Leavitt path algebras over a poset of fields

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Abstract. Let $E$ be a finite directed graph, and let $I$ be the poset obtained as the antisymmetrization of its set of vertices with respect to a pre-order $\leq$ that satisfies $v \leq w$ whenever there exists a directed path from $w$ to $v$. Assuming that $I$ is a tree, we define a poset of fields over $I$ as a family $K = \{ K_i : i \in I \}$ of fields $K_i$ such that $K_i \subseteq K_j$ if $j \leq i$. We define the concepts of a Leavitt path algebra $L_K(E)$ and a regular algebra $Q_K(E)$ over the poset of fields $K$, and we show that $Q_K(E)$ is a hereditary von Neumann regular ring, and that its monoid $V(Q_K(E))$ of isomorphism classes of finitely generated projective modules is canonically isomorphic to the graph monoid $M(E)$ of $E$.

Introduction

The work in the present paper is instrumental for the constructions developed in [4] (see also [5]). The main result in [4] states that for every finitely generated conical refinement monoid $M$ and every field $K$ there exists a von Neumann regular $K$-algebra $R$ such that $\mathcal{V}(R) \cong M$. The algebra $R$ is a certain universal localization of a precisely defined Steinberg algebra. The class of Steinberg algebras has been intensively studied in the last few years, see for instance the recent survey [17] and its references. It includes as prominent examples Leavitt path algebras of arbitrary graphs. For the proof of the main result of [4], one needs to interpret the algebra $R$ as an algebra obtained using certain building blocks, using constructions which were partially introduced in [3] and fully developed in [4]. The purpose of this paper is to introduce these building blocks in a general way, and to show their basic properties. These algebras generalize the main construction in [6] of the regular algebra $Q_K(E)$ of a finite directed graph $E$. The main features of the $K$-algebra $Q_K(E)$ are that it is von Neumann regular and that $\mathcal{V}(Q_K(E)) \cong M(E)$, where $M(E)$ is the graph monoid of $E$ (see Section 2 for the definition of the graph monoid).

This is generalized here as follows. Let $E$ be a finite graph and let $\leq$ be a pre-order on $I$ such that $v \leq w$ whenever there exists a directed path from $w$ to $v$. Let $I = E^0/\sim$ be the antisymmetrization of $E^0$ with respect to the pre-order $\leq$, endowed with its canonical partial order, also denoted by $\leq$. We assume throughout that $I$ is a tree, that is, that there exists a maximum element $i_0$ in $I$ and that for

2000 Mathematics Subject Classification. Primary 16D70; Secondary 06A12, 06F05, 46L80.
Key words and phrases. von Neumann regular ring, path algebra, Leavitt path algebra, universal localization.

Partially supported by DGI-MINECO-FEDER through the grant MTM2017-83487-P and by the Generalitat de Catalunya through the grant 2017-SGR-1725.
each $i \in I$ the interval $[i, i_0]$ is a chain. Under these assumptions, we define a poset of fields $K = \{K_i : i \in I\}$ as a collection of fields $K_i$, for $i \in I$, such that $K_i \subseteq K_j$ whenever $j \leq i$. The usual Leavitt path algebra $L_K(E)$ and regular algebra $Q_K(E)$ over a field $K$ are here generalized to a Leavitt path algebra $L_K(E)$ and a regular algebra $Q_K(E)$ with coefficients in the given poset of fields $K$. The main results state that $Q_K(E)$ is a hereditary von Neumann regular ring and that $\mathcal{V}(Q_K(E)) \cong M(E)$ canonically. Our techniques here extend the ones introduced in [7], where a more restricted scope was used to study algebras over an ordered finite sequence of fields.

For more information on the realization problem we refer the reader to [6], [3], [8], and to the survey papers [15], [2].

1. Preliminary definitions

All rings in this paper will be associative and all monoids will be abelian. A (not necessarily unital) ring $R$ is von Neumann regular if for every $a \in R$ there is $b \in R$ such that $a = aba$. Our basic reference for the theory of von Neumann regular rings is [14].

For a unital ring $R$, the monoid $\mathcal{V}(R)$ is the monoid of isomorphism classes of finitely generated projective left $R$-modules, where the operation is induced by direct sum. If $R$ is an exchange ring (in particular, if $R$ is von Neumann regular), then $\mathcal{V}(R)$ is a conical refinement monoid, see [9, Corollary 1.3].

Say that a subset $A$ of a poset $I$ is a lower subset in case $q \leq p$ and $p \in A$ imply $q \in A$. We use $I \downarrow i$ to denote the lower subset of $I$ consisting of all the elements $j$ such that $j \leq i$.

For an element $p$ of a poset $I$, write

$$L(p) = L(I, p) = \{q \in I : q < p \text{ and } [q, p] = \{q, p\}\}.$$ 

An element of $L(p)$ is called a lower cover of $p$.

In the following, $K$ will denote a field and $E = (E^0, E^1, r, s)$ a finite quiver (directed graph). Here $s(e)$ is the source vertex of the arrow $e$, and $r(e)$ is the range vertex of $e$. A path in $E$ is either an ordered sequence of arrows $\alpha = e_1 \cdots e_n$ with $r(e_t) = s(e_{t+1})$ for $1 \leq t < n$, or a path of length 0 corresponding to a vertex $v \in E^0$. The paths of length 0 are called trivial paths. A non-trivial path $\alpha = e_1 \cdots e_n$ has length $n$ and we define $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$. We will denote the length of a path $\alpha$ by $|\alpha|$, the set of all paths of length $n$ by $E^n$, for $n > 1$, and the set of all paths by $\text{Path}(E)$.

For $v, w \in E^0$, set $v \geq w$ in case there is a (directed) path from $v$ to $w$. A subset $H$ of $E^0$ is called hereditary if $v \geq w$ and $v \in H$ imply $w \in H$.

Let us recall the construction from [6] of the regular algebra $Q_K(E)$ of a quiver $E$, although we will follow the presentation in [3] rather than the used in [6]. That is, relations (CK1) and (CK2) below are reversed with respect to their counterparts in [6], so that we are led to work primarily with left modules instead of right modules. Therefore we recall the basic features of the regular algebra $Q_K(E)$ in terms of the notation used here. We will only need finite quivers in the present paper, so we restrict attention to them. The algebra $Q_K(E)$ fits into the following commutative diagram of injective algebra morphisms:
Here \( P_K(E) \) is the path \( K \)-algebra of \( E \), \( E^* \) denotes the inverse quiver of \( E \), that is, the quiver obtained by reversing the orientation of all the arrows in \( E \), \( P_K((E)) \) is the algebra of formal power series on \( E \), and \( P_K^{rat}(E) \) is the algebra of rational series, which is by definition the division closure of \( P_K(E) \) in \( P_K((E)) \) (which agrees with the rational closure \([6, Observation 1.18]\)).

