LOCAL WELL-POSEDNESS OF THE FIFTH-ORDER KDV-TYPE EQUATIONS ON THE HALF-LINE

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Abstract. This paper is a continuation of authors’ previous work [6]. We extend the argument [6] to fifth-order KdV-type equations with different nonlinearities, in specific, where the scaling argument does not hold. We establish the $X^{s,b}$ nonlinear estimates for $b < \frac{1}{2}$, which is almost optimal compared to the standard $X^{s,b}$ nonlinear estimates for $b > \frac{1}{2}$ [8, 17]. As an immediate conclusion, we prove the local well-posedness of the initial-boundary value problem (IBVP) for fifth-order KdV-type equations on the right half-line and the left half-line.

1. Introduction. This paper is a continuation of authors’ previous work [6]. In [6], the authors studied the Duhamel boundary forcing operator associated to the fifth-order linear operator, and established the local well-posedness of Kawahara equation posed on the right/left half-line. In this paper, we extend the previous study to the fifth-order KdV-type equations whose nonlinearities are different, in particular, do not satisfy the scaling symmetry. The lack of the scaling invariance cause an additional analysis on time trace estimates with a cutoff function supported on $|t| \leq T$, and to provide such an analysis is one of aims of this work. Consider the following fifth-order KdV-type equation:

$$\partial_t u - \partial_x^5 u + F(u) = 0,$$

(1)

where $u(t,x)$ is real-valued function and $F(u)$ is a nonlinearity. We, here, take $F(u) = (1 - \partial_x^2)^{\frac{1}{2}} \partial_x (u^2)$ or $F(u) = \partial_x (u^3)$.

When $F(u) = (1 - \partial_x^2)^{\frac{1}{2}} \partial_x (u^2)$, the equation (1) was introduced by Tina, Gui and Liu [32] to understand the role of dispersive and nonlinear convection effects in the fifth-order $K(m,n,p)$ equations of the form

$$\partial_t u + \beta_1 \partial_x (u^m) + \beta_2 \partial_x (u^n) + \beta_3 \partial_x (u^p) = 0$$

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when \((m, n, p) = (2, 2, 1)\), in particular, \(\beta_1 = 1\) and \(\beta_2 = \beta_3 = -1\).

When \(F(u) = \partial_x(u^3)\), the equation (1) is well-known as the modified Kawahara equation, which was proposed first by Kawahara [19]. The modified Kawahara equation arises in the theory of shallow water waves, the theory of magneto-acoustic waves in plasmas and propagation of nonlinear water-waves in the long-wavelength region as in the case of KdV equations. This equation is also regarded as a singular perturbation of KdV equation. We refer to [1, 16, 14] and references therein for more background informations.

1.1. **Main analysis.** The principal contribution in the paper is to establish the nonlinear estimates for both nonlinearities \(F(u) = (1 - \partial_x^2)^{\frac{1}{2}} \partial_x(u^2)\) and \(F(u) = \partial_x(u^3)\), in particular,

\[
\|F(u)\|_{X^{s,-b} \cap Y^{s,-b}} \lesssim \|u\|_{X^{s,b} \cap D^\alpha}^k, \quad k = 2, 3, \tag{2}
\]

for a certain regularity \(s \in \mathbb{R}, 0 < b < \frac{1}{2} < \alpha < 1 - b\). The functions spaces used in (2) are the standard \(X^{s,b}\) space\(^1\) equipped with the norm

\[
\|f\|_{X^{s,b}} = \|\langle\xi\rangle^s (\tau - \xi^5)^b \widetilde{f}\|_{L^2_{\tau,\xi}}, \tag{3}
\]

where \(\widetilde{f}\) is the space time Fourier coefficient (also denoted by \(F(f)\)) and \(\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}\), initially introduced in its current form by Bourgain [3], and its various modifications, see below for a short explanation of spaces and Section 2.3 for precise definitions of spaces.

The \(X^{s,b}\) space is known to be an appropriate device to detect dispersive phenomena in the Fourier analysis. In other words, solutions to (1) have the dispersive smoothing effect which means that the (space-time) Fourier coefficients decay far away from the characteristic curve \(\tau = \xi^b\), where \(\xi\) and \(\tau\) are the Fourier variables corresponding to \(x\) and \(t\), respectively. This property is naturally reflected in (3) as a weight \(\langle\tau - \xi^5\rangle^b\). The choice of the exponent \(b < \frac{1}{2}\) in (3) is imposed in the study on IBVP due to the presence of the Duhamel boundary forcing operator, which reflects what role boundary conditions play in the solutions (see Section 4 for more details), while the standard \(X^{s,b}\) space with \(b > \frac{1}{2}\) works well in the study on the initial value problem (IVP).

The trade-off of choosing \(b < \frac{1}{2}\) causes the lack of \(\tau\)-integrability in (2), when all functions are localized in the frequency support \(|\xi| \leq 1\). In order to resolve this problem, an additional low frequency localized space \(D^\alpha, \alpha > \frac{1}{2}\), is needed, i.e., (3) under the restriction \(|\xi| \leq 1\), precisely,

\[
\|f\|_{D^\alpha} = \|\langle\tau\rangle^b \chi_{|\xi| \leq 1} (\xi) \widetilde{f}\|_{L^2_{\tau,\xi}}.
\]

On the other hand, time trace estimates of the Duhamel parts (Lemma 5.2) in \(X^{s,b}\)-type spaces hold true for only positive regularities (see Remark 9), thus an additional introduction of the (time-adapted) Bourgain space \(Y^{s,b}\) as an intermediate norm in the iteration process, which is defined similarly as the standard \(X^{s,b}\) space but with a weight in terms of \(\tau\) instead of \(\xi\) in the sense of \(\partial_t \sim \partial_x^5\) in (1) (replacing \(\langle\xi\rangle^s\) by \(\langle\tau\rangle^\frac{5}{2}\) in (3)), precisely

\[
\|f\|_{Y^{s,b}} = \|\langle\tau\rangle^\frac{5}{2} \langle\tau - \xi^5\rangle^b \widetilde{f}\|_{L^2_{\tau,\xi}},
\]

is necessary to cover the negative regularities.

The followings are the main results established in the paper.

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\(^1\)It is called Bourgain’s space or dispersive Sobolev space.
Theorem 1.1. 
(a) For $-5/4 < s$, there exists $b = b(s) < 1/2$ such that for all $\alpha > 1/2$, we have 
\[ \|(1 - \partial_x^2)^{\alpha} \partial_x(u)\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b,\cap}D^\alpha} \|v\|_{X^{s,b,\cap}D^\alpha}. \]  
(b) For $-5/4 < s \leq 0$, there exists $b = b(s) < 1/2$ such that for all $\alpha > 1/2$, we have 
\[ \|(1 - \partial_x^2)^{\alpha} \partial_x(u)\|_{Y^{s,b}} \lesssim \|u\|_{X^{s,b,\cap}D^\alpha} \|v\|_{X^{s,b,\cap}D^\alpha}. \]

The implicit constants in (4) and (5) depend only on $s$, $b$, $\alpha$.

Theorem 1.2. 
(a) For $-1/4 \leq s$, there exists $b = b(s) < 1/2$ such that for all $\alpha > 1/2$, we have 
\[ \|\partial_x(uv)\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b,\cap}D^\alpha} \|v\|_{X^{s,b,\cap}D^\alpha}. \]  
(b) For $-1/4 \leq s \leq 0$, there exists $b = b(s) < 1/2$ such that for all $\alpha > 1/2$, we have 
\[ \|\partial_x(uv)\|_{Y^{s,b}} \lesssim \|u\|_{X^{s,b,\cap}D^\alpha} \|v\|_{X^{s,b,\cap}D^\alpha}. \]

The implicit constants in (6) and (7) depend only on $s$, $b$, $\alpha$.

Remark 1. Theorems 1.1 and 1.2 are almost sharp compared with [8] and [17], respectively, in the low regularity sense (in particular, negative regularity). We also refer to [33] for the weak ill-posedness result for the modified Kawahara equation in $H^s(\mathbb{R})$, $s < -\frac{1}{4}$.

Remark 2. Both (5) and (7) can be obtain in some positive regularity regime similarly as Proposition 5.2 in [6], but we, here, explore the nonlinear estimates for both $F(u) = (1 - \partial_x^2)^{\alpha} \partial_x(u^2)$ and $F(u) = \partial_x(u^3)$ in $Y^{s,b}$ only in the negative regularity regime, since the intermediate norm $Y^{s,b}$ occurs in the Picard iteration mechanism only in the negative regularity regime (see the proof of Lemma 5.2 (b)).

The proof of Theorems 1.1 (a) and 1.2 (a) are based on the harmonic analysis technique, in particular the Taos $[k;Z]$-multiplier norm method [29], which has become now standard to prove the multilinear estimates. Precisely, let $P_k$ be the Littlewood-Paley projection operator in terms of the frequency on a support $|\xi| \sim 2^k$, $k \in \mathbb{Z}$ (will be precisely defined in Section 2) and $f_k = P_k f$. Let further decompose $f_k$ into $f_k = \eta_j(\tau - \xi^5) f_k$ pieces, where $\eta_j(\zeta)$ is a smooth bump function supported in $|\zeta| \sim 2^j$. Then, the Littlewood-Paley decomposition (not only in terms of the frequency $|\xi|$, but also with respect to the modulation $|\tau - \xi^5|$) allows us to separate the left-hand side of (4)–(7) into each frequency part, for instance in (4)
\[ \|(1 - \partial_x^2)^{\alpha} \partial_x(u)\|_{X^{s,b}} \lesssim \sum_{k,k_1,k_2 \in \mathbb{Z}} \max(1, 2^{(1+s)k}) 2^k \|P_k(u_{k_1} v_{k_2})\|_{X^{s,b}} \]
and
\[ \|P_k(u_{k_1} v_{k_2})\|_{X^{s,b}} \lesssim \sum_{j,j_1,j_2 \geq 0} 2^{-bj} \|\eta_j(\tau - \xi^5) F(P_k(u_{j_1} v_{j_2}, v_{j_2}))\|_{L^2_t L_x}. \]

The quadratic nonlinearity consists of high × low ⇒ high, high × high ⇒ high and high × high ⇒ low cases\(^2\) (in terms of the relation among frequencies), and the cubic nonlinearity consists of more cases with various frequency relations, see the proof of Proposition 3. Thus, the task is reduced to prove multilinear estimates of each piece in $L^2$. Such $L^2$-block estimates have already been provided by Chen.

\(^2\)The low × low ⇒ low interaction case can be dealt with by the trivial bound, hence we do not comment it on here.
Li, Miao and Wu [10] for the bilinear case (Chen and Guo [7] corrected the high \times high \Rightarrow high case), and by the second author [22] for the trilinear case (in [22], the \( L^2 \)-block estimates for periodic functions in the spatial variable are given, but the proof for non-periodic functions is analogous). Performing \( L^2 \) estimates and gathering all pieces, one reaches the right-hand side of (4)–(7).

A direct proof of trilinear estimates (6) is given, while bilinear estimates and \( TT^* \) argument are used to prove the trilinear estimates in [8]. Moreover, the Strichartz estimate (with derivative gains) for the linear operator group \( \{ e^{it\partial_x^2} \} \) [13] is needed to deal with \( high \times high \times high \Rightarrow high \) interaction component, since the trilinear \( L^2 \)-block estimates (Lemma 3.2) are given for only \( high \times low \times low \Rightarrow high \) case.

The proof of Theorems 1.1 (b) and 1.2 (b) are based on the proof of Lemma 5.10 (b) in [15], but more careful examination of frequency relations is needed (also for the proof of Theorems 1.1 (a) and 1.2 (a)).

1.2. IBVP problems setting. The IBVP of (1), here, is studied as an application of Theorems 1.1 and 1.2. The precise IBVP of (1) on the right/left half-lines is set as follows:

\[
\begin{aligned}
\begin{cases}
\partial_t u - \partial_x^2 u + F(u) = 0, & (t, x) \in (0, T) \times (0, \infty), \\
u(0, x) = u_0(x), & x \in (0, \infty), \\
u(t, 0) = f(t), u_x(t, 0) = g(t) & t \in (0, T)
\end{cases}
\end{aligned}
\]  

and

\[
\begin{aligned}
\begin{cases}
\partial_t u - \partial_x^2 u + F(u) = 0, & (t, x) \in (0, T) \times (-\infty, 0), \\
u(0, x) = u_0(x), & x \in (-\infty, 0), \\
u(t, 0) = f(t), u_x(t, 0) = g(t), u_{xx}(t, 0) = h(t) & t \in (0, T).
\end{cases}
\end{aligned}
\]

The number of boundary conditions in (8) and (9) is inspired from the uniqueness issue arising in a direct calculation of \( L^2 \) integral identities for linear equations:

\[
\begin{aligned}
\int_0^\infty u^2(T, x)dx &= \int_0^\infty u^2(0, x)dx - \int_0^T (\partial_x^2 u(t, 0))^2 dt + 2 \int_0^T \partial_x^4 u(t, 0)u_x(t, 0)dt \\
&\quad - 2 \int_0^T \partial_x^4 u(t, 0)u(t, 0)dt 
\end{aligned}
\]

and

\[
\begin{aligned}
\int_{-\infty}^0 u^2(T, x)dx &= \int_{-\infty}^0 u^2(0, x)dx + \int_0^T (\partial_x^2 u(t, 0))^2 dt - 2 \int_0^T \partial_x^4 u(t, 0)\partial_x u(t, 0)dt \\
&\quad + 2 \int_0^T \partial_x^4 u(t, 0)u(t, 0)dt.
\end{aligned}
\]

We refer to [15, 6] for more expositions.

The local smoothing effect [21]

\[
\|\partial_x^j e^{it\partial_x^2} \phi\|_{L^\infty_x H^{s+2-j}(-\infty, \infty)} \leq c\|\phi\|_{H^s(\mathbb{R})}, \text{ for } j = 0, 1, 2,
\]

stipulates the appropriate spaces for the initial and boundary data, thus the initial and boundary data for (8) and (9) satisfy

\[
u_0 \in H^s(\mathbb{R}^+), \quad f(t) \in H^{\frac{s+2}{2}}(\mathbb{R}^+) \quad \text{and} \quad g(t) \in H^{\frac{s+1}{2}}(\mathbb{R}^+)
\]

and

\[
u_0 \in H^s(\mathbb{R}^-), \quad f(t) \in H^{\frac{s+2}{2}}(\mathbb{R}^+), \quad g(t) \in H^{\frac{s+1}{2}}(\mathbb{R}^+) \quad \text{and} \quad \nu(t) \in H^s(\mathbb{R}^+),
\]
respectively. On the other hand, the compatibility conditions in high regularities, for instance, $\frac{1}{2} < s < \frac{3}{2}$ or $\frac{3}{2} < s < \frac{5}{2}$, ..., are required to be considered as follows, for instance,

$$u_0(0) = f(0), \quad \text{if } \frac{1}{2} < s < \frac{3}{2}, \quad u_0(0) = f(0), \partial_x u_0(0) = g(0) \quad \text{if } \frac{3}{2} < s$$

for (8). However, our local well-posedness results for both (8) and (9) are valid only in the regularity region $s < \frac{1}{2}$ (see theorems 1.3–1.6 below), and hence the compatibility conditions for high regularities are negligible. See [6] for the comparison.

Theorems 1.1 and 1.2 in addition to the standard argument used in [6] immediately imply the local well-posedness of the IBVP for (1) on the right half-line. We state theorems separately for the sake of reader’s convenience.

**Theorem 1.3.** Let $s \in (-\frac{5}{4}, \frac{1}{2})$. For given initial-boundary data $(u_0, f, g)$ satisfying (12), there exist a positive time $T > 0$ depending on $\|u_0\|_{H^s(\mathbb{R}^+)}$, $\|f\|_{H^{\frac{s+4}{2}}(\mathbb{R}^+)}$ and $\|g\|_{H^{\frac{s+4}{2}}(\mathbb{R}^+)}$, and a solution $u(t, x) \in C((0, T); H^s(\mathbb{R}^+))$ to (8)-(12) with the nonlinearity $F(u) = (1 - \partial_x^2)\frac{1}{2}\partial_x(u^2)$ satisfying

$$u \in C(\mathbb{R}^+; H^{\frac{s-2}{2}}(0, T)) \cap X^{s,b}((0, T) \times \mathbb{R}^+) \cap D^\alpha((0, T) \times \mathbb{R}^+)$$

and

$$\partial_x u \in C(\mathbb{R}^+; H^{\frac{s+4}{2}}(0, T))$$

for some $b(s) < \frac{1}{2}$ and $\alpha(s) > \frac{1}{2}$. Moreover, the map $(u_0, f, g) \mapsto u$ is analytic from $H^s(\mathbb{R}^+) \times H^{\frac{s+4}{2}}(\mathbb{R}^+) \times H^{\frac{s+4}{2}}(\mathbb{R}^+)$ to $C((0, T); H^s(\mathbb{R}^+))$.

**Theorem 1.4.** Let $s \in [-\frac{1}{4}, \frac{1}{2})$. For given initial-boundary data $(u_0, f, g)$ satisfying (12), there exist a positive time $T > 0$ depending on $\|u_0\|_{H^s(\mathbb{R}^+)}$, $\|f\|_{H^{\frac{s+4}{2}}(\mathbb{R}^+)}$ and $\|g\|_{H^{\frac{s+4}{2}}(\mathbb{R}^+)}$, and a solution $u(t, x) \in C((0, T); H^s(\mathbb{R}^+))$ to (8)-(12) with the nonlinearity $F(u) = \partial_x(u^3)$ satisfying

$$u \in C(\mathbb{R}^+; H^{\frac{s+2}{2}}(0, T)) \cap X^{s,b}((0, T) \times \mathbb{R}^+) \cap D^\alpha((0, T) \times \mathbb{R}^+)$$

and

$$\partial_x u \in C(\mathbb{R}^+; H^{\frac{s+4}{2}}(0, T))$$

for some $b(s) < \frac{1}{2}$ and $\alpha(s) > \frac{1}{2}$. Moreover, the map $(u_0, f, g) \mapsto u$ is analytic from $H^s(\mathbb{R}^+) \times H^{\frac{s+4}{2}}(\mathbb{R}^+) \times H^{\frac{s+4}{2}}(\mathbb{R}^+)$ to $C((0, T); H^s(\mathbb{R}^+))$.

Moreover, we have the local well-posedness of the IBVP for (1) on the left half-line.

**Theorem 1.5.** Let $s \in (-\frac{5}{4}, \frac{1}{2})$. For given initial-boundary data $(u_0, f, g, h)$ satisfying (13), there exist a positive time $T$ depending on $\|u_0\|_{H^s(\mathbb{R}^-)}$, $\|f\|_{H^{\frac{s+4}{2}}(\mathbb{R}^+)}$, $\|g\|_{H^{\frac{s+4}{2}}(\mathbb{R}^+)}$ and $\|h\|_{H^{\frac{s+4}{2}}(\mathbb{R}^+)}$, and a solution $u(t, x) \in C((0, T); H^s(\mathbb{R}^-))$ to (9)-(13) with the nonlinearity $F(u) = (1 - \partial_x^2)\frac{1}{2}\partial_x(u^2)$ satisfying

$$u \in C(\mathbb{R}^-; H^{\frac{s+2}{2}}(0, T)) \cap X^{s,b}((0, T) \times \mathbb{R}^-) \cap D^\alpha((0, T) \times \mathbb{R}^-),$$

$$\partial_x u \in C(\mathbb{R}^-; H^{\frac{s+4}{2}}(0, T))$$

and

$$\partial_x^2 u \in C(\mathbb{R}^-; H^{\frac{s+4}{2}}(0, T))$$

for some $b(s) < \frac{1}{2}$ and $\alpha(s) > \frac{1}{2}$. Moreover, the map $(u_0, f, g, h) \mapsto u$ is analytic from $H^s(\mathbb{R}^-) \times H^{\frac{s+4}{2}}(\mathbb{R}^+) \times H^{\frac{s+4}{2}}(\mathbb{R}^+) \times H^{\frac{s+4}{2}}(\mathbb{R}^+)$ to $C((0, T); H^s(\mathbb{R}^-))$. 
Theorem 1.6. Let $s \in \left[ \frac{1}{3}, \frac{1}{2} \right]$. For given initial-boundary data $(u_0, f, g, h)$ satisfying (13), there exist a positive time $T$ depending on $\|u_0\|_{H^s(\mathbb{R}^{-})}$, $\|f\|_{H^{\frac{s+1}{2}}(\mathbb{R}^{+})}$, $\|g\|_{H^{\frac{s+1}{2}}(\mathbb{R}^{+})}$, and $\|h\|_{H^{\frac{s}{2}}(\mathbb{R}^{+})}$, and a solution $u(t, x) \in C((0, T); H^s(\mathbb{R}^{-}))$ to (9)-(13) with the nonlinearity $F(u) = \partial_x(u^3)$ satisfying

$$u \in C(\mathbb{R}^{-}; H^{\frac{s+1}{2}}(0, T)) \cap X^{s,b}((0, T) \times \mathbb{R}^{-}) \cap D^\alpha((0, T) \times \mathbb{R}^{-}),$$

$$\partial_t u \in C(\mathbb{R}^{-}; H^{\frac{s+1}{2}}(0, T)) \text{ and } \partial_x^2 u \in C(\mathbb{R}^{-}; H^{\frac{s}{2}}(0, T))$$

for some $b(s) < \frac{1}{2}$ and $\alpha(s) > \frac{1}{2}$. Moreover, the map $(u_0, f, g, h) \mapsto u$ is analytic from $H^s(\mathbb{R}^{-}) \times H^{\frac{s+1}{2}}(\mathbb{R}^{+}) \times H^{\frac{s+1}{2}}(\mathbb{R}^{+}) \times H^{\frac{s}{2}}(\mathbb{R}^{+})$ to $C((0, T); H^s(\mathbb{R}^{-}))$.

Remark 3. The proof of Theorems 1.3–1.6 in Section 6 claims that extension of solutions are unique under an auxiliary condition: the solution $u(t, x)$ as in (140) is unique in $Z^{s,b,\alpha,\gamma}_\lambda$, or it just guarantees the existence of a weak solution (a formulation of the integral equation). However, an analogous argument in [2, Section 4]3, in particular Proposition 4.13, Corollary 4.14 and Proposition 4.15, shows that weak solutions obtained in Theorems 1.3–1.6 are mild solutions and thus they are unique. The precise definitions and uniqueness of mild solutions are given in [2], and we do not pursue it here.

The proof of Theorems 1.3–1.6 relies on the argument introduced in Colliander-Kenig work [12], which was further developed by Holmer [15], i.e., the Fourier restriction norm method for a suitable extension of solutions ensures the local well-posedness. Precisely, the Duhamel boundary forcing operator, introduced in [12], corresponding to fifth-order KdV operator enables us to successfully construct solutions in $\mathbb{R}_x$ (or extend solutions to (1) posed on the (right/left) half-line to solutions defined in whole line $\mathbb{R}$, see Section 4). In other words, the IBVP of (1) is converted to the IVP of (1) (integral equation formula). This work has been done in our previous work [6]. After this procedure, we follows the standard iteration method in addition to the energy and nonlinear estimates (will be established in Sections 5 and 3, respectively) to show the local well-posedness of IVP of (1). The new ingredients here are the multilinear estimates for $F(u) = (1 - \partial_x^2)^{\frac{3}{2}} \partial_x(u^2)$ and $F(u) = \partial_x(u^3)$ presented in Theorems 1.1 and 1.2.

When $F(u) = (1 - \partial_x^2)^{\frac{3}{2}} \partial_x(u^3)$, the equation (1) does not admit the scaling symmetry due to the presence of the nonlocal operator $(1 - \partial_x^2)^{\frac{3}{2}}$ in the nonlinearity. When $F(u) = \partial_x(u^3)$, on the other hand, it is known that the modified Kawahara equation admits the scaling symmetry: if $u$ is a solution to (1), $u_\lambda$ defined by

$$u_\lambda(t, x) : = \lambda^2 u(\lambda^5 t, \lambda x), \quad \lambda > 0$$

is a solution to (1) as well. A straightforward calculation gives

$$\|u_{0, \lambda}\|_{H^s} + \|f_{\lambda}\|_{H^{\frac{s+1}{2}}} + \|g_{\lambda}\|_{H^{\frac{s+1}{2}}} + \|h_{\lambda}\|_{H^{\frac{s}{2}}} = \lambda^\frac{5}{2} \langle \lambda \rangle^s \|u_0\|_{H^s} + \lambda^{-\frac{s}{2}} \langle \lambda \rangle^{s+2} \|f\|_{H^{\frac{s+1}{2}}} + \lambda^\frac{s}{2} \langle \lambda \rangle^{s+1} \|g\|_{H^{\frac{s+1}{2}}} + \lambda^\frac{s}{2} \langle \lambda \rangle^s \|f\|_{H^{\frac{s}{2}}},$$

(14)

which reveals that the $\lambda$-scaled initial and boundary data cannot be small in some sense at the same time due to $\lambda^{-\frac{s}{2}} \langle \lambda \rangle^{s+2} \|f\|_{H^{\frac{s+1}{2}}}$ term in the right-hand side of (14) for $0 < \lambda \ll 1$. Thus, the scaling-rescaling argument no longer applies to the IBVP of (1) with nonlinearity both $F(u) = (1 - \partial_x^2)^{\frac{3}{2}} \partial_x(u^2)$ and $F(u) = \partial_x(u^3)$.

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3The argument introduced in [2] works well to this problem, even if the argument is concerned with the KdV equation.
To study the IBVP of (1) for arbitrary initial and boundary data, the energy estimate in a short time interval $[0,T]$, $T \ll 1$, is needed, in particular, (derivatives) time trace estimates. However, the time localized cut-off function is no longer free in the time trace norm (Lemmas 5.1 (b), 5.2 (b) and 5.3 (b)), thus, the regularity threshold, in particular the upper-bound of regularity ($s < \frac{1}{2}$) in Theorems 1.3–1.6, is restricted by the time trace norm estimates. See Section 5 for the details.

1.3. Review on the well-posedness results. Both the IVP and the IBVP of the fifth-order KdV-type equations have been extensively studied. When $F(u) = (1 - \partial_x^2)^2 \partial_x(u^2)$, the local well-posedness of (1) was first established by Tina, Gui and Liu [32] in $H^s(\mathbb{R})$, $s \geq -11/16$ by using the Fourier restriction norm method [3]. In addition to the technique Tao’s $[K;Z]$ multiplier norm method [29], Chen and Liu [9] improved the local well-posedness in $H^s(\mathbb{R})$, $s > -5/4$ and they also showed the ill-posedness, in the sense of the lack of continuity of the flow map, for $s < -5/4$. At the endpoint regularity $H^{-5/4}(\mathbb{R})$, Chen, Guo and Liu [8] proved the local well-posedness by using Besov-type function spaces. This is the optimal result until now as far as authors’ know.

When $F(u) = \partial_x^3(u^3)$, the Cauchy problem for (1) was studied by Jia and Huo [17] and Chen, Li, Miao and Wu [10], independently. They established the local well-posedness in $H^s(\mathbb{R})$, $s \geq -1/2$ by using the Fourier restriction norm method. In [10], the authors used bilinear $L^2$-block estimates and $TT^*$ argument to prove the trilinear estimate, while a direct calculation of trilinear integral operator was performed in [17]. The global well-posedness of (1) in $H^s(\mathbb{R})$, $s > -3/22$ was shown by Yan, Li and Yang [34] via the I-method [11].

The IBVP of the modified Kawahara equation posed on the right half-line in the high regularity Sobolev space $H^s(\mathbb{R}^+) \ (\frac{1}{2} \leq s < 2)$ has been studied by Tao and Lu [31]. On the other hand, the IBVP of (1) on the half-lines (both right and left), where the nonlinearity is given by both $F(u) = (1 - \partial_x^2)^2 \partial_x(u^2)$ and $F(u) = \partial_x(u^3)$, in the low regularity setting (in particular, negative regularities) is first considered here as far as we know. We end this section with referring to [25, 23, 26, 24, 6] and references therein for the IBVP results of the fifth-order KdV-type equations posed on the half-line.

1.4. Organization of the paper. The rest of paper is organized as follows: In Section 2, we mainly construct the solution space and observe several basic properties for the IBVP of (1). In Section 3, we give the proofs of Theorems 1.1 and 1.2. In Section 4, we briefly introduce the Duhamel boundary forcing operator for the fifth-order equations. In Section 5, we establish energy estimates, in particular time trace estimate, with a short time cut-off function. In Sections 6, we prove Theorems 1.3–1.6.

