Induction Functor in Non-commutative Equivariant Cohomology and Dirac Cohomology

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Dedicated to Bertram Kostant on his seventy fifth birthday

Abstract

The aim of this paper is to put some recent results of Huang-Pandžić (conjectured by Vogan) and Kostant on Dirac cohomology in a broader perspective. This is achieved by introducing an induction functor in the noncommutative equivariant cohomology. In this context, the results of Huang-Pandžić and Kostant are interpreted as special cases (corresponding to the manifold being a point) of more general results on noncommutative equivariant cohomology introduced by Alekseev-Meinrenken.

Introduction

Let $G$ be a (not necessarily connected) real Lie group and let $R$ be a closed subgroup with their complexified Lie algebras $\mathfrak{g}$ and $\mathfrak{r}$ respectively. We assume that there exists a $G$-invariant nondegenerate symmetric bilinear form $B_{\mathfrak{g}}$ on $\mathfrak{g}$ such that $B_{\mathfrak{g}|\mathfrak{r}}$ is again nondegenerate. We will impose this restriction on $G$ and $R$ throughout the paper. Let $\mathfrak{p}$ be the orthocomplement $\mathfrak{r}^\perp$ of $\mathfrak{r}$ in $\mathfrak{g}$. Then $B_{\mathfrak{g}|\mathfrak{r}}$ being nondegenerate we have $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p}$ and, moreover, $B_{\mathfrak{g}|\mathfrak{p}}$ is again nondegenerate. Further $\mathfrak{p}$ is $R$-stable under the adjoint action. For example, any compact Lie group $G$ and a closed subgroup $R$ satisfies the above restriction.

Let $M$ be a smooth $R$-manifold. Then the deRham complex $\Omega(M)$ of $M$ is canonically a $\mathbb{Z}_+$-graded (and hence $\mathbb{Z}/(2)$-graded) $R$-differential
algebra. We will only consider \( \mathbb{Z}/(2) \)-graded spaces, algebras etc., so, in the sequel, by graded we will mean \( \mathbb{Z}/(2) \)-graded. We define a certain induction functor in noncommutative equivariant cohomology which associates to the \( R \)-differential algebra \( \Omega(M) \) a differential graded algebra \( \text{Ind}_{G/R}(\Omega(M)) \). By definition,

\[
\text{Ind}_{G/R}(\Omega(M)) = (\mathcal{W}(\mathfrak{g}) \otimes \Omega(M))_R,
\]

where \( \mathcal{W}(\mathfrak{g}) := U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{g}) \) is the noncommutative Weil algebra (cf. §1), \( U(\mathfrak{g}) \) is the enveloping algebra, \( \text{Cl}(\mathfrak{g}) \) is the Clifford algebra of \( \mathfrak{g} \) with respect to the form \( B_\mathfrak{g} \), and the subscript \( R \) refers to the subspace of \('R\)-basic' elements (cf. §1). The differential graded algebra structure on \( \text{Ind}_{G/R}(\Omega(M)) \) is the restriction of the tensor product differential graded algebra structure on \( \mathcal{W}(\mathfrak{g}) \otimes \Omega(M) \). We prove that the differential graded algebra \( \text{Ind}_{G/R}(\Omega(M)) \) is canonically isomorphic (as a differential graded algebra) with the noncommutative \( G \)-equivariant Cartan model \((U(\mathfrak{g}) \otimes \Omega(M_G))^G\) of the \( G \)-manifold \( M_G := G \times^R M \) (cf. Theorem 2.2). From this isomorphism, we obtain (as an immediate corollary) that the cohomology \( H(\text{Ind}_{G/R}(\Omega(M)) \) is canonically isomorphic with the noncommutative \( G \)-equivariant cohomology \( H_G(M_G) \) as graded algebras.

We use the above isomorphism to construct a functorial graded linear cochain map \( \Phi_M : \text{Ind}_{G/R}(\Omega(M)) \rightarrow (U(\mathfrak{r}) \otimes \Omega(M))^R \), where the latter is the noncommutative \( R \)-equivariant Cartan model of the \( R \)-manifold \( M := G \times^R M \). Further, we show that \( \Phi_M \) induces an algebra isomorphism in cohomology, even though, in general, \( \Phi_M \) by itself is not an algebra homomorphism. As a corollary, we obtain a functorial graded algebra isomorphism \( \mathcal{H}_G(M_G) \simeq \mathcal{H}_R(M) \).

We now specialize the above results to the case when \( M \) is the one point manifold \( M^0 \) to obtain some important recent results of Huang-Pandžić and Kostant on Dirac cohomology ([Ko3], [HP]). In more detail, taking \( M = M^0 \),

\[
\text{Ind}_{G/R}(\Omega(M^0)) \cong (U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^R.
\]

We show that the differential \( d \) on \( \text{Ind}_{G/R}(\Omega(M^0)) \) corresponds under the above isomorphism with the differential \( \text{ad}^\mathfrak{p} D^\mathfrak{p} \) on the right side introduced by Kostant, where \( D^\mathfrak{p} \in (U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^R \) is his remarkable cubic Dirac operator \( \sum p_\ell \otimes q_\ell + 1 \otimes \gamma_\mathfrak{p} \), where \( \{p_\ell\}_\ell \) is any basis of \( \mathfrak{p} \) and \( \{q_\ell\}_\ell \) is the dual basis with respect to \( B_\mathfrak{p} \) and \( \gamma_\mathfrak{p} \) is the Cartan element in \( \wedge^3(\mathfrak{p}) \) under the standard identification \( \wedge(\mathfrak{p}) \simeq \text{Cl}(\mathfrak{p}) \). Recall that in the case when \( R \) is a maximal compact subgroup of reductive \( G \), then \( \gamma_\mathfrak{p} = 0 \) and the operator \( D^\mathfrak{p} \) reduces to the Dirac operator considered by Vogan in defining his Dirac
cohomology. Thus our theorem in the case $M = M^o$ gives that

\[(\ast) \quad H((U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^R, \text{ad} \mathcal{D}^p) \simeq \mathcal{H}_R(M^o) \simeq U(\mathfrak{r})^R,\]

which was proved by Huang-Pandžić in the case when $R$ is a maximal compact subgroup of a connected reductive $G$ and by Kostant in the general connected reductive case, i.e., when $G$ and $R$ are connected and reductive (and of course $B_{\mathfrak{g}_0}$ is nondegenerate). In fact from ($\ast$) one obtains the decomposition

$$\text{Ker}(\text{ad} \mathcal{D}^p) = \xi(Z(R)) \oplus \text{Image}(\text{ad} \mathcal{D}^p),$$

where the homomorphism $\xi : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p})$ is induced from the adjoint action of $\mathfrak{r}$ on $\mathfrak{g}$ and $\mathfrak{p}$ and $Z(R)$ is the subalgebra of $R$-invariants $U(\mathfrak{r})^R$. Also the isomorphism ($\ast$) gives rise to an algebra homomorphism $\eta_R : Z(G) \to Z(R)$. We show that, from the general properties of the Duflo isomorphism, $\eta_R$ is the unique homomorphism making the following diagram commutative:

$$
\begin{array}{ccc}
Z(G) & \xrightarrow{\eta_R} & Z(R) \\
\downarrow H_\mathfrak{g} & & \downarrow H_\mathfrak{g} \\
S(\mathfrak{g})^G & \longrightarrow & S(\mathfrak{r})^R,
\end{array}
$$

where $H_\mathfrak{g}$ is the Harish-Chandra isomorphism and the bottom horizontal map is induced by the orthogonal projection $\mathfrak{g} \to \mathfrak{r}$.

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# 1 Review of non-commutative equivariant cohomology after Alekseev-Meinrenken

Unless otherwise explicitly stated, by vector spaces we mean complex vector spaces and by linear maps complex linear maps.

Let $G$ be a (not necessarily connected) real Lie group with complexified Lie algebra $\mathfrak{g}$. We assume that $\mathfrak{g}$ has a nondegenerate symmetric $G$-invariant bilinear form $B_{\mathfrak{g}}$ on $\mathfrak{g}$, often denoted as $\langle \cdot, \cdot \rangle$. Define the $\mathbb{Z}$-graded super-Lie algebra $\hat{\mathfrak{g}}^*$ as follows. As a vector space,

$$
\hat{\mathfrak{g}}^{-1} = \mathfrak{g}^0 = \mathfrak{g}, \quad \hat{\mathfrak{g}}^1 = \mathbb{C}, \quad \hat{\mathfrak{g}}^n = 0 \text{ if } n \neq -1, 0, 1.
$$

3
For \( a \in \mathfrak{g} \), the corresponding element in \( \hat{\mathfrak{g}}^{-1} \) (resp. \( \hat{\mathfrak{g}}^0 \)) will be denoted by \( i_a \) (resp. \( L_a \)) and they represent ‘contraction’ and ‘Lie derivation’ respectively. We denote the generator of \( \hat{\mathfrak{g}}^1 \) by \( d \). The bracket relations in \( \hat{\mathfrak{g}}^* \) are defined by (for \( a, b \in \mathfrak{g} \)):

\[
[L_a, i_b] = i_{[a,b]},
\]
\[
[L_a, L_b] = L_{[a,b]},
\]
\[
[i_a, d] = L_a.
\]

By a super-space we mean a \( \mathbb{Z}/(2) \)-graded space. Any \( \mathbb{Z} \)-graded space of course has a canonical \( \mathbb{Z}/(2) \)-grading by even and odd components.

