CONSTRANDED SEMILINEAR ELLIPTIC SYSTEMS ON $\mathbb{R}^N$

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ABSTRACT. We prove the existence of solutions $u$ in $H^1(\mathbb{R}^N, \mathbb{R}^M)$ of the following strongly coupled semilinear system of second order elliptic PDEs on $\mathbb{R}^N$

$$\mathcal{P}[u] = f(x, u, \partial u), \quad x \in \mathbb{R}^N,$$

with pointwise constraints. We present the construction of the suitable topological degree which allows us to solve the above system on bounded domains. The key step in the proof consists of showing that the sequence of solutions of the truncated system is compact in $H^1$ by the use of the so-called tail estimates.

1. INTRODUCTION

The purpose of this paper is twofold. In the first place, we are going to discuss the existence of solutions $u \in H^1_0(\Omega, \mathbb{R}^M)$ to a strongly coupled system of semilinear elliptic partial differential equations

$$(\ast) \quad \mathcal{P}[u] = f(x, u, \partial u), \quad x \in \Omega$$

where $\Omega \subset \mathbb{R}^N$ (or $\Omega = \mathbb{R}^N$) and $\partial u$ is the Jacobian matrix of $u : \Omega \to \mathbb{R}^M$. Here $f : \Omega \times \mathbb{R}^M \times \mathbb{R}^{M \times N} \to \mathbb{R}^M$ is a vector-valued function and $\mathcal{P}$ is, roughly speaking, a linear second-order elliptic partial differential operator of the form $\mathcal{P}[u] = -\text{div}(A(x)\partial u)$ with the coefficient $A = [A_{kl}]_{k,l=1}^M$ being a function from $\Omega$ into $\mathbb{R}^{M \times M}$, the space of (real) $M \times M$ matrices. In applications such systems describe steady states of evolution processes involving $M$ unknown species or quantities subject to diffusion $\mathcal{P}$ and the forcing $f$ term including the advection or drift effects. The form of $\mathcal{P}$ allows interactions between species occur on the diffusion level, too.

We shall look for solutions $u$ to $(\ast)$ satisfying local constraints of the form $u(x) \in K(x)$ for a.a. (almost all) $x \in \Omega$, where the set $K(x) \subset \mathbb{R}^M$, $x \in \Omega$, is closed and convex. Constrained problems of this type arise naturally in various applications, where natural bounds for the unknown quantities are present. For instance the experimentally obtained, lower and upper threshold values $\sigma_k, \tau_k$ (depending on a position $x \in \Omega$) are often given and solutions $u$ satisfying $\sigma_k \leq u_k \leq \tau_k$ a.e. (almost everywhere) for $1 \leq k \leq M$ are sought-after. As it also seems, constrained solutions appear sometimes a posteriori as a by-product of sorts and a consequence of assumptions relaxing the standard growth condition. This is, for instance, the case when the method of sub- and supersolutions is applied. If $\Omega \subset \mathbb{R}^N$ is bounded, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is sufficiently regular and there are constants $\alpha \leq 0 \leq \beta$ such that $f(x, \alpha) \geq 0$, $f(x, \beta) \leq 0$ for all $x \in \Omega$, then there is $u \in H^1_0(\Omega)$ such that $-\Delta u(x) = f(x, u(x))$ and $\alpha \leq u(x) \leq \beta$ for a.a. $x \in \Omega$ (see also [20], [22]).

Due to the lack of compactness direct methods and not readily applicable to solve $(\ast)$ with $\Omega = \mathbb{R}^N$. We shall establish the existence of a constrained solution in this case by solving a sequence of the approximating constrained Dirichlet boundary value problems truncated to bounded parts $\Omega_n \subset \mathbb{R}^N$, e.g. $\Omega_n = B(0, R_n)$ is an open ball with $R_n \to \infty$. Then, a solution may be obtained by passing to the limit with the sequence of their solutions. The auxiliary constrained boundary value problems on $\Omega_n$ will be studied by means of the semigroup theory along with topological methods involving the so-called constrained topological degree.

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approach. The second subject of the present paper is the construction of an invariant responsible for the existence of solutions \( u \in K \) of the problem \( \mathbf{A}(u) = \mathbf{F}(u) \), where \( K \) is a closed convex subset of a Banach space \( X \), \( \mathbf{A} : D(\mathbf{A}) \to X \) is a sectorial (of angle \( < \pi/2 \)) operator in \( X \), \( \mathbf{F} : U \to X \) is a continuous map defined on an open subset \( U \) of \( K^{\alpha} := K \cap X^\alpha \) with \( X^\alpha \) being the fractional space corresponding to \( \mathbf{A} \), \( \alpha \in [0,1) \). Such problems create convenient abstract schemes for \( L^2 \)-realization of (\( * \)), where \( \mathbf{A} \) corresponds to the operator \( \mathcal{P} \), \( \mathbf{F} \) is the superposition operator generated by \( f \) and \( K = \{ u \in L^2 \mid u(x) \in K(x) \ a.e. \} \). It has to be noted here that \( K \) has empty interior and a direct use of the Leray-Schauder fixed point degree theory is not possible. Similarly, neither \( \mathbf{A} \) nor \( \mathbf{F} \) maps \( K \) (or \( U \)) into \( K \) so the Leray-Schauder fixed point index of maps on ANR-s can not be employed, too. Our construction relies on the assumption of the so-called tangency of \( \mathbf{F} \) and the invariance of \( K \) with respect to the semigroup generated by \( -\mathbf{A} \).

1.1. Preliminaries. Throughout the paper \( \mathbb{R}^N \) denotes the standard \( N \)-dimensional real Euclidean space and \( \mathbb{R}^{M \times N} \) the space of all \( (M \times N) \) real matrices. The norm in \( \mathbb{R}^N \) or \( \mathbb{R}^{M \times N} \) is denoted by \( | \cdot | \); the scalar product in \( \mathbb{R}^M \) (resp. the Frobenius product in \( \mathbb{R}^{M \times N} \)) is denoted by \( \langle \cdot, \cdot \rangle \). For example if \( \xi, \zeta \in \mathbb{R}^{M \times N} \), then \( \langle \xi, \zeta \rangle := \sum_{k=1}^M \sum_{l=1}^N \xi_{kl} \zeta_{kl} \) and \( |\xi|^2 := \langle \xi, \xi \rangle \). By \( \mathbb{T} \) we denote the transpose of a matrix \( A \).

Given a locally integrable map \( u \) from \( \Omega \) to \( \mathbb{R}^M \), \( \partial u \) is the (distributional) Jacobian matrix of \( u \), i.e.,

\[
\partial u := [\partial_i u_k(\cdot)]_{i=1,\ldots,N,k=1,\ldots,M} \in \mathbb{R}^{M \times N},
\]

where \( \partial_i u_k := \frac{\partial}{\partial x_i} u_k \) is the \( i \)-th partial derivative understood in the sense of distributions; \( \partial u := [\partial_i u_k]_{i=1}^M \) is the \( i \)-th column of \( \partial u \). Given a multi-index \( \nu \in \mathbb{Z}_+^N \),

\[
\partial^\nu := \partial^\nu_1 \cdots \partial^\nu_N
\]

and \( |\nu| = \nu_1 + \ldots + \nu_N \).

\[ L^p(\Omega, \mathbb{R}^M) \quad \text{1 \leq p < \infty, denotes the space of vector-valued functions } u = (u_1, \ldots, u_M) : \Omega \to \mathbb{R}^M \] such that each \( |u_k|^p \) is Lebesgue integrable with the standard norm \( \|u\|_{L^p} := \left( \int_\Omega |u(x)|^p \, dx \right)^{1/p} \);

\[ L^\infty(\Omega, \mathbb{R}^M) \quad \text{is the space of measurable functions } u : \Omega \to \mathbb{R}^M \text{ with } \|u\|_{L^\infty} := \text{ess sup}_{x \in \Omega} |u(x)| < \infty. \]

\[ W^{k,p}(\Omega, \mathbb{R}^M) \quad \text{(resp. } W^{0,k,p}(\Omega, \mathbb{R}^M) \text{), } k \in \mathbb{N}, \ 1 \leq p \leq \infty, \text{ stands for the Sobolev space of functions } u : \Omega \to \mathbb{R}^M \text{ having weak partial derivatives up to order } k \text{ in } L^p(\Omega, \mathbb{R}^M) \text{ (resp. having zero boundary values) with the standard norm } \]

\[
\|u\|_{W^{k,p}} := \left( \sum_{|\nu| \leq k} \|\partial^\nu u\|_{L^p}^p \right)^{1/p} \quad \text{if } p < \infty \quad \text{and } \|u\|_{W^{k,\infty}} := \sum_{|\nu| \leq k} \text{ess sup}_{x \in \Omega} |\partial^\nu u(x)| \quad \text{if } p = \infty.
\]

We write \( H^k \) (resp. \( H^k_0 \)) instead of \( W^{k,2} \) (resp. \( W^{0,k,2} \)); clearly \( H^1_0(\mathbb{R}^N, \mathbb{R}^M) = H^1(\mathbb{R}^N, \mathbb{R}^M) \) \(^1\)). It is also convenient to consider the seminorms \( | \cdot |_{j,p} \), \( 0 \leq j \leq k \), in \( W^{k,p}(\Omega, \mathbb{R}^M) \) putting

\[
|u|_{j,p} := \left( \sum_{|\nu| = j} \|\partial^\nu u\|_{L^p}^p \right)^{1/p}.
\]

1.2. The problem.

1.2.1. Problem. As mentioned in Introduction we are going to study the existence of solutions to the following system of elliptic equations

\[ (1.1) \quad \mathcal{D}[u] = f(x, u, \partial u), \]

\(^1\)If \( M = 1 \), then symbol \( \mathbb{R}^M \) will be suppressed form the notation concerning spaces.
with \(x \in \Omega\), where either \(\Omega = \mathbb{R}^N\) or \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) and \(u = (u_1, ..., u_M): \Omega \to \mathbb{R}^M\). In the case \(\Omega\) is bounded (1.1) will be studied subject to the Dirichlet boundary condition \(u|_{\partial \Omega} = 0\). In both cases we are interested in the existence of (weak) solutions \(u\) to (1.1) such that

\[
(1.2) 
\quad u(x) \in K(x) \quad \text{for a.a.} \quad x \in \Omega
\]

where \(K(x) \subset \mathbb{R}^M\) for each \(x \in \Omega\).

Assumptions \((K_1)\)–\((K_3)\) concerning \(K(\cdot)\), \((f_1)\)–\((f_3)\) concerning \(f\) and \((\mathcal{P}_1)\)–\((\mathcal{P}_3)\) concerning \(\mathcal{P}\) will be presented, discussed and illustrated in the following subsections.

1.2.2. Constraints. We assume that:

\((K_1)\) for each \(x \in \Omega\), \(K(x)\) is closed and convex subset in \(\mathbb{R}^M\);
\((K_2)\) for any open \(U \subset \mathbb{R}^M\), the set \(\{x \in \Omega \mid K(x) \cap U \neq \emptyset\}\) is (Lebesgue) measurable \(2\).
\((K_3)\) there is \(m \in L^2(\Omega)\) such that sup\(_{u \in K(x)} |u| \leq m(x)\) for a.a. \(x \in \Omega\).

For \(x \in \Omega\), let \(r(x, \cdot)\) be the metric projection of \(\mathbb{R}^M\) onto \(K(x)\):

\[
|u - r(x, u)| = d(u, K(x)) := \inf_{w \in K(x)} |u - w|, \quad u \in \mathbb{R}^M.
\]

Then \(r: \Omega \times \mathbb{R}^M \to \mathbb{R}^M\) is well-defined; for any \(x \in \Omega\), \(r(x, \cdot)\) is nonexpansive and \(|r(x, u)| \leq |u|\), \(u \in \mathbb{R}^M\). In view of \((K_2)\) and [4, Cor. 8.2.13], for any \(u \in \mathbb{R}^M\), \(r(\cdot, u)\) is measurable. In view of the Krasnosel’skiı Theorem, the Nemytskiı operator \(u \mapsto r(\cdot, u(\cdot))\) maps \(L^2(\Omega, \mathbb{R}^M)\) continuously into itself. Given \(u \in L^2(\Omega, \mathbb{R}^M)\), \(r(\cdot, u(\cdot))\) is an \(L^2\)-selection of \(K(\cdot)\), i.e., \(r(x, u(x)) \in K(x)\) for a.a. \(x \in \Omega\).

In Appendix we present some natural examples of \(K(\cdot)\) satisfying the above conditions.

1.2.3. Nonlinear term. Let \(f: \text{Gr}(K) \times \mathbb{R}^{M \times N} \to \mathbb{R}^M\), where \(\text{Gr}(K) := \{(x, u) \in \Omega \times \mathbb{R}^M \mid u \in K(x)\}\) is the graph of \(K(\cdot)\). We assume that:

\((f_1)\) \(f\) is a Carathéodory map, i.e. for a.a. \(x \in \Omega\), \(f(x, \cdot): K(x) \times \mathbb{R}^{M \times N} \to \mathbb{R}^M\) is continuous and for all \(u \in \mathbb{R}^M, \xi \in \mathbb{R}^{M \times N}\) the map \(f(\cdot, u, \xi)\) defined on \(\{x \in \Omega \mid u \in K(x)\}\) is measurable \(3\);
\((f_2)\) there are \(\beta \in L^2(\Omega), c_0 > 0, 1 \leq s < \frac{N+4}{4}\) and \(1 < q < \frac{N+4}{N+2}\) such that

\[
|f(x, u, \xi)| \leq \beta(x) + c_0(|u|^s + |\xi|^q), \quad x \in \Omega, \ u \in K(x), \ \xi \in \mathbb{R}^{M \times N};
\]
\[(f_3)\) \(f(x, \cdot, \cdot)\) is tangent to \(K(x)\), i.e. for a.a. \(x \in \Omega, u \in K(x)\) and \(\xi \in \mathbb{R}^{M \times N}\)

\[
f(x, u, \xi) \in T_{K(x)}(u),
\]

where \(T_{K(x)}(u)\) stands for the tangent cone (see Appendix for the definition of the tangent cone).

