The $B_p$ condition for Weighted inequalities in Dunkl analysis and some applications

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Abstract

In Dunkl theory on $\mathbb{R}^d$ which generalizes classical Fourier analysis, we prove weighted inequalities for the Dunkl transform when the weights belong to the well-known class $B_p$. This result is applied to derive for power weights Pitt’s theorem on $\mathbb{R}^d$ which gives an integrability theorem for this transform on a suitable Besov-type space.

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1 Introduction

A key tool in the study of special functions with reflection symmetries are Dunkl operators. The basic ingredient in the theory of these operators are root systems and finite reflection groups, acting on some Euclidean space $E$. We shall always assume that $E = \mathbb{R}^d$ with the standard Euclidean scalar product $(.,.)$. The Dunkl operators are commuting differential-difference operators $T_i, 1 \leq i \leq d$ associated to an arbitrary finite reflection group $W$ on $\mathbb{R}^d$ (see[7]). These operators attached with a root system $R$ can be considered as perturbations of the usual partial derivatives by reflection parts. These reflection parts are coupled by parameters, which are given in terms of a non negative multiplicity function $k$. Dunkl theory was further developed by several mathematicians (see [6, 9, 16, 18]) and later was applied

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and generalized in different ways by many authors (see [1, 2, 3, 21]). The Dunkl kernel $E_k$ has been introduced by C.F. Dunkl in [8]. For a family of weight functions $w_k$ invariant under a reflection group $W$, we use the Dunkl kernel and the measure $w_k(x)dx$ to define the Dunkl transform $F_k$, which enjoys properties similar to those of the classical Fourier transform. If the parameter $k \equiv 0$ then $w_k(x) = 1$, so that $F_k$ becomes the classical Fourier transform and the $T_i, 1 \leq i \leq d$ reduce to the corresponding partial derivatives $\frac{\partial}{\partial x_i}, 1 \leq i \leq d$. Therefore Dunkl analysis can be viewed as a generalization of classical Fourier analysis (see next section, Remark 1).

The classical Fourier transform behaves well with the translation operator $f \mapsto f(. - y)$, which leaves the Lebesgue measure on $\mathbb{R}^d$ invariant. However, the measure $w_k(x)dx$ is no longer invariant under the usual translation. One ends up with the Dunkl translation operators $\tau_x, x \in \mathbb{R}^d$, introduced by K. Trimèche in [22] on the space of infinitely differentiable functions on $\mathbb{R}^d$.

An explicit formula for the Dunkl translation $\tau_x$ is known. In particular, the boundedness of $\tau_x$ is established in this case. As a result one obtains a formula for the convolution $*k$ (see next section). An important motivation to study Dunkl operators originates in their relevance for the analysis of quantum many body systems of Calogero-Moser-Sutherland type. These describe algebraically integrable systems in one dimension and have gained considerable interest in mathematical physics (see [23]).

Let $\mu$ locally integrable weight functions on $(0, +\infty)$. We say that $\mu \in B_p, 1 < p < +\infty$ if there is a constant $b_p > 0$ such that for all $s > 0$

$$\int_s^{+\infty} \frac{\mu(t)}{t^p} dt \leq b_p \frac{1}{sp} \int_0^s \mu(t) dt.$$  \hspace{1cm} (1.1)

In the particular case when $\mu$ is non-increasing, one has $\mu \in B_p$.

The weighted Hardy inequality [19] (see also [10, 15]) states that if $\mu$ and $\vartheta$ are locally integrable weight functions on $(0, +\infty)$ and $1 < p \leq q < +\infty$, then there is a constant $c > 0$ such that for all non-increasing, non-negative Lebesgue measurable function $f$ on $(0, +\infty)$, the inequality

$$\left( \int_0^{+\infty} \frac{1}{t} \int_0^t f(s)ds \right)^{q \frac{1}{p}} \mu(t) dt \leq c \left( \int_0^{+\infty} (f(t))^p \vartheta(t) dt \right)^{\frac{1}{p}}$$  \hspace{1cm} (1.2)

is satisfied if and only if

$$\sup_{s > 0} \left( \int_0^s \mu(t) dt \right)^{\frac{1}{p}} \left( \int_0^s (\vartheta(t)) dt \right)^{-\frac{1}{q}} < +\infty.$$  \hspace{1cm} (1.3)
and
\[
\sup_{s>0} \left( \int_s^{+\infty} \frac{u(t)}{t^q} \frac{1}{q} \left( \int_0^s \frac{1}{t} \int_0^t \vartheta(l) \, dl \right)^{-p'} \frac{1}{p'} \vartheta(t) \, dt \right)^{\frac{1}{p'}} < +\infty. \tag{1.4}
\]

Hardy’s result still remains to be an important one as it is closely related to the Hardy-Littlewood maximal functions in harmonic analysis [20].

The aim of this paper is to prove under the $B_p$ condition and using the weight characterization of the Hardy operator, weighted Dunkl transform inequalities for general non-negative locally integrable weight functions $u, v$ on $\mathbb{R}^d$,
\[
\left( \int_{\mathbb{R}^d} |F_k(f)(y)|^q u(y) \, d\nu_k(y) \right)^{\frac{1}{q}} \leq c \left( \int_{\mathbb{R}^d} |f(x)|^p v(x) \, d\nu_k(x) \right)^{\frac{1}{p}}, \tag{1.5}
\]
where $1 < p \leq q < +\infty$ and $f \in L^p_{k,v}(\mathbb{R}^d)$ with $L^p_{k,v}(\mathbb{R}^d)$ is the space $L^p(\mathbb{R}^d, v(x) \, d\nu_k(x))$ and $\nu_k$ the weighted measure associated to the Dunkl operators defined by
\[
d\nu_k(x) := w_k(x) \, dx \quad \text{where} \quad w_k(x) = \prod_{\xi \in \mathbb{R}^d_+} |\langle \xi, x \rangle|^{2k(\xi)} , \quad x \in \mathbb{R}^d.
\]

$R_+$ being a positive root system (see next section). More precisely, under the condition $B_p$, we give sufficient conditions on the weights $u, v$ in order that (1.5) holds. These are extensions to the Dunkl analysis of some results obtained for the classical case in [4, 13]. As example, we make a study of power weights in this context. This all leads to Pitt’s inequality: for $1 < p \leq q < +\infty$, $-(2\gamma + d) < \alpha < 0$, $0 < \beta < (2\gamma + d)(p-1)$, $u(x) = ||x||^\alpha$, $v(x) = ||x||^\beta$ and $f \in L^p_{k,v}(\mathbb{R}^d)$, one has
\[
\left( \int_{\mathbb{R}^d} |F_k(f)(x)|^q ||x||^\alpha \, d\nu_k(x) \right)^{\frac{1}{q}} \leq c \left( \int_{\mathbb{R}^d} |f(x)|^p ||x||^\beta \, d\nu_k(x) \right)^{\frac{1}{p}},
\]
with the index constraint $\frac{1}{2\gamma+d}(\frac{\alpha}{q} + \frac{\beta}{p}) = 1 - \frac{1}{p} - \frac{1}{q}$.

