Magic squares with all subsquares of possible orders based on extended Langford sequences

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Abstract

A magic square of order $n$ with all subsquares of possible orders (ASMS($n$)) is a magic square which contains a general magic square of each order $k \in \{3, 4, \ldots, n-2\}$. Since the conjecture on the existence of an ASMS was proposed in 1994, much attention has been paid but very little is known except for few sporadic examples. A $k$-extended Langford sequence of defect $d$ and length $m$ is equivalent to a partition of $\{1, 2, \ldots, 2m+1\}\setminus\{k\}$ into differences $\{d, \ldots, d+m-1\}$. In this paper, a construction of ASMS based on extended Langford sequence is established. As a result, it is shown that there exists an ASMS($n$) for $n \equiv \pm 3 \pmod{18}$, which gives a partial answer to Abe’s conjecture on ASMS.

Keywords: Magic square, Extend Langford sequence, Skolem sequence

1 Introduction

An $n \times n$ matrix $A$ consisting of $n^2$ integers is a general magic square of order $n$, denoted by GMS($n$), if the sum of $n$ elements in each row, each column, main diagonal and back diagonal is the same. The sum is the magic number. A GMS($n$) is a magic square, denoted by MS($n$), if it consists of $n^2$ consecutive integers. A lot of work has been done on magic squares ($\begin{bmatrix}1&3&5\end{bmatrix}$).

An MS($n$) with all subsquares of possible orders, denoted by ASMS($n$), is an MS($n$) which contains a GMS($k$) for each integer $k$ such that $3 \leq k \leq n-2$.

The following is an ASMS(6) (see $\begin{bmatrix}1&3&5\end{bmatrix}$) in which there is a GMS(3) in the lower right corner and a GMS(4) in the upper left corner.

\[
\begin{bmatrix}
3 & 6 & 34 & 28 & 35 & 5 \\
20 & 31 & 2 & 18 & 15 & 25 \\
19 & 12 & 26 & 14 & 10 & 30 \\
29 & 22 & 9 & 11 & 13 & 27 \\
8 & 36 & 16 & 33 & 17 & 1 \\
32 & 4 & 24 & 7 & 21 & 23 \\
\end{bmatrix}
\]

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Abe gave the ASMS for $n = 6, 7, 13, 15, 17$ and Kobayashi gave the ASMS for $n = 8, 9, 10$ ([1]). Abe also proposed a conjecture on the existence of ASMS of odd $n$ as follows.

**Conjecture 2.12** ([1]) There exists an ASMS of every odd integer $n \geq 5$.

In this paper, Conjecture 2.12 is investigated and the extended Langford sequences are used to construct a quantity of submatrices (quasi-magic 2-rectangle) of an ASMS.

An $L^k_{d,m}$ is equivalent to a partition of $\{1, 2, \cdots, 2m+1\}\{k\}$ into differences $\{d, \cdots, d+m-1\}$. We refer the readers to the references [3, 7, 9, 10] for (hooked, extended) Skolem sequences and Langford sequences and their applications such as cyclic Steiner systems [11], cyclic $m$-cycle systems [4], and cyclically decomposing the complete graph into cycles [6].

A quasi-magic rectangle of size $(m, n)$, denoted by $\text{QMR}(m, n)$, is an $m \times n$ array consisting of distinct integers arranged so that the sum of the entries of each row is a constant and each column sum is another constant. A quasi-magic 3-rectangle of size $(m, n, r)$, denoted by $\text{QMR}(m, n, r)$, is a set of $r$ QMRs consisting of $mnr$ distinct integers such that if supposing the $r$ QMRs then the sum of $r$ elements in each entry is the same. A QMR consisting of consecutive integers is a magic rectangle. We refer the readers to the references [8, 12, 13] for details.

In this paper, we shall use extended Langford sequences and quasi-magic rectangles to prove the following.

**Theorem 1.1.** There exists an ASMS for $n \equiv \pm 3 \pmod{18}$.

The rest of this paper is arranged as follows. Quasi-magic rectangles based on extended Langford sequences and investigated in Section 2. Constructions and existence of ASMS are provided in Section 3.

## 2 Preliminaries

An integer set $\{a, a + 1, \cdots, b\}$ is denoted as $[a, b]$. In this section we will introduce that an $L^k_{d,m}$ gives some special quasi-magic rectangles which are the main blocks of an ASMS.