The maps \( \iota_{\Sigma} \) and \( \iota_{\Sigma_1} \) indicate universal localizations with respect to the sets \( \Sigma \) and \( \Sigma_1 \) respectively. (We refer the reader to \([13] \) and \([18] \) for the general theory of universal localization.) Here \( \Sigma \) is the set of all square matrices over \( P_K(E) \) that are sent to invertible matrices by the augmentation map \( \epsilon: P_K(E) \to K^{e_0^0} \). By \([6, Theorem 1.20]\), the algebra \( P_K^{rat}(E) \) coincides with the universal localization \( P_K(E)\Sigma^{-1} \). The set \( \Sigma_1 = \{ \mu_v \mid v \in E^0, s^{-1}(v) \neq \emptyset \} \) is the set of morphisms between finitely generated projective left \( P_K(E) \)-modules defined by \( \mu_v : P_K(E)v \to \bigoplus_{i=1}^{n_v} P_K(E)r(e_i^v) \)

for any \( v \in E^0 \) such that \( s^{-1}(v) \neq \emptyset \), where \( s^{-1}(v) = \{ e_1^v, \ldots, e_{n_v}^v \} \). By a slight abuse of notation, we use also \( \mu_v \) to denote the corresponding maps between finitely generated projective left \( P_K^{rat}(E) \)-modules and \( P_K((E)) \)-modules respectively.

The following relations hold in \( Q_K(E) \):

\[
\begin{align*}
(V) \quad & vv' = \delta_{v,v'}v \text{ for all } v,v' \in E^0. \\
(E1) \quad & s(e)e = er(e) = e \text{ for all } e \in E^1. \\
(E2) \quad & r(e)e^* = e^*s(e) = e^* \text{ for all } e \in E^1. \\
(CK1) \quad & e^*e' = \delta_{e,e'}r(e) \text{ for all } e,e' \in E^1. \\
(CK2) \quad & v = \sum_{e(v) = e} e^* \quad \text{for every } v \in E^0 \text{ that emits edges.}
\end{align*}
\]

The Leavitt path algebra \( L_K(E) = P_K(E)\Sigma^{-1} \) is the algebra generated by \( \{ v \mid v \in E^0 \} \cup \{ e, e^* \mid e \in E^1 \} \) subject to the relations \( (V)-(CK2) \) above, see \([1]\). By \([6, Theorem 4.2]\), the algebra \( Q_K(E) \) is a von Neumann regular hereditary ring and \( Q_K(E) = P_K(E)(\Sigma \cup \Sigma_1)^{-1} \). Here the set \( \Sigma \) can be clearly replaced with the set of all square matrices of the form \( I_n + B \) with \( B \in M_n(P_K(E)) \) satisfying \( \epsilon(B) = 0 \), for all \( n \geq 1 \).

2. The results

Our graphs are shaped by a poset with a special property, as follows:

**Definition 2.1.** Let \( (I, \leq) \) be a poset. We say that \( I \) is a tree in case there is a maximum element \( i_0 \in I \) and for every \( i \in I \) the interval \([i, i_0]\) := \( \{ j \in I \mid i \leq j \leq i_0 \} \) is a chain. The element \( i_0 \) will be called the root of the tree \( I \).

Let \( \leq \) be a pre-order on \( E^0 \) such that \( w \geq v \) whenever there exists a path in \( E \) from \( w \) to \( v \). Let \( I \) be the antisymmetrization of \( E^0 \), endowed with the partial order \( \leq \) induced by the pre-order on \( E^0 \). Thus, denoting by \([v]\) the class of \( v \in E^0 \) in \( I \), we have \([v] \leq [w]\) if and only if \( v \leq w \).
We will assume throughout this section that \( E \) is a finite quiver such that 
\( I := E / \sim \) is a tree.

For \( v \in E^0 \), we refer to the set \([v]\) as the component of \( v \), and we will denote by \( E[v] \) the restriction of \( E \) to \([v]\), that is, the graph with \( E[u]^0 = [v] \) and \( E[u]^1 = \{ e \in E^1 \mid s(e) \in [v] \text{ and } r(e) \in [v] \} \).

We will assume henceforth that we are given a family \( K = \{ K_i \}_{i \in I} \) of fields such that \( K_j \subseteq K_i \) if \( i \leq j \). We refer to this family as a poset of fields (over \( I \)). We will define a certain \( K_0 \)-algebra \( Q_K(E) \), where \( K_0 := K_{[v_0]} \).

Let \( i, j \in I \), and suppose that \( k \in (I \downarrow i) \cap (I \downarrow j) \). Then, since \([k, i_0]\) is a chain, we must have \( i \leq j \) or \( j \leq i \). Therefore, if \( i \) and \( j \) are incomparable elements of \( I \) then \((I \downarrow i) \cap (I \downarrow j) = \emptyset \). It follows that if \( J \) is a lower subset of \( I \), and \( x_1, \ldots, x_t \) are the maximal elements of \( J \), then \( J = \bigcup_{i=1}^t (I \downarrow x_i) \).

Given a lower subset \( J \) of \( I \) we consider the set of vertices

\[ E^0_J = \{ v \in E^0 \mid [v] \in J \}. \]

Observe that \( E^0_J \) is a hereditary subset of \( E^0 \). We will denote by \( E_J \) the restriction graph \( E|_{E^0_J} \) corresponding to the hereditary subset \( E^0_J \) of \( E^0 \), and we set \( p_J := \sum_{v \in E^0_J} v \).

We retain the above notation. We first consider an algebra of power series \( P_K((E)) \). The algebra \( P_K((E)) \) is defined as the algebra of formal power series of the form \( a = \sum_{\gamma \in \text{Path}(E)} a_{\gamma} \gamma \), where each \( a_{\gamma} \in K_{[r(\gamma)]} \). The usual multiplication of formal power series gives a structure of algebra over \( K_0 \) on \( P_K((E)) \). Observe that if \( \gamma \in K_{[r(\gamma)]} \), \( b_{\mu} \in K_{[r(\mu)]} \), and \( r(\gamma) = s(\mu) \), then since \([r(\gamma)] \geq [r(\mu)] \), we have that

\[ a_{\gamma} b_{\mu} \in K_{[r(\mu)]} = K_{[r(\gamma)]}, \]

because \( K_{[r(\gamma)]} \subseteq K_{[r(\mu)]} \).

