2-Blocks with minimal nonabelian defect groups

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Abstract
We study numerical invariants of 2-blocks with minimal nonabelian defect groups. These groups were classified by Rédei (see [41]). If the defect group is also metacyclic, then the block invariants are known (see [43]). In the remaining cases there are only two (infinite) families of “interesting” defect groups. In all other cases the blocks are nilpotent. We prove Brauer’s $k(B)$-conjecture and the Olsson-conjecture for all 2-blocks with minimal nonabelian defect groups. For one of the two families we also show that Alperin’s weight conjecture and Dade’s conjecture is satisfied. This paper is a part of the author’s PhD thesis.

Keywords: blocks of finite groups, minimal nonabelian defect groups, Alperin’s conjecture, Dade’s conjecture.

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1 Introduction

Let $R$ be a discrete complete valuation ring with quotient field $K$ of characteristic 0. Moreover, let $(\pi)$ be the maximal ideal of $R$ and $F := R/(\pi)$. We assume that $F$ is algebraically closed of characteristic 2. We fix a finite group $G$, and assume that $K$ contains all $|G|$-th roots of unity. Let $B$ be a block of $RG$ with defect group $D$. We
denote the number of irreducible ordinary characters of $B$ by $k(B)$. These characters split in $k_i(B)$ characters of height $i \in \mathbb{N}_0$. Similarly, let $k'_i(B)$ be the number of characters of defect $i \in \mathbb{N}_0$. Finally, let $l(B)$ be the number of irreducible Brauer characters of $B$. The defect group $D$ is called minimal nonabelian if every proper subgroup of $D$ is abelian, but not $D$ itself. Rédei has shown that $D$ is isomorphic to one of the following groups (see [11]):

(i) $\langle x, y \mid x^{2r} = y^{2r} = 1, xyx^{-1} = y^{1+2r^{-1}} \rangle$, where $r \geq 1$ and $s \geq 2$,

(ii) $\langle x, y \mid x^{2r} = y^{2r} = [x,y]^2 = [x,x,y] = [y,x,y] = 1 \rangle$, where $r \geq s \geq 1$, $[x,y] := xyx^{-1}y^{-1}$ and $[x,x,y] := [x,[x,y]]$.

(iii) $Q_8$.

In the first and last case $D$ is also metacyclic. In this case $B$ is well understood (see [33]). Thus, we may assume that $D$ has the form $Q_8$.

## 2 Fusion systems

To analyse the possible fusion systems on $D$ we start with a group theoretical lemma.

**Lemma 2.1.** Let $z := [x,y]$. Then the following hold:

(i) $|D| = 2^{r+s+1}$.

(ii) $\Phi(D) = \mathbb{Z}(D) = \langle x^2, y^2, z \rangle \cong C_{2^{r-1}} \times C_{2^{s-1}} \times C_2$.

(iii) $D' = \langle z \rangle \cong C_2$.

(iv) $|\text{Irr}(D)| = 5 \cdot 2^{r+s-2}$.

(v) If $r = s = 1$, then $D \cong D_8$. For $r \geq 2$ the maximal subgroups of $D$ are given by

\[
\langle x^2, y, z \rangle \cong C_{2^{r-1}} \times C_2, \times C_2,
\langle x, y^2, z \rangle \cong C_2 \times C_2 \times C_2,
\langle xy, x^2, z \rangle \cong C_2 \times C_2 \times C_{2^{s-1}}.
\]

We omit the (elementary) proof of this lemma. However, notice that $|P'| = 2$ and $|P:\Phi(P)| = |P:\mathbb{Z}(P)| = p^2$ hold for every minimal nonabelian $p$-group $P$. Rédei has also shown that for different pairs $(r,s)$ one gets nonisomorphic groups. This gives precisely $\left\lfloor \frac{n-1}{2} \right\rfloor$ isomorphism classes of these groups of order $2^n$. For $r \neq 1$ (that is $|D| \geq 16$) the structure of the maximal subgroups shows that all these groups are nonmetacyclic.

Now we investigate the automorphism groups.

**Lemma 2.2.** The automorphism group $\text{Aut}(D)$ is a 2-group, if and only if $r \neq s$ or $r = s = 1$.

**Proof.** If $r \neq s$ or $r = s = 1$, then there exists a characteristic maximal subgroup of $D$ by Lemma 2.1. Then, by these cases $\text{Aut}(D)$ must be a 2-group. Thus, we may assume $r = s \geq 2$. Then one can show that the map $x \mapsto y$, $y \mapsto x^{-1}y^{-1}$ is an automorphism of order 3.

**Lemma 2.3.** Let $P \cong C_{2^{n_1}} \times \ldots \times C_{2^{n_k}}$ with $n_1, \ldots, n_k, k \in \mathbb{N}$. Then $\text{Aut}(P)$ is a 2-group, if and only if the $n_i$ are pairwise distinct.

**Proof.** See for example Lemma 2.7 in [33].

Now we are able to decide, when a fusion system on $D$ is nilpotent.

**Theorem 2.4.** Let $\mathcal{F}$ be a fusion system on $D$. Then $\mathcal{F}$ is nilpotent or $s = 1$ or $r = s$. If $r = s \geq 2$, then $\mathcal{F}$ is controlled by $D$. 

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Proof. We assume \( s \neq 1 \). Let \( Q < D \) be an \( F \)-essential subgroup. Since \( Q \) is also \( F \)-centric, we get \( C_P(Q) = Q \). This shows that \( Q \) is a maximal subgroup of \( D \). By Lemma \( 2.1 \) and Lemma \( 2.3 \) one of the following holds:

(i) \( r = 2 \) (\( s = 1 \)) and \( Q \in \{(x^2, y, z), (x, y^2, z), (xy, x^2, z)\} \),

(ii) \( r > s = 2 \) and \( Q \in \{(x, y^2, z), (xy, x^2, z)\} \),

(iii) \( r = s + 1 \) and \( Q = \langle x^2, y, z \rangle \).

In all cases \( \Omega(Q) \subseteq \mathbb{Z}(P) \). Let us consider the action of \( \text{Aut}_F(Q) \) on \( \Omega(Q) \). The subgroup \( 1 \neq P/Q = \mathbb{N}_P(Q)/C_P(Q) \cong \text{Aut}_F(Q) \leq \text{Aut}_F(Q) \) acts trivially on \( \Omega(Q) \). On the other hand every nontrivial automorphism of odd order acts nontrivially on \( \Omega(Q) \) (see for example 8.4.3 in [19]). Hence, the kernel of this action is a nontrivial normal 2-subgroup of \( \text{Aut}_F(Q) \). In particular \( O_2(\text{Aut}_F(Q)) \neq 1 \). But then \( \text{Aut}_F(Q) \) cannot contain a strongly 2-embedded subgroup.

This shows that there are no \( F \)-essential subgroups. Now the claim follows from Lemma \( 2.2 \) and Alperin’s fusion theorem.

Now we consider a kind of converse. If \( r = s = 1 \), then there are nonnilpotent fusion systems on \( D \). In the case \( r > s = 1 \) one can construct a nonnilpotent fusion system with a suitable semidirect product (see Lemma \( 2.2 \)). We show that there is also a nonnilpotent fusion system in the case \( r > s = 1 \).

**Proposition 2.5.** If \( s = 1 \), then there exists a nonnilpotent fusion system on \( D \).

Proof. We may assume \( r \geq 2 \). Let \( A_4 \) be the alternating group of degree 4, and let \( H := \langle \bar{x} \rangle \cong C_2 \). Moreover, let \( \varphi : H \to \text{Aut}(A_4) \cong S_4 \) such that \( \varphi_2 \in \text{Aut}(A_4) \) has order 4. Write \( \tilde{y} := (12)(34) \in A_4 \) and choose \( \varphi \) such that \( \varphi_2(\tilde{y}) := (13)(24) \). Finally, let \( G := A_4 \rtimes_\varphi H \). Since all 4-cycles in \( S_4 \) are conjugate, \( G \) is uniquely determined up to isomorphism. Because \( [\bar{x}, \tilde{y}] = (13)(24)(12)(34) = (14)(23) \), we get \( \langle \bar{x}, \tilde{y} \rangle \cong D \). The fusion system \( F_G(D) \) is nonnilpotent, since \( A_4 \) (and therefore \( G \)) is not 2-nilpotent.

3 The case \( r > s = 1 \)

Now we concentrate on the case \( r > s = 1 \), i.e.

\[
D := \langle x, y \mid x^{2r} = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle
\]

with \( r \geq 2 \). As before \( z := [x, y] \). We also assume that \( B \) is a nonnilpotent block. By Lemma \( 2.2 \) \( \text{Aut}(D) \) is a 2-group, and the inertial index \( i(B) \) of \( B \) equals 1.

3.1 The \( B \)-subsections

Olsson has already obtained the conjugacy classes of so called \( B \)-subsections (see [34]). However, his results contain errors. For example he missed the necessary relations \( [x, x, y] \) and \( [y, x, y] \) in the definition of \( D \).

In the next lemma we denote by \( \text{Bl}(RH) \) the set of blocks of a finite group \( H \). If \( H \leq G \) and \( b \in \text{Bl}(RH) \), then \( b^G \) is the Brauer correspondent of \( b \) (if exists). Moreover, we use the notion of subpairs and subsections (see [36]).

**Lemma 3.1.** Let \( b \in \text{Bl}(R^D C_Q(D)) \) be a Brauer correspondent of \( B \). For \( Q \leq D \) let \( b_Q \in \text{Bl}(R^Q C_Q(Q)) \) such that \( (Q, b_Q) \leq (D, b) \). Set \( T := \mathbb{Z}(D) \cup \{x^iy^j : i, j \in \mathbb{Z}, i \text{ odd}\} \). Then

\[
\bigcup_{a \in T} \left\{(a, b_{C_{Q(a)}})\right\}
\]

is a system of representatives for the conjugacy classes of \( B \)-subsections. Moreover, \( |T| = 2^{r+1} \).

Proof. If \( r = 2 \), then the claim follows from Proposition 2.14 in [34]. For \( r \geq 3 \) the same argument works. However, Olsson refers wrongly to Proposition 2.11 (the origin of this mistake already lies in Lemma 2.8).
From now on we write $b_a := t_{C_G(a)}^{C_G(a)}$ for $a \in T$.

**Lemma 3.2.** Let $P \cong C_2^s \times C_2^{2}$ with $s \in \mathbb{N}$, and let $\alpha$ be an automorphism of $P$ of order 3. Then $C_P(\alpha) := \{ b \in P : \alpha(b) = b \} \cong C_2$. 

**Proof.** We write $P = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ with $|\langle a \rangle| = 2^s$. It is well known that the kernel of the restriction map $\text{Aut}(P) \to \text{Aut}(P/\Phi(P))$ is a 2-group. Since $|\text{Aut}(P/\Phi(P))| = |\text{GL}(3,2)| = 168 = 2^4 \cdot 3 \cdot 7$, it follows that $|\text{Aut}(P)|$ is divisible by 3 only once. In particular every automorphism of $P$ of order 3 is conjugate to $\alpha$ or $\alpha^{-1}$. Thus, we may assume $\alpha(a) = a$, $\alpha(b) = c$ and $\alpha(c) = bc$. Then $C_P(\alpha) = \langle a \rangle \cong C_2$. 

**3.2 The numbers $k(B)$, $k_i(B)$ and $l(B)$**

The next step is to determine the numbers $l(b_a)$. The case $r = 2$ needs special attention, because in this case $D$ contains an elementary abelian maximal subgroup of order 8. We denote the inertial group of a block $b \in \text{Bl}(RH)$ with $H \unlhd G$ by $T_G(b)$.

**Lemma 3.3.** There is an element $c \in Z(D)$ of order $2^{r-1}$ such that $l(b_a) = 1$ for all $a \in T \setminus \langle c \rangle$.

