Constraining the density slope of nuclear symmetry energy at subsaturation densities using electric dipole polarizability in $^{208}$Pb

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I. INTRODUCTION

Due to its multifaceted roles in nuclear physics and astrophysics, as well as new physics beyond the standard model, the symmetry energy has become a hot topic in current research frontiers of nuclear physics and astrophysics. During the last decade, a lot of experimental, observational and theoretical efforts have been devoted to constraining the magnitude $E_{\text{sym}}(\rho_c)$ and density slope $L(\rho_c)$ of the symmetry energy at nuclear saturation density $\rho_0$ ($\sim 0.16$ fm$^{-3}$), i.e., $E_{\text{sym}}(\rho_0)$ and $L(\rho_0)$. Although important progress has been made, large uncertainties on the values of $E_{\text{sym}}(\rho_0)$ and $L(\rho_0)$ still exist (See, e.g., Refs. [24, 25, 26]). For instance, while the $E_{\text{sym}}(\rho_0)$ is determined to be around $32 \pm 4$ MeV, the extracted $L(\rho_0)$ varies significantly from about 20 to 115 MeV, depending on the observables and analysis methods. To better understand the model dependence and narrow the uncertainties of the constraints is thus of extreme importance.

While many studies on heavy ion collisions and neutron stars have significantly improved our knowledge on the symmetry energy, more and more constraints on the symmetry energy have been obtained in recent years from analyzing the properties of finite nuclei, such as the nuclear binding energy [15–19], the neutron skin thickness [20–22], and the resonances and excitations [23–31]. Furthermore, it has been realized that the properties of finite nuclei usually provide more precise constraints on $E_{\text{sym}}(\rho)$ and $L(\rho)$ at subsaturation densities rather than at saturation density $\rho_0$. This feature is understandable since the characteristic (average) density of finite nuclei is less than $\rho_0$. For example, the average density of heavy nuclei (e.g., $^{208}$Pb) is about 0.11 fm$^{-3}$, and thus the properties of heavy nuclei most effectively probe the properties of nuclear matter at subsaturation densities rather than at saturation density. We demonstrate that the electric dipole polarizability $\alpha_D$ in $^{208}$Pb is sensitive to both the magnitude $E_{\text{sym}}(\rho_c)$ and density slope $L(\rho_c)$ of the symmetry energy at a subsaturation cross density $\rho_c = 0.11$ fm$^{-3}$. Using the experimental data of $\alpha_D$ in $^{208}$Pb from RCNP and the recent accurate constraint of $E_{\text{sym}}(\rho_c)$ from the binding energy difference of heavy isotope pairs, we extract a value of $L(\rho_c) = 47.3 \pm 7.8$ MeV. The implication of the present constraint of $L(\rho_c)$ to the symmetry energy at saturation density, the neutron skin thickness of $^{208}$Pb and the core-crust transition density in neutron stars is discussed.

II. MODEL AND METHOD

A. The symmetry energy and Skyrme-Hartree-Fock approach

The equation of state (EOS) of asymmetric nuclear matter, given by its binding energy per nucleon, can be written as

$$E(\rho, \delta) = E_0(\rho) + E_{\text{sym}}(\rho)\delta^2 + O(\delta^4),$$  \hspace{1cm} (1)

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where $\rho$ is the baryon density, $\delta = (\rho_n - \rho_p)/(\rho_p + \rho_n)$ is isospin asymmetry, $E_0(\rho) = E(\rho, \delta = 0)$ is the EOS of symmetric nuclear matter, and the symmetry energy is expressed as

$$E_{\text{sym}}(\rho) = \frac{1}{2} \frac{\partial^2 E(\rho, \delta)}{\partial \delta^2} |_{\delta=0}. \quad (2)$$

Around a reference density $\rho_r$, the $E_{\text{sym}}(\rho)$ can be expanded in $\chi_r = (\rho - \rho_r)/\rho_r$ as

$$E_{\text{sym}}(\rho) = E_{\text{sym}}(\rho_r) + \frac{L(\rho_r)}{3} \chi_r + O(\chi_r^2), \quad (3)$$

where $L(\rho_r) = 3 \rho_r \frac{\partial E_{\text{sym}}(\rho_r)}{\partial \rho} |_{\rho = \rho_r}$ is the density slope parameter which characterizes the density dependence of the symmetry energy around $\rho_r$.

