On $\kappa$-noncollapsed complete noncompact shrinking gradient Ricci solitons which split at infinity

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Abstract We discuss some geometric conditions under which a complete noncompact shrinking gradient Ricci soliton will split at infinity.

1 Introduction

For complete noncompact shrinking gradient Ricci solitons (GRS), global splitting theorems have been proven by Lichnerowicz [15], Fang et al. [9] and Munteanu and Wang [16,17]. In the more general context of Bakry–Emery manifolds, these works prove splitting theorems under the assumptions of the existence of geodesic lines and conditions on the potential functions $f$. As a special case, one recovers the splitting theorem of Cheeger and Gromoll [3] for complete noncompact Riemannian manifolds with nonnegative Ricci curvature.

For complete noncompact shrinking GRS one may conjecture that their curvatures must be bounded, their Ricci curvatures cannot be everywhere positive, and either they split as the Riemannian product of $\mathbb{R}$ with a one-lower-dimensional shrinking GRS or they have quadratic curvature decay. This conjecture is known to be true in dimensions at most 3, where quadratic curvature decay implies being a Gaussian shrinker, but the
conjecture remains open for dimensions at least 4. Evidence toward this conjecture are in the works of Chen [5] and Cao and Zhou [1].

In this paper, for $\kappa$-noncollapsed complete noncompact shrinking GRS with bounded curvature, we discuss a splitting theorem at infinity (Theorem 3 below), a.k.a., dimension reduction (see Hamilton [11, Sect. 22]).

In the rest of this section we collect some elementary facts regarding shrinking GRS which will be used later, the reader may see [6, Sect. 4.1] for more details. Let $(\mathcal{M}^n, g, f, \lambda)$ be a complete noncompact shrinking GRS with $f$ normalized, so that

$$\text{Rc}_g + \nabla^2 f = \frac{\lambda}{2} g \quad \text{and} \quad R_g + |\nabla_g f|^2_g = \lambda f,$$  \hspace{1cm} (1)

where $\lambda \in \mathbb{R}_+$. Here and below, $\text{Rc}$, $\nabla^2 f$, and $R$ denote the Ricci tensor, Hessian of $f$, and scalar curvature associated to a Riemannian metric, respectively. In a later application, we shall rescale the $\lambda = 1$ case. If $(\mathcal{M}^n, \bar{g}, f)$ satisfies $\text{Rc}_{\bar{g}} + \nabla^2 f = \frac{1}{2} \bar{g}$ and $R_{\bar{g}} + |\nabla_{\bar{g}} f|^2_{\bar{g}} = f$, then $(\mathcal{M}, \lambda^{-1} \bar{g}, f)$ satisfies $\text{Rc}_{\lambda^{-1} \bar{g}} + \nabla^2_{\lambda^{-1} \bar{g}} f = \frac{\lambda}{2} \cdot \lambda^{-1} \bar{g}$ and $R_{\lambda^{-1} \bar{g}} + |\nabla_{\lambda^{-1} \bar{g}} f|^2_{\lambda^{-1} \bar{g}} = \lambda f$.

For a shrinking GRS $(\mathcal{M}^n, g, f, \lambda)$ we define diffeomorphisms $\varphi_t : \mathcal{M} \to \mathcal{M}$ by

$$\frac{\partial}{\partial t} \varphi_t (x) = \frac{1}{1-t} \left( \nabla_g f \right) (\varphi_t (x)) \quad \text{and} \quad \varphi_0 = \text{id}_\mathcal{M}, \ t \in (-\infty, \lambda^{-1}).$$

Let $f(t) = f \circ \varphi_t$ and $g(t) = (1 - t) \varphi_t^* g$. Then $g(t)$ is a solution to the Ricci flow, which satisfies

$$\frac{\partial f}{\partial t} (x, t) = \frac{1}{1-\lambda t} |\nabla_g f|^2_g (\varphi_t (x)) = |\nabla_{g(t)} f(t)|^2_{g(t)}(x),$$  \hspace{1cm} (2)

$$\text{Rc}_{g(t)} + \nabla^2_{g(t)} f(t) = \frac{\lambda}{2(1-\lambda t)} g(t),$$  \hspace{1cm} (3)

$$R_{g(t)} + |\nabla_{g(t)} f(t)|^2_{g(t)} = \frac{\lambda}{1-\lambda t} f(t).$$  \hspace{1cm} (4)

By a result of Cao and Zhou [1,13], for $(\mathcal{M}, g, f, \lambda)$ we have the estimate

$$f(x) \geq \frac{1}{4} \left( \left( \lambda^{1/2} d_g (x, O) - 5n \right) \right)^2.$$  \hspace{1cm} (5)

2 Splitting at infinity for limits of shrinking GRS

We say that a sequence $\{(\mathcal{M}^n_i, g_i, f_i, \lambda_i, x_i)\}_{i=1}^\infty$ of pointed complete noncompact shrinking GRS is admissible if it satisfies the following conditions.

