Kinks in Finite Volume

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A (1+1)-dimensional quantum field theory with a degenerate vacuum (in infinite volume) can contain particles, known as kinks, which interpolate between different vacua and have nontrivial restrictions on their multi-particle Hilbert space. Assuming such a theory to be integrable, we show how to calculate the multi-kink energy levels in finite volume given its factorizable $S$-matrix. In massive theories this can be done exactly up to contributions due to off-shell and tunneling effects that fall off exponentially with volume. As a first application we compare our analytical predictions for the kink scattering theories conjectured to describe the subleading thermal and magnetic perturbations of the tricritical Ising model with numerical results from the truncated conformal space approach. In particular, for the subleading magnetic perturbation our results allow us to decide between the two different $S$-matrices proposed by Smirnov and Zamolodchikov.

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1. Introduction

A unique property of 2-dimensional Minkowski space is that a double light cone divides the set of all space-like points into two disconnected components in a Lorentz-invariant way. In a (1+1)-dimensional quantum field theory (QFT) with degenerate vacuum it is therefore possible that the expectation value of a field approaches different values at the two space-like infinities. If the corresponding excitation creates a stable particle in the theory it is generically known as a “kink”; it is labelled by the two (possibly identical) vacua between which it interpolates, in addition to its energy-momentum and perhaps other quantum numbers.

Some novel and interesting features arise when one considers the multi-particle Hilbert space of a theory with kinks. Since a multi-kink state corresponds to a sequence of vacua on a line, one cannot arbitrarily compose single kinks to obtain an allowed state. In other words, there are restrictions on the multi-kink Hilbert space — it is not a (direct sum of) bosonic or fermionic Fock spaces.

Note that the solitons of the sine-Gordon model are not quite examples of what we want to call a kink. In this theory field configurations differing by a period of the cosine potential are identified, so that there is only one type of soliton and anti-soliton with no restrictions (except the exclusion principle) on their multi-particle states. (See however sect. 5 for a discussion of some variants of the sine-Gordon model where such restrictions do arise.) We reserve the term kink for excitations with nontrivial restrictions on their multi-particle states. Recently several [1][2][3][4][5][6] factorizable scattering theories of massive kinks have been proposed to describe certain integrable perturbations [6] of conformal field theories (CFTs).

The purpose of this paper is to analyze the multi-kink spectrum of such theories in finite volume with periodic boundary conditions, i.e. on a circle of circumference $R$, say. We will show how to calculate the energy levels exactly — up to terms due to tunneling and off-shell effects that are exponentially small for large $R$ — in terms of the factorizable $S$-matrix of the theory.

A theoretical understanding of the finite-volume spectrum of a QFT provides a handle on the dynamics of a theory, allowing one, for instance, to test a conjectured exact $S$-matrix. The idea is to compare the analytical large-volume predictions obtained from the $S$-matrix with one of the available numerical methods of calculating the small-volume spectrum of a QFT, e.g. lattice simulations or the “truncated conformal space approach” (TCSA) for a perturbed CFT (cf. subsect. 3.3).
There are other methods of checking a conjectured $S$-matrix, involving comparison with the finite-volume energy levels of 0- and 1-particle states. For a generic theory of kinks these tests are not so easy to perform, and here the multi-particle spectrum can provide a very simple way of checking a proposed $S$-matrix. In most of the remainder of this introduction we will briefly review the basic physical phenomena underlying the finite-volume spectrum of a QFT, and what information about the $S$-matrix of the theory can be gained by analyzing different parts of the spectrum.

There are basically three physical effects contributing to the finite-volume energy levels of a QFT, which we will refer to as (i) tunneling, (ii) off-shell, and (iii) boundary and scattering effects.

(i): Tunneling, well-known from quantum mechanics, occurs in theories with a degenerate vacuum in infinite volume. Since the ‘energy barrier’ between the vacua is proportional to the volume of the world, there will be instantons in finite volume, tunneling between the vacua and lifting their exact degeneracy. In simple situations, like $\phi^4$ theory in $d + 1$ dimensions with ‘space’ a hypercube of volume $R^d$, the splitting (in the phase of spontaneously broken symmetry) is $\mathcal{O}(R^{-1}R^{d/2}e^{-\sigma R^d})$. Here the exponential is simply the Boltzmann factor of the instanton action, the power $R^{d/2}$ is due to zero modes, and $R^{-1}$ is due to 1-loop fluctuations. In 1+1 dimensions the exponential is $e^{-mR}$, $m$ being the mass of the kink. The prefactor in the above formula can be calculated perturbatively (see the last ref. in [9]) in a given QFT, but it is apparently not known how to express it in a “universal” manner in terms of the $S$-matrix of a generic theory — in fact, it is not clear if such a universal formula exists. If there is tunneling, it is not only relevant for the energy levels of 0-particle states; its effects on other levels will be of comparable magnitude, although we are not aware of any explicit calculations.

(ii): There are two ways in which off-shell effects influence the finite-volume spectrum. The first, relevant for multi-particle states, reflects the fact that in finite volume interactions between physical particles (due to the exchange of virtual particles) cannot be completely expressed in terms of $S$-matrix elements, which connect asymptotic states. Since in a massive theory interactions decrease exponentially with the distance between the particles, the contributions of this effect to the energy levels of stationary scattering states (cf. (iii) and subsect. 2.2) are expected to decrease exponentially with volume. The second off-shell effect is due to the fact that in finite volume virtual particles can “travel once or several times around the world” before annihilating again or being absorbed by a
real particle. The resulting volume dependence of these vacuum polarization effects again gives rise to exponentially small corrections to the energy levels. For the case of zero momentum 1-particle energies, i.e. finite-volume masses, these corrections have been studied in some detail [10] [11] for essentially arbitrary massive QFTs (not only integrable theories in 1+1 dimensions) without a degenerate vacuum in infinite volume. The leading terms contributing to finite-size mass corrections in such theories are given by a universal formula involving only S-matrix elements. It is not known if the exact finite-size mass corrections can also be expressed solely through scattering amplitudes.

For 0-particle states stronger results can be obtained in certain cases. In 1+1 dimensions there is a beautiful method, known as the thermodynamic Bethe Ansatz [12] [13] [14] [15] (TBA), to calculate the exact finite-volume ground state energy $E_0(R)$ of an integrable QFT in terms of its S-matrix. (And with some modifications this method can also be applied to certain 0-particle states above the ground state [16] [17] [18].) Although restricted to integrable theories, this method does not require the existence of a unique vacuum. The exact calculation of the ground state energy on a circle is possible due to its relation to the free energy on an infinite line at finite temperature (and zero chemical potentials). The final result for $E_0(R)$, expressed in terms of the solution of a set of coupled non-linear integral equations, can provide a very strong test of a conjectured S-matrix, since the small-R behaviour of $E_0(R)$ allows one to extract the central charge and other properties of the CFT describing the UV limit of the QFT under consideration. Unfortunately, if the S-matrix of the theory is not diagonal, the TBA requires a detailed case by case analysis (see [19] [20] for the first examples). Such an analysis can be quite nontrivial for kink theories, and therefore other methods of checking a conjectured S-matrix are desirable.

(iii): From quantum mechanics one is familiar with the fact that boundary conditions lead to the quantization of energies and momenta. For 1-particle states in a QFT this effect, together with contributions due to tunneling and vacuum polarization, completely determines the energies in finite volume. For multi-particle states one must also take scattering effects into account, which simply refer to the fact that energy eigenstates in

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1 However, in some cases $E_0(R)$ does not uniquely determine the UV CFT; this happens e.g. for certain perturbations of minimal CFTs with the same central charge but different modular invariant partition function.
finite volume are stationary scattering states. The resulting quantization conditions, which we will refer to as Bethe Ansatz equations, can be expressed solely in terms of the $S$-matrix of the (factorizable) theory if we neglect the nonzero range of the interactions. The corresponding finite-size corrections to energies are generically of $O(1/R^2)$, as we will see, and therefore much larger than the exponentially small (in massive theories, at least) contributions due to the finite interaction range, off-shell, or tunneling effects. Multi-particle energy levels are therefore ideally suited to provide a check on kink $S$-matrices.

Previously multi-particle states were only studied for theories with diagonal $S$-matrices [8] [22]; 2-particle levels were also discussed for the $O(3)$ non-linear $\sigma$-model [23], where the $S$-matrix is diagonal in a 2-particle basis of isospin eigenstates.

The paper is organized as follows. In sect. 2 we describe how to obtain the energy levels and eigenfunctions of multi-kink states in finite volume (up to the exponentially small corrections discussed above). Sects. 3 and 4 are devoted to examples. In sect. 3 we start with a description of some general features of $\phi_{1,3}$-perturbed unitary minimal CFTs, leading (in one direction) to integrable theories of massive kinks. As a warm-up, we discuss in subsect. 3.1 the simplest member of this family, the Ising field theory in the phase of spontaneously broken symmetry. For this theory the complete finite-volume spectrum is known exactly, and provides an instructive check on the large-volume Bethe Ansatz predictions (which are rather trivial to obtain in this case). Subsect. 3.2 treats the next theory in the family, the subleading thermal perturbation of the tricritical Ising CFT. In subsect. 3.3 we compare our analytical predictions for 2- and 4-kink states with numerical results of the truncated conformal space approach (TCSA). Sect. 4 deals with another family of integrable theories of kinks, the $\phi_{2,1}$-perturbed unitary minimal CFTs. We focus our attention on the subleading magnetic perturbation of the tricritical Ising CFT, for which two different $S$-matrix theories have been proposed, one by Smirnov [5] and the other by Zamolodchikov [4]. We clarify the relation between the two $S$-matrices and point out some $a$ priori problems with Zamolodchikov’s conjecture. Comparing our large-volume results for 2- and 3-kink energy levels with TCSA data provides very strong support for

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2 In integrable theories states with a definite number of particles will not mix with states of different particle number, whereas for non-integrable theories this will be true generically (i.e. in the absence of appropriate selection rules) only for 2-particle states with energies below the 3-particle threshold. See [21] for a detailed study of 2-particle levels in higher-dimensional (and therefore non-integrable) theories.
Smirnov’s proposal (although the subtle issue of “crossing factors”, cf. subsects. 2.3 and 4.1, remains open). In sect. 5 we present our conclusions, and discuss some simple modifications of the sine-Gordon model which lead to theories with kinks.

2. The $N$-Particle Transfer Matrix and the Bethe-Yang Equations

2.1. $S$-matrix conventions

In an integrable massive $(1 + 1)$-dimensional QFT all scattering amplitudes factorize into 2-particle amplitudes $S_{cd}^{ab}(\theta = |\theta_{12}|)$ describing scattering processes $a(\theta_1) + b(\theta_2) \to c(\theta_2) + d(\theta_1)$ among the different particles $a, b, \ldots$ of the theory (see fig. 1(a)).

![Diagram](image)

(a) \hspace{10cm} (b)

Fig. 1: (a) The 2-particle scattering process \( a(\theta_1) + b(\theta_2) \to c(\theta_2) + d(\theta_1) \)

(b) The 2-kink process \( K_{\alpha\gamma}(\theta_1) + K_{\gamma\beta}(\theta_2) \to K_{\alpha\delta}(\theta_2) + K_{\delta\beta}(\theta_1) \)

Here the $\theta_i$ are rapidities, parametrizing the energy-momentum of a particle on its mass shell via

\[
(p^0, p^1) = (m \cosh \theta, m \sinh \theta),
\]

and $\theta_{ij} \equiv \theta_i - \theta_j$.

We will be particularly interested in scattering theories of kinks. For such theories it is more convenient to label the $S$-matrix elements by the asymptotic vacua between which the kinks interpolate. (To be sure, by essentially just rewriting our results given below in kink language into “ordinary” $S$-matrix notation, they apply to arbitrary factorizable scattering theories.) We will consider the case of a single multiplet of kinks of mass $m > 0$ which are all bosons or all fermions, in order to keep the notation simple and transparent; this assumption can be trivially relaxed. The kink interpolating between the vacuum $\alpha$ at
As $x \to -\infty$ and $\beta$ at $x \to +\infty$ will be denoted by $K_{\alpha\beta}(\theta)$. We let $S_{\alpha\beta}^{\gamma\delta}(\theta_{12})$ describe the process $K_{\alpha\gamma}(\theta_1) + K_{\gamma\beta}(\theta_2) \to K_{\alpha\delta}(\theta_2) + K_{\delta\beta}(\theta_1)$, where $\theta_1 > \theta_2$. The latter restriction ensures the consistent ordering $\alpha, \gamma, \beta$ ($\alpha, \delta, \beta$) of the vacua along space in the far past (future), as is clear from the graphical description of the scattering process presented in fig. 1(b). (It will be convenient below to let $\alpha, \beta, \gamma, \delta$ range over all vacua and set $S_{\alpha\beta}^{\gamma\delta}(\theta) \equiv 0$ if one or more of the kinks in fig. 1(b) does not exist in the theory.)

To get used to the kink notation it is instructive to rewrite the standard requirements of $S$-matrix theory in this language. We will assume our theories to be time-reversal and parity invariant, i.e. $S_{\alpha\beta}^{\gamma\delta}(\theta) = S_{\delta\alpha}^{\gamma\beta}(\theta) = S_{\gamma\delta}^{\beta\alpha}(\theta)$. Using real analyticity $S_{\alpha\beta}^{\gamma\delta}(\theta) = (S_{\alpha\beta}^{\gamma\delta}(-\theta^*))^*$, unitarity can then be written as

$$\sum_{\mu} S_{\alpha\beta}^{\gamma\mu}(\theta)S_{\alpha\beta}^{\mu\delta}(-\theta) = \delta^{\gamma\delta} \quad \text{for all } \alpha, \beta, \gamma, \delta.$$  \hfill (2.2)

The requirement of crossing symmetry amounts to

$$S_{\alpha\beta}^{\gamma\delta}(\theta) = S_{\gamma\delta}^{\alpha\beta}(i\pi - \theta).$$  \hfill (2.3)

The factorization or Yang-Baxter equations \[24\], expressing the consistent factorization of an arbitrary amplitude into 2-particle amplitudes, read

$$\sum_{\mu} S_{\alpha\gamma}^{\beta\mu}(\theta_{12}) S_{\beta\delta}^{\gamma\epsilon}(\theta_{13}) S_{\alpha\epsilon}^{\mu\eta}(\theta_{23}) = \sum_{\mu} S_{\beta\delta}^{\gamma\mu}(\theta_{23}) S_{\alpha\mu}^{\beta\eta}(\theta_{13}) S_{\eta\delta}^{\epsilon\mu}(\theta_{12}),$$  \hfill (2.4)

which is illustrated in fig. 2.

![Diagram](image.png)

**Fig. 2:** The Yang-Baxter equation (2.5) (the sum over $\mu$ on both sides of the equation is suppressed).

\[3\] We will always use greek indices for the asymptotic vacua; this allows one to distinguish “ordinary” $S$-matrix elements, which will be adorned by latin indices for the particles, from those in kink notation.
Finally, if the kinks in a theory form (stable) bound states, as indicated by simple poles (with residues of appropriate sign) of the scattering amplitudes in the bound state region $\theta \in (0, i\pi)$, then the $S$-matrix also has to satisfy bootstrap constraints. In theories of a single multiplet of kinks this can of course happen only if the bound states of the “constituent” kinks are again kinks in the same multiplet (see sect. 4 for an example). We will treat here only this case; generalizations are straightforward.

![Fig. 3: The bootstrap equation (2.5) is obtained by taking the residues of both sides of the depicted “equality” at $\theta_1 - \theta_2 = 2\pi i/3$ (the sum over $\mu$ on the lhs is suppressed).](image)

Let $K_{\alpha\gamma}$ be a bound state of $K_{\alpha\beta}$ and $K_{\beta\gamma}$ (in the direct channel). Considering the 3-kink process $K_{\alpha\beta}(\theta_1) + K_{\beta\gamma}(\theta_2) + K_{\gamma\delta}(\theta_3) \rightarrow K_{\alpha\eta}(\theta_3) + K_{\eta\epsilon}(\theta_2) + K_{\epsilon\delta}(\theta_1)$ with $\theta_1$ and $\theta_2$ tuned to the complex values necessary to create the intermediate bound-state kink $K_{\alpha\beta}$ on shell (see fig. 3), one concludes that the following equations must hold:

$$\sum_{\mu} g_{\alpha\beta\gamma} g_{\alpha\mu\gamma} S_{\mu\delta}^{\gamma\epsilon}(\theta + \frac{i\pi}{3}) S_{\alpha\epsilon}^{\mu\eta}(\theta - \frac{i\pi}{3}) = g_{\alpha\beta\gamma} g_{\eta\epsilon\delta} S_{\alpha\delta}^{\gamma\eta}(\theta) \tag{2.5}$$

(where $g_{\alpha\beta\gamma}$ can be cancelled on both sides whenever it is nonzero). Here the couplings $g_{\alpha\beta\gamma}$ are defined (at worst up to signs) in terms of the residues at the bound-state poles,

$$\text{Res } S_{\alpha\gamma}^{\beta\mu}(\theta) \bigg|_{\theta=2\pi i/3} = i g_{\alpha\beta\gamma} g_{\alpha\mu\gamma} \tag{2.6}$$

Note that (2.3) implies that the residue of the crossed-channel pole is just the opposite of the direct-channel one (2.4),

$$\text{Res } S_{\beta\mu}^{\alpha\gamma}(\theta) \bigg|_{\theta=\pi i/3} = -i g_{\alpha\beta\gamma} g_{\alpha\mu\gamma} \tag{2.7}$$

If all couplings $g_{\alpha\beta\gamma}$ are real, the $S$-matrix is said to be “1-particle unitary”. Parity invariance, crossing symmetry, and the bootstrap equations imply that the $(g_{\alpha\beta\gamma})^2$ are
symmetric in $\alpha, \beta$ and $\gamma$, as long as all $S^{\beta\gamma}_{\alpha}(0) = -1$ (cf. [14] for an analogous statement in diagonal $S$-matrix theories of ordinary particles). The latter presumably holds in all bosonic kink theories, see eq. (2.10) below. For fermionic theories, where $S^{\beta\gamma}_{\alpha}(0) = +1$, there are some subtle signs that have to be taken into account, already in the definition of the couplings (2.6) (cf. [25] for details in the case of “ordinary” scattering theories).

