Dynamics and BPS states of AdS$_5$ supergravity with a Gauss-Bonnet term

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Abstract

Some dynamical aspects of five-dimensional supergravity as a Chern-Simons theory for the SU(2$|$2) group, are analyzed. The gravitational sector is described by the Einstein-Hilbert action with negative cosmological constant and a Gauss-Bonnet term with a fixed coupling. The interaction between matter and gravity is characterized by intricate couplings which give rise to dynamical features not present in standard theories. Depending on the location in phase space, the dynamics can possess different number of propagating degrees of freedom, including purely topological sectors. This inhomogeneity of phase space requires special care in the analysis.

Background solutions in the canonical sectors, which have regular dynamics with maximal number of degrees of freedom, are shown to exist. Within this class, explicit solutions given by locally AdS spacetimes with nontrivial gauge fields are constructed, and BPS states are identified. It is shown that the charge algebra acquires a central extension due to the presence of the matter fields. The Bogomol’nyi bound for these charges is discussed. Special attention is devoted to the N = 4 case since then the gauge group has a U(1) central charge and the phase space possesses additional irregular sectors.

1 Introduction

Standard supergravity with a negative cosmological constant is a gauge theory with fiber bundle structure only in three dimensions. Its Lagrangian is described by a Chern-Simons (CS) form for the super-AdS group Osp(p$|$2) $\otimes$ Osp(q$|$2) [1]. AdS supergravity theories sharing this powerful geometrical structure can also be formulated in five [2] and higher odd dimensions [3]. These theories are constructed assuming that the dynamical fields belong to a single connection for a supersymmetric extension of the AdS group, and consequently, the supersymmetry algebra closes automatically off-shell without requiring auxiliary fields [4]. The existence of an eleven-dimensional AdS supergravity theory which is gauge theory for OSp(32$|$1) exhibiting the features mentioned above opens up a number of new questions, and is particularly interesting due to its possible connection with M-theory [3]. This problem has been further explored in Refs. [5]-[14].

This elegant geometrical setting with its appealing gauge invariance leads, however, to a rich and quite complex dynamics involving unexpected problems. In order to understand better the subtleties, it is instructive to analyze the simplest nontrivial CS system in some detail,
which is the five dimensional case. For the purely gravitational sector, the Lagrangian in $D = 5$
dimensions contains the Gauss-Bonnet term which is quadratic in the curvature, while for $D \geq 7$,
additional terms with higher powers of the curvature and explicitly involving torsion are also
required \cite{15}. The higher powers of curvature give rise to interesting dynamical sectors within
these theories which, even at the linearized level, are beyond the notions learned from standard
supergravity.

In five dimensions, the locally supersymmetric extension of gravity with negative cosmolog-
cal constant was found in \cite{2}, and generalized in \cite{3} for higher odd dimensions. For vanishing
cosmological constant supergravity theories sharing this geometric structure have also been con-
structed in \cite{12,15,17}.

CS theories for $D \geq 5$ are not necessarily topological but contain propagating degrees of
freedom \cite{18}. Their dynamical structure changes throughout phase space, changing drastically
from purely topological sectors to others with a large number of local degrees of freedom. Sectors
where the number of degrees of freedom is less than maximal are called \textit{degenerate} and on them
additional local symmetries emerge \cite{19}.

Another unusual feature of these systems is that the symmetry generators (first class con-
straints) may become functionally dependent in some regions of phase space, called \textit{irregular}
sectors. Dirac’s canonical formalism cannot be directly applied in these sectors, obscuring the
dynamical content of CS theories \cite{20,21,22}. These irregularities also imply that the theory
is not correctly described by its linearized approximation and hence the perturbative analysis
cannot be trusted \cite{23,24,25}, the canonical analysis breaks down and it is not clear how to
identify the physical observables (propagating degrees of freedom, conserved charges, etc.).

Degeneracy and irregularity are independent features that occur in any CS theory for $D \geq 5$
but are rarely found in field theories. They arise naturally in fluid dynamics, as in the Burgers
equation \cite{26}, or in the propagation of shock waves in compressible fluids described by the
Chaplygin and Tricomi equations \cite{27}. Irregular sectors have also been found in the Plebanski
theory \cite{28}.

Fortunately, the troublesome configurations generically occur in sets of measure zero in phase
space and one can always restrict the attention to open sets where the canonical analysis holds.
Such \textit{canonical} configurations fill most of the phase space and it is desirable to know whether
among them one can find states that could be regarded as vacua around which a perturbatively
stable field theory can be built.

The presence of unbroken supersymmetries in backgrounds admitting Killing spinors implies
lower bounds for the sum of charges through the Bogomol’nyi formula. This leads to the posi-
tivity of energy in standard supergravity \cite{29,30,31,32}, which also ensures the stability of the
configurations that saturate the energy bound (BPS states). These configurations correspond
to good perturbative vacua and in this work it is shown that it is indeed possible to identify
canonical configurations which are BPS states.

In the next section, the Lagrangian of five-dimensional supergravity as a CS theory for
the supersymmetric extension of AdS$_5$, $SU(2,2|N)$, is reviewed. Special attention is devoted
to the case $N = 4$ in which the gauge group acquires a $U(1)$ central extension and the phase
space possesses additional irregular sectors. In Sect.3, the canonical representation of the charge
algebra, including its central extension, is constructed in a canonical sector previously discussed
in \cite{22}. In Sect.4 the conditions on the background manifold that allow the existence of globally
defined Killing spinors are presented. The Killing spinors are explicitly given in the canonical
background and for a spatial boundary with topology $S^1 \otimes S^1 \otimes S^1$. In Sect.5 the Bogomol’nyi
bound is established, and the conclusions and discussion are contained in Sect.6.
2 AdS$_5$ supergravity as a Chern-Simons theory

The supersymmetric extension of the AdS group in five dimensions is $SU(2,2|N)$, generated by the set $G_K = \{G_K, Z\}$, where $Z$ is the generator of the $U(1)$ subgroup, and $G_K \equiv \{J_{ab}, J_a; Q^a_s, \bar{Q}_s^\alpha; T_\Lambda \}$. Here, $J_{ab}$ and $J_a$ are the generators of the AdS group $SO(4,2)$, and $T_\Lambda$ generate the $R$-symmetry group $SU(N)$.\(^1\) The supersymmetry generators are given by $Q^a_s$ and $\bar{Q}_s^\alpha$, which transform as Dirac spinors in a vector representation of $SU(N)$, and carry $U(1)$ charges $q = \pm (\frac{1}{2} - \frac{1}{N})$. The dimension of the superalgebra $su(2,2|N)$ is $\Delta = N^2 + 8N + 15$. Its explicit form and a $(4 + N) \times (4 + N)$ matrix representation for its generators are given in the Appendix.