Recall that a \( G \)-differential space is a super-space \( B \) which is a Fréchet space, together with a graded smooth action of \( G \) on \( B \) and a super-Lie algebra homomorphism \( \theta : \hat{\mathfrak{g}}^* \to \text{End}_{\text{Cont}} B \), where \( \text{End}_{\text{Cont}} B \) denotes the continuous linear endomorphisms of \( B \). Moreover, we assume that the action of \( G \) commutes with \( d \), \( L_a \) is the derivative of the \( G \)-action and \( g i_a g^{-1} = i_{g \cdot a} \) for all \( g \in G, a \in \mathfrak{g} \).

The horizontal subspace \( B_{\text{hor}} \) is the space annihilated by \( \hat{\mathfrak{g}}^{-1} \), the invariant subspace \( B^G \) is the subspace invariant under \( G \) and the space \( B_G \) of basic elements is the intersection \( B_{\text{hor}} \cap B^G \).

A \( G \)-differential algebra is a super-algebra \( B \) together with the structure of a \( G \)-differential space on \( B \) such that \( \theta \) takes values in the superderivations of \( B \) and the action of \( G \) on \( B \) is via algebra automorphisms.

For a smooth \( G \)-manifold \( M \), \( \Omega(M) \) with the Fréchet topology provides the most important class of examples of \( G \)-differential algebras.

A homomorphism between \( G \)-differential spaces (resp. algebras) \((B_1, \theta_1)\) and \((B_2, \theta_2)\) is a continuous homomorphism of super-spaces (resp. algebras) \( \phi : B_1 \to B_2 \) such that

\[
\phi(g \cdot b) = g \cdot \phi(b), \quad \phi(\theta_1(x)b) = \theta_2(x) \phi(b), \quad \text{for } g \in G, x \in \hat{\mathfrak{g}}^* \text{ and } b \in B_1.
\]

There is the (classical) Weil algebra \( W(\mathfrak{g}) := S(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*) \) with the tensor product algebra structure. This is a \( G \)-differential algebra under the \( \mathbb{Z}_+ \)-grading

\[
W(\mathfrak{g})^n = \bigoplus_{k \geq 0} S^k(\mathfrak{g}^*) \otimes \wedge^{n-2k}(\mathfrak{g}^*).
\]

The action of \( G \) is via the coadjoint action. The operators \( L_a \) come from the coadjoint action of \( \mathfrak{g} \) on \( S(\mathfrak{g}^*) \) and \( \wedge(\mathfrak{g}^*) \). The contraction operator \( i_a \) on
\(W(\mathfrak{g})\) is defined as \(I_{S(\mathfrak{g}^\ast)} \otimes \varepsilon'_a, i'_a\) being the standard contraction operator on \(\wedge(\mathfrak{g}^\ast)\). The differential \(d\) on \(W(\mathfrak{g})\) is the unique super-derivation satisfying (for any \(f \in \mathfrak{g}^\ast\))
\[
d(1 \otimes e_f) = 1 \otimes d(\wedge e_f) + s_f \otimes 1,
\]
where \(e_f\) (resp. \(s_f\)) is the element \(f\) considered as an element of \(\mathfrak{g}^\ast \subset \wedge(\mathfrak{g}^\ast)\) (resp. \(\mathfrak{g}^\ast \subset S(\mathfrak{g}^\ast)\)) and \(d_\wedge : \wedge(\mathfrak{g}^\ast) \to \wedge(\mathfrak{g}^\ast)\) is the standard Koszul differential.

We now recall the definition of the non-commutative Weil algebra \(W(\mathfrak{g})\) from [AM], which is a \(G\)-differential algebra. Recall that the Clifford algebra \(\text{Cl}(\mathfrak{g})\) of \(\mathfrak{g}\) with respect to the bilinear form \(B_{\mathfrak{g}}\) is the quotient of the tensor algebra \(T(\mathfrak{g})\) of \(\mathfrak{g}\) by the two-sided ideal generated by \(2x \otimes x - \langle x, x \rangle\), \(x \in \mathfrak{g}\). As a super-space it is defined as the tensor product of algebras
\[
W(\mathfrak{g}) := U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{g}),
\]
where the \(\mathbb{Z}/(2)\)-grading on \(W(\mathfrak{g})\) comes from the standard grading on the Clifford algebra \(\text{Cl}(\mathfrak{g})\), and \(U(\mathfrak{g})\) is the enveloping algebra of \(\mathfrak{g}\) placed in the even degree part. Both of \(U(\mathfrak{g})\) and \(\text{Cl}(\mathfrak{g})\) are \(G\)-modules under the adjoint action and so is their tensor product. For \(a \in \mathfrak{g}\), let \(L_a\) be the adjoint action on \(W(\mathfrak{g})\).

Recall that there is a vector space isomorphism given by the symbol map
\[
\sigma : \text{Cl}(\mathfrak{g}) \to \wedge(\mathfrak{g}),
\]
where \(\sigma^{-1}\) is induced from the standard projection map \(T(\mathfrak{g}) \to \text{Cl}(\mathfrak{g})\) under the identification of \(\wedge(\mathfrak{g})\) with the skew-symmetric tensors in \(T(\mathfrak{g})\). From now on we will identify \(\text{Cl}(\mathfrak{g})\) with \(\wedge(\mathfrak{g})\) (via the symbol map) as a vector space. Under this identification, we will denote the product in \(\wedge(\mathfrak{g})\) by \(\otimes\), i.e.,
\[
x \otimes y = \sigma(\sigma^{-1}(x) \cdot \sigma^{-1}(y)), \text{ for } x, y \in \wedge(\mathfrak{g}).
\]
The exterior product in \(\wedge(\mathfrak{g})\) will be denoted by \(x \wedge y\). Recall that \(\wedge(\mathfrak{g})\) admits the contraction operator \(i_a\) (for \(a \in \mathfrak{g}\)) which is a super-derivation induced from the operator
\[
i_a b = \langle a, b \rangle, \text{ for } b \in \mathfrak{g}.
\]
Then, by [AM, Lemma 3.1], explicitly the product \(\otimes\) in \(\wedge(\mathfrak{g})\) is given by (for \(\omega, \eta \in \wedge(\mathfrak{g})\)):
\[
\omega \otimes \eta = \mu \left( \exp \left( -\frac{1}{2} \sum_{k} i^1_{ak} i^2_{bk} \right) (\omega \otimes \eta) \right),
\]
where \( \{a_k\}_k \) is any basis of \( \mathfrak{g} \) and \( \{b_k\}_k \) is the dual basis \( \{a_k, b_k\} = \delta_{k,\ell}, \mu : \wedge(\mathfrak{g}) \otimes \wedge(\mathfrak{g}) \to \wedge(\mathfrak{g}) \) is the standard wedge product, \( i^{1}_{a_k}(\omega \otimes \eta) := (\bar{i}_{a_k}\omega) \otimes \eta, i^{2}_{a_k}(\omega \otimes \eta) = \omega \otimes i_{a_k}\eta. \)

Define the operator \( i_a, a \in \mathfrak{g}, \) on \( \mathcal{W}(\mathfrak{g}) \) by
\[
i_a = I_{U(\mathfrak{g})} \otimes \bar{i}_a.
\]

For \( a \in \mathfrak{g}, \) let \( u_a \) (resp. \( c_a \)) be the corresponding element in \( \mathfrak{g} \subset U(\mathfrak{g}) \) (resp. \( \mathfrak{g} \subset \text{Cl}(\mathfrak{g}) \)). We also think of \( u_a \) (resp. \( c_a \)) as the element \( u_a \otimes 1 \) (resp. \( 1 \otimes c_a \)) of \( \mathcal{W}(\mathfrak{g}) \).

Finally, we define the differential \( d : \mathcal{W}(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g}) \) as the commutator
\[
dx = \text{ad} \, D(x),
\]
where \( D \in \mathcal{W}(\mathfrak{g}) \) is defined by
\[
D = \sum_k u_{a_k}c_{b_k} - 1 \otimes \gamma,
\]
\( \gamma = \gamma_0 \in \wedge^3(\mathfrak{g})^G \) is the \( G \)-invariant element (so called the Cartan element) defined by
\[
\gamma(a, b, c) = \langle a, [b, c] \rangle, \quad \text{for } a, b, c \in \mathfrak{g}
\]
under the identification \( \wedge(\mathfrak{g}) \simeq \wedge(\mathfrak{g}^*) \) induced from the form \( \langle \ , \ \rangle \), and \( \text{ad} \, D \) is the super-adjoint action defined by \( \text{ad} \, D(x) = Dx - xD \), for \( x \in \mathcal{W}(\mathfrak{g})^{\text{even}} \), and \( \text{ad} \, D(x) = Dx + xD \), for \( x \in \mathcal{W}(\mathfrak{g})^{\text{odd}} \).

