A continuum string model for $D > 1$

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**ABSTRACT**

The non critical string (2D gravity coupled to the matter with central charge $D$) is quantized taking care of both diffeomorphism and Weyl symmetries. In incorporating the gauge fixing with respect to the Weyl symmetry, through the condition $R_g = \text{const}$, one modifies the classical result of Distler and Kawai. In particular one obtains the real string tension for an arbitrary value of central charge $D$.

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Conformal field theory coupled to 2D Euclidean quantum gravity was solved initially by Polyakov [1] who introduced the following path integral

$$Z = \int \frac{[Dg][Dg^{ab}]}{Vol(\text{Diff}) \cdot Vol(\text{Weyl})} e^{-S_M[g;X]} - \frac{\mu_0^2}{8\pi} \int d^2\xi \sqrt{g}$$

where the matter action is

$$S_M[g;X] = \frac{1}{8\pi} \int d^2\xi \sqrt{g} g^{ab} \nabla_a X^\mu \nabla_b X^\mu$$

and an explicit bare cosmological constant term has been included. The $X^\mu$ are $D$ bosonic scalar fields taking values in $\mathbb{R}^D$.

As well known the presence of the volumes in the measure follows from the invariance of the matter action under diffeomorphisms and conformal mappings of the Riemann surface [2] [3]. Thus we are lead to analyse the symmetries of matter action. As is known $S_M[g;X]$ is invariant both under

- diffeomorphism of the word sheet

$$\xi^a \to \xi'^a = \xi^a + \epsilon^a(\xi)$$

$$\delta g_{ab} = \nabla_a \epsilon_b + \nabla_b \epsilon_a$$

- Weyl transformation, that is transformations of the metric field alone of the type

$$g'_{ab}(x) = e^{\sigma(x)} g_{ab}$$

Our purpose is to fix the gauge and define the correct determinant and Fadeev Popov ghosts for both the gauge symmetries. Furthermore we shall assume to have a string world-sheet surface of genus $h \geq 2$, i.e. an hyperbolic 2D-geometry.
To fix Weyl gauge we note that for every given hyperbolic Euclidean metric $g_{ab}$, there exists only one metric $g'_{ab}$ conformal to $g_{ab}$ having constant curvature $R_{g'} = -1$ [4] (see appendix A). So we can choose the gauge fixing condition

$$R_g = -1$$

To write the FP determinant we must calculate the formal derivative of the gauge function

$$f(g'_{ab}) = R_{g'} + 1$$

when $g'_{ab}$ is given by the Weyl transformation with a scaling parameter $k$

$$g'_{ab}(x) = e^{k\sigma(x)}g_{ab}$$

so

$$f(g') = e^{-k\sigma}(R_g - kg^{ab}\nabla_a\nabla_b\sigma) + 1$$

If we consider an infinitesimal variation $\delta\sigma$, we have

$$\delta g_{ab} = kg_{ab}\delta\sigma$$

$$R_{g'} = (1 - k\delta\sigma)(R_g - kg^{ab}\nabla_a\nabla_b\delta\sigma) = R_g - kg^{ab}\nabla_a\nabla_b\delta\sigma - kR_g\delta\sigma + o(\delta\sigma^2)$$

$$\delta R_g = -k(R_g + g^{ab}\nabla_a\nabla_b)\delta\sigma$$

$$\frac{\delta f}{\delta\sigma} = -k(R_g + g^{ab}\nabla_a\nabla_b)$$

then following Fadeev-Popov we must calculate the determinant

$$N = Vol(Weyl) \ Det\{-k(R_g + g^{ab}\nabla_a\nabla_b)\}$$

To do that we introduce the grasmaniann ghosts $\psi$ and $\bar{\psi}$ satisfying the rules

$$\{\psi, \bar{\psi}\} = 0, \ \ \psi^2 = \bar{\psi}^2 = 0$$
and we can write \( N \) in the form of path integral

\[
N = \text{Vol}(\text{Weyl}) \int [D_\psi][D\bar{\psi}] e^{\frac{1}{2k} \int d^2\xi \sqrt{\hat{g}}(\hat{\psi}^\dagger R_\psi + g^{ab}\nabla_a \nabla_b \psi)(\hat{\psi})} \equiv \text{Vol}(\text{Weyl}) \tilde{N}
\]  

