Lagrangian in quantum mechanics is a connection one-form

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We recast Dirac’s Lagrangian in quantum mechanics in the language of vector bundles and show that the action is an operator-valued connection one-form. Phases associated with change of frames of reference are seen to be total differentials in the transformation of the action. The relativistic case is discussed and we show that it gives the correct phase in the non-relativistic limit for uniform acceleration.

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I. INTRODUCTION

As expressed by Landau and Lifshitz [1] ‘Quantum theory occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, yet at the same time it requires this limiting case for its own formulation’. There can be only one correct theory in nature and that, from all indications, must be a quantum theory. Quantum theory begins with a classical Lagrangian, action or Hamiltonian formulation. But it is the classical formulation that has to be explained in terms of quantum theory. The difficulties faced by the process of quantization, that is, attempting to build a quantum theory from classical theory compel us to look for quantum origin of fundamental quantities like the Lagrangian or the Hamiltonian structure. This was the attempt made by Dirac in a remarkable paper “The Lagrangian in quantum mechanics” [2]. In this paper Dirac proposed what should be regarded as the Lagrangian (and therefore action) in quantum mechanics. As is well known this paper led to Feynman’s formulation of the path integral [3], as well as to the Schwinger action principle [4]. If one reads between the lines in Dirac’s paper the picture of a vector bundle with base space, fibre and connection can be glimpsed there. The aim of this paper is to formulate this and identify the precise mathematical nature of the Lagrangian in quantum mechanics.

There has been a recent trend to view quantum evolution as a kind of parallel transport especially since the discovery of the geometric phase by Berry [5]. It was actually Asorey et al [6] who first floated the idea of time evolution of a quantum system as a parallel transport, though their transport was limited to one dimensional manifold of time, without any of the interesting geometric consequences. This viewpoint becomes natural when we set up Dirac’s argument in differential geometric language.

The paper is organized as follows: in section 2 we present Dirac’s argument in the language of vector bundles. In section 3 we define the quantum mechanical action and demonstrate the usefulness of the formalism when applied to arbitrary frames. In section 4 we extend our definition to the relativistic case and show that it gives the correct phase in the non-relativistic limit for uniform acceleration. The minimum differential geometry needed for our construction is given in the Appendix.

II. DIRAC LAGRANGIAN AS A CONNECTION

In [2] Dirac took up the question ‘what corresponds in the quantum theory to the Lagrangian method of the classical theory?’ and he showed that the quantity \( \langle \xi_0', t+dt|\xi_0', t \rangle \) is what corresponds to \( \exp(iLt/dt) \), where \( L \) is the Lagrangian. We proceed to reformulate his argument in a differential geometric language.

Let \( \{\xi(t)\} \) be a complete set of commuting observables in the Heisenberg picture, giving a moving representation \( \langle \xi', t \rangle \) in terms of eigenvectors of \( \xi(t) \). The canonical momenta \( P(t) \) and the Hamiltonian \( H \) act on these states as

\[
-i \frac{\partial}{\partial \xi'} \langle \xi', t \rangle = \langle \xi', t | P(t) \rangle \tag{1}
\]

\[
i \frac{\partial}{\partial t} \langle \xi', t \rangle = \langle \xi', t | H \rangle \tag{2}
\]

where \( P(t) = e^{iHt}P(0)e^{-iHt} \). We assume the set of eigenvalues \( \xi' \) of \( \xi \) forms a manifold \( M \) and call \( B = M \times R \) (where \( R \) represents time) as spacetime.

The entire dynamics of the system may then be determined from an assignment of basis vectors \( \langle \xi', t \rangle \) to each point \( (\xi', t) \) of \( B \). One can think of a (generalized) Hilbert space provided at each point \( (\xi', t) \).

Let \( c : \tau \rightarrow (\xi'(\tau), t(\tau)) \) be a smooth curve in \( B \). It follows from Eqs.(1) and (2) that rate of change of the basic vector \( \langle \xi', t \rangle \) along the curve is

\[
\frac{d}{d\tau} \langle \xi'(\tau), t(\tau) \rangle = i \langle \xi'(\tau), t(\tau) | \left( \frac{dP}{d\tau}(t) - \frac{dt}{d\tau} H \right) \rangle. \tag{3}
\]

Denoting

\[
\omega_\xi \equiv -i \left( P(t)d\xi' - H dt \right), \tag{4}
\]
where the superscript $\xi$ indicates that the operator $\omega^\xi$ is determined by the choice of the complete set of commuting observables, we have
\[ d(\xi', t) = -\langle \xi', t | \omega^\xi. \] (5)

This equation can be thought of as a parallel transport, $D\langle \xi', t | = 0$, with
\[ D = d + \omega^\xi \] (6)
provided we can show that $\omega^\xi$ transforms like a connection.