Condition \((f_3)\) means that for each \(x \in \Omega, u \in K(x)\) the forcing vector field driven by \(f(x, u, \cdot)\) with its tain at \(u\) is directed inward the set \(K(x)\). This condition will be carefully illustrated below.

\(2\)This means that the set-valued map \(\Omega \ni x \mapsto K(x) \subset \mathbb{R}^M\) is measurable.

\(3\)Observe that in view of \((K_2)\) the set \(\{x \in \Omega \mid u \in K(x)\}\) is measurable for any \(x \in \Omega\); therefore the condition is correct.
1.2.4. Differential operator. We assume $\mathcal{P}$ is a linear differential operator in the divergence form

$$\mathcal{P}[u] := - \sum_{i,j=1}^{N} \partial_i \left( A^{ij}(x) \partial_j u \right) + \sum_{i=1}^{N} B^i(x) \partial_i u + C(x) u,$$

where the coefficients

$$A^{ij} = [A^{ij}_{kl}]_{k,l=1}^{M}, \quad B^i = [B^i_{kl}]_{k,l=1}^{M}, \quad C = [C_{kl}]_{k,l=1}^{M}$$

are functions from $\Omega$ into $\mathbb{R}^{M \times M}$, and $\mathcal{P}$ acts on a locally integrable (column) vector-valued function $u = (u_1, \ldots, u_M)$ in the sense of distributions returning the vector-valued function $\mathcal{P}[u] : \Omega \to \mathbb{R}^M$ with components

$$\mathcal{P}[u]_k = - \sum_{i,j=1}^{N} \sum_{l=1}^{M} \partial_i \left( A^{ij}_{kl} \partial_j u_l \right) + \sum_{i=1}^{N} \sum_{l=1}^{M} C^{i}_{kl} \partial_i u_l + \sum_{l=1}^{M} C_{kl} u_l, \quad k = 1, \ldots, M.$$

Moreover we assume that

1. for $1 \leq i, j \leq N$, $1 \leq k, l \leq M$, $A^{ij}_{kl} \in C^{0,1} \cap L^{\infty}(\mathbb{R}^N)$, $B^i_{kl}, C_{kl} \in L^{\infty}(\Omega)$;

2. the operator (1.3) is elliptic in the sense of the Legendre-Hadamard condition, i.e., there is an ellipticity constant $\theta > 0$ such that for any $\zeta \in \mathbb{R}^N$ and $p \in \mathbb{R}^M$

3. the graph $\text{Gr}(K)$ is viable (or invariant) with respect to the ‘diffusion’ flow; this means that for some $T > 0$ given $u_0 \in L^2(\Omega, \mathbb{R}^M)$ such that $u_0(x) \in K(x)$ for a.a. $x \in \Omega$ the (weak) solution $u : [0, T] \times \Omega \to \mathbb{R}^M$ of the corresponding parabolic Cauchy problem

$$u_t + \mathcal{P}[u] = 0, \quad u(0, \cdot) = u_0, \quad u|_{\partial \Omega} = 0$$

stays in $\text{Gr}(K)$, i.e., $u(t, x) \in K(x)$ for all $t \geq 0$ and a.a. $x \in \Omega$ (see paragraph 1.2.5 (ii) below).

The flow invariance granted by condition (P3) will be discussed from some different points of view below.

With $\mathcal{P}$ we associate a bilinear form $\mathcal{B}$ on $H^1_0(\Omega, \mathbb{R}^M)$ given by

$$\mathcal{B}[u, v] := \int_{\Omega} \left( \sum_{i,j=1}^{N} \langle A^{ij}_{kl} \partial_j u_l, \partial_i v \rangle + \sum_{i=1}^{N} \langle B^i \partial_i u, v \rangle + \langle C \cdot u, v \rangle \right) \, dx$$

$$= \int_{\Omega} \left( \sum_{i,j=1}^{N} \sum_{k,l=1}^{M} A^{ij}_{kl} \partial_j u_l \partial_i v_k + \sum_{i=1}^{N} \sum_{k,l=1}^{M} B^i_{kl} \partial_i u_l v_k + \sum_{k,l=1}^{M} C_{kl} u_l v_k \right) \, dx$$

for $u, v \in H^1_0(\Omega, \mathbb{R}^M)$. The use of the Poincaré inequality gives

$$|\mathcal{B}[u, v]| \leq \text{const.} \|u\|_{H^1} \|v\|_{H^1},$$

the constant depends on the $L^{\infty}(\Omega)$-norms of $A^{ij}_{kl}, B^i_{kl}$ and $C_{kl}$. Observe that in view of (P1) for $u \in H^2 \cap H^1_0(\Omega, \mathbb{R}^M)$ the expression $\mathcal{P}[u]$ makes sense and belongs to $L^2(\Omega, \mathbb{R}^M)$. In this case, by the Green identity, $\mathcal{B}[u, v] = \langle \mathcal{P}[u], v \rangle_{L^2}$ for any $v \in H^1_0(\Omega, \mathbb{R}^M)$.

Under the above assumptions we have

**Theorem 1.1.** (Gårding inequality – see [12, Theorem 3.42], [25, Theorem 4.6] and [15]) The form $\mathcal{B}$ is weakly coercive, i.e., there are $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

$$\mathcal{B}[u, u] + \omega \|u\|_{L^2} \geq \alpha \|u\|_{H^1}, \quad u \in H^1_0(\Omega, \mathbb{R}^M).$$
1.2.5. Solutions. (i) \( u \in H^1_0(\Omega, \mathbb{R}^M) \) is a weak solution of (1.1) if for all \( v \in H^1_0(\Omega, \mathbb{R}^M) \)
\[
\mathcal{B}[u, v] = \int_{\Omega} \{f(x, u, \partial u), v\} \, dx \quad (2.1).
\]
If \( u \in H^2_{loc} \cap H^1_0(\Omega, \mathbb{R}^M) \), then \( u \) is a called a strong solution if \( \mathcal{P}[u] = f(\cdot, u(\cdot), \partial u(\cdot)) \) a.e. in \( \Omega \); clearly strong solutions are weak.
(ii) \( u \in L^2(0, T; H^1_0(\Omega, \mathbb{R}^M)) \) such that \( u' \in L^2(0, T; H^{-1}(\Omega, \mathbb{R}^M)) \) is a weak solution to (1.5) if \( u(0) = u_0 \) and for any \( v \in H^1_0(\Omega, \mathbb{R}^M) \)
\[
[u'(t), v] + \mathcal{B}[u(t), v] = 0 \quad \text{for a.a } t \in [0, T],
\]
where \([\cdot, \cdot]\) stands for the duality pairing of \( H^{-1}(\Omega, \mathbb{R}^M) \) and \( H^1_0(\Omega, \mathbb{R}^M) \).

2. BOUNDED DOMAIN

In this section \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \). We are going to establish the existence of strong solutions to (1.1), (1.2), i.e.,

\[
\mathcal{P}[u] = f(x, u, \partial u), \quad u|_{\partial \Omega} = 0, \quad u(x) \in K(x) \quad \text{for a.a } x \in \Omega, \quad (2.1, 2.2)
\]

such that

\[
\mathcal{P}, f \text{ and } K(\cdot) \text{ are as above, but in addition to assumptions stated there we suppose that:}
\]

\[
(K_4) \text{ for any } u \in \mathbb{R}^M
\]

(i) \( r(\cdot, u) \in H^1(\Omega, \mathbb{R}^M) \);
(ii) \( r(\cdot, 0) \in H^1_0(\Omega, \mathbb{R}^M) \).

Condition \((K_4)\) (ii) implies that \( r(\cdot, 0)|_{\partial \Omega} = 0 \) in the sense of trace. It means that, in a certain sense, \( K(\cdot) \) has an extension onto \( \overline{\Omega} \) and \( 0 \in K(x) \) if \( x \in \partial \Omega \).

First we establish a priori bounds for solutions.

**Lemma 2.1.** There is \( M > 0 \) (depending on \( \Omega \), the ellipticity constant and the \( C^{0,1} \)-norms of \( A_{ij} \), \( 1 \leq i, j \leq N \)) such that \( \|u\|_{H^2} < M \) for every strong solution \( u \) of (2.1) satisfying (2.2).

**Proof.** Let \( u \) be a solution of (2.1) satisfying (2.2). By \((K_3)\)

\[
\|u\|_{L^2} \leq M_0 := \|m\|_{L^2}, \quad (2.3)
\]

Assumption \((f_2)\) show that \( f(\cdot, u, \partial u) \in L^2(\Omega, \mathbb{R}^M) \). Indeed

\[
\|f(\cdot, u, \partial u)\|_{L^2} \leq \|\beta\|_{L^2} + I_1 + I_2, \quad (2.4)
\]

where

\[
I_1 := \left( \int_{\Omega} |u|^{2s} \, dx \right)^{1/2}, \quad I_2 := \left( \int_{\Omega} |\partial u|^{2t} \, dx \right)^{1/2}.
\]

\(^4\)As we shall see later \( f(\cdot, u(\cdot), \partial u(\cdot)) \in L^2(\Omega, \mathbb{R}^M) \).
Since $\Omega$ is bounded, we may assume without loss of generality that $1 < q < \frac{N+4}{N+2}$ and $1 < s < \frac{N+4}{N}$. The interpolation inequality with exponents $2q \in (2, 2^*)$ (5), $\theta_0 \in (0, 1)$ such that

$$\frac{1}{2q} = \frac{1 - \theta_0}{2} + \frac{\theta_0}{2^*} \iff q \theta_0 = N^4(q-1)$$

shows that

$$I_2 = ||\partial u||^q_{L^{2q}} \leq ||\partial u||^q_{L^2} ||\partial u||^{q(1 - \theta_0)}_{L^{2^*}}.$$

Both right hand side factors will be estimated separately. By the Ehrling-Browder inequality (see [1, Thm 4.17]) (6)

$$||u||_{H^1} \leq ||u||^{1/2}_{H^2} ||u||^{1/2}_{L^2}$$

and, by (2.3), we have

$$||\partial u||^{q(1 - \theta_0)}_{L^2} \leq ||u||^{\frac{q}{2} (1 - \theta_0)}_{H^2} M_0^{\frac{q}{2} (1 - \theta_0)}.$$

Since $\partial u \in H^1(\Omega, \mathbb{R}^{M \times N})$, $||\partial u||_{H^1(\Omega, \mathbb{R}^N)} \leq ||\partial u||_{H^1(\Omega, \mathbb{R}^N)} \leq ||u||_{H^2}$ by [3, Cor. 4.6]. Therefore and by the Sobolev embedding $H^1(\Omega) \subset L^{2^*}(\Omega)$

$$||\partial u||_{L^{2^*}(\Omega)} \leq ||\partial u||_{H^1} \leq ||u||_{H^2}.$$

Summing up

(6) $$I_2 \lesssim ||u||_{H^2}^{\gamma_1}, \quad \gamma_1 := \frac{q}{2}(1 + \theta_0).$$

The inequality $q < \frac{N+4}{N+2}$ together with (2.5) yield that

(7) $$\gamma_1 = \frac{q}{2} + \frac{N}{4}(q-1) = \frac{q(N+2)}{4} - \frac{N}{4} < 1.$$

To estimate $I_1$ we consider two cases $3 \leq N \leq 4$ and $N \geq 5$ separately. If $3 \leq N \leq 4$, then

$$2s < \frac{2N + 8}{N} \leq 2^*,$$

and the interpolation inequality with exponents $2s \in (2, 2^*)$, $\theta_1 \in (0, 1)$

$$\frac{1}{2s} = \frac{1 - \theta_1}{2} + \frac{\theta_1}{2^*} \iff s \theta_1 = N^4(s-1)$$

shows

$$I_1 = ||u||_{L^{2s}} \leq ||u||_{L^2}^{s(1 - \theta_1)} ||u||_{L^{2^*}}^{\theta_1}.$$

By (2.3) and the Sobolev embedding $H^1(\Omega, \mathbb{R}^M) \subset L^{2^*}(\Omega, \mathbb{R}^M)$

$$I_1 \lesssim M_0^{s(1 - \theta_1)} ||u||_{H^1}^{\theta_1}.$$

Again by the Ehrling-Browder inequality

$$||u||_{H^1}^{\theta_1} \lesssim ||u||_{H^2}^{\theta_1} ||u||_{L^2}^{\theta_1}.$$

Hence and again from (2.3)

(9) $$I_1 \lesssim ||u||_{H^2}^{\gamma_2}, \quad \gamma_2 := \frac{s}{2} \theta_1,$$

---

5Given $1 \leq p < N$, $p^*$ stand for the Sobolev critical exponent, i.e., $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\infty}$; hence $2^* = \frac{2N}{N-2}$.

6Below and in what follows given $a, b \in \mathbb{R}$ we write $a \lesssim b$ if there is a constant $C$ such that $a \leq Cb$. 
where $\gamma_2 < 1$ since (2.8) and $s < \frac{N+4}{N}$.

If $N \geq 5$, then we use the interpolation inequality with exponents $2s \in (2, 2^*)$\(^7\), $\theta_2 \in (0, 1)$ satisfying

\begin{equation}
\frac{1}{2s} = \frac{1 - \theta_2}{2} + \frac{\theta_2}{2^*}, \text{ i.e., } s\theta_2 = \frac{N}{4}(s - 1)
\end{equation}

(2.10)

to get that

\[ I_1 = \|u\|_{L^{2s}}^s \leq \|u\|_{L^2}^{(1-\theta_2)} \|u\|_{L^{2^*}}^{\theta_2}. \]

The Sobolev embedding $H^2(\Omega, \mathbb{R}^M) \subset L^{2^*} (\Omega, \mathbb{R}^M)$ and (2.3) yield

\begin{equation}
I_1 \lesssim \|u\|_{H^2(\Omega, \mathbb{R}^M)}^{\gamma_3}, \quad \gamma_3 := s\theta_2,
\end{equation}

(2.11)

where $\gamma_3 < 1$ since (2.10) and $s < \frac{N+4}{N}$\(^8\).

