) an orthogonal complement (see Lemma 3.3 when \( x \) not Archimedean and is classic in the Archimedean case). We denote by \( \pi_{E^\perp} \) the orthogonal projection on \( E^\perp \). Then, for every non zero vector \( x \) of \( V \),

\[
\delta([x], [E]) = \frac{||\pi_{E^\perp}(x)||}{||x||}.
\]

Proof. The desired relation is trivial when \( x \in E \). From now, we assume that \( x \notin E \). We write \( x = x_1 + x_2 \), with \( 0 \neq x_1 \in E \) and \( \pi_{E^\perp}(x) = x_2 \in E^\perp \). On the one hand,

\[
\delta([x], [x_1]) = \frac{||x \wedge x_1||}{||x|| ||x_1||} = \frac{||x_2 \wedge x_1||}{||x|| ||x_1||} \leq \frac{||x_2||}{||x||}
\]

Inequality (1) is due to item 2. of Lemma 3.5. Since \( [x_1] \in P(V) \),

\[
\delta([x], [E]) = \inf_{[y] \in [E]} \delta([x], [y]) \leq \frac{||\pi_{E^\perp}(x)||}{||x||}.
\]

On the other hand, let \( B \) be an orthonormal basis of \( V \) obtained by concatenating an orthonormal basis, say \( (v_1, \ldots, v_r) \) of \( E \) and an orthonormal basis, say \( (v_{r+1}, \ldots, v_s) \), of \( E^\perp \) (the existence of such a basis is guaranteed by Lemma 3.3 when \( k \) is not Archimedean and is classic in the Archimedean case). Let \( y \in E \setminus \{0\} \). By writing \( x_1, x_2 \) and \( y \) in the basis \( B \), we see that \( x_1 \wedge y \) belongs to subspace of \( \wedge^2 V \) generated by \( (v_i \wedge v_j)_{1 \leq i < j \leq r} \) and \( x_2 \wedge y \) to the one generated by \((v_i \wedge v_j)_{(i,j) \in \{1, \ldots, r\} \times \{r+1, \ldots, s\}}\). Hence item 1. of Lemma 3.5 assures that the vectors \( x_1 \wedge y \) and \( x_2 \wedge y \) are orthogonal in \( \wedge^2 V \). This implies that \( ||x \wedge y|| = ||x_1 \wedge y + x_2 \wedge y|| \geq ||x_2 \wedge y|| \). Since \( x_2 \) and \( y \) are orthogonal in \( V \), we have by item 3. of the same lemma that \( ||x_2 \wedge y|| = ||x_2|| ||y|| \). Hence, for every \( y \in E \setminus \{0\} \), \( ||x \wedge y|| \geq ||x_2|| ||y|| \). Hence,

\[
\delta([x], [E]) = \inf_{[y] \in [E]} \frac{||x \wedge y||}{||x|| ||y||} \geq \frac{||x_2||}{||x||} = \frac{||\pi_{E^\perp}(x)||}{||x||}.
\]

The lemma is then proved. 

\[\square\]
3.2 Preliminaries on Lyapunov exponents

We recall the following crucial lemma which reduces the computation of the top Lyapunov exponent of a random walk on a group of upper triangular block matrices to the top Lyapunov exponents of the random walks induced on the diagonal part. For the reader’s convenience, we include a proof.

**Lemma 3.8.** [[FK83, Lemma 3.6], [BL85]]

Let $k$ be a local field, $V$ a finite dimensional vector space defined over $k$, $\mu$ be a probability on $\text{GL}(V)$ having a moment of order one. Consider a $G_\mu$-invariant subspace $W$ of $V$. Then the first Lyapunov exponent $\lambda_1$ of $\mu$ is given by:

$$
\lambda_1 = \max(\lambda_1(W), \lambda_1(V/W)).
$$

**Proof.** To simplify notations, we set $\lambda := \lambda_1(V)$, $\lambda' := \lambda_1(W)$, $\lambda'' := \lambda_1(V/W)$ and $\tilde{\lambda} := \max(\lambda', \lambda'')$. We want to show that $\lambda = \tilde{\lambda}$. Let $r$ be the dimension of $W$. Without loss of generality, we can assume that $V = k^d$ and that elements of $G_\mu$ are $d \times d$ invertible of the form

$$
\begin{pmatrix}
A & B \\
0 & C
\end{pmatrix}
$$

where $A \in \text{GL}_d(k)$ represents the action of $G_\mu$ on $W$, $C \in \text{GL}_{d-r}$ the action on the quotient $V/W$ and $B$ is a $(d - r) \times d$ rectangular matrix. For every $n \in \mathbb{N}^*$, let $S_n = X_n \cdots X_1$ be the random walk at time $n$. We write $S_n = \left( \begin{array}{cc} A_n & B_n \\ 0 & C_n \end{array} \right)$. For every norm $\| \cdot \|$ on the vector space $M_d(k)$ of $d \times d$ invertible matrices, we have the following almost sure limits:

$$
\lambda' = \lim_{n \to \infty} \frac{1}{n} \log \| A_n \|, \quad \lambda'' = \lim_{n \to \infty} \frac{1}{n} \log \| C_n \| \quad \text{and} \quad \lambda = \lim_{n \to \infty} \frac{1}{n} \log \| S_n \|.
$$

If we endow $M_d(k)$ with the max norm, we see that $\| S_n \| \geq \max(\| A_n \|, \| C_n \|)$, so that $\lambda \geq \tilde{\lambda}$. Now we show that $\lambda \leq \tilde{\lambda}$. Let $S_2n = X_{2n} \cdots X_{n+1}$. The relation $S_{2n} = S_n^r S_n$ gives the following three assertions:

$$
A_{2n} = A_n^r A_n, \quad C_{2n} = C_n^r C_n \quad \text{and} \quad B_{2n} = A_n^r B_n + B_n C_n^r.
$$

Let $\| \cdot \|$ be any submultiplicative matrix norm on $M_d(k)$ and $c$ a fixed positive number. Since $(X_{2n}, \cdots, X_{n+1})$ is independent of $(X_n, \cdots, X_1)$, the definitions of $\lambda', \lambda''$ and $\tilde{\lambda}$ imply that there exists $n_0 = n_0(\epsilon)$ such that for all $n \geq n_0$:

$$
\mathbb{E}(\log \| A_{2n} \|) \leq n(2\lambda + \epsilon) \leq n(2\tilde{\lambda} + \epsilon), \quad \mathbb{E}(\log \| C_{2n} \|) \leq n(2\lambda' + \epsilon) \leq n(2\tilde{\lambda} + \epsilon)
$$

and

$$
\mathbb{E}(\log \| B_{2n} \|) \leq n(\lambda' + \lambda'' + \epsilon) \leq n(2\tilde{\lambda} + \epsilon).
$$

Since all norm on $M_d(k)$ are equivalent, we deduce that for some absolute constant $C > 0$ and all $n \geq n_0$, $\mathbb{E}(\log \| S_{2n} \|) \leq C + n(2\lambda + \epsilon)$. By definition of $\lambda$, we deduce that $\lambda \leq \tilde{\lambda} + \epsilon$. This being true for every $\epsilon > 0$, we deduce that $\lambda \leq \tilde{\lambda}$. \hfill $\Box$

We easily deduce that all other Lyapunov exponents of $\mu$ can be also read on the diagonal part.

The first two Lyapunov exponents of $\mu$ will be just denoted by $\lambda_1$ and $\lambda_2$, while those of the image of $\mu$ under a linear representation of $G_\mu$ on a vector space $E$ will be denoted by $\lambda_1(E)$ and $\lambda_2(E)$.

**Corollary 3.9.** Consider the same situation as the previous lemma. Denote by $S_1$ (resp. $S_2$) the set of Lyapunov exponents associated to the probability measure induced on $W$ (resp. $V/W$). Then the set of Lyapunov exponents associated to $\mu$ is $S_1 \cup S_2$.

**Proof.** For the simplicity of the proof, we will only show the information needed for the second Lyapunov exponent of $\mu$, that is $\lambda_2$ is the second largest number of the set $S_1 \cup S_2$.

First notice that if $E$ and $F$ are two $G_\mu$-invariant finite dimensional vector spaces, then $\lambda_1(A^2 E) = \lambda_1(E) + \lambda_2(E)$ and $\lambda_1(E \otimes F) = \lambda_1(E) + \lambda_1(F)$. Now, let $W$ be a supplementary of $W$ in $V$.

We have the following observations:

- $A^2 V = A^2 W \oplus (W \otimes W) \oplus A^2 W$.
- The subspace $U := A^2 W \oplus (W \otimes W)$ of $A^2 V$ is $G_\mu$-invariant and $(A^2 V)/U$ is isomorphic to $A^2 (V/W)$, as $G_\mu$-representations.
- The subspace $A^2 W$ of $U$ is $G_\mu$-invariant and $U/A^2 W$ is isomorphic as a $G_\mu$-representation to $W \otimes (V/W)$.
Applying Lemma 3.8 twice gives the following relation:

\[ \lambda_1 + \lambda_2 = \lambda \left( \bigwedge^2 V \right) = \max \{ \lambda_1(W) + \lambda_2(W), \lambda_1(W) + \lambda_2(V/W), \lambda_1(V/W) + \lambda_2(V/W) \} \]

Since \( \lambda_1 = \max \{ \lambda_1(W), \lambda_1(V/W) \} \), this shows the desired property on \( \lambda_2 \).

\[ \square \]

### 3.3 On the subspaces \( L_\mu \) and \( U_\mu \)

Let \( \mu \) be a probability measure on \( \text{GL}(V) \). In Definition 2.1, we introduced the following subspace of \( V \):

\[ L_\mu := \sum_{S \in F, \lambda(S) < \lambda_1} S. \]

In the statement of Theorem 2.4, we introduced the following subspace of \( V \):

\[ U_\mu := \bigcap_{S \in F, \lambda(S) = \lambda_1} S, \]

and claimed that \( \{U_\mu\} \) is an attractor for the dynamics in \( \text{P}(V^*) \). In this section, we state some useful properties of these subspaces that follow immediately from their definition.

**Lemma 3.10.** \( L_\mu \) is a proper \( G_\mu \)-stable subspace of \( V \) whose Lyapunov exponent is less than \( \lambda_1 \), and is the greatest element of \( F \) with these properties.

When \( \lambda_1 > \lambda_2 \), \( U_\mu \not\subset L_\mu \). In particular, \( U_\mu \) is non zero in this case and is the smallest \( G_\mu \)-stable subspace whose Lyapunov exponent is \( \lambda_1 \).

