Definable topological dynamics for trigonalizable algebraic groups over $\mathbb{Q}_p$

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We study the flow $(G(\mathbb{Q}_p), S_G(\mathbb{Q}_p))$ of trigonalizable algebraic group acting on its type space, focusing on the problem raised in [17] of whether weakly generic types coincide with almost periodic types if the group has global definable f-generic types, equivalently whether the union of minimal subflows of a suitable type space is closed. We shall give a description of f-generic types of trigonalizable algebraic groups, and prove that every f-generic type is almost periodic.

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1 Introduction

In recent years, there has been growing interest in the interaction between topological dynamics and model theory. This approach was introduced by Newelski [10], then developed by a number of papers, including [3,6,15,17,20], and is now called definable topological dynamics. The research area of definable topological dynamics studies the action of a group $G$ definable in a structure $M$ on its type space $S_G(M)$ and aims to link the invariants suggested by topological dynamics (e.g., enveloping semigroups, minimal subflows, Ellis groups ...) with model-theoretic invariants. For a particular group $G$ there is a unique minimal flow (the generic types) in its (always unique) Ellis semigroup.

In Newelski’s seminal paper [10], almost periodic types are one of new objects suggested by the topological dynamics point of view. A type $p \in S_G(M)$ is almost periodic if the closure of its $G$-orbit is a minimal subflow. Another new object, weakly generic formulas and types were introduced in [11] as a substitute for generic formulas and types, since generic types may not always exist in a NIP environment, i.e., if the theories involved have the NIP (the negation of the independence property). Briefly, a definable set (or formula) $X \subseteq G$ is weakly generic if there exists some nongeneric definable $Y \subseteq G$ such that $X \cup Y$ is generic, where a definable set is generic if finitely many of its translates cover the whole group.

A nice observation in [10] was that the class of weak generic types in $S_G(M)$ is precisely the closure of the class of almost periodic types. When $G$ is stable, the two classes coincide. So one may naturally ask that whether the two classes coincide in some tame unstable theories. With the assumption that the theory $T$ has the NIP and the group is definably amenable, [3] gives a positive answer for Ellis group conjecture asked first by Newelski [10] and then formulated precisely by Pillay [15]. Moreover, [3] proved that weakly generic types coincide with $f$-generic types, which happens in stable context. These results make it reasonable to consider definably amenable groups as the “stable-like” groups in a NIP environment.

So [3, Question 3.32] asked whether weakly generic types coincide with almost periodic types for definably amenable groups in NIP environment. One of the main results of [17] says that the answer in negative even for o-minimal case. On the other hand, [17] proved that any definable weakly generic type is almost periodic for definably amenable groups in NIP environment. Based on this positive result, [17] conjectured that

**Conjecture 1.1** Let $G$ be a definably amenable group definable in an NIP structure $M$. If $G$ has a global definably weakly generic type, then $p \in S_G(M)$ is weakly generic if and only if it is almost periodic.
Assuming that a group has the NIP, we say that it has definable f-generics if it is definably amenable and has a definable global weakly generic type. We call such a group a dfg group. We could restate the conjecture as follows:

**Conjecture 1.2** Let \( G \) be a group definable in an NIP structure \( M \). If \( G \) has definable f-generics, then \( p \in S_G(M) \) is weakly generic if and only if it is almost periodic.

The aim of this paper is to prove the Conjecture 1.1 for the \( p \)-adic field \( \mathbb{Q}_p \). The advantage of working in the \( p \)-adic case is that we have a good understanding of dfg groups. A recent result of [16] shows that any dfg group \( G \) is eventually trigonalizable over \( \mathbb{Q}_p \), viz. there exists a finite index subgroup \( A \leq G \), a finite subgroup \( A_0 \leq A \), and a trigonalizable algebraic group \( H \) such that \( A/A_0 \) is isomorphic to an open subgroup group of \( H \). As all trigonalizable algebraic groups have definable f-generics (cf. Corollary 2.11), we see that trigonalizable algebraic groups should be the “maximal” ones among the class of dfg groups. Since [16] is currently not available to the reader, we refer to the results from [16] as “claims” rather than “theorems”. In this paper, we focus on trigonalizable algebraic groups.

We now highlight our main result as follows:

**Theorem 1.3** Let \( N \) be an elementary extension of \((\mathbb{Q}_p, +, \times, 0, 1)\), \( G \) a trigonalizable algebraic group over \( \mathbb{Q}_p \). Then \( p \in S_G(N) \) is weakly generic if and only if it is almost periodic.

Theorem 1.3 partially answers Conjecture 1.1.

Together with the following result from [16] (listed as “claim” by the above convention), we could conclude from Theorem 1.3 that Conjecture 1.1 holds in the \( p \)-adic case:

**Claim 1.4** (Pillay & Yao; [16]) Let \( G \) be a definable group over \( \mathbb{Q}_p \) with definable f-generics. Then there is a trigonalizable algebraic group \( H \) over \( \mathbb{Q}_p \), a finite index subgroup \( A \leq G \), and a finite-to-one homomorphism \( f : A \to H \) such that \( f(A) \leq H \) has finite index.

We shall assume a basic knowledge of model theory, including basic notions of model theory such as “definable type”, “heir”, “coheir”, etc. Good references for the reader who is unfamiliar with them are [8, 18].

Let \( L \) be any language and \( T \) be a complete \( L \)-theory with infinite models. We write \( \mathbb{M} \) for the monster model of \( T \), in which every type over a small subset \( A \subseteq \mathbb{M} \) is realized, where “small” means \( |A| < |\mathbb{M}| \). The symbols \( M, N, M', N' \) will denote small elementary submodels of \( \mathbb{M} \). Let \( n \in \mathbb{N} \); by \( x, y, z \) we mean arbitrary \( n \)-variables and we refer by \( a, b, c \in \mathbb{M} \) to \( n \)-tuples in \( \mathbb{M}^n \) with \( n \in \mathbb{N} \). Every formula is an \( L_M \)-formula. For an \( L_M \)-formula \( \varphi(x) \), \( \varphi(M) \) denotes the definable subset of \( M^n \) defined by \( \varphi \), and a set \( X \subseteq M^n \) is definable if there is an \( L_M \)-formula \( \varphi(x) \) such that \( X = \varphi(M) \). If \( \bar{X} \subseteq M^n \) is definable, defined with parameters from \( M \), then \( \bar{X}(M) \) will denote \( X \cap M^n \), the realizations from \( M \), which is clearly a definable subset of \( M^n \). Suppose that \( X \subseteq M^n \) is a definable set, defined with parameters from \( M \), then we write \( S_X(M) \) for the space of complete types concentrating on \( X(M) \). Let \( A, B \) be subsets of \( \mathbb{M} \), and \( p \in S(A) \), by \( p|B \) we mean the restriction of \( p \) to \( B \) if \( A \supseteq B \), and the unique heir of \( p \) over \( B \) if \( B \supseteq A \) with \( A \) a model and \( p \) definable.

