On Vaughan’s approximation: The first moment

Daniel Fiorilli

Abstract
We investigate the first moment of the difference between $\psi(x; q, a)$ and Vaughan’s approximation in a certain range of $q$. We show that this last approximation is significantly more precise than the classical $x/\phi(q)$, and that it captures the discrepancies of the distribution of primes in arithmetic progressions found in an earlier paper of the author.

1. Introduction
The moments of the error term in the prime number theorem in arithmetic progressions are a central object of study and have been extensively studied in the literature. Upper bounds for the first moment, which apply to the Titchmarsh divisor problem, were obtained by Fouvry [7], Bombieri, Friedlander and Iwaniec [1], Friedlander and Granville [9] and Friedlander, Granville, Hildebrand and Maier [10].

Theorem 1.1 [9, Theorem 1; 10, Proposition 2.1]. Let $0 < \lambda < 1/4$, $A > 0$ be given. Then uniformly for $0 < |a| < x^{\lambda}$, $2 \leq Q \leq x/3$ we have

$$
\sum_{\substack{Q < q \leq 2Q \\ (q, a) = 1}} \left( \psi(x; q, a) - \frac{\psi(x)}{\phi(q)} \right) \ll_{\lambda, A} 2^{\nu(a)} Q \log(x/Q) + \frac{x}{(\log x)^A} + Q \log |a|.
$$

These results are based on the dispersion method and deep estimates on sums of Kloosterman sums [3] and generalize to other arithmetic sequences such as friable integers in arithmetic progressions [4, 8].

In [6], the author showed that in some cases it is possible to obtain an asymptotic formula for the quantity on the left-hand side of (1).

Theorem 1.2 [6, Theorem 1.1]. Fix an integer $a \neq 0$, a positive real number $B$ and $\epsilon > 0$. Then, for $M = M(x) \leq (\log x)^B$, one has

$$
\frac{1}{\phi(a)/x} \sum_{\substack{q \leq x/M \\ (q, a) = 1}} \left( \psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) = \mu(a, M) + O_{a, \epsilon, B} \left( \frac{1}{M^{305/538-\epsilon}} \right)
$$

with

$$
\mu(a, M) := \begin{cases} 
-\frac{1}{2} \log M - C_0 & \text{if } a = \pm 1 \\
-\frac{1}{2} \log p & \text{if } a = \pm p^e \\
0 & \text{otherwise},
\end{cases}
$$

Received 28 January 2016; revised 12 July 2016; published online 9 January 2017.
2010 Mathematics Subject Classification 11N13 (primary), 11P55 (secondary).
This work was partly accomplished while the author was at the University of Michigan and at Université Paris Diderot and was supported by a Postdoctoral Fellowship from the Fondation Sciences Mathématiques de Paris and a Discovery Grant from the NSERC.
where
\[ C_0 := \frac{1}{2} \left( \log 2\pi + \gamma + \sum_{p} \frac{\log p}{p(p-1)} + 1 \right). \]

**Remark 1.3.** The exponent $205/538$ in Theorem 1.2, which comes from Huxley’s subconvexity estimate [13], can be improved to $171/448$ using Bourgain’s recent work [2].

**Remark 1.4.** In Theorem 1.2, we have excluded the first term $n = a$ of the arithmetic progression $a \mod q$; we will keep doing so and use the notation
\[ \psi^*(x; q, a) := \sum_{\substack{n \leq x \\colon n \equiv a \mod q \\colon n > a}} \Lambda(n). \]

The reason we do this is because the term $\Lambda(a)$ can have a significant contribution in this context, and this contribution is trivial to control.

One can interpret Theorem 1.2 by saying that the discrepancy of the distribution of primes in the different arithmetic progressions $a \mod q$ (with $(a, q) = 1$) is negative for $a$ having at most one prime factor and is zero otherwise. One could ask whether there exists an approximation to $\psi(x; q, a)$, superior to $\psi(x)/\phi(q)$, which has the same discrepancies as $\psi(x; q, a)$. In the present paper, we will show that Vaughan’s approximation has this property.

Vaughan introduced the following approximation to $\psi(x; q, a)$, which depends on a parameter $R \geq 1$:
\[ \rho_R(x; q, a) := \sum_{\substack{n \leq x \\colon n \equiv a \mod q \\colon n > a}} F_R(n), \]
where
\[ F_R(n) := \sum_{r \leq R} \frac{\mu(r)}{\phi(r)} \sum_{1 \leq b \leq r \atop (b, r) = 1} e(bn/r) = \sum_{r \leq R} \frac{\mu^2(r) \mu((r, n)) \phi((r, n))}{\phi(r)}. \]

The function $F_R(n)$ was motivated by the Hardy–Littlewood method, in order to remove the contribution of the major arcs. Remarkably, Vaughan showed that the second [15, Corollary 4.1] and third [16, Theorem 8] moments of $\psi(x; q, a) - \rho_R(x; q, a)$, averaged over $q \leq x/M$ with $M, R \leq (\log x)^A$, are smaller than those of $\psi(x; q, a) - \psi(x)/\phi(q)$ when $R$ is larger than $M$ (and the implied error terms are sharper than [11, Theorem 1.1] and [12, Theorems 1.2]).

Our first result shows that Vaughan’s approximation has the properties described earlier, that is it captures the discrepancies of $\psi(x; q, a)$ in the arithmetic progressions $a \mod q$ observed in Theorem 1.2. As we did with $\psi(x; q, a)$ above, we exclude the first term of the arithmetic progression $a \mod q$:
\[ \rho^*_R(x; q, a) := \sum_{\substack{n \leq x \\colon n \equiv a \mod q \\colon n > a}} F_R(n). \]

In what follows, $R$ should be thought as a fixed power of $\log x$; however, it can be even smaller when looking at moduli $q$ very close to $x$. 
Theorem 1.5. Fix $A, B \geq 1$.

(i) Uniformly for $0 < |a| \leq x/(\log x)^{A+B+2}$, $1 \leq M \leq (\log x)^A$ and $2M \leq R \leq x^{1/2}$, we have

$$
\frac{1}{x/M} \sum_{q \leq x/M} (\psi^*(x, q, a) - \rho^*_R(x, q, a)) \ll_{A,B} \frac{1}{(\log x)^B}.
$$

(ii) If in addition $2|a| M \leq R$, then restricting the sum over moduli coprime to $a$,

$$
\frac{1}{\phi(a)x} \sum_{q \leq x/M} \frac{\psi^*(x, q, a) - \rho^*_R(x, q, a)}{(q,a)=1} \ll_{A,B} \frac{1}{(\log x)^B}.
$$

Comparing with (a dyadic version of) Theorem 1.2, we deduce that $\rho^*_R(x, q, a)$ is a much better approximation to $\psi^*(x, q, a)$ than $\psi(x)/\phi(q)$, on average over $q \approx x/M$. Indeed, for $M \to \infty$, the right-hand sides of (5) and (6) are $\ll K M^{-K}$ for any $K \geq 1$ and are independent of both $a$ and $R$. They are also much smaller than (2) for fixed values of $M$.

