The Rogers–Selberg recursions, the Gordon–Andrews identities and intertwining operators

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Abstract

Using the theory of intertwining operators for vertex operator algebras we show that the graded dimensions of the principal subspaces associated to the standard modules for \( \hat{\mathfrak{sl}}(2) \) satisfy certain classical recursion formulas of Rogers and Selberg. These recursions were exploited by Andrews in connection with Gordon’s generalization of the Rogers–Ramanujan identities and with Andrews’ related identities. The present work generalizes the authors’ previous work on intertwining operators and the Rogers–Ramanujan recursion.

1 Introduction

This paper is a continuation of [CLM], to which we refer the reader for background and for our motivation.

We continue our study of a relationship between intertwining operators, in the sense of vertex operator algebra theory, associated to standard (integrable highest weight) modules for \( \hat{\mathfrak{sl}}(2) \), and the corresponding principal subspaces ([FS1]–[FS2]). Let us recall our main result in [CLM].

Consider the principal subspaces \( W(\Lambda_0) \) and \( W(\Lambda_1) \) associated to the level 1 standard \( \hat{\mathfrak{sl}}(2) \)–modules \( L(\Lambda_0) \) and \( L(\Lambda_1) \), respectively (see Section 2 below). All these spaces are (doubly) graded by the eigenvalues of the operator \( \alpha/2 \), \( \alpha \) being the positive root of \( \mathfrak{sl}(2) \), and the operator \( L(0) \) coming from the Virasoro algebra. The corresponding graded dimensions (the

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“characters”—the generating functions of the dimensions of the homogeneous subspaces) of $W(\Lambda_0)$ and $W(\Lambda_1)$ are then given by:

\[
\chi_{W(\Lambda_0)}(x, q) = \text{tr}_{W(\Lambda_0)} x^{\alpha/2} q^{L(0)},
\]

\[
\chi_{W(\Lambda_1)}(x, q) = \text{tr}_{W(\Lambda_1)} x^{\alpha/2} q^{L(0)};
\]

the variables $x$ and $q$ are formal. It is convenient to use the slightly modified graded dimension

\[
\chi'_{W(\Lambda_1)}(x, q) = x^{-1/2} q^{-1/4} \text{tr}_{W(\Lambda_1)} x^{\alpha/2} q^{L(0)},
\]

and for cosmetic reasons we correspondingly let

\[
\chi'_{W(\Lambda_0)}(x, q) = \chi_{W(\Lambda_0)}(x, q).
\]

We proved [CLM] that these graded dimensions satisfy the linear $q$–difference equation

\[
\chi'_{W(\Lambda_0)}(x, q) - \chi'_{W(\Lambda_1)}(x, q) = xq \chi'_{W(\Lambda_1)}(xq, q).
\]

We did this by constructing a certain exact sequence by means of intertwining operators associated to irreducible modules for $L(\Lambda_0)$ viewed as a vertex operator algebra. Formula (1.1) together with the formula

\[
\chi'_{W(\Lambda_1)}(x, q) = xq \chi'_{W(\Lambda_0)}(x, q)
\]

yields

\[
\chi'_{W(\Lambda_0)}(x, q) - \chi'_{W(\Lambda_0)}(xq, q) = xq \chi'_{W(\Lambda_0)}(xq^2, q),
\]

which is the classical Rogers–Ramanujan recursion (cf. [A5]), asserted for the graded dimension of $W(\Lambda_0)$. In particular, this easily implies (cf. [A5]) that

\[
\chi'_{W(\Lambda_0)}(x, q) = \sum_{n \geq 0} \frac{x^n q^{n^2}}{(q)_n},
\]

where $(q)_n = (1-q) \cdots (1-q^n)$. The specializations $x = 1$ and $x = q$ of (1.4) give the sum sides of the two Rogers–Ramanujan identities (cf. [A5]).

Gordon’s identities [G] generalize the Rogers–Ramanujan identities. They can be stated in the following form:

**Theorem 1.1** Let $l \geq 2$ and let $1 \leq t \leq l$. The number of partitions of a nonnegative integer $n$ into parts not congruent to $0, \pm t$ mod $(2l+1)$ is equal to the number of partitions $n = b_1 + \cdots + b_{s}$, $b_1 \geq b_2 \geq \cdots \geq b_{s} > 0$, such that $b_i - b_{i+t-1} \geq 2$ (the “difference two at distance $l-1$” condition) and at most $t-1$ of the $b_i$ equal 1.
In [A1] (cf. [A5]) Andrews found an elegant proof of Gordon’s identities by using certain \(q\)-hypergeometric–type series and a family of recursions ([RR], [Sel]) expressed as a system of \(q\)–difference equations ([Sel], [A1]). We will call these recursions, or equivalently, \(q\)–difference equations, the Rogers–Selberg recursions.

It is easy to see that the generating function of the number of partitions of a nonnegative integer into parts not congruent to 0, \(\pm t \mod (2l + 1)\) can be expressed as

\[
\prod_{i>0, \ i\neq 0, \ \pm t \mod (2l+1)} \frac{1}{1-q^i},
\]

the product side of Gordon’s identities. Andrews [A4] discovered an “analytic form” of (the sum sides of) these identities, and he derived it from the same difference equations mentioned above. Actually, this “analytic form” is a “multisum form,” and Andrews exploited a refined version of the \(q\)–generating function, involving the variable \(x\), giving a generating function of a refined version of the partitions counted in Gordon’s form of the identities.

We will continue to take the variables in these “analytic” expressions to be formal rather than complex. With the variable \(x\) suitably specialized, these Gordon–Andrews identities state ([A4]; cf. [A5]):

**Theorem 1.2** With \(l\) and \(t\) as in Theorem 1.1,

\[
\prod_{i>0, \ i\neq 0, \ \pm t \mod (2l+1)} \frac{1}{1-q^i} = \sum_{N_1 \geq \cdots \geq N_{l-1} \geq 0} \frac{q^{N_1^2 + \cdots + N_{l-1}^2 + N_1 + N_{l+1} + \cdots + N_{l-1}}}{(q)_{N_1-N_2} \cdots (q)_{N_{l-2}-N_{l-1}} (q)_{N_{l-1}}}.
\]

(1.5)

We emphasize that for us, \(q\) is a formal variable; in the literature, one often takes \(q\) to be a complex variable such that \(|q| < 1\).

