On the Dynamic Cumulative Past Quantile Entropy Ordering

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Abstract: In many society and natural science fields, some stochastic orders have been established in the literature to compare the variability of two random variables. For a stochastic order, if an individual (or a unit) has some property, sometimes we need to infer that the population (or a system) also has the same property. Then, we say this order has closed property. Reversely, we say this order has reversed closure. This kind of symmetry or anti-symmetry is constructive to uncertainty management. In this paper, we obtain a quantile version of DCPE, termed as the dynamic cumulative past quantile entropy (DCPQE). On the basis of the DCPQE function, we introduce two new nonparametric classes of life distributions and a new stochastic order, the dynamic cumulative past quantile entropy (DCPQE) order. Some characterization results of the new order are investigated, some closure and reversed closure properties of the DCPQE order are obtained. As applications of one of the main results, we also deal with the preservation of the DCPQE order in several stochastic models.

Keywords: dynamic cumulative past entropy; dynamic cumulative past quantile entropy; dynamic cumulative past quantile entropy order; closure property; records model

1. Introduction

In recent years, the concepts of entropies and the various kinds of stochastic order relations have not only played important roles in the risk management, but have also displayed more and more functions in all kinds of social sciences and natural sciences, such as management, physics, astronomy, geography and statistics. When we need to compare the variability of two random variables, a good way to do this thing is by a partial order relation defined on a distribution class of random variables. In recent decades, many stochastic orders were defined early or late (see Shaked and Shanthikumar [1], Nanda and Paul [2,3], Kundu et al. [4], Ghosh and Nanda [5], Gharari and Ganji [6], Ackermann et al. [7], Sbert et al. [8]). Of course, sometimes we need to infer the properties of individual (or a unit) from the properties of a population (or a system), then we say that this order has a closed property. While we do such things from the opposite direction in other times, then reversely we say this order has reversed closure. According to the definition of symmetry, a binary relation $R = S \times S$ is symmetric if, for all elements $a, b \in S$, whenever it is true that $aRb$, it is also true that $bRa$ (see Royce [9]). This case can be viewed as a kind of symmetry, and usually a partial order relation is of the anti-symmetry. This symmetry or anti-symmetry is constructive to uncertainty management.

Let $X$ be an absolutely continuous nonnegative random variable representing the random lifetime of a component or a system. Assume that $X$ has the distribution function $F_X(\cdot)$, the survival function $F_X(\cdot) = 1 - F_X(\cdot)$, and the probability density function $f_X(\cdot)$. As a baseline concept in the field of information theory, entropy was introduced by Shannon [10] and Wiener [11], and it plays an important role in information coding and
Theoretical neurobiology (see, for example, Johnson and Glants [12]). In the continuous case, the Shannon entropy is also referred to as the differential entropy, defined by:

$$H_X = -E[\ln f_X(X)] = - \int_0^{+\infty} f_X(x) \ln f_X(x) \, dx. \quad (1)$$

Since the classical contributions by Shannon [10] and Wiener [11], the properties of $H_X$ have been thoroughly investigated. Subsequently, Ebrahimi and Pellerey [13], Ebrahimi [14], Ebrahimi and Kirmani [15], Di Crescenzo and Longobardi [16], and Navarro et al. [17] investigated the differential entropy. Furthermore, several generalizations of (1) have been proposed (see Taneja [18], Nanda and Paul [2,3,19], Kundu et al. [4], Kumar and Taneja [20], Khorashadizadeh et al. [21], Kang and Yan [22], Kang [23–25], Yan and Kang [26], and the references therein). Di Crescenzo and Longobardi [16] introduced the past entropy as a dynamic generalization of differential entropy, defined by:

$$\overline{H}_X(t) = - \int_0^t f_X(x) \ln \frac{f_X(x)}{F_X(t)} \, dx. \quad (2)$$

That is, $\overline{H}_X(t) = H_{X(t)}$, where $X_{(t)}$ is the inactivity time of $X$ at time $t$ defined by $X_{(t)} = [t - X] \leq t]$. More recently, Sunoj et al. [27] obtained a quantile version of the past entropy, termed as past quantile entropy (PQE), and denoted by $\overline{F}_X(p) = H_{X[F_X^{-1}(p)]}$ for all $p \in (0,1)$. Based on the monotonicity of the PQE function, they defined two nonparametric classes of life distributions, the decreasing (increasing) past quantile entropy (DPQE (IPQE)) classes. They also defined the LPQE order by using the PQE functions. Rao et al. [28] defined a new uncertainty measure, the cumulative residual entropy (CRE), through:

$$E_X = - \int_0^{+\infty} \overline{F}_X(x) \ln \overline{F}_X(x) \, dx. \quad (3)$$

Asadi and Zohrevand [29] considered the corresponding dynamic measure, the dynamic cumulative residual entropy (DCRE), defined as the CRE of $X_t$, and denoted by $E_X(t)$. That is, $E_X(t) = \overline{E}_X(t)$. Here, $X_t$ is the residual life of $X$ at time $t$ defined by $X_t = [X - t] > 0$. Recently, Navarro et al. [17] thoroughly investigated the dynamic cumulative residual entropy and, using a similar manner to the definition of DCRE, they defined the dynamic cumulative past entropy (DCPE) of $X$ as the CRE of $X_{(t)}$, and denoted by $\overline{E}_X(t)$. Then, $\overline{E}_X(t)$ is given by:

$$\overline{E}_X(t) = E_{X(t)} = - \int_0^t F_X(x) \ln \frac{F_X(x)}{F_X(t)} \, dx. \quad (4)$$

They showed from (4) that the DCPE function can be rewritten as:

$$\overline{E}_X(t) = m_X(t) \ln F_X(t) - \frac{1}{F_X(t)} \int_0^t F_X(x) \ln F_X(x) \, dx,$$

where $m_X(t)$ is the mean inactivity time (MIT) function of $X$ defined by

$$m_X(t) = E(t - X|X < t) = \frac{1}{F_X(t)} \int_0^t F_X(x) \, dx, \quad \text{for } t \text{ such that } F_X(t) > 0. \quad (5)$$

The CPE, DCPE and past lifetime have been studied by many researchers, interested readers can refer to Di Crescenzo et al. [30], Di and Toomaj [31], Nanda et al. [32], Sunoj and Sankaran [33], Sunoj et al. [27], Li and Shaked [34], Navarro et al.[35], Kundu and Sarkar [36], Goli and Asadi [37], Ahmad et al.[38], Kayid and Ahmad [39]. By using the DCPE functions, Navarro et al. [17] defined the following classes.
Definition 1. A random variable $X$ is said to be increasing (decreasing) DCPE, denoted by $X \in$ IDCPE (DDCPE), if $\mathcal{E}_X(t) = m_X(t) - \tilde{F}_X(t)$ is an increasing (decreasing) function of $t$.

By differentiating $\mathcal{E}_X(t)$ in (4), they obtained $\mathcal{E}'_X(t) = \tau_X(t)(m_X(t) - \mathcal{E}_X(t))$, where $\tau_X(t)$ is the reversed failure (or hazard) rate defined by $\tau_X(t) = \frac{f_X(t)}{F_X(t)}$, and hence they showed that:

$$X \in$ IDCPE (DDCPE) $\iff \mathcal{E}_X(t) \leq m_X(t)$ (≥) for $t \geq 0.$

(6)

By taking $t = F_X^{-1}(p)$ in (4) yields:

$$\varphi_X(p) := \mathcal{E}_X[F_X^{-1}(p)] = -\int_0^{F_X^{-1}(p)} \frac{F_X(x)}{p} \ln \frac{F_X(x)}{p} \, dx,$$

where $F_X^{-1}$ is the quantile function of $F_X$ (or, of $X$) and is defined by:

$$F_X^{-1}(u) = \inf \{ t \mid F_X(t) \geq u \}, \quad \text{for all } u \in (0, 1).$$

We call $\varphi_X(p)$ the dynamic cumulative past quantile entropy (DCPQE) of $X$. Then, $\varphi_X(p)$ measures the uncertainty of $X$ at the age point $F_X^{-1}(p)$. Making use of the DCPQE functions, we define a new stochastic order in Section 2, termed the DCPQE order. This new stochastic order compares the uncertainties of two nonnegative random variables $X$ and $Y$ at the respective age points $F_X^{-1}(p)$ and $G_Y^{-1}(p)$; here, $G_Y^{-1}(p)$ is the quantile function of $Y$ with distribution function $G_Y(\cdot)$.

Recall that a nonnegative function $h$ defined on $[0, \infty)$ is said to be convex (concave), if for all $x, y \in [0, \infty)$ and all $\theta \in (0, 1)$, $h$ satisfies:

$$h(\theta x + (1 - \theta)y) \leq [\geq] \theta h(x) + (1 - \theta)h(y).$$

The following lemma taken from Barlow and Proschan (1981) plays a key role in the proofs of the paper.

Lemma 1. Let $W$ be a measure on the interval $(a, b)$, not necessarily nonnegative, where $-\infty \leq a < b \leq +\infty$. Let $h$ be a nonnegative function defined on $(a, b)$. If $\int_a^b dW(x) \geq 0$ for all $t \in (a, b)$ and if $h$ is decreasing, then $\int_a^b h(x) dW(x) \geq 0$.