Let $L = (l_1, l_2, \cdots, l_{2m+1})$ be an $L^k_{d,m}$. $L$ is also written as a collection of ordered pairs $\{(a_i, b_i) | i \in [d, d+m-1], b_i - a_i = i\}$ with $\bigcup_{i=d}^{d+m-1} \{a_i, b_i\} = [1, 2m+1] \{k\}$. Adding $d + m - 1$ to each number in the equality gives $\bigcup_{i=d}^{d+m-1} \{a_i + d + m - 1, b_i + d + m - 1\} = [d + m, d + 3m] \{k\}$.
\( k + d + m - 1 \). For each \( i \in [d, d + m - 1] \), adding \( i \) to the pair \((a_i + d + m - 1, b_i + d + m - 1)\) as another coordinate gives \( \bigcup_{i=d}^{d+m-1} \{i, a_i + d + m - 1, b_i + d + m - 1\} = [d, d + 3m] \setminus \{k + d + m - 1\} \). Let

\[
u_i = a_i + d + m - 1, w_i = b_i + d + m - 1, i \in [d, d + m - 1].
\]

It gives a collection of ordered triples \(((u_i, v_i, w_i): i \in [d, d + m - 1])\) with \( \bigcup_{i=1}^{m} \{u_i, v_i, w_i\} = [d, d + 3m] \setminus \{k + d + m - 1\} \) and \( u_i + v_i = w_i \). We also have a collection of ordered triples \(\{(u_i, v_i, -w_i), (-u_i, -v_i, w_i): i \in [d, d + m - 1]\}\) with \( \bigcup_{i=d}^{d+m-1} \{(u_i, v_i, -w_i)\} \cup \{-u_i, -v_i, w_i\} = [-d - 3m, -d] \cup [d, d + 3m] \setminus \{k + d + m - 1, -k - d - m + 1\} \) and \( u_i + v_i = w_i \).

Further, let

\[
\begin{align*}
x_0 &= k + d + m - 1, y_0 = -(k + d + m - 1), z_0 = 0, \\
x_i &= u_{i+d-1}, y_i = v_{i+d-1}, z_i = -w_{i+d-1}, i \in [1, m], \\
x_{-i} &= -u_{i+d-1}, y_{-i} = -v_{i+d-1}, z_{-i} = w_{i+d-1}, i \in [1, m].
\end{align*}
\]

and

\[
T_i = (x_i, y_i, z_i), i \in [-m, m].
\]

We get a partition \([d, d + 3m] \cup [-d - 3m, -d] \cup \{0\} = \bigcup_{i=-m}^{m} \{x_i, y_i, z_i\}\). Clearly, each triple has the property that the sum of the members is zero. So we have

**Lemma 2.1.** If there is an \( \mathcal{L}_{d,m}^k \) then there is a partition of the set \([d, d + 3m] \cup [-d - 3m, -d] \cup \{0\}\) into triples each having the property that the sum of the members is zero.

A QMR\((m, n)\) is denoted by QMR\(^*\)(\(m, n)\) if the row sum is zero and the column sum is also zero. A QMR\((m, n, r)\) is denoted by QMR\(^*\)(\(m, n, r)\) if it consists of \( r \) QMR\(^*\)(\(m, n)\)s and if they are supposed then the sum of \( r \) numbers in each entry is zero.

Suppose that there is an \( \mathcal{L}_{4,m}^4 \). Let \( T_i(i \in [-m, m]) \) be the triples given by (i). For any \( T_i(i \in [-m, m]) \) we define a \( 3 \times 3 \times 3 \) array \( M_i = (M_{i,1}, M_{i,2}, M_{i,3}) \), where \( M_{i,1}, M_{i,2}, M_{i,3} \) are given below.

\[
\begin{pmatrix}
9x_i + 1 & 9y_i - 4 & 9z_i + 3 \\
9z_i - 3 & 9y_i + 4 & 9x_i - 1 \\
9y_i + 2 & 9z_i & 9x_i - 2
\end{pmatrix}, \quad \begin{pmatrix}
9y_i - 3 & 9x_i + 4 & 9z_i - 1 \\
9x_i + 2 & 9z_i & 9y_i - 2 \\
9z_i + 1 & 9y_i - 4 & 9x_i + 3
\end{pmatrix}, \quad \begin{pmatrix}
9z_i + 2 & 9y_i & 9x_i - 2 \\
9y_i + 1 & 9x_i - 4 & 9z_i + 3 \\
9x_i - 3 & 9z_i + 4 & 9y_i - 1
\end{pmatrix}.
\]

It is readily verified that each \( M_i \) is a QMR\(^*\)(3, 3, 3), \( i \in [-m, m] \), and the entries of all \( M_i(i \in [-m, m]) \) run over the set \( S = [-4, 4] \cup 32, 40 + 27m] \cup [-40 - 27m, -32] \). Clearly, \( |S| = 27(2m + 1) \). So we have the following.