2. Preliminaries. Let $\mathbb{R}^+ = (0, \infty)$. For positive real numbers $x, y \in \mathbb{R}^+$, we mean $x \lesssim y$ by $x \leq C y$ for some $C > 0$. Also, $x \sim y$ means $x \lesssim y$ and $y \lesssim x$. Similarly, $\lesssim_\alpha$ and $\sim_\alpha$ can be defined, where the implicit constants depend on $\alpha$.

For a cut-off function $\psi$ given by

$$\psi \in C_0^\infty(\mathbb{R}) \quad \text{such that} \quad 0 \leq \psi \leq 1, \quad \psi \equiv 1 \quad \text{on} \quad [-1,1], \quad \psi \equiv 0, \quad |t| \geq 2, \quad (15)$$

we fix the time localized function

$$\psi_T(t) = \psi(t/T), \quad 0 < T < 1.$$
2.1. Riemann-Liouville fractional integral. A brief summary of the Riemann-Liouville fractional integral operator is, here, given, see [12, 13] for more details. Let \( t_+ \) be a function defined by
\[
  t_+ = t \quad \text{if} \quad t > 0, \quad t_+ = 0 \quad \text{if} \quad t \leq 0,
\]
and \( t_- \) can be defined by \( t_- = (-t)_+ \). Let \( \alpha \) be a complex number. For \( \Re \alpha > 0 \), the tempered distribution \( \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \) is defined as a locally integrable function by
\[
  \left\langle \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, f \right\rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f(t) \, dt.
\]
It is straightforward to obtain
\[
  \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} = \partial_k^k \left( \frac{t_+^{\alpha+k-1}}{\Gamma(\alpha+k)} \right),
\]
for all \( k \in \mathbb{N} \). The expression (16) facilitates to extend the definition of \( \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \) to all \( \alpha \in \mathbb{C} \) in the sense of distributions. The Fourier transform of \( \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \) is given by
\[
  \left( \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \right)^\wedge (\tau) = e^{-\frac{1}{2} \pi i \alpha} (\tau - i0)^{-\alpha},
\]
where \((\tau - i0)^{-\alpha}\) is the distributional limit. When \( \alpha \notin \mathbb{Z} \), (17) can be rewritten by
\[
  \left( \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \right)^\wedge (\tau) = e^{-\frac{1}{2} \alpha \pi i} |\tau|^{-\alpha} \chi_{(0,\infty)} + e^{\frac{1}{2} \alpha \pi i} |\tau|^{-\alpha} \chi_{(-\infty,0)}.
\]
Together with (17) and (18), we see
\[
  (\tau - i0)^{-\alpha} = |\tau|^{-\alpha} \chi_{(0,\infty)} + e^{\alpha \pi i} |\tau|^{-\alpha} \chi_{(-\infty,0)}.
\]
For \( f \in C_0^\infty(\mathbb{R}^+) \), we define the Riemann-Liouville fractional integral operator by
\[
  \mathcal{I}_\alpha f = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} * f,
\]
in particular,
\[
  \mathcal{I}_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds,
\]
for \( \Re \alpha > 0 \). The well-know properties are \( \mathcal{I}_0 f = f, \mathcal{I}_1 f(t) = \int_0^t f(s) \, ds, \mathcal{I}_{-1} f = f' \) and \( \mathcal{I}_a \mathcal{I}_b = \mathcal{I}_{a+b} \).

We end this subsection with introducing some lemmas associated to the Riemann-Liouville fractional integral operator \( \mathcal{I}_\alpha f \) without proofs.

**Lemma 2.1** (Lemma 2.1 in [15]). If \( f \in C_0^\infty(\mathbb{R}^+) \), then \( \mathcal{I}_\alpha f \in C_0^\infty(\mathbb{R}^+) \), for all \( \alpha \in \mathbb{C} \).

**Lemma 2.2** (Lemma 5.3 in [15]). If \( 0 \leq \Re \alpha < \infty \) and \( s \in \mathbb{R} \), then \( \|\mathcal{I}_{-\alpha} h\|_{H^0(\mathbb{R}^+)} \leq c\|h\|_{H^{\alpha}_0(\mathbb{R}^+)} \), where \( c = c(\alpha) \).

**Lemma 2.3** (Lemma 5.4 in [15]). If \( 0 \leq \Re \alpha < \infty \), \( s \in \mathbb{R} \) and \( \mu \in C_0^\infty(\mathbb{R}) \), then \( \|\mu \mathcal{I}_{\alpha} h\|_{H^0(\mathbb{R}^+)} \leq c\|h\|_{H^\alpha_0(\mathbb{R}^+)} \), where \( c = c(\mu, \alpha) \).
2.2. Oscillatory integral. Let

\[ B^{(n)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} (i\xi)^n e^{ix\xi} e^{i\xi^2} d\xi \]  

for \( n = 0, 1, \cdots \). A direct calculation (with the change of variable \( \eta = \xi^5 \), the change of contour in complex analysis and a property of the gamma function \( \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \) gives  

\[
B(0) = \frac{\cos\left(\frac{\pi}{10}\right)}{5\sin\left(\frac{\pi}{5}\right)\Gamma(4/5)}, \quad B'(0) = -\frac{\cos\left(\frac{3\pi}{10}\right)}{5\sin\left(\frac{3\pi}{5}\right)\Gamma(3/5)},
\]

\[
B''(0) = -\frac{\cos\left(\frac{3\pi}{10}\right)}{5\sin\left(\frac{3\pi}{5}\right)\Gamma(2/5)} \quad \text{and} \quad B^{(3)}(0) = \frac{\cos\left(\frac{\pi}{10}\right)}{5\sin\left(\frac{\pi}{5}\right)\Gamma(1/5)}.
\]

Moreover, we have

\[
\int_0^\infty B(y) \, dy = \frac{1}{2\pi} \left(-\frac{\pi}{5} + \pi\right) = \frac{2}{5}.
\]

We refer to [6] for the details.

We finish this subsection with introducing some lemmas associated to \( B(x) \) without proofs.

**Lemma 2.4** (Decay of oscillatory integral \( B(x) \), [28, 6]). Suppose \( x > 0 \). Then as \( x \to \infty \),

(i) \( B(x) \lesssim (x)^{-N} \) for all \( N > 0 \).
(ii) \( B(-x) \lesssim (x)^{-3/8} \).

**Lemma 2.5** (Mellin transform of \( B(x) \)).

(i) For \( \text{Re} \lambda > 0 \) we have

\[
\int_0^\infty x^{\lambda-1} B(x) \, dx = \frac{\Gamma(\lambda)\Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right)}{5\pi} \cos\left(\frac{1 + 4\lambda\pi}{10}\right).
\]

(ii) For \( 0 < \text{Re} \lambda < \frac{3}{8} \) we have

\[
\int_0^\infty x^{\lambda-1} B(-x) \, dx = \frac{\Gamma(\lambda)\Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right)}{5\pi} \cos\left(\frac{1 - 6\lambda\pi}{10}\right).
\]

We remark in (20) that \( \Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right) \) has poles at \( \lambda = 1 + 5n, n = 0, 1, 2, \cdots \), but \( \cos\left(\frac{(1+4\lambda)\pi}{10}\right) = 0 \) at the same values of \( \lambda \). Moreover, the range of \( \text{Re} \lambda \) relies on the decay rates of \( B(x) \) and \( B(-x) \) in Lemma 2.4.

2.3. Sobolev spaces on the half-line and solution spaces. Let \( s \geq 0 \). We say \( f \in H^s(\mathbb{R}^+) \) if there exists \( F \in H^s(\mathbb{R}) \) such that \( f(x) = F(x) \) for \( x > 0 \), in this case we set \( ||f||_{H^s(\mathbb{R}^+)} = \inf_F ||F||_{H^s(\mathbb{R})} \). For \( s \in \mathbb{R} \), we say \( f \in H^s_0(\mathbb{R}^+) \) if there exists \( F \in H^s(\mathbb{R}) \) such that \( F \) is the extension of \( f \) on \( \mathbb{R} \) and \( F(x) = 0 \) for \( x < 0 \). In this case, we set \( ||f||_{H^s_0(\mathbb{R}^+)} = ||F||_{H^s(\mathbb{R})} \). For \( s < 0 \), we define \( H^s(\mathbb{R}^+) \) as the dual space of \( H^{-s}_0(\mathbb{R}^+) \).

We also set \( C^\infty_0(\mathbb{R}^+) = \{ f \in C^\infty(\mathbb{R}); \text{supp} f \subset [0, \infty) \} \), and define \( C^\infty_{0,c}(\mathbb{R}^+) \) as the subset of \( C^\infty_0(\mathbb{R}^+) \), whose members have a compact support on \( (0, \infty) \). We remark that \( C^\infty_{0,c}(\mathbb{R}^+) \) is dense in \( H^{-s}_0(\mathbb{R}^+) \) for all \( s \in \mathbb{R} \).

\(^4\)Non-singularity of \( B^{(n)}(x) \) at \( x = 0 \) is needed for the continuity property of \( \partial_x^k \mathcal{L}^0 f, k = 0, 1, 2, 3 \), see Lemma 4.1.
Lemma 2.6 (Lemma 2.1 in [6]). For $-\frac{1}{2} < s < \frac{1}{2}$ and $f \in H^s(\mathbb{R})$, we have

$$\|\chi_{(0,\infty)}f\|_{H^s(\mathbb{R})} \leq c\|f\|_{H^s(\mathbb{R})}.$$ 

Lemma 2.7 (Lemma 2.2 in [6]). If $0 \leq s < \frac{1}{2}$, then $\|\psi f\|_{H^s(\mathbb{R})} \leq c\|f\|_{H^s(\mathbb{R})}$ and $\|\psi f\|_{H^{-s}(\mathbb{R})} \leq c\|f\|_{H^{-s}(\mathbb{R})}$, where the constant $c$ depends only on $s$ and $\psi$.

Remark that Lemma 2.7 is equivalent that $\|f\|_{H^s(\mathbb{R})} \sim \|f\|_{H^s(\mathbb{R})}$ for $-\frac{1}{2} < s < \frac{1}{2}$, where $f \in H^s$ with supp $f \subset [0, 1]$.

Lemma 2.8 (Proposition 2.4 in [12]). If $\frac{1}{2} < s < \frac{3}{2}$ the following statements are valid:

(a) $H^s_0(\mathbb{R}^+)$ = \{ $f \in H^s(\mathbb{R}^+); f(0) = 0$ \},

(b) If $f \in H^s(\mathbb{R}^+)$ with $f(0) = 0$, then $\|\chi_{(0,\infty)}f\|_{H^s_0(\mathbb{R}^+)} \leq c\|f\|_{H^s(\mathbb{R}^+)}$.

Lemma 2.9 (Proposition 2.5. in [12]). Let $f \in H^s_0(\mathbb{R}^+)$. For the cut-off function $\psi$ defined in (15), we have $\|\psi f\|_{H^s_0(\mathbb{R}^+)} \leq c\|f\|_{H^s_0(\mathbb{R}^+)}$ for $-\infty < s < \infty$.

Let $f \in S(\mathbb{R}^2)$. We define the Fourier transform of $f$ with respect to both spatial and time variables by

$$\hat{f}(\tau, \xi) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(t, x) \, dx \, dt,$$

and denote by $\hat{f}$ or $\mathcal{F}(f)$. We use $\mathcal{F}_x$ and $\mathcal{F}_t$ (or $\hat{\cdot}$ without distinction of variables) to denote the Fourier transform with respect to space and time variable respectively.

For $s, b \in \mathbb{R}$, the classical Bourgain space $X^{s,b}$ [3] associated to (1) is defined as the completion of $S'(\mathbb{R}^2)$ under the norm

$$\|f\|^2_{X^{s,b}} = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} (\tau - \xi^5)^{2b} |\hat{f}(\tau, \xi)|^2 \, d\xi d\tau,$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

As already mentioned in Section 1, the modifications of $X^{s,b}$ spaces are needed for our analysis due to the Duhamel boundary forcing operator and the time trace estimates. The modulation exponent $b$ of the standard $X^{s,b}$ space is forced to be taken in the range $(0, \frac{1}{2})$ from the $X^{s,b}$ estimation of the Duhamel boundary forcing terms (see Lemma 5.3 (c)). On the other hand, very low frequency interactions in the nonlinear estimates compel the exponent $b$ to be bigger than $1/2$. To balance these inter-contradiction conditions, we define the low frequency localized $X^{s,b}$-type space $D^\alpha$ as the completion of $S'(\mathbb{R}^2)$ under the norm

$$\|f\|^2_{D^\alpha} = \int_{\mathbb{R}^2} \langle \tau \rangle^{2\alpha} 1_{\{\xi: |\xi| \leq 1\}}(\xi) |\hat{f}(\tau, \xi)|^2 \, d\xi d\tau,$$

where $1_A$ is the characteristic functions on a set $A$.

Besides, the time trace estimate of the Duhamel parts

$$\|\psi_T(t)\partial_x^2 Dw(x, t)\|_{C(B^s; H^s(\mathbb{R}))} \lesssim T^\alpha \|w\|_{X^{s,-b}}$$

holds only for the positive regularity. To meet the negative regularity in the non-linear estimate (see Lemma 5.2 (b)), it is necessary to define the (time-adapted)
Bourgain space $Y^{s,b}$ associated to (1) as the completion of $S'(\mathbb{R}^2)$ under the norm
\[ \|f\|_{Y^{s,b}}^2 = \int_{\mathbb{R}^2} (\tau)^{\frac{2s}{5}} (\tau - \xi^5)^{2b} |\hat{f}(\tau, \xi)|^2 \, d\xi d\tau. \]

We make a Littlewood-Paley decomposition. Let $Z_+ = \mathbb{R} \cap [0, \infty)$. For $k \in \mathbb{Z}_+$, we set
\[ I_0 = \{ \xi \in \mathbb{R} : |\xi| \leq 2 \} \quad \text{and} \quad I_k = \{ \xi \in \mathbb{R} : 2^{k-1} \leq |\xi| \leq 2^{k+1} \}, \quad k \geq 1. \]

Let $\eta_0 : \mathbb{R} \to [0,1]$ denote a smooth bump function supported in $[-2,2]$ and equal to 1 in $[-1,1]$. For $k \in \mathbb{Z}_+$, we define
\[ \chi_0(\xi) = \eta_0(\xi), \quad \text{and} \quad \chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1}), \quad k \geq 1, \]
on the support $I_k$. Let $P_k$ denote the $L^2$ operator defined by $\hat{P_k}v(\xi) = \chi_k(\xi)\hat{v}(\xi)$.

For the modulation decomposition, we use the multiplier $\eta_j$, but the same as $\eta_j(\tau - \xi^5) = \chi_j(\tau - \xi^5)$. For $k,j \in \mathbb{Z}_+$, let
\[ D_{k,j} = \{ (\tau, \xi) \in \mathbb{R}^2 : \tau - \xi^5 \in I_j, \xi \in I_k \}, \quad D_{k,\leq j} = \cup_{l \leq j} D_{k,l}. \]

The Littlewood-Paley theory allows that
\[ \|f\|_{\chi^{s,b}}^2 \sim \sum_{k \geq 0} \sum_{j \geq 0} 2^{2sk} 2^{2bj} \|\eta_j(\tau - \xi^5)\chi_k(\xi)\hat{f}(\tau, \xi)\|_{L^2}^2 \quad (21) \]
and
\[ \|f\|_{\tilde{D}_0}^2 \sim \|P_0f\|_{X^{0,0}}^2. \]

We define the solution space denoted by $Z^{s,b,\alpha}_\ell$ under the norm\(^5\):
\[ \|f\|_{Z^{s,b,\alpha}_\ell(\mathbb{R}^2)} = \sup_{t \in \mathbb{R}} \|f(t, \cdot)\|_{H^s} + \sum_{j=0}^{\ell} \sup_{x \in \mathbb{R}} \|\partial_x^j f(t, \cdot, x)\|_{H^{s+\frac{j}{2}}} + \|f\|_{X^{s,b,\alpha} \cap D^\alpha}, \quad (22) \]
for $\ell = 1, 2$. From the boundary conditions in (12) and (13), one can see that $Z^{s,b,\alpha}_1$ ($Z^{s,b,\alpha}_2$) space is for the right half-line (left half-line) problem (see Section 6). The standard spatial and time localization of $Z^{s,b,\alpha}_\ell(\mathbb{R}^2)$ is
\[ Z^{s,b,\alpha}_\ell((0, T) \times \mathbb{R}^+) = Z^{s,b,\alpha}_\ell|_{(0, T) \times \mathbb{R}^+} \]
equipped with the norm
\[ \|f\|_{Z^{s,b,\alpha}_\ell((0, T) \times \mathbb{R}^+)} = \inf_{g \in Z^{s,b,\alpha}_\ell} \{ \|g\|_{Z^{s,b,\alpha}_\ell} : g(t,x) = f(t,x) \text{ on } (0, T) \times \mathbb{R}^+ \}. \]

3. **Nonlinear estimates.** In this section, we are going to establish nonlinear estimates, in particular, the control of $\|F(u)\|_{X^{s,-b}}$ and $\|F(u)\|_{Y^{s,-b}}$.

---

\(^5\) $Y^{s,b}$ norm plays a role of the intermediate norm in the Picard iteration argument (see Lemma 5.2 (b) and Section 3).
3.1. $L^2$-block estimates. Let $a_1, a_2, a_3 \in \mathbb{R}$. The quantities $a_{\max} \geq a_{\med} \geq a_{\min}$ can be conveniently defined to be the maximum, median and minimum values of $a_1, a_2, a_3$ respectively. Similarly, for $b_1, b_2, b_3, b_4 \in \mathbb{R}$, the quantities $b_{\max} \geq b_{\sub} \geq b_{\thd} \geq b_{\min}$ are defined to be the maximum, sub-maximum, third-maximum and minimum values of $b_1, b_2, b_3, b_4$ respectively.

For $\xi_1, \xi_2 \in \mathbb{R}$, let denote the (quadratic) resonance function by

$$H = H(\xi_1, \xi_2) = (\xi_1 + \xi_2)^5 - \xi_1^5 - \xi_2^5$$

$$= \frac{5}{2} \xi_1 \xi_2 (\xi_1 + \xi_2)(\xi_1^2 + \xi_2^2 + (\xi_1 + \xi_2)^2),$$

(23)

which plays an crucial role in the bilinear $X^s,b$-type estimates.

Let $f, g, h \in L^2(\mathbb{R}^2)$ be compactly supported functions. We define a quantity by

$$J_2(f, g, h) = \int_{\mathbb{R}^4} f(\xi_1, \xi_2) g(\xi_2, \xi_2) h(\xi_1 + \xi_2 + H(\xi_1, \xi_2), \xi_1 + \xi_2) \, d\xi_1 d\xi_2 d\xi_1 d\xi_2.$$

The change of variables in the integration yields

$$J_2(f, g, h) = J_2(g^*, h, f) = J_2(h, f^*, g),$$

where

$$f^*(\zeta, \xi) = f(-\zeta, -\xi).$$

(24)

From the identities

$$\xi_1 + \xi_2 = \xi_3$$

(25)

and

$$(\tau_1 - \xi_1^5) + (\tau_2 - \xi_2^5) = (\tau_3 - \xi_3^5) + H(\xi_1, \xi_2)$$

(26)

on the support of $J_2(f^i, g^i, h^i)$, where $f^i(\tau, \xi) = f(\tau - \xi^5, \xi)$ with the property $\|f\|_{L^2} = \|f^i\|_{L^2}$, we see that $J(f^i, g^i, h^i)$ vanishes unless

$$\frac{2^{k_{\max}} \sim 2^{k_{\med}} \sim 1}{2^{j_{\max}} \sim \max(2^{j_{\med}}, |H|)}.$$

(27)

We give the bilinear $L^2$-block estimates for the quadratic nonlinearity $F(u) = (1 - \partial_x^2)^2 \partial_x u(2u^2)$. See [10, 7] for the proof.

**Lemma 3.1.** Let $k_i \in \mathbb{Z}, j_i \in \mathbb{Z}_+, i = 1, 2, 3$. Let $f_{k_i, j_i} \in L^2(\mathbb{R} \times \mathbb{R})$ be nonnegative functions supported in $[2^{k_i-1}, 2^{k_i+1}] \times I_{j_i}$.

(a) For any $k_1, k_2, k_3 \in \mathbb{Z}$ with $|k_{\max} - k_{\min}| \leq 5$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$, then we have

$$J_2(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \leq 2^{j_{\min}/2} 2^{j_{\med}/4} 2^{k_{\max}/2} \prod_{i=1}^3 \|f_{k_i, j_i}\|_{L^2}.$$

(b) If $2^{k_{\min}} \ll 2^{k_{\med}} \sim 2^{k_{\max}}$, then for all $i = 1, 2, 3$ we have

$$J_2(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \leq 2^{(j_1 + j_2 + j_3)/2} 2^{-3k_{\max}/2} 2^{-(k_i + j_i)/2} \prod_{i=1}^3 \|f_{k_i, j_i}\|_{L^2}.$$

(c) For any $k_1, k_2, k_3 \in \mathbb{Z}$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$, then we have

$$J_2(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \leq 2^{j_{\min}/2} 2^{k_{\min}/2} \prod_{i=1}^3 \|f_{k_i, j_i}\|_{L^2}. $$
Similarly, for $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$, let
\[
G(\xi_1, \xi_2, \xi_3) = (\xi_1 + \xi_2 + \xi_3)^5 - \xi_1^5 - \xi_2^5 - \xi_3^5
\]
\[
= \frac{5}{2} (\xi_1 + \xi_2)(\xi_2 + \xi_3)(\xi_3 + \xi_1)(\xi_1^2 + \xi_2^2 + \xi_3^2 + (\xi_1 + \xi_2 + \xi_3)^2)
\]  
(28)
be the (cubic) resonance function, which plays an important role in the trilinear $X^{s,b}$-type estimates.

For compactly supported functions $f_i \in L^2(\mathbb{R} \times \mathbb{R})$, $i = 1, 2, 3, 4$, we define
\[
J_3(f_1, f_2, f_3, f_4) = \int_{\mathbb{R}} f_1(\xi_1, \tau_1) f_2(\tau_2, \xi_2) f_3(\xi_3, \tau_3) f_4(\tau_4, \xi_4) \, d\xi_1 d\tau_2 d\xi_2 d\tau_3 d\xi_3 d\tau_4 \, d\xi_4,
\]
where the $\int_\mathbb{R}$ is almost identical and easier, see [20].

The following lemma provides the trilinear $L^2$-block estimates for the cubic nonlinearity $F(u) = \partial_u u^3$.

**Lemma 3.2.** Let $k_{i, j_i} \in \mathbb{Z}_+$, $i = 1, 2, 3, 4$. Let $f_{k_{i, j_i}} \in L^2(\mathbb{R} \times \mathbb{R})$ be nonnegative functions supported in $I_{j_i} \times I_{k_{i, j_i}}$.

(a) For any $k_{i, j_i} \in \mathbb{Z}_+$, $i = 1, 2, 3, 4$, we have
\[
J_3(f_{k_{i, j_i}}, f_{k_{j, j_2}}, f_{k_{j_3, j_3}}, f_{k_{j_4, j_4}}) \lesssim 2^{(j_{min} + j_{max})/2} 2^{\frac{1}{2}(k_{min} + k_{max})/2} \prod_{i=1}^{4} \|f_{k_{i, j_i}}\|_{L^2}.
\]  
(33)

(b) Let $k_{thd} \leq k_{max} - 10$.

(b-1) If $(k_{i, j_i}) = (k_{thd}, j_{max})$ for $i = 1, 2, 3, 4$, we have
\[
J_3(f_{k_{i, j_i}}, f_{k_{j, j_2}}, f_{k_{j_3, j_3}}, f_{k_{j_4, j_4}}) \lesssim 2^{(j_{1} + j_{2} + j_{3} + j_{4})/2} 2^{-2k_{max}} 2^{k_{thd}/2} 2^{-j_{max}/2} \prod_{i=1}^{4} \|f_{k_{i, j_i}}\|_{L^2}.
\]  
(b-2) If $(k_{i, j_i}) \neq (k_{thd, j_{max}})$ for $i = 1, 2, 3, 4$, we have
\[
J_3(f_{k_{i, j_i}}, f_{k_{j, j_2}}, f_{k_{j_3, j_3}}, f_{k_{j_4, j_4}}) \lesssim 2^{(j_{1} + j_{2} + j_{3} + j_{4})/2} 2^{-2k_{max}} 2^{k_{min}/2} 2^{-j_{max}/2} \prod_{i=1}^{4} \|f_{k_{i, j_i}}\|_{L^2}.
\]

We refer to [20, 22] for the proof of Lemma 3.2. In [22], the second author established (cubic) $L^2$-block estimates for functions $f_{k_{i, j_i}} \in L^2(\mathbb{R} \times \mathbb{Z})$, but the proof, here, is almost identical and easier, see [20].
3.2. $F(u) = (1 - \partial_x^2)^{\frac{3}{2}} \partial_x(u^2)$ case. We first prove Theorem 1.1.

**Proposition 1.** For $-5/4 < s$, there exists $b = b(s) < 1/2$ such that for all $\alpha > 1/2$, we have

$$\|((1 - \partial_x^2)^{\frac{3}{2}} \partial_x(u^2))\|_{X^{s,-b}} \leq c\|u\|_{X^{s,b} \cap D^\alpha} \|u\|_{X^{s,b} \cap D^\alpha}. \tag{34}$$

**Proof.** Let

$$\tilde{f}_1(\tau_1, \xi_1) = \beta_1(\tau_1, \xi_1)\tilde{u}(\tau_1, \xi_1) \quad \text{and} \quad \tilde{f}_2(\tau_2, \xi_2) = \beta_2(\tau_2, \xi_2)\tilde{u}(\tau_2, \xi_2), \tag{35}$$

where

$$\beta_i(\tau_i, \xi_i) = (\tau_i - \xi_i^5)^b + 1_{[\xi_i, 1]}(\xi_i)\langle \tau_i \rangle^\alpha, \quad i = 1, 2. \tag{36}$$

Note that $f_1, f_2 \in L^2 \iff u, v \in X^{s,b} \cap D^\alpha$ and

$$\frac{1}{\beta_1(\tau_1, \xi_1)} \lesssim \begin{cases} \langle \tau_i - \xi_i^5 \rangle^{-b}, & \text{when } |\xi_i| > 1, \\ \langle \tau_i \rangle^{-\alpha}, & \text{when } |\xi_i| \leq 1. \end{cases} \tag{37}$$

By the duality argument, (34) is equivalent to

$$\iint_{\xi_1 + \xi_2 = \xi \atop \tau_1 + \tau_2 = \tau} \frac{\langle \xi \rangle^{s+1} \langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \tau - \xi_5 \rangle^b \beta_1(\tau_1, \xi_1)\beta_2(\tau_2, \xi_2)}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \tau - \xi_5 \rangle^b \beta_1(\tau_1, \xi_1)\beta_2(\tau_2, \xi_2)} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}. \tag{38}$$

For $k_i, j_i \in \mathbb{Z}_+$, we make the Littlewood-Paley decomposition of $f_i, i = 1, 2, 3$, into $f_{k_i,j_i}, i = 1, 2, 3$, by $f_{k_i,j_i}(\tau, \xi) = \eta_{j_i}(\tau - \xi^5)\chi_k(\xi)f_i(\tau, \xi)$. We divide the frequency regions of integration

$$\iint_{\xi_1 + \xi_2 = \xi \atop \tau_1 + \tau_2 = \tau} \frac{\langle \xi \rangle^{s+1} \langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \tau - \xi_5 \rangle^b \beta_1(\tau_1, \xi_1)\beta_2(\tau_2, \xi_2)}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \tau - \xi_5 \rangle^b \beta_1(\tau_1, \xi_1)\beta_2(\tau_2, \xi_2)} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}. \tag{39}$$

into several regions associated to the relation of frequencies to prove (38).