(2)

For what concerns the diffeomorphisms we follow the Distler-Kawai (DK) procedure [2] and introduce a set of background metrics parameterized by the moduli parameters, \( \hat{g}_{ab}(\tau) \). The gauge with respect to diffeomorphism’s invariance will be fixed parameterizing the metric by a Weyl scaling \( \phi \)

\[
g_{ab} = e^{\alpha \phi} \hat{g}_{ab}(\tau)
\]

and also the volume of diffeomorphism’s group disappears from the path integral (1) which now can be rewritten in the following way

\[
Z = \int [D_g X^\mu][D\phi][D\phi^\dagger][d\tau] \tilde{N} e^{-S_M[X;\tilde{g}] - S_{GH}[b,c;\tilde{g}] - S_L[\phi;\tilde{g}]} \]

Here

\[
S_L[\phi;\tilde{g}] = \frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}}(\hat{g}^{ab}\nabla_a \phi \nabla_b \phi - Q R\hat{g}\phi)
\]

is the Liouville action without the term \( \mu_1 \sqrt{\hat{g}} e^{\alpha \phi} \) already eliminated by setting the appropriated value to the cosmological constant \( \mu_0 \). Note also the scale parameter \( \alpha \) which has been included to have a standard kinetic term.

We can now match the two procedures paying attention to the fact that having parameterized the metric as above, we must effect the substitution

\[
g_{ab} = e^{\alpha \phi} \hat{g}_{ab}
\]

also into \( \tilde{N} \) (from now on we will omit the \( \tau \) parameter).
Our ansatz, following DK, is that the measure for Weyl ghosts transforms as the other measures, that is with a jacobian determinant given by a Liouville action

\[
[D_g \psi][D_g \bar\psi] = e^{S_L[\phi;\hat{g}]}[D_{\hat{g}} \psi][D_{\hat{g}} \bar\psi] \tag{4}
\]

under a Weyl scaling of the metric \( g_{ab} = e^{\alpha \phi} \hat{g}_{ab} \). We can now substitute (3) in the determinant (2).

The relation between curvatures and between laplacians are

\[
R_g = e^{-\alpha \phi} (R_{\hat{g}} - \alpha \hat{g}^{ab} \nabla_a \nabla_b \phi) \tag{5}
\]

\[
g^{ab} \nabla_a \nabla_b = e^{-\alpha \phi} \hat{g}^{ab} \nabla_a \nabla_b
\]

substituting these relations in the exponent in \( \tilde{N} \) we have (remember that \( \sqrt{g} = e^{\alpha \phi} \sqrt{\hat{g}} \))

\[
\frac{1}{2} k \int d^2 \xi \sqrt{\hat{g}} \hat{\psi}(\xi) (R_{\hat{g}} + g^{ab} \nabla_a \nabla_b) \psi(\xi)
\]

\[
= \frac{1}{2} k \int d^2 \xi \sqrt{\hat{g}} e^{\alpha \phi} \hat{\psi}(\xi) [e^{-\alpha \phi} (R_{\hat{g}} - \alpha \hat{g}^{ab} \nabla_a \nabla_b \phi) + e^{-\alpha \phi} \hat{g}^{ab} \nabla_a \nabla_b] \psi(\xi)
\]

\[
= \frac{1}{2} k \int d^2 \xi \sqrt{\hat{g}} \hat{\psi}(\xi) (R_{\hat{g}} - \alpha \hat{g}^{ab} \nabla_a \nabla_b \phi + \hat{g}^{ab} \nabla_a \nabla_b) \psi(\xi)
\]

