Persistence probabilities of two-sided (integrated) sums of correlated stationary Gaussian sequences

Frank Aurzada       Micha Buck
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Abstract

We study the persistence probability for some two-sided discrete-time Gaussian sequences that are discrete-time analogs of fractional Brownian motion and integrated fractional Brownian motion, respectively. Our results extend the corresponding ones in continuous-time in [11] and [12] to a wide class of discrete-time processes.

1 Introduction

Persistence concerns the probability that a stochastic process has a long negative excursion. In this paper, we are concerned mainly with two-sided discrete-time processes: If $Z = (Z_n)_{n \in \mathbb{Z}}$ is a stochastic process, we study the rate of decay of the probability

$$\mathbb{P}(Z_n \leq 0 : |n| \leq N), \quad \text{as} \quad N \to \infty.$$  

In many cases of interest, the above probability decreases polynomially, i.e., as $N^{-\theta+o(1)}$, and it is the first goal to find the persistence exponent $\theta$. For a recent overview on this subject, we refer to the surveys [9], [7], [5].

The purpose of this paper is to analyse the persistence probability for the discrete-time analogs of two-sided fractional Brownian motion (FBM) and two-sided integrated fractional Brownian motion (IFBM). Our study extends results in [11] and [12], respectively, to a wide class of discrete-time processes.

The study of persistence probabilities of FBM, IFBM and related processes has received considerable attention in theoretical physics and mathematics, recently. For instance, see [13] and [12] where a relation between the Hausdorff dimension of Lagrangian regular points for the inviscid Burgers equation with FBM initial velocity and the persistence probabilities of IFBM is established; the interest for it arises from [18] and [19]. Further, in [14] a physical model involving FBM is studied as an extension to the Sinai model; see also [2]. Here, persistence probabilities are related to scaling
properties of a quantity, called steady-state current. Moreover, persistence of non-Markovian processes that are similar to FBM are studied in \[8\] and \[4\], confirming results in \[16\] and \[10\].

Let us recall that a FBM \((W_H(t))_{t \in \mathbb{R}}\) is a centered Gaussian process with covariance

\[
E[W_H(t)W_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad t, s \in \mathbb{R},
\]

where \(0 < H < 1\) is a constant parameter, called Hurst parameter. For \(H = 1/2\) this is a usual two-sided Brownian motion. For any \(0 < H < 1\), the process has stationary increments, but no independent increments (unless \(H = 1/2\)). Furthermore, it is an \(H\)-self-similar process. An IFBM \((I_H(t))_{t \in \mathbb{R}}\) is defined by

\[
I_H(t) := \int_0^t W_H(s) \, ds
\]

and is a \((H+1)\)-self-similar process.

In order to define the discrete-time analogs, let \((\xi_n)\) be a real valued stationary centered Gaussian sequence such that

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} E[\xi_j \xi_k] \sim n^{2H} \ell(n), \quad n \to \infty,
\]

with \(0 < H < 1\) and \(\ell\) slowly varying at infinity. Here and below, we write \(f(x) \sim g(x) \, (x \to x_0)\) if \(\lim f(x)/g(x) = 1\) as \(x \to x_0\). Then, (1) implies the weak convergence result

\[
\left( \frac{1}{n^{H} \ell(n)^{1/2}} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k \right)_{t \geq 0} \Rightarrow (W_H(t))_{t \geq 0}
\]

with fractional Brownian motion \((W_H(t))\), see e.g. Theorem 4.6.1 in \[20\].

For this reason, it is natural to consider the stationary increments sequence \((S_n)_{n \in \mathbb{Z}}\) given by

\[
S_n - S_{n-1} := \xi_n \text{ for } n \in \mathbb{Z} \quad \text{and} \quad S_0 := 0
\]
as a discrete-time analog of FBM.

Now, we will define the discrete-time analog of IFBM such that symmetry properties like in the continuous-time setting are satisfied. With this in mind, a natural process is given by

\[
I_n - I_{n-1} := (S_n + S_{n-1})/2 \text{ for } n \in \mathbb{Z} \quad \text{and} \quad I_0 := 0.
\]

In Section 2, we discuss relations to the process with increments \((S_n)\) (instead of \((S_n + S_{n-1})/2\)), which may also seem natural but for which our method of proof does not apply directly due to a lack of symmetry.

In \[11\] it is shown that one has \(\mathbb{P}(W_H(t) \leq 1 : |t| \leq T) = T^{-1+o(1)}\).

Our first result, treats the discrete-time analog. The technique we use to prove the theorem is completely different from the one in \[11\].
Theorem 1. Let $(\xi_n)$ be a real valued stationary centered Gaussian sequence such that (1) holds. Then, there is a constant $c > 0$ such that, for every $N \geq 1$,

$$c^{-1} N^{-1} \leq \mathbb{P}(S_n \leq 0 : |n| \leq N) \leq N^{-1}. $$

In order to prove the corresponding result for the process $(I_n)$, we will use a change of measure argument. This argument requires an additional assumption as follows: Let $\mu$ denote the spectral measure of the sequence $(\xi_n)$, i.e.,

$$\mathbb{E} \xi_j \xi_k =: \int_{(-\pi,\pi]} e^{i(j-k)u} d\mu(u).$$

The spectral measure $\mu$ has a (possibly vanishing) component that is absolutely continuous with respect to the Lebesgue measure. Let us denote by $p$ its density, i.e., $d\mu(u) =: p(u) du + d\mu_s(u)$. We will assume that $p$ satisfies

$$p(u) \sim \ell(1/u)|u|^{1-2H}, \quad u \to 0, \quad (3)$$

where $\ell$ is a slowly varying function at infinity. It is well-known that (3) implies (1) and thus (2).

The nature of this assumption can be understood by considering the fractional Gaussian noise process, defined by

$$\xi_{\text{fgn}}_n := W_H(n) - W_H(n-1).$$

This stationary centered Gaussian sequence has an absolutely continuous spectral measure with density function $p_{\text{fgn}}$ that satisfies (see e.g. [17])

$$p_{\text{fgn}}(u) \sim m_H|u|^{1-2H}, \quad u \to 0,$$

where $m_H = \Gamma(2H + 1)\sin(\pi H)/2\pi$. So, we assume that the density of the absolutely continuous part of the spectral measure of the stationary process $(\xi_n)$ is comparable to the spectral density of fractional Gaussian noise, up to the slowly varying function $\ell$.

We are now ready to state our second main result.

