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Quickest Detection of Intermittent Signals With Application to Vision Based Aircraft Detection

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Abstract—In this paper we consider the problem of quickly detecting changes in an intermittent signal that can (repeatedly) switch between a normal and an anomalous state. We pose this intermittent signal detection problem as an optimal stopping problem and establish a quickest intermittent signal detection (ISD) rule with a threshold structure. We develop bounds to characterise the performance of our ISD rule and establish a new filter for estimating its detection delays. Finally, we examine the performance of our ISD rule in both a simulation study and an important vision based aircraft detection application where the ISD rule demonstrates improvements in detection range and false alarm rates relative to the current state of the art aircraft detection techniques.

Index Terms—Change Detection, Bayesian Quickest Change Detection, Sense and Avoid, Filtering

I. INTRODUCTION

QUICKLY detecting the presence of an anomaly condition that can repeatedly appear and disappear is important in many applications such as fault detection [2], cyber-security [3], intrusion or anomaly detection [5], and vision based aircraft detection [4]. In vision based aircraft detection, this anomaly condition represents the potential emergence of an aircraft anywhere in an image which needs to be quickly detected for collision avoidance purposes. We describe a signal containing this repeating anomaly condition as an intermittent signal. In this paper we aim to pose and solve this quickest intermittent signal detection (ISD) problem in a Bayesian setting that allows us to trade off average detection delay and false alarm probability.

In classic Bayesian quickest change detection, it is assumed that a permanent change in the statistics of a sequence of random variables occurs at some random unknown change time [5]. The classic Bayesian criterion seeks to minimise the average detection delay subject to a constraint placed on the probability of a false alarm. For this Bayesian formulation, Shiryaev established an optimal stopping rule which compares the posterior probability of a change to a threshold [5].

Inspired by classic Bayesian quickest change detection, several alternative quickest detection problems have been posed in the last decade. Incipient fault detection seeks to identify slow drifts in system parameters [6]; multi-cyclic detection seeks to identify a distant change in a stationary regime where detection procedures are reset after each false alarm [7]; quickest transient detection seeks to identify a change that occurs once for a period of time and then disappears [8, 9]; and quickest detection under transient dynamics that seeks to identify a persistent change which does not happen instantaneously, but after a series of transient phases [10]. In this paper, we consider a new quickest ISD problem where a change can repeatedly appear and disappear over time.

Our quickest ISD problem is inspired by the important vision based aircraft detection application in which a small pixel-sized aircraft can visually emerge anywhere in an image and can potentially transition in and out of view. Previous detection solutions have utilised ad hoc maximum likelihood approaches [4, 11, 12], and methods of non-Bayesian quickest change detection [13]. Here, we instead pose a quickest ISD problem and seek an optimal detection rule with the goal of quickly detecting when an aircraft emerges in an image sequence.

The key contributions of this paper are:

i) Posing the quickest ISD problem and utilising an optimal stopping framework to establish an ISD rule with a threshold structure;

ii) Introducing a new occupation time filter to estimate the detection delay of our ISD rule;

iii) Experimentally demonstrating the improvements offered by our ISD rule in the vision based aircraft detection application.

The rest of this paper is structured as follows. In Section II we pose our quickest ISD problem and associated cost criterion. In Section III we establish an optimal ISD rule. In Section IV we provide performance characteristics for our ISD rule. In Section V we examine the performance of our ISD rule in a simulation study. In Section VI we apply our ISD rule to vision based aircraft detection and examine its performance on an experimentally captured flight dataset.

II. PROBLEM FORMULATION

Let $X_k \in \{e_1, e_2\}$, for $k \geq 0$, be a sequence of random variables representing an intermittent signal that switches between a normal state $e_1$ and an anomalous state $e_2$ at (unknown) random time instances. Here $e_i \in \mathbb{R}^2$ are indicator vectors with 1 as the $i$th element and zeros elsewhere. For $k > 0$, the intermittent signal $X_k$ is hidden within measurements $y_k \in \mathbb{R}^M$, that are an independent and identically distributed (i.i.d.) sequence of random variables with (marginal) probability density functions $f_1(y_k)$ when $X_k = e_1$ and $f_2(y_k)$ when $X_k = e_2$.