Let us briefly explain why it is possible to obtain such an error term in Theorem 1.5. In Theorem 1.2, the error term comes from the cancellation of main terms in sums of a certain multiplicative function. In the corresponding situation for Theorem 1.5, we have cancellation of the whole sums of the implied multiplicative function, rather than just the main terms (see Lemmas 2.4(ii) and 2.6).

Remark 1.6. In Theorem 1.5(i), we sum over all moduli $q$, not just those coprime to $a$. The reason we do this is that when $(q, a) > 1$, both $\psi^*(x, q, a)$ and $\rho^*_R(x, q, a)$ are small. Note however that (ii) is not a direct consequence of (i), since contrary to $\psi^*(x, q, a)$, it is not trivial to handle $\rho^*_R(x, q, a)$ when $(q, a) > 1$ (see Section 6 for more details).

Things are quite different when averaging over the whole range $q \leq x/M$. Indeed in this case, we obtain non-negligible lower-order terms. This result seems to indicate that Vaughan’s approximation is better for larger values of $q$ than for more moderate ones.

Theorem 1.7. Fix $A, B \geq 1$, and $a \neq 0$.

(i) Uniformly for $M \leq (\log x)^A$ and $1 \leq M \leq x^{1/2}/(\log x)^{A+B}$, we have

$$
\frac{1}{x/M} \sum_{q \leq x/M} (\psi^*(x, q, a) - \rho^*_R(x, q, a)) = \epsilon_{a=\pm} M \frac{\log x}{R^2} + 2\gamma - 3 + O_{a} \left( \frac{M \log x}{R^{3/2} \exp \left( c \frac{(\log R)^{1/5}}{(\log \log R)^{1/7}} \right) } \right) + O_{a,A,B} \left( \frac{1}{(\log x)^B} \right),
$$

where $\epsilon_{a=\pm}$ equals 1 if $a = \pm 1$ and is zero otherwise.
(ii) Under the additional condition $|a|M \leq R$, we have that

$$
\frac{1}{\phi(|a|)} \sum_{q \leq x/M \atop (q,a)=1} (\psi^*(x; q, a) - \rho_R^*(x; q, a)) = \frac{\phi(|a|)}{|a|} \frac{M}{R} \left( \log \frac{x}{R^2} + 2\gamma - 3 + \sum_{p \mid a} \frac{p+1}{p-1} \log p \right) 
+ O_a \left( \frac{M \log x}{R^{3/2} \exp \left( \frac{c (\log R)^{3/5}}{(\log \log R)^{1/5}} \right)} \right) 
+ O_{a, A, B} \left( \frac{1}{(\log x)^B} \right).
$$

(8)

In both of these statements, $c$ is a positive absolute constant.

**Remark 1.8.** Fixing $a \not\in \{0, \pm 1\}$ and comparing (7) and (8), we see that contrary to the situation in Theorem 1.5, $\rho_R^*(x; q, a)$ has a non-trivial contribution when $(q, a) > 1$. This indicates once more that Vaughan’s approximation is more precise for larger values of $q$. We will expand on this remark in Section 6.

**Remark 1.9.** Taking $M = 1$ in Theorem 1.7(i)† and applying Lemmas 4.1 and 3.4, we recover the known estimate for the Titchmarsh divisor problem [1, 7]. Drappeau recently established [5] that the error term in this problem depends on the existence of Landau–Siegel zeros.

Comparing Theorems 1.2 and 1.7, we see that $\rho_R^*(x; q, a)$ necessarily has the same discrepancies in arithmetic progressions as $\psi^*(x; q, a)$, when averaged over $q \leq x/M$ with $M \leq (\log x)^{O(1)}$. We will show that these discrepancies persist for $M$ as large as $x^{1/2 - \epsilon}/R$, as long as $M \leq R$.

**Proposition 1.10.** Fix $\epsilon > 0$ and $a \neq 0$. Under the conditions $1 \leq |a|M \leq R \leq x^{1/2}$ and $M > |a|$, we have

$$
\frac{1}{\phi(|a|)} \sum_{q \leq x/M \atop (q,a)=1} \left( \rho_R^*(x; q, a) - \frac{x}{\phi(q)} \right) = \mu(a, M) - \frac{\phi(|a|)}{|a|} \frac{M}{R} \left( \log \frac{x}{R^2} + 2\gamma - 3 + \sum_{p \mid a} \frac{p+1}{p-1} \log p \right) 
+ O_a \left( \frac{1}{M^{171/448-\epsilon}} + \frac{M \log x}{R^{3/2} \exp \left( \frac{c (\log R)^{3/5}}{(\log \log R)^{1/5}} \right)} + \frac{RM}{x^{1/2}} \right),
$$

(9)

where $\mu(a, M)$ is defined in (3).

Note that by Lemma 7.1, the quantity $\rho_R^*(x; q, a) - x/\phi(q)$ approximately equals the discrepancy (with signs) of the distribution of $F_R(n)$ in the arithmetic progressions $a \mod q$ with $(a, q) = 1$.

†Note that this theorem itself is based on the results of [1, 7].
Remark 1.11. Combining either (13) or (23) with the formula
\[
\sum_{n \leq x} \left( \frac{1}{n} - \frac{1}{x} \right) = \log x + \gamma - 1 + \frac{1}{2x} + O \left( \frac{1}{x^2} \right) \quad (x \in \mathbb{R}_{\geq 1}),
\]
one can estimate the quantities in Theorems 1.5, 1.7 and Proposition 1.10 in the range \( R < M \leq R^{1+\delta} \), for some \( \delta > 0 \). The resulting bounds are weaker than in the case \( R \geq M \), and thus we decided not to pursue this further.

2. The dyadic average

Let us first recall two results of [15]. The proofs of these results are contained\(^\dagger\) in that of [15, Theorem 1] and will therefore be omitted.

**Lemma 2.1.** Assume that \( a, r \) and \( s \) are integers with \( r, s \geq 1 \). We have for \( a \leq y \leq x \) with \( y \geq 0 \) that
\[
\sum_{1 \leq b \leq r \atop (b, r) = 1} \sum_{y < n \leq x \atop n \equiv a \mod s} e(b n / r) = \delta_{r \mid s} \frac{x - y}{s} \frac{\mu(r/a) \phi(r)}{\phi(r/a)} + O (r \log r),
\]
where \( \delta_{r \mid s} \) equals 1 when \( r \mid s \), and 0 otherwise.

