An Application of the Nash-Moser Theorem to the Vacuum Boundary Problem of Gaseous Stars

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September 30, 2016

Abstract

We have been studying spherically symmetric motions of gaseous stars with physical vacuum boundary governed either by the Euler-Poisson equations in the non-relativistic theory or by the Einstein-Euler equations in the relativistic theory. The problems are to construct solutions whose first approximations are small time-periodic solutions to the linearized problem at an equilibrium and to construct solutions to the Cauchy problem near an equilibrium. These problems can be solved when $1/(\gamma - 1)$ is an integer, where $\gamma$ is the adiabatic exponent of the gas near the vacuum, by the formulation by R. Hamilton of the Nash-Moser theorem. We discuss on an application of the formulation by J. T. Schwartz of the Nash-Moser theorem to the case in which $1/(\gamma - 1)$ is not an integer but sufficiently large.

Key Words and Phrases. Non-linear hyperbolic equations. Nash-Moser theory. Vacuum boundary. Spherically symmetric solutions. Gaseous stars.

2010 Mathematics Subject Classification Numbers. 35L70, 35Q31, 35Q85, 76L10, 83C05

1 Introduction

In the previous works [13], [14], we investigated the time evolution of spherically symmetric gaseous stars, either in the non-relativistic case governed by the Euler-Poisson equations ([13]), or in the relativistic case governed by the Einstein-Euler equations ([14]). Our studies suppose that the gas remains to be barotropic during the evolutions. That is, the pressure $P$ is a given fixed function of the density $\rho$. We were assuming

(A): $P$ is a given smooth function of $\rho > 0$ such that $P > 0, dP/d\rho > 0$ for $\rho > 0$ and there are positive constants $A, \gamma$ such that $1 < \gamma < 2$ and an analytic

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function $\Omega$ on a neighborhood of 0 such that $\Omega(0) = 1$ and

$$P = A\rho^\gamma \Omega(\rho^{\gamma-1})$$

for $0 < \rho \ll 1$.

Assuming that there is an equilibrium with a finite radius at which the gas touches the vacuum, we investigated time-dependent solutions near this equilibrium. The existence of solutions whose first approximations are small time-periodic solutions to the linearized problem at the equilibrium, and the existence of solutions to the Cauchy problem near this equilibrium were established by applying the Nash-Moser theorem formulated by R. Hamilton, [4], which we shall call the \textit{Nash-Moser(-Hamilton) theorem}.

But in order to apply this Nash-Moser(-Hamilton) theorem, we had to put the assumption

(\textbf{B}): $N$ is an even integer.

Here the parameter $N$ is determined from the approximate adiabatic exponent $\gamma$ near the vacuum by

$$\frac{N}{2} = 1 + \frac{1}{\gamma - 1}, \quad \text{or} \quad \gamma = 1 + \frac{2}{N - 2}.$$

Under the assumption (\textbf{B}), the function $(1 - x)^{N/2}$ is analytic at $x = 1$, and the smoothness of this function plays an essential rôle for the application of the Nash-Moser(-Hamilton) theorem.

However in many physically important cases, in which, e.g., $\gamma = 5/3, 7/5$ and so on, $N/2$ is not an integer. Therefore the open problem to apply the Nash-Moser theorem for the case in which $N$ is not an even integer is very important.

The present study is a partial answer to this open problem. In fact, when $N$ is very large, that is, $\gamma - 1$ is very small, the Nash-Moser theorem formulated by J. T. Schwartz, [16], which we shall call the \textit{Nash-Moser(-Schwartz) theorem}, can be applied. To show this is the aim of this article.

The reason why we apply the Nash-Moser theorem is that we have to treat the so called ‘physical vacuum boundary’, that is, a boundary $\partial \Omega$ of the domain $\Omega$ on which $\rho > 0$ and outside which $\rho = 0$ such that

$$0 < - \frac{\partial}{\partial \mathbf{N}} \left( \frac{dP}{d\rho} \right) < +\infty,$$

where $\mathbf{N}$ is the outer normal vector of $\partial \Omega$. (Cf. [3], [5]). Because of this singularity at the physical vacuum boundary, the nonlinear hyperbolic evolution equation to be considered involves a loss of the derivative regularities by which the usual iteration does not work. (Cf. [12] p. 49). This difficulty has already
been attacked by several scholars: D. Coutand and S. Shkoller, [2], [3], J. Jang and N. Masmoudi, [5], [6], and T. Luo, Z.-P. Xin and H.-H. Zen [10]. Although they do not consider the approximation by time-periodic solutions of the linearized problem, they have already developed a powerful theoretical method based on sophisticated energy estimates using the Hardy inequality. Even for the time-periodic approximation, a work along this line has been done by J. Jang, [7]. Maybe their method, after a suitable generalization, can be applied to the Einstein-Euler system, too. The advantage of the method developed by them is the absence of additional restrictions on the number theoretic properties or the nearness to 1 of the adiabatic exponent $\gamma$, which seems inevitable if we insist on the application of the Nash-Moser technique.

Thus, in this situation, the present study may give, as another approach, a touchstone for their method.

2 Problem setting

We are going to consider the following problem.

The equations to be considered are

\[
\frac{\partial y}{\partial t} - J(x, y, x \frac{\partial y}{\partial x}) v = 0, \quad (1a)
\]

\[
\frac{\partial v}{\partial t} + H_1(x, y, x \frac{\partial y}{\partial x}, v) \mathcal{L} \left( \frac{\partial}{\partial x} \right) y + H_2(x, y, x \frac{\partial y}{\partial x}, v, x \frac{\partial v}{\partial x}) = 0 \quad (1b)
\]

with

\[
\mathcal{L} \left( \frac{d}{dx} \right) = -x(1-x) \frac{d^2}{dx^2} - \left( \frac{5}{2} (1-x) - \frac{N}{2} x \right) \frac{d}{dx} + L_1(x) \frac{d}{dx} + L_0(x). \quad (2)
\]

The independent variables $t, x$ run over $0 \leq t \leq T, 0 < x < 1$, where $T$ is a fixed number. The unknown functions are $y(t, x), v(t, x)$.

In fact, in the case of the relativistic Einstein-Euler system discussed in [14], the equations (1a) (1b) (2) are derived as follows. The metric to be found is

\[
ds^2 = e^{2\tau} c^2 dt^2 - e^{2H} dr^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

and the unknown density distribution is $\rho = \rho(t, r)$. We fix an equilibrium governed by the Tolman-Oppenheimer-Volkoff equation $\rho = \bar{\rho}(r)$ with

\[
\bar{m}(r) := 4\pi \int_0^r \bar{\rho}(r') r'^2 dr',
\]

for which the radius $r_+$ and the total mass $m_+$ are finite, and we introduce the perturbation variables $y, v$ by

\[
R = r(\bar{m})(1 + y), \quad V = r(\bar{m})v,
\]
by taking $\bar{m}(r)$ as the independent variable. Here

$$V = e^{-F} \left( \frac{\partial R}{\partial t} \right)_{r=\text{Const}}.$$ 

By a suitable change of the variable from $\bar{m}$ (running on the interval $[0, m_+]$) to $x$ (running on $[0, 1]$), the equations to be considered are reduced to (1a) (1b) (2). Actually we have

$$\rho = \bar{\rho}(1 + y)^{-2} \left( 1 + y + \frac{\partial y}{\partial \bar{r}} \right)^{-1},$$

$$e^{-F} = \sqrt{\kappa} \exp \left[ - \frac{u}{c^2} \right],$$

$$J = e^{F} (1 + P/c^2 \rho),$$

and so on. Here $\kappa := 1 - 2Gm_+/c^2r_+$ and

$$u := \int_0^\rho \frac{dP}{\rho + P/c^2}.$$ 

For the details, see [13].

In the case of the non-relativistic Euler-Poisson system, we can take $J = 1$, naturally since $c = +\infty$, and the system of the first order equations (1a) (1b) is reduced to a single second order equation of the form

$$\frac{\partial^2 y}{\partial t^2} + H_1\mathcal{L}y + H_2 = 0.$$ 

For the details, see [13].

Anyway let us list up the assumptions on the system (1a) (1b) (2), which have been ascertained for the Euler-Poisson system in [13] and for the Einstein-Euler system in [14].

First we assume

**(B0):** $N \geq 5$.

Let us denote by $\mathfrak{A}(N)$ the set of all smooth functions $f(x)$ of $x \in [0, 1]$ such that

$$f(x) = \begin{cases} [x]_0 & \text{as } x \to 0 \\ [1 - x, (1 - x)^N 0] & \text{as } x \to 1. \end{cases}$$

Here and hereafter $[X]_Q$ stands for a convergent power series of the form $\sum_{j \geq Q} a_j X^j$, and $[Z_1, Z_2]_Q$ stands for a convergent double power series of the form $\sum_{j_1 + j_2 \geq Q} a_{j_1 j_2} Z_1^{j_1} Z_2^{j_2}$. 

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By $\mathfrak{A}^Q_{(N)}(U^p)$, $U$ being a neighborhood of 0, we denote the set of all smooth functions $f(x, y_1, \ldots, y_p)$ of $x \in [0, 1], y_1, \ldots, y_p \in U$ such that there are convergent power series

$$\Phi_0(X, Y_1, \cdots, Y_p) = \sum_{k_1 + \cdots + k_p \geq Q} a_{j_1 \cdots j_p} X^{j_1} Y_1^{k_1} \cdots Y_p^{k_p}$$

and

$$\Phi_1(Z_1, Z_2, Y_1, \cdots, Y_p) = \sum_{k_1 + \cdots + k_p \geq Q} a_{j_1 j_2}^{[1]} Z_1^{j_1} Z_2^{j_2} Y_1^{k_1} \cdots Y_p^{k_p}$$

such that

$$f(x, y_1, \cdots, y_p) = \Phi_0(x, y_1, \cdots, y_p) \quad \text{for} \quad 0 < x \ll 1$$

and

$$f(x, y_1, \cdots, y_p) = \Phi_1(1 - x, (1 - x)^{N/2}, y_1, \cdots, y_p) \quad \text{for} \quad 0 < 1 - x \ll 1.$$
as $x \to 1$.

Here $z, w$ stand for $\frac{\partial y}{\partial x}, \frac{\partial v}{\partial x}$ respectively, and “$f \equiv_{(N)} 0$ as $x \to 1$” is defined as that there is a convergent power series $\Phi(Z_1, Z_2, Y_1, \cdots, Y_p)$ such that

$$f(x, y_1, \cdots, y_p) = (1 - x)\Phi(1 - x, (1 - x)^{N/2}, y_1, \cdots, y_p) \quad \text{for} \quad 0 < 1 - x \ll 1.$$ 


We assume

$$\neg(B) : N \text{ is not an even integer.}$$

So we consider the equations (1a) (1b) under the assumptions (B0), (B1; $N$), (B2), (B3; $N$) and $\neg(B)$.

Note that the result of the spectral analysis of the linear operator $L$ is the same as the case in which $N/2$ is an integer. See Appendix.

Hence the problem may be settled as follows:

Fix $T > 0$, and functions $y^*, v^* \in \mathcal{B}(N)([0, T]; U)$. Here $\mathcal{B}(N)([0, T]; U)$ stands for the set of all smooth functions $u(t, x)$ of $0 \leq t \leq T, 0 \leq x < 1$, valued in $U$, such that there are analytic functions $\Phi_0$ on $[0, T] \times \delta, \delta[2$ with $0 < \delta \ll 1$ such that

$$u(t, x) = \Phi_0(t, x) \quad \text{for} \quad 0 \leq t \leq T, 0 < x \ll 1$$

and

$$u(t, x) = \Phi_1(t, 1 - x, (1 - x)^{N/2}) \quad \text{for} \quad 0 \leq t \leq T, 0 < 1 - x \ll 1$$

Then we seek a solution $(y, v)$ of (1a) (1b) of the form

$$y = y^* + \tilde{y}, \quad v = v^* + \tilde{v} \quad (3)$$

such that

$$\tilde{y}|_{t=0} = 0, \quad \tilde{v}|_{t=0} = 0. \quad (4)$$

Actually in the application to the problem of spherically symmetric gaseous stars, we consider the following $(y^*, v^*)$:

[1]: To construct solutions whose first approximations are small time-periodic solutions of the linearized problem] Let $\lambda$ be a positive
eigenvalue of $\mathcal{L}$ and $\Phi(x)$ be the associated eigenfunction (see Appendix); put

$$Y_1(t,x) = \sin(\sqrt{\lambda} t + \Theta_0)\Phi(x),$$

$$V_1(t,x) = \frac{\sqrt{\lambda}}{J(x,0,0)} \cos(\sqrt{\lambda} t + \Theta_0)\Phi(x),$$

$\Theta_0$ being a constant; let $\varepsilon$ be small parameter; put $y^* = \varepsilon Y_1$, $v^* = \varepsilon V_1$.