so we can write \( \tilde{N} \) in the following way

\[
\tilde{N} = \int [D_g \psi][D_g \bar\psi] e^{\frac{1}{2} k \int d^2 \xi \sqrt{\hat{g}} \hat{\psi}(\xi) (R_{\hat{g}} - \alpha \hat{g}^{ab} \nabla_a \nabla_b \phi + \hat{g}^{ab} \nabla_a \nabla_b) \psi(\xi)} \tag{6}
\]

where the relation (4) is understood in eq. (6). As a consequence, we should now transform the measure in eq. (6) inserting a Liouville determinant. We shall have a unique Liouville action with undetermined parameters in the partition function.
Thus we obtain the final path integral expression for $Z$

$$Z = \int [D\hat{g}X^\mu][D\hat{g}\phi][D\hat{g}b][D\hat{g}\psi][D\hat{g}\bar{\psi}][d\tau]e^{-S_M[X;\hat{g}]-S_{GH}[b,c;\hat{g}]-S_L[\phi;\hat{g}]-S_W[\psi;\hat{g}]}$$

(7)

where we have defined the Weyl action as

$$S_W[\psi; \hat{g}] = \frac{1}{2} k \int d^2 \xi \sqrt{\hat{g}} \bar{\psi} \left\{ -(\hat{R} - \alpha \hat{g}^{ab} \nabla_a \nabla_b \phi + \hat{g}^{ab} \nabla_a \nabla_b) \psi \right\}$$

(8)

We are interested in the total action for Liouville and Weyl ghosts

$$S = S_L[\phi; \hat{g}] + S_W[\psi; \hat{g}]$$

$$= \frac{1}{8\pi} \int d^2 \xi \sqrt{\hat{g}} (\hat{g}^{ab} \nabla_a \phi \nabla_b \phi - QR_{\hat{g}} \phi) + \frac{k}{2} \int d^2 \xi \sqrt{\hat{g}} \bar{\psi} \left\{ -(\hat{R} - \alpha \hat{g}^{ab} \nabla_a \nabla_b \phi + \hat{g}^{ab} \nabla_a \nabla_b) \psi \right\}$$

$$= \frac{1}{8\pi} \int d^2 \xi \sqrt{g} (\phi \Delta_{\hat{g}} \phi - QR_{\hat{g}} \phi - q\bar{\psi} \Delta_{\hat{g}} \psi - q\bar{\psi} R_{\hat{g}} \psi + \alpha q\bar{\psi} \psi \Delta_{\hat{g}} \phi)$$

where we set

$$q = 4\pi k$$

$$\Delta_{\hat{g}} = \hat{g}^{ab} \nabla_a \nabla_b$$

that is

$$S = \frac{1}{8\pi} \int d^2 \xi \sqrt{g} [\phi \Delta_{\hat{g}} \phi - QR_{\hat{g}} \phi - q\bar{\psi} \Delta_{\hat{g}} \psi - q\bar{\psi} R_{\hat{g}} \psi + \alpha q\bar{\psi} \psi \Delta_{\hat{g}} \phi]$$

(9)

Now the term in (9) relative to the Weyl ghosts ($\bar{\psi}, \psi$) may be rewritten locally in the formal way (we adopt a complex coordinate system)

$$\frac{q}{8\pi} \sqrt{g} [(\partial \bar{\psi}) \partial \bar{\psi} - \frac{1}{\vartheta} (\partial \bar{\psi}) \psi R_{\hat{g}} + \alpha \frac{1}{\vartheta} (\partial \bar{\psi}) \psi \partial \bar{\psi}]$$

(10)

where $\partial \bar{\theta} = \Delta_{\hat{g}} |_{\text{loc}}$ and $\frac{1}{\vartheta}$ denotes the resolvent of the elliptic operator $\partial$, i.e. formally $\frac{1}{\vartheta} (\partial \text{FIELD}) \sim \text{FIELD}$.
Eq. (10) has the traditional form of a ghost bc-system after the identification

\[ b \equiv \partial \tilde{\psi} \]
\[ c \equiv \psi \]

