Adapted algebras and standard monomials

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March 29, 2022

Abstract

Let $G$ be a complex semisimple Lie group. The aim of this article is to compare two basis for $G$-modules, namely the standard monomial basis and the dual canonical basis. In particular, we give a sufficient condition for a standard monomial to be an element of the dual canonical basis and vice versa.

Introduction

Let $G$ denote a semisimple, simply connected, algebraic group defined over an algebraically closed field $k$ of arbitrary characteristic. We fix a Borel subgroup $B$ and a maximal torus $T \subset B$, denote $W$ the Weyl group of $G$ with respect to $T$. For a dominant weight $\lambda$ let $V(\lambda)$ the corresponding Weyl module and let $Q$ be the parabolic subgroup which normalizes a highest weight vector. Let $L_\lambda$ be the corresponding ample line bundle on $G/Q$. Consider the embedding $G/Q \hookrightarrow \mathbb{P}(V(\lambda))$ given by the global sections $H^0(G/Q, L_\lambda) \cong V(\lambda)^*$.

The aim of standard monomial theory is to give a presentation of the ring $R = \bigoplus_{n \geq 0} H^0(G/Q, L_{n\lambda})$, which is compatible with the natural subvarieties of $G/Q$ as for example the Schubert varieties $X(\tau)$, the opposite Schubert varieties $X^\kappa$ and the Richardson varieties $X^\kappa_\tau = X_\tau \cap X^\kappa$, $\tau, \kappa \in W/W_Q$. More precisely, the aim is to construct a basis $B \subset H^0(G/Q, L_\lambda)$ such that certain (= standard) monomials of degree $n$ in these basis elements form a basis $B(n\lambda)$ for $H^0(G/Q, L_{n\lambda})$, the relations provide an algorithm to write a non-standard monomial as linear combination of standard monomials, and the restrictions $\{m | X^\kappa_\tau : m \in B(n\lambda), m \not\equiv 0 \text{ on } X^\kappa_\tau\}$ of the standard monomials to a Richardson variety $X^\kappa_\tau$, form a basis for $H^0(X^\kappa_\tau, L_{n\lambda})$. Such a basis

*This research has been partially supported by the EC TMR network “Algebraic Lie Representations”, contract no. ERB FMRX-CT97-0100.
has turned out to be a powerful tool in the investigation of the geometry of Schubert (and related) varieties, see for example [13] and [16].

The concept of an adapted algebra has been introduced by the first author in [3, 4] in connection with the Berenstein-Zelevinsky conjecture on $q$-commuting elements of the dual canonical basis $\mathbb{C}_q[U^-]$. Let $U \subset B$ be a maximal unipotent subgroup, let $U^-$ be its opposite unipotent subgroup and let $n^-$ be its Lie algebra. Denote $U_q(n^-)$ the quantized enveloping algebra and let $B \subset U_q(n^-)$ be the canonical basis. As in the classical case, one has a non-degenerate pairing between the quantized algebra of regular functions $\mathbb{C}_q[U^-]$ on $U^-$ and $U_q(n^-)$, so $\mathbb{C}_q[U^-]$ is naturally equipped with a canonical basis, the dual $B^* \subset \mathbb{C}_q[U^-]$ of the canonical basis $B$. This basis has many nice properties, for example the natural injection $V_q(\lambda)^* \hookrightarrow \mathbb{C}_q[U^-]$ is compatible with the basis, i.e., $B^*(\lambda) = B^* \cap V_q(\lambda)^*$ is a basis for $V_q(\lambda)^*$. So the specialization at $q = 1$ provides a basis $B^*_1(\lambda) \subset H^0(G/Q, \mathcal{L}_\lambda)$, of which is known that it is compatible with Schubert (and related) varieties.

So on the one hand, $B^*_1(\lambda)$ looks like a perfect candidate as a starting basis for a standard monomial theory. The problem with this approach is that the multiplicative structure of the dual canonical basis is hardly understood, see [18]. The adapted algebras occur in this context, they are maximal subalgebras of $\mathbb{C}_q[U^-]$ which are spanned as a vector space by a subset $S \subset B^*$, and all elements of $S$ are multiplicative, i.e., with $b, b' \in S$ also $bb' \in S$. In particular, their product is again an element of the dual canonical basis! The first author has constructed in [3] for every reduced decomposition $\tilde{w}_0$ of the longest word $w_0$ in the Weyl group an adapted algebra $A_{\tilde{w}_0} \subset \mathbb{C}_q[U^-]$.

On the other hand, the construction in [18] of a standard monomial theory for the ring $R$ provides in particular a basis for $H^0(G/Q, \mathcal{L}_\lambda)$, the so-called path vectors $p_\pi$, which also has nice representation theoretic features (see for example [13, 16, 20]), and it is a natural question to ask for the relationship between these two bases. Recall that the path model theory [17, 14] provides combinatorial model [8, 11] for the crystal base theory, so we can index the the elements $b_\pi \in B^*_1(\lambda)$ and the path vectors $p_\pi$ by L-S paths of shape $\lambda$.

The main result of this article can be formulated as follows (for a more precise and more general formulation see section 3): In the $q$-commutative parts of $B^*$ given by the adapted algebras $A_{\tilde{w}_0}$, the two bases coincide up to normalization. I.e., if $b_\pi \in B^*_1(\lambda)$ is the specialization of an element of an adapted algebra $A_{\tilde{w}_0} \subset \mathbb{C}_q[U^-]$ for some reduced decomposition $\tilde{w}_0$ of $w_0$, then $b_\pi = p_\pi$, up to normalization (multiplication by a root of unity). Vice versa, if the chain $(\mathcal{L})$ of Weyl group elements in the L-S path $\pi = (\mathcal{L}, \mathcal{A})$
is compatible with some reduced decomposition of $w_0$, then $b_\pi = p_\pi$, up to normalization. In the last section, we provide some examples to illustrate the connection between the path basis and the dual canonical basis.

1 Notation

Let $G$ be a complex semisimple, simply connected algebraic group with Lie algebra $\mathfrak{g}$. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$ with fixed Cartan subalgebra $\mathfrak{h}$. Let $U^-$ be a maximal unipotent subgroup of $G$ such that $\text{Lie } U^- = \mathfrak{n}$ and let $\mathbb{C}[U^-]$ be the algebra of regular functions on $U^-$. The left multiplication of $U^-$ on itself induces a natural morphism $\phi$ from the enveloping algebra $U(\mathfrak{n}^-)$ on the algebra of differential operators on $\mathbb{C}[U^-]$. This induces a natural non-degenerate pairing on $U(\mathfrak{n}^-) \times \mathbb{C}[U^-]$, mapping $(u, f)$ to the value of $\phi(u)(f)$ at the identity of $U^-$. This pairing has an analogue in the quantum setting, i.e., one has a natural non-degenerate pairing on $U_q(\mathfrak{n}^-) \times \mathbb{C}_q[U^-]$. So the algebras can be viewed as one being the restricted dual of the other.

Kashiwara [9] and Lusztig [22] have constructed the so-called global crystal or canonical basis $\mathcal{B}$ of the algebra $U_q(\mathfrak{n}^-)$. By the dual canonical basis we mean the basis $\mathcal{B}^*$ of $\mathbb{C}_q[U^-]$ dual to $\mathcal{B}$ with respect to the pairing above.

Some elements of $\mathcal{B}^*$ can be easily described, they correspond to so-called extremal weight vectors in representations. Let $\lambda \in P^+$ be a dominant weight and let $V_q(\lambda)$ be the corresponding irreducible highest weight representation. Fix a highest weight vector $v_\lambda$, and for $\tau \in W/W_\lambda$ let $v_\tau$ be the unique weight vector in $V_q(\lambda)$ of weight $\tau(\lambda)$ obtained in the following way: let $\tau = s_{i_1} \cdots s_{i_r}$ be a reduced decomposition of $\tau$ and set

$$v_\tau = F^{(n_1)}_{i_1} \cdots F^{(n_r)}_{i_r} v_\lambda,$$

where $n_1 = \langle s_{i_2} \cdots s_{i_r}(\lambda), \alpha_{i_1}^\vee \rangle$, ..., $n_r = \langle \lambda, \alpha_{i_r}^\vee \rangle$. It follows from the Verma relations for the generators of $U_q(\mathfrak{n}^-)$ that the vector is independent of the chosen decomposition. This weight space is one dimensional, let $b^\lambda_\tau \in V_q(\lambda)^*$ be the unique element of weight $-\tau(\lambda)$ such that

$$b^\lambda_\tau(v_\tau) = 1. \hspace{1cm} (1)$$

This linear form is considered as a form on $U_q(\mathfrak{n}^-)$ by setting $b^\lambda(u) = b^\lambda(uw_\lambda)$. The compatibility property of the canonical basis with highest
weight representations shows immediately that these functions \( b^\lambda_\tau \) are elements of \( B^* \). In the following we refer to the \( b^\lambda_\tau \) as \((\lambda, q)\)-minors. For \( \mathfrak{g} = sl_n \) and \( \lambda \) a fundamental weight, these are exactly the quantum minors.

