Explicit numerical approximation for logistic models with regime switching in finite and infinite horizons

Xiaoyue Li*, Hongfu Yang†

Abstract

The stochastic logistic model with regime switching is an important model in the ecosystem. While analytic solution to this model is positive, current numerical methods are unable to preserve such boundaries in the approximation. So, proposing appropriate numerical method for solving this model which preserves positivity and dynamical behaviors of the model’s solution is very important. In this paper, we present a positivity preserving truncated Euler-Maruyama scheme for this model, which taking advantages of being explicit and easily implementable. Without additional restriction conditions, strong convergence of the numerical algorithm is studied, and 1/2 order convergence rate is obtained. In the particular case of this model without switching the first or- der strong convergence rate is obtained. Furthermore, the approximation of long-time dynamical properties is realized, including the stochastic permanence, extinctive and stability in distribution. Some simulations and examples are provided to confirm the theoretical results and demonstrate the validity of the approach.

Keywords. Stochastic logistic model, Markov chain, Explicit scheme, Strong convergence, Stochastic permanence, Stationary distribution

2000 MR Subject Classification. 60H10.

1 Introduction

In this manuscript, we consider the numerical approximation of stochastic logistic model with environmental fluctuations described by the following switching diffusion system (SDS)

\[ dx(t) = x(t)\left[ (b(r(t)) - a(r(t))x(t))dt + \sigma(r(t))dB(t) \right] \]

(1.1)
with an initial value \( x(0) = x_0 \in \mathbb{R}_+ := (0, +\infty) \), \( r(0) = \ell \in \mathbb{S} := \{1, 2, \ldots, m\} \), \( x(t) \) is the population size at time \( t \). \( r(t) \) is a right-continuous Markov chain with finite state space \( \mathbb{S} \) and the generator \( \Gamma = (\gamma_{ij})_{m \times m} \) satisfying \( \gamma_{ij} \geq 0 \) for \( i \neq j \), \( \sum_{j \in \mathbb{S}} \gamma_{ij} = 0 \) for each \( i \in \mathbb{S} \). \( B(t) \) is a scalar standard Brownian motion, which is independent of \( r(t) \). \( b(i) \) and \( a(i) \) represents the intrinsic growth rate and the intraspecific competition coefficient in regime \( i \), respectively. \( \sigma^2(i) \) is a constant representing the intensity of the white noise in regime \( i \).

This model plays an important role in biomathematics applications, the dynamical behaviors of SDS (1.1) and its related stochastic models have been investigated recently in \[10, 14, 15, 17\]. It is known that a unique strong solution exists for SDS (1.1), and that non-negativity of the initial value is preserved, see e.g. \[16, 17\]. Once we attempt to simulate SDS (1.1) using classical discretization methods, see e.g. \[12\], we face three difficulties:

- In general, these methods do not preserve positivity and therefore are not well defined when directly applied to SDS (1.1).
- The drift term is not globally Lipschitz continuous and therefore standard assumptions required for strong convergence, see e.g. \[12\], do not hold.
- Despite the convergence analysis, how to approximate long-time behaviors of SDS (1.1) is significant and challenging.

Our primary objective is to construct easily implementable preserving positivity numerical solutions and prove that they converge to the true solution of the original SDS (1.1), moreover, realize the approximation of long-time dynamical properties including the stochastic permanence, extinctive and stability in distribution.

In recent years, a few Euler-Maruyama (EM) discretization schemes have been developed for diffusion systems and SDSs including the implicit EM method \[9, 12\], the tamed EM method \[11, 25, 27, 28\], the tamed Milstein method \[30\], the stopped EM method \[19\] and the truncated EM method \[18, 22\], to mention a few. In these EM methods, the approximation can potentially escape the domain of the exact solutions of systems. Consequently, in order to close the gap, a lot of effort has focused on deriving schemes staying in restricted domains for diffusion systems with non-Lipschitz continuous coefficients \[1, 2, 6, 8, 24\]. Several modified EM methods have been developed such as the implicit schemes \[8, 24\] and the explicit EM schemes \[6, 19\], in the context of mathematical finance, a thorough overview of these can be found in \[13\]. A now classical trick is to apply a suitable Lamperti transform in order to obtain diffusion systems with constant diffusion coefficient, thereby translating all the non-smoothness to the drift. In the context of non-globally Lipschitz coefficients, this idea, introduced by Alfonsi \[1\], was further exploited in \[2, 24\] to obtain strong convergence rates for implicit “Lamperti-Euler” schemes, in particular for the CIR and the Ait-Sahalia models,
and for scalar diffusion systems with one-sided Lipschitz continuous drift and constant diffusion coefficients [24]. Recently, in the context of ecology, Mao, Wei and Wiriyakraikul [23] have established a positive preserving truncated EM method for stochastic Lotka-Volterra competition model but without any convergence rate of the algorithm. Chen, Gan and Wang [7] have proposed the Lamperti smoothing truncation scheme that can preserve the domain of the original SDEs and proved a mean-square convergence rate of order one. These modified EM methods have shown their abilities to approximate the solutions of nonlinear diffusion systems. However, to the best of our knowledge, these modified EM methods still cannot handle the convergence of nonlinear SDS (1.1).

Motivated by Lamperti transform [1,24] and truncation approaches [18,22], our key idea is to transform the original SDS (1.1) using the Lamperti transformation into an new SDS, i.e., applying Itô’s formula to $y(t) = \log x(t)$ gives

$$
\begin{align*}
    dy(t) &= \left( \beta(r(t)) - a(r(t))e^{y(t)} \right)dt + \sigma(r(t))dB(t),
\end{align*}
$$

where $\beta(i) := b(i) - \frac{\sigma^2(i)}{2}$ for any $i \in S$. The transformed SDS (1.2) is then approximated by a truncation EM scheme and transforming back yields the preserving positivity numerical schemes for the original SDS (1.1), which has a computational cost of the same order as the classical EM scheme. This allows us to prove rate of convergence for the original SDS (1.1), and the numerical solutions keep the underlying excellent properties of the exact solution of SDS (1.1). Here, we extend that work in several ways:

- Constructing an easily implementable scheme to maintain the positive path of the exact solution for nonlinear SDS (1.1). The scheme shares some of the features of the truncation schemes family.

- The explicit EM approximate solution will converge to the exact solution with order 1/2 for nonlinear SDS (1.1).

- Considering the maximum error in the discretization points, we prove that the explicit EM scheme for the stochastic logistic models without regime switching strongly converges with rate one.

- Without extra restrictions the numerical solutions of the appropriate explicit scheme stay in step of dynamical properties with the exact solutions.

The rest of the paper is organized as follows. Section 2 gives some preliminary results on certain properties of the exact solutions. Section 3 constructs an explicit scheme, and optimal convergence rate is obtained. Section 4 focuses on the analyses the stochastic permanence and extinction of the SDS (1.1). The other explicit scheme is constructed preserving the
stochastic permanence and extinction. Section 5 analyses the stability of the SDS \((1.1)\) in distribution yielding an invariant measure \(\mu(\cdot \times \cdot)\), and explicit scheme preserving the stability in distribution and a numerical invariant measure, which tends to \(\mu(\cdot \times \cdot)\) as the step size tends to 0. Section 6 presents a couple of examples to illustrate our results. Section 7 reconstructs an explicit scheme, and yields the strong convergence with rate one. Some examples are given to illustrate the availability of this scheme.

2 Preliminaries

Suppose that both \(r(\cdot)\) and \(B(\cdot)\) are defined on the complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions (i.e., it is right continuous and \(\mathcal{F}_t\) contains all \(\mathbb{P}\)-null sets). For each \(i \in \mathbb{S}\), both \(a(i)\) and \(b(i)\) are nonnegative constants, \(\sigma(i)\) is a constant. Let \(\mathbb{E}\) denotes the expectation corresponding to \(\mathbb{P}\) and \(|\cdot|\) denote the Euclidean norm in \(\mathbb{R}\). The generator of \(\{r(t)\}_{t \geq 0}\) is denoted by \(\Gamma\), so that for a sufficiently small \(\delta > 0\),

\[
\mathbb{P}\{r(t + \delta) = j | r(t) = i\} = \begin{cases} 
\gamma_{ij}\delta + o(\delta), & \text{if } i \neq j, \\
1 + \gamma_{ii}\delta + o(\delta), & \text{if } i = j,
\end{cases}
\]

where \(o(\delta)\) satisfies \(\lim_{\delta \to 0} o(\delta)/\delta = 0\). Here \(\gamma_{ij} \geq 0\) is the transition rate from \(i\) to \(j\) if \(i \neq j\) while \(\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}\). It is well known that almost every sample path of \(r(t)\) is right-continuous step functions with a finite number of simple jumps in any finite interval of \(\mathbb{R}_+\) (cf. [21]). If \(\mathcal{D}\) is a set, its indicator function is denoted by \(I_\mathcal{D}\), namely \(I_\mathcal{D}(x) = 1\) if \(x \in \mathcal{D}\) and 0 otherwise. For convenience, we let \(C_u\) and \(C\) denote two generic positive real constants respectively, whose value may change in different appearances, where \(C_u\) is dependent on \(u\). And let \(\mathcal{N}(0, 1)\) denotes the standard normal distribution. For any \(c = (c(1), \ldots, c(m))(\text{or } = (c_1, \ldots, c_m))\), define \(\hat{c} = \min_{i \in \mathbb{S}} c(i), \check{c} = \max_{i \in \mathbb{S}} c(i)\) and \(|\hat{c}| = \max_{i \in \mathbb{S}} |c(i)|\). We state a useful lemma which can be found in [16].

Lemma 2.1 ([16]) There exists a unique continuous positive solution \(x(t)\) to SDS \((1.1)\) for any initial value \(x(0) = x_0 > 0\) and \(r(0) = \ell \in \mathbb{S}\), which is global and represented by

\[
x(t) = \frac{x_0 \exp \left\{ \int_0^t \beta(r(s))ds + \int_0^t \sigma(r(s))dB(s) \right\}}{1 + x_0 \int_0^t a(r(s)) \exp \left\{ \int_0^s \beta(r(u))du + \int_0^s \sigma(r(u))dB(u) \right\} ds}.
\]

By virtue of Lemma 2.1, for any \(p > 0\), the solution \(x(t)\) of SDS \((1.1)\) with any initial value \(x_0 \in \mathbb{R}_+, \ell \in \mathbb{S}\), satisfies

\[
\sup_{0 \leq t \leq T} \mathbb{E}[x^p(t)] \leq C_T, \quad \forall \ T > 0.
\]
Now, we give the boundedness of its inverse moment. The inverse moment plays an important role in the analysis of convergence rate for the numerical scheme.

**Lemma 2.2** For any $p > 0$, then we have

$$\sup_{0 \leq t \leq T} E \left[ x^{-p}(t) \right] \leq C_T, \quad \forall \ T > 0.$$  

**Proof.** By virtue of Lemma 2.1, the solution $x(t)$ with positive initial value will remain in $\mathbb{R}_+$ for all $t \geq 0$ with probability 1. Define $U(t) = x^{-1}(t)$ on $t \geq 0$, we derive from (1.1) that

$$dU(t) = \left[ - \left( \beta(r(t)) - \frac{\sigma^2(r(t))}{2} \right) U(t) + a(r(t)) \right] dt - \sigma(r(t))U(t)dB(t).$$

Define a Lyapunov function $V(u) = (1 + u)^p$ for any $p > 0$. Using the method of Lyapunov function analysis, we could obtain the required assertion. The left proof is rather standard and hence is omitted. ■

**Remark 2.1** By virtue of Lemma 2.2 and (2.1), for any $q \in \mathbb{R}$, then the transformed SDS (1.2) has the following exponential integrability property

$$\sup_{0 \leq t \leq T} E \left[ e^{qy(t)} \right] \leq C_T, \quad \forall \ T > 0.$$  

Obviously, we also have the following property

$$\sup_{0 \leq t \leq T} E \left[ |y(t)|^p \right] \leq \sup_{0 \leq t \leq T} E \left[ e^{p|y(t)|} \right] \leq C_T,$$  

for any $p > 0$ and $T > 0$.