Then \( \mathcal{W}(\mathfrak{g}) \) with the above operators \( i_a, L_a, d = \text{ad} \, D \) and the adjoint action of \( G \) becomes a \( G \)-differential algebra called the non-commutative Weil algebra.

By [AM, Proposition 3.7 and the equation (3)], \( d \) is given by the formula
\[
d(x \otimes \omega) = - \text{ad} \, u_{a_k}(x) \otimes c_{b_k} \wedge \omega - \left( \frac{u_{a_k}x + x u_{a_k}}{2} \right) \otimes \bar{i}_{b_k} \omega + x \otimes d_A \omega + \frac{1}{4} x \otimes \bar{i}_{\gamma} \omega, \quad \text{for } x \in U(\mathfrak{g}), \omega \in \wedge(\mathfrak{g}),
\]
where \( d_A \) is the Koszul differential on \( \wedge(\mathfrak{g}) \) of degree +1 under the identification \( \wedge(\mathfrak{g}) \simeq \wedge(\mathfrak{g}^*) \).

Recall [C] that the \( G \)-equivariant cohomology \( H_G(B) \) of a \( G \)-differential algebra \( B \) is by definition the cohomology of the basic subalgebra \( (\mathcal{W}(\mathfrak{g}) \otimes B)_G \) of the tensor product \( G \)-differential algebra \( \mathcal{W}(\mathfrak{g}) \otimes B \) under the tensor product differential \( d(x \otimes y) = dx \otimes y + (-1)^{\text{deg} x} x \otimes dy, \) for \( x \in \mathcal{W}(\mathfrak{g}) \) and \( y \in B \).
Similarly, following [AM], the non-commutative $G$-equivariant cohomology $\mathcal{H}_G(B)$ is the cohomology of the basic subalgebra $(\mathcal{W}(\mathfrak{g}) \otimes B)_G$ under the tensor product differential. Then, clearly, $\mathcal{H}_G(B)$ is a super-algebra.

1.1 Proposition. For any $G$-differential algebra $B$, the projection map $\theta : \mathcal{W}(\mathfrak{g}) \otimes B \rightarrow S(\mathfrak{g}^*) \otimes B$, induced from the standard augmentation map $\wedge(\mathfrak{g}^*) \rightarrow \mathbb{C}$, induces an algebra isomorphism (again denoted by)

$$\theta : (\mathcal{W}(\mathfrak{g}) \otimes B)_G \sim \rightarrow (S(\mathfrak{g}^*) \otimes B)^G.$$ 

Under the above isomorphism, the differential $d$ corresponds to the differential $d_G$ on $(S(\mathfrak{g}^*) \otimes B)^G$ given as follows:

$$d_G = I_{S(\mathfrak{g}^*)} \otimes d - \sum_k s_{a_k^*} \otimes i_{a_k},$$

where $\{a_k\}_k$ is a basis of $\mathfrak{g}$ and $\{a_k^*\}_k$ is the dual basis of $\mathfrak{g}^*$ and $s_{a_k^*}$ denotes the operator acting on $S(\mathfrak{g}^*)$ via the multiplication by $a_k^*$.

Similarly, we have the following proposition from [AM, §4.2].

1.2 Proposition. For any $G$-differential algebra $B$, the projection map

$$\Theta : \mathcal{W}(\mathfrak{g}) \otimes B \rightarrow U(\mathfrak{g}) \otimes B,$$

induced from the standard augmentation map $\wedge(\mathfrak{g}) \rightarrow \mathbb{C}$, induces a vector space (but not in general algebra) isomorphism

$$\Theta : (\mathcal{W}(\mathfrak{g}) \otimes B)_G \sim \rightarrow (U(\mathfrak{g}) \otimes B)^G.$$ 

To distinguish, let $(U(\mathfrak{g})^\bullet \otimes B)^G$ denote the vector space $(U(\mathfrak{g}) \otimes B)^G$ with the new product $\otimes$ making $\Theta$ an algebra isomorphism.

Under the above isomorphism, the differential $d$ corresponds to the differential

$$d_G = I_{U(\mathfrak{g})} \otimes d - \frac{1}{2} \sum_k (u_{a_k}^L + u_{a_k}^R) \otimes i_{a_k} + \frac{1}{4} I_{U(\mathfrak{g})} \otimes i_{\gamma},$$

where $\gamma \in \wedge^3(\mathfrak{g})^G$ is defined earlier, $u_{a_k}^L$ (resp. $u_{a_k}^R$) denotes the left (resp. right) multiplication in $U(\mathfrak{g})$ by $u_{a_k}$ and $\{a_k\}_k, \{b_k\}_k$ are dual bases of $\mathfrak{g}$. 

7
By [AM, Proposition 4.3], explicitly the multiplication $\circ$ in $(U(\mathfrak{g}) \otimes B)^G$ is given as the restriction of the multiplication (again denoted by) $\circ$ in $U(\mathfrak{g}) \otimes B$ defined as follows. For $x, y \in U(\mathfrak{g})$, $b_1, b_2 \in B$,

\[
(x \otimes b_1) \circ (y \otimes b_2) = xy \otimes \mu \left( \exp \left( -\frac{1}{2} \sum_k i_{a_k}^1 i_{b_k}^2 (b_1 \otimes b_2) \right) \right),
\]

where $i_{a_k}^1$ and $i_{b_k}^2$ are the contraction operators on $B \otimes B$ with respect to the first and second factors respectively and $\mu : B \otimes B \to B$ is the multiplication map.

As in [AM], there exists a $G$-module isomorphism (depending only on $g$), called the quantization map, $Q = Q_g : W(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g})$ which intertwines all the operators $L_a, i_a$ and $d$. $Q|_{S(\mathfrak{g}^*)}$ is the composite of the isomorphisms $S(\mathfrak{g}^*) \to S(\mathfrak{g}) \xrightarrow{D_{\mathfrak{g}}} U(\mathfrak{g})$, where the first map is the algebra isomorphism induced from the isomorphism $\mathfrak{g}^* \to \mathfrak{g}$ (coming from $\langle , \rangle$) and $D_{\mathfrak{g}}$ is the Duflo isomorphism [D]. (Recall that $D_{\mathfrak{g}}$ is only a linear isomorphism from $S(\mathfrak{g})$ to $U(\mathfrak{g})$ but is an algebra isomorphism restricted to $S(\mathfrak{g})^\mathfrak{g}$ onto the center $U(\mathfrak{g})^\mathfrak{g}$.) Moreover, it maps isomorphically $S(\mathfrak{g})^G$ onto $U(\mathfrak{g})^G$. Also recall that, for $a \in S^p(\mathfrak{g})$, $D_{\mathfrak{g}}(a) = \Sigma(a)$ (mod $U(\mathfrak{g})^{p-1}$), where $U(\mathfrak{g})^p$ is the standard filtration of $U(\mathfrak{g})$ and $\Sigma : S(\mathfrak{g}) \to U(\mathfrak{g})$ is the standard symmetrization map. Also $Q|_{\wedge(\mathfrak{g}^*)}$ is the isomorphism (induced from $\langle , \rangle$)

\[
\wedge(\mathfrak{g}^*) \xrightarrow{\sim} \wedge(\mathfrak{g}).
\]

Of course, as earlier, we have identified $\text{Cl}(\mathfrak{g})$ with $\wedge(\mathfrak{g})$ via the symbol map $\sigma$.

However, $Q \neq Q|_{S(\mathfrak{g}^*)} \otimes Q|_{\wedge(\mathfrak{g}^*)}$ in general.

1.3 Theorem. [AM, Theorem 7.1] For any $G$-differential algebra $B$, the cochain map $Q \otimes I_B : W(\mathfrak{g}) \otimes B \to \mathcal{W}(\mathfrak{g}) \otimes B$ induces an algebra isomorphism in cohomology $Q^B : H_G(B) \xrightarrow{\sim} H_G(B)$.

1.4 Definition. Let $\hat{Q}^B_G = \hat{Q}^B : (S(\mathfrak{g}^*) \otimes B)^G \to (U(\mathfrak{g})^\bullet \otimes B)^G$ be the unique
map making the following diagram commutative:

\[
(P(g) \otimes B)_G \xrightarrow{Q \otimes I_B} (P(g) \otimes B)_G \\
\downarrow \theta \quad \quad \quad \quad \quad \downarrow \Theta \\
(S(g^*) \otimes B)^G \xrightarrow{\hat{Q}^B} (U(g) \otimes B)^G.
\]

Then clearly \(\hat{Q}^B\) is a cochain isomorphism. In general, \(\hat{Q}^B\) is not an algebra homomorphism.