[2]: To construct solutions of the Cauchy problem] Let $\psi_0(x) = y|_{t=0}$, $\psi_1(x) = v|_{t=0}$ be the smooth initial data; put

$$y^* = \psi_0(x) + tJ(x,0,0)\psi_1(x), \quad v^* = \psi_1(x).$$

Remark. In the study of the Einstein-Euler equations, \[14\], we fixed a short equilibrium $\rho(r)$, $0 < r < r_+$, for which we showed that

$$u(r) := \int_0^{\rho(r)} \frac{dP}{\rho + P/c^2}$$

is analytic in $r$ at $r = r_+$ under the assumption $(B)$. (See \[14\] Proposition 6.) However we can prove that, even if $\neg(B)$ is supposed, we have

$$u(r) = B(r_+ - r)(1 + [r_+ - r, (r_+ - r)^N]_1)$$

as $r \to r_+$ with a positive constant $B$ for any short equilibrium. Actually a proof can be found in \[13\]. (See \[13\] Theorem 4.) But at the moment, we may assume this property for the fixed equilibrium under consideration. This analytic property of the equilibrium at the vacuum boundary guarantees the validity of $(B1; N)$ and $(B3; N)$ for our application to the study of spherically symmetric gaseous stars.

3 Nash-Moser(-Schwartz) theorem

We are going to apply the following Nash-Moser(-Schwartz) theorem (See \[16\] Chapter II and also see \[17\]):

Let $J$ be a positive integer and $E_j$, $j = 0, 1, \cdots, J, F_j$, $j = 1, \cdots, J$ be Banach spaces such that $E_{j+1} \subset E_j$, $\|u\|_{E_j} \leq \|u\|_{E_{j+1}}$, $F_{j+1} \subset F_j$, $\|u\|_{F_j} \leq \|u\|_{F_{j+1}}$.

For $\theta \geq 0$ there is a linear operator $S(\theta) : E_0 \to E_J$, so called a smoothing operator, such that, if $0 \leq j \leq J$, then

$$\|S(\theta)u\|_{E_j} \leq C\theta^{J-j}\|u\|_{E_j},$$

$$\|(I - S(\theta))u\|_{E_j} \leq C\theta^{J-j}\|u\|_{E_j}.$$
\( \mathcal{F} \) is a mapping from \( V := \{ u \in E_j \mid \|u\|_{E_j} < 1 \} \) into \( F_j \) such that, for \( 1 \leq j \leq J, \mathcal{F}(V \cap E_j) \subset F_j \). For any \( u \in V \) there is a linear operator \( DF(u) : E_j \to F_j \) such that
\[
\|DF(u)h\|_{F_j} \leq C\|h\|_{E_j} \quad \text{for} \quad u \in V, h \in E_j, \\
\|F(u + h) - F(u) - DF(u)h\|_{F_j} \leq C\|h\|_{E_j}^2 \quad \text{for} \quad u, u + h \in V.
\]

For \( u \in V \) there is a linear operator \( \mathcal{I}(u) : F_j \to E_0 \) such that, for \( u \in V \cap E_j, 1 \leq j \leq J, \mathcal{I}(u)F_j \subset E_{j-1} \) and
\[
\|\mathcal{I}(u)g\|_{E_0} \leq C\|g\|_{F_j} \quad \text{for} \quad u \in V, g \in F_j, \\
DF(u)\mathcal{I}(u)g = g \quad \text{for} \quad u \in V, g \in F_j, \\
\|\mathcal{I}(u)F(u)\|_{E_{j-1}} \leq C(1 + \|u\|_j) \quad \text{for} \quad u \in V \cap E_j.
\]

Then there is a small \( \delta \) such that if \( \|F(0)\|_{E_1} \leq \delta \) then there is a solution \( u \) of the equation \( F(u) = 0 \) in \( V \), provided that \( J \geq 10 \).

We are going to apply this theorem to the equation \( \tilde{\mathcal{G}}(\tilde{w}) = 0 \), where \( \tilde{\mathcal{G}}(\tilde{w})^T = (F_1, F_2)^T \) is the left-hand side of (1H) with \( y = y^* + \tilde{g}, v = v^* + \tilde{v}, \tilde{w} = (\tilde{g}, \tilde{v})^T \).

We will take the graded Banach spaces of functions \( \bar{u}(t, x) \) of \( 0 \leq t \leq T, 0 \leq x \leq 1 \) defined by the norms
\[
\|\bar{u}\|_{E_j} = \|\bar{u}\|_{b_E+rj}^{(2)}, \quad \|\bar{u}\|_{F_j} = \|\bar{u}\|_{b_F+rj}^{(2)}.
\]
Later the positive parameters \( b_E, b_F, r \) will be chosen suitably. Here the norms \( \|\cdot\|_{b_E+r}^{(2)} \) are defined as follows:

For a function \( u(x) \) of \( 0 \leq x \leq 1 \), we put
\[
u^{[0]}(x) = \omega(x)u(x), \quad u^{[1]}(x) = (1 - \omega(x))u(x),
\]
where \( \omega \in C^\infty(\mathbb{R}) \) such that \( \omega(x) = 1 \) for \( x \leq 1/3, 0 < \omega(x) < 1 \) for \( 1/3 < x < 2/3 \), and \( \omega(x) = 0 \) for \( 2/3 \leq x \).

Let us denote
\[
\triangle_{[0]} = x \frac{d^2}{dx^2} + 5 \frac{d}{2dx}, \quad \triangle_{[1]} = X \frac{d^2}{dX^2} + \frac{N}{2} \frac{d}{dX} \quad \text{with} \quad X = 1 - x, \\
\hat{D}_{[0]} = \sqrt{2} \frac{d}{dx}, \quad \hat{D}_{[1]} = \sqrt{X} \frac{d}{dX} \quad \text{with} \quad X = 1 - x, \\
\|u\|_{[0]} = \left( \int_0^1 |u(x)|^2 x^{3/2} dx \right)^{1/2}, \\
\|u\|_{[1]} = \left( \int_0^1 |u(x)|^2 X^{N/2-1} dX \right)^{1/2}.
\]
We put, for $\mu = 0, 1$,
\[
\langle u \rangle_{[\mu]}^\ell = \begin{cases} 
\|\Delta^m_{[\mu]} u \|_{[\mu]} & \text{for } \ell = 2m, \\
\|\hat{D}_{[\mu]} \Delta^m_{[\mu]} u \|_{[\mu]} & \text{for } \ell = 2m + 1,
\end{cases}
\]
\[
\|u\|_{[\mu]_k} = \left( \sum_{0 \leq \ell \leq k} (\langle u \rangle_{[\mu]}^\ell)^2 \right)^{1/2},
\]
\[
\|\bar{u}\|_{[\mu]_k} = \left( (\|y\|_{[\mu]_{k+1}}^2 + \|v\|_{[\mu]_k}^2) \right)^{1/2}
\]
for $\bar{u} = (y, v)^T = (y(x), v(x))^T$.

For a function $\bar{u} = \bar{u}(t, x)$ of $0 \leq t \leq T, 0 \leq x \leq 1$, we put
\[
|\bar{u}; \tau, n|_{[\mu]} = \sup_{0 \leq t \leq T} \sum_{j+k \leq n} \|\partial_j^\tau \bar{u}(t, \cdot)\|_{[\mu]_k},
\]
\[
|\bar{u}; \tau, n| = \left( (|\bar{u}^{[0]}; \tau, n|_{[0]}^2 + |\bar{u}^{[1]}; \tau, n|_{[1]}^2) \right)^{1/2},
\]
\[
\|\bar{u}\|_{[\mu]n} = \left( \sum_{j+k \leq n} \int_0^T (\|\partial_j^\tau \bar{u}(t, \cdot)\|_{[\mu]_k})^2 dt \right)^{1/2},
\]
\[
\|\bar{u}\|_{n} = \left( (\|\bar{u}^{[0]}|_{[0]n}|^2 + (\|\bar{u}^{[1]}|_{[1]n})^2) \right)^{1/2}.
\]

On the other hand, we put
\[
\|u\|_{[\mu]_k}^* = \left( \sum_{0 \leq m \leq k} (\|\Delta^m_{[\mu]} u \|_{[\mu]})^2 \right)^{1/2},
\]
\[
\|u\|_{k}^* = \left( (\|u^{[0]}_{[\mu]}|_{[0]n}|^2 + (\|u^{[1]}_{[\mu]}|_{[1]n})^2) \right)^{1/2},
\]
\[
\|u\|_{[\nu]}^{(2)} = \left( \sum_{j+k \leq \nu} \int_0^T (\|(-\partial_j^2 u(t, \cdot)\|_{[\nu]_k})^2 dt \right)^{1/2},
\]
\[
\|\bar{u}\|_{[\nu]}^{(2)} = \left( (\|y\|_{[\nu]}^{(2)}|_{[0]n})^2 + (\|v\|_{[\nu]}^{(2)}|_{[1]n})^2) \right)^{1/2}
\]
for $\bar{u} = (y, v)^T$.

**Notation:** Hereafter we shall denote $X \lesssim Y$ for two quantities $X, Y$ if there is a constant $C$ such that $X \leq CY$. We shall denote $X \simeq Y$ if both $X \lesssim Y$ and $X \gtrsim Y$ hold, that is, there is a constant $C$ such that $(1/C)Y \leq X \leq CY$. 

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We can claim
\[ \|\vec{u}\|_n \lesssim |\vec{u}; T, n| \lesssim \|\vec{u}\|_{n+1}, \quad (5) \]
and
\[ \|\vec{u}\|_{2^\nu - 1} \lesssim \|\vec{u}\|^{(2)}_\nu \lesssim \|\vec{u}\|_{2^\nu}. \quad (6) \]

In fact we have
\[ \|\dot{D}_{[\mu]} u\|_{[\mu]} \lesssim \|u\|_{[\mu]} + \|\triangle_{[\mu]} u\|_{[\mu]}, \]
\[ \int_0^T (\partial_t u)^2 dt \lesssim \int_0^T u^2 dt + \int_0^T (-\partial_t^2 u)^2 dt \]
and so on.

4 Smoothing operators

We construct the smoothing operator
\[ \vec{S}(\theta)\vec{u} = (S(\theta)y, S(\theta)v)^T \quad \text{for} \quad \vec{u} = (y, v), \]
\[ S(\theta) = S_{[0]}(\theta)u^{[0]} + S_{[1]}(\theta)u^{[1]} \]
as follows.

Let us define \( S_{[1]}(\theta) \). For the simplicity, we write \( S = S_{[1]}, \triangle = \triangle_{[1]}, \| \cdot \| = \| \cdot \|_{[1]}, x \) instead of \( 1 - x \), so that, \( \triangle = xD^2 + (N/2)D, D = d/dx \), and \( u(x) \) be a function which vanishes for \( 2/3 < x \).

We fix an extension of functions \( u \in C^\infty([0,T] \times [0,1]) \) such that \( u(t,x) = 0 \) for \( 2/3 < x \) to a function \( \tilde{u} \in C^\infty([-2T, 2T] \times [0,1]) \) such that
\[ \|\tilde{u}\|^\nu_{\nu} \simeq \|u\|^{(2)}_{\nu}, \]
where
\[ \|\tilde{u}\|^\nu_{\nu} := \left( \sum_{i + \kappa \leq \nu} \int_{-2T}^{2T} \|(-\partial_t^2)^i \triangle^{\kappa} \tilde{u}(t, \cdot)\|^2 dt \right)^{1/2}. \]

See \[1\] §5.17-30.

Let \((\phi_\alpha(t))_{\alpha=1,2,...}\) be the eigenfunctions of the operator \(-\partial_t^2\) in \(L^2([-2T, 2T])\) with the Dirichlet boundary conditions at \(t = \pm 2T\) and \((\lambda_\alpha)_\alpha\) be the associated eigenvalues. That is,
\[ \phi_\alpha(t) = \frac{1}{\sqrt{2T}} \sin \frac{a\pi t}{2T}, \quad \lambda_\alpha = \left( \frac{a\pi}{2T} \right)^2. \]

Hence \(\lambda_\alpha \simeq a^2\).

Let \((\psi_b(x))_{b=1,2,...}\) be the eigenfunctions of \(-\triangle\) in \(L^2(x^{N/2-1}dx)\) with the Dirichlet boundary condition at \(x = 1\) and \((\mu_b)_b\) be the associated eigenvalues. That is, \(\psi_b(x)\) is a normalization of \(\Psi(\mu_b, x)\), where
\[ \mu_b = \left( \frac{1}{2} J_{N/2-1, b} \right)^2, \quad \Psi(z) = J_{N/2-1}(2\sqrt{z})(\sqrt{z})^{-N/2+1}. \]
Here $J_{\frac{\nu}{2}-1}$ is the Bessel function and $j_{\frac{\nu}{2}-1,b}$ is its $b$-th zero. Since it is known that $\mu_1 > 0$ and $\mu_b \sim (b\pi/4)^2$ as $b \to +\infty$, we have $\mu_b \simeq b^2$.

Now we define

$$\hat{S}(\theta)\hat{u} = \sum_{a^2 \leq \theta, b^2 \leq \theta} c_{ab} \phi_a(t) \psi_b(x)$$

for

$$\hat{u} = \sum c_{ab} \phi_a(t) \psi_b(x) \in C^\infty_0([-2T, 2T[ \times [0,1]),$$

and $S(\theta)u$ be the restriction of $\hat{S}(\theta)\hat{u}$ onto $[0,T] \times [0,1]$.