This gives that $F_k(f) \in L^q_{k}(\mathbb{R}^d)$ when $f$ belongs to the Besov-Dunkl space $BD^0_{1,\frac{(d-1)(2\gamma+d)-\alpha}{q}}$, $1 < p \leq 2$ which we introduce as the subspace of functions $f \in L^p_{k}(\mathbb{R}^d)$ such that $f \ast_k \phi_t \in L^p_{k,v}(\mathbb{R}^d)$ and satisfying the condition
\[
\int_0^{+\infty} \left\| f \ast_k \phi_t \right\|_{p,k,v} \, dt < +\infty
\]
where $\phi$ is a radial function in $\mathcal{S}(\mathbb{R}^d)$ verifying
\[
\exists \, c > 0; \quad |\mathcal{F}_k(\phi)(y)| > c \|y\|^2, \quad \forall \, y \in \left\{ z \in \mathbb{R}^d : \frac{1}{2} \leq \|z\| \leq 1 \right\}.
\]
(1.6)
$\phi_t$ being the dilation of $\phi$ given by $\phi_t(y) = \frac{1}{t^{d+\frac{d}{2}} \phi(\frac{y}{t})}$, for all $t \in (0, +\infty)$ and $y \in \mathbb{R}^d$.

The contents of this paper are as follows.

In section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operators.

The section 3 is devoted to the proofs of the weighted Dunkl transform inequalities when the weights belong to the class $B_p$. As example, we make a study of power weights in this context which gives Pitt’s theorem, including its role in the integrability for the Dunkl transform on besov spaces.

Along this paper we use $c$ to denote a suitable positive constant which is not necessarily the same in each occurrence and we write for $x \in \mathbb{R}^d, \|x\| = \sqrt{\langle x, x \rangle}$. Furthermore, we denote by

- $\mathcal{E}(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$.
- $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of functions in $\mathcal{E}(\mathbb{R}^d)$ which are rapidly decreasing as well as their derivatives.
- $\mathcal{D}(\mathbb{R}^d)$ the subspace of $\mathcal{E}(\mathbb{R}^d)$ of compactly supported functions.

2 Preliminaries

In this section, we recall some notations and results in Dunkl theory and we refer for more details to the surveys [17].

Let $W$ be a finite reflection group on $\mathbb{R}^d$, associated with a root system $R$. For $\alpha \in R$, we denote by $\mathbb{H}_\alpha$ the hyperplane orthogonal to $\alpha$. For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in R} \mathbb{H}_\alpha$, we fix a positive subsystem $R_+ = \{ \alpha \in R : \langle \alpha, \beta \rangle > 0 \}$. We denote by $k$ a nonnegative multiplicity function defined on $R$ with the property that $k$ is $W$-invariant. We associate with $k$ the index
\[
\gamma = \sum_{\xi \in R_+} k(\xi) \geq 0,
\]
and a weighted measure $\nu_k$ given by
\[
d\nu_k(x) := w_k(x)dx \quad \text{where} \quad w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d,
\]
Further, we introduce the Mehta-type constant \( c_k \) by
\[
c_k = \left( \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} w_k(x) dx \right)^{-1}.
\]

For every \( 1 \leq p \leq +\infty \), we denote respectively by \( L^p_k(\mathbb{R}^d) \), \( L^p_{k,u}(\mathbb{R}^d) \), \( L^p_{k,v}(\mathbb{R}^d) \) the spaces \( L^p(\mathbb{R}^d, d\nu_k(x)) \), \( L^p(\mathbb{R}^d, u(x)d\nu_k(x)) \), \( L^p(\mathbb{R}^d, v(x)d\nu_k(x)) \) and \( L^p_k(\mathbb{R}^d)^{rad} \) the subspace of those \( f \in L^p_k(\mathbb{R}^d) \) that are radial. We use respectively \( \| \|_{p,k} \), \( \| \|_{p,k,u} \), \( \| \|_{p,k,v} \) as a shorthand for \( \| \|_{L^p_k(\mathbb{R}^d)} \), \( \| \|_{L^p_{k,u}(\mathbb{R}^d)} \), \( \| \|_{L^p_{k,v}(\mathbb{R}^d)} \).

By using the homogeneity of degree \( 2\gamma \) of \( w_k \), it is shown in [16] that for a radial function \( f \) in \( L^1_k(\mathbb{R}^d) \), there exists a function \( F \) on \( [0, +\infty) \) such that \( f(x) = F(\|x\|) \), for all \( x \in \mathbb{R}^d \). The function \( F \) is integrable with respect to the measure \( r^{2\gamma+d-1} dr \) on \( [0, +\infty) \) and we have
\[
\int_{\mathbb{R}^d} f(x) d\nu_k(x) = \int_0^{+\infty} \left( \int_{S^{d-1}} f(ry) w_k(ry) d\sigma(y) \right) r^{d-1} dr
= \int_0^{+\infty} \left( \int_{S^{d-1}} w_k(ry) d\sigma(y) \right) F(r) r^{d-1} dr
= d_k \int_0^{+\infty} F(r) r^{2\gamma+d-1} dr, \tag{2.1}
\]
where \( S^{d-1} \) is the unit sphere on \( \mathbb{R}^d \) with the normalized surface measure \( d\sigma \) and
\[
d_k = \int_{S^{d-1}} w_k(x) d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma + \frac{d}{2})}. \tag{2.2}
\]

The Dunkl operators \( T_j, \ 1 \leq j \leq d, \) on \( \mathbb{R}^d \) associated with the reflection group \( W \) and the multiplicity function \( k \) are the first-order differential-difference operators given by
\[
T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(\rho_\alpha(x)) - f(x)}{\langle \alpha, x \rangle}, \ f \in \mathcal{E}(\mathbb{R}^d), \ x \in \mathbb{R}^d,
\]
where \( \rho_\alpha \) is the reflection on the hyperplane \( \mathbb{H}_\alpha \) and \( \alpha_j = \langle \alpha, e_j \rangle, (e_1, \ldots, e_d) \) being the canonical basis of \( \mathbb{R}^d \).

**Remark 2.1** In the case \( k \equiv 0 \), the weighted function \( w_k \equiv 1 \) and the measure \( \nu_k \) associated to the Dunkl operators coincide with the Lebesgue measure. The \( T_j \) reduce to the corresponding partial derivatives. Therefore Dunkl analysis can be viewed as a generalization of classical Fourier analysis.
For \( y \in \mathbb{C}^d \), the system
\[
\begin{align*}
T_j u(x, y) &= y_j u(x, y), \quad 1 \leq j \leq d, \\
u(0, y) &= 1.
\end{align*}
\]
admits a unique analytic solution on \( \mathbb{R}^d \), denoted by \( E_k(x, y) \) and called the Dunkl kernel. This kernel has a unique holomorphic extension to \( \mathbb{C}^d \times \mathbb{C}^d \).

