Constrained time-optimal control of double-integrator system and its application in MPC

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Abstract. The paper deals with the design of a time-optimal controller for systems subject to both state and control constraints. The focus is laid on a double-integrator system, for which the time-to-go function is calculated. The function is then used as a part of a model predictive control criterion where it represents the long-horizon part. The designed model predictive control algorithm is then used in a constrained control problem of permanent magnet synchronous motor model, which behavior can be approximated by a double integrator model. Accomplishments of the control goals are illustrated in a numerical example.

1. Introduction
Model predictive control (MPC) has gained a significant attention in the field of automatic control mainly because of its possibilities in real-world industry applications. Increasing interest in the MPC practically in the entire industry can be observed. It is a giant leap since the 1980s when the first applications of the MPC have been developed to meet the specialized control needs of power plants and petroleum refineries [1]. The current state-of-the-art of the MPC is well-presented in [2], where its future potential is outlined as well. It is also important to note that there already exist several excellent books and papers that create a quite strong background for those who want to understand the issues of the MPC, e.g. [3], [4], or [5].

In simple terms, the MPC can be characterized as a class of computer control algorithms that utilize an explicit plant model to predict the future response of a process. At each time step an MPC algorithm attempts to optimize the future plant behavior by computing a sequence of future manipulated variable adjustments. The first control action in the optimal sequence is then sent into the plant, and the entire calculation is repeated at subsequent control intervals [1]. This process is time consuming due to the optimization carried out at each time step while only the first control is utilized. To cope with such problem, the lookahead policy can be used. The lookahead policies were proposed to simplify a solution to the Bellman functional equation for a finite-time horizon problem and are based on a separation of the control horizon into a short-horizon and a long-horizon part [6]. The optimization is then simplified by approximating the long-horizon part by a precomputed value function. This paper focuses on an alternative solution. The information about the long horizon is provided by a time-to-go function [7] which contains at least a piece of the information about the future behavior of the controlled system. It was motivated by the fact that even obtaining a reasonable approximation of the long-horizon value function can be an uneasy task.
Many physical systems are also subject to constraints [7, 8] on both the control and the state variables of the system state-space model, which should be taken into account for both the short-horizon and the long-horizon parts of the MPC. In the case of continuous-time linear systems, the time-optimal control subject to the control constraints has been treated in the literature leading to the bang-bang control strategy [9]. If additionally the state constraints are considered, finding the optimal control strategy can be challenging. As the treatment of this problem for a general system is difficult, the attention will be focused on the double-integrator system. This is motivated by the fact that many physical systems possess the constrained double-integrator dynamics. It means that the relation between two quantities of the system can be investigated as the relation, for instance, between the position and the velocity. The similar idea has been presented in [7], where the fast finite-set MPC for permanent-magnet synchronous motor (PMSM) was proposed owing to such approximation.

Once the optimal control strategy is known, the time-to-go function of the double-integrator system with the control and the state constraints can be obtained. The goal of the paper is thus to obtain analytically the time-to-go function of the double-integrator system subject to both control and state constraints. Then, the calculated time-to-go function will be applied in the MPC for the long-horizon part. Application of such MPC will be illustrated in a numerical example involving the PMSM, which exhibits characteristics of the double-integrator system.

The paper is structured as follows. Section 2 is devoted to the time-optimal control of a constrained continuous-time double-integrator system. Then, the MPC is introduced in Section 3. Section 4 deals with the application of the MPC to PMSM which represents the complex system with hard constraints. The paper is concluded by final considerations in Section 5.

2. Constrained minimum-time control of double-integrator system

2.1. Problem formulation

Consider a continuous-time double-integrator system with the following state dynamics

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= K_1 u(t), \\
\frac{dx_2(t)}{dt} &= K_2 x_1(t),
\end{align*}
\]

where the state variables \(x_1(t), x_2(t)\), the control \(u(t)\) and the constants \(K_1, K_2\) meet \(x_1(t) \in [-\overline{x}_1, \overline{x}_1]\), \(x_2(t) \in \mathbb{R}, u(t) \in [-1, 1]\), \(K_1 \in \mathbb{R}_{>0}\) and \(K_2 \in \mathbb{R}_{>0}\). It means that the state \(x_1(t)\) is supposed to be constrained by a constant parameter \(\overline{x}_1 \in \mathbb{R}_{>0}\) and the control \(u(t)\) is constrained by the interval \([-1, 1]\). As can be seen, both state and control constraints are chosen as symmetrical, which is quite usual situation in many physical systems (e.g. PMSM). Nevertheless, the solution for general non-symmetrical constraints can be obtained easily. The initial state of the double-integrator system \(x_0 = [x_1(0), x_2(0)]^T\) satisfies the state constraint and is assumed to be known.

