A SUFFICIENT AND NECESSARY CONDITION OF GENERALIZED POLYNOMIAL LIÉNARD SYSTEMS WITH GLOBAL CENTERS

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Abstract. The aim of this paper is to give a sufficient and necessary condition of the generalized polynomial Liénard system with a global center (including linear type and nilpotent type). Recently, Llibre and Valls [J. Differential Equations, 330 (2022), 66-80] gave a sufficient and necessary condition of the generalized polynomial Liénard system with a linear type global center. It is easy to see that our sufficient and necessary condition is more easy by comparison. In particular, we provide the explicit expressions of all the generalized polynomial Liénard differential systems of degree 5 having a global center at the origin and the explicit expression of a generalized polynomial Liénard differential system of indefinite degree having a global center at the origin.

1. Introduction and main results

In the beginning, we refer to [15] and give the following definitions on centers and global centers.

Definition 1. An equilibrium $p$ is a center of a planar differential equation if there is a neighbourhood $U$ of $p$ is full of closed orbits.

Notice that the notion of center traces back to the investigations of Poincaré [17] and Dulac [5].

Let the period annulus of the center $p$ be the maximal connected set of periodic orbits surrounding the center $p$ and having $p$ in its boundary.

Definition 2. $p$ is a global center if and only if its period annulus is $\mathbb{R}^2 \setminus \{q\}$.

As introduced in [15], some researchers began to study global centers in the 1990s, see [3, 4, 9, 10, 11, 13, 21].

Many mathematicians such as Smale [18, 19] and Lins et. al. [14] were interested in the following polynomial Liénard system

\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -g(x) - f(x)y
\end{align*}

since system (1) has widely real world applications (see [16]) and important theoretical consequences (see [2, 20]), where

\[ g(x) = \sum_{i=r}^{m} a_i x^i, \quad f(x) = \sum_{i=s}^{n} b_i x^i \]
with \( m, n, r, s \in \mathbb{N} \) and \( a_r a_m b_r b_n \neq 0 \), and the dot represents the derivative of the independent variable \( t \). Recently, Llibre and Valls [15] gave a sufficient and necessary condition of the polynomial Liénard system (1) with a linear type global center. Moreover, Llibre and Valls [15] provided the explicit expressions of all the generalized polynomial Liénard differential systems of degree 3 having a global center at the origin, and the explicit expression of a generalized polynomial Liénard differential system of degree 5 having a global center at the origin. In [15], the sufficient and necessary condition of Liénard system (1) with a linear type global center at the origin is that the following conditions hold:

(a) \( xg(x) > 0 \) for all \( x \neq 0 \);
(b) There exist real polynomials \( h, f_1 \) and \( g_1 \) such that
\[
f(x) = f_1(h(x))h'(x), \quad g(x) = g_1(h(x))h'_1(x)
\]
with \( h'(0) = 0 \) and \( h''(0) \neq 0 \);
(c) \( \deg g = l \) is odd, and \( \deg g > 1 + \deg f \);
(d) The local phase portrait of the singular point localized at the origin of the polynomial differential system
\[
\dot{u} = uv^{l-1} - v^{l-1}, \quad \dot{v} = v^l \left( f \left( \frac{u}{v} \right) - vg \left( \frac{u}{v} \right) \right)
\]
is formed by two hyperbolic sectors.

However, it’s not easy to verify the conditions (b) and (d). Notice that the global center can have the three classifications: linear type, nilpotent type and degenerate type, where the linear type center is that the Jacobian of the system evaluated at a center has purely imaginary eigenvalues, the nilpotent type center is that it has both eigenvalues zero but its linear part is not identically zero, the degenerate type center is that it has its linear part identically zero. Notice that the global center of Liénard system (1) cannot be degenerate type since the Jacobian matrix at the origin of Liénard system (1) is
\[
\begin{bmatrix}
0, & 1 \\
-g'(0), & -f(0)
\end{bmatrix}.
\]

Naturally, based on the results of [15], we have the following three questions:

(Q1) Can we give a new sufficient and necessary condition such that the condition can be verified easier?
(Q2) Can we give a sufficient and necessary condition on the polynomial Liénard system (1) with a nilpotent type global center at the origin?
(Q3) Can we provide the explicit expressions of all the generalized polynomial Liénard differential systems of degree 5 having a global center (including linear type and nilpotent type) at the origin?

In order to answer the question (Q1), we give the following theorem.

**Theorem 1.** System (1) has a linear type global center at the origin, if and only if the following conditions hold:

(i) \( xg(x) > 0 \) for all \( x \neq 0 \);
(ii) \( r = 1, s \geq 1, a_r > 0 \);
(iii) \( m \) is odd, \( m > 2n + 1, a_m > 0 \); or \( m = 2n + 1 \) and \( 4(n+1)a_m b_n^{-2} > 1 \);
(iv) \( F(x_1) = F(x_2) \) if \( G(x_1) = G(x_2) \) for all \( x_1 < 0 < x_2 \), where \( G(x) = \int_0^x g(\xi) d\xi \).

Moreover, the global center of system (1) is as shown in Figure 1.
In order to answer the question (Q2), we give the following theorem.

**Theorem 2.** System \((1)\) has a nilpotent type global center at the origin, if and only if conditions (i), (iii), (iv) in Theorem 1 and the following condition hold:

\[(ii^*) \quad r \text{ is odd, } 2 < r < 2s + 1, a_r > 0; \text{ or } r = 2s + 1 \geq 3 \text{ and } b_s^2 - 2(r + 1)a_r < 0.\]

Moreover, the global center of system \((1)\) is as shown in Figure 1.

**Remark 1.** Notice that if condition (i) holds, we can naturally obtain that \(a_r > 0, a_m > 0\) and \(r, m\) must be odd. On the one hand, for readability, we still write “\(a_r > 0, a_m > 0\) and \(m, r\) is odd” in conditions (ii)-(iv). On the other hand, we will prove that the condition (iii) can be equivalent to that there are no orbits of system \((1)\) connecting the equilibria at infinity in the Poincaré disc.

When \(g(x)\) is odd, we can give the following corollary.

**Corollary 3.** Assume that \(g(x)\) is an odd function. System \((1)\) having a linear type global center at the origin if and only if conditions (i), (ii), (iii) in Theorem 1 and \(f(x)\) is odd; polynomial Liénard system \((1)\) having a nilpotent type global center at the origin if and only if conditions (i), (ii*), (iii) in Theorem 1 and \(f(x)\) is odd.

In order to answer the question (Q3), we give the following theorem.

**Theorem 4.** All generalized quintic Liénard systems having a linear type global center at the origin of coordinates after a rescaling of the variables \(x, y\) and \(t\) can be written as

\[
\dot{x} = y, \quad \dot{y} = -(x + ax^3 + x^5) - bxy,
\]

where \(a > -2\) and \(b \neq 0\).

All generalized quintic Liénard systems having a nilpotent type global center at the origin of coordinates after a rescaling of the variables \(x, y\) and \(t\) can be written as

\[
\dot{x} = y, \quad \dot{y} = -(x^3 + x^5) - cxy,
\]

where \(c \in (-2\sqrt{2}, 0) \cup (0, 2\sqrt{2})\).

The next proposition present a generalized polynomial Liénard system of degree \(2k + 1\) with a linear global center.
**Proposition 5.** The following generalized polynomial Liénard system of degree $2k + 1$

$$\dot{x} = y, \quad \dot{y} = -x - ax^{2k+1} - xy - bx^l y$$

has a global center at the origin of coordinates if and only if the parameters belong to any one of the following four parameter spaces

$$S_1 = \{(k, l, a, b) \in \mathbb{N}^2 \times \mathbb{R}^2 \mid k > l, l \text{ odd}, a > 0, b \neq 0\},$$
$$S_2 = \{(k, l, a, b) \in \mathbb{N}^2 \times \mathbb{R}^2 \mid k = l, l \text{ odd}, a > 0, b \neq 0, 4(l + 1)ab^{-2} > 1\},$$
$$S_3 = \{(k, l, a, b) \in \mathbb{N}^2 \times \mathbb{R}^2 \mid k > 1, a > 0, b = 0\},$$
$$S_4 = \{(k, l, a, b) \in \mathbb{N}^2 \times \mathbb{R}^2 \mid k = 1, a > 1/8, b = 0\}.$$

The organization of the rest of this paper is as follows. To prove Theorems 1 and 2 we state a preliminary result in Section 2. Theorem 1, Theorem 2 and Corollary 3 are proven in Subsections 3.1 and 3.2, respectively. Moreover, all generalized polynomial Liénard differential systems of degree 5 having a global center are given in Subsection 3.3 by Theorems 1 and 2, i.e., Theorem 4 is proven. Furthermore, we provide the explicit expression of a generalized polynomial Liénard differential system of indefinite degree $2k + 1$ in in Subsection 3.4, i.e., Proposition 5 is proven. Finally, an interesting result on the boundedness of all orbits of system (1) is given when conditions (i), (iii) in Theorem 1 hold and condition (iv) does not hold in Section 4.