Notice that the conformal dimension of \( \partial \tilde{\psi} \) and \( \psi \) are 1 and 0 respectively. We can then bosonize the action (10) following [5] by introducing a free scalar field \( \varphi \) and expressing the ghost fields \((\tilde{\psi}, \psi)\) as

\[ b = \partial \exp(i\varphi) \]
\[ c = \exp(-i\varphi) \]

In this way we found

\[ \frac{q}{8\pi} \sqrt{g} [(\partial \varphi \partial \varphi - i\varphi R_g) - iA \frac{1}{g}(\partial \varphi)R_g + i\alpha A \frac{1}{g}(\partial \varphi)\partial \partial \phi] = \frac{q}{8\pi} \sqrt{g} [-\varphi \partial \partial \varphi - i(1 + A) \varphi R_g + 2i \frac{\alpha A}{2} \varphi \partial \partial \phi] \]

where we used

\[ bc = iA(\partial \varphi) \]

Here \( A \) is a renormalization factor which shall play the role of a new bare parameter.

If now we define the free unrenormalized parameter

\[ (1 + A) \equiv \tilde{Q} \]
\[ \frac{\alpha A}{2} \equiv B \]
\[ q \equiv 8\pi \]

where \( \alpha \) is understood as a gauge (bare) parameter not to be confused with the Liouville field’s moment, the equation (10) in a general frame becomes

\[ \sqrt{g} [-\varphi \Delta_g \varphi + 2iB \varphi \Delta_g \phi - i\tilde{Q} \varphi R_g] \]

After the Wick rotation of field \( \varphi \), \( \varphi \to i\varphi \), in order to get a Liouville field, we
obtain

\[ \sqrt{g}[\varphi \Delta g \varphi - 2B \varphi \Delta g \phi + \bar{Q} \varphi R_{\bar{g}}] \]  \hspace{1cm} (15) \]

Putting eq. (15) in (9) we have the final form for the Liouville-Weyl action in matrix notation

\[ S = \int d^2 \xi \sqrt{\bar{g}}[-M_{i,j} \Phi^i \Delta \bar{g} \Phi^j - Q_i \Phi^i R_{\bar{g}}] \]

\[ \Phi^i \equiv (\Phi^1, \Phi^2) = (\phi, \varphi) \]
\[ Q_i \equiv (Q_1, Q_2) = (Q, -\bar{Q}) \]
\[ M_{i,j} \equiv \begin{pmatrix} 1 & B \\ B & -1 \end{pmatrix} \]  \hspace{1cm} (16) \]

In the following we shall regard the three parameters \( Q_1, Q_2 \) and \( B \) as renormalized parameters to be fixed by conformal invariance and by the requirement to get the DK-regime in the limit \( B \to 0 \).

The holomorphic energy-momentum tensor associated with (16) reads locally as

\[ T = -\frac{1}{2}[M_{i,j} \partial \Phi^i \partial \Phi^j + Q_i (\partial^2) \Phi^i] \]  \hspace{1cm} (17) \]

The correspondent central charge is given by

\[ c_{W+L} = 2 + 3M^{i,j}Q_iQ_j \]  \hspace{1cm} (18) \]

where \( M^{i,j} \) is the inverse matrix of \( M_{i,j} \).