For a finite change we have
\[ \langle \xi_2, t_2 | = P \left( \langle \xi_1, t_1 | \exp \left[ -\int_{t_1}^{t_2} \omega^\xi \right] \right), \] (7)
where $P$ stands for path ordering along $\tau$. In this picture $\omega^\xi$ is an operator-valued one-form which acting on a tangent to a curve determines an operator. It permits the comparison of vectors belonging to different fibres [7].

In order to see the transformation property of $\omega^\xi$, let $\langle \eta' | t' \rangle$ be another complete set of commuting observables. Then the action associated with it is $\omega^\eta$ given by
\[ d|\eta', t' \rangle = -|\eta' \rangle t' | \omega^\eta. \] (8)

By inserting a complete set $\int [\xi', t] d\xi' \langle \xi', t | = 1$ we also have
\[ d|\eta', t' \rangle = d\left( \int |\eta' \rangle t'|\xi'| \rangle t d\xi' \langle \xi', t | \right) \]
\[ = \int d\xi' (d|\eta', t'|\xi'| \rangle t d\xi' \langle \xi', t |) \]
\[ + \int d\xi' (|\eta' \rangle t'|\xi'| \rangle t d\xi' \langle \xi', t |) \]
\[ = \int d\xi' (d|\eta', t'|\xi'| \rangle t d\xi' \langle \xi', t |) \]
\[ - \int d\xi' (|\eta' \rangle t'|\xi'| \rangle t d\xi' \langle \xi', t |\omega^\xi. \]

Taking inner product with $|\eta'', t \rangle$ we get
\[ -\langle \eta', t' | \omega^\eta | \eta'', t \rangle = \int d\xi'^d|\eta', t'|\xi'| t \rangle (\xi', t | \eta'' t') \]
\[ - \int d\xi'^d|\eta', t'|\xi'| t \rangle (\xi', t |\omega^\xi | \eta'' t'). \]

Calling the unitary matrix $U_{\eta' \xi'}$ as $\omega^\eta = U_{\eta' \xi'} \omega^\xi U^{-1}_{\eta' \xi'} - dU_{\eta' \xi'} U^{-1}_{\eta' \xi'}. \]
(9)
with summation and integration understood over repeated indices. The above equation can be written in index free notation as
\[ \omega^\eta = U \omega^\xi U^{-1} + UdU^{-1}. \] (10)

This is the transformation rule for a connection.

The curvature is given by (see Appendix)
\[ \Omega = d\omega + \omega \wedge \omega \]
\[ = -i \left( \frac{\partial P(t)}{\partial t} + \frac{\partial H}{\partial \xi^\xi} + i [P, H] \right) d\xi' \wedge dt. \]
For $H = \frac{P^2}{2m} + V(\xi)$ and using the Heisenberg equation of motion for $P(t)$ we get
\[ \Omega = -i \frac{\partial V}{\partial \xi^\xi} d\xi' \wedge dt. \] (11)
which indicates that force is curvature.

Thus, by setting up a fibre bundle construction we have shown that the action in quantum mechanics must be a connection. It governs the parallel transport of Hilbert space vectors along arbitrary curves in the bundle which are lifted from the base manifold.

III. PHASE CHANGES DUE TO CHANGE OF FRAMES

In this section we use the above picture and construct a Hilbert vector bundle with physical spacetime as the base manifold. Quantum mechanically this amounts to singling out the position operators as the preferred complete set of observables. We then determine what $\omega$ should be in different frames of reference. It turns out that this formulation gives a natural explanation of the phases that must accompany changes of frames of reference.

In the usual method the transformation properties of the Schrödinger equation is first guessed and then phases are fixed to satisfy the correct Schrödinger equation. This happens for Galilean transformation as well as for accelerating frames.

Here, we show that whenever there is a change of frame of reference, treated as a change of coordinates on the base manifold, re-expression of the action connection in the coordinates of the new frame gives rise to total differentials or exact forms $d\phi(x, t)$. These exact forms become phase factors $\exp(i\phi(x, t))$ under parallel transport given by Eq. (7).