1. $\text{Rc}_{g_i} + \nabla^2_{g_i} f_i = \frac{\lambda_i}{2} g_i$, where $0 < \lambda_i \leq \lambda$ for some $\lambda < \infty$.
2. The $f_i$ are normalized by $R_{g_i} + |\nabla_{g_i} f_i|^2_{g_i} = \lambda_i f_i$.
3. $\lambda_i d_{g_i}(x_i, O_i) \to \infty$, where $O_i$ is a minimum point of $f_i$.
4. $\text{inj}_{g_i}(x_i) \geq t$, for some $t > 0$.
5. For any given radius $\rho > 0$, there exists a constant $C_1(\rho) < \infty$ such that

$$|\text{Rm}_{g_i}|_{g_i}(x) \leq C_1(\rho) \quad \text{for} \ x \in B_{\rho}^{g_i}(x_i) \quad \text{and for all} \ i.$$  \hspace{1cm} (6)
6. For any given \( \rho > 0 \), there is a constant \( C_2(\rho) < \infty \) such that Ricci tensor satisfies 
\[
|\nabla g_i R_{g_i}|g_i \leq C_2(\rho) \text{ on the ball } B_{\rho}^g(x_i) \text{ for each } i.
\]

We say that a Riemannian manifold splits if it isometric to the product of a line and a Riemannian manifold. The reason we assume that the \( \lambda_i \) are bounded in the definition of admissibility above is that otherwise we would allow for rescalings of asymptotically conical shrinking GRS (although these are not counterexamples due to conditions 5 and 6), whose corresponding limits do not split. Such a splitting result (Theorem 2 below) is the main result of this section.

We will need the following version of the compactness theorem for a sequence of Riemannian manifolds. We assume the reader is familiar with the notion of \( C^{k,\alpha} \) pointed Cheeger–Gromov convergence.

**Theorem 1** Let \( \{(M^n_i, g_{1i}, x_I)\}_{i=1}^{I=\infty} \) be a sequence of pointed smooth complete Riemannian manifolds of dimension \( n \). Suppose that:

(a) The injectivity radius \( \text{inj}_{g_I}(x_I) \geq \iota \) for all \( I \), where \( \iota \) is a positive constant;

(b) (bounded curvature at bounded distance) Given any \( \rho > 0 \), there is a constant \( C_1(\rho) \) such that the Riemann curvature tensors satisfy 
\[
|Rm|_{g_I} \leq C_1(\rho)
\]

in the ball \( B_{\rho}^g(x_I) \) for each \( I \); and

(c) Given any \( \rho > 0 \), there is a constant \( C_2(\rho) \) such that the Ricci tensors satisfy
\[
|\nabla g_I R_{g_I}|g_I \leq C_2(\rho)
\]

in the ball \( B_{\rho}^g(x_I) \) for each \( I \).

Then, for any \( \alpha \in (0, 1) \), the sequence \( \{(M^n_i, g_{1i}, x_I)\} \) subconverges in the \( C^{2,\alpha} \) pointed Cheeger–Gromov sense to a pointed \( C^{2,\alpha} \) complete Riemannian manifold \( (M^n_{\infty}, g_{\infty}, x_{\infty}) \).

**Proof** By assumption (b) and a theorem of Cheeger et al. [4], we have the following. Given any \( \rho > 0 \) there is a constant \( \iota_0 = \iota_0(\rho, \iota, n) > 0 \) such that the injectivity radius \( \text{inj}_{g_I}(x) \geq \iota_0 \) for any \( x \in B_{\rho}^g(x_I) \) and for any \( I \). Fix an \( \alpha \in (0, 1) \). Then we can use (b) and a theorem of Jost and Karcher [14] to further conclude the following. There is a constant \( r_0 = r_0(\rho, \iota, n) \in (0, \iota_0(\rho, \iota, n)) \) such that, for each \( x \in B_{r_0}^g(x_I) \), in harmonic coordinates on \( B_{r_0}^g(x) \) the metric tensor coefficients \( (g_I)_{ij} \) satisfy the following estimates:

(b1) The \( (g_I)_{ij} \) on \( B_{3r_0/4}^g(x) \) have uniformly (independent of \( I \)) bounded \( C^{1,\alpha} \) norms in the local harmonic coordinates.

(b2) For each \( y \in B_{3r_0/4}^g(x) \) we have 
\[
\frac{1}{r_0^2}(\delta_{ij}) \leq ((g_I)_{ij}(y)) \leq 10(\delta_{ij}).
\]

For any \( x \in B_{r_0}^g(x_I) \) it follows from (b1), (b2), and assumption (c) that:

(c1) The Ricci tensor coefficients \( (R_{g_I})_{ij} \), in the harmonic coordinates on \( B_{3r_0/4}^g(x) \), have uniformly (independent of \( I \)) bounded \( C^\alpha \) norm.

Since Ricci tensor coefficients in harmonic coordinates are given by
\[
(R_{g_I})_{ij} = -\frac{1}{2}(g_I)^{kl} \frac{\partial^2(g_I)_{ij}}{\partial x^k \partial x^l} + \cdots,
\]

where the dots indicate lower order terms involving at most one derivative of the metric ([8, Lemma 4.1]), by the Schauder estimates for elliptic PDE, (b1), (b2), and (c1), we have that:
(c2) The metric tensor coefficients \((g_I)_{ij}\), in harmonic coordinates, have uniform (independent of \(I\)) \(C^{2,\alpha}\) estimates on \(B^{2\lambda_i}_{\rho/2}(x)\) for any \(x \in B^{2\lambda_i}_{\rho}(x_I)\).