### 3 Convergence rate

In this section, we aim to construct an easily implementable explicit scheme and show the rate of convergence. The rate is optimal similar to the standard results of the EM scheme for SDSs with globally Lipschitz coefficients, see [20, p.115]. Given a stepsze $\Delta > 0$ and let $t_k = k\Delta$, $r_k = r(t_k)$ for $k \geq 0$, and one-step transition probability matrix $P(\Delta) = (P_{ij}(\Delta))_{m \times m} = \exp(\Delta \Gamma)$. The discrete Markov chain $\{r_k, k = 0, 1, \ldots\}$ can be simulated by the techniques in [20, p.112]. Throughout the article, $C$ and $C_T$ are independent of $\Delta$ and $k$.

To define appropriate numerical solutions, we firstly propose an explicit scheme to approximate the exact solution of SDS (1.2). For any given stepsze $\Delta \in (0, 1)$, define a truncated EM scheme by

$$\begin{align*}
Z_0 &= \log x_0, \\
\tilde{Z}_{k+1} &= Z_k + (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + \sigma(r_k)\Delta B_k, \\
Z_{k+1} &= \tilde{Z}_{k+1} \wedge \log(K\Delta^{-\theta}),
\end{align*}$$  

(3.1)
for any integer $k \geq 0$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$, $K \geq x_0 \vee 1$ is a constant independent of the iteration order $k$ and the stepsize $\Delta$, we use the convention $\theta = +\infty$ if $\bar{a} = 0$ and $\theta \in (0,1/2]$ otherwise. Transforming back, i.e.

$$X_k = e^{Z_k}, \quad k = 0, 1, \ldots, \tag{3.2}$$

gives a strictly positive approximation of the original SDS (1.1). Obviously, we have

$$Z_k = \bar{Z}_k \wedge \log(K\Delta^{-\theta}) \leq \bar{Z}_k, \quad 0 < X_k \leq K\Delta^{-\theta}. \tag{3.3}$$

To proceed, we define $Z_\Delta(t)$ and $X_\Delta(t)$ by

$Z_\Delta(t) := Z_k, X_\Delta(t) := X_k, \forall t \in [t_k, t_{k+1})$.

**Remark 3.1** The $\bar{a} = 0$ implies that $a(i) \equiv 0$, then we have $\theta = +\infty$ and $K\Delta^{-\theta} \equiv +\infty$ for any $\Delta \in (0,1]$. Thus, $\bar{Z}_k = Z_k$ and (3.3) hold always for any $\Delta \in (0,1]$.

In order to study the rate of convergence of numerical solutions $\{X_k\}_{k \geq 0}$, we first give the following lemmas.

**Lemma 3.1** For any $p > 0$, the truncated EM scheme defined by (3.1) has the property that

$$\sup_{\Delta \in (0,1)} \sup_{0 \leq k \leq [T/\Delta]} \mathbb{E}[e^{pZ_k}] \leq \sup_{\Delta \in (0,1)} \sup_{0 \leq k \leq [T/\Delta]} \mathbb{E}[e^{p\bar{Z}_k}] \leq C_T, \forall T > 0,$$

where $[T/\Delta]$ represents the integer part of $T/\Delta$.

**Proof.** Since $a(i) \geq 0$ for any $i \in S$, we know that

$$Z_{k+1} \leq Z_k + \beta(r_k)\Delta + \sigma(r_k)\Delta B_k \leq Z_k + [\beta]^{+}\Delta + \sigma(r_k)\Delta B_k.$$

Then

$$\bar{Z}_k \leq \log x_0 + [\beta]^{+}k\Delta + \sum_{j=0}^{k-1} \sigma(r_j)\Delta B_j$$

for any integer $k \geq 1$. Thus, for any $p > 0$ we have

$$\mathbb{E}[e^{p\bar{Z}_k}] \leq \mathbb{E} \left[ \exp \left( pZ_0 + p[\beta]^{+}T + p \sum_{j=0}^{k-1} \sigma(r_j)\Delta B_j \right) \right] = x_0^p e^{p[T][\beta]^{+}} \mathbb{E} \left[ \exp \left( p \sum_{j=0}^{k-1} \sigma(r_j)\Delta B_j \right) \right]$$

$$= x_0^p e^{p[T][\beta]^{+}} \mathbb{E} \left[ \exp \left( \frac{p^2}{2} \sum_{j=0}^{k-1} \sigma^2(r_j)\Delta \right) \exp \left( - \frac{p^2}{2} \sum_{j=0}^{k-1} \sigma^2(r_j)\Delta + p \sum_{j=0}^{k-1} \sigma(r_j)\Delta B_j \right) \right]$$

$$\leq x_0^p \exp \left( p[\beta]^{+}T + \frac{p^2}{2}[\bar{\sigma}]^2 \right) \mathbb{E} \left[ \exp \left( - \frac{p^2}{2} \sum_{j=0}^{k-1} \sigma^2(r_j)\Delta + p \sum_{j=0}^{k-1} \sigma(r_j)\Delta B_j \right) \right]. \tag{3.4}$$
On the other hand,

\[
\exp \left( -\frac{p^2}{2} \sum_{j=0}^{k-1} \sigma^2(r_j) \Delta + p \sum_{j=0}^{k-1} \sigma(r_j) \Delta B_j \right) = \prod_{j=0}^{k-1} \Theta_j,
\]

where

\[
\Theta_j = \exp \left( -\frac{p^2}{2} \sigma^2(r_j) \Delta + p \sigma(r_j) \Delta B_j \right).
\]

Then we have

\[
\mathbb{E} \left[ \prod_{j=0}^{k-1} \Theta_j \right] = \mathbb{E} \left[ \prod_{j=0}^{k-1} \Theta_j \mid \mathcal{F}_{t_{k-1}} \right] = \mathbb{E} \left[ \prod_{j=0}^{k-2} \Theta_j \mathbb{E} \left( \Theta_{k-1} \mid \mathcal{F}_{t_{k-1}} \right) \right].
\]

Obviously,

\[
\mathbb{E}(\Theta_{k-1} \mid \mathcal{F}_{t_{k-1}}) = \mathbb{E} \left[ \sum_{i \in \mathcal{S}} I_{\{r_{k-1} = i\}} \Theta_{k-1} \mid \mathcal{F}_{t_{k-1}} \right] = \sum_{i \in \mathcal{S}} I_{\{r_{k-1} = i\}} \mathbb{E}(\Theta_{k-1} \mid \mathcal{F}_{t_{k-1}}),
\]

where

\[
\Theta_{k-1}^i = \exp \left( -\frac{p^2}{2} \sigma^2(i) \Delta + p \sigma(i) \Delta B_{k-1} \right), \quad i \in \mathcal{S}.
\]

Note that \( \Delta B_{k-1} = B(t_k) - B(t_{k-1}) \) is independent of \( \mathcal{F}_{t_{k-1}} \), by [20, Lemma 3.2, p. 104], we can derive that

\[
\mathbb{E}(\Theta_{k-1}^i \mid \mathcal{F}_{t_{k-1}}) = \exp \left( -\frac{p^2}{2} \sigma^2(i) \Delta \right) \mathbb{E} \left[ \exp \left( p \sigma(i) \Delta B_{k-1} \right) \right] = 1.
\]

Hence, \( \mathbb{E} \left[ \prod_{j=0}^{k-2} \Theta_j \right] = \mathbb{E} \left[ \prod_{j=0}^{k-1} \Theta_j \right] \). Repeating this procedure, we obtain that

\[
\mathbb{E} \left[ \exp \left( -\frac{p^2}{2} \sum_{j=0}^{k-1} \sigma^2(r_j) \Delta + p \sum_{j=0}^{k-1} \sigma(r_j) \Delta B_j \right) \right] = 1.
\]

The above equality together with [3.4] implies

\[
\mathbb{E}[e^{p\tilde{\beta}_k}] \leq x_0^p \exp \left( p|\tilde{\beta}| + T + \frac{p^2|\tilde{\sigma}|^2}{2} T \right).
\]

The proof is complete. \( \blacksquare \)

**Lemma 3.2** For any \( L \geq 0 \) and integer \( m \geq 0 \), we have

\[
\mathbb{E} \left[ e^{L|\Delta B_k|^2} \mid \Delta B_k \right] = \sqrt{\frac{2}{\pi}} \Gamma \left( \frac{m+1}{2} \right) (4\Delta)^{\frac{m}{2}}, \quad \mathbb{E} \left[ e^{L|\Delta B_k|^m} \mid \mathcal{F}_{t_{k}} \right] \leq C \Delta^{\frac{m}{2}},
\]

where \( \Gamma(\cdot) \) is the Gamma function.
**Lemma 3.3** For any integer \( m \geq 0 \) and due to \( \Delta B_k \sim N(0, \Delta) \), we deduce that
\[
\mathbb{E}[e^{\frac{\Delta B_k^2}{4\Delta}} \mid \Delta B_k^m] = \frac{2}{\sqrt{2\pi \Delta}} \int_0^{\infty} x^m e^{-\frac{x^2}{4\Delta}} dx = \frac{4\Delta}{\sqrt{2\pi \Delta}} \int_0^{\infty} (4\Delta)^{\frac{m-1}{2}} x^{\frac{m-1}{2}} e^{-x} dx
\]
which implies that
\[
\mathbb{E}\left[ \exp \left( L \mid \Delta B_k \right) \mid \Delta B_k^m \mid \mathcal{F}_k \right] \leq \mathbb{E}\left[ \exp \left( L^2 \Delta + \frac{\Delta B_k^2}{4\Delta} \right) \mid \Delta B_k^m \mid \mathcal{F}_k \right]
\]
\[
= e^{L^2 \Delta \mathbb{E}\left[ e^{\frac{\Delta B_k^2}{4\Delta}} \mid \Delta B_k^m \right]} \leq e^{L^2 \Delta \mathbb{E}\left[ e^{\frac{\Delta B_k^2}{4\Delta}} \mid \Delta B_k^{3m} \right]} \leq C \Delta^{\frac{m}{2}}
\]
The proof is complete.  

**Proof.** For any \( p > 0 \), the truncated EM scheme defined by (3.1) has the property that
\[
\sup_{\Delta \in (0,1)} \sup_{0 \leq k \leq \lceil T/\Delta \rceil} \mathbb{E}[e^{-p Z_k}] \leq C_T, \quad \forall \ T > 0.
\]

