NORMALIZED GROUND STATES OF THE NONLINEAR SCHRÖDINGER EQUATION WITH AT LEAST MASS CRITICAL GROWTH

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ABSTRACT. We propose a simple minimization method to show the existence of least energy solutions to the normalized problem

\[
\begin{cases}
-\Delta u + \lambda u = g(u) & \text{in } \mathbb{R}^N, \ N \geq 3, \\
u \in H^1(\mathbb{R}^N), \\
\int_{\mathbb{R}^N} |u|^2 \, dx = \rho > 0,
\end{cases}
\]

where \(\rho\) is prescribed and \((\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^N)\) is to be determined. The new approach based on the direct minimization of the energy functional on the linear combination of Nehari and Pohozaev constraints intersected with the closed ball in \(L^2(\mathbb{R}^N)\) of radius \(\rho\) is demonstrated, which allows to provide general growth assumptions imposed on \(g\). We cover the most known physical examples and nonlinearities with growth considered in the literature so far as well as we admit the mass critical growth at 0.

Keywords: nonlinear scalar field equations, normalized ground states, nonlinear Schrödinger equations, Nehari manifold, Pohozaev manifold

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1. Introduction

In this paper we are looking for solutions to the following nonlinear Schrödinger problem

\[
\begin{cases}
-\Delta u + \lambda u = g(u) & \text{in } \mathbb{R}^N, \ N \geq 3, \\
u \in H^1(\mathbb{R}^N), \\
\int_{\mathbb{R}^N} |u|^2 \, dx = \rho > 0,
\end{cases}
\]

where \(\rho\) is prescribed and \((u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}\) has to be determined.

The following time-dependent, nonlinear Schrödinger equation

\[
\begin{cases}
i \frac{\partial \Psi}{\partial t}(t, x) = \Delta_x \Psi(t, x) + h(|\Psi(t, x)|)\Psi(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |\Psi(t, x)|^2 \, dx = \rho
\end{cases}
\]

with prescribed mass \(\sqrt{\rho}\) appears in nonlinear optics and the theory of Bose-Einstein condensates (see \([1, 12, 13, 18, 30]\)). Solutions \(u\) to (1.1) correspond to standing waves \(\Psi(t, x) = e^{-i\lambda t}u(x)\) of the foregoing time-dependent equation. The prescribed mass represents the power supply in nonlinear optics or the number of particles in Bose-Einstein condensates.

Let us denote

\[
S := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx = \rho \right\},
\]

where \(H^1(\mathbb{R}^N)\) is endowed with the usual norm \(\|u\| = (|\nabla u|^2 + |u|^2)^{1/2}\) and \(|\cdot|_q\) stands for the \(L^q\)-norm. Under suitable assumptions provided below, solutions to (1.1) are critical points...
of the energy functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx,$$

where $G(u) := \int_0^u g(s) \, ds$, on the constraint $\mathcal{S}$ with a Lagrange multiplier $\lambda \in \mathbb{R}$, i.e. they are critical points of the following functional

$$H^1(\mathbb{R}^N) \ni u \mapsto J(u) + \frac{\lambda}{2} \int_{\mathbb{R}^N} |u|^2 \, dx \in \mathbb{R}$$

with some $\lambda \in \mathbb{R}$. Recall that any critical point of the above functional lies in $W^{2,q}_{loc}(\mathbb{R}^N)$ for all $q < \infty$ and satisfies the following Pohozaev identity

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = 2^* \int_{\mathbb{R}^N} G(u) - \frac{\lambda}{2} |u|^2 \, dx$$

[9, 15, 20, 22]. On the other hand, all nontrivial critical points lie in the corresponding Nehari manifold, i.e.

$$J'(u)(u) + \lambda \int_{\mathbb{R}^N} |u|^2 \, dx = 0$$

and combining these two identities one can easily compute that any nontrivial solution satisfies

$$M(u) := \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{N}{2} \int_{\mathbb{R}^N} H(u) \, dx = 0,$$

where $H(u) := g(u)u - 2G(u)$, see e.g. [15]. Therefore we consider the following constraint

$$\mathcal{M} := \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : M(u) = 0 \},$$

which contains any nontrivial solution to (1.1). In our approach we also consider

$$\mathcal{D} := \left\{ u \in H^1(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} |u|^2 \, dx \leq \rho \right\}$$

and note that any nontrivial, (normalized) solution to (1.1) belongs to $\mathcal{S} \cap \mathcal{M} \subset \mathcal{D} \cap \mathcal{M}$. By a normalized ground state solution to (1.1) we mean a nontrivial solution minimizing $J$ among all nontrivial solutions. In particular, if $u$ solves (1.1) and $J(u) = \inf_{\mathcal{S} \cap \mathcal{M}} J$, then $u$ is a normalized ground state solution.

Recall that, in the case of the pure power nonlinearity

$$(1.2) \quad G(u) = \frac{1}{p} |u|^p,$$

the problem can be treated using variational methods available for the problem with fixed $\lambda > 0$ and by the scaling-type argument. This approach fails in the case of nonhomogeneous nonlinearities. In the $L^2$-subcritical case, i.e. where $G$ has growth $|u|^p$ with $2 < p < 2_* := 2 + \frac{4}{N}$, one can use a minimization on the $L^2$-sphere $\mathcal{S}$ in $H^1(\mathbb{R}^N)$ in order to obtain the existence of a global minimizer [17, 26]. In $L^2$-critical ($p = 2_*$) and $L^2$-supercritical and Sobolev-subcritical ($2_* < p < 2^* := \frac{2N}{N-2}$) cases the minimization on the $L^2$-sphere does not work, if $p > 2_*$ in (1.2), then $\inf_{\mathcal{S}} J = -\infty$, and this work is concerned with this problem. Our aim is to impose general growth condition on $g$ in the spirit of Berestycki and Lions [9, 10] and provide a new approach to study normalized ground state solution to (1.1) and similar elliptic problems.
Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth

We would like to mention that Jeanjean [15], Bartsch and Soave [4, 5], considered the problem (1.1) with the nonlinear term satisfying the following Ambrosetti-Rabinowitz-type condition that there are \( a > \frac{4}{N} \) and \( b < 2^* - 2 \) such that

\[
0 < aG(u) \leq H(u) \leq bG(u) \quad \text{for } u \in \mathbb{R} \setminus \{0\}.
\]

In [15] the solution has been found via the mountain pass argument, and in [4, 5] the authors provided a mini-max approach in \( M \) based on the \( \sigma \)-homotopy family of compact subsets of \( M \) and the minimax principle [14]. The multiplicity of solutions to (1.1) has been considered also in [3] under the condition (1.3). We would like to point out that the analysis of \( L^2 \)-mass supercritical problems and recovering the compactness of Palais-Smale sequences is usually hard, since, for instance, the embedding of radial functions \( H^1_{rad}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \) is not compact and the argument is quite involved in \( H^1_{rad}(\mathbb{R}^N) \), see e.g. [4–6, 15]. Another strategy to obtain the compactness is to show that the ground state energy map (1.8) is nonincreasing respect to \( \rho \) and strictly decreasing for some \( \rho \), see e.g. [8, 16].

In our approach we do not work in \( H^1_{rad} \), the monotonicity of the ground state energy map (1.8) is not required to deal with the lack of compactness and we do not need to work with Palais-Smale sequences, so that we avoid the mini-max approach in \( M \) involving a strong topological argument as in [4, 5, 14, 16].

We work only with a minimizing sequence of \( J \) on \( D \cap M \) as we shall see later, and a wide class of nonlinearities is considered. Moreover, if \( f \) is odd and \( u \in D \cap M \), then the projection given by (2.3) of the Schwartz symmetrization \( |u|^\sigma \) of \( |u| \) onto \( M \) remains in \( D \cap M \) and we do not encounter difficulties concerning the radial symmetry appearing on the sphere \( S \). In comparison to a very recent and interesting work [16] by Jeanjean and Lu, we require that \( H \) is of \( C^1 \)-class, however our growth conditions are more general, in particular we assume a version of (1.3) with \( a = \frac{4}{N} \) and \( b = 2^* - 2 \), which admits \( L^2 \)-growth at 0. Moreover, the strict monotonicity of (1.8) is just a simple consequence of our approach, see Step 4 below.

In order to state our assumptions, we recall the optimal constant \( C_{N,p} > 0 \) in the Gagliardo-Nirenberg inequality

\[
|u|_p \leq C_{N,p} |\nabla u|^\delta_2 |u|^{1-\delta}_2 \leq C_{N,p} \rho^{\frac{4}{p_2}} |\nabla u|^{\delta}_2, \quad \text{for } u \in H^1(\mathbb{R}^N),
\]

where \( \delta = N \left( \frac{1}{2} - \frac{1}{p} \right) \).

Given functions \( f_1, f_2 : \mathbb{R} \to \mathbb{R} \). We introduce the following notation: \( f_1(s) \preceq f_2(s) \) for \( s \in \mathbb{R} \) provided that \( f_1(s) \leq f_2(s) \) for all \( s \in \mathbb{R} \) and for any \( \gamma > 0 \) there is \( |s| < \gamma \) such that \( f_1(s) < f_2(s) \). An important property of the relation \( \preceq \) is given in Lemma 2.1.

Let us consider the following assumptions:

(A0) \( g \) and \( h := H' \) are continuous and there is \( c > 0 \) such that

\[
|h(u)| \leq c(|u| + |u|^{2^*-1}) \quad \text{for } u \in \mathbb{R}.
\]

(A1) \( \eta := \lim \sup_{|u| \to 0} G(u)/|u|^{2^*+\frac{4}{N}} < \infty \).

(A2) \( \lim_{|u| \to \infty} G(u)/|u|^{2^*} = \infty \).

(A3) \( \lim_{|u| \to \infty} G(u)/|u|^{2^*} = 0 \).

(A4) \( (2 + \frac{4}{N}) H(u) \leq h(u)u \quad \text{for } u \in \mathbb{R} \).

(A5) \( \frac{4}{N} G(u) \leq H(u) \leq (2^* - 2)G(u) \quad \text{for } u \in \mathbb{R} \).

