A SHUFFLING FRAMEWORK FOR LOCAL DIFFERENTIAL PRIVACY

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ABSTRACT

LDP deployments are vulnerable to inference attacks as an adversary can link the noisy responses to their identity and subsequently, auxiliary information using the order of the data. An alternative model, shuffle DP, prevents this by shuffling the noisy responses uniformly at random. However, this limits the data learnability – only symmetric functions (input order agnostic) can be learned. In this paper, we strike a balance and propose a generalized shuffling framework that interpolates between the two deployment models. We show that systematic shuffling of the noisy responses can thwart specific inference attacks while retaining some meaningful data learnability. To this end, we propose a novel privacy guarantee, \( d_\sigma \)-privacy, that captures the privacy of the order of a data sequence. \( d_\sigma \)-privacy allows tuning the granularity at which the ordinal information is maintained, which formalizes the degree the resistance to inference attacks trading it off with data learnability. Additionally, we propose a novel shuffling mechanism that can achieve \( d_\sigma \)-privacy and demonstrate the practicality of our mechanism via evaluation on real-world datasets.

1 Introduction

Differential Privacy (DP) and its local variant (LDP) are the most commonly accepted notions of data privacy. LDP has the significant advantage of not requiring a trusted centralized aggregator, and has become a popular model for commercial deployments, such as those of Microsoft [16], Apple [34], and Google [25, 27, 10]. Its formal guarantee asserts that an adversary cannot infer the value of an individual’s private input by observing the noisy output. However in practice, a vast amount of public auxiliary information, such as address, social media connections, court records, property records [3], income and birth dates [4], is available for every individual. An adversary, with access to such auxiliary information, can learn about an individual’s private data from several other participants’ noisy responses. We illustrate this as follows.

Problem. An analyst runs a medical survey in Alice’s community to investigate how the prevalence of a highly contagious disease changes from neighborhood to neighborhood. Community members report a binary value indicating whether they have the disease.

Next, consider the following two data reporting strategies.

Strategy 1. Each data owner passes their data through an appropriate randomizer (that flips the input bit with some probability) in their local devices and reports the noisy output to the untrusted data analyst.

Strategy 2. The noisy responses from the local devices of each of the data owners are collected by an intermediary trusted shuffler which dissociates the device IDs (metadata) from the responses and uniformly randomly shuffles them before sending them to the analyst.
Our scheme:

LDP

Uniform shuffle

\(\epsilon\)

\(1\)

\(0\)

\(1\)

\(\alpha\)

\(r\)

\(a\)

\(b\)

\(c\)

\(d\)

\(e\)

\(f\)

\(g\)

\(h\)

\(i\)

\(\text{Attack: unif. shuff.}\)

\(\text{Attack: LDP}\)

\(\text{Our scheme: } r_a\)

\(\text{Our scheme: } r_b\)

\(\text{Uniform shuffle}\)

Figure 1: Demonstration of how our proposed scheme thwarts inference attacks at different granularities. Fig. 1a depicts the original sensitive data (such as income bracket) with eight color-coded labels. The position of the points represents public information (such as home address) used to correlate them. There are three levels of granularity: warm vs. cool clusters, blue vs. green and red vs. orange crescents, and light vs. dark within each crescent. Fig. 1b depicts LDP. Fig. 1c and 1d correspond to our scheme, each with \(\alpha = 1\) (privacy parameter, Def. 3.3). The former uses a smaller distance threshold \((r_1, \text{used to delineate the granularity of grouping – see Eq. 3})\) that mostly shuffles in each cluster. The latter uses a larger distance threshold \((r_2)\) that shuffles within each cluster. Figures in the bottom row demonstrate an inference attack (uses Gaussian process correlation) on all four cases. We see that LDP reveals almost the entire dataset (Fig. 1f) while uniform shuffling prevents all classification (1i). However, the granularity can be controlled with our scheme (Figs. 1g, 1h).

**Strategy 1** corresponds to the standard LDP deployment model (for example, Apple and Microsoft’s deployments). Here the order of the noisy responses is informative of the identity of the data owners – the noisy response at index 1 corresponds to the first data owner and so on. Thus, the noisy responses can be directly linked with its associated device ID and subsequently, auxiliary information. For instance, an adversary\(^a\) may know the home addresses of the participants and use this to identify the responses of all the individuals from Alice’s household. Being highly infectious, all or most of them will have the same true value (0 or 1). So, the adversary can reliably infer Alice’s value by taking a simple majority vote of her and her household’s noisy responses. Note that this does not violate the LDP guarantee since the inputs are appropriately randomized when observed in isolation. We call such threats inference attacks – recovering an individual’s private input using all or a subset of other participants’ noisy responses. It is well known that protecting against inference attacks, that rely on underlying data correlations, is beyond the purview of DP [42, 44, 40, 52].

**Strategy 2** corresponds to the recently introduced shuffle DP model, such as Google’s Prochlo [10]. Here, the noisy responses are completely anonymized – the adversary cannot identify which LDP responses correspond to Alice and her household. Under such a model, only information that is completely order agnostic (i.e., symmetric functions that can be computed over just the bag of values, such as aggregate statistics) can be extracted. Consequently, the analyst also fails to accomplish their original goal as all the underlying data correlation is destroyed.

Thus, we see that the two models of deployment for LDP present a trade-off between vulnerability to inference attacks and scope of data learnability. In fact, as demonstrated by Kifer et. al [41], it is impossible to defend against all inference attacks while simultaneously maintaining utility for learning. In the extreme case that the adversary knows everyone in Alice’s community has the same true value (but not which one), no mechanism can prevent revelation of Alice’s datapoint short of destroying all utility of the dataset. This then begs the question: *Can we formally suppress specific inference attacks targeting each data owner while maintaining some meaningful learnability of the private data?* Referring back to our example, can we thwart attacks inferring Alice’s data using specifically her households’ responses and still allow the medical analyst to learn its target trends? Can we offer this to every data owner participating?

In this paper, we strike a balance and we propose a generalized shuffle framework for deployment that can interpolate between the two extremes. Our solution is based on the key insight is that the order of the data acts as the proxy for the identity of data owners as illustrated above. The granularity at which the ordering is maintained formalizes resistance to inference attacks while retaining some meaningful learnability of the private data. Specifically, we guarantee each data owner that their data is shuffled together with a carefully chosen group of other data owners. Revisiting our example, consider uniformly shuffling the responses from Alice’s household and her immediate neighbors. Now an adversary cannot use her household’s responses to predict her value any better than they could with a random sample of responses from this group. In the same way that LDP prevents reconstruction of her datapoint using specifically her noisy response, this scheme prevents reconstruction of her datapoint using specifically her households’ responses. The real challenge is offering such guarantees equally to every data owner. Bob, Alice’s neighbor, needs his households’ responses shuffled in with his neighbors, as does Luis, a neighbor of Bob, who is not Alice’s neighbor. In this way, we

\(^a\)The analyst and the adversary could be same, we refer to them separately for the ease of understanding.
have \( n \) data owners with \( n \) distinct groups that are most likely overlapping with each other. This disallows the trivial strategy of shuffling the noisy responses of each group uniformly. To this end, we propose shuffling the responses in a systematic manner that tunes the privacy guarantee, trading it off with data learnability. For the above example, our scheme can formally protect each data owner from inference attacks using specifically their household, while still learning how disease prevalence changes across the neighborhoods of Alice’s community. This work offers two key contributions to the machine learning privacy literature:

- **Novel privacy guarantee.** We propose a novel privacy definition, \( d_{\sigma} \)-privacy that captures the privacy of the order of a data sequence (Sec. 3.2) and formalizes the degree of resistance against inference attacks (Sec. 3.3). Intuitively, \( d_{\sigma} \)-privacy allows assigning a group, \( G_i \), for each data owner, \( DO_i \), \( i \in [n] \), and protects \( DO_i \) against inference attacks that utilize the data of any subset of members of \( G_i \). The group assignment is based on a public auxiliary information – individuals of a single group are ‘similar’ w.r.t the auxiliary information. For instance, the groups can represent individuals in the same age bracket, ‘friends’ on social media or individuals living in each other’s vicinity (as in case of Alice in our example). This grouping determines a threshold of learnability – any learning that is order agnostic within a group (disease prevalence in a neighborhood – the data analyst’s goal in our example) is utilitarian and allowed; whereas analysis that involves identifying the values of individuals within a group (disease prevalence within specific households – the adversary’s goal) is regarded as a privacy threat and protected against. See Fig. 1 for a toy demonstration of how our guarantee allows tuning the granularity at which trends can be learned.

- **Novel shuffling framework.** We propose a novel mechanism that shuffles the data systematically and achieves \( d_{\sigma} \)-privacy. This provides us a generalized shuffle framework for deployment that can interpolate between no shuffling (LDP) and uniform random shuffling (shuffle model). Our experimental results (Sec. 4) demonstrates its efficacy against realistic inference attacks.

1.1 Related Work

The shuffle model of DP [11, 13, 24] differs from our scheme as follows. These works (1) study DP benefits of shuffling where we study the inferential privacy benefits and (2) only study uniformly random shuffling where ours generalizes this to tunable, non-uniform shuffling (see App. 7.12).

A steady line of work has studied inferential privacy [37, 41, 33, 15, 21, 52]. Our work departs from those in that we focus on local inferential privacy and do so via the new angle of shuffling.

Older works such as \( k \)-anonymity [50], \( l \)-diversity [45], Anatomy [55], and others [54, 51, 56, 14, 18] have studied the privacy risk of non-sensitive auxiliary information, or ‘quasi identifiers’. These works (1) focus on the setting of dataset release, where we focus on dataset collection and (2) do not offer each data owner formal inferential guarantees, whereas this work does.

The De Finetti attack [40] shows how shuffling schemes are vulnerable to inference attacks that correlate records together to recover the original permutation of sensitive attributes. A strict instance of our privacy guarantee can thwart such attacks (at the cost of no utility, App. 7.2).

2 Background

Notations. **Boldface** (such as \( x = (x_1, \cdots, x_n) \)) denotes a data sequence (ordered list); normal font (such as \( x_1 \)) denotes individual values and \( \{ \cdot \} \) represents a set.

2.1 Local Differential Privacy

The local model consists of a set of data owners and an untrusted data aggregator (analyst); each individual perturbs their data using a LDP algorithm (randomizers) and sends it to the analyst. The LDP guarantee is formally defined as

**Definition 2.1.** [Local Differential Privacy, LDP [53, 26, 38]] A randomized algorithm \( M : \mathcal{X} \to \mathcal{Y} \) is \( \epsilon \)-locally differentially private (or \( \epsilon \)-LDP), if for any pair of private values \( x, x' \in \mathcal{X} \) and any subset of output,

\[
Pr[M(x) \in W] \leq e^\epsilon \cdot Pr[M(x') \in W]
\]

In an extension of the local model, known as the shuffle model [24, 13, 7], the data owners randomize their inputs like in the local model. Additionally, an intermediate trusted shuffler applies a uniformly random permutation to all the noisy responses before the analyst can view them. The anonymity provided by the shuffler requires less noise than the local model for achieving the same privacy.

