SAT ACTIONS OF DISCRETE QUANTUM GROUPS AND
MINIMAL AMENABLE EXTENSIONS OF QUANTUM GROUP
VON NEUMANN ALGEBRAS

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Abstract. We introduce a natural generalization of the notion of strongly
approximately transitive (SAT) states for actions of locally compact quantum
groups. In the case of discrete Kac quantum groups, we show that the existence
of unique stationary SAT states entails rigidity results concerning amenable
extensions of quantum group von Neumann algebras.

1. Introduction
The theory of boundary actions in the sense of Furstenberg [3] has been a central
concept in ergodic theory of non-amenable groups in the past few decades. Several
components of the theory have been imported to the setting of locally compact
quantum groups. In [10], Izumi introduced and studied noncommutative Poisson
boundaries of discrete quantum groups. This initiated a whole body of work on
concrete realization and applications of these boundary actions (see e.g. [11, 16, 22,
25, 27]).

Following the recent striking applications of the Furstenberg boundary actions
(the topological counterpart of the Poisson boundary) in certain problems in C∞-
algebra theory of discrete groups, the notion was extended to the discrete quantum
setting in [14].

Similarly to the commutative case, both concepts of noncommutative boundaries
mentioned above have proven to have important applications in ergodic theory of
discrete quantum groups and structure theory of their operator algebras.

The defining feature of boundary actions is the proximality property. Roughly
speaking, in the measurable setting (e.g. the Poisson boundary) this means con-
tractibility of the boundary measure (state) along paths of the underlying random
walk.

A closer look at many applications of boundary actions reveal that very often
it is the contractibility property itself that yields rigidity properties associated to
boundary actions. This is a point that we would like to emphasize in this work.

In the group setting, the notion of contractibility in the measurable setup was
formalized by Jaworski in [12] under the term strongly approximately transitive
(SAT) actions.

We introduce a natural generalization of SAT actions in the setting of actions of
locally compact quantum groups G on von Neumann algebras N.

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In the commutative case, there are examples of \( \mu \)-stationary SAT actions that are not doubly-ergodic \([1]\), hence not \( \mu \)-boundaries \([13]\). But to the best of our knowledge, those are the only known examples. This shows how close the notions are in general. Despite this, the operator theoretic nature of SAT actions offers much more flexibility, specially in the quantum setting. Moreover, SAT property can be described in commutative terms: it is irreducibility of the action of the semigroup of “quantum probabilities” on \( \mathcal{G} \) on the normal state space of the von Neumann algebra \( N \). Thus, standard operator theoretic and dynamical tools could be effective in this context.

In addition, this notion makes sense for general normal states of \( N \) and does not require stationarity. This could be particularly advantageous since requiring existence of normal stationary states on noncommutative von Neumann algebras is often too restrictive.

After proving some basic properties of SAT actions, we prove a rigidity result concerning amenable extensions of von Neumann crossed products of uniquely stationary SAT actions of discrete Kac quantum groups. These results are inspired by the recent work of Hartman and the first-named author in \([9]\), where similar results are proven for actions of locally compact groups on their boundaries.

Using the fact that Poisson boundaries are all SAT, we apply our result to recently proven examples of uniquely stationary models of noncommutative boundaries to give concrete examples where our results apply.

2. Preliminaires

In this section, we establish our notation, and briefly review some basic definitions and results concerning locally compact quantum groups and their actions. For more details we refer the reader to \([17, 18]\).

A von Neumann algebraic locally compact quantum group (lcqg) is a quadruple \( \mathbb{G} = (L^\infty(\mathbb{G}), \Delta^\mathbb{G}, \varphi^\mathbb{G}, \psi^\mathbb{G}) \), where \( L^\infty(\mathbb{G}) \) is a von Neumann algebra with a coassociative comultiplication \( \Delta^\mathbb{G} : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G}) \), and \( \varphi^\mathbb{G} \) and \( \psi^\mathbb{G} \) are, respectively, normal semifinite faithful (n.s.f.) left and right Haar weights on \( L^\infty(\mathbb{G}) \). We will denote by \( \hat{\mathbb{G}} \) and \( \mathbb{G}^{op} \), respectively, the dual quantum group and the opposite quantum group of \( \mathbb{G} \).

