Fixed-Time Synchronization of Complex Dynamical Network With Impulsive Effects

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This work was supported in part by the National Key Research and Development Program of China under Grant 2018AAA0101100, and in part by the National Natural Science Foundation of China under Grant 61973241 and Grant 61873171.

ABSTRACT

Fixed-time synchronization of complex dynamical networks with impulsive effects is investigated in this paper. First of all, a novel lemma about the fixed-time stability of the impulsive dynamical system is proposed, in which the settling time is regardless of the initial values of the considered system. Secondly, by constructing a Lyapunov function made up of the error states’ 1-norm, we design a unified controller for the network to achieve synchronization within the settling time. Moreover, the convergence time given in this paper is more accurate than that in some existing literatures. Furthermore, the nonlinear term of the dynamical behavior is assumed to be Hölder continuous, which is more general than the common Lipschitz condition. Finally, a numerical example is provided to illustrate the correctness and the effectiveness of the main result.

INDEX TERMS

Fixed-time synchronization, complex dynamical networks, impulsive effects, control.

I. INTRODUCTION

Recent years have witnessed an increasing interest in complex networks on account of its wide applications [1]–[4], such as electrical power grids, biological networks, communication networks and so on. As a typical collective behavior, synchronization has received much attention from researchers in different fields [5]–[10]. Li et al. studied the synchronizability of duplex star networks with two inter-layer links [6]. Sun et. al investigated the synchronization for different types of complex networks [7]–[9]. Besides, Tang et al. discussed the relationship among the inter-layer, intra-layer and complete synchronization [10].

However, the aforementioned papers all consider asymptotic synchronization, which can be achieved only when the time tends to infinity. Therefore, finite-time synchronization has drawn increasing attention because of its properties of limit convergence time, better robustness and disturbance rejection [11]–[15]. Thus, it can be applied to image processing and secure communication for improving the efficiency. Liu et al. proposed a nonsmooth finite-time method handling the switched coupled neural network problem [11]. Yang introduced 1-norm of the error state as the Lyapunov function to strictly analyse the finite-time synchronization for neural networks with arbitrary delays [12]. Mei et al. performed the finite-time techniques for complex networks using the periodically intermittent control [14]. Mei et al. addressed the finite-time synchronization problem by proposing optimal control [15]. However, the settling time they estimated heavily depends on the initial values of the considered system, which may be infinite or even unavailable in practice.

In recent years, fixed-time stability, a special case of finite-time stability, is proposed by Polyakov in 2012 [16]. It is worthy to be mentioned that fixed-time stability can be realized within a settling time, which is independent of any initial values of the considered network, and is only related to some known parameters. Afterwards, a great deal of results are presented using fixed-time techniques. Zuo investigated nonlinear system problem using a non-singular fixed-time terminal sliding mode in 2015 [17]. In the next year, Meng et. al. provided a nonlinear fixed-time convergence protocol to handle signed-average consensus for networks [18]. Hu et. al. proposed a new theorem to prove the fixed-time stability, which has a more accurate estimation of the settling time [19], Liu and Chen discussed the finite-time and fixed-time synchronization with or without pinning control [20] in 2017 and 2018, respectively.
As is well known to us, networks are inevitable suffering from instantaneous perturbations. That is to say, the controlled network may exhibit impulsive effects [22]–[24]. He et al. investigated the quasi-synchronization of complex dynamical networks via distributed impulsive control [22]. Lu et al. obtained a unified synchronization criteria for impulsive dynamical system [23]. Zhang et al. studied the synchronization of nonlinear networks with heterogeneous impulsive control [24]. Moreover, there have been some results about synchronization for impulsive networks using fixed-time control methods [25]–[28]. Yang et al. studied the fixed-time synchronization of complex dynamical networks with impulsive effects via non-chattering control in 2017 [25]. Li et al. proposed an improvement theorem of fixed-time stability theorem for impulsive dynamical systems, and discussed the fixed-time stability of impulsive Cohen-Grossberg BAM neural networks by designing two different control protocols in 2018 [26]. Hu and Sui and Zhang et al. investigated fixed-time synchronization of different neural networks with impulsive effects in 2019 [27], [28].

Fixed-time synchronization of impulsive complex networks is investigated in this paper. The main contributions are listed as follows: (1) A novel lemma is proposed to guarantee the fixed-time stability with impulsive dynamical systems, the settling time is not only less conservative but also more accurate than those in most existing literatures. (2) A unified controller is designed to that the synchronization of the impulsive dynamical network can be reached within a settling time, which is regardless of the initial values of the considered system. (3) The nonlinear function being Hölder continuous is assumed in this paper, which is more general than the common Lipschitz condition. As a special case, the nonlinear term satisfying Lipschitz condition is discussed in the corollary.

The paper is organized as follows: Section II presents some necessary definitions, assumptions and lemmas, which are important throughout the paper. Section III addresses the main results about fixed-time synchronization for the presented model with a unified controller used with impulsive effects. Simulation results are given in Section IV to demonstrate the effectiveness and correctness of the theoretical results. We make a conclusion in the last section.