2 An induction functor in non-commutative equivariant cohomology

Let \(G\) be a real (not necessarily connected) Lie group with complexified Lie algebra \(g\) and let \(R\) be a closed subgroup of \(G\) with complexified Lie algebra \(r\). Assume that \(g\) admits a \(G\)-invariant nondegenerate symmetric bilinear form \(B_g = \langle , \rangle\) such that \(B_{g|_r}\) is nondegenerate. We call such a pair of \((G, R)\) a quadratic pair. Thus we have the decomposition

\[g = r \oplus p, \quad p := r^\perp.\]

By the \(G\)-invariance of \(B_g\), \(p\) is \(R\)-stable under the adjoint action. Moreover, \(B_{g|_r}\) also is nondegenerate.

The following definition is influenced by the corresponding definition in the (commutative) equivariant cohomology given in [KV, Definition 32].

2.1 Definition (Induction functor). For an \(R\)-differential complex \(B\), define the cochain complex

\[\text{Ind}_{G/R}(B) = (\mathcal{W}(g) \otimes B)_R\]

equipped with the standard tensor product differential

\[d(x \otimes y) = d_{\mathcal{W}} x \otimes y + (-1)^{\deg x} x \otimes d_{B} y, \quad x \in \mathcal{W}(g), y \in B,\]

where \(d_{\mathcal{W}}\) is the differential in \(\mathcal{W}(g)\) and \(d_B\) is the differential in \(B\).

Since \(\mathcal{W}(g)\) is a \(G\) (in particular \(R\)) differential complex and \(B\) is a \(R\)-differential complex, it is easy to see that indeed \(d\) keeps the \(R\)-basic subspace of \(\mathcal{W}(g) \otimes B\) stable.

If \(B\) is a \(R\)-differential algebra, then \(\text{Ind}_{G/R}(B)\) is a differential algebra under the tensor product super-algebra structure on \(\mathcal{W}(g) \otimes B\).
Let \( M \) be a smooth real \( R \)-manifold and let \( \Omega(M) \) be the complexified deRham complex of \( M \). Consider the fiber product \( G \)-manifold \( M_G := G \times^R M \), where \( G \) acts on \( M_G \) via the left multiplication on the first factor.

2.2 Theorem. There exists a graded algebra isomorphism

\[
\psi_M : (U(\mathfrak{g}) \otimes \Omega(M))^G \rightarrow \text{Ind}_{G/R}(\Omega(M))
\]

commuting with the differentials, where \((U(\mathfrak{g}) \otimes \Omega(M))^G\) is equipped with the Cartan differential \( d \) (cf. Proposition 1.2).

Moreover, \( \psi_M \) is functorial in the sense that for any \( R \)-equivariant smooth map \( f : M \rightarrow N \), the following diagram is commutative:

\[
\begin{array}{ccc}
(U(\mathfrak{g}) \otimes \Omega(N))^G & \xrightarrow{\psi_N} & \text{Ind}_{G/R}(\Omega(N)) \\
I_{U(\mathfrak{g}) \otimes f^*_G} & \downarrow & I_{W(\mathfrak{g}) \otimes f^*} \\
(U(\mathfrak{g}) \otimes \Omega(M))^G & \xrightarrow{\psi_M} & \text{Ind}_{G/R}(\Omega(M))
\end{array}
\]

where \( f^* : \Omega(N) \rightarrow \Omega(M) \) and \( f^*_G : \Omega(N) \rightarrow \Omega(M) \) are the induced maps from \( f \).

Proof. Under the projection map \( p : G \times M \rightarrow M_G \), we can identify

\[ \Omega(M_G) \subset \Omega(G \times M). \]

For \( \omega \in \Omega(M_G) \), by \( \omega(1) \) we mean the evaluation of \( \omega \) at \( 1 \times M \). Thus \( \omega(1) \in \wedge(\mathfrak{g}^*) \otimes \Omega(M) \). Under the identification \( \wedge(\mathfrak{g}^*) \simeq \wedge(\mathfrak{g}) \) (induced from the bilinear form \( B_{\mathfrak{g}} \)), we can (and will) think of \( \omega(1) \in \wedge(\mathfrak{g}) \otimes \Omega(M) \). Thus we get the map

\[ U(\mathfrak{g}) \otimes \Omega(M_G) \rightarrow W(\mathfrak{g}) \otimes \Omega(M), \]

\[ x \otimes \omega \mapsto x \otimes \omega(1). \]

Let \( \psi_M \) be its restriction to \( (U(\mathfrak{g}) \otimes \Omega(M_G))^G \). We need to show that \( \psi_M ((U(\mathfrak{g}) \otimes \Omega(M)) \mid \Omega(M_G))^G \subset \text{Ind}_{G/R}(\Omega(M)) \).

It is easy to see that (cf. [KV, Page 147])

\[ \Omega(M_G) = C^\infty(G, (\wedge(\mathfrak{g}^*) \otimes \Omega(M))_{\text{hor}})^R, \]

where \( C^\infty(G, V) \) denotes the space of \( C^\infty \)-functions on \( G \) with values in \( V \); the \( R \)-invariants are taken with respect to the action of \( R \) on \( G \) via right multiplication, the given action of \( R \) on \( M \) and adjoint action on \( \wedge(\mathfrak{g}^*) \); the
contraction \(i_a\) (\(a \in \mathfrak{r}\)) acting on \(\wedge(\mathfrak{g}^*) \otimes \Omega(M)\) is the standard tensor product contraction. Thus

\[
(1) \quad (U(\mathfrak{g}) \otimes \Omega(M_G))^G = \left(\left(C^\infty(G, (U(\mathfrak{g}) \otimes \wedge(\mathfrak{g}^*) \otimes \Omega(M))_{\text{hor } R}\right)^R\right)^G,
\]

where \(R\) acts trivially on \(U(\mathfrak{g})\); and the \(G\)-invariants are taken with respect to the left action of \(G\) on \(U(\mathfrak{g})\), the adjoint action on \(U(\mathfrak{g})\) and the trivial action on \(\wedge(\mathfrak{g}^*) \otimes \Omega(M)\).

Take \(\tilde{\alpha} \in \left(C^\infty(G, (U(\mathfrak{g}) \otimes \wedge(\mathfrak{g}^*) \otimes \Omega(M))_{\text{hor } R}\right)^G\). Then,

\[
\tilde{\alpha}(gk^{-1}) = k \tilde{\alpha}(g), \quad \text{for } g \in G, k \in R.
\]

Writing \(\tilde{\alpha}(1) = \sum x_i \otimes \omega_i, \ x_i \in U(\mathfrak{g}), \ \omega_i \in \wedge(\mathfrak{g}^*) \otimes \Omega(M)\), since \(\tilde{\alpha}\) is \(G\)-invariant,

\[
\sum_i (\text{Ad}(gk^{-1})x_i) \otimes \omega_i = k \cdot \sum_i (\text{Ad}g x_i) \otimes \omega_i
= \sum_i (\text{Ad}g x_i) \otimes k \omega_i.
\]

Taken \(g = k\) in the above identity, we get

\[
\sum_i x_i \otimes \omega_i = \sum_i (\text{Ad} k x_i) \otimes k \omega_i.
\]

Thus \(\psi_M(\tilde{\alpha}) \in \text{Ind}_{G/R}(\Omega(M))\).

We next show that \(\psi_M\) is surjective onto \(\text{Ind}_{G/R}(\Omega(M))\). Take \(\alpha = \sum_i x_i \otimes \omega_i \in \left(U(\mathfrak{g}) \otimes \wedge(\mathfrak{g}^*) \otimes \Omega(M)\right)_R\) and define \(\tilde{\alpha} \in (U(\mathfrak{g}) \otimes \Omega(M_G))^G\), under the identification (1), by

\[
\tilde{\alpha}(g) = \sum_i (\text{Ad}g x_i) \otimes \omega_i, \quad g \in G.
\]

Clearly \(\tilde{\alpha}\) is \(G\)-invariant. Further,

\[
\tilde{\alpha} \in C^\infty\left(G, (U(\mathfrak{g}) \otimes \wedge(\mathfrak{g}^*) \otimes \Omega(M))_{\text{hor } R}\right)^R.
\]

To show this, it suffices to prove that for all \(g \in G\) and \(k \in R\),

\[
(2) \quad \sum_i (\text{Ad}(gk^{-1})x_i) \otimes \omega_i = \sum_i \text{Ad}g x_i \otimes k \omega_i.
\]

But \(\alpha\) being \(R\)-invariant,

\[
(3) \quad \sum_i x_i \otimes \omega_i = \sum_i (\text{Ad} k x_i) \otimes k \omega_i, \quad \text{for all } k \in R.
\]
Applying $g^{-1}k$ to (3) we get (2).