Our key idea is that the model (16) replaces the DK Liouville action when one consider the coupling 2D-QG to conformal matter fields with central charge \( c_m \), for any value of \( c_m \). To show that, we calculate string susceptibility of the theory and we verify that its value is real for all \( c_m \).
In particular let us consider the coupling of (16) with $D$ free massless scalar matter fields, so that before the coupling their central charge is $c_m = D$. We start by noting that, as in DK, the gravitational coupling implies that the matter vertex operator $V(a)$ had a ”gravitational dressing” given by

$$V(a) \rightarrow V(a) e^{\alpha_1 \Phi_1} \equiv V(a) e^{\alpha \phi},$$

so that, if $V(a)$ is the identity operator, conformal invariance requires that $e^{\alpha \phi}$ is a $(1,1)$ conformal field and its conformal dimension as computed by (17)

$$\Delta[e^{\alpha_1 \Phi}] = -\frac{1}{2} \alpha_1 M^{1}j(\alpha_j + Q_j),$$

must be equal to 1. This fact will be used to compute parameter $\alpha_1$.

To calculate string susceptibility we use the expression [2]

$$\Gamma = \chi(h) \frac{Q_1}{2\alpha_1} + 2$$

where $\chi(h) = 2(1-h)$ is the Euler characteristic of the surface, hence we must determinate the values of parameters $Q_1$ and $\alpha_1$. We begin by noting that, as well known, Weyl invariance of the whole system requires that

$$c_{w+L} + D - 26 = 0$$

Using (18) in eq. (22) we find

$$Q_1^2 + 2BQ_1Q_2 - Q_2^2 + \frac{1 + B^2}{3}(D - 24) = 0$$

To solve this equation we adopt the following parametrization

$$Q_2 = \sqrt{\frac{1}{3} + \lambda(B, D)}$$

$$\lim_{B \rightarrow 0} \lambda(B, D) = 0$$
In this parametrization $Q_1$ is given by (see appendix B)

$$Q_1 = -\frac{1}{\sqrt{3}} \left[ B \sqrt{1 + 3\lambda(B, D)} - \sqrt{(1 + B^2)(3\lambda(B, D) + 25 - D)} \right]$$

(25)

where the sign is chosen to have the standard semiclassical limit for $D \to -\infty$ (see appendix C).

We can calculate $\alpha_1$ using (20) (remember that $\alpha_2 = 0$).

$$-\frac{1}{2} \alpha_1 M^{1,j}(\alpha_j + Q_j) = 1$$

(26)

and we get (see appendix B)

$$\alpha_1 = -\frac{1}{2\sqrt{3}} \left[ \sqrt{(1 + B^2)(25 - D + 3\lambda(D, B))} - \sqrt{(1 + B^2)(1 - D + 3\lambda(D, B))} \right]$$

(27)

where the sign have been chosen again to have the semiclassical limit for $D \to -\infty$ (see appendix C).

String susceptibility $\Gamma$ can now be calculated by using the expression (21) and known values for $Q_1$ and $\alpha_1$ in $\lambda$ parametrization. We find

$$\Gamma = \frac{1 - h}{12(1 + B^2)} \left\{ B \sqrt{1 + B^2} \left[ \sqrt{(1 + 3\lambda)(25 - D + 3\lambda)} + \sqrt{(1 + 3\lambda)(1 - D + 3\lambda)} \right] + 
\sqrt{(25 - D + 3\lambda)(1 - D + 3\lambda)} \right\}$$

(28)

We fix a particular parametrization choosing the following form for the function $\lambda(D, B)$

$$\lambda(D, B) = \frac{B}{3} D,$$

(29)

in fact we can see that in this parametrization the reality of $\Gamma$ is ensured for all $D > 0$ simply requiring that $B > 1$, as shown in appendix D, and the DK-limit is reached for $B \to 0$. 

