ON THE RANKIN-SELBERG INTEGRAL OF KOHNEN AND SKORUPPA

AARON POLLACK AND SHRENIK SHAH

Abstract. The Rankin-Selberg integral of Kohnen and Skoruppa [7] produces the Spin $L$-function for holomorphic Siegel modular forms of genus two. In this paper, we reinterpret and extend the integral in [7] to apply to arbitrary cuspidal automorphic representations of $PGSp_4$. We show that the integral is related to a non-unique model and analyze it using the approach of Piatetski-Shapiro and Rallis.

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1. Introduction

The Rankin-Selberg integral representation of Kohnen-Skoruppa [7] produces the Spin $L$-function for holomorphic Siegel modular cusp forms on $GSp_4$. Their integral makes use of a special Siegel modular form $P_D(Z)$, which is in the Saito-Kurokawa or Maass subspace. Here $Z$ is the variable in the Siegel upper half space of genus two, and $D$ is a negative discriminant. The proof in [7] is classical, and involves global calculations with the Fourier coefficients of Siegel modular forms. In particular, Kohnen and Skoruppa make essential use of the Maass relations, which are identities satisfied by the Fourier coefficients of Siegel modular forms that are Saito-Kurokawa lifts.

The purpose of this paper is to reinterpret and extend the Rankin-Selberg integral representation in [7]. We define a class of automorphic functions $P_D^\alpha(g)$ on $GSp_4$ depending on a nonzero integer $D$ (which is allowed to be positive) and some auxiliary data $\alpha$, generalizing the $P_D(Z)$ of [7]. Denote by $Q$ the Klingen parabolic of $GSp_4$, i.e. the maximal parabolic stabilizing a line in the defining four-dimensional representation of $GSp_4$. For a cusp form $\phi$ in the space of an automorphic cuspidal representation $\pi$ of $GSp_4$ with trivial central character, the integral we consider is

$$I(\phi; s) = \int_{Z(A)GSp_4(Q)\backslash GSp_4(A)} E_Q(g; s)P_D^\alpha(g)\phi(g) \, dg$$

where $E_Q(g; s)$ is the Klingen Eisenstein series. We also consider another Rankin-Selberg integral that may be thought of as a degenerate version of $I(\phi; s)$; see Section 2. We unfold these integrals and show that they are equal to the partial Spin $L$-function $L^S(\pi, \text{Spin}, s)$ times an integral $I_S(\phi; s)$ over some set of bad places $S$. Although we do not check this in general here, it follows from Theorem 3.5 and the result of Li [9] that one can choose $D$ and the other data in the integral so that $I(\phi; s)$ is nonvanishing. When the cusp form $\phi$ comes from a level one, holomorphic Siegel...
modular form, we can even pick the data in the integral in a way that enables us to completely calculate the integral \(I_\infty(\phi; s)\) in terms of \(\Gamma\)-functions. We thus obtain a complete \(L\)-function in this case, thereby recovering the result of [7]. However, as explained below, our proof is very different.

It turns out that the integral \(I(\phi; s)\) unfolds to a \textit{non-unique model}. Recall that in the typical scenario of Rankin-Selberg integral representations, the unfolded integral depends on a model of the cuspidal representation \(\pi\), such as the Whittaker or Bessel model, that is known to be unique. The uniqueness of the model then implies that the integral is an Euler product, and one checks, place-by-place, that the local integral coming from the unfolded Rankin-Selberg convolution is equal to the local Langlands \(L\)-function. However, there are a handful of examples ([12], [3], [3], [6]) of Rankin-Selberg convolutions that unfold to a model that is not unique, yet are still known to represent \(L\)-functions. The integral in this paper is one more example of such a Rankin-Selberg convolution.

We are interested in these integrals for several reasons. One is their unusual property that by selecting suitable data they can represent the Spin \(L\)-function for all cuspidal automorphic representations, rather than either only generic or only holomorphic ones. Another is that they are members of the limited class of Rankin-Selberg integrals that have the potential to be reinterpreted in terms of geometric objects on a Shimura variety, and thus could prove useful in answering arithmetic questions. Moreover, these integrals have a higher rank generalization to \(\text{GSp}_6\) [13], yielding another pair of integrals that represent the Spin \(L\)-function there; the integrals in [13] should have connections to motivic questions.

We now define the special functions \(P^\alpha_D\). Recall that there is an isomorphism between \(\text{PGSp}_4\) and a split special orthogonal group \(\text{SO}(V_5, q)\). Here \(V_5\) is a certain 5-dimensional rational vector space and \(q\) is a quadratic form on \(V_5\). Choose \(v_D\) in \(V_5(\mathbb{Q})\) such that \(q(v_D) = D \neq 0\). Let \(\alpha = \prod_v \alpha_v\) be a factorizable function in \(S(V_5(\mathbb{A}))\), the space of Schwartz functions on \(V_5(\mathbb{A})\), equal to the characteristic function of \(V_5(\mathbb{Z})\) almost everywhere. We assume \(\text{GSp}_4\) acts on \(V_5\) on the right. The special automorphic function \(P^\alpha_D\) is

\[
P^\alpha_D(g) = \sum_{\delta \in \text{Stab}(v_D)(\mathbb{Q}) \setminus \text{GSp}_4(\mathbb{Q})} \alpha(v_D\delta g),
\]

where \(\text{Stab}(v_D)\) is the stabilizer of \(v_D\) in \(\text{GSp}_4\). (See section 2.2 for more details.) Actually, the condition that \(\alpha_\infty\) be Schwartz is too restrictive; to treat holomorphic Siegel modular forms it will behoove us to take \(\alpha_\infty\) decaying polynomially at infinity.

One important difference between this paper and [7] is in our definition of the special function \(P^\alpha_D\). In [7], \(P_D\) is defined as the lift to the Maass subspace of a certain Poincare series for Jacobi forms. The authors then use Hecke operators on Jacobi forms and the special properties of the Fourier-Jacobi expansion of elements in the Maass subspace as their key tools to relate \(I(\phi; s)\) to the Spin \(L\)-function. However, it is not all that difficult to check that the \(P_D\) of [7] may be expressed as a sum of the above the type; hence our definition of \(P^\alpha_D\). By expressing \(P^\alpha_D\) as a sum, we can use it to help unfold the global integral. Thus our unfolding is different from that in [7], as is our proof that the unfolded integral represents the Spin \(L\)-function.

We may put the functions \(P^\alpha_D\) in a larger context. For a Schwartz-Bruhat function \(\alpha\) on \(V_5(\mathbb{A})\) as above, one can consider the theta function \(\theta^\alpha(g, h)\) for the dual pair \((\text{PGSp}_4 \simeq \text{SO}(5)) \times \tilde{\text{SL}}_2\) inside \(\text{Sp}_{10}\). Taking a Fourier coefficient of \(\theta^\alpha\) with respect to \(\tilde{\text{SL}}_2\), one gets a function on \(\text{PGSp}_4\), and this function is \(P^\alpha_D(g)\). To interpret this in terms of classical Siegel modular forms, we can pick the data \(\alpha_\infty\) so that when \(D\) is negative,

\[
P_D(Z) = \sum_{v \in V_5(\mathbb{Z}), q(v) = -|D|} \frac{1}{Q_v(Z)^r}.
\]
Here $Z$ is in the Siegel upper half-space, and $Q_v$ is a certain quadratic form on the space of complex two-by-two symmetric matrices. (See Section 6.2.) As the $P_D$ are connected to the Fourier expansion of theta functions, it should come as no surprise that related functions have been considered previously by many authors in different contexts. Zagier [14] appears to have first defined the analogous functions for modular forms ($\text{PGL}_2 \simeq \text{SO}_3$) and Hilbert modular forms over a real quadratic field ($\text{PGL}^*_2, \mathbb{Q}(\sqrt{D}) \simeq \text{SO}_4$).