Using the regularity theory (see [12, Theorem 4.14], cf. Lemma 3.2 and comp. e.g., [11, Thm 8.12])

\begin{equation}
\|u\|_{H^2} \lesssim \|f(\cdot, u, \partial u)\|_{L^2} + \|u\|_{L^2},
\end{equation}

(2.12)

where the constant depends on $\Omega$, the ellipticity constant and $C^{0,1}$-norms of the coefficients of $P_0$. Therefore, in view of (2.3), (2.4), (2.6), (2.9) and (2.11), we see that there is $c > 0$ and $\theta \in (0, 1)$ such that every strong solution $u$ of (2.3) and (2.2) satisfies

\[ \|u\|_{H^2} \leq c \left(1 + \|u\|_{H^2}^\theta\right). \]

and the assertion follows. \hfill \Box

The proof of the existence of strong solutions to (2.1), (2.2) will rely on an abstract approach.

2.1. A functional setting. In order to introduce the appropriate abstract framework let

\[ V = H^1_0(\Omega, \mathbb{R}^M) \quad \text{and} \quad X := L^2(\Omega, \mathbb{R}^M) \]

2.1.1. Constraint. Define $K \subset X$ as the collection of all $L^2$-selections of $K(\cdot)$, i.e.,

\begin{equation}
K := \{u \in X \mid u(x) \in K(x) \text{ for a.a. } x \in \Omega\}.
\end{equation}

(2.13)

In view of (K1) and (K2) the set $K$ is nonempty closed and convex. If $u \in X$ and $w \in K$, then $r(\cdot, u(\cdot)) \in K$ and $|u(x) - r(x, u(x))| \leq |u(x) - w(x)|$ for a.a. $x \in \Omega$; hence $\|u - r(\cdot, u(\cdot))\|_X \leq \|u - w\|_X$. This shows that $\pi_K : X \to K$ given by

\begin{equation}
\pi_K(u) = r(\cdot, u(\cdot)), \quad u \in X,
\end{equation}

(2.14)

i.e., the Nemytski operator generated by $r$, is the metric projection of $X$ onto $K$.

### Lemma 2.2. If $u \in V$, then $\pi_K(u) \in V$.

**Proof.** Let $u \in V$. In view of (K1) (i) and [24, Lemma 5] (see also [23]) $r(\cdot, u(\cdot)) \in H^1(\Omega, \mathbb{R}^M)$. If $u \in C^{\infty}_0(\Omega, \mathbb{R}^M)$, i.e., $u$ vanishes outside a compact part $C$ of $\Omega$, then letting $w(x) := r(x, u(x)) - r(x, 0)$, $x \in \Omega$, we see that $w \in V$ and $w(x) = 0$ for $x \in \Omega \setminus C$. Therefore, in view of [9], $w \in V$ and, in view of (K1) (ii), $r(\cdot, u(\cdot)) \in V$. In general $u \in V$ is the $H^1$-limit of $u_n \in C^{\infty}_0(\Omega, \mathbb{R}^M)$, so the result follows from the $H^1$-continuity of the Nemytski operator generated by $r$ (see [21]). \hfill \Box

\(^7\)The symbol $2^*$ stands for $(2^*)^*$, i.e., $2^{**} = \frac{2N}{N-2}$.

\(^8\)Observe that constants appearing so far in the proof, aside from $M_0$, depend on $\Omega$ only.
2.1.2. Operator. It is clear that the Hilbert space \( V \) is a dense subset of the Hilbert space \( X \); moreover \( V \) is continuously embedded into \( X \). i.e. \( \|v\|_V \leq \text{const.}\|v\|_X \), \( v \in V \). The bilinear form \( B : V \times V \to \mathbb{R} \) given by (1.6) is continuous i.e. \( |B(u,v)| \leq \text{const.}\|u\|_V \|v\|_V \). In view of the Gårding inequality (see Theorem 1.1) \( B \) is weakly coercive, i.e., there are constants \( \omega \in \mathbb{R} \) and \( \alpha > 0 \) such that

\[
B[v,v] + \omega \|v\|_X^2 \geq \alpha \|v\|_V^2, \quad u \in V.
\]

Let \( A : V \to V^* \), where \( V^* \) is the dual of \( V \), be given by \([Au](v) := B(u,v)\) for any \( u, v \in V \). The part \( A := A|_X \) of \( A \) in \( X = X^* \) is given by \( Au := Au \) for \( u \in D(A) := \{u \in V \mid \text{A(u) exists}\} \). It easy to see that \( D(A) = \{u \in V \mid B(u, \cdot) \text{ is continuous on } V \text{ w.r.t. the } X \text{-norm}\} \) and \( \langle A(u), v \rangle_X = B(u,v) \) for any \( u \in D(A) \) and \( v \in V \).

By the result that apparently goes back to Lions (see e.g. [16, Sect. 7.3.2] or Theorem 2.18/Yagi):

**Lemma 2.3.** \( A \) is a sectorial operator. Consequently \(-A\) generates a holomorphic semigroup \( \{T(t)\}_{t \geq 0} \) of linear bounded operators on \( X \) such that \( \|T(t)\|_{L(X)} \leq e^\omega t \) for \( t \geq 0 \) \((9)\).

**Remark 2.4.** (i) In particular \( A \) is closed, densely defined, the resolvent set \( \rho(-A) \supset \{\lambda \in \mathbb{C} \mid \text{Re} \lambda > \omega \} \) and \( \|(\lambda I + A)^{-1}\|_{L(X)} \leq (\lambda - \omega)^{-1} \) for \( \lambda > \omega \), where \( I \) is the identity on \( X \). Given \( h > 0 \), \( h\omega < 1 \), the (modified) resolvent \( \frac{1}{h} - A = (I + hA)^{-1} \in L(X) \) is well-defined and \( \frac{1}{h} - A (X) \subset D(A) \).

(ii) In view of the regularity results in e.g. [12, Section 4.3] the domain \( D(A) = H^2 \cap H_0^1(\Omega, \mathbb{R}^M) \) and \( A(u) = \mathcal{P}[u] \) for \( u \in D(A) \), i.e., \( A \) is the \( L^2 \)-realization of \( \mathcal{P} \).

(iii) In view of the compactness of the embedding \( H_0^1(\Omega, \mathbb{R}^M) \hookrightarrow L^2(\Omega, \mathbb{R}^M) \), the semigroup \( \{T(t)\}_{t \geq 0} \) is compact and so is the resolvent \( \frac{1}{h} - A \), where \( h > 0 \), \( h\omega < 1 \).

(iv) Let us add here that if \( \Omega = \mathbb{R}^N \), then the above construction is valid, too, i.e., \( \mathcal{P} \) determines the sectorial operator \( A \) (see [19]), but the semigroup \( \{T(t)\} \) is not compact in general.

2.1.3. Nonlinearity. Consider \( f \) described from subsection 1.2.3 along with assumptions \((f_1) - (f_3)\). Set

\[
p := \max \left\{ 2q, \left( \frac{1}{2s + \frac{1}{N}} \right)^{-1} \right\}.
\]

Then assumptions \( s < \frac{N+4}{N+2} \) and \( q < \frac{N+4}{N+2} \) imply that \( p < \min\{2^*, N\} \).

The \( L^2 \)-realization \( A : D(A) \to L^2(\Omega, \mathbb{R}^M) \) of \( \mathcal{P} \) is sectorial, so, on account of [17, Thm 1.6.1], we find \( \alpha \in (0, 1) \) such that

\[
X^\alpha \subset W^{1,p}(\Omega, \mathbb{R}^M),
\]

where \( X^\alpha \) is the fractional space associated to \( A \) (precisely with positive operator \( A + \omega I \) if \( \omega > 0 \)) \((10)\).

The definition of \( p \) and the Sobolev embeddings yield

\[
X^\alpha \subset \subset W^{1,p}(\Omega, \mathbb{R}^M) \subset L^{2^*}(\Omega, \mathbb{R}^M) \cap W^{1,2q}(\Omega, \mathbb{R}^M).
\]

We define

\[
K^\alpha := K \cap X^\alpha.
\]

[Trzeba napisać gdzieś, że \( K^\alpha \) niepusty] and \( F : K^\alpha \to X \) as the superposition determined by \( f \), i.e., for a.e. \( x \in \Omega \) and \( u \in X^\alpha \)

\[
F(u)(x) = f(x, u(x), \partial u(x)).
\]

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9 A brief survey of some relevant results concerning sectorial operators is provided in the Appendix.

10 A brief survey of some relevant results concerning fractional spaces is provided in the Appendix.
The growth assumption \((f_2)\) and (2.17) show that, for \(u \in K^\alpha\),

\[
\|F(u)\|_{L^2} \lesssim \|\beta\|_{L^2} + \left(\int_{\Omega} |u|^{2s} \, dx\right)^{1/2} + \left(\int_{\Omega} |\partial u|^{2q} \, dx\right)^{1/2}
\lesssim \|\beta\|_{L^2} + \|u\|_{L^{2s}}^2 + N^{1/2q}\|u\|_{W^{1,2q}}^q \lesssim 1 + \|u\|_{\alpha}^{\max\{s,q\}}.
\]

This proves that \(F\) is well defined. The standard proof via Lebesgue’s dominated convergence theorem shows that \(F\) is continuous.

The tangency condition \((f_3)\) implies that

\[
F(u) \in T_K(u) \quad \text{for all} \quad u \in K^\alpha,
\]

where \(T_K(u)\) is the tangent cone to \(K\) at \(u\) (see Appendix).

**Remark 2.5.** Observe that \(u\) is a strong solution to (2.1) satisfying (2.2) if and only is \(u \in K \cap D(A)\) and \(Au = F(u)\)

2.1.4. **Invariance.** We shall see what is the impact and the actual meaning of assumption \((P_3)\).

**Lemma 2.6.** Condition \((P_3)\) holds if and only if \(K\) is semigroup invariant, i.e., \(T(t)K \subset K\) for any \(t \geq 0\).

**Proof.** Let \(u(t) = T(t)u_0, t \geq 0\), where \(u_0 \in K\). It is sufficient to show that \(u\) is the (unique) weak solution to (1.5). The semigroup \(\{T(t)\}_{t \geq 0}\) is analytic; hence \(u(t) \in D(A)\) for all \(t > 0\); thus, \(u \in C^1((0, +\infty), X)\) and \(u'(t) = Au(u)\) for all \(t > 0\). If \(v \in V\), then by [27, Cor. III.1.1] for all \(t > 0\).

\[
[u'(t), v] = \frac{d}{dt} \langle u(t), v \rangle_X = \langle Au(t), v \rangle_X = B[u(t), v].
\]

In view of Lemma 2.2 and Proposition 4.3 we have an additional characterization of the flow invariance given in terms of the Dirichlet form \(B\) is given by (1.6).

**Proposition 2.7.** Condition \((P_3)\) holds if and only if

\[
B[r(\cdot, u(\cdot)), u - r(\cdot, u(\cdot))] \geq 0 \quad \text{for any} \quad u \in V = H_0^1(\Omega, \mathbb{R}^M).
\]

2.2. **Existence.** We are in a position to apply the constrained degree theory introduced in Appendix. Let us collect some necessary facts:

1. \(A : D(A) \to X\) is a sectorial operator;
2. a number \(\alpha \in [0, 1)\) is fixed and the fractional space \((X^\alpha, \|\cdot\|_\alpha)\) associated with \(A\) is given;
3. \(K \subset X\) is an \(L\)-retract (as a closed convex set), \(K^\alpha \subset X^\alpha\) is a (nonempty) neighborhood retract (again as a closed convex set), \(K^\alpha \subset K\) and for all \(h > 0\) with \(h\omega < 1\), \(J_h(K) \subset K^\alpha\);
4. \(F : K^\alpha \to X\) is continuous and tangent, i.e., condition (2.19) holds. In order to apply the degree Section 4.4 we need to verify the compactness in \(K^\alpha\) of the set

\[
\text{Coin}(A, F; K^\alpha) = \{u \in D(A) \cap K^\alpha \mid Au = F(u)\}.
\]

By Lemma 2.1, there is \(M > 0\) such that for \(u \in \text{Coin}(A, F; K^\alpha)\), then

\[
\|u\|_{H^2} \leq M.
\]

The compactness of the embedding \(D(A) \subset X\), implies that \(\text{Coin}(A, F; K^\alpha)\) is relatively compact in \(K^\alpha\); this set is also closed in \(K^\alpha\), since \(F : K^\alpha \to X\) is continuous and \(A\) has a closed graph.

By the above, the constrained coincidence degree \(\text{deg}_K(A, F)\) is well-defined. Consider a homotopy \(H : K^\alpha \times [0, 1] \to X\) defined by \(H(u, t) = (1 - t)F(u)\), for \(u \in K^\alpha\). Since, for \(u \in K^\alpha\), \(F(u) \in T_K(u)\)
and the cone $T_K(u)$ is convex, we have $H(u, t) \in T_K(u)$ for every $u \in K$, $t \in [0, 1]$. $H$ is continuous and the estimates from Lemma 2.1 are also true for the solutions of the problem $Au = H(u, t)$. Hence the set

$$\bigcup_{t \in [0, 1]} \text{Coin}(A, H(t, \cdot); K)$$

is compact in $K$, i.e., $H$ is an admissible homotopy.

The homotopy invariance of $\deg_{K^\alpha}$ (see Theorem 4.9) yields

$$\deg_K(A, F) = \deg_K(A, H(\cdot, 0); K^\alpha) = \deg_K(A, H(\cdot, 1)) = \deg_K(A, 0).$$

The normalisation property of $\deg_K$ implies

$$\deg_K(A, 0) = 1,$$

since $K$ is closed and the cone.