The highlights of our research are: (1) We define a new entropy, that is the dynamic cumulative past quantile entropy (DCPQE); (2) We introduce two new nonparametric classes of life distributions, that is decreasing dynamic cumulative past quantile entropy (DDCPQE) and increasing dynamic cumulative past quantile entropy (IDCPQE); (3) We define the dynamic cumulative past quantile entropy (DCPQE) order; (4) We also deal with the preservation of the DCPQE order in proportional odds model and record values model; (5) We summarize the research results of this article, and obtained 12 results concerning anti-symmetry.

The rest of the paper is organized as follows. In Section 2, we provide some characterization results of the DCPQE order. In Section 3, we study closure properties of this order. In Section 4, we deal with the preservation of the DCPQE order in several important stochastic models, including proportional reversed failure rate, proportional odds and records models. Section 5 is the conclusion of this research.

Throughout this paper, all random variables are assumed to be absolutely continuous and nonnegative. The terms increasing and decreasing are used in a non-strict sense. Assume that all integrals involved are finite, and ratios are well defined whenever written.

2. Characterization Results of the DCPQE Odering

In this section we explore some characterizations of the DCPQE ordering.
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First, we propose a quantile version of DCPE $\mathcal{F}_X(t)$. Taking $t = F_X^{-1}(p)$ in (4) yields:

$$q_X(p) := \mathcal{F}_X[F_X^{-1}(p)] = - \int_0^{F_X^{-1}(p)} \frac{F_X(x)}{p} \ln \frac{F_X(x)}{p} \, dx, \quad \text{for all } p \in (0, 1),$$

where $F_X^{-1}$ is the quantile function of $F_X$ and is defined by:

$$F_X^{-1}(u) = \inf\{t \mid F_X(t) \geq u\}, \quad \text{for all } u \in [0, 1].$$

We call $q_X(p)$ the dynamic cumulative past quantile entropy (DCPQE) of $X$. By means of the DCPQE function, we introduce two new nonparametric classes of lifetime distributions. $X$ is said to be decreasing (increasing) DCPQE (DDCPQE) (IDCPQE) if $q_X(p)$ is decreasing (increasing) in $p \in (0, 1)$.

Let $Y$ be another absolutely continuous nonnegative random variable, assume that $Y$ has the distribution function $G_Y(\cdot)$, the survival function $S_Y(\cdot) = 1 - G_Y(\cdot)$, the probability density function $g_Y(\cdot)$, the quantile function $G_Y^{-1}(\cdot)$, and its DCPQE function $q_Y(p) := \mathcal{F}_Y[G_Y^{-1}(p)]$, respectively.

By using the DCPQE functions of $X$ and $Y$, we now define a new stochastic order as follows.

**Definition 2.** Let $X$ and $Y$ be two nonnegative continuous random variables. $X$ is said to be smaller than $Y$ in the dynamic cumulative past quantile entropy (DCPQE) order (written as $X \leq_{DCPQE} Y$) if

$$q_X(p) \leq q_Y(p), \quad \text{for all } p \in (0, 1).$$

**Remark 1.** One can verify that the DCPQE order is a partial order relation. The reflexive and transitive are evident; the anti-symmetric can be seen from Theorem 3 below.

By Definition 2, it is easy to see that:

$$X \leq_{DCPQE} Y \Leftrightarrow \int_0^{F_X^{-1}(p)} \frac{F_X(x)}{p} \ln \frac{F_X(x)}{p} \, dx \geq \int_0^{G_Y^{-1}(p)} \frac{G_Y(x)}{p} \ln \frac{G_Y(x)}{p} \, dx$$

$$\Leftrightarrow \int_0^{\mathcal{F}_X^{-1}(p)} \frac{F_X(x)}{p} \ln \frac{F_X(x)}{p} \, dx \geq \int_0^{\mathcal{F}_Y^{-1}(p)} \frac{G_Y(x)}{p} \ln \frac{G_Y(x)}{p} \, dx.$$ 

$$\Leftrightarrow \mathcal{F}_X[\mathcal{F}_X^{-1}(p)] \leq \mathcal{F}_Y[\mathcal{F}_Y^{-1}(p)], \quad \text{for all } p \in (0, 1).$$

In the below, we focus on the properties of this new order. The following theorem will be useful in the proofs of the results throughout the paper.

**Theorem 1.** $X \leq_{DCPQE} Y$ if and only if

$$\int_0^t F_X(x) \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \cdot \left[ \frac{f_X(x)}{g_Y[G_Y^{-1}(F_X(x))]} - 1 \right] \, dx \geq 0, \quad \text{for all } t \geq 0. \quad (9)$$

**Proof.** From Definition 2, we have that $X \leq_{DCPQE} Y$ if and only if

$$\int_0^{F_X^{-1}(p)} F_X(x) \ln \frac{F_X(x)}{p} \, dx \geq \int_0^{G_Y^{-1}(p)} G_Y(x) \ln \frac{G_Y(x)}{p} \, dx$$

for all $p \in (0, 1)$. Letting $x = G_Y^{-1}[F_Y(y)]$ in the right-hand side integral of the above inequality and then letting $F_X^{-1}(p) = t$ in the both sides, we see that the inequality (10) is equivalent to inequality (9), this completes the proof.  ∎
Example 1. Let $X$ and $Y$ be two exponential random variables with survival functions, respectively,

$$F_X(x) = e^{-\lambda_1 x}, \quad G_Y(x) = e^{-\lambda_2 x}, \quad \text{for all } x \geq 0.$$ 

It is easy to see that

$$\frac{f_X(x)}{g_Y \left[ G_Y^{-1}(F_X(x)) \right]} = \frac{\lambda_1}{\lambda_2}.$$

By using Theorem 1, we have that:

(a) if $0 < \lambda_2 \leq \lambda_1$, then $X \leq \text{DCPQE} Y$;

(b) if $0 < \lambda_1 \leq \lambda_2$, then $Y \leq \text{DCPQE} X$.

Before proceeding to give other characterization results of the DCPQE order, we overview several stochastic orders, one can see Shaked and Shanthikumar [1] for more details.

Definition 3. Let $X$ and $Y$ be two nonnegative continuous random variables.

(1) $X$ is said to be smaller than $Y$ in the convex order (denoted by $X \leq_c Y$) if $G_Y^{-1}[F_X(x)]$ is convex.

(2) $X$ is said to be smaller than $Y$ in the location independent riskier order (denoted by $X \leq_{lir} Y$) if

$$\int_0^{G_Y^{-1}(p)} F_X(x) dx \leq \int_0^{G_Y^{-1}(p)} G_Y(x) dx, \quad \text{for all } p \in (0, 1). \quad (11)$$

(3) $X$ is said to be smaller than $Y$ in the dispersive order (denoted by $X \leq_{\text{disp}} Y$) if

$$F_X^{-1}(\beta) - F_X^{-1}(\alpha) \leq G_Y^{-1}(\beta) - G_Y^{-1}(\alpha), \quad \text{for all } 0 < \alpha < \beta < 1.$$ 

For the convenience of use in the sequel, we give the following lemma without proof.

Lemma 2. Let $X$ and $Y$ be two nonnegative continuous random variables.

(1) $X \leq_{\text{disp}} Y$ if and only if $f_X(x) \geq g_Y \left[ G_Y^{-1}(F_X(x)) \right]$, or equivalently,

$$f_X(x) / g_Y \left[ G_Y^{-1}(F_X(x)) \right] \geq 1, \quad \text{for all } x \geq 0.$$

(2) $X \leq_c Y$ if and only if $f_X(x) / g_Y \left[ G_Y^{-1}(F_X(x)) \right]$ is increasing in $x > 0$.

(3) $X \leq_{lir} Y$ if and only if

$$\int_0^t F_X(x) \cdot \left[ f_X(x) / g_Y \left[ G_Y^{-1}(F_X(x)) \right] - 1 \right] dx \geq 0, \quad \text{for all } t \geq 0. \quad (12)$$

Proof. The proofs of results (1) and (2) can be found in Shaked and Shanthikumar [1]. We give the proof of result (3) below. By letting $x = G_Y^{-1}[F_X(y)]$ in the right-hand side integral of (11), we see that the inequality (12) is equivalent to the inequality (11). This completes the proof. □

Theorem 2. If $X \leq_{\text{disp}} Y$, then $X \leq_{\text{DCPQE}} Y$.

Proof. Suppose that $X \leq_{\text{disp}} Y$. From the definition of order $X \leq_{\text{disp}} Y$ we have

$$f_X(x) \geq g_Y \left[ G_Y^{-1}(F_X(x)) \right] \quad \text{for all } x \geq 0,$$

hence,

$$f_X(x) / g_Y \left[ G_Y^{-1}(F_X(x)) \right] - 1 \geq 0.$$
Moreover, \( \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \geq 0 \) for all \( t \geq x \geq 0 \). Now we see that the inequality (9) is valid, therefore the proof is complete. \( \square \)

Below, we provide an illustrative example that meets the DCPQE order.