A lcqg \( \mathbb{G} \) is compact if the Haar weights are finite, and is discrete if its dual \( \hat{\mathbb{G}} \) is compact.

A compact quantum group \( \mathbb{G} \) is of Kac type if \( \varphi^\mathbb{G} \) is a trace, and a discrete quantum group \( \mathbb{G} \) is of Kac type if \( \hat{\mathbb{G}} \) is of Kac type.

The set of the equivalence classes of all the irreducible representations of a compact quantum group \( \mathbb{G} \) is denoted by \( \text{Irr}(\mathbb{G}) \).

Given a discrete quantum group \( \mathbb{G} \) and \( s \in \text{Irr}(\hat{\mathbb{G}}) \) we denote by \( H_s \) the corresponding Hilbert space, by \( U_s \in B(H_s) \otimes C(\hat{\mathbb{G}}) \) the unitary representation and by \( \pi_s : \ell^\infty(\mathbb{G}) \to B(H_s) \) the corresponding representation of \( \ell^\infty(\mathbb{G}) \). There is a unique state \( \phi_s : \ell^\infty(\mathbb{G}) \to \mathbb{C} \) satisfying

\[
\phi_s(x)1_{H_s} = (\text{id} \otimes \varphi^\hat{\mathbb{G}})(U_s^*(\pi_s(x) \otimes 1)U_s)
\]

for all \( x \in \ell^\infty(\mathbb{G}) \).
A (left) action of a lcqg $G$ on a von Neumann algebra $N$ is a unital injective normal $*$-homomorphism $\alpha : N \to L^\infty(G) \otimes N$ such that $(\Delta^G \otimes \text{id}) \circ \alpha = (\text{id} \otimes \alpha) \circ \alpha$. In this case we say $N$ is a $G$-von Neumann algebra.

A weight on $\tau$ on $N$ is said to be $\alpha$-invariant if $(\text{id} \otimes \tau) \alpha(x) = \tau(x) 1_N$ for every $x \in N_+$ with $\tau(x) < \infty$.

Denote by $N^\alpha := \{x \in N \mid \alpha(x) = 1 \otimes x\}$ the fixed point algebra of the action. We say $\alpha$ is ergodic if $N^\alpha = C1_N$.

The crossed product von Neumann algebra of the action is defined $G \ltimes_\alpha N = \{\alpha(N)(L^\infty(G) \otimes \mathbb{C})\}^\prime \subseteq B(L^2(G)) \otimes N$. We often identify $N$ and $L^\infty(G)$ with their images in $G \ltimes_\alpha N$.

The crossed product $G \ltimes_\alpha N$ admits a left $G$-action
\begin{equation}
\beta : G \ltimes_\alpha N \to L^\infty(G) \otimes G \ltimes_\alpha N, \quad \beta(z) = W^G_{12} z_{13} W^G_{12}
\end{equation}
and a right $\hat{G}$-action
\begin{equation}
\hat{\alpha} : G \ltimes_\alpha N \to G \ltimes_\alpha N \otimes L^\infty(\hat{G}), \quad \hat{\alpha}(z) = V^G_{12} z_{13} V^G_{12},
\end{equation}
where $W^G$ is the left multiplicative unitary for $G$ and $V^G$ is the right multiplicative unitary for $\hat{G}$. Notice that the restriction of $\beta$ to $\alpha(N)$ is $(\Delta \otimes \text{id}) = (\text{id} \otimes \alpha)$, and the restriction of $\hat{\alpha}$ to $\alpha(N)$ is the trivial action.

If $G$ is a discrete quantum group, then the map $E = (\text{id} \otimes \varphi^G) \circ \hat{\alpha}$ defines a faithful normal conditional expectation from $G \ltimes_\alpha N$ onto $N$, where as mentioned above, we identify $N$ with its canonical image in the crossed product. We refer to $E$ as the canonical conditional expectation.