We have for all \( \lambda \in \mathbb{C} \) and \( z, z' \in \mathbb{C}^d \),
\[
E_k(z, z') = E_k(\lambda z, z'),
\]
and for \( x, y \in \mathbb{R}^d \),
\[
|E_k(x, iy)| \leq 1.
\]

The Dunkl transform \( \mathcal{F}_k \) is defined for \( f \in \mathcal{D}(\mathbb{R}^d) \) by
\[
\mathcal{F}_k(f)(x) = c_k \int_{\mathbb{R}^d} f(y) E_k(-ix, y) d\nu_k(y), \quad x \in \mathbb{R}^d.
\]

We list some known properties of this transform:

i) The Dunkl transform of a function \( f \in L^1_k(\mathbb{R}^d) \) has the following basic property
\[
\|\mathcal{F}_k(f)\|_{\infty,k} \leq \|f\|_{1,k}.
\]

ii) The Dunkl transform is an automorphism on the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \).

iii) When both \( f \) and \( \mathcal{F}_k(f) \) are in \( L^1_k(\mathbb{R}^d) \), we have the inversion formula
\[
f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(y) E_k(ix, y) d\nu_k(y), \quad x \in \mathbb{R}^d.
\]

iv) (Plancherel’s theorem) The Dunkl transform on \( \mathcal{S}(\mathbb{R}^d) \) extends uniquely to an isometric automorphism on \( L^2_k(\mathbb{R}^d) \).

Since the Dunkl transform \( \mathcal{F}_k(f) \) is of strong-type \((1, \infty)\) and \((2, 2)\), then by interpolation, we get for \( f \in L^p_k(\mathbb{R}^d) \) with \( 1 \leq p \leq 2 \) and \( p' \) such that \( \frac{1}{p} + \frac{1}{p'} = 1 \), the Hausdorff-Young inequality
\[
\|\mathcal{F}_k(f)\|_{p',k} \leq c \|f\|_{p,k}.
\]

The Dunkl transform of a function in \( L^1_k(\mathbb{R}^d)^{rad} \) is also radial. More precisely, according to ([16], proposition 2.4), we have for \( x \in \mathbb{R} \), the following results:
\[
\int_{S^{d-1}} E_k(ix, y) w_k(y) d\sigma(y) = d_k j_{\gamma + \frac{d}{2} - 1}(\|x\|),
\]
and for $f$ be in $L^1_k(\mathbb{R}^d)^{rad}$,

$$\mathcal{F}_k(f)(x) = \int_0^{+\infty} \left( \int_{S^{d-1}} E_k(-irx, y)w_k(y)\sigma(y) \right) F(r)^{2\gamma+d-1} dr$$
$$= dk \int_0^{+\infty} j_{\gamma+\frac{d}{2}-1}(r\|x\|) F(r)^{2\gamma+d-1} dr,$$

(2.3)

where $F$ is the function defined on $[0, +\infty)$ by $F(\|x\|) = f(x)$ and $j_{\gamma+\frac{d}{2}-1}$ the normalized Bessel function of the first kind and order $\gamma+\frac{d}{2}-1$ given by

$$j_{\gamma+\frac{d}{2}-1}(\lambda x) = \begin{cases} 2^{\gamma+\frac{d}{2}-1} \Gamma(\gamma+\frac{d}{2}) \frac{J_{\gamma+\frac{d}{2}-1}(\lambda x)}{(\lambda x)^{\gamma+\frac{d}{2}-1}} & \text{if } \lambda x \neq 0, \\ 1 & \text{if } \lambda x = 0, \end{cases}$$

$\lambda \in \mathbb{C}$. Here $J_{\gamma+\frac{d}{2}-1}$ is the Bessel function of first kind,

$$J_{\gamma+\frac{d}{2}-1}(t) = \frac{1}{\sqrt{\pi} \Gamma(\gamma+\frac{d}{2}-\frac{1}{2})} \int_0^\pi \cos(t \cos \theta)(\sin \theta)^{2\gamma+d-2} d\theta$$
$$= C_\gamma t^{\gamma+\frac{d}{2}-1} \int_0^{\pi/2} \cos(t \cos \theta)(\sin \theta)^{2\gamma+d-2} d\theta,$$

(2.4)

where $C_\gamma = \frac{1}{\sqrt{\pi} \sqrt{2^{\gamma+\frac{d}{2}-2}}} \Gamma(\gamma+\frac{d}{2}-\frac{1}{2})$.

K. Trimèche has introduced in [22] the Dunkl translation operators $\tau_x$, $x \in \mathbb{R}^d$, on $\mathcal{E}(\mathbb{R}^d)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, we have

$$\mathcal{F}_k(\tau_x(f))(y) = E_k(ix, y)\mathcal{F}_k(f)(y).$$

Notice that for all $x, y \in \mathbb{R}^d$, $\tau_x(f)(y) = \tau_y(f)(x)$ and for fixed $x \in \mathbb{R}^d$

$$\tau_x$$

is a continuous linear mapping from $\mathcal{E}(\mathbb{R}^d)$ into $\mathcal{E}(\mathbb{R}^d)$.

As an operator on $L^2_k(\mathbb{R}^d)$, $\tau_x$ is bounded. A priori it is not at all clear whether the translation operator can be defined for $L^p$-functions with $p$ different from 2. However, according to ([21], Theorem 3.7), the operator $\tau_x$ can be extended to the space of radial functions $L^p_k(\mathbb{R}^d)^{rad}$, $1 \leq p \leq 2$ and we have for a function $f$ in $L^p_k(\mathbb{R}^d)^{rad}$,

$$\|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k}.$$
The Dunkl convolution product $*f_k$ of two functions $f$ and $g$ in $L^2_k(\mathbb{R}^d)$ is given by
\[
(f *_k g)(x) = \int_{\mathbb{R}^d} \tau_x(f)(-y)g(y)d\nu_k(y), \quad x \in \mathbb{R}^d.
\]
The Dunkl convolution product is commutative and for $f, g \in D(\mathbb{R}^d)$, we have
\[
\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g). \quad (2.5)
\]
It was shown in ([21], Theorem 4.1) that when $g$ is a bounded radial function in $L^1_k(\mathbb{R}^d)$, then
\[
(f *_k g)(x) = \int_{\mathbb{R}^d} f(y)\tau_x(g)(-y)d\nu_k(y), \quad x \in \mathbb{R}^d,
\]
initially defined on the intersection of $L^1_k(\mathbb{R}^d)$ and $L^2_k(\mathbb{R}^d)$ extends to $L^p_k(\mathbb{R}^d)$, $1 \leq p \leq +\infty$ as a bounded operator. In particular,
\[
\|f *_k g\|_{p,k} \leq \|f\|_{p,k}\|g\|_{1,k}.
\]

3 Weighted Dunkl transform inequalities under the $B_p$ condition

In this section, we denote by $p'$ and $q'$ respectively the conjugates of $p$ and $q$ for $1 < p \leq q < +\infty$. The proof requires a useful well-known facts which we shall now state in the following.