Finally, let us mention some other integrals for the Spin $L$-function on symplectic groups. There are the integrals of Novodvorsky [10], and Piatetski-Shapiro [11] (following Andrianov [1]) on $\text{GSp}_4$, which unfold to the Whittaker, and Bessel models, respectively. There is also the integrals of Bump-Ginzburg [4] on $\text{GSp}_6, \text{GSp}_8$ and $\text{GSp}_{10}$, which unfold to the Whittaker models. The integral in this paper should be considered as a prelude to the forthcoming paper [13], in which we give a new Rankin-Selberg integral for the Spin $L$-function on $\text{GSp}_6$. Although the integral in [13] does not apply to holomorphic Siegel modular forms, it uses the same special function $P^\alpha_D$ on $\text{GSp}_4$ used here, and also unfolds to a non-unique model. The contents of the various sections is as follows: In the next section, we define the groups, the global Rankin-Selberg integrals, and the special functions that we use. In section three, we unfold the integrals, discuss the general technique of Piatetski-Shapiro and Rallis to analyze non-unique models, and state the main theorems. In section four we give the unramified calculation, and in section five, we control the local integrals at the bad finite places. In section six, we choose data appropriate for Siegel modular forms and calculate a corresponding archimedean integral $I_\infty(\phi; s)$ in terms of $\Gamma$-functions.

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2. Global constructions

In this section we define the global Rankin-Selberg integrals considered in this paper.

2.1. Groups and embeddings. We define

$$J_4 = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix}$$

and

$$\text{GSp}_4 = \{ g \in \text{GL}_4 | gJ_4g^t = \nu(g)J_4 \},$$

where $\nu : \text{GSp}_4 \to \text{GL}_1$ is the similitude. The letter $Z$ is used to denote the diagonal center of $\text{GSp}_4$.

Let $L = \mathbb{Q}(\sqrt{D})$ be a quadratic etale $\mathbb{Q}$-algebra. Denote by $\text{GL}^*_2, L$ the subgroup of $\text{GL}_2, L$ with determinant in $\mathbb{Q}$. We describe an embedding of $\text{GL}^*_2, L$ into $\text{GSp}_4$. To do this, consider $(L^2, \langle \cdot, \cdot \rangle_L)$, the two-dimensional free module over $L$ with its standard alternating bilinear form. Define the

\footnote{Denote by $R$ the parabolic of $\text{GSp}_6$ that stabilizes an isotropic flag consisting of a line inside a (maximal) three-dimensional isotropic subspace of the defining six-dimensional representation of $\text{GSp}_6$. If $U_R$ is the unipotent radical of $R$, then the integral in [13] is nonvanishing for cuspidal representations that support a Fourier coefficient defined by a nondegenerate character of $U_R$.}
Q-bilinear form $\langle , \rangle$ via $\langle x, y \rangle = \text{tr}_{L/Q}(\langle x, y \rangle_L)$. As $\text{GL}^*_{2,L}$ is the group preserving $\langle , \rangle_L$ up to $Q^*$ multiple, one obtains an inclusion $\text{GL}^*_{2,L} \subseteq \text{GSp}(\langle , \rangle)$. If the basis of $L^2$ is $\{b_1, b_2\}$, then the basis $\frac{1}{2\sqrt{D}}b_1, \frac{1}{2}b_1, \sqrt{D}b_2, b_2$ gives an isomorphism $\text{GSp}(\langle , \rangle) \cong \text{GSp}_4$. We use this identification throughout. Alternatively, with this choice of basis, the map that is multiplication by $\sqrt{D}$ is given by the matrix

$$
\begin{pmatrix}
1 & 0 \\
D & 1
\end{pmatrix},
$$

and $\text{GL}^*_{2,L}$ is the centralizer of this matrix inside $\text{GSp}_4$. One can check immediately that $\text{GL}^*_{2,L}$ is the set matrices inside $\text{GSp}_4$ that have the following form:

$$
\begin{pmatrix}
a & b & e & f \\
Db & a & f & De \\
Dc & d & x & Dy \\
d & c & y & x
\end{pmatrix}.
$$

### 2.2. The special function $P^\beta_D$. We now define the special function $P^\beta_D$ to be used in the integral representations. Let $\text{GSp}_4$ act on $W_4 = Q^4$ on the right, and denote the ordered basis of $W_4$ by $\{e_1, e_2, f_1, f_2\}$, so that $\langle e_i, f_j \rangle = \delta_{ij}$. Recalling that $\nu$ denotes the one-dimensional similitude representation, define $\wedge^2_0(W_4) = \ker\{\wedge^2 W_4 \to \langle v, w \rangle\}$ and set $V_5 = \wedge^2_0(W_4) \otimes \nu^{-1}$. As $\langle \wedge^4 W_4 \otimes \nu^{-2} \rangle$ is the trivial representation of $\text{GSp}_4$, the exterior product $\wedge : V_5 \otimes V_5 \to \wedge^4 W_4 \otimes \nu^{-2}$ defines the invariant quadratic form on $V_5$. That is, we define $\langle , \rangle$ via

$$
v \wedge w = \langle v, w \rangle e_1 \wedge e_2 \wedge f_1 \wedge f_2,
$$

where $e_1 \wedge e_2 \wedge f_1 \wedge f_2$ is here considered as an element of $\wedge^4 W_4 \otimes \nu^{-2}$. We put an integral structure on $V_5$ by picking the basis

$$
e_1 \wedge f_1 - e_2 \wedge f_2; \ e_1 \wedge e_2; \ e_1 \wedge f_2; \ e_2 \wedge f_1; \ f_1 \wedge f_2.
$$

For a nonzero integer $D$ we denote $v_D = D e_1 \wedge f_2 + e_2 \wedge f_1$ in $V_5$. Then $q(v_D) = (v_D, v_D) = D$, and one can check that the stabilizer of $v_D$ is $\text{GL}^*_{2,L}$. Now take $\alpha = \prod_v \alpha_v$ factorizable in the Schwartz space $S(V_5(A))$. We define

$$
P^\beta_D(g) = \sum_{\delta \in \text{GL}^*_{2,L}(Q) \setminus \text{GSp}_4(Q)} \alpha(v_D \delta g).
$$

In section 6.2 we will pick an $\alpha_\infty$ very convenient for computing with holomorphic Siegel modular forms.

### 2.3. The global integrals. We now define the two global Rankin-Selberg convolutions to be studied in this paper. The integral representations use the Klingen Eisenstein series on $\text{GSp}_4$. We define the Klingen parabolic, denoted $Q$, to be the stabilizer of the line $Qf_2$ via the right action. Pick a factorizable section $f(g; s) = \prod_v f_v(g; s) \in \text{Ind}(\delta_Q)$, $\delta_Q$ the modulus character of the Klingen parabolic. We require $f$ to be right $K$-finite and that $f_p(1; s) = 1$, independent of $s$, for almost all finite places $p$. We renormalize these sections by defining $f^*_{\gamma}(g; s) = \zeta_p(4s)f_p(g; s)$ for almost all finite places $p$. The normalized Eisenstein series is then

$$
E^*(g;s) = \sum_{\gamma \in Q(Q) \setminus \text{GSp}_4(Q)} f^*(\gamma g; s).
$$
Alternatively, if $\Phi \in \mathcal{S}(W_4(A))$, then

$$ f^*(g; s) = |\nu(g)|^{2s} \int_{GL_1(A)} \Phi(tf_2 g)|t|^{1s} dt $$

is a normalized section.

Suppose $\pi$ is a cuspidal automorphic representation on GSp$_4$ with trivial central character and $\phi$ is a cusp form in the space of $\pi$. The global integrals are as follows:

$$ (2) \quad I_1(\phi; s) = \int_{Z(A)GL_{2,L}(\mathbb{Q})\backslash GL_{2,L}(A)} E^*(g; s)\phi(g) \, dg $$

and

$$ (3) \quad I_2(\phi; s) = \int_{Z(A)GSp_4(\mathbb{Q})\backslash GSp_4(A)} E^*(g; s)P_D^\alpha(g)\phi(g) \, dg. $$

The main theorems of this paper, spelled out in the next section, relate both of these integrals to the partial Spin $L$-function $L^S(\pi, \text{Spin}, 2s - \frac{1}{2})$.

The integrals $I_1$ and $I_2$ are very closely related: If one unfolds the sum defining $P_D^\alpha$ in $I_2$ then the integral $I_1$ appears as an inner integral of this partially unfolded $I_2$. Note also that $I_1$ is very similar to the integrals of Andrianov [1] and Piatetski-Shapiro [11] for Spin on GSp$_4$ that unfold to the Bessel model. The difference between the integrals of [11] and [1] and $I_1$ is that [11] and [1] use an Eisenstein series for the group GL$_{2,L}$ as opposed to one for GSp$_4$. One reason for considering the integral $I_2$ and not just the slightly simpler $I_1$ is that $I_2$ is better behaved: $I_2$ produces the correct local $L$-function at primes dividing $2D$, whereas $I_1$ does not necessarily give the correct factor at these primes. (See Theorem 3.2)

3. Main theorems

In this section we give the unfoldings of the global integrals, and state all the main theorems of the paper. The calculations proving these theorems are given in Sections 4, 5, and 6. As we mentioned above, the integral unfolds to a model that is not unique. The general strategy for analyzing such integrals—which we follow—was developed by Piatetski-Shapiro and Rallis [12], and is well-known. For the convenience of the reader, we will explain this strategy in the case of the integrals $I_1(\phi; s)$ and $I_2(\phi; s)$.