In this paper, we shall assume that the intermittent signal $X_k$ is a first-order time-homogeneous Markov chain. Let $\rho$ be the...
Probability of transitioning from the normal state behavior $e_1$ to the anomalous state behavior $e_2$, and let $a$ be the probability of self transition for $e_2$. Let us denote the transition probabilities at each time instant by $A_{i,j} = P(X_{k+1} = e_i | X_k = e_j)$ for $i, j \in \{1, 2\}$ as

$$A = \begin{bmatrix} 1 - \rho & 1 - a \\ \rho & a \end{bmatrix}. \quad (1)$$

For $k \geq 0$, we can describe the intermittent signal (state process) $X_k$, as follows

$$X_{k+1} = AX_k + V_{k+1} \quad (2)$$

where $V_{k+1} \in \mathbb{R}^2$ is a martingale increment and the initial state $X_0$ has distribution $\hat{X}_0$. For the remainder of this paper we will define $X_{[0,k]} \triangleq \{X_0, \ldots, X_k\}$ and $y_{[1,k]} \triangleq \{y_1, \ldots, y_k\}$ as shorthand for sequences of these random variables.

We now introduce a probability measure space used to pose our quickest ISD problem. Similar to \cite{14} we consider the set $\Omega$ consisting of all infinite sequences $\omega \triangleq (X_{[0,k]}, \ldots, y_{[1,k]}, \ldots)$. Since $\Omega$ is separable and a complete metric space it can be endowed with a Borel $\sigma$-algebra $\mathcal{F} \triangleq \mathcal{B}\Omega$. Using Kolmogorov’s extension theorem we can now define a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We let $\mathbb{E}$ denote the expectation operation under the probability measure $\mathbb{P}$.

In this paper, our goal is to quickly detect when $X_k$ is in $e_2$ by seeking to design a stopping rule $\tau \geq 0$ that minimises the following ISD cost criterion

$$J(\tau) = \mathbb{E}\left[c \sum_{\ell=0}^{\tau-1} \langle X_\ell, e_2 \rangle + \langle X_\tau, e_1 \rangle \right], \quad (3)$$

where $\langle.,.\rangle$ denotes the inner product and $c$ is the penalty for the total amount of time spent in state $e_2$ before declaring an alert at $\tau$. This ISD cost criterion represents our desire to detect being in state $e_2$ as quickly as possible whilst avoiding false alarms (that is, avoid incorrectly declaring a stopping alert when the state is $e_1$).

III. INTERMITTENT SIGNAL DETECTION: OPTIMAL STOPPING TIME

In this section we first establish an equivalent representation of our ISD cost criterion in terms of the conditional mean estimates (CMEs) of the intermittent signal $X_k$. We then pose our quickest ISD problem as an optimal stopping problem and establish an optimal ISD rule that has a test statistic with a threshold structure. Finally we present the hidden Markov model (HMM) filter that can be used to efficiently calculate this test statistic.

A. Equivalent Representation of the ISD Cost Criterion

Lemma 1. The ISD cost criterion \cite{3} can be expressed in terms of the CMEs of the intermittent signal $X_k$ as

$$J(\tau) = \mathbb{E}\left[c \sum_{\ell=0}^{\tau-1} \hat{X}_\ell^2 + \hat{X}_\tau^1 \right] \quad (4)$$

where $\hat{X}_k^1 \triangleq \mathbb{E}[X_k^1 | y_{[1,k]}]$ denotes the CME of being in state $e_1$ given the measurements $y_{[1,k]}$.

Moreover for the case where $e_2$ is an absorbing state then the transition probability $a = 1$ and the cost criterion \cite{3} reduces to the classic Bayesian quickest change detection criterion \cite{13}

$$J^0(\tau) = c \mathbb{E}\left[(\tau - \nu)^+ \right] + \mathbb{P}(\tau < \nu), \quad (5)$$

where $\nu$ is the time of transition into the absorbing state $e_2$ and $(\tau - \nu)^+ \triangleq \max(0, \tau - \nu)$.

Proof. The ISD cost criterion \cite{3} can be expressed as

$$\mathbb{E}\left[c \sum_{\ell=0}^{\tau-1} \langle X_\ell, e_2 \rangle + \langle X_\tau, e_1 \rangle \right] = \mathbb{E}\left[c \sum_{\ell=0}^{\tau-1} \hat{X}_\ell^2 + \hat{X}_\tau^1 \right].$$

Following \cite{16} and using the tower rule for conditional expectations \cite{17} pg. 331 we obtain

$$\mathbb{E}\left[c \sum_{\ell=0}^{\tau-1} \hat{X}_\ell^2 + \hat{X}_\tau^1 \right] = \mathbb{E}\left[c \sum_{\ell=0}^{\tau-1} \mathbb{E}[X_\ell^2 | y_{[1,k]}] \right] = \mathbb{E}\left[c \sum_{\ell=0}^{\tau-1} \hat{X}_\ell^2 + \hat{X}_\tau^1 \right].$$

This proves the first lemma result.