2. Preliminaries

To prove the main results, we introduce a preliminary result about closed orbits of a generalized Liénard system in this section.

With the following global homeomorphism transformation

$$(x, y) \rightarrow (x, y - F(x)),$$

system (1) is changed into

$$\begin{aligned}
\dot{x} &= y - F(x), \\
\dot{y} &= -g(x),
\end{aligned}$$

where $F(x) := \int_0^x f(\xi)d\xi$. Here, we require that system (3) satisfies $xg(x) > 0$ for all $x \neq 0$. Define

$$w(x) = ((r + 1)G(x))^{\frac{1}{r+1}},$$

where $G(x) := \int_0^x g(\xi)d\xi$. It follows from $xg(x) > 0$ for all $x \neq 0$ that $G(x) \geq 0$ and $w(x)$ is decreasing in $x < 0$, increasing in $x > 0$. In other words, we have $w(x) \geq 0$. Thus, let $x_1(w)$ (resp. $x_2(w)$) be the branch of the inverse of $w(x)$ for $x \leq 0$ (resp. $x \geq 0$). In the zone $x \leq 0$, with the transformation $w = w(x)$, system (3) is changed into

$$\begin{aligned}
\frac{dw}{dy} &= \frac{w^{-r}g(x)dx}{dy} = \frac{F(x) - y}{w^r} = \frac{F(x_1(w)) - y}{w^r} =: \frac{F_1(w) - y}{w^r},
\end{aligned}$$

where $w \geq 0$. Similarly, in the zone $x \geq 0$, the transformation $w = w(x)$ brings system (3) to

$$\begin{aligned}
\frac{dw}{dy} &= \frac{w^{-r}g(x)dx}{dy} = \frac{F(x) - y}{w^r} = \frac{F(x_2(w)) - y}{w^r} =: \frac{F_2(w) - y}{w^r},
\end{aligned}$$

where $w \geq 0$. 

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Let \( \overline{ABC} \) be any an orbit arc surrounding the origin of system (3) in \( \mathbb{R}^2 \), as shown in Figure 2 (a). Moreover, let \( \mathcal{G} \) be a set of such all orbits \( \overline{ABC} \). Define that \( \overline{ABC} \cap \{(x,y) | x \leq 0\} \) (resp. \( \overline{ABC} \cap \{(x,y) | x \geq 0\} \)) of system (3) correspond to integral curves \( \gamma_1 \) in \( w-y \) plane of equation (5) (resp. \( \gamma_2 \) in \( w-y \) plane of equation (6)), see Figure 2(b). For simplicity, we respectively refer to the corresponding points of \( A, B, C \) in \( x-y \) plane as \( A', B', C' \) in \( w-y \) plane. Then, we can check that \( \gamma_1 \) (resp. \( \gamma_2 \)) is the integral curve \( B'C' \) (resp. \( B'A' \)) of equation (5) (resp. (6)) starting from \( B' \) and ending at \( C' \) (resp. \( A' \)) on the \( y \)-axis of \( w-y \) plane. In other words, \( \overline{ABC} \) is a closed orbit if and only if \( A' \) and \( C' \) coincide in \( w-y \) plane.

\[ \begin{align*}
\text{(a) An orbit arc } \overline{ABC} \text{ surrounding the origin} & \quad \text{(b) The orbit arcs starting from } B' \text{ for equations} \\
& \quad \text{for equations (5) and (6)} \\
\end{align*} \]

**Figure 2.** The orbit arcs for showing the change of transformation (4)

**Theorem 6.** The set \( \mathcal{G} \) is a closed-orbit set if and only if

\[ F_1(w) \equiv F_2(w) \]

for \( 0 \leq w \leq +\infty \).

**Proof.** Our proof is clearly divided into the following two steps.

**Step 1: Sufficiency.** When \( F_1(w) \equiv F_2(w) \), it is obvious that equations (5) and (6) are same. Thus, \( \gamma_1 \) and \( \gamma_2 \), which are integral curves of equations (5) and (6) starting from \( B' \) respectively, coincide by the uniqueness of solutions. Consequently, \( A' \) and \( C' \), which are respectively the intersections of \( \gamma_1 \) and \( y \)-axis, \( \gamma_2 \) and \( y \)-axis, coincide. In other words, \( \overline{ABC} \) is a closed orbit.

**Step 2: Necessity.** A straight calculation shows that

\[ \frac{dF_i(w)}{dw} = f(x(w))x_i'(w), \quad i = 1, 2. \]

According to the definition of \( x_i(w) \) \((i = 1, 2)\), we can obtain that \( x_i'(w) \) \((i = 1, 2)\) are analytic because \( g \) is polynomial. Furtherly, we have that \( dF_i(w)/dw \) \((i = 1, 2)\) are analytic. Thus, \( F_1(w) \) and \( F_2(w) \) are analytic functions. Assume that \( F_1(w) \neq F_2(w) \). Thus, we can obtain that \( \mu\{|w|F_1(w) = F_2(w)\} = 0 \) since \( F_1(w) \) and \( F_2(w) \) are both
analytic, where $\mu$ is the measure function. Otherwise, there exist two values $w_1, w_2$ and a small constant $\varepsilon$ such that $F_1(w) - F_2(w) \equiv 0$ for $w \in [w_1, w_2]$, but $F_1(w) - F_2(w) \neq 0$ for $w \in (w_1 - \varepsilon, w_1)$. Then, we can easily check that there exists $k$ such that either the $k$-th order left derivative of $F_1(w) - F_2(w)$ at $w_1$ is not zero and $k$-th order right derivative of $F_1(w) - F_2(w)$ at $w_1$ is zero, which contradicts that $F_1(w) - F_2(w)$ is analytic. In other words, all values satisfying $F_1(w) = F_2(w)$ are isolated when $F_1(w) \neq F_2(w)$.

Assume that $w_3$ is the smallest value such that $F_1(w_3) = F_2(w_3)$ if it exists. Then, either $F_1(w) > F_2(w)$ or $F_1(w) < F_2(w)$ holds for all $w \in (0, w_3)$. We only need to consider $F_1(w) < F_2(w)$ for $w \in (0, w_3)$. Otherwise, when the other case $F_1(w) > F_2(w)$ for $w \in (0, w_3)$ holds, we only need to apply a transformation $(y, t) \rightarrow (-y, -t)$ for system (3).

![Figure 3. ABC is not a closed orbit.](image)

Choose a point $P : (w^*, F_2(w^*))$ on the curve $y = F_2(w)$ such that $w^* \in (0, w_3)$. We claim that the orbit arc $ABC$ crossing $P$ of system (3) is not a closed orbit. Notice that $\gamma_1$ and $\gamma_2$ are shown in Figure 3, where $Q : (w_Q, F_2(w_Q))$ is on $y = F_1(w)$. By $F_1(w) < F_2(w)$, applying the comparison theorem (see [12, Theorem 6.1 of Chapter 1]) to (5) and (6), the integral curve $\gamma'_{BQ}$ of (5) lies on the left-hand side of the integral curve $P'Q$ of (6). Thus, $w_Q < w^*$.

On the other hand, we define $\gamma'$ to be the integral curve of equation (5) starting from $A'$, as shown in Figure 3. Using the comparison theorem again to (5) and (6), we can obtain that the integral curve $\gamma' \cap \{(w, y) | w \in (0, w_3)\}$ of (5) lies on the right-hand side of the integral curve $A'P$ of (6). Thus, $\gamma'$ must intersect the line $w = w^*$ at $D : (w^*, y_D)$. Moreover, $Q$ is the right-most point on $\gamma_1$ since $y = F_1(w)$ is the vertical isoclinic of equation (5). Therefore, $D$ does not lie on $\gamma_1$ because of $w_Q < w^*$, which means that $\gamma_1$ and $\gamma'$ do not coincide. Consequently, by uniqueness of solutions, the intersection point of $\gamma_1$ and $y$-axis cannot be $A'$, i.e., $A'$ and $C'$ do not coincide. Thus, the assertion is proven and the sufficiency is done. The proof is finished. $\Box$
3. Proofs of main results

This section is to give the proofs of our main results.

