Poisson problems involving fractional Hardy operators and measures

Huyuan Chen¹,∗, Konstantinos T Gkikas²,³, and Phuoc-Tai Nguyen⁴

¹ Department of Mathematics, Jiangxi Normal University, Nanchang, Jiangxi 330022, People’s Republic of China
² Department of Mathematics, University of the Aegean, 83200 Karlovassi, Samos, Greece
³ Department of Mathematics, National and Kapodistrian University of Athens, 15784 Athens, Greece
⁴ Department of Mathematics and Statistics, Masaryk University, Brno, Czech Republic

E-mail: chenhuyuan@yeah.net

Received 11 April 2023; revised 24 September 2023
Accepted for publication 26 October 2023
Published 17 November 2023

Recommended by Dr Susanna Terracini

Abstract
In this paper, we study the Poisson problem involving a fractional Hardy operator and a measure source. The complex interplay between the nonlocal nature of the operator, the peculiar effect of the singular potential and the measure source induces several new fundamental difficulties in comparison with the local case. To overcome these difficulties, we perform a careful analysis of the dual operator in the weighted distributional sense and establish fine properties of the associated function spaces, which in turn allow us to formulate the Poisson problem in an appropriate framework. In light of the close connection between the Poisson problem and its dual problem, we are able to establish various aspects of the theory for the Poisson problem including the solvability, a priori estimates, variants of Kato’s inequality and regularity results.

Keywords: Poisson problem, fractional hardy Laplacian, Radon measure, Kato’s inequality
Mathematics Subject Classification numbers: 35R11, 35J70, 35B40

∗ Author to whom any correspondence should be addressed.
1. Introduction

1.1. Overview of the literature

The past decades have witnessed an increasing number of significant developments in the research of elliptic equations involving Hardy type operators due to their applications in various scientific disciplines. The effect of Hardy operators is elusive and cannot be viewed simply as a lower order perturbation term of \((\Delta)^s\). In this paper, we devote special attention to the fractional Hardy operator of the form

\[ L_\mu^s := (-\Delta)^s + \frac{\mu}{|x|^{2s}} \]

which is constituted by two terms. The first one is the fractional Laplace operator \((-\Delta)^s\), \(s \in (0, 1]\), defined by

\[ (-\Delta)^s u(x) := C_{N,s} \lim_{\epsilon \to 0^+} \int_{B(0,\epsilon) \setminus B(0,\epsilon)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy. \]

Here \(B_r(x)\) is the ball with centre \(x \in \mathbb{R}^N (N \geq 2)\) and radius \(\epsilon > 0\), and

\[ C_{N,s} := 2^{2s} \pi^{-\frac{s}{2}} \frac{\Gamma \left( \frac{N+2s}{2} \right)}{\Gamma(1-s)} > 0 \]

(1.1)

with \(\Gamma\) being the Gamma function. The second term is the Hardy potential \(\frac{\mu}{|x|^{2s}}\), which is singular at the origin. The value of the parameter \(\mu \in \mathbb{R}\) has a profound influence on the analysis of \(L_\mu^s\).

The Hardy operator \(L_\mu^1 := -\Delta + \mu|x|^{-2}\), which is the local version of \(L_\mu^s\), appears in numerous contexts such as combustion models [24], quantum mechanic [30, 34] and control theory [16, 40]. The heat equation involving \(L_\mu^1\) was first studied in [41]. Sharp two-sided estimates for the heat kernel associated to \(L_\mu^1\) was established in [19]. The effect of the Hardy potential on the existence and finite time blow-up solutions to Schrödinger equations was analysed in [37, 38]. Singular solutions to semilinear elliptic equations with Hardy potentials
have been studied in many papers; see, e.g. [7, 9, 11, 28]. The topic on elliptic equations has been diversified in different directions, including [18] concerning spectral properties of Hardy potentials with multipolar inverse-square potentials, [10] for semilinear equations with Hardy potentials singular on the boundary, and [15, 25−27] for equations involving more general potentials blowing up on a submanifold.

The investigation of the fractional Hardy operator $L^s_{\mu}$, $s \in (0, 1)$, belongs to one of the hot topics in the area of PDE because of its wide-ranging interest to various fields in Mathematics and Physics. For instance, it is motivated by physical models related to relativistic Schrödinger operator with Coulomb potential (see [22, 23]) and by the study of Hardy inequalities and Hardy–Lieb–Thirring inequalities (see, e.g. [20, 21, 39]).

The operator $L^s_{\mu}$ possesses intriguing properties. On the one hand, it bears analogous properties as the classical operator $L^s_{\mu}$. More precisely, $L^s_{\mu}$ is closely related to the fractional Hardy inequality
\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(y)|^2 |x - y|^{N + 2s} \, dy \, dx + \mu_0 \int_{\mathbb{R}^N} |\varphi(x)|^2 \, dx \geq 0, \quad \forall \varphi \in C^\infty_0 (\mathbb{R}^N),
\] where the sharp constant in (1.2) is explicitly determined by (see, e.g. [20])
\[
\mu_0 = -2^{2s} \frac{\Gamma^2 \left( \frac{N + 2s}{4} \right)}{\Gamma^2 \left( \frac{2s}{4} \right)}.
\]

Therefore when $\mu \geq \mu_0$, $L^s_{\mu}$ is positive definite. Moreover, for $\mu \neq 0$, since the Hardy potential $\mu|x|^{-2s}$ is homogeneous with the same degree $-2s$ as $(-\Delta)^s$, it is critical to the validity of the classical theory. On the other hand, unlike the local case, the nonlocality of $(-\Delta)^s$ in interaction with the Hardy potential yields new types of difficulties in both methods employed and the computation level.

Further properties of $L^s_{\mu}$ can be found in [36]. Sharp estimates for the heat kernel associated to $L^s_{\mu}$ were established in [4, 29], which play an important role in the study of the corresponding Green function in the whole space $\mathbb{R}^N$ (see [3]).

Recently, it was shown in [6, proposition 1.2] that for $\mu \geq \mu_0$, the equation
\[
L^s_{\mu} u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}
\]
has two distinct radial solutions
\[
\Phi_{s,\mu}(x) := \begin{cases} 
|x|^{\tau-=(s,\mu)} & \text{if } \mu > \mu_0 \\
|x|^{-\frac{N}{2} + \frac{2s}{N} \ln |x|} & \text{if } \mu = \mu_0
\end{cases}
\text{and } \Gamma_{s,\mu}(x) := |x|^{\tau+(s,\mu)} \text{ for } x \in \mathbb{R}^N \setminus \{0\},
\]
where $\tau-=(s,\mu) \leq \tau+(s,\mu)$. The map $\mu \in [\mu_0, 2s) \mapsto \tau+(s,\mu)$ is continuous and increasing, while the map $\mu \in [\mu_0, 2s) \mapsto \tau-=(s,\mu)$ is continuous and decreasing. Moreover,
\[
\tau-=(s,\mu) + \tau+(s,\mu) = 2s - N \quad \text{for all } \mu \geq \mu_0,
\tau-(s,\mu_0) = \tau+(s,\mu_0) = \frac{2s - N}{2}, \quad \tau-(s,0) = 2s - N, \quad \tau+(s,0) = 0,
\]
\[
\lim_{\mu \to +\infty} \tau-=(s,\mu) = -N \quad \text{and } \lim_{\mu \to +\infty} \tau+(s,\mu) = 2s.
\]

In the remaining of the paper, when there is no ambiguity, we write for short $\tau_+$ and $\tau_-$ instead of $\tau+(s,\mu)$ and $\tau-(s,\mu)$.
It was also proved in [6] that
\[ \int_{\mathbb{R}^N} \Phi_{s,\mu} \left(-\Delta\right)^s_{\tau^+} \xi \, dx = c_{s,\mu} \xi(0), \quad \forall \xi \in C^2_0(\mathbb{R}^N), \]
where \(c_{s,\mu}>0\) and \((-\Delta)^s_{\tau^+}\) denotes the dual of the operator of \(L^s_{\mu}\), which is a weighted fractional Laplace operator given by
\[ (-\Delta)^s_{\tau^+} v(x) := C_N, s \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{v(x) - v(y)}{|x-y|^{N+2s}} |y|^\gamma dy. \]
In addition, via the above weighted distributional form, isolated singularities for solutions of nonhomogeneous equation
\[ L^s_{\mu} u = f \quad \text{in} \quad \Omega \setminus \{0\}, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \]
have been classified under an optimal assumption for nonnegative function \(f \in C^0_{\text{loc}}(\bar{\Omega} \setminus \{0\})\) with \(\beta \in (0,1)\).

For semilinear equations with fractional Hardy potentials, we refer to [17, 23, 35, 43].

1.2. Introduction of the problem and main results
Motivated by the above works, in the present paper, we aim to establish the existence, uniqueness and qualitative properties of solutions to the Poisson problem involving the Hardy potential
\[ \begin{cases} L^s_{\mu} u = \nu & \text{in} \ \Omega, \\ u = 0 & \text{in} \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.5} \]
where \(\Omega \subset \mathbb{R}^N (N \geq 2)\) is a bounded open set containing the origin and \(\nu\) is a Radon measure on \(\Omega\).

Problem (1.5) has the following notable features.

- To our knowledge, the existence of the Green function associated to \(L^s_{\mu}\) in \(\Omega\) has not been known in the literature, hence methods based on the Green representation cannot be applied to the study of (1.5). Our approach in this paper, inspired by [6], is to analyze the associated weighted fractional Laplace operator \((-\Delta)^s_\gamma\) which is defined by
\[ (-\Delta)^s_\gamma v(x) := C_N, s \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{v(x) - v(y)}{|x-y|^{N+2s}} |y|^\gamma dy, \]
where \(\gamma \in \left(-\frac{N-2s}{2}, 2s\right)\). Note that \((-\Delta)^s_0\) reduces to the well-known fractional Laplace operator. From the integral–differential form of the weighted fractional Laplace operator, a natural restriction for the function \(v\) is
\[ \|v\|_{L^{2s-\gamma}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|v(x)|}{(1+|x|)^{N+s-\gamma}} \, dx < +\infty. \]
In light of the crucial link between \( \mathcal{L}_\mu^s \) and \( (-\Delta)^s \), the study of problem (1.5) is closely connected to the investigation of problem

\[
\begin{aligned}
(-\Delta)^s u &= f & & \text{in } \Omega, \\
u &= 0 & & \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\tag{1.6}
\]

where \( f: \Omega \to \mathbb{R} \) is a measurable function.

- The measure source \( \nu \) requires to formulate the problem in an appropriate weak sense. Moreover, since the Hardy potential is singular at the origin, solutions to (1.5) may exhibit a singularity profile near the origin, therefore we impose a condition regarding the behaviour of test functions near the origin to guarantee the meaning of the weak formulation.

- The combined effect of the Hardy potential and the concentration of the source complicates the construction of solutions to problem (1.5). Therefore, for any given measure source on the whole domain \( \Omega \), we will decompose it into two measures: a measure concentrated away from the origin and a Dirac measure concentrated at the origin. The case of Dirac source was treated in [6], hence due to the linearity, it is sufficient to deal with measure source concentrated in \( \Omega \setminus \{0\} \).

Let us introduce the function spaces that we will work on in studying problems (1.5) and (1.6).

For \( \gamma \in [\frac{2-N}{2}, 2s) \), we denote by \( H^\gamma_0(\Omega; |x|) \) the closure of the functions in \( C_\infty(\mathbb{R}^N) \) with the compact support in \( \Omega \) under the norm

\[
\|u\|_{s, \gamma} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2\gamma}} |y| \gamma |x| \, dy \, dx \right)^{1/2}. \tag{1.7}
\]

Note that \( H^\gamma_0(\Omega; |x|) \) is a Hilbert space with the inner product

\[
\langle u, v \rangle_{s, \gamma} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2\gamma}} |y| \gamma |x| \, dy \, dx. \tag{1.8}
\]

For \( \mu \geq \mu_0 \), let \( H^\mu_{\mu,0}(\Omega) \) be the closure of the functions in \( C_\infty(\mathbb{R}^N) \) with the compact support in \( \Omega \) under the norm

\[
\|u\|_{\mu} := \left( \frac{C_{N, s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2\gamma}} \, dy \, dx + \mu \int_{\Omega} \frac{u(x)^2}{|x|^{2\gamma}} \, dx \right)^{1/2}.
\]

This is a metric space with metric induced by the following quantity

\[
\ll u, v \gg_{\mu, \mu} := \frac{C_{N, s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2\gamma}} \, dy \, dx + \mu \int_{\Omega} \frac{u(x)v(x)}{|x|^{2\gamma}} \, dx. \tag{1.9}
\]

When \( \mu > \mu_0 \), \( H^\mu_{\mu,0}(\Omega) \) is a Hilbert space with the inner product defined in (1.9). In the critical case \( \mu = \mu_0 \), \( H^\mu_{\mu_0,0}(\Omega) \) is no longer Hilbert space and we denote by the same notation \( H^\mu_{\mu_0,0}(\Omega) \) its standard completion. See subsection 2.2 for more details.

Our first main result depicts important properties of, as well as deciphers the relation between, the above spaces.

**Theorem 1.1.** Assume that \( \Omega \) is a bounded Lipschitz domain containing the origin.

(i) For any \( \gamma \in [\frac{2-N}{2}, 2s) \) and \( \mu \geq \mu_0 \), the space \( C_\infty^\gamma(\Omega \setminus \{0\}) \) is dense in \( H^\gamma_0(\Omega; |x|) \) and in \( H^\mu_{\mu,0}(\Omega) \).
(ii) For any \( \gamma \in [\frac{2N-N-2s}{2}, 2s) \), there is \( \mu \geq \mu_0 \) such that \( \tau_+(s, \mu) = \gamma \) and
\[
H^s_0(\Omega; |x|^{\gamma}) = \left\{ |x|^{-\gamma} u : u \in H^s_{\mu,0}(\Omega) \right\}.
\] (1.10)

(iii) Let \( \gamma \in [\frac{2N-N-2s}{2}, 2s) \), \( \beta < 2s \) and \( 1 \leq q < \min\left\{ \frac{2N-2\beta}{N-2}, \frac{2N}{N-2s} \right\} \). Then there exists a positive constant \( c = c(N, \Omega, s, \gamma, \beta, q) \) such that
\[
||| |\gamma| v||_{L^q(\Omega; |x|^{-\beta})} \leq c \|v\|_{s, \gamma}, \quad \forall v \in H^s_0(\Omega; |x|^{\gamma}).
\] (1.11)

The proof of statement (i) in theorem 1.1 is based on the choice of a special cut-off function and some delicate estimates, which enable us to deal with the whole range \( [\frac{2N-N-2s}{2}, 2s) \). This result is tremendously useful in our analysis as, in many places, it allows us to work on smooth functions with compact support in \( \Omega \) instead of functions in \( H^s_{\mu,0}(\Omega; |x|^{\gamma}) \) or in \( H^s_{\mu,0}(\Omega) \); hence we are able to dwindle or to avoid serious issues coming from the singularity at 0. Statement (ii) shows the one-to-one correspondence between \( H^s_{\mu,0}(\Omega; |x|^{\gamma}) \) and \( H^s_{\mu,0}(\Omega) \) under the transformation \( v = |x|^{-\gamma} u \) for \( u \in H^s_{\mu,0}(\Omega) \) and \( v \in H^s_0(\Omega; |x|^{\gamma}) \), which allows us to associate problem (1.5) to problem (1.6). Statement (iii) is derived from Hardy inequalities and the equivalence between the norm in \( H^s_0(\Omega; |x|^{\gamma}) \) and the norm in \( H^s_{\mu,0}(\Omega) \). For related results on weighted fractional spaces, we refer the reader to [12, 13].

We introduce the notion of variational solutions to (1.6).

**Definition 1.2.** Assume that \( \gamma \in [\frac{2N-N-2s}{2}, 2s) \). A function \( u \) is called a variational solution to (1.6) if \( u \in H^s_0(\Omega; |x|^{\gamma}) \) and
\[
\langle u, \xi \rangle_{s, \gamma} = (f, \xi)_{\gamma}, \quad \forall \xi \in H^s_0(\Omega; |x|^{\gamma}),
\] (1.12)
where
\[
(f, \xi)_{\gamma} := \int_{\Omega} f \xi |x|^{\gamma} \, dx.
\] (1.13)

The next theorem gives the existence of a variational solution to problem (1.6) and is obtained by using the standard variational method in combination with statement (iii) of theorem 1.1. In addition, a Kato type inequality for the variational solution is also provided, which leads to the uniqueness result.

**Theorem 1.3.** Let \( \gamma \in [\frac{2N-N-2s}{2}, 2s) \), \( \alpha \in \mathbb{R}, p > \max\left\{ \frac{2N}{N+2s}, \frac{2N+2\alpha}{N+2s}, 1 + \frac{\alpha}{2} \right\} \) and \( f \in L^p(\Omega; |x|^{\gamma}) \). Then problem (1.6) has a unique variational solution \( u \). Moreover, there exists a constant \( c = c(N, \Omega, s, \gamma, \alpha, p) \) such that
\[
\|u\|_{s, \gamma} \leq c \|f\|_{L^p(\Omega; |x|^{\gamma})}.
\] (1.14)

In addition, the following Kato type inequality holds
\[
\langle u^{+}, \xi \rangle_{s, \gamma} \leq \langle f \text{sign}^+(u), \xi \rangle_{\gamma}, \quad \forall 0 \leq \xi \in H^s_0(\Omega; |x|^{\gamma}).
\] (1.15)

We remark that if \( \alpha < 2s \) then the condition on \( p \) in theorem 1.3 is reduced to
\[
p > \max\left\{ \frac{2N}{N+2s}, \frac{2N+2\alpha}{N+2s} \right\}.
\]

Now we return to the study of problem (1.5). In order to give the definition of weak solutions, we introduce the space of test functions.
Definition 1.4. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying the exterior ball condition and containing the origin. For $b < 2s - \tau_+$, we denote by $X_\mu(\Omega; |x|^{-b})$ the space of functions $\psi$ with the following properties:

(i) $\psi \in H_0^s(\Omega; |x|^{\tau_+})$;
(ii) $(-\Delta)^s_{\tau_+} \psi$ exists a.e. in $\Omega \setminus \{0\}$ and
\[
\sup_{x \in \Omega \setminus \{0\}} |(-\Delta)^s_{\tau_+} \psi(x)| < +\infty;
\]
(iii) for any compact set $K \subset \Omega \setminus \{0\}$, there exist $\delta_0 > 0$ and $w \in L^1_{\text{loc}}(\Omega \setminus \{0\})$ such that
\[
\sup_{0 < \delta \leq \delta_0} \|(-\Delta)^s_{\tau_+} \psi\| \leq w \text{ a.e. in } K,
\]
where
\[
(-\Delta)^s_{\tau_+} \psi(x) := C_{N,s} \int_{\mathbb{R}^N \setminus B_\delta(x)} \frac{\psi(x) - \psi(y)}{|x-y|^{N+2s}} |y|^{\tau_+} \, dy \text{ for } x \in \Omega \setminus \{0\}.
\]

The space $X_\mu(\Omega; |x|^{-b})$ of test functions plays an essential role in the construction of distributional solutions and in the derivation of variants of Kato’s inequality. When $s = 1$, the solution of the problem
\[
L^*_\mu u = 1 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]
where $L^*_\mu$ is the dual operator of $L^1_\mu$ in a weighted distributional sense [7], could be used directly as a test function thanks to a careful analysis of its behaviour near the origin via the ODE method. Nevertheless, in the nonlocal setting, this strategy fails.

