Testing local Lorentz invariance with short-range gravity

V. Alan Kostelecký¹ and Matthew Mewes²

¹Physics Department, Indiana University, Bloomington, Indiana 47405, USA
²Physics Department, California Polytechnic State University, San Luis Obispo, California 93407, USA

IUHET 622, November 2016

Abstract

The Newton limit of gravity is studied in the presence of Lorentz-violating gravitational operators of arbitrary mass dimension. The linearized modified Einstein equations are obtained and the perturbative solutions are constructed and characterized. We develop a formalism for data analysis in laboratory experiments testing gravity at short range and demonstrate that these tests provide unique sensitivity to deviations from local Lorentz invariance.

General relativity (GR) is founded on the Einstein equivalence principle, which incorporates local Lorentz invariance, local position invariance, and the weak equivalence principle. GR is known to provide an excellent description of classical gravity over a broad range of length scales. However, modifications of the Einstein equivalence principle associated with local Lorentz violation may arise in an underlying framework compatible with quantum physics such as string theory [1]. Searches for Lorentz violation in gravitational experiments may thus yield clues about the nature of physics beyond GR [2, 3].

An important class of precision tests of gravity involves experiments testing its properties at short distances below about a millimeter [4]. Remarkably, even some aspects of the conventional Newton force await verification on this scale, and the presence of larger forces falling as an inverse cubic, quartic, or faster is still compatible with existing experimental data. In this work, we use a comprehensive description of possible deviations from local Lorentz invariance in the pure-gravity sector to study laboratory tests of gravity at short range and to characterize their sensitivity vis-à-vis other types of investigations. Our results also provide a formalism for the analysis of data in short-range experiments.

One approach to studying Lorentz violation in gravity is to build a specific model and study its properties. However, since no compelling signals for Lorentz violation have been uncovered to date, guidance for a broad-based experimental search is perhaps best obtained by developing instead a framework allowing all types of Lorentz violation while including accepted gravitational physics. Effective field theory is one powerful technique along these lines, as it permits a general description of emergent effects from an unobservable scale [5].

In the context of gravity, the effective field theory for Lorentz violation [6] offers a model-independent framework for exploring observables for Lorentz violation. In the pure-gravity sector in Riemann geometry, the action of this theory contains the Einstein-Hilbert action and a cosmological constant along with all coordinate-independent terms involving gravitational-field operators. The pure-gravity action is a subset of the general effective field theory describing matter and gravity known as the gravitational Standard-Model Extension (SME). A term violating Lorentz invariance in the action consist of a Lorentz-violating operator contracted with a coefficient for Lorentz violation that controls the magnitude of the resulting physical effects. It is often convenient to classify the operators according to their mass dimension \( d \) in natural units, with operators having larger \( d \) likely to induce smaller physical effects at low energies due to a greater suppression by powers of the Newton gravitational constant or, equivalently, by inverse powers of the Planck mass.

To date, comparatively few of the coefficients for Lorentz violation in the pure-gravity sector have been constrained [2]. Most remain unexplored, and some could even involve large Lorentz violation that has escaped detection so far due to "countershading" by feeble couplings [7]. For \( d = 4 \), certain Lorentz-violating operators generate noncentral orientation-dependent corrections to the inverse-square law. These have been the subject of both theoretical work [8–15] and observation [16–25] and two-sided constraints at various levels down to parts in \( 10^{11} \) have been obtained on the nine corresponding coefficients for Lorentz violation. At \( d = 6 \), many Lorentz-violating operators produce instead corrections to Newton’s law involving an inverse quartic force [26]. A variety of short-range experiments [27–29] have attained sensitivities of order \( 10^{-9} \) m² to the 14 combinations of pure-gravity coefficients controlling this type of Lorentz violation in the nonrelativistic limit, and there are excellent prospects for improved sensitivity [30]. Constraints on some operators of dimensions \( d \leq 10 \) have also been reported, based on the nonobservation of gravitational Čerenkov radiation [31, 32] and from data on gravitational waves [33], while proposals for other measurements exist [34–37].

To provide a comprehensive discussion of possible effects of Lorentz violation in the nonrelativistic limit relevant for short-
range tests of gravity, we can expand the metric $g_{\mu\nu}$ around the Minkowski spacetime metric $\eta_{\mu\nu}$ and work with the general gauge-invariant and Lorentz-violating Lagrange density $L$, restricting attention to terms quadratic in the dimensionless metric fluctuation $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ and neglecting the cosmological constant. In this limit, the Einstein-Hilbert term takes the form

$$L_0 = \frac{1}{4} g_{\munu} \epsilon^{\munu} \eta_{\lambda} \partial_\lambda \partial_\rho h^\rho_{
uln}.$$  

(1)

Incorporating both Lorentz-violating and Lorentz-invariant operators of arbitrary mass dimension $d$, the Lagrange density $L$ can be written as [33]

$$L = L_0 + \frac{1}{2} h_{\mu\nu} (\delta g_{\munu} + \delta h_{\munu} + \delta q_{\munu}) h^\rho_{
uln}.$$  

(2)

