Multiple Solutions for a Nonlocal Elliptic Problem Involving $(p(x), q(x))$-Biharmonic Operator

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In this paper, using the variational principle, the existence and multiplicity of solutions for $(p(x), q(x))$-Kirchhoff type problem with Navier boundary conditions are proved. At the same time, the sufficient conditions for the multiplicity of solutions are obtained.

1. Introduction

In this paper, we will discuss the nonlocal elliptic problem involving $(p(x), q(x))$-biharmonic operator:

\[
\begin{aligned}
\Delta_{p(x)}^2 u(x) - M_1 \left( \int_{\Omega} \frac{\left| \nabla u(x) \right|^{p(x)}}{p(x)} \right) \Delta_{p(x)} u(x) + \rho_1(x) |u|^{p(x)-2} u(x) &= \lambda F_u(x, u, v), & \text{in } \Omega, \\
\Delta_{q(x)}^2 v(x) - M_2 \left( \int_{\Omega} \frac{\left| \nabla v(x) \right|^{q(x)}}{q(x)} \right) \Delta_{q(x)} v(x) + \rho_2(x) |v|^{q(x)-2} v(x) &= \lambda F_v(x, u, v), & \text{in } \Omega, \\
\end{aligned}
\]

\[u = \Delta u = v = \Delta v = 0, \quad \text{on } \partial \Omega,\]

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a smooth bounded region. $\lambda > 0$, $\Delta_{p(x)}^2 u(x) = \Delta (|\Delta u|^{p(x)-2} \Delta u)$, $\rho_1(x), \rho_2(x) \in L^\infty(\Omega)$, $p(x), q(x) \in C(\overline{\Omega})$, $(N/2) < p = \esssup_{x \in \overline{\Omega}} p(x) \leq \esssup_{x \in \Omega} p(x) < +\infty$, and $(N/2) < q = \esssup_{x \in \overline{\Omega}} q(x) \leq q^* = \esssup_{x \in \Omega} q(x) < +\infty$. Furthermore, the function $F: \overline{\Omega} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies that $F(\cdot, s, t)$ is continuous in $\overline{\Omega}$, for all $(s, t) \in \mathbb{R}^2$, and $F(x, \cdot, \cdot)$ is $C^1$ in $\mathbb{R}^2$, for every $x \in \Omega$, and $\sup_{0 \leq t \leq m_2} |F_u(x, s, t)| + |F_v(x, s, t)| \in L^1(\Omega)$, for all $\tau > 0$, where $F_u$ and $F_v$ are partial derivatives of $u$ and $v$, respectively. Besides, the functions $M_i: [0, +\infty) \rightarrow \mathbb{R}$ are continuous and satisfy $m_0 \leq M_i(t) \leq m_1, i = 1, 2$, for every $t \geq 0$.

As we all know, the harmonic equation and biharmonic equation in partial differential equation are the most widely used equations in the field of theoretical research and engineering application. Among them, the focus and difficulty in science and engineering is to solve the boundary value problem of biharmonic equation.
Therefore, many scholars have carried out a lot of in-depth research on biharmonic problems, see [1–6]. In particular, in [3], when the nonlinear functions $F$ and $G$ satisfy certain conditions, Li and Tang studied the $(p, q)$-biharmonic problem,

\[
\begin{align*}
\Delta(|\Delta u|^{p-2}\Delta u) &= \lambda F_u(x, u, v) + \mu G_u(x, u, v), \quad x \in \Omega, \\
\Delta(|\Delta v|^{q-2}\Delta v) &= \lambda F_v(x, u, v) + \mu G_v(x, u, v), \quad x \in \Omega, \\
\Delta u = \Delta v = \Delta \nu = 0,
\end{align*}
\]

(2)

by applying the three critical point theorem by Ricceri [7].

Two years later, Masser et al. [8] supposed that the assumption in problem (2) $\mu = 0$ and imposed appropriate conditions on $\Omega$. Based on the critical point theorem introduced by Bonanno and Molica Bisci, infinitely many solutions were obtained. It was known that the variable exponent case possess more complicated properties than the constant exponent case, and some methods used in the $(p, q)$-biharmonic case cannot be applied to the $(p(x), q(x))$-biharmonic case. Therefore, Allaoui et al. [9] have made a great contribution to such problems, and they continued to extend $(p, q)$-biharmonic operator in [8] to $(p(x), q(x))$-biharmonic case, on the basis of Ricceri’s variational principle [10] and the basic theory of Sobolev space, and the following system is solved:

\[
\begin{align*}
\Delta(|\Delta u|^{(p(x))-2}\Delta u) &= \lambda F_u(x, u, v), \quad x \in \Omega, \\
\Delta(|\Delta v|^{(q(x))-2}\Delta v) &= \lambda F_v(x, u, v), \quad x \in \Omega, \\
\Delta u = \Delta v = \Delta \nu = 0,
\end{align*}
\]

(3)

With the deepening of the investigation, scholars are increasingly paying attention to the solution of nonlocal elliptic equation, see [11–15], for details.

