MAXIMUM LIKELIHOOD DRIFT ESTIMATION FOR THE MIXING OF TWO FRACTIONAL BROWNIAN MOTIONS

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Abstract. We construct the maximum likelihood estimator (MLE) of the unknown drift parameter $\theta \in \mathbb{R}$ in the linear model

$$X_t = \theta t + \sigma B^{H_1}(t) + B^{H_2}(t), \quad t \in [0, T],$$

where $B^{H_1}$ and $B^{H_2}$ are two independent fractional Brownian motions with Hurst indices $\frac{1}{2} < H_1 < H_2 < 1$. The formula for MLE is based on the solution of the integral equation with weak polar kernel.

1. Introduction. The elements of stochastic calculus for fBm

Consider the continuous-time linear model

$$X(t) = \theta t + \sigma B^{H_1}(t) + B^{H_2}(t), \quad t \in [0, T],$$

where $B^{H_1}$ and $B^{H_2}$ are two independent fractional Brownian motions with Hurst indices $\frac{1}{2} < H_1 < H_2 < 1$, $\sigma > 0$. Given the sample path of $X$ on $[0, T]$ it is required to estimate the unknown drift parameter $\theta \in \mathbb{R}$. In the case when $H_1 = \frac{1}{2}$ the problem was solved in [4] where the authors develop the basic tools for analysis of the mixed fBm based on the filtering theory of Gaussian processes. They considered the linear regression setting and demonstrated how the maximum likelihood estimator can be constructed and studied in the large sample asymptotic regime.

As the preliminaries, we give some basic facts about stochastic calculus for fractional Brownian motion (for more details see [2], [3], [7], [8], [10]). Parameter estimation for different models with long memory was studied, among others, in [1], [3], [6], [11]. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space supporting all stochastic processes considered in what follows. Introduce $\{B^H(t), t \geq 0\}$, an adapted fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$, i.e. a centered Gaussian process with the covariance function

$$\mathbb{E} [B^H(t)B^H(s)] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).$$

For a (possibly infinite) interval $[0, T]$, denote by $L^2_H[0, T]$ the completion of the space of simple functions $f : [0, T] \to \mathbb{R}$ with respect to the scalar product

$$\langle f, g \rangle^2_H := \alpha_H \int_0^T \int_0^T f(t)g(s) |t-s|^{2H-2} dsdt,$$

where $\alpha_H = H(2H - 1)$. (It is worth to mention that this completion contains not only classical functions, but also some distributions.) For a step function of the form

$$f(t) = \sum_{k=0}^{n-1} a_k \mathbb{1}_{(t_k, t_{k+1})},$$

...
where \( \{t_0 < t_1 < \cdots < t_n\} \subset [0, T] \), the integral \( I^H(f) \) of \( f \) with respect to \( B^H \) is defined by

\[
I^H(f) = \int_0^T f(t) dB^H(t) = \sum_{k=0}^{n-1} a_k \left( B^H(t_{k+1}) - B^H(t_k) \right).
\]

It can be verified that \( I^H \) maps isomorphically the space of step functions on \([0, T]\) with the scalar product \( (\cdot, \cdot)_H \) into \( L^2(\Omega) \), hence, \( I^H \) can be extended to \( L^2_H[0, T] \).

Now define a square integrable kernel

\[
K_H(t, s) = \beta_H s^{1/2-H} \int_s^t (u - s)^{H-3/2} u^{H-1/2} du,
\]

where \( \beta_H = (\frac{\alpha_H}{\Gamma(1/2-2H)})^{1/2} \), where \( B(\cdot, \cdot) \) is beta function. The map

\[
(K^*_H f)(s) = \int_s^T f(t) \partial_t K(t, s) dt = \beta_H s^{1/2-H} \int_s^T f(t) t^{H-1/2} (t - s)^{H-3/2} dt
\]

is an isometry between the space of step functions and and can be extended to a Hilbert space isomorphism between \( L^2_H[0, T] \) and \( L^2[0, T] \). This implies that the process

\[
W(t) = I^H \left( (K^*_H)^{-1} 1_{[0,t]} \right)
\]

is a standard Wiener process on \([0, T]\), moreover, for any \( f \in L^2_H[0, T] \),

\[
I^H(f) = \int_0^T (K^*_H f)(s) dW(s).
\]

In particular, putting in the last formula \( f = 1_{[0,t]} \), one gets the following well-known representation of \( B^H \):

\[
B^H(t) = \int_0^t K_H(t, s)dW(s).
\]

Finally, we define the so-called fundamental martingale, or Molchan martingale \( M^H \), for \( B^H \). In this order, introduce the kernel \( l_H(t, s) = (t - s)^{1/2-H} s^{1/2-H} \) and consider square-integrable Gaussian martingale

\[
M^H(t) = \int_0^t l_H(t, s) dB^H(s) = \gamma_H \int_0^t s^{1/2-H} dW(s),
\]

with \( \gamma_H = (2H(\frac{3}{2} - H) \Gamma(3/2 - H)^3 \Gamma(H + \frac{1}{2}) \Gamma(3 - 2H)^{-1})^{1/2} \); the last equality is due to \ref{eq:1.1}.