2.2 Local Inferential Privacy

Local inferential privacy captures what information a Bayesian adversary [42], with some prior, can learn in the LDP setting. Specifically, it measures the largest possible ratio between the adversary’s posterior and prior beliefs about an individual’s data after observing a mechanism’s output.
When the noisy response sequence \( y \) where \( \sigma \) is determined via \( \sigma(x) = (1 3 5 4 2) \) and \( x = (21, 33, 45, 65, 67) \), then \( \sigma(y) = (21, 45, 67, 65, 33) \). The Mallows model is a popular probabilistic model for permutations [46]. The mode of the distribution is given by \( \sigma \) - the probability of a permutation increases as we move closer to \( \sigma_0 \) as measured by rank distance metrics, such as the Kendall’s tau distance (Def. 7.1). The dispersion parameter \( \theta \) controls how fast this increase happens.

**Definition 2.3.** For a dispersion parameter \( \theta \), a reference permutation \( \sigma_0 \in S_n \), and a rank distance measure \( \delta : S_n \times S_n \mapsto \mathbb{R} \), \( P_{\Theta, \delta}(\sigma : \sigma_0) = \frac{1}{\psi(\theta, \delta)} e^{-\theta \delta(\sigma, \sigma_0)} \) is the Mallows model where \( \psi(\theta, \delta) = \sum_{\sigma \in S_n} e^{-\theta \delta(\sigma, \sigma_0)} \) is a normalization term and \( \sigma \in S_n \).

### 3 Data Privacy and Shuffling

In this section, we present \( d_o \)-privacy and a shuffling mechanism capable of achieving the \( d_o \)-privacy guarantee.

#### 3.1 Problem Setting

In our problem setting, we have \( n \) data owners \( \text{DO}_i, i \in [n] \) each with a private input \( x_i \in \mathcal{X} \) (Fig. 2). The data owners first randomize their inputs via a \( \epsilon \)-LDP mechanism to generate \( y_i = \mathcal{M}(x_i) \). We consider an informed adversary with public auxiliary information \( t = \langle t_1, \ldots, t_n \rangle, t_i \in \mathcal{T} \) about each individual. Additionally, just like in the shuffle model, we have a trusted shuffler. It mediates upon the noisy responses \( y = \langle y_1, \ldots, y_n \rangle \) and systematically shuffles them based on \( t \) (since \( t \) is public, it is also accessible to the shuffler) to obtain the final output sequence \( z = \mathcal{A}(y) \) (\( \mathcal{A} \) corresponds to Alg. 1) which is sent to the untrusted data analyst. Next, we formally discuss the notion of order and its implications.

**Definition 3.1.** (Order) The order of a sequence \( x = \langle x_1, \ldots, x_n \rangle \) refers to the indices of its set of values \( \{x_i\} \) and is represented by permutations from \( S_n \). When the noisy response sequence \( y = \langle y_1, \ldots, y_n \rangle \) is represented by the identity permutation \( \sigma_I = (1 2 \cdots n) \), the value at index 1 corresponds to \( \text{DO}_1 \) and so on. Standard LDP releases the identity permutation w.p. 1. The output of the shuffler, \( z \), is some permutation of the sequence \( y \), i.e.,

\[
z = \sigma(y) = \langle y_{\sigma(1)}, \ldots, y_{\sigma(n)} \rangle
\]

where \( \sigma \) is determined via \( \mathcal{A}(\cdot) \). For example, for \( \sigma = (4 5 2 3 1) \), we have \( z = \langle y_4, y_5, y_2, y_3, y_1 \rangle \) which means that the value at index 1 (\( \text{DO}_1 \)) now corresponds to that of \( \text{DO}_4 \) and so on.

#### 3.2 Definition of \( d_o \)-privacy

Inferential risk captures the threat of an adversary who infers \( \text{DO}_i \)'s private \( x_i \) using all or a subset of other data owners’ released \( y_j \)'s. Since we cannot prevent all such attacks and maintain utility, our aim is to formally limit which data owners can be leveraged in inferring \( \text{DO}_i \)'s private \( x_i \). To make this precise, each \( \text{DO}_i \) is assigned a corresponding group, \( G_i \subseteq [n] \), of data owners. Each \( G_i \) consists of all those \( \text{DO}_j \)s who are similar to \( \text{DO}_i \) w.r.t auxiliary information \( t_i, t_j \) according to some distance measure \( d : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R} \). Here, we define ‘similar’ as being under a threshold \( r \in \mathbb{R} \).

\[
G_i = \{ j \in [n] | d(t_i, t_j) \leq r \}, \forall i \in [n]; \quad \mathcal{G} = \{ G_1, \ldots, G_n \}
\]
For example, $d(\cdot)$ can be Euclidean distance if $T$ corresponds to geographical locations, thwarting inference attacks using one’s immediate neighbors. If $T$ represents a social media connectivity graph, $d(\cdot)$ can measure the path length between two nodes, thwarting inference attacks using specifically one’s friends. For the example social media connectivity graph depicted in Fig. 3, assuming distance metric path length and $r = 2$, the groups are defined as $G_1 = \{1, 7, 8, 2, 5, 6\}$, $G_2 = \{2, 1, 7, 5, 6, 3\}$ and so on.

Intuitively, $d_\sigma$-privacy protects $DO_i$ against inference attacks that leverages correlations at a finer granularity than $G_i$. In other words, under $d_\sigma$-privacy, one subset of $k$ data owners $\subset G_i$ (e.g. household) is no more useful for targeting $x_i$ than any other subset of $k$ data owners $\subset G_i$ (e.g. some combination of neighbors). This leads to the following key insight for the formal privacy definition.

**Key Insight.** Formally, our privacy goal is to prevent the leakage of ordinal information from within a group. We achieve this by systematically bounding the dependence of the mechanism’s output on the relative ordering (of data values corresponding to the data owners) within each group.

First, we introduce the notion of neighboring permutations.

**Definition 3.2.** (Neighboring Permutations) Given a group assignment $G$, two permutations $\sigma, \sigma' \in S_n$ are defined to be neighboring w.r.t. a group $G_i \in G$ (denoted as $\sigma \approx_{G_i} \sigma'$) if $\sigma(j) = \sigma'(j)$ $\forall j \notin G_i$.

Neighboring permutations differ only in the indices of its corresponding group $G_i$. For example, $\sigma = (\{1, 2, 4, 5, 7, 6, 10, 3, 8, 9\})$ and $\sigma' = (\{7, 4, 5, 6, 2, 1, 10, 8, 9\})$ are neighboring w.r.t $G_i$ (Fig.3) since they differ only in $\sigma(1), \sigma(2), \sigma(5), \sigma(6), \sigma(7)$ and $\sigma(8)$. We denote the set of all neighboring permutations as

$$N_G = \{(\sigma, \sigma') | \sigma \approx_{G_i} \sigma', \forall G_i \in G\}$$

(4)

Now, we formally define $d_\sigma$-privacy as follows.

**Definition 3.3** ($d_\sigma$-privacy). For a given group assignment $G$ on a set of $n$ entities and a privacy parameter $\alpha \in \mathbb{R}_{>0}$, a randomized mechanism $A : \mathcal{Y}^n \rightarrow \mathcal{V}$ is $(\alpha, G)$-$d_\sigma$ private if for all $y \in \mathcal{Y}^n$ and neighboring permutations $\sigma, \sigma' \in N_G$ and any subset of output $O \subseteq \mathcal{V}$, we have

$$\Pr[A(\sigma(y)) \in O] \leq e^\alpha \cdot \Pr[A(\sigma'(y)) \in O]$$

(5)

$\sigma(y)$ and $\sigma'(y)$ are defined to be neighboring sequences.

$d_\sigma$-privacy states that, for any group $G_i$, the mechanism is (almost) agnostic of the order of the data within the group. Even after observing the output, an adversary cannot learn about the relative ordering of the data within any group. Thus, two neighboring sequences are indistinguishable to an adversary.

An important property of $d_\sigma$-privacy is that post-processing computations on the output of a $d_\sigma$-private algorithm does not degrade privacy. Additionally, when applied multiple times, the privacy guarantee degrades gracefully. Both the properties are analogous to that of DP and the formal theorems are presented in App. 7.3. Interestingly, LDP mechanisms achieve a weak degree of $d_\sigma$-privacy.

**Lemma 3.1.** An $\epsilon$-LDP mechanism is $(\kappa e, G)-d_\sigma$ private for any group assignment $G$ such that $k \geq \max_{G_i \in G} |G_i|$ (proof in App. 7.3).

### 3.3 Privacy Implications

We now turn to $d_\sigma$-privacy’s semantic guarantees: what can/cannot be learned from the released sequence $z$? The group assignment $G$ delineates a threshold of learnability as follows.

- **Learning allowed.** $d_\sigma$-privacy can answer queries that are order agnostic within groups, such as aggregate statistics of a group. In Alice’s case, the analyst can estimate the disease prevalence in her neighborhood.

- **Learning disallowed.** Adversaries cannot identify (noisy) values of individuals within any group. While they may learn the disease prevalence in Alice’s neighborhood, they cannot determine the prevalence within her household and use that to target her value $x_i$.

Consider any Bayesian adversary with a prior $P$ on the joint distribution of noisy responses, $Pr_{\mathcal{P}}[y]$, modeling their beliefs on the correlation between participants in the dataset (such as the correlation between Alice and her household’s disease status). As with early DP works, such as [20], we consider an informed Bayesian adversary. With DP, informed adversaries know the private input of every data owner but $DO_i$. With $d_\sigma$-privacy, the informed adversary knows (1) the assignment of noisy values outside $G_i$, $y_{\mathcal{T},i}$, and (2) the unordered bag of noisy values in $G_i$, $\{y_{G_i}\}$. Formally,
Theorem 3.2. For a given group assignment $G$ on a set of $n$ data owners, if a shuffling mechanism $A : Y^n \rightarrow Y^n$ is $(\alpha, G)$-LDP private, then for each data owner $DO_i, i \in [n]$,
\[
\mathbb{L}_p(x) = \max_{a,b \in X} \left| \log \frac{\Pr_P[z_i = a_i \{y_{G_i}, y_{G_i}^\prime\}]}{\Pr_P[z_i = b_i \{y_{G_i}, y_{G_i}^\prime\}]} \right| \leq \alpha
\]
for a prior distribution $P$, where $z = A(y)$ and $y_{G_i}^\prime$ is the noisy sequence for all data owners outside $G_i$ (proof in App. 7.4).

