Metric curvature of infinite branched covers

Daniel Allcock*
25 May 1999

Abstract
We study branched covering spaces in several contexts, proving that under suitable circumstances the cover satisfies the same upper curvature bounds as the base space. The first context is of a branched cover of an arbitrary metric space that satisfies Alexandrov’s curvature condition CAT(κ), over an arbitrary complete convex subset. The second context is of a certain sort of branched cover of a Riemannian manifold over a family of mutually orthogonal submanifolds. In neither setting do we require that the branching be locally finite. We apply our results to hyperplane complements in several complex manifolds of nonpositive sectional curvature. This implies that two moduli spaces arising in algebraic geometry are aspherical, namely that of the smooth cubic surfaces in \( \mathbb{C}P^3 \) and that of the smooth complex Enriques surfaces.

1. Introduction
The purpose of this paper is to establish a basic result in the theory of metric space curvature in the sense of Alexandrov, together with several applications in algebraic geometry. A commonly observed phenomenon is that “taking a branched cover of almost anything can only introduce negative curvature”. One can see this phenomenon in elementary examples using Riemann surfaces, and the idea also plays a role in the construction [14] of exotic manifolds with negative sectional curvature. In this paper we work in the maximal generality in which sectional curvature bounds make sense, namely in the comparison geometry of Alexandrov. In this setting we will establish a very strong theorem concerning the persistence of upper curvature bounds in branched covers. We include examples showing that an important completeness hypothesis cannot be dropped; our examples also disprove several claims in the literature.

A simple way to build a cover \( \hat{Y} \) of a space \( \hat{X} \) branched over \( \Delta \subseteq \hat{X} \) is to take any covering space \( Y \) of \( \hat{X} - \Delta \) and define \( \hat{Y} = Y \cup \Delta \). We call \( \hat{Y} \) a simple branched cover of \( \hat{X} \) over \( \Delta \). Our main result (theorem 3.1) states that if \( \hat{X} \) satisfies Alexandrov’s CAT(\( \kappa \)) condition and \( \Delta \) is complete and convex then the natural metric on \( \hat{Y} \) also satisfies CAT(\( \kappa \)). (When \( \kappa > 0 \) we impose a minor hypothesis on the diameters of \( \hat{X} \) and \( \hat{Y} \).) See section 2 for a discussion of Alexandrov’s criteron and other background; we follow the conventions of the book [4] by Bridson and Häfliger. Most of section 3 is devoted to establishing this theorem. Only partial results can be obtained without the completeness hypothesis, and we give these results together with counterexamples when completeness is not assumed. We also give a local version, theorem 3.6, which allows one to work with branched covers more complicated than the simple sort introduced above, and also avoids any diameter constraints on \( \hat{X} \) and \( \hat{Y} \). One interesting twist is that one must take \( \Delta \) to be locally complete in order to obtain even local results.

The question which motivated this investigation is whether the moduli space of smooth cubic surfaces in \( \mathbb{C}P^3 \) is aspherical (i.e., has contractible universal cover). The answer is yes, and our argument also establishes the analogous result for the moduli space of smooth complex Enriques surfaces. To prove these claims, we use the fact that each of these moduli spaces is known

---

* Supported in part by an NSF Postdoctoral Fellowship

MSC: 53C23 (14J28, 57N65)

Keywords: branched cover, ramified cover, Alexandrov space, cubic surface, Enriques surface
to be covered by a Hermitian symmetric space with nonpositive sectional curvature, minus an arrangement of complex hyperplanes. In each case the hyperplanes have the property that any two of them are orthogonal wherever they meet. In section 5 we show that such a hyperplane complement is aspherical. We actually prove a more general result, in the setting of a complete simply connected Riemannian manifold \( \hat{M} \) of non-positive sectional curvature, minus the union \( \mathcal{H} \) of suitable submanifolds which are mutually orthogonal, complete, and totally geodesic.

The basic idea is to try to apply standard nonpositive curvature techniques like the Cartan-Hadamard theorem to the universal cover \( \hat{N} \) of \( M = \hat{M} - \mathcal{H} \). The fundamental obstruction is that \( \hat{N} \) is not metrically complete. This problem can be circumvented by passing to its metric completion \( \hat{\hat{N}} \), but this introduces problems of its own. In particular, one cannot use the techniques of Riemannian geometry. But it is still a metric space and it turns out to have curvature \( \leq 0 \), in the sense that it satisfies Alexandrov’s CAT(0) condition locally. It is then an easy matter to show that \( \hat{N} \) and \( \hat{\hat{N}} \) are contractible. In summary, to study the topology of the manifold \( N \) it turns out to be natural and useful to study the non-manifold \( \hat{N} \) and use metric-space curvature rather than Riemannian curvature.

The connection between the very general treatment of metric curvature and the applications lies in our study of the curvature of \( \hat{N} \). For this it suffices to work locally; the reader should imagine a closed ball \( B \) in \( \mathbb{C}^n \), equipped with some Riemannian metric, minus the coordinate hyperplanes. The metric completion of the universal cover of the hyperplane complement can be obtained by first taking a simple branched cover of \( B \) over one hyperplane, then taking a simple branched cover of this branched cover over (the preimage of) the second hyperplane, and so on. If the hyperplanes are mutually orthogonal and totally geodesic then our main theorem may be used inductively to study the curvature of the iterated branched cover. There are some minor technical issues, which we chase down in section 4. Note that the base space in each of the sequence of branched covers fails to be locally compact (except in the first step). This means that the inductive argument actually requires a theorem treating branched covers of spaces considerably more general than manifolds.

I would like to thank Jim Carlson and Domingo Toledo for their interest in this work, and for the collaboration [2] that suggested these problems. I would also like to thank Richard Borcherds, Misha Kapovich and Bruce Kleiner for useful conversations. Finally, I am grateful to Brian Bowditch for pointing out an error in an early version.

2. Background

Let \((X, d)\) be a metric space. A path in \(X\) is a continuous map from a nonempty compact interval to \(X\); its initial (resp. final) endpoint is the image of the least (resp. greatest) element of this interval. We sometimes describe a path as being from its initial endpoint to its final endpoint. When we wish to mention its endpoints but not worry about which is which, we describe the path as joining one endpoint and with the other. When we speak of a point of a path we mean a point in its image. If \(\gamma\) is a path in \(X\) with domain \([a, b]\) then we define its length to be

\[
\ell(\gamma) = \sup \left\{ \sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_{i})) \mid a = t_0 \leq t_1 \leq \cdots \leq t_N = b, \quad N \geq 1 \right\}.
\]

This is an element of \([0, \infty]\). We call \(X\) a length space and \(d\) a path metric if for all \(x, y \in X\) and all \(\varepsilon > 0\) there is a path of length \(< d(x, y) + \varepsilon\) joining \(x\) and \(y\). All of the spaces in this paper are length spaces. An important class of length spaces is that of connected Riemannian manifolds. Given such a manifold \(M\), one defines the ‘length’ of each piecewise differentiable path in \(M\) as a
A path $\gamma$ is called a geodesic parameterized proportionally to arclength if there exists $k \geq 0$ such that $d(\gamma(s), \gamma(t)) = k|s - t|$ for all $s$ and $t$ in the domain of $\gamma$. We call $\gamma$ a geodesic if $k = 1$. We sometimes regard two geodesics as being the same if they differ only by an isometry of their domains. For example, we use this convention in assertions about uniqueness of geodesics in $X$. Similarly, we will sometimes refer to the image of $\gamma$, rather than $\gamma$ itself, as a geodesic. Sometimes we will even refer to a path as a geodesic when it is only a geodesic parameterized proportionally to arclength. We say that $X$ is a geodesic space if any two of its points are joined by a geodesic. Most of the spaces in this paper are geodesic. A subset $Y$ of $X$ to arclength. We say that

$$\frac{\pi}{2} \leq \kappa$$

for details.

A complete simply connected length space of curvature $\kappa$ has an edge opposite it. This terminology does not reduce to the usual notion of an altitude of a triangle. We sometimes regard two geodesics as being the same if they differ only by an isometry of their domains. For example, we use this convention in assertions about uniqueness of geodesics in $X$. Similarly, we will sometimes refer to the image of $\gamma$, rather than $\gamma$ itself, as a geodesic. Sometimes we will even refer to a path as a geodesic when it is only a geodesic parameterized proportionally to arclength. We say that $X$ is a geodesic space if any two of its points are joined by a geodesic. Most of the spaces in this paper are geodesic. A subset $Y$ of $X$ is called convex (in $X$) if any two points of $Y$ are joined by a geodesic of $X$ and every such geodesic actually lies in $Y$.

A triangle $T$ in $X$ is a triple $(\gamma_1, \gamma_2, \gamma_3)$ of geodesics of $X$, called the edges of $T$, such that for each $i$, the final endpoint of $\gamma_i$ is the initial endpoint of $\gamma_{i+1}$; the vertex of $T$ opposite $\gamma_i$ is defined to be the common final endpoint of $\gamma_{i+1}$ and initial endpoint of $\gamma_{i-1}$. Here, subscripts should be read modulo 3. It is possible for a vertex to be opposite more than one edge; this occurs when an edge of $T$ has length 0. An altitude of $T$ is a geodesic of $X$ joining a vertex of $T$ and a point of an edge opposite it. This terminology does not reduce to the usual notion of an altitude of a triangle in the Euclidean plane when $X = \mathbb{R}^2$. Since we will not use the classical meaning of the term this should cause no confusion.

Now we define the notion of a metric space satisfying a bound on its curvature. This elegant idea of Alexandrov [1] captures much of the flavor of an upper bound on the sectional curvature of a Riemannian manifold, in the setting of much more general metric spaces. The idea is that triangles should be thinner than comparable triangles in some standard space like the Euclidean plane. For each $\kappa \in \mathbb{R}$, let $M_\kappa^n$ be the (unique up to isometry) complete simply connected Riemannian $n$-manifold with constant curvature $\kappa$. For $\kappa = 0$ or $\kappa > 0$ this space is $\mathbb{R}^n$ or the sphere of radius $1/\sqrt{\kappa}$. For $\kappa < 0$ it is the hyperbolic plane equipped with a suitable multiple of its standard metric. If $T$ is a triangle in $X$ then a comparison triangle $T'$ for $T$ in $M_\kappa^n$ is a triangle $(\gamma'_1, \gamma'_2, \gamma'_3)$ in $M_\kappa^n$ such that the domains of $\gamma_i$ and $\gamma'_i$ coincide for each $i$. In particular we have $\ell(\gamma_i) = \ell(\gamma'_i)$. Comparison triangles exist unless $\kappa > 0$ and $T$ has perimeter $> 2\pi/\sqrt{\kappa}$. When they exist they are unique up to isometry unless $\kappa > 0$ and $T$ has an edge of length $\pi/\sqrt{\kappa}$. We will arrange things later so that we will not need to worry about the existence or uniqueness of comparison triangles. We will follow the usual convention of taking $2\pi/\sqrt{\kappa}$ and similar expressions to represent $\infty$ when $\kappa \leq 0$. This allows many assertions to be phrased more uniformly.

For each $i$, we say that $\gamma'_i$ is the edge of $T'$ corresponding to $\gamma_i$. If $p$ is a point of $\gamma_i$ then the point $p'$ associated to $p$ on the edge $\gamma'_i$ is $\gamma'_i(t)$, where $t$ is such that $\gamma_i(t) = p$. Note that a choice of edge containing $p$ is essential for this construction, since $p$ may lie on more than one edge of $T$. We say that $T$ satisfies CAT($\kappa$) if $T$ has perimeter $< 2\pi/\sqrt{\kappa}$ and for any two edges $\alpha$ and $\beta$ of $T$ and points $p$ on $\alpha$ and $q$ on $\beta$, we have $d(p, q) \leq d(p', q')$. Here $p'$ and $q'$ are the points of $\alpha'$ and $\beta'$ corresponding to $p$ and $q$ and $\alpha'$ and $\beta'$ are the edges corresponding to $\alpha$ and $\beta$ in a comparison triangle for $T$. We say that $X$ satisfies (or is) CAT($\kappa$) if $X$ is geodesic and every triangle in $X$ of perimeter $< 2\pi/\sqrt{\kappa}$ satisfies CAT($\kappa$). The intuitive meaning of this condition is that $X$ is “at least as negatively curved” as $M_\kappa^n$.

