On the size of planarly connected crossing graphs

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Abstract

A pair of independent and crossing edges in a drawing of a graph is planarly connected if there is a crossing-free edge that connects endpoints of the crossed edges. A graph is a planarly connected crossing (PCC) graph, if it admits a drawing in which every pair of independent and crossing edges is planarly connected. We prove that a PCC graph with \( n \) vertices has \( O(n) \) edges.

1 Introduction

Throughout this paper we consider graphs with no loops or parallel edges. A topological graph is a graph drawn in the plane with its vertices as distinct points and its edges as Jordan arcs that connect the corresponding points and do not contain any other vertex as an interior point. Every pair of edges in a topological graph has a finite number of intersection points, each of which is either a vertex that is common to both edges, or a crossing point at which one edge passes from one side of the other edge to its other side. A topological graph is simple if every pair of its edges intersect at most once. A geometric graph is a (simple) topological graph in which every edge is a straight-line segment. If the vertices of a geometric graph are in convex position, then the graph is a convex geometric graph.

Call a pair of independent and crossing edges \( e \) and \( e' \) in a topological graph \( G \) planarly connected if there is a crossing-free edge in \( G \) that connects an endpoint of \( e \) and an endpoint of \( e' \). A planarly connected crossing (PCC for short) topological graph is a topological graph in which every pair of independent crossing edges is planarly connected. An abstract graph is a PCC graph if it can be drawn as a topological PCC graph.

Our motivation for studying PCC graphs comes from two examples of topological graphs that satisfy this property: A graph is \( k \)-plane if it can be drawn as a topological graph in which each edge is crossed at most \( k \) times (we call such a topological graph \( k \)-plane). Suppose that \( G \) is an \( n \)-vertex 1-planar topological graph with the maximum possible number of edges (i.e., there is no \( n \)-vertex 1-planar graph with more edges than \( G \)). Now consider a drawing \( D \) of \( G \) as a 1-plane topological graph with the least number of crossings. Then it is easy to see that \( D \) is a PCC topological graph. Indeed, if \( (u, v) \) and \( (w, z) \) are two independent edges that cross at a point \( x \) and are not planarly connected, then we can draw a crossing-free edge \( (u, w) \) that consists of the (perturbed) segments \( (u, x) \) and \( (w, x) \) of \( (u, v) \).

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1Two edges are independent if they do not share a common vertex. Note that in a simple topological graph two crossing edges must be independent.
and \((w, z)\), respectively. This way we either increase the number of edges in the graph or we are able to replace a crossed edge with a crossing-free edge and get a 1-plane drawing of \(G\) with less crossings.

Another example for PCC topological graphs are certain drawings of fan-planar graphs. A graph is called fan-planar if it can be drawn as a simple topological graph such that for every edge \(e\) all the edges that cross \(e\) share a common endpoint on the same side of \(e\). As before, it can be shown (see [9, Corollary 1]) that such an embedding of a maximum fan-planar graph with as many crossing-free edges as possible admits a PCC topological graph.

Both 1-plane topological graphs and fan-planar graphs are sparse, namely, their maximum number of edges is \(4n - 8\) [12] and \(5n - 10\) [9], respectively (where \(n\) denotes the number of vertices). Our main result shows that PCC graphs are always sparse.

**Theorem 1.** Let \(G\) be an \(n\)-vertex topological graph such that for every two independent crossing edges \(e\) and \(e'\) there is a crossing-free edge that connects an endpoint of \(e\) and an endpoint of \(e'\). Then \(G\) has at most \(cn\) edges, where \(c\) is an absolute constant.

The constant \(c\) in our proof is worse than the constants in the upper bounds on the sizes of 1-plane and fan-planar graphs. It would be interesting to improve it and find the exact maximum size of a PCC graph. We show that this value is at least \(6.6n - O(1)\) (see Section 3), which implies that not every PCC graph is a (maximum) 1-plane or fan-planar graph.

PCC graphs are also related to two other classes of topological graphs. Call a topological graph \(k\)-quasi-plane if it has no \(k\) pairwise crossing edges. According a well-known and rather old conjecture (see e.g., [6, 10]) \(k\)-quasi-plane graphs should have linearly many edges.

**Conjecture 2.** For any integer \(k \geq 2\) there is a constant \(c_k\) such that every \(n\)-vertex \(k\)-quasi-plane graph has at most \(c_kn\) edges.

