BLACK HOLE ENTROPY
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Abstract

We review black hole entropy with special reference to euclidean quantum gravity, the brick wall approach and loop quantum gravity.

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1 Introduction

In Einstein’s theory of gravitation, the gravitational field due to a point mass is described by a metric which has many interesting properties. Its black hole features have been known for a very long time, but in the seventies it began to appear that thermodynamic concepts like temperature and entropy were also associated with it. Gradually it was realized that these were quantum effects. But the degrees of freedom associated with the entropy could not be easily identified. Many suggestions have been made: we shall discuss the entanglement entropy approach and the more recent loop quantum gravity.

After summarizing black hole mechanics, we consider the euclidean quantum gravity approach for both non-extremal and extremal black holes. There are indications of discontinuity between the two kinds, which arises in one way of quantization of the classical theory. An alternative way leads to the Bekenstein-Hawking formula even for extremal black holes.

In order to understand the origin of black hole entropy, the entropy of fields in black hole backgrounds was studied. This is identified as entanglement entropy which arises because the region in the interior of the horizon has to be traced over.

More recently, attempts have been made to formulate a quantum theory of gravity itself. Black hole entropy has also been calculated in this loop quantum gravity approach. This will be discussed in detail.

2 Euclidean quantum gravity

2.1 Preliminaries

A precursor of the idea of the entropy of black holes was the area theorem: the area of the horizon of a system of black holes always increases in a class of spacetimes. There were, more generally, a set of laws of black hole mechanics analogous to laws of thermodynamics:
• zeroeth law: surface gravity $\kappa$ remains constant on the horizon of a black hole

• first law: $\frac{\kappa dA}{8\pi} = dM - \phi dQ$, where $A = \text{area of horizon}$, $\phi = \text{potential at horizon of black hole (with mass } M, \text{ charge } Q)$

• second law: the area of the horizon of a black hole system always increases in spacetimes which are predictable from partial Cauchy hypersurfaces.

For charged black holes,

$$r_\pm = M \pm \sqrt{M^2 - Q^2},$$ (1)

$$\kappa = \frac{r_+ - r_-}{2r_+^2}, \quad \phi = Q/r_+, \quad A = 4\pi r_+^2.$$ (2)

When these observations were made, there was no obvious connection with thermodynamics – only a matter of analogy. But the existence of a horizon imposes limitation on the amount of information available and hence may lead to an entropy, which should then be measured by a geometric quantity associated with the horizon, namely its area. This implies, $A \propto \text{entropy}$, $\kappa \propto \text{temperature}$.

This interpretation was not fully convincing – but quantum theory was found to cause dramatic changes in black hole spacetimes: scalar field theory in a Schwarzschild black hole background indicates the radiation of particles at a temperature

$$T = \frac{\hbar}{8\pi M} = \frac{\hbar \kappa}{2\pi}.$$ (3)

This implies the connection of the laws of black hole mechanics with thermodynamics, and fixes a scale factor, which involves Planck’s constant indicating a quantum effect.

For Schwarzschild black holes, the first law of \textit{thermodynamics} simplifies:

$$TdS = dM.$$ (4)

This can be integrated:

$$S = \frac{4\pi M^2}{\hbar} = \frac{A}{4\hbar}.$$ (5)

$T = \frac{\hbar \kappa}{2\pi}$ is generally valid for black holes having $g_{tt} \sim (1 - \frac{2M}{r})$. The first law becomes

$$Td\frac{A}{4\hbar} = dM - \phi dQ.$$ (6)

Comparison with the first law of \textit{thermodynamics}

$$TdS = dM - \phi dQ$$ (7)

leads to the identification

$$S = \frac{A}{4\hbar}.$$ (8)
2.2 Non-extremal black holes

A grand partition function may be written for euclidean charged black holes:

$$Z_{\text{grand}} \equiv e^{-\frac{M-\frac{Q^2}{r^2} + 3Q \phi}{\kappa}} \approx e^{-\frac{I}{\hbar}}. \quad (9)$$

The functional integral, over all configurations (consistent with appropriate boundary conditions), is semiclassically approximated by the maximum of the integrand. The classical action $I$ can be calculated: for a euclidean Reissner-Nordström black hole in a manifold $\mathcal{M}$ with boundary which is subsequently taken to infinity,

$$I = -\frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{g} R + \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^3x \sqrt{\gamma}(K - K_0) + \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu}. \quad (10)$$

Here $\gamma$ is the induced metric on the boundary $\partial\mathcal{M}$, and $K$ the extrinsic curvature, from which a subtraction has to be made.