Notation: \( \mathbb{R}^n \) and \( \mathbb{R}^{n \times m} \) denote the Euclidean space and the set of all \( n \times m \) real matrices, respectively. The notation 1 stands for a column vector whose elements are all 1 with compatible dimension. For a vector \( x \), \( \| x \|_1 \) means 1-norm of \( x \). Additionally, matrix \( A > 0 \) (\( A < 0 \)) denotes that \( A \) is positive (negative) definite, \( A^\top \) stands for the transposition of \( A \), and \( I_n \) is the identity matrix with \( n \)-dimension. \( C = \text{diag}\{c_1, c_2, \ldots, c_n\} \) means \( C \) is a diagonal matrix.

### II. MODEL DESCRIPTION AND PRELIMINARIES

Consider a complex dynamical network with \( N \) nodes, the dynamical equation of the coupled complex network can be described by

\[
\dot{x}_i(t) = Ax_i(t) + Bf(x_i(t)) + \sum_{j=1}^{N} u_{ij}(t) (x_j(t) - x_i(t)),
\]

where \( x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^\top \) denotes the state variables of the \( i \)-th node, \( f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t)))^\top \) is the dynamical function with \( f_j(x_{ij}(t)) \) satisfying \( f_j(0) = 0, j = 1, 2, \ldots, n \), and \( A, B \) denote constant matrices, \( \Gamma = \text{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \) means the inner link matrix satisfying \( \gamma_1 \geq \ldots \geq \gamma_2 \geq \gamma_1 > 0 \). Besides, constant \( c \) is the coupled strength, and the outer coupling matrix \( U = \{u_{ij}\} \in \mathbb{R}^{N \times N} \) satisfies: \( u_{ij} > 0 \) if there has an edge from node \( j \) to \( i \), otherwise, \( u_{ij} = 0 \). In addition, \( u_{ii} = 0 \). Next, define Laplacian matrix \( L = (l_{ij}) \in \mathbb{R}^{N \times N} \) as follows: \( l_{ij} = -u_{ij}, j \neq i \), and \( l_{ii} = \sum_{j=1,j \neq i}^{N} u_{ij}, j \) satisfying dissipative condition. Therefore, the corresponding complex network with impulsive effects can be written as

\[
\begin{align*}
\dot{x}_i(t) &= Ax_i(t) + Bf(x_i(t)) + c \sum_{j=1}^{N} u_{ij}(t) (x_j(t) - x_i(t)), \\
\Delta x_i(t_k) &= d_k^i x_i(t_k^-),
\end{align*}
\]  

for \( i = 1, 2, \ldots, N \). Here, \( \Delta x_i(t_k) = x_i(t_k^-) - x_i(t_k^+) \), \( x_i(t_k^-) = \lim_{t \to t_k^-} x_i(t) \), \( x_i(t_k^+) = \lim_{t \to t_k^+} x_i(t) \). Without loss of generality, we assume that \( x_i(t_k) = x_i(t_k^+) \). Additionally, \( d_k \) is a constant, the time sequence \( \{t_k, k \in \mathbb{N}_+\} \) denotes the impulsive instants. Note that the impulsive gain \( d_k \) depends on both node \( i \) and impulsive instants \( t_k \), which means the impulsive effects on the synchronization can be different from node to node and may be nonidentical at different impulsive instants.

**Remark 1:** As shown from impulsive network (1), we can get \( x_i(t_k) = (1 + d_k^i) x_i(t_k^-) \) from the second equation. As we all know, if \( |1 + d_k^i| < 1 \), the impulsive effects can be called synchronizing impulsive for facilitating synchronization; if \( |1 + d_k^i| = 1 \), the effects of impulsive are inactive to synchronize; otherwise, desynchronizing impulsive is presented because it makes against the synchronization of the network. In this paper, \( |1 + d_k^i| \leq 1 \) is discussed.

The synchronized target of the complex network can be presented by

\[
\dot{s}(t) = As(t) + Bf(s(t)),
\]  

where \( s(t) = (s_1(t), s_2(t), \ldots, s_N(t))^\top \) is the state of the target node, matrices \( A, B \) and dynamical function \( f(s(t)) \) are defined as the same as that in complex network (1).
Define $e_i(t) = x_i(t) - s(t)$, then the error system with controller is obtained by subtracting (2) from (1) as

\[
\begin{cases}
\dot{e}_i(t) = A e_i(t) + B \tilde{f}(e_i(t)) - c \sum_{j=1}^{N} i_j \Gamma e_j(t) + u_i(t), \\
\Delta e_i(t_k) = d_i^k e_i(t_k^-), & t = t_k,
\end{cases}
\]

for $i = 1, 2, \ldots, N$, where $\tilde{f}(e_i(t)) = f(x_i(t)) - f(s(t))$.