**Proof.** The subspace \( L_\mu \) has the claimed property because on the one hand the sum that defines it can be made a finite one and on the other hand if \( W_1 \) and \( W_2 \) are two \( G_\mu \)-stable subspaces of \( V \), then one can easily prove that \( \lambda(W_1 \cap W_2) = \max \{ \lambda(W_1), \lambda(W_2) \} \). Assume now that \( \lambda_1 > \lambda_2 \) and consider two \( G_\mu \)-stable subspaces \( W_1 \) and \( W_2 \) of \( V \) such that \( \lambda(W_1) = \lambda(W_2) = \lambda_1 \). We will prove that \( \lambda(W_1 \cap W_2) = \lambda_1 \); and the claim concerning \( U_\mu \) will immediately follow. Indeed, assume that \( \lambda(W_1 \cap W_2) < \lambda_1 \). Then by Lemma 3.8 and Corollary 3.9, we deduce that the top Lyapunov exponent of \( E := V/W_1 \cap W_2 \) is simple and is equal to \( \lambda_1 \). The same holds for the subspaces \( W_1/W_1 \cap W_2 \) and \( W_2/W_1 \cap W_2 \) of \( E \). By the simplicity of \( \lambda_1 \) in \( E \), we deduce that \( (W_1/W_1 \cap W_2) \cap (W_2/W_1 \cap W_2) \neq \{0\} \), contradiction.

The following lemma is easy to prove using the definitions of \( L_\mu \) and \( U_\mu \). It is left as an exercise.

**Lemma 3.11.** (Duality between \( L_\mu \) and \( U_\mu \))

For every \( g \in \text{GL}(V) \) and every subspace \( F \) of \( V \), denote by \( g^\dagger \) the transpose linear map on the dual \( V^* \) and \( F^\perp \) the orthogonal (annihilator) of \( F \). Let \( \mu \) be a probability measure on \( \text{GL}(V) \) such that \( \lambda_1 > \lambda_2 \). Denote by \( \mu \) the law of \( X_1 \), where \( X_1 \) has law \( \mu \). Then,

\[ L_\mu^\perp = U_{\mu^\perp}, \quad U_\mu^\perp = L_{\mu^\perp}. \]

During the proofs, we will frequently go back to the case where \( U_\mu \) is the whole subspace \( V \). We refer to three guiding examples of Section 2.2 where this condition was always satisfied, thanks to a “natural” geometric condition imposed at each time. The following lemma reformulates this condition in different ways.

**Lemma 3.12.** Assume that \( \lambda_1 > \lambda_2 \). The following properties are equivalent:

1. \( U_\mu = V \).
2. For every \( G_\mu \)-stable proper subspace \( W \) of \( V \), \( \lambda(W) < \lambda_1 \)
3. \( L_\mu \) is the greatest element of \( F \setminus \{V\} \), i.e. every \( G_\mu \)-stable subspace of \( V \) is either \( V \) or is included in \( L_\mu \).
4. \( L_{\mu^\perp} = \{0\} \).
Moreover, when one of these conditions is fulfilled, the action of $T_\mu$ on the quotient $V/L_\mu$ is strongly irreducible and proximal.

**Proof.** The equivalence between (1), (2), (3) and (4) is easy to prove by definition of $L_\mu$ and $U_\mu$, and by Lemmas 3.10 and 3.11. We prove now the last statement. Assume that (3) holds. It follows that the action of $T_\mu$ on the quotient $V/L_\mu$ is irreducible. But by Lemma 3.8 and Corollary 3.9, the top Lyapunov exponent of $V/L_\mu$ is simple. It is enough now to recall the following known result from [GR85] (see also [BL85, Theorem 6.1]): if $E$ is a vector space defined over a local field and $\eta$ is a probability measure on $GL(E)$ such that $T_\eta$ is irreducible, then $T_\eta$ is i-p if and only if the top Lyapunov exponent relative to $\eta$ is simple. This ends the proof. \( \square \)

**Remark 3.13.** If $\rho: G_\mu \to GL(U_\mu)$ is the restriction map to $U_\mu$, then it is easy to see that $U_{\rho(\mu)} = U_\mu$ and that $L_{\rho(\mu)} = L_\mu \cap U_\mu$; the latter being non zero when $\lambda_1 > \lambda_2$ by Lemma 3.10. Observe also that it follows from Lemma 3.12 that the action of $T_\mu$ on $U_\mu/L_\mu \cap U_\mu$ is strongly irreducible and proximal. We will frequently use the representation $\rho$ to go back to the case $U_\mu = V$.

**Remark 3.14.** 1. Another case for which estimates are easier to handle is the case $L_\mu = \{0\}$ (i.e. $U_\mu = V^*$). This condition appeared in [FK83, Proposition 4.1, Theorem B] (see also [Henss]), as a sufficient condition to ensure the continuity of the function $\mu \to \lambda(\mu)$. Moreover, it corresponds to a unique cocycle average (see Remark 2.3). Recall that by Section 2.2 this condition is satisfied for random walks in irreducible groups and in the affine group in the expansive case. However we insist on the fact that one of the novelty of the present paper is to give limit theorems, when $\lambda_1 > \lambda_2$, in the case $L_\mu \neq \{0\}$ (as for instance random walks on the affine group in the contracting case, see Section 2.2). We refer also to [BQ16a] where limit theorems for cocycles are given depending on their cocycle average(s).

2. If $\pi: G_\mu \to GL(V/L_\mu)$ is the morphism action on the quotient vector space $V/L_\mu$, then $L_{\pi(\mu)} = \{0\}$ and $U_{\pi(\mu)} = \pi(U_\mu)$. Observe also that if $\lambda_1 > \lambda_2$, then by Corollary 3.9 the top Lyapunov exponent of $V/L_\mu$ is equal to $\lambda_1$ and is also simple.

### 4 Stationary probability measures on the projective space

In this section, we prove Theorem 2.4. This will be done through different steps. In Section 4.1 below, we show that if a stationary measure $\nu$ on $P(V)$ such that $\nu([L_\mu]) = 0$ exists, then this determines the projective subspace generated by its support. In Section 4.2, we show the existence of such a measure via Oseledets theorem. In Section 4.3 we prove that it is unique in a constructive way. More precisely, we show in Proposition 4.6 that $\nu$ is the law of a random variable $Z(\omega) \in P(V)$ characterized in the following way: every limit point of the normalized random walk $M_n/||M_n||$ is almost surely of rank one with image that projects to $[Z(\omega)]$ in $P(V)$. Moreover, a convergence in the KAK decomposition is given.

We recall that $k$ is a local field, $V$ is a vector space over $k$ of dimension $d \geq 2$ and $P(V)$ denotes the projective subspace of $V$. We endow $V$ with the norm $|| \cdot ||$ described in Section 3.1.2. If $\mu$ is a probability measure on $GL(V)$, then $T_\mu$ (resp. $G_\mu$) denotes the sub-semigroup (resp. subgroup) of $GL(V)$ generated by the support of $\mu$. We denote by $\mathcal{F}$ the set of all $G_\mu$-stable subspaces of $V$ and for every $S \in \mathcal{F}$, $\lambda(S)$ denotes the Lyapunov exponent relative to $S$.

#### 4.1 On the support of stationary probability measures

**Proposition 4.1.** Let $\mu$ be a probability measure on $GL(V)$ such that $\lambda_1 > \lambda_2$ and $\nu$ a stationary probability measure of the projective space $P(V)$ such that $\nu([L_\mu]) = 0$. Let

$$U_\mu := \bigcap_{S \in \mathcal{F}} S.$$  

Then,
1. The projective subspace generated by the support of \( \nu \) is \([U_\mu] \).

2. The probability measure \( \nu \) is non degenerate in \([U_\mu]\) i.e. it gives zero mass to every proper projective subspace of \([U_\mu]\).

The proof of this proposition will be done through different intermediate steps. First, we give below a criterion insuring that a stationary measure on the projective space is proper. When \( G_\mu \) is strongly irreducible, Furstenberg has shown that every \( \mu \)-stationary probability measure on the projective space is proper. The proof of Furstenberg yields in fact the following general result. It will be used in Lemma 4.3 in order to identify proper stationary measures outside the strongly irreducible case.

**Lemma 4.2.** Let \( E \) be a finite dimension vector space, \( \mu \) a probability measure on \( GL(E) \) and \( \nu \) a \( \mu \)-stationary probability measure on the projective space \( P(E) \) of \( E \). Then there exists a projective subspace of \( P(E) \) whose \( \nu \)-measure is non zero, of minimal dimension and whose \( G_\mu \)-orbit is finite. Equivalently, there exists a finite index subgroup \( G_0 \) of \( G_\mu \) such that at least one of the projective subspaces of \( P(E) \) charged by \( \nu \) is stable under \( G_0 \).

**Proof.** Let \( \Lambda \) be the set of subspaces of \( E \) charged by \( \nu \) and of minimal dimension, say \( l \). Let \( r = \sup \{ \nu([W]); W \in \Lambda \} \) and \( \Gamma \) the subset of \( \Lambda \) whose elements are subspaces of \( E \) whose \( \nu \)-measure is exactly \( r \). We will show that \( \Gamma \) is a finite set and stable under \( G_\mu \), which is sufficient to show the desired lemma. By minimality of \( l \), two distinct subspaces \( W_1 \) and \( W_2 \) of \( \Gamma \) satisfy \( \nu([W_1 \cap W_2]) = 0 \). Since \( \nu \) charges equally any two elements of \( \Gamma \) and is of total mass 1, this implies that \( \Gamma \) is finite. On the other hand, since \( \nu \) is a \( \mu \)-stationary probability measure, then for every \( W \in \Gamma \) and \( n \in \mathbb{N} \):

\[
r = \nu([W]) = \int \mathbb{1}_{[W]}(g \cdot [x])d\mu(g) \ d\nu([x]) = \int \nu(g^{-1} \cdot [W]) \ d\mu^n(g). \tag{2}
\]

Let \( b \) be any probability measure on \( \mathbb{N} \). By replacing if necessary \( \mu \) by \( \sum_{i=1}^{\infty} b(i)\mu^i \) in the equality above, we can without loss of generality assume that the support of \( \mu \) is the semi-group \( T_\mu := \cup_{n \in \mathbb{N}} \text{Supp}(\mu^n) \) generated by the support of \( \mu \). By combining this remark, together with equality (2) and the maximality of \( r \), we obtain that

\[
\forall g \in T_\mu, \nu(g^{-1} \cdot [W]) = r \quad \text{i.e.} \quad g^{-1} \cdot [W] \in \Gamma.
\]

Hence for every \( g \in T_\mu \), \( g^{-1} \Gamma \subset \Gamma \). Since \( \Gamma \) is finite, we deduce that for every \( g \in T_\mu \), \( g \Gamma = \Gamma \).

It follows that \( \Gamma \) is \( G_\mu \)-stable.

We know that when \( \lambda_1 > \lambda_2 \), \( G_\mu \) is irreducible if and only if \( G_\mu \) is strongly irreducible (see [BL85, Theorem 6.1]). Here’s below a generalization.

