Novel Synchronization of Pulse-Coupled Oscillators on Time-Varying Networks

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Abstract

In this article I investigate the novel synchronization behaviors of evolving pulse-coupled oscillator networks. Unlike previous models, the time-varying mechanism is inspired by neural network development, where seldom used links die out while heavily used ones get strengthened. Even small network with all-to-all connected topology can have exotic dynamics under this circumstance. The oscillators can coevolve with the network, and finally form a co-synchronized pair. Or if we only allow a fraction of oscillators to send pulses every time, the oscillators can synchronize with the network remain chaotic.

I. INTRODUCTION

Since the discovery of small-world network [1] and scale-free network [2], the studies of complex networks has seen significant advancements. Meanwhile the fantastic phenomena of synchronization, both for phase and pulse coupled oscillators, has been studied by Kuramoto [3], Strogatz [4] and many others. These two roads of investigation quickly became blended with each other, from where rose the study of synchronization on complex networks. Unlike its predecessors, where the study focused on the complex network without dynamical process or synchronization restricted on all-to-all connected topology, this new realm showed new level of complexity: such as the coexistence of regular and irregular dynamics in one network [5,6] or the prevalence of long chaotic transients [7]. To go a step further, several groups have investigated the synchronization on time-varying networks using the tools from switched system theory [8], consensus theory [9] and graph theory [10]. But there is still not much attention paid to the situation when the change on network topology comes from oscillator dynamics. This is the major difference of the model investigated in this article from previous ones, such as Skufca and Bollt’s moving neighborhood model [11] or Belykh et al.’s blinking model [10]. They mainly focus on the influence of network structure to the oscillator dynamics, while here the influence goes both ways. Another difference worth noting is the use of pulse-coupled oscillators instead of phase-coupled ones, which better mimics the neuron dynamics behind the model, but greatly increase the difficulty of theoretical analysis.

Even at the region of small network with all-to-all connection the model exhibits exotic dynamics—the oscillators and network can undergo coevolution, which leads to the formation of co-synchronized pair. With a little modification, the oscillators can also synchronize while the links breaking and reforming chaotically. When this happens, the system shows some unique behavior that suggests the emergence of a “free energy” like quantity.

Though many interesting synchronization behaviors have already been found for non-identical chaotic oscillators [12]. Our results are for identical oscillators on globally connected networks, so the complexity comes neither from oscillators nor connection—it’s from the interplay between oscillator dynamics and network structure.

II. MODELS

The system is a random network of N oscillators with an average degree . The oscillators in this model agree with Strogatz’s integrate and fire model [4] in spirit. But instead of continuous time I use discrete tick. At every tick the phase will advance according to the equations

\[ \phi_i(0) = random[0, \lambda] \]  
\[ \phi_i(n + 1) = \phi_i(n) + f(\phi) \]  

Until it reaches the threshold and fires, at which tick the phase is reset to 0 immediately. The function satisfies .

Pulses are sent by each firing oscillator to all its neighbors, which will induce a phase jump at next tick. The pulse will also strengthen the link it passes, who comes with an inherent strength evolves according to the equations

\[ S_{ij}(0) = K \]  
\[ S_{ij}(n + 1) = \min\{S_{ij}(n) - 1 + x \cdot \epsilon, K_{max}\} \]  

Here \( x \) is the number of pulses pass through the link at one tick and \( K_{max} \) is the upper bound of the strength. If \( S_{ij}(n) = 0 \), the link will break. A new one will be generated randomly at next tick to keep the average degree constant.

The model was built using Netlogo [13] and is available at the website of Netlogo Community.

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III. Results

In this article I mainly focus on globally connected networks, where the effect of new time-varying mechanism can be demonstrated most clearly. When the parameters are tuned so that no link is breaking, the model gives exactly the same behavior as a homogeneous network of \( N \) all-to-all pulse-coupled oscillators with delayed interactions—they fall into multiple limit cycle attractors with several synchronized clusters [14]. Now if we allow the network to evolve, say, raise the threshold \( \lambda \) or lower the initial strength \( K \), how will the dynamics of oscillator and network influence each other? It turns out we enter the realm of "oscillator-network co-synchronization".

Before I describe this new synchronization behavior in detail, let’s take a side trip to see what’s the effect of link breaking. When a link breaks, a new one is guaranteed in the same place at next tick, thanks to the fully connected topology. The network structure is always preserved, so in this simple case the link breaking can be viewed as perturbation on the oscillators. The perturbation is nonzero only if oscillator \( i \) or \( j \) is firing when the link \( ij \) breaks. Let’s say oscillator \( i \) reaches threshold when link \( ij \) breaks at tick \( n \), then \( \phi_j(n+1) \) will receive a perturbation \( -\delta \) due to the lost pulse from \( i \). Again, this perturbation may or may not influence the synchronization of oscillators.