Example 2. Let \( X \) and \( Y \) be two Pareto random variables with shape parameters \( \alpha_1 > 0 \), \( \alpha_2 > 0 \) and same scale parameter \( \theta > 0 \), respectively. The survival functions of \( X \) and \( Y \) are given by,

\[
F_X(x) = \left[ \frac{\lambda}{\lambda + x} \right]^{\alpha_1} \quad \text{and} \quad G_Y(x) = \left[ \frac{\lambda}{\lambda + x} \right]^{\alpha_2}, \quad \text{for all } x > 0.
\]

Then the DCPQE order between \( X \) and \( Y \) is determined by the shape parameters \( \alpha_1 \) and \( \alpha_2 \) but irrelevant to the scale parameter \( \theta > 0 \). It is easy to see that

\[
\frac{f_X(x)}{g_Y \left[ G_Y^{-1}(F_X(x)) \right]} = \frac{\alpha_1}{\alpha_2} \left[ 1 + \frac{x}{\lambda} \right]^{\alpha_1 - 1}.
\]

By using Lemma 2 (1) and Theorem 1 we have that (a) if \( 0 < \alpha_2 \leq \alpha_1 \), then \( X \leq_{DCPQE} Y \); (b) if \( 0 < \alpha_1 \leq \alpha_2 \), then \( Y \leq_{DCPQE} X \).

From the definition of the order \( X \leq_c Y \) and (9) the following result is readily seen, the proof is omitted.

Property 1. If \( X \leq_c Y \), and \( f_X(0) \geq g_Y(0) \), then \( X \leq_{DCPQE} Y \).

Property 2. If \( X \leq_{lir} Y \), then \( X \leq_{DCPQE} Y \).

Proof. Suppose that \( X \leq_{lir} Y \). From (12) we have

\[
\int_0^t F_X(x) \cdot \left[ \frac{f_X(x)}{g_Y \left[ G_Y^{-1}(F_X(x)) \right]} - 1 \right] \, dx \geq 0, \quad \text{for all } t \geq 0. \tag{13}
\]

On the other hand, \( X \leq_{DCPQE} Y \) if and only if

\[
\int_0^t F_X(x) \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \cdot \left[ \frac{f_X(x)}{g_Y \left[ G_Y^{-1}(F_X(x)) \right]} - 1 \right] \, dx \geq 0, \quad \text{for all } t \geq 0. \tag{14}
\]

Since the function \( h(x) = \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \) is nonnegative decreasing in \( x > 0 \), from (13) and Lemma 1 we see that the inequality (14) holds. That is, \( X \leq_{DCPQE} Y \). \( \square \)

From (9) the following result is obvious.

Property 3. If \( 0 < a \leq b \), then \( aX \leq_{DCPQE} bX \).

The following result is a direct consequence of Property 3.

Corollary 1. Let \( X \) be a nonnegative random variable and \( a \) and \( b \) be two real numbers.

1. If \( b \geq 1 \), then \( X \leq_{DCPQE} bX \).
2. If \( 0 < a \leq 1 \), then \( aX \leq_{DCPQE} X \).

By means of Theorem 1, it is easy to see that the following two properties are true.
Property 4. Let $a$ and $b$ be two real numbers and $X$ and $Y$ be two nonnegative random variables such that $X \leq_{DCPQE} Y$.

1. If $b \geq 1$, then $X \leq_{DCPQE} bY$.
2. If $0 < a \leq 1$, then $aX \leq_{DCPQE} Y$.

Property 5. If $X \leq_{DCPQE} Y$, then $X \leq_{DCPQE} Y + c$ for any real number $c$.

Remark 2. Property 5 shows that the DCPQE order is location-free.

Theorem 3. $X \leq_{DCPQE} Y$ and $X \geq_{DCPQE} Y$ hold simultaneously if and only if $X =_{disp} Y$, here, $X =_{disp} Y$ means that $X =_{st} Y + k$, where $k$ is constant.

Proof. From (9) we have that $X \leq_{DCPQE} Y$ and $X \geq_{DCPQE} Y$ hold simultaneously, if and only if
\[
\int_0^t F_X(x) \ln \left( \frac{F_X(t)}{F_X(x)} \right) \left( \frac{f_X(x)}{G_Y^{-1}(F_X(x))} \right) - 1 \, dx = 0, \quad \text{for all } t \geq 0.
\]

Hence,
\[
\frac{f_X(x)}{G_Y^{-1}(F_X(x))} = 1, \quad \text{almost surely},
\]
which is equivalent to $X =_{disp} Y$. This completes the proof. \qed

3. Closure Properties of the DCPQE Order

In this section, we investigate the closure properties of the DCPQE order under several reliability operations, such as linear transform, series and random series operations, and increasing concave transform. We firstly explore the closure property of the DCPQE order under linear transform.

Making use of Property 4 and Property 5, the following theorem is easy.

Theorem 4. Let $X \leq_{DCPQE} Y$. Then, for any real numbers $a, b, c$ and $d$ with $0 < a < c, 0 \leq b \leq d$, we have:

\[
aX + b \leq_{DCPQE} cY + d.
\]

Remark 3. Theorem 4 indicates that the positive linear systems preserve the DCPQE order. Theorem 4 also states that the DCPQE order has the closure property with respect to the positive linear systems. Theorem 4 also says that the DCPQE order has the closure property under the positive linear transformations.

3.1. Closure and Reversed Closure Properties of the DCPQE Ordering with Respect to Series and Parallel Systems

Let $X$ and $Y$ be two nonnegative continuous random variables with distribution functions $F_X$ and $F_Y$, right-continuous inverse functions $F_X^{-1}$ and $F_Y^{-1}$, survival functions $F_X$ and $F_Y$, and density functions $f_X$ and $f_Y$, respectively. Assume that $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are independent and identically distributed (i.i.d.) copies of $X$ and $Y$. Denote:

\[
X_{1:n} = \min\{X_1, \ldots, X_n\}, \quad X_{n:n} = \max\{X_1, \ldots, X_n\}.
\]

Similarly, $Y_{1:n}$ and $Y_{n:n}$. Denote the survival function and the density function of $X_{1:n}$ by $F_{X_{1:n}}$ and $f_{X_{1:n}}$, respectively; Denote the distribution function and the density function of $X_{n:n}$ by $F_{X_{n:n}}$ and $f_{X_{n:n}}$, respectively. Similarly, $F_{Y_{1:n}}, f_{Y_{1:n}}, F_{Y_{n:n}}, f_{Y_{n:n}}$. We have the following result.

Theorem 5. If $X \leq_{DCPQE} Y$, then $X_{1:n} \leq_{DCPQE} Y_{1:n}$. 
Hence the function

\[ f_{X_{1:n}}(x) = \frac{f_X(x)}{g_Y[G^{-1}(F_X(x))]} = \frac{f_X(x)}{g_Y[F_X^{-1}(x)]}. \]  

(16)

Since

\[ \ln \left( \frac{F_{X_{1:n}}(t)}{F_{X_{1:n}}(x)} \right) = \ln \left( \frac{F(t)}{F_X(x)} \right) - \sum_{i=1}^{n} \frac{\sum_{k=1}^{n} (F_X(t))^{i-1}}{\sum_{k=1}^{n} (F_X(x))^{i-1}} = \ln \left( \frac{F_X(t)}{F_X(x)} \right) + \ln \left( \frac{\sum_{k=1}^{n} (F_X(t))^{i-1}}{\sum_{k=1}^{n} (F_X(x))^{i-1}} \right), \]

the function

\[ \ln \left( \frac{F_{X_{1:n}}(t)}{F_{X_{1:n}}(x)} \right) = 1 + \ln \left( \frac{F_X(t)}{F_X(x)} \right) - \ln \left( \frac{\sum_{k=1}^{n} (F_X(t))^{i-1}}{\sum_{k=1}^{n} (F_X(x))^{i-1}} \right) = 1 + \frac{\ln \left( \frac{F_X(t)}{F_X(x)} \right)}{\ln \left( \frac{\sum_{k=1}^{n} (F_X(t))^{i-1}}{\sum_{k=1}^{n} (F_X(x))^{i-1}} \right)} \]

is nonnegative decreasing in \( x \geq 0 \) for all \( t \geq 0 \). Moreover, the function

\[ \sum_{i=1}^{n} (F_X(x))^{i-1} \]

is nonnegative decreasing in \( x \geq 0 \).

Hence the function

\[ h(x) = \sum_{i=1}^{n} (F_X(x))^{i-1} \cdot \ln \left( \frac{F_{X_{1:n}}(t)}{F_{X_{1:n}}(x)} \right) / \ln \left( \frac{F_X(t)}{F_X(x)} \right) \]  

(17)

is also nonnegative decreasing in \( x \geq 0 \). By (15), (17) and Lemma 1 we obtain that

\[ \int_{0}^{t} F_{X_{1:n}}(x) \ln \left( \frac{F_{X_{1:n}}(t)}{F_{X_{1:n}}(x)} \right) \cdot \left[ \frac{f_{X_{1:n}}(x)}{g_Y[G_{X_{1:n}}^{-1}(F_{X_{1:n}}(x))]} - 1 \right] dx \geq 0 \]

for all \( t \geq 0 \), which is equivalent to \( X_{1:n} \leq_{DCPQE} Y_{1:n} \). Therefore, the proof is complete.