Theorem 2. Let $(\xi_n)$ be a real valued stationary centered Gaussian sequence such that (3) holds. Then,

$$\mathbb{P}(I_n \leq 0 : |n| \leq N) = N^{-(1-H)+o(1)}. $$

We recall that [12] considers the continuous-time case. Many arguments from that paper can be adapted to our setup. However, for instance, arguments using self-similarity need to be replaced by new ideas. Furthermore, new results concerning the change of measure are needed and may be of independent interest.

For example, as a byproduct of the change of measure techniques, we can improve Theorem 11 in [4], where the persistence problem of the one-sided discrete-time analog of FBM is considered. There it is shown that for
every real valued stationary centered Gaussian sequence \((\xi_n)_{n \in \mathbb{N}}\) such that (1) holds and every \(a > 0\), there is some constant \(c > 0\) such that

\[
e^{-N^{-1-H}} \frac{\sqrt{\ell(N)}}{\log(N)} \leq \mathbb{P}(S_n < 0 : 1 \leq n \leq N) \quad \text{and} \quad \mathbb{P}(S_n < -a : 1 \leq n \leq N) \leq cN^{-(1-H)} \sqrt{\ell(N)}.
\]

Thus, one has a lower bound for the probability \(\mathbb{P}(S_n < b : 1 \leq n \leq N)\), if \(b\) is non-negative, and an upper bound, if \(b\) is negative. In order to get both, a lower estimate and an upper estimate, for some arbitrary \(b \in \mathbb{R}\), [4] uses a change of measure argument. To get this argument to work, a strong assumption on the covariance function of \((S_n)\) is made; namely \(\inf_{n \geq 1} \mathbb{E}S_1S_n > 0\) (see also our Remark 9 below). We are able to prove upper and lower bounds whenever (3) is satisfied. We state this result as Corollary 8 below.

The outline of this paper is as follows. In Section 2, we collect some basic properties of the processes \((S_n)\) and \((I_n)\). Moreover, we present some results concerning the reproducing kernel Hilbert spaces of the considered processes that may be of independent interest. In Section 3, we give a proof of Theorem 1. Finally, in Section 4, we prove our main result, Theorem 2.

## 2 Preliminaries

Let \((W_H(t))\) be a FBM with Hurst parameter \(0 < H < 1\) and \((I_H(t))\) an IFBM. Then, unlike \((W_H(t))\), the process \((I_H(t))\) does not have stationary increments. Instead, the process satisfies for all \(t_0 \in \mathbb{R}\)

\[(I_H(t + t_0) - I_H(t_0) - tW_H(t_0))_{t \in \mathbb{R}} \overset{d}{=} (I_H(t))_{t \in \mathbb{R}}.
\]

In the discrete-time setup, we have analogous properties. From the definition of the process \((S_n)\), we straightforwardly obtain stationary increments

\[
(S_{n_0 + n} - S_{n_0})_{n \in \mathbb{Z}} \overset{d}{=} (S_n)_{n \in \mathbb{Z}} \quad \text{for all} \quad n_0 \in \mathbb{Z}.
\]

Also, it is easy to verify that we have

\[
(I_{n_0 + n} - I_{n_0} - n\tilde{S}_{n_0})_{n \in \mathbb{Z}} \overset{d}{=} (I_n)_{n \in \mathbb{Z}} \quad \text{for all} \quad n_0 \in \mathbb{Z},
\]

where \((\tilde{S}_n)_{n \in \mathbb{Z}}\) denotes the sequence given by \(\tilde{S}_n := \frac{S_n + S_{n-1}}{2}\).

Let us now recall the definition of the reproducing kernel Hilbert space (RKHS) of a centered Gaussian process \((X_t)_{t \in \mathbb{T}}\). For this purpose, let \(\mathbb{H}\) denote the \(L^2\)-closure of the set \(\text{span}\{X_t : t \in \mathbb{T}\}\). Then the RKHS \(\mathcal{H}\) of \((X_t)\) is the Hilbert space of functions

\[\mathbb{T} \ni t \mapsto \mathbb{E}[X_t h], \quad h \in \mathbb{H},\]

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with inner product $\langle \mathbb{E}[Xh_1], \mathbb{E}[Xh_2]\rangle_{\mathcal{H}} = \mathbb{E}[h_1h_2]$.

The following result, Proposition 1.6 in [3], will be an important tool throughout this work.

**Proposition 3.** Let $X$ be some centered Gaussian process with RKHS $\mathcal{H}$. Denote by $\| \cdot \|$ the norm in $\mathcal{H}$. Then, for each $f \in \mathcal{H}$ and each measurable $S$ such that $\mathbb{P}(X \in S) \in (0, 1)$, we have

$$e^{-\sqrt{2\|f\|^2 \log(1/\mathbb{P}(X \in S))} - \|f\|^2/2} \mathbb{P}(X \in S) \leq \mathbb{P}(X + f \in S).$$

(6)

If $\|f\|^2 < 2\log(1/\mathbb{P}(X \in S))$, we have in addition

$$\mathbb{P}(X + f \in S) \leq e^{\sqrt{2\|f\|^2 \log(1/\mathbb{P}(X \in S))} - \|f\|^2/2} \mathbb{P}(X \in S).$$

(7)

**Remark 4.** We want to mention that the proof of Proposition 1.6 in [3] fails if $\|f\|^2 \geq 2\log(1/\mathbb{P}(X \in S))$. Thus, unlike in [3], we have excluded this case here. In the applications of this proposition that we know of, the function $f \in \mathcal{H}$ is fixed and one is interested in the asymptotic behavior of the probabilities $\mathbb{P}(X \in S^{(N)})$ for $N \to \infty$, where $(S^{(N)})$ is a sequence of measurable sets such that $\lim_{N \to \infty} \mathbb{P}(X \in S^{(N)}) = 0$. In this case the condition is satisfied for $N$ large enough. Hence, Proposition 1.6 in [3] can be applied in the same way as before.

First, we show the existence of a function in the RKHS of $(\xi_n)_{n \in \mathbb{Z}}$ with certain asymptotic behavior.

**Proposition 5.** Let $H \in (0, 1)$, $\rho \in (-1, H - 1)$ and let $\mathcal{H}_H(\xi)$ denote the RKHS of the process $(\xi_n)_{n \in \mathbb{Z}}$. Then, if (3) is satisfied, there is an even function $h \in \mathcal{H}_H(\xi)$ such that $h > 0$ and $h(n) \sim n^\rho$.