Thus, we have proved the following

\[ \text{thm:3.1 Theorem 2.8. There is a strong solution to the system (2.1) satisfying the constraints (2.2).} \]

3. THE PROBLEM ON $\mathbb{R}^N$

In this section we study problem (1.1), (1.2), where $\Omega = \mathbb{R}^N$. We are going to apply the approximation-truncation approach sketched in Introduction. For that reason we have to enhance assumption $(P_2)$ and supplement assumption $(K_4)$:

$(P'_2)$ The coefficients $A^{ij} \in \mathbb{R}^{M \times M}$ of $P$ are constant and $P$ is strongly elliptic in the sense of the Legendre condition. i.e., there is $\theta > 0$ such that for any $\xi \in \mathbb{R}^{M \times N}$

$$\sum_{i,j=1}^{N} \sum_{k,l=1}^{M} A^{ij}_{kl} \xi_{ij} \xi_{kl} \geq \theta |\xi|^2;$$

$(K'_4)$ for any $u \in \mathbb{R}^M$, (i) $r(\cdot, u) \in H^1_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^M)$, (ii) there is a sequence $(R_n)_{n=1}^{\infty}$ such that $R_n \nearrow \infty$ and $r(\cdot, 0) \in H^1_{\text{loc}}(B_{R_n}, \mathbb{R}^M)$.

where $B_R := \{ x \in \mathbb{R}^N \mid |x| < R \}$, $R > 0$. Condition $(K'_4)$ (ii) holds for instance if $0 \in K(x)$ for all $x \in \mathbb{R}^N$ of sufficiently large norm. In what follow we assume that $R_n = n, n \geq 1$, for short.

We start we some auxiliary lemmata.

\[ \text{lem:4.1 Lemma 3.1. (i) For every } 1 \leq p < N, \text{ there is a constant } c_0(p) \text{ depending on } p \text{ only, such that for any } R \geq 1 \text{ and } v \in W^{1,p}(B_R, \mathbb{R}^M) \]

(3.1) $\|v\|_{L^{p^*}} \leq c_0(p) \|v\|_{W^{1,p}}.$

(ii) There is a constant $c_1 > 0$ such that for $R \geq 1$ and $v \in H^2(B_R, \mathbb{R}^M)$,

\[ (3.2) |v|_{1,2}^2 \leq c_1^2 (\|v\|_{L^2} |v|_{2,2} + \|v\|_{L^2}^2). \]

Proof. The results seem to be folklore: for the sake of completeness we give the proofs.

(i) By the Sobolev inequality there is $c_0 = c_0(p)$ such that

$$\|u\|_{L^{p^*}} \leq c_0 \|u\|_{W^{1,p}}$$

for any $u \in W^{1,p}(B_1, \mathbb{R})$. Take $R \geq 1$, $v \in W^{1,p}(B_R, \mathbb{R})$ and let $u(x) := v(Rx)$ for $x \in B_1$. Then clearly $u \in W^{1,p}(B_1, \mathbb{R})$; changing variables we get

$$\|u\|_{L^{p^*}} = R^{-N/p^*} \|v\|_{L^{p^*}}, \quad \|u\|_{L^p} = R^{-N} \|v\|_{L^p} \quad \text{and} \quad \|\partial_j u\|_{L^p} = R^{p-N} \|\partial_j v\|_{L^p}.$$
for any $j = 1, \ldots, N$. Hence, taking into account that $Np/p^* = N - p$, we get

$$\|v\|_{L^{p^*}} = R^{N/p^*} \|u\|_{L^{p^*}} \leq c_0 \left( R^{Np/p^* - N} \|v\|_{L^p}^p + R^{Np/p^* + p - N} \sum_{j=1}^N \|\partial_j v\|_{L^p}^p \right)^{1/p} \leq c_0 \|v\|_{W^{1,p}}. $$

If $v \in W^{1,p}(B_R, \mathbb{R}^M)$, then $w := |v| \in W^{1,p}(B_R, \mathbb{R})$ and $\|w\|_{W^{1,p}} \leq \|v\|_{W^{1,p}}$ in view of [3, Cor 4.6]. Thus

$$\|v\|_{L^{p^*}} = \|w\|_{L^{p^*}} \leq c_0 \|w\|_{W^{1,p}} \leq c_0 \|v\|_{W^{1,p}}. $$

(ii) By the Ehrlich-Browder inequalities (see [1, Corollary 4.16, Theorem 4.17]), for every $u \in H^2(B_1, \mathbb{R}^M)$

$$\|u\|_{H^2}^2 \lesssim |u|_{2,2}^2 + \|u\|_{L^2}^2 \quad \text{and} \quad |u|_{2,2}^2 \lesssim \|u\|_{H^2} \|u\|_{L^2}. $$

Combining the above inequalities we get that there is $c_1 > 0$ such that for any $u \in H^2(B_1, \mathbb{R}^M)$

$$\|u\|_{H^2}^2 \leq c_1 (|u|_{2,2}^2 + \|u\|_{L^2}^2).$$

Fix $R \geq 1$ and $v \in H^2(B_R, \mathbb{R}^M)$. Again we define $u(x) := v(Rx)$ for $x \in B_1$. Then $u \in H^2(B_1, \mathbb{R}^M)$ and, changing the variables,

$$|u|_{1,2}^2 = R^{2-N} \|v|_{1,2}^2, \quad |u|_{2,2}^2 = R^{-N/2} \|v|_{2,2}^2, \quad \|u\|_{L^2} = R^{-N/2} \|v\|_{L^2}.$$

Therefore, by (3.4)

$$|v|_{1,2}^2 \leq c_2^2 (|u|_{2,2} \|v\|_{L^2} + \|v\|_{L^2}^2).$$

**Lemma 3.2.** There is $c_2 > 0$ such that for all $R \geq 1$ and $g \in L^2(B_R, \mathbb{R}^M)$ if $v \in H^1_0(B_R, \mathbb{R}^M)$ is a weak solution of $P[v] = g$, then $v \in H^2(B_R, \mathbb{R}^M)$ and

$$\|v\|_{H^2} \leq c_2 (\|g\|_{L^2} + \|v\|_{L^2}).$$

**Proof.** Take $R \geq 1$, $g \in L^2(B_R, \mathbb{R}^M)$ and suppose $v \in H^1_0(B_R, \mathbb{R}^M)$ weakly solves $P[v] = g$ on $\Omega = B_R$, i.e., $B[v, \varphi] = \int_{B_R} \langle g, \varphi \rangle \, dx$ for any test function $\varphi \in C_0^\infty(B_R, \mathbb{R}^M)$. As before we define $f(x) := g(Rx)$, $u(x) := v(Rx)$ for $x \in B_1$ and take an arbitrary test function $\psi \in C_0^\infty(B_1, \mathbb{R}^M)$. Putting $\varphi(x) = \psi(R^{-1}x)$ for $x \in B_R$ we see that $f \in L^2(B_1, \mathbb{R}^M)$, $u \in H^1_0(B_1, \mathbb{R}^M)$ and $\varphi \in C_0^\infty(B_R, \mathbb{R}^M)$. Changing variables

$$\int_{B_R} \langle g, \varphi \rangle \, dx = R^N \int_{B_1} \langle f, \psi \rangle \, dx,$$

$$B[v, \varphi] = R^{N-2} \int_{B_1} \sum_{i,j=1}^N \langle A^{ij} \partial_j u, \partial_i \psi \rangle \, dx + R^{N-1} \int_{B_1} \sum_{i=1}^N \langle B^{ij}_R \partial_i u, \psi \rangle \, dx + R^N \int_{B_1} \langle C_R u, \psi \rangle \, dx,$$

where $B^i_R(x) := B^i(Rx)$, $C_R(x) := C(Rx)$ for $x \in B_1 (i = 1, \ldots, N)$. This shows that $u$ weakly solves the problem

$$P_0[u] = R^2 f + T[u], \quad \text{where} \quad P_0[u] := -\sum_{i,j=1}^N \partial_i (A^{ij} \partial_j u) \quad \text{and} \quad T[u] := R \sum_{i=1}^N B^{ij}_R \partial_i u + R^2 C_R u.$$

It is clear that $T[u] \in L^2(B_1, \mathbb{R}^M)$ and

$$\|T[u]\|_{L^2} \leq RN \max_{i=1,\ldots,N} \|B^i\|_{L^\infty} |u|_{1,2} + R^2 \|C\|_{L^\infty} |u|_{L^2}.$$

The regularity result (see e.g. [12, Thm 4.14]) implies that there is a constant $c_3 > 0$ such that given $h \in L^2(B_1, \mathbb{R}^M)$ if $w \in H^1_0(B_1, \mathbb{R}^M)$ is a weak solution to $P_0[w] = h$, then

$$|w|_{2,2} \leq c_3 \|h\|_{L^2}.$$
This implies that \( u \in H^2(B_1, \mathbb{R}^M) \) (in consequence \( v \in H^2(B_R, \mathbb{R}^M) \), too) and

\[
|u|_{2,2} \leq c_3(R^2\|f\|_{L^2} + ||T[u]\|_{L^2}).
\]

Hence there is a constant \( c_4 \) (depending on \( P_0 \) only) such that

\[
|u|_{2,2} \leq c_4(R^2\|f\|_{L^2} + R^2\|u\|_{L^2} + R|u|_{1,2}).
\]

As before, changing variables, we see that

\[
|u|_{2,2} = R^{2-N/2}|v|_{2,2}, \quad \|f\|_{L^2} = R^{-N/2}\|g\|_{L^2}, \quad \|u\|_{L^2} = R^{-N/2}\|v\|_{L^2} \quad \text{and} \quad |u|_{1,2} = R^{1-N/2}|v|_{1,2};
\]

hence (3.5) becomes

\[
|v|_{2,2} \leq c_4(\|g\|_{L^2} + |v|_{1,2} + \|v\|_{L^2}).
\]

By Lemma 3.2 (ii) and using the inequality \( ab \leq \varepsilon a^2 + b^2/\varepsilon \), \( a, b \geq 0 \), we have

\[
|v|_{1,2} \leq c_1(\varepsilon|v|_{2,2} + (1 + \varepsilon^{-1})\|v\|_{L^2}).
\]

Taking \( \varepsilon \) such that \( c_1c_4\varepsilon = 1/2 \) we get from (3.6) that

\[
|v|_{2,2} \leq c_5(\|g\|_{L^2} + \|v\|_{L^2})
\]

for some \( c_5 \) which, together with [1, Corollary 4.16], gives the assertion. \( \square \)

**Theorem 3.3.** Problem (1.1), (1.2), where \( \Omega = \mathbb{R}^N \), has a strong solution.

**Proof.** We consider the family of truncated problems

on \( \mathbb{R}^N \)

\[
P[u] = f(x, u, \partial u) \quad \text{on} \quad \Omega_n := B_n, \quad u|_{\partial B_n} = 0, \quad u(x) \in K(x) \quad \text{for a.e.} \quad x \in B_n,
\]

where \( n \in \mathbb{N} \). By Theorem 2.8, for every \( n \in \mathbb{N} \), there is a solution \( u_n \in H^2 \cap H_0^1(B_n, \mathbb{R}^M) \) such that \( u_n(x) \in K(x) \) a.e. on \( B_n \).

**Claim 1:** The sequence \((\|u_n\|_{H^2})_{n=1}^{\infty} \) is bounded.

In view (K3),

\[
\|u_n\|_{H^2} \leq M_0.
\]

Now we are going to establish the uniform \( H^2 \)-estimate. By Lemma 3.2 and (3.9) there is \( c_2 \) such that for all \( n \geq 1 \)

\[
\|u_n\|_{H^2}^2 \leq c_2(\|f(\cdot, u_n, \partial u_n)\|_{L^2} + M_0).
\]

Condition (f2) yields that for all \( n \geq 1 \)

\[
\|f(\cdot, u_n, \partial u_n)\|_{L^2} \leq \|\beta\|_{L^2} + c_0(I_1 + I_2),
\]

where

\[
I_1 = \left( \int_{B_n} |u_n|^{2q} \, dx \right)^{1/2} = \|u_n\|_{L^{2q}}^{s}, \quad I_2 = \left( \int_{B_n} |\partial u_n|^{2q} \, dx \right)^{1/2} = \|\partial u_n\|_{L^{2q}}^{q}.
\]

We now proceed similarly as in the proof of Lemma 2.1 but, in order to get constants independent of \( n \) we need to apply Lemmata 3.1 and 3.2.

If \( q = 1 \), then

\[
I_2 = |u_n|_{1,2} \leq \|u_n\|_{H^2}^{1/2}\|u_n\|_{L^2}^{1/2} \leq M_0^{1/2}\|u_n\|_{H^2}^{1/2}.
\]
in view of the Ehrling-Browder inequality. Let \(1 < q < \frac{N+2}{2} \), then \(2q \in (2,2^\ast)\). Let \(\theta_0 = \frac{N(q-1)}{2q}\). By interpolation and the Sobolev embedding \(H^2\) we have the interpolation

\[
I_2 \lesssim \|\partial u_n\|_{L^2}^{(1-\theta_0)} \|\partial u_n\|_{L^p}^{\theta_0} = \|u_n\|_{L^2}^{(1-\theta_0)} \|u_n\|_{L^p}^{\theta_0}.
\]

By Lemma 3.1 (ii)

\[
u_n, q^{(1-\theta_0)} \leq c_1(M_0^{1/2} \|u_n\|_{L^2}^{1/2} + M_0)q^{(1-\theta_0)} \leq c(1 + \|u_n\|_{L^2}^{(1-\theta_0)/2})
\]

for some \(c > 0\) independent of \(n\). On the other hand, by Lemma 3.1 (i),

\[
\|\partial u_n\|_{L^p} \leq c_0(2) \|\partial u_n\|_{H^1} \leq c_0(2) \|u_n\|_{H^2}.
\]

Taking (3.13), (3.14) and (3.15) into account we get

\[
I_2 \lesssim c(1 + \|u_n\|_{H^2}^{\gamma_1}),
\]

where \(\gamma_1 = q(1 + \theta_0)/2 < 1\) and the constant \(c\) does not depend on \(n\).

