Noncommutative quantum mechanics in a time-dependent background

Andreas Fring

14th International Workshop on Pseudo-Hermitian Hamiltonians
University of Ferhat Abbas (Setif, Algeria), 5-10/09/2014
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based on: Sanjib Dey and Andreas Fring; arXiv:1407.4843
Content

• Introduction to noncommutative spaces
• The 2D harmonic oscillator in a time-dependent background
• The Ermakov-Pinney equation
• The generalized uncertainty relations
• Conclusions and outlook
Noncommutative spaces

- Flat (abelian) noncommutative space:

\[ [x^\mu, x^\nu] = i\theta^{\mu\nu} \]

In 3D:

\[
\begin{align*}
[x_0, y_0] &= i\theta_1, \\
[x_0, z_0] &= i\theta_2, \\
[y_0, z_0] &= i\theta_3,
\end{align*}
\]

\[
\begin{align*}
[x_0, p_{x_0}] &= i\hbar, \\
[y_0, p_{y_0}] &= i\hbar, \\
[z_0, p_{z_0}] &= i\hbar,
\end{align*}
\]

\(\theta_1, \theta_2, \theta_3 \in \mathbb{R}\)
Noncommutative spaces

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- Snyder spaces, from twists:

\[ [x_i, x_j] = i\theta(x_i p_j - x_j p_i) \]

\[ \theta_1, \theta_2, \theta_3 \in \mathbb{R} \]
Noncommutative spaces

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• Snyder spaces, from twists:

\[ [x_i, x_j] = i\theta(x_i p_j - x_j p_i) \]

• Minimal length spaces, from q-deformed algebras:

\[ [x_i, x_j] \approx i\theta(x_j)^2 \]
Minimal lengths, areas and volumes

Uncertainty relation:

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle_\rho \right|$$

- Standard case:
  
  $[A, B] = const$; give up knowledge about $B \Rightarrow \Delta A = 0$

- Noncommutative case:
  
  $[A, B] \approx B^2$; even give up knowledge about $B \Rightarrow \Delta A \neq 0$

For instance:

$[X, P] = i \hbar (1 + \tau P^2)$

$\Rightarrow$ minimal length

$$\Delta X_{\text{min}} = \hbar \sqrt{\tau \sqrt{1 + \tau \langle P^2 \rangle_\rho}}$$

from minimizing with $(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2$
Minimal lengths, areas and volumes

Uncertainty relation:

\[ \Delta A \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle_\rho \right| \]

- Standard case:
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Minimal lengths, areas and volumes

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- For instance:
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  \( \Rightarrow \) minimal length

\[ \Delta X_{\text{min}} = \hbar \sqrt{\tau} \sqrt{1 + \tau \langle P^2 \rangle_\rho} \]

from minimizing with \((\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2\)
\( \mathcal{PT} \)-symmetric noncommutative spaces

P. Giri, P. Roy, Eur. Phys. C60 (2009) 157: \( \mathcal{PT} \)-symmetry

\[
\begin{align*}
[x_0, y_0] &= i\theta_1, \\
[x_0, z_0] &= i\theta_2, \\
[y_0, z_0] &= i\theta_3, \\
[x_0, p_x] &= i\hbar, \\
[y_0, p_y] &= i\hbar, \\
[z_0, p_z] &= i\hbar,
\end{align*}
\]

\( \theta_1, \theta_2, \theta_3 \in \mathbb{R} \)

Useful to reduce number of free parameters.

Models on these spaces will have the usual nice properties.
$\mathcal{PT}$-symmetric noncommutative spaces

P. Giri, P. Roy, Eur. Phys. C60 (2009) 157: $\mathcal{PT}$-symmetry

$[x_0, y_0] = i\theta_1, \quad [x_0, z_0] = i\theta_2, \quad [y_0, z_0] = i\theta_3, \quad \theta_1, \theta_2, \theta_3 \in \mathbb{R}$

$\mathcal{PT}_\pm : \quad x_0 \rightarrow \pm x_0, \quad y_0 \rightarrow \mp y_0, \quad z_0 \rightarrow \pm z_0, \quad i \rightarrow -i,$

$p_{x_0} \rightarrow \mp p_{x_0}, \quad p_{y_0} \rightarrow \pm p_{y_0}, \quad p_{z_0} \rightarrow \mp p_{z_0}, \quad \theta_2 = 0.$
\( \mathcal{PT} \)-symmetric noncommutative spaces