For the second result, if $e_2$ is an absorbing state, then $a = 1$ and once $X_k = e_2$ it remains in $e_2$ and the following holds,

$$\mathbb{E}\left[c \sum_{\ell=0}^{\tau-1} \langle X_\ell, e_2 \rangle \right] = c \mathbb{E}\left[(\tau - \nu)^+ \right]$$

and

$$\mathbb{E}\left[\langle X_\tau, e_1 \rangle \right] = \mathbb{P}(\tau < \nu).$$

This completes the proof. \hfill $\square$

Lemma 1 shows that our quickest ISD problem is a generalisation or relaxation of the Bayesian quickest detection problem, in the sense that, when $e_2$ becomes an absorbing state the ISD cost criterion \cite{3} reduces to the classic Bayesian quickest detection problem \cite{16}. Further, this lemma allows our proposed ISD cost criterion \cite{3} to be expressed in terms of the CMEs which will now be used to establish an optimal ISD rule.

B. Optimal ISD Rule

In the following theorem we show that an optimal solution for the ISD cost criterion is a stopping rule with a threshold structure.

Theorem 1. For the ISD cost criterion \cite{3}, there is an optimal ISD rule with stopping time $\tau^*$, and threshold point $h_s \geq 0$ given by

$$\tau^* = \inf\{k \geq 0 : \hat{X}_k^2 \geq h_s\}. \quad (6)$$

Proof. In a slight abuse notation, we let $\mathbb{E}\left[\cdot | \hat{X} \right]$ denote the expectation operation corresponding to the probability measure...
where the initial state \( X_0 \) has distribution \( \tilde{X} \). We then define a cost criterion for different initial distributions as

\[
\tilde{J}(\tau, \tilde{X}) \triangleq E \left[ e^{\sum_{\ell=0}^{\tau-1} \tilde{X}_k^2 + \tilde{X}_0^2} \right].
\]  

(7)

Noting that \( J(\tau) = \tilde{J}(\tau, \tilde{X}_0) \), we can define a value function \( V(\tilde{X}_k) \triangleq \min_{\tau} \{ J(\tau, \tilde{X}_0) \} \) for our ISD cost criterion described by the recursion \([16, \text{ pg. 156 and 258}]\)

\[
V(\tilde{X}_k) = \min \left\{ c\tilde{X}_k^2 + E \left[ V \left( \tilde{X}^+(\tilde{X}_k, y) \right) \right] | \tilde{X}_k^1 \right\},
\]

(8)

where \( \tilde{X}^+(\tilde{X}, y) = (\mathbf{1}_B(y)A\tilde{X})^{-1}B(y)A\tilde{X} \), and \( B(y) = \text{diag}(f^1(y), f^2(y)) \).

Let \( S \triangleq \{ \tilde{X}_0^2 : V(\tilde{X}_k) = \tilde{X}_0^2 \} \) denote the optimal stopping set that we are seeking. Similar to the approach in \([16, \text{ sec. 12.2.2}]\), noting that the value function \( V(\tilde{X}_k) \) is concave, Theorem 12.2.1 of \([16]\) gives that the stopping set \( S \subset [0, 1] \) is convex. This implies that \( S \) is an interval of the form \([h_s, d]\) for some \( 0 \leq h_s \leq d \leq 1 \).

We now write our value function \([8]\) when \( \tilde{X}_k = e_2 \)

\[
V(e_2) = \min \left\{ c + E \left[ V \left( \tilde{X}^+(e_2, y) \right) \right] | e_2 \right\}, 0 \right\}.
\]

(9)

Since \( V \left( \tilde{X}^+(e_2, y) \right) \) is positive then \( V(e_2) = 0 \), which shows \( \tilde{X}_k^2 = 1 \) belongs to the stopping set, thus \( d = 1 \) and \( S \) is an interval of the form \([h_s, 1]\). We can express the optimal stopping time as the first time that the stopping set \( S \) is reached, in the sense that

\[
\tau^* = \inf \{ k \geq 0: \tilde{X}_k^2 \geq h_s \}.
\]

This completes the proof.

We note that when \( a = 1 \), perhaps unsurprisingly, this ISD rule reduces to Shiryaev’s Bayesian detection (SBD) rule \([3]\).

While it is possible to write down a dynamic programming equation for the optimal threshold for the stopping time \([6]\), in practice the threshold \( h_s \) is selected to trade off the alert delay and the probability of a false alarm.

C. HMM Filter

We now present the HMM filter for \([2]\) which can be used to efficiently calculate the CME \( \tilde{X}_k = E[X_k | y_{1:k}] \), where the 2nd element \( \tilde{X}_k^2 \) can be used to implement our ISD rule \([6]\).

