Resonance-broadened Transit Time Damping of Particles in MHD Turbulence

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Abstract

As a fundamental astrophysical process, the scattering of particles by turbulent magnetic fields has its physical foundation laid by the magnetohydrodynamic (MHD) turbulence theory. In the framework of the modern theory of MHD turbulence, we derive a generalized broadened resonance function by taking into account both the magnetic fluctuations and nonlinear decorrelation of turbulent magnetic fields arising in MHD turbulence, and we specify the energy range of particles for the dominance of different broadening mechanisms. The broadened resonance allows for scattering of particles beyond the energy threshold of the linear resonance. By analytically determining the pitch-angle diffusion coefficients for transit time damping (TTD) with slow and fast modes, we demonstrate that the turbulence anisotropy of slow modes suppresses their scattering efficiency. Furthermore, we quantify the dependence of the relative importance between slow and fast modes in TTD scattering on (i) particle energy, (ii) plasma $\beta$ (the ratio of gas pressure to magnetic pressure), and (iii) damping of MHD turbulence, and we also provide the parameter space for the dominance of slow modes. To exemplify its applications, we find that among typical partially ionized interstellar phases, in the warm neutral medium slow and fast modes have comparable efficiencies in TTD scattering of cosmic rays. For low-energy particles, e.g., sub-Alfvénic charged grains, we show that slow modes always dominate TTD scattering.

Key words: cosmic rays – magnetic fields – turbulence

1. Introduction

The scattering and acceleration of particles are the central ingredients of space physics and astrophysics (Longair 1997). They play crucial roles in many phenomena in the heliosphere, the interstellar medium (ISM), and the intracluster medium (Schlickeiser 2002; Brunetti & Lazarian 2011). The scattering of particles is usually described by the pitch-angle diffusion coefficient, which is directly related to the parallel spatial diffusion and acceleration of particles.

The pitch-angle diffusion coefficient is conventionally calculated using the quasi-linear theory (QLT; Jokipii 1966), under the assumption of unperturbed orbits of particles in weakly turbulent plasmas. As a result, earlier studies of particle transport and acceleration were restricted to the case of the slab model of turbulence (Schlickeiser 1994). However, the slab model of turbulence is not supported by 3D magnetohydrodynamic (MHD) simulations (Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et al. 2002; Cho & Lazarian 2003; Kowal et al. 2012). These simulations instead support the physically motivated turbulence model developed by Goldreich & Sridhar (1995, hereafter GS95). The GS95 model provides a self-consistent interpretation of the anisotropic scaling in the frame of the local magnetic field (Lazarian & Vishniac 1999; Cho & Vishniac 2000) and the nonlinear dynamics of MHD turbulence.

The theoretical understanding of MHD turbulence based on the GS95 model significantly advances the study of astroparticle physics. By using the scalings of compressible MHD turbulence numerically derived by Cho et al. (2002) and Cho & Lazarian (2003), Yan & Lazarian (2002, 2004) demonstrated that Alfvén modes are inefficient in scattering cosmic rays (CRs) owing to their scale-dependent anisotropy (see also Chandran 2000), while fast modes dominate the CR scattering in the diffuse ISM. In addition, pioneering works on the acceleration of dust grains in MHD turbulence have been carried out by Lazarian & Yan (2002), Yan & Lazarian (2003), Yan et al. (2004), and Hoang et al. (2012), where MHD turbulence was identified as an important agent of grain acceleration.

The turbulence nature of magnetic fields entails the broadening of quasi-linear resonances. Attempts to modify the QLT to incorporate nonlinear effects are numerous (e.g., Jones et al. 1973; Völk 1973; Goldstein 1976; Shalchi et al. 2004). Yan & Lazarian (2008, hereafter YL08) first performed the nonlinear extension of QLT in the framework of a realistic and numerically tested description of MHD turbulence. They found that the broadening effect due to the perturbations of particle orbits is essential for calculating the scattering and mean free paths of CRs.

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In addition, the role of nonlinear dynamics of MHD turbulence in resonance broadening has been investigated by Chandran (2000) and Lynn et al. (2012, 2014). By considering the broadening effect due to the decorrelation of turbulent magnetic fields (Lynn et al. 2012), Lynn et al. (2014) argued for the dominance of slow modes of MHD turbulence in particle acceleration.

To identify the physical regimes for the dominance of different broadening effects, a generalized broadened resonance function should be constructed incorporating both the geometric and dynamical properties of MHD turbulence. In this work, we adopt the up-to-date model of MHD turbulence and focus on the pitch-angle scattering of particles, including both the transit time damping (TTD) and gyroresonance interactions (Schlickeiser 2002). To fully capture the physics of resonance broadening and have a general applicability of the study, we formulate the pitch-angle diffusion coefficients with a broadened resonance function applicable to a wide range of particle energies, turbulence conditions, and plasma parameters.

In this paper, Section 2 contains first the analysis on the resonance broadening and then calculations of the pitch-angle diffusion coefficients for TTD with slow and fast modes, respectively. Their relative importance in different physical conditions is also examined. In Section 3, we present the resonance-broadened gyroresonance with Alfvén, slow, and fast modes. Final conclusions follow in Section 4.

2. Resonance-broadened TTD

As the magnetic analog of Landau damping (Summers & Ma 2000), TTD is the resonant interaction of particles with compressive waves. The resulting scattering process can be described with the pitch-angle diffusion coefficient (Kulsrud & Pearce 1969; Voelk 1975),

$$D_{\mu \nu, T} = C_{\mu} \int d^3k \frac{k^2}{k^2} [J_0'(x)]^2 I(k) R(k),$$

where

$$C_{\mu} = (1 - \mu^2) \frac{\Omega^2}{B_0^2},$$

$$x = \frac{k_0 v_0}{\Omega} = \frac{k_0}{r_g},$$

with the gyrofrequency $\Omega = |q| B_0/\gamma mc$ for relativistic particles and $\Omega = |q| B_0/mc$ for nonrelativistic particles, the mean magnetic field $B_0$, the gyroradius $r_g = v_f/\Omega$, the electric charge $q$, the mass $m$, the Lorentz factor $\gamma$ of the particle, the light speed $c$, the pitch-angle cosine $\mu = v_1/v$, particle speed $v$ and its parallel and perpendicular components $v_0, v_1$, with respect to the magnetic field, and wavenumber $k$ and its parallel and perpendicular components $k_0$ and $k_\perp$. In addition, $J_0'(x)$ is the derivative of the Bessel function, $I(k)$ is the power spectrum of magnetic field fluctuations, and $R(k)$ is the resonance function.

2.1. Resonance Broadening

The standard QLT was formulated under the condition of sufficiently small turbulence amplitude (Pines & Bohm 1952; Jokipii 1966). In such weak plasma turbulence, the assumption of unperturbed particle orbits is valid.

Regarding the TTD resonance, QLT requires that $v_1$ should be matched to the wave parallel phase speed $\omega/k_0$ for the particle to resonate with the wave, where $\omega$ is the wave frequency. This resonance condition corresponds to a linear resonance function

$$R_k = \pi \delta(\omega_k - v_1 k_0),$$

resulting in a singular diffusion process and inefficient particle scattering. It entails a threshold particle speed for resonance,

$$v \geq V_{ph},$$

where $V_{ph}$ is the phase speed of the wave. It implies that the particles with the speed below the threshold speed cannot undergo the TTD resonance.

However, the restricted condition of the QLT cannot be met in most astrophysical environments, where the strong MHD turbulence is present. We note that the concept of strong turbulence discussed here is different from the strong plasma turbulence with the particle kinetic energy comparable to the wave energy. The strong MHD turbulence results from the nonlinear interaction of Alfvén waves. It is characterized by the critical balance between the turbulent eddy turnover time in the direction perpendicular to the local magnetic field and the Alfvén wave-crossing time in the direction parallel to the local magnetic field on every length scale within the inertial range of turbulence (Goldreich & Sridhar 1995; Lazarian & Vishniac 1999). As a consequence of the critical balance, (i) Alfvén turbulence has a scale-dependent anisotropy, with more elongated turbulent eddies toward smaller scales (Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et al. 2002; Cho & Lazarian 2003), and (ii) the turbulent eddy (or the wave packet) has its lifetime given by the eddy turnover time (or the Alfvén wave-crossing time).