2.2. The Bethe-Yang Equations

Consider a factorizable scattering theory on a finite space with periodic boundary conditions, in other words on a cylinder. We are interested in multi-particle energy levels as a function of the “volume of the world”, i.e. the circumference $R$ of the cylinder. In an integrable theory particle number is conserved and so we can talk about $N$-particle states for any fixed $N$ — at least as long as the notion of particles makes any sense in finite volume, namely when $R \gg 1/m$ and the particles are far apart. One should think of an $N$-particle energy eigenstate in large volume as a stationary scattering state, i.e. a superposition of $N$-kink states in our case, which is invariant under the scattering of the various kinks. Such a state is characterized by a set of rapidities $\{\theta_1, \ldots, \theta_N\}$, since they are conserved in the scattering processes. (Note that because of “mixing” between different particles of the same mass, the $\theta_i$ cannot be assigned to specific particles).

Let $\psi = \psi(x_1, \ldots, x_N)$ be the wavefunction of an $N$-particle state, and let $\psi^\alpha \equiv \psi^{\alpha_1 \ldots \alpha_N}$ denote its components with respect to the basis states $|K_{\alpha_1\alpha_2}(\theta_1)K_{\alpha_2\alpha_3}(\theta_2) \ldots K_{\alpha_N\alpha_1}(\theta_N)\rangle$. Thinking of the basis vectors as $in$-states we must have $\theta_1 > \theta_2 > \ldots > \theta_N$. The number of $N$-particle states (for fixed $\theta_i$) with periodic boundary conditions will be denoted by $d_N$.

The behaviour of $\psi$ when its arguments $x_i$ are far apart can be expressed in terms of scattering amplitudes. Periodic boundary conditions are imposed by, roughly speaking, stipulating that the wavefunction change only by a factor of $(-1)^F$ when “carrying a kink $K_{\alpha_k\alpha_{k+1}}(\theta_k)$ once around the cylinder” (Here $(-1)^F = \pm 1$ for kinks that are bosons or fermions, respectively.) This leads to the following equations for the allowed $\psi$

$$e^{iRm \sinh \theta_k} \sum_{\alpha: \; \alpha_k = \beta_{k+1}} \tilde{T}_k(\theta_1, \ldots, \theta_N)^{\beta}_\alpha \psi^\alpha = (-1)^F \psi^\beta, \quad k = 1, \ldots, N, \quad (2.8)$$

4 Of course, this should not be literally understood as a physical process, but rather as a mnemonic describing a sequence of rewritings of the wavefunction.
where

\[ \tilde{T}_k(\theta_1, \ldots, \theta_N)^\beta_\alpha = \prod_{i \neq k}^N S_{\beta_i \alpha_{i+1}}^\alpha_{\beta_i} (\theta_k - \theta_i) \quad . \]

(2.9)

Here \(\alpha_{N+1} = \alpha_1\) and \(\beta_{N+1} = \beta_1\). (A graphical illustration of a matrix very closely related to \(\tilde{T}_k\) will be given below.) Note that there is no factor corresponding to \(i = k\) in (2.9), since \(“K_{\alpha_k \alpha_{k+1}}\) does not scatter with itself when taken around the circle”. We will refer to (2.8) as the Bethe-Yang equations (cf. [27] for analogous equations in a non-relativistic theory of “ordinary” particles).

We would like to rewrite (2.8) in a more convenient way, getting rid of the restriction on the sum over the multi-index \(\alpha\). To do so, note that unitarity implies \(\sum_\delta (S_{\alpha\beta}^{\gamma\delta}(0))^2 = 1\). In all kink theories we know this is satisfied in the simplest possible way, namely the \(S\)-matrix at zero relative rapidity satisfies \(S_{\alpha\beta}^{\gamma\delta}(0) = S_{\alpha\gamma}^{\gamma\gamma}(0) \delta_{\gamma\delta} = \pm \delta_{\gamma\delta}\). [In ordinary \(S\)-matrix notation this reads \(S_{ab}^{cd}(0) = \pm \delta_{ac} \delta_{bd}\), i.e. total reflection in the low-energy limit.]

Furthermore, as a slight generalization of the arguments and observations in [13][14], we claim that all (non-vanishing) amplitudes for the scattering of kinks in the same multiplet satisfy

\[ S_{\alpha\beta}^{\gamma\gamma}(0) = \mp (-1)^F \quad , \]

(2.10)

since it amounts to an exclusion principle, i.e. \(\theta_i \neq \theta_j\) for all \(i \neq j\). Eq. (2.10) and the analogous statement in non-kink language is true in all consistent (1+1)-dimensional QFTs we are aware of, except for free bosons, which are somewhat singular from various points of view.

We can now rewrite (2.8) as

\[ e^{iRm \sinh \theta_k} \sum_\alpha T(\theta_k | \theta_1, \ldots, \theta_N)^\beta_\alpha \psi^\alpha = -\psi^\beta \quad k = 1, \ldots, N \quad , \]

(2.11)

where there is no restriction on the sum over \(\alpha\). Here \(T\) is the \(N\)-particle transfer matrix, the analog of the (inhomogeneous) row-to-row transfer matrix (of an IRF model in our kink context) in statistical mechanics [28],

\[ T(\theta)^\beta_\alpha \equiv T(\theta | \theta_1, \ldots, \theta_N)^\beta_\alpha = \prod_{i=1}^N S_{\beta_i \alpha_{i+1}}^\alpha_{\beta_i} (\theta - \theta_i) \quad , \]

(2.12)

presented graphically in fig. 4.
Fig. 4: Graphical description of the $N$-kink transfer matrix (2.10). Each vertex corresponds to a 2-kink scattering amplitude, with the time axis to be thought of as pointing in the NE-direction.

When calculating the finite-volume ground state energy of the theory using the thermodynamic Bethe Ansatz technique, it is the $N \to \infty$ limit of this transfer matrix that must be considered. Taking the thermodynamic limit, where also $R \to \infty$ with $N/R$ fixed, certain simplifications occur which often allow one to determine the form of the dominant (in the thermodynamic sense) eigenvalues, and derive integral equations for the distributions of the $\theta_i$ minimizing the free energy; see [19] [20] for details and examples.

Returning to finite $N$, for given $R$ the Bethe-Yang equations (2.11) have solutions only for special $\theta_k = \theta_k(R)$. These equations therefore determine the wavefunctions $\psi$ and the (total) energies and momenta

$$E = \sum_{k=1}^{N} m \cosh \theta_k , \quad P = \sum_{k=1}^{N} m \sinh \theta_k$$

(2.13)

dominate-volume eigenstates.

A comment on the range of validity of eq. (2.13) is necessary. The Yang-Baxter equations and unitarity imply that for any permutation $Q$

$$\tilde{T}_{Q_1}(\theta) \cdot \tilde{T}_{Q_2}(\theta) \cdot \ldots \cdot \tilde{T}_{Q_N}(\theta) = 1 ,$$

(2.14)

so that eqs. (2.8) lead to the quantization of the momentum $P$ of a state as given in (2.13),

$$e^{iPR} = ((-1)^F)^N .$$

(2.15)

This equation is in fact exact in any QFT, irrespective of the Bethe-Yang equations, because of translational invariance and periodicity. Therefore eq. (2.13) for the momentum is exact. On the other hand, the above expression for the energy of a state is of course not exact for finite $R$ — we are neglecting contributions due to the fact that the behaviour
of the wave function $\psi$ can be expressed in terms of scattering amplitudes only when all particles are far apart, as well as tunneling and polarization effects (cf. sect. 1). Fortunately, these effects decay exponentially with $R$ in a massive theory, whereas for a multi-particle state the leading finite-size corrections to the energy levels determined by (2.11)–(2.13) are $O(1/R^2)$, see subsect. 2.4. All powerlike dependence on $R$ is contained in (2.11)–(2.13).

2.3. General properties of $T(\theta)$

The most important property of the transfer matrices $T(\theta)$, eq. (2.12), is that they commute for different $\theta$, with fixed $\theta_1, \ldots, \theta_N$. This follows from the Yang-Baxter equations. Any commuting family of transfer matrices, such as that occurring in (2.11), therefore has a common set of eigenvectors $\psi^{(s)}(\theta_1, \ldots, \theta_N)$ with eigenvalues $\lambda^{(s)}(\theta|\theta_1, \ldots, \theta_N)$, $s$ labelling the different eigenvectors.\(^5\) In the cases we will consider the transfer matrix will be meromorphic in all its arguments (with no poles for real $\theta, \theta_i$).

Real analyticity and unitarity of $S(\theta)$ imply that the transfer matrix $T(\theta)$ is unitary if all $\theta - \theta_i \in \mathbb{R}$. The solutions of (2.11) we are interested in correspond to physical states, so they involve only real $\theta_i$. Hence all eigenvalues of $T$ in (2.11) are of magnitude 1. This is of course necessary for self-consistency of (2.11).

Let us denote $\{\theta_1, \ldots, \theta_N\}$ by $\theta$, and introduce the notation

$$
T_k(\theta) \equiv T(\theta_k|\theta_1, \ldots, \theta_N) \\
\lambda^{(s)}_{k}(\theta) \equiv \lambda^{(s)}(\theta_k|\theta_1, \ldots, \theta_N) \\
\delta^{(s)}_{k}(\theta) \equiv \delta^{(s)}(\theta_k|\theta_1, \ldots, \theta_N) = -i \ln \lambda^{(s)}_{k}(\theta)
$$

\(^5\) In general it is of course not easy to diagonalize the transfer matrices (note that the $S$-matrices and therefore the transfer matrices of kink theories are never diagonal). In theories with a large global symmetry, in particular non-kink theories, this task simplifies. Consider, for instance, the $O(3)$ non-linear $\sigma$-model discussed in [23], where the spectrum consists of an isospin $I=1$ triplet. In the 9-dimensional space of 2-particle states we can change to a basis of eigenstates of $I$ and $I_3$, in which the 2-particle transfer matrix is diagonal. Furthermore, since the basis change is accomplished by a matrix with rapidity-independent coefficients, the eigenvalues (which are 1-, 3- and 5-fold degenerate) will simply be linear combinations of the 2-particle $S$-matrix elements. In contrast, for a generic factorizable $S$-matrix theory there are not enough global charges to diagonalize even the 2-particle transfer matrix, and consequently the change to a diagonal basis will involve $\theta$-dependent coefficients.
for the transfer matrix, its eigenvalues and their phases, respectively. The $2\pi \mathbb{Z}$ ambiguity in the phases has to be fixed at some, say real, $\theta$ by some convention; everywhere else the phases are then uniquely determined.

Next note that real analyticity implies

$$T_k(-\theta) = T_k(\theta)^* \quad \text{for } \theta \in \mathbb{R}^N. \quad (2.17)$$

Therefore the eigenvalues come in pairs $\lambda_k(s)(\theta), \lambda_k(\bar{s})(\theta)$ related by $\theta \to -\theta$ and complex conjugation, or they are invariant under these operations (we then set $s = \bar{s}$)

$$\lambda_k(s)(-\theta) = \lambda_k(\bar{s})(\theta)^*, \quad \delta_k(s)(-\theta) = -\delta_k(\bar{s})(\theta) \pmod{2\pi} \quad \text{for } \theta \in \mathbb{R}^N. \quad (2.18)$$

Here $s = \bar{s}$ is possible only if $\lambda_k(s)(0) \in \{\pm 1\}$, obviously. That the converse is also true follows from Schwarz’s reflection principle. The same argument also shows that if we define

$$\tilde{\delta}_k(s)(\theta) \equiv \delta_k(s)(\theta) - \delta_k(s)(0), \quad (2.19)$$

where $\delta_k(s)(0) \equiv \delta_k(s)(0)$, which is clearly independent of $k$, then

$$\tilde{\delta}_k(\bar{s})(\theta) = \tilde{\delta}_k(s)(\theta) = -\tilde{\delta}_k(s)(-\theta) \quad \text{for } \theta \in \mathbb{R}^N. \quad (2.20)$$

Let $\psi(\theta)$ be a solution of (2.11), $T_k(\theta)\psi(\theta) = \lambda_k(s)(\theta)\psi(\theta)$, $k = 1, \ldots, N$. Complex conjugation, using (2.17)–(2.18), then gives for real $\theta$

$$T_k(-\theta)\psi(\theta)^* = \lambda_k(\bar{s})(-\theta)\psi(\theta)^*, \quad k = 1, \ldots, N. \quad (2.21)$$

We thus see explicitly that solutions to the Bethe-Yang equations come in pairs related by parity, as expected. Namely, if $\theta$ is a solution so is $-\theta$. These two solutions are distinct if $\theta$ and $-\theta$ are different as unordered sets; they are exactly degenerate in energy (this will also be true for the exact levels when tunneling and off-shell effects are taken into account) and belong to sectors of opposite total momentum. This is useful, for instance, in identifying degenerate and non-degenerate levels in the zero momentum sector. To give an example relevant for sect. 4.2, note that in this sector 3-particle states with rapidities $\{\theta, 0, -\theta\}$ will

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6 To prove this we need to know that the eigenvalues $\lambda_k(s)(\theta)$ are real for at least part of the imaginary $\theta$ “axis”. In the cases of physical interest, where solutions to the Bethe-Yang equations (2.8) exist for real $\theta$, and by analyticity also in a complex neighborhood of $\theta = 0$, it is clear from (2.8) that the $\lambda_k(s)(\theta)$ have this property.
not be degenerate, whereas a level $\{\theta_1, \theta_2, \theta_3\}$ with all $\theta_k \neq 0$ will be degenerate with the state with rapidities $\{-\theta_3, -\theta_2, -\theta_1\}$.

There is one more general property of the transfer matrix that we should mention. As we will see in sects. 3–4, it is of some physical interest to consider what happens if we change the $S$-matrix as follows:

$$S^\gamma_\alpha S^\delta_\beta(\theta) \rightarrow \frac{\chi^\gamma(\theta) \chi^\delta(\theta)}{\chi^\alpha(\theta) \chi^\beta(\theta)} S^\gamma_\alpha S^\delta_\beta(\theta).$$

(2.22)

Here the $\chi^\alpha(\theta)$ are functions which are, say, meromorphic in $\theta$, but otherwise arbitrary at this point. From a lattice model point of view, where the $S$-matrix elements correspond to Boltzmann weights of an IRF model, such a change might be referred to as a “gauge transformation”, since the partition function with periodic boundary conditions does not change, obviously. A slightly stronger statement is that the eigenvalues of the transfer matrix $T(\theta)$ do not change under such a transformation (the eigenvectors do, though). This is physically interesting since it implies that the finite-volume multi-kink spectrum of two scattering theories differing just by factors as in (2.22) will be identical, though perhaps only up to terms that are exponentially small for large volume.

To explicitly prove that the eigenvalues of $T(\theta)$ do not change, note that under the above transformation

$$T^\beta(\theta)^\alpha \rightarrow \prod_{i=1}^N \frac{\chi^\alpha_i(\theta - \theta_i) \chi^\beta_{i+1}(\theta - \theta_i)}{\chi^\beta_i(\theta - \theta_i) \chi^\alpha_{i+1}(\theta - \theta_i)} T^\beta(\theta)^\alpha.$$  

(2.23)

This implies that the diagonal elements of any power of $T(\theta)$ do not change. In particular, the trace of any power of $T(\theta)$ stays the same. Since the traces of the first $n$ powers of an

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7 There is another mechanism by which exactly degenerate levels can arise, namely if the common eigenspace of all $T_k(\theta)$ is 2- or higher-dimensional for some eigenvalue $\lambda_k(\theta)$ (at least on some sufficiently large submanifold of $\theta \in \mathbb{R}^N$). This happens in theories with a sufficiently large global symmetry, cf. footnote 5.

8 And using the results of [10][11] one can check that the same is true for the leading large-volume finite-size mass corrections (ignoring tunneling effects, for which we are unable to make such a statement, cf. sect. 1).
$n \times n$ matrix provide a complete set of invariants\textsuperscript{9} we have proved that the eigenvalues of $T(\theta)$ do not change.

