A survey of kernel-type estimators for copula and their applications

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Abstract. Copulas have been widely used to model nonlinear dependence structure. Main applications of copulas include areas such as finance, insurance, hydrology, rainfall to name but a few. The flexibility of copula allows researchers to model dependence structure beyond Gaussian distribution. Basically, a copula is a function that couples multivariate distribution functions to their one-dimensional marginal distribution functions. In general, there are three methods to estimate copula. These are parametric, nonparametric, and semiparametric method. In this article we survey kernel-type estimators for copula such as mirror reflection kernel, beta kernel, transformation method and local likelihood transformation method. Then, we apply these kernel methods to three stock indexes in Asia. The results of our analysis suggest that, albeit variation in information criterion values, the local likelihood transformation method performs better than the other kernel methods.

1. Introduction

Data in fields such as finance, insurance, health and hydrology are often nonlinear in nature. Researchers often interested in seeing dependence structure in these data. A commonly used distribution to model this dependence is multivariate normal. However, the multivariate normal distribution has serious shortcomings. First, it cannot capture tail dependence which plays important role in risk management. Second, the dependence for multivariate normal is summarized in correlation matrix. However, correlation is not the best measure of dependence; it is a measure of linear dependence [1]. Thus, a tool that can capture this nonlinearity and invariant under transformation is required. Copula offers flexibility in modelling this nonlinearity by combining various margins to form new distributions.

In general, copula is a function that couples multivariate distribution with their univariate margins [2]. This means that copula can model rich dependence structure. The flexibility of copula demands invariant dependence measures. Few dependence measures that are invariant under transformation are Kendall’s $\tau$, Spearman’s $\rho$, Gini’s $\gamma$ and Blomqvist’s $\beta$. For a review of some of these dependence measures see [1], [2], [3] and references therein.

Modelling using copula means modelling with many parameters. Basically these parameters consist of two parameters: the copula parameters and the marginal parameters. Methods for estimating copulas include parametric, nonparametric and semiparametric methods. For a review of these methods see [4] and [5].

This article is organized as follows. In section one we give a motivation for copula as a flexible dependence modelling. Section two introduces basic concepts of copula. A general copula estimation is briefly discussed in section three. Section four discusses kernel-type estimators of...
copulas such as mirror reflection kernel, beta kernel, transformation method and local likelihood transformation method. Application of these kernel-type estimators to Asian stock indexes data is discussed in section five. Section six concludes the article.

2. Basic concepts of copula
The concept of copula can be traced back to the seminal paper of [6]. Let $X$ and $Y$ be random variables with distribution functions $F(x)$ and $G(y)$, respectively. Let $H(x, y)$ be the joint distribution function. Now, Sklar’s Theorem (see [2]) guarantees that there exits a function $C$ such that

$$H(x, y) = C(F(x), G(y)).$$

The proof of Sklar’s Theorem in (1) and its extension to multivariate case can be found in [2].

An example of copula is Gumbel copula which is often used to model tail dependence and has the form

$$C(u, v) = \exp\{-[(-\ln u)^\theta + (-\ln v)^\theta]^\theta\}, \theta \in [1, \infty).$$

Let $C(u, v)$ be a copula. The copula density $c(u, v)$ that is associated with copula $C(u, v)$ is defined as

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}$$

provided that the derivative exists. In the next section we will discuss briefly how to estimate the copula $C(\cdot, \cdot)$ and put more emphasis on using kernel method for estimating the copula density $c(\cdot, \cdot)$.

3. Estimation of copula
In general, there are three methods that are commonly used to estimate copula. They are parametric, nonparametric and semiparametric. See [4] and [5] for details of these methods. Parametric estimation can be properly used if the margins are explicitly known. However, without prior information about the margins the parametric estimation method may lead to severe underestimation [7]. In this case, nonparametric estimation procedure is preferred.