**Lemma 4.3.** Let \( E \) be a finite dimension vector space and \( \mu \) a probability measure on \( GL(E) \) such that \( \lambda_1 > \lambda_2 \) and \( U_\mu = E \) (see Lemma 3.12). Then \( G_\mu \) cannot fix any finite union of proper vector spaces of \( E \) unless they are all included in \( L_\mu \).

In particular, a \( \mu \)-stationary probability measure \( \nu \) on \( P(E) \) is proper if and only if \( \nu([L_\mu]) = 0 \).

**Proof.** It is enough to show the first assertion, the last being a simple consequence of the first one and of Lemma 4.2. We argue by contradiction. Let \( L = \{V_1, \ldots, V_s\} \) be a finite union of \( G_\mu \)-stable subspaces of \( E \) not all included in \( L_\mu \). By considering the orbit of one element of \( L \), we can assume without loss of generality that all the \( V_i \)'s have the same dimension, say \( r \).

We can also assume that \( r \) has minimal dimension for these properties. By Lemma 3.12, our assumptions on \( \mu \) imply that \( s \geq 2 \). Let \( S \) be the non empty set below:

\[
S := \{V_i \cap V_j; 1 \leq i < j \leq s\}.
\]

It is immediate that \( S \) is a finite \( G_\mu \)-stable set. By minimality of \( r \), we deduce that

\[
\forall i \neq j, V_i \cap V_j \subset L_\mu. \tag{3}
\]
Without loss of generality, we can assume that for every \( i = 1, \cdots, s, \) \( V_i \not\subseteq L_\mu. \) For every \( i = 1, \cdots, s, \) let \( \widetilde{V}_i \) be a supplementary of \( V_i \cap L_\mu \) in \( V_i. \) By (3), the \( \{\widetilde{V}_i\}'s \) are non empty pairwise distinct closed subsets of \( P(E). \) Then, we set
\[
\alpha := \inf \{\delta(\widetilde{V}_i, [\widetilde{V}_j]); 1 \leq i < j \leq s\} > 0.
\]
Let \( x \in \widetilde{V}_1 \setminus \{0\}, \) \( y \in \widetilde{V}_2 \setminus \{0\} \) and \( g \in G_\mu. \) Since \( G_\mu \) permute the \( V_i's, \) then there exist \( 1 \leq i, j \leq s \) such that \( gx \in \widetilde{V}_i \) and \( gy \in \widetilde{V}_j. \) Let us check that \( i \neq j. \) Indeed, if \( i = j, \) then by denoting by \( k \) the unique integer such that \( gV_i = \widetilde{V}_k, \) we would have \( x \in \widetilde{V}_j \cap \widetilde{V}_k \) and \( y \in \widetilde{V}_2 \cap \widetilde{V}_k. \) By (3), either \( x \) or \( y \) would belong to \( L_\mu, \) which would contradict that \( \widetilde{V}_{i,2} \cap L_\mu = \{0\}. \) Let \( \widetilde{L}_\mu \) be a supplementary of \( L_\mu \) in \( E \) containing \( \bigoplus_{k=1}^s \widetilde{V}_k. \) We denote by \( p \) the projection on \( \widetilde{L}_\mu \) parallel to \( L_\mu. \) It is clear that \( p(gx) \in W_i \) and \( p(gy) \in W_j. \) Consequently,
\[
\forall g \in G_\mu, \ \delta (\langle p(gx), p(gy) \rangle) \geq \alpha. \tag{4}
\]
Let \( F \) be the quotient vector space \( E/L_\mu. \) For every \( v \in E, \) de denote by \( \pi \) its projection in \( F. \)
Let \( \pi : G_\mu \rightarrow GL(F) \) the morphism action. By identifying \( E/L_\mu \) with \( L_\mu, \) we can identify the vector \( p(gv) \) of \( E \) with the vector \( \pi(g)\pi \) of \( F. \) If \( \delta \) denotes the Fubini-Study metric on \( F, \) (4) becomes:
\[
\forall g \in G_\mu, \ \delta (\pi(g)\pi, \pi(g)\pi) \geq \alpha. \tag{5}
\]
But since \( L_{x,\mu} = \{0\} \) and the top Lyapunov exponent of \( F \) is simple (see Remark 3.14), we have by [FK83, Theorem 3.9] applied to to the probability measure \( \pi(\mu) \) (see Theorem 2.2 for the statement) that:
\[
\delta (\pi(S_n)\pi, \pi(S_n)\pi) \leq \frac{\|\lambda < \pi(S_n)\| \|\pi(S_n)\|}{\|\pi(S_n)\|} \rightarrow 0, \tag{6}
\]
which contradicts (5).

**Proof of Proposition 4.1.** First, we prove that \([U_\mu]\) is included in the projective subspace \( M \) of \( P(V) \) generated by the support of \( \nu. \) Let \( E \) be a \( G_\mu \)-stable subspace of \( V \) such that \( \lambda(E) = \lambda_1. \) We want to show that \( \nu([E]) = 1. \) By Lemma 3.7, there exists a subspace \( E^\perp \) which is orthogonal to \( E \) in \( V \) (see Lemma 3.3 for the non-Archimedean case), such that if \( \pi_{E^\perp} \) denotes the orthogonal projection onto \( E^\perp, \)
\[
\forall x \in P(V), \ \delta([x], [E]) = \frac{\|\pi_{E^\perp}(x)\|}{\|x\|}. \tag{6}
\]
Identify \( E^\perp \) with the quotient vector space \( V/E. \) Hence, for every \( g \in G_\mu \) and \( x \in V, \) the vector \( \pi_{E^\perp}(gx) \) of \( V \) is identified with \( \phi(g)\pi \in V/E, \) where \( \phi \) denotes the morphism action of \( G_\mu \) on \( V/E \) and \( \pi \) refers to the projection of \( x \) on the quotient. This observation, together with the inequality \( \lambda(V/E) < \lambda_1 \) give
\[
\forall x \in V \ a.s., \ \limsup \frac{1}{n} \log \|\pi_{E^\perp}(S_n x)\| < \lambda_1. \tag{7}
\]
Let us define on the same probabilistic space \( (X_i)_{i \in \mathbb{N}^*} \) a sequence of identically distributed random variables on \( GL(V) \) of law \( \mu, \) and denote by \( (S_n = X_n \cdots X_1) \) the left random walk and by \( [Z] \) a random variable on the projective space \( P(V) \) of law \( \nu \) and independent of the \( X_i's. \) On the one hand, this property of independence combined with estimate (7) give
\[
\limsup \frac{1}{n} \log \|\pi_{E^\perp}(S_n Z)\| < \lambda_1. \tag{8}
\]
On the other hand, since \( \nu([L_\mu]) = 0, \) almost surely \( Z \not\in L_\mu. \) Hence by [FK83, Theorem 3.9] (see Theorem 2.2), we have almost surely
\[
\lim \frac{1}{n} \log \|S_n Z\| = \lambda_1. \tag{9}
\]
By combining (6), (8) and (9), we obtain:
\[ \delta([S_n Z], [E]) \xrightarrow{a.s.} 0. \]

In particular, for every \( \epsilon > 0 \),
\[ \mathbb{P}(\delta([S_n Z], [E]) \leq \epsilon) \xrightarrow{n \to +\infty} 1. \]

But, for every \( n \), the law of \([S_n Z]\) is \( \nu \) (by definition of stationarity of \( \nu \)). Hence, \( \nu(\{[x]; \delta([x], [E]) \leq \epsilon\}) = 1 \) for every \( \epsilon > 0 \). Thus, \( \nu([E]) = 1 \). This being true for every such stable subspace \( E \), and since the intersection defining \( U_\mu \) can be made a finite one (the dimension of \( V \) is finite), we deduce that \( \nu([U_\mu]) = 1 \). Since \([U_\mu]\) is closed in \( \mathbb{P}(V) \), we deduce that \( M \subset [U_\mu] \). In order to prove the other inclusion, observe that the following holds
\[ \forall g \in T_\mu, g \cdot \text{Supp}(\nu) \subset \text{Supp}(\nu). \quad (10) \]

Write \( M = [E] \) for some subspace \( E \) of \( V \). It follows from (10) that for every \( g \in T_\mu \), \( gE \subset E \) and then \( gE = E \). Hence \( E \) is a \( G_\mu \)-invariant subspace of \( V \). Moreover, estimate (9) implies that the Lyapunov exponent relative to \( E \) is \( \lambda_1 \). By definition of \( U_\mu \), we deduce that \( U_\mu \subset E \) and then \([U_\mu]\) \( \subset M \). Item (1) of the proposition is then proved.

In order to prove point (2) of the proposition, we set for simplicity of notation \( E = U_\mu \) and denote by \( \rho \) the restricted representation \( G_\mu \rightarrow \text{GL}(E) \). It follows from above that \( \nu \) is a \( \rho(\mu) \)-stationary probability measure on \( \mathbb{P}(E) \). By definition of \( E \), we have the following equalities:
\[ \lambda(E) = \lambda_1, \quad L_{\rho(\mu)} = L_\mu \cap E, \quad U_{\rho(\mu)} = E. \]

By Lemma 3.12, the first and the third equalities above show that the probability measure \( \rho(\mu) \) on \( \text{GL}(E) \) satisfies the assumptions of Lemma 4.3. Since \( \nu([L_\mu]) = 0 \), the second equality above gives \( \nu([L_{\rho(\mu)})]) = 0 \). By Lemma 4.3 again, \( \nu \) is proper on \( \mathbb{P}(E) \).

\[ \square \]

### 4.2 Oseledets theorem and stationary measures

In this section, we prove that given a probability measure \( \mu \) on \( \text{GL}(V) \) such that \( \lambda_1 > \lambda_2 \), there exists a \( \mu \)-stationary probability measure \( \nu \) on the projective space \( \mathbb{P}(V) \) that satisfies the equality \( \nu([L_\mu]) = 0 \) and the conclusions of Proposition 4.1. Our proof is constructive: we use Oseledets theorem to derive a random variable \([Z]\) in \( \mathbb{P}(V) \) of law \( \nu \) from the random walk associated to \( \mu \). Since \( \lambda_1 > \lambda_2 \), such a stationary measure will immediately be a \( \mu \)-boundary.

We note that the existence of such a probability measure holds even if \( \lambda_1 = \lambda_2 \). This can be proved using the methods developed in [FKS83]. Since the framework of the latter article is very general, the method is not constructive.

**Proposition 4.4.** Let \( \mu \) be a probability measure on \( \text{GL}(V) \) such that \( \lambda_1 > \lambda_2 \). Then, there exists a \( \mu \)-stationary probability measure \( \nu \) on \( \mathbb{P}(V) \) such that \( \nu([L_\mu]) = 0 \). By Proposition 4.1, \( \nu \) is proper on \([U_\mu]\). Moreover, \((P(V) \setminus [L_\mu], \nu) \) is a \( \mu \)-boundary.