A map $\Phi : N \to M$ between $G$-von Neumann algebras $N$ and $M$ is said to be $G$-equivariant if $(\text{id} \otimes \Phi) \circ \alpha = \beta \circ \Phi$, where $\alpha : G \curvearrowright N$ and $\beta : G \curvearrowright M$ are the corresponding actions.

The following fact is used in several proofs below, it generalizes part of [14, Lemma 5.2].

**Lemma 2.1.** Let $G$ be a discrete quantum group, and let $N$ be a $G$-von Neumann algebra. The canonical conditional expectation $E : G \ltimes_\alpha N \to N$ is $G$-equivariant if and only if $G$ is of Kac type.

**Proof.** The restriction of $E$ to $N$ is the identity, hence $G$-equivariant obviously. Since $N$ is in the multiplicative domain of $E$, and $G \ltimes_\alpha N$ is generated by $N$ and $L^\infty(G)$, it is enough to show that the restriction of $E$ to $L^\infty(G)$ is $G$-equivariant. For this, let $y \in L^\infty(G)$ and compute
\[
(\text{id} \otimes E)\beta(y \otimes 1) = (\text{id} \otimes E) \left( (W^*(1 \otimes y)W) \otimes 1 \right)
\]
\[
= (\text{id} \otimes \varphi^G)(\text{id} \otimes \hat{\alpha}) \left( (W^*(1 \otimes y)W) \otimes 1 \right)
\]
\[
= (\text{id} \otimes \varphi^G) \left( (\text{id} \otimes \hat{\Delta})(W^*(1 \otimes y)W) \right)_{124}
\]
\[
= \left( (\text{id} \otimes \varphi^G)(W^*(1 \otimes y)W) \right) \otimes 1 \otimes 1.
\]

On the other hand,
\[
\beta(E(y \otimes 1)) = \beta((\text{id} \otimes \varphi^G)\Delta^G_{13}(y)_{13}) = \varphi^G(y),
\]
which show that $E$ is $G$-equivariant if and only if $\varphi^G$ is $G$-invariant. Invoking [14, Lemma 5.2] yields the result. \qed
Actions of lcqg $G$ on $C^*$-algebras are defined similarly. We refer the reader to [24] for all relevant definitions and facts, including the $C^*$-algebraic versions of crossed products and their properties.

In several results we use the notion of unitary implementation of an action in the sense of [24]. We recall that given an action $\alpha$ of a lcqg $G$ on a von Neumann algebra $N$, and an n.s.f. weight $\theta$ on $N$, the unitary implementation of the action is a unitary representation $U_\theta \in B(L^2(G) \otimes L^2(N, \theta))$ of $G$ such that $\alpha(\cdot) = U_\theta(1 \otimes \cdot)U_\theta^*$. In this setup, the adjoint unitary $U_\theta^*$ implements an action $\alpha'$ of $G^{op}$ on the commutant $N' \subset B(L^2(N, \theta))$.

3. Strongly Approximately Transitive (SAT) State

In this section we define SAT states in the setting of lcqg actions, and prove some of their main general properties. This is the natural generalization of the commutative notion, which was introduced by Jaworski in [12].

**Definition 3.1.** Let $\alpha$ be an action of a lcqg $G$ on a von Neumann algebra $N$. A normal state $\nu$ on $N$ is called strongly approximately transitive (SAT), if the norm closure of the set $\{\omega \otimes \nu \alpha \mid \omega \in L^1(G)\}$ contains all normal states on $N$.

The following is a useful characterization of SAT states; we skip the proof, it is similar to [14] Lemma 4.2 (also, cf. [12] Proposition 2.2).

**Proposition 3.2.** Let $\alpha$ be an action of a lcqg $G$ on a von Neumann algebra $N$. A normal state $\nu$ on $N$ is SAT if and only if the map $(\text{id} \otimes \nu)\alpha : N \to L^\infty(G)$ is isometric on the self-adjoint part of $N$.