The paper is organized as follows: in the rest of this introduction, we recall precise definitions and results from earlier papers, relevant to our results. In § 2.1, we shall prove some general results for the closures of the orbit of a global type. In § 2.2, we shall characterise the type-definable connected component of a trigonalizable algebraic groups, and show that every trigonalizable algebraic group has definable f-generics. § 2.3 contains the main results of the paper, where we give a description of the f-generic types of trigonalizable algebraic groups, showing that every f-generic type is almost periodic.

### 1.1 Topological dynamics and definable groups

Our reference for (abstract) topological dynamics is [1]. Given a (Hausdorff) topological group \( G \), by a \( G \)-flow we mean a continuous action \( G \times X \to X \) of \( G \) on a compact (Hausdorff) topological space \( X \). We sometimes write the flow as \((X, G)\). Often it is assumed that there is a dense orbit, and sometimes a \( G \)-flow \((X, G)\) with a distinguished point \( x \in X \) whose orbit is dense is called a \( G \)-ambit.
In spite of $p$-adic algebraic groups being nondiscrete topological groups, we shall be treating them as discrete groups so as to have their actions on type spaces being continuous. So in this background section we may assume $G$ to be a discrete group, in which case a $G$-flow is simply an action of $G$ by homeomorphisms on a compact space $X$.

By a subflow of $(X, G)$ we mean a closed $G$-invariant subspace $Y$ of $X$ (together with the action of $G$ on $Y$). $(X, G)$ will always have minimal nonempty subflows. A point $x \in X$ is almost periodic if the closure of its orbit is a minimal subflow.

Let $(X, G)$ and $(Y, G)$ be flows (with the same acting group). A $G$-homomorphism from $X$ to $Y$ is a continuous map $f : X \rightarrow Y$ such that $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

**Fact 1.5** Let $(X, G)$ and $(Y, G)$ be flows, $f : X \rightarrow Y$ a $G$-homomorphism. Then $f(x_0)$ is almost periodic whenever $x_0 \in X$ is almost periodic.

Given a flow $(X, G)$, its enveloping semigroup $E(X)$ is the closure in the space $X^X$ (with the product topology) of the set of maps $\pi_x : X \rightarrow X$, where $\pi_x(x) = gx$, equipped with composition $\circ$ (which is continuous on the left). So any $e \in E(X)$ is a map from $X$ to $X$.

**Fact 1.6** Let $X$ be a $G$-flow. Then

(i) $E(X)$ is a $G$-flow, and $E(E(X)) \cong E(X)$;

(ii) for any $x \in X$, the closure of its $G$-orbit is exactly $E(X)(x)$. Moreover, for any $f \in E(X)$, $E(X) \circ f$ is the closure of $G \cdot f$.

By a definable group $G \subseteq \mathbb{M}^n$, we mean that $G$ is a definable set with a definable map $G \times G \rightarrow G$ as its group operation. For convenience, we assume that $G$ is defined by the formula $G(x)$. We say that $G$ is $A$-definable if $G(x)$ is an $L_A$-formula and the group operation is an $A$-definable map. For any $M < \mathbb{M}$ containing the parameters in $G(x)$, $G(M) = \{g \in M^n \mid g \in G\}$ is a subgroup of $G$. It is easy to see that $S_G(M)$ is a $G(M)$-flow with a dense orbit $\{tp(g/M) \mid g \in G(M)\}$. From now on, we shall, throughout this paper, assume that every formula $\varphi(x)$, with parameters in $\mathbb{M}$, is contained in $G(x)$, namely, the subset $\varphi(\mathbb{M})$ defined by $\varphi$ is contained in $G$. Suppose that $\varphi(x)$ is an $L_M$-formula and $g \in G(M)$. Then the left translate $g\varphi(x)$ is defined to be $\varphi(g^{-1}x)$. It is easy to check that $(g\varphi)(M) = gx$ with $X = \varphi(M)$.

Let notations be as above. A definable subset $X \subseteq G$ is generic if finitely many left translates of $X$ cover $G$. Namely, there are $g_1, \ldots, g_n \in G$ such that $G = \bigcup_{i=1}^n g_i X$. A definable subset $X \subseteq G$ is weakly generic if there is a non-generic definable subset $Y$ such that $X \cup Y$ is generic. A definable subset $X \subseteq G$ is $f$-generic if for some any model $M$ over which $X$ is defined and any $g \in G$, $gX$ does not divide over $M$, i.e., for any $M$-indiscernible sequence $(g_i : i < \omega)$, with $g = g_0, [g : X : i < \omega]$ is consistent. A formula $\varphi(x)$ is generic if the definable set $\varphi(\mathbb{M})$ is generic (similarly for weakly generic and $f$-generic formulas).

A type $p \in S_G(M)$ is generic if every formula $\varphi(x) \in p$ is generic (similarly for weakly generic and $f$-generic types). A type $p \in S_G(M)$ is almost periodic if $p$ is an almost periodic point of the $G(M)$-flow $S_G(M)$.

A global type $p \in S_G(\mathbb{M})$ is strongly $f$-generic over a small model $M_0$ if every left $G$-translate of $p$ does not fork over $M_0$. A global type $p \in S_G(\mathbb{M})$ is strongly $f$-generic if it is strongly $f$-generic over some small model.

**Fact 1.7** (Newelski; [10]) Let $AP \subseteq S_G(M)$ be the space of almost periodic types, and $WG \subseteq S_G(M)$ the space of weakly generic types. Then $WG = \operatorname{cl}(AP)$.

**Fact 1.8** (Newelski; [10]) Let $G$ be a group definable in $M$, $N > M$ be any $|M|^+\text{-saturated}$ elementary extension. Let $S_{G,M}(N)$ be the space of all coheirs of types in $S_G(M)$ over $N$. Then the enveloping semigroup $E(S_{G,M}(M))$ of $S_G(M)$ is isomorphic to $(S_{G,M}(N), *)$ where $*$ is defined as following: for any $p, q \in S_{G,M}(N)$, $p * q = \operatorname{tp}(a \cdot b/N)$ with $a$ realizes $p$ and $b$ realizes $q$, and $\operatorname{tp}(a/N, b)$ is finitely satisfiable in $M$.

### 1.2 NIP, definable amenability, and connected components

Let $G \subseteq \mathbb{M}^n$ be a definable group. Recall that a type-definable over $A$ subgroup $H \subseteq \mathbb{M}^n$ is a type-definable subset of $G$, which is also a subgroup of $G$. We say that $H$ has bounded index if $|G/H| < 2^{|H|+|A|}$. For groups definable
in NIP structures, the smallest type-definable subgroup $G^{00}$ exists (cf. [4]): the intersection of all type-definable subgroups of bounded index still has bounded index. We call $G^{00}$ the type-definable connected component of $G$. Another model theoretic invariant is $G^0$, called the definable-connected of $G$, which is the intersection all definable subgroups of $G$ of finite index. Clearly, $G^{00} \subseteq G^0$.