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x_0, \ p_{x_0} &= i\hbar, & y_0, \ p_{y_0} &= i\hbar, & z_0, \ p_{z_0} &= i\hbar,
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\end{align*}
\]

\( \mathcal{PT}_{\theta\pm} \):

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\begin{align*}
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p_{x_0} &\rightarrow \mp p_{x_0}, & p_{y_0} &\rightarrow \pm p_{y_0}, & p_{z_0} &\rightarrow \mp p_{z_0}, & \theta_2 &\rightarrow -\theta_2.
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\(\mathcal{PT}\)-symmetric noncommutative spaces

P. Giri, P. Roy, Eur. Phys. C60 (2009) 157: \(\not\exists\ \mathcal{PT}\)-symmetry

\[
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\]

**\(\mathcal{PT}_{xz}\)**:

\[
x_0 \rightarrow z_0, \quad y_0 \rightarrow y_0, \quad z_0 \rightarrow x_0, \quad i \rightarrow -i,
\]

\[
p_{x_0} \rightarrow -p_{z_0}, \quad p_{y_0} \rightarrow -p_{y_0}, \quad p_{z_0} \rightarrow -p_{x_0}.
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\begin{align*}
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[x_0, p_{x_0}] &= i\hbar, & [y_0, p_{y_0}] &= i\hbar, & [z_0, p_{z_0}] &= i\hbar,
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- Useful to reduce number of free parameters.
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[x_0, p_{x_0}] &= i\hbar, & [y_0, p_{y_0}] &= i\hbar, & [z_0, p_{z_0}] &= i\hbar,
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\mathcal{PT}_\pm : & \quad x_0 \to \pm x_0, & y_0 \to \mp y_0, & z_0 \to \pm z_0, & i \to -i, \\
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\end{align*}
\]

- Useful to reduce number of free parameters.
- Models on these spaces will have the usual nice properties.
A particular $\mathcal{PT}_{\pm}$-symmetric solution
from q-deformed oscillator algebra
[S. Dey, A. Fring, L. Gouba, J. Phys. A45 (2012) 385302]

\[
\begin{align*}
[X, Y] &= i\theta_1 + i \frac{q^2 - 1}{q^2 + 1} \frac{\theta_1}{\hbar} \left[ \frac{\omega}{2\kappa_6^2} Y^2 + \frac{2\kappa_6^2}{m\omega} P_y^2 \right]
\end{align*}
\]

\[
\begin{align*}
[Y, Z] &= i\theta_3 + i \frac{q^2 - 1}{q^2 + 1} \frac{\theta_3}{\hbar} \left[ \frac{\omega}{2\kappa_6^2} Y^2 + \frac{2\kappa_6^2}{m\omega} P_y^2 \right]
\end{align*}
\]

\[
\begin{align*}
[X, P_x] &= i\hbar + i \frac{q^2 - 1}{q^2 + 1} \frac{2m\omega}{2\kappa_{11}} \left[ \frac{\kappa_{11}^2}{4} X^2 + \frac{P_x^2}{4m^2\omega^2\kappa_{11}^2} \right] + \frac{\theta_1^2}{\hbar^2} \frac{2\kappa_{11}^2 P_y^2}{\hbar^2} + \frac{\theta_1 \kappa_{11}^2}{\hbar/2} XP_y
\end{align*}
\]

\[
\begin{align*}
[Y, P_y] &= i\hbar + i \frac{q^2 - 1}{q^2 + 1} \frac{2m\omega}{2\kappa_{11}} \left[ \frac{1}{4\kappa_6^2} Y^2 + \frac{\kappa_6^2}{m^2\omega^2} P_y^2 \right]
\end{align*}
\]

\[
\begin{align*}
[Z, P_z] &= i\hbar + i \frac{q^2 - 1}{q^2 + 1} \frac{2m\omega}{2\kappa_{11}} \left[ \frac{Z^2}{4\kappa_7^2} \right] + \frac{\kappa_7^2}{m^2\omega^2} P_z^2 + \frac{\theta_3^2}{\hbar^2} \frac{2\kappa_7^2 P_y^2}{\hbar^2} - \frac{\theta_3 ZP_y}{2\hbar^2\kappa_7^2}
\end{align*}
\]
• Reduced three dimensional solution for $q \rightarrow 1$

\[
\begin{align*}
[X, Y] &= i\theta_1 (1 + \hat{r} Y^2), \\
[Y, Z] &= i\theta_3 (1 + \hat{r} Y^2), \\
[X, P_x] &= i\hbar (1 + \check{r} P_x^2), \\
[Y, P_y] &= i\hbar (1 + \hat{T} Y^2), \\
[Z, P_z] &= i\hbar (1 + \check{r} P_z^2)
\end{align*}
\]

