ON THE GEOMETRY OF GEOMETRIC RANK

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Abstract. We make a geometric study of the Geometric Rank of tensors recently introduced by Kopparty et al. Results include classification of tensors with degenerate geometric rank in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, classification of tensors with geometric rank two, and showing that upper bounds on geometric rank imply lower bounds on tensor rank.

1. Introduction and statement of main results

Tensors appear in numerous mathematical and scientific contexts. The two contexts most relevant for this paper are quantum information theory and algebraic complexity theory, especially the study of the complexity of matrix multiplication.

There are numerous notions of rank for tensors. One such, analytic rank, introduced in [15] and developed further in [22], is defined only over finite fields. In [17] they define a new kind of rank for tensors that is valid over arbitrary fields that is an asymptotic limit (as one enlarges the field) of analytic rank, that they call geometric rank (“geometric” in contrast to “analytic”), and establish basic properties of geometric rank. In this paper we begin a systematic study of geometric rank and what it reveals about the geometry of tensors.

Let $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ be a tensor and let $GR(T)$ denote the geometric rank of $T$ (see Proposition/Definition 2.4 below for the definition). For all tensors, one has $GR(T) \leq \min\{a, b, c\}$, and when $GR(T) < \min\{a, b, c\}$, we say $T$ has degenerate geometric rank. The case of geometric rank one was previously understood, see Remark 2.6.

Informally, a tensor is concise if it cannot be written as a tensor in a smaller ambient space (see Definition 2.2 below for the precise definition).

Our main results are:

- Classification of tensors with geometric rank two. In particular, in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ there are exactly two concise tensors of geometric rank two, and in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, $m > 3$, there is a unique concise tensor with geometric rank two (Theorem 3.1).

- Concise $1_{\ast}$-generic tensors (see Definition 2.7) in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ with geometric rank at most three have tensor rank at least $2m - 3$ and all other concise tensors of geometric rank at most three have tensor rank at least $m + \lfloor \frac{m-1}{2} \rfloor - 2$ (Theorem 3.3).

We also compute the geometric ranks of numerous tensors of interest in §5, and analyze the geometry associated to tensors with degenerate geometric rank in §4, where we also point out especially intriguing properties of tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of minimal border rank.

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2. Definitions and Notation

Throughout this paper we give our vector spaces names: $A = \mathbb{C}^a, B = \mathbb{C}^b, C = \mathbb{C}^c$ and we often will take $a = b = c = m$. Write $\text{End}(A)$ for the space of linear maps $A \to A$ and $GL(A)$ for the invertible linear maps. The dual space to $A$ is denoted $A^*$, its associated projective space is $\mathbb{P}A$, and for $a \in A\{0\}$, we let $[a] \in \mathbb{P}A$ be its projection to projective space. For a subspace $U \subset A$, $U^* \subset A^*$ is its annihilator. For a subset $X \subset A$, $\langle X \rangle \subset A$ denotes its linear span. We write $GL(A) \times GL(B) \times GL(C) \cdot T \subset A \otimes B \otimes C$ for the orbit of $T$, and similarly for the images of $T$ under endomorphisms. For a set $X$, $\overline{X}$ denotes its closure in the Zariski topology (which, for all examples in this paper, will also be its closure in the Euclidean topology).

Given $T \in A \otimes B \otimes C$, we let $T_A : A^* \to B \otimes C$ denote the corresponding linear map, and similarly for $T_B, T_C$. We omit the subscripts when there is no ambiguity. As examples, $T(A^*)$ means $T_A(A^*)$, and given $\beta \in B^*$, $T(\beta)$ means $T_B(\beta)$.

Fix bases $\{a_i\}, \{b_j\}, \{c_k\}$ of $A, B, C$, let $\{\alpha_i\}, \{\beta_j\}, \{\gamma_k\}$ be the corresponding dual bases of $A^*, B^*$ and $C^*$. The linear space $T(A^*) \subset B \otimes C$ is considered as a space of matrices, and is often presented as the image of a general point $\sum_{i=1}^a x_i \alpha_i \in A^*$, i.e. a $b \times c$ matrix of linear forms in variables $\{x_i\}$.

Let $T \in A \otimes B \otimes C$. $T$ has rank one if there exists nonzero $a \in A$, $b \in B$, $c \in C$ such that $T = a \otimes b \otimes c$.

For $r \leq \min\{a, b, c\}$, write $M_{(1)}^{\otimes r} = \sum_{\ell=1}^r a_\ell \otimes b_\ell \otimes c_\ell$.

We review various notions of rank for tensors:

**Definition 2.1.**

1. The smallest $r$ such that $T$ is a sum of $r$ rank one tensors is called the tensor rank (or rank) of $T$ and is denoted $\text{R}(T)$. This is the smallest $r$ such that, allowing $T$ to be in a larger space, $T \in \text{End}_r \times \text{End}_r \times \text{End}_r \cdot M_{(1)}^{\otimes r}$.

2. The smallest $r$ such that $T$ is a limit of rank $r$ tensors is called the border rank of $T$ and is denoted $\text{R}(T)$. This is the smallest $r$ such that, allowing $T$ to be in a larger space, $T \in GL_r \times GL_r \times GL_r \cdot M_{(1)}^{\otimes r}$.

3. $(\text{ml}_A, \text{ml}_B, \text{ml}_C) := (\text{rank} T_A, \text{rank} T_B, \text{rank} T_C)$ are the three multi-linear ranks of $T$.

4. The largest $r$ such that $M_{(1)}^{\otimes r} \in GL(A) \times GL(B) \times GL(C) \cdot T$ is called the border subrank of $T$ and denoted $\text{Q}(T)$.

5. The largest $r$ such that $M_{(1)}^{\otimes r} \in \text{End}(A) \times \text{End}(B) \times \text{End}(C) \cdot T$ is called the subrank of $T$ and denoted $\text{Q}(T)$. 
We have the inequalities
\[ Q(T) \leq Q_r(T) \leq \min\{m_A, m_B, m_C\} \leq \max\{m_A, m_B, m_C\} \leq R(T) \leq R(T), \]
and all inequalities may be strict. For example \( M_2 \) of Example 5.1 satisfies \( Q(M_2) = 3 \) \cite{17} and \( Q(M_2) = 2 \) \cite{6} Prop. 15] and all multilinear ranks are 4. Letting \( b \leq c \), \( T = a_1 \otimes (\sum_{j=1}^{b} b_j \otimes c_j) \) has \( m_A(T) = 1, m_B(T) = b \). A generic tensor in \( \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m \) satisfies \( m_A = m_B, = m \) and \( R(T) = O(m^2) \). The tensor \( T = a_1 b_1 c_2 + a_1 b_2 c_1 + a_2 b_1 c_1 \) satisfies \( R(T) = 2 \) and \( R(T) = 3 \). We remark that very recently Kopparty and Zuidam (personal communication) have shown that a generic tensor in \( \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m \) has subrank at most \( 3m^2 \).

In contrast, the corresponding notions for matrices all coincide.

**Definition 2.2.** A tensor \( T \in A \otimes B \otimes C \) is **concise** if \( m_A = a, m_B = b, m_C = c \).

The rank and border rank of a tensor \( T \in A \otimes B \otimes C \) measure the complexity of evaluating the corresponding bilinear map \( T : A^* \times B^* \to C \) or trilinear form \( T : A^* \times B^* \times C^* \to \mathbb{C} \). A concise tensor in \( \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m \) of rank \( m \) (resp. border rank \( m \)), is said to be of **minimal rank** (resp. minimal border rank). It is a longstanding problem to characterize tensors of minimal border rank, and how much larger the rank can be than the border rank. The largest rank of any explicitly known sequence of tensors is \( 3m - o(m) \) \cite{1}. While tests exist to bound the ranks of tensors, previous to this paper there was no general geometric criteria that would lower bound tensor rank (now see Theorem 5.3.4 below). The border rank is measured by a classical geometric object: secant varieties of Segre varieties. The border subrank, to our knowledge, has no similar classical object. In this paper we discuss how geometric rank is related to classically studied questions in algebraic geometry: linear spaces of matrices with large intersections with the variety of matrices of rank at most \( r \). See Equation (3) for a precise statement.

Another notion of rank for tensors is the **slice rank** \cite{26}, denoted by \( SR(T) \): it is the smallest \( r \) such that there exist \( r_1, r_2, r_3 \) such that \( r = r_1 + r_2 + r_3 \), \( A' \subset A \) of dimension \( r_1 \), \( B' \subset B \) of dimension \( r_2 \), and \( C' \subset C \) of dimension \( r_3 \), such that \( T \in A' \otimes B' \otimes C' \). It was originally introduced in the context of the cap set problem but has turned out (in its asymptotic version) to be important for quantum information theory and Strassen’s laser method, more precisely, Strassen’s theory of asymptotic spectra, see \cite{7}.