**Remark 4.** Theorem 5 indicates that the series systems respectively composed of n i.i.d. copies of \( X \) and \( Y \) preserve the DCPQE order. Theorem 5 also says that the DCPQE order has the closure property with respect to the series systems.

**Example 3.** Let \( X \) and \( Y \) be two exponential random variables with respective survival functions

\[ F_X(x) = e^{-2x}, \quad G_Y(x) = e^{-x}, \quad \text{for all } x \geq 0. \]

Then, \( X_{1:2} \) and \( Y_{1:2} \) have survival functions, respectively,

\[ F_{X_{1:2}}(x) = e^{-4x}, \quad G_{Y_{1:2}}(x) = e^{-2x}, \quad \text{for all } x \geq 0. \]

According to Example 1 we have \( X \leq_{DCPQE} Y \). By Theorem 5 we obtain \( X_{1:2} \leq_{DCPQE} Y_{1:2} \). In fact, the correctness of this result is obvious from Example 1.
Theorem 6. If \( X_{n:n} \leq \text{DCPQE} Y_{n:n} \), then \( X \leq \text{DCPQE} Y \).

Proof. Suppose that \( X_{n:n} \leq \text{DCPQE} Y_{n:n} \). Then we have:

\[
\int_0^t F_{X_{n:n}}(x) \ln \left( \frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)} \right) \cdot \left[ \frac{f_{X_{n:n}}(x)}{g_{Y_{n:n}}^{-1}(F_{X_{n:n}}(x))} - 1 \right] \, dx \geq 0 \tag{18}
\]
for all \( t > 0 \). It can be proven that:

\[
\frac{f_{X_{n:n}}(x)}{g_{Y_{n:n}}^{-1}(F_{X_{n:n}}(x))} = \frac{f_X(x)}{g_Y^{-1}(F_X(x))}, \quad \text{for all } x > 0. \tag{19}
\]

Moreover,

\[
F_{X_{n:n}}(x) \ln \left( \frac{F_{X_{n:n}}(t)}{F_{X_{n:n}}(x)} \right) = n[F_X(x)]^n \ln \left[ \frac{F_X(t)}{F_X(x)} \right]. \tag{20}
\]

On the other hand, \( X \leq \text{DCPQE} Y \) if and only if

\[
\int_0^t F_X(x) \ln \left( \frac{F_X(t)}{F_X(x)} \right) \cdot \left[ \frac{f_X(x)}{g_Y^{-1}(F_X(x))} - 1 \right] \, dx \geq 0 \tag{21}
\]
for all \( t > 0 \). Since the function \( h(x) = [F_X(x)]^{1/n} \) is nonnegative decreasing in \( x > 0 \), from (18), (19), (20) and Lemma 1 we see that the inequality (21) holds, that is, \( X \leq \text{DCPQE} Y \). This completes the proof. \( \square \)

Remark 5. Theorem 6 indicates that the DCPQE order has reversed closure property under the formation of parallel systems. Theorem 6 also says that parallel systems reversely preserve the DCPQE order.

Example 4. Let \( X \) and \( Y \) be two positive continuous random variables with distribution functions, respectively,

\[
F_X(x) = e^{-1/x} \quad \text{and} \quad G_Y(x) = e^{-2/x}, \quad \text{for all } x > 0.
\]

Then \( X_{2:2} \) and \( Y_{2:2} \) have respective distribution functions:

\[
F_{X_{2:2}}(x) = e^{-2/x} \quad \text{and} \quad G_{Y_{2:2}}(x) = e^{-4/x}, \quad \text{for all } x > 0.
\]

By means of Theorem 2, one can prove that \( X_{2:2} \leq \text{DCPQE} Y_{2:2} \). In view of Theorem 2 we have \( X \leq \text{DCPQE} Y \).

3.2. Closure and Reversed Closure Properties of the DCPQE Ordering with Respect to Random Series and Parallel Systems

Let \( X \) and \( Y \) be two nonnegative continuous random variables, \( X_1, X_2, \ldots \) and \( Y_1, Y_2, \ldots \) be sequences of i.i.d. copies of \( X \) and \( Y \), respectively. Assume that \( X \) and \( Y \) have distribution functions \( F_X \) and \( G_Y \), right-continuous inverses functions \( F_X^{-1} \) and \( G_Y^{-1} \), survival functions \( \overline{F}_X \) and \( \overline{G}_Y \), and density functions \( f_X \) and \( g_Y \), respectively. Let \( N \) be a positive integer-valued random variable with the probability mass function \( p_N(n) = P\{N = n\} \), \( n = 1, 2, \ldots \). Assume that \( N \) is independent of \( X \) and \( Y \). Denote by:

\[
X_{1:N} = \min\{X_1, \ldots, X_N\}, \quad Y_{1:N} = \min\{Y_1, \ldots, Y_N\},
\]

\[
X_{N:N} = \max\{X_1, \ldots, X_N\}, \quad Y_{N:N} = \max\{Y_1, \ldots, Y_N\}.
\]
It can be proven that:

$$\frac{f_{X_{1:N}}(x)}{g_{Y_{1:N}}(G_{Y_{1:N}}^{-1}(F_{X_{1:N}}(x)))} = \frac{f_X(x)}{g_Y(G_Y^{-1}(F_X(x)))}, \quad (22)$$

and

$$\frac{f_{X_{N:N}}(x)}{g_{Y_{N:N}}(G_{Y_{N:N}}^{-1}(F_{X_{N:N}}(x)))} = \frac{f_X(x)}{g_Y(G_Y^{-1}(F_X(x)))}. \quad (23)$$

Now we consider extending the results in Theorem 5 and Theorem 6 from a finite number $n$ to a random number $N$. The following Theorem 7 can be viewed as an extension of Theorem 5.

**Theorem 7.** If $X \leq_{DCPQE} Y$, then $X_{1:N} \leq_{DCPQE} Y_{1:N}$.

**Proof.** Suppose that $X \leq_{DCPQE} Y$. Then, from Theorem 4 we have, for all $t \geq 0$,

$$\int_0^t F_X(x) \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \cdot \left[ \frac{f_X(x)}{g_Y(G_Y^{-1}(F_X(x)))} - 1 \right] dx \geq 0. \quad (24)$$

A simple calculation gives

$$\ln \left[ \frac{F_{X_{1:N}}(t)}{F_{X_{1:N}(x)}} \right] = \ln \left[ \frac{F_X(t)}{F_X(x)} \right] + \ln \left[ \frac{\sum_{i=1}^{n} \left( \sum_{i=1}^{n} F_X(x) \right)^{i-1}}{\sum_{i=1}^{n} \left( \sum_{i=1}^{n} F_X(x) \right)^{i-1}} \right] \cdot p_N(n),$$

the function

$$u(x) = \frac{\ln \left[ \frac{F_{X_{1:N}}(t)}{F_{X_{1:N}(x)}} \right]}{\ln \left[ \frac{F_X(t)}{F_X(x)} \right]} = 1 + \frac{\ln \left[ \frac{\sum_{i=1}^{n} \left( \sum_{i=1}^{n} F_X(x) \right)^{i-1}}{\sum_{i=1}^{n} \left( \sum_{i=1}^{n} F_X(x) \right)^{i-1}} \right] \cdot p_N(n)}{\ln \left[ \frac{F_X(t)}{F_X(x)} \right]},$$

is nonnegative decreasing in $x \geq 0$ for all $t \geq 0$. Moreover, the function

$$v(x) = \left( \sum_{i=1}^{n} \sum_{i=1}^{n} (F_X(x))^{i-1} \right) \cdot p_N(n)$$

is nonnegative decreasing in $x \geq 0$, hence the function

$$h(x) = u(x) \cdot v(x) = \left[ \sum_{n=1}^{+\infty} \left( \sum_{i=1}^{n} (F_X(x))^{i-1} \right) \cdot p_N(n) \right] \cdot \ln \left[ \frac{F_{X_{1:N}}(t)}{F_{X_{1:N}(x)}} \right] / \ln \left[ \frac{F_X(t)}{F_X(x)} \right],$$

is also nonnegative decreasing. By (24), (22), (25) and Lemma 1 we obtain that, for all $t \geq 0$,

$$\int_0^t F_{X_{1:N}}(x) \ln \left[ \frac{F_{X_{1:N}}(t)}{F_{X_{1:N}}(x)} \right] \cdot \left[ \frac{f_{X_{1:N}}(x)}{g_{Y_{1:N}}(G_{Y_{1:N}}^{-1}(F_{X_{1:N}}(x)))} - 1 \right] dx \geq 0.$$

Again, in turn, by Theorem 1, which asserts that $X_{1:N} \leq_{DCPQE} Y_{1:N}$. The proof follows. $\Box$

**Remark 6.** Theorem 7 indicates that the random series systems respectively composed of i.i.d. copies of $X$ and $Y$ preserve the DCPQE order. Theorem 7 also indicates that the DCPQE order has a closure property with respect to random series system.
Example 5. Let $X$ and $Y$ be two exponential random variables with survival functions, respectively,
\[ F_X(x) = e^{-2x} \quad \text{and} \quad G_Y(x) = e^{-x} \quad \text{for all } x \geq 0. \]

Let again $N$ be a positive integer-valued random variable with probability mass function $P(N = 1) = 1/2$ and $P(N = 2) = 1/2$. Then, $X_{1:N}$ and $Y_{1:N}$ have respective survival functions
\[ F_{X_{1:N}}(x) = \frac{1}{2}(e^{-2x} + e^{-4x}) \quad \text{and} \quad G_{Y_{1:N}}(x) = \frac{1}{2}(e^{-x} + e^{-2x}) \quad \text{for all } x \geq 0. \]

According to Example 1 we see that $X \leq_{DCPQE} Y$. Utilizing Theorem 7 gives $X_{1:N} \leq_{DCPQE} Y_{1:N}$.

