The hole probability for Gaussian random SU(2) polynomials.
by Scott Zrebiec

Abstract

We show that for Gaussian random SU(2) polynomials of a large degree \( N \) the probability that there are no zeros in the disk of radius \( r \) is less than \( e^{-c_1 r N^2} \), and is also greater than \( e^{-c_2 r N^2} \). Enroute to this result, we also derive a more general result: probability estimates for the event that the number of complex zeros of a random polynomial of high degree deviates significantly from its mean.

1. Introduction and Notation.

In this paper we compute the hole probability of one of the key models for random holomorphic functions: random SU(2) polynomials. Various properties of the zeros of random SU(2) polynomials have been studied, in particular the zero point correlation functions have been computed. This is of particular interest in the physics literature as the zeros describe a random spin state for the Majorana representation (modulo phase), [4]. Further this choice is intuitively pleasing as the zeros are uniformly distributed on \( \mathbb{C}P^1 \) (according to the Fubini-Study metric), or alternatively the average distribution of zeros is invariant under the SU(2) action on \( \mathbb{C}P^1 \).

A hole refers to the event where a particular Gaussian random holomorphic function has no zeros in a given domain where many are expected. The order of the decay of the Hole probability has been computed in several cases including for “flat” complex Gaussian random holomorphic functions on \( \mathbb{C}^1 \), [10], using a method which shall be used here. This work was subsequently refined to cover other large deviations in the distribution of the zeros sets, [5], and generalized to \( \mathbb{C}^n \), [11]. Other results compute the hole probability for a class of complex Gaussian holomorphic functions on the unit disk, [7], and provide a weak general estimate for any one variable complex Gaussian random holomorphic functions, [8]. Additionally significant hole probability results have been discovered for real Gaussian random polynomials, (2, 6).

In this work we will consider the class of random polynomials whose zeros are distributed on \( \mathbb{C}P^1 \) according to the Fubini-Study measure. The random functions which will be studied here are called Gaussian random
SU(2) Polynomials and can be written as:

\[ \psi_{\alpha,N}(z) = \sum_{j=0}^{N} \alpha_j \sqrt{\binom{N}{j}} z^j \]  

(1)

where \( \forall j, \alpha_j \), are independent identically distributed standard complex Gaussian random variables (mean 0 and variance 1). For these Gaussian random SU(2) polynomials we will be computing the hole probability in a manner based on that used by Sodin and Tsirelson to solve the similar problem for Flat random holomorphic functions on \( \mathbb{C}^1 \), [10]. First, we shall estimate the decay rate of the probability for a more general event:

**Theorem 1.1.** Let \( \psi_{\alpha,N} \) be a degree \( N \) Gaussian random SU(2) polynomial,

\[ \psi_{\alpha,N}(z) = \sum_{j} \alpha_j \sqrt{\binom{N}{j}} z^j, \]

where \( \alpha_j \) are independent identically distributed complex Gaussian random variables, and let \( \Xi_{\alpha,r,N} \) be the number of zeros of \( \psi_{\alpha,N} \) that are in the disk of radius \( r \).

For all \( \Delta > 0 \), and \( r > 0 \) there exists \( A_{\Delta,r} \) such that

\[ \text{Prob}\left( \left\{ \left| \Xi_{\alpha,r,N} - \frac{Nr^2}{1+r^2} \right| \geq \Delta N \right\} \right) \leq e^{-A_{\Delta,r}N^2}. \]

On average, \( \psi_{\alpha,N} \) should have \( \frac{Nr^2}{1+r^2} \) zeros in the disk of radius \( r \), as is evident in the above theorem, or by other more elementary means. Theorem 1.1 gives an upper bound of the rate of decay of the hole probability, and we will be able to prove a lower bound for the decay rate of the same order:

**Theorem 1.2.** Let \( \text{Hole}_{N,r} = \{ (\alpha_j) \in \mathbb{C}^N : \forall z \in B(0,r), \psi_{\alpha,N}(z) \neq 0 \} \), then there exists \( c_{1,r} \), and \( c_{2,r} > 0 \) such that

\[ e^{-c_{2,r}N^2} \leq \text{Prob}(\text{Hole}_{N,r}) \leq e^{-c_{1,r}N^2}. \]

The techniques of this paper will generalize to higher dimensions, as in [11].