**Remark 2.3.** In \cite{12} a notion of rank for tensors inspired by invariant theory, called **G-stable rank** is introduced. Like geometric rank, it is bounded above by the slice rank and below by the border subrank. Its relation to geometric rank appears to be subtle: the \( G \)-stable rank of the matrix multiplication tensor \( M_{(n)} \) equals \( n^2 \), which is greater than the geometric rank (see Example 5.1), but the \( G \)-stable rank of \( W := a_1 b_1 c_2 + a_1 b_2 c_1 + a_2 b_1 c_1 \) is 1.5 (\( G \)-stable rank need not be integer valued), while \( GR(W) = 2 \).

Like multi-linear rank, geometric rank generalizes row rank and column rank of matrices, but unlike multi-linear rank, it salvages the fundamental theorem of linear algebra that row rank equals column rank. Let \( Seg(\mathbb{P}A^* \times \mathbb{P}B^*) \subset \mathbb{P}(A^* \otimes B^*) \) denote the **Segre variety** of rank one elements.

Let \( \Sigma^A_B = \{ [\alpha], [\beta] \} \in \mathbb{P}A^* \times \mathbb{P}B^* \mid T(\alpha, \beta, \cdot) = 0 \} \), so

\[ Seg(\Sigma^A_B) = \mathbb{P}(T(C^*)^\perp) \cap Seg(\mathbb{P}A^* \times \mathbb{P}B^*) \]
and let $\Sigma_j^A = \{[\alpha] \in \mathbb{P}A^* \mid \text{rank}(T(\alpha)) \leq \min\{b, c\} - j\}$. Let $\pi^{AB}_A : \mathbb{P}A^* \times \mathbb{P}B^* \to \mathbb{P}A^*$ denote the projection.

**Proposition/Definition 2.4.** [17] The following quantities are all equal and called the geometric rank of $T$, denoted $GR(T)$:

1. $\text{codim}(\Sigma^AB_1, \mathbb{P}A^* \times \mathbb{P}B^*)$
2. $\text{codim}(\Sigma^AC, \mathbb{P}A^* \times \mathbb{P}C^*)$
3. $\text{codim}(\Sigma^BC_1, \mathbb{P}B^* \times \mathbb{P}C^*)$
4. $a + \min\{b, c\} - 1 - \max_j(\dim \Sigma_j^A + j)$
5. $b + \min\{a, c\} - 1 - \max_j(\dim \Sigma_j^B + j)$
6. $c + \min\{a, b\} - 1 - \max_j(\dim \Sigma_j^C + j)$.

**Proof.** The classical row rank equals column rank theorem implies that when $\Sigma_j^A \neq \Sigma_{j+1}$, the fibers of $\pi^{AB}_A$ are $\mathbb{P}^{j-1}$ if $b \geq c$ and $\mathbb{P}^{j-1+b-c}$'s when $b < c$. The variety $\Sigma^AB_1$ is the union of the $(\pi^{AB}_A)^{-1}(\Sigma_j^A)$, which have dimension $\dim \Sigma_j^A + j - 1$ when $b \geq c$ and $\dim \Sigma_j^A + j - 1 + b - c$ when $b < c$. The dimension of a variety is the dimension of a largest dimensional irreducible component.

**Remark 2.5.** In [17] they work with $\hat{\Sigma}^{AB} := \{(\alpha, \beta) \in A^* \times B^* \mid T(\alpha, \beta, \cdot) = 0\}$ and define geometric rank to be $GR(T) := \text{codim}(\hat{\Sigma}^{AB}, A^* \times B^*)$. This is equivalent to our definition, which is clear except $0 \times B^*$ and $A^* \times 0$ are always contained in $\Sigma^{AB}$ which implies $GR(T) \leq \min\{a, b\}$ and by symmetry $GR(T) \leq \min\{a, b, c\}$, but there is no corresponding set in the projective variety $\Sigma^{AB}$. Since [11] implies

$$\dim \Sigma^{AB}_T \geq \dim \mathbb{P}(T(C^*)^4) + \dim \text{Seg}(\mathbb{P}A^* \times \mathbb{P}B^*) - \dim \mathbb{P}(A^* \otimes B^*)$$

$$= ab - c - 1 + a + b - 2 - (ab - 1)$$

$$= a + b - c - 2$$

we still have $GR(T) \leq a + b - c$ and by symmetry $GR(T) \leq \min\{a, b, c\}$ using our definition. We note that for tensors with more factors, one must be more careful when working projectively.

One has $Q(T) \leq GR(T) \leq SR(T)$ [17]. In particular, one may use geometric rank to bound the border subrank. An example of such a bound was an important application in [17].

**Remark 2.6.** The set of tensors with slice rank one is the set of tensors living in some $\mathbb{C}^1 \otimes B \otimes C$ (after possibly re-ordering and re-naming factors), and the same is true for tensors with geometric rank one. Therefore for any tensor $T$, $GR(T) = 1$ if and only if $SR(T) = 1$.

**Definition 2.7.** Let $a = b = c = m$. A tensor is $1_A$-generic if $T(A^*) \subset B \otimes C$ contains an element of full rank $m$, binding if it is at least two of $1_A, 1_B, 1_C$ generic, $1_{\ast}$-generic if it is $1_A, 1_B$ or $1_C$-generic, and it is $1$-generic if it is $1_A, 1_B$ and $1_C$-generic. A tensor is $1_A$-degenerate if it is not $1_A$-generic. Let $1_A - \text{degen}$ denote the variety of tensors that are not $1_A$-generic, and let $1 - \text{degen}$ the variety of tensors that are $1_A, 1_B$ and $1_C$ degenerate.
1$_A$-genericity is important in the study of tensors as Strassen’s equations [24] and more generally Koszul flattenings [21] fail to give good lower bounds for tensors that are 1-degenerate. Binding tensors are those that arise as structure tensors of algebras, see [4].

Defining equations for 1$_A$–degen are given by the module $S^m A^* \otimes \Lambda^m B^* \otimes \Lambda^m C^*$, see [18, Prop. 7.2.2.2].

**Definition 2.8.** A subspace $E \subset B \otimes C$ is of bounded rank $r$ if for all $X \in E$, $\text{rank}(X) \leq r$.

3. **Statements of main results**

Let $\mathcal{GR}_s(A \otimes B \otimes C) \subset \mathbb{P}(A \otimes B \otimes C)$ denote the set of tensors of geometric rank at most $s$ which is Zariski closed [17], and write $\mathcal{GR}_{s,m} := \mathcal{GR}_s(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m)$. By Remark 2.6, $\mathcal{GR}_1(A \otimes B \otimes C)$ is the variety of tensors that live in some $C^1 \otimes B \otimes C$, $A \otimes C^1 \otimes C$, or $A \otimes B \otimes C^1$.

In what follows, a statement of the form “there exists a unique tensor...”, or “there are exactly two tensors...”, means up to the action of $GL(A) \times GL(B) \times GL(C) \times \Theta_3$.

**Theorem 3.1.** For $a, b, c \geq 3$, the variety $\mathcal{GR}_2(A \otimes B \otimes C)$ is the variety of tensors $T$ such that $T(A^*), T(B^*), \text{or } T(C^*)$ has bounded rank 2.

There are exactly two concise tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ with $GR(T) = 2$:

1. The unique up to scale skew-symmetric tensor $T = \sum_{\sigma \in S_3} \text{sgn}(\sigma) a_{\sigma(1)} \otimes b_{\sigma(2)} \otimes c_{\sigma(3)} \in \Lambda^3 \mathbb{C}^3 \subset \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

2. $T_{\text{utriv,3}} := a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_2 + a_1 \otimes b_3 \otimes c_3 + a_2 \otimes b_1 \otimes c_2 + a_3 \otimes b_1 \otimes c_3 \in S^2 \mathbb{C}^3 \otimes \mathbb{C}^3 \subset \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

There is a unique concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ satisfying $GR(T) = 2$ when $m > 3$, namely

$$T_{\text{utriv,m}} := a_1 \otimes b_1 \otimes c_1 + \sum_{\rho = 2}^m [a_1 \otimes b_{\rho} \otimes c_{\rho} + a_{\rho} \otimes b_1 \otimes c_{\rho}].$$

This tensor satisfies $R(T_{\text{utriv,m}}) = m$ and $R(T_{\text{utriv,m}}) = 2m - 1$.

In the $m = 3$ case (1) of Theorem 3.1 we have $\Sigma^A_T \cong \Sigma^C_T \cong \Sigma^B_T \cong \mathbb{P} A^* \subset \mathbb{P} A^* \times \mathbb{P} A^*$ embedded diagonally and $\Sigma^A_T = \Sigma^B_T = \Sigma^C_T = \mathbb{P} A^*$.

