THE QUOTIENT SHAPES OF NORMED SPACES AND APPLICATION

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Abstract. The quotient shape types of normed vectorial spaces (over the same field) with respect to Banach spaces reduce to those of Banach spaces. The finite quotient shape type of normed spaces is an invariant of the (algebraic) dimension, but not conversely. The converse holds for separable normed spaces as well as for the bidual-like spaces (isomorphic to their second dual spaces). As a consequence, the Hilbert space $l_2$, or even its (countably dimensional, unitary) direct sum subspace may represent the unique quotient shape type of all $2^{\aleph_0}$-dimensional normed spaces. An application yields two extension type theorems.

1. Introduction

The shape theory began as a generalization of the homotopy theory such that the locally bad spaces can be also considered and classified in a very suitable “homotopical” way. The first step (for compacta in the Hilbert cube) had made K. Borsuk, [1]. The theory was rapidly developed and generalized by many authors. The main references are [2], [3], [5] and, especially, [10]. Although, in general, founded purely categorically, a shape theory is mostly well known only as the (standard) shape theory of topological spaces with respect to spaces having the homotopy types of polyhedra. The generalizations founded in [7] and [16] are, primarily, also on that line.

The quotient shape theory was introduced a few years ago by the author, [12]. It is, of course, a kind of the general (abstract) shape theory, [10], I. 2. However, it is possible and non-trivial, and can be...
straightforwardly developed for every concrete category \( \mathcal{C} \) and for every infinite cardinal \( \kappa \geq \aleph_0 \). Concerning a shape of objects, in general, one has to decide which ones are "nice" absolutely and/or relatively (with respect to the chosen ones). In this approach, the main principle reads as follows: An object is "nice" if it is isomorphic to a quotient object belonging to a special full subcategory and if it (its "basis") has cardinality less than (less than or equal to) a given infinite cardinal. It leads to the basic idea: to approximate a \( \mathcal{C} \)-object \( X \) by a suitable inverse system consisting of its quotient objects \( X_\lambda \) (and the quotient morphisms) which have cardinalities, or dimensions - in the case of vectorial spaces, less than (less than or equal to) \( \kappa \). Such an approximation exists in the form of any \( \kappa^- \)-expansion (\( \kappa \)-expansion) of \( X \),

\[
 p_{\kappa^-} = (p_\lambda) : X \to X_{\kappa^-} = (X_\lambda, p_{\lambda|\lambda'}, \Lambda_{\kappa^-}) \\
 p_\kappa = (p_\lambda) : X \to X_\kappa = (X_\lambda, p_{\lambda|\lambda'}, \Lambda_{\kappa}),
\]

where \( X_{\kappa^-} (X_\kappa) \) belongs to the subcategory \( pro-\mathcal{D}_{\kappa^-} \) (\( pro-\mathcal{D}_\kappa \)) of \( pro-\mathcal{D} \), and \( \mathcal{D}_{\kappa^-} (\mathcal{D}_\kappa) \) is the subcategory of \( \mathcal{D} \) determined by all the objects having cardinalities, or dimensions - for vectorial spaces, less than (less than or equal to) \( \kappa \), while \( \mathcal{D} \) is a full subcategory of \( \mathcal{C} \). Clearly, if \( X \in Ob(\mathcal{D}) \) and the cardinality \(|X| < \kappa \ (|X| \leq \kappa)\), then the rudimentary pro-morphism \([1_X] : X \to [X]\) is a \( \kappa^- \)-expansion (\( \kappa \)-expansion) of \( X \). The corresponding shape category \( Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C}) \) \( (Sh_{\mathcal{D}_\kappa}(\mathcal{C})) \) and shape functor \( S_{\kappa^-} : \mathcal{C} \to Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C}) \) \( (S_\kappa : \mathcal{C} \to Sh_{\mathcal{D}_\kappa}(\mathcal{C})) \) exist by the general (abstract) shape theory, and they have all the appropriate general properties. Moreover, there exist the relating functors \( S_{\kappa^-} : Sh_{\mathcal{D}_\kappa}(\mathcal{C}) \to Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C}) \) and \( S_{\kappa\kappa'} : Sh_{\mathcal{D}_{\kappa'}}(\mathcal{C}) \to Sh_{\mathcal{D}_\kappa}(\mathcal{C}), \kappa \leq \kappa' \), such that \( S_{\kappa^-}S_\kappa = S_{\kappa^-} \) and \( S_{\kappa\kappa'}S_{\kappa'} = S_\kappa \). Even in simplest case of \( \mathcal{D} = \mathcal{C} \), the quotient shape classifications are very often non-trivial and very interesting. In such a case we simplify the notation \( Sh_{\mathcal{D}_{\kappa^-}}(\mathcal{C}) \) \( (Sh_{\mathcal{D}_\kappa}(\mathcal{C})) \) to \( Sh_{\kappa^-}(\mathcal{C}) \) \( (Sh_\kappa(\mathcal{C})) \) or to \( Sh_{\kappa^-} (Sh_\kappa) \) when \( \mathcal{C} \) is fixed.

In [12], several well known concrete categories were considered and many examples are given which show that the quotient shape theory yields classifications strictly coarser than those by isomorphisms. In [13] and [14] were considered the quotient shapes of (purely algebraic, topological and normed) vectorial spaces and topological spaces, respectively. In the recent paper [15], we have continued the studying of quotient shapes of normed vectorial spaces of [13], Section 4.1, primarily and separately focused to the well known \( l_p \) and \( l_p \) spaces and to the Sobolev spaces \( W_p^{(k)}(\Omega_n) \) (of all real functions on \( \Omega_n \) having their supports in a domain \( \Omega_n \) and all partial derivatives up to order \( k \) continuous). The main global result of [15] is that the finite quotient shape type of a normed spaces (over the field \( F \in \{\mathbb{R}, \mathbb{C}\} \))
reduces to that of its completion (Banach) spaces, and consequently, that the quotient shape theory of \( (N\text{ Vect}_F, (N\text{ Vect}_F)_0) \) reduces to that of \( (B\text{ Vect}_F, (B\text{ Vect}_F)_0) \).

In this work we have clarified the relationship between the quotient shape theories of normed and that of Banach spaces (Theorem 1). Further, we have proven that the finite quotient shape type of normed spaces is an invariant of the (algebraic) dimension, but not conversely. The counterexamples exist in the dimensional par \( \{\aleph_0, 2^{\aleph_0}\} \). In the case of separable Banach spaces, the classifications by dimension and by the finite quotient shape (as well as by the countable quotient shape) coincide (Theorem 2). Consequently, all the infinite-dimensional separable normed spaces over the same field belong to a unique quotient shape type with respect to Banach spaces. Its representative may be, for instance, the Hilbert space \( l_2 \), or its (countably infinite dimensional, unitary) direct sum subspace \( F_N^0(2) \equiv (F_N^0, \|\cdot\|_2) \) (Corollary 1).

An application yields two extension type theorems for the category \( sB \cup bB \) (\( sB \) - separable Banach; \( bB \) - bidual-like Banach spaces, i.e., \( X \cong X^{**} \)), provided a subspace has the top-dimensional closure and a lower codimension, into lower dimensional Banach spaces (Theorems 3 and its “operable realization” - Theorem 4).

2. Preliminaries

We shall frequently use and apply in the sequel several general or special well known facts without referring to any source. So we remind a reader that
- our general shape theory technique is that of [10];
- the needed set theoretic (especially, concerning cardinals) and topological facts can be found in [4];
- the facts concerning functional analysis are taken from [8], [9] or [11];
- our category theory language follows that of [6].

For the sake of completeness, let us briefly repeat the construction of a quotient shape category and a quotient shape functor, [12]. Given a category pair \( (\mathcal{C}, \mathcal{D}) \), where \( \mathcal{D} \subseteq \mathcal{C} \) is full, and a cardinal \( \kappa \), let \( \mathcal{D}_{\kappa^-} \) denote the full subcategory of \( \mathcal{D} \) determined by all the objects having cardinalities \( \kappa^- \leq \kappa \) less than (less or equal to) \( \kappa \). By following the main principle, let \( (\mathcal{C}, \mathcal{D}_{\kappa^-}) \) \( ((\mathcal{C}, \mathcal{D}_{\kappa})) \) be such a pair of concrete categories. If
(a) every \( \mathcal{C} \)-object \( (X, \sigma) \) admits a directed set \( R(X, \sigma, \kappa^-) \equiv \Lambda_{\kappa^-} \) \( (R(X, \sigma, \kappa) \equiv \Lambda_{\kappa}) \) of equivalence relations \( \lambda \) on \( X \) such that each quotient object \( (X/\lambda, \sigma_\lambda) \) has to belong to \( \mathcal{D}_{\kappa^-} \) \( (\mathcal{D}_{\kappa}) \), while each quotient morphism \( p_\lambda : (X, \sigma) \to (X/\lambda, \sigma_\lambda) \) has to belong to \( \mathcal{C} \);
(b) the induced morphisms between quotient objects belong to $D_{k^{-}}(D_k)$;
(c) every morphism $f : (X, \sigma) \to (Y, \tau)$ of $C$, having the codomain in $D_{k^{-}}(D_k)$, factorizes uniquely through a quotient morphism $p_\lambda : (X, \sigma) \to (X/\lambda, \sigma_\lambda)$, $f = gp_\lambda$, with $g$ belonging to $D_{k^{-}}(D_k)$, then $D_{k^{-}}(D_k)$ is a pro-reflective subcategory of $C$. Consequently, there exists a (non-trivial) \textit{(quotient) shape category} $Sh_{C, D_{k^{-}}}(C) \equiv Sh_{D_{k^{-}}}(C)$ obtained by the general construction.