The injectivity of $\psi_M$ is clear from the $G$-invariance of any element in the domain of $\psi_M$. Thus $\psi_M$ is a linear isomorphism. We next show that $\psi_M$ is a cochain map.

View $\wedge(g^*)$ as the space of left invariant forms on $G$. For $x \in U(g)$, $f \in C^\infty(G)$, $\omega_1 \in \wedge(g^*)$ and $\omega_2 \in \Omega(M)$, by (1.2.1) ($\vec{d}$ being deRham differentials on $G$ and also on $M$),

$$d_G(x \otimes f \omega_1 \otimes \omega_2) = x \otimes \vec{d}f \wedge \omega_1 \otimes \omega_2 + x \otimes f \, d_A \omega_1 \otimes \omega_2$$

$$+ (-1)^{\text{deg} \omega_1} x \otimes f \omega_1 \otimes d \omega_2 - \frac{1}{2} \sum_k (u_{ak} x + xu_{ak}) \otimes f(i_{b_k} \omega_1) \otimes \omega_2$$

$$+ \frac{1}{4} x \otimes f(i_{\gamma} \omega_1) \otimes \omega_2.$$

(In fact, to be precise, in the above we should have taken $\sum_j x^j \otimes f^j \omega^j_1 \otimes \omega^j_2 \in (U(g) \otimes \Omega(M_G))^G$ instead of just a single term $x \otimes f \omega_1 \otimes \omega_2$. But, for notational convenience, we take a single term.)

Moreover, for any $x \otimes f \omega_1 \otimes \omega_2 \in (U(g) \otimes \Omega(M_G))^G$, we get (for any $a_k \in g$)

$$(u_{ak} x - xu_{ak}) \otimes f \omega_1 \otimes \omega_2 = -x \otimes a_k^* (f) \omega_1 \otimes \omega_2.$$  

Thus,

$$\psi_M d_G(x \otimes f \omega_1 \otimes \omega_2) = x \otimes (\vec{d}f)(1) \wedge \omega_1 \otimes \omega_2 + x \otimes f(1) \, d_A \omega_1 \otimes \omega_2$$

$$+ (-1)^{\text{deg} \omega_1} x \otimes f(1) \omega_1 \otimes d \omega_2$$

$$- \frac{1}{2} \sum_k (u_{ak} x + xu_{ak}) \otimes f(1)(i_{b_k} \omega_1) \otimes \omega_2$$

$$+ \frac{1}{4} x \otimes f(1)(i_{\gamma} \omega_1) \otimes \omega_2$$

$$= -\sum_k (u_{ak} x - xu_{ak}) \otimes f(1)a_k^* \wedge \omega_1 \otimes \omega_2$$

$$+ x \otimes f(1) \, d_A \omega_1 \otimes \omega_2 + (-1)^{\text{deg} \omega_1} x \otimes f(1) \omega_1 \otimes d \omega_2$$

$$- \frac{1}{2} \sum_k (u_{ak} x + xu_{ak}) \otimes f(1)(i_{b_k} \omega_1) \otimes \omega_2$$

$$+ \frac{1}{4} x \otimes f(1)(i_{\gamma} \omega_1) \otimes \omega_2,$$  

by (4),

where $\{a_k^*\}$ is the basis of $g^*$ dual to the basis $\{a_k\}$ of $g$.
On the other hand, by the expression of $d_{\mathcal{W}}$ given in Section 1,
\[
d(x \otimes f(1) \omega_1 \otimes \omega_2) = f(1) d_{\mathcal{W}}(x \otimes \omega_1) \otimes \omega_2 + (-1)^{\deg \omega_1} f(1) x \otimes \omega_1 \otimes d \omega_2
\]
\[
(6) \quad = f(1) \left( -\text{ad} u_{a_k}(x) \otimes c_{b_k} \wedge \omega_1 - \left( \frac{u_{a_k} x + x u_{a_k}}{2} \right) \otimes \bar{i}_{b_k} \omega_1 + x \otimes d \omega_1 + \frac{1}{4} x \otimes i_{\gamma} \omega_1 \right) \otimes \omega_2 + (-1)^{\deg \omega_1} f(1) x \otimes \omega_1 \otimes d \omega_2.
\]
Comparing (5) and (6) we get that $\psi_M$ commutes with the differentials.

Finally, we show that $\psi_M$ is an algebra homomorphism. Take two elements $u = x \otimes \sum_i f_i \omega_i' \otimes \omega_i''$ and $v = y \otimes \sum_j g_j \eta_j' \otimes \eta_j''$ in $U(\mathfrak{g}) \otimes \Omega(M_G)$, where $f_i, g_j \in C^\infty(G \times M)$, $x, y \in U(\mathfrak{g})$, $\omega_i', \eta_j' \in \wedge(\mathfrak{g}^*)$ and $\omega_i'', \eta_j'' \in \Omega(M)$. Then, by (1.2.2),
\[
\psi_M(u \otimes v) = \psi_M \left( \sum_{i,j} (-1)^{\deg \eta_j' \deg \omega_i''} \left( x y \otimes f_i g_j \mu \left( \text{Exp} \left( -\frac{1}{2} \sum_k i_{a_k}^1 i_{b_k}^2 \right) (\omega_i' \otimes \eta_j') \right) \right) \otimes \omega_i'' \eta_j'' \right)
\]
\[
= \sum_{i,j} (-1)^{\deg \eta_j' \deg \omega_i''} x y \otimes \mu \left( \text{Exp} \left( -\frac{1}{2} \sum_k i_{a_k}^1 i_{b_k}^2 \right) (\omega_i' \otimes \eta_j') \right)
\]
\[
\otimes f_i(1, -) g_j(1, -) \omega_i'' \eta_j''
\]
\[
= \sum_{i,j} (-1)^{\deg \eta_j' \deg \omega_i''} x y \otimes (\omega_i' \otimes \eta_j') \otimes f_i(1, -) g_j(1, -) \omega_i'' \eta_j''
\]
\[
= \psi_M(u) \cdot \psi_M(v).
\]
This completes the proof of the theorem. \hfill \Box

2.3 Corollary. For a real $R$-manifold $M$, the cochain map $\psi_M$ induces a functorial graded algebra isomorphism:
\[
\psi_M^* : \mathcal{H}_G(M_G) \simto H(\text{Ind}_{G/R}(\Omega(M))).
\]

2.4 Definition. Let $M$ be a real $R$-manifold. Define a cochain map $\Phi_M : \text{Ind}_{G/R}(\Omega(M)) \to (U(\mathfrak{r}) \otimes \Omega(M))^R$, making the following diagram commutative:
\[
\begin{align*}
(S(\mathfrak{g}^*) \otimes \Omega(M_G))^G & \xrightarrow{\alpha_M} (S(\mathfrak{r}^*) \otimes \Omega(M))^R \\
Q_G^M & \simto \hat{Q}_R^M \\
(U(\mathfrak{g}) \otimes \Omega(M_G))^G & \xrightarrow{\psi_M} \text{Ind}_{G/R}(\Omega(M)) \xrightarrow{\Phi_M} (U(\mathfrak{r}) \otimes \Omega(M))^R,
\end{align*}
\]
where $\alpha_M$ is induced from the map

$$P \otimes \eta \mapsto P_1 \otimes \eta_{1 \times M},$$

for $P \in S(\mathfrak{g}^*)$ and $\eta \in \Omega(M_G)$, where $\hat{Q}^M_G$ (and $\hat{Q}^M_R$) is the map defined in 1.4.

In fact, define

$$\Phi_M := \hat{Q}^R_M \circ \alpha_M \circ (\hat{Q}^M_G)^{-1} \circ (\psi_M)^{-1}.$$

Then, clearly it is a cochain map making the above diagram commutative. Observe that, in general, $\Phi_M$ is not an algebra homomorphism.

Further, $\Phi_M$ is functorial in the sense that for any $R$-equivariant smooth map $f : M \to N$, the following diagram is commutative:

$$\begin{array}{ccc}
\text{Ind}_{G/R}(\Omega(N)) & \xrightarrow{\Phi_N} & (U(\mathfrak{r}) \otimes \Omega(N))^R \\
\downarrow & & \downarrow \\
\text{Ind}_{G/R}(\Omega(M)) & \xrightarrow{\Phi_M} & (U(\mathfrak{r}) \otimes \Omega(M))^R,
\end{array}$$

where the vertical maps are induced canonically from the $R$-differential algebra homomorphism $f^* : \Omega(N) \to \Omega(M)$.

