Gravitational Axial Anomaly for Four Dimensional Conformal Field Theories

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Abstract

We construct the three point function involving an axial vector current and two energy-momentum tensors for four dimensional conformal field theories. Conformal symmetry determines the form of this three point function uniquely up to a constant factor if the necessary conservation conditions are imposed. The gravitational axial anomaly present on a curved space background leads to a non-zero contribution for the divergence of the axial current in this three point function even on flat space. Using techniques related to differential regularisation which guarantee that the energy-momentum tensor is conserved and traceless, we calculate the anomaly in the three point function directly. In this way we relate the overall coefficient of the three point function to the scale of the gravitational axial anomaly. We apply our results to the examples of the fermion and photon axial currents.

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1 Introduction

Conformal symmetry constrains strongly correlation functions in quantum field theory not only in two, but also in higher dimensions. A formalism for constructing conformally covariant three point functions in general dimensions $d$ involving operators of arbitrary spin has been discussed in [1], [2]. This formalism has been applied to three point functions which are determined in terms of a finite number of linearly independent forms. In particular expressions for conserved vector currents and the energy-momentum tensor were found and various Ward identities were analysed.

Even for theories which are conformally invariant on flat space, for a curved space background conformal invariance is broken by anomalies involving the curvature for theories in even dimensions. These anomalies generate a non-zero trace for the energy-momentum tensor. On flat space they lead to anomalous local contributions to the Ward identities reflecting conformal symmetry. In consequence the coefficients of the conformal anomalies are related to the coefficients of the independent forms of the conformal correlation functions. For example in two dimensions the central charge is present as coefficient of both the energy-momentum tensor two and three point functions and the the conformal trace anomaly which in two dimensions is just the curvature scalar. For the four dimensional case, the trace anomaly coefficients $c, a$ have been related to the coefficients of the three independent forms present in general in the energy-momentum tensor three point function in [2].

Here we consider the conformal three point function $\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)A_\omega(z) \rangle$ involving an axial vector current and two energy-momentum tensors in four dimensions. We show that conformal invariance and energy-momentum conservation determine the structure of this three point function uniquely up to an overall coefficient. Moreover we relate this coefficient to the coefficient of the gravitational anomaly contributing to the divergence of the axial vector current on a curved space background. This requires a careful analysis of the short-distance behaviour of the three point function, for which we follow an approach analogous to the one we have used in [2] for the axial anomaly contributing to the conformal three point function $\langle V_{\mu}(x)V_{\nu}(y)A_\omega(z) \rangle$ involving two conserved vector currents in addition to the axial current. The relation between $\langle V_{\mu}(x)V_{\nu}(y)A_\omega(z) \rangle$ and the corresponding axial anomaly was first discussed in [3].

The gravitational axial anomaly is of particular interest for supersymmetric theories since supersymmetry relates the anomalous divergence of the axial $R$ symmetry current to the anomalous trace of the energy-momentum tensor depending on $c, a$. This anomaly has been used to obtain formulae for the change in $c, a$ between fixed points [4] which strongly support the idea of a four dimensional extension of the Zamolodchikov C-theorem [5].

Recently, Pachos and Schiappa have also calculated contributions to the conformal three point function $\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)A_\omega(z) \rangle$ in a second-order perturbative approach for
the abelian Higgs model \cite{6}. These authors claim to find two independent forms in this correlation function. One of their expressions coincides with the result found in this present paper, whereas the second form they discuss does not appear to satisfy the necessary conservation condition for the energy-momentum tensor.

This paper is organised as follows. We begin by constructing the conformal three point function for an axial current and two energy-momentum tensors in four dimensions in section 2. In section 3 we derive the anomalous Ward identity for this three point function for quantum field theories coupled to a curved space background. By analysing the short-distance behaviour of this three point function we relate its coefficient to the coefficient of the gravitational axial anomaly in section 4. In section 5 we check our result by rederiving the well-known result for the axial anomaly coefficient for free fermions within our approach. As a second example we consider the case of the photon axial anomaly in section 6. Section 7 contains some concluding remarks.

2 Conformal Three Point Function

We consider here four dimensional Euclidean space, although the formalism applies in arbitrary dimensions. For three points \(x, y, z\) we may define a vector \(Z_\mu\) at \(z\) by

\[
Z_\mu = \frac{1}{2} \partial_\mu \ln \frac{(z - y)^2}{(z - x)^2} = \frac{(x - z)_\mu}{(x - z)^2} - \frac{(y - z)_\mu}{(y - z)^2}, \quad Z^2 = \frac{(x - y)^2}{(x - z)^2(y - z)^2},
\]

which transforms homogeneously under conformal transformations. \(X_\mu\) and \(Y_\mu\), which are vectors at \(x\) and \(y\), are obtained by cyclic permutation, such that

\[
X_\mu = \frac{(y - x)_\mu}{(y - x)^2} - \frac{(z - x)_\mu}{(z - x)^2}, \quad Y_\sigma = \frac{(z - y)_\sigma}{(z - y)^2} - \frac{(x - y)_\sigma}{(x - y)^2}.
\]

For constructing conformally invariant forms the inversion matrix \(I\), which is defined by

\[
I_{\mu\nu}(x) = \delta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}, \quad \det I = -1,
\]

plays an essential role since for arbitrary conformal transformations, \(I\) acts as a parallel transport. The inversion matrix and the conformal vectors (2.1) satisfy the identities

\[
I_{\mu\alpha}(x - z)Z_\alpha = -\frac{(x - y)^2}{(z - y)^2} X_\mu, \quad I_{\mu\alpha}(x - z)I_{\alpha\sigma}(z - y) = I_{\mu\sigma}(x - y) + 2(x - y)^2 X_\mu Y_\sigma,
\]

\[
I_{\sigma\alpha}(y - z)I_{\alpha\mu}(z - x) = I_{\sigma\nu}(y - x)I_{\nu\mu}(x),
\]

which will prove to be very useful in the subsequent discussion.