3.1. Proofs of Theorems 1 and 2. In this subsection, we will give the proofs of Theorems 1 and 2. Before the proofs begin, some lemmas are presented to show the necessity of system (1) with a global center. Notice that if system (1) has a global center at the origin, there is a unique equilibrium \( O : (0,0) \) at infinity.

The first lemma is characterized to obtain \( O \) is the unique equilibrium. Moreover, we can roughly determine the qualitative property of \( O \) by this lemma.

Lemma 7. System (1) has a unique equilibrium \( O : (0,0) \) which is an anti-saddle if and only if the statement (i) of Theorem 1 holds.

Proof. On the one hand, if the statement (i) of Theorem 1 holds, it is easy to check that \( O \) is the unique equilibrium. By [20, Theorems 4.1 and 4.2], the index of \( O \) is 1, which implies that \( O \) is an anti-saddle. On the other hand, if system (1) has a unique equilibrium at origin, \( g(x) \) for \( x \neq 0 \) has the following four cases:

\[
(C1) : g(x) > 0; \ (C2) : g(x) < 0; \ (C3) : xg(x) < 0; \ (C4) : xg(x) > 0.
\]

Further, by [20, Theorems 4.1 and 4.2], we calculate that the the index of \( O \) is 0 for cases (C1, C2) and −1 for case (C3), which contradicts \( O \) is an anti-saddle, whose index is 1. Thus, statement (i) of Theorem 1 must hold. The proof is done.

Secondly, we apply classical qualitative theory, which can be seen in [1], [8, Chapter 3], [20, Chapter 2] and other monographs and papers, to state a lemma of the analysis of \( O \) of system (1).

Lemma 8. \( O \) is either a linear type center or focus if and only if the statement (ii) of Theorem 1 holds, a nilpotent type center or focus if and only if the statement \((ii^*)\) of Theorem 2 holds.

Proof. We calculate that the Jacobian matrix at \( O \) as follows

\[
J := \begin{pmatrix} 0 & 1 \\ -a_1 & -b_0 \end{pmatrix}.
\]

The eigenvalues of \( J \) are \( \lambda_1 = (-b_0 + \sqrt{b_0^2 - 4a_1}) / 2 \) and \( \lambda_2 = (-b_0 - \sqrt{b_0^2 - 4a_1}) / 2 \). When statement (ii) of Theorem 1 holds, we can obtain that \( a_1 > 0 \) and \( b_0 = 0 \). Thus, \( \lambda_1 \) and \( \lambda_2 \) are a pair of purely imaginary eigenvalues. By [20, Theorem 5.1 of Chapter 2], \( O \) is a linear type center or focus.

When statement \((ii^*)\) of Theorem 2 holds, \( a_1 = b_0 = 0 \). Thus, \( \lambda_1 = \lambda_2 = 0 \). However, notice that not all the coefficients of the linear system are zero, so \( O \) is a nilpotent equilibrium in this case. Furtherly we can obtain that \( O \) is a nilpotent type center or focus by applying [20, Theorem 7.2 of Chapter 2]. We remark that statements (ii) of Theorem 1 and \((ii^*)\) of Theorem 2 are sufficient and necessary by [20, Theorem 5.1 and Theorem 7.2 of Chapter 2].
Thirdly, as shown in Figure 1 in the Poincaré disc, except for those orbits defined by 
\( z = 0 \), there are no other orbits of system (1) connecting the equilibria at infinity. The
following result provides a criterion on this.

**Lemma 9.** There is no orbit of system (1) connecting equilibria at infinity in the Poincaré disc if and only if the statement (iii) of Theorem 1 holds. Moreover, system (1) at infinity has two equilibria \( I^\pm_B \) on the y-axis and qualitative properties of system (1) near infinity are shown in Figure 7(f).

**Proof.** Without loss of generality, we assume that \( b_n > 0 \) of system (1). In fact, if \( b_n < 0 \), we can get a new \( b_n > 0 \) via the transformation \((x, y, t, b_n) \rightarrow (x, -y, -t, -b_n)\).

When \( m \neq 2n + 1 \) is odd, with a coordinate transformation

\[
(x, y, t) \rightarrow \left( a_m \frac{1}{2n+1-m} b_n^{-\frac{2}{2n+1-m}} x, a_m^{\frac{m+1}{n+1-m}} b_n^{-\frac{m+1}{n+1-m}} y, a_m^{\frac{n}{2n+1-m}} b_n^{\frac{m+1}{2n+1-m}} t \right),
\]

system (1) can be rewritten as

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\left( \epsilon x^n + \sum_{i=r}^{m-1} \hat{a}_i x^i \right) - y \left( x^n + \sum_{i=s}^{n-1} \hat{b}_i x^i \right),
\end{align*}
\]

where \( \epsilon = \text{sign}(a_m) \). When \( m \neq 2n + 1 \) is even, by the following transformation

\[
(x, y, t) \rightarrow \left( a_m^{\frac{1}{2n+1-m}} b_n^{-\frac{2}{2n+1-m}} x, a_m^{\frac{n+1}{n+1-m}} b_n^{-\frac{m+1}{n+1-m}} y, a_m^{\frac{n}{2n+1-m}} b_n^{\frac{m+1}{2n+1-m}} t \right),
\]

system (1) can be also changed into system (7) with \( \epsilon = 1 \). Finally, when \( m = 2n + 1 \), using a scaling

\[
(x, y) \rightarrow \left( b_n^{-\frac{1}{n}} x, b_n^{-\frac{1}{n}} y \right),
\]

we can change system (1) into system (7), where \( \epsilon = a_m b_n^{-2} \neq 0 \).

For the sake of simplicity, we next only need to consider system (7). According to the
result of [7], the dynamics at infinity of system (7) can be obtained directly, as shown in
Table 1. \( I^+_A \) are equilibria on the x-axis, \( I^-_B \) are equilibria on the y-axis, \( I^+_C \) are equilibria
on the line \( y = x \) and \( I^-_D \) are equilibria on the line \( y = -x \) where \( I^+_A, I^-_B, I^+_C, I^-_D \) lie in
the upper half-plane and the other four equilibria lie in the lower one. The phase portraits
in the Poincaré disc can be found in Figures 4-8, where Figure 7(f) and Figure 8(b) are
same. We can obtain that, only in the cases

\[
m = 2n + 1, \epsilon > (4n + 4)^{-1}
\]

and

\[
m > 2n + 1, \text{ } m \text{ odd, } \epsilon = 1,
\]

system (1) has no orbit connecting equilibria at infinity in the Poincaré disc. The proof is completed. \( \square \)

**Remark 2.** Notice that there are no detailed calculations about equilibria at infinity in [7].
For the completeness and readability of the article, we will show the detailed calculations
about the qualitative properties of system (7) near infinity in the two cases \( m = 2n + 1, \epsilon > (4n + 4)^{-1} \) and \( m > 2n + 1, m \text{ is odd, } \epsilon = 1 \) in the appendix.
### Table 1. Properties of equilibria at infinity