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Although we have no informations on B at this level, we see that assuming that DK regime is achieved from eq. (28) just requiring the condition $D \leq 1$, then $B$ must be a step function of $D$ as $B \sim \overline{B}\theta(D - 1)$, where $\theta(x) = 0$ for $x \leq 0$ and $\overline{B}$ is a generic real constant greater than one.
APPENDIX A

We show that for every metric $g_{ab}$ there exists just one metric $\hat{g}_{ab}$ so that

$$g_{ab} = e^\sigma \hat{g}_{ab} \text{ and } R_{\hat{g}} = -1$$

To demonstrate existence we start by writing the relation between scalar curvatures of the metrics $g_{ab}$ and $\hat{g}_{ab}$

$$R_g = e^{-\sigma}(R_{\hat{g}} - \hat{g}^{ab}\nabla_a \partial_b \sigma) \quad (A1)$$

and vice versa

$$R_{\hat{g}} = e^{\sigma}(R_g + g^{ab}\nabla_a \partial_b \sigma) \quad (A2)$$

Now by requiring that $R_{\hat{g}} = -1$ we get the equation [6]

$$R_g(x,y) + g^{ab}(x,y)\nabla_a \partial_b \sigma(x,y) = -e^{-\sigma(x,y)}$$

We can reduce complexity by considering that, at least locally, $g_{ab}$ is conformally Euclidean, that is $g_{ab} = e^\rho \delta_{ab}$. From (A2) we obtain locally

$$-1 = R_{\hat{g}} = e^{\sigma'}(R_{\delta} + \delta^{ab}\partial_a \partial_b \sigma')$$

where $\sigma' = \rho \sigma$, so, at least locally, we must deal with the solvable equation

$$(\partial_x^2 + \partial_y^2)\sigma'(x,y) = -e^{-\sigma'(x,y)}.$$ 

To verify unicity we proceed via the reductio ab absurdo, allowing the existence of two metrics $g'_{ab}$ and $g''_{ab}$ both conformal equivalents to $g_{ab}$ and having scalar
curvature equal to $-1$, that is
\[ g_{ab} = e^{\sigma'} g'_{ab} \quad \text{and} \quad R_{g'} = -1 \]
\[ g_{ab} = e^{\sigma''} g''_{ab} \quad \text{and} \quad R_{g''} = -1 \]
so we can write
\[ g''_{ab} = e^{\sigma'-\sigma''} g'_{ab} \]
Now using (A1) we can write the following relations
\[ g'_{ab} = e^{-\sigma'} g_{ab} \]
\[ R_{g'} = e^{\sigma'} (R_g + g \nabla_a \partial_b \sigma') = -1 \quad (A3) \]
\[ g''_{ab} = e^{-\sigma''} g_{ab} \]
\[ R_{g''} = e^{\sigma''} (R_g + g \nabla_a \partial_b \sigma'') = -1 \quad (A4) \]
\[ g''_{ab} = e^{\sigma'-\sigma''} g'_{ab} \]
\[ R_{g''} = e^{-(\sigma'-\sigma'')} (R_g' - g'^{ab} \nabla_a \partial_b (\sigma' - \sigma'')) = -1 \quad (A5) \]
from the first two relations we obtain
\[ R_g + \nabla_a \partial_b \sigma' = -e^{-\sigma'} \]
\[ R_g + \nabla_a \partial_b \sigma'' = -e^{\sigma''} \]
so
\[ \nabla_a \partial_b (\sigma' - \sigma'') = e^{-\sigma''} - e^{-\sigma'} \quad (A6) \]
To use the last relations we notice that
\[ \nabla_a \partial_b \sigma = g'_{ab} \nabla'_a \partial'_b \sigma \]
If the dimension of the world sheet is 2 and $\sigma$ is a conformal tensor of weight $s=0$, the latter should transform as
\[ \sigma' = \Omega^s \sigma \]
under a coordinates transformation $z^a = z^a(z')$ for which $g_{ab}' = \Omega g_{ab}$ [7]. This conditions hold in present case because we are dealing with a transformation of
the metric alone, so that $z'^a = z^a$ and being $\sigma$ a scalar field $\sigma'(z') = \sigma(z)$ so that $\sigma' = \sigma$.

