The Calogero Model: Integrable Structure and Orthogonal Basis

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Abstract. Integrability, algebraic structures and orthogonal basis of the Calogero model are studied by the quantum Lax and Dunkl operator formulations. The commutator algebra among operators including conserved operators and creation-annihilation operators has the structure of the W-algebra. Through an algebraic construction of the simultaneous eigenfunctions of all the commuting conserved operators, we show that the Hi-Jack (hidden-Jack) polynomials, which are an multi-variable generalization of the Hermite polynomials, form the orthogonal basis.

1 Introduction

In memory of the pioneering works in 70’s [1, 2, 3, 4], a class of one-dimensional quantum many-body systems with inverse-square long-range interactions are generally called the Calogero-Moser-Sutherland models. The celebrated Hamiltonians are

Calogero-Moser: \[ H_{CM} = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \frac{1}{2} \sum_{j,k=1}^{N} \frac{a^2 - \hbar a}{(x_j - x_k)^2}, \]

Calogero: \[ \hat{H}_C = \frac{1}{2} \sum_{j=1}^{N} (p_j^2 + \omega^2 x_j^2) + \frac{1}{2} \sum_{j,k=1}^{N} \frac{a^2 - \hbar a}{(x_j - x_k)^2}, \]

Sutherland: \[ \tilde{H}_S = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \frac{1}{2} \sum_{j,k=1}^{N} \frac{a^2 - \hbar a}{\sin^2(x_j - x_k)}, \]

where the constants \( N, a \) and \( \omega \) are the particle number, the coupling parameter and the strength of the external harmonic well, respectively. The momentum operator \( p_j \) is given by a differential operator, \( p_j = -i\hbar \frac{\partial}{\partial x_j} \). The Calogero and Sutherland models are a harmonic confinement and a periodic version of the Calogero-Moser model, respectively. Thus these two models have discrete energy spectra, whereas the other has continuous one. From now on, we set the Planck constant at unity, \( \hbar = 1 \).

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The Lax formulation for the classical Calogero-Moser model was discovered by Moser [M]. Let us introduce two $N \times N$ Hermitian matrices:

$$L_{ij} = p_i \delta_{ij} + a(1 - \delta_{ij}) \frac{1}{x_i - x_j},$$

$$M_{ij} = a \delta_{ij} \sum_{l=1}^{N} \frac{1}{(x_i - x_l)^2} - a(1 - \delta_{ij}) \frac{1}{(x_i - x_j)^2}.$$ 

We call them Lax pair. The classical Calogero-Moser Hamiltonian is given by eq. (1) with $p_j = \frac{dx_j}{dt}$ and $\hbar = 0$. The time evolution of the $L$-matrix is expressed as the Lax equation,

$$\frac{dL}{dt} = \{L, H_{\text{CM}}\} = [L, iM],$$

where the Poisson bracket is defined by $\{f, g\}_P \triangleq \sum_j \left( \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right)$. Thanks to the trace identity for c-number-valued matrices, $\text{Tr}AB = \text{Tr}BA$, we can easily see that the trace of the power of the $L$-matrix, $I^n_{\text{cl}} \triangleq \text{Tr}L^n$, gives the conserved quantities: $\frac{dI^n_{\text{cl}}}{dt} = \text{Tr}[L^n, iM] = 0$. Using the classical $r$-matrix [AT] or the generalized Lax equations for higher conserved quantities [BR], we can show that the conserved quantities are Poisson-commutative, $\{I^n_{\text{cl}}, I^m_{\text{cl}}\}_P = 0$, which proves the integrability of the classical Calogero-Moser model in Liouville’s sense. Natural quantization of the Lax equation for the classical case (4) by correspondence principle, $\{\hat{\mathcal{H}}, \hat{I}_{\text{CM}}^n\}_{\mathbb{P}} \rightarrow -i[\hat{\mathcal{H}}, \hat{I}_{\text{CM}}^n]$, gives an equality for the quantum Calogero-Moser Hamiltonian (1). However, the trace trick is not available to construct the commuting conserved operators for the quantum case because of the non-commutativity of the canonical conjugate variables. The initial motivation of our study was to find out a way to construct the conserved operators for the quantum models using the Lax formulation. The key of our idea is the sum-to-zero condition of the $M$-matrix:

$$\sum_{j=1}^{N} M_{jk} = 0, \quad \sum_{j=1}^{N} M_{kj} = 0, \quad \text{for } k = 1, 2, \cdots, N.$$ 

This property tells us that the (commuting) conserved operators can be obtained by summing up all the matrix elements of the powers of the $L$-matrix instead of taking traces $[UHW, UWH]$, 

$$I^n_{\text{CM}} = \sum_{j,k=1}^{N} (L^n)_{jk} \triangleq \text{T}_{\Sigma} L^n \Rightarrow [H_{\text{CM}}, I^n_{\text{CM}}] = \text{T}_{\Sigma}[L^n, M] = 0,$$

which proves the quantum integrability, or the existence of sufficiently many conserved operators, of the Calogero-Moser model. Encouraged by the result, we further investigated the integrable structure of the Calogero model (2) through the quantum Lax formulation.

