Longitudinal dielectric permeability in quantum non-degenerate and maxwellian collisional plasma with constant collision frequency

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Abstract

The formula for dielectric function of non-degenerate and maxwellian collisional plasmas is transformed to the form, convenient for research. Graphic comparison of longitudinal dielectric functions of quantum and classical non-degenerate collisional plasmas is made.

Key words: Mermin, quantum non-degenerate collisional plasma, conductance, kinetic equation.

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Introduction

In the known work of Mermin [1] by means of the analysis of nonequilibrium density matrix in $\tau$–approximation has been obtained expression for longitudinal dielectric permeability of quantum collisional plasmas for case of constant collision frequency of plasmas particles.

Earlier in the work of Klimontovich and Silin [2] and after that in the work of Lindhard [3] has been obtained expression for longitudinal and transverse dielectric permeability of quantum collisionless plasmas.

In work [1] actually without deducing it has been announced the general expression of dielectric function of quantum collisional plasmas with constant frequency of collisions.

In work [4] the general expression for dielectric function of quantum collisional plasmas with the variable frequency of collisions depending on the wave vector has been deduced.

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Then in work [5] the detailed deducing dielectric functions for quantum plasma with constant frequency of collisions has been given. In the present work the case of non-degenerate and maxwellian quantum plasma with any degree of degeneration (a case of the any temperature) is considered.

The general formula for dielectric functions from [1], [4] and [5] it becomes simpler and transformed to the convenient form for calculations and contains one quadrature.

Deducing of dielectric function of classical non-degenerate collisional plasmas is presented. It is given graphic comparison of the real and imaginary parts of dielectric function of quantum and classical plasma.

As classical non-degenerate collisional plasma we will be understand the plasma described by the kinetic Vlasov—Boltzmann equation with integral of collisions from phase space.

As quantum non-degenerate collisional plasma we will be understand the plasma described by the kinetic Schrödinger—Boltzmann equation with integral of collisions from momentum space.

Quantum plasma is studied extremely intensively. Among the big number of works we will note only some of them [6]–[20].

1. Dielectric permeability of quantum non-degenerate plasmas

We take the formula for longitudinal dielectric function of quantum non-degenerate collisional plasmas

\[ \varepsilon_l(k, \omega, \nu) = 1 + \frac{4\pi e^2 (\omega + i\nu) B(k, \omega + i\nu) B(k, 0)}{k^2 \omega B(k, 0) + i\nu B(k, \omega + i\nu)}. \]  

(1.1)

In the formula (1.1) following designations are accepted

\[ B(k, \omega + i\nu) = \int \frac{dp}{4\pi^3} \frac{f_{p+k/2} - f_{p-k/2}}{\mathcal{E}_{p-k/2} - \mathcal{E}_{p+k/2} + \hbar(\omega + i\nu)}; \]  

(1.2)

\[ B(k, 0) = \int \frac{dp}{4\pi^3} \frac{f_{p+k/2} - f_{p-k/2}}{\mathcal{E}_{p-k/2} - \mathcal{E}_{p+k/2}}, \]

\( k \) is the dimensional wave vector,

\[ \mathcal{E}_{p\pm k/2} = \frac{\hbar^2}{2m} \left( p \pm \frac{k}{2} \right)^2, \quad f_{p\pm k/2} = \frac{1}{1 + \exp \left( \frac{\mathcal{E}_{p\pm k/2} - \alpha}{k_BT} \right)}, \]
\( \alpha = \mu / (k_B T) \) is the normalized (dimensionless) chemical potential, \( k_B \) is the Boltzmann constant, \( T \) is the temperature, \( f_{p \pm k/2} \) is the Fermi–Dirac distribution function, \( e \) is the electron charge, \( \nu \) is the effective collision frequency of electrons with plasma particles, \( \omega \) is the frequency of oscillations of an electromagnetic field.

Let’s transform the formula (1.1) to the form convenient for research. We will enter thermal velocity of electrons \( v_T = 1 / \sqrt{\beta} \), \( \beta = m / (2k_B T) \). Clearly, that \( k_B T = mv_T^2 = E_T \) is thermal electrons energy.