In the $m = 3$ case (2) of Theorem 3.1 we have

$$\Sigma^A_T = \mathbb{P}(\alpha_2, \alpha_3) \times \mathbb{P}(\beta_2, \beta_3) = \mathbb{P}^1 \times \mathbb{P}^1,$$

$$\Sigma^C_T = \{([s\alpha_2 + t\alpha_3], [u\gamma_1 + v(-t\gamma_2 + s\gamma_3)]) \in \mathbb{P} A \times \mathbb{P} C | [s, t] \in \mathbb{P}^1, [u, v] \in \mathbb{P}^1\},$$

$$\Sigma^B_T = \{([s\beta_2 + t\beta_3], [u\gamma_1 + v(-t\gamma_2 + s\gamma_3)]) \in \mathbb{P} B \times \mathbb{P} C | [s, t] \in \mathbb{P}^1, [u, v] \in \mathbb{P}^1\}.$$

If one looks at the scheme structure, $\Sigma^A_T, \Sigma^B_T$ are lines with multiplicity three and $\Sigma^C_T = \mathbb{P} C^*$.

**Remark 3.2.** The tensor $T_{\text{utriv,m}}$ has appeared several times in the literature: it is the structure tensor of the trivial algebra with unit (hence the name), and it has the largest symmetry group of any binding tensor [8, Prop. 3.2]. It is also closely related to Strassen’s tensor of [23]: it is the sum of Strassen’s tensor with a rank one tensor.

Theorem 3.1 is proved in §6.4.
Theorem 3.3. Let $T \in A \otimes B \otimes C$ be concise and assume $c \geq b \geq a > 4$. If $GR(T) \leq 3$, then $R(T) \geq b + [\frac{a-1}{2}] - 2$.

If moreover $a = b = c = m$ and $T$ is $1_\ast$-generic, then $R(T) \geq 2m - 3$.

In contrast to $GR_{1,m}$ and $GR_{2,m}$, the variety $GR_{3,m}$ is not just the the set of tensors $T$ such that $T(A^\ast), T(B^\ast)$ or $T(C^\ast)$ has bounded rank 3. Other examples include the structure tensor for $2 \times 2$ matrix multiplication $M_{(2)}$ (see Example 5.1), the large and small Coppersmith-Winograd tensors (see Examples 5.5 and 5.6) and others (see §5.4).

Theorem 3.3 gives the first algebraic way to lower bound tensor rank. Previously, the only technique to bound tensor rank beyond border rank was the substitution method (see §6.2), which is not algebraic or systematically implementable.

Theorem 3.3 is proved in §6.5

4. Remarks on the geometry of geometric rank

4.1. Varieties arising in the study of geometric rank. Let $G(m, V)$ denote the Grassmannian of $m$-planes through the origin in the vector space $V$. Recall the correspondence (see, e.g., [20]):

\[
\{ A\text{-concise tensors } T \in A \otimes B \otimes C \}/\{ \text{GL}(A) \times \text{GL}(B) \times \text{GL}(C) \text{ - equivalence} \}
\]

\[\leftrightarrow \]

\[
\{ a \text{ - planes } E \in G(a, B \otimes C) \}/\{ \text{GL}(B) \times \text{GL}(C) \text{ - equivalence} \}
\]

It makes sense to study the $\Sigma_j^A$ separately, as they have different geometry. To this end define

$GR_{A,j}(T) = a + \min\{b, c\} - 1 - \dim\Sigma_j^A - j$. Let $GR_{r,A,j}(A \otimes B \otimes C) = \{ [T] \in P(A \otimes B \otimes C) \mid GR_{A,j}(T) \leq r \}$.

Let $\sigma_r(\text{Seg}(PB \times PC)) \subset P(B \otimes C)$ denote the variety of $b \times c$ matrices of rank at most $r$.

By the correspondence (2), the study of $GR_{r,A,j}(A \otimes B \otimes C)$ is the study of the variety

\[
\{ E \in G(a, B^\ast \otimes C^\ast) \mid \dim(\mathbb{P}E \cap \sigma_{\min(b, c)}(\mathbb{P}B^\ast \times \mathbb{P}C^\ast)) \geq a + \min\{b, c\} - j - 1 - r \}
\]

The following is immediate from the definitions, but since it is significant we record it:

Observation 4.1. $GR_{a-1,A,j}(A \otimes B \otimes C) = 1_A$ - degen. In particular, tensors that are $1_A$, $1_B$, or $1_C$ degenerate have degenerate geometric rank.

$GR_{a-1,A,a}(A \otimes B \otimes C)$ is the set of tensors that fail to be $A$ concise. In particular, non-concise tensors do not have maximal geometric rank.

It is classical that $\dim\sigma_{m-j}(\text{Seg}((\mathbb{P}^{m-1}) \times (\mathbb{P}^{m-1}))) = m^2 - j^2 - 1$. Thus for a general tensor in $C^m \otimes C^m \otimes C^m$, $\dim(S_j^A) = m - j^2$. In particular, it is empty when $j > \sqrt{m}$.

Observation 4.2. If $T \in C^m \otimes C^m \otimes C^m$ is concise and $GR(T) < m$, then $R(T) > m$.

Proof. If $T$ is concise $R(T) \geq m$, and for equality to hold it can be written as $\sum_{j=1}^m a_j \otimes b_j \otimes c_j$ for some bases $\{a_j\}, \{b_j\}$ and $\{c_j\}$ of $A, B$ and $C$ respectively. But $GR(\sum_{j=1}^m a_j \otimes b_j \otimes c_j) = m$. □

Question 4.3. For concise $1_\ast$-generic tensors $T \in C^m \otimes C^m \otimes C^m$, is $R(T) \geq 2m - GR(T)$?
4.2. **Tensors of minimal border rank.** If $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is concise of minimal border rank $m$, then there exist complete flags in $A^*, B^*, C^*$, $0 \subset A_1^* \subset A_2^* \subset \cdots \subset A_m^* \subset A^*$ etc. such that $T|_{A_j^* \otimes B_j^* \otimes C_j^*}$ has border rank at most $j$, see [9, Prop. 2.4]. In particular, $\dim(\mathbb{P}T(A) \cap \sigma_1(\mathbb{P}B \times \mathbb{P}C)) \geq j - 1$. If the inequality is strict for some $j$, say equal to $j - 1 + q$, we say the $(A, j)$-th flag condition for minimal border rank is passed with excess $q$.

**Observation 4.4.** The geometric rank of a concise tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is $m$ minus the largest excess of the $(A, j)$ flag conditions for minimal border rank.

We emphasize that a tensor with degenerate geometric rank need not have minimal border rank, and need not pass all the $A$-flag conditions for minimal border rank, just that one of the conditions is passed with excess.

5. **Examples of tensors with degenerate geometric ranks**

5.1. **Matrix multiplication and related tensors.**

**Example 5.1** (Matrix multiplication). Set $m = n^2$. Let $U, V, W = \mathbb{C}^n$. Write $A = U^* \otimes V$, $B = V^* \otimes W$, $C = W^* \otimes U$. The structure tensor of matrix multiplication is $T = M(n) = \text{Id}_U \otimes \text{Id}_V \otimes \text{Id}_W$ (re-ordered), where $\text{Id}_U \in U \otimes U$ is the identity. When $n = 2$, $\Sigma^{AB} = \Sigma_2^{AB} = \Sigma^{\text{Seg}(P^*U) \times I_V \times P^*W}$, where $I_V = \{[v] \times [\nu] \in P^*V \times P^*V^* \mid \nu(v) = 0\}$ has dimension 3, so $GR(M(2)) = 6 - 3 = 3$. Note that $T_{2,2} = T_{2,2} = \Sigma_2^{\text{Seg}(P^*U \times P^*V)} = T_{2,2}(P^1 \times P^1)$ (with multiplicity two). For $[\mu \otimes v] \in \Sigma_2^{AB}$, $\pi_A^{-1}[\mu \otimes v] = P(\mu \otimes v \otimes v \otimes W) \cong P^1$. Since the tensor is $\mathbb{Z}_3$-invariant the same holds for $\Sigma^{B, C}$. For larger $n$, the dimension of the fibers of $\Sigma^{AB}$ varies with the rank of $X \in \{\det_n = 0\}$. The fiber is $[X] \times \mathbb{P}(\text{Rker}(X) \otimes W)$, which has dimension $(n - \text{rank}(X))n - 1$. Write $r = \text{rank}(X)$. Each $r$ gives rise to a $(n - r)n - 1 + (2nr - r^2 - 1) = n^2 - r^2 + nr - 2$ dimensional component of $\Sigma^{AB}$. There are $n - 1$ components, the largest dimension is attained when $r = \lceil \frac{n}{2} \rceil$, where the dimension is $n^2 - \lceil \frac{n}{2} \rceil \left\lceil \frac{3}{2} \right\rceil - 2$ and we recover the result of [17] that $GR(M(n)) = \left\lfloor \frac{3}{4} n^2 \right\rfloor = \left\lceil \frac{3}{4} m \right\rceil$, caused by $\Sigma_4^{AB}$.