The above privacy loss variable differs slightly from that of Def. 2.2, since the informed adversary already knows $\{y_{G_i}\}$ and $y_{G_i}^\prime$. Equivalently, this bounds the privacy-posterior odds gap on $x_i$ –
\[
\left| \log \frac{\Pr_P[x_i = a_i \{z, \{y_{G_i}, y_{G_i}^\prime\}\}]}{\Pr_P[x_i = b_i \{z, \{y_{G_i}, y_{G_i}^\prime\}\}]} - \log \frac{\Pr_P[x_i = a_i \{y_{G_i}, y_{G_i}^\prime\}]}{\Pr_P[x_i = b_i \{y_{G_i}, y_{G_i}^\prime\}]} \right| \leq \alpha
\]

We illustrate this with the following example on $t_{e,g}$ (Fig. 3). The adversary knows (i.e. prior $\Pr_P[y|t_{e,g}]$) that data owner DO1 is strongly correlated with their close friends DO2 and DO7. Let $y = (1,1,y_3,y_4,0,1,0,y_9,y_10)$ and $y' = (0,0,y_3,y_4,1,0,1,y_9,y_10)$ represent two neighboring sequences w.r.t $G_1 = \{1,7,8,2,6,5\}$, for any $y_3,y_4,y_9,y_{10} \in \{0,1\}^4$. Under $d_\sigma$-privacy, the adversary cannot distinguish between $y$ (data sequence where DO1, DO3 and DO7 have value 1) and $y'$ (data sequence where DO1, DO2 and DO7 have value 0). Hence, after seeing the shuffled sequence, the adversary can only know the ‘bag’ of values $\{y_{G_1}\} = \{0,1,0,1,0\}$ and cannot specifically leverage DO1’s immediate friends’ responses $\{y_2,y_7\}$ to target $x_1$. However, analysts may still answer queries that are order-agnostic in $G_1$, which could not be achieved with uniform shuffling.

Note. By the post-processing [22] property of LDP, the shuffled sequence $z$ retains the $e$-LDP guarantee. The granularity of the group assignment determined by distance threshold $r$ and the privacy degree $\alpha$ act as control knobs of the privacy spectrum. For instance w.r.t. $t_{e,g}$ (Fig. 3), for $r = 0$, we have $G_i = \{i\}$ and the problem reduces to the pure LDP setting. For $r = \infty$, $\alpha = 0$, we get $G_i = [n]$ which corresponds to the case of uniform random shuffling (standard shuffle model). All other pairs of $(r, \alpha)$ represent intermediate points in the privacy spectrum which are achievable via $d_\sigma$-privacy.

3.4 Utility of a Shuffling Mechanism

We now introduce a novel metric, $(\eta, \delta)$-preservation, for assessing the utility of any shuffling mechanism. Let $S \subseteq [n]$ correspond to a set of indices in $y$. The metric is defined as follows.

Definition 3.4. ($(\eta, \delta)$-preservation). A shuffling mechanism $A : Y^n \rightarrow Y^n$ is defined to be $(\eta, \delta)$-preserving $(\eta, \delta \in [0,1])$ w.r.t to a given subset $S \subseteq [n]$, if
\[
\Pr \left[ |S_\sigma \cap S| \geq \eta \cdot |S| \right] \geq 1 - \delta, \sigma \in S_n
\]
where $z = A(y) = \sigma(y)$ and $S_\sigma = \{\sigma(i)|i \in S\}$.

For example, consider $S = \{1,4,5,7,8\}$. If $A(\cdot)$ permutes the output according to $\sigma = (5 2 6 7 9 4 1 10)$, then $S_\sigma = \{5,6,7,8,1\}$ which preserves 4 or 80% of its original indices. This means that for any data sequence $y$, at least $\eta$ fraction of its data values corresponding to the subset $S$ overlaps with that of shuffled sequence $z$ with high probability $1 - \delta$. Assuming, $(ys) = \{y_i|i \in S\}$ and $(zs) = \{z_i|i \in S\} = \{y_{\sigma(i)}(i)|i \in S\}$ denotes the set of data values corresponding to $S$ in data sequences $y$ and $z$ respectively, we have $\Pr \left[ |(ys) \cap (zs)| \geq \eta \cdot |S| \right] \geq 1 - \delta, \forall y$. For example, let $S$ be the set of individuals from Nevada. Then, for a shuffling mechanism that provides $(\eta = 0.8, \delta = 0.1)$-preservation to $S$, with probability $> 0.9$, $> 80\%$ of the values that are reported to be from Nevada in $z$ are genuinely from Nevada. The rationale behind this metric is that it captures the utility of the learning allowed by $d_\sigma$-privacy – if $S$ is equal to some group $G \in G_i$, $(\eta, \delta)$ preservation allows overall statistics of $G$ to be captured. Note that this utility metric is agnostic of both the data distribution and the analyst’s query. Hence, it is a conservative analysis of utility which serves as a lower bound for learning from $\{zs\}$.

3.5 $d_\sigma$-private Shuffling Mechanism

We now describe our novel shuffling mechanism that can achieve $d_\sigma$-privacy. In a nutshell, our mechanism samples a permutation from a suitable Mallows model and shuffles the data sequence accordingly. We can characterize the $d_\sigma$-privacy guarantee of our mechanism in the same way as that of the DP guarantee of classic mechanisms [22] – with variance and sensitivity. Intuitively, a larger dispersion parameter $\theta \in \mathbb{R}$ (Def. 2.3) reduces randomness over
permutations, increasing utility and increasing (worsening) the privacy parameter $\alpha$. The maximum value of $\theta$ for a given $\alpha$ guarantee depends on the sensitivity of the rank distance measure $\delta(\cdot)$ over all neighboring permutations $N_G$. Formally, we define the sensitivity as

$$\Delta(\sigma_0 : \delta, G) = \max_{(\sigma, \sigma') \in N_G} |\delta(\sigma_0 \sigma, \sigma_0) - \delta(\sigma_0 \sigma', \sigma_0)|,$$

the maximum change in distance $\delta(\cdot)$ from the reference permutation $\sigma_0$ for any pair of neighboring permutations $(\sigma, \sigma') \in N_G$ after applying $\sigma_0$ to them, $(\sigma_0 \sigma, \sigma_0 \sigma')$. The privacy parameter of the mechanism is then proportional to its sensitivity $\alpha = \theta \cdot \Delta(\sigma_0 : \delta, G)$.

Given $G$ and a reference permutation $\sigma_0$, the sensitivity of a rank distance measure $\delta(\cdot)$ depends on the width, $\omega_G^\sigma$, which measures how ‘spread apart’ the members of any group of $G$ are in $\sigma_0$ –

$$\omega_G^\sigma = \max_{(i, k) \in E_i \times G_i} |\sigma^{-1}(j) - \sigma^{-1}(k)|, i \in [n]$$

$$\omega_G^\sigma = \max_{G_i \in G} \omega_G^{\sigma_i}.$$

For example, for $\sigma = (1\ 3\ 7\ 8\ 4\ 5\ 2\ 9\ 10)$ and $G_1 = \{1, 7, 8, 2, 5, 6\}$, $\omega_{G_1}^\sigma = |\sigma^{-1}(1) - \sigma^{-1}(2)| = 7$. The sensitivity is an increasing function of the width. For instance, for Kendall’s $\tau$ distance $\delta(\cdot)$ we have $\Delta(\sigma_0 : \delta, G) = \omega_G^\sigma(\omega_G^\sigma + 1)/2$. If a reference permutation clusters the members of each group closely together (low width) the groups are more likely to permute within themselves. This has two benefits. First, if a group is likely to shuffle within itself, it will have better ($\eta, \delta$)-preservation (see App. 7.10 for demonstration). Second, for the same $\theta$ (always an indicator of utility as it determines the dispersion of the sampled permutation), a lower value of width gives lower $\alpha$ (better privacy).

Unfortunately, minimizing $\omega_G^\sigma$ is an NP-hard problem (Thm. 7.3 in App. 7.6). We instead estimate the optimal $\sigma_0$ using the following heuristic approach based on a graph breadth first search.

**Algorithm Description.** Alg. 1 above proceeds as follows. We first compute the group assignment, $G$, based on the public auxiliary information and desired distance threshold $r$ following Eq. 3 (Step 1). Then we construct $\sigma_0$ with a breadth first search (BFS) graph traversal.

We translate $G$ into an undirected graph $(V, E)$, where the vertices are indices $V = [n]$ and two indices $i, j$ are connected by an edge if they are both in some group (Step 2). Next, $\sigma_0$ is computed via a breadth first search traversal (Step 4) – if the $k$-th node in the traversal is $i$, then $\sigma_0(k) = i$. The rationale is that neighbors of $i$ (members of $G_i$) would be traversed in close succession. Hence, a neighboring node $j$ is likely to be traversed at some step $h$ near $k$ which means $|\sigma^{-1}_0(i) - \sigma^{-1}_0(j)| = |h - k|$ would be small (resulting in low width). Additionally, starting from the node with the highest degree (Steps 3-4) which corresponds to the largest group in $G$ (lower bound for $\omega_G^\sigma$ for any $\sigma$) helps to curtail the maximum width in $\sigma_0$

This is followed by the computation of the dispersion parameter, $\theta$, for our Mallows model (Steps 5-6). Next, we sample a permutation from the Mallows model (Step 7) $\hat{\sigma} \sim P_\theta(\sigma : \sigma_0)$ and we apply the inverse reference permutation to it, $\sigma^* = \sigma^{-1}_0 \hat{\sigma}$ to obtain the desired permutation for shuffling. Recall that $\hat{\sigma}$ is (most likely) close to $\sigma_0$, which is unrelated to the original order of the data. $\sigma^{-1}_0$ therefore brings $\sigma^*$ back to a shuffled version of the original sequence (identity permutation $\sigma_0$). Note that since Alg. 1 is publicly known, the adversary/analyst knows $\sigma_0$. Hence, even in the absence of this step from our algorithm, the adversary/analyst could perform this anyway. Finally, we permute $y$ according to $\sigma^*$ and output the result $z = \sigma^*(y)$ (Steps 9-10).

**Theorem 3.3.** Alg. 1 is $(\alpha, G)$-$d$-private where $\alpha = \theta \cdot \Delta(\sigma_0 : \delta, G)$.

The proof is in App. 7.8. Note that Alg. 1 provides the same level of privacy ($\alpha$) for any two group assignment $G, G'$ as long as they have the same sensitivity, i.e., $\Delta(\sigma_0 : \delta, G) = \Delta(\sigma_0 : \delta, G')$. This leads to the following theorem which generalizes the privacy guarantee for any group assignment.

**Theorem 3.4.** Alg. 1 satisfies $(\alpha', G')$- for any group assignment $G'$ where $\alpha' = \frac{\Delta(\sigma_0 : \delta, G')}{\Delta(\sigma_0 : \delta, G)}$. 

---

**Algorithm 1: $d$-private Shuffling Mech.**

**Input:** LDP sequence $y = (y_1, \ldots, y_n)$; 
Public aux. info. $t = (t_1, \ldots, t_n)$; 
Dist. threshold $r$; Priv. param. $\alpha$;

**Output:** z - Shuffled output sequence;

1. $G = \text{ComputeGroupAssignment}(t, r)$;
2. Construct graph $G$ with
   a) vertices $V = \{1, 2, \ldots, n\}$
   b) edges $E = \{(i, j) : i \in G_i, G_j \in G\}$
3. $\text{root} = \arg \max_{i \in [n]} |G_i|$;
4. $\sigma_0 = \text{BFS}(G, \text{root})$;
5. $\Delta = \text{ComputeSensitivity}(\sigma_0, G)$
6. $\theta = \alpha / \Delta$;
7. $\hat{\sigma} \sim P_\theta(\sigma : \sigma_0)$;
8. $\sigma^* = \sigma^{-1}_0 \hat{\sigma}$;
9. $z = (y_{\sigma^{-1}(1)} \cdot \ldots \cdot y_{\sigma^{-1}(n)})$;
10. Return $z$.
The proof is in App. 7.9. A utility theorem for Alg. 1 that formalizes the $(\eta,\delta)$-preservation for Hamming distance $\delta_H(\cdot)$ (we chose $\delta_H(\cdot)$ for the ease of numerical computation) is in App. 7.10.