| possibilities of \((m, n, \epsilon)\) | types                                                                 |
|--------------------------------------|----------------------------------------------------------------------|
| \(m, n\) even                        | \(I_B^+\): saddle, \(I_B^-\): stable node; \(I_B^0\): unstable degenerate node (shown in Figure 4(a)) |
| \(m\) odd, \(n\) even               | \(\epsilon = 1\): \(I_B^+\): saddles; \(I_B^0\): unstable degenerate nodes (shown in Figure 4(b)) \(\epsilon = -1\): \(I_B^+\): stable nodes; \(I_B^0\): unstable degenerate nodes (shown in Figure 4(c)) |
| \(m\) even, \(n\) odd               | \(\epsilon = 1\): \(I_B^+\): saddles; \(S(I_B^0)\) consists of one hyperbolic sector, \(S(I_B^-)\) consists of one elliptic sector (shown in Figure 4(d)) \(\epsilon = -1\): \(I_B^+\): stable node; \(I_B^-\): unstable node; \(S(I_B^0)\) consists of one hyperbolic sector, \(S(I_B^-)\) consists of one elliptic sector (shown in Figure 4(e)) |
| \(m = n + 1\)                        | \(\epsilon = 1\): \(I_B^+\): unstable degenerate nodes; \(I_B^0\) saddles (shown in Figure 5(a)) \(\epsilon = -1\): \(I_B^+\): unstable degenerate nodes; \(I_B^0\) stable nodes (shown in Figure 5(b)) |
| \(n\) odd                            | \(S(I_B^+):\) consists of one hyperbolic sector, \(S(I_B^0):\) consists of one elliptic sector; \(I_B^+\): saddle, \(I_B^-\): unstable node (shown in Figure 5(c)) |
| \(m, n\) even                        | \(I_B^+\): unstable degenerate node, \(S(I_B^0)\) consists of one elliptic sector and one hyperbolic sector, (shown in Figure 6(a)) |
| \(m\) odd, \(n\) even               | \(\epsilon = 1\): \(I_B^+\): saddle-nodes (shown in Figure 6(b)) \(\epsilon = -1\): \(S(I_B^0)\) consist of one elliptic sector respectively (shown in Figure 6(c)) |
| \(n + 1 < m < 2n + 1\)              | \(m\) even, \(n\) odd \(\epsilon = 1\): \(S(I_B^0)\) consists of one elliptic sector and one hyperbolic sector, (shown in Figure 6(d)) \(\epsilon = -1\): \(S(I_B^0)\) consist of one elliptic sector respectively (shown in Figure 6(f)) |
|                                       | \(m, n\) odd \(\epsilon < 0\): \(S(I_B^0)\) consist of one elliptic sector respectively (shown in Figure 7(a)) |
| \(n\) odd                            | \(0 < \epsilon < (4n + 4)^{-1}\): \(I_B^+\): saddle-nodes (shown in Figure 7(b)) \(\epsilon = (4n + 4)^{-1}\): \(I_B^+\) saddle-nodes (shown in Figure 7(c)) |
| \(m = 2n + 1\)                       | \(0 < \epsilon < (4n + 4)^{-1}\): \(S(I_B^0)\) consist of one hyperbolic sector, \(S(I_B^+):\) consists of one elliptic sector and two hyperbolic sectors (shown in Figure 7(d)) \(\epsilon = (4n + 4)^{-1}\): \(S(I_B^0)\) consist of one hyperbolic sector, \(S(I_B^+):\) consists of one elliptic sector and two hyperbolic sectors (shown in Figure 7(e)) |
| \(m > 2n + 1\)                       | \(\epsilon > (4n + 4)^{-1}\): \(S(I_B^0)\) consist of one hyperbolic sector respectively (shown in Figure 7(f)) |

Remark: \(S(I_B^+)(\text{resp. } S(I_B^-))\) stands for any a small neighborhood of the equilibrium \(I_B^+\) (resp. \(I_B^-\)).
Proof of Theorem 1. At first, we will give a proof that system (1) has a linear type global center at the origin when statements (i-iv) hold.

The proof is as follows. When statements (i-ii) of Theorem 1 hold, we can obtain that system (1) has a unique equilibrium $O$, which is a linear type center or focus by Lemmas 7 and 8. Then, we consider system (3), which is globally topologically equivalent to system (1). Choose any a point $P$ in $\mathbb{R}^2$. Without loss of generality, assume that $P$ is locate in the first quadrant. Let $\varphi(P, I^+)$ be the positive orbit of system (3) having the initial point $P$. According to the vector field $(y - F(x), -g(x))$ of system (3), it is obvious that $\varphi(P, I^+)$ has to intersect the negative $y$-axis at a first time and then return to the
positive $y$-axis. Because of arbitrariness of $P$, all orbits of system (3) belong to orbit-set $G$. As said in Section 2, for any a value $w_0 \geq 0$, we can find a unique group $(x_1, x_2) \in \mathbb{R}^2$ satisfying $x_1 < 0 < x_2, \; G(x_1) = G(x_2)$ and $w(x_1) = w(x_2) = w_0$. When statement (iv) holds, $F(x_1) = F(x_2)$. Thus, $F(x_1(w_0)) = F(x_2(w_0))$. Furtherly, considering definitions of $F_1(w)$ and $F_2(w)$, we can obtain $F_1(w_0) = F_2(w_0)$. According to the arbitrariness of $w_0$, we can obtain that $F_1(w) \equiv F_2(w)$ for all $w \geq 0$ if statement (iv) holds. Naturally, by Theorem 6 all orbits of system (3) are closed orbits, so is system (1). Finally by Lemma 9 the qualitative properties of the equilibria at infinity are shown in Figure 7(f) if statement (iii) holds.

From what has been discussed above, $O$ is a linear type global center and we can furtherly obtain the global phase portraits of system (1), as shown in Figure 1. Thus, statements (i-iv) are sufficient conditions for that system (1) has a linear type global center at the origin. The next, we will show that these statements are also necessary.

Since $O$ is the linear type global center of system (1), it is clearly evident that $O$ is the unique equilibrium which is a center. Thus, by Lemmas 7 and 8 statements (i-ii) must hold. Moreover, it is easy to check that all orbits are bounded, which implies that statement (iii) holds by Lemma 9. Furtherly, since all orbits are closed orbits, $F_1(w) \equiv F_2(w)$ for all $0 \leq w \leq +\infty$ by applying Theorem 6. Return to (4), the from of $w$, we can obtain that if $G(x_1) = G(x_2)$ for all $x_1 < 0 < x_2, \; w(x_1) = w(x_2)$ holds. Thus, $F_1(w(x_1)) = F_2(w(x_2))$, i.e. $F(x_1) = F(x_2)$. Statement (iv) holds. The proof of Theorem 1 is done. \qed

**Figure 6.** Behaviour near infinity in the Poincaré disc for $n + 1 < m < 2n + 1$. 
Proof of Theorem 2. We only need to prove that statement (ii∗) is a sufficient and necessary condition for that $O$ of system (1) is a nilpotent type center or focus. In fact, it can be obtained by Lemma 8. Like the proof of Theorem 1, we can prove similarly that the condition including statements (i), (iii), (iv) of Theorem 1 and (ii∗) of Theorem 2 is sufficient and necessary for a nilpotent type global center at the origin of system (1).
3.2. Proof of Corollary 3 Consider that \( g(x) \) is odd. Therefore, it is clear that \( G(x) \) is even. Furthermore, \( G(x) \) is strictly increasing for \( x > 0 \) and strictly decreasing for \( x < 0 \). Thus, we have \( x_1 = -x_2 \) when \( G(x_1) = G(x_2) \) for all \( x_1 < 0 < x_2 \).

When \( f(x) \) is odd, we first prove that statement (iv) of Theorem 1 holds. Since \( f(x) \) is odd, it is clear that \( F(x) \) is even. Thus, we have \( F(x_1) = F(-x_1) \) for all \( x_1 < 0 \). In other words, statement (iv) of Theorem 1 holds.

Next, when statement (iv) of Theorem 1 holds, we need to prove that \( f(x) \) is odd. Then, it is obvious that \( F(x_1) = F(-x_1) \) for all \( x_1 < 0 \) since \( F(x_1) = F(-x_1) \) for all \( x_1 < 0 \). Thus, \( F(x) \) is even. In other words, \( f(x) \) is odd.

In conclusion, the condition that \( f(x) \) is odd is equivalent to (iv) of Theorem 1. Thus, by Theorem 1 we can obtain the conclusion directly. The proof is finished.

\[ \square \]

3.3. Proof of Theorem 4 Considering quintic Liénard system (1), we can get \( m = 5 \), \( n = 4 \), or \( m = 5, n < 4 \), or \( m < 5, n = 4 \) immediately. By the statement (iii) of Theorem 1, \( m \geq 2n + 1 \). If \( n = 4 \), we have \( m \geq 9 \), which contradicts \( m \leq 5 \). On the other hand, it follows form (iii) that \( m = 5 \) and \( n \leq 2 \) furtherly. Next, we claim that the value of \( n \) must be 1. Now, we consider \( n = 2 \). However, we can obtain \( x_2 = O(|x_1|) \) when \( G(x_1) = G(x_2) \) with \( |x_1| \) sufficiently large, implying that \( F(x_1)F(x_2) < 0 \) since \( \deg(F) = 3 \). In other words, \( F(x_1) = F(x_2) \) does not hold when \( G(x_1) = G(x_2) \) for all \( x_1 < 0 < x_2 \), which contradicts statement (iv). Then, the assertion is proven. Furthermore, \( s \geq 1 \) by the statement (ii) of Theorem 1 or (ii*) of Theorem 2. Thus, \( n = s = 1 \), i.e., we can let \( f(x) = b_1x \) with \( b_1 \neq 0 \).