Chern-Simons AdS$_5$ supergravity \(^2\) is a gauge theory for the Lie-algebra-valued connection 1-form $A = A_{\mu}^K G_K dx^\mu$, with components

$$A = \frac{1}{\ell} e^a J_a + \frac{1}{2} \omega^{ab} J_{ab} + a^\Lambda T_\Lambda + (\bar{\psi}_s^\alpha Q^\alpha_s - \bar{Q}_s^\alpha \psi^\alpha_s) + b Z .$$ (1)

The bosonic sector of the theory contains the vielbein and the spin connection $(e^a, \omega^{ab})$, the $SU(N)$ gauge field $a^\Lambda$ and the $U(1)$ field $b$. The fermionic fields $\psi_s$ are $N$ complex gravitini in a vector representation of $SU(N)$.

The Lagrangian $L(A)$ satisfies

$$dL = k \langle F^3 \rangle = k g_{KLM} F^K F^L F^M,$$ (2)

where $F = dA + A^2 = F^K G_K$ is the field-strength 2-form, and $k$ is a dimensionless constant.\(^2\) The bracket $\langle \cdots \rangle$ stands for the supertrace in a representation which naturally defines the invariant tensor $g_{KLM}$ which is (anti)symmetric under permutation of (fermionic) bosonic indices \(^3\) (see Appendix). The action and its corresponding field equations are given by

$$I [A] = \int L(A) = k \int \left\langle A F^2 - \frac{1}{2} A^3 F + \frac{1}{10} A^5 \right\rangle,$$ (3)

$$\langle F^2 G_K \rangle = 0 .$$ (4)

The components of the field-strength $F$ read

$$F = F^a J_a + \frac{1}{2} F^{ab} J_{ab} + F^\Lambda T_\Lambda + (\nabla \bar{\psi}s Q^s_s - \bar{Q}_s^\alpha \nabla \psi^\alpha_s) + F^z Z ,$$ (5)

where

$$F^a = \frac{1}{2} T^a + \frac{1}{2} \bar{\psi}^s \Gamma^a \psi_s , \quad F^\Lambda = F^\Lambda + \bar{\psi}^s (\tau^s)^r \psi_r ,$$

$$F^{ab} = R^{ab} + \frac{1}{2} \epsilon^a \epsilon^b - \frac{1}{2} \bar{\psi}^s \Gamma^{ab} \psi_s , \quad F^z = f - i \bar{\psi}^s \psi_s .$$ (6)

Here the curvature and torsion two-forms have the form

$$R^{ab} = d\omega^{ab} + \omega^a \omega^{cb} , \quad T^a = de^a + \omega^a \epsilon^b ,$$ (7)

\(^1\)Hereafter, $a,b = 0, \ldots, 4$ and $\alpha = 1, \ldots, 4$ stand for vector and spinor indices in tangent space, respectively. The index $s = 1, \ldots, N$ corresponds to a vector representation of $SU(N)$, whose generators are labelled by $\Lambda = 1, \ldots, N^2 - 1$.

\(^2\)Here we omit the wedge symbol between forms for simplicity.
respectively, \( f = db \) is the \( u(1) \) field-strength, and \( a \equiv a^{\Lambda} \tau_{\Lambda}, \quad F = da + a^2 \) are the connection and curvature for \( SU(N) \), where the \( N \times N \) matrices \( \tau_{\Lambda} \) stand for the \( su(N) \) generators.

The components of the field-strength along the fermionic generators are given by the \( \text{AdS}_5 \times SU(N) \times U(1) \) covariant derivative\(^3\)

\[
\nabla \psi_s \equiv \left( D + \frac{1}{2\ell} e^a \Gamma_a \right) \psi_s - a^r_s \psi_r + i \left( \frac{1}{4} - \frac{1}{N} \right) b \psi_s,
\]

where \( D \psi_s = (d + \frac{1}{2} \omega^{ab} \Gamma_{ab}) \psi_s \) is the Lorentz covariant derivative, and \( \ell \) is the \( \text{AdS} \) radius.

The decomposition \(^6\) allows to write the Lagrangian in a manifestly Lorentz covariant way

\[
L = L_G (\omega, e) + L_{SU(N)} (a) + L_{U(1)} (\omega, e, b) + L_F (\omega, e, a, b, \psi),
\]

up to a boundary term. The gravitational sector is described by

\[
L_G = \frac{k}{8} \epsilon_{abcde} \left( \frac{1}{\ell} R^{ab} R^{cd} e^e + \frac{2}{3\ell^3} R^{ab} e^c e^d e^e + \frac{1}{5\ell^5} e^a e^b e^c e^d e^e \right),
\]

which is a linear combination of the Einstein-Hilbert Lagrangian with negative cosmological constant and the Gauss-Bonnet term which is quadratic in the curvature with a fixed coupling. The matter sector is described by

\[
L_{SU(N)} = ik \text{ Tr } (a F^2 - \frac{1}{2} a^3 F + \frac{1}{16} a^5),
\]

\[
L_{U(1)} = -k \left( \frac{1}{4} - \frac{1}{2} \frac{1}{\ell^2} \right) b (db)^2 + \frac{2k}{\ell^3} \left( T^a T_a - \frac{\ell^2}{2} R^{ab} R_{ab} - R^{ab} e_a e_b \right) b - \frac{3k}{4} F^\Lambda F_{\Lambda} b,
\]

\[
L_F = -\frac{3k}{4} \bar{\psi}^s \left( \frac{1}{\ell} T^a \Gamma_a + \frac{1}{2} \left( R^{ab} + \frac{1}{\ell^2} e^a e^b \right) \Gamma_{ab} + 2i \left( \frac{1}{N} + \frac{1}{4} \right) \frac{1}{\ell^3} \right) \bar{\psi}^s \psi_s \nabla \psi_s - \frac{3k}{4} \bar{\psi}^s \left( F^r_s - \frac{1}{2} \bar{\psi}^r \psi_s \right) \nabla \psi_r + \text{c.c.},
\]

where \( F^r_s = F^\Lambda (\tau_{\Lambda})_r^s \).

Note that the case \( N = 4 \) is exceptional and deserves special attention. As it can be seen from the covariant derivative \(^5\) and the Lagrangian \( L_{U(1)} \), in this case gravitini become neutral under \( U(1) \), and the dynamics of the \( U(1) \) field \( b \) changes because it looses the cubic kinetic term (the component \( g_{zzz} \) of the invariant tensor vanishes). This reflects the fact that for \( N = 4 \), the \( U(1) \) generator becomes a central charge in the superalgebra \( su(2, 2|4) \) (see Appendix).