2.5 Theorem. For any $R$-manifold $M$, the cochain map $\Phi_M : \text{Ind}_{G/R}(\Omega(M)) \to (U(\mathfrak{r}) \otimes \Omega(M))^R$ induces an algebra isomorphism in cohomology:

$$[\Phi_M] : H(\text{Ind}_{G/R}(\Omega(M))) \xrightarrow{\sim} \mathcal{H}_R(M).$$

Thus, by Corollary 2.3, we have a functorial algebra isomorphism

$$\mathcal{H}_G(M_G) \xrightarrow{\sim} \mathcal{H}_R(M).$$

In particular, $\mathcal{H}_G(G/R) \xrightarrow{\sim} U(\mathfrak{r})^R$.

Proof. In the first commutative diagram of (2.4), all the maps are cochain maps. Moreover, all the cochain maps $\hat{Q}^M_G$, $\psi_M$, $\hat{Q}^M_R$ are cochain isomorphisms. So it suffices to prove that $\alpha_M$ induces an isomorphism in cohomology. But this follows from [DV, Théorème 24].

Let $M$ be a $R$-manifold. Consider the $R$-module isomorphism

$$Q_g \otimes I_{\Omega(M)} : W(\mathfrak{g}) \otimes \Omega(M) \to \mathcal{W}(\mathfrak{g}) \otimes \Omega(M).$$
Since $Q_g$ commutes with the operators $L_a, i_a$ ($a \in g$) and $d$; in particular, $Q_g \otimes I_{\Omega(M)}$ induces the map

$$Q^M_{G/R} : (W(g) \otimes \Omega(M))_R \to \text{Ind}_{G/R}(\Omega(M))$$

commuting with differentials, where $W(g) \otimes \Omega(M)$ is equipped with the standard tensor product $R$-differential algebra structure.

2.6 Lemma. For any $R$-manifold $M$, the following diagram is commutative:

$$
\begin{array}{ccc}
(W(g) \otimes \Omega(M))_R & \xrightarrow{\hat{\alpha}_M} & (S(\tau^*) \otimes \Omega(M))^R \\
\Phi_{\psi_M} & \downarrow & \Phi_{\psi_M} \\
\text{Ind}_{G/R}(\Omega(M)) & \xrightarrow{\phi_M} & (U(\tau) \otimes \Omega(M))^R,
\end{array}
$$

where $\hat{\alpha}_M(P \otimes \omega \otimes \eta) = \varepsilon(\omega) P \triangledown \eta$, for $P \in S(\mathfrak{g}^*)$, $\omega \in \wedge(\mathfrak{g}^*)$, $\eta \in \Omega(M)$; $\varepsilon : \wedge(\mathfrak{g}^*) \to \mathbb{C}$ being the standard augmentation map.

Proof. From the definition of $\Phi_{\psi_M}$, it suffices to prove that the following diagram is commutative:

$$
\begin{array}{ccc}
(S(\mathfrak{g}^*) \otimes \Omega(M))^G & \xrightarrow{\tilde{\psi}_M} & (W(g) \otimes \Omega(M))_R \\
\tilde{\psi}_M & \downarrow & \tilde{\psi}_M \\
(U(\mathfrak{g}) \otimes \Omega(M))^G & \xrightarrow{\psi_M} & \text{Ind}_{G/R}(\Omega(M)).
\end{array}
$$

where $\tilde{\psi}_M$ is defined the same way as $\psi_M$. (Use the fact that $\hat{\alpha}_M \circ \tilde{\psi}_M = \alpha_{M^o}$.)

Considering the $G$-equivariant canonical projection (with $G$ acting on $G \times M$ via its left multiplication on the first factor) $G \times M \to M_G$, to prove the commutativity of $(\ast)$, we can replace $M_G$ by the $G$-manifold $G \times M$. From the definition of the various maps in $(\ast)$, we can further assume that $M$ is the one point manifold $M_0$, i.e., we are reduced to prove the commutativity of $(\ast)$ for $M_G$ replaced by $G$ (with $G$ acting on $G$ via the left multiplication).

Again using the definition of the various maps in $(\ast)$, we are reduced to proving that

$$\sigma(x) = (-1)^n x, \text{ for all } (\wedge(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*))_{\text{hor}} \text{ of total degree } n,$$

where $\sigma$ is the involution of $\wedge(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$ taking $\omega \otimes \eta \mapsto (-1)^{\deg \omega \deg \eta} \eta \otimes \omega$, and ‘hor’ is taken with respect to the standard tensor product action of $i_a$ ($a \in g$) on $\wedge(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$.  

15
Clearly, $\delta \omega_1 := \omega_1 \otimes 1 - 1 \otimes \omega_1 \in (\wedge(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*))_{\text{hor}}$, for any $\omega_1 \in \mathfrak{g}^*$. Thus the subalgebra $\mathcal{A}$ of $\wedge(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$ generated by $\{\delta \omega_1 \}_{\omega_1 \in \mathfrak{g}^*}$ is contained in $(\wedge(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*))_{\text{hor}}$. Moreover, since the projection

$$(\wedge(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)) \to \wedge(\mathfrak{g}^*), \ \omega \otimes \eta \mapsto \varepsilon(\eta) \omega,$$

induces an isomorphism

$$(\wedge(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*))_{\text{hor}} \simeq \wedge(\mathfrak{g}^*),$$

we get that

$$\mathcal{A} = (\wedge(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*))_{\text{hor}}.$$ 

Clearly (1) is satisfied for $\delta \omega_1$ and hence for each element of $\mathcal{A}$. Thus we get (1) by (2) and the lemma is proved.

2.7 Remark. (1) An appropriate analogue of Theorems (2.2) and (2.5), Corollary (2.3) and Lemma (2.6) can be proved for any $R$-differential algebra $B$ replacing $\Omega(M)$.

(2) Instead of defining $\Phi_M$ as in 2.4, we could have (uniquely) defined $\Phi_M$ satisfying the above lemma. But we find the original definition (as in 2.4) easier to work with.

3 Cubic Dirac operator and results of Huang-Pandžić and Kostant

We follow the notation and assumptions as in the beginning of Section 2. In particular, $(G,R)$ is a quadratic pair, i.e., $G$ is a real Lie group with complexified Lie algebra $\mathfrak{g}$ and $R \subset G$ is a closed subgroup with complexified Lie subalgebra $\mathfrak{r} \subset \mathfrak{g}$. We assume that $\mathfrak{g}$ has a $G$-invariant nondegenerate symmetric bilinear form $B_\mathfrak{g} = \langle , \rangle$ such that $B_{\mathfrak{g}|\mathfrak{r}}$ is nondegenerate.

We now identify the differential of Ind$_{G/R}(M)$ for $M$ a one point manifold with Kostant’s cubic Dirac operator.

3.1 Lemma. Let $M$ be the one point manifold $M^o$. Then Ind$_{G/R}(M^o)$ can canonically be identified with the super-algebra $(U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^R$ (under the tensor product algebra structure).

Moreover, the differential $d$ on Ind$_{G/R}(M)$ under the above identification is given by

$$d(x) = \text{ad} \mathcal{D}(x),$$
where

\[ D^p := \sum u_{p\ell} \otimes c_{q\ell} - 1 \otimes \gamma_p, \]

\{p_\ell\}_\ell is any basis of \( p \) and \( \{q_\ell\}_\ell \) is the dual basis with respect to the nondegenerate form \( B_{g|p} \) and \( \gamma_p \in \wedge^3(p) \simeq \wedge^3(p^*) \) is the Cartan form

\[ \gamma_p(x, y, z) = \langle x, [y, z] \rangle, \text{ for } x, y, z \in p. \]

It is easy to see that \( D^p \) is \( R \)-invariant, i.e., \( D^p \in (U(g) \otimes Cl(p))^R \).

