Peeling Close to the Orientability Threshold – Spatial Coupling in Hashing-Based Data Structures

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Abstract

Hypergraphs with random hyperedges underlie various data structures where hash functions map inputs to hyperedges, e.g. cuckoo hash tables, invertible Bloom lookup tables, retrieval data structures and perfect hash functions.

High memory efficiency and quick query times call for high hyperedge density and small hyperedge size. Moreover, orientability or even peelability of the hypergraph is required or advantageous. For \( \ell \geq 1 \), we say a hypergraph is \( \ell \)-orientable if every subhypergraph has hyperedge density at most \( \ell \). It is \( \ell \)-peelable if every subhypergraph has minimum degree at most \( \ell \).

Many families of random hypergraphs exhibit sharp density thresholds, with respect to \( \ell \)-orientability and \( \ell \)-peelability. For \( \ell \)-uniform fully random hypergraphs, the thresholds \( c^*_k,\ell \) for \( \ell \)-orientability significantly exceed the thresholds for \( \ell \)-peelability. In this paper, for every \( k \geq 2 \) and \( \ell \geq 1 \) with \( (k, \ell) \neq (2, 1) \) and every \( z > 0 \), we construct a new family of random \( k \)-uniform hypergraphs with i.i.d. random hyperedges such that both the \( \ell \)-peelability and the \( \ell \)-orientability thresholds approach \( c^*_k,\ell \) as \( z \to \infty \). In particular we achieve \( 1 \)-peelability at densities arbitrarily close to 1.

Our construction is simple: The \( N \) vertices are linearly ordered and each hyperedge selects its \( k \) elements uniformly at random from a random range of \( Nz \) consecutive vertices.

We thus exploit the phenomenon of threshold saturation via spatial coupling discovered in the context of low density parity check codes. Once the connection to data structures is in plain sight, we employ a framework by Kudekar, Richardson and Urbanke [42] to do the heavy lifting in our proof.

We demonstrate the usefulness of our construction, using our hypergraphs as a drop-in replacement in a retrieval data structure by Botelho et al. [9]. This reduces memory usage from 1.23m bits to 1.12m bits (for input size \( m \)) with no downsides. Using \( k > 3 \) attains, at small sacrifices in running time, further improvements to memory usage.

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Related Version This work is a comprehensive overhaul of [19], presented at ESA 2019. Back then, we were unaware of the general phenomenon of threshold saturation via spatial coupling [42, 43, 44] (see also [35, 62, 61, 66]). The old construction was similar, but the analysis was more ad-hoc. The results were weaker, less general and less elegant.

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# Introduction

## 1.1 Three Hypergraph Properties

Consider the following properties of a hypergraph $H = (V, E)$ with transposed incidence matrix $A \in \{0, 1\}^{E \times V}$ (i.e. for $e \in E, v \in V : A[e, v] = 1 \iff v \in e$), over the field $\mathbb{F}_2 = \{0, 1\}$.

(i) For $\ell \in \mathbb{N}$, $H$ is $\ell$-peelable if every subhypergraph of $H$ has minimum degree at most $\ell$.

Equivalently, the *peeling process* that repeatedly deletes all vertices of degree at most $\ell$ (and incident hyperedges) reaches the empty hypergraph $\emptyset$. See \[12, 35, 55\].

(ii) $H$ is *solvable* if $A$ has rank $|E|$. Note that this necessitates $|V| \geq |E|$. See \[14, 22, 57\].

(iii) For $\ell \in \mathbb{N}$, $H$ is $\ell$-orientable if every subhypergraph $H' = (V', E')$ of $H$ satisfies $|E'|/|V'| \leq \ell$. By Hall’s Theorem, this is equivalent to the existence of a map $o : E \rightarrow V$ with $o(e) \in e$ for $e \in E$ and $|o^{-1}(v)| \leq \ell$ for $v \in V$. We call $o$ an $\ell$-orientation of $H$.

See \[10, 25, 27, 28, 33, 46\].

It is a simple and well-known observation that $\ell$-peelability implies $\ell$-orientability for all $\ell \in \mathbb{N}$. Moreover, 1-peelability implies solvability, which implies 1-orientability $\dagger$.

## 1.2 Our Results

In this paper, we analyse hypergraphs with random hyperedges that are drawn independently and with identical distribution (i.i.d.). For $k \geq 2$ we find a distribution on hyperedges that yields $k$-uniform random hypergraphs with an $\ell$-peelability threshold arbitrarily close to the $\ell$-orientability threshold of fully random $k$-uniform hypergraphs, for all $\ell \geq 1$ with $(k, \ell) \neq (2, 1)$. This is achieved by *spatial coupling*. The vertices are linearly ordered and each hyperedge selects its $k$ elements uniformly at random from a random range of consecutive vertices.

Concretely:

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\[1\] A subhypergraph of $H$ is a hypergraph $H' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E \cap 2^V$.

\[2\] To give a third formulation: The $(\ell + 1)$-core of $H$ is empty. The $i$-core of $H$ is defined as the largest subhypergraph of $H$ with minimum degree at least $i$.

\[3\] Most authors directly refer to $A$ having full rank. We introduce the hypergraph property of being “solvable” for a more unified presentation.

\[4\] To see this, consider the peeling process when $H$ is $\ell$-peelable: The deletion of any vertex $v$ causes the deletion of $\ell_v \leq \ell$ hyperedges $e_1, \ldots, e_{\ell_v}$ and we define $o(e_1) = \ldots = o(e_{\ell_v}) = v$. All hyperedges are deleted eventually, so we obtain an $\ell$-orientation $o$.

\[5\] For the first implication, note that if $H$ is 1-peelable then $A$ is in row-echelon form up to rearranging rows and columns. For the second implication, use that solvability guarantees an $|E| \times |E|$ submatrix of $A$ with determinant $1$. Let $V' \subseteq V$ be the inducing vertex set. By Leibniz’s formula for the determinant, there exists a bijection $o : E \rightarrow V'$ with $\prod_{e \in E} A[e, o(e)] \neq 0$. In particular, $o$ is injective and satisfies $o(e) \in e$ for $e \in E$. Thus it is a 1-orientation. This argument can also be found in \[15\].
Definition 2 (Threshold). Let $\mathcal{P}$ be a monotone hypergraph property (meaning if $H'$ is a subhypergraph of $H$ then $H \in \mathcal{P}$ implies $H' \in \mathcal{P}$). Moreover, let $(H_{c,n})_{c \in \mathbb{R}^+_0, n \in \mathbb{N}}$ be a family of random hypergraphs. The threshold of $(H_{c,n})_{c \in \mathbb{R}^+_0, n \in \mathbb{N}}$ for $\mathcal{P}$ is defined as

$$c^* := \sup\{c \in \mathbb{R}^+_0 \mid \Pr[H_{c,n} \in \mathcal{P}] \xrightarrow{n \to \infty} 1\}.$$

Assume a context where $k, \ell \in \mathbb{N}$, $c \in \mathbb{R}^+_0$ and $n \in \mathbb{N}$ are given. By the fully random hypergraph we mean a hypergraph $H(n,k,c)$ with vertex set $[n] = \{1, \ldots, n\}$ and $cn$ hyperedges drawn independently and uniformly at random from $\binom{[n]}{k}$. Let $c_{k,\ell}^*$ be the threshold of $(H(n,k,c))_{c \in \mathbb{R}^+_0, n \in \mathbb{N}}$ for $\ell$-orientability. Note that $c_{k,\ell}^*$ is known exactly \footnote{Our definition ensures that thresholds always exist. Presumably, all our thresholds are sharp thresholds, in the sense that $c^* = \inf\{c \in \mathbb{R} \mid \Pr[H_{c,n} \in \mathcal{P}] \xrightarrow{n \to \infty} 0\}$ also holds (cf. \cite{31}). We leave this aside since our results do not hinge on this.} and $c_{k,\ell}^*/\ell$ approaches 1 as $k + \ell$ increases. We can now state our main theorem.

Theorem 3. Let $k, \ell \in \mathbb{N}$, with $k \geq 2$ and $k + \ell \geq 4$. 

\begin{itemize}
  \item[(i)] $c < c_{k,\ell}^* \Rightarrow \forall z \in \mathbb{R}^+ : \Pr[F_{n}] is \ell$-peelable $\xrightarrow{n \to \infty} 1$.
  \item[(ii)] $c > c_{k,\ell}^* \Rightarrow \exists z^* \in \mathbb{R}^+ : \forall z \geq z^* : \Pr[F_n] is \ell$-orientable $\xrightarrow{n \to \infty} 0$.
\end{itemize}

Let us distil the main takeaways from these claims.

Corollary 4. Let $k, \ell \in \mathbb{N}$ with $k \geq 2$ and $k + \ell \geq 4$. For $z \in \mathbb{R}^+$ consider the family $(F_{c,n} = F(n,k,c,z))_{c \in \mathbb{R}^+_0, n \in \mathbb{N}}$. Let $f_{k,\ell,z}$ be its threshold for $\ell$-peelability and $f_{k,\ell,z}^*$ its threshold for $\ell$-orientability. Then we have:

\begin{itemize}
  \item[(i)] $\forall z \in \mathbb{R}^+ : f_{k,\ell,z} \geq c_{k,\ell}^*$.
  \item[(ii)] $\limsup_{z \to \infty} f_{k,\ell,z}^* \leq c_{k,\ell}^*$.
  \item[(iii)] Let $f_{k,\ell} = \lim_{z \to \infty} f_{k,\ell,z}$ and $f_{k,\ell}^* = \lim_{z \to \infty} f_{k,\ell,z}^*$. Then $f_{k,\ell} = f_{k,\ell}^* = c_{k,\ell}^*$.
  \item[(iv)] There exists a family $(F_{c,n})_{c \in \mathbb{R}^+_0, n \in \mathbb{N}}$ of random hypergraphs with threshold $c_{k,\ell}^*$ for $\ell$-peelability. The hypergraph $F_{c,n}$ has i.i.d. random hyperedges of size $k$ and hyperedge density $c$.
\end{itemize}

Here, (i) and (ii) are immediate consequences of the claims from Theorem 3. Since $f_{k,\ell,z} \leq f_{k,\ell,z}^*$ (since $\ell$-peelability implies $\ell$-orientability) we conclude (iii). Lastly, (iv) is obtained by defining a “diagonal family” where $z$ depends on $c$. Concretely, for $c < c_{k,\ell}^*$ and $n \in \mathbb{N}$, we can use $F_{c,n} := F(n,k,c,z(c))$ where $z = z(c)$ is large enough to fulfill $c + \frac{1}{2} < f_{k,\ell,z(c)}$. For $c \geq c_{k,\ell}^*$ use any (non-peatble) random hypergraph to complete the definition.