The path algebra \( P_K(E) \) is defined as the subalgebra of \( P_K((E)) \) consisting of all the series in \( P_K((E)) \) having finite support.

For \( i \in I \), let \( K_i \) be the system of fields defined over \( I \downarrow i \) by \( K_i = \{ K_j \}_{j \leq i} \). Then one may define the algebras \( P_{K_i}((E_{I \downarrow i})) \) and \( P_{K_i}(E_{I \downarrow i}) \). Note that for \( i \in I \) we have

\[ P_{K_i}(E_{I \downarrow i}) = P_{K_i}(E_{I \downarrow i}) + \left( \bigoplus_{j \in I \downarrow (i, i)} P_{K_i}(E_{I \downarrow i}) P_{K_j}(E_{I \downarrow j}) \right). \]

We consider the commutative algebra \( E := \bigoplus_{i \in I} \bigoplus_{v \in X} K_i v \), which is a subalgebra of \( P_K(E) \). Note that there is an augmentation map \( \epsilon : P_K((E)) \to E \), and that a square matrix \( A \in M_n(P_K((E))) \) is invertible if and only if \( \epsilon(A) \) is invertible in \( M_n(E) \) (see e.g. the proof of [6, Lemma 1.8]).

Recall from Section 1 the definition and basic properties of the algebra \( P_{rat}^E(K) \) of rational series, for a field \( K \). We are now ready to define the algebra of rational series in our setting. Let \( E \) and \( K = \{ K_i \}_{i \in I} \) be as above. We will define inductively \( P_{rat}^J(E) \) for any lower subset \( J \) of \( I \).

We first set

\[ P_{rat}^J(E) = \bigoplus_{i \in S} P_{K_i}^J(E_{I \downarrow i}), \]

for any non-empty subset \( S \) of minimal elements of \( I \). Now assume that \( J \) is a non-empty lower subset of \( E \) and that we have defined the algebras \( P_{rat}^J(E) \) for all lower subsets \( J' \) of \( J \). Assuming that \( J \neq E^0 \), let \( i \) be a minimal element in \( E^0 \setminus J \).
We now define $P^\text{rat}_{J'}(E)$, where $J' = J \cup \{i\}$, as

$$P^\text{rat}_{J'}(E) = P^\text{rat}_{K'_i}(E_{I,i}) + P^\text{rat}_{K'_i}(E_{I,i})P^\text{rat}_{J'}(E) = P^\text{rat}_{J'}(E).$$

This defines inductively (and unambiguously) $P^\text{rat}_{J'}(E)$ for any non-empty lower subset $J$ of $I$. Note that if $i_1, \ldots, i_r$ are the maximal elements of $J$ then

$$P^\text{rat}_{J'}(E) = \bigoplus_{k=1}^r P^\text{rat}_{I_{\downarrow i_k}}(E).$$

We now define $P^\text{rat}_K(E) := P^\text{rat}_{E^0}(E)$. Observe that $P^\text{rat}_K(E)$ is a $K_0$-subalgebra of $P_K((E))$.

The following generalizes [6, Theorem 1.20].

**Theorem 2.2.** Let $E$ and $K = \{K_i\}_{i \in I}$ be as above. Let $\Sigma$ denote the set of matrices over $P_K(E)$ that are sent to invertible matrices by $\epsilon$. Then $P^\text{rat}_K(E)$ is the rational closure of $P_K(E)$ in $P_K((E))$, and the natural map $P_K(E)^{\Sigma^{-1}} \to P^\text{rat}_K(E)$ is an isomorphism.

**Proof.** We will prove by induction that $P^\text{rat}_{I_{\downarrow i}}(E)$ is the rational closure of $P_{K_{i'}}(E_i)$ in $P_{K_i}((E_i))$, where $E_i := E_{I,i}$ is the restriction graph of $E$ to $E_{I_i}$. Let $i_1, \ldots, i_r$ be the set of maximal elements of $(I \downarrow i) \setminus \{i\}$, and assume that the result is known for the graphs $E_{i_k} = E_{I_{\downarrow i_k}}$, for $k = 1, \ldots, r$. Changing notation we may furthermore assume that $i = i_0$ is the maximal element of $I$.

Therefore, we have to show that

$$S := P^\text{rat}_K(E) = P^\text{rat}_{K_0}(E) + P^\text{rat}_{K_0}(E)\left(\bigoplus_{k=1}^r P^\text{rat}_{K_{i_k}}(E_{i_k})\right)$$

is the rational closure of $R := P_K(E)$ in $P_K((E))$. Write $R$ for this rational closure, and recall that we are assuming that each $P^\text{rat}_{K_{i_k}}(E_{i_k})$ is the rational closure of $P_{K_{i_k}}(E_{i_k})$ in $P_{K_{i_k}}((E_{i_k}))$.

It is convenient to introduce some additional notation. Let $H$ be the hereditary subset of $E^0$ generated by $i_1, \ldots, i_k$, i.e., $H = \bigsqcup_{k=1}^r E_{I_{\downarrow i_k}}$. Recall also the notation $p_H = \sum_{v \in H} v$. Also, to somewhat simplify the notation we set $K := K_0$.

We start by showing that $S \subseteq R$. Since $P^\text{rat}_K(E)$ is the rational closure of $P_K(E)$ inside $P_K((E))$ ([6, Theorem 1.20]), we see that $P^\text{rat}_K(E) \subseteq R$. Also, note that the algebra $p_HR = p_HR_{K,H}$ is rationally closed in $p_HP_{K,H}((E_H))$ and contains $p_HP_{K,H}(E_H)$, so it must contain the rational closure of $p_HP_{K,H}(E_H)$ in $p_HP_{K,H}(E_H)$ which is $p_HP_{K_{i_k}}(E_{i_k})$ by the induction hypothesis. It follows that $P^\text{rat}_{K}(E)$ and $p_HP^\text{rat}_{K,H}(E_H)$ are both contained in $R$. Since $R$ is a ring, we get $S \subseteq R$.