As it will be shown later by (4.6) and (4.15), for any $\psi \in X_\mu(\Omega; |x|^{-b})$, there holds
\[
|\psi(x)| \leq Cd(x)^s \quad \text{for a.e. } x \in \Omega,
\]
where $d(x) = \text{dist}(x, \partial \Omega)$. Moreover, by lemmas 4.4–4.6 below, for $\alpha \in (0,s)$,
\[
C^2_0(\Omega) \subset X_\mu(\Omega; |x|^{-b}) \subset L^\infty(\Omega) \cap C^1_{\text{loc}}(\Omega \setminus \{0\}) \cap C^{2\alpha}_{\text{loc}}(\Omega \setminus \{0\}).
\]

For $\alpha \in \mathbb{R}$, we denote by $\mathcal{M}(\Omega; d(x)^s|x|^\alpha)$ (resp. $\mathcal{M}(\Omega \setminus \{0\}; d(x)^s|x|^\alpha)$) the space of Radon measures $\nu$ on $\Omega$ (resp. $\Omega \setminus \{0\}$) such that
\[
\|\nu\|_{\mathcal{M}(\Omega; d(x)^s|x|^\alpha)} := \int_{\Omega} d(x)^s|x|^\alpha \, d|\nu| < +\infty,
\]
\[
\text{(resp. } \|\nu\|_{\mathcal{M}(\Omega \setminus \{0\}; d(x)^s|x|^\alpha)} := \int_{\Omega \setminus \{0\}} d(x)^s|x|^\alpha \, d|\nu| < +\infty)\]
and by $\mathcal{M}^+(\Omega; d(x)^s|x|^\alpha)$ (resp. $\mathcal{M}^+(\Omega \setminus \{0\}; d(x)^s|x|^\alpha)$) its positive cone.

Definition 1.5. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying the exterior ball condition and containing the origin.

(i) Suppose $f \in L^1(\Omega; d(x)^s|x|^\tau_+)$. A function $u$ is called a weak solution to problem
\[
\begin{cases}
L^*_\mu u = f & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

1797
Suppose (1.20) and (1.23) still hold true.

More precisely, if $\Omega$ is an open bounded domain containing the origin and $f$ is a measure source. Then problem (1.18) admits a unique weak solution $u = u_f$. For any $b < 2s - \tau_+$, there exists a positive constant $c = c(N, \Omega, s, \mu, b)$ such that
\[
\|u\|_{L^1(\Omega; |x|^{-b})} \leq c \|f\|_{L^1(\Omega; d(x)\, |x|^{-b})}.
\]

The next result deals with the case of $L^1$ source.

**Theorem 1.6.** Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying the exterior ball condition and containing the origin and $f \in L^1(\Omega; d(x)^s \, |x|^{-b})$. Then problem (1.18) admits a unique weak solution $u = u_f$. For any $b < 2s - \tau_+$, there exists a positive constant $c = c(N, \Omega, s, \mu, b)$ such that
\[
\|u\|_{L^1(\Omega; |x|^{-b})} \leq c \|f\|_{L^1(\Omega; d(x)^s \, |x|^{-b})}. \tag{1.20}
\]

Furthermore, there holds
\[
\int_{\Omega} u^+( -\Delta)^{s_+}_\tau \psi \, dx \leq \int_{\Omega} f \text{sign}^+(u) \psi \, dx, \quad \forall \psi \in X_{\mu} (\Omega; |x|^{-b}) \tag{1.21}
\]
and
\[
\int_{\Omega} |u| ( -\Delta)^{s_+}_\tau \psi \, dx \leq \int_{\Omega} f \text{sign}(u) \psi \, dx, \quad \forall \psi \in X_{\mu} (\Omega; |x|^{-b}). \tag{1.22}
\]

As a consequence, the mapping $f \mapsto u_f$ is nondecreasing. In particular, if $f \geq 0$ then $u \geq 0$ a.e. in $\Omega \setminus \{0\}$.

The variants of Kato’s inequality (1.21) and (1.22) are a main thrust of the present paper. The idea of the proof is to reduce problem (1.18) to the associated problem (1.6) and then to make use of Kato type inequality (1.15). Nevertheless, the derivation of (1.21) and (1.22) does not follow straightforward from estimate (1.15), but contains intermediate steps regarding an approximation procedure and a perturbation process.

The existence and uniqueness result still holds when the source is a measure, as pointed out in the next result.

**Theorem 1.7.** Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying the exterior ball condition and containing the origin and $\nu \in \mathcal{M}(\Omega \setminus \{0\}; d(x)^s \, |x|^{-b})$. Then problem (1.19) admits a unique weak solution $u = u_\nu$. For any $b < 2s - \tau_+$, there exists a positive constant $c = c(N, \Omega, s, \mu, b)$ such that
\[
\|u\|_{L^1(\Omega; |x|^{-b})} \leq c \|\nu\|_{\mathcal{M}(\Omega \setminus \{0\}; d(x)^s \, |x|^{-b})}. \tag{1.23}
\]

Moreover, the mapping $\nu \mapsto u$ is nondecreasing. In particular, if $\nu \geq 0$ then $u \geq 0$ a.e. in $\Omega \setminus \{0\}$.

We note that the exterior ball condition can be relaxed if the weight $d(x)$ is not involved in the space of measure source. More precisely, if $\Omega$ is an open bounded domain containing the origin, for any $\nu \in \mathcal{M}(\Omega \setminus \{0\}; |x|^{-b})$, then theorems 1.6 and 1.7 still hold true.
Moreover, the mapping the semilinear problem a.e. in $c$ a unique weak solution $u$ containing the origin, if for any $\Omega$ containing the origin, theorem 4.14, there exist a positive constant $c = c(N,s,\mu)$ and a nonnegative function $\Phi_{s,\mu} \in W^{2,2}_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ such that $\Phi_{s,\mu} = 0$ in $\mathbb{R}^N \setminus \Omega$,
\[
\lim_{|x| \to 0} \frac{\Phi_{s,\mu}(x)}{x} = 1 \quad \text{and} \quad \Phi_{s,\mu}(x) \leq c|x|^{-s(\rho,\mu)} \quad \forall x \in \Omega \setminus \{0\}  \tag{1.24}
\]
Moreover
\[
\ll \Phi_{s,\mu}^2, \phi \gg_\mu = 0, \quad \forall \phi \in C^\infty_0(\Omega \setminus \{0\})
\]
and
\[
\int_{\Omega} \Phi_{s,\mu}^2(0) \frac{\partial^2}{\partial^2_{x^+} x} \frac{\partial^2}{\partial^2_{x^+} x} \psi \, dx = c_{s,\mu} \psi(0), \quad \forall \psi \in C^{1,1}_0(\Omega),
\]
where $c_{s,\mu}$ is a constant given in [6, (1.15)].

We note that in [6, theorem 4.14] the fundamental solution $\Phi_{s,\mu}$ is constructed under the assumption that $\Omega$ is $C^2$. In fact, the $C^2$ smoothness of $\Omega$ can be relaxed and the assumption that $\Omega$ satisfies the exterior ball condition is sufficient for the existence of $\Phi_{s,\mu}$ due to the method of the super and sub solutions.

**Definition 1.8.** Assume that $\Omega$ is a bounded domain satisfying the exterior ball condition and containing the origin, $\nu \in \mathcal{M}(\Omega \setminus \{0\}; d(x^+|x^+))$ and $\ell \in \mathbb{R}$. We will say that $u$ is a weak solution to
\[
\begin{cases}
\mathcal{L}_{s,\mu} u = \nu + \ell \delta_0 & \text{in } \Omega \\
u \leq 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases} \tag{1.25}
\]
if for any $b < 2s - \tau_+$, $u \in L^1(\Omega; |x|^{-b})$ and
\[
\int_{\Omega} u (\Delta)^{s} \psi \, dx = \int_{\Omega \setminus \{0\}} \psi |x|^s \, d\nu + \int_{\Omega} \Phi_{s,\mu} \frac{\partial^2}{\partial^2_{x^+} x} \psi \, dx, \quad \forall \psi \in X_{s,\mu}(\Omega; |x|^{-b}). \tag{1.26}
\]

The following result states the solvability of problem (1.25).

**Theorem 1.9.** Assume that $\Omega$ is a bounded domain satisfying the exterior ball condition and containing the origin, $\nu \in \mathcal{M}(\Omega \setminus \{0\}; d(x^+|x^+))$ and $\ell \in \mathbb{R}$. Then problem (1.25) admits a unique weak solution $u = u(\nu, \ell)$. For any $b < 2s - \tau_+$, there exists a positive constant $c = c(N,\Omega,s,\mu,b)$ such that
\[
\|u\|_{L^1(\Omega; |x|^{-b})} \leq c \left( \|\nu\|_{\mathcal{M}(\Omega \setminus \{0\}; d(x^+|x^+))} + \ell \right).
\]
Moreover, the mapping $(\nu, \ell) \mapsto u$ is nondecreasing. In particular, if $\nu \geq 0$ and $\ell \geq 0$ then $u \geq 0$ a.e. in $\Omega \setminus \{0\}$.

In a forthcoming article we will develop our observations to study qualitative properties of the semilinear problem
\[
\begin{cases}
\mathcal{L}_{s,\mu} u + g(u) = \nu & \text{in } \Omega \\
u \leq 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

7199
where \( \nu \) is a Radon measure defined in an appropriate framework and \( g : \mathbb{R} \to \mathbb{R} \) is a non-decreasing continuous function such that \( g(0) = 0 \).

**Organization of the paper.** The rest of this paper is organized as follows. In section 2, we prove density properties of \( H_0^s(\Omega; |x|^{\gamma}) \) and in \( H_{\mu,0}^s(\Omega) \) and the relation between these spaces (theorem 1.1). We also establish the related continuous and compact embeddings. Section 3 is devoted to the study of variational solutions of Poisson problems involving the dual operators (theorem 1.3). In section 4, we address the solvability for Poisson equations in \( H_0^s(\Omega; |x|^{\gamma}) \) and show local regularity results. The Poisson problem is treated in section 5. In particular, we prove theorem 1.6 in section 5.1 and demonstrate theorems 1.7 and 1.9 in section 5.2.

**Notation.** Throughout this paper, unless otherwise specified, we assume that \( \Omega \subset \mathbb{R}^N \) \( (N \geq 2) \) is a bounded domain containing the origin and \( d(x) \) is the distance from \( x \in \Omega \) to \( \mathbb{R}^N \setminus \Omega \). We denote by \( c, c_1, c_2, \cdots \) positive constants that may vary from one appearance to another and depend only on the data. The notation \( c = c(a, b, \cdots) \) indicates the dependence of the constant \( c \) on \( a, b, \cdots \). The constant \( C_{N,s} \) is given by (1.1). For a function \( u \), we denote that \( u^+ = \max\{u, 0\} \) and \( u^- = \max\{-u, 0\} \). For a set \( A \subset \mathbb{R}^N \), the function \( 1_A \) denotes the indicator function of \( A \).

## 2. Function setting

In this section, we provide important properties of function spaces that we work on.

### 2.1. Space \( H_0^s(\Omega; |x|^{\gamma}) \)

We start with some estimates which are derived from Hardy inequalities.

For each \( \gamma \in \left[ \frac{2-N}{2}, 2s \right) \), by (1.4), there exists \( \mu \geq \mu_0 \) such that \( \gamma = \tau_+(s, \mu) \). For \( u \in C_0^\infty(\Omega) \), put \( v = \frac{1}{|x|^{-\gamma}}u \in C(\Omega \setminus \{0\}) \), it can be checked that

\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dydx + \mu \int_{\mathbb{R}^N} \frac{u(x)^2}{|x|^{2s}} dx
= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dydx |y| |v(x) - v(y)| |x|^{\gamma} dx.
\]

(2.1)

If \( \mu > \mu_0 \) then \( \gamma > \frac{2N-N}{2s} \). We infer from (1.2) that

\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dydx + \mu \int_{\mathbb{R}^N} \frac{u(x)^2}{|x|^{2s}} dx \geq (\mu_0 - \mu_0) \int_{\mathbb{R}^N} \frac{u(x)^2}{|x|^{2s}} dx,
\]

which implies (note that \( v \) has compact support in \( \Omega \))

\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dydx |y| |v(x) - v(y)| |x|^{\gamma} dx \geq (\mu_0 - \mu_0) \int_{\mathbb{R}^N} v(x)^2 |x|^{\gamma-2s} dx
\geq C(\Omega, s) (\mu_0 - \mu_0) \int_{\mathbb{R}^N} v(x)^2 |x|^{\gamma} dx.
\]

(2.2)

If \( \mu = \mu_0 \) then \( \gamma = \frac{2-N}{2s} \). Put \( D_\Omega := \sup_{x \in \Omega} |x| \) and denote

\[
X(t) := \left( 1 - \ln \frac{t}{D_\Omega} \right)^{-\frac{1}{2}} \quad \text{for } t > 0.
\]

(2.3)
By [39, theorem 5],
\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dy \, dx + \mu_0 \int_{\mathbb{R}^N} \frac{u(x)^2}{|x|^{2s}} \, dx \geq C(N,s) \int_{\mathbb{R}^N} \frac{u(x)^2 X(|x|)^{2}}{|x|^{2s}} \, dx,
\]
which implies
\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} \, dy \, dx \geq C(N,s) \int_{\mathbb{R}^N} v(x)^2 X(|x|)^{2} |x|^{-N} \, dx \geq C(N,\Omega,s) \int_{\mathbb{R}^N} v(x)^2 |x|^{2s-N} \, dx.
\]
From (2.2) and (2.5), we may define the space \( H_0^s(\Omega \setminus \{0\}; |x|^{\gamma}) \) as the closure of the space of functions in \( C^\infty(\mathbb{R}^N) \) with compact support in \( \Omega \setminus \{0\} \) under the norm
\[
\|u\|_{s,\gamma} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} |y|^{\gamma} |y|^{\gamma} \, dx \right)^{\frac{1}{2}}.
\]
Consequently, \( H_0^s(\Omega \setminus \{0\}; |x|^{\gamma}) \) is a Hilbert space with inner product
\[
\langle u, v \rangle_{s,\gamma} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} |y|^{\gamma} |y|^{\gamma} \, dx.
\]
In the sequel, we will use the norm (2.6) and the inner product (2.7) in \( H_0^s(\Omega \setminus \{0\}; |x|^{\gamma}) \).

**Proposition 2.1.** Assume that \( \gamma \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\geq 0} \). Then \( C^\infty_0(\Omega \setminus \{0\}) \) is dense in \( H_0^s(\Omega; |x|^\gamma) \). Consequently, \( H_0^s(\Omega \setminus \{0\}; |x|^{\gamma}) = H_0^s(\Omega; |x|^{\gamma}) \).

**Proof.** The proof is split into two steps.

**Step 1.** We show that \( C^1_0(\Omega \setminus \{0\}) \) is dense in \( H_0^s(\Omega; |x|^\gamma) \). For \( u \in H_0^s(\Omega; |x|^\gamma) \) and \( \varepsilon > 0 \), we will show that there is a function \( w \in C^1_0(\Omega \setminus \{0\}) \) with support in \( \Omega \setminus \{0\} \) such that
\[
\|u - w\|_{s,\gamma} < \varepsilon.
\]
Since \( u \in H_0^s(\Omega; |x|^\gamma) \), by density, there exists \( v \in C^\infty_0(\Omega) \) such that
\[
\|u - v\|_{s,\gamma} < \frac{\varepsilon}{3}.
\]
For any \( 0 < h < \min \left\{ \frac{1}{2}, \frac{\text{dist}(0, \partial \Omega)}{4} \right\} \), we consider the function
\[
\eta_h(x) := \begin{cases} 
1 & \text{if } |x| > h, \\
1 - \frac{1}{\ln h} \ln \left( \frac{|x|}{h} \right) & \text{if } h^2 \leq |x| \leq h, \\
0 & \text{if } |x| < h^2,
\end{cases}
\]
which is in \( C^{0,1}(\mathbb{R}^N) \), and set \( v_h := \eta_h v \).
Now we will show that
\[
\lim_{h \to 0} \|v - v_h\|_{s,\gamma} = 0.
\]
Indeed, by the definition of the norm $\| \cdot \|_{s, \gamma}$, we have
\[
\| v - v_h \|_{s, \gamma}^2 = \int_{\{ |x| \leq 2h \}} \int_{\{|y| \leq h \}} \frac{|v(x) - v_h(x) + v_h(y) - v(y)|^2}{|x-y|^{N+2s}} |y|^{\gamma} dy |x|^{\gamma} dx + 2 \int_{\{ |x| > 2h \}} \int_{\{ |y| \leq h \}} \frac{|v(x) - v_h(x) + v_h(y) - v(y)|^2}{|x-y|^{N+2s}} |y|^{\gamma} dy |x|^{\gamma} dx. \tag{2.12}
\]

In the remaining of the proof, $c$ denotes a generic constant which is dependent on $N, s$ and $\gamma$, but independent of $h$ and $v$, and which may vary from one appearance to another.

The second term on the right-hand side of (2.12) is estimated as
\[
2 \int_{\{ |x| > 2h \}} \int_{\{ |y| \leq h \}} \frac{|v(x) - v_h(x) + v_h(y) - v(y)|^2}{|x-y|^{N+2s}} |y|^{\gamma} dy |x|^{\gamma} dx \leq 2 h^{-2s} \int_{\{ |y| \leq h \}} |v(y) - v(y)|^2 |y|^{\gamma} dy \leq c \| v \|_{L^\infty(\Omega)}^2 h^{N+2s-2} \|1 - \eta_h(y)\|_{L^\infty(\Omega)} h^{-2} \leq c \| v \|_{L^\infty(\Omega)}^2 h^{N+2s-2} \ln h^{-2},
\]
where $2 \gamma + N - 2s \geq 0$ and $\| v \|_{L^\infty(\Omega)}$ could be large, but it is independent of $h$.

