Introduction to Pseudo-Group

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Abstract

This paper introduces the idea of pseudo-group. Applications of pseudo-groups in Group Theory and Symmetry Breaking in Particle Physics and Cosmology are considered.

1 Definitions

A group \( G \) is a set with a binary operation which satisfies the four group axioms: distributivity, associativity, existence of inverse, and existence of identity.

Definition 1 (Pseudo-Group)

A pseudo-group \( G(\rho_1, \rho_2, \ldots, \rho_k) \) of a group \( G \) is a set with a binary operation which gradually acquires the group properties of \( G \) and gradually satisfies the group axioms as some of the parameters \( \rho_1, \rho_2, \ldots, \rho_k \) of the set approach certain limiting values or tend asymptotically to infinity.

We shall study how concrete representations of pseudo-groups can be constructed, and consider their applications in Group Theory and Symmetry Breaking in Particle Physics and Cosmology.

2 Construction of Pseudo-group Representations

A method of constructing pseudo-group representations is by means of a variant of Fractional Derivative.

2.1 Method of Fractional Derivative

It is possible to classify Fractional Derivative into two types, each supporting a different view or school of thought.

Definition 2 (Type I Fractional Derivative)

Adopt the view that fractional derivative is an abstract analytic extension of ordinary derivative without invoking any physical picture. Fractional derivatives can be non-commutative, and the fractional derivative of a constant can be non-zero.
For instance, the Riemann-Liouville Fractional Derivative belongs to Type I. The Riemann-Liouville Fractional Derivative \[ \text{Type I} \] begins with
\[
\int_a^x f(\hat{x}) (d\hat{x})^n \equiv \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_2} f(x_1) \, dx_1 \, dx_2 \cdots dx_{n-1} \, dx_n \tag{1}
\]
for \( n \in \mathbb{Z}^+ \) as the fundamental defining expression, and it can be shown \[ \text{I. p. 38} \] to be equal to the Cauchy formula for repeated integration,
\[
\frac{1}{\Gamma(n)} \int_a^x \frac{f(t)}{(x-t)^{1-n}} \, dt . \tag{2}
\]

The Riemann-Liouville fractional integral is analytically extended from (2) as
\[
D_0^\sigma f(x) = \frac{d^\sigma}{dx^\sigma} f(x) = \int_a^x f(x)(dx)^{-\sigma} = \frac{1}{\Gamma(-\sigma)} \int_a^x \frac{f(t)}{(x-t)^{1+\sigma}} \, dt \quad (\sigma < 0, \, a \in \mathbb{R}) \text{ by (2)}, \tag{3}
\]
and the R-L fractional derivative is in turn derived from the R-L fractional integral (3) by ordinary differentiation:
\[
D_0^\sigma f(x) = D_0^m \left( D_0^{-(m-\sigma)} f(x) \right) \quad (\sigma > 0, \, m \in \mathbb{Z}^+) \tag{4}
\]
where \( m \) is chosen such that \( m > 1 + \sigma, \, \sigma > 0 \).

**Lemma 1** The equation (4) is independent of the choice of \( m \) for \( m > 1 + \sigma, \, \sigma > 0 \).

**Proof**

For \( m > 1+\sigma, \, m \in \mathbb{Z}^+, \, \sigma > 0 \), we have \(-(m-\sigma)) < -1 < 0 \text{ and } (m-\sigma-1) > 0 \text{. The first condition, } -(m-\sigma) < 0 \text{, allows us to use the equation (3) to write}
\[
D_0^{-(m-\sigma)} f(x) = \frac{1}{\Gamma(m-\sigma)} \int_a^x \frac{f(t)}{(x-t)^{1+\sigma-m}} \, dt .
\]

From (4),
\[
D_0^m \left( D_0^{-(m-\sigma)} f(x) \right) = \frac{d^m}{dx^m} \left( \frac{1}{\Gamma(m-\sigma)} \int_a^x \frac{f(t)}{(x-t)^{1+\sigma-m}} \, dt \right) = \frac{1}{\Gamma(m-\sigma)} \int_a^x f(t) \left( \frac{d^m}{dx^m}(x-t)^{m-\sigma-1} \right) \, dt . \tag{5}
\]