**Lemma 2.2.** Let \( a \in \mathbb{Z} \) and \( s \in \mathbb{Z}_{\geq 1} \). If \( a \leq y \leq x \) and \( y \geq 0 \), then
\[
F_R(n) = \frac{x - y}{s} \sum_{r \leq R \atop r \mid q} \frac{\mu^2(r) \mu((r, a)) \phi((r, a))}{\phi(r)} + O(R).
\]

**Remark 2.3.** Lemma 2.2 implies that
\[
\rho_R^*(x; q, a) = \frac{x}{q} \sum_{r \leq R \atop r \mid q} \frac{\mu^2(r) \mu((r, a)) \phi((r, a))}{\phi(r)} + O(R).
\]
This expression is precise when \( q \) is small compared to \( x \) (cf. [15, Theorem 1, Corollaries 1.1–1.2]); for example, when \( q \leq R \) it takes the form
\[
\rho_R^*(x; q, a) = \delta_{(q, a) = 1} \frac{x}{\phi(q)} + O(R).
\]
However, (12) is not accurate when \( q \) is close to \( x \). Nevertheless, we will see by a different approach (see, for instance, the proof of Theorem 1.5(i)) that on average over large \( q \), \( \rho_R^*(x; q, a) \) is much closer to \( \psi^*(x; q, a) \) than to \( \delta_{(q, a) = 1} x / \phi(q) \).

We will average \( \psi(x; q, a) \) and \( \rho_R^*(x; q, a) \) over \( q \) close to \( x \) separately. We begin with \( \rho_R^*(x; q, a) \).

**Lemma 2.4.** For \( 0 < |a| < x/N \) and \( 1 \leq N, R \leq x \), we have
\[
\sum_{q \neq q \leq x} \rho_R^*(x; q, a) = x \sum_{s \leq N} \frac{1}{s} \left( 1 - \frac{s}{N} \right) \sum_{r \leq R \atop r \mid s} \frac{\mu^2(r) \mu((r, a)) \phi((r, a))}{\phi(r)} + O(RN + |a| (\log N)^2).
\]
\(^\dagger\)In Lemma 2.2, we have used the identity \( \mu(r) \mu(r / (r, a)) / \phi(r / (r, a)) = \mu^2(r) \mu((r, a)) \phi((r, a)) / \phi(r) \).
Remark 2.5. Under the additional condition $N \leq R$, the first term on the right-hand side of (13) equals
\[ x \sum_{s \leq N} \frac{1}{\phi(s)} \left(1 - \frac{s}{N}\right) + O(|a| \log N + RN), \]
and the expression $|a|(\log N)^2$ can be replaced with $|a| \log N$ in the error term.

Proof of Lemma 2.4. We rewrite the conditions $n \equiv a \mod q; n > a; x/N < q \leq x$ as $n = a + qs$, with $1 \leq s < N - aN/x$ and $a + sx/N < n \leq x$. We obtain that
\[ \sum_{x/N < q \leq x} \rho^*_R(x; q, a) = \sum_{1 \leq s < N - aN/x} \sum_{a + sx/N < n \leq x \text{ mod } s} F_R(n) - \sum_{x + a < n \leq x} F_R(n). \tag{14} \]
If $a < 0$, then Lemma 2.2 implies that
\[ \sum_{x + a < n \leq x} F_R(n) = |a| + O(R), \]
which is an admissible error term. Applying Lemma 2.2 with $y = a + sx/N > 0$, we see that the first term on the right-hand side of (14) is
\[ x \sum_{1 \leq s < N - aN/x} \frac{1}{s} \left(1 - \frac{s}{N} - \frac{a}{x}\right) \left(\frac{1}{s} - \frac{s}{N}\right) + O(RN) \tag{15} \]
\[ = x \sum_{1 \leq s < N - aN/x} \frac{1}{s} \left(1 - \frac{s}{N}\right) \sum_{r \leq R \mid s} \frac{\mu^2(r) \mu((r, a)) \phi((r, a)) \phi(r)}{\phi(r)} + O(RN + |a|(\log N)^2) \]
\[ = x \sum_{1 \leq s \leq N} \frac{1}{s} \left(1 - \frac{s}{N}\right) \sum_{r \leq R \mid s} \frac{\mu^2(r) \mu((r, a)) \phi((r, a)) \phi(r)}{\phi(r)} + O(RN + |a|(\log N)^2), \]
since for $s \in (N - |a|N/x, N + |a|N/x)$ we have that $|1 - s/N| < |a|/x$. The estimate (13) follows.

We now average $\psi^*(x; q, a)$ over $q$ close to $x$.

Lemma 2.6. Fix $A, B \geq 1$. In the range $1 \leq N \leq (\log x)^A$ and for $0 < |a| < x/N$, we have
\[ \sum_{x/N < q \leq x} \psi^*(x; q, a) = x \sum_{s \leq N} \frac{1}{\phi(s)} \left(1 - \frac{s}{N}\right) + O_{A, B}\left(\frac{x}{(\log x)^B}\right) + O(|a| \log x). \tag{16} \]

Proof. The proof is achieved by swapping moduli as in the proof of Lemma 2.4 and applying the Siegel–Walfisz theorem. We have
\[ \sum_{x/N < q \leq x} \psi^*(x; q, a) = \sum_{1 \leq s < N - aN/x} \sum_{a + sx/N < n \leq x \text{ mod } s} \Lambda(n) + O(|a| \log x) \]
\[ \sum_{1 \leq s < N - aN/x} \frac{x - (a + sx/N)}{\phi(s)} + O_{A,B} \left( \frac{x}{(\log x)^B} \right) + O(|a| \log x) \]

\[ = x \sum_{1 \leq s \leq N \atop (s,a)=1} \frac{1}{\phi(s)} \left( 1 - \frac{s}{N} \right) + O_{A,B} \left( \frac{x}{(\log x)^B} \right) + O(|a| \log x). \]

**Corollary 2.7.** Fix \( A, B \geq 1 \). For \( 0 < |a| < x/N \), \( 1 \leq N \leq (\log x)^A \) and \( N \leq R \leq x/(\log x)^{A+B} \), we have

\[ \sum_{x/N < q \leq x} (\psi(x; q, a) - \rho_R(x; q, a)) = O_{A,B} \left( \frac{x}{(\log x)^B} \right) + O(|a| \log x + RN). \]

**Proof.** We combine Lemmas 2.4 and 2.6 (see Remark 2.5), in which the main terms are identical. \( \square \)

We are ready to prove Theorem 1.5(i).

**Proof of Theorem 1.5(i).** Take \( N = 2M \) and \( N = M \) in Corollary 2.7, and subtract the resulting expressions. \( \square \)

### 3. Averages of multiplicative functions

In this section, we give estimates on averages of multiplicative functions which will be needed in Sections 4 and 5 to average \( \rho(x; q, a) \) over the full range \( q \leq x/M \). The following two constants will appear repeatedly:

\[ C_1(a) := \frac{\zeta(2) \zeta(3) \phi(a)}{\zeta(6)} a \prod_{p|a} \left( 1 - \frac{1}{p^2 - p + 1} \right), \]

\[ C_2(a) := C_1(a) \left( \gamma - 1 - \sum_p \frac{\log p}{p^2 - p + 1} + \sum_{p|a} \frac{p^2 \log p}{(p-1)(p^2 - p + 1)} \right). \]

Moreover, we define for \( c > 0 \)

\[ \eta_c(x) := \exp \left( c(\log x)^{3/5}(\log \log x)^{-1/5} \right). \]