**Proof.**

**Case 1:** $a \in Z(D)$.

Then $b_a = b_D^{C_G(a)}$ is a block with defect group $D$ and Brauer correspondent $b_D \in \text{Bl}(RD\text{C}_{C_G(a)}(D))$. Let $M := \langle x^2, y, z \rangle \cong C_{2^{r-1}} \times C_2^2$. Since $D$ is nonnilpotent, there exists an element $\alpha \in T_{N_G(M)}(b_M)$ such that $\alpha C_G(M) \in T_{N_G(M)}(b_M) / C_G(M)$ has order $q \in \{3,7\}$. We will exclude the case $q = 7$. In this case $r = 2$ and $T_{N_G(M)}(b_M) / C_G(M)$ is isomorphic to a subgroup of $\text{Aut}(M) \cong \text{GL}(3,2)$. Since

$$(M, d_{b_M}) = d(M, b_M) \leq d(D, b_D) = (D, b_D)$$

for all $d \in D$, we have $D \subseteq T_{N_G(M)}(b_M)$. This implies $T_{N_G(M)}(b_M) / C_G(M) \cong \text{GL}(3,2)$, because $\text{GL}(3,2)$ is simple. By Satz 1 in [2], this contradicts the fact that $T_{N_G(M)}(b_M) / C_G(M)$ contains a strongly 2-embedded subgroup (of course this can be shown “by hand” without invoking [2]). Thus, we have shown $q = 3$. Now

$$T_{N_G(M)}(b_M) / C_G(M) \cong S_3$$

follows easily. By Lemma 3.2, there is an element $c := x^2y^i z^k \in C_M(\alpha) (i, j, k \in Z)$ of order $2^{r-1}$. Let us assume that $j$ is odd. Since $xax \equiv xox^{-1} \equiv \alpha^{-1}$ (mod $C_G(M)$) we get

$$\alpha(x^2y^i z^{k+1})(\alpha^{-1}) = \alpha(x^2y^i z^k)x^{-1} = x^{-1} (x^2y^i z^k) x^{-1} = x^2y^i z^{k+1}.$$

But this contradicts Lemma 3.2. Hence, we have proved that $j$ is even. In particular $c \in Z(D)$. For $a \notin \langle c \rangle$ we have $a \notin C_G(a)$ and $l(b_a) = 1$. While in the case $a \in \langle c \rangle$ we get $a \in C_G(a)$, and $b_a$ is nonnilpotent. Thus, in this case $l(b_a)$ remains unknown.

**Case 2:** $a \notin Z(D)$.

Let $C_D(a) = \langle Z(D), a \rangle =: M$. Since $(M, b_M)$ is a Brauer subpair, $b_M$ has defect group $M$. It follows from $(M, b_M) \leq (D, b_D)$ that also $b_a$ has defect group $M$ and Brauer correspondent $b_M$. In case $M \cong C_{2^r} \times C_2$ we get $l(b_a) = 1$. Now let us assume $M \cong C_{2^{r-1}} \times C_2^2$. As in the first case, we choose $\alpha \in T_{N_G(M)}(b_M)$ such that $\alpha C_G(M) \in T_{N_G(M)}(b_M) / C_G(M)$ has order 3. Since $a \notin Z(D)$, we derive $a \notin C_G(a)$ and $l(b_a) = l(b_a) = 1$. 

We denote by $\text{IBr}(b_a) := \{ \varphi_a \}$ for $u \in T \setminus \langle c \rangle$ the irreducible Brauer character of $b_a$. Then the generalized decomposition numbers $d_{\chi \varphi u}^u$ for $\chi \in \text{Irr}(B)$ form a column $d(u)$. Let $2^k$ be the order of $u$, and let $\zeta := \zeta_{2^k}$ be a primitive $2^k$-th root of unity. Then the entries of $d(u)$ lie in the ring of integers $Z[\zeta]$. Hence, there exist integers $a^u_i(\chi) \in Z$ such that

$$d_{\chi \varphi u}^u = \sum_{i=0}^{2^k-1} a^u_i(\chi) \zeta^i.$$
We expand this by

\[ a_{i+2k-1}^u := -a_i^u \]

for all \( i \in \mathbb{Z} \).

Let \( |G| = 2^m \) where \( 2 \nmid m \). We may assume \( \mathbb{Q}(\zeta_{|G|}) \subseteq K \). Then \( \mathbb{Q}(\zeta_{|G|})|\mathbb{Q}(\zeta_m) \) is a Galois extension, and we denote the corresponding Galois group by

\[ G := \text{Gal}(\mathbb{Q}(\zeta_{|G|})|\mathbb{Q}(\zeta_m)). \]

Restriction gives an isomorphism

\[ G \cong \text{Gal}(\mathbb{Q}(\zeta_{2^s})|\mathbb{Q}). \]

In particular \( |G| = 2^{s-1} \). For every \( \gamma \in G \) there is a number \( \bar{\gamma} \in \mathbb{N} \) such that \( \gcd(\bar{\gamma}, |G|) = 1, \bar{\gamma} \equiv 1 \pmod{m} \), and \( \gamma(\zeta_{|G|}) = \zeta_{\bar{\gamma}|G|} \) hold. Then \( G \) acts on the set of subsections by

\[ \gamma(u, b) := (u^{\bar{\gamma}}, b). \]

For every \( \gamma \in G \) we get

\[ d(u^{\bar{\gamma}}) = \sum_{s \in S} a_s^u \zeta_{2k}^s \]

for every system \( S \) of representatives of the cosets of \( 2^{k-1}\mathbb{Z} \) in \( \mathbb{Z} \). It follows that

\[ a_s^u = 2^{1-a} \sum_{\gamma \in G} d(u^{\bar{\gamma}}) \zeta_{2k}^{-s} \quad (1) \]

for \( s \in S \).

Now let \( u \in T \setminus Z(D) \) and \( M := C_D(u) \). Then \( b_u \) and \( b_M^{T_{NGC}(M)(bar) \cap NGC(u)} \) have \( M \) as defect group, because \( D \not\subset N_{GC}(u) \). By (6B) in [6] it follows that the \( 2^{s-1} \) distinct \( B \)-subsections of the form \( \gamma(u, b_u) \) with \( \gamma \in G \) are pairwise nonconjugate. The same holds for \( u \in Z(D) \setminus \{1\} \). Using this and equation (1) we can adapt Lemma 3.9 in [33].

**Lemma 3.4.** Let \( c \in Z(D) \) as in Lemma 3.3, and let \( u, v \in T \setminus \{c\} \) with \( |\langle u \rangle| = 2^k \) and \( |\langle v \rangle| = 2^l \). Moreover, let \( i \in \{0, 1, \ldots, 2^{k-1} - 1\} \) and \( j \in \{0, 1, \ldots, 2^{l-1} - 1\} \). If there exist \( \gamma \in G \) and \( g \in G \) such that \( g(u, b_u) = \gamma(v, b_v) \), then

\[ (a_i^u, a_j^v) = \begin{cases} 2^{d(B)-k+1} & \text{if } u \in Z(D) \text{ and } j^{-a} - i \equiv 0 \pmod{2^k} \\ -2^{d(B)-k+1} & \text{if } u \in Z(D) \text{ and } j^{-a} - i \equiv 2^{k-1} \pmod{2^k} \\ 0 & \text{if } u \notin Z(D) \text{ and } j^{-a} - i \equiv 0 \pmod{2^k} \\ -2^{d(B)-k} & \text{if } u \notin Z(D) \text{ and } j^{-a} - i \equiv 2^{k-1} \pmod{2^k} \end{cases} \]

Otherwise \( (a_i^u, a_j^v) = 0 \). In particular \( (a_0^u, a_0^v) = 0 \) if \( k \neq l \).

Using the theory of contributions we can also carry over Lemma (6.E) in [20].

**Lemma 3.5.** Let \( u \in Z(D) \) with \( l(b_u) = 1 \). If \( u \) has order \( 2^k \), then for every \( \chi \in \text{Irr}(B) \) holds:

\[ (i) \ 2^{k(\chi)} | a_i^u(\chi) \text{ for } i = 0, \ldots, 2^{k-1} - 1, \]

\[ (ii) \ \sum_{i=0}^{2^k-1} a_i^u(\chi) \equiv 2^k(\chi) \pmod{2^{k(\chi)+1}}. \]

By Lemma 1.1 in [39] we have

\[ k(B) \leq \sum_{i=0}^{\infty} 2^{2i}k_i(B) \leq |D|. \quad (2) \]

In particular Brauer’s \( k(B) \)-conjecture holds. Olsson’s conjecture

\[ k_0(B) \leq |D : D'| = 2^{r+1} \quad (3) \]

follows by Theorem 3.1 in [39]. Now we are able to calculate the numbers \( k(B), k_i(B) \) and \( l(B) \).
Theorem 3.6. We have
\[ k(B) = 5 \cdot 2^{r-1} = |\text{Irr}(D)|, \quad k_0(B) = 2^{r+1} = |D : D'|, \quad k_1(B) = 2^{r-1}, \quad l(B) = 2. \]

Proof. We argue by induction on \( r \). Let \( r = 2 \), and let \( c \in Z(D) \) as in Lemma 3.3. By way of contradiction we assume \( c = z \). If \( \alpha \) and \( M \) are defined as in the proof of Lemma 3.3, then \( \alpha \) acts nontrivially on \( M/\langle z \rangle \cong C_2^2 \). On the other hand \( x \) acts trivially on \( M/\langle z \rangle \). This contradicts \( xax^{-1} \alpha \in C_G(M) \).

This shows \( c \in \{x^2, x^2z\} \) and \( D/\langle c \rangle \cong D_8 \). Thus, we can apply Theorem 2 in [3]. For this let \( M_1 := \left\{ \langle x, z \rangle \mid c = x^2 \right\} \cup \left\{ \langle xy, z \rangle \mid c = x^2z \right\} \). Then \( M \neq M_1 \cong C_4 \times C_2 \) and \( \overline{M} := M/\langle c \rangle \cong C_2^2 \cong M_1/\langle c \rangle =: \overline{M}_1 \). Let \( \beta \) be the block of \( R\overline{C_G(c)} := R[C_G(c)/\langle c \rangle] \) which is dominated by \( b_c \). By Theorem 1.5 in [33] we have
\[ 3 \mid |T_{N_{\overline{C_G(c)}}(\beta \overline{M}/\beta \overline{M}_1)}/C_G(c)(\overline{M})| \]
and
\[ 3 \mid |T_{N_{\overline{C_G(c)}}(\beta \overline{M}/\beta \overline{M}_1)}/C_G(c)(\overline{M}_1)|, \]
where \((\overline{M}, \beta \overline{M}_1)\) and \((\overline{M}_1, \beta \overline{M}_1)\) are \( \beta \)-subpairs. This shows that case \((ab)\) in Theorem 2 in [3] occurs. Hence, \( l(b_c) = l(\beta) = 2 \). Now Lemma 3.3 yields
\[ k(B) \geq 1 + k(B) - l(B) = 9. \]

It is well known that \( k_0(B) \) is divisible by 4. Thus, the equations (2) and (3) imply \( k_0(B) = 8 \). Moreover,
\[ d_{x^2}^\chi = a_0^\chi = \pm 1 \]
holds for every \( \chi \in \text{Irr}(B) \) with \( b(\chi) = 0 \). This shows \( 4k_1(B) \leq |D| - k_0(B) = 8 \). It follows that \( k_1(B) = l(B) = 2 \).

Now we consider the case \( r \geq 3 \). Since \( z \) is not a square in \( D \), we have \( z \notin \langle c \rangle \). Let \( a \in \langle c \rangle \) such that \( |\langle a \rangle| = 2^k \). If \( k = r - 1 \), then \( l(b_a) = 2 \) as before. Now let \( k < r - 1 \). Then \( D/\langle a \rangle \) has the same isomorphism type as \( D \), but one has to replace \( r \) by \( r - k \). By induction we get \( l(b_a) = 2 \) for \( k \geq 1 \). This shows
\[ k(B) \geq 1 + k(B) - l(B) = 2^{r+1} + 2^{r-1} - 1. \]

Equation (2) yields
\[ 2^{r+2} - 4 = 2^{r+1} + 4(2^{r-1} - 1) \leq k_0(B) + 4(k(B) - k_0(B)) \]
\[ \leq \sum_{i=0}^{\infty} 2^{2i} k_i(B) \leq |D| = 2^{r+2}. \]

Now the conclusion follows easily.