The first term of the action vanishes by Einstein’s equations ($R = 0$). To evaluate the second term, one takes the boundary of the manifold at $r = r_B \to \infty$.

$$K = -\frac{1}{\sqrt{g} dr^2} \frac{1}{\sqrt{g} r} \frac{d}{dr}(\sqrt{g} r^2)$$

$$= -\frac{1}{r^2} \frac{d}{dr}[(1 - \frac{M}{r} + \cdots)r^2]$$

$$= -\frac{1}{r^2} \frac{d}{dr}(r^2 - Mr), \quad (11)$$

So $\int d^3x \sqrt{g} K$ diverges as $r \to \infty$: this can be cured by subtracting from $K$ its flat space contribution $K_0 = \frac{1}{r^2} \frac{d}{dr}r^2$. The second piece of the action becomes

$$-\frac{1}{8\pi} \int dt(1 - \frac{M}{r} + \cdots)4\pi r^2 \frac{d}{dr}(-Mr)|_{r=r_B \to \infty}$$

$$= -\frac{1}{2} \int dt (-M) = \frac{1}{2} \beta M. \quad (13)$$

The euclidean time $t$ has to go over one period $0 \to \beta = \frac{2\pi}{\kappa}$ to avoid a conical singularity at the horizon.

The third term becomes

$$-\frac{1}{16\pi} \int dt 4\pi \int dr r^2 2 \frac{Q^2}{r^4}$$

$$= -\frac{1}{2} \int dt \frac{Q^2}{r^4}$$

$$= -\frac{1}{2} \beta Q \phi, \quad (14)$$
where $\phi$ is the electrostatic potential. The negative sign is for the euclidean solution.

Finally,

$$I = \frac{1}{2} \beta (M - Q \phi) = \frac{A}{4}, \quad (15)$$

$$M = T(S + \frac{I}{\hbar}) + \phi Q = T(S + \frac{A}{4\hbar}) + \phi Q. \quad (16)$$

Now the Smarr formula reads

$$M = \frac{\kappa A}{4\pi} + \phi Q = T \frac{A}{2\hbar} + \phi Q, \quad (17)$$

implying $S = \frac{A}{4\hbar}$.

### 2.3 Extremal black holes

Extremal black holes have $r_+ = r_-, Q = M, \phi = 1$. They are of special interest because the topology changes discontinuously in the passage from the (euclidean) non-extremal to the extremal case.

The action

$$I = \frac{1}{2} \beta (M - Q \phi) = 0, \quad (18)$$

$$M = T(S + \frac{I}{\hbar}) + \phi Q = TS + M \Rightarrow S = 0 \quad (19)$$

where $\beta$ has been assumed finite; note that $\lim_{Q \rightarrow M} \beta = \infty$ but there is no conical singularity in the extremal case, so there is no reason to fix the euclidean temperature, which can be arbitrary.

Here, the quantum theory is based exclusively on extremal topology. There is a more natural method of quantization: sum over topologies. Here the temperature $\beta^{-1}$ and the chemical potential $\Phi$ are specified as inputs at the boundary $r_B$ of the manifold, while the mass $M$ and the charge $Q$ of the black hole are calculated as functional integral averages. The definition of extremality $Q = M$ is imposed on these, making it a case of extremalization after quantization, as opposed to quantization after extremalization.

A spherically symmetric class of metrics is considered; the boundary conditions are:

$$g_{tt}(r_+) = 0, \quad 2\pi \sqrt{g_{tt}(r_B)} = \beta. \quad (20)$$

$$A_t(r_+) = 0, \quad A_t(r_B) = \frac{\beta \Phi}{2\pi i}, \quad (21)$$

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(the vector potential is taken to be zero). Another boundary condition reflects the extremal/non-extremal nature:

\[
\frac{1}{\sqrt{g_{rr}(r_+)} } \frac{d}{dr_+} \sqrt{g_{tt}(r_+)} = 1 \quad \text{in non-extremal case,}
\]

\[
\text{but} = 0 \quad \text{in extremal case.}
\] (22)

Variation of the action together with boundary conditions leads to reduced versions of Einstein - Maxwell equations, whose solution has a mass parameter \( m \) and a charge \( q \).

\[
I = \beta (m - q\Phi) - \pi (m + \sqrt{m^2 - q^2})^2 \quad \text{for non-extremal bc,}
\]

\[
I = \beta (m - q\Phi) \quad \text{for extremal bc.}
\] (23)

The partition function is of the form

\[
\sum_{\text{topologies}} \int d\mu(m) \int d\mu(q) e^{-I(q,m)},
\] (24)

with \( I \) appropriate for non-extremal/extremal \( q \).