By construction, the action is invariant under diffeomorphisms and under infinitesimal gauge transformations \( \delta_A A = -\nabla \lambda \), where \( \lambda \) is a Lie algebra valued zero form. Local supersymmetry transformations can be obtained as a particular case choosing the parameter as \( \lambda = \bar{e}^s Q_s - \bar{Q}^s e_s \), from which one obtains

\[
\delta_A e^a = -\frac{1}{2} \left( \bar{\psi}^s \Gamma^a \epsilon_s - \bar{e}^s \Gamma^a \psi_s \right), \quad \delta_A \psi_s = -\nabla \epsilon_s, \quad \delta_A \omega^{ab} = \frac{1}{4} \left( \bar{\psi}^s \Gamma^{ab} \epsilon_s - \bar{e}^s \Gamma^{ab} \psi_s \right), \quad \delta_A \bar{\psi}^s = -\nabla \epsilon_s, \quad \delta_A a^\Lambda = \bar{\psi}^s (\tau^\Lambda)^r_s \epsilon_r - \bar{e}^s (\tau^\Lambda)^r_s \psi_r, \quad \delta_A b = i \left( \bar{\psi}^s \epsilon_s - \bar{e}^s \psi_s \right).
\]

Note that, as a consequence, the supersymmetry algebra closes \textit{off-shell} by construction, without requiring auxiliary fields \(^4\).

\(^3\)The covariant derivative acts on a Lie-algebra valued \( p \)-form \( X_p \) as \( \nabla X_p = X_p + [A, X_p] \), where \([A, X_p] = AX_p - (-)^p X_p A\).
3 Charges and their algebra in the canonical sectors

In order to have a bona fide BPS bound, a canonical realization of the supersymmetry algebra is needed. We follow the time-honored formalism of Dirac for constrained systems since it ensures by construction the closure of the canonical generators algebra. However, the standard Dirac procedure, required to identify the physical observables (propagating degrees of freedom, conserved charges, etc.), is not directly applicable around irregular backgrounds. Indeed, the naive linearization of the theory fails to provide a good approximation to the full theory around those backgrounds [21, 22, 23].

Thus, we analyze the system around background solutions in the canonical sectors, namely, sectors possessing maximal number of degrees of freedom where all constraints are functionally independent. The action (3) can be seen to belong to the class of theories studied in [22], for which a family of backgrounds in the canonical sectors were identified. It is worth mentioning that for \( N = 4 \) the theory contains additional irregular sectors which do not exist otherwise, and which require special attention.

As shown in [22], configurations where the only nonvanishing components of \( F^K \) is\(^4\)

\[ F^K_{12} \, dx^1 \, dx^2 \neq 0, \]

for at least one \( K \) and

\[ F^z_{34} = 0, \quad \text{with} \quad \det (F^z_{ij}) \neq 0, \]

turn out to be canonical for any \( N \). Therefore, around this kind of background solutions the counting of degrees of freedom can be safely done following the standard procedure [20, 35]. In this case, the number is \( \Delta - 2 = N^2 + 8N + 13 \) (see [18]).

3.1 Charge algebra

The advantage of the class of canonical sectors described above, is that the splitting between first and second class constraints, which is in general an extremely difficult task, can be performed explicitly. As a consequence, the conserved charges and their algebra can be obtained following the Regge-Teitelboim approach [36], and as shown in [22], they turn out to be

\[ Q [\lambda] = -3k \int_{\partial \Sigma} g_{KLM} \lambda^K \bar{F}^L A^M. \]

Here \( \bar{F} \) is the background field strength and the parameters \( \lambda^K(x) \) approach covariantly constant fields at the boundary. According to the Brown-Henneaux theorem, in general the charge algebra is a central extension of the gauge algebra [37],

\[ \{ Q [\lambda], Q [\eta] \} = Q [[\lambda, \eta]] + C [\lambda, \eta]. \]

In the present case the central charge is

\[ C [\lambda, \eta] = 3k \int_{\partial \Sigma} g_{KLM} \lambda^K \bar{F}^L d\eta^M. \]

\(^4\)The five-dimensional manifold is assumed to be topologically \( \mathbb{R} \otimes \Sigma \), and the coordinates are chosen as \( x^\mu = (x^0, x^i) \), where \( x^i \), with \( i = 1, \ldots, 4 \) correspond to the space-like section \( \Sigma \).
The charge algebra can be recognized as the WZW extension of the full gauge group \( \mathbb{Z}_2 \). In an irregular sector the charges are not well defined and the naive application of the Dirac formalism would at best lead to a charge algebra associated to a subgroup of \( G \).

Having obtained the canonical realization of the symmetry algebra, allows one to proceed with construction of the BPS bound as well as the states that saturate it. In the next section we find explicit BPS solutions within the class of canonical configurations given by Eqs. (13) and (14), and in Sect.5, we explicitly obtain the Bogomol’nyi bound for states in the neighborhood of a BPS state.

4 Background solutions

The simplest background solutions are purely bosonic \((\psi_s = 0)\), for which the field equations become

\[
\varepsilon_{abcde} \left( R^{bc} R^{de} + \frac{2}{\ell^2} R^{be} e^d e^e + \frac{1}{\ell^4} e^b e^c e^d e^e \right) + \frac{4}{\ell} T_a f = 0, \tag{18}
\]

\[
\varepsilon_{abcde} \left( R^{cd} + \frac{1}{\ell^2} e^c e^d \right) T^e + \ell \left( R_{ab} + \frac{1}{\ell^2} e_a e_b \right) f = 0, \tag{19}
\]

\[
\frac{1}{4\ell^2} \left( \frac{\ell^2}{2} R^{ab} R_{ab} + R^{ab} e_a e_b - T^a T_a \right) + \frac{1}{N} F^A F_A - \left( \frac{1}{N^2} - \frac{1}{4\ell^2} \right) f f = 0, \tag{20}
\]

\[
\gamma_{\Lambda_1 \Lambda_2} F^A F^A - \frac{2}{N} F_A f = 0. \tag{21}
\]

Assuming spacetime to be locally AdS, so that \( F^{ab} = R^{ab} + \frac{1}{\ell^2} e^a e^b = 0 \), the torsion vanishes. Therefore, the modified Einstein and torsion Eqs. (18) are trivially satisfied, and the first term in Eq. (20) vanishes, as well.

Note that in the absence of matter fields, any locally AdS spacetime solves the bosonic fields equations. However, this kind of backgrounds are maximally degenerate and irregular. It is noteworthy that in this case it is possible to overcome degeneracy and irregularity by switching on matter fields which do not have a back reaction on the metric.\(^5\)

It must be emphasized that locally AdS spacetime configurations require the presence of nontrivial \( SU(N) \) and \( U(1) \) fields. Indeed, it might seem as if a simpler solution could be obtained for \( N = 4 \) by turning off the \( SU(4) \) curvature, \( F^A = 0 \) in Eqs. (22) \( \text{[22]} \). That solution is, however, irregular.