To show the reverse inclusion $R \subseteq S$, take any element $a$ in $R$. By [13, Theorem 7.1.2], there exist a row $\lambda \in nR$, a column $\rho \in R^n$ and a matrix $B \in M_n(R)$ with $\epsilon(B) = 0$ such that

$$a = \lambda(I - B)^{-1}\rho. \quad (2.1)$$

Now the matrix $B$ can be written as $B = B_1 + B_2$, where $B_1 \in M_n(P_{K_{i_0}}(E)) \subseteq M_n(R)$ and $B_2 \in M_n(R)$ satisfy that $\epsilon(B_1) = \epsilon(B_2) = 0$, all the entries of $B_1$ are supported on paths ending in $E^0 \setminus H = [v_0]$ and all the entries of $B_2$ are supported on paths ending in $H$. Note that, since $H$ is hereditary, this implies that all the
paths in the support of the entries of $B_1$ start in $E^0 \setminus H$ and thus $B_2B_1 = 0$. It follows that
\begin{equation}
(I - B)^{-1} = (I - B_1 - B_2)^{-1} = (I - B_1)^{-1}(I - B_2)^{-1},
\end{equation}
and therefore $(I - B)^{-1} = (I - B_1)^{-1} + (I - B_1)^{-1}B_2(I - B_2)^{-1} \in M_n(S)$. It follows from (2.1) that $a \in S$, as desired.

Since the set $\Sigma$ is precisely the set of square matrices over $R$ which are invertible over $P_K((E))$, we get from a well-known general result (see for instance [16, Lemma 10.35(3)]) that there is a surjective $K$-algebra homomorphism $\phi$: $R\Sigma^{-1} \to \mathcal{R}$.

The rest of the proof is devoted to show that $\phi$ is injective. We have a commutative diagram
\begin{equation}
\begin{array}{c}
P_K(E)\Sigma(\epsilon_K)^{-1} \\
\downarrow \phi_K \quad \cong \\
P_K(E) \downarrow \phi
\end{array} \longrightarrow \ R\Sigma^{-1} \longrightarrow \mathcal{R}
\end{equation}
where the map $\phi_K$ is an isomorphism by [6, Theorem 1.20]. The map $P_K(E)\Sigma(\epsilon_K)^{-1} \to \mathcal{R}$ is injective, so the map $P_K(E)\Sigma(\epsilon_K)^{-1} \to R\Sigma^{-1}$ must also be injective. Hence the $K$-subalgebra of $R\Sigma^{-1}$ generated by $P_K(E)$ and the entries of the inverses of matrices in $\Sigma(\epsilon_K)$ is isomorphic to $P_K^{rat}(E)$. Observe that we can replace $\Sigma$ by the set of matrices of the form $I - B$, where $B$ is a square matrix over $R$ with $\epsilon(B) = 0$. An element $x$ in $R\Sigma^{-1}$ is of the form
\begin{equation}
x = \lambda(I - B)^{-1}\rho
\end{equation}
with $\lambda \in \mathbb{R}^n$ and $\rho \in \mathbb{R}^n$, and $B \in M_n(R)$ satisfies $\epsilon(B) = 0$.

**Claim 1.** We have
\[ p_HR\Sigma^{-1} \cong p_H P_K^{rat}(E_H) \cong P_K(E_H)\Sigma(\epsilon_{K_H})^{-1}. \]

**Proof of Claim 1.** Observe first that we have a natural homomorphism
\[ \psi: P_K(E_H)\Sigma(\epsilon_{K_H})^{-1} \longrightarrow p_HR\Sigma^{-1}. \]
By the induction hypothesis, the composition of $\psi$ with the map $\phi$ is injective, because it coincides with the canonical map from $P_K(E_H)\Sigma(\epsilon_{K_H})^{-1}$ onto $p_H P_K^{rat}(E_H) \cong P_K^{rat}(E_H)$. Hence $\phi$ induces an isomorphism from
\[ \mathfrak{G} := \psi(P_K(E_H)\Sigma(\epsilon_{K_H})^{-1}) \]
on to $p_H P_K^{rat}(E_H)$. Note that $\mathfrak{G}$ is precisely the subalgebra of $p_H R\Sigma^{-1}$ generated by $p_H P_K(E_H)$ and the entries of the inverses of matrices of the form $p_HI - B$, with $B$ a square matrix over $p_H P_K(E_H)$ with $\epsilon(B) = 0$. For an element $x$ in $R\Sigma^{-1}$, we write it in its canonical form (2.4) and we write $B = B_1 + B_2$ with all the entries in $B_1$ ending in $E^0 \setminus H$ and all the entries of $B_2$ ending in $H$.

Now multiply (2.4) on the left by $p_H$ and use (2.2) to get
\[ p_Hx = p_H\lambda(I - B_1)^{-1}\rho + p_H\lambda(I - B_1)^{-1}B_2(I - B_2)^{-1}\rho \]
\[ = p_H\lambda p_H(I - B_1)^{-1}\rho + p_H\lambda p_H(I - B_1)^{-1}B_2(I - B_2)^{-1}\rho \]
\[ = p_H\lambda p_H \rho + p_H\lambda p_H B_2p_H(I - B_2)^{-1}\rho. \]
Write $B_2 = B_2' + B_2''$, where all the entries of $B_2'$ start in $E^0 \setminus H$ and all the entries in $B_2''$ start in $H$ (and so end in $H$ as well). Note that $(I - B_2'')^{-1} = I + B_2'$,
because $B_2^2 = 0$, so that $p_H(I-B_2)^{-1} = p_H$. Since $B_2^n B_2^m = 0$ we have $(I-B_2)^{-1} = (I-B_2)^{-1}(I-B_2)^{-1}$, and thus

$$p_H x = p_H \lambda p_H p_H + p_H \lambda p_H B_2 p_H (I-B_2)^{-1} p_H p_H.$$ 

It follows that $p_H x \in \mathcal{S}$, and so $p_H R \Sigma^{-1} = \mathcal{S}$, as wanted. □

Assume now that $x \in \ker(R \Sigma^{-1} \to R) = \ker(R \Sigma^{-1} \to P_K^{rat}(E))$ and write $x$ as in (2.4), with $B = B_1 + B_2$ as before. Then

$$x = \lambda(I-B_1)^{-1} \rho + \lambda(I-B_1)^{-1}B_2(I-B_2)^{-1} \rho.$$ 

Multiplying on the right by $1-p_H$, we get

$$x(1-p_H) = \lambda(I-B_1)^{-1} \rho(1-p_H) = \lambda(1-p_H)(I-B_1)^{-1} \rho(1-p_H) \in P_K(E) \Sigma(\epsilon_K)^{-1}$$

and $0 = \phi(x(1-p_H)) = \phi_K(x(1-p_H))$. Since $\phi_K$ is an isomorphism, we get $x(1-p_H) = 0$.