In order to estimate \(I_1\) we distinguish three cases. If \(s = 1\), then \(I_1 = \|u_n\|_{L^2} \leq M_0\). Let \(s > 1\) and \(N = 3\) or \(4\). Then \(2 < 2s < 2^\ast\); taking \(\theta_1 = \frac{N(s-1)}{2s}\), by interpolation and Lemma 3.1 (i)

\[
I_1 = \|u_n\|_{L^2}^s \lesssim \|u_n\|_{L^2}^{(1-\theta_1)} \|\partial u_n\|_{L^2}^{\theta_1} \leq M_0^{s(1-\theta_1)} [c_0(2) \|u_n\|_{H^1}]^{s\theta_1};
\]

additionally

\[
\|\partial u_n\|_{H^1}^{s\theta_1} \leq (M_0 + \|u_n\|_{L^2}^{\theta_1}) \|\partial u_n\|_{L^2}^{s\theta_1} \leq c(1 + \|u_n\|_{L^2}^{\theta_1})
\]

for some constant \(c\) independent of \(n\). Next, similarly as in (3.14), via Lemma 3.1 (ii)

\[
\|u_n\|_{L^2}^{\theta_1} \leq c(1 + \|u_n\|_{L^2}^{\theta_1}/2)
\]

and, finally,

\[
I_1 \lesssim c(1 + \|u_n\|_{H^2}^{\gamma_2}),
\]

where \(c\) does not depend on \(n\) and \(\gamma_2 = s\theta_1/2 < 1\). If \(s > 1\) but \(N \geq 5\), then \(2 < 2s < 2^\ast = \frac{2N}{N-1}\) and may write

\[
I_1 = \|u_n\|_{L^2}^s \lesssim \|u_n\|_{L^2}^{(1-\theta_2)} \|\partial u_n\|_{L^2}^{\theta_2}
\]

where \(s = \frac{N(s-1)}{4s}\). Observe that \(2^\ast < N\), hence we may use Lemma 3.1 (i) to get

\[
\|u_n\|_{L^2}^{\theta_2} \leq c_0(2^\ast) \|u_n\|_{W^{1,2^\ast}}.
\]

We have

\[
\|u_n\|_{W^{1,2^\ast}}^2 = \sum_{|\alpha| \leq 1} \|\partial^\alpha u_n\|_{L^2}^2 \leq \sum_{|\alpha| \leq 1} \|\partial^\alpha u_n\|_{H^1}^2 \leq c_0(2)(N+1) \|u_n\|_{H^2}^2.
\]

Thus

\[
I_1 \lesssim M_0^{s(1-\theta_2)} c\|u_n\|_{H^2}^{\gamma_3};
\]

where \(c := [c_0(2^\ast) c_0(2)(N+1)^{1/2^\ast}]^{\theta_2}\) is independent of \(n\) and \(\gamma_3 = s\theta_2 < 1\).

In view of (3.11), (3.12), (3.16), (3.17) and (3.19)

\[
|f| \lesssim (1 + \|u_n\|_{H^2}^2),
\]

where constants \(c > 0\) and \(0 < \gamma < 1\) do not depend on \(n \geq 1\). This together with (3.10) show the Claim, i.e., there is \(M_1 > 0\) such that

\[
\sup_{n \geq 1} \|u_n\|_{H^2} \leq M_1.
\]
From now on let us think of each $u_n$ as being extended to zero outside $B_n$. Since $u_n \in H^1_0(B_n, \mathbb{R}^M)$, we may assume that actually $u_n \in H^1(B_n, \mathbb{R}^M)$ (11).

Claim 2: The set $\{u_n\}_{n \geq 1}$ is relatively compact in $H^1(B_n, \mathbb{R}^M)$.

The idea is to decompose $\{u_n\} \subset \{\chi_{B_R} u_n + \chi_{\mathbb{R}^N \setminus B_R} u_n\}$ and show that the first set, being bounded in $H^2$, is compact in the $H^1$-sense due to the Rellich–Kondrachov theorem, while the second one is contained in the arbitrarily small ball, provided $R$ is large enough. In general, however, $\chi_{B_R} u_n \notin H^1(B_n, \mathbb{R}^M)$. We introduce instead a function $\varphi_R: \mathbb{R}^N \to [0, 1]$ having properties similar to those of $\chi_{\mathbb{R}^N \setminus B_R}$.

Consider a smooth function $\varphi \in C^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi(t) = 0$ for $t \leq 1$ and $\varphi(t) = 1$ for $t \geq 4$. For $R > 0$, let $\varphi_R: \mathbb{R}^N \to \mathbb{R}$ be given by

$$\varphi_R(x) := \varphi(R^{-2}|x|^2), \quad x \in \mathbb{R}^N.$$  

Then $\varphi_R \in C^\infty(\mathbb{R}^N)$, $0 \leq \varphi_R \leq 1$, $\varphi_R(x) = 0$ for $x \in B_R$ and $\varphi_R(x) = 1$ for $x \in \mathbb{R}^N \setminus B_{2R}$.

For any $R > 0$ and $n \in \mathbb{N}$, $\varphi_R u_n \in H^1_0(B_n, \mathbb{R}^M)$ so we may take it to test (3.8) and get

$$\mathbb{B}[u_n, \varphi_R u_n] = \int_{\mathbb{R}^N} \varphi_R \sum_{i,j=1}^N \langle A^{ij} \partial_j u_n, \partial_i u_n \rangle + \sum_{i,j=1}^N \partial_i \varphi_R \langle A^{ij} \partial_j u_n, u_n \rangle + \sum_{i=1}^N \varphi_R \langle B^i \partial_i u_n, u_n \rangle + \varphi_R (C u_n, u_n) \, dx = \int_{\mathbb{R}^N} \varphi_R (f(x, u_n, \partial u_n), u_n) \, dx$$

where we integrate over $\mathbb{R}^N$ since $\text{supp} \varphi_R u_n \subset \overline{B}_n$. The strong ellipticity in $(P'_2)$ implies that

$$\int_{\mathbb{R}^N} \theta \varphi_R |\partial u_n|^2 \, dx \leq \int_{\mathbb{R}^N} \varphi_R \sum_{i,j=1}^N \langle A^{ij} \partial_j u_n, \partial_i u_n \rangle \, dx = \int_{\mathbb{R}^N} \varphi_R (f(x, u_n, \partial u_n), u_n) \, dx +$$

$$- \int_{\mathbb{R}^N} \sum_{i,j=1}^N \partial_i \varphi_R \langle A^{ij} \partial_j u_n, u_n \rangle \, dx - \int_{\mathbb{R}^N} \left( \sum_{i=1}^N \varphi_R \langle B^i \partial_i u_n, u_n \rangle - \varphi_R (C u_n, u_n) \right) \, dx$$

Consequently

$$\theta \int_{|x| \geq 2R} |\partial u_n|^2 \, dx \leq \int_{\mathbb{R}^N} \theta \varphi_R |\partial u_n|^2 \, dx \leq I_0(n, R) + I_1(n, R) + I_2(n, R),$$

(3.22)

where

$$I_0(n, R) := \int_{|x| \geq R} |f(x, u_n, \partial u_n)| |u_n| \, dx, \quad I_1(n, R) := \int_{R \leq |x| \leq 2R} |u_n| \sum_{i,j=1}^N |A^{ij}| |\partial_j u_n| |\partial_i \varphi_R| \, dx,$$

$$I_2(n, R) := \int_{|x| \geq R} |u_n| \sum_{i=1}^N |B^i| |\partial_i u_n| + |C||u_n|^2 \, dx.$$

We estimate the right hand side summands. Firstly,

$$I_0(n, R) = \int_{|x| \geq R} |f(x, u_n, \partial u_n)| |u_n| \, dx = \int_{B_n \setminus B_c} |f(x, u_n, \partial u_n)| |u_n| \, dx \leq \|f(\cdot, u_n, \partial u_n)\|_{L^2(B_n, \mathbb{R}^M)} \left( \int_{|x| \geq R} m^2(x) \, dx \right)^{1/2} \to 0 \text{ as } r \to \infty,$$

(3.23)

Note that in general $u_n \notin H^2(\mathbb{R}^N, \mathbb{R}^M)$; this is the reason of some technical difficulties in the sequel.
since, in view of (3.19) and (3.20), the first factor above is bounded.

By the properties of $\varphi_R$

$$I_1(n, R) \leq \max_{1 \leq i, j \leq N} |A^{ij}| \int_{R \leq |x| \leq 2R} \left( \sum_{j=1}^{N} |u_n||\partial_j u_n| \right) \left( \sum_{i=1}^{N} |\partial_i \varphi_R| \right) dx \leq$$

$$\leq N \max_{i,j} |A^{ij}| \int_{R \leq |x| \leq 2R} |u_n||\partial u_n| |\partial \varphi_R| dx \leq N \sup_{t \in \mathbb{R}} |\varphi'(t)| \max_{i,j} |A^{ij}| \int_{R \leq |x| \leq 2R} |u_n||\partial u_n| \frac{2|x|}{R^2} dx \leq$$

$$\leq \frac{4N}{R} \sup_{t \in \mathbb{R}} |\varphi'(t)| \max_{i,j} |A^{ij}| \|u_n\|_{L^2} \|u_n\|_{H^1} \leq \frac{4N}{R} \sup_{t \in \mathbb{R}} |\varphi'(t)| \max_{i,j} |A^{ij}| M_0 M_1^2,$$

since $\|u_n\|_{H^1(B_R \setminus B_R M)} = \|u_n\|_{H^1(B_R \setminus B_R M)} \leq \|u_n\|_{H^2} \leq M_1$ in view of 3.21. Hence

$$I_1(n, R) \to 0 \text{ as } R \to \infty \text{ uniformly w.r.t. } n \in \mathbb{N}.$$  

Finally

$$I_2(n, R) \leq \max_i \|B^i\|_{L^\infty} \int_{|x| \geq R} |u_n| \sum_{i=1}^{N} |\partial_i u_n| \, dx + C \|L^\infty\| \int_{|x| \geq R} |u_n|^2 \, dx \leq$$

$$\leq \sqrt{N} \max_i \|B^i\|_{L^\infty} \int_{|x| \geq R} |u_n| |\partial u| \, dx + \|C\|_{L^\infty} \int_{|x| \geq R} m^2(x) \, dx \leq$$

$$\leq \left( \sqrt{N} \max_i \|B^i\|_{L^\infty} M_1 + \|C\|_{L^\infty} \right) \int_{|x| \geq R} m^2(x) \, dx \to 0 \text{ as } R \to \infty$$

uniformly w.r.t. $n \in \mathbb{N}.$

By (3.22), (3.24), (3.25) and (3.23), we find that $\sup_{n \geq 1} \|u_n\|_{1,2;R^N \setminus B_R} \to 0 \text{ as } R \to \infty.$ Hence, again by (3.9)

$$\sup_{n \geq 1} \|u_n\|_{H^1(B_R \setminus B_R M)} \leq \sup_{n \geq 1} \left( \|m\|^2_{L^2} + |u_n|^2_{1,2;R^N \setminus B_R} \right) \to 0 \text{ as } R \to \infty.$$  

Take an arbitrary $\varepsilon > 0$ and choose $R_0 > 0$ such that for $R \geq R_0$

$$\sup_{n \geq 1} \|u_n\|_{H^1(B_R \setminus B_R M)} < \varepsilon.$$  

Clearly, for all $n \geq 1$ and some $c > 0,$

$$\|\varphi_{R_0} u_n\|_{H^1} = c \|f_{R_0}\|_{W^{1,\infty}} \|u_n\|_{H^1} < \varepsilon c(1 + \|f\|_{L^\infty}).$$  

Similarly, if $n \geq 2R_0,$ then

$$\|((1 - \varphi_{R_0}) u_n)_{H^2}(B_{2R_0} \setminus B_R M) \| \lesssim \|\varphi\|_{W^{2,\infty}} \|u_n\|_{H^1(B_{2R_0} \setminus B_R M)} \lesssim \|f\|_{W^{2,\infty}} M_1.$$  

The set $\{(1 - \varphi_{R_0}) u_n\}_{n \geq 2R_0}$ being bounded in $H^2$-sense, is relatively compact in $H^1(B_{2R_0}, \mathbb{R}^M)$ in view of the Rellich-Kondrachov theorem. At the same time this set is contained in $H^1_0(B_{2R_0}, \mathbb{R}^M).$ This space (if we think of its elements as being extended onto $\mathbb{R}^N$) is closed in $H^1(\mathbb{R}^N, \mathbb{R}^M);$ therefore $\{(1 - \varphi_{R_0}) u_n\}_{n \geq 2R_0}$ is relatively compact in $H^1(\mathbb{R}^N, \mathbb{R}^M).$ Summing up

$$\{u_n\}_{n \geq 1} \subset \{(1 - \varphi_{R_0}) u_n\}_{n \geq 2R_0} \cup \{\varphi_{R_0} u_n\}_{n \geq 1},$$

where the first is relatively compact while the second is contained in the ball of a arbitrarily small radius. This show proves Claim 2.

**Claim 3:** If $u_0$ is a cluster point of $\{u_n\},$ then $u_0$ is a strong solution to (1.1), (1.2).

Without loss of generality we may assume that $u_n \to u_0$ in $H^1$- and $L^2^*$-sense. Therefore $u_n(x) \to u_0(x)$ and $\partial u_n(x) \to \partial u_0(x)$ for a.a. $x \in \mathbb{R}^N;$ moreover there are $h_0 \in L^2^*(\mathbb{R}^N)$ and $h_1 \in L^2(\mathbb{R}^N)$ such
that \(|u_n|, |u_0| \leq h_0\) and \(|\partial u_n|, |\partial u_0| \leq h_1\) a.e. on \(\mathbb{R}^N\).