Recall also that a Keisler measure over $M$ on $X$, with $X$ a definable subset of $M^n$, is a finitely additive measure on the Boolean algebra of $M$-definable subsets of $X$. When we take the monster model, i.e., $M = \mathbb{M}$, we call it a global Keisler measure. A definable group $G$ is said to be definably amenable if it admits a global (left) $G$-invariant probability Keisler measure.

**Fact 1.9** (Chernikov & Simon; [3]) If $\mathbb{M}$ has the NIP and $G \subseteq \mathbb{M}^n$ is a definable group. Then the following are equivalent:

(i) $G$ is definably amenable;

(ii) $G$ admits a global type $p \in S_G(\mathbb{M})$ with bounded $G$-orbit;

(iii) $G$ admits a strongly $f$-generic type.

Moreover,

**Theorem 1.10** (Chernikov & Simon; [3]) For a definably amenable NIP group $G$, we have that

(i) weakly generic definable subsets, formulas, and types coincide with $f$-generic definable subsets, formulas, and types, respectively,

(ii) $p \in S_G(\mathbb{M})$ is $f$-generic if and only if it has bounded $G$-orbit,

(iii) $p \in S_G(\mathbb{M})$ is $f$-generic if and only if it is $G^{00}$-invariant,

(iv) a type-definable subgroup $H$ fixing a global $f$-generic type is exactly $G^{00}$,

(v) $G/G^{00}$ is isomorphic to the Ellis subgroup of $S_G(M)$ for any $M < \mathbb{M}$.

Note that Theorem 1.10(v) proves the Ellis group conjecture formulated by Newelski and Pillay [10, 15].

**Fact 1.11** Suppose that $G \subseteq \mathbb{M}^n$ is a definably amenable NIP group. Then every $f$-generic type $p \in S_G(N)$ has an $f$-generic global extension.

**Proof.** It is easy to see that the collection of all weakly generic definable subset of $G$ forms an ideal of the Boolean algebra of the collection of all definable subset of $G$. So every weakly generic type $p \in S_G(N)$ has a weakly generic global extension $\bar{p} \in S_G(\mathbb{M})$. This completes the proof as the concepts of being weakly generic and being $f$-generic coincide.

**Fact 1.12** (Pillay & Yao; [17]) If $p \in S_G(\mathbb{M})$ (as a $G(\mathbb{M})$-flow) is almost periodic, then $p|N \in S_G(N)$ (as a $G(N)$-flow) is almost periodic for any $N < \mathbb{M}$.

### 1.3 Groups definable in $(\mathbb{Q}_p, +, \times, 0, 1)$

We refer to the field $\mathbb{Q}_p$ as “the $p$-adics” and denote by $M$ the structure $(\mathbb{Q}_p, +, \times, 0, 1)$ with multiplicative group $\mathbb{Q}_p^\times = \mathbb{Q}_p \setminus \{0\}$. As usual, $\mathbb{Z}$ is the ordered additive group of integers, the value group of $\mathbb{Q}_p$. The group homomorphism $v : \mathbb{Q}_p^\times \to \mathbb{Z}$ is the valuation map. By $\mathbb{M}$, we denote a very saturated elementary extension $\langle \mathbb{K}, +, \times, 0, 1 \rangle$ of $M$ and by $\Gamma_{\mathbb{K}}$, we denote the value group of $\mathbb{K}$. Similarly, $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$ is the multiplicative group. We sometimes write $\mathbb{Q}_p^\times$ for $M$ and $\mathbb{K}$ for $\mathbb{M}$.

For convenience, we use $\mathbb{G}_a$ and $\mathbb{G}_m$ denote the additive group and multiplicative group of field $\mathbb{M}$ respectively. So $\mathbb{G}_a(M)$ (or $\mathbb{G}_a(\mathbb{Q}_p)$) and $\mathbb{G}_m(M)$ (or $\mathbb{G}_m(\mathbb{Q}_p)$) are $(\mathbb{Q}_p, +)$ and $(\mathbb{Q}_p^\times, \times)$ respectively.

We shall be referring a lot to the comprehensive survey [2] for the basic model theory of the $p$-adics. A key point is MacIntyre’s theorem that $\text{Th}(\mathbb{Q}_p, +, \times, 0, 1)$ has quantifier elimination in the language of rings $L_{\text{ring}}$ together with new predicates $P_n(x)$ for the $n$-th powers for each $n \in \mathbb{N}^+$ [7]. Moreover, the valuation is quantifier-free definable in the MacIntyre’s language $L_{\text{ring}} \cup \{ P_n \mid n \in \mathbb{N}^+ \}$; in particular, it is definable in the language of rings. (Cf. [2, § 3.2].)
The valuation map $v$ endows $\mathbb{Q}_p$ with an absolute valuation $|\cdot|$: for each $x \in \mathbb{Q}_p$, $|x| = p^{-v(x)}$ if $x \leq 0$ and $|x| = 0$ otherwise. The absolute valuation makes the $p$-adic field $\mathbb{Q}_p$ a locally compact topological field, with topology base given by the sets of $x$ such that $v(x-a) \geq n$ for $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$. For any $X \subseteq \mathbb{Q}_p^n$, the “topological dimension”, denoted by $\dim(X)$, is the greatest $k \leq n$ such that the image of $X$ under some projection from $M^n$ to $M^k$ contains an open subset of $\mathbb{Q}_p^k$. On the other side, as model-theoretic algebraic closure coincides with field-theoretic algebraic closure [5, Proposition 2.11], we see that for any model of $N$ of Th($M$) the algebraic closure satisfies exchange (so gives a so-called pregeometry on $N$) and there is a finite bound on the sizes of finite sets in uniformly definable families. If $a$ is a finite tuple from $N \models \text{Th}(M)$ and $B$ a subset of $N$ then the algebraic dimension of $a$ over $B$, denoted by $\text{dim}(a/B)$, is the size of a maximal subtuple of $a$ which is algebraically independent over $B$. When $X \subseteq \mathbb{Q}_p^n$ is definable, the algebraic dimension of $X$, denoted by $\text{alg-dim}(X)$, is the maximal $\text{dim}(a/B)$ such that $a \in X$ and $B$ contains the parameters over which $X$ is defined. It is important to know that when $X \subseteq \mathbb{Q}_p^n$ is definable, then its algebraic-dimension coincides with its topological dimension, namely $\text{dim}(X) = \text{alg-dim}(X)$. As a conclusion, for any definable $X \subseteq \mathbb{Q}_p^n$, $\dim(X)$ is exactly the algebraic geometric dimension of its Zariski closure.