Random polynomials of the form studied here are the simplest examples of a class of natural random holomorphic sections of large \( N \) powers of a positive line bundle on a compact Kähler manifold. Most of the results stated in this paper may be restated in terms of Szegő kernels, which exhibit universal behavior in the large \( N \) limit. Hopefully, this paper may provide insight into proving a similar decay rate for this more general setting. This
has already been done for other properties of random holomorphic sections, e.g. correlation functions, \[.\]

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2. **SU(2) Invariance.**

We begin by calling the set of polynomials in one variable whose degree is less than or equal to \(N\), \(\text{Poly}_N\). \(\text{Poly}_N\) becomes a Hilbert space with respect to the SU(2) invariant norm, \[[1], [9]:\]

\[
\|f\|_N^2 := \frac{N + 1}{\pi} \int_{z \in \mathbb{C}} |f(z)|^2 \frac{dm(z)}{(1 + |z|^2)^{N+2}},
\]

where \(dm\) is just the usual Lebesque measure. For this norm \(\{\sqrt(N)z^j\}\) is an orthonormal basis, as is \(\{\sqrt(N)j(z+b)^j(-\overline{b}z + a)^{N-j}\}\), where \(|a|^2 + |b|^2 = 1\). Specifically, one orthonormal basis which will be useful in subsequent work is for any \(\zeta \in \mathbb{C}\), \(\{\sqrt(N)j(z-\overline{\zeta})^j(1+\overline{\zeta}z)^{N-j}\}/(1+|\zeta|^2)^{N/2}\).

Clearly, by line (1), a Gaussian random SU(2) polynomial is defined as, \(\psi_{\alpha,N}(z) = \sum_{j=0}^{N} \alpha_j \psi_j(z)\), where \(\alpha_j\) are i.i.d. standard complex Gaussian random variables, and \(\{\psi_j\}\) is a particular orthonormal basis. Any basis for \(\text{Poly}_N\) could have been used and the Gaussian random SU(2) polynomials would be probabilistically identical, \([9]\), as for \((\alpha_0, \alpha_1, \ldots, \alpha_N) \in \mathbb{C}^{N+1}\) there exists \(U\) a unitary matrix such that for \((\alpha')^T = U \cdot \alpha^T\)

\[
\psi_{\alpha,N}(z) = \sum \alpha_j \sqrt(N-j)z^j = \sum \alpha_j \sqrt(N-j)(z-\overline{\zeta})^j(1+\overline{\zeta}z)^{N-j}/(1+|\zeta|^2)^{N/2}. \quad (2)
\]

3. **Large deviations of the Maximum of a random SU(2) polynomial.**

We will use following elementary estimates to compute upper and lower bounds for the probability of several events:

**Proposition 3.1.** Let \(\alpha\) be a standard complex Gaussian random variable, then

i) \(\text{Prob}(\{|\alpha| \geq \lambda\}) = e^{-\lambda^2}\)

ii) \(\text{Prob}(\{|\alpha| \leq \lambda\}) = 1 - e^{-\lambda^2} \in [\lambda^2, \lambda^2], if \lambda \leq 1\)
This next lemma is key as it states that the maximum of the norm of a random SU(2) polynomial on the disk of radius $r$ tends to not be too far from its expected value.

**Lemma 3.2.** For all $\delta \in (0, 1]$, and for all $r > 0$ there exists $a_{r, \delta} > 0$ such that

$$
Prob \left( \left\{ \max_{B(0,r)} |\psi_{\alpha,N}(z)| \not\in \left[ (1 + r^2)^{\frac{N}{2}} (1 - \delta)^{\frac{N}{2}}, (1 + r^2)^{\frac{N}{2}} (1 + \delta)^{\frac{N}{2}} \right] \right\} \right) < e^{-a_{r,\delta} N^2}
$$

**Proof.** First we shall prove that for all $\varepsilon > 0$,

$$
Prob \left( \left\{ \max_{B(0,r)} |\psi_{\alpha,N}(z)| > (1 + r^2)^{\frac{N}{2}} (1 + \delta)^{\frac{N}{2}} \right\} \right) < e^{-(1-\varepsilon) N^2}.
$$

To do this we consider the event $\Omega_N := \{ \forall j, |\alpha_j| \leq N \}$, the complement of which has probability $\leq (N + 1) e^{-N^2}$, by Proposition 3.1.