We introduce the nondimensional vector \( \mathbf{P} = \frac{\mathbf{p}}{k_T} \), where \( k_T = \frac{p_T}{\hbar} = \frac{mv_T}{\hbar} \) is the thermal wave number, \( p_T \) is the thermal momentum of electrons.

Let’s find difference of energies
\[
\mathcal{E}_{p-k/2} - \mathcal{E}_{p+k/2} = -2\mathcal{E}_T P_x q, \quad q = q(1,0,0), \quad q = \frac{k}{k_T},
\]
\( q \) is the nondimensional wave number.

The denominator from expression (1.2) is equal
\[
\mathcal{E}_{p-k/2} - \mathcal{E}_{p+k/2} + \hbar(\omega + i\nu) = -\frac{\hbar^2 k_T^2}{m} q \left( P_x - \frac{z}{q} \right) = -2\mathcal{E}_T q (P_x - \frac{z}{q}).
\]

Here dimensionless frequencies are entered
\[
z = x + iy = \frac{\omega + i\nu}{k_T v_T}, \quad x = \frac{\omega}{k_T v_T}, \quad y = \frac{\nu}{k_T v_T}.
\]

Now the integral (1.2) is equal
\[
B(k, \omega + i\nu) = -\frac{k_T^3}{2\mathcal{E}_T q} \int \frac{d^3P f_{p+q/2} - f_{p-q/2}}{4\pi^3} \frac{P_x - z/q}{P_x - z/q}.
\]

Here
\[
f_{p\pm q/2} = \frac{1}{1 + \exp \left[ \left( \mathbf{P} \pm \frac{\mathbf{q}}{2} \right)^2 - \alpha \right]} \equiv f_{P\pm q/2}.
\]

Let’s designate
\[
B(q, z) = \int \frac{d^3P f_{p+q/2} - f_{p-q/2}}{4\pi^3} \frac{P_x - z/q}{P_x - z/q}.
\]

Then expression (1.3) will be rewritten in the form
\[
B(k, \omega + i\nu) = -\frac{k_T^3}{2\mathcal{E}_T q} B(q, z).
\]
Let’s consider integral (1.4). We will present this integral in the form of difference of integrals. In each of integrals it is realizable the obvious linear replacement of variables. As result we receive

$$B(q, z) = \int \frac{d^3 P}{4\pi^3} \frac{f_0(P, \alpha)}{P_x - z/q - q/2} - \int \frac{d^3 P}{4\pi^3} \frac{f_0(P, \alpha)}{P_x - z/q + q/2}. \quad (1.5)$$

where

$$f_0(P, \alpha) = \frac{1}{1 + e^{P^2 - \alpha}}.$$  

According to (1.5) we receive

$$B(k, \omega + i\nu) = -\frac{k_3^2}{2\mathcal{E}_T} b(q, z), \quad (1.6)$$

where

$$b(q, z) = \int \frac{d^3 P}{4\pi^3} \frac{f_0(P, \alpha)}{(P_x - z/q)^2 - (q/2)^2}.$$

Now expression (1.1) for dielectric function will be transformed to the form

$$\varepsilon_l(x, y, q) = 1 - \frac{4\pi e^2}{q^2 k_T^3} \frac{k_3^3}{2\mathcal{E}_T} \frac{(x + iy)b(q, z)b(q, 0)}{xb(q, 0) + iyb(q, z)}. \quad (1.7)$$

It is easy to find, that numerical density of particles of plasma (its concentration) in an equilibrium condition is equal

$$N = \int f_0(P, \alpha) \frac{2m^3 d^3 v}{(2\pi \hbar)^3} = \frac{2m^3 v_T^3}{8\pi^3 \hbar^3} \int f_0(P, \alpha) d^3 P = \frac{k_3^3}{\pi^2} f_2(\alpha), \quad (1.8)$$

where

$$f_2(\alpha) = \int_0^\infty f_0(P, \alpha) P^2 dP.$$  

Let’s calculate internal integral on $dP_y dP_z$ in expression for $b(q, z)$. We have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dP_y dP_z}{1 + \exp(P_x^2 + P_y^2 + P_z^2 - \alpha)} = \pi \ln(1 + e^{-P_z^2}).$$