The "oscillator-network co-synchronization" is due to the determining effect of oscillator dynamics on the network topology. This enables the system to control the perturbation at almost any point of the parameter space. When the perturbation is mild, it can be completely overcome by the synchronization of oscillators. The links will break and reform at early stage, but the network become stabilized as soon as the oscillators find the right limit cycle attractors [Figure 1]. After that the system behaves exactly the same as the case of static network. But do note the synchronization of oscillators is no guarantee to a stable network, as indicated by the two metastable states in Figure 1. Also, the network doesn’t have to be static for oscillators to become synchronized, which we show in the next paragraph.

When the perturbation is strong the network never return to static state, instead, it becomes synchronized together with the oscillators [Figure 2]. This phenomena can be viewed as a co-evolution process. At first, the network evolution disrupt the oscillator synchronization, and the lack of pattern in oscillator firing makes the network evolve chaotically. But as the simulation goes on, these two become more and more compatible with each other, until they finally form a synchronized pair that can coexist. This searching process can be very nontrivial, with the cycle length of co-synchronized state easily reach several hundred ticks. Apart from the intuitive explanation above, we still need a theoretical framework to give us more precise description of how the system converges to that attractor, and how stable it is. There are some nice results for the synchronization of phase-coupled oscillators on time-varying network, such as Jadbabaie et al.’s results for periodically connected networks [15]. But the graph laplacian or Perron matrix used there are not applicable for pulse-coupled oscillators.

Until now, we have assumed the oscillators firing in the same tick can all make their full impact. What if we add the restriction that no neighbor oscillators can send pulse within the same tick? That is, if oscillator \( i \) and \( j \) are connected at tick \( n \) and both of them...
Figure 2: Second example of oscillator-network co-synchronization. (a) shows a whole period of synchronized oscillators and the first half period of network evolution. The yellow nodes represent firing oscillators. Note that though the oscillators are synchronized, the dynamics is more complex than the limit cycle attractors in the static network. Here the synchronized clusters can merge and separate with each other. You can also view this as the mixture of three different but similar common synchronized states (each row for one state). (b) and (c) shows the transition from desynchronized to synchronized state. The red dots correspond to the first period of oscillators and network after transition. There are three details need to be noted. First, the dynamics of oscillators and network just before the transition is very similar to the synchronized state. Second, the synchronization of oscillators comes before the synchronization of network, with a 8-tick delay between two events. Third, one period of the synchronized network is composed of two similar half-cycles, each corresponds to a full cycle of synchronized oscillators. ($N = 6$, $M = 5$, $\delta = 1$, $\epsilon = 1$, $K = 5$, $K_{\text{max}} = 6$, $\lambda = 10$)
reach the threshold, then only one of them can send the pulse. The other will be reset without producing any influence to the rest of the system. When the network is static, the rule gives rise to the same behavior as the original model. This may seems a bit surprising, given that only a fraction of oscillators can contribute to the synchronization every time. But from Stilwell et al.’s results [16] we know synchronization is a very robust phenomena—oscillators can synchronize even if at every tick the network is insufficiently connected to achieve synchronization.

Now if we allow the network to evolve, under appropriate condition the oscillators can still synchronize, but no longer able to control the network. This new type of synchronization answers the question “Can synchronization happen before the network settle down?” with an resounding Yes. Since the oscillators are firing in synchronized clusters but the network dynamics remain chaotic [Figure 3].

More interestingly, in some region of the parameter space, those clusters have a strong tendency to become equal in size. Namely with a network of 6 oscillators you will get $6 = 2 + 2 + 2$ (three clusters with two oscillators each) but never $6 = 1 + 2 + 3$. This reflected in the graph of number of firing oscillators is a sharp convergence on oscillation amplitude [Figure 3a]. Recall the identification of link breaking as perturbation at the beginning of this section, it is natural to assume there is a “free energy” like quantity out there, which measures the “unevenness” between synchronized clusters. From this point of view, the dominating limit cycle attractors on static networks are just local minimums on the whole landscape. Once there is appropriate perturbation, the system will undergoes phase transitions and find the ground state like a protein folds to its native structure. Though the links are breaking in an unpredictable manner, so long as the perturbation stays moderate, the system will not leave the ground state and synchronization will persists. When the perturbation do gets strong that even ground state becomes unstable, we can observe spontaneous phase transitions caused by system switching among different basins. For example, the system can stay synchronized at $9 = 3 + 3 + 3$ for some time, then suddenly switch to $9 = 2 + 3 + 4$. Another more common transition happens at the same size division (e.g. within $9 = 2 + 3 + 4$), where the clusters can exchange oscillators with each other. But why does “even division” has the lowest free energy? What can be used to quantify perturbation? These are questions for future inquiry.

There is also a state of “fuzzy synchronization” when the network is not globally connected. There the system is divided into a stable, synchronized core, and a few free, unsynchronized oscillators. The free ones keep attaching and detaching from the core. This is somewhat like the situation observed by Zumdieck and his colleagues in randomly diluted networks of pulse-coupled oscillators [7].