Applying the statement (iv) of Theorem 1 again, \( g(x) \) must be odd since \( f(x) \) is odd. Moreover, since \( f(x) = b_1x \) and \( s = 1 \), we can get that \( r = 1 \) or \( r = 3 \) by the statement (ii) of Theorem 1 or (ii*) of Theorem 2. To obtain the concrete form of system (1), we distinguish two cases: \( r = 1 \) and \( r = 3 \). In the first case, the global center is linear type, and in the second case, it is nilpotent type.

Case 1: \( r = 1 \). System (1) has the following form,

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -(a_1 x + a_3 x^3 + a_5 x^5) - b_1 x y,
\end{align*}
\]

with \( a_1 a_5 b_1 \neq 0 \). Besides, it follows from \( x g(x) > 0 \) for \( x \neq 0 \) that \( a_1 > 0 \) and \( a_5 > 0 \). A scaling transformation

\[
(x, y, t) \rightarrow \left( a_1^{-\frac{1}{4}} a_5^{-\frac{1}{2}} x, a_1^{\frac{3}{4}} a_5^{-\frac{1}{2}} y, a_1^{-\frac{1}{2}} t \right)
\]

brings system (8) to

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -(x + a x^3 + x^5) - b x y,
\end{align*}
\]

where \( a = a_1^{-1/2} a_3 a_5^{-1/2} \) and \( b = b_1 a_1^{-1/4} a_5^{-1/4} \neq 0 \). Further, we can obtain that \( a > -2 \) from \( x g(x) = x^2 + a x^4 + x^6 > 0 \) for \( x \neq 0 \). Thus, system (9) with \( a > -2 \) and \( b \neq 0 \) has a linear type global center at the origin.

Case 2: \( r = 3 \). Rewrite system (1) as

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -(a_3 x^3 + a_5 x^5) - b_1 x y,
\end{align*}
\]
where $a_3 a_5 b_1 \neq 0$. Similarly, we have $a_3 > 0$ and $a_5 > 0$. With a transformation
\[
(x, y, t) \rightarrow \left( a_3^{-\frac{1}{2}} x, a_5^{-\frac{1}{2}} y, a_3^{-1} a_5^{-\frac{1}{2}} t \right),
\]
system (10) is changed into
\[
\begin{cases}
\dot{x} = y, \\
\dot{y} = -(x^3 + x^5) - cxy,
\end{cases}
\]
where $c = b_1 a_3^{-1/2} \neq 0$. Moreover, by the statement (ii*) of Theorem 2, $c^2 - 8 < 0$ and $c \neq 0$, i.e. $c \in (-2\sqrt{2}, 0) \cup (0, 2\sqrt{2})$. Thus, system (11) with $c \in (-2\sqrt{2}, 0) \cup (0, 2\sqrt{2})$ has a nilpotent type global center at the origin. The proof is finished.

3.4. Proof of Proposition 5. Considering Liénard system (2) of degree $2k + 1$, let $g(x) := x + ax^{2k+1}$ and $f(x) := x + bx^l$. Thus, $r = 1$, $m = 2k + 1$, $s = 1$, $n = l$ for $b \neq 0$, $n = 1$ for $b = 0$, where $m, r, n, s$ stand respectively for the highest and lowest orders of $g(x)$ and $f(x)$.

First we shall prove that system (2) has a global center in one of $S_1, \ldots, S_4$. In these spaces, $xg(x) > 0$ for all $x \neq 0$ since $a > 0$. One can check that the statement (i) of Theorem 1 holds. Since $xg(x) > 0$ for $a > 0$, statement (ii) of Theorem 1 holds. Moreover, we can obtain that $g(x)$ and $f(x)$ are odd functions when $(m, n, a, b) \in S_i$, where $i = 1, 2, 3, 4$. Then, we will check that statement (iii) of Theorem 1 holds in $S_1, \ldots, S_4$.

Case 1: $(m, n, a, b) \in S_1$. In this case, $m = 2k + 1$ is odd and $n = l$. Thus, $m > 2n + 1$ for $k > l$. Then statement (iii) of Theorem 1 holds since $a > 0$ and $k > l$.

Case 2: $(m, n, a, b) \in S_2$. In this case, $m = 2k + 1$ is odd, $n = l$, $a_m = a$ and $b_n = b$. It follows from $k = l$ and $4(n + 1)ab^{-2} > 1$ that statement (iii) of Theorem 1 holds.

Case 3: $(m, n, a, b) \in S_3$. In this case, $m = 2k + 1$ is odd, $n = 1$. Thus, $2k + 1 > 3$ for $k > 1$. Then statement (iii) of Theorem 1 holds since $a > 0$ and $k > 1$.

Case 4: $(m, n, a, b) \in S_4$. In this case $m = 3$, $n = 1$, $a_m = a$ and $b_n = 1$. When $a > 1/8$, $4(n + 1)a_m b_n^{-2} > 1$ holds. Consequently, statement (iii) of Theorem 1 holds.

In summary, statement (iii) of Theorem 1 holds for $(m, n, a, b) \in S_i$ where $i = 1, 2, 3, 4$. Thus, by Corollary 3, system (2) has a linear type global center at the origin.

Our next task now in this proof is to show that the parameters of system (2) belong to any one of spaces $S_1, \ldots, S_4$ when system (2) has a global center at the origin. Since $g(x)$ is odd, by Corollary 3, statements (i)-(iii) of Theorem 1 hold and $f(x)$ is an odd function.

It is easy to check statement (ii) holds for all $(k, l, a, b) \in \mathbb{N}^2 \times \mathbb{R}^2$. Then we can obtain $a \geq 0$ from statement (i). Moreover, $a \neq 0$ because the degree of system (2) is $2k + 1$. Thus, $a > 0$. Consider $b \neq 0$. In this case, $l$ is odd since $f(x)$ is an odd function. It follows from statement (iii) that either $k > l$ or $k = l$, $4(l + 1)ab^{-2} > 1$. Therefore, $(k, l, a, b) \in S_1$ or $(k, l, a, b) \in S_2$. Consider $b = 0$. In this case, $f(x) = x$ and $n = 1$. We can obtain that $k > 1$ or $k = 1$, $8a > 1$ from statement (iii). Thus $(k, l, a, b) \in S_3$ or $(k, l, a, b) \in S_4$. In summary, if system (2) has a global center at the origin, the parameters belong to any one of spaces $S_1, \ldots, S_4$. The proof is complete. □
4. Remark   conclusions

By Theorem 6, we can obtain that all orbits are bounded of system (1) when condition (iv) of Theorem 1 holds. Naturally, we want to know the properties of boundedness of orbits of system (1) when condition (iv) of Theorem 1 does not hold. First, we give the following lemma.

Lemma 10. Assume that conditions (i) and (iii) of Theorem 1 hold. there is a value \( \hat{w} \geq 0 \) such that \( F_1(\hat{w}) = F_2(\hat{w}) \) and either \( F_1(w) < F_2(w) \) or \( F_1(w) > F_2(w) \) for \( w > \hat{w} \) when condition (iv) of Theorem 1 does not hold. Naturally, we want to know the properties of boundedness of system (1) when condition (iv) of Theorem 1 does not hold. First, we give the following lemma.

Proof. It is obvious that \( F_1(w) \neq F_2(w) \) when condition (iv) of Theorem 1 does not hold. Then, we can assume that \( \hat{w} \geq 0 \) is the largest value satisfying \( F_1(w) = F_2(w) \) since \( F(x) \) and \( G(x) \) are polynomial. In other words, we have either \( F_1(w) < F_2(w) \) or \( F_1(w) > F_2(w) \) for \( w > \hat{w} \). Otherwise, there are two values \( w_4 > \hat{w} \) and \( w_5 > \hat{w} \) satisfying \( F_1(w_4) < F_2(w_4) \) and \( F_1(w_5) > F_2(w_5) \). On the one hand, we can find a value \( w_b \in (\min\{w_4,w_5\}, \max\{w_4,w_5\}) \) satisfying \( F_1(w_b) = F_2(w_b) \) by intermediate value theorem. On the other hand, \( \hat{w} \geq 0 \) is the largest value satisfying \( F_1(w) = F_2(w) \). This is a contradiction. The proof is finished.

The following proposition will show an interesting result when condition (iv) of Theorem 1 does not hold. Naturally, by Lemma 10 there is a value \( \hat{w} \geq 0 \) such that \( F_1(\hat{w}) = F_2(\hat{w}) \) and either \( F_1(w) < F_2(w) \) or \( F_1(w) > F_2(w) \) for \( w > \hat{w} \).