We claim

**Proposition 1** If $0 \leq \nu \leq \bar{\nu}$, then

$$\|\hat{S}(\theta)\hat{u}\|_{\nu}^2 \lesssim \theta^{p-\nu} \|\hat{u}\|_{\bar{\nu}}^2$$

and

$$\|(I - \hat{S}(\theta))\hat{u}\|_{\nu} \lesssim \theta^{p-\bar{\nu}} \|\hat{u}\|_{\bar{\nu}}$$

Proof. Note that

$$\left(\|\hat{u}\|_{\bar{\nu}}^2 \right)^2 = \sum_{j+k \leq \nu} \int_{-2T}^{2T} \|(-\partial_t^2)^j (-\Delta)^k \hat{u}(t, \cdot)\|^2 dt$$

$$\simeq \sum_{a,b} \sum_{j+k=\nu} |c_{ab}|^2 \lambda_a^{2j} \mu_b^{2k}$$

$$\simeq \sum_{a,b} \sum_{j+k=\nu} |c_{ab}|^2 (a^j b^k)^4.$$ 

Note that, if $0 \leq X, Y \leq 1$ and $0 \leq \nu \leq \bar{\nu}$, then

$$\sum_{j+k=\nu} X^j Y^k \leq (\bar{\nu} - \nu + 1) \sum_{j+k=\nu} X^j Y^k.$$ 

Hence

$$\left(\|\hat{S}(\theta)\hat{u}\|_{\bar{\nu}}^2 \right)^2 \simeq \sum_{a^2 \leq \theta, b^2 \leq \theta} \sum_{j+k=\nu} |c_{ab}|^2 (a^j b^k)^4$$

$$= \theta^{2\bar{\nu}} \sum_{a^2 \leq \theta, b^2 \leq \theta} \sum_{j+k=\nu} |c_{ab}|^2 \left(\frac{a^j}{\theta^j} \right) \left(\frac{b^k}{\theta^k} \right)$$

$$\lesssim \theta^{2\bar{\nu}} \sum_{a,b} \sum_{j+k=\nu} |c_{ab}|^2 \left(\frac{a^j}{\theta^j} \right) \left(\frac{b^k}{\theta^k} \right)$$

$$= \theta^{2(\bar{\nu} - \nu)} \sum_{a,b} \sum_{j+k=\nu} |c_{ab}|^2 (a^j b^k)^4$$

$$\simeq \theta^{2(\bar{\nu} - \nu)} (\|\hat{u}\|_{\bar{\nu}}^2),$$

where

$$\|\hat{u}\|_{\bar{\nu}}^2 = \sum_{a^2 \leq \theta, b^2 \leq \theta} \sum_{j+k=\nu} |c_{ab}|^2 (a^j b^k)^2.$$
that is, (7) holds.

On the other hand,

\[
\| (I - \tilde{S}(\theta)) \tilde{u} \|_{\nu}^2 \simeq \sum_{(\theta < a^2) \lor (\theta < b^2)} \sum_{j + k = \nu} |c_{ab}|^2 (a^2 b^k)^4.
\]

But

\[
\sum_{\theta < a^2} \sum_{j + k = \nu} |c_{ab}|^2 (a^2 b^k)^4 \leq \sum_{\theta < a^2} \sum_{j + k = \nu} |c_{ab}|^2 (a^2 b^k)^4 \leq \sum_{a,b} \sum_{j + k = \nu} |c_{ab}|^2 (a^2 b^k)^4 \simeq \theta^{2(\nu - \bar{\nu})} \| \tilde{u} \|_{\nu}^2.
\]

We see the same estimate for \( \sum_{\theta < b^2} \sum_{j + k = \nu} \). Hence (8) holds. 

We can define \( S_{[0]}(\theta) \) similarly, replacing \( N \) by 5.

This conclusions lead us to

**Proposition 2** If \( 0 \leq \nu \leq \bar{\nu} \), then

\[
\| \tilde{S}(\theta) \tilde{u} \|_{\nu}^2 \lesssim \theta^{\bar{\nu} - \nu} \| \tilde{u} \|_{\nu}^2
\]

and

\[
\| (I - \tilde{S}(\theta)) \tilde{u} \|_{\nu}^2 \lesssim \theta^{\nu - \bar{\nu}} \| \tilde{u} \|_{\nu}^2.
\]

## 5 Estimate of \( \mathcal{F}(\vec{w}) \) by \( \vec{w} \)

We are going to estimate \( \| \mathcal{F}(\vec{w}) \|_{\nu}^2 \) by \( \| \vec{w} \|_{\nu+1}^2 \).

Let us consider

\[
\begin{aligned}
\Delta &= \Delta_{[1]} = X \frac{d^2}{dX^2} + \frac{N}{2} \frac{d}{dX}, \\
D &= D_{[1]} = \frac{d}{dX}, \\
\dot{D} &= \dot{D}_{[1]} = \sqrt{X} \frac{d}{dX}
\end{aligned}
\]

with \( X = 1 - x \). Let us write \( x \) instead of \( X = 1 - x \) and write \( \| \cdot \|_k \) instead of \( \| \cdot \|_{[1]k} \).

Then the first Sobolev’s imbedding for functions \( u(x) \) reads

\[
\| u \|_{L^\infty} \lesssim \| u \|_{s_N},
\]

where

\[
s_N := \left\lceil \frac{N}{2} \right\rceil + 1 = \min \left\{ s \in \mathbb{N} \mid s > \frac{N}{2} \right\}.
\]
Proposition 3 We have
\[ \| \Delta^m D^j u \| \lesssim \| \Delta^{m+j} u \|, \]  
(13a)
\[ \| \dot{\Delta}^m D^j u \| \lesssim \| \dot{\Delta}^{m+j} u \|. \]  
(13b)

Therefore we can claim
\[ \| D^j u \|_k \lesssim \| u \|_{k+2j}. \]  
(14)

Proof. As [12, Proposition 3] we start from the formula
\[ \Delta^m Du(x) = x - N^{m+1} \int_0^x \Delta^{m+1} u(x') (x') \frac{N^{m+1}}{x'} dx'. \]
Then we can estimate
\[ \| \Delta^m Du \|^2 = \int_0^1 x - N^{m+1} \left( \int_0^x \Delta^{m+1} u(x') (x') \frac{N^{m+1}}{x'} dx' \right)^2 dx \]
by
\[ \frac{1}{N^{m+1}} \int_0^1 x - N^{m+1} \int_0^x \left| \Delta^{m+1} u(x') \right|^2 (x') \frac{N^{m+1}}{x'} dx' dx, \]
using the Schwartz’ inequality. It is estimated by
\[ \frac{1}{N^{m+1}} \int_0^1 x - N^{m+1} \| \Delta^{m+1} u \|^2, \]
using the Fubini’s change of the order of integrations. This implies (13a) for \( j = 1 \). Repeating this argument, we get (13a) for \( j \geq 2 \), too. Let us omit the proof of (13b). ■

Proposition 4 We have
\[ \| \Delta^m \dot{D}^j u \| \lesssim \| u \|_{2m+k+2j}, \]  
(15a)
\[ \| \dot{\Delta}^m \dot{D}^j u \| \lesssim \| u \|_{2m+1+k+2j}, \]  
(15b)

therefore
\[ \| \dot{D}^j u \|_n \lesssim \| u \|_{n+k+2j}. \]  
(16)

Proof. First we consider the case when \( m = 0 \). We start from the formula [12 (B.1)]:
\[ \dot{D}^k u(x) = x - \frac{N^{k+k}}{x} \int_0^x \dot{D}^k \Delta u(x') (x') \frac{N^{k+k}}{x'} dx'. \]
Using the Schwartz’ inequality and the Fubini’s theorem, we can verify
\[ \| \dot{D}^k u \| \leq \frac{2}{\sqrt{(N+k)^k}} \| \dot{D}^k \Delta u \|. \]
On the other hand, using \( \dot{D}^2 = \Delta - \frac{\Delta - 1}{4} D \) and \([D, \Delta] = D^2\), we can verify that
\[
\dot{D}^{2K}\Delta = \sum_{k+\alpha = K+1} C_{k\alpha} \Delta^\alpha D^k,
\]
with some constants \( C_{k\alpha} \). Hence
\[
\|\dot{D}^{2K}\Delta u\| \lesssim \|u\|_{2K+2}
\]
by Proposition 3, and so on. The conclusion is
\[
\|\dot{D}^{k}D^j u\| \lesssim \|u\|_{k+2j},
\]
that is, (15a) holds at least for \( m = 0 \).

Using \([D, \Delta] = D^2\), we can show
\[
D^j \Delta^k = \sum_{\alpha+\beta = k+j} C_{\alpha\beta} \Delta^\alpha D^\beta.
\]
(17)
with some constants \( C_{\alpha\beta} \). Then, using (17) and
\[
[D, \dot{D}] = \frac{N-1}{4} E^{1/2} D,
\]
where \( E = x^{-1} \), we have
\[
[D, \dot{D}] = \sum C_{k\alpha\beta} E^{k+\frac{1}{2}} \Delta^\alpha D^\beta,
\]
(18)
where the summation runs over \( k + \alpha + \beta = m, \beta \geq 1 \). On the other hand, we have
\[
s = k \in \mathbb{N} \Rightarrow \|E^s u\| \lesssim \|D^k u\|,
\]
(19a)
\[
s = k + \frac{1}{2} \in \mathbb{N} + \frac{1}{2} \Rightarrow \|E^s u\| \lesssim \|\dot{D}^k u\|.
\]
(19b)
In fact, if \( s \in \mathbb{N}/2 \), then
\[
\|E^s u\| \leq \frac{1}{\frac{N}{4} - s} \|E^{s-1} Du\|,
\]
(20)
for the integration by parts leads us to
\[
\|E^s u\| = \frac{1}{\frac{N}{4} - s} \|E^{s-1} Du\|
\]
for the integration by parts leads us to
\[
\|E^s u\|^2 = \int_0^1 u(x)^2 x^{-2s+\frac{N}{2}-1} dx
\]
\[= - \frac{1}{\frac{N}{4} - s} \int_0^1 u \cdot Du \cdot x^{-2s+\frac{N}{2}} dx
\]
\[\leq \frac{1}{\frac{N}{4} - s} \|x^{-s} u\| \|x^{-s+1} Du\|
\]
provided that \( u \in C_0^\infty ([0,1]) \). Note that \( N/2 \notin \mathbb{N} \) is supposed by \( \neg (B) \) so that \( \frac{N}{4} - s \neq 0 \) for \( s \in \mathbb{N}/2 \). Therefore, if \( k + \alpha + \beta = m \), then
\[
\| E^{k+\alpha} D^\beta u \| \lesssim \| \hat{D} D^k \Delta^\alpha D^\beta u \|
\lesssim \sum_{\nu=k+\alpha} \| \hat{D} \Delta^\nu D^\beta u \|
\lesssim \sum_{\nu=m} \| \hat{D} \Delta^\nu D^\beta u \|
\lesssim \| \hat{D} \Delta^m u \|.
\]
Hence
\[
\| \Delta^m \hat{D} u \| \lesssim \| \hat{D} \Delta^m u \|. \tag{21}
\]
This can be used to consider the case of odd \( k \) in (15a). If \( k = 2K \), the identities \( \hat{D}^2 = \Delta - \frac{N}{2} D \) and (17) are sufficient.

**Proposition 5** Suppose that
\[
\ell_1 + \cdots + \ell_p + 2(j_1 + \cdots + j_p) = n.
\]
Then
\[
\| (\hat{D}^{\ell_1} D^{j_1} u_1) \cdots (\hat{D}^{\ell_p} D^{j_p} u_p) \| \lesssim 1 + \| u_1 \|_n + \cdots + \| u_p \|_n,
\]
provided that \( \| u_\beta \|_n \lesssim 1 \) \( \forall \beta \).

Proof. By the Sobolev’s imbedding and Proposition 4, (16), we have
\[
\| (\hat{D}^{\ell_1} D^{j_1} u_1) \cdots (\hat{D}^{\ell_p} D^{j_p} u_p) \| \lesssim \| u_1 \|_{s_N + \ell_1 + 2j_1} \cdots \| u_p \|_{s_N + \ell_p + 2j_p}.
\]
Suppose \( n > s_N \). (Otherwise, we can claim that \( \| \| \lesssim \min_\beta \| u_\beta \|_n \), provided that \( \| u_\beta \|_n \lesssim 1 \).) Suppose that \( \ell_p + 2j_p \geq s_N \). (Otherwise, we can assume that \( \ell_\beta + 2j_\beta < s_N \) for \( \forall \beta \), and \( \| \| \lesssim \min_\beta \| u_\beta \|_n \), provided that \( \| u_\beta \|_n \lesssim 1 \) \( \forall \beta \).) by interpolation we have
\[
\| u_\beta \|_{s_N + \ell_\beta + 2j_\beta} \lesssim \| u_\beta \|_{s_N}^{1-\theta_\beta} \| u_\beta \|_{n}^{\theta_\beta}
\]
for \( \beta \leq p - 1 \) with \( \theta_\beta = \frac{\ell_\beta + 2j_\beta}{n - s_N} \), and
\[
\| u_p \|_{\ell_p + 2j_p} \lesssim \| u_p \|_{s_N}^{n-(\ell_p + 2j_p)} \| u_p \|_n^{n-(\ell_p + 2j_p - s_N)}.\]
Note that \( s_N \leq \ell_p + 2j_p \leq n \) and
\[
\frac{\ell_p + 2j_p - s_N}{n - s_N} = \frac{n - \sum_{\beta=1}^{p-1}(\ell_\beta + 2j_\beta) - s_N}{n - s_N} = 1 - \sum_{\beta=1}^{p-1} \theta_\beta.
\]
Therefore
\[ \ast \lesssim X_1^{\theta_1} \cdots X_{p-1}^{\theta_{p-1}} X_p^{\theta_p} \]
with \( X_\beta = \|u_\beta\|_n \), provided that \( \|u_\beta\|_{2n} \lesssim 1 \). It is easy to show that
\[ X_1^{\theta_1} \cdots X_{p-1}^{\theta_{p-1}} X_p^{\theta_p} \leq X_1 + \cdots + X_{p-1} + X_p \]
provided that \( 0 \leq \theta_\beta \leq 1, \sum_{\beta=1}^n \theta_\beta = 1 \). \[ \Box \]

**Proposition 6**
Let \( F \in C^\infty(\mathbb{R}^p) \). Then
\[ \| F(u_1, \cdots, u_p) \|_n \lesssim 1 + \|u_1\|_n + \cdots + \|u_p\|_n, \]
provided that \( \|u_\beta\|_{2n} \lesssim 1 \) \( \forall \beta \).