**Proposition 3.1** *(see [19])* Let $1 < p < +\infty$ and $v$ be a weight function on $(0, +\infty)$. The following are equivalent:

i) $v \in B_p$.

ii) There is a positive constant $c$ such that for all $s > 0$,
\[
\left( \int_0^s v(t)dt \right)^{\frac{1}{p}} \left( \int_0^s \left( \frac{1}{t} \int_0^t v(l)dl \right)^{1-p'}dl \right)^{\frac{1}{p'}} \leq c \cdot s. \quad (3.1)
\]

**Remark 3.1**
1/ (see [5]) (Hardy’s Lemma) Let \( f \) and \( g \) be non-negative Lebesgue measurable functions on \((0, +\infty)\), and assume
\[
\int_0^t f(s)ds \leq \int_0^t g(s)ds
\]
for all \( t \geq 0 \). If \( \varphi \) is a non-negative and decreasing function on \((0, +\infty)\), then
\[
\int_0^{+\infty} f(s)\varphi(s)ds \leq \int_0^{+\infty} g(s)\varphi(s)ds.
\] (3.2)

2/ Let \( f \) be a complex-valued \( \nu_k \)-measurable function on \( \mathbb{R}^d \). The distribution function \( D_f \) of \( f \) is defined for all \( s \geq 0 \) by
\[
D_f(s) = \nu_k(\{x \in \mathbb{R}^d : |f(x)| > s\}).
\]
The decreasing rearrangement of \( f \) is the function \( f^* \) given for all \( t \geq 0 \) by
\[
f^*(t) = \inf\{s \geq 0 : D_f(s) \leq t\}.
\]
We have the following results:

- Let \( f \in L^p_k(\mathbb{R}^d) \), \( 1 \leq p < +\infty \) then
\[
\int_{\mathbb{R}^d} |f(x)|^p d\nu_k(x) = p \int_0^{+\infty} s^{p-1} D_f(s)ds = \int_0^{+\infty} (f^*(t))^p dt.
\] (3.3)

- (see [14], Theorems 4.6 and 4.7) Let \( q \geq 2 \), then there exists a constant \( c > 0 \) such that, for all \( f \in L^1_k(\mathbb{R}^d) + L^2_k(\mathbb{R}^d) \) and \( s \geq 0 \),
\[
\int_0^s (\mathcal{F}_k(f^*)(t))^q dt \leq c \int_0^s \left( \int_0^{1/2} f^*(y)dy \right)^q dt.
\] (3.4)

- (see [5, 11, 12]) (Hardy-Littlewood rearrangement inequality) Let \( f \) and \( \vartheta \) be non negative \( \nu_k \)-measurable functions on \( \mathbb{R}^d \), then
\[
\int_{\mathbb{R}^d} f(x)\vartheta(x)d\nu_k(x) \leq \int_0^{+\infty} f^*(t)\vartheta^*(t)dt
\] (3.5)

and
\[
\int_0^{+\infty} f^*(t)\left[\left(\frac{1}{\vartheta}\right)^*\right]^{-1} dt \leq \int_{\mathbb{R}^d} f(x)\vartheta(x)d\nu_k(x).
\] (3.6)
Example 3.1 Let \( \vartheta(x) = \|x\|^{\beta} \), \( x \in \mathbb{R}^d \) with \( \beta > 0 \) and \( f(x) = \chi_{(0,r)}(\|x\|) \) for \( r > 0 \). Using (2.1) and (2.2), we have for \( s \geq 0 \),

\[
D_{\frac{1}{\beta}}(s) = \nu_k(\{x \in \mathbb{R}^d : \|x\|^{-\beta} > s\})
= \frac{d_k}{2\gamma + d} s^{-\frac{2\gamma + d}{\beta}},
\]

which gives for \( t \geq 0 \),

\[
(\frac{1}{\beta})^{*}(t) = \inf\{s \geq 0 : D_{\frac{1}{\beta}}(s) \leq t\} = \left(\frac{2\gamma + d}{d_k}\right)^{-\frac{\beta}{2\gamma + d}} t^{\frac{-\beta}{2\gamma + d}}.
\]

The distribution function of \( f \) is

\[
D_{f}(s) = \nu_k(\{x \in \mathbb{R}^d : \chi_{(0,1)}(\|x\|) > s\})
= \nu_k(B(0,1)) r^{2\gamma + d} \chi_{(0,1)}(s)
= \frac{d_k}{2\gamma + d} r^{2\gamma + d} \chi_{(0,1)}(s),
\]

then we get for \( t \geq 0 \), \( f^{*}(t) = \chi_{(0,R)}(t) \) where \( R = \frac{d_k}{2\gamma + d} r^{2\gamma + d} \). Hence, using (2.1) and (2.2) again, we obtain

\[
\int_{0}^{+\infty} f^{*}(t) \left(\frac{1}{\beta}\right)^{*}(t) dt = \frac{d_k}{\beta + 2\gamma + d} r^{\beta + 2\gamma + d}
= \int_{\mathbb{R}^d} f(x) \vartheta(x) d\nu_k(x),
\]

which gives (3.6).

Theorem 3.1 Let \( u, v \) be non-negative \( \nu_k \)-locally integrable weight functions on \( \mathbb{R}^d \), \( 1 < p \leq q < +\infty \) with \( q \geq 2 \) and assume \( \frac{1}{v} \in B_p \). Then there exists a constant \( c > 0 \) such that for all \( f \in L^1_k(\mathbb{R}^d) + L^2_k(\mathbb{R}^d) \), the inequality

\[
\left(\int_{0}^{+\infty} \left((F_k(f))^{*}(t)\right)^{\frac{p}{q}} u^{*}(t) dt\right)^{\frac{1}{q}} \leq c \left(\int_{0}^{+\infty} \left(f^{*}(t)\right)^{p} \left[\left(\frac{1}{v}\right)^{*}(t)\right]^{-1} dt\right)^{\frac{1}{p}}
\]

holds if and only if

\[
\sup_{s > 0} s^{\frac{1}{q}} \left(\int_{0}^{s} u^{*}(t) dt\right)^{\frac{1}{q}} \left(\int_{0}^{s} \left[\left(\frac{1}{v}\right)^{*}(t)\right]^{-1} dt\right)^{\frac{1}{p}} < +\infty.
\]
Proof. We begin with the necessity part. Put for any fixed \( r > 0 \),

\[
R = \left( r \frac{\nu_k(B(0,1))}{1 + (\nu_k(B(0,1))^2)} \right)^{1/2},
\]

and we take \( f = \chi_{(0,R)} \) in (3.7), where \( \chi_{(0,R)} \) is the characteristic function of the interval \((0,R)\). For \( s \geq 0 \) and by (2.1) and (2.2), the distribution function of \( f \) is

\[
D_f(s) = \nu_k(\{x \in \mathbb{R}^d : \chi_{(0,R)}(\|x\|) > s\})
= \frac{d_k}{2\gamma + d} R^{2\gamma + d} \chi_{(0,1)}(s)
= \nu_k(B(0,1)) R^{2\gamma + d} \chi_{(0,1)}(s)
= r' \chi_{(0,1)}(s),
\]

where we set

\[
r' = r \frac{(\nu_k(B(0,1))^2}{1 + (\nu_k(B(0,1))^2)}.
\]