Such a measure will be obtained thanks to Oseledets theorem, and more precisely the equivariance equality we recall below.

**Theorem 4.5.** [Ose68] Let \( (\Omega, \theta, \mathbb{P}) \) be an ergodic dynamical system. Let \( A : \Omega \rightarrow \text{GL}(E) \) be a measurable application such that \( \log(\|A\|) \) and \( \log(\|A^{-1}\|) \) are integrable. Then there exist \( l \in \mathbb{N}^* \), \( m_1, \cdots, m_l \in \mathbb{N}^* \) and real numbers \( \lambda_1 = \cdots = \lambda_{m_1} > \cdots > \lambda_{m_1+1} = \cdots = \lambda_{m_l} \) such that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), there exist subspaces \( E = E_1 \supset E_2 \supset \cdots \supset E_{m_1} \supset E_{m_1+1} = \{0\} \) such that:

1. **Equivariance equality:** for every \( 1 \leq i \leq l \), \( A(\omega) \cdot E_i = E_{i+1}(\omega) \)
2. for every \( i = 1, \cdots, l \) and every non zero vector \( v \) of \( E_i \setminus E_{i+1} \) if and only if \( \lim \frac{1}{n} \log(\|A(\theta^n(\omega)) \cdots A(\theta(\omega))A(\omega)\|) = \lambda_m \).
3. \( m_i = \dim(E_i) - \dim(E_{i+1}) \), for every \( i = 1, \cdots, l \).
Proof of Proposition 4.4. Let $d = \text{dim}(V)$. Lyapunov exponents $\lambda_1 \geq \cdots \geq \lambda_d$ of the probability measure $\mu$ coincide with the exponents $\lambda_{m_i}$ given by Oseledets theorem applied to the canonical probabilistic space $\Omega = G^{\mathbb{N}}$, $\mathbb{P} = \mu^{\otimes \mathbb{N}}$, $\theta$ being the shift operator and $A : \Omega \to G, \omega = (g_i)_{i \in \mathbb{N}} \mapsto g_1$ the projection on the first component.

The theorem is not useful in this form because the equivariance equality can't be used to construct stationary measures. We will rather consider the reflected probability measure $\tilde{\mu}$ (law of $g_1^{-1}$) on $G$, which is equivalent to keep the same probabilistic space and replace $A$ by $\bar{A} : \Omega \to G, \omega = (g_i)_{i \in \mathbb{N}} \mapsto g_1^{-1}$. In this case, Lyapunov exponents are $\bar{\lambda}_i = -\lambda_{d-i+1}$. Oseledets theorem gives then, for the same integers $m_1, \cdots, m_l$ relative to the measure $\mu$, a new random filtration $\mathcal{F}_0^\omega = \{0\} \subset \mathcal{F}_1^\omega \subset \cdots \subset \mathcal{F}_l^\omega = V$ such that for $\mathbb{P}$-almost every $\omega = (g_i)_{i \in \mathbb{N}} \in \Omega$:

1. $F_n^\omega = g_1 \cdot F_{n-1}^\omega$, \hspace{1cm} (11)
   for every $1 \leq i \leq l$.
2. For every non zero vector $v$ of $V$ and every $i = 1, \cdots, l$:
   \[ v \in \mathcal{F}_i^\omega \backslash \mathcal{F}_{i-1}^\omega \iff \lim \frac{1}{n} \log \|M_n^{-1} v\| = -\lambda_{m_i}, \] \hspace{1cm} (12)
   where $M_n = X_1 \cdots X_n$ is the right random walk.
3. For every $i = 1, \cdots, l$,
   \[ m_i = \text{dim}(F_i^\omega) - \text{dim}(F_{i-1}^\omega). \] \hspace{1cm} (13)

Under the assumption $\lambda_1 > \lambda_2$, we have by (13) $m_1 = 1$ so that $kZ(\omega) := F_1^\omega$ is a line for $\mathbb{P}$-almost every $\omega \in \Omega$. Let $\nu$ be the law of the random variable $Z : \Omega \to \mathbb{P}(V), \omega \mapsto [Z(\omega)]$ on the projective space. The probability $\nu$ is $\mu$-stationary. Indeed, for every real valued measurable function $f$ on $\mathbb{P}(V)$,

\[ \int_{\mathbb{P}(V)} f([x]) d\nu([x]) = \int_\Omega f(F_1^\omega) d\mathbb{P}(\omega) \]
\[ = \int_\Omega f(g_1 \cdot F_{\theta(\omega)}^\omega) d\mathbb{P}(\omega) \]
\[ = \int_G \left[ \int_\Omega f(\gamma \cdot F_{\theta(\omega)}^\omega) d\mathbb{P}(\omega) \right] d\mu(\gamma) \]
\[ = \int_{G \times \mathbb{P}(V)} f(\gamma \cdot x) d\mu(\gamma) d\nu([x]). \]
\hspace{1cm} (16)

Equality (14) is straight-forward consequence of the equivariance equality (11); (15) is due to the independence of $g_1$ and $\theta(\omega) = (g_2, g_2, \cdots)$ while (16) is true because $\theta$ preserves the measure $\mathbb{P}$.

Finally, we show that $\nu([E]) = 0$. Let $E$ be a proper subspace such that $\lambda(E) < \lambda_1$. Fix $\omega \in \Omega$. Then
\[ \forall v \in E, \lim \frac{1}{n} \log \|M_n^{-1}(\omega)v\| \geq \lambda_{\text{dim}(E)}(E) = -\lambda(E) > -\lambda_1. \]

Taking if necessary $\omega$ in a measurable subset of $\Omega$ of $\mathbb{P}$-probability 1, assertion (12) gives
\[ \forall v \in E, v \notin kZ(\omega), \text{ i.e. } [Z(\omega)] \neq [v]. \]

hence, $\nu([E]) = 0$.

The fact that $\nu$ is a $\mu$-boundary is also a consequence of the equivariance equality (see for example [Kai00], [Led85], [BS11]).

4.3 Uniqueness of the stationary measure

In this section, we prove that the stationary measure given by Proposition 4.4 is unique. We fix a basis $(e_1, \cdots, e_d)$ of $V$. The dual vector space $V^*$ of $V$ will be equipped with the dual norm and with the dual basis $(e^*_1, \cdots, e^*_d)$. For every $g \in GL(V)$, we denote by $g' \in GL(V^*)$ the transpose linear map on $V^*$ and for every subspace $H$ of $V$, we denote by $H^\perp$ the subspace
of $V^*$; annihilator of $H$.

Recall that if $K$ denotes the isometry group of $(V, ||\cdot||)$ and $A$ the subgroup of $\text{GL}(V)$ consisting of diagonal matrices in the chosen basis, then the following decomposition holds $G = KAK$ (called KAK decomposition). For $g \in \text{GL}_d(k)$, we write $g = k(g)a(g)u(g)$ a KAK decomposition of $g$. Notice that $g' = u(g')a(g')k(g')$ is a KAK decomposition of $g'$ in $\text{GL}(V^*)$.

Finally, if $\mu$ is a probability measure on $\text{GL}(V)$, we denote by $\bar{\mu}$ the pushforward measure of $\mu$ on $\text{GL}(V^*)$ by the map $g \mapsto g'$.

**Proposition 4.6.** Let $\mu$ be a probability measure on $P(V)$ such that $\lambda_1 > \lambda_2$ and $\nu$ a $\mu$-stationary probability on $P(V)$ such that $\nu([L_\mu]) = 0$. Then there exists a random variable $\omega \mapsto [Z(\omega)] \in P(V)$ of law $\nu$ such that:

1. almost surely, every limit point of $\frac{M_n}{||M_n||}$ in $\text{End}(V)$ is a matrix of rank one whose image in $P(V)$ is equal to $[Z]$.
2. $k(M_n)e_1$ converges almost surely to $[Z]$.

In particular, $\nu$ is unique.

**Proof.** In item i. below we prove the proposition in the particular case $U_\mu = V$. In item ii we check that this is enough to deduce the uniqueness of the stationary measure on $P(V) \setminus [L_\mu]$. Finally, in item iii. we prove the limit theorems claimed in the proposition in the general case.

i. Assume first that $U_\mu = V$.

By Proposition 4.4, there exists $\mu$-stationary probability measure $\nu$ on $P(V)$ such that $\nu([L_\mu]) = 0$. Since $U_\mu = V$, then Proposition 4.3 gives that $\nu$ is proper on $P(V)$. Let $\omega \in \Omega$ and $A(\omega)$ a limit point of $\frac{M_n(\omega)}{||M_n(\omega)||}$. Let $\phi : \mathbb{N} \to \mathbb{N}$ such that $\frac{M_{\phi(n)}(\omega)}{||M_{\phi(n)}(\omega)||}$ converges to $A(\omega)$. Since $\nu$ is proper, the pushforward measure $A(\omega)\nu$ on $P(V)$ is well defined and we have the following vague convergence:

$$M_{\phi(n)}(\omega)\nu \xrightarrow{\nu \text{ vague}} A(\omega)\nu. \quad (17)$$

But using Doob’s theorem on convergence of bounded martingales, Furstenberg showed in [Fur63] that there exists for $\mathbb{P}$-almost every $\omega$, a probability measure $\nu(\omega)$ on $P(V)$ such that

$$M_n(\omega)\nu \xrightarrow{n \to \infty} \nu(\omega). \quad (18)$$

and

$$\forall f \in \mathcal{C}(P(V)), \quad \mathbb{E}\left(\int f d\nu(\omega)\right) = \int f d\nu. \quad (19)$$

By (17) and (18), we obtain the following relation:

$$\nu(\omega) = \delta_{[Z(\omega)]}. \quad (20)$$

In particular $[Z(\omega)]$ does not depend on the subsequence $\phi$. By (19), $\nu$ is the law of the random variable $\omega \mapsto [Z(\omega)]$ on $P(V)$. This proves the uniqueness of $\nu$, together with item 1 in the case $U_\mu = V$. Item 2 is an immediate consequence of the KAK decomposition.

ii. Now if $U_\mu \neq V$, we apply the previous part for the restriction $\rho : T_\mu \to \text{GL}(U_\mu)$ on $U_\mu$. Since $U_{\rho(\mu)} = U_\mu$, $L_{\rho(\mu)} = U_\mu \cap L_\mu$ (see Remark 3.13) and since the top Lyapunov exponent of $\rho(\mu)$ is simple, we obtain using item i. a unique $\mu$-invariant probability measure on $[U_\mu] \setminus [U_\mu \cap L_\mu]$. But by Proposition 4.1, any $\mu$-invariant probability measure on $P(V) \setminus [L_\mu]$ gives total mass to $[U_\mu]$, then such a probability measure is unique.
iii. It is left to prove the limit theorems in the first and second claims of Proposition 4.6 even if \( U_\mu \neq V \). For every \( n \in \mathbb{N} \), write

\[
M_n = k_n a_n u_n.
\]

in the KAK decomposition of \( GL(V) \), with \( a_n = \text{diag} (a_1(n), \ldots, a_d(n)) \). For every \( x \in V \), the following holds:

\[
\frac{M_n x}{||\rho(M_n)||} = e^*_1(u_n x) \frac{||M_n||}{||\rho(M_n)||} k_n e_1 + O \left( \frac{a_2(n)}{||\rho(M_n)||} \right).
\]