**Lemma 2.2.** If there is an \( \mathcal{L}_{4,m}^4 \) then there are \( 2m + 1 \) QMR\(^*\)(3, 3, 3)s with the entries run over the set \( S \).
We should point out that each \(-M_i(i \in [1, m])\) is also a QMR*\((3,3,3)\) and the element set of \(-M_i\) is exactly the element set of \(M_{-i}\). Thus \(M_0, \pm M_1, \cdots, \pm M_m\) are also the QMR*\((3,3,3)\)s with the property mentioned in Lemma 2.2. Further, the sum of the elements in the main diagonal of each \(\pm M_i, 2(i \in [0, m])\) is zero.

**Example 1** The sequence \(L = (l_1, l_2, \cdots, l_{11})\) listed below is an \(L_{4,5}^6\).

\[
\begin{array}{cccccccccc}
l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 & l_9 & l_{10} & l_{11} \\
8 & 6 & 4 & 7 & 5 & 4 & 6 & 8 & 5 & 7 \\
\end{array}
\]

By Lemma 2.1 there is a partition of the set \([4, 4 + 3m] \cup [-4 - 3m, -4] \cup \{0\}\) into triples as follows.

\[
T_0 = (14, -14, 0), \\
T_1 = (4, 11, -15), \\
T_{-1} = (-4, -11, 15), \\
T_2 = (5, 13, -18), \\
T_{-2} = (-5, -13, 18), \\
T_3 = (6, 10, -16), \\
T_{-3} = (-6, -10, 16), \\
T_4 = (7, 12, -19), \\
T_{-4} = (-7, -12, 19), \\
T_5 = (8, 9, -17), \\
T_{-5} = (-8, -9, 17).
\]

By Lemma 2.2 we have a QMR*\((3,3,3)\), \(M_0 = (M_{01}, M_{02}, M_{03})\), where

\[
M_{0,1} = \begin{pmatrix}
127 & -4 & -123 \\
-3 & -122 & 125 \\
-124 & 126 & -2
\end{pmatrix}, \\
M_{0,2} = \begin{pmatrix}
-129 & 130 & -1 \\
128 & 0 & -128 \\
1 & -130 & 129
\end{pmatrix}, \\
M_{0,3} = \begin{pmatrix}
2 & -126 & 124 \\
-125 & 122 & 3 \\
123 & 4 & -127
\end{pmatrix}.
\]

One can easily list the other 10 QMR*\((3,3,3)\)s \(\pm M_i, i \in [1, 5]\). The entries of these 11 QMR*\((3,3,3)\)s run over the set \([-4, 4] \cup [32, 175] \cup [-175, -32]\).

### 3 Proof of Theorem 1.1

Let \(n \equiv 3 \pmod{18}, n \geq 21\). Denote \(n = 18u + 3\) and \(\lambda = (n^2 - 1)/2\). We shall construct an ASMS(\(n\)) \(A = (a_{ij})\), \(i, j \in [1, n]\) over the set \([-\lambda, \lambda]\).

Suppose that there is an \(L_{m,n}^k\). Let \(T_i = (x_i, y_i, z_i)(i \in [-m, m])\) be the ordered triples defined as (i) and let \(M_0, M_i, -M_i, i \in [1, m]\) be the \(2m + 1\) QMR*\((3,3,3)\)s defined as (ii). The entries of these QMR*\((3,3,3)\)s run over the set \([-4, 4] \cup [32, 40 + 27m] \cup [-40 - 27m, -32]\) by Lemma 2.2. Note that \(|S| = 27(2m + 1)\).

Let \(m = 6u^2 \pm 2u - 3\). Then \(27(2m + 1) = n^2 - 144\), i.e., \(m = (n^2 - 171)/54\). So, \(S \subset [-\lambda, \lambda]\), \(|[-\lambda, \lambda]| - |S| = 144\).