This yields for \( t \geq 0 \),

\[
f^*(t) = \inf\{s \geq 0 : D_f(s) \leq t\}
= \chi_{(0,r')}(t).
\]

Observe that \( r' < r \), hence we have

\[
\left( \int_0^{+\infty} ((\mathcal{F}_k(f))^*(t))^q u^*(t)dt \right)^{\frac{1}{q}} \leq c \left( \int_0^{r'} \left[ \left( \frac{1}{v} \right)^*(t) \right]^{-1} dt \right)^{\frac{1}{q}}
\leq c \left( \int_0^{r'} \left[ \left( \frac{1}{v} \right)^*(t) \right]^{-1} dt \right)^{\frac{1}{q}}.
\]

(3.10)

According to (2.3), for \( x \in \mathbb{R}^d \), we can assert that

\[
\mathcal{F}_k(f)(x) = c_k^{-1} \int_0^R J_{\gamma + \frac{d}{2} - 1}(\|x\|t) \frac{t^{2\gamma + d - 1}}{2^{\gamma + \frac{d}{2} - 1} \Gamma(\gamma + \frac{d}{2})} dt
= c_k^{-1} \|x\|^{\frac{2\gamma + d - 1}{2}} \int_0^R J_{\gamma + \frac{d}{2} - 1}(\|x\|t) t^{\frac{2\gamma + d}{2}} dt.
\]

(3.11)
Since \( \cos(t\|x\| \cos \theta) \geq \cos 1 > \frac{1}{2} \), for \( t \in (0, R) \), \( \|x\| \in (0, \frac{1}{R}) \) and \( \theta \in (0, \frac{\pi}{2}) \), then we obtain from (2.4), the estimate

\[
J_{\gamma}^{\frac{d}{2}} (\|x\|t) > \frac{1}{2} C_{\gamma} (\|x\|t)^{\gamma + \frac{d}{2} - 1} \int_{0}^{\frac{\pi}{2}} (\sin \theta)^{2\gamma + d - 2} d\theta
\]

\[
= \frac{1}{2} C_{\gamma} (\|x\|t)^{\gamma + \frac{d}{2} - 1} \frac{\sqrt{\pi} \Gamma(\gamma + \frac{d}{2} - \frac{1}{2})}{2\Gamma(\gamma + \frac{d}{2})}
\]

\[
= \frac{(\|x\|t)^{2\gamma + d - 2}}{2^{2\gamma + d} \Gamma(\gamma + \frac{d}{2})},
\]

which gives by (2.1), (2.2), (3.9), (3.11) and for \( \|x\| \in (0, \frac{1}{R}) \)

\[
\mathcal{F}_k(f)(x) > C_k^{-1} \|x\|^{2 - 2\gamma - d} \int_{0}^{R} \frac{(\|x\|t)^{2\gamma + d - 2}}{2^{2\gamma + d} \Gamma(\gamma + \frac{d}{2})} t^{2\gamma + d - 1} dt
\]

\[
= \frac{C_k^{-1}}{2^{2\gamma + d} \Gamma(\gamma + \frac{d}{2})} \int_{0}^{R} t^{2\gamma + d - 1} dt
\]

\[
= \frac{r'}{2}.
\]

By the fact that

\[
\{ t \in (0, \frac{1}{r}) : (\mathcal{F}_k(f)^*) (t) > s \} = \{ t \in (0, \frac{1}{r}) : D \mathcal{F}_k(f)(s) > t \},
\]

we have from (3.3)

\[
\left( \int_{0}^{+\infty} ((\mathcal{F}_k(f)^*)(t))^q u^*(t) dt \right)^{\frac{1}{q}}
\]

\[
\geq \left( \int_{0}^{+\infty} \left( \int_{\{t \in (0, \frac{1}{r}) : (\mathcal{F}_k(f)^*)(t) > s \}} u^*(t) dt \right) ds \right)^{\frac{1}{q}}
\]

\[
= \left( q \int_{0}^{+\infty} s^{q-1} \left( \int_{\{t \in (0, \frac{1}{r}) : (\mathcal{F}_k(f)^*)(t) > s \}} u^*(t) dt \right) ds \right)^{\frac{1}{q}}
\]

\[
= \left( q \int_{0}^{+\infty} s^{q-1} \left( \int_{0}^{\min(D \mathcal{F}_k(f)(s), \frac{r'}{2})} u^*(t) dt \right) ds \right)^{\frac{1}{q}}.
\]

If \( s < \frac{r'}{2} \), then by (3.12)

\[
B(0, \frac{1}{R}) \subseteq \{ x \in \mathbb{R}^d : |\mathcal{F}_k(f)(x)| > \frac{r'}{2} \} \subseteq \{ x \in \mathbb{R}^d : |\mathcal{F}_k(f)(x)| > s \},
\]
thus using (2.1) and (2.2), we have

\[
D_{\mathcal{F}_k(f)}(s) = \int_{\{x \in \mathbb{R}^d : |\mathcal{F}_k(f)(x)| > s\}} w_k(x) \, dx \\
\geq d_k \int_0^{\frac{1}{s}} \rho^{2\gamma + d - 1} \, d\rho = \frac{d_k}{2\gamma + d} R^{-2\gamma - d} \\
= \frac{1}{r} \left( 1 + (\nu_k(B(0, 1)))^2 \right) > \frac{1}{r},
\]

which gives that

\[
\left( \int_0^{+\infty} \left( \mathcal{F}_k(f)^*(t) \right)^q u^*(t) \, dt \right)^{\frac{1}{q}} \geq \left( \frac{r'}{2} \left( \int_0^{\frac{1}{s}} u^*(t) \, dt \right)^{\frac{1}{2}} \right)^{\frac{1}{q}} \\
= \left( \frac{r'}{2} \left( \int_0^{\frac{1}{s}} u^*(t) \, dt \right)^{\frac{1}{2}} \right)^{\frac{1}{q}} \\
= \frac{(\nu_k(B(0, 1))^2}{2[1 + (\nu_k(B(0, 1)))^2]} \left( \int_0^{\frac{1}{s}} u^*(t) \, dt \right)^{\frac{1}{q}}.
\]