Here, the derivative operators $\delta q_{\munu}$, $\delta g_{\munu}$, and $\delta h_{\munu}$ can be expanded as sums of constant cartesian coefficients $\delta (d)_{\mu
u\rho\sigma}$, $\delta (d)_{\mu\nu\rho\sigma}$, $\delta (d)_{\mu\nu\rho\sigma}$ for Lorentz violation contracted with factors of derivatives $\partial_\mu$, $\partial_\mu$, $\partial_\mu$ as

$$\delta q_{\munu} = \sum_{d=\text{even}} \delta (d)_{\mu
u\rho\sigma} \partial^{d-5},$$

$$\delta g_{\munu} = \sum_{d=\text{odd}} \delta (d)_{\mu
u\rho\sigma} \partial^{d-4},$$

$$\delta h_{\munu} = \sum_{d=\text{even}} \delta (d)_{\mu
u\rho\sigma} \partial^{d-5},$$  

(3)

where a circle index $\circ$ denotes an index contracted into a derivative, and where $n$-fold contractions are written as $\circ^n$. The operator $\delta q_{\munu}$ is antisymmetric in both the first and second pairs of indices, while $\delta \hat{q}_{\munu}$ is antisymmetric in the first pair and symmetric in the second, and $\delta \hat{h}_{\munu}$ is totally symmetric. Contracting any one of these operators with a derivative produces zero. Note that the $d = 4$ piece of $\delta q_{\munu}$ includes a term of the same form as $L_0$ with an overall scaling factor, which can be set to zero if desired.

In studying the nonrelativistic limit, it is convenient to work with the trace-reverse metric fluctuation

$$\overline{\eta}_{\mu\nu} = r_{\mu\nu\rho\sigma} h^\rho_{
uln},$$  

(4)

where

$$r_{\mu\nu\rho\sigma} = \frac{1}{4} (\eta_{\rho\sigma} g^{\mu\nu} + \eta_{\mu\nu} g^{\rho\sigma} - 2 \eta_{\mu\nu} \eta^{\rho\sigma})$$  

(5)

is the trace-reverse operator. The modified linearized Einstein tensor obtained by the variation of $L$ can be written as the sum of the usual linearized Einstein tensor $G_{\munu}^L$ and a correction $\delta G_{\munu}^L$:

$$G_{\munu}^L + \delta G_{\munu}^L = \frac{1}{4} \left( \partial_\mu \partial_\nu \overline{\eta}_{\rho\sigma} - \eta_{\rho\sigma} \partial_\mu \partial_\nu \overline{\eta}_{\rho\sigma} - \partial_\mu \partial_\nu \overline{\eta}_{\rho\sigma} \right) + \delta \overline{G}_{\munu}^L = -\frac{1}{2} \partial_\mu \partial_\nu \overline{\eta}_{\rho\sigma} + \delta \overline{G}_{\munu}^L,$$

(6)

where in the last line we adopt the Hilbert gauge, $\partial_\mu \overline{\eta}_{\rho\sigma} = 0$. The correction $\delta \overline{G}_{\munu}^L$ can be expressed as the action of a combination of derivative operators on $\overline{h}_{\mu\nu}$,

$$\delta \overline{G}_{\munu}^L = \delta M_{\munu\rho\sigma} \overline{h}_{\rho\sigma},$$  

(7)

where

$$\delta M_{\munu\rho\sigma} = \delta M^{\xi\lambda\nu\rho\sigma} r_{\xi\lambda\mu\nu}$$  

(8)

with

$$\delta M^{\mu\nu\rho\sigma} = -\frac{1}{2} (\delta q_{\mu\rho\nu} + \delta q_{\mu\nu\rho}) - \frac{1}{2} \delta \hat{q}_{\mu\nu\rho}$$

(9)

being expressed in terms of the operators appearing in the Lagrange density (2).

The modified linearized Einstein equation takes the form

$$G_{\munu}^L + \delta G_{\munu}^L = 8\pi G N T_{\munu}^M,$$

(10)

where $T_{\munu}^M$ is the energy-momentum tensor. The trace-reversed metric fluctuation can be expanded as $\overline{\eta}_{\rho\sigma} = \overline{\eta}_{\rho\sigma}^0 + \delta \overline{\eta}_{\rho\sigma}$, where $\overline{\eta}_{\rho\sigma}^0$ is a conventional Lorentz-invariant solution and $\delta \overline{\eta}_{\rho\sigma}$ is the perturbation arising from the correction $\delta G_{\munu}^L$. Solving Eq. (10) at first order then reduces to solving the coupled set of equations

$$\delta \overline{\eta}_{\rho\sigma}^0 = -16\pi G N T_{\rho\sigma}^M, \quad \delta^2 \overline{\eta}_{\rho\sigma} = 2\delta M_{\rho\sigma} \overline{\eta}_{\rho\sigma}^0.$$  

(11)

In the static limit, the zeroth-order solution satisfies the usual Poisson equation $\nabla^2 \overline{\eta}_{\rho\sigma}^0 = -16\pi G N T_{\rho\sigma}^M$ and takes the standard form

$$\overline{\eta}_{\rho\sigma}^0 (x) = 4\pi G N \int d^3 \chi' T_{\rho\sigma}^M (x'),$$

(12)

while the first-order solution is found to be

$$\delta \overline{\eta}_{\rho\sigma} = 4\pi G N \delta \overline{M}_{\rho\sigma} \int d^3 \chi' |x - x'| T_{\rho\sigma}^M (x').$$

(13)

Note that this solution is compatible with the Hilbert gauge because $\partial_\mu \delta M_{\rho\sigma} = 0$.

