Minimum-cost integer circulations in given homology classes

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Abstract. Let $D$ be a directed graph cellularly embedded on a surface together with costs and capacities on its arcs. Given any integer circulation in $D$, we study the problem of finding a minimum-cost integer circulation in $D$ that is homologous (over the integers) to the given circulation and respects the capacities. It has been recently shown that the stable set problem for graphs with bounded genus and bounded odd cycle packing number can be efficiently reduced to this problem, in which the surface is non-orientable.

For orientable surfaces, polynomial-time algorithms have been obtained for different variants of this problem. We complement these findings by showing that the convex hull of feasible solutions has a very simple polyhedral description.

In contrast, only little seems to be known about the case of non-orientable surfaces. We show that the problem is strongly NP-hard in this case. For surfaces of fixed genus, we obtain that the problem can be recast as an integer linear program with a coefficient matrix of bounded subdeterminants, which yields a polynomial-time algorithm for the projective plane. Moreover, we describe a pseudo-polynomial time algorithm for the case in which the surface has fixed genus and the circulation is only restricted to be non-negative.

Keywords: minimum-cost circulations · surfaces · homology classes.

1 Introduction

Finding optimal subgraphs of a surface-embedded graph that satisfy certain topological properties is a basic subject in topological graph theory and an important ingredient in many algorithms, see, e.g., [5, §1]. Motivated by recent work on the stable set problem [3], we study a variant of the minimum-cost circulation problem with such an additional topological constraint. While the standard minimum-cost circulation problem is among the most-studied problems in combinatorial optimization, much less seems to be known about the following version.

Problem 1. Given a directed graph $D$ cellularly embedded on a surface together with costs and (lower and upper) capacities on (some) arcs, and any integer circulation $y$ in $D$, find a minimum-cost integer circulation in $D$ that is $\mathbb{Z}$-homologous to $y$ and respects the capacities.
Here, a circulation \( x \) is said to be \( \mathbb{Z} \)-homologous to \( y \) if their difference \( x - y \) is a linear combination of facial circulations with integer coefficients, where a facial circulation is a circulation that sends one unit along the boundary of a single face, see Figure 1. If \( x - y \) is a linear combination of facial circulations with real coefficients, we say that \( x \) is \( \mathbb{R} \)-homologous to \( y \). We will provide more formal definitions and explanations later. In recent work of Conforti, Fiorini, Joret, Huynh, and the second author [3] it was crucially exploited that the stable set problem for graphs with bounded genus and bounded odd cycle packing number can be efficiently reduced to the following special case of Problem 1.

**Problem 2.** Given a directed graph \( D \) cellarily embedded on a surface together with costs on its arcs and any integer circulation \( y \) in \( D \), find a minimum-cost non-negative integer circulation in \( D \) that is \( \mathbb{Z} \)-homologous to \( y \).

As an illustration, if the costs are non-negative and \( y \) is the all-zero circulation, then \( y \) itself is clearly an optimal solution to Problem 2. However, for general \( y \) the all-zero circulation might not be feasible. In fact, if the surface is different from the sphere and the projective plane, then there are actually infinitely many homology classes, and their characterization is a basic subject in algebraic topology.

![Fig. 1. The circulations that send one unit along the blue directed cycles on the torus are \( \mathbb{Z} \)-homologous. In fact, their difference is the sum of three facial circulations, which are depicted in orange.](image)

The aim of this work is to introduce this problem to the combinatorial optimization community, with a particular emphasis on its polyhedral investigation. On the one hand, we complement existing results for the case of orientable surfaces and show that the underlying polyhedra are actually easy to describe. On the other hand, we will derive new complexity-theoretic results for the case of a non-orientable surface, for which only little seems to be known, but which arise in recent connections to the stable set problem [3].

For the case of orientable surfaces, Chambers, Erickson, and Nayyeri [2] show that Problem 2 can be solved in polynomial time. Their approach is based on an exponential-size linear program that can be solved using the ellipsoid method in near-linear time, provided that the surface has small genus. However, it is not clear how to adapt their approach to the case of arbitrary capacities (and
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Dey, Hirani, and Krishnamoorthy [4] consider a variant of Problem 1 defined on simplicial complexes of arbitrary dimension, in which the (weighted) $l_1$-norm of a chain homologous to $y$ is to be minimized. For the case of an orientable surface, they derive a polynomial-time algorithm that is based on a linear program defined by a totally unimodular matrix. We complement these existing results by showing that the convex hull of feasible solutions to Problem 1 has a very simple polyhedral description.

To this end, note that Problem 1 asks for minimizing a linear objective over the convex hull of all integer circulations in $D$ that are $\mathbb{Z}$-homologous to $y$ and that respect some lower and upper capacities $\ell$ and $u$, respectively. We will denote this polyhedron by $P(D, \ell, u, y)$. Moreover, let $P(D, \ell, u)$ be the convex hull of integer circulations in $D$ that respect $\ell$ and $u$. Note that any integer circulation $x$ that is $\mathbb{Z}$-homologous to $y$ must be also $\mathbb{R}$-homologous to $y$. In other words, $x$ must be contained in the affine subspace of all circulations that are $\mathbb{R}$-homologous to $y$, which we denote by $L(D, y)$. Surprisingly, it turns out that it suffices to add the equations defining $L(D, y)$ to a description of $P(D, \ell, u)$ to obtain one for $P(D, \ell, u, y)$.