Discussion. Our construction is in the spirit of a well-known technique from coding theory. Namely, our hypergraphs arise from the fully random hypergraphs via spatial coupling (see e.g. \cite{42,44}) along a one-dimensional coupling dimension (the interval $X = [0, z + 1]$).

Note that results similar to ours can already be found in \cite{37} and \cite{35}, however, the goals of these papers are very different, concerning the Maxwell conjecture and the structure of the set of solutions to XORSAT formulae, respectively. Relative to these results, our paper offers: (1) A generalisation to $\ell > 1$. (2) A more elegant construction using the updated tools from \cite{42} (continuous coupling dimension). (3) A framing with data structures in mind and a demonstration of practical benefits for data structures.

\footnote{Using $F_{c,n} = F(n,k,c,z)$ for some constant $z \in \mathbb{R}$ is not allowed because $F_{c,n}$ has hyperedge density $c_{k,\ell}^*/\ell \neq c$.}

\footnote{In the coupling dimension is discrete. In our own terms, this means that the set of admissible positions of a hyperedge is $Y \cap \{\frac{k}{w} Z\}$ for some constant $w \in \mathbb{N}$. Our construction is attained with $w = n$.}
1.3 The Data Structure Perspective (HBDS)

Hypergraphs underlie many hashing based data structures (HBDS) that exploit the “power of multiple choices” paradigm \[54\]. Vertices correspond to buckets where data can be stored – usually array cells indexed by \([n] = \{1, \ldots, n\}\). We are given a set \(S\) of objects from some universe \(U\). Each \(x \in S\) is associated with several buckets by a constant number \(k \geq 2\) of hash functions \(h_1, \ldots, h_k : U \to [n]\). Thus, \(x\) gives rise to a hyperedge \(e(x) := \{h_1(x), \ldots, h_k(x)\} \subseteq [n]\) in the hypergraph \(H = ([n], \{e(x) \mid x \in S\})\). We give a few examples and explain how properties (i), (ii) and (iii) of \(H\) come into play.

Cuckoo Hash Table \[21, 50, 56\]. This implements a set or dictionary data structure with key set \(S\). Each \(x \in S\) (and, possibly, associated data) should be stored in exactly one bucket \(o(x)\) and each bucket can hold up to \(\ell\) objects. To allow for constant-time queries, we demand \(o(x) \in e(x)\). Clearly, this asks for an \(\ell\)-orientation of \(H\). If \(H\) is also \(\ell\)-peelable, then the \(\ell\)-orientation can be constructed greedily in linear time. Otherwise, linear time constructions are only known in some cases.\[9\]

Invertible Bloom Lookup Table (IBLT) \[36\]. Among other things, IBLTs have been used to construct error correcting codes \[53\] and solve the set reconciliation and straggler identification problem \[23\]. The data structure is inspired by Bloom Filters \[5\] and Bloomer Filters \[11\].

In IBLTs, each bucket \(v \in [n]\) stores \(\bigoplus_{x \in N(v)} x\), the bit-wise XOR of (the bit representations of) the objects \(N(v) := \{x \in S \mid v \in e(x)\}\) incident to \(v\), as well as the degree \(|N(v)|\). Note that this data structure is easy to maintain when insertions or deletions modify \(S\), even through phases with \(|S| > n\). Importantly, a ListEntries operation can be supported that recovers \(S\) if \(H\) is 1-peelable.

Retrieval \[15, 18, 20, 34, 59\]. Here, we are given a function \(f : S \to \{0, 1\}\) and want a data structure that reproduces \(f(x)\) for any query \(x \in S\). Note that naively storing \(f\) as a set of pairs requires \(|S| \cdot (1 + \log |U|)\) bits. If \(H\) is solvable, however, we can find a function \(b : V \to \{0, 1\}\) with \(\bigoplus_{e(x)} b(v) = f(x)\) for \(x \in S\), by solving the linear system \(A\vec{b} = \vec{f}\). To answer queries it then suffices to store \(b\) (a bit vector of length \(|V|\)) as well as \(h_1, \ldots, h_k\). Constructions with constant time queries and \(|V| = (1 + \varepsilon)|S|\) (and even \(|V| = (1 + o(1))|S|\)) exist. Note that membership queries “\(x \in S\)” are not supported. Similar as in cuckoo hashing, if \(H\) is 1-peelable, construction time is linear (see also Section \[5\]). Retrieval data structures are used to implement approximate membership queries similar to Bloom filters, see e.g. \[65\], and to build perfect hash functions as follows.

Perfect Hash Function via Retrieval \[3, 7, 8, 9, 34, 49\]. We wish for an injective function \(p : S \to [n]\) where \(n = (1 + \varepsilon)|S|\) and \(p\) is efficient to store and evaluate. Assume that \(k = 4\) and that \(H\) is 1-orientable via \(o : E \to V\). Since \(o(e(x)) \in e(x) = \{h_1(x), \ldots, h_4(x)\}\) there is \(f : S \to \{1, 2, 3, 4\}\) such that \(o(e(x)) = h_{f(x)}(x)\) for \(x \in S\). Thus we only need to store \(f\) with a (two-bit) retrieval data structure (see above) as well as \(h_1, \ldots, h_4\) to be able to evaluate \(p(x) := h_{f(x)}(x)\).

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\[9\] Consider fully random \(k\)-uniform hypergraphs and a hyperedge density below the orientability threshold \(c_{k,\ell}\). Then, for graphs (i.e. \(k = 2\)) and \(\ell \geq 2\), linear time algorithms to construct orientations are described by \[10, 23\]. For \(\ell = 1\) and \(k \geq 3\), consider \[59\]. It is empirically plausible that random walk insertion can maintain an orientation in a dynamic setting with expected constant time per update for any \(k\) and \(\ell\) – a partial answer is given in \[31\].
1.4 Coding Theory Perspective (LDPC codes)

The binary erasure channel (BEC) constitutes a simple but important setting. We recommend [59] Chapter 3 for an excellent introduction to this subject. When a sequence \( (x_1, \ldots, x_m) \in \{0, 1\}^m \) is sent over the BEC, the receiver sees a sequence \( (y_1, \ldots, y_m) \in \{0, 1, ?\}^m \) where for each \( i \in [m] \) independently, the \( i \)-th bit is erased \( (y_i = ?) \) with probability \( \varepsilon \in [0, 1] \) and unchanged \( (y_i = x_i) \) with probability \( 1 - \varepsilon \). For reliable communication over such channels, redundancy is introduced. In linear codes, several parity conditions are each specified by a set \( P \subseteq [m] \) and require \( \bigoplus_{i \in P} x_i \) to be zero. The set of admissible messages (codewords) then forms a linear subspace of \( \{0, 1\}^m \).

To relate this to hypergraphs, let \( V \) be the set of all parity conditions and let \( E^+ = \{e_1, \ldots, e_m\} \) where \( v \in e_i \) if \( x_i \) is involved in parity condition \( v \). The incidence graph of \( H^+ = (V, E^+) \) is known as the Tanner graph [63]. In low density parity check (LDPC) codes the Tanner graph is sparse.

After transmission, bits corresponding to some set \( E \subseteq E^+ \) are erased and we consider \( H = (V, E) \). When decoding, we seek an assignment \( x_{\text{dec}} : E \rightarrow \{0, 1\} \) such that for \( v \in V \) we have \( \bigoplus_{e_i \ni v} x_{\text{dec}}(e_i) = c_v \) where \( c_v \) is the parity of the successfully transferred bits involved in parity condition \( v \). The existence of a solution is guaranteed by construction, namely \( x_{\text{dec}}(e_i) = x_i \) for \( e_i \in E \). Uniqueness of the solution and thus success of the ideal maximum a posteriori probability decoder (MAP-decoder) requires the kernel of the incidence matrix of \( H \) to be trivial – which is equivalent to \( H \) being solvable.

Success of the linear time belief propagation decoder (BP-decoder) requires 1-peelability of \( H \). This decoder iteratively identifies a parity condition where all but one of the involved bits are known, and then decodes the unknown bit.

1.5 Aligning both Perspectives

We now explain how the goals in HBDS are sufficiently similar to those in LDPC decoding in order for the techniques from LDPC codes to be useful in HBDS.

Hyperedge Size. The (average) hyperedge size \( k \) is, in HBDS, related to (average) query time and (average) number of cache faults per query. In LDPC codes, \( k \) is the (average) number of parity conditions relating to each message bit and contributes to overall encoding and decoding time. Thus, small \( k \) is good.

