Partial Differential Equations

Spectral instability of some non-selfadjoint anharmonic oscillators

Instabilité spectrale de certains oscillateurs anharmoniques non-autoadjoints

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ABSTRACT

The purpose of this Note is to highlight the spectral instability of some non-selfadjoint differential operators, by studying the growth rate of the norms of the spectral projections $\Pi_n$ associated with their eigenvalues. More precisely, we are concerned with some anharmonic oscillators $A(m, \theta) = -\frac{d^2}{dx^2} + e^{i \theta} |x|^m$ with $|\theta| < \min\left\{ \frac{(m+2)\pi}{4}, \frac{(m+2)\pi}{2m} \right\}$, defined on $L^2(\mathbb{R})$. We get asymptotic expansions for the norm of the spectral projections associated with the large eigenvalues of $A(1, \theta)$ and $A(2k, \theta)$, $k \geq 1$, extending the results of Davies (2000) [4] and Davies and Kuijlaars (2004) [5].

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1. Spectral instability and pseudospectra

It is well known that the spectral theorem implies some control of stability for the spectrum of selfadjoint operators: if $A$ is a selfadjoint operator acting on the Hilbert space $\mathcal{H}$, the spectrum of its perturbations $A + \varepsilon B$, with $\varepsilon > 0$ and any $B \in \mathcal{L}(\mathcal{H})$, $\|B\| \leq 1$, lies entirely inside an $\varepsilon$-neighborhood of the spectrum $\sigma(A)$. In other words, the norm of the resolvent of $A$ near the spectrum blows up like the inverse distance to the spectrum. It has also been known for several years (see for instance [13]) that such a behavior could not be expected in general in the case of non-selfadjoint operators. One can understand it thanks to the notion of $\varepsilon$-pseudospectra of an operator $A$, defined as the family of sets $\sigma_{\varepsilon}(A)$, indexed by $\varepsilon > 0$,

$$\sigma_{\varepsilon}(A) = \left\{ \xi \in \rho(A) : \| (A - \xi)^{-1} \| > \frac{1}{\varepsilon} \right\} \cup \sigma(A).$$

The link between spectral instability and pseudospectra appears more clearly in the following equivalent formulation, which is a weak version of the Roch and Silbermann theorem [11]:
\[ \sigma_\varepsilon(A) = \bigcup_{\omega \in \mathcal{C}(\varepsilon)} \sigma(A + \varepsilon B) \]

(see also [12] and references therein).

In the following, we deal with the instability indices associated with an isolated eigenvalue \( \lambda \in \sigma(A) \). The instability index associated with \( \lambda \) is defined as \( \kappa(\lambda) = \|\Pi(\lambda)\| \), where \( \Pi(\lambda) \) denotes the spectral projection associated with \( \lambda \). Of course \( \kappa(\lambda) \geq 1 \) in any case, and \( \kappa(\lambda) = 1 \) when \( A \) is self-adjoint. These numbers \( \kappa(\lambda) \) are closely related to the size of \( \varepsilon \)-pseudospectra around \( \lambda \). Indeed, if \( \sigma^1_\varepsilon \) denotes the connected component of \( \sigma_\varepsilon(A) \) containing \( \lambda \), and if we assume for simplicity that \( \sigma^1_\varepsilon \cap \sigma(A) = \{\lambda\} \) and \( \sigma^1_\varepsilon \) is bounded, then the perimeter \( |\partial \sigma^1_\varepsilon| \) of \( \sigma^1_\varepsilon \) satisfies (see [3])

\[ |\partial \sigma^1_\varepsilon| \geq 2\pi \varepsilon \kappa(\lambda). \tag{1} \]

In the finite dimensional setting at least, instability indices give a better description of pseudospectra: if \( A \in \mathcal{M}_n(\mathbb{C}) \) is a diagonalizable matrix with distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \), Embree and Trefethen show [13] that there exists \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon < \varepsilon_0 \),

\[ \bigcup_{\lambda_k \in \sigma(A)} \Delta(\lambda_k, \varepsilon, \Pi(\lambda_k)) \cup \mathcal{O}(\varepsilon^2) \subset \sigma_\varepsilon(A) \subset \bigcup_{\lambda_k \in \sigma(A)} \Delta(\lambda_k, \varepsilon, \Pi(\lambda_k)) + \mathcal{O}(\varepsilon^2). \tag{2} \]

In the case of an infinite dimensional space, the validity of this statement should be investigated, as well as the dependance on \( \lambda_k \) of the \( \mathcal{O}(\varepsilon^2) \) terms.