The semiclassical approximation involves replacing the double integral by the maximum value of the integrand, \( i.e., \) by \( e^{-I_{\text{min}}} \), where \( I_{\text{min}} \) is the classical action for the non-extremal case, minimized with respect to \( q, m \), yielding a function of \( \beta, \Phi \), and implying \( S = A/4 \) for all values of \( \beta, \Phi \). The averages \( Q, M \), are calculated from \( \beta, \Phi \). The extremal limit is reached for limiting values

\[
\beta \to \infty, \quad |\Phi| \to 1, \quad \text{with} \quad \gamma \equiv \beta (1 - |\Phi|) = 2\pi M \text{(finite)}
\] (25)

for the ensemble parameters \( \beta, \Phi \). Then

\[
I = \frac{\gamma^2}{4\pi} = \pi M^2,
\] (26)

\[
Z \equiv e^{S - \gamma M/\hbar} = e^{-\pi M^2/\hbar},
\] (27)

continuing to correspond to \( S = \frac{A}{4\hbar} \).

3 Matter in black hole background

To study the entropy of a scalar field in the background provided by a black hole, one may employ brick-wall boundary conditions, where the wave function is cut off just outside the horizon

\[
\varphi(x) = 0 \quad \text{at} \quad r = r_h + \epsilon
\] (28)
with $\epsilon$ an ultraviolet cut-off. One also needs an infrared cut-off (box):

$$\varphi(x) = 0 \quad \text{at } r = L >> r_h$$

We use a static, spherically symmetric black hole spacetime

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{\theta\theta}(r)d\Omega^2.$$ (30)

An $r$-dependent radial wave number is defined for particles with mass $m$, energy $E$ and orbital angular momentum $l$:

$$k_r^2(r, l, E) = g_{rr}[-g^{tt}E^2 - l(l + 1)g^{\theta\theta} - m^2] \geq 0$$ (31)

One imposes on it the (semiclassical) quantization condition

$$\frac{1}{\pi} \int_{r_h + \epsilon}^{L} dr \ k_r(r, l, E) = n_r \text{ integral.}$$ (32)

The free energy $F$ at inverse temperature $\beta$ is given by a sum over single particle states:

$$\beta F = \sum_{n_r, l, m_l} \log(1 - e^{-\beta E})$$

$$\approx \int dl \ (2l + 1) \int dn_r \log(1 - e^{-\beta E})$$

$$= -\int dl \ (2l + 1) \int d(\beta E) \ (e^{\beta E} - 1)^{-1} n_r$$

$$= -\frac{\beta}{\pi} \int dl \ (2l + 1) \int dE \ (e^{\beta E} - 1)^{-1} \int_{r_h + \epsilon}^{L} dr \ g_r^{1/2}$$

$$\sqrt{-g^{tt}E^2 - l(l + 1)g^{\theta\theta} - m^2}$$

$$= \frac{2\beta}{3\pi} \int_{r_h + \epsilon}^{L} dr \ g_r^{1/2} g^{\theta\theta}(-g_{tt})^{-3/2}$$

$$\int dE \ (e^{\beta E} - 1)^{-1}[E^2 + g_{tt}m^2]^{3/2}.$$ (33)

The limits of integration for $l, E$ are such that the arguments of the square roots are nonnegative. $l$ integration is then explicit, while the $E$ integral has to be approximated. The contribution to the $r$ integral from large $r$ is also present in flat spacetime:

$$F_0 = -\frac{2}{9\pi} L^3 \int_m^{\infty} dE \frac{(E^2 - m^2)^{3/2}}{e^{\beta E} - 1}$$ (34)

and is not relevant. The contribution from small $r$ is singular in the limit $\epsilon \to 0$.