As required by (13), locally AdS spacetime configurations must have the \( SU(N) \) field-strength \( F^{12}_{\Lambda} \) switched on. It is easy to see that this configuration solves the remaining field equations (20) and (21), provided the \( U(1) \) field \( b \) has a field-strength satisfying \( F^b_{34} = 0 \), while the remaining components are arbitrary, and \( F^z_{ij} = \partial_i b_j - \partial_j b_i \) can be assumed to be invertible.

In sum, the bosonic solutions given by

\[
R^{ab} = -\frac{1}{\ell^2} e^a e^b, \quad T^a = 0, \quad F^A = F^A_{12} dx^1 dx^2 \neq 0, \quad F^z_{34} = 0, \quad \text{with } \det \left( F^z_{ij} \right) \neq 0, \tag{22}
\]

\(^5\)Matter fields may not produce back reaction as a result of non-minimal couplings. This has been observed in very simple systems, such as general relativity with scalar fields \( \text{[39, 40]} \).
provide canonical backgrounds for any $N$.

One then concludes that in this supergravity theory, constant curvature spacetimes can be embedded in a canonical sector since they can be consistently combed with nontrivial $SU(N)$ and $U(1)$ fields. This includes AdS spacetime and quotients of it, as in Refs. [41], giving rise to a wide class of solutions with different topologies.

In what follows we will look for solutions of the form (22) admitting Killing spinors.

4.1 BPS states

Bosonic solutions of the field equations which are left invariant under globally defined supersymmetry transformations (BPS states), by virtue of Eqs. [12] must satisfy $\delta_\epsilon \psi_s = - \nabla \epsilon_s = 0$. Hence, Killing spinors $\epsilon_s$ solve the equation

$$\nabla \epsilon_s = \left( d + A_{AdS} + i \left( \frac{1}{4} - \frac{1}{N} \right) b \right) \epsilon_s - a^r_s \epsilon_r = 0 ,$$

where $a^r_s = a^\Lambda (\tau_\Lambda)^s_r$, and the AdS connection is given by $A_{AdS} = \frac{1}{4} \omega^{ab} \Gamma_{ab} + \frac{1}{2\ell} e^a \Gamma_a$. The consistency condition of the Killing spinor Eq. (23), $\nabla \nabla \epsilon_s = 0$, reads

$$\left( F_{AdS} + i \left( \frac{1}{4} - \frac{1}{N} \right) f \right) \epsilon_s - \mathcal{F}_s^r \epsilon_r = 0 ,$$

where the AdS curvature is

$$F_{AdS} = \frac{1}{4} \left( R^{ab} + \frac{1}{\ell^2} e^a e^b \right) \Gamma_{ab} + \frac{1}{2\ell^2} T^a \Gamma_a .$$

Note that for $N = 4$, neither the Killing spinor equation (23) nor the consistency condition (24) involve the $U(1)$ field. Hence, for simplicity, we will focus on this case in what follows.

4.1.1 $N = 4$

For $N = 4$, equation (23) reduces to

$$\left[ (d + A_{AdS}) \delta^r_s - a^r_s \right] \epsilon_r = 0 ,$$

and since the AdS curvature $F_{AdS}$ vanishes for the class of background solutions under consideration given by (22), the consistency condition simply reads

$$\mathcal{F}_s^r \epsilon_r = 0 .$$

For the canonical class of solutions given by (22), the consistency condition (26) means that the Killing spinors must be zero modes of the $SU(4)$ field strength. Hence, $\mathcal{F}^\Lambda$ must be nonvanishing for more than one value of the index $\Lambda$, so that the contributions of all components cancel.

As an example, taking advantage of the isomorphism $su(4) \simeq so(6)$, the $SU(4)$ curvature can be expressed as $\mathcal{F}_s^r = \frac{1}{2} \mathcal{F}^{IJ} (\tau_{IJ})^s_r$, where the $so(6)$ generators

$$\tau_{IJ} = \frac{1}{2} \hat{\Gamma}_{IJ} , \quad (I, J = 1, \ldots, 6)$$

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$$\tau_{IJ} = \frac{1}{2} \hat{\Gamma}_{IJ} , \quad (I, J = 1, \ldots, 6)$$
are given in terms of the Euclidean Dirac matrices $\hat{\Gamma}_I$, with $\hat{\Gamma}_{IJ} = \frac{1}{2} \left[ \hat{\Gamma}_I, \hat{\Gamma}_J \right]$. The commuting matrices $\tau_{12}$ and $\tau_{34}$ generate a $U(1) \otimes U(1)$ subgroup for $SU(4)$, and since $(\tau_{12})^2 = (\tau_{34})^2 = -\frac{1}{4}$, the eigenvalues of $\tau_{12}$ and $\tau_{34}$ are $\pm i/2$.

For simplicity, one can make use of a “twisted” configuration for which the only nonvanishing $U(1) \otimes U(1)$ components of the $SU(4)$ curvature are given by $F^{12} = da_{12}$ and $F^{34} = da_{34}$, and the Killing spinor $\epsilon_s$ is assumed to satisfy

$$\left(\tau_{12}\right)_s^r \epsilon_r = \frac{i}{2} \epsilon_s, \quad \left(\tau_{34}\right)_s^r \epsilon_r = -\frac{i}{2} \epsilon_s.$$  \hspace{1cm} (28)

Therefore, the consistency condition (26) becomes

$$\frac{i}{2} \left( F^{12} - F^{34} \right) \epsilon_s = 0,$$

and is solved by $F^{12} = F^{34}$. Then the $SU(4)$ field satisfy

$$a^{12} = a^{34} + d\theta,$$  \hspace{1cm} (29)

where $\theta = \theta (x)$ is an arbitrary phase. Hence, the twisted $SU(4)$ configuration satisfies $a_s^r \epsilon_s = \frac{i}{2} d\theta \epsilon_r$, and the Killing equation (25) reduces to

$$\left( d + A_{AdS} - \frac{i}{2} d\theta \right) \epsilon_s = 0.$$  \hspace{1cm} (30)

The solution of this last equation is given by

$$\epsilon_s = e^{-\frac{i}{2} \theta} \eta_s,$$  \hspace{1cm} (31)

where $\eta_s$ satisfies the twisting conditions (28), and is a nontrivial solution of the Killing spinor equation in vacuum

$$(d + A_{AdS}) \eta_s = 0.$$  \hspace{1cm} (32)

Locally AdS spacetimes admitting Killing spinors have been extensively discussed in the literature \[41\]. We now consider a particular geometry which is simple and allows to deal with a nontrivial topology at the boundary.

