On the definition of velocity in theories with two observer-independent scales

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Abstract

We argue that a consistent definition of the velocity of a particle in generalizations of special relativity with two observer-independent scales should be independent from the mass of the particle. This request rules out the definition $v_i = \partial p_0 / \partial p_i$, but allows for other definitions proposed in the literature.

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1 Introduction

Recently, following a suggestion of Amelino-Camelia \cite{1}, large interest has been devoted to modifications of special relativity admitting two observer-independent fundamental scales, the speed of light and the Planck energy. These models aim to describe the dynamics of particles up to the Planck region, where the structure of spacetime may change due to quantum gravity effects.

The existence of two observer-independent scales is implemented in the theory through a nonlinear action of the Lorentz group on momentum space, whose main consequence is a deformation of the dispersion relations of special relativity, which are recovered only in the low-energy limit. Of course, it is possible to construct several different models obeying these postulates. Historically, the first example was given by the quantum Poincaré algebra of Lukierski, Nowicki and Ruegg (LNR) \cite{2}. More recently, an algebraically simpler model has been introduced by Magueijo and Smolin (MS) \cite{3}.

The identification of physical quantities in these models may lead to problems, since the full range of validity of the theory is not accessible to experiments, and different definitions may lead to the same low-energy limit. In absence of experimental tests, one has to resort to requirements of consistency.

A debated problem is for example the correct definition of the velocity of a particle. Several different proposal have been advanced in the literature [1-7]. Apparently, the most natural proposal is to define the velocity like in Hamiltonian mechanics as $v^H \equiv \frac{\partial E}{\partial p}$ \cite{1}. However, this proposal gives rise to complicated addition laws for velocity and, as we shall see, does not seem to be fully consistent. Other proposal have been advanced [4-6], which instead predict the classical addition law of special relativity.

In this note, we suggest that a good definition of velocity should be such that any particles having the same velocity in a reference frame must have the same velocity in any other frame. In particular, the Lorentz transformation between the rest frame of a particle and a frame where it moves with velocity $v$, should depend only on $v$ and not on the mass of the particle. This request singles out the definitions of refs. [4-6]. In particular, the hamiltonian definition $v = v_H$ is ruled out. Similar conclusions have been reached in ref. \cite{7} starting from a totally different point of view.

In the following, we consider for simplicity of notation a two-dimensional spacetime (generalization to four dimensions is trivial). We use $(+, -)$ signa-
ture and denote with \((E, p)\) the components of the 2-momentum \(p_a, a = 0, 1\).

\[ \text{2 The MS model} \]

We start the discussion from the MS model [3]. In this case, a boost of rapidity parameter \(\xi\) is assumed to transform the 2-momentum \((E_0, p_0)\) into \((E, p)\), where

\[
E = \frac{E_0 \cosh \xi - p_0 \sinh \xi}{\Delta}, \quad p = \frac{p_0 \cosh \xi + E_0 \sinh \xi}{\Delta},
\]

with \(\Delta = 1 + \frac{(E_0(\cosh \xi - 1) - p_0 \sinh \xi)}{\kappa}\).

The Casimir mass \(m\), defined as

\[
m^2 = \frac{E^2 - p^2}{\left(1 - \frac{E}{\kappa}\right)^2},
\]

is invariant under the transformations (1), and is related to the rest energy \(m_0\) of the particle by

\[
m_0 = \frac{m}{1 + \frac{m}{\kappa}}.
\]

Consider now a particle of Casimir mass \(m\) at rest in an inertial frame. Its 2-momentum is given by \((m_0, 0)\). From (1), we can derive the energy \(E\) and the momentum \(p\) of the particle in a frame related to the first by a boost of parameter \(\xi\):

\[
E = \frac{m \cosh \xi}{1 + \frac{m}{\kappa} \cosh \xi}, \quad p = \frac{m \sinh \xi}{1 + \frac{m}{\kappa} \cosh \xi}.
\]

One can then express \(\cosh \xi\) in terms of \(E\) and \(m\) [8]:

\[
\cosh \xi = \frac{E}{m \left(1 - \frac{E}{\kappa}\right)},
\]

to be compared with the classical result, \(\cosh \xi = E/m\).

However, we are interested in the relation between the rapidity \(\xi\) and the velocity \(v\) of the particle in the moving frame. If the velocity of the particle is
defined by the relation $v_H = \frac{\partial E}{\partial p}$, one can derive its expression differentiating the mass-shell constraint (2), $p^2 = E^2 - m^2 \left(1 - \frac{E}{\kappa}\right)^2$. One has

$$v_H = \frac{p}{E + \frac{m^2}{\kappa} \left(1 - \frac{E}{\kappa}\right)},$$

and hence, from (4),

$$v_H = \frac{\sinh \xi}{\cosh \xi + \frac{m}{\kappa}}.\quad (7)$$

Inverting (7) one obtains the rapidity parameter in terms of the velocity:

$$\cosh \xi = \frac{\frac{m}{\kappa} v_H^2 + \sqrt{1 - \left(1 - \frac{m^2}{\kappa^2}\right) v_H^2}}{1 - v_H^2}.\quad (8)$$

Hence, the rapidity parameter of the boost that relates the rest frame to the frame where the particle has velocity $v_H$ depends on the mass of the particle and particles of different masses at rest in one frame would have different velocities in another frame. Moreover, there is no definite relation between $\xi$ and the velocity of the moving frame.