Hence we have

$$x = \lambda(I-B_1)^{-1} \rho_2 + \lambda(I-B_1)^{-1}B_2(I-B_2)^{-1} \rho_2,$$

where $\rho = \rho_1 + \rho_2$ with $\rho_1$ ending in $E^0 \setminus H$ and $\rho_2$ ending in $H$. By Claim 1 we have $p_H x = 0$, because $\phi$ is an isomorphism when restricted to $p_H R \Sigma^{-1}$. Now we are going to find a suitable expression for $x = (1-p_H)xp_H$. Write $\lambda = \lambda_1 + \lambda_2$ with $\lambda_1 = (1-p_H) \lambda$ and $\lambda_2 = p_H \lambda$. Then

$$(1-p_H) \lambda(I-B_1)^{-1} \rho_2 = \lambda_1(I-B_1)^{-1} \rho_2.$$ 

Similarly $(1-p_H) \lambda(I-B_1)^{-1} B_2(I-B_2)^{-1} \rho_2 = \lambda_1(I-B_1)^{-1} B_2(I-B_2)^{-1} \rho_2$. Write $B_2 = B_2' + B_2''$, with $B_2'$ starting in $E^0 \setminus H$ and $B_2''$ starting in $H$. Then $B_2' B_2'' = 0$ and $(I-B_2)^{-1} = (I-B_2')^{-1}(I-B_2'')^{-1}$, so that

$$(1-p_H) \lambda(I-B_1)^{-1} B_2(I-B_2)^{-1} \rho_2 = \lambda_1(I-B_1)^{-1} B_2(I-B_2)^{-1} \rho_2$$

and

$$= \lambda_1(I-B_1)^{-1} B_2(I-B_2') (I-B_2'')^{-1} \rho_2$$

From (2.7), (2.8) and (2.6) we get

$$x = (1-p_H)xp_H = \lambda_1(I-B_1)^{-1} \rho_1 + \lambda_1(I-B_1)^{-1}B_2(I-B_2')^{-1} \rho_2.$$ 

It follows that $x \in \sum_{i=1}^k P_{K}^{rat}(E/H)e_i P_{K}^{rat}(E_H)$, where $e_1, \ldots, e_k$ is the family of crossing edges, that is, the family of edges $e \in E^1$ such that $s(e) \in E^0 \setminus H$ and $r(e) \in H$. Write $x = \sum_{i=1}^k \sum_{j=1}^{m_i} a_{ij} e_i b_{ij}$ for certain $a_{ij} \in P_{K}^{rat}(E/H)$ and $b_{ij} \in P_{K}^{rat}(E_H)$. Then we have

$$0 = \phi(x) = \sum_{i=1}^k \sum_{j=1}^{m_i} a_{ij} e_i b_{ij},$$

this element being now in $P_{K}((E))$. Clearly this implies that $\sum_{j=1}^{m_i} a_{ij} e_i b_{ij} = 0$ in $P_{K}((E))$ for all $i = 1, \ldots, k$. So the result follows from the following claim:

**Claim 2.** Let $e$ be a crossing edge, so that $s(e) \in E^0 \setminus H$ and $r(e) \in H$. Assume that $b_1, \ldots, b_m \in r(e) P_{K}^{rat}(E_H)$ are $K$-linearly independent elements, and assume that $a_1 e, \ldots, a_m e$ are not all 0, where $a_1, \ldots, a_m \in P_{K}((E \setminus H))$. Then $\sum_{i=1}^m a_i e b_i \neq 0$ in $P_{K}((E))$.

**Proof of Claim 2.** By way of contradiction, suppose that $\sum_{i=1}^m a_i e b_i = 0$. We may assume that $a_1 e \neq 0$. Let $\gamma$ be a path in the support of $a_1$ such that $r(\gamma) = s(e).$
For every path $\mu$ with $s(\mu) = r(e)$ we have that the coefficient of $\gamma e\mu$ in $a_e b_i$ is $a_i(\gamma) b_i(\mu)$, so that $\sum_{i=1}^{m} a_i(\gamma) b_i(\mu) = 0$ for every $\mu$ such that $s(\mu) = r(e)$. Since every path in the support of each $b_i$ starts with $r(e)$, we get that
\[ \sum_{i=1}^{m} a_i(\gamma) b_i = 0 \]
with $a_1(\gamma) \neq 0$, which contradicts the linear independence over $K$ of $b_1, \ldots, b_m$. □

This concludes the proof of the theorem.

Following [6, Section 2], we define, for $e \in E^1$, the right transduction $\tilde{\delta}_e : P_K((E)) \rightarrow P_K((E))$ corresponding to $e$ by
\[ \tilde{\delta}_e(\sum_{\alpha \in E^*} \lambda_{\alpha} \alpha) = \sum_{s(\alpha) = r(e)} \lambda_{\alpha} \alpha. \]
Note that the left transductions are not even defined on $P_K((E))$ in general.

Observe that $R := P_K(E)$ is closed under all the right transductions, i.e. $\tilde{\delta}_e(R) \subseteq R$. Some of the proofs in [6] make use of the fact that the usual path algebra $P_K(E)$ is closed under left and right transductions. Fortunately we have been able to overcome the potential problems arising from the failure of invariance of $R$ under left transductions by using alternative arguments.

We are now ready to get a description of the algebra $Q_K(E)$ as a universal localization of the path algebra $P_K(E)$.

Write $R := P_K(E)$. For any $v \in E^0$ such that $s^{-1}(v) \neq \emptyset$ we put $s^{-1}(v) = \{e_1^v, \ldots, e_{n_v}^v\}$, and we consider the left $R$-module homomorphism
\[ \mu_v : Rv \rightarrow \bigoplus_{i=1}^{n_v} R r(e_i^v) \]
\[ r \mapsto (r e_1^v, \ldots, r e_{n_v}^v) \]
Write $\Sigma_1 = \{\mu_v \mid v \in E^0, s^{-1}(v) \neq \emptyset\}$.