Let a \(T\) be the time, in which the control system starting from \(x_0\) reaches a required state \(x_r = [x_{1,r}, x_{2,r}]^T\), i.e. \(x_{1,r} = x_1(T)\) and \(x_{2,r} = x_2(T)\), where again \(x_{1,r} \in [-\overline{x}_1, \overline{x}_1]\) and \(x_{2,r} \in \mathbb{R}\). Then the problem of seeking a minimum time, in which the required state is reachable from the initial state, can be formally expressed as

\[
t_f (x_0, \overline{x}_1) = \min T \\
\text{subject to} \quad (1),
\]

\[
x_1(t) \in [-\overline{x}_1, \overline{x}_1],
\]

\[
u(t) \in [-1, 1],
\]

where \(t_f (x_0, \overline{x}_1)\) is called time-to-go function. The form of the time-to-go function can be derived if the optimal control strategy \(u^*(t) = \gamma(x(t))\), which governs the system from its initial state to the required state in minimum time, is known.
2.2. Time-optimal control strategy

The control strategy ensuring time optimality can be determined via Pontryagin’s minimum principle [10]. The solution can be divided into two special cases. In the first special case, the optimization problem can be solved regardless the state constraint. Then, the bang-bang control strategy is obtained [9]. The second case leads to the bang-off-bang control strategy because in this case the state reaches the region, in which its constraint has to be taken into account.

Both the bang-bang and bang-off-bang control strategies require specification of a switching curve, which governs switching between the minimum and the maximum admissible control strategies. For both control strategies, the switching curve has the form [9]

$$x_2(t) = \frac{K_2}{2K_1} \text{sign}( -x_1(t) + x_{1,r} ) \left( x_1^2(t) - x_{1,r}^2 \right) + x_{2,r}.$$  (3)

If the initial state lies on the switching curve and meets the state constraint, it is obvious that the switching is not necessary and only the optimum admissible control depending on the $x_{1,r}$ and $x_{1,0}$ locations is applied. If the initial state does not lie on the switching curve, in the first part of the problem the switching between the minimum and the maximum of admissible controls or vice versa has to be performed to get the optimal solution. The switching is performed when the state of the controlled system reaches the switching curve. As the first control, the minimum admissible control ($u(t) = -1$) is applied if the state is above the switching curve and the maximum admissible control ($u(t) = 1$) is used if the state is below the switching curve. The special bang-bang control case is illustrated in Figure 1.

![Figure 1: The time-optimal bang-bang control of the double-integrator system $(K_1 = 1, K_2 = 1)$ from the initial states $x_0 = [-0.8, 1.8]^T$ (blue circle) and $x_0 = [1, 2]^T$ (red circle) to the required state $x_r = [0, 2]^T$ (black asterisk). In this case, the state constraints given by $\bar{x}_1 = 1$ cannot be reached. Red (blue) lines illustrate whether $u(t) = -1$ or $u(t) = 1$ control was used. Dash-dotted line denotes the switching curve. Shading delineates the region in which the initial condition has to lie to use the bang-bang control.](image)

As can be seen, if the minimum or the maximum admissible control is applied, the state trajectory in the phase plane can be described using the following parabolic curves

$$x_2(t) = \begin{cases} -\frac{K_2}{2K_1} \left( x_1^2(t) - x_{1,0}^2 \right) + x_{2,0}, & \text{if } u(t) = -1, \\ \frac{K_2}{2K_1} \left( x_1^2(t) - x_{1,0}^2 \right) + x_{2,0}, & \text{if } u(t) = 1. \end{cases}$$  (4)