The first term on the right-hand side of (2.12) is estimated as
\[
\int_{\{ |x| \leq 2h \}} \int_{\{ |y| \leq h \}} \frac{|v(x) - v_h(x) + v_h(y) - v(y)|^2}{|x-y|^{N+2s}} |y|^{\gamma} dy |x|^{\gamma} dx \leq 2 \int_{\{ |x| \leq 2h \}} \int_{\{ |y| \leq h \}} \frac{(1 - \eta_h(x))^2 |v(x) - v(y)|^2}{|x-y|^{N+2s}} |y|^{\gamma} dy |x|^{\gamma} dx + 2 \int_{\{ |x| \leq 2h \}} \int_{\{ |y| \leq h \}} \frac{v(y)^2 |\eta_h(x) - \eta_h(y)|^2}{|x-y|^{N+2s}} |y|^{\gamma} dy |x|^{\gamma} dx \leq 2 \int_{\{ |x| \leq 2h \}} \int_{\{ |y| \leq h \}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} |y|^{\gamma} dy |x|^{\gamma} dx + 2 \int_{\{ |x| \leq 2h \}} \int_{\{ |y| \leq h \}} \frac{|\eta_h(x) - \eta_h(y)|^2}{|x-y|^{N+2s}} |y|^{\gamma} dy |x|^{\gamma} dx.
\]

By the dominated convergence theorem we have
\[
\lim_{h \to 0} \int_{\{ |x| \leq 2h \}} \int_{\{ |y| \leq h \}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} |y|^{\gamma} dy |x|^{\gamma} dx = 0. \tag{2.15}
\]
Next we estimate the last term in (2.14) by splitting it into several terms as follows

\[ \int_{|x|<2h} \int_{|y|<2h} \frac{|\eta_h(x) - \eta_h(y)|^2}{|x-y|^{N+2\gamma}} |y|^\gamma |x|^\gamma \, dx \]

\[ = \int_{|x|<h^2} \int_{|y|<h^2} \cdots \, dx + \int_{|x|<h^2} \int_{0 \leq |y| < 2h^2} \cdots \, dx 
+ \int_{|x|<h^2} \int_{2h^2 \leq |y| < h} \cdots \, dx + \int_{|x|<h^2} \int_{|y| \leq 2h^2} \cdots \, dx 
+ \int_{h^2 \leq |x| < h} \int_{|y|<h^2} \cdots \, dx + \int_{h^2 \leq |x| < h} \int_{0 \leq |y| < 2h^2} \cdots \, dx 
+ \int_{h^2 \leq |x| < h} \int_{2h^2 \leq |y| < h} \cdots \, dx + \int_{h^2 \leq |x| < h} \int_{|y| \leq 2h^2} \cdots \, dx 
+ \int_{h \leq |x| \leq 2h} \int_{h^2 \leq |y| < h} \cdots \, dx + \int_{h \leq |x| \leq 2h} \int_{0 \leq |y| < 2h^2} \cdots \, dx 
+ \int_{h \leq |x| \leq 2h} \int_{h \leq |y| \leq 2h^2} \cdots \, dx \]

(2.16)

\[ =: A_{h,1} + A_{h,2} + A_{h,3} + A_{h,4} + A_{h,5} + A_{h,6} + A_{h,7} + A_{h,8} + A_{h,9} + A_{h,10} \]

We will estimate \( A_{h,i} \), \( i \in \{1, \cdots, 10\} \), successively.

**Estimate of \( A_{h,1} \).** We note that \( \eta_h(x) = 0 \) for any \( |x| < h^2 \), hence

\[ A_{h,1} = 0. \]

**Estimate of \( A_{h,2} \).** By the definition of \( \eta_h \) in (2.10), the inequality \( \ln t \leq t - 1 \) for any \( t > 0 \) and the assumption that \( \frac{2N}{2} \leq \gamma < 2s \), we have

\[ A_{h,2} = |\ln h|^{-2} \int_{|x|<h^2} \int_{|y|<h^2} \frac{1}{|x-y|^{N+2\gamma}} |y|^\gamma |x|^\gamma \, dx \]

\[ \leq h^{-4} |\ln h|^{-2} \int_{|x|<h^2} \int_{|y|<h^2} \frac{|y|^2}{|x-y|^{N+2\gamma}} |y|^\gamma |x|^\gamma \, dx \]

\[ \leq h^{-4+2\gamma} |\ln h|^{-2} \int_{|x|<h^2} \int_{|y|<h^2} \frac{|y|^2}{|x-y|^{N+2\gamma}} |y|^\gamma |x|^\gamma \, dx \]

\[ \leq c h^{2(\gamma+2\gamma-2s)} |\ln h|^{-2}. \]

Here in the last estimate in (2.17), we have used the inequalities

\[ |y|^2 \leq (|y| - |x|)^2 \leq |y-x|^2 \quad \text{for} \quad |x| < h^2 \leq |y|. \]

**Estimate of \( A_{h,3} \).** By the definition of \( \eta_h \) in (2.10) and since \( |x-y| \geq \frac{|x|}{2} \) for any \( |x| < h^2 \) and \( 2h^2 \leq |y| < h \), we obtain

\[ A_{h,3} = |\ln h|^{-2} \int_{|x|<h^2} \int_{2h^2 \leq |y| < h} \frac{1}{|x-y|^{N+2\gamma}} |y|^\gamma |x|^\gamma \, dx \]

\[ \leq c h^{2(\gamma+2\gamma)} |\ln h|^{-2} \int_{2h^2 \leq |y| < h} \frac{|y|^2}{h^2} |y|^\gamma |x|^\gamma \, dy \]

\[ \leq c h^{2(\gamma+2\gamma-2s)} |\ln h|^{-2}. \]

(2.18)
Estimate of $A_{h,4}$. By the definition of $\eta_h$ in (2.10), we have

$$A_{h,4} = \int_{|x| < h} \int_{h^2 \leq |y| < 2h} \frac{1}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx \leq c h^{-2+\gamma+2(N+\gamma)}.$$  

Estimate of $A_{h,5}$. By Fubini’s theorem and by exchanging variables $x$ and $y$ in $A_{h,5}$, we obtain that

$$A_{h,5} = \int_{h^2 \leq |x| < h} \int_{|y| < h} \frac{|\eta_h(x) - \eta_h(y)|^2}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx = A_{h,2} + A_{h,3} \leq c h^{2(N+2\gamma-2)} |\ln h|^{-2}. $$

Estimate of $A_{h,6}$. By the definition of $\eta_h$ in (2.10), we obtain

$$A_{h,6} = |\ln h|^{-2} \int_{h^2 \leq |x| < h} \int_{|y| < h} \frac{|\ln |x| - \ln |y||^2}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx \leq 2 |\ln h|^{-2} \int_{h^2 \leq |x| < h} \int_{|y| < h} \frac{|\ln |x| - \ln |y||^2}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx \leq 2 |\ln h|^{-2} \int_{h^2 \leq |x| < h} \int_{|y| < h} \frac{|\ln |x| - \ln |y||^2}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx + 2 |\ln h|^{-2} \int_{h^2 \leq |x| < h} \int_{|y| < h} \frac{|\ln |x| - \ln |y||^2}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx =: A_{h,6,1} + A_{h,6,2}. $$

Since $|\ln |x| - \ln |y|| \leq \frac{|x-y|}{|x|}$, we find

$$A_{h,6,1} \leq 2 |\ln h|^{-2} \int_{h^2 \leq |x| < h} \int_{|y| < h} \frac{|x-y|^{N-2\gamma+2}|y|^\gamma dy |x|^\gamma dx \leq c h^{2-2\gamma} |\ln h|^{-2} \int_{h^2 \leq |x| < h} |x|^{2\gamma-2\gamma} dx \leq \begin{cases} c h^{2-2\gamma} |\ln h|^{-1} & \text{if } \gamma = \frac{2s-N}{2}, \\
 c h^{N+2\gamma-2-4\gamma} |\ln h|^{-2} & \text{if } \gamma > \frac{2s-N}{2}. \end{cases}$$

Also,

$$A_{h,6,2} \leq 2 |\ln h|^{-2} \int_{h^2 \leq |x| < h} \int_{|y| < h} \frac{|\ln |x| - \ln |y||^2}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx \leq c |\ln h|^{-2} \int_{h^2 \leq |x| < h} \int_{|y| < h} \frac{|\ln |x| |y|^{\gamma} dy |x|^\gamma dx \leq c |\ln h|^{-2} \int_{h^2 \leq |x| < h} |x|^{2\gamma-2\gamma} dx \leq \begin{cases} c |\ln h|^{-1} & \text{if } \gamma = \frac{2s-N}{2} \\
 c h^{N+2\gamma-2} |\ln h|^{-2} & \text{if } \gamma > \frac{2s-N}{2}. \end{cases}$$

7204
Combining the above estimates, we deduce

\[ A_{h,6} \leq c |\ln h|^{-1}. \]

**Estimate of** \( A_{h,7} \). By the definition of \( \eta_h \) in (2.10), we have

\[
A_{h,7} = |\ln h|^{-2} \int_{\{h^2 \leq |x| < h \}} \int_{\{|h| \leq 2h \}} \frac{|\ln \frac{|x|}{h}|^2}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx
\]

\[
\leq |\ln h|^{-2} \int_{\{h^2 \leq |x| < \frac{h}{2} \}} \int_{\{|h| \leq 2h \}} \frac{|\ln \frac{|x|}{h}|^2}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx
\]

\[
+ |\ln h|^{-2} \int_{\{\frac{h}{2} \leq |x| \leq h \}} \int_{\{|h| \leq 2h \}} \frac{|\ln \frac{|x|}{h}|^2}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx
\]

\[=: A_{h,7,1} + A_{h,7,2}. \]

We have

\[ A_{h,7,1} \leq c h^{N+2\gamma-2\alpha} |\ln h|^{-2} \int_{\{h^2 \leq |x| < \frac{h}{2} \}} \frac{|x|^\gamma}{h^\alpha} |x|^\gamma dx \leq c h^{N+2\gamma-2\alpha} |\ln h|^{-2}. \]

By using the estimate \( |\ln \frac{|x|}{h}| \leq 2 |\frac{|x|-|h|}{h}| \) for \( \frac{h}{2} \leq |x| \leq |h| \), we obtain

\[ A_{h,7,2} \leq c h^{N+2\gamma-2\alpha} |\ln h|^{-2}. \]

From the above estimates, we derive

\[ A_{h,7} \leq c h^{N+2\gamma-2\alpha} |\ln h|^{-2}. \]

**Estimate of** \( A_{h,8} \). Using Fubini’s theorem and exchanging variables \( x \) and \( y \) in \( A_{h,8} \) lead to

\[ A_{h,8} = A_{h,4} \leq c h^{-2\alpha+\gamma+2(N+\gamma)}. \]

**Estimate of** \( A_{h,9} \). Similarly, using Fubini’s theorem and exchanging variables \( x \) and \( y \) in \( A_{h,9} \) imply

\[ A_{h,9} = A_{h,7} \leq c h^{N+2\gamma-2\alpha} |\ln h|^{-2}. \]

**Estimate of** \( A_{h,10} \). By the definition of \( \vartheta_h \) in (2.10),

\[ A_{h,10} = 0. \]

Finally, by plugging the estimates of \( A_{h,i}, i \in \{1, \cdots, 10\} \) into (2.16), we derive, for \( h \) small,

\[
\int_{\{|x| \leq 2h \}} \int_{\{|y| \leq 2h \}} \frac{|\vartheta_h(x) - \vartheta_h(y)|^2}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx \leq c |\ln h|^{-1}. \tag{2.19}
\]

Combining (2.12)–(2.15) and (2.19), we deduce (2.11).
As a consequence, there exists $h_0$ such that
\[ \|v - v_{h_0}\|_{x, \gamma} < \frac{\epsilon}{3}. \]  
(2.20)

From (2.9) and (2.20), we deduce
\[ \|u - v_{h_0}\|_{x, \gamma} < \frac{2\epsilon}{3}. \]

Since $v_{h_0} \in C^{0,1}_0(\Omega \setminus \{0\})$, we obtain (2.8) with $w = v_{h_0}$. Thus $C^{0,1}_0(\Omega \setminus \{0\})$ is dense in $H_0^1(\Omega; |x|^{\gamma})$.

**Step 2.** We show that $C^{0,\infty}_0(\Omega \setminus \{0\})$ is dense in $H_0^1(\Omega; |x|^{\gamma})$. For $u \in H_0^1(\Omega; |x|^{\gamma})$ and $\epsilon > 0$, we will show that there is a function $w \in C^{0,\infty}_0(\Omega \setminus \{0\})$ with support in $\Omega \setminus \{0\}$ such that
\[ \|u - w\|_{x, \gamma} < \epsilon. \]
(2.21)

Consider a sequence of mollifiers $\{\zeta_n\}_{n \in \mathbb{N}}$. Let $n_0 \in \mathbb{N}$ large enough such that
\[ \text{supp} \ (\zeta_n * v_{h_0}) \cup \text{supp} \ (v_{h_0}) \subset \Omega_1 \subseteq \Omega_2 \subseteq \Omega \setminus \{0\}, \]
where $\Omega_1$ and $\Omega_2$ are open sets. We will show that
\[ \lim_{n \to \infty} \|v_{h_0} - \zeta_n * v_{h_0}\|_{x, \gamma} = 0. \]  
(2.22)

Indeed, we write
\[
\begin{align*}
\|v_{h_0} - \zeta_n * v_{h_0}\|_{x, \gamma}^2 &= \int_{\Omega_2} \int_{\Omega_2} \frac{|v_{h_0}(x) - \zeta_n * v_{h_0}(x) + \zeta_n * v_{h_0}(y) - v_{h_0}(y)|^2}{|x - y|^{N+2s}} |y| \gamma \, dy \, |x| \gamma \, dx \\
&\quad + 2 \int_{\mathbb{R}^n \setminus \Omega_1} \int_{\Omega_2} \frac{|v_{h_0}(x) - \zeta_n * v_{h_0}(x) + \zeta_n * v_{h_0}(y) - v_{h_0}(y)|^2}{|x - y|^{N+2s}} |y| \gamma \, dy \, |x| \gamma \, dx.
\end{align*}
\]

Since $\zeta_n * v_{h_0} \to v_{h_0}$ in $L^2(\Omega_2)$ as $n \to \infty$, we have
\[
\int_{\mathbb{R}^n \setminus \Omega_1} \int_{\Omega_2} \frac{|v_{h_0}(x) - \zeta_n * v_{h_0}(x) + \zeta_n * v_{h_0}(y) - v_{h_0}(y)|^2}{|x - y|^{N+2s}} |y| \gamma \, dy \, |x| \gamma \, dx \\
\leq c \int_{\Omega_2} |\zeta_n * v_{h_0}(y) - v_{h_0}(y)|^2 \, dy \to 0 \quad \text{as } n \to \infty,
\]
where $c > 0$ depends on $N, s, \gamma, \Omega_1, \Omega_2$.

We also find that
\[
\int_{\Omega_2} \int_{\Omega_2} \frac{|v_{h_0}(x) - \zeta_n * v_{h_0}(x) + \zeta_n * v_{h_0}(y) - v_{h_0}(y)|^2}{|x - y|^{N+2s}} |y| \gamma \, dy \, |x| \gamma \, dx \\
\leq c \int_{\Omega_2} \int_{\Omega_2} \frac{|v_{h_0}(x) - \zeta_n * v_{h_0}(x) + \zeta_n * v_{h_0}(y) - v_{h_0}(y)|^2}{|x - y|^{N+2s}} \, dy \, dx \to 0 \quad \text{as } n \to \infty.
\]

From the above estimates, we get (2.22). Consequently, there exists $n_0 \in \mathbb{N}$ such that
\[ \|v_{h_0} - \zeta_{n_0} * v_{h_0}\|_{x, \gamma} < \frac{\epsilon}{3}. \]  
(2.23)
Finally combining (2.9), (2.20) and (2.23) yields
\[ \|u - \zeta_m \ast v_h\|_{\mathcal{H}} < \varepsilon. \]

Therefore, we obtain (2.21) with \( w = \zeta_m \ast v_h \). The proof is complete. \( \square \)

2.2. The space \( \mathcal{H}^s_{\mu,0}(\Omega) \)

For any \( \mu > \mu_0 \), we deduce from (1.2) that
\[ \frac{C_{N,s}}{2} \left( 1 + \frac{\mu - \mu_0}{\mu_0} \right) \|u\|_{\mathcal{H}}^2 \leq \frac{C_{N,s}}{2} \left( 1 - \frac{\mu - \mu_0}{\mu_0} \right) \|u\|_{\mathcal{H}}^2, \quad \forall u \in C^\infty_0(\Omega). \quad (2.24) \]

Therefore, for \( \mu > \mu_0 \), the space \( \mathcal{H}^s_{\mu,0}(\Omega) \) defined in the section 1 is a Hilbert space.

Nevertheless, when \( \mu = \mu_0 \), the space \( \mathcal{H}^s_{\mu_0,0}(\Omega) \) is no longer Hilbert. We point out below that \( \mathcal{H}^s_{\mu_0,0}(\Omega) \) can be associated with \( H^s_0(\Omega; |x|^{-\frac{n-2}{2}}) \).

Let \( Z \) be the space of all Cauchy sequences \( \{u_n\} \) in \( \mathcal{H}^s_{\mu_0,0}(\Omega) \). We introduce the equivalence relation \( \sim \) on \( Z \): for any \( \{u_n\}, \{u'_n\} \in Z \), we write \( \{u_n\} \sim \{u'_n\} \) if and only if \( \lim_{n \to \infty} \|u_n - u'_n\|_{\mu_0} = 0 \). The quotient map associated with \( \sim \) is the following surjective map
\[ Q: Z \to \mathcal{H}^s_{\mu_0,0}(\Omega) := Z / \sim \]
\[ \{u_n\} \mapsto \{\{u_n\}\}. \]

Then the quotient space \( \mathcal{H}^s_{\mu_0,0}(\Omega) \) endowed with the inner product
\[ \rho([\{u_n\}], [\{u'_n\}]) := \lim_{n \to \infty} \ll u_n, u'_n \gg_{\mu_0}, \]
is a Hilbert space.