**Note.** Producing $\sigma^*$ is completely data ($y$) independent. It only requires access to the public auxiliary information $t$. Hence, Steps 1 – 6 can be performed in a pre-processing phase and do not contribute to the actual running time. See App. 7.7 for an illustration of Alg. 1.

4 Evaluation

![diagram](image)

Figure 4: Our scheme interpolates between standard LDP (orange line) and uniform shuffling (blue line) in both privacy and data learnability. All plots increase group size along x-axis (except (d)). (a) → (c): The fraction of participants vulnerable to an inferential attack. (d): Attack success with varying $\alpha$ for a fixed $r$. (e) → (g): The accuracy of a calibration model trained on $z$ predicting the distribution of LDP outputs at any point $t \in T$, such as the distribution of medical insurance types used specifically in the Houston area (not possible when uniformly shuffling across Texas). (h): Test accuracy of a classifier trained on $z$ for the synthetic dataset in Fig. 1.

The previous sections describe how our shuffling framework interpolates between standard LDP and uniform random shuffling. We now experimentally evaluate this asking the following two questions –

**Q1.** Does the Alg. 1 mechanism protect against realistic inference attacks?

**Q2.** How well can Alg. 1 tune a model’s ability to learn trends within the shuffled data i.e. tune data learnability?

We evaluate on four datasets. We are not aware of any prior work that provides comparable local inferential privacy. Hence, we baseline our mechanism with the two extremes: standard LDP and uniform random shuffling. For concreteness, we detail our procedure with the PUDF dataset [2] (license), which comprises $n \approx 29k$ psychiatric patient records from Texas. Each data owner’s sensitive value $x_i$ is their medical payment method, which is reflective of socioeconomic class (such as medicaid or charity). Public auxiliary information $t \in T$ is the hospital’s geolocation. Such information is used for understanding how payment methods (and payment amounts) vary from town to town for insurances in practice [23]. Uniform shuffling across Texas precludes such analyses. Standard LDP risks inference attacks, since patients attending hospitals in the same neighborhood have similar socioeconomic standing and use similar payment methods, allowing an adversary to correlate their noisy $y_i$’s. To trade these off, we apply Alg. 1 with $d(\cdot)$ being distance (km) between hospitals, $\alpha = 4$ and Kendall’s $\tau$ rank distance measure for permutations.

Our inference attack predicts $DO_i$’s $x_i$ by taking a majority vote of the $z_j$ values of the 25 data owners within $r^*$ of $t_i$ and who are most similar to $DO_i$ w.r.t some additional privileged auxiliary information $t_{ij} \in T_p$. For PUDF, this includes the 25 data owners who attended hospitals that are within $r^*$ km of $DO_i$’s hospital, and are most similar in payment amount $t_{ij}$. Using an $\epsilon = 2.5$ randomized response mechanism, we resample the LDP sequence $y$ 50 times, and apply Alg. 1’s chosen permutation to each, producing 50 $z$’s. We then mount the majority vote attack on each $x_i$ for each $z$. If the attack on a given $x_i$ is successful across $\geq 90\%$ of these LDP trials, we mark that data owner as vulnerable – although they randomize with LDP, there is a $\geq 90\%$ chance that a simple inference attack can recover their true value. We record the fraction of vulnerable data owners as $\rho$. We report 1-standard deviation error bars over 10 trials.
Additionally, we evaluate data learnability – how well the underlying statistics of the dataset are preserved across $\mathcal{T}$. For PUDF, this means training a model on the shuffled $z$ to predict the distribution of payment methods used near, for instance, $t_i = \text{Houston}$ for DO$_i$. For this, we train a calibrated model, $\mathcal{C}_1 : \mathcal{T} \rightarrow \mathcal{D}_x$, on the shuffled outputs where $\mathcal{D}_x$ is the set of all distributions on the domain of sensitive attributes $X$. We implement $\mathcal{C}_1$ as a gradient boosted decision tree (GBDT) model [29] calibrated with Platt scaling [47]. For each location $t_i$, we treat the empirical distribution of $x_i$ values within $r^*$ as the ground truth distribution at $t_i$, denoted by $\mathcal{E}(t_i) \in \mathcal{D}_x$. Then, for each $t_i$, we measure the Total Variation error between the predicted and ground truth distributions $\text{TV}(\mathcal{E}(t_i), \mathcal{C}_1(x_i(t_i)))$. We then report $\lambda(r)$ – the average TV error for distributions predicted at each $t_i \in \mathcal{T}$ normalized by the TV error of naively guessing the uniform distribution at each $t_i$. With standard LDP, this task can be performed relatively well at the risk of inference attacks. With uniformly shuffled data, it is impossible to make geographically localized predictions unless the distribution of payment methods is identical in every Texas locale.

We additionally perform the above experiments on the following three datasets

- **Adult** [19]. This dataset is derived from the 1994 Census and has $\approx 33$K records. Whether DO$_i$’s annual income is $\geq 50$k is considered private, $X = \{ \geq 50k, < 50k \}$. $\mathcal{T} = [17, 90]$ is age and $\mathcal{T}_P$ is the individual’s marriage status.
- **Twitch** [48]. This dataset, gathered from the Twitch social media platform, includes a graph of $\approx 9$K edges (mutual friendships) along with node features. The user’s history of explicit language is private $X = \{0, 1\}$. $\mathcal{T}$ is a user’s mutual friendships, i.e. $t_i$ is the $i$’th row of the graph’s adjacency matrix. We do not have any $\mathcal{T}_P$ here, and select the 25 nearest neighbors randomly.
- **Syn.** This is a synthetic dataset of size $20K$ which can be classified at three granularities – 8-way, 4-way and 2-way (Fig. 1a shows a scaled down version of the dataset). The eight color labels are private $X = [8]$; the 2D-positions are public $\mathcal{T} = \mathbb{R}^2$. For learnability, we measure the accuracy of 8-way, 4-way and 2-way GBDT models trained on $z$ on an equal sized test set at each $r$.

**Experimental Results.**

**Q1.** Our formal guarantee on the inferential privacy loss (Thm. 3.2) is described w.r.t to a ‘strong’ adversary (with access to $\{y_{C_i}, y_{\mathcal{T}_C}\}$). Here, we test how well does our proposed scheme (Alg. 1) protect against inference attacks on real-world datasets without any such assumptions. Additionally, to make our attack more realistic, the adversary has access to extra privileged auxiliary information $\mathcal{T}_P$ which is not used by Alg. 10. Fig. 4a→4c show that our scheme significantly reduces the attack efficacy. For instance, $\rho$ is reduced by $2.7X$ at the attack distance threshold $r^*$ for PUDF. Additionally, $\rho$ for our scheme varies from that of LDP$^b$ (minimum privacy) to uniform shuffle (maximum privacy) with increasing $r$ (equivalently group size as in Fig. 4c) thereby spanning the entire privacy spectrum. As expected, $\rho$ decreases with decreasing privacy parameter $\alpha$ (Fig. 4d).

**Q2.** Fig. 4e→4g show that $\lambda$ varies from that of LDP (maximum learnability) to that of uniform shuffle (minimum learnability) with increasing $r$ (equivalently group size), thereby providing tunability. Interestingly, for Adult our scheme reduces $\lambda$ by $1.7X$ at the same $\lambda$ as that of LDP for $r = 1$ (Fig. 4f). Fig. 4h shows that the distance threshold $r$ defines the granularity at which the data can be classified. LDP allows 8-way classification while uniform shuffling allows none. The granularity of classification can be tuned by our scheme – $r_8$, $r_4$ and $r_2$ mark the thresholds for 8-way, 4-way and 2-way classifications, respectively. Experiments on evaluation of $(\eta, \delta)$-preservation are in App. 7.11.

5 Conclusion

In this paper, we propose a generalized shuffling framework that interpolates between standard LDP and uniform random shuffling. We establish a new privacy definition, $\varphi_{\sigma}$-privacy, which casts new light on the inferential privacy benefits of shuffling.

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$^b$Our scheme gives lower $\rho$ than LDP at $r = 0$ because the resulting groups are non-singletons. For instance, for PUDF, $G_i$ includes all individuals with the same zipcode as DO$_i$. 

9
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7 Appendix

7.1 Background Cntd.

Here we define two rank distance measures

**Definition 7.1** (Kendall’s τ Distance). For any two permutations, \( \sigma, \pi \in S_n \), the Kendall’s τ distance \( \delta_\tau(\sigma, \pi) \) counts the number of pairwise disagreements between \( \sigma \) and \( \pi \), i.e., the number of item pairs that have a relative order in one permutation and a different order in the other. Formally,

\[
\delta_\tau(\sigma, \pi) = \left| \{ (i, j) : i < j, [\sigma(i) > \sigma(j) \land \pi(i) < \pi(j)] \lor [\sigma(i) < \sigma(j) \land \pi(i) > \pi(j)] \} \right|
\]

(7)

For example, if \( \sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10) \) and \( \pi = (1 \ 2 \ 3 \ 6 \ 5 \ 4 \ 7 \ 8 \ 9 \ 10) \), then \( \delta_\tau(\sigma, \pi) = 3 \).

Next, Hamming distance measure is defined as follows.

**Definition 7.2** (Hamming Distance). For any two permutations, \( \sigma, \pi \in S_n \), the Hamming distance \( \delta_H(\sigma, \pi) \) counts the number of positions in which the two permutations disagree. Formally,

\[
\delta_H(\sigma, \pi) = \left| \{ i \in [n] : \sigma(i) \neq \pi(i) \} \right|
\]

Repeating the above example, if \( \sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10) \) and \( \pi = (1 \ 2 \ 3 \ 6 \ 5 \ 4 \ 7 \ 8 \ 9 \ 10) \), then \( \delta_H(\sigma, \pi) = 2 \).