where $\hat{r} = \tau m\omega / \hbar$, $\check{r} = \tau / (m\omega\hbar)$
• Reduced three dimensional solution for $q \to 1$

\[
[X, Y] = i\theta_1 \left(1 + \hat{\tau} Y^2\right), \quad [Y, Z] = i\theta_3 \left(1 + \hat{\tau} Y^2\right),
\]
\[
[X, P_x] = i\hbar \left(1 + \tilde{\tau} P_x^2\right), \quad [Y, P_y] = i\hbar \left(1 + \tilde{\tau} Y^2\right),
\]
\[
[Z, P_z] = i\hbar \left(1 + \tilde{\tau} P_z^2\right)
\]

where $\hat{\tau} = \tau m\omega/\hbar, \tilde{\tau} = \tau/(m\omega\hbar)$

• Representation in flat noncommutative space:

\[
X = (1 + \tilde{\tau} p_{x_0}^2)x_0 + \frac{\theta_1}{\hbar} \left(\tilde{\tau} p_{x_0}^2 - \hat{\tau} y_0^2\right) p_{y_0}, \quad P_x = p_{x_0},
\]
\[
Z = (1 + \tilde{\tau} p_{z_0}^2)z_0 + \frac{\theta_3}{\hbar} \left(\hat{\tau} y_0^2 - \tilde{\tau} p_{z_0}^2\right) p_{y_0}, \quad P_z = p_{z_0},
\]
\[
P_y = (1 + \hat{\tau} y_0^2)p_{y_0}, \quad Y = y_0.
\]
• Reduced three dimensional solution for $q \rightarrow 1$

\[
\begin{align*}
[X, Y] &= i\theta_1 (1 + \hat{\tau} Y^2), \\
[Y, Z] &= i\theta_3 (1 + \hat{\tau} Y^2), \\
[X, P_x] &= i\hbar (1 + \tilde{\tau} P_x^2), \\
[Y, P_y] &= i\hbar (1 + \tilde{\tau} Y^2), \\
[Z, P_z] &= i\hbar (1 + \tilde{\tau} P_z^2)
\end{align*}
\]

where $\hat{\tau} = \tau m\omega/\hbar$, $\tilde{\tau} = \tau/(m\omega\hbar)$

• Representation in flat noncommutative space:

\[
\begin{align*}
X &= (1 + \tilde{\tau} p_{x_0}^2) x_0 + \frac{\theta_1}{\hbar} \left( \tilde{\tau} p_{x_0}^2 - \hat{\tau} y_0^2 \right) p_{y_0}, \\
Z &= (1 + \tilde{\tau} p_{z_0}^2) z_0 + \frac{\theta_3}{\hbar} \left( \hat{\tau} y_0^2 - \tilde{\tau} p_{z_0}^2 \right) p_{y_0}, \\
P_y &= (1 + \hat{\tau} y_0^2) p_{y_0}, \\
P_x &= p_{x_0}, \\
P_z &= p_{z_0}, \\
Y &= y_0.
\end{align*}
\]

• Bopp-shift to standard canonical variables:

\[
\begin{align*}
x_0 &\rightarrow x_s - \frac{\theta_1}{\hbar} p_{y_s}, \\
y_0 &\rightarrow y_s, \\
z_0 &\rightarrow z_s + \frac{\theta_3}{\hbar} p_{y_s}, \\
p_{x_0} &\rightarrow p_{x_s}, \\
p_{y_0} &\rightarrow p_{y_s}, \\
p_{z_0} &\rightarrow p_{z_s}
\end{align*}
\]
- Dyson map: \[ \eta = \eta_y \eta_{p_{x_0}} \eta_{p_{z_0}} \]

\[ \eta_y = (1 + \hat{\tau} y_0^2)^{-1/2}, \quad \eta_{p_{x_0}} = (1 + \hat{\tau} p_{x_0}^2)^{-1/2}, \quad \eta_{p_{z_0}} = (1 + \hat{\tau} p_{z_0}^2)^{-1/2} \]