For applications to short-range experiments, which involve nonrelativistic sources, $T_{\munu}^M$ is well approximated by its energy-density component $T_{\mu0} = \rho (x)$, where $\rho (x)$ is the local mass density. We disregard here possible Lorentz-violating modifications to the dispersion relations for various SME matter species [11, 38], which generate geodesics on Finsler spacetimes [39, 40]. Also, the components of the metric fluctuation can be expressed in terms of a modified gravitational potential $U (x)$ producing a modified gravitational acceleration $g (x) = \nabla U$,

$$h_{\mu0}^0 = \frac{1}{\sqrt{\overline{\eta}^0}} = 2U, \quad h^i = \frac{1}{\sqrt{\overline{\eta}^0}} \delta^i = 2U \delta^i.$$  

(14)

Expanding $U (x) = U_0 (x) + \delta U (x)$ as the sum of the usual gravitational potential $U_0$ and the perturbation $\delta U$ then yields

$$U_0 (x) = G N \int d^3 \chi' \frac{\rho (x')}{|x - x'|},$$

(15)

as expected. The Lorentz-violating modification to the potential is given by

$$\delta U (x) = \frac{1}{4} \delta h_{\mu0} = \frac{1}{4} \sqrt{\overline{\eta}^0} \delta \overline{\eta}_{\rho\sigma} = 2G N \sqrt{\overline{M}}_{0000} \int d^3 \chi' |x - x'| \rho (x'),$$

(16)

where for convenience we define the double trace-reversed operator

$$\delta \overline{M}_{0000} = r_{\rho0\nu0} r_{\rho0\sigma0} \delta M_{\rho\nu\sigma\rho} = -\frac{1}{2} \delta M_{\rho\nu\sigma\rho}$$  

(17)
Note the noncovariant traces.

The last expression in Eq. (17) reveals that terms in $L$ involving the CPT-odd operator $\bar{q}^{\mu\nu\rho}$ produce no effects on short-range experiments. Modifications of the potential in this limit therefore arise only from operators of even dimension $d$, which contain an even number $d - 2$ of derivatives $\nabla$. Note also that the expression (16) for the Lorentz-violating potential holds in regions sufficiently far from source masses for the perturbative approach to be valid.

To make further progress in characterizing the result (16), it is convenient to perform Fourier transforms and work in momentum space, where we can identify $\delta_\mu \rightarrow ip_\mu$. In the quasistatic nonrelativistic limit we can neglect the frequencies $p_0$, so each derivative contraction can be taken as a contraction with the three-momentum $p^j$. In this limit, the operators $\bar{q}^{\mu\nu\rho\sigma}$ and $\bar{k}^{\mu\nu\rho\sigma}$ in Eq. (17) reduce to $d - 2$ symmetrized three-momenta contracted with constant coefficients for Lorentz violation. We can therefore perform a spherical-harmonic expansion,

$$\delta M_{\text{tot}} = \sum_{d,j,m} \rho^{d-2}Y_{jm}(\hat{p})K^{(d)}_{jm},$$

(18)

which captures the rotational properties of the perturbation that are essential for short-range experiments. Here, the sum ranges over $d = 4, 6, 8, \ldots$ and $j = 0, 2, 4, \ldots, d - 2$, and we write $p = |\hat{p}|$ for later convenience. The spherical coefficients $K^{(d)}_{jm}$ are constants controlling the magnitudes of the perturbative effects, and they are linear combinations of the cartesian coefficients appearing in Eq. (3).

When applied to the perturbed potential (16), the expansion (18) offers some direct insights into the nature of the perturbation effects. The maximum angular momentum $j_{\text{max}} = d - 2$ arises from the totally traceless and symmetric combination of cartesian coefficients contained in the expressions (3), which contain no Laplace operator $\nabla^2$. The contributions with $j = j_{\text{max}} = d - 2 = d - 4$ arise from coefficient combinations involving one spatial trace and therefore only a single Laplace operator. All other terms in the expansion (18) involve two or more spatial traces, producing two or more Laplace operators, and these cannot correct the gravitational potential because $\nabla^4|x - x'| = 0$ outside the source. In short, for fixed $d$ $\delta U$ acquires contributions only for $j = d - 2$ and $j = d - 4$, involving a total of $4d - 10$ independent coefficients $K^{(d)}_{jm}$. This implies that for fixed $d$ only $4d - 10$ independent physical effects can modify the Newton potential up to an overall scaling factor. For $d = 4$, this reproduces the degrees of freedom found in the modified potential in Eq. (137) of Ref. [8], while for $d = 6$ it matches the counting obtained in Eq. (7) of Ref. [26] when the Lorentz-invariant trace is removed.

To gain further insight, we can calculate $\delta U(x)$ for a point source, working in momentum space for convenience. This involves the Fourier transform of the modulus $r = |x - x'|$ of the displacement vector $r = x - x'$ from the source mass at $x'$ to the point $x$,

$$r = |r| = |x - x'| = \int d^dp \, \tilde{r}(p)e^{ip\cdot r}.$$  

(19)

This expression contains an infrared divergence because $r$ grows at infinity. Moreover, the momentum-space expression for $\delta U(x)$ involves $d - 2$ derivatives of $r$, which introduces ultraviolet divergences as well. We can control all the divergences by introducing a regulated version of the Fourier transform $\tilde{K}(p)$, given by

$$\tilde{r}(p; \epsilon, \Lambda) = -\frac{\partial}{\partial \epsilon} \frac{g^{(d)}(\epsilon)}{\pi^2(p^2 + \epsilon^2)^{d/2}},$$