**Proof.** Using the Taylor formula, we obtain that
\[
e^{-Z_{k+1}} \leq e^{-Z_k} e^{-Z_k(\bar{Z}_{k+1} - Z_k)} + \frac{1}{2} e^{-Z_k(\bar{Z}_{k+1} - Z_k)^2} + \frac{1}{6} e^{-Z_k} e^{Z_{k+1} - Z_k} \mid \bar{Z}_{k+1} - Z_k \mid^3
\]
\[
\leq e^{-Z_k} \left[ 1 + (a(r_k)e^{Z_k} - \beta(r_k)) \Delta - \sigma(r_k) \Delta B_k + \frac{1}{2}(\bar{Z}_{k+1} - Z_k)^2
\right.
\]
\[
+ \frac{1}{6} \exp \left( (\bar{a}e^{Z_k} + |\bar{\beta}|) \Delta + |\bar{\sigma}| \| \Delta B_k \| \right) \left( (\bar{a}e^{Z_k} + |\bar{\beta}|) \Delta + |\bar{\sigma}| \| \Delta B_k \| \right)^3 \]
\[
\leq e^{-Z_k} \left[ 1 + (a(r_k)e^{Z_k} - \beta(r_k)) \Delta - \sigma(r_k) \Delta B_k + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2}
\right.
\]
\[
+ (\bar{a}K \Delta^{-\theta} + |\bar{\beta}|)^2 \Delta^2 - \sigma(r_k) \Delta B_k (a(r_k)e^{Z_k} - \beta(r_k)) \Delta
\]
\[
+ \frac{1}{6} \exp \left( (\bar{a}K \Delta^{-\theta} + |\bar{\beta}|) \Delta + |\bar{\sigma}| \| \Delta B_k \| \right) \left( (\bar{a}K \Delta^{-\theta} + |\bar{\beta}|) \Delta + |\bar{\sigma}| \| \Delta B_k \| \right)^3 \]
\[
\leq e^{-Z_k} \left[ 1 + (a(r_k)e^{Z_k} - \beta(r_k)) \Delta + C \Delta^{2(1-\theta)} + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} - \sigma(r_k) \Delta B_k
\right.
\]
\[
- \sigma(r_k) \Delta B_k (a(r_k)e^{Z_k} - \beta(r_k)) \Delta
\]
\[
+ \frac{1}{6} \exp \left( (\bar{a}K \vee |\bar{\beta}|) \Delta^{1-\theta} \exp \left( |\bar{\sigma}| \| \Delta B_k \| \right) \left( \Delta^{1-\theta} + |\Delta B_k| \right)^3 \right]
\]
\[
\leq e^{-Z_k} \left[ 1 + (a(r_k)e^{Z_k} - \beta(r_k)) \Delta + C \Delta^{2(1-\theta)} + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} - \sigma(r_k) \Delta B_k
\right.
\]
\[
- \sigma(r_k) \Delta B_k (a(r_k)e^{Z_k} - \beta(r_k)) \Delta + \mathcal{U}_k \right],
\]
where
\[ U_k = C \exp \left( |\tilde{\sigma}| |\Delta B_k| \right) \left( \Delta^{3(1-\theta)} + |\Delta B_k|^3 \right), \]
which implies that
\[ (1 + e^{-Z_{k+1}})^p \leq (1 + e^{-Z_k})^p (1 + \zeta_k)^p, \quad (3.6) \]
where
\[ \zeta_k = \frac{e^{-Z_k}}{1 + e^{-Z_k}} \left[ (a(r_k)e^{Z_k} - \beta(r_k))\Delta + C\Delta^{2(1-\theta)} + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} - \sigma(r_k)\Delta B_k \right. 
\[ \left. - \sigma(r_k)\Delta B_k (a(r_k)e^{Z_k} - \beta(r_k))\Delta + U_k \right], \]
and we can see that \( \zeta_k > -1 \). For the given constant \( p > 2 \), choose an integer \( m \) such that \( 2m < p \leq 2(m + 1) \). It follows from Lemma 3.3 and (4.21) that
\[ \mathbb{E}\left( (1 + e^{-Z_{k+1}})^p \right) \leq \left( 1 + e^{-Z_k} \right)^p \left( 1 + p\mathbb{E}[\zeta_k | F_{t_k}] + \frac{p(p-1)}{2} \mathbb{E}[\zeta_k^2 | F_{t_k}] + \mathbb{E}[P_m(\zeta_k)\zeta_k^3 | F_{t_k}] \right), \quad (3.7) \]
where \( P_m(x) \) represents a \( m \)th-order polynomial of \( x \) with coefficients depending only on \( p \), and \( m \) is an integer. Noticing that the increment \( \Delta B_k \) is independent of \( F_{t_k} \), we derive that
\[ \mathbb{E}\left( |\Delta B_k|^{2j} | F_{t_k} \right) = (2j - 1)!\Delta^j, \quad \mathbb{E}\left( |\Delta B_k|^{2j-1} | F_{t_k} \right) = 0, \quad j = 1, 2, \ldots \quad (3.8) \]
and using (3.3) and Lemma 3.2, we compute
\[ \mathbb{E}[\zeta_k | F_{t_k}] = \frac{e^{-Z_k}}{1 + e^{-Z_k}} \left[ (a(r_k)e^{Z_k} - \beta(r_k))\Delta + C\Delta^{2(1-\theta)} + \frac{\sigma^2(r_k)\Delta}{2} + \mathbb{E}[U_k | F_{t_k}] \right] \leq \frac{e^{-Z_k}}{1 + e^{-Z_k}} \left( a(r_k)e^{Z_k} \Delta + C\Delta \right) \leq C\Delta. \quad (3.9) \]
and
\[ \mathbb{E}[\zeta_k^2 | F_{t_k}] = \frac{e^{-2Z_k}}{(1 + e^{-Z_k})^2} \mathbb{E}\left[ (a(r_k)e^{Z_k} - \beta(r_k))\Delta + C\Delta^{2(1-\theta)} + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} \right. 
\[ \left. - \sigma(r_k)\Delta B_k \Delta B_k (a(r_k)e^{Z_k} - \beta(r_k))\Delta + U_k \right] \leq \frac{Ce^{-2Z_k}}{(1 + e^{-Z_k})^2} \left[ a^2(r_k)e^{2Z_k} \Delta^2 + \Delta^2 + \sigma^2(r_k)\Delta + \mathbb{E}[U_k^2 | F_{t_k}] \right] \leq C\Delta. \quad (3.10) \]
To estimate \( \mathbb{E}\left[ P_m(\zeta_k)\zeta_k^3 | F_{t_k} \right] \), we begin with \( \mathbb{E}[\zeta_k^3 | F_{t_k}] \). Using (3.3), (3.8) and Lemma 3.2 we obtain
\[ \mathbb{E}[\zeta_k^3 | F_{t_k}] = \frac{e^{-3Z_k}}{(1 + e^{-Z_k})^3} \mathbb{E}\left[ (a(r_k)e^{Z_k} - \beta(r_k))\Delta + C\Delta^{2(1-\theta)} + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} \right. 
\[ \left. - \sigma(r_k)\Delta B_k \Delta B_k (a(r_k)e^{Z_k} - \beta(r_k))\Delta + U_k \right] \leq \frac{Ce^{-3Z_k}}{(1 + e^{-Z_k})^3} \left[ a^3(r_k)e^{3Z_k} \Delta^3 + \Delta^3 + \sigma^2(r_k)\Delta + \mathbb{E}[U_k^3 | F_{t_k}] \right] \leq C\Delta. \]
\[ -\sigma(r_k)\Delta B_k - \sigma(r_k)\Delta B_k(a(r_k)e^{Z_k} - \beta(r_k))\Delta + U_k \triangleq F_{t_k} \]
\[ \leq \frac{C e^{-3Z_k}}{(1 + e^{-Z_k})^3} \left( a^3(r_k)e^{3Z_k} \Delta^3 + \Delta^3 + \mathbb{E}[U_k^3|F_{t_k}] \right) \leq C \Delta^\frac{3}{2}. \]

On the other hand, we can use the same method to derive that
\[ \mathbb{E}[\zeta_k^3|F_{t_k}] \geq -C \Delta^\frac{3}{2}. \]

Thus, both of the above inequalities imply \( \mathbb{E}[c_0\zeta_k^3|F_{t_k}] \leq o(\Delta) \) for any constant \( c_0 \), where \( c_j \) represents the coefficient of \( \zeta_j \) term in polynomial \( P_m(\zeta_k) \). We can also show that
\[ \mathbb{E}[|c_j\zeta_k^{3+j}||F_{t_k}] \leq o(\Delta) \]
for any \( j \geq 1 \). These imply
\[ \mathbb{E}[P_m(\zeta_k)\zeta_k^3|F_{t_k}] \leq o(\Delta). \]  

Combining (3.7), (3.9), (3.10) and (3.11), we obtain that
\[ \mathbb{E}\left[(1 + e^{-Z_k})^p|F_{t_k}\right] \leq (1 + e^{-Z_k})^p (1 + C\Delta) \]
for any integer \( 0 \leq k \leq \lfloor T/\Delta \rfloor \). Obviously,
\[ \mathbb{E}\left[(1 + e^{-Z_k})^p\right] \leq (1 + C\Delta) \mathbb{E}\left[(1 + e^{-Z_k})^p\right]. \]  

Define \( \Omega_k = \{ \bar{Z}_k > \log(K\Delta^{-\theta}) \} \). Using the Chebyshev inequality, we can see that
\[ \mathbb{E}\left[(1 + e^{-Z_k})^p\right] = \mathbb{E}\left[(1 + e^{-Z_k})^pI_{\Omega_k}\right] + \mathbb{E}\left[(1 + e^{-Z_k})^pI_{\Omega_k^c}\right] \]
\[ = \mathbb{E}\left[(1 + e^{-Z_k})^pI_{\Omega_k}\right] + \mathbb{E}\left[(1 + K^{-1}\Delta)^pI_{\Omega_k}\right] \]
\[ \leq \mathbb{E}\left[(1 + e^{-Z_k})^p\right] + 2^p \mathbb{P}\{ \bar{Z}_k > \log(K\Delta^{-\theta}) \} \]
\[ \leq (1 + C\Delta) \mathbb{E}\left[(1 + e^{-Z_{k-1}})^p\right] + 2^p \frac{\mathbb{E}e^{Z_k/\theta}}{K^{1/\theta} \Delta^{-1}}. \]

It follows from the result of Lemma 3.1 that
\[ \mathbb{E}\left[(1 + e^{-Z_k})^p\right] \leq (1 + C\Delta) \mathbb{E}\left[(1 + e^{-Z_{k-1}})^p\right] + C_T\Delta \]
\[ \leq (1 + C\Delta)^k (1 + x_0^{-1})^p + C_T\Delta \sum_{j=0}^{k-1} (1 + C\Delta)^j \]
\[ \leq e^{C k \Delta} (1 + x_0^{-1})^p + C_T k \Delta e^{C k \Delta} \leq e^{C_T} \left[(1 + x_0^{-1})^p + C_T\right]. \]

The proof is complete.
Remark 3.2 By virtue of Lemmas 3.1 and 3.3, for any \( q \in \mathbb{R} \), then the truncated EM scheme defined by (3.1) has the following exponential integrability property

\[
\sup_{\Delta \in (0,1)} \sup_{0 \leq k \leq \lfloor T/\Delta \rfloor} \mathbb{E}[e^{qZ_k}] \leq C_T, \quad \forall \ T > 0.
\]

Obviously, we also have the following property

\[
\sup_{\Delta \in (0,1)} \sup_{0 \leq k \leq \lfloor T/\Delta \rfloor} \mathbb{E}[|Z_k|^p] \leq \sup_{\Delta \in (0,1)} \sup_{0 \leq k \leq \lfloor T/\Delta \rfloor} \mathbb{E}[e^{p|Z_k|}] \leq C_T
\]

for any \( p > 0 \) and \( T > 0 \).

In order to show that the numerical scheme defined by (3.2) perform the dynamical behaviors of exact solutions perfectly, we further require the chosen \( \theta \in (0,1/2) \) if \( \bar{a} > 0 \). By (3.3), for any \( p > 0 \),

\[
e^{Z_k} \leq e^{\bar{Z}_k}, \quad \bar{Z}_k^p = [\bar{Z}_k \wedge \log(K\Delta^{-\theta})]^p \leq \bar{Z}_k^p, \quad X_k = e^{Z_k} \leq K\Delta^{-\theta}.
\]

Moreover, to study the rate of convergence of numerical solutions \( \{X_k\}_{k \geq 0} \), we also need to study the EM method to (1.2), which are defined as follows: For any given stepsize \( \Delta \in (0,1] \),

\[
\begin{aligned}
Y_0 &= \log x_0, \\
Y_{k+1} &= Y_k + (\beta(r_k) - a(r_k)e^{Y_k})\Delta + \sigma(r_k)\Delta B_k,
\end{aligned}
\]

for any integer \( k \geq 0 \), where \( \Delta B_k = B(t_{k+1}) - B(t_k) \). Transforming back, i.e.

\[
X_k^E = e^{Y_k}, \quad k = 0, 1, \ldots,
\]

gives a strictly positive approximation of the original SDS (1.1). In addition, we also need the following lemma, the proof of which can be found in Appendix A.

Lemma 3.4 The EM method defined by (3.14) has the property that

\[
\sup_{0 \leq k \leq \lfloor T/\Delta \rfloor} \mathbb{E}\left[ |Y_k - y(t_k)|^2 \right] \leq C_T \Delta
\]

for any \( T > 0 \) and \( \Delta \in (0,1] \).