**Case I.** high $\times$ high $\Rightarrow$ high ($k_3 \geq 10, |k_3 - k_1|, |k_3 - k_2| \leq 5$). From (27) and (23), $j_{\max} \geq 5k_3 - 5$ holds in this case. The change of variables yields that (39) is bounded by

$$\sum_{k_3 \geq 10} \sum_{j_1, j_2, j_3 \geq 0 \atop \substack{|k_3 - k_1| \leq 5 \\ |k_3 - k_2| \leq 5}} 2^{(2-s)k_3}2^{-b(j_1 + j_2 + j_3)}J_2(f_{k_1,j_1}^4, f_{k_2,j_2}^4, f_{k_3,j_3}^4).$$

By applying Lemma 3.1 (a) to $J_2(f_{k_1,j_1}^4, f_{k_2,j_2}^4, f_{k_3,j_3}^4)$, and using the Cauchy-Schwarz inequality and (21), it suffices to show

$$\sum_{k_3 \geq 10} \sum_{j_1, j_2, j_3 \geq 0 \atop \substack{|k_3 - k_1| \leq 5 \\ |k_3 - k_2| \leq 5}} 2^{(2-s)k_3}2^{-b(j_1 + j_2 + j_3)}2^{j_{\min}}2^{j_{\max}/2}2^{-\frac{3}{2}k_{\max}} \lesssim 1. \tag{40}$$

Without loss of generality, we may assume that $j_1 \leq j_2 \leq j_3$. Given $-5/4 < s$, we can choose $\max(3, 8, 3/20 - s/5) < b < 1/2$. A computation of the summation over $0 \leq j_1 \leq j_2 \leq j_3$ with $5k_3 - 5 \leq j_3$ and $j_3 = 1, 2, 3$ gives

$$\text{LHS of (40)} \lesssim \sum_{k_3 \geq 10} 2^{(1/2 - 2s)k_3}2^{-10bk_3} \lesssim 1,$n which completes the proof of (40).
of this observation, the argument used in the proof of Proposition 5.1 in [6] causes a cancels two derivatives in high frequency (two derivative gains). As a consequence of this, we may assume that $j_2 \leq j_3$. By (37) and Lemma 3.1 (b), (39) on this case is dominated by

$$\sum_{k_3 \geq 10} \sum_{j_1, j_2, j_3 \geq 0} 2^{2k_3} 2^{-\alpha j_1 - b j_2 - c j_3} 2^{(j_1 + j_2 + j_3)/2} 2^{-3k_{\text{max}}/2} 2^{-(k_3 + j_3)/2} 3 \prod_{l=1}^{3} \|f_{k_l, j_l}\|_{L^2}.$$ (41)

Note that

$$\sum_{0 \leq j_1} \sum_{0 \leq j_1} 2^{(1-2\alpha) j_1} 2^{(1-2b) j_2} 2^{-2b j_3} \lesssim \sum_{0 \leq j_1, j_3} 2^{(1-2\alpha) j_1} 2^{(1-2b) j_3} \lesssim 1,$$

whenever we choose $1/4 < b < 1/2$ and for all $\alpha > 1/2$. We use the Cauchy-Schwarz inequality to obtain

$$(41) \lesssim \|f_1\|_{L^2} \sum_{k_3 \geq 10} \|f_{k_3}\|_{L^2} \sum_{|k_3 - k'| \leq 5} \|f_{k'}\|_{L^2} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}.$$

**Remark 4.** The proof of Proposition 1 is indeed analogous to the proof of Proposition 5.1 in [6]. However, the high-low interaction component with very low frequency ($|\xi_1| \leq 1$) (Case II-a) above of $(1 - \partial_x^2)^{1/2} \partial_x (\bar{u}^2)$ is slightly worse than the same one of $\partial_x (\bar{u}^2)$ in some sense, since the high-low bilinear local smoothing effect exactly cancels two derivatives in high frequency (two derivative gains). As a consequence of this observation, the argument used in the proof of Proposition 5.1 in [6] causes a logarithmic divergence in $k_3$-summation, and thus more delicate computation, here, is required as above compared with Case II-a in the proof of Proposition 5.1 in [6].

**Case II-b** $k_1 \geq 1$. In this case, we have from (27) and (23) that $j_{\text{max}} \geq 4k_3 + k_1 - 5$. By (37) and Lemma 3.1 (b), it suffices, similarly as Case I, to show

$$\sum_{k_3 \geq 10} \sum_{1 \leq k_1 \leq k_3, 0} 2^{4k_3} 2^{-2k_1} 2^{-2b(j_1 + j_2 + j_3)} 2^{(j_1 + j_2 + j_3)/2} 2^{-3k_{\text{max}}/2} 2^{-(k_3 + j_3)} \lesssim 1. \quad (42)$$

Without loss of generality, we may assume that $j_2 \leq j_3$.

If $j_1 \neq j_{\text{max}}$, given $-5/2 < s$, by choosing $\max((5 - s)/15, 1/3) < b < 1/2$. We perform the summation over $0 \leq j_1, j_2 \leq j_3$ with the fact $4k_3 + k_1 - 5 \leq j_3$ after choosing $(k_1, j_1) = (k_3, j_3)$ to obtain

$$\text{LHS of (42)} \lesssim \sum_{k_3 \geq 10} \sum_{1 \leq k_1 \leq k_3} 2^{8 - 24b)k_3/2} 2^{-(2 - 6b - 2s)k_1} \lesssim 1.$$

If $j_1 = j_{\text{max}}$, given $-5/2 < s$, we can choose $\max((5 - s)/15, 7/24) < b < 1/2$. We perform the summation over $0 \leq j_1, j_2 \leq j_3$ with the fact $4k_3 + k_1 - 5 \leq j_3$ after choosing $(k_1, j_1) = (k_1, j_1)$ to obtain

$$\text{LHS of (42)} \lesssim \sum_{k_3 \geq 10} \sum_{1 \leq k_1 \leq k_3} 2^{9 - 24b)k_3/2} 2^{-(1 - 6b - 2s)k_1} \lesssim 1.$$

Thus, given $-5/2 < s$, we choose $\max((5 - s)/15, 7/24) < b < 1/2$ such that (42) holds.

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6 We may assume that $\xi_1$ the low frequency without loss of generality due to the symmetry.
Remark 5. In view of Lemma 3.1, the case when the low frequency mode has the largest modulation \((k_i, j_i) = (k_{\text{min}}, j_{\text{max}})\) is the worst among other cases.

**Case III.** \(\text{high} \times \text{high} \Rightarrow \text{low}\) \((k_2 \geq 10, |k_1 - k_2| \leq 5, 0 \leq k_3 \leq k_2 - 5)\). We, similarly, further divide the case into two cases: \(k_3 = 0\) and \(k_3 \geq 1\).

**Case III-a** \(k_3 = 0\). In this case, we know \((\xi_3) \sim 1\). We further decompose the low frequency component \(f_3 = \sum_{l \leq 0} f_{3,l}\) with \(f_{3,l} = F^{-1} 1_{|\xi| \sim 2} F f_3\). Then, from (37), (39) is bounded by

\[
\sum_{k_2 \geq 10} \sum_{l \leq 0} \sum_{j_1, j_2, j_3 \geq 0} 2^{-2a k_2} 2^l 2^{-b(j_1 + j_2 + j_3)} J(f_{k_1, j_1}^2 f_{k_2, j_2}^2 f_{l, j_3}^2),
\]

where \(f_{l,j_3}(\tau, \xi) = \eta_{j_3}(\tau - \xi^5) f_{3,l}(\tau, \xi)\). Without loss of generality, we may assume that \(j_1 \leq j_2\).

From Remark 5, the worst case occurs when \(j_3 = j_{\text{max}}\). Lemma 3.1 (b) in \(J(f_{k_1, j_1}^2 f_{k_2, j_2}^2 f_{l, j_3}^2)\) and the Cauchy-Schwarz inequality in terms of \(k_2, l, j_i\)s yield

\[
(43) \lesssim \sum_{k_2 \geq 10} \sum_{l \leq 0} \sum_{j_1, j_2, j_3 \geq 0} 2^{-4a k_2} 2^l 2^{-2b(j_1 + j_2 + j_3)} 2^{j_1 + j_2 + j_3} 2^{-3k_{\text{max}}} 2^{-(l + j_3)}.
\]

Given \(-7/4 < s\), we can choose \(\max((5 - 4s)/24, 1/3) < b < 1/2\). Since \(j_{\text{max}} \geq 4k_2 + l - 5\) and \(1/3 < b < 1/2\), we have

\[
\sum_{0 \leq j_1, j_2 \leq j_3} 2^{(1-2b)j_1} 2^{(1-2b)j_2} 2^{-2b j_3} \lesssim 2^{(2-6b)(4k_2+l)},
\]

which implies

\[
(43) \lesssim \sum_{k_2 \geq 10} \sum_{l \leq 0} 2^{(5-4s-24b) k_2} 2^{(3-6b) l} \lesssim 1.
\]

**Case III-b** \(k_3 \geq 1\). From (27) and (23), we know \(j_{\text{max}} \geq 4k_2 + k_3 - 5\) in this case. Similarly, it suffices from (37) and Lemma 3.1 (b) to show

\[
\sum_{k_2 \geq 10} \sum_{l \leq 0} \sum_{j_1, j_2, j_3 \geq 0} 2^{(2+s) k_2} 2^{-4a k_2} 2^{-2b(j_1 + j_2 + j_3)} 2^{(j_1 + j_2 + j_3)} 2^{-3k_{\text{max}}} 2^{-(l + j_3)} \lesssim 1.
\]

Without loss of generality, we may assume that \(j_1 \leq j_2\).

Similarly, it suffice to consider the case when \(j_3 = j_{\text{max}}\). For given \(-7/4 < s\), we can choose \(\max((5 - 4s)/24, (5 + 2s)/6) < b < 1/2\). We perform the summation over \(0 \leq j_1 \leq j_2 \leq j_3\) in addition to \(4k_2 + k_3 - 5 \leq j_3\) after choosing \((k_i, j_i) = (k_3, j_3)\) to obtain

\[
\text{LHS of (44)} \lesssim \sum_{k_2 \geq 10} \sum_{l \leq 0} \sum_{j_1, j_2, j_3 \geq 0} 2^{(5-4s-24b) k_2} 2^{(5+2s-6b) k_3} \lesssim 1,
\]

which completes the proof of (44).

The \(\text{low} \times \text{low} \Rightarrow \text{low}\) interaction component can be directly controlled by the Cauchy-Schwarz inequality, since the low frequency localized space \(D^\alpha\) with \(\alpha > 1/2\) allows the \(L^2\) integrability with respect to \(\tau\)-variables.

Therefore, the proof of (34) is completed. \(\square\)

\(^7\)Similarly, when \(s \geq -5/4\), the \((5 + 2s)/6 < b\) implies \((5 - s)/15 < b\), which guarantees \(10 - 2s - 30b < 0\).
Remark 6. In view of the proof of Proposition 5.1 in [6], one can see that the regularity threshold appears in the high × high ⇒ low interaction component, which is the well-known worst component of quadratic nonlinearity (for semi-linear "dispersive" equations), while the regularity threshold −5/4, here, occurs in the high × high ⇒ high interaction case. It is because the high × high ⇒ low interaction component of \( (1 - \partial_x^2)^2 \partial_x(u^2) \) is no longer different from \( \partial_x(u^2) \) (roughly, \( (1 - \partial_x^2)^2 \sim 1 \)).

Proposition 2. For \(-5/4 < s \leq 0\), there exists \( b = b(s) < 1/2\) such that for all \( \alpha > 1/2\), we have

\[
\| (1 - \partial_x^2)^{1/2} \partial_x(uv) \|_{\dot{Y}_{s, \leq b}} \lesssim c \| u \|_{X_{s, b \cap D^s}} \| v \|_{X_{s, b \cap D^s}}.
\]  

(45)

We state the elementary integral estimates without proof.

Lemma 3.3 (Lemmas 5.12, 5.13 in [15]). Let \( \alpha, \beta \in \mathbb{R} \).

(a) If \( \frac{1}{4} < b < \frac{1}{2} \), then

\[
\int_{-\infty}^{\infty} \frac{dx}{(x - \alpha)^{2b} (x - \beta)^{2b}} \leq \frac{c}{(\alpha - \beta)^{4b - 1}}.
\]  

(46)

(b) If \( b < \frac{1}{4} \), then

\[
\int_{| x | \leq \beta} \frac{dx}{(x)^{4b - 1} | x - \alpha |^{1/2}} \leq \frac{c (1 + \beta)^{2 - 4b}}{(\alpha)^{1/2}}.
\]  

(47)

(c) Moreover, if \( \alpha \in \mathbb{R} \) and \( \frac{1}{4} < b < \frac{1}{2} \), we have

\[
\int_{-\infty}^{\infty} \frac{dx}{(x)^{2b} (x - \alpha)^{4b - 1}} \leq \frac{c}{(\alpha)^{6b - 2}}.
\]  

(48)

The proof of (48) is almost identical to the proof of (46) and (47), hence we omit the detail.

Proof of Proposition 2. We may assume that \( | \tau | \leq \frac{1}{32} | \xi |^5 \) for \(-5/4 < s \leq 0\), otherwise, it follows (34) in the proof of Proposition 1 due to \( \langle \tau \rangle^s \lesssim \langle \xi \rangle^s \). A direct calculation gives

\[
\frac{31}{32} | \xi |^5 \leq \frac{31}{32} | \xi |^5 - (| \tau | - \frac{1}{32} | \xi |^5) = | \xi |^5 - | \tau | \leq | \tau - \xi^5 | \leq | \tau | + | \xi |^5 \leq \frac{33}{32} | \xi |^5,
\]

which implies

\[
| \tau - \xi^5 | \sim | \xi |^5
\]  

(49)

under the assumption \( | \tau | \leq \frac{1}{32} | \xi |^5 \). Moreover, we have

\[
| \tau - \frac{1}{16} \xi^5 | \sim | \xi |^5.
\]  

(50)

We use the same notation \( f_i \), defined as in (35) under (36) and (37). Then, (45) is equivalent to

\[
\iint_{\xi_1 + \xi_2 = \xi} \frac{| \xi | \langle \xi \rangle \langle \tau \rangle^5 \tilde{f}_1(\tau_1, \xi_1) \tilde{f}_2(\tau_2, \xi_2) \tilde{f}_3(\tau, \xi)}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi \rangle^s \beta_1(\tau_1, \xi_1) \beta_2(\tau_2, \xi_2)} \lesssim \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2}.
\]  

(51)

We may assume from the symmetry that \( | \xi_1 | \leq | \xi_2 | \) without loss of generality.
Case I $|\xi_2| < 1$. From the identity (25), we know $|\xi| < 1$ in this case, which implies $|\tau| \lesssim 1$. Then, the left-hand side of (51) is equivalent to

$$\int \int_{\xi_1 + \xi_2 = \xi, |\xi_1|, |\xi_2|, |\xi| < 1} f_1(\tau_1, \xi_1)^\alpha f_2(\tau_2, \xi_2)^\beta f_3(\tau, \xi)^\gamma \, d\tau_1 d\tau_2 d\xi_1 d\xi_2.$$

The Cauchy-Schwarz inequality yields (51) thanks to $\alpha > 1/2$.

Case II $|\xi_2| \geq 1$. We further split the region of $\xi_1$ into two regions.

Case II-1 $|\xi_1| < 1$. From the identity (25), we know $|\xi| \geq 1$. Moreover, $(\tau)^{s/5} \lesssim 1$ in the negative regularity regime. Then, the left-hand side of (51) is bounded by

$$\int \int_{\xi_2 = \xi, |\xi_1| < 1} f_1(\tau_1, \xi_1)^\alpha f_2(\tau_2, \xi_2)^\beta f_3(\tau, \xi)^\gamma \, d\tau_1 d\tau_2 d\xi_1 d\xi_2.$$

where

$$\ast = \{ (\tau_1, \tau_2, \tau, \xi_1, \xi_2) \in \mathbb{R}^6 : \xi_1 + \xi_2 = \xi, \tau_1 + \tau_2 = \tau, |\xi_1| < 1, |\xi_2|, |\xi| \geq 1 \}. $$

From (49) and (23) under the assumption $|\xi_1| < 1 \leq |\xi|$, we know

$$|\tau - \xi^5| \sim |\xi|^5 \gg |\xi_1||\xi|^4 \sim |H|.$$

From (27), we, thus, divide this case into the following two cases:

$$|\tau - \xi^5| \sim |\tau_1 - \xi_1^5| \quad \text{or} \quad |\tau - \xi^5| \sim |\tau_2 - \xi_2^5| \gg |\tau_1 - \xi_1^5|.$$

For the first case, we denote the region of $\xi_1$ in the integral by $A = \{ \xi_1 : |\xi_1| \leq |\xi|^{-2} \} \cup \{ \xi_1 : |\xi_1|^{-2} < |\xi_1| \leq 1 \} =: A_1 \cup A_2$.

On $A_1$, for given $-3/2 < s \leq 0$ we can choose $b = b(s)$ satisfying $\frac{1-s}{2} < b < \frac{1}{2}$. Since $|\tau_2 - \xi_2^5|^{-b} \lesssim 1$, the Cauchy-Schwarz inequality with respect to $\xi_1, \xi_2, \tau_1, \tau_2$ yields

$$(52) \lesssim |\xi|^{2-s-\frac{5b}{2}} \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2} \lesssim \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2}.$$ 

On $A_2$, from (26) and (53), we always have

$$|\tau_2 - \xi_2^5| = |\tau - \xi^5 - (\tau_1 - \xi_1^5) + H| \gtrsim |H| \sim |\xi_1||\xi|^4 \gtrsim |\xi|^2,$$

which guarantees $(\tau_2 - \xi_2^5)^{-b} \lesssim |\xi|^{-2b}$. For given $-3/2 < s \leq 0$, we can choose $b = b(s)$ satisfying $\frac{2-s}{2} < b < \frac{1}{2}$. Then, the Cauchy-Schwarz inequality with respect to $\xi_1, \xi_2, \tau_1, \tau_2$ yields

$$(52) \lesssim |\xi|^{2-s-2b} \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2} \lesssim \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2}.$$ 

For the second case $(|\tau - \xi^5| \sim |\tau_2 - \xi_2^5| \gg |\tau_1 - \xi_1^5|)$, we know

$$|\tau_2 - \xi_2^5| \sim |\tau - \xi^5| \sim |\xi|^5.$$

For given $-3 < s \leq 0$ we can choose $b = b(s)$ satisfying $\frac{2-s}{10} < b < \frac{1}{2}$. Then, the Cauchy-Schwarz inequality with respect to $\xi_1, \xi_2, \tau_1, \tau_2$ yields

$$(52) \lesssim |\xi|^{2-s-10b} \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2} \lesssim \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2}.$$ 

Case II-2 $1 \leq |\xi_1| \leq |\xi_2|$. We may further assume that $|\tau_1 - \xi_1^5| \leq |\tau_2 - \xi_2^5|$ due to the symmetry.
and hence (55) can be controlled by
\[
\sup_{\xi, \tau \in \mathbb{R}} \left( \int \int_{|\tau| \leq \frac{1}{2}|\xi|^b} \frac{|\xi|^2|\xi|^2(\tau)^{\frac{2s}{b}}}{(\xi_1)^{2s}(\xi_2)^{2s}|\xi|^{10b/|\tau_1-\xi|^b}d\xi_1 d\tau_1} \right)^{1/2} \leq c. \tag{54}
\]
Under the assumption, we only consider the case when $|\xi| \geq 1$. Otherwise, (49) implies $|\tau_1 - \xi_1| \lesssim 1$, and hence we have (51) similarly as Case II-1 for $-2 \leq s \leq 0$. Indeed, from the identity (26) under this condition, we know $|H| \lesssim 1$. Since
\[
|H| = \frac{5}{2} |\xi_1||\xi_2| |(\xi_1^2 + \xi_2^2 + \xi^2)| \geq 5|\xi_1|^2|\xi_2|^2|\xi|,
\]
we have $|\xi_1|^{-s}|\xi_2|^{-s} \lesssim |\xi|^s$, and hence $|\xi|^{1+s/2} \leq 1$ for $-2 \leq s \leq 0$. The Cauchy-Schwarz inequality with respect to $\xi, \tau_1, \tau$ guarantees (51).

We now consider (54) on the case when $|\xi| \geq 1$. We use (46) in addition to (26) so that the left-hand side of (54) is bounded by
\[
\left| \int_{\xi_1 + \xi_2 = \xi} \frac{d\xi_1}{(\xi_1)^{2s}(\xi_2)^{2s}(\tau - \xi_5 + H)^{4b-1}} \right|^{1/2}. \tag{55}
\]
The support property ($|\tau - \xi_5| \gtrsim |H|$) and (49) implies
\[
|\xi_1|^{-2s}|\xi_2|^{-2s} \lesssim |\xi|^{-4s},
\]
and hence (55) can be controlled by
\[
\left| \int_{\xi_1 + \xi_2 = \xi} \frac{d\xi_1}{(\tau - \xi_5 + H)^{4b-1}} \right|^{1/2}. \tag{56}
\]
Let $\mu = \tau - \xi_5 + H$. Note that $|\mu| \leq 2|\tau - \xi_5|$ in this case. Then, by the direct calculation, we know
\[
\mu - (\tau - \frac{1}{16}\xi^5) = -\frac{5}{16}\xi(\xi - 2\xi_1)^2(2\xi_2^2 + (\xi - 2\xi_1)^2)
\]
and
\[
d\mu = \frac{5}{2}\xi(\xi^2 + (\xi - 2\xi_1)^2)(\xi - 2\xi_1) \, d\xi_1.
\]

Since
\[
|\xi|^{\frac{s}{2}}|\mu - (\tau - \frac{1}{16}\xi^5)|^{\frac{s}{2}} \leq |\xi||\xi - 2\xi_1||2\xi_2^2 + (\xi - 2\xi_1)^2| \leq 2|\xi||\xi - 2\xi_1||\xi^2 + (\xi - 2\xi_1)^2|
\]
we can reduce (56) by
\[
\left| \int_{|\mu| \leq |\tau - \xi_5|} \frac{d\mu}{(\mu)^{4b-1}|\mu - (\tau - \frac{1}{16}\xi^5)|^{1/2}} \right|^{1/2}. \tag{57}
\]
By (47), (57) is bounded by
\[
\left| \int_{|\mu| \leq |\tau - \xi_5|} \frac{d\mu}{(\mu)^{4b}|\mu - (\tau - \frac{1}{16}\xi^5)|^{1/4}} \right|^{1/2}. \tag{58}
\]
For given $-5/4 < s \leq 0$, we choose $b = b(s)$ satisfying $\frac{5-2s}{15} \leq b < \frac{1}{2}$. From (49) and (50) with $|\xi| \geq 1$ and $s \leq 0$, we obtain
\[
\left| \int_{|\mu| \leq |\tau - \xi_5|} \frac{d\mu}{(\mu)^{4b}|\mu - (\tau - \frac{1}{16}\xi^5)|^{1/4}} \right|^{1/2} \lesssim |\xi|^{5-2s-15b} \lesssim 1.
\]
Case II-2.b \(|\tau - \xi^5| \leq \frac{1}{1000000} |\tau_2 - \xi^5_2|\). In this case, it suffices to show from the Cauchy-Schwarz inequality that
\[
\sup_{\xi_1, \tau_1, \xi_2, \tau_2} \left( \int_{\xi_1 + \xi_2 = \xi}^{\xi_1 + \xi_2 = \xi} \int_{\tau_1 + \tau_2 = \tau} \frac{|\xi|^2 |\xi|^2 (\tau)^{\frac{5}{2}}}{(\xi_1)^{2s} (\xi_2)^{2s} (\tau)^{10b} (\tau_1 - \xi^5_1)^{2b} (\tau_2 - \xi^5_2)^{2b}} \, d\xi \, d\tau \right)^{1/2} \leq c. \tag{58}
\]
In this case we fix \(-2 < s \leq 0\). Since \(-5/2 < -2 < s\), we can choose \(b = b(s)\) satisfying \(-s/5 \leq b < 1/2\). From the fact that
\[
\langle \tau \rangle^{5/2 + 2b} \lesssim (\xi^5)^{5/2 + 2b} \sim (\xi)^{2s + 10b},
\]
the left-hand side of (58) is bounded by
\[
\frac{1}{(\tau_2 - \xi^5_2)^{b}} \left( \int_{\xi_1 + \xi_2 = \xi}^{\xi_1 + \xi_2 = \xi} \int_{\tau_1 + \tau_2 = \tau} \frac{|\xi|^2 |\xi|^2 (\tau)^{2s + 2}}{(\xi_1)^{2s} (\xi_2)^{2s} (\tau)^{2b} (\tau_1 - \xi^5_1)^{2b} (\tau_2 - \xi^5_2)^{2b}} \, d\xi \, d\tau \right)^{1/2}. \tag{59}
\]
When \(|H| \leq \frac{1}{2} |\tau_2 - \xi^5_2|\), we can know the following facts:
\begin{itemize}
  \item \(|\tau - \xi^5| \ll |\tau_2 - \xi^5_2|\) and \(|\tau| \leq \frac{1}{32} |\xi^5|\) imply \(|\xi^5_1| \ll |\tau_2 - \xi^5_2|\).
  \item \(|\tau_2 - \xi^5_2 - H + \xi^5| \sim (\tau_2 - \xi^5_2).
  \item \(|\xi_1|^{-2s} (\xi_2)^{-2s} \lesssim |\tau_2 - \xi^5_2|^{-s} |\xi^5|^4\).
\end{itemize}
We perform the integration in (59) in terms of \(\tau\) variable by using (46), then (59) is bounded by
\[
\frac{1}{(\tau_2 - \xi^5_2)^{b}} \left( \int_{R} \frac{|\xi|^2 |\xi|^2 (\tau)^{2s} (\tau_2 - \xi^5_2 - H + \xi^5)^{4b - 1}}{(\xi_1)^{2s} (\xi_2)^{2s} (\tau)^{2s} (\tau_1 - \xi^5_1)^{2b} (\tau_2 - \xi^5_2)^{2b}} \, d\xi \right)^{1/2}
\lesssim \frac{1}{(\tau_2 - \xi^5_2)^{3b - 1/2}} \left( \int_{|\xi| \leq |\tau_2 - \xi^5_2|^{1/5}} \frac{|\xi|^2 |\xi|^2 (\tau)^{2s + 2}}{(\xi_1)^{2s} (\xi_2)^{2s} (\tau_1 - \xi^5_1)^{2b} (\tau_2 - \xi^5_2)^{2b}} \, d\xi \right)^{1/2}
\lesssim \frac{|\tau_2 - \xi^5_2|^{-s/2}}{(\tau_2 - \xi^5_2)^{3b - 1/2}} \left( \int_{|\xi| \leq |\tau_2 - \xi^5_2|^{1/5}} \frac{|\xi|^2 |\xi|^2 (\tau)^{2s + 2}}{(\xi_1)^{2s} (\xi_2)^{2s} (\tau_1 - \xi^5_1)^{2b} (\tau_2 - \xi^5_2)^{2b}} \, d\xi \right)^{1/2}.
\]
For given \(-5/2 < -2 < s \leq 0\), we choose can \(b = b(s)\) satisfying \(\frac{5-s}{18} < b < \frac{1}{2}\).
Then, by performing integration in terms of \(\xi\), we have
\[
\frac{|\tau_2 - \xi^5_2|^{-s/2}}{(\tau_2 - \xi^5_2)^{3b - 1/2}} \left( \int_{|\xi| \leq |\tau_2 - \xi^5_2|^{1/5}} \frac{|\xi|^{2s} (\tau)^{2s + 2}}{(\xi_1)^{2s} (\xi_2)^{2s} (\tau_1 - \xi^5_1)^{2b} (\tau_2 - \xi^5_2)^{2b}} \, d\xi \right)^{1/2} \lesssim (\tau_2 - \xi^5_2)^{3b(10 - 2s - 30b)} \lesssim 1.
\]
For the other case \(|H| > \frac{1}{2} |\tau_2 - \xi^5_2|\), we can know the following facts:
\begin{itemize}
  \item \(10|\xi| \leq |\xi_1| \sim |\xi_2|\).
  \item \(|\xi - \xi_2| \sim |\xi_2|\).
  \item \(|\xi| \sim \frac{|\tau_2 - \xi^5_2|}{|\xi_2|^4}\). \tag{60}
  \item \(|\xi|^5 \ll |\tau_2 - \xi^5_2|\).
  \item \(|\xi_1|^{-2s} (\xi_2)^{-2s} \lesssim |\tau_2 - \xi^5_2|^{-s} |\xi|^s\).
\end{itemize}
\footnote{The strict inequality \(\frac{5-s}{18} < b\) covers the logarithmic divergence when \(s = \frac{5}{2}\).}
To verify the first one in (60), suppose that $|\xi_1| \leq 10|\xi|$. From (25), we know $|\xi_2| \leq 11|\xi|$. Then,

$$|H| = \frac{5}{2}|\xi_1||\xi_2||\xi_1||\xi_1^2 + |\xi_2|^2 + |\xi|^2|$$

$$\leq 30525|\xi|^5 \leq \frac{976800}{31}|\tau - \xi^5| \leq \frac{1}{3}|\tau_2 - \xi_2^5|,$$

which contradicts the assumption $|H| > \frac{1}{2}|\tau_2 - \xi_2^5|$. 