Let \( i: \mathcal{H}^s_{\mu_0,0}(\Omega) \to \mathcal{H}^s_{\mu_0,0}(\Omega) \) be such that
\[ i(u) = [\{u\}], \quad \forall u \in \mathcal{H}^s_{\mu_0,0}(\Omega), \]
where \([u]\) denotes the constant sequence in \( \mathcal{H}^s_{\mu_0,0}(\Omega) \). Then \( i(\mathcal{H}^s_{\mu_0,0}(\Omega)) \) is dense in \( \mathcal{H}^s_{\mu_0,0}(\Omega) \).

We have the following observations.

(a) For each \( u \in \mathcal{H}^s_{\mu_0,0}(\Omega) \), by proposition 2.3, we can easily show that there exists a Cauchy sequence \( \{u_n\} \subset C^\infty_0(\Omega \setminus \{0\}) \) such that \( u = [\{u_n\}] \). Set \( \gamma = -\frac{N-2}{2} \) and \( v_n = |x|^{-\gamma}u_n \).

Then by (2.1), we can easily see that \( \{v_n\} \) is a Cauchy sequence in \( H^s_0(\Omega; |x|^\gamma) \). This implies the existence of a function \( v \in H^s_0(\Omega; |x|^\gamma) \) such that \( v_n \to v \) in \( H^s_0(\Omega; |x|^\gamma) \) and
\[ \rho(u, u) = \lim_{n \to \infty} \|u_n\|_{\mu_0}^2 = \lim_{n \to \infty} \frac{C_{N,s}}{2} \|v_n\|_{\mathcal{H}}^2 = \frac{C_{N,s}}{2} \|v\|_{\mathcal{H}}^2. \quad (2.25) \]

(b) Conversely, for any \( v \in H^s_0(\Omega; |x|^\gamma) \), by proposition 2.1, there exists a sequence \( \{v_n\} \subset C^\infty_0(\Omega \setminus \{0\}) \) such that \( v_n \to v \) in \( H^s_0(\Omega; |x|^\gamma) \). Set \( u_n = |x|^\gamma v_n \), then by (2.1), \( \{u_n\} \subset C^\infty_0(\Omega \setminus \{0\}) \) is a Cauchy sequence in \( \mathcal{H}^s_{\mu_0,0}(\Omega) \). Hence there exists \( u \in \mathcal{H}^s_{\mu_0,0}(\Omega) \) such that \( u = [\{u_n\}] \) and (2.25) holds.
In light of the above observations, we may identify the space \( \tilde{H}_{\mu,0}^s(\Omega) \) with the space
\[
\mathcal{H}_0(\Omega) := \{ |x|^{-\gamma} v : v \in H^s_0(\Omega; |x|^{\gamma}) \}
\]
endowed with the norm
\[
\|u\|_{\mathcal{H}_0(\Omega)} := \sqrt{\frac{C_{N,s}}{2}} \|v\|_{s,\gamma}.
\]

**Remark 2.2.** Let \( \gamma = -\frac{N - 2s}{2} \) and \( v \in C_0^\infty(\Omega) \) such that \( v = 1 \) in \( B_\varepsilon(0) \) for some \( \varepsilon > 0 \) small enough such that \( B_{4\varepsilon}(0) \subset \Omega \). Then \( v \in H^s_0(\Omega; |x|^{\gamma}) \), and hence \( u = |x|^{\gamma} v \in \tilde{H}_{\mu,0}^s(\Omega) \).

**Proposition 2.3.** For any \( \mu \geq \mu_0 \), \( C_0^\infty(\Omega \setminus \{0\}) \) is dense in \( \tilde{H}_{\mu,0}^s(\Omega) \).

**Proof.** For any \( u \in \tilde{H}_{\mu,0}^s(\Omega) \) and \( \varepsilon > 0 \), by the definition of \( \tilde{H}_{\mu,0}^s(\Omega) \), there exists \( u_\varepsilon \in C_0^\infty(\Omega) \) such that
\[
\|u - u_\varepsilon\|_\mu < \frac{\varepsilon}{2}.
\]  
(2.26)

Let \( v_\varepsilon = |x|^{-\gamma} u_\varepsilon \in C_0^\infty(\Omega \setminus \{0\}) \) with \( \gamma = \tau_+ (s, \mu) \), then, by (2.1), \( v_\varepsilon \in H^s_0(\Omega; |x|^{\gamma}) \) and
\[
\|u - |x|^{\gamma} v_\varepsilon\|_\mu < \frac{\varepsilon}{2}.
\]
From the proof of proposition 2.1, there exists \( \tilde{v}_\varepsilon \in C_0^\infty(\Omega \setminus \{0\}) \) such that
\[
\sqrt{\frac{C_{N,s}}{2}} \|\tilde{v}_\varepsilon - v_\varepsilon\|_{s,\gamma} < \frac{\varepsilon}{2}.
\]
Put \( \tilde{u}_\varepsilon = |x|^{\gamma} \tilde{v}_\varepsilon \) then \( \tilde{u}_\varepsilon \in C_0^\infty(\Omega \setminus \{0\}) \) and from the equality (2.1), we have
\[
\|\tilde{u}_\varepsilon - u_\varepsilon\|_\mu < \frac{\varepsilon}{2}.
\]  
(2.27)

Combining (2.26) and (2.27) implies
\[
\|\tilde{u}_\varepsilon - u\|_\mu < \varepsilon.
\]
By the arbitrariness of \( \varepsilon > 0 \), we derive that \( C_0^\infty(\Omega \setminus \{0\}) \) is dense in \( \tilde{H}_{\mu,0}^s(\Omega) \). \( \square \)

Recall that the fractional Sobolev exponent is
\[
2^*_s = \frac{2N}{N - 2s}.
\]  
(2.28)

**Lemma 2.4.** Assume that \( \mu \geq \mu_0 \), \( \alpha < 2s \) and \( 1 \leq q < \min \left\{ 2^*_s, \frac{2N - 2\alpha}{N - 2s} \right\} \). Then \( \tilde{H}_{\mu,0}^s(\Omega) \) continuously and compactly embedded into \( L^q(\Omega; |x|^{-\alpha}) \). Moreover, there exists a positive constant \( c = c(N, \Omega, s, \mu, \alpha, q) \) such that
\[
\|u\|_{L^q(\Omega; |x|^{-\alpha})} \leq c \|u\|_\mu, \quad \forall u \in \tilde{H}_{\mu,0}^s(\Omega).
\]  
(2.29)
Proof. We consider two cases.

Case 1: \( \alpha \leq 0 \). If \( \mu > \mu_0 \), we infer from (2.24) that \( H_{\mu,0}^s(\Omega) = H_0^s(\Omega) \). This and the well-known fractional embedding (see [14]) imply that the embedding \( H_\mu^s(\Omega) \hookrightarrow L^q(\Omega) \) is continuous for any \( q \in [1, 2^*_\mu] \) and compact for any \( q \in [1, 2^*_\mu) \). If \( \mu = \mu_0 \) then \( H_0^s(\Omega) \subsetneq H_{\mu_0,0}^s(\Omega) \) and the embedding \( H_{\mu,0}^s(\Omega) \hookrightarrow L^q(\Omega) \) is continuous and compact for any \( q \in [1, 2^*_\mu) \) (see [21]). Therefore, for any \( q \in [1, 2^*_\mu) \), there holds

\[
\|u\|_{L^q(\Omega; |x|^{-\alpha})} \leq c \|u\|_{L^q(\Omega)} \leq c \|u\|_\mu, \quad \forall u \in H_{\mu,0}^s(\Omega),
\]

(2.30)

where \( c = c(N, \Omega, s, \mu, \alpha, q) \).

Case 2: \( 0 < \alpha < 2s \). Let \( \tilde{\alpha} \in (\alpha, 2s) \) be close enough to \( 2s \), then by using Hölder inequality and estimate (2.4), we obtain

\[
\int_\Omega |x|^{-\alpha} |u(x)|^q \, dx = \int_\Omega |x|^{-\alpha} |u(x)|^{\frac{2q}{\alpha}} |u(x)|^{q - \frac{2q}{\alpha}} \, dx
\]

\[
\leq \left( \int_\Omega |x|^{-\tilde{\alpha}} |u(x)|^2 \, dx \right)^{\frac{2q}{\alpha}} \left( \int_\Omega |u(x)|^{q - \frac{2q}{\alpha}} \, dx \right)^{1 - \frac{2q}{\alpha}} \leq c \|u\|_\mu^q.
\]

(2.31)

Here in the second estimate, we have used the inequality that \( \frac{\tilde{\alpha}}{\alpha}(q - \frac{2q}{\alpha}) < 2s^* \) due to the choice that \( \tilde{\alpha} \) is close enough to \( 2s \) and the assumption \( q < \frac{2s^* - 2s}{N - 2s} \). Thus \( H_{\mu,0}^s(\Omega) \) is continuous embedded into \( L^q(\Omega; |x|^{-\alpha}) \) and (2.29) follows by (2.31).

Let \( u_n \to 0 \) weakly in \( H_{\mu,0}^s(\Omega) \) as \( n \to +\infty \). Then \( \{u_n\} \) is uniformly bounded in \( H_{\mu,0}^s(\Omega) \) and by the compactness embedding \( H_{\mu,0}^s(\Omega) \hookrightarrow L^q(\Omega; |x|^{-\alpha}) \),

\[
\int_\Omega |u_n(x)|^{\frac{2q}{\alpha}} (q - \frac{2q}{\alpha}) \, dx \to 0
\]

due to the inequality \( \frac{\tilde{\alpha}}{\alpha}(q - \frac{2q}{\alpha}) < 2s^* \) in (2.31). Consequently, by (2.31), we obtain

\[
\int_\Omega |x|^{-\alpha} |u_n(x)|^q \, dx \to 0 \quad \text{as } n \to +\infty.
\]

Therefore \( H_{\mu,0}^s(\Omega) \) is compactly embedded in \( L^q(\Omega; |x|^{-\alpha}) \).

Proof of theorem 1.1. (i) Statement (i) follows from proposition 2.1 and proposition 2.3.

(ii) From the property of \( \tau_+(s, \mu) \), for any \( \gamma \in \left[ \frac{2s^*-N}{2s}, 2s \right) \), there exists \( \mu \geq \mu_0 \) such that \( \tau_+(s, \mu) = \gamma \). Therefore, for any \( u, v \in C_0^\infty(\Omega \setminus \{0\}) \), such that \( u = |x|^{-\gamma} v \in C_0^\infty(\Omega \setminus \{0\}) \), we have

\[
\|u\|_\mu^2 = \frac{C_N \gamma}{2} \|v\|_{\mu, \gamma}^2,
\]

(2.32)

which, together with proposition 2.1 and proposition 2.3, implies that

\[
H_{\mu,0}^s(\Omega; |x|^{-\gamma}) = \left\{ |x|^{-\gamma} u : u \in H_{\mu,0}^s(\Omega) \right\}.
\]

(iii) For any \( v \in H_{\mu,0}^s(\Omega; |x|^{-\gamma}) \), by (ii), \( u = |x|^{-\gamma} v \in H_{\mu,0}^s(\Omega) \) and

\[
\int \Omega |x|^{-\beta+\gamma}|v(x)|^q \, dx = \int \Omega |x|^{-\beta}|u(x)|^q \, dx,
\]

7209
it follows from (2.29) that
\[ \| \cdot \gamma \|_{L^p(\Omega; |x|^{-\beta})} \leq c \| v \|_{s, \gamma}, \quad \forall v \in H^s_0(\Omega; |x|^{\gamma}) \]
for
\[ q < \frac{2N - 2\beta}{N - 2s} \quad \text{if} \quad \beta \in (0, 2s) \quad \text{and} \quad q < 2_s^* \quad \text{if} \quad \beta \leq 0, \]
where \( c = c(N, \Omega, s, \gamma, \beta, q) \). The proof is complete.

When \( \gamma = \frac{2s - N}{2} \), we show that the exponent \( 2_s^* \) is involved with a logarithmic correction.

**Corollary 2.5.** The embedding
\[ H^s_0(\Omega; |x|^{\frac{2s - N}{2}}) \hookrightarrow L^{2_s^*}(\Omega; |x|^{-N}X(|x|^{\frac{2s-N}{2s}})) \]
is continuous, where the function \( X \) is defined in (2.3). Moreover, there exists a positive constant \( c = c(N, s) \) such that
\[ \| v \|_{L^{2_s^*}(\Omega; |x|^{-N}X(|x|^{\frac{2s-N}{2s}}))} \leq c \| v \|_{s, \frac{2s-N}{2s}}, \quad \forall v \in H^s_0(\Omega; |x|^{\frac{2s-N}{2s}}). \quad (2.33) \]

**Proof.** Recall that \( \tau_+ = \frac{2s-N}{2} \). Take \( v \in H^s_0(\Omega; |x|^{\frac{2s-N}{2s}}) \) and put \( u = |x|^\frac{2s-N}{2} v \). Invoking [39, theorem 3], we obtain
\[ \left( \int_{\Omega} X(|x|^{\frac{2s-N}{2}})|u(x)|^{2_s^*} \, dx \right)^{\frac{2}{2_s^*}} \leq c \| u \|_{\mu_0}^2, \]
where \( c = c(N, s) \), which implies (2.33). The proof is complete.

3. Dual problems: variational solutions

We will show below the existence of a solution to the dual problem (1.6) by using a variational method. The highlight of this section is a variant of Kato’s inequality which implies the uniqueness of problem (1.6).

**Proof of theorem 1.3.** Existence. Consider the functional
\[ I(\varphi) := \frac{1}{2} \| \varphi \|_{s, \gamma}^2 - (f, \varphi)_\gamma, \quad \forall \varphi \in H^s_0(\Omega; |x|^{\gamma}). \]
We first see that \( I \) is \( C^1 \) in \( H^s_0(\Omega; |x|^{\gamma}) \).

Since
\[ p > \max \left\{ \frac{2N}{N + 2s}, \frac{2N + 2\alpha_0}{N + 2s}, 1 + \frac{\alpha}{2s} \right\}, \quad (3.1) \]
it follows that \( p' \leq \min \left\{ \frac{2N}{N - 2s}, \frac{2N - 2\alpha_0}{N - 2s} \right\} \) and \( \alpha p' \leq 2s \). By using Hölder’s inequality and theorem 1.1 (iii) with \( q \) replaced by \( p' \) and \( \beta \) replaced by \( \alpha p' \), we obtain that, for each \( \varphi \in H^s_0(\Omega; |x|^{\gamma}) \),
\[ |(f, \varphi)_\gamma| \leq \|f\|_{L^2(\Omega; |x|^{\gamma})} \cdot |\gamma \varphi| \|\varphi\|_{L^\infty(\Omega; |x|^{-\alpha} \gamma)} \leq c \|f\|_{L^2(\Omega; |x|^{\gamma})} \|\varphi\|_{s,\gamma}, \quad (3.2) \]

where \( c = c(N, \Omega, s, \gamma, \alpha, p) \). This implies that \( \mathcal{I} \) is coercive on \( H_0^s(\Omega; |x|^{\gamma}) \).

Next, we will show that \( \mathcal{I} \) is weakly lower semicontinuous on \( H_0^s(\Omega; |x|^{\gamma}) \). Let \( \{\varphi_n\} \subset H_0^s(\Omega; |x|^{\gamma}) \) such that \( \varphi_n \rightharpoonup \varphi \) weakly in \( H_0^s(\Omega; |x|^{\gamma}) \). By (3.2), the linear operator \( T(\varphi) := (f, \varphi)_\gamma \) belongs to the dual of \( H_0^s(\Omega; |x|^{\gamma}) \). Hence, we have that

\[ |T(\varphi_n - \varphi)| = |(f, \varphi_n - \varphi)_\gamma| \to 0 \quad \text{as} \quad n \to \infty. \]

Next, we see that

\[ \|\varphi_n\|_{s,\gamma}^2 - \|\varphi\|_{s,\gamma}^2 = \|\varphi_n - \varphi\|_{s,\gamma}^2 + 2\langle \varphi, \varphi_n - \varphi \rangle_{s,\gamma}, \]

which gives

\[ \mathcal{I}(\varphi_n) - \mathcal{I}(\varphi) = \|\varphi_n - \varphi\|_{s,\gamma}^2 + 2\langle \varphi, \varphi_n - \varphi \rangle_{s,\gamma} - (f, \varphi_n - \varphi)_\gamma, \]

This yields

\[ \liminf_{n \to \infty} \mathcal{I}(\varphi_n) \geq \mathcal{I}(\varphi). \]

Therefore \( \mathcal{I} \) has a critical point \( u \in H_0^s(\Omega; |x|^{\gamma}) \). It can be checked that \( u \) is a variational solution of (1.6).

**Uniqueness.** The uniqueness follows from Kato type inequality (1.15).

**A priori estimate.** Estimate (1.14) can be obtained by taking \( \xi = u \) in (1.12) and using estimate (3.2) with \( \varphi = u \).

**Kato type inequality.** The proof is in the spirit of [32]. Assume that \( u \in H_0^s(\Omega; |x|^{\gamma}) \) is a variational solution of (1.6). Let \( \varepsilon > 0 \) and \( 0 \leq \zeta < c_0^\infty(\Omega \setminus \{0\}) \). Put

\[ \eta_\varepsilon = \min \{1, \varepsilon^{-1} u^+\} \quad \text{and} \quad \phi = \eta_\varepsilon \zeta. \]

Note that \( \phi \in H_0^s(\Omega; |x|^{\gamma}) \), hence by taking \( \phi \) as a test function in (1.12), we have that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y)) (\eta_\varepsilon(x) \zeta(x) - \eta_\varepsilon(y) \zeta(y))}{|x - y|^{N+2\gamma}} |y|^{\gamma} dy |x|^{\gamma} dx = \int_{\Omega} f(x) \eta_\varepsilon(x) \zeta(x) |x|^{\gamma} dx. \quad (3.3)
\]

We will estimate the left-hand side of (3.3) from below by dividing \( \mathbb{R}^N \times \mathbb{R}^N \) into several subsets based on the value of \( u \) in comparison with 0 and \( \varepsilon \).