According to (3.10), we deduce that

\[
r \left( \int_0^{\frac{1}{s}} u^*(t) \, dt \right)^{\frac{1}{q}} \left( \int_0^{r} \left[ \left( \frac{1}{s} \right)^* (t) \right]^{-1} \, dt \right)^{-\frac{1}{p}} \leq c \left( \int_0^{+\infty} \left( \mathcal{F}_k(f)^*(t) \right)^q u^*(t) \, dt \right)^{\frac{1}{q}} \\
\times \left( \int_0^{r} \left[ \left( \frac{1}{s} \right)^* (t) \right]^{-1} \, dt \right)^{-\frac{1}{p}} \leq c.
\]

For the sufficiency part, we take \( f \in L_k^1(\mathbb{R}^d) + L_k^2(\mathbb{R}^d) \), then using (3.2) and (3.4), we obtain

\[
\left( \int_0^{+\infty} \left( \mathcal{F}_k(f)^*(t) \right)^q u^*(t) \, dt \right)^{\frac{1}{q}} \leq c \left( \int_0^{+\infty} \left( \mathcal{F}_k(f)^*(t) \right)^q u^*(t) \, dt \right)^{\frac{1}{q}}.
\]

If we make the change of variable \( t = \frac{1}{s} \) on the right side, we get

\[
\left( \int_0^{+\infty} \left( \mathcal{F}_k(f)^*(t) \right)^q u^*(t) \, dt \right)^{\frac{1}{q}} \leq c \left( \int_0^{+\infty} \left( \frac{1}{s} \int_0^s f^*(t) \, dt \right)^q u^*(t) \, dt \right)^{\frac{1}{q}},
\]
which gives from (1.2), (1.3) and (1.4), that the inequality (3.7) is satisfied if and only if
\[
\sup_{s>0} \left( \int_0^s \frac{u^*(\frac{1}{t})}{t^2-q} \, dt \right)^{\frac{1}{q}} \left( \int_0^s \left[ \left( \frac{1}{v} \right)^* (t) \right]^{-1} \, dt \right)^{-\frac{1}{p}} < +\infty
\]
and
\[
\sup_{s>0} \left( \int_0^{+\infty} \frac{u^*(\frac{1}{t})}{t^2-q} \, dt \right)^{\frac{1}{q}} \left( \int_0^s \left[ \left( \frac{1}{v} \right)^* (t) \right]^{-1} \, dt \right)^{-\frac{1}{p}} < +\infty.
\]
In order to complete the proof, we must verify that (3.8) implies these two conditions between the weights \( u^* \) and \( \left( \frac{1}{v} \right)^* \). This follows closely the argumentations of [4]. More precisely, since \( u^* \) is non-increasing, then \( u^* \in B\), and by (1.1), it yields
\[
\int_0^s u^*(\frac{1}{t})t^{q-2} \, dt = \int_{s^{-\frac{1}{q}}}^{+\infty} \frac{u^*(t)}{t^q} \, dt \leq b_{p,q}^{-\frac{1}{q}} \int_0^s u^*(t) \, dt.
\]
Hence by (3.8), we get
\[
\left( \int_0^s u^*(\frac{1}{t})t^{q-2} \, dt \right)^{\frac{1}{q}} \left( \int_0^s \left[ \left( \frac{1}{v} \right)^* (t) \right]^{-1} \, dt \right)^{-\frac{1}{p}} \leq b_{p,q}^{\frac{1}{q}} s \left( \int_0^s u^*(t) \, dt \right)^{\frac{1}{q}} \times \left( \int_0^s \left[ \left( \frac{1}{v} \right)^* (t) \right]^{-1} \, dt \right)^{-\frac{1}{p}} < +\infty,
\]
and so we obtain the first condition.

To show that the second condition is satisfied, observe that by means of a change of variable, we have
\[
\left( \int_0^{+\infty} \frac{u^*(\frac{1}{t})}{t^2-q} \, dt \right)^{\frac{1}{q}} = \left( \int_0^1 \frac{u^*(t)}{t^q} \, dt \right)^{\frac{1}{q}}.
\]
(3.13)

Now, define the function \( G \) by
\[
G(s) = \left( \int_0^s \left( \frac{1}{l} \right)^* (l) \, dl \right)^{-p'} \left[ \left( \frac{1}{v} \right)^* (t) \right]^{-1} \, dt \right)^{\frac{1}{p'}} + s^{p'} \left( \int_0^s \left[ \left( \frac{1}{v} \right)^* (t) \right]^{-1} \, dt \right)^{1-p'} \left( \int_0^1 \frac{u^*(t)}{t^q} \, dt \right)^{\frac{1}{q}}
\]
then by integration by parts, we get
\[
G(s) = \left[ p' G(s)^{p'} + s^{p'} \left( \int_0^s \left[ \left( \frac{1}{v} \right)^* (t) \right]^{-1} \, dt \right)^{1-p'} \right]^{\frac{1}{p'}} - p' \int_0^s \left( \frac{1}{l} \right)^* (l) \, dl \left[ \int_0^s \left[ \left( \frac{1}{v} \right)^* (t) \right]^{-1} \, dt \right]^{1-p'} \, dt \right)^{\frac{1}{p'}}.
\]
which implies
\[(p' - 1)G(s)^{p'} \leq p' \int_0^s \left( \frac{1}{t} \int_0^t \left[ \left( \frac{1}{v} \right)^*(l) \right]^{-1} dl \right)^{1-p'} dt,\]
and so
\[G(s) \leq \left( \frac{p'}{p' - 1} \int_0^s \left( \frac{1}{t} \int_0^t \left[ \left( \frac{1}{v} \right)^*(l) \right]^{-1} dl \right)^{1-p'} dt \right) \frac{1}{p'}.\]

Since \(\frac{1}{v} \in B_p\), we can invoke (3.1) and we obtain
\[\left( \int_0^s \left( \frac{1}{t} \int_0^t \left[ \left( \frac{1}{v} \right)^*(l) \right]^{-1} dl \right)^{-p'} \left( \left( \frac{1}{v} \right)^*(t) \right)^{-1} dt \right)^{\frac{1}{p'}} \leq c s \left( \int_0^s \left[ \left( \frac{1}{v} \right)^*(t) \right]^{-1} dt \right)^{\frac{1}{p'}}.\]

Combining this inequality and (3.13), we deduce our result. This completes the proof. □