Now, under the conditions (60), we control the following integral:

$$\int_{\tau_1 + \tau_2 = \tau} \int_{\xi_1 + \xi_2 = \xi} \frac{|\xi|^2 (\xi_1)^2 (\tau_2 - \xi_2^5)^{-2s}}{(\tau - \xi^5)^{2b}} d\tau d\xi. \quad (61)$$

When $|\xi| \leq 1$, (60) and (46) yield

$$\int_{|\xi| \leq 1} \frac{|\xi|^2 (\xi_1)^2 (\tau_2 - \xi_2^5)^{-2s}}{(\tau - \xi^5)^{2b}} d\tau d\xi \lesssim \int_{|\xi| \leq 1} \frac{1}{(\tau_2 - \xi_2^5)^{s-2b-1} (\xi_2)^{4b-1}} d\xi.$$

Let $\mu = \tau_2 - \xi_2^5 - H + \xi^5$, then we have $d\mu = 5(\xi - \xi_2)^4 d\xi$. From the facts (60) with $|\xi| \leq 1$, since $|\xi_2|^{-1} \lesssim |\tau_2 - \xi_2^5|^{-1}$, the change of variable enables us to get

$$\int_{|\mu| \leq |\tau_2 - \xi_2^5|} \frac{1}{(\mu)^{4b-1}} d\mu \lesssim \int_{|\mu| \leq |\tau_2 - \xi_2^5|} (\tau_2 - \xi_2^5)^{-s-2b-1} (\xi_2)^{4b-1} d\mu.$$

for $-2 < s \leq 0$. For given $-2 < s \leq 0$, we can choose $b = b(s)$ satisfying $\frac{1-s}{6} \leq b < \frac{1}{2}$. Then, by performing the integration in terms of $\mu$, we have

$$\int_{|\mu| \leq |\tau_2 - \xi_2^5|} \frac{1}{(\mu)^{4b-1}} d\mu \lesssim (\tau_2 - \xi_2^5)^{-s-2b+1} \lesssim 1.$$ 

Now, we focus on the case when $|\xi| > 1$. Similarly as before, (61) can be reduced by

$$\int_{|\xi| > 1} \frac{|\xi|^{2+3s} (\tau_2 - \xi_2^5)^{-s-2b}}{(\tau_2 - \xi_2^5 - H + \xi^5)^{4b-1}} d\xi. \quad (62)$$

We use the change of variable $\mu = \tau_2 - \xi_2^5 - H + \xi^5$ with

$$d\mu = 5(\xi - \xi_2)^4 d\xi.$$ 

If $-2 < s \leq -5/3$, since

$$|\xi_2|^{-4} \sim |\xi_2|^{5-3s} \lesssim 1 \quad (\Rightarrow |\xi_2|^{5+3s} \lesssim 1),$$

we have

$$\int_{|\mu| < |\tau_2 - \xi_2^5|} \frac{1}{(\mu)^{4b-1}} d\mu \lesssim (\tau_2 - \xi_2^5)^{-s-2b+1} \lesssim 1$$

by choosing $b = b(s)$ satisfying $(1-s)/6 < b < 1/2$. 

---

It is not difficult to verify the others.
Otherwise \((-5/3 < s \leq 0)\), we can choose \(b = b(s)\) satisfying \(\frac{5 - s}{15} \leq b < \frac{1}{2}\). Then, from the fact \(|\xi| \ll |\tau_2 - \xi_2|^{1/5}\), we obtain
\[
(62) \lesssim \int_{|\mu| \leq |\tau_2 - \xi_2|} \frac{(\tau_2 - \xi_2)^{\frac{5 + 3s}{15} - s - 2b - 1}}{|\mu|^{4b - 1}} \, d\mu \\
\lesssim (\tau_2 - \xi_2)^{\frac{10 - 2s - 3b}{4}} \lesssim 1.
\]

Therefore, we complete the proof of Proposition 2. \(\square\)

3.3. \(F(u) = \partial_x(u^2)\) case. We now prove Theorem 1.2.

**Proposition 3.** For \(-1/4 \leq s\), there exists \(b = b(s) < 1/2\) such that for all \(\alpha > 1/2\), we have
\[
\|\partial_x(uvw)\|_{X^{s,\beta}} \leq c\|u\|_{X^{s,\beta} \cap D^\alpha} \|v\|_{X^{s,\beta} \cap D^\alpha} \|w\|_{X^{s,\beta} \cap D^\alpha}.
\]

Before proving Proposition 3, we bring the Strichartz estimates for the fifth-order dispersive equations.

**Lemma 3.4 (Strichartz estimates for \(e^{itD^5_x}\) operator [13]).** Assume that \(-1 < \sigma \leq \frac{3}{2}\) and \(0 \leq \theta \leq 1\). Then there exists \(C > 0\) depending on \(\sigma\) and \(\theta\) such that
\[
\|D_x^\frac{\sigma}{2} e^{itD^5_x} \varphi \|_{L^6_t L^6_x} \leq C \|\varphi\|_{L^2}
\]
for \(\varphi \in L^2\), where \(p = \frac{2}{1 + \theta}\) and \(q = \frac{10}{3\sigma + 17}\). In particular, we have
\[
\|e^{itD_x^5} P_k \varphi\|_{L^6_t L^6_x} \lesssim 2^{-k/2} \|P_k \varphi\|_{L^2}, \quad k \geq 1.
\]

**Proof of Proposition 3.** Similar mechanism as in the proof of Proposition 1 will be used. Let
\[
\tilde{f}_1(\tau_1, \xi_1) = \beta_1(\tau_1, \xi_1) \tilde{u}(\tau_1, \xi_1), \quad \tilde{f}_2(\tau_2, \xi_2) = \beta_2(\tau_2, \xi_2) \tilde{v}(\tau_2, \xi_2)
\]
\[
\text{and} \quad \tilde{f}_3(\tau_3, \xi_3) = \beta_3(\tau_3, \xi_3) \tilde{w}(\tau_3, \xi_3),
\]
where
\[
\beta_i(\tau_i, \xi_i) = \langle \tau_i - \xi_2^5 \rangle^b + 1_{|\xi_i| \leq 1}(\xi_i)^{\alpha}, \quad i = 1, 2, 3
\]
satisfying
\[
\frac{1}{\beta_i(\tau_i, \xi_i)} \lesssim \begin{cases} 
\langle \tau_i - \xi_i^5 \rangle^{-b}, & \text{when } |\xi_i| > 1, \\
(\tau_i)^{-\alpha}, & \text{when } |\xi_i| \leq 1.
\end{cases}
\]

Note that \(f_1, f_2, f_3 \in L^2 \Leftrightarrow u, v, w \in X^{s,\beta} \cap D^\alpha\). By the duality argument, (63) is equivalent to
\[
\left( \int \int_{\xi_1 + \xi_2 + \xi_3 = 0} \frac{|\xi| (\langle \xi \rangle^s \tilde{f}_1(\tau_1, \xi_1) \tilde{f}_2(\tau_2, \xi_2) \tilde{f}_3(\tau_3, \xi_3) \tilde{f}_4(\tau, \xi))^s (\tau - \xi_2^5)^b \beta_1(\tau_1, \xi_1) \beta_2(\tau_2, \xi_2) \beta_3(\tau_3, \xi_3)}{\langle \xi_1 \rangle^{s} \langle \xi_2 \rangle^{s} \langle \xi_3 \rangle^{s} (\tau - \xi_2^5)^b \beta_1(\tau_1, \xi_1) \beta_2(\tau_2, \xi_2) \beta_3(\tau_3, \xi_3)} \right) \lesssim \prod_{i=1}^{4} \|f_i\|_{L^2}.
\]

Let \(k_i, j_i \in \mathbb{Z}_+\). We decompose \(f_i\), \(i = 1, 2, 3, 4\), into \(f_{k_i, j_i}\), \(i = 1, 2, 3, 4\), by \(f_{k_i, j_i}(\tau, \xi) = \eta_{j_i}(\tau - \xi^5) \chi_{k_i}(\xi) \tilde{f}_i(\tau, \xi)\). We divide the frequency regions of integration
\[
\left( \int \int_{\xi_1 + \xi_2 + \xi_3 = 0} \frac{|\xi| (\langle \xi \rangle^s \tilde{f}_1(\tau_1, \xi_1) \tilde{f}_2(\tau_2, \xi_2) \tilde{f}_3(\tau_3, \xi_3) \tilde{f}_4(\tau, \xi))^s (\tau - \xi_2^5)^b \beta_1(\tau_1, \xi_1) \beta_2(\tau_2, \xi_2) \beta_3(\tau_3, \xi_3)}{\langle \xi_1 \rangle^{s} \langle \xi_2 \rangle^{s} \langle \xi_3 \rangle^{s} (\tau - \xi_2^5)^b \beta_1(\tau_1, \xi_1) \beta_2(\tau_2, \xi_2) \beta_3(\tau_3, \xi_3)} \right)
\]
into several regions associated to the relation of frequencies to prove (68).
The choice of high-high-high \(\Rightarrow\) high \((k_4 \geq 10\ and\ |k_1 - k_4|, |k_2 - k_4|, |k_3 - k_4| \leq 5)\). Without loss of generality, we may assume \(j_4 = j_{\text{max}}\). The change of variables yields in this case that (69) is bounded by

\[
\sum_{k_4 \geq 10}^{k_4 \geq 10} \sum_{|k_4 - k_i| \leq 5, i=1,2,3}^{k_4 \geq 10} 2^{(1-2b)k_4} 2^{-b(j_1+j_2+j_3+j_4)} f_3(f_{k_4, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4}).
\]

On the other hand, since \(f_{k_4, j_3}^2 (\tau, \xi) = f_{k_4, j_3} (\tau - \xi^5, \xi)\), we get

\[
\left| J_3(f_{k_4, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4}) \right| = \left| \int (f_{k_4, j_1} * f_{k_2, j_2} * f_{k_3, j_3}) f_{k_4, j_4} \right| \lesssim \| f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3} \|_{L^2} \| f_{k_4, j_4} \|_{L^2} \lesssim 3 \| F^{-1}(f_{k_1, j_1}) \|_{L^6} \| f_{k_4, j_4} \|_{L^2}. \tag{70}
\]

The Fourier inversion formula, Minkowski inequality, (64) and the Cauchy-Schwarz inequality yield

\[
\| F^{-1}(f_{k_1, j_1}) \|_{L^6} = \left\| \int e^{it\tau} e^{ix\xi} e^{ix^5} f_{k_1, j_1}(\tau, \xi) d\xi d\tau \right\|_{L^6} \lesssim \int \left( \left\| \int e^{ix\xi} e^{ix^5} f_{k_1, j_1}(\tau, \xi) d\xi \right\|^2 d\tau \right) \lesssim 2^{-k_1/2} 2^{j_1/2} \| f_{k_1, j_1} \|_{L^2}.
\]

Using this, we estimate (70) by

\[
\sum_{k_4 \geq 10}^{k_4 \geq 10} \sum_{|k_4 - k_i| \leq 5, i=1,2,3}^{k_4 \geq 10} 2^{(1-2b)k_4} 2^{-b(j_1+j_2+j_3+j_4)} 2^{-j_4/2} \prod_{i=1}^{4} \| f_{k_1, j_i} \|_{L^2}. \tag{71}
\]

The choice of \(\frac{3}{8} < b < \frac{1}{2}\) ensures the \(l^2\)-summability of \(2^{(1-2b)(j_1+j_2+j_3+j_4)} 2^{-j_4/2}\) over \(0 \leq j_1 \leq j_2 \leq j_3 \leq j_4\). On the other hand, we see that the frequency summation includes only one infinite sum as

\[
\sum_{k_4 \geq 10}^{k_4 \geq 10} = \sum_{k_4 \geq 10}^{k_4 \geq 10} \sum_{k_4 - k_i \leq 5, i=1,2,3}^{k_4 \geq 10} \sum_{k_4 - k_i \leq 5, i=1,2,3}^{k_4 \geq 10} \sum_{k_4 - k_i \leq 5, i=1,2,3}^{k_4 \geq 10}.
\]

We therefore have for \(s \geq -\frac{1}{4}\) that

\[
(71) \lesssim \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2} \| f_4 \|_{L^2}.
\]

**Case II.** high-high-low \(\Rightarrow\) high \((k_4 \geq 10, |k_2 - k_4|, |k_3 - k_4| \leq 5\) and \(k_1 \leq k_4 - 10)\). In this case, we know from (31) and (28) that \(j_{\text{max}} \geq 5k_4\). We further divide the case into two cases: \(k_1 = 0\) and \(k_1 \geq 1\).

**Case II-a.** \(k_1 = 0\). It suffices to consider

\[
\sum_{k_4 \geq 10}^{k_4 \geq 10} \sum_{j_1,j_2,j_3,j_4}^{k_4 \geq 10} 2^{(1-s)k_4} 2^{-b(j_2+j_3+j_4)} f_3(f_{k_2, j_1}, f_{k_3, j_2}, f_{k_3, j_3}, f_{k_4, j_4}).
\]

\[\text{We may assume that } \xi_1 \text{ is the lowest frequency without loss of generality due to the symmetry.}\]
By (33), we can control \( J_3(f_{0,j_1}^1, f_{k_2,j_2}^2, f_{k_3,j_3}^2, f_{k_4,j_4}^2) \), and hence it suffices to show
\[
\sum_{k_4 \geq 10} \sum_{|k_4-k_i| \leq 5, i=2,3} \sum_{j_1, j_2, j_3, j_4 \geq 0} 2((1-s)k_4) 2(1-2\alpha)j_1 2(1-2b)(j_2+j_3+j_4) 2k_4 2^{-(j_{\text{sub}}+j_{\max})} \lesssim 1.
\] (72)

Without loss of generality, we may assume \( j_2 \leq j_3 \leq j_4 \). When \( j_4 = j_{\max} \), we know \( j_3 \leq j_{\text{sub}} \). For \( s > -1 \), by choosing \( \max(\frac{1-2s}{10}, \frac{1}{b}) < b < \frac{1}{2} \), we have
\[
\text{LHS of (72)} \lesssim \sum_{k_4 \geq 10} \sum_{|k_4-k_i| \leq 5, i=2,3} \sum_{0 \leq j_1} 2(3-2s)k_4 2(1-2\alpha)j_1 2(1-2b)(j_2+j_3+j_4)
\[
\lesssim \sum_{k_4 \geq 10} \sum_{|k_4-k_i| \leq 5, i=2,3} \sum_{j_1, j_2, j_3} 2(3-2s)k_4 2(1-2\alpha)j_1 2(1-4b)j_3 2-2bj_4
\[
\lesssim \sum_{k_4 \geq 10} 2(3-2s-10b)k_4 \lesssim 1.
\]
whenever \( \alpha > \frac{1}{2} \). When \( j_4 \neq j_{\max} \), we know \( j_3 \leq j_1 \). Since \( j_{\max} \geq 5k_4 \), we have
\[
\text{LHS of (72)} \lesssim \sum_{k_4 \geq 10} \sum_{0 \leq j_1, j_2, j_3 \leq j_4} 2(1-2\alpha)j_1 2(1-2b)(j_2+j_3+j_4) 2^{(2-6b)j_4} \lesssim 1.
\]
whenever \( s > -1 \), \( \alpha > \frac{1}{2} \) and \( \frac{1}{b} < b < \frac{1}{2} \).

**Case II-b.** \( k_1 \geq 1 \). Without loss of generality, we may assume that \( j_1 \leq j_2 \leq j_3 \leq j_4 \). Similarly as before, it suffices to show
\[
\sum_{k_4 \geq 10} \sum_{j_1, j_2, j_3, j_4 \geq 0} 2((1-s)k_4) 2^{-s(k_1)} 2^{-b(j_1+j_2+j_3+j_4)} J_3(f_{k_1,j_1}^2, f_{k_2,j_2}^2, f_{k_3,j_3}^2, f_{k_4,j_4}^2)
\[
\lesssim \prod_{i=1}^{4} \| f_{k_i,j_i} \|_{L^2}.
\] (73)

If \( j_1 = j_{\max} \), we apply the argument used in **Case I** to \( J_3(f_{k_1,j_1}^2, f_{k_2,j_2}^2, f_{k_3,j_3}^2, f_{k_4,j_4}^2) \) by changing the role of \( f_{k_1,j_1} \) and \( f_{k_4,j_4} \). It is possible thanks to (32). Similarly as before, we have
\[
\text{LHS of (73)} \lesssim \sum_{k_4 \geq 10} \sum_{j_1, j_2, j_3 \geq 0} 2^{-(1/2+s)k_4} 2^{-s(k_1)} 2^{(1/2-b)(j_2+j_3+j_4)} 2^{-bj_1}\prod_{i=1}^{4} \| f_{k_i,j_i} \|_{L^2}.
\]
Since the frequency summation includes only two infinite sums (but one of them is for low frequency mode), for \( s \geq -\frac{1}{4} \), by choosing \( \frac{3}{4} < b < \frac{1}{2} \), we can have
\[
\text{LHS of (73)} \lesssim \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2} \| f_4 \|_{L^2}.
\] (74)
If \( j_1 \neq j_{\text{max}} \) (we assume \( j_4 = j_{\text{max}} \)), we can obtain

\[
J_3(f_{k_1,j_1}^2, f_{k_2,j_2}^2, f_{k_3,j_3}^2, f_{k_4,j_4}^2) \lesssim 2^{-\frac{2}{3}k_4 \cdot 2^{(j_1+j_2+j_3)/2}} \prod_{i=1}^{4} \| f_{k_i,j_i} \|_{L^2}. \tag{75}
\]

Then, similarly as the case when \( j_1 = j_{\text{max}} \), we have (74). Now it remains to show (75). It suffices to show

\[
\int_{\mathbb{R}^3} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g_4(G(\xi_1, \xi_2, \xi_3), \xi_1 + \xi_2 + \xi_3) \, d\xi_1 d\xi_2 d\xi_3 \lesssim 2^{-\frac{2}{3}k_4} \prod_{i=1}^{4} \| g_i \|_{L^2} \tag{76}
\]

for \( L^2 \)-functions \( g_i : \mathbb{R} \to \mathbb{R}_{\geq 0} \) supported in \( I_{k_i}, \, i = 1, 2, 3, \) and \( g_4 : \mathbb{R}^2 \to \mathbb{R}_{\geq 0} \) supported in \( I_{k_4} \times I_{k_1} \), where \( G \) is defined as in (28). Indeed, if (76) holds true, then

\[
\begin{align*}
J_3(f_{k_1,j_1}^2, f_{k_2,j_2}^2, f_{k_3,j_3}^2, f_{k_4,j_4}^2) & = \int_{\mathbb{R}} f_{k_1,j_1}^2(\tau_1, \xi_1)f_{k_2,j_2}^2(\tau_2, \xi_2)f_{k_3,j_3}^2(\tau_3, \xi_3)f_{k_4,j_4}^2(\tau_4, \xi_4) \\
& \quad \quad \cdot (\tau_1 + \tau_2 + \tau_3 + G(\xi_1, \xi_2, \xi_3), \xi_1 + \xi_2 + \xi_3) \\
& \lesssim 2^{-\frac{2}{3}k_4} \| f_{k_4,j_4} \|_{L^2} \int_{\mathbb{R}^3} \| f_{k_1,j_1}(\tau_1) \|_{L^2_{\xi_1}} \| f_{k_2,j_2}(\tau_2) \|_{L^2_{\xi_2}} \| f_{k_3,j_3}(\tau_3) \|_{L^2_{\xi_3}} \, d\tau_1 d\tau_2 d\tau_3 \\
& \lesssim 2^{-\frac{2}{3}k_4} 2^{(j_1+j_2+j_3)/2} \prod_{i=1}^{4} \| f_{k_i,j_i} \|_{L^2}.
\end{align*}
\]

The change of variables \( (\xi'_1 = \xi_1, \xi'_2 = \xi_1 + \xi_2, \xi'_3 = \xi_3) \) gives

LHS of (76) = \( \int g_1(\xi_1)g_2(\xi_2 - \xi_1)g_3(\xi_3)g_4(G(\xi_1, \xi_2 - \xi_1, \xi_3), \xi_2 + \xi_3) \, d\xi_1 d\xi_2 d\xi_3. \)

Note that \( |\xi_i| \sim 2^k_i, \, i = 1, 2, 3, \) still holds. A direct calculation gives

\[|\partial_{\xi_i} G(\xi_1, \xi_2 - \xi_1, \xi_3)| = | - 5\xi_1^4 + 5(\xi_2 - \xi_1)^4| \sim 2^{4k_4},\]

and then the Cauchy-Schwarz inequality with respect to \( \xi_1 \) and \( \xi_2 \), and the change of variable \( (\mu = G(\xi_1, \xi_2 - \xi_1, \xi_3)) \) ensure

LHS of (76) \( \lesssim 2^{-2k_4} \int g_3(\xi_3) \| g_1 \|_{L^2} \| g_2 \|_{L^2} \| g_4 \|_{L^2} \, d\xi_3 \)

\( \lesssim 2^{-2k_4} 2^{k_4/2} \prod_{i=1}^{4} \| g_i \|_{L^2}, \)

which completes the proof of (76). Thanks to (32), our assumption \( j_4 = j_{\text{max}} \) does not lose the generality.

**Case III.** high-high-high ⇒ low \( (k_3 \geq 10, \, |k_1-k_3|, \, |k_2-k_3| \leq 5 \) and \( k_4 \leq k_3-10). \)

In this case, we also have \( j_{\text{max}} \geq 5k_4 \) similarly as **Case II.** It suffices to show

\[
\begin{align*}
\sum_{|k_3 - k_i| \leq 5, \, i = 1, 2} \sum_{0 \leq k_4 \leq k_3 - 10} 2^{(1-s)k_4} 2^{-3sk_4} 2^{-b(j_1+j_2+j_3+j_4)} J_3(f_{k_1,j_1}^2, f_{k_2,j_2}^2, f_{k_3,j_3}^2, f_{k_4,j_4}^2) \\
\lesssim 4 \prod_{i=1}^{4} \| f_{k_i,j_i} \|_{L^2}. \tag{77}
\end{align*}
\]
We further assume that $k_1 < k_2$ (by replacing the role of $j_1$ and $j_4$) can be applied to the left-hand side of (77) and hence, for $s \geq -1/4$, by choosing $\frac{3}{8} < b < \frac{1}{2}$, we prove (77).

**Case IV.** high-low-low $\Rightarrow$ high ($k_4 \geq 10$, $|k_3 - k_4| \leq 5$ and $k_1, k_2 \leq k_1 - 10$).

We further assume that $k_1 \leq k_2$ without loss of generality.

**Case IV-a.** $k_2 = 0$. By Lemma 3.2 (b-2) and the Cauchy-Schwarz inequality, it suffices to show
\[ \sum_{k_3 \geq 10} \sum_{j_1, j_2, j_3, j_4 \geq 0} 2^{k_4} 2^{-4k_4 z_2 (1-2b)(j_1 + j_2) / 2 - j_{max} / 2} \leq 1. \tag{78} \]

Without loss of generality, we may assume $j_3 \leq j_4$. Since $\alpha > \frac{1}{2}$, by choosing $\frac{1}{4} < b < \frac{1}{2}$, we can show (78) for any $s \in \mathbb{R}$.

**Case IV-b.** $k_2 \geq 1$ and $k_1 = 0$. It suffices to consider
\[ \sum_{k_4 \geq 10} \sum_{j_1, j_2, j_3, j_4 \geq 0} 2^{k_4} 2^{-4k_4 z_2 (1-2b)(j_1 + j_2 + j_3 + j_4) / 2} \leq 1. \tag{79} \]

If $j_2 = j_{max}$, we have
\[ J_3(f_{0, j_1}^k, f_{k_2, j_2}^k, f_{k_3, j_3}^k, f_{k_4, j_4}^k) \lesssim 2^{-2k_4 z_2 (1-2b)(j_1 + j_3 + j_4) / 2} \| f_{0, j_1} \|_{L^2} \prod_{i=2}^4 \| f_{k_i, j_i} \|_{L^2}, \]

thanks to Lemma 3.2 (b-1). Otherwise, we have
\[ J_3(f_{0, j_1}^k, f_{k_2, j_2}^k, f_{k_3, j_3}^k, f_{k_4, j_4}^k) \lesssim 2^{-2k_4 z_2 (1-2b)(j_1 + j_2 + j_3 + j_4) / 2} \| f_{0, j_1} \|_{L^2} \prod_{i=2}^4 \| f_{k_i, j_i} \|_{L^2}, \]

thanks to Lemma 3.2 (b-2). In both cases, for $s \geq -\frac{1}{4}$, by choosing $\frac{1}{4} < b < \frac{1}{2}$, we have
\[ (79) \lesssim \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_3 \|_{L^2} \| f_4 \|_{L^2}, \]

whenever $\alpha > \frac{1}{2}$.

**Case IV-c.** $k_1 \geq 1$. It suffices to consider
\[ \sum_{k_4 \geq 10} \sum_{j_1, j_2, j_3, j_4 \geq 0} 2^{k_4} 2^{-4k_4 z_2 (1-2b)(j_1 + j_2 + j_3 + j_4) / 2} \| f_{0, j_1} \|_{L^2} \prod_{i=2}^4 \| f_{k_i, j_i} \|_{L^2}. \tag{80} \]

Since the worst bound of $J_3(f_{k_1, j_1}^k, f_{k_2, j_2}^k, f_{k_3, j_3}^k, f_{k_4, j_4}^k)$ is
\[ 2^{-2k_4 z_2 (1-2b)(j_1 + j_2 + j_3 + j_4) / 2} \| f_{0, j_1} \|_{L^2} \prod_{i=2}^4 \| f_{k_i, j_i} \|_{L^2}, \]

for $s \geq -\frac{1}{4}$, by choosing $\frac{3}{8} < b < \frac{1}{2}$, we have
\[ (80) \lesssim \sup_{k_4 \geq 10} 2^{-\left(\frac{1}{4} + 2b\right)k_4} \sum_{j_{max} \geq 0} 2^{(\frac{3}{8} - 4b)j_{max}} \prod_{i=1}^4 \| f_i \|_{L^2} \leq 2^{-\left(\frac{1}{4} + 2b\right)k_4} \sum_{j_{max} \geq 0} 2^{(\frac{3}{8} - 4b)j_{max}} \prod_{i=1}^4 \| f_i \|_{L^2}. \]

\[ 11 \text{Due to the symmetry, the assumption } |\xi_1|, |\xi_2| \leq |\xi_3| \text{ does not lose the generality.} \]

\[ 12 \text{Since } \xi_1 \text{ and } \xi_2 \text{ are comparable, we can avoid the case when } (k_i, j_i) = (k_{2hd}, j_{max}), i = 1, 2. \]
Case V. high-high-low \( \Rightarrow \) low \( (k_3 \geq 10, \ |k_2-k_3| \leq 5 \text{ and } k_1, k_4 \leq k_3 - 10) \).

We further divide the case in two cases \( k_1 = 0 \) and \( k_1 \geq 1 \).

Case V-a. \( k_1 = 0 \). It suffices to consider
\[
\sum_{k_3 \geq 10} \sum_{j_1, j_2, j_3, j_4 \geq 0 \atop |k_2-k_3| \leq 5 \atop 0 \leq k_4 \leq k_3 - 10} 2^{(1+s)k_3} 2^{-2sk_3} 2^{-\alpha j_1} 2^{-b(j_2+j_3+j_4)} J_3(f_{0,j_1}^d, f_{k_2,j_2}^d, f_{k_3,j_3}^d, f_{k_4,j_4}^d).
\]
(81)

The worst case happens when \( k_4 \geq 1 \) and \( j_4 = j_{\text{max}} \). By Lemma 3.2 (b-1), we have for \( s \geq -\frac{1}{2} \) that
\[
(81) \lesssim \sum_{k_3 \geq 10} \sum_{j_1, j_2, j_3, j_4 \geq 0 \atop |k_2-k_3| \leq 5 \atop 0 \leq k_4 \leq k_3 - 10} 2^{(\frac{1}{2}+s)k_3} 2^{-(2+2s)k_3} 2^{(\frac{1}{2}-\alpha)j_1} 2^{(\frac{1}{2}-b)(j_2+j_3)+b j_4}
\]
\[
\cdot \| f_{0,j_1} \|_{L^2} \prod_{i=2}^{4} \| f_{k_i,j_i} \|_{L^2} \lesssim \prod_{i=1}^{4} \| f_i \|_{L^2},
\]
by choosing \( \frac{1}{3} < b < \frac{1}{2} \) and \( \alpha > \frac{1}{2} \).

Case V-b. \( k_1 \geq 1 \). Similarly as before, the worst bound of
\[
J_3(f_{0,j_1}^d, f_{k_2,j_2}^d, f_{k_3,j_3}^d, f_{k_4,j_4}^d)
\]
is
\[
2^{-2k_32^k_{\text{bd}}/2^g(j_1+j_2+j_3+j_4)/2g-j_{\text{max}}/2}
\]
thanks to Lemma 3.2 (b-1). Hence, for \( s \geq -\frac{1}{4} \), by choosing \( \frac{1}{3} < b < \frac{1}{2} \), we can obtain
\[
\sum_{k_3 \geq 10} \sum_{j_1, j_2, j_3, j_4 \geq 0 \atop 1 \leq k_1, k_4 \leq k_3 - 10} 2^{(1+s)k_3} 2^{-2sk_3} 2^{-sk_3} 2^{-b(j_1+j_2+j_3+j_4)}
\]
\[
\cdot J_3(f_{k_1,j_1}^d, f_{k_2,j_2}^d, f_{k_3,j_3}^d, f_{k_4,j_4}^d) \lesssim \prod_{i=1}^{4} \| f_i \|_{L^2}.
\]

The low \( \times \) low \( \times \) low \( \Rightarrow \) low interaction component can be directly controlled by
the Cauchy-Schwarz inequality, since the low frequency localized space \( D^\alpha \) with \( \alpha > 1/2 \) allows the \( L^2 \) integrability with respect to \( \tau \)-variables.