For the construction of conformal three point functions in Euclidean space we use the formalism developed in \cite{1, 2}, according to which conformal covariance restricts the three
point function for quasi-primary operators $O^i(x)$ of arbitrary spin to be of the form

$$
\langle O_1^i(x) O_2^j(y) O_3^k(z) \rangle = \frac{1}{(x - z)^{2\eta_1} (y - z)^{2\eta_2}} D_1^{i,j}(I(x - z)) D_2^{j,i}(I(y - z)) t_{i,j,k}^{123,3}(Z). \tag{2.5}
$$

Here $\eta_1, \eta_2, \eta_3$ denote the scale dimensions of the operators $O$. The indices $i, j, k$ denote components for the vector spaces $V_1, V_2, V_3$ to which the fields $O_1, O_2, O_3$ belong and which define representations of the group $O(4)$. $D_{i,j}(I)$ denotes the matrix acting on $V_1$ associated with an inversion. In (2.5) the tensor $t_{i,j,k}^{123,3}(Z)$, belonging to $V_1 \otimes V_2 \otimes V_3$, is a homogeneous function of the conformal vector $Z$ defined in (2.1),

$$
\frac{t_{i,j,k}^{123,3}(\lambda Z)}{I^{\lambda}} = \lambda^{\eta_1 - m - \eta_2} t_{i,j,k}^{123,3}(Z),
$$

$$
D_1^{i,j}(I) D_2^{j,i}(R) D_3^{k,i',k}(R) t_{i,j,k}^{123,3}(Z) = t_{i,j,k}^{123,3}(RZ) \text{ for all } R \in O(d). \tag{2.6}
$$

The general expression (2.5) treats the three operators in an equivalent way. To demonstrate this we note that (2.5) may be written equivalently in the form

$$
\langle O_1^i(x) O_2^j(y) O_3^k(z) \rangle = \frac{1}{(x - y)^{2\eta_2} (x - z)^{2\eta_3}} D_2^{j,i}(I(y - x)) D_3^{k,i}(I(z - x)) t_{i,j,k}^{123,1}(X), \tag{2.7}
$$

where

$$
i_{i,j,k}^{123,1}(X) = (X^2)^{i,j,k} D_2^{j,i}(I(X)) t_{i,j,k}^{123,3}(-X), \tag{2.8}
$$

with $X$ as in (2.2).

We now apply these general results to the three point function involving the axial vector current $A_\omega$ as well as two energy-momentum tensors $T_{\mu\nu}$. On flat space these operators satisfy

$$
\partial_\omega A_\omega = 0, \quad \partial_\mu T_{\mu\nu} = 0, \quad T_{\mu\nu} = T_{\nu\mu}, \quad T_{\mu\mu} = 0, \tag{2.9}
$$

discarding any anomalies depending on the gauge fields which do not contribute to the three point function considered here. In order to be conformal quasi-primary fields, $A_\omega$ and $T_{\mu\nu}$ must have scale dimension 3 and 4 respectively. The general expression (2.5) for the three point function becomes in this special case

$$
\langle T_{\mu\nu}(x) T_{\sigma\rho}(y) A_\omega(z) \rangle = \frac{I^T_{\mu\nu,\sigma\rho}(x - z) I^T_{\sigma\rho,\mu\nu}(y - z)}{(x - z)^8(y - z)^8} t_{\mu\nu,\sigma\rho,\omega}^{TTA}(Z), \tag{2.10}
$$

where $I^T$ is the inversion on $V_T$, the space of traceless symmetric tensors,

$$
I^T_{\mu\nu,\sigma\rho}(x) = E^T_{\mu\nu,\alpha\beta} I_{\alpha\sigma}(x) I_{\beta\rho}(x), \tag{2.11}
$$

$$
E^T_{\mu\nu,\alpha\beta} = \frac{1}{2} (\delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha}) - \frac{1}{4} \delta_{\mu\nu} \delta_{\alpha\beta}. \tag{2.12}
$$
with $\mathcal{E}_{\mu,\nu,\alpha,\beta}^T$ the projection operator onto $V_T$. Bose symmetry imposes the condition
\begin{equation}
t^{TTA}_{\mu\nu\rho\omega}(Z) = t^{TTA}_{\sigma\mu\nu\omega}(-Z)
\end{equation}
on (2.13). The most general solution of (2.13), which has also the correct dimension and reflects the symmetry and tracelessness properties of the energy-momentum tensor and takes account of the odd parity properties of the axial current, is given by
\begin{equation}
t^{TTA}_{\mu\nu\sigma\rho\omega}(Z) = \frac{1}{Z^6}(A \mathcal{E}_{\mu,\nu,\eta,\xi}^T \mathcal{E}_{\sigma,\rho,\kappa,\varepsilon}^T \varepsilon_{\eta\kappa\omega\lambda} Z_\lambda \\
+ B \mathcal{E}_{\mu,\nu,\eta,\gamma}^T \mathcal{E}_{\sigma,\rho,\kappa,\delta}^T \varepsilon_{\eta\kappa\omega\lambda} Z_\gamma Z_\delta Z_\lambda Z^{-2}),
\end{equation}
where $A$, $B$ are independent coefficients. This satisfies
\begin{equation}
I^{TTA}_{\mu,\nu,\mu',\nu'}(Z) I^{TTA}_{\sigma,\rho,\sigma',\rho'}(Z) I^{TTA}_{\omega,\omega'}(Z) = I^{TTA}_{\mu',\nu',\mu,\nu'}(-Z),
\end{equation}
as a consequence of
\begin{equation}
I_{\mu\nu'} I_{\sigma\rho'} I_{\alpha\beta'} I_{\beta\alpha'} = \text{det} I \varepsilon_{\mu\nu\alpha\beta} = -\varepsilon_{\mu\nu\alpha\beta}.
\end{equation}