Therefore, a $k^{-}$-shape morphism $F_{k^{-}} : (X, \sigma) \to (Y, \tau)$ is represented by a diagram (in pro-$C$)
\[
\begin{array}{ccc}
(X, \sigma)_{k^{-}} & \xrightarrow{p_{k^{-}}} & (X, \sigma) \\
\downarrow f_{k^{-}} & & \\
(Y, \tau)_{k^{-}} & \xrightarrow{q_{k^{-}}} & (Y, \tau)
\end{array}
\]
(with $p_{k^{-}}$ and $q_{k^{-}}$ - a pair of appropriate expansions), and similarly for a $k$-shape morphism $F_{k} : (X, \sigma) \to (Y, \tau)$. Since all $D_{k^{-}}$-expansions ($D_{k}$-expansions) of a $C$-object are mutually isomorphic objects of pro-$D_{k}$ (pro-$D_k$), the composition and identities follow straightforwardly. Observe that every quotient morphism $p_{\lambda}$ is an effective epimorphism. (If $U$ is the forgetful functor, then $U(p_{\lambda})$ is a surjection), and thus condition (E2) for an expansion follows trivially.

The corresponding \textit{“quotient shape” functors} $S_{k^{-}} : C \to Sh_{D_{k^{-}}}(C)$ and $S_{k} : C \to Sh_{D_{k}}(C)$ are defined in the same general manner. That means,

\[
S_{k^{-}}(X, \sigma) = S_{k}(X, \sigma) = (X, \sigma);
\]

if $f : (X, \sigma) \to (Y, \tau)$ is a $C$-morphism, then, for every $\mu \in M_{k^{-}}$, the composite $g_{\mu}f : (Y, \tau) \to (Y_\mu, \tau_\mu)$ factorizes (uniquely) through a $p_{\lambda(\mu)} : (X, \sigma) \to (X_{\lambda(\mu)}, \sigma_{\lambda(\mu)})$, and thus, the correspondence $\mu \mapsto \lambda(\mu)$ yields a function $\phi : M_{k^{-}} \to \Lambda_{k^{-}}$ and a family of $D_{k^{-}}$-morphisms $f_{\mu} : (X_{\phi(\mu)}, \sigma_{\phi(\mu)}) \to (Y_\mu, \tau_\mu)$ such that $q_{\mu}f = f_{\mu}p_{\phi(\mu)}$;

one easily shows that $(\phi, f_{\mu}) : (X, \sigma)_{k^{-}} \to (Y, \tau)_{k^{-}}$ is a morphism of $inv-D_{k^{-}}$, so the equivalence class $f_{k^{-}} = [(\phi, f_{\mu})] : (X, \sigma)_{k^{-}} \to (Y, \tau)_{k^{-}}$ is a morphism of pro-$D_{k^{-}}$;

then we put $S_{k^{-}}(f) = (f_{k^{-}}) \equiv F_{k^{-}} : (X, \sigma) \to (Y, \tau)$ in $Sh_{D_{k^{-}}}(C)$.

The identities and composition are obviously preserved. In the same way one defines the functor $S_{k}$.

Furthermore, since $(X, \sigma)_{k^{-}}$ is a subsystem of $(X, \sigma)_{k}$ (more precisely, $(X, \sigma)_{k}$ is a subobject of $(X, \sigma)_{k^{-}}$ in pro-$D$), one easily shows that there exists a functor $S_{k^{-}} : Sh_{D_{k}}(C) \to Sh_{D_{k^{-}}}(C)$ such that $S_{k^{-}}S_{k} = S_{k^{-}}$, i.e., the diagram
commutes. Moreover, an analogous functor $S_{\kappa'} : Sh_{\mathcal{D}_{\kappa'}}(\mathcal{C}) \to Sh_{\mathcal{D}_{\kappa}}(\mathcal{C})$, satisfying $S_{\kappa'}S_{\kappa'} = S_{\kappa}$, exists for every pair of infinite cardinals $\kappa \leq \kappa'$.

Generally, in the case of $\kappa = \aleph_0$, the $\kappa$-shape is said to be the finite (quotient) shape, because all the objects in the expansions are of finite (bases) cardinalities, and the category is denoted by $Sh_{\mathcal{D}_0}(\mathcal{C})$ or by $Sh_0(\mathcal{C}) \equiv Sh_{\mathcal{D}_0}$ only, whenever $\mathcal{D} = \mathcal{C}$.

Let us finally notice that, though $\mathcal{D} \not\subseteq \mathcal{C}$, the quotient shape category $Sh_{\mathcal{C}_\kappa}(\mathcal{D}) (Sh_{\mathcal{C}_\kappa}(\mathcal{D}))$ exists as a full subcategory of $Sh_{\mathcal{C}_\kappa}(\mathcal{D}) (Sh_{\mathcal{C}_\kappa}(\mathcal{D}))$, and, if $\mathcal{D}$ is closed with respect to quotients, then $Sh_{\mathcal{C}_\kappa}(\mathcal{D}) = Sh_{\mathcal{D}_\kappa}(\mathcal{D}) (Sh_{\mathcal{C}_\kappa}(\mathcal{D}) = Sh_{\mathcal{D}_\kappa}(\mathcal{D}))$.

3. THE QUOTIENT SHAPES OF NORMED AND BANACH SPACES

Let $\mathcal{N}$ denote the category of all normed (vectorial) spaces over the field $F \in \{\mathbb{R}, \mathbb{C}\}$ together with all corresponding continuous linear functions. Let $\mathcal{H} \subseteq \mathcal{B} \subseteq \mathcal{N}$ denote its full subcategories of all Hilbert and all Banach spaces over the same $F$, respectively, and let $s\mathcal{N} (s\mathcal{N} \subseteq \mathcal{B})$ denote full subcategory of all separable normed (bidual-like) spaces over the same $F$. Further, given an infinite cardinal $\kappa \geq \aleph_0$, let $\mathcal{H}_\kappa \subseteq \mathcal{B}_\kappa \subseteq \mathcal{N}_\kappa$ denote the corresponding full subcategories determined by all the objects having algebraic dimensions less than (less or equal to) $\kappa$.

The next lemma and theorem slightly reinforce Theorem 3 of [15].

**Lemma 1.** Let $Z$ be a dense subspace of a normed space $X$ and let $\dim X \geq \kappa \geq \aleph_0$. Then, for every $\mathcal{B}_\kappa$-expansion $p_{\kappa} : (p_\lambda : X \to X_{\kappa'} = (X_\lambda, p_{\lambda'}, \Lambda_{\kappa'})$ of $X$, the pro-morphism $q_{\kappa} = (q_\lambda = p_\lambda j : Z \to X_{\kappa'}$, $(j : Z \hookrightarrow X$ is the inclusion) is a $\mathcal{B}_\kappa$-expansion of $Z$. Especially, every continuous linear function $f : Z \to Y$, where $Y$ is a Banach space over the same field having $\dim Y < \kappa$, and its extension $\bar{f} : X \to Y$ factorize uniquely (linearly and continuously) trough a Banach space $X_\lambda$, $\dim X_\lambda < \kappa$, $\lambda \in \Lambda_{\kappa'}$, and if $\dim Y < 2^{\aleph_0} \leq \kappa$, then $\dim X_\lambda < \aleph_0$. The quite analogous statements hold in the $\kappa$-case, $\aleph_0 \leq \kappa < \dim X$.

**Proof.** One readily sees that $q_{\kappa} = (q_\lambda = p_\lambda j : Z \to X_{\kappa'}$, is a $\mathcal{B}_\kappa$-expansion of $Z$. We need to verify the mentioned factorization property. Let $Y$ be a Banach space over the same field, $\dim Y < \kappa$, let
Let $f : X \to Y$ be a continuous linear function and let $\bar{f} : Cl(X) \to Y$ be the continuous extension of $f$, that is also linear. Since $p_\kappa : X \to X_\kappa$ is a $B_\kappa$-expansion of $X$, there exist a $\lambda \in \Lambda_\kappa$ and a unique continuous linear function $f^\lambda : X_\lambda \to Z$ such that $f^\lambda p_\lambda = \bar{f}$. Then $f^\lambda p_\lambda j = \bar{f} j = f$. If a $g^\lambda : X_\lambda \to Z$ has the same property, i.e., $g^\lambda p_\lambda j = f$, then $g^\lambda p_\lambda j = \bar{f} j$. Since every continuous extension of $Z$ onto $X$ is unique, it follows that $g^\lambda p_\lambda = \bar{f}$. Finally, the factorization through an expansion term is also unique, hence $g^\lambda = f^\lambda$. The dimension property holds because there is no countably infinite-dimensional Banach space. □

**Lemma 2.** For every vectorial space $X \neq \{\emptyset\}$ over $F \in \{\mathbb{R}, \mathbb{C}\}$,
(i) $\dim X_0^N > \dim X \Leftrightarrow \dim X < \aleph_0$,
where $X_0^N$ is the direct sum space. Equivalently,
(i)' $\dim X_0^N = \dim X \Leftrightarrow \dim X \geq \aleph_0$,
and consequently,
(ii) $X \cong X_0^N \cong F_0^n \Leftrightarrow \dim X = \aleph_0$.
If, in addition, $X$ is a normed space and $Cl(X)$ is its completion in the second dual space, then
(iii) $\dim Cl(X) > \dim X \Leftrightarrow \dim X = \aleph_0$.