At time \( k > 0 \), let \( B(y_k) = \text{diag}(f^1(y_k), f^2(y_k)) \) denote the diagonal matrix of output probability densities. We can now calculate the CME \( \tilde{X}_k \) at time \( k \), via the HMM filter \([17]\)

\[
\tilde{X}_k = N_k B(y_k) A \tilde{X}_{k-1},
\]

(9)

with initial condition \( \tilde{X}_0 \) and where \( N_k \) are scalar normalisation factors defined by

\[
N_k^{-1} = (\mathbf{1}_B(y_k) A \tilde{X}_{k-1}).
\]

(10)

IV. PERFORMANCE BOUND AND DELAY ESTIMATION

In this section we provide a bound for the probability of a false alarm (PFA) for our ISD rule. We then propose a new occupation time filter that can be used to estimate how long has been spent in the anomalous state \( e_2 \) before an alert is declared. Finally we establish some stability results for our proposed occupation time filter.

A. Bound on Probability of False Alarm

For a given threshold \( h_s \) used in our ISD rule \([4]\), we define the PFA as the probability that the system is in the normal state \( e_1 \) when an alert is declared, that is \( \text{PFA}(\tau) \triangleq P(X_\tau = e_1) \).

We can then bound the PFA as follows

\[
\text{PFA}(\tau) = P (X_\tau = e_1) = 1 - P (X_\tau = e_2)
\]

\[
= 1 - E \left[ \tilde{X}_\tau^2 \right]
\]

\[
\leq 1 - h_s.
\]

In the second line we have followed \([18]\) and used the tower rule for conditional expectations \([17, \text{ pg. 331}]\). In the third line we have used the fact that the CME \( \tilde{X}_\tau^2 = P(X_\tau = e_2 | y_{0:\tau}) \). Finally we use the definition of the stopping time \([6]\).

B. Occupation Time Filter

At time \( k > 0 \), for \( i \in \{1, 2\} \) we define the state occupation time \( \hat{O}_k^i \in \mathbb{R} \) as

\[
\hat{O}_k^i \triangleq \sum_{n=0}^{k-1} (X_n, e_i).
\]

(11)

We also define the CME of the occupation time \( \tilde{O}_k^i \triangleq E[\hat{O}_k^i | y_{1:k}] \), and the CME of the occupation time ending in state \( X_k \) as \( \tilde{O}_k^{i,X} \triangleq E[\hat{O}_k^i X_k | y_{1:k}] \in \mathbb{R}^2 \). Filters for these CMEs are established in the next lemma.

Lemma 2. For \( k > 0 \), an occupation time filter \( \hat{O}_k^i \) for all \( i \in \{1, 2\} \) is given by

\[
\hat{O}_k^{i,X} = N_k B(y_k) A (\hat{O}_{k-1}^{i,X} + \tilde{X}_{k-1} e_i)
\]

\[
\hat{O}_k^i = (\mathbf{1}_B(y_k) A \hat{O}_{k-1}^i),
\]

(12)

with initial conditions \( \hat{O}_0^{i,X} = 0 \) and \( \tilde{X}_0 \) is given by \([9]\).

Proof. See appendix for proof.

By setting \( i = 2 \), this lemma lets us estimate how long has been spent in the anomalous state \( e_2 \) when our ISD rule declares an alert. We highlight that similar occupation time filters are presented in \([17]\) for a delayed measurement model.

C. Occupation Time Filter Stability

We now present results characterising the stability of our proposed occupation time filter with respect to initial conditions.

We first introduce some required concepts before we present our proof. Let \( \hat{O}_k^i(X_0) \) and \( \tilde{X}_k(X_0) \) denote the occupation time CME filter and the HMM filter respectively with
initial condition $\hat{X}_0$. We now define an average error rate
$R_k^E(\hat{X}_0, \hat{X}_0)$ between correct and misspecified initial conditions $\hat{X}_0$ and $\hat{X}_0$, respectively, as

$$R_k^E(\hat{X}_0, \hat{X}_0) = \frac{|\hat{O}_{k}^{\hat{X}_{0}}(\hat{X}_0) - \hat{O}_{k}^{\hat{X}_{0}}(\hat{X}_0)|}{k}. \quad (13)$$

Finally, a function $\phi$ is said to be of class $\mathcal{K}$ if it is strictly increasing, continuous and $\phi(0) = 0$. A function $\beta$ is said to be of class $\mathcal{KL}$ if for each $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each $s > 0$, $\beta(s, \cdot)$ is decreasing to zero.