Because of the state constraint, there are initial conditions, for which the switching curve cannot be reached by the bang-bang control strategy. From the intersections of the switching curve and the state constraint (denoted by triangles in Figure 1), it can be easily calculated that the region, in which the bang-bang control is sufficient, is given by

$$\mathcal{R} = \left\{ x(t) \in \mathbb{R}^2 : x_1(t) \in [-\bar{x}_1, \bar{x}_1] \wedge \frac{K_2}{2K_1} \left( x_1(t)^2 + x_{1,r}^2 \right) - \frac{K_2}{K_1} \bar{x}_1^2 + x_{2,r} \leq x_2(t) \wedge x_2(t) \leq -\frac{K_2}{2K_1} \left( x_1(t)^2 + x_{1,r}^2 \right) + \frac{K_2}{K_1} \bar{x}_1^2 + x_{2,r} \right\}.$$  (5)
For the initial state lying outside the region (5), the bang-off-bang control strategy for time-optimal control has to be applied. The strategy consists in bringing the state to its constraint by the same control as in the first part of the solution. Once the state reaches the constraint, the zero control is being applied till the switching curve is reached. Then, again the opposite value of admissible control is used to bring the system to the required state. An example of the bang-off-bang control is illustrated in Figure 2.

Finally, the complete time-optimal control strategy for the constrained double-integrator system (1) can be expressed as

\[
u^*(t) = \begin{cases} 
1, & \text{if } \begin{cases} x_1(t) \in (-\overline{x}_1, \overline{x}_1) \end{cases} \land \begin{cases} x_2(t) < \frac{K_2}{2K_1} \text{sign}(-x_1(t) + x_{1,r}) \left(x_1^2(t) - x_{1,r}^2\right) + x_{2,r} \\
\text{from (2a) below the switching curve (3)}
\end{cases} \\
\lor \begin{cases} x_1(t) \in [-\overline{x}_1, \overline{x}_1] \end{cases} \land \begin{cases} x_2(t) = \frac{K_2}{2K_1} \left(x_1^2(t) - x_{1,r}^2\right) + x_{2,r} \\
\text{on the corresponding part of switching curve (3)}
\end{cases} \\
0, & \text{if } \begin{cases} x_1(t) = -\overline{x}_1 \end{cases} \land \begin{cases} x_2(t) \neq \frac{K_2}{2K_1} \left(x_1^2(t) - x_{1,r}^2\right) + x_{2,r} \\
\text{on the lower constraint and not on the corresponding part of switching curve (3)}
\end{cases} \\
\lor \begin{cases} x_1(t) = x_{1,r} \end{cases} \land \begin{cases} x_2(t) = x_{2,r} \\
\text{at the required state}
\end{cases} \\
\lor \begin{cases} x_1(t) = \overline{x}_1 \end{cases} \land \begin{cases} x_2(t) \neq -\frac{K_2}{2K_1} \left(x_1^2(t) - x_{1,r}^2\right) + x_{2,r} \\
\text{on the upper constraint and not on the corresponding part of switching curve (3)}
\end{cases} \\
-1, & \text{if } \begin{cases} x_1(t) \in (-\overline{x}_1, \overline{x}_1) \end{cases} \land \begin{cases} x_2(t) > \frac{K_2}{2K_1} \text{sign}(-x_1(t) + x_{1,r}) \left(x_1^2(t) - x_{1,r}^2\right) + x_{2,r} \\
\text{from (2a) above the switching curve (3)}
\end{cases} \\
\lor \begin{cases} x_1(t) \in [-\overline{x}_1, \overline{x}_1] \end{cases} \land \begin{cases} x_2(t) > x_{1,r} \end{cases} \land \begin{cases} x_2(t) = -\frac{K_2}{2K_1} \left(x_1^2(t) - x_{1,r}^2\right) + x_{2,r} \\
\text{on the corresponding part of switching curve (3)}
\end{cases}
\end{cases}
\]

Figure 2: The time-optimal bang-off-bang control of the double-integrator system \((K_1 = 1, K_2 = 1)\) from the initial states \(x_0^1 = [-0.8, 1]^T\) (blue circle) and \(x_0^2 = [1, 3]^T\) (red circle) to the required state \(x_r = [0, 2]^T\) (black asterisk). In this case, the zero control (green line) is applied when the constraint given by \(\overline{x}_1 = 1\) is reached. Red (blue) lines illustrate whether \(u(t) = -1\) or \(u(t) = 1\) control was used. Dash-dotted line delineates the region \(\mathcal{R}\).
2.3. Time-to-go function

Once the strategy ensuring the minimum time to achieve the desired state $x_r$ is known, the time-to-go function $t_f$ in the sense of (2) can be determined. In simple terms, the determination of the time-to-go function is based on the computation of switching time instants. For this purpose, in the argument of the function (2) the initial conditions are substituted by the state $x = [x_1, x_2]^T$. This leads to a general specification of the problem with arbitrary initial state $x$, which enables calculation of the time-to-go function.