In [16], Ferrara et al. researched the problem:

\[
\begin{align*}
\Delta u + \rho |u|^{p-2} u &= \lambda f(x, u), \quad x \in \Omega, \\
\end{align*}
\]

(4)

2. Preliminaries

Define the space as follows:

\[ L^{p(x)}(\Omega) = \left\{ u: \Omega \rightarrow \mathbb{R} \right\} \text{measurable}, \quad \int_{\Omega} |u(x)|^{p(x)} dx < +\infty \}
\]

(5)

which has the norm

\[ |u|_{p(x)} = \inf \left\{ \mu > 0: \int_{\Omega} \frac{|u(x)|^{p(x)}}{\mu} dx \leq 1 \right\} \]

(6)

The Sobolev space with variable exponents is defined as

\[ W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega): D^k u \in L^{p(x)}(\Omega), |\sigma| \leq k, k \in \mathbb{Z}, x \right\}, \]

(7)

which has the norm

\[ |u|_{k,p(x)} = \sum_{|\sigma| \leq k} |D^\sigma u|_{p(x)}, \]

(8)

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a multi-index; moreover, $|\sigma| = \sum_{j=1}^{n} \sigma_j$ was established.

The closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(x)}(\Omega)$ is the $W_0^{k,p(x)}(\Omega)$. Besides, from [20], we know that $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are Banach space and satisfy separability, uniform convexity, and reflexivity.

**Proposition 1** (see [20]). Assume $(1/p_1(x)) + (1/p_2(x)) = 1$, then, $L^{p_1(x)}(\Omega)$ and $L^{p_2(x)}(\Omega)$ are conjugate spaces and satisfy the Holder inequality:
\[
\int_{\Omega} u \psi dx \leq \left( \frac{1}{p_1} + \frac{1}{p_2} \right) |u|_{p_1(x)} |\psi|_{p_2(x)}, \quad (9)
\]

where \( u \in L^{p_1(x)}(\Omega) \) and \( \psi \in L^{p_2(x)}(\Omega) \).

Denote \( X = (W^{1,p(x)}_0(\Omega) \cap W^{2,p(x)}(\Omega)) \times (W^{1,q(x)}_0(\Omega) \cap W^{2,q(x)}(\Omega)) \), which is a separable and reflexive Banach space endowed with the norm:

\[
\|(u, \psi)\| = \|u\|_{p(x)} + \|\psi\|_{q(x)}, \quad (10)
\]

where

\[
\|u\|_{p(x)} = \inf \left\{ \omega > 0 : \int_{\Omega} \left( \frac{|\Delta u(x)|^{p(x)}_\omega}{\omega} + \frac{|u(x)|^{p(x)}_\omega}{\omega} \right) dx \leq 1 \right\},
\]

\[
\|\psi\|_{q(x)} = \inf \left\{ \omega > 0 : \int_{\Omega} \left( \frac{|\Delta \psi(x)|^{q(x)}_\omega}{\omega} + \frac{|\psi(x)|^{q(x)}_\omega}{\omega} \right) dx \leq 1 \right\}. \quad (11)
\]

From [21], we know that \( |\cdot|_{2,p(\cdot)}, \|\cdot\|_{p(\cdot)}, \) and \( |\Delta \cdot|_{p(\cdot)} \) are equivalent norms of \( X \).

\[
K = \max \left\{ \frac{\sup_{x \in \Omega} |u(x)|}{\|u\|_{p(x)}}, \frac{\sup_{x \in \Omega} |v(x)|}{\|v\|_{q(x)}} \right\} < +\infty. \quad (12)
\]

**Proposition 3** (see [9, 19]). Let \( f(u) = \int_{\Omega} |\Delta u|^{p(x)} dx \), for \( u \in W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega) \); then, we deduce

\( (h1) \|u\|_{p(x)} < 1 \Rightarrow f(u) < 1 \)

\( (h2) \|u\|_{p(x)} \geq 1 \Rightarrow \|u\|_{p(x)} \geq f(u) \) \( \leq \|u\|_{p(x)} \]

\[
\int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \phi dx + \int_{\Omega} |\nabla \psi|^{q(x)-2} \nabla \psi dx + M_1 \left( \int_{\Omega} \frac{|\nabla u|^{p(x)}_{p(x)}}{p(x)} dx \right) \times \int_{\Omega} \frac{|\nabla \psi|^{q(x)}_{q(x)}}{q(x)} dx
\]

\[
+ M_2 \left( \int_{\Omega} \frac{|\nabla \psi|^{q(x)}_{q(x)}}{q(x)} dx \right) \times \int_{\Omega} |\nabla \psi|^{q(x)}_{q(x)} dx + \int_{\Omega} \rho_1(x)|u|^{p(x)-2} u \phi dx + \int_{\Omega} \rho_2(x)|\psi|^{q(x)-2} \psi dx \quad (13)
\]

holds, for all \( (\phi, \psi) \in X \), then \( (u, \psi) \in X \) is a weak solution of problem (1).

