More on Phase Structure of Nonlocal 2D Generalized Yang-Mills Theories (nlgYM$_2$’s)

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Abstract

We study the phase structure of nonlocal two dimensional generalized Yang-Mills theories (nlgYM$_2$) and it is shown that all order of $\phi^{2k}$ model of these theories has phase transition only on compact manifold with $g = 0$ (on sphere), and the order of phase transition is 3. Also it is shown that the $\phi^2 + \frac{2k}{3} \phi^3$ model of nlgYM$_2$ has third order phase transition on any compact manifold with $1 < g < 1 + \frac{1}{|\eta|}$, and has no phase transition on sphere.

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1 Introduction

The two dimensional Yang-Mills theory ($YM_2$) is a theoretical laboratory for understanding the main theory of particle physics, $QCD_4$. In recent years there have been much effort to analyze the different aspects of this theory [1-8]. It is well known that $YM_2$ is defined by the Lagrangian $\text{tr}(\frac{1}{4}F^2)$ on a Riemann surface where $F$ is the field strength tensor. This theory have certain properties, such as invariance under area preserving diffeomorphism and lack of any propagating degrees of freedom [9]. In a $YM_2$ one starts from a B-F theory in which a Lagrangian of the form $i\text{tr}(BF) + \text{tr}(B^2)$ is used. There are, however, the many way to generalized these theories without losing properties. One way is so called generalized Yang - Mills theory (gYM$_2$’s). These theories are defined by replacing the term $\text{tr}(B^2)$ in the $YM_2$ Lagrangian by an arbitrary class function of $B$ [10]. It is worthy of mention that for gYM$_2$, one can not eliminate the auxiliary field that obtain a Lagrangian for the gauge field. Remarkable that the large gauge group limit of YM$_2$ and gYM$_2$ theories is also interesting. Several aspects of this theory have been studied in [13 - 19]. There is another way to generalize YM$_2$ and gYM$_2$ and that is to use a nonlocal action for the auxiliary field, leading to the so - called nonlocal YM$_2$ (nlYM$_2$) and nonlocal gYM$_2$(nlgYM$_2$) theories, respectively [11]. It is remarkable that, the action of nlYM$_2$ and nlgYM$_2$ is no extensive [11]. Several aspects of nlgYM$_2$, such as, wave function, partition function, generating functional, and also large $N$ limit of it, have been studied on sphere [12]. The authors of [12] obtained that $\phi^4$ model of this theory (nlgYM$_2$) on sphere has third order phase transition. The authors of [20] have studied the large-$N$ limit of YM$_2$ and some features of it on cylinder.

The scheme of this paper is the following. In section 2 we briefly review the large-$N$ limit of (nlgYM$_2$)theories for $U(N)$ gauge group. In section 3, we study the phase structure of the theory for $\phi^{2k}$ in all order and $\phi^2 + \frac{2\alpha}{3}\phi^3$ models on arbitrary compact manifold with $g \neq 1$. 

2
2 Preliminaries

The partition function of two dimensional nonlocal generalized Yang- Mills theories (nlgYM$^2$) on a compact manifold $\Sigma_g$ with genus $g$ and area $A$ is given by the exact formula as:

$$Z_{\Sigma_g}(g, A) = \sum_R d^2 - 2g \exp\{\omega[-A\Lambda(R)]\},$$

(1)

where $R$’s label the irreducible representation of the gauge group, $d_R$ is the dimension of the representation $R$ and $\Lambda(R)$ is

$$\Lambda(R) = \sum_{k=1}^p \frac{\alpha_j}{N^{k-1}} C_k(R).$$

(2)

Here $C_k$ is the $k$’th Casimir of gauge group, $\alpha_j$’s are arbitrary constant. We consider the case that the gauge group is $U(N)$. The representation of this gauge group are labelled by $N$ integers $n_i$ satisfying $n_i \geq n_j (i \leq j)$ and it is found that

$$d_R = \prod_{1 \leq i \leq j \leq N} (1 + \frac{n_i - n_j}{j - i}),$$

(3)

$$C_k(R) = \sum_{i=1}^N [(n_i + N - i)^k - (N - i)^k].$$

(4)

Now we redefine the function $\omega$ as:

$$-N^2V[A \sum_{k=1}^p \alpha_k \hat{C}_k(R)] := \omega[-A\Lambda(R)],$$

(5)

where

$$\hat{C}_k(R) = \frac{1}{N^{k+1}} \sum_{i=1}^N (n_i + N - i)^k.$$  

(6)