It is often more convenient to forget in the notation the stabilizer \( W_\lambda \) and, by abuse of notation, just write \( b^\lambda_\tau \) for \( \tau \in W \) and \( \lambda \in P^+ \). In this way we can formulate the following simple product rule (see \cite{6}):

\[
\forall \lambda, \mu \in P^+ : \quad b^\lambda_\tau b^\mu_\tau = b^{\lambda+\mu}_\tau \quad \text{in} \quad \mathbb{C}_q[U^-].
\] (2)

More generally, for \( \xi \in V_q(\lambda)^* \) let \( c^\lambda_\xi \) be the linear form on \( U_q(n^-) \) obtained by setting \( c^\lambda_\xi(u) := c^\lambda(u\nu_\lambda) \). The following commutation relation can be found for example in \cite{6}, it is a direct consequence of \cite{15}, where the expression of the \( R \)-matrix is made explicit: suppose \( \xi, \eta \) are of weight \( \nu_1, \nu_2 \). Then

\[
c^\lambda_\xi c^\lambda_\eta = q^{-(\lambda, \lambda)+(\nu_1, \nu_2)} c^\lambda_\eta c^\lambda_\xi + \sum_i c^\lambda_{\eta_i} c^\lambda_{\xi_i}
\] (3)

where there exist \( p_i \in \mathbb{C}(q) \) and some non-constant monomials \( M_i \) such that

\[
\eta_i \otimes \xi_i = p_i(q)(M_i(E) \otimes M_i(F))\eta \otimes \xi.
\]

2 \((\lambda, q)\)-minors and adapted algebras

Recall that two elements \( a, b \in \mathbb{C}_q[U^-] \) are said to \( q \)-commute if \( ab = q^m ba \) for some \( m \in \mathbb{Z} \). Two elements \( b, b' \in B^* \) are called multiplicative if the product is a multiple of an element of the canonical basis, i.e., \( bb' = rb'' \) for some \( b'' \in B^* \), \( r \in \mathbb{C}(q) \). The elements of the dual canonical basis are in general neither multiplicative nor \( q \)-commuting. Let \( \varpi_1, \ldots, \varpi_n \) be the set of fundamental weights and fix a reduced decomposition \( w_0 = s_{i_1} \cdots s_{i_N} \) of the longest word in \( W \), we write \( \tilde{w}_0 \) to refer to this decomposition. Set

\[
y_j = s_{i_1} \cdots s_{i_j} \quad \text{and} \quad b_j := b_{\varpi_i}^{\varpi_j} \in B^*.
\] (4)

Remark 1 Since \( b^\lambda_\tau = b^\lambda_{\tau'} \) for \( \tau \equiv \tau' \mod \ W_\lambda \), it is easy to verify the following description of the elements above:

\[
\{b_1, \ldots, b_N\} = \{b_{\varpi_i}^{\varpi_j} \mid 1 \leq i \leq n, 1 \leq j \leq N\}
\]

These elements have been studied in great detail by the first author in \cite{3}, where he shows that they have the following remarkable properties:

i) they are multiplicative and \( q \)-commute, and, more precisely,
ii) the monomials \( h_1^{m_1} \cdots h_N^{m_N} \) are, up to a power of \( q \), elements of \( B^* \).

Let \( S_{\bar{a}_0} \) be the set of the monomials above and let \( A_{\bar{a}_0} \) be the subalgebra of \( \mathbb{C}_q[U^\sim] \) spanned by \( S_{\bar{a}_0} \). Actually, \( A_{\bar{a}_0} \) is generated as a space by \( S_{\bar{a}_0} \).

These properties motivate the following definition: a subalgebra \( A \) of \( \mathbb{C}_q[U^\sim] \) is called adapted if it is spanned by a subset \( P^* \subset B^* \) such that the elements in \( P^* \) are multiplicative and \( A \) is maximal with this property, i.e., for all \( b \in B^* - P^* \) there exist a \( p \in P^* \) such that \( b \) and \( p \) are not multiplicative.

**Theorem 2** (\[3\]) The subalgebra \( A_{\bar{a}_0} \) is adapted with the elements of \( S_{\bar{a}_0} \) (up to a rescaling by a power of \( q \)) as spanning set. The set \( S_{\bar{a}_0} \) is an Ore set in \( \mathbb{C}_q[U^\sim] \), and for the localizations one has: \( S_{\bar{a}_0}^{-1} \mathbb{C}_q[U^\sim] = S_{\bar{a}_0}^{-1} A_{\bar{a}_0} \).

### 3 Lusztig’s Frobenius maps

Let \( \mathcal{R} \) be the ring of Laurent polynomials \( \mathbb{Z}[q, q^{-1}] \). We denote \( U_{\mathcal{R}}(n^-) \) the \( \mathcal{R} \)-form of \( U_q(n^-) \) generated by the divided powers \( F_i^{(n)}(\mathcal{R}) \). Fix a primitive \( 2\ell \)-th root of unity \( \xi \), where \( \ell \) itself is even, and denote \( \mathbb{Z}_\xi \) the ring \( \mathbb{Z}[\xi] \).

Let \( U(n^-) \) be the \( \mathbb{Z}_\xi \)-form of the enveloping algebra of \( n^- \), i.e., it is obtained from the Kostant \( \mathbb{Z} \)-form by the ring extension \( \mathbb{Z} \hookrightarrow \mathbb{Z}_\xi \).

Let \( U_{\xi}(n^-) \) be the \( \mathbb{Z}_\xi \)-algebra obtained from \( U_{\mathcal{R}}(n^-) \) by specialization at \( q = \xi \), i.e., it is obtained from the Lusztig \( \mathcal{R} \)-form \( U_{\mathcal{R}}(n^-) \) by base change with respect to \( \mathcal{R} \hookrightarrow \mathbb{Z}_\xi \), \( q \mapsto \xi \).

Let \( A = (a_{i,j})_{1 \leq i,j \leq n} \) be the Cartan matrix of \( G \) and denote by \( t A = (\overline{a}_{i,j}) \), \( \overline{a}_{i,j} := a_{j,i} \), the transposed matrix. Let \( d = (d_1, \ldots, d_n) \), \( d_i \in \mathbb{N} \), be minimal such that the matrix \( (d_i a_{i,j}) \) is symmetric. We denote by \( d \) the smallest common multiple of the \( d_j \). To distinguish between the objects associated to \( A \) and \( t A \), we add always a \( t \) in the notation. For example, we write \( t g = t n^- \oplus t h \oplus t n \) for the Cartan decomposition of the semisimple Lie algebra associated to \( t A \), and we write \( t F_i \) for the generators of \( U_q(t n^-) \).

In the following we assume that \( \ell \) is divisible by \( 2d \) and we set \( \bar{\ell} = \ell/d \) and let \( \ell_i \in \mathbb{N} \) be minimal such that \( \ell_i d_j \in \ell \mathbb{Z} \). In \[22\], Chapter 35, Lusztig defines two algebra homomorphisms:

\[
\text{Fr} : t U_{\xi}(n^-) \to U(n^-) \quad \text{Fr}^{(m)}_{t F_i} : t F_i^{(m)} \to F_i^{(m/\ell_i)}
\]

where we set \( F_i^{(m/\ell_i)} := 0 \) if \( \ell_i \) does not divide \( m \). The first map is in fact a Hopf algebra homomorphism, but the second is not.
The corresponding $R$-form of $C_q[U^-]$ is denoted $R_q[U^-]$, and its specializations at $\xi$ respectively $1$ are denoted $R_{\xi}[U^-]$ respectively $R_1[U^-]$. The two algebra homomorphisms $Fr$ and $Fr'$ induce dual maps:

$$Fr^*: R_1[U^-] \to R_{\xi}[U^-] \quad Fr'^*: R_{\xi}[U^-] \to R_1[U^-].$$

Note that $Fr^*$ is an algebra homomorphism since $Fr$ is a Hopf algebra homomorphism. Further, since $Fr \circ Fr'$ is the identity map it follows that $Fr'^* \circ Fr^*$ is the identity. In particular, $Fr^*$ is injective and maps $R_1[U^-]$ isomorphically onto a commutative subalgebra of $R_{\xi}[U^-]$.