### 4.1.2 Explicit BPS solutions in the canonical sectors

The local coordinates on $\mathcal{M}$ are chosen as $x^\mu = (t, \rho, \varphi^p)$, where $\varphi^p$ ($p = 2, 3, 4$) parametrize the boundary $\partial \Sigma$, placed at the infinity of the coordinate $\rho$ ($\rho \geq 0$).

The globally AdS space-time ($F^{AB} = 0$) can be described by the metric

$$ds^2_{AdS} = \ell^2 \left( d\rho^2 + e^{2\rho} \eta_{\bar{p}q} dx^\bar{p} dx^q \right),$$  \hspace{1cm} (33)

where $x^\bar{p} = (t, \varphi^p)$ and $\eta_{\bar{p}q} = \text{diag} (-, +, +, +)$. The AdS connection is

$$A_{AdS} = - \frac{\ell}{2} \omega^{ab} \Gamma_{ab} + \frac{1}{\ell} \epsilon^a \Gamma_a = - \frac{\ell}{2} \left[ e^1 \Gamma_1 + e^\bar{p} \Gamma_{\bar{p}} (1 + \Gamma_1) \right].$$

The Killing spinors for the metric (33) solving equation (32), have the form \[42\]

$$\eta_s = e^{-\frac{i}{2} \Gamma_1} \left[ 1 - x^\bar{p} \Gamma_{\bar{p}} (1 + \Gamma_1) \right] \eta_{0s}.$$  \hspace{1cm} (34)
where \( \eta_0 s \) is a constant spinor. This gives a solution to the Killing spinor equation of the form \( ^{30} \), provided \( \eta_0 s \) satisfies the twisting conditions \( ^{28} \).

Assuming the boundary of the spatial section to be topologically \( \partial \Sigma \simeq S^1 \otimes S^1 \otimes S^1 \), the spinor \( \eta_s \) must be antichiral under the action of \( \Gamma_1 \) in order to be globally defined. Then the solution becomes,

\[
\epsilon_s = e^{\frac{i}{2} (i\theta + \rho)} \eta_0 s, \quad \text{with} \quad (1 + \Gamma_1) \eta_0 s = 0.
\]

Chirality and twisting conditions give \( \frac{1}{8} \times 16 = 2 \) unbroken supersymmetries.

The remaining \( SU(4) \) and \( U(1) \) fields can be chosen as follows,

\[
\begin{align*}
\bar{a}^{12} &= h d\rho, \\
\bar{a}^{34} &= h d\rho - d\theta, \\
\bar{b} &= E \varphi^3 d\rho + B \varphi^4 d\varphi^2,
\end{align*}
\]

where \( h = h(\varphi^2) \) is an arbitrary function, and \( E, B \) are nonvanishing constants. Then \( \bar{f}_{34} = 0 \), and \( \det (\bar{f}_{ij}) = (BE)^2 \neq 0 \), as required by \( ^{22} \).

Various topologies. For chosen boundary conditions \( A \rightarrow \bar{A} \) and \( \lambda \rightarrow \bar{\lambda} \), where \( \nabla \bar{\lambda} = 0 \), the most general central charge is given by \( ^{17} \). In locally AdS spacetimes, it takes the form

\[
C [\lambda, \eta] = 6k \int_{\partial \Sigma} \langle \lambda \bar{F}_{U(1)} d\eta \rangle + 6k \int_{\partial \Sigma} \langle \lambda \bar{F}_{SU(4)} d\eta \rangle,
\]

so that the charge only acquires contributions from the internal bosonic subgroup \( SU(4) \otimes U(1) \).

If the topology of the boundary is isomorphic to \( S^1 \otimes S^1 \otimes S^1 \), then the first term in \( ^{39} \) gives a non-trivial contribution to the central charge since \( \pi_1 (U(1)) = \mathbb{Z} \), while the second term vanishes because of \( \pi_1 (SU(4)) = 0 \). When \( SU(4) \) is explicitly broken into \( U(1) \otimes U(1) \otimes U(1) \), it can give non-trivial contribution, as well.

In the case of \( S^1 \otimes S^2 \), since \( \pi_2 (G) = 0 \) for all Lie groups, a non-trivial central charge can be obtained only if the Abelian field \( b \), or some other corresponding to a \( U(1) \) subset of \( SU(4) \), winds around \( S^1 \).

If the topology is \( S^3 \), the non-trivial \( SU(4) \) field must have three non-vanishing components at \( \partial \Sigma \), \( F_{pq}^A \neq 0 \), where it is a solution if constraints. Then the first term in the central charge vanishes due to \( \pi_3 (U(1)) = 0 \), but considering that \( \pi_3 (SU(4)) = \mathbb{Z} \), the second term may give a non-vanishing result.

5 Bogomol’nyi bound

In this section we establish the Bogomol’nyi bound \( ^{43} \) for configurations in the neighborhood of a BPS state. Due to the presence of a central charge in this case the bound cannot be obtained naively from the original supersymmetry algebra.

Around the background \( ^{33, 36, 38} \), the charges \( ^{15} \) take the form\(^{6} \)

\[
Q [\lambda] = \int_{\partial \Sigma} \frac{d^3 \varphi}{(2\pi)^3} \lambda^K q_K, \quad q_K = \frac{3kB}{2} \gamma_{KL} A^L_3.
\]

\(^{6}\)The normalization of the volume element of \( \partial \Sigma = S^1 \otimes S^1 \otimes S^1 \) has been chosen as \( d^3 x = \frac{d^3 \varphi}{(2\pi)^3} \).
Note that the class of solutions considered here has identically vanishing $u(1)$ charge since $\gamma_{zz} = \gamma_{Kz} = 0$ (see Appendix). Thus, the canonical algebra \([10]\) is a central extension of $psu(2, 2\mid 4)$, and the central charge \([17]\) is just

$$C [\lambda, \eta] = -\frac{3kB}{2} \int \frac{d^3\varphi}{(2\pi)^3} \gamma_{KL} \lambda^K \partial_3 \eta^L. \tag{41}$$

In particular, $C [\lambda^z, \eta^K] \equiv 0$, which implies that there is no $u(1)$ central extension.

Demanding the charges to vanish on the BPS background, we obtain \(^7\)

\begin{align*}
\tilde{Q}_{u(1)} &= Q [\tilde{\lambda}^z, \tilde{A}] = 0, \\
\tilde{Q}_{AdS} &= Q [\tilde{\lambda}^{AB}, \tilde{A}] = \frac{3kB}{2} (\tilde{\lambda}^{31} - \tilde{\lambda}^{35}) e^\rho = 0, \tag{42} \\
\tilde{Q}_{su(4)} &= Q [\tilde{\lambda}^{IJ}, \tilde{A}] = -\frac{3kB}{2} \tilde{\lambda}^{34} \int \frac{d^3\varphi}{(2\pi)^3} \partial_3 \theta (\rho, \varphi).
\end{align*}

Therefore, the phase $\theta$ is periodic in $\varphi^3$ in order to have a vanishing $su(4)$ charge.