**Example 5.2** (Structure Tensor of $\mathfrak{sl}_n$). Set $m = n^2 - 1$. Let $U = \mathbb{C}^n$, let $A = B = C = \mathfrak{sl}_n = \mathfrak{sl}(U)$. For $a, b \in \mathfrak{sl}_n$, $[a, b]$ denotes their commutator. Let $T_{\mathfrak{sl}_n} \in \mathfrak{sl}_n(\mathbb{C})^{\otimes 3}$ be the structure tensor of $\mathfrak{sl}_n$: $T_{\mathfrak{sl}_n} = \sum_{i,j=1}^{n-1} a_i \otimes b_j \otimes [a_i, b_j]$. Then $\hat{\Sigma}^{AB} = \{(x, y) \in A^* \times B^* \mid [x, y] = 0\}$.

Let $C(2, n) = \{(x, y) \in U^* \otimes U^* \otimes U^* \mid xy = yx\}$. In [23] it was shown that $C(2, n)$ is irreducible. Its dimension is $n^2 + n$, which was computed in [16, Prop. 6]. Therefore $\hat{\Sigma}^{AB} = (\mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})) \cap C(2, n)$ has dimension $n^2 + n - 2$, and $GR(T_{\mathfrak{sl}_n}) = \dim(\mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})) - \dim \hat{\Sigma}^{AB} = n^2 - n - m - \sqrt{m + 1}$.

**Example 5.3** (Symmetrized Matrix Multiplication). Set $m = n^2$. Let $A = B = C = U^* \otimes U$, with $\dim U = n$. Let $T = SM_{cn} \in (U^* \otimes U)^{\otimes 3}$ be the symmetrized matrix multiplication tensor: $SM_{cn}(X, Y, Z) = \text{tr}(XYZ) + \text{tr}(YXZ)$. In [5] it was shown that the exponent of $SM_{cn}$ equals the exponent of matrix multiplication. On the other hand, $SM_{(n)}$ is a cubic polynomial and thus may be studied with more tools from classical algebraic geometry, which raises the hope of new paths towards determining the exponent. Note that $SM_{cn}(X, Y, \cdot) = 0$ if and only if $XY + YX = 0$. So $\hat{\Sigma}^{AB} = \{(X, Y) \in U^* \otimes U \times U^* \otimes U \mid XY + YX = 0\}$.
Fix any matrix $X$, let $M_X$ and $M_{-X}$ be two copies of $\mathbb{C}^n$ with $\mathbb{C}[t]$-module structures: $t \cdot v := Xv, \forall v \in M_X$ and $t \cdot w := -Xw, \forall w \in M_{-X}$, where $\mathbb{C}[t]$ is the polynomial ring.

For any linear map $\varphi : M_X \to M_{-X}$,

$$\varphi \in \text{Hom}_{\mathbb{C}[t]}(M_X, M_{-X}) \iff \varphi(tv) = t\varphi(v), \forall v \in M_X$$

$$\iff \varphi(Xv) = -X\varphi(v), \forall v \in M_X$$

$$\iff \varphi X = -X\varphi.$$ 

This gives a vector space isomorphism $(\pi_A^{AB})^{-1}(X) := \{Y | XY + YX = 0\} \cong \text{Hom}_{\mathbb{C}[t]}(M_X, M_{-X}).$

By the structure theorem of finitely generated modules over principal ideal domains, $M_X$ has a primary decomposition:

$$M_X \cong \mathbb{C}[t]/(t - \lambda_1)^{r_1} \oplus \cdots \oplus \mathbb{C}[t]/(t - \lambda_k)^{r_k}$$

for some $\lambda_i \in \mathbb{C}$ and $\sum r_i = n$. Replacing $t$ with $-t$ we get a decomposition of $M_{-X}$:

$$M_{-X} \cong \mathbb{C}[t]/(t + \lambda_1)^{r_1} \oplus \cdots \oplus \mathbb{C}[t]/(t + \lambda_k)^{r_k}.$$

We have the decomposition $\text{Hom}_{\mathbb{C}[t]}(M_X, M_{-X}) \cong \bigoplus_{i,j} \text{Hom}_{\mathbb{C}[t]}(\mathbb{C}[t]/(t - \lambda_i)^{r_i}, \mathbb{C}[t]/(t + \lambda_j)^{r_j}).$

For each $i, j$:

$$\text{Hom}_{\mathbb{C}[t]}(\mathbb{C}[t]/(t - \lambda_i)^{r_i}, \mathbb{C}[t]/(t + \lambda_j)^{r_j}) = \begin{cases} 
\{ 1 \mapsto (t - \lambda_i)^l \mid 0 \leq l \leq r_j - 1 \} & \text{if } \lambda_i + \lambda_j = 0 \text{ and } r_i \geq r_j; \\
\{ 1 \mapsto (t - \lambda_i)^l \mid r_j - r_i \leq l \leq r_j - 1 \} & \text{if } \lambda_i + \lambda_j = 0 \text{ and } r_i < r_j; \\
0 & \text{otherwise.}
\end{cases}$$

Let $d_{ij}(X)$ denote its dimension, then $d_{ij}(X) = \left\{ \begin{array}{ll} \min\{r_i, r_j\} & \text{if } \lambda_i + \lambda_j = 0; \\
0 & \text{otherwise.} \end{array} \right.$ Thus $\dim((\pi_A^{AB})^{-1}(X)) = \sum_{i,j} d_{ij}(X)$.

Each direct summand $\mathbb{C}[t]/(t - \lambda_i)^{r_i}$ of $M_X$ corresponds to a Jordan block of the Jordan canonical form of $X$ with size $r_i$ and eigenvalue $\lambda_i$, denoted as $J_{\lambda_i}(r_i)$.

Assume $X$ has eigenvalues $\pm \lambda_1, \ldots, \pm \lambda_k, \pm \lambda_{k+1}, \ldots, \pm \lambda_l$ such that $\lambda_i \neq \pm \lambda_j$ whenever $i \neq j$. Let $q_{X,1}(\lambda) \geq q_{X,2}(\lambda) \geq \cdots$ be the decreasing sequence of sizes of the Jordan blocks of $X$ corresponding to the eigenvalue $\lambda$. Let $W(X)$ be the set of matrices $X'$ with eigenvalues $\pm \lambda'_1, \ldots, \pm \lambda'_l, \lambda'_{k+1}, \ldots, \lambda'_l$ such that $\lambda'_i \neq \pm \lambda'_j$ whenever $i \neq j$, and $q_{X,j}(\pm \lambda_i) = q_{X',j}(\pm \lambda'_i) \forall i, j$. Then $W(X)$ is quasi-projective and irreducible of dimension $\dim W(X) = \dim\{P^{-1}XP \mid \det P \neq 0\} + l$, and $(\pi_A^{AB})^{-1}(X')$ is of the same dimension as $(\pi_A^{AB})^{-1}(X)$ for all $X' \in W(X)$.

By results in [11], the codimension of the orbit of $X$ under the adjoint action of $GL(U)$ is $c_{Jor}(X) := \sum_{\lambda}[q_{X,1}(\lambda) + 3q_{X,2}(\lambda) + 5q_{X,3}(\lambda) + \cdots]$. Then

$$\dim \Sigma^{AB} = \max_X (\dim W(X) + \dim \pi_1^{-1}(X)) = \max_X (n^2 - c_{Jor}(X) + \dim (\pi_A^{AB})^{-1}(X) + l)$$

because $\Sigma^{AB} = \cup_X (\pi_A^{AB})^{-1}(W(X))$ is a finite union.
It is easy to show that \( \dim(p_A^{AB})^{-1}(X) - c_{tor}(X) \) takes maximum 0 when for every eigenvalue \( \lambda_i \) of \( X \), \( -\lambda_i \) is an eigenvalue of \( X \) and \( q_{X,j}(-\lambda_j) = q_{X,j}(\lambda_j), \forall i, j \). So the total maximum is achieved when \( X \) has the maximum possible number of distinct pairs \( \pm \lambda_i \), i.e.,

\[
X \simeq \begin{cases} 
\text{diag}(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \ldots, \lambda_n, -\lambda_n) & \text{if } n \text{ is even;} \\
\text{diag}(\lambda_1, -\lambda_1, \lambda_2 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_{n-1}, 0) & \text{if } n \text{ is odd.}
\end{cases}
\]

In both cases \( \dim \Sigma^{AB} = n^2 + \left\lfloor \frac{n}{2} \right\rfloor \). We conclude that \( GR(sM(n)) = n^2 - \left\lfloor \frac{n}{2} \right\rfloor = m - \left\lfloor \frac{m}{2} \right\rfloor \).

5.2. Large border rank and small geometric rank. The following example shows that border rank can be quite large while geometric rank is small:

**Example 5.4.** Let \( T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m \) have the form \( T = a_1 \otimes (b_1 \otimes c_1 + \cdots + b_m \otimes c_m) + T' \) where \( T' \in A' \otimes B' \otimes C' := \text{span} \{a_2, \ldots, a_m \} \otimes \text{span} \{b_1, \ldots, b_m \} \otimes \text{span} \{c_1, \ldots, c_m \} \), where \( T' \) is generic. It was shown in [20] that \( R(T) = R(T') + m \) and \( R(T) \geq \frac{m^2}{8} \).