Define \( \tau_L = \inf\{k\Delta \geq 0 : x(t_k) \geq e^L\} \), \( \rho_\Delta = \inf\{k\Delta \geq 0 : \bar{Z}_\Delta(t_k) \geq \log(K\Delta^{-\theta})\} \). By (2.1) we have

\[
e^{pL}P\{\tau_L \leq T\} \leq \mathbb{E}\left[ x^p(T \wedge \tau_L) I_{\{\tau_L \leq T\}} \right] + \mathbb{E}\left[ x^p(T \wedge \tau_L) I_{\{\tau_L > T\}} \right] = \mathbb{E}x^p(T \wedge \tau_L) \leq C_T.
\]

It is easy to see that

\[
P\{\tau_L \leq T\} \leq C_T e^{-pL}, \quad \forall \ p > 0,
\]

for any \( p > 0 \).
where $C_T$ is a positive constant independent of $L$. Moreover, by virtue of Lemma 3.1 we have

$$
\mathbb{E}\left[ \exp \left( p \log(K\Delta^{-\theta}) \right) I_{(\rho_{k\Delta} \leq T)} \right] \leq \mathbb{E}\left[ \exp \left( p\bar{Z}_{\Delta}(T \wedge \rho_{\Delta}) \right) \right] = \mathbb{E}\left[ \exp \left( p\bar{Z}_{\left[\frac{T}{\Delta}\rho_{\Delta}\right]} \right) \right]
$$

$$
\leq \sup_{0 \leq k \leq [T/\Delta]} \mathbb{E}\left[ \exp \left( p\bar{Z}_k \right) \right] \leq C_T,
$$

implies that

$$
\mathbb{P}\left\{ \rho_{\Delta} \leq T \right\} \leq C_T K^{-p}\Delta^{-\theta p}, \quad \forall \ p > 0.
$$

(3.18)

**Theorem 3.1** The truncated EM scheme defined by (3.1) has the property that

$$
\sup_{0 \leq k \leq [T/\Delta]} \mathbb{E}\left[ |Z_k - y(t_k)|^2 \right] \leq C_T \Delta
$$

for any $\Delta \in (0, 1)$ and $T > 0$.

**Proof.** Define $\bar{\theta}_\Delta = \tau_{\log(K\Delta^{-\theta}) \wedge \rho_{\Delta}}, \Omega_1 := \{ \omega : \bar{\theta}_\Delta > T \}$, $\bar{u}_k = Z_k - y(t_k)$, for any $k\Delta \in [0, T]$, where $\tau_k$ and $\rho_{\Delta}$ are defined by (3.17), and (3.18), respectively. For any $\bar{p} > 2$, using the Young inequality we obtain that

$$
\mathbb{E}|\bar{u}_k|^2 = \mathbb{E}\left( |\bar{u}_k|^2 I_{\Omega_1} \right) + \mathbb{E}\left( |\bar{u}_k|^2 I_{\Omega_1^c} \right)
$$

$$
\leq \mathbb{E}\left( |\bar{u}_k|^2 I_{\Omega_1} \right) + \frac{2\Delta}{\bar{p}} \mathbb{E}\left( |\bar{u}_k|^\bar{p} \right) + \frac{\bar{p} - 2}{\bar{p}\Delta^2/(\bar{p} - 2)} \mathbb{P}(\Omega_1^c).
$$

(3.19)

It follows from the results of (2.2) and (3.13) that

$$
\frac{2\Delta}{\bar{p}} \mathbb{E}\left( |\bar{u}_k|^\bar{p} \right) \leq C_T \Delta.
$$

(3.20)

It follows from (3.17), and (3.18) that

$$
\frac{\bar{p} - 2}{\bar{p}\Delta^2/(\bar{p} - 2)} \mathbb{P}(\Omega_1^c) \leq \frac{\bar{p} - q}{\bar{p}\Delta^2/(\bar{p} - 2)} \left( \mathbb{P}\left\{ \tau_{\log(K\Delta^{-\theta})} \leq T \right\} + \mathbb{P}\left\{ \rho_{\Delta} \leq T \right\} \right)
$$

$$
\leq \frac{2(\bar{p} - 2)}{\bar{p}\Delta^2/(\bar{p} - 2)} C_T \leq \frac{2C_T(\bar{p} - 2)}{K^\delta \Delta^{-\delta \theta}} \Delta^{-\frac{2}{\bar{p} - 2}} \leq C_T \Delta,
$$

(3.21)

where $\delta \geq \bar{p}/\theta(\bar{p} - 2)$. Inserting (3.20), (3.21) and (3.16) into (3.19) yields

$$
\mathbb{E}|\bar{u}_k|^2 \leq \mathbb{E}\left( |\bar{u}_k|^2 I_{\Omega_1} \right) + C_T \Delta \leq \sup_{0 \leq k \leq [\frac{T}{\Delta}]} \mathbb{E}\left[ |Z_k - y(t_k)|^2 \right] + C_T \Delta
$$

$$
= \sup_{0 \leq k \leq [\frac{T}{\Delta}]} \mathbb{E}\left[ |Y_k - y(t_k)|^2 \right] + C_T \Delta \leq C_T \Delta.
$$

The proof is complete. \[\blacksquare\]
As a consequence we also obtain the same convergence order for the approximation of the original SDS (1.1) by $X_k := e^{Z_k}$.

**Theorem 3.2** For any $0 < p < 1$, there exists a constant $C_T > 0$ such that

$$
\sup_{0 \leq k \leq \lfloor T/\Delta \rfloor} \mathbb{E} \left[ |x(t_k) - X_k|^{2p} \right] \leq C_T \Delta^p
$$

for any $\Delta \in (0, 1)$ and $T > 0$.

**Proof.** Using Hölder’s inequality, we have

$$
\mathbb{E} \left[ |x(t_k) - e^{Z_k}|^{2p} \right] = \mathbb{E} \left[ |e^{y(t_k)} - e^{Z_k}|^{2p} \right] \leq \mathbb{E} \left[ (e^{2py(t_k)} + e^{2pZ_k}) |y(t_k) - Z_k|^{2p} \right] \\
\leq \left[ \mathbb{E} \left( e^{2py(t_k)} + e^{2pZ_k} \right) \right]^{1/(1-p)} \left[ \mathbb{E} |y(t_k) - Z_k|^{2p} \right].
$$

Thus, by applying Theorem 3.1, Lemma 3.1 and (2.1), we infer that

$$
\mathbb{E} \left[ |x(t_k) - e^{Z_k}|^{2p} \right] \leq C_T \Delta^p
$$

for any $\Delta \in (0, 1)$. The proof is complete. \(\blacksquare\)

**Remark 3.3** As a consequence we also obtain the same convergence order for the approximation of the original SDS (1.1) by $X_k^E := e^{Y_k}$, where $Y_k$ is defined by (3.14). Again, for numerical solutions $\{X_k^E\}_{k \geq 0}$, by applying Lemmas 3.4 and A.1 for any $0 < p < 1$ we have

$$
\sup_{k=0,\ldots,\lfloor T/\Delta \rfloor} \mathbb{E} \left[ |x(t_k) - X_k^E|^{2p} \right] \leq C_T \Delta^p
$$

for any $\Delta \in (0, 1]$ and $T > 0$.

## 4 Stochastic permanence

In this section, we focus on the stochastic permanence and extinction. Firstly, we establish the criterion on the stochastic permanence and extinction of the exact solution of SDS (1.1). Then we show that the numerical scheme defined by (3.2) keep this property very well. From this section we always assume $r(t)$ is irreducible, namely, the following linear equation

$$
\pi \Gamma = 0, \quad \sum_{i=1}^{m} \pi_i = 1,
$$

(4.1)
has a unique solution \( \pi = (\pi_1, \ldots, \pi_m) \in \mathbb{R}^{1 \times m} \) satisfying \( \pi_i > 0 \) for each \( i \in \mathcal{S} \). This solution is termed a stationary distribution. Then the rank of \( \Gamma \) is \( m - 1 \). It follows that null space of \( \Gamma \) is one dimensional spanned by \( \mathbb{I}_m := (1, \cdots, 1)^T \in \mathbb{R}^m \). Consider the linear equation

\[
\Gamma \nu = \xi, \tag{4.2}
\]

where \( \nu \) and \( \xi \in \mathbb{R}^m \).

**Lemma 4.1 (32)** The following assertions hold.

1. Equation \( (4.2) \) has a solution if and only if \( \pi \xi = 0 \).
2. Suppose that \( \nu_1 \) and \( \nu_2 \) are two solutions of \( (4.2) \). Then \( \nu_1 - \nu_2 = \kappa \mathbb{I}_m \) for some \( \kappa \in \mathbb{R} \).
3. Any solution of \( (4.2) \) can be written as \( \nu = \kappa \mathbb{I}_m + h_0 \), where \( \kappa \in \mathbb{R} \) is an arbitrary constant, and \( h_0 \in \mathbb{R}^m \) is the unique solution of \( (4.2) \) satisfying \( \pi h_0 = 0 \).

For the definitions of stochastic permanence and its relatives (see e.g., Li and Yin [17, Definitions 2.1-2.3]). We begin with the following lemmas and make use of it to obtain the stochastically ultimate upper boundedness of SDS \( (1.1) \).

**Lemma 4.2 (29)** If \( \pi a :\sum_{i \in \mathcal{S}} \pi_i a_i > 0 \) hold, for any \( \eta \in (0, 1) \) sufficiently small, the solution \( x(t) \) of SDS \( (1.1) \) with any initial value \( x_0 \in \mathbb{R}_+ \), \( \ell \in \mathcal{S} \) has the property that

\[
\limsup_{t \to \infty} \mathbb{E}[\log^\eta (x(t) \vee 1)] \leq C.
\]

In the special case where \( \hat{a} > 0 \), we cite the following lemma from literature.

**Lemma 4.3 (16)** If \( \hat{a} > 0 \), for any \( p > 0 \), the solution \( x(t) \) of SDS \( (1.1) \) with any initial value \( x_0 \in \mathbb{R}_+, \ell \in \mathcal{S} \) has the property that

\[
\limsup_{t \to \infty} \mathbb{E}[x^p(t)] \leq C.
\]

Next we continue to consider the case \( \pi a = 0 \). As we know, either \( \pi a > 0 \) or \( \pi a = 0 \) because of each \( a(i) \equiv 0 \). The \( \pi a = 0 \) implies that \( \hat{a} = 0 \), then SDS \( (1.1) \) degenerates into

\[
dx(t) = b(r(t))x(t)dt + \sigma(r(t))x(t)dB(t), \tag{4.3}
\]

We give the limit of the moment of linear SDS \( (4.3) \) for small \( p \), which is stronger than the stochastically ultimate upper boundedness.
Lemma 4.4 ([17]) If \( \pi \beta := \sum_{i \in S} \pi_i \beta_i < 0 \) hold, then for any \( \rho \in (0, 1) \) sufficiently small, the solution \( x(t) \) of SDS (4.3) has the property that
\[
\lim_{t \to \infty} \mathbb{E} [x^\rho(t)] = 0.
\]

Moreover, the following result is a direct consequence of (1.2).

Lemma 4.5 ([17]) The solution \( x(t) \) of SDS (1.1) with any initial value \( x_0 \in \mathbb{R}_+, \ell \in S \) satisfies
\[
\limsup_{t \to \infty} \frac{y(t)}{t} = \limsup_{t \to \infty} \frac{\log x(t)}{t} \leq \pi \beta \quad \text{a.s.}
\]
In particular, when \( \pi a = 0 \),
\[
\limsup_{t \to \infty} \frac{y(t)}{t} = \limsup_{t \to \infty} \frac{\log x(t)}{t} = \pi \beta \quad \text{a.s.}
\]

Now we look for the ultimate lower boundary of the moment of solutions in order to obtain the stochastic permanence.

Lemma 4.6 ([17]) If \( \pi \beta := \sum_{i \in S} \pi_i \beta_i > 0 \) hold, for any \( \vartheta \in (0, 1) \) sufficiently small, the solution \( x(t) \) of SDS (1.1) has the property that
\[
\limsup_{t \to \infty} \mathbb{E} [x^{-\vartheta}(t)] \leq C. \tag{4.4}
\]

Theorem 4.1 ([17]) If \( \pi a > 0 \) and \( \pi \beta > 0 \) hold, SDS (1.1) is stochastically permanent.

Theorem 4.2 ([17]) Suppose that \( \pi \beta \neq 0 \). Then,

- the solution of (1.1) are stochastically permanent if and only if \( \pi a > 0, \pi \beta > 0 \);
- the solution of (1.1) are almost surely extinctive if and only if \( \pi \beta < 0 \);
- almost all paths of (1.1) increase at an exponential rate if and only if \( \pi a = 0, \pi \beta > 0 \).