Our calculations in the present work are based on the Skyrme-Hartree-Fock (SHF) approach with the so-called standard Skyrme force (see, e.g., Ref. [13–15]) which includes 10 parameters, i.e., the 9 Skyrme force parameters $\sigma, t_0 - t_3, x_0 - x_3$, and the spin-orbit coupling constant $W_0$. Instead of directly using the 9 Skyrme force parameters, we can express them explicitly in terms of 9 macroscopic quantities, i.e., $\rho_0$, $E_0(\rho_0)$, the incompressibility $K_0$, the isoscalar effective mass $m_{t,0}^*$, the isovector effective mass $m_{v,0}^*$, $E_{\text{sym}}(\rho_r)$, $L(\rho_r)$, the gradient coefficient $G_S$, and the symmetry-gradient coefficient $G_V$. In this case, we can examine the correlation of properties of finite nuclei with each individual macroscopic quantity by varying individually these macroscopic quantities within their empirical ranges. Recently, this correlation analysis method has been successfully applied to study nuclear matter properties from analyzing nuclear structure observables [21, 32, 36, 46, 47], and will also be used in this work.

B. Random-phase approximation and electric dipole polarizability

The random-phase approximation (RPA) provides an important microscopic approach to calculate the electric dipole polarizability in finite nuclei. Within the framework of RPA theory, for a given excitation operator $\hat{F}_{JM}$, the reduced transition probability from RPA ground state $\lvert \bar{0} \rangle$ to RPA excitation state $\lvert \nu \rangle$ is given by:

$$B(EJ : \bar{0} \rightarrow \nu) = \langle \nu | \hat{F}_J | \bar{0} \rangle^2 = \sum_{m(i)} \langle X^{\nu}_m + Y^{\nu}_m | \langle m | \hat{F}_J | i \rangle \rangle^2, \quad (4)$$

where $m(i)$ denotes the unoccupied (occupied) single nucleon state; $\langle m | \hat{F}_J | i \rangle$ is the reduced matrix element of $\hat{F}_{JM}$; and $X^{\nu}_m$ and $Y^{\nu}_m$ are the RPA amplitudes. The strength function then can be calculated as:

$$S(E) = \sum_{\nu} |\langle \nu | \hat{F}_J | \bar{0} \rangle|^2 \delta(E - E_{\nu}), \quad (5)$$

where $E_{\nu}$ is the energy of RPA excitation state $\lvert \nu \rangle$. Thus the moments of strength function can be obtained as:

$$m_k = \int dE E^k S(E) = \sum_{\nu} |\langle \nu | \hat{F}_J | \bar{0} \rangle|^2 E_{\nu}^k. \quad (6)$$

In the case of electric dipole (E1) response, the excitation operator is defined as:

$$\hat{F}_{1M} = \frac{eN}{A} \sum_{i=1}^Z r_i Y_{1M} (\hat{r}_i) - \frac{eZ}{A} \sum_{i=1}^N r_i Y_{1M} (\hat{r}_i), \quad (7)$$

where $Z, N$ and $A$ are proton, neutron and mass number, respectively; $r_i$ is the nucleon’s radial coordinate; $Y_{1M}(\hat{r}_i)$ is the corresponding spherical harmonic function. For a given Skyrme interaction, we can calculate the inverse energy-weight moment $m_{-1}$ using the HF-RPA method, and then obtain the electric dipole polarizability $\alpha_D$ as

$$\alpha_D = \frac{8\pi}{9} e^2 \int dE E^{-1} S(E) = \frac{8\pi}{9} e^2 m_{-1}. \quad (8)$$

For the theoretical calculations of electric dipole polarizability in $^{208}$Pb in the present work, we employ the Skyrme-RPA program by Colò et al. [48]. In this program, the SHF mean field and the RPA excitations is fully self-consistent. In particular, we calculate the isovector dipole strength in $^{208}$Pb with a spherical box extending up to 24 fm, a radial mesh of 0.1 fm and a cutoff energy of $E_C = 150$ MeV which denotes the maximum energy of the unoccupied single-particle states in the RPA model space. Then the inverse energy-weight moment $m_{-1}$ is evaluated with an upper integration limit of 130 MeV according to the experimental energy range [30], and thus the electric dipole polarizability $\alpha_D$ can be calculated invoking Eq. (5).