To proceed the process, we design the following controller

\[
u_i(t) = -ke_i(t) - \alpha \text{sgn}(e_i(t))|e_i(t)|^\theta - \beta \text{sgn}(e_i(t))|e_i(t)|^{\gamma + \text{sgn}(\|e_i(t)\|_1)},
\]

where $0 < \theta < 1, \gamma > 1$, the notation $e_i(t) = (e_i^1(t), e_i^2(t), \ldots, e_i^n(t))^T$, $\text{sgn}(e_i(t)) = \text{diag}(\text{sgn}(e_i^1(t)), \text{sgn}(e_i^2(t)), \ldots, \text{sgn}(e_i^n(t)))$, $|e_i(t)|^\xi = (|e_i^1(t)|^\xi, |e_i^2(t)|^\xi, \ldots, |e_i^n(t)|^\xi)^T$, $\xi = \{\theta, \gamma + \text{sgn}(\|e_i(t)\|_1) - 1\}$, and $k, \alpha, \beta$ are several positive constants to be determined.

Before moving on, the following definitions, assumptions and lemmas are made, which will be used in deriving the main result of this paper.

**Definition 1 [29]:** A function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is said to belong to class $\mathcal{V}$, if it satisfies

(i) $V$ is continuous on each of the sets $[t_{k-1}, t_k)$, $k \in \mathbb{N}$, and $\lim_{t \rightarrow t_k^-} V(x(t_k^-))$ exists.

(ii) $V$ is locally Lipschitzian in $x$ and $V(0) \equiv 0$.

**Definition 2 [30]:** The origin of system (3) is said to be globally finite-time stable if it is Lyapunov stable and finite-time convergent. Finite-time convergence means for any initial state $e_0 \in \mathbb{R}^n$, there is a function $T : \mathbb{R}^n \cap \{0\} \rightarrow (0, +\infty)$, called the settling-time function, such that $\lim_{t \rightarrow T(e_0)} e(t, e_0) = 0$ and $z(t, e_0) = 0$ for all $t \geq T(e_0)$.

**Definition 3 [16]:** The origin of system (3) is said to be globally finite-time stable if it is globally finite-time stable and the settling-time function $T(e_0)$ is bounded, i.e., $\exists T_{\max} > 0$ such that $T(e_0) \leq T_{\max}, \forall e_0 \in \mathbb{R}^n$.

**Assumption 1 [31]:** There exist non-negative constant $\rho_i$, $i = 1, 2, \ldots, n$ and positive constant $\eta$ such that

$$(f(x) - f(y)) \leq \rho_i |x - y|^\theta,$$

for all $x, y \in \mathbb{R}$ and $f(\cdot)$ is continuous.

**Remark 2:** Nonlinear function $f(\cdot)$ in complex network (1) satisfies Assumption 1, which is H"older continuous with $\eta > 0$. Therefore, Assumption 1 is more general than the common Lipschitz condition.

**Assumption 2 [25]:** The impulse sequence $\{t_k, k \in \mathbb{N}_+\}$ belongs to $D(\tau_{\min}, \tau_{\max})$ if

$$0 = t_0 < t_1 < \ldots < t_k < \ldots < t_k = +\infty, \tau_{\min} \leq t_k - t_{k-1} \leq \tau_{\max},$$

where $\tau_{\min} \leq \tau_{\max}$ and $\tau_{\min}$, $\tau_{\max}$ are positive constants.

Let $\varrho(s, t)$ be the number of impulse times on the time interval $(s, t)$; then, a time sequence $\{t_k, k \in \mathbb{N}_+\}$ belonging to $D(\tau_{\min}, \tau_{\max})$ satisfies $\frac{t - s}{\tau_{\max}} - 1 \leq \varrho(s, t) \leq \frac{t - s}{\tau_{\min}}$.

**Lemma 1 [32]:** For any vector $z \in \mathbb{R}^n$ and $0 < r < 1$

$$\|z\|_r \leq \frac{n^r - 1}{n^{r-1} - 1} \|z\|_r,$$

where $\|z\|_\sigma = \left( \sum_{i=1}^{n} |z_i|^{\sigma} \right)^{\frac{1}{\sigma}}$, in which $\sigma = r, 1$.

**Lemma 2 [26]:** Let a function $V(e(t)) \in \mathcal{V}$ be positive definite and radially unbounded, such that any solution $e(t)$ of (5) satisfies:

$$\dot{V}(e(t)) \leq -\mu V^\gamma(e(t)) - \lambda V^\delta(e(t)), \quad t \in [t_k, t_{k+1}), \quad t \in \mathbb{R}_+, \quad V(e(t_k)) \leq V(e(t_k^-)), \quad k \in \mathbb{N},$$

where $\lambda > 0, \mu > 0, 0 < \delta < 1, \gamma > 1$, then the system (5) is fixed-time stable, and the settling time $T$ is estimated by

$$T = \frac{1}{\lambda(1 - \delta)} + \frac{1}{\mu(\gamma - 1)}.$$

**Lemma 3:** Let a function $V(e(t)) \in \mathcal{V}$ be positive definite and radially unbounded, such that any solution $e(t)$ of (5) satisfies:

$$\dot{V}(e(t)) \leq -\mu V^\gamma + \text{sgn}(V(e(t))-1)(e(t)) - \lambda V^\delta(e(t)), \quad t \in [t_k, t_{k+1}), \quad t \in \mathbb{R}_+, \quad V(e(t_k)) \leq V(e(t_k^-)), \quad k \in \mathbb{N},$$

where $\lambda > 0, \mu > 0, 0 < \alpha \leq 1, 0 < \delta < 1, \gamma > 1$, then the system (5) is fixed-time stable within the settling time $T = T_1 + T_2$, in which

$$T_1 = \begin{cases} \frac{\ln(1 - \frac{\gamma - \alpha}{\mu(\gamma - 1)} \alpha)}, \quad \text{if } 0 < \alpha < 1, \\
\frac{\mu^\gamma}{\gamma}, \quad \text{if } \alpha = 1, \\
\frac{\ln((1 - \delta)\alpha)}{\ln(1 - \delta)}, \quad \text{if } 0 < \alpha < 1, \\
\frac{\lambda(1 - \delta)\alpha^{1-\delta} - (1 - \delta)\ln(\alpha)}, \quad \text{if } \alpha = 1 \end{cases}$$

$$T_2 = \begin{cases} \frac{\ln((1 - \delta)\alpha)}{\ln(1 - \delta)}, \quad \text{if } 0 < \alpha < 1, \\
\frac{1}{\lambda(1 - \delta)}, \quad \text{if } \alpha = 1. \end{cases}$$

**Proof:** Consider the following comparison system:

$$\begin{cases} \dot{x}(t) = -\mu(r(t))^{\gamma + \text{sgn}(r(t))-1}, \quad r(t) \geq 1, \quad t \neq t_k, \\
-\lambda(r(t))^{\delta}, \quad 0 \leq r(t) < 1, \quad t \neq t_k, \\
r(t_k) = \alpha r(t_k^-), \quad t = t_k, \\
r(0) = r_0, \end{cases}$$

(6)

Compared with Eqs. (5) and (6), we can easily obtain $0 \leq V(t) \leq r(t)$. Therefore, if there exists a $T > 0$ such that $\lim_{t \rightarrow T} r(t) = 0$ and $r(t) \equiv 0$, $\forall t \geq T$, then we can get that $\lim_{t \rightarrow T} V(t) = 0$ and $V(t) \equiv 0$, $\forall t \geq T$. In what follows, we investigate the fixed-time stability of system (6).

From Eq. (6), we know that $r(t)$ is strictly decreasing in $(t_{k-1}, t_k)$. On account of $0 < \alpha \leq 1$, $r(t)$ is strictly decreasing in $[0, +\infty)$. When $r(t_k^-) > 1$, the value of $r(t_k)$ may be larger.
than 1, smaller than 1, or even equal to 1. Anyway, there exists $\bar{t} \in (t_0, t_0 + \epsilon)$ such that $r(\bar{t}) \leq 1$ because of the strictly decreasing property of $r(t)$, and $r(t) \leq 1$ for all $t \in \bar{t}, +\infty$.

When $r(t) \geq 1$, $\dot{r}(t) = -\mu r(t)\gamma + \sigma\gamma^{(r(t)-1)}$. Take $z(t) = (r(t))^{-\gamma}$. Then we have $z(t) \rightarrow 0$ when $r(t) \rightarrow +\infty$, and $z(t) \rightarrow 1$ when $r(t) \rightarrow 1^+$. Besides, $\dot{z}(t) = -\gamma r(t)(r(t))^{-\gamma-1} \left[ -\mu r(t)^{\gamma} + \sigma\gamma^{(r(t)-1)} \right] = \mu \gamma r(t)^{-\gamma}(r(t))^{-\gamma-1} \# \mu \gamma$ no matter whether $r(t) > 1$ or $r(t) = 1$. Then the equation of $z(t)$ can be obtained as follows:

$$
\begin{align*}
\dot{z}(t) &= \mu \gamma, \quad t \neq t_k, \quad 0 < z(t) \leq 1, \\
\dot{z}(t_k) &= \tilde{a} z(t_k^-), \quad t = t_k, \\
z(0) &= z_0,
\end{align*}
$$
where $\tilde{a} = a^{-\gamma}$ and $z_0 = r_0^{-\gamma}$.

Similarly, take $z(t) = (r(t))^{1-\delta}$ when $0 \leq r(t) < 1$. Thus, $z(t) \rightarrow 1$ when $r(t) \rightarrow 1$, and $z(t) \rightarrow 0$ when $r(t) \rightarrow 0$. Additionally, the corresponding equation of $z(t)$ is presented as

$$
\begin{align*}
\dot{z}(t) &= -\lambda (1 - \delta), \quad t \neq t_k, \quad 0 < z(t) \leq 1, \\
\dot{z}(t_k) &= \tilde{a} z(t_k^-), \quad t = t_k, \\
z(0) &= z_0,
\end{align*}
$$
in which $\tilde{a} = a^{1-\delta}$.