Define the functional \( I_\lambda : X \rightarrow \mathbb{R} \):

\[
I_\lambda(u, \psi) = \Phi(u, \psi) - \lambda \Psi(u, \psi),
\]

\[
\Phi(u, \psi) = \int_{\Omega} \frac{|\Delta u|^{p(x)}_{p(x)}}{p(x)} dx + \tilde{M}_1 \left( \int_{\Omega} \frac{|\nabla u|^{p(x)}_{p(x)}}{p(x)} dx \right) + \int_{\Omega} \rho_1(x)|u|^{p(x)}_{p(x)} dx
\]

\[
+ \int_{\Omega} |\Delta \psi|^{q(x)}_{q(x)} dx + \tilde{M}_2 \left( \int_{\Omega} \frac{|\nabla \psi|^{q(x)}_{q(x)}}{q(x)} dx \right) + \int_{\Omega} \rho_2(x)|\psi|^{q(x)}_{q(x)} dx,
\]

\[
\Psi(u, \psi) = \int_{\Omega} F(x, u, \psi) dx,
\]

where \( \Phi \) is a convex, lower semi-continuous and quasi-convex functional, \( \Psi \) is a convex, upper semi-continuous and quasi-convex functional, and \( \Phi \) and \( \Psi \) are Lipschitz continuous in \( u \) and \( \psi \) respectively.
The functional \( \Phi, \Psi : X \rightarrow \mathbb{R} \) are well defined and Gateaux differentiable functions for \( \forall (u, v) \in X \); we have

\[
\begin{align*}
\langle \Phi'(u, v), (\psi, \phi) \rangle &= \int_\Omega |\nabla u|^p(x) \Delta u \phi \, dx + M_1 \left( \int_\Omega \frac{|\nabla u|^p(x)}{p(x)} \, dx \right) \times \int_\Omega \frac{|\nabla u|^p(x)}{p(x)} \, dx \times \int_\Omega \nabla u \nabla \psi \, dx \\
&\quad + \int_\Omega |\nabla \phi|^2 \Delta v \, dx + M_2 \left( \int_\Omega \frac{|\nabla \phi|^2}{q(x)} \, dx \right) \times \int_\Omega \frac{|\nabla \phi|^2}{q(x)} \, dx \times \int_\Omega \nabla \phi \nabla \psi \, dx, \\
\langle \Psi'(u, v), (\psi, \phi) \rangle &= \int_\Omega F_u(x, u, v) \phi \, dx + \int_\Omega F_v(x, u, v) \psi \, dx.
\end{align*}
\]

Hence, according to Definition 1, we know that if \((u, v) \in X\) is the weak solution of problem (1), it is equivalent to that \((u, v)\) is the critical point of \( I_1 \). In addition, since the space \( X \) to \( C(\bar{\Omega}) \times C(\bar{\Omega}) \) is a compact embedding, so \( \Phi, \Psi : X \rightarrow \mathbb{R} \) are sequentially weakly lower semicontinuous, and it is clear that \( \Phi \) is coercive.

**Theorem 1** (see [10]). Let \( X \) be a reflexive real Banach space. \( \Phi, \Psi : X \rightarrow \mathbb{R} \) are Gateaux differential functionals; \( \Phi \) satisfies coercive and sequentially weakly lower semicontinuity and \( \Psi \) satisfies sequentially weakly upper semicontinuity. If \( r > \inf_X \Phi \), denotes

\[
\varphi(r) := \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\sup_{v \in \Phi^{-1}((-\infty, r))} \Psi(v) - \Psi(u)}{r - \Phi(u)},
\]

\[
y := \lim_{r \rightarrow +\infty} \inf \varphi(r),
\]

\[
\delta := \lim_{r \rightarrow (\fin \Phi)} \inf \varphi(r),
\]

then one has

(a) For every \( r > \inf_X \Phi \) and every \( \lambda \in (0, (1/\varphi(r))) \), the functional \( I_1 = \Phi - \lambda \Psi \) has a global minimum in \( \Phi^{-1}((-\infty, r)) \), which is a critical point (local minimum) of \( I_1 \) in \( X \).