In physical applications the interest in (2.22) is due to the following. Methods like the quantum-group approach (cf. sect. 3 for some brief remarks) allow one to conjecture “proto-$S$-matrices” for certain QFTs. These objects are real analytic and satisfy the Yang-Baxter equations, but are not necessarily unitary or crossing symmetric. Unitarity can often be restored by multiplying the “proto-$S$-matrix” by a suitably chosen function of $\theta$. The crossing properties can be changed by factors as in (2.22). For the transformed $S$-matrix to still satisfy the Yang-Baxter equations and be real analytic, the $\chi_\alpha(\theta)$ have to be of the form

$$
\chi_\alpha(\theta) = \frac{\rho_\alpha}{\rho_\gamma} \frac{\rho_\delta}{\rho_\beta} S_{\alpha\beta}(i\pi - \theta) \rightarrow \frac{\rho_\alpha\rho_\beta}{\rho_\gamma\rho_\delta} \frac{S_{\alpha\beta}(i\pi - \theta)}{S_{\gamma\delta}(i\pi - \theta)}.
$$

Note that for theories in which the kinks are bound states of themselves the couplings defined in (2.6) transform as $g_{\alpha\beta\gamma} \rightarrow (\frac{\rho_\alpha\rho_\beta}{\rho_\gamma\rho_\delta})^{1/6} g_{\alpha\beta\gamma}$ under (2.22), assuming the above form of the “crossing factors” $\chi_\alpha(\theta)$. In particular, the symmetry of the couplings in $\alpha, \beta, \gamma$, and the equality (up to a sign) of the direct and crossed channel residues of bound-state poles, are valid only if strict crossing symmetry (2.3) holds. On the other hand, one can check that the bootstrap equations (2.5) are not affected, given the above $\chi_\alpha(\theta)$. Examples where the crossing factors are important will be encountered in sects. 3–4.

2.4. Solution of the Bethe Ansatz equations

For any given $\theta = \{\theta_1, \ldots, \theta_N\}$ the $N$ transfer matrices $T_k(\theta)$ will have $d_N$ common eigenvectors $\psi^{(s)}(\theta)$ with eigenvalues $\lambda_{k}^{(s)}(\theta)$ of magnitude 1. For a given $R$ and eigenvector $\psi^{(s)}(\theta)$ the Bethe-Yang equations restrict the allowed $\theta_k$ of a physical state to the solutions of

$$
r \sinh \theta_k + \delta^{(s)}(\theta_k|\theta_1, \ldots, \theta_N) = 2\pi n_k, \quad n_k \in \mathbb{Z} + \frac{1}{2}, \quad k = 1, \ldots, N,
$$

\textsuperscript{9} Except possibly for a discrete set of exceptional values of the traces. But in our case, where the eigenvalues are (branches of) meromorphic functions, the possibility of such exceptional values can be ignored.
which we will refer to as the Bethe Ansatz (BA) equations, to distinguish them from the Bethe-Yang equations (2.11). Here we introduced the dimensionless “volume” of the world $r = m R$, measured in units of the Compton wavelength of the kinks.

We assume that for given $s$ and $\{n_k\}$ eqs. (2.25) have a unique solution $\theta_k = \theta_k(r)$, $k = 1, \ldots, N$. At least physically this is clear, by “continuity of the infinite-volume limit”: As $r \to \infty$ all $\theta_k \to 0$, and the unique solution is $\theta_k = (2\pi n_k - \delta^{(s)}(0))/r + \mathcal{O}(1/r^2)$.

The level of “type” $s$ and “quantum numbers” $n_k$ will be denoted by $s(n_1, \ldots, n_N)$. The allowed $n_1, \ldots, n_k \in \mathbb{Z} + \frac{1}{2}$ are subject to the constraints that in a sector of total momentum $P = 2\pi n/R$

\[ n + \frac{N}{2\pi} \delta^{(s)}(0) = \sum_{k=1}^{N} n_k, \tag{2.26} \]

and $n_1 > n_2 > \ldots > n_N$, to comply with the exclusion principle.

It will often be more convenient to label a state not by the quantum numbers $n_k$, but rather by $\tilde{n}_k \equiv n_k - (2\pi)^{-1}\delta^{(s)}(0) \in \mathbb{Z} + \frac{1}{2} - (2\pi)^{-1}\delta^{(s)}(0)$. With these quantum numbers on the r.h.s of (2.25), one should use the phase shifts $\tilde{\delta}_k^{(s)}(\theta)$ of (2.19) on the l.h.s. The advantage of the $\tilde{n}_k$ is that now simply $n = \sum_{k=1}^{N} \tilde{n}_k$. We will denote a level $s(n_1, \ldots, n_N)$ equivalently as $s(\tilde{n}_1, \ldots, \tilde{n}_N)$.

How does one solve eq. (2.25) to obtain the $\theta_k = \theta_k(r)$ characterizing a given level? In general this is of course only possible numerically. One can proceed as follows: For given $r$, type $s$, and allowed $\{n_k\}$, rewrite eqs. (2.25) as

\[ \theta_k = \arcsinh\left(\frac{2\pi n_k - \delta^{(s)}(\theta_k|\theta_1, \ldots, \theta_N)}{r}\right) \quad k = 1, \ldots, N, \tag{2.27} \]

and then iterate these equations till they converge to a fixed point. One subtlety is that when the phases $\tilde{\delta}_k^{(s)}(\theta)$ are determined from a numerical diagonalization of $T_k(\theta)$, one must make sure not to “loose track” of a phase when it intersects other phases, or ranges over a region larger than $2\pi$. For the iteration to converge it is perhaps necessary to modify the most naive recipe just described on a case by case basis.

A very simple example, that in practice is quite important and can be treated rather explicitly, is that of 2-kink states in the sector of zero total momentum. With $n \equiv n_1 \in \mathbb{Z} + \frac{1}{2}$, $\theta \equiv \theta_1$, $\delta(\theta) \equiv \delta^{(s)}(\theta|\theta, -\theta)$, we have the following parametric representation of the 2-kink energy in finite volume

\[ (r, E)(\theta) = \left(\frac{2\pi n - \delta(\theta)}{\sinh \theta}, 2m \cosh \theta\right). \tag{2.28} \]
Solving for $\theta = \theta(r)$ in a $\frac{1}{r}$ expansion we find

$$\frac{E(r)}{m} = 2 + \left(\frac{2\pi \tilde{n}}{r}\right)^2 \left[1 - \frac{2\delta'}{r} + \frac{3\delta'^2 - \pi^2 \tilde{n}^2}{r^2} - \frac{4\delta'^3}{r^3} + \frac{(4\pi^2 \tilde{n}^2 / 3)(\delta''' - 4\delta''')}{r^3}\right] + \mathcal{O}\left(\frac{1}{r^6}\right)$$

(2.29)

where $2\pi \tilde{n} = 2\pi n - \delta$, and $\delta, \delta', \delta'''$ are the phase shift and its derivatives evaluated at $\theta = 0$ (all even derivatives vanish there). This expansion can of course easily be extended to higher orders. Even though this expansion has a finite radius of convergence around $\frac{1}{r} = 0$, it is usually not particularly helpful in practice when comparing with numerical results for the finite-volume spectrum obtained using a “small-volume method” like the TCSA, since the latter typically becomes inaccurate before the above expansion is useful. Instead, one should employ (2.28) directly, which is correct for all $r$ up to terms exponentially small in $r$; then there is usually a sizable “window of overlap” with the numerical small-volume methods, as we will see in the next sections.

Note that for arbitrary levels the first nontrivial term in the large-$r$ expansion is $\mathcal{O}(r^{-2})$, because of the exclusion principle. Two different levels with the same $\{\tilde{n}_k\}$ will generically be split at $\mathcal{O}(r^{-3})$, which always dominates off-shell and tunneling contributions.

Finally, we remark that even though energywise the infinite-volume limit is rather trivial — any given level with $N$ particles approaches $Nm$ — the corresponding eigenstates, being eigenvectors of $T(0|0)$, remain nontrivial and distinct, generically. Note that $T(0|0)$ is a $d_N \times d_N$ permutation matrix; its eigenvalues are therefore some collection of $d_N$-th roots of unity.

3. $\phi_{1,3}$-Perturbed Unitary Minimal CFTs

As for any other perturbed CFTs, we have to specify the partition function of the unperturbed CFT and the direction of the perturbation. This is crucial for determining the nature of the theory, in particular its global symmetry (which may be spontaneously broken in infinite volume) and whether the theory is massive or massless. To simplify the discussion below we will use the following notation: Let $X_m \pm \phi_{p,q}$ denote the $\phi_{p,q}$-perturbation, in the positive or negative direction respectively, of the unitary minimal
CFT \cite{29} of central charge $c_m = 1 - \frac{6}{m(m+1)}$ and modular invariant partition function (MIPF) in the $X = A, D, E$ series \cite{30} \cite{10}.

Let us first briefly mention the different types of arguments which are helpful in discovering the kink structure of perturbed CFTs. They give us a better qualitative understanding of the QFT, and sometimes even provide crucial hints for constructing its $S$-matrix.

(i) Landau-Ginzburg (LG) formulation: It has proven useful in many cases to describe CFTs by LG potentials involving a small number of “fundamental” fields \cite{31} \cite{32} \cite{33} \cite{34}. The perturbed CFT is then obtained by adding to the unperturbed potential a term corresponding to the LG realization (in the CFT) of the perturbing field. In the case of supersymmetry-preserving integrable perturbations of $N=2$ super-CFTs (for which the potential is believed not to renormalize), the resulting equations of motion can be treated like classical field equations and analyzed for kink solutions. For several models \cite{4} \cite{20} the results have been used directly to construct the exact quantum spectrum and $S$-matrix. For theories without $N=2$ SUSY the LG formulation is of more limited use, since the structure of composite operators and the renormalization properties of the theories are more complicated. Nevertheless, in some cases it does provide intuition about the kink structure \cite{1}.

(ii) Quantum-group motivated restrictions: In refs. \cite{2} \cite{35} \cite{5} certain perturbed CFTs are identified as restricted lagrangian QFTs having $SL_q(2)$ symmetry. The restrictions are made at specific couplings in the lagrangian which correspond to $q$ being a root of unity. The kink structure of the restricted theory is then dictated by the tensor product rules of the (finite number of) non-singular finite-dimensional highest-weight irreps of the algebra $U_q[su(2)]$ (see e.g. \cite{36}). Namely, the degenerate vacua are labelled by the spins $j$ of (possibly a subset of) these representations, and a multiplet of kinks forms a representation of some spin $s$; kinks in the multiplet interpolate between the vacuum $j_1$ and vacua $j_2$ appearing in the decomposition of $j_1 \otimes s$ into $U_q[su(2)]$ irreps. The quantum group machinery enables one to go far beyond the determination of the kink structure. Its disadvantage is that starting from a given perturbed CFT, it is not always straightforward

\footnote{Since certain $\phi_{p,q}$ are doubled in the $D_m$ unperturbed CFTs, one has to specify more precisely the perturbation in these cases. The only case this subtlety arises below is that of $D_5 \pm \phi_{1,3}$, by which we will mean the perturbation by the $\mathbb{Z}_2$-even combination of the two $\phi_{1,3}$ fields. Also, the sign choice for the direction of a perturbation is just a convention for the sign of the perturbing field. We use the conventions that are by now standard in the literature.}
and easy to construct the restricted QFT which it is to be identified with (the intuition for choosing the unrestricted theory, coming from the Feigin-Fuchs construction of the CFT, is not always clear).

(iii) Statistical Mechanics: Since perturbed CFTs describe two-dimensional lattice models in the scaling region around a critical point, knowledge of the phase structure of the off-critical lattice models can give useful hints about the spectrum of the corresponding perturbed CFT.

Returning to the family of $\phi_{1,3}$-perturbed minimal CFTs, we will now briefly review some of the known facts about the theories $X_m \pm \phi_{1,3}$. $A_3 \pm \phi_{1,3}$ ($= A_3 \pm \phi_{2,1}$) are the two phases of the so-called Ising field theory (IFT) \cite{37}, which can be obtained by taking the scaling limit of the Ising model at zero magnetic field from above or below the critical temperature, respectively. (From the viewpoint of the underlying lattice model, the two theories $A_3 \pm \phi_{1,3}$ are therefore related by duality.) $A_3 + \phi_{1,3}$ is an interacting $\mathbb{Z}_2$-symmetric theory of a single massive boson with factorizable $S$-matrix $S(\theta) = -1$. In $A_3 - \phi_{1,3}$ the $\mathbb{Z}_2$ symmetry is spontaneously broken, and the theory is described by the simplest (factorizable) $S$-matrix theory of a pair of kinks which interpolate between two degenerate vacua. Though very simple, it is instructive to discuss this theory in some detail, which we will do in subsect. 3.1.

The theories $A_m + \phi_{1,3}$, $m \geq 4$, are believed \cite{38,32,33,39,17} to be massless and flow to the CFTs $A_{m-1}$ in the infrared (IR). It has recently been argued \cite{40,41} that similarly $X_m + \phi_{1,3}$ flows to $X_{m-1}$ for $X = D$ ($m \geq 6$) and $X = E$ ($m = 12, 18, 30$), while $D_5 + \phi_{1,3}$ flows to the CFT $A_4$ \cite{12,88,11}. On the other hand, $X_m - \phi_{1,3}$ is believed to be a massive theory for any $m \geq 4$ (and any $X$ for which this perturbed CFT is defined). Factorizable $S$-matrix theories of a multiplet of $2(m - 2)$ kinks have been proposed for $A_m - \phi_{1,3}$ independently in \cite{1} (for $m = 4, 6$) and \cite{2} (for $m \geq 4$).\cite{11}

In \cite{2} the scattering theories are constructed by a truncation of the Hilbert space of the sine-Gordon theory at certain couplings, based on the quantum group ($SL_q(2)$) symmetry

\footnote{S-matrices for the theories $X_m - \phi_{1,3}$ with $X \neq A$ have not been explicitly discussed in the literature, except for $D_6 - \phi_{1,3}$ for which an $S$-matrix related through “orbifolding” (cf. \cite{43} for a statistical mechanics analogy) to that of $A_6 - \phi_{1,3}$ was constructed \cite{1}. They can presumably \cite{20} be constructed (for $m$ even) by “unitarizing and crossing-symmetrizing” the Boltzmann weights of the critical RSOS lattice models of \cite{14}.}
of the model. The restriction leading to $A_m - \phi_{1,3}$ is obtained when $q = \exp(i\pi\frac{m+1}{m})$, and so the vacua $\alpha$ are labelled by the spins $0, 1/2, \ldots, (m/2) - 1$ of the $U_q[sl(2)]$ highest weight irreps. A kink carries spin 1/2, and so only vacua $\alpha$ and $\beta$ such that $|\alpha - \beta| = 1/2$ can be linked by a single kink $K_{\alpha\beta}(\theta)$. This situation is graphically described by the Dynkin diagram of the simple Lie algebra $sl(m)$ where the dots stand for the $m - 1$ vacua and the links for the pair of kinks interpolating between them. The number of types of $N$-kink states on a circle, $d_N$, can easily be expressed in terms of the incidence matrix $I$ of this diagram: Defining $I_{\alpha\beta} = 1$ (0) if the vacua $\alpha$ and $\beta$ are (not) linked, we have $d_N = \text{tr}(I^N) = \sum \lambda^N$, where $\lambda$ are the eigenvalues of $I$. For the theory $A_m - \phi_{1,3}$, therefore, $d_N = \sum_{k=1}^{m-1} [2 \cos(\pi k/m)]^N$ (note that $d_N = 0$ if $N$ is odd, as it should).

The $S$-matrix conjectured for $A_m - \phi_{1,3}$ is given by

$$S_{\alpha\beta}^\gamma\delta = -\frac{U(\theta)}{\pi i} \left( \frac{[2\gamma + 1][2\delta + 1]}{[2\alpha + 1][2\beta + 1]} \right)^{-\frac{q}{2\pi i}} \times \left\{ \left( \frac{[2\gamma + 1][2\delta + 1]}{[2\alpha + 1][2\beta + 1]} \right)^{1/2} \sin \left( \frac{\theta}{m} \right) \delta_{\alpha\beta} + \sinh \left( \frac{\pi i - \theta}{m} \right) \delta_{\gamma\delta} \right\},$$

where

$$U(\theta) = \Gamma \left( \frac{1}{m} \right) \Gamma \left( 1 + \frac{i\theta}{\pi m} \right) \Gamma \left( 1 + \frac{i(\pi i - \theta)}{\pi m} \right),$$

$$R_k(\theta) = \frac{\Gamma \left( \frac{2k + 1}{m} + \frac{i\theta}{\pi m} \right) \Gamma \left( 1 + \frac{2k}{m} + \frac{i\theta}{\pi m} \right)}{\Gamma \left( \frac{2k + 1}{m} + \frac{i\theta}{\pi m} \right) \Gamma \left( 1 + \frac{2k - 1}{m} + \frac{i\theta}{\pi m} \right)},$$

and we use the $q$-number notation

$$[a] = [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}} = (-1)^{a-1} \frac{\sin(\frac{\pi a}{m})}{\sin(\frac{\pi}{m})}$$

(which is unambiguous for integer $a$). The $S$-matrix (3.1) is invariant under the $\mathbb{Z}_2$-symmetry operation that takes any vacuum $\alpha$ to $(m/2) - 1 - \alpha$. Note the “crossing factors” in the $S$-matrix (3.1) (cf. the discussion in subsect. 2.3) which ensure strict crossing symmetry (2.3).

