Some unlikely intersections between the Torelli locus and Newton strata in $A_g$

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Abstract

Let $p$ be an odd prime. What are the possible Newton polygons for a curve in characteristic $p^2$? Equivalently, which Newton strata intersect the Torelli locus in $A_g$? In this note, we study the Newton polygons of certain curves with $\mathbb{Z}/p\mathbb{Z}$-actions. Many of these curves exhibit unlikely intersections between the Torelli locus and the Newton stratification in $A_g$. Here is one example of particular interest: fix a genus $g$. We show that for any $k$ with $2g - \frac{2p(p-1)}{3} \geq 2k(p-1)$, there exists a curve of genus $g$ whose Newton polygon has slopes $\{0, 1\}^{g-k(p-1)} \sqcup \{\frac{1}{2}\}^{2k(p-1)}$. This provides evidence for Oort’s conjecture that the amalgamation of the Newton polygons of two curves is again the Newton polygon of a curve. We also construct families of curves $\{C_g\}_{g \geq 1}$, where $C_g$ is a curve of genus $g$, whose Newton polygons have interesting asymptotic properties. For example, we construct a family of curves whose Newton polygons are asymptotically bounded below by the graph $y = \frac{x^2}{4g}$. The proof uses a Newton-over-Hodge result for $\mathbb{Z}/p\mathbb{Z}$-covers of curves due to the author, in addition to recent work of Booher-Pries on the realization of this Hodge bound.

1 Introduction

Let $p$ be an odd prime. By a curve, we will always mean a smooth proper irreducible curve. What are the possible Newton polygons for a curve of genus $g$ in characteristic $p^2$? In general, it seems difficult to answer this question for all $g$. However, the following conjecture of Oort offers some guidance on what to expect.

Conjecture 1.1. (Oort, see [3, Conjecture 8.5.7]) Let $C$ (resp. $C'$) be a curve of genus $g$ (resp. $g'$) with Newton polygon $P$ (resp. $P'$). Then there exists a curve $C''$ of genus $g + g'$ whose Newton polygon $P''$ is the amalgamation of $P$ and $P'$ (i.e. the slopes of $P''$ are the disjoint union of the slopes of $P$ and $P'$).

This conjecture implies, for example, that there exists irreducible supersingular curves of every genus. This currently only know in characteristic 2, due to a theorem of van der Geer and van der Vlugt (see [5]). Another approach to studying Newton polygons of curves is to ask if there are curves $C_g$ of every genus $g \in \mathbb{Z}_{\geq 0}$ whose Newton polygons approach some limit asymptotically. This motivates the following questions.

Question 1.2. Let $m_1, \ldots, m_{2r} \in \mathbb{Q} \cap [0, 1]$ such that $m_k = 1 - m_{2r-k}$. Does there exist a family of curves $\{C_g\}_{g \in \mathbb{Z}_{\geq 0}}$, where $C_g$ has genus $g$, such that the Newton polygon of $C_g$ consists only of the slopes $m_1, \ldots, m_{2r}$ and each $m_i$ occurs with multiplicity close to $\frac{2g^2}{g^2}$?
Question 1.3. Let $P$ be the graph of a continuous function $f : [0, 2] \to [0, 1]$. Does there exist a family of curves $\{C_g\}_{g \in \mathbb{Z}_{\geq 0}}$, where $C_g$ has genus $g$, such that the Newton polygon of $C_g$, scaled by a factor of $\frac{1}{g}$, approaches or lies above $P$ as $g \to \infty$?

In this article we study Conjecture 1.1, Question 1.2, and Question 1.3 by considering $\mathbb{Z}/p\mathbb{Z}$-covers of curves.