(A6) \( H(\zeta_0) > 0 \) for some \( \zeta_0 \neq 0 \).
Note that (A0) implies that $J$ and $M$ are of class $C^1$. Moreover, assuming in addition (A2) and (A5), $G(u) > 0$ and $H(u) > 0$ for $u \neq 0$, in particular (A6) holds. Indeed, in view of (A5) $(G(u)/u^2)' \geq 0$ and $(G(u)/u^2)' \leq 0$ for $u > 0$, thus

\[
u^2 G(1) \leq G(u) \leq u^2 G(1), \quad \text{if } 0 < u < 1,
\]

\[
u^2 G(1) \leq G(u) \leq u^2 G(1), \quad \text{if } u \geq 1.
\]

Moreover, (A2) and (A5) imply that $G(1) > 0$, hence $G(u) > 0$ and $H(u) > 0$ for $u > 0$. We argue similarly if $u < 0$. Then $\eta \geq 0$ if (A1) holds. Now we show that $\mathcal{M}$ is a nonempty $C^1$-manifold, since $\mathcal{M}'(u) \neq 0$ for $u \in \mathcal{M}$, cf. [22]. Indeed, if $\mathcal{M}'(u) = 0$, then $u$ solves $-\Delta u = \frac{\nu}{4} h(u)$ and satisfies the Pohozaev identity $\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = 2^* \frac{\nu}{4} \int_{\mathbb{R}^N} H(u) \, dx$. Since $u \in \mathcal{M}$, we infer that $u = 0$.

Observe that (A1) admits $L^2$-critical growth of $G$ close to 0, however (A2) excludes the pure $L^2$-critical case, e.g. (1.2) with $p = 2$. Moreover (A3) excludes (1.2) with $p = 2^*$. In our considerations we will also consider more restrictive inequalities $\leq_1 \in (A4)$ and (A5) denoted by $(A4, \leq)$ and $(A5, \leq)$ respectively. Note that $(A5, \leq)$ is a weaker variant of (1.3). $(A5, \leq)$ plays an important role in Lemma 2.8.

Note that if (A1) holds with $\eta = 0$ and (A2) is satisfied, and the inequality in (A4) is strict for $u \neq 0$, then $\frac{\nu}{4} G(u) < H(u)$ for $u \neq 0$ according to [16, Lemma 2.3] and we cover the growth conditions considered recently in [16]. Finally, if (A5) holds with the first inequality replaced by $\leq$, then (A6) is clearly satisfied. If $\eta = 0$, then arguing similarly as in [16, Lemma 2.3], we can show that (A0), (A2) and (A4, $\leq$) imply that $\frac{\nu}{4} G(u) \leq H(u)$ for $u \in \mathbb{R}$.

We recall the following definition of radial symmetry with respect to an affine subspace, cf. [19]. Fix an affine subspace $V$ of $\mathbb{R}^N$ and a function $u : \mathbb{R}^N \to \mathbb{R}$. Let $p_V : \mathbb{R}^N \to V$ denote the projection onto $V$. We say that $u$ is radially symmetric with respect to $V$ if there is $\tilde{u} : V \times [0, \infty) \to \mathbb{R}$ such that $u(x) = \tilde{u}(p_V(x), |x - p_V(x)|)$ for all $x \in \mathbb{R}^N$. If, in particular, $V = \{0\}$ then $u$ is radially symmetric.

The main result reads as follows.

**Theorem 1.1.** Suppose that (A0)–(A5) are satisfied and

\[(1.5) \quad 2^* \eta C_{N,2}^2 \rho^{\frac{2}{N}} < 1\]

holds. Then there is $u \in \mathcal{D} \cap \mathcal{M}$ such that

\[(1.6) \quad J(u) = \inf_{\mathcal{D} \cap \mathcal{M}} \ J > 0, \]

and if, in addition, $g$ is odd, then $u$ is radially symmetric. Suppose that $(A5, \leq)$ is satisfied.

(a) If $g(s) = o(1)$ as $s \to 0$, then $\inf_{\mathcal{D} \cap \mathcal{M}} J = \inf_{\mathcal{S} \cap \mathcal{M}} J$ and $u \in \mathcal{S} \cap \mathcal{M}$ is a normalized ground state solution to (1.1). Moreover $u$ is radially symmetric with respect to some one-dimensional affine subspace $V$ in $\mathbb{R}^N$.

(b) If $g$ is odd, then $\inf_{\mathcal{D} \cap \mathcal{M}} J = \inf_{\mathcal{S} \cap \mathcal{M}} J$ and $u \in \mathcal{S} \cap \mathcal{M}$ is a positive and radially symmetric normalized ground state solution to (1.1). If $N \in \{3, 4\}$, then it is sufficient to assume only that $H(s) \leq (2^* - 2)G(s)$ for $s \in \mathbb{R}$ in (A5).

In order to illustrate Theorem 1.1 we provide the following examples and properties with regard to our assumptions.
(E1) Suppose that $g$ satisfies (A0)–(A5) and $g$ is odd, e.g. (1.2) with $2_* < p < 2^*$. Then $g$ is of class $C^1$ on $(-\infty, 0) \cup (0, \infty)$ and note that $g'(\zeta) > 0$ for some $\zeta > 0$. Assume for simplicity that $\zeta = 1$. We define $\tilde{g} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{g}(0) = 0$ and

$$\tilde{g}'(s) = \begin{cases} g'(1)|s|^{2^*-2} & |s| \leq 1 \\ g'(s) & |s| > 1, \end{cases}$$

Then $\tilde{G}(u) = \int_0^u \tilde{g}(s) \, ds$ and $\tilde{H}(u) := \tilde{g}(u)u - 2\tilde{G}(u)$ satisfy (A0)–(A5). If, in addition, $g$ satisfies (A4, $\preceq$) or $\frac{i}{\tilde{g}}\tilde{G}(u) \preceq H(u)$ for $u \in \mathbb{R}$, then $\tilde{g}$ satisfies the analogous assumptions, and Theorem 1.1 (b) applies to both $g$ and $\tilde{g}$ provided that $N \in \{3, 4\}$.

(E2) Let $M > 0$ and consider a sequence of disjoint and closed intervals $(I_j)_{j=1}^{\infty}$ in $(0, M)$ such that $\sup I_{j+1} < \inf I_j$ for all $j \geq 1$. Take any decreasing sequence of positive numbers $(a_j)_{j=1}^{\infty}$ and we define $g'(s) = a_j |s|^{2_*-2}$ for $|s| \in I_j$, $j \geq 1$ and $g'(s) = C|s|^{p-2}$ for $|s| \geq M$, $2_* < p < 2^*$ and properly chosen $C > 0$. We extend $g'(s)/|s|^{2^*-2}$ linearly on $\mathbb{R}$ to a continuous and even function. Note that (A0)–(A3), (A4, $\preceq$) and (A5, $\preceq$) are satisfied with $\eta = (2_*(2_* - 1))^{-1} \lim_{j \to \infty} a_j$.

(E3) Suppose that $g$ satisfies (A0)–(A5) and $g$ is odd. Similarly as in (E1) we find an interval $[a, b] \subset (0, \infty)$ such that $g'(\zeta) > 0$ for $\zeta \in [a, b]$. Assume for simplicity that $a = 1$. We define $\tilde{g} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{g}(0) = 0$ and

$$\tilde{g}'(s) = \begin{cases} g'(s) & |s| < 1, \\ \frac{g'(1)|s|^{2^*-2} - g'(b)}{g'(b)} & 1 \leq |s| \leq b, \\ \frac{g'(1)|s|^{2^*-2}}{g'(b)} & |s| > b, \end{cases}$$

Then $\tilde{G}(u) = \int_0^u \tilde{g}(s) \, ds$ and $\tilde{H}(u) := \tilde{g}(u)u - 2\tilde{G}(u)$ satisfy (A0)–(A5). If, in addition, (A4, $\preceq$) or (A5, $\preceq$) holds for $g$, then the same condition holds for $\tilde{g}$.

(E4) Suppose that $g$ satisfies (A0)–(A5) with some $\eta$ in (A1). Then $\tilde{G}(u) = \mu|u|^{2_*} + G(u)$, $\mu \geq 0$ and $\tilde{H}(u) := \tilde{g}(u)u - 2\tilde{G}(u)$ satisfy (A0)–(A5) with $\mu + \eta$ in (A1). If, in addition, (A4, $\preceq$) or (A5, $\preceq$) holds for $g$, then the same condition holds for $\tilde{g}$. In particular, we can deal with $\mu|s|^{2^*-2}u + |s|^{p-2}u$, $2_* < p < 2^*$ as in [24, Theorem 1.6].

Now we sketch our strategy to find normalized ground state solutions to (1.1). We believe that the following procedure can be applied to similar variational problems with different differential operators. Contrary to previous works we consider the minimization problem on the closed $L^2$-ball in $H^1(\mathbb{R}^N)$ of radius $\rho$ (instead of the sphere $S$) intersected with $\mathcal{M}$.

**Step 1.** We show that $J$ is bounded away from 0 on $\mathcal{D} \cap \mathcal{M}$. Here the Gagliardo-Nirenberg inequality (1.4) as well as (1.5) play an important role.

**Step 2.** $J$ is coercive on $\mathcal{D} \cap \mathcal{M}$. Here (A4) and the weak monotonicity of $H(u)/|u|^{2_*}$ is important. We adapt some ideas of [16, 28], however we do not require the existence of the continuous projection of $H^1(\mathbb{R}^N) \setminus \{0\}$ onto $\mathcal{M}$ preserving the $L^2$-norm, so the argument is more delicate.

**Step 3.** If $(u_n) \subset \mathcal{D} \cap \mathcal{M}$ is a minimizing sequence, then by means of the profile decomposition Theorem 2.6 ([20, Theorem 1.4]) we may find a sequence of translations $(y_n) \subset \mathbb{R}^N$ such that $u_n(\cdot + y_n)$ weakly and a.e. converges to a minimizer $u$ of $J$ on $\mathcal{D} \cap \mathcal{M}$. Here a standard one-step concentration-compactness approach in the spirit of Lions [17] seem to be insufficient, since $u$ may be outside $\mathcal{M}$. We need to find a full (possibly infinite) decomposition of $(u_n)$ in order to find a weak limit point in $\mathcal{M}$ up to a
proper translations. If \( g \) is odd, then working on the ball \( D \) allows us to use easily
the Schwartz symmetrization and we infer that we may find nonnegative and radially
symmetric minimizer. The symmetrization approach directly on \( S \cap M \) seems to be
cumbersome even for the simplest particular nonlinearities (1.2) as in [4–7, 24].

