Relative Chern character number and super-connection

Dexie Lin†

Abstract For two complex vector bundles admitting a homomorphism with odd-dimensional sphere singularity between them, we give a formula to compute the relative Chern characteristic number of these two complex vector bundles. As a consequence, for the spin manifold admitting some bundle structure, we give a formula to express the index of twisted Dirac operator by using localization method.

Keywords Characteristic Classes, Super-connection, Chern character class

AMS Subject Classifications 53C23 55Q05 58A12

1 Introduction

In this paper, we regard the relative Chern character as the following map

\[ K(M, M\setminus U) \overset{\text{ch}}{\longrightarrow} H^{\text{even}}(M, M\setminus U), \]

where \( M \) is the closed oriented even dimensional manifold and \( U \) is an open submanifold of \( M \).

Since any element \([E]\) of \( K(M, M\setminus U)\) can be represented by the following data \([1, \text{Lemma 8.4}]:\)

\[ 0 \rightarrow E_+ \overset{v}{\longrightarrow} E_- \rightarrow 0, \]

where \( E_+ \) and \( E_- \) are two complex vector bundles of the same rank, and \( v \) is a complex linear homomorphism between the two vector bundles which is isomorphic on \( M \setminus U \).

If we assume that \( U \) is a neighborhood of a closed submanifold \( Y \), then by homotopy deformation, we can assume that the singularity of \( v \) (i.e. the set of points where \( v \) is not isomorphism) is a closed submanifold \( Y \) without of loss of generality.

The action between relative Chern character class \( \text{ch}(E_+, E_-, v) \) and the fundamental class \([M]\) is equal to \( \langle \text{ch}(E_+) - \text{ch}(E_-), [M] \rangle \). Here we call it relative Chern character number.

It is clear that the relative Chern character number depends on the neighborhood of the submanifold \( Y \), however the explicit relationship between the relative Chern character number and the neighborhood is not clear.

The notion of super-connection was introduced by Quillen [13], as a generalization of the notion of connection to the category of \( \mathbb{Z}_2 \)-graded vector bundles with odd endomorphism. By the super-connection method, in different conditions the relative Chern character number can be localized to the different topological terms.

1) When \( Y \) is a set of finitely many points. Feng, Li and Zhang [8] gave a formula to count the relative Chern character number.

†Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan. E-mail: dexielin@ms.u-tokyo.ac.jp
2) When $M$ and $Y$ are closed complex manifolds and $Y \hookrightarrow M$ is an holomorphic embedding. Bismut (1990) gave several estimates of the convergence of the relative Chern character w.r.t. deformation variable of the super-connection, under some assumptions ("quantization assumption" [6, Section 1 a. Assumptions and notations]). And such estimate implies the localization of the relative Chern character number to some topological invariants of $Y$.

In this paper, we assume that $M$ is a closed oriented $2n$ dimensional manifold.

Let $v$ be a homomorphism between two complex vector bundles $E_+, E_-$ with the same rank. We suppose that $v$ satisfies the following basic assumption.

**Assumption 1.1** Suppose there are finite imbedding $i_i : S^{2k_i-1} \hookrightarrow M$, $i = 1, \ldots, l$, such that

- the normal bundle of each $S^{2k_i-1}$ is trivial,
- the image of these imbedding are mutually disjoint, i.e. $i_i(S^{2k_i-1}) \cap i_j(S^{2k_j-1}) = \emptyset$, $i \neq j$.

We suppose that the singular point set of $v$, denoted by $\text{Sing}(v)$, to the union of these images, i.e. $\text{Sing}(v) = \bigcup_{1 \leq i \leq l} \text{Im}(i_i)$.

For the convenience, we also use $S^{2k_i-1}$ to represent the $\text{Im}(i_i)$. Let $N(S^{2k_i-1})$ denote the tubular neighborhood of $S^{2k_i-1}$, by the hypothesis, $N(S^{2k_i-1}) \cong S^{2k_i-1} \times D^{2n-2k_i+1}$, where $D^{2n-2k_i+1}$ denotes the standard unit $2n - 2k_i + 1$ dimensional ball. The boundary $\partial N(S^{2k_i-1}) \cong S^{2k_i-1} \times S^{2n-2k_i}$.