Before embarking on a discussion of specific models in the series $A_m - \phi_{1,3}$, let us point out one feature that is common to the multi-particle spectrum of all these theories. Namely, as we now show, the eigenvalues of the $N$-particle transfer matrix always come in pairs of opposite sign: Note that any $N$-kink state, written as $\alpha_1 \alpha_2 \ldots \alpha_N$ as a shorthand
for $|K_{\alpha_1 \alpha_2}(\theta_1) \ldots K_{\alpha_N \alpha_1}(\theta_N)\rangle$, has integer (half-integer) $\alpha_i$ either in all even or in all odd positions. In other words, there are two classes of vacuum configurations. Let us order our basis states such that the first half have integer $\alpha_i$ in the even positions $i = 2, 4, \ldots$, say, and the second half have integer $\alpha_i$ in the odd positions. Due to the restrictions on neighboring vacua it is clear that in this basis the transfer matrix is of off-diagonal block form $\begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}$. This implies that for any eigenvector of the transfer matrix, written in block form as $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, there is another one $\begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}$ of opposite eigenvalue, as claimed. When $m$ is odd the two classes of vacuum configurations are actually exchanged by the $\mathbb{Z}_2$ symmetry of the theory, and so the basis of states can be chosen so that $v_1 = v_2$ are eigenvectors of $A_1 = A_2$; each pair of eigenstates consists therefore of a $\mathbb{Z}_2$-even and a $\mathbb{Z}_2$-odd state. This is not the case when $m$ is even, where the two eigenstates in each pair are of the same parity.

### 3.1. Ising Field Theory in the Phase of Spontaneously Broken Symmetry

This is the case $m = 3$ of the above family of theories. There are only two vacua, labelled by $\alpha = 0, 1/2$, which are exchanged by the (spontaneously broken in infinite volume) $\mathbb{Z}_2$ symmetry. These vacua correspond to the all-spins-up and all-spins-down degenerate ground states of the underlying Ising lattice model. There are only two allowed types of 2-kink states, $|K_{0,1/2}(\theta_1)K_{1/2,0}(\theta_2)\rangle$ and $|K_{1/2,0}(\theta_1)K_{0,1/2}(\theta_2)\rangle$, and the two allowed scattering processes are given by the amplitudes $S_{0,1/2,1/2}^{1/2}(\theta) = S_{1/2,0,0}^{0,1/2}(\theta) = -1$.

In general, there are two types of states of any even number $N$ of kinks on the circle, $|K_{0,1/2}K_{1/2,0} \ldots K_{1/2,0}\rangle$ and $|K_{1/2,0}K_{0,1/2} \ldots K_{0,1/2}\rangle$, and none when $N$ is odd. So $d_N = 2$ (0) for $N$ even (odd). The above two types for $N$ even transform into each other under the $\mathbb{Z}_2$ symmetry. Their symmetric and antisymmetric superpositions are the eigenvectors of the transfer matrix $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with eigenvalues 1 and $-1$ respectively. We therefore have $\delta^{(1)} = 0$ and $\delta^{(2)} = \pi$. The BA equations (2.25) immediately lead to the quantization $m \sinh \theta_k = 2\pi \tilde{n}_k / R$ of the momenta in any allowed multi-kink state, where the $\tilde{n}_k$ are half-odd-integers for the $\mathbb{Z}_2$-even sector (type $s = 1$) and integers for the $\mathbb{Z}_2$-odd sector ($s = 2$). The corresponding energies and momenta are therefore simply

$$E(R) = \sum_{k=1}^{N} \sqrt{m^2 + \left(\frac{2\pi \tilde{n}_k}{R}\right)^2} \quad , \quad P = \frac{2\pi}{R} \sum_{k=1}^{N} \tilde{n}_k \quad , \quad (3.4)$$
where \( N \) is restricted to be even and the \( \tilde{n}_k \) are non-coinciding integers or half-odd-integers, depending on the sector as above (\( P \) is therefore always an integral multiple of \( 2\pi/R \)). In addition to these “peculiar” restrictions, the difference between the energy levels \( (3.4) \) and those of a free particle is that the energy in \( (3.4) \) is not exact for the Ising field theory (IFT) when off-shell effects are taken into account.

The Ising field theory in its high- and low-temperature phases (denoted henceforth as \( \text{IFT}^{(\pm)} \)) is to our knowledge the only nontrivial massive QFTs for which the exact energy levels are known. They can be obtained from the partition function on a cylinder of circumference \( R \) and finite length \( L \), with periodic boundary conditions on the spin field in both directions, which can be written as

\[
Z^{(\pm)}(q; r) = \frac{1}{2} q^{e_0(r)} \left\{ \prod_{n \in \mathbb{Z}+1/2} (1 + q^{\epsilon_n(r)}) + \prod_{n \in \mathbb{Z}+1/2} (1 - q^{\epsilon_n(r)}) \right. \\
+ \left. q^{\hat{e}_1(r)} \left[ \prod_{n \in \mathbb{Z}} (1 + q^{\epsilon_n(r)}) \mp \prod_{n \in \mathbb{Z}} (1 - q^{\epsilon_n(r)}) \right] \right\},
\]

(3.5)

Here \( q = e^{-2\pi L/R} \), \( r = mR \) (\( m \) being the mass of the particle/kink in the +/- case), \( \epsilon_n(r) = \sqrt{(\frac{r}{2\pi})^2 + n^2} \), \( \hat{e}_1(r) = e_1(r) - e_0(r) \), and

\[
e_{0,1}(r) = -\frac{r}{4\pi^2} \int_{-\infty}^\infty d\theta \cosh \theta \ln(1 \pm e^{-r \cosh \theta})
\]

(3.6)

are the scaling functions corresponding to the ground state and the first excitation energies \( E_{0,1}(R) = (2\pi/R)e_{0,1}(r) \) in \( \text{IFT}^{(-)} \) (both decaying exponentially to zero as \( r \to \infty \)).

Expanding the expression inside the braces in \( (3.5) \) as \( \sum q^{\hat{e}(r)} \), one can read off all the scaled energy gaps

\[
\hat{e}^{(\pm)}(r) = e^{(\pm)}(r) - e_0(r) = \sum_{n \in \mathcal{N}^{(\pm)}} \epsilon_n(r) + \begin{cases} 0 & \mathbb{Z}_2\text{-even sector} \\ \hat{e}_1(r) & \mathbb{Z}_2\text{-odd sector} \end{cases}.
\]

(3.7)

Here for the \( \mathbb{Z}_2\)-even sector, which is the same in both phases of the theory, \( \mathcal{N}^{(\pm)} \) is any set of an even number \( (|\mathcal{N}^{(\pm)}| = 0, 2, 4, \ldots) \) of distinct half-odd-integers. For the \( \mathbb{Z}_2\)-odd sector the \( \mathcal{N}^{(\pm)} \) are sets of distinct integers with \( |\mathcal{N}^{(+)}| = 1, 3, 5, \ldots \) but \( |\mathcal{N}^{(-)}| = 0, 2, 4, \ldots \). Comparing these expressions with \( (3.4) \), we see that the latter gives the exact energy gaps in the \( \mathbb{Z}_2\)-even sector of IFT\(^{(-)} \). In the odd sector, on the other hand, the results \( (3.4) \) are not exact. But the exponentially small deviation \( (2\pi/R)\hat{e}_1(mR) = \sqrt{\frac{2m}{\pi R}} e^{-mR} [1 + \mathcal{O}((mR)^{-2})] \) is the same for all levels. This feature is apparently special to the IFT.
It is interesting to find the correspondence between the multi-particle states in IFT\((\pm)\), specified by the BA quantum numbers \(\tilde{n}_k \in \mathcal{N}(\pm)\), and the conformal states of the Ising CFT. To do so, recall \([40]\) that the UV limit of the exact scaled energy gaps, \(\hat{e}(0)\), are scaling dimensions in the UV CFT. We therefore compare (3.7) for large \(r\), where it agrees with the BA results (3.4), with its \(r \to 0\) limit. The only “nontrivial” fact we need is \(\hat{e}_1(0) = 1/8\), the rest follows from \(\epsilon_n(0) = |n|\). Concentrating on the kink phase IFT\((-)\), one concludes that the UV scaling dimensions \(\hat{e}^{(-)}(0)\) of \(Z_2\)-even states are integers, thus corresponding to fields in the conformal families of \([\phi_{1,1} = 1]\) and \([\phi_{1,3} = \varepsilon]\) (the dimensions of the ancestor primary fields are \(d_1 = 0\) and \(d_{\varepsilon} = 1\)). More specifically, \(Z_2\)-even \(N\)-kink states in the momentum sector \(P\) “come from” \([I]\) \([\varepsilon]\) if \(N/2 + R_P \pi P\) is even (odd). This decomposition of the \(Z_2\)-even sector into two subsectors reflects the fact that \([I]\) \([\varepsilon]\) is even (odd) with respect to the Kramers-Wannier duality. On the other hand, the whole \(Z_2\)-odd sector originates from \([\phi_{1,2} = \sigma]\) in the UV, as \(d_{\sigma} = 1/8\).

The characterization of multi-particle states in terms of their quantum numbers \(\tilde{n}_k\) on the one hand and their UV limits on the other hand, leads to the following identities for the level degeneracy in \(c = 1/2\) unitary highest weight irreps of the Virasoro algebra:

\[
\chi_n(0) = \text{(number of partitions of } n \text{ into distinct positive half odd integers)} \\
\chi_n(1/2) = \text{(number of partitions of } n + \frac{1}{2} \text{ into distinct positive half odd integers)} \\
\chi_n(1/16) = \text{(number of partitions of } n \text{ into distinct positive integers)} .
\]

Here \(n\) is a non-negative integer (we set \(\chi_0(0) = \chi_0(1/16) = 1\), and \(\chi^{(\Delta)}(q) = q^{\Delta - 1/48} \sum_{n=0}^{\infty} \chi_n^{(\Delta)} q^n\) is the character of the Virasoro irrep of highest weight \(\Delta\) \([17]\).

What allows one to obtain the exact partition function (3.5), and consequently (3.8), is the fact that the IFT can be constructed from the theory of a free massive Majorana fermion by appropriately summing over the four different boundary conditions for the fermion field, cf. eq. (3.3). The case of other integrable perturbed CFTs is much harder; the exact (in any analytic form) full finite-volume spectrum is certainly out of reach by present methods. However, obtaining the correspondence between multi-particle states and UV conformal fields, and ultimately the analog of (3.8), does not really require the knowledge of the exact spectrum at all volumes. The (numerical) TCSA studies together with the large-volume BA analysis described in sect. 2 allow us to find this correspondence for the low-lying levels, as we will see in the examples studied in the rest of this paper.
Our results will extend earlier empirical observations \cite{22} for certain perturbed CFTs described by diagonal scattering theories of “ordinary” particles: It was found that in certain such cases (though not all) \( P=0 \) states of two lightest particles all seem to come from a single conformal family in the UV CFT. Furthermore, in these cases the BA results rather unexpectedly become exact in the \( R \to 0 \) limit. In IFT\(^{(+)} \) this phenomenon is explicitly demonstrated by eqs. (3.4) and (3.7): Zero momentum 2-particle states are in the \( \mathbb{Z}_2 \)-even sector, the BA equations give their exact energies at any \( R \), and from the more general comments above we identify all these states as originating from \( [\varepsilon] \) in the UV.

3.2. The Subleading Thermal Perturbation of the Tricritical Ising CFT

We turn now to the theory \( A_4 - \phi_{1,3} \). Let us label the vacua by \( \alpha = -, 0, + \) instead of 0, 1/2, 1, respectively, as in (3.1). We also let \( \sigma, \sigma' \) take only the values +, −. There are eight possible types of 2-kink processes, described by the amplitudes \( S_{00}^{\sigma\sigma'}(\theta) \) and \( S_{\sigma\sigma'}^{00}(\theta) \) which can be written explicitly as

\[
S_{00}^{\sigma\sigma'}(\theta) = -2^{\frac{\theta}{\pi}} R(\theta) W^{\sigma\sigma'}(\theta) , \quad S_{\sigma\sigma'}^{00}(\theta) = -2^{-\frac{\theta}{\pi}} R(\theta) W_{\sigma\sigma'}(\theta) , \quad (3.9)
\]

where

\[
W^{\pm\pm}(\theta) = \frac{1}{\sqrt{2}} W_{\pm\pm}(\pi i - \theta) = \cosh \frac{\theta}{4} \quad (3.10)
\]

\[
W^{\pm\mp}(\theta) = \frac{1}{\sqrt{2}} W_{\pm\mp}(\pi i - \theta) = -i \sinh \frac{\theta}{4} ,
\]

and \cite{19} (see the first reference in \cite{3} for the factors \( (k - \frac{1}{2}) \))

\[
R(\theta) = \prod_{k=1}^{\infty} (k - \frac{1}{2}) \frac{\Gamma(k + \frac{\theta}{2\pi}) \Gamma(k - \frac{1}{2} - \frac{\theta}{2\pi})}{\Gamma(k - \frac{\theta}{2\pi}) \Gamma(k + \frac{1}{2} + \frac{\theta}{2\pi})} = \frac{1}{\sqrt{\cosh(\theta/2)}} \exp \left\{ \frac{i}{4} \int_{0}^{\infty} \frac{dx}{x} \frac{\sin(x\theta/\pi)}{\cosh^2(x/2)} \right\} . \quad (3.11)
\]

(For explicit computations the alternative form

\[
R(\theta) = \frac{1}{\sqrt{\cosh(\theta/2)}} \exp \left\{ i \int_{0}^{\theta} \frac{d\theta'}{2\pi} \frac{\theta'}{\sinh \theta'} \right\} \quad (3.12)
\]

is most convenient.) Eq. (3.11) is the “minimal” solution to the constraints \( R(\theta) = R(\pi i - \theta) = [\cosh(\theta/2)R(-\theta)]^{-1} \).
The number of types of $N$-kink states in finite volume with periodic boundary conditions is 0 if $N$ is odd, and $d_N = 2^{(N/2)+1}$ if $N > 0$ is even, of which half are even and half odd with respect to the $\mathbb{Z}_2$ symmetry which exchanges the $\sigma = +, -$ vacua.

Now consider 2-kink states of zero momentum, for which we use the basis

$$\{ |K_0(\theta)K_0(-\theta)\rangle, |K_0(\theta)K_0^+(-\theta)\rangle, |K_0(-\theta)K_0^-(\theta)\rangle, |K_0^+(\theta)K_0^-(\theta)\rangle \} \quad (3.13)$$

where $\theta > 0$. In this basis the 2-kink transfer matrix appearing in the first of the two Bethe-Yang equations (2.11) is

$$T(\theta|\theta,-\theta) = (T(-\theta|\theta,-\theta))^{-1} = \begin{pmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \\ a & b & 0 & 0 \\ b & a & 0 & 0 \end{pmatrix}$$

$$\equiv -\begin{pmatrix} 0 & 0 & S^{00}_{-} & 0 \\ 0 & 0 & 0 & S^{00}_{+} \\ S^{0-}_{0} & S^{0+}_{0} & 0 & 0 \\ S^{1-}_{0} & S^{1+}_{0} & 0 & 0 \end{pmatrix} (2\theta) . \quad (3.14)$$

The $\mathbb{Z}_2$-odd eigenvectors are $(1,-1,\pm\sqrt{(a-b)/c}, \mp\sqrt{(a-b)/c})^T$, while the $\mathbb{Z}_2$-even ones are $(1,1,\pm\sqrt{(a+b)/c}, \pm\sqrt{(a+b)/c})^T$. The corresponding eigenvalues are given by $\pm\sqrt{(a-b)c}$ and $\pm\sqrt{(a+b)c}$, respectively, which explicitly read $\pm R(2\theta)\sqrt{\cosh \theta}$ and $\pm i R(2\theta)\sqrt{\cosh \theta f_{1/2}(\theta)}$ where $f_{\alpha}(\theta) = \sinh(\frac{\theta+i\alpha\pi}{2})/\sinh(\frac{\theta-i\alpha\pi}{2})$. As shown in general earlier, the phases come in two pairs, with a difference of $\pi$ between the phases in a pair. Explicitly, one of the phases in the $\mathbb{Z}_2$-odd sector is

$$\delta^{(A)}(\theta) = \delta_R(2\theta), \quad (3.15)$$

while for the $\mathbb{Z}_2$-even sector

$$\delta^{(B)}(\theta) = \delta_R(2\theta) - \frac{1}{2}\arctan(\sinh \theta), \quad (3.16)$$

where $\delta_R(\theta)$, the phase of $R(\theta)$ with the branch choice $\delta_R(0) = 0$, can be easily read off from (3.12). These phases are shown in fig. 5. Note that $\delta^{(A)}(\pm\infty) = -\delta^{(B)}(\pm\infty) = \pm\frac{\pi}{8}$.