**Step 4.** Next we show that for \( v \in (D \setminus S) \cap M \) the following crucial inequality holds
\[
\inf_{S \cap M} J < J(v),
\]
thus the minimizer \( u \) of \( J \) on \( D \cap M \) is attained in fact in \( S \cap M \).

**Step 5.** Analysis of Lagrange multipliers \( \lambda \) and \( \mu \) for constraints \( S \) and \( M \) respectively,
implies that \( \mu = 0 \) and we conclude that \( u \) is a normalized ground state solution to
(1.1).

Observe that, an important consequence of **Step 4** is that the
ground state energy map
\[
\rho \mapsto \inf_{S \cap M} J
\]
defined for \( \rho > 0 \) satisfying (1.5) is strictly decreasing. For a particular power-type nonli-
earity and the Schrödinger-Poisson problem in \( \mathbb{R}^3 \), the monotonicity has been investigated in [8].
For more general nonlinearity in [16] the authors also proved the strict monotonicity of the
ground state energy map. Their proof is technical and uses the existence of the continuous
projection of \( H^1(\mathbb{R}^N) \setminus \{0\} \) onto \( M \) preserving the \( L^2 \)-norm, which seems to be not present
in our situation. The crucial inequality (1.5) provides the strict monotonicity immediately.
In Proposition 2.9 we show also the continuity of the map and the further properties.

## 2. Proof of Theorem 1.1

Here and in the sequel \( C \) denotes a generic positive constant which may vary from one
equation to another.

**Lemma 2.1.** Let \( f_1, f_2 \in C(\mathbb{R}) \) such that \( f_1(s) \leq f_2(s) \) and \( |f_1(s)| + |f_2(s)| \leq C(|s|^2 + |s|^{2^*}) \)
for any \( s \in \mathbb{R} \) and some constant \( C > 0 \). Then, \( f_1(s) \leq f_2(s) \) for \( s \in \mathbb{R} \) if and only if
\[
\int_{\mathbb{R}^N} f_1(u) - f_2(u) \, dx < 0
\]
for any \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \).

**Proof.** Suppose that there is a sequence \( (s_n) \subset (0, \infty) \) such that \( f_1(s_n) < f_2(s_n) \) and \( s_n \to 0 \)
as \( n \to \infty \). Note that for any \( n \), we find an open interval \( I_n \subset (0, 1/n) \) such that \( f_1(s) < f_2(s) \)
for \( s \in I_n \). We may assume that \( I_n \) are pairwise disjoint. Fix \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \) and let
\[
\Omega := \left\{ x \in \text{ess supp } u : |u(x)| \in \bigcup_{n \geq 1} I_n \right\}
\]
and suppose that \( |\Omega| = 0 \). Then
\[
(0, \infty) \setminus \bigcup_{n \geq 1} I_n = \bigcup_{n \geq 1} J_n
\]
is a union of closed and disjoint intervals \( J_n \). Note that \( \left\{ x \in \text{ess supp } u : |u(x)| \in J_{n_0} \right\} \) has
a positive measure for some \( n \geq 1 \). We choose \( J_{n_0} \) with the largest left endpoint \( a := \inf J_{n_0} \).
such that $\Omega' := \{x \in \text{ess supp} \ u : |u(x)| \in J_n\}$ has a positive measure. Let $b := \sup \{s < a : s \in \bigcup_{n \geq 1} J_n\}$. Observe that $0 < b < a$ and

$$\int_{\mathbb{R}^N} |u(x+h) - u(x)|^2 \ dx \geq (a - b)^2 \int_{\mathbb{R}^N} |\chi_{\Omega'}(x+h) - \chi_{\Omega'}(x)|^2 \ dx$$

for a.e. $h \in \mathbb{R}^N$, where $\chi_{\Omega'}$ is the characteristic function of $\Omega'$. Indeed, note that

$$|\chi_{\Omega'}(x+h) - \chi_{\Omega'}(x)| > 0$$

if and only if $x + h \in \Omega'$ and $x \notin \Omega'$, or $x + h \notin \Omega'$ and $x \in \Omega'$. If the latter conditions hold, then $|u(x+h)| \geq a$ and $|u(x)| \leq b$, or $|u(x+h)| \leq b$ and $|u(x)| \geq a$. Then we obtain (2.2).

In view of [31][Theorem 2.1.6] we infer that $\chi_{\Omega'} \in H^1(\mathbb{R}^N)$ and we get the contradiction, thus $|\Omega| > 0$. Therefore

$$\int_{\mathbb{R}^N} f_1(u) \ dx = \int_{\mathbb{R}^N \setminus \Omega} f_1(u) \ dx + \int_{\Omega} f_1(u) \ dx < \int_{\mathbb{R}^N \setminus \Omega} f_2(u) \ dx + \int_{\Omega} f_2(u) \ dx = \int_{\mathbb{R}^N} f_2(u) \ dx.$$

Now suppose that there is a sequence $(s_n) \subset (-\infty, 0)$ such that $f_1(s_n) < f_2(s_n)$ and $s_n \to 0$ as $n \to \infty$. Then $f_1(-(-s_n)) < f_2(-(-s_n)), (-s_n) \subset (0, \infty)$ and by the above proof applied to $f_1(-\cdot)$, $f_2(-\cdot)$ and $-u$ we obtain

$$\int_{\mathbb{R}^N} f_1(u) \ dx = \int_{\mathbb{R}^N} f_1(-(-u)) \ dx < \int_{\mathbb{R}^N} f_2(-(-u)) \ dx = \int_{\mathbb{R}^N} f_2(u) \ dx.$$

Therefore (2.1) holds provided that $f_1(s) \leq f_2(s)$ for $s \in \mathbb{R}$.

On the other hand, suppose by contradiction that (2.1) holds for every $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, $f_1(s) \leq f_2(s)$ for all $s \in \mathbb{R}$ and there is an open interval $I \subset \mathbb{R}$ such that $0 \in T$ and $f_1(s) = f_2(s)$ on $I$. We may assume that $\sup I > 0$. Take $a > 0$ such that $a \in I$ and let

$$\varphi(x) := a \exp \left(-\frac{|x|^2}{1 - |x|^2}\right) \chi_{[0,1]}(|x|), \quad x \in \mathbb{R}^N.$$

Then $\varphi \in C_0^\infty(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$ is such that $\varphi(\mathbb{R}^N) \subset T$. Hence $f_1(\varphi(x)) = f_2(\varphi(x))$ for all $x \in \mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} f_1(\varphi) - f_2(\varphi) \ dx = 0,$$

and we obtain a contradiction with (2.1).

In view of (A6) and arguing as in [9, page 325], for any $R > 0$ one can find a radial function $u \in H^1_0(B(0, R)) \cap L^\infty(B(0, R))$ such that $\int_{\mathbb{R}^N} H(u) \ dx > 0$. Then $u(r(\cdot)) \in \mathcal{M}$ for

$$r(u) := \left(\frac{\sqrt{N} \int_{\mathbb{R}^N} H(u) \ dx}{\int_{\mathbb{R}^N} |\nabla u|^2 \ dx}\right)^{1/2},$$

so that $\mathcal{M}$ is nonempty.

**Lemma 2.2.** Suppose that (A0), (A1), (A3), (A5), (A6) and (1.5) are satisfied. There holds

$$\inf_{u \in D \cap \mathcal{M}} |\nabla u|_2 > 0.$$
Proof. Take any $2 + \frac{4}{N} < p < 2^*$. In view of (A1), (A3) and (A5) for any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that
\[
H(u) \leq (2^* - 2)G(u) \leq (2^* - 2)\left(\varepsilon|u|^{2^*} + (\varepsilon + \eta)|u|^{2 + \frac{4}{N}} + c_\varepsilon|u|^p\right)
\]
for any $u \in \mathbb{R}$. From the Gagliardo-Nirenberg inequality
\[
|u|_p \leq C_{N,p}|\nabla u|_2^\delta |u|_2^{1-\delta} \leq C_{N,p} \rho^{(1-\delta)p/2} |\nabla u|_2^\delta,
\]
where $\delta = N\left(\frac{1}{2} - \frac{1}{p}\right)$. Note that
\[
\delta p = N\left(\frac{p}{2} - 1\right) > N\left(1 + \frac{2}{N} - 1\right) = 2.
\]
Since $u \in \mathcal{D} \cap \mathcal{M}$, we get
\[
|\nabla u|_2^2 = \frac{N}{2} \int_{\mathbb{R}^N} H(u) \, dx \leq \frac{N}{2} (2^* - 2) \left(\varepsilon\left(|u|_2^{2^*} + |u|_2^{2 + \frac{4}{N}}\right) + \eta|u|_2^{2 + \frac{4}{N}} + c_\varepsilon C_{N,p}^{\frac{1-\delta}{2}} |\nabla u|_2^{\delta p}\right) = 2^* \left(\varepsilon\left(|u|_2^{2^*} + |u|_2^{2 + \frac{4}{N}}\right) + \eta|u|_2^{2 + \frac{4}{N}} + c_\varepsilon C_{N,p}^{\frac{1-\delta}{2}} |\nabla u|_2^{\delta p}\right) \leq \varepsilon C(|\nabla u|_2^{2^*} + |\nabla u|_2^2) + Cc_\varepsilon |\nabla u|_2^{\delta p} + 2^* \eta C_{N,2}^{2^*} \rho^{\frac{2}{N}} |\nabla u|_2^2
\]
for a constant $C > 0$. Taking $\varepsilon < \frac{1}{2} \left(1 - 2^* \eta C_{N,2}^{2^*} \rho^{\frac{2}{N}}\right)$ we obtain that $|\nabla u|_2^2$ is bounded away from 0 on $\mathcal{D} \cap \mathcal{M}$ provided that
\[
2^* \eta C_{N,2}^{2^*} \rho^{\frac{2}{N}} < 1.
\]

Let $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ be such that
\[
(2.4) \quad 2\eta C_{N,2}^{2^*} \left(\int_{\mathbb{R}^N} |u|^2 \, dx\right)^{2/N} < 1.
\]
Define
\[
\varphi(\lambda) := J(\lambda^{\frac{N}{2}} u(\cdot)), \quad \lambda \in (0, \infty).
\]
In particular we can consider $u \in \mathcal{D}$ such that (1.5) holds or $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ if $\eta = 0$.