**Theorem 1.** Let $D$ be a graph that is cellulary embedded on an orientable surface together with capacities $\ell, u$ on its arcs, and let $y$ be any integer circulation in $D$. Then, we have $P(D, \ell, u, y) = P(D, \ell, u) \cap L(D, y)$.

We remark that the above description does not result in a totally unimodular system (unless the surface is a sphere). We will also provide an explicit description for $L(D, y)$ later.

While the above results apply to orientable surfaces, the surfaces arising in [3] are non-orientable, and only little seems to be known about this case. As a first observation, it turns out that Theorem 1 does not hold for non-orientable surfaces. In fact, our second contribution is to show that Problem 1 becomes inherently more difficult in this case.

**Theorem 2.** Problem 2 (and hence also Problem 1) is strongly NP-hard on non-orientable surfaces.

While the authors in [4] show that variants of Problem 1 become NP-hard on general 3-dimensional simplicial complexes, their findings do not seem to apply to surfaces. We will derive the above result by exploiting connections to the stable set problem drawn in [3] and the fact that planar stable set is NP-hard.

On the positive side, we show that Problem 1 becomes more tractable when dealing with (non-orientable) surfaces of small genus. Let us say that an integer linear program $\max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n\}$ with $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ has sub-determinant $\delta$ if the largest absolute value of the determinant of any square submatrix of $A$ is equal to $\delta$. It has been recently shown by Artmann, Weismantel, and Zenklusen [1] that such problems can be solved in polynomial-time for $\delta \leq 2$, and it is an open question whether they are polynomially-solvable whenever $\delta$ is a constant. We show that new instances for the latter question can be obtained from Problem 1 on surfaces of fixed genus.
Theorem 3. Problem 1 can be formulated as an integer linear program with sub-determinant $\delta = 2^g$, where $g$ is the Euler genus of the surface.

Together with the algorithm in [1] this implies a polynomial-time algorithm for the projective plane, which has Euler genus 1 and is the simplest non-orientable surface.

Corollary 1. On the projective plane, Problem 1 can be solved in polynomial time.

Finally, we provide an algorithm for Problem 2 for non-orientable surfaces of fixed genus whose running time depends polynomially on the binary encoding of the input data, but also on the absolute values of the entries in $y$.

Theorem 4. Problem 2 can be solved in pseudo-polynomial time on general surfaces of fixed genus.

We remark that our algorithm is a slight extension of the algorithm given in [3], which was originally designed for instances of Problem 1 that arise from specific instances of stable set problems.

We leave it as an open problem to find a proper polynomial-time algorithm for Problem 1 (or at least Problem 2) on non-orientable surfaces of fixed genus. Moreover, in view of Theorem 1 and Corollary 1, it is natural to ask about a complete inequality description of $P(D, \ell, u, y)$ in the case of the projective plane, which we are not aware of.

Outline In Section 2 we provide a simple introduction to surfaces, graph embeddings, and homology. For more background and details, we refer to the books of Mohar & Thomassen [8] and Hatcher [7].

2 Surfaces and homology

We begin our paper by providing a brief introduction to surfaces, graph embeddings, and homology. We will also provide some first insights on a formulation of Problem 1 that will serve as a basis for the subsequent sections. The case of orientable surfaces is then discussed in Section 3, in which we also provide a proof of Theorem 1. The case of non-orientable surfaces is treated in Section 4. The proofs of Theorems 2, 3, and 4 can be found in Sections 4.1, 4.2, and 4.3, respectively.

2.1 Graphs embedded on surfaces

A surface is a non-empty connected compact Hausdorff topological space in which each point has an open neighborhood homeomorphic to the open unit disc in the plane. We stress that these surfaces have no boundary. Examples of such surfaces are the sphere, the torus, and the projective plane. While the first two are orientable surfaces, the latter one is non-orientable. Up to homeomorphism,
each surface $S$ can be characterized by a single non-negative integer called the
Euler genus $g$ of $S$. If $S$ is orientable, then $g$ is even and $S$ can be obtained
from the sphere by deleting $g/2$ pairs of open discs and, for each pair, identifying
their boundaries in opposite directions ("glueing handles"). Otherwise, $S$ is non-
orientable and can be obtained from the sphere by deleting $g \geq 1$ open discs
and, for each disc, identifying the antipodal points on its boundary ("glueing
Möbius bands"), see Figure 2 for an illustration.

![Diagram](image.png)

**Fig. 2.** A graph embedded on the Klein bottle, the non-orientable surface of Euler
genus 2. On the left, the surface is embedded in 3-dimensional space. Recall that the
Klein bottle is obtained from the sphere by deleting two open discs and, for each disc,
identifying the antipodal points on its boundary. On the right, an equivalent embedding
of the same graph is shown, where these discs are depicted in gray.

In this work, we consider (undirected and directed) graphs $G = (V, E)$ em-
bedded in a surface (with non-crossing edges). We will require that every face
of the embedding is homeomorphic to an open disc, which is called a cellular
embedding. An embedding is cellular if the graph is embedded in a surface of
minimum genus. Regardless of the (global) orientability of a surface, one can
define a local orientation around each node $v$ of $G$. If the surface is orientable,
these local orientations can be chosen in a way that they are consistent along each
dge. In non-orientable surfaces this is not possible. To keep track of these inconsist-
encies, any cellular embedding can be represented by an embedding scheme $\Pi = (\pi, \lambda)$, where $\pi$ denotes the rotation system describing, for all nodes, a
cyclic permutation of the edges around a node induced by the local orientation,
and the signature $\lambda \in \{-1, +1\}^E$ indicates, for every edge, whether the two local
orientations (clockwise vs. anti-clockwise) of the adjacent nodes agree (+1) or
not \((-1)\), see Figure 3. If nothing else is stated, we assume that a graph is always given together with its embedding and embedding scheme.