Density. Let the normalised hyperedge density of \( H \) be \( \hat{c} = |E|/(\ell|V|) \) when discussing \( \ell \)-peelability and \( \ell \)-orientability, and \( \hat{c} = c = |E|/|V| \) when discussing solvability. In HBDS, high density means accommodating many objects in little space (high memory efficiency), while in LDPC codes it means recovering many erased bits from little redundancy (high rate). Thus, large \( \hat{c} \) is good. In both cases, \( \hat{c} = 1 \) is an obvious information theoretic upper bound.

Peelability, Solvability, Orientability. As far as we are aware, among our list of properties (i), (ii), (iii) important in HBDS, only 1-peelability and solvability play a role for LDPC codes. Luckily, in the cases we consider, the thresholds for solvability and 1-orientability coincide. Moreover, the generalisations to \( \ell > 1 \) are easily established.

In HBDS, hyperedges cannot be usefully related. An LDPC code is given by a fixed hypergraph \( H^+ \). We are free to design it, for instance we might give all vertices the same degree. Since \( H \) arises from a random \( \varepsilon \)-fraction of the hyperedges of \( H^+ \), this gives us control (proportional to \( \varepsilon \)) on \( H \) as well.

We argue heuristically that, when building a HBDS, we are more restricted. Recall that we do not control the data set \( S \subseteq U \) and that we need to evaluate \( e(x) \subseteq [m] \) for
x ∈ U. If e were specifically tailored to S, it is unclear how e(x) could be evaluated without the need for another data structure relating to S (in that case our data structure would somehow not be self-sufficient\footnote{For instance, e could be a perfect hash function, i.e. an injection e : S → [m]. In this case our argument would apply to the “more fundamental” data structure storing e.}). Now assume e to be independent of S and consider the case where |U| ≫ |S| and the elements of S are chosen independently at random from U (we may assume repetitions do not occur). Then the hyperedges of H are stochastically independent with a distribution implicit in e. So the use case with i.i.d. random hyperedges is unavoidable. We simplify our job by turning all use cases into this case by assuming \{e(s) | s ∈ U\} to be an independent family with a distribution we control. This assumption is vindicated in practice by good hash functions and auxiliary constructions (e.g. \cite{16}).

Given this restriction to i.i.d. random hyperedges, the random hypergraph families suitable for HBDS are a proper subset of those considered for LDPC codes. For instance, the degree of any vertex v ∈ V is necessarily random with distribution Bin(\(|S|, p_v\)) for some p_v ∈ [0, 1] (typically well approximated by Po(\(\lambda_v\)) for some \(\lambda_v = |S|/p_v\)). Of course, the general techniques from LDPC codes still apply in this special case.

1.6 Comparison with Known Results

We collect known trade-offs between threshold densities \(c^*\) for properties (i), (ii), (iii) and (average) hyperedge size \(k\), as achieved by hypergraph families with i.i.d. hyperedges, see Figure 1. A dot at \((k, c^*)\) ∈ \(\mathbb{R}^2\) indicates the existence of a family \((H_{c,n})_{c∈\mathbb{R}^+, n∈\mathbb{N}}\) of random hypergraphs where \(H_{c,n}\) has \(n\) vertices, \(\lfloor cn \rfloor\) random independent hyperedges and expected hyperedge size \(k\). The value \(c^*\) is the threshold.

The 1-orientability thresholds of fully random \(k\)-uniform hypergraphs (\(\bullet\)) \cite{29, 32} and the solvability thresholds \cite{22, 57} of the same family and are known to coincide \cite{14}. They are relevant for cuckoo hashing and retrieval. The 1-peelability thresholds (\(\bullet\)) \cite{55} on the other hand are decreasing in \(k\) and thus only \(k = 3\) is of interest.

Peelability thresholds of a non-uniform construction (\(\bullet\)) \cite{17}, famous in coding theory, approach \(c = 1\) for \(k → \infty\). Further trade-offs (\(\circ\)) were examined by \cite{60}.

The construction in this paper yields \(\ell\)-peelable families in the positions of the best known \(\ell\)-orientable families (\(\bullet\)) for general \(\ell\) (only \(\ell = 1\) is shown).
1.7 The Technique of Spatial Coupling

Our hypergraphs are spatially coupled along the “coupling dimension” $X = [0, z + 1)$. In the peeling process, vertices with a position close to the borders 0 or $z + 1$ tend to be deleted early on, while vertices in the denser, central parts remain stable. But gradually, deletions at the border “expose” vertices further on the inside and the whole hypergraph “erodes” from the outside in. This does not happen in the more symmetric construction when $X$ is glued into a circle (i.e. for all $\varepsilon \in [0, 1)$ the positions $\varepsilon$ and $z + \varepsilon$ are identified).

The authors of [41, 44] liken the phenomenon to water that is super-cooled to below 0°C in a smooth container. It will not freeze unless a nucleus for crystallization is introduced. Once this is done all water crystallizes quickly, starting from that nucleus. In our construction, the borders play the role of such a nucleus.

When introducing a linear geometry in the way we did, the 1-peelability threshold of the resulting (coupled) hypergraph family approaches the solvability thresholds of the underlying uncoupled construction, in a wide range of cases. In coding theory, this phenomenon is known as threshold saturation.

We leave a summary of the field to the experts [42, 44]. Put briefly, the phenomenon was discovered in the form of convolutional codes [24], then rigorously explained, first in a special case [43], then more generally [44], later accounting for continuous coupling dimensions (and even multiple dimensions) [42], a form we will exploit in this paper.

1.8 Outline

The paper is organised as follows. In Section 2 we idealise the peeling process by switching to a tree-like distributional limit of our hypergraphs, and capture the essential behaviour of the process in terms of an operator $\hat{\mathbf{P}}$ acting on functions $q : \mathbb{R} \to [0, 1]$. In Section 3 we analyse the effect of iterated application of $\hat{\mathbf{P}}$ to functions using the rich toolbox from [42]. This is the main ingredient to proving part (i) of Theorem 3 in Section 4. The comparatively simple part (ii) is independent of these considerations and is proved in Section 5. Finally, in Section 6 we demonstrate how using our hypergraphs can improve the performance of practical retrieval data structures.

2 The Peeling Process and Idealised Peeling Operators

The goal of this section, is to understand how the probabilities for vertices of $F_n$ to “survive” $r \in \mathbb{N}$ rounds of peeling change from one round to the next. In the classical setting this could be described by a function, mapping the old survival probability to the new one [55]. In our case, however, there are distinct survival probabilities $q(x)$ depending on the position $x$ of the vertex. Thus we need a corresponding operator $\hat{\mathbf{P}}$ that acts on such functions $q$.

We almost always suppress $k, \ell, c, z$ in notation outside of definitions. Big-$\mathcal{O}$ notation refers to $n \to \infty$ while $k, \ell, c, z$ are constant.

Consider the parallel peeling process peel$(F_n, \ell)$ on $F_n = F(n, k, c, z)$. In each round of peel$(F_n)$, all vertices of degree at most $\ell$ are determined and then deleted simultaneously. Deleting a vertex implicitly deletes all incident hyperedges. We also define the $r$-round rooted peeling process peel$_v, r(F_n, \ell)$ for any vertex $v \in V$ and $r \in \mathbb{N}$. In round 1 $\leq r' \leq r - 1$ of peel$_v, r(F_n)$, only vertices with distance $r - r'$ from $v$ are considered for deletion. Moreover, in round $r$, the root vertex $v$ is only deleted if it has degree at most $\ell - 1$, not if it has degree $\ell$.

For any vertex position $x \in X = [0, z + 1)$ and $r \in \mathbb{N}$ we let $q^{(r)}(x) = q^{(r)}(x, n, k, \ell, c, z)$ be the probability that the vertex $v = \lfloor xn \rfloor$ survives peel$_v, r(F_n)$, i.e. is not deleted. It
is convenient to define \( q^{(0)}(x) = 1 \) for all \( x \in X \), i.e. every vertex survives the “0-round peeling process”. Even though \( q^{(i)} \) is essentially discrete in \( x \), we will later see that it has a continuous limit for \( n \to \infty \).

Whether a vertex \( v \) at position \( x \) survives \( \text{peel}_{x,r} \) is a function of its \( r \)-neighbourhood \( F_n(x,r) \), i.e. the subhypergraph of \( F_n \) that can be reached from \( v \) by traversing at most \( r \) hyperedges.

It is natural to consider the distributional limit of \( F_n(x,r) \) to get a grip on \( q^{(r)}(x) \). In the spirit of the objective method \( \square \), we identify a (possibly infinite) random tree \( T_x \) that captures the local characteristics of \( F_n(x,r) \) for \( n \to \infty \). In the following, \( Po(\lambda) \) refers to the Poisson distribution with mean \( \lambda \in \mathbb{R}^+ \).

**Definition 5 (Limiting Tree).** Let \( k \in \mathbb{N}, c, z \in \mathbb{R}^+, X = [0,z+1], Y = \left[ \frac{1}{2},z + \frac{1}{2} \right] \) and \( x \in X \). The random (possibly infinite) hypertree \( T_x = T_x(k,c,z) \) is distributed as follows.

\( T_x \) has a root vertex \( \text{root}(T_x) \) at position \( x \), which for \( Y_x := [x - \frac{1}{2}, x + \frac{1}{2}] \cap Y \) has \( d_v \sim \text{Po}(ck|Y_x|) \) child hyperedges with positions uniformly distributed in \( Y_x \) \( \square \). Each child hyperedge at position \( y \) is incident to \( k - 1 \) (fresh) child vertices of its own, each with a uniformly random position \( x' \in [y - \frac{1}{2}, y + \frac{1}{2}] \). The sub-hypertree at such a child vertex at position \( x' \) is distributed recursively (and independently of its sibling-subtrees) according to \( T_{x'} \).