First, since \( \eta_\varepsilon(x) = 0 \) when \( u(x) \leq 0 \), it is easy to see that

\[
\int_{\{u(x) \leq 0\}} \int_{\{u(y) \leq 0\}} \frac{(u(x) - u(y)) (\eta_\varepsilon(x) \zeta(x) - \eta_\varepsilon(y) \zeta(y))}{|x - y|^{N+2\gamma}} |y|^{\gamma} dy |x|^{\gamma} dx = 0.
\]

Next, since \( \eta_\varepsilon(x) = 0 \) when \( u(x) \leq 0 \) and \( \eta_\varepsilon(y) = \varepsilon^{-1} u(y) \) when \( 0 < u(y) < \varepsilon \), we obtain

\[
\int_{\{u(x) \leq 0\}} \int_{\{0 < u(y) < \varepsilon\}} \frac{(u(x) - u(y)) (\eta_\varepsilon(x) \zeta(x) - \eta_\varepsilon(y) \zeta(y))}{|x - y|^{N+2\gamma}} |y|^{\gamma} dy |x|^{\gamma} dx \geq \varepsilon^{-1} \int_{\{u(x) \leq 0\}} \int_{\{0 < u(y) < \varepsilon\}} \frac{(u(y) - u(x)) u(y) \zeta(y)}{|x - y|^{N+2\gamma}} |y|^{\gamma} dy |x|^{\gamma} dx \geq 0. \quad (3.4)
\]
We note that $\eta_c(x) = 0$ when $u(x) \leq 0$ and $\eta_c(y) = 1$ when $u(y) \geq \varepsilon$, hence
\[
\int_{\{u(x) \leq 0\}} \int_{\{u(y) \geq \varepsilon\}} \frac{(u(x) - u(y)) (\eta_c(x) \zeta(x) - \eta_c(y) \zeta(y))}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx
\]
\[
= \int_{\{u(x) \leq 0\}} \int_{\{u(y) \geq \varepsilon\}} \frac{(u(y) - u(x)) \zeta(y)}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx
\]
\[
\geq \int_{\{u(x) \leq 0\}} \int_{\{u(y) \geq \varepsilon\}} \frac{(u^+(x) - u_+(y)) (\zeta(x) - \zeta(y))}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx.
\]
By symmetry, as in (3.4), we have
\[
\int_{\{0 < u(x) < \varepsilon\}} \int_{\{u(y) \leq 0\}} \frac{(u(x) - u(y)) (\eta_c(x) \zeta(x) - \eta_c(y) \zeta(y))}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx \geq 0.
\]
Again, by using the fact that $\eta_c(x) = \varepsilon^{-1} u(x)$ when $0 < u(x) < \varepsilon$, we derive
\[
\int_{\{0 < u(x) < \varepsilon\}} \int_{\{0 < u(y) < \varepsilon\}} \frac{(u(x) - u(y)) (\eta_c(x) \zeta(x) - \eta_c(y) \zeta(y))}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx
\]
\[
= \varepsilon^{-1} \int_{\{0 < u(x) < \varepsilon\}} \int_{\{0 < u(y) < \varepsilon\}} \frac{(u(x) - u(y)) \zeta(x)}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx
\]
\[
= \varepsilon^{-1} \int_{\{0 < u(x) < \varepsilon\}} \int_{\{0 < u(y) < \varepsilon\}} \frac{(u(x) - u(y))^2 \zeta(x)}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx
\]
\[
+ \varepsilon^{-1} \int_{\{0 < u(x) < \varepsilon\}} \int_{\{0 < u(y) < \varepsilon\}} \frac{(u(x) - u(y)) (u(y) \zeta(x) - \zeta(y))}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx
\]
\[
\geq \varepsilon^{-1} \int_{\{0 < u(x) < \varepsilon\}} \int_{\{0 < u(y) < \varepsilon\}} \frac{(u(x) - u(y)) u(y) (\zeta(x) - \zeta(y))}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx.
\]
Since $u \in H_0^s(\Omega; |x|^\gamma)$ and $\zeta \in C_0^\infty(\Omega \setminus \{0\})$, by using the dominated convergence theorem, we deduce that
\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\{0 < u(x) < \varepsilon\}} \int_{\{0 < u(y) < \varepsilon\}} \frac{(u(x) - u(y)) u(y) (\zeta(x) - \zeta(y))}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx = 0.
\]
Next, as $\eta_c(x) = \varepsilon^{-1} u(x)$ when $0 < u(x) < \varepsilon$ and $\eta_c(y) = 1$ when $u(y) \geq \varepsilon$, it follows that
\[
\int_{\{0 < u(x) < \varepsilon\}} \int_{\{u(y) \geq \varepsilon\}} \frac{(u(x) - u(y)) (\eta_c(x) \zeta(x) - \eta_c(y) \zeta(y))}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx
\]
\[
= \varepsilon^{-1} \int_{\{0 < u(x) < \varepsilon\}} \int_{\{u(y) \geq \varepsilon\}} \frac{(u(x) - u(y)) \zeta(x) - \varepsilon \zeta(y)}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx
\]
\[
= \varepsilon^{-1} \int_{\{0 < u(x) < \varepsilon\}} \int_{\{u(y) \geq \varepsilon\}} \frac{(u(x) - u(y)) (\zeta(x) - \zeta(y)) u(x)}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx
\]
\[
+ \varepsilon^{-1} \int_{\{0 < u(x) < \varepsilon\}} \int_{\{u(y) \geq \varepsilon\}} \frac{(u(x) - u(y)) (u(x) - \varepsilon) \zeta(x)}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx
\]
\[
\geq \varepsilon^{-1} \int_{\{0 < u(x) < \varepsilon\}} \int_{\{u(y) \geq \varepsilon\}} \frac{(u(x) - u(y)) (\zeta(x) - \zeta(y)) u(x)}{|x - y|^{N + 2s}} |y|^\gamma dy |x|^\gamma dx.
\]
Noting that \( u \in H^1_0(\Omega; |x|^\gamma) \) and \( \zeta \in C^\infty_0(\Omega \setminus \{0\}) \), by using the dominated convergence theorem, we obtain

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\{0 < u(x) < \varepsilon\}} \int_{\{u(y) \geq \varepsilon\}} \frac{(u(x) - u(y))(\zeta(x) - \zeta(y))u(x)}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx = 0.
\]

By symmetry, proceeding as in (3.5), we get

\[
\int_{\{u(x) \geq \varepsilon\}} \int_{\{0 < u(y) < \varepsilon\}} \frac{(u(x) - u(y))(\eta_\varepsilon(x)\zeta(x) - \eta_\varepsilon(y)\zeta(y))}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx
\]

and, as in (3.5),

\[
\int_{\{u(x) \geq \varepsilon\}} \int_{\{0 < u(y) < \varepsilon\}} \frac{(u(x) - u(y))(\eta_\varepsilon(x)\zeta(x) - \eta_\varepsilon(y)\zeta(y))}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx
\]

where

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\{0 < u(y) < \varepsilon\}} \int_{\{u(x) \geq \varepsilon\}} \frac{(u(x) - u(y))(\zeta(x) - \zeta(y))u(y)}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx = 0.
\]

Combining all the above estimates and then letting \( \varepsilon \to 0 \), we deduce that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u^+(x) - u^+(y))(\zeta(x) - \zeta(y))}{|x-y|^{N+2\gamma}} |y|^\gamma dy |x|^\gamma dx
\]

\[
\leq \int_{\Omega} f(x) \text{sign}^+(u(x))\zeta(x) |x|^\gamma dx
\]

for any \( 0 \leq \zeta \in C^\infty_0(\Omega \setminus \{0\}) \). By proposition 2.1, we conclude that (3.6) holds true for any \( 0 \leq \zeta \in H^1_0(\Omega; |x|^\gamma) \). The proof is complete.

\[\Box\]

4. Nonhomogeneous linear equations

4.1. Existence, uniqueness and a priori estimates

Assume that \( N \geq 2 \) and \( \Omega \subset \mathbb{R}^N \) is an open set. We denote by \( H_0^N(\Omega) \) the Banach space

\[
H_0^N(\Omega) := \left\{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\gamma}} dy dx < \infty \right\}
\]

endowed with the norm

\[
\|u\|_{H^N_0(\Omega)}^2 := \int_{\Omega} |u|^2 dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\gamma}} dy dx.
\]

We say that \( u \in H^N_0(\Omega) \) if \( u \in H^N(\Omega') \) for any domain \( \Omega' \Subset \Omega \).

In the following, unless otherwise stated, we assume that \( \Omega \subset \mathbb{R}^N \) is an open set containing the origin.
Definition 4.1. Let \( f \in L^1_{\text{loc}}(\Omega \setminus \{0\}) \). A function \( u \) is called a weak solution of
\[
L_p^\mu u = f \quad \text{in} \ \Omega \tag{4.1}
\]
if \( u \in H^1_{\text{loc}}(\Omega \setminus \{0\}) \cap L^1(\mathbb{R}^N; (1 + |x|^{N+2r})^{-1}) \) and
\[
\langle u, \phi \rangle = \int_{\Omega} f \phi \, dx, \quad \forall \phi \in C^\infty_0(\Omega \setminus \{0\}). \tag{4.2}
\]

The solvability and a priori estimate for solutions of (4.1) in \( H^1_{\mu,0}(\Omega) \) is provided in the next result.

Proposition 4.2. Let \( f \in L^2(\Omega; |x|^a) \) for some \( a < 2s \). Then there exists a unique weak solution \( u \) of (4.1) such that \( u \in H^1_{\mu,0}(\Omega) \). Furthermore, there exists a positive constant \( c = c(N, \Omega, s, \mu, a) \) such that
\[
\|u\|_{L^\infty} \leq c \|f\|_{L^2(\Omega; |x|^a)} \tag{4.3}
\]

Proof. Existence. Note that \( \tau_+ \in [\frac{2-N}{2}, 2s) \). Since \( f \in L^2(\Omega; |x|^a) \) with \( a < 2s \), by theorem 1.3, there exists a unique variational \( v \in H^1_0(\Omega; |x|^{\tau_+}) \) of the equation
\[
\langle v, \zeta \rangle_{s, \tau_+} = (f, \zeta)_{s, \tau_+}, \quad \forall \zeta \in H^1_0(\Omega; |x|^{\tau_+}) \tag{4.4}
\]
Moreover,
\[
\|v\|_{s, \tau_+} \leq c \|f\|_{L^2(\Omega; |x|^a)} \tag{4.5}
\]
Put \( u := |x|^{\tau_+} v \) then \( u \in H^1_{\mu,0}(\Omega) \) due to theorem 1.1 (ii). Let \( \phi \in C^\infty_0(\Omega \setminus \{0\}) \) and put \( \zeta := |x|^{\tau_+} \phi \). Since \( v, \zeta \) satisfy (4.4), it follows that \( u, \phi \) satisfy (4.2), hence \( u \) is a weak solution of (4.1).

Uniqueness. Let \( \phi \in C^\infty_0(\Omega \setminus \{0\}) \). Assume that \( u \) is a weak solution of (4.1) such that \( u \in H^1_{\mu,0}(\Omega) \). Put \( v = |x|^{\tau_+} u \) then \( v \in H^1_0(\Omega; |x|^{\tau_+}) \) by theorem 1.1 (ii). Take \( \zeta = c_0^{\infty}(\Omega \setminus \{0\}) \) and put \( \phi = |x|^{\tau_+} \zeta \in C^\infty_0(\Omega \setminus \{0\}) \). Since \( u, \phi \) satisfy (4.2), it follows that \( v, \zeta \) satisfy (4.4). In light of theorem 2.1, we deduce (4.4). Since \( v \) is the unique variational solution of (4.4), we find that \( u \) is the unique weak solution of (4.1).

A priori estimate. Estimate (4.3) follows from (4.5) and (2.32). \( \square \)

Denote
\[
L^\infty(\Omega; |x|^b) := \{ u \in L^\infty_{\text{loc}}(\Omega) : |x|^b u \in L^\infty(\Omega) \},
\]
with the norm
\[
\|u\|_{L^\infty(\Omega; |x|^b)} = \text{esssup}_{x \in \Omega} (|u(x)| |x|^b).
\]

Remark 4.3. We note that, for any \( b < 2s - \tau_+ \), there exists \( 0 \leq a = a(N, b, \tau_+, \mu) < 2s \) such that \( L^\infty(\Omega; |x|^b) \subset L^2(\Omega; |x|^a) \). Indeed, since \( 2b - N < 2s \), we can choose \( a = a(N, b, \tau_+, \mu) \in [0, 2s) \) such that \( a > 2b - N \). Then for any \( f \in L^\infty(\Omega; |x|^b) \), we have
\[
\int_{\Omega} |f(x)|^2 |x|^a \, dx \leq \|f\|^2_{L^\infty(\Omega; |x|^b)} \int_{\Omega} |x|^{a-2b} \, dx = c \|f\|^2_{L^\infty(\Omega; |x|^b)},
\]
therefore \( L^\infty(\Omega; |x|^b) \subset L^2(\Omega; |x|^a) \). In case \( b < \frac{N}{2} \), we can choose \( a \leq 0 \), hence \( L^\infty(\Omega; |x|^b) \subset L^2(\Omega) \).
Lemma 4.4. Let $b < 2s - \tau_+(s, \mu)$ and $f \in L^\infty(\Omega; |x|^b)$. Assume that $u$ is a weak solution of (4.1) such that $u \in H^s_{\mu, 0}(\Omega)$. Then there exists a positive constant $c = c(N, \Omega, s, \mu, b)$ such that

$$|u(x)| \leq c \|f\|_{L^\infty(\Omega; |x|^b)} |x|^\tau_+(s, \mu) \quad \text{for a.e. } x \in \Omega \setminus \{0\}. \quad (4.6)$$

Proof. In this proof, we will modify $\mu$, hence we will use the notation $\tau_+(s, \mu)$ to avoid confusion.

Since $u$ is a solution of (4.1) such that $u \in H^s_{\mu, 0}(\Omega)$, by putting $v = |x|^{-\tau_+(s, \mu)} u$ then $v \in H^s_{\mu}(\Omega; |x|^\tau_+(s, \mu))$ and, as in the proof of the uniqueness part of lemma 4.2, it satisfies

$$\langle v, \zeta \rangle_{H^s_{\mu}(\Omega; |x|^\tau_+(s, \mu))} = (f, \zeta)_{H^s_{\mu}(\Omega; |x|^\tau_+(s, \mu))}, \quad \forall \zeta \in H^s_{\mu}(\Omega; |x|^\tau_+(s, \mu)). \quad (4.7)$$

Next, we construct an upper bound for $v$, which will yield an upper bound for $u$. Let $\varepsilon > 0$ small which will be determined later. By [6, proposition 1.2], we have

$$L^s_{\mu}(\Omega; |x|^\tau_+(s, \mu)) = -\varepsilon |x|^\tau_+(s, \mu) - 2s \leq 0 \quad \text{in } \Omega \setminus \{0\}. \quad (4.8)$$

Let $\eta \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$, $\eta(x) = 1$ if $|x| \leq 1$ and $\eta(x) = 0$ if $|x| \geq 2$. For $R > 1$, set $\eta_R(x) = \eta(\frac{x}{R})$. Then from (4.8), we derive

$$L^s_{\mu}(\eta_R |x|^\tau_+(s, \mu)) = -\varepsilon \eta_R |x|^\tau_+(s, \mu) - 2s + \int_{\mathbb{R}^N} |x|^\tau_+(s, \mu + \varepsilon) \eta_R(x) - \eta_R(y) \ dy,$$

$$= -\varepsilon \eta_R |x|^\tau_+(s, \mu) - 2s + f_1(x).$$

We see that

$$|f_1(x)| \leq c R^{-2s + \tau_+(s, \mu + \varepsilon)}, \quad \forall x \in \Omega \quad \text{and} \quad \forall R > 4 \max \{\text{diam}(\Omega), 1\},$$

therefore there exists $R_0 = R_0(\Omega, s, \mu, \varepsilon) > 4 \max \{\text{diam}(\Omega), 1\}$ such that

$$L^s_{\mu}(\eta_{R_0} |x|^\tau_+(s, \mu)) = -\varepsilon \eta_{R_0} |x|^\tau_+(s, \mu + \varepsilon) - 2s + f_1(x) \leq -\frac{\varepsilon}{2} |x|^\tau_+(s, \mu + \varepsilon) - 2s,$$

$$\forall x \in \Omega \setminus \{0\}. \quad (4.9)$$

Set $t_0 := 2e^{-\frac{1}{2}} \|f\|_{L^\infty(\Omega; |x|^p)} \text{diam}(\Omega)^{2s - \tau_+(s, \mu + \varepsilon) - b}$ then from (4.9), we get

$$L^s_{\mu}(t_0 \eta_{R_0} |x|^\tau_+(s, \mu + \varepsilon)) \leq -\|f\|_{L^\infty(\Omega; |x|^p)} (\text{diam}(\Omega) \frac{2s - \tau_+(s, \mu + \varepsilon) - b}{|x|}) |x|^{-b},$$

$$\forall x \in \Omega \setminus \{0\}. \quad (4.10)$$

We choose $\varepsilon > 0$ small such that $2s - \tau_+(s, \mu + \varepsilon) - b > 0$. Set

$$\psi(x) := t_0 (2R_0 + 1)^{\tau_+(s, \mu + \varepsilon) - \tau_+(s, \mu)} |x|^\tau_+(s, \mu) - t_0 \eta_{R_0}(x) |x|^\tau_+(s, \mu + \varepsilon), \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Then by using the definition of $\eta_{R_0}$ and $t_0$, we see that

$$t_0 (2R_0 + 1)^{\tau_+(s, \mu + \varepsilon) - \tau_+(s, \mu)} |x|^\tau_+(s, \mu) \leq \psi(x) \leq t_0 (2R_0 + 1)^{\tau_+(s, \mu + \varepsilon) - \tau_+(s, \mu)} |x|^\tau_+(s, \mu),$$

7215
for any \( x \in \mathbb{R}^N \setminus \{0\} \). Moreover, by (4.10) and the fact that \( \mathcal{L}_\mu^t (|x|^{\tau_+(r, \mu)}) = 0 \) in \( \Omega \setminus \{0\} \), we have

\[
\mathcal{L}_\mu^t \psi = -\mathcal{L}_\mu^t \left( b \log |x|^{\tau_+(r, \mu) + \varepsilon} \right) \geq \|f\|_{L^\infty(\Omega; |x|^p)} \left( \frac{\text{diam}(\Omega)}{|x|} \right)^{2s - \tau_+(r, \mu) + \varepsilon - b} |x|^{-b},
\]

\( \forall x \in \Omega \setminus \{0\} \).