Proposition 11. Assume that conditions (i) and (iii) of Theorem 1 hold. Then, all orbits of system (1) are positive bounded (resp. negative bounded) if and only if there exists a value \( \hat{w} \geq 0 \) such that \( F_1(w) < (\text{resp.} >) F_2(w) \) for all \( w > \hat{w} \).

Proof. On the one hand, considering condition (i) of Theorem 1, system (1) has a unique equilibrium, which is an anti-saddle by Lemma 4. On the other hand, considering condition (iii) of Theorem 1, system (1) has no orbits connecting the equilibria at infinity in the Poincaré disc by Lemma 9. In what follows, for convenience, we consider system (3) because it is equivalent to system (1).

Firstly, we show the sufficiency of this proposition. It suffices to prove this proposition in the case \( F_1(w) < F_2(w) \) and we can similarly obtain the results in the case \( F_1(w) > F_2(w) \).

Consider an orbit segment of system (3) with a point \( A \) in the positive y-axis, where the ordinate \( y_A \) of \( A \) is sufficiently large. It follows from nonexistence of orbits connecting the equilibria at infinity that this orbit must cross the positive x-axis, intersect with the negative y-axis at a point \( B : (0, y_B) \) and then intersect with the positive y-axis again at a point \( C \), as shown in Figure 2(a). We can choose a point \( A \) such that \( |y_B| \) is also sufficiently large, where \( y_A \) is large enough.

By the transformation (4), let the orbit segment \( \hat{ABC} \cap \{(x, y)| x \geq 0 \} \) be corresponded to \( \hat{A}'PB' \) in \( w-y \) plane, where \( P : (w_P,y_P) \) lies on \( y = F_2(w) \) and \( A_1 : (\hat{w}, y_{A_1}) \), \( B_2 : (\hat{w}, y_{B_2}) \) are two points on \( w = \hat{w} \), as shown in Figure 3.

Consider an integral curve \( \hat{A}'A_1P_1 \) of (5), where \( A_1 : (\hat{w}, y_{A_1}) \) lies on \( w = \hat{w} \) and \( P_1 : (w_{P_1}, y_{P_1}) \) lies on \( y = F_2(w) \). We claim that \( w_{P_1} > w_P \). In fact, since integral curve
Figure 9. Orbits in $w$-$y$ plane.

$\overrightarrow{A_2A}$ (resp. $\overrightarrow{A_1A}$) satisfies equation (6) (resp. (5)), we can obtain that

$$y_{A_2} - y_{A'} = \int_{0}^{\hat{w}} \frac{w^r}{F_2(w) - y_2(w)} dw$$

(resp. $y_{A_1} - y_{A'} = \int_{0}^{\hat{w}} \frac{w^r}{F_1(w) - y_1(w)} dw$),

where $y_2(w)$ (resp. $y_1(w)$) represents the integral curve $\overrightarrow{A_2A_1}$ $P$ (resp. $\overrightarrow{A_1A_1}$ $P_1$). Since $y_{A'} = y_A$ is sufficiently large, then there exists a sufficiently small constant $\varepsilon > 0$ such that

$$y_{A_2} - y_{A'} < \varepsilon/2$$

and

$$y_{A_1} - y_{A'} < \varepsilon/2.$$ 

Thus, we can obtain

$$|y_{A_2} - y_{A_1}| < \varepsilon.$$

Let $\varphi(w) = y_1(w) - y_2(w)$. Thus, $|\varphi(\hat{w})| = |y_{A_2} - y_{A_1}| < \varepsilon$. For $w \in (\hat{w}, \min\{w_P, w_{P_1}\})$, it follows from that

$$\varphi(w) = \int_{\hat{w}}^{w} \frac{z^r}{F_1(z) - y_1(z)} dz - \int_{\hat{w}}^{w} \frac{z^r}{F_2(z) - y_2(z)} dz + \varphi(\hat{w})$$

$$= \int_{\hat{w}}^{w} \frac{z^r(F_2(z) - y_2(z))}{(F_1(z) - y_1(z))(F_2(z) - y_2(z))} dz$$

$$+ \int_{\hat{w}}^{w} \frac{z^r(y_1(z) - y_2(z))}{(F_1(z) - y_1(z))(F_2(z) - y_2(z))} dz + \varphi(\hat{w})$$

$$= :H_1(w) + \int_{\hat{w}}^{w} H_2(z) \varphi(z) dz,$$

where

$$H_1(w) = \varphi(\hat{w}) + \int_{\hat{w}}^{w} \frac{z^r(F_2(z) - F_1(z))}{(F_1(z) - y_1(z))(F_2(z) - y_2(z))} dz.$$
and
\[ H_2(w) = \frac{w^r}{(F_1(w) - y_1(w))(F_2(w) - y_2(w))}. \]

Letting \( H(w) = \int_{\hat{w}}^{w} H_2(z) \varphi(z) dz \), it follows from (12) that
\[ \frac{dH(w)}{dw} = H_2(w) \varphi(w) = H_2(w)H_1(w) + H_2(w)H(w). \]

Using the constant variation formula to solve equation (13), we obtain that
\[ H(w) = \int_{\hat{w}}^{w} H_2(z)H_1(z) \exp\left( \int_{z}^{w} H_2(s) ds \right) dz. \]

Thus,
\[ \varphi(w) = H_1(w) + H(w) \]
\[ = H_1(w) + \int_{\hat{w}}^{w} H_2(z)H_1(z) \exp\left( \int_{z}^{w} H_2(s) ds \right) dz \]
\[ = H_1(\hat{w}) \exp\left( \int_{\hat{w}}^{w} H_2(s) ds \right) + \int_{\hat{w}}^{w} H_1'(z) \exp\left( \int_{z}^{w} H_2(s) ds \right) dz \]
\[ = \varphi(\hat{w}) \exp\left( \int_{\hat{w}}^{w} H_2(s) ds \right) + \int_{\hat{w}}^{w} H_1'(z) \exp\left( \int_{z}^{w} H_2(s) ds \right) dz. \]

Since
\[ H'_1(w) = \frac{w^r(F_2(w) - F_1(w))}{(F_1(w) - y_1(w))(F_2(w) - y_2(w))} > 0 \]
for \( w \in (\hat{w}, \min\{w_P, w_{P_1}\}) \), we can have that
\[ \int_{\hat{w}}^{w} H_1'(z) \exp\left( \int_{z}^{w} H_2(s) ds \right) dz > 0. \]

Moreover, \( |\varphi(\hat{w})| < \varepsilon \). One can check that \( \varphi(\hat{w}) > 0 \), where \( \hat{w} = \min\{w_P, w_{P_1}\} \). Consequently, we have \( w_{P_1} > w_P \).

Consider \( B'B_1Q_1 \), the integral curve starting from \( B' \) of \( [5] \). We can similarly prove \( w_{Q_1} < w_Q \), where \( Q_1, Q \) are lie on \( y = F_1(w) \) and \( w_{Q_1}, w_Q \) are respectively abscissas of \( Q_1, Q \). Returning to the \( x-y \) plane of system (3), we get that its all orbits are positive bounded. The proof of the sufficiency is done.

Next, we consider the necessity. Similarly, we only need to prove that there exists a value \( \hat{w} \geq 0 \) such that \( F_1(w) < F_2(w) \) for all \( w > \hat{w} \) when all orbits of system (1) are positive bounded. We choose \( \hat{w} \) to be the largest value satisfying \( F_1(w) = F_2(w) \). Using contradiction, according to the proof Lemma [10] \( F_1(w) > F_2(w) \) for all \( w > \hat{w} \) is the inverse of \( F_1(w) < F_2(w) \) for all \( w > \hat{w} \). As proven above, we can obtain that all orbits of system (1) are negative bounded if there exists a value \( \hat{w} \geq 0 \) such that \( F_1(w) > F_2(w) \) for all \( w > \hat{w} \), which is a contradiction. The proof of the necessity is finished, so is this proposition.

\[ \square \]

**APPENDIX**

For the sake of completeness, we will show how to obtain the qualitative properties of system (7) at infinity in following two cases: (C1) \( m = 2n + 1, \epsilon > (4n + 4)^{-1} \), (C2) \( m > 2n + 1, m \) is odd, \( \epsilon = 1 \) in the appendix.
Consider the case (C1). By a Poincaré transformation
\[ x = \frac{1}{z}, \quad y = \frac{u}{z}, \]
system (7) is changed to

\[
\begin{aligned}
\frac{du}{d\tau} = \epsilon + u^2 z^{2n} + \sum_{i=r}^{2n} \hat{a}_i u^{i+1} z^{2n+1-i} + u z^n + \sum_{i=s}^{n-1} \hat{b}_i u z^{2n-i}, \\
\frac{dz}{d\tau} = u z^{2n+1},
\end{aligned}
\]  

(14)

where \( d\tau = -dt/z^{2n} \). It is easy to check that system (14) has no equilibria on \( z = 0 \).