**Lemma 3.1.** There exists an absolute constant \( c \) such that for \( x \in \mathbb{R}_{\geq 3} \) and \( \ell \in \mathbb{Z}_{\geq 1} \) with \( \ell \leq x^{10} \),

\[ \sum_{n > \ell \atop (n, \ell) = 1} \mu^2(n) \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)x} \prod_{p|\ell} \left( 1 + \frac{1}{p} \right)^{-1} + O \left( \frac{1}{x^{3/2} \eta_c(x)} \right). \]

**Proof.** We first record the unconditional bound on the Mertens function, which follows from the Korobov–Vinogradov zero-free region for \( \zeta(s) \):

\[ \sum_{n \leq x} \mu(n) \ll \frac{x}{\eta_c(x)}. \]
Proceeding as in [14, Exercise 6.2.19], we recover the classical estimate
\[ \sum_{n \leq x} \mu^2(n) = \frac{x}{\zeta(2)} + O \left( \frac{x^{1/2}}{\eta_c(x)} \right). \]

Combining this with the identity\(^1\)
\[ \sum_{n \leq x} \mu^2(n) = \sum_{d \mid \ell^\infty} \lambda(d) \sum_{m \leq x/d} \mu^2(m), \]
we obtain that
\[ \sum_{n \leq x} \mu^2(n) = \frac{x}{\zeta(2)} \prod_{p \mid \ell} \left( 1 + \frac{1}{p} \right)^{-1} + O \left( \frac{x^{1/2}}{\eta_c(x)} \sum_{d \mid \ell^\infty} \frac{1}{d} \right). \]

The sum in the error term is easily shown to be bounded by a constant times \((\log x)^2\). The estimate (17) follows from applying summation by parts. \(\square\)

**Lemma 3.2.** There exists an absolute constant \(c > 0\) such that for \(R \in \mathbb{R}_{\geq 3}\); we have the estimates
\[
\sum_{r > R} \frac{\mu^2(r)}{r \phi(r)} = \frac{1}{R} + O \left( \frac{1}{R^{3/2}\eta_c(R)} \right);
\]
\[
\sum_{r > R} \frac{\mu^2(r) \log r}{r \phi(r)} = \frac{\log R + 1}{R} + O \left( \frac{1}{R^{3/2}\eta_c(R)} \right).
\]

**Proof.** Using the convolution identity \(r/\phi(r) = \sum_{d \mid r} \mu^2(d)/\phi(d)\) and applying Lemma 3.1, we have that
\[
\sum_{r > R} \frac{\mu^2(r)}{r \phi(r)} = \sum_{d \geq 1} \frac{\mu^2(d)}{\phi(d)} \sum_{m > R/d} \frac{\mu^2(dm)}{d^2m^2} + O \left( \frac{1}{R^{8/5}} \right)
\]
\[
= \sum_{d \leq R^{1/5}} \frac{\mu^2(d)}{d^2\phi(d)} \sum_{m > R/d} \frac{\mu^2(m)}{m^2} + O \left( \frac{1}{R^{3/2}\eta_c(R)} \right) + \sum_{d > R^{1/5}} \frac{\mu^2(d)}{d^{1/2}\phi(d)} + \frac{1}{R^{8/5}}
\]
\[
= \frac{1}{R\zeta(2)} \sum_{d \leq R^{1/5}} \frac{\mu^2(d)}{d\phi(d)} \prod_{p \mid d} \left( 1 + \frac{1}{p} \right)^{-1} + O \left( \frac{1}{R^{3/2}\eta_c(R)} \right) + \sum_{d > R^{1/5}} \frac{\mu^2(d)}{d^{1/2}\phi(d)} + \frac{1}{R^{8/5}}
\]
\[
= \frac{1}{R\zeta(2)} \sum_{d \geq 1} \frac{\mu^2(d)}{d\phi(d)} \prod_{p \mid d} \left( 1 + \frac{1}{p} \right)^{-1} + O \left( \frac{1}{R^{3/2}\eta_c(R)} \right).
\]

The first result follows from a straightforward computation, and the second from a summation by parts. \(\square\)

\(^1\)By \(d \mid \ell^\infty\), we mean that \(d\) is a positive integer such that each of its prime factors divides \(\ell\).
Lemma 3.3. There exists an absolute constant $c > 0$ such that for $a \in \mathbb{Z}_{\neq 0}$ and $R \in \mathbb{R}_{\geq 3}$, we have the estimates

$$
\sum_{r \leq R, (r,a)=1} \frac{\mu^2(r)}{r \phi(r)} = \frac{|a|}{\phi(|a|)} C_1(a) - \frac{\phi(|a|)}{|a|} \frac{1}{R} + O \left( \frac{\prod_{p|a} \left( 1 + \frac{1}{p^{1/3}} \right)}{R^{3/2} \eta_c(R)} \right);
$$

$$
\sum_{r \leq R, (r,a)=1} \frac{\mu^2(r) \log r}{r \phi(r)} = \frac{|a|}{\phi(|a|)} C_1(a) \sum_{p|a} \frac{\log p}{p^2 - p + 1} - \frac{\phi(|a|) \log R + 1}{R} + O \left( \frac{\prod_{p|a} \left( 1 + \frac{1}{p^{1/3}} \right)}{R^{3/2} \eta_c(R)} \right).
$$

Proof. We only prove the first of these estimates. We have the identity

$$
Z := C_1(a) - \frac{\phi(|a|)}{|a|} \frac{1}{R} + O \left( \frac{\prod_{p|a} \left( 1 + \frac{1}{p^{1/3}} \right)}{R^{3/2} \eta_c(R)} \right);
$$

which combined with Lemma 3.2 gives that for some $c > 0$,

$$
\sum_{r > R, (r,a)=1} \frac{\mu^2(r)}{r \phi(r)} = \frac{1}{R} \sum_{d|a} \frac{\lambda(d)}{d} \prod_{p} ^{\nu} \frac{1}{p-1)^\nu} \sum_{m > R/d} \frac{\mu^2(m)}{m \phi(m)},
$$

which follows from a straightforward computation.

The second estimate follows from a straightforward computation.

Lemma 3.4. There exists an absolute constant $c > 0$ such that for $a \in \mathbb{Z}_{\neq 0}$ and $R \in \mathbb{R}_{\geq 3}$ are such that $a_R := \prod_{p \leq R} p \leq a \leq R / \log R$, then we have

$$
\sum_{r \leq R} \frac{\mu^2(r) \mu((r,a)) \phi((r,a))}{r \phi(r)} = C_1(a) - \frac{\epsilon_{a=\pm 1}}{R} + O \left( \frac{a_R^{1/2} \prod_{p|a} \left( 1 + \frac{2}{p^{1/3}} \right)}{R^{3/2} \eta_c(R/a_R)} \right);
$$

$$
\sum_{r \leq R} \frac{\mu^2(r) \mu((r,a)) \phi((r,a))}{r \phi(r)} \log r = (\gamma - 1) C_1(a) - C_2(a) - \frac{\epsilon_{a=\pm 1}(\log R + 1)}{R}
$$

$$
+ O \left( \frac{a_R^{1/2} \prod_{p|a} \left( 1 + \frac{2}{p^{1/3}} \right)}{R^{3/2} \eta_c(R/a_R)} \right);
$$

Proof. The first of these estimates follows from writing

$$
\sum_{r \leq R} \frac{\mu^2(r) \mu((r,a)) \phi((r,a))}{r \phi(r)} = \sum_{d|a} \frac{\mu(d)}{d} \sum_{m \leq R/d, (m,a)=1} \frac{\mu^2(m)}{m \phi(m)},
$$

applying Lemma 3.3 and performing a straightforward calculation.
4. The sum over all moduli

In order to prove Theorem 1.7, we need to understand the quantity $\rho^*_R(x; q, a)$ for more moderate values of $q$.