Imposing conservation of the energy-momentum tensor,
\begin{equation}
\partial^\mu \langle T^\mu_{\nu}(x) T_{\sigma\rho}(y) A_\omega(z) \rangle = 0, \quad x \neq y, z,
\end{equation}
implies the condition
\begin{equation}
B = -6A
\end{equation}
on the parameters in (2.14). Thus the three point function is entirely determined by conformal invariance and by energy-momentum conservation up to one overall constant, such that there is only one independent form\footnote{In a recent paper by Pachos and Schiappa \cite{ref6} it was argued within a perturbative study of the abelian Higgs model that there are two independent forms in the three point function (2.10). However one of the two forms proposed by these authors coincides with the form obtained here, whereas the second does not satisfy the conservation condition (2.17), (2.18) for the energy-momentum tensor.}. Moreover the three point function (2.10) automatically satisfies conservation with respect to the axial vector leg for non-coincident points.

3 Ward Identities

For massless quantum field theories on a curved space background in four dimensions the axial vector current may have an anomalous divergence given by \cite{ref4,ref1,ref2}
\begin{equation}
\nabla_\omega \langle A_\omega(z) \rangle = \frac{1}{2} c_A \varepsilon_{\mu\nu\rho} R^{\mu\nu\alpha\beta} R^\rho_{\alpha\beta}.
\end{equation}
Here $R_{\mu \nu \sigma \rho}$ is the curvature tensor for the background metric $g^{\mu \nu}$, and $c_A$ is a model-dependent coefficient.

Furthermore on curved space the energy-momentum tensor may be defined by

$$\langle T_{\mu \nu}(x) \rangle = - \frac{2}{\sqrt{g}} \frac{\delta W}{\delta g^{\mu \nu}(x)},$$  \hspace{1cm} (3.2)

with $W$ the generating functional for connected Green functions. In consequence we have

$$\langle T_{\mu \nu}(x) T_{\sigma \rho}(y) A_{\omega}(z) \rangle = \frac{4}{\sqrt{g(x)} \sqrt{g(y)} \sqrt{g(z)}} \frac{\delta}{\delta g^{\mu \nu}(x)} \frac{\delta}{\delta g^{\sigma \rho}(y)} \left( \sqrt{g(z)} \langle A_{\omega}(z) \rangle \right).$$  \hspace{1cm} (3.3)

Also the usual diffeomorphism invariance can here be expressed as

$$\nabla^\omega \langle T_{\mu \nu}(x) A_{\omega}(z) \rangle = \nabla_\nu \left( \delta^{(4)}(x, z) g_{\mu \omega} \right) \langle A^\mu(z) \rangle - \nabla_\mu \left( \delta^{(4)}(x, z) g_{\nu \omega} \langle A^\mu(z) \rangle \right).$$  \hspace{1cm} (3.4)

Using (3.3) the anomalous divergence (3.1) leads to the anomalous Ward identity on flat space when $g_{\mu \nu} = \delta_{\mu \nu}$ and we may then identify up and down indices.

$$\frac{\partial}{\partial z} \langle T_{\mu \nu}(x) T_{\sigma \rho}(y) A_{\omega}(z) \rangle = 4 c_A E^{\mu \nu, \eta \epsilon} E^{\sigma \rho, \kappa \epsilon} \varepsilon_{\eta \kappa \beta \alpha} \partial_\alpha \partial_\lambda \delta^{(4)}(x - z) \partial_\beta \partial_\lambda \delta^{(4)}(y - z)$$  \hspace{1cm} (3.5)

for the three point function on flat space. Furthermore (3.4) has no anomaly and (2.17) must hold even for $x = y, z$.

In our discussion it is more convenient to replace the Riemann tensor by the Weyl tensor defined by

$$C_{\mu \nu \sigma \rho} = R_{\mu \nu \sigma \rho} - (g_{\mu [\sigma} R_{\rho] \nu} - g_{\nu [\sigma} R_{\rho] \mu}) + \frac{1}{3} g_{\mu [\sigma} g_{\rho] \nu} R. \hspace{1cm} (3.6)$$

The Weyl tensor belongs to the space $V^C = \{ C_{\mu \nu \sigma \rho} \}$ of tensors defined by

$$C_{\mu \nu \sigma \rho} = C_{[\mu \nu] [\sigma \rho]}, \hspace{1cm} C_{\mu [\nu [\sigma \rho]} = 0, \hspace{1cm} g^{\mu \nu} C_{\mu \nu \sigma \rho} = 0. \hspace{1cm} (3.7)$$

It is important to recognise that

$$C_{\mu \nu \sigma \rho} \in V^C \Rightarrow C^*_{\mu \nu \sigma \rho} \equiv \frac{1}{2} \varepsilon_{\mu \nu \alpha \beta} C^{\alpha \beta}_{\sigma \rho} \in V^C. \hspace{1cm} (3.8)$$