1.1 Some previous results

There are two main approaches when trying to find curves with certain Newton polygons. The first is to consider $\mathbb{Z}/n\mathbb{Z}$-covers $f : X \to \mathbb{P}^1$. When $n$ is a power of $p$ and $f$ is ramified at a single point, the Newton polygon of $X$ has been studied extensively by Robba, Zhu, Blache-Ferard, and Liu-Wei, using $p$-adic methods pioneered by Dwork (see [14], [2], [11], and [16]). Also, the work of van der Geer and van der Vlugt (see [15]) studies $\mathbb{Z}/p\mathbb{Z}$-covers with additional structure when $p = 2$ by analyzing their Jacobians. When $n$ is coprime to $p$, the number of slope zero segments (i.e. the $p$-rank) was determined in many cases by Bouw (see [4]). Determining the higher slopes appears to be more difficult and is the subject of work by Li-Mantovan-Pries-Tang (see [9]). The second approach is to use clutching morphisms between moduli spaces of curves. This technique was used by Achter-Pries in [1] to show that for $g \geq 4$, there exists a genus $g$ curve whose Newton polygon has slopes $\{0, 1\}^{g-4} \sqcup \{\frac{1}{2}\}^4$. More recently, these two techniques were combined in work of Li-Mantovan-Pries-Tang by studying clutching morphisms for tame covers of $\mathbb{P}^1$. Their work constructs many interesting families of curves whose Newton polygons are far from ordinary and follow certain patterns. However, most of these families do not include curves of every genus. Instead these families include curves whose genera satisfy certain congruence conditions. We describe some of these families in Remark 2.12 and Examples 4.5-4.6.

1.2 $\mathbb{Z}/p\mathbb{Z}$-covers with many branch points

In §4 we study $\mathbb{Z}/p\mathbb{Z}$-covers of curves with many branch points of fixed Swan conductor. Let us state a specific case of our general result (see Theorem 4.1 for the general statement).

Theorem 1.4. For any $g \geq 0$ and $k$ with $\frac{2g}{3} - \frac{2p(p-1)}{3} \geq 2k(p-1)$, there exists a curve $C_{g,k}$ of genus $g$ whose Newton polygon has slopes $\{0, 1\}^{g-k(p-1)} \sqcup \{\frac{1}{2}\}^{2k(p-1)}$.

From Theorem 1.4 we may deduce Conjecture 1.1 for many interesting examples. Indeed, if $C = C_{g,k}$ and $C' = C_{g',k'}$, then the Newton polygon of $C'' = C_{g+g',k+k'}$ is the amalgamation of the Newton polygons of $C$ and $C'$. When $k$ is not small relative to $g$, we see that $C_{g,k}$ demonstrates an unlikely intersection between a Newton stratum and the Torelli locus in the Siegel modular variety (see Corollary 4.4). Furthermore, we may use Theorem 1.4 to study Question 1.2. By letting $k$ be as large as possible, we see that there are curves of every genus consisting of slopes $0, \frac{1}{2}, 1$, where each slope occurs with multiplicity approximately $\frac{2g}{3}$. This was previously only known under the assumption of $p \equiv 2 \pmod{3}$ (see [9, Corollary 9.4]).

1.3 $\mathbb{Z}/p\mathbb{Z}$-covers of curves with large Swan conductor

In §5 we study curves with a small number of branch points and large Swan conductors. We construct a family of curves $\{C_g\}_{g \in \mathbb{Z}_{\geq 0}}$, where $C_g$ has genus $g$, such that their Newton polygons...
asymptotically lie above or on the graph \( y = \frac{x^2}{4g} \) (see Definition 2.7 and Theorem 5.1 for a precise statement). To the best of our knowledge, it is unknown if there is a family of curves containing a curve of every genus whose Newton polygons asymptotically lie strictly above \( y = \frac{x^2}{4g} \). This would certainly follow from Conjecture 1.1. However, all of the families constructed in §4 and the families constructed in [9] lie well below \( y = \frac{x^2}{4g} \) when the genus is large.

\[ \text{1.4 Method of proof} \]

To prove Theorem 1.4 and Theorem 5.1 we use a Newton-over-Hodge result due to the author (see [8] or Theorem 3.1). This theorem gives a lower bound for the Newton polygon of a \( \mathbb{Z}/p\mathbb{Z} \)-cover \( C \to X \) in terms of local Swan conductors. By considering covers of an arbitrary curve \( X \) instead of only \( \mathbb{P}^1 \), we are able to obtain curves of any genus (e.g. if \( X \) has genus \( i \), the genus of \( C \) is of the form \( i + k \frac{2 - 1}{2} \) by Riemann-Hurwitz). This is a clear advantage over earlier techniques. For the general statement of Theorem 1.4, we also need to use recent work of Booher-Pries (see [3]). This work shows the lower bound in [8] is realized when certain congruence conditions between \( p \) and the Swan conductors hold.