By a definable manifold $X \subseteq \mathbb{Q}_p^n$ over a subset $A \subseteq \mathbb{Q}_p$, we mean a topological space $X$ with a covering by finitely many open subsets $U_1, \ldots, U_m$, and homeomorphisms of $U_i$ with some definable open $V_i \subseteq \mathbb{Q}_p^n$ for $i = 1, \ldots, m$, such that the transition maps are $A$-definable and continuous. If the transition maps are $C^k$, then we call $X$ a definable $C^k$ manifold over $\mathbb{Q}_p$, of dimension $n$. A definable group $G \subseteq \mathbb{Q}_p^n$ can be equipped uniquely with the structure of a definable manifold over $K$ such that the group operation is $C^\infty$ (cf. [12, 14]). The facts described above work for any $N \models \text{Th}(M)$.

### 1.4 One-dimensional groups over $\mathbb{Q}_p$

Let $M = (\mathbb{K}, \times, +, 0, 1)$ be a saturated extension of $M = (\mathbb{Q}_p, \times, +, 0, 1)$, and $(\Gamma, \times, +, <) \models (\mathbb{Z}, +, <)$ be its valuation group. Recall that $G_m = (\mathbb{K}^*, \times)$ is the multiplicative group and $G_a = (\mathbb{K}, +)$ is the additive group. Both $G_m$ and $G_a$ are commutative, and hence definably amenable.

**Fact 1.13** (Penazzi, Pillay, & Yao; [13]) The complete $1$-types over $M$ (or $\mathbb{Q}_p$) are precisely the following:

(i) The realized types $\text{tp}(a/M)$ for each $a \in \mathbb{Q}_p$.

(ii) for each $a \in \mathbb{Q}_p$ and coset $C$ of $\mathbb{G}_m^n$, the type $p_{n,C}$ saying that $x$ is infinitesimally close to $a$ (i.e., $v(x-a) > n$ for each $n \in \mathbb{N}$), and $(x-a) \in C$.

(iii) for each coset $C$ as above the type $p_{\infty,C}$ saying that $x \in C$ and $v(x) < n$ for all $n \in \mathbb{Z}$.

**Fact 1.14** (Penazzi, Pillay, & Yao; [13]) Let $p \in S_1(M)$, with the notations as above, then we have

(i) $p$ is an $f$-generic/weakly generic type of $S_{G_m}(M)$ iff $p$ is of the form $p_{\infty,C}$.

(ii) $p$ is an $f$-generic/weakly generic type of $S_{G_a}(M)$ iff $p$ is of the form $p_{\infty,C}$ or $p_{0,C}$.

It is well-known that every complete type over $M$ is definable.

**Fact 1.15** (Penazzi, Pillay, & Yao; [13]) Let $p \in S_1(M)$, with the notations as above, then we have

(i) $p$ is an $f$-generic/weakly generic type of $S_{G_a}(N)$ (or $S_{G_m}(N)$). Then

(ii) $p$ is $\emptyset$-definable, and is the unique heir of $p|N$.

(iii) $p|N$ is an $f$-generic/weakly generic type of $S_{G_m}(M)$ (or $S_{G_a}(M)$, respectively).

One concludes directly from Fact 1.16 that both $G_a$ and $G_m$ have definable $f$-generics.
2 Main results

2.1 Closure of the orbit of a global type

Lemma 2.1 Let $T$ be any first-order theory, and $\mathbb{M}$ a very saturated model of $T$. Let $G \subseteq \mathbb{M}^n$ be any definable group and $p \in S_G(\mathbb{M})$. Then the closure of the $G(\mathbb{M})$-orbit of $p$ is

$$cl(G(\mathbb{M}) \cdot p) = \{tp(a \cdot b/\mathbb{M}) \mid a, b \in G(\overline{\mathbb{M}}), b \models p, \text{ and } tp(b/\mathbb{M}, a) \text{ is an heir of } tp(b/\mathbb{M})\},$$

where $\overline{\mathbb{M}}$ is some $|\mathbb{M}|^+$-saturated elementary extension of $\mathbb{M}$.

Proof. Let $S_{G,\overline{\mathbb{M}}}(\overline{\mathbb{M}})$ be the collection of all the coheir extensions of types in $S_G(\mathbb{M})$. By Fact 1.8, $S_{G,\overline{\mathbb{M}}}(\overline{\mathbb{M}})$ is the enveloping semigroup of $S_G(\mathbb{M})$, which is also a $G(\mathbb{M})$-flow. Let $\bar{p} \in S_{G,\overline{\mathbb{M}}}(\overline{\mathbb{M}})$ be any extension of $p$. Then, by Fact 1.6 and Fact 1.8, the closure of $G(\mathbb{M})$-orbit of $\bar{p}$ is $\bar{p} = \{q \ast \bar{p} \mid q \in S_{G,\overline{\mathbb{M}}}(\overline{\mathbb{M}})\}$, where $q \ast \bar{p} = tp(\bar{a} \cdot \bar{b}/\overline{\mathbb{M}})$ with $\bar{a} \models q$, $\bar{b} \models \bar{p}$, and $tp(\bar{a}/\overline{\mathbb{M}}, \bar{b})$ is finitely satisfiable in $\overline{\mathbb{M}}$.

It is easy to see that $\pi : S_{G,\overline{\mathbb{M}}}(\overline{\mathbb{M}}) \to S_G(\mathbb{M})$ defined by $p \mapsto p|\mathbb{M}$ is a $(G(\mathbb{M}))$-homomorphism.

Claim 2.2 $cl(G(\mathbb{M}) \cdot p) = \pi(cl(G(\mathbb{M}) \cdot \bar{p}))$

Proof. Since both $S_{G,\overline{\mathbb{M}}}(\overline{\mathbb{M}})$ and $S_G(\mathbb{M})$ are compact and Hausdorff, we see that $\pi(cl(G(\mathbb{M}) \cdot \bar{p}))$ is closed. Since $G(\mathbb{M}) \cdot p \subseteq \pi(cl(G(\mathbb{M}) \cdot \bar{p}))$, we have $cl(G(\mathbb{M}) \cdot p) \subseteq \pi(cl(G(\mathbb{M}) \cdot \bar{p}))$. Conversely, if $\bar{q} \in cl(G(\mathbb{M}) \cdot \bar{p})$, then for any $L_\mathbb{M}$-formula $\varphi \in \bar{q}$ there is $g \in G(\mathbb{M})$ such that $\varphi \models g\bar{p}$. Take any $L_\mathbb{M}$-formula $\psi \in \pi(\bar{q}) = \bar{q}|\mathbb{M}$, there is $g \in G(\mathbb{M})$ such that $\psi \models g\bar{p}$, thus $\psi \models g\bar{p}|\mathbb{M} = gp$. So $\pi(\bar{q}) \subseteq cl(G(\mathbb{M}) \cdot p)$ as required.