Note that Greene and Wu, and separately Peters (see Greene’s survey [10]), proved the following theorem. If a sequence of Riemannian manifolds \(\{(M^n_I, g_I)\}_{I=1}^{\infty}\) satisfies the following conditions:

(a1) The injectivity radius \(\text{inj}_{g_I} \geq \iota > 0\) for all \(I\),
(b3) For some constant \(C_1\) the Riemann curvature tensor \(|Rm_{g_I}|_{g_I} \leq C_1\) on \(M^n_I\) for each \(I\), and
(d) (uniformly bounded diameter) For some constant \(C_3\) the diameter \(\text{diam}(M^n_I, g_I) \leq C_3\) for each \(I\),

then the sequence \(\{(M^n_I, g_I))\}_{I=1}^{\infty}\) subconverges in the \(C^{1,\alpha}\) Cheeger–Gromov sense to a \(C^{1,\alpha}\) Riemannian manifold \((M^n_\infty, g_\infty)\) of dimension \(n\).

If the Riemannian manifolds in Theorem 1 have uniformly bounded diameter, then we can use (c2) and the above compactness result to conclude Theorem 1.

If the Riemannian manifolds in Theorem 1 do not have uniformly bounded diameter, then we can adjust slightly Hamilton’s proof of a compactness result for pointed Riemannian manifolds under the assumption that all derivatives of curvature tensor are bounded [12] to address the noncompact limit. Regarding this, a key modification is to replace the normal coordinates used in constructing the limit manifolds by the harmonic coordinates constructed above. We omit the details here. \(\square\)

Now we can prove a splitting theorem at infinity for the limit of a sequence of shrinking GRS.

**Theorem 2** Let \(\{(M^n_I, g_I, f_i, \lambda_i, x_i)\}\) be an admissible sequence of pointed complete noncompact shrinking GRS. Let \((M^n_\infty, g_\infty, x_\infty)\) be the pointed \(C^{2,\alpha}\) Cheeger–Gromov limit of some subsequence of \(\{(M^n_I, g_I, x_i)\}\) given by Theorem 1. Then \((M^n_\infty, g_\infty)\) is isometric to the product of \(\mathbb{R}\) with a complete \(C^{2,\alpha}\) Riemannian manifold \((N^{n-1}, h)\).

**Remark 1** Note that \(h\) is possibly flat. In particular, this is true for noncompact shrinking GRS \(\{(M^n, g, f_1, x_i)\}\) with \(x_i \to \infty\) under condition that \(|\text{Rc}| \to 0\), in which case the shrinking GRS is asymptotically conical.

**Proof** Define the function

\[
F^{(i)}(x) \doteq 2\lambda_i^{-1/2} \left( \sqrt{f_i}(x) - \sqrt{f_i}(x_i) \right) \quad \text{for} \quad x \in M_i.
\]

Using the diffeomorphisms in the definition of Cheeger–Gromov convergence, we can transplant \(F^{(i)}\) to a sequence of functions which are defined on balls \(B^{2\lambda_i}_{\rho}(x_\infty) \subset M_\infty\) for arbitrary \(\rho > 0\), as long as we choose \(i\) large enough. The basic idea of the proof is that the limit of a subsequence of these transplanted functions provides the splitting of \((M_\infty, g_\infty)\). A version of this idea was introduced in [6, p. 383] by the first named author of this article.

By (1), the gradient of \(F^{(i)}\) satisfies

\[
|dF^{(i)}|_{g_i} = \lambda_i^{-1/2} \cdot \frac{|df_i|_{g_l}}{f_i^{1/2}} = \left(1 - \frac{R_{g_l}}{\lambda_i f_i} \right)^{1/2} \leq 1 \quad \text{on} \quad M_i,
\]

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since $R_{g_i} \geq 0$ by Chen [5]. Fix any $\rho > 0$ and let $y \in B_{\rho}^{g_i}(x_i)$. Using (5), we have

\[
\begin{align*}
fi(y) & \geq \frac{1}{4} \left( \lambda_i^{1/2}d_{g_i}(y, O_i) - 5n \right)^2 \\
& \geq \frac{1}{4} \left( \lambda_i^{1/2}(d_{g_i}(x_i, O_i) - \rho) - 5n \right)^2 \\
& \geq \frac{1}{4} \left( \lambda_i^{1/2}d_{g_i}(x_i, O_i) - \lambda^{1/2}\rho - 5n \right)^2,
\end{align*}
\]

for $i$ large enough such that $\lambda_i^{1/2}d_{g_i}(x_i, O_i) - \lambda^{1/2}\rho - 5n > 0$. Here we have used $\l_i \leq \lambda$. Hence, we have

\[
\begin{align*}
\frac{R_{g_i}(y)}{\lambda_i fi(y)} & \leq \frac{C_1(\rho)}{4 \left( \lambda_i d_{g_i}(x_i, O_i) - \lambda\rho - 5n\lambda^{1/2} \right)^2}.
\end{align*}
\]

By the admissible assumption that $\lambda_i d_{g_i}(x_i, O_i) \to \infty$, we get that for any fixed $\rho > 0$

\[
\lim_{i \to \infty} \sup_{y \in B_{\rho}^{g_i}(x_i)} \frac{R_{g_i}(y)}{\lambda_i fi(y)} = 0.
\]

Hence we get

\[
\lim_{i \to \infty} \sup_{y \in B_{\rho}^{g_i}(x_i)} |dF^{(i)}|_{g_i}(y) = 1. \tag{8}
\]