Pseudo-Alfvén modes in incompressible MHD turbulence and slow modes in compressible MHD turbulence have their dynamics governed by Alfvén modes and have the same anisotropic scaling as Alfvén modes (Lithwick & Goldreich 2001; Cho & Lazarian 2005). By contrast, fast modes in compressible MHD turbulence have isotropic scaling and independent energy cascade with a much slower cascading rate compared to other modes (Cho & Lazarian 2002).

Due to the parallel magnetic perturbations induced by such pseudo-Alfvén modes or compressible fast and slow modes, particles moving along the fluctuating magnetic field have a variation in $v_1$ (Voelk 1975),

$$\Delta v_1 \approx v_1 \left( \frac{\delta B_1^2}{B_0^2} \right)^{1/4},$$

where $\delta B_1$ represents the parallel magnetic perturbation. It arises as a result of the conservation of the magnetic moment $J_1 \propto v_1^2/B$ when the magnetic field variation over a gyroperiod is insignificant. The existence of $\Delta v_1$ gives rise to the deviation from the unperturbed orbit and thus the broadening of the wave–particle resonance (YL08).

Another source of broadening is the dynamical decorrelation of MHD turbulence (Lynn et al. 2012). It leads to the deviation from the linearized particle trajectory in the direction perpendicular to the local magnetic field with the characteristic timescale as the turbulent cascading time $t_{\text{lat}}^{-1}$. 

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By taking into account both broadening effects as discussed above, we have a general form of the broadened resonance function

\[ R_B = \text{Re} \int_0^\infty dt \exp \left[ i(\omega - v_{|k|})t - \frac{1}{2} (k \cdot \delta \mathbf{r}(t))^2 \right] \]

\[ = \text{Re} \int_0^\infty dt \exp \left[ i(\omega - v_{|k|})t - \frac{1}{2} (k \cdot \Delta \mathbf{v}) + \omega_{\text{tur}} \right] t^2 \]

\[ = \frac{\sqrt{2\pi}}{2(\Delta \mathbf{v} |k| + \omega_{\text{tur}})} \exp \left[ - \frac{(\omega - v_{|k|})^2}{2(\Delta \mathbf{v} |k| + \omega_{\text{tur}})^2} \right]. \]  

The two terms \( \Delta \mathbf{v} |k| \) and \( \omega_{\text{tur}} \) come from the broadening effects due to magnetic perturbations and the dynamical decorrelation of MHD turbulence, respectively. We can easily see that high-energy particles with \( v \gg V_{ph} \) mainly sample magnetic fluctuations when propagating through the quasi-static magnetic field. By contrast, low-energy particles with \( v \ll V_{ph} \) can sample uncorrelated magnetic fields when being quasi-static relative to turbulent magnetic fields. More specifically, when

\[ \Delta \mathbf{v} |k| \gg \omega_{\text{tur}}, \]  

the broadening is primarily caused by the propagation of particles and magnetic perturbations. Accordingly, the general expression of \( R_B \) reduces to

\[ R_{B1} = \frac{\sqrt{2\pi}}{2 \omega_{\text{tur}}} \exp \left[ - \frac{(\omega - v_{|k|})^2}{2 \omega_{\text{tur}}^2} \right]. \]  

A similar form of resonance function was introduced in YL08 for studying the scattering of CRs. In the opposite limit of

\[ \Delta \mathbf{v} |k| \ll \omega_{\text{tur}}, \]  

the broadening is dominated by the decorrelation of turbulent magnetic fields, and \( R_B \) approximately becomes

\[ R_{B2} = \frac{\sqrt{2\pi}}{2 \omega_{\text{tur}}} \exp \left[ - \frac{(\omega - v_{|k|})^2}{2 \omega_{\text{tur}}^2} \right]. \]  

A similar Gaussian resonance function involving wave decoherence was earlier proposed by Lynn et al. (2012). We will next clarify the relative importance between different broadening effects and the applicability of \( R_{B1} \) and \( R_{B2} \) in different regimes.

2.2. Pitch-angle Diffusion Coefficients

Compressible MHD turbulence can be decomposed into Alfvén, slow, and fast modes (Cho & Lazarian 2003). Both compressible slow and fast modes contribute to the TTD scattering. The corresponding \( D_{\mu, \nu, T} \) has two components associated with slow and fast modes, respectively,

\[ D_{\mu, T} = D_{\mu, s, T} + D_{\mu, f, T}. \]  

In what follows, we separately calculate \( D_{\mu, s, T} \) and \( D_{\mu, f, T} \) with resonance broadening and then compare their relative importance under different conditions.

2.2.1. Slow Modes

As mentioned above, slow modes have the same anisotropic scaling as Alfvén modes, which is reflected in their magnetic energy spectrum (Cho et al. 2002),

\[ L_s(k) = C_s k_{\perp}^{\beta_s} \exp \left[ - \frac{L_s k_{\perp}}{k_{\perp}^{\beta_s}} \right], \]  

where \( L \) is the injection scale of strong MHD turbulence. The normalization factor is

\[ C_s = \frac{1}{6\pi} \delta B_s L^{-\frac{7}{2}}, \]  

and thus

\[ \frac{\delta B_s}{B_0} \sim \frac{V_{La}}{V_A} \]  

in the high-\( \beta \) case and

\[ \frac{\delta B_s}{B_0} \sim \frac{c_s V_{La}}{V_A} \]  

in the low-\( \beta \) case. Here \( V_{La} \) is the turbulent speed of slow modes at \( L \). It shows that the magnetic fluctuations of slow modes in a high-\( \beta \) medium are much larger than those in a low-\( \beta \) medium.

On the other hand, the wave-like motions of slow modes propagate along the magnetic field, so there is

\[ \omega_{s} = V_{ph} k_{||}, \]  

where \( V_{ph} \approx V_A \) in the high-\( \beta \) case and \( V_{ph} \approx c_s \) in the low-\( \beta \) case. In what follows we analytically calculate the pitch-angle diffusion coefficients of slow modes with the linear and broadened resonance functions.

(1) Linear resonance: By inserting Equations (4) and (13) into Equation (1), we can derive the pitch-angle diffusion coefficient of slow modes by using the discrete resonance of the QLT,

\[ D_{\mu, s, L, T} = \frac{\pi^2}{2} \frac{C_\mu C_s V_{ph}^2}{\Omega^2} \delta (V_{ph} - v_{|}) \times \int \frac{k_{\perp}^{max}}{k_{\perp}^{min}} \frac{k_{||}^{max}}{k_{||}^{min}} k_{||} \exp \left[ - \frac{L S k_{\perp}^{2}}{1 - k_{||}^{2}} \right] dk_{||} dk_{\perp} \]

\[ = \frac{\pi^2}{2} \frac{C_\mu C_s L^{-\frac{7}{4}}}{B_0^2} \ln \left( \frac{L}{L_{||, \text{min}}} \right) v^2 (1 - \mu^2)^{3} \delta (v_{||} - V_{ph}). \]

Since particles are insensitive to the averaged magnetic fields on length scales below \( r_g \), we only consider the magnetic fluctuations on scales larger than \( r_g \) and thus \( x < 1 \) as a
simplifying assumption. At $x < 1$, there is

$$[J_0'(x)]^2 = [J_0(x)]^2 \sim \frac{x^2}{4}. \quad (20)$$

Correspondingly, the upper limit of integration in Equation (1) is determined by the larger value between the dissipation scale $l_d$ of magnetic fluctuations and $r_e$, i.e., $l_{i,\text{min}} = 1/k_{i,\text{max}} = \max[l_d, r_e]$. In addition, since TTD is dominated by the interaction of particles with the magnetic fluctuations at $l_{i,\text{min}}$, where the turbulence anisotropy is prominent with $k_i \gg k_{i,\text{min}}$, we assume $k^2 = k_{i,\text{min}}^2 + k_{\perp}^2 \approx k_{i,\text{min}}^2$ when calculating the above integral.