For $\alpha \in \Omega_N$ we will now estimate $\max_{B(0,r)} |\psi_{\alpha,N}|$:

$$
\max_{z \in B(0,r)} |\psi_{\alpha,N}(z)| = \max_{z \in B(0,r)} \left| \sum_{j} \alpha_j \binom{N}{j}^{\frac{1}{2}} (z)^j \right|
$$

$$
\leq \max_{z \in B(0,r)} \sum_{j} |\alpha_j| \binom{N}{j}^{\frac{1}{2}} |z|^j
$$

$$
\leq \max_{z \in B(0,r)} N \sqrt{N + 1} (1 + |z|^2)^{\frac{N}{2}}, \text{ by the Schwartz inequality.}
$$

$$
= N \sqrt{N + 1} (1 + r^2)^{\frac{N}{2}}
$$

For all $\delta > 0$, $\lim_{N \to \infty} \frac{N \sqrt{N + 1} (1 + \delta)^{\frac{N}{2}}}{(1+\delta)^{\frac{N}{2}}} = 0$, therefore there exists $N_\delta > 0$ such that if $N > N_\delta$ then $\frac{N \sqrt{N + 1} (1 + \delta)^{\frac{N}{2}}}{(1+\delta)^{\frac{N}{2}}} < 1$.

Hence if $N > N_\delta$ and if $\max_{B(0,r)} |\psi_{\alpha,N}(z)| > (1 + r^2)^{\frac{N}{2}} (1 + \delta)^{\frac{N}{2}}$ then $\alpha \in \Omega_{c,N}$. This then guarantees that the event of all such $\alpha$ is a subset of $\Omega_{c,N}$ and thus has probability equal to $(N + 1) e^{-N^2} < e^{-(1-\varepsilon) N^2}$, for large $N$. This decay rate is independent of $\delta$ and $r$.

We complete the proof by showing that:

$$
Prob \left( \left\{ \max_{B(0,r)} |\psi_{\alpha,N}(z)| < (1 + r^2)^{\frac{N}{2}} (1 - \delta)^{\frac{N}{2}} \right\} \right) < e^{-a_{r,\delta} N^2}.
$$
Consider the event where \( \max_{B(0,r)} |\psi_{\alpha,N}(z)| = M < (1 + r^2)^{\frac{N}{2}} (1 - \delta)^{\frac{N}{2}} \). The Cauchy estimates for a holomorphic function state that: \( |\psi_{\alpha,N}(0)| \leq j! \frac{M}{r^j} \).

Differentiating equation (1) yields that \( \sqrt{\binom{N}{j} j!} \alpha_j = \psi_{\alpha,N}(0) \). Combining this with Stirling’s formula \( \sqrt{2\pi j^j e^{-j} < j! < \sqrt{2\pi j^j e^{-j} e^{\frac{1}{12}}}} \) we get that:

\[
|\alpha_j| \leq \frac{(1 + r^2)^{\frac{N}{2}} (1 - \delta)^{\frac{N}{2}}}{r^j \sqrt{\binom{N}{j}}} \leq e^{\frac{1}{12}} \left( 1 + r^2 \right)^{\frac{N}{2}} (1 - \delta)^{\frac{N}{2}} \frac{\binom{N}{j} j!}{r^j N^{\frac{N}{2}}} \frac{r^j N^{\frac{N+1}{2}}}{\binom{N}{j} (N-j)^{\frac{N}{2}} (N-j)^{\frac{N}{2}}} \cdot \left( 1 + r^2 \right)^{\frac{N}{2}} (1 - \delta)^{\frac{N}{2}} \frac{\binom{N}{j} j!}{r^j N^{\frac{N}{2}}} \frac{r^j N^{\frac{N+1}{2}}}{\binom{N}{j} (N-j)^{\frac{N}{2}} (N-j)^{\frac{N}{2}}}.
\]

For the time being we focus on the term in the second parenthesis which we call \( A \). Writing \( j = xN, \ x \in (0,1) \), and we now have:

\[
A = (1 - \delta)^{\frac{N}{2}} \left( \frac{1 + r^2}{r^{2x}} \left( \frac{x}{1-x} \right)^x (1 - x) \right)^{\frac{N}{2}}
\]

If \( x = \frac{r^2}{1+r^2} \) then \( A = (1 - \delta)^{\frac{N}{2}} \), which inspires the next useful estimate:

Claim: Let \( m_r = \min \left\{ \frac{(2^\frac{1}{2} - 1)r^2}{1+r^2}, \frac{(2^\frac{1}{2} - 1)}{1+r^2} \right\} \).