We may rewrite (3.1) as

$$\nabla_\omega \langle A^\omega(z) \rangle = c_A C^{\mu \nu \sigma \rho} C^*_{\mu \nu \sigma \rho}. \hspace{1cm} (3.9)$$

If we define a projection operator $E^C$ onto tensors in $V^C$ then to first order in an expansion about flat space, $g_{\alpha \beta} = \delta_{\alpha \beta} + h_{\alpha \beta}$, the the Weyl tensor is simply expressed as

$$C_{\mu \nu \sigma \rho} = 2 E^C_{\mu \nu \sigma \rho, \alpha \beta} \partial_\alpha \partial_\beta h_{\alpha \beta}. \hspace{1cm} (3.10)$$

An explicit expression for $E^C$ may be found in [2]. With (3.10) the flat space Ward identity (3.5) may be rewritten equivalently as

$$\partial_\omega \langle T_{\mu \nu}(x) T_{\sigma \rho}(y) A_{\omega}(z) \rangle = 16 c_A E^C_{\mu \rho \sigma \nu, \alpha \beta \gamma \delta} \varepsilon_{\delta \beta \gamma \epsilon} \partial_\alpha \partial_\beta \delta^{(4)}(x - z) \partial_\gamma \partial_\delta \delta^{(4)}(y - z). \hspace{1cm} (3.11)$$
4 Anomaly Calculation

Given the explicit form for the three point function provided by (2.10), (2.14) and (2.18) it is natural to endeavour to demonstrate the explicit appearance of the anomaly as in (3.11) and obtain $c_A$ explicitly in terms of $A$. However a direct calculation does not give a unique answer without further analysis due to the very singular behaviour of the three point function for $x \sim y \sim z$, due to which for instance its Fourier transform is ill-defined. This singular behaviour may lead to anomalies in the conservation equations for the energy-momentum tensor in general. Therefore the axial anomaly is only unambiguous when the energy-momentum tensor conservation equations are imposed even including contributions with support at $x = y = z$. To achieve this we make use of the result that if, on flat space, 

$$T_{\mu\nu} = \partial_\sigma \partial_\rho C_{\mu\sigma\rho\nu}, \quad C_{\mu\sigma\rho\nu} \in V^C,$$

then $T_{\mu\nu}$ is automatically conserved and traceless. Thus we write

$$\langle T_{\mu\nu}(x)T_{\alpha\beta}(y)A_\omega(z) \rangle = \partial^x_\sigma \partial^y_\rho \partial^y_\gamma \partial^y_\delta \Gamma^{CCA}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta,\omega}(x,y,z),$$

where $\Gamma^{CCA}(x,y,z)$ is a tensor belonging to $V^C \otimes V^C \otimes V^C$, with $V$ the space of vector fields, $C_{\mu\sigma\rho\nu}$ is constructed according to the general prescription for conformal three point functions regarding $C_{\mu\sigma\rho\nu}$ as a quasi-primary operator of dimension two. Writing the three point function in the form (4.2) amounts to a form of differential regularisation [10]. A similar, albeit simpler, calculation of the axial anomaly for a theory coupled to vector fields, maintaining the vector current conservation equations was given in [2]. For symmetry we must require in (4.3)

$$\Gamma^{CCA}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta,\omega}(x,y,z) = \Gamma^{CCA}_{\alpha\gamma\delta\beta,\mu\sigma\rho\nu,\omega}(y,x,z),$$

and also for conservation of the axial current

$$\partial^z_\omega \Gamma^{CCA}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta,\omega}(x,y,z) = 0, \quad z \neq x, y,$$

A form for $\Gamma^{CCA}$ with the desired properties which has overall odd parity is given by

$$\Gamma^{CCA}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta,\omega}(x,y,z) = \frac{1}{8} A \frac{T^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta,\omega}(x,y)}{(x-y)^4} \varepsilon_{\delta\beta} Z_2 Z_\omega,$$

where

$$T^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x) = \mathcal{E}^C_{\mu\sigma\rho\nu,\alpha'\gamma'\delta'\beta'} I_{\alpha\alpha'}(x) I_{\gamma\gamma'}(x) I_{\delta\delta'}(x) I_{\beta\beta'}(x),$$

is the representation of inversions on $V^C$. The symmetry constraint (1.3) then follows from $\det I = -1$ once more. The conformal three point function (1.3) may also be written in a form in which its consistency with the general formalism (2.3) is obvious,

$$\Gamma^{CCA}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta,\omega}(x,y,z) = \frac{T^C_{\mu\sigma\rho\nu,\alpha'\gamma'\delta'\beta'}(x-z)T^C_{\alpha'\gamma'\delta'\beta',\alpha\gamma\delta\beta}(y-z)}{(x-z)^4(y-z)^4} \epsilon^{CCA}_{\alpha'\gamma'\delta'\beta',\omega,\alpha'\gamma'\delta'\beta'}(Z),$$