Based on the obtained control strategy, the time-to-go function can be described as follows

$$t_f(x, x_1) = \begin{cases} 
-K_2(x_1 + x_{1,r}) + \frac{2}{x_1^2 - x_{1,r}^2} + K_1 B \left( \frac{2}{x_1^2 - x_{1,r}^2} \right), & \text{if } A \land (x_2 \leq B) \land (x_2 \geq C) \\
-K_2(x_1 + x_{1,r}) + \frac{2}{x_1^2 - x_{1,r}^2} + K_1 C \left( \frac{2}{x_1^2 - x_{1,r}^2} \right), & \text{if } A \land (x_2 \leq B) \land (x_2 < C) \\
-K_2(x_1 + x_{1,r}) + \frac{2}{x_1^2 - x_{1,r}^2} + K_1 D \left( \frac{2}{x_1^2 - x_{1,r}^2} \right), & \text{if } A \land (x_2 > B) \land (x_2 \leq D) \\
-K_2(x_1 + x_{1,r}) + \frac{2}{x_1^2 - x_{1,r}^2} + K_1 \left( \frac{2}{x_1^2 - x_{1,r}^2} \right), & \text{otherwise} 
\end{cases} \tag{7}$$

where

$$A: x_1, r \in [-x_1, x_1] \land x_2, r \in \mathbb{R} \land x_1 \in [-x_1, x_1],$$

$$B = \frac{K_2}{2x_1^2} \text{sign} \left( -x_1 + x_{1,r} \right) \left( x_1^2 - x_{1,r}^2 \right) + x_2, r,$$

$$C = \frac{K_2}{2x_1^2} \left( x_1^2 + x_{1,r}^2 \right) - \frac{K_2}{x_1^2} x_2, r,$$

$$D = -\frac{K_2}{2x_1^2} \left( x_1^2 + x_{1,r}^2 \right) - \frac{K_2}{x_1^2} x_1 + x_2, r,$$

and $p$ is a penalty for violating the constraints. Therefore, it should be chosen according to requirements of the problem that is treated. In the paper $p = \infty$ will be considered. An example of the time-to-go function is illustrated in Figure 3.

![Figure 3: The example of the time-to-go function $t_f(x, 1)$, where the required state was $x_r = [0, 2]^T$ and the parameters were chosen as $K_1 = 1$ and $K_2 = 1$.](image)

2.4. Vehicle moving along a single axis

To demonstrate possibilities of the obtained time-to-go function for time-optimal control problems, a simple illustration example will be presented.

Consider the problem of control of a vehicle moving only along single axis and its state being given by its velocity $x_1(t)$ and its position $x_2(t)$. The state is supposed to be measured with the constant sampling period $T_s = 0.01 [s]$. For simplicity, the mass of the vehicle is considered to be $m = 1 [kg]$. Then, the
control $u(t)$ represents directly an actuation, a force driving the vehicle. The dynamics of the vehicle is described by the following continuous-time model

$$\frac{dx_1(t)}{dt} = u(t),$$  \hfill (8a)

$$\frac{dx_2(t)}{dt} = x_1(t).$$  \hfill (8b)

A discrete-time model of the system (8) based on the zero order hold is

$$x(k+1) = f(x(k), u(k)) = \begin{bmatrix} 1 & 0 \\ Ts & 1 \end{bmatrix} x(k) + \begin{bmatrix} Ts \\ \frac{T_s}{2} x_2 \end{bmatrix} u(k),$$  \hfill (9)

where $k \in \mathbb{N}_0$ represents the time index.