Proof. As [12, (B.5)] we have
\[ \Delta^m = \sum_{\mu+j=m} C_{\mu j} \hat{D}^{\alpha_\mu} \hat{D}^{\beta_\mu}. \]
Therefore \( \Delta^m F(u_1, \cdots, u_p), m \geq 1 \) consists of terms of the following types:

(I)
\[ (D^{\alpha_1} u_{\beta_1}) \cdots (D^{\alpha_q} u_{\beta_q}) \partial_{\beta_1} \cdots \partial_{\beta_1} F \]
with \( \alpha_i \geq 1 \) for \( \forall i, \alpha_1 + \cdots + \alpha_q = m \). Here \( \partial_{\beta} = \partial/\partial u_\beta \).

(II)
\[ (\hat{D}^{\gamma_1} D^{\alpha_1} u_{\beta_1}) \cdots (\hat{D}^{\gamma_q} D^{\alpha_q} u_{\beta_q}) \partial_{\beta_1} \cdots \partial_{\beta_1} F \]
with \( \alpha_i \geq 1 \) for \( \forall i, \gamma_1 + \cdots + \gamma_q = 2 \mu, \alpha_1 + \cdots + \alpha_q = j \), \( \mu + j = m \).

(III)
\[ (\hat{D}^{\gamma_1} D^{\alpha_1} u_{\beta_1}) \cdots (\hat{D}^{\gamma_q} D^{\alpha_q} u_{\beta_q}) (\hat{D}^{\gamma_{q+1}} u_{\delta_1}) \cdots (\hat{D}^{\gamma_{q+r}} u_{\delta_r}) \partial_{\delta_1} \cdots \partial_{\delta_1} F \]
with \( \alpha_i \geq 1 \) for \( \forall i \leq q, \gamma_k \geq 0 \) for \( \forall k \leq q, \gamma_k \geq 1 \) for \( \forall k \geq q + 1 \),
\( \gamma_1 + \cdots + \gamma_q + \gamma_{q+1} + \cdots + \gamma_{q+r} = 2 \mu, \alpha_1 + \cdots + \alpha_q = j \), \( \mu + j = m \).

(IV)
\[ (\hat{D}^{\gamma_1} u_{\delta_1}) \cdots (\hat{D}^{\gamma_r} u_{\delta_r}) \partial_{\delta_1} \cdots \partial_{\delta_1} F \]
with \( \gamma_k \geq 1 \) for \( \forall k \leq r, \gamma_1 + \cdots + \gamma_r = 2m \).

Then we can apply Proposition 5 to each term. \[ \Box \]

Now let us consider \( \tilde{\mathcal{F}}(\mathbf{w}) = (\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)^T \).
Put \( \tilde{\mathcal{F}}^{[0]}_1 = \omega(x)\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}^{[1]}_1 = (1 - \omega(x))\tilde{\mathcal{F}}_1 \) and so on.
We are going to estimate
\[ \left( \|\tilde{\mathcal{F}}(\mathbf{w})\|^{(2)}_n \right)^2 = \left( \|\tilde{\mathcal{F}}_1\|^{(2)}_n \right)^2 + \left( \|\tilde{\mathcal{F}}_2\|^{(2)}_n \right)^2. \]
Let us observe \( \mathfrak{F}_2 \). Note that
\[
\mathcal{L}y = \mathcal{L}y[0] + \mathcal{L}y[1]
\]
\[
= -(1 - x)\Delta[0]y[0] + \left( \frac{N}{2} + L_1 \right) D_y[0] + L_0y[0] + x\Delta[1]y[1] + \left( -\frac{5}{2}(1 - x) + L_1 \right) D_y[1] + L_0y[1].
\]

Therefore \( \mathfrak{F}_2 \) is a smooth function of
\[
y[0], y[1], D_y[0], D_y[1], \Delta[0]y[0], \Delta[1]y[1], \nu[0], \nu[1], D\nu[0], D\nu[1].
\]

Let us observe
\[
(||\mathfrak{F}_2||_0^2)^2 = \sum_{j + k < \nu} \int_0^T ||(-\partial_t^2)^j \mathfrak{F}_2(t, \cdot)\|_k^2 dt
\]
\[
= \sum_{j + k < \nu} \int_0^T (||(-\partial_t^2)^j \mathfrak{F}_2[0](t, \cdot)||_0^2)^2 dt + \sum_{j + k < \nu} \int_0^T (||(-\partial_t^2)^j \mathfrak{F}_2[1](t, \cdot)||_1^2)^2 dt.
\]

When we estimate
\[
||\Delta[1]y[1]||_1 = ||\Delta[1](1 - \omega(x))\mathfrak{F}_2||_1
\]
we see that \( \Delta[1](1 - \omega)\mathfrak{F}_2 \) consists of terms of the form
\[
\spadesuit = (\hat{D}^\gamma D^{\alpha_1} u_{\beta_1}) \cdots (\hat{D}^\gamma D^{\alpha_p} u_{\beta_p}) \partial_{\beta_p} \cdots \partial_{\beta_1} ((1 - \omega)F_2),
\]
where \( F_2 \) is a smooth function of \( u_1, \ldots, u_p \). We have to estimate \( ||u_{\beta,||_1}||_k \).

Consider the case in which some of \( u_{\beta} \) is either \( y[0], D_y[0], \Delta[0]y[0], \nu[0], D\nu[0] \).

E.g., if \( u_{\beta_1} = y[0] \), its support is included in \([0, 2/3] \). On the other hand the support of \((1 - \omega)F_2 \) is included in \([1/3, 1] \). Therefore, if \( \chi \in C^\infty(\mathbb{R}) \) satisfies
\[
\chi(x) = 0 \text{ for } x \leq 1/6, 0 < \chi(x) < 1 \text{ for } 1/6 < x < 1/3, \chi(x) = 1 \text{ for } 1/3 \leq x \leq 2/3, 0 < \chi(x) < 1 \text{ for } 2/3 < x < 5/6 \text{ and } \chi(x) = 0 \text{ for } 5/6 \leq x,
\]
we have the equality
\[
\spadesuit = (\hat{D}^\gamma D^{\alpha_1} \chi(x)y[0]) \cdots \partial_{\beta_1} (1 - \omega)F_2
\]
holds everywhere on \( 0 \leq x \leq 1 \). Since the support of \( \chi(x)y[0] \) is included in \([1/6, 5/6] \), we see
\[
||\chi(x)y[0]||_1 \leq ||\chi(x)y[0]||_{0} \leq ||y[0]||_0 \leq ||y||_k
\]
Thus we can claim
Proposition 7 If \( \| (1 - x)^N/2 \|_{2sN+2} < \infty \) and if \( \| \vec{w} \|_{2sN+2} \lesssim 1 \), then
\[
\| \mathcal{F}(\vec{w}) \|_n \lesssim 1 + \| \vec{w} \|_{n+2},
\]
provided that \( \| (1 - x)^N/2 \|_n < \infty \).

Note that \( \| (1 - x)^N/2 \|_{2sN+2} < \infty \) if \( N > 12 \).

\[\begin{align*}
\bullet \quad &\| \vec{w} \|_{E_1} \lesssim 1 \text{ should imply } \| \vec{w} \|_{2sN+2} \lesssim 1. \text{ This requires that } \\
&\| \vec{w} \|_{(b_E + \tau)^2} \lesssim \| \vec{w} \|_{2(b_E + \tau)} \lesssim 1 \text{ should imply } \| \vec{w} \|_{2sN+2} \lesssim 1. \text{ Hence we require that } \\
&b_E + \tau \geq s_N + 2. \quad (22)
\end{align*}\]

\[\begin{align*}
\bullet \quad &\text{On the other hand, } \\
&\| \mathcal{F}(\vec{w}) \|_n \lesssim 1 + \| \vec{w} \|_{n+2}
\end{align*}\]

implies, by dint of (6), that
\[
\| \mathcal{F}(\vec{w}) \|_{(2 \nu)^2} \lesssim 1 + \| \vec{w} \|_{(2 \nu)^2}
\]
with \( n = 2 \nu \). But it is required that
\[
\| \mathcal{F}(\vec{w}) \|_{E_j} \lesssim 1 + \| \vec{w} \|_{E_j}.
\]
This requires
\[
b_F + \tau j \leq b_E + \tau j - 2,
\]
or
\[
b_F \leq b_E - 2. \quad (23)
\]

\[\begin{align*}
\bullet \quad &\text{The estimate should be applied for } n = 2(b_F + 10\tau), \text{ therefore it should hold that } \\
&(1 - x)^N/2 \|_{2(b_F + 10\tau)} < \infty, \text{ that is, } \\
&\frac{3N}{2} > 2(b_F + 10\tau). \quad (24)
\end{align*}\]

6 Existence of the inverse \( \mathcal{I}(\vec{w}) \) of the the Fréchet derivative \( D\mathcal{F}(\vec{w}) \)

We have to analyze the Fréchet derivative \( D\mathcal{F}(\vec{w}) \) of the mapping \( \mathcal{F} \) at a given small \( \vec{w} = (\vec{g}, \vec{v})^T \). For \( \vec{h} = (h, k)^T \) we have \( D\mathcal{F}(\vec{w})\vec{h} = ((DF)_1, (DF)_2)^T \), where
\[
(DF)_1 = \frac{\partial h}{\partial t} - Jk - \left( (\partial_y J)v + (\partial_z J)x \frac{\partial}{\partial x} \right) h,
\]
\[
(DF)_2 = \frac{\partial k}{\partial t} + H_1 \mathcal{L} h + \\
+ \left( (\partial_y H_1) \mathcal{L} y + \partial_y H_2 + ((\partial_z H_1) \mathcal{L} y + \partial_z H_2) x \frac{\partial}{\partial x} \right) h + \\
+ \left( (\partial_v H_1) \mathcal{L} y + \partial_v H_2 + (\partial_w H_2) x \frac{\partial}{\partial x} \right) k.
\]
Thanks to the assumption (B3; N), we see that there are functions \( a_{01}, a_{00}, a_{11}, a_{10}, a_{21}, a_{20} \) of class \( \mathcal{A}_{(N)}([0, T] \times U^5) \) of \( t, x, y(= y^* + \tilde{y}), Dy, D^2y, v(= v^* + \tilde{v}), Dv \), where \( D = \partial/\partial x \), such that

\[
\begin{align*}
(DF)_1 &= \frac{\partial h}{\partial t} - Jk + (a_{01}x(1-x)D + a_{00})h, \quad (25a) \\
(DF)_2 &= \frac{\partial k}{\partial t} + H_1Lh + (a_{11}x(1-x)D + a_{10})h + (a_{21}x(1-x)D + a_{20})k. \quad (25b)
\end{align*}
\]

Here a smooth function \( a(t, x, y_1, \ldots, y_p) \) of \( [0, T] \times [0, 1] \times U \) \( \times \cdots \times U \) \( \times \cdots \times U \) belongs to \( \mathfrak{A}_{(N)}([0, T] \times U^p) \) if there are analytic functions \( \Phi_0 \) on \( [0, T] \times -\delta, \delta|\times U^p \) and \( \Phi_1 \) on \( [0, T] \times -\delta, \delta|\times U^p \) with \( 0 < \delta \ll 1 \) such that

\[
\begin{align*}
a(t, x, y_1, \ldots) &= \Phi_0(t, x, y_1, \ldots) \quad \text{for} \quad 0 < x \ll 1, \\
a(t, x, y_1, \ldots) &= \Phi_1(t, 1-x, (1-x)^{N/2}, y_1, \ldots) \quad \text{for} \quad 0 < 1-x \ll 1.
\end{align*}
\]

We put

\[
\begin{align*}
\mathfrak{x} &:= L^2([0, 1]; x^{3/2}(1-x)^{N/2-1}dx), \\
\mathfrak{x}^1 &:= \{ \phi \in \mathfrak{x} \mid \hat{D}\phi := \sqrt{x(1-x)}\frac{d\phi}{dx} \in \mathfrak{x} \}, \\
\mathfrak{x}^2 &:= \{ \phi \in \mathfrak{x}^1 \mid -\Lambda \phi \in \mathfrak{x} \},
\end{align*}
\]

with

\[
\Lambda = x(1-x)\frac{d^2}{dx^2} + \left( \frac{5}{2}(1-x) - \frac{N}{2} \right) \frac{d}{dx}.
\]

We claim

**Proposition 8** Given \( \tilde{g} \in C([0, T], \mathfrak{x}^1 \times \mathfrak{x}) \), the equation \( D\tilde{\Phi} = \tilde{g} \) admits a unique solution \( \tilde{h} \in C([0, T], \mathfrak{x}^2 \times \mathfrak{x}^1) \) such that \( \tilde{h} |_{t=0} = 0 \).