3.1. Unfolding. Recall that the standard Siegel parabolic $P$ on GSp$_4$ is the stabilizer of the isotropic subspace of $W_4$ spanned by $f_1, f_2$. Denote by $U$ its unipotent radical, and by $M$ the Levi consisting of elements of GSp$_4$ whose lower left and upper right two-by-two blocks are zero. We denote by $N$ the unipotent radical of the standard Borel in GL$_{2,L}$. Then, under the embedding GL$_{2,L} \to \text{GSp}_4$, $N$ is contained in $U$.

In fact, define a unitary character $\chi : U(A) \to \mathbb{C}^\times$ as follows. As is usual, pick once and for all an additive character $\psi : \mathbb{Q}\backslash A \to \mathbb{C}^\times$, with conductor equal to one at all finite places. If

$$ u = \begin{pmatrix} 1 & u_{11} & u_{12} \\ 1 & u_{12} & u_{22} \\ 1 & 1 & 1 \end{pmatrix}, $$

then set $\chi(u) = \psi(-Du_{11} + u_{22})$. Note that $\chi$ is trivial on $N$. With this definition of $\chi$, define

$$ \phi_{\chi}(g) := \int_{U(\mathbb{Q})\backslash U(A)} \chi^{-1}(u)\phi(ug) \, du $$
and
\[ \alpha_\chi(g) := \int_{N(A)\backslash U(A)} \chi(u)\alpha(vDug)\,du. \]

Then

**Proposition 3.1.** The global integrals \( I_1 \) and \( I_2 \) unfold:
\[ I_1(\phi, s) = \int_{N(A)Z(A)\backslash GL_{2,L}(A)} f^*(g; s)\phi_\chi(g)\,dg \]
and
\[ I_2(\phi, s) = \int_{U(A)Z(A)\backslash GSp_4(A)} f^*(g; s)\phi_\chi(g)\alpha_\chi(g)\,dg. \]

**Proof.** We will unfold \( I_2 \). The unfolding of \( I_1 \) is almost identical, but simpler.

Assume for now that \( L \) is a field, and not the split quadratic etale \( Q \) algebra. Then \( GL_{2,L}^* \) acts transitively on \( W_4 \), and the stabilizer \( B' \) of the line \( Qf_2 \) consists of the elements of \( GSp_4 \) of the form \( (1) \) with \( c = d = y = b = 0 \). Hence
\[ I_2(\phi; s) = \int_{B'(Q)Z(A)\backslash GSp_4(A)} \alpha(vDg)f^*(g; s)\phi(g). \]

Now
\[ \int_{N(Q)\backslash N(A)} \phi(n)\,dn = \sum_{\gamma\in Z(Q)\backslash N(Q)\backslash B'(Q)} \phi_\chi(\gamma g), \]
from which the proposition follows in this case.

In case \( L = Q \times Q \) is split, then \( GL_{2,L}^* \) acts on the lines in \( W_4 \) with three orbits, one represented by \( Qf_2 \) and the other two by \( Q(f_1 \pm f_2) \). After a change of variables, one can check that the integrals associated to \( f_1 \pm f_2 \) vanish by cuspidality of \( \phi \). The unfolding of the integral associated to \( Qf_2 \) proceeds as above. \( \square \)

### 3.2. Non-unique models.

The Fourier coefficient \( \phi_\chi \) does not factorize for general cusps forms \( \phi \). Thus, it is not at all clear that the integrals \( I_1 \) and \( I_2 \) have the structure of an Euler product. However, Piatetski-Shapiro and Rallis in \([12]\) introduced \( f \) or a strategy for analyzing exactly this sort of phenomenon.

Suppose \((\pi_v, V)\) is an irreducible admissible representation of \( GSp_4(Q_v) \). Define a \((U, \chi)\) model to be a linear functional \( \ell : V \to \mathbb{C} \) that satisfies \( \ell(\pi(u)v) = \chi(u)\ell(v) \) for all \( u \in U(Q_v) \) and \( v \in V \). If one replaced \( U \) in this definition by \( U_B \), the unipotent radical of a Borel of \( GSp_4 \), and \( \chi \) by a nondegenerate character of \( U_B \), then one would get the definition of the Whittaker model. However, unlike the Whittaker model, the space of \((U, \chi)\) models will usually be infinite dimensional. Consequently, the Fourier coefficient \( \phi_\chi \) will not factorize. However, the following theorem allows one to recover the Euler product nature of the global integral nonetheless.

**Theorem 3.2.** Suppose all the data are unramified: \( p \) is finite, \( \pi_p \) is unramified, \( f^* \) is right \( GSp_4(Z_p) \) invariant, and \( \alpha_p \) is the characteristic function of \( V_5(Z_p) \). Let \( v_0 \) denote a nonzero spherical vector in \( \pi_p \). Then, for all \((U, \chi)\) models \( \ell \),
\[ I_{2,p}(\ell; s) := \int_{U(Q_p)Z(Q_0)\backslash GSp_4(Q_0)} f^*_p(g; s)\alpha_{\chi,p}(g)\ell(\pi(g)v_0)\,dg = \ell(v_0)L(\pi_p, \text{Spin}, 2s - \frac{1}{2}) \]
when \( \text{Re}(s) \) is sufficiently large. Furthermore, if \( p \) does not divide \( 2D \), then
\[ I_{1,p}(\ell; s) := \int_{N(Q_p)Z(Q_0)\backslash GL_{2,L}^*(Q_p)} f^*_p(g; s)\ell(\pi(g)v_0)\,dg = \ell(v_0)L\left(\pi_p, \text{Spin}, 2s - \frac{1}{2}\right) \]
for \( \text{Re}(s) \) sufficiently large.
We will prove the part of this theorem concerning $I_{2,p}$ in the next section; the part concerning $I_1$ is similar, but easier, so we omit it. Let us explain how one uses it to analyze the global integral. For a finite set of places $\Omega$, put
\[
I_{2,\Omega}(\phi; s) = \int_{U(\mathcal{A}_\Omega)Z(\mathcal{A}_\Omega)\setminus \text{GSp}_4(\mathcal{A}_\Omega)} f_\Omega^s(g; s)\alpha_{\chi,\Omega}(g)\phi_\chi(g) \, dg,
\]
where for a linear algebraic group $H$ we write $H(\mathcal{A}_\Omega) = \prod_{v \in \Omega} H(\mathcal{Q}_v)$ and $f_\Omega$ and $\alpha_{\chi,\Omega}$ respectively denote the product of the factors of $f^*$ and $\alpha_\chi$ for places in $\Omega$. Then
\[
I_2(\phi; s) = \lim_{\Omega} I_{2,\Omega}(\phi; s).
\]

Suppose $S$ is a finite set of places containing all the bad places, i.e. all the places excluded in the theorem above, and $\Omega \supseteq S$. Then the theorem allows one to prove

**Lemma 3.3.** $I_{\Omega,p}(\phi; s) = L(\pi_p, \text{Spin}, 2s - \frac{1}{2})I_\Omega(\phi; s)$.

*Proof.* The integral $I_{\Omega,p}(\phi; s)$ is equal to
\[
\int_{U(\mathcal{A}_\Omega)Z(\mathcal{A}_\Omega)\setminus \text{GSp}_4(\mathcal{A}_\Omega)} f_\Omega^s(g; s)\alpha_{\chi,\Omega}(g) \int_{U(\mathcal{Q}_p)Z(\mathcal{Q}_p)\setminus \text{GSp}_4(\mathcal{Q}_p)} f_p^s(g_p; s)\alpha_{\chi,p}(g_p)\phi_\chi(gg_p) \, dg_p \, dg.
\]
But for each $g \in \text{GSp}_4(\mathcal{A}_\Omega)$, the function $g_p \mapsto \phi_\chi(gg_p)$ is of the form $\ell(\pi_p(g_p)v_0)$ for some $(U, \chi)$ model $\ell$ and spherical vector $v_0$. Furthermore $\ell(v_0) = \phi_\chi(g)$. Hence the lemma follows from Theorem 3.2.