**Proof.** Let $X \neq \{\emptyset\}$. If $\dim X < \aleph_0$, then $X \cong F^n$, for some $n \in \mathbb{N}$, and the conclusion $\dim X_0^N = \dim F_0^n = \aleph_0 > n = \dim X$ follows straightforwardly. Conversely, let $\dim X_0^N > \dim X$. Let us assume to the contrary, i.e., that $\dim X \geq \aleph_0$. If $\dim X = \aleph_0$, then one readily sees that $\dim X_0^N = \aleph_0 = \dim X$. Thus, it remains that $\dim X > 2^\aleph_0$, i.e., $\dim X = 2^{\aleph_k}$, $k \geq 0$ (an ordinal). Then, by Lemma 3.2 (iv) of [13], $\dim X = |X|$, and hence,

$$2^{\aleph_k} = \dim X \leq \dim X_0^N = \dim X_0^N = \aleph_0 \leq \dim X = |X| = |X|^{|X|} = (2^{\aleph_k})^{\aleph_0} = 2^{\aleph_k} = \dim X$$

- a contradiction again. This proves equivalences (i) and (i)', and the consequence (ii) follows. Let $X$ be a normed space such that $\dim X = \aleph_0$. Then $X$ is not a Banach space. Thus its completion $Cl(X)$, being a Banach space, must increase the algebraic dimension, i.e., $\dim Cl(X) > \dim X$. Conversely, let $\dim Cl(X) > \dim X$. Assume to the contrary, i.e., that either $\dim X < \aleph_0$ or $\dim X > \aleph_0$. If $\dim X < \aleph_0$, then $X \cong F^n$ for some $n \in \mathbb{N}$, and hence, $Cl(X) = X$ - a contradiction. It remains that $\dim X > \aleph_0$, i.e., $\dim X = 2^{\aleph_k}$, $k \geq 0$. Then

$$2^{\aleph_k} = \dim X \leq \dim Cl(X) \leq \dim X = |X| = |X|^{|X|} = (2^{\aleph_k})^{\aleph_0} = 2^{\aleph_k} = \dim X$$

- a contradiction again, and equivalence (iii) is proven. □

Recall that every normed space is dense in its Banach completion. Further, since the embedding into Banach completion is an isometry, all
dense subspaces of a normed space have the same Banach completion (in the second dual space). Since there is no Banach space of the
countably infinite (algebraic) dimension, the following theorem is an
immediate consequence of Lemma 1 (see also Theorem 4 of [13]) and
Lemma 2.

**Theorem 1.** (i) The quotient shape theory of
(i) \((\mathcal{N}, \mathcal{N}_0)\) (and of \((\mathcal{N}, \mathcal{B}_{\mathcal{N}_0})\) as well) reduces to that of \((\mathcal{B}, \mathcal{B}_0)\);
(ii) \((\mathcal{N}, \mathcal{B}_{\kappa^+})\) \(((\mathcal{N}, \mathcal{B}_{\kappa})\)) reduces to that of \((\mathcal{B}, \mathcal{B}_\kappa)\) \(((\mathcal{B}, \mathcal{B}_\kappa)\)).

The following lemma is a generalization of [15], Proposition 1 (which
was a correction of incorrectly formulated [13], Corollary 4.4).

**Lemma 3.** Let \(X\) and \(Y\) be normed spaces over the same field such
that \(\dim X = \dim Y \equiv \kappa\). Then

(i) \(Sh_0(X) = Sh_0(Y)\)
and there exist an isomorphism \(F : X \to Y\) of \(Sh_0(\mathcal{N})\) that is induced
by an isomorphism \(f' : X' \to Y'\) of \(\text{pro-}\mathcal{H}_0\). If \(\kappa > \aleph_0\), then

\[\text{Sh}_{\aleph_0}(X) = \text{Sh}_{\aleph_0}(Y)\]
with respect to Banach spaces,
and there exists an isomorphism \(F' : X \to Y\) of \(\text{Sh}_{\aleph_0}(\mathcal{N})\) with respect
to Banach spaces, that is induced by the same isomorphism \(f' : X' \to Y'\)
of \(\text{pro-}\mathcal{H}_0\).

(ii) If \(\kappa > \aleph_0\) and, in addition, \(X\) and \(Y\) are Banach spaces such that
there exists a closed embedding \(e : X \to Y\) so that \(\dim (Y/e[X]) < \kappa\),
then

\[\text{Sh}_{\kappa^-}(X) = \text{Sh}_{\kappa^-}(Y)\]
and there exists an isomorphism \(F : X \to Y\) of \(\text{Sh}_{\kappa^-}(\mathcal{B})\) that is induced
by \(e\).

**Proof.** (i). If \(\kappa < \aleph_0\), then \(X\) and \(Y\) are isomorphic to an \(F^n\), and thus,
the statement is trivially true. Let \(\kappa \geq \aleph_0\). Clearly, one may assume
that \(X = (V, |||)\) and \(Y = (V, |||')\), \(\dim V = \kappa\). Let us firstly construct
a desired \(f : X_0 \to Y_0\) of \(\text{pro-}\mathcal{B}_0\). By a careful examining of the proof
of [15], Proposition 1, one notices that there assumed continuity of
\(1_V : X \to Y\) does not play any essential role. Namely, instead of by
the identity \(1_V\) induced pro-morphism, one can construct a morphism

\[f = [(f_\mu)] : X_0 = (X_\lambda, p_{\lambda\lambda'}, \Lambda_0) \to (Y_\mu, q_{\mu\mu'}, M_0) = Y_0\]
of \(\text{pro-}\mathcal{N}_0\) (actually, of \(\text{pro-}\mathcal{B}_0\)) in the same way as

\[g = [(g_\lambda)] : Y_0 \to X_0,\]
(in that proof) is constructed. Mor precisely, by [9], Section 8. 11, (b),
p. 440, every closed subspace \(Z_\lambda \leq X\) such that \(\dim Z_\lambda = \dim X = \dim V\)
and \(\dim (X/Z_\lambda) < \aleph_0\), induces a direct sum presentation \(X = Z_\lambda + W_\lambda, W_\lambda \leq X\) closed. Clearly, \(W_\lambda \cong X/Z_\lambda = X_\lambda\). And similarly,
every closed subspace \(Z_\mu \leq Y\) such that \(\dim Z_\mu = \dim Y = \dim V\)
and $\dim(Y/Z_\mu) < \aleph_0$ induces a direct sum presentation $Y = Z_\mu + W_\mu$,
$W_\mu \leq Y$ closed, and $W_\mu \cong Y/Z_\mu = Y_\mu$. For each $\lambda \in \Lambda_0$ and each
$\mu \in M_0$, choose and fix such a $W_\lambda$ and a $W_\mu$ respectively. Recall
that the morphisms $p_\lambda : X \rightarrow X_\lambda$, $p_{\lambda \mu} : X_{\lambda \mu} \rightarrow X_\lambda$, $\lambda \leq \lambda'$, and
$q_\mu : Y \rightarrow Y_\mu$, $q_{\mu \mu'} : X_{\mu \mu'} \rightarrow Y_\mu$, $\mu \leq \mu'$, are the corresponding quotient
functions, which all are linear and continuous. Observe that every
finite-dimensional subspace $W \leq V$ is closed in the both $X$ and $Y$.
Therefore, by [9], Section 8. 11, (c), p. 440, given a $\mu \in M_0$, i.e., a
$Y_\mu$, for a chosen $W_\mu$, there exists a closed subspace $Z_{\lambda_\mu}$ of $X$ that is a
direct complement of $W_\mu$ and of the chosen $W_{\lambda_\mu}$ as well. Clearly, for
every $\mu \in M_0$,
$$X_{\lambda_\mu} = X/Z_{\lambda_\mu} \cong W_{\lambda_\mu} \leq W_\mu \cong Y/Z_\mu = Y_\mu,$$
and
$$Y = Z_\mu + W_\mu, \quad X = Z_{\lambda_\mu} + W_\mu.$$ It implies that each $[v = y]_\mu \in Y_\mu$ is represented by a unique $w \in W_\mu$
and conversely, and that each $[v = x]_{\lambda_\mu} \in X_{\lambda_\mu}$ is represented by a
unique $w' \in W_\mu$ and conversely.
Let us define
$$\phi_\mu : W_\mu \rightarrow X_{\lambda_\mu}, \quad \phi_\mu(w) = [w]_{\lambda_\mu} = w + Z_{\lambda_\mu},$$
$$\psi_\mu : W_\mu \rightarrow Y_\mu, \quad \psi_\mu(w) = [w]_\mu = w + Z_\mu.$$ Since the elements of $W_\mu$ bijectively represent the element-classes of $Y_\mu$
and of $X_{\lambda_\mu}$, it follows that $\phi_\lambda$ and $\psi_\lambda$ are linear bijections. Since all
the spaces are finite-dimensional, $\phi_\lambda$, $\phi_\lambda^{-1}$, $\psi_\lambda$ and $\psi_\lambda^{-1}$ are continuous,
and thus, they are the isomorphisms of Banach spaces. Hence, the composite
$$\phi_\mu \psi_\mu^{-1} : X_{\lambda_\mu} \rightarrow Y_\mu, \quad \phi_\mu \psi_\mu^{-1}([x = w]_{\lambda_\mu}) = [w = y]_\mu,$$
is an isomorphism of Banach spaces. Put
$$f : M_0 \rightarrow \Lambda_0, \quad f(\mu) = \lambda_\mu, \quad \text{and}$$
$$\tilde{f}_\mu : X_{f(\mu)} \rightarrow Y_\mu, \quad \tilde{f}_\mu = \phi_\mu \psi_\mu^{-1}.$$ Then
$$(f, \tilde{f}_\mu) : X_0 \rightarrow Y_0$$
is a morphism of $inv-B_0 \subseteq inv-N_0$. Indeed, for every related pair
$\mu \leq \mu'$, i.e., $Z_{\mu'} \leq Z_\mu$, there exists a $\lambda \geq f(\mu), f(\mu')$, and since each
$[x]_\lambda = [w]_\lambda \in X_\lambda$ for one and only one $w \in W_\lambda$, it follows that
$$q_{\mu \mu'} \tilde{f}_{\mu'} f(\mu)'([x = w]_{\lambda}) = q_{\mu \mu'} \tilde{f}_{\mu'} ([w]_{f(\mu)'}),$$
$$= q_{\mu \mu'} ([w]_\mu) = [w = y]_\mu, \quad \text{and}$$
$$f_\mu f(\mu) ([x = w]_{\lambda}) = f_\mu ([w]_{f(\mu)}) = \phi_\mu \psi_\mu^{-1}([w]_{f(\mu)}) = [w = y]_\mu.$$ Denote by
$$f = [(f, \tilde{f}_\mu) : X_0 \rightarrow Y_0$$
the induced morphism of $pro-N_0 = pro-B_0$. One can now construct, in
the same way, a morphism
$$g = [(g, g_\lambda) : Y_0 \rightarrow X_0$$
and straightforwardly prove that $g^{-1} = f$. However, it is more convenient to observe that the index function $f : \mathcal{M}_0 \to \Lambda_0$ is cofinal, i.e., that every $\lambda$ admits a $\mu$ such that $f(\mu) \geq \lambda$. Namely, one readily sees that $X = Z_\lambda + W_\lambda$ admits