The second condition, \( (m-\sigma-1) > 0 \), and the condition \( m > 0 \) allow us to use the second case of (4). Thus, (3) becomes
\[
\frac{1}{\Gamma(m-\sigma)} \int_a^x f(t) \frac{\Gamma(m-\sigma)}{\Gamma(-\sigma)} (x-t)^{-(1+\sigma)} \, dt = \frac{1}{\Gamma(-\sigma)} \int_a^x \frac{f(t)}{(x-t)^{1+\sigma}} \, dt = D_0^\sigma f(x) .
\]
When $a = 0$, (3) for $f(x) = x^r$ is well-defined only for the half plane $r > -1$. Consequently, in the Riemann-Liouville Fractional Derivative, $D^a_{x|a} x^r$ is well-defined only for the half plane $r > -1$.

$$D^a_{x|a} x^r = \begin{cases} \frac{\Gamma(1+r)}{\Gamma(1+r-\sigma)} x^{r-\sigma} & (\sigma > 0, r > -1) \\ x^r & (\sigma = 0, \forall r) \\ \frac{\Gamma(1+r)}{\Gamma(1+r-\sigma)} \frac{x^{r-\sigma}}{a} & (\sigma < 0, r > -1) \end{cases}.$$  \hspace{1cm} (6)

**Definition 3 (Type II Fractional Derivative)**
The ordinary derivative (derivative of integer order) of a constant is zero and ordinary derivatives commute. The fractional derivative is to inherit these properties from the ordinary derivative, i.e. fractional derivatives commute, and the fractional derivative of a constant is zero.

For instance, given that $c$ is an arbitrary constant and $D^1_x c = 0$,

$$D^\sigma_x c = D^\sigma_x (D^{-1}_x c) = D^\sigma_x 0 = 0 \quad (\sigma > 1).$$  \hspace{1cm} (7)

Since $D^x_x x^r = \Gamma(1+r) \quad (r \geq 0)$ and $D^\sigma_x c = 0$, by continuity,

$$D^\sigma_x x^r = 0 \quad (\sigma > r, r \geq 0)$$

where the commutativity of fractional derivatives is preserved. Hence, a Type II Fractional Derivative with commuting fractional derivatives is

$$D^\sigma_x x^r = \begin{cases} 0 & (\sigma > r, r \geq 0) \\ \frac{\Gamma(1+r)}{\Gamma(1+r-\sigma)} x^{r-\sigma} & (0 < \sigma \leq r, r \geq 0) \text{ and } (\sigma > 0, -1 < r < 0) \\ x^r & (\sigma = 0, \forall r) \\ \frac{\Gamma(1+r)}{\Gamma(1+r-\sigma)} \frac{x^{r-\sigma}}{a} & (\sigma < 0, r > -1) \end{cases}.$$  \hspace{1cm} (7)

Note that there is a mutual trade-off between the properties of analyticity and commutativity in these two types:

*Type I*: we choose to preserve *global analyticity* and lose *global commutativity*.  
*Type II*: we choose to preserve *global commutativity* and lose *global analyticity*.

### 2.2 An Application of Type II: Power Series of Non-integer Order

For *Type I*, we write the power series for $\exp(x)$ as

$$\exp(x) = \lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} \frac{1}{\Gamma(1+\epsilon+k)} x^{k+\epsilon},$$  \hspace{1cm} (8)
and
\[ D_x^\sigma \lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} \frac{1}{\Gamma(1+\epsilon+k)} x^k = \lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} \frac{1}{\Gamma(1+\epsilon+k-\sigma)} x^{k-\sigma+\epsilon} \quad (\sigma > 0) . \quad (9) \]

For Type II, we write it as
\[ \exp(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+k)} x^k , \quad (10) \]

and
\[ D_x^\sigma \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+k)} x^k = \sum_{k=\lceil \sigma \rceil}^{\infty} \frac{1}{\Gamma(1+k-\sigma)} x^{k-\sigma} \quad (\sigma > 0) \quad (11) \]

where \( \lceil \cdot \rceil \) denotes taking the integer ceiling, since \( D_x^\sigma x^r = 0 \quad (\sigma > r, \ r \geq 0) \).