One concludes that $I_2(\phi; s) = I_S(\phi; s)L^S(\pi, \text{Spin}, 2s - \frac{1}{2})$. The integration in $I_S$ over the bad finite places may be controlled in a standard way using the following easy proposition, which is proved in Section 5.

**Proposition 3.4.** For any $v_0$ in $V_{\pi_p}$, there exists a section of $f_p^*$ of $\text{Ind}(\delta_Q^s)$, a Schwartz function $\alpha_p$ on $V_5(\mathcal{Q}_p)$, and a vector $v$ in $V_{\pi_p}$, so that for all $(U, \chi)$ models $\ell$,
\[
\int_{U(\mathcal{Q}_p)Z(\mathcal{Q}_p)\setminus \text{GSp}_4(\mathcal{Q}_p)} f_p^s(g; s)\alpha_{\chi,p}(g)\ell(\pi(g)v) \, dg = \ell(v_0).
\]

We also prove the analogous statement for $I_1$. From the proposition, we see that the data $(f^*, \phi_1, \alpha)$ for the integral may be chosen so that
\[
I_1(\phi_1; s) = I_{1,\Omega}(\phi; s)L^S(\pi, \text{Spin}, 2s - \frac{1}{2}).
\]

### 3.3. Main theorems.

We summarize the findings above in the first main theorem.

**Theorem 3.5.** Suppose $\pi$ is an automorphic cuspidal representation of $\text{PGSp}_4(\mathcal{A})$, and $\phi$ is a cusp form in the space of $\pi$. Then the data $\alpha_f$, $f^*$, and $\phi_1 \in V_1$ may be chosen so that
\[
I_1(\phi_1; s) = I_{1,\infty}(\phi; s)L^S(\pi, \text{Spin}, 2s - \frac{1}{2})
\]
for a sufficiently large set of finite primes $S$. Here,
\[
I_{1,\infty}(\phi; s) = \int_{N(\mathcal{R})Z(\mathcal{R})\setminus \text{GL}_{2,L}(\mathcal{R})} f_{\infty}^*(g; s)\phi_\chi(g) \, dg.
\]

Similarly, the data may be chosen so that
\[
I_2(\phi_1; s) = I_{2,\infty}(\phi; s)L^S(\pi, \text{Spin}, 2s - \frac{1}{2}).
\]
for a sufficiently large set of finite primes $S$, where

$$I_{2,\infty}(\phi; s) = \int_{U(\mathbb{R})Z(\mathbb{R})\backslash GSp_4(\mathbb{R})} f_{\infty}^*(g; s)\phi_\chi(g)\alpha_{\chi,\infty}(g)\,dg.$$  

Remark 3.6. Note that the archimedean integrals $I_{\infty}(\phi; s)$ involve the actual cusp form $\phi$, not some arbitrary $(U, \chi)$ model $\ell$.

If $\phi$ comes from a level one holomorphic Siegel modular form of weight $r \geq 6$ (see section 6 for a precise statement), we can even choose the data $\alpha_\infty, f_{\infty}^*$ in such a way that we can calculate $I_{2,\infty}(\phi; s)$ explicitly in terms of $\Gamma$-functions.

Theorem 3.7. Suppose $\phi$ comes from a level one holomorphic Siegel modular form $f_\phi$ of weight $r \geq 6$, as made precise in section 6. Denote by $a((-D_1))$ the Fourier coefficient of $f_\phi$ corresponding to the two-by-two symmetric matrix $(-D_1)$. Then $\alpha_\infty, f_{\infty}^*$ may be chosen so that

$$I_{2,\infty}(\phi; s) = a((-D_1))\pi^{-2s}(4\pi)^{-(2s+r-2)}\Gamma(2s)\Gamma(2s + r - 2).$$

This theorem will be proved in Section 6. We thus obtain

Theorem 3.8. Suppose $\phi$ and $a((-D_1))$ are as in Theorem 3.7. Then the data $\alpha, f^*$ may be chosen so that

$$I(\phi; s) = a((-D_1))\pi^{-2s}(4\pi)^{-(2s+r-2)}\Gamma(2s)\Gamma(2s + r - 2)L(\pi, Spin, 2s - \frac{1}{2}).$$

In the following three sections we will find it convenient to have a notation for the characteristic function of different sets. If $\mathcal{P}$ is a condition on, say, the set of four-by-four matrices, then $\text{char}(\mathcal{P})$ denotes the characteristic function of the set where $\mathcal{P}$ is satisfied. For instance, $\text{char}(\det(g) \neq 0)$ denotes the characteristic function of $GL_4$ inside $M_4$. The exact domain of definition of the functions $\text{char}(\mathcal{P})$ will always be clear from the context.

4. Unramified Calculation

In this section we will prove Theorem 3.2. Again, the general strategy for proving this type of result goes back to Piatetski-Shapiro and Rallis [12]. One starts with the explicit determination of $L(\pi_p, Spin, s)$ in terms of Hecke operators, which is codified in the following statement.

Proposition 4.1. Suppose $\omega_p(g)$ is the Macdonald spherical function for the unramified representation $\pi_p$, normalized so that $\omega_p(1) = 1$. Define $\Delta^s(g) = |v(g)|^s \text{char}(g)$, where $\text{char}(g) = \text{char}(g \in M_4(\mathbb{Z}_p))$ is the characteristic function of $M_4(\mathbb{Z}_p)$. Then

$$\int_{GSp_4(\mathbb{Q}_p)} \Delta^s(g)\omega_p(g)\,dg = (1 - \omega_\pi(p)p^{2-2s})L(\pi_p, Spin, s - \frac{3}{2}),$$

where $\omega_\pi$ is the central character of $\pi_p$.

Proof. This is a classical result of Shimura in the theory of the rank two symplectic group. See, for instance, [2].

Suppose now that $\ell$ is a $(U, \chi)$ model. If $v_0$ is a spherical vector for $\pi_p$, it follows from Proposition 4.1 that

$$\int_{GSp_4(\mathbb{Q}_p)} \Delta^s(g) \ell(\pi(g)v_0)\,dg = \ell(v_0) \int_{GSp_4(\mathbb{Q}_p)} \Delta^s(g)\omega_p(g)\,dg = \ell(v_0)(1 - \omega_\pi(p)p^{2-2s})L(\pi_p, Spin, s - \frac{3}{2}).$$
Thus, to prove \( I_p(\ell; s) = \ell(v_0) L(\pi_p, \text{Spin}, 2s - \frac{1}{2}) \), it suffices to prove

\[
(1 - \omega_\pi(p)p^{2-2s}) I_p \left( \ell, \frac{s - 1}{2} \right) = \int_{\text{GSp}_4(\mathbb{Q}_p)} \Delta^s(g) \ell(\pi(g)v_0) \, dg.
\]

We will prove this by explicitly calculating both sides.

To begin, we make the calculation of \( \alpha_{X,v} \) almost everywhere. To succinctly express the result, we first make a definition.

**Definition 4.2.** Any \( m \in \text{GL}_2(\mathbb{Q}_p) \) is right \( \text{GL}_2(\mathbb{Z}_p) \)-equivalent to a matrix of the form \( \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \).

Define

\[
\Delta'_0(m) = \begin{cases} 1 & \text{if } m_{22}|m_{11}, m_{22}|m_{12}, \text{ and } (\frac{m_{22}}{m_{22}})^2 \equiv D \mod \frac{m_{22}}{m_{22}} \\ 0 & \text{otherwise} \end{cases}
\]

This definition is independent of the choice of \( m_{11}, m_{12}, \) and \( m_{22} \). For a matrix

\[
m = \begin{pmatrix} m_t \\ m_b \end{pmatrix}
\]

in the Levi of the Siegel parabolic of \( \text{GSp}_4 \), so that \( m_t, m_b \) are two-by-two matrices, define

\[
\Delta_0(m) = \text{char} \left( \frac{\det(m_t)}{\det(m_b)} \leq 1 \right) \left| \frac{\det(m_t)}{\det(m_b)} \right|^{1/2} \Delta'_0(m_b).
\]

**Proposition 4.3.** Suppose that \( \alpha_p \) is the characteristic function of \( V_2(\mathbb{Z}_p) \). Then if \( m \) is in the Levi of the Siegel parabolic, \( \alpha_{X,p}(m) = \Delta_0(m) \).