From (A5) we can now write

$$\nabla_a \partial_b (\sigma' - \sigma'') = e^{(\sigma' - \sigma'')} - 1$$

which compared with (A6) gives

$$e^{-\sigma''} - e^{-\sigma'} = e^{(\sigma' - \sigma'')} - 1$$

we can solve this equation by setting

$$x = e^{-\sigma'}$$
$$y = e^{-\sigma''}$$

so we obtain

$$y - x = \frac{y - x}{x}$$

which solutions are $x=y$ or $x=1$, that is

$$\sigma' = \sigma''$$

or

$$\sigma' = 0$$

which both confirm absurdity of initial assumption and complete the demonstration of unicity.
APPENDIX B

To proceed to the determination of parameters $Q_1$, $Q_2$ and $\alpha_1$ we must first determine the inverse matrix $M^{i,j}$ of $M_{i,j}$ defined in (16)

$$M^{i,j} = \frac{1}{1 + B^2} \begin{pmatrix} 1 & B \\ B & -1 \end{pmatrix}$$

(B1)

Then we can develop equations (23) and (26) obtaining

$$Q_1 = -BQ_2 \pm \frac{1}{\sqrt{3}} \sqrt{(1 + B^2)(3Q_2^2 + 24 - D)}$$

$$\alpha_1^2 + (Q_1 + BQ_2)\alpha_1 + 2(1 + B^2) = 0$$

$$\alpha_1 = -\frac{1}{2\sqrt{3}} \left[ \pm \sqrt{(1 + B^2)(3Q_2^2 + 24 - D)} \mp \sqrt{(1 + B^2)(3Q_2^2 - D)} \right]$$

Now inserting the parametrization (24) we get equations (25) and (27).
APPENDIX C

We verify that the parametrization (24) is compatible with DK-regime. Specifically we get DK-regime in the limit for $B \to 0$.

For $B \to 0$ we have
\[
\begin{align*}
\lambda & \to 0 \\
Q_2 & = \frac{1}{3} \\
Q_1 & \to \sqrt{\frac{25 - D}{3}}
\end{align*}
\]
which is exactly DK values for the central charge. Moreover we get the following values for $\alpha_1$ and $\Gamma$
\[
\begin{align*}
\alpha_1 & = -\frac{1}{2\sqrt{3}} \left[ \sqrt{25 - D} - \sqrt{1 - D} \right] \\
\Gamma & = \frac{(1 - h)}{12} \left[ D - 25 - \sqrt{(25 - D)(1 - D)} \right]
\end{align*}
\]
which are again DK values.
APPENDIX D

We want to verify the reality of $\Gamma$ as given by (28), when $\lambda$ is fixed by the parametrization (29). Reality of $\Gamma$ is obtained by simultaneously satisfying

\[
(1 + DB)(25 - (1 - B)) \geq 0 \\
(1 + DB)(1 - (1 - B)) \geq 0 \\
(1 - (1 - B))(25 - (1 - B)) \geq 0
\]

requiring that $B > 1$ and introducing the parameter

\[
a = B \\
b = B - 1
\]

we can solve the system

\[
(1 + aD)(25 + bD) \geq 0 \\
(1 + aD)(1 + bD) \geq 0 \\
(1 + bD)(25 + bD) \geq 0
\]

where the following relations between the parameters hold

\[
a > b > 0 \\
\frac{-25}{b} < -\frac{1}{b} < -\frac{1}{a} < 0
\]

We can so write the solutions of the equations in system (D1)

\[
D \leq -\frac{25}{b} ; \quad D \geq -\frac{1}{a} \\
D \leq -\frac{1}{b} ; \quad D \geq -\frac{1}{a} \\
D \leq \frac{25}{b} ; \quad D \geq -\frac{1}{b}
\]
and the solution of the system is

\[ D \leq -\frac{25}{b}; \ D \geq -\frac{1}{a} \]

we are interested in positive solutions, that is for \( D \geq -\frac{1}{a} \), which traduces in

\[ D \geq -\frac{1}{B} \]

so being \( B > 1 \) we have verified that system \((D1)\) is satisfied for all \( D > 0 \).
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