(20)

where $\epsilon$ regulates the infrared divergence and $\Lambda$ regulates the ultraviolet divergences. The function $f(x)$ is taken as a generic even smoothing function of $x$ that is assumed to obey $f(x) \rightarrow 1$ when $x \rightarrow 0$ and $f(x) \rightarrow 0$ when $x \rightarrow \pm\infty$ and that vanishes sufficiently rapidly to suppress any relevant divergences. In imposing the limiting condition for $x \rightarrow \pm\infty$, we are allowing for an extension of the range of $p = |\hat{p}| \geq 0$ to the full real line for later convenience. The physical result can ultimately be obtained by taking the limits $\epsilon \rightarrow 0$ and $\Lambda \rightarrow \infty$. Adapting this regularization, the perturbation of the gravitational potential due to a single point source mass $m_0$, becomes

$$\delta U(r) = -2Gm_0 \sum_{d,j,m} \rho^{d-1}Y_{jm}(\hat{p}) \frac{g^{(d)}(\epsilon)}{\pi^2(p^2 + \epsilon^2)^{d/2}}e^{ip\cdot r},$$

(21)

where $u = \hat{r} \cdot \hat{p}$. Since $d \geq 4$, the infrared divergence is no longer an issue, so at this point we can take the limit $\epsilon \rightarrow 0$.

To perform the integral explicitly, it is useful to work in a chosen “apparatus” frame in which the $x_3$ axis points along a symmetry axis of the system. Note that this frame typically differs from the canonical laboratory frame used in Lorentz-violation studies [41]. Here, we adopt an apparatus frame in which $r = r\hat{x}_3$ is aligned with the $x_3$ axis. The spherical harmonics then contribute only for $m = 0$ and so reduce to Legendre polynomials, giving

$$\delta U^{\text{app}}(r\hat{x}_3) = -\frac{2Gm_{0}}{\pi} \sum_{d,j} \rho^{d-1} \frac{g^{(d)}(\epsilon)}{\pi^{2}} \frac{2\pi}{4 \pi} (i\epsilon)^{d-4}$$

$$\times \int_{u=1}^{\infty} du \, \Phi(u) \left( \frac{\partial}{\partial u} \right)^{d-4} \int_{-\infty}^{\infty} dp \, f(p) e^{ip\cdot r},$$

(22)

where we have used the evenness of $d$ and $j$ to extend the $p$ integral over the entire real line. In the limit $\Lambda \rightarrow \infty$, the $p$ integral becomes $2\pi\delta(u)$. Performing the $u$ integral using this delta function yields

$$\delta U^{\text{app}}(r\hat{x}_3) = -4Gm_{0} \sum_{d,j} \rho^{d-1} \frac{g^{(d)}(\epsilon)}{\pi^{2}}$$

$$\times (-1)^{d/2} \frac{2\pi^{d-1}}{4 \pi} r^{d-3} \frac{1}{\pi^{2}} f^{(d-4)}(0),$$

(23)

where $f^{(d)}(0)$ denotes the $r$-th derivative of the Legendre polynomial $P_r(x)$. Notice that this result is zero unless either $j = d - 2$ or $j = d - 4$, as expected, because the Legendre polynomial $P_j$ is of order $j$. Evaluation of the $(d - 4)$ derivatives of the Legendre polynomial gives

$$f^{(d-4)}(0) = \frac{(-1)^{(d+1)/2}(j + d - 4)!}{2^{j}\left(\frac{j}{2}(j - d + 4)\right)!\left(\frac{1}{2}(j + d - 4)\right)!}. $$

(24)
showing that they are nonvanishing for the range of indices of interest here. The correction to the Newton gravitational potential in the apparatus frame containing all contributing effects for Lorentz violation can therefore be written in the compact form

$$\delta U^{\text{app}}(r x_3) = \sum_{d,j} \frac{G N_m}{d! 3^d} \sqrt{\frac{2 \pi}{d^2}} \delta^{(d)}_{jm} \delta_{jm}^{(d)},$$

(25)

where now the sum over $d$ includes even values $d \geq 4$, and the allowed values of $j$ are $j = d - 2$ and $d - 4$. In this equation, we have introduced reduced spherical coefficients for Lorentz violation defined as

$$k^{(d)}_{jm} \equiv 4(-1)^{d/2} p_j^{d-4}(0) \mathcal{K}^{(d)}_{jm},$$

(26)

Note that this definition holds in any frame.

To apply this result in realistic circumstances, we must reconstruct the gravitational potential in the canonical laboratory frame [41]. In this frame, the $z$ axis points towards the zenith and the $x$ axis lies at an angle $\varphi$ east of south. Using standard angles for spherical polar coordinates, we can write the components of the displacement vector as $r = r(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$. The linear combination of spherical coefficients in the laboratory frame producing the spherical coefficients in the apparatus frame involves a rotation and can be expressed as

$$k^{(d)}_{jm}^{\text{app}} = \sum_m e^{i m \rho} e^{i m \chi} d_{jm}^{(d)}(\rho) k^{(d)}_{jm}^{\text{lab}},$$

(27)

where $\gamma$ is an Euler angle relating the two frames that plays no physical role because the sum (25) involves only $m = 0$, and where the quantities $d_{jm}^{(d)}(\rho)$ are the little Wigner matrices given by Eq. (136) of Ref. [42]. Using the identity

$$Y_{jm}(\theta, \phi) = \sqrt{\frac{2j+1}{4\pi}} e^{im\phi} d_{jm}^{(0)}(\theta),$$