\[ \text{1.5 Acknowledgments} \]

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\[ \text{2 Newton polygons and unlikely intersections in } A_g \]

\[ \text{2.1 Conventions on Newton polygons} \]

Let \( \alpha \in \mathbb{Z}_{\geq 0} \cup \infty \) and let \( f : [0, \alpha] \to \mathbb{R} \) be a continuous convex function. We let \( P = P(f, \alpha) \) refer to the graph of \( f \) in the \( xy \)-plane. We refer to the points \((0, f(0))\) and \((\alpha, f(\alpha))\) as the endpoints of \( P \). We say that \( P \) is a symmetric if \( f(\alpha - x) = f(\alpha) - f(x) \) for all \( x \in [0, \alpha] \). The following two types of graphs are of particular interest.

**Definition 2.1.** A graph \( P = P(f, 2) \) is called basic if \( P \) is symmetric and the endpoints of \( P \) are \((0, 0)\) and \((2, 1)\).

**Definition 2.2.** We say \( P(f, 2g) \) is a Newton polygon of height \( 2g \) if \( f \) satisfies the following:

1. \( f \) is symmetric and the endpoints are \((0, 0)\) and \((2g, g)\).
2. For any integer \( i \in (0, 2g) \), the function \( f(x) \) is linear on the domain \( x \in [i - 1, i] \) with slope \( m_i \geq 0 \).

We will refer to the multiset \( \{m_i\}_{1 \leq i \leq 2g} \) as the slope-set of \( P \) and its elements as the slopes of \( P \). Note that the slope-set completely determines the Newton polygon.

For \( i = 1, 2 \), consider graphs \( P_1 = P(f_i, \alpha) \). We write \( P_1 \geq P_2 \) if \( f_1(x) \geq f_2(x) \) for all \( 0 \leq x \leq \alpha \). When \( P_1 \geq P_2 \), we say that \( P_1 \) lies above \( P_2 \). If \( P_2 \) is a Newton polygon with slope-set \( N \), we will occasionally write \( P_1 \geq N \) instead of \( P_1 \geq P_2 \). Finally, for any \( c \geq 0 \), we define the scaled graph \( cP_1 = \{(cx, cy) | (x, y) \in P_1\} \).
2.2 The Newton stratification of \( A_g \)

Let \( A_g \) denote the moduli space of principally polarized abelian varieties of dimension \( g \) and let \( X \to A_g \) be the universal abelian scheme. For each closed \( x \in A_g \), we obtain an Abelian variety \( X_x \). Let \( NP_x \) denote the Newton polygon of height \( 2g \) associated to \( X_x \) (see [12, (1.2)]). We remark that the vertices of \( NP_x \) have integer coordinates. This greatly restricts the possibilities for \( NP_x \).

**Definition 2.3.** Let \( P \) be a Newton polygon of height \( 2g \). Let \( W_P \subset A_g \) (resp. \( W_P^0 \)) denote the locus of principally polarized abelian varieties \( X_g \) with \( NP_x \geq P \) (resp. \( NP_x = P \)). Note that \( W_P \) is closed in \( A_g \) and \( W_P^0 \) is open in \( W_P \) (see [12, §4]).

**Theorem 2.4.** The codimension of \( W_P \) in \( A_g \) is at least \( \#\Omega(P) \), where

\[
\Omega(P) = \{ (x, y) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid 0 \leq x \leq g \text{ and } (x, y) \text{ lies strictly below } P \}.
\]

Furthermore, if \( P \) has integer vertices, the codimension of \( W_P \) in \( A_g \) is exactly \( \#\Omega(P) \).

**Proof.** This follows from [12, Theorem 4.1], by noting that \( W_P = \bigcup P' W_{P'} \), where the union runs over all Newton polygons of height \( 2g \) with integer vertices lying above \( P \). \( \square \)

2.3 The Torelli locus

The Torelli map \( \iota : M_g \to A_g \) sends a curve \( C \) of genus \( g \) to its Jacobian. Let \( T_g \) denote the image of the Torelli map. It is a closed subscheme of dimension \( 3g - 3 \). The Newton polygon \( NP(C) \) of \( C \) is defined to be the Newton polygon of the corresponding point in \( A_g \). If \( C \) is defined over a finite field \( \mathbb{F}_q \), then \( NP(C) \) is equal to the \( q \)-adic Newton polygon of the numerator of its zeta function. We define the *scaled Newton polygon* to be

\[
sNP(C) = \frac{1}{g} NP(C).
\]

Note that \( sNP(C) \) is a basic graph (see Definition 2.1). We are interested in the following questions.