We have

\[
T(A^*) \subseteq \begin{pmatrix} x_1 & \cdots & x_1 \\
* & \cdots & * & x_1 \\
\vdots & \vdots & \vdots & \vdots \\
* & \cdots & * & x_1 \\
\end{pmatrix}.
\]

Setting \( x_1 = 0 \), we see a component of \( \Sigma^A_{(\frac{m}{2})} \subset \mathbb{P}A^* \) is a hyperplane so \( GR(T) \leq \left\lfloor \frac{m}{2} \right\rfloor + 1 \).

5.3. Tensors arising in Strassen’s laser method.

**Example 5.5** (Big Coppersmith-Winograd tensor). The following tensor has been used to obtain every new upper bound on the exponent of matrix multiplication since 1988:

\[
T_{CW,q} = \sum_{j=1}^q a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0 + a_0 \otimes b_0 \otimes c_{q+1} + a_0 \otimes b_{q+1} \otimes c_0 + a_{q+1} \otimes b_0 \otimes c_0.
\]

One has \( R(T_{CW,q}) = 2q + 3 = 2m - 1 \) [20 Prop. 7.1] and \( R(T_{CW,q}) = q + 2 = m \) [10]. Note

\[
T_{CW,q}(A^*) = \begin{pmatrix} x_{q+1} & x_1 & \cdots & x_q & x_0 \\
x_1 & x_0 & \cdots & \cdot & \cdot \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
x_q & x_0 & \cdots & \cdot & \cdot \\
x_0 & 0 & \cdots & \cdot & \cdot \\
\end{pmatrix} \simeq \begin{pmatrix} x_0 & x_1 & \cdots & x_q & x_{q+1} \\
x_0 & x_1 & \cdots & \cdot & \cdot \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
x_0 & x_q & \cdots & \cdot & \cdot \\
x_0 & 0 & \cdots & \cdot & \cdot \\
\end{pmatrix}
\]

where \( \simeq \) means equal up to changes of bases. So we have \( \Sigma^A_1 = \Sigma^A_2 = \cdots = \Sigma^A_q = \{x_0 = 0\} \) and \( \Sigma^A_{q+1} = \{x_0 = \cdots = x_q = 0\} \). Therefore \( GR(T_{CW,q}) = 2(q + 2) - 1 - (\dim \Sigma^A_q + q) = 3 \).

**Example 5.6** (Small Coppersmith-Winograd tensor). The following tensor was the second tensor used in the laser method and for \( 2 \leq q \leq 10 \), it could potentially prove the exponent is less than 2.3: \( T_{cw,q} = \sum_{j=1}^q a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0 \). It satisfies \( R(T_{cw,q}) = 2q + 1 = 2m - 1 \) [20 Prop. 7.1] and \( R(T_{cw,q}) = q + 2 = m + 1 \) [10]. We again have \( GR(T_{cw,q}) = 3 \) as e.g., \( \Sigma^{AB} = \{x_0 = y_0 = \sum_{j \geq 1} x_j y_j = 0\} \cup \{\forall j \geq 1, x_j = y_j = 0\} \).
Example 5.7 (Strassen’s tensor). The following is the first tensor that was used in the laser method: \( T_{\text{str},q} = \sum_{j=1}^{q} a_j \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j \in \mathbb{C}^{q+1} \otimes \mathbb{C}^q \otimes \mathbb{C}^q \). It satisfies \( R(T_{\text{str},q}) = q + 1 \) and \( R(T_{\text{str},q}) = 2q \) \cite{20}. Since

\[
T_{\text{str},q}(A^*) = \begin{pmatrix}
x_1 & \cdots & x_q \\
x_0 & \ddots & \\
x_0 & & x_0
\end{pmatrix}
\]

We see \( GR(T_{\text{str},q}) = 2 \) caused by \( \Sigma^A_q = \mathbb{P}(\alpha_1, \ldots, \alpha_q) \).

5.4. Additional examples of tensors with geometric rank 3.

Example 5.8. The following tensor was shown in \cite{20} to take minimal values for Strassen’s functional (called maximal compressibility in \cite{20}):

\[
T_{\maxsym}\text{compr},m = a_1 \otimes b_1 \otimes c_1 + \sum_{\rho=2}^{m} a_1 \otimes b_\rho \otimes c_\rho + a_\rho \otimes b_1 \otimes c_\rho + a_\rho \otimes b_\rho \otimes c_1.
\]

Note

\[
T_{\maxsym}\text{compr},m(A^*) = \begin{pmatrix}
x_1 & x_2 & \cdots & x_m \\
x_2 & x_1 & \cdots & 0 \\
x_3 & 0 & x_1 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
x_m & 0 & \cdots & 0 & x_1
\end{pmatrix}.
\]

Restrict to the hyperplane \( \alpha_1 = 0 \), we obtain a space of bounded rank two, i.e., \( \Sigma^A_{m-2} \subset \mathbb{P}A^* \) is a hyperplane. We conclude, assuming \( m \geq 3 \), that \( GR(T) = 3 \).

Example 5.9. Let \( m = 2q \) and let

\[
T_{\text{gr},3,1\text{deg},2q} := \sum_{s=1}^{q} a_s \otimes b_1 \otimes c_s + \sum_{t=2}^{q} a_{t+q-1} \otimes b_1 \otimes c_1 + a_m \otimes (\sum_{u=q+1}^{m} b_u \otimes c_u),
\]

so

\[
T_{\text{gr},3,1\text{deg},m}(A^*) = \begin{pmatrix}
x_1 & x_2 & \cdots & x_q & 0 & \cdots & 0 \\
x_{q+1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
x_{m-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & x_m & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & x_m & \cdots & \ddots
\end{pmatrix}.
\]

Then \( GR(T_{\text{gr},3,1\text{deg},m}) = 3 \) (set \( x_m = 0 \)) and \( R(T) = \frac{3}{2}m - 1 \), the upper bound is clear from the expression the lower bound is given in Example \( 5.3 \).

Example 5.10. Let \( m = 2q - 1 \) and let

\[
T_{\text{gr},3,1\text{deg},2q-1} := \sum_{s=2}^{q} a_s \otimes b_1 \otimes c_s + a_{s+q-1} \otimes b_{s} \otimes c_1 + a_1 \otimes (\sum_{u=q+1}^{m} b_u \otimes c_u),
\]
A large class of spaces of matrices of bounded rank $E \subset B \otimes C$ are the compression spaces. In bases, the space takes the block format

$$E = \begin{pmatrix} \ast & \ast \\ \ast & 0 \end{pmatrix}$$

where if the 0 is of size $(b - k) \times (c - \ell)$, the space is of bounded rank $k + \ell$. 

5.5. Kronecker powers of tensors with degenerate geometric rank. For tensors $T \in A \otimes B \otimes C$ and $T' \in A' \otimes B' \otimes C'$, the Kronecker product of $T$ and $T'$ is the tensor $T \otimes T' := T \otimes T' \in (A \otimes A') \otimes (B \otimes B') \otimes (C \otimes C')$, regarded as a 3-way tensor. Given $T \in A \otimes B \otimes C$, the Kronecker powers of $T$ are $T \otimes \cdots \otimes T$, defined iteratively. Rank and border rank are submultiplicative under the Kronecker product, while subrank and border subrank are supermultiplicative under the Kronecker product.

Geometric rank is neither sub- nor super-multiplicative under the Kronecker product. We already saw the lack of sub-multiplicativity with $M_{(n)}$ (recall $M_{(n)} = M_{(n,2)}$): $2n^2 = GR(M_{(n)}^2) > \frac{2}{9}n^2 = GR(M_{(n)})^2$. An indirect example of the failure of super-multiplicativity is given in [17] where they point out that some power of $W := a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1$ is strictly sub-multiplicative. We make this explicit:

**Example 5.11.** With basis indices ordered 22, 21, 12, 11 for $B \otimes 2, C \otimes 2$, we have

$$W \otimes 2(A \otimes 2^*) = \begin{pmatrix} x_{11} & x_{12} & x_{21} & x_{22} \\ 0 & x_{11} & 0 & x_{21} \\ 0 & 0 & x_{11} & x_{12} \\ 0 & 0 & 0 & x_{11} \end{pmatrix}$$

which is $T_{CW,2}$ after permuting basis vectors (see Example 5.5) so $GR(W \otimes 2) = 3 < 4 = GR(W)^2$. 

6. Proofs of main theorems

In this section, after reviewing facts about spaces of matrices of bounded rank and the substitution method for bounding tensor rank, we prove a result lower-bounding the tensor rank of tensors associated to compression spaces (Proposition 6.2), a lemma on linear sections of $\sigma_3(\text{Seg}(PB \times PC))$ (Lemma 6.5), and Theorems 6.1 and 6.3.