A closed walk in \(G\) is called two-sided if the number of appearances of edges with negative signature is even, otherwise it is called one-sided. As an example, a two-sided cycle has a neighborhood that is homeomorphic to an annulus, while a one-sided cycle has a neighborhood that is homeomorphic to an open Möbius band.

For any graph with embedding scheme \(\Pi = (\pi, \lambda)\), let \(F\) be the set of \(\pi\)-facial walks, which contains one closed walk along each face boundary whose direction is determined by \(\Pi\). We observe that every edge is either contained twice in one \(\pi\)-facial walk or in exactly two \(\pi\)-facial walks. Moreover, the signature of an edge is positive if it is used in opposite direction in the two corresponding \(\pi\)-facial walks and negative if it appears with the same direction, as illustrated in Figure 3. The \(\pi\)-facial walks of an embedded digraph are walks in the underlying undirected graph. Now, for graphs with a cellular embedding Euler’s Formula states

\[ |V| - |E| + |F| = 2 - g. \]  

(1)

Given the cellular embedding of a graph \(G\), one can define a dual graph \(G^*\). Each node of \(G^*\) corresponds to a \(\pi\)-facial walk \(f\) of \(G\), and we will denote this dual node by \(f^*\). If a graph is directed, we define the dual graph as the dual graph of the underlying undirected graph. The dual edge of an arc \(a\) is then denoted by \(a^*\). As there is a one-to-one correspondence between dual edges and arcs in the digraph, the dual of the dual edge \(\{f^*, g^*\}\) is the unique arc \(a\) lying in the face boundary of the faces that correspond to the \(\pi\)-facial walks \(f\) and \(g\). Moreover, we assume that the dual graph is canonically embedded, which means that the traversing directions of the \(\pi\)-facial walks directly correspond to the dual rotation system and the dual signature corresponds to the signature on \(G\).

Let \(D = (V, A)\) be a digraph with underlying undirected graph \(G = (V, E)\) and dual \(G^*\). For any walk \(W = (v_1, e_1, v_2, e_2, \ldots, e_{\ell-1}, v_\ell)\) in \(G\), we define the corresponding characteristic flow \(\chi(W) \in \mathbb{Z}^A\) as follows. The characteristic flow is an assignment vector on the arcs of \(D\) indicating the total flow over the arcs when sending one unit of flow along the walk \(W\). This means \(\chi(W)((v, w))\), for \((v, w) \in A\), equals the number of appearances of the sub-sequence \((v, \{v, w\}, w)\) in \(W\) minus the number of appearances of \((w, \{v, w\}, v)\). We observe that \(\chi(W)\) is a circulation, if \(W\) is a closed walk. We call \(\chi(f)\), for a \(\pi\)-facial walk \(f\), a facial circulation.

Finally, for a walk \(H\) in the dual graph \(G^*\), we will consider the vector \(\xi(H) \in \mathbb{Z}^A\) defined as follows. Intuitively, we think of \(\xi(H)\) as a flow that sends one unit along the edges in \(H\). Whenever a unit is sent along a dual edge, we will account it for the corresponding arc in \(D\). However, the sign of this value will depend on the direction of this arc relative to the way we traverse \(H\). To make this formal, consider any arc \(a = (v, w) \in A\) and let \(f, g\) be any \(\pi\)-facial walks of \(G\). We set \(s(a; f, g) \in \{-1, 0, 1\}\) to be non-zero in the case that \(f \neq g\) and the edge \(\{v, w\}\) appears in both \(f\) and \(g\), otherwise we set \(s(a; f, g) = 0\). If \(f\) traverses
the edge from \( v \) to \( w \), we set \( s(a; f, g) = 1 \), otherwise \( s(a; f, g) = -1 \). For instance, in Figure 3 \( s((v, w); f, g) = 1 \). Now, for a walk \( H = (f_1^*, e_1^*, f_2^*, \ldots, e_{\ell-1}^*, f_\ell^*) \) in \( G^* \) and an arc \( a \in A \) we define

\[
\xi(H)(a) := \sum_{i=1}^{\ell-1} \lambda(e_1) \cdots \lambda(e_{i-1}) s(a; f_i, f_{i+1}).
\]

We observe that \( \langle z, \xi(v^*) \rangle = 0 \), for any circulation \( z \) in \( D \) and any \( \pi \)-facial walk \( v^* \) in \( G^* \).

![Fig. 3. An extract of an embedded graph. The embedding scheme is depicted in green: green arrows around the nodes indicate the local orientations, and the numbers on the edges give the induced signature. The facial walks are drawn in blue.](image)

### 2.2 Homology

We are ready to give a more formal definition of being \( \mathbb{R} \)-homologous and \( \mathbb{Z} \)-homologous, respectively. Given a directed graph \( D = (V, A) \) cellularly embedded in \( S \), we call a circulation \( z \in \mathbb{R}^4 \) a boundary circulation if \( z \) is a linear combination of facial circulations. That is, there exists an assignment vector \( \alpha \in \mathbb{R}^F \) with an entry \( \alpha_f \in \mathbb{R} \), for each facial walk \( f \), such that \( z = \sum_{f \in F} \alpha_f \chi(f) \). Two circulations \( x, y \in \mathbb{R}^4 \) are said to be \( \mathbb{R} \)-homologous if \( x - y \) is a boundary circulation. This relation is an equivalence relation, and equipped with the standard addition this results in a group that (up to isomorphism) only depends on \( S \), and is known as the first homology group over the reals of \( S \).