(b) If \( \gamma < +\infty \), then \( \forall \lambda \in (0, (1/\gamma)) \); the following alternative holds: either

(b1) \( I_1 \) has a global minimum or

(b2) \( I_1 \) has a series of critical points (local minimum) defined as \( \{ u_n \} \) satisfying \( \lim_{n \rightarrow +\infty} \Phi(u_n) = +\infty. \)

(c) If \( \delta < +\infty \), then \( \forall \lambda \in (0, 1/\delta) \); the following alternative holds: either

(c1) \( \Phi \) has a global minimum which is a local minimum of \( I_1 \) or

(c2) \( I_1 \) has a series of pairwise distinct critical points (local minimum) which weakly converges to global minimum of \( \Phi \).

**3. Main Results and Proof**

**Theorem 2.** Suppose

(A1) \( \forall (x, s, t) \in \Omega \times [0, +\infty)^2, F(x, s, t) \geq 0 \).

(A2) \( \exists x_0 \in \Omega \) and two real numbers \( O_2 > O_1 > 0 \) satisfy \( R(x_0, O_2) \subseteq \Omega \). If we put

\[
\alpha = \lim_{b \rightarrow +\infty} \inf \int_\Omega \sup_{|s| \leq b} F(x, s, t) \, dx \frac{1}{\min_{|p'|} |p'|},
\]

\[
\beta = \lim_{s, t \rightarrow +\infty} \sup \frac{\int_\Omega F(x, s, t) \, dx}{(s^q / p') + (t^q / q')},
\]

one has

\[
\alpha < L \beta,
\]

where
\[ L := \min[L_1, L_2], \]
\[
L_1 = \frac{1}{\left[ K\left( \left( \frac{p^*}{M^-} \right)^{1/p^*} + \left( \frac{q^*}{M^-} \right)^{1/q^*} \right) \right]^{\min\{p^-, q^*\}}} \frac{1}{M^+(\theta_1 + \theta_2 + |\rho_1(x)|, \theta_3)}.
\]
\[
L_2 = \frac{1}{\left[ K\left( \left( \frac{p^*}{M^-} \right)^{1/p^*} + \left( \frac{q^*}{M^-} \right)^{1/q^*} \right) \right]^{\min\{p^-, q^*\}}} \frac{1}{M^+(\theta_1 + \theta_2 + |\rho_2(x)|, \theta_3)}.
\]
\[
\theta_1 = \frac{2\pi(N/2)}{N\Gamma(N/2)} \left( \frac{2N}{O_1^N - O_1^N} \right)^{p^*} (O_1^N - O_1^N),
\]
\[
\theta_2 = \frac{2\pi(N/2)}{\Gamma(N/2)} \left( \frac{2}{O_2^N - O_2^N} \right)^{p^*} \frac{O_2^N - O_1^N}{p^* + N},
\]
\[
\theta_3 = \frac{2\pi(N/2)}{\Gamma(N/2)} \left( \frac{O_1^N}{N} + \left( \frac{1}{O_2^N - O_2^N} \right)^{p^*} \int_{O_1}^{O_2} \frac{O_2^N - r^N}{r^N - r^N} dr \right),
\]
\[
M^+ = \max\{1, m_1\},
\]
\[
M^- = \min\{1, m_0\}.
\]

Then, for every \( \lambda \in \Lambda := \left( \frac{1}{K\left( \left( \frac{p^*}{M^-} \right)^{1/p^*} + \left( \frac{q^*}{M^-} \right)^{1/q^*} \right) \right]^{\min\{p^-, q^*\}}} \frac{1}{L^2} \frac{1}{\alpha} \right)
\]

Problem (1) has a sequence of solutions \((u_n, v_n)\) such that \( \lim_{n \to +\infty} \Phi(u_n, v_n) = +\infty \).

Proof. Since \( \Phi(0,0) = 0 \) and \( \Psi(0,0) \geq 0 \), if \( r > 0 \), then

\[ \phi(r) = \inf_{(u,z) \in \Phi^{-1}((\infty,0))} \frac{\sup_{(u,z) \in \Phi^{-1}((\infty,0))} \Psi(u, v) - \Psi(w, z)}{r - \Phi(w, z)} \leq \sup_{(u,v) \in \Omega} F(x, u, v) dx \]

From \((A_2)\), we know that there exists a sequence \( \{\xi_n\} \)

\[ \lim_{n \to +\infty} \frac{\int_\Omega \sup_{|s| < |t| < |\xi_n|} F(x, s, t) dx}{\xi_n^{\min\{p^-, q^*\}}} = \alpha < +\infty. \]

Put

\[ r_n = \frac{\xi_n^{\min\{p^-, q^*\}}}{K\left( \left( \frac{p^*}{M^-} \right)^{1/p^*} + \left( \frac{q^*}{M^-} \right)^{1/q^*} \right) \right]^{\min\{p^-, q^*\}}}.
\]

for all \( n \in \mathbb{N} \). When \( (u,v) \in \Phi^{-1}((\infty, r_n)) \), we have \( \Phi(u,v) < r_n \).