### 2 Integrability and Algebraic Structure

Let us start from the Lax equation of the Calogero Hamiltonian (2),

$$-i\frac{dL^\pm}{dt} = [\hat{H}_C, L^\pm] = [L^\pm, M] \pm \omega L^\pm,$$
where the new matrices $L^\pm$ are defined by $L^\pm \overset{\text{def}}{=} L + Q, Q_{jk} \overset{\text{def}}{=} i x_j \delta_{jk}$. Using the sum-to-zero trick, we can get the conserved operators of the Calogero model as follows:

$$\hat{I}_n \overset{\text{def}}{=} T \Sigma (L^+ - L^-)^n, \quad [\hat{H}_C, \hat{I}_n] = T \Sigma [(L^+ - L^-)^n, M] = 0,$$

$$\hat{I}_1 = 2\hat{H}_C - N\omega(Na + (1 - a)), \quad \hat{I}_n = \sum_{j=1}^{N} p_j^{2n} + \cdots.$$

Mutual commutativity of the above conserved operators is verified rather easily by the Dunkl operator formulation [3, 4]. Introducing the coordinate exchange operator,

$$(K_{lk} f)(\cdots, x_l, \cdots, x_k, \cdots) = f(\cdots, x_k, \cdots, x_l, \cdots),$$

we define the creation-annihilation like operators as

$$c_l^\dagger = p_l + ia \sum_{k=1 \atop k \neq l}^{N} \frac{1}{x_l - x_k} K_{lk} + i\omega x_l,$$

$$c_l = p_l + ia \sum_{k=1 \atop k \neq l}^{N} \frac{1}{x_l - x_k} K_{lk} - i\omega x_l. \quad (5)$$

Commutation relations among the creation-annihilation operators are

$$[c_l^\dagger, c_m^\dagger] = 0, \quad [c_l, c_m] = 0,$$

$$[c_l, c_m^\dagger] = 2\omega \delta_{lm} (1 + a \sum_{k \neq l}^{N} K_{lk}) - 2\omega a (1 - \delta_{lm}) K_{lm},$$

which prove that the Hermitian operators $I_n \overset{\text{def}}{=} \sum_{j=1}^{N} (c_j^\dagger c_j)^n$, are commuting operators. We denote the restriction of the operand to the space of symmetric functions by $\big|_{\text{Sym}}$. Under the restriction, the conserved operators $\hat{I}_n$ and commuting Hermitian operators $I_n$ are considered to be the same, $\hat{I}_n \big|_{\text{Sym}} = I_n \big|_{\text{Sym}}$. Thus we have proved the quantum integrability of the Calogero model.

We can recursively construct generalized Lax equations for a family of operators $O^p_m$, $m, p = 1, 2, \cdots$, which reveal the W-algebraic structure of the Calogero model [5, 6]. The operators are defined by the sum of all the matrix elements of the Weyl ordered product of $p L^+$s and $m L^-$s:

$$O^p_m \overset{\text{def}}{=} T \Sigma [(L^+)^p (L^-)^m]_W,$$

$$[(L^+)^p (L^-)^m]_W \overset{\text{def}}{=} \frac{p! m!}{(p + m)!} \sum_{\text{all possible order}} (L^+)^p (L^-)^m.$$

The generalized Lax equations for the operators $O^p_m$ are

$$[O^p_m, L^\pm] = [(L^\pm, M^p_m) + m\omega(1 \pm 1)[(L^+)^p (L^-)^{m-1}]_W \quad - \quad p\omega(1 \mp 1)[(L^+)^{p-1} (L^-)^m]_W, \quad (6)$$
and the $M^p_m$ matrices satisfy the sum-to-zero condition:

$$\sum_{j=1}^{N} (M^p_m)_{jk} = 0, \quad \sum_{j=1}^{N} (M^p_m)_{kj} = 0.$$ 

The Hamiltonian $\hat{H}_C$ belongs to the operator family, $2\hat{H}_C = O_1^1$. The operators $O^n_n$ are conserved operators, though they do not commute each other. The family has two interesting subsets of commuting non-Hermitian operators,

$$B_n^\dagger = O^n_0, \quad B_n = O^n_n, \quad n = 1, 2, \cdots,$$

which we call power-sum creation-annihilation operators. They will play an important role in the algebraic construction of the energy eigenfunctions in the next section.