On the one hand, the Lyapunov exponent of \( \rho \) is \( \lambda_1 \) and \( \frac{\log a_2(n)}{n} \) converges almost surely to \( \lambda_2 < \lambda_1 \). Hence \( \frac{a_2(n)}{||\rho(M_n)||} \) converges (exponentially fast) to zero almost surely. Hence, almost surely,

\[
\forall x \in V, \quad \frac{M_n x}{||\rho(M_n)||} = e^*_1(u_n x) \frac{||M_n||}{||\rho(M_n)||} k_n e_1 + o(1). \tag{21}
\]

Let \( \omega \in \Omega \) and \( k_\infty \) be a limit point of \( k_n \). We write \( k_\infty = \lim k_{\psi(n)} \) for some increasing function \( n \to \psi(n) \). Take now a limit point \( A(\omega) \in \text{End}(U_\mu) \) of \( \frac{\rho(M_{\psi(n)}\omega)}{||\rho(M_{\psi(n)}\omega)||} \). To simplify the notation, we will still write \( \frac{\rho(M_{\psi(n)}\omega)}{||\rho(M_{\psi(n)}\omega)||} \to A(\omega) \). Recall that, by part i, \( A(\omega) \) is of rank one with image \( k Z(\omega) \) such that the random variable \( \omega \to [Z(\omega)] \) has law \( \nu \). Choose any \( x \in U_\mu \) such that \( x \not\in \text{Ker}(A(\omega)) \). Hence \( \frac{M_{\psi(n)} x}{\rho(M_{\psi(n)} \omega)} \xrightarrow[n \to + \infty]{} A(\omega) x \neq 0 \). Comparing with (21) gives that

\[
|e^*_1(u_n x)| \frac{||M_{\psi(n)}\omega||}{||\rho(M_{\psi(n)}\omega)||} \xrightarrow[n \to + \infty]{} ||A(\omega)x|| > 0 \quad \text{and} \quad \frac{M_{\psi(n)} x}{||\rho(M_{\psi(n)}\omega)||} \xrightarrow[n \to + \infty]{} ||A(\omega)x|| k_\infty e_1.
\]

In particular, \( M_{\psi(n)}|x| \xrightarrow[k_\infty]{} k_\infty e_1 \). But since \( x \not\in \text{Ker}(A(\omega)) \), item i. shows that \( M_{\psi(n)}|x| \xrightarrow[k_\infty]{} |Z(\omega)| \). Hence \( k_\infty e_1 = |Z(\omega)| \). This proves part 2 of the proposition in the general case. Since \( \lambda_1 > \lambda_2 \), part 1 is an easy consequence of the KAK decomposition.

\[
\square
\]

Corollary 4.7. Let \( \mu \) be a probability measure on \( GL(V) \) such that \( \lambda_1 > \lambda_2 \). Then,

1. For every sequence \( \{\{x_n\}\}_n \) in \( P(V) \) that converges to some \( [x] \in P(V) \setminus [L_\mu] \), we have almost surely,

\[
\inf_{n \in \mathbb{N}^*} \frac{||S_n x_n||}{||S_n||} > 0.
\]

2. \( \frac{1}{n} \mathbb{E}(\log \frac{||S_n x_n||}{||S_n||}) \) converges to \( \lambda_1 \) uniformly on compact subsets of \( P(V) \setminus [L_\mu] \).

Remark 4.8. We deduce from the previous corollary that: if \( \lambda_1 > \lambda_2 \), then

1. \( \sup_{[x] \in P(V)} \frac{1}{n} \mathbb{E}(\log \frac{||S_n x||}{||S_n||}) = \lambda_1 \).

2. if \( L_\mu = \{0\} \), then \( \inf_{[x] \in P(V)} \frac{1}{n} \mathbb{E}(\log \frac{||S_n x||}{||S_n||}) = \lambda_1 \). This is coherent with the result of Furstenberg-Kifer saying that \( L_\mu = \{0\} \) if and only if there exists a unique cocycle average (see Remark 2.3).

Proof. Let \( \{\{x_n\}\}_n \in \mathbb{N}^* \) be a sequence in \( P(V) \) that converges to \( [x] \in P(V) \setminus [L_\mu] \). Write \( S_n = K_n A_n U_n \) the KAK decomposition of \( S_n \). Since \( \lambda_1 > \lambda_2 \),

\[
\frac{||S_n x_n||}{||S_n||} = \frac{||A_n U_n x_n||}{||A_1(n)||} = ||(U_\mu^* e_1^*)(x_n)|| + o(1).
\]

Since \( U_\mu^* = k(S_\mu) = k(X_1^* \cdots X_d^*) \), and since the Lyapunov exponents of \( \mu \) coincide with those of \( \mu \), item 2 of Proposition 4.6 applied to \( \mu \) shows then that

\[
\frac{||S_n x_n||}{||S_n||} \xrightarrow[n \to + \infty]{} ||\hat{Z}(x)||,
\]

21
with $||\hat{Z}|| = 1$ and $[\hat{Z}]$ being a random variable on $P(V^*)$ with law the unique $\hat{\nu}$-stationary probability measure $\hat{\nu}$ on $P(V^*) \setminus \{L_0\}$. Let $H \subset P(V^*)$ be the hyperplane orthogonal to $x$. Since $x \not\in L_\mu$, $L_\mu^\perp \not\subset H$, i.e. by Lemma 3.11 $U_{\hat{\mu}} \not\subset H$. Since, by proposition 4.1 $\hat{\nu}(U_{\hat{\mu}}) = 1$ and $\hat{\nu}$ is proper on $[U_{\hat{\mu}}]$, we deduce that

$$\hat{\nu}([H]) = \hat{\nu}([H \cap U_{\hat{\mu}}]) = 0.$$ 

Hence, almost surely, $|\hat{Z}(x)| \neq 0$. Item 1. is then proved. To prove item 2, take a compact subset $K$ of $P(V) \setminus \{L_\mu\}$. By compactness of $K$, it is enough to show that for any sequence $(\{x_n\})_n$ in $K$ that converges to some $[x] \in K$, one has that $\frac{1}{n}\mathbb{E}||\frac{S_n x_n}{||S_n x_n||}|| \to \lambda_1$. By the previous item 1., we deduce that $\frac{1}{n}\log ||S_n x_n||$ converges to $\lambda_1$. But by the law of large numbers, it is easy to see that the sequence $\{\frac{1}{n}\log ||S_n x_n||, n \geq 1\}$ is uniformly integrable. This is enough to conclude.

5 The limit set and the support of the stationary measure

In this section, we understand further the support of the unique $\mu$-stationary measure given by Theorem 2.4, by relating it to the limit set of $T = T_\mu$. We will adapt the proof of [GG96] to our setting. Finally we give two concrete examples by simulating the limit set of two non irreducible subgroups of $GL_3(\mathbb{R})$.

5.1 Proof of Theorem 2.6

We keep the same notation as in Section 2.3 concerning the set $Q$ of quasi-projective maps of $P(V)$, the limit set $\Lambda(T) \subset [U]$ of $T$ relative to the subspaces $L$ and $U$, and the subsets $p^+(T_\alpha)$ (resp. $p^+(T^n_\alpha)$) of $P(V)$ of attractive points of proximal elements of $T$ in $[U]$ (resp. in $[U \setminus L]$).

First we check that following property that we claimed to hold:

**Lemma 5.1.** $\Lambda(T)$ is a closed $T$-invariant subset of $[U]$.

**Proof.** Only the closed part needs a proof. Let $y_i \in \Lambda(T)$ be a sequence in $\Lambda(T)$ that converges in $P(V)$ to some $y$. Clearly $y \in [U]$. For each $y_i = p(\delta_i)$, find a projective subspace $S_i$ of $P(V)$, a sequence of projective maps $\{g_{i,n}\}_{n \in \mathbb{N}}$ such that $g_{i,n}$ converges pointwise, when $n$ tends to infinity, to $\delta_i$ with $\delta_i$ that maps $P(V) \setminus [S_i]$ to $y_i$. Since by [GM89, Lemma 2.10, 1.], $Q$ is sequentially compact for the topology of pointwise convergence, there exists a subsequence of the $\delta_i$’s that converges to some quasi-projective map $\delta$. To simplify notations, we will write $\delta = \lim_{i \to +\infty} \delta_i$. Let

$$S := \lim_{i \to +\infty} \inf S_i = \{x \in P(V); \exists i(x); \forall i \geq i(x), x \in S_i\} = \bigcup_{n \geq k \geq 1} S_k.$$ 

It is clear that $S$ is a projective subspace of $P(V)$ and hence that the union above is a finite one. Taking the latter fact into account and the fact that $[U] \not\subset S$, for every $i$, we deduce that $U \not\subset S$. Let now $x \not\in S$. By definition of $S$, one can find an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that $x \not\in S_{\phi(i)}$ for every $i$. Hence

$$\delta x = \lim_{i \to +\infty} \delta_i x = \lim_{i \to +\infty} \delta_{\phi(i)} x = \lim_{i \to +\infty} y_i = y.$$ 

It is left to show that $\delta$ is a pointwise limit of projective transformations that belong to $PT \subset PGL(V)$. Since each $\delta_i$ is such a map and since $\delta$ is the limit of the $\delta_i$’s, this follows from [GM89, Lemma 2.10, 2.].

We are now able to prove Theorem 2.6.