The above result clearly shows that only the particles exactly matching the linear resonance condition can be scattered. (2) Broadened resonance: With the use of $R_\phi$ in Equation (7), the resonance-broadened diffusion coefficient can be expressed as

$$D_{\mu,\text{res,}T} = \frac{\sqrt{2\pi}}{8} \frac{C_i}{\Omega_c} \frac{V_A^2}{\Omega_e^2} (\Delta v_\parallel + V_A)^{-1} \exp \left[-\frac{(V_{ph} - v_j)^2}{2(\Delta v_\parallel + V_A)^2}\right] \times \int d^3k k_{i,\text{min}}^2 + k_{\perp}^2 \exp \left[-\frac{L^2 k_{i,\text{min}}^2}{k_{\perp}^2}\right]. \quad (21)$$

Here we adopt

$$\omega_{\text{turb,}s} = V_A k_{i,\text{min}} \quad (22)$$

based on the critical balance of strong MHD turbulence. In Appendix A, we also provide the analysis with $\omega_{\text{turb,}s}$ given by the eddy turnover rate. We further derive its approximate analytical result by using the same simplifying assumptions for deriving Equation (19),

$$D_{\mu,\text{res,}T} \approx \frac{\sqrt{2\pi}}{4} \frac{C_i}{B_0} \frac{V_A^2}{\Omega_e^2} L^{-\frac{3}{2}} \ln \left(\frac{L}{l_{i,\text{min}}}\right) v^2 \times (1 - \mu^2)^2 \exp \left[-\frac{(V_{ph} - v_j)^2}{2(\Delta v_\parallel + V_A)^2}\right]. \quad (23)$$

By comparing with $D_{\mu,\text{res,}T}$ in Equation (19), we see that the delta function of $v_j$ in $D_{\mu,\text{res,}T}$ is replaced by a Gaussian function with the variance $(\Delta v_\parallel + V_A)^2$. With the broadened resonance, particles with a broad range of $v$ including $v < V_{ph}$ can be effectively scattered.

$\Delta v_\parallel$ and $V_A$ in $D_{\mu,\text{res,}T}$ correspond to the two broadening effects as discussed in Section 2.1. For high-energy particles, $\Delta v_\parallel(v_{ph}, \text{Equation (6)})$ tends to be much larger than $V_A$. The broadening due to magnetic perturbations is dominant, and the general form of $D_{\mu,\text{res,}T}$ in Equation (23) can be simplified as

$$D_{\mu,\text{res,}T} \approx \frac{\sqrt{2\pi}}{4} \frac{C_i}{B_0} \left(\frac{\delta B_0^2}{B_0}\right)^{\frac{3}{2}} \frac{L^{-\frac{3}{2}} \ln \left(\frac{L}{l_{i,\text{min}}}\right) v^2}{\Omega_c^2} \times (1 - \mu^2)^2 \exp \left[-\frac{(V_{ph} - v_j)^2}{2(\Delta v_\parallel + V_A)^2}\right]. \quad (24)$$

where Equation (6) is inserted. Since slow modes have $V_{ph} \ll V_A$, for high-energy particles the above equation can be further simplified as

$$D_{\mu,\text{res,}T} \approx \frac{\sqrt{2\pi}}{4} \frac{C_i}{B_0} \left(\frac{\delta B_0^2}{B_0}\right)^{\frac{3}{2}} \frac{L^{-\frac{3}{2}} \ln \left(\frac{L}{l_{i,\text{min}}}\right) v^2}{\Omega_c^2} \times (1 - \mu^2)^2 \exp \left[-\frac{(V_{ph} - v_j)^2}{2(\Delta v_\parallel + V_A)^2}\right]. \quad (25)$$

Hence, $D_{\mu,\text{res,}T}$ is approximant constant at a small $\mu$ and significantly drops when $\mu$ approaches 1.

For sufficiently low energy particles with $\Delta v_\parallel \ll V_A$, the broadening due to turbulent decorrelation is dominant. In this case $D_{\mu,\text{res,}T}$ can be approximated by

$$D_{\mu,\text{res,}T} \approx \frac{\sqrt{2\pi}}{4} \frac{C_i}{B_0} \frac{V_A^2}{\Omega_e^2} L^{-\frac{3}{2}} \ln \left(\frac{L}{l_{i,\text{min}}}\right) v^2 \times (1 - \mu^2)^2 \exp \left[-\frac{(V_{ph} - v_j)^2}{2V_A^2}\right], \quad (27)$$

which can be further cast as

$$D_{\mu,\text{res,}T} \approx \frac{\sqrt{2\pi}}{4} \frac{C_i}{B_0} \frac{V_A^2}{\Omega_e^2} L^{-\frac{3}{2}} \ln \left(\frac{L}{l_{i,\text{min}}}\right) v^2 \times (1 - \mu^2)^2 \exp \left[-\frac{V_{ph}^2}{2V_A^2}\right]. \quad (28)$$

when $v_\parallel \ll V_{ph}$. With $V_{ph} \ll V_A$ for slow modes, the exponential term is of the order of unity.

The above analysis shows that owing to the broadened resonance, both high- and low-energy particles can be effectively scattered at pitch angles away from $\mu = 1$.

### 2.2.2. Fast Modes

Fast modes are weakly coupled with Alfvén and slow modes. They have an isotropic scaling and an energy spectrum (Cho & Lazarian 2002)

$$I_j(k) = C_j k^{-2}. \quad (29)$$

The normalization constant is

$$C_j = \frac{1}{16\pi} \delta B_0^2 L^{-\frac{3}{2}}, \quad (30)$$
so that there is
\[
\frac{\delta B_f^2}{2} = \int d^3k J_f(k). \tag{31}
\]
The rms strength of magnetic fluctuations of fast modes also depends on the plasma $\beta$. It is approximately given by (Cho & Lazarian 2003)
\[
\frac{\delta B_f}{B_0} \sim \frac{V_{lf}}{c_s} \tag{32}
\]
in a high-$\beta$ medium and
\[
\frac{\delta B_f}{B_0} \sim \frac{V_{lf}}{V_A} \tag{33}
\]
in a low-$\beta$ medium, where $V_{lf}$ is the turbulent speed of fast modes at $L$. By comparing with $\delta B_z$ in Equations (16) and (17), we find that, provided that $V_{ls} \sim V_{lf}$, $\delta B_f$ is much smaller than $\delta B_z$ in a high-$\beta$ medium, but much larger than $\delta B_z$ in a low-$\beta$ medium.

In addition, fast modes have the dispersion relation as
\[
\omega_k = V_{ph} k, \tag{34}
\]
where $V_{ph} \approx c_s$ in a high-$\beta$ medium and $V_{ph} \approx V_A$ in a low-$\beta$ medium. Next, we will investigate the pitch-angle diffusion coefficients of fast modes and the effect of resonance broadening on particle scattering.

(1) Linear resonance: The diffusion coefficient with the linear resonance function is (Equations (1), (4), and (29))
\[
D_{\mu,fi,T} = \pi^2 C_p C_f \frac{V_{ph}^2}{\Omega^2} \frac{1}{|v||} \int \int_{L} \int_{k} k^{-\frac{3}{2}} \int_{0}^{k_{max}} \eta^2 (1 - \eta^2) \delta \left( \eta - \frac{V_{ph}}{v||} \right) d\eta dk, \tag{35}
\]
where $\eta = \cos \theta$ and $\theta$ is the angle between $k$ and $B$. We again use the approximation in Equation (20) and obtain the analytical expression
\[
D_{\mu,fi,T} = 2H^* \pi^2 C_p C_f \left( l^{\frac{1}{2}}_{\min} - l^{-\frac{1}{2}} \right) \frac{V_{ph}^2}{\Omega^2} \frac{1}{2} \left( \frac{V_{ph}}{v||} \right)^2 \left( 1 - \frac{V_{ph}}{v||} \right)^2, \tag{36}
\]
where $l_{min} = 1/k_{max} = \max[l_{\mu}, r_g]$, and
\[
H^* = H \begin{pmatrix} 1 - \frac{V_{ph}}{v||} \end{pmatrix} \begin{pmatrix} \frac{V_{ph}}{v||} \end{pmatrix} = 0, \quad v|| < V_{ph}, \tag{37a}
\]
\[
H^* = H \begin{pmatrix} 1 - \frac{V_{ph}}{v||} \end{pmatrix} \begin{pmatrix} \frac{V_{ph}}{v||} \end{pmatrix} = \frac{1}{2}, \quad v|| = V_{ph}, \tag{37b}
\]
\[
H^* = H \begin{pmatrix} 1 - \frac{V_{ph}}{v||} \end{pmatrix} \begin{pmatrix} \frac{V_{ph}}{v||} \end{pmatrix} = 1, \quad v|| > V_{ph}. \tag{37c}
\]
with the Heaviside step function $H$. Different from $D_{\mu,fi,L,T}$ (Equation (19)), $D_{\mu,fi,T}$ does not contain a delta function. The particles satisfying $v|| \gg V_{ph}$ can all be scattered. When $v|| \gg V_{ph}$, it scales with $\mu$ as
\[
D_{\mu,fi,T} \propto \frac{(1 - \mu^2)^2}{\mu^2}. \tag{38}
\]
$D_{\mu,fi,T}$ decreases with $\mu$ as $\mu^{-3}$ at a relatively small $\mu$. Moreover, different from $D_{\mu,fi,L,T} \propto \ln(L/l_{\min})$, $D_{\mu,fi,T}$ depends on $(L/l_{\min})^{1/2}$.