As a consequence, Brauer’s height zero conjecture and the Alperin-McKay-conjecture hold for \( B \).

3.3 Generalized decomposition numbers

Now we will determine some of the generalized decomposition numbers. Again let \( c \in Z(D) \) as in Lemma 3.3 and let \( u \in Z(D) \setminus \langle c \rangle \) with \( |\langle u \rangle| = 2^k \). Then \( (a_u^r, a_v^r) = 2^{r+3-k} \) and \( 2 | a_v^r(\chi) \) for \( b(\chi) = 1 \) and \( i = 0, \ldots, 2^{k-1} - 1 \). This gives
\[ |\{ \chi \in \text{Irr}(B) : a_i^u(\chi) \neq 0 \}| \leq 2^{r+3-k} - 3|\{ \chi \in \text{Irr}(B) : b(\chi) = 1, a_i^u(\chi) \neq 0 \}|. \]
Moreover, for every character \( \chi \in \text{Irr}(B) \) there exists \( i \in \{0, \ldots, 2^{k-1} - 1 \} \) such that \( a_i^u(\chi) \neq 0 \). Hence,

\[
k(B) \leq \sum_{i=0}^{2^{k-1}-1} \sum_{\chi \in \text{Irr}(B), \ a_i^u(\chi) \neq 0} 1 \leq \sum_{i=0}^{2^{k-1}-1} \left( 2^{r+3-k} - 3 \sum_{\chi \in \text{Irr}(B), \ h(\chi)=1, \ a_i^u(\chi) \neq 0} 1 \right) = |D| - 3 \sum_{i=0}^{2^{k-1}-1} \sum_{\chi \in \text{Irr}(B), \ h(\chi)=1, \ a_i^u(\chi) \neq 0} 1 \leq |D| - 3k_1(B) = k(B).
\]

This shows that for every \( \chi \in \text{Irr}(B) \) there exists \( i(\chi) \in \{0, \ldots, 2^{k-1} - 1 \} \) such that

\[
d_{\chi^u}^u = \begin{cases} \pm \zeta_2^{i(\chi)} & \text{if } h(\chi) = 0 \\ \pm 2 \zeta_2^{-i(\chi)} & \text{if } h(\chi) = 1 \end{cases}
\]

In particular

\[
d_{\chi^u}^u = a_0^u(\chi) = \begin{cases} \pm 1 & \text{if } h(\chi) = 0 \\ \pm 2 & \text{if } h(\chi) = 1 \end{cases}
\]

for \( k = 1 \).

By Lemma 3.4 we have \( (a_i^u, a_0^u) = 4 \) for \( u \in T \setminus Z(D) \) and \( i = 0, \ldots, 2^{r-1} - 1 \). If \( a_i^u \) has only one nonvanishing entry, then \( a_i^u \) would not be orthogonal to \( a_0^u \). Hence, \( a_i^u \) has up to ordering the form

\[
(\pm 1, \pm 1, \pm 1, 0, \ldots, 0)^T,
\]

where the signs are independent of each other. The proof of Theorem 3.1 in [39] gives

\[
|d_{\chi^u}^u| = 1
\]

for \( u \in T \setminus Z(D) \) and \( \chi \in \text{Irr}(B) \) with \( h(\chi) = 0 \). In particular \( d_{\chi^u}^u = 0 \) for characters \( \chi \in \text{Irr}(B) \) of height 1.

By suitable ordering we get

\[
a_i^u(\chi_j) = \begin{cases} \pm 1 & \text{if } j - 4i \in \{1, \ldots, 4\} \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
d_{\chi^u}^u = \begin{cases} \pm \zeta_2^{i(\chi_j)} & \text{if } 1 \leq j \leq k_0(B) \\ 0 & \text{if } k_0(B) < j \leq k(B) \end{cases}
\]

for \( i = 0, \ldots, 2^{r-1} - 1 \), where \( \chi_1, \ldots, \chi_{k_0(B)} \) are the characters of height 0.

Now let \( \text{Irr}(b_c) := \{\varphi_1, \varphi_2\} \). We determine the numbers \( d_{\chi^\varphi_1}^c, d_{\chi^\varphi_2}^c \in \mathbb{Z}[\zeta_{2^r-1}] \). By (4C) in [6] we have \( d_{\chi^\varphi_1}^c \neq 0 \) or \( d_{\chi^\varphi_2}^c \neq 0 \) for all \( \chi \in \text{Irr}(B) \). As in the proof of Theorem 3.6, \( b_c \) dominates a block \( b_c \in \text{Bl}(R[C_G(c)/c]) \) with defect group \( D_k \). The table at the end of [14] shows that the Cartan matrix of \( b_c \) has the form

\[
\begin{pmatrix}
8 & 4 & 3 \\
4 & 2 & 3
\end{pmatrix}
\]

We label these possibilities as the “first” and the “second” case. The Cartan matrix of \( b_c \) is

\[
2^{r-1} \begin{pmatrix}
8 & 4 & 3 \\
4 & 2 & 3
\end{pmatrix}
\]

respectively. The inverses of these matrices are

\[
2^{-r-2} \begin{pmatrix}
3 & -4 & -8 \\
-4 & -2 & 4
\end{pmatrix}
\]

respectively. The inverses of these matrices are

\[
2^{-r-2} \begin{pmatrix}
3 & -4 & -8 \\
-4 & -2 & 4
\end{pmatrix}
\]

Let \( m_{\chi^\psi}^{(c,b_c)} \) be the contribution of \( \chi, \psi \in \text{Irr}(B) \) with respect to the subsection \( (c, b_c) \) (see [6]). Then we have

\[
|D|m_{\chi^\psi}^{(c,b_c)} = 3d_{\chi^\varphi_1}^c d_{\psi^\varphi_1}^c - 4(d_{\chi^\varphi_1}^c d_{\psi^\varphi_2}^c + d_{\chi^\varphi_2}^c d_{\psi^\varphi_1}^c) + 8d_{\chi^\varphi_2}^c d_{\psi^\varphi_2}^c
\]

or

\[
|D|m_{\chi^\psi}^{(c,b_c)} = 3d_{\chi^\varphi_1}^c d_{\psi^\varphi_1}^c - 2(d_{\chi^\varphi_1}^c d_{\psi^\varphi_2}^c + d_{\chi^\varphi_2}^c d_{\psi^\varphi_1}^c) + 4d_{\chi^\varphi_2}^c d_{\psi^\varphi_2}^c
\]
respectively. For a character \( \chi \in \text{Irr}(B) \) with height 0 we get

\[
0 = h(\chi) = \nu\left( (D)c_{\chi}^{c,b} \right) = \nu(3d_{\chi \varphi_1}^c) = \nu(d_{\chi \varphi_1}^c)
\]
by (5H) in [6]. In particular \( d_{\chi \varphi_1}^c \neq 0 \). We define \( c_i^j \in \mathbb{Z}^{k(B)} \) by

\[
d_{\chi \varphi_1}^c = \sum_{i=0}^{2^{r-2}-1} c_i^j(\chi)\zeta_{2^r-1}^j
\]
for \( j = 1, 2 \). Then

\[
(c_1^j, c_2^j) = \begin{cases} 
\delta_{ij}16 & \text{first case} \\
\delta_{ij}8 & \text{second case}
\end{cases}
\]

as in Lemma 3.4 (Since the \( 2^{r-2} \) \( B \)-subsections of the form \( c(c, b_\gamma) \) for \( \gamma \in \mathcal{G} \) are pairwise nonconjugate, one can argue like in Lemma 3.4) Hence, in the second case

\[
d_{\chi \varphi_1}^c = \begin{cases} 
\pm \zeta_{2^r-1}^j & \text{if } h(\chi) = 0 \\
\pm 2\zeta_{2^r-1}^j & \text{if } h(\chi) = 1
\end{cases}
\]

holds for a suitable arrangement. Again \( \chi_1, \ldots, \chi_{k(B)} \) are the characters of height 0. In the first case

\[
1 = h(\psi) = \nu\left( (D)c_{\chi}^{c,b} \right) = \nu(3d_{\chi \varphi_1}^c) = \nu(d_{\chi \varphi_1}^c)
\]
by (5G) in [6] for \( h(\psi) = 1 \) and \( h(\chi) = 0 \). As in Lemma 3.5 we also have \( 2 \mid c_i^j(\psi) \) for \( h(\psi) = 1 \) and \( i = 0, \ldots, 2^{r-2} - 1 \). Analogously as in the case \( u \in \mathbb{Z}(D) \setminus \{c\} \) we conclude

\[
d_{\chi \varphi_1}^c = \begin{cases} 
\pm \zeta_{2^r-1} & \text{if } h(\chi) = 0 \\
\pm 2\zeta_{2^r-1} & \text{if } h(\chi) = 1
\end{cases}
\]

(5)

We show that the latter possibility does not occur. In the second case for every character \( \chi \in \text{Irr}(B) \) with height 1 there exists \( i \in \{0, \ldots, 2^{r-2} - 1\} \) such that \( c_i^j(\chi) \neq 0 \). In this case we get

\[
d_{\chi \varphi_2}^c = \begin{cases} 
\pm \zeta_{2^r-1} & \text{if } 1 \leq i \leq 2^r \\
0 & \text{if } 2^r < i \leq k_0(B) \\
\pm \zeta_{2^r-1} & \text{if } k_0(B) < i \leq k(B)
\end{cases}
\]

where \( \chi_1, \ldots, \chi_{k_0(B)} \) are again the characters of height 0. Now let us consider the first case. Since \( (c_1^j, c_2^j) = \delta_{ij}6 \), the value \( \pm 2 \) must occur in every column \( c_i^j \) for \( i = 0, \ldots, 2^{r-2} - 1 \) at least twice. Obviously exactly two entries have to be \( \pm 2 \). Thus, one can improve equation \( \text{5} \) to

\[
d_{\chi \varphi_1}^c = \begin{cases} 
\pm \zeta_{2^r-1} & \text{if } 1 \leq i \leq k_0(B) \\
0 & \text{if } 2^r < i \leq k_0(B) \\
\pm \zeta_{2^r-1} & \text{if } k_0(B) < i \leq k(B)
\end{cases}
\]

It follows

\[
d_{\chi \varphi_2}^c = \begin{cases} 
\pm \zeta_{2^r-1} & \text{if } 1 \leq i \leq 2^r \\
0 & \text{if } 2^r < i \leq k_0(B) \\
\pm \zeta_{2^r-1} & \text{if } k_0(B) < i \leq k(B)
\end{cases}
\]

Hence, the numbers \( d_{\chi \varphi_2}^c \) are independent of the case. Of course, one gets similar results for \( d_{\chi \varphi_1}^u \) with \( \langle u \rangle = \langle c \rangle \).
3.4 The Cartan matrix

Now we investigate the Cartan matrix of $B$.

**Lemma 3.7.** The elementary divisors of the Cartan matrix of $B$ are $2^{r-1}$ and $|D|$.

**Proof.** Let $C$ be the Cartan matrix of $B$. Since $l(B) = 2$, it suffices to show that $2^{r-1}$ occurs as elementary divisor of $C$ at least once. In order to prove this, we use the notion of lower defect groups (see [35]). Let $(u, b)$ be a $B$-subsection with $|(u)| = 2^{r-1}$ and $l(b) = 2$. Let $b_1 := b^{N_G(u)}$. Then $b_1$ has also defect group $D$, and $l(b_1) = 2$ holds. Moreover, $u^{2^{r-2}} \in Z(N_G(u))$. Let $\overline{b_1} \in Bl(R[N_G(u)/\langle u^{2^{r-2}} \rangle])$ be the block which is covered by $b_1$. Then $\overline{b_1}$ has defect group $D/\langle u^{2^{r-2}} \rangle$. We argue by induction on $r$. Thus, let $r = 2$. Then $b = b_1$ and $D/\langle u^{2^{r-2}} \rangle = D/\langle u \rangle \cong D_8$. By Proposition (5G) in [8] the Cartan matrix of $\overline{b}$ has the elementary divisors $1$ and $8$. Hence, $2 = 2^{r-1}$ and $16 = |D|$ are the elementary divisors of the Cartan matrix of $b$. Hence, the claim follows from Theorem 7.2 in [35].