Let us introduce operators $W^{(s)}_n$:

$$W^{(s)}_n \overset{\text{def}}{=} \frac{1}{4\omega} O^{s-n-1}_{s+n-1}, \quad s \geq |n| + 1,$$

where the indices $n$ and $s$ are integer or half odd integer, and respectively correspond to the Laurent mode and the conformal spin. The commutator among the operators above is

$$[W^{(s)}_n, W^{(t)}_m] = (n(t-1) - m(s-1)) W^{(s+t-2)}_{n+m} + P_{n,m}^{(s,t)}(W^{(u)}_l),$$

where $P_{n,m}^{(s,t)}(W^{(u)}_l)$ is a polynomial of $W^{(u)}_l$, $u \leq s + t - 3$, $l \leq n + m$. The polynomial is generated while the products of $L^\pm$-matrices are rearranged into the Weyl ordered products by replacements of $L^+$ and $L^-$,

$$[L^+, L^-] = 2\omega((a-1)\mathbf{1} - aT), \quad 1_{jk} = \delta_{jk}, \quad T_{jk} = 1.$$ 

In terms of the W-operators, conserved operators and power-sum creation-annihilation operators are respectively expressed as $O^n_n \propto W^{n+1}_{0}, B_n^\dagger \propto W^{(\frac{n+1}{2})}_{-\frac{n}{2}}$ and $B_n \propto W^{(\frac{n}{2}+1)}_{\frac{n}{2}}$.

For the classical case, the W-algebraic structure of the Calogero model was discovered by the collective field theory [AJ1] and the classical $r$-matrix method [A]. The quantum collective field theory also possesses the W-algebraic structure [AJ2, AJ3], though its relationship with the quantum Calogero model is not directly confirmed. An $SU(\nu)$ generalization of our approach is presented in [UW2].

### 3 Perelomov Basis

The eigenvalue problem of the Calogero model was first solved by Calogero [C]. Later, inspired by the simple form of its energy spectrum, Perelomov tried an algebraic construction of the energy eigenfunctions [Pe]. In what follows, we shall complete the Perelomov’s approach [UW1]. The generalized Lax equations (6) for the power-sum creation operators (7) yield the following commutators:

$$[\hat{H}_C, B_n^\dagger] = n\omega B_n^\dagger, \quad [B_n^\dagger, B_m^\dagger] = 0, \quad n, m = 1, 2, \cdots, N. \quad (8)$$
To construct all the eigenfunctions for the \( N \)-body Calogero model, we need \( N \) creation operators. By straightforward calculations of the commutators \( \mathcal{B} \), Perelomov presented three creation operators, \( B_n^\dagger, n = 2, 3, 4 \). The quantum Lax formulation provides an easy way to make sufficient number of such operators. By successive operations of the power-sum creation operators on the ground state wave function, we can get all the excited states. The Calogero Hamiltonian is cast into the following form,

\[
\hat{H}_C = \frac{1}{2} \sum_{L} L^+ L^- + \frac{1}{2} N\omega(Na + (1 - a)) = \frac{1}{2} \sum_{j=1}^{N} h_j^\dagger h_j + E_g,
\]

where the operators \( h_j^\dagger \) and \( h_j \) are defined as

\[
h_j^\dagger \triangleq \sum_{k=1}^{N} L_{kj}^+ = p_j + i \omega x_j - ia \sum_{k=1 \atop k \neq j}^{N} \frac{1}{x_j - x_k},
\]

\[
h_j \triangleq \sum_{k=1}^{N} L_{jk}^- = p_j - i \omega x_j + ia \sum_{k=1 \atop k \neq j}^{N} \frac{1}{x_j - x_k}.
\]

Thus the differential equations, \( h_j |0\rangle = 0, j = 1, 2, \cdots, N \), are the sufficient conditions for the ground state \( |0\rangle \). In the coordinate representation, the solution is expressed as the (real) Laughlin wave function:

\[
\langle x | 0 \rangle = \prod_{1 \leq j < k \leq N} |x_j - x_k|^a \exp\left(-\frac{1}{2} \omega \sum_{j=1}^{N} x_j^2\right). \tag{9}
\]

As is similar to the free boson case, the excited states are labeled by the Young diagram,

\[
\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0\} \in Y_N,
\]

where \( \lambda_k, k = 1, 2, \cdots, N \), are nonnegative integers. Conventionally, we omit zeroes and use superscript for a sequence of the same numbers, e.g., \( \{4,1^2\} = \{4,1,1,0,\cdots,0\} \). The excited state labeled by the Young diagram \( \lambda \) is given by

\[
|\lambda\rangle = \prod_{k=1}^{N} (B_k^\dagger)^{\lambda_k-\lambda_{k+1}}|0\rangle, \quad \lambda_{N+1} = 0,
\]

\[
\hat{H}_C = (|\lambda|\omega + E_g)|\lambda\rangle \overset{\text{def}}{=} E(\lambda)|\lambda\rangle, \tag{10}
\]

where \( |\lambda\rangle \) denotes the weight of the Young diagram, \( |\lambda\rangle \overset{\text{def}}{=} \sum_{k=1}^{N} \lambda_k \). The above energy spectrum has the same form as that of non-interacting bosons confined in an external harmonic well up to the ground state energy. In other words, the inverse-square interactions just shift the ground state energy. The multiplicity of the \( n \)-th energy level, \( n\omega + E_g \), is equal to the number of the Young diagrams of the weight \( n \), \( \#\{\lambda ||\lambda| = n\} \). This is also the same as that of the non-interacting case. Thus we have algebraically constructed a basis of the eigenfunctions of the Calogero Hamiltonian.
4 Diagonalization of $\hat{I}_2$

