Linking disjoint axis-parallel segments into a simple polygon is hard too

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Abstract
Deciding whether a family of disjoint axis-parallel line segments in the plane can be linked into a simple polygon (or a simple polygonal chain) by adding segments between their endpoints is NP-hard.

1 Introduction

Given a family $S$ of $n$ closed line segments in the plane, SIMPLE CIRCUIT (respectively, SIMPLE PATH) is the problem of deciding whether these segments can be linked into a simple polygon (respectively, a simple polygonal chain) by adding segments between their endpoints.

Rappaport [4] proved that SIMPLE CIRCUIT is NP-hard if the segments in $S$ are allowed to intersect at their common endpoints, and asked whether the problem remains NP-hard when the segments are disjoint. Later, Bose, Houle, and Toussaint [2] asked whether the related problem SIMPLE PATH is NP-hard when the segments in $S$ are disjoint. Later, Tóth [6] asked about the complexity of SIMPLE PATH again, with a special interest in the case when the segments in $S$ are both disjoint and axis-parallel.

Recently, Akitaya et al. [1] proved that SIMPLE CIRCUIT is NP-hard when the segments in $S$ are disjoint and have only four distinct orientations. Subsequently, the authors of this paper came up with a similar construction [3], and proved that both SIMPLE CIRCUIT and SIMPLE PATH are NP-hard, when the segments in $S$ are disjoint and have only four distinct orientations. Akitaya et al. [1] asked whether SIMPLE CIRCUIT is NP-hard when the segments in $S$ are disjoint and axis-parallel.

In this paper, we prove the following theorem:

**Theorem 1.** SIMPLE CIRCUIT and SIMPLE PATH are both NP-hard even if the segments in $S$ are disjoint and axis-parallel.

We prove the theorem in two steps. First, we modify the construction in Rappaport’s proof of NP-hardness of SIMPLE CIRCUIT on not necessarily disjoint axis-parallel segments [4] to show that the problem remains NP-hard on disjoint axis-parallel segments. Next, we modify the construction further to prove that SIMPLE PATH is also NP-hard on disjoint axis-parallel segments.

Some of the ideas behind our construction in this paper are also used in our previous construction [3]; interested readers may read the short proof there as a warm-up exercise. Following the same setup there, we briefly review Rappaport’s proof in the following, which is based on a polynomial reduction from the NP-hard problem HAMILTONIAN PATH in planar cubic graphs.

For any family $S$ of closed segments in the plane, denote by $V(S)$ the set of endpoints of the segments in $S$. For any two endpoints $p$ and $q$ in $V(S)$, we call the open segment $pq$ a visibility edge if it does not intersect any closed segment in $S$.

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Given a planar cubic graph $G$ with $n \geq 4$ vertices, the reduction [4] first obtains a rectilinear planar layout of the planar graph using an algorithm of Rosenstiehl and Tarjan [5], then constructs a family $S$ of $O(n)$ axis-parallel segments following the rectilinear planar layout, such that $G$ admits a Hamiltonian path if and only if $S$ can be linked into a simple polygon by adding visibility edges between endpoints in $V(S)$. The segments in $S$ are axis-parallel and interior-disjoint, but may intersect at common endpoints. Each endpoint in $V(S)$ is incident to at most one horizontal segment and at most one vertical segment in $S$. Since the rectilinear planar layout has width at most $2n - 4$ and height at most $n$ [5], all coordinates of endpoints in $V(S)$ are integers of magnitude $O(n^2)$; indeed a closer look at the construction [4, Figure 8] shows that $V(S) \subseteq [1, 22n^2] \times [1, 11n]$. The reduction is hence strongly polynomial. Consequently, SIMPLE CIRCUIT is strongly NP-hard, on not necessarily disjoint axis-parallel segments.