**Theorem 2.3.** Let $K$ and $E$ be as before. Let $\Sigma$ denote the set of matrices over $P_K(E)$ that are sent to invertible matrices by $e$ and let $\Sigma_1$ be the set of maps defined above. Then we have that $Q_K(E) := P_K(E)(\Sigma \cup \Sigma_1)^{-1}$ is a hereditary von Neumann regular ring and all finitely generated projective $Q_K(E)$-modules are induced from $P_K^{rat}(E)$. Moreover each element of $Q_K(E)$ can be written as a finite sum
\[ \sum_{\gamma \in \text{Path}(E)} a_{\gamma} \gamma^*, \]
where $a_{\gamma} \in P_K^{rat}(E)r(\gamma)$.

**Proof.** First observe that $P_K(E)$ is a hereditary ring and that $V(P_K(E)) = (\mathbb{Z}^+)^d$, where $|E^0| = d$. This follows by successive use of [11, Theorem 5.3].

In order to get that the right transduction $\tilde{\delta}_e : P_K((E)) \rightarrow P_K((E))$ corresponding to $e$ is a right $\tau_e$-derivation, that is,
\[ (2.10) \tilde{\delta}_e(rs) = \tilde{\delta}_e(r)s + \tau_e(r)\tilde{\delta}_e(s) \]
for all $r, s \in P_K((E))$, we have to modify slightly the definition of $\tau_e$ given in [6, page 220]. Concretely we define $\tau_e$ as the endomorphism of $P_K((E))$ given by the
composition

\[ P_K((E)) \to \prod_{v \in E^0} K_{[v]}^v \to \prod_{v \in E^0} K_{[v]}^v \to P_K((E)), \]

where the first map is the augmentation homomorphism, the third map is the canonical inclusion, and the middle map is the \( K_0 \)-lineal map given by sending \( s(e) \) to \( r(e) \), and any other idempotent \( v \) with \( v \neq s(e) \) to \( 0 \). Observe that this restricts to an endomorphism of \( P_K((E)) \) and that the proof in [6, Lemma 2.4] gives the desired formula (2.10) for \( r, s \in P_K((E)) \) and, in particular for \( r, s \in P_K(E) \).

Note that, by the argument given after the proof of Proposition 2.7 in [6], the algebras \( P_K^{rat}(E) \) are stable under all the right transductions. Hence, the constructions in [6, Section 2] apply to \( R := P_K^{rat}(E) \) (with some minor changes), and we get that \( RS^{-1} = R \langle \tau; \delta \rangle / I \), where \( I \) is the ideal of \( R \langle \tau; \delta \rangle \) generated by the idempotents \( q_v := v - \sum_{e \in \delta^{-1}(v)} ee^* \) for \( v \notin \text{Sink}(E) \).

By Theorem 2.2, we have

\[ Q_K(E) = P_K(E)(\Sigma \cup \Sigma_1) = (P_K(E)\Sigma^{-1})\Sigma_1^{-1} \]

\[ = P_K^{rat}(E)(\Sigma_1^{-1} = (P_K^{rat}(E))(\tau; \delta)/I. \]

By a result of Bergman and Dicks [12] any universal localization of a hereditary ring is hereditary, thus we get that both \( P_K^{rat}(E) \) and \( Q_K(E) \) are hereditary rings. Since \( P_K^{rat}(E) \) is hereditary, closed under inversion in \( P_K((E)) \) (by Theorem 2.2), and closed under all the right transductions \( \delta_e \), for \( e \in E^1 \), the proof of [6, Theorem 2.16] gives that \( Q_K(E) \) is von Neumann regular and that every finitely generated projective module is induced from \( P_K^{rat}(E) \).

The last statement follows from [6, Proposition 2.5(ii)]. This concludes the proof of the theorem.

\[ \square \]

**Remark 2.1.** Theorem 2.16 in [6] is stated for a subalgebra \( R \) of \( P_K((E)) \) which is closed under all left and right transductions (and which is inversion closed in \( P_K((E)) \)). However the invariance under right transductions is only used in the proof of that result to ensure that the ring \( R \) is left semihereditary. Since we are using the opposite notation concerning (CK1) and (CK2), the above hypothesis translates in our setting into the condition that \( P_K(E) \) and \( P_K^{rat}(E) \) should be invariant under all left transductions, which is not true in general as we observed above. We overcome this problem by the use of the result of Bergman and Dicks ([12]), which guarantees that \( P_K(E) \) and \( P_K^{rat}(E) \) are indeed right and left hereditary (see the proof of Theorem 2.3).

For our last result, we need an auxiliary lemma.

**Lemma 2.4.** Let \( (I, \leq) \) be a finite poset which is a tree, with maximum element \( i_0 \). Let \( K = \{ K_i \}_{i \in I} \) be a poset of fields, so that \( K_i \subseteq K_j \) if \( j \leq i \) in \( I \). Then there exists a field \( K \) and an embedding of \( K \) into the constant system \( K \) over \( I \), i.e., there is a collection of field morphisms \( \varphi_i : K_i \to K \) such that \( \varphi_i|_{K_j} = \varphi_j \) for all \( i \leq j \).

**Proof.** For a finite poset \( I \) as in the statement, define the depth of \( I \) as the maximum of the lengths of maximal chains \( i_k < i_{k-1} < \cdots < i_0 \). We prove the result by induction on the depth of \( I \). If the depth of \( I \) is 0, there is nothing to prove. Assume the result is true for finite posets of depth at most \( r \) satisfying the hypothesis in the statement and let \( (I, \leq) \) such a finite poset of depth \( r + 1 \),
where \( r \geq 0 \). Write \( L(I,i_0) = \{ a_1, \ldots, a_n \} \). Then the depth of each of the posets \( I \downarrow a_i \) is at most \( r \), so that there are embeddings \( \psi_i: \mathbb{K}_{I \downarrow a_i} \to L_t \) for some fields \( L_t \), for \( t = 1, \ldots, r \). This means that there are field embeddings \( \psi_{t,i}: K_i \to L_t \) for each \( i \leq a_i \), such that \( \psi_{t,i}|_{K_j} = \psi_{t,j} \) for \( i \leq j \leq a_i \). Now, setting \( K_0 := K_{i_0} \), there exist a field \( K \) and embeddings \( \delta_t: L_t \to K \) for \( t = 1, \ldots, n \) such that \( \delta_t \circ ((\psi_{t,a_t})|_{K_0}) = \delta_{t'} \circ (\psi_{t,a_t'})|_{K_0} \) for \( 1 \leq t, t' \leq n \). Define \( \varphi_i: K_i \to K \) to be \( \delta_t \circ \psi_{t,i} \) if \( i \leq a_i \). (Observe that this is well-defined by our hypothesis on \( I \).) Finally define \( \varphi_{i_0}: K_0 \to K \) as \( \varphi_{i_0} := \delta_t \circ ((\psi_{t,a_t})|_{K_0}) \), which is independent of the choice of \( t \) by the above.