**Theorem 3.2** Let \(u, v\) be non-negative \(\nu_\kappa\)-locally integrable weight functions on \(\mathbb{R}^d\) and suppose \(1 < p \leq q < +\infty\). Then there exists a constant \(c > 0\) such that for all \(f \in L^1_k(\mathbb{R}^d) + L^2_k(\mathbb{R}^d)\), the inequality
\[\left( \int_{\mathbb{R}^d} |f_k(f)(x)|^p u(x) d\nu_\kappa(x) \right)^{\frac{1}{p}} \leq c \left( \int_{\mathbb{R}^d} |f(x)|^p v(x) d\nu_\kappa(x) \right)^{\frac{1}{p}} \quad (3.14)\]
holds with the following hypotheses on \(u\) and \(v\):

i) for \(q \geq 2\):
\[\frac{1}{v} \in B_p\] and
\[\sup_{s > 0} s \left( \int_0^s u^*(t) dt \right)^{\frac{1}{q}} \left( \int_0^s \left[ \left( \frac{1}{v} \right)^*(t) \right]^{-1} dt \right)^{-\frac{1}{p}} < +\infty, \quad (3.15)\]

ii) for \(q < 2\):
\[\frac{1}{u^*(v)q'-1} \in B_{q'}\] and
\[\sup_{s > 0} s \left( \int_0^s (u^*(t))^{1-q'} dt \right)^{-\frac{1}{q'}} \left( \int_0^s \left[ \left( \frac{1}{v} \right)^*(t) \right]^{q'-1} dt \right)^{\frac{1}{p'}} < +\infty. \quad (3.16)\]
Proof. We start with the proof of part i). Since \( q \geq 2 \), we can use Theorem 1 and we have
\[
\left( \int_0^{+\infty} \left( (\mathcal{F}_k(f))^*(t) \right)^q u^*(t) dt \right)^{\frac{1}{q}} \leq c \left( \int_0^{+\infty} \left( f^*(t) \right)^p \left[ \left( \frac{1}{v} \right)^*(t) \right]^{-1} dt \right)^{\frac{1}{p}}.
\]
Note that \( (|f|^p)^* = (f^*)^p \) and \( (|\mathcal{F}_k(f)|^q)^* = (|\mathcal{F}_k(f)|^*)^q \), then applying (3.5) and (3.6) for this inequality, we obtain (3.14).

Now, we consider the case \( q < 2 \) in part ii). By definition of \( L_{k,u}^q(\mathbb{R}^d) \) and the Hahn-Banach theorem, we have
\[
\left( \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)|^q u(x) d\nu_k(x) \right)^{\frac{1}{q}} = \sup_{\|g\|_{L_{k,u}^q(\mathbb{R}^d)} = 1} \left| \int_{\mathbb{R}^d} \mathcal{F}_k(f)(x) \overline{g(x)} d\nu_k(x) \right|.
\]
(3.17)

The dual space \((L_{k,u}^q(\mathbb{R}^d))^'*\) of \( L_{k,u}^q(\mathbb{R}^d) \) can be identified with \( L_{k,\eta}^{q'}(\mathbb{R}^d) \) where \( \eta(x) = u(x)^{-q'/q} \). Using the Parseval relation over the Schwarz space \( S(\mathbb{R}^d) \) and Hölder’s inequality, we obtain
\[
\left| \int_{\mathbb{R}^d} \mathcal{F}_k(f)(x) \overline{g(x)} d\nu_k(x) \right| = \left| \int_{\mathbb{R}^d} f(x) \overline{\mathcal{F}_k(g)(-x)} d\nu_k(x) \right| \leq \|f\|_{p,k,v} \left( \int_{\mathbb{R}^d} |\mathcal{F}_k(g)(-x)|^{p'} v(x)^{1-p'} d\nu_k(x) \right)^{\frac{1}{p'}}.
\]
(3.18)

Since \( p' \geq 2 \) and \( q' \leq p' \), we can use part i) of the proof with \( q, p, u(x) \) and \( v(x) \) replaced respectively by \( p', q', v(x)^{1-p'} \) and \( u(x)^{1-q'} \). More precisely, observe that \( \frac{1}{(u^*)^{q'-1}} \in B_{q'} \) gives \( \frac{1}{(u^{q'-1})^*} \in B_p \) and by (3.16), we have
\[
\sup_{s > 0} \frac{1}{s} \left( \int_0^s \left( \frac{1}{v^{p'-1}} \right)^*(t) dt \right)^{\frac{1}{p'}} \left( \int_0^s \left[ (u^{q'-1})^*(t) \right]^{-1} dt \right)^{-\frac{1}{q'}} = \sup_{s > 0} \frac{1}{s} \left( \int_0^s \left( \frac{1}{v} \right)^*(t) dt \right)^{\frac{1}{p'}} \left( \int_0^s \left( u^*(t) \right)^{1-q'} dt \right)^{-\frac{1}{q'}} < +\infty,
\]
then we can deduce from part i) that
\[
\left( \int_{\mathbb{R}^d} |\mathcal{F}_k(g)(-x)|^{p'} v(x)^{1-p'} d\nu_k(x) \right)^{\frac{1}{p'}} \leq c \left( \int_{\mathbb{R}^d} |g(-x)|^{q'} u(x)^{1-q'} d\nu_k(x) \right)^{\frac{1}{q'}},
\]
and, hence, by density of \( S(\mathbb{R}^d) \) in \( L_{k,v}^p(\mathbb{R}^d) \) and \( L_{k,\eta}^{q'}(\mathbb{R}^d) \), we obtain from (3.17), (3.18) and the fact that \( 1 - q' = -\frac{q'}{q} \).
Weighted inequalities in Dunkl analysis

\[
\left( \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)|^q u(x) \, dv_k(x) \right)^{\frac{1}{q}} \\
\leq c \|f\|_{p,k,v} \sup_{\|g\|_{L^q_k}} \left( \int_{\mathbb{R}^d} |g(-x)|^q u^{\frac{q'}{q}}(x) \, dv_k(x) \right)^{\frac{1}{q'}} \\
\leq c \|f\|_{p,k,v} = c \left( \int_{\mathbb{R}^d} |f(x)|^p v(x) \, dv_k(x) \right)^{\frac{1}{p}}.
\]

This completes the proof. \(\square\)

**Example 3.2** Let \(u(x) = \|x\|^\alpha, v(x) = \|x\|^\beta, x \in \mathbb{R}^d\) with \(\alpha < 0\) and \(\beta > 0\). Using (2.1) and (2.2), we have for \(s \geq 0\)

\[
D_u(s) = \nu_k(\{x \in \mathbb{R}^d : \|x\|^\alpha > s\}) = \nu_k(B(0, s^{\frac{1}{\alpha}})) = \frac{d_k}{2\gamma + d} s^{\frac{2\gamma + d}{\alpha}},
\]

which gives for \(t \geq 0\)

\[
u^*(t) = \inf\{s \geq 0 : D_u(s) \leq t\} = \left( \frac{2\gamma + d}{d_k} \right)^{\frac{\alpha}{2\gamma + d}} t^{\frac{\alpha}{2\gamma + d}}.
\]