Collecting all, we therefore complete the proof of Proposition 3. \( \square \)

Proposition 4. For \( -1/4 \leq s \leq 0 \), there exists \( b = b(s) < 1/2 \) such that for all \( \alpha > 1/2 \), we have
\[
\| \partial_x (uvw) \|_{X^{s,b} \cap B^\alpha} \leq c \| u \|_{X^{s,b} \cap D^\alpha} \| v \|_{X^{s,b} \cap D^\alpha} \| w \|_{X^{s,b} \cap D^\alpha}.
\]
(82)

Proof. Similarly as in the proof of Proposition 2, it is enough to consider the case when \( |\tau| \leq \frac{1}{2} |\xi|^5 \), which ensures (49) (we recall here)
\[
|\tau - \xi^5| \sim |\xi|^5.
\]
(83)

\(^{13}\)We may assume that \( \xi_1 \) is the lowest frequency among \( \xi_1, \xi_2, \xi_3 \), without loss of generality due to the symmetry.
Let \( f_i, i = 1, 2, 3, \) be \( L^2 \)-functions defined in (65) under (66) and (67). Then, (82) is equivalent to

\[
\iint_{\xi_1 + \xi_2 + \xi_3 = \xi} \frac{\xi(\tau)^2 \tilde{f}_1(\tau_1, \xi_1)\tilde{f}_2(\tau_2, \xi_2)\tilde{f}_3(\tau_3, \xi_3)\tilde{f}_4(\tau, \xi)}{(\xi_1)^a(\xi_2)^a(\xi_3)^a(\xi)^a(\beta_1(\tau_1, \xi_1)\beta_2(\tau_2, \xi_2)\beta_2(\tau_3, \xi_3))} \lesssim \prod_{i=1}^4 \|f_i\|_{L^2}. \tag{84}
\]

Due to the symmetry, we may assume \( |\xi_1| \leq |\xi_2| \leq |\xi_3| \) without loss of generality.

**Case I** (high × high × high ⇒ high). \( |\xi_1| > 1, |\xi_1| \sim |\xi_3| \sim |\xi| \). From (83) and (28), we know

\[
|\tau - \xi^5| \sim |\xi|^5 \gg |\xi|^2|\xi_1 + \xi_2 + \xi_3 + \xi_1| \sim |G|.
\]

Taking the Cauchy-Schwarz inequality to the left-hand side of (84) in addition to (30), it suffices to show

\[
\sup_{\xi, \tau \in \mathbb{R}} \frac{\xi}{|\tau|} \leq \frac{1}{2} |\xi|^5 \left( \iint_{\xi_1, \xi_2 \in \mathbb{R}} \frac{d\xi_1 d\xi_2 d\tau_1 d\tau_2}{|\tau_1 - \xi_1^5|^{2b}(\tau_2 - \xi_2^5)^{2b}(\tau_1 - \xi_1^5 + (\tau_2 - \xi_2^5) - \Sigma_1^{2b})} \right)^{1/2} \leq c,
\]

where

\[
\Sigma_1 = \tau - \xi^5 + G(\xi_1, \xi_2, \xi - \xi_1 - \xi_2). \tag{85}
\]

Since

\[
\iint_{\tau_1, \tau_2 \in \mathbb{R}} \frac{d\tau_1 d\tau_2}{|\tau_1 - \xi_1^5|^{2b}(\tau_2 - \xi_2^5)^{2b}(\tau_1 - \xi_1^5 + (\tau_2 - \xi_2^5) - \Sigma_1^{2b})} \lesssim \langle \tau - \xi^5 \rangle^{2-6b} \sim |\xi|^5(2-6b)
\]

thanks to (46) and (48) for \( \frac{1}{3} < b < \frac{1}{2} \), for \( -1 < s \leq 0 \), the choice \( \max(\frac{1}{3}, \frac{7-3a}{20}) < b < \frac{1}{2} \) ensures

\[
(84) \lesssim \sup_{\xi \in \mathbb{R}} |\xi|^{-3s-5b} \prod_{i=1}^4 \|f_i\|_{L^2} \lesssim \prod_{i=1}^4 \|f_i\|_{L^2}.
\]

**Case II** (low × high × high ⇒ high). \( |\xi_2| > 1, |\xi_1| \ll |\xi_2| \sim |\xi_3| \sim |\xi| \). When \( |\xi| \leq 1 \), the left-hand side of (84) is bounded by

\[
\iint_{*} \frac{|\xi|^{-2s-5b} \tilde{f}_1(\tau_1, \xi_1)\tilde{f}_2(\tau_2, \xi_2)\tilde{f}_3(\tau_3, \xi_3)\tilde{f}_4(\tau, \xi)}{(\tau_1)^a(\tau_2 - \xi_2^5)^b(\tau_3 - \xi_3^5)^b} \tag{86}
\]

where

\[
* = \{(\tau_1, \tau_2, \tau_3, \tau, \xi_1, \xi_2, \xi_3, \xi) \in \mathbb{R}^8 : \xi_1 + \xi_2 + \xi_3 = \xi, \tau_1 + \tau_2 + \tau_3 = \tau, |\xi_1| < 1 |\xi_2| \sim |\xi_3| \sim |\xi| \geq 1 \}.
\]

From (30) and (46), we have for fixed \( \tau, \xi, \tau_1, \xi_1 \) that

\[
\iint_{|\xi_2| \sim |\xi|, \tau_2} \frac{\tilde{f}_2(\tau_2, \xi_2)}{\langle \tau_2 - \xi_2^5 \rangle^b \langle \tau_3 - \xi_3^5 \rangle^b} \lesssim \|f_2\|_{L^2} \left( \iint_{|\xi_2| \sim |\xi|, \tau_2} \frac{d\xi_2 d\tau_2}{\langle \tau_2 - \xi_2^5 \rangle^{2b}(\tau_2 - \xi_2^5 + \Sigma_2)^{2b}} \right)^{1/2} \lesssim \|f_2\|_{L^2} \left( \int_{|\xi_2| \sim |\xi|} \langle \Sigma_2 \rangle^{1-4b} d\xi_2 \right)^{1/2} \lesssim |\xi|^b \|f_2\|_{L^2},
\]
for $\frac{1}{4} < b < \frac{1}{2}$, i.e., $(\Sigma_2)^{1-4b} \lesssim 1$, where
\[
\Sigma_2 = (\tau_1 - \xi_1^5) - (\tau - \xi^5) - G(\xi_1, \xi, \xi_1 - \xi_2).
\]
Hence, for $-\frac{1}{2} < s \leq 0$, the choice $\max(\frac{1}{4}, \frac{3-4s}{10}) < b < \frac{1}{2}$ in addition to $\alpha > \frac{1}{2}$ yields
\[
(86) \lesssim |f_1|_{L^2} |f_2|_{L^2} |f_3|_{L^2} |f_4|_{L^2}.
\]
When $|\xi_1| > 1$, we know from (83) and (28) that
\[
|\xi| \sim |\xi|^5 \sim |\tau - \xi^5|.
\]
Taking the Cauchy-Schwarz inequality in addition to (30), it suffices to show
\[
\sup_{\xi, \tau \in \mathbb{R}} \frac{|\xi|^{1-3s-5b}}{|\tau - \xi^5 + G(\xi_1, \xi_2, \xi - \xi_1 - \xi_2)|^{6b-2}} \sim \int_{|\mu| \leq |\tau - \xi^5|} \frac{|\xi|^{-4} d\mu}{(\tau - \xi^5 + G(\xi_1, \xi_2, \xi - \xi_1 - \xi_2))^{6b-2}} \lesssim |\xi|^{-4} |\tau - \xi^5|^{3-6b} \sim |\xi|^{11-30b},
\]
where $\Sigma_1$ is defined in (85). Let $\mu = \Sigma_1$. Since
\[
|\partial_\tau \Sigma_1| = | - 5\xi_1^4 + 5(\xi - \xi_1 - \xi_2)^4 | \sim |\xi|^4,
\]
we have for $\frac{1}{3} < b < \frac{1}{2}$ that
\[
\int_{|\xi_1| \leq |\xi|} \frac{d\xi_1}{(\tau - \xi^5 + G(\xi_1, \xi_2, \xi - \xi_1 - \xi_2))^{6b-2}} \sim \int_{|\mu| \leq |\tau - \xi^5|} \frac{|\xi|^{-4} d\mu}{(\tau - \xi^5 + G(\xi_1, \xi_2, \xi - \xi_1 - \xi_2))^{6b-2}} \lesssim |\xi|^{-4} |\tau - \xi^5|^{3-6b} \sim |\xi|^{11-30b},
\]
where $\tau, \xi_2, \xi$ are fixed. Hence, by (46), (48), (88) and the Cauchy-Schwarz inequality, we have
\[
\left( \int_{|\xi_1| \leq |\xi|} \left( \int_{\tau_1 \in \mathbb{R}} \frac{d\xi_1 d\xi_2 d\tau_1 d\tau_2}{(\tau_1 - \xi_1^5)^{2b}(\tau_2 - \xi_2^5)^{2b}(\tau_1 - \xi_1^5 + (\tau_2 - \xi_2^5) - \Sigma_1)^{2b}} \right)^{1/2} \right)^2 \lesssim \left( \int_{|\xi_1| \leq |\xi|} \int_{\tau_1 \in \mathbb{R}} \frac{d\xi_1 d\xi_2}{(\tau - \xi^5 + G(\xi_1, \xi_2, \xi - \xi_1 - \xi_2))^{6b-2}} \right)^{1/2} \lesssim |\xi|^{6-15b},
\]
for $\frac{1}{4} < b < \frac{1}{2}$, which implies that for $-1 < s \leq 0$, the choice $\max(\frac{1}{4}, \frac{7-3s}{20}) < b < \frac{1}{2}$ ensures (87).

**Case III** (high × high × high ⇒ low). $|\xi_1| > 1$, $|\xi| \ll |\xi_1| \sim |\xi_3|$. When $|\xi| \leq 1$, we know from (83) and (28) that
\[
|\tau - \xi^5| \lesssim 1 \ll |\xi_3|^5 \sim |G|.
\]
We assume that $|\tau_1 - \xi_1^5| \leq |\tau_2 - \xi_2^5| \leq |\tau_3 - \xi_3^5|$. Taking the Cauchy-Schwarz inequality to the left-hand side of (84) in addition to (30), it suffices to show
\[
\sup_{|\xi_1| > 1, \tau \in \mathbb{R}} \frac{|\xi_1|^{-3s}}{(\tau_3 - \xi_3^5)^b} \left( \int_{|\xi| \leq |\xi_1|} \int_{\tau, \tau_1 \in \mathbb{R}} \frac{d\xi d\tau d\tau_1}{(\tau_1 - \xi_1^5)^{2b}(\tau - \xi^5)^{2b}(\tau - \xi_1^5 - (\tau_1 - \xi_1^5) - \Sigma_3)^{2b}} \right)^{1/2} \leq c, (89)
\]
due to $|\tau - \xi^5| \sim 1$, where
\[
\Sigma_3 = \tau_3 - \xi_3^5 - G(\xi_1, \xi - \xi_1 - \xi_3, \xi_3).
\]
Using (46) and (48) for $\frac{1}{3} < b < \frac{1}{2}$, we have

$$\text{LHS of (89)} \lesssim \sup_{|\xi_3| > |\xi_3| \neq 0} \frac{|\xi_3|^{-3s}}{1 + |\xi_3|} \left( \int |d\xi d\xi_1 (\tau_3 - \xi_3) + G(\xi_1, \xi - \xi_3)|^6 \right) \lesssim 1.$$  

If $|\tau_3 - \xi_3^5| \gg |G| \sim |\xi_3|^5$, for $\frac{3}{2} < s \leq 0$, we choose $\max(\frac{1}{3}, \frac{11}{20} - \frac{3s}{20}) < b < \frac{1}{2}$ so that the Cauchy-Schwarz inequality yields

$$\text{LHS of (89)} \lesssim \sup_{|\xi_3| > |\xi_3| \neq 0} |\xi_3|^{-3s} (\tau_3 - \xi_3^5)^{1-4b}|\xi_3|^{\frac{b}{2}} \lesssim \sup_{|\xi_3| > |\xi_3|} |\xi_3|^{\frac{b}{2} - 3s - 20b} \lesssim 1.$$  

Otherwise ($|\tau_3 - \xi_3^5| \sim |G| \sim |\xi_3|^5$), let $\mu = \Sigma_3$. Since

$$|\partial_\xi \Sigma_3| = |5\xi^4 - 5(\xi - \xi_3 - \xi_3^4)| \sim |\xi_3|^4,$$

we have for $\frac{1}{3} < b < \frac{1}{2}$ that

$$\int_{|\xi| \leq 1, |\xi_1| \sim |\xi_3|} \frac{d\xi d\xi_1}{(\tau_3 - \xi_3^5 + G(\xi_1, \xi - \xi_3^5))^6} \lesssim \int_{|\xi_1| \sim |\xi_3|} \int_{|\mu| \leq |\xi_3|^5} |\xi_3|^{-4} \frac{d\mu}{|\mu|^6} \lesssim |\xi_3|^{-4} |\xi_3|^{-3|6|} |\xi_3| \sim |\xi_3|^{12 - 30b}.$$  

For $-\frac{1}{2} < s \leq 0$, the choice $\max(\frac{1}{3}, \frac{6 - 4s}{20}) < b < \frac{1}{2}$ in addition to (91) yields

$$\text{LHS of (89)} \lesssim \sup_{|\xi_3| > |\xi_3|} |\xi_3|^{-3s} |\xi_3|^{-5b} |\xi_3|^{6 - 15b} \lesssim 1.$$  

We remark in the above argument that our assumption $|\tau_1 - \xi_1^5| \leq |\tau_2 - \xi_2^5| \leq |\tau_3 - \xi_3^5|$ does not lose the generality.

When $|\xi| > 1$, a direct computation for $-\frac{5}{2} < -1 < s \leq 0$ gives

$$\frac{|\xi|(|\tau|)^{\frac{5}{2}}}{|\tau|^{5b}} \lesssim \frac{|\xi|^{1+s}}{\langle |\tau| \rangle^{b}}.$$  

We assume $|\tau_1 - \xi_1^5| \leq |\tau_2 - \xi_2^5| \leq |\tau_3 - \xi_3^5|$. If $|\tau_3 - \xi_3^5| \gg |G|$, we know

$$\langle -\xi^5 - (\tau_3 - \xi_3^5) + G \rangle \sim |\tau_3 - \xi_3^5| \quad \text{and} \quad |\xi|^2 |\xi_3| \ll |G| \ll |\tau_3 - \xi_3^5|.$$  

Similarly, thanks to (46) and (48) for $\frac{1}{3} < b < \frac{1}{2}$, it suffices to show

$$\sup_{|\xi_3| > |\xi_3|} \frac{|\xi_3|^{-3s}}{(\tau_3 - \xi_3^5)^{\frac{b}{2}} \langle |\xi|^{\frac{1}{2}} \rangle^{\frac{1}{2}}} \lesssim 1.$$  

For $-\frac{7}{18} < s \leq 0$, the choice $\frac{7}{18} < b < \frac{1}{2}$ ensures

$$\text{LHS of (94)} \lesssim \sup_{|\xi_3| > |\xi_3|} |\xi_3|^{-3s + \frac{5}{2} - \frac{5}{2}(\frac{1}{18} + \frac{1}{2})} |\tau_3 - \xi_3^5|^{1 - 4b + \frac{1}{4b}} \lesssim 1.$$  

Otherwise ($|\tau_3 - \xi_3^5| \sim |G| \sim |\xi_3|^5$), similarly as before, it suffices to show

$$\sup_{|\xi_3| > |\xi_3|} |\xi_3|^{-3s - 5b} \left( \int_{1 < |\xi| \leq |\xi_3|} \frac{|\xi|^{1+s} d\xi d\xi_1}{|\xi|^2 |\xi_3|} \right)^{1/2} \lesssim 1.$$  

for $\frac{1}{3} < b < \frac{1}{2}$. Let $\mu = -\xi^5 - \Sigma_3$. Since $|\mu| \lesssim |G| \sim |\xi_3|^b$ and

$$|\partial_\xi (-\xi^5 - \Sigma_3)| = |5(\xi - \xi_3 - \xi_3^4)| \sim |\xi_3|^4,$$
we have for \( \frac{1}{3} < b < \frac{1}{2} \) that
\[
\int_{|\xi_1|\sim |\xi_3|} d\xi_1 \frac{d\xi_2}{(-\xi^5 - \Sigma_3)^{6b-2}} \sim \int_{|\xi_1|\sim |\xi_3|} \int_{|\mu|\leq |\xi_3|^4} d\tau_1 d\tau_2 d\xi_2 \langle \tau_1 \rangle^{2\alpha} \langle \tau_2 - \xi^5 \rangle^{2b} \langle \tau_2 - \xi^5 + \Sigma_4 \rangle^{2b} \]  
\[
\lesssim |\xi_3|^{-4} |\xi_3|^{5(3-6b)} |\xi_3| \sim |\xi_3|^{12-30b}.
\]
For \(-\frac{3}{2} < s \leq 0\), the choice \(\frac{1}{2}, \frac{7-2n}{20}\) \(< b < \frac{1}{2}\) in addition to (95) ensures
\[
\text{LHS of (94)} \lesssim \sup_{|\xi_1|>1} |\xi_3|^{7-2s-20b} \lesssim 1.
\]

**Case IV** (low \(\times\) low \(\times\) high \(\Rightarrow\) high). \(|\xi_3| > 1\), \(|\xi_2| \ll |\xi_3| \sim |\xi_3|\). When \(|\xi_2| \leq 1\), the left-hand side of (84) is bounded by
\[
\int \int \int \sum_{\xi_1, \xi_2, \xi_3, \xi} \xi_{\xi_1, \xi_2, \xi_3, \xi} \langle \tau_1 \rangle^{2\alpha} \langle \tau_2, \xi_2, \xi_3 \rangle \langle \tau_3, \xi_3 \rangle \langle \tau, \xi \rangle \langle \tau_4, \xi \rangle \]  
\[
\lesssim |\xi_3|^{-x-5b} \langle \tau_1 \rangle \langle \tau_2 \rangle \langle \tau_3 \rangle \langle \tau_4 \rangle \]  
\[
\times \langle \tau_2 - \xi^5 \rangle^{2b} \langle \tau_2 - \xi^5 \rangle^{2b} \langle \tau_2 - \xi^5 + \Sigma_4 \rangle^{2b} \]
\[
\lesssim \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}.
\]

When \(|\xi_1| \leq |\xi_2|\), the exact same argument used in the **Case II** for \(|\xi_1| \leq 1\) can be directly applied to this case, hence we omit the detail.

When \(|\xi_2| \leq 1\), we know from (28) and (85) that
\[|G| \sim |\xi_3^4| |\xi_2 + |\xi_3| < |\xi_3^5 \sim |\tau - \xi^5|\).

The similar argument used in **Case I** can be applied to this case, hence we omit the detail.

**Case V (low \(\times\) high \(\times\) high \(\Rightarrow\) low).** \(|\xi_2| > 1\), \(|\xi_1|, |\xi| \ll |\xi_2| \sim |\xi_2|\). We may assume that
\[|\tau_2 - \xi^5_2| \leq |\tau_3 - \xi^5_3|\]
without loss of generality. We split this case into several cases.

**Case V-a** \(|\xi_3|, |\xi_1| \leq 1\). In this case, we know \(|\tau| \leq 1\) and \(|\tau - \xi^5| \sim 1\). When \(|\xi_2 + \xi_3| \leq |\xi_3|^{-2}\), we have from (46) and the Cauchy-Schwarz inequality that
\[
\text{LHS of (84)} \lesssim \sup_{|\xi_3| \geq 1} |\xi_3|^{-2s} \int \int \frac{d\tau_1 d\tau_2 d\xi_2}{(\tau_1)^{2\alpha} (\tau_2 - \xi_3^{5})^{2b} (\tau_2 - \xi_3^{5} + \Sigma_4)^{2b}} \lesssim \sup_{|\xi_3| \geq 1} \|f_i\|_{L^2} \lesssim \sup_{|\xi_3| \geq 1} \|f_i\|_{L^2},
\]
whenever \(-\frac{3}{8} \leq s \leq 0\), \(\frac{1}{2} < b < \frac{1}{2}\) and \(\alpha > \frac{1}{2}\), where
\[
\Sigma_4 = (\tau_1 - \xi_1^5) + (\tau_3 - \xi_3^5) - G(\xi_1, \xi_2, \xi_3).
\]
Otherwise \(|\xi_2 + \xi_3| > |\xi_3|^{-2}\), we know from (28) that
\[|G| \sim |\xi_3^4| |\xi_2 + \xi_3| > |\xi_3|^{\frac{5}{2}}.
\]
whenever $\alpha > -\frac{3}{8}$, we know from (30) that $|\tau_1| \sim |G| > |\xi_3|^\frac{5}{8}$. Similarly as before, we have from (46) and the Cauchy-Schwarz inequality that

LHS of (84)

$$\lesssim \sup_{|\xi_3| \geq 1} |\xi_3|^{-2s} \left( \int_{|\xi_1| \leq 1} \int_{|\xi_2| \leq 1} \int \frac{d\tau_1 d\xi_1 d\xi_2}{(\tau_1)^{2\alpha}(\tau_2 - \xi_2^2)^{2b}(\tau_3 - \xi_3^2)^{2b}} \right)^\frac{1}{2} \prod_{i=1}^{4} \|f_i\|_{L^2}$$

$$\lesssim \sup_{|\xi_3| \geq 1} |\xi_3|^{-2s-\frac{3}{8}} \prod_{i=1}^{4} \|f_i\|_{L^2} \lesssim \prod_{i=1}^{4} \|f_i\|_{L^2},$$

whenever $-\frac{3}{8} \leq s \leq 0$, $0 < b < \frac{1}{2}$ and $\alpha > \frac{1}{2}$. If $|\tau_3 - \xi_3^2| \gtrsim |G| > |\xi_3|^\frac{5}{8}$, for $-\frac{3}{8} < s$, by choosing $-\frac{4s}{5} < b < \frac{1}{2}$, we, similarly, have from (46) and the Cauchy-Schwarz inequality that

LHS of (84) $\lesssim \sup_{|\xi_3| \geq 1} |\xi_3|^{-2s} \left( \int_{|\xi_1| \leq 1} \int_{|\xi_2| \leq 1} \int \frac{d\tau_1 d\xi_1 d\xi_2}{(\tau_1)^{2\alpha}(\tau_2 - \xi_2^2)^{2b}(\tau_3 - \xi_3^2)^{2b}} \right)^\frac{1}{2} \prod_{i=1}^{4} \|f_i\|_{L^2}$

$$\lesssim \sup_{|\xi_3| \geq 1} |\xi_3|^{-2s-\frac{3}{8}} \prod_{i=1}^{4} \|f_i\|_{L^2} \lesssim \prod_{i=1}^{4} \|f_i\|_{L^2},$$

whenever $\alpha > \frac{1}{2}$.

**Case V-b** $|\xi_1| \leq 1 < |\xi|$. We use (92) for $-\frac{5}{2} < -1 < s \leq 0$. We know from (28) and (83) that

$$|\tau - \xi^5| \sim |\xi|^5 \ll |\xi_3|^4|\xi| \sim |G|.$$  

If $|\tau_3 - \xi_3^2| \gtrsim |G|$, since $|\xi| \lesssim |\tau_3 - \xi_3^2|^\frac{1}{8}$, for $-\frac{3}{8} < s \leq 0$, by choosing $\max(\frac{1}{4}, \frac{3+2s}{10}, \frac{3-5s}{10}) < b < \frac{1}{2}$, we have from (46) and the Cauchy-Schwarz inequality that

LHS of (84)

$$\lesssim \sup_{|\xi_3| \geq 1} \left|\xi_3\right|^{-2s} \left( \int_{|\tau_1| \leq 1} \int_{|\tau_2| \leq 1} \int \frac{\left|\xi_1\right|^{2s+2} d\tau_1 d\xi_1 d\xi_2}{(\tau_1)^{2b}(\tau_2 - \xi_2^2)^{2b}(\tau_3 - \xi_3^2)^{2b}} \right)^\frac{1}{2} \prod_{i=1}^{4} \|f_i\|_{L^2}$$

$$\lesssim \sup_{|\xi_3| \geq 1} \left|\xi_3\right|^{-2s-b} \prod_{i=1}^{4} \|f_i\|_{L^2} \lesssim \prod_{i=1}^{4} \|f_i\|_{L^2},$$

where

$$\Sigma_5 = -\xi^5 - (\tau_1 - \xi_1^5) - (\tau_3 - \xi_3^5) + G(\xi_1, \xi - \xi_1, \xi_2, \xi_3).$$
Otherwise \(|\tau_3 - \xi_3^4| \ll |G|\), we know from (30) under (97) and (98) that \(|\tau_1 - \xi_1^4| \sim |\tau_1| \sim |G| > |\xi_3|^4\). Then, the left-hand side of (84) is bounded by

\[
\sup_{|\xi_3| \geq 1} |\xi_3|^{-2s} \left( \iint \frac{|\xi_1|^{-2\alpha}d\xi_2 d\tau d\tau}{|\xi_3|^{36\alpha - 2}} \right) \lesssim \tau \xi_3^{3} \sum_{i=1}^{4} \|f_i\|_L^2
\]  

(99)

where

\[
\Sigma_6 = -\xi_5^5 - (\tau_1 - \xi_1^4) + G(\xi_1, \xi_2, \xi - \xi_1 - \xi_2).
\]  

Let \(\mu = \Sigma_6\). Since \(|\mu| \lesssim |\tau_1|\)

we have for \(\frac{1}{4} < b < \frac{1}{2}\) and \(\alpha > \frac{1}{2}\) that

\[
\iint \frac{|\tau_1|^{-2\alpha}d\xi_2 d\tau}{|\Sigma_3|^{36\alpha - 2}} \lesssim \iint \frac{|\tau_1|^{-2\alpha}|\xi_3|^{-4} d\mu}{|\Sigma_3|^{36\alpha - 2}} d\xi_2 d\tau
\]  

(100)

\[
\lesssim |\xi_3|^{-4}|\xi_3|^{4(3-6\alpha - 2\alpha)}|\xi_3| \sim |\xi_3|^{9-24\alpha - 8\alpha}.
\]

For \(-\frac{7}{4} \leq s \leq 0\) and \(\alpha > \frac{1}{2}\), the choice \(\max(\frac{1}{4}, \frac{7-2s}{24}, \frac{5-4s}{24}) < b < \frac{1}{2}\) in addition to (100) ensures

\[
(99) \lesssim \sup_{|\xi_3| \geq 1} |\xi_3|^{-2s} \max(|\xi_3|^{1+s}, 1)|\xi_3|^{-\frac{9-24\alpha - 8\alpha}{4}} \prod_{i=1}^{4} \|f_i\|_L^2 \lesssim \prod_{i=1}^{4} \|f_i\|_L^2.
\]

**Case V-c** \(|\xi| \leq 1 < |\xi_1|\). We know from (83) and (28) that

\[
|\tau| \lesssim 1, \quad |\tau - \xi_1^5| \lesssim 1 \ll |\xi_3|^4 |\xi_1| \sim |G|.
\]

If \(|\tau_1 - \xi_1^5| \sim |G| \gg |\tau_3 - \xi_3^4|\), since \(|\tau - \xi_5^5| = 1\) and \(|\tau_3 - \xi_3^5|^{-b} \lesssim 1\), the left-hand side of (84) is bounded by

\[
\sup_{|\xi_3| \geq 1} |\xi_3|^{-2s} \left( \iint \frac{|\xi_1|^{-2\alpha}d\xi_1 d\tau d\tau_1}{|\tau_3|^{3(1-3\alpha)\xi_3}} \right) \lesssim \tau \xi_3^{3} \sum_{i=1}^{4} \|f_i\|_L^2
\]  