**Definition 1.** (Asymptotic stability with respect to initial conditions) The HMM filter $\hat{X}_k(\cdot)$ is said to be asymptotically stable with respect to initial conditions if there exists a $\beta(\cdot, \cdot) \in \mathcal{KL}$ such that for any $\hat{X}_0$ and $\hat{X}_0$,

$$|\hat{X}_k(\hat{X}_0) - \hat{X}_k(\hat{X}_0)| \leq \beta(|\hat{X}_k(\hat{X}_0) - \hat{X}_k(\hat{X}_0)|, k). \quad (14)$$

**Lemma 3.** Assume that the HMM filter $\hat{X}_k(\cdot)$ is asymptotically stable with respect to initial conditions. Then, the occupation time filter $[14]$ is (average error rate) practically stable in the sense that for any given $0 < \delta \leq 1$, there is a $\mathcal{H}$ such that for all $k > \mathcal{H}$, and for any $\hat{X}_0$ and $\hat{X}_0$, we have

$$R_k^E(\hat{X}_0, \hat{X}_0) \leq \delta + \frac{\mathcal{H}}{k}. \quad (15)$$

**Proof.** See appendix for proof. \hfill \Box

**Remark 1.** There are standard mild conditions under which the HMM filter $\hat{X}_k(\cdot)$ is asymptotically stable in the sense of Definition 1 see [19] for more information.

V. SIMULATION RESULTS

In this section we examine the performance of our ISD rule (6) and occupation time filter (12) in simulation.

A. Illustrative Example of the ISD and SBD Optimal Stopping Rules

We simulated a hand-crafted intermittent signal $X_k$ which switched between normal $e_1$ and anomalous $e_2$ states. The measurements $y_k$ are i.i.d. with marginal probability densities $f^1(y) = \psi(y - 1)$ when $X_k = e_1$, and $f^2(y) = \psi(y - 2)$ when $X_k = e_2$, where $\psi(\cdot)$ is zero-mean Gaussian probability density function with variance $\sigma^2$. The ISD rule (6) with $\rho = 0.01$ and $a = 0.99$, and the SBD rule (6) with $\rho = 0.01$ and $a = 1$ were both applied to the simulated observation data with $\sigma^2 = 5$.

From top to bottom, Figure 1 gives an illustrative example of the intermittent signal $X_k$, the measurements $y_k$, and a comparison of the ISD and the SBD test statistics against a threshold of $h_s = 0.7$. In this example the underlying intermittent signal switches into the anomalous state at $k = 600$. Our ISD rule (correctly) declares an alert at $k = 617$ with no false alarms. The SBD test statistic also exceeds the threshold at $k = 617$, however the SBD rule declares an alert at $k = 223$ corresponding to a false alarm.

B. Performance of Stopping rules in Monte Carlo Study

We simulated an intermittent signal $X_k$, the measurements $y_k$, and considered the ISD and SBD rules as described in the previous simulation study. We compared the performance of the ISD and SBD rules over a range of different thresholds $h_s$ to examine the trade off between the false alarms and the alert delay (AD). For a set threshold we applied both rules for 1000 Monte Carlo cases and determined the mean AD and mean number of false alarms. The maximum standard error of the delays is 0.024. Our ISD rule appears to outperform the SBD rule over a range of different thresholds.

Fig. 1. From top to bottom: the intermittent signal $X_k$, the measurements $y_k$ and a comparison of the ISD and the SBD optimal stopping rules for an arbitrarily selected threshold of $h_s = 0.7$. Our ISD rule alerts at $k = 617$, while the SBD rule alerts at $k = 223$ (corresponding to a false alarm).

Fig. 2. The mean alert delay versus the mean number of false alarms of the ISD and the SBD optimal stopping rules for range of different thresholds $h_s$. The maximum standard error of the delays is 0.024. Our ISD rule appears to outperform the SBD rule over a range of different thresholds.
in the image (that is, the aircraft can visually emerge anywhere as it approaches from a distance).

Between consecutive frames the aircraft can transition between different Markov states. The likelihood of state transitions depends on expected aircraft motion and are modelled by the HMM transition probabilities $A^{i,j}$ for $1 \leq i, j \leq N + 1$. Within the image, the possible aircraft inter-frame motion can be represented by a transition patch (see [21] for detailed explanation of patches). State transitions that would cross the image boundary will transition to the NVA state. An aircraft located in the NVA state is able to transition to any pixel in the image (that is, the aircraft can visually emerge anywhere as it approaches from a distance).