We say that $X$ is locally CAT($\kappa$), or has curvature $\leq \kappa$, if each point of $X$ has a convex CAT($\kappa$) neighborhood. Our interest in spaces with curvature bounded above stems from the following very general version of the Cartan-Hadamard theorem, proven by Bridson and Haefliger [4, II.5.1].

**Theorem 2.1.** A complete simply connected length space of curvature $\leq \kappa \leq 0$ is CAT($\kappa$).
Note that part of the conclusion of theorem 2.1 is that the space is geodesic, which is very important and not at all obvious. This statement of the theorem implies the version quoted in the introduction: contractibility follows from the CAT(κ) condition for κ ≤ 0, and if the given metric isn’t a path metric then it induces one and the two metrics define the same topology. Another very important theorem in Alexandrov’s subdivision lemma, a proof of which appears in [4].

**Theorem 2.2 (Alexandrov).** Let T be a triangle in a metric space, with an altitude α. If T has perimeter < 2π/√κ and both of the triangles into which A subdivides T satisfy CAT(κ), then T also satisfies CAT(κ).

Sometimes this is stated with the conditions that the two subtriangles have perimeters < 2π/√κ, but we have made this condition a part of the definition of CAT(κ).

3. Simple branched covers

The purpose of this section is to show that under very general conditions a branched cover satisfies the same upper bounds on curvature as its base space. Our precise formulation of this idea is theorem 3.1. The statement is slightly stronger than the version given in the introduction because it turns out that the completeness of the branch locus is needed for the existence of geodesics but not for the fact that all triangles in the cover satisfy CAT(κ). The other result of this section is a local version of this result, theorem 3.6. Since we do not need this result we will merely state it and give the idea of its proof.

The branched covering spaces we treat here are what we call simple branched covers. The basic idea is very simple: one removes a closed subset Δ from a length space X, takes a cover of what is left, and then attaches a copy of Δ in the obvious way. Formally, if X is a length space and Δ is a closed subset of X then we say that π : Y → X is a simple branched cover of X over Δ if Y is a length space and π satisfies the following two conditions. First, the restriction of π to π−1(Δ) must be a locally isometric covering map. Second, we require that d(y, z) = d(πy, πz) if at least one of y, z ∈ Y lies in π−1(Δ). It follows from the second condition that the restriction of π to π−1(Δ) is an isometry. We will identify Δ with its preimage under π and write X and Y for X − Δ and Y − Δ, respectively. It is easy to see that any simple branched cover is distance non-increasing.

If we are given X and Δ as above, and Y is any covering space of X = X − Δ, then there is a unique metric on Y = Y ∪ Δ such that the obvious map π : Y → X is a simple branched cover of X over Δ. This may be constructed as follows. First, each component of X carries a unique path metric under which its inclusion into X is a local isometry. (This uses the fact that Δ is closed.) Second, each component of Y carries a natural path metric, the unique such metric under which the covering map is a local isometry. Third, for y, z ∈ Y with at least one of them in Δ we define

\[ d(y, z) = d(\pi y, \pi z). \]

Finally, if x, z ∈ Y then we define d(x, z) as

\[ \inf \left( \{ d(x, y) + d(y, z) \mid y \in \Delta \} \cup \{ \ell(\gamma) \mid \gamma \text{ is a path in a component of } Y \text{ joining } x \text{ and } z \} \right). \]

One can check that d is a path metric on Y and that π is a simple branched covering. Our main theorem is a sufficient condition for Y to be CAT(κ):

**Theorem 3.1.** Suppose Δ is a closed convex subset of a CAT(κ) space X and let π : Y → X be a simple branched cover of X over Δ. If κ > 0 then assume also that Diam(X) < π/2√κ and Diam(Y) < 2π/3√κ. Then

(i) Every triangle in Y satisfies CAT(κ).
(ii) If $\Delta$ is complete then $\hat{Y}$ is geodesic and hence $\text{CAT}(\kappa)$.

Example: The completeness condition in (ii) cannot be dropped, because of the following example. Take $\hat{X}$ to be the set of points $(x, y) \in \mathbb{R}^2$ with $x \geq 0$ and $y > 0$, together with the point $(1, 0)$. Let $\Delta$ be the positive $y$-axis. Then $\hat{X}$ is a convex subset of $\mathbb{R}^2$, hence $\text{CAT}(0)$, and $\Delta$ is a closed convex subset of $\hat{X}$. The set $X = \hat{X} - \Delta$ is contractible, so any cover of it is a union of disjoint copies of it. Taking $Y$ to be the cover with 2 sheets, $\hat{Y}$ is isometric to the upper half plane in $\mathbb{R}^2$ together with the points $(\pm 1, 0)$. There is no geodesic joining these two points, so $\hat{Y}$ is not a geodesic space. This provides a counterexample to several assertions in the literature, such as [8, 4.3–4.4], [10, Lemma 1.1] and [6, Lemma 2.4].

Example: Although the space $\hat{Y}$ of the previous example is not geodesic, it still has curvature $\leq 0$. The following example shows that even this may fail if $\Delta$ is not complete. We take $\hat{X}$ to be the set of points $(x, y, z) \in \mathbb{R}^3$ whose first nonzero coordinate is positive, together with the origin. That is, $\hat{X}$ is the union of an open half-space together with an open half-plane in its boundary, together with a ray in its boundary. We take $\Delta$ to be the set of points of $\hat{X}$ with vanishing $x$-coordinate, which is the union of the open half-plane and the ray. Then $\hat{X}$ is a convex subset of $\mathbb{R}^3$ and $\Delta$ is closed and convex in $\hat{X}$. As before, any cover of $X = \hat{X} - \Delta$ is a union of copies of $X$, and we take $Y$ to be the cover with 2 sheets. Then $\hat{Y}$ is isometric to the subset of $\mathbb{R}^3$ given by

$$\hat{Y} = \Delta \cup \{(x, y, z) \in \mathbb{R}^3 \mid x \neq 0\},$$

equipped with the path metric induced by the Euclidean metric. It is easy to see that for each $n \geq 1$ the points $(\pm 1/n, -1/n, -1/n)$ are joined by no geodesic of $\hat{Y}$. Since every neighborhood of 0 contains such a pair of points, 0 has no geodesic neighborhood.

In the proofs below we will use the following two facts about $\hat{X}$. First, geodesics are characterized by their endpoints. Second, geodesics vary continuously with respect to their endpoints, by which we mean that for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $d(x, x') < \delta$ and $d(y, y') < \delta$ for $x, y, x', y' \in \hat{X}$ then the geodesic from $x$ to $y$ is uniformly within $\varepsilon$ of the geodesic from $x'$ to $y'$. These facts follow from the $\text{CAT}(\kappa)$ inequalities and the fact that $diam(\hat{X}) < \pi/\sqrt{\kappa}$. We will ignore all conditions about perimeters of triangles in $\hat{Y}$ being less than $2\pi/\sqrt{\kappa}$ because we have bounded $\text{Diam}(\hat{Y})$ in order to guarantee that all triangles in $\hat{Y}$ satisfy this condition. We chose the bounds on $\text{Diam}(\hat{X})$ and $\text{Diam}(\hat{Y})$ out of convenience; one could probably weaken them, although most theorems about $\text{CAT}(\kappa)$ spaces require some sort of extra condition when $\kappa > 0$. Of course, if $\text{Diam}(\hat{X}) < \pi/3\sqrt{\kappa}$ then the condition on $\text{Diam}(\hat{Y})$ follows automatically.

We begin with some elementary properties of geodesics in $\hat{Y}$, and then show that under special circumstances they vary continuously with respect to their endpoints.

Lemma 3.2. Under the hypotheses of theorem 3.1, we have the following:

(i) If $\gamma$ is a geodesic of $\hat{X}$ meeting $\Delta$ and $w, z \in \hat{Y}$ lie over the endpoints of $\gamma$, then there is a unique path in $\hat{Y}$ joining $w$ with $z$ and projecting to $\gamma$.

(ii) A path in $\hat{Y}$ projecting to a geodesic of $\hat{X}$ is the unique geodesic with its endpoints.

(iii) If $x \in \hat{Y}$ and $y \in \Delta$ then there is a unique geodesic of $\hat{Y}$ joining them, and it projects to a geodesic of $\hat{X}$.

(iv) A geodesic of $\hat{Y}$ that misses $\Delta$ projects to a geodesic of $\hat{X}$.

(v) If $x \in \hat{Y}$ then the set of points of $\hat{Y}$ that may be joined with $x$ by a geodesic of $\hat{Y}$ that misses $\Delta$ is open.

Proof: (i) Because $\Delta$ is convex in $\hat{X}$, $\gamma$ meets $\Delta$ in an interval, with endpoints say $x$ and $y$. There is clearly a unique lift of this interval. If $\pi w \neq x$ then by covering space theory the half-open
segment of $\gamma$ from $\pi w$ to $x$ has a unique lift to $Y$ beginning at $w$. Similarly, there is a unique lift of the segment from $y$ to $\pi z$. It is obvious that these lifts fit together to form a lift of $\gamma$.

(ii) This follows from the uniqueness of geodesics in $\tilde{X}$ and the uniqueness of their lifts with specified endpoints in $\tilde{Y}$, which in turn follows from (i) for geodesics that meet $\Delta$ and from covering space theory for those that do not.

(iii) This follows from (i) and (ii) by lifting a geodesic of $\tilde{X}$ joining $\pi x$ and $\pi y$.

(iv) Suppose $\gamma$ is a geodesic of $\tilde{Y}$ from $x$ to $z$ that misses $\Delta$, and that $\pi \gamma$ is not a geodesic of $\tilde{X}$. Consider the homotopy $\Gamma$ from $\pi \gamma$ to the constant path at $\pi x$ given by retraction along geodesics. Suppose first that $\Gamma$ meets $\Delta$. Then there is a point $y$ of $\pi \gamma$ that is joined with $\pi x$ by a geodesic $\beta$ of $\tilde{X}$ that meets $\Delta$. By (i), there is a lift $\tilde{\beta}$ of $\beta$ from $x$ to the point $\tilde{y}$ of $\gamma$ lying over $y$. By (ii), $\tilde{\beta}$ is the unique geodesic of $\tilde{Y}$ with these endpoints. But then the subsegment of $\gamma$ from $x$ to $\tilde{y}$ must coincide with $\tilde{\beta}$, contradicting the fact that $\gamma$ misses $\Delta$. Now suppose $\Gamma$ misses $\Delta$. Consider the geodesic $\delta$ of $\tilde{X}$ from $\pi x$ to $\pi z$, which is shorter than $\gamma$. Since $\delta$ is the track of $\pi z$, we may regard $\Gamma$ as a homotopy rel endpoints between $\pi \gamma$ and $\delta$. This lifts to a homotopy between $\gamma$ and a lift $\tilde{\delta}$ of $\delta$ that joins $x$ and $z$. Then $\ell(\tilde{\delta}) < \ell(\gamma)$, contradicting the hypothesis that $\gamma$ is a geodesic.