In particular, Proposition 3.2 provides us with a source of natural examples of SAT states. Namely, assume $G$ is a discrete quantum group, and $N$ a weak* closed operator subsystem of $L^\infty(G)$ with $\Delta^G(N) \subset L^\infty(G) \otimes N$. If $N$ admits a multiplication that turns it into a von Neumann algebra, then considered with the $G$-action defined by $\Delta^G$, the restriction of $\epsilon^G$ to $N$ is a SAT state, where $\epsilon^G$ is the unit of $L^1(G)$. Indeed, in this case, the map $(\text{id} \otimes \epsilon^G) \circ \alpha : N \to L^\infty(G)$ is just the inclusion, hence isometric.

An important class of examples of the above setup are noncommutative Poisson boundaries in the sense of Izumi [10], defined as follows (see [15] for the general locally compact case).

Given a lcqg $G$ and a normal state $\mu \in L^1(G)$, we denote $H^\infty(G, \mu) = \{x \in L^\infty(G) \mid (\text{id} \otimes \mu)\Delta^G(x) = x\}$ for the space of all $\mu$-harmonic elements in $L^\infty(G)$. The space $H^\infty(G, \mu)$ is a weak* closed operator subsystem of $L^\infty(G)$ which admits a canonical multiplication, turning it into a von Neumann algebra, called the noncommutative Poisson boundary of the pair $(G, \mu)$. The co-multiplication $\Delta^G$ restricts to an action of $G$ on $H^\infty(G, \mu)$, turning it into a $G$-von Neumann algebra. We refer the reader to [10] and [15] for more details.

We frequently use the fact that the crossed product $G \ltimes_{\Delta^G} H^\infty(G, \mu)$ is an injective von Neumann algebra [10, Corollary 2.5].

So, by the above comments, if $G$ is a discrete quantum group and $\mu \in L^1(G)$ any normal state, then the restriction of $\epsilon^G$ to $H^\infty(G, \mu)$ is SAT.

**Proposition 3.3.** Let $\alpha$ be an action of a lcqg $G$ on a von Neumann algebra $N$ and suppose that $N$ admits a SAT state. Then $\alpha$ is ergodic.
Proof. Let \( \nu \) be a SAT state. For every \( x \in N_\alpha \) we have \((id \otimes \nu)\alpha(x) = \nu(x)1\). Since \( N_\alpha \) is a von Neumann subalgebra of \( N \), it is the span of its self-adjoint part, thus, it follows from Proposition 3.2 that \( N_\alpha = C1 \).

Measurable boundaries are considered to be the opposite type of actions to measure-preserving ones. The same is true for SAT actions. This was made precise in the case of locally compact group actions in [12, Proposition 2.6]. We generalize the result to the quantum case.

Recall that a von Neumann algebra \( N \) is called purely atomic if every projection in \( N \) has a minimal subprojection.

**Theorem 3.4.** Let \( \alpha \) be an action of a lcqg \( G \) on a von Neumann algebra \( N \). Suppose that \( N \) admits an \( \alpha \)-invariant n.s.f. tracial weight and a normal faithful SAT state. Then \( N \) is purely atomic.

Proof. Let \( \tau \) be an \( \alpha \)-invariant n.s.f. tracial weight and \( \nu \) be a normal faithful SAT state on \( N \).

We claim that there is \( \varepsilon > 0 \) such that \( \nu(x) < \frac{1}{2} \) for all positive \( x \) in the unit ball of \( N \) with \( \tau(x) < \varepsilon \). Indeed, since \( \nu \) is normal, there is a positive element \( a \in N \) such that \( \tau(a) < \infty \) and \( \|\tau(a \cdot) - \nu(\cdot)\|_{N_*} < \frac{1}{4} \). Let \( \varepsilon = \frac{1}{4\|a\|} \). If \( 0 \leq x \leq 1 \) and \( \tau(x) < \varepsilon \), then

\[
\nu(x) = \nu(x) - \tau(ax) + \tau(ax) \leq \frac{1}{4} + \|a\|\tau(x) < \frac{1}{2},
\]

and the claim follows.