Now assume that the claim already holds for $r - 1 \geq 2$. By induction the elementary divisors of the Cartan matrix of $\overline{b_1}$ are $2^{r-2}$ and $|D|/2$. The claim follows easily as before. \qed

Now we are in a position to calculate the Cartan matrix $C$ up to equivalence of quadratic forms. Here we call two matrices $M_1, M_2 \in \mathbb{Z}^{2 \times 2}$ equivalent if there exists a matrix $S \in GL(l, \mathbb{Z})$ such that $A = SBS^T$, where $S^T$ denotes the transpose of $S$.

By Lemma 3.7 all entries of $C$ are divisible by $2^{r-1}$. Thus, we can consider $\tilde{C} := 2^{1-r}C \in \mathbb{Z}^{2 \times 2}$. Then det $\tilde{C} = 8$ and the elementary divisors of $\tilde{C}$ are $1$ and $8$. If we write

$$\tilde{C} = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix},$$

then $\tilde{C}$ corresponds to the positive definite binary quadratic form $q(x_1, x_2) := c_1 x_1^2 + 2c_2 x_1 x_2 + c_3 x_2^2$. Obviously, $\gcd(c_1, c_2, c_3) = 1$. If one reduces the entries of $\tilde{C}$ modulo 2, then one gets a matrix of rank 1 (this is just the multiplicity of the elementary divisor 1). This shows that $c_1$ or $c_3$ must be odd. Hence, $\gcd(c_1, 2c_2, c_3) = 1$, i.e. $q$ is primitive (see [10] for example). Moreover, $\Delta := -4 \det \tilde{C} = -32$ is the discriminant of $q$. Now it is easy to see that $q$ (and $\tilde{C}$) is equivalent to exactly one of the following matrices (see page 20 in [10]):

$$\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

The Cartan matrices for the block $\overline{b_1}$ with defect group $D_8$ (used before) satisfy

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 8 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Hence, only the second matrix occurs up to equivalence. We show that this holds also for the block $B$.

**Theorem 3.8.** The Cartan matrix of $B$ is equivalent to

$$2^{r-1} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

**Proof.** We argue by induction on $r$. The smallest case was already considered by $b_1$ (this would correspond to $r = 1$). Thus, we may assume $r \geq 2$ (as usual). First, we determine the generalized decomposition numbers $d_{\chi, u}$ for $u \in \langle c \rangle \setminus \{1\}$ with $|(u)| = 2^k < 2^{r-1}$. As in the proof of Theorem 3.6, the group $D/\langle u \rangle$ has the same isomorphism type as $D$, but one has to replace $r$ by $r - k$. Hence, by induction we may assume that $b_u$ has a Cartan matrix which is equivalent to the matrix given in the statement of the theorem. Let $C_u$ be the Cartan matrix of $b_u$, and let $S_u \in GL(2, \mathbb{Z})$ such that

$$C_u = 2^{r-1} S_u^T \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix} S_u,$$
i.e. with the notations of the previous section, we assume that the “second case” occurs. (This is allowed, since we can only compute the generalized decomposition numbers up to multiplication with $S_u$ anyway.) As before we write $\text{IBr}(b_u) = \{\varphi_1, \varphi_2\}$, $D_u := (d^u_{\chi, \varphi_1})$ and $(\tilde{d}^u_{\chi, \varphi_2}) := D_u S_u^{-1}$. The consideration in the previous section carries over, and one gets

$$\tilde{d}^u_{\chi, \varphi_1} = \begin{cases} \pm \zeta_i^{\frac{i-1}{4}} & \text{if } 1 \leq i \leq k_0(B) \\ 0 & \text{if } k_0(B) < i \leq k(B) \end{cases}$$

and

$$\tilde{d}^u_{\chi, \varphi_2} = \begin{cases} \pm \zeta_i^{\frac{i-1}{4}} & \text{if } 1 \leq i \leq 2^r \\ 0 & \text{if } 2^r < i \leq k_0(B) \\ \pm \zeta_i^{\frac{i-1}{2^r-4}} & \text{if } k_0(B) < i \leq k(B) \end{cases}$$

where $\chi_1, \ldots, \chi_{k_0(B)}$ are the characters of height 0. But notice that the ordering of those characters for $\varphi_1$ and $\varphi_2$ is different.

Now assume that there is a matrix $S \in \text{GL}(2, \mathbb{Z})$ such that

$$C = 2r^{-1} S^T \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} S.$$

If $Q$ denotes the decomposition matrix of $B$, we set $(\tilde{d}^u_{\chi, \varphi}) := Q S^{-1}$ for $\text{IBr}(B) = \{\varphi_1, \varphi_2\}$. Then we have

$$|D|m^{(1, B)}_{\chi \varphi} = 8 \tilde{d}^u_{\chi \varphi_1} \tilde{d}^u_{\varphi \varphi_2} + \tilde{d}^u_{\chi \varphi_2} \tilde{d}^u_{\varphi \varphi_1} \text{ for } \chi, \psi \in \text{Irr}(B).$$

In particular $|D|m^{(1, B)}_{\chi \varphi} \equiv 1 \pmod{4}$ for a character $\chi \in \text{Irr}(B)$ of height 0. For $u \in T \setminus Z(D)$ we have $|D|m^{(u, b_u)}_{\chi \varphi} = 2$, and for $u \in Z(D) \setminus \langle c \rangle$ we have $|D|m^{(u, b_u)}_{\chi \varphi} = 1$. Let $u \in \langle c \rangle \setminus \{1\}$. Equation (4) and the considerations above imply $|D|m^{(u, b_u)}_{\chi \varphi} \equiv 3 \pmod{4}$. Now (5B) in [4] reveals the contradiction

$$|D| = \sum_{u \in T} |D|m^{(u, b_u)}_{\chi \varphi} \equiv |D|m^{(1, B)}_{\chi \varphi} + 2r^2 + 2r^{-1} + 3 \cdot (2r^{-1} - 1) \equiv 2 \pmod{4}. \quad \square$$

With the proof of the last theorem we can also obtain the ordinary decomposition numbers (up to multiplication with an invertible matrix):

$$d^u_{\chi, \varphi_1} = \begin{cases} \pm 1 & \text{if } h(\chi) = 0 \\ 0 & \text{if } h(\chi) = 1 \end{cases}, \quad d^u_{\chi, \varphi_2} = \begin{cases} \pm 1 & \text{if } 0 \leq i \leq 2^r \\ 0 & \text{if } 2^r < i \leq k_0(B) \\ \pm 1 & \text{if } k_0(B) < i \leq k(B) \end{cases}.$$ Again $\chi_1, \ldots, \chi_{k_0(B)}$ are the characters of height 0.

Since we know how $G$ acts on the $B$-subsections, we can investigate the action of $G$ on $\text{Irr}(B)$.

**Theorem 3.9.** The irreducible characters of height 0 of $B$ split in $2(r + 1)$ families of $2$-conjugate characters. These families have sizes $1, 1, 1, 2, 2, 4, 4, \ldots, 2r^{-1}, 2r^{-1}$ respectively. The characters of height 1 split in $r$ families with sizes $1, 1, 2, 2, \ldots, 2r^{-2}$ respectively. In particular there are exactly six $2$-rational characters in $\text{Irr}(B)$.

**Proof.** We start by determining the number of orbits of the action of $G$ on the columns of the generalized decomposition matrix. The columns $\{d^u_{\chi, \varphi_1} : \chi \in \text{Irr}(B)\}$ with $u \in T \setminus Z(D)$ split in two orbits of length $2r^{-1}$. For $i = 1, 2$ the columns $\{d^u_{\chi, \varphi_2} : \chi \in \text{Irr}(B)\}$ with $u \in \langle c \rangle$ split in $r$ orbits of lengths $1, 2, 2, 4, \ldots, 2r^{-2}$ respectively. Finally, the columns $\{d^u_{\chi, \varphi_3} : \chi \in \text{Irr}(B)\}$ with $u \in Z(D) \setminus \langle c \rangle$ consist of $r$ orbits of lengths $1, 2, 2, 4, \ldots, 2r^{-2}$ respectively. This gives $3r + 2$ orbits altogether. By Theorem 11 in [3] there exist exactly $3r + 2$ families of $2$-conjugate characters. (Since $G$ is noncyclic, one cannot conclude a priori that also the lengths of the orbits of these two actions coincide.)

By considering the column $\{d^u_{\chi, \varphi_2} : \chi \in \text{Irr}(B)\}$, we see that the irreducible characters of height 0 split in at most $2(r + 1)$ orbits of lengths $1, 1, 1, 2, 2, 4, 4, \ldots, 2r^{-1}, 2r^{-1}$ respectively. Similarly the column $\{d^u_{\chi, \varphi_2} : \chi \in \text{Irr}(B)\}$ shows that there are at most $r$ orbits of lengths $1, 2, 2, 4, \ldots, 2r^{-2}$ of characters of height 1. Since $2(r + 1) + r = 3r + 2$, these orbits do not merge further, and the claim is proved. \qed
Let $M = \langle x^2, y, z \rangle$ as in Lemma 3.3. Then $D \subseteq T_{NG(M)}(b_M)$. Since $e(B) = 1$, Alperin’s fusion theorem implies that $T_{NG(M)}(b_M)$ controls the fusion of $B$-subpairs. By Lemma 3.3 we also have $T_{NG(M)}(b_M) \subseteq CG(c)$ for a $c \in Z(D)$. This shows that $B$ is a so-called “centrally controlled block” (see [22]). In [22] it was shown that then the centers of the blocks $B$ and $b_c$ (regarded as blocks of $FG$) are isomorphic.

### 3.5 Dade’s conjecture

In this section we will verify Dade’s (ordinary) conjecture for the block $B$ (see [12]). First, we need a lemma.

**Lemma 3.10.** Let $\tilde{B}$ be a block of $RG$ with defect group $\tilde{D} \cong C_2 \times C_2^s$ ($s \in \mathbb{N}_0$) and inertial index 3. Then $k(\tilde{B}) = k_0(\tilde{B}) = |\tilde{D}| = 2^{s+2}$ and $l(\tilde{B}) = 3$ hold.

**Proof.** Let $\alpha$ be an automorphism of $\tilde{D}$ of order 3 which is induced by the inertial group. By Lemma 3.2 we have $C_{\tilde{D}}(\alpha) \cong C_2$. We choose a system of representatives $x_1, \ldots, x_k$ for the orbits of $\tilde{D} \setminus C_{\tilde{D}}(\alpha)$ under $\alpha$. Then $k = 2^s$. If $b_i \in \text{Bl}(RC_G(x_i))$ for $i = 1, \ldots, k$ and $b_u \in \text{Bl}(RC_G(u))$ for $u \in C_{\tilde{D}}(\alpha)$ are Brauer correspondents of $\tilde{B}$, then

$$
\bigcup_{i=1}^k \{(x_i, b_i)\} \cup \bigcup_{u \in C_{\tilde{D}}(\alpha)} \{(u, b_u)\}
$$

is a system of representatives for the conjugacy classes of $\tilde{B}$-subsections. Since $\alpha \notin CG(x_i)$, we have $l(b_i) = 1$ for $i = 1, \ldots, k$. In particular $k(\tilde{B}) \leq 2^{s+2}$ holds. Now we show the opposite inequality by induction on $s$.