In the following, we study the instability indices of simple non-selfadjoint differential operators introduced by Davies in [4], for which the instability phenomenon described above will appear clearly. Let us define the anharmonic oscillators

\[ A(m, \theta) = -\frac{d^2}{dx^2} + e^{i\theta}|x|^m \tag{3} \]

with

\[ |\theta| < \min \left\{ \frac{(m+2)\pi}{4}, \frac{(m+2)\pi}{2m} \right\}, \tag{4} \]

de ned on \( L^2(\mathbb{R}) \) in [4] by taking the closure of the associated quadratic form defined on \( C^2_0(\mathbb{R}) \), which is sectorial if \( \theta \) satisfies [4]. According to [4], its spectrum consists of a sequence of discrete simple eigenvalues, denoted in nondecreasing modulus order by \( \lambda_n = \lambda_n(m, \theta), |\lambda_n| \to +\infty \). The associated spectral projections are of rank 1, and E.-B. Davies showed in [4] that for all \( m \in [0, +\infty[ \) and \( \theta \neq 0 \) satisfying (4), for all \( \alpha > 0 \), there exists \( N = N(m, \theta, \alpha) \geq 0 \) such that the instability indices \( \kappa_n(m, \theta) \) of \( A(m, \theta) \) satisfy \( \kappa_n(m, \theta) \geq n^\alpha \) for \( n \geq N \). This statement has been improved in the case \( m = 2 \) of the harmonic oscillator (sometimes referred as the Davies operator), since E.-B. Davies and A. Kuijlaars showed [5] that \( \kappa_n(2, \theta) \) grows exponentially fast as \( n \to +\infty \), with an explicit rate \( c(\theta) \): there exists an explicit \( c(\theta) \) such that

\[ \lim_{n \to +\infty} \frac{1}{n} \log \kappa_n(2, \theta) = c(\theta). \tag{5} \]

The purpose of this Note is to prove that this statement actually holds for the so-called complex Airy operator \( A(1, \theta) \) and for the even anharmonic oscillators \( A(2k, \theta) \), \( k \geq 1 \).

2. Non-selfadjoint anharmonic oscillators

We first deal with the complex Airy operator \( A(1, \theta) \) defined in [3]. We show that the corresponding instability indices \( \kappa_n(1, \theta) \) grow like in (5) as \( n \to +\infty \). More precisely, we get asymptotic expansions in powers of \( n^{-1} \) as \( n \to +\infty \).

**Theorem 2.1.** Let \( 0 < |\theta| < 3\pi/4 \). There exists a real sequence \( (\alpha_j(\theta))_{j \geq 1} \) such that the instability indices \( \kappa_n(1, \theta) \) of \( A(1, \theta) \) satisfy, as \( n \to +\infty \),

\[ \exp(-C(\theta)(n-1/2))\kappa_n(1, \theta) = \frac{K(\theta)}{\sqrt{n}} \left( 1 + \sum_{j=1}^{+\infty} \alpha_j(\theta)n^{-j} \right) + \mathcal{O}(n^{-\infty}), \tag{6} \]

where

\[ C(\theta) = \pi m_\theta^{3/2} |\sin \theta| \quad \text{and} \quad K(\theta) = \frac{1}{2\sqrt{3}|\sin \theta|m_\theta^{1/4}} \]

with

\[ m_\theta = \sqrt{\frac{1 + \frac{\sin^2(2\theta/3)}{\sin^2 \theta} - 2\cos(\theta/3)\sin(2\theta/3)}{\sin \theta} > 0}. \]
Sketch of the proof. Let us first recall that all the eigenvalues of $A(m, \theta)$, $m \in \mathbb{N}$, have associated spectral projections of rank 1, see Lemma 5 in [4]. Hence, one can easily check that [3]

$$
\kappa_n(m, \theta) = \frac{\|u_n\|^2}{(u_n, u_n)},
$$

(7)

where $u_n$ denotes an eigenfunction associated with the $n$-th eigenvalue of $A(m, \theta)$.