In general, for Type II,
\[ D_x^\sigma \sum_{k=0}^{\infty} a_k x^k = \sum_{k=\lceil \sigma \rceil}^{\infty} a_k \frac{\Gamma(1+k)}{\Gamma(1+k-\sigma)} x^{k-\sigma} \quad (\sigma > 0) \quad (12) \]

where \( \lceil \cdot \rceil \) denotes taking the integer ceiling.

When \( \sigma \) is not an integer, the series on the right of (12) forms a power series of non-integer order.

**Definition 4 (Notation)**

Define the notation
\[ f(\sigma, x) \equiv D_x^\sigma f(x) . \quad (13) \]

Think of \( \sigma \) in the following way: the one-variable function \( f(x) \) is extended to a two-variable function \( f(\sigma, x) \) in which \( \sigma \) has now become a variable of the extended function.

For instance,
\[ \cos(\sigma, x) = D_x^\sigma \cos(x) = D_x^\sigma \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \sum_{k=Wc(\sigma)}^{\infty} \frac{(-1)^k}{(2k)!} \frac{\Gamma(1+2k)}{\Gamma(1+2k-\sigma)} x^{2k-\sigma} \quad (\sigma > 0) \quad (14) \]

where \( Wc(\sigma) = \lceil \frac{\sigma}{2} + 1 \rceil - 1 \),

\[ \sin(\sigma, x) = D_x^\sigma \sin(x) = D_x^\sigma \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{k=Ws(\sigma)}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{\Gamma(2k+1)}{\Gamma(2k+1-\sigma)} x^{2k+1-\sigma} \quad (\sigma > 0) \quad (15) \]

where \( Ws(\sigma) = \lceil \frac{\sigma}{2} + \frac{1}{2} \rceil - 1 \).
and

\[ \exp(\sigma, x) = D_x^\sigma \exp(x) = D_x^\sigma \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k = \sum_{k=\lceil \sigma \rceil}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(1+k)}{\Gamma(1+k-\sigma)} x^{k-\sigma} \text{ (}\sigma > 0\text{)} . \tag{16} \]

| $\sigma$ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | ... |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $W_c(\sigma)$ | 0.0 | 1.0 | 1.1 | 1.2 | 2.0 | 2.2 | 2.2 | 2.2 | 3.0 | 3.0 | ... |
| $W_s(\sigma)$ | 0.0 | 0.0 | 0.1 | 0.1 | 1.1 | 1.1 | 2.0 | 2.2 | 2.2 | 2.2 | ... |

Table 1: Some tabulated values of $W_c(\sigma)$ and $W_s(\sigma)$.

Note that

\[
\begin{align*}
\cos(\sigma, 0) &= \begin{cases} 
2 \mathbb{Z} & \text{if } \sigma \in \mathbb{Z} \\
2 \mathbb{Z} + 1 & \text{otherwise;}
\end{cases} \\
\sin(\sigma, 0) &= \exp(\sigma, 0) \\
\cos(\sigma, x) &= \pm \sin(\sigma \pm 1, x) = -\cos(\sigma \pm 2, x) , \\
\sin(\sigma, x) &= \mp \cos(\sigma \pm 1, x) = -\sin(\sigma \pm 2, x) , \\
\exp(\sigma, x) &= \exp(\sigma \pm 1, x) ,
\end{align*}
\]

in agreement with the definitions of $\cos(x)$, $\sin(x)$ and $\exp(x)$ when $\sigma \in \mathbb{Z}$.