For \( x \in X \) and \( r \in \mathbb{N} \), let \( F_n(x,r) \) and \( T_x(r) \) denote the \( r \)-neighbourhoods of vertex \( v = \lfloor xn \rfloor \) in \( F_n \) and \( \text{root}(T_x) \) in \( T_x(r) \), respectively. In the following, \( H \) is an arbitrary fixed rooted hypergraph and equality of hypergraphs indicates a root-preserving isomorphism.

**Lemma 6.** \( \forall x \in X, r \in \mathbb{N}, H : \lim_{n \to \infty} \Pr[F_n(x,r) = H] = \Pr[T_x(r) = H] \).

**Sketch of Proof.** We only sketch some broad strokes required in a proof. Consider a vertex \( v \) at position \( x \) in \( F_n \). By construction, any hyperedge containing \( v \) must have a position \( y \in [x - \frac{1}{2}, x + \frac{1}{2}] \). For \( x \in [0,1) \) or \( x \in [z,z+1) \) the potential positions are further restricted by the upper and lower bounds on hyperedge positions, i.e. we have \( y \in Y_x := [x - \frac{1}{2}, x + \frac{1}{2}] \cap Y \).

In order for a random hyperedge \( e \) to contain \( v \), two things have to work out:

1. The position of \( e \) must fall within \( Y_x \). The probability for this is \( |Y_x|/|Y| = |Y_x|/z \).
2. At least one of the \( k \) incidences of \( e \) must turn out to be to \( v \). The probability for this is \( 1 - (1 - \frac{1}{n})^k \).

Since there are \( czn \) hyperedges in total, we obtain a binomial distribution \( \text{deg}(v) \sim \text{Bin}(czn, |Y_x|/z(1 - (1 - \frac{1}{n})^k)) \). This distribution converges, for \( n \to \infty \), to \( \text{Po}(ck|Y_x|) \), which is the distribution of \( \text{deg}(\text{root}(T_x)) \). To see the correspondence between the distributions of \( F_n(x,r) \) and \( T_x(r) \) for \( r > 1 \), we may reveal \( F_n(x,r) \) and \( T_x(r) \) vertex by vertex in breadth-first-search order and argue by induction. Conditioned on \( F_n(x,r) \) and \( T_x(r) \) matching up to a certain step, the distributions of what is revealed in the next step coincide up to terms of order \( o(1) \). There are three complications to deal with: (i) Vertex positions in \( F_n \) are restricted to integer multiples of \( \frac{1}{n} \). (ii) \( F_n(x,r) \) may contain cycles. (iii) There are slight dependencies between vertex degrees in \( F_n(x,r) \). It should be intuitively plausible that these problems vanish in the limit. We refer to \( \square \) for a full argument showing a similar convergence. See also \( \square \) for the related technique of Poissonisation. \( \square \)

\( \square \) In other words: The positions of the child hyperedges are a Poisson point field on \( Y_x \) with intensity \( ck \). By \( [I] \) for an interval \( I = [a, b] \) we mean \( b - a \).

Note also that the position is now a property of a vertex, not an identifying feature. Possibly (though with probability 0) the tree \( T_x \) may contain several vertices at the same position.
We now direct our attention to survival probabilities in the idealised peeling processes \( \text{peel}_{\text{root}(T_x),r}(T_x) \) for \( x \in X \), which are easier to analyse than those of \( \text{peel}_{v,r}(F_n) \).

\[ q^{(r+1)}_I(x) = Q \left( c_k \int_{[x-\frac{1}{2},x+\frac{1}{2}] \cap Y} \left( \int_{y-\frac{1}{2}}^{y+\frac{1}{2}} q_I^{(r)}(x') dx' \right)^{k-1} dy, \ell \right) \quad \text{for } x \in X. \]

where \( Q(\lambda, \ell) = 1 - \sum_{i=0}^{\ell - \lambda} \frac{\lambda^i}{i!} = \Pr[\text{Po} (\lambda) \geq \ell] \), the latter term slightly abusing notation.

**Proof.** Let \( x \in X \) and \( v = \text{root}(T_x) \). Assume \( y \in [x - \frac{1}{2}, x + \frac{1}{2}] \cap Y \) is the type of some hyperedge \( e \) incident to \( v \). Hyperedge \( e \) survives \( r \) rounds of \( \text{peel}_{v,r+1}(T_x) \) if and only if all of its incident vertices survive these \( r \) rounds. Since \( v \) itself may only be deleted in round \( r + 1 \), the relevant vertices are the \( k-1 \) child vertices \( w_1, \ldots, w_{k-1} \) with positions uniformly distributed in \([y - \frac{1}{2}, y + \frac{1}{2}] \). Let \( W_i \) be the subtree rooted at \( w_i \) for \( 1 \leq i \leq k \). Consider the peeling process \( \text{peel}_{v,r}(W_i) \). Assume the process deletes \( w_i \) in round \( r \), meaning \( w_i \) has degree at most \( \ell - 1 \) at the start of round \( r \). Then \( w_i \) has degree at most \( \ell \) at the start of round \( r \) in \( \text{peel}_{v,r+1}(T_x) \), meaning \( \text{peel}_{v,r+1}(T_x) \) deletes \( e \) in round \( r \). Conversely, if none of \( \text{peel}_{w_1,r}(W_1), \ldots, \text{peel}_{w_{k-1},r}(W_{k-1}) \) delete their root vertex within \( r \) rounds, then \( w_1, \ldots, w_{k-1} \) have degree at least \( \ell + 1 \) after round \( r \) of \( \text{peel}_{v,r+1}(T_x) \) and \( e \) survives round \( r \) of \( \text{peel}_{v,r+1}(T_x) \). Since the position of each \( w_i \) is independent and uniformly distributed in \([y - \frac{1}{2}, y + \frac{1}{2}] \), the probability for \( e \) to survive is \( p_y := \int_{y-\frac{1}{2}}^{y+\frac{1}{2}} q_I^{(r)}(x') dx' \). Since the positions of the hyperedges incident to \( v \) are Poisson point field on \([x - \frac{1}{2}, x + \frac{1}{2}] \cap Y \) with intensity \( c_k \), the number of incident hyperedges surviving round \( r \) of \( \text{peel}_{v,r+1}(T_x) \) has Poisson distribution with mean \( \lambda := \int_{[x-\frac{1}{2},x+\frac{1}{2}] \cap Y} c_k p_y dy \).

The claim now follows by observing that \( v \) survives \( r + 1 \) rounds of \( \text{peel}_{v,r+1}(T_x) \) if it is incident to at least \( \ell \) hyperedges surviving \( r \) rounds. The probability for this is \( Q(\lambda, \ell) \).

For convenience we define the operator \( P = P(k, \ell, c, z) \), which maps any (measurable) \( q : X \to [0,1] \) to \( Pq : X \to [0,1] \) with

\[ (Pq)(x) = Q \left( c_k \int_{[x-\frac{1}{2},x+\frac{1}{2}] \cap Y} \left( \int_{y-\frac{1}{2}}^{y+\frac{1}{2}} q(x') dx' \right)^{k-1} dy, \ell \right) \quad \text{for } x \in X. \]

Together Lemmas 6 and 7 imply that \( P \) can be used to approximate survival probabilities.

\[ P^r q^{(0)}(x) \overset{\text{def}}{=} P^r q^{(0)}_I(x) \overset{\text{lem}}{=} q^{(r)}(x) \overset{\text{lem}}{=} q^{(r)}(x) \pm o(1). \]

To obtain upper bounds on survival probabilities, we may remove the awkward restriction "\( \cap Y \)" in the definition of \( \hat{P} \). We define \( \hat{P} = \hat{P}(k,\ell,c) \) as mapping any \( q : \mathbb{R} \to [0,1] \) to \( \hat{P}q : \mathbb{R} \to [0,1] \) with

\[ (\hat{P}q)(x) = Q \left( c_k \int_{[x-\frac{1}{2},x+\frac{1}{2}] \cap Y} \left( \int_{y-\frac{1}{2}}^{y+\frac{1}{2}} q(x') dx' \right)^{k-1} dy, \ell \right) \quad \text{for } x \in \mathbb{R}. \]
Note that $\hat{P}$ does not depend on $z$ or $n$. To simplify notation, we assume that the old operator $P$ also acts on functions $q : \mathbb{R} \rightarrow [0, 1]$, ignoring $g(x)$ for $x \notin X$, and producing $Pq : \mathbb{R} \rightarrow [0, 1]$ with $Pq(x) = 0$ for $x \notin X$. We also extend $q^{(0)}$ to be $1_X : \mathbb{R} \rightarrow [0, 1]$, i.e., the characteristic function on $X$, essentially introducing vertices at positions $x \notin X$ which are, however, already deleted with probability 1 before the first round begins. Note that while $q^{(r)}(x)$ and $q^{(r)}_t(x)$ are by definition non-increasing in $r$, this is not the case for $(\hat{P}^rq^{(0)})(x)$. For instance, $\hat{P}^r q^{(0)}$ has support $(-r, z+1+r)$, which grows with $r$\footnote{It is still possible to interpret $\hat{P}^r q^{(0)}(x)$ as survival probabilities in more symmetric, extended versions $T_{x,c}$ of the tree $T_x$, but we will not pursue this.}. The following lemma lists a few easily verified properties of $\hat{P}$. All inequalities between functions should be interpreted point-wise.

\begin{lemma}
(i) $\forall q : \mathbb{R} \rightarrow [0, 1] : Pq \leq \hat{P}q$.
(ii) $P$ and $\hat{P}$ are monotonic, i.e. $\forall q, q' : \mathbb{R} \rightarrow [0, 1] : q \leq q' \Rightarrow Pq \leq Pq' \leq \hat{P}q$.
(iii) $P$ and $\hat{P}$ are continuous, i.e. pointwise convergence of $(q_i)_{i \in \mathbb{N}}$ to $q^*$ implies pointwise convergence of $(Pq_i)_{i \in \mathbb{N}}$ and $(\hat{P}q_i)_{i \in \mathbb{N}}$ to $Pq^*$ and $\hat{P}q^*$, respectively.
\end{lemma}

3 Analysis of Iterated Peeling

The goal of this section is to prove the following Lemma.

\begin{proposition}
(i) For $c < \epsilon_{k,t}^*$ and any $z \in \mathbb{R}^+$, we have $(P^r q_0)(x) \xrightarrow{r \rightarrow \infty} 0$ for all $x \in X$.
(ii) For $c > \epsilon_{k,t}^*$ and large $z$, we have $(P^r q_0)(x) \xrightarrow{r \rightarrow \infty} q^*(x)$ for all $x \in X$ and some $q^* \neq 0$.
\end{proposition}

The intuition is that for $c > \epsilon_{k,t}^*$ the peeling process gets stuck, while for $c < \epsilon_{k,t}^*$ all vertices are eventually peeled.