4.1 Stochastic permanence of numerical solution

In order to approximate the stochastic permanence of SDS (1.1) we need to construct the appropriate scheme such that the numerical solutions must be both stochastically ultimately upper bounded and lower bounded. In this subsection, we first give the definitions of the stochastic permanence and the stochastically ultimate boundedness of numerical solutions to the SDS (1.1).
Definition 4.1 A time-discretization \( \{X_k\} \), with stepsize \( \Delta \in (0, 1] \), of the solution to the SDS (1.1) is said to be stochastically ultimately upper bounded, if for any \( \nu \in (0, 1) \), there exist a positive constant \( \chi = \chi(\nu) \) such that for any initial value \( x_0 \in \mathbb{R}_+, \ell \in S \) satisfies

\[
\limsup_{k \to +\infty} \mathbb{P}\{X_k > \chi\} < \nu.
\]

Definition 4.2 A time-discretization \( \{X_k\} \), with stepsize \( \Delta \in (0, 1] \), of the solution to the SDS (1.1) is said to be stochastically ultimately lower bounded, if for any \( \nu \in (0, 1) \), there exist a positive constant \( \chi = \chi(\nu) \) such that for any initial value \( x_0 \in \mathbb{R}_+, \ell \in S \) satisfies

\[
\limsup_{k \to +\infty} \mathbb{P}\{X_k < \chi\} < \nu.
\]

Definition 4.3 A time-discretization \( \{X_k\} \), with stepsize \( \Delta \in (0, 1] \), of the solution to the SDS (1.1) is said to be stochastically permanent if its time-discretization solutions are both stochastically ultimately upper bounded and stochastically ultimately lower bounded.

For convenience, denote by \( \mathcal{G}_{t_k} \) the \( \sigma \)-algebra generated by \( \{\mathcal{F}_{t_k}, r_{k+1}\} \). Obviously, \( \mathcal{F}_{t_k} \subseteq \mathcal{G}_{t_k} \). We begin with a criterion on asymptotic upper boundedness of the moment, and make use of it to obtain the stochastically ultimate upper boundedness of the numerical solutions.

Lemma 4.7 Under the condition of Lemma 4.2, there is a constant \( \Delta^*_1 \in (0, 1) \) such that the scheme (3.2) has the property that

\[
\sup_{k \geq 0} \mathbb{E}\left[\log^n (X_k \vee 1)\right] = \sup_{k \geq 0} \mathbb{E}\left[(Z_k \vee 0)^\eta\right] \leq \sup_{k \geq 0} \mathbb{E}\left[(\bar{Z}_k \vee 0)^\eta\right] \leq C
\]

for any \( \Delta \in (0, \Delta^*_1] \), where \( \eta \) is defined in Lemma 4.2.

Proof. Note that

\[
\pi[-a + (\pi a)I_m] = 0, \quad \sum_{i=1}^m \pi_i = 1.
\]

It follows from Lemma 4.1(1) that the equation

\[
\Gamma c = -a + (\pi a)I_m
\]

has a solution \( c = (c_1, \cdots, c_m)^T \in \mathbb{R}^m \). Thus we have

\[
a(i) + \sum_{j=1}^m \gamma_{ij}c_j = \pi a > 0, \quad i \in S.
\]
Using the well-known Taylor formula we get
\[ e^{\bar{Z}_{k+1}} = e^{Z_k} + e^{Z_k}(\bar{Z}_{k+1} - Z_k) + \frac{1}{2}e^{Z_k}(\bar{Z}_{k+1} - Z_k)^2 + \frac{1}{6}e^{\xi_k}(\bar{Z}_{k+1} - Z_k)^3, \] (4.6)
where \( \xi_k \in (Z_{k+1} \land Z_k, Z_{k+1} \lor Z_k) \). For \( \omega \in \{Z_{k+1} < Z_k\} \), we have
\[ e^{\xi_k}(\bar{Z}_{k+1} - Z_k)^3 \leq 0. \] (4.7)

On the other hand, for \( \omega \in \{\bar{Z}_{k+1} \geq Z_k\} \), we have
\[ \frac{1}{6}e^{\xi_k}(\bar{Z}_{k+1} - Z_k)^3 \leq \frac{1}{6}e^{\bar{Z}_{k+1}}(\bar{Z}_{k+1} - Z_k)^3 = \frac{1}{6}e^{Z_k}e^{\bar{Z}_{k+1}-Z_k}(\bar{Z}_{k+1} - Z_k)^3 \]
\[ = \frac{1}{6}e^{Z_k}\exp\left((\beta(r_k) - a(r_k)e^{Z_k})\Delta + \sigma(r_k)\Delta B_k\right)\left((\beta(r_k) - a(r_k)e^{Z_k})\Delta + \sigma(r_k)\Delta B_k\right)^3 \]
\[ \leq \frac{1}{3}e^{Z_k}\exp\left(|\bar{\beta}|\Delta + |\bar{\sigma}|\right)^3 \exp\left(|\bar{\sigma}|\Delta B_k\right)\left(\Delta^3 + |\Delta B_k|^3\right) =: e^{Z_k}\bar{U}_k > 0, \] (4.8)
where
\[ \bar{U}_k := \frac{2}{3}e^{\beta\Delta}(|\bar{\beta}| \lor |\bar{\sigma}|)^3 \exp\left(|\bar{\sigma}|\Delta B_k\right)\left(\Delta^3 + |\Delta B_k|^3\right). \]

Therefore, we derive from (4.6)-(4.8) that for any integer \( k \geq 0 \),
\[ e^{\bar{Z}_{k+1}} - e^{Z_k} \leq e^{Z_k}\left[\bar{Z}_{k+1} - Z_k + \frac{1}{2}(\bar{Z}_{k+1} - Z_k)^2 + \bar{U}_k\right] \]
\[ = e^{Z_k}\left[(\beta(r_k) - a(r_k)e^{Z_k})\Delta + \sigma(r_k)\Delta B_k + \frac{\sigma^2(r_k)}{2}(\Delta B_k)^2 + \sigma(r_k)\Delta B_k(\beta(r_k) - a(r_k)e^{Z_k})\Delta + \frac{1}{2}(\beta(r_k) - a(r_k)e^{Z_k})^2\Delta^2 + \bar{U}_k\right]. \] (4.9)

Choose a constant \( 0 < \eta_0 \leq 1 \) such that for each \( 0 < \eta \leq \eta_0 \),
\[ \xi_i^{\eta,c} := 1 - c_i\eta > 0, \quad i = 1, \ldots, m. \]

By the Markov property (see, [31] Lemma 3.2 for more details), we derive that
\[ \mathbb{E}\left[e^{r_{jk}^c} \mid \mathcal{F}_k\right] = s_k^{\eta,c} + \sum_{j \in \mathcal{S}} \xi_j^{\eta,c} (\gamma_{rk,j} \Delta + o(\Delta)). \] (4.10)

Moreover, one observes
\[ \log (1 + e^{\bar{Z}_{k+1}}) \leq \log (1 + e^{Z_k}) + \frac{e^{\bar{Z}_{k+1}} - e^{Z_k}}{1 + e^{Z_k}}. \]

Then using the above inequality and (4.9), we have
\[ [1 + \log (1 + e^{\bar{Z}_{k+1}})]^\eta \leq [1 + \log (1 + e^{Z_k})]^\eta (1 + \varsigma_k)^\eta, \] (4.11)
where
\[ s_k = [1 + \log (1 + e^{Z_k})]^{-1} \frac{e^{Z_k}}{1 + e^{Z_k}} \left( (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + \sigma(r_k) \Delta B_k + \frac{\sigma^2(r_k)}{2} (\Delta B_k)^2 \right. \]
\[ \left. + \sigma(r_k) \Delta B_k (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + \frac{1}{2} (\beta(r_k) - a(r_k)e^{Z_k})^2 \Delta^2 + \bar{U}_k \right], \]
and we can see that \( s_k > -1 \). For any \( 0 < \eta \leq 1 \), by virtue of Lemma 3.3, we derive from (4.11) that
\[
E \left[ \left[ 1 + \log (1 + e^{Z_{k+1}}) \right]^\eta \xi_{\eta,c}^{r_{k+1}} \mid \mathcal{F}_{T_k} \right] \]
\[
\leq \left[ 1 + \log (1 + e^{Z_k}) \right]^\eta \left\{ E \left[ \xi_{\eta,c}^{r_{k+1}} \mid \mathcal{F}_{T_k} \right] + \eta E \left[ s_k \xi_{\eta,c}^{r_{k+1}} \mid \mathcal{F}_{T_k} \right] + \frac{\eta(\eta - 1)}{2} E \left[ \xi_{\eta,c}^{r_{k+1}} \mid \mathcal{F}_{T_k} \right] \right\} \]
\[
\leq \left[ 1 + \log (1 + e^{Z_k}) \right]^\eta \left\{ E \left[ \xi_{\eta,c}^{r_{k+1}} \mid \mathcal{F}_{T_k} \right] + \eta E \left[ s_k \xi_{\eta,c}^{r_{k+1}} \mid \mathcal{F}_{T_k} \right] + \frac{\eta(\eta - 1)(\eta - 2)}{6} E \left[ \xi_{\eta,c}^{r_{k+1}} \mid \mathcal{F}_{T_k} \right] \right\}, \quad (4.12) \]
Using the properties
\[
E(|\Delta B_k|^{2j} \mid \mathcal{G}_{T_k}) = (2j - 1)!! \Delta^j, \quad E((\Delta B_k)^{2j-1} \mid \mathcal{G}_{T_k}) = 0, \quad j = 1, 2, \ldots \quad (4.13) \]
we deduce that
\[
E[s_k \mid \mathcal{G}_{T_k}] = \left[ 1 + \log (1 + e^{Z_k}) \right]^{-1} \frac{e^{Z_k}}{1 + e^{Z_k}} \left( (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + \frac{\sigma^2(r_k)}{2} \Delta \right.
\]
\[ \left. + \frac{1}{2} (\beta(r_k) - a(r_k)e^{Z_k})^2 \Delta^2 + E(\bar{U}_k \mid \mathcal{G}_{T_k}) \right] \]
\[ = \left[ 1 + \log (1 + e^{Z_k}) \right]^{-1} \frac{e^{Z_k}}{1 + e^{Z_k}} \left( (\beta(r_k) - a(r_k)e^{Z_k}) \Delta \right.
\]
\[ \left. + \left( |\beta|^2 + a^2 \Delta^2 \right) \Delta^2 + E(\bar{U}_k \mid \mathcal{G}_{T_k}) \right]. \]
By virtue of Lemma 3.2, we obtain \( E(\bar{U}_k \mid \mathcal{G}_{T_k}) \leq C \Delta^2 \). Then
\[
E[s_k \mid \mathcal{G}_{T_k}] \leq \left[ 1 + \log (1 + e^{Z_k}) \right]^{-1} \frac{e^{Z_k}}{1 + e^{Z_k}} \left( (\beta - a(r_k)e^{Z_k}) \Delta + C \Delta^2(1-\theta) + C \Delta^2 \right]. \]
Making use of the above inequality and (4.10) yields
\[
E[\xi_{\eta,c}^{r_{k+1}} \mid \mathcal{F}_{T_k}] = E[\xi_{\eta,c}^{r_{k+1}} \mathbb{E}(s_k \mid \mathcal{G}_{T_k}) \mid \mathcal{F}_{T_k}] \]
\[
\leq \left[ 1 + \log (1 + e^{Z_k}) \right]^{-1} \frac{e^{Z_k}}{1 + e^{Z_k}} \left( (\beta - a(r_k)e^{Z_k}) \Delta + C \Delta^2(1-\theta) + C \Delta^2 \right] \mathbb{E}[\xi_{\eta,c}^{r_{k+1}} \mid \mathcal{F}_{T_k}] \]
\[ = \left[ 1 + \log (1 + e^{Z_k}) \right]^{-1} \frac{e^{Z_k}}{1 + e^{Z_k}} \left( (\beta - a(r_k)e^{Z_k}) \Delta \right] \]
On the other hand, it follows from (4.11) that

\[ E \leq [1 + \log(1 + e^{Z_k})] \left[ \left( \hat{b} - a(r_k)e^{Z_k} \right) \xi_{r_k}^c + C_\Delta^2 \sum_{j \in S} \xi_j^c \left( \gamma_{r_k,j} \Delta + o(\Delta) \right) \right] \]

\[ \leq [1 + \log(1 + e^{Z_k})]^{-1} \frac{e^{Z_k}}{1 + e^{Z_k}} \left[ \left( \hat{b} - a(r_k)e^{Z_k} \right) \xi_{r_k}^c + C_\Delta^2 \sum_{j \in S} \xi_j^c \left( \gamma_{r_k,j} \Delta + o(\Delta) \right) \right]. \]  