A nonclassical approach to Gordon’s identities was initiated and developed by Lepowsky and Wilson in [LW2]–[LW4] by means of “twisted vertex operators” and “\(Z\)–algebras”; they used this to give a construction of bases for all the standard \(\hat{\mathfrak{sl}}(2)\)–modules (or more particularly, bases for the “vacuum spaces” for the “principal Heisenberg subalgebra” \(\hat{\mathfrak{sl}}(2)\) of \(\hat{\mathfrak{sl}}(2)\)) compatible with Gordon’s identities in the case of the odd levels, and with certain generalizations of Gordon’s identities due to Andrews ([A2], [A3]) and Bressoud ([B1], [B2]) in the case of the even levels. Analogously, but with a somewhat different flavor, in [LP1] and [LP2], Lepowsky and Primc used untwisted vertex operators and \(Z\)–algebras to construct bases for all
the standard $\hat{\mathfrak{sl}}(2)$–modules. These twisted and untwisted $Z$–algebra constructions ([LW2]–[LW4], [LP1]–[LP2]) both led naturally to the “differencetwo–at–a–distance” condition that had appeared in Gordon’s identities; in the untwisted case [LP2] it led to a new version of this condition.

Then Meurman and Primc [MP1] used $Z$–algebras to complete a new proof of Gordon’s identities in the setting of [LW2]–[LW4] (Lepowsky and Wilson had obtained a $Z$–algebra proof of the identities in special cases, including the Rogers–Ramanujan identities, in [LW2]–[LW3]). On the other hand, Feigin and Stoyanovsky ([FS1]–[FS2]) obtained the sum sides of Gordon’s identities as the graded dimensions of what they called the “principal subspaces” associated to the standard $\hat{\mathfrak{sl}}(2)$–modules of level $l−1$ (which we shall also write as $k$, below). In order to compute these “characters,” Feigin and Stoyanovsky considered “quasi–particles,” which are the expansion coefficients of certain generalized “fields” (cf. Chapter 13 in [DL]). This approach was further developed by Georgiev in [Ge]. In addition, Feigin and Stoyanovsky interpreted the product side of (1.5) by using a geometric approach, via infinite–dimensional analysis on a flag manifold. This approach was further extended in [FL] (see also [FKLMM1]–[FKLMM3]).

All these developments have concerned $\hat{\mathfrak{sl}}(2)$–modules; there have been many other developments in this spirit (see [FF], etc.).

In this paper we analyze the structure of the principal subspaces associated to the standard modules for $\hat{\mathfrak{sl}}(2)$ by using the theory of vertex operator algebras and intertwining operators ([FLM], [FHL]). Our main result is a construction of $l$ exact sequences that yield a linear system of $q$–difference equations equivalent to the Rogers–Selberg recursions ([RR], [Sel]) mentioned above (see also [A1]). It is important to say that, in order to obtain these exact sequences, we use certain aspects of the theory of intertwining operators for vertex operator algebras. As in [A1] and [A1] (cf. [A5]), these recursions in turn yield the sum sides of Gordon’s identities (see Theorem 1.1) and of Andrews’ identities (see Theorem 1.2).

2 Principal subspaces

The setting and notation are as in [CLM]. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ be the 3–dimensional complex simple Lie algebra with a standard basis \{\(h, x_\alpha, x_{-\alpha}\}\} and bracket relations

\[
[h, x_\alpha] = 2x_\alpha, \ [h, x_{-\alpha}] = -2x_{-\alpha}, \ [x_\alpha, x_{-\alpha}] = h.
\]
We fix the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$, which we identify with its dual by means of the form $\langle x, y \rangle = \text{tr}(xy)$ for $x, y \in \mathfrak{g}$. Take $\alpha \in \mathfrak{h}$ to be the root corresponding to the root vector $x_\alpha$, and take this root to be positive; then $\langle \alpha, \alpha \rangle = 2$ and we have the root space decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where $\mathfrak{n}_\pm = \mathbb{C}x_{\pm \alpha}$. Note that under our identifications, $\mathfrak{h} = \alpha$.

We shall use the affine Lie algebra

$$\hat{\mathfrak{sl}}(2) = \mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $c$ is a nonzero central element and

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y \rangle m \delta_{m+n,0} c$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.

For a dominant weight $\Lambda \in (\mathfrak{h} \oplus \mathbb{C}c)^*$, let $L(\Lambda)$ be the corresponding standard (= integrable highest weight) $\hat{\mathfrak{sl}}(2)$--module with highest weight $\Lambda$ (cf. [K]). We say that an $\hat{\mathfrak{sl}}(2)$--module has level $k \in \mathbb{C}$ if $c$ acts as multiplication by $k$. The standard module $L(\Lambda)$ (which is irreducible) has nonnegative integral level, given by $\langle \Lambda, c \rangle$. (The evaluation map $\langle \cdot, \cdot \rangle$ of $(\mathfrak{h} \oplus \mathbb{C}c)^*$ on $\mathfrak{h} \oplus \mathbb{C}c$ extends the form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h} = \mathfrak{h}^*$; also, for $\mu, \nu \in (\mathfrak{h} \oplus \mathbb{C}c)^*$, $\langle \mu, \nu \rangle$ is defined to be $\langle \mu \vert_\mathfrak{h}, \nu \vert_\mathfrak{h} \rangle$.) The highest weights of the level 1 standard modules are the fundamental weights $\Lambda_0$ and $\Lambda_1$, defined by: $\langle \Lambda_i, c \rangle = 1$, $\langle \Lambda_i, h \rangle = 1$ for $i = 0, 1$. The highest weights of the level $k$ standard modules are given by

$$k_0 \Lambda_0 + k_1 \Lambda_1, \quad k_0 + k_1 = k,$$

where $k_0$ and $k_1$ are nonnegative integers.