Conveniently, iterations such as the one given by $P$ and $\hat{P}$ were extensively studied in a stunning paper by Kudekar, Richardson and Urbanke \cite{22}. For some initial function $f^{(0)} : \mathbb{R} \rightarrow [0, 1]$ and non-decreasing functions $h_f, h_g : [0, 1] \rightarrow [0, 1]$ they study the sequence of functions

\begin{align}
g^{(r)}(y) &:= h_g((f^{(r)} \otimes \omega)(y)) \quad (1) \\
f^{(r+1)}(x) &:= h_f((g^{(r)} \otimes \omega)(x))
\end{align}

where $\omega$ is an averaging kernel, i.e. an even non-negative function with integral 1 and $\otimes$ is the convolution operator. To apply the theory to our case, we use:

\begin{align}
h_f(u) &:= Q(cku, \ell) \quad h_g(v) := v^{k-1} \quad \omega(x) = 1_{|x| \leq \frac{1}{2}} \quad (2)
\end{align}

With these substitutions the iteration \footnote{There is a corresponding flexibility in Definition 1. Instead of a hyperedge at position $y$ choosing its incident vertices uniformly at random from $[y - \frac{1}{2}, y + \frac{1}{2}]$, we can use an almost arbitrary bounded density function that is symmetric around $y$. For details consider \cite{22} Definition 2.} satisfies $\hat{P}f^{(r)} = f^{(r+1)}$. If we force the functions $g^{(r)}$, $r \in \mathbb{N}$, to be zero outside of $Y = [\frac{1}{2}, z + \frac{1}{2}]$ by replacing \footnote{It is still possible to interpret $\hat{P}^r q^{(0)}(x)$ as survival probabilities in more symmetric, extended versions $T_{x,c}$ of the tree $T_x$, but we will not pursue this.} $1_Y$ with $g^{(r)}(y) := \min\{1_Y(y), h_g((f^{(r)} \otimes \omega)(y))\}$ we get the system with two-sided termination. In this case $\hat{P}f^{(r)} = f^{(r+1)}$. The system with one-sided termination is defined similarly with $Y = [\frac{1}{2}, \infty)$.

We remark that nothing in the following depends on the choice of $\omega$.\footnote{It is still possible to interpret $\hat{P}^r q^{(0)}(x)$ as survival probabilities in more symmetric, extended versions $T_{x,c}$ of the tree $T_x$, but we will not pursue this.}
We plan to delegate the proof of Proposition 10 to theorems from [42]. For this, we need to examine the solutions to Equation (3). The potential \( \phi(u, v) \) can be visualised as the sum of three areas as shown. The significance of the threshold \( c^*_{k,\ell} \) is that the two areas enclosed by the two curves have exactly the same size, or put differently, \( \phi(u_2, v_2) = 0 \).

### 3.1 Unleashing Heavy Machinery from Coding Theory

We plan to delegate the proof of Proposition 10 to theorems from [42]. For this, we need to examine the solutions to Equation (3). The potential \( \phi(u, v) \) can be visualised as the sum of three areas as shown. The significance of the threshold \( c^*_{k,\ell} \) is that the two areas enclosed by the two curves have exactly the same size, or put differently, \( \phi(u_2, v_2) = 0 \).

**Lemma 11.**

(i) Every local minimum \((u, v)\) of \( \phi \) is a solution to Equation (3).

(ii) If Equation (3) has at least one non-trivial solution, then the smallest non-trivial solution \((u_1, v_1)\) has potential \( \phi(u_1, v_1) > 0 \).

(iii) Equation (3) has at most two non-trivial solutions.

(iv) For \( c = c^*_{k,\ell} \) there is a non-trivial solution \((u_2, v_2)\) of Equation (3) with \( \phi(u_2, v_2) = 0 \).

In this case \((0, 0)\) and \((u_2, v_2)\) are the only minima of \( \phi \).

(v) For \( c < c^*_{k,\ell} \) we have \( \phi(u, v) > 0 \) for \((u, v)\neq (0,0)\).

(vi) For \( c > c^*_{k,\ell} \) Equation (3) has two non-trivial solutions \((u_1, v_1) < (u_2, v_2)\). They satisfy \( \phi(u_2, v_2) < \phi(0,0) = 0 < \phi(u_1, v_1) \).

**Proof.**

(i) The partial derivatives of \( \phi \) are \( \nabla \phi(u, v) = (h_g^{-1}(u) - v, h_f^{-1}(v) - u) \). Therefore, the only candidates for local minima of \( \phi \) are the solutions to Equation (3) (it is easy to check that except for \((u, v) = (0,0)\) there are no local minima at the borders).

(ii) Assume \((u_1, v_1)\) is the smallest non-trivial solution to Equation (3). Considering Figure 2, we see that \( \phi(u_1, v_1) \) is the area enclosed by \( h_f(u) \) and \( h_g^{-1}(u) \) for \( u \in [0, u_1] \). To see that the sign of \( \phi(u_1, v_1) \) is positive, observe that for small values of \( u \) we have \( h_f(u) = Q(cku, \ell) = O(u^{k}) \) while \( h_g^{-1}(u) = \Omega(u^{1/(k-1)}) \) and thus \( h_f(\varepsilon) < h_g^{-1}(\varepsilon) \) for \( \varepsilon \in (0, u_1) \). This uses \( \ell \geq 1, k \geq 2 \) and \((k, \ell) \neq (2,1)\).
(iii) By expanding $h_f$ and $h_g$ and substituting $\xi = c k v^{k-1}$ we get for $(u, v) \neq (0, 0)$:

$$(u, v) = (h_g(v), h_f(u)) \Leftrightarrow v = Q(c k v^{k-1}, \ell) \Leftrightarrow \frac{\xi}{c k} = Q(\xi, \ell)^{k-1} \Leftrightarrow \frac{\xi}{Q(\xi, \ell)^{k-1}} = c k.$$

To show that the right-most equation has at most two solutions it suffices to show that $\xi / Q(\xi, \ell)^{k-1}$ has at most one local extremum. If $\xi$ is such an extremum, we get

$$\frac{d}{d \xi} \frac{\xi}{Q(\xi, \ell)^{k-1}} = 0 \Rightarrow Q(\xi, \ell)^{k-1} - \xi(k - 1)Q(\xi, \ell)^{k-2}Q'(\xi, \ell) = 0 \Rightarrow Q(\xi, \ell) - \xi(k - 1)Q'(\xi, \ell) = 0 \Rightarrow \sum_{i \geq \ell} \xi_i - (k - 1)\xi(\ell - 1)! = 0 \Rightarrow \sum_{i > 0} \xi_i (i + \ell)! = (k - 1)(\ell - 1)!$$

Since the left hand side is increasing in $\xi$ for $\xi > 0$ while the right hand side is constant, there is exactly one solution $\xi$ as claimed.

(iv) Recall that $c$ occurs in the definition of $h_f$ and note that $\phi$ is monotonically decreasing in $c$. It is easy to see that $\phi$ is nowhere negative for small values of $c$, and negative for some $(u, v)$ if $c$ is large. For continuity reasons and because $\phi(u, v) \geq 0$ for $u, v \in [0, c]$ with $\varepsilon = \varepsilon(c)$ small enough (using similar arguments as in (iii)), there must be some intermediate value $c$ where $\phi(u_2, v_2) = 0$ for a local minimum $(u_2, v_2) \neq (0, 0)$ of $\phi$. By (ii), $(u_2, v_2)$ is a solution of Equation (3). By (ii), there must be a smaller solution $(u_1, v_1)$ with $\phi(u_1, v_1) > 0$. Now by (i), and (iii), there cannot be minima of $\phi$ in addition to $(0, 0)$ and $(u_2, v_2)$. The only thing left to show is $c = c_{k, \ell}$.