2 Modification for SIMPLE CIRCUIT

To show that SIMPLE CIRCUIT remains strongly NP-hard on disjoint segments, we will transform the family $S$ of interior-disjoint axis-parallel segments, which Rappaport constructed, into a family $S'$ of disjoint axis-parallel segments in polynomial time, such that $S$ can be linked into a simple polygon if and only if $S'$ can be linked into a simple polygon. This transformation can be viewed as a reduction from SIMPLE CIRCUIT on one type of input to the same problem on another type of input.

To obtain $S'$ from $S$, we first scale the integer coordinates of all segment endpoints by a factor of 42, then locally modify each intersection between a horizontal segment and a vertical segment into a gadget. The gadgets come in four variants, one for each possible orientation of an intersection; see Figure 1.
Figure 1: Gadgets of four different orientations for the four intersections (i.e., the four corners) of a unit square, scaled up by a factor of 42 and illustrated on a $42 \times 42$ grid.
Figure 2: The gadget for the intersection \( o \) between a horizontal segment \( oa \) and a vertical segment \( ob \) illustrated on a \( 21 \times 21 \) grid. Left: The common endpoint \( o = (0,0) \) is split to \( o' = (-2,0) \), \( o'' = (0,1) \). The two endpoints illustrated as black dots are \( u' = (6,4-\delta) \) and \( v' = (15,17+\frac{1}{3}+\delta) \). The 16 added segments include the left group of five vertical segments, the middle group of five horizontal segments, the right group of three vertical segments, the top horizontal segment by itself, and the left group of two horizontal segments. The top horizontal segment has left endpoint \( (1,19) \) and right endpoint \( (20,19) \). Right: An alternating path of segments and visibility edges in the gadget.

By symmetry, it suffices to describe in detail only one variant of the gadget. Refer to Figure 2. For the intersection \( o \) between a horizontal segment \( oa \) and a vertical segment \( ob \), where \( o \) is the left endpoint of \( oa \) and the lower endpoint of \( ob \), the corresponding gadget is constructed as follows:

- Separate the two segments \( oa \) and \( ob \) into two disjoint segments \( o' a \) and \( o'' b \), by splitting their common endpoint \( o = (0,0) \), then moving one to \( o' = (-2,0) \) and the other to \( o'' = (0,1) \).
- Add 16 segments with integer coordinates as indicated by the grid lines in the figure, except that
  - the lower endpoint of the vertical segment at \( x = 6 \) is \( u' = (6,4-\delta) \),
  - the upper endpoint of the vertical segment at \( x = 15 \) is \( v' = (15,17+\frac{1}{3}+\delta) \),

where \( \delta \) is determined by Lemma 1 below.

In the presence of \( S' \), we say that two points \( p \) and \( q \) can see each other if the open segment \( pq \) is disjoint from all closed segments in \( S' \), and we say that two segments \( A \) and \( B \) in \( S' \) can see each other if at least one of the four pairs of endpoints, one of \( A \) and one of \( B \), can see each other. The gadgets we constructed have the following property of mutual invisibility:

**Lemma 1.** With \( \delta = 1/(c \cdot n^2) \) for a sufficiently large integer constant \( c > 0 \), the endpoint \( u' = (6,4-\delta) \) cannot see any endpoints outside the gadget through the gap between \( o' = (-2,0) \) and \( o'' = (0,1) \), and the endpoint \( v' = (15,17+\frac{1}{3}+\delta) \) cannot see any endpoints outside the gadget through the gap between \( (17,18) \) and \( (20,19) \).

**Proof.** Refer to Figure 3. Let \( u = (6,4) \), which is collinear with \( o' = (-2,0) \) and \( o'' = (0,1) \). Note that \( |uu'| = \delta \), \( |o''u'| > 6 \), and \( \angle o''u'u > \pi/2 \). Thus