The idea to localize the relative Chern character number is to use the odd Chern character on the product sphere, i.e. $S^{2k_i-1} \times S^{2n-2k_i}$.

In Feng, Li and Zhang’s paper [8], if the $\text{Sing}(v)$ consists of the finite points on $M$, then the relative Chern character number is related to the $v$ restricted on the boundary of $D(p)$. That is to say the homomorphism map

$$\big|v|_{\partial D(p)} : E_+|_{\partial D(p)} \rightarrow E_-|_{\partial D(p)},$$

where $D(p)$ denotes a small ball in $M$, determines a number in $K^1(S^{2n-1})$, which was showed in Getzler’s paper [9, Section 1, P. 492] and was denoted by $\deg(p)$.

By a similar method and homotopy exact sequence, we will see that the homotopy map

$$\big|v|_{\partial N(S^{2k_i-1})} : E_+|_{\partial N(S^{2k_i-1})} \rightarrow E_-|_{\partial N(S^{2k_i-1})}$$

also determines a number, which we denote by $\deg^*(v_i)$.

The main theorem of this paper is:

**Theorem 1.2** Suppose $M$ is an oriented closed $2n$-dimensional manifold and under the assumption 1.1, the following formula holds

$$\langle \text{ch}(E_+) - \text{ch}(E_-), [M] \rangle = (-1)^{n+1} \sum_{1 \leq i \leq k} \deg^*(v_{x_i}). \quad (1.2)$$

The direct application of the above formula is to localize the index of twisted Dirac operator for some spin manifold with some vector bundle $F$.

To be more specific, suppose $M$ is closed $2n$-dimensional spin manifold admitting the odd dimensional sphere bundle structure, and the homomorphic map $v$ between a trivial vector
bundle (i.e. $E_+$ is trivial) and another vector bundle $F(E_− = F)$. Moreover, we assume that the homology class $[\text{Sing}(v)]$ belongs to the torsion part of $H_*(M; \mathbb{Z})$. Let $D^{\otimes F}$ be the twisted Dirac operator, i.e. Dirac operator of the $F$ tensered spinor bundle. Then, the index can be localized as follows

$$\text{Ind}(D^{\otimes F}) = (-1)^n \sum_{1 \leq i \leq k} \deg^*(v_x_i).$$

The organization of the paper is as follows:

In Section 2, we review some properties odd Chern characteristic class. In Section 3, we show the local calculation near the singularites. In Section 4, we give the proof of the Main Theorem in this paper. In Section 5, we will see how to apply such formula to localize the index of the twisted Dirac operator on some special spin manifold.

2 Review of odd Chern character

Let $X$ be a closed manifold.

We recall the notion of odd Chern character given by Getzler [9, P.490-495]. If $\nabla_0$ and $\nabla_1$ are two connections on a complex vector bundle $E$, their Chern-Simons form is the differential form of the odd degree

$$\text{cs}(\nabla_0, \nabla_1) = \int_0^1 \text{Tr}(\nabla_u e^{\nabla^2_u}) du,$$

where $\nabla_u = (1 - u)\nabla_0 + u\nabla_1$.

By the projection $[0,1] \times X \to X$, we can define the connection on the product space $[0,1] \times X$,

$$\tilde{\nabla} := du \frac{\partial}{\partial u} + \nabla_u.$$

Definition 2.1 The odd Chern character $\text{Ch}(g)$ of a differentiable map $g : X \to GL_N(\mathbb{C})$ is the odd differential form $\text{cs}(d, d + g^{-1} dg)$.

Proposition 2.2 (cf. [9, Proposition 1.2]) The odd Chern character is a closed differential form of odd degree, given by the formula

$$\text{Ch}(g) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k + 1)!} \text{Tr}(\omega^{2k+1}),$$

where $\omega = g^{-1} dg$.