We rewrite the potential at $(u_2, v_2)$, using Equation (3):

$$\phi(u_2, v_2) = \int_0^{u_2} h_g^{-1}(u)du + \int_0^{v_2} h_f^{-1}(v)dv - u_2v_2$$

$$= (u_2v_2 - \int_0^{v_2} h_g(v)dv) + (u_2v_2 - \int_0^{u_2} h_f(u)du) - u_2v_2$$

$$= v_2h_g(v_2) - H_g(v_2) - H_f(h_g(v_2)),$$

where $H_g$ and $H_f$ are antiderivatives of $h_g$ and $h_f$, i.e:

$$H_g(v) = \int h_g(v)dv = \frac{1}{k} v^k \quad H_f(u) = \int h_f(u)du = u - \frac{1}{c k} \sum_{i = 1}^{\ell} Q(c k v^{2k-1}, i).$$

The fact that $\int_0^1 Q(x, \ell)dx = \lambda - \sum_{i = 1}^{\ell} Q(\lambda, i)$ can be seen by induction on $\ell$. We now examine the implications of $\phi(u_2, v_2) = 0$. In the following calculation let $\xi := c k v^{2k-1}$ which implies $Q(\xi, \ell) = v_2$.

$$0 = \phi(u_2, v_2) = v_2h_g(v_2) - H_g(v_2) - H_f(h_g(v_2))$$

$$= v_2^k - v_2^k/k - v_2^{k-1} + \frac{1}{c k} \sum_{i = 1}^{\ell} Q(c k v^{2k-1}, i)$$

$$\Rightarrow 0 = \xi v_2 - \xi v_2/k - \xi + \sum_{i = 1}^{\ell} Q(\xi, i) = \xi Q(\xi, \ell) - \xi Q(\xi, \ell)/k - \xi + \sum_{i = 1}^{\ell} Q(\xi, i)$$

$$\Rightarrow \xi Q(\xi, \ell)/k = \xi(Q(\xi, \ell) - 1) + \sum_{i = 1}^{\ell} Q(\xi, i) = -e^{-\xi} \sum_{j = 0}^{\ell - 1} \frac{\xi^{j+1}}{j!} + \sum_{i = 1}^{\ell} Q(\xi, i)$$

$$= \ell - e^{-\xi} \sum_{j = 0}^{\ell - 1} \frac{\xi^{j+1}}{j!} + (\ell - j) \frac{\xi^j}{j!} = \ell - e^{-\xi} \left( \frac{\xi^{j+1}}{(\ell - 1)!} + \sum_{j = 0}^{\ell - 1} \frac{\xi^j}{j!} \right)$$
We are now ready to prove Proposition 10.

See previous footnote.

Strictly speaking, the theorem requires functions \( h_f \) and \( h_g \) with \( h_f(0) = h_g(0) = 0 \) and \( h_f(1) = h_g(1) = 1 \). As the authors of [42] point out themselves, this is purely to simplify notation. We can apply the theorem to our \( h_f : [0, u_2] \rightarrow [0, u_2] \) and \( h_g : [0, v_2] \rightarrow [0, v_2] \) with \( h_f(0) = h_g(0) = 0 \) and \( h_f(u_2) = v_2, h_g(v_2) = u_2 \) after rescaling the axes so \((u_2, v_2)\) becomes \((1, 1)\). We will not do so explicitly.

We now make the dependence of \( \phi_c(u, v) \) on \( c \) explicit. For monotonicity reasons we have \( \phi_c(u, v) > \phi_c(u', v) \) whenever \( c < c' \) and \( v \neq 0 \). Since \( \phi_{c^*_{c, \ell}} \) is positive except for its two roots at \((0, 0)\) and \((u_2, v_2)\), for \( c < c^*_{c, \ell} \) the potential \( \phi_c \) is positive except at \((0, 0)\).

Since \( \phi_{c^*_{c, \ell}} \) has a non-trivial root, \( \phi_c \) attains negative values for monotonicity reasons. By (vi), the potential attains its (negative) minimum at a non-trivial solution to Equation (3), and by (i) it attains a positive value at the smallest non-trivial solution. Thus, the claim follows.

We are now ready to prove Proposition 10.

Proof of Proposition 10 First note that we have \( q_0 \geq P q_0 \) by definition, which implies \( P^r q_0 \geq P^{r+1} q_0 \) by monotonicity of \( P \) and induction on \( r \). Thus, \( P^r q_0 \) is pointwise bounded and decreasing and must converge to a limit \( q^* \). As \( P \) is continuous (see Lemma 9) we have \( P q^* = q^* \).

(i) Let \( 1 : \mathbb{R} \rightarrow \{1\} \) be the 1-function. First note that for any \( x \in X \) we have, using properties from Lemma 9 and monotonicity of \( h_f \) and \( h_g \)

\[
(P^r q_0)(x) \leq (P^r 1)(x) = (h_f \circ h_g)^r(1) \xrightarrow{r \to \infty} \max\{u \in [0, 1] \mid h_f(h_g(u)) = u\}.
\]

So if the only solution of \( h_f(h_g(u)) = u \) is \( u = 0 \), then we get \( P^r q_0 \xrightarrow{r \to \infty} 0 \) from this alone. Otherwise, by Lemma 11(iii), there are one or two non-trivial solutions, the larger one we denote by \((u_2, v_2)\).

We now apply [42] Theorem 10 [16]. It requires \( \phi(u, v) > 0 \) for \( 0 \neq (u, v) \in [0, u_2] \times [0, v_2] \), which we have shown in Lemma 11(i). The theorem asserts pointwise convergence of \( f^{(r)} \) to zero for any \( f^{(0)} : \mathbb{R} \rightarrow [0, u_2] \) in the case of one-sided termination. Clearly this implies convergence to zero in the case of two-sided termination as well, i.e. \( P^r f^{(0)} \xrightarrow{r \to \infty} 0 \).

Choosing \( f^{(0)} = 1 \cdot u_2 \) we get

\[
\lim_{r \to \infty} (P^r q_0) = \lim_{r \to \infty} P^r \lim_{r \to \infty} P^r q_0 \leq \lim_{r \to \infty} P^r f^{(0)} \equiv 0.
\]

(ii) Using Lemma 11(vi) and (iii), we know there are exactly three solutions \((0, 0) < (u_1, v_1) < (u_2, v_2)\) to Equation (3) and the signs of their potentials are zero, positive and negative, respectively. This is sufficient to apply [42] Theorem 14 [16]. The theorem asserts the existence of a solution \( q^* : X \rightarrow [0, u_2] \) of \( P q^* = q^* \) with \( q^*(z(\varepsilon - \frac{1}{2})) = u_2 - \varepsilon \) for any \( \varepsilon > 0 \), assuming \( z = z(\varepsilon) \) is large enough.

By monotonicity of \( P \) we have \( \lim_{r \to \infty} P^r q_0 \geq \lim_{r \to \infty} P^r q^* = q^* \). □

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15 Strictly speaking, the theorem requires functions \( h_f \) and \( h_g \) with \( h_f(0) = h_g(0) = 0 \) and \( h_f(1) = h_g(1) = 1 \). As the authors of [42] point out themselves, this is purely to simplify notation. We can apply the theorem to our \( h_f : [0, u_2] \rightarrow [0, u_2] \) and \( h_g : [0, v_2] \rightarrow [0, v_2] \) with \( h_f(0) = h_g(0) = 0 \) and \( h_f(u_2) = v_2, h_g(v_2) = u_2 \) after rescaling the axes so \((u_2, v_2)\) becomes \((1, 1)\). We will not do so explicitly.

16 See previous footnote.
Peeling Close to the Orientability Threshold

## 4 Peelability of $F_n$ below $c^*_k,\ell$

We now connect the behaviour of system (1) to the survival probabilities $q^{(R)}(x)$ we were originally interested in. For $c < c^*_k,\ell$ and any $z \in \mathbb{N}$, they can be made smaller than any $\delta > 0$ in $R = R(\delta, k, \ell, z, c)$ rounds.

**Lemma 12.** If $c < c^*_k,\ell$ then $\forall z \in \mathbb{R}^+, \delta > 0$: $\exists R, N \in \mathbb{N}$: $\forall n \geq N, x \in X : q^{(R)}(x) < \delta$.

**Proof.** Let $z \in \mathbb{R}^+$ and $\delta > 0$ be arbitrary constants. At first, Proposition [16] implies only pointwise convergence $\mathbb{P}^r q^{(0)}(x) \xrightarrow{r \to \infty} 0$ for all $x \in X$. However, since $X$ is compact, $\mathbb{P}^r q^{(0)}$ is continuous for $r > 0$ and the all-zero limit is obviously continuous, basic calculus [17] implies uniform convergence, i.e. there is a constant $R$ such that $\mathbb{P}^R q^{(0)}(x) \leq \delta/2$ for all $x \in X$. Therefore for $x \in X$:

$$q^{(R)}(x) \xrightarrow{\mathbb{P}^R} (\mathbb{P}^R q^{(0)})(x) + o(1) \leq \delta/2 + o(1) \leq \delta.$$

In the last step we simply choose $N \in \mathbb{N}$ large enough.

Lemma [6] only allows us to track $q^{(R)}$ via $\mathbb{P}^R q_0$ for a constant number of rounds $R$. Therefore, we need to accompany Lemma [12] with the following combinatorial argument that shows that if all but a $\delta$-fraction of the vertices are peeled, then with high probability [18] (whp) the rest is peeled as well. Arguments such as these are standard, many similar ones can be found for instance in [27, 29, 38, 46, 48, 51, 55].

**Lemma 13.** Let $c \in [0, \ell]$. There exists $\delta = \delta(k, \ell, z) > 0$ such that, whp, any subhypergraph of $F_n = F(n, k, c, z)$ induced by at most $\delta |V(F_n)|$ vertices has minimum degree at most $\ell$.

**Proof.** In the course of the proof we will implicitly encounter positive upper bounds on $\delta$ in terms of $k$ and $z$. Any $\delta > 0$ small enough to respect these bounds is suitable. We consider the bad events $W_{s,t}$ that some small set $V' \subseteq V$ of size $s$ induces $t$ hyperedges for $k \leq s \leq \delta |V|, \frac{(t+1)s}{k} \leq t \leq |E|$. If none of these events occur, then all such $V'$ induce less than $(t+1)|V'|/k$ hyperedges and therefore induce hypergraphs with average degree less than $\ell + 1$, so a vertex of degree at most $\ell$ exists in each of them.