It is easy to see that if \(G\) is a simple PCC topological graph, then \(G\) is 13-quasi-plane: Suppose for contradiction that there are 13 pairwise crossing edges in \(G\). Then, since \(G\) is a simple topological graph, their 26 endpoints are distinct. Thus, for every pair of crossing edges there is a distinct crossing-free edge that connects endpoints of the two crossing edges. Therefore, there are at least \(\binom{13}{2} = 78\) crossing-free edges between those 26 vertices. However, a planar graph with 26 vertices contains at most \(3 \cdot 26 - 6 = 72\) edges.

Therefore, Conjecture 2, if true, would immediately imply Theorem 1 for simple topological graphs. However, this conjecture was only verified for \(k = 3\) [4, 5, 11], for \(k = 4\) [1], and (for any \(k\)) for convex geometric graphs [7]. For \(k \geq 5\) the currently best upper bounds on the size of \(n\)-vertex \(k\)-quasi-plane graphs are \(n(\log n)^{O(\log k)}\) by Fox and Pach [8], and \(O_k(n \log n)\) for simple topological graphs by Suk and Walczak [14].

Another conjecture that implies Theorem 1 (also for not necessarily simple topological graphs) is related to grids in topological graphs. A \(k\)-grid in a topological graph is a pair of edge subsets \(E_1, E_2\) such that \(|E_1| = |E_2| = k\), and every edge in \(E_1\) crosses every edge in \(E_2\). Ackerman et al. [2] proved that every \(n\)-vertex topological graph that does not contain a \(k\)-grid with distinct vertices has at most \(O_k(n \log^* n)\) edges and conjectured that this upper bound can be improved to \(O_k(n)\). It is not hard to show, as before, that a PCC graph does not contain a 12-grid with distinct vertices. Therefore, this conjecture, if true, would also imply Theorem 1.

**Outline.** We prove Theorem 1 in the following section. In Section 3 we give a lower bound on the maximum size of a PCC graph, generalize the notion of planarly connected edges, and conclude with some open problems.
2 Proof of Theorem 1

Let $G = (V, E)$ be an $n$-vertex topological graph such that for every two independent crossing edges $e$ and $e'$ there is a crossing-free edge that connects an endpoint of $e$ and an endpoint of $e'$. Denote by $E' \subseteq E$ the set of crossing-free (planar) edges in $G$, and by $E'' = E \setminus E'$ the set of crossed edges in $G$. Let $G'_1 = (V_1, E'_1), \ldots, G'_k = (V_k, E'_k)$ be the connected components of the graph $G' = (V, E')$, and let $E''_{i,j} = \{(u, v) \in E'' \mid u \in V_i \text{ and } v \in V_j\}$.

**Lemma 2.1.** $|E''_{i,j}| \leq 96|V_i|$ for every $1 \leq i \leq k$.

**Proof.** Assume without loss of generality that $i = 1$ and consider the graph $G'_1$. Let $f_1, \ldots, f_\ell$ be the faces of the plane graph that is induced by $G'_1$. For a face $f_j$, let $V(f_j)$ be the vertices that are incident to $f_j$, and let $E''(f_j)$ be the edges in $E''_{1,j}$ that lie within $f_j$ (thus, their endpoints are in $V(f_j)$). Denote by $|f_j|$ the size of $f_j$, that is, the length of the shortest closed walk that visits every edge on the boundary of $f_j$.

**Proposition 2.2.** There are no 9 pairwise independent and crossing edges in $E''(f_j)$.

**Proof.** Suppose for contradiction that $(x_1, y_1), \ldots, (x_9, y_9) \in E''(f_j)$ are pairwise independent and crossing edges. Thus, for every pair of crossing edges there is a distinct crossing-free edge that connects endpoints of the two crossing edges. Therefore, there are $\binom{9}{2} = 36$ crossing-free edges that connect vertices in $\{x_1, \ldots, x_9, y_1, \ldots, y_9\}$. Note that all of these edges lie outside of $f_j$ or on its boundary, and therefore they define an outer-planar graph. However, an outer-planar graph with 18 vertices contains at most $2 \cdot 18 - 3 = 33$ edges. \hfill \Box