The paper is organized as follows. In Section 2 we reduce the main problem to the solution of the integral equation with the weak polar kernel and establish the existence-uniqueness result for this equation. Section 3 contains an auxiliary result concerning the existence and uniqueness of the solution of the corresponding integral equation of the 1st kind. We prove this fact directly, by constructing the unique solution.
2. Main Problem

Now, let \(\frac{1}{2} < H_1 < H_2 < 1\), \(\left\{ \widetilde{B}^{H_1}(t), B^{H_2}(t), \ t \geq 0 \right\}\), \(i = 1, 2\), be two processes defined on the space \((\Omega, \mathcal{F}, (\mathcal{F}_t))\) and \(P_\theta\) be a probability measure under which \(\widetilde{B}^{H_1}\) and \(B^{H_2}\) are independent, \(B^{H_2}\) is a fractional Brownian motion with Hurst parameter \(H_2\), and \(\widetilde{B}^{H_1}\) is a fractional Brownian motion with Hurst parameter \(H_1\) and with drift \(\frac{\theta}{2}\), i.e.,

\[
\sigma \widetilde{B}^{H_1}(t) = \theta t + \sigma B^{H_1}(t).
\]

The probability measure \(P_\theta\) corresponds to the case when \(\theta = 0\). Our main problem is the construction of maximum likelihood estimator for \(\theta \in \mathbb{R}\) by the observations of the process \(Z(t) = \theta t + \sigma B^{H_1}(t) + B^{H_2}(t), \ t \in [0, T]\). However, the form of the process \(Z\) (two fBm’s with different Hurst indices) does not allow to construct the estimator immediately. To simplify the construction, we apply to \(Z\) the linear transformation of the following form:

\[
Y(t) = \int_0^t l_{H_1}(t, s)dZ(s) = \theta B\left(\frac{3}{2} - H_1, \frac{3}{2} - H_1\right)t^{2-2H_1} + \sigma M^{H_1}(t)
\]

\[
+ \int_0^t l_{H_1}(t, s)dB^{H_2}(s).
\]

(2.1)

this process is preferable since it involves Gaussian martingale \(M^H\).

Lemma 2.1. The linear transformation (2.1) is correctly defined.

Proof. It is sufficient to establish the existence of the integral \(\int_0^t l_{H_1}(t, s)dB^{H_2}(s)\) for any \(t \in [0, T]\). But we have that for any \(u, s \in [0, t]\)

\[
|u - s|^{2H_2 - 2} \leq t^{2H_2 - 2H_1}|u - s|^{2H_1 - 2},
\]

therefore

\[
\left|l_{H_1}(t, \cdot)\right|_{H_2}^2 := \alpha_{H_2} \int_0^t \int_0^t l_{H_1}(t, s)l_{H_1}(t, u) |u - s|^{2H_2 - 2} dsdu
\]

\[
\leq \alpha_{H_2} t^{2H_2 - 2H_1} \int_0^t \int_0^t l_{H_1}(t, s)l_{H_1}(t, u) |u - s|^{2H_1 - 2} dsdu
\]

\[
= \alpha_{H_2} t^{2H_2 - 2H_1} \left|l_{H_1}(t, \cdot)\right|_{H_1}^2 = \alpha_{H_2} t^{2H_2 - 2H_1} E|\gamma_{H_1}^2|^{2H_2 - 2H_1} E|\gamma_{H_1}^2|^{2H_1 - 2H_1}
\]

\[
= \frac{\alpha_{H_2} \gamma_{H_1}^2 t^{2H_2 - 2H_1}}{2 - 2H_1} < \infty,
\]

whence the proof follows.

As it was mentioned, process \(Y\) is more convenient to deal with since it involves martingale with a drift. Furthermore, it follows from the next result that processes \(Z\) and \(Y\) are observed simultaneously, so, we can reduce the original problem to the equivalent problem of the construction of maximum likelihood estimator of \(\theta \in \mathbb{R}\) basing on the linear transformation \(Y\).

Lemma 2.2. Processes \(Z\) and \(Y\) are observed simultaneously.

Proof. Taking into account (2.1), it is enough to present \(Z\) via \(Y\). But it follows from (2.1), from Fubini theorem for integrals w.r.t fBm (Theorem 2.6.5 [3]), and
from elementary integral transformations, that for any \( t \in [0, T] \)
\[
\int_0^t (t - s)^{H_1 - \frac{3}{2}} \int_0^s l_{H_1}(s, u)dZ(u)ds = \int_0^t (t - s)^{H_1 - \frac{3}{2}}dZ(u)
\]
\[
= B\left(H_1 - \frac{1}{2}, \frac{3}{2} - H_1\right) \int_0^t u^{\frac{1}{2} - H_1}dZ(u) = \int_0^t (t - s)^{H_1 - \frac{3}{2}}Y(s)ds
\]

whence
\[
Z(t) = B\left(H_1 - \frac{1}{2}, \frac{3}{2} - H_1\right)^{-1} \int_0^t \int_0^s (u - s)^{H_1 - 3/2}u^{H_1 - 1/2}dudY(s),
\]
and the proof follows. \( \square \)

Denote for simplicity \( \mathcal{B}_{H_1} := B\left(\frac{3}{2} - H_1, \frac{3}{2} - H_1\right) \). Now the main problem can be formulated as follows. Let \( \frac{1}{2} < H_1 < H_2 < 1 \), \( \{X_1(t) = \tilde{M}^{H_1}(t), X_2(t) := \int_0^t l_{H_1}(t, s)dB^{H_2}(s), \ t \geq 0\} \), \( i = 1, 2 \), be two processes defined on the space \( (\Omega, \mathcal{F}) \) and \( P_\theta \) be a probability measure under which \( \tilde{X}_1 \) and \( \tilde{X}_2 \) are independent, \( B^{H_2} \) is a fractional Brownian motion with Hurst parameter \( H_2 \), and \( \tilde{X}_1 \) is a martingale with square characteristics \( \langle \tilde{X}_1 \rangle(t) = \frac{\gamma_{H_1}^2}{2 - 2H_1}t^{2 - 2H_1} \) and with drift \( \frac{\theta \mathcal{B}_{H_1}}{\sigma}t^{2 - 2H_1} \), i.e.,
\[
\tilde{X}_1(t) = \tilde{M}^{H_1}(t) = \frac{\theta \mathcal{B}_{H_1}}{\sigma}t^{2 - 2H_1} + M^{H_1}(t).
\]