(v) Suppose $\gamma$ is a geodesic of $\tilde{Y}$ from $x$ to some point $y$ of $\tilde{Y}$, and that $\gamma$ misses $\Delta$. By (iv), $\pi \gamma$ is a geodesic of $\tilde{X}$. Since geodesics of $\tilde{X}$ depend continuously upon their endpoints, there is an open ball $U$ of radius $r > 0$ about $\pi y$ such that the geodesics of $\tilde{X}$ from $\pi x$ to the various points of $U$ are all uniformly within $d(\text{image}(\gamma), \Delta)$ of $\pi \gamma$. By replacing $r$ by a smaller number if necessary, we may also suppose that the open $r$-ball $\tilde{U}$ about $y$ maps isometrically onto its image. Now, if $y'$ lies in $\tilde{U}$ then consider the homotopy along geodesics from $\pi \gamma$ to the geodesic $\beta$ from $x$ to $\pi y'$. This misses $\Delta$, so it lifts to a homotopy from $\gamma$ to a lift $\tilde{\beta}$ of $\beta$. Considering the length of the track of $y$, we see that the final endpoint of $\tilde{\beta}$ lies within $r$ of $y$, so that it must coincide with $y'$. Finally, $\beta$ is a geodesic by (ii). This shows that every point of $\tilde{U}$ is joined with $x$ by a geodesic that misses $\Delta$.

Lemma 3.3. Under the hypotheses of theorem 3.1, suppose $x_n$ and $y_n$ are sequences in $Y$ converging to points $x$ and $y$ of $\tilde{Y}$, respectively. Suppose also that for each $n$ there is a geodesic of $\tilde{Y}$ from $x_n$ to $y_n$ that misses $\Delta$, and let $\gamma_n : [0, 1] \to \tilde{Y}$ be a parameterization of this geodesic proportional to arclength. Then there is a geodesic of $\tilde{Y}$ from $x$ to $y$, and the $\gamma_n$ converge uniformly to (the obvious reparameterization of) it.

Proof: By lemma 3.2(iv), each $\pi \gamma_n$ is a geodesic of $\tilde{X}$. By the continuous dependence of geodesics in $\tilde{X}$ on their endpoints, the $\pi \gamma_n$ converge uniformly to a (suitably parameterized) geodesic $\beta$ from $\pi x$ to $\pi y$. We distinguish two cases. First, suppose that $\beta$ misses $\Delta$. Then there is a unique lift $\tilde{\beta}$ of $\beta$ with $\tilde{\beta}(0) = x$, and this is a (reparameterized) geodesic by 3.2(ii). We claim that the $\gamma_n$ converge uniformly to $\tilde{\beta}$ and that $\tilde{\beta}(1) = y$, so that $\tilde{\beta}$ is the required geodesic. The second claim follows from the first. To see convergence, first choose a constant $\delta > 0$ such that $\delta < d(\text{image}(\beta), \Delta)$ and such that the open ball of radius $\delta$ about $x$ maps isometrically onto its image. By discarding finitely many terms of the sequence, we may suppose that all the $x_n$ are within $\delta$ of $x$ and that all the $\pi \gamma_n$ are uniformly within $\delta$ of $\beta$.

We claim that for each $n$, the uniform distance between $\gamma_n$ and $\tilde{\beta}$ equals the uniform distance between $\pi \gamma_n$ and $\beta$. This clearly implies that the $\gamma_n$ converge uniformly to $\tilde{\beta}$. To see the claim, simply construct the homotopy along geodesics in $\tilde{X}$ from $\pi \gamma_n$ to $\beta$ and lift this to a homotopy from $\gamma_n$ to some lift of $\beta$, and then argue as in lemma 3.2(v) that this lift coincides with $\tilde{\beta}$.

On the other hand, suppose that $\beta$ meets $\Delta$. Then parts (i) and (ii) of lemma 3.2 show that there is a unique lift $\tilde{\beta}$ of $\beta$ with $\tilde{\beta}(0) = x$ and $\tilde{\beta}(1) = y$, and that this is a (reparameterized) geodesic of $\tilde{Y}$. We must show that the $\gamma_n$ converge uniformly to $\tilde{\beta}$. Let $u$ (resp. $w$) be the least
(resp. greatest) element of $[0,1]$ whose image under $\beta$ lies in $\Delta$. It is easy to see that the $\gamma_n$ converge uniformly to $\tilde{\beta}$ on $[u,w]$. (One just uses the fact that the distance between an element of $\tilde{\mathbf{Y}}$ and an element of $\Delta \subseteq \tilde{\mathbf{Y}}$ coincides with the distance between their projections.) From this and the fact that the $\gamma_n$ are all parameterized proportionally to arclength, one obtains the following slightly stronger statement: for each $\varepsilon > 0$ there exist an $N$ and a $\delta > 0$ such that for all $n > N$ and all $v \in [u - \delta, w + \delta]$, $d(\gamma_n(v), \tilde{\beta}(v)) < \varepsilon$. Therefore it suffices to prove that for all $\delta > 0$, the $\gamma_n$ converge uniformly on $[0, u - \delta]$ and on $[w + \delta, 1]$. Since the images of these intervals under $\beta$ are disjoint from $\Delta$ we can use the same argument as in the case that $\beta$ missed $\Delta$.

Lemma 3.4. A triangle of $\tilde{\mathbf{Y}}$ with an edge in $\Delta$ satisfies CAT$(\kappa)$.

Proof: We write $T$ for the triangle, and represent the edge in $\Delta$ as $A : [0,1] \to \Delta$, a geodesic parameterized proportionally to arclength. Let $a$ be the vertex of $T$ opposite $A$, and for $t \in [0,1]$ let $B^t$ be the geodesic from $a$ to $A(t)$, parameterized proportionally to arclength. In particular, the edges of $T$ are $A$, $B^0$ and $B^1$. Let $T'$ be a comparison triangle in $M^2_\kappa$ for $T$, and suppose that $p$ and $q$ are points of given edges of $T$. Let $p'$ and $q'$ be the corresponding points of the corresponding edges of $T'$, and let $k = d_{T'}(p', q')$. We must show $d_{\tilde{\mathbf{X}}}(p,q) \leq k$. If one of $p$ and $q$ lies on $A$ then we are done, because

$$d_{\tilde{\mathbf{X}}}(p,q) = d_{\tilde{\mathbf{X}}}(\pi p, \pi q) \leq k,$$

where we have used the definition of a simple branched cover, the fact that $\tilde{\mathbf{X}}$ is CAT$(\kappa)$, and the fact that $T'$ is a comparison triangle for $\pi T$ as well as for $T$.

Now we consider the case in which neither $p$ nor $q$ is given as lying on $A$. To avoid trivialities we suppose that they lie on different edges of $T$, so we may suppose that $p$ lies on $B^0$ and $q$ on $B^1$. The obvious idea is to construct the geodesic in $\tilde{\mathbf{X}}$ joining $x_0 = \pi p$ with $x_1 = \pi q$, and then lift it to a geodesic of $\tilde{\mathbf{Y}}$. The problem is that while we may always lift the geodesic, there is no guarantee that the lift will join $p$ and $q$. We will circumvent this problem by joining $x_0$ to $x_1$ by a path $\alpha$ that may fail to be a geodesic, but will have length $\leq k$. Our path will have the virtue of lying in the ‘surface’ $S$ swept out by the geodesics $\pi B^t$, which will allow us to lift it to a path from $p$ to $q$. By the continuous dependence of geodesics on their endpoints, $(t,z) \mapsto \pi B^t(z)$ is continuous on $[0,1] \times [0,1]$, and so $S$ is compact.

We will need some “comparison complexes” $\tilde{K}_n$ as well as the comparison triangle $T'$. For $0 \leq t \leq u \leq 1$ we define $T(t,u)$ to be the triangle with edges $B^t$, $B^u$ and $A|_{[t,u]}$. For each $n = 0,1,2,\ldots$ we let $D_n$ be the set of dyadic rational numbers in $[0,1]$ of the form $t_{n,i} = i/2^n$. For each $i = 1,\ldots,2^n$ we define $\tilde{T}_{n,i}$ to be the comparison triangle in $M^2_\kappa$ for $T(t_{n,i-1},t_{n,i})$. We write $\tilde{B}^-_{n,i}$ and $\tilde{B}^+_{n,i}$ for the edges of $\tilde{T}_{n,i}$ corresponding to $B^{t_{n,i-1}}$ and $B^{t_{n,i}}$, $\tilde{A}_{n,i}$ for the edge corresponding to $A|_{[t_{n,i-1},t_{n,i}]}$, and $\tilde{a}_{n,i}$ for the vertex corresponding to $a$. We take $\tilde{U}_{n,i}$ to be the convex hull of $\tilde{T}_{n,i}$ in $M^2_\kappa$. Finally, we define $\tilde{K}_n$ as the union of disjoint copies of the $\tilde{U}_{n,i}$, subject to the identification of the the segment $\tilde{B}^+_{n,i}$ in $\tilde{U}_{n,i}$ with $\tilde{B}^-_{n,i+1}$ in $\tilde{U}_{n,i+1}$, in such a way that $\tilde{a}_{n,i}$ is identified with $\tilde{a}_{n,i+1}$, for each $i = 1,\ldots,2^n - 1$. In short, $\tilde{K}_n$ is a ‘fan’ of $2^n$ triangular pieces cut from $M^2_\kappa$, although some of these pieces may degenerate to segments. We equip $\tilde{K}_n$ with its natural path metric. The paths $\tilde{A}_{n,i}$ in the $\tilde{U}_{n,i}$ fit together to form a path $\tilde{A}_n : [0,1] \to \tilde{K}_n$. The vertices $\tilde{a}_{n,i}$ are identified with each other, resulting in a single point $\tilde{a}_n$ of $\tilde{K}_n$. If $t \in D_n$ then we let $\tilde{B}^t_n : [0,1] \to \tilde{K}_n$ be the geodesic from $\tilde{a}_n$ to $\tilde{A}_n(t)$, parameterized proportionally to arclength. These paths, together with $\tilde{A}_n$, form the ‘1-skeleton’ of $\tilde{K}_n$.

We claim that $\tilde{A}_n$ is a geodesic of $\tilde{K}_n$ for all $n$. Otherwise, a simple application of the CAT($\kappa$) property of $\tilde{\mathbf{X}}$ would show that $\pi \tilde{A}$ failed to be a geodesic. We use this to deduce that points of the ‘1-skeleton’ of $\tilde{K}_n$ are at least as far apart as the corresponding points of $\tilde{K}_{n+1}$. To make this precise, observe that if $t \in D_n$ then $\pi B^t$ and $\tilde{B}^t_n$ have the same length. Therefore to each point $x$ of
\(\pi B^t\) we may associate a point \( \bar{x} \) of \( \hat{B}^t_n \) and vice-versa. The relationship is \( d(\bar{x}, a) = d_{\hat{K}_n}(\bar{a}, \bar{x}) \). If \( t \) also lies in \( D_m \) then we can identify \( \hat{B}^t_n \) with \( \hat{B}^t_m \) in a similar way. We claim that if \( u \) and \( w \) lie in \( D_{n-1} \) and \( b \) and \( c \) are points on \( \hat{B}^u_n \) and \( \hat{B}^w_n \), with corresponding points \( \beta \) and \( \gamma \) on \( \hat{B}^u_n \) and \( \hat{B}^w_n \), then
\[
d_{\hat{K}_n}(\beta, \gamma) \leq d_{\hat{K}_{n-1}}(b, c)
\] (3.1)
To prove this it suffices to treat the case in which \( u \) and \( w \) are consecutive elements of \( D_{n-1} \). Since \( A_{n|[u,w]} \) is a geodesic, we may consider the geodesic triangle in \( \hat{K}_n \) with this edge together with \( \hat{B}^u_n \) and \( \hat{B}^w_n \). This satisfies CAT(\( \kappa \)) because we may subdivide it along the altitude \( \hat{B}^v_n \) (where \( v = (u + w)/2 \)), into two triangles which are pieces of \( M^2_\kappa \) and therefore obviously CAT(\( \kappa \)). Furthermore, as a comparison triangle we may take the triangle in \( \hat{K}_{n-1} \) bounded by \( \hat{A}_{n-1|[u,w]}, \hat{B}^u_{n-1} \) and \( \hat{B}^w_{n-1} \), since this triangle is also a piece of \( M^2_\kappa \). Then (3.1) follows immediately.