If \( x \in \left[ \frac{r^2}{1+r^2} - m_r \delta, \frac{r^2}{1+r^2} + m_r \delta \right] \) then \( (1 + r^2) \left( \frac{x}{r^2(1-x)} \right)^x (1 - x) < (1 - \delta)^{-\frac{1}{2}} \).

Note: that \( x \in \left[ \frac{r^2}{1+r^2} - m_r \delta, \frac{r^2}{1+r^2} + m_r \delta \right] \subset (0,1) \)

Proof: Using the concavity of \( x^{\frac{1}{2}} \), a Taylor’s series and that \( \delta < 1 \), we see that \( (\frac{1}{1-\delta})^{\frac{1}{2}} \geq (2^\frac{1}{2} - 1) \delta + 1 \).

We then set \( x = (1 + \Delta) \frac{r^2}{1+r^2} \), therefore

\[
\Delta \in \left[ - \min \left\{ \frac{(2^\frac{1}{2} - 1)}{r^2}, (2^\frac{1}{2} - 1) \right\} \delta, \min \left\{ \frac{(2^\frac{1}{2} - 1)}{r^2}, (2^\frac{1}{2} - 1) \right\} \delta \right].
\]

Thus, \( 1 - x = \frac{1 - \Delta r^2}{1+r^2} \), and from this we compute that:
\[
(1 + r^2) \left( \frac{x}{r^2(1 - x)} \right)^x (1 - x) = (1 - \Delta r^2) \left( \frac{1 + \Delta}{1 - \Delta r^2} \right)^x
\]
\[
= (1 + \Delta)(1 - \Delta r^2)^{1-x}
\leq \max\{1 - \Delta r^2, 1 + \Delta\}, \text{ as } x \in (0, 1).
\leq 1 + (2 \delta - 1)\delta
\]

Proving the claim.

Therefore for \(x \in \left[ \frac{r^2}{1 + r^2 - m_r \delta}, \frac{r^2}{1 + r^2 + m_r \delta} \right] \), \(A < (1 - \delta)\frac{3N}{\delta} \). This then in turn guarantees that \(|\alpha_j| < e^{\frac{1}{12} \sqrt{2\pi N (1 - \delta) \frac{3N}{\delta}}} \). The probability this occurs for a single \(\alpha_j \) is less than or equal to \(e^{\frac{1}{12} \sqrt{2\pi N (1 - \delta) \frac{3N}{\delta}}} \). Thus the chance it occurs for all \(\alpha_j, j \in \left[ \left( \frac{r^2}{1 + r^2 - m_r \delta}, \frac{r^2}{1 + r^2 + m_r \delta} \right) \right] \), is less than or equal to
\[
\left( e^{\frac{1}{12} \sqrt{2\pi N (1 - \delta) \frac{3N}{\delta}}} \right)^{4[Nm_r \delta]}.
\]
The result follows as there exists \(a_{r,\delta} \) such that
\[
\left( e^{\frac{1}{12} \sqrt{2\pi N (1 - \delta) \frac{3N}{\delta}}} \right)^{4[Nm_r \delta]} < e^{-a_{r,\delta}N^2}
\]

A nice application of this lemma, along with line (2) is the following:

**Lemma 3.3.** For all \(\Delta \in (0, 1] \), and \(\zeta \in \mathbb{C}\setminus\{0\} \) there exists \(N_{\Delta,|\zeta|} \) and \(c_{\Delta, \zeta} > 0 \), such that if \(N > N_{\Delta,|\zeta|} \) then
\[
\text{Prob}\{ \max_{z \in B(0,\Delta)} |\psi_{\alpha,N}(z - \zeta)| < (1 + |\zeta|^2)\frac{N}{\delta}(1 - \Delta)\frac{N}{\delta} \} < e^{-c_{\Delta, \zeta}N^2}
\]

**Proof.** Let \(\Delta < 1 \), and set \(\delta = \frac{\Delta}{2(1 + |\zeta|^2 + |\zeta|^2)} \).

\[
(1 - \delta)^{\frac{N}{\delta}} \leq \max_{\partial B(0,\delta)} \frac{\sum \alpha_j' \sqrt{\binom{N}{j}(z - \zeta)^j(1 + \zeta z)^{N-j}}}{(1 + |\zeta|^2)\frac{N}{\delta}(1 + \delta^2)\frac{N}{\delta}}, \text{ by Lemma 3.2 and line (2), except for an event whose probability is less than } e^{-c_{\Delta, \zeta}N^2}.
\]