**Proof.** Recall that $h \in \mathcal{H}_H(\xi)$ if and only if there is a function $\varphi \in L^2(\mu)$ with $h(n) = \int_{(-\pi, \pi]} \varphi(u)e^{-inu}d\mu(u)$, see e.g. Comment 2.2.2 (c) in [1]. In order to prove the Proposition, we will first consider a function $\varphi_1 \in L^2(\mu)$ such that the corresponding function $h_1 \in \mathcal{H}_H(\xi)$ has the correct asymptotic behavior. This function can attain non-positive values at finitely many times. To fix this, we will construct afterwards another function $\varphi_2 \in L^2(\mu)$ such that the corresponding function $h_2 \in \mathcal{H}_H(\xi)$ is non-negative, takes positive values when $h_1$ takes non-positive values and decays faster than $h_1$. Then, for suitable constants $c_1, c_2 > 0$, the function $h = c_1h_1 + c_2h_2$ has the required properties.

**Construction of $h_1$:** Due to (3), there is a function $\tilde{\ell}$ and a constant $u_0 > 0$ such that $p_u(u) = \tilde{\ell}(u)|u|^{1-2H}$ for $u \in [-u_0, u_0]$ and $\tilde{\ell}$ is slowly varying at zero. By Potter’s theorem, see Theorem 1.5.6 in [6], $u_0$ can be chosen such that $\tilde{\ell}(u_0)/\tilde{\ell}(u) \leq A \left(\frac{|u|}{u_0}\right)^{-\delta}$ for $|u| < u_0$, fixed $A > 1$ and fixed
\[ 0 < \delta < 2(H - 1 - \rho). \] We set
\[
\varphi_1(u) := \begin{cases} 
|u|^{2H-2-\rho}/\ell(u), & u \in [-u_0, u_0] \cap \text{supp}(\mu_*)^C, \\
0, & \text{otherwise}.
\end{cases}
\]

Then, \( \varphi_1 \in L^2(\mu) \) because
\[
\int_{(-\pi, \pi]} |\varphi_1(u)|^2 \, d\mu(u) = \int_{-u_0}^{u_0} \frac{|u|^{2H-3-2\rho}}{\ell(u)} \, du \\
\leq \frac{A}{\ell(u_0)} \int_{-u_0}^{u_0} |u|^{2H-3-2\rho} \left( \frac{|u|}{u_0} \right)^{-\delta} \, du < \infty.
\]
Here we used that \( 2H - 3 - 2\rho - \delta > -1 \). Moreover,
\[
\int_{(-\pi, \pi]} \cos(nu) \varphi_1(u) \, d\mu(u) = \int_{-u_0}^{u_0} \cos(nu)|u|^{-\rho-1} \, du \\
= n^{\rho} \int_{-nu_0}^{nu_0} \cos(v)|v|^{-\rho-1} \, dv \\
= 2n^{\rho} \int_{0}^{nu_0} \cos(v)|v|^{-\rho-1} \, dv.
\]
Since \(-\rho - 1 < 0\), it is easy to show, using the Leibniz criterion and the concavity of \((-\cdot)^{-\rho-1}\), that the latter integral converges to a constant \( c/2 > 0 \). Thus,
\[
h_1(n) = \int_{(-\pi, \pi]} \varphi_1(u)e^{-inu} \, d\mu(u) \sim cn^\rho.
\]

Construction of \( h_2 \): Choose \( n_0 \) such that \( h_1 \) attains only positive values for \(|n| > n_0\). Let \( g \in C^1 \) be an even real-valued function with support contained in \([-u_0/2, u_0/2]\) such that the Fourier coefficients for \(|n| \leq n_0\) do not vanish, e.g. take any smooth even function \( g \) with \( g(u) > 0 \) for \(|u| < \min(u_0/2, \pi/(2n_0)) \) and \( g(u) = 0 \) otherwise. Then, the function \( f \) given by \( f(u) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v)g(u-v) \, dv \) has Fourier coefficients \( \hat{f}_n = |\hat{g}_n|^2 \). In particular \( \hat{f}_n > 0 \) for \(|n| \leq n_0 \). Moreover, \( f \in C^2 \) because \( f \) is a convolution of two differentiable functions. Thus, we have
\[
0 \leq \hat{f}_n = \frac{1}{(in)^2} \hat{(f''u)}_n \leq \sup_{x \in (-\pi, \pi]} |f''(x)| \frac{1}{|n|^2} \quad \text{for} \quad n \in \mathbb{Z} \setminus \{0\}.
\]

Now, we consider the function
\[
\varphi_2(u) := \begin{cases} 
\frac{f(u)}{|u|^{2H} \ell(u)}, & u \in [-u_0, u_0] \cap \text{supp}(\mu_*)^C, \\
0, & \text{otherwise}.
\end{cases}
\]
Let $M$ denote the maximum of $f$, then
\[
\int_{(-\pi, \pi]} |\varphi_2(u)|^2 \, d\mu(u) \leq \int_{-u_0}^{u_0} \frac{M^2}{|u|^{1-2H} \ell(u)} \, du \\
\leq \frac{A}{\ell(u_0)} \int_{-u_0}^{u_0} \frac{M^2}{|u|^{1-2H}} \left( \frac{|u|}{u_0} \right)^{-\delta} \, du < \infty,
\]

since $2H - 1 - \delta > -1$. Furthermore, we have by construction of $\varphi_2$
\[
b_2(n) = \int_{(-\pi, \pi]} \varphi_2(u)e^{-iu} \, d\mu(u) = \int_{-\pi}^{\pi} f(u)e^{-iu} \, du = \tilde{f}_n.
\]

As a corollary of Proposition 5 we show the existence of functions with certain asymptotic behavior in the RKHSs of $(S_n)_{n \in \mathbb{Z}}$ and $(I_n)_{n \in \mathbb{Z}}$, respectively.

**Corollary 6.** Let $H \in (0, 1)$, $\rho \in (-1, H - 1)$ and let $\mathcal{H}_H(S)$ and $\mathcal{H}_H(I)$ denote the RKHS of the processes $(S_n)_{n \in \mathbb{Z}}$ and $(I_n)_{n \in \mathbb{Z}}$, respectively. Then, if (3) is satisfied, there are functions $f \in \mathcal{H}_H(S)$, $g \in \mathcal{H}_H(I)$ such that $f$ is odd with $f(n) > 0$ for $n > 0$ and $f(n) \sim n^{\rho+1}$ as $n \to \infty$ whereas $g$ is even and positive on $\mathbb{Z} \setminus \{0\}$ with $g(n) \sim n^{\rho+2}$ as $n \to \infty$.