Proof. Let \( \{r_m\} \) be a basis of \( r \) and let \( \{s_m\} \) be the dual basis of \( r \) under \( B_{g|r} \). Then, of course, \( \{r_m\}_m \cup \{p_\ell\}_\ell \) is a basis of \( g \) and \( \{s_m\}_m \cup \{q_\ell\}_\ell \) is the dual basis of \( g \). Thus the element \( D \in \mathcal{W}(g) \) as in Section (1) is given by

\[ D = \sum_m u_{r_m} \otimes c_{s_m} + \sum_\ell u_{p_\ell} \otimes c_{q_\ell} - 1 \otimes \gamma_g. \]

Now

\[ \mathcal{W}(g)_R = (U(g) \otimes Cl(p))^R_{\text{hor}} \]

\[ \simeq (U(g) \otimes Cl(p))^R. \]

The differential \( d \) in \( \mathcal{W}(g) \) is given by \( dx = \text{ad} D(x) \). Moreover, \( d \) keeps the subspace \( \mathcal{W}(g)_R \) stable. From this it is easy to see that, for \( x \in \mathcal{W}(g)_R \), \( dx = \text{ad} D^p(x) \). This proves the lemma. \( \square \)

3.2 Definition. As in [Ko1, §1.5], the adjoint representation of \( R \) on \( p \) gives rise to the Lie algebra homomorphism

\[ \alpha : r \to Cl(p)^{\text{even}} \text{ satisfying } \]

\[ [\alpha(x), y] = [x, y], \text{ for } x \in r \text{ and } y \in p, \]

where the bracket on the left side is commutation in \( Cl(p) \). Then, \( \alpha \) is an \( R \)-module map under the adjoint actions. In particular, for \( x_1, x_2 \in r \),

\[ \alpha[x_1, x_2] = x_1 \cdot \alpha(x_2). \]

Thus, we get an algebra homomorphism

\[ \xi : U(r) \to U(g) \otimes Cl(p), \]

so that \( \xi(x) = x \otimes 1 + 1 \otimes \alpha(x) \), for \( x \in r \). It is easy to see that \( \xi \) is injective. Moreover, the earlier given \( R \)-module structure on \( U(g) \otimes Cl(p) \) (obtained
from the adjoint action) is compatible with $\xi$. In particular, for $x \in \mathfrak{r}$ and $a \in U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p})$,

$$x \cdot a = \xi(x)a - a\xi(x).$$

Let $Z(G)$ (resp. $Z(R)$) be the subalgebra of invariants $U(\mathfrak{g})^G$ (resp. $U(\mathfrak{r})^R$). Then, $Z(G) \otimes 1$ and $\xi(Z(R))$ are subalgebras of $(U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^R$. Further, for $d = \text{ad} \mathcal{D}^p$,

$$d|_{Z(G)\otimes 1} \equiv 0,$$

and

$$d|_{\xi(U(\mathfrak{r}))} \equiv 0,$$

since $\xi(U(\mathfrak{r}))$ commutes with any element in $(U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^R$ by (2). Thus, by Theorem (2.5) and Lemma (3.1), we get algebra homomorphisms

$$Z(G) \to H(((U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^R, \text{ad} \mathcal{D}^p) = H(\text{Ind}_{G/R}(\mathbb{C})[\Phi_{M\mathfrak{o}}]) \sim Z(R),$$

where, as earlier, $M^o$ is the one point manifold and the first map is induced from the map $z \mapsto z \otimes 1$. Let $\eta_R$ be the composite algebra homomorphism

$$\eta_R : Z(G) \to Z(R).$$

Define a $\mathcal{R}$-differential algebra homomorphism $F = F^\mathcal{R}_\mathfrak{r} : W(\mathfrak{r}) \to W(\mathfrak{g})$ by

$$F(\lambda \otimes 1) = \bar{\lambda} \otimes 1 - 1 \otimes \delta(\lambda) \quad \text{and} \quad F(1 \otimes \lambda) = 1 \otimes \bar{\lambda}, \quad \text{for} \ \lambda \in \mathfrak{r}^*,$$

where $\bar{\lambda} \in \mathfrak{g}^*$ is defined by $\lambda|_{\mathfrak{k}} = \lambda$ and $\bar{\lambda}|_{\mathfrak{p}} \equiv 0$, and $\delta : \mathfrak{r}^* \to \wedge^2(\mathfrak{g}^*)$ is defined by

$$\delta(\lambda)(y,z) = \bar{\lambda}([y,z]) \quad \text{for} \ y,z \in \mathfrak{p},$$

$$= 0 \quad \text{if at least one of} \ y,z \in \mathfrak{r}.$$ 

Similarly, define a $\mathcal{R}$-differential algebra homomorphism $\mathcal{F} = \mathcal{F}^\mathcal{R}_\mathfrak{r} : \mathcal{W}(\mathfrak{r}) \to \mathcal{W}(\mathfrak{g})$ by

$$\mathcal{F}(x \otimes 1) = x \otimes 1 + 1 \otimes \alpha(x), \quad \text{and} \quad \mathcal{F}(1 \otimes x) = 1 \otimes x, \quad \text{for} \ x \in \mathfrak{r}.$$ 

Clearly,

$$\mathcal{F}|_{U(\mathfrak{r})} = \xi.$$
Then, interestingly, as proved by Alkseev-Meinrenken (private communication), we have:

\[ Q_g \circ F = F \circ Q_r, \]

i.e., the following diagram is commutative:

\[
\begin{array}{ccc}
W(r) & \xrightarrow{F} & W(g) \\
\downarrow Q_r & & \downarrow Q_g \\
W(r) & \xrightarrow{F} & W(g).
\end{array}
\]

As a corollary of Theorem (2.5) we get the following. This was conjectured by Vogan (actually Vogan conjectured a slightly weaker version) and proved by Huang-Pandžić [HP, Theorem 3.4] in the case \( R \) is a maximal compact subgroup of a connected reductive \( G \). The case when \( G \) and \( R \) are connected and reductive was proved by Kostant [Ko3, §4.1].

3.3 Theorem. For the differential \( d := ad \mathcal{D}^p \) on \((U(g) \otimes Cl(p))^R\),

\[ \text{Ker } d = \xi(Z(R)) \oplus \text{Im } d. \]

In particular, \( \xi(Z(R)) \simeq H \left( (U(g) \otimes Cl(p))^R, ad \mathcal{D}^p \right) \).

Proof. We first prove that the composite map \( \Phi_{M^o} \circ \xi: Z(R) \xrightarrow{\xi} (U(g) \otimes Cl(p))^R \xrightarrow{\Phi_{M^o}} Z(R) \)

is an isomorphism. (In fact, we will see during the proof of the next theorem that \( \Phi_{M^o} \circ \xi \) is the identity map.) As earlier, let \( \{U(\tau)^p\}_{p \geq 0} \) be the standard filtration of the enveloping algebra \( U(\tau) \) and let \( Z(R)^p := U(\tau)^p \cap Z(R) \). By the definition of the map \( \xi \), for \( a \in Z(R)^p \setminus Z(R)^{p-1} \),

\[ \xi(a) = a \otimes 1 + x, \]

for some \( x \in (U(g)^{p-1} \otimes Cl(p))^R \). Thus, from the definition of the map \( \Phi_{M^o} \) and the description of the isomorphism \( \hat{Q}^{MG}_G \) as in [AM, Proposition 6.5],

\[ \Phi_{M^o} \circ \xi(a) = a \mod Z(R)^{p-1}. \]

From this we see that \( \Phi_{M^o} \circ \xi \) is an isomorphism.

Since \( \Phi_{M^o} \) induces an isomorphism in cohomology by Theorem (2.5), we get that the induced cohomology map

\[ [\xi]: Z(R) \rightarrow H \left( (U(g) \otimes Cl(p))^R, ad \mathcal{D}^p \right) \]

is an isomorphism. From this of course (1) follows immediately. \( \square \)
3.4 Theorem. The algebra homomorphism \( \eta_R : Z(G) \to Z(R) \) is the unique homomorphism making the following diagram commutative:

\[
\begin{array}{ccc}
Z(G) & \xrightarrow{\eta_R} & Z(R) \\
H_G & \downarrow & \downarrow H_R \\
S(\mathfrak{g})^G & \xrightarrow{\beta_R} & S(\mathfrak{r})^R,
\end{array}
\]

where \( \beta_R \) is the restriction map under the identification \( S(\mathfrak{g}) \simeq S(\mathfrak{g}^*) \), \( S(\mathfrak{r}) \simeq S(\mathfrak{r}^*) \) induced by the bilinear form \( B_{\mathfrak{g}} \), and \( H_G \) (resp. \( H_R \)) is the inverse of the Duflo isomorphism of \( \mathfrak{g} \) (resp. \( \mathfrak{r} \)) restricted to \( S(\mathfrak{g})^G \) (resp. \( S(\mathfrak{r})^R \)). (Recall that for reductive \( G \), \( H_G \) coincides with the Harish-Chandra isomorphism.)

Thus, for \( z \in Z(G) \),

\[
(1) \quad z \otimes 1 - \xi(\eta_R(z)) = D^p a_z + a_z D^p,
\]

for some \( a_z \in (U(\mathfrak{g}) \otimes \text{Cl}(p)^{\text{odd}})^R \).

Proof. With the notation as in the first diagram of Definition 2.4, for any \( z \in Z(G) \), \( z \otimes 1 \in (U(\mathfrak{g}) \otimes \Omega(M_G^*))^G \) and, moreover,

\[
\psi_M(z \otimes 1) = z \otimes 1.
\]

Thus, by [AM, Proposition 6.5],

\[
\Phi_{M^o}(z \otimes 1) = D_\xi((D_{\mathfrak{g}^{-1}}(z))|_{\mathfrak{r}}),
\]

where \( D_{\mathfrak{g}} : S(\mathfrak{g}) \to U(\mathfrak{g}) \) is the Duflo isomorphism under the identification \( S(\mathfrak{g}) \simeq S(\mathfrak{g}^*) \), and similarly for \( D_\xi \). This gives that

\[
\eta_R(z) = [\Phi_{M^o}](z \otimes 1) = D_\xi \circ \beta_R \circ (D_{\mathfrak{g}^{-1}})(z).
\]

From this the first part of the theorem follows.