From the above discussion, the fixed-time stability of Eq. (6) can be transformed to the following two problems needed to approach: (i) the solution of Eq. (7) approaches 1 in a fixed time $T_1$; (ii) the solution of (8) converges to 0 from 1 in a fixed time $T_2$. Specially, if the initial value of (6) is smaller than 1, then the fixed-time stability of (6) is directly equivalent to the corresponding problem of (8). In a word, the fixed-time stability of (6) can be achieved within $T = T_1 + T_2$. We consider problems (i) and (ii) in the following two cases: $0 < \alpha < 1$ and $\alpha = 1$.

Case 1: $0 < \alpha < 1$. In this case, $\tilde{a} > 1$ and $0 < \tilde{a} < 1$.

If $r(t_k) > 1$ and $r(t_k) \geq 1$, we have $0 < z(t) \leq 1$, the solution of Eq. (7) is [33]

$$
z(t) = \tilde{a} \bar{e}^{(0, t)} z(0) + \mu \gamma \int_0^t \tilde{a} \bar{e}^{(s, t)} ds.
$$

According to Assumption 2, $\tilde{a} \bar{e}^{t_{\max}} \leq \tilde{a} \bar{e}^{(t)} \leq \tilde{a} \bar{e}^{t_{\max}}$ is obtained. Since $z(0) < 1$, $z(t)$ is increasing and $\lim_{t \rightarrow +\infty} z(t) = +\infty$ from Eq. (9), there exists $T_1 > 0$ such that $\lim_{t \rightarrow +\infty} z(t) = 1$ and $0 < z(t) < 1$, $0 < t < T_1$, that is, $\tilde{a} \bar{e}^{(0, t)} z(0) + \mu \gamma \int_0^t \tilde{a} \bar{e}^{(s, t)} ds = 1$. Furthermore,

$$
\mu \gamma \int_0^t \tilde{a} \bar{e}^{(s, t)} ds = 1
\begin{align*}
\mu \gamma \int_0^t \tilde{a} \bar{e}^{(s, t)} ds \leq 1 \\
\mu \gamma \int_0^t \tilde{a} \bar{e}^{(s, t)} ds \leq 1 \\
t \leq T_1
\end{align*}
$$

Due to $\tilde{a} = a^{-\gamma}$, $T_1 = \frac{\tau_{\max}}{-\gamma \ln \tilde{a}} \ln (1 + \frac{\tilde{a} \ln \tilde{a}}{\mu \gamma \tau_{\max}}) \geq T_1$.

Similar to (9), the solution of Eq. (8) can be described by

$$
z(t) = \tilde{a} \bar{e}^{(0, t)} - \lambda (1 - \delta) \int_0^t \tilde{a} \bar{e}^{(s, t)} ds.
$$

Since $0 < \tilde{a} < 1$, we have $\tilde{a} \bar{e}^{t_{\max}} \leq \tilde{a} \bar{e}^{(t)} \leq \tilde{a} \bar{e}^{t_{\min}}$. Actually, $z(t) = 0$ is desirable finally, then

$$
0 = \tilde{a} \bar{e}^{(0, t)} - \lambda (1 - \delta) \int_0^t \tilde{a} \bar{e}^{(s, t)} ds
\begin{align*}
\leq \tilde{a} \bar{e}^{t_{\min}} - \lambda (1 - \delta) \int_0^t \tilde{a} \bar{e}^{(s, t)} ds \\
= \tilde{a} \bar{e}^{t_{\min}} - \lambda (1 - \delta) \gamma \int_0^t \tilde{a} \bar{e}^{(s, t)} ds
\end{align*}
$$

Let $h(t) = [\tilde{a} \bar{e}^{t_{\min}} - \lambda (1 - \delta) \gamma \int_0^t \tilde{a} \bar{e}^{(s, t)} ds] \frac{t}{\ln \tilde{a}} + \lambda (1 - \delta) \gamma \int_0^t \tilde{a} \bar{e}^{(s, t)} ds$.

Therefore, there exists only one $T_3 > 0$ such that $h(T_3) = 0$ and $T_2 \geq \frac{\tau_{\max}}{\ln \tilde{a}} \ln \frac{\lambda (1 - \delta) \gamma \int_0^t \tilde{a} \bar{e}^{(s, t)} ds}{\ln \tilde{a}}$. Furthermore, substitute $\tilde{a} = a^{1-\delta}$ into the above equation, $T_3 = \frac{\tau_{\max}}{(1-\delta) \lambda \gamma \int_0^t \tilde{a} \bar{e}^{(s, t)} ds}$.

When $r(t_k) > 1$ and $r(t_k) < 1$, the interval of $r(t)$ from $r_0$ to 1 is obviously smaller than $T_1$. Hence, when $0 < \alpha < 1$, the fixed-time stability of (6) can be guaranteed within $T_1 + T_2$.