Throughout this paper we will write $x(n)$ for the action of $x \otimes t^n \in \hat{\mathfrak{sl}}(2)$ on an $\hat{\mathfrak{sl}}(2)$--module, for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$.

Let $P = \frac{1}{2} \mathbb{Z} \alpha$ be the weight lattice and $Q = \mathbb{Z} \alpha$ the root lattice in $\mathfrak{h}$. Let $\mathbb{C}[P]$ and $\mathbb{C}[Q]$ be the corresponding group algebras, with bases $\{e^\mu \mid \mu \in P\}$ and $\{e^\mu \mid \mu \in Q\}$. Consider the subalgebra

$$\hat{\mathfrak{h}}_\mathbb{Z} = \prod_{m \in \mathbb{Z} \setminus \{0\}} \mathfrak{h} \otimes t^m \oplus \mathbb{C}c$$
of $\widehat{\mathfrak{sl}}(2)$, a Heisenberg algebra, meaning that its commutator subalgebra is equal to its center, which is one–dimensional (namely, $\mathbb{C}c$). We will also need the subalgebra

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c.$$  \hspace{1cm} (2.1)

We shall consider the $\hat{\mathfrak{h}}$–module

$$M(1) = U(\hat{\mathfrak{h}}) \otimes U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c) \mathbb{C},$$

where $\mathfrak{h} \otimes \mathbb{C}[t]$ acts trivially on the one–dimensional module $\mathbb{C}$ and $c$ acts as 1. It is well known ([FK], [Seg]; cf. [FLM]) that

$$V_P = M(1) \otimes \mathbb{C}[P]$$

and its subspaces

$$V_Q = M(1) \otimes \mathbb{C}[Q]$$

and

$$V_{Q+\alpha/2} = M(1) \otimes e^{\alpha/2} \mathbb{C}[Q]$$

admit natural $\mathfrak{sl}(2)$–module structure, via certain vertex operators (recalled in [CLM]), and that as $\mathfrak{sl}(2)$–modules,

$$V_Q \cong L(\Lambda_0) \text{ and } V_{Q+\alpha/2} \cong L(\Lambda_1).$$

It is much harder to obtain a similar construction for the higher level standard $\mathfrak{sl}(2)$–modules; this was done in [LP2]. By tensoring $k$ level 1 standard modules one obtains a level $k$ module which is completely reducible and whose irreducible components are level $k$ standard modules (cf. [K]); what was hard was to construct bases of such irreducible modules and to determine their precise structure. The action of $g \in \mathfrak{sl}(2)$ on a vector

$$v \in L(\Lambda_{i_1}) \otimes \cdots \otimes L(\Lambda_{i_k}),$$

where each $i_r$ is either 0 or 1, is given by the usual comultiplication

$$g \cdot v = \Delta(g)v = (g \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes g)v, \hspace{1cm} (2.2)$$

and this action of course extends to $U(\mathfrak{sl}(2))$.

Consider the subalgebra

$$\hat{n}_+ = n_+ \otimes \mathbb{C}[t, t^{-1}]$$  \hspace{1cm} (2.3)
of $\widehat{\mathfrak{sl}}(2)$. The principal subspace $W(\Lambda)$ associated to $L(\Lambda)$, $\Lambda$ a dominant weight, is defined as

$$W(\Lambda) = U(\widehat{\mathfrak{n}}^+\times) \cdot v_\Lambda \subset L(\Lambda),$$

where $v_\Lambda$ is a highest weight vector (which is unique up to a nonzero multiple).

To study the principal subspaces it will be convenient, as in [CLM], to use the polynomial algebra

$$A = \mathbb{C}[y_{-1}, y_{-2}, \ldots],$$

where $y_{-1}, y_{-2}, \ldots$ are (commuting, independent) formal variables. For an $\widehat{\mathfrak{sl}}(2)$–module $M$, consider the algebra map

$$A \rightarrow \text{End } M$$

$$y_{-j} \mapsto x_\alpha(-j) \quad (2.4)$$

($j > 0$); this map is well defined because the operators $x_\alpha(-j)$ commute. For a dominant weight $\Lambda$, define the linear surjection

$$f_\Lambda : A \rightarrow W(\Lambda)$$

$$p(y_{-1}, y_{-2}, \ldots) \mapsto p(x_\alpha(-1), x_\alpha(-2), \ldots) \cdot v_\Lambda, \quad (2.5)$$

where $p$ is a polynomial and where as above, $v_\Lambda$ is a highest weight vector, and set

$$A_\Lambda = \text{Ker } f_\Lambda,$$

an ideal in $A$. Then we have

$$A/A_\Lambda \cong W(\Lambda). \quad (2.6)$$

From now on we fix a positive integer $k$ (which corresponds to the integer $l - 1$ in the Introduction).

Our main goal is to derive explicit formulas for certain graded dimensions of principal subspaces via certain systems of $q$–difference equations. To achieve this goal we shall need an explicit description of the ideals $A_\Lambda$. The following result was (essentially) proved in [FS1]–[FS2]:

**Theorem 2.1** For every $i$ with $0 \leq i \leq k$,

$$A_{(k-i)\Lambda_0 + i\Lambda_1} = A_{y_{-1}^{k-i+1}} + A_{k\Lambda_0}. \quad (2.7)$$
Proof: From \([FS1]\) it follows that the ideal

\[ \mathcal{A}_{(k-i)\Lambda_0 + i\Lambda_1} \]

is generated by the elements

\[ r_n^{(k)} = \sum_{i_1,\ldots,i_{k+1} > 0 \atop i_1 + \cdots + i_{k+1} = -n} y_{-i_1} \cdots y_{-i_{k+1}} \]

for \(n \leq -(k+1)\), and

\[ y_{k+1-i}^{k+1-i}. \]

This fact immediately implies the statement. ■

3 Intertwining operators for vertex operator algebras and fusion rules

In this section we will be using the theory of vertex operator algebras and intertwining operators, as developed in \([FLM]\), \([FHL]\) and \([DL]\). The reader unfamiliar with the theory of modules and intertwining operators for vertex operator algebras may consult our previous paper \([CLM]\), where we recalled, and motivated, the definition of intertwining operator (see \([FHL]\) for more details).