If \( N \) is not purely atomic, then \( N \) contains a non-zero projection \( p \) such that \( \tau(p) < \varepsilon \). Let \( \rho \in N_* \) be the normal state \( \rho(\cdot) = \frac{1}{\tau(p)} \tau(p \cdot) \). Since \( \nu \) is SAT, there is a state \( \omega \in L^1(G) \) such that

\[
|\rho(p) - (\omega \otimes \nu)\alpha(p)| < \frac{1}{2},
\]

which implies \( \nu((\omega \otimes id)\alpha(p)) > \frac{1}{2} \).

On the other hand, since \( \tau \) is \( \alpha \)-invariant, we have

\[
\tau((\omega \otimes id)\alpha(p)) = \tau(p) < \varepsilon.
\]

This contradicts the claim we established above, and hence completes the proof. \( \square \)

4. Unique stationary SAT states

The importance of unique stationarity in measurable boundaries was known from the beginning of the theory. The study of these systems was initiated by Furstenberg in [4], where they were called \( \mu \)-proximal actions. They were further studied in [15,19].

Unique stationarity in the setting of group actions on \( C^* \)-algebras was studied in [8], where a characterization of \( C^* \)-simplicity was proved in terms of unique stationarity of the canonical trace.

So, it should not come as a surprise that actions admitting unique stationary SAT states behave similarly. This section is concerned with such actions of discrete quantum groups.

Let \( G \) be a lcqg, let \( \mu \in L^1(G) \) be a state, and let \( \alpha \) be an action of \( G \) on a unital \( C^* \)-algebra \( A \). A state \( \nu \) on \( A \) is called \( \mu \)-stationary if \((\mu \otimes \nu)\alpha = \nu \). It can
be easily observed that for every state $\mu \in L^1(G)$, the set of $\mu$-stationary states on $A$ is non-empty.

**Proposition 4.1.** Let $G$ be a lcrg, $\mu \in L^1(G)$ be a state, and let $N$ be a $G$-von Neumann algebra that admits a $\mu$-stationary normal SAT state $\nu$. Assume $A$ is a $G$-invariant unital $C^*$-subalgebra of $N$ such that $\nu|_A$ is the unique $\mu$-stationary state on $A$, then the identity map is the unique $G$-equivariant u.c.p. map from $A \to N$.

**Proof.** Let $\Phi : A \to N$ be a $G$-equivariant u.c.p. map, and let $\Phi^* : N^* \to A^*$ be the adjoint map. Since $\Phi$ is $G$-equivariant,

$$(\mu \otimes \Phi^*(\nu))\alpha = (\mu \otimes \nu)(\id \otimes \Phi)\alpha = (\mu \otimes \nu)\alpha \Phi = \nu \circ \Phi = \Phi^*(\nu),$$

which shows $\Phi^*(\nu)$ is a $\mu$-stationary state on $A$, hence $\Phi^*(\nu) = \nu|_A$ by the uniqueness assumption. Moreover for every $\omega \in L^1(G)$,

$$\Phi^*((\omega \otimes \nu)\alpha) = (\omega \otimes \nu)\alpha \Phi = (\omega \otimes \Phi^*(\nu))\alpha = (\omega \otimes \nu|_A)\alpha = (\langle \omega \otimes \nu \rangle\alpha)|_A.$$ 

Since $\nu$ is SAT, the space $\{\langle \omega \otimes \nu \rangle\alpha \mid \omega \in L^1(G)\}$ is norm-dense in $N_\ast$. This implies $\Phi^*(\rho) = \rho|_A$ for all $\rho \in N_\ast$. By weak*-continuity of $\Phi^*$, we conclude $\Phi^*$ is the restriction map, hence $\Phi = \id$. $\square$

The above proposition generalizes [9] Theorem 3.4], where the similar result proved for measurable boundary actions with unique stationary compact models. The proof of the latter is based on a result of Margulis [19 Corollary 2.10(a)], which allows to pass from morphisms at level of function algebras to maps at level of underlying sets, and then use rigidity properties of boundaries. Such an argument is obviously not very quantizable: there is no underlying sets involved. Our result above shows that it is indeed the SAT property of the boundary that is behind the above rigidity. And the proof is the manifestation of our remarks in the introduction that the operator theoretic nature of the SAT property can be a significant advantage compared to boundary actions in the noncommutative setting.