**Lemma 4.1.** Uniformly for $0 < |a| \leq x^{1/2}$ and $R \leq x^{1/2}$, we have

$$
\sum_{q \leq x} \rho^*_R(x; q, a) = x \sum_{r \leq R} \frac{\mu^2(r) \mu((r, a)) \phi((r, a))}{r \phi(r)} \left( \log \frac{x}{r^2} + 2 \gamma - 1 \right) + O(Rx^{1/2} + |a| \log x^2).
$$

**Proof.** For those $q$ in the interval $(x^{1/2}, x]$, we apply Lemma 2.4 to obtain that

$$
\sum_{x^{1/2} < q \leq x} \rho^*_R(x; q, a) = x \sum_{r \leq R} \frac{\mu^2(r) \mu((r, a)) \phi((r, a))}{r \phi(r)} \sum_{t \leq x^{1/2}/r} 1 \left( 1 - \frac{t}{x^{1/2}/r} \right) + O(Rx^{1/2} + |a| \log x^2).
$$

As for the remaining values of $q$, we take $y = a_+ := \max\{0, a\}$ in Lemma 2.2 and obtain

$$
\sum_{q \leq x^{1/2}} \rho^*_R(x; q, a) = \sum_{q \leq x^{1/2}} \frac{x - a_+}{q} \sum_{r | q} \frac{\mu^2(r) \mu((r, a)) \phi((r, a))}{r \phi(r)} + O(Rx^{1/2})
$$

$$
= x \sum_{r \leq R} \frac{\mu^2(r) \mu((r, a)) \phi((r, a))}{r \phi(r)} \sum_{t \leq x^{1/2}/r} 1 + O(Rx^{1/2} + |a| \log x^2).
$$

The desired estimate follows from applying the standard estimate on the harmonic sum. \qed

In the following lemma, we show that the average of $\psi^*(x; q, a)$ is very small when $(q, a) > 1$.

**Lemma 4.2.** We have that

$$
\sum_{q \leq x, (q, a) > 1} \psi^*(x; q, a) \ll x^{1/2} \omega(a) \log x.
$$

**Proof.** We write

$$
\sum_{q \leq x, (q, a) > 1} \psi^*(x; q, a) \leq \sum_{q \leq x^{1/2}, (q, a) > 1} \psi^*(x; q, a) + \sum_{s \leq x^{1/2} - ax^{-1/2}} \sum_{n \leq x, n \equiv a \mod s} \Lambda(n)
$$

$$
\ll \sum_{q \leq x^{1/2}} \sum_{p | a} \sum_{k \leq (\log x / \log p)} \log p + \sum_{s \leq x^{1/2}} \sum_{p | a} \sum_{k \leq (\log x / \log p)} \log p \leq 2x^{1/2} \omega(a) \log x.
$$

\qed

We are now ready to estimate the average of $\psi^*(x; q, a) - \rho^*_R(x; q, a)$ over $q \leq x/M$. 
Proposition 4.3. Fix $A, B \geq 1$ and $0 < \lambda < 1/4$. We have for $0 < |a| \leq x^\lambda$, $1 \leq M \leq R \leq x^{1/2}$ and $M \leq (\log x)^A$ that

$$\sum_{q \leq x/M} (\psi^*(x; q, a) - \rho^*_R(x; q, a))$$

$$= x \left[ C_1(a) \log x + C_1(a) + 2C_2(a) - \sum_{r \leq R} \frac{\mu^2(r)\mu((r, a))\phi((r, a))}{r\phi(r)} \left( \log \frac{x}{r^2} + 2\gamma - 1 \right) \right]$$

$$+ O(Rx^{1/2}) + O_{A, B, \lambda} \left( \frac{2^{\omega(a)} x}{(\log x)^B} \right).$$

Proof. Applying [6, Proposition 6.1] (which is based on the works [1, 7, 9, 10]) and the elementary estimate [6, Lemma 5.2] (see also [10, Lemma 13.1]), we obtain that

$$\sum_{q \leq x/M} \psi^*(x; q, a) = x \left[ C_1(a) \log x + C_1(a) + 2C_2(a) - \sum_{s \leq M} \frac{1}{\phi(s)} \left( 1 - \frac{s}{M} \right) \right]$$

$$+ O(2^{\omega(a)} M \log x) + O_{A, B, \lambda} \left( \frac{2^{\omega(a)} x}{(\log x)^B} \right). \quad (18)$$

Note that [6, Proposition 6.1] has the extra condition that $M$ should be an integer; however, going through the proof, we see that in general we have

$$\sum_{s < M - aM/x} \frac{1}{\phi(s)} \left( 1 - \frac{s}{M} \right) = \sum_{s \leq M} \frac{1}{\phi(s)} \left( 1 - \frac{s}{M} \right) + O \left( \frac{|a|}{x} \right),$$

and hence this extra condition can be removed at the cost of an admissible error term. Moreover, by Lemma 4.2, we can remove the condition $(q, a) = 1$ at the cost of the error term $O(x^{1/2}(\log x)^2)$. Finally, we combine Lemmas 2.4(ii) and 4.1 to obtain that

$$\sum_{q \leq x/M} \rho^*_R(x; q, a) = x \left[ \sum_{r \leq R} \frac{\mu^2(r)\mu((r, a))\phi((r, a))}{r\phi(r)} \left( \log \frac{x}{r^2} + 2\gamma - 1 \right) \right.$$

$$\left. - \sum_{s \leq M} \frac{1}{\phi(s)} \left( 1 - \frac{s}{M} \right) \right] + O(Rx^{1/2} + |a|(\log x)^2). \quad (19)$$

Subtracting this from (18) gives the desired result. \[\Box\]

Proof of Theorem 1.7(i). The result follows from combining Proposition 4.3 with Lemma 3.4, and a straightforward calculation. \[\Box\]

5. The coprimality condition

In this section we prove Theorems 1.5(ii) and 1.7(ii). This amounts to controlling the contribution of $\rho^*_R(x; q, a)$ with $(q, a) > 1$ (this is much easier for $\psi^*(x; q, a)$ and was already
done in Lemma 4.2). The condition \((q, a) = 1\) is easier to treat than the condition \((q, a) > 1\), and hence we will estimate sums over \((q, a) = 1\) directly.

Theorem 1.5(ii) will follow from the following analogue of Lemma 2.4.