### 5.1 Mode expansion

Since the canonical BPS background has a boundary with topology $\partial \Sigma \simeq S^1 \otimes S^1 \otimes S^1$, any variable $X (\rho, \varphi)$, periodic or anti-periodic in $\varphi = (\varphi^2, \varphi^3, \varphi^4) \in [0, 2\pi)$, can be expanded in Fourier series as

$$X (\rho, \varphi) = \sum_{\bar{\nu}} X_{\bar{\nu}} (\rho) e^{i \bar{\nu} \cdot \varphi}, \quad X_{\bar{\nu}} (\rho) = \int \frac{d^3\varphi}{(2\pi)^3} X (\rho, \varphi) e^{-i \bar{\nu} \cdot \varphi}, \tag{43}$$

where $\bar{\nu} \equiv (\nu_2, \nu_3, \nu_4)$ are winding numbers. Bosonic modes are periodic in $\varphi$ and the numbers $\bar{\nu}$ are integers. Fermionic modes can be periodic (Ramond (R) sector), or anti-periodic (Neveu-Schwartz (NS) sector) in any of the angular coordinates $\varphi$, and the corresponding winding numbers are integers ($\nu_i \in \mathbb{Z}$) or half-integers ($\nu_i + 1/2 \in \mathbb{Z}$), respectively, giving rise to eight possible sectors $R_2-R_3-R_4$, $R_2-R_3-NS_4$, etc.

The mode expansion of the charges \([40]\) is

$$Q [\lambda] = \sum_{\bar{\nu}} \lambda^K_{\bar{\nu}} q_{K,-\bar{\nu}}, \quad q_{K,\bar{\nu}} = \frac{3kB}{2} \gamma_{KL} A^L_{3,\bar{\nu}}, \tag{44}$$

and the central charges \([41]\) are

$$C [\lambda, \eta] = -\frac{3kB}{2} \gamma_{KL} \sum_{\bar{\nu}, \bar{\mu}} \mu_3 \lambda^K_{\bar{\nu}} \eta^L_{\bar{\mu}} \delta_{\bar{\nu} + \bar{\mu}, 0}. \tag{45}$$

Finally, the charge algebra acquires the form

$$\text{alg} \{q_{K,\bar{\nu}}, q_{L,\bar{\mu}}\} = f_{KL}^M q_{M,\bar{\nu}+\bar{\mu}} = \frac{3kB}{2} \nu_3 \gamma_{KL} \delta_{\bar{\nu}+\bar{\mu}, 0}. \tag{46}$$

\(^7\)The covariantly constant AdS vectors $\tilde{\lambda}^{AB}$ are solutions of $\nabla_{AdS} \tilde{\lambda}^{AB} = 0$ with $A_{AdS}$ given by \([33]\). They have the form $\tilde{\lambda}^{31} = \tilde{\lambda}^{35} = V^\rho e^\rho$ ($V^\rho = \text{Const.}$). For the $su(4)$ connection \([36, 37]\), the nonvanishing covariantly constant vectors, $\nabla_{su(4)} \tilde{\lambda}^{IJ} = 0$, are $\tilde{\lambda}^{12}$, $\tilde{\lambda}^{31}$, $\tilde{\lambda}^{36} = \text{Const.}$, and they describe the unbroken symmetry $U(1) \otimes U(1) \subset SU(4)$ of the background.
The algebra \( \mathfrak{psu} \) is a supersymmetric extension of the WZW algebra \( \mathfrak{psu} \). It has a nontrivial central extension for \( psu(2,2|4) \) which depends only on \( u(1) \) flux determined by \( B \). Note that the modes \( q_{K,\tilde{K}} \) with \( \tilde{\nu} = (0, \nu_3, 0) \) form a Kac-Moody subalgebra with the central charge \( c = -\frac{3kB}{2} \), while the modes with \( \tilde{\nu} = (\nu_2, 0, 0) \) and \( (0, 0, \nu_4) \) form Kac-Moody subalgebras without central charges.

### 5.2 Bogomol’nyi bound

In the case of fermionic charges, the algebra \( \mathfrak{psu} \) reads

\[
\{ q^a_{r,\tilde{r}}, \bar{q}^s_{\beta,\tilde{\beta},\mu} \} = -\frac{1}{2} \delta^s_r (\Gamma^0)^\beta_{\beta} q_{a,\tilde{r}+\mu} + \frac{1}{4} \delta^s_r (\Gamma^{ab})^\alpha_{\beta} q_{ab,\tilde{r}+\mu} - \frac{1}{2} \bar{\delta}^s_\beta \left( \tau^{IJ} \right)^s_r q_{IJ,\tilde{r}+\mu} + \frac{3ikB}{2} \nu_3 \delta^s_r \delta^\alpha_\beta \delta_{\tilde{r}+\mu,\tilde{\mu}} .
\]

(Multiplying (47) by \( \Gamma^0 \), and using the fact that the operator \( \{ q^a_{r,\tilde{r}}, (q^\dagger)^s_{\beta,\tilde{\beta},\mu} \} \) is positive semidefinite, we have the bound

\[
-\frac{1}{2} \delta^s_r (\Gamma^0)^0_{\beta} q_{a,\tilde{r}} + \frac{1}{4} \delta^s_r (\Gamma^{ab} \Gamma^0)^0_{\beta} q_{ab,\tilde{r}} - \frac{1}{2} (\Gamma^0)^0_{\beta} (\tau^{IJ})^s_r q_{IJ,\tilde{r}} + \frac{3ikB}{2} \nu_3 (\Gamma^0)^0_{\beta} \delta^s_r \geq 0 .
\]

Decomposing the AdS boost charge as \( q_{a,\tilde{r}} = (q_{a,\tilde{r}}, q_{a,\tilde{r}}) \), one finds

\[
-\frac{1}{2} \delta^s_r (\Gamma^0)^0_{\beta} q_{a,\tilde{r}} = \frac{1}{2} \delta^s_r \delta^\alpha_\beta E - \frac{1}{2} \delta^s_r (\Gamma^0)^0_{\beta} q_{a,\tilde{r}},
\]

where the energy is identified as \( E = q_{a,\tilde{r}} \). Then (48) can be rewritten as

\[
\delta^s_r \delta^\alpha_\beta E \geq \delta^s_r (\Gamma^0)^0_{\beta} q_{a,\tilde{r}} - \frac{1}{2} \delta^s_r (\Gamma^{ab} \Gamma^0)^0_{\beta} q_{ab,\tilde{r}} + (\Gamma^0)^0_{\beta} (\tau^{IJ})^s_r q_{IJ,\tilde{r}} - 3ikB \nu_3 (\Gamma^0)^0_{\beta} \delta^s_r .
\]