Proof. We can rewrite

\[
(DF)_2 = \frac{\partial k}{\partial t} - H_1Dh + b_1\hat{D}h + b_0h + a_{21}\hat{D}k + a_{20}k,
\]

where

\[
\begin{align*}
\hat{D} &:= x(1-x)D, \\
b_1 &:= H_1L_1 x(1-x) + a_{11}, \\
b_0 &:= H_1L_0 + a_{10}.
\end{align*}
\]

Of course \( b_1, b_0 \) are also analytic on a neighborhood of \( [0, 1] \times \{0\} \times \cdots \). Then we can write the equation \( D\tilde{\Phi}(\tilde{w})\tilde{h} = \tilde{g}(= (g_1, g_2)^T) \) as

\[
\frac{\partial}{\partial t} \begin{bmatrix} h \\ k \end{bmatrix} + \begin{bmatrix} a_1 & -J \\ A & a_2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \tag{27}
\]
Here
\begin{align*}
a_1 &= a_{01} \hat{D} + a_{00}, \\
a_2 &= a_{21} \hat{D} + a_{20}, \\
A &= -H_1 \Lambda + b_1 \hat{D} + b_0.
\end{align*}

The standard calculation gives
\[
\frac{1}{2} \frac{d}{dt} \left( \|k\|^2 + ((H_1/J) \hat{D}h|\hat{D}h) \right) + \\
+ (\beta_1 \hat{D}h|\hat{D}h) + (\beta_2 \hat{D}h|h) + (\beta_3 \hat{D}h|k) + (\beta_4 h|k) + (\beta_5 k|k) = \\
= ((H_1/J) \hat{D}h|\hat{D}g_1) + (k|g_2),
\]
(28)

where \( \hat{D} = \sqrt{x(1-x)}D \) and
\[
\beta_1 = -\frac{1}{4}(3 + (N + 3)x + 2\hat{D})(H_1/J)a_{01} - \frac{1}{2} \frac{\partial(H_1/J)}{\partial t} + (H_1/J)(\hat{D}a_{01} + a_{00}), \\
\beta_2 = (H_1/J)\hat{D}a_{00}, \\
\beta_3 = -(H_1/J)\hat{D}J + \hat{D}H_1 + \sqrt{x(1-x)}(b_1 + a_{21}), \\
\beta_4 = b_0, \quad \beta_5 = a_{20}.
\]

We assume that \( \beta_j \in C([0,T] \times [0,1]) \) for \( j = 1, 2, \ldots, 5 \).

Of course \( (\cdot|\cdot) \) and \( \| \cdot \| \) stand for the inner product and the norm of the Hilbert space \( X \). We have used the following formulas:

**Formula 1:** If \( \phi \in X^2, \psi \in X^1, \alpha \in C^1([0,1]), \) then
\[
(-\alpha \Lambda \phi|\psi) = (\alpha \hat{D}\phi|\hat{D}\psi) + ((D\alpha)\hat{D}\phi|\psi).
\]
(29)

**Formula 2:** If \( \phi \in X^2 \) and \( \alpha \in C^1([0,1]), \) then
\[
(\alpha \hat{D}\phi|\hat{D}\hat{D}\phi) = (\alpha^* \hat{D}\phi|\hat{D}\phi),
\]
(30)
with
\[
\alpha^* = -\frac{1}{4}(3 + (N + 3)x + 2\hat{D})\alpha.
\]

Since \( \vec{w} = (y^* + \tilde{y}, v^* + \tilde{v})^T \) is supposed to be small, we can assume\[
\frac{1}{M_0} < J < M_0, \quad \frac{1}{M_0} < H_1 < M_0\]

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with a constant $M_0$ independent of $\vec{w}$ thanks to the assumption (B2). Now the energy
\[ \mathcal{E} := \|k\|^2 + ((H_1/J)\dot{D}h)\dot{D}h) \]
enjoys the inequality
\[ \frac{1}{2} \frac{d\mathcal{E}}{dt} \leq M(\|\vec{h}\|_2^2 + \|\vec{h}\|_B\|\vec{g}\|_B), \]
where $\vec{h} = \mathcal{X}^1 \times \mathcal{X}$ and
\[ \|(\phi, \psi)^T\|_{\vec{h}}^2 = \|\phi\|_{\mathcal{X}^1}^2 + \|\psi\|_{\mathcal{X}} = \|\phi\|^2 + \|\dot{D}\phi\|^2 + \|\psi\|^2 \]
and
\[ M = \sum_{j=1}^{5} \|\beta_j\|_{L^\infty} + (M_0)^2 + 1. \]
Since $\mathcal{E}$ is equivalent to $\|k\|^2 + \|\dot{D}h\|^2$, the Gronwall argument and application of the Kato’s theory ([8]) deduce the conclusion. Here $\|\vec{h}\|$ should be estimated by $\mathcal{E}$ as follows: The first component of (27) implies
\[ h(t) = \int_0^t (-a_{01}Dh - a_{00}h + Jk + g_1)(t')dt', \]
therefore
\[ \|h(t)\| \leq C \int_0^t \|h(t')\|dt' + \int_0^t (C\mathcal{E}(t') + \|g_1(t')\|)dt', \]
where $C = \max(\|a_{00}\|_{L^\infty}, \|a_{01}\|_{L^\infty}M_0^2 + M_0)$, which implies, through the Gronwall’s argument,
\[ \|h(t)\| \leq \int_0^t (e^{C(t-t')} - 1)(C\mathcal{E}(t') + \|g_1(t')\|)dt'. \]
As the result the solution enjoys
\[ \|\vec{h}(t)\|_B \leq C \int_0^t \|\vec{g}(t')\|_B dt'. \]

Here, in order to make sure, let us sketch proofs of Formula 1, Formula 2, and (28).

Proof of (29): if $\psi \in \mathcal{X}^1$, then
\[ \psi(1/2) + \int_{1/2}^x \frac{\dot{D}\psi(x')}{\sqrt{x'(1-x')}}dx'. \]
implies
\[ |\psi(x)| \leq Cx^{-3/4}(1 - x)^{-N/4+1/2}, \]
and, if \( \phi \in X^2 \), then
\[ x^{5/2}(1 - x)^{N/2} \frac{d\phi}{dx} = x^{5/2}(1 - x)^{N/2} \frac{d\phi}{dx} |_{x=1/2} + \]
\[ - \int_{1/2}^{x} \Lambda \phi(x')x^{3/2}(1 - x')^{N/2-1} dx' \]
implies
\[ \left| \frac{d\phi}{dx} \right| \leq Cx^{-5/4}(1 - x)^{-N/4}. \]
Actually the finite constant
\[ x^{5/2}(1 - x)^{N/2} D\phi|_{x=1/2} + \int_{0}^{1/2} \Lambda \phi(x')x^{3/2}(1 - x')^{N/2-1} dx' \]
should vanish in order to \( \dot{D}\phi \in X \) and so on. Therefore the boundary terms in the integration by parts vanish at \( x = +0, 1-0 \) and we get the desired equality.

Proof of (30): We see
\[ (\alpha \dot{D}\phi | \dot{D} \dot{D}\phi) = \int_{0}^{1} \alpha x(1 - x)(D\phi)D(x(1-x)D\phi)x^{3/2}(1 - x)^{N/2-1} dx \]
\[ = I + (\alpha(1 - 2x)\dot{D}\phi | \dot{D} \dot{D}\phi), \]
where
\[ I := \int_{0}^{1} \alpha(x(1 - x))^2 (D\phi)(D^2 \phi)x^{3/2}(1 - x)N/2 - 1 dx \]
\[ = \int_{0}^{1} \frac{\alpha}{2} D(D\phi)^2 x^{7/2}(1 - x)^{N/2+1} dx \]
\[ = - \int_{0}^{1} D \left( \frac{\alpha}{2} x^{7/2}(1 - x)^{N/2+1} \right) (D\phi)^2 dx. \]
Here the integration by parts has been done by using
\[ |D\phi| \leq Cx^{-5/4}(1 - x)^{-N/4} \]
which holds for \( \phi \in X^2 \). Then we see
\[ I = - \left( \left( \frac{\dot{D}\alpha}{2} + \frac{\alpha}{2} \left( \frac{7}{2} (1 - x) + \left( \frac{N}{2} + 1 \right)x \right) \right) \dot{D}\phi | \dot{D} \dot{D}\phi \right), \]
and get (30).
Proof of (28): multiplying the second component of the equation (27) by $k$ and integrating it, we get
\[
\frac{1}{2} \frac{d}{dt} \|k\|^2 - (H_1 \Delta h | k) + (b_1 \hat{D}h | k) + (b_0 h | k) + (a_{21} \hat{D}h | k) + (a_{20} k | k) = (g_2 | k).
\]

By Formula 1 we see
\[
\frac{1}{2} \frac{d}{dt} \|k\|^2 + (H_1 \hat{D}h \hat{D}k) + ((DH_1) \hat{D}h | k) + (b_1 \hat{D}h | k) + (b_0 h | k) + (a_{21} \hat{D}h | k) + (a_{20} k | k) = (g_2 | k).
\]

On the other hand, operating $\hat{D}$ on the first component of (27), we get
\[
\hat{D}k = \frac{1}{J} \partial_t (\hat{D}h) + \frac{a_{01}}{J} \hat{D}h + \frac{1}{J} (\hat{D}a_{01} + a_{00}) \hat{D}h + \frac{\hat{D}a_{00}}{J} h - \frac{\hat{D}J}{J} h - \frac{\hat{D}J}{J} k - \frac{1}{J} \hat{D}g_1.
\]

Inserting this into the second term of the left-hand side of (31), we get
\[
\frac{1}{2} \frac{d}{dt} \|k\|^2 + \frac{1}{2} \frac{d}{dt} ((H_1 / J) \hat{D}h \hat{D}h) + ((H_1 a_{01} / J) \hat{D}h \hat{D}h) + ((1/2)(H_1 / J) + (H_1 / J) (\hat{D}a_{01} + a_{00})) \hat{D}h \hat{D}h + ((H_1 / J) (\hat{D}a_{00}) \hat{D}h | h) + (H_1 \hat{D}J / J) \hat{D}h | k - ((H_1 / J) \hat{D}h | \hat{D}g_1) + ((b_1 + a_{21}) \hat{D}h | k) + (b_0 h | k) + (a_{20} k | k) = (g_2 | k).
\]

Applying Formula 2 to the third term of the left-hand side, we get the desired (28).

Of course, when we apply the Nash-Moser(-Schwartz) theorem, we take $\mathcal{I}(\vec{w}) : \vec{g} \mapsto \vec{h}$, where $\vec{h}$ is the solution of
\[
D\mathcal{F}(\vec{w}) \vec{h} = \vec{g}, \quad \vec{h}_{|t=0} = \vec{0}.
\]

7 Tame estimate of the inverse $\mathcal{J}(\vec{w})$ of the the Fréchet derivative $D\mathcal{F}(\vec{w})$

Now we investigate the equation $D\mathcal{F}(\vec{w}) \vec{h} = \vec{g}$ on $0 \leq t \leq T$. The equation can be written as:
\[
\frac{\partial}{\partial t} \begin{bmatrix} h_{[\mu]} \\ k_{[\nu]} \end{bmatrix} + \begin{bmatrix} a_1^{[\mu]} \\ a_2^{[\mu]} \end{bmatrix} \begin{bmatrix} h_{[\nu]} \\ k_{[\nu]} \end{bmatrix} = \begin{bmatrix} g_1^{[\mu]} \\ g_2^{[\mu]} \end{bmatrix} + (-1)^\mu \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} h_{[1-\mu]} \\ k_{[1-\mu]} \end{bmatrix}, \quad \mu = 0, 1,
\]

(32)
where
\[ a_1^{[\mu]} = a_{01} \hat{D} + a_{00} = a_{01} \hat{D} + a_{00} - (1)^{\mu} a_{01} \hat{D} \omega, \]
\[ a_2^{[\mu]} = a_{21} \hat{D} + a_{20} = a_{21} \hat{D} + a_{20} - (1)^{\mu} a_{21} \hat{D} \omega, \]
\[ A^{[\mu]} = -H_1 \Lambda + (b_1 + (1)^{\mu} 2H_1(D\omega)) \hat{D} + b_0 + (1)^{\mu} (H_1 \Lambda - b_1 \hat{D}) \omega, \]
\[ c_{11} = a_{01} \hat{D} \omega, \]
\[ c_{21} = -2H_1(D\omega) \hat{D} + b_1 \hat{D} \omega - H_1 \Lambda \omega, \]
\[ c_{22} = a_{21} \hat{D} \omega. \]