**Proof.** Suppose

\[
m = \begin{pmatrix} \lambda \\ \lambda \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ -c \\ -b \\ a \end{pmatrix}
\]

and

\[
n_{24}(z) = \begin{pmatrix} 1 \\ z \\ 1 \\ 1 \end{pmatrix}.
\]

Then \( v_D n_{24}(z)m \)

\[
= \delta^{-1}(cd - Da)(e_1 \wedge f_1 - e_2 \wedge f_2) + \delta^{-1}(Da^2 - c^2)e_1 \wedge f_2 + \delta^{-1}(d^2 - Db^2)e_2 \wedge f_1 - \frac{z}{\lambda}f_1 \wedge f_2,
\]

where \( \delta = ad - bc \). To calculate \( \alpha_X(m) \), by right \( \text{GSp}_4(\mathbb{Z}_p) \) invariance we may assume \( b = 0 \). Then \( v_D n_{24}(z)m \)

\[
= \frac{c}{a}(e_1 \wedge f_1 - e_2 \wedge f_2) + \frac{Da^2 - c^2}{ad}e_1 \wedge f_2 + \frac{d}{a}e_2 \wedge f_1 - \frac{z}{\lambda}f_1 \wedge f_2.
\]

Thus for the integral

\[
\int_{\mathbb{Q}_p} \psi(z) \alpha_p(v_D n_{24}(z)m) \, dz = \alpha_{X,p}(m)
\]

to be nonzero, we need to have \( |\lambda| \leq 1 \) and the other three coefficients above must be integral. When all these conditions are met, the integral is \( |\lambda| \). The conditions for the first three coefficients are exactly the conditions defining \( \Delta_0(m_b) \). Since \( |\lambda| = \left| \frac{\det(m_b)}{\det(m_b)} \right|^{1/2} \), the proposition follows. \( \square \)
As remarked above, the normalized sections \( f_p^*(g; s) \) appearing in the definition of the Eisenstein series may be constructed in the following way. Set \( \Phi \) to be the characteristic function of \( W_4(\mathbb{Z}_p) \). Then
\[
f_p^*(g; s) = |\nu(g)|^{2s} \int_{\text{GL}_4(\mathbb{Q}_p)} \Phi(t f_2 g)|t|^{4s} \, dt.
\]

We may now compute
\[
I_{p,2}(\ell; s) = \int_{U(\mathbb{Q}_p) Z(\mathbb{Q}_p) \setminus \text{GSp}_4(\mathbb{Q}_p)} f^*(g; s) \alpha_\chi(g) \ell(\pi(g)v_0) \, dg
\]
\[
= \int_{Z(\mathbb{Q}_p) \setminus M(\mathbb{Q}_p)} \delta_P^{-1}(g) f_p^*(g; s) \alpha_\chi(g) \ell(\pi(g)v_0) \, dg
\]
\[
= \int_{M(\mathbb{Q}_p)} \delta_P^{-1}(g) |\nu(g)|^{2s} \Phi(f_2 g) \alpha_\chi(g) \ell(\pi(g)v_0) \, dg
\]
\[
= \int_{M(\mathbb{Q}_p)} \delta_P^{-1}(g) |\nu(g)|^{2s} \Delta_0(g) \text{char}(\pi(g)v_0) \, dg.
\]
The last equality is because from Definition 4.2, one checks that if \( \Delta_0(g) \neq 0 \), then \( \Phi(f_2 g) \neq 0 \) if and only if \( \text{char}(g) \neq 0 \). For an element \( g \) of \( \text{GL}_4(\mathbb{Q}_p) \), we define \( \text{val}_p(g) \) to be the unique integer such that \( g = p^{\text{val}_p(g)} g_0 \) with \( g_0 \) in \( M_4(\mathbb{Z}_p) \setminus p M_4(\mathbb{Z}_p) \). Then
\[
I_{p,2}(\ell; s) = \sum_{k \geq 0} (|p|^{4s} \omega_\pi(p))^k \int_{M(\mathbb{Q}_p)} \delta_P^{-1}(g) |\nu(g)|^{2s} \Delta_0(g) \text{char}(\text{val}_p(g) = 0) \ell(\pi(g)v_0) \, dg
\]
\[
= (1 - \omega_\pi(p)p^{-4s})^{-1} \int_{M(\mathbb{Q}_p)} \delta_P^{-1}(g) |\nu(g)|^{2s} \Delta_0(g) \text{char}(\text{val}_p(g) = 0) \ell(\pi(g)v_0) \, dg.
\]

Hence
\[
(1 - \omega_\pi(p)p^{2-2s})I_p(\ell; s - 1) = \int_{M(\mathbb{Q}_p)} \delta_P^{-1}(g) |\nu(g)|^{s-1} \Delta_0(g) \text{char}(\text{val}_p(g) = 0) \ell(\pi(g)v_0) \, dg.
\]

Now let us compute
\[
\int_{\text{GSp}_4(\mathbb{Q}_p)} \Delta^s(g) \ell(\pi(g)v_0) \, dg = \int_{M(\mathbb{Q}_p)} \delta_P^{-1}(g) |\nu(g)|^s \ell(\pi(g)v_0) \left\{ \int_{U(\mathbb{Q}_p)} \chi(u) \text{char}(ug) \, du \right\} \, dg.
\]

**Lemma 4.4.** Define \( K_M = \text{GSp}_4(\mathbb{Z}_p) \cap M(\mathbb{Q}_p) \), let \( g \) be in \( M(\mathbb{Q}_p) \), and suppose that \( g \) is right \( K_M \)-equivalent to a matrix with bottom right \( 2 \times 2 \) block of the form \( \begin{pmatrix} p^a & b \\ p^b & p^c \end{pmatrix} \). Then
\[
\int_{U(\mathbb{Q}_p)} \chi(u) \text{char}(ug) \, du
\]
is 0 unless \( c = 0 \) and \( b^2 \equiv D \text{ modulo } p^a \), in which case the integral is \( p^a \).

As a function of \( g \in M(\mathbb{Q}_p) \), the integral is certainly right \( K_M \)-invariant, hence there is no loss of generality in assuming \( g \) is of the special form above. Note also that when \( p \) is inert in \( L \), the only \( 2 \times 2 \) matrix satisfying these conditions is the identity.

**Proof.** The set \( U_g := \{ u \in U(\mathbb{Q}_p) | \text{char}(ug) \neq 0 \} \) forms an abelian group. Hence the integral vanishes unless \( \chi \) is identically 1 on \( U_g \). If \( c > 0 \) then the matrix with upper right block \( \begin{pmatrix} p^{-a} \\ -b p^{-a} \\ b^2 p^{-a} \end{pmatrix} \) is in \( U_g \), so the integral vanishes. So now assume \( c = 0 \). Then the matrix with upper right block \( \begin{pmatrix} p^{-a} \\ -b p^{-a} \\ b^2 p^{-a} \end{pmatrix} \) is in \( U_g \). As \( \chi \) of this matrix is \( \psi(\frac{b^2 - D}{p}) \), we get the second condition. Finally, when both conditions are satisfied, suppose \( u \) has upper right block \( \begin{pmatrix} a_{11} & a_{12} \\ u_{12} & u_{22} \end{pmatrix} \). Then one sees for \( u \) to be
in \( U_g \) we must have \( u_{11} \in p^{-a} \mathbb{Z}_p \), and then that \( u_{12}, u_{22} \) are determined modulo \( \mathbb{Z}_p \). Finally, the second condition forces \( \chi \) to be trivial on \( U_g \), so the measure of \( U_g \), which is \( p^a \).

**Lemma 4.5.** If \( g \in M \), and \( \ell (\pi(g)v_0) \neq 0 \), then
\[
\int_{U(\mathbb{Q}_p)} \chi(u) \text{char} (u g) \, du = |\nu(g)|^{-1} \Delta_0(g) \text{char} (\text{val}_p(g) = 0)
\]

**Proof.** This is an immediate consequence of the previous lemma, except for the fact that, in the notation above, \( \ell (\pi(g)v_0) \neq 0 \) forces \( |\det(m_t)|/|\det(m_b)| \leq 1 \). This latter fact may be seen by taking \( u \in U(\mathbb{Z}_p) \), and considering the equation
\[
\ell (\pi(g)v_0) = \ell (\pi(g)u v_0) = \chi (g u g^{-1}) \ell (\pi(g)v_0).
\]

One deduces that \( \ell (\pi(g)v_0) \neq 0 \) implies \( \chi (g u g^{-1}) = 1 \) for all \( u \in U(\mathbb{Z}_p) \), from which one can check \( |\det(m_t)|/|\det(m_b)| \leq 1 \).