(2) Broadened resonance: The turbulent cascade of fast modes is slow, with the cascading rate much smaller than that of Alfvén and slow modes (Cho & Lazarian 2002),
\[
\omega_{ur,t} = \frac{v_k}{V_{ph}} (kv_k) = \frac{V_{lf}^2}{V_{ph}} L^{-\frac{1}{2}} k^\frac{1}{2}. \tag{39}
\]
We first consider the case of high-energy particles. As there generally exists $\omega_{ur,s} > \omega_{ur,t}$, if there is
\[
k||/\Delta v|| \gg \omega_{ur,s} \tag{40}
\]
for slow modes, then the condition for the dominance of the broadening due to magnetic perturbations is naturally satisfied for fast modes. The corresponding diffusion coefficient is (Equations (1), (9), and (29))
\[
D_{\mu,fi,1,T} \approx \frac{\sqrt{2}}{4} \pi^2 C_p C_f \frac{v_{ph}^2}{\Omega^2} \frac{1}{2} \int_0^{k_{max}} k^{-\frac{3}{2}} \int_0^{k} \int_0^{l_{max}} \int \int k^{-\frac{1}{2}} \cos \theta \sin^3 \theta \exp \left[ -\frac{(kv_{ph} - k||v||)^2}{2k||^2/\Delta v||^2} \right]. \tag{41}
\]
For high-energy particles with $v|| \gg V_{ph}$, from the above equation we can approximately derive
\[
D_{\mu,fi,1,T} \approx \frac{\sqrt{2}}{4} \pi^2 C_p C_f \frac{v_{ph}^2}{\Omega^2} \frac{1}{2} \exp \left[ -\frac{v_{ph}^2}{2\Delta v||^2} \right] \times \int_{L_{\min}}^{L_{\max}} k^{-\frac{3}{2}} k^{-\frac{1}{2}} \int_0^{k_{max}} \int_0^{l_{max}} \int \int \frac{\delta B_f^2}{B_0} \left( \frac{\delta B_f^2}{B_0} \right) \int \left( l_{\min}^{-\frac{1}{2}} - L^{-\frac{1}{2}} \right) \left( 1 - \mu^2 \right)^2 \frac{v_{ph}^2}{2\Delta v||^2}, \tag{42}
\]
where $\zeta = \sin \theta$. By comparing with $D_{\mu,fi,T}$ in Equation (36), which is proportional to $(V_{ph}/v||)^2$ and thus has a small value when $v|| \gg V_{ph}$, $D_{\mu,fi,1,T}$ depends on $v_{ph}$ because of the broadening effect, and consequently high-energy particles can be much more effectively scattered.

Compared to $D_{\mu,fi,l,T}$ in Equation (25), we see that, similar to the case with the linear resonance, $D_{\mu,fi,1,T}$ depends on $(L/l_{\min})^{1/2}$, while $D_{\mu,fi,1,T}$ depends on $\ln(L/l_{\min})$. This difference is not related to the broadening effect. The logarithmic dependence on $l_{\min}$ of $D_{\mu,fi,L,T}$ and $D_{\mu,fi,1,T}$ comes from the anisotropy of slow modes. The energy distribution is increasingly anisotropic toward small scales. With insignificant magnetic fluctuations parallel to the magnetic field on small scales, TTD with slow modes tends to be less effective compared to that with fast modes. As a test, in Appendix B we adopt an isotropic spectrum but with the same spectral index as that of slow modes to calculate the diffusion.
coefficient of fast modes, it again shows a power-law dependence on $l_{\text{min}}$, but with a different scaling due to the different spectral index used.

For sufficiently low energy particles, there can be (Equation 10))

$$k \Delta v || \ll \omega_{\text{tur},f}. \quad (43)$$

Then, the broadening is dominated by the turbulent decorrelation, and the diffusion coefficient is (Equations (1), (11), and (29))

$$D_{\mu\mu,2,T} \approx \frac{\sqrt{2}}{4} \pi^2 C_p C_f \frac{V_{\text{ph}}}{V_{\text{Lj}}} \frac{V_{\perp}^2}{\Omega^2} \times \int \cos^2 \theta \sin^3 \theta \exp \left[ -\frac{(k V_{\text{ph}} - k \Delta v ||)^2}{2 V_{\text{ph}}^2 V_{\perp}^2 L^{-1} k} \right]. \quad (44)$$

We consider $v || \ll V_{\text{ph}}$ and approximately have

$$D_{\mu\mu,2,T} \approx \frac{\sqrt{2}}{2} \pi^2 C_p C_f \frac{V_{\text{ph}}}{V_{\text{Lj}}} L^2 \frac{V_{\perp}^2}{\Omega^2} \times \int_{k_{\text{min}}}^{k_{\text{max}}} \int_0^1 \eta^2 (1 - \eta^2) d\eta \exp \left[ -\frac{V_{\text{ph}}^4}{2 V_{\text{Lj}}^2 L^{-1} k} \right] dk$$

$$\approx \frac{2 \sqrt{2}}{15} \pi^2 C_p \frac{V_{\perp}^2}{B_0^2 V_{\text{ph}}} L^{-1} \frac{V_{\perp}^2}{\Omega^2} (1 - \mu^2)^2 \exp \left[ -\frac{V_{\text{ph}}^4}{2 V_{\text{Lj}}^2} \right]. \quad (45)$$

As the main contribution to the integral comes from $k = 1/L$, the consequent $D_{\mu\mu,2,T}$ does not depend on $l_{\text{min}}$.

Due to the particularly slow cascade of fast modes and the scaling $\omega_{\text{tur},f} \propto k^{1/2}$, it is likely that the condition in Equation (43) breaks down on large wavenumbers. In this case we should use the general form of the broadened resonance function $R_B$ in Equation (7). At $v || \ll V_{\text{ph}}$, $R_B$ is

$$R_B = \frac{\sqrt{2}}{2(k || \Delta v || + \omega_{\text{tur},f})} \exp \left[ -\frac{V_{\text{ph}}^2 k^2}{2(k || \Delta v || + \omega_{\text{tur},f})^2} \right]. \quad (46)$$

which is approximately

$$R_B \approx \frac{\sqrt{2}}{2 \omega_{\text{tur},f}} \exp \left[ -\frac{V_{\text{ph}}^2 k^2}{2 \omega_{\text{tur},f}^2} \right] \quad (47)$$

at small wavenumbers and

$$R_B \approx \frac{\sqrt{2}}{2 k || \Delta v ||} \exp \left[ -\frac{V_{\text{ph}}^2 k^2}{2(k || \Delta v ||)^2} \right] \quad (48)$$

at large wavenumbers. Obviously, $R_B$ at large wavenumbers is small owing to $\Delta v || \ll V_{\text{ph}}$ for very low energy particles, while $R_B$ at small wavenumbers decreases with increasing $k$. So even when the condition in Equation (43) cannot hold on large wavenumbers, $k = 1/L$ still mainly contributes to $D_{\mu\mu,2,T}$, and its expression in Equation (45) is still valid for sufficiently low energy particles.

Owing to the broadening effect, particles in the range $v || < V_{\text{ph}}$ can also be scattered with a nonzero pitch-angle diffusion coefficient. In the limiting case with $v || \ll V_{\text{ph}}$, $D_{\mu\mu,2,T}$ (Equation (45)) does not have the dependence on $(L/l_{\text{min}})^{1/2}$ when the broadening is dominated by the turbulent decorrelation effect.