Case 2: When $\alpha = 1$, we have $\tilde{a} = \tilde{a} = 1$. Clearly, the solution of Eq. (9) turned to

$$
r(t) = r(0) + \mu \gamma t,
$$
and in this situation, $T_1 \leq \frac{1}{\mu \gamma}$. Similarly, Eq. (10) can be turned to

$$
r(t) = 1 - \lambda (1 - \delta) t,
$$
and $T_2 \leq \frac{1}{\lambda (1-\delta)}$. Therefore, the fixed-time stability of the solution (6) is realized within $T = T_1 + T_2 = \frac{1}{\mu \gamma} + \frac{1}{\lambda (1-\delta)}$.

Remark 3: Only $\alpha = 1$ is considered in the fixed-time synchronization with impulsive effects in Lemma 2. This paper discusses fixed-time synchronization with impulsive effects in two cases of $0 < \alpha < 1$ and $\alpha = 1$ using another method, and the settling time is correspondingly estimated, which is tighter than that in Lemma 2 [26]-[28]. Besides, controller (4) has a faster convergence time to realize the fixed-time synchronization of the complex network and the target one based on Eq. (5) in Lemma 3 than that in [25].

III. MAIN RESULTS

In the following, some criteria are provided to guarantee the stability of error network (3) within a settling time under controller (4), which is equivalent to the fixed-time synchronization between complex network (1) and target system (2).
Theorem 1: Suppose that Assumptions 1 and 2 hold. If there exist some coefficients $k$, $\alpha$, $\beta$ in controller (4) satisfying

\begin{align*}
\|A\|_1 - c\psi - k &\leq 0, \\
\|e\| &< \min[\alpha, \beta (Nn)^{-\gamma}], \\
\|I_N + D^k\|_1 &\leq 1,
\end{align*}

in which

$$
\psi = \begin{cases} 
\gamma n \min_{j} \sum_{i=1}^{N} l_{ij}, & \text{if } \min_{j} \sum_{i=1}^{N} l_{ij} \leq 0, \\
\gamma l \min_{i} \sum_{j=1}^{N} l_{ij}, & \text{if } \min_{i} \sum_{j=1}^{N} l_{ij} > 0,
\end{cases}
$$

$$
\varepsilon = \max_{l} \{ \sum_{j=1}^{N} |b_{jl}| \rho_l \}, \quad \varphi = \max_{l} \{ (Nn)^{1-\eta} \}. 
$$

Then the considered complex network (1) will synchronize with time $t_{\max}$. Define $\sigma = \min\{\beta (Nn)^{-\gamma-\varphi}, \beta \cdot \max\{1, (Nn)^{2-\gamma}\}\}$ and $\omega = \|N + D^k\|_1$, then the settling time is estimated according to Lemma 3 as $T_3 = T_3 + T_4$, which is

$$
T_3 = \begin{cases} 
\frac{\tau_{\max} \ln (1 - \gamma \omega^{-\gamma} \ln \omega)}{\varphi \gamma^{\varphi \max}}, & \text{if } 0 < \omega < 1, \\
1, & \text{if } \omega = 1,
\end{cases}
$$

$$
T_4 = \begin{cases} 
\frac{\tau_{\max} \ln (\frac{(\alpha - \varepsilon \varphi)(1-\theta)\omega^{1-\theta} \tau_{\min} -(1-\theta) \ln \omega)}{(1-\theta) \ln \omega}}{(\alpha - \varepsilon \varphi)(1-\theta)}, & \text{if } 0 < \omega < 1, \\
1, & \text{if } \omega = 1.
\end{cases}
$$

Proof: Construct a proper Lyapunov function

$$
V(e(t)) = \|e(t)\|_1 = \sum_{i=1}^{N} 1^\top \text{sgn}(e_i(t)) e_i(t). \tag{14}
$$

When $t \neq t_k$, we have the derivative of $V(t)$ along with error network (3) as

$$
\dot{V}(e(t)) = \sum_{i=1}^{N} 1^\top \text{sgn}(e_i(t)) \left[ A e_i(t) + B \tilde{f}(e_i(t)) \right] - c \sum_{j=1}^{N} l_{ij} e_j(t) + u(t)
$$

$$
= \sum_{i=1}^{N} 1^\top \text{sgn}(e_i(t)) A e_i(t) + \sum_{i=1}^{N} 1^\top \text{sgn}(e_i(t)) B \tilde{f}(e_i(t))
$$

$$
- c \sum_{j=1}^{N} l_{ij} \text{sgn}(e_j(t)) e_j(t) - k \sum_{i=1}^{N} \text{sgn}(e_i(t)) e_i(t)
$$

$$
- \alpha \sum_{i=1}^{N} \text{sgn}(e_i(t)) \text{sgn}(e_i(t)) \text{sgn}(e_i(t)) e_i(t)\theta
$$