$$Y = Z_{\mu_\lambda} + W_\lambda = (Z_{\mu} + W) + W_\lambda = Z_{\mu} + (W + W_\lambda) \equiv Z_{\mu} + W_\mu$$

such that $Z_\mu$ is closed in $Y$, $\dim Z_\mu = \dim Y$ and $\dim W < \aleph_0$. Then, by the canonical construction and the definition of $f$, it follows that $f(\mu) = \lambda_\mu \geq \lambda$. Now, the conclusion that $f : X_0 \to Y_0$ is an isomorphism follows by the fact that each its term $f_\mu : X_{\phi(\mu)} \to Y_\mu$, $\mu \in \mathcal{M}_0$, is an isomorphisms of Banach spaces. Then $F = \langle f \rangle \in SH_0(X, Y)$ is a desired finite quotient shape isomorphism. If, in addition, $\dim V > \aleph_0$, i.e. $\langle CH \text{ accepted} \rangle$, $\dim V \geq 2^{\aleph_0}$, then, by Lemma 2 (iii), $\dim Cl(X) = \dim X = \dim V = \dim Y = \dim Cl(Y) \geq 2^{\aleph_0}$, and the conclusion about the countable quotient shapes and the isomorphism $F' = \langle f' \rangle : X \to Y$ of $SH_{\aleph_0}(\mathcal{N})$ follows by Theorem 1 and Lemma 1. Let us now find a representative of the quotient shape isomorphism $F : X \to Y$ of $SH_0(\mathcal{N})$ ($F' : X \to Y$ of $SH_{\aleph_0}(\mathcal{N})$ with respect to Banach spaces, whenever $\kappa > \aleph_0$) belonging to pro-$\mathcal{H}_0$. If $\dim V < \aleph_0$, the canonical rudimentary identity “expansions” may be replaced by the isomorphic ones with the Hilbert codomains $F^n(2)$, where $n = \dim V \in \mathbb{N}$. Let $\dim V > \aleph_0$. Firstly, we are to construct an inverse system

$$X' = (X'_1, \varepsilon'_\lambda, \lambda')$$

in $\mathcal{H}_0$, and an isomorphism

$$u : X_0 \to X'$$

of pro-$\mathcal{B}_0$. By [10], I.1.2, Theorem 2, we may assume, without loss of generality, that $\Lambda_0$ is cofinite (every $\lambda \in \Lambda_0$ admits at most finitely many predecessors). Then the construction goes by induction on $|\lambda| \in \{0\} \cup \mathbb{N}$. Let $|\lambda| = 0$. Then $X_{\lambda} \cong F$ and, by the canonical construction, no pair $\lambda, \lambda'$ is related whenever $|\lambda| = |\lambda'| = 0$. Put $X'_1 = F$ and $p'_{\lambda\lambda} = 1_{X'_1}$, and choose an isomorphism $u_{\lambda} : X_{\lambda} \to X'_{\lambda}$. Let $n \in \mathbb{N}$, and assume that, for all $\lambda \in \Lambda_0$ such that $|\lambda| < n$, the construction is made, i.e., for all $\lambda' \leq \lambda$, the Hilbert spaces $X'_{\lambda'}$ and the isomorphisms $u_{\lambda'} : X_{\lambda'} \to X'_{\lambda'}$ are chosen and, for every related pair $\lambda_1 \leq \lambda_2 \leq \lambda$ and every related triple $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda$ the bonds $p'_{\lambda_1, \lambda_2}, p'_{\lambda_2, \lambda_3}, p'_{\lambda_3, \lambda_3}$ are defined according to commutativity conditions $p'_{\lambda_1, \lambda_2} u_{\lambda_2} = u_{\lambda_1} p'_{\lambda_1, \lambda_2}$ and $p'_{\lambda_1, \lambda_2} p'_{\lambda_2, \lambda_3} = p'_{\lambda_1, \lambda_3}$. Let $\lambda \in \Lambda_0$ such that $|\lambda| = n \in \mathbb{N}$, and let $\lambda_1, \ldots, \lambda_n \leq \lambda$ be all the predecessors of $\lambda$. Then, for each $i = 1, \ldots, n$, $|\lambda_i| < n$ holds. Thus, by the inductive assumption, for all $\lambda_i$ and all their predecessors the construction is already made. By the canonical construction of a quotient expansion, $X_{\lambda} \cong F^{k(\lambda)}$, $k(\lambda) \in \mathbb{N}$. Put $X'_\lambda = F^{k(\lambda)}$ and $p'_{\lambda\lambda} = 1_{X'_\lambda}$, and choose an isomorphism $u_{\lambda} : X_{\lambda} \to X'_{\lambda}$.
Now define, for each \(i = 1, \ldots, n\), \(p'_{\lambda, i} = u_{\lambda, i}p_{\lambda, i}u_{\lambda}^{-1}\). A straightforward verification shows that, for every \(i\), \(p'_{\lambda, i}u_{\lambda} = u_{\lambda}p_{\lambda, i}\), and that, for every related pair \(\lambda, \lambda'\), \(p'_{\lambda, \lambda'}, p'_{\lambda', \lambda} = p'_{\lambda, \lambda}\) holds. This completes the inductive construction of an inverse system \(X' \in \text{Ob}(\text{pro-}\mathcal{H}_0)\) and a pro-morphism \(u = [\{1_{A_0}, u_{\lambda}\}] : X_0 \to X'\) (of \(\text{pro-}\mathcal{B}_0\)). Since the index function \(A_0\) is, obviously, cofinal and each \(u_{\lambda}\) is an isomorphism of the Banach spaces, it follows that \(u\) is an isomorphism of \(\text{pro-}\mathcal{B}_0\). In the same way one can construct an isomorphism \(v : Y_0 \to Y'\) of \(\text{pro-}\mathcal{B}_0\), where \(Y' \in \text{Ob}(\text{pro-}\mathcal{H}_0)\). Clearly,

\[up_0 : X \to X'\] and \(vq_0 : Y \to Y'\)

are \(\mathcal{B}_0\)-expansions of \(X\) and \(Y\), respectively, having the expansion systems in \(\mathcal{H}_0\). The proof of the first statement of (i) is complete by putting \(f' = vf u^{-1}\). If \(\kappa > \aleph_0\), then the countable quotient shape with respect to Banach spaces reduces to the finite one (Theorem 1 (i)), and the conclusion follows as previously.

(ii). We may assume that \(X\) is a closed subspace of \(Y\) such that \(\dim X = \dim Y = \kappa \geq 2^{\aleph_0}\) and \(\dim(Y/X) < \kappa\). Recall that \(\kappa = 2^{\aleph_0}\) implies \(\text{Sh}_{\kappa^{-}}(\mathcal{B}) = \text{Sh}_0(\mathcal{B})\). Let

\[p_{\kappa^{-}} = (p_{\lambda}) : X \to X_{\kappa^{-}} = (X_{\lambda}, p_{\lambda}, A_{\kappa^{-}}),\]

\[q_{\kappa^{-}} = (q_{\mu}) : Y \to Y_{\kappa^{-}} = (Y_{\mu}, q_{\mu}, M_{\kappa^{-}})\]

be the canonical \(\mathcal{B}_{\kappa^{-}}\)-expansions of \(X\), \(Y\) respectively. Let

\[(\varphi, i_{\mu}) : X_{\kappa^{-}} \to Y_{\kappa^{-}}\]

be by the inclusion \(i : X \to Y\) induced morphism of \(\text{inv-}\mathcal{B}_{\kappa}\). By the canonical construction of these \(\kappa^{-}\)-expansions, if the index function \(\varphi : M_{\kappa^{-}} \to A_{\kappa^{-}}\) is cofinal (i.e., if each \(\lambda \in A_{\kappa^{-}}\) admits a \(\mu \in M_{\kappa^{-}}\) such that \(\varphi(\mu) \geq \lambda\)), then the equivalence class

\[[\varphi, i_{\mu}] : x_{\kappa^{-}} \to Y_{\kappa^{-}}\]

is an isomorphism of \(\text{pro-}\mathcal{B}_{\kappa}\), implying \(\text{Sh}_{\kappa^{-}}(X) = \text{Sh}_{\kappa^{-}}(Y)\). Therefore, according to the previous case, the proof reduces to the verification of the following claim:

For every closed subspace \(W \subseteq Y\) such that \(\dim W = \dim Y\) and \(\dim(Y/W) < \kappa\), there exists a closed subspace \(Z \subseteq X\) such that \(\dim Z = \dim X\) and \(\dim(X/Z) < \kappa\), and in addition, \(Z \subseteq W\), \(\dim Z = \dim Y\) and \(\dim(Y/Z) < \kappa\).