**Question 2.5.** How does the Torelli locus interact with the Newton stratification? More specifically, let \( P \) be a Newton polygon of height \( 2g \). Does there exist a curve \( C \) of genus \( g \) with \( NP(C) = P \) (resp. \( NP(C) \geq P \)). Equivalently, is \( W_P^0 \cap T_g \) (resp. \( W_P \cap T_g \)) nonempty?

This question appears to be very difficult in general. Instead, one may ask for asymptotic behavior as \( g \) gets large. One way to do this is to study the behavior of \( sNP(C) \) as \( C \) varies over a collection of curves. This prompts the following definitions.

**Definition 2.6.** A *family of curves* is a collection \( \mathcal{C} = \{ C_g \}_{g \in S} \) where \( S \subset \mathbb{Z}_{\geq 0} \) and \( C_g \) is a curve of genus \( g \). We say that \( \mathcal{C} \) is *full* if \( S = \mathbb{Z}_{\geq 0} \) and we say that \( \mathcal{C} \) is *arithmetic* if \( S \) is the union of finitely many arithmetic progressions.

**Definition 2.7.** Let \( \mathcal{C} = \{ C_g \}_{g \in S} \) be a family of curves and let \( P = P(f, 2) \) be a basic graph. For each \( g \in S \), let \( f_g \) be the piecewise linear function such that \( sNP(C_g) = P(f_g, 2) \). We write \( sNP(\mathcal{C}) \geq P \) if

\[
\liminf_{g \in S} \min_{x \in [0,2]} \{ f_g(x) - f(x) \} \geq 0.
\]
**Question 2.8.** Let $P$ be a basic graph. Does there exist a family of curves $\mathcal{C}$ such that $sNP(\mathcal{C}) \succeq P$? Can we take this family to be arithmetic or full?

Another natural question is to ask for families of curves where certain slopes occur with some specified frequency. For example, we may ask for a family of curves $\{C_g\}_{g \in S}$ where $NP(C_g)$ only has slopes $0, \frac{1}{2}, 1$, and each slope occurs with approximately equal frequency. This prompts the following definition.

**Definition 2.9.** Let $\mathcal{C} = \{C_g\}_{g \in S}$ be a family of curves. Let $\mathcal{N} = \{m_i\}_{i=1}^{2r}$ be the slope-set of a Newton polygon with $m_i \in [0, 1] \cap \mathbb{Q}$. We write $NP(\mathcal{C}) \sim \mathcal{N}$ if there exists $\epsilon > 0$ such that

$$NP(C_g) = \bigsqcup_{i=1}^{2r} \left\{m_i\right\}_{i}^{2r+i\epsilon(g)},$$

with $|\epsilon_i(g)| < \epsilon$. Informally, this means that $NP(C_g)$ has slopes $m_1, \ldots, m_{2r}$, each occurring with approximately the same frequency.

**Question 2.10.** Let $\mathcal{N} = \{m_i\}_{i=1}^{2r}$ be as in Definition 2.9. Is there a full or arithmetic family $\mathcal{C}$ with $NP(\mathcal{C}) \sim \mathcal{N}$?

**Remark 2.11.** If one does not require the family of curves to be arithmetic or full, it is much easier to find families with interesting asymptotic properties. For example, in [13] Corollary 2.6] the authors construct an infinite family of supersingular curves. However, this family is much too sparse to be arithmetic.