III. RESULTS AND DISCUSSIONS

To examine the correlation of the $\alpha_D$ in $^{208}$Pb with each macroscopic quantity, especially on $E_{\text{sym}}(\rho_c)$ and $L(\rho_c)$ at $\rho_c = 0.11$ fm$^{-3}$, we show in Fig. 4 the $\alpha_D$ in $^{208}$Pb from SHF with the Skyrme force MSL0 [21] by varying individually $L(\rho_c)$, $G_V$, $G_S$, $E_0(\rho_0)$, $E_{\text{sym}}(\rho_c)$, $K_0$, $m_{t,0}^*$, $m_{v,0}^*$, $\rho_0$, and $W_0$ within their empirical uncertain ranges, namely, varying one quantity at a time while keeping all others at their default values in MSL0. It is seen from Fig. 4 that the $\alpha_D$ in $^{208}$Pb exhibits strong correlations with both $L(\rho_c)$ and $E_{\text{sym}}(\rho_c)$, while much weaker correlation with other macroscopic quantities. Particularly, the $\alpha_D$ decreases sensitively with $E_{\text{sym}}(\rho_c)$ while increases rapidly with $L(\rho_c)$, implying a fixed value of $\alpha_D$ will lead to a strong positive correlation between $E_{\text{sym}}(\rho_c)$ and $L(\rho_c)$. Given that the symmetry energy at $\rho_c = 0.11$ fm$^{-3}$ has been well constrained as $E_{\text{sym}}(\rho_c) = 26.65 \pm 0.20$ MeV, one thus expects the electric dipole polarizability $\alpha_D$ in $^{208}$Pb can constrain stringently the parameter $L(\rho_c)$. 
Fixing the values of other 8 macroscopic quantities, i.e., $G_V$, $G_S$, $E_0(\rho_0)$, $K_0$, $m_{\sigma,0}$, $m_{\rho,0}$, $\rho_0$ and $W_0$ at their default values in MSL0, we illustrate in Fig. 2 by open up-triangles (down-triangles) the $\alpha_D$ in $^{208}$Pb as a function of $L(\rho_c)$ for $E_{\text{sym}}(\rho_c) = 26.45$ (26.85) MeV. As expected, it is seen from Fig. 2 that the $\alpha_D$ in $^{208}$Pb increases (decreases) with $L(\rho_c)$ ($E_{\text{sym}}(\rho_c)$) for a fixed $E_{\text{sym}}(\rho_c)$ ($L(\rho_c)$). By comparing with the experimental data $\alpha_D = 20.1 \pm 0.6$ fm$^3$, one can extract a strong constraint of $L(\rho_c) = 48.6 \pm 7.9$ MeV.

The above constraint of $L(\rho_c) = 48.6 \pm 7.9$ MeV has been obtained by neglecting the weak correlations between the $\alpha_D$ in $^{208}$Pb and other 8 macroscopic quantities. To test the robustness of this constraint and to obtain a more precise constraint, we optimize all other 8 parameters, instead of simply fixing them at their default values in MSL0, by minimizing the weighted sum of $\chi^2$ evaluated from the difference between SHF prediction and the experimental data for some selected observables using the simulated annealing technique [49].

In the optimization, we chose some experimental data of spherical even-even nuclei, i.e., (i) the binding energy $E_B$ of $^{16}$O, $^{40,48}$Ca, $^{56,68}$Ni, $^{88}$Sr, $^{90}$Zr, $^{100,116,132}$Sn, $^{144}$Sm, $^{208}$Pb [50]; (ii) the charge rms radii $r_C$ of $^{16}$O, $^{40,48}$Ca, $^{56}$Ni, $^{88}$Sr, $^{90}$Zr, $^{116,132}$Sn, $^{144}$Sm, $^{208}$Pb [51, 52]; (iii) the bremsstrahlung mode energy $E_0$ of $^{80}$Zr, $^{116}$Sn, $^{144}$Sm and $^{208}$Pb [53]. In the calculation of the bremsstrahlung mode energy $E_0$, we use the inverse energy-weighted sum rule $m_1/m-1$, we evaluate the inverse energy-weighted sum rule $m_1$ with the constrained Hartree-Fock (CHF) method and obtain the energy-weighted sum rule $m_1$ using the double commutator sum rule [54–57]. In addition, in the optimization, we constrain the macroscopic parameters by requiring that (i) the neutron 3$p_{1/2} - 3p_{3/2}$ energy level splitting in $^{208}$Pb should lie in the range of $0.8 \sim 1.0$ MeV; (ii) $m_{\sigma,0}^* - m_{\rho,0}^*$ should be greater than $m_{\sigma,0}$ and here we set $m_{\sigma,0}^* - m_{\rho,0} = 0.1m$ (m is nucleon mass in vacuum) to be consistent with the extraction from global nucleon optical potentials constrained by world data on nucleon-nucleus and (p,n) charge-exchange reactions [58]. As usual, in the optimization, we assign a theoretical error 1.2 MeV to $E_B$, 0.025 fm to $r_C$ while use the experimental error for breath mode energy $E_0$ with a weight factor 0.08, so that the respective $\chi^2$ evaluated from each sort of experimental data is roughly equal to the number of the corresponding data points [59].