(101)

where \(\Sigma_3\) is defined in (90). Note that \(|\Sigma_3| \sim |G|\). For \(-\frac{3}{4} < s \leq 0\), we choose \(\max(\frac{1}{4}, \frac{10}{27} - \frac{s}{8}) < b < \frac{1}{2}\). Using (46) and (48), and taking the Cauchy-Schwarz inequality with respect to \(\xi, \xi_1\), we have

\[
\text{LHS of (101)} \lesssim \sup_{|\xi_3| \geq 1} |\xi_3|^{-2s} |\xi_3|^{4(1-3\alpha)} |\xi_3|^{-s + \frac{3}{4} - 3b} \prod_{i=1}^{4} \|f_i\|_L^2 \lesssim \prod_{i=1}^{4} \|f_i\|_L^2.
\]

Otherwise \(|\tau_3 - \xi_3^4| \sim |G|\), let \(\mu = \Sigma_3\). Since \(|\mu| \lesssim |\tau_3 - \xi_3^4|\) and

\[
|\partial_\xi \Sigma_3| = | -5\xi_3^5 + 5(\xi - \xi_1 - \xi_3)| \sim |\xi_3|^4,
\]  

we have
we have for $\frac{1}{3} < b < \frac{1}{2}$ that

$$
\int_{|\xi| \leq 1} \frac{d\xi \, d\xi_1}{(\Sigma_3)^{6b-2}} \sim \int_{|\xi| \leq 1} \int_{|\mu| \leq |\tau_3 - \xi_3^5|} \frac{|\xi_3|^{-4} \, d\mu \, d\xi}{(\mu)^{6b-2}} \lesssim |\xi_3|^{-4}|\tau_3 - \xi_3^5|^{3-6b}.
$$

(102)

For $-\frac{2}{3} \leq s \leq 0$, the choice $\frac{3}{2} < b < \frac{1}{2}$ in addition to the Cauchy-Schwarz inequality, (46), (48) and (102) ensures

LHS of (84)

$$
\lesssim \sup_{|\xi_3| \geq 1, |\tau_3 - \xi_3^5| \geq 1} \frac{|\xi_3|^{-3s}}{|\tau_3 - \xi_3^5|^{6}} \left( \int_{|\xi| \leq 1} \int_{|\xi| \leq |\xi_1|} \frac{d\xi \, d\xi_1 \, d\tau_1 \, d\tau_1}{(\tau - \xi_3^5)^{2b}(|\tau_1 - \xi_1^5|)^{2b}(|\tau_1 - \xi_1^5 - (\tau - \xi_3^5)| - \Sigma_3)^{2b}} \right)^{\frac{1}{2}} \leq \frac{4}{3} \prod_{i=1}^{4} \|f_i\|_{L^2}.
$$

Case V-d $1 < |\xi_1|, |\xi_2|$. In this case, we know from (28) that $|G| \sim |\xi_1|^4|\xi_2 + \xi_3|$. We first consider the case when $|\tau - \xi_3^5| \sim |\xi_5^5| \geq |G| \geq |\tau_3 - \xi_3^5|$. If $|\xi| \leq |\xi_3|^{\frac{1}{4}}$, from (92) for $-\frac{3}{2} < s \leq 0$, it suffices to show

$$
\sup_{|\xi_3| \geq 1} \frac{|\xi_3|^{-3s+\frac{s}{4}(1+s)}}{|\tau_3 - \xi_3^5|^{6}} \left( \int_{|\xi| \leq 1} \int_{|\xi| \leq |\xi_1|} \frac{d\xi \, d\xi_1 \, d\tau_1 \, d\tau_1}{(\tau - \xi_3^5)^{2b}(|\tau_1 - \xi_1^5|)^{2b}(|\tau_1 - \xi_1^5 - (\tau - \xi_3^5)| - \Sigma_3)^{2b}} \right)^{\frac{1}{2}} \leq c,
$$

(103)

where $\Sigma_3$ is defined in (90) and whenever $-1 < s \leq 0$. Let $\mu = \xi_5^5 + \Sigma_3$. Since $|\mu| \lesssim |\xi_5^5| \leq |\xi_3|^{\frac{3}{4}}$ and

$$
|\partial_{\xi_1} (\xi_5^5 + \Sigma_3)| = |-5\xi_5^5 + 5(\xi - \xi_1 - \xi_3)^4| \sim |\xi_3|^4,
$$

we have for $\frac{1}{3} < b < \frac{1}{2}$ that

$$
\int_{|\xi| \leq |\xi_3|^{\frac{3}{4}}, |\xi_1| \ll |\xi_3|} \frac{d\xi \, d\xi_1}{(\Sigma_3)^{6b-2}} \sim \int_{|\xi| \leq |\xi_3|^{\frac{3}{4}}} \int_{|\mu| \leq |\xi_3|^4} \frac{|\xi_3|^{-4} \, d\mu \, d\xi}{(\mu)^{6b-2}} \lesssim |\xi_3|^{-4}|\xi_3|^{2(3-6b)}|\xi_3|^\frac{s}{4}.
$$

(104)

For $-\frac{4}{3} < s \leq 0$, the choice $\max(\frac{1}{4}, \frac{26-11s}{60}) < b < \frac{1}{2}$ in addition to the Cauchy-Schwarz inequality, (46), (48) and (104) ensures

LHS of (103) \lesssim \sup_{|\xi_3| \geq 1} \frac{|\xi_3|^{-3s+\frac{s}{4}(1+s)}|\xi_3|^{-2}|\xi_3|^{2(3-6b)}|\xi_3|^\frac{s}{4} \lesssim 1.

Otherwise ($|\xi_3|^{\frac{3}{4}} < |\xi_3|$), we know $|\xi| \ll |\xi_3| \leq |\xi_3|^{\frac{3}{4}}$. Then, it suffices to show

$$
\sup_{|\xi| \geq 1} \frac{|\xi|^{-5b}|\xi|^{-15b}}{|\xi|^{\frac{s}{4}}}
$$
\[ \left( \int \int \frac{d\xi_1 d\xi_2 d\tau_1 d\tau_2}{(\tau_1 - \xi_1^5)^{2b}(\tau_2 - \xi_2^5)^{2b}(\tau_1 - \xi_1^5 + (\tau_2 - \xi_2^5) - \Sigma_1)^{2b}} \right)^{\frac{1}{4}} \leq c, \]

where \( \Sigma_1 \) is defined in (85). Let \( \mu = \Sigma_1 \). Since \( |\mu| \lesssim |\xi|^5 \) and
\[ |\partial_\xi \Sigma_1| = |5\xi_1^4 + 5(\xi - \xi_1) - \xi_3^4| \sim |\xi_3|^4, \]
we have for \( \frac{1}{3} < b < \frac{1}{2} \) that
\[ \int_{|\xi_2| \leq |\xi|^{\frac{2}{5}}, |\xi_1| \ll |\xi_3|} \frac{d\xi_1 d\xi_2}{(\Sigma_1)^{6b-2}} \sim \int_{|\xi_2| \leq |\xi|^{\frac{2}{5}}} \int_{|\mu| \ll |\xi|^5} \frac{|\xi_1|^{-4}}{(\mu)^{6b-2}} d\mu d\xi \lesssim |\xi|^{-4} |\xi|^{5(3-6b)} |\xi|^{\frac{5}{3}}. \]

For \( -\frac{24}{25} < s \leq 0 \), the choice \( \max\left( \frac{1}{10}, \frac{57 - 30s}{100} \right) < b < \frac{1}{2} \) in addition to the Cauchy-Schwarz inequality, (46), (48) and (106) ensures
\[ \text{LHS of (105)} \lesssim \sup_{|\xi| \geq 1} |\xi|^{1-5b} |\xi|^{-\frac{12s}{5}} |\xi|^{-2} |\xi|^{\frac{5}{3}(3-6b)} |\xi|^{\frac{5}{3}} \lesssim 1. \]

Now we consider the case when \( |\tau - \xi_1^5| \sim |\xi|^5 \ll |\xi_3|^4 |\xi_2 + \xi_3| \sim |G| \). If \( |\tau_3 - \xi_3^5| \ll |G| \), we know from (30) and (97) that \( |\tau_1 - \xi_1^5| \sim |G| \). Similarly as before, we can obtain for \( \frac{5}{8} < b < \frac{1}{2} \) that
\[ \sup_{|\tau_1 - \xi_1^5| \geq 1} |\tau_1 - \xi_1^5|^{1-2b} \]
\[ \cdot \left( \int \int_{|\tau_1 - \xi_1^5| \sim |G|} \frac{d\tau_1 d\tau_2 d\xi}{(\tau_1 - \xi_1^5)^{2b}(\tau_2 - \xi_2^5)^{2b}(\tau - \xi_1^5 - (\tau_2 - \xi_2^5) - \Sigma_\tau)^{2b}} \right)^{\frac{1}{4}} \lesssim |\xi_3|^{-4} |\tau_1 - \xi_1^5|^{3-6b} \lesssim |\xi_3|^{-4}, \]
where \( \Sigma_\tau = \tau_1 - \xi_1^5 - G(\xi_1, \xi_2, \xi - \xi_1 - \xi_2) \),
due to
\[ |\partial_\xi (-\xi_1^5 - \Sigma)| = |5(\xi_1 - \xi_1) - \xi_3^4| \sim |\xi_3|^4. \]
This in addition to (92) implies
\[ \text{LHS of (84)} \lesssim \sup_{|\xi_1| \geq 1} |\xi_1|^{1-2s} |\xi_3|^{-2} \left( \int_{|\xi_1| \ll |\xi_3|} d\xi_1 \right)^{\frac{1}{4}} \prod_{i=1}^{4} \|f_i\|_{L^2} \prod_{i=1}^{4} \|f_i\|_{L^2} \]
\[ \lesssim \prod_{i=1}^{4} \|f_i\|_{L^2}, \]
for \( -\frac{1}{3} \leq s \leq 0 \). Note that the above argument does not depend on the choice of the maximum modulation among \( |\tau_i - \xi_i^5|, i = 1, 2, 3 \), and hence we can have
\[ \text{LHS of (84)} \prod_{i=1}^{4} \|f_i\|_{L^2}, \]
for the case when \( |\tau_3 - \xi_3^5| \gg |G| \).
Case VI (low $\times$ low $\times$ low $\Rightarrow$ low) $|\xi_3| < 1$. We know from the identity (29) that $|\xi| < 1$, which implies $|\tau| \lesssim 1$. Then, the left-hand side of (84) is equivalent to
\[
\int \frac{f_1(\tau_1, \xi_1) f_2(\tau_2, \xi_2) f_3(\tau_3, \xi_3) f_4(\tau, \xi)}{\langle \tau_1 \rangle^\alpha \langle \tau_2 \rangle^\alpha \langle \tau_3 \rangle^\alpha},
\]
with $\xi_1 + \xi_2 + \xi_3 = \xi$, $\tau_1 + \tau_2 + \tau_3 = \tau$, and $|\xi_1|, |\xi_2|, |\xi_3|, |\tau| < 1$.

which is bounded by
\[
|f_1| |f_2| |f_3| |f_4| L^2 \| f_3 \| L^2 \| f_4 \| L^2,
\]
by taking the Cauchy-Schwarz inequality, since $\alpha > 1/2$.

Therefore we complete the proof of Proposition 4.

4. Duhamel boundary forcing operator. In this section, we introduce the Duhamel boundary forcing operator, which was introduced by Colliander and Kenig [12] and further developed by several researchers [15, 5, 4, 6], which helps to construct the solution operator involving the boundary forcing conditions. We, particularly, refer to [6] for the fifth-order KdV-type equation.

4.1. Duhamel boundary forcing operator class. We introduce the Duhamel boundary forcing operator associated to the linear fifth-order equation. Let
\[
M = \frac{1}{B(0)\Gamma(4/5)}.
\]

For $f \in C_0^\infty(\mathbb{R}^+)$, define the boundary forcing operator $L^0$ of order 0
\[
L^0 f(t, x) := M \int_0^t e^{(t-t')/\alpha} \partial_x^5 \delta_0(x) \tilde{\mathcal{I}}_{-4/5} \tilde{f}(t') dt'.
\]

By the change of variable and (19), we represent (108) by
\[
L^0 f(t, x) = M \int_0^t B \left( \frac{x}{(t-t')^{1/5}} \right) \tilde{\mathcal{I}}_{-4/5} \tilde{f}(t') dt'.
\]

Moreover, a straightforward calculation gives
\[
L^0(\partial_x f)(t, x) = M \delta_0(x) \tilde{\mathcal{I}}_{-4/5} \tilde{f}(t) + \partial_x^5 L^0 f(t, x).
\]

We state the several lemmas associated to $L^0 f$ defined as in (108). We refer to [6] and references therein for the proofs.

**Lemma 4.1** (Continuity and decay property of $L^0 f$ [6]). Let $f \in C_0^{\infty}(\mathbb{R}^+)$. (a) For fixed $0 \leq t \leq 1$, $\partial_x^k L^0 f(t, x)$, $k = 0, 1, 2, 3$, is continuous in $x \in \mathbb{R}$ and has the decay property in terms of the spatial variable as follows:
\[
|\partial_x^k L^0 f(t, x)| \lesssim_N \| f \|_{L^{\infty} + k} \langle x \rangle^{-N}, \quad N \geq 0.
\]
(b) For fixed $0 \leq t \leq 1$, $\partial_x^4 L^0 f(t, x)$ is continuous in $x$ for $x \neq 0$ and is discontinuous at $x = 0$ satisfying
\[
\lim_{x \to 0^-} \partial_x^4 L^0 f(t, x) = c_1 \mathcal{I}_{-4/5} f(t), \quad \lim_{x \to 0^+} \partial_x^4 L^0 f(t, x) = c_2 \mathcal{I}_{-4/5} f(t)
\]
for $c_1 \neq c_2$. $\partial_x^4 L^0 f(t, x)$ also has the decay property in terms of the spatial variable
\[
|\partial_x^4 L^0 f(t, x)| \lesssim_N \| f \|_{L^{\infty} + 4} \langle x \rangle^{-N}, \quad N \geq 0.
\]

In particular, we have $L^0 f(t, 0) = f(t)$. 

\[\Box\]
In the following, we give the generalization of the boundary forcing operator \( \mathcal{L}^0 f \) and its properties introduced in [6].

Let \( \text{Re} \lambda > 0 \) and \( g \in C_0^\infty(\mathbb{R}^+) \) be given. Define
\[
\mathcal{L}_\pm^h g(t, x) = \left[ \frac{x_\pm^{(\lambda+5)-1}}{\Gamma(\lambda)} \ast \mathcal{L}^0(I_{-\mp} g)(t, \cdot) \right](x),
\]
where \( \ast \) denotes the convolution operator. Note that \( \mathcal{L}_\pm^h \) is for the right / left half-line problem, respectively. With \( x_\pm^{(\lambda+5)-1}/\Gamma(\lambda) = (\mp x)^{\lambda+1}/\Gamma(\lambda) \) (in the sense of distribution), we represent each of them by
\[
\mathcal{L}_\pm^h g(t, x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (y - x)^{\lambda-1} \mathcal{L}^0(I_{-\mp} g)(t, y)dy. \tag{110}
\]
and
\[
\mathcal{L}_\pm^h g(t, x) = \frac{1}{\Gamma(\lambda)} \int_{-\infty}^x (x - y)^{\lambda-1} \mathcal{L}^0(I_{-\mp} g)(t, y)dy. \tag{111}
\]

For \( \text{Re}\lambda > -5 \), the integration by parts in (110) and (111), the decay property in Lemma 4.1 and (109) yield
\[
\mathcal{L}_-^h g(t, x) = \frac{x_+^{(\lambda+5)-1}}{\Gamma(\lambda+5)} \ast \partial_+^h \mathcal{L}^0(I_{-\mp} g)(t, \cdot) \tag{112}
\]
and
\[
\mathcal{L}_+^h g(t, x) = \frac{x_-^{(\lambda+5)-1}}{\Gamma(\lambda+5)} \ast \partial_-^h \mathcal{L}^0(I_{-\mp} g)(t, \cdot)
\]
respectively. It, thus, immediately satisfies (in the sense of distributions)
\[
(\partial_t - \partial_+^h) \mathcal{L}_-^h g(t, x) = M \frac{x^{\lambda-1}}{\Gamma(\lambda)} I_{-\mp} - \mp g(t)
\]
and
\[
(\partial_t - \partial_-^h) \mathcal{L}_+^h g(t, x) = M \frac{x^{\lambda-1}}{\Gamma(\lambda)} I_{-\mp} - \mp g(t).
\]

**Lemma 4.2** (Spatial continuity and decay properties for \( \mathcal{L}_\pm^h g(t, x) \) [6]). Let \( g \in C_0^\infty(\mathbb{R}^+) \) and \( M \) be as in (107). Then, we have
\[
\mathcal{L}_\pm^h g = \partial_\pm^h \mathcal{L}^0 I_{\pm}^h g, \quad k = 0, 1, 2, 3, 4.
\]
Moreover, \( \mathcal{L}_\pm^h g(t, x) \) is continuous in \( x \in \mathbb{R} \setminus \{0\} \) and has a step discontinuity of size \( Mg(t) \) at \( x = 0 \). For \( \lambda > -4 \), \( \mathcal{L}_\pm^h g(t, x) \) is continuous in \( x \in \mathbb{R} \). For \( -4 \leq \lambda \leq 1 \) and \( 0 \leq t \leq 1 \), \( \mathcal{L}_\pm^h g(t, x) \) satisfies the following decay bounds:
\[
|\mathcal{L}_-^h g(t, x)| \leq c_{m, \lambda, g}(x)^{-m}, \quad \text{for all} \quad x \leq 0 \quad \text{and} \quad m \geq 0,
\]
\[
|\mathcal{L}_+^h g(t, x)| \leq c_{\lambda, g}(x)^{\lambda-1}, \quad \text{for all} \quad x \geq 0.
\]
\[
|\mathcal{L}_\pm^h g(t, x)| \leq c_{m, \lambda, g}(x)^{-m}, \quad \text{for all} \quad x \geq 0 \quad \text{and} \quad m \geq 0,
\]
and

\[ |L_\lambda^+ g(t, x)| \leq c_{\lambda, g} (x)^{\lambda-1}, \quad \text{for all} \quad x \leq 0. \]

**Lemma 4.3** (Values of \(L_\lambda^+ f(t, 0)\) and \(L_\lambda^- f(t, 0)\) [6]). For \(\Re \lambda > -4\),

\[
L_\lambda^+ f(t, 0) = \frac{1}{B(0)\Gamma(4/5)} \frac{\cos \left( \frac{(1+4\lambda)\pi}{10} \right)}{5 \sin \left( \frac{(1-\lambda)\pi}{5} \right)} f(t)
\]

and

\[
L_\lambda^- f(t, 0) = \frac{1}{B(0)\Gamma(4/5)} \frac{\cos \left( \frac{(1-6\lambda)\pi}{10} \right)}{5 \sin \left( \frac{(1-\lambda)\pi}{5} \right)} f(t)
\]

### 4.2. Linear version

We consider the linearized equation of (1).

\[
\partial_t u - \partial_x^5 u = 0.
\]

The unitary group associated to (113) as

\[
e^{it\partial_t^5} \phi(x) = \frac{1}{2\pi} \int e^{ix\xi} e^{it\xi} \hat{\phi}(\xi) d\xi,
\]

allows

\[
\begin{cases}
(\partial_t - \partial_x^5) e^{it\partial_t^5} \phi(x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\
e^{it\partial_t^5} \phi(x)|_{t=0} = \phi(x), & x \in \mathbb{R}.
\end{cases}
\]

Recall \(L_\lambda^+\) in (112) for the right half-line problem. Let \(a_j\) and \(b_j\) be constants depending on \(\lambda_j\), \(j = 1, 2\), given by

\[
a_j = \frac{1}{B(0)\Gamma \left( \frac{j}{5} \right)} \frac{\cos \left( \frac{(1+4\lambda_j)\pi}{10} \right)}{5 \sin \left( \frac{(1-\lambda_j)\pi}{5} \right)} \quad \text{and} \quad b_j = \frac{1}{B(0)\Gamma \left( \frac{j}{5} \right)} \frac{\cos \left( \frac{(4\lambda_j-3)\pi}{10} \right)}{5 \sin \left( \frac{(2-\lambda_j)\pi}{5} \right)}.
\]

Let us choose \(\gamma_1\) and \(\gamma_2\) satisfying

\[
\begin{bmatrix}
f(t) \\
I_\lambda^+ g(t)
\end{bmatrix} = A
\begin{bmatrix}
\gamma_1(t) \\
\gamma_2(t)
\end{bmatrix},
\]

where

\[
A(\lambda_1, \lambda_2) = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}.
\]

We choose an appropriate \(\lambda_j\), \(j = 1, 2\), such \(A\) is invertible. Then, \(u\) defined by

\[
u(t, x) = L_\lambda^+ \gamma_1(t, x) + L_\lambda^- \gamma_2(t, x),
\]

solves

\[
\begin{cases}
(\partial_t - \partial_x^5) u = 0, \\
u(0, x) = 0, \\
u(t, 0) = f(t), \quad \partial_x u(t, 0) = g(t).
\end{cases}
\]

See Section 3 in [6] for more details.
4.3. Nonlinear version. The Duhamel inhomogeneous solution operator $D$

$$Dw(t,x) = \int_0^t e^{(t-t')\partial_x^5}w(t',x)dt'$$

solves

$$\begin{cases}
\left(\partial_t - \partial_x^5\right) Dw(t,x) = w(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\
Dw(x,0) = 0, \quad x \in \mathbb{R}
\end{cases}$$

By choosing a suitable $\gamma_1$ and $\gamma_2$ depending on not only $f$ and $g$, but also $e^{t\partial_x^5}\phi(x)$ and $Dw$, $u$ defined by

$$u(t,x) = L^{\gamma_1}_+\gamma_1(t,x) + L^{\gamma_2}_+\gamma_2(t,x) + e^{t\partial_x^5}\phi(x) + Dw$$

solves

$$\begin{cases}
\left(\partial_t - \partial_x^5\right) u = w, \\
u(0,x) = \phi(x), \\
u(t,0) = f(t), \quad \partial_x u(t,0) = g(t)
\end{cases}$$

We refer to [6] for more details.

5. Energy estimates. We are going to give the fundamental energy estimates.

**Lemma 5.1.** Let $s \in \mathbb{R}$ and $0 < T \leq 1$. If $\phi \in H^s(\mathbb{R})$, then

(a) (Space traces) $\|\psi_T(t)e^{t\partial_x^5}\phi(x)\|_{C(\mathbb{R};H^s(\mathbb{R}))} \lesssim \|\phi\|_{H^s(\mathbb{R})}$;

(b) ((Derivatives) Time traces) In particular, for $-\frac{9}{2} + j \leq s \leq \frac{1}{2} + j$,

$$\|\psi_T(t)\partial_x^j e^{t\partial_x^5}\phi(x)\|_{C(\mathbb{R};L^{\frac{s+j-\frac{3}{2}}{s}}(\mathbb{R}))} \lesssim \|\phi\|_{H^s(\mathbb{R})}, \quad j \in \{0,1,2\};$$

(c) (Bourgain spaces) For $b \leq \frac{1}{2}$ and $\alpha > \frac{1}{2}$, we have

$$\|\psi_T(t)e^{t\partial_x^5}\phi(x)\|_{X^{s,b} \cap D^{s \alpha}} \lesssim T^{\frac{1}{2} - \alpha}\|\phi\|_{H^s(\mathbb{R})}.$$  

The implicit constants do not depend on $0 < T \leq 1$ but $\psi$.

**Remark 7.** In contrast to IVP, the time localization may restrict the regularity range (both upper- and lower-bounds) due to the (derivatives) time trace estimates. Similar phenomenon can be seen in $X^{s,b}$ estimates (see, in particular, Lemma 2.11 in [30]), while the modulation exponent $b$ is affected to be taken by the cut-off function. See (b) and (c) in Lemma 5.1 for the comparison.

**Proof.** The proofs of (a) and (c) are standard, hence we omit the details and refer to [30].

(b). Let $\phi = \phi_1 + \phi_2$, where $\widehat{\phi_1}(\xi) = \chi_{\leq 1}(\xi)\widehat{\phi}(\xi)$. For $\phi_1$, we observe

$$\mathcal{F}_t[\psi_T\partial_x^j e^{t\partial_x^5}\phi_1](\tau,x) = \int e^{ix\xi}(i\xi)^j\widehat{\phi_1}(\xi)\widehat{\psi_T}(\tau - \xi^5)\ d\xi,$$

which yields

$$\|\psi_T\partial_x^j e^{t\partial_x^5}\phi_1\|_{H^{s+j-\frac{3}{2}}} = \left(\int \left|\int e^{ix\xi}(i\xi)^j\widehat{\phi_1}(\xi)\widehat{\psi_T}(\tau - \xi^5)\ d\xi\right|^2 d\tau\right)^{\frac{1}{2}}.$$  

When $|\tau| \leq 1$, it directly follows from $|\widehat{\psi_T}(\tau - \xi^5)| \lesssim T$ and the Cauchy-Schwarz inequality that

$$\text{RHS of (114)} \lesssim T\|\phi_1\|_{H^s} \lesssim \|\phi\|_{H^s}. $$
When $|\tau| > 1$, we know $|\tilde{\psi}_T(\tau - \xi^5)| \sim |\tilde{\psi}_T(\tau)|$, and hence we have

$$\text{RHS of (114)} \lesssim \left( \int_{|\tau| > 1} |\tau|^{\frac{s+2-j}{5}} |\tilde{\psi}_T(\tau)|^2 \, d\tau \right) \frac{1}{\sqrt{T}} \|\phi_1\|_{H^s}.$$  

Note that

$$\int_{|\tau| > 1} |\tau|^2 |\tilde{\psi}_T(\tau)|^2 \, d\tau = T^{1-2\sigma} \int_{|\tau| > T} |\tau|^{2\sigma} |\tilde{\psi}(\tau)|^2 \, d\tau. \quad (115)$$

Hence, for $\frac{s+2-j}{5} \leq \frac{1}{2}$, we have

$$\text{RHS of (114)} \lesssim \|\phi\|_{H^s}.$$ 

For $\phi_2$, observe that

$$\partial_x^s e^{it\xi^5} \phi_2(x) = \int e^{ix\xi} (i\xi)^j e^{it\xi^5} \tilde{\phi}_2(\xi) \, d\xi = \int e^{it\eta} e^{ix\eta^5} (i\eta)^j \tilde{\phi}_2(\eta^5) \frac{d\eta}{5\eta^x}.$$ 

It implies

$$\|\psi_T \partial_x^s e^{it\xi^5} \phi_2\|_{H^s} \lesssim \left( \int (\tau)^{\frac{2(s+2-j)}{5}} \left| \int e^{ix\eta^5} (i\eta)^j \tilde{\phi}_2(\eta^5) \tilde{\psi}_T(\tau - \eta) \frac{d\eta}{5\eta^x} \right|^2 \, d\tau \right)^{\frac{1}{2}}. \quad (116)$$

It is known from the support of $\phi_2$ that $|\eta| > 1$. For $|\tau| \leq 1$, the weight $(\tau)^{\frac{2(s+2-j)}{5}}$ and the integration with respect to $\tau$ are negligible. If $|T\eta| \leq 1$, we know $|\tilde{\psi}_T(\tau - \eta)| \lesssim T$. Since

$$\left| \int_{1<|\eta|<1/T} |\eta|^j \tilde{\phi}_2(\eta^5) \frac{d\eta}{5\eta^x} \right| \lesssim \max(1, T^{\frac{s+2-j}{5} - \frac{1}{2}}) \|\phi_2\|_{H^s},$$

for $0 < T \leq 1$, we have

$$\|\psi_T \partial_x^s e^{it\xi^5} \phi_2\|_{H^s} \lesssim T^{\frac{s+2-j}{5}} \|\phi\|_{H^s} \lesssim \|\phi\|_{H^s},$$

for $\frac{1}{2} + \frac{s+2-j}{5} \geq 0$. If $|T\eta| > 1$, we know $|\tilde{\psi}_T(\tau - \eta)| \lesssim |\eta|^{1-s}$. Since

$$\left| \int_{1/T < |\eta|} |\eta|^{-\frac{1}{2}} \tilde{\phi}_2(\eta^5) \frac{d\eta}{5\eta^x} \right| \lesssim \left( \int_{|\eta| > 1/T} |\eta|^{-\frac{2(s+2-j)}{5} - 2} \, d\eta \right)^{\frac{1}{2}} \|\phi\|_{H^s},$$

we have

$$\text{LHS of (116)} \lesssim T^{\frac{s+2-j}{5}} \|\phi\|_{H^s} \lesssim \|\phi\|_{H^s},$$

for $\frac{1}{2} + \frac{s+2-j}{5} \geq 0$.