We write \( D = \cup D_n \) for the set of all dyadic rational numbers in \([0,1] \). We will use the \( \hat{K}_n \) to construct a point \( x_u \) on \( \pi B^u_n \) for each \( u \in D \). Then we will string the \( x_u \) together to build the path \( \alpha \). We have already defined \( x_0 = \pi p \) and \( x_1 = \pi q \). We will sometimes write \( x_{n,i} \) for \( x_{i/2^n} \), so we have just defined \( x_{0,0} = x_0 \) and \( x_{0,1} = x_1 \). Supposing that all the \( x_{n-1,i} \) have been defined, we define the \( x_{n,i,j} \) as follows. We have already defined the \( x_{n,j} \) for even \( j \), namely \( x_{n,j} = x_{n-1,j/2} \). If \( j \) is odd then take \( u = (j - 1)/2^n, v = j/2^n \) and \( w = (j + 1)/2^n \), and consider the points \( \bar{x}_u \) and \( \bar{x}_v \) of \( \hat{K}_n \) that lie on \( \hat{B}^u_n \) and \( \hat{B}^w_n \) and correspond to \( x_u \) and \( x_v \). We construct the geodesic of \( \hat{K}_n \) joining \( \bar{x}_u \) and \( \bar{x}_v \), and let \( \bar{x}_v \) be any point of \( \hat{B}^v_n \) that it meets. Such an intersection point exists by the construction of \( \hat{K}_n \). (Typically, there will be a unique intersection point, but this can fail if one of \( \bar{T}_{n,j} \) and \( \bar{T}_{n,j+1} \) degenerates to a segment.) We take \( x_v \) to be the point of \( \pi B^n \) corresponding to \( \bar{x}_v \).

For each \( n \) we write \( \bar{x}_{n,i} \) for the point of \( \hat{B}^u_n \) corresponding to \( x_u \), where \( u = i/2^n \). (In particular, \( \bar{x}_{n,i} \) and \( \bar{x}_{n+1,2i} \) are points of different spaces, but both correspond to \( x_u \).) Consider the sum
\[
k_n = \sum_{i=1}^{2^n} d_{\hat{K}_n}(\bar{x}_{n,i-1}, \bar{x}_{n,i})
\] (3.2)
We claim that \( k_n \leq k_{n-1} \). To see this, observe that
\[
k_n = \sum_{i=2,4,\ldots,2^n} d_{\hat{K}_n}(\bar{x}_{n,i-2}, \bar{x}_{n,i})
\]
because of the construction of the \( x_{n,i} \) for odd \( i \). By (3.1), when \( i \) is even we have
\[
d_{\hat{K}_n}(\bar{x}_{n,i-2}, \bar{x}_{n,i}) \leq d_{\hat{K}_{n-1}}(\bar{x}_{n-1,i-2}, \bar{x}_{n-1,i})
\]
and \( k_n \leq k_{n-1} \) follows. We immediately obtain \( k_n \leq k_0 = k \) for all \( n \). By applying the CAT(\( \kappa \)) inequality for \( \hat{X} \) we see that for all \( n \),
\[
\sum_{i=1}^{2^n} d(x_{n,i-1}, x_{n,i}) \leq k.
\]
It follows immediately that if \( u_0, \ldots, u_j \) is any increasing sequence of dyadic rationals, then
\[
d(x_{u_0}, x_{u_1}) + \cdots + d(x_{u_{j-1}}, x_{u_j}) \leq k.
\] (3.3)
We are now ready to string the \( x_u \)'s together into a path. There is a technical complication, which we will work around in the next few paragraphs. Specifically, the map \( u \mapsto x_u \) might not
be continuous on $D$. This can happen if some of the comparison triangles $\hat{T}_{n,i}$ are degenerate. However, if $t \in (0,1]$ then as the $u \in D$ approach $t$ from the left, the $x_u$ do converge to a limit $L(t)$. Similarly, if $t \in [0,1)$ then as the $u \in D$ approach $t$ from the right, the $x_u$ converge to a limit $R(t)$. We will treat $L(t)$; the discussion of $R(t)$ is similar. Certainly there is some sequence $u_i$ of dyadic rationals approaching $t$ from below, such that the $x_{u_i}$ converge, since the $x_u$ all lie in the compact set $S$. We will call this limit $L(t)$. Now we show that if $u_i'$ is any sequence in $D$ approaching $t$ from below then the $x_{u_i'}$ converge to $L(t)$. For otherwise we could suppose (by passing to a subsequence) that the $x_{u_i'}$ converge to some other other limit. Then by interleaving of terms of the sequences $u_i$ and $u_i'$ we could violate (3.3). This establishes the existence of the left and right limits $L(t)$ and $R(t)$. It is obvious that $L(t)$ and $R(t)$ lie on $\pi B^t$. For completeness we define $L(0) = x_0$ and $R(1) = x_1$.

Next, we claim that if $t_1,t_2,\ldots$ is an increasing sequence in $[0,1]$ with limit $t$, then $L(t) = \lim_{n \to \infty} R(t_n)$. If this failed then there would be such a sequence that converged to some point other than $L(t)$. But then for each $n$ we could choose $u_n \in D$ such that $t_n < u_n < t_{n+1}$ and $d(x_{u_n}, R(t_n)) < 1/n$. Then the $x_{u_n}$ would converge to a point other than $L(t)$, while the $u_n$ approach $t$ from below, a contradiction. A symmetric argument shows that if $t_1,t_2,\ldots$ is a decreasing sequence with limit $t$ then $R(t) = \lim_{n \to \infty} L(t_n)$.

Our path $\alpha$ will pass through all the points $L(t)$ and $R(t)$ in order. To accomplish this, we define for $t \in (0,1]$ the quantity

$$l^-(t) = \sup \left\{ \sum_{i=1}^{n} d(L(t_{i-1}), R(t_{i-1})) + d(R(t_{i-1}), L(t_i)) \right\},$$

where the supremum is over all increasing sequences $0 = t_0 < \cdots < t_n = t$. For completeness we define $l^-(0) = 0$. Then for $t \in [0,1]$ we define $l^+(t) = l^-(t) + d(L(t), R(t))$. To motivate these definitions, we mention that the length of the subpath of $\alpha$ from $x_0$ to $L(t)$ (resp. $R(t)$) will be $l^-(t)$ (resp. $l^+(t)$). It is obvious that if $t < t'$ then $l^-(t) \leq l^+(t) \leq l^-(t') \leq l^+(t')$. Furthermore, $l = l^+(1)$ satisfies $l \leq k$. To see this, consider an increasing sequence $0 = t_0 < \cdots < t_n = 1$ such that

$$d(L(t_0), R(t_0)) + d(R(t_0), L(t_1)) + \cdots + d(L(t_n), R(t_n))$$

approximates $l$. We may approximate each $t_i$ (except $t_0$) by a dyadic rational $u_i$ smaller than $t_i$, and each $t_i$ (except $t_n$) by a dyadic rational $v_i$ larger than $t_i$. We may do this in such a way that the sequence $0, v_0, u_1, v_1, \ldots, v_{n-1}, u_{n-1}, u_n, 1$ is increasing. Then the $u_i$ approximate the $L(t_i)$ and the $v_i$ approximate the $R(t_i)$. It follows that

$$d(x_0, x_{v_0}) + d(x_{v_0}, x_{u_1}) + d(x_{u_1}, x_{v_1}) + \cdots + d(x_{u_n}, x_1)$$

approximates $l$, and then $l \leq k$ follows from (3.3). Finally, we claim that if $t \in (0,1]$ then $l^-(t) = \sup_{t' < t} l^+(t')$ and if $t \in [0,1]$ then $l^+(t) = \inf_{t' > t} l^-(t')$. This follows from the relationship between the $L(t)$ and the $R(t)$, together with the fact that $l$ is finite.

Now we build $\alpha$. One can check that there is a unique function $\alpha : [0,l] \to \tilde{X}$ such that for each $t \in [0,1]$ the restriction of $\alpha$ to $[l^-(t), l^+(t)]$ is the geodesic from $L(t)$ to $R(t)$. It follows from the relations between the $L(t)$ and the $R(t)$ that $\alpha$ is continuous, and from the definitions of $l^\pm(t)$ that $\alpha$ is parameterized by arclength. In particular, $\ell(\alpha) = l \leq k$. Finally, $\alpha$ lies in $S$ since $L(t)$ and $R(t)$ lie on $\pi B^t$ for each $t$.

Now we will lift $\alpha$ to $\tilde{Y}$. Suppose first that $\alpha$ misses $\Delta$. If a point $x$ of $\alpha$ lies on $\pi B^t$, then the subsegment of $\pi B^t$ from $\pi a$ to $x$ misses $\Delta$, for otherwise to convexity of $\Delta$ would force $x \in \Delta$. We may regard the retraction of $\alpha$ along geodesics to $\pi a$ as a homotopy rel endpoints between $\alpha$
and the path \( \beta \) which travels along \( \pi B^0 \) from \( x_0 \) to \( \pi a \) and then along \( \pi B^1 \) from \( \pi a \) to \( x_1 \). Of course \( \beta \) lifts to a path \( \tilde{\beta} \) from \( p \) to \( q \), and since the homotopy misses \( \Delta \) it may also be lifted. Therefore there is a lift \( \tilde{\alpha} \) of \( \alpha \) from \( p \) to \( q \), with length \( l \leq k \), as desired. On the other hand, if \( \alpha \) meets \( \Delta \) then the lifting is even easier. One defines \( \tilde{\alpha} \) on \( \tilde{\alpha}^{-1}(\Delta) \) in the obvious way, and then one defines the rest of \( \tilde{\alpha} \) by lifting each component of \( \alpha^{-1}(\hat{X} - \Delta) \) however one desires, subject to the conditions \( \tilde{\alpha}(0) = p \) and \( \tilde{\alpha}(l) = q \).

\[ \square \]

**Lemma 3.5.** At most one geodesic joins any two given points of \( \hat{Y} \).

**Proof:** Suppose \( x, y \in \hat{Y} \); we claim that there is at most one geodesic joining them. If either \( x \) or \( y \) lies in \( \Delta \) then lemma 3.2(iii) applies. If they are joined by a geodesic missing \( \Delta \) then its uniqueness follows from lemma 3.2(iv) and (ii). So it suffices to consider the case with \( x, y \in Y \) such that every geodesic joining them meets \( \Delta \). Let \( \gamma \) and \( \delta \) be two such geodesics, meeting \( \Delta \) in points \( c \) and \( d \) respectively; we will prove \( \gamma = \delta \). The (unique) geodesic triangle with vertices \( x, c \) and \( y \) satisfies \( \text{CAT}(\kappa) \) by lemma 3.4. So does the geodesic triangle with vertices \( y, c \) and \( d \).

We now apply Alexandrov’s subdivision lemma to the ‘bigon’ formed by \( \gamma \) and \( \delta \). Taking \( T \) to be the triangle with edges \( \delta \) and the subsegments of \( \gamma \) joining \( x \) to each of \( x \) and \( y \), and the altitude to be the geodesic joining \( c \) and \( d \), we see that \( T \) satisfies \( \text{CAT}(\kappa) \). Since \( \ell(\gamma) = \ell(\delta) \), the comparison triangle degenerates to a segment, and the \( \text{CAT}(\kappa) \) inequality immediately implies \( \gamma = \delta \).

\[ \square \]

The proofs of the two parts of theorem 3.1 are independent of each other.

**Proof of theorem 3.1:** Cases (A)–(G) below show that various sorts of triangles in \( \hat{Y} \) satisfy \( \text{CAT}(\kappa) \). These constitute a proof because every triangle is treated either by case (A) or by case (G). In each case \( T \) is a triangle with vertices \( A \), \( B \) and \( C \). For two points \( P \) and \( Q \) of \( \hat{Y} \) that are joined by a geodesic we write \( \overline{PQ} \) for the geodesic joining them. For three points \( P \), \( Q \) and \( R \) of \( \hat{Y} \), any two of which are joined by a geodesic, we write \( \triangle PQR \) for the geodesic triangle with edges \( \overline{PQ} \), \( \overline{QR} \) and \( \overline{RP} \). Figure 3.1 illustrates the arguments for a few of the cases. Most of the cases use Alexandrov’s lemma. Because \( \hat{Y} \) might not be a geodesic space we have to prove the existence of all the geodesics we introduce, which complicates the argument. At a first reading one should simply assume that all needed geodesics exist.