For $s = 0$ the claim is well known. Let $s \geq 1$. By induction $l(b_u) = 3$ for $u \in C_{\tilde{D}}(\alpha) \setminus \{1\}$. This shows $k(\tilde{B}) - l(\tilde{B}) = k + (2^s - 1)3 = 2^{s+2} - 3$ and $l(\tilde{B}) = 3$. An inspection of the numbers $d_{x,\varphi}^B$ implies $k(\tilde{B}) = k_0(\tilde{B}) = 2^{s+2} = |\tilde{D}|$ and $l(\tilde{B}) = 3$. This would also follow from Theorem 1 in [10].

Now assume $O_2(G) = 1$ (this is a hypothesis of Dade’s conjecture). In order to prove Dade’s conjecture it suffices to consider chains

$$
\sigma : P_1 < P_2 < \ldots < P_n
$$

of nontrivial elementary abelian 2-subgroups of $G$ (see [12]). (Note that also the empty chain is allowed.) In particular $P_i \leq P_n$ and $P_n \leq N_G(\sigma)$ for $i = 1, \ldots, n$. Hence, for a block $b \in \text{Bl}(RN_G(\sigma))$ with $b^2 = B$ and defect group $Q$ we have $P_n \leq Q$. Moreover, there exists a $g \in G$ such that $Q < D$. Thus, by conjugation with $g$ we may assume $P_n \leq Q \leq D$ (see also Lemma 6.9 in [12]). This shows $n \leq 3$.

In the case $|P_n| = 8$ we have $P_n = \langle x^{2^{n-1}}, y, z \rangle = 1'$, because this is the only elementary abelian subgroup of order 8 in $D$. Let $b \in \text{Bl}(RN_G(\sigma))$ with $b^2 = B$. We choose a defect group $Q$ of $\tilde{B} := b^{N_G(E)}$. Since $\Omega(Q) = P_n$, we get $N_G(Q) \leq N_G(E)$. Then Brauer’s first main theorem implies $Q = D$. Hence, $\tilde{B}$ is the unique Brauer correspondent of $B$ in $RN_G(E)$. For $M := \langle x^2, y, z \rangle \leq D$ we also have $N_G(M) \leq N_G(\Omega(M)) = N_G(E)$. Hence, $\tilde{B}$ is nonnilpotent. Now consider the chain

$$
\tilde{\sigma} : \begin{cases} 
\varnothing & \text{if } n = 1 \\
\{P_1\} & \text{if } n = 2 \\
P_1 < P_2 & \text{if } n = 3
\end{cases}
$$

for the group $\tilde{G} := N_G(E)$. Then $N_G(\sigma) = N_G(\tilde{\sigma})$ and

$$
\sum_{b \in \text{Bl}(RN_G(\sigma)), \ b^2 = B} k^i(b) = \sum_{b \in \text{Bl}(RN_G(\tilde{\sigma})), \ b^2 = \tilde{B}} k^i(b).
$$

The chains $\sigma$ and $\tilde{\sigma}$ account for all possible chains of $G$. Moreover, the lengths of $\sigma$ and $\tilde{\sigma}$ have opposite parity. Thus, it seems plausible that the contributions of $\sigma$ and $\tilde{\sigma}$ in the alternating sum cancel out each other (this would imply Dade’s conjecture). The question which remains is: Can we replace $(\tilde{G}, \tilde{B}, \tilde{\sigma})$ by $(G, B, \sigma)$? We make this more precise in the following lemma.
Lemma 3.11. Let \( Q \) be a system of representatives for the \( G \)-conjugacy classes of pairs \((\sigma, b)\), where \( \sigma \) is a chain \((G)\) of length \( n \) with \( P_n < E \) and \( b \in \text{Bl}(R_N G(\sigma)) \) is a Brauer correspondent of \( B \). Similarly, let \( \tilde{Q} \) be a system of representatives for the \( \tilde{G} \)-conjugacy classes of pairs \((\tilde{\sigma}, \tilde{b})\), where \( \tilde{\sigma} \) is a chain \((\tilde{G})\) of length \( n \) with \( P_n < E \) and \( \tilde{b} \in \text{Bl}(R_N \tilde{G}(\tilde{\sigma})) \) is a Brauer correspondent of \( \tilde{B} \). Then there exists a bijection between \( Q \) and \( \tilde{Q} \) which preserves the numbers \( k'(b) \).

**Proof.** Let \( b_D \in \text{Bl}(R_N G(D)) \) be a Brauer correspondent of \( B \). We consider chains of \( B \)-subpairs

\[
\sigma : (P_1, b_1) < (P_2, b_2) < \ldots < (P_n, b_n) < (D, b_D),
\]

where the \( P_i \) are nontrivial elementary abelian \( 2 \)-subgroups such that \( P_n < E \). Then \( \sigma \) is uniquely determined by these subgroups \( P_i, \ldots, P_n \) (see Theorem 1.7 in [36]). Moreover, the empty chain is also allowed. Let \( U \) be a system of representatives for \( G \)-conjugacy classes of such chains. For every chain \( \sigma \in U \) we define

\[
\tilde{\sigma} : (P_1, \tilde{b}_1) < (P_2, \tilde{b}_2) < \ldots < (P_n, \tilde{b}_n) < (D, b_D)
\]

with \( \tilde{b}_i \in \text{Bl}(R_C \tilde{G}(P_i)) \) for \( i = 1, \ldots, n \). Finally we set \( \tilde{U} := \{ \tilde{\sigma} : \sigma \in U \} \). By Alperin’s fusion theorem \( \tilde{U} \) is a system of representatives for the \( \tilde{G} \)-conjugacy classes of corresponding chains for the \( \tilde{B} \). Hence, it suffices to show the existence of bijections \( f \) (resp. \( \tilde{f} \)) between \( U \) (resp. \( \tilde{U} \)) and \( Q \) (resp. \( \tilde{Q} \)) such that the following property is satisfied: If \( f(\sigma) = (\tau, b) \) and \( \tilde{f}(\tilde{\sigma}) = (\tilde{\tau}, \tilde{b}) \), then \( k'(b) = k'(\tilde{b}) \) for all \( i \in [n] \).

Let \( \sigma \in U \). Then we define the chain \( \tau \) by only considering the subgroups of \( \sigma \), i.e. \( \tau : P_i \ldots < P_n \). This gives \( C_G(P_n) \subseteq N_G(\tau) \), and we can define

\[
f : U \rightarrow Q, \quad \sigma \mapsto (\tau, b_n^{N_G(\tau)}).
\]

Now let \((\sigma, b) \in Q\) arbitrary. We write \( \sigma : P_i \ldots < P_n \). By Theorem 5.5.15 in [29] there exists a Brauer correspondent \( \beta_n \in \text{Bl}(R_C \tilde{G}(P_i)) \) of \( b \). Since \( (P_n, \beta_n) \) is a \( B \)-subpair, we may assume \( (P_n, \beta_n) < (D, b_D) \) after a suitable conjugation. Then there are uniquely determined blocks \( \beta_i \in \text{Bl}(R_C \tilde{G}(P_i)) \) for \( i = 1, \ldots, n-1 \) such that

\[
(P_1, \beta_1) < (P_2, \beta_2) < \ldots < (P_n, \beta_n) < (D, b_D).
\]

This shows that \( f \) is surjective.

Now let \( \sigma_1, \sigma_2 \in U \) be given. We write

\[
\sigma_1 : (P_1^i, \beta_1^i) \ldots < (P_n^i, \beta_n^i)
\]

for \( i = 1, 2 \). Let us assume that \( f(\sigma_1) = (\tau_1, b_1) \) and \( f(\sigma_2) = (\tau_2, b_2) \) are conjugate in \( G \), i.e. there is a \( g \in G \) such that

\[
(\tau_2, (g \beta_n^1)^{N_G(\tau_2)}) = g(\tau_1, b_1) = (\tau_2, (\beta_n^2)^{N_G(\tau_2)}).
\]

Since \( g \beta_n^1 \in \text{Bl}(R_C \tilde{G}(P_i)) \) and \( \beta_n^2 \) are covered by \( b_2 \), there is \( h \in N_G(\tau_2) \) with \( b_2 h \beta_n^1 = \beta_n^2 \). Then

\[
b_2 (P_n^1, \beta_n^1) = (P_n^2, \beta_n^2).
\]

Since the blocks \( \beta_j^i \) for \( i = 1, 2 \) and \( j = 1, \ldots, n-1 \) are uniquely determined by \( P_i^j \), we also have \( g^h \sigma_1 = \sigma_2 = \sigma_1 \). This proves the injectivity of \( f \). Analogously, we define the map \( \tilde{f} \).

It remains to show that \( f \) and \( \tilde{f} \) satisfy the property given above. For this let \( \sigma \in U \) with \( (P_1, b_1) < \ldots < (P_n, b_n), \sigma : (P_1, \tilde{b}_1) \ldots < (P_n, \tilde{b}_n), f(\sigma) = (\tau, b_n^{N_G(\tau)}), \tilde{f}(\tilde{\sigma}) = (\tilde{\tau}, b_n^{N_G(\tau)}). \) We have to prove \( k'(b_n^{N_G(\tau)}) = k'(\tilde{b}_n^{N_G(\tau)}) \) for \( i \in [n] \).

Let \( Q \) be a defect group of \( b_n^{N_G(\tau)} \). Then \( Q \subseteq \text{G}(Q) \subseteq N_G(\tau) \), and there is a Brauer correspondent \( \beta_n \in \text{Bl}(R_Q \text{G}(Q)) \) of \( b_n^{N_G(\tau)} \). In particular \((Q, \beta_n) \) is a \( B \)-Brauer subpair. As in Lemma 3.1 we may assume \( Q \in \{D, M, (x, z), (xy, z)\} \). The same considerations also work for the defect group \( \tilde{Q} \) of \( b_n^{N_G(\tau)} \). Since \( b_n^{D_C_G(P_n)} = b_D^{C_G(P_n)} = b_n^{D_C_G(P_n)} \), we get:

\[
Q = D \iff D \subseteq N_G(\tau) \iff D \subseteq \tilde{N}_G(\tau) \iff \tilde{Q} = D.
\]
Let us consider the case \( Q = D \) (\( \equiv \tilde{Q} \)). Let \( b_M \in \text{Bl}(RC_G(M)) \) such that \((M, b_M) \leq (D, b_D)\) and \( \alpha \in T_{N_G(M)}(b_M) \setminus DCG(M) \subseteq N_G(M) \subseteq \tilde{G} \). Then:

\[
b_n^{N_G(\tau)} \text{ is nilpotent} \iff \alpha \notin N_G(\tau) \iff \alpha \notin N_G(\tau) \iff b_n^{N_G(\tau)} \text{ is nilpotent}.
\]

Thus, the claim holds in this case. Now let \( Q < D \) (and \( \tilde{Q} < D \)). Then we have \( QC_G(Q) = C_G(Q) \subseteq C_G(P_n) \). Since \( \beta_{C_G(P_n)} \) is also a Brauer correspondent of \( b_n^{N_G(\tau)} \), the blocks \( \beta_{C_G(P_n)} \) and \( b_n \) are conjugate. In particular \( b_n \) (and \( \tilde{b}_n \)) has defect group \( Q \). Hence, we obtain \( Q = \tilde{Q} \). If \( Q \in \{ \langle x, z \rangle, \langle xy, z \rangle \} \), then \( b_n^{N_G(\tau)} \) and \( \tilde{b}_n^{N_G(\tau)} \) are nilpotent, and the claim holds. Thus, we may assume \( Q = M \). Then as before:

\[
b_n^{N_G(\tau)} \text{ is nilpotent} \iff \alpha \notin N_G(\tau) \iff \alpha \notin N_G(\tau) \iff b_n^{N_G(\tau)} \text{ is nilpotent}.
\]

We may assume that the nonnilpotent case occurs. Then \( t(b_n^{N_G(\tau)}) = t(\tilde{b}_n^{N_G(\tau)}) = 3 \), and the claim follows from Lemma 3.10.