On the other hand, it follows from (4.11) that

\[ \mathbb{E}[\xi_k^3 | G_{t_k}] = [1 + \log(1 + e^{Z_k})]^{-3} \frac{e^{3Z_k}}{(1 + e^{Z_k})^3} \mathbb{E} \left\{ 3\sigma^2(r_k)(\Delta B_k)^2 \left( 1 + (\beta(r_k) - a(r_k)e^{Z_k}) \Delta \right) \right. \]
\[ + \left. 3\sigma^2(r_k)\Delta B_k \left( 1 + (\beta(r_k) - a(r_k)e^{Z_k}) \Delta \right) \left[ (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + \frac{\sigma^2(r_k)}{2} (\Delta B_k)^2 + \frac{1}{2} (\beta(r_k) - a(r_k)e^{Z_k})^2 \Delta^2 + \bar{U}_k \right] \right. \]
\[ + \left. \left[ (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + \frac{\sigma^2(r_k)}{2} (\Delta B_k)^2 + \frac{1}{2} (\beta(r_k) - a(r_k)e^{Z_k})^2 \Delta^2 + \bar{U}_k \right]^{3} | G_{t_k} \right\}. \]

Using the properties (4.13) and

\[ \mathbb{E} \left[ (\Delta B_k) e^{3|\Delta B_k|} (\Delta^3 + |\Delta B_k|^3) \right] = 0, \] \[ \mathbb{E} \left[ (\Delta B_k) e^{2|\Delta B_k|} (\Delta^3 + |\Delta B_k|^2)^2 \right] = 0, \]

we deduce that

\[ \mathbb{E}[\xi_k^3 | G_{t_k}] = [1 + \log(1 + e^{Z_k})]^{-3} \frac{e^{3Z_k}}{(1 + e^{Z_k})^3} \mathbb{E} \left\{ 3\sigma^2(r_k)(\Delta B_k)^2 \left( 1 + (\beta(r_k) - a(r_k)e^{Z_k}) \Delta \right)^2 \right. \]
\[ \times \left. \left[ (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + \frac{\sigma^2(r_k)}{2} (\Delta B_k)^2 + \frac{1}{2} (\beta(r_k) - a(r_k)e^{Z_k})^2 \Delta^2 + \bar{U}_k \right] \right. \]
\[ + \left. \left[ (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + \frac{\sigma^2(r_k)}{2} (\Delta B_k)^2 + \frac{1}{2} (\beta(r_k) - a(r_k)e^{Z_k})^2 \Delta^2 + \bar{U}_k \right]^{3} | G_{t_k} \right\} \]
\[ \leq [1 + \log(1 + e^{Z_k})]^{-3} \mathbb{E} \left\{ 9|\bar{\sigma}|^2 (\Delta B_k)^2 \left( 1 + |\bar{\beta}|^2 \Delta^2 + \bar{a}^2 K^2 \Delta^2 (1 - \theta) \right) \right. \]
\[ \times \left. \left[ |\bar{\beta}| \Delta + \frac{\sigma^2(r_k)}{2} (\Delta B_k)^2 + |\bar{\beta}|^2 \Delta^2 + \bar{a}^2 K^2 \Delta^2 (1 - \theta) + \bar{U}_k \right] \right. \]
\[ + \left. \left[ |\bar{\beta}| \Delta + \frac{\sigma^2(r_k)}{2} (\Delta B_k)^2 + |\bar{\beta}|^2 \Delta^2 + \bar{a}^2 K^2 \Delta^2 (1 - \theta) + \bar{U}_k \right]^{3} | G_{t_k} \right\} \]
\[ \leq [1 + \log(1 + e^{Z_k})]^{-3} \left( C \Delta^2 + 9|\bar{\sigma}|^2 \mathbb{E} \left[ (\Delta B_k)^2 \bar{U}_k \right] + 10 \mathbb{E} [\bar{U}_k^3] \right). \]
By virtue of Lemma 3.2, we obtain

$$ \mathbb{E} \left[ (\Delta B_k)^2 \mathcal{U}_k \mid \mathcal{G}_{t_k} \right] \leq C \left\{ \Delta^3 \mathbb{E} \left[ e^{\frac{|\Delta B_k|^2}{4\Delta}} \right] + \mathbb{E} \left[ e^{\frac{|\Delta B_k|^2}{4\Delta}} \right] \right\} \leq C \Delta^\frac{5}{2}, $$

and

$$ \mathbb{E} \left[ \mathcal{U}_k^3 \mid \mathcal{G}_{t_k} \right] \leq C \mathbb{E} \left[ e^{\frac{|\Delta B_k|^2}{4\Delta}} \right] \leq C \Delta^\frac{9}{2}, $$

which implies

$$ \mathbb{E} \left[ s_k^{\eta,c} \mid \mathcal{F}_{t_k} \right] = \mathbb{E} \left[ s_k^{\eta,c} \mathbb{E} \left( s_k^{\eta,c} \mid \mathcal{G}_{t_k} \right) \mid \mathcal{F}_{t_k} \right] \leq C \left[ 1 + \log (1 + e^{Z_k}) \right]^{-3} \Delta^2 \mathbb{E} \left[ s_k^{\eta,c} \mid \mathcal{F}_{t_k} \right] \leq C \left[ 1 + \log (1 + e^{Z_k}) \right]^{-3} \Delta^2. \quad (4.15) $$

Combining (4.12), (4.14) and (4.15), we derive from (4.10) that

$$ \mathbb{E} \left[ [1 + \log (1 + e^{Z_k})]^{\eta} s_k^{\eta,c} \mid \mathcal{F}_{t_k} \right] \leq [1 + \log (1 + e^{Z_k})]^{\eta} \left\{ \xi_k^{\eta,c} + \sum_{j \in \mathcal{S}} \xi_j^{\eta,c} (\gamma_{r_k,j} \Delta + o(\Delta)) + C [1 + \log (1 + e^{Z_k})]^{-3} \Delta^2 \right\} 
+ \eta [1 + \log (1 + e^{Z_k})]^{-1} \frac{e^{Z_k}}{1 + e^{Z_k}} \left[ \hat{b} - a(r_k) e^{Z_k} \right] \eta \xi_k^{\eta,c} \Delta + C \Delta^2 (1 - \theta) + C \Delta^2 \right\} 
\leq \xi_k^{\eta,c} [1 + \log (1 + e^{Z_k})]^{\eta} \left\{ 1 + \frac{1}{\xi_k^{\eta,c}} \sum_{j \in \mathcal{S}} \xi_j^{\eta,c} \gamma_{r_k,j} \Delta + o(\Delta) \right\} 
- \eta [1 + \log (1 + e^{Z_k})]^{-1} a(r_k) e^{Z_k} \Delta + \eta [1 + \log (1 + e^{Z_k})]^{-1} \frac{\hat{b} e^{Z_k}}{1 + e^{Z_k}} \Delta \right\} 
\leq \xi_k^{\eta,c} [1 + \log (1 + e^{Z_k})]^{\eta} \left\{ 1 + \frac{1}{\xi_k^{\eta,c}} \sum_{j \in \mathcal{S}} \xi_j^{\eta,c} \gamma_{r_k,j} \Delta + o(\Delta) \right\} 
- \eta a(r_k) [1 + \log (1 + e^{Z_k})]^{-1} (1 + e^{Z_k}) \Delta + 2 \eta a(r_k) [1 + \log (1 + e^{Z_k})]^{-1} \Delta 
- \eta a(r_k) [1 + \log (1 + e^{Z_k})]^{-1} \frac{1}{1 + e^{Z_k}} \Delta + \eta \hat{b} [1 + \log (1 + e^{Z_k})]^{-1} \Delta \right\}.$$
\[ + 2\eta a(r_k) \left[ 1 + \log \left( 1 + e^{\bar{Z}_k} \right) \right]^{-1} \Delta + \eta \bar{b}[1 + \log \left( 1 + e^{\bar{Z}_k} \right)]^{-1} \Delta \right) \\
= \zeta_{r_k}^{\eta,c}[1 + \log (1 + e^{\bar{Z}_k})]^{\eta} \left\{ 1 + \frac{1}{\xi_{r_k}^{\eta,c}} \sum_{j \in S} \zeta_{r_k}^{\eta,c} \gamma_{r_k j} \Delta - \eta a(r_k) \Delta + \eta \bar{b}[1 + \log \left( 1 + e^{\bar{Z}_k} \right)]^{-1} \Delta \\
+ o(\Delta) + 3\eta a(r_k) \left[ 1 + \log \left( 1 + e^{\bar{Z}_k} \right) \right]^{-1} \Delta \right\}. \tag{4.16} \]

By the properties of the generator, we have

\[
\frac{1}{\zeta_{r_k}^{\eta,c}} \sum_{j \in S} \zeta_{r_k}^{\eta,c} \gamma_{ij} = \frac{1}{1 - c_i \eta} \sum_{j=1}^{m} (1 - c_j \eta) \gamma_{ij} \\
= \frac{\eta}{1 - c_i \eta} \sum_{j \neq i} \gamma_{ij} (c_i - c_j) \\
= - \frac{\eta}{1 - c_i \eta} \sum_{j=1}^{m} \gamma_{ij} c_j \\
= - \eta \left( \sum_{j=1}^{m} \gamma_{ij} c_j + \frac{c_i \eta}{1 - c_i \eta} \sum_{j=1}^{m} \gamma_{ij} c_j \right). \tag{4.17} \]

It follows from (4.5), (4.16) and (4.17) that

\[
\mathbb{E} \left[ \left[ 1 + \log \left( 1 + e^{\bar{Z}_{k+1}} \right) \right]^{\eta} \zeta_{r_{k+1}}^{\eta,c} |\mathcal{F}_{t_k} \right] \\
\leq \zeta_{r_k}^{\eta,c}[1 + \log (1 + e^{\bar{Z}_k})]^{\eta} \left\{ 1 - \eta \left( \pi a + \frac{c_k \eta}{1 - c_k \eta} \sum_{j=1}^{m} \gamma_{r_k j} c_j \right) \Delta + o(\Delta) \\
+ \eta \bar{b}[1 + \log \left( 1 + e^{\bar{Z}_k} \right)]^{-1} \Delta + 3\eta a(r_k) \left[ 1 + \log \left( 1 + e^{\bar{Z}_k} \right) \right]^{-1} \Delta \right\}. \tag{4.18} \]

Choose a constant \( 0 < \eta_1 \leq \eta_0 \) such that for any \( 0 < \eta \leq \eta_1 \),

\[
\pi a + \frac{c_i \eta}{1 - c_i \eta} \sum_{j=1}^{m} \gamma_{ij} c_j > 0, \quad i \in S. \]

Now, choose a positive constant \( \kappa = (\kappa(\eta)) < 1 \) sufficiently small such that it satisfies

\[
h_i := \pi a + \frac{c_i \eta}{1 - c_i \eta} \sum_{j=1}^{m} \gamma_{ij} c_j - \frac{\kappa}{\eta} > 0. \tag{4.19} \]

Then, by (4.18) and (4.19), we have

\[
\mathbb{E} \left[ \left[ 1 + \log \left( 1 + e^{\bar{Z}_{k+1}} \right) \right]^{\eta} \zeta_{r_{k+1}}^{\eta,c} |\mathcal{F}_{t_k} \right] \\
\leq \zeta_{r_k}^{\eta,c}[1 + \log (1 + e^{\bar{Z}_k})]^{\eta} \left\{ 1 - \kappa \Delta - \eta h_k \Delta + \eta \bar{b}[1 + \log \left( 1 + e^{\bar{Z}_k} \right)]^{-1} \Delta \right\}. 
\]

21
\[ + o(\Delta) + 3\eta a(r_k) [1 + \log (1 + e^{Z_k})]^{-1} \Delta \].