Next we consider the Hessian of $F^{(i)}$. In local coordinates and using (1), we compute that

\[
\begin{align*}
\left| \nabla_{g_i}^2 F^{(i)} \right|_{g_i} & = \lambda_i^{-1/2} \left| \nabla_{g_i}^2 fi - \frac{1}{2} \frac{df_i \otimes df_i}{f_i^{3/2}} \right|_{g_i} \\
& = \lambda_i^{-1/2} \left( -\text{Rc}_{g_i} + \frac{\lambda_i}{2} g_i \right)_{g_i} + \frac{1}{2} \left| \frac{|df_i|^2_{g_i}}{f_i^{3/2}} \right|_{g_i} \\
& \leq \frac{|\text{Rc}_{g_i}|_{g_i} + \sqrt{n+1} \lambda_i}{(\lambda_i f_i)^{1/2}}.
\end{align*}
\]

Fix any $\rho > 0$ and let $y \in B_{\rho}^{g_i}(x_i)$. Using (7), we have

\[
\left| \nabla_{g_i}^2 F^{(i)} \right|_{g_i}(y) \leq \frac{C_1(\rho) + \sqrt{n+1} \lambda}{2 \left( \lambda_i d_{g_i}(x_i, O_i) - \lambda\rho - 5n\lambda^{1/2} \right)^2}. \tag{9}
\]
Hence, we get
\[ \lim_{i \to \infty} \sup_{y \in B^g_i (x_i)} \left| \nabla^3_{g_i} F^{(i)} \right|_{g_i} (y) = 0. \]  

(10)

Thirdly, we consider the third covariant derivatives of \( F^{(i)} \). Again, using local coordinates, we compute that
\[
\left| \nabla^3_{g_i} F^{(i)} \right|_{g_i} = \lambda_i^{-1/2} \left[ \frac{\nabla^3_{\ell k} f_i}{f_i^{1/2}} - \frac{\nabla^2_{jk} f_i \nabla_{\ell} f_i + \nabla^2_{\ell k} f_i \nabla_j f_i + \nabla^2_{\ell j} f_i \nabla_k f_i - \lambda_i 2 f_i^{3/2}}{2 f_i^{5/2}} + \frac{3 \nabla_{\ell} f_i \nabla_j f_i \nabla_k f_i}{f_i^{5/2}} \right]_{g_i} \\
\leq \lambda_i^{-1/2} \left( \frac{|\nabla g_i \text{Rc}_{g_i}|_{g_i}}{f_i^{1/2}} + \frac{3 \left( |\text{Rc}_{g_i}|_{g_i} + \frac{\lambda_i \sqrt{n}}{2} \right) |\nabla g_i f_i|_{g_i}}{2 f_i^{3/2}} + \frac{3 \left| \nabla g_i f_i^3 \right|_{g_i}}{4 f_i} \right) \\
\leq \frac{|\nabla g_i \text{Rc}_{g_i}|_{g_i}}{(\lambda_i f_i)^{1/2}} + \frac{3 \lambda_i^{1/2} (2 |\text{Rc}_{g_i}|_{g_i} + \lambda_i (\sqrt{n} + 1))}{4 f_i},
\]

where we used \( \text{Rc}_{g_i} + \nabla^2_{g_i} f_i = \frac{\lambda_i}{2} g_i \) and \( |\nabla g_i f_i|_{g_i} \leq \lambda_i f_i \). Fix any \( \rho > 0 \) and let \( y \in B^g_\rho (x_i) \). Using (7) and condition 6 in the admissible assumption, we have
\[
\left| \nabla^3_{g_i} F^{(i)} \right|_{g_i} (y) \leq \frac{C_2 (\rho)}{2 \left( \lambda_i d_{g_i} (x_i, O_i) - \lambda_i \rho - 5n \lambda_i^{1/2} \right)} \\
+ \frac{6C_1 (\rho) \lambda_i^{3/2} + 3 (\sqrt{n} + 1) \lambda_i^{5/2}}{\left( \lambda_i d_{g_i} (x_i, O_i) - \lambda_i \rho - 5n \lambda_i^{1/2} \right)^2}.
\]

(11)

Hence, we get
\[ \lim_{i \to \infty} \sup_{y \in B^g_\rho (x_i)} \left| \nabla^3_{g_i} F^{(i)} \right|_{g_i} (y) = 0. \]  

(12)

By the admissible assumption and Theorem 1, we know that \( \{(M_i, g_i, x_i)\} \) subconverges to \( (M_\infty, g_\infty, x_\infty) \) in the \( C^{2, \alpha} \) Cheeger–Gromov sense for any \( \alpha \in (0, 1) \). There exists an exhaustion \( \{U_i\} \) of \( M_\infty \) by relatively compact open sets and embeddings \( \psi_i : U_i \to M_i \) such that \( \overline{U}_i \subset U_{i+1} \), \( \psi_i (x_\infty) = x_i \), and \( \psi_i^* g_i \to g_\infty \) in the \( C^{2, \alpha} \) topology on any compact subset of \( M_\infty \).