Let $N \subset M$ be an inclusion of von Neumann algebras. We say $M$ is a minimal injective extension of $N$, if $M$ is injective and no intermediate von Neumann algebra $N \subset Q \subset M$ is injective.

**Theorem 4.2.** Let $G$ be a discrete quantum group of Kac type, and $\alpha$ an action of $G$ on the von Neumann algebra $N$. Assume

(i) the von Neumann algebra $G \ltimes_\alpha N$ is injective;

(ii) $N$ admits a normal SAT state $\nu$ that is $\mu$-stationary for some state $\mu \in L^1(G)$;

(iii) $N$ contains a weak* dense $G$-invariant unital $C^*$-subalgebra $A$ such that the restriction of $\nu$ to $A$ is the unique $\mu$-stationary state on $A$.

Then

(1) $G \ltimes_\alpha N$ is the minimal injective von Neumann algebra extension of $L^\infty(\hat{G})$.

(2) Let $\theta$ be an n.s.f. weight on $N$, and $U_\theta$ be the unitary implementation of $\alpha$. Then the von Neumann algebra $U_\theta(G^{op} \ltimes_\alpha N')U_\theta^\ast$ is maximal injective in $L^\infty(\hat{G})' \overline{\otimes} B(H_\theta)$.

**Proof.** (1): Recall the left action $\beta$ of $G$ on $G \ltimes_\alpha N$ defined in (2.1). Suppose that $M$ is an injective von Neumann algebra and $L^\infty(\hat{G}) \subseteq M \subseteq G \ltimes_\alpha N$. We claim
that \( M = G \ltimes_{\alpha} N \). By injectivity of \( M \), there exists a conditional expectation
\[
P : G \ltimes_{\alpha} N \to M.
\]
Since \( L^\infty(\hat{G}) \subseteq M \) and \( W \in \ell^\infty(G) \otimes L^\infty(\hat{G}) \), it follows that \( W_{12} \) is in the multiplicative domain of \( \id \otimes P \), hence
\[
(id \otimes P)(\beta(y)) = (id \otimes P)(W_{12} y_{23} W_{12}) = W_{12} (id \otimes P)(y_{23}) W_{12} = \beta(P(y)).
\]
for all \( y \in G \ltimes_{\alpha} N \), and therefore \( P \) is \( G \)-equivariant.

Let \( A \subseteq N \) be as in the assumption (iii). Define the map \( \Phi : A \to N \) to be the composition \( \Phi := E \circ P|_A \), where \( E : G \ltimes_{\alpha} N \to \alpha(N) \) is the canonical conditional expectation. Since \( G \) is of Kac type, \( E \) is \( G \)-equivariant by Lemma 3.3. Thus, \( \Phi \) is \( G \)-equivariant and ucp, hence \( \Phi = \id \) by Proposition 4.1.

Since \( E \) is faithful, it follows that \( P \) restricts to the identity map on \( A \) \cite[Lemma 3.3]{1}. Hence, \( A \subseteq M \), and therefore also \( N \subseteq M \) by weak* closedness of \( M \).

Since also \( L^\infty(\hat{G}) \subseteq M \) and \( G \ltimes_{\alpha} N \) is generated by \( L^\infty(\hat{G}) \) and \( N \), the assertion (1) follows.