This implies

\[
\langle \psi, \phi \rangle \geq \|f\|_{L^\infty(\Omega; |x|^p)} \int_\Omega \left( \frac{\text{diam}(\Omega)}{|x|} \right)^{2s - \tau_+(r, \mu) + \varepsilon - b} |x|^{-b} \phi(x) \, dx,
\]

\[\forall 0 \leq \phi \in C^\infty_0 (\Omega \setminus \{0\}). \tag{4.11}\]

Set \( \bar{\psi} = |x|^{-\tau_+(r, \mu)} \psi \) and for \( 0 \leq \phi \in C^\infty_0 (\Omega \setminus \{0\}) \), set \( \zeta = |x|^{-\tau_+(r, \mu)} \phi \). We deduce from (4.11) that

\[
\langle \bar{\psi}, \zeta \rangle_{s, \tau_+(r, \mu)} \geq \|f\|_{L^\infty(\Omega; |x|^p)} \int_\Omega \left( \frac{\text{diam}(\Omega)}{|x|} \right)^{2s - \tau_+(r, \mu) + \varepsilon - b} \zeta(x) |x|^{-b} |x|^{\tau_+(r, \mu)} \, dx. \tag{4.12}\]

Subtracting term by term (4.7) and (4.12) yields

\[
\langle v - \bar{\psi}, \zeta \rangle_{s, \tau_+(r, \mu)} \leq 0, \quad \forall 0 \leq \zeta \in C^\infty_0 (\Omega \setminus \{0\}). \tag{4.13}\]

Note that \( v \in H^s_0(\Omega; |x|^{\tau_+(r, \mu)}) \) and \( \|\bar{\psi}\|_{s, \tau_+(r, \mu)} < +\infty \), hence from proposition 2.1, (4.13) is valid for any \( \zeta \in H^s_0(\Omega; |x|^{\tau_+(r, \mu)}) \). In addition, using again proposition 2.1, we may show that \( (v - \bar{\psi})^+ \in H^s_0(\Omega; |x|^{\tau_+(r, \mu)}) \). By taking \( \zeta = (v - \bar{\psi})^+ \) in (4.13), we obtain

\[
\langle v - \bar{\psi}, (v - \bar{\psi})^+ \rangle_{s, \tau_+(r, \mu)} \leq 0.
\]

This implies \( (v - \bar{\psi})^+ = 0 \) a.e. in \( \mathbb{R}^N \), i.e. \( v \leq \bar{\psi} \) a.e. in \( \mathbb{R}^N \). Therefore

\[
u(x) \leq c \|f\|_{L^\infty(\Omega; |x|^p)} |x|^{\tau_+(r, \mu)} \quad \text{for a.e. } x \in \Omega \setminus \{0\},
\]

where \( c = c(N, \Omega, s, \mu, b) \).

Similarly, for \( -u \) in place of \( u \), we obtain the lower bound, i.e.

\[
-u(x) \leq c \|f\|_{L^\infty(\Omega; |x|^p)} |x|^{\tau_+(r, \mu)} \quad \text{for a.e. } x \in \Omega \setminus \{0\}.
\]

Combining the above estimates yields (4.6). The proof is complete. \( \square \)

### 4.2. Regularity

In the sequel, we assume that \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is an open bounded set containing the origin. Recall that \( d(x) = \text{dist}(x, \partial \Omega) \).

**Lemma 4.5.** Let \( b < 2s - \tau_+ \) and \( f \in L^\infty(\Omega; |x|^p) \cap L^2(\Omega) \). Assume that \( u \) is a weak solution of (4.1) such that \( u \in H^s_{-\mu, 0}(\Omega) \). Then the following regularity properties hold.
(i) \( u \in C^\beta_{\text{loc}}(\Omega \setminus \{0\}) \) for any \( \beta \in (0, 2s) \). In addition, for any open set \( D \subset \Omega \setminus \{0\} \), there exists a constant \( c \) depending only on \( N, \Omega, s, \mu, \beta \) and dist \( (D, \partial \Omega \cup \{0\}) \) such that
\[
\|u\|_{C^\beta(D)} \leq c \|f\|_{L^\infty(\Omega; |x|^\beta)}.
\] (4.14)

(ii) We additionally assume that \( \Omega \) satisfies the exterior ball condition. Then for any \( \epsilon > 0 \) such that \( B_{4\epsilon}(0) \subset \Omega \) there exists \( c = c(N, \Omega, s, \mu, \beta, b, \epsilon) \) such that
\[
\|d^{-1}u\|_{L^\infty(\Omega; B_{4\epsilon}(0))} \leq c \|f\|_{L^\infty(\Omega; |x|^\beta)}.
\] (4.15)

(iii) In addition, if \( \Omega \) is \( C^{1,1} \) then there exist \( 0 < \alpha < \min\{s, 1-s\} \) and \( c > 0 \) depending on \( N, \Omega, s, \mu, b, \epsilon \) such that
\[
\|d^{-1}u\|_{C^\alpha(\Omega; B_{4\epsilon}(0))} + \|u\|_{C(\Omega; B_{4\epsilon}(0))} \leq c \|f\|_{L^\infty(\Omega; |x|^\beta)}.
\] (4.16)

**Proof.** (i) Since \( u \) is a weak solution of (4.1), we have
\[
\langle u, \phi \rangle \geq \mu = \int_{\Omega} f \phi \, dx, \quad \forall \phi \in C^0_0(\Omega \setminus \{0\}).
\]

Let \( x_0 \in \Omega \setminus \{0\} \), then there exists \( \rho > 0 \) such that \( B_{2\rho}(x_0) \subset \Omega \setminus \{0\} \). Let \( \{\zeta_\delta\} \) be the sequence of standard mollifiers, then for \( \delta > 0 \) small enough, the function \( u_\delta = \zeta_\delta \ast u \) satisfies
\[
(-\Delta)^s u_\delta = H_\delta \quad \text{in} \ B_{\rho}(x_0),
\] (4.17)

where
\[
H_\delta := \zeta_\delta \ast \left( f - \mu \frac{u}{|x|^2} \right).
\] (4.18)

By (4.6), we have that \( u_\delta \in C^\infty(\mathbb{R}^N) \) and there exists a positive constant \( c \) depending on \( N, s, \mu, B_{\rho}(x_0), \Omega, b \) such that
\[
|H_\delta(x)| + |u_\delta(x)| \leq c \|f\|_{L^\infty(\Omega; |x|^\beta)}, \quad \forall x \in B_{\rho}(x_0).
\]

Furthermore, by (4.6), we deduce that
\[
\int_{\mathbb{R}^N} \frac{|u_\delta(x)|}{1 + |x|^{N+2s}} \, dx \leq c \|f\|_{L^\infty(\Omega; |x|^\beta)}.
\]

Hence, by [42, corollary 2.5], for any \( \beta \in (0, 2s) \), there holds
\[
\|u_\delta\|_{C^0(B_{\rho}(x_0))} \leq c \|f\|_{L^\infty(\Omega; |x|^\beta)},
\] (4.19)

where the constant \( c > 0 \) depends only on \( N, s, \mu, x_0, \Omega, \beta, b \). By the Arzelà–Ascoli theorem, there exists a sequence \( \{\delta_n\} \) converging to 0 such that \( u_{\delta_n} \to u \) in \( C^0(B_{\rho}(x_0)) \), hence
\[
\|u\|_{C^0(B_{\rho}(x_0))} \leq c \|f\|_{L^\infty(\Omega; |x|^\beta)}
\] (4.20)

for any \( \beta \in (0, 2s) \). By the standard covering argument, we derive \( u \in C^\beta_{\text{loc}}(\Omega \setminus \{0\}) \) and estimate (4.14).
(ii) Let $0 \leq \eta \leq 1$ be a smooth function such that $\eta = 0$ for any $|t| \leq \frac{1}{2}$ and $\eta = 1$ for any $|t| \geq 1$. Set $\eta_\varepsilon(x) = \eta(\varepsilon^{-1}|x|)$ for any $\varepsilon > 0$. If $\varepsilon$ is small enough such that $B_{\varepsilon}(0) \subset \Omega$, we can easily show that the function $\eta_\varepsilon u$ satisfies

$$
\langle \eta_\varepsilon u, \phi \rangle_{\mu} = \int_{\Omega} \eta_\varepsilon f \phi \, dx + \int_{\Omega} \phi \, \partial_{\mu} \phi \, dx, \quad \forall \phi \in C_0^\infty(\Omega \setminus B_\varepsilon(0)),
$$

where

$$
\varphi(x) := C_{N, s} \lim_{\delta \to 0} \int_{\mathbb{R}^n \setminus B_{\delta}(x)} \frac{(\eta_\varepsilon(x) - \eta_\varepsilon(y))(u(y) - u(x))}{|x - y|^{N + 2s}} \, dy, \quad \forall x \in \Omega \setminus B_\varepsilon(0).
$$

Now, for any $x \in \Omega \setminus B_\varepsilon(0)$, we have

$$
\varphi(x) = C_{N, s} \lim_{\delta \to 0} \int_{\mathbb{R}^n \setminus B_{\delta}(x)} \frac{(\eta_\varepsilon(x) - \eta_\varepsilon(y))}{|x - y|^{N + 2s}} \, dy + u(x) C_{N, s} \lim_{\delta \to 0} \int_{\mathbb{R}^n \setminus B_{\delta}(x)} \eta_\varepsilon(y) \, dy.
$$

By (4.6) and (4.20), we may show that

$$
|\varphi(x)| \leq c \|f\|_{L^\infty(\Omega; |x|^s)}, \quad \forall x \in \Omega \setminus B_\varepsilon(0).
$$

Set

$$
h(x) := f(x) \eta_\varepsilon(x) - \mu \frac{\eta_\varepsilon}{|x|^s} + \varphi(x),
$$

then there exists a positive constant $c = c(N, \Omega, s, \mu, \varepsilon)$ such that

$$
|h(x)| \leq c \|f\|_{L^\infty(\Omega; |x|^s)}, \quad \forall x \in \Omega \setminus B_\varepsilon(0).
$$

Using [42, lemma 2.7], taking into account that $\Omega$ satisfies the exterior ball condition (which is needed to apply [42, lemma 2.7]) and (4.21), we obtain (4.15).

(iii) By [42, proposition 1.1 and theorem 1.2], the assumption that $\Omega$ is a $C^{1,1}$ bounded domain (which is required to apply [42, theorem 1.2]) and (4.21), we derive (4.16).

The next result provides a higher Hölder regularity of weak solutions to equation (4.1).

**Lemma 4.6.** Let $b < 2s - \tau_+ + \delta$, $\theta \in (0, 1)$ and $f \in L^\infty(\Omega; |x|^\theta) \cap C_{loc}^\delta(\Omega \setminus \{0\})$. Assume that $u$ is a weak solution of (4.1) such that $u \in H^\delta_{\theta, 0}(\Omega)$. Then $u \in C_{loc}^{\delta + \beta_0}(\Omega \setminus \{0\})$ for some $\beta_0 > 0$. Furthermore, for any open set $D \Subset \Omega \setminus \{0\}$, there exists a positive constant $c$ depending only on $N, s, \Omega, \mu, \beta_0$ and $\text{dist}(D, \partial \Omega \cup \{0\})$ such that

$$
\|u\|_{C^{\delta + \beta_0}(\Omega \setminus \{0\})} \leq c \left( \|f\|_{L^\infty(\Omega; |x|^\theta)} + \|f\|_{C^\delta(\Omega)} \right).
$$

**Proof.** First we note that

$$
\langle u, \phi \rangle_{\mu} = \int_{\Omega} f \phi \, dx, \quad \forall \phi \in C_0^\infty(\Omega \setminus \{0\}).
$$

Let $x_0 \in \Omega \setminus \{0\}$ then there exists $\rho > 0$ such that $B_{2\rho}(x_0) \subset \Omega \setminus \{0\}$. Consider the mollifiers $\zeta_\delta$, then for $\delta > 0$ small enough and put $u_\delta = \zeta_\delta * u$. Then $u_\delta$ solves equation (4.17) with $H_\delta$
as in (4.18). By repeating the argument after (4.18) in part (i) of the proof of lemma 4.5, we obtain (4.19). This, together with [42, corollary 2.4], implies the existence of a constant $0 < \beta_0 < \min \{\theta, 2s\}$ such that

$$
\|v_{\delta}\|_{C^{\beta_0}(\overline{\Omega}(\delta))} \leq C \left( \|v\|_{C^\theta(\overline{\Omega}(\delta))} + \|v\|_{L^\infty(\Omega; |x|^{\gamma})} \right),
$$

where the constant $C$ depends only on $N, s, \mu, |x_0|, d(x_0), \Omega, \beta_0$. By the Arzelá–Ascoli theorem, there exists a subsequence $\{\delta_n\}$ converging to 0 such that $u_{\delta_n} \to u$ in $C^{2^s + \beta_0}(\overline{\Omega}(\delta_n))$, hence

$$
\|u\|_{C^{2^s + \beta_0}(\overline{\Omega}(\delta_n))} \leq C \left( \|f\|_{C^\theta(\overline{\Omega}(\delta))} + \|f\|_{L^\infty(\Omega; |x|^{\gamma})} \right). \tag{4.23}
$$

The desired results follow by the above inequality and a standard covering argument. \qed

As a consequence of the above results, we obtain the following result

**Corollary 4.7.** Assume that $b < 2s - \tau_\gamma$, $\theta \in (0, 1)$ and $f \in L^\infty(\Omega; |x|^{\beta}) \cap C^0_{\text{loc}}(\Omega \setminus \{0\})$. Then the weak solution of (1.18) belongs to $X_\mu(\Omega; |x|^{-b})$.

## 5. Poisson problems

In this section, we study problem (1.25).

### 5.1. $L^1$ sources

We start with the case of $L^1$ source.

**Proof of theorem 1.6.** Existence. Let $\theta \in (0, 1)$ and $\{f_n\} \subset C^\theta(\Omega)$ such that $f_n \to f$ in $L^1(\Omega; d(x)^\gamma |x|^{\tau_\gamma})$. By proposition 4.2, there exists a unique function $v_n \in H^0_\gamma(\Omega; |x|^{\tau_\gamma})$ such that

$$
\langle v_n, \phi \rangle_{s, \tau_\gamma} = \int_\Omega f_n \phi |x|^{\tau_\gamma} \, dx, \quad \forall \phi \in H^0_\gamma(\Omega; |x|^{\tau_\gamma}). \tag{5.1}
$$

Since $v_n \in H^0_\gamma(\Omega; |x|^{\tau_\gamma})$, by proposition 2.1, there exists $v_{n,m} \in C^\infty_c(\Omega \setminus \{0\})$ such that

$$
\lim_{m \to \infty} \|v_n - v_{n,m}\|_{s, \tau_\gamma} = 0.
$$

Let $\psi \in X_\mu(\Omega; |x|^{-b})$. By property (iii) in the definition of $X_\mu(\Omega; |x|^{-b})$ (see definition 1.4), we have

$$
\langle v_{n,m}, \psi \rangle_{s, \tau_\gamma} = \int_\Omega v_{n,m}(x) (\hat{-\Delta})^s_{\tau_\gamma} \psi(x) |x|^{\tau_\gamma} \, dx. \tag{5.2}
$$

Since $|(-\Delta)^s_{\tau_\gamma} \psi(x)| \leq C |x|^{-b}$ for a.e. $x \in \Omega \setminus \{0\}$ due to property (ii) in definition 1.4, by (2.2) and (2.5), we have

$$
\lim_{m \to \infty} \int_\Omega v_{n,m}(x) (\hat{-\Delta})^s_{\tau_\gamma} \psi(x) |x|^{\tau_\gamma} \, dx = \int_\Omega \psi(x) \, dx. \tag{5.3}
$$

Combining (5.1)–(5.3) yields

$$
\int_\Omega \psi(x) (\hat{-\Delta})^s_{\tau_\gamma} \psi(x) |x|^{\tau_\gamma} \, dx = \langle v_n, \psi \rangle_{s, \gamma} = \int_\Omega f_n(x) \psi(x) |x|^{\tau_\gamma} \, dx. \tag{5.4}
$$
By lemma 4.5 (i), \( v_n \in C^\beta(\Omega \setminus \{0\}) \) for any \( \beta \in (0, 2\alpha) \).

Let \( h_m : \mathbb{R} \to \mathbb{R} \) be a smooth function such that \( h_m(t) \to \text{sign}(t) \) and \( |h_m(t)| \leq 1 \) for any \( t \in \mathbb{R} \). By proposition 4.2 and remark 4.3, there exists a unique weak solution \( U_m \in H^1_{\mu, \beta}(\Omega) \) of

\[
L_{\mu}^\beta U = |x|^{-\beta} h_m(v_n) \quad \text{in} \quad \Omega
\]

in the sense of definition 4.1. Put \( \tilde{U}_m = |x|^{-\beta} U_m \), then \( \tilde{U}_m \in H^1_0(\Omega; |x|^{-\beta}) \) and

\[
\langle \tilde{U}, \zeta \rangle_{x, \tau_+} = (h_m(v_n), \zeta)_{\tau_+ - \beta}, \quad \forall \zeta \in H^1_0(\Omega; |x|^{-\beta}). \tag{5.5}
\]

By (4.6) and (4.15), we have

\[
|\tilde{U}_m(x)| \leq c d(x)^\gamma \quad \text{for a.e.} \ x \in \Omega,
\]

where the constant \( c \) is independent of \( \tilde{U}_m \). Furthermore in view of lemma 4.6, we can easily show that \( \tilde{U}_m \in X^{\mu}(\Omega; |x|^{-\beta}) \). By taking \( \psi = \tilde{U} \) in (5.4), we derive

\[
\int_{\Omega} v_n(x) (-\Delta)^{\tau_+} \tilde{U}(x) |x|\beta dx = \langle v_n, \tilde{U} \rangle_{x, \tau_+} = \int_{\Omega} f_n(x) \tilde{U}(x) |x|\beta dx.
\]

Taking \( \zeta = v_n \) as a test function in (5.5), we have

\[
\langle \tilde{U}, v_n \rangle_{x, \tau_+} = \int_{\Omega} h_m(v_n) v_n |x|^{\beta - \beta} dx.
\]

From the two preceding equalities and (5.6), we derive

\[
\int_{\Omega} v_n(x) h_m(v_n(x)) |x|^{\beta - \beta} dx \leq c \int_{\Omega} |f_n(x)| d(x)^\gamma |x|^{\beta} dx.
\]

By (2.2) and (2.5), the fact that \( h_m(t) \to \text{sign}(t) \) as \( m \to \infty \), and the dominated convergence theorem, we have

\[
\int_{\Omega} |v_n(x)| |x|^{\beta - \beta} dx \leq c \int_{\Omega} |f_n(x)| d(x)^\gamma |x|^{\beta} dx. \tag{5.7}
\]

Using a similar argument, we can show that

\[
\|v_n - v_k\|_{L^1(\Omega; |x|^{\beta - \beta})} \leq c \|f_n - f_k\|_{L^1(\Omega; d(x)^\gamma |x|^{\beta})}, \quad \forall n, k \in \mathbb{N}. \tag{5.8}
\]

Since \( \{f_n\} \) is a convergent sequence in \( L^1(\Omega; d(x)^\gamma |x|^{\beta}) \), we deduce from (5.8) that \( \{v_n\} \) is a Cauchy sequence in \( L^1(\Omega; |x|^{\beta - \beta}) \), which in turn implies that there exists \( v \in L^1(\Omega; |x|^{\beta - \beta}) \) such that \( v_n \to v \) in \( L^1(\Omega; |x|^{\beta - \beta}) \).