Let us consider a one-particle system and let the position degrees of freedom constitute a complete set of commuting operators. Let us identify the eigenvalues of the position operators with physical space. Consider the vector bundle whose typical fibre is Hilbert space $H$, base manifold is spacetime and the structure group is the group of all unitary operators on $H$. Let $P_\mu = (H, -\vec{P})$ be the generators of spacetime translation operators which act on the Hilbert space at each point, where $\vec{P} = P^\mu = -P_\mu$ is the physical momentum.

We now propose that: the operator-valued connection one-form $\omega = iP_\mu dx^\mu$ be called the quantum mechanical action.

In the following we investigate the transformation of $\omega$ under change of frames of reference.
A. Uniformly moving frames

Uniformly moving frames can be accommodated by a change of coordinate from static to moving ones.

\[ x'^0 = x^0 = t \] (12)
\[ x'^i = x^i - v^i t \] (13)

The connection form \( \omega \) when expressed in terms of \( x' \) looks like

\[ i\omega = p_1 dx'^i - \frac{1}{2m} \left( p^2 - 2mv^i p^i \right) dt'. \] (14)

The momentum in the primed coordinates must be \( p'^i = p^i - mv^i \). We next put \( i\omega \) in a form which emphasizes the equivalence of the two coordinate systems, as follows

\[ i\omega = p'^i dx'^i - \frac{p'^2}{2m} + d \left( mv^i x'^i + \frac{1}{2} mv^2 t' \right) \]
\[ = i\omega' + d \left[ mv^i x'^i + \frac{1}{2} mv^2 t' \right] \]

where \( i\omega' = p'^i dx'^i - p'^2/2m \). Thus, the action connection as seen in the two frames differ by an exact differential. This exact differential shows up in the wavefunction as a phase

\[ \psi(x, t) = \exp \left[ i \left( mv^i x'^i + \frac{1}{2} mv^2 t' \right) \right] \psi'(x', t'). \] (15)

Therefore, parallel transport in the \((x', t')\) coordinates takes place with an additional factor. This phase is well known from the representation theory of the Galilean group \([8]\). It is precisely the phase that must be multiplied to the wavefunction in order to make the Schrödinger equation transform covariantly under Galilean transformation. It is worth emphasizing that no such assumption regarding how the Schrödinger equation must transform is required.

B. Uniformly accelerated frames

We can deal with accelerating frames also by simply treating them as change of coordinates and imposing energy-momentum relation in the new frame. It is sufficient to consider acceleration in the \(x'\) direction.

\[ x' = x - \frac{1}{2} g t^2, \]
\[ t' = t. \] (17)

Proceeding in a similar manner as in the last section this gives

\[ i\omega = p dx' + pgt' dt' - \frac{p^2}{2m} dt'. \] (18)

The momentum in the primed coordinates must be \( p'^i = p^i - mg t' \). Completing the square for the coefficient of \( dt' \), which amounts to imposing the mass-shell condition in the accelerated frame we get

\[ i\omega = p'^i dx'^i - \left( \frac{p'^2}{2m} + mg x' \right) dt' \]
\[ + d \left[ mg(t' x' + \frac{1}{6} g t'^3) \right]. \] (19)

Thus by simply re-expressing the action in the new frame and imposing the energy-momentum relation we have obtained the pseudo-gravitational force field given by the potential \( mg x' \) and the phase \( mg(t' x' + \frac{1}{6} g t'^3) \). It follows that the equivalence principle must hold in quantum mechanics as a natural consequence of the transformation property of the action.

It must be pointed out that in the conventional approach \([9]\) one demands that the transformation of the Schrödinger equation have an intuitive linear “gravitational potential term” and then it is found that the wavefunction must pick up the time dependent phase given above. The extra linear potential has been experimentally verified \([10]\).

It is interesting to see the group property of linear acceleration in the same direction. Let

\[ x'' = x'^1 - \frac{1}{2} g't'^2 \] (20)
\[ t'' = t. \] (21)

The exact forms add giving

\[ mg' (x'' t'' + \frac{1}{6} g' t''' + mg (x' t' + \frac{1}{6} g t'^3) \]
\[ = m(g + g') x'' t'' \]
\[ + \frac{1}{6} mt''' (g'^2 + g^2 + 3gg'). \] (22)

From the time component of the connection we get

\[ - mg x' dt' - mg' x'' dt'' = -m(g + g') x'' dt'' - \frac{1}{6} m g g' t''' . \] (23)

which gives the combined transformation with acceleration \( g + g' \).