Let us further assume that the control satisfies $u(k) \in \{-1, 0, 1\}$. It is also assumed that the speed limit of the vehicle is given by $1 \left[ m \cdot s^{-1} \right]$. At the initial time, the vehicle is located at the position $x_2(0) = -3 \left[ m \right]$ and moves with the maximal admissible velocity $x_1(0) = -1 \left[ m \cdot s^{-1} \right]$. The task is to park $x_1(t_f) = 0 \left[ m \cdot s^{-1} \right]$ the vehicle at the position $x_2(t_f) = 1 \left[ m \right]$ in minimum time (the situation is illustrated in Figure 4).

![Figure 4: The initial state (solid vehicle) and the required state (dash-dotted vehicle) of the system.](image)

The model of the considered system corresponds to the double-integrator system (1) where $K_1 = K_2 = 1$ and the constraint of the state representing the speed limit is $\overline{x}_1 = 1$. The obtained time-to-go function (7) with the corresponding parameters can be seen as the value function $V$ defined as

$$V(x(k)) \triangleq t_f (x(k), 1).$$  \hfill (10)

Thus, the control action in time step $k$ is chosen as

$$u^*(k) = \arg \min_{u(k) \in \{-1, 0, 1\}} V(f(x(k), u(k))),$$  \hfill (11)

The results of the control (11) applied to the system (9) are illustrated in Figures 5 and 6. As can be seen, the required state was successfully reached at time $t_f = 6.54 \left[ s \right]$ by meeting all requirements. The process of the proposed control was a little bit slower than the pre-calculated continuous-time control strategy (6) ($6.50 \left[ s \right]$). It is mainly caused by discretization of the problem i.e. the control can be changed only at sampling times. For instance, if the actual state lies close to its constraint, the predictions for some control actions fall behind the constraint at the next time step and thus the control algorithm would rather choose the control ensuring non-exceeding of the constraint, however, these control actions would be possible in the continuous case. It is obvious that this property can be adjusted by the choice of the sampling period.
3. MPC with constraints

In this section, the general problem of the MPC subjected to constraints will be formulated. The theoretical relations will be later applied in a practical problem of the PMSM control.

Consider the general problem of control of a system given by the following state-space model

\[
x(k + 1) = f(x(k), u(k)),
\]

where \(x \in \mathbb{X} \subset \mathbb{R}^n\), \(u \in \mathbb{U} \subset \mathbb{R}^m\), and \(f: \mathbb{X} \times \mathbb{U} \to \mathbb{X}\), which means that the state and control are constrained by \(\mathbb{X}\) and \(\mathbb{U}\), respectively. Basically, the idea of the MPC is to find a sequence of inputs \(\{u(j)\}_{j=k}^{N+k-1} \in \mathbb{U}^N\) that minimizes the criterion

\[
J(x(k), \{u(j)\}_{j=k}^{N+k-1}) = g_N(x(N + k)) + \sum_{i=k}^{N+k-1} g_{i-k}(x(i), u(i)),
\]

where \(g_N: \mathbb{X} \to \mathbb{R}\) and \(g_l: \mathbb{X} \times \mathbb{U} \to \mathbb{R}\) are given cost functions, subject to the constraints. Only the first control action \(u^*(k)\) is used once the optimal control strategy \(\{u^*(i)\}_{j=k}^{N+k-1}\) is obtained. Then, at all subsequent time steps \(k + 1, k + 2, \ldots\) the MPC repeats the optimization and application of the first control action.

It is important to mention that the minimization problem can be separated into several stages in the sense of

\[
J^*(x(k)) = \min_{u(k)} \left( g_0(x(k), u(k)) + \min_{u(k+1)} \left( g_1(x(k+1), u(k+1)) + \ldots + V_{k+l}(x(k+l)) \right) \right)
\]

Figure 5: Time evolution of velocity (top) with the detail (bottom) (red line - continuous pre-calculated time-optimal control, blue stair-step line - proposed discrete-time controller (11)).

Figure 6: Time evolution of position (top) with the detail (bottom) (red line - continuous pre-calculated time-optimal control, blue stair-step line - proposed discrete-time controller (11)).
subject to the constraints. Then, the first $l$, $l < N$, stages are called short horizon and the rest represented by the value function $V$ in eq. (14), defined as

$$V_{k+l}(x(k+l)) = \min_{\{r_i(x(i))\}_{i=1}^{N+k-1}} \sum_{i=k+1}^{N+k-1} g_{i-k}(x(i), y_{i-k}(x(i))) + g_{N+k}(x(N+k))$$

subject to the constraints, are called long horizon. The reason why the separation is mentioned here is because the optimization with respect to the whole control strategy $\{u(i)\}_{i=1}^{N+k-1}$ at each time instant can be computationally very intensive depending on $N$ and functions $f$ and $g$. This is the major reason why the approximation (15) is used. Unfortunately, for the complex systems even obtaining the long horizon can be problematic. Therefore, in the example introduced in the next section, the long-horizon part of the criterion will be substituted by the time-to-go function derived in Section 2.