We can write
\[ A^{[\mu]} = -b^{[\mu]}_2 \Delta^{[\mu]} + b^{[\mu]}_1 \hat{D}^{[\mu]} + b^{[\mu]}_0, \]
where
\[ \hat{D}^{[0]} = x \frac{\partial}{\partial x}, \quad \hat{D}^{[1]} = X \frac{\partial}{\partial X} \quad \text{with} \quad X = 1 - x, \]
\[ b^{[0]}_2 = H_1 \cdot (1 - x), \quad b^{[0]}_1 = \frac{N}{2} H_1 + (b_1 + 2H_1(D\omega))(1 - x), \]
\[ b^{[1]}_2 = H_1 \cdot x, \quad b^{[1]}_1 = \frac{5}{2} H_1 - (b_1 - 2H_1(D\omega))x, \]
\[ b^{[\mu]}_0 = b_0 - (1)^{\mu} (H_1 \Lambda - b_1 \hat{D}) \omega. \]

First we consider the component \( \vec{h} = \vec{h}^{[\mu]}, \mu = 0, 1, \) which satisfies
\[ \frac{\partial \vec{h}}{\partial t} + A \vec{h} = \vec{f}, \]
with
\[ A = A^{[\mu]} = \begin{bmatrix} a_1^{[\mu]} & -J \\ A^{[\mu]} & a_2^{[\mu]} \end{bmatrix}, \]
\[ \vec{f} = f^{[\mu]} = \begin{bmatrix} f^{[\mu]}_1 \\ f^{[\mu]}_2 \end{bmatrix} = \begin{bmatrix} g^{[\mu]}_1 \\ g^{[\mu]}_2 \end{bmatrix} + (1)^{\mu} \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} h^{[1-\mu]}_1 \\ h^{[1-\mu]}_2 \end{bmatrix}. \]

We consider \( \mu = 1. \) Let us write \( x \) instead of \( X = 1 - x. \)

First we prepare the second Sobolev’s imbedding:

**Proposition 9** If \( s \in \mathbb{N}, s < \frac{N}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}, \) \( \frac{s}{N} \leq \frac{1}{p} \leq \frac{1}{2}, \) then \( \|u; L^p\| \lesssim \|u\|_s. \)

(Note that \( s \leq N/2 \iff s < s_N \) for \( N/2 \notin \mathbb{N}. \))

Here we denote the \( L^p \)-norm with respect to the measure \( x^{N/2-1} \) by
\[ \|u; L^p\| := \left( \int_0^1 |u(x)|^p x^{N/2-1} dx \right)^{1/p}. \]
Proof. We may assume that $u \in C_0^\infty([0,1])$, by dint of the extension technique. Supposet that $s \geq 1$, $\frac{1}{2} - \frac{s}{N} < \frac{1}{p} \leq \frac{1}{2}$. Let

$$u(x) = \sum_{n=1}^{\infty} c_n \psi_n(x),$$

where

$$\psi_n(x) = \frac{\Psi_\nu(\lambda_n x)}{\|\Psi_\nu(\lambda_n x)\|},$$

$$\Psi_\nu\left(\frac{r^2}{4}\right) = J_\nu(r)\left(\frac{r}{2}\right)^{-\nu}, \quad \nu = \frac{N}{2} - 1,$$

$$\lambda_n = \left(\frac{j_{\nu,n}}{2}\right)^2,$$

Here $J_\nu$ is the Bessel function, and $j_{\nu,n}$ is its $n$-th positive zero. See [12, Appendix A]. We know

$$\|\Psi_\nu(\lambda_n x)\|^{-1} \simeq n^{\frac{N-1}{2}}.$$

Since

$$\int_0^1 |\Psi_\nu(\lambda_n x)|^p x^{\nu} \, dx = \frac{1}{\lambda_n^{p+1}} I,$$

with

$$I = 2^{\nu(p-2)-1} \int_{0}^{j_{\nu,n}} |J_\nu(r)|r^{-\nu(p-2)+1} \, dr,$$

and since

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos \left(\frac{r \nu}{2} \pi - \frac{\pi}{4}\right) + O\left(\frac{1}{r^{3/2}}\right),$$

we see

(1): $I \lesssim 1$, provided that

$$-\frac{N-1}{2} (p-2) < -1;$$

(2): $I \lesssim \log n$, provided that

$$-\frac{N-1}{2} (p-2) = -1;$$

(3): $I \lesssim n^{-\frac{N-1}{2} (p-2)+1}$, provided that

$$-\frac{N-1}{2} (p-2) > -1.$$

Since $\lambda_n \simeq n^2$, we have

$$\|\psi_n; L^p\| \lesssim n^{\frac{N-1}{2} - \frac{N}{p}}.$$
for the case (1):
\[ \| \psi_n; L^p \| \lesssim n^{\frac{N}{2} + (N+\epsilon)^{1/p}}, \text{ with } 0 < \epsilon \ll 1 \]
for the case (2); and
\[ \| \psi_n; L^p \| \lesssim 1 \]
for the case (3). Therefore, keeping in mind that \( \| u \|_s \simeq \sqrt{|c_n|^2\lambda_n} \), we have
\[ \| u; L^p \| \lesssim \| u \|_s \sqrt{\sum n^{2(\frac{N}{2} + (N+\epsilon)^{1/p} - s)}} \]
for the case (1);
\[ \| u; L^p \| \lesssim \| u \|_s \sqrt{\sum n^{2(\frac{N}{2} + (N+\epsilon)^{1/p} - s)}} \]
for the case (2); and
\[ \| u; L^p \| \lesssim \| u \|_s \sqrt{n - 2s} \]
for the case (3). Then we see
\[ 2 \left( \frac{N - 1}{2} - \frac{N}{p} - s \right) < 2 \left( \frac{N - 1}{2} - (N - \epsilon)^{1/p} - s \right) < -1 \]
and \(-2s < -1\), provided that \( s \geq 1 \) and \( \frac{1}{2} - \frac{s}{N} \leq \frac{1}{p}, \epsilon \ll 1 \). So, in every case \( \sqrt{\sum \cdots} < \infty \). □

This implies:

**Proposition 10** If \( s_1, s_2, k \in \mathbb{N} \) satisfy \( s_1 \geq k, s_2 \geq k, s_1 + s_2 \geq s_N + k \), then it holds that
\[ \| f \cdot g \|_k \lesssim \| f \|_{s_1} \| g \|_{s_2}. \]
(Note that \( s_1 + s_2 > \frac{N}{2} + k \leftrightarrow s_1 + s_2 \geq s_N + k \) for \( \frac{N}{2} \notin \mathbb{N} \).)

Proof. First consider the case with \( k = 0 \). Suppose \( s_1 < N/2, s_2 < N/2 \). Hölder’s inequality gives
\[ \| f \cdot g \| \leq \| f; L^p \| \| g; L^q \|, \]
provided that \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \). Choose \( p \) such that
\[ \frac{1}{2} - \frac{s_1}{N} \leq \frac{1}{p} \leq \frac{s_2}{N} \quad \left( < \frac{1}{2} \right). \]
This possible, since \( s_1 + s_2 > N/2 \). Then Proposition 8 gives \( \| f; L^p \| \lesssim \| f \|_{s_1} \), and
\[ \frac{1}{2} - \frac{s_2}{N} \leq \frac{1}{q} = \frac{1}{2} - \frac{1}{p} \leq \frac{s_1}{N}, \]
which gives \( \|g; L^q\| \lesssim \|g\|_{s_2} \). That is done. If \( s_1 \geq N/2 \), then \( s_1 > N/2 \), and

\[
\|f; L^\infty\| \leq \|f\|_{s_1}, \quad \|g; L^2\| \lesssim \|g\|_{s_2}, \quad \|f \cdot g\| \lesssim \|f; L^\infty\| \|g; L^2\|
\]

will work for \( s_2 \geq 0 \). Thus the proof for \( k = 0 \) is done.

Next consider the case \( k = 2m \geq 2 \) with \( m \in \mathbb{N} \). We have to estimate \( \|\Delta^m(f \cdot g)\| \). But, according to [12, (8.5)], \( \Delta^m(f \cdot g) \) is a linear combination of terms

\[
\circ = (\hat{D}^{\ell'}D^{j'} f) \cdot (\hat{D}^{\ell}D^j g)
\]

with \( \ell + \ell' + 2(j' + j) = 2m \). In order to estimate \( \|\circ\| \), we put \( t_1 = s_1 - (\ell' + 2j'), t_2 = s_2 - (\ell + 2j) \). Then \( s_1 \geq 2m, s_2 \geq 2m, s_1 + s_2 \geq s_N + 2m \) imply \( t_1 \geq 0, t_2 \geq 0, t_2 + t_2 \geq s_N \). Therefore the inequality for \( k = 0 \) can be applied to get

\[
\|\circ\| \lesssim \|\hat{D}^{\ell'}D^{j'} f\|_{t_1}\|\hat{D}^{\ell}D^j g\|_{t_2} \lesssim \|f\|_{s_1}\|g\|_{s_2}.
\]

We omit the rest of the proof. \( \blacksquare \)

Let us deduce an elliptic estimate. Let \( u, \sigma \in \mathbb{N} \) satisfy \( s_N + 2 \leq \sigma, n + 2 \leq \sigma \).

We know

\[
\|[\Delta^m, \mathcal{A}]u\| \lesssim C(m, u), \tag{33}
\]

with

\[
C(m, u) := \sum \|\Delta^\gamma D^\delta b_i(\hat{D}\Delta^j u)\| + \|(\Delta^\gamma D^\delta b_i)(\Delta^j u)\| + \|\hat{D}\Delta^\gamma D^\delta b_i(\Delta^j u)\|. \tag{34}
\]

Here the summation runs over

\[
i = 0, 1, 2, \quad \gamma + \delta \leq k + 1, \quad k + j = m.
\]

See [12, Appendix D]. Similarly

\[
\|\hat{D}[\Delta^m, \mathcal{A}]u\| \lesssim C^\delta(m, u), \tag{35}
\]

with

\[
C^\delta(m, u) := C(m, u) + \sum \|\Delta^\gamma D^\delta b_i(\Delta^{j+1} u)\| + \|\hat{D}\Delta^\gamma D^\delta b_i(\Delta^j u)\|. \tag{36}
\]

The range of the summation is the same as above.

Using \( \Delta^m[\Delta, \mathcal{A}] = [\Delta^{m+1}, \mathcal{A}] - [\Delta^m, \mathcal{A}]\Delta \) and noting that \( C(m, \Delta u) \leq C(m + 1, u) \), we have

\[
\|\Delta^m[\Delta, \mathcal{A}]u\| \lesssim C(m + 1, u).
\]

We can estimate \( \|\hat{D}\Delta^m[\Delta, \mathcal{A}]u\| \) similarly, and we see

\[
\|[\Delta, \mathcal{A}]u\|_n \lesssim K(n + 2, u), \tag{37}
\]

27
where

\[ K(n, u) := \begin{cases} 
C(m, u) & \text{for} \quad n = 2^m \\
C^\sigma(m, u) & \text{for} \quad n = 2^m + 1 
\end{cases} \tag{38} \]

Let us estimate \( C(m, u) \). For example, consider

\[ \mathcal{A} = \| (\Delta \gamma D^\delta b_i)(\check{D} \Delta^j u) \|, \]

where \( i = 0, 1, 2, \gamma + \delta \leq k + 1, k + j = m \). Suppose that \( s_N + 2 \leq \sigma, n + 2 = 2m + 2 \leq \sigma \). Put \( s_1 = \sigma - 2 - 2k, s_2 = 2k \). Then \( s_1 \geq 0, s_2 \geq 0, s_1 + s_2 = \sigma - 2 \geq s_N \).

Therefore Proposition 10 can be applied to get

\[ \mathcal{A} \lesssim \| \vec{b} \|_s \| u \|_{n+1}, \tag{39} \]

provided that \( s_N + 2 \leq \sigma, n + 2 \leq \sigma \). Then we have

**Proposition 11** If \( n, \sigma \in \mathbb{N} \) satisfy \( s_N + 2 \leq \sigma, n + 2 \leq \sigma \), then it holds that

\[ \| u \|_{n+2} \lesssim \| Au \|_n + \| u \|_1, \tag{40} \]

provided that \( \| \vec{b} \|_\sigma \lesssim 1 \).

Proof. \( \Delta u = -\frac{1}{\nu_2} (Au - b_1 \check{D}u - b_0 u) \) implies

\[ \| u \|_2 \lesssim \| Au \| + \| u \|_1, \]
\[ \| u \|_3 \lesssim \| Au \|_1 + \| u \|_1. \]

See [12, pp.83-84]. Keeping in mind that \( \| u \|_1 \lesssim K(n, u) \) for \( n \geq 0 \), we can verify

\[ \| u \|_{n+2} \lesssim \| Au \|_n + K(n, u) \]

by induction, using the fact \( K(n-2, \Delta u) \leq K(n, u) \) and the interpolation

\[ \| u \|_{n+2} \lesssim \| u \|_n + \| \Delta u \|_n. \]

Then (39) implies

\[ \| u \|_{n+2} \lesssim \| Au \|_n + \| u \|_{n+1}, \]

provided that \( \| \vec{b} \|_\sigma \lesssim 1 \). By induction, we can replace \( \| u \|_{n+1} \) in the left-hand side by \( \| u \|_1 \). \( \blacksquare \).