In order to describe the set of circulations that are \( \mathbb{R} \)-homologous to a given circulation, it is useful to consider the boundary matrix \( \partial = \partial_D \in \mathbb{Z}^{A \times F} \) defined via

\[
\partial_{a,f} := \chi(f)(a) \quad \text{for all} \ a \in A, \ f \in F.
\]
With this notation, circulations $x, y$ are $\mathbb{R}$-homologous if we can write
\[ x = y + \partial \alpha, \]
for some $\alpha \in \mathbb{R}^F$. If $x$ and $y$ are both integer circulations that are $\mathbb{R}$-homologous, it may happen that each entry of $\alpha$ is an integer, i.e., $\alpha \in \mathbb{Z}^F$. In this case, we call $x$ and $y$ to be $\mathbb{Z}$-homologous. Again, this notion yields an equivalence relation that, together with the addition, forms the first homology group over the integers of $\mathbb{S}$.

Using the boundary matrix, we can formulate Problem 1 as the integer linear program
\[
\min \{ c^\top x : x = y + \partial \alpha, \ell \leq x \leq u, x \in \mathbb{Z}^A, \alpha \in \mathbb{Z}^F \},
\]
where $\ell, u \in (\mathbb{Z} \cup \{-\infty, \infty\})^A$ represent the lower and upper capacities, and $c$ the costs.

### 3 The orientable case

It is a basic fact that for any digraph $D$ and capacities $\ell, u \in (\mathbb{Z} \cup \{-\infty, \infty\})^A$, the convex hull of integer circulations in $D$ that respect $\ell$ and $u$, which we denote by $P(D, \ell, u)$, is actually equal to the set of all circulations in $D$ that respect $\ell$ and $u$. Thus, this polyhedron can be simply described as the set of all $x \in [\ell, u]^A$ that further satisfy the “flow conservation” constraints.

However, regarding Problem 1 we are interested in the convex hull of only those integer circulations in $P(D, \ell, u)$ that are $\mathbb{Z}$-homologous to a given integer circulation $y$, and we denote the respective polyhedron by $P(D, \ell, u, y)$. As we will see in the next section, a description of this set might be complicated for general surfaces. The purpose of this section is to show that a description of $P(D, \ell, u, y)$ can be easily obtained in the orientable case.

To this end, recall that if the graph $D$ is embedded in an orientable surface, we can choose the local orientations around the nodes in a way that they are consistent along each edge. This implies that, for each arc $a \in A$, there exists exactly one $\pi$-facial walk that traverses $a$ in its direction, and exactly one $\pi$-facial walk that traverses $a$ in the opposite direction. This means that for each arc $a$, the corresponding row in the boundary matrix $\partial$ is either all-zero, or contains exactly one $+1$ and one $-1$. Such matrices are well-known to be totally unimodular [11, §19.3] and hence we obtain the following observation, which has been a key ingredient in the work of [4].

**Lemma 1.** If $D$ is a digraph cellularly embedded in an orientable surface, then $\partial_D$ is totally unimodular.

Clearly, this shows that the integer linear program in (2), and hence also Problem 1, can be solved in polynomial-time (if the surface is orientable). Here, we would like to elaborate on another consequence concerning the description of $P(D, \ell, u, y)$. By the discussion in the previous section, we already know that
\[
P(D, \ell, u, y) = \text{conv} \{ x \in \mathbb{Z}^A : x = y + \partial \alpha, \ell \leq x \leq u, \alpha \in \mathbb{Z}^F \}
\]
\[
= \text{conv} \{ x \in \mathbb{R}^A : x = y + \partial \alpha, \ell \leq x \leq u, \alpha \in \mathbb{R}^F \},
\]
where the second equality follows from the integrality of the latter polyhedron, which is a consequence of Lemma 1. This means that \( P(D, \ell, u, y) \) is the set of all circulations in \( D \) that are \( \mathbb{R} \)-homologous to \( y \) and respect the capacities \( \ell, u \). Thus, denoting by \( L(D, y) \) the set of all circulations in \( D \) that are \( \mathbb{R} \)-homologous to \( y \), we obtain

\[
P(D, \ell, u, y) = P(D, \ell, u) \cap L(D, y),
\]

and hence Theorem 1. To obtain an even more explicit description of \( P(D, \ell, u, y) \), observe that \( \chi(C) \) formula yields that we need \( g \) additional constraints to obtain \( L(D, y) \).

These constraints can be obtained by the following construction, which can be already found in [2]. Pick any spanning tree \( F \) in \( G \) and observe that \( G^* \setminus F^* \) is still connected. Hence, there exists a spanning tree \( T \) in \( G^* \setminus F^* \). By Euler’s formula, there exist exactly \( g \) edges \( e_1, \ldots, e_g \) in \( G \) that are not contained in \( F \) and whose dual edges \( e_1^*, \ldots, e_g^* \) are not contained in \( T \). For each \( i \in [g] \), we define the cycle \( C_i \) as the unique (dual) cycle in \( T \cup \{e_i^* \} \). These \( g \) cycles will yield the additional constraints needed to describe \( L(D, y) \).