**Proposition 2.3.** $|E''(f_j)| \leq 16|f_j|$, for every $1 \leq j \leq \ell$.

**Proof.** Define first an auxiliary graph $\hat{G}_j$ as follows. When traveling along the boundary of $f_j$ in clockwise direction, we meet every vertex in $V(f_j)$ at least once and possibly several times if the boundary of $f_j$ is not a simple cycle. Let $v_1, v_2, \ldots, v_{|f_j|}$ be the list of vertices as they appear along the boundary of $f_j$, where a new instance of a vertex is introduced whenever a visited vertex is revisited. The edge set of $\hat{G}_j$ corresponds to $E''(f_j)$, however, we make sure to pick the “correct” instance of a vertex in $v_1, v_2, \ldots, v_{|f_j|}$ for a vertex in $V(f_j)$ that was visited more than once when traveling along the boundary of $f_j$ (see Figure 1 for an example).

Clearly, two independent edges in $\hat{G}_j$ are crossing if and only if their corresponding edges cross in $G$. Therefore, $\hat{G}_j$ does not contain 9 pairwise independent and crossing edges by Proposition 2.2. We now realize the underlying abstract graph of $\hat{G}_j$ as a convex geometric
crossing edges \((v_i, v_j)\) in the realization. Then it follows that \(G\) contains two independent and crossing edges. Then it follows that \(G\) contains two independent and crossing edges \((v_i, v_j)\) and \((v_c, v_d)\). Since these two edges are planarly connected, there should...
be a crossing-free edge that connects a vertex in \( \{v_a, v_b\} \) with vertex in \( \{v_c, v_d\} \). However, this is impossible since these four vertices belong to distinct connected components of \( G' \).

Finally, a graph that can be drawn such that each crossing is between two edges that share a common vertex is planar: this follows from the strong Hanai-Tutte Theorem (see, e.g., [13]) and is also easy to show by locally redrawing the edges at crossing points such that all crossings are eliminated.

\[ \square \]

**Lemma 2.5.** \( |E''_{i,j}| \leq 8(|V_{i,j}| + |V_{j,i}|) \) for every \( 1 \leq i < j \leq k \).

*Proof.* Since \( G'_i \) and \( G'_j \) are planar graphs, we can properly color their vertices with four colors. Denote the colors by 1, 2, 3, 4, and let \( V^c_{i,j} \) (resp., \( V^c_{j,i} \)) be the vertices of color \( c \) in \( V_{i,j} \) (resp., \( V_{j,i} \)). We claim that the number of edges in \( E''_{i,j} \) that connect a vertex from \( V^c_{i,j} \) and a vertex from \( V^c_{j,i} \) is at most \( 2(|V^c_{i,j}| + |V^c_{j,i}|) \) for every \( c, c' \in \{1, 2, 3, 4\} \). Indeed, denote the graph that consists of these edges by \( G^* \) and consider its drawing as inherited from \( G \).

It is not hard to see that \( G^* \) is a planar graph: Suppose that two edges in \( G^* \) cross and denote them by \((u, v)\) and \((x, y)\) such that \( u, x \in V^c_{i,j} \) and \( v, y \in V^c_{j,i} \). Since \( u \) and \( x \) are both of color \( c \), there is no crossing-free edge in \( G'_i \) that connects them. Similarly, there is no crossing-free edge in \( G'_j \) that connects \( v \) and \( y \). Since there are also no crossing-free edges in \( E''_{i,j} \), it follows that \((u, v)\) and \((x, y)\) are not independent. Since \( G^* \) has no independent crossing edges, it follows that \( G^* \) is a planar graph. Because \( G^* \) is also bipartite, its number of edges is at most twice its number of vertices. Thus,

\[
|E''_{i,j}| \leq 2 \sum_{1 \leq c \leq 4} \sum_{1 \leq c' \leq 4} (|V^c_{i,j}| + |V^{c'}_{j,i}|) = 8(|V_{i,j}| + |V_{j,i}|),
\]

and the lemma follows. \[ \square \]

**Lemma 2.6.** \( \sum_{j \neq i} |V_{i,j}| \leq 3(|V_i| + 4 \deg_H(u_i)) \) for every \( 1 \leq i \leq k \).

*Proof.* We use again ideas from the proofs of Lemma 2.4 and Lemma 2.5. Assume without loss of generality that \( i = 1 \) and consider the graph \( G'_1 \). Since \( G'_1 \) is a planar graph, we can properly color its vertices with four colors. Denote the colors by 1, 2, 3, 4, and let \( V^c_{1,j} \) (resp., \( V^c_{i,j} \)) be the vertices of color \( c \) in \( V_j \) (resp., \( V_{i,j} \)). Clearly, \( \sum_{j=2}^{k} |V_{i,j}| = \sum_{c=1}^{4} \sum_{j=2}^{k} |V^c_{i,j}|. \)

Therefore it is enough to consider \( \sum_{j=2}^{k} |V^c_{1,j}| \) for a fixed color \( c \).