In Table 1, we summarize $D_{\mu\mu,T}$ for both slow and fast modes, where the effect of resonance broadening on TTD scattering of both low- and high-energy particles can be clearly seen.

2.2.3. Comparison between Slow and Fast Modes

With the above analytical expressions of the pitch-angle diffusion coefficients of the resonance-broadened TTD, we are able to examine the relative importance between slow and fast modes in TTD scattering of particles in different regimes.

(1) High-energy particles ($v || >> V_{\text{ph}}$): For high-energy particles satisfying the condition in Equation (8), the broadening due to magnetic fluctuations is dominant. The total $D_{\mu\mu,T}$ (Equation (12)) can be approximately given by (Equations (25) and (42))

$$D_{\mu\mu,T} \approx D_{\mu\mu,1,T} + D_{\mu\mu,1,T'}$$

$$= \frac{\sqrt{2}}{4} \pi^2 \frac{1}{B_0^2} \left( \frac{\delta B_z^2}{B_0^2} \right)^{1/2} \exp \left[ -\frac{v^2}{2 \Delta v ||} \right]$$

$$\times \left[ C_t L^{-2} \ln \left( \frac{L}{l_{\text{min},s}} \right) + C_f \left( l_{\text{min},s}^{1/2} - L^{-1/2} \right) \right]. \quad (49)$$

To compare the relative importance between $D_{\mu\mu,1,T}$ and $D_{\mu\mu,1,T'}$, from the above equation we find their ratio as (Equations (14) and (30))

$$\frac{D_{\mu\mu,1,T}}{D_{\mu\mu,1,T'}} = \frac{C_t L^{-2} \ln \left( \frac{L}{l_{\text{min},s}} \right)}{C_f \left( l_{\text{min},s}^{1/2} - L^{-1/2} \right)} = \frac{8 \beta B_z^2}{3 \delta B_f^2} \ln \left( \frac{L}{l_{\text{min},s}} \right) - 1, \quad (50)$$

where we assume $v || >> V_{\text{ph}}$. In terms of $\beta$, the above ratio is

$$\frac{D_{\mu\mu,1,T}}{D_{\mu\mu,1,T'}} \approx \frac{4 V_{Ls}}{3 V_{Lj}} \ln \left( \frac{L}{l_{\text{min},s}} \right) \beta \approx \frac{4}{3} \frac{L}{l_{\text{min},s}} \beta. \quad (51)$$

in both high- and low-$\beta$ media, where we also assume that the turbulent energies contained in slow and fast modes are comparable, i.e., $V_Ls \approx V_{Lj}$. 

### Table 1

|                | Linear resonance | Broadened resonance |
|----------------|------------------|---------------------|
| $v || << V_{ph}$ | $\propto \Delta v ||$ (Equation (28)) | $\propto \Delta v ||$ (Equation (25)) |
| $v || >> V_{ph}$ | $\propto \Delta v ||$ (Equation (45)) | $\propto \Delta v ||$ (Equation (42)) |
For particles with \( r_g \) larger than \( l_d \) of both slow and fast modes, the ratio in Equation (51) is in fact

\[
\frac{D_{\nu\nu,1,T}}{D_{\nu\nu,1,F \beta}} \approx \frac{4}{3} \left( \frac{L}{\gamma} \right) \beta \approx \frac{4}{3} \left( \frac{L}{\gamma} \right) \beta
\]

under the condition \( L \gg r_g \). We can then immediately see that when

\[
\beta < \frac{3}{4} \left( \frac{L}{\gamma} \right),
\]

fast modes dominate the TTD resonance of energetic particles, while TTD with slow modes becomes more important at a larger \( \beta \). The relation in Equation (53) is illustrated in Figure 1, and \( E_{CR} \) is the energy of CR protons corresponding to \( r_g \) at \( \mu = 0 \). The parameters used are \( L = 30 \) pc and \( B_0 = 3 \) \( \mu G \).

In reality, the damping of turbulence should be taken into account. Fast modes are usually subjected to a strong damping effect and have a large \( l_d \). For particles with \( r_g \) larger than \( l_{d,s} \) of slow modes but smaller than \( l_{d,f} \) of fast modes, the ratio in Equation (51) can be rewritten as

\[
\frac{D_{\nu\nu,1,T}}{D_{\nu\nu,1,F \beta}} \approx \frac{4}{3} \left( \frac{L}{\gamma} \right) \beta
\]

when \( L \gg l_{d,f} \). If the particles have \( r_g \) smaller than both \( l_{d,s} \) and \( l_{d,f} \), then there is

\[
\frac{D_{\nu\nu,1,T}}{D_{\nu\nu,1,F \beta}} \approx \frac{4}{3} \left( \frac{L}{\gamma} \right) \beta
\]

when \( L \gg l_{d,f} \). Therefore, the above results can be summarized in a more general form,
the strong damping effect makes the TTD with fast modes less efficient.

(2) Low-energy particles ($v_l \ll V_{ph}$): For very low energy particles, the broadening mainly comes from the turbulent decorrelation. The total $D_{mu,T}$ (Equation (12)) has an approximate expression (Equations (28) and (45))

$$D_{mu,T} \approx D_{mu,s,T} + D_{mu,f,T}$$

$$= \sqrt{\pi} \frac{1}{2} B_0^2 \left( 1 - \mu^2 \right)^2$$

$$\times \left\{ \frac{C}{4} V_A^2 L^{-2} \ln \left( \frac{L}{l_{min}} \right) \exp \left[ \frac{V_{ph,s}^2}{2 V_A^2} \right] \right. + \frac{2}{15} \frac{C}{10} V_L^2 \frac{L^{-2}}{V_L^2} \exp \left[ \frac{V_{ph,f}^4}{2 V_L^4} \right] \right\}$$

(57)

Then, we can easily compare $D_{mu,s,T}$ with $D_{mu,f,T}$ (Equations (14) and (30)),

$$\frac{D_{mu,s,T}}{D_{mu,f,T}} = \frac{5 \beta B_0^2 V_{ph,s}^2}{2 V_L^2} \frac{V_L}{V_L} \ln \left( \frac{L}{l_{min}} \right) \exp \left[ \frac{V_{ph,s}^4}{2 V_L^4} - \frac{V_{ph,s}^4}{2 V_A^4} \right]$$

(58)

where the relation $v_l \ll V_{ph}$ is used. In a high-$\beta$ medium, the above expression becomes

$$\frac{D_{mu,s,T}}{D_{mu,f,T}} \approx \frac{5 V_L^4}{2 V_L^4} \frac{2 \beta V_L^2}{4} \ln \left( \frac{L}{l_{min}} \right) \exp \left[ \frac{V_{ph,s}^4}{2 V_L^4} - \frac{1}{2} \right]$$

(59)

We further assume $V_L^2 \sim V_L^2 \sim V_A^2 / 3$, so that it reduces to

$$\frac{D_{mu,s,T}}{D_{mu,f,T}} \approx 2.7 \ln \left( \frac{L}{l_{min}} \right) \beta^2 \exp \left( \frac{9}{8} \beta^2 \right)$$

(60)

It clearly shows that $D_{mu,s,T}$ is significantly larger than $D_{mu,f,T}$.

In a low-$\beta$ medium, we rewrite Equation (58) as

$$\frac{D_{mu,s,T}}{D_{mu,f,T}} = \frac{5 V_L^4}{2 V_L^4} \frac{2 \beta V_L^2}{4} \ln \left( \frac{L}{l_{min}} \right) \exp \left[ \frac{V_{ph,s}^4}{2 V_L^4} - \frac{1}{2} \right]$$

(61)

We again assume $V_L^2 \sim V_L^2 \sim V_A^2 / 3$ and obtain

$$\frac{D_{mu,s,T}}{D_{mu,f,T}} \approx 15 \frac{2}{2} \beta \ln \left( \frac{L}{l_{min}} \right) \exp \left( \frac{9}{2} \right)$$

$$\approx 675 \ln \left( \frac{L}{l_{min}} \right) \beta$$

(62)

Even with a small $\beta$, $D_{mu,s,T}$ still tends to be larger than $D_{mu,f,T}$.

The above comparisons show that despite the anisotropic energy distribution, slow modes are more important than fast modes in TTD resonance of very low energy particles. The inefficiency of scattering by fast modes is attributed to the long decorrelation timescale, i.e., $\omega_{tur,f}$, of fast modes.