The maps $Fr$ respectively its dual $Fr^*$ are called the Frobenius maps, and the maps $Fr'^*$ and $Fr^*$ are called the Frobenius splittings (for the connection with the algebraic geometric notion of Frobenius splitting in positive characteristic see [11] and [12]).

Of special interest are for us the $(\lambda, q)$–minors:

**Proposition 3** The Frobenius map $Fr^*$ is an algebra homomorphism which, restricted to the $(\lambda, q)$–minors, is the $\ell$-th power map, i.e.,

$$Fr^*(b_\lambda^\ell) = (b_\lambda^\ell)^\ell = b_\lambda^{\ell^2}.$$  

The splitting $Fr'^*$ is a $\mathbb{Z}_\xi$-module homomorphism which commutes with the image of $Fr^*$, i.e., for $f \in R_1[U^-]$ and $g \in R_{\xi}[U^-]$ we have

$$Fr'^*(Fr^*(f)g) = fFr'^*(g).$$

In particular,

$$Fr'^*(b_{m_1}^{n_1}\cdots b_{m_r}^{n_r}) = b_{m_1}^{n_1} \cdots b_{m_r}^{n_r}.$$  

**Proof.** The first part follows immediately from [18], we sketch for convenience the arguments: let $V_{\xi}(\overline{\ell}\lambda)$ be the $U_{\xi}(\ell\mathfrak{g})$–Weyl module for the highest weight $\overline{\ell}\lambda$, and let $L_{\xi}(\ell\lambda)$ be the simple module of highest weight $\ell\lambda$. Then $L_{\xi}(\ell\lambda)$ is nothing but the Weyl module $V(\lambda)$ for $U(\mathfrak{g})$, viewed as quantum module via the (extension of the) Frobenius map $Fr : U_{\xi}(\ell\mathfrak{g}) \to U(\mathfrak{g})$. One sees easily that this induces a dual map (also) denoted $Fr^*: V(\lambda)^* \to V_{\xi}(\overline{\ell}\lambda)^*$ which sends extremal weight vectors of $V(\lambda)^*$ to the corresponding extremal weight vectors of $V_{\xi}(\overline{\ell}\lambda)^*$. In particular, the $b_\lambda^\ell$ is sent to $b_{\overline{\ell}}^{\ell^2}$, which is equal to $(b_\lambda^\ell)^\ell$ by (2). The commutation part follows from [11], see also [12]. More precisely, the assumptions made on $\ell$ in these articles are, as mentioned in their introductions, just made for convenience. Suppose $f \in R_1[U^-]$ and
g ∈ ℜξ[ℓU−]. By choosing λ ∈ P+ generic enough, we can assume that we can identify f and g as elements of representations, i.e.,

\[ f \in V(\lambda)^* \hookrightarrow ℜ₁[U⁻], \quad g \in V(\ell\lambda)^* \hookrightarrow ℜξ[ℓU⁻], \]

and Fr∗(Fr∗(f)g) ∈ V(2\lambda)^* ↪ ℜ₁[U⁻]. Now the same arguments as in [1], section 4, or [3], section 6, go through to prove the compatibility Fr∗(Fr∗(f)g) = fFr∗(g). The rest is obvious since Fr∗ is a splitting.

The canonical basis \( \mathcal{B} \subset U_q(\mathfrak{n}⁻) \) spans \( U_{R}(\mathfrak{n}⁻) \) as an \( R \)-module, correspondingly its dual basis \( \mathcal{B}^* \) spans \( R[U⁻] \) as an \( R \)-module. It follows that the images of this basis in \( ℜξ[U⁻] \) respectively \( ℜ₁[U⁻] \) span the corresponding specializations.

Fix a reduced decomposition \( w₀ = s_{i₁} \cdots s_{i_N} \) of the longest word in \( W \). As in section 2 set \( b_j := b_{ij} \in \mathcal{B}^* \subset U_q(\mathfrak{g}) \), let \( S_{i₀} \) be the set of the monomials in the \( b_j \). Still denote \( A_{i₀} \) the \( R \)-module generated by \( S_{i₀} \). The \( R \)-algebra \( A_{i₀} \) specializes to a subalgebra \( A_{i₀}^\xi \subset ℜξ[U⁻] \) respectively \( A_{i₀}^1 \subset ℜ₁[U⁻] \). In both cases the algebra has the (images of the) monomials of \( S_{i₀} \) as a basis as \( Z₀ \)-module.

As an immediate consequence of the proposition above one sees:

**Corollary 4** Fr∗ induces an inclusion of algebras \( \text{Fr}^*: A_{i₀}^1 \to A_{i₀}^\xi \) such that \( \text{Fr}^*(\ell b_1^{m_1} \cdots b_N^{m_N}) = \ell b_1^{m_1} \cdots b_N^{m_N} \).

### 4 The path model, (\( \lambda, q \))–minors and SMT

We come now first back to representations of \( G \) respectively \( U(\mathfrak{g}) \). A combinatorial tool for the analysis of these representations is the path model, of which we recall quickly the most important features. Let \( \lambda \in P^+ \) be a dominant weight.

Let \( \tau = (\tau₀, \ldots, \tau_r) \) be a strictly decreasing sequence (with respect to the Bruhat order) of elements of \( W/W_\lambda \), and let \( a = (a₁, \ldots, a_r) \) be a strictly increasing sequence of rational numbers such that \( 0 < a₁ < \ldots < a_r < 1 \). The pair \( \tau = (\tau, a) \) is called a \( L-S \) path of shape \( \lambda \) if the pair satisfies the following integrality condition. For all \( i = 1, \ldots, r \):

- set \( r_i = l(\tau_i−1) − l(\tau_i) \). There exists a sequence \( \beta₁, \ldots, \beta_{r_i} \) of positive roots joining \( \tau_i−1 \) and \( \tau_i \) by the corresponding reflections, i.e.,

\[ \tau_i−1 > s\beta₁ \tau_i−1 > s\beta₂ s\beta₁ \tau_i−1 > \ldots > s\beta_{r_i} \cdots s\beta₁ \tau_{i−1} = \tau_i, \]

and \( a_i \langle \tau_i−1(\lambda), s\beta₁ \cdots s\beta_j−1(\beta_j) \rangle ∈ ℤ \) for all \( j = 1, \ldots, r_i \).
In the following formulas we set always $a_0 = 0$ and $a_{r+1} = 1$. The weight of an L-S path $\pi = (\tau, a)$ of shape $\lambda$ is the convex linear combination 
\[
\pi(1) = \sum_{i=0}^{r} x_{i+1} \tau_{i}(\lambda), \quad \text{where} \quad x_{i} = a_{i} - a_{i-1} \quad \text{for} \quad 1 \leq i \leq r + 1.
\]

For more details on the combinatorics of L-S paths we refer to [17]. Let $B(\lambda)$ be the set of L-S paths of shape $\lambda$, and let $B(\lambda)_{\tau}$ be the set of L-S paths $\pi = (\tau, a)$ of shape $\lambda$ such that $\tau \geq \tau_{0}$. The character of the Weyl module $V(\lambda)$ of highest weight $\lambda$ and the Demazure module $V(\lambda)_{\tau}$ can be calculated using the L-S paths:

**Theorem 5 ([17])**

\[
\text{Char } V(\lambda) = \sum_{\pi \in B(\lambda)} e^{\pi(1)} \quad \text{and} \quad \text{Char } V(\lambda)_{\tau} = \sum_{\pi \in B(\lambda)_{\tau}} e^{\pi(1)}.
\]