**Remark 2.12.** The work of [9] proves the existence of many families $\mathcal{C}$ satisfying $NP(\mathcal{C}) \sim \mathcal{N}$ where $\mathcal{N}$ is some interesting slope-set. Most of these families are arithmetic, although a few special cases are full. Here is one particularly interesting example: assume that $p \equiv 4 \mod 5$. They prove the existence of a family $\mathcal{C} = \{C_g\}_{g \in S}$ where $S = \{10n - 4\}_{n \geq 1}$ and

$$NP(\mathcal{C}) \sim \{0\}^3 \sqcup \left\{\frac{1}{2}\right\}^4 \sqcup \{1\}^3.$$

The key technical aspect of their work is an analysis of clutching morphisms for moduli of tame cyclic covers of $\mathbb{P}^1$. Using this analysis, they give an inductive process to construct arithmetic families of curves with prescribed Newton polygon. Combining this with their previous results on special subvarieties of Shimura varieties (see [10]) gives many interesting examples.

### 2.4 Unlikely intersections on $A_g$

Let $X$ be a variety of dimension $d$. Let $V_1$ and $V_2$ be subvarieties of $X$ with codimensions $c_1$ and $c_2$. If $c_1 + c_2 \leq d$, then we expect the intersection $V_1 \cap V_2$ to be nonempty and have dimension $d - c_1 - c_2 \geq 0$. However, if $c_1 + c_2 > d$, then $V_1 \cap V_2$ will typically be empty. We say that $V_1$ and $V_2$ have an *unlikely intersection* if $V_1 \cap V_2$ is nonempty and $c_1 + c_2 > d$. For example, let $A_g^{ss} \subset A_g$ denote the supersingular locus. We know that $\dim(A_g^{ss}) = \left\lfloor \frac{g^2}{2} \right\rfloor$ and $\dim(T_g) = 3g - 3$. Thus, the existence of a supersingular curve of genus $g > 3$ implies $A_g^{ss}$ and $T_g$ have an unlikely intersection. More generally, a high genus curve that is sufficiently far from being ordinary implies an unlikely intersection between the Torelli locus and a Newton stratum. We point the reader to Oort’s article in [5] for more background.
Definition 2.13. Let $C$ be a curve of genus $g$ and let $P = NP(C)$. We say that $C$ has an unlikely Newton polygon if $W^0_p$ and $T_g$ have an unlikely intersection. We say a family of curves $\mathcal{C} = \{C_g\}_{g \in S}$ is unlikely if for $g \gg 0$ the curve $C_g$ has an unlikely Newton polygon.

Lemma 2.14. Let $\mathcal{C} = \{C_g\}_{g \in S}$ be a family of curves and let $P = P(f, 2)$ be a basic graph. If $f(1) > 0$ and $sNP(C) \geq P$, then $\mathcal{C}$ is an unlikely family.

Proof. We may replace $P$ with a slightly lower basic graph so that $sNP(C_g) \geq P$ for large $g$. By lowering $P$ more we may assume that $P$ consists of three line segments with slopes $0, \frac{1}{2},$ and $1$. The codimension of $T_g$ in $A_g$ is $\frac{g(g+1)}{2} - 3g - 3$, so it suffices to show that the codimension of $W^0_p$ in $A_g$ grows quadratically in $g$. Define $R_g = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 0 \leq x \leq g \text{ and } 0 \leq y < gf\left(\frac{x}{g}\right)\}$. By Theorem 2.14 it suffices to show that $\#((\mathbb{Z} \times \mathbb{Z}) \cap R_g)$ grows quadratically in $g$. This follows by observing that the $R_g$ are similar triangles whose side lengths grow linearly in $g$.

Corollary 2.15. Let $\mathcal{C}$ be a family of curves and let $\mathcal{N}$ be a Newton polygon of height $2g$ that is non-ordinary. If $NP(C) \sim \mathcal{N}$, then $\mathcal{C}$ is an unlikely family.

Proof. Let $P(f, 2) = \frac{1}{g}N$. Note that $P(f, 2)$ is a basic graph. The non-ordinary condition implies $f(1) > 0$ and the condition $NP(C) \sim \mathcal{N}$ implies $sNP(C) \geq P(f, 2)$. The corollary follows from Lemma 2.14.