- Hermitian variables:

\[
\begin{align*}
x & := \eta X \eta^{-1} = \eta_{p_{x_0}} \left( x_0 + \frac{\theta_1}{\hbar} \right) \eta_{p_{x_0}} - \frac{\theta_1}{\hbar} \eta_{y_0} p_y \eta_{y_0} = x^\dagger \\
y & := \eta Y \eta^{-1} = y_0 = y^\dagger \\
z & := \eta Z \eta^{-1} = \eta_{p_{z_0}} \left( z_0 - \frac{\theta_3}{\hbar} \right) \eta_{p_{z_0}} + \frac{\theta_3}{\hbar} \eta_{y_0} p_y \eta_{y_0} = z^\dagger \\
p_x & := \eta P_x \eta^{-1} = p_{x_0} = p_x^\dagger \\
p_y & := \eta P_y \eta^{-1} = \eta_{y_0}^{-1} p_y \eta_{y_0}^{-1} = p_y^\dagger \\
p_z & := \eta P_z \eta^{-1} = p_{z_0} = p_z^\dagger 
\end{align*}
\]
• Dyson map: \[ \eta = \eta_y \eta_{p_x} \eta_{p_z} \]

\[ \eta_y = (1 + \hat{\gamma} y_0^2)^{-1/2}, \quad \eta_{p_x} = (1 + \hat{\gamma} p_x^2)^{-1/2}, \quad \eta_{p_z} = (1 + \hat{\gamma} p_z^2)^{-1/2} \]

• Hermitian variables:

\[
\begin{align*}
    x & := \eta X \eta^{-1} = \eta_{p_x}^{-1} \left( x_0 + \frac{\theta_1}{\hbar} \right) \eta_{p_x}^{-1} - \frac{\theta_1}{\hbar} \eta_{y_0}^{-1} \eta_{p_x}^{-1} p_y \eta_{y_0}^{-1} = x^\dagger \\
    y & := \eta Y \eta^{-1} = y_0 = y^\dagger \\
    z & := \eta Z \eta^{-1} = \eta_{p_z}^{-1} \left( z_0 - \frac{\theta_3}{\hbar} \right) \eta_{p_z}^{-1} + \frac{\theta_3}{\hbar} \eta_{y_0}^{-1} \eta_{p_x}^{-1} p_y \eta_{y_0}^{-1} = z^\dagger \\
    p_x & := \eta P_x \eta^{-1} = p_{x_0} = p_x^\dagger \\
    p_y & := \eta P_y \eta^{-1} = \eta_{y_0}^{-1} p_y \eta_{y_0}^{-1} = p_y^\dagger \\
    p_z & := \eta P_z \eta^{-1} = p_{z_0} = p_z^\dagger
\end{align*}
\]

• Isospectral Hermitian counterpart:

\[ H(X, Y, Z, P_x, P_y, P_z) \neq H^\dagger(X, Y, Z, P_x, P_y, P_z) \Rightarrow h = \eta H \eta^{-1} = h^\dagger \]

• Metric: \[ \rho = \eta^2 \]
Different types of representations (1D)

\[
[X, P] = i\hbar \left( 1 + \tilde{\tau} P^2 \right)
\]

non-Hermitian: \( X(1) = (1 + \tilde{\tau} p^2)x \), \( P(1) = p \)
Different types of representations (1D)

\[
[X, P] = i\hbar (1 + \tilde{\tau} P^2)
\]

non-Hermitian: \( X_{(1)} = (1 + \tilde{\tau} p^2)x, \quad P_{(1)} = p \)

non-Hermitian: \( X_{(4)} = ix(1 + \tilde{\tau} p^2)^{1/2}, \quad P_{(4)} = -ip(1 + \tilde{\tau} p^2)^{-1/2} \)
Different types of representations (1D)

\[ [X, P] = i\hbar (1 + \tilde{\tau} P^2) \]

\begin{align*}
\text{non-Hermitian: } & \quad X(1) = (1 + \tilde{\tau} p^2)x, \quad P(1) = p \\
\text{non-Hermitian: } & \quad X(4) = ix(1 + \tilde{\tau} p^2)^{1/2}, \quad P(4) = -ip(1 + \tilde{\tau} p^2)^{-1/2} \\
\text{Hermitian: } & \quad X(2) = (1 + \tilde{\tau} p^2)^{1/2}x(1 + \tilde{\tau} p^2)^{1/2}, \quad P(2) = p
\end{align*}
Different types of representations (1D)

\[ [X, P] = i\hbar \left( 1 + \gamma P^2 \right) \]

non-Hermitian: \( X_{(1)} = (1 + \gamma p^2)x, \quad P_{(1)} = p \)

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Hermitian: \( X_{(2)} = (1 + \gamma p^2)^{1/2}x(1 + \gamma p^2)^{1/2}, \quad P_{(2)} = p \)