It can be observed from Figures 1 and 2 that there exist asymptotic limits

\[
\begin{align*}
\cos(\sigma, x) &\sim \cos(x + \frac{\pi}{2} \sigma) \\
\sin(\sigma, x) &\sim \sin(x + \frac{\pi}{2} \sigma) \\
\exp(\sigma, x) &\sim \exp(x)
\end{align*} \quad \forall \sigma \text{ as } x \to \infty . \tag{17}
\]

In general, any function in a power series representation can similarly have a power series of non-integer order generalization.
2.3 Concrete Representations of Pseudo-Groups

Consider extending the group elements of $SO(2)$ (the group of rotation in a plane) as follows:

$$R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad \text{where } \theta \in [0, 2\pi)$$

$$\xrightarrow{\sigma, \theta} R(\sigma, \theta) = \begin{pmatrix} \cos(\sigma, \theta) & \sin(\sigma, \theta) \\ -\sin(\sigma, \theta) & \cos(\sigma, \theta) \end{pmatrix} \quad \text{where } \theta \in [0, \infty). \quad (18)$$

$R(\sigma, \theta)$ forms a family of sets, parametrized at 2 levels. The family of sets is parametrized by $\sigma$, and each of these sets is further parametrized by $\theta$. Denote the family of these sets as $SO(2; \sigma, \theta)$.

Since $R(0, \theta) \in SO(2) \ \forall \ \theta$ and $R(\sigma, \theta) \sim R(0, \theta + \pi\sigma/2)$ as $\theta \to \infty$, we are motivated to introduce the idea of pseudo-groups as in Definition [4].

$SO(2; \sigma, \theta)$ is a pseudo-group of $SO(2)$ since it is isomorphic to $SO(2)$ either when the parameter $\sigma \to n \in \mathbb{Z}$ while $\theta$ varies freely in the interval $[0, \infty)$,

$$R(\sigma, \theta) R(\sigma', \theta) \sim R(\sigma + \sigma', \theta) \quad \text{as } \sigma, \sigma' \to n, n' \in \mathbb{Z}$$

$$\Rightarrow \lim_{\sigma \to n \in \mathbb{Z}} SO(2; \sigma, \theta) \cong SO(2) \quad (19)$$

or when the parameter $\theta \to \infty$ while $\sigma$ varies freely in the interval $(0, 2)$,

$$R(\sigma, \theta) R(\sigma, \theta') \sim R(0, \theta + \frac{\pi}{2\sigma}) R(0, \theta' + \frac{\pi}{2\sigma})$$
\[
\left( \exp(\sigma, x) - \exp(x) \right)
\]

Figure 2: Asymptotic limit of \( \exp(\sigma, x) - \exp(x) \to 0 \) as \( x \to \infty \).

\[
= R(0, \theta + \theta' + \pi \sigma) \quad \text{as} \quad \theta, \theta' \to \infty
\]

\[
\Rightarrow \lim_{\theta \to \infty} SO(2; \sigma, \theta) \cong SO(2) \quad (20)
\]
as shown in Figure 3.

Figure 3: \( SO(2; \sigma, \theta) \) plane diagram.

3 Measure of Group Property Deviation

Definition 5 (A Measure of Group Property Deviation)

We define a measure of group property deviation, \( W(G(\sigma, x), G | \sigma, x) \), for a pseudo-group \( G(\sigma, x) \), as the measure of how much group property the pseudo-group has lost or deviated from the associated "parent" group \( G \) from which it is analytically extended. When the pseudo-group becomes isomorphic to the parent group for certain values of the parameter, the measure should be zero.

For the case of pseudo-group \( SO(2; \sigma, \theta) \),

\[
W(SO(2; \sigma, \theta), SO(2) | \sigma, \theta)
\]
\[
\begin{align*}
\left| R(\sigma, \theta) - R(\theta + \frac{\pi}{2}\sigma) \right| & = \\
\begin{vmatrix}
\cos(\sigma, \theta) - \cos(\theta + \frac{\pi}{2}\sigma) & \sin(\sigma, \theta) - \sin(\theta + \frac{\pi}{2}\sigma) \\
-\left( \sin(\sigma, \theta) - \sin(\theta + \frac{\pi}{2}\sigma) \right) & \cos(\sigma, \theta) - \cos(\theta + \frac{\pi}{2}\sigma)
\end{vmatrix} & = \\
\begin{vmatrix}
\delta \cos & \delta \sin \\
-\delta \sin & \delta \cos
\end{vmatrix} & = \\
\sqrt{(\delta \cos + \delta \sin)^2 + (-\delta \sin + \delta \cos)^2} & = \\
\sqrt{2(\delta \cos^2 + \delta \sin^2)} & = \\
\frac{2 \left( \left( \cos(\sigma, \theta) - \cos(\theta + \frac{\pi}{2}\sigma) \right)^2 + \left( \sin(\sigma, \theta) - \sin(\theta + \frac{\pi}{2}\sigma) \right)^2 \right)}{(21)}
\end{align*}
\]

satisfies the requirement in Definition 5. See Figure 4 for the plot of this measure.