B. Aircraft Observations

At each time $k > 0$ we obtain a noise corrupted morphologically processed greyscale images of an aircraft $y_k$, as in [4], [11]. We denote the measurement of the $i$th pixel at time $k$ as $y_k^i$. Following [21] we let $p(y_k^i)$ denote the probability density of pixels occupied by an aircraft and $q(y_k^i)$ denote the probability density of pixels not occupied by an aircraft. That is, for $1 \leq i \leq N$

$$y_k^i \sim \begin{cases} p(y_k^i) & \text{for } Z_k = \mathcal{E}_i \\ q(y_k^i) & \text{for } Z_k \neq \mathcal{E}_i. \end{cases}$$

Recalling that we consider single pixel sized aircraft, we assume that these densities are statistically independent in the sense that $p(y_k^m, y_k^n) = p(y_k^m)p(y_k^n)$ and $q(y_k^m, y_k^n) = q(y_k^m)q(y_k^n)$ for $1 \leq m, n \leq N + 1, m \neq n$. Hence we have that the probability of receiving an image $y_k$ when the aircraft is in the $i$th pixel is

$$b^i(y_k) \triangleq p(y_k) \prod_{j=1}^{N} q(y_k^j) = \frac{p(y_k^i) \prod_{j=1, j\neq i}^{N} q(y_k^j)}{q(y_k^i) \prod_{j=1}^{N} q(y_k^j)}.$$ 

Noting that $\prod_{j=1}^{N} q(y_k^j)$ is a common factor for all $1 \leq i \leq N$, we can instead consider the unnormalised $\tilde{b}^i(y_k) = \frac{p(y_k^i) \prod_{j=1, j\neq i}^{N} q(y_k^j)}{q(y_k^i)}$. However, as $p(y_k^i)$ and $q(y_k^i)$ are not known a priori, we follow [4] and use the approximation $\tilde{b}^i(y_k) = y_k^i + 1$ for $1 \leq i \leq N$. When the aircraft is in the NVA state, i.e. for $i = N + 1$, then the whole image would consist of noise with no aircraft $b^{N+1}(y_k) = \prod_{j=1}^{N} q(y_k^j)$, giving the unnormalised
\[ \hat{b}^{N+1}(y_k) = 1. \] Our diagonal matrix of (unnormalised) output densities is then given by
\[
B^{ij}(y_k) = \begin{cases} \hat{b}(y_k) & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}
\]
for \(1 \leq i, j \leq N + 1.\)

C. Applying our ISD Optimal Stopping Rule

Recall that our goal is to quickly detect when an aircraft emerges in an image sequence, specifically, when the aircraft leaves the NVA state and appears in any of the pixels in the image frame. Hence, we seek to design a rule \(\tau \geq 0\) for stopping that minimises the following cost criterion
\[
J(\tau) = \mathbb{E}\left[ e^{\sum_{i=1}^{N} \sum_{\ell=0}^{\tau-1} \langle Z_{\ell}, \mathcal{E}_i \rangle + \langle Z_{\tau}, \mathcal{E}_{N+1} \rangle} \right], \quad (16)
\]
which represents our desire to detect when the aircraft appears at any pixel as quickly as possible whilst avoiding false alarms.

Consider two possible detection states: a no aircraft (normal) state \(e_1\) and an aircraft (anomalous) state \(e_2\). We can construct these states by equating \(\langle X_k, e_1 \rangle = \langle Z_k, \mathcal{E}_{N+1} \rangle\) and \(\langle X_k, e_2 \rangle = \sum_{i=1}^{N} \langle Z_k, \mathcal{E}_i \rangle\) through aggregating our first \(N\) image states (see [22] for information on state aggregation). The cost criterion (16) now reduces to our ISD cost criterion (4) allowing the use of our ISD rule (6)
\[
\tau^* = \inf\{k \geq 0 : \hat{\zeta}_k \geq h_c\},
\]
where \(\hat{\zeta}_k = 1 - \hat{Z}_k^{N+1}\) and \(h_c\) is a threshold chosen to trade off the alert delay and probability of false alarm. The CME \(\hat{Z}_k^{N+1}\) of the NVA state can be efficiently calculated via the HMM filter [17]
\[
\hat{Z}_k = N_k B(y_k) A \hat{Z}_{k-1},
\]
where
\[
N_k^{-1} = (A^2 B(y_k) A \hat{Z}_{k-1}).
\]

D. Performance Study

In this section we evaluate our proposed ISD rule in an application study on an experimentally captured flight dataset. We will compare the performance of our proposed rule to a baseline system developed in [4]. We will denote this baseline rule smoothed normalisation thresholding (SNT-4). We highlight that the baseline SNT-4 rule employs a filter bank (4 HMM filters) while our proposed ISD rule just uses a single filter (with \(A\) having the patch from [4] that allows motion in the up direction for transitions within the image and an equal probability of 0.1/N for transitions from the NVA state to each pixel in the image).

Detection performance will be evaluated on the 15 head-on near collision course encounters reported in [20] where we have maintained their numbering convention for comparison purposes.

1) Detection Range Study: Note that detection range and false alarm performance varies with the choice of the threshold parameters. Here, we will identify the lowest thresholds \(h_c\) for each algorithm that achieve zero false alarms (ZFAs) for this dataset. We will compare the two rule on the basis of their resulting ZFA detection ranges (the ability to achieve low false alarm rates is consistent with findings in [4], [11]). In practice, detection thresholds could be adaptively selected on the basis of scene difficulty such as proposed in [23].