4. Application of the MPC with constraints to the PMSM

The control of the PMSM is a very challenging task because the system has to satisfy hard constraints. A strong emphasis is also placed on the speed of control, since the sampling period has to be small and a control algorithm is usually implemented in a hardware with limited computational power. For this reason, the control of the PMSM will serve as an ideal example for employing the proposed MPC to a system with a complex dynamics.

The conventional model of the PMSM in the $d$-$q$ reference frame [11] has the form

$$\begin{align*}
\frac{di_d(t)}{dt} &= \frac{R_s}{L_d} i_d(t) + \frac{L_q}{L_d} i_q(t) \omega(t) + \frac{1}{L_d} u_d(t), \\
\frac{di_q(t)}{dt} &= -\frac{R_s}{L_q} i_q(t) - \frac{\Psi_{PM}}{L_q} \omega(t) - \frac{L_d}{L_q} i_d(t) \omega(t) + \frac{1}{L_q} u_q(t), \\
\frac{d\omega(t)}{dt} &= \frac{P_p}{J} (T_e(t) - T_L(t)), \\
\frac{d\theta(t)}{dt} &= \omega(t), \\
\frac{dT_L(t)}{dt} &= 0,
\end{align*}$$

where $i_d(t)$, $i_q(t)$, $u_d(t)$, and $u_q(t)$ represent components of the stator current and voltage vector in the rotating reference frame, respectively. The variable $\omega(t)$ is the electrical rotor speed and $\theta(t)$ is the electrical rotor position. The variable $T_L(t)$ represents the load torque. The system parameters are: the components of stator inductance $L_d$ and $L_q$, the stator resistance $R_s$, the moment of inertia $J$, the number of pole pairs $p_p$, the flux linkage excited by permanent magnets on the rotor $\Psi_{PM}$, and the Park constant $k_p$.

For the purpose of the MPC, an approximate discrete-time model has to be obtained. This can be done by using the Taylor series based discretization method [12], which can be for the state $x_p(t)$ of the PMSM given by $x_p(t) = [i_d(t), i_q(t), \omega(t), \theta(t), T_L(t)]^T$ and for the input $u(t)$ represented by the voltage vector $u(t) = [u_d(t), u_q(t)]^T$ generally expressed as

$$x_p(k+1) = x_p(k) + \sum_{i=1}^{Z} A_i(x_p(k), u(k)) \frac{T_r}{l!}.$$  (17)

where $A_i(x_p, u)$ is determined recursively as

$$\begin{align*}
A_1(x_p, u) &= f(x_p) + g(x_p) u, \\
A_{i+1}(x_p, u) &= \frac{\partial A_i(x_p, u)}{\partial x_p} (f(x_p) + g(x_p) u),
\end{align*}$$  (17a)
where the form of the functions \( f(x_p) \) and \( g(x_p) \) can be determined from the PMSM model (16). It is also important to mention how to choose the truncation of the series given by the parameter \( Z \). For the stator current dynamics, where the input has an immediate effect on the derivative, it is sufficient to set \( Z = 1 \). On the other hand, for the mechanical variables, higher expansions, at least \( Z = 2 \), are required [13].

4.1. Constraints of the PMSM

From the control point of view, the biggest challenge is represented by the constraint on currents. This constraint can formally be expressed as

\[
i_d^2(t) + i_q^2(t) \leq I_{\text{max}}^2,
\]

where \( I_{\text{max}} \) is a given maximum current. Besides current, the input voltage is also constrained. Since a finite set of control actions is required for the computational costs reason and these control actions have to satisfy some other conditions too, the usual way of control generation is to add a voltage source inverter (VSI) to the PMSM [7], [8]. The VSI can generate stepwise six active and two zero input voltage vectors according to the relation

\[
u(k) = \frac{2}{3} U_{dc} \begin{bmatrix} \cos(\vartheta(k)) & \sin(\vartheta(k)) \\ -\sin(\vartheta(k)) & \cos(\vartheta(k)) \end{bmatrix} \begin{bmatrix} 1 & \cos\left(\frac{2}{3}\pi\right) \\ 0 & \sin\left(\frac{2}{3}\pi\right) \end{bmatrix} \begin{bmatrix} S_x(k) \\ S_y(k) \end{bmatrix},
\]

where \( U_{dc} \) is the DC-link voltage and all possible input actions are generated through all possible switching combinations given by \( S_x(k) \in \{0, 1\} \).