By Claim 2.2, we conclude that $cl(G(\mathbb{M}) \cdot p) \subseteq \{tp(a \cdot b/\mathbb{M}) \mid a, b \in G(\overline{\mathbb{M}}), b \models p, \text{ and } tp(b/\mathbb{M}, a) \text{ is an heir of } tp(b/\mathbb{M})\}$. For the other inclusion, suppose that $p \models \pi cl(G(\mathbb{M}) \cdot p)$ is finitely satisfiable in $\mathbb{M}$. We now show that $tp(a \cdot b/\mathbb{M}) \in cl(G(\mathbb{M}) \cdot p)$.

Let $\bar{p} = tp(\bar{b}/\overline{\mathbb{M}}) \in S_{G,\overline{\mathbb{M}}}(\overline{\mathbb{M}})$ be an extension of $p$ and $tp(\bar{a}/\overline{\mathbb{M}}) \in S_{G,\overline{\mathbb{M}}}(\overline{\mathbb{M}})$ an extension of $tp(\bar{a}/\overline{\mathbb{M}})$ such that $tp(\bar{a} \cdot \bar{b}/\overline{\mathbb{M}})$ is finitely satisfiable in $\mathbb{M}$. Then $tp(\bar{a} \cdot \bar{b}/\overline{\mathbb{M}}) = tp(\bar{a}/\overline{\mathbb{M}}) \ast tp(\bar{b}/\overline{\mathbb{M}}) \in cl(G(\mathbb{M}) \cdot \bar{p})$ and $\pi(tp(\bar{a} \cdot \bar{b}/\overline{\mathbb{M}})) = tp(\bar{a} \cdot \bar{b}/\overline{\mathbb{M}}) \in cl(G(\mathbb{M}) \cdot p)$.

Let $\varphi(x, y) \in tp(a, b/\mathbb{M})$, and $N > \overline{\mathbb{M}}$ be any $|\mathbb{M}|^+$-saturated model, we see that $N \models \varphi(a, b)$ implies that $\varphi(x, b) \in tp(a/\mathbb{M}, b)$ which implies that $\varphi(x, b) \in tp(\bar{a}/\overline{\mathbb{M}}, \bar{b})$.

Since $tp(\bar{a}/\overline{\mathbb{M}}, \bar{b})$ is finitely satisfiable in $\mathbb{M}$, $tp(\bar{a}/\overline{\mathbb{M}}, \bar{b})$ is $M$-invariant. As $\bar{b}/\overline{\mathbb{M}} = \bar{b}/\overline{\mathbb{M}}$, we have $\varphi(x, \bar{b}) \in tp(\bar{a}/\overline{\mathbb{M}}, \bar{b})$, and hence $N \models \varphi(\bar{a}, \bar{b})$. This implies that $tp(a, b/\mathbb{M}) = tp(\bar{a}, \bar{b}/\overline{\mathbb{M}})$. So $tp(a \cdot b/\mathbb{M}) = tp(\bar{a} \cdot \bar{b}/\overline{\mathbb{M}}) \subseteq cl(G(\mathbb{M}) \cdot p)$, as required.

Lemma 2.3 Let $\mathbb{M}$ be a saturated model with NIP, and $G \subseteq \mathbb{M}^n$ a definably amenable group. If every heir of $p \in S_G(\mathbb{M})$ is $f$-generic, then $G(\mathbb{M}) \cdot p$ is closed. In particular, $p$ is almost periodic.

Proof. Let $\overline{\mathbb{M}} \succ \mathbb{M}$ be the any $|\mathbb{M}|^+$-saturated model. By Lemma 2.1, $cl(G(\mathbb{M}) \cdot p)$ is the collection of types of form $tp(a \cdot b/\mathbb{M})$ with $a, b \in G(\overline{\mathbb{M}})$, $b \models p$, and $tp(b/\mathbb{M}, a)$ is a heir of $p$. Let $tp(\bar{b}/\overline{\mathbb{M}})$ be an heir extension of $p$ such that $tp(b/\mathbb{M}, a) \subseteq tp(\bar{b}/\overline{\mathbb{M}})$. Since $tp(b/\mathbb{M}, a)$ is $f$-generic, it is $G^{00}(\mathbb{M})$-invariant. Let $a' \in G(\mathbb{M})$ such that $a'G^{00}(\overline{\mathbb{M}}) = aG^{00}(\overline{\mathbb{M}})$, we have $tp(a' \cdot \bar{b}/\overline{\mathbb{M}}) = tp(a' \cdot \bar{b}/\overline{\mathbb{M}})$. This implies that $tp(a \cdot b/\mathbb{M}) = tp(a \cdot \bar{b}/\overline{\mathbb{M}}) = tp(a' \cdot \bar{b}/\overline{\mathbb{M}}) = a' \cdot tp(b/\overline{\mathbb{M}}) = a' \cdot p \in G(\mathbb{M}) \cdot p$.

2.2 $G^{00}$ and global definable $f$-generic types

Let $\mathbb{M} = (\mathbb{K}, x, +, 0, 1)$ now be a very elementary saturated extension of $M = (\mathbb{Q}_p, x, +, 0, 1)$, and $(\Gamma_k, +, <) \to (\mathbb{Z}_p, +, <)$ be its valuation group. Every definable group $G$ is defined in the saturated model $\mathbb{M}$, and defined by the formula $G(x)$ with parameters from $\mathbb{Q}_p$. For any $N > M$, $G(N)$ is the set of realizations of $G(x)$ in $N$, which is a definable group in $N$. We say that a type $p \in S_G(M)$ is $G(M)$-invariant type if $gp = p$ for all $g \in G(M)$. It is easy to see that every heir of $p$ over any model $N > M$ is $G(N)$-invariant. We say that $G$ has a $G$-invariant type if there is $p \in S_G(M)$ which is $G(M)$-invariant. By Theorem 1.10, $G$ has a $G$-invariant type if and only if $G$ is definably amenable and $G^{00} = G$.

Lemma 2.4 Let $G \subseteq \mathbb{M}^n$, $H \subseteq \mathbb{M}^k$, and $T \subseteq \mathbb{M}^l$ be definable groups such that $H \subseteq G$ and $1 \to H \to G \to T \to 1$
is a short exact sequence. Suppose that both \( H \) and \( T \) have definable \( f \)-generics, and \( H \) has an \( H \)-invariant type. Then \( G \) has definable \( f \)-generics. Moreover \( H \leq G^{00} \) and \( \pi(G^{00}) = T^{00} \).

**Proof.** Let \( p \in S_H(M) \) be an \( f \)-generic type of \( H \), and \( q \in S_T(M) \) a \( f \)-generic type of \( T \), both of them definable over a small submodel \( N \). We now define a type \( r \in S_G(M) \) as follows: for any \( L_M \)-formula \( \varphi(x) \), we say that \( \varphi(x) \in r \) if and only if \( \{g \in G \mid g^{-1}\varphi(x) \in p\} \subset q \).