3. $\mathbb{Z}/p\mathbb{Z}$-covers of curves

Let $X$ be a curve of genus $g$ over a finite field $\mathbb{F}_q$ and let $r : C \to X$ be a $\mathbb{Z}/p\mathbb{Z}$-cover. Let $\tau_1, \ldots, \tau_m \in X$ be the points where $r$ is ramified. Let $d_i$ be the Swan conductor of $r$ at $\tau_i$. We may describe $d_i$ as follows. Let $t_i$ be a local parameter at $\tau_i$. Locally the cover $r$ is given by an Artin-Schreier equation $Y^p - Y = g_i$, where $g_i \in \mathbb{F}_q(t_i)$. We may assume that $g_i$ has a pole whose order is coprime to $p$. The order of this pole is equal to the Swan conductor. That is, $g_i = \sum_{n \geq -d_i} a_n t_i^n$ and $a_{-d_i} \neq 0$. Now, let $f$ be a rational function on $X$ and assume that $C$ is given by the equation $Y^p - Y = f$. Then $-\text{ord}_{\tau_i}(f) \geq d_i$ with equality if and only if $\gcd(\text{ord}_{\tau_i}(f), p) = 1$.

Theorem 3.1. We have

$$NP(C) \geq \left\{0, 1\right\}^{pg+m(p-1)} \bigcup \left\{\frac{1}{d_1}, \ldots, \frac{d_1 - 1}{d_1}, \ldots, \frac{1}{d_m}, \ldots, \frac{d_m - 1}{d_m}\right\}^{p-1}.$$  

Proof. This is an earlier theorem of the author. See [3, Corollary 1.3]. The proof uses the Monsky trace formula and some delicate $p$-adic analysis.

Corollary 3.2. Assume $X$ is ordinary and each Swan conductor is equal to 2. Then

$$NP(C) = \left\{0, 1\right\}^{pg+m(p-1)} \bigcup \left\{\frac{1}{2}\right\}^{m(p-1)}.$$  

Proof. From Theorem 3.1 we know $NP(C) \geq \left\{0, 1\right\}^{pg+m(p-1)} \bigcup \left\{\frac{1}{2}\right\}^{m(p-1)}$. By the Deuring-Shafarevich formula (see e.g. [6]), we see that $\left\{0, 1\right\}^{pg+m(p-1)} \bigcup \left\{\frac{1}{2}\right\}^{m(p-1)}$ has the correct number of slope zero segments and slope one segments. As the remaining slopes are $1/2$, the two Newton polygons must be equal.
In general, the bound in Theorem 3.1 will not be attained. However, if \( p \equiv 1 \mod d_i \) for each \( i \), recent work of Booher-Pries shows this bound is optimal.

**Theorem 3.3.** (Booher-Pries) Assume \( X \) is ordinary and let \( d_1, \ldots, d_m \in \mathbb{Z}_{\geq 1} \) such that \( p \equiv 1 \mod d_i \) for each \( i \). There exists a \( \mathbb{Z}/p\mathbb{Z} \)-cover of \( X \), which is ramified at the points \( \tau_1, \ldots, \tau_m \) with Swan conductor \( d_i \), such that

\[
NP(C) = \left\{ \begin{array}{ll} 0, 1 \\ \cup \left\{ \frac{1}{d_1}, \ldots, \frac{d_1-1}{d_1}, \ldots, \frac{d_m-1}{d_m} \right\} \right\}^{p-1}.
\]

**Proof.** See [3, Corollary 4.3]. The main idea is as follows: work of Blache-Ferard computes the Newton polygon of a generic \( \mathbb{Z}/p\mathbb{Z} \)-cover of \( \mathbb{P}^1 \) ramified only at \( \infty \) when the Swan conductor is less than \( 3p \). Booher and Pries use this to construct a \( \mathbb{Z}/p\mathbb{Z} \)-cover of singular curves \( C_0 \to X_0 \) and calculate the Newton polygon of \( C_0 \). Using a formal patching argument, they show that \( C_0 \to X_0 \) deforms to a family of \( \mathbb{Z}/p\mathbb{Z} \)-covers \( C_0 \to X_0 \) that generically gives a cover of \( X \). By Grothendieck’s specialization theorem (see [7]), this gives an upper bound for the Newton polygon of a generic cover in this family. This upper bound is precisely the lower bound of Theorem 3.1 when \( p \equiv 1 \mod d_i \).