Figure 4: Measure \( W \) of \( SO(2; \sigma, \theta) \).

Similarly for the case of \( U(1; \sigma, ix) \), a pseudo-group of \( U(1) \) where \( x \in \mathbb{R} \),

\[
\exp(\sigma, ix_1) \exp(\sigma', ix_2) \sim \exp(\sigma, i(x_1 + x_2)) \quad \text{as} \quad \sigma, \sigma' \to n, n' \in \mathbb{Z}
\]

\[
\Rightarrow \quad \lim_{\sigma \to n \in \mathbb{Z}} U(1; \sigma, x) \cong U(1),
\]

\[
\exp(\sigma, ix_1) \exp(\sigma, ix_2) \sim \exp(\sigma, i(x_1 + x_2)) \quad \text{as} \quad x_1, x_2 \to \infty
\]

\[
\Rightarrow \quad \lim_{\theta \to \infty} U(1; \sigma, x) \cong U(1),
\]

\[
W(U(1; \sigma, ix), U(1) | \sigma, ix) = \exp(\sigma, ix) - \exp(ix).
\]

This measure was plotted in Figure 2.
4 Rotations and Deformations of $SO(2; \sigma, \theta)$

Figure 5 shows the effect of planar rotations and deformations of $SO(2; \sigma, \theta)$ on a square with vertices \{(1, -1), (1, 1), (-1, 1), (-1, -1)\} in a sequence of \((x, y)\) planes clipped by square windows of size $x \in [-2, 2]$, $y \in [-2, 2]$. The deformation effects can be seen as a combination of rotations and contractions/dilations.

```plaintext
\[
\begin{array}{ccccccccc}
\sigma \setminus \theta & \approx 0 & \pi/16 & \pi/8 & \pi/4 & \pi/2 & 3\pi/4 & \pi & 3\pi/2 & 2\pi & 4\pi & 6\pi & 8\pi \\
0 & \& \& \& \& \& \& \& \& \& \& \& \& \\
0.001 & \& \& \& \& \& \& \& \& \& \& \& \& \\
0.25 & \& \& \& \& \& \& \& \& \& \& \& \& \\
0.5 & \& \& \& \& \& \& \& \& \& \& \& \& \\
0.75 & \& \& \& \& \& \& \& \& \& \& \& \& \\
1 & \& \& \& \& \& \& \& \& \& \& \& \& \\
1.25 & \& \& \& \& \& \& \& \& \& \& \& \& \\
1.5 & \& \& \& \& \& \& \& \& \& \& \& \& \\
1.75 & \& \& \& \& \& \& \& \& \& \& \& \& \\
2 & \& \& \& \& \& \& \& \& \& \& \& \& \\
3 & \& \& \& \& \& \& \& \& \& \& \& \& \\
4 & \& \& \& \& \& \& \& \& \& \& \& \& \\
\end{array}
\]
```

Figure 5: $SO(2; \sigma, \theta)$ rotations and deformations

5 Symmetry Breaking/Deforming via Pseudo-Group

In the Higgs mechanism of Spontaneous Symmetry Breaking in Particle Physics and Cosmology,

1. the symmetry of the effective potential $V_{eff}$ in a Lagrangian density $\mathcal{L}$ with respect to a gauge group $G$ is preserved, while
2. the symmetry of the Quantum state $\psi$ satisfying the equations of motion derived from $\mathcal{L}$ is broken and reduced from $G$ to a subgroup, $H \subset G$.

The profile of $V_{\text{eff}}$ changes with energy or temperature. At high energy or temperature, the symmetry of $\psi$ is restored from $H \rightarrow G$.