We next prove that

\[
(2) \quad \Phi_{M^o} \circ \xi|_{Z(R)} = I,
\]

where \( \xi : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes \text{Cl}(p) \) is defined in §3.2. By Lemma (2.6), and the identities (3.2.5), (3.2.6), for \( x \in S(\mathfrak{r})^R \),

\[
\Phi_{M^o} \circ \xi \circ Q_\xi(x) = \hat{Q}_R^{M^o} \circ \hat{\alpha}_{M^o} \circ F(x)
\]

\[
= Q_\xi \circ \hat{\alpha}_{M^o} \circ F(x), \quad \text{since } Q_{\xi S(\mathfrak{r})^R} = \hat{Q}_R^{M^o}
\]

\[
= Q_\xi(x), \quad \text{from the definition of } F \text{ and } \hat{\alpha}_{M^o}.
\]
Since \( Q_{r|S(r^*)}^R \) is an isomorphism onto \( Z(R) \), this proves (2).

From (2) we easily see that, for \( z \in Z(G) \),

\[
\Phi_{M^o}(z \otimes 1) = \Phi_{M^o}(\xi(\eta_R(z))) = \eta_R(z),
\]

and, moreover, by (3.2.3), (3.2.4), both of \( z \otimes 1 \) and \( \xi(\eta_R(z)) \) are cycles under \( \text{ad} D^p \). Thus they differ by a coboundary, proving (1). This proves the theorem.

Alternatively, we can also obtain (1) in the special (but important) case where \( G \) and \( R \) are connected reductive groups (and \( B_{\mathfrak{g}_c} \) is nondegenerate) by using a result of Kostant as follows.

By virtue of Theorem (3.3), define the map \( \hat{\eta}_R : Z(G) \to Z(R) \) such that \( z \otimes 1 - \xi(\hat{\eta}_R(z)) \in \text{Im} d \). Then it is easy to see that \( \hat{\eta}_R \) is an algebra homomorphism. Moreover, by Kostant [Ko3, Theorem 4.2] (generalizing the corresponding result in the case when \( R \) is a maximal compact subgroup of \( G \) by Huang-Pandžić [HP, Theorem 5.5]), \( \hat{\eta}_R \) replacing \( \eta_R \) also makes the diagram (D) commutative. Thus \( \eta_R = \hat{\eta}_R \), proving (1).

3.5 Definition. Let \( S \) be the space of spinors for \( \text{Cl}(\mathfrak{p}) \), which is a simple module of \( \text{Cl}(\mathfrak{p}) \). Then, for any \( U(\mathfrak{g}) \)-module \( V \), \( V \otimes S \) is a \( U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}) \)-module under the componentwise action. In particular, the element \( D^p \in (U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^\tau \) defined in Lemma (3.1) acts as a linear endomorphism \( D^p_V \) on \( V \otimes S \).

Following Vogan, define the Dirac cohomology

\[
H_D(\mathfrak{g}, \tau; V) = \frac{\text{Ker} D^p_V}{\text{Ker} D^p_V \cap \text{Im} D^p_V}.
\]

Since the element \( D^p \) commutes with \( \xi(U(\tau)) \) (cf. §3.2), both of \( \text{Ker} D^p \) and \( \text{Im} D^p \) are \( \tau \)-submodules of \( V \otimes S \) via \( \xi \). Thus \( H_D(\mathfrak{g}, \tau; V) \) has a canonical \( \tau \)-module structure.

Let \( \chi : Z(\mathfrak{g}) \to \mathbb{C} \) be an algebra homomorphism, where \( Z(\mathfrak{g}) \) is the center of \( U(\mathfrak{g}) \). Recall that a \( U(\mathfrak{g}) \)-module \( V \) is said to have central character \( \chi \) if, for all \( v \in V \) and \( z \in Z(\mathfrak{g}) \),

\[
z \cdot v = \chi(z)v.
\]

As an immediate consequence of Theorem (3.4), one gets the following corollary. Recall that this corollary was conjectured by Vogan in the case \( R \) is a maximal compact subgroup of \( G \) and proved in this case by Huang-Pandžić [HP] and proved for general reductive pairs by Kostant [Ko3].
3.6 Corollary. Let $V$ be a $U(\mathfrak{g})$-module with central character $\chi$. Then, for any $v \in H_D(\mathfrak{g}, \mathfrak{r}; V)$ and $z \in Z(\mathfrak{g})$,

$$\chi(z)v = \eta_R(z)v.$$ 

Of course, the homomorphism $\eta_R : Z(\mathfrak{g}) \rightarrow Z(\mathfrak{r})$ is completely determined from the diagram $(D)$ of Theorem (3.4).

Loosely speaking, the corollary asserts that the central character of any irreducible $\mathfrak{r}$-submodule of $H_D(\mathfrak{g}, \mathfrak{r}; V)$ (if nonzero) determines the central character of $V$.

Proof. We can clearly assume that $G$ and $R$ are connected and thus $Z(\mathfrak{g}) = Z(G)$ and $Z(\mathfrak{r}) = Z(R)$. By (3.4.1),

$$z \otimes 1 - \xi(\eta_R(z)) = D^p a_z + a_z D^p,$$

for some $a_z \in (U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^R$. Thus, for any $v_o \in \ker D^p_U$,

$$ (z \otimes 1)v_o - \eta_R(z)v_o \in \text{Im} D^p_U \cap \ker D^p_V,$$

since $\eta_R(z)v_o \in \ker D^p_V$. Thus, $\chi(z)v = \eta_R(z)v$ in $H_D(\mathfrak{g}, \mathfrak{r}; V)$. 

Applying the definition of $\psi_M$ as in Theorem (2.2), for the case $R = G$ and a $G$-manifold $M$, interestingly we get an explicit expression for the inverse of the isomorphism $\Theta$.

3.7 Lemma. Take $R = G$ and a $G$-manifold $M$. Then the inverse of the isomorphism

$$\Theta = \Theta_M : (W(\mathfrak{g}) \otimes \Omega(M))_G \rightarrow (U(\mathfrak{g}) ^{\bullet} \otimes \Omega(M))^G$$

(cf. Proposition 1.2) is given by the composition

$$(U(\mathfrak{g}) ^{\bullet} \otimes \Omega(M))^G \overset{I \otimes \mu^*}{\sim} (U(\mathfrak{g}) ^{\bullet} \otimes \Omega(M_G))^G \overset{\psi_M}{\rightarrow} (W(\mathfrak{g}) \otimes \Omega(M))_G,$$

where $\mu^* : \Omega(M) \rightarrow \Omega(M_G)$ is the $G$-module map induced from the $G$-equivariant smooth map

$$\mu : G \times^G M \rightarrow M, \quad (g, m) \mapsto g \cdot m.$$

Proof. Since $\Theta_M$ is a vector space isomorphism, it suffices to prove that

$$\psi_M \circ (I \otimes \mu^*) \circ \Theta_M = I.$$
From the functoriality of $\Theta$, we have the following commutative diagram:

\[
\begin{array}{ccc}
(W(g) \otimes \Omega(M))_G & \xrightarrow{I \otimes \mu^*} & (W(g) \otimes \Omega(M_G))_G \\
\downarrow_{\Theta_M} & & \downarrow_{\Theta_{M_G}} \\
(U(g) \otimes \Omega(M))^G & \xrightarrow{I \otimes \mu^*} & (U(g) \otimes \Omega(M_G))^G.
\end{array}
\]

Take $a = \sum_i x_i \otimes \omega_i \in (U(g) \otimes \Omega(M))^G$. Then from the above commutative diagram:

\[
\Theta_M \circ \psi_M \circ \Theta_{M_G} \circ (I \otimes \mu^*) \circ \Theta_M^{-1}(a) = \Theta_M \circ \psi_M \circ (I \otimes \mu^*)(a) \\
= \Theta_M \circ \psi_M \left( \sum_i x_i \otimes (\mu^* \omega_i) \right) \\
= \sum_i x_i \otimes ((\mu^* \omega_i)|_{1 \times M}) \\
= \sum_i x_i \otimes \omega_i = a.
\]

This gives

\[
\psi_M \circ \Theta_{M_G} \circ (I \otimes \mu^*) \circ \Theta_M^{-1} = \Theta_M^{-1}.
\]

Thus $\psi_M \circ \Theta_{M_G} \circ (I \otimes \mu^*) = I$ and hence, from the above commutative diagram again, $\psi_M \circ (I \otimes \mu^*) \circ \Theta_M = I$. This proves the lemma.

3.8 Remark. After an earlier version of this paper was distributed, E. Meinrenken informed me that he and Alekseev have obtained some results (unpublished) which overlaps with our work. In particular, they also have obtained Theorems (3.3) and (3.4).

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23
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