Since
\[
\Phi(u, v) \geq \frac{1}{p'} \left( \int_\Omega |\Delta u|^{p(x)} dx + m_0 \int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega \rho_1(x)|u|^{p(x)} dx \right) \\
+ \frac{1}{q'} \left( \int_\Omega |\Delta v|^{q(x)} dx + m_0 \int_\Omega |\nabla v|^{q(x)} dx + \int_\Omega \rho_2(x)|v|^{q(x)} dx \right) \\
\geq \frac{M^-}{p'} \left[ \int_\Omega (|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + \rho_1(x)|u|^{p(x)}) dx \right] \\
+ \frac{M^-}{q'} \left[ \int_\Omega (|\Delta v|^{q(x)} + |\nabla v|^{q(x)} + \rho_2(x)|v|^{q(x)}) dx \right]
\]

(24)

according to Proposition 3, for \( u \in X \), we infer that

If \( \|u\|_{p(x)} \geq 1, \|u\|_{p(x)}^{p^-} \leq \int_\Omega (|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + \rho_1(x)|u|^{p(x)}) dx \leq \|u\|_{p(x)}^{p^-}, \)

(25)

if \( \|u\|_{p(x)} \leq 1, \|u\|_{p(x)}^{p^-} \leq \int_\Omega (|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + \rho_1(x)|u|^{p(x)}) dx \leq \|u\|_{p(x)}^{p^-}. \)

(26)

Then,

\[
\frac{M^-}{p'} \min \left\{ \|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^-} \right\} + \frac{M^-}{q'} \min \left\{ \|v\|_{q(x)}^{q^-}, \|v\|_{q(x)}^{q^-} \right\} < r_n
\]

(27)

Thus,

\[
\frac{M^-}{p'} \min \left\{ \|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^-} \right\} < r_n,
\]

(28)

\[
\frac{M^-}{q'} \min \left\{ \|v\|_{q(x)}^{q^-}, \|v\|_{q(x)}^{q^-} \right\} < r_n.
\]

(29)

From (12)

\[
\|u(x)\| < K \left( \frac{r_n p^+}{M^-} \right)^{(1/p^-)},
\]

(30)

\[
\|v(x)\| < K \left( \frac{r_n q^+}{M^-} \right)^{(1/q^-)},
\]

thus

\[
|u(x)| + |v(x)| < K \left( \left( \frac{p^+}{M^-} \right)^{(1/p^-)} + \left( \frac{q^+}{M^-} \right)^{(1/q^-)} \right) r_n^{1 \min \{p^- q^-\}} = \xi_n.
\]

(31)

Then, the inclusion of sets is valid:

\[
\Phi^{-1}((-\infty, r_n]) \subseteq \{ (u, v) \in X | |u(x)| + |v(x)| < \xi_n \}.
\]

(32)

\[
\varphi(r_n) \leq \frac{\sup_{\Phi(u, v) < \xi_n} \int_\Omega F(x, u, v) dx}{r_n} \leq \left[ K \left( \left( \frac{p^+}{M^-} \right)^{(1/p^-)} + \left( \frac{q^+}{M^-} \right)^{(1/q^-)} \right) \right]^{\min \{p^- q^-\}} \frac{\sup_{|u|, |v| < \xi_n} \int_\Omega F(x, s, t) dx}{\xi_n^{\min \{p^- q^-\}}}
\]

(33)

Put

\[
y = \lim_{r \to \infty} \inf \varphi(r).
\]

(34)
From (22) and (33), we can deduce

\[
\gamma \leq \lim_{n \to \infty} \inf \varphi(r_n) \leq \left[ K \left( \left( \frac{p^*}{M^*} \right)^{(1/p^*)} + \left( \frac{q^*}{M^*} \right)^{(1/q^*)} \right) \right]^{\min\{p^*, q^*\}} \cdot \lim_{n \to \infty} \frac{\int_\Omega \sup_{|s|, |t| < \xi_n} F(x, s, t) \, dx}{\xi_n^{\min\{p^*, q^*\}}} \\
= a \left[ K \left( \left( \frac{p^*}{M^*} \right)^{(1/p^*)} + \left( \frac{q^*}{M^*} \right)^{(1/q^*)} \right) \right]^{\min\{p^*, q^*\}} < +\infty.
\]

(35)