Let $g_t, t \in [0,1]$, be a family of differential maps from $X$ to $GL_N(\mathbb{C})$. This defines a differential map $\tilde{g} : [0,1] \times X \to GL_N(\mathbb{C})$. So, the odd Chern character $\text{Ch}(\tilde{g})$ can be decomposed into

$$\text{Ch}(\tilde{g}) = \text{Ch}(g_t) + dt \wedge \text{Ch}(g_t).$$

Proposition 2.3 (cf. [9, Proposition 1.3]) $\text{Ch}(g_t)$ can be expressed by the formula

$$\text{Ch}(g_t) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k)!} \text{Tr}(g_t^{-1} \dot{g}_t \wedge \omega_t^{2k}),$$

and satisfies the transgression formula

$$\frac{\partial}{\partial t} \text{Ch}(g_t) = d\text{Ch}(g_t),$$
The above lemma says that the cohomology class \([\text{Ch}(g)]\) depends only on the homotopy class \([g] \in [X, GL_N(\mathbb{C})]\).

**Lemma 2.4** (cf. [9, Proposition 1.4]) If \(g : S^{2k-1} \to GL_N(\mathbb{C})\) is a differential map, then we have the integral

\[
\frac{1}{(-2\pi)^{k}} \int_{S^{2k-1}} \text{Ch}(g) = -\text{deg}^{\text{top}}(g),
\]

where \(\text{deg}^{\text{top}}(g)\) denotes the topological degree of the homotopy class \([g] \in \pi_{2k-1}(GL_N(\mathbb{C}))\).

In this paper, we denote the integral \(\frac{1}{(-2\pi)^{k}} \int_{S^{2k-1}} \text{Ch}(g)\) by \(\text{deg}(g)\).

## 3 Local calculation on singularity

Now we consider the local model of each \(S^{2k-1}\). To simplify the notation, we drop the subscript \(i\).

The following technique is well-known in the homotopy theory, which will be applied to split the elements in \([S^{2n-2k} \times S^{2k-1}, GL_N(\mathbb{C})]\).

**Lemma 3.1** For any homotopy class \([g] \in [S^{2n-2k} \times S^{2k-1}, GL_N(\mathbb{C})]\) has a representation as a matrix product of two mappings \(f : S^{2k-1} \to GL_N(\mathbb{C})\) and \(h : S^{2n-1} \to GL_N(\mathbb{C})\), i.e. there is \(g \in [g]\), such that \(g = \text{pr}_2^* f \cdot \phi h\), where \(\text{pr}_2^*\) denotes the canonical projection \(S^{2n-2k} \times S^{2k-1} \to S^{2k-1}\) and \(\phi\) denotes the suspension map \(S^{2n-2k} \times S^{2k-1} \triangleright S^{2n-1}\).

**Proof** By the cofibration sequence,

\[
S^{p+q-1} \to S^p \vee S^q \to S^p \times S^q \to S^{p+q-1} \to S^{p+1} \vee S^{q+1},
\]

where the first map denotes the attaching and the forth map denotes the suspension, therefore null-homotopic. Let \(p = 2n - 2k\) and \(q = 2k - 1\), we have the exact short sequence

\[1 \to \pi_{2n-1}(GL_N(\mathbb{C})) \to [S^{2n-2k} \times S^{2k-1}, GL_N(\mathbb{C})] \to \pi_{2n-2k}(GL_N(\mathbb{C})) \oplus \pi_{2k-1}(GL_N(\mathbb{C})) \to 1.\]

Since \(\pi_{2n-2k}(GL_N(\mathbb{C})) = 0\), hence we can find a homotopic equivalent map

\[g_1 : S^{2n-2k} \times S^{2k-1} \to GL_N(\mathbb{C}),\]

such that \(g_1|_{S^{2n-2k} \times q} = \text{Id}_N\),

where \(q\) is a fixed point in \(S^{2k-1}\).