Write \(w = \frac{3}{2} = 6u \pm 1\). Partition \(A\) into submatrices of order 3, \(A = (A_{st}), s, t \in [1, w]\). We use binary Cartesian product to denote the index set of \(A_{st}\), i.e., \((s, t) \in [1, w] \times [1, w]\). Divide \([1, w] \times [1, w]\) into four parts as follows.
\[ V_1 = [1, 4] \times [1, 4] \cup \{(w, w)\} \setminus \{(4, 4)\}, \quad V_2 = [1, 2] \times [5, w], \]
\[ V_3 = [5, w] \times [1, 2], \quad V_4 = [3, w] \times [3, w] \setminus \{(3, 3), (3, 4), (4, 3), (w, w)\}. \]

By calculation we have \([-\lambda, \lambda] \setminus S = [5, 31] \cup [-31, -5] \cup [\lambda - 44, \lambda] \cup [-\lambda, 44 - \lambda]\) and \([[-\lambda, \lambda] \setminus S] = 144\). We shall fill these 144 numbers in \(V_1\) and the rest numbers of \([-\lambda, \lambda]\) in \(V_2, V_3\) and \(V_4\) in a proper way. The construction follows the following four steps.

**Step 1** The following table gives the blocks \(A_{sxt}\) with indices \((s, t) \in V_1 \setminus \{(w, w)\}\).

| \(\lambda - \lambda\) | \(-25\) | \(29\) | \(24 - \lambda\) | \(12 - \lambda\) | \(6 - \lambda\) | \(\lambda - 22\) | \(42 - \lambda\) | \(\lambda - 20\) |
|----------------------|--------|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(17 - \lambda\)     | \(-17\) | \(25\) | \(-29\) | \(\lambda - 24\) | \(20 - \lambda\) | \(\lambda - 12\) | \(26 - \lambda\) | \(\lambda - 6\) | \(22 - \lambda\) | \(\lambda - 42\) | \(20\) |
| \(-22\)              | \(22\) | \(14\) | \(\lambda - 9\) | \(13 - \lambda\) | \(\lambda - 29\) | \(1 - \lambda\) | \(37 - \lambda\) | \(\lambda - 27\) | \(11 - \lambda\) | \(\lambda - 21\) | \(10\) |
| \(-30\)              | \(30\) | \(9 - \lambda\) | \(-14\) | \(\lambda - 13\) | \(29 - \lambda\) | \(-1\) | \(\lambda - 37\) | \(27 - \lambda\) | \(\lambda - 11\) | \(21 - \lambda\) | \(-10\) |
| \(-23\)              | \(23\) | \(\lambda - 5\) | \(5 - \lambda\) | \(-11\) | \(8 - \lambda\) | \(-16\) | \(\lambda - 15\) | \(34 - \lambda\) | \(\lambda - 4\) | \(10 - \lambda\) | \(-6\) |
| \(-26\)              | \(-26\) | \(\lambda - 38\) | \(38 - \lambda\) | \(\lambda - 8\) | \(11\) | \(16 - \lambda\) | \(\lambda - 15\) | \(\lambda - 34\) | \(4 - \lambda\) | \(\lambda - 10\) | \(6\) |
| \(\lambda\)          | \(0\) | \(-36 - \lambda\) | \(36 - \lambda\) | \(3 - \lambda\) | \(-3\) | \(-5\) | \(\lambda - 39\) | \(44 - \lambda\) | \(-18\) | \(-13\) | \(31\) |
| \(\lambda - 32\)    | \(32 - \lambda\) | \(\lambda - 41\) | \(41 - \lambda\) | \(30 - \lambda\) | \(\lambda - 30\) | \(7 - \lambda\) | \(21\) | \(\lambda - 28\) | \(-9\) | \(28\) | \(-19\) |
| \(\lambda - 35\)    | \(35 - \lambda\) | \(\lambda - 25\) | \(\lambda - 25\) | \(\lambda - 14\) | \(14 - \lambda\) | \(\lambda - 2\) | \(18 - \lambda\) | \(-16\) | \(27\) | \(-15\) | \(-12\) |
| \(\lambda - 43\)    | \(43 - \lambda\) | \(\lambda - 31\) | \(31 - \lambda\) | \(\lambda - 40\) | \(\lambda - 40\) | \(18\) | \(13\) | \(-31\) |
| \(19 - \lambda\)    | \(-19\) | \(23 - \lambda\) | \(\lambda - 23\) | \(\lambda - 33\) | \(33 - \lambda\) | \(9\) | \(-28\) | \(19\) |
| \(-24\)             | \(24\) | \(8\) | \(-8\) | \(-7\) | \(7\) | \(-27\) | \(15\) | \(12\) |