It is well known that \(L(k\Lambda_0)\) (see the previous section) has a natural vertex operator algebra structure. In addition, all the level \(k\) standard modules are modules for this vertex operator algebra \(L(k\Lambda_0)\), and conversely, these are all the irreducible \(L(k\Lambda_0)\)-modules up to equivalence (see \([FZ]\); cf. \([Li1]\), \([DL]\), \([MP2]\) and \([LL]\)). Let \(L(\Lambda)\) be one of these irreducible \(L(k\Lambda_0)\)-modules. It is known (cf. \([FZ]\)) that \(L(\Lambda)\) is graded with respect to a standard action of the Virasoro algebra element \(L(0)\) (not to be confused with the trivial \(\mathfrak{sl}(2)\)-module!) and that it decomposes as

\[ L(\Lambda) = \bigoplus_{s \geq 0} L(\Lambda)_{s+h_\Lambda}, \]

where

\[ L(\Lambda)_\lambda = \{ v \in L(\Lambda) \mid L(0) \cdot v = \lambda v \} \]
is the weight space of $L(\Lambda)$ of weight $\lambda \in \mathbb{C}$, and where

$$h_\Lambda = \frac{\langle \Lambda, \Lambda + \alpha \rangle}{2(k + 2)}$$

(cf. [K], [DL], [LL]). In addition, $L(\Lambda)$ has second, compatible, grading, by charge, given by the eigenvalues of the operator $\alpha(0)/2 = h(0)/2$. It is important to notice that the principal subspace $W(\Lambda)$ is doubly graded as well. We will denote by $L(\Lambda)_{r+(\alpha/2, \Lambda), s+h_\Lambda}$ the subspace of $L(\Lambda)$ consisting of the vectors of charge $r + (\alpha/2, \Lambda)$ and weight $s + h_\Lambda$, so that

$$L(\Lambda) = \bigsqcup_{r \in \mathbb{Z}, s \in \mathbb{N}} L(\Lambda)_{r+(\alpha/2, \Lambda), s+h_\Lambda}.$$

Clearly, $W(\Lambda)$ also admits a decomposition

$$W(\Lambda) = \bigsqcup_{r, s \in \mathbb{N}} W(\Lambda)_{r+(\alpha/2, \Lambda), s+h_\Lambda}.$$

Now let $V$ be an arbitrary vertex operator algebra and let $W_1$, $W_2$ and $W_3$ be $V$–modules. We denote by

$$I \left( \begin{array}{c} W_3 \\ W_1 & W_2 \end{array} \right)$$

the vector space of all intertwining operators of type $(W_3 \ W_1 \ W_2)$ (see [FHL]). In many cases these spaces are finite–dimensional. Their dimensions

$$\mathcal{N}^{W_3}_{W_1 \ W_2} = \dim I \left( \begin{array}{c} W_3 \\ W_1 & W_2 \end{array} \right)$$

are the so–called fusion coefficients or fusion rules. A formula for the fusion rules for irreducible modules for $L(k\Lambda_0)$ is due to Frenkel and Zhu [FZ] (for clarification and generalization of Frenkel–Zhu’s formula see [Li2]). We should mention that the same result was obtained previously in [TK], but not in the setting of vertex operator algebras. The following result is from [FZ]:

**Proposition 3.1** Let

$$\mathcal{N}^{i_3}_{i_1, i_2} = \dim I \left( \begin{array}{c} L((k - i_3)\Lambda_0 + i_3\Lambda_1) \\ L((k - i_1)\Lambda_0 + i_1\Lambda_1) & L((k - i_2)\Lambda_0 + i_2\Lambda_1) \end{array} \right),$$
where $0 \leq i_j \leq k$, $i_j \in \mathbb{N}$, $j = 1, 2, 3$. Then

$$N_{i_1, i_2}^{i_3} = 1$$

if and only if

$$i_1 + i_2 + i_3 \in 2\mathbb{Z}, \ i_1 + i_2 + i_3 \leq 2k, \ |i_1 - i_2| \leq i_3 \leq i_1 + i_2.$$

Otherwise, $N_{i_1, i_2}^{i_3} = 0$.

For example, if $k = 1$, the nontrivial intertwining operators are of the following types:

$$\begin{pmatrix} L(\Lambda_0) \\ L(\Lambda_0) \end{pmatrix}, \begin{pmatrix} L(\Lambda_1) \\ L(\Lambda_0) \end{pmatrix}, \begin{pmatrix} L(\Lambda_0) \\ L(\Lambda_1) \end{pmatrix}, \begin{pmatrix} L(\Lambda_1) \\ L(\Lambda_1) \end{pmatrix}.$$

In order to proceed we will need a more detailed description of the intertwining operators indicated in Proposition 3.1. Let $W_1$, $W_2$ and $W_3$ be irreducible $L(k\Lambda_0)$–modules. Then it is not hard to see (cf. [FHL], [FZ]) that every intertwining operator $\mathcal{Y}(\cdot, x)$ of type

$$\begin{pmatrix} W_3 \\ W_1 \end{pmatrix} \begin{pmatrix} W_2 \end{pmatrix}$$

satisfies the condition

$$\mathcal{Y}(w_1, x) \in x^{h_{W_3} - h_{W_1} - h_{W_2}} \text{End}(W_2, W_3)[[x, x^{-1}]]$$

for every $w_1 \in W_1$. We will write (cf. [FZ])

$$\mathcal{Y}(w_1, x) = x^{h_{W_3} - h_{W_1} - h_{W_2}} \sum_{n \in \mathbb{Z}} (w_1)[n] x^{-n-1},$$

$$(w_1)[n] \in \text{End}(W_2, W_3).$$

Also, for every $w_1 \in W_1$, let

$$o_{\mathcal{Y}}(w_1) = \text{Coeff}_{x^0} x^{-h_{W_3} + h_{W_1} + h_{W_2}} \mathcal{Y}(w_1, x) = (w_1)_{[-1]}.$$

(Note: If $h_{W_1} + h_{W_2} = h_{W_3}$, such as when $W_2 = L(\Lambda_0)$ and $W_1 = W_3$, then $o_{\mathcal{Y}}(w_1)$ is simply the constant term of the intertwining operator $\mathcal{Y}(w_1, x)$.)