For non-extremal black holes, $g_{rr} \propto (r - r_h)^{-1}$, $g_{tt} \propto (r - r_h)$, while $g_{\theta\theta}$ is regular:

$$F_{\text{sing}} \approx -\frac{2\pi^3}{9\epsilon^3} (r - r_h)^{1/2}(-\frac{g_{tt}}{r - r_h})^{-3/2} g_{\theta\theta}|_{r=r_h}.$$ (35)
with corrections involving $m^2 \beta^2$. The entropy

$$S_{\text{sing}} = \beta^2 \frac{\partial F_{\text{sing}}}{\partial \beta} = \frac{8\pi^3}{45\beta^3} \epsilon (r - r_h) g_{rr}^{1/2}(r - r_h)^{-3/2} g_{\theta\theta}|_{r=r_h}. \quad (36)$$

Using the Hawking temperature

$$\frac{1}{\beta} = \frac{1}{2\pi} (g_{rr})^{-1/2} \left( -g_{tt} \right)^{1/2}|_{r=r_h} \quad (37)$$

and the proper radial width (defined through $d\tilde{r}^2 \equiv g_{rr} dr^2$)

$$\tilde{c} = \tilde{r}(r_h + \epsilon) - \tilde{r}(r_H) \approx 2\epsilon^{1/2} [(r - r_h) g_{rr}]^{1/2}|_{r=r_h}, \quad (38)$$

$$S_{\text{sing}} = \frac{1}{90\pi^2} g_{\theta\theta}|_{r=r_h} = \frac{1}{360\pi^2} \text{Area}. \quad (39)$$

This area factor crucially depends on the behaviour of the metric near the horizon and is valid only for non-extremal black holes; it does not emerge in the extremal case.

### 4 Loop quantum gravity

#### 4.1 Preliminaries

This is an approach to a quantum theory of gravity called loop quantum gravity or quantum geometry. A classical “isolated horizon” is the starting point: quantum states are built up by associating spin variables with “punctures” on this horizon. The entropy is obtained by counting all possible states consistent with a given area, more specifically, a particular eigenvalue of the area operator.

A generic configuration may be taken to have $s_j$ punctures with spin $j$, $j = 1/2, 1, 3/2,...$. Then

$$2 \sum_j s_j \sqrt{j(j+1)} = A, \quad (40)$$

the horizon area in special units

$$4\pi \gamma \ell_p^2 = 1,$$

where $\gamma$ is the so-called Barbero-Immirzi parameter and $\ell_p$ the Planck length. There is a spin projection constraint

$$\sum m = 0, \quad \text{over all punctures}$$

$$m \in \{-j, -j + 1, ...j\}$$

for puncture with spin $j$. 

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4.2 Spin 1/2

For simplicity first consider spin 1/2 on each puncture. The punctures have to be considered as distinguishable. The number of punctures \( n \) with spin 1/2 is given by

\[ 2n\sqrt{\frac{3}{4}} = A, \]

so if we neglect the spin projection constraint, the entropy

\[ n \ln 2 = \frac{A \ln 2}{\sqrt{3}} = \frac{A \ln 2}{4\sqrt{3}\pi \ell_p^2}. \]

It involves \( \gamma \), which can be chosen to yield the Bekenstein-Hawking entropy:

\[ \frac{A}{4\ell_p^2} \Rightarrow \gamma = \frac{\ln 2}{\sqrt{3}} \pi. \]

To implement the \( m \) constraint, the number \( 2^n \) of states is written as

\[ 2^n = 1 + ^nC_2 + \ldots + ^nC_n : \]

with the \( (r+1)^{\text{th}} \) term counting states with \( r \) up spins. For zero total projection, \( m = 0 \), \( n/2 \) spins are up. If \( n \) is odd, there is no such state, but if \( n \) is even, the number of states \( =^nC_{n/2} \). For large \( n \), the Stirling approximation is

\[ \ln n! \simeq \ln[\sqrt{2\pi n}(\frac{n}{e})^n] = n \ln n - n + \frac{1}{2} \ln(2\pi n), \]

\[ \frac{n!}{(n/2)!((n/2)!)!} \simeq n \ln 2 - \frac{1}{2} \ln n + \frac{1}{2} \ln 2 - \frac{1}{2} \ln \pi. \]

If the \( n \) independent piece is neglected for large \( n \), the entropy is

\[ \frac{A}{4\ell_p^2} - \frac{1}{2} \ln A. \]