The algebraic construction à la Perelomov generates a basis of the eigenfunctions of the Calogero Hamiltonian. Unfortunately, the basis is not orthogonal. To make an orthogonal basis from a basis, we usually try the Gram-Schmidt method. Here we take another way. As we have confirmed before, the Calogero model has a set of commuting conserved operators, which means existence of simultaneous eigenfunctions for them. The simultaneous eigenfunctions should be the orthogonal basis because they must be non-degenerate eigenfunctions of Hermitian operators. As the first step of our approach, we get some simultaneous eigenfunctions of the Hamiltonian and the second conserved operator $\hat{I}_2$, and observe their properties [UW3].

Since the Hamiltonian and $\hat{I}_2$ commute, the matrix representation of $\hat{I}_2$ on the Perelomov basis has a block-diagonalized form and each block consists of the wave functions of a weight (energy eigenvalue). By a straightforward calculation of commutators between $\hat{I}_2$ and $B_n^1$, we calculated the first seven blocks, whose weights are from zero to six, of the matrix representation and their eigenvalues. The eigenvalues imply the general form of the eigenvalue of $\hat{E}_2$:

$$\hat{E}_2(\lambda) = 4\omega^2 \sum_{k=1}^{N} ((\lambda_k)^2 + a(N + 1 - 2k)\lambda_k).$$

Though a combination of $E(\lambda)$ and $\hat{E}_2(\lambda)$ removes most of the degeneracies, there still remain degeneracies for the states whose weights larger than or equal to six. For example, the following two pairs of the Young diagrams with the weight six give such degeneracies:

$\{4, 1^2\}, \{3^2\} \rightarrow \hat{E}_2 = 4\omega^2(18 + 6a(N - 2))$,

$\{3, 1^3\}, \{2^3\} \rightarrow \hat{E}_2 = 4\omega^2(12 + 6a(N - 3)).$

It is interesting that the pairs have a common property. We cannot compare two Young diagrams of each pair by the dominance order. The dominance order $D \leq$, sometimes called the natural partial order, is defined as follows:

$$\mu \leq \lambda \Leftrightarrow |\mu| = |\lambda| \text{ and } \sum_{k=1}^{l} \mu_k \leq \sum_{k=1}^{l} \lambda_k \text{ for all } l.$$  

We can readily confirm that the Young diagrams of each pair are incomparable in the dominance order,

$\{4, 1^2\} \not\leq \{3^2\} \text{ and } \{3^2\} \not\leq \{4, 1^2\},$

$\{3, 1^3\} \not\leq \{2^3\} \text{ and } \{2^3\} \not\leq \{3, 1^3\}.$

The specific observation above is, in fact, a general fact. We cannot define the dominance order between any pair of distinct Young diagrams $\lambda$ and $\mu$ of a weight that share the common eigenvalue $\hat{E}_2$ [Sl, UW6], i.e.,

$$|\lambda| = |\mu| \text{ and } \hat{E}_2(\lambda) = \hat{E}_2(\mu) \Rightarrow \lambda \not\leq \mu \text{ and } \mu \not\leq \lambda.$$
We calculated the eigenvectors of the blocks with weights up to three in the matrix representation of $\hat{I}_2$. The eigenvectors correspond to seven simultaneous eigenfunctions of $\hat{H}_C$ and $\hat{I}_2$. Since the eigenvalues $E$ and $\hat{E}_2$ for the seven functions have no degeneracy, they belong to the orthogonal basis and also to the simultaneous eigenfunctions of all the commuting conserved operators $\hat{I}_n$ of the Calogero model. The eigenfunction of the Calogero model is factorized into the ground state wave function $|0\rangle$ and a symmetric polynomial. Symmetric polynomial parts of the seven simultaneous eigenfunctions, which we denote by $[\lambda]$, are

$$[0] = 1, \quad [1] = m_1, \quad [1^2] = m_{12} + \frac{a}{2\omega} \frac{N(N-1)}{2},$$

$$[2] = (1 + a)m_2 + 2am_{12} - \frac{1}{2\omega} N(Na + 1),$$

$$[1^3] = m_{13} + \frac{1}{2\omega} a \frac{(N-1)(N-2)}{2} m_1,$$

$$[2, 1] = (2a + 1)m_{2,1} + 6am_{13} - \frac{1}{2\omega} (1 - a)(N - 1)(Na + 1)m_1,$$