The following Theorem 8 can be viewed as an extension of Theorem 6.

Theorem 8. If $X_{N:N} \leq_{DCPQE} Y_{N:N}$, then $X \leq_{DCPQE} Y$.

Proof. Suppose that $X_{N:N} \leq_{DCPQE} Y_{N:N}$. Then, from (9) we have, for all $t \geq 0$,
\[ \int_0^t F_{X_{N:N}}(x) \ln \left[ \frac{F_{X_{N:N}}(t)}{F_{X_{N:N}}(x)} \right] \cdot \frac{f_{X_{N:N}}(x)}{G_{Y_{N:N}}^{-1}(F_{X_{N:N}}(x))} - 1 \, dx \geq 0. \quad (26) \]

It is easy to see that:
\[ F_{X_{N:N}}(x) = \sum_{n=1}^{+\infty} [F_X(x)]^n p_N(n) = F_X(x) \cdot \left( \sum_{n=1}^{+\infty} [F_X(x)]^n p_N(n) \right). \quad (27) \]

From (27) we have, for all $t \geq 0$,
\[ \ln \left[ \frac{F_{X_{N:N}}(t)}{F_{X_{N:N}}(x)} \right] = \ln \left[ \frac{F_X(t)}{F_X(x)} \cdot \frac{\sum_{n=1}^{+\infty} [F_X(t)]^n p_N(n)}{\sum_{n=1}^{+\infty} [F_X(x)]^n p_N(n)} \right] \]
\[ = \ln \left[ \frac{F_X(t)}{F_X(x)} \right] + \ln \left[ \frac{\sum_{n=1}^{+\infty} [F_X(t)]^n p_N(n)}{\sum_{n=1}^{+\infty} [F_X(x)]^n p_N(n)} \right], \]
so the function
\[ \varphi(x) = \ln \left[ \frac{F_X(t)}{F_X(x)} \right] = \ln \left[ \frac{F_X(t)}{F_X(x)} \right] + \ln \left[ \frac{\sum_{n=1}^{+\infty} [F_X(t)]^n p_N(n)}{\sum_{n=1}^{+\infty} [F_X(x)]^n p_N(n)} \right] \]
\[ = \left[ 1 + \frac{\sum_{n=1}^{+\infty} [F_X(t)]^n p_N(n)}{\sum_{n=1}^{+\infty} [F_X(x)]^n p_N(n)} \right]^{-1} \quad (28) \]

is nonnegative decreasing in $x \geq 0$ for all $t \geq 0$. Moreover, the function
\[ \psi(x) = \left[ \sum_{n=1}^{+\infty} [F_X(x)]^n p_N(n) \right]^{-1} \]
is nonnegative decreasing in \( x \geq 0 \), hence the function

\[
 h(x) = \phi(x) \cdot \psi(x) = \left( \sum_{n=1}^{\infty} [F_X(x)]^n p_N(n) \right)^{-1} \left( 1 + \frac{\ln \left( \sum_{n=1}^{\infty} [F_X(t)]^n p_N(n) \right)}{\ln \left( \frac{F_X(x)}{F_X(t)} \right)} \right)^{-1}
\]

(29)
is also nonnegative decreasing in \( x \geq 0 \). By (26), (23), (29), and Lemma 1 we obtain that:

\[
 \int_0^t F_X(x) \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \cdot \left[ \frac{f_X(x)}{g_Y [G_Y^{-1}(F_X(x))]} - 1 \right] dx \geq 0, \quad \text{for all } t \geq 0,
\]

(30)
which asserts that \( X \leq_{\text{DCPQE}} Y \). Therefore the proof of the theorem is complete. \( \square \)

**Remark 7.** Theorem 8 shows that the random parallel systems respectively composed of i.i.d. copies of \( X \) and \( Y \) reversely preserve the DCPQE order. Theorem 8 also indicates that the DCPQE order has reversed closure property with respect to a random series system.

### 3.3. Closure and Reversed Closure Properties of the DCPQE Ordering under Increasing Convex and Concave Transforms

Let \( X \) and \( Y \) be two nonnegative continuous random variables. \( X \) is said to be smaller than \( Y \) in the usual stochastic order (denoted by \( X \leq_{\text{st}} Y \)) if \( F_X(x) \leq G_Y(x) \) for all \( x \geq 0 \).

**Theorem 9.** Let \( \phi(\cdot) \) be an increasing concave function with \( \phi(0) = 0 \). Assume that \( X \geq_{\text{st}} Y \). If \( X \leq_{\text{DCPQE}} Y \), then \( \phi(X) \leq_{\text{DCPQE}} \phi(Y) \).

**Proof.** Suppose that \( X \leq_{\text{DCPQE}} Y \). Then from Theorem 1 we have, for all \( t \geq 0 \),

\[
 \int_0^t F_X(x) \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \cdot \left[ \frac{f_X(x)}{g_Y [G_Y^{-1}(F_X(x))]} - 1 \right] dx \geq 0.
\]

(31)
Since \( \phi(\cdot) \) is increasing, concave implies that \( \phi'(\cdot) \) is nonnegative decreasing, using (31) and Lemma 1 we get:

\[
 \int_0^t \phi'(x) F_X(x) \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \cdot \left[ \frac{f_X(x)}{g_Y [G_Y^{-1}(F_X(x))]} - 1 \right] dx \geq 0.
\]

(32)
By Theorem 1, \( \phi(X) \leq_{\text{DCPQE}} \phi(Y) \) if and only if

\[
 \int_0^t F_{\phi(X)}(x) \ln \left[ \frac{F_{\phi(X)}(t)}{F_{\phi(X)}(x)} \right] \cdot \left[ \frac{f_{\phi(X)}(x)}{g_{\phi(Y)} [G_{\phi(Y)}^{-1}(F_{\phi(X)}(x))]} - 1 \right] dx \geq 0.
\]

(33)
Moreover,

\[
 F_{\phi(X)}(x) = F_X(\phi^{-1}(x)), \quad f_{\phi(X)}(x) = \frac{f_X(\phi^{-1}(x))}{\phi'(\phi^{-1}(x))},
\]

(34)
then

\[
 \frac{f_{\phi(X)}(x)}{g_{\phi(Y)} [G_{\phi(Y)}^{-1}(F_{\phi(X)}(x))]} = \frac{f_X(\phi^{-1}(x))}{g_Y [G_Y^{-1}(F_X(\phi^{-1}(x)))]} \cdot \frac{\phi'[G_Y^{-1}(F_X(\phi^{-1}(x)))]}{\phi'(\phi^{-1}(x))}.
\]

(35)
Hence, from (33), \(\phi(X) \leq_{\text{DCPQE}} \phi(Y)\) if and only if, for all \(t \geq 0\),

\[
\int_0^t F_X(x) \ln \left[ \frac{F_X(\phi^{-1}(x))}{F_X(\phi^{-1}(x))} \right] dx =
\int_0^t F_X(x) \ln \left[ \frac{F_X(\phi^{-1}(x))}{F_X(\phi^{-1}(x))} \right] dx =
\int_0^t \phi'(x) F_X(x) \ln \left[ \frac{F_X(x)}{F_X(\phi^{-1}(x))} \right] dx.
\]

(36)

In light of \(X \geq_{st} Y\), implying \(G_Y^{-1}[F_X(x)] \leq x\) and \(\phi(\cdot)\) is an increasing concave function implies that \(\phi'(\cdot)\) is nonnegative decreasing, so \(\phi'[G_Y^{-1}(F_X(x))] / \phi'(x) \geq 1\). Thus,

\[
\int_0^t \phi'(x) F_X(x) \ln \left[ \frac{F_X(x)}{F_X(\phi^{-1}(x))} \right] dx \geq 0.
\]

(37)

Making use of (32) and (37) we see that inequality (36) is valid, which asserts that

\(\phi(X) \leq_{\text{DCPQE}} \phi(Y)\).

This completes the proof. \(\Box\)

**Example 6.** Let \(X\) and \(Y\) be two uniform random variables with distribution functions, respectively,

\[ F_X(x) = x - 1 \quad \text{for all } x \in (1, 2) \quad \text{and} \quad G_Y(x) = x/2 \quad \text{for all } x \in (0, 2). \]

It is not difficult to verify that \(X \geq_{st} Y\) and that \(X \leq_{\text{DCPQE}} Y\). Take \(\phi(x) = x^{1/2}\), then \(\phi(x)\) is increasing concave for all \(x \geq 0\). The distribution functions of \(\phi(X)\) and \(\phi(Y)\) are given by:

\[ F_{\phi(X)}(x) = \sqrt{x - 1} \quad \text{for all } x \in (1, 4) \quad \text{and} \quad G_{\phi(Y)}(x) = \sqrt{x}/2 \quad \text{for all } x \in (0, 4). \]

Hence, by making use of Theorem 7 we get that \(\phi(X) \leq_{\text{DCPQE}} \phi(Y)\).