Since \( p \) is definable, \( D_{\varphi} = \{g \in G \mid g^{-1}\varphi(x, b) \in p\} \) is a definable subset of \( G \), hence \( \pi(D_{\varphi}) \) is a definable subset of \( T \), and we identify it with a formula defining it. Since \( p \) is \( H \)-invariant, we see that \( D_{\varphi} = H D_{\varphi} \) and thus \( T \setminus \pi(D_{\varphi}) = \pi(D_{\varphi}) \). So \( r \) is well-defined and complete. Since \( q \) is also definable over \( N \), \( r \) is definable over \( N \).

Let \( M \) be an \(|M|^\ast\)-saturated elementary extension of \( M \), and \( \tilde{M} \) an \(|\tilde{M}|^\ast\)-saturated elementary extension of \( \tilde{M} \). For any \( \tilde{g} \in G(M) \) such that \( \pi(\tilde{g}) \) realizes \( q \) and any \( \tilde{h} \in H(\tilde{M}) \) such that \( \tilde{h} \) realizes \( p|\tilde{M} \) (the unique heir of \( p \) over \( \tilde{M} \)), we claim that:

**Claim 2.5** \( r = \text{tp}(\tilde{g}\tilde{h}/M) \).

**Proof.** Let \( \varphi(x) \in r \), and \( \varphi(x, y) \) be the formula \( y^{-1}\varphi(x) \). As \( p \) is definable, there is a formula \( \vartheta(y) \) such that \( \vartheta(x, y) \in p \iff \vartheta(y) \) holds. Since \( \vartheta(\varphi(x)) \in q \), we see that \( \vartheta(\tilde{g}) \) holds in \( \tilde{M} \) and thus \( \varphi(\tilde{g}, x) \in p|\tilde{M} \). So \( \varphi(\tilde{g}, \tilde{h}) \) holds in \( \tilde{M} \), which means that \( \varphi(\tilde{g}\tilde{h}) \) holds as required. \( \square \)

**Claim 2.6** \( r \) is invariant under \( \pi^{-1}(T^{00}) \).

**Proof.** Let \( r = \text{tp}(\tilde{g}\tilde{h}/M) \), with \( \tilde{g} \) and \( \tilde{h} \) as in the above Claim. By the above Claim, we only need to check that for any \( g \in \pi^{-1}(T^{00}) \), \( \pi(g \tilde{g}) \) realizes \( q \). Since \( \pi(g \tilde{g}) = \pi(g) \pi(\tilde{g}) \) and \( q \) is \( T^{00} \), \( q = \text{tp}(\pi(g) \pi(\tilde{g})/M) = \text{tp}(\pi(g \tilde{g})/M) \) as required. \( \square \)

Since \( T^{00} \) has bounded index in \( T \), we see that \( \pi^{-1}(T^{00}) \) has bounded index in \( G \). So \( r \) has a bounded orbit and hence is an \( f \)-generic type of \( G \) by Theorem 1.10 (ii). This also implies that \( G^{00} = \pi^{-1}(T^{00}) \) by Theorem 1.10 (iv). So we conclude that \( H \leq G^{00} \) and \( \pi(G^{00}) = T^{00} \). \( \square \)

Recall from \cite[Chapter 16]{[9]} that a connected algebraic group \( G \) is trigonalizable over \( \mathbb{Q}_p \) if it admits a short exact sequence

\[ 1 \rightarrow U \overset{i}{\rightarrow} G \overset{\pi}{\rightarrow} T \rightarrow 1, \]

where \( U \subseteq M^n \) is the maximal unipotent subgroup of \( G \) and \( T \subseteq M^n \) is a split torus, namely, isomorphic to \( \mathbb{G}_m^n \). \( U \) is connected as \( \text{char } \mathbb{Q}_p = 0 \), and a connected unipotent group admits a normal sequence

\[ 1 = U_0 \leq \cdots \leq U_i \leq U_{i+1} \leq U_k = U \]

such that each \( U_{i+1}/U_i \) has dimension one for each \( i \leq k \). Since any connected one-dimensional unipotent algebraic group is isomorphic to \( \mathbb{G}_a \), we see that \( U_{i+1}/U_i \cong \mathbb{G}_a = (\mathbb{K}, +) \) for each \( i \leq k \), and thus \( U \) splits over \( \mathbb{Q}_p \). By \cite[Theorem 17.26]{[9]}, the short exact sequence

\[ 1 \rightarrow U \overset{i}{\rightarrow} G \overset{\pi}{\rightarrow} T \rightarrow 1, \]

splits. So \( G \) is definably isomorphic to the semidirect product \( T \rtimes U \).

By induction on \( \text{dim}(U) \), we obtain the next result easily.

**Corollary 2.7** If \( U \) is a unipotent algebraic group, then \( U^{00} = U \).

**Lemma 2.8** Let \( G \) and \( H \) be definable groups such that both \( G \) and \( H \) have definable \( f \)-generics. Then the direct product \( G \times H \) has definable \( f \)-generics.

**Proof.** Let \( p = \text{tp}(g/M) \in S_G(M) \) be an \( f \)-generic type of \( H \) definable over \( M \) and \( q = \text{tp}(h/M) \in S_H(M) \) a \( f \)-generic type of \( H \) definable over \( M \), where \( M \) is an \(|M|^\ast\)-saturated elementary extension of \( M \). Let \( r = \text{tp}(g, h)/M \in S_{G \times H}(M) \). An analogous argument as in the proof of Lemma 2.4 shows that \( r \) is definable, and for any \( g' \) that realizes \( p \) and any \( h' \) that realizes \( q|M \), we have \( r = \text{tp}((g', h')/M) \). Since \( p \) is \( G^{00} \)-invariant and \( q \) is \( H^{00} \)-invariant, we see that \( r \) is \( G^{00} \times H^{00} \)-invariant. Since \( G^{00} \times H^{00} \) has bounded index in \( G \times H \), we see that \( r \) has bounded index and hence an \( f \)-generic type by Theorem 1.10 (ii), and \( G^{00} \times H^{00} = (G \times H)^{00} \) by Theorem 1.10 (iv). \( \square \)
Corollary 2.9 $G^a_m$ has definable f-generics, and $(G^m)^00 = (G^m^0)^a = (G^m^0)_a$.

Proof. Follow directly from Fact 1.13 & Lemma 2.4. □

Fact 2.10 (Yao; [19]) Let $N > M$. If $p \in S^2_{G_m}(N)$ is f-generic, then $p$ is definable over $\emptyset$.