Hence, the integral $b(q, z)$ is expressed through the one-dimensional integral

$$b(q, z) = \frac{1}{4\pi^2} l(q, z), \quad (1.9)$$
where

\[ l(q, z) = \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha - \mu^2})d\mu}{(\mu - z/q)^2 - (q/2)^2}. \]

Longitudinal dielectric function (1.7) according to (1.8) and (1.9) we rewrite in the form

\[ \varepsilon_l(x, y, q) = 1 - \frac{x_p^2}{4q^2f_2(\alpha)} \frac{(x + iy)l(q, z)l(q, 0)}{xl(q, 0) + iyl(q, z)}. \] (1.10)

In (1.10) \( x_p \) is the nondimensional plasma frequency,

\[ x_p = \frac{\omega_p}{kTvT}, \]

\( \omega_p \) is the dimensional plasma (Langmuir) frequency,

\[ \omega_p = \frac{4\pi e^2N}{m}. \]

2. Dielectric permeability of classical non-degenerate plasmas

We take the kinetic Vlasov—Boltzmann equation for collisional Fermi—Dirac plasmas with arbitrary temperature

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} + eE(r, t) \frac{\partial f}{\partial p} = \nu [f_{eq} - f]. \] (2.1)

Here \( f_{eq} \) is the local equilibrium distribution electrons function of Fermi—Dirac (local Fermian)

\[ f_{eq} = \frac{1}{1 + \exp \left( \frac{mv^2}{2k_BT} - \frac{\mu(r)}{k_BT} \right)}, \] (2.2)

\( k_B \) is the Boltzmann constant, \( T \) is the plasma temperature, \( \nu \) is the electron collisional frequency with plasma particles, \( p = mv \) is the electron momentum, \( e \) is the electron charge, \( \mu(r) \) is the plasma chemical potential. We present the chemical potential in linear approximation as

\[ \mu(r) = \mu + \delta\mu(r), \quad \mu = \text{const}. \]

We introduce the nondimensional parameters

\[ t_1 = \nu t, \quad P = \frac{v}{v_T} = \frac{p}{p_T}, \quad v_T = \frac{1}{\sqrt{\beta}}, \]
Let’s rewrite equation (2.1) in the following form
\[
\frac{\partial f}{\partial t_1} + P \frac{\partial f}{\partial r_1} + \frac{e\tau}{mv_T} E(r_1, t_1) = f_{eq}(P, \alpha(r_1)) - f(r_1, P, t_1).
\] (2.2)

Here
\[
f_{eq}(P, \alpha(r_1)) = \frac{1}{1 + \exp(P^2 - \alpha(r_1))}, \quad \alpha(r_1) = \alpha + \delta\alpha.
\]

We find the distribution function in the form
\[
f = f_0(P, \alpha) + g(P, \alpha)h(r_1, P, t_1),
\]
where
\[
f_0(P, \alpha) = \frac{1}{1 + e^{P^2 - \alpha}}, \quad g(P, \alpha) = \frac{e^{P^2 - \alpha}}{(1 + e^{P^2 - \alpha})^2}.
\]

Let’s linearize the equilibrium function \(f_{eq}\):
\[
f_{eq} = f_0(P, \alpha) + g(P, \alpha)\delta\alpha.
\]

In linear approximation we have
\[
E \frac{\partial f}{\partial P} = E \frac{\partial f_0}{\partial P} \frac{1}{P_T} = -E(r_1, t_1)g(P, \alpha)2P \frac{2P}{P_T}.
\]

Let’s substitute (2.3) into (2.2) and then we obtain
\[
\frac{\partial h}{\partial t_1} + P \frac{\partial h}{\partial r_1} + h(r_1, P, t_1) = \frac{2e\tau P}{P_T} E(r_1, t_1) + \delta\alpha.
\] (2.4)

Let’s consider the law of preservation of number of particles
\[
\int (f - f_{eq})d\Omega = 0, \quad d\Omega = \frac{2m^3d^3v}{(2\pi\hbar)^3}.
\]