Let \(X\) be a nonnegative continuous random variable and \(\phi(\cdot)\) an increasing function defined on \([0, \infty)\) with \(\phi(0) = 0\). We call \(\phi(X)\) as the generalized scale transform of \(X\). If a function \(\phi(\cdot)\) is increasing convex with the derivative function \(\phi'(\cdot)\) being continuous and \(\phi(0) = 0\), then \(\phi(\cdot)\) is called a risk preference function, and \(\phi(X)\) is called the risk preference transform of \(X\). If \(\phi(\cdot)\) is increasing concave with the derivative function \(\phi'(\cdot)\) being continuous and \(\phi(0) = 0\), then \(\phi(\cdot)\) is called a risk aversion function, and \(\phi(X)\) is called the risk aversion transform of \(X\).

**Remark 8.** Theorem 9 says, with the condition of \(X \geq_{st} Y\), that the DCPQE order has the closure property under the concave generalized scale transforms. Theorem 9 also indicates, with the condition of \(X \geq_{st} Y\), that the DCPQE order has the closure property under the risk aversion transforms.

**Example 7.** Let \(X\) and \(Y\) be two uniform random variables with distribution functions, respectively,

\[ F_X(x) = x - 1 \quad \text{for all } x \in (1, 2) \quad \text{and} \quad G_Y(x) = x/2 \quad \text{for all } x \in (0, 2). \]
Take \( \phi(x) = x^{1/2} \), then \( \phi(x) \) is increasing and concave for all \( x \geq 0 \). It is easy to see that \( X \geq_{st} Y \). Moreover, it is not difficult to verify that \( X \leq_{DCPQE} Y \). Thus, making use of Theorem 8 gets \( \phi(X) \leq_{DCPQE} \phi(Y) \).

**Remark 9.** In Theorem 9, the condition \( X \geq_{st} Y \) is only a sufficient condition, but is not necessary.

**Counterexample 1.** Let \( X \) and \( Y \) be two exponential random variables with survival functions, respectively,
\[
F_X(x) = e^{-2x} \quad \text{and} \quad G_Y(x) = e^{-x}, \quad \text{for all} \ x \geq 0.
\]

According to Example 1 we know that \( X \leq_{DCPQE} Y \). Take \( \phi(x) = x^{1/2} \), then \( \phi(x) \) is increasing and concave for all \( x \geq 0 \). Moreover, \( \phi(X) \) and \( \phi(Y) \) have their survival functions, respectively,
\[
F_X(x) = e^{-2x^2} \quad \text{and} \quad G_Y(x) = e^{-x^2}, \quad \text{for all} \ x \geq 0.
\]

One can verify that \( \phi(X) \leq_{DCPQE} \phi(Y) \) by Theorem 2. Obviously, \( X \geq_{st} Y \) does not hold. Hence, the condition \( X \geq_{st} Y \) in Theorem 9 is only a sufficient condition, but not necessary.

**Remark 10.** In Theorem 9, the condition ‘\( \phi(\cdot) \) is concave’ is only a sufficient condition, but not necessary.

**Counterexample 2.** Let \( X \) and \( Y \) be two uniform random variables with distribution functions, respectively,
\[
F_X(x) = x - 1 \quad \text{for all} \ x \in (1,2) \quad \text{and} \quad G_Y(x) = x/2, \quad \text{for all} \ x \in (0,2).
\]

Take \( \phi(x) = 1 + (x - 1)^2 \), for all \( x \geq 0 \). Evidently, \( \phi(0) = 0 \), and \( \phi(x) \) is increasing but not concave for all \( x \geq 0 \). As shown in Example 6, \( X \leq_{DCPQE} Y \) and \( X \geq_{st} Y \). However, one can prove that \( \phi(X) \leq_{DCPQE} \phi(Y) \) by means of Theorem 2. This indicates that the condition ‘\( \phi(\cdot) \) is concave’ is only a sufficient condition, but not necessary.

On using a similar manner to above Theorem 9 we easily have the following theorem.

**Theorem 10.** Let \( \phi(\cdot) \) be an increasing convex function with the derivative \( \phi'(\cdot) \) being continuous and \( \phi(0) = 0 \). Assume that \( X \geq_{st} Y \). If \( \phi(X) \leq_{DCPQE} \phi(Y) \), then \( X \leq_{DCPQE} Y \).

**Remark 11.** Theorem 10 says, with the condition of \( X \geq_{st} Y \), that the DCPQE order has the reversed closure property under the convex generalized scale transforms. Theorem 10 also indicates, with the condition of \( X \geq_{st} Y \), that the DCPQE order has the reversed closure property under the risk preference transforms.

**Remark 12.** In Theorem 10, the condition \( X \geq_{st} Y \) is only a sufficient condition, but not necessary. The condition ‘\( \phi(\cdot) \) is convex’ is also a sufficient condition, but not necessary.

4. Preservation of the DCPQE Order in Several Stochastic Models

As applications of the main result of Theorem 1, in this section we deal with the preservation of the DCPQE ordering in several important stochastic models, including the proportional reversed hazard rate, the proportional odds and the records models.

4.1. Preservation of the DCPQE Order in Proportional Reversed Hazard Rate Model

First, we consider the preservation of the DCPQE order in the proportional reversed hazard rate model. For more details about the proportional reversed hazard rate model, one may refer to Gupta and Gupta [40] and Di Crescenzo and Longobardi [41].
Let $X$ and $Y$ be two absolutely continuous nonnegative random variables with respective distribution functions $F_X$ and $G_Y$. For $\theta > 0$, let $X(\theta)$ and $Y(\theta)$ denote another two random variables with respective distribution functions $(F_X)^{\theta}$ and $(G_Y)^{\theta}$. Suppose that $X$ and $Y$ have 0 as the common left endpoint of their supports. Then we have the following result.

**Theorem 11.** Let $X, Y, X(\theta)$ and $Y(\theta)$ be nonnegative random variables as described above.

(a) If $0 < \theta \leq 1$, then $X \leq_{\text{DCPQE}} Y \implies X(\theta) \leq_{\text{DCPQE}} Y(\theta)$.

(b) If $\theta \geq 1$, then $X(\theta) \leq_{\text{DCPQE}} Y(\theta) \implies X \leq_{\text{DCPQE}} Y$.

**Proof.** Suppose that $X, Y, X(\theta)$ and $Y(\theta)$ have respective distribution functions $F_X, G_Y, F_{X(\theta)}$ and $G_{Y(\theta)}$, the density functions $f_X, g_Y, f_{X(\theta)}$ and $g_{Y(\theta)}$, and the quantile functions $F_X^{-1}, G_Y^{-1}, F_{X(\theta)}^{-1}$ and $G_{Y(\theta)}^{-1}$, respectively. It can be proven that

$$
\begin{align*}
&\frac{f_{X(\theta)}(x)}{g_Y^{-1}\left(G_Y^{-1}\left(f_{X(\theta)}(x)\right)\right)} = \frac{f_X(x)}{g_Y^{-1}(F_X(x))}. \\
\end{align*}
$$

(38)

According to Theorem 1, $X \leq_{\text{DCPQE}} Y$ if and only if the inequality

$$
\int_0^t F_X(x) \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \cdot \left[ \frac{f_X(x)}{g_Y^{-1}(F_X(x))} - 1 \right] dx \geq 0
$$

(39)

holds for all $t \geq 0$; and $X(\theta) \leq_{\text{DCPQE}} Y(\theta)$ if and only if the inequality

$$
\int_0^t F_{X(\theta)}(x) \ln \left[ \frac{F_{X(\theta)}(t)}{F_{X(\theta)}(x)} \right] \cdot \left[ \frac{f_{X(\theta)}(x)}{g_Y^{-1}\left(F_{X(\theta)}(x)\right)} - 1 \right] dx \geq 0
$$

holds for all $t \geq 0$. Noting that $F_{X(\theta)}(x) = [F_X(x)]^{\theta}$ and making use of (38), $X(\theta) \leq_{\text{DCPQE}} Y(\theta)$ if and only if the inequality

$$
\int_0^t \theta[F_X(x)]^{\theta} \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \cdot \left[ \frac{f_X(x)}{g_Y^{-1}(F_X(x))} - 1 \right] dx \geq 0
$$

(40)

holds for all $t \geq 0$.

(a) Assume that $X \leq_{\text{DCPQE}} Y$. If $0 < \theta \leq 1$, then the function $h(x) = \theta[F_X(x)]^{\theta-1}$ is nonnegative decreasing in $x > 0$. By Lemma 1 and (39), we see that the inequality (40) is valid. Thus $X(\theta) \leq_{\text{DCPQE}} Y(\theta)$.