Hermitian: \( X_{(3)} = x, \quad P_{(3)} = \frac{1}{\sqrt{\gamma}} \tan \left( \sqrt{\gamma} p \right) \)
Different types of representations (1D)

\[ [X, P] = i\hbar (1 + \tau P^2) \]

non-Hermitian: \( X_{(1)} = (1 + \tau p^2)x, \quad P_{(1)} = p \)

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Hermitian: \( X_{(3)} = x, \quad P_{(3)} = \frac{1}{\sqrt{\tau}} \tan(\sqrt{\tau}p) \)

The physics is the same for all representations

\[
\langle \psi_{(i)} | F(P_{(i)}, X_{(i)}) \psi_{(i)} \rangle_{\rho_{(i)}} = \frac{1}{N} \int_{-1}^{1} F \left[ \frac{z}{\sqrt{\tau}(1 - z^2)}, i\hbar \sqrt{\tau}(1 - z^2) \partial_z \right] \left| P_{m-\mu_-}^\mu_-(z) \right|^2 \, dz
\]

[S. Dey, A. Fring, B. Khantoul, J. Phys. A46 (2013) 335304]
Time-dependent noncommutativity (2D)

\[
\begin{align*}
[X, Y] &= i\theta(t) \\
[P_x, P_y] &= i\Omega(t), \\
[X, P_x] &= i\hbar + i\frac{\theta(t)\Omega(t)}{4\hbar} \\
[Y, P_y] &= i\hbar + i\frac{\theta(t)\Omega(t)}{4\hbar}
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⇒ time-dependent Hamiltonians \( H(X, Y, P_x, P_y) \rightarrow H(t) \).
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⇒ time-dependent Hamiltonians \( H(X, Y, P_x, P_y) \rightarrow H(t) \).

Representation:

\[
\begin{align*}
X &= x - \frac{\theta(t)}{2\hbar} p_y, \quad Y = y + \frac{\theta(t)}{2\hbar} p_x, \\
P_x &= p_x + \frac{\Omega(t)}{2\hbar} y, \quad P_y = p_y - \frac{\Omega(t)}{2\hbar} x.
\end{align*}
\]

with nonvanishing commutators \([x, p_x] = [y, p_y] = i\hbar\)
Lewis-Riesenfeld theory of invariants

Aim: solve time-dependent Schrödinger equation

\[ i\hbar \partial_t |\psi_n\rangle = H(t) |\psi_n\rangle \]
Lewis-Riesenfeld theory of invariants

Aim: solve time-dependent Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} |\psi_n\rangle = H(t) |\psi_n\rangle \]

Step 1: Construct Hermitian time-dependent invariant \( I(t) \)

\[ \frac{dI(t)}{dt} = \partial_t I(t) + \frac{1}{i\hbar} [I(t), H(t)] = 0. \]

Step 2: Solve eigensystem for \( I(t) \)

\[ I(t) |\phi_n\rangle = \lambda |\phi_n\rangle, \]

Step 3: Determine the phase of \( |\psi_n\rangle = e^{i\alpha(t)} |\phi_n\rangle \)

\[ d\alpha(t) = \frac{\hbar}{\lambda} \langle \phi_n | i\hbar \frac{\partial}{\partial t} - H(t) | \phi_n \rangle. \]
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[H. Lewis, W. Riesenfeld, J. Math. Phys. 10, 1458 (1969)]
Example: The 2D harmonic oscillator

\[ H(X, Y, P_x, P_y) = \frac{1}{2m} (P_x^2 + P_y^2) + \frac{m\omega^2}{2} (X^2 + Y^2), \]

Using the above representation

\[ H(t) = \frac{1}{2} a(t) (p_x^2 + p_y^2) + \frac{1}{2} b(t) (x^2 + y^2) + c(t) (p_x y - x p_y) \]

\[ = \frac{1}{2} a(t) \left( p_r^2 + \frac{p_\theta^2}{r^2} - \frac{\hbar^2}{4r^2} \right) + \frac{1}{2} b(t) r^2 - c(t) p_\theta \]

with coefficients

\[ a(t) = \frac{1}{m} + \frac{m\omega^2}{4\hbar^2} \theta^2(t), \quad b(t) = m\omega^2 + \frac{\Omega^2(t)}{4m\hbar^2}, \quad c(t) = \frac{m\omega^2 \theta(t)}{2\hbar} + \frac{\Omega(t)}{2\hbar m} \]
Step 1 in LR-theory