Let such a \(W\) be given. Put \(Z = X \cap W\). Then \(Z \subseteq W\) is a closed subspace of the both \(X\) and \(Y\). Since the codimensions of \(X\) and \(W\) are less than \(\kappa\), it follows (see also Lemma 3.8 (iii) of [13]) that

\(\dim Z = \dim(X \cap W) = \dim X = \dim Y = \kappa\).

Further, since, in addition, \(\dim(Y/X) < \kappa\) and \(\dim(Y/W) < \kappa\), where \(\kappa = 2^{\aleph_k}\), for some ordinal \(k \geq 1\) (GCH accepted), it follows that
\begin{align*}
\dim(X/Z) = \dim(X/(X \cap W)) & \leq \dim(Y/(X \cap W)) = \dim(Y/Z) < \kappa. \\
\text{So the claim is verified, and the proof is completed.} \qed
\end{align*}

Given a vectorial space \( V \), denote
\[ W(V) = \{ W \leq V \mid \dim W < \aleph_0 \}, \]
\[ Z(V) = \{ Z \leq X \mid \dim Z = \dim X \wedge \dim(X/Z) < \aleph_0 \}. \]

Further, given a normed space \( X \), denote
\[ \mathcal{Z}_{cl}(X) = \{ Z \leq X \mid \text{Cl}(Z) = Z \wedge \dim Z = \dim X \wedge \dim(X/Z) < \aleph_0 \}. \]

We shall need the following general facts.

**Lemma 4.** Let \( V \) be a vectorial spaces over \( F \in \{ \mathbb{R}, \mathbb{C} \} \). If \( \dim V > \aleph_0 \), then
\begin{enumerate}[(i)]
  \item \(|W(V)| = |V| = \dim V < 2^{|V|} = |Z(V)|.\]
  Further, if \( X = (V, \| \cdot \|) \) is a separable or bidual-like normed space \( (X^{**} \cong X) \), then
  \item \(|W(X)| = |\mathcal{Z}_{cl}(X)| = |X| = \dim X.\]
\end{enumerate}

**Proof.** The equality \(|V| = \dim V\), and thus \(|X| = \dim X\) as well, follows by [13], Lemma 3.2 (iv). Notice that \( W(V) \) is the disjoint union of all \( W_n(V) \), \( n \in \{ 0 \} \cup \mathbb{N} \), where
\[ W_n(V) = \{ W \leq V \mid \dim W = n \}. \]

Hereby, \( W_0(V) = \{ \theta \} \). The same holds for \( W(X) \). (Recall that every finite-dimensional subspace \( W \) of \( X \) is closed.) Similarly, \( Z(V) \) is the disjoint union of all \( Z_n(V) \), \( n \in \{ 0 \} \cup \mathbb{N} \), where
\[ Z_n(V) = \{ Z \leq V \mid \dim Z = \dim V \wedge \dim(V/Z) = n \}. \]

Hereby \( Z_0(V) = \{ V \}. \) In the same way, \( Z_{cl}(X) \) is the disjoint union of all \( Z_{cl,n}(X) \), \( n \in \{ 0 \} \cup \mathbb{N} \), where
\[ Z_{cl,n}(X) = \{ Z \leq X \mid \text{Cl}(Z) = Z \wedge \dim Z = \dim X \wedge \dim(X/Z) = n \} \]
and \( Z_{cl,0}(X) = \{ X \}. \) Observe that \(|W_1(V)| = |V|\), and \(|W_1(X)| = |X|\) as well, hold because of \(|V| \geq 2^{\aleph_0}\). Further, one readily sees that, for every \( n \neq 0 \), \( \dim V > n \) implies \(|W_n(V)| = |W_1(V)| \geq 2^{\aleph_0}\). The same holds true for \( X \). Now observe that
\[ |Z_0(V) \cup Z_1(V)| = |V^+|, \]
where \( V^+ \) denotes the (algebraic) dual of \( V \), while
\[ |Z_0(X) \cup Z_1(X)| = |X^*|, \]
where \( X^* \) denotes the (normed) dual of \( X \). Then, it is easy to see that, for every \( n \neq 0 \), \(|Z_n(V)| = |Z_1(V)|\) and \(|Z_{cl,n}(X)| = |Z_{cl,1}(X)|\). Consequently, statement (i) reduces to
\[ |W_1(V)| = |V| = \dim V < 2^{|V|} = |V^+| = |Z_1(V)|, \]
that holds true because of \( \dim V \geq 2^{\aleph_0}\). Similarly, concerning (ii), it suffices to prove that
\[ |W_1(X)| = |X| = |X^*| = |Z_{cl,1}(X)|, \]
where the first and third equality hold already. It remains to prove the second one. Clearly, \(|X| \leq |X^*|\). Assume, firstly, that \(X\) is a separable normed space. Then \(|X| = 2^\aleph_0\), while the cardinality of the set \(F^X\) of all functions of \(X\) to \(F\) is \(|F|^{|X|} = 2^\aleph_1\) (GCH accepted). Since \(X\) is separable, the cardinality of the set \(c(X, F)\) of all continuous functions of \(X\) to \(F\) is determined by countability of a dense subset on \(X\). This implies that \(|c(X, F)| = 2^\aleph_0 = |X|\). Since \(|X^*| \leq |c(X, F)|\), the conclusion follows. Finally, if \(X\) is a bidual-like normed space, i.e., \(X^{**} \cong X\), then \(|X^{**}| = |X|\), implying \(|X^*| = |X|\). \(\square\)

**Remark 1.** If \(\dim X = \aleph_0\), i.e., \(X \cong (F_0^\aleph_0, ||\cdot||)\) (the direct sum), then the cardinalities considered in Lemma 4 are \(|W(F_0^\aleph_0)| = |Z_{cl}(F_0^\aleph_0)| = 2^\aleph_0 = |F_0^\aleph_0| > \dim X\). Further, in all finite-dimensional cases, i.e., \(X \cong F^n, n \in \mathbb{N}\), and \(|W(F^n)| = |F^n| = |F| = 2^\aleph_0\), while \(|Z_{cl}(F^n)| = 1\) because \(Z = \{X\}\). Notice that, for the full subcategory \(\mathcal{U} \subseteq \mathcal{N}\) (unitary; all the continuous linear functions included), it holds \(X \cong Y\) in \(\mathcal{U}\) if and only if \(X \cong Y\) in \(\mathcal{N}\). Namely, though the restriction to the linear inner-product preserving functions is convenient for making quotients spaces, it breaks the shape relationship with \(\mathcal{N}\) (hereby a full subcategory is needed!). In other words, the restriction to the inner product preserving morphisms, would lead to a new quotient shape theory of unitary (Hilbert) spaces and linear inner-product preserving functions.

Though the separability assumption of a non-bidual-like space \(X\) in our proof of Lemma 4 (ii) is essential, the following question still makes sense:

**Question 1.** Does Lemma 4 (ii) hold true for every normed (Banach) space \(X\) having \(\dim X \neq \aleph_0\)?

Namely, the author can prove that \(X = l_\infty\) (non-separable and non-bidual-like) is an example towards the affirmative answer.

**Theorem 2.** The finite quotient shape type of normed spaces over the same field is a strict invariant of the (algebraic) dimension, i.e.,

(i) \((\dim X = \dim Y) \Rightarrow (\Sh_\varnothing(X) = \Sh_\varnothing(Y))\).

(and with respect to \(\mathcal{B}_0\) as well). Furthermore, there exists a quotient shape isomorphism \(F = (f) : X \rightarrow Y\) of \(\Sh_\varnothing(N\VectlF)\) induced by an isomorphism \(f\) of \(\text{pro}-\mathcal{H}_0\).

Further, either \(\max\{\dim X, \dim Y\} \leq \aleph_0\) or \(X, Y \in \Ob(s\mathcal{B} \cup b\mathcal{B})\) such that \(\min\{\dim X, \dim Y\} > \aleph_0\), then

(ii) \((\dim X = \dim Y) \Leftrightarrow (\Sh_\varnothing(X) = \Sh_\varnothing(Y)) \Leftrightarrow (\Sh_{\aleph_0}(X) = \Sh_{\aleph_0}(Y))\)
hold true. Consequently, the classifications on $s\mathcal{B} \cup b\mathcal{B}$ by (algebraic) dimension, by the finite quotient shape and by the countable quotient shape coincide.

Furthermore, for the bidual-like Banach spaces, if there exists a closed embedding $e : X \to Y$ such that $\dim(Y/e[X]) < \dim Y = \kappa$, then

(iii) $(\dim X = \dim Y) \iff (Sh_{\kappa-}(X) = Sh_{\kappa-}(Y))$.

Proof. Let $X$ and $Y$ be normed spaces over the same field such that $\dim X = \dim Y$. Since one may assume that $X = (V, \|\cdot\|)$ and $Y = (V, \|\cdot\|')$, the implication (i), and the necessity parts in (ii) follow by Theorem 1 and Lemma 3, while the second sufficiency in (ii) holds trivially.