With the other Poincaré transformation
\[ x = \frac{v}{z}, \quad y = \frac{1}{z}, \]
system (7) is written as

\[
\begin{aligned}
\frac{dv}{d\tau} = z^{2n} + \epsilon v^{2n+2} + \sum_{i=r}^{2n} \hat{a}_i v^{i+1} z^{2n+1-i} + v^n z^n + \sum_{i=s}^{n-1} \hat{b}_i v^i z^{2n-i}, \\
\frac{dz}{d\tau} = \epsilon v^{2n+1} z + \sum_{i=r}^{2n} \hat{a}_i v^{i+1} z^{2n+2-i} + v^{n+1} z^{n+1} + \sum_{i=s}^{n-1} \hat{b}_i v^i z^{2n+1-i},
\end{aligned}
\]  

(15)

where \( d\tau = dt/z^{2n} \). Notice that system (15) has a unique equilibrium \( B : (0, 0) \) on \( z = 0 \).

Furtherly, we can obtain that \( B \) is a degenerate equilibrium. With a polar transformation \((v, z) = (r \cos \theta, r \sin \theta)\), system (15) can be written as

\[
\frac{1}{r} \frac{dr}{d\theta} = \frac{H_1(\theta) + O(r)}{G_1(\theta) + O(r)},
\]

where \( G_1(\theta) = -\sin^{2n+1} \theta \) and \( H_1(\theta) = \cos \theta \sin^{2n} \theta \). A necessary condition on existence of exceptional directions is \( G_1(\theta) = 0 \) by [20, Chapter 2]. Obviously, \( G_1(\theta) = 0 \) has exactly two roots 0, \( \pi \) in \( \theta \in [0, 2\pi] \). However, \( H_1(0) = H_1(\pi) = 0 \). Thus, we cannot apply the normal sector method (see [20, Chapter 2]) to analyze the two exceptional directions \( \theta = 0, \pi \) of \( B \) for system (15). Instead, we adopt Briot–Bouquet transformations to blow up the two directions.

With the Briot–Bouquet transformation \( z = \tilde{z}v \), system (15) is changed into

\[
\begin{aligned}
\frac{dv}{d\delta} = v^{2n} + \epsilon v^3 + \sum_{i=r}^{2n} \hat{a}_i v^{i+1} z^{2n+1-i} + v^2 z^n + \sum_{i=s}^{n-1} \hat{b}_i v^i z^{2n-i}, \\
\frac{d\tilde{z}}{d\delta} = -\tilde{z}^{2n+1},
\end{aligned}
\]  

(16)

where \( d\delta = v^{2n-1} dt \). System (16) has a unique equilibrium \( E : (0, 0) \), which is degenerate. The polar change of variables \((v, \tilde{z}) = (r \cos \theta, r \sin \theta)\) sends system (16) to

\[
\frac{1}{r} \frac{dr}{d\theta} = \frac{H_2(\theta) + O(r)}{G_2(\theta) + O(r)},
\]
where

\[ G_2(\theta) = \begin{cases} 
- \sin \theta \cos \theta (2 \sin^2 \theta + \sin \theta \cos \theta + \epsilon \cos^2 \theta), & \text{if } n = 1, \\
- \epsilon \sin \theta \cos^3 \theta, & \text{if } n > 1,
\end{cases} \]

and

\[ H_2(\theta) = \begin{cases} 
- \sin^4 \theta + \sin^2 \theta \cos^2 \theta + \sin \theta \cos^3 \theta + \epsilon \cos^4 \theta, & \text{if } n = 1, \\
\epsilon \cos^4 \theta, & \text{if } n > 1.
\end{cases} \]

Firstly, consider \( n = 1 \). Since \( \epsilon > 1/8 \), the equation \( 2 \sin^2 \theta + \sin \theta \cos \theta + \epsilon \cos^2 \theta = 0 \) has no real roots. Thus, \( G_2(\theta) \) has four simple zeros \( \theta = 0, \pi/2, \pi, 3\pi/2 \) in \([0, 2\pi]\). Moreover, we have

\[ G_2'(0)H_2(0) = G_2'(\pi)H_2(\pi) = -\epsilon^2 < 0 \]

and

\[ G_2'(\pi/2)H_2(\pi/2) = G_2'(3\pi/2)H_2(3\pi/2) = -2 < 0. \]

Thus, by

\[ H_2(0) = H_2(\pi) = \epsilon > 0, H_2(\pi/2) = H_2(3\pi/2) = -1 < 0 \]

and [20] Theorem 3.7 of Chapter 2], system (16) has a unique orbit connecting \( E \) in respectively the directions \( \theta = \pi/2 \) and \( 3\pi/2 \) as \( \delta \to +\infty \), and a unique orbit connecting \( E \) in respectively the directions \( \theta = 0 \) and \( \pi \) as \( \delta \to -\infty \), see Figure 10(a). Therefore, blowing down the equilibrium \( E \) of system (16) to \( B \) of system (15) yields that there is a unique orbit connecting \( B \) in the direction \( \theta = 0 \) as \( \tau \to -\infty \), and a unique orbit connecting \( B \) in the direction \( \pi \) as \( \tau \to +\infty \), see Figure 10(b).

![Figure 10](image_url)

**Figure 10.** Orbits changing under the Briot-Bouquet transformation for \( n = 1 \).

Then, we consider \( n > 1 \). It is easy to check that \( G_2(\theta) \) has exactly two simple zeros \( 0, \pi \) and two trible zeros \( \pi/2, 3\pi/2 \) in \([0, 2\pi]\). Easy calculation gives that \( G_2'(0)H_2(0) = G_2'(\pi)H_2(\pi) = -\epsilon^2 < 0 \). One can obtain that there is a unique orbit connecting \( E \) in respectively the directions \( \theta = 0 \) and \( \pi \) as \( \delta \to -\infty \) by \( H_2(0) = H_2(\pi) = \epsilon > 0 \) and [20] Theorem 3.7 of Chapter 2]. However, since \( H_2(\pi/2) = H_2(3\pi/2) = 0 \), we need to blow up the two directions \( \theta = \pi/2, 3\pi/2 \).
With the quasihomogeneous blow-up \( v = \tilde{z}^{n-1} \tilde{v} \), (see [8 Chapter 3.3], or [6]), we change system \[ (16) \]
\[
\begin{align*}
\frac{d\tilde{v}}{ds} &= n\tilde{z}^2 + c\tilde{v}^3 + \sum_{i=r}^{2n} \hat{a}_i \tilde{v}^{2n+1-i} + \tilde{v}^2 \tilde{z} + \sum_{i=s}^{n-1} \hat{b}_i \tilde{v}^{2n+1-i}, \\
\frac{d\tilde{z}}{ds} &= -\tilde{z}^3,
\end{align*}
\]
where \( ds = \tilde{z}^{2n-2}d\delta \). System \[ (17) \] has a unique equilibrium \( F : (0,0) \). Similarly, transforming system \[ (17) \] into equation
\[
\frac{1}{r} \frac{dr}{d\theta} = \frac{H_3(\theta) + O(r)}{G_3(\theta) + O(r)},
\]
by a polar transformation \((\tilde{v}, \tilde{z}) = (r \cos \theta, r \sin \theta)\), where
\[
G_3(\theta) = -\sin \cos \theta((n + 1) \sin^2 \theta + \sin \cos \theta + \epsilon \cos^2 \theta)
\]
and
\[
H_3(\theta) = -\sin^4 \theta + n \sin^2 \theta \cos^2 \theta + \sin \cos^3 \theta + \epsilon \cos^4 \theta.
\]
It follows from \( \epsilon > 1/(4n+4) \) that
\[
(n + 1) \sin^2 \theta + \sin \cos \theta + \epsilon \cos^2 \theta = 0
\]
has no roots. Thus, \( G_3(\theta) \) has four simple zeros \( \theta = 0, \pi/2, \pi, 3\pi/2 \) in \([0, 2\pi)\). Moreover, we can check that
\[
G_3'(0)H_3(0) = G_3'(\pi)H_3(\pi) = -\epsilon^2 < 0
\]
and
\[
G_3'(\pi/2)H_3(\pi/2) = G_3'(3\pi/2)H_3(3\pi/2) = -(n + 1) < 0.
\]
By
\[
H_3(0) = H_3(\pi) = \epsilon > 0, H_3(\pi/2) = H_3(3\pi/2) = -1 < 0
\]
and [20]. Theorem 3.7 of Chapter 2], there is a unique orbit connecting \( F \) in respectively the directions \( \theta = \pi/2 \) and \( 3\pi/2 \) as \( s \to +\infty \), and a unique orbit connecting \( F \) in respectively the directions \( \theta = 0 \) and \( \pi \) as \( s \to -\infty \), see Figure \[ 11(a) \]. Furthermore, we can obtain that there is a unique orbit connecting \( E \) in respectively the directions \( \theta = \pi/2 \) and \( 3\pi/2 \) as \( \delta \to +\infty \). The qualitative properties of \( E \) are shown in Figure \[ 11(b) \]. Then, by blowing down the equilibrium \( E \) of system \[ (16) \] to \( B \) of system \[ (15) \], there is a unique orbit connecting \( B \) in the direction \( \theta = 0 \) as \( \tau \to -\infty \), and a unique orbit connecting \( B \) in the direction \( \pi \) as \( \tau \to +\infty \), see Figure \[ 11(c) \].