Hence
\[
\int_{\text{GSp}_4(\mathbb{Q}_p)} \Delta^4(g) \ell (\pi(g)v_0) \, dg = \int_{M(\mathbb{Q}_p)} \delta_P^{-1}(g) |\nu(g)|^4 \ell (\pi(g)v_0) \left\{ \int_{U(\mathbb{Q}_p)} \chi(u) \text{char} (u g) \, du \right\} \, dg
\]
\[
= \int_{M(\mathbb{Q}_p)} \delta_P^{-1}(g) |\nu(g)|^4 |\nu(g)|^{-1} \Delta_0(g) \text{char} (\text{val}_p(g) = 0) \ell (\pi(g)v_0) \, dg.
\]

This last line is what we obtained for \((1 - \omega_\pi(p)p^{2s}) I_p(\ell; \frac{s - 1}{2})\), and thus
\[
\int_{\text{GSp}_4(\mathbb{Q}_p)} \Delta^4(g) \ell (\pi(g)v_0) \, dg = (1 - \omega_\pi(p)p^{2s}) I_p \left( \ell; \frac{s - 1}{2} \right),
\]
completing the unramified calculation.

## 5. Ramified Calculation

In this section we prove Proposition 5.1. We first do the easier case for the integral \( I_1 \):

**Proposition 5.1.** For any \( v_0 \) in \( V_{\pi_p} \), there exists a section \( f_p^* \) of \( \text{Ind}(\delta_Q^*) \) and a vector \( v \) in \( V_{\pi_p} \), such that for all \((U, \chi)\) models \( \ell \)
\[
\int_{\mathbb{N}(\mathbb{Q}_p) \mathbb{Z}(\mathbb{Q}_p) \cap \text{GL}_2(\mathbb{Q}_p)} f_p^* (g; s) \ell (\pi(g)v) \, dg = \ell (v_0).
\]

**Proof.** Denote by \( K_N = \{1 + p^N M_4(\mathbb{Z}_p)\} \cap \text{GSp}_4(\mathbb{Z}_p) \) the congruence subgroups of \( \text{GSp}_4(\mathbb{Z}_p) \). The vector \( v_0 \) is stabilized by some \( K_N, N \gg 0 \). Define \( \varphi_1 \in C_c^\infty(\mathbb{Q}_p) \) by \( \varphi_1(z) = \psi(-z)|p|^N \text{char}(p^N z \in \mathbb{Z}_p) \). Now set
\[
v_1 = \int_{G_0(\mathbb{Q}_p)} \varphi_1(z) \pi(u(z)) v_0 \, dz,
\]
where
\[
u(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & z \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Note that if
\[
m = \begin{pmatrix} tx & -ty \\ -tDy & tx \\ x & Dy \\ y & x \end{pmatrix},
\]
so that \( m \in T_L(\mathbb{Q}_p) \), where \( T_L \) denotes the diagonal torus of \( \text{GL}_2^*(\mathbb{Q}_p) \), then

\[
\ell(\pi(m)v_1) = \left\{ \int_{G_u(\mathbb{Q}_p)} \varphi_1(z) \chi(mu(z)m^{-1}) \, dz \right\} \ell(mv_0)
\]

and \( \chi(mu(z)m^{-1}) = \psi(tz) \). Since

\[
\int_{G_u(\mathbb{Q}_p)} \varphi_1(z) \psi(vz) \, dz = \text{char}(v \in 1 + p^N \mathbb{Z}_p),
\]

we conclude that

\[
\ell(\pi(m)v_1) = \text{char}(t \in 1 + p^N \mathbb{Z}_p) \ell(\pi(m)v_0).
\]

Now \( v_1 \) is stable by some congruence subgroup \( K_{N_1} \). We pick a section \( f^* \) supported on \( QK_{N_1} \). For example, one may take

\[
f^*(g;s) = \int_{Q(\mathbb{Q}_p)} \delta_Q^{-s}(q) \text{char}(qq \in K_{N_1}) \, dq.
\]

Recall that we defined \( B' = Q \cap \text{GL}_2^*, \) and set \( K_{N_1}' = K_{N_1} \cap \text{GL}_2^* \). We then get, for some positive constants \( C_i \)

\[
\int_{N(\mathbb{Q}_p)Z(\mathbb{Q}_p) \backslash \text{GL}_2^*(\mathbb{Q}_p)} f^*(g;s) \ell(\pi(g)v_1) \, dg = \int_{N(\mathbb{Q}_p)Z(\mathbb{Q}_p) \backslash B'(\mathbb{Q}_p)K_{N_1}'} f^*(g;s) \ell(\pi(g)v_1) \, dg
\]

\[
= C_1 \int_{N(\mathbb{Q}_p)Z(\mathbb{Q}_p) \backslash B'(\mathbb{Q}_p)} f^*(g;s) \ell(\pi(g)v_1) \, dg
\]

\[
= C_1 \int_{Z(\mathbb{Q}_p) \backslash T'(\mathbb{Q}_p)} \delta_B^{-1}(m)f^*(m;s) \ell(\pi(m)v_1) \, dm.
\]

Here \( T' = T_L \cap B' \), and the first equality is because, as one may check, \( QK_{N_1} \cap \text{GL}_2^* = (Q \cap \text{GL}_2^*)(K_{N_1} \cap \text{GL}_2^*) \). This last line above is equal to

\[
C_2 \int_{Z(\mathbb{Q}_p) \backslash T'(\mathbb{Q}_p)} \delta_B^{-1}(m) \delta_Q^2(m) \, \text{char}_N(m) \ell(\pi(m)v_0) \, dm,
\]

where \( \text{char}_N(m) = \text{char}(t \in 1 + p^N \mathbb{Z}_p) \). But when restricted to \( T' \), the support of \( \text{char}_N \) is contained in \( Z(\mathbb{Q}_p)K_N \). Furthermore, \( \delta_Q \) and \( \delta_B \) are identically one on this subset of \( T' \). Hence the above is equal to \( C_3 \ell(v_0) \). Setting \( v = C_3^{-1}v_1 \) gives the proposition. \( \square \)

**Proof of Proposition 3.4.** Pick \( \alpha_p \) so that \( \alpha_p(v_Dg) = \text{char}(g \in \text{GL}_2^* K_N) \) for \( N \) sufficiently large. Then we are reduced to the case in Proposition 5.1. \( \square \)

### 6. Holomorphic Siegel Modular Forms

In this section, we choose the data \( \alpha_{\infty} \) for \( P_0^D \) and \( \Phi_{\infty} \) for \( E_\infty(g;s) \) so that the archimedean integral \( I_\infty(\phi;s) \) may be computed explicitly when \( \phi \) comes from a level one holomorphic Siegel modular form of weight \( r \geq 6 \). We will see that for such a \( \phi \), and with our choice of data, the archimedean integral is equal to \( \pi^{-2s}(4\pi)^{-2s} \Gamma(2s+2r-2) \Gamma(2s+2s+r-2) \), up to a constant. Throughout this entire section, \( D = -|D| \) is negative, so that \( L = Q(\sqrt{D}) \) is an imaginary quadratic field. The paper \( S \) of Kudla was very helpful for us in making the correct choice of \( \alpha_{\infty} \).
6.1. Preliminaries. We now make our assumptions more precise, and pick notation. Denote by \( \mathcal{H}_2 \) the upper half space of symmetric two-by-two complex matrices, with positive definite imaginary parts. We write \( \text{GSp}_4^+ (\mathbb{R}) \) for the elements of \( \text{GSp}_4 (\mathbb{R}) \) with positive similitude. Then \( \text{GSp}_4^+ (\mathbb{R}) \) acts transitively on \( \mathcal{H}_2 \) via the formula

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} z = (Az + B)(Cz + D)^{-1}.
\]

We will write \( i \) for the element of \( \mathbb{C} \) satisfying \( i^2 = -1 \) with positive imaginary part and also for the two-by-two matrix \( iI_2 \in \mathcal{H}_2 \). This slight abuse of notation will not cause any confusion. Denote by \( K_\infty \) the subgroup of \( \text{Sp}_4 (\mathbb{R}) \) that stabilizes \( i \). Then

\[
K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A + iB \in U(2) \right\}.
\]

For \( \gamma \in \text{GSp}_4 (\mathbb{R}) \), \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) and \( z \in \mathcal{H}_2 \), we write \( j(\gamma, z) = \det(Cz + D) \). In this section, we assume our cusp form \( \phi \) satisfies

1. \( \phi(\gamma k) = j(k, i)^{-r} \phi(g) \) for all \( k \in \prod_{p < \infty} \text{GSp}_4 (\mathbb{Z}_p) \) and \( k_\infty \in K_\infty \)
2. The function \( f_\phi : \mathcal{H}_2 \to \mathbb{C} \) (well-)defined by \( f_\phi(g) = \nu(g_\infty)^{-r} j(g, i)^{r} \phi(g_\infty) \) for all \( g_\infty \in \text{GSp}_4^+ (\mathbb{R}) \) is a classical holomorphic Siegel modular form.