Let \(\phi(z) = \frac{z - \zeta}{1 + \zeta z} \), so that we may rewrite the previous equations as:
\[(1 - \delta)^{\frac{N}{2}} \leq \left( \max_{\partial B(0, \delta)} \frac{|1 + \xi z|^N}{(1 + |\xi|^2)^{\frac{N}{2}} (1 + \delta^2)^{\frac{N}{2}}} \right) \left( \max_{B(0, \delta)} |\psi_{\alpha', N}(\phi(z))| \right) \]
\leq \left( \frac{(1 + |\xi|\delta)^N}{((1 + |\xi|)^2(1 + \delta^2))^{\frac{N}{2}}} \right) \left( \max_{B(-\xi, (2 + 2|\xi|)^\delta)} |\psi_{\alpha', N}(z)| \right), \text{ since the image of } \phi_{|B(0, \delta)} \subset B(-\xi, (2 + 2|\xi|)^\delta).\]

Rearranging the previous sets of equations we get the result:
\[
\max_{B(-\xi, \Delta)} |\psi_{\alpha', N}(z)| \geq \frac{(1 + |\xi|^2)^{\frac{N}{2}} (1 + (2 + 2|\xi|)\delta)^{\frac{N}{2}}}{(1 + |\xi|\delta)^N} \cdot (1 - \delta)^{\frac{N}{2}}
\geq (1 - \Delta)^{\frac{N}{2}} (1 + |\xi|^2)^{\frac{N}{2}}
\]

4. Second key lemma.

The goal of this section will be to estimate \( \int \log |\psi_{\alpha, N}(re^{i\theta})| \frac{d\theta}{2\pi} \), which will be accomplished when we prove lemma 4.2, using the same techniques as in [10]. As \( \log(x) \) becomes unbounded near 0, we will first prove a deviation result for the event where the \( L^1 \) norm of \( \log |\psi_{\alpha, N}| \) is significantly larger than its max on the same region.

**Lemma 4.1.** For all \( r > 0 \) there exists \( N_1 \) and \( c_r > 0 \) such that for all \( N > N_1 \),
\[
\text{Prob} \left\{ \int_{\theta \in \mathbb{T}} |\log(|\psi_{\alpha, N}(re^{i\theta})|)| \frac{d\theta}{2\pi} \leq 5N \log ((2)(1 + r^2)) \right\} < e^{-c_r N^2}
\]

**Proof.** By Lemma 3.2, \( \exists N_1 \) such that if \( N > N_1 \), then, with the exception of an event, \( \Omega, \) whose probability is less than \( e^{-c_r N^2} \), there exists \( \zeta_0 \in \partial B(0, \frac{1}{2}r) \) such that \( \log(|\psi_{\alpha, N}(\zeta_0)|) > 0 \). This also implies that:
\[
\int_{\theta = 0}^{2\pi} P_r(\zeta_0, re^{i\theta}) \log(|\psi_{\alpha, N}(re^{i\theta})|) \frac{d\theta}{2\pi} \geq \log(|\psi(\zeta_0)|) \geq 0,
\]

Where \( P_r \) is the Poisson kernel: \( P_r(\zeta, z) = \frac{e^{2z - |\zeta|^2}}{|z - \zeta|^2} \). Hence,
\[
\int_{\theta = 0}^{2\pi} P_r(\zeta_0, re^{i\theta}) \log^+(|\psi_{\alpha, N}(re^{i\theta})|) \frac{d\theta}{2\pi} \leq \int_{\theta = 0}^{2\pi} P(\zeta_0, re^{i\theta}) \log^+(|\psi_{\alpha, N}(re^{i\theta})|) \frac{d\theta}{2\pi}
\]
Now given the event where $\log \max_{B(0,r)} |\psi_\alpha(z)| < \frac{N}{2} \log \left((2)(1 + r^2)\right)$, (whose complement for $N > N_\delta$ has probability less than $e^{-(1-\epsilon)N^2}$), we see that
\[
\int_{\theta=0}^{2\pi} \log^+ (|\psi_{\alpha,N}(re^{i\theta})|) \frac{d\theta}{2\pi} \leq \frac{N}{2} \log \left((2)(1 + r^2)\right).
\]