where

$$\epsilon^{CCA}_{\alpha'\gamma'\delta'\beta',\omega,\alpha'\gamma'\delta'\beta'}(Z) = \frac{1}{8} A \frac{T^C_{\mu'\sigma'\rho'\nu',\alpha'\gamma'\delta'\beta'}(y-z)}{(x-z)^4(y-z)^4} \epsilon^{CCA}_{\mu'\sigma'\rho'\nu',\alpha'\gamma'\delta'\beta',\omega}(Z),$$
with
\[ t^{CCA}_{\mu \sigma \rho \nu, \alpha \gamma \delta \beta \omega} (Z) = - \frac{1}{8} A \frac{Z_{\omega}}{Z^2} E^{C}_{\mu \sigma \rho \nu, \mu' \rho' \nu'} E^{C}_{\alpha \gamma \delta \beta, \alpha' \gamma' \delta' \beta'} \varepsilon^{\delta' \beta' \kappa \lambda} I_{\mu' \alpha'}(Z) I_{\sigma' \gamma'}(Z) I_{\rho' \kappa}(Z) I_{\nu' \lambda}(Z), \]
which may be obtained from (4.5) by repeated use of (2.4) and (2.16). From the results of [2]
\[ \Gamma^{TTA}_{\mu \nu \sigma \rho \omega} (Z) = \partial^Z_{\sigma} \partial^Z_{\rho} \partial^Z_{\gamma} \partial^Z_{\delta} t^{CCA}_{\mu \sigma \rho \nu, \alpha \gamma \delta \beta \omega} (Z). \]
We have checked that inserting (4.8) into (4.9) and calculating the four derivatives using the algebraic computing program FORM [11], we obtain (2.10) with (2.18) as expected.

The expression in (4.5) for \( \Gamma^{CCA}(x, y, z) \) is homogeneous of degree \(-8\) in \( x, y, z \). Any ambiguity is proportional to \( \delta^{(4)}(x - z) \delta^{(4)}(y - z) \), without derivatives, but the presence of such a term is precluded by the tensorial structure. Thus the divergence of the axial vector current in (4.4) may be unambiguously calculated even including contributions with support at \( z = x, y \). For this purpose we generalise results for distributions derived in [2], which rely on the general discussion of [12]. First we use that from (2.1) we have
\[ \partial^x_{\sigma} \partial^x_{\rho} \frac{\mathcal{I}_{\mu \sigma \rho \nu, \alpha \gamma \delta \beta}(x)}{(x^2)^2} = \frac{\pi^2}{6} \mathcal{E}_{\mu \sigma \rho \nu, \alpha \gamma \delta \beta} \partial^x_{\sigma} \partial^x_{\rho} \delta^{(4)}(x), \]
in four dimensions. To obtain this we make use of the results for the singular behaviour as \( \lambda \to 0 \)
\[ \partial^x_{\sigma} \partial^x_{\rho} \frac{\mathcal{I}_{\mu \sigma \rho \nu, \alpha \gamma \delta \beta}(x)}{(x^2)^{(4-\lambda)/2}} = \frac{\lambda(\lambda + 2)}{(4-\lambda)(6-\lambda)} \mathcal{E}_{\mu \sigma \rho \nu, \alpha \gamma \delta \beta} \partial^x_{\sigma} \partial^x_{\rho} \frac{1}{(x^2)^{(4-\lambda)/2}} \]
\[ \sim \frac{\pi^2}{6} \mathcal{E}_{\mu \sigma \rho \nu, \alpha \gamma \delta \beta} \partial^x_{\sigma} \partial^x_{\rho} \delta^{(4)}(x), \]
which depends on
\[ \frac{1}{(x^2)^{(4-\lambda)/2}} \sim \frac{2\pi^2}{\lambda} \delta^{(4)}(x). \]
Furthermore in the same fashion for the singular term as \( \lambda \to 0 \)
\[ \frac{\mathcal{I}_{\mu \sigma \rho \nu, \alpha \gamma \delta \beta}(x)}{(x^2)^{(4-\lambda)/2}} \sim 0, \]
so that the left hand side is well defined as a distribution at \( \lambda = 0 \) and (4.11) is unambiguous. Thus we obtain
\[ \partial^z_{\omega} \langle T_{\mu \nu}(x) T_{\alpha \beta}(y) A_{\omega}(z) \rangle = \frac{1}{12} \pi^4 A \mathcal{E}_{\mu \sigma \rho \nu, \alpha \gamma \delta \beta} \varepsilon^{\delta \beta \gamma \delta'} \partial_{\sigma} \partial_{\rho} \delta^{(4)}(x - z) \partial_{\gamma} \partial_{\delta} \delta^{(4)}(y - z). \]
Comparison with (3.11) yields
\[ c_A = \frac{1}{192\pi^2} A, \] (4.16)
which relates the anomaly coefficient to the coefficient of the conformal three point function (2.10).

5 Free Fermions

In order to check the result (4.16), we use it to rederive the well-known result for \( c_A \) in the case of free fermions, for which
\[ T_{\mu\nu} = \bar{\psi} \gamma_\mu (\begin{pmatrix} \frac{x}{2} \\ \sigma \end{pmatrix} \partial_\nu) \psi, \quad \frac{\partial_\nu}{\partial_\mu} = \frac{1}{2} (\partial_\nu - \frac{\partial_\mu}{\partial_\nu}), \quad A_\omega = \bar{\psi} \gamma_\omega \gamma_5 \psi, \] (5.1)
with conventions so that \( \text{tr} (\gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_5) = 4 \varepsilon_{\alpha\beta\gamma\delta} \). The basic two point function is
\[ \langle \bar{\psi}(x) \psi(0) \rangle = \frac{1}{2\pi^2} \frac{\gamma \cdot x}{x^4} \] (5.2)
in four dimensions. This yields
\[ \langle T_{\mu\nu}(x) T_{\alpha\beta}(y) A_\omega(z) \rangle = -\frac{1}{(2\pi)^3} \text{tr} \left[ \gamma_\omega \gamma_5 \gamma_\mu (\begin{pmatrix} \frac{x}{2} \\ \sigma \end{pmatrix} \partial_\nu) \frac{\gamma \cdot (x-y)}{x^4} \frac{\gamma \cdot (y-z)}{y^4} \gamma_\sigma \partial_\rho \gamma_\gamma \gamma_5 \gamma_\rho \right] \] (5.3)
This expression is most easily evaluated in a frame in which \( x, y, z \) are constrained to lie on a straight line. The result is of the form (2.10) with (2.14) and satisfies the conservation condition (2.18). Comparison of (5.3) with (2.10) yields
\[ A = \frac{1}{\pi^6} \] (5.4)
for free fermions, which according to (4.16) corresponds to
\[ c_A = \frac{1}{192\pi^2} \] (5.5)
for the fermion triangle anomaly. This agrees with the old result of Eguchi and Freund \[ [7] \], which provides a check on our calculation.
6 Photons