2.3. Resonance-broadened TTD Scattering of CRs and Charged Grains

To illustrate the relative importance between the two resonance broadening effects for slow and fast modes and for different energies of particles, we next consider the TTD scattering of CR protons and nonrelativistic charged grains in the diffuse warm ionized medium (WIM).

The typical parameters of the WIM are listed in Table 3 (see Drake & Lazarian 1998), where $n_H$ is the number density of gas and $T$ is the temperature. For the interstellar turbulence, we assume that the turbulent energy is injected at a large scale by, e.g., supernova explosions, with the turbulent speed comparable to $v_A$ and the turbulent energy equally distributed into three modes, i.e., $V_L \sim V_L \sim V_A / \sqrt{3}$. Thus, slow and fast modes have comparable magnetic fluctuations at $\beta \sim 1$. We also assume for simplicity that $\delta B^2$ in $\Delta \Omega_l$ (Equation (6)) is equal to the sum of $\delta B^2$ and $\delta B^2$. In addition, $L_\parallel$ of Alfvén and slow modes is the viscous damping scale (Yan & Lazarian 2004), which is consistent with the inner scale of the observed power-law spectrum of electron density fluctuations in the WIM (Armstrong et al. 1995; Chepurnov & Lazarian 2010); $L_\parallel$ of fast modes is determined based on the calculations in Yan & Lazarian (2004).

(1) TeV CR protons: For high-energy CRs interacting with slow modes, the condition $\Delta \Omega_l \gg v_A$ is satisfied for most pitch angles except for $\mu \rightarrow 1$, implying the dominance of the broadening due to magnetic fluctuations. Figure 2(a) displays the numerically calculated $D_{mu,s,T}$ with $R_B$ (solid line) and $D_{mu,s,T}$ with $R_B$ (dot-dashed line), which are overlapped as expected. Corresponding analytical approximations are Equation (23) (crosses) and Equation (25) (squares), where $l_{min} = r_s$ as $r_s \gg L_s$ (e.g., $r_s$ at $\mu = 0$ is $1.1 \times 10^{15}$ cm).

For TTD scattering by fast modes, Figure 2(b) presents the overlapping $D_{mu,f,T}$ and $D_{mu,f,T}$, which are numerically calculated with $R_B$ and $R_{Al,F}$ used, respectively. The analytical result for the latter case is given by Equation (42), where $l_{min} = r_s$ is used as $r_s > L_s$.

The above results confirm that for TTD scattering of high-energy particles, the broadening due to magnetic fluctuations is dominant for both slow and fast modes. The damping of turbulence does not affect the scattering efficiency as long as $r_s$ is larger than $L_s$. In addition, the ratio $D_{mu,s,T}/D_{mu,f,T}$ can be quantified by using Equation (56c), which provides $D_{mu,s,T}/D_{mu,f,T} \approx 0.05$, in a good agreement with the numerical result. It shows the dominance of fast modes in TTD scattering of TeV CRs in the WIM.

(2) Low-energy charged grains: Here we consider graphite grains with the density $\rho_d = 2.2$ g cm$^{-3}$, the charge $Z_d = 2$, and the size $a = 10^{-6}$ cm (Hoang et al. 2012). We adopt $v = 0.1V_A$ to exemplify the TTD scattering of very low energy particles.

In the case of slow modes, as $\Delta \Omega_l \ll V_A$, the broadening due to turbulent decorrelation is dominant. Figure 2(c) shows that the numerically calculated $D_{mu,s,T}$ with $R_B$ (solid line) coincides with the numerically calculated $D_{mu,s,T}$ with $R_{B2}$ (dashed line). Corresponding analytical approximations are taken from Equation (23) (crosses) and Equation (28) (open circles), where $l_{min} = r_s$ as $r_s \gg L_s$ (e.g., $r_s = 0 = 2.0 \times 10^{15}$ cm). In addition, we present the numerically (dotted line) and analytically (Equation (85); filled circles) calculated $D_{mu,s,T}$ using the eddy turnover rate as $\omega_{tur,s}$, which are close to other results using the Alfvén wave-crossing rate because of the critical balance in strong MHD turbulence.

As regards the fast modes, the numerical result of $D_{mu,f,T}$ with $R_B$ is presented in Figure 2(d) (solid line). The numerically (dashed line) and analytically (Equation (45); open circles)
derived $D_{\text{lin},T}$ with $R_{B2}$ have a smaller value. This implies that although the main contribution comes from the broadening due to turbulent decorrelation, both broadening effects should be taken into account to obtain the accurate result.

The above results confirm that as a consequence of the decorrelation of turbulent magnetic fields, particles with $v \ll V_{\text{ph}}$ can also undergo the TTD resonance. We also see that as analyzed in Section 2.2.3, for TTD scattering of low-energy particles, slow modes are much more efficient than fast modes.

(3) Charged grains with super-Alfvénic speeds: We again consider graphite grains but with $v = 1.5V_A$ as an example for TTD scattering of moderate-energy particles. As $v$ is comparable to $V_A$, both broadening effects are important for TTD with slow modes, and thus the general expression of $D_{\text{lin},T}$ should be used. In Figure 2(e), we see that $D_{\text{lin},1,T}$ with $R_{B1}$ and $D_{\text{lin},2,T}$ with $R_{B2}$ have similar values. They are both larger than the general $D_{\text{lin},T}$ with $R_{B}$. The analytical expressions used here are Equation (23) for $D_{\text{lin},1,T}$, Equation (24) for $D_{\text{lin},2,T}$ and Equation (27) for $D_{\text{lin},1,T}$, where $l_{\text{min}} = r_g$ as $r_g \gg l_{d,s}$ (e.g., $r_{\rho,p} = 3.0 \times 10^{14}$ cm).

For TTD with fast modes, $v \sim V_A$ indicates $\Delta v||k|| \sim V_A$, $k || \omega_{\text{lin}}, > \omega_{\text{tur}}$, and the dominance of the broadening due to magnetic fluctuations. As expected, Figure 2(f) shows that $D_{\text{lin},1,T}$ with $R_{B}$ and $D_{\text{lin},1,T}$ with $R_{B1}$ are the same. The analytical approximation of $D_{\text{lin},1,T}$ is derived from Equation (41) in the asymptotic limit $v || \ll V_{\text{ph}}$.

$$D_{\text{lin},1,T} \approx \frac{\sqrt{2}}{4\pi} \frac{C_{\rho}}{B_0} \left( \frac{\langle \delta B_{\rho}^2 \rangle}{B_0^2} \right)^{\frac{1}{2}} (l_{\text{min}} - L^{-\frac{3}{2}})^v \times (1 - \mu^2)^{\frac{3}{2}} \exp \left[ -\frac{V_{\text{ph}}^2}{\Delta v ||} \right],$$

(63)

where $l_{\text{min}} = r_g$ as $r_g > l_{d,f}$. It only serves as a rough estimate of $D_{\text{lin},1,T}$. By comparing with slow modes, we see that TTD with fast modes is more efficient in scattering moderate-energy particles in the WIM.

In addition, by comparing the pitch-angle diffusion coefficients at different particle energies, we find that as $D_{\text{lin},T}$ depends on $v$ (see Table 1), CRs are much more efficiently scattered than charged grains by both slow and fast modes.

### 3. Resonance-broadened Gyroresonance

We have detailed the resonance-broadened TTD above; next, we will briefly discuss the effect of resonance broadening on the gyroresonant pitch-angle scattering.

The linear resonance function of gyroresonance scattering is

$$R_L = \pi \delta (\omega_k - v||k|| + n\Omega),$$

(64)

where $n$ is a nonzero integer. The strong scattering requires that $\omega_k$ is Doppler-shifted to the gyrofrequency $\Omega$ of the particle or its harmonics. In MHD turbulence, similar to the broadening of TTD resonance, the broadened resonance function for gyroresonance is

$$R_B = \frac{\sqrt{2\pi}}{2(\Delta v||k|| + \omega_{\text{tur}})} \exp \left[ -\frac{(\omega_k - v||k|| + n\Omega)^2}{2(\Delta v||k|| + \omega_{\text{tur}})^2} \right].$$

(65)

For high-energy particles with $v \gg V_{\text{ph}}$ and $\Delta v||k|| \gg \omega_{\text{tur}}$, the broadening induced by magnetic fluctuations is dominant. The above $R_B$ can be simplified,

$$R_{B1} \approx \frac{\sqrt{2\pi}}{2\omega_{\text{tur}}} \exp \left[ -\frac{(\omega_k - \Omega)^2}{2\omega_{\text{tur}}^2} \right].$$

(66)

with $n = 1$ for the dominant resonance. As the magnetic perturbations on scales smaller than $r_g$ are unimportant in scattering particles, we consider $k \lesssim 1/r_g$. At a small $k$, the exponential term in $R_{B1}$ becomes small, and the scattering becomes inefficient. We see that only when $\mu$ is small can the broadening make a significant difference compared to the linear resonance (Yan & Lazarian 2008; Xu et al. 2016).