Also, denote \( X_1(t) = M^{H_1}(t) \). Our main problem is the construction of maximum likelihood estimator for \( \theta \in \mathbb{R} \) by the observations of the process
\[
Y(t) = \theta \mathcal{B}_{H_1}t^{2 - 2H_1} + \sigma X_1(t) + X_2(t).
\]

Note that under measure \( P_\theta \) the process
\[
\tilde{W}(t) := W(t) + \frac{\theta(2 - 2H_1)\mathcal{B}_{H_1}}{\sigma \gamma_{H_1}} t^{\frac{3}{2} - H_1}
\]
is a Wiener process with drift. Denote \( \delta_{H_1} = \frac{(2 - 2H_1)\mathcal{B}_{H_1}}{\sigma \gamma_{H_1}} \).

By Girsanov’s theorem and independence of \( X_1 \) and \( X_2 \),
\[
\frac{dP_\theta}{dP_0} = \exp \left\{ \theta \delta_{H_1} \int_0^T s^{\frac{1}{2} - H_1}d\tilde{W}(s) - \frac{\theta^2 \delta_{H_1}^2}{4(1 - H_1)} T^{2 - 2H_1} \right\}
\]
\[
= \exp \left\{ \theta \delta_{H_1} \tilde{X}_1(T) - \frac{\theta^2 \delta_{H_1}^2}{4(1 - H_1)} T^{2 - 2H_1} \right\}.
\]

For technical simplicity, we put \( \sigma = 1 \) in what follows. Note that, similarly to the linear combination of Wiener process and fBm, considered in [4], this derivative is not the likelihood for the problem at hand, because it is not measurable with respect to the observed \( \sigma \)-algebra
\[
\mathfrak{F}_Y^Y := \sigma\{Y(t), t \in [0, T]\} = \mathfrak{F}_Y^X := \sigma\{X(t), t \in [0, T]\},
\]
where \( X(t) = X_1(t) + X_2(t) \).

We shall proceed as in [4]: let \( \mu_\theta \) be the probability measure induced by \( Y \) on the space of continuous functions with the supremum topology under probability \( P_\theta \).
Then for any measurable set \( A \mu_\theta(A) = \int_A \Phi(x)\mu_0dx \), where \( \Phi(x) \) is such measurable functional that \( \Phi(X) = E_0\left(\frac{dP_\theta}{dP_0}\mid \mathcal{F}_T^X\right) \). The latter means that \( \mu_\theta \ll \mu_0 \) for any \( \theta \in \mathbb{R} \).

Taking into account that \( X_1 = X_1 \) under \( P_0 \) and the fact that the vector process \((X_1, X)\) is Gaussian, we get that the corresponding likelihood function is given by

\[
L_T(X, \theta) = E_0\left(\frac{dP_\theta}{dP_0}\mid \mathcal{F}_T^X\right) = E_0\left(\exp\left\{\theta\delta_{H_1}X_1(T) - \frac{\theta^2\delta_{H_1}^2}{4(1 - H_1)}T^{2 - 2H_1}\right\}\mid \mathcal{F}_T^X\right)
\]

(2.2)

\[
= \exp\left\{\theta\delta_{H_1}E_0(X_1(T)\mid \mathcal{F}_T^X) + \frac{\theta^2\delta_{H_1}^2}{2}\left(V(T) - \frac{T^{2 - 2H_1}}{2 - 2H_1}\right)\right\},
\]

where \( V(t) = E_0(X_1(t) - E_0(X_1(t)\mid \mathcal{F}_T^X))^2 \), \( t \in [0, T] \).

Thus, we arrive at the following problem: to find the projection \( P_XX_1(T) \) of \( X_1(T) \) onto \( \{X(t) = X_1(t) + X_2(t), t \in [0, T]\} \). We recall from Section I that

\[
W_i(t) = \int_0^t ((K_{H_i}^*)^{-1}1_{[0, t]}\right)dB^{H_i}(s), \ i = 1, 2,
\]

are standard Wiener processes, which are obviously independent. Also from Section I, we have

\[
X_1(t) = \gamma_{H_1} \int_0^t s^{1/2 - H_1}dW_1(s), \ B^{H_2}(t) = \int_0^t K_{H_2}(t, s)dW_2(s).
\]

Then, using (1.1), we can write

\[
X_2(t) = \int_0^t K_{H_1, H_2}(t, s)dW_2(s),
\]

where

\[
K_{H_1, H_2}(t, s) = \beta_{H_2}s^{1/2 - H_2} \int_s^t (t - u)^{1/2 - H_1}u^H_1 - (u - s)^{H_2 - 3/2}du.
\]

Similarly to (1.1), we have for \( f \in L_2^{H_2}[0, T] \)

\[
\int_0^T f(s)dX_2(s) = \int_0^T (K_{H_1, H_2}^*f)(s)dW_2(s),
\]

where

\[
(K_{H_1, H_2}^*f)(s) = \int_s^T f(t)\partial_tK_{H_1, H_2}(t, s)dt.
\]

The projection of \( X_1(T) \) onto \( \{X(t), t \in [0, T]\} \) is a centered \( X \)-measurable Gaussian random variable, therefore, it has a form

\[
P_XX_1(T) = \int_0^T h_T(t)dX(t)
\]

with \( h_T \in L_2^{H_1}[0, T] \). Note that \( h_T \) still can be a distribution. This projection for all \( u \in [0, T] \) must satisfy

\[
E[X(u)P_XX_1(T)] = E[X(u)X_1(T)].
\]

Using (2.6) together with independency of \( X_1 \) and \( X_2 \), we arrive at

\[
E\left[X_1(u) \int_0^T h_T(t)dX_1(t) + X_2(u) \int_0^T h_T(t)dX_2(t)\right] = E[X_1(u)X_1(T)] = \varepsilon_{H_1}u^{2 - 2H_1},
\]

(2.7)
where \( \varepsilon_H = \gamma_H^2/(2 - 2H) \).