$$
- \beta \sum_{i=1}^{N} 1^\top \text{sgn}(e_i(t)) \text{sgn}(e_i(t)) |e_i(t)|^{\gamma + \text{sgn}(|e_i(t)|)\gamma - 1}
$$

$$
\leq \sum_{i=1}^{N} \|A\|_1 \cdot \|e_i(t)\|_1 + \vartheta_1 + \vartheta_2 - k \sum_{i=1}^{N} \|e_i(t)\|_1,
$$

$$
- \alpha \sum_{i=1}^{N} \sum_{j=1}^{n} |e_j(t)|^{\theta} - \beta \sum_{i=1}^{N} \sum_{j=1}^{n} |e_j(t)|^{\gamma + \text{sgn}(|e_i(t)|)\gamma - 1},
$$

in which $\vartheta_1 = \sum_{i=1}^{N} 1^\top \text{sgn}(e_i(t)) B \tilde{f}(e_i(t))$, and

$$
\vartheta_2 = -c \sum_{i=1}^{N} \sum_{j=1}^{n} l_{ij} 1^\top \text{sgn}(e_i(t)) \Gamma e_j(t) \text{ for notational simplicity. Thus, they are discussed as follows.}
$$

According to Assumption 1, we have

$$
\vartheta_1 \leq \sum_{i=1}^{N} 1^\top \text{sgn}(e_i(t)) B \tilde{f}(e_i(t)) \leq \sum_{i=1}^{N} \sum_{j=1}^{n} |b_{ji}| \cdot \rho_l |e_i(t)|^{\eta} \leq \sum_{i=1}^{N} \sum_{j=1}^{n} |e_j(t)|^{\eta},
$$

Furthermore, $\vartheta_1$ can be estimated based on Lemma 1 for different $\eta$.

$$
\vartheta_1 \leq e(Nn)^{1-\eta} V^{\eta}(e(t)), \quad 0 < \eta < 1
$$

$$
V^{\eta}(e(t)), \quad \eta \geq 1
$$

considering the definition of $\vartheta_1$ in Theorem 1, we can get

$$
\vartheta_1 \leq e \psi V^{\eta}(e(t)).
$$

Besides, $\vartheta_2$ is discussed based on the definition of $L$,

$$
\vartheta_2 = -c \sum_{i=1}^{N} \sum_{j=1}^{n} l_{ij} \text{sgn}(e_i(t)) \gamma \eta e_j(t)
$$

$$
- e \sum_{i=1}^{N} \sum_{j=1}^{n} l_{ij} \gamma |e_j(t)|
$$

$$
\leq -e \sum_{i=1}^{N} \sum_{j=1}^{n} l_{ij} \gamma |e_j(t)|
$$

$$
\leq -e \psi \sum_{i=1}^{N} \sum_{j=1}^{n} |e_j(t)| = -c \psi V(e(t)).
$$

Therefore, it is found from the above two inequalities and criteria (11) in Theorem 1 that

$$
\dot{V}(e(t))
$$

$$
\leq \|A\|_1 \|V(e(t))\| + e \psi V^{\eta}(e(t)) - c \psi V(e(t)) - k V(e(t))
$$

$$
- \alpha \sum_{i=1}^{N} \sum_{j=1}^{n} |e_j(t)|^{\theta} - \beta \sum_{i=1}^{N} \sum_{j=1}^{n} |e_j(t)|^{\gamma + \text{sgn}(|e_i(t)|)\gamma - 1}
$$

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Thus \[ V(t) \leq \varepsilon V^\eta(t) - \alpha V^\theta(t) \]
\[ \quad - \beta \max\{1, (Nn)^{1-\gamma} \sum_{i=1}^{N} |e^i(t)|^{\gamma}\} |\text{sign}(e(t))| \]
\[ \leq \varepsilon V^\eta(t) - \alpha V^\theta(t) \]
\[ \quad - \beta \cdot \max\{1, (Nn)^{1-\gamma} \sum_{i=1}^{N} |e^i(t)|^{\gamma}\} |\text{sign}(e(t))|, \]

Remark 4: The criteria in Theorem 1 are easy to realize. Eq. (11) and (12) will be satisfied if we take some proper \( k, \alpha, \beta \); In addition, Assumption 1 is more general compared to Lipschitz condition, so there is an extra condition (12) to handle it. Condition (13) decides that the effects of impulses are synchronizing or inactive in synchronization.

As a special case, the nonlinear function \( f(\cdot) \) satisfies the usual Lipschitz condition.