In the large $N$- limit, the above summation is replaced by a path integration over the continuous function

$$\phi(x) = -n(x) + x - 1,$$

(7)

where

$$0 \leq x := \frac{i}{N} \leq 1 \quad \text{and} \quad n(x) := \frac{n_i}{N}.$$  

(8)
The partition function can be rewritten as:

$$Z_{\Sigma} = \int D\phi(x) \exp \{-N^2 S(\phi)\},$$  \hspace{1cm} (9)

where

$$S(\phi) = V \left( A \int_0^1 W[\phi(x)] dx \right) + (1 - g) \int_0^1 dx \int_0^1 dy \log |\phi(x) - \phi(y)|,$$  \hspace{1cm} (10)

and

$$W(\phi) := \sum_{k=1} (-1)^k \alpha_k \phi^k.$$  \hspace{1cm} (11)

Introducing the density function as

$$u(\phi) := \frac{dx(\phi)}{d\phi} [16],$$

it is seen that it satisfies

$$\int_b^a u(z) dz = 1,$$  \hspace{1cm} (12)

where $[b, a]$ is the interval corresponding to values of $\phi(x)$ and also the condition $n_i \geq n_j$ demands

$$u(z) \leq 1.$$  \hspace{1cm} (13)

As $N \to \infty$, only the configuration of $\phi$ contributes to the partition function that minimizes $S$. To find this representation we put variation of $S$ with respect to $\phi$ equal to zero. So one can arrive at

$$h(z) = P \int_b^a \frac{u(z') dz'}{z - z'},$$  \hspace{1cm} (14)

where $P$ indicates the principal value of integral and

$$h(z) = \frac{\hat{A}}{2(1 - g)} W'(z),$$  \hspace{1cm} (15)

and

$$\hat{A} := AV'[\left\{ A \int_0^1 W[\phi(x)] dx \right\}].$$

The free energy of the theory is defined as

$$F := S|_{\phi_{\text{cla.}}},$$  \hspace{1cm} (17)
It is seen that
\[ F'(A) = \frac{\hat{A}}{A} \int_b^a u(z)W(z)dz. \] (18)

Using the standard method of solving the integral equation (14), the density function \( u(z) \) is obtained in terms of the parameters \( a \) and \( b \). One can arrive at [12].

\[ u(z) = \frac{\sqrt{(a-z)(z-b)}}{\pi} \sum_{n,m,q=0}^\infty \frac{(2n-1)!!(2q-1)!!}{2^{n+q}n!q!(n+m+q+1)!}a^nb^n z^m h^{(n+m+q+1)}(0), \] (19)

and the values of \( a \) and \( b \) are determined from [12] and

\[ \sum_{n,q=0}^\infty \frac{(2n-1)!!(2q-1)!!}{2^{n+q}n!q!(n+q)!}a^nb^n h^{(n+q)}(0) = 0. \] (20)

The density function, \( u(z) \) found from (19), depend on the modified area \( \hat{A} \), and therefore \( A \). As \( A \) increases, a situation is encountered where \( u \) exceeds 1. So, the density function \( u(z) \) violates the condition (13) for \( A \)'s larger than some critical value \( A_c \). \( A_c \) is the value of \( A \) at which the maximum of \( u \) becomes 1 (\( u_{\text{max}}(A_c) = 1 \)). The region \( A < A_c \) is called the weak coupling phase (WCP) regime, and the region \( A > A_c \) is called the strong coupling phase (SCP) regime.

3 Phase transition of nlgYM on a compact manifold with 
\[ g \neq 1 \]

3.1 \( \phi^{2k}(k > 1) \) model.

In this case \( W(\phi) \) is an even function of \( \phi \), therefore the density function in WCP regime, \( u_w(z) \), is even, then \( b = -a \). So by rewriting (19), we have

\[ u_w(z) = k\eta \sqrt{a^2-z^2} \sum_{n=0}^\infty \frac{(2n-1)!!}{2^n n!}a^k z^{2k-2n-2}, \] (21)

where \( \eta = \frac{\hat{A}}{1-g} \) and \( a_k \) is obtained from (12) as:

\[ a_k = \left[ \frac{2^k(k-1)!!}{(2k-1)!!\eta} \right]^\frac{1}{k}. \] (22)
Equation (21) has three extremum points at $z = 0$ and $z_{1,2} = \pm a_k \sqrt{\zeta_k}$ [19], in which $\zeta_k$ is independent of $a_k$ and is determined from

$$\sum_{n=0}^{k-2} \frac{(2n)!}{2^{n+1}(n+1)} \zeta_k^{(n+1)} = 1. \quad (23)$$

By institute (22) in (21), we have

$$u_w(z) = \eta \frac{1}{2\pi} f(k) \quad (24)$$

where $f(k)$ is independent of $\eta (\hat{A} \text{or} A)$. The value of $A_c$ is obtained from