Fixing a point \(p\) in \(S^{2n-2k}\), we set \(f = g_1|_{p \times S^{2k-1}}\). Denote by \(p_2 : S^{2n-2k} \times S^{2k-1} \to p \times S^{2k-1}\), then \((p_2 f)^{-1} g_1\) is null-homotopy in the above exact sequence, hence it equals to a map \(h : S^{2n-1} \to GL_N(\mathbb{C})\).

By the lemmas 2.4 and 3.1, we can calculate the integral on the product space.

**Proposition 3.2** If \([g] \in [S^{2n-2k} \times S^{2k-1}, GL_N(\mathbb{C})]\), then

\[
\frac{1}{(-2\pi)^{n}} \int_{S^{2n-2k} \times S^{2k-1}} \text{Ch}(g) = \text{deg}(h),
\]

where \(h\) is the map from the sphere \(S^{2n-1}\) as in the above lemma.
Here, we denote the integral \( \frac{1}{(−2π\sqrt{−1})^n} \int_{S^{2n−2k} × S^{2k−1}} \text{Ch}(g) \) by \( \deg^*(g) \).

**Proof** We denote the smashing map
\[
S^{2n−2k} × S^{2k−1} \xrightarrow{\phi} S^{n−2k} \wedge S^{2k−1} = S^{2n−1}
\]
by \( φ \) and the canonical projection
\[
S^{2n−2k} × S^{2k−1} \rightarrow S^{2k−1}
\]
by \( pr_2 \).

By the lemma 3.1, we can find a representation \( g : S^{2n−2k} × S^{2k−1} \rightarrow GL_N(\mathbb{C}) \) as the product of the two mappings
\[
g = pr_2^*f \cdot φ^*h,
\]
where \( f \in π_{2k−1}(GL_N(\mathbb{C})) \) and \( h \in π_{2n−1}(GL_N(\mathbb{C})) \) respectively.

It is sufficient to show that the integral is independent on the map \( f \).

Note that \( GL_N(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}) \), the image of \( pr_2^*f \cdot φ^*h \) is
\[
\left( \begin{array}{c}
pr_2^*f \cdot φ^*h \\
Id_N
\end{array} \right) \in GL_N(\mathbb{C}),
\]
which is homotopic to
\[
\left( \begin{array}{c}
pr_2^*f \\
φ^*h
\end{array} \right).
\]

Hence,
\[
[\text{Ch}(g)] = [\text{Ch}(pr_2^*f \cdot φ^*h)] = [pr_2^*\text{Ch}(f)] + [φ^*\text{Ch}(h)].
\]

Moreover, we know that \( 2k−1 < 2n−1 \) and the differential form \( f \) is just on the \( S^{2k−1} \) component, so the integral \( \int_{S^{2n−2k} × S^{2k−1}} \text{Ch}(f) = 0 \).

Thus the integral (3.1) equals to
\[
\frac{1}{(−2π\sqrt{−1})^n} \int_{S^{2n−2k} × S^{2k−1}} \text{Ch}(g) = \frac{1}{(−2π\sqrt{−1})^n} \int_{S^{2n−2k} × S^{2k−1}} φ^*\text{Ch}(h) + pr_2^*\text{Ch}(f)
\]
\[
= \frac{1}{(−2π\sqrt{−1})^n} \int_{S^{2n−2k} × S^{2k−1}} φ^*\text{Ch}(h)
\]
\[
= \frac{1}{(−2π\sqrt{−1})^n} \int_{φ(S^{2n−2k} × S^{2k−1})} \text{Ch}(h)
\]
\[
= \frac{1}{(−2π\sqrt{−1})^n} \int_{S^{2n−1}} \text{Ch}(h) = \deg(h).
\]

\( \square \)

**4 Proof of the Theorem 1.2**

Without loss of generality, we can suppose the rank of \( E_\pm \) is arbitrary large, if necessary we can take direct sum with some trivial bundle on \( E_+ \) and \( E_- \).