7.2 \( d_\sigma \)-privacy and the De Finetti attack

We now show that a strict instance of \( d_\sigma \)-privacy is sufficient for thwarting any de Finetti attack [40] on individuals. The de Finetti attack involves a Bayesian adversary, who, assuming some degree of correlation between data owners, attempts to recover the true permutation from the shuffled data. As written, the de Finetti attack assumes the sequence of sensitive attributes and side information \((x_1, t_1), \ldots, (x_n, t_n)\) are exchangeable: any ordering of them is equally likely. By the de Finetti theorem, this implies that they are i.i.d. conditioned on some latent measure \( \theta \). To balance privacy with utility, the \( x \) sequence is non-uniformly randomly shuffled w.r.t. the \( t \) sequence producing a shuffled sequence \( z \), which the adversary observes. Conditioning on \( z \) the adversary updates their posterior on \( \theta \) (i.e. posterior on a model predicting \( x_i | t_i \)) and thereby their posterior predictive on the true \( x \). The definition of privacy in [40] holds that the adversary’s posterior beliefs are close to their prior beliefs by some metric on distributions in \( X \). \( \delta(\cdot, \cdot) \):

\[
\delta\left( \Pr[x_i], \Pr[x_i | z] \right) \leq \alpha
\]

We now translate the de Finetti attack to our setting. First, to align notation with the rest of the paper we provide privacy to the sequence of LDP values \( y \) since we shuffle those instead of the \( x \) values as in [40]. We use max divergence (multiplicative bound on events used in DP) for \( \delta \):

\[
\Pr[y_i \in O] \leq e^\alpha \Pr[y_i \in O | z] \\
\Pr[y_i \in O | z] \leq e^\alpha \Pr[y_i \in O]
\]

which, for compactness, we write as

\[
\Pr[y_i \in O] \approx_\alpha \Pr[y_i \in O | z]. \tag{8}
\]

We restrict ourselves to shuffling mechanisms, where we only randomize the order of sensitive values. By learning the unordered values \( \{y\} \) alone, an adversary may have arbitrarily large updates to its posterior (e.g. if all values are identical), breaking the privacy requirement above. With this in mind, we assume the adversary already knows the unordered sequence of values \( \{y\} \) (which they will learn anyway), and has a prior on permutations \( \sigma \) allocating values from that sequence to individuals. We then generalize the de Finetti problem to an adversary with an arbitrary prior on the true permutation \( \sigma \), and observes a randomize permutation \( \sigma' \) from the shuffling mechanism. We require that the adversary’s prior belief that \( \sigma(i) = j \) is close to their posterior belief for all \( i, j \in [n] \):

\[
\Pr[\sigma \in \Sigma_{i,j}] \approx_\alpha \Pr[\sigma \in \Sigma_{i,j} | \sigma'] \quad \forall i, j \in [n], \forall \sigma' \in S_n
\]

(9)

where \( \Sigma_{i,j} = \{ \sigma \in S_n : \sigma(i) = j \} \), the set of permutations assigning element \( j \) to \( DO_i \). Conditioning on any unordered sequence \( \{y\} \) with all unique values, the above condition is necessary to satisfy Eq. (8) for events of the
form \( O = \{y_i = a\} \), since \( \{y_i = a\} = \{\Sigma_{i,j}\} \) for some \( j \in [n] \). For any \( \{y\} \) with repeat values, it is sufficient since \( \Pr[y_i = a] \) is the sum of probabilities of disjoint events of the form \( \Pr[\sigma \in \Sigma_{i,k}] \) for various \( k \in [n] \) values.

We now show that a strict instance of \( d_{\sigma} \)-privacy satisfies Eq. (9). Let \( \mathcal{G} \) be any group assignment such that at least one \( G_i \in \mathcal{G} \) includes all data owners, \( G_i = \{1, 2, \ldots, n\} \).

**Property 1.** A \((\mathcal{G}, \alpha)\)-\( d_{\sigma} \)-private shuffling mechanism \( \sigma' \sim \mathcal{A} \) satisfies

\[
\Pr[\sigma \in \Sigma_{i,j}] \approx_{\alpha} \Pr[\sigma \in \Sigma_{i,j} | \sigma']
\]

for all \( i, j \in [n] \) and all priors on permutations \( \Pr[\sigma] \).

**Proof.**

**Lemma 1.** For any prior \( \Pr[\sigma] \), Eq. (9) is equivalent to the condition

\[
\frac{\sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma] \Pr[\sigma' | \sigma]}{\sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma] \Pr[\sigma' | \sigma']} \approx_{\alpha} \frac{\sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma]}{\sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma']}
\]

where the set \( \Sigma_{i,j} \) is the complement of \( \Sigma_{i,j} \).

Under grouping \( \mathcal{G} \), every permutation \( \sigma_a \in \Sigma_{i,j} \) neighbors every permutation \( \sigma_b \in \Sigma_{i,j} \), \( \sigma_a \approx \mathcal{G} \sigma_b \), for any \( i, j \). By the definition of \( d_{\sigma} \)-privacy, we have that for any observed permutation \( \sigma' \) output by the mechanism:

\[
\Pr[\sigma' | \sigma = \sigma_a] \approx_{\alpha} \Pr[\sigma' | \sigma = \sigma_b] \quad \forall \sigma_a \in \Sigma_{i,j}, \sigma_b \in \Sigma_{i,j}, \sigma' \in S_n.
\]

This implies Eq. 10. Thus, \((\mathcal{G}, \alpha)\)-\( d_{\sigma} \)-privacy implies Eq. 10, which implies Eq. 9, thus proving the property.

Using Lemma 1, we may also show that this strict instance of \( d_{\sigma} \)-privacy is necessary to block all de Finetti attacks:

**Property 2.** A \((\mathcal{G}, \alpha)\)-\( d_{\sigma} \)-private shuffling mechanism \( \sigma' \sim \mathcal{A} \) is necessary to satisfy

\[
\Pr[\sigma \in \Sigma_{i,j}] \approx_{\alpha} \Pr[\sigma \in \Sigma_{i,j} | \sigma']
\]

for all \( i, j \in [n] \) and all priors on permutations \( \Pr[\sigma] \).

**Proof.** If our mechanism \( \mathcal{A} \) is not \((\mathcal{G}, \alpha)\)-\( d_{\sigma} \)-private, then for some pair of true (input) permutations \( \sigma_a \neq \sigma_b \) and some released permutation \( \sigma' \sim \mathcal{A} \), we have that

\[
\Pr[\sigma | \sigma_b] \geq e^\alpha \Pr[\sigma | \sigma_a] .
\]

Under \( \mathcal{G} \), all permutations neighbor each other, so \( \sigma_a \approx \mathcal{G} \sigma_b \). Since \( \sigma_a \neq \sigma_b \), then for some \( i, j \in [n] \), \( \sigma_a \in \Sigma_{i,j} \) and \( \sigma_b \in \Sigma_{i,j} \): one of the two permutations assigns some \( j \) to some DO; and the other does not. Given this, we may construct a bimodal prior on the true \( \sigma \) that assigns half its probability mass to \( \sigma_a \) and the rest to \( \sigma_b \),

\[
\Pr[\sigma_a] = \Pr[\sigma_b] = \frac{1}{2} .
\]

Therefore, for released permutation \( \sigma' \), the RHS of Eq. 10 is 1, and the LHS is

\[
\frac{\sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma] \Pr[\sigma' | \sigma]}{\sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma] \Pr[\sigma' | \sigma']} \geq e^\alpha ,
\]

violating Eq. 10, thus violating Eq. 9, and failing to prevent de Finetti attacks against this bimodal prior.

Ultimately, unless we satisfy \( d_{\sigma} \)-privacy shuffling the entire dataset, there exists some prior on the true permutation \( \Pr[\sigma] \) such that after observing the shuffled \( z \) permuted by \( \sigma' \), the adversary’s posterior belief on one permutation is larger than their prior belief by a factor \( \geq e^\alpha \). If we suppose that the set of values \( \{y\} \) are all distinct, this means that for some \( a \in \{y\} \), the adversary’s belief that \( y_i = a \) is significantly larger after observing \( z \) than it was before.

Now to prove Lemma 1:
Proof.

\[
\Pr[\sigma \in \Sigma_{i,j}] \approx_a \Pr[\sigma \in \Sigma_{i,j} | \sigma'] \\
\Pr[\sigma \in \Sigma_{i,j}] \approx_a \frac{\Pr[\sigma' | \sigma \in \Sigma_{i,j}] \Pr[\sigma \in \Sigma_{i,j}]}{\sum_{\sigma \in S_n} \Pr[\sigma] \Pr[\sigma' | \sigma]}
\]

\[
\sum_{\sigma \in S_n} \Pr[\sigma] \Pr[\sigma' | \sigma] \approx_a \Pr[\sigma' | \sigma \in \Sigma_{i,j}] \\
\sum_{\sigma \in S_n} \Pr[\sigma] \Pr[\sigma' | \sigma] \approx_a \Pr[\sigma' | \sigma \in \Sigma_{i,j}^{-1}] \sum_{\sigma \in S_n} \Pr[\sigma] \Pr[\sigma' | \sigma] \\
\sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma] \Pr[\sigma' | \sigma] + \sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma] \Pr[\sigma' | \sigma] \approx_a \Pr[\sigma' | \sigma \in \Sigma_{i,j}^{-1}] \sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma] \Pr[\sigma' | \sigma] \\
\sum_{\sigma \in \Sigma_{i,j}} \sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma] \Pr[\sigma' | \sigma] \approx_a \sum_{\sigma \in \Sigma_{i,j}} \Pr[\sigma] \Pr[\sigma' | \sigma] \left( \frac{1}{\Pr[\sigma' | \sigma \in \Sigma_{i,j}]} - 1 \right)
\]

As such, a strict instance of \(d_\alpha\)-privacy can defend against any de Finetti attack (i.e. for any prior \(\Pr[\sigma]\) on permutations), wherein at least one group \(G_i \in \mathcal{G}\) includes all data owners. Furthermore, it is necessary. This makes sense. In order to defend against any prior, we need to significantly shuffle the entire dataset. Without a restriction of priors as in Pufferfish [42], the de Finetti attack (i.e. uninformed Bayesian adversaries) is an indelicate metric for evaluating the privacy of shuffling mechanisms: to achieve significant privacy, we must sacrifice all utility. This in many regards is reminiscent of the no free lunch for privacy theorem established in [41]. As such, there is a need for more flexible privacy definitions for shuffling mechanisms.

7.3 Additional Properties of \(d_\alpha\)-privacy

Lemma 2 (Convexity). Let \(A_1, \ldots, A_k : \mathcal{Y}^n \rightarrow \mathcal{V}\) be a collection of \(k\) \((\alpha, \mathcal{G})\)-\(d_\alpha\)-private mechanisms for a given group assignment \(\mathcal{G}\) on a set of \(n\) entities. Let \(A : \mathcal{Y}^m \rightarrow \mathcal{V}\) be a convex combination of these \(k\) mechanisms, where the probability of releasing the output of mechanism \(A_i\) is \(p_i\), and \(\sum_{i=1}^k p_i = 1\). \(A\) is also \((\alpha, \mathcal{G})\)-\(d_\alpha\)-private w.r.t. \(\mathcal{G}\).

Proof. For any \((\sigma, \sigma') \in N_\mathcal{G}\) and \(y \in \mathcal{Y}\):

\[
\Pr[A(\sigma(y)) \in O] = \sum_{i=1}^k p_i \Pr[A_i(\sigma(y)) \in O] \\
\leq e^\alpha \sum_{i=1}^k p_i \Pr[A_i(\sigma'(y)) \in O] \\
= \Pr[A(\sigma'(y)) \in O]
\]

Theorem 7.1 (Post-processing). Let \(A : \mathcal{Y}^m \rightarrow \mathcal{V}\) be \((\alpha, \mathcal{G})\)-\(d_\alpha\)-private for a given group assignment \(\mathcal{G}\) on a set of \(n\) entities. Let \(f : \mathcal{V} \rightarrow \mathcal{V}'\) be an arbitrary randomized mapping. Then \(f \circ A : \mathcal{Y}^m \rightarrow \mathcal{V}'\) is also \((\alpha, \mathcal{G})\)-\(d_\alpha\)-private.