For low-energy particles with $v \ll V_{\text{ph}}$ and $\Delta v||k|| \ll \omega_{\text{tur}}$, there is

$$R_{B2} \approx \frac{\sqrt{2\pi}}{2\omega_{\text{tur}}} \exp \left[ -\frac{(\omega_k - \Omega)^2}{2\omega_{\text{tur}}^2} \right].$$

(67)

with $n = -1$ for the dominant resonance. In the case of fast modes with a very small $\omega_{\text{tur}}$, the broadening effect is negligible.

We next provide the formulae of pitch-angle diffusion coefficients for gyroresonance.

(1) Alfvén modes: The pitch-angle diffusion coefficient for the gyroresonance with Alfvén modes is (Kulsrud & Pearce 1969; Voelk 1975)

$$D_{\text{lin},A,G} = C_{\mu} \int d^3k x^{-2} [J_1(x)]^2 I(k) R(k),$$

(68)

where $C_{\mu}$ is given in Equation (2). The anisotropic magnetic energy spectrum is (Cho et al. 2002)

$$I_A(k) = C_A \frac{k^{10}}{k^2} \exp \left( -L \frac{k}{k^2} \right),$$

(69)

where

$$C_A = \frac{1}{6\pi} \delta B_{\rho}^2 L^{-\frac{3}{2}},$$

(70)

and

$$\frac{\delta B_{\rho}}{B_0} \sim \frac{V_{\text{ph}}}{V_A} \sim 1.$$
Figure 2. $D_{\mu, T}$ for TTD with slow and fast modes in the WIM. “N” and “A” represent numerical and analytical results. For slow modes, $R_{B_2, A}$ and $R_{B_2, e}$ denote the resonance function $R_{B_2}$ with the Alfvén wave-crossing time and eddy turnover time used as the turbulent decorrelation time, respectively.
for strong MHD turbulence. Based on the critical balance, we also have
\[ \omega_k = \omega_{\text{tur.A}} = V_A k_i, \]
which is used in the broadened resonance function \( R_{\text{B2}} \). (2) Slow and fast modes: For gyroresonance with compressive modes, the pitch-angle diffusion coefficient is (Kulsrud & Pearce 1969; Voelk 1975)
\[ D_{\mu,\text{S.F.G}} = C_{\mu} \int k^2 \langle J_1'(x) \rangle^2 I(k) R(k), \]
where
\[ J_1'(x) = \frac{1}{2} [J_0(x) - J_2(x)]. \]
\( I(k) \) is given in Equations (13) and (29) for slow and fast modes, respectively.

For illustrative purposes, here we consider broadened gyroresonance scattering of TeV CRs and charged grains in MHD turbulence in the WIM, in comparison with the TTD scattering as shown in Figure 2. The parameters used are the same as those in Section 2.3. In the case of TeV CR protons (see Figure 3(a)), due to the anisotropic scaling of Alfvén and slow modes, the turbulent eddy at \( k_{\perp} \sim r_g^{-1} \) has \( k_{\parallel} \gg r_g^{-1} \). The interactions with many uncorrelated turbulent eddies within one gyro-orbit make the gyroresonance scattering inefficient (Chandran 2000; Yan & Lazarian 2002). By contrast, we see that the gyroresonance with isotropic fast modes is much more efficient, which is consistent with the earlier finding in Yan & Lazarian (2008). We also obtain the analytical result from Equation (73) for gyroresonance with fast modes by using the linear resonance function (squares in Figure 3(a); see also Xu et al. 2016).
\[ D_{\mu,\text{F.G}} \approx \frac{2\pi^2}{7} \frac{C_{\mu}}{B_0^2} \Omega^2 \hbar^2 (1 - \mu^2)^{3/2}. \]
By comparing with \( D_{\mu,\text{S.F.T}} \) in Equation (42), we find
\[ D_{\mu,\text{F.G}} \approx \frac{7\sqrt{2}}{8\sqrt{\pi}} \left( \frac{\langle B_0^2 \rangle}{B_0^2} \right)^{-4} \exp \left[ -\frac{v_{\gamma}^2}{2\Delta v_{\gamma}^2} \right] \frac{(1 - \mu^2)^{3/2}}{\sqrt{\mu^2}}, \]
where we assume \( L \gg l_{\text{min}} = r_g \). This shows that TTD and gyroresonance scattering of high-energy particles with fast modes have comparable efficiencies at a moderate \( \mu \), but TTD dominates over gyroresonance at a small \( \mu \), and gyroresonance dominates over TTD at a large \( \mu \).

In the case of low-energy graphite grains with \( v = 0.1V_A \), the resonance scale \( k_{\perp}^{-1}(or k^{-1}) = (V_{ph} / v) r_g \) can be much larger than \( r_g \). Figure 3(b) shows that the gyroresonance with fast modes dominates the scattering. By using the linear resonance function, we obtain the analytical result for gyroresonance with fast modes (squares in Figure 3(b)),
\[ D_{\mu,\text{F.G}} \approx \frac{\pi^2}{3} \frac{C_{\mu}}{B_0^2} \Omega^2 v_{\text{ph,G}}^2 (1 - \mu^2). \]
It agrees well with the numerical result with the broadened resonance function, indicating the negligible role of broadening for gyroresonance with fast modes as mentioned above.

By comparing \( D_{\mu,\text{F.G}} \) with \( D_{\mu,\text{T.G}} \) in Equation (28), we find
\[ D_{\mu,\text{T.G}} \approx 2\sqrt{2} \pi^{-3/2} \frac{B_0^2}{\delta B_0^2} \left( \frac{V_{ph}}{V_A} \right)^{1/2} \left( \frac{v}{v_{\gamma}} \right)^{3/2} \times \exp \left[ -\frac{v_{\gamma}^2}{2\Delta v_{\gamma}^2} \right] \frac{(1 - \mu^2)^{3/2}}{\sqrt{\mu^2}}, \]
For low-energy particles with \( r_g \ll L \) and \( v \ll V_A \), the above ratio is small. As \( D_{\mu,\text{T.G}} \) is even smaller than \( D_{\mu,\text{T.T.D.}} \), it shows that for low-energy particles in the WIM, gyroresonance with fast modes dominates over TTD with both slow and fast modes at all pitch angles.

In Figure 3(c), we also present the results for graphite grains with \( v = 1.5V_A \). We see that TTD with fast modes becomes comparably important to gyroresonance with fast modes. Approximately, we have (Equations (63) and (77))
\[ D_{\mu,\text{T.G}} \approx \frac{3\sqrt{2}}{8\sqrt{\pi}} \left( \frac{\langle B_0^2 \rangle}{B_0^2} \right)^{-4} \exp \left[ -\frac{v_{\gamma}^2}{2\Delta v_{\gamma}^2} \right] \frac{(1 - \mu^2)^{3/2}}{\sqrt{\mu^2}}, \]
where we again assume \( L \gg l_{\text{min}} = r_g \). The above ratio is of the order of unity.

4. Conclusions

The key element of studying the scattering and acceleration of particles is a physically realistic model of MHD turbulence. Its characteristics, including the anisotropic scaling of Alfvén and slow modes with respect to the local magnetic field, are natural consequences of the nonlinear dynamics of MHD turbulence.