C. Uniformly rotating frame

In this case we restrict ourselves to two space dimensions for simplicity. Let the coordinates of the rotating frame be given by

\[ x' = x \cos \omega t + y \sin \omega t \]
\[ y' = -x \sin \omega t + y \cos \omega t \]
\[ t' = t, \]
where \( \tilde{\omega} \) being the angular velocity. Then
\[
\begin{align*}
    dx &= \cos \tilde{\omega}t' dx' - \sin \tilde{\omega}t' dy' - \tilde{\omega} y dt', \\
    dy &= \sin \tilde{\omega}t' dx' + \cos \tilde{\omega}t' dy' + \tilde{\omega} x dt'.
\end{align*}
\]
The action connection in the new coordinates is
\[
i\omega = \tilde{p}' . d\tilde{x}' - \left( \frac{\tilde{p}'^2}{2m} - \tilde{\omega} J \right) dt'.
\]
where the momentum in the rotated frame is \( p' = R p, \) \( R \) being the rotation matrix and \( J = (x' p'^2 - y' p'^1) \). The terms in \( J \) can be combined with the Hamiltonian \( \tilde{p}'^2 / 2m \) to recover Coriolis and centrifugal force terms
\[
i\omega = \tilde{p}' . d\tilde{x}' - \left( \frac{(p'^1 + m \tilde{\omega} x')^2}{2m} + \frac{(p'^2 - m \tilde{\omega} y')^2}{2m} - \frac{1}{2} m \tilde{\omega}^2 \tilde{x}'^2 \right) dt'.
\]
The Coriolis force does no work as it is perpendicular to velocity. So it does not appear as a potential term, rather, it appears as a connection or a vector potential in momentum in the Hamiltonian.

\[\text{IV. RELATIVISTIC UNIFORMLY ACCELERATING FRAME AND ITS NON-RELATIVISTIC LIMIT}\]

Let the action connection in relativistic quantum mechanics be defined by the same formula as in the non-relativistic case:
\[
- i\omega = P_\mu dx^\mu
\]
where \( P_\mu \)'s are the translation operators of Minkowski spacetime. We are not requiring that a complete set of commuting observables be provided as the basic ingredient for defining action. In fact we can no longer make such a demand because in a relativistic setup, the position operators are not defined \[\text{[11]}\]. Our basic assumption is that Minkowski spacetime be given and we construct the Hilbert vector bundle with it is the base manifold. We transform to a uniformly accelerating frame. On taking the non-relativistic limit of the action we expect that we should obtain Eq. (17). We show that this is indeed so.

Consider Hilbert space consisting of momentum wave functions \( \psi(p) \) of a spinless particle of mass \( m \) with the inner product
\[
(\psi, \phi) = \int \frac{d^3p}{2p_0} \psi(p)^* \phi(p).
\]
We explicitly introduce the velocity of light \( c \) on the formulas for convenience of taking the non-relativistic limit.
\[
\begin{align*}
    x^\mu &= (ct, x^1, x^2, x^3) \\
    p^\mu &= (E, p^1 c, p^2 c, p^3 c).
\end{align*}
\]
Let us consider an observer which is uniformly accelerating along the \( x \)-axis. This means that the acceleration
\[\text{in the ‘co-moving’ frame is a fixed number} \ g. \ \text{It can be shown} \ [\text{[12]}] \ \text{that such an observer follows a trajectory}
\[
x = A \cosh \tau, \quad ct = A \sinh \tau.
\]
where \( A \) is related to the acceleration \( g \) by \( g = c^2 / A \) and \( \tau / g \) is the proper time in the ‘co-moving’ frame. In order to treat change of frames of reference as a change of coordinates we must consider a continuum of observers with \( \tau \) and \( A \) as variables.

Thus we obtain a family of trajectories, one for each value of the acceleration \( g \) or the relativistic accelerated space coordinate \( A \) and parametrized by \( \tau \). At \( \tau = 0 \) we have \( x = A \). At low velocities the non-relativistic limit for the trajectory is given by
\[
x' = x - \frac{1}{2} g \tau^2.
\]
This corresponds to large values of \( A \).