Denote by \( T = (-D, 1) \) the two-by-two symmetric matrix corresponding to the character \( \chi \). It follows from these assumptions that for \( g \in \text{GSp}_4^+ (\mathbb{R}) \),

\[
(4) \quad \phi_\chi(g_\infty) = a(T) \nu(g)^{r} j(g, i)^{-r} e^{2\pi i \text{tr}(Tg(i))}
\]

where \( a(T) \in \mathbb{C} \) is the (constant) Fourier coefficient of \( f_\phi \) associated to \( T \).

We will be in need of \( K_\infty \)-invariant norms on \( W_4 (\mathbb{R}) \) and \( V_5 (\mathbb{R}) \). We denote by \( \| \| \) the norm on \( W_4 (\mathbb{R}) \) that comes from considering \( e_1, f_2 \) orthonormal. That is, \( \| a e_1 + b e_2 + c f_1 + d f_2 \| = a^2 + b^2 + c^2 + d^2 \). We abuse notation and also write \( \| \| \) for the norm on \( V_5 (\mathbb{R}) \) induced from this inner product on \( W_4 (\mathbb{R}) \). Then

\[
\| A(e_1 \wedge f_1 - e_2 \wedge f_2) + B e_1 \wedge f_2 + C e_2 \wedge f_1 + G e_1 \wedge e_2 + E f_1 \wedge f_2 \| = 2A^2 + B^2 + C^2 + E^2 + G^2.
\]

Both of these inner products are \( K_\infty \)-invariant.

In this section, we will diverge slightly from the rest of the paper, and instead consider the global integral to be

\[
I(\phi; s) = \int_{\text{GSp}_4 (\mathbb{Q}) \backslash \text{GSp}_4 (\mathbb{A})} E^s(g; s) \frac{\alpha_f(v_\Delta g gf)}{(r_*, v_\Delta g g_\infty)^r} \frac{1}{(r_*, v_\Delta g g_\infty)^r} dg,
\]

i.e. we replace \( P_\Delta \) with its complex conjugate.

6.2. Definition of and results on \( P_\Delta \). We now define \( \alpha_\infty \). Denote by

\[
r_* = -(e_1 - if_1) \wedge (e_2 - if_2) \in V_5 (\mathbb{C}).
\]

Then for \( g \in \text{GSp}_4^+ (\mathbb{R}) \) we define \( \alpha_\infty (g) = (r_*, v_\Delta g)^{-r} \), and for \( g \in \text{GSp}_4 (\mathbb{R}) \) with negative similitude, we define \( \alpha_\infty (g) = (r_*, v_\Delta g (-1^2_{1^2}))^{-r} \). Recall that \( (\ , \ ) \) is the PGS\( \text{p}_4 \) invariant linear form on \( V_5 \), and here we have extended scalars to \( \mathbb{C} \). We will see shortly that the hypothesis \( q(v) < 0 \) implies \( (r_*, v) \neq 0 \), and hence \( \alpha_\infty \) is well-defined. Note that \( \alpha_\infty \) is right invariant under \( \left( \begin{smallmatrix} -1 \\ 1 \\ -1 \\ 2 \\ 1 \\ 2 \\ 1 \end{smallmatrix} \right) \).

Thus, if \( \nu(g_\infty) > 0 \),

\[
P_\Delta^\alpha (g) = \sum_{\gamma \in \text{GL}_2 (\mathbb{Q}) \backslash \text{GSp}_4 (\mathbb{Q})} \alpha_f(v_\Delta g g_\gamma gf) \frac{1}{(r_*, v_\Delta g g_\infty)^r}.
\]
Note that we have not picked \( \alpha_\infty \) to come from a Schwartz function on \( V_5(\mathbb{R}) \), but we will shortly see that the weight \( r \geq 6 \) implies \( P_D \) is nonetheless absolutely convergent, and defines a function of moderate growth. The following lemma summarizes the properties of \( r_* \) that we need.

**Lemma 6.1.** The element \( r_* = -(e_1 - i f_1) \wedge (e_2 - i f_2) \in V_5(\mathbb{C}) \) has the following properties.

1. \( r_* k_\infty = j(k_\infty, i)^{-1} r_* \) for all \( k_\infty \in K_\infty \).
2. For \( v \in V_5(\mathbb{R}) \), \( |(r_*, v)| = ||v||^2 - (v, v) \), and hence \( (r_*, v) \) is not zero when \( q(v) < 0 \).
3. Suppose \( g \in GSp_5^+(\mathbb{R}) \), \( v \in V_5(\mathbb{R}) \). Then \( j(g, i)^{-1} \nu(g) (r_*, v) \), as a function of \( g \), is right-invariant under \( Z(\mathbb{R}) K_\infty \), and hence descends to a function of \( H_2 \). If \( v = A(e_1 \wedge f_1 - e_2 \wedge f_2) + Be_1 \wedge f_2 + Ce_2 \wedge f_1 + Ge_1 \wedge e_2 + Ef_1 \wedge f_2 \), and \( Z = g(i) \), then

\[
  j(g, i)^{-1} \nu(g) (r_*, v) = -G \det(Z) + \text{tr} \left( \begin{pmatrix} -B & A \\ A & C \end{pmatrix} Z \right) - E.
\]

**Proof.** These are all straightforward computations. The first item is very simple. For the second, assume \( v = A(e_1 \wedge f_1 - e_2 \wedge f_2) + Be_1 \wedge f_2 + Ce_2 \wedge f_1 + Ge_1 \wedge e_2 + Ef_1 \wedge f_2 \). Then one computes

\[
  (r_*, v) = (E - G) + i(B - C), \quad (v, v) = 2A^2 + 2BC + 2GE
\]

from which the desired equality follows. For the third item, it is immediate to see, using the first part, that \( \nu(g) j(g, i)^{-1} r_* g^{-1} \) is a right \( Z(\mathbb{R}) K_\infty \)-invariant function, thus descends to \( H_2 \). Hence to compute it, we may assume \( g \) is in the Siegel parabolic, i.e. has lower left two-by-two block equal to zero. A comparison between the expressions \( g(i) = Z \) and \( \nu(g) j(g, i)^{-1} r_* g^{-1} \) yields the formula

\[
  \nu(g) j(g, i)^{-1} r_* g^{-1} = -e_1 \wedge e_2 + z_{12}(e_1 \wedge f_1 - e_2 \wedge f_2) - z_{11} e_2 \wedge f_1 + z_{22} e_1 \wedge f_2 - \det(Z) f_1 \wedge f_2.
\]

More precisely, one first writes a general element of the Siegel parabolic in the form

\[
  g = \left( \begin{array}{cc} 1 & X \\ & 1 \end{array} \right) \cdot \left( \begin{array}{cc} \lambda & 1 \\ & 1 \end{array} \right) \cdot \left( \begin{array}{cccc} a & b & & \\ c & d & & \\ & & d & -c \\ & & -b & a \end{array} \right)
\]

for \( X = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \). Let \( \delta = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then \( g(i) = X + iY = Z \), where

\[
  Y = \frac{\lambda}{\delta} \begin{pmatrix} a^2 + b^2 & & & \\ & ac + bd & & \\ & & a^2 + c^2 + d^2 & \\ & & & \end{pmatrix}.
\]