Choose \( \Delta_1^* \in (0, 1) \) sufficiently small such that \( \Delta_1^* < 2/\kappa \) and \( o(\Delta_1^*) \leq \kappa \eta \Delta_1^*/2 \). For any \( \Delta \in (0, \Delta_1^*) \), yields

\[
\mathbb{E}\left[ [1 + \log (1 + e^{\tilde{Z}_{k+1}})]^{\eta} \xi_{\tau_{k+1}}^{\eta,c} | F_k \right] \\
\leq \xi_{\tau_k}^{\eta,c} \left[ [1 + \log (1 + e^{Z_k})]^{\eta} \left( 1 - \frac{\kappa}{2} \Delta \right) \right] \\
- \eta \xi_{\tau_k}^{\eta,c} \left\{ h_k [1 + \log (1 + e^{Z_k})]^{\eta} - (3\tilde{a} + \tilde{b}) [1 + \log (1 + e^{Z_k})]^{\eta-1} \right\} \Delta \\
\leq \left( 1 - \frac{\kappa}{2} \Delta \right) [1 + \log (1 + e^{\tilde{Z}_k})]^{\eta} \xi_{\tau_k}^{\eta,c} + C \Delta
\]

for any integer \( k \geq 0 \). Repeating this procedure arrives at

\[
\mathbb{E}\left[ (1 + \log (1 + e^{\tilde{Z}_k}))^{\eta} \xi_{\tau_k}^{\eta,c} \right] \leq e^{-\frac{\Delta}{2}k} \left( 1 + \log (1 + x_0) \right)^{\eta} \xi_{\tau_k}^{\eta,c} + \frac{2C}{\kappa} \left[ 1 - \left( 1 - \frac{\kappa}{2} \Delta \right)^k \right].
\]

Therefore, \( \sup_{k \geq 0} \mathbb{E}\left[ \log^\eta (1 + e^{\tilde{Z}_k}) \right] \leq C \). The desired assertion follows.

The proofs of both below lemmas can be found in Appendix B.

**Lemma 4.8** If \( \tilde{a} > 0 \), for any \( p > 0 \) and \( \Delta \in (0, 1) \), the truncated EM scheme defined by (3.2) has the property that

\[
\sup_{k \geq 0} \mathbb{E}[X^p_k] = \sup_{k \geq 0} \mathbb{E}[e^{p\tilde{Z}_k}] \leq \sup_{k \geq 0} \mathbb{E}[e^{p\tilde{Z}_k}] \leq C. \tag{4.20}
\]

**Theorem 4.3** Under the condition of Lemma 4.7, the numerical solutions \( X_k \) are stochastically ultimately upper bounded.

**Proof.** The proof is an application of Chebyshev’s inequality, so we omit it.

Next we continue to consider the case \( \pi a = 0 \), we can get the following results.

**Lemma 4.9** Under the condition of Lemma 4.4, there is a constant \( \Delta_1^* \in (0, 1) \) such that the scheme (3.2) has the property that

\[
\lim_{k \to \infty} \mathbb{E}[X^p_k] = \lim_{k \to \infty} \mathbb{E}[e^{p\tilde{Z}_k}] = 0
\]

for any \( \Delta \in (0, \Delta_1^*) \), where \( p \) is defined in Lemma 4.4.

Moreover, we can also get the following result.
Lemma 4.10 For any $\Delta \in (0,1)$ and initial value $(x_0, \ell) \in \mathbb{R}_+ \times \mathcal{S}$, the scheme (3.2) has the property that

$$\limsup_{k \to \infty} \frac{\log(X_k)}{k\Delta} = \limsup_{k \to \infty} \frac{Z_k}{k\Delta} \leq \limsup_{k \to \infty} \frac{\bar{Z}_k}{k\Delta} \leq \pi \beta \quad \text{a.s.}$$

In particularly, when $\pi a = 0$,

$$\limsup_{k \to \infty} \frac{\log(X_k)}{k\Delta} = \limsup_{k \to \infty} \frac{Z_k}{k\Delta} = \limsup_{k \to \infty} \frac{\bar{Z}_k}{k\Delta} = \pi \beta \quad \text{a.s.}$$

Proof. By the scheme (3.1), we have

$$\bar{Z}_{k+1} = Z_k + \left(\beta(r_k) - a(r_k)e^{Z_k}\right)\Delta + \sigma(r_k)\Delta B_k \leq \bar{Z}_k + \beta(r_k)\Delta + \sigma(r_k)\Delta B_k$$

$$\leq Z_{k-1} + \beta(r_{k-1})\Delta + \sigma(r_{k-1})\Delta B_{k-1} + \beta(r_k)\Delta + \sigma(r_k)\Delta B_k$$

$$\leq Z_0 + \sum_{j=0}^{k-1} \beta(r_j)\Delta + \sum_{j=0}^{k-1} \sigma(r_j)\Delta B_j.$$

By the strong law of large numbers for martingales (see [20, Theorem 1.6]), we have

$$\lim_{k \to \infty} \frac{\sum_{j=0}^{k-1} \sigma(r_j)\Delta B_j}{k\Delta} = 0 \quad \text{a.s.}$$

Then, by the ergodic property of the Markov chain (see, e.g., [3]), we compute

$$\lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \beta(r_j)\Delta = \sum_{i \in \mathcal{S}} \pi_i \beta(i)\Delta = \pi \beta \Delta \quad \text{a.s.}$$

which implies

$$\limsup_{k \to \infty} \frac{Z_k}{k\Delta} \leq \pi \beta \quad \text{a.s.}$$

Particularly, when $\pi a = 0$, we have $a(\cdot) \equiv 0$ and

$$Z_{k+1} = Z_k + \beta(r_k)\Delta + \sigma(r_k)\Delta B_k = Z_0 + \sum_{i=0}^{k} \beta(r_k)\Delta + \sum_{i=0}^{k} \sigma(r_k)\Delta B_i,$$

then the required assertion follows. \hfill \blacksquare

On the other hand, to show the numerical solutions $X_k$ defined by (3.2) is stochastically ultimately lower bounded, we need the following lemma.

Lemma 4.11 Under the condition of Lemma 4.6 and $\hat{a} > 0$, there is a constant $\Delta^{*}_3 \in (0,1)$ such that the scheme (3.2) has the property that

$$\sup_{k \geq 0} \mathbb{E}[X^{-\vartheta}_k] = \sup_{k \geq 0} \mathbb{E}[e^{-\vartheta Z_k}] \leq C$$

for any $\Delta \in (0, \Delta^{*}_3]$, where $\vartheta$ is defined in Lemma 4.6.
Proof. By (3.5), we have
\[(1 + e^{-Z_{k+1}}) \leq (1 + e^{-Z_k})(1 + \zeta_k), \tag{4.21}\]
where \(\zeta_k\) is defined by (3.5). It follows from Lemma 4.1 (1) that the system of equations
\[\Gamma d = -\beta + (\pi \beta)\mathbb{I}_m\]
has a solution \(d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m\). Then we have
\[\beta(i) + \sum_{j=1}^{m} \gamma_{ij} d_j = \pi \beta > 0. \tag{4.22}\]
Choose a constant \(0 < \vartheta_1 < 1\) such that for each \(0 < \vartheta \leq \vartheta_1\),
\[\xi_{\vartheta, r_k}^d := 1 - d_i \vartheta > 0, \quad i \in S.\]
For any \(0 < \vartheta < 1\), by virtue of \([31, \text{Lemma 3.3}]\), it follows from (4.21) that
\[\mathbb{E}\left[ (1 + e^{-Z_{k+1}})^\vartheta s_{r_{k+1}}^d | \mathcal{F}_{t_k} \right] \leq (1 + e^{-Z_k})^\vartheta \left\{ \mathbb{E}[s_{r_{k+1}}^d | \mathcal{F}_{t_k}] + \vartheta \mathbb{E}[\zeta_k s_{r_{k+1}}^d | \mathcal{F}_{t_k}] + \frac{\vartheta(\vartheta - 1)}{2} \mathbb{E}[\zeta_k^2 s_{r_{k+1}}^d | \mathcal{F}_{t_k}] ight. \\
+ \left. \frac{\vartheta(\vartheta - 1)(\vartheta - 2)}{6} \mathbb{E}[\zeta_k^3 s_{r_{k+1}}^d | \mathcal{F}_{t_k}] \right\}. \tag{4.23}\]
Then, making use of the techniques in the proof of Lemma 4.9 as well as Lemma 3.2 and (4.13) yields
\[\mathbb{E}[U_k | \mathcal{G}_{t_k}] \leq C \left( \Delta^{3(1-\theta)} + \Delta^2 \right),\]
and
\[\mathbb{E}[\zeta_k | \mathcal{G}_{t_k}] \leq \frac{e^{-Z_k}}{1 + e^{-Z_k}} \left( (\alpha(r_k) e^{Z_k} - \beta(r_k)) \Delta + C \Delta^{2(1-\theta)} + \frac{\sigma^2(r_k) \Delta^2}{2} + \mathbb{E}[U_k | \mathcal{G}_{t_k}] \right).\]
By the Markov property, we derive that
\[\mathbb{E}[\zeta_k s_{r_{k+1}}^d | \mathcal{F}_{t_k}] \leq \frac{e^{-Z_k}}{1 + e^{-Z_k}} \left[ (\alpha(r_k) e^{Z_k} - \beta(r_k) + \frac{\sigma^2(r_k)}{2}) \Delta + C \Delta^{2(1-\theta)} \right] \mathbb{E}[s_{r_{k+1}}^d | \mathcal{F}_{t_k}] \leq \frac{e^{-Z_k}}{1 + e^{-Z_k}} \left[ \xi_{r_k}^d (\alpha(r_k) e^{Z_k} - \beta(r_k) + \frac{\sigma^2(r_k)}{2}) \Delta \right] + C \Delta^{2(1-\theta)}
\leq \xi_{r_k}^d (-\beta(r_k) + \frac{\sigma^2(r_k)}{2}) \Delta - \frac{1}{1 + e^{-Z_k}} \xi_{r_k}^d (-\beta(r_k) + \frac{\sigma^2(r_k)}{2}) \Delta \leq \frac{a(r_k)}{1 + e^{-Z_k}} \xi_{r_k}^d \Delta + C \Delta^{2(1-\theta)} \right].
Using the techniques in the proof of Lemma 4.7 we show that

\[ \mathbb{E} [\zeta_k^2 | \mathcal{G}_{tk}] \geq \frac{e^{-2Z_k}}{(1 + e^{-Z_k})^2} \mathbb{E} \left[ \left( a(r_k)e^{Z_k} - \beta(r_k) \right) \Delta + C \Delta^{2(1-\theta)} \right] \]

which implies that

\[ \mathbb{E} [\zeta_k^2 \zeta_{r_k,i}^d | \mathcal{F}_{tk}] \geq \frac{(1 + e^{-Z_k} - 1)^2}{(1 + e^{-Z_k})^2} \sigma^2(r_k) \zeta_{r_k,i}^d \Delta - C \Delta^{2-\theta} \]

\[ = \left( 1 - \frac{2}{1 + e^{-Z_k}} + \frac{1}{(1 + e^{-Z_k})^2} \right) \sigma^2(r_k) \zeta_{r_k,i}^d \Delta - C \Delta^{2-\theta}. \] (4.25)

In addition,

\[ \mathbb{E} \left[ (\Delta B_k) \mathcal{U}_k | \mathcal{G}_{tk} \right] = 0, \quad \mathbb{E} \left[ (\Delta B_k)^2 \mathcal{U}_k^2 | \mathcal{G}_{tk} \right] = 0. \]