Since $\zeta_0 \in \partial B(0, \frac{1}{2}r)$ and $z = re^{i\theta}$, we have: $\frac{1}{2}r \leq |z - \zeta_0| \leq \frac{3}{2}r$. Hence, by using the formula for the Poisson Kernel, $\frac{1}{3} \leq P(\zeta, z) \leq 3$. Putting the pieces together proves the result:

\[
\int_{\theta=0}^{2\pi} P_r(\zeta_0, re^{i\theta}) \log^+ (|\psi_{\alpha,N}(re^{i\theta})|) \frac{d\theta}{2\pi} \leq \frac{1}{\min P(\zeta_0, z)} \int_{\theta=0}^{2\pi} P_r(\zeta_0, re^{i\theta}) \log^+ (|\psi_{\alpha,N}|) \frac{d\theta}{2\pi} \\
\leq 3 \int_{\theta=0}^{2\pi} P_r(\zeta_0, re^{i\theta}) \log^+ (|\psi_{\alpha,N}(re^{i\theta})|) \frac{d\theta}{2\pi} \\
\leq \frac{9}{2} N \log \left(2(1 + r^2)\right)
\]

□

We now arrive at them main result of this section:

**Lemma 4.2.** For all $\Delta \in (0, 1)$, and for all $r > 0$ there exists $c_{\Delta,r} > 0$ such that

\[
\text{Prob} \left( \left\{ \int_{\theta=0}^{2\pi} \log (|\psi_{\alpha,N}(re^{i\theta})|) \frac{d\theta}{2\pi} < \frac{N}{2} \log \left((1 + r^2)(1 - \Delta)\right) \right\} \right) < e^{-c_{\Delta,r} N^2}
\]

**Proof.** With out loss of generality let $\Delta < \Delta_{0,r}$. Set $\delta = \frac{\Delta^4}{81} < 1$. Let $m = \lceil \frac{1}{\delta} \rceil$, and let $\kappa = 1 - \delta^{\frac{1}{2}} = 1 - \Delta$.

We partition the circle of radius $\kappa r$, into $m$ disjoint even length segments. We choose a $\zeta_j$ within $\delta r$ of the midpoint of each of these segments such that

\[
\log(|\psi_{\alpha,N}(\zeta_j)|) > \frac{N}{2} \log \left((1 + \kappa^2 r^2)(1 - \delta r)\right), \tag{3}
\]

which by Lemma 3.3 may be done except for an event whose probability is less than $e^{-c_{\delta,r} N^2}$. Therefore there exists $N_\Delta$ such that if $N > N_\Delta$ the union of these $m$ events has probability less than or equal to $\left(\lceil \frac{m}{2} \rceil + 1\right)e^{-c_{\delta,r} N^2} < e^{-c_{\delta,r} N^2}$. We now turn to investigating the average of $\log |\psi_\alpha(z)|$ on the unit circle by using Riemann integration and line (3):
\[ \frac{N}{2} \log((1 + \frac{1}{2} \kappa^2 r^2)(1 - \delta)) \leq \frac{1}{m} \sum_{j=1}^{j=m} \log |\psi_{\omega,N}(\zeta_j)| \]

\[ \leq \int_{\theta=0}^{2\pi} \left( \sum_{j} \frac{1}{m} P_r(\zeta_j, re^{i\theta}) \log(|\psi_{\omega,N}(re^{i\theta})|) \frac{d\theta}{2\pi} \right) \]

\[ = \int_{\theta=0}^{2\pi} \left( \sum_{j} \frac{1}{m} (P_r(\zeta_j, re^{i\theta}) - 1) \right) \log(|\psi_{\omega,N}(re^{i\theta})|) \frac{d\theta}{2\pi} \]

\[ + \int_{\theta=0}^{2\pi} \log(|\psi_{\omega,N}(re^{i\theta})|) \frac{d\theta}{2\pi} \]

This will simplify to:

\[ \int \log(|\psi_{\alpha,N}(re^{i\theta})|) \frac{d\theta}{2\pi} \geq \frac{N}{2} \log((1 + \kappa^2 r^2)(1 - \delta r)) \]

\[ - (\int |\log|\psi_{\alpha,N}(re^{i\theta})|| \frac{d\theta}{2\pi}) \max_{\theta \in \mathbb{T}} \left| \sum_{j} \frac{1}{m} (P_r(\zeta_j, re^{i\theta}) - 1) \right| \]

In [10], it was computed that in exactly this situation that:

\[ \max_{\theta \in [0, 2\pi]} \left| \sum_{j} \frac{1}{m} (P_r(\zeta_j, re^{i\theta}) - 1) \right| \leq C \delta^{\frac{1}{2}} \]