(b) Assume that $X(\theta) \leq_{\text{DCPQE}} Y(\theta)$. If $\theta \geq 1$, then the function $h(x) = \frac{1}{\theta}[F_X(x)]^{1-\theta}$ is nonnegative decreasing in $x > 0$. By Lemma 1 and (40), we see that the inequality (39) is valid. That is, $X \leq_{\text{DCPQE}} Y$. Therefore, the proof is complete. \(\square\)

### 4.2. Preservation of the DCPQE Order in Proportional Odds Model

Marshall and Olkin (1997), Sankaran and Jayakumar (2008) and Navarro et al. [17] studied the following proportional odds models. Let $X$ be a nonnegative continuous random variable with the distribution function $F_X$ and density function $f_X$. The proportional odds random variable, denoted by $X_p$, is defined by the distribution function

$$
F_{X_p}(x) = \frac{\theta F_X(x)}{1 - (1 - \theta) F_X(x)}
$$
for \( \theta > 0 \), where \( \theta \) is a proportional constant. Let \( Y \) be another nonnegative continuous random variable with distribution function \( G_Y \) and density function \( g_Y \). Similarly, define the proportional odds random variable \( Y_p \) of \( Y \) by the distribution function

\[
G_{Y_p}(x) = \frac{\theta G_Y(x)}{1 - (1 - \theta)G_Y(x)}
\]

for \( \theta > 0 \), where the proportional constant \( \theta \) is the same as above.

It is easy to see that the reversed hazard rate function of \( X_p \) is

\[
a_{X_p}(x) = \frac{f_{X_p}(x)}{F_{X_p}(x)} = \frac{a_X(x)}{1 - (1 - \theta)F_X(x)}.
\]

Clearly,

\[
\frac{a_{X_p}(x)}{a_X(x)} = \frac{1}{1 - (1 - \theta)F_X(x)}.
\]

Thus, we have reached the following result:

**Lemma 3.** Let \( X \) and \( X_p \) be as described above.

(a) If \( \theta \geq 1 \), then \( a_{X_p}(x)/a_X(x) \) is decreasing in \( x \geq 0 \).
(b) If \( 0 < \theta \leq 1 \), then \( a_{X_p}(x)/a_X(x) \) is increasing in \( x \geq 0 \).

A real-valued function on \( D \subseteq \mathbb{R}^r \) is called increasing [decreasing] if it is increasing [decreasing] in each variable when the other variables are held fixed. For the convenience of citation, we introduce the following lemma which will be useful in the proofs of upcoming theorems. This result is motivated by Lemma 2.2 of Khaledi et al. [42], and the proof utilizes a similar manner there.

**Lemma 4.** Let \( X \) and \( Y \) be two nonnegative random variables with corresponding reversed hazard rate functions \( a_X \) and \( a_Y \). If \( a_Y(u)/a_X(u) \) is increasing in \( u \geq 0 \), then the function:

\[
\varphi(x, y) = \frac{\ln G_Y(x) - \ln G_Y(y)}{\ln F_X(x) - \ln F_X(y)}
\]

is increasing in \( (x, y) \in \{(u, v) : 0 \leq u \leq v\} \).

**Proof.** Denote \( h_i(x) = a_X(x) \), \( h_2(x) = a_Y(x) \), and define

\[
\psi_i(x, y) = \int_0^{\infty} 1_{\{x \leq u \leq y\}} h_i(u) du, \quad i \in \{1, 2\},
\]

where \( 1_A \) is the indicator function of set \( A \). Since

\[
\ln F_Y(y) - \ln F_Y(x) = \int_0^{\infty} 1_{\{x \leq u \leq y\}} a_X(u) du
\]

and

\[
\ln G_Y(y) - \ln G_Y(x) = \int_0^{\infty} 1_{\{x \leq u \leq y\}} a_Y(u) du,
\]

we get that

\[
\varphi(x, y) = \frac{\psi_2(x, y)}{\psi_1(x, y)}.
\]

Note that \( a_Y(u)/a_X(u) \) is increasing in \( u \geq 0 \), meaning that \( h_i(u) \) is TP2 in \( (i, u) \in \{1, 2\} \times \mathbb{R}_+ \). It is easy to verify that \( 1_{\{x \leq u \leq y\}} \) is TP2 in \((u, x) \in \mathbb{R}_+ \times [0, y] \) for any \( y \in \mathbb{R}_+ \), and is TP2 in \((u, y) \in \mathbb{R}_+ \times [x, +\infty) \) for any \( x \in \mathbb{R}_+ \). Utilizing these facts, by using the basic composition formula, we conclude that \( \psi_i(x, y) \) is TP2 in \((i, x) \in \{1, 2\} \times [0, y] \) for
Let \( X \) be decreasing in inequality (45) holds, which asserts by Theorem 1 that \( X \) is decreasing in \( u \) concave in \( u \). Hence, the density functions of \( X_p \) and \( Y_p \) are, respectively, \( f_{X_p}(x) = h'(F_X(x))f_X(x) \) and \( g_{Y_p}(x) = h'(G_Y(x))g_Y(x) \), for all \( x \geq 0 \).

It can be proven that \( G_Y^{-1}(F_{X_p}(x)) = G_Y^{-1}(F_X(x)) \). By differentiating this equation we get that:

\[
\frac{f_{X_p}(x)}{g_{Y_p}(G_Y^{-1}(F_{X_p}(x)))} = \frac{f_X(x)}{g_Y(G_Y^{-1}(F_X(x)))}.
\]

From (9), we have that \( X \leq_{DCPQE} Y \) if and only if

\[
\int_0^t F_X(x) \ln \left( \frac{F_X(t)}{F_X(x)} \right) \cdot \left( \frac{f_X(x)}{g_Y(G_Y^{-1}(F_X(x)))} - 1 \right) \, dx \geq 0, \quad \text{for all } t \geq 0;
\]

and that \( X_p \leq_{DCPQE} Y_p \) if and only if

\[
\int_0^t F_{X_p}(x) \ln \left( \frac{F_{X_p}(t)}{F_{X_p}(x)} \right) \cdot \left( \frac{f_{X_p}(x)}{g_{Y_p}(G_{Y_p}^{-1}(F_{X_p}(x)))} - 1 \right) \, dx \geq 0, \quad \text{for all } t \geq 0,
\]

or, from (42) and (43), equivalently,

\[
\int_0^t h[F_X(t)] \ln \left( \frac{h[F_X(t)]}{h[F_X(x)]} \right) \cdot \left( \frac{f_X(x)}{g_Y(G_Y^{-1}(F_X(x)))} - 1 \right) \, dx \geq 0, \quad \text{for all } t \geq 0.
\]

(a) Assume that \( X \leq_{DCPQE} Y \). If \( \theta \geq 1 \), since \( h(u) \) is nonnegative and increasing concave in \( u \in [0,1] \), then the function \( \frac{h[F_X(t)]}{h[F_X(x)]}f_X(x) = h[F_X(x)]/F_X(x) \) is nonnegative decreasing in \( x \geq 0 \). Furthermore, by Lemmas 3 and 4 we know that \( \ln \left[ \frac{h[F_X(t)]}{h[F_X(x)]} \right] / \ln \left[ \frac{F_X(t)}{F_X(x)} \right] \) is nonnegative decreasing due to Lemma 3 (a). On using (44) and Lemma 1, we see that inequality (45) holds, which asserts by Theorem 1 that \( X_p \leq_{DCPQE} Y_p \).

(b) Assume that \( X_p \leq_{DCPQE} Y_p \). If \( \theta \leq 1 \), since \( h(u) \) is nonnegative and increasing convex in \( u \in [0,1] \), then the function \( f_X(x)/F_X(x) = F_X(x)/h[F_X(x)] \) is nonnegative decreasing in \( x \geq 0 \). Furthermore, by Lemma 3 and Lemma 4 we know that
Therefore, the proof is complete. □

4.3. Preservation of the DCPQE Order in Proportional Record Values Model

Then, we investigate the preservation of the DCPQE ordering in the record values model. Chandler [43] introduced and studied many basic properties of records. For more details about record values and their applications, one may refer to Khaledi et al. [42], Kundu et al. [44], Zhao and Balakrishnan [45], Zarezadeh and Asadi [46], Li and Zhang [47], and the references therein.

Let \( \{X_i, i \geq 1\} \) be a sequence of independent and identically distributed random variables (rv’s) from an absolutely continuous nonnegative random variable \( X \) with the density \( f_X(x) \) and the survival function \( S_X(x) = 1 - F_X(x) \). The rv’s \( T_{n_1}^X = 1 \) and 
\[ T_{n_1}^X = \min\{j > T_n^X : X_j > X_{T_n^X}\}, \quad n \geq 1 \]
are called the \( n \)-th record times, and the quantities \( X_{T_n^X} \), denoted by \( R_n^X \), are termed the \( n \)-th record values. For a detailed discussion on record values one may refer to Arnold et al. (1998).

It can be proven that the probability density, distribution and reversed hazard rate functions of \( R_n^X \) are given, respectively, by:
\[ f_{R_n^X}(x) = \frac{1}{1!} \Lambda_{X}^{n-1}(x) f_X(x), \]
\[ F_{R_n^X}(x) = \mathbb{P}(X \leq x) = \sum_{j=n}^{+\infty} \frac{(\Lambda_X(x))^j}{j!} = \Gamma_n(\Lambda_X(x)), \]
\[ a_{R_n^X}(x) = \frac{1}{\Gamma(n)} \frac{f_X(x)}{F_X(x)} \frac{\Lambda_X^{n-1}(x)}{n} = \frac{1}{\Gamma(n) \Gamma(n)} \Lambda_X^{n-1}(x) f_X(x) \]
for all \( x \geq 0 \), where \( \Gamma_n(\cdot) \) is the distribution function of a Gamma random variable with a shape parameter \( n \) and a scale parameter 1, and \( \Lambda_X(x) = -\ln F_X(x) \) is the cumulative failure rate function of \( X \).