As explained in the beginning of the section, the Dade conjecture follows.

**Theorem 3.12.** The Dade conjecture holds for \( B \).

### 3.6 Alperin’s weight conjecture

In this section we prove Alperin’s weight conjecture for \( B \). Let \((P, \beta)\) be a weight for \( B \), i.e. \( P \) is a 2-subgroup of \( G \) and \( \beta \) is a block of \( R[N_G(P)/P] \) with defect 0. Moreover, \( \beta \) is dominated by a Brauer correspondent \( b \in \text{Bl}(RN_G(P)) \) of \( B \). As usual, one can assume \( P \leq D \). If \( \text{Aut}(P) \) is a 2-group, then \( N_G(P)/C_G(P) \) is also a 2-group. Then \( P \) is a defect group of \( b \), since \( \beta \) has defect 0. Moreover, \( \beta \) is uniquely determined by \( b \). By Alperin’s first main theorem we have \( P = D \). Thus, in this case there is exactly one weight for \( B \) up to conjugation.

Now let us assume that \( \text{Aut}(P) \) is not a 2-group (in particular \( P < D \)). As usual, \( \beta \) covers a block \( \beta_1 \in \text{Bl}(RC_G(P)/P)) \). By the Fong-Reynolds theorem (see [29] for example) also \( \beta_1 \) has defect 0. Hence, \( \beta_1 \) is dominated by exactly one block \( b_1 \in \text{Bl}(RC_G(P)) \) with defect group \( P \). Since \( \beta \beta_1 \neq 0 \), we also have \( bb_1 \neq 0 \), i.e. \( b \) covers \( b_1 \). Thus, the situation is as follows:

\[
\begin{align*}
\beta & \in \text{Bl}(R[N_G(P)/P]) \quad b \in \text{Bl}(RN_G(P)) \\
\beta_1 & \in \text{Bl}(R[C_G(P)/P]) \quad b_1 \in \text{Bl}(RC_G(P))
\end{align*}
\]

By Theorem 5.5.15 in [29] we have \( b_1^{N_G(P)} = b \) and \( b_1^2 = B \). This shows that \( (P, b_1) \) is a \( B \)-Brauer subpair. Then \( P = M \) (\( = \langle x^2, y, z \rangle \)) follows. By Alperin’s first main theorem \( b_1 \) is uniquely determined (independent of \( \beta \)). Now we prove that also \( \beta \) is uniquely determined by \( b \).

In order to do so it suffices to show that \( \beta \) is the only block with defect 0 which covers \( \beta_1 \). By the Fong-Reynolds theorem it suffices to show that \( \beta_1 \) is covered by only one block of \( RT_{N_G(M)/M}(\beta_1) = R[T_{N_G(M)}(b_1)/M] \) with defect 0. For convenience we write \( \overline{C_G(M)} := C_G(M)/M, N_G(M) := N_G(M)/M \) and \( T := T_{N_G(M)}(b_1)/M \). Let \( \chi \in \text{Irr}(\beta_1) \). The irreducible constituents of \( \text{Ind}_{C_G(M)}^T(\chi) \) belong to blocks which covers \( \beta_1 \) (where \( \text{Ind} \) denote induction). Conversely, every block of \( RT \) which covers \( \beta_1 \) arises in this way (see Lemma 5.5.7 in [29]). Let

\[
\text{Ind}_{C_G(M)}^T(\chi) = \sum_{i=1}^t e_i \psi_i
\]

with \( \psi_i \in \text{Irr}(T) \) and \( e_i \in \mathbb{N} \) for \( i = 1, \ldots, t \). Then

\[
\sum_{i=1}^t e_i^2 = |T : \overline{C_G(M)}| = |T_{N_G(M)}(b_1) : C_G(M)| = 6
\]
(see page 84 in [17]). Thus, there is some \( i \in \{1, \ldots, t \} \) with \( e_i = 1 \), i.e. \( \chi \) is extendible to \( \overline{T} \). We may assume \( e_1 = 1 \). By Corollary 6.17 in [17] it follows that \( t = |\text{Irr}(\overline{T}/C_G(M))| = |\text{Irr}(S_3)| = 3 \) and
\[
\{\psi_1, \psi_2, \psi_3\} = \{\psi_1 \tau : \tau \in \text{Irr}(\overline{T}/C_G(M))\},
\]
where the characters in \( \text{Irr}(\overline{T}/C_G(M)) \) were identified with their inflations in \( \text{Irr}(T) \). Thus, we may assume \( e_2 = 1 \) and \( e_3 = 2 \). Then it is easy to see that \( \psi_1 \) and \( \psi_2 \) belong to blocks with defect at least 1. Hence, only the block with contains \( \psi_3 \) is allowed. This shows uniqueness.

Finally we show that there is in fact a weight of the form \((M, \beta)\). For this we choose \( b, b_1, \beta_1, \chi \) and \( \psi_i \) as above. Then \( \chi \) vanishes on all nontrivial 2-elements. Moreover, \( \psi_1 \) is an extension of \( \chi \). Let \( \tau \in \text{Irr}(\overline{T}/C_G(M)) \) be the character of degree 2. Then \( \tau \) vanishes on all nontrivial 2-elements of \( \overline{T}/C_G(M) \). Hence, \( \psi_3 = \psi_1 \tau \) vanishes on all nontrivial 2-elements of \( \overline{T} \). This shows that \( \psi_3 \) belongs in fact to a block \( \beta \in \text{Bl}(\overline{RT}) \) with defect 0. Then \((M, \beta)^{N_G(M)}\) is the desired weight for \( B \).

Hence, we have shown that there are exactly two weights for \( B \) up to conjugation. Since \( l(B) = 2 \), Alperin’s weight conjecture is satisfied.

**Theorem 3.13.** Alperin’s weight conjecture holds for \( B \).

### 3.7 The gluing problem

Finally we show that the gluing problem (see Conjecture 4.2 in [26]) for the block \( B \) has a unique solution. We will not recall the very technical statement of the gluing problem. Instead we refer to [37] for most of the notations. Observe that the field \( F \) is denoted by \( k \) in [37].

**Theorem 3.14.** The gluing problem for \( B \) has a unique solution.

**Proof.** As in [37] we denote the fusion system induced by \( B \) with \( F \). Then the \( F \)-centric subgroups of \( D \) are given by \( M_1 := \langle x^2, y, z \rangle, M_2 := \langle x, z \rangle, M_3 := \langle xy, z \rangle \) and \( D \). We have seen so far that \( \text{Aut}_F(M_i) \cong \text{Out}_F(M_1) \cong S_3, \text{Aut}_F(M_i) \cong D/M_i \cong C_2 \) for \( i = 2, 3 \) and \( \text{Aut}_F(D) \cong D/Z(D) \cong C_2^2 \) (see proof of Lemma 3.3). Using this, we get \( \text{H}^1(\text{Aut}_F(\sigma), F_\sigma^\chi) = 0 \) for \( i = 1, 2 \) and every chain \( \sigma \) of \( F \)-centric subgroups (see proof of Corollary 2.2 in [37]). Hence, \( \text{H}^1([S(F^c)], A_{F^c}^\chi) = \text{H}^1([S(F^c)], A_{F^c}^\chi) = 0 \). Now the claim follows from Theorem 1.1 in [37]. \( \square \)

### 4 The case \( r = s > 1 \)

In the section we assume that \( B \) is a nonnilpotent block of \( RG \) with defect group
\[
D := \langle x, y \mid x^r = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1 \rangle
\]
for \( r \geq 2 \). As before we define \( z := [x, y] \). Since \( |D/\Phi(D)| = 4 \), 2 and 3 are the only prime divisors of \( |\text{Aut}(D)| \).

In particular \( t(B) \in \{1, 3\} \). If \( t(B) = 1 \), then \( B \) would be nilpotent by Theorem 2.4. Thus, we have \( t(B) = 3 \).

#### 4.1 The \( B \)-subsections

We investigate the automorphism group of \( D \).

**Lemma 4.1.** Let \( \alpha \in \text{Aut}(D) \) be an automorphism of order 3. Then \( z \) is the only nontrivial fixed-point of \( Z(D) \) under \( \alpha \).

**Proof.** Since \( D' = \langle z \rangle \), \( z \) remains fixed under all automorphisms of \( D \). Moreover, \( \alpha(x) \in yZ(D) \cup xyZ(D) \), because \( \alpha \) acts nontrivially on \( D/Z(D) \). In both cases we have \( \alpha(x^2) \neq x^2 \). This shows that \( \alpha|_{Z(D)} \in \text{Aut}(Z(D)) \) is also an automorphism of order 3. Obviously \( \alpha \) induces an automorphism of order 3 on \( Z(D)/\langle z \rangle \cong C_{2^{r-1}}^2 \). But this automorphism is fixed-point-free (see Lemma 1 in [27]). The claim follows. \( \square \)
Using this, we can find a system of representatives for the conjugacy classes of $B$-subsections.

**Lemma 4.2.** Let $b \in \text{Bl}(RD C_G(D))$ be a Brauer correspondent of $B$, and for $Q \leq D$ let $b_Q$ be the unique block of $RQ C_G(Q)$ with $(Q, b_Q) \leq (D, b)$. We choose a system $S \subseteq Z(D)$ of representatives for the orbits of $Z(D)$ under the action of $T_{NC_G(D)}(b)$. We set $T := S \cup \{y^t z^j : i, j \in \mathbb{Z}, i \text{ odd}\}$. Then

$$ \bigcup_{a \in T} \{(a, b^S_{C_G(a)})\} $$

is a system of representatives for the conjugacy classes of $B$-subsections. Moreover,

$$ |T| = \frac{5 \cdot 2^{2(r-1)} + 4}{3}. $$

**Proof.** Proposition 2.12.(ii) in [34] states the desired system wrongly. More precisely the claim $I_D = Z(D)$ in the proof is false. Indeed Lemma 4.1 shows $I_D = S$. Now the claim follows easily. \( \square \)

From now on we write $b_a := b^S_{C_G(a)}$ for $a \in T$. We are able to determine the difference $k(B) - l(B)$.

**Proposition 4.3.** We have

$$ k(B) - l(B) = \frac{5 \cdot 2^{2(r-1)} + 7}{3}. $$

**Proof.** Consider $l(b_a)$ for $1 \neq a \in T$.

**Case 1:** $a \in Z(D)$.

Then $b_a$ is a block with defect group $D$. Moreover, $b_a$ and $B$ have a common Brauer correspondent in $\text{Bl}(RD C_G(a)(D)) = \text{Bl}(RD C_G(D))$. In case $a \neq z$ we have $t(b_a) = 1$ by Lemma 4.1. Hence, $b_a$ is nilpotent and $l(b_a) = 1$. Now let $a = z$. Then there exists a block $b_z$ of $C_G(z)/\langle z \rangle$ with defect group $D/\langle z \rangle \cong C_2^r$ and $l(b_z) = l(b_z)$. By Theorem 1.5(iv) in [33], $t(b_z) = t(b_z) = 3$ holds. Thus, Theorem 2 in [33] implies $l(b_z) = l(b_z) = 3$.

**Case 2:** $a \notin Z(D)$.

Then $b_{C_F(a)} = b_M$ is a block with defect group $M := \langle x^2, y, z \rangle$. Since $b_M^{C_G(M)} = b_D^{C_G(M)}$, also $b_M^{C_G(a)} = b_a$ has defect group $M$. For every automorphism $\alpha \in \text{Aut}(D)$ of order 3 we have $\alpha(M) \neq M$. Since $D$ controls the fusion of $B$-subpairs, we get $t(b_a) = l(b_a) = 1$.

Now the conclusion follows from $k(B) = \sum_{a \in T} l(b_a)$.

The next result concerns the Cartan matrix of $B$.

**Lemma 4.4.** The elementary divisors of the Cartan matrix of $B$ are contained in $\{1, 2, |D|\}$. The elementary divisor 2 occurs twice and $|D|$ occurs once (as usual). In particular $l(B) \geq 3$.