Let $(X_1, Y_1)^T, \ldots, (X_n, Y_n)^T$ be independent and identically distributed samples from bivariate random vector $(X, Y)^T$ with joint distribution function $H$ and marginal distribution functions $F$ and $G$. Based on this random sample we can define empirical distribution function of the form

$$H_n(x, y) = \frac{1}{n} \sum_{i=1}^{n} I_{\{X_i \leq x, Y_i \leq y\}}, -\infty < x, y < \infty$$

where $I_{\{A\}}$ is an indicator function. Let $F_n(x)$ and $G_n(x)$ be marginal distribution functions that associated with $H_n$ that is $F_n(x) = H_n(x, +\infty)$ and $G_n(y) = H_n(+\infty, y)$ for $-\infty < x, y < +\infty$. Empirical copula $C_n$ is defined as [8]

$$C_n(u, v) = H_n(F^{-1}_n(u), G^{-1}_n(v)), 0 \leq u, v \leq 1$$

and empirical copula process is defined as

$$Z_n(u, v) = \sqrt{n}(C_n - C)(u, v), 0 \leq u, v \leq 1. \quad (2)$$
Nonparametric estimator for copula in (2) is proposed by [9] and one popular estimator has the form

\[ \hat{C}_n(u, v) = \frac{1}{n} \sum_{i=1}^{n} I(\hat{U}_i \leq u, \hat{V}_i \leq v) \]

with \( \hat{U}_i = F_n(x_i) \) and \( \hat{V}_i = G_n(y_i) \). Here \( F_n \) and \( G_n \) are empirical cumulative distribution functions of the marginals.

Another approach of estimating copula is proposed by [10] by inversion of marginal cumulative distribution functions and probability density functions.

4. Kernel-type estimators for copula density

In the previous section we discussed estimation of copula distribution \( \hat{C}_n \). However, copula distribution cannot capture invaluable information such as mode, tail, or mode near boundary [11]. This information can be captured by copula density \( \hat{c}_n \). Nonparametric estimation of copula density can be done via kernel, wavelets, spline, or Bernstein polynomials. In this section we survey some kernel-type estimators for copula density. The use of kernel is motivated by the fact that a density should be estimated by a density and kernel provides a natural way of estimating a density.

4.1. Mirror reflection kernel

Kernel estimation for copula is known to suffer from boundary bias. One way of overcoming this problem is by reflecting all data points with respect to all corners and edges of the unit square (see [7] and [12]). This idea was introduced by [13] and the technique is known as mirror reflection. The mirror reflection (MR) kernel has the form [13]

\[ \hat{c}_n^{(\text{MR})} = \frac{1}{n} \sum_{i=1}^{n} \sum_{l=1}^{9} \left[ K \left( \frac{u - \hat{U}_i^{(l)}}{h_n} \right) - K \left( -\frac{\hat{U}_i^{(l)}}{h_n} \right) \right] \left[ K \left( \frac{v - \hat{V}_i^{(l)}}{h_n} \right) - K \left( -\frac{\hat{V}_i^{(l)}}{h_n} \right) \right] \]  

with \( \{(\hat{U}_i^{(l)}, \hat{V}_i^{(l)}), i = 1, \ldots, n; l = 1, \ldots, 9\} = \{(\pm \hat{U}_i, \pm \hat{V}_i), (\pm \hat{U}_i, 2 - \hat{V}_i), (2 - \hat{U}_i, \pm \hat{V}_i), (2 - \hat{U}_i, 2 - \hat{V}_i), i = 1, \ldots, n\} \). An improved version of the MR kernel in (3) is suggested by [14] by adding a quantity \( r(w) = \min(\sqrt{w}, \sqrt{1-w}) \) to the kernel so that it will shrink bandwidth to zero near the corner of the unit rectangle. The bias and the variance of the MR kernel can be found in [12].