It is clear that \(u_0(x) \in K(x)\) for a.a. \(x \in \mathbb{R}^N\). To see that \(u_0\) is a weak solution take an arbitrary \(\psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^M)\). The continuity of \(B\) implies that

\[
\mathcal{B}[u_n, \psi] \to \mathcal{B}[u_0, \psi] \quad \text{as} \quad n \to \infty.
\] (3.28)

The continuity of \(f(x, \cdot, \cdot)\) for a.a. \(x \in \mathbb{R}^N\) implies that

\[
f(x, u_n(x), \partial u_n(x)) \to f(x, u_0(x), \partial u_0(x)) \quad \text{as} \quad n \to \infty \quad \text{a.e.}
\]

The growth conditions imply also that

\[
|f(x, u_n(x), \partial u_n(x)) - f(x, u_n(x), \partial u_n(x))f(x, u_0(x), \partial u_0(x))| \leq \gamma(x) = \beta(x) + h_0^\delta(x) + h_1^\delta(x) \text{ a.e.}
\]

Hölder’s inequality with suitable exponents shows that \(\gamma(\cdot)|\psi| \in L^1(\mathbb{R}^N)\). The Lebesgue theorem show now that

\[
\int_{\mathbb{R}^N} f(x, u_n, \partial u_n), \psi \rangle \, dx \to \int_{\mathbb{R}^N} f(x, u_0, \partial u_0), \psi \rangle \, dx \quad \text{as} \quad n \to \infty.
\]

This shows that \(u_0\) is a weak solution to (1.1), (1.2). Now take an arbitrary bounded \(\Omega \subset \mathbb{R}^N\) and \(R > 0\) such that \(\Omega \subset \mathbb{B}_R\). It is clear that \(\psi u_n \to \psi u_0\) as \(n \to \infty\), where \(\psi := 1 - \varphi_R\). Observe that \(w := \psi u_0\) is a weak solution to \(\mathcal{P}[w] = g\), where

\[
g_k = \psi f_k(x, w, \partial w) - \sum_{i,j=1}^{N} \sum_{l=1}^{M} A^{ij}_{kl}((\partial_i \partial_j \psi)u_l + (\partial_i \psi)(\partial_j u_l) + (\partial_j \psi)(\partial_i u_l)) + \sum_{i=1}^{N} \sum_{l=1}^{M} B^{ij}_{kl}(\partial_i \psi)u_l.
\]

Estimating as in (3.20) we infer that \(g \in L^2(\mathbb{B}_{2R}, \mathbb{R}^M)\). Hence by the regularity \(w \in H^2(\mathbb{B}_{R}, \mathbb{R}^M)\), i.e., \(u_0 \in H^2(\Omega, \mathbb{R}^M)\).

\[
\Box
\]

4. Appendix

Here we collect some relevant facts used throughout the paper; we discuss assumptions and provide some examples and general results as well as we present the construction of the coincidence degree.

4.1. Sectorial operators (see e.g. [6, Chapter 1.3]). Let \((X, \| \cdot \|)\) be a (real) Banach space. A closed densely defined linear operator \(A : X \ni D(A) \to X\) is a sectorial (of angle \(< \pi/2\)) if there are \(0 < \phi < \pi/2\), \(M \geq 1\) and \(\lambda_0 \in \mathbb{R}\) such that the spectrum \(\sigma(A)\) of \(A\) is contained in the sector \(S_{\phi, \lambda_0} := \{\lambda \in \mathbb{C} \mid \lambda = \lambda_0 + re^{i\theta}, r > 0, |\theta| < \phi\} \cup \{\lambda_0\}\) and for \(\lambda \notin S_{\phi, \lambda_0}\)

\[
\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq M|\lambda - \lambda_0|^{-1}\quad (12).
\]

It is well-known that \(A\) is a sectorial operator if and only if \(-A\) generates the holomorphic semigroup \(\{T(t)\}_{t \geq 0}\); one has \(\|T(t)\| \leq M e^{-\lambda_0 t}\) for \(t \geq 0\).

If a sectorial operator \(A\) is positive, i.e., \(\text{Re} \lambda > 0\) for \(\lambda \in \sigma(A)\), then for \(\alpha > 0\) the improper integral

\[
A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) \, dt
\]

is convergent in the norm topology in \(\mathcal{L}(X)\); \(A^{-\alpha}\) is injective. Let \(A^{\alpha} := (A^{-\alpha})^{-1} : X^{\alpha} \to X\), where \(X^{\alpha} := A^{-\alpha}(X)\) is the fractional space associated with \(A\). \(X^{\alpha}\) is a Banach space endowed with the graph norm \(\|x\|_\alpha := \|A^\alpha x\|, \, x \in X^{\alpha}\). We also put \(X^0 := X\) and \(A^0 := I\), the identity on \(X\). For each \(\alpha \geq 0\), \(A^{\alpha}\) is a densely defined closed linear operator; for all \(0 \leq \alpha \leq \beta\), the embedding \(X^{\beta} \hookrightarrow X^{\alpha}\) is dense and continuous; it is compact provided \(A\) has compact resolvent. Observe that \(X^1 = D(A)\).

\[12\mathcal{L}(X)\] denotes the space of bounded linear operators on \(X\).
If $A$ is a sectorial operator, then there is $c \in \mathbb{R}$ such that $A_c := A + cI$ is positive (e.g. $c > -\lambda_0$).
Hence, given $\alpha \geq 0$, we may consider the fractional space $X^\alpha$ associated to $A_c$.

Remark 4.1. Different choices of $c$ give equivalent norms (see [17, Theorem 1.4.6]) on $X^\alpha$. This implies that for a sectorial operator $A$ the fractional space $X^\alpha$ is uniquely defined as a topological vector space: regardless the choice of the norm there is no ambiguity in topological terminology.

If $A : D(A) \to X$ is a sectorial operator, then the resolvent set $\rho(-A) \supset \{ \lambda \in \mathbb{C} \mid \Re \lambda > -\omega \}$. Let $\omega := -\lambda_0$. Given $h > 0$ with $h\omega < 1$,

\[ J_h = J_h^A = (I + hA)^{-1} : X \to X \]

is well-defined and

\[ J_h(X) \subset D(A). \]

Let us collect several well-known properties of $J_h$.

Lemma 4.2. If $h > 0$ and $h\omega < 1$, then

(i) if $h' > 0$ and $h'\omega < 1$, then $J_h = J_{h'} \left( \frac{h'}{h} I + \left( 1 - \frac{h'}{h} \right) J_h \right)$.  
(ii) $\| J_h \|_{L(X^\alpha)} \leq M(1 - h\omega)^{-1}$ for every $\alpha \geq 0$;  
(iii) $\| J_h \|_{L(X,X^\alpha)} \leq \frac{h\omega}{h}$.

Moreover for all $\alpha \in [0,1]$

(j) $\| J_h \|_{L(X,X^\alpha)} \leq h^{-\alpha}(1 - h\omega)^{\alpha-1}$;  
(jj) if $A$ has compact resolvent then $J_h \in L(X,X^\alpha)$ is compact, for every $h > 0$, $hc < 1$;  
(jjj) $\| J_h x - x \|_\alpha \to 0$ as $h \to 0^+$, for every $x \in X^\alpha$;  
(iv) the map $X \times (0,\omega_0) \ni (x,t) \mapsto J_h x \in X^\alpha$, where $\omega_0 := \infty$ if $\omega \leq 0$ and $\omega_0 := \omega^{-1}$ if $\omega > 0$, is continuous. \hfill $\Box$

Sectorial operators in Hilbert spaces are generated in a way described in subsection 2.1.2 (see [16, Sect. 7.3.2, Cor. 7.3.5]): let a Hilbert space $V$ be a dense subset of a Hilbert space $X$, assume that $\| v \|_X \lesssim \| v \|_V$ for $v \in V$, i.e. the embedding $V \hookrightarrow X$ is continuous, and let a bilinear form $B : V \times V \to \mathbb{R}$ be continuous and weakly coercive, i.e. there are $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

\[ B[v,v] + \omega \| v \|_X^\alpha \geq \alpha \| v \|_V^\alpha \] for $v \in V$.

If $A : V \to V^*$ is given by $[Au](v) := B(u,v)$, $u,v \in V$, then the part $A := A|_X$ of $A$ in $X = X^*$, given by $A u := Au$ for $u \in D(A) := \{ u \in V \mid Au \in X^* \}$ is a sectorial operator (with $\lambda_0 = \omega$). Clearly, for $u \in D(A)$ and $v \in V$, $\langle Au,v \rangle_X = B[u,v]$. The analytic semigroup generated by $-A$ is denoted by $\{ T(t) \}_{t \geq 0}$.

4.1.1. Invariance. Let $K \subset X$ be closed and convex; let $\pi_K : X \to K$ be the metric projection onto $K$, i.e., for $u \in X$, $\| u - \pi_K(u) \|_X = d(u,K) = \inf_{w \in K} \| u - w \|_X$; equivalently $v = \pi_K(u)$ is uniquely characterized by

\[ \langle u - v, w - v \rangle_X \leq 0 \text{ for any } w \in K. \]

We will give means to prove Proposition 2.7 and explain the invariance of $K$ with respect to the semigroup $\{ T(t) \}$ generated by the sectorial operator associated to a coercive bilinear form $B$ as above.
First let us observe that the so-called Post-Widder formula [10, Cor. III.5.5] and the integral representation of resolvents of \(-A\) in terms of the semigroup (see [10, (II.1.13)]) imply that the semigroup invariance of \(K\) is equivalent to its resolvent invariance, i.e., \(J_h(K) \subset K\) for \(h > 0\) with \(h\omega < 1\) (see also [10, Th. VI.1.8]). Observe that in fact the resolvent invariance of \(K\) means that

\[
J_h(K) \subset K \cap D(A) \subset K^\alpha.
\]

**Proposition 4.3.** The set \(K\) is invariant if and only if

\[
\pi_K(V) \subset V \quad \text{and} \quad B[\pi_K(u), u - \pi_K(u)] \geq 0 \quad \text{for any} \quad u \in V.
\]

**Proof.** Assume that \(K\) is resolvent invariant, take \(u \in V\) and let \(v := \pi_K(u)\). For any \(h > 0\), \(h\omega < 1\), \(J_hv \in D(A) \cap K\) and \(AJ_hv = h^{-1}(v - J_hv)\). Therefore \(B[J_hv, J_hv - v] = (AJ_hv, J_hv - v)_X = -h^{-1}\|v - J_hv\|^2_X \leq 0\). Thus \(B[J_hv, J_hv] \leq B[J_hv, v]\) and by (4.3) and

\[
\alpha\|J_hv\|^2_h \leq B[J_hv, J_hv] + \omega\|J_hv\|^2_X \leq B[J_hv, v] + \|J_hv\|^2_X \leq \text{const.}\|J_hv\|v\|v\| + \|J_hv\|^2_X.
\]

Take a sequence \(h_n \to 0^+\). By Lemma 4.2 (iii), \(J_{h_n}v \to v\) in \(X\). Hence the sequence \((J_{h_n}v)\) is bounded in \(V\) and weakly convergent to some \(w \in V\). The continuity \(V \to X\) implies actually that \(v = w\). This shows that \(v = \pi_K(u) \in V\). Next, in view of (4.4) and since \(J_{h_n}v \in K\) we have that for any \(n \geq 1\)

\[
B[J_{h_n}v, u - v] = (A J_{h_n}v, u - v)_X = h_n^{-1}(v - J_{h_n}v, u - v)_X \geq 0.
\]

The weak continuity of \(B[\cdot, u - v]\) implies \(B[v, u - v] = \lim_{n \to \infty} B[J_{h_n}v, u - v] \geq 0\).

Conversely assume (4.5), take \(h > 0\), \(h\omega < 1\) and \(u \in K\). Let \(y = J_hu\). Then then, by (4.5), (4.2), \(\pi_K(y) \in K \cap V\) and \(u = y + hA\). In view of (4.4), (4.5) and (2.15)

\[
0 \geq \langle u - \pi_K(y), y - \pi_K(y) \rangle_X = \langle y - \pi_K(y) + hA\,y, y - \pi_K(y) \rangle_X = \|y - \pi_K(y)\|^2_X + hB[y, y - \pi_K(y)] = \|y - \pi_K(y)\|^2_X + hB[y, y - \pi_K(y)] + hB[\pi_K(y), y - \pi_K(y)] \geq (1 - h\omega)\|y - \pi_K(y)\|^2_X + h\|y - \pi_K(y)\|^2_Y \geq 0.
\]

This shows that \(y = \pi_K(y) \in K\). \(\square\)

### 4.2. Tangent cones

(see e.g. [4, Chapter 4]). If \(K\) is a closed subset of a Banach space \(X(\| \cdot \|)\), then

\[
d_K(u) = \inf_{w \in K} \|u - w\|.
\]

Let \(x \in K\); the Bouligand (or contingent) cone

\[
T_K(x) := \{v \in X \mid \lim_{h \to 0^+} h^{-1}d_K(x + hv) = 0\}.
\]

The Clarke (or circatangent) cone

\[
C_K(x) := \{v \in X \mid \lim_{h \to 0^+, y \to x, y \in K} h^{-1}d_K(y + hv) = 0\}.
\]

Both these sets are cones, \(C_K(x) \subset T_K(x)\) and, in general, the inclusion is strict. However, in view of [4, Theorem 4.1.9]

\[
\liminf_{y \to x, y \in K} T_K(y) \subset C_K(x),
\]

where \(\liminf\) is the lower limit of sets in the sense of Kuratowski (see [4, Def. 1.4.6]). If \(K\) is convex, then

\[
C_K(x) = T_K(x) = \bigcup_{h > 0} h^{-1}(K - x).
\]

In such a case if \(v \in X\), then \(v \in T_K(x)\) if and only if \(p(v) \leq 0\) for any \(p \in X^*\) such that \(p(w - x) \leq 0\) for all \(w \in K\).
4.3. Examples of constraints. We are going to provide some examples of constraint \( K : \Omega \rightarrow \mathbb{R}^M \), where \( \Omega \subset \mathbb{R}^N \) having the properties studied above.