The resulting ZFA detection ranges are presented in Figure 4. The mean detection distance and standard error was 2227m and 52m for the ISD rule and 2076m and 42m for the baseline SNT-4 rule.
TABLE I
THE MEAN ZFA DETECTION RANGES AND STANDARD ERRORS FOR ALL 15 CASES.

| Detection rule | Mean ZFA detection range (m) |
|----------------|-----------------------------|
| SNT-4          | 2216 ± 62                   |
| SNT-4I         | 2216 ± 62                   |
| NCD-4I         | 2216 ± 62                   |
| ISD            | 2227 ± 62                   |
| ISD-4I         | 2376 ± 82                   |

2) System Operating Characteristics Analysis: We next composed system operating characteristic (SOC) curves for our proposed ISD rule and baseline SNT-4 rule (SOC curves examine detection range and false alarm performance for different thresholds). Figure 5 presents the mean detection range for all 15 cases versus the mean false alarms per hour. The maximum standard error of the mean detection ranges is 72m for our proposed ISD rule and 108m for the baseline SNT-4 rule. Figure 5 illustrates longer detection ranges for our proposed system whilst maintaining lower false alarm rates across all tested thresholds.

E. Advanced Detection Rule Study

We compare the performance of our proposed ISD rule with 4 other detection rules. We modified the baseline SNT-4 rule to have individual thresholds for each of the filters in the bank (we denote this SNT-4I). We also considered a 4 filter bank version of the normalisation change detection (NCD) approach [13] with individual thresholds (we denote this NCD-4I). Finally, we implemented a 4 filter bank version of our proposed ISD rule with individual thresholds (we denote this ISD-4I).

Table I presents the mean detection ranges and standard errors for the five compared detection rules. We highlight that our proposed ISD and ISD-4I rules illustrate longer mean detection ranges than all other rules. Additionally our proposed ISD rules have the benefit of not requiring an estimate of aircraft and non-aircraft densities (this is required in the NCD-4I rule).

VII. CONCLUSION

In this paper we examined the problem of quickly detecting changes in an intermittent signal. We first posed the quickest ISD problem and established an optimal ISD rule. We then developed techniques and bounds for characterising the performance of our ISD rule. Finally, we investigated the performance of our ISD rule in both a simulation study and in the important vision based aircraft detection application. We were able to show that our ISD rule improves detection performance by at least 3.6%, at a significance level of 0.05, relative to the current state of the art vision based aircraft detection technique.

APPENDIX

Proof of Lemma 2

We first introduce some measure change concepts, see [17] for more details. Let us define a new probability measure \( \overline{\mathbb{P}} \) on \((\Omega, \mathcal{F})\) under which \( y_k \) becomes a sequence of i.i.d. random variables with probability density function \( \psi(\cdot) \). Let \( \mathbb{E}[\cdot] \) denote the expectation operation defined by \( \overline{\mathbb{P}} \). Let \( \mathcal{F}_k \) denote the complete filtration generated by \((X_{[0,k]}, y_{[1,k]})\). We can define a measure change between \( \mathbb{P} \) and \( \overline{\mathbb{P}} \) via the Radon-Nikodym derivative \( \frac{d\overline{\mathbb{P}}}{d\mathbb{P}} |_{\mathcal{F}_k} = \Lambda_k \) as follows (see [17, 24])

\[
\Lambda_k = \prod_{\ell=1}^{k} \lambda_{\ell}, \quad \lambda_{\ell} = \frac{f_1(y_{\ell})}{f(y_{\ell})} X_{\ell}.
\]

We can now introduce an unnormalised CME of the state, \( \hat{X}_k = \frac{X_k}{\mathbb{E}[\Lambda_k|y_{[1,k]}]} \) which is related to our normalised estimate via the conditional Bayes theorem [17, Thm 2.3.2] as follows

\[
\hat{X}_k = \mathbb{E}[\Lambda_k|y_{[1,k]}].
\]

Similarly, we can introduce an unnormalised CME of the occupation time ending in state \( X_k, X_kO_k \) as \( \hat{O}^i_{k,X} = \mathbb{E}[\Lambda_kX_kO_k|y_{[1,k]}] \) which is related to our normalised estimate via the conditional Bayes theorem as

\[
\hat{O}^i_{k,X} = \mathbb{E}[\Lambda_k|y_{[1,k]}].
\]

Before we present our main argument we note that martingale increment properties of \( V_k \) gives that

\[
\mathbb{E}[\Lambda_{k-1}V_k(X_{\ell-1}, e_i)|X_{[0,k-1]}, y_{[1,k]}] = 0 \quad \mathbb{P} \text{ a.s. for any } \ell \leq k \text{ and any } i.
\]