Lemma 2.3. Suppose that $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ satisfies (2.4). Assume moreover that (A0), (A1), (A3)–(A6) hold. Then there is an interval $[a, b] \subset (0, \infty)$ such that $\varphi$ is constant on $[a, b]$, $\lambda^{\frac{N}{2}} u(\lambda \cdot) \in \mathcal{M}$ and $\varphi(\lambda) \geq \varphi(\lambda')$ for any $\lambda \in [a, b]$ and $\lambda' \in (0, \infty)$ and the strict inequality holds for $\lambda' \in (0, \infty) \setminus [a, b]$. Moreover if $u \in \mathcal{M}$, then $1 \in [a, b]$.

Proof. Fix $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ satisfying (2.4). Observe that, from (A1),
\[
\varphi(\lambda) = \frac{\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(\lambda \frac{\lambda^N}{2} u) \, dx \to 0
\]
as $\lambda \to 0^+$. Let $R := |u|^2 = |\lambda^{N/2} u(\lambda \cdot)|^2 > 0$. Note that, from (A1), (A3), (A5) and the Gagliardo-Nirenberg inequality, for every $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that
\[
\int_{\mathbb{R}^N} G(u) \, dx \leq (\varepsilon + \eta)|u|_2^{2 + \frac{4}{N}} + c_\varepsilon |u|_2^{2^*} \leq (\varepsilon + \eta) C_{N,2}^{2^*} |\nabla u|_2^2 R^\frac{4}{N} + C_\varepsilon C |\nabla u|_2^{2^*}.
\]
Hence
\[
\frac{\varphi(\lambda)}{\lambda^2} = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{\lambda^2} \int_{\mathbb{R}^N} G \left( \frac{\lambda}{2} u(\lambda x) \right) \, dx \\
\geq \frac{1}{2} |\nabla u|^2 - (\varepsilon + \eta) C^2_{N/2} |\nabla u|^2 R^{\frac{2}{N}} - C \lambda^{2^* - 2} |\nabla u|^{2^*} \\
= \frac{|\nabla u|^2}{2} \left( 1 - 2(\eta + \varepsilon) C^2_{N/2} R^{\frac{2}{N}} \right) + o(1)
\]
as \lambda \to 0^+, and $\varphi(\lambda) > 0$ for sufficiently small $\lambda > 0$. Moreover from (A2) there follows that
\[
\frac{\varphi(\lambda)}{\lambda^2} = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} \frac{G(\lambda u(x))}{\lambda u(x)} \, dx \to -\infty
\]
as $\lambda \to \infty$. Hence $\varphi$ has a maximum at some $\lambda_0 > 0$. In particular $\varphi'(\lambda_0) = 0$, so that
\[
0 = \varphi'(\lambda_0) = \lambda_0 \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{N}{2} \int_{\mathbb{R}^N} H(\lambda_0^{N/2} u(x)) \, dx \right)
\]
and
\[
\lambda_0^{\frac{N}{2}} - \frac{N}{2} \int_{\mathbb{R}^N} H\left( \lambda_0^{\frac{N}{2}} u(x) \right) \, dx. \text{ Hence } \lambda_0^{\frac{N}{2}} u(x) \in \mathcal{M}. \text{ From (A4) there follows that the function }
\]
\[
(0, \infty) \ni \lambda \mapsto \int_{\mathbb{R}^N} H(\lambda^{N/2} u(x)) \, dx \in \mathbb{R}
\]
is nonincreasing. Moreover, from (A2) and (A5), $\int_{\mathbb{R}^N} H(\lambda^{N/2} u(x)) \, dx \to \infty$ as $\lambda \to \infty$. Hence there is an interval $[a, b]$ such that $\varphi'(\lambda) = 0$ for $\lambda \in [a, b]$. In particular, $\lambda_0^{\frac{N}{2}} u(x) \in \mathcal{M}$ and $\varphi(\lambda) \geq \varphi(\lambda')$ for any $\lambda \in [a, b]$ and $\lambda' \in (0, \infty)$ and the strict inequality holds for $\lambda' \in (0, \infty) \setminus [a, b]$. Since $\varphi'(\lambda) = 0$ for $\lambda \in [a, b]$ then $\varphi$ is constant on $[a, b]$. If, in addition, $u \in \mathcal{M}$, then
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{N}{2} \int_{\mathbb{R}^N} H(u) \, dx,
\]
so that $\varphi'(1) = 0$ and $1 \in [a, b]$.

**Lemma 2.4.** Suppose that (A0)–(A5) and (1.5) hold. Then $J$ is coercive on $\mathcal{D} \cap \mathcal{M}$.

**Proof.** Observe that for $u \in \mathcal{D} \cap \mathcal{M}$, taking (A5) into account, we have
\[
J(u) = J(u) - M(u) = \frac{N}{4} \int_{\mathbb{R}^N} H(u) - \frac{4}{N} G(u) \, dx \geq 0.
\]
Hence $J$ is bounded from below on $\mathcal{D} \cap \mathcal{M}$. Now we follow similar arguments as in [16, Lemma 2.5], [28, Proposition 2.7]. Suppose that $(u_n) \subset \mathcal{D} \cap \mathcal{M}$ is a sequence such that $\|u_n\| \to \infty$ and $J(u_n)$ is bounded from above. Since $u_n \in \mathcal{D}$ we see that $|\nabla u_n|^2 \to \infty$. Put
\[
\lambda_n := \frac{1}{|\nabla u_n|^2} > 0
\]
and define
\[
v_n := \lambda_n^{N/2} u_n (\lambda_n). \]
Note that $\lambda_n \to 0^+$ as $n \to \infty$. Then
\[
\int_{\mathbb{R}^N} |v_n|^2 \, dx = \int_{\mathbb{R}^N} |u_n|^2 \, dx \leq \rho,
\]
so that \( v_n \in D \). Moreover
\[
|\nabla v_n|^2 = \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx = \lambda_n^N \lambda_n^{-N+2} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = \lambda_n^2 \lambda_n^{-2} = 1.
\]
In particular, \((v_n)\) is bounded in \( H^1(\mathbb{R}^N) \). Suppose that
\[
\limsup_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |v_n|^2 \, dx \right) > 0.
\]
Then, up to a subsequence, we can find translations \((z_n) \subset \mathbb{R}^N\) such that
\[
v_n(\cdot + z_n) \rightharpoonup v \neq 0 \text{ in } H^1(\mathbb{R}^N)
\]
and \(v_n(x + z_n) \to v(x)\) for a.e. \(x \in \mathbb{R}^N\). Then by (A2)
\[
0 \leq \frac{J(u_n)}{|\nabla u_n|^2} = \frac{1}{2} - \int_{\mathbb{R}^N} \frac{G(u_n)}{|\nabla u_n|^2} \, dx = \frac{1}{2} - \lambda_n^N \lambda_n^2 \int_{\mathbb{R}^N} G(u_n(\lambda_n x)) \, dx
\]
\[
= \frac{1}{2} - \lambda_n^N + \int_{\mathbb{R}^N} G(\lambda_n^{-N/2} v_n) \, dx = \frac{1}{2} - \lambda_n^N + \int_{\mathbb{R}^N} \frac{G(\lambda_n^{-N/2} v_n)}{|\lambda_n^{-N/2} v_n|^{2 + \frac{4}{N}}} |\lambda_n^{-N/2} v_n|^{2 + \frac{4}{N}} \, dx
\]
\[
= \frac{1}{2} - \int_{\mathbb{R}^N} \frac{G(\lambda_n^{-N/2} v_n(x + z_n))}{|\lambda_n^{-N/2} v_n(x + z_n)|^{2 + \frac{4}{N}}} |\lambda_n^{-N/2} v_n(x + z_n)|^{2 + \frac{4}{N}} \, dx \to -\infty
\]
and we obtain a contradiction. Hence we may assume that
\[
\sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |v_n|^2 \, dx \to 0
\]
and from Lion’s lemma \( v_n \to 0 \) in \( L^{2+\frac{4}{N}}(\mathbb{R}^N) \). Observe that
\[
u_n = \lambda_n^{-N/2} v_n \left( \frac{\cdot}{\lambda_n} \right) \in \mathcal{M}.
\]
Since \( u_n = \lambda_n^{-N/2} v_n \left( \frac{\cdot}{\lambda_n} \right) \in D \) and (1.5) holds, \( u_n \) satisfies also (2.4). Hence, from Lemma 2.3, for any \( \lambda > 0 \) there holds
\[
J(u_n) = J(\lambda_n^{-N/2} v_n \left( \frac{\cdot}{\lambda_n} \right)) \geq J(\lambda_n^{-N/2} v_n(\lambda \cdot)) = \frac{\lambda_n^2}{2} - \lambda_n^{-N} \int_{\mathbb{R}^N} G(\lambda_n^{N/2} v_n) \, dx
\]
and \( \lambda_n^{-N} \int_{\mathbb{R}^N} G(\lambda_n^{N/2} v_n) \, dx \to 0 \) as \( n \to \infty \). Thus we obtain a contradiction by taking sufficiently large \( \lambda > 0 \).

\begin{lemma}
Suppose that (A0), (A1), (A3)–(A6) and (1.5) hold. There holds
\[
c := \inf_{D \cap \mathcal{M}} J > 0
\]
\end{lemma}
Proof. We will show that for $\rho > 0$ satisfying (1.5) there is $\delta > 0$ such that

\begin{equation}
\frac{1}{2N} |\nabla u|_2^2 \leq J(u)
\end{equation}

for $u \in D$ such that $|\nabla u|_2 \leq \delta$. From the Gagliardo-Nirenberg inequality we obtain

\[
\int_{\mathbb{R}^N} G(u) \, dx \leq (\varepsilon + \eta)|u|^{2+\frac{4}{N}} \, dx + C_{\varepsilon}|u|^{2^*} \leq (\varepsilon + \eta)C_{N,2}^2, \rho^\frac{4}{N} |\nabla u|_2^2 + C_{\varepsilon}C_{N,2}^{2^*} |\nabla u|_2^{2^*}
\]

\[
= (\varepsilon C_{N,2}^{2^*}, \rho^\frac{4}{N} + C_{\varepsilon}C_{N,2}^{2^*} |\nabla u|_2^{2^*} + \frac{4}{2^*} |\nabla u|_2^2
\]

\[
< \left( \varepsilon C_{N,2}^{2^*}, \rho^\frac{4}{N} + C_{\varepsilon}C_{N,2}^{2^*} |\nabla u|_2^{2^*} + \frac{1}{2} \right) |\nabla u|_2^2
\]