$$u_w(z_0) = 1, \quad (25)$$

hence $z_0$’s are those extremum points, which $u_w(z)$ is maximum. One can arrive at

$$A_c V'(\frac{\hat{A}}{A_c}) = \frac{1 - g}{f(k) \pi}. \quad (26)$$

It is also easy to obtain $u''_w(z_0)$. Using $u'_w(z_0) = 0$, one can see

$$u''_w(z_0) = \frac{k \eta z_0 a^{2k-3}(2k-1)}{\pi \sqrt{a^2 - z_0^2}} \left[ \sum_{n=0}^{k-2} \frac{(2n)!}{2^{n+1}(n+1)!a^{2k-2n-5}} - (2k-1)a^{3-2k}z_0^{-2} \right],$$

$$= -\frac{k \eta a^{2k-2}(2k-1)}{\pi \sqrt{a^2 - z_0^2}} \sum_{n=0}^{k-2} \frac{(2n)!}{2^{n+1}(n+1)!}\frac{z_0^{2k-2n-4}}{a^{2k-2n-4}}. \quad (27)$$

When $g > 1$, then $\eta$ is negative, so that the $\phi^{2k}$ model has no phase transition on compact manifold with $g > 1$. But for the case $g = 0$ ($\eta = \hat{A}$), (27) is clearly negative and density function in WCP regime, $u_w(z)$, has a minimum at $z = 0$ and two absolute maxima at $z_{1,2} = \pm a_k \sqrt{\zeta_k}$. Thus all $\phi^{2k}$’s models has phase transition only on sphere. For areas slightly more than the critical area, we can write the density function in WCP, $u_w(z)$, as:

$$u_w(z) = u_w(z_{1,2}) - \frac{1}{2} |u''_w(z_{1,2})|(z - z_{1,2})^2. \quad (28)$$

In the adjacent of critical point, $A_c$, $u_w(z) \geq 1$, and it is found the point $z'$ which satisfying $u_w(z') = 1$. Then

$$|z' - z_{1,2}| = \sqrt{\frac{2}{u''_w(z_{1,2})}} \xi, \quad (29)$$
where by using of (24) \( \xi \) is
\[
\xi = u_w(z,1,2) - 1 = \hat{A} \frac{f(k)}{f(k)} - 1. \tag{30}
\]

By the same procedure which used in [15], we can obtain that the difference of free energy in SCP and WCP regime for nlgYM\(_2\) is
\[
F_s - F_w \simeq \xi^3. \tag{31}
\]

One can expand \( \xi(A) \) about \( A_c \) as:
\[
\xi(A) = \xi(A_c) + \xi'(A_c)(A - A_c) + \ldots, \tag{32}
\]
where
\[
\xi'(A_c) = \frac{1}{2k} \left( \frac{d\hat{A}}{dA} \right)_c \frac{1}{A_c V''(\frac{A}{A_c})}. \tag{33}
\]
So that, for the case \( (\frac{d\hat{A}}{dA})_c \neq 0 \), \( \xi'(A) \) is nonzero and \( \xi(A_c) = 0 \), then
\[
F_s - F_w = \beta (A - A_c)^3. \tag{34}
\]
Hence \( \beta \) is a constant which independent of modified area of manifold, \( \hat{A} \), ( or \( A \)). Therefore we conclude that the nlgYM\(_2\) theories for all order of \( W(\phi) = \phi^{2k} \) models have third order phase transition only on sphere \((g = 0)\).

### 3.2 \( W(\phi) = \phi^2 + \frac{2\alpha}{3} \phi^3 \) models

We see that in the some nonlocal generalized case the constraint (13) can be satisfy also for negative \( \eta = \frac{\hat{A}}{1-g} \) by an appropriate choice of the coupling constant \( \alpha_k \), which implies the existence of phase transition at higher genera except the torus. Starting from (19) for \( W(\phi) = \phi^2 + \frac{2\alpha}{3} \phi^3 \) model and obtain
\[
uw(z) = \frac{\eta}{\pi} \left( 1 + \frac{\alpha}{2}(a + b) + \alpha z \right) \sqrt{(z - b)(a - z)}, \tag{35}
\]
and the value of $a$ and $b$ are determined from