In order to prove that the converse of (i) does not hold, let us consider the direct sum vectorial (algebraic) space $F_0^N(\leq l_\beta \leq F^N)$ and the corresponding normed subspaces $F_0^N(p)$ (of $l_\beta$), $1 \leq p \leq \infty$, that all are of dimension $\dim F_0^N = \aleph_0$. Since $Cl(F_0^N(p)) = l_\beta$, $1 \leq p < \infty$, and $Cl(F_0^N(\infty)) = l_\beta(\infty)$ in $l_\beta(\infty)$, and $Cl(F_0^N(\infty)) = c_0$ in $l_\beta(\infty)$ (see also [15], Section 4), and since $dim l_\beta = dim c_0 = 2^{\aleph_0} > dim(F_0^N(p))$ and $Sh_0(l_\beta) = Sh_0(F_0^N(p)) = Sh_0(c_0)$ (Lemma 1), it follows that there is a lot of counterexamples in the dimensional pair $\{\aleph_0, 2^{\aleph_0}\}$.

Let us now prove the sufficiency part of the first equivalence in (ii). Let $X$, $Y$ be a pair of normed spaces over the same field such that either the both $\dim X, \dim Y \leq \aleph_0$ or $X, Y \in Ob(s\mathcal{N} \cup b\mathcal{N})$ having $\dim X, \dim Y \geq 2^{\aleph_0}$ ($CH$ accepted), and let us assume that $Sh_0(X) = Sh_0(Y)$. If $\dim X < \aleph_0$ and $\dim Y < \aleph_0$, then the both $X$ and $Y$ have to be isomorphic to an $F^n$, $n \in \mathbb{N}$, and hence, $\dim X = \dim Y$. Further, $Sh_0(X) = Sh_0(Y)$ and $n = \dim X < \dim Y = \aleph_0$ (or $n = \dim Y < \dim X = \aleph_0$) immediately leads to a contradiction. It remains to prove the statement in the case of $X, Y \in Ob(s\mathcal{N} \cup b\mathcal{N})$ having $\dim X \geq 2^{\aleph_0}$ and $\dim Y \geq 2^{\aleph_0}$. Let

$$p_0 = (p_\lambda) : X \to X_0 = (X_\lambda, p_{\lambda^*}, \lambda_0),$$

$$q_0 = (q_\mu) : Y \to Y_0 = (Y_\mu, q_{\mu^*}, M_0)$$

be the canonical $N_0$-expansions of $X$, $Y$ respectively, that also are the $B_0$-expansions. Then $X_0 \cong Y_0$ in $pro-B_0$. According to [10], I.1.2, Theorem 2, we may pass to the associated cofinite (i.e., with cofinite index sets) inverse systems

$$X' = (X'_\lambda, p'_{\lambda^*}, \Lambda)$$

$$Y' = (Y'_\mu, q'_{\mu^*}, M)$$

such that $|\Lambda| = |\Lambda_0|$, $|M| = |M_0|$ and $X' \cong X_0$, $Y' \cong Y_0$ in $pro-B_0$. Then $X' \cong Y'$ in $pro-B_0$. Let $f : X' \to Y'$ be an isomorphism. Choose a special representative $(\phi, f_{\bar{\mu}})$ of $f$ (having the index function $\phi$ increasing, [10], I. 1.2, Lemma 3). Then, for every related pair $\bar{\mu} \leq \bar{\mu}'$,}
Similarly, there exists a special representative \((\psi, g_\lambda)\) of \(f^{-1} : Y' \to X'\)
(\(\psi\) is increasing) such that, for every related pair \(\bar\lambda \leq \bar\lambda'\),
\[
g_\lambda q'_{\psi(\lambda)\psi(\lambda')} = p'_{\lambda\lambda'} g_{\mu'}.
\]
Since
\[
(\phi, \bar f_\mu) \circ (\psi, g_\lambda) = (\psi \phi, f_\mu g_{\phi(\bar\mu)}) \sim (1_{M_{\bar\mu}}, 1_{Y_\mu}),
\]
we conclude that, for every \(\bar\mu \in \bar M\), there exists a \(\bar\mu' \geq \bar\mu\), \(\psi(\bar\mu) = \bar\lambda_0\) such that
\[
f_\mu g_{\phi(\bar\mu)} q'_{\psi(\bar\mu)\psi(\bar\mu')} = q'_{\mu\mu'}.
\]
Recall that all \(q_{\mu\mu'}\) are epimorphisms, and thus such are all \(q'_{\mu\mu'}\) (\(q'_{\mu\mu'} =
q'_{\mu\psi(\bar\mu)} q'_{\psi(\bar\mu)\psi(\bar\mu')}, \) by the choice of the special representatives). Therefore,
\[
(\forall \bar\mu \in \bar M_{\bar\kappa}) \quad f_\mu g_{\phi(\bar\mu)} = q'_{\mu\psi(\bar\mu)}.
\]
Now, assume to the contrary, i.e., that \(\dim X \neq \dim Y\). We may assume, without loss of
generality, that \(\dim X = \kappa \geq 2^{\aleph_0}, \dim Y = \kappa' > \kappa\), i.e. \((\text{GCH accepted})\), that \(\kappa' \geq \kappa\). Then, by Lemma 4 (ii),
\[
|\Lambda_0| = |Z_{cl}(X)| = \kappa \text{ and } |M_0| = |Z_{cl}(Y)| = |\kappa'|.
\]
Since
\[
|\bar M| = |M_0| = \kappa' \geq 2^\kappa > \kappa = |\Lambda_0| = |\bar\Lambda| \geq 2^{\aleph_0},
\]
there exists a \(\bar\lambda_0 \in \bar\Lambda\) and there exists infinitely many (actually, \(\kappa'\)
many) elements \(\bar\mu \in \bar M\) such that \(\phi(\bar\mu) = \bar\lambda_0\). Put \(\bar\mu_0 = \psi(\bar\lambda_0) \in \bar M\).
Then,
\[
\bar\mu_0 = \psi(\bar\mu) \geq \bar\mu
\]
for infinitely many \(\bar\mu \in \bar M\). It follows that \(\bar M\) is not cofinite - a
contradiction. Finally, the statement concerning separable Banach spaces
follows by Theorem 1 (i).

It remains to prove the last statement. Since \(Sh_{\kappa^-}(X) = Sh_{\kappa^-}(Y)\)
implies \(Sh_0(X) = Sh_0(Y)\), the sufficiency part of (iii) holds by the
sufficiency part of (ii) in general, i.e., without any additional assumption.
Conversely, let \(X, Y \in \text{ObbB} \) such that \(\dim X = \dim Y \equiv \kappa\) and let there exist a closed embedding \(e : X \to Y\) such that \(\dim(Y/e[X]) < \kappa\).
If \(\kappa \leq 2^{\aleph_0}\), then the \(\kappa^-\)-quotient shape reduces to 0-quotient shape, and
the conclusion follows, in general, by the necessity part of (ii). Finally,
in the case of \(\kappa > 2^{\aleph_0}\), the conclusion follows by Lemma 3 (ii).

**Corollary 1.** All infinite-dimensional separable normed spaces over \(F\)
(especially, all the direct sum spaces \(F_0^n, \|\cdot\|\)) and all the \(C_p(n)\) spaces,
\(1 \leq p < \infty, n \in \mathbb{N}\) and all \(2^{\aleph_0}\)-dimensional normed spaces over \(F\)
(especially, the Banach spaces \(L_p\) and \(L_p(n)\), \(1 \leq p \leq \infty, n \in \mathbb{N}\),
the subspaces \(c_0 \leq c \leq l_\infty\) and Sobolev spaces \(W_p^{(k)}(\Omega_n)\)) belong to
the same and unique non-trivial quotient shape type with respect to
Banach spaces, that is the finite one. A representing space may be the
\(\aleph_0\)-dimensional unitary direct sum normed subspace \(F_0^{W}(2)\) of \(l_2\) as well as the Hilbert space \(l_2\).

**Proof.** Every infinite-dimensional separable Banach space has the (algebraic) dimension \(2^{\aleph_0}\). If an infinite-dimensional separable normed space \(X\) is not complete, then one may use the inclusion \(j_X : X \to Cl(X)\), where \(X\) is isometrically embedded into its second dual space. Then \(Cl(X)\) is an infinite-dimensional separable Banach space, and hence, \(\dim Cl(X) = 2^{\aleph_0}\). All the considered concrete spaces belong to the mentioned classes. Thus, the conclusion follows by Lemma 1 (or Theorem 3 [15]) and Theorem 2. \(\Box\)

**Remark 2.** (i) A counterexample for the converse in Theorem 2 (i), in the dimensional pair \(\{\aleph_0, 2^{\aleph_0}\}\), is \(X = (F_0^{W}, \| \cdot \|)\) and \(Y = Cl(X)\) - the Banach completion of \(X\) in the second dual space of \(X\). Namely, by Lemmata 1 and 2 (CH accepted) \(Sh_0(X) = Sh_0(Y)\) and \(\dim X = \aleph_0 < 2^{\aleph_0} = \dim Y\).