Recall that in the proof of Lemma 2.4 for every \( 1 \leq i \leq k \), we have identified \( u_i \) with one of the vertices of \( G'_i \) and denoted by \( T_i \) a spanning tree of \( G'_i \). We define a graph \( H^c \) whose vertex set consists of \( V^c_{i} \) and the vertices \( u_i \) that are adjacent to \( u_i \) in \( H \). For every vertex \( v_1 \in V^c_{i} \) that is connected by at least one edge to a vertex in \( V_j \), pick arbitrarily such an edge, say \((v_1, v_j)\), and draw an edge \((v_1, u_j)\) as follows: \((v_1, u_j)\) consists of the edge \((v_1, v_j)\) in \( G \) and the unique path in \( T_j \) from \( v_j \) to \( u_j \).

Observe that \( H^c \) is a simple graph (i.e., it has no parallel edges or self-loops). Moreover, in the drawing of \( H^c \) that is obtained as above, all the crossing points are inherited from \( G \), however, there are overlaps between edges. Still, each such (maximal) overlap contains an endpoint of an edge, and thus, as in the proof of Lemma 2.4, the edges of \( H^c \) can be slightly perturbed such that all the overlaps are removed and no new crossings are introduced.

Consider such a drawing of \( H^c \) and observe that if two edges cross in this drawing, then they must share a common endpoint. Indeed, suppose for contradiction that \((v_1, v_a)\) and \((v'_1, u_b)\) are two independent and crossing edges. Then it follows that \( G \) contains two independent and crossing edges \((v_1, v_a)\) and \((v'_1, v_b)\), such that \( v_1, v'_1 \in V_1, v_a \in V_a, \) and \( v_b \in V_b \). Since these two edges are planarly connected, there should be a crossing-free edge that connects a vertex in \( \{v_1, v_a\} \) with a vertex in \( \{v'_1, v_b\} \). However, this is impossible because
there is no crossing-free edge between two vertices from different connected components of \( G' \) and there is also no crossing-free edge \((v_1, v'_1)\) since both \(v_1\) and \(v'_1\) are of color \(c\).

This implies that \( H' \) is a planar graph. Observe that \( \sum_{j=2}^{k} |V_{i,j}| \) is precisely the number of edges in \( H' \). Thus, \( \sum_{j=2}^{k} |V_{i,j}| \leq 3(|V_i| + \deg_H(u_1)) \), and it follows that \( \sum_{j=2}^{k} |V_{i,j}| = \sum_{c=1}^{4} \sum_{j=2}^{k} |V_{1,j}| \leq 3|V_1| + 12 \deg_H(u_1) \). \( \square \)

Recall that it remains to show that \( |E''| = O(n) \):

\[
|E''| = \sum_{1 \leq i < k} |E''_{i,i}| + \sum_{1 \leq i < j \leq k} |E''_{i,j}|
\leq 96n + 8 \sum_{1 \leq i < j \leq k} (|V_{i,j}| + |V_{j,i}|)
= 96n + 4 \sum_{1 \leq i \leq k} \sum_{j \neq i} |V_{i,j}|
\leq 96n + 12 \sum_{1 \leq i \leq k} (|V_i| + 4 \deg_H(u_i))
\leq 96n + 12n + 288n = 396n.
\]

We conclude that \( |E| = |E'| + |E''| \leq 399n \). Theorem 1 is proved.

3 Discussion

It would be interesting to find the maximum size of an \( n \)-vertex PCC graph. Theorem 1 shows that this quantity is at most 399\( n \), but we believe that a linear bound with a much smaller multiplicative constant holds.