The Ansatz:

\[ I(t) = \alpha(t)p_r^2 + \beta(t)r^2 + \gamma(t)\{r, p_r\} + \delta(t)\frac{p_\theta^2}{r^2} + \varepsilon(t)\frac{p_\theta}{r^2} + \phi(t)\frac{1}{r^2} \]

leads to the set of coupled differential equations

\[ \dot{\alpha} = -2a\gamma, \quad \dot{\beta} = 2b\gamma, \quad \dot{\gamma} = b\alpha - a\beta \]

\[ \dot{\delta}p_\theta^2 + \dot{\varepsilon}p_\theta + \dot{\phi} = \hbar^2 a\gamma - 2a\gamma p_\theta^2, \quad (\delta - \alpha) p_\theta^2 + \varepsilon p_\theta + \phi + \frac{\alpha\hbar^2}{4} = 0 \]
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which are solved by

\[ I(t) = \frac{\tau}{\sigma^2}r^2 + \left(\sigma p_r - \frac{\dot{\sigma}}{a}r\right)^2 + \frac{\sigma^2 p_\theta^2}{r^2} - \frac{\sigma^2\hbar^2}{4r^2} \]

\[ \tau = \text{const}, \sigma(t) \text{ solves the Ermakov-Pinney equation} \]

\[ \ddot{\sigma} - \frac{a}{\sigma}\dot{\sigma} + ab\sigma = \tau \frac{a^2}{\sigma^3} \]
Step 2 in LR-theory

Rewrite $I(t)$ in terms of time-dependent creation and annihilation operators

$$\hbar \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) - p_\theta = \frac{1}{4} I(t) - \frac{1}{2} p_\theta =: \hat{I}(t)$$

$$\hat{a}(t) = \frac{1}{2\sqrt{\hbar}} \left[ \left( \sigma p_r - \frac{\dot{\sigma}}{a} r \right) - i \left( \frac{r}{\sigma} + \frac{\sigma}{r} \left( p_\theta + \frac{\hbar}{2} \right) \right) \right] e^{-i\theta}$$

$$\hat{a}^\dagger(t) = \frac{1}{2\sqrt{\hbar}} e^{i\theta} \left[ \left( \sigma p_r - \frac{\dot{\sigma}}{a} r \right) + i \left( \frac{r}{\sigma} + \frac{\sigma}{r} \left( p_\theta + \frac{\hbar}{2} \right) \right) \right]$$
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From standard arguments:

$$\psi_{n,m-n} = \lambda_n \left( \frac{i\hbar^{1/2}}{2} \sigma \right)^m r^{n-m} e^{i\theta(m-n) - \frac{a-\sigma \dot{\sigma}}{2a\hbar \sigma^2} r^2} U\left(-m, 1-m+n, \frac{r^2}{\hbar \sigma^2}\right)$$

with normalization constant

$$\lambda_n^2 = \frac{1}{\pi n! (\hbar \sigma^2)^{(1+n)}}$$
Step 3 in LR-theory

We fix the phase by solving:

\[ \dot{\alpha}_{n,\ell} = \frac{1}{\hbar} \langle n, \ell | i\hbar \partial_t - H | n, \ell \rangle \]

to

\[ \alpha_{n,\ell}(t) = (n + \ell) \int^t \left( c(s) - \frac{a(s)}{\sigma^2(s)} \right) ds \]
The Ermakov-Pinney equation

\[ \ddot{\sigma} - \frac{\dot{a}}{a} \dot{\sigma} + ab\sigma = \tau \frac{a^2}{\sigma^3} \]

For \( \dot{a} = 0 \), i.e. \( \theta(t) = \text{const} \), particular solutions are known

\[ \sigma = \sqrt{u_1^2 + \tau a^2 u_2^2} \]

where \( u_1, u_2 \) solve \( \ddot{u} + ab(t)u = 0 \) and

\[ W = u_1 \dot{u}_2 - \dot{u}_1 u_2 \]

\[ [E. Pinney, Proc. Amer. Math. Soc. 1, 681(1) (1950)] \]