**Proposition 1.** Let \( D \) be a digraph cellularly embedded in an orientable surface, \( y \) an integer circulation in \( D \), and let \( C_1, \ldots, C_g \) be the (dual) cycles defined above. Then,

\[
L(D, y) = \left\{ x \in \mathbb{R}^A : x \text{ is a circulation and } \langle x, \xi(C_i) \rangle = \langle y, \xi(C_i) \rangle \forall i \in [g] \right\}.
\]

**Proof.** We may assume that \( y = 0 \). Let \( L \) denote the linear subspace on the right-hand side and let us first show that \( L(D, 0) \subseteq L \). Every \( x \in L(D, 0) \) is of the form \( x = \sum_{f \in F} \alpha_f \chi(f) \), for appropriate coefficients \( \alpha_f \in \mathbb{R} \). For every cycle \( H = (f_1^*, f_2^*, \ldots, f_k^*) \) in the dual graph \( G^* \), we have \( \langle x, \xi(H) \rangle = \sum_{i=1}^k \alpha_{f_i} - \alpha_{f_{i+1}} = 0 \), where \( f_{k+1} = f_1 \). This shows that \( x \in L \), and hence \( L(D, 0) \subseteq L \).

It remains to show that \( \dim(L) \leq \dim(L(D, 0)) \). To this end, recall that \( \dim(L(D, 0)) = |F| - 1 \) and that the space of all circulations has dimension \(|F| - 1 + g \). We claim that with each constraint \( \langle x, \xi(C_i) \rangle = 0 \) that we iteratively add to the space of all circulations, the dimension drops by one. Indeed, for each \( i \in [g] \) there is a (unique) cycle \( H_i \) in \( F \cup \{e_i \} \), and for this \( H_i \), the circulation \( \chi(H_i) \) satisfies all constraints \( \langle \chi(H), \xi(C_j) \rangle = 0 \), for \( j \neq i \), but \( \langle \chi(H_i), \xi(C_i) \rangle \neq 0 \). This means that \( \dim(L) \leq (|F| - 1 + g) - g = |F| - 1 = \dim(L(D, 0)) \).

**4 The non-orientable case**

Now, we turn to the case where the digraph is embedded in a non-orientable surface. In contrast to the orientable case, Problem 2 becomes difficult on general non-orientable surfaces. However, we will provide some evidence that the problem might be tractable on surfaces of fixed genus.
4.1 Hardness

In this section, we will show that the following problem is NP-hard.

**Problem 3.** Given a digraph \( D = (V, A) \) cellularly embedded in a non-orientable surface of genus \( g \leq |A| \) with arc costs \( c \in \{0, \frac{1}{2}, 1\}^A \) and an integer \( k \), decide whether there is a non-negative integer circulation in \( D \) that is \( \mathbb{Z} \)-homologous to the circulation \( 1 \in \mathbb{Z}^A \) and has cost less than \( k \).

Note that the above problem is a special case of (the decision version of) Problem 2, and hence a hardness result for the above problem will imply Theorem 2.

We will also see that the above problem remains hard if we restrict to circulations in \( \{0, 1\}^A \). In what follows, we will show that the planar stable set problem can be efficiently reduced to Problem 3. To this end, we will transform any instance of planar stable set to another “special” instance of a weighted stable set problem in which the graph is embedded in a very particular way, and then we will show that this problem is equivalent to Problem 3. Let us begin with the latter reduction.

**Reducing special instances of stable set to Problem 3.** In this part, we will exploit that if a graph is embedded in a non-orientable surface in a particular way, then there is a bijection between assignments of its nodes to integers (e.g., 0/1-values given by stable sets) and certain integer circulations in its dual graph. This connection was established in [3] and is described next. Suppose that \( G = (V, E) \) is a non-bipartite, 2-connected graph cellularly embedded in a non-orientable surface in a way that every closed walk of odd length is one-sided. (3)

For such an (arguably special) embedding, we can direct the edges of the dual graph to obtain a digraph \( D^* \) with arc set \( A \) whose facial walks are actually directed walks in \( D^* \). Equivalently, the arcs incident to each node alternate between incoming and outgoing in the order given by the rotation system. Consider the affine map \( \sigma : \mathbb{R}^V \to \mathbb{R}^E \) with \( \sigma(x) = 1 - Mx \), where \( M \) denotes the edge-node incidence matrix of \( G \). In other words, for each edge \( \{i, j\} \in E \), we have \( \sigma(x)\{i, j\} = 1 - x(i) - x(j) \). As there is a one-to-one correspondence between \( E \) and \( A \), \( \sigma \) may also be seen as a map from \( \mathbb{R}^V \) to \( \mathbb{R}^A \). With this identification, each vector \( x \in \mathbb{Z}^V \) is mapped to \( 1 - \sum_v x(v)\chi(v^*) \), where \( \chi(v^*) \) is the facial circulation associated to \( v^* \). Note that, due to the choice of \( D^* \), \( 1 \) is also a circulation. Thus, the image of \( \mathbb{Z}^V \) under \( \sigma \) is equal to the set of integer circulations in \( D^* \) that are \( \mathbb{Z} \)-homologous to \( 1 \). Moreover, a vector \( x \in \mathbb{Z}^V \) satisfies \( Mx \leq 1 \) if and only if its image under \( \sigma \) is non-negative. These observations are summarized in the next lemma. Formal proofs can be found in Proposition 14, Lemma 17, and Corollary 21 in [3].