In these coordinates \( \omega \) is expressed as
\[
- i\omega = P_\tau d\tau + P_A dA
\]
where
\[
\begin{align*}
P_\tau &= A (P_0 \cosh \tau + P_1 \sinh \tau) \\
P_A &= P_0 \sinh \tau + P_1 \cosh \tau.
\end{align*}
\]
In order to take non-relativistic limit we choose a suitably large \( A \) such that
\[
A = A_0 + x', \quad A_0 = \frac{c^2}{g_0},
\]
where \( A_0 \gg x' \) and \( g_0 \) is the constant nonrelativistic acceleration. In the following we use \( g \) for \( g_0 \). Then \((x', t)\) are non-relativistic uniformly accelerating coordinates and \( \omega \) must be expressed in terms of these coordinates in the non-relativistic limit. We must make the following approximations
\[
P_0 = mc^2 + \frac{p^2}{2m}, \quad P_1 = -p
\]
Thus the non-relativistic action connection is obtained to get the correct expression for the action.

To get non-relativistic limit we must keep terms upto zeroth order only in the approximations. Since the terms \( mc^2 \) and \( A \) are each of order \(-1\) in degree of smallness we must at least keep terms upto second order before we multiply out all factors and throw away terms higher than zeroth order. Thus with these approximations we get

\[
P_\tau d\tau = \left( mc^2 + \frac{p^2}{2m} - pgt \right) cdt
\]

\[
+ mgx' cdt - mgx' cdt - mgt cdx' = \left( mc^2 + \frac{p^2}{2m} - pgt + mgx' \right) cdt - cd(mgx' t). 
\]

(34)

Upon completing the square to get \((p - mgt)^2/2m\) we get

\[
P_\tau d\tau = \left( mc^2 + \frac{(p - mgt)^2}{2m} + mgx' \right) cdt 
\]

\[-cd \left[ mg(x't + \frac{1}{6}gt^3) \right]. 
\]

(35)

The other term in the action is

\[
P_{AdA} = -\left(p - mgt\right) cdx'. 
\]

(36)

Thus the non-relativistic action connection is obtained to be

\[
-i\omega = \left( mc^2 + \frac{(p - mgt)^2}{2m} + mgx' \right) cdt 
\]

\[-\left(p - mgt\right) cdx' - cd \left[ mg(x't + \frac{1}{6}gt^3) \right]. 
\]

(37)

Comparison with Eq. (17) immediately tells that this is the correct expression for the action.

APPENDIX A: GEOMETRIC SETTING

The material in this appendix is only for fixing notation. The geometry is explained in well known books, see for example Chern et al [13].

1. The bundle and connection

Consider a vector bundle \( E \) with base manifold \( M \), Hilbert space \( \mathcal{H} \) as fibre and the group \( \mathcal{U} \) of all unitary transformations on \( \mathcal{H} \) as the structure group.

Let \( \phi_n(x) \) be a set of smooth sections such that it forms an orthonormal basis in the fibre at \( x \).

\[
(\phi_n(x), \phi_m(x)) = \delta_{nm}. 
\]

(33)

Any arbitrary section \( \psi(x) \) can then be written as

\[
\psi(x) = c_n(x) \phi_n(x) 
\]

(34)

where \( c_n(x) \) are the complex coefficients of expansion and we use Einstein summation convention. Let \( \Gamma \) be the vector space of all sections. They can be added pointwise.

\[
(\psi_1 + \psi_2)(x) = \psi_1(x) + \psi_2(x) 
\]

(35)

and multiplied with smooth functions

\[
(c\psi)(x) = c(x)\psi(x). 
\]

(36)

Let \( \Lambda \otimes \Gamma \) be the tensor product of the space \( \Lambda \) of all one-forms on the base \( M \) and \( \Gamma \). A connection on this bundle is a mapping \( D : \Gamma \to \Lambda \otimes \Gamma \) such that

\[
D(\psi_1 + \psi_2) = D\psi_1 + D\psi_2 
\]

(37)

and

\[
D(c\psi) = cD\psi + dc \otimes \psi. 
\]

(38)

As \( \phi_n(x) \) is a basis in \( \Gamma \) we can express \( D(\phi_n) \) in terms of the basis \( dx^\mu \otimes \phi_m \) in \( \Lambda \otimes \Gamma \) as

\[
(D\phi_n)(x) = \phi_m(x) \Gamma_{\mu mn} dx^\mu 
\]