6.1. Spaces of matrices of bounded rank. Spaces of matrices of bounded rank (Definition 2.8) is a classical subject dating back at least to [14]. The results most relevant here are from [3] and [2], and they were recast in the language of algebraic geometry in [13]. We review notions relevant for our discussion.

A large class of spaces of matrices of bounded rank $E \subset B \otimes C$ are the compression spaces. In bases, the space takes the block format

$$E = \begin{pmatrix} \ast & \ast \\ \ast & 0 \end{pmatrix}$$

where if the 0 is of size $(b - k) \times (c - \ell)$, the space is of bounded rank $k + \ell$. 

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If \( m \) is odd, then any linear subspace of \( \Lambda^2 \mathbb{C}^m \) is of bounded rank \( m - 1 \). More generally one can use the multiplication in any graded algebra to obtain spaces of bounded rank, the case of \( \Lambda^2 \mathbb{C}^m \) being the exterior algebra.

Spaces of bounded rank at most three are classified in [2]: For three dimensional rank two spaces there are only the compression spaces and the skew symmetric matrices \( \Lambda^2 \mathbb{C}^3 \subset \mathbb{C}^3 \otimes \mathbb{C}^3 \).

6.2. Review of the substitution method.

**Proposition 6.1.** [H, Appendix B] Let \( T \in A \otimes B \otimes C \). Fix a basis \( a_1, \ldots, a_n \) of \( A \), with dual basis \( \alpha^1, \ldots, \alpha^n \). Write \( T = \sum_{i=1}^n a_i \otimes M_i \), where \( M_i = T(\alpha_i) \in B \otimes C \). Let \( R(T) = r \) and \( M_1 \neq 0 \). Then there exist constants \( \lambda_2, \ldots, \lambda_n \), such that the tensor

\[
T' := \sum_{j=2}^n a_j \otimes (M_j - \lambda_j M_1) \in \text{span}\{a_2, \ldots, a_n\} \otimes B \otimes C,
\]

has rank at most \( r - 1 \). I.e., \( R(T) \geq 1 + R(T') \).

The analogous assertions hold exchanging the role of \( A \) with that of \( B \) or \( C \).

A visual tool for using the substitution method is to write \( T(B^*) \) as a matrix of linear forms. Then the \( i \)-th row of \( T(B^*) \) corresponds to a tensor \( a_i \otimes M_i \in \mathbb{C}^1 \otimes B \otimes C \). One adds unknown multiples of the first row of \( T(B^*) \) to all other rows, and deletes the first row, then the resulting matrix is \( T'(B^*) \in \text{span}\{a_2, \ldots, a_n\} \otimes C \).

In practice one applies Proposition 6.1 iteratively, obtaining a sequence of tensors in spaces of shrinking dimensions. See [19, §5.3] for a discussion.

For a positive integer \( k \leq b \), if the last \( k \) rows of \( T(A^*) \) are linearly independent, then one can apply Proposition 6.1 \( k \) times on the last \( k \) rows. In this way, the first \( b - k \) rows are modified by unknown linear combinations of the last \( k \) rows, and the last \( k \) rows are deleted. Then one obtains a tensor \( T' \in A \otimes \text{span}\{b_1, \ldots, b_{b-k}\} \otimes C \) such that \( R(T') \leq R(T) - k \).

**Proposition 6.2.** Let \( T \in A \otimes B \otimes C \) be a concise tensor with \( T(A^*) \) a bounded rank \( \rho \) compression space. Then \( R(T) \geq b + c - \rho \).

**Proof.** Consider [H]. Add to the first \( k \) rows of \( T(A^*) \) unknown linear combinations of the last \( b - k \) rows, each of which is nonzero by conciseness. Then delete the last \( b - k \) rows. Note that the last \( c - \ell \) columns are untouched, and (assuming the most disadvantageous combinations are chosen) we obtain a tensor \( T' \in A \otimes \mathbb{C}^{c-k} \otimes C \) satisfying \( R(T) \geq (b - k) + R(T') \). Next add to the first \( \ell \) columns of \( T'(A^*) \) unknown linear combinations of the last \( c - \ell \) columns, then delete the last \( c - \ell \) columns. The resulting tensor \( T'' \) could very well be zero, but we nonetheless have \( R(T') \geq (c - \ell) + R(T'') \) and thus \( R(T) \geq (b - k) + (c - \ell) = b + c - \rho \). \( \square \)

Here are the promised lower bounds for \( T_{gr;3,1;deg,m} \):

**Example 6.3.** Consider [H]. Add to the first row unknown linear combinations of the last \( m - 1 \) rows then delete the last \( m - 1 \) rows. The resulting tensor is still \( (a_1, \ldots, a_q) \)-concise so we have \( R(T_{gr;3,1;deg,2q}) \geq m - 1 + \frac{m}{2} \). The case of \( T_{gr;3,1;deg,2q-1} \) is similar.
Lemma 6.4. Let $E \subset B \otimes C$ be a linear subspace. If $\mathbb{P} E \cap \sigma_r(Seg(\mathbb{P} B \times \mathbb{P} C))$ is a hypersurface in $\mathbb{P} E$ of degree $r + 1$ (counted with multiplicity) and does not contain any hyperplane of $\mathbb{P} E$, then $\mathbb{P} E \subset \sigma_{r+1}(Seg(\mathbb{P} B \times \mathbb{P} C))$.

Proof. Write $E = (y^i_j)$ where $y^i_j = y^i_j(x_1, \ldots, x_q)$, $1 \leq i \leq b$, $1 \leq j \leq c$ and $q = \dim E$. By hypothesis, all size $r + 1$ minors are up to scale equal to a polynomial $S$ of degree $r + 1$. No linear polynomial divides $S$ since otherwise the intersection would contain a hyperplane. Since $\mathbb{P} E \notin \sigma_r(Seg(\mathbb{P} B \times \mathbb{P} C))$, there must be a size $r + 1$ minor that is nonzero restricted to $\mathbb{P} E$. Assume it is the $(1, \ldots, r + 1) \times (1, \ldots, r + 1)$-minor.

Consider the vector consisting of the first $r + 1$ entries of the $(r + 2)$-st column. In order that all size $r + 1$ minors of the upper left $(r + 1) \times (r + 2)$ block equal to multiples of $S$, this vector must be a linear combination of the vectors corresponding to the first $r + 1$ entries of the first $r + 1$ columns. By adding linear combinations of the first $r + 1$ columns, we may make these entries zero. Similarly, we may make all other entries in the first $r + 1$ rows zero. By the same argument, we may do the same for the first $r + 1$ columns. We have

$$
\begin{pmatrix}
  y^1_1 & \cdots & y^1_{r+1} & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  y^{r+1}_1 & \cdots & y^{r+1}_{r+1} & 0 & \cdots & 0 \\
  0 & \cdots & 0 & y^{r+2}_r & \cdots & y^{r+2}_c \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & \cdots & 0 & y^{r+2}_r & \cdots & y^{r+2}_c
\end{pmatrix}
$$

If $\mathbb{P} E \notin \sigma_{r+1}(Seg(\mathbb{P} B \times \mathbb{P} C))$, some entry in the lower $(b - r - 1) \times (c - r - 1)$ block is nonzero. Take one such and the minor with it and a $r \times r$ submatrix of the upper left minor. We obtain a polynomial that has a linear factor, so it cannot be a multiple of $S$, giving a contradiction. \qed

Lemma 6.5. Let $b, c > 4$. Let $E \subset B \otimes C$ be a linear subspace of dimension $q > 4$. Say $\dim(\mathbb{P} E \cap \sigma_r(Seg(\mathbb{P} B \times \mathbb{P} C))) = q - 2$ and $\mathbb{P} E \notin \sigma_3(Seg(\mathbb{P} B \times \mathbb{P} C))$. Then either all components of $\mathbb{P} E \cap \sigma_2(Seg(\mathbb{P} B \times \mathbb{P} C))$ are linear $\mathbb{P}^{q-2}$‘s, or $E \subset \mathbb{C}^5 \otimes \mathbb{C}^5$.

The proof is similar to the argument for Lemma 6.4, except that we work in a local ring.

Proof. Write $E = (y^i_j)$ where $y^i_j = y^i_j(x_1, \ldots, x_q)$, $1 \leq i \leq b$, $1 \leq j \leq c$. Assume, to get a contradiction, that there is an irreducible component of degree greater than one in the intersection, given by an irreducible polynomial $S$ of degree two or three that divides all size 3 minors. By Lemma 6.4, $\deg(S) = 2$. Since $\mathbb{P} E \notin \sigma_2(Seg(\mathbb{P} B \times \mathbb{P} C))$, there must be some size 3 minor that is nonzero restricted to $\mathbb{P} E$. Assume it is the $(123) \times (123)$ minor.