If in addition one wants the total angular momentum to vanish, the number of states with total projection 1 must be subtracted:

\[ ^nC_{n/2} - ^nC_{n/2+1} = ^nC_{n/2}(1 - \frac{n/2}{n/2+1}) \Rightarrow \frac{A}{4\ell_p^2} - \frac{3}{2} \ln A. \]

4.3 General spin

So far only \( j = 1/2 \) spins have been considered at each puncture. If spin \( j \) occur at all punctures, an area \( A \) needs \( n = A/[2\sqrt{j(j+1)}] \) punctures. The number of states is \( (2j+1)^n \) if the \( m \) constraint is neglected. This yields

\[ n \ln(2j+1) = A \ln(2j+1)/[2\sqrt{j(j+1)}]. \]
This decreases with increasing $j$ (because $\ln(2j + 1)$ increases slowly compared with $\sqrt{j(j+1)}$). Higher spins contribute less to entropy.

A general configuration may be taken to have $s_j$ punctures with spin $j$.

$$N = \frac{(\sum_j s_j)!}{\prod_j s_j!} \prod_j (2j + 1)^{s_j}$$  \hspace{1cm} (47)

if the $m$ constraint is neglected (the first factor gives the number of ways of choosing locations of spins, the second factor counts the numbers of spin states at the punctures). One must sum $N$ over all nonnegative $s_j$ consistent with a given $A$. We estimate the sum by maximizing $\ln N$ w.r.t. $s_j$ subject to fixed $A$. Using Stirling again, and neglecting the last piece therein,

$$\ln N = \sum_j s_j \ln \frac{2j + 1}{s_j} + (\sum_j s_j) \ln(\sum_j s_j),$$  \hspace{1cm} (48)

$$\delta \ln N = \sum_j \delta s_j \left[ \ln(2j + 1) - \ln s_j + \ln(\sum_k s_k) \right],$$  \hspace{1cm} (49)

With some Lagrange multiplier $\lambda$ to implement the area constraint,

$$\ln(2j + 1) - \ln s_j + \ln(\sum_k s_k) = \lambda \sqrt{j(j+1)}.$$  \hspace{1cm} (50)

$$s_j = (2j + 1) \exp \left[ -\lambda \sqrt{j(j+1)} \right] \sum_k s_k.$$  \hspace{1cm} (51)

Summing over $j$,

$$\sum_j (2j + 1) \exp \left[ -\lambda \sqrt{j(j+1)} \right] = 1,$$  \hspace{1cm} (52)

which determines $\lambda \approx 1.72$.

$$\ln N = \lambda A / 2.$$  \hspace{1cm} (53)

To make this $\frac{A}{\sqrt{F}}$ (with $4\pi\gamma \ell_P^3 = 1$) the Barbero-Immirzi parameter becomes

$$\gamma = \frac{\lambda}{(2\pi)} \approx 0.274.$$  \hspace{1cm} (54)

Summing over $s_j$ may raise this value, while the projection constraint may lower it.
4.4 Disregarding \( j \) labels

An alternative counting criterion regards states with the same \( m \) on punctures but having different \( j \) to be equivalent, thus yielding fewer states. It yields

\[
\sum_j (2 + \delta_j) \exp[-\tilde{\lambda} \sqrt{j(j+1)}] = 1. \tag{55}
\]

whence \( \tilde{\lambda} \approx 1.58 \). The rationale behind this criterion is supposed to be that \( m \) is defined in a ‘surface Hilbert space’ while \( j \) is defined in a ‘volume Hilbert space’. However, the area of the horizon involves \( j \) and both quantum numbers are associated with punctures on the horizon. As the area cannot be expressed in terms of the \( m \), this counting is more complicated and many attempts to implement it contain inaccuracies. This holds for recipes as well as results given in the literature.