Let \( Y \) be another absolutely continuous nonnegative random variable with survival function \( S_Y(x) = 1 - F_Y(x) \) and density function \( g_Y(x) \), the corresponding \( n \)-th record values are denoted by \( R_n^Y \).

Now we recall some stochastic orders which will be used in the following. One can refer to Shaked and Shanthikumar [1] for more details.

**Definition 4.** Let \( X \) and \( Y \) be two nonnegative continuous random variables with the density functions \( f_X \) and \( g_Y \) and the distribution functions \( F_X \) and \( F_Y \), respectively.

(a) \( X \) is said to be smaller than \( Y \) in the likelihood ratio order (denoted by \( X \leq_{lr} Y \)) if \( g_Y(x)/f_Y(x) \) is increasing in \( x \geq 0 \).

(b) \( X \) is said to be smaller than \( Y \) in the reversed hazard rate order (denoted by \( X \leq_{rh} Y \)) if \( G_Y(x)/F_X(x) \) is increasing in \( x \geq 0 \).

It is well-known that:
\[ X \leq_{lr} Y \iff X \leq_{rh} Y. \]

For the preservation property of the DCPQE ordering in the record values models, we obtain the following two results.
Theorem 13. Let X and Y be two absolutely continuous and nonnegative random variables, and let m and n be positive integers. Then

$$R_n^X \leq_{\text{DCPQE}} R_m^X \implies R_n^X \leq_{\text{DCPQE}} R_m^Y$$

for all $n > m \geq 1$.

Proof. Suppose that $R_n^X \leq_{\text{DCPQE}} R_m^X$. Then from Theorem 1 we have, for all $t \geq 0$,

$$\int_0^t F_{R_n^X}(x) \ln \left[ \frac{f_{R_n^X}(x)}{F_{R_n^X}(x)} \right] \cdot \left[ \frac{f_{R_n^X}(x)}{G_{R_n^X}^{-1}(F_{R_n^X}(x))} - 1 \right] \, dx \geq 0. \quad (49)$$

On using (46) and (47) one can prove that, for all positive integers $n > m \geq 1$ and all $x \geq 0$,

$$\frac{f_{R_n^X}(x)}{G_{R_n^X}^{-1}(F_{R_n^X}(x))} = \frac{f_{R_m^X}(x)}{G_{R_m^X}^{-1}(F_{R_m^X}(x))}.$$ \quad (50)

From (46) we see that the function:

$$\frac{f_{R_n^X}(x)}{f_{R_m^X}(x)} = \frac{\Gamma(m)}{\Gamma(n)} (\Lambda_X(x))^{n-m} \text{ is increasing in } x \geq 0 \text{ for } n > m.$$

Hence, we have $R_m^X \leq_{\text{DCPQE}} R_n^X$. So, $R_m^X \leq_{\text{DCPQE}} R_n^X$. Thus, we obtain that:

$$\frac{f_{R_n^X}(x)}{f_{R_m^X}(x)} \text{ is decreasing in } x \geq 0. \quad (51)$$

On the other hand, in view of (48) the ratio of reversed failure rates of $R_n^X$ and $R_m^X$,

$$\frac{a_{R_n^X}(x)}{a_{R_m^X}(x)} = \frac{\Gamma(m)}{\Gamma(n)} \sum_{i=n}^{\infty} \frac{(\Lambda_X(x))^i}{(i-(n-m))!} \text{ is increasing in } x > 0$$

by means of Lemma 2.1 in Kundu et al. [44]. On using Lemma 4 we get that:

$$\frac{\ln \frac{F_{R_n^X}(t)}{F_{R_m^X}(t)}}{\ln \frac{F_{R_n^X}(x)}{F_{R_m^X}(x)}} = \frac{\ln F_{R_n^X}(x) - \ln F_{R_n^X}(t)}{\ln F_{R_m^X}(x) - \ln F_{R_m^X}(t)} \text{ is increasing in } x \text{ such that } 0 < x \leq t.$$

That is,

$$\frac{\ln \frac{F_{R_n^X}(t)}{F_{R_n^X}(x)}}{\ln \frac{F_{R_m^X}(t)}{F_{R_m^X}(x)}} = \frac{\ln F_{R_n^X}(x) - \ln F_{R_n^X}(t)}{\ln F_{R_m^X}(x) - \ln F_{R_m^X}(t)} \text{ is decreasing in } x \text{ such that } 0 < x \leq t. \quad (52)$$

Making use of (49)–(52) and Lemma 1 we obtain that:

$$\int_0^t F_{R_n^X}(x) \ln \left[ \frac{f_{R_n^X}(t)}{F_{R_n^X}(x)} \right] \cdot \left[ \frac{f_{R_n^X}(x)}{G_{R_n^X}^{-1}(F_{R_n^X}(x))} - 1 \right] \, dx \geq 0$$

holds for all $t \geq 0$. Again, in turn, by Theorem 1 this gives that $R_m^X \leq_{\text{DCPQE}} R_n^Y$. Therefore, the desired result follows. \square
Theorem 14. Let $X$ and $Y$ be two absolutely continuous and nonnegative random variables, and let $n$ be a positive integer. Then,

$$X \leq_{DCPQE} Y \implies R_n^X \leq_{DCPQE} R_n^Y,$$

for all $n \geq 1$.

Proof. The proof is similar to that of Theorem 13, and hence is omitted. \qed

5. Conclusions

In this paper, we obtain a quantile version of DCPE, that is, DCPQE. Based on the DCPQE function, we define the DDCPQE and IDCPQE life distribution classes and the DCPQE Order. Some characterization results of the DCPQE order are given, meanwhile we mainly investigate some closure and reversed closure properties of the DCPQE order. We also consider the preservation of the DCPQE order in several stochastic models.

We get that:
(1) If $X \leq_{disp} Y$, then $X \leq_{DCPQE} Y$.
(2) If $X \leq_{lir} Y$, then $X \leq_{DCPQE} Y$.
(3) Let $a, b, c$ and $d$ be real numbers with $0 < a \leq c$, $0 \leq b \leq d$. Then

$$X \leq_{DCPQE} Y \implies aX + b \leq_{DCPQE} cY + d.$$

We get that the DCPQE order is:
(4) closed respect to a series system (see Theorem 5); but
(i) the inverse proposition of Theorem 5 does not hold.
(ii) not closed respect to a parallel system.
These two cases can all be viewed as a kind of anti-symmetry.
(5) reversely closed with respect to a parallel system (see Theorem 6); but
(i) the inverse proposition of Theorem 6 does not hold.
(ii) not reversely closed respect to a series system.
These two cases can all be viewed as a kind of anti-symmetry.
(6) closed respect to a random series system (see Theorem 7); but
(i) the inverse proposition of Theorem 7 does not hold.
(ii) not closed respect to a random parallel system.
These two cases can all be viewed as a kind of anti-symmetry.
(7) reversely closed respect to a random parallel system (see Theorem 8); but
(i) the inverse proposition of Theorem 8 does not hold.
(ii) not reversely closed respect to a random series system.
These two cases can all be viewed as a kind of anti-symmetry.
(8) closed under a nonnegative increasing concave transform (see Theorem 9); but
(i) the inverse proposition of Theorem 9 does not hold.
(ii) not reversely closed under a nonnegative increasing concave transform.
These two cases can all be viewed as a kind of anti-symmetry.
(9) reversely closed under a nonnegative increasing convex transform (see Theorem 10); but
(i) the inverse proposition of Theorem 10 does not hold.
(ii) not closed under a nonnegative increasing convex transform.
These two cases can all be viewed as a kind of anti-symmetry.
(10) closed but not reversely closed under some appropriate conditions in the proportional reversed hazard rate models (see Theorem 11 (a)). This case can be viewed as a kind of anti-symmetry.
(11) reversely closed but not closed under other appropriate conditions in the proportional reversed hazard rate model (see Theorem 11 (b)). This case can be viewed as a kind of anti-symmetry.
(12) closed but not reversely closed under some appropriate conditions in the proportional odds model (see Theorem 12 (a)). This case can be viewed as a kind of anti-symmetry.
(13) reversely closed but not closed under other appropriate conditions in the proportional odds model (see Theorem 12 (b)). This case can be viewed as a kind of anti-symmetry.

(14) reversely closed in the record-value models (see Theorem 13); but the inverse proposition of Theorem 13 does not hold. This case can be viewed as a kind of anti-symmetry.

(15) closed in the record-value models (see Theorem 14); but the inverse proposition of Theorem 14 does not hold. This case can be viewed as a kind of anti-symmetry.

Some generalizations of DCPE and applications of the DCQPE order will further be considered for us in the future. In this respect, readers can see Di Crescenzo et al. [30], Di and Toomaj [31].

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