\( \text{przykład } \) Example 4.4. (1) (Moving rectangle) Assume that \( \sigma, \tau \in H^1(\Omega, \mathbb{R}^M) \), \( \sigma \leq 0 \leq \tau \) and let

\[
K(x) := [\sigma(x); \tau(x)] = \{ w \in \mathbb{R}^M \mid \sigma_k(x) \leq w_k \leq \tau_k(x), k = 1, \ldots, M \}, \quad x \in \Omega.
\]

It is immediate to see that assumptions \((K_1), (K_2)\) and \((K_3)\) are satisfied. For each \( x \in \Omega \), the projection \( r(x, \cdot) \) of \( \mathbb{R}^M \) onto \( K(x) \) is given by \( r(x, \cdot) = (r_1(x, \cdot), \ldots, r_M(x, \cdot)) \), where for \( k = 1, \ldots, M \) and \( u = (u_1, \ldots, u_m) \in \mathbb{R}^M \) and

\[
r_k(x, u) = (\tau_k - \sigma_k - (u_k - \tau_k)^-) + \sigma_k = \begin{cases} 
\sigma_k(x) & \text{if } u_k < \sigma_k(x) \\
u_k & \text{if } \sigma_k(x) \leq u_k \leq \tau_k(x) \\
\tau_k(x) & \text{if } \tau_k(x) < u_k.
\end{cases}
\]

In view of [11, Section 7.4] \( r_k(\cdot, u) \in H^1(\Omega) \), i.e., \((K_4)\) (i) is satisfied. If \( \sigma_k|_{\partial \Omega} \leq 0 \) and \( \tau_k|_{\partial \Omega} \geq 0 \) in the sense of trace \((k = 1, \ldots, M)\), then assumption \((K_4)\) (ii) holds, too. Observe that for a.a. \( x \in \Omega \), if \( w \in K(x) \), then \( T_K(x)(w) = \mathbb{R}^M \) if \( \sigma_k(x) < w_k < \tau_k(x) \) \((k = 1, \ldots, M)\) and

\[
v = (v_1, \ldots, v_M) \in T_K(x)(w) \iff \begin{cases} 
v_k \geq 0 & \text{if } w_k = \sigma_k(x), \\
v_k \leq 0 & \text{if } w_k = \tau_k(x).
\end{cases}
\]

(2) (Tube) Let \( K \subset \mathbb{R}^M \) be closed and convex, \( b \in H^1(\mathbb{R}^N, \mathbb{R}^M) \), \( \alpha \in H^1(\mathbb{R}^N) \) with \( \text{ess inf } \alpha > 0 \) and \( K(x) = b(x) + \alpha(x)K, \ x \in \Omega \).

\( \text{tube } \) If \( s : \mathbb{R}^M \rightarrow K \) is the metric projection onto \( K \), then

\[
r(x, u) = b(x) + \alpha(x)s(\alpha(x)^{-1}(u - b(x))), \quad x \in \Omega, \ u \in \mathbb{R}^M.
\]

Clearly \( K(\cdot) \) is closed convex and satisfies assumptions \((K_1), (K_2)\) and \((K_4)\) (i); condition \((K_4)\) (ii) holds if \( -\alpha^{-1}(x)b(x) \in K \) for \( x \in \partial \Omega \) in the trace sense. Condition \((K_3)\) holds if \( K \) is bounded. It is easy to see that, for a.a. \( x \in \Omega \) and \( w = b(x) + \alpha(x)u \in K(x) \), where \( u \in K, T_K(x)(w) = T_K(u) \).

(3) (Funnel) Let \( K \) be the closed unit ball in \( \mathbb{R}^M \)-closed convex, let all entries of a matrix-valued map \( E : \Omega \rightarrow \mathbb{R}^{M \times M} \) belong to \( H^1(\Omega) \) and let \( \text{ess inf } x \in \Omega \det E(x) > 0 \). One shows that for any \( u \in \mathbb{R}^M \) the map

\[
\Omega \ni x \mapsto y(x, u) = \arg \min_{y \in K} \left\{ \frac{1}{2} \langle E(x)E(x)y, y \rangle - \langle E(x)u, y \rangle \right\},
\]

is in \( H^1(\Omega, \mathbb{R}^M) \). The funnel

\[
K(x) := E(x)B, \quad x \in \Omega,
\]

satisfies conditions \((K_1), (K_2), (K_3)\) and \((K_4)\) (i), (ii), since the projection \( r(x, u) = E(x)y(x, u), x \in \Omega, u \in \mathbb{R}^M \). If \( K \) is bounded and \( \|E(\cdot)\| \in L^2(\Omega) \), then condition \((K_3)\) holds, too. Moreover, for a.a. \( x \in \Omega \) and \( u \in K(x), v \in T_K(x)(u) \) if and only if \( \langle E(x)^{-1}v, E(x)^{-1}u \rangle \leq 0 \).

(4) (Moving polyhedron) Suppose a set \( P \subset \{ p \in \mathbb{R}^M \mid |p| = 1 \} \) is at most countable and consider

\[
K(x) := \bigcup_{p \in P} K_p(x), \quad K_p(x) := \{ u \in \mathbb{R}^M \mid \langle p, u \rangle \leq \xi_p(x) \}, \ x \in \Omega,
\]

where \( \xi_p \in H^1(\Omega) \) and \( \xi_p|_{\partial \Omega} \geq 0 \) in the trace sense. Properties \((K_1), (K_2)\) and \((K_4)\) are satisfied. For a.a. \( x \in \Omega \) and \( u \in K(x) \), let \( P(u) = \{ p \in P \mid \langle p, u \rangle = \xi_p(x) \} \). Then for a.a. \( x \in \Omega \), \( T_K(x)(u) = \{ v \in \mathbb{R}^M \mid \langle p, v \rangle \leq 0 \ \forall \ p \in P(u) \} \). □
Proposition 4.5. Recall the operator $\mathcal{P}$ defined by (1.3) satisfying conditions $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$. Let $K(\cdot)$ be defined as in example (4) above and Assumption $(\mathcal{P}_3)$ is fulfilled if for all $1 \leq i,j \leq N$, any $p \in P$ is an eigenvector of transposed matrices $^tA^{ij}$, $^tB^i$ and $^tC$, i.e.,

$$
^tA^{ij}(x)p = a^{ij}(x)p, \quad ^tB^i(x)p = b^i(x)p, \quad ^tC(x)p = c(x)p
$$

for a.a. $x \in \Omega$ and some functions $a^{ij}, b^i, c : \Omega \to \mathbb{R}$, and, for any $p \in P$, $\mathcal{B}[\xi_p(\cdot)p, \eta(\cdot)p] \geq 0$ for any $\eta \in H^1_0(\Omega), \eta \geq 0$.

Proof. Let $p \in P$. It is clear that $K_p(\cdot)$ satisfies conditions $(K_1), (K_2)$; in view of [4, Th. 8.2.4], so does $K(\cdot)$. It is clear that $K = \bigcap_{p \in P} K_p$, where $K_p = \{ u \in L^2(\Omega, \mathbb{R}^M) \mid u(x) \in K_p(x) \text{ a.e.} \}$. Hence in order to show the invariance of $K$ it is sufficient to show the invariance of $K_p$. The projection onto $K_p(x)$ is given by

$$
r(x, u) = u - (\langle u, p \rangle - \xi(x))^+ p, \quad u \in \mathbb{R}^M, \quad x \in \Omega, u \in \mathbb{R}^M.
$$

Thus $r(\cdot, u) \in H^1(\Omega, \mathbb{R}^M)$ if and only if $\xi \in H^1(\Omega), i.e., K_p(\cdot)$ satisfies assumption $(K_3)$, too. We may rely on Lemma 4.3 by showing that for any $u \in H^1_0(\Omega, \mathbb{R}^M)$, $\mathcal{B}[\pi(u), u - \pi(u)] \geq 0$, where the metric projection $\pi = \pi_{K_p}$ onto $K_p$ is given by the formula

$$
\pi(u) = u - (\langle u, p(\cdot) \rangle - \xi_p)^+ p, \quad u \in L^2(\Omega, \mathbb{R}^M).
$$

To simplify the notation let $v := (\langle p, u(\cdot) \rangle - \xi_p)^+; \text{ clearly } v \in H^1_0(\Omega), v \geq 0 \text{ and } v = 0 \text{ off the set } \Omega_0 := \{ x \in \Omega \mid \langle p, u(\cdot) \rangle > \xi_p \}$. In view of [11, Section 7.4] for any $i = 1, \ldots, N$, $\partial_i v = \langle p, \partial_i u(\cdot) \rangle - \partial_i \xi_p$ on $\Omega_0$ and 0 elsewhere. In view of our assumptions we have that on $\Omega_0$

$$
\sum_{i,j=1}^N \langle A^{ij} \partial_j u - \partial_j v(\cdot)p, \partial_i v(\cdot)p \rangle + \sum_{i=1}^N \langle B^i \partial_i u - \partial_i v(\cdot)p, v(\cdot)p \rangle + \langle C(\cdot - v(\cdot)p, v(\cdot)p) \\
= \sum_{i,j=1}^N \langle \partial_j u - \partial_j v(\cdot)p, \partial_i v(\cdot)p \rangle A^{ij} + \sum_{i=1}^N \langle \partial_i u - \partial_i v(\cdot)p, v(\cdot)p \rangle B^i + \langle u, v(\cdot)^t C p \rangle \\
= \sum_{i,j=1}^N a^{ij} \partial_j \xi_p \partial_i v + \sum_{i=1}^N b_i \partial_i \xi_p v + c \xi_p v \\
= \sum_{i,j=1}^N \langle A^{ij} \partial_j (\cdot)p, \partial_i v(\cdot)p \rangle + \sum_{i=1}^N \langle B^i \partial_j \xi_p(\cdot)p, v(\cdot)p \rangle + \langle C \xi_p(\cdot)p, v(\cdot)p \rangle.
$$

This implies that

$$
\mathcal{B}[\pi(u), u - \pi(u)] = \mathcal{B}[u - v(\cdot)p, v(\cdot)p] = \mathcal{B}[\xi_p(\cdot)p, v(\cdot)p] \geq 0. \quad \square
$$

Corollary 4.6. Suppose $K(\cdot)$ is given by Example 4.4. If the operator $\mathcal{P}$ is diagonal, i.e., for each $1 \leq i,j \leq N$, matrices of coefficients $A^{ij}, B^i$ and $C$ are diagonal, $A^{ij}_{kl} = \delta_{k} a_{ij}^{kl}, B^i_{kl} = \delta_{kl} b^i_k$ and $C_{kl} = \delta_{kl} c_k$ for $1 \leq k,l \leq M$, then condition $(\mathcal{P}_3)$ is satisfied if $\mathcal{B}_k[\sigma_k, v] \leq 0 \text{ and } \mathcal{B}_k[\tau_k, v] \geq 0 \text{ for any } v \in H^1_0(\Omega)$, where

$$
\mathcal{B}_k[u, v] := \int_{\Omega} \left( \sum_{i,j=1}^N \delta_{ij} \partial_j \partial_i v + \sum_{i=1}^N b^i_k \partial_i u v + c_k u v \right) \ dx, \quad u, v \in H^1_0(\Omega).
$$
Remark 4.7. A closed set $K$ of a Banach space $E$ is called an $\mathcal{L}$-retract (see e.g. [5]) if there is $\eta > 0$, a continuous map $r : B(K, \eta) \to K$, where $B(K, \eta) := \{ x \in E \mid d_K(x) < \eta \}$, and $L \geq 1$ such that $\| r(x) - x \| \leq L d_K(x)$ for any $x \in B(K, \eta)$. $\mathcal{L}$-retracts constitute a broad subclass of neighborhood retracts containing most classes of sets usually considered a constraint sets; in particular any closed convex set $K \subset E$ is an $\mathcal{L}$-retract; for more details and examples of $\mathcal{L}$-retracts - see [5].

(2) Assumption (2) along with (1) implies that if $K$ is bounded, then the set $K^\alpha$ is of finite homological type, i.e., for each $q \geq 0$ the vector space $H_q(K^\alpha)$, where $H_*(\cdot)$ stands for the singular homology functor with the rational coefficients, is finite dimensional and $H_q(K^\alpha) = 0$ for almost all $q \geq 0$. To see this let $U := B(K, \eta) \cap X^\alpha$ and let $\phi(x) := J_h \circ r(x)$, $x \in U$, where $0 \leq h \leq h_0$ is fixed. Then $\phi : U \to K^\alpha$ is a well-defined continuous compact map. Let $j : K^\alpha \to U$ be the embedding and $\phi := j \circ \phi : U \to U$. In view of the so-called normalization property of the Leray-Schauder fixed point index $\tilde{\phi}$ is a Lefschetz map (see [13, Theorem (7.1)]). The commutativity of the following diagram

\[
\begin{array}{ccc}
U & \xleftarrow{\phi} & K^\alpha \\
\downarrow{\phi} & & \downarrow{\phi|_{K^\alpha}} \\
U & \xrightarrow{j} & K^\alpha
\end{array}
\]
suppose now to the contrary that there are a sequences

\[ h_n \rightarrow h \in (0, h_0) \]

then by Lemma 3.3 (ii) and (iii),

\[ \| \Phi(x_n, t_n) - \Phi(x_0, 0) \| = \| J_{t_n h} x_n - x_0 \| \] \tag{4.14}

\[ \leq \frac{M}{1 - t_n h \omega} \| x_n - x_0 \| + \| J_{t_n h} x_0 - x_0 \| \to 0, \quad n \to \infty. \]

(3) Observe that if \( F \) satisfies (4.13), then \( F \) is tangent in the sense of Clarke (see section 4.2), i.e., for any \( x \in U \cap \mathbb{K}^\alpha \),

\[ F(x) \in C_{\mathbb{K}}(x) := \{ v \in \mathbb{K} \mid \lim_{y \to x, \bar{y} \in \mathbb{K}} \frac{h^{-1} d_{\mathbb{K}}(y + hv)}{h} = 0 \}; \]

This is a consequence of the continuity of \( F \) and (4.6).

Let us now present the steps of the construction:

**Step 1:** Let \( \overline{F} : U \to \mathbb{X} \) be a continuous extension of \( F \) (\( K^\alpha \) is closed in \( X^\alpha \); hence \( F \) is defined on a closed subset of \( U \); \( \overline{F} \) exists in view of the Dugundji extension theorem);

**Step 2:** Fix \( \eta > 0, L \geq 1 \) and a retraction \( r : B_X(K, \eta) \to K \) such that

\[ \| r(x) - x \| \leq L d_{\mathbb{K}}(x), \quad x \in B_X(K, \eta). \]

**Step 3:** Since \( C \) is compact and \( \overline{F} \) is continuous, one can find an open bounded subset \( V \subset X^\alpha \) such that

\[ C \subset V \subset \mathbb{X} \subset U \cap (B(K, \eta/2) \cap X^\alpha), \]

and \( \overline{F}(\mathbb{X}) \) is bounded in \( X \) (recall that the embedding \( X^\alpha \subset X \) is continuous, i.e., \( B(K, \eta/2) \cap X^\alpha \) is open and contains \( C \)).