The path model of a representation is closely connected to the combinatorics of the crystal base. For the irreducible representation $V_{q}(\lambda)$ fix a highest weight vector $v_{\lambda}$ and let $B(\lambda) = \{bv_{\lambda} \mid b \in B, bv_{\lambda} \neq 0\}$ be the canonical basis of $V_{q}(\lambda)$. Denote $L_{q}(\lambda)$ the $\mathbb{C}[q]$–submodule of $V_{q}(\lambda)$ spanned by the elements of the canonical basis $B(\lambda)$ and set

\[
B_{q}(\lambda) = \{b \mod L_{q}(\lambda) \mid b \in B(\lambda)\}.
\]

Recall ([10, 8]) that the Kashiwara crystal $B_{q}(\lambda)$ (with the Kashiwara operators $f_{\alpha}, e_{\alpha}$) is isomorphic to the crystal given by the path model $B(\lambda)$ (with the root operators). In the following we will often identify these two crystals and just write $B(\lambda)$. For an element $b \in B(\lambda)$ we write $[b]^{\lambda}$ or just $[b]$ for its class in $B(\lambda)$, and for an element $\pi \in B(\lambda)$ we write $b_{\pi}$ for the element in $B(\lambda)$ such that $[b_{\pi}] = \pi$. The tensor products of crystals can be defined in the path model in terms of concatenations of paths or in terms of *standard tuples*:

**Definition 6** Let $\pi_{1} = (\tau, a) \in B(\lambda_{1}), \pi_{2} = (\kappa, b) \in B(\lambda_{2}), \ldots$, respectively $\pi_{m} = (\sigma, c) \in B(\lambda_{m})$ be L-S paths of shape $\lambda_{1}, \lambda_{2}, \ldots$, respectively $\lambda_{m}$, where

\[
\tau = (\tau_{0}, \ldots, \tau_{r}), \quad \kappa = (\kappa_{0}, \ldots, \kappa_{s}), \ldots, \quad \sigma = (\sigma_{0}, \ldots, \sigma_{t}).
\]

The tuple $\bar{\pi} = (\pi_{1}, \pi_{2}, \ldots, \pi_{m})$ is called a *standard tuple of shape* $\bar{\lambda} = (\lambda_{1}, \ldots, \lambda_{m})$ if there exist representatives $\bar{\tau}_{0}, \ldots, \bar{\tau}_{r} \in W$ of the $\tau_{0}, \ldots, \tau_{r} \in$
$W/W_\lambda$, and representatives $\tilde{\kappa}_0, \ldots, \tilde{\kappa}_s \in W$ of the $\kappa_0, \ldots, \kappa_s \in W/W_\lambda$, etc., such that

$$\tilde{\tau}_0 \geq \ldots \geq \tilde{\tau}_r \geq \tilde{\kappa}_0 \geq \ldots \geq \tilde{\kappa}_s \geq \ldots \geq \tilde{\sigma}_0 \geq \ldots \geq \tilde{\sigma}_t,$$

Such a sequence $(\tilde{\tau}_0, \ldots, \tilde{\tau}_r, \tilde{\kappa}_0, \ldots, \tilde{\kappa}_s, \ldots, \tilde{\sigma}_0, \ldots, \tilde{\sigma}_t)$ of representatives is called a defining chain. The weight $\pi(1)$ is the sum $\pi_1(1) + \ldots + \pi_m(1)$ of the weights of the L–S paths.

**Remark 7** If $\lambda_1 = \ldots = \lambda_m$, the condition on standardness reduces to the condition $\tau_r \geq \kappa_0 \geq \kappa_s \geq \ldots \geq \sigma_0 \geq \ldots \geq \sigma_t$ in the Bruhat ordering on $W/W_\lambda$.

By the independence of the crystal structure of path models [19], we have a canonical bijection between the set $B(\lambda)$ of standard tuples of shape $\lambda$ and the set of L–S paths $B(\lambda)$ of shape $\lambda = \lambda_1 + \ldots + \lambda_m$. In fact, the standard tuples correspond exactly to the elements in the Cartan component of the tensor product of the crystals. The notation $b_\pi \in B(\lambda)$ for an element of the canonical basis corresponding to an element of a path model generalizes in the obvious way.

The following is a translation of [14], Proposition 33, into the language of standard tuples:

**Proposition 8** Let $b_i \in B(\lambda_i)$ be elements of the canonical basis and let $\pi_i \in B(\lambda_i)$ be the L–S path such that $[b_i] = \pi_i$, $i = 1, \ldots, m$. Suppose the tuple $\pi = (\pi_1, \ldots, \pi_m)$ is standard, let $\pi \in B(\lambda)$, $\lambda = \sum_{i=1}^m \lambda_i$, be the L–S path of shape $\lambda$ that identifies with the standard tuple, and let $b_\pi = b_{\pi} \in B(\lambda)$ be such that $[b_\pi] = \pi$. Then $[b_1] \otimes \ldots \otimes [b_m]$ identifies with $[b_\pi]$, and for the dual basis elements there exists a $s \in \mathbb{N}$ such that

$$q^s b_1^* b_2^* \ldots b_m^* = b_\pi^* \text{ mod } qL_q^*(\lambda).$$

The dual representation $V(\lambda)^*$ can be geometrically realized as the space of global sections $H^0(G/B, L_\lambda)$ of the line bundle $L_\lambda$ on $G/B$. We recall now the construction of a basis of $H^0(G/B, L_\lambda)$ using $(\lambda, q)$–minors. The restriction map induces a map between (tensor products of) dual Weyl modules of the quantum group $U_\xi(tg)$:

$$V_\xi(\lambda)^* \otimes \ldots \otimes V_\xi(\lambda)^* \longrightarrow V_\xi(\ell\lambda)^*. $$

We write shortly $b_\tau^\lambda \ldots b_\kappa^\lambda$ for the image of $b_\tau^\lambda \otimes \ldots \otimes b_\kappa^\lambda$. The dual of the Frobenius splitting induces a map between dual Weyl modules of the quantum group $U_\xi(tg)$ and $U(g)$ (see [13]):

$$Fr^*: V_\xi(\ell\lambda)^* \longrightarrow V(\lambda)^*. $$

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Note that these maps are actually defined over $\mathbb{Z}_\xi$.

Let now $\pi = (\tau, a) \in B(\lambda)$ be an L-S path of shape $\lambda$, fix $\ell \in \mathbb{N}$, $\ell > 0$ minimal such that $2d$ divides $\ell$ and $\ell a_i \in \mathbb{N}$ for all $i = 1, \ldots, r$, and let $\xi$ be a corresponding primitive $2\ell$-th root of unity.

**Definition 9** The path vector $p_\pi \in H^0(G/B, \mathcal{L}_\lambda)$, $\pi = (\tau, a)$, is defined as:

$$p_\pi := \mathrm{Fr}' \ast \left( (b_{\tau_0}^\lambda)^{\overline{\ell} x_1} (b_{\tau_1}^\lambda)^{\overline{\ell} x_2} \cdots (b_{\tau_{r-1}}^\lambda)^{\overline{\ell} x_r} (b_{\tau_r}^\lambda)^{\overline{\ell} x_{r+1}} \right),$$

where $x_i = a_i - a_{i-1}$ for $1 \leq i \leq r + 1$, and the $b_{\tau_j}^\lambda$ are $(\lambda, q)$-minors in $V_\xi(\lambda)^*$ for $j = 0, \ldots, r$.

To distinguish between extremal weight vectors for $U(g)$ and $U_\xi(\ell g)$ we keep the notation $b_{\tau_j}^\lambda$ in the quantum case and write $p_{\tau_j}^\lambda$ for the classical case. The reader should think of the $p_\pi$ as a kind of algebraic approximation of $p_\pi \sim \ell \sqrt{p_{\tau_0} x_1^\ell p_{\tau_1} x_2^\ell \cdots p_{\tau_r} x_{r+1}^\ell}.$

Though the expression above does not make sense in general, we will see soon that at least in some special cases the expression can be given a useful interpretation.