\[
\angle uo''u' < \tan \angle uo''u' < \frac{|uu'|}{|o''u'|} < \frac{\delta}{6}.
\]
Consider the ray that starts from \( o'' \) and goes through \( o' \). Rotate this ray clockwise about \( o'' \) for a positive angle till it goes through another integer point \( p \) in \( V(S') \). The area of the triangle \( o'o''p \) is at least \( \frac{1}{2} \) since all three endpoints have integer coordinates. Recall that the endpoints of all segments in \( S \) are in the range \([1, 22n^2] \times [1, 11n]\), and is scaled by a factor of 42 by the transformation from \( S \) to \( S' \). Thus we have
\[
|po''| < 42 \cdot (22n^2 + 11n) < 50 \cdot 33n^2.
\]
Also note that \( |oo'| < 3 \). Thus
\[
\angle oo'p > \sin \angle oo''p = \frac{1}{2} \cdot \frac{\text{area}(oo''p)}{|oo'| \cdot |po''|} > \frac{1}{3 \cdot 50 \cdot 33n^2} > \frac{1}{5000n^2}.
\]

Let \( \delta = 1/(c \cdot n^2) \) for a sufficiently large integer \( c > 0 \) such that \( \angle oo'u < \angle oo''p \). Then the ray \( u'oo'' \) splits the angle \( \angle oo''p \). Thus the cone with the angle \( \angle oo'u' \) does not contain any integer endpoints in \( V(S') \). Thus \( u' \) cannot see any integer endpoints through the gap between \( o' \) and \( o'' \).

Let \( v = (15, 17 + \frac{1}{7}) \), which is collinear with \((17, 18)\) and \((20, 19)\). Then \( |vv'| = \delta \). By a similar analysis, we can also guarantee that \( v' \) cannot see any integer endpoints from other gadgets through the gap between \((17, 18)\) and \((20, 19)\).

Moreover, for an intersection of the orientation as illustrated in Figure 2, the slope of a viewing line through \( u' \) and its gap is around \( \frac{1}{2} \), and the slope of a viewing line through \( v' \) and its gap is around \( \frac{1}{5} \). With \( \delta = 1/(c \cdot n^2) \) for a sufficiently large integer \( c > 0 \), no two endpoints \( u' \) or \( v' \) from two different gadgets can see each other, because the eight narrow ranges of slopes of viewing lines, as illustrated by the red lines in Figure 1, are disjoint for intersections of different orientations.

The following lemma shows that our local transformation preserves linkability:

**Lemma 2.** \( S \) can be linked into a simple polygon if and only if \( S' \) can be linked into a simple polygon.

**Proof.** We first prove the direct implication. Suppose that \( S \) can be linked into a simple polygon. For each visibility edge of this polygon between two endpoints in \( V(S) \), we add a visibility edge between the corresponding endpoints in \( V(S') \). For each endpoint \( o \) incident to two segments \( oa \) and \( ob \) in \( S \), we link the segments in the corresponding gadget in \( S' \) following the alternating path as illustrated in Figure 2 right. Then \( S' \) is also linked into a simple polygon.

We next prove the reverse implication. Suppose that \( S' \) can be linked into a simple polygon. Refer to Figures 4, 5, 6, 7, 8, 9. The linking of the segments along the sequence of lengths as illustrated in Figures 4 through 9 is inevitable in each gadget. Then, following the other visibility edges of the simple polygon through \( S' \), which are outside and between the gadgets, \( S \) can be linked into a simple polygon too.
Figure 4: Linking the middle group of five horizontal segments. Each of the two length-1 segments can only see the length-7 segment between them and one other segment. Thus the five segments must be linked consecutively, forming a sequence of 3, 1, 7, 1, 3 in lengths. There is some flexibility in the choice of which endpoints to link between two consecutive segments in the sequence. The combination of visibility edges illustrated here and in subsequent figures may be one of many possibilities unless specified.