(28)

we find that the correction to the gravitational potential in terms of the spherical coordinates $r$, $\theta$, and $\phi$ in the laboratory frame is

$$\delta U(r) = \sum_{d,j,m} \frac{G N_m}{d! 3^d} Y_{jm}(\theta, \phi) k^{(d)}_{jm}^{\text{lab}},$$

(29)

where $d \geq 4$ is even, $j = d - 2$ or $j = d - 4$, and $m = -j, \ldots, j$. The corresponding correction $\delta g(r)$ to the gravitational acceleration is given by

$$\delta g(r) = \nabla \delta U.$$

(30)

The coefficients $k^{(d)}_{jm}$ for Lorentz violation are frame-dependent quantities, so an inertial frame must be specified in reporting their measurement. The laboratory frame is noninertial due to the rotation of the Earth, so it is unsuitable for this purpose. The canonical inertial frame adopted in the literature is the Sun-centered celestrial-equatorial frame [41], which is conventionally defined using cartesian coordinates $(T, X, Y, Z)$. The origin for $T$ is fixed as the 2000 vernal equinox, at which time the $X$ axis lies along the line from the Earth to the Sun. The $Z$ axis is aligned with the rotation axis of the Earth, and the $Y$ axis forms a right-handed coordinate system. To a sufficient approximation, the Sun-centered frame is inertial over the time scale of typical laboratory experiments, and in this frame the coefficients for Lorentz violation can be taken as spacetime constants [43]. The Earth rotation therefore induces variations with sidereal time of the coefficients in the laboratory frame, which implies that sidereal variations can appear in experimental data [44]. The result (29) must therefore be expressed in terms of coefficients in the Sun-centered frame when performing an experimental analysis. This conversion involves a rotation that depends on sidereal time [41]. Standard methods [42] can be applied to obtain the relationship

$$k^{(d)}_{jm}^{\text{lab}} = \sum_{m'} e^{im'} e^{im} Y_{jm}(\rho, \chi) k^{(d)}_{jm}^{\text{app}},$$

(31)

between the Newton spherical coefficients $k^{(d)}_{jm}^{\text{app}}$ in the laboratory frame and the Newton spherical coefficients $k^{(d)}_{jm}$ in the Sun-centered frame. Here, $\omega_0 \equiv 2\pi/(23.5 \times 56)$ is the sidereal angular rotation rate of the Earth, and $\chi$ is the colatitude of the laboratory. Also, $T_0$ is the local laboratory sidereal time, which differs from the time $T$ by a constant offset [45].

The coefficients $k^{(d)}_{jm}$ in the canonical Sun-centered frame are the ultimate target of experimental analyses. Using the result (29) from a point source of mass $m$, together with spherical coefficients expressed in the Sun-centered frame according to Eq. (31), the gravitational potential and hence the force due to an extended source mass can be obtained. The inverse-power corrections appearing in the potential (29) imply that experiments testing gravity at short range have maximal sensitivity to these Lorentz-violating effects. In practical applications, numerical methods are likely to be required to calculate the gravitational potential from a test mass of finite extent [27–30]. Nonetheless, the equations derived here via the spherical decomposition provide a clean separation of the observable harmonics in sidereal time and therefore offer a direct path for analyses seeking effects of Lorentz violation at arbitrary $d$.

The methodology developed here also permits sensitivity comparisons between short-range experiments and other types of investigations. For example, an earlier analysis has provided a complete characterization of coefficients for Lorentz violation that are accessible to experiments involving gravitational waves [33]. This work reveals that Lorentz violation in gravitational radiation is controlled by four sets of vacuum spherical coefficients, $k_{(1)jm}$, $k_{(2)jm}$, $k_{(3)jm}$, and $k_{(4)jm}$. Data from the observation of gravitational waves and the absence of gravitational Čerenkov radiation already place significant constraints on a subset of these coefficients [31, 33]. It is therefore of definite interest to establish the relationship between the Newton spherical coefficients $k^{(d)}_{jm}$ and the vacuum spherical coefficients in the canonical Sun-centered frame. Here, we demonstrate that a partial overlap exists: short-range experiments are sensitive to certain types of Lorentz violation that are inaccessible to analyses using gravitational radiation, and vice versa.

Table 1 provides a summary of the Newton and vacuum spherical coefficients. The first two columns of this table identify the type of coefficients and list their components. The third
Table 1: Summary of Newton and vacuum spherical coefficients.

| Type         | Coefficient | Parity | $d$ | $j$ | Number | $d = 4$ | $d = 5$ | $d = 6$ | $d = 7$ | $d = 8$ |
|--------------|-------------|--------|-----|-----|--------|--------|--------|--------|--------|--------|
| Newton       | $k_{jm}^{N(4)}$ | $E$     | even, $\geq 4$ | $d - 4, d - 2$ | $(d - 1)^2$ | 4 | 6 | 10 | 14 | 22 |
| vacuum       | $k_{(1)jm}^{d}$ | $E$     | even, $\geq 4$ | 0, 1, $\ldots$, $d - 2$ | $(d - 1)^2$ | 9 | 25 | 49 |  |  |
| vacuum       | $k_{(2)jm}^{d}$ | $E$     | even, $\geq 6$ | 4, 5, $\ldots$, $d - 2$ | $(d - 1)^2 - 16$ |  | 9 | 33 |  |  |
| vacuum       | $k_{(3)jm}^{d}$ | $B$     | even, $\geq 6$ | 4, 5, $\ldots$, $d - 2$ | $(d - 1)^2 - 16$ | 9 | 33 |  |  |  |
| vacuum       | $k_{(4)jm}^{d}$ | $B$     | odd, $\geq 5$ | 0, 1, $\ldots$, $d - 2$ | $(d - 1)^2$ | 16 | 36 |  |  |  |