Let \( \bar{g}_i \doteq \psi_i^* g_i \) be metrics on \( M_\infty \) and let \( \bar{F}^{(i)} \doteq F^{(i)} \circ \psi_i \) be the transplanted functions on \( M_\infty \). Fix any \( \rho > 0 \). By the convergence of \( \bar{g}_i \) to \( g_\infty \) and by Eqs. (8), (9), and (11), there is a constant \( C_3 > 1 \) such that the following bounds hold for any \( y \in B^g_\rho (x_\infty) \):
\[
\bar{F}_i (x_\infty) = 0, \\
|d \bar{F}^{(i)}|_{g_\infty} (y) \leq C_3.
\]  

(13)  

(14)
\[ \left| \nabla^2_{\tilde{g}_t} \tilde{F}^{(i)} \right|_{g_\infty} (y) \leq \frac{1}{2} \left( \lambda_i d_{\tilde{g}_t}(x_i, O_i) - \lambda_0 - 5n \lambda^{1/2} \right) \]

\[ \left| \nabla^3_{\tilde{g}_t} \tilde{F}^{(i)} \right|_{g_\infty} (y) \leq \frac{1}{2} \left( \lambda_i d_{\tilde{g}_t}(x_i, O_i) - \lambda_0 - 5n \lambda^{1/2} \right) + \frac{C_3 (6C_1(\rho)\lambda^{3/2} + 3(\sqrt{n} + 1)\lambda^{5/2})}{(\lambda_i d_{\tilde{g}_t}(x_i, O_i) - \lambda_0 - 5n \lambda^{1/2})^2}. \]

From (13) and (14), we know that \( \tilde{F}^{(i)} \) and \( d\tilde{F}^{(i)} \) are uniformly bounded on \( (B^g_{\rho}(x_\infty), g_\infty) \).

In local coordinates, we compute that the components of Hessian are

\[ \left( \nabla^2_{\tilde{g}_t} \tilde{F}^{(i)} \right)_{kl} = \frac{\partial^2 \tilde{F}^{(i)}}{\partial x^k \partial x^l} - \Gamma^p_{kl}(\tilde{g}_t) \frac{\partial \tilde{F}^{(i)}}{\partial x^p}. \]

By the \( C^{2,\alpha} \) convergence of \( \tilde{g}_t \) to \( g_\infty \), we know that the \( \Gamma^p_{kl}(\tilde{g}_t) \) are uniformly bounded on \( (B^g_{\rho}(x_\infty), g_\infty) \), and hence the \( \frac{\partial^2 \tilde{F}^{(i)}}{\partial x^k \partial x^l} \) are uniformly bounded on \( (B^g_{\rho}(x_\infty), g_\infty) \), so the \( \nabla^2_{\tilde{g}_t} \tilde{F}^{(i)} \) are uniformly bounded on \( (B^g_{\rho}(x_\infty), g_\infty) \).

Equation (16) implies that components \( \left( \nabla^2_{\tilde{g}_t} \tilde{F}^{(i)} \right)_{kl} \) have uniformly bounded \( C^\alpha \) norm with respect to the metric \( g_\infty \). By (17) and that the \( \Gamma^p_{kl}(\tilde{g}_t) \) have uniformly bounded \( C^\alpha \) norm, we conclude that the \( \frac{\partial^2 \tilde{F}^{(i)}}{\partial x^k \partial x^l} \) are uniformly bounded in \( C^\alpha \) norm on \( (B^g_{\rho}(x_\infty), g_\infty) \), so the \( \nabla^2_{\tilde{g}_t} \tilde{F}^{(i)} \) are uniformly bounded in the \( C^\alpha \) norm on \( (B^g_{\rho}(x_\infty), g_\infty) \).

Hence we can conclude that \( \tilde{F}^{(i)} \) subconverges to a function \( F_\infty \) in the \( C^{2,\alpha} \) norm on \( (\mathcal{M}_\infty, g_\infty) \). It follows from (8), (10), and (12) that the limit function \( F_\infty \) on \( \mathcal{M}_\infty \) satisfies \( F_\infty(x_\infty) = 0, |\nabla F_\infty|_{g_\infty} \equiv 1, \) and \( |\nabla^2 F_\infty|_{g_\infty} \equiv 0. \) This yields a Riemannian splitting for the limit. Note that if the metric \( g_\infty \) is smooth, then \( F_\infty \) is also smooth.

\[ \square \]

The following is a parabolic version of Theorem 2 under a stronger assumption.

**Corollary 1** Let \( \{(\mathcal{M}_i^n, g_i, f_i, \lambda_i, x_i)\} \) be a sequence of pointed complete noncompact shrinking GRS with \( f_i \) normalized so that \( \text{Re}_{g_i} + \nabla^2_{g_i} f_i = \frac{\lambda_i}{2} g_i \), where \( 0 < \lambda_i \leq \lambda \) for some \( \lambda < \infty \) and where \( \text{Re}_{g_i} + \nabla_{g_i} f_i \geq \lambda_i f_i \). We assume that the \( (\mathcal{M}_i^n, g_i) \), \( i = 1, 2, \ldots \), have uniformly bounded curvatures \( |\text{Rm}_{g_i}|_{g_i} \leq C < \infty \) and are uniformly \( \kappa \)-noncollapsed (below a fixed scale) for some \( \kappa > 0 \). Let \( (\mathcal{M}_i^n, g_i(t)) \) be the Ricci flow solution in canonical form associated to the shrinking GRS \( (\mathcal{M}_i^n, g_i, f_i, \lambda_i) \) as defined preceding (2). Then the sequence \( \{(\mathcal{M}_i^n, g_i(t), x_i)\} \) subconverges in the \( C^\infty \) pointed Cheeger–Gromov sense to an ancient solution of the Ricci flow \( (\mathcal{M}_\infty^n, g_\infty(t), x_\infty), t \in (-\infty, \lambda^{-1}) \). Moreover, \( (\mathcal{M}_\infty^n, g_\infty(t)) \) is isometric to the product of \( \mathbb{R} \) with a complete \( \kappa \)-noncollapsed (below some fixed scale) Type I (at \( t = -\infty \)) ancient solution (\( N^{n-1}, h(t) \)). Note that \( h(t) \) is possibly flat.
Proof Since $|Rm_{gi_i}|_{gi_i} \leq C$, from the discussion preceding (2), we have

$$|Rm_{gi_i(t)}|_{gi_i(t)} \leq \frac{C}{1 - \lambda_i t} \leq \frac{C}{1 - \lambda t}.$$ 