Let $\Delta^I_j$ denote a (signed) size 3 minor restricted to $E$, where $I = (i_1i_2i_3)$, $J = (j_1j_2j_3)$, so $\Delta^I_j = L^I_j S$, for some $L^I_j \in E^*$. Set $I_0 = (123)$. Consider the $(st4) \times I_0$ minors, where $1 \leq s < t \leq 3$. Using the Laplace expansion, we may write them as

$$
\begin{pmatrix}
  \Delta^I_0|_1 \\
  \Delta^I_0|_2 \\
  \Delta^I_0|_3
\end{pmatrix}
\begin{pmatrix}
  \Delta^I_0|_1 \\
  \Delta^I_0|_2 \\
  \Delta^I_0|_3
\end{pmatrix}
\begin{pmatrix}
  y^I_1 \\
  y^I_2 \\
  y^I_3
\end{pmatrix}
= S
\begin{pmatrix}
  L^{234} \\
  L^{134} \\
  L^{124}
\end{pmatrix}
$$
Choosing signs properly, the matrix on the left is just the cofactor matrix of the $(123) \times (123)$ submatrix, so its inverse is the transpose of the original submatrix divided by the determinant (which is nonzero by hypothesis). Thus we may write

$$
\begin{pmatrix}
y_1^4 \\
y_2^4 \\
y_3^4
\end{pmatrix} = \frac{S}{\Delta_{123}} \begin{pmatrix}
y_1^1 & y_2^1 & y_3^1 & 0 & \ldots & 0 \\
y_1^2 & y_2^2 & y_3^2 & 0 & \ldots & 0 \\
y_1^3 & y_2^3 & y_3^3 & 0 & \ldots & 0 \\
0 & 0 & 0 & y_4^4 & \ldots & y^4_c \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & y_b^4 & \ldots & y^b_c
\end{pmatrix}.
$$

In particular, each $y_s^4$, $1 \leq s \leq 3$, is a rational function of $L_{I_0}^{(I_0 \setminus t), 4}$ and $y_v^u$, $1 \leq u, v \leq 3$.

$$
(y_1^4, y_2^4, y_3^4) = \sum_{t=1}^3 \frac{L_{I_0}^{(I_0 \setminus t), 4}}{L_{I_0}} (y_t^1, y_t^2, y_t^3).
$$

Note that the coefficients $\frac{L_{I_0}^{(I_0 \setminus t), 4}}{L_{I_0}}$ are degree zero rational functions in $L_{I_0}^{(I_0 \setminus t), 4}$ and $y_v^u$, $1 \leq u, v \leq 3$.

The same is true for all $(y_{1}^{t}, y_{2}^{t}, y_{3}^{t})$ for $\ell \geq 4$. Similarly, working with the first 3 rows we get $(y_{1}^{t}, y_{2}^{t}, y_{3}^{t})$ written in terms of the $(y_{1}^{t}, y_{2}^{t}, y_{3}^{t})$ with coefficients degree zero rational functions in the $y_s^t$. Restricting to the Zariski open subset of $E$ where $L_{I_0}^{I_0} \neq 0$, we may subtract rational multiples of the first three rows and columns to normalize our space to (9):

$$
\begin{pmatrix}
y_1^1 & y_2^1 & y_3^1 & 0 & \ldots & 0 \\
y_1^2 & y_2^2 & y_3^2 & 0 & \ldots & 0 \\
y_1^3 & y_2^3 & y_3^3 & 0 & \ldots & 0 \\
0 & 0 & 0 & y_4^4 & \ldots & y^4_c \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & y_b^4 & \ldots & y^b_c
\end{pmatrix}.
$$

Since a Zariski open subset of $\mathbb{P}E$ is not contained in $\sigma_2(Seg(\mathbb{P}B \times \mathbb{P}C))$, at least one entry in the lower right block must be nonzero. Say it is $y_4^1$.

On the Zariski open set $L_{I_0}^{I_0} \neq 0$, for all $1 \leq s < t \leq 3$, $1 \leq u < v \leq 3$, we have $y_4^1 \Delta_{uv}^{st} = Q_{uv}^{st} S/L_{I_0}^{I_0}$, where $Q_{uv}^{st}$ is a quadratic polynomial (when $\Delta_{uv}^{st} \neq 0$) or zero (when $\Delta_{uv}^{st} = 0$). Then either $y_4^1$ is a nonzero multiple of $S/L_{I_0}^{I_0}$, or all $\Delta_{uv}^{st}$'s are multiples of $S$, because $S$ is irreducible.

If all $\Delta_{uv}^{st}$'s are multiples of $S$, at least one must be nonzero, say $\Delta_{12}^{12} \neq 0$. Then by a change of bases we set $y_3^1, y_2^1, y_1^1, y_3^2$ to zero. At this point, for all $1 \leq \alpha, \beta \leq 2$, $\Delta_{\alpha \beta}^{33}$ becomes $y_\beta^3 y_3^3$. By hypothesis $\Delta_{\alpha \beta}^{33}$ is a multiple of the irreducible quadratic polynomial $S$, so $y_\beta^3 y_3^3 = 0$. Therefore either all $y_\beta^\alpha$'s are zero, which contradicts with $\Delta_{12}^{12} \neq 0$, or $y_3^3 = 0$, in which case all entries in the third row and the third column are zero, contradicting our assumption that the first $3 \times 3$ minor is nonzero.

If there exists $\Delta_{uv}^{st} \neq 0$ that is not a multiple of $S$, change bases such that it is $\Delta_{12}^{12}$. Note that $y_4^1 = \frac{\Delta_{12}^{124}}{\Delta_{123}^{12}} = (\Delta_{12}^{124}/S)/L_{I_0}^{I_0}$ where $(\Delta_{12}^{124}/S)$ is a quadratic polynomial. By hypothesis $S$ divides $\Delta_{12}^{124} = \Delta_{12}^{124}$. Since $S$ is irreducible and $\Delta_{12}^{12}$ is not a multiple of $S$, $\Delta_{12}^{124}/S$ must be a multiple of $S$. Therefore $\Delta_{12}^{124}$ is a multiple of $S^2$. This is true for all size 4 minors, therefore we can apply 6.4. By the proof of 6.4, all entries of $E$ can be set to zero except those in the upper left $5 \times 5$ block, so $E \subset \mathbb{C}^5 \otimes \mathbb{C}^5$. \hfill \Box
Remark 6.6. The normalization in the case $\deg (S) = 2$ is not possible in general without the restriction to the open subset where $L_{i_0} \neq 0$. Consider $T(A^*)$ such that the upper $3 \times 3$ block is

$$
\begin{pmatrix}
x_1 & x_2 & x_3 \\
-x_2 & x_1 & x_4 \\
-x_3 & -x_4 & x_1
\end{pmatrix}.
$$

Then the possible entries in the first three columns of the fourth row are not limited to the span of the first three rows. The element $(x_4, -x_3, x_2)$ is also possible.

6.4. Proof of Theorem 3.1. We recall the statement:

**Theorem 3.1.** For $a, b, c \geq 3$, the variety $\mathcal{GR}_2(A \otimes B \otimes C)$ is the variety of tensors $T$ such that $T(A^*)$, $T(B^*)$, or $T(C^*)$ has bounded rank 2.

There are exactly two concise tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ with $\text{GR}(T) = 2$:

1. The unique up to scale skew-symmetric tensor $T = \sum_{\sigma \in S_3} \text{sgn}(\sigma) a_{\sigma(1)} \otimes b_{\sigma(2)} \otimes c_{\sigma(3)} \in A^3 \mathbb{C}^3 \otimes B^3 \mathbb{C}^3 \otimes C^3 \mathbb{C}^3$

   and

2. $T_{\text{utriv},3} := a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_2 + a_1 \otimes b_3 \otimes c_3 + a_2 \otimes b_1 \otimes c_2 + a_3 \otimes b_1 \otimes c_3 \in S^2 \mathbb{C}^3 \otimes B^3 \mathbb{C}^3 \otimes C^3 \mathbb{C}^3$.

There is a unique concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ satisfying $\text{GR}(T) = 2$ when $m > 3$, namely

$$T_{\text{utriv},m} := a_1 \otimes b_1 \otimes c_1 + \sum_{\rho=2}^m [a_1 \otimes b_\rho \otimes c_\rho + a_\rho \otimes b_1 \otimes c_\rho].$$

This tensor satisfies $R(T_{\text{utriv},m}) = m$ and $R(T_{\text{utriv},m}) = 2m - 1$.