**Lemma 5.1.** Let \(R, N \leq x\), and \(|a| < x/N\) be such that \(R \geq |a|N\). Then we have

\[
\sum_{x/N < q \leq x \atop (q, a) = 1} \rho_R^*(x; q, a) = x \sum_{s \leq N \atop (s, a) = 1} \frac{1}{\phi(s)} \left(1 - \frac{s}{N}\right) + O(2^{\omega(a)}RN + |a| \log N). \tag{20}
\]

**Proof.** Following the proof of Lemma 2.4, we write for \(a > 0\)

\[
\sum_{x/N < q \leq x \atop (q, a) = 1} \rho_R^*(x; q, a) = \sum_{1 \leq s < N-aN/x} \sum_{\substack{n \equiv a \mod s \\ ((n-a)/s, a) = 1}} F_R(n) + O(|a| + R). \tag{21}
\]

Applying Möbius inversion and Lemma 2.2, we see that the inner sum equals

\[
\sum_{d \mid a} \mu(d) \sum_{a+sx/N < n \leq x \atop n \equiv a \mod ds} F_R(n) = x - \left(\frac{a}{x} + \frac{sx}{N}\right) \sum_{d \mid a} \mu(d) \sum_{r \leq R \atop r \mid ds} \frac{\mu^2(r)\mu((r,a))\phi((r,a))}{\phi(r)} + O(2^{\omega(a)}R). \tag{22}
\]

Hence, the first term on the right-hand side of (21) equals

\[
x \sum_{1 \leq s < N-aN/x} \frac{1}{s} \left(1 - \frac{s}{N} - \frac{a}{x}\right) \sum_{d \mid a} \mu(d) \sum_{r \leq R \atop r \mid ds} \frac{\mu^2(r)\mu((r,a))\phi((r,a))}{\phi(r)} + O(2^{\omega(a)}RN). \tag{23}
\]

Since \(|a|N \leq R\), for \(a > 0\) the innermost sums equals

\[
\sum_{r \mid ds} \frac{\mu^2(r)\mu((r,a))\phi((r,a))}{\phi(r)} = \begin{cases} 
\frac{ds}{\phi(ds)} & \text{if } (ds, a) = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

If \(a < 0\), then we need to add an error term for the term \(d = a\); this error term is easily seen to sum to \(O(1)\). Therefore, the first term on the right-hand side of (21) is

\[
= x \sum_{1 \leq s < N-aN/x \atop (s, a) = 1} \frac{1}{s} \left(1 - \frac{s}{N} - \frac{a}{x}\right) \sum_{d \mid a \atop (d, a) = 1} \frac{\mu(d)}{d} \frac{ds}{\phi(ds)} + O(2^{\omega(a)}RN)
\]

\[
= x \sum_{s \leq N \atop (s, a) = 1} \frac{1}{\phi(s)} \left(1 - \frac{s}{N}\right) + O(2^{\omega(a)}RN + |a| \log N). \tag*{□}
\]

**Proof of Theorem 1.5(ii).** Combine Lemmas 2.6, 4.2 and 5.1. \(\Box\)

In order to prove Theorem 1.7(ii), we need to have an estimate on the sum of \(\rho_R^*(x; q, a)\) over all \(q \leq x\) coprime to \(a\). We start with an elementary lemma.
Lemma 5.2. If \( a \neq 0 \) and \( r \geq 1 \) are integers, then for \( y \in \mathbb{R}_{\geq 1/2} \) we have the estimate

\[
\sum_{n \leq y} \frac{1}{n} = \delta_{(r,a)=1} \left( \frac{\phi(a)}{ar} \left( \log \frac{y}{r} + \gamma + \sum_{p|a} \frac{\log p}{p-1} \right) + O \left( \frac{2^{\omega(a)}}{y} \right) \right),
\]

where \( \delta_{(r,a)=1} \) equals 1 when \( (r,a) = 1 \), and is zero otherwise.

Proof. If \( (r,a) > 1 \), then the sum on the left-hand side is clearly zero. Otherwise, we apply Möbius inversion and the standard estimate on the harmonic sum to obtain that

\[
\sum_{n \leq y} \frac{1}{n} = \frac{1}{r} \sum_{d|a} \mu(d) \left( \log \frac{y}{rd} + \gamma + \sum_{t \leq y/rd} \frac{\mu^2(t)}{t \phi(t)} \right) + O \left( \frac{\delta(a)}{y} \right).
\]

The proof follows from a standard calculation.

The next lemma is an analogue of Lemma 4.1.

Lemma 5.3. For \( 0 \neq |a| < x^{1/2} \) and \( R \leq x^{1/2} \), the following holds:

\[
\sum_{q \leq x} \rho_R(x; q, a) = x \frac{\phi(|a|)}{|a|} \sum_{r \leq R} \frac{\mu^2(r)}{r \phi(r)} \left( \log \frac{x}{r^2} + 2\gamma - 1 + \sum_{p|a} \frac{\log p}{p-1} \right)
\]

\[
+ \frac{\phi(|a|)}{|a|} \sum_{p|a} \log p \sum_{t \leq R/p} \frac{\mu^2(t)}{t \phi(t)} + O(|a| \log x + 2^\omega(a) R x^{1/2}).
\]

Proof. Arguing as in Lemma 4.1, we cut the sum at \( q = x^{1/2} \) and exchange divisors. Applying Möbius inversion, setting \( y = a_+ := \max\{0, a\} \) in Lemma 2.2 and applying (22), we compute

\[
\sum_{q \leq x} \rho_R(x; q, a) = \sum_{q \leq x^{1/2}} \sum_{n \leq x \mod q} \frac{\mu^2(r) \mu((r,a)) \phi((r,a))}{\phi(r)} + O(R x^{1/2} + |a|)
\]

\[
+ \sum_{s \leq x^{1/2} - ax^{-1/2}} \frac{\mu^2(r) \mu((r,a)) \phi((r,a))}{\phi(r)} + O(2^\omega(a) R x^{1/2})
\]

\[= I + II.\]
To evaluate the first term, we apply Lemma 5.2 and obtain that

\[ I = x \frac{\phi(|a|)}{|a|} \sum_{r \leq R, (r,a) = 1} \frac{\mu^2(r)}{r \phi(r)} \left( \log \frac{x^{1/2}}{r} + \gamma + \sum_{p | a} \frac{\log p}{p - 1} \right) + O(|a| \log x + 2^{\omega(a)} R x^{1/2}). \]

As for the second, we note that \( r | ds \) if and only if \( r/(r,d) | s \), and thus

\[ II = x \sum_{r \leq R} \frac{\mu^2(r)\mu((r,a))\phi((r,a))}{\phi(r)} \sum_{d | a} \frac{\mu(d)}{d} \sum_{s < x^{1/2}, x^{1/2}-s < x^{-1/2}} \frac{1 - ax^{-1} - sx^{-1/2}}{s} + O(2^{\omega(a)} R x^{1/2}) \]

\[ = x \sum_{r \leq R} \frac{\mu^2(r)\mu((r,a))\phi((r,a))}{\phi(r)} \sum_{d | a} \frac{\mu(d)}{d} \frac{(r,d)}{r} \times \left[ \log \frac{x^{1/2}(r,d)}{r} + \gamma - 1 + O \left( \frac{|a|}{x} \log x + \frac{r}{(r,d)x^{1/2}} \right) \right] \]

\[ + O(2^{\omega(a)} R x^{1/2}). \]