From (2.3) - (2.7) we get
\[
(2.8) \quad \varepsilon_{H_1} u^{1-2H_1} = \gamma_{H_1}^2 \int_0^u h_T(s)s^{1-2H_1}ds + \int_0^T h_T(s)r_{H_1,H_2}(s,u)ds,
\]
where
\[
r_{H_1,H_2}(s,u) = \int_0^{s\wedge u} \partial_s K_{H_1,H_2}(s,v)K_{H_1,H_2}(u,v)dv.
\]
This kernel can be written alternatively as \( r_{H_1,H_2}(t,s) = \partial_t R_{H_1,H_2}(t,s) \), where
\[
R_{H_1,H_2}(t,s) = \int_0^t K_{H_1,H_2}(t,u)K_{H_1,H_2}(s,u)du = E[X_2(t)X_2(s)]
\]
\[
= \alpha_{H_2} \int_0^t \int_0^s (t-u)^{1/2-H_1}u^{1/2-H_1}(s-v)^{1/2-H_1}v^{1/2-H_1}|u-v|^{2H_2-2}dv du.
\]
Differentiating (2.8) with respect to \( u \), we arrive to
\[
(2.9) \quad \gamma_{H_1}^2 u^{1-2H_1} = \gamma_{H_1}^2 h_T(u)u^{1-2H_1} + \int_0^T h_T(s)k(s,u)ds,
\]
where
\[
k(s,u) = \partial_u r_{H_1,H_2}(s,u) = \int_0^{s\wedge u} \partial_s K_{H_1,H_2}(s,v)\partial_u K_{H_1,H_2}(u,v)dv.
\]

**Theorem 2.3.** Let \( H_2 - H_1 > \frac{1}{3} \). Then there exists a sequence \( T_n \to \infty \) such that integral equation (2.9) has unique solution \( h_T \) on any interval \([0,T_n]\) and \( h_{T_n}(\cdot)^{1/2-H_1} \in L_2[0,n] \).

**Proof.** We denote \( C_{H_1,H_2} \) constants which values are not so important; their values can change from line to line. At first, we can apply the changing of variables \( u = s + (t-s)z \) to (2.4) and transform the kernel \( K_{H_1,H_2}(t,s) \) from (2.4) to the following form:
\[
(2.11) \quad K_{H_1,H_2}(t,s) = \beta_{H_2}s^{\frac{1}{2}-H_2}(t-s)^{H_2-H_1} \int_0^1 (1-z)^{\frac{1}{2}-H_1}(s+(t-s)z)^{H_2-H_1}z^{\frac{1}{2}-H_2}dz.
\]
Then we can differentiate (2.11) and after inverse changing of variables we get that
\[
\frac{\partial_t K_{H_1,H_2}(t,s)}{t-s} = (H_2 - H_1) \left( \frac{K_{H_1,H_2}(t,s)}{t-s} \right)
\]
\[
+ \beta_{H_2}s^{\frac{1}{2}-H_2} \frac{1}{t-s} \int_0^t (t-r)^{\frac{1}{2}-H_1}r^{H_2-H_1-1}(r-s)^{H_2-H_1}dr.
\]
Further, we have the following bound for kernel \( K_{H_1,H_2}(t,s) \) on the interval \([0,T]\):
\[
(2.13) \quad 0 \leq K_{H_1,H_2}(t,s) \leq \beta_{H_2}B\left(\frac{3}{2} - H_1, H_2 - \frac{1}{2}\right)t^{H_2-H_1}s^{\frac{1}{2}-H_2}(t-s)^{H_2-H_1},
\]
and it follows from (2.12) and (2.13) that
\[
(2.14) \quad 0 \leq \partial_t K_{H_1,H_2}(t,s) \leq \beta_{H_2}s^{\frac{1}{2}-H_2}\left( B\left(\frac{3}{2} - H_1, H_2 - \frac{1}{2}\right)t^{H_2-H_1}(t-s)^{H_2-H_1-1} + B\left(\frac{3}{2} - H_1, H_2 + \frac{1}{2}\right)t^{H_2-H_1-1}(t-s)^{H_2-H_1} \right) \leq C_{H_1,H_2}s^{\frac{1}{2}-H_2}t^{H_2-H_1}(t-s)^{H_2-H_1-1}.
\]
Now we can substitute the bound from \((2.14)\) into \((2.17)\) and get that
\[
0 \leq k(s, u) \leq C_{H_1, H_2} u^{H_2 - H_1} s^{H_2 - H_1} \int_0^s v^{-2H_2} (u - v)^{H_2 - H_1 - 1} (s - v)^{H_2 - H_1 - 1} dv.
\]
Let, for example, \(s < u\). Note that
\[
(u - v)^{H_2 - H_1 - 1} = (u - v)^{H_2 + H_1 - 2} (u - v)^{1 - 2H_1} \leq (u - v)^{H_2 + H_1 - 2} (u - s)^{1 - 2H_1}.
\]
Then it follows from \((2.15)\) that
\[
0 \leq k(s, u) \leq C_{H_1, H_2} u^{H_2 - H_1} s^{H_2 - H_1} (u - s)^{1 - 2H_1}
\]
and it follows from \((2.18)\) that for \(s < u\)
\[
0 \leq k(s, u) \leq C_{H_1, H_2} s^{1 - 2H_1} (u - s)^{2H_2 - 2H_1 - 1}.
\]
Evidently, for \(u < s\)
\[
0 \leq k(s, u) \leq C_{H_1, H_2} u^{1 - 2H_1} (s - u)^{2H_2 - 2H_1 - 1}.
\]
Now we rewrite equation \((2.9)\) in the equivalent form
\[
\gamma_{H_1}^2 u^{1 - H_1} = \gamma_{H_1}^2 h(u) u^{1 - H_1} + \int_0^T h(s) s^{\frac{1}{2} - H_1} s^{H_1 - \frac{1}{2}} u^{\frac{1}{2} - H_1} k(s, u) ds,
\]
or
\[
\gamma_{H_1}^2 u^{1 - H_1} = \gamma_{H_1}^2 \tilde{h}_T(u) + \int_0^T \tilde{h}_T(s) k_1(s, u) ds,
\]
where \(k_1(s, u) = s^{H_1 - \frac{1}{2}} u^{\frac{1}{2} - H_1} k(s, u), \tilde{h}_T(u) = h(u) u^{\frac{1}{2} - H_1}\) and it follows from \((2.18)\) and \((2.19)\) that for \(s < u\)
\[
k_1(s, u) \leq C_{H_1, H_2} u^{H_1 - \frac{1}{2}} s^{\frac{1}{2} - H_1} (u - s)^{2H_2 - 2H_1 - 1}
\]
and for \(u < s\)
\[
k_1(s, u) \leq C_{H_1, H_2} s^{H_1 - \frac{1}{2}} u^{\frac{1}{2} - H_1} (s - u)^{2H_2 - 2H_1 - 1}.
\]
Therefore, taking into account that for $H_2 - H_1 > \frac{1}{4}$ we have that $4H_2 - 4H_1 - 2 > -1$, it is possible to bound $L_2[0, T]^2$ - norm of the kernel:

\[
\|k_1\|_{L_2[0, T]^2}^2 = \int_0^T \int_0^T k_1^2(s, u)dsdu = \int_0^T \int_0^u k_1^2(s, u)dsdu + \int_0^T \int_u^T k_1^2(s, u)dsdu
\]

\[
\leq C_{H_1, H_2} \left( \int_0^T \int_0^u u^{2H_1 - 1} s^{1-2H_1} (u-s)^{4H_2 - 4H_1 - 2} dsdu + \int_0^T \int_u^T s^{2H_1 - 1} u^{1-2H_1} (s-u)^{4H_2 - 4H_1 - 2} dsdu \right)
\]

\[
\leq C_{H_1, H_2} \left( \int_0^T u^{4H_2 - 4H_1 - 1} du + T^{2H_1 - 1} \int_0^T u^{1-2H_1} (T-u)^{4H_2 - 4H_1 - 1} du \right)
\]

\[
\leq C_{H_1, H_2} T^{4H_2 - 4H_1} < \infty.
\]

It means that the integral operator $K_T f(u) = \int_0^T k_1(s, u) f(u)du$ is compact linear self-adjoint operator from $L_2[0, T]$ into $L_2[0, T]$ and Fredholm alternative can be applied to equation (2.21). To avoid the question concerning eigenvalues and eigenfunctions, we produce the following trick.

It is very easy to see that for any $a > 0$

\[
K(ta, sa) = K(t, s)a^{\frac{1}{2} + H_2 - 2H_1}, \quad \partial_t K_{H_1, H_2}(ta, sa) = \partial_t K(t, s)a^{-\frac{1}{2} + H_2 - 2H_1},
\]

whence

\[
k_1(ta, sa) = k_1(t, s)a^{2H_2 - 2H_1 - 1}.
\]

Therefore we can put in equation (2.21) $s = s'T, u = u'T$ and $h_T(z) = \tilde{h}_T(Tz)$, and equation (2.21) will be reduced to the equivalent form (we omit superscripts)

\[
(uT)^{\frac{1}{2} - H_1} = \tilde{h}_T(u) + T^{2H_2 - 2H_1} \gamma_{H_1}^{-2} \int_0^1 \tilde{h}_T(s) k_1(s, u) ds = \tilde{h}_T(u)
\]

\[
+ \lambda \int_0^1 \tilde{h}_T(s) k_1(s, u) ds,
\]

with $\lambda = T^{2H_2 - 2H_1} \gamma_{H_1}^{-2}$. Since operator $K_1$ is compact linear self-adjoint operator from $L_2[0, T]$ into $L_2[0, T]$, as it was mentioned above, it has no more than countable number of eigenvalues any of them are real numbers, and with only one possible condensation point 0. Taking the sequence $T_n \to \infty$ in such a way that

\[
\lambda_n = T_n^{2H_1 - 2H_2} \gamma_{H_1}^{-2}
\]

will be not an eigenvalue, we get that equation (2.24) with $T_n$ as upper bound of integration has unique solution whence the proof follows. \(\square\)

Now we establish the form of maximum likelihood estimate.