$$[3] = (a^2 + 3a + 2)m_3 + 3a(a + 1)m_{2,1} + 6am_{13}$$

$$- \frac{3}{2\omega} (a^2N^2 + 3aN + 2)m_1,$$

where $m_\lambda$ is the monomial symmetric polynomial defined by

$$m_\lambda(x_1, \cdots, x_N) = \sum_{\sigma \in S_N, \text{distinct permutations}} (x_{\sigma(1)})^{\lambda_1} \cdots (x_{\sigma(N)})^{\lambda_N}.$$

Note that the sum over $S_N$ is performed so that any monomial in the summand appears only once. In the above expressions, we notice that all the seven symmetric polynomials share a common property, triangularity. Namely, the seven polynomials $[\lambda]$ are expanded by the monomial symmetric polynomials $m_\mu$ whose Young diagram $\mu$ is smaller than or equal to the Young diagram $\lambda$ in the weak dominance order $\leq$, i.e.,

$$\mu \leq \lambda \iff \sum_{k=1}^l \mu_k \leq \sum_{k=1}^l \lambda_k \quad \text{for all } l.$$  

The observation means that we can uniquely identify the simultaneous eigenfunctions of the first two conserved operators of the Calogero model just by the first two eigenvalues and triangularity up to normalization. We shall confirm the existence of such functions by algebraically constructing them.

5 Hi-Jack Polynomials

Since our interest now concentrates on the symmetric polynomial parts of the simultaneous eigenfunctions, we modify some operators to make them suitable for the aim. A gauge transformation of the creation-annihilation-like operators (7) yields the following
Dunkl operators:

\[\alpha_i^\dagger \overset{\text{def}}{=} \langle x|0\rangle \left( -\frac{i}{2\omega} c_i^\dagger \frac{1}{\langle x|0\rangle} \right) = -\frac{i}{2\omega}(p_i + ia\sum_{k=1, k\neq i}^{N} x_l - x_k (K_{lk} - 1) + 2i\omega x_i),\]

\[\alpha_i \overset{\text{def}}{=} \langle x|0\rangle i c_i \frac{1}{\langle x|0\rangle} = i(p_i + ia\sum_{k=1, k\neq i}^{N} x_l - x_k (K_{lk} - 1)),\]

\[d_i \overset{\text{def}}{=} \alpha_i^\dagger \alpha_i. \tag{11}\]

The gauge transformation above removes the action on the ground state wave function from the operators. Note that the definition of Hermiticity of such gauge-transformed operators is modified and different from the ordinary one. Using the \(d_i\)-operators, we define the normalized conserved operators:

\[I_n \overset{\text{def}}{=} \sum_{l=1}^{N} (d_i)^n \left| \text{Sym} \right|, \quad \frac{1}{\langle x|0\rangle} \left| \text{Sym} \right| \left( x|0\rangle \right) = \omega I_1 + E_g. \tag{12}\]

We note that the Dunkl operators (11) reduce to those for the Sutherland model (3) in the limit, \(\omega \to \infty\):

\[\alpha_i^\dagger \to z_i, \quad \alpha_i \to \nabla_i = i(p_{zi} + ia\sum_{k=1, k\neq i}^{N} \frac{1}{z_l - z_k} (K_{lk} - 1)), \quad p_{zi} \overset{\text{def}}{=} -i\frac{\partial}{\partial z_i}, \quad d_i \to D_i = z_i \nabla_i. \tag{13}\]

We change the variables by

\[\exp 2i\pi x_j = z_j, \quad j = 1, 2, \cdots, N,\]

and denote the ground state wave function and the ground state energy of the Sutherland model by

\[\tilde{\psi}_g = \prod_{1 \leq j < k \leq N} |z_j - z_k|^a \prod_{j=1}^{N} z_j^{-\frac{1}{2}a(N-1)}, \quad \epsilon_g = \frac{1}{6}a^2 N(N-1)(N+1).\]

Then the Sutherland Hamiltonian (3) is gauge-transformed to and related with the \(D\)-operator by

\[H_S - \epsilon_g = \tilde{\psi}_g^{-1}(\tilde{H}_S - \epsilon_g)\tilde{\psi}_g\]
\[ = -2 \sum_{j=1}^{N} (z_j p_zj)^2 + i a \sum_{j,k=1 \atop j \neq k}^{N} \frac{z_j + z_k}{z_j - z_k} (z_j p_zj - z_k p_zk) \]
\[ = 2 \sum_{l=1}^{N} (D_l)^2 \bigg|_{\text{Sym}}. \]