Taking

\[
\varepsilon := \frac{1}{4NC_{N,2}^{2^*}, \rho^\frac{4}{N}} > 0, \quad \delta := \left( \frac{1}{4NC_{N,2}^{2^*}, \rho^\frac{4}{N}} \right)^\frac{N-2}{2-N} > 0
\]

we obtain that

\[
\int_{\mathbb{R}^N} G(u) \, dx \leq \left( \frac{1}{4N} + \frac{1}{4N} + \frac{1}{2} - \frac{1}{N} \right) |\nabla u|_2^2 = \left( \frac{1}{2} - \frac{1}{2N} \right) |\nabla u|_2^2.
\]

Hence

\[
J(u) = \frac{1}{2} |\nabla u|_2^2 - \int_{\mathbb{R}^N} G(u) \, dx \geq \frac{1}{2} |\nabla u|_2^2 - \left( \frac{1}{2} - \frac{1}{2N} \right) |\nabla u|_2^2 = \frac{1}{2N} |\nabla u|_2^2.
\]

Fix $u \in D \cap M$. In view of (1.5), $u$ clearly satisfies the inequality (2.4). Then, from Lemma 2.3, for every $\lambda > 0$ there holds

\[
J(u) \geq J(\lambda^{N/2} u(\cdot)).
\]

Choose $\lambda := \frac{4}{|\nabla u|_2^2} > 0$, where $\delta > 0$ is chosen so that (2.5) holds, and let $v := \lambda^{N/2} u(\cdot)$. Obviously $|v|_2 = |u|_2$ so that $v \in D$. Moreover $|\nabla v|_2 = \delta$. Then

\[
J(u) \geq J(v) \geq \frac{1}{2N} |\nabla v|_2^2 = \frac{1}{2N} \delta^2 > 0.
\]

Before we show that $\inf_{D \cap M} J$ is attained, we need the following profile decomposition result obtained in [20, Theorem 1.4] applied to $H$ satisfying

\[
\lim_{u \to 0} H(u)/|u|^2 = \lim_{|u| \to \infty} H(u)/|u|^{2^*} = 0.
\]

**Theorem 2.6.** Suppose that $(u_n) \subset H^1(\mathbb{R}^N)$ is bounded. Then there are sequences $(\tilde{u}_i)_{i=0}^\infty \subset H^1(\mathbb{R}^N)$, $(y_n^i)_{i=0}^\infty \subset \mathbb{R}^N$ for any $n \geq 1$, such that $y_n^0 = 0$, $|y_n^i - y_n^j| \to \infty$ as $n \to \infty$ for $i \neq j$, and passing to a subsequence, the following conditions hold for any $i \geq 0$:

\[
u_n(\cdot + y_n^i) \rightharpoonup \tilde{u}_i \text{ in } H^1(\mathbb{R}^N) \text{ as } n \to \infty,
\]

\begin{equation}
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = \sum_{j=0}^i \int_{\mathbb{R}^N} |\nabla \tilde{u}_j|^2 \, dx + \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx,
\end{equation}
where \( v_n^i := u_n - \sum_{j=0}^i \tilde{u}_j(\cdot - y_n^i) \) and 

\[
(2.7) \quad \lim_{n \to \infty} \sup_{\mathbb{R}^N} \int H(u_n) \, dx = \sum_{j=0}^\infty \int_{\mathbb{R}^N} H(\tilde{u}_j) \, dx.
\]

**Lemma 2.7.** Suppose that (A0)–(A5) and (1.5) hold. Then \( c = \inf_{\mathcal{D} \cap \mathcal{M}} J \) is attained. If, in addition, \( g \) is odd, then \( c \) is attained by a nonnegative and radially symmetric function in \( \mathcal{D} \cap \mathcal{M} \).

**Proof.** Take any sequence \((u_n) \subset \mathcal{D} \cap \mathcal{M}\) such that \( J(u_n) \to c \) and by Lemma 2.4, \((u_n)\) is bounded in \( H^1(\mathbb{R}^N) \). Note that by (A1), (A3) and (A5), we may apply Theorem 2.6 we find a profile decomposition of \((u_n)\) satisfying (2.6) and (2.7). We show that 

\[
0 < \int_{\mathbb{R}^N} |\nabla \tilde{u}_i|^2 \, dx \leq \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}_i) \, dx
\]

for some \( i \geq 0 \). Let 

\[
I := \{ i \geq 0 : \tilde{u}_i \neq 0 \}.
\]

In view of Lemma 2.2 and (2.7), \( I \neq \emptyset \). Suppose that 

\[
\int_{\mathbb{R}^N} |\nabla \tilde{u}_i|^2 \, dx > \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}_i) \, dx
\]

for all \( i \in I \). Then by (2.6) and (2.7)

\[
\limsup_{n \to \infty} \frac{N}{2} \int_{\mathbb{R}^N} H(u_n) \, dx = \limsup_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \geq \sum_{j=0}^\infty \int_{\mathbb{R}^N} |\nabla \tilde{u}_j|^2 \, dx = \sum_{j \in I} \int_{\mathbb{R}^N} |\nabla \tilde{u}_j|^2 \, dx
\]

\[
> \sum_{j=0}^\infty \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}_j) \, dx = \limsup_{n \to \infty} \frac{N}{2} \int_{\mathbb{R}^N} H(u_n) \, dx,
\]

which is a contradiction. Therefore there is \( i \in I \) such that \( r(\tilde{u}_i) \geq 1 \) defined as in (2.3) and \( \tilde{u}_i(r(\tilde{u}_i) \cdot) \in \mathcal{M} \). Moreover

\[
\int_{\mathbb{R}^N} |\tilde{u}_i(r(\tilde{u}_i) \cdot)|^2 \, dx = r(\tilde{u}_i)^{-N} \int_{\mathbb{R}^N} |\tilde{u}_i|^2 \, dx \leq r(\tilde{u}_i)^{-N} \rho \leq \rho,
\]

hence \( \tilde{u}_i(r(\tilde{u}_i) \cdot) \in \mathcal{D} \cap \mathcal{M} \). If \( r(\tilde{u}_i) > 1 \), then passing to a subsequence \( u_n(x + y_n^i) \to \tilde{u}_i(x) \) for a.e. \( x \in \mathbb{R}^N \) and by Fatou’s lemma

\[
0 < \inf_{\mathcal{D} \cap \mathcal{M}} J \leq J(\tilde{u}_i(r(\tilde{u}_i) \cdot)) = r(\tilde{u}_i)^{-N} \frac{N}{4} \int_{\mathbb{R}^N} H(\tilde{u}_i) - \frac{4}{N} G(\tilde{u}_i) \, dx
\]

\[
< \frac{N}{4} \int_{\mathbb{R}^N} H(\tilde{u}_i) - \frac{4}{N} G(\tilde{u}_i) \, dx
\]

\[
\leq \liminf_{n \to \infty} \frac{N}{4} \int_{\mathbb{R}^N} H(u_n(\cdot + y_n^i)) - \frac{4}{N} G(u_n(\cdot + y_n^i)) \, dx
\]

\[
= \liminf_{n \to \infty} J(u_n) = c = \inf_{\mathcal{D} \cap \mathcal{M}} J.
\]
and again we get a contradiction. Therefore \( r(\tilde{u}_i) = 1 \), \( \tilde{u}_i \in \mathcal{D} \cap \mathcal{M} \) and

\[
J(\tilde{u}_i) = \frac{N}{4} \int_{\mathbb{R}^N} H(\tilde{u}_i) - \frac{4}{N} G(\tilde{u}_i) \, dx \\
\leq \liminf_{n \to \infty} \frac{N}{4} \int_{\mathbb{R}^N} H(u_n(\cdot + y_n^i)) - \frac{4}{N} G(u_n(\cdot + y_n^i)) \, dx \\
= \liminf_{n \to \infty} J(u_n) = c.
\]

Thus \( J(\tilde{u}_i) = c \).

Suppose that \( g \) is odd. Then \( G \) and \( H \) are even, so that \( G(|u|) = G(u) \) and \( H(|u|) = H(u) \) for all \( u \in H^1(\mathbb{R}^N) \). We define \( \tilde{v}_i := |\tilde{u}_i|^* \) as the Schwarz symmetrization of \( |\tilde{u}_i| \). Then \( |\tilde{v}_i|_2 = |\tilde{u}_i|_2 \), hence \( \tilde{v}_i \in \mathcal{D} \). Moreover, since

\[
\int_{\mathbb{R}^N} |\nabla \tilde{v}_i|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla \tilde{u}_i|^2 \, dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}_i) \, dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{v}_i) \, dx,
\]

we obtain that \( r(\tilde{v}_i) \geq 1 \), where \( r \) is given by (2.3) and \( \tilde{v}_i(r(\tilde{v}_i)) \in \mathcal{M} \). Suppose that \( r(\tilde{v}_i) > 1 \). Then

\[
\inf_{\mathcal{D} \setminus \mathcal{M}} J \leq J(\tilde{v}_i(r(\tilde{v}_i))) = r(\tilde{v}_i)^{-N} \int_{\mathbb{R}^N} H(\tilde{v}_i) - \frac{4}{N} G(\tilde{v}_i) \, dx \\
< \frac{N}{4} \int_{\mathbb{R}^N} H(\tilde{v}_i) - \frac{4}{N} G(\tilde{v}_i) \, dx = \frac{N}{4} \int_{\mathbb{R}^N} H(\tilde{u}_i) - \frac{4}{N} G(\tilde{u}_i) \, dx = J(\tilde{u}_i) = \inf_{\mathcal{D} \setminus \mathcal{M}} J,
\]

which is a contradiction. Hence \( r(\tilde{v}_i) = 1 \) and \( \tilde{v}_i \in \mathcal{M} \). Obviously \( J(\tilde{v}_i) = \inf_{\mathcal{D} \setminus \mathcal{M}} J, \tilde{v}_i \geq 0 \) and \( \tilde{v}_i \) is radially symmetric. Hence \( r(\tilde{v}_i) = 1 \) and \( \tilde{v}_i \in \mathcal{M} \).

\[\square\]

**Lemma 2.8.** Suppose that \((A0) – (A5), (1.5)\) hold. Assume moreover that

(a) \((A5, \leq)\) hold

or

(b) \(\frac{4}{N}G(s) \leq H(s)\) for \(s \in \mathbb{R}\), \(g\) is odd and \(N \in \{3,4\}\).