This implies

\[ 28 \]
Proposition 12 If \( n, \sigma \in \mathbb{N} \) satisfy \( s_N + 2 \leq \sigma, n + 2 \leq \sigma \), then it holds that
\[
\| \vec{u} \|_{n+1} \lesssim \| \mathcal{A} \vec{u} \|_n + \| \vec{u} \|_1,
\]
provided that \( \| \vec{a} \|_\sigma \lesssim 1 \).

Here \( \vec{a} = \vec{a}^{[\mu]} = (b_0^{[\mu]}, b_1^{[\mu]}, b_2^{[\mu]}, a_{01}, a_{00}, a_{21}, a_{20}, J) \).

In order to derive Proposition 12 from Proposition 11, it is sufficient to note that
\[
\| \phi \cdot u \|_n \lesssim \| \phi \|_{\sigma'} \| u \|_n,
\]
provided that \( s_N \leq \sigma', n \leq \sigma' \). We omit the proof.

Inversely

Proposition 13 If \( n, \sigma \in \mathbb{N} \) satisfy \( s_N + 2 \leq \sigma, n + 2 \leq \sigma \), then it holds that
\[
\| \mathcal{A} \vec{u} \|_n \lesssim \| \vec{u} \|_{n+1}
\]
for \( 0 \leq t \leq \tau \), provided that \( |\vec{a}; \tau, \sigma| \lesssim 1 \).

Here we recall the definition
\[
|\phi; \tau, \nu| = |\phi; \tau, \nu|_{[\mu]} := \sup_{0 \leq t \leq \tau} \sum_{\iota + \kappa \leq \nu} \| \partial^\iota_t \phi(t, \cdot) \|_{[\mu]_\kappa}.
\]

The proof can be reduced to
\[
\| \mathcal{A} u \|_n \lesssim \| u \|_{n+2} + K(n, u).
\]
Let us omit the details.

Let \( n, \sigma \in \mathbb{N} \) satisfy \( s_N + 1 \leq \sigma, n \leq \sigma \). Then we claim
\[
\| \| (\partial_t^j, \mathcal{A}) u \|_k \| \lesssim \| \vec{b}; \tau, \sigma \| \cdot \| u; \tau, n + 1 \|
\]
for \( j + k = n, 0 \leq t \leq \tau \).

In fact, let us consider, e.g.,
\[
[\partial_t^j, b_2 \triangle] u = \sum C_{\alpha \beta}(\partial_t^\alpha b_2)(\partial_t^\beta \triangle u).
\]

The summation runs over
\[
\alpha + \beta = j, \quad \alpha \geq 1.
\]

Put \( t_1 = \sigma - \alpha, t_2 = n - 1 - \beta = \alpha - 1 \). Then \( t_1 - k = \sigma - n + \beta \geq 0, t_2 - k = \alpha - 1 \geq 0, t_1 + t_2 - s_N - k = \sigma - 1 - s_N \geq 0 \). Proposition 10 can be applied to get
\[
\| (\partial_t^\beta b_2)(\partial_t^\beta \triangle u) \|_k \lesssim \| \partial_t^\alpha b_2 \|_{\sigma - \alpha} \| \partial_t^\beta \triangle u \|_{n-1-\beta}
\]
\[
\leq |b_2; \tau, \sigma| \cdot \| u; \tau, n + 1 \|,
\]
and so on. This observation implies
Proposition 14 If \( s_N + 2 \leq \sigma, n + 1 \leq \sigma \), then it holds that
\[
\| [\partial^j_t, \mathfrak{A}] \bar{u} \|_k \lesssim |\bar{a}; \tau, \sigma| \cdot |\bar{u}; \tau, n|
\] (45)
for \( j + k = n, 0 \leq t \leq \tau \).

Now we are going to find estimates of the solution \( \vec{h} = \vec{h}[\mu] \) of the problem
\[
\frac{\partial \vec{h}}{\partial t} + \mathfrak{A} \vec{h} = \vec{f}, \quad \vec{h}|_{t=0} = \vec{0},
\]
where \( \mathfrak{A} = \mathfrak{A}[\mu], \vec{f} = \vec{f}[\mu] \).

Consider \( \mu = 1 \). Recall that a solution of
\[
\frac{\partial \vec{H}}{\partial t} + \mathfrak{A} \vec{H} = \vec{F}
\]
enjoys the energy estimate
\[
\| \vec{H}(t) \| \lesssim \| \vec{H}(0) \| + \int_0^t \| \vec{F}(t') \| dt'.
\]

We put
\[
X(\vec{H}; j, k) := \| \partial^j_t \vec{H} \|_k, \quad X_0(\vec{H}; j, k) := X(\vec{H}; j, k)|_{t=0},
\]
\[
Z(\vec{H}; n) := \sum_{j+k=n} X(\vec{H}; j, k), \quad Z_0(\vec{H}; n) := Z(\vec{H}; n)|_{t=0},
\]
\[
W(\vec{H}; n) := \sum_{j+k \leq n} X(\vec{H}; j, k) = \sum_{\nu \leq n} Z(\vec{H}; \nu), \quad W_0(\vec{H}; n) := W(\vec{H}; n)|_{t=0}.
\]

First we note
\[
Z(\vec{h}; n + 1) = Z(\partial_t \vec{h}; n) + \| \vec{h} \|_{n+1}. \tag{46}
\]
Suppose that \( s_N + 2 \leq \sigma, n + 2 \leq \sigma, |\bar{a}; \tau, \sigma| \lesssim 1, 0 \leq t \leq \tau \). We claim
\[
Z(\vec{h}; n + 1) \lesssim \| \partial^{n+1}_t \vec{h} \| + \sum_{0 \leq \nu \leq n} \| \partial^\nu_{\tau'} \vec{h} \| + \| \vec{f} \|_n. \tag{47}
\]

(47) is true for \( n = 0 \), since
\[
Z(\vec{h}; 1) = \| \partial_t \vec{h} \| + \| \vec{h} \|_1
\]
\[
\lesssim \| \partial_t \vec{h} \| + \| \mathfrak{A} \vec{h} \| + \| \vec{h} \|
\]
\[
= \| \partial_t \vec{h} \| + \| \partial \vec{h} - \vec{f} \| + \| \vec{h} \|
\]
\[
\lesssim \| \partial_t \vec{h} \| + \| \vec{h} \| + \| \vec{f} \|,
\]

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thanks to Proposition 12. Suppose (47) is true for \( n \) such that \( n + 2 \leq \sigma \).

Then, thanks to Proposition 12, we see

\[
Z(\vec{h}; n + 1) = Z(\partial_t \vec{h}, n) + \| \vec{h} \|_{n + 1}
\]

\[
\lesssim \| \partial_t^{n+1} \vec{h} \| + \sum_{\nu \leq n-1} \| \partial_t^{\nu+1} \vec{h} \| + \| \vec{f} \|_{n-1}
\]

\[
+ \| \mathfrak{A} \vec{h} \|_n + \| \vec{h} \|
\]

\[
\lesssim \| \partial_t^{n+1} \vec{h} \| + \sum_{\nu \leq n-1} \| \partial_t^{\nu+1} \vec{h} \| + \| \vec{f} \|_{n-1}
\]

\[
+ \| \partial_t \vec{h} - \vec{f} \|_n + \| \vec{h} \|
\]

\[
\lesssim \| \partial_t^{n+1} \vec{h} \| + \sum_{\nu \leq n} \| \partial_t^{\nu+1} \vec{h} \| + \| \vec{f} \|_n,
\]

that is, (47) holds for \( n \). Hence (47) holds for \( n \leq \sigma - 2 \).

(47) implies

\[
W(\vec{h}; n + 1) \lesssim \| \partial_t^{n+1} \vec{h} \| + W(\vec{h}; n) + \| \vec{f} \|_n.
\]

By induction on \( n \), we have

\[
W(\vec{h}; n + 1) \lesssim \| \partial_t^{n+1} \vec{h} \| + \| \vec{h} \| + \| \vec{f} \|_n.
\]

(48)

On the other hand, \( \vec{H} = \partial_t^{n+1} \vec{h} \) satisfies

\[
\frac{\partial \vec{H}}{\partial t} + \mathfrak{A} \vec{H} = \vec{f}_{n+1}
\]

with

\[
\vec{f}_{n+1} := \partial_t^{n+1} \vec{f} - [\partial_t^{n+1}, \mathfrak{A}] \vec{h}.
\]

The energy estimate gives

\[
\| \partial_t^{n+1} \vec{h} \| \lesssim \| \partial_t^{n+1} \vec{h} \|_{t=0} + \int_0^t \| \vec{f}_{n+1}(t') \| dt'.
\]

(49)

Let us estimate \( \| \partial_t^{n+1} \vec{h} \|_{t=0} \).

First we note

\[
X_0(\vec{h}; 0, k) = \| \vec{h} \|_{t=0} \|_k = \| \vec{0} \|_k = 0 \quad \forall k.
\]

(50)

Let \( j + k = n \). Then

\[
X_0(j + 1, k) = \| \partial_t^{j+1} \vec{h} \|_{t=0} \|_k
\]

\[
= \| - \partial_t^j \mathfrak{A} \vec{h} + \partial_t^j \vec{f}_{t=0} \|_k
\]

\[
= \| - \mathfrak{A} (\partial_t^j \vec{h}) + [\partial_t^j, \mathfrak{A}] \vec{h} + \partial_t^j \vec{f}_{t=0} \|_k
\]

\[
\leq X_0(j, k + 1) + W_0(\vec{h}; n) + W_0(\vec{f}; n),
\]

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thanks to Propositions 13, 14. This and (50) imply
\[ X_0(\vec{h}; j, k) \lesssim W_0(\vec{h}; n) + W_0(\vec{f}; n) \] (51)
for \( j + k = n + 1 \). Hence
\[ W_0(\vec{h}; n + 1) \lesssim W_0(\vec{h}; n) + W_0(\vec{f}; n), \]
which implies
\[ W_0(\vec{h}; n + 1) \lesssim W_0(\vec{f}; n), \]
since \( W_0(\vec{h}; 0) = 0 \). Since \( \|\partial^{n+1}_t \vec{h}\| \leq W_0(\vec{h}; n + 1) \), (49) reads
\[ \|\partial^{n+1}_t \vec{h}\| \lesssim W_0(\vec{f}; n) + \int_0^t \|\vec{f}_{n+1}(t')\| dt'. \] (52)

On the other hand,
\[ \|\vec{f}_{n+1}(t)\| = \|\partial^{n+1}_t \vec{f} - \partial^{n+1}_t [\vec{h}]\| \lesssim \|\partial^{n+1}_t \vec{f}\| + |\vec{h}; t, n + 1|, \] (53)
thanks to Proposition 14. Here we recall
\[ |\vec{h}; \tau, n| = \sup_{0 \leq t \leq \tau} W(\vec{h}; n). \]

Summing up (48), (52), (53), we have
\[ |\vec{h}; t, n + 1| \lesssim 1 + W_0(\vec{f}; n) + \|\vec{f}\|_n + \int_0^t \|\partial^{n+1}_t \vec{f}(t')\| dt' + \int_0^t |\vec{h}; t', n + 1| dt' \] (54)
Here we have supposed
\[ \|\vec{h}^{[\mu]}\| \lesssim \int_0^t \|\vec{f}^{[\mu]}\| \lesssim \int_0^t \|\vec{g}^{[\mu]}\| + \|\vec{h}^{[1-\mu]}\| \lesssim \int_0^t \|\vec{g}\| \lesssim 1, \]
recalling
\[ \vec{f}^{[\mu]} = \vec{g}^{[\mu]} + \begin{pmatrix} (-1)^{\mu} c_{11} & 0 \\ (-1)^{\mu} c_{21} A + (-1)^{\mu} c_{20} & c_{22} \end{pmatrix} \vec{h}^{[1-\mu]}. \]
It follows from (54) by the Gronwall’s argument that
\[ |\vec{h}^{[\mu]}; t, n + 1| \lesssim 1 + |\vec{f}^{[\mu]}; t, n| + \int_0^t |\vec{f}^{[\mu]}; t', n + 1| dt'. \] (55)
This implies
\[ |\vec{h}; T, n + 1| \lesssim 1 + |\vec{g}; T, n| + \int_0^T |\vec{g}; t, n + 1| dt, \]
or, keeping in mind (5), we have

**Proposition 15** It holds that
\[ \|\vec{f}\|_n \lesssim 1 + \|\vec{g}\|_{n+1}, \]
provided that
\[ s_N + 2 \leq \sigma, \quad n + 2 \leq \sigma, \quad \|(1 - x)^{N/2}\|_{\sigma} < \infty, \quad \|\vec{w}\|_{\sigma+3} \lesssim 1, \quad \|\vec{g}\|_1 \lesssim 1. \]
8 Main result

Now we are ready to apply the Nash-Moser(-Schwartz) theorem to our problem.