Commutation relations among the Dunkl operators (11) and the action of \( \alpha_l \) on 1 are

\[
[\alpha_l, \alpha_m] = 0, \quad [\alpha_l^\dagger, \alpha_m^\dagger] = 0, \quad [\alpha_l, \alpha_m^\dagger] = \delta_{lm} (1 + a \sum_{k=1 \atop k \neq l}^{N} K_{lk}) - a(1 - \delta_{lm}) K_{lm},
\]
\[
[d_l, d_m] = a(d_m - d_l) K_{lm}, \quad \alpha_l \cdot 1 = 0. \tag{14}
\]

We should remark that the above relations do not explicitly depend on the parameter \( \omega \), which implies the Dunkl operators for the Sutherland model (13) also satisfy the above relations. Hence the Calogero and Sutherland models share the same algebraic structure [UW4, UW5, U1]. To put it another way, the theory of the Calogero model is a one-parameter deformation of that of the Sutherland model. Thus the simultaneous eigenfunction of the Calogero model is expected to be a one-parameter deformation of that of the Sutherland model, which is known to be the Jack polynomial [J]. In the following, we call the simultaneous eigenfunction of the Calogero model Hi-Jack (hidden-Jack) polynomial.

Using the normalized conserved operators (12), we define the Hi-Jack polynomials \( j_\lambda(x; \omega, 1/a) \) in a similar fashion to a definition of the Jack polynomials:

\[
I_1 j_\lambda(x; \omega, 1/a) = \sum_{k=1}^{N} \lambda_k j_\lambda(x; \omega, 1/a) \overset{\text{def}}{=} E_1(\lambda) j_\lambda(x; \omega, 1/a), \tag{15}
\]
\[
I_2 j_\lambda(x; \omega, 1/a) = \sum_{k=1}^{N} (\lambda_k^2 + a(N + 1 - 2k)\lambda_k) j_\lambda(x; \omega, 1/a) \overset{\text{def}}{=} E_2(\lambda) j_\lambda(x; \omega, 1/a), \tag{16}
\]
\[
j_\lambda(x; \omega, 1/a) = \sum_{\mu \leq \lambda} w_{\lambda \mu}(a, 1/2\omega) m_\mu(x), \tag{17}
\]
\[
w_{\lambda \lambda}(a, \omega) = 1. \tag{18}
\]

We can prove the existence of the Hi-Jack polynomials by explicit construction. Following the results by Lapointe and Vinet on the Jack polynomials [LV], we introduce the raising operators for the Hi-Jack polynomials,

\[
b_k^+ = \sum_{J \subseteq \{1, 2, \ldots, N\} \atop |J| = k} \alpha^J \cdot d_{1,J}, \quad \text{for } k = 1, 2, \ldots, N - 1,
\]
\[
b_N^+ = \alpha_1^+ \alpha_2^+ \cdots \alpha_N^+.
\]

The operators, \( \alpha_j^+ \) and \( d_{1,J} \), stand for

\[
\alpha_j^+ = \prod_{J \subseteq \{1, 2, \ldots, N\} \atop \lambda \in J} \alpha_\lambda^J,
\]
\[
d_{1,J} = (d_{j_1} + a)(d_{j_2} + 2a) \cdots (d_{j_k} + ka),
\]
where \( J \) is a subset of a set \( \{1, 2, \ldots, N\} \) whose number of elements \(|J|\) is equal to \( k \), \( J \subseteq \{1, 2, \ldots, N\} \), \(|J| = k\). From eq. (14), we can verify an identity,

\[
(d_i + ma)(d_j + (m + 1)a) |_{\text{Sym}}^{\{i,j\}} = (d_j + ma)(d_i + (m + 1)a) |_{\text{Sym}}^{\{i,j\}},
\]

where \( m \) is some integer. The symbol \( |^{\{J\}}_{\text{Sym}} \) where \( J \) is some set of integers means that the operands are restricted to the space which is symmetric with respect to the exchanges of any indices in the set \( J \). This identity (19) guarantees that the operator \( d_{1,J} \) does not depend on the order of the elements of a set \( J \) when it acts on symmetric functions and hence operation of the raising operators on symmetric functions yields symmetric functions. The function generated by the following Rodrigues formula,

\[
j_\lambda(x; \omega; 1/a) = C_\lambda^{-1}(b_N^+)^{\lambda_N}(b_{N-1}^+)^{\lambda_{N-1}-\lambda_N} \cdots (b_1^+)^{\lambda_1-\lambda_2} \cdot 1,
\]

with the normalization constant \( C_\lambda \) given by

\[
C_\lambda = \prod_{k=1}^{N-1} C_k(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}; a),
\]

where

\[
C_k(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}; a) = (a)_{\lambda_k-\lambda_{k+1}} (2a + \lambda_{k-1} - \lambda_k)_{\lambda_k-\lambda_{k+1}} \cdots (ka + \lambda_1 - \lambda_k)_{\lambda_k-\lambda_{k+1}},
\]

satisfies the definition of the Hi-Jack polynomial \( j_\lambda(x; \omega; 1/a) \). The symbol \((\beta)_n\) in the above expression is the Pochhammer symbol, that is, \((\beta)_n = \beta(\beta + 1) \cdots (\beta + n - 1)\), \((\beta)_0 \overset{\text{def}}{=} 1\).