By property (ii) in definition 1.4, (1.16) and the dominated convergence theorem, we deduce from (5.4) that

\[
\int_{\Omega} v(-\Delta)^{\tau_+} \psi |x|^{\beta} dx = \int_{\Omega} f(x) \psi(x) |x|^{\beta} dx.
\]

Put \( u = |x|^{\beta - \beta} v \) then \( u \in L^1(\Omega; |x|^{-\beta}) \) and \( u \) is a weak solution of (1.18).

**A priori estimate.** By letting \( n \to \infty \) in (5.7) and using \( u = |x|^{\beta - \beta} v \), we deduce (1.20).

**Kato type inequality.** First we prove estimate (1.21). Let \( \{\Omega_l\}_{l \in \mathbb{N}} \) be a smooth exhaustion of \( \Omega \), namely \( \Omega_l \Subset \Omega_{l+1} \) and \( \cup_{l \in \mathbb{N}} \Omega_l = \Omega \). Let \( \phi_l \in C_0^\infty(\Omega) \) be such that \( \phi_l = 1 \) on \( \Omega_l \) and \( 0 \leq \phi_l \leq 1 \).

Put \( \zeta_l = (-\Delta)^{\tau_+} \phi_l \).
For $\varepsilon > 0$, since $(v_n - \varepsilon \phi_l)^+ \in H_0^1(\Omega; |x|^{r_+})$, by theorem 2.1, there exists a sequence $0 \leq \nu_{n,m} \in C_0^\infty(\Omega \setminus \{0\})$, such that
\[
\lim_{n \to \infty} \left\| (v_n - \varepsilon \phi_l)^+ - \nu_{n,m} \right\|_{H^1_0(\Omega; |x|^{r_+})} = 0.
\]

Proceeding as above, for any $\psi \in X_\mu(\Omega; |x|^{-b})$, we have that
\[
\langle (v_n - \varepsilon \phi_l)^+, \psi \rangle_{x,\tau_+} = \lim_{n \to \infty} \langle \nu_{n,m}, \psi \rangle_{x,\tau_+} = \lim_{n \to \infty} \int_\Omega \nu_{n,m} (-\Delta)^{r_+}_\tau \psi |x|^{r_+} \, dx = \int_\Omega (v_n - \varepsilon \phi_l)^+ (-\Delta)^{r_+}_\tau \psi |x|^{r_+} \, dx.
\]

By theorem 1.3, there holds
\[
\int_\Omega (v_n - \varepsilon \phi_l)^+ (-\Delta)^{r_+}_\tau \psi |x|^{r_+} \, dx \leq \int_\Omega \sign^+(v_n - \varepsilon \phi_l) \psi (f_n - \varepsilon \xi) |x|^{r_+} \, dx
\]
\[
\leq \int_\Omega \sign^+(v_n - \varepsilon) \psi (f_n - \varepsilon \xi) |x|^{r_+} \, dx + \int_{\Omega \setminus \Omega_l} \sign^+(f_n - \varepsilon \xi) |x|^{r_+} \, dx,
\]
for any nonnegative $\psi \in X_\mu(\Omega; |x|^{-b})$.

Let $\varepsilon_m \downarrow 0$ be such that $\{x \in \Omega : v = \varepsilon_m\} = 0$, then $\sign^+(v_n - \varepsilon_m) \to \sign^+(v - \varepsilon_m)$ a.e. in $\Omega$. Replacing $\varepsilon$ by $\varepsilon_n$ in (5.9) and using the estimate $|\psi(x)| \leq c d(x)^r$ for a.e. $x \in \Omega$, we obtain
\[
\int_\Omega (v_n - \varepsilon_m \phi_l)^+ (-\Delta)^{r_+}_\tau \psi |x|^{r_+} \, dx \leq \int_\Omega \sign^+(v_n - \varepsilon_m) \psi (f_n - \varepsilon_m \xi) |x|^{r_+} \, dx
\]
\[
+ c \int_{\Omega \setminus \Omega_l} |f_n - \varepsilon_m \xi| d(x)^r |x|^{r_+} \, dx.
\]
Letting $n \to \infty$ and $m \to \infty$ in (5.9) successively, we obtain
\[
\int_\Omega \varepsilon^+ (-\Delta)^{r_+}_\tau \psi |x|^{r_+} \, dx \leq \int_\Omega \sign^+(v) \psi f |x|^{r_+} \, dx + c \int_{\Omega \setminus \Omega_l} |f| d(x)^r |x|^{r_+} \, dx.
\]
Using the relation $u = |x|^{r_+} v$ and letting $l \to +\infty$ in the above estimate yield (1.21).

By employing (1.21) with $-v$ in place of $v$, we obtain
\[
\int_\Omega (-v)^+ (-\Delta)^{r_+}_\tau \psi |x|^{r_+} \, dx \leq - \int_\Omega \sign^+(v) \psi f |x|^{r_+} \, dx.
\]
Adding term by term of the above inequality and (1.21), and then using the relation $u = |x|^{r_+} v$, we deduce (1.22).

**Monotonicity and uniqueness.** Let $f_1, f_2 \in L^1(\Omega; d(x)^r |x|^{r_+})$ such that $f_1 \leq f_2$ a.e. in $\Omega$ and let $u_i, i = 1, 2$, is a weak solution to (1.18) with $f = f_i$. By (1.21) with $u$ replaced by $u_1 - u_2$ and $f$ replaced by $f_1 - f_2$, we obtain
\[
\int_\Omega (u_1 - u_2)^+ (-\Delta)^{r_+}_\tau \psi \, dx \leq \int_\Omega (f_1 - f_2) \sign^+(u_1 - u_2) \psi |x|^{r_+} \, dx \leq 0,
\]
\[
\forall 0 \leq \psi \in X_\mu(\Omega; |x|^{-b}).
\]

(5.10)
Let $\xi_0$ be the weak solution to
\begin{equation}
\begin{cases}
\mathcal{L}_\mu \xi = |x|^{-b} & \text{in } \Omega, \\
\xi = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\end{equation}
(5.11)

Then $\xi_0 \in X_s(\Omega; |x|^{-b})$ due to corollary 4.7 and hence by (1.16), $|\xi_0| \leq C d'$ a.e. in $\Omega$. Taking $\psi = \xi_0$ in (5.10), we deduce that $(u_1 - u_2)^+ = 0$ a.e. in $\Omega \setminus \{0\}$, namely $u_1 \leq u_2$ a.e. in $\Omega \setminus \{0\}$. Thus the mapping $f \mapsto u_f$ is nondecreasing.

The uniqueness follows straightforward from the monotonicity. In fact, let $f \in L^1(\Omega; d(x)|x|^{b+})$ be nonnegative and $u$ be the unique weak solution to problem (1.18). Since the mapping $f \mapsto u_f$ is nondecreasing, $f \geq 0$, and $0$ is the unique weak solution to problem (1.18) with zero source, we deduce that $u \geq 0$ a.e. in $\Omega \setminus \{0\}$. 

\[ \square \]

### 5.2. Measure sources

We start this subsection by noting that if $\mu_0 < \mu$ then

\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dy \, dx + \mu \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}} \, dx \geq \frac{C_{N,s}}{2} \frac{\mu_0 - \mu}{\mu_0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dy \, dx \\
\geq C(N,s,\mu,\mu_0) \left( \int_{\mathbb{R}^N} \frac{|u|^{\frac{2s}{1-s}}}{|x|^{\frac{N}{1-s}} \, dx} \right)^{\frac{1-s}{N-2s}},
\]

for any $u \in C_0^\infty(\Omega)$. Setting $u = |x|^{b+} v$, we derive
\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} \, dy \, dx + \mu_0 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}} \, dx \\
\geq C(N,s) \left( \int_{\mathbb{R}^N} \frac{|v|^{\frac{2s}{1-s}} X(|x|) \frac{|v|^2}{|x|^{N+2s}}}{|x|^{\frac{N}{1-s}} \, dx} \right)^{\frac{1-s}{N-2s}},
\]

for any $u \in C_0^\infty(\Omega)$, where $X$ is defined in (2.3). Setting $u(x) = |x|^\frac{2s}{1-s} v(x)$, we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} \frac{|v|^2}{|x|^{N+2s}} \, dy \, dx \\
\geq C(N,s) \left( \int_{\mathbb{R}^N} \frac{|v|^{\frac{2s}{1-s}} |x|^{-N} X(|x|) \frac{|v|^2}{|x|^{N+2s}}}{|x|^{\frac{N}{1-s}} \, dx} \right)^{\frac{1-s}{N-2s}}.
\]

By theorem 2.1, inequality (5.12) is valid for any $v \in H^s_0(\Omega; |x|^{b+})$.

By theorem 2.1, inequality (5.13) is valid for any $v \in H^s_0(\Omega; |x|^{b+})$. 

7222
From (5.12) and (5.13), we see that, for any $\mu \geq \mu_0$,
\[
\frac{C_N s}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+s}} |y|^\tau \, dy \, |x|^\tau \, dx \\
\geq C(\Omega, N, s, \mu, \mu_0) \left( \int_\Omega |v|^{\frac{N}{N-s}} |x|^{\frac{\tau}{N-s}} X(|x|)^{\frac{2(N-s)}{\tau-s}} \, dx \right)^{\frac{N-s}{N}}, \quad \forall v \in H^s(\Omega; |x|^\tau).
\] (5.14)

We need the following a priori Lebesgue estimate on weak solutions to equation (4.1).

We recall the definition of weak Lebesgue spaces. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. The weak Lebesgue space $L^q_s(\Omega)$, $1 \leq q < \infty$, is defined by
\[
L^q_s(\Omega) := \left\{ u \in L^1_{\text{loc}}(\Omega) : \sup_{\lambda > 0} \lambda^q \int_\Omega 1_{\{x \in \Omega: |u(x)| > \lambda\}} \, dx < +\infty \right\},
\]
where $1_A$ denotes the indicator function of a set $A \subset \mathbb{R}^N$. Put
\[
\|u\|_{L^q_s(\Omega)} := \left( \sup_{\lambda > 0} \lambda^q \int_\Omega 1_{\{x \in \Omega: |u(x)| > \lambda\}} \, dx \right)^{\frac{1}{q}}.
\]
We note that $\| \cdot \|_{L^q_s(\Omega)}$ is not a norm, but for $q > 1$, it is equivalent to the norm
\[
\|u\|_{L^q_s(\Omega)} := \sup \left\{ \frac{\int_\Omega |u|^q \, dx}{(\int_\Omega 1_A \, dx)^{\frac{1}{q}}} : A \subset \Omega, A \text{ measurable}, |A| > 0 \right\}.
\]
In fact, there hold
\[
\|u\|_{L^q_s(\Omega)} \leq \|u\|_{L^q_s(\Omega)} \leq \frac{q}{q-1} \|u\|_{L^q_s(\Omega)}, \quad \forall u \in L^q_s(\Omega).
\] (5.15)

It is well-known that the following embeddings hold
\[
L^q_s(\Omega) \subset L^{q_0}(\Omega) \subset L^r(\Omega), \quad \forall r \in [1, q).
\]

**Lemma 5.1.** Assume that $\mu \geq \mu_0$, $f \in L^2(\Omega)$ and $u \in H^s_{\mu, \Omega}(\Omega)$ is the unique weak solution of (4.1). For any $r > 0$ and $q \in (1, 2^*_s)$, there exists $C = C(N, \Omega, s, \mu, r, q) > 0$ such that
\[
\|u\|_{L^q_s(\Omega)} \leq C \|f\|_{L^1_s(\Omega; |x|^\tau)}.
\] (5.16)

**Proof.** Since $f \in L^2(\Omega)$, the existence and uniqueness of a weak solution $u \in H^s_{\mu, \Omega}(\Omega)$ of (4.1) is guaranteed by proposition 4.2. Put $v = |x|^{-\tau_+} u$, then $v \in H^s(\Omega; |x|^\tau)$ by theorem 1.1 (ii) and
\[
\langle v, \phi \rangle_{s, \tau_+} = \int_\Omega f(x) \phi(x) |x|^\tau \, dx, \quad \forall \phi \in H^s_0(\Omega; |x|^\tau).
\] (5.17)

Since $f \in L^2(\Omega)$ and $\tau_+ \geq \frac{2N}{2-N}$, by the Hölder inequality, we deduce that $f \in L^1(\Omega; |x|^\tau)$. Let $\lambda > 0$, taking $v_\lambda := \max\{-\lambda, \min\{v, \lambda\}\}$ as a test function in (5.17), we have
\[
\langle v, v_\lambda \rangle_{s, \tau_+} = \int_\Omega f(x) v_\lambda(x) |x|^\tau \, dx \leq \lambda \int_\Omega |f(x)| |x|^\tau \, dx.
\]
We see that
\[(v(x) - v(y)) (v_\lambda(x) - v_\lambda(y)) \geq (v_\lambda(x) - v_\lambda(y))^2, \quad \forall x, y \in \mathbb{R}^N.\]

Hence from the two proceeding inequalities, we obtain
\[\langle v_\lambda, v_\lambda \rangle_{r, \tau^+} \leq \lambda \int_{\Omega} |f(x)||x|^{\tau^+} \, dx.\]

Therefore for \(r > 0\) such that \(B_r(0) \subset \Omega\),
\[
\{x \in \Omega \setminus B_r(0) : |v(x)| \geq \lambda \} = \{x \in \Omega \setminus B_r(0) : |v_\lambda(x)| \geq \lambda r^{-\tau^+} \}
\subset \{x \in \Omega \setminus B_r(0) : |v_\lambda(x)| \geq a \lambda \}
\subset \{x \in \Omega \setminus B_r(0) : |v_\lambda(x)| \geq \min \{a, 1\} \lambda \},
\]
where \(a = a(N, \Omega, s, \mu, r)\). This and (5.14) imply
\[
|\{x \in \Omega \setminus B_r(0) : |v(x)| \geq \lambda \}| \leq C(N, s, \mu, r) \lambda^{-2^*_\tau} \int_{\Omega \setminus B_r(0)} |v_\lambda|^2 \, dx
\leq C(N, s, \mu, r) \lambda^{-2^*_\tau} \int_{\Omega} |v_\lambda|^2 |x|^{2^*_\tau} X(|x|)^{2(N - 1) \tau / (N - 2\tau)} \, dx
\leq C(N, s, \mu, r) \lambda^{-2^*_\tau} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_\lambda(x) - v_\lambda(y))^2}{|x - y|^{N + 2\tau}} |x|^{\tau^+} |y|^{\tau} \, dx \, dy \right)^{\frac{2^*_\tau}{2^*}}
\leq C(N, s, \mu, r) \lambda^{-2^*_\tau} \left( \int_{\Omega} |f(x)||x|^{\tau^+} \, dx \right)^{\frac{2^*_\tau}{2^*}}, \quad \forall \lambda > 0.
\]

Therefore,
\[\|u\|_{L^{2^*_\tau}(\Omega \setminus B_r(0))} \leq C(N, s, \mu, r) \|f\|_{L^{2^*_\tau}((\Omega \setminus B_r(0)))},\]
which leads to (5.16) due to the continuous embedding \(L^q(\Omega \setminus B_r(0)) \subset L^{2^*_\tau}(\Omega \setminus B_r(0))\) for any \(q \in (1, 2^*_\tau)\).

First we will treat the case where the measure source is concentrated in \(\Omega \setminus \{0\}\).

**Proof of theorem 1.7.** Existence. By the linearity, we may assume that \(\nu \geq 0\). First we assume that \(\nu\) has compact support in \(\Omega \setminus \{0\}\). Consider a sequence of standard mollifiers \(\{\nu_\delta\}_{\delta > 0}\) such that \(\nu_\delta = \zeta_\delta \ast \nu \in C_0^\infty(D)\), where \(D \Subset \Omega \setminus \{0\}\) is an open set. Then
\[
\int_{\Omega} d(x)^+ |x|^{\tau^+} \nu_\delta \, dx \to \int_{\Omega \setminus \{0\}} d(x)^+ |x|^{\tau^+} \, d\nu \quad \text{as} \ \delta \to 0
\]
and
\[\|\nu_\delta\|_{L^1((\Omega \setminus \{0\}))} \leq C \|\nu\|_{L^1((\Omega \setminus \{0\}), d(x)^+ |x|^{\tau^+})}, \quad \forall \delta > 0.\]
Let \( u_\delta \) be the nonnegative weak solution of (1.18) with \( f = \nu_\delta \), namely \( u_\delta \in L^1(\Omega; |x|^{-b}) \) and
\[
\int_\Omega u_\delta (-\Delta)_x \psi \, dx = \int_\Omega \psi |x|^{-\gamma} \nu_\delta \, dx, \quad \forall \psi \in X_\mu (\Omega; |x|^{-b}).
\] (5.20)

In view of the proof of theorem 1.6, \( u_\delta \in H^\mu(\Omega) \) and there holds
\[
\ll u_\delta, \phi \gg_\mu = \int_\Omega \nu_\delta \phi \, dx, \quad \forall \phi \in C_0^\infty (\Omega \setminus \{0\}).
\]

Furthermore, by (1.20) and (5.19), for any \( n \in \mathbb{N} \), there holds
\[
\|u_\delta\|_{L^1(\Omega; |x|^{-b})} \leq c \|\nu_\delta\|_{L^1(\Omega; \delta(x)|x|^{-b})} \leq c \|\nu\|_{W^1(\Omega; \delta(x)|x|^{-b})},
\] (5.21)

where \( c = c(N, \Omega, s, \mu, b) > 0 \).