$$
(a + b)(1 + \frac{\alpha}{2}(a + b)) + \frac{\alpha}{4}(a + b)^2 = 0,
$$

$$
\eta(a - b)^2(1 + \alpha(a + b)) = 8.
$$

By solving these equations, we obtain

$$
b = \frac{2\lambda - 1}{2\alpha} - 2\sqrt{\frac{\lambda}{\eta}},
$$

$$
a = \frac{2\lambda - 1}{2\alpha} + 2\sqrt{\frac{\lambda}{\eta}},
$$

with

$$
\lambda^3 - \frac{1}{4} \lambda + \frac{\alpha^2}{2\eta} = 0.
$$

By substitute (38) and (39) in (35), we have

$$
u_w(y) = \frac{\eta}{\pi} \{2\lambda + y\} \sqrt{4\lambda/\eta - y^2},
$$

where

$$
y = z - \frac{2\lambda - 1}{2\alpha}.
$$

The density function $u_w(y)$, (42), has two extremum points as:

$$
y_1 = (Q - 1)\frac{\lambda}{2},
$$

$$
y_2 = -(Q + 1)\frac{\lambda}{2},
$$

here $Q = \sqrt{1 + \frac{\alpha^2}{2\eta}} > 1$. At the negative $\eta$ ($g > 1$), we should take the $\lambda < 0$ solution which exists for $\eta < -6\sqrt{3}\alpha^2$. So from equations (38), (39) and (42) we conclude that $y$ take the values in interval $[-2\sqrt{\frac{\lambda}{\eta}}, 2\sqrt{\frac{\lambda}{\eta}}]$, and the only extremum point which repose in $[-2\sqrt{\frac{\lambda}{\eta}}, 2\sqrt{\frac{\lambda}{\eta}}]$ is $y_1$. It is clearly seen that $u_w(y)$ has an absolute maximum in $[-2\sqrt{\frac{\lambda}{\eta}}, 2\sqrt{\frac{\lambda}{\eta}}]$ at $y_1$. So that

$$
u''_w(y_1) < 0.
$$
From (42), we have
\[ u_w(y_1) = \frac{\lambda \eta}{2\pi} (Q + 3) \sqrt{\frac{4\lambda}{\eta} - \frac{\lambda^2}{4} (Q - 1)^2}. \]  
\[ (45) \]

\( \eta_c \) and therefore the critical value for modified area of manifold, \( \hat{A}_c \), or \( A_c \) is determined by \( u_w(y_1) = 1 \), and it is found that
\[ \frac{\lambda_c \eta_c}{2\pi} (Q_c + 3) \sqrt{\frac{4\lambda_c}{\eta_c} - \frac{\lambda^2}{4} (Q_c - 1)^2} = 1. \]  
\[ (46) \]

By making use of (32), we have
\[ \xi'(A_c) = \frac{d}{dA} [u_w(y_1) - 1]|_{A=A_c}, \]
\[ = \left[ 1 - \frac{1}{2(1 - \frac{\lambda_c \eta_c (Q_c - 1)^2}{16})} \right] \frac{d}{dA} \ln \hat{A}_{A=A_c}, \]
\[ (47) \]
where if \( \frac{dA}{dA_c} \neq 0 \), then \( \xi'(A_c) \neq 0 \). So by the same procedure in the previous subsection, one can obtain
\[ F_s - F_w = \gamma \xi^3. \]  
\[ (48) \]

By substitute (47) in (32) and then in (48), we have
\[ F_s - F_w = \gamma' (A - A_c)^3, \]  
\[ (49) \]
where \( \gamma' \) is a constant which is independent of \( \hat{A} \) and therefore of \( A \), so the order of transition of this model is 3.

### 4 Conclusion

We study \( \phi^{2k} \) and \( \phi^2 + \frac{2a}{3} \phi^3 \) models for nlgYM\(_2\) theories, and obtained that all order of \( \phi^{2k} \) model has phase transition only on sphere and the order of this transition is three. Also by considering the \( \phi^2 + \frac{2a}{3} \phi^3 \) model of nlgYM\(_2\), we found that, this theory has third order phase transition on compact manifold with \( g > 1 \) and there is no phase transition on sphere. Note that the whole reasoning is independent of the number of points at them \( u_w \) attains...
its absolute maximum. It is clear that similar situation prevails for the cases which \( u_w \) has many absolute maximum. In this case, one can easily realize the WCP regime as the same technique in Preliminaries section, and then obtain the phase transition of theory in the multi - critical points. Also remark that the critical value \( \eta_c \) for any particular model is fixed, therefore, increasing \( \hat{A} \) the genus increases proportionally in order to keep fixed the critical number of handles per area, \( \eta_c \). So if we fixed the modified area \( \hat{A} \) of the surface, then with the increase of the genus the number of multi - critical points decreases and for \( g > 1 + \frac{\hat{A}}{\eta_c} \) there is no phase transition.

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