**Proof.** Let $h \in \mathcal{H}_H(\xi)$ be the positive and even function in Proposition 5 with $h(n) \sim n^\rho$. Then, by the definition of the RKHS, there is a random variable $X$ in the $L^2$-closure of the set $\text{span}\{\xi_n : n \in \mathbb{Z}\}$ with $h(n) = \mathbb{E}[\xi_nX]$. Now, let the functions $f, g$ be given by $f(n) = (\rho + 1)\mathbb{E}[S_nX]$ and $g(n) = (\rho + 1)(\rho + 2)\mathbb{E}[I_nX]$, respectively. Since the sets $\text{span}\{\xi_n : n \in \mathbb{Z}\}$, $\text{span}\{S_n : n \in \mathbb{Z}\}$ and $\text{span}\{I_n : n \in \mathbb{Z}\}$ coincide, we have $f \in \mathcal{H}_H(S)$ and $g \in \mathcal{H}_H(I)$. By $h(n) \sim n^\rho$ and the symmetry of $h$, we have $-f(-n) = f(n) = (\rho + 1)\sum_{k=1}^{n} h(k) \sim n^{\rho+1}$ as $n \to \infty$. Thus, we have further $g(-n) = g(n) = (\rho + 1)(\rho + 2)\sum_{k=1}^{n-1} \mathbb{E}[S_kX] + \mathbb{E}[S_nX]/2 \sim n^{\rho+2}$ as $n \to \infty$.

As a first application of Corollary 6 we compare the persistence probabilities of $(I_n)$ to a closely related process. Let $(\tilde{I}_n)_{n \in \mathbb{Z}}$ be the sequence given by $\tilde{I}_n - I_{n-1} := S_n$ for $n \in \mathbb{Z}$ and $\tilde{I}_0 := 0$. This process is related to the process $(I_n)$ by the identity $\tilde{I}_n = I_n + S_n/2$. Both processes are defined as integrals of stationary increments sequences that have FBM as scaling limit. In the context of this paper, the major difference between these processes is that $(I_n)$ vanishes only at 0 and satisfies $(I_n) \overset{d}{=} (I_{-n})$ whereas $\tilde{I}_n - I_0 = 0$ and $\tilde{I}_1$ does not vanish. The symmetry property of $(I_n)$ resembles the continuous-time case and is needed in the proof of Theorem 2. In the following corollary, we relate the persistence probabilities of both processes.
Corollary 7. Let $(\xi_n)$ be a real valued stationary centered Gaussian sequence such that (3) holds. Then,

$$\mathbb{P}(\bar{I}_n \leq 0 : -N-1 \leq n \leq N) \leq \mathbb{P}(I_n \leq 0 : |n| \leq N).$$

If in addition $\mathbb{E}[\bar{I}_n I_m] \geq 0$ for all $n, m \in \mathbb{Z}$, then one has

$$\mathbb{P}(I_n \leq 0 : |n| \leq N) \leq \mathbb{P}(\bar{I}_n \leq 0 : |n| \leq N) \ell_0(N),$$

where $\ell_0$ denotes a slowly varying function at infinity.

Proof. The first inequality follows directly from the definitions of the processes, since one has $I_n = (\bar{I}_n + \bar{I}_{n-1})/2$ for all $n \in \mathbb{Z}$. Using Slepian’s Lemma and the additional assumption about the correlations of $(\bar{I}_n)$, we obtain

$$\mathbb{P}(\bar{I}_n \leq 0 : |n| \leq N) \geq \mathbb{P}(\bar{I}_n \leq 0 : |n| \leq \log(N)) \cdot \mathbb{P}(\bar{I}_n \leq 0 : \log(N) < |n| \leq N). \quad (8)$$

By the same argument and Theorem 1, we have

$$\mathbb{P}(\bar{I}_n \leq 0 : |n| \leq \log(N)) \geq \mathbb{P}(\bar{I}_n \leq 0 : 0 \leq n \leq \log(N)) \cdot \mathbb{P}(\bar{I}_n \leq 0 : -\log(N) \leq n < 0) \geq \mathbb{P}(S_n \leq 0 : 0 \leq n \leq \log(N)) \cdot \mathbb{P}(S_n \geq 0 : -\log(N) \leq n < 0) \geq c^{-2} \log(N)^{-2}.$$

Thus, the first factor on the right hand side in (8) can be estimated by a slowly varying function at infinity. It remains to relate the second factor on the right hand side in (8) to the probability $\mathbb{P}(I_n \leq 0 : |n| \leq N)$.

By Corollary 6 for $\varepsilon \in (0, 1/4)$, there is a symmetric function $f \in \mathcal{H}_H(I)$ such that $f(n) \geq |n|^{1+H-\varepsilon}$ for all $n \in \mathbb{Z}$. Obviously, we have

$$\mathbb{P}(I_n \leq -n^{1+H-\varepsilon} : \log(N) < |n| \leq N) \leq \mathbb{P}(\bar{I}_n \leq 0 : \log(N) < |n| \leq N) + \mathbb{P}(\exists n : \bar{I}_n - I_n > n^{1+H-\varepsilon}, \log(N) < |n| \leq N). \quad (9)$$

We will see that the second term on the right hand side is of lower order, while the term on the left hand side can be related to $\mathbb{P}(I_n \leq 0 : |n| \leq N)$. For this purpose, let $X$ denote a standard normal random variable. Then, by using $\bar{I}_n - I_n = S_n/2$ in the first step and (1) in the second step, we have
for $N$ large enough
\[
\mathbb{P} \left( \exists n : \bar{I}_n - I_n > n^{1+H-\varepsilon}, \log(N) < |n| \leq N \right) \\
\leq 2 \sum_{n=\lceil \log(N) \rceil}^{N} \mathbb{P}\left( S_n/2 > n^{1+H-\varepsilon} \right) \\
\leq 2 \sum_{n=\lceil \log(N) \rceil}^{N} \mathbb{P}\left( n^{H+\varepsilon} X > n^{1+H-\varepsilon} \right) \\
\leq 2N \mathbb{P}\left( X \geq \log(N)^{1-2\varepsilon} \right) \\
\leq 2Ne^{-\log(N)^2/2} \\
\leq 2N^{-2}. \tag{10}
\]