**Theorem 2.4.** Let $H_2 - H_1 > \frac{1}{4}$. Then the likelihood function has a form

\[
L_T(X, \theta) = \exp\{\theta \delta_{H_1} N(T) - \frac{1}{2} \theta^2 \delta_{H_1}^2 \langle N \rangle(T)\},
\]

and maximum likelihood estimate has a form

\[
\hat{\theta}(T) = \frac{N(T)}{\delta_{H_1} \langle N \rangle(T)},
\]
where \( N(t) = E_0(X_1(t) | \overline{\mathcal{F}}_t^X) \) is a square integrable Gaussian \( \overline{\mathcal{F}}_t^X \)-martingale, \( N(T) = \int_0^T h_T(t) dX(t) \) with \( h_T(t) \in L_2[0, T] \), \( h_T(t) \) be a unique solution to (2.29) and \( \langle N \rangle(T) = \gamma_{\text{H}} \int_0^T h_T(t) t^{1-2H_1} dt \).

Proof. We start with (2.22). Consider Gaussian process \( N(t) = E_0(X_1(t) | \overline{\mathcal{F}}_t^X) \).
Since \( X_1(t) \) is \( \overline{\mathcal{F}}_t \)-martingale and \( \overline{\mathcal{F}}_t^X \subset \overline{\mathcal{F}}_t \), the process \( N \) is a \( \mathcal{F}_t^X \)-martingale with respect to probability measure \( P_0 \). Furthermore, we can present \( V(t) \) as \( V(t) = E_0(X_1^2(t)) | \overline{\mathcal{F}}_t^X) - N^2(t) \). Note that \( X_1^2(t) - \frac{t^{2-2H}}{2-2H} \) is \( \mathcal{F}_t \)-martingale. Therefore,
\[
E_0 \left( N^2(t) - \left( \frac{t^{2-2H}}{2-2H} - V(t) \right) | \overline{\mathcal{F}}_t^X \right) = E_0 \left( E_0(X_1^2(t) | \overline{\mathcal{F}}_t^X) - \frac{t^{2-2H}}{2-2H} | \overline{\mathcal{F}}_t^X \right) \]
\[
= E_0 \left( X_1^2(t) - \frac{t^{2-2H}}{2-2H} | \overline{\mathcal{F}}_t^X \right) = E_0(\langle N \rangle(s)) - \frac{s^{2-2H}}{2-2H} \]
and this means that the quadratic variation of the martingale \( N \) equals \( \langle N \rangle(t) = t^{2-2H} - V(t) \), and the likelihood ratio is reduced to
\[
\langle N \rangle(T) = E_0(N^2(T)) = E_0 \left( \int_0^T h_T(t) dX(t) \right)^2 = E_0 \left( \int_0^T h_T(t) d(X_1(t) + X_2(t)) \right)^2 \]
\[
= E_0 \left( \int_0^T h_T(t) dX_1(t) \right)^2 + E_0 \left( \int_0^T h_T(t) dX_2(t) \right)^2 = \gamma_{\text{H}} \int_0^T h_T^2(t) t^{1-2H_1} dt \]
\[
+ \int_0^T \int_t^T h_T(u) \partial_u K_{H_1, H_2}(u, t) du \int_t^T h_T(s) \partial_s K_{H_1, H_2}(s, t) ds dt \]
\[
= \gamma_{\text{H}}^2 \int_0^T h_T^2(t) t^{1-2H_1} dt + \int_0^T h_T(u) h_T(s) \int_0^{\hat{\mathcal{K}_\alpha}} \partial_s K_{H_1, H_2}(s, t) \partial_u K_{H_1, H_2}(u, t) dtdsdu \]
\[
= \gamma_{\text{H}}^2 \int_0^T h_T(t) t^{1-2H_1} dt, \]
whence the proof follows. \( \square \)

In what follows, saying “\( T \to \infty \)” we have in mind that the corresponding property holds for any sequence \( T_n \to \infty \) that has only finite common points with the sequence of eigenvalues of operator \( K_1 \). Proof of the following result repeats the proof of the corresponding statements from [14] so is omitted.

**Theorem 2.5.** The estimator \( \hat{\theta}_T \) is unbiased and the corresponding estimation error is normal
\[
\hat{\theta}_T - \theta \sim N \left( 0, \frac{1}{\int_0^T h_T(s) s^{1-2H_1} ds} \right) .
\]

Now we establish the asymptotic behavior of the estimator.
Theorem 2.6. Let $H_2 - H_1 > \frac{1}{2}$. Estimator $\hat{\theta}_T$ is strongly consistent and

$$
\lim_{T \to \infty} T^{2-2H_2} E_\theta (\hat{\theta}_T - \theta)^2 = \frac{1}{\int_0^1 h_0(u)u^{1-H_1}du}.
$$

**Proof.** At first we rewrite equation (2.21) in the equivalent form, changing $u = u'T, s = s'T$ and omitting superscripts:

(2.29) $\gamma_{H_1}^2 u^{1-H_1} T^{1-H_1} = \gamma_{H_1}^2 \tilde{h}_T(uT) + T^{2H_2-2H_1} \int_0^1 \tilde{h}_T(sT) k_1(s, u)ds,$

or

(2.30) $\gamma_{H_1}^2 u^{1-H_1} = \gamma_{H_1}^2 \tilde{h}_T(uT) T^{H_1-1/2} + T^{2H_2-2H_1} \int_0^1 \tilde{h}_T(sT) T^{H_1-1/2} k_1(s, u)ds,$

Denote $\mu = T^{2H_2-2H_1}$. Let $h_\mu(u) = \mu \tilde{h}_T(uT) T^{H_1-1/2}$. Then, taking into account (2.23), equation (2.29) can be rewritten as

(2.31) $\gamma_{H_1}^2 u^{1-H_1} = \frac{1}{\mu} \gamma_{H_1}^2 h_\mu(u) + \int_0^1 h_\mu(s) k_1(s, u)ds.$