**Lemma 2.** If \( G \) satisfies (3), then there exists an orientation of the dual graph, call this dual digraph \( D^* \), such that

\[
P(D^*, 0, \infty, 1) = \sigma(\text{conv}\{x \in \mathbb{Z}^V : Mx \leq 1\}).
\]
Recall that the weighted stable set problem essentially asks for minimizing a linear functional over \( \{ x \in \mathbb{Z}^V : Mx \leq 1, x \geq 0 \} \), and observe that the preimage of \( P(D^*, 0, \infty, 1) \) relates to this set by dropping the non-negativity constraints. While the non-negativity constraints cannot be dropped in general, it turns out that they are redundant in the case that the node weights \( w \in \mathbb{R}^V \) are edge-induced, that is, there exist edge costs \( c \in \mathbb{R}^E \geq 0 \) with \( \sum_{e \in \delta(v)} c_e = w_v \), for all \( v \in V \). (4)

The following fact is a summary of Proposition 12 and Observation 13 in [3].

**Lemma 3.** Let \( G \) be a graph as in Lemma 2 with node weights \( w \) that are edge-induced by \( c \). Then, the following holds true. First, \[
\max\{ w^\top x : Mx \leq 1, x \in \{0, 1\}^V \} = \max\{ w^\top x : Mx \leq 1, x \in \mathbb{Z}^V \}.
\]
Second, for any \( x \in \mathbb{Z}^V \) with \( Mx \leq 1 \) and \( y = \sigma(x) \) it holds that \[
w^\top x = c^\top 1 - c^\top y.
\]

Because of the one-to-one correspondence between \( E \) and \( A \), the edge costs \( c \) may also be seen as costs on the dual arcs \( A \). Now, the main implication of Lemma 2 and Lemma 3 is the following. For any instance of the stable set problem that satisfies (3) and (4), we always have
\[
\max\{ w^\top x : Mx \leq 1, x \in \{0, 1\}^V \} = -\min\{ c^\top y : y \in P(D^*, 0, \infty, 1) \} + c^\top 1,
\]
where \( D^* \) is the corresponding directed dual graph and \( M \) the edge-node incidence matrix. In this case, we will even see that we can choose \( c \) to be a vector in \( \{0, \frac{1}{2}, 1\}^A \). This means that, for any instance satisfying (3) and (4), the weighted stable set problem can be reduced to Problem 3.

**From planar stable set to special instances.** In the following, we will explain how a general instance of planar stable set can be reduced to a stable set problem satisfying (3) and (4) (with \( c \in \{0, \frac{1}{2}, 1\}^E \)). To this end, recall the planar stable set problem, which is known to be \( \text{NP-hard} \) [6].

**Problem 4.** Given a graph \( G \) embedded in the sphere and an integer \( k \). Decide whether there is a stable set in \( G \) with cardinality greater than \( k \).

First, let us consider Property (4). In the above problem, the weights \( w = 1 \) are not necessarily edge-induced. However, we can easily reduce to this case as shown by the following two statements. Given a graph \( G = (V, E) \), we say that a vector \( x \in \mathbb{R}^V \) is an optimal solution for the fractional stable set problem on \( G \) if it is an optimal solution for \( \max\{ 1^\top x : Mx \leq 1, x \in [0, 1]^V \} \), where \( M \) is again the edge-node incidence matrix of \( G \).

**Lemma 4.** If \( \frac{1}{2} 1 \) is an optimal solution for the fractional stable set problem, then \( 1 \) is edge-induced by some \( c \in \{0, \frac{1}{2}, 1\}^E \).
Proof. Note that $\frac{1}{2}1$ is also optimal with respect to $\max \{1^T x : Mx \leq 1 \}$ and hence, by strong LP duality, the vector 1 is contained in the convex cone generated by the rows of $M$. In other words, the set $F := \{y \in \mathbb{R}^E_{\geq 0} : \sum_{e \in \delta(v)} y(e) = 1, \text{ for all } v \in V \}$ is non-empty, and 1 is edge-induced by each point in $F$. Finally, observe that $F$ is the fractional perfect matching polytope, whose vertices are known to be in $\{0, \frac{1}{2}, 1\}^E$ [12, Thm. 30.2].

Lemma 5 (Nemhauser & Trotter [9,10]). In polynomial time, we can compute a partition of the node set $V$ into $V_0, V_{1/2}, V_1$ such that

- there is a stable set $S$ in $G$ of maximum cardinality that satisfies $S \cap (V_0 \cup V_1) = V_1$,
- the nodes in $V_{1/2}$ are not adjacent to nodes in $V_1$, and
- $\frac{1}{2}1$ is an optimal solution for the fractional stable set problem on the subgraph of $G$ induced by $V_{1/2}$.

Note that Lemma 5 states that, in order to find a stable set of maximum cardinality, we can first compute the above sets $V_0, V_{1/2}, V_1$, next compute a maximum stable set $S$ in the (planar) subgraph $G$ induced by $V_{1/2}$, and then return $S \cup V_1$.

Thus, by Lemma 4 we may assume that the vector 1 is edge-induced by some $c \in \{0, \frac{1}{2}, 1\}^E$. In particular, Property (4) is satisfied.