We may assume, from now on, that $|\tau|, |\eta| > 1$. For given $\tau$, we further divide the region in $\eta$ into the following:

$$\text{I. } |\eta| < \frac{1}{2} |\tau|, \quad \text{II. } 2|\tau| < |\eta|, \quad \text{III. } \frac{1}{2} |\tau| \leq |\eta| \leq 2|\eta|. \quad (117)$$

The way to divide the integration region as (117) will be used repeatedly.

\footnotetext{14}{When \( \frac{s+2-j}{5} = \frac{1}{2} \), the constant depending on \( T \) is \( \log \frac{1}{T} \) instead of \( T^{\frac{s+2-j}{5} - \frac{1}{2}} \), but it does not influence on our analysis.}
I. $|\eta| < \frac{1}{2}|\tau|$. If $T|\tau| \leq 1$, we know $|\hat{\psi}_T(\tau - \eta)| \lesssim T$. Since

$$\left| \int_{|\eta| \leq |\tau|} |\eta|^\frac{s}{2} \hat{\phi}_2(\eta) \frac{d\eta}{5\eta^\frac{s}{2}} \right|^2 \lesssim |\tau|^{-\frac{s}{2}(s+2-j)+1} \|\phi_2\|_{H^s}^2,$$  \hspace{1cm} (118)

we have

$$\text{LHS of (116)} \lesssim T \left( \int_{1<|\tau| \leq 1/T} |\tau| \left\| f \right\|_{H^s} \right)^{\frac{1}{2}} \lesssim \|\phi\|_{H^s}.$$  

If $T|\tau| > 1$, we know $|\hat{\psi}_T(\tau - \eta)| \lesssim T^{1-k}|\tau|^{-k}$, for any positive $k$. For $k > 1$, we have from (118) that

$$\text{LHS of (116)} \lesssim T^{1-k} \left( \int_{1/T < |\tau|} |\tau|^{-2k+1} d\tau \right)^{\frac{1}{2}} \|\phi\|_{H^s} \lesssim \|\phi\|_{H^s}.$$  

We remark that the smoothness of $\psi$ guarantees not the good bound of $T$, but the integrability, in other words, we only need a large $k$ for

$$\int_{1/T < |\tau|} |\tau|^{-2k+1} d\tau < \infty.$$  

II. $2|\tau| < |\eta|$. If $T|\eta| < 1$, we know $|\hat{\psi}_T(\tau - \eta)| \lesssim T$. Then, the right-hand side of (116) is bounded by

$$T \left( \int_{|\tau| \leq 1/T} \int_{|\eta| \leq 1/T} |\eta|^{s+2-j} |\eta|^\frac{s}{2} \hat{\phi}_2(\eta) \frac{d\eta}{5\eta^\frac{s}{2}} \right)^{1/2}.$$  \hspace{1cm} (119)

Since

$$\int \left( |\eta|^{s+2-j} |\eta|^\frac{s}{2} \hat{\phi}_2(\eta) \frac{d\eta}{5\eta^\frac{s}{2}} \right)^2 d\eta = c \int |\xi|^{2s} \hat{\phi}_2^2 d\xi,$$

the Cauchy-Schwarz inequality with respect to $\eta$ and $\tau$ yields (119) $\lesssim \|\phi\|_{H^s}$. If $T|\eta| > 1$, we know $|\hat{\psi}_T(\tau - \eta)| \lesssim T^{1-k} |\eta|^k$, for any positive $k$. On the region $|\tau| < 1/T$, similarly as before, we have

$$T^{1-k} \left( \int_{|\tau| \leq 1/T} \int_{|\eta| > 1/T} |\eta|^{-k} |\eta|^{s+2-j} |\eta|^\frac{s}{2} \hat{\phi}_2(\eta) \frac{d\eta}{5\eta^\frac{s}{2}} \right)^{1/2},$$

which implies from the Cauchy-Schwarz inequality that

$$\text{LHS of (116)} \lesssim T^{1-k} T^{-\frac{1}{2}} \left( \int_{|\eta| > 1/T} |\eta|^{-2k} d\eta \right)^{\frac{1}{2}} \|\phi\|_{H^s} \lesssim \|\phi\|_{H^s}.$$  

On the region $|\tau| \geq 1/T$, it suffices to control

$$T^{1-k} \left( \int_{|\tau| > 1/T} \int_{|\tau| < |\eta|} |\eta|^{-k} |\eta|^{s+2-j} |\eta|^\frac{s}{2} \hat{\phi}_2(\eta) \frac{d\eta}{5\eta^\frac{s}{2}} \right)^{1/2},$$

since

$$\int_{|\tau| < |\eta|} |\eta|^{-2k} d\eta \lesssim |\tau|^{-2k+1}.$$
the Cauchy-Schwarz inequality and the change of variable yield
\[
\text{LHS of (116)} \lesssim T^{1-k} \left( \int_{|\tau| > 1/T} |\tau|^{-2k+1} \, d\eta \right)^{\frac{1}{2}} \|\phi\|_{H^s} \lesssim \|\phi\|_{H^s}. \tag{120}
\]

**III.** $\frac{1}{2} |\tau| < |\eta| < 2|\tau|$. In this case, we know $|\tau|^{-\frac{2}{1-k} - \frac{1}{p}} |\eta|^{-\frac{1}{2}} \sim |\eta|^{-\frac{1}{2}}$ in the integrand of the right-hand side of (116). We further assume $|\tau|, |\eta| > 1/T$, otherwise we can use the same way to control (119). If $\tau \cdot \eta < 0$, we know $|\hat{\psi}_T(\tau - \eta)| \lesssim T^{1-k}|\tau|^k$. Since
\[
\left( \int_{|\eta|^{-1/|\tau|}}^1 d\eta \right)^{\frac{1}{2}} \lesssim |\tau|^{\frac{1}{2}},
\]
we have LHS of (116) $\lesssim \|\phi\|_{H^s}$, similarly as (120). If $\tau \cdot \eta > 0$, we further divide the case into $|\tau - \eta| < 1/T$ and $|\tau - \eta| > 1/T$. For the former case, let $\Phi(\eta) = |\eta|^{\frac{2}{1-k}} \hat{\phi}_2(\eta^{\frac{1}{2}})$. We note that $\|\Phi\|_{L^2} \sim \|\phi_2\|_{H^s}$. Since $|\hat{\psi}_T(\tau - \eta)| \lesssim T$, we have
\[
\left| \int_{|\eta - \tau| < 1/T} |\eta|^{-\frac{2}{1-k}} \hat{\phi}_2(\eta^{\frac{1}{2}}) \hat{\psi}_T(\tau - \eta) \, d\eta \right| \lesssim T \int_{|\eta - \tau| < 1/T} \Phi(\eta) \, d\eta \lesssim M\Phi(\tau),
\]
where $Mf(x)$ is the Hardy-Littlewood maximal function of $f$. Since $\|Mf\|_{L^p} \lesssim \|f\|_{L^p}$ for $1 < p \leq \infty$ (see, in particular, [27]), we have
\[
\text{LHS of (116)} \lesssim \left( \int |M\Phi(\eta)|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim \|\Phi\|_{L^2} \lesssim \|\phi\|_{H^s}.
\]
For the latter case, the integration region in $\eta$ can be reduced to $\tau + 1/T < \eta < 2\tau$ for positive $\tau$ and $\eta$, since the exact same argument can be applied to the other regions.\(^{15}\) Since $|\hat{\psi}_T(\tau - \eta)| \lesssim T^{1-k}|\tau - \eta|^{-k}$ in this case, the left-hand side of (116) is bounded by
\[
T^{1-k} \left( \int_{|\tau| > 1/T} \int_{|\tau - \eta| > 1/T} \frac{\Phi(\eta)}{|\eta|^{k}} \, d\eta \, d\tau \right)^{\frac{1}{2},}
\]
where $\Phi$ is defined as in (121). Let $\epsilon = (k - 1)/2$ for $k > 1$. Then, the change of variable, the Cauchy-Schwarz inequality and the Fubini theorem yield
\[
(122) \lesssim T^{1-k} \left( \int_{|\tau| > 1/T} \int_{|\tau - \eta| > 1/T} \frac{\Phi(\tau + h)}{|h|^{k}} \, d\tau \, dh \right)^{\frac{1}{2}},
\]
\[
\lesssim T^{1-k} T^{\epsilon} \left( \int_{|\tau| > 1/T} \int_{|h| > 1/T} \frac{|\Phi(\tau + h)|^2}{|h|^{2k-1-2\epsilon}} \, dh \, d\tau \right)^{\frac{1}{2}}
\]
\[
\lesssim \|\Phi\|_{L^2} \lesssim \|\phi\|_{H^s}.
\]
Therefore, we complete the proof of (b). \(\square\)

\(^{15}\)Indeed, we only have four regions; $\tau + \frac{1}{2} < \eta < 2\tau$ and $\frac{1}{2} \tau < \eta < \tau - \frac{1}{2}$ for positive $\tau, \eta$, and $\tau + \frac{1}{2} < \eta < \frac{1}{2}\tau$ and $2\tau < \eta < \tau - \frac{1}{2}$ for negative $\tau, \eta$, and the same argument can be applied on each region.
Remark 8. The proof of Lemma 5.1 (b) exactly shows the proof of
\[ \| \psi_T(t)f(t) \|_{H^\sigma} \lesssim \| f \|_{H^\sigma}, \]
whenever \(-\frac{1}{2} \leq \sigma \leq \frac{1}{2}\). With this, we have a variant of Lemma 2.9
\[ \| \psi_T(t)f(t) \|_{H^\sigma_0} \lesssim \| f(t) \|_{H^\sigma_0}, \]
whenever \(-\frac{1}{2} < \frac{s+2}{5} < \frac{1}{2}\).

**Lemma 5.2.** Let \( s \in \mathbb{R} \) and \( 0 < T \leq 1 \). For \( 0 < b < \frac{1}{2} < \alpha < 1 - b \), there exists \( \theta = \theta(s,j,b,\alpha) \) such that

(a) (Space traces)
\[ \| \psi_T(t)Dw(x,t) \|_{C(\mathbb{R}; H^{s-b}(\mathbb{R}))} \lesssim T^\theta \| w \|_{X^{s,-b}}; \]

(b) (Derivatives) Time traces) for \(-\frac{9}{2} + j \leq s \leq \frac{1}{2} + j, j = 0,1,2,\)
\[ \| \psi_T(t)\partial_t^j Dw(x,t) \|_{C(\mathbb{R}; H^{s+\frac{j}{2}-1}(\mathbb{R}))} \lesssim \begin{cases} T^\theta \| w \|_{X^{s,-b}}, & \text{if } 0 \leq \frac{s+2-j}{5} \leq \frac{1}{2}, \\ T^\theta \| \| w \|_{X^{s,-b}} + \| w \|_{Y^{s,-b}}, & \text{if } -\frac{1}{2} \leq \frac{s+2-j}{5} \leq 0; \end{cases} \]

(c) (Bourgain spaces estimates)
\[ \| \psi_T(t)Dw(x,t) \|_{X^{s,b} \cap D^0} \lesssim T^\theta \| w \|_{X^{s,-b}}. \]

Remark 9. In view of the proof of Lemma 5.2 (b), the intermediate norm \( Y^{s,b} \) is needed only for the regularity region \(-\frac{1}{2} \leq \frac{s+2-j}{5} \leq 0\) (equivalently, \(-\frac{9}{2} + j \leq s \leq j - 2\), and thus, the nonlinear estimates in \( Y^{s,b} \) norm (Theorems 1.1 (b) and 1.2 (b)) for the negative regularities are enough for our analysis.

Remark 10. In view of the proof of Lemma 5.2, under the condition \(-\frac{1}{2} \leq \frac{s+2-j}{5} \leq \frac{1}{2}, j = 0,1,2,\) we can choose \( \theta = 1 - \alpha - b \) uniformly in \( s \) and \( j \) for \( b < \frac{1}{2} < \alpha < \frac{3}{4} - b \) such that \( T^{\frac{1}{2} - 2\alpha - b} \) can be small enough (by choosing small \( T \ll 1 \)) to close the iteration argument. See Section 6.

**Proof of Lemma 5.2.**

(a) A direct calculation gives
\[ \mathcal{F}[\psi_T Dw](\tau, \xi) = c \int \tilde{w}(\tau', \xi) \frac{\hat{\psi}_T(\tau - \tau') - \hat{\psi}_T(\tau - \xi^5)}{i(\tau' - \xi^5)} d\tau'. \tag{123} \]
Since \( \| \psi_T Dw \|_{C_t H^s} \lesssim \| \langle \xi \rangle^s \mathcal{F}[\psi_T Dw](\tau, \xi) \|_{L^2_x L^1_t}, \) it suffices to control
\[ \left( \int \langle \xi \rangle^{2s} \left| \int |\tilde{w}(\tau', \xi)| \left| \frac{\hat{\psi}_T(\tau - \tau') - \hat{\psi}_T(\tau - \xi^5)}{|\tau' - \xi^5|} \right|^2 d\tau' \right|^2 d\xi \right)^{\frac{1}{2}}, \tag{124} \]
due to (123). On the region \(|\tau' - \xi^5| \leq T^{-1}, \) we note from mean value theorem that
\[ \frac{|\hat{\psi}_T(\tau - \tau') - \hat{\psi}_T(\tau - \xi^5)|}{|\tau' - \xi^5|} = T^2 |\hat{\psi}'(T(\tau - \xi^5) + \sigma)|, \]
for small $\sigma$ depending on $\tau'$ and $\xi^5$. Since $T\hat{\psi}'(T\tau)$ is $L^1$ integrable with respect to $\tau$, the Cauchy-Schwarz inequality yields

$$\text{(124)} \lesssim T \left( \int |\xi|^{2s} \left| \int_{|\tau'-\xi^5| \leq 1/T} |\bar{w}(\tau', \xi)| d\tau' \right|^2 d\xi \right)^{1/2} \lesssim T^{\frac{1}{2} - b} (1 + T^{\frac{1}{2} + b}) \|w\|_{X^s, -b}.$$ 

On the other hand, on the region $|\tau' - \xi^5| > T^{-1}$, we use the $L^1$ integrability of $\hat{\psi}_T$, so that

$$\text{(124)} \lesssim T \left( \int |\xi|^{2s} \left| \int_{|\tau'-\xi^5| > 1/T} \frac{1}{|\tau' - \xi^5|} \hat{\psi}_T d\tau' d\xi \right|^2 d\xi \right)^{1/2} \lesssim T^{\frac{1}{2} - b} \|w\|_{X^s, -b}.$$ 

(b). A direct calculate gives

$$\psi_T \partial_x^j Dw(t, x) = c \int e^{ix\xi} e^{it\xi^5} (i\xi)^j \psi_T(t) \int \bar{w}(\tau', \xi) e^{it(\tau' - \xi^5)} - \frac{1}{i(\tau' - \xi^5)} d\tau' d\xi. \quad \text{(125)}$$

We denote by $w = w_1 + w_2$, where

$$\tilde{w}_1(\tau, \xi) = \chi_{\leq 1/T} (\tau - \xi^5) \bar{w}(\tau, \xi),$$

for a characteristic function $\chi$.

For $w_1$, we use the Taylor expansion of $e^x$ at $x = 0$. Then, we can rewrite (125) for $w_1$ as

$$\psi_T \partial_x^j Dw(t, x) = c \int e^{ix\xi} e^{it\xi^5} (i\xi)^j \psi_T(t) \int \tilde{w}_1(\tau', \xi) e^{it(\tau' - \xi^5)} - \frac{1}{i(\tau' - \xi^5)} d\tau' d\xi$$

$$= c T \sum_{k=1}^{\infty} \frac{i^{k-1}}{k!} \psi_T^k(t) \int e^{ix\xi} e^{it\xi^5} (i\xi)^j \hat{F}_1^k(\xi) d\xi$$

$$= c T \sum_{k=1}^{\infty} \frac{i^{k-1}}{k!} \psi_T^k(t) \partial_x^j e^{it\xi^5} F_1^k(x),$$

where $\psi^k(t) = t^k \psi(t)$ and

$$\hat{F}_1^k(\xi) = \int \tilde{w}_1(\tau, \xi) (T(\tau - \xi^5))^{k-1} d\tau.$$  

Since

$$\|F_1^k\|_{H^s} = \left( \int |\xi|^{2s} \left| \int \tilde{w}_1(\tau, \xi) (T(\tau - \xi^5))^{k-1} d\tau \right|^2 d\xi \right)^{1/2} \lesssim (1 + T^{-\frac{1}{2} - b}) \|w\|_{X^s, -b},$$

we have from Lemma 5.1 (b) that

$$\|\psi_T \partial_x^j Dw\|_{L^\infty_t H^s_x} \lesssim T \sum_{k=1}^{\infty} \frac{1}{k!} \|F_1^k\|_{H^s_x}$$

$$\lesssim T (1 + T^{-\frac{1}{2} - b}) \|w\|_{X^s, -b} \sum_{k=1}^{\infty} \frac{1}{k!}$$

$$\lesssim T^{\frac{1}{2} - b} \|w\|_{X^s, -b},$$
when \(-\frac{3}{2} + j \leq s \leq \frac{1}{2} + j\).

For \(w_2\), recall (125)

\[
\psi_T \partial_x^2 Dw(t, x) = \int e^{ix\xi} e^{i\xi^5} (i\xi)^j \psi_T(t) \int \frac{\widehat{w}(\tau, \xi)}{i(\tau - \xi^5)} \left(e^{i(\tau' - \xi^5)} - 1\right) d\tau' d\xi
= I - II.
\]

We first consider II. Let

\[
\widehat{W}(\xi) = \int \frac{\widehat{w}(\tau, \xi)}{i(\tau - \xi^5)} d\tau.
\]

Note that

\[
||W||_{H^s} \leq T^{\frac{1}{2} - b} ||w_2||_{X^{s-b}}.
\]

Then, it immediately follows from

\[
II = \psi_T(t) \partial_x^j e^{i\xi^5} F(x)
\]

and Lemma 5.1 (b) that

\[
||\psi_T \partial_x^j e^{i\xi^5} F||_{C_t H^{\frac{s+2}{2} - 1}} \lesssim ||F||_{H^s} \lesssim T^{\frac{1}{2} - b} ||w||_{X^{s-b}},
\]

when \(-\frac{3}{2} + j \leq s \leq \frac{1}{2} + j\).

Now it remains to deal with I. Taking the Fourier transform to \(I\) with respect to \(t\) variable, we have

\[
\int e^{ix\xi} (i\xi)^j \int \frac{\widehat{w}(\tau', \xi)}{i(\tau' - \xi^5)} \widehat{\psi_T}(\tau - \tau') d\tau' d\xi,
\]

and hence it suffices to control

\[
\left(\int \langle \tau \rangle^\frac{2(\frac{s+2}{2} - 1)}{5} \left|\int e^{ix\xi} (i\xi)^j \int \frac{\widehat{w}(\tau', \xi)}{i(\tau' - \xi^5)} \widehat{\psi_T}(\tau - \tau') d\tau' d\xi\right|^2 d\tau\right)^\frac{1}{2}.
\]

The argument is very similar used in the proof of Lemma 5.1 (b), while the relation among \(|\tau|, |\tau'|\) and \(|\xi|^5\) should be taken into account carefully. Hence, we only give, here, a short idea on each case. We first split the region in \(\tau\) as follows:

**Case I.** \(|\tau| \leq 1\), \hspace{1cm} **Case II.** \(1 < |\tau| \leq \frac{1}{T}\), \hspace{1cm} **Case III.** \(\frac{1}{T} < |\tau|\).

**Case I.** \(|\tau| \leq 1\). In this case, the weight \(\langle \tau \rangle^\frac{2(\frac{s+2}{2} - 1)}{5}\) and the integration with respect to \(\tau\) can be negligible. If \(|\xi|^5 \leq 1\), the weight \(|\xi|^j\) and the integration with respect to \(\xi\) is negligible as well. Moreover, we know \(|\widehat{w}(\tau - \tau')| \lesssim T^{1-k}|\tau' - \xi^5|^{-k}\) for any positive \(k\), since \(|\tau' - \xi^5| > 1/T\). Then, the Cauchy-Schwarz inequality gives

\[
(126) \lesssim T^{\frac{3}{2} - b} ||w_2||_{X^{s-b}}.
\]

When \(1 < |\xi|^5\), we further divide the case into \(1 < |\xi|^5 < 1/T\) and \(1/T < |\xi|^5\). For the former case, we still have \(|\widehat{w}(\tau - \tau')| \lesssim T^{1-k}|\tau' - \xi^5|^{-k}\). The Cauchy-Schwarz inequality gives

\[
(126) \lesssim T^{1-k} \left(\int_{|\xi|^5 < 1/T} |\xi|^{2j-2s} \int_{|\tau' - \xi^5| > 1/T} |\tau' - \xi^5|^{-2k-2+2b} d\xi d\tau\right)^\frac{1}{2} ||w_2||_{X^{s-b}}
\]

\[
\lesssim T^{\frac{3}{2} - b} \max(1, T^{-\frac{1}{2} + \frac{s+2}{10}}) ||w_2||_{X^{s-b}},
\]

which enables us to obtain \(T^\theta\) for positive \(\theta > 0\) when \(5(b-1) - 2 + j < s\) (roughly \(-\frac{1}{2} \leq \frac{s+2-j}{5}\)).
For the latter case, we split the region in $\tau'$ similarly as (117) (with corresponding variables $|\xi|^5$ and $|\tau' - \xi|^5$). Then, we can apply the same argument to each case. Indeed, for $|\tau' - \xi|^5 < \frac{1}{2}|\xi|^5$, we can control (126) by using $|\hat{\psi}_T(\tau - \tau')| \lesssim T^{1-k}|\xi|^{-5k}$, while we use $|\hat{\psi}_T(\tau - \tau')| \lesssim T^{1-k}|\tau' - \xi|^5$ in the case when $2|\xi|^5 < |\tau' - \xi|^5$. Hence, we have for both cases that

$$\text{(126)} \lesssim T^{1-b + \frac{s+2-j}{b}}\|w_2\|_{X^{s,b}}.$$

As seen in the proof of Lemma 5.1 (b), the case when $\frac{1}{2}|\xi|^5 < |\tau' - \xi|^5 < 2|\xi|^5$ is more complicated. Since $|\tau| \leq 1$, this case is equivalent to the case when $\frac{1}{2}|\tau - \xi|^5 < |\tau' - \xi|^5 < 2|\tau - \xi|^5$. If $(\tau' - \xi|^5) \cdot (\tau - \xi)| < 0$, by using the facts that $|\hat{\psi}_T(\tau - \tau')| \lesssim T^{1-k}|\xi|^{-5k}$ and

$$\int_{|\tau' - \xi|^5 < |\xi|^5} 1 \, d\tau' \lesssim |\xi|^5,$$

we obtain

$$\text{(126)} \lesssim T^{1-b + \frac{s+2-j}{b}}\|w_2\|_{X^{s,b}}.$$

Otherwise $(|\tau' - \xi|^5) \cdot (\tau - \xi)| > 0)$, we use the Hardy-Littlewood maximal function of $|\tau' - \xi|^5 - \frac{1}{b}w_2(\tau', \xi)$ for $|\tau - \tau'| < 1/T$, and the smoothness of $\psi$ to control (126) for $1/T < |\tau - \tau'| < |\tau - \xi|^5$, so that

$$\text{(126)} \lesssim T^{\frac{1}{2} - b + \frac{s+2-j}{b}}\|w_2\|_{X^{s,b}},$$

which imposes the regularity restriction $0 \leq \frac{s+2-j}{b}$. In order to cover $-\frac{1}{2} \leq \frac{s+2-j}{b} < 0$ regime, we use $Y^{s,b}$ space for the case when $\frac{1}{2}|\xi|^5 < |\tau' - \xi|^5 < 2|\xi|^5$. It suffice to consider

$$\left(\int_{|\tau| \leq 1} \int_{|\xi|^5 \leq 1/T} |\xi|^7 \int_{|\tau' - \xi|^5 > 1/T} \frac{\hat{w}_2(\tau', \xi)}{|\tau' - \xi|^5} \langle \xi \rangle^\frac{1}{2} \langle \tau' \rangle^\frac{7}{2} |\hat{\psi}_T(\tau - \tau')| \, d\tau' \, d\xi \right)^{\frac{1}{2}}. \quad \text{(127)}$$

We may assume $|\tau'| > 2$, otherwise, we use $\langle \tau' \rangle^\frac{7}{2} \sim 1$ and $|\hat{\psi}_T(\tau - \tau')| \lesssim T$ to obtain

$$\text{(127)} \lesssim T^{\frac{3}{2} - b - \frac{j}{2}}\|w_2\|_{Y^{s,b}}.$$ 

Since $\langle \tau' \rangle \sim (\tau - \tau')$, we have from (115) that

$$\text{(127)} \lesssim \left| \int_{|\xi|^5 \leq 1/T} |\xi|^{j-5+5b} \int_{|\tau'| > 2} \langle \tau' \rangle^\frac{7}{2} \langle -b \rangle \langle \tau'\rangle \langle \tau' - \xi \rangle^\frac{-5}{2} \langle \hat{\psi}_T(\tau - \tau') \rangle \, d\tau' \, d\xi \right| \lesssim \left| \int_{|\tau'| \geq 1} |\tau|^{-\frac{5}{2}} |\hat{\psi}_T(\tau)|^2 \, d\tau \right|^{\frac{1}{2}} \|w_2\|_{Y^{s,b}} \lesssim T^{\frac{3}{2} - b + \frac{5}{2}j} T^{\frac{3}{2} + \frac{j}{2}}\|w_2\|_{Y^{s,b}}.$$

Thus we cover $-\frac{1}{2} \leq \frac{s+2-j}{b} < 0$.

**Case II.** $1 < |\tau| \leq \frac{1}{T}$. For $|\xi|^5 < 1/T$, we can apply the same argument in **Case I**, since, roughly speaking, the space bound $T^{\frac{3}{2} + \frac{5}{2}j}$ obtained in **Case I** controls

$$\left(\int_{|\tau| < 1/T} |\tau|^{\frac{2(s+2-j)}{b}} \, d\tau \right)^{\frac{1}{2}}.$$
Moreover, the case when $1/T < |\xi|^5$, and $|\tau - \xi| < \frac{1}{2} |\xi|^5$ or $2|\xi|^5 < |\tau - \xi|$, can be dealt with similarly. Hence, we have for these cases that

$$\tag{126} \|w_2\|_{X_{s,-b}} \lesssim T^{\frac{1}{2} - b}.$$

For the rest case, in view of the proof, we can see that $L^2$ integral with respect to $\tau$ is performed for $w_2$. Moreover, the weight $|\tau|^{\frac{s+2-j}{s}} \lesssim T^{-\frac{s+2-j}{s}}$ (when $0 \leq \frac{s+2-j}{s}$), which is killed by the spare bound $T^{\frac{s+2-j}{s}}$. Hence we have for the rest case that

$$\tag{126} \|w_2\|_{X_{s,-b}} \lesssim T^{\frac{1}{2} - b}.$$

Otherwise (when $-\frac{1}{2} \leq \frac{s+2-j}{s} < 0$), we have similarly as before that

$$\tag{127} \|w_2\|_{Y_{s,-b}} \lesssim T^{1-b+\frac{s+2-j}{s}}.$$

**Case III.** $1/T < |\tau|$. This case is much more complicated. When $|\xi|^5 < 1/T$, $\xi^5$ is negligible compared with $\tau$ and $\tau'$, and hence (126) is reduced to

$$\max \left\{ 1, T^{\frac{s+2-j}{s}-\frac{1}{2}} \right\} \left( \left\| \langle \tau \rangle^{\frac{2(s+2-j)}{s}} \left| \int \frac{\widetilde{w}_2(\tau')}{|\tau'|} \psi_T(\tau - \tau') \, d\tau' \right|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim T^{1-b+\frac{s+2-j}{s}} \|w_2\|_{X_{s,-b}},$$

where $\widetilde{w}_2(\tau') = \|\cdot\|_2 w_2(\cdot, \tau')$. Then, the following cases can be treated via the similar way:

- **III.a** $|\tau'| < \frac{1}{2} |\tau|$, in this case we use $|\widetilde{\psi}_T(\tau - \tau')| \lesssim T^{1-k}|\tau|^{-k}$,
- **III.b** $2|\tau| < |\tau'|$, in this case we use $|\widetilde{\psi}_T(\tau - \tau')| \lesssim T^{1-k}|\tau'|^{-k}$,
- **III.c** $\frac{1}{2} |\tau| < |\tau'| < 2|\tau|$ and $\tau \cdot \tau' < 0$, in this case we use $|\psi_T(\tau - \tau')| \lesssim T^{1-k}|\tau|^{-k}$.