A second field theory example in which there is an anomaly in the conservation of an axial current for curved space backgrounds concerns the Chern-Simons current in abelian gauge theories as described in [13]. The relevant current in terms of the gauge field is

\[ K_\omega = \varepsilon_{\omega\alpha\beta\gamma} a_\alpha F_{\beta\gamma}, \quad F_{\beta\gamma} = \partial_\beta a_\gamma - \partial_\gamma a_\beta. \]  

(6.1)

Formally this satisfies

\[ \partial_\omega K_\omega = F_{\beta\gamma} F_{\beta\gamma}^*, \]  

(6.2)

but there are potential anomalies arising from the photon triangle diagram. The current \( K_\omega \), defined in (6.1) so as to satisfy (6.2), may be regarded as arbitrary up to

\[ K_\omega \rightarrow K_\omega + \partial_\alpha W_{\omega\alpha}, \quad W_{\omega\alpha} = -W_{\alpha\omega}. \]  

(6.3)

This allows for the freedom of gauge transformations since if \( a_\mu \rightarrow a_\mu + \partial_\mu \Phi \) the current transforms as in (6.3) with \( W_{\omega\alpha} = \varepsilon_{\omega\alpha\beta\gamma} \Phi F_{\beta\gamma} \).

For the gauge field the gauge invariant energy-momentum tensor is given by

\[ T^\mu_{\nu\mu} = F_{\mu\lambda} F_{\nu\lambda} - \frac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} = \mathcal{E}^T_{\mu\nu,\mu'} F_{\mu'\lambda} F_{\nu'\lambda}, \]  

(6.4)

which is of course traceless and conserved subject to \( \partial_\mu F_{\mu\nu} = 0 \). In general with gauge fixing this need not be conserved even for free fields but in a BRS formalism the full canonical conserved energy momentum tensor, which need not be traceless, may be written as

\[ T^\text{can}_{\mu\nu} = T^\mu_{\nu\mu} + sX_{\mu\nu}, \]  

(6.5)

where \( s^2 = 0 \). BRS invariance for a general three point function may be expressed as

\[ \langle sA(x) B(y) C(z) \rangle + (-1)^{\varepsilon_A}\langle A(x) sB(y) C(z) \rangle + (-1)^{\varepsilon_A+\varepsilon_B}\langle A(x) B(y) sC(z) \rangle = 0, \]  

(6.6)

where \( \varepsilon_A, \varepsilon_B \) are the BRS grading of the operators \( A, B \). For BRS invariant operators \( sO_i = 0 \) this shows that the three point function is invariant under the freedom \( O_i \rightarrow O_i + sX_i \). In the following we endeavour to analyse the odd parity three point function formed by two energy momentum tensors and the axial current \( K_\omega \) to see whether the anomalous divergence suggested in [13] can be derived in a similar fashion to the previous discussion for the axial fermion current. The current is not BRS closed but in general we may write

\[ sK_\omega = \partial_\alpha W_{\omega\alpha}. \]  

(6.7)
tensor conservation equations, so long as we allow for the freedom represented by (6.3) to obtain an expression satisfying these conditions. It is important to confirm whether this three point function is BRS non trivial and hence cannot be transformed to zero.

In consequence it is easy to see using (6.6) that we must have
\[
\langle T^\text{can}_{\mu\nu}(x)T^\text{can}_{\sigma\rho}(y)K_\omega(z) \rangle = \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)K_\omega(z) \rangle + \partial^\alpha_\omega W_{\mu\nu,\sigma\rho,\omega}(x, y, z),
\]
with \(W_{\mu\nu,\sigma\rho,\omega}(x, y, z) = W_{(\mu\nu), (\sigma\rho), \omega}(x, y, z) = W_{\sigma\rho,\mu\nu,\omega}(y, x, z)\). It is therefore sufficient to consider just \(\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)K_\omega(z) \rangle\), although this need not satisfy the energy momentum tensor conservation equations, so long as we allow for the freedom represented by (6.3) to obtain an expression satisfying these conditions. It is important to confirm whether this three point function is BRST non trivial and hence cannot be transformed to zero.