Note however that

\[ \sum_{d | a} \frac{\mu(d)(r,d)}{d} = \delta_{(r,a) = 1} \frac{\phi(|a|)}{|a|} \cdot \sum_{d | a} \frac{\mu(d)(r,d)}{d} \log(r,d) = \begin{cases} -\frac{\phi(|a|) p \log p}{|a|} & \text{if } (a,r) = p^k \\ 0 & \text{otherwise}, \end{cases} \]

and hence

\[ II = x \frac{\phi(|a|)}{|a|} \sum_{r \leq R, (r,a) = 1} \frac{\mu^2(r)}{r \phi(r)} \left( \log \frac{x^{1/2}}{r} + \gamma - 1 \right) + x \frac{\phi(|a|)}{|a|} \sum_{p | a} \frac{\log p}{p - 1} \sum_{t \leq R/p, (t,a) = 1} \frac{\mu^2(t)}{t \phi(t)} \]

\[ + O(2^{\omega(a)} R x^{1/2}). \]

\[ \square \]

**Proposition 5.4.** Fix \( A, B \geq 1 \) and \( 0 < \lambda < 1/4 \). For \( 0 \neq |a| \leq x^\lambda, 1 \leq |a|M \leq R \leq x^{1/2} \) and \( M \leq (\log x)^A \), we have that

\[ \sum_{q \leq x/M, (q,a) = 1} (\psi^*(x; q,a) - \rho^*_R(x; q,a)) = x \left[ C_1(a) \log x + C_1(a) + 2C_2(a) \right. \]

\[ - \frac{\phi(|a|)}{|a|} \sum_{r \leq R, (r,a) = 1} \frac{\mu^2(r)}{r \phi(r)} \left( \log \frac{x}{r^2} + 2\gamma - 1 + \sum_{p | a} \frac{\log p}{p - 1} \right) \]

\[ - \frac{\phi(|a|)}{|a|} \sum_{p | a} \frac{\log p}{p - 1} \sum_{t \leq R/p, (t,a) = 1} \frac{\mu^2(t)}{t \phi(t)} \]

\[ + O(2^{\omega(a)} R x^{1/2}) + O_{A,B,\lambda} \left( 2^{\omega(a)} \frac{x}{(\log x)^B} \right). \]
ON VAUGHAN’S APPROXIMATION: THE FIRST MOMENT

Proof. Combining Lemmas 5.1 and 5.3 gives that

\[
\sum_{q \leq x/M} \rho^*_R(x; q, a) = x \frac{\phi(|a|)}{|a|} \sum_{r \leq R} \frac{\mu^2(r)}{r \phi(r)} \left( \log \frac{x}{r^2} + 2\gamma - 1 + \sum_{p \mid a} \frac{\log p}{p - 1} \right) \\
+ x \frac{\phi(|a|)}{|a|} \sum_{p \mid a} \frac{\log p}{p - 1} \sum_{t \leq R/p} \frac{\mu^2(t)}{t \phi(t)} - x \sum_{s \leq M} \frac{1}{\phi(s)} \left( 1 - \frac{s}{M} \right) \\
+ O(|a| \log x + 2^{\omega(a)} R x^{1/2}).
\]

(24)

Applying (18) then yields the desired result. □

Proof of Theorem 1.7(ii). The proof follows from combining Proposition 5.4 with Lemma 3.3. □

6. The quantity \( \rho^*_R(x; q, a) \) when \( (q, a) > 1 \)

Comparing Theorem 1.7(i) and (ii), we see that the main terms agree when \( a = \pm 1 \) (since the sums on the left-hand side coincide), but they are very different when \( \omega(a) \geq 1 \). More precisely, combining Lemmas 2.4, 4.1, 5.1 and 5.3 we see that for \( 0 < |a| < x^{1/2} \) and \( 1 \leq |a| N \leq R \leq x^{1/2} \),

\[
\sum_{q \leq x/N} \rho^*_R(x; q, a) = x \sum_{r \leq R} \frac{\mu^2(r) \mu((r, a)) \phi((r, a))}{r \phi(r)} \left( \log \frac{x}{r^2} + 2\gamma - 1 \right) \\
- x \frac{\phi(|a|)}{|a|} \sum_{r \leq R} \frac{\mu^2(r)}{r \phi(r)} \left( \log \frac{x}{r^2} + 2\gamma - 1 + \sum_{p \mid a} \frac{\log p}{p - 1} \right) \\
- x \frac{\phi(|a|)}{|a|} \sum_{p \mid a} \frac{\log p}{p - 1} \sum_{t \leq R/p} \frac{\mu^2(t)}{t \phi(t)} + O_a(R x^{1/2}).
\]

(25)

It is not surprising that the main terms in this estimate are independent of \( N \). Indeed applying Lemmas 2.4 and 5.1 directly shows that

\[
\sum_{x/N < q \leq x} \rho^*_R(x; q, a) \ll 2^{\omega(a)} R N + |a| \log N.
\]

One can evaluate the sums in (25) using Lemmas 3.3 and 3.4, resulting in the expression

\[
\sum_{q \leq x/N} \rho^*_R(x; q, a) = \left( \frac{\phi(|a|)}{|a|} \right)^2 \frac{x}{R} \left[ \log \frac{x}{R^2} + 2\gamma - 3 + \sum_{p \mid a} \frac{p + 1}{p - 1} \log p \right] \\
+ O_{e,a} \left( \frac{x \log x}{R^{3/2} \eta_e(R)} + R x^{1/2} \right).
\]
Hence, the term $\rho_R^*(x; q, a)$ is on average of order $(N/R)(\log(x/R^2) + 1)$. However, the mass in this average is contained in the terms $q \ll x/R$, and thus it is more accurate to say that this term is of order $(\log(x/R^2) + 1)$ on average for $q \ll x/R$ and is small for larger moduli.

In conclusion, while being quite small when $(q, a) > 1$, the quantity $\rho_R^*(x; q, a)$ is not completely negligible and can be evaluated asymptotically on average over those values of $q$.

7. Further proofs

The goal of this section is to prove Proposition 1.10. We first show in Lemma 7.1 that the total mass of $\rho_R^*(x; q, a)$ over all arithmetic progressions modulo $q$ is about $x$, and that this mass is concentrated in the invertible residue classes. It follows that $\rho_R^*(x; q, a) - x/\phi(q)$ is the approximate discrepancy of $\rho_R^*(x; q, a)$ in the invertible residue classes modulo $q$.