![Figure 3: An example of synchronization on chaotic network. (a) shows the quick and steady convergence towards even division of synchronized clusters. It should be noted that the oscillators are also synced in the early state of uneven division. The lack of pattern at (b) indicates the chaotic dynamics of network structure. ($N = 6$, $M = 5$, $\delta = 1$, $\epsilon = 1$, $K = 5$, $K_{\text{max}} = 8$, $\lambda = 8$)](image)

### IV. Discussion

In this article I demonstrated some novel synchronization properties of the pulse-coupled oscillators on globally connected network with breakable links. The model itself is very rich and can do much more than that. For example, it can be used to investigate the influence of complex network’s statistical properties on synchronization. What’s the difference on synchronization transition time with small-world network and scale-free network as initial topology? Random network and clustered network, which is easier to settle down in this model? These are all questions waiting to be answered. Also, will the model show any self-
organization behavior? It has been shown that the brain networks possess lots of universal structures, such as short path length, high clustering, modular community structure, etc. Can the interplay between oscillator dynamics and network topology in our model produce some of these structures under suitable condition? This relies heavily on how the new links are generated, and further investigation on how the modularity function, clustering coefficient and degree distribution change along the course of simulation is also essential.

To get a bit ahead of time, there are some possible extensions of the model that may prove fruitful. What if the strength decay course is nonlinear, will the networks become easier to synchronize just as oscillators do? What if the system is expandable, how will this impact the self-organization and synchronization? We can also introduce recovery period, use directed links, etc.

The picture provided in this paper is still somehow incomplete, more systematic results about the behavior of the system on parameter space and theoretical analysis of the attractor stability as well rigorous argument about “free energy” will be included in the next update.

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References

[1] D. J. Watts and S. H. Strogatz, “Collective dynamics of ‘small-world’networks,” nature, vol. 393, no. 6684, pp. 440–442, 1998.

[2] A.-L. Barabási and R. Albert, “Emergence of scaling in random networks,” science, vol. 286, no. 5439, pp. 509–512, 1999.

[3] S. H. Strogatz, “From kuramoto to crawford: exploring the onset of synchronization in populations of coupled oscillators,” Physica D: Nonlinear Phenomena, vol. 143, no. 1, pp. 1–20, 2000.

[4] R. E. Mirollo and S. H. Strogatz, “Synchronization of pulse-coupled biological oscillators,” SIAM Journal on Applied Mathematics, vol. 50, no. 6, pp. 1645–1662, 1990.

[5] M. Timme, F. Wolf, and T. Geisel, “Coexistence of regular and irregular dynamics in complex networks of pulse-coupled oscillators,” Physical review letters, vol. 89, no. 25, p. 258701, 2002.

[6] D. M. Abrams and S. H. Strogatz, “Chimera states for coupled oscillators,” Physical review letters, vol. 93, no. 17, p. 174102, 2004.

[7] A. Zumdieck, M. Timme, T. Geisel, and F. Wolf, “Long chaotic transients in complex networks,” Physical review letters, vol. 93, no. 24, p. 244103, 2004.

[8] J. Zhao, D. J. Hill, and T. Liu, “Synchronization of complex dynamical networks with switching topology: a switched system point of view,” Automatica, vol. 45, no. 11, pp. 2502–2511, 2009.

[9] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and cooperation in networked multi-agent systems,” Proceedings of the IEEE, vol. 95, no. 1, pp. 215–233, 2007.

[10] I. V. Belykh, V. N. Belykh, and M. Hasler, “Blinking model and synchronization in small-world networks with a time-varying coupling,” Physica D: Nonlinear Phenomena, vol. 195, no. 1, pp. 188–206, 2004.

[11] J. D. Skufca and E. M. Bollt, “Communication and synchronization in disconnected networks with dynamic topology: Moving neighborhood networks,” arXiv preprint nlin/0307010, 2003.

[12] S. Boccaletti, J. Kurths, G. Osipov, D. Valladares, and C. Zhou, “The synchronization of chaotic systems,” Physics Reports, vol. 366, no. 1, pp. 1–101, 2002.

[13] U. Wilensky, “{NetLogo},” 1999.

[14] U. Ernst, K. Pawelzik, and T. Geisel, “Synchronization induced by temporal delays in pulse-coupled oscillators,” Physical Review Letters, vol. 74, no. 9, p. 1570, 1995.

[15] A. Jadabaie, J. Lin, and A. S. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” Automatic Control, IEEE Transactions on, vol. 48, no. 6, pp. 988–1001, 2003.

[16] D. J. Stilwell, E. M. Bollt, and D. G. Roberson, “Sufficient conditions for fast switching synchronization in time-varying network topologies,” SIAM Journal on Applied Dynamical Systems, vol. 5, no. 1, pp. 140–156, 2006.

[17] E. Bullmore and O. Sporns, “Complex brain networks: graph theoretical analysis of structural and functional systems,” Nature Reviews Neuroscience, vol. 10, no. 3, pp. 186–198, 2009.