In the fourth step above, we used the standard estimate $\mathbb{P}(X > x) \leq e^{-x^2/2}$ for $x \geq 1$. Finally, using Proposition 3, we obtain for $N$ large enough
\[
\mathbb{P}(I_n \leq 0 : |n| \leq N) \leq \mathbb{P}(I_n \leq 0 : \log(N) < |n| \leq N) \\
\leq \mathbb{P}(I_n \leq -f(n) : \log(N) < |n| \leq N) \\
\cdot e^{\sqrt{2\|f\|^2 \log(1/\mathbb{P}(I_n \leq 0 : |n| \leq N))}-\|f\|^2/2} \\
\leq \mathbb{P}(I_n \leq -n^{1+H-\varepsilon} : \log(N) < |n| \leq N) \\
\cdot e^{\sqrt{2\|f\|^2 \log(1/\mathbb{P}(I_n \leq 0 : |n| \leq N))}-\|f\|^2/2}.
\]

This, together with (9), (10) and Theorem 2 finishes the proof. \qed

As another application of Corollary 6, we can give an improvement of Theorem 11 in [4]:

**Corollary 8.** Let $(\xi_n)$ be a real valued stationary centered Gaussian sequence such that (3) holds. Then, for every $b \in \mathbb{R}$ there is some constant $c > 0$ such that
\[
N^{-(1-H)} \sqrt{\ell(N)} e^{-c\sqrt{\log(N)}} \leq \mathbb{P}\left( \max_{1 \leq n \leq N} S_n \leq b \right) \\
\leq N^{-(1-H)} \sqrt{\ell(N)} e^{c\sqrt{\log(N)}} \forall N \in \mathbb{N}.
\]

**Proof.** Let $a > 0$. By Corollary 6 there is a function $f \in \mathcal{H}_H(S)$ with $f(n) \geq 2a$ for all $n \geq 1$. Further, using the lower estimate in [4], we have for $N$ large enough
\[
N^{-1} \leq \mathbb{P}(S_n \leq a : 1 \leq n \leq N).
\]

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This together with Proposition 3 yields for $N$ large enough

$$
\mathbb{P}(S_n \leq -a : 1 \leq n \leq N) = \mathbb{P}(S_n + f(n) \leq -a + f(n) : 1 \leq n \leq N) \\
\geq \mathbb{P}(S_n + f(n) \leq a : 1 \leq n \leq N) \\
\geq \mathbb{P}(S_n \leq a : 1 \leq n \leq N) e^{-\sqrt{2\|f\|^2\log(N)} - \|f\|^2/2}.
$$

Combining this with (4) finishes the proof.

Remark 9. In Theorem 11 in [4], the authors assume $\inf_{n \geq 1} \mathbb{E}S_nS_1 > 0$ to get the change of measure argument to work. For instance, the fractional Gaussian noise process $(\xi_{\text{FGN}})$ satisfies this assumption. This can be easily verified by using that $\mathbb{E}S_n^2 = n^{2H}$. In general, this does not remain true if one only has (1). For example, consider the case where $\ell(x) = 1 + \cos(\pi x)/\log(x)$ in (1). Then, one has $\sum_{j=1}^{n} \sum_{k=1}^{n} \mathbb{E}\xi_j\xi_k \sim n^{2H}$ but the function $\mathbb{E}S_nS_1$ attains infinitely often positive and negative values.

Remark 10. Consider the function $f : \mathbb{N} \to \mathbb{R}$ with $f(n) = 1_{n=1}$. Clearly, $f$ is in the RKHS of the process $(\xi_n)_{n \geq 1}$ if and only if $\xi_1 \notin H_2$, where $H_2$ denotes the $L^2$-closure of the set $\text{span}\{\xi_n : n \geq 2\}$. It is well known that this condition is equivalent to the Kolmogorov condition

$$
\int_{-\pi}^{\pi} \log(p(u)) \, du > -\infty,
$$

where $p$ denotes the density of the component of the spectral measure of $(\xi_n)$ that is absolutely continuous with respect to the Lebesgue measure, see e.g. Theorem 2.5.4 in [1]. In this case, all constant functions are in the RKHS of the process $(S_n)_{n \geq 1}$. Hence, the proof of Corollary 3 still works if we replace condition (3) by (1) and (11).

3 Proof of Theorem 1

Upper bound

Let $T_N$ denote the time where the process $(S_n)_{n \in \mathbb{Z}}$ attains its maximum on $\{0, 1, \ldots, N\}$. Since $(S_n)_{n \in \mathbb{Z}}$ has stationary increments and $\mathbb{P}(S_j = S_k) = 0$
for \( j \neq k \), the upper bound follows from

\[
N \cdot \mathbb{P}(S_n \leq 0 : -N \leq n \leq N) \leq \sum_{k=1}^{N} \mathbb{P}(S_n \leq 0 : -k \leq n \leq N - k)
\]
\[
= \sum_{k=1}^{N} \mathbb{P}(S_n \leq S_k : 0 \leq n \leq N)
\]
\[
= \sum_{k=1}^{N} \mathbb{P}(T_N = k)
\]
\[
\leq 1.
\]

**Lower bound**

Using again the stationary increments of \((S_n)_{n \in \mathbb{Z}}\), we obtain

\[
(N + 1) \cdot \mathbb{P}(S_n \leq 0 : -N \leq n \leq N)
\]
\[
\geq \sum_{k=0}^{N} \mathbb{P}(S_n \leq 0 : -N - k \leq n \leq 2N - k)
\]
\[
= \sum_{k=0}^{N} \mathbb{P}(S_n \leq S_{N+k} : 0 \leq n \leq 3N) \quad (12)
\]
\[
= \sum_{k=0}^{N} \mathbb{P}(T_{3N} = N + k)
\]
\[
= \mathbb{P}(T_{3N} \in [N, 2N]).
\]

Now, we consider the continuous functional \( F: (D([0, 1]), \| \cdot \|_{\infty}) \to (\mathbb{R}, | \cdot |) \) given by

\[
F(g) = \left( \sup_{x \in \left(\frac{1}{3}, \frac{2}{3}\right)} g(x) - \sup_{x \in \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right)} g(x) \right) \wedge 1,
\]
where \((x)_{+} := \max(x, 0)\) for \( x \in \mathbb{R} \) and \( D([0, 1]) \) denotes the set of all càdlàg functions on \([0, 1]\). We set

\[
Y_N(t) = \frac{1}{N^{\frac{1}{2}}} \sum_{k=1}^{[N t]} \xi_k.
\]

Due to (12), it follows that

\[
\mathbb{P}(T_{3N} \in [N, 2N]) = \mathbb{E} \left[ \mathbb{I}_{T_{3N} \in [N, 2N]} \right] \geq \mathbb{E} F(Y_N) \to c_0 > 0, \quad \text{as } N \to \infty.
\]