Next, let us consider Property (3). Clearly, we may assume that $G$ is non-bipartite (otherwise we can efficiently find a stable set of maximum cardinality) and connected (otherwise we treat each component separately). We may even assume that $G$ is 2-connected: If not, we can make it 2-connected by connecting its 2-connected components by paths of length 2 in a way that preserves planarity. In order to not affect the optimum value of a stable set, we set the weights to be zero on the additional nodes. Moreover, we extend the cost vector $c$ and set it to zero on the new edges. Observe, that with this extension the weights on the nodes stay edge-induced by some vector $c \in \{0, \frac{1}{2}, 1\}^E$. In particular, Property (3) is satisfied.

Since $G'$ is still embedded in the sphere, we may assume that the local orientations around the nodes are consistent along each edge, and hence every edge has signature +1. Next, we will modify the surface and the embedding in a way that all edges in $E$ become signature $-1$. For each original edge $e \in E$, we can choose a closed disc $C_e$ that only intersects $e$ and no other edge. By removing the interior of each $C_e$ and identifying the antipodal points of its boundary, we obtain an embedding of $G'$ in a non-orientable surface of genus $|E|$, see Figure 4.

We claim that the embedding is cellular. To check this, we observe that only two kinds of faces appear in the embedding. First, faces whose boundary consists
only of edges in $E' \setminus E$. These faces are homeomorphic to the respective faces in the embedding of $G'$ in the sphere, which hence are still homeomorphic to an open disc. Second, there are faces whose boundary consists of one edge in $E$ and the four new edges surrounding it. It is easy to check, that these faces are also homeomorphic to an open disc.

It remains to show that in this embedding, each closed walk of odd length is one-sided. To see this, observe that each edge in $E$ has now signature $-1$ and each edge in $E' \setminus E$ has still signature $+1$. Consider any closed walk $W$ of odd length. Since all edges in $E' \setminus E$ belong to the additional paths of length 2, it is clear that $W$ uses an even number of edges from $E' \setminus E$. Thus, since $W$ has odd length, it uses an odd number of edges from $E$. This means that $W$ uses an odd number of edges with negative signature, which means that $W$ is indeed one-sided.

### 4.2 Sub-determinants of the boundary matrix

In Section 3 we have seen that, for an orientable surface, the boundary matrix $\partial$ is always totally unimodular, see Lemma 1. In non-orientable surfaces this is no longer true, but we can give a bound on the absolute value of all sub-determinants (determinants of square submatrices) in terms of the Euler genus.

**Proposition 2.** If $D$ is a digraph cellularly embedded in a non-orientable surface of Euler genus $g$, then all sub-determinants of the corresponding boundary matrix $\partial_D$ are bounded by $2^g$ in absolute value.

In view of the integer linear program in (2), we thus see that Problem (1) can be formulated as an integer linear program with sub-determinant $2^g$, which proves Theorem 3.

**Proof of Proposition 2.** Let $A$ be any submatrix of $\partial$. Recall that the rows of $A$ correspond to arcs of $D$, which we identify with dual edges, and that the columns...
of $A$ correspond to $\pi$-facial walks, which we identify with dual nodes. Let us denote by $k(A)$ the maximum number of disjoint one-sided cycles (including loops) in $G^*$ consisting of only edges that correspond to the rows of $A$ and nodes that correspond to the columns of $A$.

We claim that $|\det(A)| \leq 2^{k(A)}$ holds for any square submatrix of $\partial$. Since a non-orientable surface of genus $g$ does not contain more than $g$ disjoint one-sided simple closed curves [8, Lem. 4.2.4], this yields $k(A) \leq g$ and hence $|\det(A)| \leq 2^g$.

We prove the claim by induction on the size of $A$. If $A$ consists of a single entry $a$, then the claim is trivially fulfilled if $a \in \{0, \pm 1\}$. Otherwise, we have $a \in \{\pm 2\}$ in which the edge corresponding to this row and the node corresponding to the column form a loop, which must be one-sided. In this case, we have $|\det(A)| = 2 = 2^{k(A)}$.

Now, suppose that $A$ is an $(n \times n)$-matrix with $n \geq 2$. First, let us consider the case in which $A$ consists of a column or row $z$ with at most one non-zero entry. If $z$ is all-zero, we have $\det(A) = 0$ in which case the claim is trivially fulfilled. If the non-zero entry in $z$ is $\pm 1$, we can expand along $z$ and obtain $|\det(A)| = |\det(A')|$ for some proper submatrix of $A$. The claim follows by induction since $k(A') \leq k(A)$. Otherwise, the non-zero entry in $z$ is $\pm 2$. Again, in this case the non-zero entry corresponds to a loop in $G^*$, which is one-sided. By expanding along the row containing this entry, we see that $|\det(A)| = 2|\det(A')|$, where $A'$ arises from $A$ by deleting the column and row corresponding to that entry. Observe that $k(A') \leq k(A) - 1$ and hence $|\det(A)| \leq 2 \cdot 2^{k(A')} \leq 2^{k(A)}$.

We are left with the case that each column and row in $A$ contains at least two non-zero entries. Since each row of $\partial$ contains at most two non-zero entries, we have that each row of $A$ contains exactly two non-zero entries. This implies that also every column contains exactly two non-zero entries. Moreover, all entries in $A$ are $0$ or $\pm 1$. We obtain that the columns and rows of $A$ form a subgraph of $G^*$ that is a disjoint union of cycles $C_1, \ldots, C_\ell$. Thus, we can write $A$ as a block matrix with blocks $A_1, \ldots, A_\ell$ where each $A_i$ corresponds to the cycle $C_i$. Note that $|\det(A)| = |\det(A_1)| \cdots |\det(A_\ell)|$ and that $k(A_1) + \cdots + k(A_\ell) \leq k(A)$. The claim now follows from the observation that if $B$ is a submatrix of $\partial$ that corresponds to a cycle $C$ in $G^*$, then $|\det(B)| = 2$ if $C$ is one-sided, and $|\det(B)| = 0$ if $C$ is two-sided, which we leave to the reader.