Taking \(A_{ww} = -A_{33}\) together with the above table gives all the blocks \(A_{sxt}\) with indices \((s, t) \in V_1\). Since \(n \geq 21\) the element set of the blocks \(A_{sxt}\) with indices \((s, t) \in V_1\) is exactly \([-\lambda, \lambda] \setminus S\).

**Step 2** Performing row permutation \((\frac{1}{2} \frac{3}{4} \frac{5}{6})\) and column permutation \((\frac{1}{2} \frac{5}{3} \frac{4}{1})\) on the matrix \(\begin{pmatrix} M_{1,2} \\ -M_{i,2} \end{pmatrix}\) gives a \(6 \times 3\) matrix as follows.

\[
P_i = \begin{pmatrix}
1 - 9z_i & 3 - 9y_i & -9x_i - 4 \\
9z_i - 1 & 9y_i - 3 & 9x_i + 4 \\
2 - 9y_i & -9x_i - 2 & -9z_i \\
9y_i - 2 & 9x_i + 2 & 9z_i \\
-9x_i - 3 & -9z_i - 4 & 4 - 9y_i \\
9x_i + 3 & 9z_i + 1 & 9y_i - 4
\end{pmatrix}, \quad i \in [1, w - 4].
\]

Taking

\[
\begin{pmatrix}
A_{1, i + 4} \\
A_{2, i + 4}
\end{pmatrix} = P_i, \quad i \in [1, w - 4]
\]
gives all the blocks \(A_{s,t}\) with \((s, t) \in V_2\).

**Step 3** Performing column permutation \((\frac{1}{2} \frac{3}{4} \frac{5}{6})\) on the matrix \((M_{i,3}^T, -M_{i,3}^T)\) gives a \(3 \times 6\) matrix as follows.

\[
Q_i = \begin{pmatrix}
9z_i + 2 & -9z_i - 2 & 9y_i + 1 & -9y_i - 1 & 9x_i - 3 & 3 - 9x_i \\
9y_i & -9y_i & 9x_i - 4 & 4 - 9x_i & 9z_i + 4 & -9z_i - 4 \\
9x_i - 2 & 2 - 9x_i & 9z_i + 3 & -9z_i - 3 & 9y_i - 1 & 1 - 9y_i
\end{pmatrix}, \quad i \in [1, w - 4].
\]

Taking
\[(A_{i+4,1}, A_{i+4,2}) = Q_1, \ i \in [1, w - 4] \]
gives all the blocks \(A_{s,t}\) with \((s,t) \in V_3\).

**Step 4** Let \(V_{4,1} = \{(w - 2, w), (w - 1, w - 1), (w, w - 2)\}\), and
\[V_{4,2} = \{(h - 1, h + 1), (h, h), (h + 1, h - 1), h = 4, 5, \ldots, w - 2\},\]
and \(V_{4,3} = V_4 \setminus (V_{4,1} \cup V_{4,2})\). Taking
\[A_{w-2,w} = M_{0,1}, A_{w-1,w-1} = M_{0,2}, A_{w-2,w} = M_{0,3},\] (iii)
gives the blocks \(A_{s,t}\) with \((s,t) \in V_{4,1}\). Taking
\[A_{h-1,h+1} = M_{m-(h-4),1}, A_{h,h} = M_{m-(h-4),2}, A_{h+1,h-1} = M_{m-(h-4),3},\]
\[A_{h,h+2} = -M_{m-(h-4),1}, A_{h+1,h+1} = -M_{m-(h-4),2}, A_{h+2,h} = -M_{m-(h-4),3}.\] (iv)
gives the blocks \(A_{s,t}\) with \((s,t) \in V_{4,2}\). Note that \(w - 5\) is even since \(n \equiv \pm 3 \pmod{18}\).