Let $v_{W_i}$ be a highest weight vector of $W_i$, $i = 1, 2, 3$. Then for every nonzero $\mathcal{Y} \in \mathcal{I}(W_1 W_2)$,

$$o_{\mathcal{Y}}(v_{W_i})v_{W_2} = (v_{W_1})_{[-1]} v_{W_2}, \quad (3.1)$$

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a highest weight vector of $W_3$ or zero; it will be nonzero for our cases below. Also, from the commutator formula for intertwining operators [FHL], it follows that

$$[x_\alpha(n), \mathcal{Y}(v_{W_1}, x)] = 0,$$

for every $n \in \mathbb{Z}$. Therefore,

$$[U(\widehat{n}), \mathcal{Y}(v_{W_1}, x)] = 0.$$  \hspace{1cm} (3.3)

So far we have treated irreducible $L(k\Lambda_0)$–modules abstractly. We have already outlined (cf. Section 2) that the level 1 standard modules admit an explicit construction as subspaces of $V_P$ (where $P = \frac{1}{2}\mathbb{Z}\alpha$ is the weight lattice of $\mathfrak{sl}(2)$). We can realize the level $k$ standard modules as submodules of the tensor products of $k$ level 1 standard modules (cf. Section 2). Let

$$L = \underbrace{P \oplus \cdots \oplus P}_{k \text{ times}}.$$  

Consider the space $V_P = V_Q \oplus V_{Q+\alpha/2}$ (cf. [FLM], [CLM]). Let

$$V_L = \underbrace{V_P \otimes \cdots \otimes V_P}_{k \text{ times}};$$

$V_L$ is naturally an $\widehat{\mathfrak{sl}}(2)$–module. Now, inside $V_P \cong L(\Lambda_0) \oplus L(\Lambda_1)$, we make the following identifications:

$$1 = v_{\Lambda_0} \quad \text{and} \quad e^{\alpha/2} = v_{\Lambda_1}.$$  

There are of course many ways to embed $L(k_0\Lambda_0 + k_1\Lambda_1)$ ($k_0 + k_1 = k$) inside $V_L$. Let $(i_1, \ldots, i_k)$ be a $k$–tuple such that $i_j \in \{0, 1\}$, $j = 1, \ldots, k$. Consider a vector of the form

$$v_{i_1,\ldots,i_k} = v_{\Lambda_{i_1}} \otimes \cdots \otimes v_{\Lambda_{i_k}} \in V_L,$$

where exactly $k_0$ indices (i.e., $i_j$'s) are equal to 0 (and exactly $k_1$ are equal to 1). This is certainly a highest weight vector for $\widehat{\mathfrak{sl}}(2)$, and

$$L(k_0\Lambda_0 + k_1\Lambda_1) \cong U(\widehat{\mathfrak{sl}}(2)) \cdot v_{i_1,\ldots,i_k} \subset V_L$$  

(cf. [K]). Let us denote by $\iota_{i_1,\ldots,i_k}$ the embedding

$$\iota_{i_1,\ldots,i_k} : L(k_0\Lambda_0 + k_1\Lambda_1) \rightarrow V_L,$$  

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uniquely determined by the identification

\[ v_{k_0\Lambda_0+k_1\Lambda_1} = v_{i_1,...,i_k}. \]

In parallel with [CLM], for \( \lambda \in P \), we will consider

\[ e^\lambda_{(k)} : V_L \rightarrow V_L, \]

\[ e^\lambda_{(k)} = e^\lambda \otimes \cdots \otimes e^\lambda, \]

a linear automorphism of \( V_L \), with inverse \( e^{-\lambda}_{(k)} \). Then it follows (cf. [Ge], [CLM]) that

\[ e^\lambda_{(k)} x_\alpha(-m_1) \cdots x_\alpha(-m_s) \cdot v = x_\alpha(-m_1 - \langle \lambda, \alpha \rangle) \cdots x_\alpha(-m_s - \langle \lambda, \alpha \rangle) e^\lambda_{(k)} \cdot v, \]

for every \( v \in V_L \) and \( m_1, \ldots, m_s \in \mathbb{Z} \). Also,

\[ e^\lambda_{(k)} x_{-\alpha}(-m_1) \cdots x_{-\alpha}(-m_s) \cdot v = x_{-\alpha}(-m_1 + \langle \lambda, \alpha \rangle) \cdots x_{-\alpha}(-m_s + \langle \lambda, \alpha \rangle) e^\lambda_{(k)} \cdot v \]

and if each \( m_i \neq 0 \),

\[ e^\lambda_{(k)} h(-m_1) \cdots h(-m_s) \cdot v = h(-m_1) \cdots h(-m_s) e^\lambda_{(k)} \cdot v. \]

The following lemma describes the action of \( e^{\alpha/2}_{(k)} \) on certain irreducible \( \mathfrak{sl}(2) \)-submodules of \( V_L \) (i.e., irreducible \( L(k\Lambda_0) \)-modules) and the corresponding principal subspaces:

**Lemma 3.2**  
(a) Consider the subspace \( L(k_0\Lambda_0 + k_1\Lambda_1) \subset V_L \) embedded via \( \iota_{i_1,...,i_k} \) as above. The image of the restriction map \( e^{\alpha/2}_{(k)} |_{L(k_0\Lambda_0 + k_1\Lambda_1)} \) lies in \( L(k_1\Lambda_0 + k_0\Lambda_1) \), embedded via \( \iota_{1-i_1,...,1-i_k} \). In particular, \( e^{\alpha/2}_{(k)} \) defines maps interchanging \( L(k_0\Lambda_0 + k_1\Lambda_1) \) and \( L(k_1\Lambda_0 + k_0\Lambda_1) \).