For any \(u \in (\mathcal{D} \setminus \mathcal{S}) \cap \mathcal{M}\) there holds

\[
\inf_{\mathcal{S} \cap \mathcal{M}} J < J(u).
\]

**Proof.** Suppose by contradiction that there is \(\tilde{u} \in \mathcal{M}\) such that \(\int_{\mathbb{R}^N} |\tilde{u}|^2 \, dx < \rho\) and

\[
c = J(\tilde{u}) \leq \inf_{\mathcal{S} \cap \mathcal{M}} J.
\]

Hence \(\tilde{u}\) is a local minimizer for \(J\) on \(\mathcal{D} \setminus \mathcal{M}\). Since \(\mathcal{D} \setminus \mathcal{S}\) is an open set in \(\mathcal{M}\), we see that \(\tilde{u}\) is a local minimizer of \(J\) on \(\mathcal{M}\). Hence there is a Lagrange multiplier \(\mu \in \mathbb{R}\) such that

\[
J'(\tilde{u})(v) + \mu \left( \int_{\mathbb{R}^N} \nabla \tilde{u} \nabla v \, dx - \frac{N}{4} \int_{\mathbb{R}^N} h(\tilde{u}) v \, dx \right) = 0
\]

for any \(v \in H^1(\mathbb{R}^N)\), i.e. \(\tilde{u}\) is a weak solution to

\[
-\Delta \tilde{u} - g(\tilde{u}) + \mu \left( -\Delta \tilde{u} - \frac{N}{4} h(\tilde{u}) \right) = 0
\]

or equivalently

\[
-(1 + \mu) \Delta \tilde{u} = g(\tilde{u}) + \frac{N}{4} \mu h(\tilde{u}).
\]
In particular \( \tilde{u} \) satisfies the following Nehari-type identity

\[
(1 + \mu) \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx = \int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} + \frac{N}{4} \mu h(\tilde{u}) \tilde{u} \, dx.
\]

If \( \mu = -1 \) we obtain that

\[
\int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} - \frac{N}{4} h(\tilde{u}) \tilde{u} \, dx = 0.
\]

On the other hand, by (A4), (A5), \( \frac{4}{N} G(s) \leq H(s) \) for \( s \in \mathbb{R} \), and Lemma 2.1

\[
\int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} - \frac{N}{4} h(\tilde{u}) \tilde{u} \, dx = -\frac{N}{4} \int_{\mathbb{R}^N} h(\tilde{u}) \tilde{u} - \frac{4}{N} g(\tilde{u}) \tilde{u} \, dx
\]

\[
\leq -\frac{N}{4} \int_{\mathbb{R}^N} \left(2 + \frac{4}{N}\right) H(\tilde{u}) - \frac{4}{N} g(\tilde{u}) \tilde{u} \, dx
\]

\[
= -\frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}) - \frac{4}{N} G(\tilde{u}) \, dx < 0,
\]

and we obtain a contradiction. Hence \( \mu \neq -1 \). Since \( \tilde{u} \in \mathcal{M} \) we obtain

\[
\int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}) \, dx.
\]

On the other hand \( \tilde{u} \) satisfies Pohozaev and Nehari identities. Thus

\[
(1 + \mu) \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}) + \frac{N}{4} \mu (h(\tilde{u}) \tilde{u} - 2H(\tilde{u})) \, dx.
\]

Combining these two identities we get

\[
(1 + \mu) \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}) \, dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}) + \frac{N}{4} \mu (h(\tilde{u}) \tilde{u} - 2H(\tilde{u})) \, dx.
\]

Thus

\[
\mu \int_{\mathbb{R}^N} H(\tilde{u}) \, dx = \frac{N}{4} \mu \int_{\mathbb{R}^N} h(\tilde{u}) \tilde{u} - 2H(\tilde{u}) \, dx.
\]

If \( \mu \neq 0 \), then

\[
\int_{\mathbb{R}^N} h(\tilde{u}) \tilde{u} - \left(2 + \frac{4}{N}\right) H(\tilde{u}) \, dx = 0.
\]

From the elliptic regularity theory we may assume that \( \tilde{u} \) is continuous and \( h(\tilde{u}(x)) \tilde{u}(x) - \left(2 + \frac{4}{N}\right) H(\tilde{u}(x)) = 0 \) for all \( x \in \mathbb{R}^N \). Since \( \tilde{u} \in H^1(\mathbb{R}^N) \), we know that \( \tilde{u}(x) \to 0 \) as \( |x| \to \infty \).

In particular, there is an open interval \( I \) such that \( 0 \in \mathcal{T} \) and \( h(u)u - \left(2 + \frac{4}{N}\right) H(u) = 0 \) for \( u \in \mathcal{T} \). Hence \( H(u) = C|u|^{2 + \frac{4}{N}} \) for some \( C > 0 \) and \( u \in \mathcal{T} \), which is a contradiction with the first inequality in (A5, \( \leq \)). Thus we have \( \mu = 0 \) and \( \tilde{u} \) is a weak solution to

\[
-\Delta \tilde{u} = g(\tilde{u}).
\]

From the Nehari-type identity

\[
\int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx = \int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} \, dx
\]

and since \( \tilde{u} \in \mathcal{M} \) we obtain

\[
\int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} \, dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}) \, dx.
\]
and
\[ \int_{\mathbb{R}^N} 2^* G(\tilde{u}) - g(\tilde{u})\tilde{u} \, dx = 0. \]

By the elliptic regularity theory, \( \tilde{u} \) is continuous and in view of (A5)
\[ 2^* G(\tilde{u}(x)) = g(\tilde{u}(x))\tilde{u}(x) \]
for \( x \in \mathbb{R}^N \). Since \( \tilde{u} \in H^1(\mathbb{R}^N) \), there is an open interval \( I \subset \mathbb{R} \) such that \( 0 \in \overline{I} \) and \( 2^* G(u) = g(u)u \) for \( u \in \overline{I} \). Then there is \( C > 0 \) such that \( G(u) = C|u|^{2^*} \) for \( u \in \overline{I} \). Now we need to consider two cases.

(a) If the inequality \((A5, \preceq)\) holds, then we obtain a contradiction immediately. (b) If \( g \) is odd, then in view of Lemma 2.7, we may assume that \( \tilde{u} \) is nonnegative and radially symmetric. Moreover \( \tilde{u} \) solves
\[ -(\Delta \tilde{u}) = (2^* C)|\tilde{u}|^{2^* - 2} \tilde{u}, \]
and we get a contradiction, since the nonnegative and radial solution to problem (2.8) is an Aubin-Talenti instanton, up to a scaling and a translation, which is not \( L^2 \)-integrable if \( N \in \{3, 4\} \), see [11, Section 6.2], cf. [2, 29].

\[ \square \]

**Proof of Theorem 1.1.** In view of Lemma 2.7 and Lemma 2.8 we infer that \( c = \inf_{S \cap M} J \) is attained. Now we find Lagrange multipliers \( \lambda, \mu \in \mathbb{R} \) such that \( \tilde{u} \in S \cap M \) solves
\[ -\Delta \tilde{u} - g(\tilde{u}) + \lambda \tilde{u} + \mu \left( -\Delta \tilde{u} - \frac{N}{4} h(\tilde{u}) \right) = 0, \]
that is
\[ -(1 + \mu) \Delta \tilde{u} + \lambda \tilde{u} = g(\tilde{u}) + \frac{N}{4} \mu h(\tilde{u}). \]

Suppose that \( \mu = -1 \) and consider two cases.

(a) Suppose that \( g'(u) = o(1) \) as \( u \to 0 \). Then, from (A1) and (A5) there follows that \( g(u) = o(u) \) and \( h(u) = o(u) \) as \( u \to 0 \). Note that by (A4), the first inequality in \((A5, \preceq)\) and Lemma 2.1
\[ \lambda \int_{\mathbb{R}^N} |\tilde{u}|^2 \, dx \leq \int_{\mathbb{R}^N} g(\tilde{u})\tilde{u} + \frac{N}{4} \mu h(\tilde{u})\tilde{u} \, dx = \frac{N}{4} \int_{\mathbb{R}^N} \frac{4}{N} g(\tilde{u})\tilde{u} - h(\tilde{u})\tilde{u} \, dx < 0, \]
hence \( \lambda < 0 \). On the other hand, take
\[ \Sigma := \{ x \in \mathbb{R}^N : \lambda \tilde{u}(x) = g(\tilde{u}(x)) - \frac{N}{4} h(\tilde{u}(x)) \} \]
and note that the measure of \( \Omega := \{ x \in \Sigma : \tilde{u}(x) \neq 0 \} \) is nonzero. Suppose that \( \delta := \text{ess inf}_{x \in \Omega} |\tilde{u}(x)| > 0 \). Since \( \tilde{u} \in L^2(\mathbb{R}^N) \setminus \{0\} \), we infer that \( \Omega \) has finite positive measure and observe that
\[ \int_{\mathbb{R}^N} |\tilde{u}(x + h) - \tilde{u}(x)|^2 \, dx \geq \delta^2 \int_{\mathbb{R}^N} |\chi_\Omega(x + h) - \chi_\Omega(x)|^2 \, dx \quad \text{for any} \ h \in \mathbb{R}^N, \]
where \( \chi_\Omega \) is the characteristic function of \( \Omega \). In view of [31, Theorem 2.1.6] we infer that 
\( \chi_\Omega \in H^1(\mathbb{R}^N) \), hence we get a contradiction. Therefore we find a sequence \( (x_n) \subset \Omega \) such that 
\( \tilde{u}(x_n) \to 0 \) and
\[
\lambda = \frac{g(\tilde{u}(x_n))\tilde{u}(x_n) - \frac{N}{4} h(\tilde{u}(x_n)) \tilde{u}(x_n)}{|\tilde{u}(x_n)|^2}
\]
for any \( n \geq 1 \). From (A5) there follows that
\[
\frac{g(\tilde{u}(x_n))\tilde{u}(x_n)}{|\tilde{u}(x_n)|^2} = \frac{H(\tilde{u}(x_n))}{|\tilde{u}(x_n)|^2} + \frac{2G(\tilde{u}(x_n))}{|\tilde{u}(x_n)|^2} \to 0
\]
as \( n \to \infty \). Hence
\[
\lambda = -\lim_{n \to \infty} \frac{\frac{N}{4} h(\tilde{u}(x_n)) \tilde{u}(x_n)}{|\tilde{u}(x_n)|^2} = -\frac{N}{4} \lim_{n \to \infty} \left( \frac{g(\tilde{u}(x_n)) - g(\tilde{u}(x_n)) \tilde{u}(x_n)}{|\tilde{u}(x_n)|^2} \right) = 0
\]
and we obtain a contradiction.