(ii) Notice that, although each direct sum normed space \((F_0^{W}, \| \cdot \|)\) may represent the (unique) finite quotient shape type of all \(2^{\aleph_0}\)-dimensional normed spaces considered in Corollary 1, none of \((F_0^{W}, \| \cdot \|)\) can represent any (but its own) of their countable quotient shape types with respect to normed spaces. Namely, \((F_0^{W}, \| \cdot \|)\) itself represents its own countable quotient shape type, while the countable quotient shape types of all normed (Banach) spaces \(X, \dim X = 2^{\aleph_0}\), are non-rudimentary. (They reduce to their unique non-rudimentary finite quotient shape type). Hence, “philosophically” speaking, in the “world of Banach spaces” there is no “fine/close” approximation of a \(2^{\aleph_0}\)-object by the “shape-like” \(\aleph_0\)-objects, i.e., there is only a “coarse” approximation by the finite-dimensional objects. It might be the main cause for the general difficulties in a practical application, especially, in solving of partial differential equations! So the theoretical “advantage” (there is no \(\aleph_0\)-dimensional Banach space) can turn back to be a practical disadvantage.

**Question 2.** Given a Banach space \(X\) and its closed subspace \(Z\) such that \(\dim Z = \dim X \geq 2^{\aleph_k}\) and \(\dim (X/Z) \leq \aleph_k\) (\(k \geq 1; GCH\) accepted), is there a closed complement of \(Z\) in \(X\)?

**4. Application**

It is well known that, in general, a continuous linear function of a (closed) subspace of a normed space into a Banach space of dimension \(\dim \geq 2\) does not admit a continuous linear extension on the whole space. We shall prove that under certain dimensional conditions such
an extension exists. Clearly, if the dimension of a codomain space is \( \text{dim} = 1 \), then an extension exists without any additional condition (the Hahn-Banach theorem).

**Theorem 3.** Let \( X \) be a normed space having \( \text{dim} X \geq \kappa \geq \aleph_0 \), and let \( Z \subseteq X \) be a (normed) subspace. If, by the inclusion \( e : Cl(Z) \hookrightarrow X \), induced, quotient shape morphism \( S_{\kappa^{-}}(e) : Cl(Z) \rightarrow X \) is an isomorphism of \( S_{\kappa^{-}}(N) \), then for every Banach space \( Y \), over the same field, such that \( \text{dim} Y < \kappa \), every continuous linear function \( f : Z \rightarrow Y \) admits a continuous linear extension \( \bar{f} : X \rightarrow Y \). Moreover, \( \|\bar{f}\| = \|f\| \).

**Proof.** Let \( S_{\kappa^{-}}(e) : Cl(Z) \rightarrow X \) be an isomorphism of \( Sh_{\kappa^{-}}(N) \). Let \( Y \in \text{Ob}(B_{\kappa^{-}}) \) and let \( f \in \mathcal{N}(Z,Y) \). We may assume, without loss of generality, that \( S_{\kappa^{-}}(e) \) is represented by a level morphism

\[
(1_N, u_\nu) : Z_{\kappa^{-}} = (Z_{\nu}', r_{\nu}' \circ p_{\nu}', N) \rightarrow (X_{\nu}', p_{\nu}', N) = X_{\kappa^{-}}'
\]

of \( \text{inv-N}_{\kappa^{-}} \), where \( Z_{\kappa^{-}}\) and \( X_{\kappa^{-}}\) are the expansion objects of \( \kappa^{-} \)-expansions

\[
r_{\kappa^{-}}(e) = (r_{\nu}') : Cl(Z) \rightarrow Z_{\kappa^{-}} \quad \text{and} \quad p_{\kappa^{-}}(e) = (p_{\nu}') : X \rightarrow X_{\kappa^{-}}'
\]

of \( Cl(Z) \) and \( X \) respectively. Then, for every \( \nu \in N, p_{\nu}' e = u_\nu r_{\nu}' \) and, for every related pair \( \nu \leq \nu' \in N, u_\nu r_{\nu}' = p_{\nu}' u_{\nu'} \). By the well-known Morita lemma, for every \( \nu \in N \), there exist a \( \nu' \geq \nu \) and a \( u_\nu \in \mathcal{N}(X_{\nu}', Z_{\nu}') \) such that the diagram

\[
\begin{array}{ccc}
Z_{\nu}' & \xrightarrow{r_{\nu}'} & Z_{\nu}' \\
\downarrow u_{\nu} & & \downarrow u_{\nu'} \\
X_{\nu}' & \xrightarrow{p_{\nu}'} & X_{\nu}'
\end{array}
\]

in \( \mathcal{N} \) commutes, i.e., \( u_\nu u_{\nu'} = p_{\nu}' u_{\nu'} \) and \( u_\nu u_{\nu'} = r_{\nu}' \). Denote by \( i : Z \hookrightarrow Cl(Z) \) the inclusion, and by \( f' : Cl(Z) \rightarrow Y \) the unique continuous linear extension of \( f \), i.e., \( f' i = f \) (Lemma 1). Since \( r' \) is \( \kappa^{-} \)-expansion of \( Cl(Z) \) and \( \text{dim} Y < \kappa \), there exist a \( \nu \in N \) and an \( f_\nu' \in \mathcal{N}(Z_{\nu}', Y) \) such that \( f' r_\nu' = f' \). Put

\[
\bar{f} = f'^* v_\nu p_{\nu}' : X \rightarrow Y.
\]

Then

\[
\bar{f} e = f'^* v_\nu p_{\nu}' e = f'^* v_\nu u_{\nu'} r_{\nu}' = f'^* r_{\nu}' v_{\nu'} f' = f'.
\]

Consequently, \( \bar{f}(ei) = (\bar{f} e)i = f' i = f \), that is a desired extension. Recall that all the projections and bonding morphisms in a canonical expansion are the appropriate non-trivial quotient morphisms (the initial one onto \( \{\theta\} \) may be dropped and ignored), and hence, their norm is 1. Further, since \( e : Cl(Z) \hookrightarrow X \) is the inclusion, the induced canonical \((\varphi, e_\lambda)\) inv-morphism consists of the quotient morphisms as well, and thus, \( \|e_\lambda\| = 1, \lambda \in \Lambda_{\kappa^{-}} \) (those onto the initial term or from the
initial term may be dropped and ignored). Further, the construction of a level morphism uses the already existing morphisms only. Therefore, \( \|u_\nu\| = \|u_\nu'\| = \|v_\nu\| = 1, \nu, \nu' \in N \).

Finally, \( \|e\| = 1 \), and the extension \( f' \) of \( Z \) on \( Cl(Z) \) does not affect the norm \( \|f\| \). Consequently, \( \|\bar{f}\| = \|f_\nu v_\nu'\| = \|f_\nu'\| = \|f\| \), that completes the proof.

\( \square \)

**Theorem 4.** Let \( X \) be a normed space having \( \dim X = \kappa \geq \aleph_0 \), and let \( Z \) be a (normed) subspace such that \( \dim(Cl(Z)) = \dim X \) and \( \dim(X/Cl(Z)) < \kappa \), and let \( Y \) be a Banach space (over the same field) having \( \dim Y < \kappa \).

(i) If \( X \) is a separable Banach space and \( \max\{\dim(X/Cl(Z)), \dim Y\} \leq \aleph_0 \), then every continuous linear function \( f : Z \to Y \) admits a continuous linear extension \( \bar{f} : X \to Y \) such that \( \|\bar{f}\| = \|f\| \).

(ii) If \( X \) is a bidual-like Banach space, then every continuous linear function \( f : Z \to Y \) admits a continuous linear extension \( \bar{f} : X \to Y \) such that \( \|\bar{f}\| = \|f\| \).

**Proof.** Firstly notice that, in general, a desired (unique) continuous linear extension on \( Cl(Z) \) exists by Lemma 1. Further, in the case of \( \dim(X/Cl(Z)) < \aleph_0 \), there is no need for the assumption \( \dim(Cl(Z)) = \dim X \). Namely, \( \dim X \geq \aleph_0 \) and \( \dim(X/Cl(Z)) < \aleph_0 \) imply \( \dim(Cl(Z)) = \dim X \). Furthermore, there is a rather simple proof of that special case without using Theorem 3. Nevertheless, we want to use Theorem 3 in our proof. Denote by \( e : Cl(Z) \hookrightarrow X \) the (closed continuous linear) inclusion. By Theorem 2 (i), it follows that \( Sh_0(Cl(Z)) = Sh_0(X) \).

(i). Since \( X \in Ob(sB) \), it follows that either \( \dim X < \aleph_0 \) or \( \dim X = 2^{\aleph_0} \). In the first (finite-dimensional) case, the statement is obviously true. Let \( \dim X = 2^{\aleph_0} \). Since \( Y \) is a Banach space having \( \dim Y < \dim X \), it follows that \( \dim Y < \aleph_0 \). According to Theorem 3, it suffices to prove that the induced quotient shape morphism

\[
F \equiv S_0(e) : Cl(Z) \to X
\]

of \( Sh_0(B) \subseteq Sh_0(N) \) is an isomorphism. We shall prove this by proving the analogue claim in the proof of statement (ii).