This and (12) show the lower bound.
4 Proof of Theorem 2

The proof is structured as follows: We first consider the functional

$$ F_N := \sum_{k=1}^{N-1} \left( \gamma_{k,k}^--\gamma_{k,N-k}^+ \right)_+ , $$

where for $k \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$ \gamma_{k,m}^- := \min_{1 \leq n \leq m} \frac{I_k - I_{k-n}}{n} \quad \text{and} \quad \gamma_{k,m}^+ := \max_{1 \leq n \leq m} \frac{I_{k+n} - I_k}{n} , $$

and determine the polynomial order of $\mathbb{E}F_N$ as $N \to \infty$. Then, we relate the quantity $\mathbb{E}F_N$ to the probability

$$ \tilde{p}_N := \mathbb{P}(I_n + |n| \leq 0, |n| \leq N) . \quad (13) $$

Finally, we obtain the asymptotic order of

$$ p_N := \mathbb{P}(I_n \leq 0 : |n| \leq N) \quad (14) $$

from (13) by using a change of measure argument (Proposition 3 and Corollary 6).

Upper bound for $\mathbb{E}F_N$

In the following, we fix $N$ and write $\gamma_k^- = \gamma_{k,k}$ and $\gamma_k^+ = \gamma_{k,N-k}$ to ease notation. Let $C_N: [0, N] \to \mathbb{R}$ denote the concave majorant of $I_n$ on $[0, N]$, i.e., $C_N$ is the smallest concave function with $I_n \leq C_N(n)$. Obviously, $C_N$ is a piecewise linear function and we denote by $\{k_1, k_2, \ldots \}$ (depending on $N$) its nodal points. At these points the slope on the left is $\gamma_{k_i}$ and the slope on the right is $\gamma_{k_i}^+$. Further, we note that $\gamma_k^- - \gamma_k^+ \geq 0$ if and only if $k$ is a nodal point of $C_N$. In that case one has $\gamma_k^+ = \gamma_{k+1}^-$. Thus,

$$ F_N = \sum_{k=1}^{N-1} (\gamma_k^- - \gamma_k^+) = \sum_{i} (\gamma_{k_i}^- - \gamma_{k_{i+1}}^-) = \gamma_0^+ - \gamma_N^- . $$

By $\mathbb{E}\tilde{S}_N = 0$, (5) and $(I_n) \overset{d}{=} (I_{-n})$, we have

$$ \mathbb{E}[-\gamma_N^+] = \mathbb{E} \left[ - \min_{1 \leq n \leq N} \frac{I_N - I_{N-n}}{n} \right] \\
= \mathbb{E} \left[ \max_{1 \leq n \leq N} \frac{I_{N-n} - I_N - (-n)\tilde{S}_N}{n} \right] \\
= \mathbb{E} \left[ \max_{1 \leq n \leq N} \frac{I_{-n}}{n} \right] = \mathbb{E} \left[ \max_{1 \leq n \leq N} \frac{I_n}{n} \right] = \mathbb{E}\gamma_0^+ . $$

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Therefore,  
\[ EF_N = 2E\gamma_0^+. \]  
(15)

Due to (15), one obtains the upper estimate

\[
EF_N = 2E \left[ \max_{1 \leq n \leq N} \frac{\sum_{k=1}^n \tilde{S}_k}{n} \right] 
\leq 2E \left[ \max_{1 \leq n \leq N} \frac{\sum_{k=1}^n \max_{1 \leq j \leq N} \tilde{S}_j}{n} \right] = 2E \left[ \max_{1 \leq j \leq N} \tilde{S}_j \right].
\]

It can be obtained from (2) that

\[
\frac{1}{N^{H/\ell}(N)^{1/2}} E \left[ \max_{1 \leq n \leq N} \tilde{S}_n \right] \to E \left[ \sup_{t \in [0,1]} W_H(t) \right] \in (0, \infty),
\]

where \((W_H(t))\) is a fractional Brownian motion, see e.g. proof of Theorem 11 in [4]. Thus, there is a constant \(c\) such that for all \(N\)

\[
EF_N \leq c \ell(N)^{1/2} N^H. 
\]

(16)

In the following, \(c\) will denote a varying positive constant independent of \(N\) for ease of notation.

**Lower bound for \(EF_N\)**

Since \((\xi_n)\) is a stationary process, we have

\[
E S_j S_k = \frac{1}{2} \left( ES_j^2 + ES_k^2 - ES_{|j-k|}^2 \right).
\]

Consequently,

\[
E \left( I_N + \frac{S_N}{2} \right)^2 = \sum_{j=1}^N \sum_{k=1}^N ES_j S_k = \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N \left( ES_j^2 + ES_k^2 - ES_{|j-k|}^2 \right).
\]

Counting how often \(ES_k^2\) is added, yields

\[
E \left( I_N + \frac{S_N}{2} \right)^2 = \frac{1}{2} \sum_{k=1}^N N\mathbb{E}S_k^2 + \frac{1}{2} \sum_{k=1}^N N\mathbb{E}S_k^2 - \sum_{k=1}^N (N-k)\mathbb{E}S_k^2 
\]

\[
= \sum_{k=1}^N k\mathbb{E}S_k^2.
\]

Since \(ES_n^2 \sim n^{2H} \ell(n)\), we can apply Proposition 1.5.8 in [6] to obtain

\[
E \left( I_N + \frac{S_N}{2} \right)^2 \sim N^{2H+2} \ell(N)/(2H+2). 
\]

(17)
Now, using the Cauchy-Schwarz Inequality, we have $|\mathbb{E} S_N I_N| \leq \sqrt{\mathbb{E} S_N^2} \mathbb{E} I_N^2$ and we can thus conclude from (17) that

$$\mathbb{E} I_N^2 \sim N^{2H+2} \ell(N)/(2H + 2). \quad (18)$$

In the following, we let $\| \cdot \|_2$ denote the norm $\| X \|_2 = \mathbb{E} [ |X|^2 ]^{1/2}$. Moreover, we recall the identity $\mathbb{E} X_+ = (2\pi)^{-1/2} \| X \|_2$ for a centered normal random variable $X$. Now, we can give a lower bound for $\mathbb{E} F_N$. By (15) and $\mathbb{E} I_1 = 0$, we have

$$\mathbb{E} F_N = 2\mathbb{E} \left[ \max_{1 \leq n \leq N} I_n \right] = 2\mathbb{E} \left[ \max_{1 \leq n \leq N} \frac{I_n}{n} - I_1 \right]$$

$$= 2\mathbb{E} \left( \max_{1 \leq n \leq N} \frac{I_n}{n} - I_1 \right) \geq 2\mathbb{E} \left( \frac{I_N}{N} - I_1 \right) + \frac{\sqrt{2/\pi}}{\| I_N/N - I_1 \|_2}.$$