Let \( x \in \Omega \) and \( r > 0 \) be such that \( B_{r/2}(x) \subset \Omega \setminus \{0\} \). We assume that \( \varepsilon > 0 \) is small enough and we set \( u_{\delta, \varepsilon} = \zeta_\varepsilon \ast u_\delta \in C_0^\infty (\mathbb{R}^N) \). In addition, \( u_{\delta, \varepsilon} \) satisfies
\[
\begin{cases}
(-\Delta)_x u = h_\varepsilon \quad &\text{in } B_r(x) \\
u_\delta \ast u_{\delta, \varepsilon} \quad &\text{in } \mathbb{R}^N \setminus B_r(x),
\end{cases}
\]

where \( h_\varepsilon = \zeta_\varepsilon \ast (\nu_\delta - \mu \frac{u_\delta}{|u_\delta|_L^\infty}) \). Set \( v_\varepsilon = u_{\delta, \varepsilon} - \bar{u}_{\delta, \varepsilon} \), where \( \bar{u}_{\delta, \varepsilon} \) is a solution of
\[
\begin{cases}
(-\Delta)_x u = 0 \quad &\text{in } B_r(x) \\
u_\delta \ast u_{\delta, \varepsilon} \quad &\text{in } \mathbb{R}^N \setminus B_r(x).
\end{cases}
\]

By [8, proposition 2.5], for any \( p \in (1, \frac{N}{N-2}) \) and \( \gamma > \frac{N}{p} \), there holds
\[
\|v_\varepsilon\|_{W^{2-\gamma,p}(B_r(x))} \leq C(r, p, \gamma) \int_{B_r(x)} h_\varepsilon \, dx
\] (5.22)
\[
\leq C(r, p, \gamma) \int_{\Omega} |\nu_\delta| \, dx \leq C(r, p, \gamma) \|\nu\|_{W^1(\Omega; \delta(x)|x|^{-b})},
\]

where in the last two inequalities we have used (5.21).

By [5, theorem 2.10], we have that
\[
\bar{u}_{\delta, \varepsilon}(y) = c(N, s) \int_{\mathbb{R}^N \setminus B_r(x)} \left( \frac{r^2 - |y|^2}{|z|^2 - r^2} \right)^{\gamma} |y - z|^{-N} u_{\delta, \varepsilon}(z) \, dz, \quad \forall y \in B_r(x).
\]

By (5.21) and the above equality, we can easily show that
\[
\|\nabla \bar{u}_{\delta, \varepsilon}\|_{L^p(B_r^-(x))} \leq C(r, \kappa, N, s) \|\nu\|_{W^1(\Omega; \delta(x)|x|^{-b})}, \quad \forall \kappa > 1.
\] (5.23)

Taking into account (5.22), (5.23) and using a standard covering argument, we can easily show that for any open set \( D \subset \Omega \setminus \{0\} \) there exists a positive constant \( C = C(r, \kappa, N, s) \) such that
\[
\|u_{\delta, \varepsilon}\|_{W^{2-\gamma,p}(D)} \leq C \|\nu\|_{W^1(\Omega; \delta(x)|x|^{-b})}, \quad \forall \delta > 0, \varepsilon > 0.
\] (5.24)

Since \( u_{\delta, \varepsilon} \to u_\delta \) weakly in \( W^{2-\gamma,p}(D) \), it follows that
\[
\|u_\delta\|_{W^{2-\gamma,p}(D)} = \liminf_{\varepsilon \to 0} \|u_{\delta, \varepsilon}\|_{W^{2-\gamma,p}(D)} \leq C \|\nu\|_{W^1(\Omega; \delta(x)|x|^{-b})}, \quad \forall \delta > 0.
\] (5.25)
This implies that there exists a subsequence, denoted by the same index $\delta$, and $u \in L^1_{\text{loc}}(\Omega \setminus \{0\})$ such that $u_\delta \to u$ a.e. in $\Omega$. Furthermore, by (5.16), we can easily show that for any $r > 0$ and $q \in [1, 2^*_r)$, $u_\delta \to u$ in $L^q(\Omega \setminus B_\delta(0))$.

Let $\sigma > 0$ and $b < a < 2s - \tau_+$. Then by (5.21), for any $\delta > 0$, we have

$$\int_{B_\epsilon(0)} |u_\delta| |x|^{-b} \, dx \leq c \epsilon^{-b} \int_{B_\epsilon(0)} |u_\delta| |x|^{-a} \, dx \leq c \epsilon^{-a-b} \int_{\Omega \setminus \{0\}} d(x)^s |x|^{-s} \, dv. \quad (5.26)$$

Hence there exists $\epsilon_0$ such that for any $\epsilon \leq \epsilon_0$, there holds

$$\int_{B_{\epsilon(0)}} |u_\delta| |x|^{-b} \, dx \leq \frac{\sigma}{4}. \quad (5.27)$$

Since $u_\delta \to u$ in $L^1(\Omega \setminus B_\epsilon(0))$, there exists $\delta_0 > 0$ such that

$$\int_{\Omega \setminus B_\epsilon(0)} |u_\delta - u| |x|^{-b} \, dx \leq \frac{\sigma}{2}, \quad \forall \delta \in (0, \delta_0).$$

Letting $\delta \to 0$ in (5.27) and using Fatou’s lemma, we obtain

$$\int_{B_{\epsilon(0)}} |u| |x|^{-b} \, dx \leq \frac{\sigma}{4}. \quad (5.28)$$

Therefore, for any $\delta < \delta_0$,

$$\int_{\Omega \setminus \{0\}} |u_\delta - u| |x|^{-b} \, dx \leq \sigma,$$

which implies that $u_\delta \to u$ in $L^1(\Omega; |x|^{-b})$. This, together with the convergence (5.18) and estimate (1.16), enables us to pass to the limit in (5.20) to obtain

$$\int_{\Omega} u (-\Delta)^s_{\tau_+} \psi \, dx = \int_{\Omega \setminus \{0\}} \psi |x|^{-s} \, d\nu, \quad \forall \psi \in X_{\mu}(\Omega; |x|^{-b}), \quad (5.29)$$

namely $u$ is a weak solution of (1.25). Since $u_\delta \geq 0$ a.e. in $\Omega \setminus \{0\}$ for any $\delta > 0$, it follows that $u \geq 0$ a.e. in $\Omega \setminus \{0\}$. Moreover, $u$ is the unique solution to (1.25).

As for the general case $\nu \geq 0$, we consider a smooth exhaustion of $\Omega \setminus \{0\}$, i.e. smooth open sets $\{O_t\}_{t \in N}$ such that

$$O_t \subset O_{t+1} \subset \Omega \setminus \{0\} \quad \text{and} \quad \bigcup_{t \in N} O_t = \Omega \setminus \{0\}.$$ 

Set $\nu_t = 1_{\overline{O}_t}$ and let $u_t \in L^1(\Omega; |x|^{-b})$ be the nonnegative weak solution to (1.25), namely

$$\int_{\Omega} u_t (-\Delta)^s_{\tau_+} \psi \, dx = \int_{\Omega \setminus \{0\}} \psi |x|^{-s} \, d\nu_t, \quad \forall \psi \in X_{\mu}(\Omega; |x|^{-b}). \quad (5.30)$$

For $l > I'$, since $\nu_t - \nu_l' \geq 0$, it follows that $u_t \geq u_l' \text{ a.e. in } \Omega \setminus \{0\}$. We have

$$\int_{\Omega} (u_t - u_l') (-\Delta)^s_{\tau_+} \psi \, dx = \int_{\Omega \setminus \{0\}} \psi |x|^{-s} d(\nu_t - \nu_l'), \quad \forall \psi \in X_{\mu}(\Omega; |x|^{-b}). \quad (5.31)$$
By taking $\psi = \xi_b$ (where $\xi_b$ is the solution to (5.11)) in (5.30), we deduce
\[
\|u_l - u_{l'}\|_{L^1(\Omega; |x|^{-b})} = \int_\Omega (u_l - u_{l'}) |x|^{-b} \, dx = \int_{\Omega \setminus \{0\}} \xi_b |x|^{-b} d(u_l - u_{l'}) \leq c \|u_l - u_{l'}\|_{\mathcal{M}(\Omega \setminus \{0\}; \mathcal{L}(|x|^{-b}))},
\]
where in the last inequality we have used (1.16). Since $u_l \to u$ strongly in $\mathcal{M}(\Omega \setminus \{0\}; |x|^{-b})$ as $l \to \infty$, from the above estimates, we see that $\{u_l\}_{l \in \mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega; |x|^{-b})$, which implies that there exists a function $u$ such that $u_l \to u$ in $L^1(\Omega; |x|^{-b})$. Letting $l \to \infty$ in (5.29) yields (5.28). Therefore $u$ is a weak solution of (1.18).

A priori estimate. Taking $\psi = \xi_b$ (where $\xi_b$ is the solution to (5.11)) as a test function in (1.26), we obtain (1.23).

Monotonicity. Assume that $\nu_1, \nu_2 \in \mathcal{M}(\Omega \setminus \{0\}; |x|^{-b})$ such that $\nu_1 \leq \nu_2$ and let $u_l$ be the unique weak solution to (1.18) with $\nu$ replaced by $\nu_i$, $i = 1, 2$. From the above construction, we see that $u_l \leq u_2$ a.e. in $\Omega \setminus \{0\}$. Consequently, we see that if $\nu \geq 0$ then the weak solution $u$ to (1.25) satisfies $u \geq 0$ a.e. in $\Omega \setminus \{0\}$.

Uniqueness. The uniqueness is a consequence of the monotonicity.

Next we treat the case where the source is a measure on the whole domain $\Omega$. Recall that function $\Phi^\Omega_{\beta, \mu}$ satisfies (1.24). In view of the proof of lemmas 4.5 and 4.6, $\Phi^\Omega_{\beta, \mu} \in C^{\beta+\beta_0}(\Omega \setminus \{0\})$ for some $\beta_0 > 0$ and for any $\varepsilon > 0$ such that $B_{\varepsilon}(0) \subset \Omega$ there holds
\[
\left\|d^{-s}\Phi^\Omega_{\beta, \mu}\right\|_{C^\alpha(\overline{\Omega \setminus B_\varepsilon(0)})} + \left\|\Phi^\Omega_{\beta, \mu}\right\|_{C(\overline{\Omega \setminus B_\varepsilon(0)})} \leq c,
\]
where $c > 0$ depends only on $\Omega, s, \mu, \varepsilon$.

Proof of theorem 1.9. By virtue of theorem 1.7, there exists a unique weak solution $u_{\mu} \in L^1(\Omega; |x|^{-b})$ of problem (1.18), namely
\[
\int_\Omega u_{\mu} (-\Delta s)_+ \psi \, dx = \int_{\Omega \setminus \{0\}} \psi |x|^{-b} \, d\nu, \quad \forall \psi \in \mathbf{X}_\mu(\Omega; |x|^{-b}). \tag{5.31}
\]
Put $u_{\mu, \ell} = u_{\mu} + \ell \Phi^\Omega_{\beta, \mu}$. Take $\psi \in \mathbf{X}_\mu(\Omega; |x|^{-b})$, then $|(-\Delta s)_+ \psi(x)| \leq C \psi |x|^{-b} \text{ for a.e. } x \in \Omega \setminus \{0\}$. By noticing that $\tau_- - b > -N$, we deduce that
\[
\left| \int_{\Omega \setminus \{0\}} \Phi^\Omega_{\beta, \mu}(x) (-\Delta s)_+ \psi \, dx \right| \leq C \psi \int_{\Omega} |x|^{-b} \, dx < +\infty.
\]
Therefore, from (5.31), we obtain (1.26).

The uniqueness of the weak solution to (1.18) follows from theorem 1.6.

Data availability statement

No new data were created or analysed in this study.
Acknowledgments

H Chen is supported by NSFC, Nos. 12071189, 12361043, by Jiangxi Province Science Fund No. 20232ACB201001.

K T Gkikas is supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the ‘2nd Call for H.F.R.I. Research Projects to support Post-Doctoral Researchers’ (Project Number: 59).

P-T Nguyen is supported by Czech Science Foundation, Project GA22-17403S.

References

[1] Abdellaoui B, Medina M, Peral I and Primo A 2016 The effect of the Hardy potential in some Calderón–Zygmund properties for the fractional Laplacian J. Differ. Equ. 260 8160–206
[2] Barrios B, Medina M and Peral I 2014 Some remarks on the solvability of non-local elliptic problems with the Hardy potential Commun. Contemp. Math. 16 1350046
[3] Bhakta M, Biswas A, Ganguly D and Montoro L 2020 Integral representation of solutions using Green function for fractional Hardy equations J. Differ. Equ. 269 5573–94
[4] Bogdan K, Grzywny T, Jakubowski T and Pilarczyk D 2019 Fractional Laplacian with Hardy potential Commun. PDE 44 20–50
[5] Bucur C 2016 Some observations on the Green function for the ball in the fractional Laplace framework Commun. Pure Appl. Anal. 15 657–99
[6] Chen H and Weth T 2021 The Poisson problem for the fractional Hardy operator: distributional identities and singular solutions Trans. Am. Math. Soc. 374 6881–925
[7] Chen H, Quaas A and Zhou F 2021 On nonhomogeneous elliptic equations with the Hardy-Leray potentials J. Anal. Math. 144 305–34
[8] Chen H and Véron L 2014 Semilinear fractional elliptic equations involving measures J. Differ. Equ. 257 1457–86
[9] Chen H and Véron L 2019 Weak solutions of semilinear elliptic equations with Leray-Hardy potential and measure data Math. Eng. 1 391–418
[10] Chen H and Véron L 2020 Semilinear elliptic equations with Leray-Hardy potential singular on the boundary and measure data J. Differ. Equ. 269 2091–131
[11] Cîrstea F 2014 A complete classification of the isolated singularities for nonlinear elliptic equations with inverse square potentials Mem. Am. Math. Soc. 227 1–97
[12] Dipierro S, Montoro L, Peral I and Sciunzi B 2016 Qualitative properties of positive solutions to nonlocal critical problems involving the Hardy-Leray potential Calc. Var. PDE 55 99
[13] Dipierro S and Valdinoci E 2015 A density property for fractional weighted Sobolev spaces Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 26 397–422
[14] Di Nezza E, Palatucci G and Valdinoci E 2012 Hitchhiker’s guide to the fractional Sobolev spaces Bull. Sci. Math. 136 521–73
[15] Dupaigne L 2002 A nonlinear elliptic PDE with the inverse square potential J. Anal. Math. 86 359–98
[16] Ervedoza S 2008 Control and stabilization properties for a singular heat equation with an inverse-square potential Commun. PDE 33 996–2019
[17] Fall M 2020 Semilinear elliptic equations for the fractional Laplacian with Hardy potential Nonlinear Anal. 193 111311
[18] Felli V, Marchini E M and Terracini S 2007 On Schrödinger operators with multipolar inverse-square potentials J. Funct. Anal. 250 265–316
[19] Filippas S, Moschini L and Terracini A 2007 Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains Commun. Math. Phys. 273 237–81
[20] Frank R L, Lieb E H and Seiringer R 2008 Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators J. Am. Math. Soc. 21 925–50
[21] Frank R L 2009 A simple proof of Hardy–Lieb–Thirring inequalities Commun. Math. Phys. 290 789–800
[22] Frank R, Lieb E and Seiringer R 2007 Stability of relativistic matter with magnetic fields for nuclear charges up to the critical value Commun. Math. Phys. 275 479–89
[23] Ghoussoub N, Robert F, Shakerian S and Zhao M 2018 Mass and asymptotics associated to fractional Hardy–Schrödinger operators in critical regimes Commun. PDE 43 859–92
[24] Gel’fand I M 1959 Some problems in the theory of quasi-linear equations Usp. Mat. Nauk. 14 87–158
[25] Gkikas K T and Nguyen P-T 2022 Martin kernel of Schrödinger operators with singular potentials and applications to B.V.P. for linear elliptic equations Calc. Var. PDE 61 1
[26] Gkikas K T and Nguyen P-T 2022 Semilinear elliptic Schrödinger equations with singular potentials and absorption terms (arXiv:2203.01266)
[27] Gkikas K T and Nguyen P-T 2022 Semilinear elliptic Schrödinger equations involving singular potentials and source terms (arXiv:2203.01328)
[28] Guerch B and Véron L. 1991 Local properties of stationary solutions of some nonlinear singular Schrödinger equations Rev. Mat. Iberoam. 7 65–114
[29] Jakubowski T and Wang J 2020 Heat kernel estimates of fractional Schrödinger operators with negative Hardy potential Potential Anal. 53 997–1024
[30] Kalf H, Schmincke U W, Walter J and Wüst R 1975 On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials Spectral Theory and Differential Equations (Lecture Notes in Mathematics vol 448) (Springer) pp 182–226
[31] Korvenpää J, Kuusi T and Palatucci G 2016 The obstacle problem for nonlinear integro-differential operators Calc. Var. PDE 55 63
[32] Korvenpää J, Kuusi T and Palatucci G 2016 A note on fractional supersolutions Electron. J. Differ. Equ. 263 1–9
[33] Kuusi T, Mingione G and Sire Y 2015 Nonlocal equations with measure data Commun. Math. Phys. 337 1317–68
[34] Lévy-Leblond J M 1967 Electron capture by polar molecules Phys. Rev. 153 1–4
[35] Malick A 2019 Extremals for fractional order Hardy–Sobolev–Maz’ya inequality Calc. Var. PDE 58 37
[36] Mizutani H and Yao X 2021 Kato smoothing: Strichartz and uniform Sobolev estimates for fractional operators with sharp Hardy potentials Commun. Math. Phys. 388 581–623
[37] Mukherjee D, Nam P T and Nguyen P-T 2021 Uniqueness of ground state and minimal-mass blow-up solutions for focusing NLS with Hardy potential J. Funct. Anal. 281 109092
[38] Suzuki T 2016 Solvability of nonlinear Schrödinger equations with some critical singular potential via generalized Hardy–Rellich inequalities Funkc. Ekvacioj 59 1–34
[39] Tzirakis K 2016 Sharp trace Hardy-Sobolev inequalities and fractional Hardy-Sobolev inequalities J. Funct. Anal. 270 4513–39
[40] Vancostenoble J and Zuazua E 2008 Null controllability for the heat equation with singular inverse-square potentials J. Funct. Anal. 254 1864–902
[41] Vazquez J L and Zuazua E 2000 The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential J. Funct. Anal. 173 103–53
[42] Ros-Oton X and Serra J 2014 The Dirichlet problem for the fractional Laplacian: regularity up to the boundary J. Math. Pures Appl. 101 275–302
[43] Wang C, Yang J and Zhou J 2021 Solutions for a nonlocal problem involving a Hardy potential and critical growth J. Anal. Math. 144 261–303