The parametric form of the zero momentum 2-kink energies is then

$$(r,E) = \left( \frac{2\pi n - \delta^{(s)}(\theta)}{\sinh \theta}, 2m \cosh \theta \right), \quad s \in \{A,B\}, \quad n \in \frac{1}{2}\mathbb{N}. \quad (3.17)$$

Remember that allowing (positive) integer quantum numbers $n$ as well as half-odd-integers in the above is due to the fact that the eigenvalues of (3.14) come in pairs of opposite sign.
We now briefly discuss 4-particle levels. There are eight ($\theta$-dependent) 4-kink states, so we have four pairs of phases, differing by $\pi$ within a pair. Allowing the $n_k$ labelling a state to be either all integers or half-odd-integers, we only have to consider four types of phases, which we will denote by $s = A, B, C, D$. Types $A$ and $C$ correspond to $\mathbb{Z}_2$-even states whereas $B$ and $D$ to $\mathbb{Z}_2$-odd ones. We will not present analytic expressions for these phases, since they are rather messy. Instead, in fig. 6 we show a typical “cross section” of the diagonal phases. Note that if we use the quantum numbers $\tilde{n}_k$ (cf. subsect. 2.4), then all $\tilde{n}_k \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ for type $A, B, C$, whereas for type $D$ all $\tilde{n}_k \in \mathbb{Z} + \frac{1}{4}$ or $\mathbb{Z} - \frac{1}{4}$. In particular, in the zero momentum sector all levels of type $D$ will be doubly degenerate. The explicit discussion of the lowest 4-particle levels will be presented in the next subsection where we compare them with numerical results.

3.3. Comparison with the TCSA

The truncated conformal space approach [8] (TCSA) is a non-perturbative method to calculate the low-lying energy levels of a perturbed CFT in finite volume. The idea is to truncate the Hilbert space of the CFT to a finite-dimensional subspace on which the hamiltonian of the perturbed theory can be diagonalized numerically. This is to be contrasted with conformal perturbation theory (CPT), where the theory is treated perturbatively in the coupling constant $\lambda$, but the Hilbert space is not truncated. The hamiltonian, obtained from the perturbed CFT action $A_{CFT} + \lambda \int \Phi$ (where the integration is over a cylinder of circumference $R$), simply acts on the truncated Hilbert space as an ordinary matrix

$$\langle j | (H_0 + \lambda V)(R) | i \rangle = \frac{2\pi}{R} \left( (\Delta_i + \bar{\Delta}_i - \frac{c}{12}) \delta_{ij} + (2\pi)^{1-y} \lambda R^y C_{j\Phi i} \delta_{\Delta_i - \bar{\Delta}_i, \Delta_j - \bar{\Delta}_j} \right). \quad (3.18)$$

Here $\{|i\rangle\}$ are (orthonormalized) conformal states, created by fields of left (right) dimension $\Delta_i$ ($\bar{\Delta}_i$), $y = 2 - (\Delta_\Phi + \bar{\Delta}_\Phi)$ is the RG eigenvalue of the perturbing field $\Phi$ which is assumed to be relevant and spinless, i.e. $y > 0$ and $\Delta_\Phi = \bar{\Delta}_\Phi$, and $C_{j\Phi i}$ are the OPE coefficients [29] of the corresponding fields on the plane.

As the truncation is removed the eigenvalues $E_i(R)$ of the above matrix should converge to the exact finite-volume energy levels. There is one subtlety [17] though. If $y \leq 1$ the (bare) hamiltonian $H_0 + \lambda V$ will suffer from UV divergences, so that all of its eigenvalues actually diverge as the truncation is removed. If one wants to avoid the explicit (and hard) renormalization of these divergences, one can still calculate energy gaps $\hat{E}_i(R) = E_i(R) - E_0(R)$ using the TCSA, since the divergences cancel in the differences.
of eigenvalues. In view of this fact it is not surprising that even if \( y > 1 \) energy gaps seem to converge faster than the energy levels as the truncation is removed. We will therefore always calculate gaps. Note that \( y = \frac{4}{5} \) for \( A_4 - \phi_{1,3} \), so this is a case where the energy levels themselves will not converge as the truncation is removed.

The truncation of the Hilbert space is conveniently performed by \( e.g. \) ignoring all states above a certain scaling dimension or level. Truncation effects become important when considering high levels, or if the interaction term gets large compared to the unperturbed term in (3.18). Since the interaction term is proportional to \( \lambda R^y \), the TCSA will deteriorate with increasing \( R \).

The perturbed theory is necessarily non-scale-invariant, \( |\lambda|^{1/y} \) being related to the mass scale \( m \) of the theory. In the cases we will consider, \( m \) will be chosen to be the mass of the kink-multiplet. The coefficient \( \kappa \) relating \( \lambda \) to \( m \), \( |\lambda| = \kappa m^y \), has been determined for some perturbed CFTs to high precision by comparison with the thermodynamic Bethe Ansatz (TBA), in particular \( \kappa = |\lambda|m^{-4/5} = 0.148695516112(3) \) for \( A_4 - \phi_{1,3} \) [19]. In other cases the mass gap has to be determined by estimating the \( R \to \infty \) limit of levels corresponding to 1- or 2-particle states. For more details about the TCSA and some case studies we refer the reader to [8][48][49][11][17] (in particular, TCSA results for \( A_4 - \phi_{1,3} \) were first obtained in [48]).

In fig. 7 we show the first 35 scaled energy gaps \( \hat{e}_i(r) = \frac{R}{2\pi} \hat{E}_i(R) \), \( r = mR \), for \( A_4 - \phi_{1,3} \) as obtained from the TCSA in the zero momentum sector, together with our BA results for 2- and 4-kink levels. Note that at \( r = 0 \) the TCSA is trivially exact, the \( \hat{e}_i(0) \) being the scaling dimensions \( d_i = \Delta_i + \bar{\Delta}_i \) of fields in the UV CFT, and that the UV fields relevant for the 0-momentum sector are all spinless, since up to a factor of \( 2\pi/R \) the momentum of a state on the cylinder becomes the Lorentz spin of the corresponding field in the UV limit. The analytical BA results were obtained as described in subsect. 2.4.

The first two gaps correspond to 0-particle states. In infinite volume there are three degenerate vacua, but in finite volume they split exponentially, due to tunneling between the three wells of the potential. In [17] we presented strong evidence for an exact expression for the first gap \( \hat{e}_1(r) \) in terms of the solution of a TBA-like integral equation (cf. also [16]). As this is, in particular, an example where the precise form of the large-\( r \) behaviour of a 0-particle level is believed to be known, let us mention that this integral equation predicts

\[
\hat{e}_1(r) = \frac{r}{\sqrt{2\pi^2}} K_1(r) - \frac{e^{-2r}}{4\pi^2} \left( \sqrt{\pi r} - \frac{1}{2} - \frac{1}{\pi} + \frac{3}{16} \sqrt{\frac{\pi}{r}} + O\left(\frac{1}{r}\right) \right).
\]  

(3.19)
Unfortunately, as one sees in the figure, the large-$r$ behaviour is swamped by truncation errors.

The next few levels are the first of an infinite series of 2-kink states, which soon begin to overlap with 4-kink states (states of 6 or more kinks appear only at larger energies not shown in the figure). The first of each of the “doublets” of 2-kink states correspond to type $A$, the second to $B$, with the same quantum number $n$; BA results are given for the $n = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ levels. The agreement between the TCSA and the BA results is quite impressive for the low-lying levels; for small $r$, large $r$, and higher levels, deviations should be expected due to the limitations of the two methods. In particular, although we have not indicated any estimated errors for the TCSA results in the figure, we have checked that these errors (as estimated from the variation with truncation level) can explain all deviations at large $r$. Surprisingly, for some levels the BA results are actually exact in the $r \to 0$ limit. We will come back to this point below.

The first 4-kink level corresponds to $\tilde{A}(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$, and the analytical results agree quite spectacularly with the TCSA. For the next 4-kink level, $\tilde{B}(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$, the moderate but noticeable deviations from the TCSA are mainly due to the (in)accuracy of the latter. We identify the next two TCSA levels shown as corresponding to BA type $\tilde{C}(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$, and the degenerate pair $\tilde{D}(\frac{7}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{9}{4}), \tilde{D}(\frac{9}{4}, \frac{1}{4}, -\frac{3}{4}, -\frac{7}{4})$, respectively. That the latter level is exactly degenerate is also clear from the perturbed CFT point of view, see below. Due to the technical difficulties involved in keeping track of the phases of type $C$ and $D$ (cf. the crossings in fig. 6) we have not systematically calculated these energy levels, but just computed them for a few selected $r$ values to convince ourselves that the above identification is correct. It is actually rather clear from fig. 6 that levels of type $B$ and $C$ of the same quantum numbers will be almost degenerate, with level $C$ a bit higher, and that a level of type $D$ will be quite a bit higher than the former for large $r$.

The next three 4-kink levels are all of type $A$. The first one corresponds to $\tilde{A}(2, 1, -1, -2)$, the next one to $\tilde{A}(\frac{5}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{5}{2})$, the last one to the degenerate pair $\tilde{A}(2, 1, 0, -3), \tilde{A}(3, 0, -1, -2)$. The BA results shown for the first two of these levels are somewhat lower than the TCSA results, for which we blame the latter. (As one can see also for 2-kink levels, the general tendency of the TCSA is to give too high estimates for the energies as one goes to large $r$ and higher levels). Above these levels there is a “combinatorial explosion” of 4-kink states; unfortunately the TCSA is not accurate enough anymore to warrant a further comparison with the BA classification (which can be continued ad nauseam).
We should briefly comment on the errors of the TCSA results. Experience has shown that for large \( r \), and in particular when \( y < 1 \), TCSA results can exhibit certain spurious features, even qualitatively: In fig. 7, for example, the fact that the second TCSA gap increases for \( r > 3 \) and the crossing of the first two TCSA 2-particle levels at \( r \approx 5.6 \) are due to truncation errors. Also, since we generically expect the TCSA to become increasingly inaccurate for higher levels, it seems somewhat surprising how good the first 4-kink level agrees with the BA results in fig. 7; for large \( r \), where we can presumably trust the BA results, the agreement is much better than for 2-particle levels of comparable energy. We will find a similar feature, even more pronounced, in the example studied in sect. 4. It is therefore probably not an accident. Instead, it seems to indicate that the volume at which the TCSA begins to deteriorate for a given level is not so much determined by the value of the energy at that volume, but rather by the UV scaling dimension of the level. This is in fact quite plausible, since the TCSA is non-perturbative in \( r \) and the only input required for this method are CFT data.

In table 1 we compare the UV (CFT) and IR (BA) classification of the first 21 levels in \( A_4 - \phi_{1.3} \). For the fields creating the conformal states in the UV limit we use the following notation, for any perturbed CFT. The operators \( L^{(\Delta)}_{n,i} \), where \( n \in \mathbb{N} \) and \( i = 1, \ldots, \chi^{(\Delta)}_n \) (=the degeneracy of level \( n \) of the Virasoro irrep of highest weight \( \Delta \) and central charge \( c \)), are certain linear combinations of strings of Virasoro generators \( L_{-n_1} \ldots L_{-n_p} \) with \( n_1 \geq \ldots \geq n_p \geq 1 \) and \( \sum_{j=1}^p n_j = n \). The coefficients in these linear combinations depend on \( n, \Delta, c \), and the perturbing field \( \Phi \). They are chosen such that the states \( |i\rangle = L^{(\Delta)}_{n,i} |\Delta\rangle \) form a basis of the irrep of highest weight state \( |\Delta\rangle \) at level \( n \) in which \( \Phi \) is diagonal, namely \( C_{i_1,i_2} \Phi_{i_2} = 0 \) if \( i_1 \neq i_2 \). (The latter requirement is necessary to ensure that the UV states in the table are the \( \lambda \to 0 \) limit of energy-eigenstates in the perturbed theory.) The operators \( L^{(\Delta)}_{n,i} \) are defined analogously.

There are two noteworthy features of the results in table 1. The first is the correspondence between the mechanisms by which exact degeneracies arise from the conformal and BA points of view. From the UV point of view, the degeneracy between states \( L^{(\Delta)}_{n,i_1} L^{(\Delta)}_{n,i_2} \phi \) and \( L^{(\Delta)}_{n,i_2} L^{(\Delta)}_{n,i_1} \phi \) with \( i_1 \neq i_2 \) (where \( \phi \) is a spinless primary field of dimensions \( \Delta = \tilde{\Delta} \)) will be preserved to all orders of conformal perturbation theory (CPT) due to parity invariance, and is therefore exact for all \( r \) (this is of course also clear from the TCSA point of view). From the BA point of view, on the other hand, levels with quantum numbers \( (\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N) \neq (\tilde{n}_N, \ldots, \tilde{n}_2, -\tilde{n}_1) \) will be degenerate, and again because of parity invariance this degeneracy will be exact for all \( r \), even when the exponential corrections are
taken into account. We can form parity eigenstates by (anti-)symmetrizing over the two degenerate BA states, whose UV limits are then identified with superpositions of definite parity of the corresponding conformal states (and this is the way in which the UV↔IR identification of degenerate levels should be made in table 1). Note that from both viewpoints all degeneracies of this kind are two-fold. Higher degeneracies are possible if a theory has additional symmetries.

The second feature is the pattern of differences between the BA \( \hat{e}_i(r) \) and the exact ones. We will restrict our remarks mainly to 2-kink levels, although qualitatively the same picture might be true for states with more kinks. Because the tricritical Ising CFT \( A_4 \) has a finite number (six) of primary fields, the pattern of UV scaling dimensions \( d_i \) repeats itself mod 2 (though with varying degeneracies) in any sector of fixed spin. Similarly, for 2-kink levels in a sector of fixed total momentum the BA \( \hat{e}_i(0) \) repeat themselves mod 1 (for generic scattering theories without the pairing of eigenvalues it would be mod 2). The first four 2-kink levels correspond to the fields \( \sigma', \varepsilon', L_{-1}\bar{L}_{-1} \sigma, L_{-1}\bar{L}_{-1} \varepsilon \) in the UV CFT, and the differences between their scaling dimensions and the corresponding BA \( \hat{e}_i(0) \) are \( 0, \frac{3}{40}, \frac{1}{5}, \frac{3}{40} \), respectively, cf. table 1. This pattern then repeats itself, presumably forever, with certain descendents in the same families as the above fields and BA 2-kink levels with higher quantum numbers. All other spinless conformal fields lead to 4- and higher multi-kink levels.

Recalling our discussion (subsect. 3.1) of the BA and the exact energy levels in \( A_3 - \phi_{1,3} \), one is naturally led to ask if also here the differences between the exact and the BA 2-kink levels are just four “universal” functions of \( r \), in the pattern observed above for \( r = 0 \). (In particular, one might wonder if the BA 2-kink levels with UV limit \( \hat{e}_i(0) = \frac{7}{8} \) (mod 2) are in fact exact.) This is not the case, though, as an analysis of eq. (3.17) reveals: Since the large-\( \theta \) expansion of the phase shifts \((3.15)-(3.16)\) involves powers of \( e^{-\theta} \) multiplied by \( \theta \), the scaled BA levels have a small \( r \) expansion in powers of \( r \) multiplied by powers of \( \ln r \). Furthermore, the log terms do not cancel in differences of \( \hat{e}_i(r) \) corresponding to BA levels of the same type, differing only in their quantum numbers. In contrast, the small-\( r \) expansion of the exact \( \hat{e}_i(r) \), that can be obtained from CPT, is in powers of \( r^{y} = r^{4/5} \), without any log terms. [In theories with diagonal S-matrices, where the scattering amplitudes are products of certain universal building blocks (see e.g. \[14\]), the phases entering the 2-particle BA equations can be expanded in powers of \( e^{-2\theta} \). Consequently, the scaled BA energies have a small-\( r \) expansion in powers of \( r^{2} \), which again contradicts CPT predictions, in general. The Ising field theory is an exception: The coincidence of the
BA and the exact results for the $\mathbb{Z}_2$-even sector in this case is consistent with the above remarks because $y = 1$ and Kramers-Wannier duality forces all odd terms in the expansion in $r^y$ to vanish.]