Thus, by (18), we have

$$\mathbb{E} F_N \geq c^{-1} \ell(N)^{1/2} N^H. \quad (19)$$

**Upper bound for $\tilde{p}_N$**

In order to get an upper bound for the probability in (13), it is convenient to consider the random variable

$$\vartheta_N := \left( \gamma_{-0,N} - \gamma_{0,N}^+ \right).$$

We have

$$\mathbb{E} (\gamma_k^+ - \gamma_k^-)_+ \geq \mathbb{E} \vartheta_N. \quad (20)$$

To see this, note that by using (4), we obtain

$$\gamma_{k,N}^- - \gamma_{k,N}^+ = \min_{1 \leq n \leq N} \frac{I_k - n \tilde{S}_k}{n} - \min_{1 \leq n \leq N} \frac{I_k - n \tilde{S}_k}{n} = \gamma_{0,N}^+ - \gamma_{0,N}^-.$$
Using (16), (20), and (21), we thus obtain
\[ c \ell(N)^{1/2} N^H \geq \mathbb{E} F_N = \sum_{k=1}^{N-1} \mathbb{E} \left( \gamma_k^- - \gamma_k^+ \right)_+ \]
\[ \geq (N-1) \mathbb{E} \vartheta_N \geq 2(N-1) \mathbb{P}(\vartheta_N \geq 2) \]
\[ \geq 2(N-1) \mathbb{P} \left( \gamma_{0,N}^- \geq 1, \gamma_{0,N}^+ \leq -1 \right) \]
\[ = 2(N-1) \mathbb{P} (I_n + |n| \leq 0 : |n| \leq N). \]

Hence, we have for any \( N \)
\[ \mathbb{P}(I_n + |n| \leq 0 : |n| \leq N) \leq c \ell(N)^{1/2} N^{-(1-H)}. \] (23)

**Lower bound for \( \tilde{p}_N \)**

Along the lines of the proof of (20), one gets an analogous estimate when replacing \( N \) by \( \tilde{k} := \min(k, N - k) \); namely
\[ \mathbb{E} \left( \gamma_k^- - \gamma_k^+ \right)_+ \leq \mathbb{E} \vartheta_k. \] (24)

Now, let \( 1 - H < \alpha < 1 \). Then, we have by the monotonicity of \( \vartheta_N \) for \( N^\alpha \leq k \leq N - N^\alpha \)
\[ \mathbb{E} \vartheta_k \leq \mathbb{E} \vartheta_{[N^\alpha]} \]. (25)

Thus, by using (24) and (25), we obtain
\[ \mathbb{E} F_N = \sum_{k=1}^{N-1} \mathbb{E} \left( \gamma_k^- - \gamma_k^+ \right)_+ \]
\[ \leq (N - 2 \lfloor N^\alpha \rfloor) \mathbb{E} \vartheta_{\lfloor N^\alpha \rfloor} + 2 \sum_{k=1}^{\lfloor N^\alpha \rfloor} \mathbb{E} \vartheta_k. \]

Moreover, we know from (22), that we have for all \( k \)
\[ \mathbb{E} \vartheta_k \leq c \ell(k)^{1/2} k^{H-1}. \] (26)

Hence, by (26) and Proposition 1.5.8 in [6], we obtain
\[ \sum_{k=1}^{\lfloor N^\alpha \rfloor} \mathbb{E} \vartheta_k \leq c \ell(\lfloor N^\alpha \rfloor)^{1/2} N^\alpha H. \]

Thus, we have
\[ c^{-1} \ell(N)^{1/2} N^H \leq \mathbb{E} F_N \leq N \mathbb{E} \vartheta_{\lfloor N^\alpha \rfloor} + c \ell(\lfloor N^\alpha \rfloor)^{1/2} N^\alpha H. \]
Since $\alpha H < 1$, we obtain
\[ e^{-1}\ell(N)^{1/2} N^{H-1} \leq \mathbb{E} \vartheta_{\lfloor N^\alpha \rfloor}. \]
Replacing $N$ by $\lceil N^{1/\alpha} \rceil$ yields
\[ \ell_1(N) N^{-(1-H)/\alpha} \leq \mathbb{E} \vartheta_N, \]
where $\ell_1$ is a slowly varying function at infinity.

Fix $q > 1$ to be chosen later and let $\| \cdot \|_q$ denote the norm $\mathbb{E} \| X \|^q / q$ for some random variable $X$. Then, using $\vartheta_N \leq \vartheta_1$ and Hölder’s Inequality, we have
\[ \mathbb{E} \vartheta_N = \mathbb{E} \vartheta_N \mathbb{1}_{\vartheta_N > 0} \leq \| \vartheta_1 \|_q \mathbb{P} (\vartheta_N > 0)^{1-1/q}. \]
Further,
\[ \| \vartheta_1 \|_q \leq \| I_{-1} - I_1 \|_q \leq c \sqrt{q}, \]
using that $I_{-1} - I_1$ is a Gaussian random variable. So, we have
\[ \frac{\ell_1(N)}{c \sqrt{q}} N^{-(1-H)/\alpha} \leq \mathbb{P} (\vartheta_N > 0)^{1-1/q}. \]
Now, setting $q := \log(N) + 1$ yields
\[ \ell_2(N) N^{-(1-H)/\alpha} \leq \mathbb{P} (\vartheta_N > 0), \tag{27} \]
where $\ell_2$ is a slowly varying function at infinity.

In the following, we will relate the probability $\mathbb{P} (\vartheta_N > 0)$ to the probability in (13). This is divided into four steps.

Step 1: We start with a change of measure argument. By Corollary 6, we can find a function $f \in \mathcal{H}_H(I)$ such that $f(n) \geq \frac{3}{2} |n|$ for all $n \in \mathbb{Z}$. Then, using (5), we obtain
\[
-\sqrt{2} \| f \|^2 \log(1/\mathbb{P}(\vartheta_N > 0)) - \| f \|^2/2 \mathbb{P} (\vartheta_N > 0)
\]
\[
\leq \mathbb{P} \left( \min_{-N \leq n \leq -1} \frac{I_n + f(n)}{n} - \max_{1 \leq n \leq N} \frac{I_n + f(n)}{n} > 0 \right)
\]
\[
\leq \mathbb{P} \left( \min_{-N \leq n \leq -1} \frac{I_n + \frac{3}{2} |n|}{n} - \max_{1 \leq n \leq N} \frac{I_n + \frac{3}{2} |n|}{n} > 0 \right)
\]
\[
= \mathbb{P} \left( \min_{-N \leq n \leq -1} \frac{I_n}{n} - \max_{1 \leq n \leq N} \frac{I_n}{n} > 3 \right)
\]
\[
= \mathbb{P} (\vartheta_N > 3). \tag{28}
\]
So, by (27), $\mathbb{P} (\vartheta_N > 0)$ and $\mathbb{P} (\vartheta_N > 3)$ differ by less than a slowly varying function at infinity.