In the case of symmetry breaking/deforming via pseudo-group, the situation is very different from the Higgs mechanism. The symmetry in a group $G$ itself is broken to a subgroup $H \subset G$ or “deformed” into a pseudo-group with an approximate symmetry of $G$.

Consider $SO(3; (\sigma_1, x_1), (\sigma_2, x_2))$, a pseudo-group of $SO(3)$, as an example.

When both $\sigma_1, \sigma_2 = 0$, $SO(3; (\sigma_1, x_1), (\sigma_2, x_2)) \sim SO(3)$.

However, when $\sigma_1 \notin \mathbb{Z}$ and $\sigma_2 = 0$, the $SO(3)$ symmetry is broken, approximate or restored, for small, intermediate, or large values of $x$ respectively. The $SO(3)$ symmetry in a sphere is “deformed” to an approximate $SO(3)$ symmetry or completely broken to $SO(2)$ in a plane depending on the chosen values of $\sigma_1$ and $x$.

When both $\sigma_1, \sigma_2 \notin \mathbb{Z}$, the $SO(2)$ is further “deformed” to an approximate $SO(2)$ symmetry or broken to identity.

The symmetry breaking/deforming sequence is thus

$$
SO(3; (\sigma_1, x_1), (\sigma_2, x_2)) \xrightarrow{\sigma_1, \sigma_2 = 0} SO(3) \xrightarrow{\sigma_1 \notin \mathbb{Z}, \sigma_2 = 0} SO(2) \xrightarrow{\sigma_1, \sigma_2 \notin \mathbb{Z}} 1.
$$

(22)

Similarly for $SU(N; (\sigma_1, x_1), (\sigma_2, x_2), \cdots, (\sigma_N, x_N))$, the symmetry breaking/deforming sequence is

$$
SU(N; (\sigma_1, x_1), (\sigma_2, x_2), \cdots, (\sigma_N, x_N)) \xrightarrow{\sigma_1, \sigma_2, \cdots, \sigma_N = 0} SU(N) \xrightarrow{\sigma_1 \notin \mathbb{Z}, \sigma_2, \cdots, \sigma_N = 0} SU(N-1) \xrightarrow{\vdots} \cdots \xrightarrow{\sigma_1, \cdots, \sigma_{N-2} \notin \mathbb{Z}, \sigma_{N-1}, \sigma_N = 0} SU(2) \xrightarrow{\sigma_1, \cdots, \sigma_{N-1} \notin \mathbb{Z}, \sigma_N = 0} U(1) \xrightarrow{\sigma_1, \sigma_2, \cdots, \sigma_N \notin \mathbb{Z}} 1.
$$

This mode of symmetry breaking/deforming seems to have useful implications for models in Particle Physics and Cosmology. The challenge is to develop the full theory of symmetry breaking/deforming via pseudo-group to model the approximate symmetry or gauge groups in the Universe we live in today.

In Particle Physics, the symmetry of the flavor of quarks are not exact symmetry but only approximate symmetry of Gell-Mann’s Eightfold way $SU(3)$ or Georgi-Sheldon’s GUT $SU(5)$ because quarks of different flavors have
different masses, and light quarks do not transform in the mixing matrix exactly into heavy quarks. The effects from the presence of massive gluons interactions and glueballs also perturb the symmetry away from an exact symmetry. It is possible that $SU(3)$ and $SU(5)$ may be “deformed” into pseudo-groups with approximate symmetries that will fit the phenomenological data better.

In the context of Cosmology, consider taking a pseudo-group to describe the product of the residual exact and approximate symmetries in the present day Universe. Let’s set the pseudo-group variable $x \propto T$, the temperature of the Universe. As we go back in time, the temperature $T$ goes up, $x$ goes up, and we find that the approximate and other broken symmetries are gradually being restored. The rate at which the symmetries are being restored will be dependent on the values of $\sigma_1, \sigma_2, \ldots, \sigma_N$, the parameters of the pseudo-group. The fully restored symmetry will be the symmetry of the parent group of the pseudo-group. Qualitatively, this model resembles the unification of gauge groups in Cosmology.