We can also show that the six Newton spherical coefficients $k_{jm}^{N(4)}$ are related to the dual cartesian coefficients $\bar{\Gamma}_{\xi\lambda}^{(4)}$ according to

$$k_{(1)jm}^{N(4)} = \frac{1}{\sqrt{2}}(-1)^{j+d/2}k_{jm}^{N(4)},$$

(32)

which is a one-to-one relationship.

For the case $d = 4$, the only coefficients for Lorentz violation appearing in the Lagrange density (2) are the cartesian coefficients $s^{(4)apovert}$. These contain ten independent components, which can conveniently be packaged in the dual cartesian coefficients

$$\bar{\Gamma}_{\xi\lambda}^{(4)} \equiv -\frac{1}{36} \epsilon_{apovert} \epsilon_{\xi\lambda\eta\delta} s^{(4)apovert}. $$

Note that the cartesian indices distinguish these dual coefficients from the related spherical coefficients appearing in Eq. (32). Calculation shows that the nine vacuum spherical coefficients $k_{(1)jm}^{(d)}$ are given in terms of the dual cartesian coefficients $\bar{\Gamma}_{\xi\lambda}^{(4)}$ by

$$k_{(1)jm}^{(4)} = -\frac{1}{36} \sqrt{\frac{5}{2}} \bar{\Gamma}_{\xi\lambda}^{(4)}.$$

(33)

The nine vacuum spherical coefficients $k_{(1)jm}^{(4)}$ therefore span the coefficient space for $d = 4$, and so the six Newton spherical coefficients $k_{jm}^{N(4)}$ are completely determined by the vacuum spherical coefficients $k_{(1)jm}^{(4)}$. Since tight two-sided bounds on $k_{(1)jm}^{(4)}$ have been obtained from the absence of gravitational Čerenkov radiation [31], we can conclude that short-range tests of gravity searching for anisotropic effects cannot yield unique information about the $d = 4$ coefficients.

The situation for $d = 6$ is more involved. Here, there are 84 independent cartesian coefficients $s^{(6)\mu_1...\mu_6}$ and also 105 independent cartesian coefficients $k^{(6)\mu_1...\mu_6}$, for a total of 189 degrees of freedom. For $d = 6$, Table 1 shows that the vacuum spherical coefficients include 25 independent components of $k_{(1)jm}^{(6)}$ governing nonbirefringent effects, 9 independent components $k_{(E)jm}^{(6)}$ controlling $E$-parity birefringent effects, and 9 independent components $k_{(B)jm}^{(6)}$ determining $B$-parity birefringent effects. The Newton spherical coefficients $k_{jm}^{N(6)}$ include 14 independent coefficients controlling $E$-parity Lorentz-violating operators. Explicit expressions for all 57 of these spherical coefficients in terms of the 189 independent cartesian coefficients are lengthy and so are omitted here. However, some calculation yields a relationship among the $j = 4$ components of the spherical coefficients controlling $E$-parity effects,
This reveals that the nine Newton spherical coefficients \( k_{4m}^{(N(6))} \) with \( j = 4 \) are completely determined by vacuum spherical coefficients. In contrast, the five Newton spherical coefficients \( k_{2m}^{(N(6))} \) with \( j = 2 \) are independent of the vacuum spherical coefficients.

At present, the coefficients \( k_{4m}^{(N(6))} \) are tightly constrained [31], but the limits on the coefficients \( k_{4m}^{(E4m)} \) from gravitational waves [33] are one to three orders of magnitude weaker than the best current bounds from short-range experiments [29]. This difference in sensitivity can be traced to the inverse-quartic behavior of the modified gravitational force and the consequent gain in reach for tests at short range. However, even if future techniques for gravitational radiation are developed that permit vastly improved sensitivities to the coefficients \( k_{4m}^{(E4m)} \), the analysis performed here demonstrates that the Newton spherical coefficients \( k_{2m}^{(N(6))} \) with \( j = 2 \) will remain unconstrained by gravitational-radiation studies while being accessible in short-range experiments. Also, many vacuum spherical coefficients that can be studied using gravitational waves are inaccessible to short-range experiments, so the two extremes of Newton and relativistic experiments provide a complementary sensitivity to violations of Lorentz invariance in the gravity sector. Note also that the 48 independent degrees of freedom spanned in total by the vacuum and Newton spherical coefficients leave a 141-dimensional coefficient space at \( d = 6 \) that is untouched by studies of Lorentz violation using gravitational radiation or short-range tests of gravity. Identifying experimental tests with sensitivity to these many unconstrained effects is an interesting and worthwhile open problem.