Indeed, we roughly have

$$\left( \int \langle \tau \rangle^{\frac{2(s+2-j)}{s}} \left| \int \frac{\widetilde{w}_2(\tau')}{|\tau'|} \psi_T(\tau - \tau') \, d\tau' \right|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim T^{1-b+\frac{s+2-j}{s}} \|w_2\|_{X_{s,-b}},$$

which, together with (128), implies

$$\tag{126} \|w_2\|_{X_{s,-b}} \lesssim T^{\frac{1}{2} - b}.$$

On the other hand, when $(s + 2 - j)/5 > 1/2$ ($s > 0$), we have from (128) that

$$\left( \int \langle \tau \rangle^{\frac{2(s+2-j)}{s}} \left| \int \frac{\widetilde{w}_2(\tau')}{|\tau'|} \psi_T(\tau - \tau') \, d\tau' \right|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim T^{1-b+\frac{s+2-j}{s}} \|w_2\|_{Y_{s,-b}},$$

which does not guarantee (129). However, it is possible to obtain

$$\tag{126} \|w_2\|_{Y_{s,-b}} \lesssim T^b \|w_2\|_{Y_{s,-b}},$$

for positive $\theta > 0$. Precisely, for III.b and III.c, we have

$$\tag{126} \lesssim \left( \int_{1/T < |\tau|} \langle \tau \rangle^{\frac{2(s+2-j)}{s}} \left| \int_{|\xi|^5 \leq 1/T} |\xi|^j \int \frac{\widetilde{w}_2(\tau', \xi)}{i(\tau' - \xi^5)} \psi_T(\tau - \tau') \, d\tau' \, d\xi \right|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim T^{\frac{2s-j}{5}} \left( \int \langle \tau \rangle^{\frac{2(s+2-j)}{s}} \left| \int \frac{\langle \tau \rangle^{j/2} \widetilde{w}_2(\tau')}{|\tau'|} \psi_T(\tau - \tau') \, d\tau' \right|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim T^{\frac{1}{2} - b} \|w_2\|_{Y_{s,-b}}.$$

\footnote{When $\frac{2s-j}{5} = \frac{1}{2}$, the constant depending on $T$ is $-\log T$ instead of $T^{\frac{s+2-j}{s} - \frac{1}{2}}$, but it does not influence on our analysis.}
For III.a, it follows from
\[
\left( \int |(\tau)^{-\frac{2s+2-j}{2}} \left| \frac{w_2 (\tau') \tilde{\psi}_T (\tau - \tau')}{|\tau'|} \right|^2 d\tau' \right)^{\frac{1}{2}} \lesssim T^{1-b-\frac{2s-j}{b}} \| w_2 \|_{Y^{s,-b}} ,
\]
thanks to
\[
\int_{|\tau'| < |\tau|} |\tau'|^{-2-\frac{b}{2}+2b} d\tau' \lesssim T^{1+b-2b} .
\]

For the case when \( \tau \cdot \tau' > 0 \), we similarly split the case into \( |\tau - \tau'| < 1/T \) and \( |\tau - \tau'| > 1/T \). Then, by using the Hardy-Littlewood maximal function of \( |\tau'|^{-b} \tilde{w}_2 (\tau') \) for \( |\tau - \tau'| < 1/T \), and the smoothness of \( \psi (\tilde{\psi}_T (\tau - \tau')) \lesssim |\tau - \tau'|^{-1} \) for \( 1/T < |\tau - \tau'| < |\tau| \) similarly as before, we have for the rest case that
\[
(126) \lesssim T^{\frac{1}{2}-b} \| w_2 \|_{Y^{s,-b}} .
\]

By the same reason, we have
\[
(126) \lesssim T^{\frac{1}{2}-b} \| w_2 \|_{Y^{s,-b}} ,
\]
when \( (s+2-j)/5 > 1/2 \).

Now we consider the case when \( |\xi|^5 > 1/T \). For given \( \tau, \xi \), we further divide the case into \( |\tau' - \xi|^5 \leq \frac{1}{2} |\tau - \xi|^5 \), \( 2 |\tau - \xi|^5 \leq |\tau' - \xi|^5 \) and \( \frac{1}{2} |\tau - \xi|^5 < |\tau' - \xi|^5 \) and \( |\tau' - \xi|^5 > 2 |\tau - \xi|^5 \).

For the case when \( |\tau' - \xi|^5 \leq \frac{1}{2} |\tau - \xi|^5 \), we know \( |\tau - \xi|^5 > 1/T \) and \( \tilde{\psi}_T (\tau - \tau') \lesssim T^{1-k} |\tau - \xi|^5^{-k} \). Moreover, the region of \( \xi \) can be expressed as \( \cup_{j=1}^4 A_j \), where
\[
A_1 = \left\{ \xi : |\xi|^5 > \frac{1}{T}, 2 |\tau| < |\xi|^5 \right\} ,
\]
\[
A_2 = \left\{ \xi : |\xi|^5 > \frac{1}{T}, |\xi|^5 < \frac{1}{2} |\tau| \right\} ,
\]
\[
A_3 = \left\{ \xi : |\xi|^5 > \frac{1}{T}, \frac{1}{2} |\tau| \leq |\xi|^5 \leq 2 |\tau|, \tau \cdot \xi^5 < 0 \right\}
\]
and
\[
A_4 = \left\{ \xi : |\xi|^5 > \frac{1}{T}, \frac{1}{2} |\tau| \leq |\xi|^5 \leq 2 |\tau|, \tau \cdot \xi^5 > 0 \right\} .
\]

On \( A_1 \), we have \( |\tau|^{\frac{s+2-j}{s-j}} \lesssim |\xi|^{s+2-j} \), \( |\tau - \xi|^5 \sim |\xi|^5 \) and \( \tilde{\psi}_T (\tau - \tau') \lesssim T^{1-k} |\xi|^{-5k} \) for \( k > 1 \). Let
\[
\tilde{W}^*(\tau', \xi) = \langle \tau' - \xi|^5 - b(\xi) \rangle w_2 (\tau', \xi) ,
\]
\[\text{This property restricts the regularity condition as } \frac{s+2-j}{s-j} > 0 \text{. However, in the case when } \frac{s+2-j}{s-s} > 0 \text{, since } |\tau|^{\frac{s+2-j}{s-j}} \lesssim T^{1-\frac{s+2-j}{s-j}} , \text{ the same argument yields}
(126) \lesssim T^{\frac{1}{2}-b} \| w_2 \|_{X^{s,-b}} .\]
Note that $||W^*||_{L^2_{T,x}} = ||w_2||_{X^{s,-b}}$. Then, we have

\[
(126) \lesssim T^{1-k} \left( \int_{|\tau| > 1/T} \left| \int_{|\tau'| < |\tau - \xi|^5} |\tau' - \xi|^5 |^{1+b} \overline{W}^*(\tau', \xi) \, d\tau' \right| \right)^{1/2} \\
\lesssim T^{1-k}T^{1-b} \left( \int_{|\tau| > 1/T} |\tau|^{1-2k} \, d\tau \right)^{1/2} ||W^*||_{L^2} \\
\lesssim T^{1-b} ||w_2||_{X^{s,-b}}.
\]

On $A_3$, we have $|\tau - \xi|^5 \sim |\tau|$ and $|\hat{\psi}_T(\tau - \tau')| \lesssim T^{1-k}|\tau|^{-k}$ for $k > 1$. Then, similarly as (130), we have

\[
(126) \lesssim T^{1-b} ||w_2||_{X^{s,-b}}.
\]

On $A_3$, since $|\tau - \xi|^5 \sim |\tau| \sim |\xi|^5$, we have

\[
(126) \lesssim T^{1-b} ||w_2||_{X^{s,-b}},
\]

similarly as on $A_1$ or $A_2$. On $A_4$, we have $|\tau|^{-\frac{5}{2}} \sim |\xi|^{s+2-j}$ and $|\hat{\psi}_T(\tau - \tau')| \lesssim T^{1-k}|\tau - \xi|^5 |\tau - \xi|^5$ for $k > 1$. Moreover, it is enough to consider the region $\tau + \frac{\xi}{T} < \xi^5 < 2\tau$ due to the footnote 15 in the proof of Lemma 5.1 (b). Then, we have

\[
(126) \lesssim T^{1-b} \left( \int_{|\tau| > 1/T} \left| \int_{|\tau'| < |\tau - \xi|^5} |\tau' - \xi|^5 |^{1+b} \overline{W}^*(\tau', \xi) \, d\tau' \right| \right)^{1/2} \\
\lesssim T^{1-k}T^{1-b} \left( \int_{|\tau| > 1/T} \left( \int_{|\tau'| < |\tau - \xi|^5} |\tau' - \xi|^5 |^{1+b} \overline{W}^*(\tau', \xi) \, d\tau' \right)^{1/2} \right)^{1/2} \\
\lesssim T^{1-k}T^{1-b} \left( \int_{|\tau| > 1/T} \left( \int_{|\tau'| < |\tau - \xi|^5} |\tau' - \xi|^5 |^{1+b} \overline{W}^*(\tau', \xi) \, d\tau' \right)^{1/2} \right)^{1/2} \\
\lesssim T^{1-k}T^{1-b} \left( \int_{|\tau| > 1/T} \left( \int_{|\tau'| < |\tau - \xi|^5} |\tau' - \xi|^5 |^{1+b} \overline{W}^*(\tau', \xi) \, d\tau' \right)^{1/2} \right)^{1/2} \\
\lesssim T^{1-b} ||w_2||_{X^{s,-b}},
\]

for small $0 < \epsilon \ll 1$, which implies

\[
(126) \lesssim T^{1-b} ||w_2||_{X^{s,-b}}.
\]

For the case when $2|\tau - \xi|^5 \leq |\tau' - \xi|^5$, the region of $\xi$ can be further divided by

\[
B_1 = \left\{ \xi : |\xi|^5 > \frac{1}{T}, |\tau - \xi|^5 < \frac{1}{T} \right\}
\]

and

\[
B_2 = \left\{ \xi : |\xi|^5 > \frac{1}{T}, |\tau - \xi|^5 \geq \frac{1}{T} \right\}.
\]
On $B_1$, we know $|\tau|^{2 + \frac{4}{b}} \sim |\xi|^{s + 2 - j}$. Since $|\hat{\psi}_T(\tau - \tau')| \lesssim T$ and
\[
\int_{|\tau'| < 1/|\tau, \xi|} |\tau' - \xi^5|^{-1 + b} \hat{W}^*(\tau', \xi) \, d\tau' \lesssim T^{1 - 2k} \hat{W}^*(\xi),
\]
we have from the change of variable ($\eta = \xi^5$) that
\[
(126) \lesssim T^{\frac{1}{2} - b} \left( \int_{|\tau| > 1/T} \left| \int_{|\eta - \tau| < 1/T} \hat{W}^*(\eta^\frac{1}{5}) \eta^{-\frac{2}{5}} \, d\eta \right|^2 \, d\tau \right)^{\frac{1}{2}}
\]
\[
\lesssim T^{\frac{1}{2} - b} \left( \int_{|\tau| > 1/T} |M \hat{W}^*(\tau)|^2 \, d\tau \right)^{\frac{1}{2}},
\]
where $\tilde{W}^*(\eta) = \tilde{W}^*(\eta^\frac{1}{5}) \eta^{-\frac{2}{5}}$. Note that $\|\tilde{W}^*\|_{L^2} = c \|w_2\|_{X^{s, -b}}$. Therefore, we have
\[
(126) \lesssim T^{\frac{1}{2} - b} \|w_2\|_{X^{s, -b}}.
\]
On $B_2$, by dividing the region of $\xi$ as $A_j$, $j = 1, 2, 3, 4$, we have similarly
\[
(126) \lesssim T^{\frac{1}{2} - b} \|w_2\|_{X^{s, -b}}.
\]
For the rest case ($\frac{1}{2} |\tau - \xi^5| < |\tau' - \xi^5| < 2|\tau - \xi^5|$), we further divide the region of $\tau'$ as $C_1 \cup C_2$, where
\[
C_1 = \left\{ \tau' : |\tau'| > 1/T, \frac{1}{2} |\tau - \xi^5| < |\tau' - \xi^5| < 2|\tau - \xi^5|, (\tau' - \xi^5) \cdot (\tau - \xi^5) < 0 \right\}
\]
and
\[
C_2 = \left\{ \tau' : |\tau'| > 1/T, \frac{1}{2} |\tau - \xi^5| < |\tau' - \xi^5| < 2|\tau - \xi^5|, (\tau' - \xi^5) \cdot (\tau - \xi^5) > 0 \right\}.
\]
On $C_1$, since
\[
|\hat{\psi}_T(\tau - \tau')| \lesssim T^{1 - k} |\tau - \xi^5|^{-k} \sim T^{1 - k} |\tau - \xi^5|^{-k},
\]
for $k \geq 0$, by dividing the region of $\xi$ as $A_j$, $j = 1, 2, 3, 4$, we have similarly
\[
(126) \lesssim T^{\frac{1}{2} - b} \|w_2\|_{X^{s, -b}}.
\]
On the other hand, we further split the set $C_2$ by
\[
C_{21} = \left\{ \tau' : |\tau'| > 1/T, \frac{1}{2} |\tau - \xi^5| < |\tau' - \xi^5| < 2|\tau - \xi^5|,
(\tau' - \xi^5) \cdot (\tau - \xi^5) > 0, \left\| \tau - \tau' < \frac{1}{T} \right\}
\]
and
\[
C_{22} = \left\{ \tau' : |\tau'| > 1/T, \frac{1}{2} |\tau - \xi^5| < |\tau' - \xi^5| < 2|\tau - \xi^5|,
(\tau' - \xi^5) \cdot (\tau - \xi^5) > 0, \left\| \tau - \tau' > \frac{1}{T} \right\}
\]
On $C_{21}$, $(126)$ is reduced by
\[
\left( \int_{|\tau'| > 1/T} |\tau|^{8 \times (s + 2 - j)} \int_{|\xi'| > 1/T} |\xi|^{j \cdot s} |\tau - \xi^5|^{-1 + b} M \hat{W}^*(\tau, \xi) d\xi \right)^{\frac{1}{2}}.
\]
Then, by dividing the region of $\xi$ in $(132)$ as $A_j$, $j = 1, 2, 3, 4$, we have similarly
\[
(126) \lesssim T^{\frac{1}{2} - b} \|w_2\|_{X^{s, -b}}.
On \( C_{22} \), we know \(|\hat{\psi}_T(\tau - \tau')| \lesssim T^{1-k}|\tau - \tau'|^{-k} \) for \( k \geq 0 \). Then, (126) is reduced by

\[
T^{1-k+\epsilon} \left( \int_{|\tau| > 1/T} |\tau|^{2(\alpha + 2 - 1)/\beta} \right) \\
\times \left| \int_{|\xi| > 1/T} |\xi|^a |\tau - \xi^5|^{-1+b} \left( \int_{|\tau|} ^{T-\xi^5} |h|^{-2k+1+2b}|\hat{W}_\tau^\ast(\tau + h, \xi)|^2 \, dh \right)^{\frac{1}{2}} \, d\xi \right|^2 \, d\tau^{\frac{1}{2}},
\]

(133)

for small \( 0 < \epsilon \ll 1 \). Then, for \( k \gg 1 \) large enough, by dividing the region of \( \xi \) in (133) as \( A_j, j = 1, 2, 3, 4 \), we have similarly

\[
(126) \lesssim T^{1-k-b} \|w_2\|_{X^{s,-b}}.
\]

Therefore, we have for \( -\frac{a}{2} + j \leq s \leq \frac{1}{2} + j \) that

\[
\|\psi_T(t) \partial_{\xi}^j T^{1-b} D w(x,t)\|_{C(\mathbb{R} x; H^{\frac{1+b}{2} - 1}(\mathbb{R}))} \lesssim T^\theta (\|w\|_{X^{s,-b}} + \|w\|_{X^{Y,-b}}),
\]

for some \( \theta = \theta(s, b) > 0 \).

(e). Since

\[
\mathcal{F} [\psi_T D w](\tau, \xi) = \int \hat{w}(\tau', \xi) \frac{\hat{\psi}_T(\tau - \tau') - \hat{\psi}_T(\tau - \xi^5)}{i(\tau' - \xi^5)} \, d\tau',
\]

it suffices to show

\[
\left( \int_{|\xi| \leq 1} \langle \xi \rangle^{2a} \left| \int \hat{w}(\tau', \xi) \frac{\hat{\psi}_T(\tau - \tau') - \hat{\psi}_T(\tau - \xi^5)}{i(\tau' - \xi^5)} \, d\tau' \right|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim T^\theta \|w\|_{X^{s,-b}}
\]

(134)

and

\[
\left( \int_{|\xi| > 1} |\xi|^{2s} \langle \tau - \xi^5 \rangle^{2b} \left| \int \hat{w}(\tau', \xi) \frac{\hat{\psi}_T(\tau - \tau') - \hat{\psi}_T(\tau - \xi^5)}{i(\tau' - \xi^5)} \, d\tau' \right|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim T^\theta \|w\|_{X^{s,-b}},
\]

(135)

for some \( \theta = \theta(\alpha, b) > 0 \). For \( T|\tau' - \xi^5| \leq 1 \), we use the mean value theorem in order to deal with

\[
\frac{|\hat{\psi}_T(\tau - \tau') - \hat{\psi}_T(\tau - \xi^5)|}{|\tau' - \xi^5|} \lesssim T^2 |\hat{\psi}(T(\tau - \xi^5) + \delta)|,
\]

for some \( |\delta| \leq 1 \), in the left-hand side of (134) and (135). Then, since

\[
\left( \int \langle \tau \rangle^{2\sigma} T^4 |\hat{\psi}'(T\tau)|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim T^{\frac{4}{2} - \sigma} \|\langle \tau \rangle^\sigma \hat{\psi}'(\tau)\|_{L^2}
\]

and

\[
\left( \int |\xi|^{2s} \left| \int_{|\tau' - \xi^5| \leq 1/T} \hat{w}(\tau', \xi) \, d\tau' \right|^2 \, d\xi \right)^{\frac{1}{2}} \lesssim T^{-\frac{1}{2} - b} \|w\|_{X^{s,-b}},
\]

we have

\[
\left( \int_{|\xi| \leq 1} \langle \tau \rangle^{2a} \left| \int_{|\tau' - \xi^5| \leq 1/T} \hat{w}(\tau', \xi) \frac{\hat{\psi}_T(\tau - \tau') - \hat{\psi}_T(\tau - \xi^5)}{i(\tau' - \xi^5)} \, d\tau' \right|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim T^{1-a-b} \|w\|_{X^{s,-b}}
\]

and

\[
\left( \int_{|\xi| > 1} |\xi|^{2s} \langle \tau - \xi^5 \rangle^{2b} \left| \int_{|\tau' - \xi^5| \leq 1/T} \hat{w}(\tau', \xi) \frac{\hat{\psi}_T(\tau - \tau') - \hat{\psi}_T(\tau - \xi^5)}{i(\tau' - \xi^5)} \, d\tau' \right|^2 \, d\tau \right)^{\frac{1}{2}} \lesssim T^{1-a-b} \|w\|_{X^{s,-b}}.
\]
and

\[
\left( \int_{|\xi|>1} |\xi|^{2s} \left( \int_{|\tau-\xi^{5}|\leq 1/T} \int_{|\tau' - \xi|\leq 1/T} \tilde{w}(\tau', \xi) \frac{\tilde{\psi}_T(\tau - \tau') - \tilde{\psi}_T(\tau - \xi^{5})}{i(\tau' - \xi^{5})} \right)^{2} d\tau' d\tau \xi^{2} \right)^{\frac{1}{2}} \lesssim T^{1-2b} \|w\|_{X^{s,-b}}.
\]

Otherwise \( (T|\tau' - \xi^{5}| > 1) \), since

\[
\left( \int \langle \tau \rangle^{2\sigma} |\tilde{\psi}_T(\tau)|^{2} d\tau \right)^{\frac{1}{2}} \lesssim T^{\frac{1}{2} - \sigma} \|\psi\|_{H^{s}}
\]

and

\[
\left( \int_{|\xi|>1} |\xi|^{2s} \left( \int_{|\tau-\xi^{5}|>1/T} \int_{|\tau' - \xi|>1/T} \tilde{w}(\tau', \xi) \frac{\tilde{\psi}_T(\tau - \tau') - \tilde{\psi}_T(\tau - \xi^{5})}{i(\tau' - \xi^{5})} \right)^{2} d\tau' d\tau \xi^{2} \right)^{\frac{1}{2}} \lesssim T^{\frac{1}{2} - b} \|w\|_{X^{s,-b}},
\]

it suffices to show

\[
\left( \int_{|\xi|\leq 1} \langle \tau \rangle^{2\alpha} \left( \int_{|\tau-\xi^{5}|>1/T} \int_{|\tau' - \xi|>1/T} \tilde{w}(\tau', \xi) \frac{\tilde{\psi}_T(\tau - \tau')}{i(\tau' - \xi^{5})} \right)^{2} d\tau' d\tau \xi^{2} \right)^{\frac{1}{2}} \lesssim T^{\theta} \|w\|_{X^{s,-b}},
\]

(136)

and

\[
\left( \int_{|\xi|>1} |\xi|^{2s} \left( \int_{|\tau-\xi^{5}|>1/T} \int_{|\tau' - \xi|>1/T} \tilde{w}(\tau', \xi) \frac{\tilde{\psi}_T(\tau - \tau')}{} d\tau' d\tau \xi^{2} \right)^{\frac{1}{2}} \lesssim T^{\theta} \|w\|_{X^{s,-b}},
\]

(137)

for some \( \theta > 0 \). It follows the similar way used in the proof of (b). In fact, the proofs of (136) and (137) are much simpler and easier than the proof of (b), since \( L^{2} \) integral with respect to \( \xi \) is negligible and hence it is enough to consider the relation between \( \tau - \xi^{5} \) and \( \tau' - \xi^{5} \). Thus, we omit the details and we have

\[
\|\tilde{\psi}_T D^{a}w\|_{D^{a}} \lesssim T^{1-a-b} \|w\|_{X^{s,-b}}
\]

and

\[
\|\tilde{\psi}_T D^{a}w\|_{X^{s,b}} \lesssim T^{1-2b} \|w\|_{X^{s,-b}}.
\]

Lemma 5.3. Let \(-\frac{5}{2} < s < \frac{1}{2}\).

(a) (Space traces) For \( \max(s - \frac{9}{2}, -4) < \lambda < \min(s + \frac{1}{2}, \frac{1}{2}) \), we have

\[
\|\tilde{\psi}_T(t) L^{\lambda}_{x} f(t, x)\|_{C(\mathbb{R}; H^{s}(\mathbb{R}))} \leq c \|f\|_{H^{\frac{\alpha+2}{2}}_{0}(\mathbb{R}^{+})};
\]

(b) (Derivatives) Time traces) For \(-4 + j < \lambda < 1 + j, j = 0, 1, 2 \), we have

\[
\|\tilde{\psi}_T(t) D^{j}_{x} L^{\lambda}_{x} f(t, x)\|_{C(\mathbb{R}; H^{\frac{s+2-\lambda}{2}}_{0}(\mathbb{R}))} \leq c \|f\|_{H^{\frac{s+2}{2}}_{0}(\mathbb{R}^{+})};
\]

(c) (Bourgain spaces) For \( b < \frac{1}{2} < \alpha < 1 - b \) and \( \max(s - 2, -\frac{13}{2}) < \lambda < \min(s + \frac{1}{2}, \frac{1}{2}) \), we have

\[
\|\tilde{\psi}_T(t) L^{\lambda}_{x} f(t, x)\|_{X^{s,b} \cap D^{a}} \leq c \|f\|_{H^{\frac{s+2}{2}}_{0}(\mathbb{R})};
\]

\[\text{18}\text{The restriction of regularity makes the range of } \lambda \text{ below non-empty.}\]
Proof. From the fact \( \psi_T(t) = \psi_2(t) \psi_T(t) \) for \( 0 < T \leq 1 \) and the definition of \( \mathcal{L}_T^1 \), it suffices to consider
\[
\psi_2(t) \mathcal{L}_T^1 (\psi_T f(t, x))
\]
instead of
\[
\psi_T(t) \mathcal{L}_T^1 f(t, x).
\]
Then, by Lemma 4.3 in [6] and Remark 8 (a variant of Lemma 2.9), we have Lemma 5.3. We omit the details. \( \square \)

6. Proof of Theorems 1.3 – 1.6. The proofs of Theorems 1.3 – 1.6 are based on the argument in [6], while the scaling argument does not hold here as mentioned in Section 1. Hence, we only provide a sketch of the proof of Theorem 1.3.

We fix \(-\frac{3}{2} < s < \frac{1}{2}\). Recall from [6]
\[
a_j = \frac{1}{5B(0)\Gamma \left(\frac{1}{2}\right)} \frac{\cos \left(\frac{(1+4\lambda_j)\pi}{10}\right)}{\sin \left(\frac{(1-\lambda_j)\pi}{2}\right)} \quad \text{and} \quad b_j = \frac{1}{5B(0)\Gamma \left(\frac{1}{2}\right)} \frac{\cos \left(\frac{(4\lambda_j-3)\pi}{10}\right)}{\sin \left(\frac{(2-\lambda_j)\pi}{2}\right)}
\]
and define a matrix
\[
A(\lambda_1, \lambda_2) = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}.
\]
We note that when \(-\frac{3}{2} < s < \frac{1}{2}\), the parameters \( \lambda_1 \) and \( \lambda_2 \) satisfying
\[
\max(s-2, -3) < \lambda_j < \min \left(\frac{1}{2}, s + \frac{1}{2}\right), \quad j = 1, 2,
\]
and
\[
\lambda_1 - \lambda_2 \neq 5n, \quad n \in \mathbb{Z},
\]
facilitate that Lemma 5.3 holds and \( A \) is invertible.

We fix \( \lambda_j, \quad j = 1, 2 \), satisfying (138) and (139). We bring the solution operator on \([0, T]\) from [6] as follows:
\[
A u(t, x) = \psi_T(t) \mathcal{L}_T^1 \gamma_1(t, x) + \psi_T(t) \mathcal{L}_T^1 \gamma_2(t, x) + \psi_T(t) F(t, x),
\]
(140)
where
\[
\begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix} = A^{-1} \begin{bmatrix} f(t) - F(t, 0) \\ \mathcal{L}_2 g(t) - \mathcal{L}_2 \partial_x F(t, 0) \end{bmatrix},
\]
and \( F(t, x) = e^{it \partial_x^2} u_0 - (1 - \partial_x^2) \frac{1}{\pi} \partial_x (u_0^2)(t, x) \).

For given initial and boundary data \( u_0, f \) and \( g \), we fix \( 0 < T < 1 \) such that
\[
4C^2 T^{\frac{3}{2} - 4\alpha - b} \left( \| u_0 \|_{H^s(R^+)} + \| f \|_{H^{\frac{4\alpha}{5}+2}(R^+)} + \| g \|_{H^{\frac{4\alpha}{5}+1}(R^+)} \right) < \frac{1}{2},
\]
(141)
where \( C \) is the maximum constant among other implicit constants appeared in all estimates in Sections 5 and 3. Note that \( \frac{1}{2} < \frac{3}{4} - \frac{6}{5} \) holds for \( b < \frac{3}{4} \), and hence it is possible to choose a small \( T > 0 \) satisfying (141), since \( \frac{3}{4} - 2\alpha - b > 0 \) when \( b < \frac{3}{4} < \alpha < \frac{3}{4} - \frac{2}{5} \).

Recall the \( Z_{1,\alpha}^{s,\alpha} \)-norm defined in (22). All estimates obtained in Sections 5 and 3 yield
\[
\| A u \|_{Z_{1,\alpha}^{s,\alpha}} \leq C T^{\frac{3}{2} - \alpha} \left( \| u_0 \|_{H^s(R^+)} + \| f \|_{H^{\frac{4\alpha}{5}+2}(R^+)} + \| g \|_{H^{\frac{4\alpha}{5}+1}(R^+)} \right) + C T^{1-\alpha-b} \| u \|_{Z_{1,\alpha}^{s,\alpha}}^2.
\]
Similarly,
\[
\| A u_1 - A u_2 \|_{Z_{1,\alpha}^{s,\alpha}} \leq C T^{1-\alpha-b} (\| u_1 \|_{Z_{1,\alpha}^{s,\alpha}} + \| u_2 \|_{Z_{1,\alpha}^{s,\alpha}}) \| u_1 - u_2 \|_{Z_{1,\alpha}^{s,\alpha}}.
\]
for \( u_1(0,x) = u_2(0,x) \). These immediately imply that \( \Lambda \) is a contraction map on
\[
\{ u \in Z^s_{1,a,b} : \|u\|_{Z^s_{1,a,b}} < 2CT^{\frac{1}{2} - \alpha}(\|u_0\|_{H^s(\mathbb{R}^+)} + \|f\|_{H^{s+\frac{1}{2}}(\mathbb{R}^+)} + \|g\|_{H^{s+1}(\mathbb{R}^+)}) \},
\]
and it completes the proof.

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