Proof. Let \(g : \mathcal{V} \rightarrow \mathcal{V}'\) be a deterministic, measurable function. For any output event \(\mathcal{Z} \subset \mathcal{V}'\), let \(\mathcal{W}\) be its preimage: \(\mathcal{W} = \{v \in \mathcal{V} | g(v) \in \mathcal{Z}\}\). Then, for any \((\sigma, \sigma') \in N_\mathcal{G},

\[
\Pr[g(A(\sigma(y))) \in \mathcal{Z}] = \Pr[A(\sigma(y)) \in \mathcal{W}] \\
\leq e^\alpha \cdot \Pr[A(\sigma'(y)) \in \mathcal{W}] \\
= e^\alpha \cdot \Pr[g(A(\sigma'(y))) \in \mathcal{Z}]
\]
This concludes our proof because any randomized mapping can be decomposed into a convex combination of measurable, deterministic functions [22], and as Lemma 2 shows, a convex combination of \((\alpha, G)\)-\(d_\sigma\)-private mechanisms is also \((\alpha, G)\)-\(d_\sigma\)-private.

\[\boxdot\]

**Theorem 7.2 (Sequential Composition).** If \(A_1\) and \(A_2\) are \((\alpha_1, G)\)- and \((\alpha_2, G)\)-\(d_\sigma\)-private mechanisms, respectively, that use independent randomness, then releasing the outputs \((A_1(y), A_2(y))\) satisfies \((\alpha_1 + \alpha_2, G)\)-\(d_\sigma\)-privacy.

**Proof.** We have that \(A_1 : \mathcal{Y}^n \to \mathcal{Y}'\) and \(A_1 : \mathcal{Y}^n \to \mathcal{Y}''\) each satisfy \(d_\sigma\)-privacy for different \(\alpha\) values. Let \(A : \mathcal{Y}^n \to (\mathcal{Y}' \times \mathcal{Y}'')\) output \((A_1(y), A_2(y))\). Then, we may write any event \(Z \in (\mathcal{V}' \times \mathcal{V}'')\) as \(Z' \times Z''\), where \(Z' \in \mathcal{V}'\) and \(Z'' \in \mathcal{V}''\). We have for any \((\sigma, \sigma') \in N_G\),

\[
\Pr[A(\sigma(y)) \in Z] = \Pr[(A_1(\sigma(y)), A_2(\sigma(y))) \in Z] \\
= \Pr[(A_1(\sigma(y)) \in Z') \cap (A_2(\sigma(y)) \in Z'')] \\
= \Pr[(A_1(\sigma(y)) \in Z')] \Pr[(A_2(\sigma(y)) \in Z'')] \\
\leq e^{\alpha_1 + \alpha_2} \Pr[(A_1(\sigma'(y)) \in Z')] \Pr[(A_2(\sigma'(y)) \in Z'')] \\
= e^{\alpha_1 + \alpha_2} \Pr[A(\sigma'(y)) \in Z']
\]

\[\boxdot\]

**Proof of Lemma 3.1**

**Lemma 3.1** An \(\varepsilon\)-LDP mechanism is \((k\varepsilon, G)\)-\(d_\sigma\)-private for any group assignment \(G\) such that \(k \geq \max_{G_i \in G} |G_i|\)

**Proof.** This follows from \(k\)-group privacy [22]. \(y\) are \(\varepsilon\)-LDP outputs \(A_{LDP}(x)\) from input sequence \(x\). For any \(\sigma \approx G_i\), \(\sigma'\), we know by definition that \(\sigma(j) = \sigma'(j)\) for all \(j \notin G_i\). As such, the permuted sequences \(\sigma(x)_j = \sigma'(x)_j\) for all \(j \notin G_i\), and differ in at most \(|G_i|\) entries. In other words,

\[\delta_H(\sigma(x), \sigma'(x)) \leq |G_i|\]

Using this fact, we have from the \(k\)-group property of LDP that

\[
\Pr[A_{LDP}(\sigma(x)) \in O] \leq e^{k|G_i|} \Pr[A_{LDP}(\sigma'(x)) \in O]
\]

and thus if \(k \geq \max_{G_i \in G} |G_i|\),

\[
\Pr[A_{LDP}(\sigma(x)) \in O] \leq e^{k\varepsilon} \Pr[A_{LDP}(\sigma'(x)) \in O]
\]

for all \((\sigma, \sigma') \in N_G\).

\[\boxdot\]

### 7.4 Proof for Thm. 3.2

**Theorem 3.2** For a given group assignment \(G\) on a set of \(n\) data owners, if a shuffling mechanism \(A : \mathcal{Y}^n \to \mathcal{Y}^n\) is \((\alpha, G)\)-\(d_\sigma\)-private, then for each data owner \(DO_i\), \(i \in [n]\),

\[
\mathbb{E}_P(x) = \max_{a, b \in \mathcal{X}} \left| \log \frac{\Pr_P[z|x_i = a, \{y_{G_i}\}, y_{\overline{G_i}}]}{\Pr_P[z|x_i = b, \{y_{G_i}\}, y_{\overline{G_i}}]} \right| \leq \alpha
\]

for a prior distribution \(P\), where \(z = A(y)\) and \(y_{\overline{G_i}}\) is the noisy sequence for all data owners outside \(G_i\).
We refer to Mallows model \( y \).

The fifth line uses the fact that for any \( \alpha \), and then define all inputs \( i \in G_i \) identically, and then define all inputs \( i \in G_i \) differently as needed. As such, they are neighboring w.r.t. \( G_i \).

Recalling that Alg. 1 applies \( \sigma_0^{-1} \) to the sampled permutation, we must sample \( \sigma_0 \sigma' \) (for some \( \sigma' \in \Sigma_1 \)) for the mechanism to produce \( z \) from \( \sigma_1(y) \). Formally, since \( \sigma'_1 \sigma_1(y) = z \) we must sample \( \sigma_0 \sigma'_1 \) to get \( z \) since we are going

\begin{align*}
\text{Proof.} \\
\Pr_{\mathcal{F}}[z | x_i = a, \{ y_{G_i} \}, y_{\overline{G_i}}] \\
\Pr_{\mathcal{F}}[z | x_i = b, \{ y_{G_i} \}, y_{\overline{G_i}}] \\
= \int \Pr_{\mathcal{F}}[y | x_i = a, \{ y_{G_i} \}, y_{\overline{G_i}}] \Pr_{\mathcal{A}}[z | y] dy \\
= \int \Pr_{\mathcal{F}}[y | x_i = b, \{ y_{G_i} \}, y_{\overline{G_i}}] \Pr_{\mathcal{A}}[z | y] dy \\
= \sum_{\sigma \in S_m} \Pr_{\mathcal{F}}[\sigma(y_{G_i}^*), | x_i = a, y_{\overline{G_i}}] \Pr_{\mathcal{A}}[z | \sigma(y_{G_i}^*), y_{\overline{G_i}}] \\
\leq \max_{\{ \sigma, \sigma' \in S_m \}} \Pr_{\mathcal{A}}[z | \sigma(y)] \\
\leq e^\alpha \\
\end{align*}

The second line simply marginalizes out the full noisy sequence \( y \). The third line reduces this to a sum over permutations of of \( y_{G_i} \), where \( m = |G_i| \) and \( y_{G_i}^* \) is any fixed permutation of values \( \{ y_{G_i} \} \). This is possible since we are given the values outside the group, \( y_{\overline{G_i}} \), and the unordered set of values inside the group, \( \{ y_{G_i} \} \).

The fourth line uses the fact that the numerator and denominator are both convex combinations of \( \Pr_{\mathcal{A}}[z | \sigma(y_{G_i}^*), y_{\overline{G_i}}] \) over all \( \sigma \in S_m \).

The fifth line uses the fact that for any \( y_{\overline{G_i}} \),

\begin{align*}
(\sigma(y_{G_i}^*), y_{\overline{G_i}}) \approx_{G_i} (\sigma'(y_{G_i}^*), y_{\overline{G_i}}) .
\end{align*}

This allows a further upper bound over all neighboring sequences w.r.t. \( G_i \), and thus over any permutation of \( y_{\overline{G_i}} \), as long as it is the same in the numerator and denominator. \( \square \)

7.5 Discussion on Properties of Mallows Mechanism

Property 3. For group assignment \( G \), a mechanism \( \mathcal{A}(\cdot) \) that shuffles according to a permutation sampled from the Mallows model \( \mathcal{P}_{\theta, \delta}(\cdot) \), satisfies \( (\alpha, G) \)-\( \delta \)-privacy where

\begin{align*}
\Delta(\sigma_0 : \delta, G) &= \max_{(\sigma, \sigma') \in \mathcal{N}_G} |\delta(\sigma_0 \sigma, \sigma_0) - \delta(\sigma_0 \sigma', \sigma_0)| \\
\alpha &= \theta \cdot \Delta(\sigma_0 : \delta, G) \\
\end{align*}

We refer to \( \Delta(\sigma_0 : \delta, G) \) as the sensitivity of the rank-distance measure \( \delta(\cdot) \)

Proof. Consider two permutations of the initial sequence \( y \), \( \sigma_1(y) \), \( \sigma_2(y) \) that are neighboring w.r.t. some group \( G_i \in G \), \( \sigma_1 \approx_{G_i} \sigma_2 \). Additionally consider any fixed released shuffled sequence \( z \). Let \( \Sigma_1, \Sigma_2 \) be the set of permutations that turn \( \sigma_1(y), \sigma_2(y) \) into \( z \), respectively:

\begin{align*}
\Sigma_1 &= \{ \sigma \in S_n : \sigma \sigma_1(y) = z \} \\
\Sigma_2 &= \{ \sigma \in S_n : \sigma \sigma_2(y) = z \} .
\end{align*}

In the case that \( \{ y \} \) consists entirely of unique values, \( \Sigma_1, \Sigma_2 \) will each contain exactly one permutation, since only one permutation can map \( \sigma_i(y) \) to \( z \).

Lemma 3. For each permutation \( \sigma'_1 \in \Sigma_1 \) there exists a permutation in \( \sigma'_2 \in \Sigma_2 \) such that

\begin{align*}
\sigma'_1 \approx_{G_i} \sigma'_2 .
\end{align*}

Proof follows from the fact that — since only the elements \( j \in G_i \) differ in \( \sigma_1(y) \) and \( \sigma_2(y) \) — only those elements need to differ to achieve the same output permutation. In other words, we may define \( \sigma'_1, \sigma'_2 \) at all inputs \( i \notin G_i \) identically, and then define all inputs \( i \in G_i \) differently as needed. As such, they are neighboring w.r.t. \( G_i \).
Therefore, setting \( \alpha = \Delta \), we achieve \((\alpha, \mathcal{G})\)-d\(\sigma\) privacy.

**Property 4.** The sensitivity of a rank-distance is an increasing function of the width \( \omega_{\mathcal{G}}^{\sigma_{0}} \). For instance, for Kendall’s \( \tau \) distance \( d_{\tau}(\cdot) \), we have \( \Delta(\sigma_{0} : d_{\tau}, \mathcal{G}) = \frac{\omega_{\mathcal{G}}^{\sigma_{0}}(\omega_{\mathcal{G}}^{\sigma_{0}} + 1)}{2} \).

To show the sensitivity of Kendall’s \( \tau \), we make use of its triangle inequality.