The remaining \(M_{i,j}\)s are put in \(V_{4,3}\) so that they satisfy the following property.

if \(A_{s,t} = M_{i,j}\) then \(A_{t,s} = -M_{i,j}\). (vi)

The following table is used to show the positions of \(P_i, Q_i\) and the blocks in \(V_{4,1}\) and \(V_{4,2}\).

|     | \(P_1\) | \(P_2\) | \(\ldots\) | \(P_{w-6}\) | \(P_{w-5}\) | \(P_{w-4}\) |
|-----|--------|--------|------------|------------|------------|------------|
|     |        |        |            |            |            |            |
| \(M_{0,3}\) |        |        |            |            |            |            |
| \(Q_1\) | \(M_{m,3}\) |        |            |            |            |            |
| \(Q_2\) |        |        |            |            |            |            |
| \(\ldots\) |        |        |            |            |            |            |
| \(Q_{w-6}\) |        |        |            |            |            |            |
| \(Q_{w-5}\) |        |        |            |            |            |            |
| \(Q_{w-4}\) |        |        |            |            |            |            |

**Lemma 3.1.** \(P_i, Q_i(i \in [1, w - 4])\) have the following properties.

(I) \(P_i\) is a QMR\(^*\)(6, 3), the last four rows of \(P_i\) form a QMR\(^*\)(4, 3), and the last two rows of \(P_i\) form a QMR\(^*\)(2, 3).

(II) \(Q_i\) is a QMR\(^*\)(3, 6)s, the last four columns of \(Q_i\) form a QMR\(^*\)(3, 4), and the last two columns of \(Q_i\) form a QMR\(^*\)(3, 2).

**Proof.** The conclusions follow from a direct calculation. □

A matrix \(A\) constructed by taking the above four steps has the following properties.
**Lemma 3.2.** A has the following useful properties.

1. Each block $A_{s,t}$ with $(s,t) \in [3,w] \times [3,w]$ is a QMR$^+(3,3)$.
2. The sum of the main diagonal of $A_{s,s}$ is zero for each $s \in [3,w]$.
3. The equality $\sum_{i \in [3,w]} a_{i,i} = 0$ holds for each $h \in [20,n+7]$.

**Proof.** (III) $A_{3,3}, A_{3,4}, A_{4,3}$ and $A_{w,w}$ are QMR$^+(3,3)$s by using direct calculation. For $(s,t) \in V_4$ the blocks $A_{s,t}$ are all QMR$^+(3,3)$s by (iii)-(vi).

(IV) Clearly, the sum of the main diagonal of $A_{s,s}$ is zero when $s = 3$ and $s = w$. By (ii) the sum of the main diagonal of each QMR$^+(3,3)$ $M_{i,2}(i \in [0,m])$ is zero. By (iii) and (iv) the sum of the elements in the main diagonal of $A_{s,s}$ is zero when $s \in [4,w-1]$.

(V) Let $h \in [20,n+7]$. For $i \in [7,h-7]$, let $s_0 = \lceil \frac{i}{3} \rceil, t_0 = \lceil \frac{h-i}{3} \rceil$, where $[a]$ is the smallest integer $x$ such that $x \geq a$. Then $(s_0, t_0) \in V_4 \cup \{(3,4),(4,3)\}$ and $a_{i,h-i}$ is an element of $A_{s_0,t_0}$.

If $(s_0, t_0) \in V_{4,1} \cup V_{4,2}$ then $s_0 + t_0$ is even, denoted by $2d_0$. So $a_{i,h-i}$ belongs to one of $A_{d_0-1,d_0+1}, A_{d_0,d_0}$ and $A_{d_0+1,d_0-1}$. Since $M_0, M_1, -M_i(i \in [1,m])$ are QMR$^+(3,3)$s the sum of the $(i \ (mod 3), (h-i) \ (mod 3))$-entries of the blocks $A_{d_0-1,d_0+1}, A_{d_0,d_0}, A_{d_0+1,d_0-1}$ is zero by (iii), (iv), (v).

If $(s_0, t_0) \notin V_{4,1} \cup V_{4,2}$ then the sum of the $(i \ (mod 3), (h-i) \ (mod 3))$-entries of the blocks $A_{s_0,t_0}$ and $A_{t_0,s_0}$ is zero by (vi). The proof is completed.

**Theorem 3.3.** Let $n \equiv \pm 3 \ (mod\ 18)$ and $m = (n^2 - 171)/54$. If there is an $L_{4,m}^k$ then there is an $ASMS(n)$.