4.2. Approximation of the PMSM by the double-integrator system

In this subsection an approximation of the PMSM model will be presented. The approximation will later be used to obtain an approximate time-to-go function for the PMSM model.

The behavior of the PMSM torque and angular speed roughly corresponds to the behavior of the double-integrator system (1), where \( x_1(t) \approx T_x(t) \) and \( x_2(t) \approx \omega(t) \) [7]. The key question is how to choose the parameters of double integrator \( K_1, K_2 \) and how to fit the currents constraint (18) to the state constraint \( \Pi_1 \). Since the torque is approximated by the first state of the double-integrator system \( x_1(t) \), the parameter \( K_1 \) can be obtained from the time derivative of the torque \( T_x(t) \). This derivative is

\[
dT_x(t) \approx k_p P_p \left( \Psi_{PM} \frac{di_q(t)}{dt} + (L_d - L_q) \left( \frac{di_d(t)}{dt} - i_q(t) + \frac{di_q(t)}{dt} i_d(t) \right) \right).
\]

In many practical applications the values of the inductances \( L_d \) and \( L_q \) are close to each other. Therefore, assuming \( L_d \approx L_q \) and neglecting the state-variables in (16b) depending only on the last control, the following approximation of \( K_1 \)

\[
K_1 \approx \frac{k_p P_p \Psi_{PM}}{L_q}
\]

seems to be reasonable. The approximation of \( K_2 \) can be based on neglecting the load torque, so it is assumed as

\[
K_2 \approx \frac{P_p}{J}.
\]

The constraint for the currents \( i_d \) and \( i_q \) (18) should be transformed to a constraint for the torque. Unfortunately, the constraint for the torque varies with the currents. Hence, the state constraint appearing
in the time-to-go function will also vary with the currents. If the value of current $i_d(k)$ is known, the relation for the current $i_q(k)$ on the boundary of the constraint (18) can be expressed as

$$i_q(k) = \pm \sqrt{(I_{max}^2 - i_d^2(k))}.$$  

(23)

By substituting relation (23) into eq. (16f), the torque constraint at time step $k$ with respect to known current $i_d(k)$ can be described by

$$\bar{x}_i(k) = k_p p_p \left[ (L_d - L_q) i_d(k) \sqrt{I_{max}^2 - i_d^2(k)} + \Psi_{PM} \sqrt{I_{max}^2 - i_d^2(k)} \right].$$  

(24)

Such choice of the state constraint ensures that the currents will always satisfy the constraint (18).

### 4.3. Control strategy

For the MPC of the PMSM, only the horizon $N = 1$ in criterion (13) is considered. The reason is that the control of PMSM has to be very fast to achieve good performance.

For the short horizon part, the cost function $g_0$ in criterion (13) was chosen as

$$g_0(x_P(k), u(k)) = c_1 \left( \omega'(k + 1) - \omega_r \right)^2 + c_2 \left( i_d'(k + 1) - i_{d,r} \right)^2$$  

(25)

saying that, in addition to reach the required speed, the current $i_d$ has to be choke, which is a reasonable requirement [8]. The weights $c_1$ and $c_2$ in the cost function (25) allow to set compromise between the requirements. Apostrophes in the cost function (25) denote predictions of the state variables based on the discretized model of system (16).

However, even when considering $N = 1$, it is desirable to include some information about the long horizon at least in some way. Thus, the following choice of cost function $g_1(x'_P(k + 1))$ in criterion (13) employing the obtained time-to-go function for constrained double-integrator system is assumed

$$g_1(x'_P(k + 1)) = t_f \left( [T'_i(k + 1), \omega'(k + 1)]^T, \bar{x}_i(k + 1) \right),$$  

(26)

where the parameters $K_1$ and $K_2$ are given by relations (21) and (22), respectively. Such choice of the cost function utilizes the knowledge of dynamics between the torque and speed in the sense of providing the rough information about regulation time. Thus, this information can represent, to a certain extent, the long horizon. The time-to-go function also contains the transversality conditions within, so there is a huge emphasis on reaching the required state. All is in compliance with the constraints.