**4 Letting the number of branch points tend to infinity**

**Theorem 4.1.** Let \( d \geq 2 \) with \( p \nmid d \). Set \( \delta \) to be 1 if \( d \) is odd and 2 if \( d \) is even. For \( g \geq 1 \) and \( k \) satisfying

\[
\frac{2g}{d+1} - \frac{2p(p-1)}{d+1} \geq k\delta(p-1),
\]

there exists a curve \( C_{g,k} \) of genus \( g \) such that

\[
NP(C_{g,k}) \geq \left\{ \begin{array}{ll} 0, 1 \\ \cup \left\{ \frac{1}{d}, \ldots, \frac{d-1}{d} \right\}^{k\delta(p-1)} \right\}^{g-\frac{1}{2}}.
\]

If \( p \equiv 1 \mod d \), then we may choose \( C_{g,k} \) so that (3) is an equality of Newton polygons.

**Proof.** Write \( g = i + m(p-1) \), where \( 0 \leq i < p-1 \), and define

\[
A = g - ip + (p-1) - \frac{k\delta(p-1)(d+1)}{2}.
\]

By (2) we know \( A \geq 0 \). Also, we see that \( (p-1)|A \). Let \( j \geq 0 \) with \( A = j(p-1) \). Choose an ordinary curve \( X_i \) with genus \( i \). Then choose an Artin-Schreier cover \( C_{g,k} \to X_i \) ramified at \( j + \delta k \) points, such that \( j \) points have Swan conductor 1 and \( \delta k \) points have Swan conductor \( d \). By Riemann-Hurwitz we know \( C_{g,k} \) has genus \( g \). We then apply Theorem 3.1 to obtain the bound (3). If \( p \equiv 1 \mod d \) we can use Theorem 3.3 (or Corollary 3.2 when \( d = 2 \)) to make sure the Newton polygons in (3) are equal.

**Corollary 4.2.** Let \( d \geq 2 \) with \( p \nmid d \). Let \( N \) be the Newton polygon with slopes \( \{0, 1\}^{u} \cup \left\{ \frac{1}{d}, \ldots, \frac{d-1}{d} \right\}^{v} \) where \( u \geq v \) and let \( P = \frac{1}{2u+(d-1)v}N \) be the scaled Newton polygon. There exists a full family of curves \( C \) such that \( sNP(C) \geq P \). In particular, \( C \) is an unlikely family. If \( p \equiv 1 \mod d \), we may choose \( C \) so that \( NP(C) \sim N \).
Proof. The existence of $\mathcal{C}$ follows from Theorem 4.4. From Lemma 2.4 and Corollary 2.15 we know $\mathcal{C}$ is an unlikely family. \hfill $\square$

Corollary 4.3. Let $d \geq 2$ and assume $p \equiv 1 \mod d$. Let $g, g' \geq 1$. Let $C_{g,k}$ and $C_{g',k'}$ be curves as in Theorem 4.4. Then Oort’s conjecture (see Conjecture 1.1) holds for $C_{g,k}$ and $C_{g',k'}$. That is, there exists a curve $C$ of genus $g + g'$ whose Newton polygon is $NP(C_{g,k}) \cup NP(C_{g',k'})$.

Proof. Take $C$ to be $C_{g+g',k+k'}$. \hfill $\square$

Corollary 4.4. Set $d = 2$. Let $g$ and $k$ be as in Theorem 4.4. Define $m = \frac{p-1}{2}$. Let $N_{g,k}$ denote the Newton polygon with slopes $\{0, 1\}^{g-k(p-1)} \cup \{2\}^{2k(p-1)}$. If $km(km-1) > 3g-g$, then $W_{N_{g,k}}$ and $T_g$ have an unlikely intersection.

Proof. By Theorem 4.4 we know $W_{N_{g,k}}$ and $T_g$ have a nonempty intersection. We will compute the codimension of $W_{N_{g,k}}$ in $A_g$ using Theorem 2.4. Let $T_{g,k}$ be the triangle whose vertices are $(g-k(p-1),0)$, $(g,0)$ and $(g, \frac{2k(p-1)}{2})$. We compute $\#(T_{g,k} \cap \mathbb{Z} \times \mathbb{Z}) = (km)^2$. Also, there are $km$ lattice points on the hypotenuse of $T_{g,k}$. The codimension of $W_{N_{g,k}}$ in $A_g$ is thus $km(km-1)$. The codimension of $T_g$ in $A_g$ is $\frac{g(g-1)}{2} - 3g - 3$, which proves the corollary. \hfill $\square$