Note that

$$
\langle N \rangle(T) = \int_0^T h_T(s)s^{1-2H_1}ds = \int_0^T \tilde{h}_T(s)s^{1-H_1}ds = T^{2-2H_2} \int_0^1 h_\mu(u)u^{1-H_1}du.
$$

Define the operator $K$

$$(Kf)(u) = \int_0^1 f(s)k_1(s, u)ds$$

and the scalar product $\langle f, g \rangle = \int_0^1 f(s)g(s)ds, f, g \in L_2[0, 1]$. Then equation (2.31) can be rewritten as

(2.32) $\gamma_{H_1}^2 u^{1-H_1} = \frac{1}{\mu} \gamma_{H_1}^2 h_\mu(u) + Kh_\mu(u).$

Note that

(2.33)

$$
\langle Kf, f \rangle = \int_0^1 (Kf)(t)f(t)dt = \int_0^1 (\int_0^1 f(s)k_1(t, s)ds)f(t)dt
$$

$$
= \int_0^1 \int_0^1 f(t)t^{H_1-1/2} f(s)s^{H_1-1/2} \int_0^s t^{2H_1-1/2} \partial_s K_{H_1, H_2}(s, v)\partial_t K_{H_1, H_2}(t, v)dvdsdt
$$

$$
= \int_0^1 dv \int_0^1 \partial_s K_{H_1, H_2}(s, v) f(s)s^{H_1-1/2}ds \int_v^1 \partial_t K_{H_1, H_2}(t, v)f(t)t^{H_1-1/2}dt \geq 0.
$$

Introduce corresponding the first type auxiliary integral equation

(2.34) $\gamma_{H_1}^2 u^{1-H_1} = (Kh)(u).$

It follows from Lemma 3.1 that (2.31) has the unique solution, say, $h_0$, obviously, not depending on $\mu$. The function $\delta_\mu = h_\mu - h_0$ satisfies two equations $K\delta_\mu + \frac{1}{\mu} \gamma_{H_1}^2 h_\mu = 0$ and $K\delta_\mu + \frac{1}{\mu} \gamma_{H_1}^2 \delta_\mu = -\frac{h_0}{\mu}$. Multiplying the 2nd equation by $\delta_\mu$ and integrating, we get

(2.35) $\langle K\delta_\mu, \delta_\mu \rangle + \frac{1}{\mu} \gamma_{H_1}^2 \|\delta_\mu\|^2 = \frac{1}{\mu} \langle h_0, \delta_\mu \rangle,$
and it follows from (2.35) and (2.33) that $\gamma_{H_1}^2 \Vert \delta_\mu \Vert^2 \leq |\langle h_0, \delta_\mu \rangle| \leq \Vert h_0 \Vert \Vert \delta_\mu \Vert$, which implies that $\Vert \delta_\mu \Vert \leq \Vert h_0 \Vert$. Multiplying the 1st equation by $h_0$ and integrating we get

$$\langle K\delta_\mu, h_0 \rangle + \frac{1}{\mu} \gamma_{H_1}^2 \langle h_\mu, h_0 \rangle = 0.$$  

Note that inequality $\Vert \delta_\mu \Vert \leq \Vert h_0 \Vert$ implies that $|\langle h_\mu, h_0 \rangle| \leq |\langle \delta_\mu, h_0 \rangle| + \Vert h_0 \Vert^2 \leq 2 \Vert h_0 \Vert^2 < \infty$, and hence

$$\gamma_{H_1}^2 |\langle \delta_\mu, u^{1/2-H_1} \rangle| = |\langle \delta_\mu, Kh_0 \rangle| = |\langle K\delta_\mu, h_0 \rangle| = \frac{1}{\mu} \gamma_{H_1}^2 \langle h_\mu, h_0 \rangle \to 0$$

as $T \to \infty$. It means that $\lim_{T \to \infty} \int_0^1 h_\mu(u)u^{1/2-H_1}du = \int_0^1 h_0(u)u^{1/2-H_1}du$. Therefore

$$T^{2-2H_2} E_\theta(\hat{\theta}_T - \theta)^2 = \frac{1}{\int_0^1 h_\mu(u)u^{1/2-H_1}du} \to \frac{1}{\int_0^1 h_0(u)u^{1/2-H_1}du},$$

whence the proof follows.

□

Remark 2.7. In outline, our method of proof follows the method of the corresponding result from [4], however Lemma 3.1 is specific to our case.

3. Appendix

We recall some notions from fractional calculus. For the details see [12]. Fractional integrals are defined as

$$(I^\alpha_+ f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}dt$$

and

$$(I^\alpha_- f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1}dt,$$

while fractional derivatives are defined as

$$(D^\alpha_+ f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x f(t)(x-t)^{-\alpha}dt$$

and

$$(D^\alpha_- f)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b f(t)(t-x)^{-\alpha}dt.$$  

Fractional differentiation and integration are inverse operators on the appropriate functional classes. Also, we shall use the following integration by parts formula for fractional derivatives,

$$\int_a^b (D^\alpha_+ f)(x)g(x)dx = \int_a^b f(x)(D^\alpha_- g)(x)dx.$$

Lemma 3.1. For any constant $C > 0$ integral equation

$$u^{1/2-H_1} = C(Kh)(u), \quad u \in (0, 1)$$

of the 1st kind has the unique solution.
Proof. We can present equation (3.1) in equivalent form
\[ u^{1/2-H_1} = C \int_0^1 h(s)k_1(s,u)ds, \quad u \in (0,1), \]
or
\[ u^{1-2H_1} = C \int_0^1 \tilde{h}(s) \int_0^{s/u} \partial_s K_{H_1,H_2}(s,v)\partial_u K_{H_1,H_2}(u,v)dvds, \quad u \in (0,1), \]
where \( \tilde{h}(s) = h(s)s^{1/2-H_1} \), or, at last,
\[ u^{1-2H_1} = C \int_0^u \left( \int_0^1 \tilde{h}(s)\partial_s K_{H_1,H_2}(s,v)ds \right)\partial_u K_{H_1,H_2}(u,v)dv. \]