(b) Consider the principal subspace \( W(k_0\Lambda_0 + k_1\Lambda_1) \subset L(k_0\Lambda_0 + k_1\Lambda_1) \). The image of the restriction map \( e^{\alpha/2}_{(k)} |_{W(k_0\Lambda_0 + k_1\Lambda_1)} \) lies in \( W(k_1\Lambda_0 + k_0\Lambda_1) \).

(c) If \( k_0 = k \) (or \( k_1 = 0 \)), the map in (b) is surjective and in particular is a linear isomorphism from \( W(k\Lambda_0) \) to \( W(k\Lambda_1) \).
Proof: First we prove (a). From the formulas preceding the lemma it follows that
\[
e^{\alpha/2}_k \pi(U(\widehat{\mathfrak{sl}}(2))) e^{-\alpha/2}_k = \pi(U(\widehat{\mathfrak{sl}}(2))),
\]
where \( \pi \) is our representation of \( U(\widehat{\mathfrak{sl}}(2)) \) on \( V_L \). Therefore, we need only show that
\[
e^{\alpha/2}_k \cdot v_{i_1, \ldots, i_k} \in U(\widehat{\mathfrak{sl}}(2)) \cdot v_{1-i_1, \ldots, 1-i_k}.
\]
For convenience let us assume that \( i_1 = i_2 = \cdots = i_{k_1} = 1 \). By using our identifications we have
\[
e^{\alpha/2}_k v_{\Lambda_1} = e^{\alpha} (= e^\alpha v_{\Lambda_0}) = x_\alpha(-1)v_{\Lambda_0} \in V_P
\]
(cf. [CLM]). Then
\[
= x_\alpha(-1)v_{\Lambda_0} \otimes \cdots \otimes x_\alpha(-1)v_{\Lambda_0} \otimes v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}.
\]
Now, because
\[
x_\alpha(-1)v_{\Lambda_1} = 0 \quad \text{and} \quad x_\alpha^2(-1)v_{\Lambda_0} = 0
\]
(again cf. [CLM]), it follows that
\[
= \frac{1}{k_1!} \Delta(x_\alpha(-1)^{k_1}) (v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0} \otimes v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}),
\]
so that
\[
e^{\alpha/2}_{(k)} \cdot v_{i_1, \ldots, i_k} \in U(\widehat{\mathfrak{sl}}(2)) \cdot v_{1-i_1, \ldots, 1-i_k},
\]
as desired.

The same reasoning with \( U(\widehat{\mathfrak{sl}}(2)) \) replaced by \( U(\widehat{\mathfrak{n}}_+) \) proves (b).

To prove (c) it is enough to notice that
\[
e^{\alpha/2}_{(k)} v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0} = v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}.
\]

4 The main theorem and consequences

We continue to fix a positive integer $k$. Let $\Lambda$ be a dominant weight such that $\langle \Lambda, c \rangle = k$. We shall determine the following graded dimensions (generating functions of the dimensions of homogeneous subspaces) of the spaces $W(\Lambda)$:

$$\chi_W(\Lambda)(x, q) = \dim_s(W(\Lambda), x, q) = \text{tr}|_{W(\Lambda)} x^{\alpha/2} q^{\ell(0)},$$

where we are using the formal variables $x$ and $q$. To avoid the multiplicative factor $x^{\langle \alpha/2, \Lambda \rangle} q^{h\Lambda}$ it is convenient to use slightly modified graded dimensions:

We define

$$\chi'_W(\Lambda)(x, q) = x^{\langle \alpha/2, \Lambda \rangle} q^{-h\Lambda} \chi_W(\Lambda)(x, q),$$

so that

$$\chi'_W(\Lambda)(x, q) \in \mathbb{C}[[x, q]],$$

and in fact,

$$\chi'_W(\Lambda)(x, q) \in 1 + xq\mathbb{C}[[x, q]].$$

It is a nontrivial task to find explicit formulas for the $\chi'_W(\Lambda)(x, q)$. As in [CLM], we will first derive certain representation–theoretic results and convert these results into statements about graded traces and $q$–difference equations.

The simplest $q$–difference equation follows easily from Lemma 3.2, part (c), just as in formula (3.26) of [CLM]:

**Proposition 4.1** We have

$$\chi_W(k\Lambda_1)(x, q) = x^{k/2} q^{k/4} \chi_W(k\Lambda_0)(xq, q) = x^{k/2} q^{h\Lambda_1} \chi_W(k\Lambda_0)(xq, q),$$

or simply

$$\chi'_W(k\Lambda_1)(x, q) = \chi'_W(k\Lambda_0)(xq, q).$$

Our main result is:

**Theorem 4.2** Let $1 \leq i \leq k$, let

$$Y(\cdot, x) \in I \left( \begin{array}{cc} L((i-1)\Lambda_0 + (k-i+1)\Lambda_1) & L((k-1)\Lambda_0 + \Lambda_1) \\ L(i\Lambda_0 + (k-i)\Lambda_1) & L(i\Lambda_0 + (k-i)\Lambda_1) \end{array} \right)$$

be a nonzero intertwining operator and let $\alpha_Y(v_{(k-1)\Lambda_0+\Lambda_1})$ be as in [CLM]. Then the sequence

$$0 \rightarrow W((k-i)\Lambda_0 + i\Lambda_1) \overset{\epsilon_{(k,i)}^{\alpha/2}}{\longrightarrow} W(i\Lambda_0 + (k-i)\Lambda_1) \overset{\alpha_Y(v_{(k-1)\Lambda_0+\Lambda_1})}{\longrightarrow} W((i-1)\Lambda_0 + (k-i+1)\Lambda_1) \rightarrow 0$$

(4.4)
is exact. Also,

\[ 0 \longrightarrow W(k\Lambda_0) \xrightarrow{e^{\alpha/2}_{(k)}} W(k\Lambda_1) \longrightarrow 0 \]  

(4.5)

is exact.

**Remark 4.3** When \( k = i = 1 \), this theorem is equivalent to the main theorem in [CLM], which yielded the Rogers–Ramanujan recursion.