The eigenvalues \( \lambda_i \) of the matrix \( M_{r\beta}^{s\alpha} \) on the r.h.s. of (49) can be calculated from \( \sum_i \lambda_i^2 = \text{Tr}(M)^2 \), using the orthogonality of the group generators. The result is\(^8\) \( \lambda^2 = p^2 + (3kB)^2 \nu_3^2 \), with \( p^2 \equiv \sum_K (q_{K,\tilde{K}})^2 \). The requirement that the energy is not smaller than the largest eigenvalue of \( M \), namely \( \delta^s_r \delta^\alpha_\beta E \geq M_{r\beta}^{s\alpha} \), leads to the Bogomol’nyi bound

\[
E \geq \sqrt{p^2 + (3kB)^2 \nu_3^2} .
\]

This bound is saturated for the BPS states, \( E_{BPS} = |3kB \nu_3| \). In the NS3 sector where \( (\nu_3)_{\text{min}} = 1 \), the energy is \( E_{BPS} = |\frac{3kB}{2}| \), while in the R3 sector, \( (\nu_3)_{\text{min}} = 0 \), \( E_{BPS} = 0 \). The ground state \( \mathfrak{psu} \) is a R3 state, consistent with the fact that \( \theta \) is periodic in \( \varphi^3 \).

### 6 Conclusions

Degeneracy and irregularity are largely unexplored phenomena in dynamical systems. Although these features are rarely found in standard field theories, they are unavoidable in higher dimensional gravity theories of current interest such as those described by the Gauss-Bonnet and

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\(^8\)In this signature, \( (\varphi)^2 = -1 \).

\(^9\)Apart from the traces of unit matrices, the other non-vanishing traces are \( \text{Tr}(\varphi^{ab}) = 4\eta^{ab} \), \( \text{Tr}(\varphi^{ab} \varphi^{cd}) = -4\eta^{[ab]} \eta^{[cd]} \) and \( \text{Tr}(\varphi^{IJ} \varphi^{I'J'}) = -\delta^{IJ} \delta^{I'J'} \).

---
Lovelock actions. Irregularities imply that the degrees of freedom of the linearized approximation do not correspond to those of the full theory and hence one needs to go beyond the perturbative analysis. Moreover, the canonical analysis breaks down and since it is not clear how to identify the conserved charges—and physical observables in general—the possibility of finding a concrete expression of their algebra is severely limited.

Although canonical sectors, which are nondegenerate and regular, fill open sets of phase space, they may not be not easily identified. The rich geometric structure of the supergravity theory considered here helps in this task, as well as in obtaining a canonical representation of the conserved charges. It is found that, unlike the situation in standard theories, the resulting charge algebra turns out to be a nontrivial central extension of the symmetry algebra, in an analogous way as it occurs in the case of asymptotically AdS gravity in three dimensions [47]. In this case, the central charge is nonzero thanks to the presence of matter fields with a nontrivial winding. Interestingly, these matter fields have a nontrivial field strength but nevertheless, due the nonminimal coupling, produce no back reaction on the geometry.

The BPS bound is constructed here in the canonical sector. The canonical realization of the algebra guarantees the stability of the theory, which would not be achieved through the naive bound, constructed purely from the symmetry algebra. BPS states in the canonical sectors saturating the bound are explicitly found.

Conserved charges for Chern-Simons gravity theories in higher dimensions were constructed using a background-independent approach and have been shown to be well defined even for degenerate and irregular configurations, including black holes [48]. These charges were shown to be related to the notion of transgression forms [49]. Alternative expressions for conserved charges also based on the idea of transgression forms have been constructed in Refs. [50, 51, 52, 53]. It would be interesting to see whether the centrally extended algebra constructed here can be reproduced by those methods.

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### A Supersymmetric extension of AdS$5$, $SU(2,2|N)$

The supersymmetric extension of the AdS group in five dimensions, $SO(2,4)$, is the super unitary group $SU(2,2|N)$ [33, 34], containing supermatrices of unit superdeterminant which leave invariant the (real) quadratic form

$$q = \theta^\alpha G_{\alpha\beta}\theta^\beta + z^r g_{rs} z^s \quad (\alpha = 1, \ldots, 4; \ r = 1, \ldots, N).$$

(51)

Here $\theta^\alpha$ are complex Grassman numbers (with complex conjugation defined as $(\theta^\alpha\theta^\beta)^* = \theta^\beta\theta^\alpha$), and $G_{\alpha\beta}$ and $g_{rs}$ are Hermitean matrices, antisymmetric and symmetric respectively, which can be chosen as

$$G_{\alpha\beta} = i (\Gamma_0)_{\alpha\beta}, \quad g_{rs} = \delta_{rs}.$$  

(52)
The bosonic sector of this supergroup is
\[ SU(2, 2) \otimes SU(N) \otimes U(1) \subset SU(2, 2|N), \] (53)
where the AdS group is present on the basis of the isomorphism \( SU(2, 2) \simeq SO(2, 4) \). Therefore, the generators of \( su(2, 2|N) \) algebra are

\[
\begin{align*}
so(2, 4) : & \quad J_{AB} = (J_{ab}, J_a), \quad (A, B = 0, \ldots, 5), \\
su(N) : & \quad T_\Lambda, \quad (\Lambda = 1, \ldots, N^2 - 1), \\
SU(1) : & \quad Q_s^\alpha, \bar{Q}_s^\alpha, \quad (\alpha = 1, \ldots, 4; \ s = 1, \ldots, N), \\
u(1) : & \quad Z,
\end{align*}
\] (54)

where \( \eta_{AB} = \text{diag} (-, +, +, +, +, -) \), and AdS rotations and translations are \( J_{ab} \) and \( J_a \equiv J_{a5} \ (a, b = 0, \ldots, 4) \). The dimension of this superalgebra is \( \Delta = N^2 + 8N + 15 \). For \( N = 1 \), the generators \( T_\Lambda \) are absent, and the bosonic sector is given by \( \text{AdS}_5 \otimes u(1) \) algebras.