(b) Suppose that \( g \) is odd. Then we may assume that \( \tilde{u} \) is positive and radially symmetric. Then from Strauss lemma ([25, Radial Lemma 1]) we may assume that \( \tilde{u} \) is continuous and from (2.9)
\[
\lambda \tilde{u}(x) = g(\tilde{u}(x)) - \frac{N}{4} h(\tilde{u}(x))
\]
holds for \( x \in \mathbb{R}^N \). Since \( \tilde{u} \) is continuous and \( \tilde{u} \in H^1(\mathbb{R}^N) \), there is an interval \( I \) such that 
\( 0 \in I \) and
\[
\lambda u = g(u) - \frac{N}{4} h(u) \quad \text{for } u \in I.
\]
From the definition of \( h \) we obtain that
\[
\lambda u = \left( 1 + \frac{N}{4} \right) g(u) - \frac{N}{4} g'(u) u \quad \text{for } u \in I.
\]
Hence
\[
g(u) = C_1 |u|^{\frac{2}{1+\mu}} u + C_2 u, \quad u \in I
\]
for some \( C_1, C_2 \in \mathbb{R} \). In particular \( G(u) = \frac{C_1}{2(1+\mu)} |u|^{2+\frac{2}{1+\mu}} + \frac{C_2}{2} u^2 \). From (A1) there follows that 
\( C_2 = 0, C_1 \geq 0 \) and we obtain a contradiction with the first inequality in (A5, \( \preceq \)).

Therefore \( \mu \neq -1 \) and taking into account Nehari and Pohozaev identities for (2.9), we obtain
\[
\frac{1}{2} (1 + \mu) \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 + \lambda |\tilde{u}|^2 \, dx = \int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} + \frac{N}{4} \mu h(\tilde{u}) \tilde{u} \, dx,
\]
\[
(1 + \mu) \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 + \frac{2}{2} \lambda |\tilde{u}|^2 \, dx = 2^* \int_{\mathbb{R}^N} G(\tilde{u}) + \frac{N}{4} \mu H(\tilde{u}) \, dx,
\]
thus
\[
(2.10) \quad (1 + \mu) \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}) + \frac{N}{4} \mu \left( h(\tilde{u}) \tilde{u} - 2H(\tilde{u}) \right) \, dx.
\]
Since \( \tilde{u} \in \mathcal{M} \) we get
\[
(1 + \mu) \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}) \, dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}) + \frac{N}{4} \mu \left( h(\tilde{u}) \tilde{u} - 2H(\tilde{u}) \right) \, dx
\]
and
\[
\mu \int_{\mathbb{R}^N} h(\tilde{u}) \tilde{u} - \left( 2 + \frac{4}{N} \right) H(\tilde{u}) \, dx = 0.
\]
In view of (A4) and since $\frac{4}{N}G(u) \leq H(u)$ for $u \in \mathbb{R}$, then similarly as in proof of Lemma 2.8 we obtain $\mu = 0$. Therefore $\tilde{u}$ solves (1.1). In the case (b) we already know that $\tilde{u}$ is nonnegative and radially symmetric. Hence, from the maximum principle, $\tilde{u}$ is positive and the proof is completed. In the case (a) note that our solution $\tilde{u}$ is a minimizer of $J$ subject to the following constraints

\begin{align}
(2.11) & \int_{\mathbb{R}^N} |\tilde{u}|^2 \, dx = \rho > 0, \\
(2.12) & \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 + |\tilde{u}|^2 - \frac{N}{2} H(\tilde{u}) \, dx = \rho > 0.
\end{align}

From the regularity theory we know that every minimizer of (1.1) with respect to (2.11) and (2.12) is of class $C^1$ (see [27, Appendix B]). Hence, from [19, Theorem 2] there follows that $\tilde{u}$ is radially symmetric with respect to a one-dimensional affine subspace $V$ in $\mathbb{R}^N$. $\square$

With the aid of Lemmas 2.7 and 2.8 we easy infer that the ground state energy map (1.8) is strictly decreasing. The further properties are given as follow.

**Proposition 2.9.** Under the assumptions of Theorem 1.1, the ground state energy map $\rho \mapsto \inf_{S \cap M} J$ is continuous, strictly decreasing and $\inf_{S \cap M} J \to \infty$ as $\rho \to 0^+$. If $\eta = 0$ and

\begin{equation}
(2.13) \quad \lim_{u \to 0} G(u)/|u|^{2^*} = \infty,
\end{equation}

and $\rho \to \infty$, then $\inf_{S \cap M} J \to 0^+$.

**Proof.** Let us denote

$\mathcal{D}_\rho := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx \leq \rho \right\}$ and $\mathcal{S}_\rho := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx = \rho \right\}$.

Suppose that $\rho_n \to \rho^+$ as $n \to \infty$, and let $J(u_n) = \inf_{\mathcal{D}_{\rho_n} \cap M} J$ for some $u_n \in \mathcal{D}_{\rho_n} \cap M$. Arguing as in proof of Lemma 2.7, $u_n \to \tilde{u}$ such that $r(\tilde{u}) \geq 1$ up to a translation and up to a subsequence. If $r(\tilde{u}) > 1$, then

$$\begin{align*}
\inf_{\mathcal{D}_\rho \cap M} J & \leq J(\tilde{u}(r(\tilde{u}) \cdot )) = r(\tilde{u})^{-N} \frac{N}{4} \int_{\mathbb{R}^N} H(\tilde{u}) - \frac{4}{N} G(\tilde{u}) \, dx \\
& < \frac{N}{4} \int_{\mathbb{R}^N} H(\tilde{u}) - \frac{4}{N} G(\tilde{u}) \, dx \\
& \leq \liminf_{n \to \infty} \frac{N}{4} \int_{\mathbb{R}^N} H(u_n) - \frac{4}{N} G(u_n) \, dx \\
& = \liminf_{n \to \infty} J(u_n) \leq \inf_{\mathcal{D}_\rho \cap M} J,
\end{align*}$$

where the last inequality holds, since $\mathcal{D}_\rho \cap M \subset \mathcal{D}_{\rho_n} \cap M$. We get a contradiction and $r(\tilde{u}) = 1$ and as in proof of Lemma 2.7 we infer that $J(\tilde{u}) = \inf_{\mathcal{D}_\rho \cap M} J = \lim_{n \to \infty} \inf_{\mathcal{D}_{\rho_n} \cap M} J$. Suppose that $\rho_n \to \rho^-$ as $n \to \infty$, and choose $u \in \mathcal{D}_\rho \cap M$ so that $J(u) = \inf_{\mathcal{D}_\rho \cap M} J$. Similarly as in [16, Lemma 3.1] we consider $s_n := \sqrt{\rho_n/\rho}$, $v_n := s_n u$ and in view of Lemma 2.3 we find $\lambda_n$ such that $\lambda_n^{N/2} v_n(\lambda_n \cdot ) \in M$, however $\lambda_n$ need not be unique and $(\lambda_n)$ may be divergent. Note that $|\lambda_n^{N/2} v_n(\lambda_n \cdot )|_2 = |v_n|_2 = \rho_n$. If $\lambda_n \to \infty$ passing to a subsequence, then
by (A2)
\[
\frac{s_n^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{N}{2} \int_{\mathbb{R}^N} \frac{H(\lambda_n^{N/2} s_n u)}{(\lambda_n^{N/2})^2} \, dx \to \infty,
\]
which is a contradiction with \(s_n \to 1\), as \(n \to \infty\). Similarly (A3) exclude \(\lambda_n \to 0\) passing to a subsequence. Therefore, passing to subsequence \(\lambda_n \to \lambda > 0\), \(\lambda_n^{N/2} u(\lambda \cdot) \in \mathcal{M}\) and
\[
\lim_{n \to \infty} J(\lambda_n^{N/2} u_n(\lambda \cdot)) = \frac{\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} G(\lambda^{N/2} u) \lambda^{-N} \, dx = J(\lambda^{N/2} u(\cdot)) = J(u),
\]
where the last equality follows from Lemma 2.3, hence \(\lim \sup_{n \to \infty} \inf_{\mathcal{D}_{\rho_n} \cap \mathcal{M}} J \leq \inf_{\mathcal{D}_{\rho} \cap \mathcal{M}} J\) and taking into account that \(\mathcal{D}_{\rho_n} \cap \mathcal{M} \subset \mathcal{D}_{\rho} \cap \mathcal{M}\) we conclude the continuity of the ground state energy map.