Let \( E = E_+ \oplus E_- \) be the \( \mathbb{Z}_2 \)-graded complex vector bundle over \( M \). Choose two unitary connections \( \nabla^{E_+} \) and \( \nabla^{E_-} \) on \( E_+ \) and \( E_- \) respectively, such that near a small tubular neighborhood of the singularity \( Sing(v) \), the connections are both trivial. Let \( \nabla^E = \nabla^{E_+} \oplus \nabla^{E_-} \) be the \( \mathbb{Z}_2 \)-graded connection on \( E \).
Let \( v : E_+ \to E_- \) extend to an endomorphism of \( E \), by acting as zero on \( E_- \). Let \( v^* : E_- \to E_+ \) be the adjoint of \( v \) w.r.t. the Hermitian metrics on \( E_\pm \) respectively. Set \( V = v + v^* \), then \( V \) is an odd endomorphism of \( E \) and \( V^2 \) is fiberwise positive over \( M_1 \).

For the convenience of the computation, we define a function \( \varphi : \Omega^*(M) \to \Omega^*(M) \), by

\[
\varphi(\alpha) = (2\pi \sqrt{-1})^{-\frac{k}{2}} \alpha,
\]

where \( \alpha \) is a differential form of degree \( k \).

For any \( T \in \mathbb{R}_{\geq 0} \), let \( A_T \) be the superconnection on \( E \), in the sense of Quillen(cf. [13, Section 2]), defined by

\[
A_T := \nabla_E + TV.
\] (4.1)

Let \( \text{ch}(E, A_T) \) be the associated Chern character form defined by

\[
\text{ch}(E, A_T) = \varphi(\text{Tr}[e^{-A_T^2}]).
\] (4.2)

By the straightforward calculation, one can derive the following lemmas.

**Lemma 4.1** (cf. [8, (2.6) and (2.5)], [13, Proposition 2])

\[
\frac{\partial \text{ch}(E, A_T)}{\partial T} = -\frac{1}{\sqrt{2\pi \sqrt{-1}}} \varphi \text{Tr}[V e^{-A_T^2}].
\] (4.3)

For \( T > 0 \), we set

\[
\gamma(T) = \frac{1}{\sqrt{2\pi \sqrt{-1}}} \varphi \int_0^T \text{Tr}(V e^{-A_T^2}) dt.
\] (4.4)

Then, the following formula holds

\[
\text{ch}(E, A_0) - \text{ch}(E, A_T) = d\gamma(T).
\] (4.5)

**Lemma 4.2** (cf. [13, Section 4]) *The following formula holds uniformly on \( M_1 *\)

\[
\lim_{T \to +\infty} \text{ch}(E, A_T) = 0.
\] (4.6)

**Lemma 4.3** (cf. [8, Lemma 2.1]) *The following identity holds,

\[
\langle \text{ch}(E_+) - \text{ch}(E_-), [M] \rangle = - \sum_{1 \leq i \leq l} \lim_{T \to \infty} \int_{\partial N(S_{x_i})} \gamma(T).
\] (4.7)

**Proof**(of the Theorem 1.2)

For any \( x_i \), since when restricted on the \( \partial N(S_{x_i}) \), the homomorphism \( v \) has deformed to be unitary, we get that \( v^* = v^{-1} \) and \( V^2 \) is the identity map acting on \( E |_{\partial N(S_{x_i})} \). Also, since \( \nabla^E \) is trivial over \( N(S_{x_i}) \), hence we can choose a local trivial connection denoted by \( d \). And on \( N(S_{x_i}) \), one gets

\[
A_t = d + tV, \quad A_t^2 = d^2 + t^2V^2 + t[d, V] = t^2 Id_E + tdV.
\]