By using the optimized parameters, we evaluate the electric dipole polarizability $\alpha_D$ in $^{208}$Pb as a function of $L(\rho_c)$ for a fixed $E_{\text{sym}}(\rho_c)$, and the results are shown in Fig. 2 by solid up-triangles (down-triangles) for $E_{\text{sym}}(\rho_c) = 26.45$ (26.85) MeV. It is interesting to see that the values of $\alpha_D$ with optimization are quite consistent with the results using the default values in MSL0 without optimization and only show a small upward shift compared with the latter. Comparing the results from optimization to the experimental data, one can obtain a constraint of $L(\rho_c) = 47.3 \pm 7.8$ MeV, which is again in good agreement with the constraint $L(\rho_c) = 48.6 \pm 7.9$ MeV extracted using the default values in MSL0. These features demonstrate the validity of neglecting the weak correlations between the $\alpha_D$ in $^{208}$Pb and other 8 macroscopic quantities. The present constraint on $L(\rho_c)$ further agrees very well with the constraint $L(\rho_c) = 46.0 \pm 4.5$ MeV extracted from analyzing the experimental data on the neutron skin thickness of Sn isotopes [30]. This is a very interesting finding since these two constraints are obtained from two completely independent experimental observables.

In addition, using the constrained $E_{\text{sym}}(\rho_c)$ and $L(\rho_c)$ together with the 8 other optimized quantities, one can easily extract the $E_{\text{sym}}(\rho)$ and $L(\rho)$ at saturation density.
\[ \rho_0, \text{ and the results are } E_{\text{sym}}(\rho_0) = 32.7 \pm 1.7 \text{ MeV and } L(\rho_0) = 47.1 \pm 17.6 \text{ MeV, which are essentially consistent with } \]
\[ \text{other constraints extracted from terrestrial experiments, astrophysical observations, and theoretical calculations with controlled uncertainties}. \]

Especially, our present results agree surprisingly well with the constraint of \( E_{\text{sym}}(\rho_0) = 31.2-34.3 \text{ MeV and } L(\rho_0) = 36-55 \text{ MeV} \) at 95\% confidence level obtained from analyzing the mass and radius of neutron stars \[61\] as well as that of \( E_{\text{sym}}(\rho_0) = 29.0-32.7 \text{ MeV and } L(\rho_0) = 40.5-61.9 \text{ MeV} \) extracted from the experimental, theoretical and observational analyses \[11\]. Our results are also in agreement with the constraint of \( E_{\text{sym}}(\rho_0) = 32.0 \pm 1.8 \text{ MeV and } L(\rho_0) = 43.1 \pm 15 \text{ MeV} \) from analyzing pygmy dipole resonances (PDR) of \(^{130,132}\)Sn \[24\] and that of \( E_{\text{sym}}(\rho_0) = 32.3 \pm 1.3 \text{ MeV and } L(\rho_0) = 64.8 \pm 15.7 \text{ MeV} \) from analyzing PDR of \(^{68}\)Ni and \(^{132}\)Sn \[23\]. In addition, our results are further consistent with the constraint of \( E_{\text{sym}}(\rho_0) = 32.3 \pm 1.0 \text{ MeV and } L(\rho_0) = 45.2 \pm 10.0 \text{ MeV} \) extracted from analyzing the experimental data of the binding energy difference of heavy isotope pairs and the neutron skins of Sn isotopes \[32\] as well as the constraint of \( E_{\text{sym}}(\rho_0) = 32.5 \pm 0.5 \text{ MeV and } L(\rho_0) = 70 \pm 15 \text{ MeV} \) from a new finite-range droplet model analysis of the nuclear mass \[18\].