By Shi’s derivative estimates, on any compact subinterval $I$ of $(-\infty, \lambda^{-1})$ there exist constants $C_{k,n,I}$ (independent of $i$) such that $|\nabla^k_{gi_i(t)} Rm_{gi_i(t)}|_{gi_i(t)} \leq C_{n,k,I}$ on $M_i$ for $k \in \mathbb{N}$, $t \in I$, and all $i$.

It follows from the uniformly $\kappa$-noncollapsed assumption and the uniform curvature bound assumption that there is a constant $\iota > 0$ such that $\text{inj}_{gi_i(t)}(x_i) \geq \iota$ for all $i$. By Hamilton’s Cheeger–Gromov compactness theorem for solutions of Ricci flow and by passing to a subsequence if necessary, $\{(M^n_i, gi_i(t), x_i)\}$, $t \in (-\infty, \lambda^{-1})$, converges to an ancient solution $(M^n_\infty, g_\infty(t), x_\infty)$, $t \in (-\infty, \lambda^{-1})$, of the Ricci flow satisfying $|Rm_{g_\infty(t)}|_{g_\infty(t)} \leq \frac{C}{1 - \lambda t}$. This implies that $g_\infty(t)$ is Type I at $t = -\infty$.

Fix a $t \in (-\infty, \lambda^{-1})$. By (3), $gi_i(t)$ is a shrinking GRS. It is easy to check that $\{(M^n_i, gi_i(t), f_i(t), 1/2, 1/\lambda_i t, x_i)\}$ is an admissible sequence of pointed complete noncompact shrinking GRS. In particular, condition 3 in the definition of admissibility follows from

$$C^{-1} d_{gi_i}(x_i, O_i) \leq d_{gi_i}(x_i, O_i) \leq C d_{gi_i(t)}(x_i, O_i)$$

for some constant $C$ independent of $i$. This inequality is due to the fact that a uniform curvature bound implies uniform metric equivalence for the Ricci flow (see, for example, [6, Lemma 6.10]). The uniform curvature bound also implies uniform volume equivalence of unit balls. Hence, condition 4 that $\text{inj}_{gi_i(t)}(x_i) \geq \iota_1 > 0$ for an admissible sequence follows from a theorem of Cheeger et al. [4]. Now we can apply Theorem 2 to the sequence $\{(M^n_i, gi_i(t), f_i(t), 1/\lambda_i t, x_i)\}$ to conclude that the limit $(M^n_\infty, g_\infty(t))$ is isometric to the product of $\mathbb{R}$ with a complete Riemannian manifold $(N^{n-1}, h(t))$ for each $t$. The constancy of the splitting of $M^n_\infty$ into $\mathbb{R} \times N^{n-1}$ for different $t$ follows from the smooth dependence on $t$ of the functions

$$F^{(i)}(x, t) = 2 \left( \frac{\lambda_i}{1 - \lambda_i t} \right)^{-1/2} \cdot \left( \sqrt{f_i(x, t)} - \sqrt{f_i(x_i, t)} \right),$$

whose limits are used to define the splitting. Now the theorem is proved. \hfill \Box

Remark 2 If we combine Deng and Zhu’s discussion ([7, p. 12]) and Corollary 1 and apply them to a complete noncompact shrinking Kähler GRS, then we obtain a splitting of the form $(\mathbb{R} \times S^1) \times W^{n-1}$ or $\mathbb{C} \times W^{n-1}$, where $W$ is of complex dimension $n - 1$.

3 Splitting at infinity of shrinking GRS with bounded curvature

From Corollary 1, we obtain our main theorem, which has some flavor of the canonical neighborhood property.
Theorem 3 Let \( (\mathcal{M}^n, g(t), f(t)) \), \( t \in (-\infty, 1) \), be a complete noncompact \( \kappa \)-noncollapsed (below a fixed scale) shrinking GRS in canonical form with bounded curvature. Then, for any \( \varepsilon > 0 \) there is a compact set \( K_\varepsilon \subset \mathcal{M} \) such that for each \( x \in \mathcal{M} - K_\varepsilon \) there is an open set \( U \subset \mathcal{M} \) such that \((U, g(t), x)\) is \( \varepsilon \)-close for \( t \in (\varepsilon^{-1}, 1 + \varepsilon) \) in the \( C^{[\varepsilon^{-1}]} \)-pointed Cheeger–Gromov sense to \((B_{1/\varepsilon}^g(L_{1/\varepsilon}) \times (\varepsilon^{-1}, \varepsilon^{-1}), h(t) + ds^2, (y, 0))\), where \( y \in N^{n-1} \) and where \((N, h(t)), t \in (-\infty, 1)\), is a complete (possibly flat) \( \kappa \)-noncollapsed (below some fixed scale) Type I ancient solution of Ricci flow with \( |Rm_{h(t)}| \leq \frac{C}{1-t} \).