For instance for \( a(t) = \alpha \) and \( b(t) = \beta e^{\gamma t} \), \( \alpha, \beta, \gamma \in \mathbb{R} \)

\[ \sigma(t) = \sqrt{\pi^2 \alpha^2 \tau \gamma^2} c_1 Y_2^0 \left( 2 \sqrt{\alpha \beta} e^{\gamma t} / 2 \right) + c_2 J_2^0 \left( 2 \sqrt{\alpha \beta} e^{\gamma t} / 2 \right) \]

with \( J_0, Y_0 \) Bessel functions of first and second kind.
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For instance for \(a(t) = \alpha\) and \(b(t) = \beta e^{\gamma t}\), \(\alpha, \beta, \gamma \in \mathbb{R}\)

\[ \sigma(t) = \sqrt{\frac{\pi^2 \alpha^2 \tau}{\gamma^2 c_1^2}} Y_0^2 \left( \frac{2\sqrt{\alpha \beta e^{\gamma t/2}}}{\gamma} \right) + c_1^2 J_0^2 \left( \frac{2\sqrt{\alpha \beta e^{\gamma t/2}}}{\gamma} \right), \]

with \(J_0, Y_0\) Bessel functions of first and second kind.
• When $\dot{a} \neq 0$ no general solution is known. Special solution:

$$\frac{1}{\lambda_k} \int_{\sigma}^{\sigma} \frac{\dot{a}s^3}{\tau a^3 - a^2 b s^4} ds = t \quad \lambda_{\kappa}^\pm = \frac{-1 \pm \sqrt{1 - 4\kappa}}{2\kappa}$$

when Chiellini integrability condition holds

$$\frac{d}{d\sigma} \left( \frac{\dot{a}s^3}{\tau a^3 - a^2 b s^4} \right) = -\kappa \frac{\dot{a}}{a} \quad \kappa \in \mathbb{R}$$
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$\Rightarrow$ Does not allow to pre-select $\Theta(t)$ and $\Omega(t)$. 
• When $\dot{a} \neq 0$ no general solution is known. Special solution:

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\]

⇒ Does not allow to pre-select $\Theta(t)$ and $\Omega(t)$.
⇒ Resort to numerical solutions.
Sample solutions:

(a) Exactly integrable solution (red, dashed) versus a non-Chiellini integrable solution for pre-selected exponential backgrounds $\theta(t) = e^{-\gamma t}$ and $\Omega(t) = e^{\gamma t}$ (black, solid).

$\alpha = 5$, $\beta = 2$, $\gamma = 2$, $m = \hbar = \tau = \omega = 1$, $\kappa = 1/4$, $\mu = \sqrt{5/3}$.
Sample solutions:

(b) Non-Chiellini integrable solution for pre-selected sinusoidal background \( \theta(t) = \alpha \sin(\gamma t) \) and \( \Omega(t) = \beta \sin(\gamma t/2) \).

\[
\alpha = 5, \quad \beta = 2, \quad \gamma = 2, \quad m = \hbar = \tau = \omega = 1, \quad \kappa = 1/4, \quad \mu = \sqrt{5/3}
\]
Generalized uncertainty relations

In general we have:

\[ \Delta A \Delta B \big|_\psi \geq \frac{1}{2} |\langle \psi | [A, B] |\psi \rangle| \]
Generalized uncertainty relations

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$$\Delta A \Delta B |_\psi \geq \frac{1}{2} | \langle \psi | [A, B] | \psi \rangle |$$

We can compute all required expectation values, such that:

$$\Delta X \Delta Y |_{\psi_{n,m-n}} = \frac{n - m}{2} \theta(t) + \frac{n + m + 1}{8\hbar} \left[ 4\hbar\sigma^2 + \left( \frac{1}{\sigma^2} + \frac{\dot{\sigma}^2}{a^2} \right) \theta^2(t) \right]$$
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\[ \geq \frac{\theta(t)}{2} \]
Generalized uncertainty relations

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\[ \Delta P_x \Delta P_y |_{\psi_{n,m-n}} = \frac{n - m}{2} \Omega(t) + \frac{\hbar}{2} (n + m + 1) \left[ \frac{\sigma^2 \Omega^2(t)}{4} + \left( \frac{1}{\sigma^2} + \frac{\dot{\sigma}^2}{a^2} \right) \right] \geq \frac{\Omega(t)}{2} \]

\[ \Delta X \Delta P_x |_{\psi_{n,m-n}} = \Delta Y \Delta P_y |_{\psi_{n,m-n}} \geq \frac{\hbar}{2} + \frac{\theta(t) \Omega(t)}{8\hbar} \]
(a) for background fields \( \theta(t) = \alpha e^{-\gamma t} \) and \( \Omega(t) = \beta e^{\gamma t} \)
\( \alpha = 5, \beta = 2, \gamma = 2, \quad m = \hbar = \tau = \omega = 1, \quad \kappa = 1/4, \quad \mu = \sqrt{5/3} \)
(b) for background fields $\theta(t) = \alpha \sin(\gamma t)$, $\Omega(t) = \beta \sin(\gamma t/2)$