Then, on each component of the singularity \( \text{Sing}(v) \), we have

\[
\int_{\partial N(S^{2\nu_i-1})} \gamma(T) = \frac{1}{\sqrt{2\pi \sqrt{-1}}} \int_{\partial N(S^{2\nu_i-1})} \varphi \int_0^T \text{Tr}(V e^{-A_t^2}) dt
\]
3.1 An application to the index of twisted Dirac operator

Theorem 4.4

In this section, we assume the $M$ is a spin manifold, and satisfies the following assumption.
**Assumption 5.1** M admits the odd-dimensional sphere bundle structure over a manifold \( B \), i.e.

\[
\begin{array}{c}
M \\
\pi \\
\downarrow \\
S \\
B
\end{array}
\]

where \( B \) and \( S \) are closed manifolds. And \( M \) satisfies the following conditions:

1) \( M \) is locally trivial, i.e. each \( x \in B \) having small neighborhood \( U(x) \) in \( B \), such that \( \pi^{-1}(U(x)) \) is diffeomorphic to \( U(x) \times S \), we denote the fiber over \( x \in B \) by \( S_x \).

2) \( S \) is diffeomorphic to the odd dimensional sphere \( S^{2k-1} \).

Moreover, we suppose that homology class of \( \text{Sing}(v) \) belongs to the torsion part of the \( H_*(M) \), i.e. \( [\text{Sing}(v)] \in \text{Tor}(H_*(M;\mathbb{Z})) \). Then, we show that the index twisted Dirac operator can be localized to the relative Chern character number.

We need the following lemma to show the proof.

**Lemma 5.2** Let \( X \) and \( Y \) be two closed manifolds, and \( \iota \) denote the imbedding \( Y \hookrightarrow X \). Suppose \( Y \) is a rational homology sphere and \( [Y] \in \text{Tor}(H_*(X;\mathbb{Z})) \). For any \( d \)-closed form \( \omega \) on \( X \) without the zero degree component, then \( \omega|_Y \) is \( d \)-exact, where \( N \) denotes a small tubular neighborhood of the \( Y \).

**Proof** We can choose the tubular neighborhood of \( Y \) by equipping a metric on \( X \), and by the pull-back of the projective map \( N \to Y \), we need only to show that \( \omega|_Y \) is \( d \)-exact.

By the Poincaré duality, we need to show the following formula

\[
0 = \langle \omega, \iota_* [Z] \rangle,
\]

for any element \( [Z] \in H_*(Y;\mathbb{Z}) \).

Since, \( Y \) is a rational homology sphere and \( [Y] \in \text{Tor}(H_*(X;\mathbb{Z})) \), the above formula holds.

**Proposition 5.3** Let \( M \) be a \( 2n \)-dimensional closed, oriented spin manifold. We assume that \( M \) satisfies the Assumption 5.1. If there is homomorphic mapping \( \bigoplus^r \overset{\psi}{\to} F \) with the same rank \( r \), and the singularity of \( v \) satisfies the Assumption 1.1, i.e. \( \text{Sing}(v) = \bigsqcup_{1 \leq i \leq l} S^{2k-1} \) and \( [\text{Sing}(v)] \in \text{Tor}(H_*(M;\mathbb{Z})) \), then

\[
\text{Ind}(D^{\psi}_F) = (-1)^n \sum_{1 \leq i \leq k} \deg^*(v_x).
\]

**Proof** Since the fiber \( S \) can be viewed as the foliation of the manifold \( M \), and it admits a positive scalar curvature, by Connes Vanishing Theorem [7, Theorem 02] we know \( \langle \hat{A}(TM), [M] \rangle = 0 \). Recently Zhang [15] gave a geometric proof assuming the whole manifold \( X \) is spin instead the foliation.

In our setting, we can use adiabatic limits method of Dirac operator as in Bismut’s paper [5, Proposition 5.2] to show the \( \hat{A}(TM) \) is vanishing.

Let \( E := \bigoplus^r \otimes F \) be the \( \mathbb{Z}_2 \) vector bundle.