Fully expanding the expression \( \nu(g) j(g, i)^{-1} r_* g^{-1} \) in terms of \( a, b, c, d, X, \delta, \) and \( \lambda \), we obtain the expression [5] after substituting the formula for entries of \( Y \) above. The calculation can be simplified by noting that the coefficient of \( f_1 \wedge f_2 \) is determined by the others due to the fact that \( r_* \) and thus \( \nu(g) j(g, i)^{-1} r_* g^{-1} \) is isotropic for the pairing \((\cdot, \cdot)\). The third item now follows. \( \square \)

It is now very easy to check that the sum defining \( P_D \) converges absolutely and defines a function of moderate growth. Indeed, we have

\[
  |P_D^\alpha(g f g_\infty)| \leq \sum_{\gamma} |a_f(\nu_D \gamma g_f)| \frac{1}{(||\nu_D \gamma g_\infty||^2 + |D|)^{r/2}}
\]

since \( q(\nu_D \gamma g_\infty) = -|D| \) for all \( \gamma, g \). And since \( a_f(\nu_D \gamma g_f) \) is nonzero only for \( \nu_D \gamma \) in some lattice \( \Lambda \) of \( V_5(\mathbb{R}) \), we have

\[
  |P_D^\alpha(g f g_\infty)| \leq \sum_{v \in \Lambda, q(v) = D} \frac{1}{||v g_\infty||^r},
\]

from which the absolute convergence and moderate growth follow immediately.
We remark that in classical notation, we have

\[ P_D(Z) = \sum_{v \in V_5(Z), \delta(v) = -|D|} \frac{1}{Q_v(Z)^r}, \]

where for

\[ v = A(e_1 \wedge f_1 - e_2 \wedge f_2) + Be_1 \wedge f_2 + Ce_2 \wedge f_1 + Ge_1 \wedge e_2 + Ef_1 \wedge f_2, \]

we set

\[ Q_v(Z) = -G \det(Z) + \text{tr} \left( \begin{pmatrix} -B & A \\ A & C \end{pmatrix} Z \right) - E. \]

This is the analogue for GSp\(_4\) of the modular and Hilbert modular forms defined by Zagier in [14]. For a 2 × 2 symmetric matrix X, denote by \(u(X) \in U\) the element \(\Phi(\chi)\) of GSp\(_4\). The following lemma is the key to the calculation of \(I_\infty(\phi; s)\) below. It is the analogue of Proposition 4.3 above.

**Lemma 6.2.** Recall that \(T = \begin{pmatrix} -D \\ 1 \end{pmatrix}\) denotes the two-by-two symmetric matrix corresponding to the character \(\chi\). Assume that \(g \in \text{GSp}_4^+ \cap M(R)\) and set \(Y = g(i) = \text{Im}(g(i)).\) Then

\[ (6) \quad \alpha_\infty(g) = \int_{N(R) \setminus U(R)} e^{-2\pi i \text{tr}(TX)} \alpha_\infty(v_D u(X) g) dX = \nu(g)^r j(g, i)^{-r} \left( \frac{2\pi i}{r} \right)^r e^{-2\pi \text{tr}(TY)}. \]

**Proof.** We have

\[ \nu(g)^r j(g, i)^r \int_{N(R) \setminus U(R)} e^{-2\pi i \text{tr}(TX)} \alpha_\infty(u(X) g) dX = \int_{N(R) \setminus U(R)} e^{-2\pi i \text{tr}(TX)} \alpha_\infty(u(X) g) \nu(u(X) g)^{-r} j(u(X) g, i)^r dX. \]

By the third part of Lemma 6.1,

\[ \alpha_\infty(v_D u(X) g) \nu(u(X) g)^{-r} j(u(X) g, i)^r = \text{tr}(Tu(X) g(i))^{-r}. \]

Set \(z = \text{tr}(Tu(X) g(i))\) and define \(x\) and \(y\) by \(z = x + iy.\) Then we have

\[ \nu(g)^r j(g, i)^r \int_{N(R) \setminus U(R)} e^{-2\pi i \text{tr}(TX)} \alpha_\infty(u(X) g) dX = \int_R e^{-2\pi i x} \frac{1}{(x + iy)^r} dx = e^{-2\pi \text{tr}(TY)} \int_{\text{Im}(z) = y > 0} e^{-2\pi iz} z^{-r} dz. \]

The last equality is standard and comes from computing the residue of the integrand around 0. The lemma follows. \(\Box\)

6.3. **The Eisenstein series and the calculation of \(I_\infty.\)** For \(w \in W_4(R)\), we define \(\Phi_\infty(w) = e^{-\pi \|w\|^2}.\) Then for \(g \in \text{GSp}_4(R)\),

\[ f^*_\infty(g; s) = |\nu(g)|^{2s} \int_{\text{GL}_1(R)} \Phi_\infty(tf_2 g) |t|^{4s} dt. \]

Then \(f^*_\infty\) is right invariant under \(Z(R) K_\infty\), hence descends to a function on \(H_2\) for \(g \in \text{GSp}_4^+ (R)\). In fact, if \(g \in \text{GSp}_4^+(R)\) and \(g(i) = X + iY\), then

\[ (7) \quad f^*_\infty(g; s) = \pi^{-2s} \Gamma(2s) |y_{11}|^{-2s} |\det(Y)|^{2s}, \]

where \(y_{ij}\) denotes the entry of \(Y\) in position \((i,j)\).
We now have all the ingredients necessary to compute the archimedean integral,
\[ I_\infty(\phi; s) = \int_{Z(\mathbb{R})N(\mathbb{R}) \backslash GSp_4(\mathbb{R})} f_\infty^*(g; s)\alpha_\infty(g)\phi_\chi(g) \, dg. \]
We will compute it up to a nonzero constant. Integrating over \( N(\mathbb{R}) \setminus U(\mathbb{R}) \) this becomes
\[ \int_{Z(\mathbb{R}) \setminus M(\mathbb{R})} \delta_P^{-1}(m)f_\infty^*(m; s)\alpha_\infty(m)\phi_\chi(m) \, dm. \]
As all the terms in the integrand are right invariant under \((-1, 1)\) and \( K_\infty \), we may rewrite this as an integral over the purely imaginary part of the upper half space. We first use the formulas (4), (7), and (8) to obtain
\[ a(T)\frac{(-2\pi i)^r}{(r-1)!}\pi^{-2s}\Gamma(2s) \int_{Z(\mathbb{R}) \setminus M(\mathbb{R})} \delta_P^{-1}(m)|\det(Y)|^{y_{11}}|\nu(m)|^{2r}|j(m, i)|^{-2r}e^{-4\pi \tr(TY)} \, dm. \]
Set \( dY = dy_{11}dy_{12}dy_{22} \). Then we can replace \( \delta_P(m)dm \) with \( dY \) and use the relations
\[ \delta_P(m) = |\det(Y)|^{3/2} \quad \text{and} \quad |\nu(m)|^{2r}|j(m, i)|^{-2r} = |\det(Y)|^r \]
to transform \( I_\infty(\phi; s) \) into, up to a nonzero constant,
\[ a(T)\pi^{-2s}\Gamma(2s) \int_Y |\det(Y)|^{y_{11}}2^{s+r-3}y_{11}^{-3}e^{-4\pi \tr(TY)} dY \]
where is the integration is over the set of two-by-two real symmetric positive-definite matrices. Following Kohnen-Skoruppa now, we make the variable change \( t = \det(Y)/y_{11} = y_{22} - y_{12}^2/y_{11} \). We obtain
\[ a(T)\pi^{-2s}\Gamma(2s) \int_{0>y_{11}>0,y_{12}\in \mathbb{R}} 2^{s+r-2}e^{-4\pi(t+y_{12}^2/y_{11}+|D|y_{11})} dy_{11}dy_{12} \frac{dt}{t}. \]
Up to a nonzero constant (depending on \(|D|\)), this is
\[ a(T)\pi^{-2s}(4\pi)^{-(2s+r-2)}\Gamma(2s)\Gamma(2s + r - 2). \]
Thus we have proved

**Proposition 6.3.** With assumptions as above, \( I_\infty(\phi; s) \) is equal, up to a nonzero constant depending on \( r \) and \(|D|\), to
\[ a(T)\pi^{-2s}(4\pi)^{-(2s+r-2)}\Gamma(2s)\Gamma(2s + r - 2). \]

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**Stanford University, Stanford, CA USA**

*E-mail address:* aaronjp@stanford.edu

**Columbia University, New York, NY USA**

*E-mail address:* snshah@math.columbia.edu