Proof Suppose that the theorem is false. Then there exists \( \varepsilon > 0 \) and a sequence of points \( x_i \to \infty \) without the stated property. On the other hand, by Corollary 1 \((M, g(t), x_i))\), \( t \in (-\infty, 1) \), subconverges to the product of \( \mathbb{R} \) with a complete \( \kappa \)-noncollapsed (below some fixed scale) Type I ancient solution \((\Lambda^{n-1}, h(t))\). Choosing \( i \) large enough leads to a contradiction. \( \Box \)

Remark 3 As explained in the introduction, in general the best result in the direction of Theorem 3 one can hope for is that if curvature tensor \( Rm \) of a complete noncompact shrinking GRS does not limit to 0 at infinity, then \((M^n, g)\) splits as the product of \( \mathbb{R} \) and a Riemannian manifold \((\Lambda^{n-1}, h)\).

Now we give some conclusions related to Theorem 3 assuming that the solution is \( \kappa \)-noncollapsed below all scales. First, in dimension \( n = 4 \), the sectional curvature of \( h(t) \) is nonnegative by Chen [5]. If \( N^3 \) is compact, then by Ni [19], \((N, h(t))\) is a spherical space form (note that a compact quotient of \( \mathbb{R}^3 \) and \( S^2 \times \mathbb{R} \) are not \( \kappa \)-noncollapsed). If \( N^3 \) is noncompact, then \( h(t) \) is either \( \mathbb{R}^3 \), \( S^2 \times \mathbb{R} \), its \( \mathbb{Z}_2 \)-quotient, or has positive sectional curvature.

For the general case \( n \geq 4 \), we have outside a compact set \( K \) that \( \nabla f \neq 0 \) and \( v = \frac{\sqrt{f}}{\nabla f} \) is well defined. Let \( \Sigma_c = \{ f = c \} \), which is a \( C^{\infty} \) hypersurface, and let \( g^T = g - v^* \otimes v^* \). Let \( F^{(i)}(x) = 2(\sqrt{f}(x) - \sqrt{f}(x_i)) \). By the Gauss equations,

\[
\begin{align*}
f^{1/2} \nabla^2 F^{(i)} &= \nabla^2 f - \frac{1}{2} \nabla f \otimes \nabla f - \frac{1}{2} \frac{\nabla f \otimes \nabla f}{f} \\
&= -R_{c,T} + \frac{1}{2} g^T - R_{g} (v, v) v^* \otimes v^* - R_{m}(v, \cdot, \cdot, v) \\
&\quad + HII - II^2 + \frac{R}{2} v^* \otimes v^*,
\end{align*}
\]

where \( II \) and \( H \) are the second fundamental form and mean curvature of \( \Sigma_c \). By the theorem above, on \( \mathcal{N} \) the limit \( \lim_{i \to \infty} (f^{1/2} \nabla^2 F^{(i)}) = \frac{1}{2} h - R_{c,h} \) exists using Cheeger–Gromov convergence. Let \( p \in \mathcal{M} \). We have \( \frac{1}{2} - \nabla^2 f(v, v) = \frac{1}{2|\nabla f|} \nabla R \cdot v = o(d(\cdot, p)^{-1}) \) and \( \nabla^2 f(X, v) = -\frac{1}{2|\nabla f|} \nabla R \cdot X = O(d(\cdot, p)^{-1}) \) for unit \( X \in T \Sigma_c \).

Example Let \((\Lambda^{n-1}, \tilde{g}, \tilde{f})\) be a shrinking GRS. Define the shrinking GRS \((\mathcal{M}^n, g, f)\) by \( \mathcal{M} = \mathcal{N} \times \mathbb{R} \), \( g = \tilde{g} + ds \otimes ds \), and \( f(y, s) = \tilde{f}(y) + \frac{s^2}{4} \). Then \( f^{1/2} \nabla^2 F^{(i)} = \frac{1}{2} - R_{c,\tilde{g}} + \frac{R}{2f} v^* \otimes v^* = \tilde{\nabla}^2 \tilde{f} + O(d(\cdot, p)^{-1}) \).

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There are implications of the works of Naber [18] and Cao and Zhang [2] on Type I ancient solutions. In particular, Cao and Zhang proved that, for any $\kappa$-noncollapsed complete Type I ancient solution with nonnegative curvature operator, Naber’s complete shrinking GRS backward limit is necessarily nonflat. In view of Lemma 8.27 in [6], we may apply this to those shrinking GRS which are singularity models with nonnegative curvature operator.

4 Standard point picking

The results of this section are known consequences of standard techniques, going back to Hamilton. For convenience, we include the statements in the form we use. Let $(\mathcal{M}^n, g)$ be a complete noncompact Riemannian manifold satisfying

$$\text{ACR} \equiv \lim_{\rho \to \infty} \sup_{x \in B_\rho(O)} d(O, x)^2 \cdot |\text{Rm}(x)| = \infty \text{ for some } O \in \mathcal{M}. \quad (18)$$

We now describe the point picking argument, which enables us to construct sequences $x_i \to \infty$ such that $g_i \equiv |\text{Rm}_g(x_i)| \cdot g$ satisfies (6) in the admissible assumption. Let $\rho_i \to \infty$ be a sequence. Choose $x_i$ with $r_i \equiv d_g(O, x_i)$ so that

$$\frac{r_i (\rho_i - r_i) \sqrt{|\text{Rm}_g(x_i)|}}{\sup_{x \in B_{\rho_i}(O)} d_g(O, x) \left(\rho_i - d_g(O, x)\right) \sqrt{|\text{Rm}_g(x)|}} \leq 1 - \delta_i \to 1. \quad (19)$$