Hence, except for an event of probability \( e^{-cN^2} \), by lemma [11],

\[ \int \log(|\psi_{\alpha,N}|) \frac{d\theta}{2\pi} \geq \frac{N}{2} \log((1 + \kappa^2 r^2)(1 - \delta r)) - 7N \log(2(1 + r^2)) \cdot C \delta^{\frac{1}{2}}, \]

\[ \geq \frac{N}{2} \log((1 + r^2)(1 - 3\delta^{\frac{1}{2}})). \]

\[ \square \]

5. Main Results.

We now have all the tools needed to prove the two main results of this paper, starting with theorem [11].

**Proof.** (of theorem [11].) Let \( \Xi_{\alpha,r,N} = \int_{B(0,r)} Z_{\psi_{\alpha,N}} \) be the number of zeros inside the disk of radius \( r \) counted with multiplicity. We ignore the null event where there is a zero on the boundary for a particular Gaussian random SU(2) polynomial.

It suffices to prove the result for small \( \Delta \). Let \( \delta = \frac{\Delta^2}{4} < 1 \). Let \( \kappa = 1 + \sqrt{\delta} = 1 + \frac{\Delta}{2} \). Let \( a_j \) denote the zeros of a fixed \( \psi_{\alpha,N} \) that are inside \( B(0,r) \), and \( b_j \) those in \( B(0,\kappa r) \backslash B(0,r) \).
We start the proof by recalling Jensen’s formula:
\[
\log |\psi_{\alpha,N}(0)| = - \sum \log \left( \frac{\kappa r}{|b_j|} \right) - \sum \log \left( \frac{\kappa r}{|a_j|} \right) + \int_{\theta=0}^{2\pi} \log |\psi_{\alpha,N}(\kappa r e^{i\theta})| \frac{d\theta}{2\pi}.
\]
\[
\log |\psi_{\alpha,N}(0)| = - \sum \log \left( \frac{r}{|a_j|} \right) + \int_{\theta=0}^{2\pi} \log |\psi_{\alpha,N}(r e^{i\theta})| \frac{d\theta}{2\pi}.
\]

Except for an event of probability \( \leq e^{-aN^2} \), after subtracting the second line from the first we get that,
\[
\Xi_{\alpha,r,N} \log(\kappa) \leq \int_{\theta=0}^{2\pi} \log |\psi_{\alpha,N}(\kappa r e^{i\theta})| \frac{d\theta}{2\pi} - \int_{\theta=0}^{2\pi} \log |\psi_{\alpha,N}(r e^{i\theta})| \frac{d\theta}{2\pi}.
\]
by Lemma 3.2
\[
\leq \frac{N}{2} \left( \log \left( (1 + \kappa^2 r^2)(1 + \delta) \right) - \int_{\theta=0}^{2\pi} \log |\psi_{\alpha}(r e^{i\theta})| \frac{d\theta}{2\pi} \right),
\]
by Lemma 4.2
\[
\leq \frac{N}{2} \left( \log \left( (1 + \kappa^2 r^2)(1 + \delta) \right) - \frac{N}{2} \log \left( (1 + r^2)(1 - \delta) \right) \right).
\]

Therefore,
\[
\Xi_{\alpha,r,N} \leq N \left( \frac{r^2 + \frac{1}{2}\sqrt{\delta}r + \sqrt{\delta} + \sqrt{\delta}r^2}{(1 + r^2)} - \frac{\sqrt{\delta}r^4}{(1 + r^2)^2} + O(\delta^2) \right) \left( 1 + \frac{\sqrt{\delta}}{2} + O(\delta) \right)
\]
\[
\leq \frac{N r^2}{1 + r^2} + 2N \sqrt{\delta} + O(\delta)
\]