This problem is not present if one defines the velocity as in (6) (but this definition was already implicit in (3)),

$$v_G = \frac{p}{E}.\quad (9)$$

In fact, in view of eq. (4),

$$v_G = \tanh \xi, \quad (10)$$

from which follows the classical relation of special relativity,

$$\cosh \xi = \frac{1}{\sqrt{1 - v_G^2}},\quad (11)$$

which is of course independent from the particle mass.

### 3 The LNR model

We pass now to consider the LNR model. In this case, the transformation laws of the 2-momentum are

$$e^{\kappa} = \Delta e^{\kappa}, \quad p = \frac{p_0 \cosh \xi + \kappa \left(1 - \cosh \frac{m}{\kappa} e^{-\frac{E}{\kappa}}\right) \sinh \xi}{\Delta},$$

$$e^{\kappa} = \Delta e^{\kappa}, \quad p = \frac{p_0 \cosh \xi + \kappa \left(1 - \cosh \frac{m}{\kappa} e^{-\frac{E}{\kappa}}\right) \sinh \xi}{\Delta}.\quad (12)$$
where
\[ \Delta = \cosh \frac{m_0}{\kappa} e^{-\frac{E}{\kappa}} + \left(1 - \cosh \frac{m_0}{\kappa} e^{-\frac{E}{\kappa}}\right) \cosh \xi + \frac{p}{\kappa} \sinh \xi, \] (13)
and \( m_0 \) is the rest energy of the particle, which in terms of the invariant Casimir mass \( m \), defined as
\[ m^2 = \kappa^2 \left( e^{\frac{E}{\kappa}} - 2 + e^{-\frac{E}{\kappa}}\right) - p^2 e^{\frac{E}{\kappa}}, \] (14)
is given by
\[ \cosh \frac{m_0}{\kappa} = 1 + \frac{m^2}{2\kappa^2}. \] (15)

Consider now a particle at rest in an inertial frame, and derive from (12) its energy \( E \) and momentum \( p \) in a frame related to the first by a boost of rapidity parameter \( \xi \):
\[ e^{\frac{E}{\kappa}} = \cosh \frac{m_0}{\kappa} + \sinh \frac{m_0}{\kappa} \cosh \xi, \] (16)
\[ p = \frac{\sinh \frac{m_0}{\kappa} \sinh \xi}{\cosh \frac{m_0}{\kappa} + \sinh \frac{m_0}{\kappa} \cosh \xi}. \] (17)

From (16), one can express \( \cosh \xi \) in terms of \( E \) and \( m_0 \) [8, 10]:
\[ \cosh \xi = \frac{e^{\frac{E}{\kappa}} - \cosh \frac{m_0}{\kappa}}{\sinh \frac{m_0}{\kappa}}. \] (18)

One can derive now the relation between the rapidity parameter \( \xi \) and the velocity \( v \) of the particle. If the velocity is defined by the relation \( v_H = \frac{\partial E}{\partial p} \), one can obtain its expression differentiating the mass-shell constraint (14), \( p^2 = \kappa^2 \left(1 - e^{-\frac{E}{\kappa}}\right)^2 - m^2 e^{-\frac{E}{\kappa}} \). One has
\[ v_H = \frac{p e^{\frac{E}{\kappa}}}{\kappa \left(\cosh \frac{m_0}{\kappa} - e^{-\frac{E}{\kappa}}\right)}, \] (19)
and hence, from (16) and (17)
\[ v_H = \frac{\cosh \frac{m_0}{\kappa} + \sinh \frac{m_0}{\kappa} \cosh \xi}{\sinh \frac{m_0}{\kappa} + \cosh \frac{m_0}{\kappa} \cosh \xi} \sinh \xi. \] (20)
Of course, inverting (20) one obtains that the rapidity parameter depends both on the velocity and the mass of the particle, leading to the same problems as with the MS model.

Also in this case these problems can be avoided adopting a different definition for the velocity of a particle. Namely, one can adopt the right velocity introduced in [4, 5],

\[ v_R \equiv \frac{v_H}{1 + \frac{p v_H}{\kappa}} = \frac{p e^\frac{E}{\kappa}}{\kappa (e^\frac{E}{\kappa} - \cosh \frac{m_0}{\kappa})}. \]  

(21)

After substituting (16) and (17), in fact, one obtains

\[ v_R = \tanh \xi, \quad \cosh \xi = \frac{1}{\sqrt{1 - v^2_R}}, \]  

(22)

which are again the classical relations, independent from the particle mass.

4 Conclusions

We have shown that in order to obtain a definite relation between the relative velocity of two reference frames and the rapidity of a boost relating them, one cannot define the velocity of a particle of energy \( E \) and momentum \( p \) as \( v = \frac{\partial E}{\partial p} \). A suitable definition seems to be model-dependent (see however [7]), but in the known cases always satisfies the special relativistic relations, and in particular the addition law of velocity [6, 5]. Although this is of course a sufficient condition for avoiding the problems discussed in this paper, it does not appear to be also necessary in principle.

It is also interesting to notice that the correct definition of velocity implies that the velocity of a massless particle is always equal to \( c \). In our opinion, this is a very basic prediction and we see no reason for introducing a variable speed of light. First of all, this would be at odds with the spirit of the model, which is based on the existence of two invariant fundamental scales. A more serious problem is that a variable speed of light would destroy the causal structure of special relativity, leading to great difficulties with the physical interpretation.

Of course, it would be useful to build a suitable hamiltonian formalism that predicts the correct velocities. This seems to be possible only if one uses deformed Poisson brackets and noncommuting spacetime coordinates [11, 5, 6].
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