**Proof of Theorem 4.2:** The exactness of (4.5) has been proved in Lemma 3.2. Now we prove the exactness of (4.4). First note that a nonzero intertwining operator \( Y(\cdot, x) \) exists by Proposition 3.1. We recall (see (3.1)) that

\[ oY(v((k-1)\Lambda_0 + \Lambda_1))v(\Lambda_0 \otimes \cdots \otimes \Lambda_1) \]

is a nonzero multiple of \( v((i-1)\Lambda_0 + (k-i)\Lambda_1) \). In addition, \( oY(v((k-1)\Lambda_0 + \Lambda_1)) \) commutes with the action of \( U(\hat{n}_+) \); thus \( oY(v((k-1)\Lambda_0 + \Lambda_1)) \) is surjective. We already know that \( e^{\alpha/2}_{(k)} \) is injective and that it maps \( W((k-i)\Lambda_0 + i\Lambda_1) \) into \( W(i\Lambda_0 + (k-i)\Lambda_1) \) (Lemma 3.2).

Let us prove the chain property. First, we have

\[ e^{\alpha/2}_{(k)} x_\alpha(-m_1) \cdots x_\alpha(-m_n) = x_\alpha(-m_1 - 1) \cdots x_\alpha(-m_n - 1)e^{\alpha/2}_{(k)} \]  

(4.6)

for \( m_j \in \mathbb{Z} \). By combining (4.6) with the proof of Lemma 3.2 we see that the image of \( e^{\alpha/2}_{(k)} \) is the \( U(\hat{n}_+) \)-submodule of \( W(i\Lambda_0 + (k-i)\Lambda_1) \) generated by

\[ \Delta(x_\alpha(-1)^i)\left(v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0} \otimes v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}\right). \]

But by (3.2) (or (3.3)) and (3.1),

\[ oY(v((k-1)\Lambda_0 + \Lambda_1))\Delta(x_\alpha(-1)^i)\left(v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0} \otimes v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}\right) \]

\[ = \Delta(x_\alpha(-1)^i)oY(v((k-1)\Lambda_0 + \Lambda_1))\left(v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0} \otimes v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}\right) \]

\[ = a\Delta(x_\alpha(-1)^i)\left(v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0} \otimes v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}\right), \]

where \( a \) is a nonzero constant, and this equals 0 in view of (the easy part of) Theorem 2.1. The chain property now follows by another use of (3.2).
Finally, to prove the exactness, we continue to follow [CLM]. First we characterize the kernel of the map $o_Y(v(k-1)\Lambda_0 + \Lambda_1)$. Suppose that $p$ is a polynomial in $A$, so that $f_{i\Lambda_0 + (k-i)\Lambda_1}(p)$ is a general element of $W(i\Lambda_0 + (k-i)\Lambda_1)$ (recall (2.5)). Then

$$o_Y(v(k-1)\Lambda_0 + \Lambda_1)(f_{i\Lambda_0 + (k-i)\Lambda_1}(p)) = 0$$

if and only if

$$p \in \text{Ker} f_{i\Lambda_0 + (k-i+1)\Lambda_1} = A_{i\Lambda_0 + (k-i+1)\Lambda_1}.$$  

Now, the description of the ideals in (2.7) implies that

$$A_{i\Lambda_0 + (k-i+1)\Lambda_1} = \mathbb{C}[y_2, y_3, \ldots] y_{i-1} + A_{i\Lambda_0 + (k-i)\Lambda_1}.$$  

Thus

$$o_Y(v(k-1)\Lambda_0 + \Lambda_1)(f_{i\Lambda_0 + (k-i)\Lambda_1}(p)) = 0 \iff p \in \mathbb{C}[y_2, y_3, \ldots] y_i + A_{i\Lambda_0 + (k-i)\Lambda_1}.$$  

(4.7)

Next we characterize the image of

$$e_{\alpha/2}^{(k)} : W((k-i)\Lambda_0 + i\Lambda_1) \rightarrow W(i\Lambda_0 + (k-i)\Lambda_1).$$

Notice that $f_{i\Lambda_0 + (k-i)\Lambda_1}(p)$ is of the form $e_{\alpha/2}^{(k)}(w)$ for some $w \in W((k-i)\Lambda_0 + i\Lambda_1)$ if and only if for some $q = q(y_2, y_3, \ldots) \in A$,

$$f_{i\Lambda_0 + (k-i)\Lambda_1}(p) = e_{\alpha/2}^{(k)}(f_{(k-i)\Lambda_0 + i\Lambda_1}(q)) = q(x_1(-2), x_1(-3), \ldots) e_{\alpha/2}^{(k)} v_{(k-i)\Lambda_0 + i\Lambda_1} = \frac{1}{i!} q(x_1(-2), x_1(-3), \ldots) x_1(-1)^i \cdot v_{i\Lambda_0 + (k-i)\Lambda_1} = \frac{1}{i!} f_{i\Lambda_0 + (k-i)\Lambda_1}(q(y_2, y_3, \ldots) y_i).$$

But this holds if and only if

$$p - q(y_2, y_3, \ldots) y_i \in A_{i\Lambda_0 + (k-i)\Lambda_1}.$$  

In other words,

$$f_{(k-i)\Lambda_0 + i\Lambda_1}(p) \in \text{Im}(e_{\alpha/2}^{(k)}) \iff p \in \mathbb{C}[y_2, y_3, \ldots] y_i + A_{i\Lambda_0 + (k-i)\Lambda_1}.$$  

(4.8)
The exactness now follows from \((4.7)\) and \((4.8)\).