(2): Let
\[
U_0(G^\op \ltimes_{\alpha'} N')U_0^* \subseteq M \subseteq L^\infty(\hat{G}) \otimes B(H_0)
\]
be inclusions of von Neumann algebras, and assume that \( M \) is injective. By \cite[Theorem 11.7 (ii)]{2} the commutant of \( G \ltimes_{\alpha} N \) in \( B(L^2(\hat{G}) \otimes H_0) \) is \( U_0(G^\op \ltimes_{\alpha'} N')U_0^* \). Hence, taking commutants inside \( B(L^2(\hat{G}) \otimes H_0) \) of the von Neumann algebras in inclusions \( \eqref{4.1} \), we get the inclusions \( L^\infty(\hat{G}) \subseteq M' \subseteq G \ltimes_{\alpha} N \), and \( M' \) is injective.

By the first part of Theorem 4.2 \( G \ltimes_{\alpha} N \) is the minimal injective extension of \( L^\infty(\hat{G}) \), therefore \( M' = G \ltimes_{\alpha} N \). Taking commutant in \( B(L^2(\hat{G}) \otimes H_0) \) once again, we conclude that \( M = U_0(G^\op \ltimes_{\alpha'} N')U_0^* \).

\[\square\]

5. Examples and applications

In this section we apply Theorem 4.2 to certain classes of discrete quantum groups for which concrete realization of Poisson boundaries with uniquely stationary models have been obtained.

First, let us recall some standard facts about noncommutative Poisson boundaries of discrete quantum groups \( G \). Given a probability measure \( \mu \) on the set \( \Irr(\hat{G}) \), let \( \phi_\mu := \sum_{s \in \Irr(\hat{G})} \phi_s \in \ell^1(G) \), and let \( P_\mu : \ell^\infty(\hat{G}) \to \ell^\infty(G) \) be the map \( x \mapsto (id \otimes \phi_\mu) \Delta^G(x) \). The map \( P_\mu \) canonically yields a Markov operator on \( \ell^\infty(\Irr(\hat{G})) \), hence a classical random walk on the set \( \Irr(\hat{G}) \). We say \( \mu \) is generating if the corresponding random walk on \( \Irr(\hat{G}) \) is irreducible, and we say \( \mu \) is ergodic if the Poisson boundary of the random walk is trivial.

If \( \mu \) is generating and ergodic, then
\[
H^\infty(G, \phi_\mu) = \{ x \in \ell^\infty(G) \big| (id \otimes \phi_s) \Delta^G(x) = x \text{ for all } s \in \Irr(\hat{G}) \}.
\]
In particular, the noncommutative Poisson boundary of the pair \( (G, \phi_\mu) \) is independent of \( \mu \), we denote it by \( H^\infty(G) \).
5.1. Orthogonal free quantum groups. First, we consider Van Daele and Wang’s orthogonal free discrete quantum groups $G = FO_N$.

In [27] a Gromov boundary $C^*$-algebra was constructed for $FO_N$, which was then shown in [14], by means of a unique stationarity result, to be also a topological boundary in the sense of [14].

We denote by $O^+_N = FO_N$ the free orthogonal compact quantum groups.

**Theorem 5.1.** Let $G = FO_N$ be Van Daele and Wang’s orthogonal free discrete quantum group, with $N \geq 3$. Let $\theta$ be an n.s.f. weight on $H^\infty(G)$. Then the following hold.

1. $G \ltimes H^\infty(G)$ is a minimal injective extension of $L^\infty(O^+_N)$.
2. $U_\theta(G^{op} \ltimes_{\alpha'} H^\infty(G))U_\theta^*$ is maximal injective in $L^\infty(O^+_N)' \otimes B(H_\theta)$, where $\theta$ is an n.s.f weight on $H^\infty(G)$, $H_\theta$ is its GNS Hilbert space, and $U_\theta \in B(\ell^2(G) \otimes H_\theta)$ is the unitary implementation of the action of $G$ on $H^\infty(G)$.