**Proof of Theorem 2.6.** 1. We use the same notation as Section 4.2. Using Oseledets theorem, we know that the $GL(V)$-valued cocyle $M^{-1}_n$ can be written as $M^{-1}_n = (\phi \circ \theta^n) \Delta_n \phi^{-1}$ where $\phi(\omega) \in GL(V)$ is a random automorphism with $\phi(\omega)e_1$ is a least expanding vector of $M^{-1}_n$ and $\Delta_n$ is a block diagonal matrix with a block structure defined by the increasing
Lyapunov filtration of $M_n^{-1}$. Since $\lambda_2 < \lambda_1$, we have $\Delta_n e_1 = \lambda_n^1 e_1$ with $-\frac{1}{2} \log |\lambda_n^1| = \lambda_1$ and $\Delta'_n = \lambda_n^1 \Delta_n^{-1}$ converges to the canonical projection $\Delta'$ on $ke_1$. We have

$$\phi^{-1}(\omega)M_n(\omega)\phi(\omega) = \Delta_n^{-1}v_n(\omega),$$

(22)

with $v_n(\omega) = \phi^{-1}(\theta^n \omega)\phi(\omega)$. Also:

$$||\Delta_n v_n - \Delta'|| \leq ||v_n - I|| ||\Delta'_n - \Delta'||.$$

(23)

As in [Gui90, Proposition 3], using Poincaré recurrence theorem, we can find a subsequence $n_k(\omega) = n_k$ such that $\lim_{k \to +\infty} v_{n_k}(\omega) = I$; so that by (23)

$$\lim \Delta'_{n_k} v_{n_k} = \Delta'.$$

We deduce that

$$|\lambda_{n_k}^1| M_{n_k} \xrightarrow{k \to +\infty} \phi(\omega) A(\omega),$$

(24)

where $A(\omega) := \phi(\omega) \Delta' \phi(\omega)^{-1}$ is a projection endomorphism on the line $\phi(\omega)[e_1]$. In particular, $A(\omega)$ is a proximal one endomorphism of $V$. By a perturbation argument, $M_{n_k}(\omega)$ has a dominant and simple eigenvalue for all large $k$ with a dominant eigenvector $p^+(M_{n_k})$ close to $\phi(\omega)[e_1]$. The proof of Proposition 4.4 shows that the direction of $\phi(\omega)[e_1]$ is $\{Z(\omega)\} \not\in L$. Hence for all large $k$, $p^+(M_{n_k}) \not\in L$. Finally, we check that $p^+(\lambda_{n_k}) \in U$. Indeed, the largest eigenvalue of $M_{n_k}$ is either an eigenvalue of its restriction to $U$ with its corresponding eigenvector being that of the restriction operator, or is an eigenvalue of its projection on $V/U$. But the latter eigenvalue grows at most as $\exp(n_k \lambda_2)$, while it follows from (24) that the spectral radius of $M_{n_k}$ grows as the norm of $||M_{n_k}||$, i.e. as $\exp(n_k \lambda_1)$. Since $\lambda_2 < \lambda_1$, we deduce that $p^+(M_{n_k}) \in U$, for all large $k$. Hence $M_{n_k} \in T_0^\circ$ so that $\{Z(\omega)\} \in p^+(T_0^\circ)$. In particular, $p^+(T_0^\circ) \not\subset \emptyset$. Since $\{Z\}$ has law $\nu$, we deduce that

$$\nu(p^+(T_0^\circ)) = \mathbb{P}\{\{Z\} \in p^+(T_0^\circ)\} = 1,$$

so that

$$\text{Supp}(\nu) \subset p^+(T_0^\circ) \subset p^+(T_0).$$

(25)

Conversely, let $h \in T_0$. Then $h^\nu/||h^\nu||$ converges to the projection $\eta$ on the line generated by $p^+(h) \in U$ and parallel to some $h$-invariant subspace of $V$. In particular, $U \not\subset \text{Ker}(\eta)$. Since by Theorem 2.4 $\nu$ is proper in $[U]$, we have that $\nu([U] \cap \text{Ker}(\eta)) = 0$ so that

$$h^\nu \xrightarrow{n \to +\infty} \delta_{p^+(h)}.$$

Since $\text{Supp}(h^\nu) \subset \text{Supp}(\nu)$ for every $n$, we get $p^+(h) \in \text{Supp}(\nu)$, hence $p^+(T_0^\circ) \subset \text{Supp}(\nu)$ and

$$\text{Supp}(\nu) \supset p^+(T_0) \supset p^+(T_0^\circ).$$

(26)

Inclusions (25) and (26) show item 1.

2. By the previous item, we know that there exists a random variable $\omega \mapsto [Z(\omega)] \in [U]$ of law $\nu$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$, there exists a random subsequence $\{n_k\}_k$ such that $\lambda_{n_k}^1 M_{n_k}(\omega)$ converges in $\text{End}(V)$ to a rank one endomorphism $A(\omega) = \phi(\omega) \Delta(\omega)^{-1}$. Let $S := \text{Ker}(A(\omega))$. By item 1, $A(\omega)$ is also proximal, so that $A(\omega)^2 \not= 0$, i.e. $\text{Im}(A(\omega)) \not\subset S$. Since $\text{Im}(A(\omega)) = [Z(\omega)] \in [U]$, we deduce that $[U] \not\subset S$. By finite normalizations of $M_{n_k}$ on $S$, we can make the projective map associated to $M_{n_k}$ converge pointwise to a quasi projective map $\delta$ such that $\delta$ maps $P(V) \setminus S$ to $p(\delta) := [Z(\omega)] \in [U]$. Hence $\delta \in \hat{T}$ and $[Z(\omega)] \in \Lambda(T)$. Since by Lemma 5.1 $\Lambda(T)$ is closed in $P(V)$ and since $[Z]$ has law $\nu$, we deduce that

$$\text{Supp}(\nu) \subset \Lambda(T).$$

Conversely, let $y = p(\delta) \in \Lambda(T)$ and $\{g_n\}$ a sequence of projective maps converging pointwise to $\delta$, together with a projective subspace $S$ of $P(V)$ that does not contain $[U]$ and such that with $\delta$ maps $P(V) \setminus S$ to the point $y$ of $P(V)$. Since $\nu$ is proper on $[U]$ and $S$ does not contain $[U]$, we deduce that $\nu([U] \cap S) = 0$ so that $\delta \nu$ is the Dirac measure on $y$. We conclude that $g_n^\nu \xrightarrow{n \to +\infty} \delta_y$. Since $\text{Supp}(g_n^\nu) \subset \text{Supp}(\nu)$ for every $n$, we deduce that $y \in \text{Supp}(\nu)$. Consequently,

$$\text{Supp}(\nu) \supset \Lambda(T).$$
Item 2 is then proved.

3. Let \( x \in P(V) \setminus [L] \) and \([Z] \in P(V)\) be the random variable described above in item 1. By item 2, it is enough to show that \( P(Z \in T \cdot x) = 1 \). One can prove using Proposition 4.6, that \( M_n x \) converges in probability to \([Z]\). We will refrain from including the details since in Theorem 6.2 we will prove the almost sure convergence under an exponential moment assumption. We refer to [BL85, Theorem 4.3] for the same result under the i-p assumption. Hence there exists almost surely a random subsequence \( n \mapsto \phi(n) \) such that \( M_{\phi(n)}[x] \) converges to \( Z \). Therefore, almost surely, \( Z \in T \cdot x \).

We deduce easily the proof of Corollary 2.7 stated in Section 2.3.

**Proof of Corollary 2.7:** The implication \( 1 \implies 2 \) follows immediately from item 3. of Theorem 2.6. The implication \( 2 \implies 1 \) is an easy consequence of the fact the support of \( \nu \) is \( T \) -invariant. The equivalence between 1 and 3 follow directly from item 2. of Theorem 2.6.

5.2 Simulations

In this section we give a simulation of the limit set of two subgroups of \( GL_3(\mathbb{R}) \). Let \( U = V = \mathbb{R}^3 \) and \( L = \mathbb{R}(1,0,0) \). The open subset \( F = P(V) \setminus [L] \) of \( P(V) \) is homeomorphic to a suitable quotient of the cylinder \( L \times \pi_1(V/L) \cong \mathbb{R} \times S^1 \) (see Section 2.2).

5.2.1 An example with non compact support in \( F \)

Consider the following matrices in \( GL_3(\mathbb{R}) \):

\[
g_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Denote by \( T \) be the sub-semigroup of \( GL_3(\mathbb{R}) \) generated by \( g_1 \) and \( g_2 \). Since the determinant of the restriction of \( g_1, g_2 \) to \( L \) is in absolute value equal to that of its projection on \( V/L \), we deduce that \( T \) can actually be seen as a sub-semigroup of the automorphism group of the Heisenberg group (see Section 2.2). We want to simulate \( \Lambda(T) \) and \( \Lambda^\alpha(T) \). Let then \( \mu \) be the uniform probability measure on \( \{g_1, g_2\} \), i.e. \( \mu = \frac{1}{2} \delta_{g_1} + \frac{1}{2} \delta_{g_2} \). By Furstenberg theorem [Fur63], we deduce that \( \lambda_1 > 0 \) and \( \lambda_2 = 0 \) (see Lemma 3.12). Hence, we are in the situation of our paper with \( L_\mu = L \) and \( U_\mu = V \). By theorem 2.4 and Section 2.2, there exists a \( \mu \)-invariant stationary measure \( \nu \) on the cylinder \( S^1 \times \mathbb{R} \) (not necessarily unique if we don’t impose the invariance under the action of \( x \mapsto -x \)). By Theorem 2.6, \( \text{Supp}(\nu) = \Lambda(T) \). Hence its enough to simulate \( \nu \), which can be done by simulating the points \( Z(\omega), \omega \in \Omega \). Here is a picture of \( \nu \) and its projection on \( V/L \).

![Figure 1: Simulation of \( \nu \).](image1.png)

![Figure 2: Projection of \( \nu \) on \( V/L \).](image2.png)
We observe the fibered structure as described in Section 2.2. Each fiber is contained in an affine line with (horizontal) direction $L$. The base is the (vertical) circle and corresponds to the limit set of the i-p semi-group of $\text{SL}_2(\mathbb{R})$ generated by $\mathcal{F}$ and $\mathcal{F}$ (Figure 2). The picture suggests that $\text{Supp}(\nu)$ not to be compact in $F$. This observation can be proved by means of Corollary 2.7 and checking that the orbit of any point in $F$ under the cyclic group generated by $g_1$ is non compact. Hence, this example justifies point 2 of Remark 2.8.

5.2.2 An example with compact support in $F$

We replace $g_1$ and $g_2$ by

$$g_1 = \begin{pmatrix} 0.5 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0.5 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

instead of $g_1$ and $g_2$. We have also $\lambda_1 > 0 = \lambda_2$, $L_\mu = L$ and $U_\mu = V$. We obtain the following simul...

The picture suggests that the support of $\nu$ is compact in $F$. Note that the projection of $\nu'$ on $V/L$ should coincide with that of $\nu$.