We will show $\Pr[\bigcup_{s,t} W_{s,t}] = O(1/n)$ using a first moment argument. First note that $F_n$ contains three copies of the same hyperedge with probability at most $\left(\frac{czn}{\delta}\right)(n^k)^{-2} = O(n^{-2k+3}) = O(n^{-1})$, so we restrict our attention to $F_n$ without triplicate hyperedges. Given $s$ and $t$ there are $t+1)^s$ ways to choose $V'$. Since there are $s^k$ ways to form $k$-tuples from vertices of $V'$ and each hyperedge occurs at most twice, there are at most $\left(\binom{s}{k}\right)^2$ multisets of hyperedges that $V'$ could induce. The probability that any given $k$-tuple actually does induce a hyperedge is either zero if the $k$ vertices are too far apart or $1 - (1 - n^{-k})^{czn} \leq \frac{czn}{n^k}$. Similarly, it induces a duplicate hyperedge with probability at most $\left(\frac{t+1}{n^k}\right)^2$. Since the presence of hyperedges is negatively correlated we may obtain an upper bound on the probability of the event that a set of hyperedges are all simultaneously present by taking the product of the events for the presence of the individual hyperedges. Thus, using constants

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17 Sometimes referred to as Dini’s Theorem after Ulisse Dini (1848 – 1918).
18 Meaning with probability approaching 1 as $n \to \infty$. 
$C, C', C'' \in \mathbb{R}^+$ (that may depend on $k, \ell$ and $z$) where precise values do not matter, we get

\[
\Pr\bigg[ \bigcup_{s=k}^{\delta |V|} \sum_{t=(\ell+1)s}^{\delta |E|} W_{s,t} \bigg] \leq \sum_{s=k}^{\delta |V|} \sum_{t=(\ell+1)s}^{\delta |E|} \frac{(z+1)^n}{s} \left( \frac{2e\ell z k}{tn^{k-1}} \right)^t \leq \sum_{s=k}^{\delta |V|} \sum_{t=(\ell+1)s}^{\delta |E|} \left( \frac{n}{s} \right)^s \left( \frac{C''}{s} \right)^{k-1} \left( \frac{k-1}{n} \right)^t \leq 2 \sum_{s=k}^{\delta |V|} \left( C'' \frac{n}{s} \right)^{(k-1)(\ell+1)-k} \frac{z^t}{n^{k-1}} \leq \sum_{s=k}^{\delta |V|} \left( C'' \frac{n}{s} \right)^{\ell-1} \frac{z^t}{n^{k-1}}.
\]

To get rid of the summation over $t$, we assumed $(s/n)^{k-1} \leq \delta^{k-1} \leq \frac{1}{\ell n}$, in the last step we used $k \geq 2$, $\ell \geq 1$ and $(k, \ell) \neq (2, 1)$. Elementary arguments show that in the resulting bound, the contribution of summands for $s \in \{k, \ldots, 2k\}$ is of order $O\left( \frac{1}{n} \right)$, the contribution of the summands with $s \in \{2k+1, \ldots, O(\log n)\}$ are of order $O\left( \frac{\log n}{n} \right)$ (using $\frac{s}{n} \leq \frac{\log n}{n}$) and the contribution of the remaining terms with $s \geq 3 \log_2 n$ is of order $O(2^{-\log_2 n}) = O\left( \frac{1}{n} \right)$ (using $C'' \frac{n}{s} \leq C''(z+1) \leq \frac{1}{2}$). This gives $\Pr[\bigcup_{s,t} W_{s,t}] = O(n^{-1})$, proving the claim.

We are now ready to prove the first half of Theorem 3.

**Proof of Theorem 3(i).** Let $c < c^*_{k,\ell}$ and $z \in \mathbb{R}^+$. We need to show that $F_n$ is $\ell$-peelable whp.

First, let $\delta = \delta(k, \ell, z)$ be the constant from Lemma 13 and $R = R(\delta/2)$ as well as $N$ the corresponding constants from Lemma 12.

Assuming $n \geq N$ we have $q(R)(x) \leq \delta/2$ for all $x \in \mathbb{R}$, meaning any vertex $v$ from $F_n$ is not deleted within $R$ rounds of peel$_{c,R}(F_n)$ with probability at most $\delta/2$. Since peel$_{c,R}(F_n)$ deletes in $R$ rounds at least the vertices that any peel$_{c,R}(F_n)$ for $v \in V$ deletes in $R$ rounds, the expected number of vertices not deleted by peel$_{c,R}(F_n)$ within $R$ rounds is at most $\delta|V|/2$.

Now standard arguments using Azuma’s inequality (see e.g. [32, Theorem 13.7]) suffice to conclude that whp at most $\delta|V|$ vertices are not deleted by peel$_{c,R}(F_n)$ within $R$ rounds.

By Lemma 13, whp, neither the remaining $\delta|V|$ vertices, nor any subset induces a hypergraph of minimum degree $\ell + 1$). Therefore peel$_{c,R}(F_n)$ deletes all vertices whp.

5 Non-Orientability of $F_n$ above $c^*_{k,\ell}$

To show that $F_n$ is not $\ell$-peelable whp for $c > c^*_{k,\ell}$ we argue that $F_n$ is even not $\ell$-orientable whp\footnote{Alternatively, one could try to base a proof on Proposition 10 (ii), possibly by going through similar motions as [35, Lemma 4]. If successful, this might give a detailed characterisation of the $(\ell+1)$-core of $F_n$ – the largest subhypergraph of $F_n$ with minimum degree $\ell + 1$. Presumably, the $(\ell+1)$-core contains roughly a $q'(x)$-fraction of the vertices with position roughly at $x \in X$. We leave this aside. Our approach has the upside of establishing a connection between orientability thresholds and peelability thresholds.}. Our proof relies on local weak convergence theory, a subject we danced around in Section 2. There are three ingredients.

**Ingredient 1: Identical weak limits.** For a finite graph $G$, let $G(v)$ be the random rooted graph obtained by designating a root at random. For a rooted (possibly infinite) graph $T$, let $T(r)$ be the $r$-neighbourhood of the root.
Definition 14 (Random Weak Limit [46]). Let \((G_n)_{n \in \mathbb{N}}\) be a sequence of (fixed) graphs and \(T\) a random (possibly infinite) rooted graph. We say that \((G_n)_{n \in \mathbb{N}}\) has random weak limit \(T\) if \(G_n(v)(r)\) converges in distribution to \(T(r)\) as \(n \to \infty\), for all \(r \in \mathbb{N}\).

For example, for \(c \in \mathbb{R}^+, k \in \mathbb{N}\) and \(n \in \mathbb{N}\), consider the fully random \(k\)-uniform hypergraph \(H_n\) with \(n\) vertices and \(cn\) independent and uniformly random hyperedges of size \(k\). Let \(G^H_n\) be the incidence graph of \(H_n\). In particular, \(G^H_n\) is bipartite with \(cn\) vertices of degree \(k\) that correspond to hyperedges in \(H_n\) and \(n\) vertices (of varying degrees) that correspond to vertices in \(H_n\). Moreover, consider the random (possibly infinite) tree \(T_{\text{vert}}\) generated as follows. The root vertex is on level zero. A vertex \(v\) at an even level is given a random number \(X_v \sim \text{Po}(c)\) of children on the next level. A vertex at an odd level is given \(k - 1\) children on the next level. Let further \(T_{\text{edge}}\) be the random tree with a root connected to the roots of \(k\) independently sampled copies of \(T_{\text{vert}}\). Lastly, let \(T\) be the random tree obtained by taking a copy of \(T_{\text{vert}}\) with probability \(\frac{1}{1+e}\) and a copy of \(T_{\text{edge}}\) with probability \(\frac{e}{1+e}\).

The following claim is standard.

Fact 15. Almost surely, the sequence \((G^H_n)_{n \in \mathbb{N}}\) has random weak limit \(T\).

Now, let also \(z \in \mathbb{R}^+\) and let \(F_n = F(n, k, c, z)\) be the random hypergraph from Definition 1. We define \(\bar{F}_n\) to be a “seamless” version of \(F_n\) where the vertices \(i\) and \(i+nz\) for all \(i \in [n]\) are merged, “glueing” the right-most \(n\) vertices of \(F_n\) on top of the left-most \(n\) vertices of \(F_n\). Moreover, let \(\bar{G}^F_n\) be the incidence graph of \(\bar{F}_n\).

Techniques from [45] suffice to prove that we get the same random weak limit.

Fact 16. Almost surely, the sequence \((\bar{G}^F_n)_{n \in \mathbb{N}}\) has random weak limit \(T\).

Ingredient 2: Lelarge’s Theorem [46]. To clarify its role in our proof, we restate a remarkable theorem due to Lelarge [46] in weaker form.

Theorem 17 (Lelarge [46] Theorem 4.1). Let \((G_n = (A_n, B_n, E_n))_{n \in \mathbb{N}}\) be a sequence of bipartite graphs with \(|E_n| = o(|A_n|)\). Let further \(M(G_n)\) be the maximum size of a set \(E' \subseteq E_n\) with \(\text{deg}_{G_n}(a) \leq 1\) for \(a \in A_n\) and \(\text{deg}_{G_n}(b) \leq \ell\) for \(b \in B_n\). If \((G_n)_{n \in \mathbb{N}}\) has random weak limit \(T^*\) and \(T^*\) satisfies certain natural properties, then \(\lim_{n \to \infty} \frac{M(G_n)}{|A_n|}\) exists and depends only on \(T^*\).

A graph \(G\) in the theorem should be interpreted as the incidence graph of a hypergraph \(H\) with vertex set \(B\) and hyperedge set \(A\). Then \(M(G)\) is the size of a largest set \(A' \subseteq A\) such that the subhypergraph \((B, A')\) of \(H\) is \(\ell\)-orientable. In other words, \(M(G)\) is the size of the largest partial \(\ell\)-orientation of \(H\).

\[\text{16 Peeling Close to the Orientability Threshold}\]

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20 The name random weak limit comes from [19]. The notion is also known as Benjamini-Schramm limit [4]. Aldous and Steele [1] call it the standard construction.