Note that from (2.2), \(d_k = \frac{c_k^{-1}}{2^{\gamma + \frac{d}{2}} - 1} \Gamma(\gamma + \frac{d}{2})\), this yields

\[
\frac{d_k}{2\gamma + d} = \frac{c_k^{-1}}{2^{\gamma + \frac{d}{2}} \Gamma(\gamma + \frac{d}{2} + 1)}.
\]

On the other hand, we have from Example 1

\[
\left( \frac{1}{v^*} \right)^{\alpha} = \left( \frac{2\gamma + d}{d_k} \right)^{-\frac{\alpha}{2\gamma + d}} t^{-\frac{\beta}{2\gamma + d}}.
\]

For these weights and \(1 < p \leq q < +\infty\), the hypothesis for \(q \geq 2\) of Theorem 2, i) gives respectively that the integrals in the \(B_p\)-inequality (1.1) for \(\frac{1}{(v^{\beta})^*}\) are finite and the boundedness condition (3.15) is valid if and only if \(0 < \beta < (2\gamma + d)(p - 1)\) and

\[
\begin{align*}
-(2\gamma + d) &< \alpha < 0, \\
\frac{1}{2\gamma + d} \left( \frac{\alpha}{q} + \frac{\beta}{p} \right) &= 1 - \frac{1}{p} - \frac{1}{q}.
\end{align*}
\]
By the same manner, the hypothesis of Theorem 2, ii) gives for $q < 2$ respectively that
\[ \frac{1}{(q^*)^q - 1} \in B_{q'} \] and the boundedness condition (3.16) is valid if and only if
\[-(2\gamma + d) < \alpha < 0 \] and
\[ 0 < \beta < (2\gamma + d)(p - 1), \]
\[ \frac{1}{2\gamma + d} (\frac{\alpha}{q} + \frac{\beta}{p}) = 1 - \frac{1}{p} - \frac{1}{q}. \]

These are the same conditions verified in parts i) and ii) of Theorem 2. Under these conditions and index constraints, we obtain from Theorem 2, Pitt’s inequality for $f \in L_{p,k,v}^p(\mathbb{R}^d)$,
\[ (\int_{\mathbb{R}^d} |F_k(f)(x)|^q \|x\|^\alpha d\nu_k(x))^{\frac{1}{q}} \leq c \left( \int_{\mathbb{R}^d} |f(x)|^p \|x\|^\beta d\nu_k(x) \right)^{\frac{1}{p}}. \] (3.19)

Now by applying Pitt’s inequality, we establish further results concerning integrability of the Dunkl transform of function $f$ on $\mathbb{R}^d$ when $f$ is in a suitable Besov-Dunkl space.

**Theorem 3.3** Let $1 < p \leq 2$, $p \leq q < +\infty$, $-(2\gamma + d) < \alpha < 0$ and $0 < \beta < (2\gamma + d)(p - 1)$. If $u(x) = \|x\|^\alpha$, $v(x) = \|x\|^\beta$ and $f \in BD_{1-(q-1)(2\gamma + d)-\alpha}^{p,k,v}$ with the index constraints $\frac{p + \alpha}{p - 1} < 2\gamma + d$ and $\frac{1}{2\gamma + d} (\frac{\alpha}{q} + \frac{\beta}{p}) = 1 - \frac{1}{p} - \frac{1}{q}$, then
\[ F_k(f) \in L_{k,v}^1(\mathbb{R}^d). \]

**Proof.** Let $f \in BD_{1-(q-1)(2\gamma + d)-\alpha}^{p,k,v}$. Since $F_k(\phi_t)(x) = F_k(\phi)(tx)$ for $t \in (0, +\infty)$, then from (2.5), we can write
\[ F_k(f * k \phi_t)(x) = F_k(f)(x)F_k(\phi_t)(x) = F_k(f)(x)F_k(\phi)(tx) \quad a.e \ x \in \mathbb{R}^d, \]

hence, we obtain by (3.19)
\[ \int_{\mathbb{R}^d} |F_k(f)(x)|^q |F_k(\phi)(tx)|^q \|x\|^\alpha d\nu_k(x) \leq c \left( \int_{\mathbb{R}^d} |f * k \phi_t|^p(x)\|x\|^\beta d\nu_k(x) \right)^{\frac{q}{p}} \leq c \|f * k \phi_t\|_{p,k,v}^q. \] (3.20)

From (1.6) and (3.20), we get
\[ t^2 \left( \int_{\frac{1}{2} \leq \|x\| \leq T} |F_k(f)(x)|^q \|x\|^{\alpha + 2q} d\nu_k(x) \right)^{\frac{1}{q}} \leq c \|f * k \phi_t\|_{p,k,v}. \]
which gives by Hölder’s inequality and (2.1),
\[
\int_{\frac{1}{2}t \leq \|x\| \leq t} |\mathcal{F}_k(f)(x)||x||d\nu_k(x) \leq c \frac{\|f \ast_k \phi_t\|_{p,k,v}}{t^2} \left( \int_{\frac{1}{2}t}^t r^{\frac{1}{p} - \frac{1}{1-q} + \frac{1}{q} - \frac{1}{r}} r^{2\gamma + d - 1} dr \right)^{\frac{1}{q}'} \\
\leq c \frac{\|f \ast_k \phi_t\|_{p,k,v}}{t^{\frac{1}{q} - \frac{1}{q} + \frac{1}{q} - \frac{1}{r}}}.
\]
Integrating with respect to \(t\) over \((0, +\infty)\) and applying Fubini’s theorem, it yields
\[
\int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)||d\nu_k(x) \leq c \int_0^{+\infty} \frac{\|f \ast_k \phi_t\|_{p,k,v}}{t^{\frac{1}{q} - \frac{1}{q} + \frac{1}{q} - \frac{1}{r}}} dt < +\infty,
\]
by the definition of \(BD^{p,k,v}_{1,\frac{(q-1)(2\gamma + d) - \alpha}{q}}\). This complete the proof. \(\square\)

**Remark 3.2** The case \(\beta = 0, \alpha = (2\gamma + d)(p - 2)\) and \(1 < p = q \leq 2\) was obtained in ([1], Section 4, Lemma 1) and gives the Hardy-Littlewood-Paley inequality
\[
\left( \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)||x|^{(2\gamma + d)(p-2)} d\nu_k(x) \right)^{\frac{1}{p}} \leq c \left( \int_{\mathbb{R}^d} |f(x)|^p d\nu_k(x) \right)^{\frac{1}{p}}. \tag{3.21}
\]
Using (3.21), it was shown in ([1], Section 4, Theorem 4) the following result:
Let \(f \in BD^{p,k,v}_{1,\frac{2\gamma + d}{p}}\), for some \(1 < p \leq 2\). If \(p < 2\gamma + d\) then
\[
\mathcal{F}_k(f) \in L^1_k(\mathbb{R}^d).
\]

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