6 Regularity of the stationary measure

6.1 Introduction

Let $V$ be a vector space over the local field $k$ of dimension $d \geq 2$. We use the same notation as Section 3.1.2 concerning the choice of the norm on $V$ and the Fubini-Study metric on $P(V)$. In this section, we prove that under an exponential moment of $\mu$, the stationary measure given by Theorem 2.4 has Hölder regularity, and more precisely the following

**Theorem 6.1.** Let $\mu$ be a probability measure on $GL(V)$ with an exponential moment such that $\lambda_1 > \lambda_2$. Let $\nu$ be the unique $\mu$-stationary probability measure on $P(V)$ such that $\nu([L_\mu]) = 0$ (Theorem 2.4). For every compact subset $K$ of $P(V^*) \setminus [L_{\mu}]$, let $\eta(K)$ be the following finite quantity

$$\eta(K) := \sup_{f \in K} \left( \frac{1}{\delta^{-\gamma}([f],[L_{\mu}])} \right)$$

Then there exists $c > 0$ such that,

$$\sup_K \sup_{\mathcal{F} \in P(V^*) \setminus [L_{\mu}]} \int_{F(V)} \delta^{-\gamma}([x],[\text{Ker}(f)]) \, d\nu([x]) < +\infty.$$ (27)

In particular, the Hausdorff dimension of $\nu$ is positive unless $[U_{\mu}]$ is a point.
A crucial step is to show that the random walk converges exponentially fast towards its invariant measure, namely:

**Theorem 6.2.** Let $\mu$ be a probability measure on $GL(V)$ with an exponential moment such that $\lambda_1 > \lambda_2$. Let $\nu$ be the unique $\mu$-stationary measure on $P(V)$ such that $\nu([L_\mu]) = 0$. For every compact subset $K$ of $P(V) \setminus [L_\mu]$, we denote by $\eta(K)$ the following finite quantity $\eta(K) := \sup_{|x| \in K} \left( \frac{1}{\nu([x],[L_\mu])] \right)$. Then, there exists a random variable $[Z]$ on the projective space $P(V)$ with law $\nu$, there exist $C > 0$ and $n_0 \in \mathbb{N}^*$ such that for every $n \geq n_0$, for every compact subset $K$ of $P(V) \setminus [L_\mu]$, the following holds:

$$\sup_{|x| \in K} \mathbb{E}(\delta(M_n[x],[Z])) \leq \eta(K) \exp(-nC).$$  \hfill (28)

**Remark 6.3.** The above statement is stronger than just saying that $\nu$ is a $\mu$-boundary. Indeed, it implies that, for any probability measure $\eta$ on $P(V)$ that gives zero mass to $[L_\mu]$, $\mu^n * \eta \xrightarrow{w} \nu$.

When $T_\mu$ is irreducible (or equivalently i-p), Theorem 6.1 was shown in [Gui90] using the spectral gap property [LP82]. Other alternative proofs were then proposed [Aou13], [BQ16b]. When $T_\mu$ is a non degenerate sub-semigroup of the affine group of $k^d$, our result on Hausdorff dimension is new. Here are the main ingredients of the proof.

A first step is Theorem 6.2 above. It consists of showing that $M_n[x]$ converges exponentially fast towards the stationary measure, with exponential speed and uniformly on compact subsets of $P(V) \setminus [L_\mu]$. In the i-p case, this is known (see [BL85] for the convergence and [Aou11] for the speed). For affine groups in the contracting setting, this is straightforward by direct computation. When $\lambda_1 > \lambda_2$ and $G_\mu$ is any group of upper triangular matrix blocs, such as a subgroup of the automorphism of the Heisenberg group (see Section 2.2), this result is new.

The second step is the deterministic Lemma 6.6. This lemma will imply that estimating the distance from $M_n[x]$ to a fixed hyperplane $H$ consists, with probability exponentially close to one, of establishing large deviation estimates of the ratio of norms $\frac{||M_n x||}{||f||}$ uniformly on $f \in V^*$.

In both steps, we need large deviation inequalities for norms ratios. This is done using a classical cocycle lemma (see Lemma 6.4 below). Since we do not need the more delicate large deviation estimates for the norms themselves, we do not aim to give the optimal formulation for the concerning statement (see Corollary 6.5). We refer to [BQ16a] for related estimates for cocycles.

In terms of techniques, we note that even though our result applies to the interesting case $L_\mu \neq \{0\}$ (as the contracting case in the context of affine groups), our proof uses heavily different passages through the easier case $L_\mu = \{0\}$ (as the expansive case for affine groups or the irreducible groups) via group representations. We refer to Remark 3.14 for more on this condition.

### 6.2 Cocycles

We begin by recalling a cocycle lemma: Lemma 6.4 below. The case a) allows us to obtain large deviations estimates of cocycles whose average is negative. It is due to Le Page [LP82] and was crucial in order to establish fine limit theorems for the norm of matrices. Case b) treats the case where the average of the cocycle is zero and appears in [Gui90], [Aou11].

**Lemma 6.4.** (Cocycle)/[LP82, Aou11] Let $G$ be a semi-group acting on a space $X$, $s$ an additive cocycle on $G \times X$, $\mu$ a probability measure on $G$ such that: for $r(g) = \sup_{x \in X} |s(g,x)|$, there exists $\tau > 0$ such that

$$E(exp(\tau r(X_1))) < \infty.$$ \hfill (29)

Set $l = \lim_{n \to \infty} \frac{1}{n} \sup_{x \in X} E(s(M_n, x)).$
(Controlling ratio norms) Let \( \eta \) be a probability measure on GL(V) such that \( \lambda_1 > \lambda_2 \). For every compact subset \( K \) of \( P(V) \setminus [L_\mu] \), we denote by \( \eta(K) \) the following finite quantity:

\[
\eta(K) := \sup_{x \in K} \mathbb{E}_{g} \left[ \frac{||g(x)||}{||g||} \right].
\]

Then, for every \( \epsilon > 0 \) and \( n > n(\epsilon) \), \( \sup_{x \in \mathcal{F}} \mathbb{E} \left[ \exp(\epsilon(s(M_n, x))) \right] \leq (1 + \epsilon)^n \).

**Proof.** For every \( x \in V \), denote by \( \pi \) its projection on the quotient vector space \( V/L_\mu \). Let \( \pi \) be the morphism action of \( G_\mu \) on \( V/L_\mu \), so that \( \pi(g) \pi = \pi g \) for every \( g \in G_\mu \) and \( x \in V \). Let \( G_\mu \) acts naturally on the product space \( X := P(V/L_\mu) \times P(V) \) and denote by \( s \) the function defined on \( G_\mu \times X \) by:

\[
s(g, ([x], [y])) := \log \frac{||\pi([x]), [y])||}{||\pi([x]), [y])||}.\]

It is immediate to see that \( s \) is a cocycle. Since \( \mu \) has an exponential moment, then condition (29) of Lemma 6.4 is satisfied. With the notations of the aforementioned lemma, let us show that \( l = 0 \). Since \( L_{x(\mu)} = \{0\} \) (see Remark 3.14) and \( \lambda_1 (\pi(\mu)) = \lambda_1 (\mu) = \lambda \), then Corollary 4.7 (and Remark 4.8 part 2.) show that:

\[
\inf_{[x] \in \mathcal{F}} \sup_{[y] \in P(V)} \frac{1}{n} \mathbb{E} \left( \log \frac{||\pi(S_n)x||}{||\pi(x)||} \right) = \lambda_1.
\]

Moreover, by Remark 4.8 part 1., \( \sup_{[y] \in P(V)} \frac{1}{n} \mathbb{E} \left( \log \frac{||S_n y||}{||y||} \right) = \lambda_1 (\mu) \). Hence:

\[
l := \lim_{n \to \infty} \frac{1}{n} \sup_{x \in X} \mathbb{E}(s(M_n, x)) = 0.
\]

Let \( \gamma > 0 \). The cocycle lemma shows then that there exist \( r(\gamma) > 0, n(\gamma) \in \mathbb{N}^* \) such that for every \( 0 < r < r(\gamma) \) and \( n > n(\gamma) \),

\[
\sup_{([x], [y]) \in X} \mathbb{E} \left[ \left( \frac{||\pi([x]), [y])||}{||\pi([x]), [y])||} \right)^r \right] \leq (1 + r\gamma)^n.
\]

Let now \( K \) be compact subset of \( P(V) \setminus [L_\mu] \). Since for every \( x \in V \setminus \{0\} \), \( ||gx|| \geq ||\pi(g)x|| \) and \( \delta([x], [L_\mu]) = \frac{||x||}{||x||} \), estimate (31) shows that for every \( 0 < r < r(\gamma) \),

\[
\sup_{x \in K} \mathbb{E} \left[ \left( \frac{||x||}{||S_n x||} \right)^r \right] \leq \eta(K)^r (1 + r\gamma)^n.
\]

Now let \( \epsilon > 0 \) and fix \( [x] \in K \). Chose \( \gamma = \epsilon/2 \) and the corresponding \( r(\gamma) \) given by Lemma 6.4. Then for every \( 0 < r < \min\{r(\gamma), 1\} \),

\[
\mathbb{P} \left( \frac{||S_n x||}{||S_n x||} \leq \exp(-ne) \right) = \mathbb{P} \left[ \left( \frac{||S_n x||}{||S_n x||} \right)^r \geq \exp(ner) \right] \leq \exp(-ner) \mathbb{E} \left( \left( \frac{||S_n x||}{||S_n x||} \right)^r \right) \leq \eta(K)^r \exp(-ner)(1 + re/2)^n \leq \eta(K) \exp(-nre/2).
\]

Inequality (33) follows from Markov’s inequality, (34) follows from (32) while (35) is true since on the one hand, \( 1 + a \leq \exp(a) \) for every real number \( a \) and on the other hand, \( \eta(K) \geq 1 \) and \( r \leq 1 \). The proposition is then proved.

\[
\square
\]
6.3 Exponential convergence in direction

In this section, we prove Theorem 6.2 stated above.

Proof. **Step 1:** First, we check that it is enough to show the following statement: there exists \( C > 0, n_0 \in \mathbb{N}^* \) such that for every compact subset \( K \) of \( P(V) \setminus [L_\mu] \), and every \( n \geq n_0 \),

\[
\sup_{[x] \in K} \mathbb{E}(\delta(M_n[x], M_{n+1}[x])) \leq \eta(K) \exp(-nC).
\]  

(36)

Indeed (36) would imply that for every \( x \notin L_\mu, (M_n[x])_{n \in \mathbb{N}^*} \) is almost surely a Cauchy sequence in the complete space \( P(V) \). Hence, it converges to a random variable \([Z_\alpha] \in P(V)\). The latter is in the image of every convergent subsequence of \((M_n)_{n \in \mathbb{N}^*}\). By Lemma 4.6, \([Z_\alpha]\) is almost surely independent of \( x \) and has law \( \nu \). Now (28) would follow immediately from (36) by applying Fatou’s lemma and the triangular inequality.