Thus, \( \Lambda \subset (0, (1/\gamma)) \). Next, it will prove when \( \lambda \in \Lambda \); \( I_\lambda \) is unbounded from below. Indeed, on account of

\[
\frac{1}{\lambda} < a \left[ K \left( \left( \frac{p^*}{M^*} \right)^{(1/p^*)} + \left( \frac{q^*}{M^*} \right)^{(1/q^*)} \right) \right]^{\min\{p^*, q^*\}} \cdot L \beta.
\]

(36)

There exists a column of positive real numbers \( \{\eta_n\} \) and \( \theta > 0 \) such that \( \lim_{n \to \infty} \eta_n = +\infty \) and

\[
\int_\Omega \Delta u_n \phi(x) \, dx = \int_{R(x_0, O_2) \setminus R(x_0, O_1)} \Delta u_n \phi(x) \, dx \leq \theta_1 \eta_n^{p^*}, \\
\int_\Omega \nabla u_n \phi(x) \, dx = \int_{R(x_0, O_2) \setminus R(x_0, O_1)} \nabla u_n \phi(x) \, dx \leq \theta_2 \eta_n^{p^*}, \\
\int_\Omega u_n \phi(x) \, dx = \int_{R(x_0, O_2) \setminus R(x_0, O_1)} u_n \phi(x) \, dx + \int_{R(x_0, O_2) \setminus R(x_0, O_1)} \eta_n \phi(x) \, dx \leq \theta_3 \eta_n^{p^*}.
\]

(39)

Thus,

\[
\int_\Omega |\Delta u_n|^{p(x)} \, dx = \int_{R(x_0, O_2) \setminus R(x_0, O_1)} |\Delta u_n|^{p(x)} \, dx \leq \theta_1 \eta_n^{p^*}, \\
\int_\Omega |\nabla u_n|^{p(x)} \, dx = \int_{R(x_0, O_2) \setminus R(x_0, O_1)} |\nabla u_n|^{p(x)} \, dx \leq \theta_2 \eta_n^{p^*}, \\
\int_\Omega |u_n|^{p(x)} \, dx = \int_{R(x_0, O_2) \setminus R(x_0, O_1)} |u_n|^{p(x)} \, dx + \int_{R(x_0, O_2) \setminus R(x_0, O_1)} \eta_n \phi(x) \, dx \leq \theta_3 \eta_n^{p^*}.
\]

(38)

Now, we define \( u_n \in X \) as follows:

\[
\left\{ \begin{array}{ll}
0, & x \in \Omega \setminus R(x_0, O_2), \\
\eta_n, & x \in R(x_0, O_1),
\end{array} \right.
\]

\[
\frac{\eta_n}{O_2^2 - O_1^2} (O_2^2 - |x - x_0|^2), & x \in R(x_0, O_2) \setminus R(x_0, O_1).
\]

(37)
\[
\Phi(u_n, u_n) \leq \frac{1}{p} \left( \int_{\Omega} |\Delta u_n|^{p(x)} \, dx + m_1 \int_{\Omega} |\nabla u_n|^{p(x)} \, dx + \int_{\Omega} \rho_1(x)|u_n|^{p(x)} \, dx \right) \\
+ \frac{1}{q} \left( \int_{\Omega} |\Delta u_n|^{q(x)} \, dx + m_1 \int_{\Omega} |\nabla u_n|^{q(x)} \, dx + \int_{\Omega} \rho_2(x)|u_n|^{q(x)} \, dx \right) \\
\leq \frac{M^+}{p} (\theta_1 + 2\|\rho_1\|_{\infty} \theta_3) \eta_n^p + \frac{M^+}{q} (\theta_1 + 2\|\rho_2\|_{\infty} \theta_2) \eta_n^q \\
\leq \frac{1}{\left[ K\left( (p^+/M^-)^{(1/p)} + (q^+/M^-)^{(1/q)} \right) \right]^{\min\{p, q\}} \cdot \left( \frac{\eta_n^p}{p L_1} + \frac{\eta_n^q}{q L_2} \right)}
\]

From (A1), we have
\[
\Psi(u_n, u_n) = \int_{\Omega} F(x, u_n, u_n) \, dx \geq \int_{R(x_0, \Omega)} F(x, \eta_n, \eta_n) \, dx.
\]

When \( n \) is large enough, from (37), (40), and (41), we can obtain
\[
I_{\lambda}(u_n, u_n) = \Phi(u_n, u_n) - \lambda\Psi(u_n, u_n)
\]

so
\[
I_{\lambda}(u_n, u_n) = -\infty, \quad \text{as } n \to +\infty.
\]

According to Theorem 1 (b), we know that Theorem 2 is established.