(ii). Since \( X \in Ob(bB) \) and \( e : Cl(Z) \to X \) a closed embedding such that \( \dim(Cl(Z)) = \dim X \) and \( \dim(X/Cl(Z)) < \dim X = \kappa \), Theorem 2 (iii) implies that \( Sh_{\kappa^-}(Cl(Z)) = Sh_{\kappa^-}(X) \). According to Theorem 3, it remains to prove that the quotient shape morphism

\[
F \equiv S_{\kappa^-}(e) : Cl(Z) \to X
\]

of \( S_{\kappa^-}(bB) \subseteq S_{\kappa^-}(N) \) is an isomorphism. In order to cover the both statements, assume that \( X \) is a Banach space. Let

\[
r_{\kappa^-} = (r_\mu) : Cl(Z) \to Z_{\kappa^-} = (Z_\mu, r_\mu'_{\kappa}, M_{\kappa^-}) \quad \text{and}
\]
be the canonical $N_{\kappa}^\prime$-expansions of $\text{Cl}(Z)$ and $X$ respectively. (They are, actually, the $B_{\kappa}$-expansions, because $\text{Cl}(Z)$ is a Banach space as well, and, consequently, all the quotient spaces by closed subspaces are Banach spaces.) Then the induced pro-morphism $e_{\kappa^-} : Z_{\kappa^-} \to X_{\kappa^-}$ of $e$ is the equivalence class of the inv-morphism

$$(\varphi, e_{\lambda}) : Z_{\kappa^-} \to X_{\kappa^-},$$

where $\varphi : \Lambda_{\kappa^-} \to M_{\kappa^-}$ is obtained by the expansion factorization property (E1) for each $p_{\lambda}e = e_{\lambda}r_{\varphi(\lambda)}$. Therefore, by the construction of a canonical $N_{\kappa}^\prime$-expansion, for every $\lambda \in \Lambda_{\kappa^-}$, corresponding to a closed subspace $U_{\lambda} \subseteq X$ such that $\dim U_{\lambda} = \dim X$ and $\dim(X/U_{\lambda}) < \kappa$, the class $[z]_{\varphi(\lambda)} = z + (\text{Cl}(Z) \cap U_{\lambda}) \in Z_{\varphi(\lambda)}$ goes by $e_{\lambda} : Z_{\varphi(\lambda)} \to X_{\lambda}$ to the class $[z]_{\lambda} = z + U_{\lambda} \subseteq X_{\lambda}$. Further, if $\lambda \leq \lambda'$, then $U_{\lambda'} \subseteq U_{\lambda}$ (having the same dimensional properties), and hence, $\text{Cl}(Z) \cap U_{\lambda'} \subseteq \text{Cl}(Z) \cap U_{\lambda}$ implying $\varphi(\lambda) \leq \varphi(\lambda')$ and $e_{\lambda}r_{\varphi(\lambda')\varphi(\lambda')} = p_{\lambda\lambda'}e_{\varphi(\lambda')}$. On the other side, every $\mu \in M_{\kappa^-}$, corresponding to a closed subspace $V_{\mu} \subseteq \text{Cl}(Z)$ such that $\dim V_{\mu} = \dim \text{Cl}(Z)$ and $\dim(\text{Cl}(Z)/V_{\mu}) < \kappa$, is a $\lambda_{\mu} \in \Lambda_{\kappa^-}$, corresponding to a closed $U_{\lambda_{\mu}} = V_{\mu} \subseteq X$ such that $\dim U_{\lambda_{\mu}} = \dim X$ and $\dim(X/U_{\lambda_{\mu}}) < \kappa$. Thus, there exists a canonical “inclusion” $\psi : M_{\kappa^-} \hookrightarrow \Lambda_{\kappa^-}$, $\psi(\mu) = \lambda_{\mu}$. Put $\psi[M_{\kappa^-}] \equiv \Lambda'_{\kappa^-} \subseteq \Lambda_{\kappa^-}$. It remains to prove that $\psi$ is a cofinal function. Indeed, in that case, the codomain restriction $e_{\kappa^-}' : Z_{\kappa^-} \to X'_{\kappa^-}$ of $e_{\kappa^-}$, i.e., the equivalence class of the codomain restriction

$$(\varphi', e_{\lambda}) : Z_{\kappa^-} \to X'_{\kappa^-},$$

of $(\varphi, e_{\lambda})$, where the bijection $\varphi' : \Lambda'_{\kappa^-} \to M_{\kappa^-}$, $\varphi'(\lambda = \lambda_{\mu}) = \mu$, is the domain restriction of the index function $\varphi$, will be an isomorphism of pro-$B$, and consequently,

$S_{\kappa^-}(e) = \langle e_{\kappa^-} \rangle = \langle e_{\kappa^-}' \rangle : \text{Cl}(Z) \to X$

will be a desired quotient shape isomorphism. Namely, in that case, the restriction

$p_{\kappa^-}' = (p_{\lambda}) : X \to X'_{\kappa^-} = (X_{\lambda}, p_{\lambda\lambda'}, \Lambda'_{\kappa^-})$

will be an $N_{\kappa^-}$-expansion of $X$. The proof now reduces to the following claim:

For every closed subspace $U \subseteq X$ such that $\dim U = \dim X$ and $\dim(X/U) < \kappa$, there exists a closed subspace $V \subseteq \text{Cl}(Z)$ such that $\dim V = \dim \text{Cl}(Z)$ and $\dim(\text{Cl}(Z)/V) < \kappa$, and in addition, $V \subseteq U$, $\dim V = \dim X$ and $\dim(X/V) < \kappa$. Given such a $U$, put $V = \text{Cl}(Z) \cap U$, and the verification works in the same way as in the proof of Lemma 3 (ii). \hfill \Box

At the end, we give a rather general example as a corollary.
Corollary 2. (i) Given a $p, 1 \leq p < \infty$, let $X$ be a separable Banach space that admits a continuous injection $i : l_p \to X$ such that $\dim(X/\text{Cl}(i[l_p])) \leq \aleph_0$, and let $Y$ be a finite-dimensional normed space over the same field. Then, for each $r$, $1 \leq r \leq p$, every continuous linear function $f : F_0^N(r) \to Y$ ($f : l_r \to Y$ as well), admits a continuous linear extension $\hat{f} : Y \to Y$ through all the $l_r$ ($l_s, r \leq s \leq p$);

(ii) Let $X$ be a separable Banach space that admits a continuous injection $i : c_0 \to X$ such that $\dim(X/\text{Cl}(i[c_0])) \leq \aleph_0$, and let $Y$ be a finite-dimensional normed space over the same field. Then every continuous linear function $f : F_0^N(\infty) \to Y$ ($f : l_p(\infty) \to Y$ as well), $1 \leq p < \infty$, admits a continuous linear extension $\hat{f} : Y \to Y$ through all the $l_p(\infty)$, $1 \leq p < \infty$ ($l_p(\infty), p \leq p' < \infty$).

Proof. (i) Observe that the inclusions $i^{0r}_p : F_0^N(r) \to l_p$ and $i^r_p : l_r \to l_p$, $1 \leq r \leq p < \infty$, are continuous, and that, in $l_p$, $\text{Cl}(F_0^N(r)) = \text{Cl}(l_r) = l_p$. Now apply Theorem 4 to $Z = F_0^N(r)$ ($Z = l_r$).

(ii) Notice that $F_0^N(\infty) \subseteq l_1(\infty) \subseteq \cdots \subseteq l_p(\infty) \subseteq \cdots \subseteq c_0$ are proper subspaces of $c_0$ and that $\text{Cl}(F_0^N(\infty)) = l_p(\infty)$ in $l_p(\infty)$, and $\text{Cl}(F_0^N(\infty)) = \text{Cl}(l_p(\infty)) = c_0$ in $l_\infty$, $1 \leq p < \infty$. By applying Theorem 4 to $Z = F_0^N(\infty)$ and $Z = l_p(\infty)$ respectively ($Y$ is a Banach space), the conclusion follows.

(Observe that, in $l_\infty$, $\text{Cl}(F_0^N(p)) = \text{Cl}(l_p) = c_0$, and that $\dim(c/c_0) = 1$, where $c \subseteq l_\infty$ is the subspace of all convergent sequences in $F$, which is closed. Hence, $X = c$ is a concrete example for (i) and (ii.).) \hfill \Box

References

[1] K. Borsuk, Concerning homotopy properties of compacta, Fund. Math. 62 (1968), 223-254.
[2] K. Borsuk, Theory of Shape, Monografie Matematyczne 59, Polish Scientific Publishers, Warszawa, 1975.
[3] J.-M. Cordier and T. Porter, Shape Theory: Categorical Methods of Approximation, Ellis Horwood Ltd., Chichester, 1989. (Dover edition, 2008.)
[4] J. Dugundji, Topology, Allyn and Bacon, Inc. Boston, 1978.
[5] Dydak and J. Segal, Shape theory: An introduction, Lecture Notes in Math. 688, Springer-Verlag, Berlin, 1978.
[6] H. Herlich and G. E. Strecker, Category Theory, An Introduction, Allyn and Bacon Inc., Boston, 1973.
[7] N. Koceić Bilan and N. Uglešić, The coarse shape, Glasnik. Mat. 42(62) (2007), 145-187.
[8] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York, 1989.
[9] S. Kurepa, *Funkcionalna analiza: elementi teorije operatora*, Školska knjiga, Zagreb, 1990.
[10] S. Mardešić and J. Segal, *Shape Theory*, North-Holland, Amsterdam, 1982.
[11] W. Rudin, *Functional Analysis, Second Edition*, McGraw-Hill, Inc., New York, 1991.
[12] N. Uglešić, *The shapes in a concrete category*, Glasnik. Mat. Ser. III 51(71) (2016), 255-306.
[13] N. Uglešić, *On the quotient shape of vectorial spaces*, Rad HAZU - Matematičke znanosti, Vol. 21 = 532 (2017), 179-203.
[14] N. Uglešić, *On the quotient shapes of topological spaces*, Top. Appl. 239 (2018), 142-151.
[15] N. Uglešić, *The quotient shapes of $l_p$ and $L_p$ spaces*, Rad HAZU. Matematičke znanosti, Vol. 27 = 536 (2018), 149-174.
[16] N. Uglešić and B. Ćervar, *The concept of a weak shape type*, International J. of Pure and Applied Math. 39 (2007), 363-428.

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