**Proof.** Let $C$ be the Cartan matrix of $B$. As in Lemma 3.7 we use the notion of lower defect groups. For this let $P \leq D$ such that $|P| \geq 4$, and let $b \in \text{Bl}(RN_C(P))$ be a Brauer correspondent of $B$ with defect group $Q \leq D$. Brauer’s first main theorem implies $P < Q$. By Proposition 1.3 in [33] there exists a block $\beta \in \text{Bl}(RNC_G(P))$ with $\beta^{NC_G(P)} = b$ such that at most $l(\beta)$ lower defect groups of $b$ contain a conjugate of $P$. Let $S \leq Q$ be a defect group of $\beta$. First, we consider the case $S = D$. Then $P \subseteq Z(D)$. By Lemma 4.1 we have $l(b) = 1$, since $|P| \geq 4$. It follows that $m_B^1(P) = m_B(P) = 0$, because $P$ is contained in the (lower) defect group $Q$ of $b$.

Now assume $S < D$. In particular $S$ is abelian. If $S$ is even metacyclic, then $l(\beta) = 1$ and $m_B^1(P) = 0$, since $P \subseteq Z(C_G(P))$. Thus, let us assume that $S$ is nonmetacyclic. By (3C) in [33], $x^2 \in Z(D)$ is conjugate to an element of $Z(S)$. This shows $S \cong C_{2^k} \times C_{2^l} \times C_2$ with $k \in \{r, r - 1\}$ and $1 \leq l \leq r$. If $1, k, l$ are pairwise distinct, then $l(\beta) = 1$ and $m_B^1(P) = 0$ follow from Lemma 2.3. Let $k = l$. Then every automorphism of $S$ of order 3 has only one nontrivial fixed-point. Since $|P| \geq 4$, it follows again that $l(\beta) = 1$ and $m_B^1(P) = 0$.

Now let $S \cong C_{2^k} \times C_{2^l}^2$ with $2 \leq k \in \{r - 1, r\}$. Assume first that $P$ is noncyclic. Then $S/P$ is metacyclic. If $S/P$ is not a product of two isomorphic cyclic groups, then $l(\beta) = 1$ and $m_B^1(P) = 0$. Hence, we may assume
It is easy to see that there exists a subgroup $P_1 \leq P$ with $S/P_1 \cong C_4 \times C_2$. We get $l(\beta) = 1$ and $m^1_b(P) = 0$ also in this case.

Finally, let $P = \langle u \rangle$ be cyclic. Then $(u, \beta)$ is a $B$-subsection. Since $|P| \geq 4$, $u$ is not conjugate to $z$. As in the proof of Proposition 4.3, we have $l(\beta) = 1$ and $m^1_b(P) = 0$. This shows $m^1_b(P) = 0$. Since $P$ was arbitrary, the multiplicity of $|P|$ as an elementary divisor of $C$ is 0.

It remains to consider the case $|P| = 2$. We write $P = \langle u \rangle \leq D$. As before let $b \in \text{Bl}(RN_G(P))$ be a Brauer correspondent of $B$. Then $(u, b)$ is a $B$-subsection. If $(u, b)$ is not conjugate to $(z, b_z)$, then $l(b) = 1$ and $m^1_b(P) = 0$ as in the proof of Proposition 4.3. Since we can replace $P$ by a conjugate, we may assume $P = (z)$ and $(u, b) = (z, b_z)$. Then $l(b) = 3$ and $D$ is a defect group of $b$. Now let $\overline{b} \in \text{Bl}(RN_G(P)/P)$ be the block which is dominated by $b$. By Corollary 1 in [16], the elementary divisors of the Cartan matrix of $\overline{b}$ are 1, 1, $|D|/2$.

Hence, the elementary divisors of the Cartan matrix of $b$ are $2, 2, |D|$. This shows

$$2 = \sum_{Q \in \mathcal{P}(N_G(P)), |Q| = 2} m^1_b(Q),$$

where $\mathcal{P}(N_G(P))$ is a system of representatives for the conjugacy classes of $p$-subgroups of $N_G(P)$. The same arguments applied to $b$ imply $m^1_b(Q) = 0$ for $P \neq Q \leq N_G(P)$ with $|Q| = 2$. Hence, $2 = m^1_b(P) = m^2_b(P)$, and 2 occurs as elementary divisors of $C$ twice.

As in Section 3, we write $\text{IBr}(b_u) = \{\varphi_u\}$ for $u \in T \setminus \{z\}$. In a similar manner we define the integers $a^n_i$. If $u \in T \setminus \{z\}$ with $|\langle u \rangle| = 2^k > 2$, then the $2^k - 1$ distinct subsections of the form $\gamma(u, b_u)$ for $\gamma \in G$ are pairwise nonconjugate (same argument as in the case $r > s = 2$). Hence, Lemma 3.4 carries over in a corresponding form. Apart from that we can also carry over Lemma 6(B) in [20]:

**Lemma 4.5.** Let $\chi \in \text{Irr}(B)$ and $u \in T \setminus Z(D)$. Then $\chi$ has height 0 if and only if the sum

$$2^{r-1-1} \sum_{i=0}^{2^{r-1}-1} a^n_i(\chi)$$

is odd.

**Proof.** If $\chi$ has height 0, the sum is odd by Proposition 1 in [39]. The other implication follows easily from (5G) in [6].

The next lemma is the analogon to Lemma 3.5.

**Lemma 4.6.** Let $u \in Z(D) \setminus \{z\}$ of order $2^k$. Then for all $\chi \in \text{Irr}(B)$ we have:

(i) $2^k(\chi) \mid a^n_i(\chi)$ for $i = 0, \ldots, 2^k - 1$,

(ii) $\sum_{i=0}^{2^k-1} a^n_i(\chi) \equiv 2^k(\chi) \mod 2^{k+1}(\chi+1)$.

As in the case $r > s = 1$, Lemma 1.1 in [39] implies

$$k(B) \leq \sum_{i=0}^{\infty} 2^{2i} k_i(B) \leq |D|. \tag{6}$$

In particular Brauer’s $k(B)$-conjecture holds. Moreover, Theorem 3.1 in [39] gives $k_0(B) \leq |D|/2 = |D : D'|$, i.e. Olsson’s conjecture is satisfied. Using this, we can improve the inequality (6) to

$$|D| \geq 4k_0(B) + 4(k(B) - k_0(B)) = 4k(B) - 3k_0(B) \geq 4k(B) - \frac{3|D|}{2}$$

and

$$\frac{5 \cdot 2^{2(r-1)} + 16}{3} \leq k(B) - l(B) + l(B) = k(B) \leq \frac{5|D|}{8} = 5 \cdot 2^{2(r-1)}.$$
We will improve this further. Let $\overline{b}_2$ be the block of $\text{Bl}(RC_G(z)/\langle z \rangle)$ which is dominated by $b_2$. Then $\overline{b}_2$ has defect group $D/\langle z \rangle \cong C_2^2$. Using the existence of a perfect isometry (see [13], [15], [33]), one can show that the Cartan matrix of $\overline{b}_2$ is equivalent to

$$\mathcal{C} := \frac{1}{3} \begin{pmatrix} 2^{2r+2} & 2^{2r} - 1 & 2^{2r-1} \\ 2^{2r} - 1 & 2^{2r+2} & 2^{2r-1} \\ 2^{2r-1} & 2^{2r} - 1 & 2^{2r+2} \end{pmatrix}.$$ 

Hence, the Cartan matrix of $b_2$ is equivalent to $2\mathcal{C}$. Now inequality $(\ast \ast)$ in [24] yields

$$k(B) \leq 2 \frac{2^r + 8}{3} = |D| + \frac{16}{3}.$$ 

(Notice that the proof of Theorem A in [24] also works for $b_2$ instead of $B$, since the generalized decomposition numbers corresponding to $(z, b_2)$ are integral. See also Lemma 3 in [12].)

In addition we have

$$k_i(B) = 0 \text{ for } i \geq 4$$

by Corollary (6D) in [7]. This means that the heights of the characters in $\text{Irr}(B)$ are bounded independently of $r$. We remark also that Alperin’s weight conjecture is equivalent to

$$l(B) = l(b)$$

for the Brauer correspondent $b \in \text{Bl}(RN_G(D))$ of $B$ (see Consequence 5 in [1]). Since $z \in Z(N_G(D))$, $l(B) = l(b) = 3$ and $k(B) = (5 \cdot 2^{(r-1)} + 16)/3$ would follow in this case (see proof of Proposition 4.3).

### 4.2 The gluing problem

As in section 3.7 we use the notations of [37].

**Theorem 4.7.** The gluing problem for $B$ has a unique solution.

**Proof.** Let $F$ be the fusion system induced by $B$. Then the $\mathcal{F}$-centric subgroups of $D$ are given by $M := \langle x^2, y, z \rangle$ and $D$ (up to conjugation in $F$). We have $\text{Aut}_F(M) \cong D/M \cong C_2$ and $\text{Aut}_F(D) \cong A_4$. This shows $H^2(\text{Aut}_F(\sigma), F^\times) = 0$ for every chain $\sigma$ of $\mathcal{F}$-centric subgroups. Consequently, $H^0([S(F^\circ)], A_2^\circ) = 0$. On the other hand, we have $H^1(\text{Aut}_F(D), F^\times) \cong H^1(C_3, F^\times) \cong C_3$ and $H^2(\text{Aut}_F(\sigma), F^\times) = 0$ for all chains $\sigma \neq D$. Hence, the situation is as in Case 3 of the proof of Theorem 1.2 in [37]. However, the proof in [37] is pretty short. For the convenience of the reader, we give a more complete argument.

Since $[S(F^\circ)]$ is partially ordered by taking subchains, one can view $[S(F^\circ)]$ as a category, where the morphisms are given by the pairs of ordered chains. In particular $[S(F^\circ)]$ has exactly five morphisms. With the notations of [37] the functor $A_2^\circ$ is a representation of $[S(F^\circ)]$ over $\mathbb{Z}$. Hence, we can view $A_2^\circ$ as a module $\mathcal{M}$ over the incidence algebra of $[S(F^\circ)]$. More precisely, we have

$$\mathcal{M} := \bigoplus_{a \in \text{Ob}[S(F^\circ)]} A_2^\circ(a) = A_2^\circ(D) \cong C_3.$$ 

Now we can determine $H^1([S(F^\circ)], A_2^\circ)$ using Lemma 6.2(2) in [37]. For this let $d : \text{Hom}[S(F^\circ)] \to \mathcal{M}$ a derivation. Then we have $d(\alpha) = 0$ for all $\alpha \in \text{Hom}[S(F^\circ)]$ with $\alpha \neq (D, D) =: \alpha_1$. However,

$$d(\alpha_1) = d(\alpha_1 \alpha_1) = (A_2^\circ(\alpha_1))(d(\alpha_1)) + d(\alpha_1) = 2d(\alpha_1) = 0.$$ 

Hence, $H^1([S(F^\circ)], A_2^\circ) = 0$. \qed
4.3 Special cases

Since the general methods do not suffice to compute the invariants of $B$, we restrict ourself to certain special situations.

Proposition 4.8. If $O_2(G) \neq 1$, then

$$k(B) = \frac{5 \cdot 2^{(r-1)} + 16}{3}, \quad k_0(B) \geq \frac{2^{2r} + 8}{3}, \quad l(B) = 3.$$

Proof. Let $1 \neq Q := O_2(G).$ Then $Q \subseteq D.$ In the case $Q = D'$ we have $C_G(z) = N_G(Q) = G$ and $B = b_z.$ Then the assertions on $k(B)$ and $l(B)$ are clear. Moreover, $b_z$ dominates a block $\beta_2 \in \text{Bl}(RC_G(z)/(z))$ with defect group $C_2^2.$ By Theorem 2 in [13] we have

$$k_0(B) \geq k_0(b_z) = k(b_z) = \frac{2^{2r} + 8}{3}.$$

Hence, we may assume $Q \neq D'.$ With the same argument we may also assume $Q < D.$ In particular $Q$ is abelian. We consider a $B$-subpair $(Q, b_Q).$ Then $D$ or $M$ is a defect group of $b_Q$ (see proof of Lemma 4.2). If $D$ is a defect group of $b_Q,$ then $D \subseteq C_G(Q)$ and $Q \subseteq Z(D).$ By Lemma 4.1 it follows that $b_Q$ is nilpotent.