**Theorem 3.** Presume (A1) holds and

- (B1), \( \forall x \in \Omega, F(x, 0, 0) = 0 \)
- (B2), \( \exists x_0 \in \Omega, O_2 > O_1 > 0 \)

and one has
\[
\alpha^0 < L^0 \beta^0,
\]

where
\[ L^0 = \min \{ L_3, L_4 \}, \]
\[ L_3 = \frac{1}{K \left( \left( p^+ / M^+ \right)^{1/p^+} + \left( q^+ / M^- \right)^{1/q^+} \right) \max \{ p^+, q^+ \}} \cdot \frac{1}{M^+ (\partial_1 + \partial_2 + |\rho_1 (x)| \partial_3)}, \]
\[ L_4 = \frac{1}{K \left( \left( p^+ / M^+ \right)^{1/p^+} + \left( q^+ / M^- \right)^{1/q^+} \right) \max \{ p^+, q^+ \}} \cdot \frac{1}{M^+ (\partial_1 + \partial_2 + |\rho_2 (x)| \partial_3)}, \]
\[ \bar{\partial}_1 = \frac{2\pi (N/2)}{N \Gamma (N/2)} \left( \frac{2N}{O_2^N - O_1^N} \right) \left( O_2^N - O_1^N \right), \]
\[ \bar{\partial}_2 = \frac{2\pi (N/2)}{\Gamma (N/2)} \left( \frac{2}{O_2^N - O_1^N} \right) \left( O_2^N - O_1^N \right), \]
\[ \bar{\partial}_3 = \frac{2\pi (N/2)}{\Gamma (N/2)} \left( \frac{O_2^N}{N} + \left( \frac{1}{O_2^N - O_1^N} \right) \int_{O_1}^{O_2} \left( O_2^N - r^2 \right)^{p-1} \cdot r^{N-1} \, dr \right), \]
\[ M^+ = \max \{ 1, m_1 \}, \]
\[ M^- = \min \{ 1, m_0 \}. \]

Then, for every

\[ \lambda \in \Lambda := \frac{1}{K \left( \left( p^+ / M^+ \right)^{1/p^+} + \left( q^+ / M^+ \right)^{1/q^+} \right) \max \{ p^+, q^+ \}} \left( \frac{1}{L^0 \beta^0} \beta^0 \right), \]

problem (1) has infinity solutions converging to 0.

Proof. Obviously, \( \min_{x} \Phi = \Phi (0, 0) = 0. \)

There is a real sequence \( \{ \xi_n \} \) satisfying \( \xi_n \to 0^+ \) as \( n \to +\infty \)
and
\[ \lim_{n \to +\infty} \int_{\Omega} \sup_{|\phi| \leq \xi_n} F (x, s, t) \, dx = a^0 < +\infty. \]

Put \( r_n = (\xi_n \max \{ p^+, q^+ \} \left( K \left( \left( p^+ / M^+ \right)^{1/p^+} + \left( q^+ / M^+ \right)^{1/q^+} \right) \right) \max \{ p^+, q^+ \}) \), for all \( n \in \mathbb{N} \) and \( \delta = \lim_{n \to +\infty} \inf \Phi (r); \)
then, by assumption \((B_2)\), we have

\[ \delta \leq \lim_{n \to +\infty} \inf \Phi (r_n) \leq \left[ K \left( \left( p^+ / M^+ \right)^{1/p^+} + \left( q^+ / M^+ \right)^{1/q^+} \right) \right] \max \{ p^+, q^+ \} \lim_{n \to +\infty} \int_{\Omega} \sup_{|\phi| \leq \xi_n} F (x, s, t) \, dx \]
\[ = a^0 \left[ K \left( \left( p^+ / M^+ \right)^{1/p^+} + \left( q^+ / M^+ \right)^{1/q^+} \right) \right] \max \{ p^+, q^+ \} < +\infty. \]

Hence, \( \Lambda \subseteq (0, (1/\delta)). \)

In the next step, we prove that \( L_3 \) does not have the local minimum at 0. For \( \lambda \in \Lambda \), we have

\[ \frac{1}{\lambda} \leq K \left( \left( p^+ / M^+ \right)^{1/p^+} + \left( q^+ / M^+ \right)^{1/q^+} \right) \max \{ p^+, q^+ \} \cdot L^0 \beta^0, \]

(50)
and there exists a column of positive real numbers \( \{\eta_n\} \longrightarrow 0^+ \) as \( n \longrightarrow +\infty \) and \( \theta > 0 \) such that

\[
\frac{1}{\lambda} < \theta < L^0 \left[ K \left( \left( \frac{p^+}{M^-} \right)^{\left(1/p^+\right)} + \left( \frac{q^+}{M^-} \right)^{\left(1/q^+\right)} \right) \right]^{\max\{p^-,q^+\}} \cdot \frac{\int_{R(x_0,O_t)} F(x,\eta_n,\eta_n) dx}{(\eta_n^p/p^+) + (\eta_n^q/q^+)}.
\]