If \( i \leq j \leq a_i \) for some \( t \), then

\[
\varphi_i|_{K_j} = (\delta_t \circ \psi_{t,i})|_{K_j} = \delta_t \circ (\psi_{t,i}|_{K_j}) = \delta_t \circ \psi_{t,j} = \varphi_j.
\]

If \( i < i_0 \), then there is a unique \( t \in \{1, \ldots, n\} \) such that \( i \leq a_t < i_0 \), and so

\[
\varphi_i|_{K_0} = (\delta_t \circ \psi_{t,i})|_{K_0} = \delta_t \circ (\psi_{t,i}|_{K_{a_t}})|_{K_0} = \delta_t \circ (\psi_{t,a_t})|_{K_0} = \varphi_{i_0}.
\]

This completes the proof. \( \square \)

Define the Leavitt path algebra \( L_K(E) \) associated to the poset of fields \( K \) and the finite quiver \( E \) as the universal localization of \( P_K(E) \) with respect to the set \( \Sigma_1 \). Let \( M(E) \) be the abelian monoid with generators \( E^{\mathbb{Z}} \) and relations given by \( v = \sum_{e \in s^{-1}(v)} r(e) \), see [10] and [1, Chapter 3].

**Theorem 2.5.** With the above notation, we have natural isomorphisms

\[
M(E) \cong \mathbb{V}(L_K(E)) \cong \mathbb{V}(Q_K(E)).
\]

**Proof.** The proof that \( M(E) \cong \mathbb{V}(L_K(E)) \) follows from Bergman’s results [11], as in [10, Theorem 3.5].

Note that \( R := P_K^{\text{gt}}(E) \) is semiperfect. Thus we get \( \mathbb{V}(R) \cong (\mathbb{Z}^+)^{|E_0|} \) in the natural way, that is the generators of \( \mathbb{V}(R) \) correspond to the projective modules \( Rv \) for \( v \in E^{\mathbb{Z}} \). By Theorem 2.3, we get that the natural map \( M(E) \to \mathbb{V}(Q_K(E)) \) is surjective. To show injectivity, let \( K \) and \( \{ \varphi_i \}_{i \in I} \) be a field and maps satisfying the hypothesis of Lemma 2.4. Then it is easily seen that there is a unital ring homomorphism \( \varphi: Q_K(E) \to Q_K(E) \). Indeed, one directly verifies that there is well-defined homomorphism \( \varphi: P_K(E) \to P_K(E) \) given by the rule

\[
\varphi(\sum a_{\gamma}) = \sum \varphi(\gamma)(a_\gamma) \gamma.
\]

Let \( \Sigma \) (respectively \( \Sigma_K \)) be the set of all square matrices over \( P_K(E) \) (respectively over \( P_K(E) \)) which are sent to invertible matrices through the augmentation map \( \epsilon \). Since \( \epsilon(\varphi(A)) = \varphi(\epsilon(A)) \), it is obvious that \( \varphi(\Sigma) \subseteq \Sigma_K \), and consequently the map \( \varphi \) can be uniquely extended to a homomorphism (also denoted by \( \varphi \)) from \( Q_K(E) = P_K(E)(\Sigma \cup \Sigma_1)^{-1} \) to \( Q_K(E) = P_K(E)(\Sigma_K \cup \Sigma_1)^{-1} \). By [6, Theorem 4.2], we have \( M(E) \cong \mathbb{V}(Q_K(E)) \) and \( M(E) \cong \mathbb{V}(Q_K(E)) \) canonically, so that we get

\[
M(E) \cong \mathbb{V}(Q_K(E)) \to \mathbb{V}(Q_K(E)) \to \mathbb{V}(Q_K(E)) \cong M(E),
\]

and the composition of the maps above is the identity. It follows that the map \( M(E) \to \mathbb{V}(Q_K(E)) \) is injective and so it must be a monoid isomorphism. \( \square \)

We adopt now the specific setting of [4] for our graphs in order to obtain a property of \( Q_K(E) \) needed in [4]. So in addition to our previous assumptions on \( E \), we make the following requirements:

1. There is a partition \( I = I_{\text{free}} \sqcup I_{\text{reg}} \) (where \( I = E/\sim \)).
(2) For each \( v \in E^0 \) such that \([v]\) is not minimal in \( I \) and \([v]\) \( \in I_{\text{free}} \), we have that \( E[v] \) is a graph with a single vertex \( v \) and a single arrow \( \alpha \), with \( s(\alpha) = r(\alpha) = v \).

(3) For each \( v \in E^0 \) such that \([v]\) \( \in I_{\text{reg}} \), we have that \( E[v] \) is a graph with at least two edges.

(4) If \([v]\) is minimal in \( I \), then either \( v \) is a sink or \([v]\) \( \in I_{\text{reg}} \).

With these conditions at hand, we finally observe that we can obtain \( Q_K(E) \) by inverting a smaller set of matrices over \( P_K(E) \). For each non-minimal \([v]\) \( \in I_{\text{free}} \), let

\[
\Sigma_i := \{ p(\alpha^n) \mid p(x) \in K[x] \text{ and } p(0) \neq 0 \},
\]

be the set of polynomials in \( \alpha^n \) with coefficients in \( K[x] \) with nonzero constant term. Set \( p_i = \sum_{v \in I} v \) for \( i \in I \). For each \( i \in I_{\text{reg}} \), let \( \Sigma_i \) be the set of square matrices \( M \) over \( p_i P_K(E) p_i \) such that \( \epsilon_i(M) \) is an invertible matrix over \( \prod_{i \in I} K_i \), where \( p_i P_K(E) p_i \to \prod_{i \in I} K_i \) is the augmentation homomorphism. Finally set \( \Sigma' = \bigcup_{i \in I} \Sigma_i' \), where, for \( i \in I \), \( \Sigma_i' = \{ M + (1 - p_i) I_m \mid M \in M_m(P_K(E)) \cap \Sigma_i \} \).

**Proposition 2.6.** Let \( E \) and \( \Sigma, \Sigma' \) be as above. Then we have \( Q_K(E) = L_K(E)(\Sigma')^{-1} \).