From this law we obtain that
\[
\delta\alpha = \int g(P, \alpha)h(r_1, P, t_1)d^3P = \frac{1}{2\pi f_0(\alpha)} \int g(P, \alpha)h(r_1, P, t_1)d^3P.
\]

Here
\[
f_0(\alpha) = \int_{0}^{\infty} f_0(P, \alpha)dP.
\]
This equality becomes simpler
\[ \delta \alpha = \frac{1}{2\pi f_0(\alpha)} \int_{-\infty}^{\infty} f_0(\mu', \alpha) h(r_1, \mu', t_1) d\mu'. \]

Let’s consider further that
\[ h(r_1, P, t_1) = h(x_1, P_x, t_1), \quad E(r_1, t_1) = e^{i(k_1 x_1 - \omega_1 t_1)}(1, 0, 0). \]

We notice that \( k_1 x_1 - \omega_1 t_1 = k x - \omega t \), because \( k_1 = kl_T, x_1 = x/l_T, \omega_1 = \omega \tau, t_1 = \nu t = t/\tau. \)

Thus, for function \( h \) the following kinetic equation is received
\[
\frac{\partial h}{\partial t_1} + P_x \frac{\partial h}{\partial x_1} + h(x_1, P_x, t_1) = \frac{2e\tau}{p_T} P_x E_x(x_1, t_1) + \int_{-\infty}^{\infty} K(\mu', \alpha) h(x_1, \mu', t_1) d\mu'.
\]

Here
\[ K(\mu, \alpha) = \frac{f_0(\mu', \alpha)}{2f_0(\alpha)}. \]

For function \( h \) we will search in the form \( h(x_1, \mu, t_1) = \psi(\mu)e^{i(k_1 x_1 - \omega_1 t_1)}. \)

From equation (2.4) we find that
\[ \psi(\mu) = \frac{e^{-i(k_1 x_1 - \omega_1 t_1)} \delta \alpha + (el_T/E_T) \mu}{1 - i\omega_1 + ik_1 \mu}, \quad \mu = P_x. \quad (2.6) \]

Substituting (2.6) in (2.5), we receive:
\[ e^{-i(k_1 x_1 - \omega_1 t_1)} \delta \alpha = \frac{el_T}{E_T} \frac{B_1(k_1, \omega_1)}{1 - B_0(k_1, \omega_1)}. \]

Here
\[
B_1(k_1, \omega_1) = \frac{1}{2f_0(\alpha)} \int_{-\infty}^{\infty} \frac{\mu f_0(\mu, \alpha) d\mu}{1 - i\omega_1 + ik_1 \mu},
\]
\[
B_0(k_1, \omega_1) = \frac{1}{2f_0(\alpha)} \int_{-\infty}^{\infty} \frac{f_0(\mu, \alpha) d\mu}{1 - i\omega_1 + ik_1 \mu}.
\]
It means that according to (2.6) the function $\psi$ is constructed

$$
\psi(\mu) = \frac{e^{lT} B_1/(1 - B_0) + \mu}{e^{-lT} \frac{1}{1 - i\omega\tau + ik_1\mu}}, \quad \mu = P_x.
$$

(2.7)

From definition of density of a current follows, that

$$
\mathbf{j} = \sigma_l e^{i(kr - \omega t)} = e \int \mathbf{v} f d\Omega = e \int v_x e^{i(kr - \omega t)} g(P, \alpha)\psi(\mu) d\Omega.
$$

From here for electrical conductivity we receive

$$
\sigma_l = e \int v_x g(P, \alpha)\psi(\mu) d\Omega.
$$

Let’s substitute (2.7) in this equality and we will receive expression for longitudinal conductivity

$$
\frac{\sigma_l}{\sigma_0} = \frac{1}{2f_2(\alpha)} \int_{-\infty}^{\infty} \frac{\mu^2 + \mu B_1/(1 - B_0)}{1 - i\omega_1 + ik_1\mu} f_0(\mu, \alpha) d\mu = \frac{1}{2f_2(\alpha)} \left[ \int_{-\infty}^{\infty} \frac{\mu^2 f_0(\mu, \alpha) d\mu}{1 - i\omega_1 + ik_1\mu} + \frac{B_1}{1 - B_0} \int_{-\infty}^{\infty} \frac{\mu f_0(\mu, \alpha) d\mu}{1 - i\omega_1 + ik_1\mu} \right],
$$

or

$$
\frac{\sigma_l}{\sigma_0} = \frac{f_0(\alpha)}{f_2(\alpha)} \left[ B_2 + \frac{B_1^2}{1 - B_0} \right].
$$