Due to the magnetic fluctuations and nonlinear decorrelation of turbulence, the broadening effects arising from MHD turbulence play an important role in enhancing the pitch-angle scattering of particles in MHD turbulence. When addressing the broadening effect induced by magnetic fluctuations, it is noteworthy that \( \delta B_0 \) in Equation (6) refers to (i) the parallel component of magnetic fields; (ii) the compressive component of MHD turbulence, or pseudo-Alfvén modes in incompressible MHD turbulence; and (iii) the magnetic perturbation averaged over the parallel mean free path of particles when it is smaller than \( L \), which should be all considered in a more accurate treatment of the broadening. As regards the broadening effect caused by the turbulent decorrelation, the long decorrelation timescale of fast modes originates from not only the spectral scaling (Equation (29)) but also the factor \( v_{ph} / v_{\gamma} \) in the cascading rate (Equation (39)). The latter was not taken into account in Lynn et al. (2012, 2014).

We mainly focused on TTD scattering and clarified that although the magnetic fluctuations on all scales larger than \( r_g \) can contribute to TTD, the main contribution comes from those on \( r_g \) (or \( l_d \) when \( l_d > r_g \)). The pitch-angle diffusion coefficient depends on the ratio \( L/r_g \) (or \( l_d > r_g \)). In addition, it is known that gyroresonance with slow modes is inefficient because of the anisotropy of slow modes (Yan & Lazarian 2002). Here we demonstrate that TTD with slow modes also suffers from
the turbulence anisotropy, irrespective of the resonance broadening.

With the broadened resonance, both high- \((v_\parallel \gg V_{ph})\) and low-energy \((v_\parallel \ll V_{ph})\) particles can undergo TTD scattering with both slow and fast modes (see Table 1). For high-energy particles, the broadening due to magnetic fluctuations is dominant. The relative importance between slow and fast modes depends on their magnetic fluctuations and the plasma damping (Equation (56)). Despite the anisotropy of slow modes, they can be comparably or even more efficient than fast modes in TTD scattering in the following scenarios: (i) in a high-\(\beta\) medium, e.g., the intracluster medium; (ii) when fast modes are severely damped in, e.g., partially ionized WNM (Figure 1(b)); and (iii) when the particle energy is sufficiently high (Figure 1(a)). For very low energy particles, the broadening is determined by the turbulent decorrelation, and slow modes always dominate over fast modes in TTD. For moderate-energy particles with \(v_\parallel \sim V_{ph}\), the two broadening effects can both be important for slow modes, while the broadening due to magnetic fluctuations becomes dominant for fast modes. Important implications concerning, e.g., dynamics of charged grains, heating of particles, and TTD damping of MHD turbulence will be investigated in our future work.

Compared with TTD by fast modes, the resonance broadening has an insignificant effect on gyroresonance with fast modes. For high-energy particles, the relative importance between TTD and gyroresonance with fast modes depends on \(\mu\) (Equation (76)). For very low energy particles, gyroresonance with fast modes tends to be dominant in scattering at all pitch angles. These findings agree with earlier studies, e.g., Yan & Lazarian (2003, 2008). For moderate-energy particles, TTD and gyroresonance with fast modes are comparably important.

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Appendix A

**D$_{\rho s,2,T}$ Calculated Using the Turbulent Eddy Turnover Rate as $\omega_{\text{turb}}$**

Following the turbulent energy cascade, $\omega_{\text{turb}}$ increases toward smaller scales. It can be formulated as

$$\omega_{\text{turb}} = k_l u_k = V_A L^{-\frac{5}{2}}k_L^2$$  \hspace{1cm} (80)

according to the Kolmogorov scaling in the direction perpendicular to the local magnetic field. $V_A$ is the turbulent speed of strong Alfvénic turbulence at $L$. Alfvén modes govern the energy cascade of slow modes. With the above expression of $\omega_{\text{turb}}$, we have

$$D_{\rho s,2,T} = \frac{\sqrt{2\pi}}{8} C_s C_v^2 \frac{\nu^2}{\Omega^2} V_A^{-1} L^{\frac{7}{2}} \int d^3 k$$

$$\times \left[ k_L^2 \frac{1}{k_L^2 + k_L^2} \exp \left[ -L^2 \frac{k_L^2}{k_L^2} - \frac{(V_{ph} - \nu_l^2)k_L^2}{2V_A^2 L^{-\frac{5}{2}}k_L^2} \right] \right].$$

(81)

When $\nu_l \sim V_{ph}$, we obtain the approximate expression

$$D_{\rho s,2,T} \approx \frac{\sqrt{2}}{2} \frac{\pi}{\nu_L^2} \frac{C_s}{B_0^2} V_A^{-1} L^{-\frac{7}{2}}$$

$$\times \ln \left( \frac{L}{l_{\text{min}}} \right) \nu_l^2 (1 - \mu^2) \left[ 1 - \frac{6(V_{ph} - \nu_l^2)}{V_A^2} \right].$$

(82)

We notice that for particles with $\nu_l \sim V_{ph}$, the expression of $D_{\rho s,2,T}$ using $\omega_{\text{turb}}$ in Equation (27) can be approximated by

$$D_{\rho s,2,T} \approx \frac{\sqrt{2}}{4} \frac{\pi}{\nu_L^2} \frac{C_s}{B_0^2} V_A^{-1} L^{-\frac{7}{2}}$$

$$\times \ln \left( \frac{L}{l_{\text{min}}} \right) \nu_l^2 (1 - \mu^2)^2 \left[ 1 - \frac{(V_{ph} - \nu_l^2)^2}{2V_A^2} \right].$$

(83)

which has a form similar to Equation (82).

When $\nu_l$ is significantly away from $V_{ph}$, i.e., $\nu_l \ll V_{ph}$, we drop the first term of the exponential function in Equation (81) and find

$$D_{\rho s,2,T} \approx \frac{\sqrt{2}}{4} \frac{\pi}{\nu_L^2} \frac{C_s}{B_0^2} V_A^{2} L^{-\frac{7}{2}}$$

$$\times \ln \left( \frac{L}{l_{\text{min}}} \right) \nu_l^2 (1 - \mu^2)^2 |V_{ph} - \nu_l|^3.$$  \hspace{1cm} (84)

A similar scaling relation between $D_{\rho s,2,T}$ and $\nu_l$ was earlier found by Lynn et al. (2012). To make it converge with Equation (82) at $\nu_l = V_{ph}$, we modify the above expression to

$$D_{\rho s,2,T} \approx \frac{\sqrt{2}}{2} \frac{\pi}{\nu_L^2} \frac{C_s}{B_0^2} V_A^{-1} L^{-\frac{7}{2}}$$

$$\times \ln \left( \frac{L}{l_{\text{min}}} \right) \nu_l^2 (1 - \mu^2)^2 \left[ 1 + \frac{|V_{ph} - \nu_l|^3}{V_A} \right].$$

(85)

Appendix B

**$D_{\rho s,1,T}$ Calculated with a Different Power-law Spectrum**

We assume that the power spectrum of fast modes is

$$I_f(k) = C_f k^{-\frac{3}{4}},$$

(86)

with the normalization constant

$$C_f = \frac{1}{12\pi} \delta B_0^2 L^{-\frac{7}{2}}.$$  \hspace{1cm} (87)

It has the same power-law index as that of slow modes in terms of $k_L$ (Equation (13)), but it has an isotropic energy distribution. Combining Equations (1), (9), and (86), we approximately have

$$D_{\rho s,1,T}^n = \frac{3\sqrt{2}}{8} \frac{\pi}{\nu^2} \frac{C_f}{B_0^2} \left( \frac{\delta B_0^2}{B_0^2} \right)^{-\frac{1}{2}}$$

$$\times \left( \frac{L}{l_{\text{min}}} \right)^{-\frac{1}{2}} \nu_l^2 (1 - \mu^2)^\frac{3}{4} \exp \left[ -\frac{\nu_l^2}{2\Delta L} \right].$$  \hspace{1cm} (88)

for high-energy particles with $\nu_l \gg V_{ph}$.

By comparing with $D_{\rho s,1,T}$ in Equation (24), we find that $D_{\rho s,1,T}$ depends on $(L/l_{\text{min}})^{-1/2}$, while $D_{\rho s,1,T}$ depends on $ln(L/l_{\text{min}})$. In the case of $L \gg l_{\text{min}}$, fast modes with the same spectral index as that of slow modes are still much more effective in TTD scattering of energetic particles.

By comparing with $D_{\rho s,1,T}$ in Equation (42), which depends on $(L/l_{\text{min}})^{-1/2}$, we see that because of the steeper spectrum used here, $D_{\rho s,1,T}$ is smaller than $D_{\rho s,1,T}$ owing to the relatively small magnetic perturbations on small scales.

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