21 I cannot find a crystal clear reference for this, but [19] [46] consider it to be standard with reference to [39]. Since the language differs significantly, I consider [45] itself to be a better reference, since an arguably more complicated case is treated in detail.

22 It is easy to get confused here because we implicitly “cast” the sequence \((G^H_n)_{n \in \mathbb{N}}\) of random variables on graphs into a sequence of graphs. To reiterate: Having a certain random weak limit is a property of a sequence of graphs (not of distributions). The claim is that when sampling a sequence of graphs by independently sampling each element \(G^H_n\) of the sequence as explained above, then the resulting sequence of graphs will have the property almost surely, i.e. with probability 1.

23 Note that the limit \(T\) does not depend on \(z\).

24 The limit \(T^*\) must be a bipartite unimodular Galton-Watson tree, see [39] for an explanation. It is clear that \(T^* = T\) has the required properties.
We used our implementation we mean it is not the case that $H_n$ is $\ell$-orientable whp. More strongly however, it is known [27, 46] that there exists a constant $\delta = \delta(\varepsilon) > 0$ such that the largest partial $\ell$-orientation of $H_n$ has size $(1 - \delta)cn + o(n)$ whp. In the terms of Theorem 17 this means $\lim_{n \to \infty} M(G^H_n)/(cn) = 1 - \delta$ almost surely. We now put all three ingredients together.

**Proof of Theorem 3(ii).** Let $c = c^{\ast,k}_{k,\ell} + \varepsilon$ and $\delta = \delta(\varepsilon)$ as above. We pick $z \geq z^\ast := \frac{2\delta}{\varepsilon} + 1$.

Since $(G^H_n)_{n \in \mathbb{N}}$ and $(\bar{G}^F_n)_{n \in \mathbb{N}}$ almost surely share the random weak limit $T$ by Facts 15 and 16 we conclude from Theorem 17 that the orientability gap carries over from $H_n$ to $F_n$, i.e. $\lim_{n \to \infty} M(\bar{G}^F_n)/(cn) = 1 - \delta$ almost surely (recall that $F_n$ contains $cn$ hyperedges edges).

In particular, the size of the largest partial orientation of $\bar{F}_n$ is $(1 - \delta)cn + o(n)$ whp. Switching from $\bar{F}_n$ back to $F_n$ can increase the size of a largest partial orientation by at most $\ln n$ to $(1 - \delta + \frac{2}{cz})cn + o(n) \leq (1 - \frac{1}{2})cn + o(n)$ whp. Thus $F_n$ is not $\ell$-orientable whp. \hfill \qed

# 6 Experiments

We used our 1-peekable hypergraph families to implement retrieval data structures and compare these to existing implementations.

A 1-bit retrieval data structure for a universe $U$ is a pair of algorithms construct and query, where the input of construct is a set $S \subseteq U$ of size $m = |S|$ and $f : S \to \{0, 1\}$. If construct succeeds, then the output is a data structure $D_f$ such that query($D_f, x$) = $f(x)$ for all $x \in S$. The output of query($D_f, y$) for $y \in U \setminus S$ may yield an arbitrary element of $\{0, 1\}$. The interesting setting is when the data structure may only occupy $O(m)$ bits.

Recall the well-known approach [8, 9, 13, 18, 20, 34, 58] explained in Section 1.3. We map each element $x \in S$ to a set $c(x) \subseteq [N]$ via a hash function, where $N = m/c$ for some desired hyperedge density $c$. We then seek a solution $b : [N] \to \{0, 1\}$ satisfying $\bigoplus_{v \in c(x)} b(v) = f(x)$ for all $x \in S$. The bit vector $b$ and the hash function then form $D_f$. A query simply evaluates the left hand side of the equation for $x$ to recover $f(x)$. To compute $b$, we consider the hypergraph $H = ([N], \{c(x), x \in S\})$. A vertex $v \in [N]$ only contained in one hyperedge $c(x)$ corresponds to a variable $b(v)$ only occurring in the equation associated with $x$. It is thus easy to see that if $H$ is 1-peekable, repeated elimination and back-substitution yields $b$ in $O(m)$ time.

We implemented the following variations and report results in Table 1. By the overhead of an implementation we mean $\frac{N'}{m} - 1$ where $N' \geq N$ is the total number of bits used, including auxiliary data structures.

**Botelho et al.** [9] $H$ is a fully random 3-ary hypergraph with a hyperedge density below the 1-peekability threshold $c_{3,1}^\ast \approx 0.818$. Construction via peeling and queries are very fast, but the overhead of 23% is sizeable ($D_f$ occupies roughly 1.23$m$ bits).

**This work.** The hyperedges are distributed such that $H = F(N/z, k, c, z)$. Recall that the hyperedge density is $c = \frac{1}{z + 1}$. Note that $z$ should be large to keep the density close to $c$, but not too large, as our construction relies on $n \gg z$.

**Luby et al.** [47] The hyperedges are distributed such that $H$ is the 1-peekable hypergraph from [47] already mentioned in Section 1.6. To our knowledge, these hypergraphs have not been considered in the context of retrieval. They seem to be particularly well suited to achieve very small overheads at the cost of larger construction times and larger average query times compared to our other approaches. Note, however, that the largest hyperedge size $D + 4$ is exponential in the average hyperedge size. Therefore, the worst-case query time is much larger than the reported average query time.
| Paper                  | Configuration | Overhead | construct [µs/key] | query [ns] |
|------------------------|---------------|----------|-------------------|------------|
| Botelho et al. [9]     | c = 0.81      | 23.5%    | 0.31              | 35         |
| (this work)            | c = 0.910, k = 3, z = 50 | 12.1%    | 0.26              | 27         |
| (this work)            | c = 0.960, k = 4, z = 67 | 5.7%     | 0.26              | 32         |
| (this work)            | c = 0.985, k = 7, z = 72 | 2.9%     | 0.38              | 43         |
| Luby et al. [17]       | c = 0.9, D = 12 | 11.1%    | 0.76              | 69         |
| Luby et al. [17]       | c = 0.99, D = 150 | 1.1%     | 0.85              | 82         |
| Genuzio et al. [34]    | c = 0.91, k = 3, C = 10^4 | 10.2%    | 1.43              | 36         |
| Genuzio et al. [34]    | c = 0.97, k = 4, C = 10^4 | 3.4%     | 2.32              | 41         |
| Dietzfelbinger and W. [18] | c = 0.9995, block size = 16, C = 10^4 | 0.24%    | 2.60              | 37         |

Table 1: Overheads and average running times per key of various practical retrieval data structures.

Genuzio et al. [34], Dietzfelbinger and W. [18]. For reference, we also implemented two recent retrieval data structures that do not rely on peeling but solve linear systems. There, to counteract cubic solving time, the input set is partitioned into chunks of size $C$ by some hash function with range $[m/C]$. Especially [18] achieves much smaller overheads than what is feasible with peeling approaches, with the downside of having much larger construction times and being more complicated.

Experiments are performed on a desktop computer with an Intel® Core i7-2600 Processor @ 3.40GHz. In all cases, the data set $S$ contains $10^7$ random 64 bit integers. The function $f: S \to \{0, 1\}$ is the parity of the integer. As hash function we use the 2-independent multiply-shift scheme developed in [13] and crisply explained in [64] to produce 128 bit hashes. If more bits are needed, techniques resembling double-hashing are used to avoid further evaluations of the hash function. Query times are averages obtained by querying all elements of the data set once. The reported numbers are averages of 10 executions.

Overall, it seems using spatial coupling in retrieval data structures can outperform existing approaches when moderate memory overheads of $\approx 5\%$ are acceptable.

However, more research is required to explore the complex space of possible input sizes, configurations of the data structures and trade-offs between overhead and running time. Our implementations are configured reasonably, but arbitrary in some aspects. A full exploration is beyond the scope of this more theoretically oriented paper.

7 Conclusion

We have constructed families of $k$-uniform random hypergraphs with i.i.d. random hyperedges and an $\ell$-peelability threshold that is (asymptotically for $z \to \infty$) equal to the $\ell$-orientability threshold $c_{k,\ell}^*$ of fully random $k$-uniform hypergraphs.

We conjecture that this is best possible, i.e. no family of $k$-uniform random hypergraphs with i.i.d. random hyperedges has an $\ell$-peelability threshold exceeding $c_{k,\ell}^*$. In fact, even achieving $\ell$-orientability beyond $c_{k,\ell}^*$ seems unlikely.

25 When using the first $m = 10^7$ URLs from the eu-2015-host dataset gathered by [6] with $\approx 80$ bytes per key, we get similar results, except that all query times increase by roughly 25ns due to the cost for evaluating the hash function on these large keys. In this case we used MurmurHash3_x64_128 [2].

26 Note that the number and type of operation performed by construct and query is completely independent of $f$. Thus $f$ does not affect our measurements.
We demonstrated the usefulness of our construction for hashing based data structure using the example of retrieval data structures. Of course, the applicability of peelable hypergraphs is much wider than this, and whether using our construction yields significant improvements needs to be explored case by case and in detail. For instance, the stronger locality of the hyperedges might turn out to be advantageous in some settings while the higher number of rounds required by the (parallel) peeling process might be a problem in others.

We exploited the phenomenon of “threshold saturation via spatial coupling” that was discovered in coding theory and our proof borrows the powerful methods that were developed in the area. We are very pleased to so effortlessly obtain improvements in hashing based data structures and are curious to see whether this connection might be fruitful in other ways as well.

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