For completeness, we can also explicitly relate the 14 \( d = 6 \) Newton spherical coefficients \( k_{jm}^{(N(6))} \) to the 14 \( d = 6 \) effective cartesian coefficients \( \tilde{k}_{JKLM}^{(4)} \) adopted in the recent literature discussing searches for Lorentz violation with experiments on short-range gravity [26–29]. We find the correspondence

\[
\begin{align*}
k_{20}^{(N(6))} & = \frac{36}{\sqrt{7}} \left( (\tilde{k}_{LM})_{XXIJ} + (\tilde{k}_{LM})_{YYIJ} \right), \\
\text{Re} k_{21}^{(N(6))} & = \frac{12}{\sqrt{7}} \sqrt{2} (\tilde{k}_{LM})_{XZJJ}, \\
\text{Im} k_{21}^{(N(6))} & = -\frac{12}{\sqrt{7}} \sqrt{2} (\tilde{k}_{LM})_{YZZJ}, \\
\text{Re} k_{22}^{(N(6))} & = -\frac{6}{\sqrt{7}} \sqrt{2} (\tilde{k}_{LM})_{XXJJ} - (\tilde{k}_{LM})_{YYJJ}, \\
\text{Im} k_{22}^{(N(6))} & = \frac{12}{\sqrt{7}} \sqrt{2} (\tilde{k}_{LM})_{XZJJ}, \\
k_{40}^{(N(6))} & = \frac{3}{2} \sqrt{7} \left( (\tilde{k}_{LM})_{XXJJ} + (\tilde{k}_{LM})_{YYJJ} + 7(\tilde{k}_{LM})_{XZJJ} + 7(\tilde{k}_{LM})_{YZZJ} \right), \\
\text{Re} k_{41}^{(N(6))} & = \frac{2}{2} \sqrt{75} \left( 3(\tilde{k}_{LM})_{XXJJ} - 7(\tilde{k}_{LM})_{YZZJ} \right), \\
\text{Im} k_{41}^{(N(6))} & = -\frac{2}{2} \sqrt{75} \left( 3(\tilde{k}_{LM})_{YZZJ} - 7(\tilde{k}_{LM})_{XXJJ} \right), \\
\text{Re} k_{42}^{(N(6))} & = -\frac{2}{2} \sqrt{75} \left( 3(\tilde{k}_{LM})_{XXJJ} - 7(\tilde{k}_{LM})_{YYJJ} \right), \\
\text{Im} k_{42}^{(N(6))} & = -\frac{2}{2} \sqrt{75} \left( 3(\tilde{k}_{LM})_{YYJJ} - 7(\tilde{k}_{LM})_{XXJJ} \right), \\
\text{Re} k_{43}^{(N(6))} & = -\frac{2}{2} \sqrt{75} \left( (\tilde{k}_{LM})_{XXJJ} - 3(\tilde{k}_{LM})_{YZZJ} \right), \\
\text{Re} k_{44}^{(N(6))} & = -\frac{2}{2} \sqrt{75} \left( (\tilde{k}_{LM})_{YYJJ} - 3(\tilde{k}_{LM})_{XXJJ} \right).
\end{align*}
\]

This yields the values given in Table 2. Note that other existing results for cartesian coefficients with \( d = 4 \) [16–25] and \( d = 6 \) [27–29] can also be converted to measurements of spherical coefficients using the correspondences (36) and (38). Moreover, certain experiments studying Lorentz-invariant short-range gravity [46–48] and conceivably others designed to search for large Lorentz-invariant forces at short distances [49–52] may have sensitivity to the Newton spherical coefficients \( k_{jm}^{(N(6))} \) through the perturbation (29) as well.

For even \( d \geq 8 \) the calculations are more challenging, but we conjecture a similar relationship to the result (37),

\[
k_{d-2,m}^{(N(6))} = a_d k_{(d-2,m)}^{(4)} + b_d k_{(d,|d-2,m|)}^{(4)}.
\]

where \( a_d \) and \( b_d \) are real constants. For example, we expect that for \( d = 8 \) the 115 independent vacuum spherical coefficients and the 22 independent Newton spherical coefficients can be expressed in terms of the 270+630=900 independent cartesian coefficients, with 13 of the 22 Newton coefficients determined in terms of vacuum spherical coefficients according to Eq. (39) and with short-range tests of gravity offering unique access to the remaining nine Newton spherical coefficients \( k_{4m}^{(N(6))} \).

| Coefficient | Measurement |
|-------------|-------------|
| \( k_{20}^{(N(6))} \) | \((3 \pm 23) \times 10^{-8} \text{ m}^2\) |
| \( \text{Re} k_{21}^{(N(6))} \) | \((-4 \pm 4) \times 10^{-8} \text{ m}^2\) |
| \( \text{Im} k_{21}^{(N(6))} \) | \((-2 \pm 4) \times 10^{-8} \text{ m}^2\) |
| \( \text{Re} k_{22}^{(N(6))} \) | \((0 \pm 9) \times 10^{-8} \text{ m}^2\) |
| \( \text{Im} k_{22}^{(N(6))} \) | \((1 \pm 4) \times 10^{-8} \text{ m}^2\) |
| \( \text{Re} k_{41}^{(N(6))} \) | \((3 \pm 5) \times 10^{-8} \text{ m}^2\) |
| \( \text{Im} k_{41}^{(N(6))} \) | \((1 \pm 5) \times 10^{-8} \text{ m}^2\) |
| \( \text{Re} k_{42}^{(N(6))} \) | \((0 \pm 12) \times 10^{-8} \text{ m}^2\) |
| \( \text{Im} k_{42}^{(N(6))} \) | \((2 \pm 2) \times 10^{-8} \text{ m}^2\) |
| \( \text{Re} k_{43}^{(N(6))} \) | \((0 \pm 1) \times 10^{-8} \text{ m}^2\) |
| \( \text{Im} k_{43}^{(N(6))} \) | \((1 \pm 1) \times 10^{-8} \text{ m}^2\) |
| \( \text{Re} k_{44}^{(N(6))} \) | \((2 \pm 9) \times 10^{-8} \text{ m}^2\) |
| \( \text{Im} k_{44}^{(N(6))} \) | \((2 \pm 5) \times 10^{-8} \text{ m}^2\) |
To summarize, we have developed in this work a convenient formalism for analyzing short-range tests of gravity for general signals of Lorentz violation. The procedure adopts a spherical decomposition to enable a treatment of Lorentz-violating operators of arbitrary mass dimension and to provide a comparatively simple description of the predicted sidereal variations in gravity and ones that are unique to short-range tests of gravity. The presence of the latter for all $d \geq 6$ and the exceptional sensitivity of short-range tests to the associated inverse-power modifications of the gravitational potential of a point source imply a promising future for this class of laboratory experiments.