**Proof.** The quantum groups $FO_N$ are of Kac type. Let $B_\infty$ be the Gromov boundary of $G$ in the sense of Vaes–Vergnioux [27]. We recall that $B_\infty$ is a unital $G$-C*-algebra. By [14] there is a unique $\mu$-stationary state $\nu$ on $B_\infty$, and by [27] there is a canonical $G$-von Neumann isomorphism $\pi_\nu(B_\infty)' \cong H^\infty(G, \mu)$. Since the crossed product von Neumann algebra $G \ltimes H^\infty(G, \mu)$ is injective, the conditions (i)-(iii) of Theorem 4.2 hold for the $G$-von Neumann algebra $H^\infty(G, \mu)$. Hence, both assertions in the statement follow. □

5.2. $\hat{G}$-injectivity. We prove a rigidity property for $\hat{G}$-injective extensions of $L^\infty(\hat{G})$, similar to Theorem 4.2 for a much wider class of discrete quantum groups.

Recall the action of $\hat{G}$ on the crossed product $G \ltimes H^\infty(G)$, [2.2] The fact that $G \ltimes H^\infty(G)$ is $\hat{G}$-injective follows from [20] Theorem 5.3 & Corollary 7.7.

For the proof, we need the notion of Drinfeld doubles, and particularly the properties of Furstenberg boundaries from the recent work of Habbestad–Hataishi–Neshveyev [6]. We recall that given a lcqg $G$, its associated Drinfeld double is a lcqg $D(G)$ which contains (and is canonically generated by) both $G$ and $\hat{G}$ as closed quantum subgroups. We refer the reader to [6][21] for the relevant definitions, properties and boundary theory of Drinfeld doubles.

**Theorem 5.2.** Let $G$ be a discrete quantum group of Kac type. Assume that $\text{Irr}(\hat{G})$ admits a generating ergodic probability measure. Then the crossed product $G \ltimes H^\infty(G)$ is the minimal $\hat{G}$-injective von Neumann algebra extension of $L^\infty(\hat{G})$.

**Proof.** The idea of the proof is similar to Theorem 4.2 with use of a modified version of Proposition 4.1 which we argue below. Also, the required boundary properties are afforded to us by the results of [6].

Let $\mu \in \text{Prob}(\text{Irr}(\hat{G}))$ be generating, and let $D(G)$ be the Drinfeld double associated to $G$. The combination of [6] Proposition 3.3, Theorem 3.7, Theorem 3.10 imply that $H^\infty(G)$ is a $D(G)$-von Neumann algebra containing a unital $D(G)$-invariant $C^*$-subalgebra $A$ with the property that $A$ admits a unique $\hat{G}$-invariant $\mu$-stationary state $\nu$ such that there is a canonical $G$-von Neumann isomorphism $\pi_\nu(A)' \cong H^\infty(G)$. Now, similarly to the proof of Proposition 4.1 it follows that the inclusion map is the unique $D(G)$-equivariant ucp map from $A$ to $H^\infty(G)$.

Let $M$ be a $\hat{G}$-injective von Neumann algebra such that

$L^\infty(\hat{G}) \subseteq M \subseteq G \ltimes H^\infty(G)$. 

By $\hat{G}$-injectivity of $M$ there exists a $\hat{G}$-equivariant conditional expectation $P : G \ltimes H^\infty(\hat{G}) \to M$. Since $L^\infty(\hat{G}) \subseteq M$, as we noticed in the proof of Theorem 4.2, $P$ is moreover $G$-equivariant. It follows $P$ is $D(G)$-equivariant.

Since $G$ is of Kac type, by Lemma 2.1 the canonical conditional expectation $E : G \ltimes H^\infty(G) \to H^\infty(G)$ is $G$-equivariant. Clearly, $E$ is also $\hat{G}$-equivariant, so it is $D(G)$-equivariant.

Thus, $E \circ P|_A : A \to H^\infty(G)$ is a $D(G)$-equivariant map so it is identity by the above remarks. Since $E$ is faithful, it follows $P|_A = \text{id}_A$, hence $A \subseteq M$. This implies $M = G \ltimes H^\infty(\hat{G})$. □

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