4.5 Logarithmic corrections

Let us impose the constraint of zero angular momentum projection. Let \( s_{j,m} \) punctures carry spin \( j \) and projection \( m \). Then \( s_j = \sum_m s_{j,m} \) is involved in the area constraint, while \( \sum_{j,m} ms_{j,m} = 0 \). The total number of ways of distributing these spins

\[
N_{\text{cor}} = \frac{(\sum_j s_j)!}{\prod_j s_j!} \prod_j \frac{s_j!}{\prod_m s_{j,m}!} = \frac{(\sum_{j,m} s_{j,m})!}{\prod_{j,m} s_{j,m}!}. \tag{56}
\]

To extremize the variation of \( \ln N_{\text{cor}} \) with two Lagrange multipliers \( \lambda, \alpha \) to implement the constraints,

\[
-\ln s_{j,m} + \ln \sum_{k,n} s_{k,n} = \lambda \sqrt{j(j+1)} + \alpha m, \tag{57}
\]

\[
\frac{s_{j,m}}{\sum_{k,n} s_{k,n}} = \exp[-\lambda \sqrt{j(j+1)} - \alpha m]. \tag{58}
\]

The projection constraint requires \( \sum_m \exp[-\alpha m] = 0 \), i.e., \( \alpha = 0 \), so that the distribution is the same as before.

To estimate the sum over \( s_{j,m} \) configurations, one approximates the sum by an integral. To study the variation of \( \ln N_{\text{cor}} \) with \( s_{j,m} \), one notes that

\[
\ln(s + \delta s)! \simeq (s + \delta s) \ln(s + \delta s) - (s + \delta s) \simeq s \ln s - s + (\ln s)\delta s + (\delta s)^2/(2s).
\]

Terms linear in \( \delta s \) cancel out because of extremization, while the quadratic part on exponentiation leads to factors of

\[
\exp[-(\delta s_{j,m})^2/(2s_{j,m})].
\]
Factors of $1/\sqrt{2\pi s_{j,m}}$ from Stirling’s formula are cancelled by $\sqrt{2\pi s_{j,m}}$ from gaussian integrations:

$$\int_{-\infty}^{\infty} d(\delta s_{j,m}) \exp \left[ -\frac{(\delta s_{j,m})^2}{2s_{j,m}} \right] = \sqrt{2\pi s_{j,m}}.$$ \hspace{1cm} (59)

Each $\sqrt{s_{j,m}} \propto \sqrt{A}$. The area constraint and the projection constraint reduce the number of summations, hence reducing the number of factors of $\sqrt{A}$ by two. But the numerator has an extra factor $\left( \sum s_{j,m} \right)^{1/2} \propto \sqrt{A}$. A net factor $1/\sqrt{A}$ survives, and the entropy

$$\ln \sum N_{cor} \simeq \lambda A/2 - \frac{1}{2} \ln A.$$ \hspace{1cm} (60)

This is the same log correction as for spin $1/2$.

4.6 Departure from linearity

The linearity of $S$ with $A$ is not borne out by numerical investigations which fix the area sharply. This can be understood by realizing the nature of the area constraint.

$$A = 2 \sum_j s_j \sqrt{j(j+1)} = [s_{1/2} \sqrt{3} + 2s_1 \sqrt{2} + s_{3/2} \sqrt{15} + \ldots],$$ \hspace{1cm} (61)

so if the natural numbers $s_j$ change, $s_{1/2}, s_1, s_{3/2}, \ldots$ cannot mix, but some mixing is still possible:

$$A = [s_{1/2} \sqrt{3} + 4s_3 \sqrt{3} + 15s_{25/2} \sqrt{3} + \ldots]$$

$$+ [2s_1 \sqrt{2} + 12s_8 \sqrt{2} + 70s_{49} \sqrt{2} + \ldots] + \cdots ;$$ \hspace{1cm} (62)

each set must be separately constant. One finds sets of compatible spins $N_1 \equiv \{1/2, 3, 25/2, \ldots\}, N_2 \equiv \{1, 8, 49, \ldots\}, \ldots$. There may be several constraints, the number depending on the area,

$$A = \sum_N A_N \equiv 2 \sum_{j \in N} s_j \sqrt{j(j+1)}.$$ \hspace{1cm} (63)

Corresponding Lagrange multipliers $\lambda_N$ satisfy

$$\sum_N \sum_{j \in N} (2j + 1) \exp \left[ -\lambda_N \sqrt{j(j+1)} \right] = 1,$$ \hspace{1cm} (64)

$$S = \sum_N \lambda_N A_N / 2 = \bar{\lambda} A/2,$$ \hspace{1cm} (65)

$\bar{\lambda} \leq 1.72$, depending on the ratios $A_1 : A_2 : \cdots$. $S$ reaches $1.72A/2$ only at special values of $A$ and is generally smaller.
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