We will also consider **standard monomials** in the path vectors:

**Definition 10** Let $\pi = (\pi_1, \ldots, \pi_m)$ be a tuple of shape $\lambda = (\lambda_1, \ldots, \lambda_m)$. We associate to $\pi$ the monomial $p_{\pi} = p_{\pi_1} p_{\pi_2} \cdots p_{\pi_m}$ in $H^0(G/Q, \mathcal{L}_\lambda)$, where $\lambda = \lambda_1 + \cdots + \lambda_m$. The monomial $p_{\pi}$ is called a **standard monomial of shape $\lambda$** if $\pi$ is a standard tuple.

The path vectors are defined over $\mathbb{Z}_\xi$ for some appropriate root of unity, so the collection $\mathbb{B}(\lambda)$ of all standard monomials of shape $\lambda$ is defined over the ring $S \subset \mathbb{C}$ generated by $\mathbb{Z}$ and all roots of unity.

**Theorem 11** ([18]) The set of standard monomials $\mathbb{B}(\lambda)$ of shape $\lambda$ forms a basis for the space of sections $H^0(G/B, \mathcal{L}_\lambda)$, $\lambda = \lambda_1 + \cdots + \lambda_m$.

For more information about applications of standard monomial theory (singularities of Schubert varieties, generators and relations for homogeneous coordinate rings of Schubert varieties, Koszul property, Pieri-Chevalley type formula, ...) see for example [13], [18], [16].
5 The path vectors and the dual canonical basis

Given an L-S path of shape \( \lambda \), we use the same notation \( p_\pi \) for the path vector in \( H^0(G/B, L_\lambda) \) as well as the element \( c^\lambda_{p_\pi} \in \mathbb{C}[U^-] \). The dual canonical basis \( B^* \) is compatible with the injection \( H^0(G/B, L_\lambda) \hookrightarrow \mathbb{C}[U^-] \), i.e., a subset of \( B^* \) forms a basis of the image of \( H^0(G/B, L_\lambda) \).

So it is natural to ask for a description of the transformation matrix between the basis given by the path vectors and the basis given by elements of the dual canonical basis.

Let \( \pi = (\pi_1, \ldots, \pi_m) \) be an L-S path of shape \( \lambda \). We identify an element \( \tau \in W/W_\lambda \) with its minimal representative in \( W \). Fix a reduced decomposition \( w_0 = s_{i_1} \cdots s_{i_N} \) of the longest word of the Weyl group, we write \( \tilde{w}_0 \) to refer to this decomposition. For \( j = 1, \ldots, N \), set \( y_j = s_{i_1} \cdots s_{i_j} \) as in (4).

**Definition 12** We say that \( \pi \) is compatible with \( \tilde{w}_0 \) if \( \tau \subset \{ y_1, \ldots, y_N \} \), i.e., \( \{ \tilde{\tau}_0, \ldots, \tilde{\tau}_r, \tilde{\kappa}_0, \ldots, \tilde{\kappa}_s, \ldots, \tilde{\sigma}_0, \ldots, \tilde{\sigma}_t \} \subset \{ y_1, \ldots, y_N \} \).

Let now \( \underline{\pi} = (\pi_1, \ldots, \pi_m) \) be a standard tuple of shape \( \underline{\lambda} = (\lambda_1, \ldots, \lambda_m) \) with a defining chain (see Definition 6).

\[
\tilde{\tau}_0 \geq \ldots \geq \tilde{\tau}_r \geq \tilde{\kappa}_0 \geq \ldots \geq \tilde{\kappa}_s \geq \ldots \geq \tilde{\sigma}_0 \geq \ldots \geq \tilde{\sigma}_t,
\]

**Definition 13** We say that \( \underline{\pi} \) is compatible with \( \tilde{w}_0 \) if the defining chain can be chosen such that \( \{ \tilde{\tau}_0, \ldots, \tilde{\tau}_r, \tilde{\kappa}_0, \ldots, \tilde{\kappa}_s, \ldots, \tilde{\sigma}_0, \ldots, \tilde{\sigma}_t \} \subset \{ y_1, \ldots, y_N \} \).

The following theorem provides the connection between certain standard monomials and the adapted algebras:

**Theorem 14** Let \( p_{\underline{\pi}} = p_{\pi_1} \cdots p_{\pi_m} \in H^0(G/B, L_\lambda) \) be a standard monomial monomial of shape \( \underline{\lambda} = (\lambda_1, \ldots, \lambda_m) \). If there exists a defining chain \( (\tilde{\tau}_0, \ldots, \tilde{\tau}_r, \tilde{\kappa}_0, \ldots, \tilde{\kappa}_s, \ldots, \tilde{\sigma}_0, \ldots, \tilde{\sigma}_t) \) which is compatible with some reduced decomposition of \( w_0 \), then the inclusion \( H^0(G/B, L_\lambda) \hookrightarrow \mathbb{C}[U^-] \), \( \lambda = \lambda_1 + \ldots + \lambda_m \), maps the standard monomial \( p_{\underline{\pi}} \) onto the element \( b^*_\underline{\lambda} \) of the dual canonical basis, up to multiplication by a root of unity.

**Proof.** We fix a reduced decomposition \( \tilde{w}_0 \) of \( w_0 \) and assume that the standard tuple \( \underline{\pi} = (\pi_1, \ldots, \pi_m) \) has a defining chain compatible with \( \tilde{w}_0 \). Let \( A^1_{\tilde{w}_0} \subset R_1[U^-] \subset \mathbb{C}[U^-] \) be the corresponding (specialization of the) adapted algebra defined in section 2. We will show that the \( p_{\pi_j} \) are elements of the set of monomials \( S_{\tilde{w}_0} \) spanning \( A^1_{\tilde{w}_0} \). This proves the theorem by Theorem 3.
So in the following it suffices to consider only one path vector \( p_\pi \). Fix a
appropriate \( \ell \in \mathbb{N} \) as in section \( \text{[3]} \). Definition \( \text{[3]} \), for \( \pi = (\tau_0, \ldots, \tau_r; a_1, \ldots, a_r) \),
and let \( \xi \) be a primitive \( 2\ell \)-th root of unity. Set
\[
  m_\pi := (b^\lambda_{\tau_0})^{\ell a_1} (b^\lambda_{\tau_1})^{\ell (a_2 - a_1)} \cdots (b^\lambda_{\tau_{r-1}})^{\ell (a_r - a_{r-1})} (b^\lambda_{\tau_r})^{\ell (1 - a_r)},
\]
where the \( b^\lambda_{\tau_j} \) are \((\lambda, \xi)\)-minors in \( V_2(\lambda)^* \) for \( j = 0, \ldots, r \). By definition, we
have \( p_\pi = \text{Fr}^\ell (m_\pi) \). Let \( \lambda = \sum_{j=1}^{r} \lambda_j \varpi_j \) be the expression of \( \lambda \) as a linear
combination of fundamental weights. Then, by \( \text{[2]} \), we know for arbitrary
\( \tau \in W \):
\[
  b^\lambda_{\tau} = \prod_{i=1}^{n} (b^\varpi_i)^{\lambda_i},
\]
So we can rewrite \( m_\pi \) as:
\[
  m_\pi = \left( \prod_{j=1}^{n} (b^\varpi_i)^{\ell \lambda_i} \right) \left( \prod_{j=1}^{n} (b^\varpi_i)^{\ell (a_1 - a_i)} \right) \cdots \left( \prod_{j=1}^{n} (b^\varpi_i)^{\ell (1 - a_r)} \right).
\]
The defining chain of \( \pi \) is compatible with \( \tilde{w}_0 \), it follows (see Remark \( \text{[I]} \)) for
the lifts \( \tilde{\tau}_j \) of the \( \tau_j \) that the \( b^\varpi_j \) are generators of \( tA_{\tilde{\tau}_0} \subset R^{[\ell]} \). Note if
\( \tilde{\tau} \in W \) is a lift for \( \tau \in W/W_\lambda \), then \( (b^\varpi_j)^{\lambda_j} = (b^\varpi_j)^{\lambda_j} \). So \( b_\pi \in tA_{\tilde{w}_0} \), and
by reordering the elements we can write with the notation as in \( \text{[3]} \):
\[
  m_\pi = c b_1^{r_1} \cdots b_N^{r_N},
\]
where \( c \) is some root of unity. If all the exponents \( r_j \) are divisible by \( \ell \), then
we know by Proposition \( \text{[3]} \) that
\[
  p_\pi = \text{Fr}^\ell (m_\pi) = c \text{Fr}^\ell (b_1^{r_1} \cdots b_N^{r_N}) = c b_1^{r_1/\ell} \cdots b_N^{r_N/\ell},
\]
is an element of \( S_{\tilde{w}_0} \), the spanning set for \( A_{\tilde{w}_0} \).
To prove the divisibility property, let \( \pi = (\tau_0, \ldots, \tau_r; a_1, \ldots, a_r) \) be an
L-S path of shape \( \lambda \) compatible with \( \tilde{w}_0 \) and consider the following projection:
write \( \lambda \) as a linear combination \( \sum_{\alpha} \lambda_\alpha \varpi_\alpha \) of fundamental weights, fix
a simple root \( \alpha \), let \( \tau_\alpha \) be the class of \( \tau \) in \( W/W_{\varpi_\alpha} \), and set
\[
  \pi_\alpha = (\tau_0, \ldots, \tau_r; a_1, \ldots, a_r) = (\sigma_0, \ldots, \sigma_{r}; c_1, \ldots, c_t),
\]
where the second pair of sequences is obtained from the first by omitting
repetitions among the \( \tau_j \) as well as the corresponding \( b_j \), i.e., if
\[
  \sigma_0 = \tau_0 \ldots = \tau_{j-1} > \sigma_1 = \tau_j = \ldots = \tau_{k-1} > \sigma_2 = \tau_k = \ldots = \tau_{m-1} > \ldots
\]
then \( c_1 = a_j, c_2 = a_k, c_3 = a_m, \ldots \). Set