This proves the probability estimate when the number of zeros in the lower hemisphere is significantly larger than expected. If we choose new random variables by taking the unitary transformation which reverses the order of the random variables, \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N) \), \( U\alpha = (\alpha_N, \alpha_{N-1}, \ldots, \alpha_0) = \alpha' \), and assemble a random SU(2) polynomial from these Gaussian Random variables, \( \psi_{\alpha',N}(z) \), a direct computation shows that for \( z_0 \neq 0 \), \( \psi_{\alpha,N}(z_0) = 0 \Leftrightarrow \psi_{\alpha',N}(\frac{1}{z_0}) = 0 \). Thus,
\[
\left\{ \alpha' : \Xi_{\alpha',r} - \frac{N}{1 + r^2} < \Delta N \right\} = \left\{ \alpha : \Xi_{\alpha,r} - \frac{N r^2}{1 + r^2} \geq \Delta N \right\}.
\]
As, \( \{\alpha'\} \) are all i.i.d. Gaussian random events we may apply the first part of this proof implies that these two events have probability less than or equal to \( e^{-aN^2} \), and the result follows immediately.
\[\square\]
We have just implicitly proven an upper bound on the order of the decay of the hole probability. We will now compute the lower bound to finish the proof of theorem 1.2.

Proof. (of theorem 1.2). If \( \forall z \in B(0,r), \psi_{\alpha,N}(z) \neq 0 \) then \( \Xi_{\alpha,r} = 0 \), therefore by the previous theorem, \( \text{Hole}_{N,r} \) is contained in the event:

\[
\left\{ \left| \Xi_{\alpha,r} - \frac{N r^2}{1 + r^2} \right| \geq \frac{r^2}{1 + r^2} N \right\}.
\]

In other words events of the type whose probability has just been given an upper bound in theorem 1.1. For this event there exists an \( N_r \), such that for all \( N > N_r \), the probability is less than \( e^{-aN^2} \).

We must still prove the lower bound, for which we start by considering the event, \( \Omega \) which consists of \( \alpha_j \) where:

\[
|\alpha_0| \geq N \\
|\alpha_j| < \left(\frac{N}{j}\right)^{\frac{1}{2}} r^{-j
\]

If \( \alpha \in \Omega \), then \( |\alpha_0| > \sum_{j>0} |\alpha_j| \left(\frac{N}{j}\right)^{\frac{1}{2}} r^j \). Hence \( \forall z \in B(0,r) \psi_{\alpha,N}(z) \neq 0 \Rightarrow \Omega \subset \text{Hole}_{N,r} \).

We will now approximate the probability of \( \Omega \) in order to get a lower bound for the probability of \( \text{Hole}_{N,r} \):

\[
\text{Prob} \left( \left\{ |\alpha_j| < \left(\frac{N}{j}\right)^{\frac{1}{2}} r^{-j}\right\}\right) \geq \frac{1}{2} \left(\frac{N}{j}\right)^{\frac{1}{2}} r^{-2j}, \text{ if } \left(\frac{N}{j}\right) \geq \frac{1}{r^2} \\
\geq \frac{1}{2} \left(\frac{N-j}{j}\right)! j! r^{-2j} \\
\geq \frac{\sqrt{\pi N}}{2^{N+\frac{1}{2}}} r^{-2j} e^{-\frac{j}{2}} \\
\geq \frac{e^{\frac{j}{2}}}{(2)^N} \min \left\{ \frac{1}{r}, 1 \right\}, \text{ if } N > 1 \text{ and } j \geq 1
\]

\[
\text{Prob}(\{|\alpha_0| > N\}) = e^{-N^2} \\
\text{Hence, } \text{Prob}(\Omega) \geq e^{-N^2 - N \log(2) - N \max\{\log(r),2N \log(r)\} - \frac{N}{12}} = e^{-c_r N^2}
\]

6. Generalizations to \( m \) dimensions.

In this paper we have just computed the decay rate of the hole probability for Gaussian random SU(2) polynomials. This work may be adapted to the
case of m variables or random SU(m+1) polynomials:

$$\psi_{\alpha,N,m}(z_1, \ldots, z_m) = \sum_{j=(j_1, \ldots, j_m)} \alpha_j \sqrt{N_{j_1, \ldots, j_m}} z^j$$

where $\alpha_j$ are i.i.d. standard complex Gaussian random variables. For this system, by following the steps in this paper and in [11], the following theorem will be proven in subsequent work.

**Theorem 6.1.** Let $\text{Hole}_{N,r,m}$ be the event where for all $z \in B(0,r), \psi_{\alpha,N,m} \neq 0$. Then there exists $N_{r,m}, c_{r,m} > 0$ such that for all $N > N_{r,m}$

$$e^{-c_{r,m}N^{m+1}} \leq \text{Prob}(\text{Hole}_{N,r,m}) \leq e^{-c'_{r,m}N^{m+1}}$$

Also, a result similar to theorem 1.1, but with an “$e^{-cN^{m+1}}$” rate of decay could be proven.

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