(2.8)

We notice that

$$
B_1 = \frac{1}{ik_1} - \frac{1 - i\omega\tau}{ik_1} B_0, \quad B_2 = -\frac{1 - i\omega\tau}{ik_1} B_1.
$$

According to (2.8)

$$
\frac{\sigma_l}{\sigma_0} = \frac{f_0(\alpha)}{f_2(\alpha)} \frac{(\omega/kv_T) B_1}{1 - B_0} = -\frac{i}{q^2} \frac{xy f_0(\alpha)}{f_2(\alpha)} \frac{1 + (z/q)b(z/q)}{1 + (iy/q)b(z/q)}.
$$

(2.9)

Here

$$
b(z/q) = \frac{1}{2f_0(\alpha)} \int_{-\infty}^{\infty} \frac{f_0(\mu, \alpha) d\mu}{\mu - z/q}.
$$

On the basis of (2.9) we receive expression for the longitudinal dielectric function of classical non-degenerate plasmas

$$
\varepsilon_l = 1 + \frac{x_p^2 f_0(\alpha)}{q^2 f_2(\alpha)} \frac{1 + (z/q)b(z/q)}{1 + (iy/q)b(z/q)}.
$$

(2.10)
3. Maxwell quantum and classical plasmas

Passing to the limit at \( \alpha \to -\infty \) in the formula (1.10), we receive expression for longitudinal dielectric function of quantum Maxwell collisional plasmas

\[
\varepsilon_l(q, x, y) = 1 - \frac{x_p (x + iy) l_0(q, z) l_0(q, 0)}{q^2 x l_0(q, 0) + iy l_0(q, z)}.
\] (3.1)

In formula (3.1) the designation is accepted

\[
l_0(q, z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{(\mu - z/q)^2 - (q/2)^2}.
\]

Passing to the limit at \( \alpha \to -\infty \) in the formula (2.10), we receive expression for longitudinal dielectric function of classical Maxwell collisional plasmas

\[
\varepsilon_l(q, x, y) = 1 + \frac{2x_p}{q^2} \frac{1 + (z/q)t(z/q)}{1 + (iy/q)t(z/q)}.
\] (3.2)

In formula (3.2) the designation is accepted

\[
t(z/q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{\mu - z/q}.
\]

4. Comparison of quantum and classical plasma

Curves 1 and 2 on figs. 1–8 correspond to quantum and classical plasmas. Dimensionless chemical potential on figs. 1–8 equals to zero: \( \alpha = 0 \). Dimensionless plasma frequency equals to unit: \( x_p = 1 \).
Fig. 1. Real parts of dielectric function, $x_p = 1$, $x = 0.5$, $y = 0.01$.

Fig. 2. Imaginary parts of dielectric function, $x_p = 1$, $x = 0.5$, $y = 0.01$. 
Fig. 3. Real parts of dielectric function, \( x_p = 1, x = 1, y = 0.01 \).

Fig. 4. Imaginary parts of dielectric function, \( x_p = 1, x = 1, y = 0.01 \).
Fig. 5. Real parts of dielectric function, $x_p = 1$, $q = 0.5$, $y = 0.001$.

Fig. 6. Imaginary parts of dielectric function, $x_p = 1$, $q = 0.5$, $y = 0.001$. 
Fig. 7. Real parts of dielectric function, $x_p = 1$, $q = 1$, $y = 0.001$.

Fig. 8. Imaginary parts of dielectric function, $x_p = 1$, $q = 1$, $y = 0.001$. 
5. Conclusion

The formula for dielectric function of non-degenerate and maxwellian collisional plasmas is transformed to the form, convenient for research. Graphic comparison of longitudinal dielectric functions of quantum and classical non-degenerate collisional plasmas is made.

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