Example 4.5. Consider the case where $d = 2$ and let $N = \{0, \frac{1}{2}, 1\}$. Corollary 4.2 tells us there exists a full family of curves $\mathcal{C} = \{C_g\}$, where $NP(C_g)$ only has slopes $0, 1$, and $\frac{1}{2}$, and each occurs about a third of the time (with a constant error term). This was previously only known under the assumption $p \equiv 2 \mod 3$ (see [9, Corollary 9.4]). More generally, let $\epsilon \leq \frac{1}{3}$ be a rational number. There exists a full family $\mathcal{C} = \{C_g\}$, such that the Newton polygon of $C_g$ consists only of slopes $0, 1$, and $\frac{1}{2}$ and the multiplicity of $\frac{1}{2}$ is approximately $2g\epsilon$ (with a constant error term).

Example 4.6. Consider the case where $d = 3$ and $p \equiv 1 \mod 3$. Let $N = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Corollary 4.2 gives a full family of curves $\mathcal{C} = \{C_g\}$, where $NP(C_g)$ only has slopes $0, \frac{1}{3}, \frac{2}{3}$, and $1$. Furthermore, each slope occurs with approximately equal frequency. To the best of our knowledge, there were no previous examples of full or arithmetic families of curves with this property. There were, however, examples of arithmetic families of curves where the slopes $\frac{1}{3}, \frac{2}{3}$ occurred with smaller frequencies (see [9, §9.2]).

5 Letting the ramification break tend to infinity

Theorem 5.1. There exists a full family $\mathcal{C} = \{C_g\}$ such that

$$sNP(\mathcal{C}) \geq P\left(\frac{x^2}{4}, 2\right).$$

In particular $\mathcal{C}$ is an unlikely family.

Proof. Let $m = \frac{p-1}{2}$ and for each $i = 0, \ldots, m - 1$, choose $u_i, v_i$ such that

$$pu_i - (p - 1) = i + mv_i.$$

Let $X_i$ be a smooth ordinary curve of genus $u_i$. For $g \gg 0$, write $g = i + km$. We define a genus $g$ curve $C_g$ as follows:
I. If \( d = k - 1 - v_i \) is relatively prime to \( p \), we choose a rational function \( f_g \) on \( X_i \) that has exactly one pole of order \( d \) (this is possible by Riemann-Roch). We let \( C_g \) be the curve defined by the Artin-Schreier equation \( y^p - y = f_g \). By the Riemann-Hurwitz formula we know \( C_g \) has genus \( g \).

II. If \( d = k - 1 - v_i \) is divisible by \( p \), we choose a rational function \( f_g \) that has exactly two poles: one pole of order \( d - 2 \) and one pole of order 2 (again, this is possible by Riemann-Roch). We let \( C_g \) be the curve defined by the Artin-Schreier equation \( y^p - y = f_g \). By the Riemann-Hurwitz formula we know \( C_g \) has genus \( g \).

The theorem then follows from Theorem 3.1. Indeed, by Theorem 3.1 we know that \( NP(C_g) \) is very close to the Newton polygon \( \{ 0, \frac{1}{g}, \ldots, \frac{g-1}{g} \} \). The latter polygon, when scaled by \( \frac{1}{g} \), approaches the graph \( P(\frac{x^2}{4}, 2) \) as \( g \) tends to infinity.

**Question 5.2.** Theorem 5.1 allows for the possibility that \( sNP(C_g) \) lies well above \( P(\frac{x^2}{4}, 2) \). Does there exist a full family of curves \( C = \{ C_g \}_{g \in \mathbb{Z} \geq 0} \) such that \( sNP(C_g) \) converges uniformly to \( P(\frac{x^2}{4}, 2) \)?

**Question 5.3.** Does there exist a full or arithmetic family of curves \( C \) and a basic graph \( P_0 \) that lies strictly above \( P(\frac{x^2}{4}, 2) \) such that \( sNP(C) \geq P_0 \)?

To the best of our knowledge, both questions are unknown even for arithmetic families. The arithmetic families constructed in [9] have scaled Newton polygons whose limits are well below \( P(\frac{x^2}{4}, 2) \). Similarly, the bounds in Theorem 4.2 are well below \( P(\frac{x^2}{4}, 2) \).

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