\[
m_{\pi_\alpha} = (b_{\sigma_0}^{\omega_\alpha})^{\tilde{\lambda}_\alpha(c_1 - 1)} \cdots (b_{\sigma_t}^{\omega_\alpha})^{\tilde{\lambda}_\alpha(c_2 - c_1)} \in R_{\xi}[U^-]
\]

Then \( m_\pi = \xi^p m_{\pi_1}, m_{\pi_2}, \ldots, m_{\pi_n} \in R_{\xi}[U^-] \) for some power \( p \) of \( \xi \). So the divisibility property, and hence the first part of the theorem, is a consequence of the following lemma, and the fact that \( p_\pi \) and the canonical basis element \( b_\pi \) coincide (up to multiplication by an appropriate root of unity) follows from Proposition 8.

**Lemma 15** Set \( c_0 = 0 \) and \( c_{r+1} = 1 \). The exponents have the property \( \lambda_\alpha(c_{j+1} - c_j) \in \mathbb{Z} \) for all \( j = 0, \ldots, r \).

**Proof.** Let \( \pi_\alpha = (\tau_0, \ldots, \tau_r; a_1, \ldots, a_r) = (\sigma_0, \ldots, \sigma_{t}; c_1, \ldots, c_t) \) be as above and suppose

\[
\sigma_{i-1} < \tau_{j-1} = \sigma_i = \tau_j = \tau_{j+1} = \cdots = \tau_{k-1} > \sigma_{k+1} = \tau_k
\]

Corresponding to the fixed reduced decomposition \( \tilde{w}_0 \) let \( y_j = s_{i_1} \cdots s_{i_j} \) be as in [4]. By assumption, \( \pi \) is compatible with \( \tilde{w}_0 \), so \( \{\tau_0, \ldots, \tau_r\} \subset \{y_1, \ldots, y_N\} \). It follows: there exist indices such that

\[
\tau_k \leq y_t < y_ts_\alpha \leq \tau_{k-1} < \cdots < \tau_j \leq y_v < y_vs_\alpha \leq \tau_{j-1}
\]

and \( y_ts_\alpha \) and \( y_vs_\alpha \) are elements of \( \{y_1, \ldots, y_N\} \). Set \( \beta = y_t(\alpha) \), so \( s_\beta(y_t s_\alpha) = y_t \). The definition of an L-S path implies:

\[
c_{i+1} \lambda_\alpha = a_k \lambda_\alpha = a_k \langle \lambda, \alpha^\vee \rangle = a_k \langle y_t(\lambda), \beta^\vee \rangle \in \mathbb{Z}.
\]

Similarly, set \( \delta = y_v(\alpha) \), so \( s_\delta(y_v s_\alpha) = y_v \). The definition of an L-S path implies:

\[
c_i \lambda_\alpha = a_j \lambda_\alpha = a_j \langle \lambda, \alpha^\vee \rangle = a_j \langle y_v(\lambda), \delta^\vee \rangle \in \mathbb{Z}.
\]

It follows: \( \lambda_\alpha(c_{i+1} - c_i) \in \mathbb{Z} \).

6 Examples

In this section, we study some examples which illustrate our main theorem. Let \( g \) be semisimple Lie algebra, let \( \mathcal{B} \subset U_q(n^-) \) be the canonical basis and denote \( L_q(n^-) \) the \( \mathbb{C}[q] \)–submodule of \( U_q(n^-) \) spanned by the elements of
the canonical basis $B$. Let $(B(\infty), L_q(n^-))$ be the crystal of $U_q(n^-)$ (see for example [3], or [4]), consisting of the $\mathbb{C}[q]$-module $L_q(n^-)$, the basis

$$B(\infty) = \{ b \mod L_q(n^-) \mid b \in B \},$$

and the Kashiwara operators $\{ f_\alpha, e_\alpha \mid \alpha \text{ a simple root} \}$ on $B(\infty) \cup \{ 0 \}$.

In the same way as the dual canonical basis $B^*$ is compatible with the representations $V_q(\lambda)^*$, the crystal of $U_q(n^-)$ can be seen as the limit of the crystals of the representations $V_q(\lambda)$. Recall (section 4) that the path model can also be understood as a combinatorial model for the crystal of the representations.

**Example 16** Let $\mathfrak{g}$ be of type $A_2$, denote $\alpha_1, \alpha_2$ the simple roots, then every element $b \in B^* \subset \mathbb{C}[U^-]$ of the dual canonical basis is the image of some appropriately chosen standard monomial $p_{\underline{\pi}}$.

**Proof.** Let $u_\infty \in B(\infty)$ be the highest weight element of $B(\infty)$ and let $u_b$ be the class of $b$ in $B(\infty)$. The element $u_b \in B(\infty)$ can be written as

$$u_b = f^r_{\alpha_1} f^m_{\alpha_2} f^n_{\alpha_1} u_\infty, \quad \text{where } m \geq n, \quad \text{or } u_b = f^s_{\alpha_2} f^t_{\alpha_1} f^\ell u_\infty \quad \text{where } s \geq t,$$

and $s = \ell + n$, $r = \max\{n, m - \ell\}$ and $t = \min\{m - n, \ell\}$ (see [21] or [2]).

Note if $u_b = f^r_{\alpha_1} f^m_{\alpha_2} f^n_{\alpha_1} u_\infty$ is such that $\ell < m - n$, then in the corresponding expression $u_b = f^s_{\alpha_2} f^t_{\alpha_1} f^\ell u_\infty$ we have $r = m - \ell > n = \ell + n - \ell = s - t$.

So in the following we just write $u_b = f^\ell f^m f^n u_\infty$ and suppose that $i$ and $j$ are chosen such that $\ell \geq m - n \geq 0$.

Set $\lambda = (\ell + 2n - m) \varpi_i + (m - n) \varpi_j$, and let $\underline{\pi}_\lambda = (id, id, id)$ be the unique standard tuple of shape $\underline{\lambda} = (n \varpi_i, (m - n) \varpi_j, (\ell + n - m) \varpi_i)$ of weight $\underline{\pi}_\lambda(1) = \lambda$. By applying the root operators (see [19]), we get for $\overline{\pi} = f^\ell f^m f^n \pi_\lambda$ the following standard sequence of shape $\lambda$:

$$\overline{\pi} = (\pi_1 = (s_j s_i), \pi_2 = (s_i s_j), \pi_3 = (s_i)).$$

Note that the sequence $(s_j s_i, s_i s_j, s_i)$ is a defining chain for $\mathfrak{g}$ which is obviously compatible with the reduced decomposition $w_0 = s_i s_j s_i$. It follows that the image of the standard monomial $p_{\underline{\pi}} = p_{s_j s_i} p_{s_i s_j} p_{s_i}$ of shape $\underline{\lambda}$ in $\mathbb{C}[U^-]$ is, up to multiplication by a root of unity, the element $b = b_{\underline{\pi}}$ of the dual canonical basis $B^*$.