**A** Suppose no altitude from \( A \) meets \( \Delta \). The set of points of \( \overline{BC} \) to which there is a geodesic from \( A \) is nonempty because it contains \( C \). Because no altitude from \( A \) meets \( \Delta \), this set is open by lemma 3.2(v). It is also closed (lemma 3.3), hence all of \( \overline{BC} \). For each \( P \) in \( \overline{BC} \), let \( \gamma_P : [0, 1] \to \hat{Y} \) be a parameterization proportional to arclength of the geodesic \( \overline{AP} \) from \( A \) to \( P \). For each fixed \( s \in [0, 1] \) the map \( \gamma : \overline{BC} \times [0, 1] \to \hat{Y} \) given by \( (P, s) \mapsto \gamma_P(s) \) is continuous in \( P \), by lemma 3.3.
For each fixed $P$ the map is Lipschitz as a function of $s$, with Lipschitz constant $\text{Diam}(T)$. It follows that $\Gamma$ is jointly continuous in $P$ and $s$. The fact that $T$ satisfies CAT($\kappa$) now follows from a standard subdivision argument, like that of [12, p. 328].

(B) Suppose that the only altitude from $A$ meeting $\Delta$ is $\overline{AB}$. If $B = C$ then $T$ degenerates to a segment (by the uniqueness of geodesics) and therefore automatically satisfies CAT($\kappa$). If $B \neq C$ then arguing as in the previous case we see that each point $P$ of $\overline{BC}$ is joined with $A$ by a geodesic. We choose a sequence of points $B_n$ of $\overline{BC} - \{B\}$ approaching $B$. For each $n$, $\triangle AB_nC$ satisfies CAT($\kappa$) by case (A). By lemma 3.3, the geodesics $\overline{AB_n}$ converge uniformly to $\overline{AB}$. As a uniform limit of triangles that satisfy CAT($\kappa$), $\triangle ABC$ does also.

(C) Suppose $\Delta$ contains two vertices of $T$. This is lemma 3.4.

(D) Suppose $\Delta$ contains a vertex of $T$ and also a point of an opposite side. Then there is a geodesic joining these points, by lemma 3.2(iii). Subdivide $T$ along this altitude and apply case (C) to each of the resulting triangles.

(E) Suppose $\Delta$ contains a vertex (say $B$) of $T$. If $\overline{AC}$ meets $\Delta$ then apply the previous case. So suppose $\overline{AC}$ misses $\Delta$ and consider the set of points $P$ of $\overline{BC}$ that are not joined to $A$ by a geodesic that misses $\Delta$. This set is closed by lemma 3.2(v) and nonempty because it contains $B$; we let $B'$ be a point of this set closest to $C$. Since $B' \neq C$ there is a sequence of points in the interior of $\overline{BC}$ approaching $B'$, each of which is joined with $A$ by a geodesic missing $\Delta$. By lemma 3.3, there is a geodesic $\overline{AB'}$. By the construction of $B'$, $\overline{AB'}$ meets $\Delta$. Subdivide $T$ along this altitude and apply case (B) to $\triangle AB'C$ and case (D) to $\triangle ABB'$, which of course reduces in turn to two applications of case (C).

(F) Suppose $\Delta$ contains a point of $T$. By lemma 3.2(iii) there is a geodesic joining this point with a vertex opposite it. Subdivide $T$ along this altitude and apply case (E) to each of the resulting triangles.

(G) Suppose an altitude of $T$ meets $\Delta$. Subdivide $T$ along this altitude and apply case (F) to each of the resulting triangles.

Proof of theorem 3.1(ii): Suppose that $\Delta$ is complete; we must show that $\hat{\Gamma}$ is geodesic. In light of lemma 3.2(iii) it suffices to show that any two points $x, z$ of $Y$ are joined by a geodesic. We write $D$ for $d(x, z)$. Suppose first that there exists a sequence $y_i$ in $\Delta$ such that the sequence $d(x, y_i) + d(y_i, z)$ converges to $D$. Then for each $i$ there is a geodesic $\alpha_i$ (resp. $\beta_i$) from $x$ (resp. $z$) to $y_i$ and we write $a_i$ (resp. $b_i$) for its length. By passing to a subsequence we may suppose that the $a_i$ converge to a limit, say $a$. Then the $b_i$ converge to $b = D - a$. Since neither $x$ nor $z$ lies in the closed set $\Delta$, the $a_i$ and $b_i$ are bounded away from 0, so $a > 0$ and $b > 0$. We will show that the $y_i$ form a Cauchy sequence.

We let $\delta > 0$ be sufficiently small, by which we mean that $\delta < a$, $\delta < b$, $a + \delta < \pi/2\sqrt{\kappa}$ and $b + \delta < \pi/2\sqrt{\kappa}$. This is possible because each $a_i$ and $b_i$, hence each of $a$ and $b$, is bounded above by $\text{Diam}(\hat{\Delta}) < \pi/2\sqrt{\kappa}$. For such $\delta$, let $Y_\delta = \{ y_i \mid |a_i - a| < \delta \text{ and } |b_i - b| < \delta \}$. Now suppose $y_i, y_j \in Y_\delta$ and let $\gamma$ be the geodesic joining them. Since $\gamma$ lies in $\Delta$, the geodesic triangle formed by $\alpha_i$, $\alpha_j$ and $\gamma$ satisfies CAT($\kappa$), by lemma 3.4. Similarly, the triangle formed by $\beta_i$, $\beta_j$ and $\gamma$ also satisfies CAT($\kappa$). We will show that if $\gamma$ were very long then its midpoint would be problematic. Let $A$ be a closed annulus in $M^2_\kappa$ with inner radius $a - \delta$ and outer radius $a + \delta$. The conditions we have imposed on $\delta$ guarantee that the inner radius is positive and (if $\kappa > 0$) that $A$ lies in an open hemisphere. Let $f(\delta)$ be the maximum of the lengths of geodesic segments of $M^2_\kappa$ that lie entirely in $A$. We observe that $f(\delta)$ tends to 0 as $\delta$ does. Also, given any geodesic of $M^2_\kappa$ with length > $2f(\delta)$ and endpoints in $A$, its midpoint does not lie in $A$ and hence lies at distance < $a$ from the center of $A$. (By ‘the’ center of $A$ when $\kappa > 0$ we mean the center closer to $A$.) We define $g(\delta)$ in a similar way, with $b$ in place of $a$. Now, if $\gamma$ were longer than max($2f(\delta), 2g(\delta)$), then by the CAT($\kappa$) inequalities its midpoint $y'$ would satisfy $d(x, y') < a$ and $d(y', z) < b$, a contradiction.
of the fact that \( d(x, z) = a + b \).

We have shown that any two elements of \( Y_\delta \) have distance bounded by \( \max(2f(\delta), 2g(\delta)) \). Since this bound tends to 0 as \( \delta \) does, the \( y_i \) form a Cauchy sequence and hence converge. Since the limit lies in \( \Delta \), there are geodesics joining it with \( x \) and with \( z \). Concatenating these yields the required geodesic.

Now suppose that there is no such sequence \( y_i \). Then there is a positive number \( k \) such that \( d(x, y) + d(y, z) > D + k \) for all \( y \) in \( \Delta \), and there is also a sequence of paths in \( Y \) from \( x \) to \( z \) with lengths tending to \( D \). From the first of these facts we deduce that no path from \( x \) to \( z \) of length \( < D + k \) meets \( \Delta \). So let \( \gamma : [0, 1] \to Y \) be a path from \( x \) to \( y \) of length \( < D + k \) and let \( U \) be the set of \( t \in [0, 1] \) for which there is not only a geodesic of \( \hat{Y} \) from \( x \) to \( \gamma(t) \) but even one that misses \( \Delta \). The theorem follows because \( U \) contains 0 and is open (lemma 3.2(v)) and closed. To see that it is closed, let a sequence \( t_i \) in \( U \) converge to a point \( t \) of \( [0, 1] \), and let \( \beta_i \) be a geodesic from \( x \) to \( \gamma(t_i) \) that misses \( \Delta \). By lemma 3.3, the \( \beta_i \) converge to a geodesic \( \beta \) from \( x \) to \( \gamma(t) \). The concatenation of \( \beta \) and \( \gamma([t, 1]) \) has length bounded by that of \( \gamma \), which is less than \( D + k \). It follows that the concatenation does not meet \( \Delta \). In particular, \( \beta \) misses \( \Delta \) and hence \( t \in U \). \( \square \)

**Example:** In some circumstances one can show that a branched cover is \( \text{CAT}(\kappa) \), even when the branch locus is not complete. To illustrate this, we take \( \hat{X} \) to be the open unit ball in \( \mathbb{R}^3 \) and \( \Delta \) to be a diameter. With \( Y \) as the universal cover of \( X = \hat{X} - \Delta \), we can deduce that \( \hat{Y} = Y \cup \Delta \) is geodesic, even though \( \Delta \) is not complete. One simply takes \( \Delta \) to be the metric completion of \( \Delta \), \( \hat{X} = X \cup \Delta \), and considers the the branched cover of \( \hat{X} \) over \( \Delta \), where \( X = \hat{X} - \Delta \) is the same as before, \( Y \) is the universal cover of \( X \), and \( \hat{Y} = Y \cup \Delta \). That is, we add the endpoints of the diameter, then remove them along with the diameter, take the cover as before, and then glue the diameter and its endpoints back in. Theorem 3.1 shows that \( \hat{Y} \) is \( \text{CAT}(0) \). Then, as an open ball in the \( \text{CAT}(0) \) space \( \hat{Y} \), \( \hat{Y} \) is convex and hence also \( \text{CAT}(0) \).

We now present a local form of theorem 3.1. The main new feature is that the projection map \( \hat{Y} \to \hat{X} \) is no longer required to be 1-1 on the branch locus. It also allows us to dispense with the explicit diameter bounds for \( \hat{X} \) and \( \hat{Y} \). We say that a metric space \( \Delta \) is locally complete if each of its points has a neighborhood whose closure is metrically complete. This is equivalent to \( \Delta \) being an open subset of its completion. The pathological local properties of the second example following theorem 3.1 stem from the fact that the branch locus used there is not locally complete. If \( \hat{X} \) is a metric space and \( \Delta \subseteq \hat{X} \) then we say that \( \Delta \) is locally convex (in \( \hat{X} \)) if each point has a neighborhood \( V \) such that \( V \cap \Delta \) is convex in \( \hat{X} \).

**Theorem 3.6.** Suppose \( \hat{X} \) is a metric space of curvature \( \leq \kappa \) for some \( \kappa \in \mathbb{R} \), and that \( \Delta \) is a locally convex, locally complete subset of \( \hat{X} \). Suppose \( \hat{Y} \) is a metric space and that \( \pi : \hat{Y} \to \hat{X} \) has the following properties. First, each element of \( \Delta = \pi^{-1}(\Delta) \) has a neighborhood \( V \) such that \( \pi|_V \) is a simple branched cover of its image \( \pi(V) \), over \( \pi(V) \cap \Delta \). Second, \( \pi \) is a local isometry on \( Y = \hat{Y} - \Delta \). Then \( \hat{Y} \) has curvature \( \leq \kappa \). \( \square \)

We omit the proof because we do not need the result and the argument is a straightforward application of theorem 3.1 and the idea used in the example above.