4. $\phi_{2,1}$-Perturbed Unitary Minimal CFTs

The theories $X_m \pm \phi_{2,1}$ (in the notation of sect. 3) appear to be massive for all $m$. In $A_m \pm \phi_{2,1}$ with $m$ even the perturbation is “magnetic”, manifestly breaking the $\mathbb{Z}_2$ symmetry of $A_m$, and so the sign of the perturbation is immaterial. For $m$ odd, on the other hand, the two theories $A_m \pm \phi_{2,1}$ are different, related to each other by duality. Smirnov proposed \[5\] factorizable $S$-matrices for the theories $A_m - \phi_{2,1}$, $m \geq 4$ (as well as for the closely related $A_m - \phi_{1,2}$ theories), based on $SL_q(2)$-restrictions of the imaginary-coupling $A_2^{(2)}$ affine Toda field theory. Depending on $m$, the spectrum contains kinks (belonging to one or more multiplets) and possibly bound states of kinks ("breathers"). In the case of $m = 4$ the spectrum consists of a single multiplet of three kinks. For this theory, describing the subleading magnetic perturbation of the tricritical Ising CFT, Zamolodchikov has independently proposed \[6\] a scattering theory with the same spectrum and kink structure but different amplitudes. One of the purposes of our paper is to decide which (if any) of the two $S$-matrices for this theory is correct.\[12\]

The CFTs $D_m$ ($m \geq 5$ odd) and $E_m$ ($m = 11, 17, 29$) can also be perturbed by $\phi_{2,1}$. Smirnov has shown \[5\] that by an appropriate “orbifolding” of the $S$-matrix he proposed for $A_5 - \phi_{2,1}$, one can reconstruct the $\mathbb{Z}_3$-symmetric diagonal $S$-matrix of \[51\]; this scattering theory of an ordinary particle and its antiparticle was previously proposed \[52\] for the theory $D_5 + \phi_{2,1}$. Except for this case, we are not aware of any explicit discussion of scattering theories for $X_m \pm \phi_{2,1}$, $X \neq A$, in the literature. In general, we think that “dual” relations between various particle scattering theories and kink theories, and between different kink theories, deserve further investigation.

\[12\] An attempt to resolve this issue has recently been made in \[50\] by comparing TCSA data for finite-size mass corrections with analytical expressions. However, in this case tunneling effects have to be taken into account (cf. sect. 1), and in \[50\] an ad hoc form of the corresponding contributions was assumed without justification. Therefore we do not find the analysis of ref. \[50\] satisfactory.
4.1. The $\phi_{2,1}$-Perturbation of the Tricritical Ising CFT and its S-Matrix

From here on we concentrate on the theory $A_4 - \phi_{2,1}$. The first information about the kink structure of this theory came from the TCSA analysis of [48]. The large-volume spectrum in the zero momentum sector indicated two degenerate vacua — even though there is no $\mathbb{Z}_2$ symmetry that relates them — and a single 1-particle state of exactly half (within the numerical accuracy) the energy of the 2-particle threshold. It was therefore concluded that the spectrum of the S-matrix theory consists of two kinks, interpolating between the two vacua, and only one bound state of the kinks ("propagating only in one of the two vacua") which is degenerate with them in mass. Alternatively, it is more instructive to think about the spectrum as consisting of a single multiplet of three kinks $K_{01}, K_{10}$, and $K_{11}$, so that the incidence matrix describing the linking between the two vacua labelled by 0 and 1 is $I = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. [This is the natural notation from the point of view of the $SL_q(2)$ structure of the theory, as later proposed by Smirnov [3], where the kinks carry spin 1 and the vacua are characterized by spins 0 and 1 ($q = e^{i\pi/5}$ in our conventions embodied in eq. (3.3)).]

Based on the above spectrum and observations, Zamolodchikov proposed [4] to construct the kink-type S-matrix of the theory using the Boltzmann weights of the critical hard-hexagon lattice model [28]. The motivation comes from the fact that the "spin" variables of this IRF model obey exactly the desired restriction imposed on the kinks (namely the "spins" can be chosen to take the values 0 and 1, and neighboring "spins" are not allowed to be both 0). The nonvanishing Boltzmann weights corresponding to a face of the square lattice with the "spins" $\alpha, \gamma, \beta, \delta$ at its vertices (ordered anti-clockwise starting at the upper left vertex) are

$$W_{\alpha \gamma, \beta}^\delta(u) = \frac{\sin(u - u)}{\sin \frac{2\pi}{5}} \delta_{\gamma \delta} + \frac{\sin u}{\sin \frac{2\pi}{5}} \sqrt{\frac{s_{\gamma s_{\delta}}}{s_{\alpha s_{\beta}}}} \delta_{\alpha \beta}, \quad s_{\alpha} = [2\alpha + 1]_{q=e^{i\pi/5}}, \ (4.1)$$

where $u$ is the spectral parameter. Explicitly, this is

$$W_{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}^\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}(u) = \frac{\sin(u + \frac{2\pi}{5})}{\sin \frac{2\pi}{5}}$$

$$W_{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}^\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}(u) = \frac{\sin u}{\sqrt{\sin \frac{\pi}{5} \sin \frac{2\pi}{5}}}$$

$$W_{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}^\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}(u) = -\frac{\sin(u - \frac{\pi}{5})}{\sin \frac{\pi}{5}}$$

$$W_{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}(u) = -\frac{\sin(u - \frac{2\pi}{5})}{\sin \frac{2\pi}{5}}$$

(4.2)
These Boltzmann weights satisfy the Yang-Baxter equation.

Zamolodchikov’s proposal is to look for a kink $S$-matrix of the form

$$S^{\gamma \delta}_{\alpha \beta} (\theta) = \left( \frac{\rho_\gamma \rho_\delta}{\rho_\alpha \rho_\beta} \right)^{-\frac{\theta}{2\pi i}} W^{\alpha \beta}_{\gamma \delta} (\lambda \theta),$$

(4.3)

with $\lambda, \rho_\alpha, \rho_\gamma, \rho_\delta$ suitably chosen to satisfy unitarity (2.2), crossing symmetry (2.3), and lead to a direct-channel simple pole corresponding to one of the kinks at $\theta = \frac{2\pi i}{3}$ in all scattering amplitudes, except $S_{00}^{11}(\theta)$ (since there is no kink $K_{00}$).

Actually, a stronger version of this “pole constraint” must hold, namely the bootstrap equations (2.5) should be satisfied. It is convenient to postpone the discussion of the bootstrap, since already the weaker version of the pole constraint together with crossing symmetry can be satisfied only if $\rho_\alpha = s_\alpha$ (ignoring an irrelevant $\alpha$-independent overall factor), and (i) $R(i\pi - \theta) = +R(\theta)$ and $\lambda \in i(\frac{9}{5} + 6\mathbb{Z})$ or (ii) $R(i\pi - \theta) = -R(\theta)$ and $\lambda \in i(-\frac{6}{5} + 6\mathbb{Z})$.

Unitarity implies the constraint

$$R(\theta) R(-\theta) = \frac{\sin^2 \frac{\pi}{5}}{\sinh(\frac{\lambda \theta}{i} - \pi i \frac{5}{9}) \sinh(\frac{\lambda \theta}{i} + \frac{\pi i}{5})}. \quad (4.4)$$

Zamolodchikov chooses $\lambda = -\frac{6i}{5}$. In this case the constraints on $R(\theta)$ have the “minimal solution”

$$R_Z(\theta) = \frac{-i \sin \frac{\pi}{5}}{\sinh(\frac{6\theta + 5\pi i}{5})} f_\frac{9}{5} \left( \frac{12\theta}{5} \right), \quad (4.5)$$

taken in [3]. Here $f_\alpha(\theta) = \sinh(\frac{\theta + i\pi \alpha}{2})/\sinh(\frac{\theta - i\pi \alpha}{2})$.

But $\lambda = -6i/5$ is not the only choice. Choosing (with hindsight) $\lambda = 9i/5$, the “simplest” solution for $R(\theta)$ is

$$R(\theta) = \frac{\pm i \sin \frac{\pi}{5}}{\sinh(\frac{9\theta - 5\pi i}{5})} f_{\frac{9}{5}} \left( \frac{9\theta}{5} \right) f_{\frac{9}{5}} \left( \frac{9\theta}{5} \right). \quad (4.6)$$

A similar ambiguity in the analog of Zamolodchikov’s $\lambda$ shows up in Smirnov’s quantum-group approach. Since he is working with the $A_2^{(2)}$ affine Toda field theory for arbitrary imaginary coupling, he can fix it as follows: For generic imaginary coupling the model contains a (lightest) breather, a kink-kink bound state which is an “ordinary” particle. Smirnov demands that its scattering amplitude with itself be the analytic continuation of that of the single boson in the real-coupling $A_2^{(2)}$ Toda theory of ref. [23]. (It is known that the analogous analytic continuation indeed relates the scattering amplitude of the single boson in the sinh-Gordon model to that of the first breather in the sine-Gordon model.)
However, the resulting $S$-matrix (4.3) exhibits some unwanted simple poles in the physical strip, namely at $\theta = \frac{8\pi i}{9}$ and $\theta = \frac{3\pi i}{9}$. There is a cure, though: One can always multiply the above $R(\theta)$ by arbitrary “CDD factors” [24] of the form $\prod_{\alpha} F_{\alpha}(\theta)$, where $F_{\alpha}(\theta) = -f_{\alpha}(\theta)f_{\alpha}(\pi i - \theta)$, which do not affect the algebraic constraints on $R(\theta)$. Multiplying by $F_{\theta/9}(\theta)$ cancels the unwanted poles, and a further factor of $F_{2/9}(\theta)$ cancels the zeros of $f_{-2/5}(9\theta/5)$ and $f_{3/5}(9\theta/5)$ in the physical strip without introducing new unwanted poles.

Choosing the ‘+’ sign in (4.6) to get the right signs for the residues of the poles in the full $S$-matrix, cf. (2.6), we see that all physical constraints are satisfied by

$$R_S(\theta) = \frac{i \sin \frac{\pi}{5}}{\sinh \left( \frac{9\theta - \pi i}{5} \right)} \left( \frac{9\theta}{5} \right) \left( \frac{9\theta}{5} \right) F_{\theta/9}(\theta) F_{2/9}(\theta) \ ,$$

in a “minimal” way, in the sense that (4.7) has the smallest number of poles and zeros in the physical strip. And this solution corresponds to Smirnov’s proposal [14]. Smirnov arrived at his proposal from a completely different direction [5], and we — being aware of his result — just rederived it using considerations employed by Zamolodchikov in order to clarify the relation between the two approaches.

The function $R_S(\theta)$ looks somewhat complicated, in particular since we had to include the factors $F_{\theta/9}(\theta)$ and $F_{2/9}(\theta)$. The presence of these factors becomes less mysterious when considering the bootstrap equations (2.5). One can check that they indeed hold for Smirnov’s $S$-matrix (with or without the crossing factors), using

$$R_S(\theta + \frac{i\pi}{3}) R_S(\theta - \frac{i\pi}{3}) = -\frac{R_S^2(\theta)}{2 \cos \frac{\pi}{5} S_{11}(\theta)} \ .$$

Given the factor $F_{\theta/9}(\theta)$, necessary to eliminate the unwanted poles, it is crucial to also include the factor $F_{2/9}(\theta)$ in $R_S(\theta)$, otherwise the bootstrap equations are not satisfied.

There is one more important point we have to discuss in connection with Smirnov’s $S$-matrix. In ref. [3] there are no crossing factors, i.e. all $\rho_\alpha = 1$ in (4.3). Smirnov claims that the resulting violation of crossing symmetry does not make the theory inconsistent. However, he has not explained how such a violation could arise in a QFT, which is indeed rather difficult to see. It would be nice, to perform a direct check of the presence or absence of the crossing factors. Unfortunately, as we saw in sect. 2.3, these factors do not affect

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14 In [3] $R_S(\theta)$ was given in an integral representation; it was rewritten in (essentially) the form (4.7) in ref. [3].
the finite-volume spectrum of a QFT (modulo, possibly, exponentially small terms), so
comparison of BA with TCSA results will not reveal if they are present or not.

Returning to Zamolodchikov’s $S$-matrix, we note that for $R_Z(\theta)$ eq. (4.8) holds with
a plus sign on the rhs, which leads to a violation of all the bootstrap equations (2.5) by a
sign. Since it is just one overall sign, we can restore the bootstrap equations by reversing
the sign of all $S$-matrix elements. (It would have been a serious problem if some bootstrap
equations were satisfied and others violated by a sign, because unitarity, crossing and the
Yang-Baxter equations fix all relative signs.) The overall sign that Zamolodchikov chose [6]
was presumably motivated by the fact that it leads to positive (imaginary) residues for the
direct-channel poles of $S$-matrix elements of the form $S^{\gamma\gamma}_{\alpha\beta}(\theta)$. Such residues correspond to
real couplings $g_{\alpha\gamma\beta}$ in (2.6), as expected for a unitary theory. On the other hand, his sign
is such that $S^{\gamma\gamma}_{\alpha\beta}(0) = +1$ for all allowed $\alpha, \beta, \gamma$, which according to (2.10) means that all
kinks are fermions. But then it is hard to see how they can be bound states of each other.
Changing the overall sign of Zamolodchikov’s $S$-matrix solves this problem (since then
all kinks are bosons), in addition to restoring the bootstrap equations. The $S$-matrix will
however now violate 1-particle unitarity, i.e. some couplings $g_{\alpha\gamma\beta}$ will be purely imaginary.
According to previous experience [54] this suggests that Zamolodchikov’s $S$-matrix — if it
describes any consistent theory — might be that of a perturbed non-unitary CFT.

Still, since the a priori problems with Zamolodchikov’s $S$-matrix are due to signs, a
somewhat subtle issue in $S$-matrix theory (cf. [24][14][11]), and since problems with one
conjecture do not prove that another (Smirnov’s) is correct, it is important to provide a
more direct check of these proposals. This will be done in the next subsections.

4.2. Multi-kink states

Imposing periodic boundary conditions the number of different types of $N$-kink states
is $d_N = tr \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^N = \left(\frac{1+\sqrt{5}}{2}\right)^N + \left(\frac{1-\sqrt{5}}{2}\right)^N$, or recursively $d_N = d_{N-1} + d_{N-2}$ with $d_0 = 2$
and $d_1 = 1$. Note that states with an odd number of kinks are allowed here, contrary to
the theories $A_m - \phi_{1,3}$ of sect. 3. In particular, there is one type of 1-kink states $|K_{11}(\theta)\rangle$, with $\theta$ quantized in finite volume $R$ according to $m \sinh \theta = 2\pi \tilde{n}/R$ with $\tilde{n} \in \mathbb{Z}$. Up to
exponentially small corrections the corresponding energy is just $\sqrt{m^2 + (2\pi \tilde{n}/R)^2}$.

For 2-kink states of zero momentum on a circle we use the basis

$$\left\{ |K_{01}(\theta)K_{10}(-\theta)\rangle, \, |K_{10}(\theta)K_{01}(-\theta)\rangle, \, |K_{11}(\theta)K_{11}(-\theta)\rangle \right\}. \quad (4.9)$$
The notation
\[ a(\theta) = \tilde{a}(\theta) = S_{11}^{11}(\theta) \]
\[ b(\theta) = \rho_1^{\theta/(2\pi i)} \tilde{b}(\theta) = S_{11}^{01}(\theta) = S_{11}^{10}(\theta) \]
\[ c(\theta) = \rho_1^{-\theta/(2\pi i)} \tilde{c}(\theta) = S_{01}^{11}(\theta) = S_{10}^{11}(\theta) \]
\[ d(\theta) = \rho_1^{-\theta/(\pi i)} \tilde{d}(\theta) = S_{00}^{11}(\theta) \]
\[ e(\theta) = \rho_1^{\theta/(\pi i)} \tilde{e}(\theta) = S_{11}^{00}(\theta) \]

will be useful below. Here \( \rho_1 = 2 \cos \frac{\pi}{5} \) for Zamolodchikov’s or the crossing-symmetrized Smirnov proposal, whereas \( \rho_1 = 1 \) for Smirnov’s original proposal. Note that the tilded quantities would be the \( S \)-matrix elements if there were no crossing factors.

The 2-kink transfer matrix now reads
\[
T(\theta|\theta,-\theta) = - \begin{pmatrix} 0 & e(2\theta) & b(2\theta) \\ d(2\theta) & 0 & 0 \\ 0 & b(2\theta) & a(2\theta) \end{pmatrix}.
\]

Its three eigenvalues
\[ \lambda(\theta) = -\tilde{d}(2\theta) \]
\[ \lambda_{\pm}(\theta) = \frac{1}{2} \left( \tilde{d} - \frac{\sqrt{\tilde{d}^2 + 4\tilde{a}\tilde{c}}}{\tilde{a}} \right)(2\theta) \]

corespond to the eigenvectors
\[
v(\theta) = \left( \rho_1^{-\theta/(\pi i)} \quad \rho_1^{\theta/(\pi i)} \quad \rho_1^{1/2} \right)^T,
\]
\[ v_{\pm}(\theta) = (1/\sqrt{\chi_{\pm}}, \sqrt{\chi_{\pm}}, \sqrt{\chi_{\pm}} z_{\pm})(\theta)^T. \]

Here
\[ \chi_{\pm}(\theta) = -\frac{d(2\theta)}{\lambda_{\pm}(\theta)} \quad \text{and} \quad z_{\pm}(\theta) = -\frac{b(2\theta)}{\lambda_{\pm}(\theta) + a(2\theta)} \]

Note that in agreement with our general proof in sect. 2.3 the eigenvalues are independent of the crossing factors, whereas the eigenvectors are not.

Let us denote the phases of \( \lambda(\theta), \lambda_{\pm}(\theta) \) by \( \delta^{(A)}(\theta), \delta^{(B)}(\theta), \delta^{(C)}(\theta) \), respectively. Fig. 8 shows a plot of the three types of phase shifts for Smirnov’s \( S \)-matrix. In terms of these phases we have the standard parametric form (2.28) of the 2-kink energy levels.