The first seven Hi-Jack polynomials are, for instance, given as follows:

\[
\begin{align*}
j_0(x; \omega; 1/a) &= J_0(x; 1/a) = m_0(x) = 1, \\
j_1(x; \omega; 1/a) &= J_1(x; 1/a) = m_1(x), \\
j_{12}(x; \omega; 1/a) &= J_{12}(x; 1/a) + \frac{a}{2\omega} \frac{N(N-1)}{2} J_0(x; 1/a) \\
&= m_{12}(x) + \frac{a}{2\omega} \frac{N(N-1)}{2} m_0(x), \\
(a + 1)j_2(x; \omega; 1/a) &= (a + 1) J_2(x; 1/a) - \frac{1}{2\omega} N(Na + 1) J_0(x; 1/a) \\
&= (a + 1)m_2(x) + 2am_{12}(x) - \frac{1}{2\omega} N(Na + 1) m_0(x), \\
j_{13}(x; \omega; 1/a) &= J_{13}(x; 1/a) + \frac{1}{2\omega} \frac{(N-1)(N-2)}{2} J_1(x; 1/a) \\
&= m_{13}(x) + \frac{1}{2\omega} \frac{(N-1)(N-2)}{2} m_1(x), \\
(2a + 1)j_{2,1}(x; \omega; 1/a) &= (2a + 1) J_{2,1}(x; 1/a) - \frac{1}{2\omega} (1 - a)(Na + 1) J_1(x; 1/a) \\
&= (2a + 1)m_{2,1}(x) + 6am_{13}(x).
\end{align*}
\]
\[-\frac{1}{2\omega}(1-a)(N-1)(Na+1)m_1(x),
\]
\[(a^2 + 3a + 2)j_3(x; \omega, 1/a) = (a^2 + 3a + 2)J_3(x; 1/a) - \frac{3}{2\omega}(a^2 N^2 + 3aN + 2)J_1(x; 1/a)
\]
\[(a^2 + 3a + 2)m_3(x) + 3a(a+1)m_{2,1}(x) + 6a^2 m_{13}(x)
\]
\[-\frac{3}{2\omega}(a^2 N^2 + 3aN + 2)m_1(x),
\]
where the symbol \(J_3(x; 1/a)\) denotes the Jack polynomial. The explicit forms also show the fact that the Hi-Jack polynomial reduces to the Jack polynomial in the limit, \(\omega \to \infty\),

\[j_\lambda(x; \omega = \infty, 1/a) = J_\lambda(x; 1/a). \quad (20)\]

Besides the above relation, we have some other relations between the Hi-Jack and Jack polynomials \([UW3, U1, U2]\). While the Hi-Jack polynomial is a one-parameter deformation of the Jack polynomial, we can get the Hi-Jack polynomial from the Jack polynomial by the following formula,

\[J_\lambda(\alpha^\dagger_1, \alpha^\dagger_2, \ldots, \alpha^\dagger_N; 1/a) \cdot 1 = j_\lambda(x; \omega, 1/a),\]

which gives another relation between the Jack polynomials and the Hi-Jack polynomials.

In the above expansion, we have an observation, and it is generally true, that increasing the order of \(1/2\omega\) by one causes decreasing of the weight of the symmetrized monomial by two. The fact yields an stronger form of the triangularity:

\[j_\lambda(x; \omega, 1/a) = \sum_{\mu \leq \lambda \text{ and } |\mu| \equiv |\lambda| (\text{mod} 2)} \left(\frac{1}{2\omega}\right)^{(|\lambda| - |\mu|)/2} w_{\lambda\mu}(a)m_\mu(x),\]

\[w_{\lambda\lambda}(a) = 1. \quad (21)\]

Combining eqs. (20) and (21), we have the following expansion form of the Hi-Jack polynomial with respect to the Jack polynomial:

\[j_\lambda(x; \omega, 1/a) = J_\lambda(x; \omega, 1/a) + \sum_{\mu \leq \lambda \text{ and } |\mu| < |\lambda| \text{ and } |\mu| \equiv |\lambda| (\text{mod} 2)} \left(\frac{1}{2\omega}\right)^{(|\lambda| - |\mu|)/2} w_{\lambda\mu}(a)J_\mu(x; 1/a).\]