Consider the case (C2). Similarly by a Poincaré transformation

\[
x = \frac{1}{\tilde{z}}, \quad y = \frac{u}{\tilde{z}},
\]

system \[ (7) \] is changed to
\[ (18) \]
\[
\begin{align*}
\frac{du}{d\tau} &= 1 + u^2 \tilde{z} m^{-1} + \sum_{i=r}^{m-1} \hat{a}_i \tilde{z}^{m-1-i} + uz^{m-1-n} + \sum_{i=s}^{n-1} \hat{b}_i uz^{m-1-i}, \\
\frac{dz}{d\tau} &= uz^{m},
\end{align*}
\]
where \( d\tau = -dt/\tilde{z}^{m-1} \). There is no equilibrium on \( z = 0 \) of system \[ (18) \].
By another Poincaré transformation
\[ x = \frac{v}{z}, \quad y = \frac{1}{z}, \]
system (7) is written as
\[
\begin{aligned}
\frac{dv}{d\tau} &= z^{m-1} + v^{m+1} + \sum_{i=r}^{m-1} \hat{a}_i v^{i+1} z^{m-1-i} + v^{n+1} z^{m-1-n} + \sum_{i=s}^{n-1} \hat{b}_i v^{i+1} z^{m-1-i}, \\
\frac{dz}{d\tau} &= v^m z + \sum_{i=r}^{2n} \hat{a}_i v^i z^{m+1-i} + v^n z^{m-n} + \sum_{i=s}^{n-1} \hat{b}_i v^i z^{m-i},
\end{aligned}
\]
where \( d\tau = dt/z^{m-1} \). Then, \( B : (0, 0) \) is the unique equilibrium of system (19) on \( z = 0 \). Considering a polar transformation \((v, z) = (r \cos \theta, r \sin \theta)\), system (15) can be written as
\[
\begin{aligned}
\frac{1}{r} \frac{dr}{d\theta} &= \frac{H_4(\theta) + O(r)}{G_4(\theta) + O(r)},
\end{aligned}
\]
where $G_4(\theta) = -\sin^m \theta$ and $H_4(\theta) = \cos \theta \sin^{m-1} \theta$. It is easy to check that $G_4(\theta) = 0$ has two roots 0 and $\pi$ in $\theta \in [0, 2\pi)$. However, since $H(0) = H(\pi) = 0$, we need to desingularize $B$ furtherly.

With the Briot–Bouquet transformation $z = \tilde{z}v$, system (19) is rewritten as

\[
\begin{align*}
\frac{dv}{d\delta} &= v_2z^{m-1} + v^3 + \sum_{i=r}^{m-1} \hat{a}_i v^3z^{m-i} + v^2z^{m-1-n} + \sum_{i=s}^{n-1} \hat{b}_i v^2z^{m-1-i}, \\
\frac{d\tilde{z}}{d\delta} &= -\tilde{z}^m,
\end{align*}
\]

where $d\delta = v^{m-2}d\tau$. Obviously, $E : (0, 0)$ of system (20) is the equilibrium. It follows from $m > 2n+1$ and $n \geq 1$ that $m > 3$. Using a polar coordinate $(v, \tilde{z}) = (r \cos \theta, r \sin \theta)$, system (20) is transformed into the following polar form

\[
\frac{1}{r} \frac{dr}{d\theta} = \frac{H_5(\theta) + O(r)}{G_5(\theta) + O(r)},
\]

$G_5(\theta) = -\sin \theta \cos^3 \theta$ and $H_5(\theta) = \cos^4 \theta$. $G_5(\theta)$ has two simple zeros $\theta = 0, \pi$ and two triple zeros $\pi/2, 3\pi/2$ in $[0, 2\pi)$. Moreover, we can check that

$G'_5(0)H_5(0) = G'_5(\pi)H_5(\pi) = -1 < 0$.

Thus, by $H_5(0) = H_2(\pi) = 1$ and [20] Theorem 3.7 of Chapter 2], system (20) has a unique orbit connecting $E$ in respectively the directions $\theta = 0$ and $\pi$ as $\delta \to -\infty$. However, since $H_5(\pi/2) = H_2(3\pi/2) = 0$, we need to blow up the two directions.

The quasihomogeneous blow-up $v = \tilde{z}^{(m-3)/2}\tilde{v}$ sends system (20) to

\[
\begin{align*}
\frac{d\tilde{v}}{ds} &= \frac{m-1}{2} \tilde{v}^2 + \tilde{v}^3 + \sum_{i=r}^{m-1} \hat{a}_i \tilde{v}^3\tilde{z}^{m-i} + \tilde{v}^2\tilde{z}^{m-2n+1} + \sum_{i=s}^{n-1} \hat{b}_i \tilde{v}^2\tilde{z}^{m-2i+1}, \\
\frac{d\tilde{z}}{ds} &= -\tilde{z}^3,
\end{align*}
\]

where $ds = \tilde{z}^{m-3}d\delta$. It is clear that $F : (0, 0)$ is an equilibrium. Using a polar coordinate $(\tilde{v}, \tilde{z}) = (r \cos \theta, r \sin \theta)$, system (21) is transformed into the following polar form

\[
\frac{1}{r} \frac{dr}{d\theta} = \frac{H_6(\theta) + O(r)}{G_6(\theta) + O(r)},
\]

where

$G_6(\theta) = -\sin \theta \cos (r \theta) \left( \frac{m+1}{2} \sin^2 \theta + \cos^2 \theta \right)$

and

$H_6(\theta) = -\sin^4 \theta + \frac{m-1}{2} \sin^2 \theta \cos^2 \theta + \cos^4 \theta$.

$\theta = 0, \pi/2, \pi, 3\pi/2$ are four simple zeros of $G_6(\theta)$ in $[0, 2\pi)$. Moreover,

$G'_6(0)H_6(0) = G'_6(\pi)H_6(\pi) = -1 < 0$

and

$G'_6(\pi/2)H_6(\pi/2) = G'_6(3\pi/2)H_6(3\pi/2) = -(m+1)/2 < 0$.

By

$H_6(0) = H_6(\pi) = 1 > 0, H_6(\pi/2) = H_6(3\pi/2) = -1 < 0$.  

and [20, Theorem 3.7 of Chapter 2], there is a unique orbit connecting $F$ in respectively the directions $\theta = \pi/2$ and $3\pi/2$ as $s \to +\infty$, and a unique orbit connecting $F$ in respectively the directions $\theta = 0$ and $\pi$ as $s \to -\infty$, as also seen in Figure 11(a). Furtherly, we can obtain that there is a unique orbit connecting $E$ in respectively the directions $\theta = \pi/2$ and $3\pi/2$ as $\delta \to +\infty$. The qualitative properties of $E$ are also shown in Figure 11(b). Then, by blowing down the equilibrium $E$ of system [10] to $B$ of system [15], there is a unique orbit connecting $B$ in the direction $\theta = 0$ as $\tau \to -\infty$, and a unique orbit connecting $B$ in the direction $\pi$ as $\tau \to +\infty$, see Figure 11(c).

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