**Proof.** Recall from the proof of the previous property that the expression \( \delta(\sigma, \sigma_{0}) = \delta(\sigma_{0}, \sigma, \sigma_{0}) \), where \( \delta \) is the actual rank distance measure e.g. Kendall’s \( \tau \). As such, we require that

\[
|\delta(\sigma_{0}\sigma_{a}, \sigma_{0}) - \delta(\sigma_{0}\sigma_{b}, \sigma_{0})| \leq \frac{\omega_{\mathcal{G}}^{\sigma_{0}}(\omega_{\mathcal{G}}^{\sigma_{0}} + 1)}{2}
\]

for any pair of permutations \((\sigma_{a}, \sigma_{b}) \in N_{\mathcal{G}}\).

For any group \( G_{i} \in \mathcal{G} \), let \( W_{i} \subseteq n \) represent the smallest contiguous subsequence of indices in \( \sigma_{0} \) that contains all of \( G_{i} \).

For instance, if \( \sigma_{0} = [2, 4, 6, 8, 1, 3, 5, 7] \) and \( G_{i} = \{2, 6, 8\} \), then \( W_{i} = \{2, 4, 6, 8\} \). Then the group width is \( \omega_{i} = |W_{i}| - 1 = 3 \). Now consider two permutations neighboring w.r.t. \( G_{i}, \sigma_{a} \approx_{G_{i}} \sigma_{b} \), so only the elements of \( G_{i} \) are shuffled between them. We want to bound

\[
|\delta(\sigma_{0}\sigma_{a}, \sigma_{0}) - \delta(\sigma_{0}\sigma_{b}, \sigma_{0})|
\]

For this, we use a pair of triangle inequalities:

\[
\delta(\sigma_{0}\sigma_{a}, \sigma_{0}\sigma_{b}) \geq \delta(\sigma_{0}\sigma_{a}, \sigma_{0}) - \delta(\sigma_{0}\sigma_{b}, \sigma_{0}) \quad \& \quad \delta(\sigma_{0}\sigma_{a}, \sigma_{0}\sigma_{b}) \geq \delta(\sigma_{0}\sigma_{b}, \sigma_{0}) - \delta(\sigma_{0}\sigma_{a}, \sigma_{0})
\]

so,

\[
|\delta(\sigma_{0}\sigma_{a}, \sigma_{0}) - \delta(\sigma_{0}\sigma_{b}, \sigma_{0})| \leq \delta(\sigma_{0}\sigma_{a}, \sigma_{0})
\]

Since \( \sigma_{0}\sigma_{a} \) and \( \sigma_{0}\sigma_{b} \) only differ in the contiguous subset \( W_{i} \), the largest number of discordant pairs between them is given by the maximum Kendall’s \( \tau \) distance between two permutations of size \( \omega_{i} + 1 \):

\[
|\delta(\sigma_{0}\sigma_{a}, \sigma_{0}\sigma_{b})| \leq \frac{\omega_{i}(\omega_{i} + 1)}{2}
\]

Since \( \omega_{\mathcal{G}}^{\sigma_{0}} \geq \omega_{i} \) for all \( G_{i} \in \mathcal{G} \), we have that

\[
\Delta(\sigma_{0} : \delta, \mathcal{G}) \leq \frac{\omega_{\mathcal{G}}^{\sigma_{0}}(\omega_{\mathcal{G}}^{\sigma_{0}} + 1)}{2}
\]

\( \square \)
7.6 Hardness of Computing The Optimum Reference Permutation

**Theorem 7.3.** The problem of finding the optimum reference permutation, i.e., \( \sigma_0 = \arg \min_{\sigma \in S_n} \omega^\sigma_G \) is NP-hard.

**Proof.** We start with the formal representation of the problem as follows.

**Optimum Reference Permutation Problem.** Given \( n \) subsets \( G = \{ G_i \in 2^{[n]}, i \in [n] \} \), find the permutation \( \sigma_0 = \arg \min_{\sigma \in S_n} \omega^\sigma_G \).

Now, consider the following job-shop scheduling problem.

**Job Shop Scheduling.** There is one job \( J \) with \( n \) operations \( o_i, i \in [n] \) and \( n \) machines such that \( o_i \) needs to run on machine \( M_i \). Additionally, each machine has a sequence dependent processing time \( p_i \). Let \( S \) be the sequence till \( \sigma_0 \). For instance, \( S = \{ 3, 5, 7, 1 \} \) is the shortest subsequence such that it contains all the elements in \( S \). For example for \( S = \{ 3, 5, 7, 1 \} \), \( P_{2_4} = 345 \).

The objective is to find a scheduling for \( J \) such that it minimizes the makespan, i.e., the completion time of the job. Note that \( p_n = \max_i p_i \), hence the problem reduces to minimizing \( p_n \).

**Lemma 4.** The aforementioned job shop scheduling problem with sequence-dependent processing time is NP-hard.

**Proof.** Consider the following instantiation of the sequence-dependent job shop scheduling problem where the processing time is given by \( p_t = p_{t-1} + w_kt, p_1 = 0 \) where \( S[i-1] = k, S[i] = l \) and \( w_{ij}, j \in S \) represents some associated weight. This problem is equivalent to the travelling salesman problem (TSP) [5] and is therefore, NP-hard. Thus, our aforementioned job shop scheduling problem is also clearly NP-hard.

**Reduction:** Let the \( n \) subsets \( S_i \) correspond to the groups in \( G \). Clearly, minimizing \( \omega^\sigma_G \) minimizes \( p_n \). Hence, the optimal reference permutation gives the solution to the scheduling problem as well.

\[ \square \]

7.7 Illustration of Alg. 1

We now provide a small-scale step-by-step example of how Alg. 1 operates.

Fig. 5a is an example of a grouping \( G \) on a dataset of \( n = 8 \) elements. The group of \( DO_i \) includes \( i \) and its neighbors. For instance, \( G_8 = \{ 8, 3, 5 \} \). To build a reference permutation, Alg. 1 starts at the index with the largest group, \( i = 5 \) (highlighted in purple), with \( G_5 = \{ 5, 2, 3, 8, 4 \} \). As shown in Figure 5b, the \( \sigma_0 \) is then constructed by following a BFS traversal from \( i = 5 \). Each \( j \in G_5 \) is visited, queuing up the neighbors of each \( j \) in \( G_5 \) that haven’t been visited along the way, and so on. The algorithm completes after the entire graph has been visited.

The goal is to produce a reference permutation in place, where the width of each group in the reference permutation \( \omega_i \) is small. In this case, the width of the largest group \( G_5 \) is as small as it can be \( \omega_5 = 5 - 1 = 4 \). However, the width of \( G_4 = \{ 4, 5, 7 \} \) is the maximum possible since \( \sigma^{-1}(5) = 1 \) and \( \sigma^{-1}(7) = 8 \), so \( \omega_4 = 7 \). This is difficult to avoid when the maximum group size is large as compared to the full dataset size \( n \). Realistically, we expect \( n \) to be significantly larger, leading to relatively smaller groups.

With the reference permutation in place, we compute the sensitivity:

\[ \Delta(\sigma_0 : \delta, G) = \frac{\omega_4(\omega_4 + 1)}{2} \]

Which lets us set \( \theta = \frac{\alpha}{28} \) for any given \( \alpha \) privacy value. To reiterate, lower \( \theta \) results in more randomness in the mechanism.

We then sample the permutation \( \tilde{\sigma} = \mathbb{P}_{\theta, \delta}(\sigma_0) \). Suppose

\[ \tilde{\sigma} = [3 \ 2 \ 5 \ 4 \ 8 \ 1 \ 7 \ 6] \]
Then, the released \( z \) is given as
\[
z = \sigma^* = \sigma^{-1} \hat{\sigma}(y) = [y_1 \ y_2 \ y_5 \ y_8 \ y_3 \ y_7 \ y_6 \ y_4]
\]
One can think of the above operation as follows. What was 5 in the reference permutation \( \sigma_0(1) = 5 \) is 3 in the sampled permutation \( \hat{\sigma}(1) = 3 \). So, index 5 corresponding to \( \text{DO}_5 \) now holds \( \text{DO}_3 \)'s noisy data \( y_3 \). As such, we shuffle mostly between members of the same group, and minimally between groups.

The runtime of this mechanism is dominated by the Repeated Insertion Model sampler [17], which takes \( O(n^2) \) time. It is very possible that there are more efficient samplers available, but RIM is a standard and simple to implement for this first proposed mechanism. Additionally, the majority of this is spent computing sampling parameters which can be stored in advance with \( O(n^2) \) memory. Furthermore, sampling from a Mallows model with some reference permutation \( \sigma_0 \) is equivalent to sampling from a Mallows model with the identity permutation and applying it to \( \sigma_0 \). As such, permutations may be sampled in advance, and the runtime is dominated by computation of \( \sigma_0 \) which takes \( O(|V| + |E|) \) time (the number of vertices and edges in the graph).

### 7.8 Proof of Thm. 3.3

**Theorem 3.3** Alg. 1 is \((\alpha, \mathcal{G})\)-d\(_{\sigma}\) private.
Proof. The proof follows from Prop. 3. Having computed the sensitivity of the reference permutation $\sigma_0$, $\Delta$, and set $\theta = \alpha/\Delta$, we are guaranteed by Property 3 that shuffling according to the permutation $\hat{\sigma}$ guarantees $(\alpha, \mathcal{G})$-$d_s$-privacy.

\[ \square \]

7.9 Proof of Thm. 3.4

**Theorem 3.4** Alg. 1 satisfies $(\alpha', \mathcal{G}')$-$d_s$-privacy for any group assignment $\mathcal{G}'$ where $\alpha' = \frac{\Delta(\sigma_0 : \mathcal{G} \setminus \mathcal{G}')}{\Delta(\sigma_0 : \mathcal{G})}$

**Proof.** Recall from Property 3 that we satisfy $(\alpha, \mathcal{G})$-$d_s$-privacy by setting $\theta = \frac{\alpha}{\Delta(\sigma_0 : \mathcal{G})}$. Given alternative grouping $\mathcal{G}'$ with sensitivity $\Delta(\sigma_0 : \mathcal{G}, \mathcal{G}')$, this same mechanism provides

\[
\alpha' = \frac{\theta}{\Delta(\sigma_0 : \mathcal{G} \setminus \mathcal{G}')}
\]

\[ = \frac{\alpha}{\Delta(\sigma_0 : \mathcal{G})} \]

\[ = \frac{\Delta(\sigma_0 : \mathcal{G} \setminus \mathcal{G}')}{\Delta(\sigma_0 : \mathcal{G})} \]

\[ \square \]

7.10 Formal Utility Analysis of Alg. 1

**Theorem 7.4.** For a given set $S \subset [n]$ and Hamming distance metric, $\delta_H(\cdot)$, Alg. 1 is $(\eta, \delta)$-preserving for $\delta = \frac{1}{\exp(\delta_H)} \sum_{k=2k+1}^{n} c_h$ where $k = [(1-\eta) \cdot |S|]$ and $c_h$ is the number of permutations with hamming distance $h$ from the reference permutation that do not preserve $\eta\%$ of $S$ and is given by

\[
c_h = \sum_{j=\max(l_s,|/h/2|)}^{\min(1,|/h/2|)} (l_s - j) \cdot f(i,j) \cdot (n - l_s - j) \cdot f(h - 2j - i, j)!
\]

\[
f(i, j) = \frac{j!}{q!} \cdot \frac{j!}{q!} \cdot \frac{j!}{q!} \cdot f(i - q, q)
\]

\[ l_s = |S|, k = (1-\eta) \cdot l_s, n = \left[ \frac{n!}{e} + \frac{1}{2} \right] \]

**Proof.** Let $l_s = |S|$ denote the size of the set $S$ and $k = [(1-\eta) \cdot l_s]$ denote the maximum number of correct values that can be missing from $S$. Now, for a given permutation $\sigma \in S_n$, let $h$ denote its Hamming distance from the reference permutation $\sigma_0$, i.e., $h = \delta_H(\sigma, \sigma_0)$. This means that $\sigma$ and $\sigma_0$ differ in $h$ indices. Now, $h$ can be analysed in the the following two cases,

Case I. $h \leq 2k + 1$

For $(1-\eta)$ fraction of indices to be removed from $S$, we need at least $k + 1$ indices from $S$ to be replaced by $k + 1$ values from outside $S$. This is clearly not possible for $h \leq 2k + 1$. Hence, here $c_h = 0$.