Let us now turn to 3-kink levels. For the four-dimensional space of 3-kink states on a circle we use the basis
\[
\{ |K_{01}(\theta_1)K_{11}(\theta_2)K_{10}(\theta_3)\rangle, |K_{11}(\theta_1)K_{10}(\theta_2)K_{01}(\theta_3)\rangle, |K_{10}(\theta_1)K_{01}(\theta_2)K_{11}(\theta_3)\rangle, |K_{11}(\theta_1)K_{11}(\theta_2)K_{11}(\theta_3)\rangle \}.
\]
It is possible to give explicit analytic expression for the eigenvalue phases of the 3-kink transfer matrix for general $\theta_1, \theta_2, \theta_3$. However, except for one of the eigenvalues (see below) they seem to be rather complicated; we will not write them down, since in the end the BA equations have to be solved numerically anyhow. Instead, we show in fig. 9 a representative “cross section” of the four phase functions for Smirnov’s $S$-matrix.

For a special class of 3-kink states the relevant phases have very simple expressions, as we will presently discuss. For reference, let us first write down the transfer matrix relevant for general 3-kink states:

$$
T(\theta|\theta_1, \theta_2, \theta_3) = \begin{pmatrix}
0 & c & c & e & d & b & b & c & a & b \\
b & b & d & 0 & c & e & c & a & b & c \\
c & c & b & d & b & 0 & b & c & a & b \\
b & a & c & a & c & b & c & b & a & a & a
\end{pmatrix}, 
$$

(4.16)

where the $i$-th factor in each of the triplets of functions is to be evaluated at $\theta - \theta_i$. Note that for $\theta \in \{\theta_1, \theta_2, \theta_3\}$ the structure of this matrix simplifies since $b(0) = 0$.

One would expect that in the zero momentum sector there are solutions of the Bethe-Yang equations with $\theta_2 = 0$, and, in fact, that the lowest 3-kink levels are of this form. This is indeed true. $\theta_2 = 0$ means that we have to find eigenvectors of $T(0|\theta, 0, -\theta)$ with eigenvalue equal to $-1$, where we have set $\theta \equiv \theta_1 = -\theta_3$. This eigenvalue turns out to have a two-dimensional eigenspace for all $\theta$, which we can write (projectively) as $(z, 1, 1, (zc(\theta) - e(\theta)) / b(\theta))$ in terms of a free parameter $z$. Since transfer matrices $T(\theta)$ with different $\theta$ can be simultaneously diagonalized, there must be (at least) two values of $z$ where this vector is also an eigenvector of $T(\theta|\theta, 0, -\theta)$. We find that these values are

$$
z_\pm(\theta) = -\frac{\lambda_\pm(\theta)}{c(\theta)c(2\theta)} 
$$

(4.17)

corresponding to the eigenvalues

$$
\lambda_\pm(\theta) = \bar{c}(\theta)\bar{c}(2\theta) - \frac{\bar{a}(\theta)\bar{b}(2\theta)\bar{c}(\theta)}{\bar{b}(\theta)} 
$$

(4.18)

\lambda_-(\theta) = -\bar{c}(\theta)\bar{c}(2\theta)

of $T(\theta|\theta, 0, -\theta)$. These eigenvalues correspond to the $\theta_2 = 0$ section of the phase shift functions labelled $A$ and $B$, respectively, in fig. 9.\textsuperscript{15}

\textsuperscript{15} And this is perhaps the appropriate moment to remark that for general $\theta$, the eigenvalue of type $B$ of $T_k(\theta_1, \theta_2, \theta_3)$ is $\lambda^{(B)}_k(\theta_1, \theta_2, \theta_3) = -\bar{c}(\theta_{ki})\bar{c}(\theta_{kj})$, where $k, i, j \in \{1, 2, 3\}$ are all distinct.
In terms of the corresponding phases $\tilde{\delta}^{(s)}(\theta)$, (2.19), the parametric form for the 3-kink energies of type $A,B$ with $\theta_2 = 0$ is

$$(r, E) = \left( \frac{2\pi \tilde{n} - \tilde{\delta}^{(s)}(\theta)}{\sinh \theta}, m \left( 1 + 2 \cosh \theta \right) \right), \quad s \in \{A, B\}, \quad \tilde{n} \in \mathbb{N}.$$  (4.19)

All other zero momentum 3-kink levels, for which we do not give explicit expressions, come in degenerate pairs. They are either the pairs $\tilde{s}(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ and $\tilde{s}(-\tilde{n}_3, -\tilde{n}_2, -\tilde{n}_1)$ with $s = A, B$ and the $\tilde{n}_i$ distinct nonzero integers that sum up to zero, or $\tilde{C}(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ and $\tilde{D}(-\tilde{n}_3, -\tilde{n}_2, -\tilde{n}_1)$ with distinct $\tilde{n}_i \in \mathbb{Z}$ whose sum vanishes.

### 4.3. Comparison with the TCSA

For $A_4 - \phi_{2,1}$ the coefficient $\kappa = |\lambda| m^{-9/8}$ has not yet been determined to high precision by comparing CPT with the TBA. Performing the TCSA with $\lambda = \frac{1}{2\pi}$ we estimate $m = 0.99(1)$ (cf. [50]) by looking at the variation (with truncation level) of the 1-kink energy for moderate to large volume. Hence $\kappa = 0.161(2)$. Compared to the case of $A_4 - \phi_{1,3}$ we will see that the greater inaccuracy in $\kappa$ for $A_4 - \phi_{2,1}$ will be more than compensated by the larger value of $y$, so that the TCSA is accurate up to larger values of $r$.

In fig. 10 we show the first 21 scaled energy gaps in the zero momentum sector obtained from the TCSA for the $\phi_{2,1}$-perturbed tricritical Ising CFT, together with our BA results for 2- and 3-kinks as derived from Smirnov’s $S$-matrix. What is clear right away is that the latter $S$-matrix receives very strong support from our results; the $S$-matrix of [6] gives quite a different picture, definitely not describing $A_4 - \phi_{2,1}$.

Since there are two degenerate vacua in infinite volume, there is now only one gap approaching zero energy exponentially. The next gap corresponds to the kink $K_{11}$ at zero momentum, which is followed by a series of 2- and 3-kink states, and we can also see the first three 4-kink levels (for which we have not performed a BA analysis, mainly out of laziness). The UV and IR classification of levels is shown in table 2.

Because of our detailed discussion of the $A_4 - \phi_{1,3}$ case we can be more brief here. Now the differences between the 2-kink BA $\hat{e}_i(r)$ and the exact gaps at $r = 0$ are $\frac{3}{40}, 0, \frac{3}{40}$, a pattern which is repeated mod 2. So it appears that 2-kink states of type $A, B, C$ correspond to fields in the families $[\sigma'], [\varepsilon]', [\sigma]$, respectively, of the tricritical Ising CFT. One can similarly speculate how the 3-kink states of different types correspond to specific conformal families: The 3-kink levels of types $A$ and $B$ with $\theta_2 = 0$ correspond to the families $[\varepsilon]$ and $[\varepsilon''']$, respectively, whereas the “conjugate” types $C$ and $D$ are associated to
both of the families $[\sigma]$ and $[\varepsilon']$. (It is not so clear, though, what the UV-IR correspondence is for 3-kink levels of type $A$ and $B$ with $\theta_2 \neq 0$.) Again, as in subsect. 3.3, one can show that the differences between the exact and the BA energy levels can not be some finite number of “universal” functions (corresponding to the types of BA levels, independent of the $\{n_k\}$), as they are in the Ising field theory of sect. 3.1.

5. Discussion

The multi-particle finite-volume spectrum provides a very characteristic “fingerprint” of a QFT. We discussed how to calculate this spectrum for large volume in integrable scattering theories of kinks. For simplicity we considered theories with a particle spectrum consisting of a single multiplet of kinks, in finite volume with periodic boundary conditions; these assumptions can easily be generalized. As comparison with numerical results from the truncated conformal space approach has shown in several cases, the analytical large-volume predictions are usually even better than one could have hoped for, in that they are accurate up to quite small volumes (in fact, for some levels they become exact again at zero volume).

Applying our results to the case of the subleading magnetic perturbation of the tricritical Ising CFT, comparison with TCSA data provides very strong support for the $S$-matrix conjectured by Smirnov [5]; the $S$-matrix conjectured by Zamolodchikov [6] definitely does not describe this theory (as suggested in subsect. 4.1, it might describe a perturbation of a non-unitary CFT). There remains, however, one subtle question that cannot be answered by this comparison. Namely, if the original $S$-matrix of Smirnov should be multiplied by “crossing factors” in order to restore crossing symmetry in the “traditional” form (2.3). These crossing factors do not affect the multi-particle spectrum (at least not the dominant terms). The same is true for 1-particle levels, and the exact ground state energy which can be calculated [55] using the thermodynamic Bethe Ansatz is easily seen not to depend on the crossing factors for any volume. Nevertheless, the $S$-matrices are different and can in fact be distinguished “experimentally”, by measuring, for example, time delays (which are proportional to phase shift derivatives) in scattering experiments. Since it is hard to see how a non-crossing-symmetric $S$-matrix could correspond to a consistent QFT, one might favor the crossing-symmetrized $S$-matrix as the correct one. Still, it would be nice to perform some explicit check.
Shifting to more general issues, an important open problem in the study of (massive) integrable perturbations of CFTs is the relation between the UV CFT classification of states and the IR Bethe Ansatz classification. For the examples we have studied, our results show explicitly how this “UV to IR map” works for the lowest 20 levels or so in the zero momentum sector. In addition to extending this map to arbitrary levels and figuring out the combinatorics involved (will this lead to new ways of writing Virasoro characters?), one wonders if it is possible to understand more conceptually why a given multi-particle state corresponds to a certain conformal field in the UV. Finally, it remains to be seen if the simple universal pattern that holds for the differences between the exact and the Bethe Ansatz finite-volume levels in the Ising field theory generalizes to a more sophisticated pattern in other integrable QFTs.

Throughout the paper we considered only integrable theories. At least for 2-particle states this restriction is not really necessary. By increasing the volume the energy of any level in a generic non-integrable theory will eventually drop below the threshold for particle creation, with only 2-particle states scattering elastically among themselves remaining. The results of sect. 2 then apply without any modification. Of course, for a non-integrable theory there is not much hope of ever knowing the exact $S$-matrix. Therefore one might want to use our results “in reverse”, namely extract the scattering amplitudes by comparison with numerical data for the energy levels \[21\,23\,22\] (and this is just as interesting for integrable theories whose $S$-matrix is not known). Looking at 2-particle levels in the zero momentum sector, one can directly only obtain the diagonal phases $\delta^{(s)}(\theta) = \delta^{(s)}(\theta|\theta, -\theta)$. Unfortunately, if one is just given the energy levels, one does not know how to relate the diagonal to the asymptotic 2-particle basis of states. Except in rather special cases, where the theory is a priori known to have a large (global) symmetry so that the relation between these bases is very simple, it will be quite difficult if not impossible to extract the physical scattering amplitudes from the diagonal phases $\delta^{(s)}(\theta)$.

We mentioned in the introduction that the sine-Gordon (SG) model contains solitons which are not kinks, in that there are no nontrivial restrictions on the multi-particle Hilbert space. We would like to conclude with some remarks on certain variants of the SG model that do contain kinks. Note that to define the (classical) SG theory we do not only have to specify its lagrangian, \[\frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - (m_0/\beta)^2 (1 - \cos \beta \varphi),\] but also the “target space” on which the field $\varphi$ lives. For the standard SG model $\varphi$ lives on a circle of radius $r' = \frac{1}{\beta}$, i.e. one identifies field configurations differing by a period of the potential. But what if we let $\varphi$ live on a circle of radius $r' = \frac{k}{\beta}$, for some generic $k \in \mathbb{N}$?\[16\] Let us denote the

\[\text{This possibility was also noticed by Swieca } 20\]
corresponding theory by $SG(\beta, k)$. All properties of the ordinary SG model $SG(\beta, 1)$ that rely only on local properties of the field $\varphi$, like classical integrability, will also hold for $SG(\beta, k)$. We expect similar statements to be true also at the quantum level (provided that $\beta^2 < 8\pi$). In a semi-classical framework it is clear that the quantum theory $SG(\beta, k)$ will have a $k$-fold degenerate vacuum. Labelling the vacua by $\alpha = 0, 1, \ldots, k - 1$, the theory will have kinks $K_{\alpha \alpha'}$ with $|\alpha - \alpha'| = \frac{1}{2}$ (or $\frac{k-1}{2}$). As in the ordinary SG model there may also be bound states of kinks.

The $S$-matrices of $SG(\beta, k)$ with different $k$ are also closely related: Using the obvious $Z_k$ symmetry, as well as time-reversal and parity invariance, we see that every non-vanishing 2-kink amplitude is equal to one of the three ($\alpha$-independent) amplitudes $S^{\alpha+\alpha +}_{\alpha+\alpha +}(\theta)$, $S^{\alpha+\alpha +}_{\alpha \alpha}(\theta)$, $S^{\alpha+\alpha +}_{\alpha+\alpha -}(\theta)$, where $\alpha_{\pm} = \alpha \pm \frac{1}{2}$, $\alpha_{++} = \alpha + 1 \pmod{k}$. These should be independent of the global properties of the field $\varphi$ and therefore equal to the soliton-soliton, anti-soliton-soliton reflection, and anti-soliton-soliton transmission amplitudes, respectively, of the standard SG model \cite{24}. Nevertheless, the global properties of $\varphi$, reflected in the fact that there are restrictions on multi-particle states, do give rise to differences between the theories with different $k$, e.g. in their finite-volume spectra, as is clear by just contemplating the analysis of sect. 2 for different $k$.

Finally, it is illuminating to consider the UV limit of the theories $SG(\beta, k)$. Changing to standard CFT conventions, where the massless limit of the (euclidean) SG field $\sqrt{\pi} \varphi$ is denoted by $X$, we see that the UV limit of $SG(\beta, k)$ is the $c = 1$ gaussian CFT at radius $r = \sqrt{\pi}k/\beta$ (in the notation of \cite{57}).\footnote{Note that all these CFTs can be obtained from the one with $r = \sqrt{\beta}/\pi$ by “orbifolding” with respect to the $Z_k$ subgroup of the $U(1)$ (“winding”) symmetry of the gaussian model. By analogy, $SG(\beta, k)$ can be thought of as a $Z_k$ “massive orbifold” of $SG(\beta, 1)$.} Gaussian models at different $r$ differ in their field content, and considering $SG(\beta, k)$ as a perturbed CFT, namely the perturbation by the dimension $\frac{1}{4}(\frac{k}{r})^2 = \frac{\beta^2}{4\pi}$ spinless vertex operator $\cos(kX/r)$, it is again clear that theories with different $k$ will have different finite-volume spectra (although the ground state energy is the same). We plan to provide a more detailed analysis of various aspects of the theories $SG(\beta, k)$ elsewhere.