**Step 4:** Since the set \( \overline{F}(\mathbb{X}) \) is bounded we may assume that \( \| h \overline{F}(x) \| \leq \eta/2 \) for \( h \in (0, h_0) \) and \( x \in \mathbb{X} \). By (4.16), for any \( x \in V, d(x, \mathbb{K}) < \eta/2 \) and, thus, \( x + h \overline{F}(x) \in B(K, \eta) \). Therefore and in view of 4.12 the map \( \phi_h : V \to X^\alpha \), where \( h \in (0, h_0) \), given by the formula

\[ \phi_h(x) = J_h \circ r(x + h \overline{F}(x)), \quad x \in \mathbb{X}, \]

is well-defined. Observe that if \( h > 0 \) is small enough, then \( \phi_h : \mathbb{X} \to X^\alpha \) is a compact map such that \( \text{Fix}(\phi_h) := \{ x \in \mathbb{X} \mid \phi_h(x) = x \} \subset V \). The compactness of \( \phi_h \) follows from the resolvent compactness of \( A \) and the fact that \( r \) maps bounded sets into bounded ones. First observe that if \( x \in \text{Fix}(\phi_h) \), then \( x \in \cap D(A) \cap K^\alpha \) since \( J_h(K) \subset D(A) \cap K \subset K^\alpha \) and, obviously, \( x \in \mathbb{X} \). Hence \( x = J_h(r(x + h \overline{F}(x))) \).

Suppose now to the contrary that there are a sequences \( h_n \searrow 0 \) and \( (x_n) \subset \partial V \cap \text{Fix}(\phi_h) \). Hence \( x_n + h_n A(x_n) = r(x_n + h_n \overline{F}(x_n)) \). Arguing as in the proof of [8, Lemma 3.3] (with obvious modifications) we gather that, after passing to a sequence if necessary, \( x_n \to x_0 \in C \cap \partial V \). This proves our assertion.

**Step 5:** Without loss of generality we may assume that \( \phi_h \) is well-defined and compact and \( \text{Fix}(\phi_h) \subset V \).
for any \( h \in (0, h_0) \). This implies that for any such \( h \), the \textit{Leray-Schauder fixed point index} \( \operatorname{Ind}_{LS}(\phi_h, V) \) is well-defined (see [13, Sect. 7, 8]). Finally we are in a position to define the \textit{constrained topological degree} \( \operatorname{deg}_K(A, F; U) \) of coincidence between \( A \) and \( F \) on \( U \) as follows:

\[
\operatorname{deg}_K(A, F; U) := \lim_{h \to 0^+} \operatorname{Ind}_{LS}(\phi_h, V),
\]

Definition (4.17) is correct: it stabilizes, i.e., \( \operatorname{Ind}_{LS}(\phi_{h_1}, V) = \operatorname{Ind}(\phi_{h_2}, V) \) for sufficiently small \( h_1, h_2 \), it depends neither on the choice of an \( \mathcal{L} \)-retraction \( r \) nor the choice of a bounded open neighbourhood \( V \) of \( C \) or the extension \( \tilde{F} \). These issues can be shown in a similar manner as in the proof of [8, Lemma 3.5 and Lemma 3.6]. Definition (4.17) does not depend on the choice of \( c \), i.e., the constant such that \( A_c = A + cI \) determines the norm \( \| \cdot \|_a \). If we take another \( \hat{c} > -\omega \), then the identity provides a (topological) homeomorphism between \( (X^\alpha, \| \cdot \|_a) \) and \( (X^\alpha, \| \cdot \|_a) \) in view of Remark 4.1. The claim follows now immediately from the so-called \textit{commutativity property} of the Leray-Schauder index (see [13, Theorem (7.1)]).

By an \textit{admissible homotopy} we understand a continuous map \( H : U \cap K^\alpha \times [0, 1] \to X \) such that tangent \( H(x, t) \in T_K(x) \) for all \( x \in U \cap K^\alpha \), \( t \in [0, 1] \), and the set \( \operatorname{Coin}(A, H; U \cap K^\alpha) \) is compact.

\textbf{Remark 4.8.} Arguing as above we easily see that one can find an open bounded \( V \subset X^\alpha \) such that \( \operatorname{Coin}(A, H; U \cap K^\alpha) \subset V \subset U \cap (B(K, \eta/2) \cap X^\alpha) \) and \( H(V \times [0, 1]) \) is bounded in \( X \). Then, for sufficiently small \( h > 0 \) the map \( \phi_h : V \times [0, 1] \to X^\alpha \) given by \( \phi_h(x, t) := J_h \circ r(x + hH(x, t)) \) for \( x \in V \), \( t \in [0, 1] \), where \( \tilde{H} : U \to X \) is a continuous extension of \( H \), is well-defined compact and

\[
\{ x \in V \mid \exists t \in [0, 1], \phi_h(x, t) = x \} \subset V.
\]

\textbf{Theorem 4.9.} The degree from Definition 4.17 has the following properties:

\textbf{(Existence)} If \( \operatorname{deg}_K(A, F; U) \neq 0 \) then there is \( x \in U \cap D(A) \) such that \( A x = F(x) \).

\textbf{(Additivity)} If \( U_1, U_2 \subset U \) are open disjoint and \( \operatorname{Coin}(A, F; U) \subset (U_1 \cup U_2) \setminus (U_1 \cap U_2) \), then

\[
\operatorname{deg}_K(A, F; U) = \operatorname{deg}_K(A, F; U_1) + \operatorname{deg}_K(A, F, U_2).
\]

\textbf{(Homotopy invariance)} If \( H : U \times [0, 1] \to X \) is an admissible homotopy, then

\[
\operatorname{deg}_K(A, H(\cdot, 0), U) = \operatorname{deg}_K(A, H(\cdot, 1), U).
\]

\textbf{(Normalisation)} If \( K \) is bounded, \( F : K^\alpha \to X \) and \( F(K^\alpha) \) is bounded in \( X \), then for any \( U \supset K^\alpha \)

\[
\operatorname{deg}_K(A, F, U) = \chi(K^\alpha).
\]

\textbf{Proof.} (Existence) By definition (4.17), given a sequence \( h_n \searrow 0 \) (with sufficiently small entries), we have \( \operatorname{Ind}(\phi_{h_n}, V) \neq 0 \). The existence property of the Leray-Schauder index implies the existence of a sequence \( (x_n) \) in \( V \) such that \( \phi_{h_n}(x_n) = x_n \), i.e., \( x_n \in D(A) \) and \( x_n + h_n A x_n = r(x_n + h_n F(x_n)) \). The above equality yields

\[
\| A x_n - F(x_n) \| = \frac{1}{h_n} \| x_n + h_n A x_n - x_n - h_n F(x_n) \|
\]

\[
= \frac{1}{h_n} \| r(x_n + h_n F(x_n)) - (x_n + h_n F(x_n)) \| \leq L \frac{d_K(x_n + h_n F(x_n))}{h_n},
\]

where in the last inequality we used the fact that \( r \) is \( \mathcal{L} \)-retraction. Since \( x_n \in V \cap K \), we get

\[
\frac{d_K(x_n + h_n F(x_n))}{h_n} = \frac{d_K(x_n + h_n F(x_n)) - d_K(x_n)}{h_n} \leq \| F(x_n) \| \leq R,
\]
for a constant $R$ such that $\sup_{x \in \overline{V}} \| F(x) \| = R$. Eq. (4.18) combined with (4.19) yields that $\{ \| Ax_n \| \}_{n \geq 1}$ is bounded. For any $n \gg 1$, $x_n = J_{h_0}(x_n + h_0Ax_n)$. By Lemma 4.2 (jj), the set $\{ x_n \}_{n \geq 1}$ is relatively compact (in $X^\alpha$). Passing to a subsequence if necessary, we have $x_n \rightarrow x_0 \in \overline{V} \cap K^\alpha \subset U \cap K^\alpha$. By (4.18), we have

$$\| Ax_n - F(x_n) \| \leq L \frac{d_K(x_n + h_nF(x_0))}{h_n} + L\| F(x_0) - F(x_n) \|$$

Letting $n \rightarrow \infty$, using the continuity and the tangency (4.15) of $F$, we see that $\lim_{n \rightarrow \infty} \| Ax_n + F(x_n) \| = 0$. Thus, $A x_n \rightarrow F(x_0)$ and, since $A$ is closed, $x_0 \in D(A)$ with $A x_0 = F(x_0)$. Thus, $x_0 \in D(A) \cap U$ satisfies $A x_0 = F(x_0)$.

(Additivity) follows by Definition 4.17 from the additivity property of the Leray-Schauder index.

(Homotopy invariance) is a consequence of the homotopy invariance of the Leray-Schauder index, Definition 4.17 and Remark 4.8.

(Normalization) The independence of $\deg_{K}(A, F; U)$ of $U$ follows immediately form the additivity property. Take an open $V$ such that $K^\alpha \subset \overline{V} \subset U \cap (B(K^\alpha, \eta/2) \cap X^\alpha)$ and a continuous extension $\overline{F} : X^\alpha \rightarrow X$ of $F$ being bounded on $\overline{V}$. Let $h \in (0, h_0]$ be such that $\deg_{K}(A, F; U) = \Ind_{LS}(\phi_h, V)$. Recall that $\phi_h(V) \subset K^\alpha$. Hence we may treat $\phi_h : V \rightarrow K^\alpha$. Denoting the embedding $K^\alpha \subset V$ by $j$ and $\tilde{\phi}_h := \phi_h \circ j$, we see that $\tilde{\phi}_h$ is a Lefschetz map and

\[
\text{(4.20)} \quad \Ind_{LS}(\phi_h, V) = \Ind_{LS}(\tilde{\phi}_h, V) = \Lambda(\tilde{\phi}_h)
\]

is the (generalized) Lefschetz number of $\tilde{\phi}_h$. The argument is now similar to that from Remark 4.7 (2). We have the commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\phi_h} & K^\alpha \\
\downarrow{\phi_h} & \ & \downarrow{\phi_h|_K^\alpha} \\
V & \xrightarrow{\phi_h} & K^\alpha \\
\end{array}
\]

Again, by [13, Lemma (3.1)], $\phi_h|_K^\alpha$ is a Lefschetz map and

\[
\text{(4.21)} \quad \Lambda(\phi_h|_K^\alpha) = \Lambda(\tilde{\phi}_h).
\]

Now we shall show that $\phi_h|_K^\alpha$ is homotopic to the identity $id : K^\alpha \rightarrow K^\alpha$. To this end, let us define $\Phi : K^\alpha \times [0, 1] \rightarrow K^\alpha$ by

$$\Phi(x, t) := \begin{cases} 
\phi_h(x), & \text{for } x \in K^\alpha, \ t \in (0, 1], \\
x, & \text{for } x \in K^\alpha, \ t = 0.
\end{cases}$$

By Lemma 4.2 (jv), $\Phi$ is continuous on $K^\alpha \times (0, 1]$. Let $x_n \rightarrow x_0$ in $K^\alpha$ and $t_n \searrow 0$. Then

$$\|\Phi(x_n, t_n) - \Phi(x_0, 0)\| = \|\phi_{t_n h}(x_n) - x_0\|_\alpha \leq S_1(n) + S_2(n),$$

where

\[
S_1(n) = \|J_{t_n h} r(x_n + h_nF(x_n)) - J_{t_n h}(x_n + t_n hF(x_n))\|_\alpha, \quad S_2(n) = \|J_{t_n h}(x_n + t_n hF(x_n)) - x_0\|_\alpha.
\]
By Lemma 4.2 (j), there is $C_\alpha > 0$ such that

$$S_1(n) \leq \frac{C_\alpha}{(t_n h)^\alpha (1 - t_n h \omega)^1 - \alpha} \|r(x_n + t_n h F(x_n)) - (x_n + t_n h F(x_n))\| \leq$$

$$\leq \frac{C_\alpha}{(t_n h)^\alpha (1 - t_n h \omega)^1 - \alpha} L d_{K}(x_n + t_n h F(x_n)) \leq \frac{C_\alpha L}{(t_n h)^\alpha (1 - t_n h \omega)^1 - \alpha} t_n h \|F(x_n)\| \leq$$

$$\leq \frac{C_\alpha L}{(1 - t_n h \omega)^1 - \alpha} (t_n h)^{1 - \alpha} \|F(x_n)\| \to 0, \quad n \to \infty,$$

since the sequence $\|F(x_n)\|$ is bounded. To estimate $S_2(n)$ note that

$$S_2(n) \leq \|J_{t_n h}(x_n + t_n h F(x_n)) - J_{t_n h}(x_n)\|_\alpha + \|J_{t_n h}(x_n) - x_0\|_\alpha.$$

The first summand satisfies

$$\|J_{t_n h}(x_n + t_n h F(x_n)) - J_{t_n h}(x_n)\|_\alpha \leq \frac{C_\alpha}{(1 - t_n h \omega)^1 - \alpha} (t_n h)^{1 - \alpha} \|F(x_n)\| \to 0, \quad n \to \infty,$$

and, as in (4.14), $\|J_{t_n h}(x_n) - x_0\|_\alpha \to 0$ as $n \to \infty$.

Now it is clear that

$$\Lambda(\phi_h | K^\alpha) = \Lambda(id) = \lambda(id)$$

the (ordinary) Lefschetz number of id. But, by the very definition $\lambda(id) = \chi(K^\alpha)$. In view of (4.20), (4.21) and (4.22) we conclude the proof of the normalization property and the theorem.

\[\square\]

\textbf{Remark 4.10.} If $F$ is defined on $K^\alpha$, i.e., $K^\alpha \subset U$, then $\deg_K(A, F; U)$ does not depend on $U$ and we may suppress it from the notation and write $\deg_K(A, F)$.  

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