Suppose that \(\rho_n \to 0^+\) and let \(J(u_n) = \inf_{\mathcal{D}_{\rho_n} \cap \mathcal{M}} J\) for some \(u_n \in \mathcal{S}_{\rho_n}\). We follow the ideas from [16, Lemma 3.5]. Put \(\lambda_n := \frac{1}{|\nabla u_n|^2} > 0\) and \(v_n := \lambda_n^{N/2} u_n(\lambda_n \cdot)\). Then \(|\nabla v_n|^2 = 1\), \(|v_n|^2 = |u_n|^2 = \rho_n \to 0^+\), \(u_n = \lambda_n^{-N/2} v_n(\lambda_n \cdot) \in \mathcal{M}\) and \((v_n)\) is bounded in \(H^1(\mathbb{R}^N)\). In particular, \((v_n)\) is bounded in \(L^{2^*}(\mathbb{R}^N)\) and from the interpolation inequality there holds
\[
|v_n|^2 \leq |v_n|^2 |v_n|^2 = \rho_n^{2-N} |v_n|^2 \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence \(|v_n|^2 \to 0\) and \(\int_{\mathbb{R}^N} G(\lambda_n^{N/2} v_n) \lambda_n^{-N} \, dx \to 0\) as \(n \to \infty\) for any fixed \(\lambda > 0\). From Lemma 2.3 there follows that
\[
J(u_n) = J\left(\lambda_n^{N/2} u_n(\lambda_n^{-1})\right) \geq J\left(\lambda_n^{N/2} v_n(\lambda)\right) = \frac{\lambda^2}{2} - \int_{\mathbb{R}^N} G(\lambda v_n) \lambda_n^{-N} \, dx = \frac{\lambda^2}{2} + o(1)
\]
for any \(\lambda > 0\). Hence \(J(u_n) \to \infty\).

Suppose now that \(\eta = 0\). Then the ground state energy map (1.8) is well-defined for all \(\rho > 0\). Suppose that \(\rho_n \to \infty\). Take \(u \in H^1(\mathbb{R}^N)\) as a ground state solution for the problem with \(\rho = 1\), i.e. \(J(u) = \inf_{\mathcal{D}_1 \cap \mathcal{M}} J = \inf_{\mathcal{S}_1 \cap \mathcal{M}} J\). From the regularity theory we know that \(u\) is continuous, and therefore \(u \in L^\infty(\mathbb{R}^N)\). Without loss of generality we may assume that \(\rho_n > 1\) and, as in [16, Lemma 3.6], define \(u_n := \sqrt{\rho_n} u\). Then \(u_n \in \mathcal{S}_{\rho_n} \subset \mathcal{D}_{\rho_n}\). From Lemma 2.3 there is \(\lambda_n > 0\) such that \(v_n := \lambda_n^{N/2} u_n(\lambda_n^{-1}) \in \mathcal{M}\). In general, \(\lambda_n\) is not unique. Moreover \(|u_n|^2 = |v_n|^2\) so that \(v_n \in \mathcal{D}_{\rho_n} \cap \mathcal{M}\). Hence
\[
0 < \inf_{\mathcal{D}_{\rho_n} \cap \mathcal{M}} J \leq J(v_n) \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx = \frac{1}{2} \lambda_n^2 \rho_n \int_{\mathbb{R}^N} |\nabla u|^2 \, dx,
\]
so it is enough to show that \(\lambda_n \sqrt{\rho_n} \to 0\). Note that
\[
\lambda_n^2 \rho_n \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx = \frac{N}{2} \int_{\mathbb{R}^N} H(v_n) \, dx = \frac{N}{2} \lambda_n^{-N} \int_{\mathbb{R}^N} H(\lambda_n^{N/2} \sqrt{\rho_n} u) \, dx
\]
and
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{N}{2} \lambda_n^{-N-2} \rho_n^{-1} \int_{\mathbb{R}^N} H(\lambda_n^{N/2} \sqrt{\rho_n} u) \, dx = \frac{N}{2} \rho_n^{2/N} \int_{\mathbb{R}^N} \frac{H(\lambda_n^{N/2} \sqrt{\rho_n} u)}{\lambda_n^{N/2} \sqrt{\rho_n} u} |u|^{2+\frac{4}{N}} \, dx
\]
\[
\int_{\mathbb{R}^N} \frac{H(\lambda_n^{N/2} \sqrt{\rho_n} u)}{\lambda_n^{N/2} \sqrt{\rho_n} u} |u|^{2+\frac{4}{N}} \, dx \to 0 \quad \text{as} \quad n \to \infty,
\]
and \(\lambda_n^{N/2} \sqrt{\rho_n} \to 0\).
Fix $\varepsilon > 0$. Then, from (A5) and (2.13) there follows that
\[
H(s) \geq \frac{4}{N} G(s) \geq \varepsilon^{-1} |s|^{2^*}
\]
for sufficiently small $|s|$. Then, taking into account that $u \in L^\infty(\mathbb{R}^N)$, for sufficiently large $n$
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{N}{2} \lambda_n^{N-2} \frac{1}{\rho_n} \int_{\mathbb{R}^N} H(\lambda_n^{N/2} \sqrt{\rho_n} u) \, dx \geq \varepsilon^{-1} \frac{N}{2} \lambda_n^{N-2} \frac{1}{\rho_n} |\lambda_n^{N/2} \sqrt{\rho_n}|^{2^*} |u|_{2^*}^{2^*}
\]
\[
= \varepsilon^{-1} \frac{N}{2} \lambda_n^{N-2} \rho_n^{\frac{2}{N-2}} |u|_{2^*}^{2^*} = \varepsilon^{-1} \frac{N}{2} (\lambda_n^2 \rho_n)^{\frac{2}{N-2}} |u|_{2^*}^{2^*},
\]
and $\lambda_n^2 \rho_n \to 0$ as $n \to \infty$, which completes the proof. \qed

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**References**

[1] N. Akhmediev, A. Ankiewicz: *Partially coherent solitons on a finite background*, Phys.Rev. Lett. **82** (13) (1999), 2661–2664.
[2] T. Aubin: *Problèmes isopérimétriques et espaces de Sobolev*, J. Differ. Geometry **11** (1976), 573–598.
[3] T. Bartsch, S. de Valeriola: *Normalized solutions of nonlinear Schrödinger equations*, Arch. Math., **100** (1) (2012), 75–83.
[4] T. Bartsch and N. Soave: *A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems*, J. Funct. Anal. **272** (12), (2017), 4998–5037.
[5] T. Bartsch and N. Soave: Corrigendum: *Correction to: A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems*, J. Funct. Anal., **275** (2), (2018), 516–521.
[6] T. Bartsch, L. Jeanjean, N. Soave: *Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$*, J. Math. Pures Appl., **106** (4) (2016), 383–614.
[7] J. Bellazzini, V. Georgiev, N. Visciglia: *Long time dynamics for semi-relativistic NLS and half wave in arbitrary dimension*, Math. Ann. **371** (2018), no. 1-2, 707–740.
[8] J. Bellazzini, L. Jeanjean, T. Luo: *Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations*, Proc. Lond. Math. Soc. (3) **107** (2013), no. 2, 303–339.
[9] H. Berestycki, P.L. Lions: *Nonlinear scalar field equations. I - existence of a ground state*, Arch. Ration. Mech. Anal. **82** (1983), 313–345.
[10] H. Berestycki, P.L. Lions: *Nonlinear scalar field equations. II. Existence of infinitely many solutions*, Arch. Ration. Mech. Anal. **82** (1983), 347–375.
[11] M. Del Pino: *New entire solutions to some classical semilinear elliptic problems*, In R. Bhatia, A. Pal, G. Rangarajan, V. Srinivas, M. Vanninathan (Eds.), *Proceedings of the International Congress of Mathematicians 2010*, ICM 2010, pp. 1934–1957, World Scientific Publishing.
[12] B.D. Esry, Chris H. Greene, James P. Burke, Jr., and John L. Bohn: *Hartree-Fock Theory for Double Condensates*, Phys. Rev. Lett. **78** (19) (1997), 3594–3597.
[13] D.J. Frantzeskakis: *Dark solitons in atomic Bose-Einstein condensates: from theory to experiments.*, J. Phys. A: Math. Theor. **43** (2010).
[14] N. Ghoussoub: *Duality and Perturbation Methods in Critical Point Theory*, Cambridge Tracts in Mathematics, vol. 107, Cambridge University Press, Cambridge (1993).
[15] L. Jeanjean: *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal. **28** (10) (1997), 1633–1659.
[16] L. Jeanjean, S.-S. Lu: *A mass supercritical problem revisited*, Calc. Var. Partial Differential Equations **59** (2020), no. 5, Paper No. 174, 43 pp.
[17] P.-L. Lions: The concentration-compactness principle in the calculus of variations. The locally compact case. Part I and II, Ann. Inst. H. Poincaré, Anal. Non Linéaire, 1, (1984), 109–145; and 223–283.
[18] B. Malomed: Multi-component Bose-Einstein condensates: Theory in: P.G. Kevrekidis, D.J. Frantzeskakis, R. Carretero-Gonzalez (Eds.): Emergent Nonlinear Phenomena in Bose-Einstein Condensation, Springer-Verlag, Berlin, 2008, 287–305.
[19] M. Mariš: On the symmetry of minimizers, Arch. Ration. Mech. Anal. 192 (2009), no. 2, 311–330.
[20] J. Mederski: Nonradial solutions for nonlinear scalar field equations, Nonlinearity 33 (2020), no. 12, 6349–6381.
[21] J. Mederski: General class of optimal Sobolev inequalities and nonlinear scalar field equations, submitted arXiv:1812.11451
[22] J. Shatah: Unstable ground state of nonlinear Klein–Gordon equations, Trans. Amer. Math. Soc., 290 (2) (1985), 701–710.
[23] M. Shibata: Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term, Manuscripta Math. 143, (2014) 221–237.
[24] N. Soave: Normalized ground states for the NLS equation with combined nonlinearities, J. Differential Equations 269 (2020), no. 9, 6941–6987.
[25] W.A. Strauss: Existence of solitary waves in higher dimensions, Commun. Math. Phys. 55, (1977), 149–162.
[26] C.A. Stuart: Bifurcation for Dirichlet problems without eigenvalues, Proc. Lond. Math. Soc. 45 (1982), 169–192.
[27] M. Struwe: Variational Methods, Springer 2008.
[28] A. Szulkin, T. Weth: Ground state solutions for some indefinite variational problems, J. Funct. Anal. 257 (2009), no. 12, 3802–3822.
[29] G. Talenti: Best constants in Sobolev inequality, Annali di Matematica 10 (1976), 353–372.
[30] E. Timmermans: Phase Separation of Bose-Einstein Condensates, Phys. Rev. Lett. 81 (26) (1998), 5718–5721.
[31] W. P. Ziemer: Weakly differentiable functions. Sobolev spaces and functions of bounded variation, Graduate Texts in Mathematics, 120. Springer-Verlag, New York (1989).

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