In a standard covariant gauge the basic two point function for the gauge field is given by
\[
\langle a_\mu(x)a_\sigma(y) \rangle_1 = \frac{1}{8\pi^2} \left( (1 + \alpha) \frac{\delta_{\mu\nu}}{(x-y)^2} + 2(1 - \alpha) \frac{(x-y)_\mu(x-y)_\nu}{(x-y)^4} \right),
\]
with \(\alpha\) a gauge parameter, is not conformally covariant. However the two point function for the gauge invariant field strength,
\[
\langle F_{\mu\nu}(x)F_{\sigma\rho}(y) \rangle = \frac{2}{\pi^2} \frac{\mathcal{I}^F_{\mu\nu,\sigma\rho}(x-y)}{(x-y)^4},
\]
with
\[
\mathcal{I}^F_{\mu\nu,\sigma\rho}(x-y) = \frac{1}{2}(I_{\mu\sigma}(x-y)I_{\nu\rho}(x-y) - I_{\mu\rho}(x-y)I_{\nu\sigma}(x-y))
\]
the inversion on \(V_F\), the space of antisymmetric tensor fields, is manifestly conformally covariant as expected since \(F_{\mu\nu}\) is quasi-primary. Under a general gauge transformation we expect
\[
\langle a_\mu(x)a_\sigma(y) \rangle \to \langle a_\mu(x)a_\sigma(y) \rangle + \partial^\mu_\sigma \Lambda(x, y) + \bar{\Lambda}_\mu(x, y) \frac{\partial^\mu_\sigma}{x, y, z, \omega, \mu, \nu, \sigma, \rho, \omega, \alpha}.
\]
If we choose in this
\[
\Lambda(x, y) = -\frac{1}{4\pi^2} \left( \ln(x-y)^2 - \frac{1}{2} \ln(x-z)^2 \right) \frac{(z-y)_\sigma}{(z-y)^2},
\]
\[
\bar{\Lambda}_\mu(x, y) = -\frac{1}{4\pi^2} \left( \frac{(z-x)_\mu}{(z-x)^2} \right) \left( \ln(x-y)^2 - \frac{1}{2} \ln(y-z)^2 \right),
\]
we get, with \(X, Y\) as in (2.3),
\[
\langle a_\mu(x)a_\sigma(y) \rangle_2 = \frac{1}{8\pi^2} \left( (1 + \alpha) \frac{I_{\mu\sigma}(x-y)}{(x-y)^2} + 4X_\mu Y_\sigma \right),
\]
where the point \(z\) here plays the role of a gauge parameter. This reduces to (6.3) for \(z = \infty\). Since \(I_{\mu\sigma}(x-y)/(x-y)^2 = -\frac{1}{2} \partial^\mu_\sigma \ln(x-y)^2 \partial^\nu_\sigma\) the first term, depending on \(\alpha\), is a pure gauge. The expression in (6.13) is conformally covariant if conformal transformations are extended to the three points \(x, y, z\). Similar non-local gauge transformations and
essentially the same result for the photon propagator were described in [14]. Using the result (6.13) we then have

\[ \langle F_{\mu\nu}(x)a_\sigma(y) \rangle_2 = \frac{1}{\pi^2} \frac{X_{|\mu} I_{\nu|\sigma}(x-y)}{(x-y)^2}, \quad (6.14) \]

which satisfies

\[ \partial_\mu \langle F_{\mu\nu}(x)a_\sigma(y) \rangle_2 = \frac{1}{2\pi^2} X^2 \left( \frac{I_{\nu\sigma}(x-y)}{(x-y)^2} - 2X_\nu Y_\sigma \right) = \frac{1}{2\pi^2} \left( X^2 X_\nu \right) \delta_{\mu\nu}. \quad (6.15) \]

With these results, in any gauge, we have

\[ \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)F_{\beta\gamma}F^{\rho\beta\gamma}(z) \rangle = 0. \quad (6.16) \]

As a consequence \( \partial^\omega \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)K_\omega(z) \rangle = 0 \) for \( z \neq x, y \). If we use the covariant form (6.9) to compute the three point function of the current \( K_\omega \) with two energy momentum tensors the photon triangle diagrams give

\[ \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)K_\omega(z) \rangle_1 = -\frac{16}{\pi^6} \varepsilon_{\omega\beta\gamma} \varepsilon_{\mu\alpha\nu\kappa} \varepsilon_{\sigma\rho\alpha\sigma'} \delta_{\alpha\beta'\lambda}(z-x)_\lambda \frac{T^{\sigma\rho\mu\nu}(x-y)}{(x-y)^4} \frac{T^{\nu\kappa}(y-z)}{(y-z)^4} + \{(x, \mu, \nu) \leftrightarrow (y, \sigma, \rho)\}, \quad (6.17) \]

which is not conformally invariant. However by using the conformal results (6.10) and (6.14) we obtain the gauge equivalent expression

\[ \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)K_\omega(z) \rangle_2 = \frac{8}{\pi^6} \frac{T^{\sigma\rho\mu\nu}(x-z)T^{\sigma\rho\alpha\sigma'}(y-z)}{(x-z)^8(y-z)^8} \varepsilon_{\mu\alpha\kappa} \varepsilon_{\sigma\rho\sigma'} Z_\lambda \frac{Z_{\nu\rho}}{Z^2} \delta_{\lambda\alpha} - 2Z_{\nu\rho}Z_{\nu\rho} \], \quad (6.18) 

which is of the form (2.10) required for conformal invariance. However neither (6.17) nor (6.18) satisfy the requirements arising from \( \partial^\sigma T_{\mu\nu}(x) = 0, \partial^\mu T_{\sigma\rho}(y) = 0 \), as shown by the disagreement of (6.18) with (2.14) and (2.18). This is a consequence of the non-zero result in (6.13). Nevertheless an expression in which energy-momentum is conserved may be obtained by performing a further transformation of the current \( K_\omega \) of the form shown in (6.3), so that

\[ \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)K_\omega(z) \rangle_3 = \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)K_\omega(z) \rangle_2 + \partial^\omega W_{\mu\nu,\sigma\rho,\alpha\omega}(x, y, z). \quad (6.19) \]