We get rid of the singularity of the potential at $x = 0$ by decomposing $A(1, \theta)$ into its Dirichlet and Neumann realizations $A^D(1, \theta)$ and $A^N(1, \theta)$ on $\mathbb{R}^+$. We then compute their instability indices

$$
\kappa_n^{D/N}(1, \theta) = \frac{\int_{\mathbb{R}^+} |\phi_n^{D/N} + e^{i\theta/3}x|^2 \, dx}{\int_{\mathbb{R}^+} |\phi_n^{D/N} + e^{i\theta/3}x|^2 \, dx},
$$

(8)

given by formula (7), where $x \mapsto \phi_n^{D/N}(x) + e^{i\theta/3}x$ is the $n$-th eigenfunction of $A^{D/N}(1, \theta)$, $\mu_n^D$ (resp. $\mu_n^N$) being the $n$-th (negative) zero (resp. critical point) of the Airy function $Ai$ (see [2,8]). We estimate the numerator in (8) by using the well-known asymptotic expansion of $Ai$ at infinity in the complex plane (see [1]), and the Laplace method brings an $\exp(\text{const} |\mu_n^{D/N}|^{3/2})$ term in $\kappa_n^{D/N}(1, \theta)$, $\text{const} > 0$. The integral in the denominator of (8), after deformation of the path of integration by homotopy, is equal to

$$
\int_{\mu_n^{D/N}}^{+\infty} Ai^2(x) \, dx = Ai^2(\mu_n^{D/N})
$$

(9)

(it is indeed straightforward, using Airy equation, to check that $x \mapsto x\bar{A}^2(x) - Ai^2(x)$ is a primitive for $Ai^2$). Hence the expansion of $\bar{A}$ as $\theta \to +\infty$, given in [1], provides an asymptotic expansion for (9) in powers of $|\mu_n^{D/N}|^{-3/2}$. The statement follows from the behavior of $\mu_n^{D/N}$ as $n \to +\infty$, since we have asymptotic expansions for $(n - 1/4)^{-2/3}|\mu_n^D|^{-3/2}$ (resp. $(n - 3/4)^{-2/3}|\mu_n^N|^{-3/2}$) in powers of $(n - 1/4)^{-2}$ (resp. $(n - 3/4)^{-2}$).

Notice that the exponential instability appears as soon as $\theta \neq 0$.

We have a similar statement for even anharmonic oscillators:

**Theorem 2.2.** Let $k \in \mathbb{N}^*$ and $\theta$ be such that $0 < |\theta| < \frac{(k+1)\pi}{2k}$. If $\kappa_n(2k, \theta)$ denotes the $n$-th instability index of $A(2k, \theta) = -\frac{d^2}{dx^2} + e^{i\theta} x^{2k}$, then there exist $K(2k, \theta) > 0$ and a real sequence $(C^j(2k, \theta))_{j \geq 1}$ such that

$$
e^{-c_k(\theta)n} \kappa_{n}(2k, \theta) = \frac{K(2k, \theta)}{\sqrt{n}} \left(1 + \sum_{j=1}^{+\infty} C^j(2k, \theta)n^{-j}\right) + \mathcal{O}(n^{-\infty})
$$

(10)

as $n \to +\infty$, with

$$
c_k(\theta) = \frac{2(k+1)\sqrt{\pi} \Gamma(\frac{k+1}{2k})\psi_k(\xi_k)}{\Gamma(\frac{1}{2k})} > 0
$$

(11)

where

$$
\xi_k = \left(\frac{\tan(|\theta|/(k+1))}{\sin(|\theta|/(k+1)) + \cos(|\theta|/(k+1)) \tan(|\theta|/(k+1))}\right)^{1/2},
$$

(12)

$$
\psi_k(\xi) = \lim_{t \to 0} \left(1 - t^{2k}\right)^{1/2} dt.
$$

(13)

**Sketch of the proof.** We first perform an analytic dilation and a scale change to recover the semiclassical selfadjoint anharmonic oscillator $P_\hbar(2k) = -\hbar^2 \frac{d^2}{dx^2} + x^{2k} - 1$, with $\hbar = h_n = |\lambda_n(2k, \theta)|^{-\frac{k+1}{2k}}$. The $n$-th instability index of $A(2k, \theta)$ then writes

$$
\kappa_n(2k, \theta) = \frac{\int_{\mathbb{R}} |\psi_\hbar(e^{i\theta\xi/\sqrt{\pi^{k+1}}} x)|^2 \, dx}{\int_{\mathbb{R}} |\psi_\hbar(x)|^2 \, dx}
$$

(14)

where $\psi_\hbar$ solves $P_\hbar(2k)\psi_\hbar = 0$, $\psi_\hbar \in L^2(\mathbb{R})$ (see (7), after deformation of the integration path in the denominator). The complex WKB method (see [10,14,6]) and the analysis of the Stokes lines of $P_\hbar(2k)$ provide an asymptotic expansion of
ψ_θ(e^{i(t-T)A}) as $h \to 0$, which enables us to determine the asymptotic behaviour of the numerator in (14), using again the Laplace method.