**Example 17** The case $\mathfrak{g} = sl_3$ is very special, in general it is not possible to find for every element in $B(\infty)$ an “appropriate” standard tuple of L-S.
paths with a defining chain compatible with a reduced decomposition of $w_0$. An example is for $\mathfrak{g} = \mathfrak{sl}_4$ the element $b \in B^*$ such that for its class in $B(\infty)$ holds $u_b = f_{\alpha_2}f_{\alpha_1}f_{\alpha_3}u_\infty$. Nevertheless, it is easy to see by Theorem 14, that the other elements of $B(\infty)$ of the same weight

$$u_{b_1} = f_{\alpha_1}f_{\alpha_2}f_{\alpha_3}u_\infty, \quad u_{b_2} = f_{\alpha_3}f_{\alpha_2}f_{\alpha_1}u_\infty, \quad u_{b_3} = f_{\alpha_3}f_{\alpha_1}f_{\alpha_1}u_\infty$$

correspond all to elements of the dual canonical basis which are “standard monomials”: $b_1 = p_{(s_1s_2s_3)} \in H^0(G/B, \mathcal{L}_{\omega_3})$, $b_2 = p_{(s_3s_2s_1)} \in H^0(G/B, \mathcal{L}_{\omega_1})$ and $b_3 = p_{(s_1s_3s_2)} \in H^0(G/B, \mathcal{L}_{\omega_3})$.

But let now $\lambda = \omega_1 + \omega_3$, then there exists an L–S path of shape $\lambda$ that corresponds under the identification between crystal graphs and path models to $u_b$, it is the path $\pi = (s_2s_3s_1, s_3s_1; \frac{1}{2})$. The $\mathfrak{g}$-module $V(\lambda)$ is isomorphic to $\mathfrak{g}$ endowed with the adjoint action. Via this isomorphism, the canonical basis of $V(\lambda)$ is given (up to scalar multiples) by a Chevalley basis. As a function, $p_\pi$ is the dual of the canonical basis element $h_{\alpha_2}$, thus $b = p_\pi$ can again be constructed as a “standard monomial” for an appropriate choice of $\lambda$, but this time not in the context of Theorem 14.

**Example 18** We suppose again $\mathfrak{g} = \mathfrak{sl}_3$. Set $\lambda = 2\omega_1 + 2\omega_2$. We shall explicit the $p_\pi$ corresponding to the zero weight space $V(\lambda)_0$ of $V(\lambda)$.

Let’s introduce some notation. Let

$$a = p_{\omega_1}^{\omega_1}, \quad b = p_{\omega_2}^{\omega_2}, \quad c = p_{s_2s_1}^{\omega_1}, \quad d = p_{s_1s_2}^{\omega_2},$$

be the elements of $\mathbb{C}[U^-]$ as defined in section 4. It is well known (see [3]), that the dual canonical basis is given by $\{a^kb^lc^md^t, k, l, s, t \in \mathbb{N}, kl = 0\}$. Moreover, it is easy to see that $ab = c + d$. Indeed, $ab$ decomposes in the base $\{c, d\}$ of the subspace of $\mathbb{C}[U^-]$ of weight $\alpha_1 + \alpha_2$. Both coefficients in this base are equal by symmetry and equal to one by [3], Theorem 2.3.

The LS paths corresponding to $V(\lambda)_0$ are

$$\pi_1 = (s_2s_1, s_1; \frac{1}{2}), \quad \pi_2 = (w_0, \text{Id}; \frac{1}{2}), \quad \pi_3 = (s_1s_2, s_2; \frac{1}{2}).$$

Remark that $\pi_2$ is compatible with a reduced decomposition while $\pi_1$ and $\pi_3$ are not. As in the proof of Theorem 9, with $l = 2$, we obtain, up to a root of one,

$$p_{\pi_1} = cba, \quad p_{\pi_2} = cd, \quad p_{\pi_3} = dab.$$

Hence,

$$p_{\pi_1} = c^2 + cd, \quad p_{\pi_2} = cd, \quad p_{\pi_3} = cd + d^2.$$
Remark that \( p_{\pi_2} \) is an element of the dual canonical basis while \( p_{\pi_1} \) and \( p_{\pi_3} \) are not. Under the identification of the path model with crystal basis elements we have in the dual canonical basis: \( b_{\pi_1} = c^2 \), \( b_{\pi_2} = cd \) and \( b_{\pi_3} = d^2 \), so the transformation matrix reads as: \( p_{\pi_1} = b_{\pi_1} + b_{\pi_2} \), \( p_{\pi_2} = b_{\pi_2} \) and \( p_{\pi_3} = b_{\pi_3} + b_{\pi_2} \).

**Example 19** We suppose again \( \mathfrak{g} = \mathfrak{sl}_3 \), set now \( \lambda = m\varpi_1 + m\varpi_2 \). To simplify the notation of the L–S paths, we still write \( \pi = (\tau, a) \) for \( a = (a_1, a_2, a_3) \), even if we have equality in \( 0 \leq a_1 \leq a_2 \leq a_3 \leq 1 \). In case of equality, the reader has to omit the corresponding parameters and Weyl group elements to avoid double counting.

We claim that the \( p_{\pi} \) have a simple expression in terms of the canonical basis: if \( \pi = (\tau, a) \) is compatible with some reduced decomposition, then \( p_{\pi} = b_{\pi} \) is an element of the dual canonical basis (after a possible renormalization with a root of unity).

If \( \pi = (\tau, a) \) is not compatible with a reduced decomposition, then let \( \ell \) be even and minimal with the property that \( \ell a_i \in \mathbb{N} \) for all \( i \). Again, after a possible renormalization (multiplication with a root of unity):

**Proposition 20** If \( \pi = (\tau, a) \) is not compatible with a reduced decomposition, then set \( t = \min\{m(a_2 - a_1), m(a_3 - a_2)\} \). The following transition relation holds between the canonical basis and the path vectors:

\[
p(\tau, a) = \sum_{j=0}^{[t]} \binom{t\ell}{j\ell} b_{(\tau, a_1 + \frac{t}{\ell}, a_2, a_3 - \frac{t}{\ell})}
\]

**Proof.** To prove the statement, we divide the set of L–S paths in groups. The first group is the set \( B(\lambda)^1 = B(\lambda)^{1,1} \cup B(\lambda)^{1,2} \), where:

\[
B(\lambda)^{1,1} = \{ \pi = (w_0, s_1 s_2, s_1, id; a_1, a_2, a_3) \mid ma_1, ma_2, ma_3 \in \mathbb{N} \}
B(\lambda)^{1,2} = \{ \pi = (w_0, s_2 s_1, s_2, id; a_1, a_2, a_3) \mid ma_1, ma_2, ma_3 \in \mathbb{N} \}
\]

These paths are all compatible with a reduced decomposition, so the elements \( p_{\pi} \) are canonical basis elements: Set \( x_1 = a_1 \), \( x_2 = a_2 - a_1 \), \( x_3 = a_3 - a_2 \), then, after renormalizing the \( p_{\pi} \) (multiplication by a root of unity), we have (with same notation as above):

\[
b_{\pi} = p_{\pi} = a^{m(x_2 + x_3)} e^{m(x_1 + x_2)} d^{m(x_1 + x_2)} \quad \text{for} \quad \pi \in B(\lambda)^{1,1}
b_{\pi} = p_{\pi} = b^{m(x_2 + x_3)} e^{m(x_1 + x_2)} d^{m(x_1 + x_2)} \quad \text{for} \quad \pi \in B(\lambda)^{1,2}
\]
In other words, let $a^v c^w d^w$ (respectively $b^v c^w d^w$) be an element of the dual canonical basis such that
\[ v \leq w \leq u + v \leq m \] (respectively: $w \leq v \leq u + w \leq m$)

In both cases, this canonical basis element is equal to $p_r$, where
\[
\pi = \begin{cases} 
(w_0, s_1 s_2, s_1, id; \frac{w}{m}, \frac{w}{m}, \frac{w}{m} + \frac{w}{m}) & \text{for } a^v c^w d^w \\
(w_0, s_2 s_1, s_2, id; \frac{w}{m}, \frac{w}{m}, \frac{w}{m} + \frac{w}{m}) & \text{for } b^v c^w d^w 
\end{cases}
\]