6 Towards a General Theory

Definition 6 (General Definition of Pseudo-Group and Measure $W$)

An element of any Lie Group $G$ can be expressed as $\exp(g.x)$ where $g$ is the generator of $G$ and $x$ is the parameter of $G$.

The element of the pseudo-group of $G$ is thus

$$\exp(\sigma, g.x) = D^\sigma \exp(g.x) = D^\sigma \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (g.x)^k$$

$$= \sum_{k=\lfloor \sigma \rfloor}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(1+k)}{\Gamma(1+k-\sigma)} (g.x)^{k-\sigma} \quad (\sigma \in \mathbb{R}, \sigma > 0). \quad (23)$$

In the matrix representation of the generator $g$, $(g.x)$ is a matrix and $(g.x)^{k-\sigma}$ is an analytic extension of the matrix $(g.x)$. We shall study the analytic extension of matrices and outline the general steps in evaluating analytic extended matrices in the next paper.

References

[1] K.B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[2] P.W. Higgs, Phys. Lett., 12, 132 (1964).
[3] M. Gell-Mann and Y. Ne’eman, The Eightfold Way (Benjamin, New York, 1964).
[4] H. Georgi and S.L. Glashow, Phys. Rev. Lett., 32, 438 (1974).

Appendix
A Extension of Type II from $\sigma, r \in \mathbb{R}$ to $s, w \in \mathbb{C}$

The differential operator $d/dx$ commutes with itself and with its inverse, the integral operator $\int dx$. For Type II, commutativity is preserved. The Type II commutative operator $D^s$, $s \in \mathbb{C}$, acting on a function $f$ can be split into two parts with each acting on the function independently,

$$D^s f = D^{\sigma+it} f = D^\sigma D^{it} f = D^{it} D^\sigma f \quad (s = \sigma + it, \sigma, t \in \mathbb{R}) \quad (24)$$

as illustrated in the commutative diagram in Figure 6 with all the limits, if any, taken only after the actions of all the operators have been performed.

Consider $D^s_x x^w$, $s = \sigma + it$, $w = u + iv$.

For $\sigma > u, u \geq 0$,

$$D^\sigma_x x^u = 0 \quad \Rightarrow \quad D^{it} D^\sigma_x x^u = D^s_x x^u = 0 \ . \quad (25)$$

Similarly, for $t > v, v \geq 0$,

$$D^{it} D^\sigma_x x^{iv} = 0 \quad \Rightarrow \quad D^\sigma_x D_x^it x^{iv} = D^s_x x^{iv} = 0 \ . \quad (26)$$

Figure 7: The zero space wedges in $(s, w)$ diagrams where $D^s_x x^w$ in Type II.
We can think of the following pictures. $D^it$ in (25) expanding the triangular zero region of the $(\sigma, u)$ plane into a wedge-shaped volume of infinite length along the $t$-direction. Similarly $D^\sigma$ in (26) expands the triangular zero region of the $(it, iv)$ plane into another wedge-shaped volume of infinite length along the $\sigma$-direction as shown in Figure 4. Both these wedge-shaped volume defines the zero space in which $D^\sigma x^w = 0$.

**B Extension of Groups – $\mathbb{R}(\text{mod } n)$ Groups**

The differential operator, and its inverse — integral operator, can act on different sets to generate different *discrete* groups. These are groups of operators, ie. groups with operators as elements.

\[
\frac{d^n}{dx^n} f(x) = f(x)
\]

| Order | $f(x)$ | Symmetry Group |
|-------|--------|---------------|
| $n=1$ | $\exp(x)$ | $Z_1 = \{1\}$, $\frac{d}{dx} \equiv 1 = \frac{d^0}{dx^0}$ |
| $n=2$ | $\{\cosh(x), \sinh(x)\}$ | $Z_2 = \{1, \frac{d}{dx}\}$, $\frac{d^2}{dx^2} \equiv 1$ |
| $n=4$ | $\{\pm \cos(x), \pm \sin(x)\}$ | $Z_4 = \{1, \frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}\}$, $\frac{d^4}{dx^4} \equiv 1$ |

Table 2: Groups of $\frac{d}{dx}$ acting on exponential, trigonometric and hyperbolic functions.