4. Iterated branched covers of Riemannian manifolds

In this section we define precisely what we mean by a branched cover which is locally an iterated branched cover of a manifold over a family of mutually orthogonal totally geodesic submanifolds. Then we show that such a branched cover satisfies the same upper bounds on local curvature as the base manifold. We prove this only in the case of nonpositive curvature, but we indicate what else is needed in the general case.
We say that a collection $S_0$ of subspaces of a real vector space $A$ is normal if the intersection of any $k \geq 1$ members of $S_0$ has codimension $2k$. This means that each subspace has codimension 2 and that they are as transverse as possible to each other. The basic example is a subset of the coordinate hyperplanes in $\mathbb{C}^n$, with $A$ being the underlying real vector space. This is essentially the only example, in the following sense. If $S_1, \ldots, S_n$ are the elements of $S_0$ then we may introduce a basis $w_1, \ldots, w_m, x_1, y_1, \ldots, x_n, y_n$ of $A$ such that each $S_i$ is the span of $w_1, \ldots, w_m$ and those $x_j$ and $y_j$ with $j \neq i$. We will write $S$ for the union of the elements of $S_0$.

Now suppose $\mathcal{H}_0$ is a family of immersed submanifolds of a Riemannian manifold $\hat{M}$ with union $\mathcal{H}$. We say that $\mathcal{H}_0$ is normal at $x \in \hat{M}$ if there is a family $S_0$ of orthogonal subspaces of $T_x \hat{M}$ that are normal in the sense above and have the following property. We require that there be an open ball $U$ about $0$ in $T_x \hat{M}$ which the exponential map $\exp_x$ carries diffeomorphically onto its image $V$, such that $V \cap \mathcal{H} = \exp_x(U \cap \mathcal{S})$, and such that $\exp(S \cap U)$ is a convex subset of $V$ for each $S \in S_0$. We say that $\mathcal{H}_0$ is normal if it is normal at each $x \in M$. In this case, each element of $\mathcal{H}_0$ is totally geodesic, distinct elements of $\mathcal{H}_0$ meet orthogonally everywhere along their intersection, and each self-intersection of an element of $\mathcal{H}_0$ is also orthogonal.

Let $x \in M$, $U$, $V$, and $\mathcal{H}$ be as above, and write $S_1, \ldots, S_n$ for the elements of $S_0$. Then
\[
\pi_1(V - \mathcal{H}) \cong \pi_1(U - \mathcal{S}) \cong \pi_1(T_x \hat{M} - \mathcal{S}) \cong \mathbb{Z}^n ;
\]
the first two isomorphisms are obvious and canonical, and the last follows from the explicit description of $S_0$ given above. That is, $T_x \hat{M} - \mathcal{S}$ is a product of $n$ punctured planes and a Euclidean space. We choose generators $\sigma_1, \ldots, \sigma_n$ for $\pi_1(T_x \hat{M} - \mathcal{S})$ by taking a representative for $\sigma_i$ to be a simple circular loop that links $S_i$ but none of the other $S_j$. We say that a connected covering space $T_x \hat{M} - \mathcal{S}$ is standard if the subgroup of $\mathbb{Z}^n$ to which it corresponds is generated by $\sigma_1^{d_1}, \ldots, \sigma_n^{d_n}$ for some $d_1, \ldots, d_n \in \mathbb{Z}$. We apply the same terminology to the corresponding cover of $V - \mathcal{H}$. In particular, the universal cover is standard. An arbitrary covering space of $V - \mathcal{H}$ is called standard if each of its components is.

Now suppose $\hat{M}$ is a Riemannian manifold and $\mathcal{H}_0$ is a normal family of immersed submanifolds. We write $M$ for $\hat{M} - \mathcal{H}$. If $\pi : N \to M$ is a covering space then we say that $N$ is a standard cover of $M$ if for each $x \in M$ with $V$ as above, $\pi : \pi^{-1}(V - \mathcal{H}) \to V - \mathcal{H}$ is a standard covering in the above sense. In this case, we take $\hat{N}$ to be a certain subset of the metric completion of $N$: those points of the completion which map to points of $\hat{M}$ under the completion of $\pi$. In particular, if $\hat{M}$ is complete then $\hat{N}$ is the completion of $N$. We write $\hat{\pi}$ for the natural extension $\hat{N} \to \hat{M}$ of $\pi$, and we say that this map is a standard branched covering of $\hat{M}$ over $\mathcal{H}_0$.

The simplest example of a standard branched cover is when $M = \mathbb{C}^n$, $\mathcal{S}_0$ is the set of coordinate hyperplanes, $M$ is their complement and $\pi : N \to M$ is the covering space with $N = \mathbb{C}^n - \mathcal{H}$ and $\pi : (z_1, \ldots, z_n) \mapsto (z_1^{d_1}, \ldots, z_n^{d_n})$. The generalization to the case of locally infinite branching requires the more complicated discussion in terms of metric completions of covering spaces. It is also possible that different components of the preimage of $V - \mathcal{H}$ are inequivalent covering spaces of $V - \mathcal{H}$. This can happen when $N \to M$ is an irregular cover.

We will need the following two general lemmas, whose proofs should be skipped on a first reading. The first simplifies the task of establishing local curvature conditions and the second says that one may often ignore the added points when taking a metric completion of a length space.

**Lemma 4.1.** Let $X$ be a length space with metric $d$. Let $Y$ be a path-connected subset of $X$ with the property that
\[
\delta(y, z) = \inf \{ \ell(\gamma) \mid \gamma \text{ a path in } Y \text{ joining } y \text{ and } z \}
\]
is finite for all $y, z \in Y$, so that $(Y, \delta)$ is a length space. Suppose also that $(Y, \delta)$ is $\text{CAT}(\kappa)$ for some $\kappa \in \mathbb{R}$. Then any point of the interior of $Y$ admits a neighborhood which is convex in $X$ and also $\text{CAT}(\kappa)$.
Proof: We will define three open balls; all are balls with respect to the metric \( d \), rather than \( \delta \). Suppose \( x \) lies in the interior of \( Y \) and that \( U \) is an open ball centered at \( x \) and lying in \( Y \). Let \( r \) be the radius of \( U \), and let \( V \) be the open ball with center \( x \) and radius \( r/2 \). A simple argument shows that \( d(y, z) = \delta(y, z) \) for all \( y, z \in V \). Let \( W \) be the open ball with center \( x \) and radius \( r' = \min(r/4, \pi/\sqrt{n}) \). Any two points \( y, z \) of \( W \) are joined by a path \( \gamma \) in \( Y \) that is a geodesic with respect to \( \delta \). Now, \( \gamma \) lies in \( V \) by the triangle inequality, so \( \gamma \) is also a geodesic with respect to \( d \), and \( d(x, t) = \delta(x, t) \) for all points \( t \) of \( \gamma \). By the CAT(\( \kappa \)) inequality in \((Y, \delta)\), applied to a triangle obtained by joining \( y \) and \( z \) to \( x \) with geodesics, we see that \( \delta(x, t) < r' \) for all \( t \). This shows that \( \gamma \) lies in \( W \). Since the same argument applies to every geodesic of \( X \) joining \( y \) and \( z \), we see that \( W \) is convex in \( X \). Since \( d \) and \( \delta \) coincide on \( W \), \( W \) is CAT(\( \kappa \)).

We say that the interior of a path \( \gamma \) with domain \([a, b] \) lies in a subset \( Z \) of \( \hat{X} \) if \( \gamma((a, b)) \subseteq Z \).

**Lemma 4.2.** Let \( X \) be a length space with metric \( d \) and let \( \hat{X} \) be its metric completion. For any \( x, y \in \hat{X} \) there are paths joining \( x \) and \( y \) with interiors in \( X \) and lengths arbitrarily close to \( d(x, y) \).

Furthermore, the intersection of \( X \) with any open ball in \( \hat{X} \) is path-connected.

**Proof:** Choose a sequence of points \( x_1 \in X \) that tend to \( x \), such that \( \sum d(x_i, x_{i+1}) < \infty \). By choosing short paths in \( X \) joining each \( x_i \) to \( x_{i+1} \) and concatenating them, we obtain an open path \( \gamma : [0, 1) \to X \) of finite length, which can be extended to \([0, 1] \) by defining \( \gamma(1) = x \). The extension is continuous because \( \gamma \) has finite length. By taking subpaths of \( \gamma \) we see that for all \( \varepsilon > 0 \) there is a path of length \( < \varepsilon \) from some point \( x' \) of \( X \) to \( x \), with interior in \( X \). The same result holds with \( y \) in place of \( x \). Given \( \varepsilon > 0 \), choose such paths of lengths \( < \varepsilon/4 \). Then \( x' \) and \( y' \) may be joined by a path in \( X \) of length \( < d(x, y) + 2\varepsilon/4 \). Putting the three paths together establishes the first claim.

Now suppose \( U \subseteq \hat{X} \) is the open ball of radius \( r \) and center \( x \), and that \( y, z \in U \cap X \). By the above, there are paths from \( y \) and \( z \) to \( x \) with lengths \( < r \) and interiors in \( X \). These paths obviously lie in \( U \). If \( y' \) and \( z' \) are points in \( U \cap X \) on these paths at distance \( < r/2 \) from \( x \), then they may be joined by a path in \( X \) of length \( < r \). Such a path lies in \( U \) by the triangle inequality. We have joined \( y \) to \( y' \), \( y' \) to \( z' \) and \( z' \) to \( z \) by paths in \( U \cap X \), establishing the second claim.

**Theorem 4.3.** If a Riemannian manifold \( \hat{M} \) has sectional curvature bounded above by \( \kappa \leq 0 \) and \( \hat{\pi} : \hat{N} \to \hat{M} \) is a standard branched cover over a normal family \( \mathcal{H}_0 \) of immersed submanifolds of \( \hat{M} \), then \( \hat{N} \) is locally CAT(\( \kappa \)).

**Remark:** For a global version of this result see theorem 5.1.

**Proof:** We will write \( \hat{\mathcal{H}} \) for \( \hat{\pi}^{-1}(\mathcal{H}) \). Let \( \tilde{x} \in \hat{N} \), \( x = \hat{\pi}(\tilde{x}) \), and let \( U \), \( V \), \( S_0 \) and \( S \) be as in the definition of the normality of \( \mathcal{H}_0 \) at \( x \). We may suppose without loss of generality that \( S_0 \neq \emptyset \). Let \( r \) be the common radius of \( U \) and \( V \). Without loss of generality we may take \( r \) small enough so that \( V \) and all smaller balls centered at \( x \) are convex in \( \hat{M} \). We write \( S_1, \ldots, S_n \) for the elements of \( S_0 \), and \( T_i \) for \( \text{exp}_x(U \cap S_i) \subseteq V \). We choose \( 0 < r' \) such that the orthogonal projection maps from the closed \( r' \)-ball \( B \) about \( x \) to the \( T_i \) are well-behaved. By this we mean that for each \( i \), there is a fiberwise starshaped (about \( 0 \)) set in the restriction to \( T_i \cap B \) of the normal bundle of \( T_i \), which is carried diffeomorphically onto \( B \) by the exponential map. The orthogonal projection maps \( B \to B \cap T_i \) are then obtained by applying the inverse of this diffeomorphism followed by the natural projection of the normal bundle to \( T_i \). These maps will not be used until late in the proof.

For \( t \in [0, 1] \) and \( p \in V \) let \( t.p \) denote the point in \( V \) on the radial segment from \( x \) to \( p \) at distance \( t \cdot d(x, p) \) from \( x \). Then the radial homotopy \( \Gamma : [0, 1] \times V \to V \) given by \( (t, p) \mapsto (1 - t).p \) is a deformation retraction of \( V \) to \( \{x\} \). Observe that if \( p \in V - \mathcal{H} \) then \( t.p \) also lies in \( V - \mathcal{H} \) for all \( t \neq 0 \). We may therefore lift \( \Gamma|_{[0, 1] \times (V - \mathcal{H})} \) to an ‘open homotopy’ \([0, 1] \times \pi^{-1}(\mathcal{H}) \to \)
\(\pi^{-1}(V - \mathcal{H})\) in the obvious way. This is a lipschitz map and therefore extends to a homotopy \(\tilde{\Gamma} : [0, 1] \times \pi^{-1}(V) \to \pi^{-1}(V)\). We call this the radial homotopy; its tracks are geodesics of \(\tilde{N}\) and project to radial segments of \(V\).