Assumption 3 [34]: For \( x, y \in \mathbb{R} \), there exist some positive constants \( \kappa_j \), \( j = 1, 2, \ldots, n \) such that
\[ |f_j(y) - f_j(x)| \leq \kappa_j |y - x|. \]

Corollary 1: Suppose that Assumptions 2 and 3 hold. If there exists \( k > 0 \) in controller (4) satisfying
\[ \|A\|_1 + \varepsilon - c \psi < k \leq \frac{\alpha}{\varepsilon} - c \psi - k \leq \frac{\alpha}{\varepsilon} \]
\[ \|I_N + D^k\|_1 \leq 1, \]
where \( \varepsilon = \max\{\sum_{j=1}^{n} |b_{ij}| \} \). Additionally, the constants \( \psi, \omega \), matrix \( D^k \) are defined as the same as Theorem 1. Then the considered complex network (1) and target system (2) will realize the fixed-time synchronization. According to Lemma 3, the settling time is
\[ T_6 = \frac{\tau_{\max}}{\beta} \left( 1 - \frac{\gamma \omega - \gamma \ln \omega}{\beta (Nn)^{1-\gamma} \gamma \tau_{\max}} \right), \]
\[ T_7 = \frac{\tau_{\max}}{\alpha} \ln \left( \frac{\alpha(1-\theta)\omega^{1-\theta} \tau_{\min}}{(1-\theta) \ln \omega} \right), \]
\[ \text{if } 0 < \omega < 1, \]
\[ \frac{1}{\alpha(1-\theta)}, \quad \text{if } \omega = 1. \]

Remark 5: The proof of Corollary 1 is similar to that in Theorem 1 except for \( \theta_1 \), which can be enlarged by applying Lipschitz condition as: \( \theta_1 < \sum_{j=1}^{n} \sum_{i=1}^{N} |b_{ij}| \cdot |e^i(t)| \leq \varepsilon V(e(t)) \), and that is the origin of criteria (15). It is worthy mentioning that parameters \( \alpha, \beta > 0 \) will be fine in this case.

IV. ILLUSTRATIVE EXAMPLE

In this section, an example is proposed in order to illustrate the correctness of our result. Without loss of generality, a common system is presented to satisfy Lipschitz condition. Consider a Chua’s circuits described by [22]
\[ \dot{s}(t) = As(t) + Bf(s(t)), \]
where \( s(t) = (s_1(t), s_2(t), s_3(t))^T \), \( f(s(t)) = \left( \frac{1}{2}(|s_1(t) + 1| - |s_1(t) - 1|), 0, 0 \right)^T \), and
\[ A = \begin{pmatrix} -2.5 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -18 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 25 \\ 6 \\ 0 \end{pmatrix}, \]
then the considered network is presented as
\[ \dot{x}_i(t) = Ax_i(t) + Bf(x_i(t)) - c \sum_{j=1}^{N} I_{ij} y_j(t), \]
\[ i = 1, 2, \ldots, 10. \]
Assume complex network (18) is strongly connected and the Laplacian matrix is

$$
L = \begin{pmatrix}
5 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \\
-1 & 5 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & -1 & 6 & -1 & -1 & 0 & -1 & 0 & 0 & -1 \\
0 & -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 2 & -1 & 6 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 & 0 & 6 & -1 & -1 & -1 & -1 \\
0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & -1 & 5 & 1 & -1 & -1 \\
-1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 5 & -1 \\
0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 5
\end{pmatrix}
$$

which does not need to be symmetrical, and the coupling strength is $c = 1$.

From Assumption 1 and the definition of $f(x_i(t))$, we can easily get $k_1 = 1$, $k_2 = k_3 = 0$ and $\eta = 1$. According to the definitions of $\psi$, $\epsilon$ in Theorem 1 and Corollary 1, $\psi = -1$ and $\epsilon = \frac{35}{3}$ are obtained.

**Example 1:** In this part, take $k = 36$ to satisfy condition (15) through simple computation. Since $\eta = 1$, any $\alpha$, $\beta > 0$ is fine. The impulsive instant is random. Without loss of generality, we define the impulsive gain is $d_i = -0.1$ to satisfy condition (16). The initial values of networks (17) and (18) are some random numbers. Then some corresponding figures are presented when $\alpha = 0.5$, $\beta = 0.1$, $\theta = 0.5$ and $\gamma = 1.1$ as follows:

Figure 1 depicts complex network (18). The random impulsive instant “*” is shown in Figure 2, we let “y = 1” once impulsive instant happened. Figure 3 shows that the curve evolution of the error state between network (18) and target network (17). From figure 3, we can obtain that complex networks (17) and (18) will achieve synchronization within $t = 0.15s$. Numerical simulations illustrate the validity of Theorem 1.

**V. CONCLUSION**

Fixed-time synchronization for complex dynamical networks with impulsive effects has been investigated in this paper. Firstly, a novel fixed-time stability theorem with impulsive effects has been proposed, which is less conservative than those in most existing literatures. Then, the nonlinear part of the dynamical behavior has been assumed to be Hölder continuous, which is more general than the most commonly used Lipschitz condition. In addition, some criteria have been provided for the considered network to achieve fixed-time synchronization by designing a unified controller and constructing a Lyapunov function composed of the 1-norm of the error state. As a special case, a corollary has been obtained when we regard Lipschitz condition as the assumption. Finally, a numerical example has been performed to illustrate the correctness and effectiveness of our result.

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