Let us describe a construction of a topological PCC graph with 6.6\( n - O(1) \) edges. Consider first a drawing of \( K_9 \) as a geometric graph with vertices \( v_0, \ldots, v_8 \), such that \( v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \) are the vertices of a regular hexagon, listed in their clockwise order, and for \( i = 1, 4, 7 \), \( v_i \) is in the interior of this hexagon very close to the middle of the edge \((v_{i-1}, v_{i+1})\), see Figure 3(a). Observe that if we draw all the possible edges as straight-line segments, then we get a PCC graph. Indeed, note first that the crossing-free edges in this drawing are the edges of the hexagon as well as the edges \((v_i, v_{i+1})\) (addition is modulo 9) for every \( 0 \leq i \leq 8 \). Consider two edges \((x, y)\) and \((w, z)\). We may assume without loss of generality that \( x = v_0 \), since at least one endpoint of these edges is not an interior vertex. Denote \( y = v_j \) and observe that if \( j \leq 5 \) (resp., \( j \geq 5 \)), then each of \( v_1, \ldots, v_{j-1} \) (resp., \( v_{j+1}, \ldots, v_8 \)) is adjacent by a crossing-free edge to either \( v_0 \) or \( v_j \). It follows that \((x, y)\) and \((w, z)\) are planarly connected.

To complete the construction, for an integer \( l \geq 2 \), we tile a vertical cylindrical surface with \( l - 1 \) horizontal layers each consisting of three hexagonal faces that are wrapped around the cylinder. Each of these faces corresponds to a drawing of \( K_9 \) as above. The top and bottom of the cylinder are also tiled with hexagonal faces (see Figure 3(b)). Note that in both of these faces there are three vertices that are incident to three hexagons \((v_2, v_5, v_8 \) in Figure 3(b)), and three vertices that are incident to just two hexagons. The top and bottom hexagons correspond to the drawing of \( K_9 \) minus the three edges that connect the three vertices that are incident to three hexagons (to avoid parallel edges). Therefore, there are \( 6(l-2) \) vertices whose degree is 21, 6 vertices whose degree is 19, 6 vertices whose degree is 16, and \( 9(l-1) + 6 \) (“inner”) vertices whose degree is 8. Thus, the number of vertices is \( n = 15l - 3 \) and the number of edges is \( 99l - 33 \geq 6.6n - O(1) \).
The notion of planarly connected edges can be generalized as follows. For an integer $k \geq 0$, we say that two crossing edges $e$ and $e'$ in a topological graph $G$ are \emph{$k$-planarly connected} if there is a path of at most $k$ crossing-free edges in $G$ that connects an endpoint of $e$ with an endpoint of $e'$. Call a graph \emph{$k$-planarly connected crossing} ($k$-PCC for short) graph if it can be drawn as a topological graph in which every pair of crossing edges is $k$-planarly connected. Thus, PCC graphs are 1-PCC graphs.

For $k = 0$, graphs that can be drawn as topological graphs in which every pair of crossing edges share a common vertex are actually planar graphs, as noted in the proof of Lemma 2.4. For $k \geq 2$ we can no longer claim that a $k$-PCC graph is sparse. Indeed, it is easy to see that $K_n$ is a 2-PCC graph: simply pick a vertex $v$ and draw it with all of its neighbors as a crossing-free star. Now every remaining edge can be drawn such that we get a simple topological graph in which for every two crossing edges there is a path (through $v$) of two crossing-free edges that connects their endpoints.

Note that if $G$ is a $k$-PCC graph and $G'$ is a subgraph of $G$, then this does not imply that $G'$ is also a $k$-PCC graph. For example, it is not hard to see that for any $k$ there is a (sparse) graph that is not $k$-PCC: simply replace every edge of $K_5$ (or any non-planar graph) with a path of length $k + 1$. Clearly, any drawing of the resulting graph must contain two independent and crossing edges such that there is no path of length at most $k$ between their endpoints.

We conclude with a few interesting questions one can ask about the notion of planarly connected crossings: Is it possible to construct for any $n$ and $k$ a dense graph which is not $k$-PCC? Can we recognize $(k)$-PCC graphs efficiently? Given that a graph is a $(k)$-PCC graph, is it possible to find efficiently such an embedding?

\textbf{Acknowledgments.} Most of this work was done during a visit of the first author to the Rényi Institute that was partially supported by Hungarian National Science Fund (OTKA),
under grant PD 108406 and by ERC Advanced Research Grant no. 267165 (DISCONV).

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