On the real axis, $\psi_\theta$ is treated separately in its oscillatory region $[-1 + \delta, 1 - \delta]$, $\delta > 0$, and in the neighbourhood of the turning points $\pm 1$. Hence, the stationary phase method leads to an asymptotic expansion in powers of $h$ of the denominator in (14). Finally, the statement follows from the Bohr–Sommerfeld quantization rule for $h_\text{e}$ (see [7, Exercise 12.3]) or Weyl formula [9]. □

In the harmonic case $k = 1$ (Davies operator), the first term in (10) yields the Davies–Kuijlaars theorem [5]:

$$\lim_{n \to +\infty} \frac{1}{n} \log \|\Pi_n\| = c_1(\theta) = 4\varphi_1\left(\frac{1}{\sqrt{2\cos(\theta/2)}}\right) = 2 \Re f\left(\frac{e^{i\theta}/A}{\sqrt{2\cos(\theta/2)}}\right)$$

where $f(z) = \log(z + \sqrt{z^2 - 1}) - z\sqrt{z^2 - 1}$.

3. Eigenfunctions and semigroups

The following theorem has been proved in [2] in the case of complex Airy operator $A(1, \theta)$, and in [4] in the harmonic case ($k = 1$), as well as for $A(2k, \theta)$, $k \geq 2$, $|\theta| < \frac{\pi}{4}$. The proof actually extends to any operator $A(2k, \theta)$ with $|\theta| < \frac{k+1}{2k}$:

**Theorem 3.1.** For any $m = 1, 2k, k \geq 1$, and any $\theta$ satisfying (4), the eigenfunctions of $A(m, \theta)$ form a complete set of the space $L^2(\mathbb{R})$.

Notice however that the eigenfunctions of $A(1, \theta)$ and $A(2k, \theta)$, $k \geq 1$, cannot form a Riesz basis because of the growth of the instability indices as $n \to +\infty$.

**Theorem 3.1** and the previous estimates enable us to study the convergence of the operator series defining the semigroup $e^{-tA(m,\theta)}$ associated with $A(m, \theta)$ when decomposed along the projections $\Pi_n$.

The following statement extends the result of [5] in the harmonic case.

**Corollary 3.2.** Let $|\theta| \leq \pi/2$ and $e^{-tA(m,\theta)}$ be the semigroup generated by $A(m, \theta)$, $\lambda_n = \lambda_n(m, \theta)$ the eigenvalues of $A(m, \theta)$, and $\Pi_n = \Pi_n(m, \theta)$ the associated spectral projections.

Let $T(\theta) = c_1(\theta)/\cos(\theta/2)$, where $c_1(\theta)$ is the constant in (11). The series $\Sigma_{m, \theta}(t) = \sum_{n = 0}^{+\infty} e^{-t\lambda_n(m, \theta)}\Pi_n(m, \theta)$ is not normally convergent in cases $m = 1$ for any $t > 0$, and $m = 2$ for $t < T(\theta)$; in cases $m = 2$ for $t > T(\theta)$, and $m = 2k$ for any $t > 0, k \geq 2$, the series converges normally towards $e^{-tA(m,\theta)}$ and, for $N$ sufficiently large and for some constants $C_1 = C_1(k, \theta)$ and $C_2 = C_2(\theta)$, the following estimate holds

$$\|e^{-tA(m,\theta)}(I - \Pi_{< N})\| \leq \begin{cases} C_1 \sqrt{\frac{C_2}{N}} e^{c_2(\theta)m} \exp(-t \Re \lambda_N), & k \geq 2, \\
C_1 \sqrt{\frac{C_2}{N}} \exp(-t \cos(\theta/2)(t - T(\theta)))N, & k = 1, t > T \end{cases}$$

where $\Pi_{< N} = \Pi_1 + \cdots + \Pi_{N-1}$ denote the projection on the first $N - 1$ eigenspaces.

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