Figure 8 shows the $Z_4$ group flow.

\[
\begin{align*}
\frac{d}{dx} & \cos(x) & \frac{d}{dx} \\
\sin(x) & -\sin(x) & \\
\frac{d}{dx} & -\cos(x) & \frac{d}{dx}
\end{align*}
\]

Figure 8: $Z_4$ group flow diagram of $\frac{d}{dx}$ acting on set $\{\pm \cos(x), \pm \sin(x)\}$.

If elements of the set are extended from functions $f(x)$ to their analytic extensions $f(\sigma, x) = D^\sigma f(x)$ with real $\sigma$, operators $D^\sigma$ acting on these extended sets will generate *continuous* groups or *Lie* groups, eg., $D_2^\sigma$ acting on the set

\[
\left\{ \cos(\sigma, x) \mid \sigma \in [0, 4), x \in [0, \infty) \right\}
\]
generates a natural analytic extension of the $\mathbb{Z}_4$ group,

$$\left\{ D_x^{\sigma'} \mid \sigma' \in [0, 4) \right\}, \quad D_x^4 \equiv 1$$

as illustrated in Figure 9.

By analogy to the concept of $(\text{mod } n)$ congruence in Number Theory, we denote this analytically extended group $R_{(\text{mod } 4)}$.

$$D_x^{\sigma'} f(\sigma, x) = f(\sigma, x)$$

| Order $\sigma'$ | $f(x)$ | Symmetry Group $R_{(\text{mod } m)}$ |
|-----------------|-------|-----------------------------------|
| $\sigma' = 1$   | $\exp(\sigma, x)$ | $\left\{ D_x^{\sigma'} \mid \sigma' \in [0, 1) \right\}, \quad D_x^1 \equiv 1$ |
| $\sigma' = 2$   | $\left\{ \cosh(\sigma, x), \sinh(\sigma, x) \right\}$ | $R_{(\text{mod } 2)} = \left\{ D_x^{\sigma'} \mid \sigma' \in [0, 2) \right\}, \quad D_x^2 \equiv 1$ |
| $\sigma' = 4$   | $\left\{ \pm \cos(\sigma, x), \pm \sin(\sigma, x) \right\}$ | $R_{(\text{mod } 4)} = \left\{ D_x^{\sigma'} \mid \sigma' \in [0, 4) \right\}, \quad D_x^4 \equiv 1$ |

Table 3: Analytically extended groups of $D_x^{\sigma'}$ acting on analytically extended exponential, trigonometric and hyperbolic functions.

For complex $s = \sigma + it$, $s' = \sigma' + it'$, $D_x^{s'}$ acting on set

$$\left\{ \cos(s, x) \mid \sigma \in [0, 4), \quad t \in \mathbb{R}, \quad x \in [0, \infty) \right\}$$

generates a Lie group $R_{(\text{mod } 4)} \times \mathbb{R}$ since $D_x^{it'}$ commutes with $D_x^{\sigma'}$ and so $D_x^{it'}$ acts independently from $D_x^{\sigma'}$.

In general, the topology of such analytically extended groups progresses from sets of points on a circle $S^1$ for discrete groups generated by $d/dx$, to a circle $S^1$ for Lie groups generated by $D_x^{\sigma'}$, and to a 2-dimensional cylinder $S^1 \times \mathbb{R}$ for Lie group generated by $D_x^{\sigma'+it'}$ as illustrated in Figure 10.

C Open Problems
1. Develop the full theory of the symmetry breaking/deforming of the exact symmetry or gauge groups (e.g., Eightfold way $SU(3)$ and GUT $SU(5)$) to pseudo-groups with approximate symmetries that will fit the phenomenological data better.

2. As a mathematical curiosity, establish whether a meaningful extension between groups of the same family but of different dimensions, e.g., $SU(\mathbb{Z}) \mapsto SU(\mathbb{R}) \mapsto SU(\mathbb{C})$ is possible.

Figure 10: Topology change of groups.