Given that the neutron skin thickness \( \Delta r_{np} \) of \(^{208}\)Pb is uniquely fixed by the slope parameter \( L(\rho_c) \) at \( \rho_c = 0.11 \) fm\(^{-3}\) \[66\], we can also extract a constraint \( \Delta r_{np} = 0.176 \pm 0.027 \text{ fm for } ^{208}\)Pb by using the optimized parameters together with \( E_{\text{sym}}(\rho_c) = 26.65 \pm 0.20 \text{ MeV and } L(\rho_c) = 47.3 \pm 7.8 \text{ MeV} \). Our result is consistent with the estimated range \( \Delta r_{np} = 0.165 \pm (0.009)_{\text{exp}} \pm (0.013)_{\text{theor}} \pm (0.021)_{\text{syst}} \) fm in Ref. \[31\] obtained by analyzing the experimental data of \(^{208}\)Pb with an empirical range of \( E_{\text{sym}}(\rho_0) = 31 \pm (2) \) fm\(^{-3}\). One can see that our present constraint on \( \Delta r_{np} \) of \(^{208}\)Pb has higher precision, indicating a more precise constraint on the symmetry energy at a sub-saturation density is very helpful to extract \( \Delta r_{np} \) of \(^{208}\)Pb from the electric dipole polarizability. Our result further agrees with the constraint \( \Delta r_{np} = 0.156^{+0.025}_{-0.021} \text{ fm} \) obtained from the \(^{208}\)Pb dipole polarizability by using an empirical correlation between \( \alpha_D \) and \( \Delta r_{np} \) of \(^{208}\)Pb \[20\], the constraint \( \Delta r_{np} = 0.15 \pm 0.03(\text{stat})^{+0.01}_{-0.01}(\text{syst}) \) fm extracted very recently from coherent pion photoproduction cross sections \[62\], and within the experimental error bar the constraint \( \Delta r_{np} = 0.33^{+0.16}_{-0.18} \text{ fm} \) extracted from the PREX at JLab \[63\].

Furthermore, we notice there exists a strong correlation between \( L(\rho_c) \) and the core-crust transition density \( \rho_t \) of neutron stars, which plays a crucial role in neutron star properties \[11\]. Using a dynamical approach \[\text{(See, e.g., Ref. \[64\]}\] with the optimized parameters as well as \( E_{\text{sym}}(\rho_0) = 26.65 \pm 0.20 \text{ MeV and } L(\rho_0) = 47.3 \pm 7.8 \text{ MeV} \), we obtain a value of \( \rho_t = 0.084 \pm 0.009 \text{ fm}^{-3} \), which agrees well with the empirical values \[1\].

### IV. SUMMARY AND OUTLOOK

In summary, we have demonstrated that the electric dipole polarizability \( \alpha_D \) in \(^{208}\)Pb is sensitive to both the magnitude \( E_{\text{sym}}(\rho_c) \) and density slope \( L(\rho_c) \) of the symmetry energy at a sub-saturation cross density \( \rho_c = 0.11 \text{ fm}^{-3} \), and it decreases (increases) with \( E_{\text{sym}}(\rho_c) \) (\( L(\rho_c) \)), leading to a positive correlation between \( L(\rho_c) \) and \( E_{\text{sym}}(\rho_c) \) for a fixed value of \( \alpha_D \) in \(^{208}\)Pb. Using the experimental value of \( \alpha_D \) in \(^{208}\)Pb measured at RCNP and the very well-constrained range of \( E_{\text{sym}}(\rho_c) \), we have obtained a strong constraint on the slope parameter \( L(\rho_c) = 47.3 \pm 7.8 \text{ MeV} \). This constraint is in surprisingly good agreement with the previous solely existing constraint \( L(\rho_c) = 46.0 \pm 4.5 \text{ MeV} \) from neutron skin data of Sn isotopes, demonstrating the robustness of these constraints on the value of the \( L(\rho_c) \) parameter.

The present constraint of \( L(\rho_c) \) further leads to \( E_{\text{sym}}(\rho_0) = 32.7 \pm 1.7 \text{ MeV and } L(\rho_0) = 47.1 \pm 17.7 \text{ MeV} \) for the symmetry energy at saturation density, the neutron skin thickness \( \Delta r_{np} = 0.176 \pm 0.027 \text{ fm for } ^{208}\)Pb, and \( \rho_t = 0.084 \pm 0.009 \text{ fm}^{-3} \) for the core-crust transition density of neutron stars. These results are nicely consistent with other constraints extracted from terrestrial experiments, astrophysical observations, and theoretical calculations with controlled uncertainties.

Our present results are based on the standard SHF energy density functional. It will be interesting to see how the results change if different energy-density functionals, e.g., the relativistic mean field model or the extended non-standard SHF energy density functional, are applied. These works are in progress and will be reported elsewhere.

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