The first consequence of this analysis is that \(\pi^{-1}(V)\) is the union of the open \(r\)-balls about the points of \(\pi^{-1}(x)\). The second consequence is that distinct preimages of \(x\) lies at distance \(\geq 2r\) from each other. For a path of length \(< 2r\) between two preimages would lie in \(\pi^{-1}(V)\) and then the deformation retraction of \(\pi^{-1}(V)\) to \(\pi^{-1}(x)\) shows that the endpoints of the path coincide. This implies that the open \(r\)-ball \(\tilde{V}\) about \(\tilde{x}\) is a component of \(\pi^{-1}(V)\), and therefore the restriction of \(\pi\) to \(\tilde{V}\) is a covering map. We note that \(\tilde{V} - \mathcal{H}\) is connected, by lemma 4.2. The radial homotopy also shows that the closed \(r'\)-ball \(B\) about \(\tilde{x}\) is the preimage in \(\tilde{V}\) of \(B\), that \(\tilde{B}\) is path-connected, and that for all \(\tilde{y}, \tilde{z} \in \tilde{B}\),

\[
\delta(\tilde{y}, \tilde{z}) = \inf \left\{ \ell(\gamma) \mid \gamma \text{ a path in } \tilde{B} \text{ joining } \tilde{y} \text{ and } \tilde{z} \right\}
\]

is finite. Finally, since \(\tilde{V} - \mathcal{H}\) is connected, the radial homotopy shows that \(\tilde{B} - \mathcal{H}\) is also connected.

To show that \(\tilde{x}\) admits a convex \(\text{CAT}(\kappa)\) neighborhood it suffices by lemma 4.1 to show that \((\tilde{B}, \delta)\) is \(\text{CAT}(\kappa)\). We will prove this by realizing \(\tilde{B}\) as an iterated simple branched cover of \(B\). We let \(\sigma_1, \ldots, \sigma_n\) denoted generators for \(G = \pi_1(B - \mathcal{H}) = \pi_1(T_\nu \tilde{M} - \mathcal{S}) \cong \mathbb{Z}^n\) of the sort discussed above. Since \(\pi : N \to M\) is a standard covering, there are \(d_1, \ldots, d_n \in \mathbb{Z}\) such that the subgroup of \(G\) associated to the covering \(B - \mathcal{H} \to B - \mathcal{H}\) is generated by \(\sigma_1^{d_1}, \ldots, \sigma_n^{d_n}\). For each \(k = 0, \ldots, n\), let \(G_k\) be the subgroup generated by \(\sigma_1^{d_1}, \ldots, \sigma_k^{d_k}, \sigma_{k+1}, \ldots, \sigma_n\). We let \(B_k\) be the metric completion of the cover of \(B - \mathcal{H}\) associated to \(G_k\), equipped with the natural path metric. Then \(B_k\) is the standard branched cover of \(B\), branched over the \(T_i \cap B\), with branching indices \(d_1, \ldots, d_k, 1, \ldots, 1\). In particular, \(B_0 = B\) and \(B_n = (\tilde{B}, \delta)\).

We write \(p_k\) for the natural projection \(B_k \to B\) obtained by extending the covering map to a map of metric completions. Because \(G_{k+1} \subseteq G_k\), there is a covering map \(B_{k+1} - p_{k+1}^{-1}(\mathcal{H}) \to B_k - p_k^{-1}(\mathcal{H})\) whose completion \(q_{k+1} : B_{k+1} \to B_k\) satisfies \(p_k \circ q_{k+1} = p_{k+1}\). For each \(k = 0, \ldots, n - 1\) we let \(\Delta_k = p_k^{-1}(T_{k+1})\). We will show that \(q_{k+1}\) is a simple branched covering map with branch locus \(\Delta_k\), for each \(k = 0, \ldots, n - 1\).

First we claim that \(q_{k+1}\) carries \(q_k^{-1}(\Delta_k)\) bijectively to \(\Delta_k\) and is a covering map on the complement of \(q_k^{-1}(\Delta_k)\). To see this, observe that \(B\) is bilipschitz to the metric product \(A_1 \times \cdots \times A_n \times D\) of \(n\) closed Euclidean disks and a closed Euclidean ball, such that \(T_i \cap B\) is identified with the set of points in the product whose \(i\)th coordinate is the center (say 0) of \(A_i\). This identifies each \(B_k\) with \(\tilde{A}_1 \times \cdots \times \tilde{A}_k \times A_{k+1} \times \cdots \times A_n \times D\), where \(\tilde{A}_j\) is the metric completion of the \(d_j\)-fold cover of \(A_j - \{0\}\) (or the universal cover if \(d_j = 0\)). The metric completion of this cover of \(A_j - \{0\}\) is obtained from the cover by adjoining a single point, which lies over 0. Then \(q_{k+1} : B_{k+1} \to B_k\) is given by the branched cover \(A_{k+1} \to A_{k+1}\) and the identity maps on \(A_1, \ldots, A_{k+1}\). The claim is now obvious.

Now we show that \(q_{k+1}\) is a local isometry away from \(q_k^{-1}(\Delta_k)\). If \(\tilde{y} \in B_{k+1} - q_k^{-1}(\Delta_k)\) then by the previous paragraph there is an \(s > 0\) and a neighborhood \(E\) of \(\tilde{y}\) such that \(q_{k+1}\) carries \(E\) homeomorphically onto its image, which is the \(s\)-ball about \(q_{k+1}(\tilde{y})\). It is then easy to see that \(q_{k+1}\) carries the open \((s/2)\)-ball about \(\tilde{y}\) isometrically onto its image.

To complete the proof that \(q_{k+1}\) is a simple branched cover, we need only show that \(d(\tilde{y}, \tilde{z}) = d(y, z)\), when \(\tilde{y}, \tilde{z} \in B_{k+1}\) have images \(y\) and \(z\) under \(q_{k+1}\), and at least one of \(y\) and \(z\) lies in \(\Delta_k\). Without loss of generality we may take \(z \in \Delta_k\), and by continuity it suffices to treat the case \(y \notin \Delta_k\). It is obvious that \(d(y, z) \leq d(\tilde{y}, \tilde{z})\). To see the converse, let \(\gamma\) be a sequence of paths in \(B_k\) from \(y\) to \(z\), with interiors in \(B_k - \Delta_k\) and lengths approaching \(d(y, z)\). This is possible by lemma 4.2. By lifting each path except for its final endpoint, and extending using completeness,
we obtain paths \( \tilde{\gamma}_i \) from \( \tilde{y} \) to points of \( B_{k+1} \) lying over \( z \), with the same lengths as the \( \gamma_i \). By the injectivity of \( q_{k+1} \) on \( q_{k+1}^{-1}(\Delta_k) \), all of these are paths from \( \tilde{y} \) to \( \tilde{z} \), establishing the claim.

In order to use theorem 3.1 inductively, we will need to know that \( \Delta_k \) is a convex subset of \( B_k \). We will use the orthogonal projections introduced earlier. Each of these projections \( B \to B \cap T_j \) may be realized by a deformation retraction along geodesics. The retraction is distance non-increasing, since each \( T_j \) is totally geodesic and \( M \) has sectional curvature \( \leq 0 \). Because \( T_1, \ldots, T_{k-1} \) are totally geodesic and orthogonal to \( T_k \), the track of the deformation retraction to \( T_k \) starting at a point outside \( T_1 \cup \cdots \cup T_{k-1} \) misses \( T_1 \cup \cdots \cup T_{k-1} \) entirely. Therefore the deformation lifts to a deformation retraction of \( B_k \to B_k \cup \cdots \cup T_{k-1} \) to \( \Delta_k \to \Delta_k \cup \cdots \cup T_{k-1} \). This extends to a distance nonincreasing retraction \( B_k \to \Delta_k \), which we will also call an orthogonal projection.

Now we prove by simultaneous induction that \( B_k \) is \( \text{CAT}(\kappa) \) and that \( \Delta_k \) is convex in \( B_k \). The fact that \( B_0 = B \) is \( \text{CAT}(\kappa) \) follows from its convexity in \( V \) and the fact that \( V \) is \( \text{CAT}(\kappa) \), which in turn follows from the proof of [15, theorem 12]. That theorem asserts that simply connected complete Riemannian manifolds of sectional curvature \( \leq \kappa \) are \( \text{CAT}(\kappa) \), but the proof shows that the completeness condition may be replaced by the weaker condition that the manifold be geodesic.

The convexity of \( \Delta_0 = T_1 \cap B \) in \( B \) follows from the convexity of \( T_1 \) in \( V \). Now the inductive step is easy. If \( B_k \) is \( \text{CAT}(\kappa) \) and \( \Delta_k \) is convex in \( B_k \) then \( B_{k+1} \) is \( \text{CAT}(\kappa) \) by theorem 3.1. In particular, geodesics in \( B_{k+1} \) are unique. Then if \( \gamma \) is a geodesic of \( B_{k+1} \) with endpoints in \( \Delta_{k+1} \), the orthogonal projection to \( \Delta_{k+1} \) carries \( \gamma \) to a path of length \( \ell(\gamma) \) with the same endpoints. By the uniqueness of geodesics, \( \gamma \) lies in \( \Delta_{k+1} \), so we have proven that \( \Delta_{k+1} \) is convex in \( B_{k+1} \).

The theorem follows by induction.

Remark: We indicate here the additional work required to prove the theorem when \( \kappa > 0 \). A minor point is that one should take \( r < \pi/\sqrt{\kappa} \), so that all of the \( B_k \) have diameters \(< \pi/2\sqrt{\kappa} \). A more substantial change is required where we used the fact that the orthogonal projection maps \( B \to B \cap T_j \) are distance non-increasing, because this fails in the presence of positive curvature. All that is important here is that the length of a path in \( B \) with \textit{endpoints} in \( T_k \) does not increase under projection to \( T_k \). Even this is not true, but we only need the result for paths of length \( < 2r \). One should choose \( r' \) small enough so that any path in \( B \) of length \( < 2r \) with endpoints in \( T_k \) grows no longer under the projection to \( T_k \). Presumably this can be done but I have not checked the details.

Theorem 4.3 has been widely believed, but ours seems to be the first proof. The theorem overlaps partly with theorem 5.3 of [5], which considers locally finite branched covers of Riemannian manifolds over subsets considerably more complicated than mutually orthogonal submanifolds. Unfortunately there is a gap in the proof of that theorem which I do not know how to bridge (lemma 5.7 does not seem to follow from lemma 5.6). Nevertheless I regard the ‘infinitesimal’ \( \text{CAT}(\kappa) \) condition (condition 3 of theorem 5.3) as very natural, and expect that the theorem not only holds but extends to the case of locally infinite branching.

5. Applications

In this section we solve the problems which motivated our investigation, concerning the asphericity of certain moduli spaces. By using known models for the moduli spaces of cubic surfaces in \( \mathbb{C}P^3 \) and Enriques surfaces we will show that both of these spaces have contractible universal covers. In both cases a key ingredient is the following theorem, which is a sort of global version of theorem 4.3.

**Theorem 5.1.** Let \( \hat{M} \) be a complete simply connected Riemannian manifold with section curvature bounded above by \( \kappa \leq 0 \). Let \( \mathcal{H}_0 \) be a family of complete submanifolds which are normal in the sense of section 4, and let \( \mathcal{H} \) be the union of the members of \( \mathcal{H}_0 \). Then the metric completion \( \hat{N} \) of the universal cover \( N \) of \( \hat{M} - \mathcal{H} \) is \( \text{CAT}(\kappa) \), and \( N \) and \( \hat{N} \) are contractible.
Corollary 5.2. The moduli space $\mathcal{M}$ of smooth cubic surfaces in $\mathbb{C}P^3$ is aspherical.