(51)

Let \( \{u_n(x)\} \) be defined by (38); then, we can obtain

\[
\Phi(u_n,u_n) \leq \left( \frac{\eta_n^p}{p} + \frac{\eta_n^q}{q} \right) \left[ K \left( \left( \frac{p^+}{M^-} \right)^{\left(1/p^+\right)} + \left( \frac{q^+}{M^-} \right)^{\left(1/q^+\right)} \right) \right]^{\max\{p^-,q^+\}} \cdot \frac{1}{\lambda} \int_{R(x_0,O_t)} F(x,\eta_n,\eta_n) dx
\]

(52)

Combining (41), (51), and (52), we have

\[
I_1(u_n,u_n) = \Phi(u_n,u_n) - \lambda \Psi(u_n,u_n)
\]

\[
\leq \frac{1}{\lambda} \left( \frac{\eta_n^p}{p} + \frac{\eta_n^q}{q} \right) \left[ K \left( \left( \frac{p^+}{M^-} \right)^{\left(1/p^+\right)} + \left( \frac{q^+}{M^-} \right)^{\left(1/q^+\right)} \right) \right]^{\max\{p^-,q^+\}} - \lambda \int_{R(x_0,O_t)} F(x,\eta_n,\eta_n) dx
\]

\[
\leq 1 - \lambda \theta \left( \frac{\eta_n^p}{p} + \frac{\eta_n^q}{q} \right) \left[ K \left( \left( \frac{p^+}{M^-} \right)^{\left(1/p^+\right)} + \left( \frac{q^+}{M^-} \right)^{\left(1/q^+\right)} \right) \right]^{\max\{p^-,q^+\}}
\]

\[
\leq 0 = I_1(0,0).
\]

Obviously, according to Theorem 1 (c), we have completed the proof of Theorem 3. Moreover, \( (u_n,u_n) \) is the solution satisfying the conditions and \( \|u_n,u_n\| \longrightarrow 0. \)

Example 1. Let \( \Omega = [(-1, 1)]^2; \) for all \( t > 0, M_1(t) = a + bt \) and \( M_2(t) = c + dt, \) where \( a, b, c, \) and \( d \) are positive

\[
F(s,t) = \begin{cases} 
(a_{n+1})^6 e^{-\left(1/\left(1-(s-a_{n+1})^2-(t-a_{n+1})^2\right)\right)}, & \text{if } (s,t) \in \cup_{n\in{\mathbb{N}}^+} R((a_{n+1},a_{n+1}),1), \\
0, & \text{otherwise},
\end{cases}
\]

where \( a_1 = 3 \) and \( a_{n+1} = n(a_1)^3 + 3(n \geq 1) \) and \( R((a_{n+1},a_{n+1}),1) \) is an open unit ball with its center point at \( (a_{n+1},a_{n+1}). \)

It is clear that \( F \geq 0 \) and \( F \in C^1(\mathbb{R}^2), \) for all \( n \in \mathbb{N}^+. \) The maximum of \( F \) on \( R((a_{n+1},a_{n+1}),1) \) is \( F(a_{n+1},a_{n+1}) = (a_{n+1})^6 e^{-1}. \) Hence,

\[
\lim_{n \to +\infty} \sup_{s,t} \frac{F(a_{n+1},a_{n+1})}{(a_{n+1}^4/2) + (a_{n+1}^5/t^3)} = +\infty.
\]

(56)

numbers and \( m_0 = \min\{a,c\}. \) On \( \Omega, \) \( p(x) \) and \( q(x) \) are defined as

\[
p(x) = x^2 + y^2 + 2, \quad q(x) = x^2 + y^2 + 3.
\]

Define the function \( F: \mathbb{R}^2 \longrightarrow \mathbb{R}\) by

\[
\beta := \lim_{s,t \to +\infty} \sup \frac{\int_{R(x_0,O_t)} F(s,t) dx}{(s^4/2) + (t^4/3)}
\]

\[
= |R(x_0,O_t)| \cdot \lim_{s,t \to +\infty} \frac{F(s,t)}{(s^4/2) + (t^4/3)} = +\infty,
\]

(57)

where \( |R(x_0,O_t)| \) is the measure of \( R(x_0,O_t). \) At the time, \( \forall n \in \mathbb{N}^+, \) and we deduce
have a sequence of unbounded weak solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

Each part of this paper is the result of the joint efforts of QZ and QM. They contributed equally to the final version of the paper. All the authors have read and approved the final manuscript.

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