Corollary 2.11 If $G$ is trigonalizable over $\mathbb{Q}_p$, then $G$ has definable f-generics. Moreover, if $G \cong T \ltimes U$ with $U$ the maximal unipotent subgroup of $G$ and $T \cong G^m_0$ the torus, we have $G^{00} \cong T^{00} \ltimes U = T^0 \ltimes U$.

Proof. Follow directly from Lemma 2.4 and Corollaries 2.7 & 2.9. □

2.3 f-generic types and almost periodic

We now suppose that $G$ is a trigonalizable algebraic group over $\mathbb{Q}_p$, which is isomorphic to the semiproduct $G^a_m \ltimes U$, with $U \subseteq M^m$ a unipotent algebraic group over $\mathbb{Q}_p$. We now denote $G^a_m$ by $T$ and identify $G$ with $T \ltimes U$, and every element $g \in G$ with the unique pair $(t, u)$, where $t \in T, u \in U$, and $g = tu$. It is easy to see that the group operation is given by

$$(t_1, u_1)(t_2, u_2) = t_1 t_2 t_1^{-1} u_1 t_2 u_2 = (t_1 t_2, t_2^{-1} u_1 t_2 u_2) = (t_1 t_2, u_1^0 u_2).$$

The groups $G, T,$ and $U$ are defined by the formulas $G(x, y), T(x), a$ and $U(y)$, respectively.

For any $t = (t_1, \ldots, t_i) \in T$, and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, by $t_m$ we mean $\prod_{i=1}^m t_i$. For any $m = (m_1, \ldots, m_n)$ and $m' = (m'_1, \ldots, m'_n)$ in $\mathbb{Z}^n$, by $m - m'$ we mean $(m_1 - m'_1, \ldots, m_n - m'_n)$. Let $\Gamma_1 > \Gamma_0 > (\mathbb{Z}, +, <)$, we say that $\gamma \in \Gamma_1$ is bounded over $\Gamma_0$ if there are $c, d \in \Gamma_0$ such that $c \leq \gamma \leq d$ and unbounded over $\Gamma_0$ if otherwise.

Proof. Let $N = (K, +, \times, 0, 1)$ be an elementary extension of $M$, and $\Gamma_K$ is the valuation group of $K$.

(i) $\text{tp}(t, u/N) = \text{tp}(t, b'u/N)$ for any $b \in U(N)$;

(ii) For any $m \in \mathbb{Z}^n, b \in U(N)$, and $N$-definable function $f : G(N) \to N, v(t^m) + v(f(b'u))$ is unbounded over $\Gamma_K$ whenever $m \neq (0, \ldots, 0)$.

Proof. (i) Since any f-generic extension $\bar{p} \in S_C(M)$ of $\text{tp}(t, u/N)$ is $U$-invariant, we see that $\text{tp}(t, u/N)$ is $U(N)$-invariant, and hence $\text{tp}(t, b'u/N) = (1, b) \cdot \text{tp}(t, u/N) = \text{tp}(t, u/N)$.

(ii) Suppose for a contradiction that there are $c, d \in \Gamma_K, 0 \neq m \in \mathbb{Z}^n$, and a function $f$ definable over $N$ such that $c \leq f(t^m) + v(f(b'u)) \leq d$. Then for any $a \in T^{00}$, we see that $\text{tp}(t, b'u/N) = \text{tp}(t, u/N) = \text{tp}(at, b'u/N)$. So we have $c \leq v(t^m) + v(f(b'u)) \leq d$. Moreover, $c \leq v(at^m) + v(f(b'u)) \leq d$. If $m \neq 0$, then we can take some $a \in T^{00}$ such that $v(a^m) = \sum_{i=1}^m m_i v(a_i) > d - c$. But $c \leq v((at)^m) + v(f(b'u)) = v(a^m) + v(t^m) + v(f(b'u)) \leq d$ implies that $v(a^m) \leq d - c$, which is a contradiction.

Lemma 2.13 Let $N = (K, +, \times, 0, 1)$ be an elementary extension of $M, t \in T, u \in U$. If the following conditions

(i) $\text{tp}(t, u/N) = \text{tp}(t, b'u/N)$ for any $b \in U(N)$

(ii) For any $m \in \mathbb{Z}^n, b \in U(N)$, and $N$-definable function $f : G(N) \to N, v(t^m) + v(f(b'u))$ is unbounded over $\Gamma_K$ whenever $m \neq (0, \ldots, 0)$.

hold for $(t, u, N)$. Then $\text{tp}(t, u/N)$ is an f-generic type of $G$ over $N$.

Proof. Let $N > M$ and $(t, u) \in T \ltimes U$ such that the above two conditions hold for $(t, u, N)$.

Claim 2.14 Let $\bar{t} \bar{u}/[M]$ be an heir of $\text{tp}(t, u/N)$ over $M$, where $\bar{t} \in T([M])$ and $\bar{u} \in U([M])$ with $[M]$ an $\langle M \rangle^+$-saturated extension of $M$. Then conditions (i) and (ii) hold for $(\bar{t}, \bar{u}, [M])$.

Proof. It is easy to see that condition (i) holds for $(t, u, N)$ iff for any $L_N$-formula $\varphi(x, y)$ and any $b \in U(N)$

$$(\varphi(x, y) \iff \varphi(x, b'y)) \in \text{tp}(t, u/N),$$

(1)
and condition (ii) holds for \((t, u, N)\) iff for any \(N\)-definable function \(f : G(N) \to N\), any \(b \in U(N)\), any \(m \neq (0, \ldots, 0)\), and any \(L_N\)-formula \(\varphi(x, y) \in p\)

\[
\begin{align*}
&\text{either } N \models \forall z((z \neq 0) \to \exists x, y (\varphi(x, y) \land (\varphi(x^m) + \varphi(f(b^ty)) > \varphi(z)))) \\
&\text{or } N \models \forall z((z \neq 0) \to \exists x, y (\varphi(x, y) \land (\varphi(x^m) + \varphi(f(b^ty)) < \varphi(z))).
\end{align*}
\]

\tag{2}

Let \( tp(\vec{t}, \vec{u}/\mathbb{M}) \) be as in the claim. Then, by the definition of heir, it is easy to see that (1) and (2) hold when we replace \(t, u, N\) by \((\vec{t}, \vec{u}, \mathbb{M})\).

Now (8) and (9) imply (3) as required.