We need to show that

\[
\int_M \hat{A}(TM)(\text{ch}(E)) = \int_M \text{ch}(E).
\]
Since we can write
\[ \hat{A}(TM) = \sqrt{\det \left( \frac{R^TM/2}{\sinh(R^TM/2)} \right)} = 1 + \beta, \]
where \( R^TM \) denotes the curvature of \( TM \) and \( \beta \) denotes the high degree form part.

We show the more general formula,
\[ \langle \omega \wedge \text{ch}(E), [M] \rangle = \langle \omega^0 \wedge \text{ch}(E), [M] \rangle, \quad \text{for any } d - \text{closed form } \omega, \]
where \( \omega^0 \) denotes the 0-degree part of the form \( \omega \). Find a cut-off function \( \rho \) whose support locates in \( N \). By the former lemma, we know that there exists a form \( \alpha \) whose support locates inside \( N \) such that \( \rho \cdot \omega^* = d\alpha \), where \( \omega^* \) denotes \( \omega - \omega^0 \).

Hence,
\[ \langle \omega^* \wedge \text{ch}(E), [M] \rangle - \langle d\alpha \wedge \text{ch}(E), [M] \rangle = \int_M (1 - \rho)\omega^* \wedge \text{ch}(E) + \left( \int_M \rho\omega^* \wedge \text{ch}(E) - d\alpha \wedge \text{ch}(E) \right) \]
\[ = \int_M \omega^* \wedge \text{ch}(E, A_T) + \int_N (\rho\omega^* - d\alpha) \wedge \text{ch}(E), \]
by the vanishing property of (4.6) and \( \rho \omega = d\alpha \) we can say the above formula equals 0.

So,
\[ \text{Ind}(D^{\otimes F}_+) = -\left( \int_M \hat{A}(TM) \text{ch}(\mathcal{C}) - \int_M \hat{A}(TM) \text{ch}(F) \right) \]
\[ = -\int_M \text{ch}(E) = (-1)^n \sum_{1 \leq i \leq k} \deg^*(s_{x_i}). \]

**Corollary 5.4** Let \( M \) be a closed oriented 2n dimensional satisfies the Assumption 5.1 and \( k \geq 2 \). Suppose a complex line bundle \( L \) over \( M \) admits a global section \( s \in \Gamma(M, L) \) whose zero point set consists of the disjoint union on the fibers over \( \{ x_i \}_{1 \leq i \leq l} \) points in \( B \), i.e. \( s^{-1}(0) = \prod_{1 \leq i \leq k} S_{x_i} \), and \( [s^{-1}] \in \text{Tor}(H_*(M; \mathbb{Z})) \), then
\[ \text{Ind}(D^{\otimes L}_+) = (-1)^n \sum_{1 \leq i \leq l} \deg^*(s_{x_i}). \quad (5.1) \]

**Proof**

The global section \( s \) can viewed as the complex homomorphism between the trivial line bundle and \( L \), i.e.
\[ \mathbb{C} \xrightarrow{s} L. \]

By the (1.2), we have
\[ -\text{Ind}(D^{\otimes L}_+) = \int_M (\text{ch}(\mathcal{C}) - \text{ch}(L)) = (-1)^{n+1} \sum_{1 \leq i \leq k} \deg^*(s_{x_i}), \]
\[ \text{Ind}(D^{\otimes L}_+) = (-1)^n \sum_{1 \leq i \leq k} \deg^*(s_{x_i}). \]

\[ \square \]
Even though, in this paper we often consider the odd-dimensional singularity case, at the end of this paper, we see a special even-dimensional singularity case.