Let us derive a few consequences of the theorem. As above, we denote by

\[ W(\Lambda)_{r+\langle\alpha/2,\Lambda\rangle, s+h_\Lambda} \subset W(\Lambda) \]

the subspace of \(W(\Lambda)\) consisting of the vectors of charge \(r+\langle\alpha/2,\Lambda\rangle\) and of weight \(s+h_\Lambda\). Write

\[ W'(\Lambda)_{r,s} = W(\Lambda)_{r+\langle\alpha/2,\Lambda\rangle, s+h_\Lambda}. \]

From our construction it is clear that for \(i = 0, \ldots, k\), \(e_{(k)}^{\alpha/2}\) maps \(W'((k - i)\Lambda_0 + i\Lambda_1)_{r,s}\) to \(W'((i\Lambda_0 + (k - i)\Lambda_1)_{r,i,s+r+i}\). Also, for \(i = 1, \ldots, k\), \(o_y(v_{(k-1)\Lambda_0+\Lambda_1})\) maps \(W'((i\Lambda_0 + (k - i)\Lambda_1)_{r,i}s+r+i)\) to \(W'((i - 1)\Lambda_0 + (k - i + 1)\Lambda_1)_{r,i,s+r+i}\). Now, because of the exactness, for \(i = 1, \ldots, k\),

\[
\frac{W'(i\Lambda_0 + (k - i)\Lambda_1)_{r,i,s+r+i}}{e_{(k)}^{\alpha/2}W'((k - i)\Lambda_0 + i\Lambda_1)_{r,s}} \cong W'((i - 1)\Lambda_0 + (k - i + 1)\Lambda_1)_{r,i,s+r+i}
\]

(and similarly for \(i = 0\), and so

\[
\dim \{W'(i\Lambda_0 + (k - i)\Lambda_1)_{r,i,s+r+i}\} - \dim \{W'((k - i)\Lambda_0 + i\Lambda_1)_{r,s}\} = \dim \{W'((i - 1)\Lambda_0 + (k - i + 1)\Lambda_1)_{r,i,s+r+i}\},
\]

for every \(r\) and \(s\). The last formula written in generating function form yields

\[
x^{-i} \chi'_{W((i\Lambda_0 + (k - i)\Lambda_1)}(x/q, q) - \chi'_{W((k - i)\Lambda_0 + i\Lambda_1)}(x, q) = x^{-i} \chi'_{W((i - 1)\Lambda_0 + (k - i + 1)\Lambda_1)}(x/q, q).
\]

If we multiply this equation by \(x^i\) and substitute \(xq\) for \(x\), we obtain

\[
\chi'_{W((i\Lambda_0 + (k - i)\Lambda_1)}(x, q) - (xq)^i \chi'_{W((k - i)\Lambda_0 + i\Lambda_1)}(xq, q) = \chi'_{W((i - 1)\Lambda_0 + (k - i + 1)\Lambda_1)}(x, q), \tag{4.9}
\]

for every \(1 \leq i \leq k\), and

\[
\chi'_{W((k\Lambda_1)}(x, q) = \chi'_{W((k\Lambda_0)}(xq, q). \tag{4.10}
\]

These are exactly the Rogers–Selberg recursion formulas mentioned in the Introduction; cf. A1 and Lemma 7.2 of A5, with our \(\chi'_{W((k\Lambda_0 + (k - i)\Lambda_1)}(x, q)\) playing the role of \(J_{k+1,i+1}(0; x, q)\) in that Lemma. Of course, in these earlier works, these recursions applied to series in \(x\) and \(q\), not to the graded dimensions of graded vector spaces.
Remark 4.4 Historically, recursion formulas equivalent to (4.9) appeared for the first time in [RR]. They were independently rediscovered by Selberg [Sel]. In [A1] Andrews used these recursions to give a new proof of the Gordon identities, and in [A4], to prove his “analytic form” (multisum form) of the Gordon identities; cf. [A5].

It is easy to see, as in [A5], that the recursion relations (4.9) and (4.10), together with the initial conditions \( \lim_{x \to 0} \chi'_{W(\Lambda)}(x, q) = 1 \) and \( \lim_{q \to 0} \chi'_{W(\Lambda)}(x, q) = 1 \) (here the limits stand for the formal substitutions), uniquely determine the graded dimensions \( \chi'_{W(i\Lambda_0+(k-i)\Lambda_1)}(x, q), \quad i = 0, \ldots, k \), as elements of \( \mathbb{C}[[x, q]] \).

Example 4.5 Let \( k = 2 \). We have three \( q \)-difference equations:

\[
\begin{align*}
\chi'_{W(\Lambda_0+\Lambda_1)}(x, q) - xq\chi'_{W(\Lambda_0+\Lambda_1)}(xq, q) &= \chi'_{W(2\Lambda_1)}(x, q), \\
\chi'_{W(2\Lambda_0)}(x, q) - (xq)^2\chi'_{W(2\Lambda_1)}(xq, q) &= \chi'_{W(\Lambda_0+\Lambda_1)}(x, q)
\end{align*}
\]

and

\[
\chi'_{W(2\Lambda_0)}(xq, q) = \chi'_{W(2\Lambda_1)}(x, q).
\]

We finally have:

Corollary 4.6 For every \( i = 0, \ldots, k \),

\[
\chi'_{W(i\Lambda_0+(k-i)\Lambda_1)}(x, q) = \sum_{m \geq 0} \sum_{\frac{N_1 + \cdots + N_k = m}{N_1 \geq \cdots \geq N_k \geq 0}} \frac{x^m q^{N_1^2 + \cdots + N_k^2 + N_1 + \cdots + N_k}}{(q)_{N_1-N_2} \cdots (q)_{N_k-1-N_k} (q)_{N_k}}.
\]

or

\[
\chi_{W(i\Lambda_0+(k-i)\Lambda_1)}(x, q) = \sum_{m \geq 0} \sum_{\frac{N_1 + \cdots + N_k = m}{N_1 \geq \cdots \geq N_k \geq 0}} \frac{x^{m+(k-i)/2} q^{h_{i\Lambda_0+(k-i)\Lambda_1} + N_1^2 + \cdots + N_k^2 + N_1 + \cdots + N_k}}{(q)_{N_1-N_2} \cdots (q)_{N_k-1-N_k} (q)_{N_k}}.
\]

Proof: As in [A4], [A5], the expressions on the right–hand side of (4.11) satisfy the system of recursions (4.9), (4.10). Because of the uniqueness the result follows.

The last corollary gives the Feigin–Stoyanovsky character formulas obtained in [FS1] (formula 2.3.3') and in [Ge].
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