**Step 2:** Next, we give an upper bound of the left side of assertion (36). We denote by \( \eta : G_\mu \to \text{GL}(V/L_\mu) \) the morphism action of \( G_\mu \) on \( V/L_\mu \). Fix a compact subset \( K \) of \( P(V) \setminus [L_\mu] \) and \([x] \in K\). Then we have for every \( n \in \mathbb{N}^* \), the following almost sure estimates:

\[
\delta(M_n[x], M_{n+1}[x]) = \delta(M_n[x], M_nX_{n+1}[x])
\]

\[
= \frac{||A^2 M_n(x \wedge X_{n+1})||}{||M_n|| ||M_nX_{n+1}||}
\]

\[
\leq \frac{||A^2 M_n(x \wedge X_{n+1})||}{||\pi(M_n)|| ||\pi(M_n)\pi(X_{n+1})||}
\]

(37)

We let \( G_\mu \) act naturally on the compact space \( M := P(A^2 V) \times P(V/L_\mu)^2 \) and set, for every \( m = ([a], [b], [\pi], [\pi]) \in M \) and every \( g \in G_\mu \),

\[
s(g, m) := \log \frac{||A^2 g(a \wedge b)|| ||\pi|| ||\pi||}{||a \wedge b|| ||\pi(g)\pi|| ||\pi(g)||}.
\]

Hence if \( m_n \) denotes the following random variable in \( M, m_n := ([x], [X_{n+1}], [\pi], [\pi(X_{n+1})\pi]), \) (37) becomes,

\[
\delta(M_n[x], M_{n+1}[x]) \leq \exp(s(M_n, m_n)) \times \frac{||x \wedge X_{n+1}||}{||\pi|| ||\pi(X_{n+1})||}.
\]

(38)

By combining (39), the equality \( \delta(x, L_\mu) = ||\pi|| \) and the inequalities \( ||x \wedge y|| \leq ||x|| ||y||, \)

\[
||gx|| \geq \frac{||x||}{||g||}, \quad ||\pi(g)|| \leq ||g|| \quad \text{for every } x, y \in V \quad \text{and } g \in G_\mu,
\]

we obtain the following almost sure inequality \( (x) \) is always fixed:

\[
\delta(M_n[x], M_{n+1}[x]) \leq \eta(K)^2 ||X_{n+1}|| ||X_{n+1}^{-1}|| \exp(s(M_n, m_n)).
\]

(39)

Using Cauchy-Schwartz inequality and the fact that \( m_n \) and \( M_n \) are independent random variables, we deduce that for every \( \alpha > 0 \) (to be chosen in Step 3 below),

\[
\mathbb{E}\left(\delta^{n/2}(M_n[x], M_{n+1}[x])\right) \leq \eta(K)^\alpha \sqrt{\mathbb{E}\left(||X_{n+1}||^\alpha ||X_{n+1}^{-1}||^\alpha\right)} \sup_{m \in M} \mathbb{E}(\exp(s(M_n, m))).
\]

(40)

**Step 3:** Finally, we check that we are in the case b) of the cocycle lemma (Lemma 6.4). The map \( s : G \times M \to \mathbb{R} \) is clearly a cocycle on \( G \times M \). Since \( \mu \) has an exponential moment, condition 29 is fulfilled. Moreover the representation \( \pi \) satisfies \( L_{\pi(\mu)} = 0 \). Hence, by Corollary 4.7, \( \inf_{[\pi] \in P(V/L_\mu)} \frac{||\pi(M_n)||}{||\pi||} \to \lambda_1 \). Consequently, \( l \leq \lambda_2 - \lambda_1 < 0 \). Cocycle lemma gives then \( \alpha_1 > 0 \) such that for every \( \alpha \in [0, \alpha_1] \) and every large \( n \),

\[
\sup_{m \in M} \mathbb{E}(\exp(s(M_n, m))) \leq \exp(-Cn).
\]

(41)

Since \( \mu \) has an exponential moment, there exists \( \alpha_2 > 0 \) such that for every \( \alpha \in [0, \alpha_2], \)

\[
\mathbb{E}(||X_{n+1}||^\alpha ||X_{n+1}^{-1}||^\alpha) < +\infty.
\]

Apply now (40) for \( \alpha = \min(\alpha_1, \alpha_2, 1) \). Since the Fubini-Study metric \( \delta \) is bounded from above by one and \( \eta(K) \leq 1 \), we obtain the desired estimate (36). Theorem 6.2 is then proved.

\( \square \)
6.4 Proof of the regularity of the stationary measure

We begin with the following deterministic lemma.

**Lemma 6.6.** Let $k$ be a local field, $V$ a vector space over $k$ of dimension $d \geq 2$ endowed with the norm described in Section 3.1.2, $L$ a subspace of $V$ and $F$ be a basis of any supplementary of $L$ in $V$ with vectors of norm $1$. Let $C(k,d) = \frac{1}{\sqrt{d}}$ if $k$ is Archimedean and $C(k,d) = 1$ otherwise. Then for any $g \in GL(V)$ such that $g(L) = L$ and for any $f \in V^* \setminus \{0\}$, there exists $x \in F$ such that:

$$\delta (g[x], Ker(f)) \geq C \left( \frac{||gf||}{||g||} \right)^{1-c \frac{||g||}{||f||}} \cdot \max_{i=1}^d \frac{|f(e_i)|}{||g||}$$

Proof. Let $d' = \dim(L)$ and $B = \{e_1, \ldots, e_d\}$ be a basis of $V$ where $||e_i|| = 1$ for every $i$, $B' := \{e_1, \ldots, e_d'\}$ is a basis of $L$ and $F = B \setminus B'$ a basis for some supplementary of $L$ in $V$. Assume first that $k$ is non-Archimedean. Then the following relation is true for every $g \in GL(V)$,

$$\frac{||gf||}{||g||} = \max_{1 \leq i \leq d'} \frac{|f(e_i)|}{||g||}$$

$$\leq \max_{1 \leq i \leq d'} \frac{||g||}{||g||} \cdot \max_{i=1}^d \frac{|f(e_i)|}{||g||} \cdot \max_{i=1}^d \frac{||g||}{||g||} = \max_{i=1}^d\delta(g[e_i], Ker(f))$$

Equality (42) holds because $||g'|| = ||g||$ and inequality (43) is true because for any $x \in V$, $\frac{||f(x)||}{||f||} \leq ||x||$ and

$$\delta (g[x], Ker(f)) = \frac{|f(gx)|}{||g||} \leq \frac{|f(gx)|}{||g||}.$$

Estimate (43) shows that the lemma is true for $C = 1$. When $k$ is Archimedean, estimate (43) is replaced by:

$$\left( \frac{||gf||}{||g||} \right)^2 \leq \left( \frac{||g||}{||g||} \right)^2 + \sum_{i=d'+1}^d \delta (g[e_i], Ker(f))^2.$$

Hence, for any $C < 1$,

$$C \frac{||gf||}{||g||} \geq \frac{||g||}{||g||} \Rightarrow \max_{1 \leq i \leq d'} \delta (g[e_i], Ker(f)) \geq \sqrt{1 - C^2 \frac{||gf||}{||g||}}.$$

The constant $C(k,d) := \frac{1}{\sqrt{d}}$ solves the equation $C = \sqrt{1 - C^2}$. \qed

**Proof of Theorem 6.1:** Let $[Z] \in P(V)$ be the random variable given by Theorem 6.2. Let $K$ be a compact subset of $P(V^* \setminus [L_\mu])$, $f \in K$ and $H := Ker(f)$. Since the Lyapunov exponent of the restriction to $L_\mu$ is less than $\lambda_1$, one can show using the same techniques as the proof of Corollary 6.5 that there exists $\rho_1 \in (0,1)$ such that $\frac{||M_n||}{||M_n||} \leq \rho_1^n$, with probability tending to one exponentially fast. Corollary 6.5 applied for the measure $\mu$ shows then that for any $C > 0$, there exists $\rho_2 \in (0,1)$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$P \left( C \frac{||M_n f||}{||M_n f||} \geq \frac{||M_n L||}{||M_n L||} \right) \geq 1 - \eta(K) \rho_2^n.$$

Take now $C$ to be the constant $C(k,d)$ given by Lemma 6.6. The aforementioned lemma together with estimate (44) imply that, for $\rho_3 := \max\{\rho_1, \rho_2\}$, the following is true for every $n \geq n_0$:

$$P \left( \exists x \in F; \delta (M_n[x], [H]) \geq C \frac{||M_n f||}{||M_n f||} \right) \geq 1 - \eta(K) \rho_3^n.$$

Hence, for every $t \in [0,1]$. 29
\[ P(\delta([Z],[H]) \leq t^n) \leq \eta(K)\rho_3^n + \sum_{x \in F} P \left( C \frac{||M^*_n f||}{||M^*_n||||f||} \leq t^n + \delta(M_n[x],[Z]) \right). \]

But by Theorem 6.2 and the Markov’s inequality, one deduces that there exists \( \rho_4, \rho_5 \in (0,1) \) such that:

\[ P(\delta([Z],[H]) \leq t^n) \leq \eta(K)\rho_4^n + \sum_{x \in F} P \left( \frac{||M^*_n f||}{||M^*_n||||f||} \leq \rho_5^n \right). \]

Using Corollary 6.5, we deduce finally that for every \( t \in (0,1) \) and \( n_0 \in \mathbb{N} \) such that for every \( n \geq n_0 \) and every compact \( K \) of \( P(V^*) \setminus [L_\mu] \), one has:

\[ \sup_{f \in K} P(\delta([Z],[\text{Ker}(f)]) \leq t^n) \leq \eta(K)t^{n_0}. \]

(45)

Fix now such a \( t \). Let \( A_n = \{ x \in P(V) ; \delta(x,[H]) \in (t^{n+1},t^n) \} \). Since \( (A_n)_{n \in \mathbb{N}} \) cover \( P(V) \), estimate (45) gives for any \( c > 0 \),

\[
\int_{P(V)} \delta^{-c}(x,[H]) \, dv(x) = \sum_{n=0}^{n_0-1} \int_{A_n} \delta^{-c}(x,[H]) \, dv(x) + \sum_{n=n_0}^{+\infty} \int_{A_n} \delta^{-c}(x,[H]) \, dv(x)
\]

\[
\leq \eta(K) \sum_{n=n_0}^{+\infty} t^{-c(n+1)} t^n
\]

\[
\leq \eta(K) \left( \frac{t^{n_0}}{t^n} + \frac{t^{-c}}{t^n} \sum_{n=n_0}^{+\infty} \left( \frac{t'}{t^n} \right)^n \right)
\]

Hence \( \eta(K)^{-1} \int_{P(V)} \delta^{-c}(x,[H]) \, dv(x) \) is finite (and independent of the compact \( K \) and of \( f \in V^* \)) as soon as \( 0 < c < \frac{\log t'}{\log t} \). \( \square \)

Finally, we show how to conclude easily from Theorem 6.1 the proof of some results stated in Section 2.4. Proposition 2.12 concerning the exponential decay of the probability of hitting a hyperplane follows immediately from Theorem 6.1 and Theorem 6.2 proved above. Also, Corollary 2.10 concerning the positivity of the Hausdorff dimension of the unique stationary measure in the context of affine groups in the contracting case follows also from Theorem 6.1 and Example 2 of Section 2.2.2.

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