The second set of L–S paths of shape $\lambda$ is in this example the union $B(\lambda)^2 = B(\lambda)^{2,1} \cup B(\lambda)^{2,2}$, where:
\[
B(\lambda)^{2,1} = \{ \pi = (w_0, s_2 s_1, s_1, id; a_1, a_2, a_3) \mid ma_1, 2ma_2, ma_3 \in \mathbb{N} \} \\
B(\lambda)^{2,2} = \{ \pi = (w_0, s_1 s_2, s_2, id; a_1, a_2, a_3) \mid ma_1, 2ma_2, ma_3 \in \mathbb{N} \}
\]

By construction, we get (up to multiplication by a root of unity): Let $\ell$ be minimal and even such that $\ell a_i \in \mathbb{N}$ for all $i$, then, for $\pi \in B(\lambda)^{2,1}$, we have up to multiplication by a root of unity $\tilde{r}$:
\[
P_\pi = \begin{cases} 
\tilde{r} \text{Fr}^* (c^{m x_1} \ell, d^{m x_2} \ell, \lambda(x_2 - x_1)) & \text{if } x_2 \geq x_3 \\
\tilde{r} \text{Fr}^* (c^{m x_1} \ell, d^{m x_2} \ell, \lambda(2x_2 - x_1)) & \text{if } x_3 \geq x_2
\end{cases}
\]

and hence by Proposition 3 for some roots of unity $r'$:
\[
P_\pi = \begin{cases} 
r' \text{Fr}^* (c^{m x_2 - x_3}, m x_1, \lambda(x_2 - x_1)) & \text{if } x_2 \geq x_3 \\
r' \text{Fr}^* (c^{m x_2 - x_3}, m x_1, \lambda(2x_2 - x_1)) & \text{if } x_3 \geq x_2
\end{cases}
\]

Replacing $ab$ by $(c + d)$ (note that $c$ and $d$ commute) we get after renormalizing (by a root of unity):
\[
P_\pi = \begin{cases} 
\sum_{j=0}^{m x_2} (m x_2) b^{m(x_2 - x_3)} c^{m x_1} d^{m x_1} \text{Fr}^* (c^{m(x_2 + x_3)} - j d) & \text{if } x_2 \geq x_3 \\
\sum_{j=0}^{m x_2} (m x_2) a^{m(x_3 - x_2)} c^{m x_1} d^{m x_1} \text{Fr}^* (c^{m 2x_2 - j d}) & \text{if } x_3 \geq x_2
\end{cases}
\]

Since $2m x_2$ respectively $m(x_2 + x_3)$ are integers, it follows that Fr$^*$ maps a summand to zero for $j$ not divisible by $\ell$, and hence:
\[
P_\pi = \begin{cases} 
\sum_{j=0}^{m x_3} (m x_3) b^{m(x_2 - x_3)} c^{m(x_2 + x_3) - j d} & \text{if } x_2 \geq x_3 \\
\sum_{j=0}^{m x_2} (m x_2) a^{m(x_3 - x_2)} c^{m x_1} d^{m x_1} e^{m 2x_2 - j d} & \text{if } x_3 \geq x_2
\end{cases}
\]

and thus, after reordering, we get for $\pi \in B(\lambda)^{2,1}$:
\[
P_\pi = \begin{cases} 
\sum_{j=0}^{m x_3} (m x_3) b^{m(x_2 - x_3)} c^{m(x_2 + x_3) - j d} & \text{if } x_2 \geq x_3 \\
\sum_{j=0}^{m x_2} (m x_2) a^{m(x_3 - x_2)} c^{m(x_2 + x_3) - j d} & \text{if } x_3 \geq x_2
\end{cases}
\]
One sees easily that in terms of the root operators one has
\[
\pi = f_1^{m(x_1+2x_2)} f_2^{m(2x_1+x_2+x_3)} f_1^{mx_1}(id),
\]
so, by Example 9, the corresponding element \(b_\pi\) in the dual canonical basis is
\[
b_\pi = \begin{cases} 
    b^{m(x_2-x_3)} c^{m(x_1+x_2+x_3)} d^{mx_1} & \text{if } x_2 \geq x_3 \\
    a^{m(x_3-x_2)} c^{m(x_1+2x_2)} d^{mx_1} & \text{if } x_3 \geq x_2.
\end{cases}
\]
i.e., \(b_\pi\) is the summand for \(j = 0\) in the expression above. Note that the exponents in the summands for \(p_\pi\) in (3) satisfy all the following inequalities:
\[
2w \leq u + v + w \leq 2v \leq 2m \quad \text{for } b^u c^v d^w \\
2w \leq v + w \leq 2(u + v) \leq 2m \quad \text{for } a^u c^v d^w.
\]
On the other hand, any triple \((u, v, w)\) of integers satisfying these inequalities gives an element of the dual canonical basis of \(V(\lambda)^*\) and is hence equal to \(b_\eta\) for some L–S path \(\eta\) of shape \(\lambda\). We obtain the corresponding path by the following rules:
\[
\eta = \begin{cases} 
    (w_0, s_2 s_1, s_1, id; \frac{w}{m}, \frac{u+v+w}{m}) & \text{for } b_\eta = b^u c^v d^w \\
    (w_0, s_2 s_1, s_1, id; \frac{w}{m}, \frac{u+v+w}{2m}, \frac{w+z}{m}) & \text{for } b_\eta = a^u c^v d^w
\end{cases}
\]
In the same way one gets for \(\pi \in B(\lambda)^{2,2}\):
\[
p_\pi = \begin{cases} 
    \sum_{j=0}^{mx_1} (m^{x_3})^{j} a^{m(x_2-x_3)} c^{mx_1+j} d^{m(x_1+x_2+x_3)-j} & \text{if } x_2 \geq x_3 \\
    \sum_{j=0}^{mx_1} (m^{x_2})^{j} b^{m(x_3-x_2)} c^{mx_1+j} d^{m(x_1+2x_2)-j} & \text{if } x_3 \geq x_2
\end{cases} \tag{6}
\]
As above, in terms of the root operators one has
\[
\pi = f_2^{m(x_1+2x_2)} f_1^{m(2x_1+x_2+x_3)} f_2^{mx_1}(id),
\]
so the corresponding element \(b_\pi\) in the dual canonical basis is the summand for \(j = 0\) in the expression above:
\[
b_\pi = \begin{cases} 
    a^{m(x_2-x_3)} c^{mx_1} d^{m(x_1+x_2+x_3)} & \text{if } x_2 \geq x_3 \\
    b^{m(x_3-x_2)} c^{mx_1} d^{m(x_1+2x_2)} & \text{if } x_3 \geq x_2.
\end{cases}
\]
The exponents in the expression for \(p_\pi\) in (3) satisfy the following inequalities:
\[
2v \leq u + v + w \leq 2w \leq 2m \quad \text{for } b^u c^v d^w \\
2v \leq v + w \leq 2(u + w) \leq 2m \quad \text{for } a^u c^v d^w.
\]
and for any triple \((u, v, w)\) of integers satisfying these inequalities, we can obtain the corresponding path by the following rules:
\[
\eta = \begin{cases} 
(w_0, s_1 s_2, id; \frac{v}{m}, \frac{u+w}{m}, \frac{w}{m}) & \text{for } b_\eta = a^u c^v d^w \\
(w_0, s_1 s_2, s_2, id; \frac{v}{m}, \frac{u+w}{2m}, \frac{w}{m}) & \text{for } b_\eta = b^u c^v d^w
\end{cases}
\]

As a consequence we can now describe in terms of the path model the sum expressions in (5) and (6): For all \(\pi = (\tau, a) \in B(\lambda)^2\), \(a = (a_1, a_2, a_3)\), we get for \(t = \min\{mx_2, mx_3\}\)
\[
p(\pi; a) = \sum_{j=0}^{\lfloor t \rfloor} \binom{t\ell}{j\ell} b(\ell, a_1 + \frac{j\ell}{m}, a_2, a_3 - \frac{j\ell}{m})
\]  
(7)

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Keywords: Quantum groups, Canonical basis, Standard Monomial Theory