Now, taking into account the transition from equation (2.8) to (2.9) with the help of differentiation, we can perform the inverse operation and get from (3.2) the following equivalent equation
\[ u^{2-2H_1} = C(2 - 2H_1) \int_0^u K_{H_1,H_2}(u,v) \left( \int_0^1 \tilde{h}(s)\partial_s K_{H_1,H_2}(s,v)ds \right)dv, \quad u \in [0,1]. \]
The right-hand side of equation (3.3) can be rewritten as
\[ C(2 - 2H_1) \int_0^u K_{H_1,H_2}(u,v)q(v)dv, \]
where \( q(v) = \int_v^1 \tilde{h}(s)\partial_s K_{H_1,H_2}(s,v)ds \). At first, solve the equation
\[ u^{2-2H_1} = C_1 \int_0^u K_{H_1,H_2}(u,v)q(v)dv, \]
with \( C_1 = C(2 - 2H_1) \). Taking into account (2.4), the latter equation can be rewritten in equivalent form
\[ u^{2-2H_1} = C_1 \beta_{H_2} \int_0^u v^{1/2-H_2} \int_v^u (u - z)^{1/2-H_1}z^{H_2-H_1}(z - v)^{H_2-3/2}dzq(v)dv, \]
or
\[ u^{2-2H_1} = C_1 \beta_{H_2} \int_0^u z^{H_2-H_1}(u - z)^{1/2-H_1} \int_0^z v^{1/2-H_2}(z - v)^{H_2-3/2}q(v)dvdz, \]
or, at last,
\[ u^{2-2H_1} = C_2 \beta_{H_2} \Gamma(3/2 - H_2) \]and
\[ p(z) = z^{H_2-H_1} \int_0^z v^{1/2-H_2}(z - v)^{H_2-3/2}q(v)dv. \]
It means that
\[ p(u) = C_2^{-1}(D_{0+}^{3/2-H_1}(u^{2-2H_1}))(u) \]
\[ = (C_2\Gamma(H_1 - 1/2))^{-1} \left( \int_0^u (u - t)^{H_1-3/2}t^{2-2H_1}dt \right) = C_3 u^{1/2-H_1}, \]
where \( C_3 = \frac{(3/2-H_1)\Gamma(H_1-1/2)}{C_2\Gamma(H_1-1/2)} \). Furthermore, comparing (3.4) and (3.5), we get that
\[ C_4 z^{1/2-H_2} = \int_0^z v^{1/2-H_2}(z - v)^{H_2-3/2}q(v)dv = \Gamma(H_2 - 1/2)(I_{0+}^{H_2-1/2}(1/2-H_2q))(z), \]
whence
\[ v^{1/2-H_2} q(v) = C_3 (\Gamma(H_2 - 1/2))^{-1}(D_{0+}^{H_2-1/2,1/2-H_2})(v) \]
\[ = C_3 (\Gamma(H_2 - 1/2)\Gamma(3/2 - H_2))^{-1} \left( \int_0^v (v-t)^{1/2-H_2} t^{1/2-H_2} dt \right)_v \]
\[ = C_4 v^{1-2H_2}, \]
where \( C_4 = C_3 (2 - 2H_2)(\Gamma(H_2 - 1/2)\Gamma(3/2 - H_2))^{-1} \). Obviously, \( q(v) = C_4 v^{1/2-H_2} \), and we arrive at the equation
\[
(3.6) \quad C_4 v^{1/2-H_2} = \int_0^1 \tilde{h}(s) \partial_s K_{H_1,H_2}(s,v) ds.
\]
Note that
\[ \partial_s K_{H_1,H_2}(s,v) = \beta H_2 \Gamma(3/2 - H_1) v^{1/2-H_2} (D_{v+}^{H_1-1/2} (H_2-H_1 (\cdot - v)^{H_2-3/2}))(s), \]
so, with the help of integration by parts formula, equation (3.6) can be rewritten as
\[
(3.7) \quad C_5 = \int_0^1 \tilde{h}(s) \left( D_{v+}^{H_1-1/2} (H_2-H_1 (\cdot - v)^{H_2-3/2}) \right)(s) ds
\]
\[ = \left( (\Gamma(H_2 - 1/2))^{-1}(I_{1-}^{H_2-1/2} (H_1-1/2\tilde{h})(H_2-H_1))(v), \right) \]
where \( C_5 = C_4 (\beta H_2 \Gamma(3/2 - H_1))^{-1} \). The latter equation means that
\[ \left( D_{1-}^{H_1-1/2} \tilde{h} \right)(v) v^{H_2-H_1} = C_6 (1 - v)^{1/2-H_1}, \]
\[ C_6 = \frac{C_5 \Gamma(H_1-1/2)}{\Gamma(3/2-H_1)}. \]
At last, we get that
\[ h(v) = v^{H_1-1/2} \tilde{h}(v) = C_6 v^{H_1-1/2} (I_{1-}^{H_1-1/2} (H_1-H_2 (1 - \cdot)^{1/2-H_1}))(v), \]
and this solution of equation (3.1) is unique. \( \square \)

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