Lemma 7.1. The total mass of $F_R(n)$ for $q < n \leq x$ in all residue classes modulo $q$ equals

$$\sum_{1 \leq a \leq q} \rho_R^*(x; q, a) = x - q + O(R),$$

and its mass in the invertible residue classes modulo $q$ is given by

$$\sum_{1 \leq a \leq q \atop (a, q) = 1} \rho_R^*(x; q, a) = x - q + O \left( \frac{x}{R} \prod_{p|q} \left( 2 + \frac{1}{p} \right) + 2^{\omega(q)} R \right).$$

Proof. The first estimate follows from a direct application of Lemma 2.2:

$$\sum_{a=1}^q \rho_R^*(x; q, a) = \sum_{q < n \leq x} F_R(n) = (x - q) \sum_{r \leq R \atop r|1} \frac{\mu^2(r) \mu((r, n)) \phi((r, n))}{\phi(r)} + O(R)$$

$$= x - q + O(R).$$

To prove the second, we first use M"obius inversion, and then apply Lemma 2.2 with $a = 0$. This gives the estimate

$$\sum_{1 \leq a \leq q \atop (a, q) = 1} \rho_R^*(x; q, a) = \sum_{q < n \leq x \atop (n, q) = 1} F_R(n) = \sum_{d|q} \mu(d) \sum_{q < n \leq x \atop n \equiv 0 \mod d} F_R(n)$$

$$= (x - q) \sum_{d|q} \frac{\mu(d)}{d} \sum_{r \leq R \atop r|d} \mu(r) + O(2^{\omega(q)} R)$$

$$= (x - q + (x - q) \sum_{d|q} \frac{\mu(d)}{d} \sum_{r \leq R \atop r|d} \mu(r) + O(2^{\omega(q)} R).$$

Now, writing $q' := \prod_{p|q} p$, we have that

$$\sum_{d|q} \frac{\mu(d)}{d} \sum_{r \leq R \atop r|d} \mu(r) \ll \sum_{d|q'} \sum_{r > R \atop r|d} \frac{1}{d}$$
ON VAUGHAN’S APPROXIMATION: THE FIRST MOMENT

\[ \prod_{p|q} \left( 1 + \frac{1}{p} \right) \sum_{\substack{r|q \ r > R}} \prod_{p|r} \left( 1 + \frac{1}{p} \right)^{-1} \]

\[ \leq \frac{1}{R} \prod_{p|q} \left( 1 + \frac{1}{p} \right) \sum_{\substack{r|q \ r = 1}} \prod_{p|r} \left( 1 + \frac{1}{p} \right)^{-1}. \]

The proof follows by multiplicativity. \[ \square \]

We now come back to the discrepancies of \( \rho^*_R(x; q, a) \) in arithmetic progressions.

**Proof of Proposition 1.10.** Combining Lemmas 5.1 and 5.3 with Lemma 5.9\(^\dagger\) gives that

\[ \sum_{q \leq x/M \ (q,a)=1} \rho^*_R(x; q, a) = x \left( C_1(a) \log \frac{x}{M} + C_1(a) + C_2(a) + \frac{\phi(|a|) \mu(a,M)}{|a|} \right) \]

\[ - \left( \frac{\phi(|a|)}{|a|} \right)^2 \frac{x}{R} \left( \log \frac{x}{R^2} + 2\gamma - 3 + \sum_{p|a} \frac{p+1}{p-1} \log p \right) \]

\[ + O_{a,\varepsilon} \left( Rx^{1/2} + \frac{x}{M^{19/448}} + \frac{x \log x}{R^{3/2} \eta_c(R)} \right). \]

The result follows from subtracting the following classical elementary estimate (see for instance \[10\], Lemma 13.1), in which we can replace \( \tau(a) \) by \( 2^{\omega(a)} \):

\[ \sum_{q \leq x/M \ (q,a)=1} \frac{x}{\phi(q)} = x \left[ C_1(a) \log \frac{x}{M} + C_1(a) + C_2(a) + O \left( 2^{\omega(a)} M \frac{\log x}{x} \right) \right]. \] \[ \square \]

**Acknowledgements.** I would like to thank Robert C. Vaughan for introducing me to his approximation as well as Régis de la Bretèche and James Maynard for fruitful conversations.

**References**

1. E. Bombieri, J. B. Friedlander and H. Iwaniec, ‘Primes in arithmetic progressions to large moduli’, Acta Math. 156 (1986) 203–251.
2. J. Bourgain, ‘Decoupling, exponential sums and the Riemann zeta function’, J. Amer. Math. Soc. 30 (2017) 205–224.
3. J.-M. Deshouillers and H. Iwaniec, ‘Kloosterman sums and Fourier coefficients of cusp forms’, Invent. Math. 70 (1982/83) 219–288.
4. S. Drappeau, ‘Théorèmes de type Fouvry–Iwaniec pour les entiers friables’, Compos. Math. 151 (2015) 828–862.
5. S. Drappeau, ‘Sums of Kloosterman sums in arithmetic progressions, and the error term in the dispersion method’, Proc. London Math. Soc., to appear.
6. D. Fiorilli, ‘Residue classes containing an unexpected number of primes’, Duke Math. J. 161 (2012) 2923–2943.
7. É. Fouvry, ‘Sur le problème des diviseurs de Titchmarsh’ J. reine angew. Math. 357 (1985) 51–76.
8. É. Fouvry and G. Tenenbaum, ‘Diviseurs de Titchmarsh des entiers sans grand facteur premier’, Analytic number theory (Tokyo, 1988), Lecture Notes in Mathematics 1434 (Springer, Berlin, 1990) 86–102.
9. J. B. Friedlander and A. Granville, ‘Relevance of the residue class to the abundance of primes’, Proceedings of the Amalfi Conference on Analytic Number Theory, Maiori, 1989 (University of Salerno, Salerno, 1992) 95–103.

\(^\dagger\)The exponent 205/538 in this estimate can be replaced with 171/448 thanks to Bourgain’s result [2].
10. J. B. Friedlander, A. Granville, A. Hildebrand and H. Maier, ‘Oscillation theorems for primes in arithmetic progressions and for sieving functions’, J. Amer. Math. Soc. 4 (1991) 25–86.

11. D. A. Goldston and R. C. Vaughan, ‘On the Montgomery-Hooley asymptotic formula’, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), London Mathematics Society Lecture Note Series 237 (eds G. R. H. Greaves, G. Harman and M. N. Huxley; Cambridge University Press, Cambridge, 1997) 117–142.

12. C. Hooley, ‘On the Barban-Davenport-Halberstam theorem. VIII’, J. reine angew. Math. 499 (1998) 1–46.

13. M. N. Huxley, ‘Exponential sums and the Riemann zeta function. V’, Proc. Lond. Math. Soc. (3) 90 (2005) 1–41.

14. H. L. Montgomery and R. C. Vaughan, Multiplicative number theory. I. Classical theory, Cambridge Studies in Advanced Mathematics 97 (Cambridge University Press, Cambridge, 2007) xviii+552 pp.

15. R. C. Vaughan, ‘Moments for primes in arithmetic progressions. I’, Duke Math. J. 120 (2003) 371–383.

16. R. C. Vaughan, ‘Moments for primes in arithmetic progressions. II’, Duke Math. J. 120 (2003) 385–403.

Daniel Fiorilli
Département de mathématiques et de statistique
Université d’Ottawa
Ottawa, Ontario
Canada K1N 6N5
daniel.fiorilli@uottawa.ca