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References

[1] A.B. Zamolodchikov, Landau Institute preprint (1989)

[2] A. LeClair, Phys. Lett. 230B (1989) 103;
   D. Bernard and A. LeClair, Nucl. Phys. B340 (1990) 721;
   N.Yu. Reshetikhin and F.A. Smirnov, Comm. Math. Phys. 131 (1990) 157

[3] K. Schoutens, Nucl. Phys. B344 (1990) 665; C. Ahn, Nucl. Phys. B354 (1991) 57

[4] P. Fendley, S.D. Mathur, C. Vafa and N.P. Warner, Phys. Lett. 243B (1990) 257

[5] F.A. Smirnov, Int. J. Mod. Phys. A6 (1991) 1407

[6] A.B. Zamolodchikov, “S-matrix of the Subleading Magnetic Perturbation Of the Tri-
critical Ising Model”, Princeton preprint PUPT 1195 (1990)

[7] A.B. Zamolodchikov, Adv. Stud. Pure Math. 19 (1989) 1

[8] V.P. Yurov and Al.B. Zamolodchikov, Int. J. Mod. Phys. A5 (1990) 3221, and Paris
   preprint ENS-LPS-273 (1990)

[9] V. Privman and M.E. Fisher, J. Stat. Phys. 33 (1983) 285;
   E. Brézin and J. Zinn-Justin, Nucl. Phys. B257 (1985) 867;
   G. Münster, Nucl. Phys. B324 (1989) 630

[10] M. Lüscher, in: Progress in Gauge Field Theory (Cargese 1983), ed. G. ’t Hooft et al
    (Plenum, New York, 1984), and Commun. Math. Phys. 104 (1986) 177

[11] T.R. Klassen and E. Melzer, Nucl. Phys. B362 (1991) 329

[12] C.N. Yang and C.P. Yang, J. Math. Phys. 10 (1969) 1115

[13] Al.B. Zamolodchikov, Nucl. Phys. B342 (1990) 695

[14] T.R. Klassen and E. Melzer, Nucl. Phys. B338 (1990) 485

[15] T.R. Klassen and E. Melzer, Nucl. Phys. B350 (1991) 635

[16] M.J. Martins, Phys. Lett. 257B (1991) 317, and Phys. Rev. Lett. 67 (1991) 419

[17] T.R. Klassen and E. Melzer, “Spectral Flow between Conformal Field Theories in 1+1
   Dimensions”, Chicago/Miami preprint EFI 91-17/UMTG-162 (1991), Nucl. Phys. B
   (in press)

[18] P. Fendley, “Excited-state Thermodynamics”, Boston preprint BUHEP-91-16 (1991)

[19] Al.B. Zamolodchikov, Nucl. Phys. B358 (1991) 497

[20] P. Fendley and K. Intriligator, “Scattering and Thermodynamics of Fractionally-
    Charged Supersymmetric Solitons”, Boston/Harvard preprint BUHEP-91-17/HUTP-
    91-A043 (1991), “Scattering and Thermodynamics in Integrable N = 2 Theories”,
    BUHEP-92-5/HUTP-91-A067 (1992)

[21] M. Lüscher, Commun. Math. Phys. 105 (1986) 153, and Nucl. Phys. B354 (1991) 531

[22] M. Lässig and M.J. Martins, Nucl. Phys. B354 (1991) 666;
    M.J. Martins, Phys. Lett. 262B (1991) 39

[23] M. Lüscher and U. Wolff, Nucl. Phys. B339 (1990) 222

[24] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1980) 253
[25] M. Karowski, Nucl. Phys. B153 (1979) 244
[26] B. Schroer, T.T. Truong and P. Weisz, Phys. Lett. 63B (1976) 422
[27] C.N. Yang, Phys. Rev. Lett. 19 (1967) 1312
[28] R.J. Baxter, *Exactly Solved Models in Statistical mechanics* (Academic Press, London, 1982); R.J. Baxter and P.A. Pearce, J. Phys. A15 (1982) 897
[29] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333;
    D. Friedan, Z. Qiu and S.H. Shenker, Phys. Rev. Lett. 52 (1984) 1575;
    P. Goddard, A. Kent and D. Olive, Phys. Lett. 152B (1985) 88
[30] J.L. Cardy, Nucl. Phys. B270 (1986) 186; *ibid.* B275 (1986) 200;
    A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. B280 (1987) 445;
    D. Gepner, Nucl. Phys. B287 (1987) 111
[31] A.B. Zamolodchikov, Sov. J. Nucl. Phys. 44 (1986) 529
[32] A.W.W. Ludwig and J.L. Cardy, Nucl. Phys. B285 (1987) 687
[33] D.A. Kastor, E.J. Martinec and S.H. Shenker, Nucl. Phys. B316 (1989) 590
[34] E.J. Martinec, Phys. Lett. 217B (1989) 431;
    C. Vafa and N.P. Warner, Phys. Lett. 218B (1989) 51
[35] T. Eguchi and S.-K. Yang, Phys. Lett. 235B (1990) 282
[36] L. Alvarez-Gaumé, C. Gomez and G. Sierra, Phys. Lett. 220B (1989) 142
[37] T.T. Wu, B.M. McCoy, C.A. Tracy and E. Baruch, Phys. Rev. B13 (1976) 316;
    M. Sato, T. Miwa and M. Jimbo, Proc. Japan Acad. 53A (1977) 6, 147, 153, 183, 219;
    B. Schroer and T.T. Truong, Nucl. Phys. B144 (1978) 80
[38] A.B. Zamolodchikov, Sov. J. Nucl. Phys. 46 (1987) 1090
[39] A.B. Zamolodchikov, Nucl. Phys. B358 (1991) 524
[40] F. Ravanini, “RG Flows of Non-diagonal Minimal Models Perturbed by $\phi_{1,3}$”, Paris
    preprint SPhT/91-147 (1991)
[41] T.R. Klassen and E. Melzer, “RG Flows in the $D$-Series of Minimal CFTs”, Cornell/Stony
    Brook preprint CLNS-91-1111/ITP-SB-91-57 (1991)
[42] J.L. Cardy, in: *Fields, Strings, and Critical Phenomena*, Les Houches 1988, ed.
    E. Brézin and J. Zinn-Justin, (North Holland, Amsterdam, 1989)
[43] P. Fendley and P. Ginsparg, Nucl. Phys. B324 (1989) 549
[44] V. Pasquier, Nucl. Phys. B285 (1987) 162
[45] A.E. Ferdinand and M.E. Fisher, Phys. Rev. 185 (1969) 832;
    H. Saleur and C. Itzykson, J. Stat. Phys. 48 (1987) 449
[46] J.L. Cardy, J. Phys. A17 (1984) L385
[47] A. Rocha-Caridi, in: *Vertex Operators in Mathematics and Physics*, ed. J. Lepowsky
    et al (Springer, New York, 1985)
[48] M. Lässig, G. Mussardo and J.L. Cardy, Nucl. Phys. B348 (1991) 591
[49] M. Lässig and G. Mussardo, Computer Phys. Comm. 66 (1991) 71

42
[50] F. Colomo, A. Koubek and G. Mussardo, “On the S-matrix of the Subleading Magnetic Deformation of the Tricritical Ising Model in Two Dimensions”, preprint ISAS/94/91/EP (1991)
[51] R. Köberle and J.A. Swieca, Phys. Lett. 86B (1979) 209
[52] A.B. Zamolodchikov, Int. J. Mod. Phys. A3 (1988) 743
[53] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, Phys. Lett. 87B (1979) 389
[54] J.L. Cardy and G. Mussardo, Phys. Lett. 225B (1989) 275;
    P.G.O. Freund, T.R. Klassen and E. Melzer, Phys. Lett. 229B (1989) 243
[55] P. Fendley, in preparation
[56] J.A. Swieca, Phys. Rev. D13 (1976) 312
[57] P. Ginsparg, Nucl. Phys. B295 (1988) 153
\[ \begin{array}{|c|c|c|c|}
\hline
\text{UV field}\ \phi & d_\phi & \text{BA } \tilde{s}(\tilde{n}_1, \ldots, \tilde{n}_N) \text{ BA } \tilde{e}_i(0) \\
\hline
1 = \phi_{1,1} & 0 & \text{0-kink state} \\
\sigma = \phi_{2,2} & \frac{3}{40} & \text{0-kink state} \\
\varepsilon = \phi_{3,3} & \frac{1}{5} & \text{0-kink state} \\
\sigma' = \phi_{2,1} & \frac{7}{8} & \tilde{A}(\frac{1}{2}, -\frac{1}{2}) \quad 1 - \frac{1}{8} \\
\varepsilon' = \phi_{1,3} & \frac{6}{5} & \tilde{B}(\frac{1}{2}, -\frac{1}{2}) \quad 1 + \frac{1}{8} \\
L_{-1} \tilde{L}_{-1} \sigma & 2 + \frac{3}{40} & \tilde{A}(1, -1) \quad 2 - \frac{1}{8} \\
L_{-1} \tilde{L}_{-1} \varepsilon & 2 + \frac{1}{5} & \tilde{B}(1, -1) \quad 2 + \frac{1}{8} \\
L_{-1} \tilde{L}_{-1} \sigma' & 2 + \frac{7}{8} & \tilde{A}(\frac{3}{2}, -\frac{3}{2}) \quad 3 - \frac{1}{8} \\
\varepsilon'' = \phi_{3,1} & 3 & \tilde{A}(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}) \quad 3 \\
L_{-1} \tilde{L}_{-1} \varepsilon' & 2 + \frac{6}{5} & \tilde{B}(\frac{3}{2}, -\frac{3}{2}) \quad 3 + \frac{1}{8} \\
L_{-2} \tilde{L}_{-2} 1 & 4 & \tilde{C}(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}) \quad 4 \\
L_{2,1}^{(3/80)} \tilde{L}_{2,1}^{(3/80)} \sigma & 4 + \frac{3}{40} & \tilde{A}(2, -2) \quad 4 - \frac{1}{8} \\
L_{2,1}^{(3/80)} \tilde{L}_{2,2}^{(3/80)} \sigma & 4 + \frac{3}{40} & \tilde{B}(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}) \quad 4 \\
L_{2,2}^{(3/80)} \tilde{L}_{2,2}^{(3/80)} \sigma & 4 + \frac{3}{40} & \tilde{B}(\frac{7}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{9}{4}) \quad \simeq 4 \\
L_{2,1}^{(3/80)} \tilde{L}_{2,1}^{(3/80)} \sigma & 4 + \frac{3}{40} & \tilde{B}(\frac{9}{4}, \frac{1}{4}, -\frac{3}{4}, -\frac{7}{4}) \quad \simeq 4 \\
L_{-2} \tilde{L}_{-2} \varepsilon & 4 + \frac{1}{5} & \tilde{B}(2, -2) \quad 4 + \frac{1}{8} \\
L_{-2} \tilde{L}_{-2} \sigma' & 4 + \frac{7}{8} & \tilde{A}(\frac{5}{2}, -\frac{5}{2}) \quad 5 - \frac{1}{8} \\
L_{-1} \tilde{L}_{-1} \varepsilon'' & 2 + 3 & \tilde{A}(2, 1, -1, -2) \quad 5 \\
L_{2,1}^{(3/5)} \tilde{L}_{2,1}^{(3/5)} \varepsilon' & 4 + \frac{6}{5} & \tilde{A}(\frac{5}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{5}{2}) \quad 5 \\
L_{2,2}^{(3/5)} \tilde{L}_{2,2}^{(3/5)} \varepsilon' & 4 + \frac{6}{5} & \tilde{B}(\frac{5}{2}, -\frac{5}{2}) \quad 5 + \frac{1}{8} \\
L_{2,1}^{(3/5)} \tilde{L}_{2,2}^{(3/5)} \varepsilon' & 4 + \frac{6}{5} & \tilde{A}(2, 0, -3) \quad \simeq 5 \\
L_{2,2}^{(3/5)} \tilde{L}_{2,1}^{(3/5)} \varepsilon' & 4 + \frac{6}{5} & \tilde{A}(3, 0, -1, -2) \quad \simeq 5 \\
\hline
\end{array} \]

\textbf{Table 1:} UV and IR classification of levels in } A_4 - \phi_{1,3}. 

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| UV field $\phi$ | $d_\phi$ | BA $\bar{s}(\bar{n}_1, \ldots, \bar{n}_N)$ | BA $\hat{e}_i(0)$ |
|----------------|---------|---------------------------------|------------------|
| $I = \phi_{1,1}$ | 0       | 0-kink state                     |                  |
| $\sigma = \phi_{2,2}$ | $\frac{3}{40}$ | 0-kink state                     |                  |
| $\varepsilon = \phi_{3,3}$ | $\frac{1}{5}$ | 1-kink state                     |                  |
| $\sigma' = \phi_{2,1}$ | $\frac{7}{5}$ | $A(\frac{1}{2}, -\frac{1}{2})$ | $\frac{4}{5}$   |
| $\varepsilon' = \phi_{1,3}$ | $\frac{6}{5}$ | $B(\frac{1}{2}, -\frac{1}{2})$ | $\frac{6}{5}$   |
| $L_{-1}\bar{L}_{-1}$ $\sigma$ | $2 + \frac{3}{40}$ | $\tilde{C}(1, -1)$ | 2                |
| $L_{-1}\bar{L}_{-1}$ $\varepsilon$ | $2 + \frac{1}{5}$ | $\tilde{A}(1, 0, -1)$ | 2                |
| $L_{-1}\bar{L}_{-1}$ $\sigma'$ | $2 + \frac{7}{5}$ | $A(\frac{3}{2}, -\frac{3}{2})$ | $2 + \frac{4}{5}$ |
| $\varepsilon'' = \phi_{3,1}$ | 3       | $\tilde{B}(1, 0, -1)$ | $2 + \frac{4}{5}$ |
| $L_{-1}\bar{L}_{-1}$ $\varepsilon'$ | $2 + \frac{6}{5}$ | $B(\frac{3}{2}, -\frac{3}{2})$ | $2 + \frac{6}{5}$ |
| $L_{-2}\bar{L}_{-2}$ $I$ | 4       | 1st 4-kink state                 |                  |
| $L^{(3/80)}_{2,1}$ $L^{(3/80)}_{2,2}$ $\sigma$ | $4 + \frac{3}{40}$ | $\tilde{C}(\frac{4}{3}, \frac{1}{3}, -\frac{5}{3})$ | 4                |
| $L^{(3/80)}_{2,2}$ $L^{(3/80)}_{2,1}$ $\sigma$ | $4 + \frac{3}{40}$ | $\tilde{D}(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3})$ | 4                |
| $L^{(3/80)}_{2,1}$ $L^{(3/80)}_{2,1}$ $\sigma$ | $4 + \frac{3}{40}$ | $\tilde{C}(2, -2)$ | 4                |
| $L^{(3/80)}_{2,2}$ $L^{(3/80)}_{2,2}$ $\sigma$ | $4 + \frac{3}{40}$ | 2nd 4-kink state                 |                  |
| $L_{-2}\bar{L}_{-2}$ $\varepsilon$ | $4 + \frac{1}{5}$ | $\tilde{A}(2, 0, -2)$ | 4                |
| $L_{-2}\bar{L}_{-2}$ $\sigma'$ | $4 + \frac{7}{5}$ | $A(\frac{5}{7}, -\frac{5}{7})$ | $4 + \frac{4}{5}$ |
| $L_{-1}\bar{L}_{-1}$ $\varepsilon''$ | 2+3     | $\tilde{B}(2, 0, -2)$ | $4 + \frac{4}{5}$ |
| $L^{(3/5)}_{2,1}$ $L^{(3/5)}_{2,2}$ $\varepsilon'$ | $4 + \frac{6}{5}$ | $\tilde{D}(\frac{5}{3}, \frac{2}{3}, -\frac{7}{3})$ | $4 + \frac{6}{5}$ |
| $L^{(3/5)}_{2,2}$ $L^{(3/5)}_{2,1}$ $\varepsilon'$ | $4 + \frac{6}{5}$ | $\tilde{C}(\frac{2}{3}, -\frac{2}{3}, -\frac{5}{3})$ | $4 + \frac{6}{5}$ |
| $L^{(3/5)}_{2,1}$ $L^{(3/5)}_{2,1}$ $\varepsilon'$ | $4 + \frac{6}{5}$ | $B(\frac{5}{2}, -\frac{5}{2})$ | $4 + \frac{6}{5}$ |
| $L^{(3/5)}_{2,2}$ $L^{(3/5)}_{2,2}$ $\varepsilon'$ | $4 + \frac{6}{5}$ | 3rd 4-kink state                 |                  |

**Table 2:** UV and IR classification of levels in $A_4 - \phi_{2,1}$. 

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Figure Captions

Fig. 5: Diagonal 2-kink phases $\delta^{(s)}(\theta) = \delta^{(s)}(\theta|\theta, -\theta)$ (in units of $\pi$) for the $S$-matrix (3.9)–(3.11), conjectured for $A_4 - \phi_{1,3}$. Shown are type $A$, eq. (3.17), as the upper line, and type $B$, eq. (3.16), as the lower line.

Fig. 6: The “cross section” $\theta \mapsto \frac{1}{\pi} \delta^{(s)}(\theta|\theta, \frac{1}{3}\theta, -\frac{1}{3}\theta, -\theta)$ of the diagonal 4-kink phases of type $s = A, B, C, D$ (labelled from top to bottom for small $\theta$) for the $S$-matrix (3.9)–(3.11).

Fig. 7: Scaled energy gaps $\hat{e}_i(r)$ in the zero momentum sector as obtained from the TCSA for the $\phi_{1,3}$-perturbed tricritical Ising CFT and the BA using the $S$-matrix (3.9)–(3.11). Shown are the first 35 gaps calculated with the TCSA using the 228 states in the CFT up to level 5 (solid lines), the first ten 2-kink levels from the BA (dotted lines), as well as BA results for the first 4-kink level of type $B$ (short dashed line) and the first three 4-kink levels of type $A$ (dashed lines). The next 4-kink level, for which no BA results are shown, is doubly degenerate.

Fig. 8: The three types $s = A, B, C$ (from top to bottom) of diagonal 2-kink phases $\delta^{(s)}(\theta) = \delta^{(s)}(\theta|\theta, -\theta)$ (in units of $\pi$) for Smirnov’s $S$-matrix conjectured to describe the $\phi_{2,1}$-perturbed tricritical Ising CFT.

Fig. 9: The cross section $\theta \mapsto \frac{1}{2\pi} \delta^{(s)}(\theta|\theta, \frac{1}{2}\theta, \frac{1}{2}\theta, \theta_3(\theta))$ where $\sinh \theta_3(\theta) = - (\sinh \theta + \sinh \frac{1}{2}\theta)$, for the four types $s = C, D, A, B$ (from top to bottom) of diagonal 3-kink phases in $A_4 - \phi_{2,1}$ (using Smirnov’s $S$-matrix).

Fig. 10: Scaled energy gaps $\hat{e}_i(r)$ in the zero momentum sector as obtained from the TCSA for the $\phi_{2,1}$-perturbed tricritical Ising model and the BA based on Smirnov’s $S$-matrix. In addition to the first 21 TCSA gaps calculated with 228 states (solid lines), we show the first eight 2-kink levels from the BA (dotted), and the BA results for the first four 3-kink levels with $\theta_2 = 0$ (dot-dashed), which are alternatingly of type $A$ and $B$, as well as the first two degenerate pairs of 3-kink levels of type $C$ and $D$ (dashed). The levels for which no BA results are shown are the first three 4-kink levels.

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