**Proof.** Let $n$ and $m$ be as assumption. Suppose that there is an $L_{4,m}^k$. Let $T_i = (x_i, y_i, z_i)(i \in [-m, m])$ be the ordered triple defined as (i) and let $M_0, M_1, -M_i, i \in [1, m]$ be the $2m+1$ QMR$^+(3,3)$s defined as (ii). The element set of the matrix $A$ under the above construction is exactly $[-\lambda, \lambda]$. We now prove that $A$ is an ASMS($n$).

The blocks $A_{s,t}$ with $(s,t) \in [1,4] \times [1,4]$ form a QMR$^+(12,12)$ by using direct calculation and (III). The blocks $A_{s,t}$ with $(s,t) \in [1,2] \times [5,w]$ form a QMR$^+(6,n-12)$ by (I), and the blocks $A_{s,t}$ with $(s,t) \in [5,w] \times [1,2]$ form a QMR$^+(6,n-12)$ by (II). Combining with (III) we know that $A$ is a QMR$^+(n,n)$. The sum of the first six elements $17, -17, 14, -14, -11, 11$ is zero in the main diagonal of $A$ from the upper left to the lower right is zero. Combining with (IV) the main diagonal of $A$ has zero sum. The first six elements of back diagonal of $A$ from upper right to lower left are $-9x_{w-4} - 4, 9y_{w-4} - 3, 2 - 9y_{w-4}, 9z_{w-5} - 9z_{w-5} - 1, 9x_{w-5} + 3,$ and the last six elements are $9x_{w-4} - 2, -9y_{w-4}, 9y_{w-4} + 1, -9z_{w-5} - 3, 9z_{w-5} + 4, 3 - 9x_{w-5}$. The sum of the above 12 numbers is 0. Combining with (V) the back diagonal of $A$ has zero sum. Thus $A$ is an MS($n$).

Now we prove that $A$ contains a GMS($k$) for $3 \leq k \leq n - 2$. Since $M_{0,2}$ is a GMS($3$), $A_{w-1,w-1}$ is a GMS($3$) by (iii). The matrix $(a_{ij})$ with $(i,j) \in [1,4] \times [8,11]$ is a GMS($4$) by
where \( h \) that

It is readily verified that \( B \) has three elements, \( 9x_{w-5} - 2, -9y_{w-5}, 9y_{w-5} + 1 \). \( Q_{w-6} \) has three elements, 

\[ P_{w-6} \] has three elements \( 9z_{w-6} - 9z_{w-6} - 1, 9x_{w-6} + 3 \). The sum of the above 12 elements is zero. Combining with \( V \) we know that the sum of the elements of the back diagonal of \( C \) is zero. So, \( C \) is a GMS\((n - 3)\).

Let \( h \in [5, n - 2]\setminus\{n - 3\} \). We have

1. if \( h \equiv 0 \pmod{3} \) then the matrix \( (a_{ij})_{h\times h} \) with \( (i, j) \in [7, 6 + h] \times [7, 6 + h] \) is a GMS\((h)\);
2. if \( h \equiv 1 \pmod{3} \) then the matrix \( (a_{ij})_{h\times h} \) with \( (i, j) \in [3, 2 + h] \times [3, 2 + h] \) is a GMS\((h)\);
3. if \( h \equiv 2 \pmod{3} \) then the matrix \( (a_{ij})_{h\times h} \) with \( (i, j) \in [5, 4 + h] \times [5, 4 + h] \) is a GMS\((h)\).

In fact, for the case (1), \( h \in \{6, 9, \ldots, n - 6\} \). By (III)-(V) \( (a_{ij}) \) with \( (i, j) \in [7, 6 + h] \times [7, 6 + h] \) is a GMS\((h)\).

For Case (2), \( h \in \{7, 10, \ldots, n - 2\} \). Let \( B(h) = (a_{ij})_{h\times h} \) where \( (i, j) \in [3, 2 + h] \times [3, 2 + h] \). It is readily verified that \( B(h) \) is a GMS\((h)\) for \( h = 7, 10 \) by using direct calculation together with properties (III) and (IV). Let \( h \geq 13 \). By direct calculation we have \( \sum_{j=3}^{12} a_{i,j} = 0, i \in [3, 6] \) and \( \sum_{i=3}^{12} a_{i,j} = 0, j \in [3, 6] \). Combining with (I)-(III) we know that \( B(h) \) is a GMS\((h)\), the first four elements of the main diagonal of \( B(h) \) from upper left have zero sum. By (IV) we know that the main diagonal of \( B(h) \) has zero sum. Now we check the back diagonal of \( B(h) \). Note that \( h \geq 13 \). The first four elements and the last four elements of the back diagonal of \( B(h) \) from the upper right to the lower left are 