A representation of the superalgebra acting in \( (4 + N) \)-dimensional superspace \( (\theta^a, y^s) \) is given by the \( (4 + N) \times (4 + N) \) supermatrices

\[
\begin{align*}
J_{AB} &= \begin{pmatrix} \frac{1}{2} (\Gamma_{AB})_\alpha^\beta & 0 \\ 0 & 0 \end{pmatrix}, & Q_s^\alpha &= \begin{pmatrix} 0 & 0 \\ -\delta_s^\alpha \delta_\beta^\alpha & 0 \end{pmatrix}, & Z &= \begin{pmatrix} \frac{i}{4} \delta_\alpha^\beta & 0 \\ 0 & \frac{i}{N} \delta_\alpha^\beta \end{pmatrix}, \\
T_\Lambda &= \begin{pmatrix} 0 & (\tau_\Lambda)_s^\alpha \\ 0 & (\tau_\Lambda)_s^\alpha \end{pmatrix}, & \bar{Q}_s^\alpha &= \begin{pmatrix} 0 & \delta_s^\alpha \delta_\beta^\alpha \\ 0 & 0 \end{pmatrix},
\end{align*}
\] (55)

where the \( 4 \times 4 \) matrices \( \Gamma_{AB} \) are defined as
\[
\Gamma_{ab} = \frac{1}{2} [\Gamma_a, \Gamma_b], \quad \Gamma_{a5} = \Gamma_a,
\] (56)

\( \Gamma_a \) are the Dirac matrices in five dimensions with the signature \((- + + + +)\), and \( \tau_\Lambda \) are anti-Hermitian generators of \( su(N) \) acting in \( N \)-dimensional space \( y^s \).

From the given representation of supermatrices it is straightforward to find the explicit form of the corresponding Lie algebra. The commutators of the bosonic generators \( J_{AB} \), \( T_\Lambda \) and \( Z \) closes the algebra \( su(2, 2) \otimes su(N) \otimes u(1) \), while the supersymmetry generators transforms as spinors under AdS and as vectors under \( su(N) \),

\[
\begin{align*}
[J_{AB}, Q_s^\alpha] &= -\frac{i}{2} (\Gamma_{AB})_\beta^\alpha Q_s^\beta, & [T_\Lambda, Q_s^\alpha] &= (\tau_\Lambda)_s^\alpha Q_r^\alpha, \\
[J_{AB}, \bar{Q}_s^\alpha] &= \frac{i}{2} \bar{Q}_s^\beta (\Gamma_{AB})_\alpha^\beta, & [T_\Lambda, \bar{Q}_s^\alpha] &= -\bar{Q}_s^\alpha (\tau_\Lambda)_r^s,
\end{align*}
\] (57)

and they carry \( u(1) \) charges,

\[
\begin{align*}
[Z, Q_s^\alpha] &= -i \left(\frac{1}{4} - \frac{1}{N}\right) Q_s^\alpha, & [Z, \bar{Q}_s^\alpha] &= i \left(\frac{1}{4} - \frac{1}{N}\right) \bar{Q}_s^\alpha.
\end{align*}
\] (58)

The anticommutator of the supersymmetry generators has the form
\[
\{Q_s^\alpha, \bar{Q}_s^\beta\} = \frac{1}{4} \delta_s^\alpha (\Gamma^{AB})_\beta^\alpha J_{AB} - \delta_\beta^\alpha (\tau_\Lambda)_s^\alpha T_\Lambda + i \delta_\beta^\alpha \delta_s^\alpha Z.
\] (59)

From these expressions it is clear that for \( N = 4 \) the \( U(1) \) generator \( Z \) becomes a central charge and the algebra becomes a central extension of \( PSU(2, 2 | 4) \).
An invariant third rank tensor, completely symmetric in bosonic and antisymmetric in fermionic indices, can be constructed as

\[ i g_{KLM} \equiv (G_K G_L G_M) = \frac{1}{2} \text{Str} \left[ \left( G_K G_L + (-)^{K\mathcal{L}} G_L G_K \right) G_M \right] , \]

with the following non-vanishing components:

\[
\begin{align*}
    g_{[AB][CD][EF]} &= \frac{1}{2} \varepsilon_{ABCDEF}, \\
    g_{\Lambda_1 \Lambda_2 \Lambda_3} &= -\gamma_{\Lambda_1 \Lambda_2 \Lambda_3}, \\
    g_{\gamma(\gamma)(\gamma)} &= \frac{1}{4} (\Gamma_{AB})_{\beta\gamma}^\alpha \delta_{z}^s, \\
    g_{\gamma(\gamma)} &= \frac{1}{2} \delta_{\beta}^\alpha (\tau_{\Lambda})_{r}^s, \\
    g_{zzz} &= \frac{1}{2} \delta_{\beta}^\alpha (\tau_{\Lambda})_{r}^s,
\end{align*}
\]

Here \( \eta_{[AB][CD]} \equiv \eta_{AC} \eta_{BD} - \eta_{AD} \eta_{BC} \) and \( \gamma_{KL} \) are the Killing metrics of \( SO(2,4) \) and \( SU(N) \), respectively. The symmetric third rank invariant tensor for \( su(N) \) is \( \gamma_{\Lambda_1 \Lambda_2 \Lambda_3} \equiv \frac{1}{N} \text{Tr}_N \left( \left\{ \tau_{\Lambda_1}, \tau_{\Lambda_2} \right\} \tau_{\Lambda_3} \right) \), and the \( \Gamma \)-matrices are normalized so that

\[
\text{Tr}_4 \left( \Gamma_a \Gamma_b \Gamma_c \Gamma_d \Gamma_e \right) = -4i \varepsilon_{abcde}, \quad (\varepsilon_{abcde}^5 \equiv \varepsilon_{abcde}, \quad \varepsilon_{012345} = 1).
\]

Splitting the generators as \( G_K = (G_K, Z) \), it can be seen that the invariant tensor for \( SU(2|2|N) \) fulfills the conditions: (i) \( g_{KLz} \) is invertible, and (ii) \( g_{zzz} \) vanishes. Then, as shown in [22], it is easy to identify generic configuration and those satisfying eq. (60) are canonical.

In the special case \( N = 4 \), the invariant tensor \( g_{KLM} \) of \( SU(2,2|4) \) simplifies to:

\[
\begin{align*}
    g_{[AB][CD][EF]} &= \frac{1}{2} \varepsilon_{ABCDEF}, \\
    g_{\Lambda_1 \Lambda_2 \Lambda_3} &= -\gamma_{\Lambda_1 \Lambda_2 \Lambda_3}, \\
    g_{\gamma(\gamma)} &= \frac{1}{4} (\Gamma_{AB})_{\beta\gamma}^\alpha \delta_{r}^s, \\
    g_{\gamma(\gamma)(\gamma)} &= \frac{1}{2} \delta_{\beta}^\alpha (\tau_{\Lambda})_{r}^s, \\
    g_{zzz} &= 0 \quad \text{and} \quad \gamma_{KL} \text{ is the Killing metric for } PSU(2,2|4). \quad \text{In this case, the dimension of the group is } \Delta = 63.
\end{align*}
\]

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