This work was supported in part by the U.S. Department of Energy under grant no. DE-SC0010120, by the U.S. National Science Foundation under grant no. PHY-1520570, and by the Indiana University Center for Spacetime Symmetries.

References

[1] V.A. Kostelecký and S. Samuel, Phys. Rev. D 39, 683 (1989); V.A. Kostelecký and R. Potting, Nucl. Phys. B 359, 545 (1991); Phys. Rev. D 51, 3923 (1995).

[2] V.A. Kostelecký and N. Russell, Data Tables for Lorentz and CPT Violation, 2016 edition, arXiv:0801.0287v9.

[3] See, e.g., A. Hees, Q.G. Bailey, A. Bourgoin, H. Pihan-LeBars, C. Guerin, and C. Le Poncin-Lafitte, arXiv:1610.04682; J.D. Tasson, Rept. Prog. Phys. 77, 056901 (2014); C.M. Will, Liv. Rev. Rel. 17, 4 (2014); R. Bluhm, Lect. Notes Phys. 702, 191 (2006).

[4] See, e.g., J. Murata and S. Tanaka, Class. Quant. Grav. 32 033001 (2015); J. Jaeckel and A. Ringwald, Ann. Rev. Nucl. Part. Sci. 60, 405 (2010); E.G. Adelberger, J.H. Gundlach, B.R. Heckel, S. Hoeld, and S. Schlamminger, Prog. Part. Nucl. Phys. 62, 102 (2009); E. Fischbach and C. Talmadge, The Search for Non-Newtonian Gravity, Springer-Verlag, 1999.

[5] See, e.g., S. Weinberg, Proc. Sci. CD 09, 001 (2009).

[6] V.A. Kostelecký, Phys. Rev. D 69, 105009 (2004).

[7] V.A. Kostelecký and J.D. Tasson, Phys. Rev. Lett. 102, 010402 (2009).

[8] Q.G. Bailey and V.A. Kostelecký, Phys. Rev. D 74, 045001 (2006).

[9] M.D. Seifert, Phys. Rev. D 79, 124012 (2009); Phys. Rev. D 81, 065010 (2010).

[10] B. Alschohl, Q.G. Bailey, and V.A. Kostelecký, Phys. Rev. D 81, 065028 (2010).

[11] V.A. Kostelecký and J. Tasson, Phys. Rev. D 83, 016013 (2011).

[12] Q.G. Bailey and R. Tso, Phys. Rev. D 84, 085025 (2011).

[13] J.D. Tasson, Phys. Rev. D 86, 124021 (2012).

[14] Y. Bonder, Phys. Rev. D 91, 125002 (2015).

[15] R.J. Jennings and J.D. Tasson, Phys. Rev. D 92, 125028 (2015).

[16] J.B.R. Battat, J.F. Chandler, and C.W. Stubbs, Phys. Rev. Lett. 99, 241103 (2007).

[17] H. Müller, S.-w. Chiow, S. Herrmann, S. Chu, and K.-Y. Chung, Phys. Rev. Lett. 100, 031101 (2008).

[18] K-Y. Chung, S.-w. Chiow, S. Herrmann, S. Chu, and H. Müller, Phys. Rev. D 80, 016002 (2009).

[19] D. Bennett, V. Skavysh, and J. Long, in V.A. Kostelecký, ed., CPT and Lorentz Symmetry V, World Scientific, Singapore 2011.

[20] L. Iorio, Class. Quant. Grav. 29, 175007 (2012).

[21] Q.G. Bailey, R.D. Everett, and J.M. Overduin, Phys. Rev. D 88, 102001 (2013).

[22] L. Shao, Phys. Rev. Lett. 112, 111103 (2014); Phys. Rev. D 90, 122009 (2014).

[23] A. Hees, Q.G. Bailey, C. Le Poncin-Lafitte, A. Bourgoin, A. Rivoldini, B. Lamine, F. Meynadier, C. Guerin, and P. Welt, Phys. Rev. D 92, 064049 (2015).

[24] C. Le Poncin-Lafitte, A. Hees and S. Lambert, arXiv:1604.01663.

[25] A. Bourgoin, A. Hees, S. Bouquillon, C. Le Poncin-Lafitte, G. Francou, and M.-C. Angomin, arXiv:1607.00294.