Figure 5: Linking the left group of five vertical segments. Among the unlinked neighbors, each of the two length-1 segments can only see the length-17 segment between them and one other segment. Thus the five segments must also be linked consecutively, forming a sequence of 5, 1, 17, 1, 3 in lengths.
Figure 6: Linking the right group of three vertical segments. The lowest segment in the middle group of five horizontal segments must now be linked to the lower endpoint of the length-17 vertical segment. Consequently, the length-1 segment in the right group of three vertical segments can only be linked to the other two segments in the same group, and the sequence extends to 3, 1, 7, 1, 3, 17, 1, 5.+

Figure 7: Linking the top horizontal segment. To avoid creating a loop or a dead end, the highest segment in the middle group of five horizontal segments must now be linked to the rightmost segment in the left group of five vertical segments, and the two sequences merge into a single sequence 5, 1, 17, 1, 3, 3, 1, 7, 1, 3, 17, 1, 5. Then the endpoint $v' = (15, 17 + \frac{1}{2} + \delta)$ must be linked to the right endpoint (20, 19) of the top horizontal segment; linking it to the left endpoint (1, 19) would block further linking to the length-1 segment in the left group of two horizontal segments.
Figure 8: Linking the left group of two horizontal segments. As required by the length-1 horizontal segment in the left group, the sequence must extend to $5^+, 1, 17, 1, 3, 1, 7, 1, 3, 17, 1, 5^+, 19, 1, 3$.

Figure 9: Linking the horizontal segment $\sigma' a$ and the vertical segment $\sigma'' b$. Finally, the sequence extends at both ends to $\sigma'$ and $\sigma''$. In particular, the length-3 horizontal segment in the left group must be linked to $\sigma'' = (0, 1)$, and the length-5$^+$ vertical segment in the left group of five vertical segments must be linked to $\sigma' = (-2, 0)$ through its lower endpoint $u' = (6, 4 - \delta)$. 
Recall that the coordinates of endpoints in \( V(S) \) are integers of magnitude \( O(n^2) \). After the transformation, all endpoints in \( V(S') \) except the black dots have integer coordinates too. We can scale all coordinates by another factor of \( O(1/\delta) = O(n^2) \), so that all endpoints in \( V(S') \) including the black dots have integer coordinates of magnitude \( O(n^4) \), and the reduction remains strongly polynomial. Thus \textsc{Simple Circuit} is strongly NP-hard, even if the input segments are disjoint and axis-parallel.

3 Modification for \textsc{Simple Path}

To prove that \textsc{Simple Path} is also NP-hard, we use almost the same transformation from \( S \) to \( S' \) as before, except two changes:

1. Increase the initial scaling factor from 42 to 80, and correspondingly decrease the distance \( \delta \) when constructing the gadgets, to ensure that Lemma 1 still holds.

2. Select an arbitrary gadget, and replace it by an extended gadget of the same orientation. For example, if the gadget is as illustrated in Figure 2, then the extended gadget is as illustrated in Figure 10. A closer look at the construction [4, Figure 8] shows that there is at least one intersection in \( S \); correspondingly, we have at least one gadget in \( S' \) ready for this upgrade.

The endpoints of the segments inside the extended gadget have integer coordinates as indicated by the grid lines, except six endpoints illustrated as black dots:

- the lower endpoint of the vertical segment at \( x = 6 \) is \( u' = (6, 4 - \delta) \),
- the two vertical segments at \( x = 8 \) and \( x = 38 \) have lower endpoints at \( y = \epsilon \) and upper endpoints at \( y = 18 - \epsilon \), where \( \epsilon = \frac{1}{1000} \),
- the upper endpoint of the vertical segment at \( x = 36 \) is \( v'' = (36, 18 - 3\epsilon + \delta) \).

The purpose of the first change is to make room for additional segments in the extended gadget. Note that the endpoint \( v'' = (39, 18 - 3\epsilon + \delta) \) is at a distance of \( \delta \) above the point \( (36, 18 - 3\epsilon) \) which is collinear with the two endpoints \( (38, 18 - \epsilon) \) and \( (39, 18) \). By an analogous argument as in Lemma 1, with \( \delta = 1/(c \cdot n^2) \) for a sufficiently large integer constant \( c > 0 \), we can ensure that \( u' \) and \( v'' \) cannot see any endpoints outside the extended gadget. Clearly, the reduction remains strongly polynomial.

The following lemma is analogous to Lemma 2:

\textbf{Lemma 3.} \( S \) can be linked into a simple polygon if and only if \( S' \) can be linked into a simple polygonal chain.