Then using this result within the tower rule for conditional expectations [17, pg. 331] shows that, for any \( \ell \leq k \) and any \( i \),

\[
\mathbb{E}[\Lambda_{k-1}V_k(X_{\ell-1}, e_i)|y_{[1,k]}] = 0 \quad \mathbb{P} \text{ a.s.}
\]

Simple algebra also gives

\[
\bar{\lambda}_kX_k = \frac{B(y_k)}{\psi(y_k)} X_k.
\]

For \( i \in \{1,2\} \), we can rewrite the unnormalised estimate \( \overline{\mathbb{P}} \text{ a.s. as} \)

\[
\hat{O}^i_{k,X} = \mathbb{E}[\Lambda_kX_kO_k|y_{[1,k]}]^{-1}
\]

where in the third line we have used \([20]\), in the fourth line we have used \([2]\), in the fifth line we have used the definition of \( O^i_{k} \) and \([19]\), in the sixth line we have used that
Finally, an inner product gives our occupation time CME
\[ \hat{O}_{k}^{i,X} = N_{1,k}(\hat{X}_{0})U_{1,k}\hat{O}_{k}^{i,X}. \]

This completes the proof. \(\square\)

**Proof of Lemma 3**

Let us define \( U_{1,k} \equiv B(y_{k})AB(y_{k-1})A, ..., B(y_{1})A \) and \( N_{1,k}(\hat{X}_{0}) \equiv \left(1'U_{1,k}\hat{X}_{0}\right)^{-1} \). Note that we can write the CME as \( \hat{X}_{k}(\hat{X}_{0}) = N_{1,k}(\hat{X}_{0})U_{1,k}\hat{X}_{0}. \)

For \( k > 0 \) we can write our occupation time filter \( \hat{O}_{k}^{i,X} \) in terms of an initial condition \( \hat{X}_{0} \), as follows
\[
\hat{O}_{k}^{i,X}(\hat{X}_{0}) = N_{1,k}(\hat{X}_{0})U_{1,k}\hat{O}_{k}^{i,X} + \sum_{j=1}^{k} N_{j,k}(\hat{X}_{j-1}(\hat{X}_{0}))U_{j,k}\hat{X}_{j}(\hat{X}_{0})e_{i}.
\]

Noting the initialisation \( \hat{O}_{k}^{i,X} = \emptyset \), for any \( i \), we have
\[
\left| \hat{O}_{k}^{i,X}(\hat{X}_{0}) - \hat{O}_{k}^{i,X}(\hat{X}_{0}) \right| \leq \sum_{j=1}^{k} C_{j,k}(\hat{X}_{0}, \hat{X}_{0})
\]
where \( C_{j,k}(\hat{X}_{0}, \hat{X}_{0}) \equiv \left| N_{j,k}(\hat{X}_{j-1}(\hat{X}_{0}))U_{j,k}\hat{X}_{j}(\hat{X}_{0})e_{i} - N_{j,k}(\hat{X}_{j-1}(\hat{X}_{0}))U_{j,k}\hat{X}_{j}(\hat{X}_{0})e_{i} \right|. \) Under our lemma assumption, for \( k > j \), we can now write
\[
C_{j,k}(\hat{X}_{0}, \hat{X}_{0}) \leq \beta \left( \left| \hat{X}_{j}(\hat{X}_{0})e_{i} - \hat{X}_{j}(\hat{X}_{0})e_{i} \right|, k-j \right).
\]

Given that \( \beta \left( \left| \hat{X}_{j}(\hat{X}_{0})e_{i} - \hat{X}_{j}(\hat{X}_{0})e_{i} \right|, k-j \right) < \beta(1, k-j) \), we note that for any given \( \delta \) there is a \( H \) such that for sufficiently large \( k \) we can write
\[
C_{j,k}(\hat{X}_{0}, \hat{X}_{0}) < \begin{cases} \delta & \text{for } k-j \geq H \\ 1 & \text{for } k-j < H. \end{cases}
\]

Finally, because there are \( H \) terms that are bounded by \( 1 \) and \( k-H \) terms bounded by \( \delta \), we can bound our error as
\[
\left| \hat{O}_{k}^{i,X}(\hat{X}_{0}) - \hat{O}_{k}^{i,X}(\hat{X}_{0}) \right| \leq \delta(k-H) + H.
\]

Dividing by \( k \) and using that \( \delta < 1 \) gives
\[
R_{k}^{E}(\hat{X}_{0}, \hat{X}_{0}) \leq \frac{\delta + H}{k},
\]
and this completes the proof. \(\square\)

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