4.3 A pseudo-polynomial time algorithm

While we have shown that Problem 1 and Problem 2 are NP-hard on general non-orientable surfaces, our reduction does not rule out polynomial-time algorithms for instances on surfaces of fixed genus. Unfortunately, we are not aware of such algorithms, yet. As a partial result in that direction, let us briefly describe a pseudo-polynomial time algorithm for Problem 2 on non-orientable surfaces of fixed genus, which is based on the algorithm given in [3]. A more detailed description including proofs will be given in an extended version of this paper.

Recall that in the orientable case, two integer circulations are $\mathbb{Z}$-homologous if and only if they are $\mathbb{R}$-homologous. In the non-orientable case, $\mathbb{R}$-homology does
not imply $\mathbb{Z}$-homology. The next lemma, however, yields an additional simple criterion that finally yields $\mathbb{Z}$-homology.

**Lemma 6.** Let $D$ be a digraph cellularly embedded in a non-orientable surface, and let $C$ be a one-sided cycle in the dual graph $G^*$. Furthermore, let $x$ and $y$ be two integer circulations in $D$ that are $\mathbb{R}$-homologous. Then, $x$ is $\mathbb{Z}$-homologous to $y$, if and only if $\langle \xi(C), x \rangle - \langle \xi(C), y \rangle$ is even.

Besides this characterization of $\mathbb{Z}$-homology, it will be important to have an explicit description of $L(D, y)$ in the non-orientable case, analogous to Proposition \footnote{4} for the orientable case. We remark that in a non-orientable surface, the facial circulations span a space of dimension $|F|$. Therefore, again by Euler’s formula \footnote{1}, we need to add $g - 1$ constraints to the circulation constraints to obtain $L(D, y)$.

**Lemma 7.** Let $D$ be a digraph cellularly embedded in a non-orientable surface of genus $g$ and $y$ a circulation in $D$. Then there exist two-sided closed walks $W_1, \ldots, W_{g-1}$ in the dual graph $G^*$ such that

$$L(D, y) = \{ x \in \mathbb{R}^A : x \text{ is a circulation and } \langle x, \xi(W_i) \rangle = \langle y, \xi(W_i) \rangle \forall i \in [g-1] \}.$$

The construction of the walks $W_1, \ldots, W_{g-1}$ works in a similar fashion as for the orientable case, and can be done efficiently. Moreover, each such walk uses an edge in $G^*$ at most two times and hence the vectors $\xi(W_i)$ have entries in $\{0, \pm 1, \pm 2\}$.

For the sake of notation, let us define the function $h : \mathbb{Z}^A \to \mathbb{Z}_2 \times \mathbb{Z}^{g-1}$ via

$$h(z) := (\langle \xi(C), z \rangle \mod 2, \langle \xi(W_1), z \rangle, \langle \xi(W_2), z \rangle, \ldots, \langle \xi(W_{g-1}), z \rangle),$$

where $W_1, \ldots, W_{g-1}$ are closed walks as in Lemma \footnote{7} and $C$ any one-sided cycle in $G^*$. Now, Lemma \footnote{6} and Lemma \footnote{7} state that two integer circulations $x, y \in \mathbb{Z}^A$ are $\mathbb{Z}$-homologous if and only if $h(x) = h(y)$.

Another key ingredient is that every vertex of the polyhedron $P(D, 0, \infty, y)$ can be decomposed in a particular way.

**Lemma 8.** Every vertex $\bar{x}$ of $P(D, 0, \infty, y)$ can be written as $\bar{x} = \chi(H_1) + \chi(H_2) + \cdots + \chi(H_\ell)$, where $\ell \leq O(g)$ and $H_1, \ldots, H_\ell$ are directed closed walks in $D$ of length at most $\|y\|_1 \cdot |A|^{O(g)}$.

While the main part of the above statement is identical to Lemma 22 in \footnote{3}, the bound on the length of the walks can be obtained by bounds on the maximum entries of vertices of $P(D, 0, \infty, y)$. Note that, since $h$ is linear, we must have $h(\chi(H_1)) + \cdots + h(\chi(H_\ell)) = h(y)$ in the above lemma. Note also that the bound on the length of each $H_i$ implies a bound on the entries of $h(\chi(H_i))$.

These observations allow to guess the vectors $h_1 = h(\chi(H_1)), \ldots, h_\ell = h(\chi(H_\ell))$ in pseudo-polynomial time. For each such guess, it remains to compute a closed directed walk $H_1, \ldots, H_\ell$ with $h(\chi(H_i)) = h_i$ whose length is bounded by $\|y\|_1 |A|^{O(g)}$ such that the cost of each $\chi(H_i)$ is minimum possible. This task can be performed by a sequence of shortest path computations in an auxiliary graph that keeps track of the intermediate values of $h(\cdot)$ while traversing a walk, see \footnote{3}.
Acknowledgements We would like to thank Ulrich Bauer and Joseph Paat for valuable discussions on this topic.

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