\(^3\text{In the covariant gauge, with the photon two point function given by (6.9), the equation of motion becomes } \partial_\mu F_{\mu\nu} + \partial_\nu b = 0 \text{ where } b = \partial a/\alpha. \text{ The associated conserved energy momentum tensor is then given by (6.7) } X_{\mu\nu} = -2\partial_\lambda c_{\alpha\nu} + \partial_\nu (\partial \varepsilon a + \frac{1}{2} \alpha \bar{\partial} b), \text{ which is not traceless. The BRS action in this abelian theory is } sa_\mu = \partial_\mu c, \text{ sc = 0, sc = b, sb = 0. The additional gauge dependent terms in the energy momentum tensor contribute in the three point function involving the gauge dependent current } K_\omega.\)
Assuming the conformally covariant form
\[ W_{\mu\nu,\sigma,\rho,\omega}(x, y, z) = \frac{\mathcal{I}^{T}_{\mu\nu,\mu'(x - z)} \mathcal{I}^{T}_{\sigma,\rho',\omega}(y - z)}{(x - z)^8(y - z)^8} t^{TTW}_{\mu',\nu',\sigma',\rho',\omega}(Z), \] (6.20)

with
\[ t^{TTW}_{\mu\nu,\sigma,\rho,\omega}(Z) = C \frac{1}{Z_{10}^4} \epsilon^{T}_{\mu\nu,\nu'} \epsilon^{T}_{\sigma,\rho',\omega} \left( \varepsilon_{\omega_{\alpha}\beta_{\mu'}} Z_{\beta} Z_{\rho'} Z_{\omega'} + \varepsilon_{\omega_{\alpha}\beta_{\sigma'}} Z_{\beta} Z_{\rho'} Z_{\omega'} \right), \] (6.21)

which is the unique form satisfying the necessary symmetry condition
\[ t^{TTW}_{\mu\nu,\sigma,\rho,\omega}(Z) = t^{TTW}_{\sigma,\rho,\mu,\omega}(Z), \] (6.22)
gives
\[ \partial_{\alpha} W_{\mu\nu,\sigma,\rho,\omega}(x, y, z) = 4 C \frac{\mathcal{I}^{T}_{\mu\nu,\nu}(x - z) \mathcal{I}^{T}_{\sigma,\rho',\omega}(y - z)}{(x - z)^8(y - z)^8} \varepsilon_{\mu',\omega',\lambda} \frac{Z_{\lambda} Z_{\nu'} Z_{\rho'}}{Z^8}. \] (6.23)

Hence choosing \( C = -8/\pi^6 \) leads in (6.19) to
\[ \langle T_{\mu\nu}(x) T_{\nu}(y) K_{\omega}(z) \rangle_3 = \frac{8}{\pi^6} \frac{\mathcal{I}^{T}_{\mu\nu,\nu}(x - z) \mathcal{I}^{T}_{\sigma,\rho',\omega}(y - z)}{(x - z)^8(y - z)^8} \varepsilon_{\omega_{\mu'}\rho',\lambda} \frac{Z_{\lambda} Z_{\nu'} Z_{\rho'}}{Z^8} \left( \delta_{\nu'} - 6 Z_{\nu'} Z_{\rho'} Z_{\omega'} \right), \] (6.24)

which agrees with (2.10) and with (2.14) and so satisfies the energy-momentum conservation condition (2.18). By comparing the coefficients we read off that
\[ A = \frac{8}{\pi^6}, \] (6.25)

for photons. Since (6.24) is conserved, the anomaly discussion of section 4 above applies to this example as well, such that from (4.10) the anomaly coefficient in (3.1) is
\[ c_A = \frac{1}{24\pi^2}, \] (6.26)

for the photon triangle anomaly. This result agrees with the result of [13], according to which the Chern-Simons current (with the scale set by (6.1)) has an anomaly
\[ \nabla_{\omega} \langle K_{\omega} \rangle = \frac{1}{48\pi^2} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}, \] (6.27)
on curved space, up to a factor of 2 for the numerical value of the coefficient.\footnote{It is difficult to discern the origin for this numerical discrepancy, since the curved space calculation of [13] is very different from the flat space triangle diagram calculation performed here.}

Thus this second example confirms that there is a unique conserved form for the conformal three point function, which is uniquely related to the axial anomaly present on curved space. It should be noted that since the \( \partial_{\omega} \) derivative of (6.23) vanishes identically, the transformations performed do not alter the anomaly.
7 Conclusion

We have constructed the three point function involving the axial vector current as well as two energy-momentum tensors for four dimensional conformal field theories and shown that this three point function is determined by conformal invariance up to an overall factor. By analysing the short-distance behaviour we have related this factor to the coefficient of the gravitational axial anomaly.

Our result is of particular interest for supersymmetric theories, for which it applies to the three point function \( \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)R_\omega(z) \rangle \) involving the axial \( R \) symmetry current. In this case there is a gravitational axial anomaly whose coefficient satisfies

\[
c_A \propto c - a ,
\]

with \( c, a \) the coefficients of the Weyl and Euler densities in the curved space trace anomaly respectively.

A direct consequence of our result (4.16) is that for theories for which the gravitational axial anomaly is absent, \( c_A = 0 \), the three point function \( \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)A_\omega(z) \rangle \) vanishes altogether, even without taking the divergence with respect to the axial vector leg. This applies in particular to the three point function \( \langle T_{\mu\nu}(x)T_{\sigma\rho}(y)R_\omega(z) \rangle \) for specific supersymmetric theories for which \( c = a \). For such theories a similar result was found in [15] by considering the operator product expansion. Examples for theories with \( c = a \) have been considered recently in the context of the AdS/CFT correspondence, for instance in the references listed in [16].

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