The complete algorithm of the MPC for the PMSM based on the approximation by the double-integrator system can be summarized as follows.

**Step 1:** Obtain the value of the actual state variables of the controlled PMSM $x_P(k)$.

**Step 2:** Compute the set of seven possible control actions generated by the VSI (19) and make predictions of the state for the next time step $x'_P(k + 1)$ based on the approximate discrete-time model.

**Step 3:** From the set of the control actions find the action $u^*(k)$ minimizing

$$J(x_P(k), u(k)) = c \cdot g_1(x'_P(k + 1)) + g_0(x(k), u(k)),$$  

(27)

where the approximation of $g_1$ is done as in eq. (26) and $c \in \mathbb{R}_+$ is a constant allowing eventually tune the influence of the long-horizon information represented by the time-to-go function.

**Step 4:** Apply the optimal control $u^*(k)$.

**Step 5:** Set $k = k + 1$ and return to Step 1.
Table 1: The numerical example parameters.

| Parameter | Value     | Parameter | Value     |
|-----------|-----------|-----------|-----------|
| $R_s$     | 0.28 [Ω]  | $P_p$     | 4         |
| $L_d$     | 0.003465 [H] | $J$     | 0.0098 [kg · m²] |
| $L_q$     | 0.004465 [H] | $U_{dc}$ | 200 [V]  |
| $\Psi_{PM}$ | 0.1989 [Wb] | $I_{max}$ | $\sqrt{10}$ [A] |
| $k_p$     | 1.5       | $T_s$     | $50 \cdot 10^{-6}$ [s] |

4.4. Numerical results

The proposed control algorithm is illustrated using a PMSM model (16) with parameters listed in Table 1. It is supposed that the PMSM starts from its standstill mode ($x_p(0) = 0$) and that the required state is given by speed $100 \, [rad \cdot s^{-1}]$ ($x_r = [0, 0, 100, 0, 0]^T$). The tune constants in criterion (27) are chosen as $c = 1$, $c_1 = 0.1$, and $c_2 = 0.005$.

![Figure 7: The time evolution of angular speed (red dashed line - required speed, blue line - true speed).](image1)

![Figure 8: The time evolution of currents $i_d(t)$ and $i_q(t)$ during the process of speed control (blue line - $i_d(t)$, red line - $i_q(t)$).](image2)

It can be seen that the required speed is successfully reached (see Figure 7) while the constraints are satisfied (see Figure 9). This demonstrates satisfactory control quality of the MPC of the PMSM with the long-horizon part of the criterion approximated by the time-optimal control obtained for a double-integrator dynamics. The time evolution of the currents $i_d$ and $i_q$ and the used control actions are illustrated in Figure 8 and Figure 10, respectively.

Simulations (not provided here for space reasons) also showed better properties of the designed MPC compared to the simple MPC with criterion (27) where the cost function $g_1$ is equal to zero. This simple MPC was not able to reach the required speed at all and the only way to reach the required speed was to increase the constant $c_1$ penalizing the square of deviation of the speeds in loss function (25). However, this leads to an undesirable increase in the current $i_d$. For this reason, the designed MPC represents the good compromise between the performance and the demands. The use of the time-to-go function directly as the criterion for MPC (assuming the function $g_0$ in (27) is equal to zero) is also interesting. In such a case, the controller is unable to reach high speed requirements ($10^2 [rad \cdot s^{-1}]$). It is caused by the fact that the constraint (24) and consequently the time-to-go function vary with each state prediction.
5. Conclusion
The paper focused on a design of a time-optimal controller of a double-integrator system subject to both state and control constraints. The derived time-to-go function was then utilized as a long-horizon part of a criterion for the model predictive control problem. The obtained controller was then applied to a control problem involving a permanent magnet synchronous motor model. As this model exhibits the double-integrator behavior, the derived time-to-go function was convenient for it. Satisfactory performance and achievement of the control goals were illustrated using a numerical example.

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