\[ -9z_f, 9x_f + 2, -9x_f - 3, d_1, d_2, 9x_f - 3, 4 - 9x_f, 9z_f + 3, \]

where \( f = \frac{h-10}{3} \). If \( h \geq 16 \) then \( d_1 = 1 - 9y_f - 1 \) and \( d_2 = 9y_f - 1 - 4 \). The sum of these eight numbers is zero. If \( h = 13 \) then \( d_1 = 6 \) and \( d_2 = 7 \). The sum of these eight numbers is 16. But the remaining five elements are \(-13, -9, -16, 13, 9\), which have the sum \(-16\). Combining with \( V \) we know that the back diagonal of \( B(h) \) has zero sum for \( h \in [13, n - 2] \) and \( h \equiv 1 \pmod{3} \).

Thus \( B(h) \) is a GMS\((h)\).

Now we consider Case (3). Clearly, \( h \in \{5, 8, \ldots, n - 4\} \). Let \( E(h) = (a_{ij})_{h\times h} \), where \( (i, j) \in [5, 4 + h] \times [5, 4 + h] \). It is readily verified that \( E(h) \) is a GMS\((h)\) for \( h = 5, 8 \) by using direct calculation together with properties (III) and (IV). Let \( h \geq 11 \). By direct calculation we have \( \sum_{j=5}^{12} a_{i,j} = 0, i \in [5, 6] \) and \( \sum_{i=5}^{12} a_{i,j} = 0, j \in [5, 6] \). Combining with (I)-(IV) we know that \( E(h) \) is a GMS\((h)\), and the main diagonal of \( E(h) \) has zero sum. The first two elements and the last two elements of the back diagonal of \( E(h) \) from the upper right to lower left are 

\[ 4 - 9y_f, 9z_f + 1, -9z_f - 4, 9y_f - 1, \]

where \( f = \frac{h-8}{3} \). The sum of the four numbers is zero. Combining with \( V \) we know that the back diagonal of \( E(h) \) has zero sum. Thus \( E(h) \) is a GMS\((h)\). Therefore \( A \) is an ASMS\((n)\). The proof is completed. \( \square \)
Theorem 3.3. The proof is completed.

Acknowledgements

There exists an ASMS\((n,2)\) to exist is\((n,k)\equiv (0,1),(1,0),(2,0),(3,1)\) \((\text{mod} (4,2))\), on condition that \(m \geq 5\) and \(\frac{\pi}{2}(7-m)+1 \leq k \leq \frac{\pi}{2}(m-3)+1\).

Theorem 7.2 in \([10]\) provided the following.

**Lemma 3.4.** \((\text{10})\) There exists an \(L^k_{4,m}\) whenever \((m,k)\equiv (0,1),(1,0),(2,0),(3,1)\) \((\text{mod} (4,2))\),

**Theorem 1.1** There exists an ASMS\((n)\) for \(n \equiv \pm 3\) \((\text{mod} 18)\).

**Proof.** The necessary condition for an ASMS\((n)\) to exist is \(n \geq 5\) by definition. Let \(n \equiv 3, 15\) \((\text{mod} 18)\). An ASMS\((15)\) was given by Abe \([1]\). For \(n \geq 21\), let \(m = (n^2 - 171)/54\), then \(m \geq 5\). Denote \(n = 3(6u \pm 1)\), we have \(m = 6u^2 \pm 2u - 3 = 4u^2 + 2(u \pm 1) - 3 \equiv 1\) \((\text{mod} 4)\).

By Lemma 3.2 \([\text{3.2}]\) there exists an \(L^k_{4,m}\) for some \(k \equiv 0\) \((\text{mod} 2)\). So, there exists an ASMS\((n)\) by Theorem 3.3. The proof is completed.

**Concluding remarks**

The existence of an ASMS\((n)\) for \(n \equiv \pm 3\) \((\text{mod} 18)\) is completely solved. To give a complete solution of Abe's conjecture 2.12, the following cases should be considered: \((1) n \equiv 9\) \((\text{mod} 18)\); \((2) n \equiv \pm\) \((\text{mod} 6)\).

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