4.2. Beta kernel

Another approach of estimating the density of a copula is by using kernel whose support matches the support of the density we wish to estimate [15]. Beta kernel is proposed by [16] for univariate density estimation. Kernel estimator for copula density \( \hat{c}_n \) with bandwidth parameter \( b_n \) is defined as (see [7] and [12])

\[ \hat{c}^{(\text{Beta})}(u, v) = \frac{1}{n} \sum_{i=1}^{n} K \left( \frac{U_i}{b_n} + 1, \frac{V_i}{b_n} + 1 \right) K \left( \frac{\hat{U}_i}{b_n} + 1, \frac{\hat{V}_i}{b_n} + 1 \right) \]

for all \((u, v) \in [0, 1]^2\). Here \( K(x, a, b) \) is the density of Beta(a, b) random variable. Derivation of asymptotic integrated mean square error (AMISE) and bandwidth selection can be found in detail in [12].
4.3. Transformation method
The idea behind the transformation method is as follows (see [12]). Rather than having supports on the unit square, the data is transformed such that it has full support on $R^2$. Then, standard kernel techniques can be applied to the transformed domain. Finally, apply back-transformation to estimate the copula density. One popular choice is by using standard normal cumulative distribution function. Let $\Phi$ be the standard normal distribution function and $\phi$ be the standard normal density function. The random vector $(X, Y) = (\Phi^{-1}(U), \Phi^{-1}(V))$ has normal margin and full support on $R^2$. By using the Sklar’s Theorem the joint density function can be written as

$$f(x, y) = c(\Phi(x), \Phi(y))\phi(x)\phi(y).$$

Now, using the kernel estimator of the form

$$\hat{f}_n = \frac{1}{n} \sum_{i=1}^{n} K_{b_n}(x - X_i)K_{b_n}(y - Y_i)$$

for all $(x, y) \in R^2$, we obtain the following transformation estimator

$$\hat{c}^{(T)}_n = \frac{1}{n\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))} \sum_{i=1}^{n} K_{b_n}(\Phi^{-1}(u) - \Phi^{-1}(U_i))K_{b_n}(\Phi^{-1}(v) - \Phi^{-1}(V_i))$$

for all $(u, v) \in [0, 1]^2$. Asymptotic properties and bandwidth selection for this estimator can be found in [12]. Another extension is proposed by [17] by tapering the back-transformation in the tails. Recent extension using logit transformation kernel estimator is proposed by [11].

4.4. Local likelihood transformation method
Extension of transformation method is proposed by [18] by fitting polynomial locally to the log-density (TLL1) and by quadratic polynomials (TLL2). Let $Z_i = (U_i, V_i)$ and let $\Delta_{k_n}$ be Euclidean distance between $(x, y)$ and the $k$th closest observation between $(\Phi^{-1}(Z_i))_{i=1,...,n} = (\Phi^{-1}(U_i), \Phi^{-1}(V_i))_{i=1,...,n}$. For all $(u, v) \in [0, 1]^2$, the local likelihood transformation of the copula density $c(u, v)$ with nearest-neighbour factor $\Delta_{k_n}$ and bandwidth matrix $B$ has the form

$$\hat{c}^{(TLL)}_n(u, v) = \frac{\exp[\hat{\alpha}(\Phi^{-1}(u), \Phi^{-1}(v))]\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$

where the expression of $\hat{\alpha}(\cdot, \cdot)$ can be found in [18] or [12].

5. Application
Various applications of the kernel-type estimators in previous section can be found in [11], [18], [12] and [17]. In this section we apply the kernel estimators to three weekly log return of stock indexes in Asia (STI, HSI and AORD) for period 3 January 2000 to 7 March 2016. We use the R package kdecopula of [15].

Figure 1 shows the kernel density estimation for empirical copula of stock indexes pairs using local likelihood transformation kernel. AIC values can be seen in Table 1. Here we use the following abbreviation: A = AORD, H = HSI and S = STI.

6. Conclusion
In this article we have reviewed kernel-type estimators for copula density estimation. We conclude that the transformation likelihood (TLL2) performs better than the other estimators since its AIC values are the smallest among the other four kernel estimators (highlighted in bold in Table 1).
Figure 1. Kernel density estimation of stock indexes using local likelihood transformation kernel (TLL2).

Table 1. AIC values for kernel estimators for stock indexes data

|      | MR   | Beta | T    | TLL1 | TLL2 |
|------|------|------|------|------|------|
| S vs H | -586.12 | -614.50 | -654.03 | -641.20 | **-667.94** |
| S vs A | -398.33 | -437.68 | -471.11 | -486.86 | **-489.93** |
| H vs A | -352.01 | -379.03 | -416.23 | -400.61 | **-434.49** |

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