$\alpha = 5$, $\beta = 2$, $\gamma = 2$, $m = \hbar = \tau = \omega = 1$, $\kappa = 1/4$, $\mu = \sqrt{5/3}$
Examples:

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GUR for coherent states

Glauber coherent states

\[ |\alpha, t\rangle := D(\alpha, t)|0, 0\rangle, \quad \text{with} \quad D(\alpha, t) := e^{\alpha \hat{a}^+(t) - \alpha^* \hat{a}(t)} \]

yield

\[ \Delta X|_{\alpha, t}\rangle^2 = \Delta Y|_{\alpha, t}\rangle^2 = \Delta X|_{\psi_{0,0}}\rangle^2, \quad \Delta P_x|_{\alpha, t}\rangle^2 = \Delta P_y|_{\alpha, t}\rangle^2 = \Delta P_x|_{\psi_{0,0}}\rangle^2 \]

Squeezed coherent states

\[ |\alpha, \beta, t\rangle := S(\beta, t)D(\alpha, t)|0, 0\rangle, \quad \text{with} \quad S(\beta, t) := e^{\frac{\beta}{2} [\hat{a}^2(t) - \hat{a}^+\hat{a}(t)]} \]

yield for instance

\[ \Delta X|_{\alpha, \beta, t}\rangle^2 = \]

\[ \frac{\hbar}{2} \left[ \sigma^2 e^\beta + \frac{\theta^2(t)}{4\hbar^2} \left( \frac{1}{\sigma^2} e^\beta + \frac{\hat{\sigma}^2}{a^2} e^{-\beta} \right) \right] \cosh \beta + \frac{\theta(t)}{4} (1 - e^{2\beta}) \]
Use $\beta$ to minimise uncertainties:

- Possible for generic $t$ for $\Delta x \Delta p_x|_{\alpha, \beta, t}$:

$$\beta(t) = \beta_{\text{min}}(t) = \frac{1}{2} \ln \left[ \frac{a \sqrt{a^2 + 8 \sigma^2 \dot{\sigma}^2} - a^2}{4 \sigma^2 \dot{\sigma}^2} \right]$$

such that $\Delta x \Delta p_x|_{\alpha, \beta_{\text{min}}, t} < \Delta x \Delta p_x|_{\alpha, t}$

- Possible for specific $t$ for $\Delta X \Delta P_x|_{\alpha, \beta, t}$, e.g. $t = 4$ for background fields $\theta(t) = \alpha \sin(\gamma t)$, $\Omega(t) = \beta \sin(\gamma t/2)$
Use $\beta$ to minimise uncertainties:

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\[
\beta(t) = \beta_{\text{min}}(t) = \frac{1}{2} \ln \left( \frac{a \sqrt{a^2 + 8\sigma^2 \dot{\sigma}^2} - a^2}{4\sigma^2 \dot{\sigma}^2} \right)
\]

such that $\Delta x \Delta p_x \mid_{\alpha, \beta_{\text{min}}, t} < \Delta x \Delta p_x \mid_{\alpha, t}$

- Possible for specific $t$ for $\Delta X \Delta P_x \mid_{\alpha, \beta, t}$, e.g. $t = 4$:

\[
\Phi = \psi_{\alpha, t} = \mid \alpha, t \rangle
\]

for background fields $\theta(t) = \alpha \sin(\gamma t)$, $\Omega(t) = \beta \sin(\gamma t/2)$
Glauber versus Gaussian Klauder coherent states

\[ |n, m_0, \phi_0, s\rangle := \frac{1}{\sqrt{N(m_0)}} \sum_{m=0}^{\infty} \exp \left[ -\frac{(m - m_0)^2}{4s^2} \right] e^{im\phi_0} |n, m - n\rangle \]

Quality depends on background fields:

(a) for background fields \( \theta(t) = \alpha e^{-\gamma t} \) and \( \Omega(t) = \beta e^{\gamma t} \)
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Quality depends on background fields:

(b) for background fields \( \theta(t) = \alpha \sin(\gamma t) \), \( \Omega(t) = \beta \sin(\gamma t/2) \)
Conclusions

- Models on time-dependent background solvable with LR-theory
- Implemented solution from EP-equation into discussion
- Glauber coherent states $\sim$ in quality to ground state
- Squeezing is possible for specific times
- Quality $|GK\rangle$ versus standard states depends on background

Outlook

- Investigate different types of backgrounds
- Investigate different types of models
- Sovability?

Thank you for your attention
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Thank you for your attention