By quantifier elimination, we only need to check that for every polynomial \(g(x, y) \in \mathbb{K}[x, y]\), and every predicate \(P_n\), we have

\[
\mathbb{M} \models P_n(g(\vec{t}, \vec{u})) \iff \mathbb{M} \models P_n(g(a\vec{t}, b\vec{u})).
\]

\tag{3}

Suppose that \(g(x, y) = \sum_{i=1}^{k} g_i(y)x^{m_i}\), where \(g_i \in \mathbb{K}[y]\), and \(m_i = (m_{i1}, \ldots, m_{in}) \in \mathbb{N}^n\). Now \(m_i \neq m_j\) whenever \(i \neq j \leq k\). By condition (ii), we have

\[
v(g_j \cdot (b\vec{u})^{\vec{m}_j}) < v(g_i(b\vec{u})^{\vec{m}_i}) + \Gamma_{\mathbb{K}}
\]

is unbounded over \(\Gamma_{\mathbb{K}}\) for \(i \neq j\). So there is a unique \(j^* \leq k\) such that

\[
v(g_j \cdot (b\vec{u})^{\vec{m}_j}) < v(g_i(b\vec{u})^{\vec{m}_i}) + \Gamma_{\mathbb{K}}
\]

for all \(i \neq j^*\). Since \(v(a^m) \in \Gamma_{\mathbb{K}}\) is bounded for all \(m \in \mathbb{Z}^n\), and \(v(g_i(b\vec{u})^{\vec{m}_i}) = v(g_i(b\vec{u})a^{m_i}) = v(g_i(b\vec{u})^{\vec{m}_i}) + v(a^{m_i})\) for all \(i \leq k\), we see that

\[
v(g_j \cdot (b\vec{u})^{\vec{m}_j}) < v(g_i(b\vec{u})^{\vec{m}_i}) + \Gamma_{\mathbb{K}}\] for all \(i \neq j^*\). \hfill \(5\)

By condition (i) and equation (4), we have

\[
v(g_j \cdot (b\vec{u})^{\vec{m}_j}) < v(g_i(b\vec{u})^{\vec{m}_i}) + \Gamma_{\mathbb{K}}\] for all \(i \neq j^*\). \hfill \(6\)

Moreover, \(g_j \cdot (b\vec{u})^{\vec{m}_j}\) and \(g_j \cdot (b\vec{u})^{\vec{m}_j}\) are in the same coset of \(P_n(\mathbb{K}^*)\) as \(tp(g_j \cdot (b\vec{u})^{\vec{m}_j}/\mathbb{M}) = tp(g_j \cdot (b\vec{u})^{\vec{m}_j}/\mathbb{M})\). Let \(\lambda \in \mathbb{Q}_p^+\) such that

\[
\mathbb{M} \models \lambda P_n(g_j \cdot (b\vec{u})^{\vec{m}_j}) \text{ and } \mathbb{M} \models \lambda P_n(g_j \cdot (b\vec{u})^{\vec{m}_j}).
\]

\tag{7}

We see that

\[
\mathbb{M} \models P_n(g(\vec{t}, \vec{u})) \iff \mathbb{M} \models \lambda^{-1} P_n(\frac{g(\vec{t}, \vec{u})}{g_j \cdot (b\vec{u})^{\vec{m}_j}}).
\]

By (6),

\[
\frac{g(\vec{t}, \vec{u})}{g_j \cdot (b\vec{u})^{\vec{m}_j}} = 1 + \mu,
\]

with \(v(\mu) > \Gamma_{\mathbb{K}}\). Namely \(\frac{g(\vec{t}, \vec{u})}{g_j \cdot (b\vec{u})^{\vec{m}_j}}\) is infinitesimally close to 1 over \(\mathbb{K}\). As \(P_n(\mathbb{Q}_p^*)\) is open and \(1 \in P_n(\mathbb{Q}_p^*)\), \(1 + \mu \in P_n(\mathbb{K}^*)\). So we conclude that

\[
\mathbb{M} \models P_n(g(\vec{t}, \vec{u})) \iff \mathbb{M} \models \lambda^{-1} P_n(1 + \mu) \iff \mathbb{M} \models P_n(\lambda).
\]

\tag{8}

By (5) and (7), a similar argument showing that

\[
\mathbb{M} \models P_n(g(a\vec{t}, b\vec{u})) \iff \mathbb{M} \models P_n(\lambda).
\]

\tag{9}

Now (8) and (9) imply (3) as required. 

\hfill \square
By Lemmas 2.12 & 2.13, we conclude directly that

**Proposition 2.15** Let \( N = (K, +, \times, 0, 1) \) be an elementary extension of \( M \), and \( \Gamma_K \) the valuation group of \( K \). If \( t \in \mathbb{G}_m^N(M) \) and \( u \in U(M) \). Then \( tp(t, u/N) \) is an \( f \)-generic type of \( G \) if and only if the following hold.

(i) \( tp(t, u/N) = tp(t, b'u/N) \) for any \( b \in U(N) \)

(ii) For any \( m \in \mathbb{Z}^n, b \in U(N) \), and \( N \)-definable function \( f : G(N) \to N, v(t^m) + v(f(b'u)) \) is unbounded over \( \Gamma_K \) whenever \( m \neq (0, \ldots, 0) \).

**Corollary 2.16** Let \( N = (K, +, \times, 0, 1) \) be an elementary extension of \( M \). If \( p \in S_G(N) \) is \( f \)-generic, then every heir of \( p \) over any \( N' \supset N \) is \( f \)-generic.

**Proof.** Let \( p(x, y) = tp(t, u/N) \) with \( t \in T \) and \( u \in U \). By Proposition 2.15, we have

(i) \( tp(t, u/N) = tp(t, b'u/N) \) for any \( b \in U(N) \)

(ii) For any \( m \in \mathbb{Z}^n, b \in U(N) \), and \( N \)-definable function \( f : G(N) \to N, v(t^m) + v(f(b'u)) \) is unbounded over \( \Gamma_K \) whenever \( m \neq (0, \ldots, 0) \).

Let \( N' \supset N, t \in T \) and \( u \in U \) such that \( tp(t, u/N) \) is an heir of \( p \) over \( N' \). Then, as we proved in Lemma 2.13, conditions (i) and (ii) hold when we replace \( t, u, N \) by \( t, u, N' \) respectively. So \( tp(t, u/N') \) is \( f \)-generic over \( N' \) by Proposition 2.15. \( \square \)

**Theorem 2.17** Let \( N = (K, +, \times, 0, 1) \) be an elementary extension of \( M \). Then \( p \in S_G(N) \) is \( f \)-generic if and only if \( p \) is almost periodic.

**Proof.** Clearly, every almost periodic type is \( f \)-generic by Fact 1.7. Let \( p = tp(t, u/N) \) be an \( f \)-generic type of \( G \) over \( N \). Let \( p' \in S_G(M) \) be any heir of \( p \). Then, by Corollary 2.16, \( p' \) is \( f \)-generic and every heir of \( p' \) is \( f \)-generic. By Lemma 2.3, \( p' \) is almost periodic. By Fact 1.12, \( p = p'|N \) is almost periodic. \( \square \)

**Question 2.18** Let \( U \) and \( T \) are as above. By Fact 2.10, every \( f \)-generic type of \( T \) over any model \( N \) is definable over \( \emptyset \). Is it true that every \( f \)-generic type of \( U \) is definable over \( \emptyset \)?

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