Case II. $h > 2k$

For the following analysis we consider we treat the permutations as strings (multi-digit numbers are treated as a single string character). Now, Let $S_{\sigma_0}$ denote the non-contiguous substring of $\sigma_0$ such that it consists of all the elements of $S$, i.e.,

\[ |S| = l_s \]

\[ \forall i \in [l_s], S_{\sigma_0}[i] \in S \]

(12)

(13)
Let $S_{\sigma}$ denote the substring corresponding to the positions occupied by $S_{\sigma_0}$ in $\sigma$. Formally,

$$|S_{\sigma}| = l_S \quad \forall i \in [l_S], E_{\sigma_0}[i] = \sigma(\sigma^{-1}_0(E_{\sigma_0}[i]))$$  \hspace{1cm} (14)

For example, for $\sigma_0 = (1 \ 2 \ 3 \ 5 \ 4 \ 7 \ 8 \ 10 \ 9 \ 6)$, $\sigma = (1 \ 3 \ 2 \ 7 \ 8 \ 5 \ 4 \ 6 \ 1 \ 0 \ 9)$ and $S = \{2, 4, 5, 8\}$, we have $S_{\sigma_0} = 2548$ and $S_{\sigma} = 3784$ when $h = \delta_H(\sigma, \sigma_0) = 9$. Let $|S_{\sigma}|$ denote the set of the positions of strings $S_{\sigma}$. Let $A$ be the set of characters in $S_{\sigma}$ such that they do not belong to $S$, i.e., $A = \{S_{\sigma}[i] | S_{\sigma}[i] \notin S, i \in [l_S]\}$. Let $B$ be the set of characters in $S_{\sigma}$ that belong to $S$ but differ from $S_{\sigma_0}$ in position, i.e., $B = \{S_{\sigma}[i] | S_{\sigma}[i] \in S, S_{\sigma}[i] \neq S_{\sigma_0}[i], i \in [l_S]\}$. Additionally, let $C = S - \{S_{\sigma}\}$. For instance, in the above example, $A = \{3, 7\}, B = \{4, 8\}, C = \{2, 5\}$. Now consider an initial arrangement of $p + m$ distinct objects that are subdivided into two types $- p$ objects of Type A and $m$ objects of Type B. Let $f(p, m)$ denote the number of permutations of these $p + m$ objects such that the $m$ Type B objects can occupy any position but no object of Type A can occupy its original position. For example, for $f(p, 0)$ this becomes the number of derangements $[1]$ denoted as $!p = \left[\frac{e^p}{p!} + \frac{1}{2}\right]$. Therefore, $f(|B|, |A|)$ denotes the number of permutations of $S_{\sigma}$ such that $\delta_H(S_{\sigma_0}, S_{\sigma}) = |A| + |B|$. This is because if elements of $B$ are allowed to occupy their original position then this will reduce the Hamming distance.

Now, let $S_{\sigma}(S_{\sigma_0})$ denote the substring left out after extracting from $S_{\sigma}(S_{\sigma_0})$ from $\sigma$ ($\sigma_0$). For example, $S_{\sigma} = 1256109$ and $S_{\sigma_0} = 1371096$ in the above example. Let $D$ be the set of elements outside of $S$ and $A$ that occupy different positions in $S_{\sigma}$ and $S_{\sigma_0}$ (thereby contributing to the Hamming distance), i.e., $D = \{S_{\sigma_0}[i] | S_{\sigma_0}[i] \notin S, S_{\sigma}[i] \neq S_{\sigma_0}[i], i \in [n - l_S]\}$. For instance, in the above example $D = \{9, 6, 10\}$. Hence, $h = \delta_H(S_{\sigma_0}, S_{\sigma}) = |A| + |B| + |C| + |D|$ and clearly $f(|D|, |C|)$ represents the number of permutations of $S_{\sigma}$ such that $\delta_H(S_{\sigma_0}, S_{\sigma}) = |C| + |D|$. Finally, we have

$$c_h = \max(\frac{\min(l_s, h/2)}{j + 1}) \cdot \frac{(l_s)}{j} \cdot \frac{(n - l_s)}{j} \cdot \left[\sum_{i=0}^{\min(l_s - j, h - 2j)} \frac{(l_s - j)}{i} \cdot f(i, j) \cdot \frac{(n - l_s - j)}{h - 2j} \cdot f(h - 2j - i, j)\right]$$

Now, for $f(i, j)$ let $E$ be the set of original positions of Type A that are occupied by Type B objects in the resulting permutation. Additionally, let $F$ be the set of the original positions of Type B objects that are still occupied by some Type B object. Clearly, Type B objects can occupy these $|E| + |F| = m$ in any way they like. However, the type A objects can only result in $f(p - q, q)$ permutations. Therefore, $f(p, m)$ is given by the following recursive function

$$f(p, 0) =!p$$

$$f(0, m) = m!$$

$$f(p, m) = \sum_{q=0}^{\min(p, m)} \frac{m! \cdot f(p - q, q)}{\binom{p}{q} \cdot \binom{m - q}{m - q}}$$

Thus, the total probability of failure is given by

$$\delta = \frac{1}{\psi(\theta, \delta_H)} \sum_{h=2k+2}^{n} (e^{-\theta h} \cdot c_h)$$  \hspace{1cm} (16)
7.11 Additional Experimental Details

7.11.1 Evaluation of \((\eta, \delta)\)-preservation

In this section, we evaluate the characteristics of the \((\eta, \delta)\)-preservation for Kendall’s \(\tau\) distance \(d_\tau(\cdot, \cdot)\).

Each sweep of Fig. 6 fixes \(\delta = 0.01\), and observes \(\eta\). We consider a dataset of size \(n = 10K\) and a subset \(S\) of size \(l_S\) corresponding to the indices in the middle of the reference permutation \(\sigma_0\) (the actual value of the reference permutation is not significant for measuring preservation). For the rest of the discussion, we denote the width of a permutation by \(\omega\) for notational brevity. For each value of the independent axis, we generate 50 trials of the permutation \(\sigma\) from a Mallows model with the appropriate \(\theta\) (given the \(\omega\) and \(\alpha\) parameters). We then report the largest \(\eta\) (fraction of subset preserved) that at least 99% of trials satisfy.

In Fig. 6a, we see that preservation is highest for higher \(\alpha\) and increases gradually with declining width \(\omega\) and increasing subset size \(l_S\).

Fig. 6b demonstrates that preservation declines with increasing width. \(\Delta\) and increasing \(\alpha\) with \(\delta\), resulting in declining \(\theta\) and increasing randomness. We also see that larger subset sizes result in a more gradual decline in \(\eta\). This is due to the fact that the worst-case preservation (uniform random shuffling) is better for larger subsets. i.e. we cannot do worse than 80% preservation for a subset that is 80% of indices.

Finally, Fig. 6c demonstrates how preservation grows rapidly with increasing subset size. For large widths, we are nearly uniformly randomly permuting, so preservation will equal the size of the subset relative to the dataset size. For smaller widths, we see that preservation offers diminishing returns as we grow subset size past some critical \(l_S\). For \(\omega = 30\), we see that subset sizes much larger than a quarter of the dataset gain little in preservation.

7.12 Additional Related Work

In this section, we discuss the relevant existing work.

The anonymization of noisy responses to improve differential privacy was first proposed by Bittau et al. [11] who proposed a principled system architecture for shuffling. This model was formally studied later in [24, 13]. Erlingsson et al. [24] showed that for arbitrary \(\epsilon\)-LDP randomizers, random shuffling results in privacy amplification. Cheu et al. [13] formally defined the shuffle \(\text{DP}\) model and analyzed the privacy guarantees of the binary randomized response in this model. The shuffle \(\text{DP}\) model differs from our approach in two ways. First, it focuses completely on the \(\text{DP}\) guarantee. The privacy amplification is manifested in the from of a lower \(\epsilon\) (roughly a factor of \(\sqrt{n}\)) when viewed in an alternative \(\text{DP}\) model known as the central \(\text{DP}\) model. [24, 13, 7, 28, 11, 6]. However, our result caters to local inferential privacy. Second, the shuffle model involves an uniform random shuffling of the entire dataset. In contrast, our approach the granularity at which the data is shuffled is tunable which delineates a threshold for the learnability of the data.

A steady line of work has sujded the inferential privacy setting [37, 41, 33, 15, 21, 52]. Kifer et al. [41] formally studied privacy degradation in the face of data correlations and later proposed a privacy framework. Pufferfish [42, 49, 36], for analyzing inferential privacy. Subsequently, several other privacy definitions have also been proposed for the inferential privacy setting [44, 57, 12, 59, 8]. For instance, Gehrke et al. proposed a zero-knowledge privacy [31, 30] which is based on simulation semantics. Bhaskar et al. proposed noiseless privacy [9, 35] by restricting the set of prior distributions that the adversary may have access to. A recent work by Zhang et al. proposes attribute privacy [58] which focuses on the sensitive properties of a whole dataset. In another recent work, Ligett et al. study a relaxation of
DP that accounts for mechanisms that leak some additional, bounded information about the database [43]. Some early work in local inferential privacy include profile-based privacy [32] by Gehmke et al. where the problem setting comes with a graph of data generating distributions, whose edges encode sensitive pairs of distributions that should be made indistinguishable. In another work by Kawamoto et al., the authors propose distribution privacy [39] – local differential privacy for probability distributions. The major difference between our work and prior research is that we provide local inferential privacy through a new angle – data shuffling.

Finally, older works such as \(k\)-anonymity [50], \(l\)-diversity [45], and Anatomy [55] and other [54, 51, 56, 14, 18] have studied the privacy risk of non-sensitive auxiliary information, or ‘quasi identifiers’ (QIs). In practice, these works focus on the setting of dataset release, where we focus on dataset collection. As such, QIs can be manipulated and controlled, whereas we place no restriction on the amount or type of auxiliary information accessible to the adversary, nor do we control it. Additionally, our work offers each individual formal inferential guarantees against informed adversaries, whereas those works do not. We emphasize this last point since formalized guarantees are critical for providing meaningful privacy definitions. As established by Kifer and Lin in *An Axiomatic View of Statistical Privacy and Utility* (2012), privacy definitions ought to at least satisfy post-processing and convexity properties which our formal definition does.