Now let us assume that $M$ is a defect group of $b_Q.$ Since $D$ controls the fusions of $B$-subpairs, we have $l(b_Q) = 1$ (see Case 2 in the proof of Proposition 1.3). Hence, again $b_Q$ is nilpotent. Thus, in both cases $B$ is an extension of a nilpotent block of $BL(RC_G(Q)).$ In this situation the Külshammer-Puig theorem applies. In particular we can replace $B$ by a block with normal defect group (see [23]). Hence, $B = b_z,$ and the claim follows as before. \qed

Since $N_G(D) \subseteq C_G(z),$ $B$ is a $B$-subpair (see [22]). In [22] it was shown that then an epimorphism $Z(B) \to Z(b_z)$ exists, where one has to regard $B$ (resp. $b_z$) as blocks of $FG$ (resp. $F C_G(z)).$ Moreover, we conjecture that the blocks $B$ and $b_z$ are Morita equivalent. For the similar defect group $Q_b$ this holds in fact (see [13]). In this context the work [11] is also interesting. There is was shown that there is a perfect isometry between any two blocks with the same quaternion group as defect group and the same fusion of subpairs. Thus, it would be also possible that there is a perfect isometry between $B$ and $b_z.$

Proposition 4.9. In order to determine $k(B)$ (and thus also $l(B)$), we may assume that $O_2(G)$ is trivial and $O_2(G) = Z(G) = F(G)$ is cyclic. Moreover, we can assume that $G$ is an extension of a solvable group by a quasisimple group. In particular $G$ has only one nonabelian composition factor.

Proof. By Proposition 4.8 we may assume $O_2(G) = 1.$ Now we consider $O(G) := O_2(G).$ Using Clifford theory we may assume that $O(G)$ is central and cyclic (see e.g. Theorem X.1.2 in [15]). Since $O_2(G) = 1,$ we get $O(G) = Z(G).$ Let $E(G)$ be the normal subgroup of $G$ generated by the components. As usual, $B$ covers a block $b$ of $E(G).$ By Fong-Reynolds we can assume that $b$ is stable in $G.$ Then $d := D \cap E(G)$ is a defect group of $b.$ By the Külshammer-Puig result we may assume that $b$ is nonnilpotent. In particular $d$ has rank at least 2. Let $C_1, \ldots, C_n$ be the components of $G.$ Then $E(G)$ is the central product of $C_1, \ldots, C_n.$ Since $[C_i, C_j] = 1$ for $i \neq j,$ $b$ covers exactly one block $\beta_i$ of $RC_i$ for $i = 1, \ldots, n.$ Then $b$ is dominated by the block $\beta_1 \otimes \ldots \otimes \beta_n$ of $R(C_1 \times \ldots \times C_n).$ Since $Z(C_1)$ is abelian and subnormal in $G,$ it must have odd order. Hence, we may identify $b$ with $\beta_1 \otimes \ldots \otimes \beta_n$ (see Proposition 1.5 in [13]). In particular $d = \delta_1 \times \ldots \times \delta_n,$ where $\delta_i := d \cap C_i$ is a defect group of $\beta_i$ for $i = 1, \ldots, n.$ Assume that $\delta_i$ is cyclic. Then $\beta_i$ is nilpotent and isomorphic to $(R\delta_i)^m \times m$ for some $m \in \mathbb{N}$ by Puig. Let $\{C_1, \ldots, C_k\}$ be the orbit of $C_1$ under the conjugation action of $G (k \leq n).$ Then $\beta_1 \otimes \ldots \otimes \beta_k \cong (R\delta_1)^{m_1 \times m_k}$ for some $m_1, m_k \in \mathbb{N}$ is a block of $R(C_1 \ldots C_k)$ with $l(\beta_1 \otimes \ldots \otimes \beta_k) = 1.$ Lemma 2.1 [14] implies $k \leq 2$ or $k = 3$ and $|\delta_1| = 2.$ In the first case Theorem 2 in [13] shows that $\beta_1 \otimes \ldots \otimes \beta_k$ is nilpotent. This also holds in the second case by [23]. Since $C_1 \ldots C_k \leq G,$ $B$ is an extension of a nilpotent block. This shows that we can assume that the groups $\delta_i$ are noncyclic for $i = 1, \ldots, n.$ By Lemma 2.1 [14], $d$ has rank at most 3. Hence, $n = 1$ and $E(G) = C_1.$

That means in order to determine the invariants of the block $B$ we may assume that $G$ contains only one component. Let $F(G)$ (resp. $F^*(G)$) be the Fitting subgroup (resp. generalized Fitting subgroup) of $G.$ Since $F(G) = Z(G),$ we have $C_G(E(G)) = C_G(F^*(G)) \leq F(G).$ Hence, $C_G(E(G))$ is nilpotent. On the other hand, the quotient $G/C_G(E(G))$ is isomorphic to a subgroup of the automorphism group of the quasisimple group $E(G).$
Consider the canonical map \( f : \text{Aut}(E(G)) \to \text{Aut}(E(G)/Z(E(G))) \). Let \( \alpha \in \ker f \). Then \( \alpha(g)g^{-1} \in Z(E(G)) \) for all \( g \in E(G) \). Hence, we get a map \( \beta : E(G) \to Z(E(G)) \), \( g \mapsto \alpha(g)g^{-1} \). Moreover, it is easy to see that \( \beta \) is a homomorphism. Since \( E(G) \) is perfect, we get \( \beta = 1 \) and thus \( \alpha = 1 \). This shows \( \text{Aut}(E(G)) \leq \text{Aut}(E(G)/Z(E(G))) \).

By Schreier’s conjecture (which can be proven using the classification) \( \text{Aut}(E(G)/Z(E(G))) \) is an extension of the solvable group \( \text{Out}(E(G)/Z(E(G))) \) by the simple group \( \text{Inn}(E(G)/Z(E(G))) \cong E(G)/Z(E(G)) \). Taking these facts together, we see that \( G \) has only one nonabelian composition factor. In particular \( G \) is an extension of a soluble group by a quasisimple group.

Now we consider blocks of maximal defect, i.e. \( D \) is a Sylow 2-subgroup of \( G \). These include principal blocks.

**Proposition 4.10.** If \( B \) has maximal defect, then \( G \) is solvable. In particular Alperin’s weight conjecture is satisfied, and we have

\[
\begin{align*}
  k(B) &= \frac{5 \cdot 2^{2(r-1)} + 16}{3}, \\
  k_0(B) &= \frac{2^{2r} + 8}{3}, \\
  k_1(B) &= \frac{2^{2(r-1)} + 8}{3}, \\
  l(B) &= 3.
\end{align*}
\]

**Proof.** By Feit-Thompson we may assume \( O_2'(G) = 1 \) in order to show that \( G \) is soluble. We apply the \( Z^* \)-theorem. For this let \( g \in G \) such that \( g z \in D \). Since all involutions of \( D \) are central (in \( D \)), we get \( g z \in Z(D) \).

By Burnside’s fusion theorem there exists \( h \in N_G(D) \) such that \( h z = g z \). (For principal blocks this would also follow from the fact that \( D \) controls fusion.) Since \( D' = \langle z \rangle \), we have \( g z = z \). Now the \( Z^* \)-theorem implies \( z \in Z(G) \). Then \( D/\langle z \rangle \cong C_2^r \) is a Sylow 2-subgroup of \( G/\langle z \rangle \). By Theorem 1 in [4], \( G/\langle z \rangle \) is solvable. Hence, also \( G \) is solvable. Since Alperin’s weight conjecture holds for soluble groups, we obtain the numbers \( k(B) \) and \( l(B) \).

It is also known that the Alperin-McKay-conjecture holds for soluble groups (see [32]). Thus, in order to determine \( k_0(B) \) we may assume \( D \leq G \). Then we can apply the results of [21]. For this let \( L := D \rtimes C_3 \). Then \( B \cong (RL)^{n \times n} \) for some \( n \in \mathbb{N} \). Hence, \( k_0(B) \) is just the number of irreducible characters of \( L \) with odd degree.

By Clifford, every irreducible character of \( L \) is an extension or an induction of a character of \( D \). Thus, it suffices to count the characters of \( L \) which arise from linear characters of \( D \). These linear characters of \( D \) are just the inflations of \( \text{Irr}(D/D') \). They split into the trivial character and orbits of length 3 under the action of \( L \) by Brauer’s permutation lemma. The three inflations of \( \text{Irr}(L/D) \) are the extensions of the trivial character of \( D \). The other linear characters of \( D \) remain irreducible after induction. Characters in the same orbit amount to the same character of \( L \). This shows

\[
k_0(B) = 3 + \frac{|D/D'| - 1}{3} = \frac{2^{2r} + 8}{3}.
\]

By Theorem 1.4 in [28] we have \( k_i(B) = 0 \) for \( i \geq 2 \). We conclude

\[
k_1(B) = k(B) - k_0(B) = \frac{5 \cdot 2^{2(r-1)} + 16}{3} - \frac{2^{2r} + 8}{3} = \frac{2^{2(r-1)} + 8}{3}.
\]

The last result implies that Brauer’s height zero conjecture is also satisfied for blocks of maximal defect. Moreover, the Dade-conjecture holds for soluble groups (see [10]).

Finally we consider the case \( r = 2 \) (i.e. \( |D| = 32 \)) for arbitrary groups \( G \).

**Proposition 4.11.** If \( r = 2 \), we have

\[
k(B) = 12, \quad k_0(B) = 8, \quad k_1(B) = 4, \quad l(B) = 3.
\]

There are two pairs of \( 2 \)-conjugate characters of height 0. The remaining characters are \( 2 \)-rational. Moreover, the Cartan matrix of \( B \) is equivalent to

\[
\begin{pmatrix}
  4 & 2 & 2 \\
  2 & 4 & 2 \\
  2 & 2 & 12
\end{pmatrix}.
\]

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Proof. The proof is somewhat lengthy and consists entirely of technical calculations. For this reason we will only outline the argumentation. Since \( k_0(B) \) is divisible by 4, inequality (6) implies \( k_0(B) \geq 8 \). Since there are exactly two pairs of 2-conjugate \( B \)-subsections, Brauer’s permutation lemma implies that we also have two pairs of 2-conjugate characters. Hence, the column \( a^1_y \) contains at most four nonvanishing entries. Since \( (a^1_y, a^1_y) = 8 \), there are just two nonvanishing entries, both are \( \pm 2 \). Now Lemma 4.5 implies \( k_0(B) = 8 \). This shows \( (k(B), k_1(B), l(B)) \in \{(12, 4, 3), (14, 6, 5)\} \).

By way of contradiction, we assume \( k(B) = 14 \). Then one can determine the numbers \( d^u_{\nu \varphi} \) for \( u \neq 1 \) with the help of the contributions. However, there are many possibilities. The ordinary decomposition matrix \( Q \) can be computed as the orthogonal space of the other columns of the generalized decomposition matrix. Finally we obtain the Cartan matrix of \( B \) as \( C = Q^TQ \). In all cases is turns out that \( C \) has the wrong determinant (see Lemma 4.4). This shows \( k(B) = 12, k_1(B) = 4 \) and \( l(B) = 3 \).

Again we can determine the numbers \( d^u_{\nu \varphi} \) for \( u \neq 1 \). This yields the heights of the 2-conjugate characters. We also obtain some informations about the Cartan invariants in this way. We regard the Cartan matrix \( C \) as a quadratic form. Using the tables [31, 30] we conclude that \( C \) has the form given in the statement of the proposition.

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