Step 2: Let
\[ A_0^{(N)} := \{(x_{-N}, \ldots, x_{-1}, x_1, \ldots, x_N) \in \mathbb{R}^{2N} : x_n \leq -|n|, 1 \leq |n| \leq N\}, \]
In inequality. It is well known that for any convex subsets we make use of an argument that is commonly used to prove Anderson’s. Here, we used that one has 

\[ \text{is a centered Gaussian random variable, we can choose a constant } \lambda \] 

get. Altogether we thus obtain

\[ \text{where } \mu \] 

In the following, we write \( I \in A_m^{(N)} \) instead of \((I_{-N}, \ldots, I_{-1}, I_1, \ldots, I_N) \in A_m^{(N)} \) for ease of notation. We will show that \( \{ \vartheta_N > 3 \} \subseteq \cup_{m \in \mathbb{Z}} \{ I \in A_m^{(N)} \}. \) For this purpose, let \( m^{(N)} \) be an integer-valued random variable such that \( \min_{-N \leq n \leq -1} \frac{I_n}{n} \in [m^{(N)} + 1, m^{(N)} + 2) \). Then, we obviously have \( I_n \leq (m^{(N)} + 1)n \) for \(-N \leq n \leq -1 \). Furthermore, assuming \( \vartheta_N > 3 \), we can conclude that

\[ \max_{1 \leq n \leq N} \frac{I_n}{n} - \min_{-N \leq n \leq -1} \frac{I_n}{n} - 3 < m^{(N)} - 1. \]

Step 3: We show that \( P(I \in A_0^{(N)}) \geq P(I \in A_m^{(N)}) \). For this purpose, we make use of an argument that is commonly used to prove Anderson’s Inequality. It is well known that for any convex subsets \( A, B \subseteq \mathbb{R}^{2N} \) and \( 0 < \lambda < 1 \), one has

\[ \mu (\lambda A + (1 - \lambda)B) \geq \mu (A)^{\lambda} \mu (B)^{1-\lambda}, \]

where \( \mu \) is a centered Gaussian measure on \( \mathbb{R}^{2N} \), see e.g. Theorem 2 in [15]. Since \((I_{-N}, \ldots, I_{-1}, I_1, \ldots, I_N) \) is a centered Gaussian random variable, by setting \( \lambda = \frac{1}{2} \), we obtain

\[ P(I \in A_0^{(N)}) = P(I \in \frac{1}{2} A_{-m}^{(N)} + \frac{1}{2} A_m^{(N)}) \geq P(I \in A_{-m}^{(N)})^{1/2} P(I \in A_m^{(N)})^{1/2} = P(I \in A_m^{(N)}). \]

Here, we used that one has \( A_0^{(N)} = \frac{1}{2} A_{-m}^{(N)} + \frac{1}{2} A_m^{(N)} \) and, by symmetry of the process \((I_n)\), \( P(I \in A_{-m}^{(N)}) = P(I \in A_{m}^{(N)}) \).

Step 4: Now, we relate the quantities \( P(\vartheta_N > 3) \) and \( \tilde{p}_N \). Since \( I_{-1} \) is a centered Gaussian random variable, we can choose a constant \( c_0 \) such that \( P(I_{-1} \leq -(a_N + 1)) \in o(N^{-1}) \) for \( a_N = \sqrt{c_0 \log(N)} \). Further, by \( P(\cup_{m \geq a_N} A_m) \leq P(I_{-1} \leq -(a_N + 1)) \) and symmetry of the process \((I_n)\), we get

\[ P(\cup_{|m| \geq a_N} A_m) \in o(N^{-1}). \]

Altogether we thus obtain

\[ P(\vartheta_N > 3) \leq P(\cup_{m \in \mathbb{Z}} A_m) \leq \sum_{|m| < a_N} P(A_m) + P(\cup_{|m| \geq a_N} A_m) \leq 2a_N P(A_0) + 2P(I_{-1} \leq -(a_N + 1)) = 2a_N P(I_n + |n| \leq 0 : |n| \leq N) + o(N^{-1}). \]
Putting this together with (27) and (28), we get
\[ \ell_3(N)N^{-(1-H)/\alpha} \leq \mathbb{P}(I_n + |n| \leq 0 : |n| \leq N), \]
where \( \ell_3 \) denotes a slowly varying function at infinity.

**Polynomial rate of \( p_N \)**

Clearly, we have from (29)
\[ \ell_3(N)N^{-(1-H)/\alpha} \leq \mathbb{P}(I_n \leq 0 : |n| \leq N) \leq \mathbb{P}(I_n \leq 0 : |n| \leq N) = p_N. \]

In particular, \( p_N \geq c^{-1}N^{-1} \) for some suitable constant \( c \). This estimate will be used in the following change of measure argument. Due to Corollary 6, we can choose a function \( f \in \mathcal{H}_H(I) \) with \( f(n) \geq |n| \) for all \( n \in \mathbb{Z} \). Then, by (29) and Proposition 4, we obtain
\[ c \ell(N)^{1/2}N^{-(1-H)} \geq \mathbb{P}(I_n + |n| \leq 0 : |n| \leq N) \geq \mathbb{P}(I_n + f(n) \leq 0 : |n| \leq N) \geq \mathbb{P}(I_n \leq 0 : |n| \leq N) e^{-\sqrt{2\|f\|^2 \log(1/p_N)} - \|f\|^2/2} \]
\[ \geq \mathbb{P}(I_n \leq 0 : |n| \leq N) e^{-\sqrt{2\|f\|^2 \log(cN)} - \|f\|^2/2}. \]

Finally, we take \( \log \) in (30), (31) and divide by \( \log(N) \). Then, taking \( \limsup_N \) and \( \liminf_N \), respectively, and letting \( \alpha \nearrow 1 \) yields
\[ \lim_{N \to \infty} \frac{\log(\mathbb{P}(I_n \leq 0 : |n| \leq N))}{\log(N)} = H - 1. \]

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