(39)

where coefficients \( \Gamma_{\mu mn}(x) \) are the Christoffel symbols with respect to the basis \( dx^\mu \otimes \phi_m \). We write this equation as

\[
(D(\phi_n))(x) = \phi_m \omega_{mn}(x), 
\]

(40)

where the complex matrix \( \omega_{mn} \) can be obtained by taking inner product with \( \phi_m \) in Eq. (6).

\[
\omega_{\phi mn} = (\phi_m, D\phi_n). 
\]

(41)

This matrix of one-forms is called the connection matrix. For \( \psi(x) = c_m(x) \phi_n(x) \) we have

\[
(\phi_n, D\psi) = dc_n + \Gamma_{\mu mn} c_m dx^\mu = dc_n + \omega_{mn} c_m. 
\]

(42)

We require \( D \) to satisfy the Leibniz rule

\[
D(\phi, \psi) = (D\phi, \psi) + (\phi, D\psi) = d(\phi, \psi), 
\]

(43)

which when applied to \( \delta_{mn} = (\phi_m, \phi_n) \) shows that \( \omega^\phi \) is an anti-hermitian matrix.
Under a change of basis
\[ \chi_n(x) = U(x)\phi_n(x) \quad (A12) \]
we have
\[ \chi_n(x) = \phi_n(x) \left( \phi_m(x), U(x)\phi_n(x) \right) = \phi_n(x)U_{mn}(x). \quad (A13) \]

Omitting the base point \( x \) for simplicity of notation
\[ D\chi_n = D(\phi_s U_{sn}) = \phi_r \omega_r^\phi U_{sn} + \phi_s dU_{sn} = \chi_m \omega_m^\chi = \phi_r U_{mn} \omega_m \chi \quad (A14) \]
Or
\[ \omega_m^\chi = U_{mn}^{-1} \omega_r^\phi U_{sn} + U_{mn}^{-1} dU_{rn}. \quad (A15) \]
Omitting matrix indices, we have
\[ \omega^\chi = U^{-1} \omega^\phi U + U^{-1} dU. \quad (A16) \]
The curvature two-form for the connection is given by
\[ \Omega^\phi = d\omega^\phi + \omega^\phi \wedge \omega^\phi, \quad (A17) \]
which transforms as
\[ \Omega^\chi = U^{-1} \Lambda^\phi U. \quad (A18) \]

2. Parallel transport

Let
\[ x^\mu = c^\mu(\tau) \quad (A19) \]
be a curve in the base manifold. Let \( \psi(x) \) be a (Hilbert) vector field parallel transported along \( c^\mu(\tau) \). This means
\[ < \dot{c}, D\psi > = 0 \quad \text{for all } \tau. \quad (A20) \]

Then
\[ < \dot{c}, D\psi > = \frac{dc^\mu}{d\tau} \frac{\partial}{\partial x^\mu} d_{a_m} \phi_m + i \phi_m P_{\mu mn} a_n > \]
\[ = \frac{dc^\mu}{d\tau} \left( \frac{\partial d_{a_m}}{\partial x^\mu} \phi_m + i \phi_m P_{\mu mn} a_n \right) \]
\[ = 0 \quad (A21) \]
which gives
\[ i \partial_\mu a_m = P_{\mu mn} a_n. \quad (A22) \]
Or,
\[ a_m(y) = P \left( \exp \left( -i \int_x^y P_{\mu dx^\mu} \right) a_n(x) \right) \quad (A23) \]
where \( P \) stands for path ordering.

When the constant basis in the common Hilbert space is understood we need not specify vectors and operators as matrices and use operator notation. For example the equation above can be written as
\[ \psi(y) = P \left( \exp \left( -i \int_x^y P_{\mu dx^\mu} \right) \psi(x) \right). \quad (A24) \]

3. Curvature

If the connection is written as
\[ \omega = i P_{\mu dx^\mu} \]
then the curvature is
\[ \Omega = i \frac{1}{2} F_{\mu\nu} dx^\nu \wedge dx^\mu \quad (A25) \]
where
\[ F_{\mu\nu} = \partial_\mu P_{\nu} - \partial_\nu P_{\mu} + i [P_{\mu}, P_{\nu}]. \quad (A26) \]

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