Proof: We begin by recalling the main result of [2]. We let $\omega$ be a primitive cube root of unity and set $\mathcal{E} = \mathbb{Z} [\omega]$, a discrete subring of $\mathbb{C}$. Let $L$ be the lattice $\mathcal{E}^5$ equipped with the Hermitian inner product

$$h(x, y) = -x_0 y_0 + x_1 y_1 + \cdots + x_4 y_4.$$ 

Then the complex hyperbolic space $\mathbb{C}H^4$ may be taken to be the set of lines in $\mathbb{C}^5$ on which $h$ is negative-definite, so that $\mathbb{C}H^4$ is a subset of $\mathbb{C}P^4$. Let $\mathcal{K}_0$ be the set of (complex) hyperplanes in $\mathbb{C}H^4$ which are the orthogonal complements of those $r \in L$ with $h(r, r) = 1$. Let $\Gamma$ be the unitary group of $L$, which is obviously discrete in $U(4, 1)$. By [2], $(\mathbb{C}H^4 - \mathcal{K})/\Gamma$ is isomorphic as an orbifold to $\mathcal{M}$.

To see that $\mathcal{K}_0$ is locally finite, observe that $\Gamma$ contains a complex reflection in each element $H$ of $\mathcal{K}_0$. That is, if $H$ corresponds to $r \in L$ with $h(r, r) = 1$, then $\Gamma$ contains an element fixing $r^\perp$ (and hence $H$) pointwise and multiplying $r$ by $-1$. If $\mathcal{K}_0$ failed to be locally finite then the existence of these reflections would contradict the discreteness of $\Gamma$. Now consider two elements of $\mathcal{K}_0$ that meet in $\mathbb{C}H^4$, and suppose that they are associated to $r, r' \in L$. Since they meet, $h$ is positive-definite on the span of $r$ and $r'$. Since $h(r, r) = h(r', r') = 1$, positive-definiteness requires $|h(r, r')| < 1$. Since $h(r, r') \in \mathcal{E}$ we must have $h(r, r') = 0$. This shows that any two elements of $\mathcal{K}_0$ that meet do so orthogonally.

Since $\mathbb{C}H^4$ has negative sectional curvature, theorem 5.1 implies that $\mathbb{C}H^4 - \mathcal{K}$ has contractible universal cover. This is also the orbifold universal cover of $(\mathbb{C}H^4 - \mathcal{K})/\Gamma$, so the result follows. 

Corollary 5.3. The period space for smooth complex Enriques surfaces (defined below) is aspherical.

Proof: This is similar to the previous proof. By the Torelli theorem for Enriques surfaces ([9] and [11]), the isomorphism classes of smooth complex Enriques surfaces are in 1-1 correspondence with the points of the period space $(\mathcal{D} - \mathcal{K})/\Gamma$. Here $\mathcal{D}$ is the symmetric space for $O(2, 10)$, $\Gamma$ is a certain discrete subgroup, and $\mathcal{K}_0$ is a certain $\Gamma$-invariant arrangement of complex hyperplanes. By [3], $\Gamma$ may be taken to be the isometry group of the lattice $L$ which is $\mathbb{Z}^{12}$ equipped with the inner product

$$x \cdot y = x_1 y_1 + x_2 y_2 - x_3 y_3 - \cdots - x_{12} y_{12}.$$ 

A concrete model for $\mathcal{D}$ is the set of $v \in L \otimes \mathbb{C}$ satisfying $v \cdot v = 0$ and $v \cdot \bar{v} > 0$. $\mathcal{K}_0$ may be taken to be the set of (complex) hyperplanes in $\mathcal{D}$ which are the orthogonal complements of the norm $-1$ vectors of $L$. The arguments of the previous proof show that $\mathcal{K}_0$ is normal. As a symmetric space of noncompact type, $\mathcal{D}$ has sectional curvature $\leq 0$, so that theorem 5.1 applies and $\mathcal{D} - \mathcal{K}_0$ is aspherical. It follows that the period space, the orbifold $(\mathcal{D} - \mathcal{K})/\Gamma$, is also aspherical.

Remark: We have referred to the period space of Enriques surfaces rather than to a moduli space. This is because it is highly nontrivial to assemble the isomorphism classes of Enriques surfaces into a moduli space $\mathcal{M}$. One way to do this is to equip the surfaces with suitable extra structure and then use geometric invariant theory, as in [13]. Then $\mathcal{M}$ has a natural topology and is identified with some finite cover $C$ of $(\mathcal{D} - \mathcal{K})/\Gamma$. Strictly speaking, one should also impose additional structure to make sure that $C$ is a manifold and not just an orbifold. The reason is that the orbifold structure on $C$ is not terribly relevant to $\mathcal{M}$. (There are Enriques surfaces with infinitely many automorphisms, as well as those with only finitely many, so there is not much hope of a reasonable orbifold structure on any $\mathcal{M}$.) But if $C$ is a manifold then the homeomorphism of $C$ with $\mathcal{M}$ shows that $\mathcal{M}$ is aspherical.

Now we will prove theorem 5.1. In the proofs we will conform to the notation of section 4 by writing $M$ for $\hat{\mathcal{M}} - \mathcal{K}$, $\pi$ for the covering map $N \to M$, $\hat{\pi}$ for its completion, and $\hat{\mathcal{K}}$ for $\hat{\pi}^{-1}(\mathcal{K}) \subseteq \hat{N}$. 

17
Lemma 5.4. The map $\pi$ is a standard covering and the map $\tilde{\pi}$ is a standard branched cover over $\mathcal{H}$. More precisely, suppose $\tilde{x} \in \tilde{N}$, $x = \pi(\tilde{x})$, $V$ is an open ball about $x$ that meets no element of $\mathcal{H}_0$ except those passing through $x$, and $\tilde{V}$ is the ball of the same radius about $\tilde{x}$. Then $\tilde{V} - \mathcal{H}$ is a copy of the universal cover of $V - \mathcal{H}$.

Proof: We prove only the last assertion. Let $\mathcal{J}_0$ be the set of elements of $\mathcal{H}_0$ which contain $x$, and let $\mathcal{I}$ be their union. Then there is a natural sequence of group homomorphisms

$$\pi_1(V - \mathcal{I}) \to \pi_1(\tilde{M} - \mathcal{H}) \to \pi_1(\tilde{M} - \mathcal{I}) \to \pi_1(V - \mathcal{I})$$

The first and second maps are induced by inclusions of the indicated spaces and the third is induced by a retraction of $\tilde{M} - \mathcal{I}$ to $V - \mathcal{I}$ along geodesic rays based at $x$. The composition is obviously the identity map, which shows that $\pi_1(V - \mathcal{I}) \to \pi_1(\tilde{M} - \mathcal{H})$ is injective. Therefore each component of the preimage in $N$ of $V - \mathcal{H}$ is a copy of the universal cover of $V - \mathcal{H}$. By lemma 4.2, $\tilde{V} - \tilde{\mathcal{H}}$ is connected, and as in the proof of theorem 4.3 it is a component of $\pi^{-1}(V - \mathcal{H})$. This completes the proof.

Lemma 5.5. The inclusion $N \to \tilde{N}$ is a weak homotopy equivalence.

Proof: First we show that for each $\tilde{x} \in \tilde{N}$ there is a homotopy of $\tilde{N}$ to itself that carries $N$ into itself and also some neighborhood of $\tilde{x}$ into $N$. We write $n$ for the number of hyperplanes passing through $x = \pi(\tilde{x})$. By the previous lemma and the ideas of the proof of theorem 4.3, there is a closed neighborhood $\tilde{V}$ of $\tilde{x}$ which is homeomorphic to the metric product $\tilde{A}^n \times D$, where $\tilde{A}$ is the metric completion of the universal cover of a closed Euclidean disk minus its center and $D$ is a closed Euclidean ball. It is easy to see that $\tilde{A}$ is homeomorphic to a ‘wedge’ in the plane, by which we mean

$$\tilde{A} \cong \{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 \mid 0 < x, \ |y| < x, \ x^2 + y^2 \leq 1\}$$

There is obviously a homotopy of $\tilde{A}$ into $\tilde{A} - \{(0,0)\}$ which is supported on a small neighborhood of $(0,0)$. We obtain the desired homotopy of $\tilde{N}$ by applying this homotopy to each factor $\tilde{A}$ of $\tilde{V}$ and fixing each point of $N - V$.

Now, if $f : S^k \to \tilde{N}$ represents any element of the homotopy group $\pi_k(\tilde{N})$ then we may cover $f(S^k)$ with finitely many open sets, each of which is carried into $N$ by some homotopy of $\tilde{N}$ that also carries all of $N$ into $\tilde{N}$. Applying these homotopies one after another shows that $f$ is homotopic to a map $S^k \to N$. This shows that $\pi_k(N)$ surjects onto $\pi_k(\tilde{N})$ for all $k$. The same argument applied to balls rather than spheres shows that $\pi_k(N)$ also injects, completing the proof.

Proof of theorem 5.1: By lemma 5.4, $\tilde{N}$ is a standard branched cover of $\tilde{M}$ over the normal family $\mathcal{H}_0$. Since $\tilde{M}$ has sectional curvature $\leq \kappa \leq 0$, theorem 4.3 shows that $\tilde{N}$ is locally $\text{CAT}(\kappa)$. Since $N$ is simply connected lemma 5.5 implies that $\tilde{N}$ is also. Theorem 2.1 now implies that $\tilde{N}$ is $\text{CAT}(\kappa)$ and hence contractible. In particular, all of its homotopy groups vanish, and by another application of lemma 5.5 the same is true of $N$. As a manifold all of whose homotopy groups vanish, $N$ is contractible.

In the introduction we promised to show that the inclusion $N \to \tilde{N}$ is a homotopy equivalence, but all we have established so far is a weak homotopy equivalence. The stronger result follows immediately from theorem 5.1, since any inclusion of one contractible space into another is a homotopy equivalence.
References

[1] A. D. Alexandrov. A theorem on triangles in a metric space and some of its applications. 
Trudy Math. Inst. Steks., 38:5–23, 1951.
[2] D. Allcock, J. Carlson, and D. Toledo. The complex hyperbolic geometry of the moduli space 
of cubic surfaces. Preprint, 1998.
[3] D. J. Allcock. The period lattice for Enriques surfaces. Preprint 1999.
[4] M. Bridson and A. Haefliger. Metric spaces of non-positive curvature. Book preprint, 1995.
[5] R. Charney and M. Davis. Singular metrics of nonpositive curvature on branched covers of 
Riemannian manifolds. Am. J. Math., 115:929–1009, 1993.
[6] M. Davis and T. Januszkiewicz. Hyperbolization of polyhedra. J. Diff. Geom., 34:347–388, 
1991.
[7] E. Ghys et al., editors. Group Theory from a Geometrical Viewpoint. World Scientific, 1991.
[8] M. Gromov. Hyperbolic groups. In S. M. Gersten, editor, Essays in Group Theory, volume 8 
of MSRI Publications, pages 75–263. Springer-Verlag, 1987.
[9] E. Horikawa. On the periods of Enriques surfaces. I. Math. Ann., 234:73–88, 1978.
[10] T. Januszkiewicz. Hyperbolizations. In Ghys et al. [7], pages 464–490.
[11] Y. Namikawa. Periods of Enriques surfaces. Math. Ann., 270:201–222, 1985.
[12] F. Paulin. Construction of hyperbolic groups via hyperbolizations of polyhedra. In Ghys et al. 
[7], pages 313–372.
[13] J. Shah. Projective degenerations of Enriques’ surfaces. Math. Ann., 256(4):475–495, 1981.
[14] W. Thurston and M. Gromov. Pinching constants for hyperbolic manifolds. Invent. Math., 
89(1):1–12, 1987.
[15] M. Troyanov. Espaces à courbure négative et groups hyperboliques. In E. Ghys and P. de la 
Harpe, editors, Sur les Groupes Hyperboliques d’après Mikhael Gromov, volume 83 of Progress 
in Mathematics, pages 47–66. Birkhäuser, 1990.

Department of Mathematics, Harvard University
One Oxford Street, Cambridge, MA 02138.
email: allcock@math.harvard.edu
web page: [http://www.math.harvard.edu/~allcock](http://www.math.harvard.edu/~allcock)