Claim Assume (18). Then

$$\alpha_i \equiv r_i \sqrt{|\text{Rm}_g(x_i)|} \to \infty \text{ and } \omega_i \equiv (\rho_i - r_i) \sqrt{|\text{Rm}_g(x_i)|} \to \infty.$$  

Proof of the claim We compute that

$$\frac{1}{\alpha_i^{-1} + \omega_i^{-1}} = \frac{\alpha_i \omega_i}{\alpha_i + \omega_i} = \frac{r_i (\rho_i - r_i) \sqrt{|\text{Rm}_g(x_i)|}}{\rho_i} = (1 - \delta_i) \cdot \frac{\sup_{x \in B_{\rho_i/2}(O)} d_g(O, x) \left(\rho_i - d_g(O, x)\right) \sqrt{|\text{Rm}_g(x)|}}{\rho_i} \geq \frac{1 - \delta_i}{2} \cdot \sup_{x \in B_{\rho_i/2}(O)} d_g(O, x) \sqrt{|\text{Rm}_g(x)|} \geq \frac{1}{3} \sup_{x \in B_{\rho_i/2}(O)} d_g(O, x) \sqrt{|\text{Rm}_g(x)|} \to \infty \text{ as } i \to \infty.$$

Hence $\alpha_i \to \infty$ and $\omega_i \to \infty$. \qed
Then, for \( x \in B^g_{\rho_i}(O) = B^{g_i}_{\rho_i}\sqrt{|R_{g_i}(x_i)|}(O), \)

\[
\sqrt{|R_{g_i}(x)|} \leq \frac{1}{1 - \delta_i} \cdot \frac{r_i (\rho_i - r_i)}{d_g(O, x) (\rho_i - d_g(O, x))} \cdot \frac{\alpha_i}{\alpha_i - \rho} \cdot \frac{\omega_i}{\omega_i - \rho},
\]

where the RHS tends to \( 1 \) as \( i \to \infty \).

Let \( \rho \in (0, \infty) \). Suppose that \( x \in B^g_{\rho_i}(x_i) \), which is equivalent to \( x \in B^g_{\rho/\sqrt{|R_{g_i}(x_i)|}}(x_i) \). This implies that

\[
-\rho/\sqrt{|R_{g_i}(x_i)|} \leq d_g(O, x) - r_i \leq \rho/\sqrt{|R_{g_i}(x_i)|},
\]

i.e.,

\[
-\rho \leq (d_g(O, x) - r_i) \sqrt{|R_{g_i}(x_i)|} \leq \rho.
\]

Thus:

**Lemma 1** Assume (18) and choose \( x_i \) as in (19). If \( x \in B^g_{\rho_i}(x_i) \), then

\[
\sqrt{|R_{g_i}(x)|} \leq \frac{1}{1 - \delta_i} \cdot \frac{\alpha_i}{\alpha_i - \rho} \cdot \frac{\omega_i}{\omega_i - \rho},
\]

where the RHS tends to \( 1 \) as \( i \to \infty \).

**Corollary 2** Let \((M^n, g, f, 1)\) be a complete noncompact \( \kappa \)-noncollapsed (on all scales) shrinking GRS satisfying (18) and \(|Rm|(x) = o(d(O, x)^2)\), where \( O \) is a minimum point of \( f \). Let \( x_i \to \infty \) be chosen as in (19). Assume that \( K_i \equiv |Rm|(x_i) \geq c > 0 \) and

\[
\sup_i \left( K_i^{-3/2} \cdot \sup_{x \in B^g_{\rho}(x_i)} |\nabla R_c| \right) < \infty \quad \text{for all} \quad \rho > 0. \tag{20}
\]

Define \( g_i \equiv K_i g \). Let \( \alpha \) be any constant in \((0, 1)\). Then \((M^n, g_i, x_i)\) converges in the \( C^{2,\alpha} \) pointed Cheeger–Gromov sense to a complete \( C^{2,\alpha} \) Riemannian manifold \((M^n_{\infty}, g_{\infty}, x_{\infty})\), and \((M_{\infty}, g_{\infty})\) is isometric to the product of \( \mathbb{R} \) with a complete nonflat Riemannian manifold \((N^{n-1}, h)\).

**Proof** The corollary follows from the claim that \( \{(M, g_i, f, K_i^{-1}, x_i)\} \) is an admissible sequence of complete noncompact shrinking GRS and from Theorem 2.
To see the claim, note that condition 5 follows from Lemma 1. Condition 4 follows from condition 5 and the $\kappa$-noncollapsing assumption. Condition 6 follows from assumption (20). Conditions 1 and 2 follow from the definition and $\lambda_i = K_i^{-1} \leq c^{-1}$.

Finally, to see condition 3, we compute that

$$\lambda_i d_{g_i}(x_i, O) = K_i^{-1} \cdot K_i^{1/2} d_g(x_i, O) = \frac{d_g(x_i, O)}{K_i^{1/2}} = \frac{d_g(x_i, O)}{o(d_g(x_i, O))} \to \infty,$$

where we have used $|Rm|(x) = o(d(x, O)^2)$ in the last equality. \hfill \box

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