**Corollary 5.5** Let $M$ be a closed oriented $2n(n \geq 3)$ dimensional spin manifold and admit a $S^{2n-2}$-fiber bundle structure, i.e. $M \xrightarrow{\pi} B \rightarrow S^{2n-2}$ for some closed manifold $B$. Suppose a complex line bundle $L$ over $M$ admits a global section $s \in \Gamma(M, L)$, such that the zero-point set consists of finite disjoint union of the $2n-2$ dimensional sphere and each such sphere belongs to the torsion part of $H_*(M; \mathbb{Z})$, i.e. $s^{-1}(0) = \bigsqcup_{1 \leq i \leq l} S^{2n-2}_i$, $[S^{2n-2}_i] \in Tor(H_*(M; \mathbb{Z}))$, $\forall i$. Then

$$\text{Ind}(D^\otimes L) = (-1)^n \sum_{1 \leq i \leq l} \deg^*(s_i).$$

**Proof** Since $n \geq 3$, any complex line bundle on $S^{2n-2}$ is trivial. We can choose a connection of $L$, such that near $S^{2n-2}$ it is trivial. $M$ is oriented, we can say each normal bundle of $S^{2n-2}_i$ is also orientable, hence it admits a complex structure by $SO(2) \cong U(1)$, which means it also trivial.

So, the tubular neighborhood of $S^{2n-2}_i$ is diffeomorphic to $D^2 \times S^{2n-2}$, whose boundary is diffeomorphic to $S^1 \times S^{2n-2}$. Thus, the above arguments also work. By similar method, we directly deduce

$$\text{Ind}(D^\otimes L) = (-1)^n \sum_{1 \leq i \leq l} \deg^*(s_i).$$

\hfill $\square$

**Acknowledgement:** First, the author wants to show the best gratitude to Prof. Huitao Feng, who brought this problem to his focus, and for his generality. The author also would like to express the special thanks to Prof. Mikio Furuta, for the long time discussion and helping, and comments that greatly improved the manuscript. This research is supported by the Todai Fellowship of the University of Tokyo.

**References**

[1] Atiyah, M. F., Bott, R. and Shapiro, A.: Clifford modules. Topology, 3, 3-38 (1964)

[2] Atiyah, M. F., Patodi, V. K. and Singer, I. M.: Spectral asymmetry and Riemannian Geometry I, Math. Proc. Camb. Phil. Soc. 77, 43-69 (1975)

[3] Atiyah, M. F. and Singer, I. M.: The index of the elliptic operators on compact manifolds. Bull. Amer. Math. Soc. 69, 422-433 (1963)

[4] Berline, N., Getzler, E. and Vergne, M.: Heat Kernels and Dirac Operators. Springer-Verlag, Paris and Chicago (2003)

[5] Bismut, J. M.: The Atiyah-Singer index theorem for families of Dirac operators: Two heat equation proofs. Invent. Math., 83, 91-151 (1986)

[6] Bismut, J. M.: Superconnection and Complex Immersions. Invent. Math. 99, P. 59-113 (1990)

[7] Connes, A.: Cyclic cohomology and the transverse fundamental class of a foliation. in Geometric Methods in Operator Algebras. H. Araki eds., Pitman Res. Notes in Math. Series, 123, 52-144 (1986)

[8] Feng, H., Li, W. and Zhang, W.: A Poincaré Hopf type formula for Chern character numbers. Mathematische Zeitschrift, 269, 401-411 (2011)

[9] Getzler, E.: The odd Chern character in cyclic homology and spectral flow. Topology 32, 489-507 (1993)
[10] Jacobowicz, H.: Non-vanishing complex vector fields and Euler Characteristic. Proc. Am. Math. Soc. 137, 3163-3165 (2009)
[11] Lichnerowicz, A.: Spineurs harmoniques. C.R. Acad. Sci. Paris, Ser. A-B 257, 7-9 (1963)
[12] Lawson, H. B. and Michelsohn, M. L.: Spin Geometry. Pric. Uni. Pres., Princeton (1989)
[13] Quillen, D.: Superconnections and the Chern Character. Topology, 24, 89-95 (1985)
[14] Zhang, W.: Lectures on Chern-Weil Theory and Witten Deformations. N.T.M. vol 4 World Scientific, Singapore (2001)
[15] Zhang, W.: Positive scalar curvature on foliations. arXiv:1508.04503 (2015)