\textit{Proof.} We first prove the direct implication. Suppose that \( S \) can be linked into a simple polygon. We link the segments in \( S' \) as before, except that in the extended gadget we link the segments as illustrated in Figure 14. Then \( S' \) is linked into a simple polygonal chain starting and ending in the extended gadget.

We next prove the reverse implication. Suppose that \( S' \) can be linked into a simple polygonal chain. As before, the linking of the segments in each ordinary gadget into an alternating path is inevitable. Refer to Figures 11, 12, 13, 14. The linking of the segments in the extended gadget into two disjoint alternating paths is also inevitable. Then, following the other visibility edges of the simple polygonal chain through \( S' \), which are outside and between the gadgets, \( S \) can be linked into a simple polygon.

Thus \textsc{Simple Path} is also NP-hard. This completes the proof of Theorem 1.
Figure 10: The extended gadget for the intersection $o$ between a horizontal segment $oa$ and a vertical segment $ob$ illustrated on a $40 \times 20$ grid (due to the scaling factor of 80, there is a free space of $40 \times 40$ for each gadget, which is sufficient for both the $40 \times 20$ grid of the extended gadget and the $21 \times 21$ grid of the ordinary gadgets). The common endpoint $o = (0, 0)$ is split to $o' = (-2, 0), o'' = (0, 1)$. The top horizontal segment has left endpoint $(1, 18)$ and right endpoint $(39, 18)$. The six endpoints illustrated as black dots are, from left to right, $u' = (6, 4 - \delta), (8, \epsilon), (8, 18 - \epsilon), v'' = (36, 18 - 3\epsilon + \delta), (38, \epsilon), (38, 18 - \epsilon)$. Except the leftmost black dot $u'$, the other five black dots are illustrated with integer $y$-coordinates here and in subsequent figures for visual clarity.

Figure 11: Consider the left group of five horizontal segments and the group of seven vertical segments to their left. There are five length-1 segments among these 12 segments. Each of these length-1 segments can see either one or both of the length-$(18 - 2\epsilon)$ vertical segment and the length-7 horizontal segment, plus one other segment. Since the two long segments can accommodate at most four neighbors, at least one of the five length-1 segments must be linked to only one neighbor, and hence is either the starting or the ending segment of the polygonal chain. The situation is similar for the middle group of five horizontal segments and the group of seven vertical segments to their right. Among these 12 segments, there are also five length-1 segments, and one of the five must be either the starting or the ending segment of the polygonal chain. For example, the polygonal chain could start and end at the two intervals illustrated in red, then the other eight length-1 segments must be linked to the two long vertical segments and the two long horizontal segments, which become unavailable for further linking. In particular, the length-$(18 - 2\epsilon)$ vertical segment on the left is now a barrier.
Figure 12: Since the starting and the ending segments of the polygonal chain have both been accounted for, each remaining segment must be linked to two neighbors. Consider the right group of five horizontal segments and the group of five vertical segments to their right. Among these 10 segments, there are four length-1 segments which must be linked to the length-7 horizontal segment and the length-(18 − 2ε) vertical segment. Then this length-(18 − 2ε) vertical segment also becomes a barrier. Since the two length-(18 − 2ε) vertical segments at $x = 8$ and $x = 38$ are only a distance of $ε = \frac{1}{1000}$ away from the two horizontal segments at $y = 0$ and $y = 18$, the segments bounded by them are all isolated, except the black dot $v'' = (36, 18 − 3ε + δ)$, which must be linked to the right endpoint $(39, 18)$ of the top horizontal segment.

Figure 13: The isolated segments have only two ways out, one through the length-(18 − 2ε) vertical segment on the left, and the other through the endpoint $v''$ on the right. Thus all of them must be linked internally into two disjoint alternating paths.
Figure 14: The two disjoint alternating paths must then extend through the remaining segments, and finally exit the extended gadget at $o'$ and $o''$, and will be joined into a single polygonal chain through the segments outside the extended gadget.

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