Take $2\nu = n + 1, \sigma = 2\nu + 2 = n + 3$. Then $n + 2 \leq \sigma$ is satisfied, and $s_N + 2 \leq \sigma \iff s_N \leq 2\nu$. Proposition 15 reads that

$$\|\vec{h}\|_{\nu}^{(2)} \lesssim 1 + \|\vec{g}\|_{\nu+1},$$

provided that

$$(\|\vec{w}\|_{\sigma + 3} \lesssim \|\vec{w}\|_{\nu+3} \lesssim 1, \quad \|\vec{g}\|_{1}^{(2)} \lesssim 1, \quad \|(1 - x)^{N/2}\|_{2\nu + 2} < \infty).$$

(Recall (5)(6).) Since the Nash-Moser(-Schwartz) theorem requires that

$$\|\vec{w}\|_{E_j} \lesssim 1, \quad \|\vec{g}\|_{F_j} < \infty \Rightarrow \|\vec{h}\|_{E_{j-1}} < \infty,$$

we should guarantee that $\nu = b_E + r(j - 1)$ satisfy $\nu + 1 \leq b_F + rj$ and $\nu + 3 \leq b_E + rj$. That is, we require

$$b_E + r(j - 1) + 1 \leq b_F + rj$$

and

$$b_E + r(j - 1) + 3 \leq b_E + rj.$$  (57)

Now (57) is satisfied, if we take $r = 3$ to fix the idea. (56) reads

$$b_E - 2 \leq b_F$$

for $r = 3$. Recall that (254) required

$$b_F \leq b_E - 2.$$

So, (254) and (56) are satisfied if we take

$$b_F = b_E - 2.$$

We should have

$$s_N \leq 2\nu = 2(b_E + r(j - 1))$$

for $j = 1, \cdots, 10$. It holds if $2b_E \geq s_N$. If $N > 6$, then $s_N \geq 4$, and $2b_E \geq s_N$ and (25) are satisfied, if we take

$$b_E = s_N - 2.$$

Summing up, we take

$$b_E = s_N - 2, \quad b_F = s_N - 4, \quad r = 3,$$

provided that $N > 6$.

Finally $\|(1 - x)^{N/2}\|_{2\nu + 2} < \infty$ should hold for $\nu = b_E + r(J - 1) = s_N + 25$, where $J = 10$. This means

$$\frac{3N}{2} > 2s_N + 52,$$

which is equivalent to (24). This condition (65) is satisfied if $N > 108$.

Thus we have
Theorem 1  If $N > 108$, the Nash-Moser(-Schwartz) theorem can be applied: If $\|\mathfrak{F}(\bar{0})\|_{s_{N+1}}^{(2)}$ is sufficiently small, there exists a solution $\vec{w} = (\tilde{y}, \tilde{v})^T$ with small $\|\vec{w}\|_{s_{N+1}}^{(2)}$. Note that

$$
\mathfrak{F}(\bar{0}) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},
$$

$$
c_1 = -\frac{\partial y^*}{\partial t} + J\left(x, y^*, x \frac{\partial y^*}{\partial x}\right) u^*,
$$

$$
c_2 = -\frac{\partial v^*}{\partial t} - H_1\left(x, y^*, x \frac{\partial y^*}{\partial x}, v^*\right) \mathcal{L} y^* - H_2\left(x, y^*, x \frac{\partial y^*}{\partial x}, v^*, x \frac{\partial v^*}{\partial x}\right).
$$

9 Remark for the application to the gaseous stars

If we consider the gaseous stars governed by the Euler-Poisson equations under the exact $\gamma$-law $P = A\rho^\gamma$, the equilibrium governed by the Lane-Emden equation has a finite radius if and only if $\gamma > 6/5$, that is, $N < 12$. Therefore when $N$ is large, the equilibria cannot have finite radii. But the equation of state is not exact $\gamma$-law, equilibria can have finite radii, even if $\gamma - 1$ near the vacuum is very small. We shall show this.

Let us consider the Tolman-Oppenheimer-Volkoff equation

$$
\frac{dm}{dr} = 4\pi r^2 \rho, \quad \frac{du}{dr} = -\frac{G(m + 4\pi r^2 P/c^2)}{r^2(1 - 2Gm/c^2)},
$$

where $\rho$ and $P$ are given functions of $u$. Since we define

$$
u = \int_0^\rho \frac{dP}{\rho + P/c^2},
$$

when the function $\rho \to P$ is given so that $dP/d\rho = O(\rho^{\gamma-1})$ with $\gamma > 1$, we should put

$$
P = \int_0^u \varphi(u') e^{(u-u')/c^2} du',
$$

when $\varphi(u)$ is given. Here $\varphi(u)$ is a positive smooth function of $u > 0$ such that $D\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to 0$ as $u \to +0$. Suppose that there is a positive smooth function $\nu(u)$ of $u > 0$ such that $\nu(u) = \nu_0 + [u]_1$ as $u \to +0$ with a constant $\nu_0 > 1$ such that

$$
\rho = K^* \exp \left[ \int_{u_*}^u \frac{\nu(u')}{u'} du' \right].
$$

It holds for $\rho = \varphi(u)$, when $\nu(u) = uD\varphi(u)/\varphi(u)$, $K^* = \varphi(u^*)$, $u^* > 0$ being arbitrary. Note that if

$$
\nu(u) = \nu_0 = \frac{N_0}{2} - 1 = \frac{1}{\gamma_0 - 1} = \text{Const.}
$$
for \(0 < u \ll 1\), then we have \(\rho \propto u^{\nu_0}\) and \(P \propto \rho^{\gamma_0}(1 + [\rho^{\gamma_0-1}]_1)\) as \(\rho \to +0\).

Now let us consider \(\rho = \rho_1(u) = u^{\nu_1}\) with a constant \(\nu_1 = 1/(\gamma_1 - 1)\) such that \(1 < \nu_1 < 3\) or \(4/3 < \gamma_1 < 2\). Let \((m, u) = (m_1(r), u_1(r))\) be the solution such that \((m, u) = (0, u_{1c})\) at \(r = 0\). Since \(P \propto \rho^{\gamma_1}(1 + O(\rho^{\gamma_1-1}))\) with \(4/3 < \gamma_1 < 2\), Theorem I] says that this solution is short, that is, there is a finite \(r_{1+}\) such that \(u_1(r) \to 0\) as \(r \to r_{1+}\). Hence we can find \(r^* \in ]0, r_{1+}\] such that \(x_1(r^*) > 1/G\), where \(x = x_1(u)\) is defined by \(x = -m/ru\) along the solution \(u = u_1(r)\). Of course we assume that \(u_{1c}\) is so small that \(dP/d\rho < c^2\) for \(0 < u < 2u_{1c}\). Moreover we can assume that \(r^*\) is independent of large \(c\), for, as \(c \to +\infty\), the Tolman-Oppenheimer-Volkoff equation approaches the Lane-Emden equation

\[
\frac{dm}{dr} = 4\pi r^2 \rho^{\nu_1}, \quad \frac{du}{dr} = -\frac{Gm}{r^2}.
\]

Clearly there is a positive smooth function \(\tilde{\nu}(u)\) of \(u > 0\) such that \(\tilde{\nu}(u) = \nu_1\) for \(u \geq u^* := u_1(r^*)\) and \(\tilde{\nu}(u) = \nu_0\) for \(u \leq u^*/2\). Here we take \(N_0\) arbitrarily large and put \(\nu_0 = \frac{N_0}{2} - 1 = \frac{1}{\gamma_0 - 1}\).

Put

\[
\tilde{\rho}(u) = \rho_1(u^*) \exp \left[ \int_{u^*}^u \tilde{\nu}(u') - \frac{u}{u'} du' \right],
\]

\[
\tilde{P}(u) = \int_0^u \tilde{\rho}(u') e^{(u-u')/c^2} du'.
\]

Let \((m, u) = (m_0(r), u_0(r))\) be the solution for \(\rho = \tilde{\rho}(u), P = \tilde{P}(u)\) such that \((m, u) = (0, u_{1c})\). Since \(\tilde{\rho}(u) = \rho_1(u)\) for \(u \geq u^*\) and \(\tilde{P}(u) = O(1), d\tilde{P}/d\tilde{\rho} = O(1)\) as \(c \to +\infty\), we have \(d\tilde{P}/d\tilde{\rho} < c^2\) and \(|m_0(r) - m_1(r)| + |u_0(r) - u_1(r)| = O(1/c^2)\) for \(0 \leq r \leq r^*\), so, \(x_0(r^*) > 1/G\) provided that \(c\) is sufficiently large. In fact, as \(c \to +\infty\), the Tolman-Oppenheimer-Volkoff equation approaches the generalized Lane-Emden equation

\[
\frac{dm}{dr} = 4\pi r^2 \tilde{\rho}(u), \quad \frac{du}{dr} = -\frac{Gm}{r^2}
\]

with error \(O(1/c^2)\). Then by [11, p.61, Remark] we can claim that the solution \(u_0(r)\) is short, that is, there is a finite \(r_{0+} < r^* \exp[1/(Gx_0(r^*) - 1)]\) such that \(u_0(r) \to 0\) as \(r \to r_{0+}\). Since \(P \propto \rho^{\gamma_0}(1 + [\rho^{\gamma_0-1}]_1)\) with \(N_0\) arbitrarily large, this is a desired example.

**Acknowledgment**

A part of this work was done during the stay of the author at Nara Women's University at the opportunity of the Workshop 'Free Boundary Problems in Fluid and Plasma Dynamics' in February, 2016. The stay was financially supported by JSPS KAKENHI Grant Number 15K04957. The author would like
to express his sincere thanks to the organizer Professor Taku Yanagisawa (Nara WU) for his kind hospitality, and Professor Paolo Secchi (Brescia University) for his kind advices during discussions in the workshop. Moreover, the author would like to express his sincere thanks to the anonymous referee for pointing insufficiency of the original exposition. Thanks to his/her suggestions, the author could supplement the manuscript.

**Appendix**

Let us describe the spectral properties of the linear operator $L$. We write

$$L = -\frac{1}{b(x)} \frac{d}{dx} a(x) \frac{d}{dx} + L_0(x),$$

where

$$a(x) = x^{5/2} (1-x)^{N/2} M(x),$$
$$b(x) = x^{3/2} (1-x)^{N/2-1} M(x),$$
$$M(x) = \exp \left[ \int_0^x \frac{L_1(x')}{x'(1-x')} dx' \right].$$

Note that $M(x)$ is a smooth function of $0 \leq x < 1$ and enjoys

$$M(x) = \begin{cases} M_0 + [x]_1 & \text{as } x \to 0 \\ M_1 + [1-x, (1-x)^{N/2}]_1 & \text{as } x \to 1 \end{cases}$$

with positive constants $M_0, M_1$.

The Liouville transformation

$$\xi = \int_{1/2}^x \frac{dx}{\sqrt{x(1-x)}} = \arcsin(2x - 1),$$
$$y = x^{-1} (1-x)^{-\frac{(N-1)}{4}} M(x)^{-1/2} \eta$$

turns the equation

$$L y = \lambda y + f$$

to the standard form

$$-\frac{d^2 \eta}{dx^2} + q \eta = \lambda \eta + \hat{f},$$

where

$$\hat{f}(x) = x^{-1/2} (1-x)^{-\frac{(N-3)}{4}} M(x)^{-1/2} f(x),$$

and

$$q = L_0 + \frac{1}{4} a \left(D \left( \frac{Da}{a} + \frac{Db}{b} \right) - \frac{1}{4} \left( \frac{Da}{a} + \frac{Db}{b} \right)^2 + \frac{Da}{a} \left( \frac{Da}{a} + \frac{Db}{b} \right) \right).$$

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Note that $x = 0, 1/2, 1$ are mapped to $\xi = -\pi/2, 0, \pi/2$ and
\[
x \sim \begin{cases}
\frac{1}{4} \left( \frac{1}{\xi + \pi/2} \right)^2 & \text{as } x \to 0 \\
\frac{1}{4} \left( \frac{1}{\pi/2 - \xi} \right)^2 & \text{as } x \to 1.
\end{cases}
\]
We see
\[
q \sim \begin{cases}
\frac{2}{(\xi + \pi/2)^2} & \text{as } \xi \to -\pi/2 \\
\frac{(N-1)(N-3)}{4} \frac{1}{(\pi/2 - \xi)^2} & \text{as } \xi \to \pi/2.
\end{cases}
\]
Note that $(N-1)(N-3)/4 > 3/4$ under the assumption (B0): $N \geq 5$.

Therefore we have

*The operator $\mathcal{S}_0, D(\mathcal{S}_0) = C_0^\infty (-\pi/2, \pi/2)$, $\mathcal{S}_0 \eta = -d^2 \eta/dx^2 + q \eta$ in $L^2(-\pi/2, \pi/2)$ has the Friedrichs extension $\mathcal{S}$, a self-adjoint operator, whose spectrum consists of simple eigenvalues $\lambda_1 < \cdots < \lambda_n < \lambda_{n+1} < \cdots \to +\infty$. Thus the operator $\mathcal{T}_0, D(\mathcal{T}_0) = C_0^\infty (0, 1)$, $\mathcal{T}_0 y = \mathcal{L} y$ in $L^2([0, 1], x^{N/2}(1-x)^{(N-1)/2} dx)$ has the self-adjoint extension $\mathcal{T}$ with eigenvalues $(\lambda_n)$.*

Moreover we can claim

*If $\Phi(x)$ is an eigenfunction of the operator $\mathcal{T}$, we have
\[
\Phi(x) = \begin{cases}
C_0 + [x]_1 & \text{as } x \to 0 \\
C_1 + [1-x, (1-x)^{N/2}]_1 & \text{as } x \to 1,
\end{cases}
\]
where $C_0, C_1$ are non-zero constants.*

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