Relationship between the Hi-Jack polynomials and the Perelomov basis \([\Pi]\) is given as follows. The power-sum creation operator \(B^+_k\) is cast into the power-sum of \(\alpha^\dagger_l\)-operators,

\[B^+_k = (2i\omega)^k \sum_{l=1}^N (\alpha^\dagger_l)^k_{\text{Sym}} \overset{\text{def}}{=} (2i\omega)^k p_k(\alpha^\dagger)_{\text{Sym}},\]

and the Perelomov basis is expressed by the power-sum of the \(\alpha^\dagger_l\)-operators as

\[\langle x|\lambda \rangle = (2i\omega)^{\lambda|}\prod_{k=1}^N (p_k(\alpha^\dagger))^{\lambda_k - \lambda_{k+1}} \cdot 1 = (2i\omega)^{\lambda|} p_{\lambda}(\alpha^\dagger) \cdot 1.\]
Thus the transition matrix between the power-sums and Jack polynomials $M(J, p)$,

$$J_\lambda(x, 1/a) = \sum_{|\mu|=|\lambda|} M(J, p)_{\lambda\mu} p_\mu(x),$$

gives a relation between the Hi-Jack polynomials and the Perelomov basis,

$$j_\lambda(x; \omega, 1/a) = (2i\omega)^{-|\lambda|} \sum_{|\mu|=|\lambda|} M(J, p)_{\lambda\mu} \langle x|\mu\rangle \langle \mu|0 \rangle.$$

We have introduced the Hi-Jack polynomials as the simultaneous eigenfunctions for the first two commuting conserved operators with the triangularity. As we shall see shortly, they are non-degenerate simultaneous eigenfunctions for all the commuting conserved operators of the Calogero model. From a calculation of the action of $d_i$ operator on a symmetrized monomial of $\alpha_k^\dagger$'s, $m_\lambda(\alpha_1^\dagger, \cdots, \alpha_N^\dagger)$, we can prove the following expression:

$$I_n j_\lambda(x; \omega, 1/a) = \sum_{\mu \leq \lambda \atop |\mu|<|\lambda|} w'_{\lambda,\mu}(a, 1/2\omega) m_\mu(x). \quad (22)$$

This means that operation of the conserved operators on the Hi-Jack polynomials keeps their triangularity. Since the $n$-th conserved operator commutes with the first and second conserved operators, $[I_1, I_n] = [I_2, I_n] = 0$, we can easily verify,

$$I_1 I_n j_\lambda(x; \omega, 1/a) = E_1(\lambda) I_n j_\lambda(x; \omega, 1/a), \quad (23)$$
$$I_2 I_n j_\lambda(x; \omega, 1/a) = E_2(\lambda) I_n j_\lambda(x; \omega, 1/a). \quad (24)$$

Equations (23), (24) and (22) for $I_n j_\lambda$ are respectively the same as eqs. (15), (16) and (17) for the Hi-Jack polynomial $j_\lambda$, which means $I_n j_\lambda$ satisfies the definition of the Hi-Jack polynomial except for normalization. Our definition of the Hi-Jack polynomial uniquely specifies the Hi-Jack polynomial. So we conclude that $I_n j_\lambda$ must coincide with $j_\lambda$ up to normalization. Thus we confirm that the Hi-Jack polynomials $j_\lambda$ simultaneously diagonalize all the commuting conserved operators $I_n, n = 1, \cdots, N$. The eigenvalues of the conserved operators,

$$I_n j_\lambda(x; \omega, 1/a) = E_n(\lambda) j_\lambda(x; \omega, 1/a),$$

are generally polynomials of the coupling parameter $a$:

$$E_n(a) = e_n^{(0)}(\lambda) + e_n^{(1)}(\lambda) a + \cdots.$$

It is easy to get the constant ($a$-independent) term $e_n^{(0)}(\lambda)$ because the term corresponds to the $n$-th eigenvalue for $N$ non-interacting bosons confined in an external harmonic well:

$$e_n^{(0)}(\lambda) = \sum_{k=1}^N (\lambda_k)^n.$$
It is clear that there is no degeneracy in the constant terms of the eigenvalues \( \{ e_n^{(0)}(\lambda) | n = 1, \cdots, N \} \). Since the conserved operators \( I_n \) are Hermitian operators concerning the inner product,

\[
\langle j_\lambda, j_\mu \rangle = \int_{-\infty}^{\infty} \prod_{k=1}^{N} \, dx_k |\langle x | 0 \rangle|^2 j_\lambda j_\mu \propto \delta_{\lambda,\mu},
\]

the Hi-Jack polynomials are the orthogonal symmetric polynomials with respect to the above inner product. From the explicit form of the weight function,

\[
|\langle x | 0 \rangle|^2 = \prod_{1 \leq j < k \leq N} |x_j - x_k|^2 a \exp \left( -\omega \sum_{l=1}^{N} x_l^2 \right),
\]

we conclude that the Hi-Jack polynomial is a multivariable generalization of the Hermite polynomial [La].

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