**Proof.** For simplicity, assume $a \leq b \leq c$. A tensor $T \in A \otimes B \otimes C$ has geometric rank 2 if and only if $\mathbb{P} T(A^*) \not\subseteq \text{Seg}(\mathbb{P} B \times \mathbb{P} C)$, and either $\mathbb{P} T(A^*) \cap \text{Seg}(\mathbb{P} B \times \mathbb{P} C)$ has dimension $a - 2$ or $\mathbb{P} T(A^*) \subset \sigma_2 (\text{Seg}(\mathbb{P} B \times \mathbb{P} C))$. For the case $\mathbb{P} T(A^*) \subset \sigma_2 (\text{Seg}(\mathbb{P} B \times \mathbb{P} C))$, $T(A^*)$ is of bounded rank 2. By the classification of spaces of bounded rank 2, up to equivalence $T(A^*)$ must be in one of the following forms:

$$
\begin{pmatrix}
* & \ldots & * \\
* & \ldots & * \\
0 & \ldots & 0
\end{pmatrix} \quad \text{or} \quad
\begin{pmatrix}
0 & x & y & \ldots & 0 \\
-x & 0 & z & \ldots & 0 \\
-y & -z & 0 & \ldots & 0
\end{pmatrix}.
$$

When $T$ is concise, it must be of the second form or the third with $a = b = c = m = 3$. If it is the third, $T$ is the unique up to scale skew-symmetric tensor in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. If it is of the second form and $a = b = c = m$, the entries in the first column must be linearly independent, as well as the entries in the first row. Thus we may choose a basis of $A$ such that $T = a_1 \otimes b_1 \otimes c_1 + \sum_{i=1}^m y_i \otimes b_i \otimes c_1 + \sum_{i=1}^m z_i \otimes b_1 \otimes c_i$ where $y_i$'s and $z_i$'s are linear combinations of $a_2, \ldots, a_m$. Then by a change of basis in $b_2, \ldots, b_m$ and $c_2, \ldots, c_m$ respectively, we obtain $T_{\text{utriv},m}$.

For the case $\dim (\mathbb{P} T(A^*) \cap \text{Seg}(\mathbb{P} B \times \mathbb{P} C)) = a - 2$, by Lemma 6.4, if this intersection is an irreducible quadric, we are reduced to the case $\mathbb{P} T(A^*) \subset \sigma_2 (\text{Seg}(\mathbb{P} B \times \mathbb{P} C))$. Thus all $2 \times$
2 minors of $T(A^*)$ have a common linear factor. Assume the common factor is $x_1$. Then $\mathbb{P}T(A^*) \cap \text{Seg}(\mathbb{P}B \times \mathbb{P}C) \ni \{x_1 = 0\}$. Hence $\mathbb{P}T((\alpha_2, \ldots, \alpha_d)) \subset \text{Seg}(\mathbb{P}B \times \mathbb{P}C)$, i.e. $T((\alpha_2, \ldots, \alpha_d))$ is of bounded rank one. By a change of bases in $B, C$ and exchanging $B$ and $C$, all entries but the first row of $T((\alpha_2, \ldots, \alpha_d))$ becomes zero. Then all entries but the first column and the first row of $T(C^*)$ are zero, so $T(C^*)$ is of bounded rank 2.

When $T$ is concise and $a = b = c = m$, the change of bases as in the case when $T(A^*)$ is of bounded rank 2 shows that up to a reordering of $A, B$ and $C$ we obtain $T_{\text{utriv}, m}$. \hfill \Box

6.5. \textbf{Proof of Theorem 3.3}. We recall the statement:

\textbf{Theorem 3.3.} Let $T \in A \otimes B \otimes C$ be concise and assume $c \geq b \geq a > 4$. If $GR(T) \leq 3$, then $R(T) \geq b + \left\lceil \frac{a - 1}{2} \right\rceil - 2$.

If moreover $a = b = c = m$ and $T$ is $1_*$-generic, then $R(T) \geq 2m - 3$.

\textbf{Proof.} In order for $GR(T) = 3$, either $\mathbb{P}T(A^*) \subset \sigma_3(\text{Seg}(\mathbb{P}B \times \mathbb{P}C))$, $\mathbb{P}T(A^*) \cap \sigma_2(\text{Seg}(\mathbb{P}B \times \mathbb{P}C))$ has dimension $a - 2$, or $\mathbb{P}T(A^*) \cap \text{Seg}(\mathbb{P}B \times \mathbb{P}C)$ has dimension $a - 3$.

Case $\mathbb{P}T(A^*) \subset \sigma_3(\text{Seg}(\mathbb{P}B \times \mathbb{P}C))$: Since $a > 3$, it must be a compression space. We conclude by Proposition 6.2.

Case $\mathbb{P}T(A^*) \cap \sigma_2(\text{Seg}(\mathbb{P}B \times \mathbb{P}C))$ has dimension $a - 2$: By Lemma 6.5 there exists $a \in A$ such that $\mathbb{P}T(a^i) \subset \sigma_2(\text{Seg}(\mathbb{P}B \times \mathbb{P}C))$. Write $T(A^*) = x_1Z + U$, where $Z$ is a matrix of scalars and $U = U(x_2, \ldots, x_a)$ is a matrix of linear forms of bounded rank two. As discussed in \S 6.1 there are two possible normal forms for $U$ up to symmetry.

If $U$ is zero outside of the first two rows, add to the first two rows an unknown combination of the last $b - 2$ rows (each of which is nonzero by conciseness), so that the resulting tensor $T'$ satisfies $R(T) \geq b - 2 + R(T')$. Now the last $b - 2$ rows only contained multiples of $x_1$ so $T'$ restricted to $a_1^i$ is $(a_2, \ldots, a_a)$-concise and thus of rank at least $a - 1$, so $R(T) \geq a + b - 3$.

Now say $U$ is zero outside its first row and column.

Subcase: $a = b = c$ and $T$ is $1_*$-generic. Then either $T$ is $1_A$-generic, or the first row or column of $U$ consists of linearly independent entries. If $T$ is $1_A$-generic, we may change bases so that $Z$ is of full rank. Consider $T(B^*)$. Its first row consists of linearly independent entries. Write $a = b = c = m$ and apply the substitution method to delete the last $m - 1$ rows (each of which is nonzero by conciseness). Call the resulting tensor $T''$, so $R(T) \geq R(T'') + m - 1$. Let $T'' = T''|_{A^* \otimes \text{span}(\beta_2, \ldots, \beta_m) \otimes \text{span}(\gamma_2, \ldots, \gamma_m)}$. Then $T''(A^*)$ equals to the matrix obtained by removing the first column and the first row from $x_1Z$, so $R(T'') \geq R(x_1Z) - 2 = m - 2$. Thus $R(T) \geq (m - 1) + m - 2$ and we conclude. If the first row of $U$ consists of linearly independent entries, then the same argument, using $T(A^*)$, gives the bound.

Subcase: $T$ is 1-degenerate or $a, b, c$ are not all equal. By $A$-conciseness, either the first row or column must have at least $\left\lceil \frac{a - 1}{2} \right\rceil$ independent entries of $\text{span}(x_2, \ldots, x_a)$. Say it is the first row. Then applying the substitution method, adding an unknown combination of the last $b - 1$ rows to the first, then deleting the last $b - 1$ rows. Note that all entries in the first row except the $(1, 1)$ entry are only altered by multiples of $x_1$, so there are at least $\left\lceil \frac{a - 1}{2} \right\rceil - 1$ linearly independent entries in the resulting matrix. We obtain $R(T) \geq b - 1 + \left\lceil \frac{a - 1}{2} \right\rceil - 1$. 
Case \( \dim(\mathbb{P}T(A^*) \cap \text{Seg}(\mathbb{P}B \times \mathbb{P}C)) = a - 3 \): We split this into three sub-cases based on the dimension of the span of the intersection:

1. \( \dim(\mathbb{P}T(A^*) \cap \text{Seg}(\mathbb{P}B \times \mathbb{P}C)) = a - 3 \)
2. \( \dim(\mathbb{P}T(A^*) \cap \text{Seg}(\mathbb{P}B \times \mathbb{P}C)) = a - 2 \)
3. \( \dim(\mathbb{P}T(A^*) \cap \text{Seg}(\mathbb{P}B \times \mathbb{P}C)) = a - 1 \)

Sub-case (1): the intersection must be a linear space. We may choose bases such that

\[
T(A^*) = \begin{pmatrix}
0 & 0 & x_3 & \cdots & x_a \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & & \ddots & & \vdots \\
0 & & & \ddots & \\
0 & & & & 0
\end{pmatrix} + x_1Z_1 + x_2Z_2
\]

where \( Z_1, Z_2 \) are \( b \times c \) scalar matrices. Add to the first row a linear combination of the last \( b - 1 \) rows (each of which is nonzero by conciseness) to obtain a tensor of rank at least \( b - 2 \), giving \( \text{R}(T) \geq a + b - 3 \).

Sub-case (2): By Lemma 6.4 the intersection must contain a \( \mathbb{P}^{a-2} \), and one argues as in case (1), except there is just \( x_1Z_1 \) and \( x_2 \) also appears in the first row.

Sub-case (3): \( T(A^*) \) must have a basis of elements of rank one. The only way \( T \) can be concise, is for \( a = b = c = m \) and \( m \) elements of the \( B \) factor form a basis and same for the \( C \) factor. Changing bases, we have \( T = \sum_{j=1}^{m} a_j \otimes b_j \otimes c_j \) which just intersects the Segre in points, so this case cannot occur. \( \square \)

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