**Proof.** Since \( Q_K(E) = L_K(E)(\Sigma')^{-1} \), we only have to show that every element of \( \Sigma \) is invertible over \( L_K(E)(\Sigma')^{-1} \). For this, it is enough to show that any matrix of the form \( I - A \) is invertible over \( L_K(E)(\Sigma')^{-1} \), where \( A \) is a square matrix over \( P_K(E) \) such that \( \epsilon(A) = 0 \).

Now given such a matrix \( A \), we can uniquely write it in the form

\[
A = A_0 + B + A_1 + \cdots + A_l,
\]

where \( L(I, i_0) = \{ i_1, \ldots, i_l \} \), \( \epsilon(B) = \epsilon(A_k) = 0 \) for \( k = 0, \ldots, l \), \( A_0 \) is a square matrix over \( p_{i_0} P_K(E) p_{i_0} \), all paths in the support of \( B \) start in a vertex in the component of \( i_0 \) and end in a vertex in the component of some \( i \in I \), where \( i < i_0 \), and each \( A_k \) is a square matrix over \( P_{K_{i_k}}(E_{i_k}) \). Here, \( E_{i_k} \) is the hereditary subset \( E_{i_1 \cdots i_k} \) and \( K_{i_k} \) is the restriction of the poset of fields \( K \) to \( I \downarrow i_k \).

Observe that \( (B + \sum_{k=1}^l A_k)A_0 = 0 \), so that we obtain

\[
(I - A)^{-1} = (I - A_0)^{-1}(I - (B + \sum_{k=1}^l A_k))^{-1}
\]

in any ring where the matrices on the right hand side are invertible. Now observe that \( (\sum_{k=1}^l A_k)B = 0 \), because all the paths in the support of \( B \) start at the component \( i_0 \), so we get

\[
(I - (B + \sum_{k=1}^l A_k))^{-1} = (I - B)^{-1}(I - \sum_{k=1}^l A_k)^{-1}
\]

in any ring where the matrices on the right hand side are invertible. Now since

\[
(I \downarrow i_k) \cap (I \downarrow i_{k'}) = \emptyset
\]

for \( k \neq k' \) we get that

\[
(I - \sum_{k=1}^l A_k)^{-1} = \prod_{k=1}^l (I - A_k)^{-1}
\]

again in any ring where all matrices on the right hand side are invertible.
Continuing in this way, we obtain that the matrix \((I - A)^{-1}\) can be expressed as a product of matrices of the forms \((I - A')^{-1}\), where \(A'\) is a matrix over \(p_i P_K(E)p_i\) such that \(\epsilon(A') = 0\), and \((I - B')^{-1}\), where \(B'\) is a matrix such that all paths in the support of \(B'\) start at a vertex in a fixed component \(i \in I\) and end at a component which is strictly less than \(i\). Now observe that \((B')^2 = 0\) and so \((I - B')^{-1} = I + B'\) is a matrix over \(P_K(E)\). Therefore, since all the matrices \(I - A'\) and \(I - B'\) as above are invertible in \(L_K(E)(\Sigma')^{-1}\), we see that \(I - A\) is also invertible, and the proof is complete. (Note that, in case \(i = [v] \notin I_{\text{free}}\) is non-minimal, the determinant of the matrix \(I_v - A'\) is of the form \(v + p(\alpha^v)\), where \(p(0) = 0\), so the matrix \(I - A'\) is invertible over \(L_K(E)(\Sigma')^{-1}\).)

\[\Box\]

References

[1] G. Abrams, P. Ara, M. Siles Molina, Leavitt Path Algebras, Springer Lecture Notes in Mathematics, vol. 2191, 2017.
[2] P. Ara, The realization problem for von Neumann regular rings, Ring Theory 2007. Proceedings of the Fifth China-Japan-Korea Conference, (eds. H. Marubayashi, K. Masaike, K. Oshiro, M. Sato); World Scientific, 2009, pp. 21–37.
[3] P. Ara, The regular algebra of a poset, Trans. Amer. Math. Soc., 362 (2010), 1505–1546.
[4] P. Ara, J. Bosa, E. Pardo, The realization problem for finitely generated refinement monoids. In preparation.
[5] P. Ara, J. Bosa, E. Pardo, A. Simis, The groupoids of adaptable separated graphs and their type semigroups, arXiv:1904.05197 [math.OA], 2019.
[6] P. Ara, M. Brustenga, The regular algebra of a quiver, J. Algebra, 309 (2007), 207–235.
[7] P. Ara, M. Brustenga, Mixed quiver algebras, arXiv:0909.0421 [math.RA], 2009.
[8] P. Ara, K. R. Goodearl, The realization problem for some wild monoids and the Atiyah problem, Trans. Amer. Math. Soc., 369 (2017), 5665–5710.
[9] P. Ara, K. R. Goodearl, K.C. O’Meara, E. Pardo, Separative cancellation for projective modules over exchange rings, Israel J. Math., 105 (1998), 105–137.
[10] P. Ara, M. A. Moreno, E. Pardo, Nonstable K-theory for graph algebras, Algebras Repr. Theory, 10 (2007), 157–178.
[11] G. M. Bergman, Coproducts and some universal ring constructions, Trans. Amer. Math. Soc., 200 (1974), 33–88.
[12] G. M. Bergman, W. Dicks, Universal derivations and universal ring constructions, Pacific J. Math., 79 (1978), 293–337.
[13] P. M. Cohn, “Free ideal rings and localization in general rings”, New Mathematical Monographs, 3. Cambridge University Press, Cambridge, 2006.
[14] K. R. Goodearl, “Von Neumann Regular Rings”, Pitman, London 1979; Second Ed., Krieger, Malabar, Fl., 1991.
[15] K. R. Goodearl, “von Neumann regular rings and direct sum decomposition problems”, Abelian groups and modules (Padova, 1994), Math. Appl. 343, 249–255, Kluwer Acad. Publ., Dordrecht, 1995.
[16] W. Lück, \(L^2\)-invariants: theory and applications to geometry and \(K\)-theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, 44. Springer-Verlag, Berlin, 2002.
[17] S. W. Rigby, The groupoid approach to Leavitt path algebras. Preprint, 2018.
[18] A. H. Schofield, “Representations of Rings over Skew Fields”, LMS Lecture Notes Series 92, Cambridge Univ. Press, Cambridge, UK, 1985.

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