This together with (4.13) as well as Lemma 3.2 yields

\[ \mathbb{E} [\zeta_k^2 | \mathcal{G}_{tk}] = \frac{e^{-2Z_k}}{(1 + e^{-Z_k})^3} \mathbb{E} \left[ \left( a(r_k)e^{Z_k} - \beta(r_k) \right) \Delta + C \Delta^{2(1-\theta)} + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} \right] \]

\[ \quad \times \left( (a(r_k)e^{Z_k} - \beta(r_k)) \Delta + C \Delta^{2(1-\theta)} + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} + \mathcal{U}_k \right) \]

\[ - 3\sigma(r_k)(\Delta B_k) \left( 1 + (a(r_k)e^{Z_k} - \beta(r_k)) \Delta \right) \]
\[ \times \left( (a(r_k)e^{Z_k} - \beta(r_k))\Delta + C\Delta^2(1-\theta) + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} + \mu_k \right)^2 + \left( (a(r_k)e^{Z_k} - \beta(r_k))\Delta + C\Delta^2(1-\theta) + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} + \mu_k \right)^3 \right|_{t_k}^{G_k} \]

\[ = \frac{e^{-3Z_k}}{(1 + e^{-Z_k})^3} \mathbb{E} \left[ 3\sigma^2(r_k)(\Delta B_k)^2 \left( 1 + (a(r_k)e^{Z_k} - \beta(r_k))\Delta \right)^2 \times \left( (a(r_k)e^{Z_k} - \beta(r_k))\Delta + C\Delta^2(1-\theta) + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} + \mu_k \right)^3 \right|_{t_k}^{G_k} \]

\[ \leq \frac{e^{-3Z_k}}{(1 + e^{-Z_k})^3} \left[ 3\sigma^2(r_k) \left( 1 + (a(r_k)e^{Z_k} - \beta(r_k))\Delta \right)^2 \times \left( (a(r_k)e^{Z_k} - \beta(r_k))\Delta + C\Delta^2(1-\theta) + \frac{\sigma^2(r_k)(\Delta B_k)^2}{2} + \mu_k \right)^3 \right|_{t_k}^{G_k} \]

\[ \leq \frac{e^{-3Z_k}}{(1 + e^{-Z_k})^3} \left[ C \left( \Delta^{2-\theta} + \Delta^{3-2\theta} + \Delta^2 + \Delta^{4-3\theta} + \Delta^3 \right) + C \left( \Delta^{3(1-\theta)} + \Delta^6(1-\theta) + \Delta^3 + \Delta^{9(1-\theta)} + \Delta^2 \right) \right], \]

which implies that

\[ \mathbb{E} \left[ \zeta_{t_k}^3 \xi_{\varphi, r_{k+1}}^d \big| \mathcal{F}_{t_k} \right] \leq C \left( \Delta^{2-\theta} + \Delta^{3(1-\theta)} \right). \quad (4.26) \]

Substituting (4.24), (4.25) and (4.26) into (4.23), we derive that

\[ \mathbb{E} \left[ (1 + e^{-Z_{k+1}})^\theta \xi_{\varphi, r_{k+1}}^d \big| \mathcal{F}_{t_k} \right] \]

\[ \leq (1 + e^{-Z_k})^\theta \left\{ \xi_{r_k}^d + \sum_{j \in S} s_j^d (\gamma_{r_k} \Delta + o(\Delta)) + C\Delta^2(1-\theta) + C\Delta^{2-\theta} \right. \]

\[ - \frac{\vartheta}{1 + e^{-Z_k}} \left( \beta(r_k) - \frac{\sigma^2(r_k)}{2} \right) \Delta + \frac{\vartheta \beta(r_k) + a(r_k)}{1 + e^{-Z_k}} \xi_{r_k}^d \Delta \]

\[ + \frac{\vartheta (\vartheta - 1)}{2} \left( 1 - \frac{2}{1 + e^{-Z_k}} + \frac{1}{(1 + e^{-Z_k})^2} \right) \sigma^2(r_k) \xi_{r_k}^d \Delta \]

\[ \leq (1 + e^{-Z_k})^\theta \left\{ \xi_{r_k}^d + \sum_{j \in S} s_j^d (\gamma_{r_k} \Delta + o(\Delta)) - \vartheta \xi_{r_k}^d \left( \beta(r_k) - \frac{\sigma^2(r_k)}{2} \right) - \frac{\vartheta (\vartheta - 1)}{2} \sigma^2(r_k) \right\} \Delta \]

\[ + \frac{\vartheta}{1 + e^{-Z_k}} \left( \beta(r_k) + a(r_k) - (\vartheta - 1)\Delta \right) \xi_{r_k}^d \Delta + \frac{\vartheta (\vartheta - 1)\Delta \sigma^2(r_k)}{2(1 + e^{-Z_k})^2} \xi_{r_k}^d \Delta \]

\[ \leq (1 + e^{-Z_k})^\theta \left\{ 1 + \vartheta \left( \frac{1}{\vartheta \xi_{r_k}^d} \sum_{j \in S} s_j^d (\gamma_{r_k} \Delta - \beta(r_k)) + \frac{\vartheta \sigma^2(r_k)}{2} \right) \Delta + o(\Delta) \right\} \]

\[ + \vartheta (a(r_k) + \beta(r_k) - (\vartheta - 1)\Delta) \xi_{r_k}^d \Delta + \frac{\vartheta (\vartheta - 1)\sigma^2(r_k)}{2(1 + e^{-Z_k})^2} \xi_{r_k}^d \Delta. \quad (4.27) \]
By the properties of the generator, we have

$$\frac{1}{\vartheta} \sum_{j \in S} \xi_j \gamma_{ij} = -\left( \sum_{j=1}^{m} \gamma_{ij} d_j + \frac{d_i \vartheta}{1 - d_i \vartheta} \sum_{j=1}^{m} \gamma_{ij} d_j \right) \tag{4.28}$$

It follows from (4.22), (4.27) and (4.28) that

$$\mathbb{E}\left[ (1 + e^{-Z_{k+1}})^\vartheta \xi_{r_{k+1}} \mid \mathcal{F}_k \right] \leq (1 + e^{-Z_k})^\vartheta \xi_{r_k} \left[ 1 - \vartheta \left( \sum_{j=1}^{m} \gamma_{r_k j} d_j + \frac{d_r \vartheta}{1 - d_r \vartheta} \sum_{j=1}^{m} \gamma_{r_k j} d_j + \beta(r_k) - \frac{\vartheta \sigma^2(r_k)}{2} \right) \Delta + o(\Delta) \right]$$

$$+ \vartheta \left( a(r_k) + \beta(r_k) - (\vartheta - 1) \sigma^2(r_k) \right) \left( 1 + e^{-Z_k} \right)^{\vartheta - 1} \xi_{r_k} \Delta + \frac{\vartheta (\vartheta - 1) \sigma^2(r_k)}{2 (1 + e^{-Z_k})^{2 - \vartheta}} \xi_{r_k} \Delta.$$

Choose a small constant $0 < \vartheta_1 \leq \vartheta_0$ such that for any $0 < \vartheta \leq \vartheta_1$,

$$\pi \beta + \vartheta \left( \frac{d_i}{1 - d_i \vartheta} \sum_{j=1}^{m} \gamma_{ij} d_j - \frac{\sigma^2(i)}{2} \right) > 0, \quad i \in S.$$ 

Now, choose a positive constant $\bar{\lambda} = \bar{\lambda}(\vartheta) < 1$ sufficiently small such that it satisfies

$$\pi \beta + \vartheta \left( \frac{d_i}{1 - d_i \vartheta} \sum_{j=1}^{m} \gamma_{ij} d_j - \frac{\sigma^2(i)}{2} \right) - \frac{\bar{\lambda}}{\vartheta} > 0, \quad i \in S.$$ 

Then we have

$$\mathbb{E}\left[ (1 + e^{-Z_{k+1}})^\vartheta \xi_{r_{k+1}} \mid \mathcal{F}_k \right] \leq (1 + e^{-Z_k})^\vartheta \xi_{r_k} \left[ 1 - \frac{\bar{\lambda}}{2} \Delta + o(\Delta) \right] - \frac{\bar{\lambda}}{2} (1 + e^{-Z_k})^{\vartheta} \xi_{r_k} \Delta$$

$$+ \vartheta \left( a(r_k) + \beta(r_k) - (\vartheta - 1) \sigma^2(r_k) \right) \left( 1 + e^{-Z_k} \right)^{\vartheta - 1} \xi_{r_k} \Delta + \frac{\vartheta (\vartheta - 1) \sigma^2(r_k)}{2 (1 + e^{-Z_k})^{2 - \vartheta}} \xi_{r_k} \Delta$$

$$\leq \left( 1 - \frac{\bar{\lambda}}{2} \Delta + o(\Delta) \right) \left( 1 + e^{-Z_k} \right)^{\vartheta} \xi_{r_k} \Delta.$$ 

Choose $\Delta_3^* \in (0, 1)$ sufficiently small such that $\Delta_3^* < 4/\bar{\lambda}$ and $o(\Delta_3^*) \leq \bar{\lambda} \Delta_3^*/4$. For any $\Delta \in (0, \Delta_3^*)$ yields

$$\mathbb{E}\left[ (1 + e^{-Z_{k+1}})^\vartheta \xi_{r_{k+1}} \mid \mathcal{F}_k \right] \leq \left( 1 - \frac{\bar{\lambda}}{4} \Delta \right) \left( 1 + e^{-Z_k} \right)^{\vartheta} \xi_{r_k} \Delta$$

27
for any integer $k \geq 0$. Obviously,

$$
E\left[ (1 + e^{-Z_{k+1}}) \xi_{r_k}^d \right] \leq \left( 1 - \frac{\bar{\lambda}}{4} \Delta \right) E\left[ (1 + e^{-Z_k}) \xi_{r_k}^d \right] + C \Delta.
$$

Define $\Omega_k = \{ \bar{Z}_k > \log(K\Delta^{-\theta}) \}$, we have

$$
(1 + e^{-Z_k}) \xi_{r_k}^d = (1 + e^{-Z_k}) \xi_{r_k}^d I_{\Omega_k} + (1 + e^{-Z_k}) \xi_{r_k}^d I_{\Omega_k^c}
\leq (1 + K^{-1} \Delta^{-\theta}) \xi_{r_k}^d I_{\Omega_k} + (1 + e^{-Z_k}) \xi_{r_k}^d.
$$

By Chebyshev’s inequality and Lemma 4.8

$$
E\left[ (1 + e^{-Z_k}) \xi_{r_k}^d \right] \leq C E\left[ I_{\Omega_k} \right] + E\left[ (1 + e^{-Z_k}) \xi_{r_k}^d \right]
\leq \left( 1 - \frac{\bar{\lambda}}{4} \Delta \right) E\left[ (1 + e^{-Z_{k-1}}) \xi_{r_k-1}^d \right] + C \Delta + C \mathbb{P}\{ \bar{Z}_k > \log(K\Delta^{-\theta}) \}
\leq \left( 1 - \frac{\bar{\lambda}}{4} \Delta \right) E\left[ (1 + e^{-Z_{k-1}}) \xi_{r_k-1}^d \right] + C \left( \Delta + \frac{E\left[ e^{Z_k/\theta} \right]}{K^{1/\theta} \Delta^{-1}} \right)
\leq \left( 1 - \frac{\bar{\lambda}}{4} \Delta \right) E\left[ (1 + e^{-Z_{k-1}}) \xi_{r_k-1}^d \right] + C \Delta.
$$

Repeating this procedure arrives at

$$
E\left[ (1 + e^{-Z_k}) \xi_{r_k}^d \right] \leq e^{-\frac{\Delta}{4} k} (1 + x_0^{-1}) \xi_{r_k}^d + \frac{4C}{\bar{\lambda}} \left[ 1 - \left( 1 - \frac{\bar{\lambda}}{4} \Delta \right)^k \right] \leq C
$$

for any integer $k \geq 0$. Therefore, $\sup_{k \geq 0} E\left[ e^{-\theta Z_k} \right] \leq C$. The desired assertion follows.

**Theorem 4.4** Under the condition of Lemma 4.11, the numerical solutions $X_k$ are stochastically ultimately lower bounded.

**Proof.** The proof is an application of Chebyshev’s inequality, so we omit it. ■

**Theorem 4.5** Under the condition of Theorem 4.2 and $\hat{a} > 0$, for any $\Delta \in (0, \Delta^*_1 \wedge \Delta^*_3]$, the numerical solutions $X_k$ are stochastically permanent.

Moreover, we obtain the following improved necessary and sufficient conditions for the dynamical behaviors of the numerical solutions $X_k$ defined by (3.2).

**Theorem 4.6** Suppose that $\pi \beta \neq 0$. For any $\Delta \in (0, \Delta^*_1 \wedge \Delta^*_3]$, if $\hat{a} > 0$,

- the numerical solutions $X_k$ are stochastically permanent if and only if $\pi \beta > 0$;
the numerical solutions $X_k$ are almost surely extinctive if and only if $\pi \beta < 0$.

In particular, if $\pi a = 0$,

- the numerical solutions $X_k$ are almost surely extinctive if and only if $\pi \beta < 0$;
- almost all paths of $X_k$ increase at an exponential rate if and only if $\pi \beta > 0$.

5 Stability in distribution

In this section, we first give sufficient conditions that guarantee SDS (1.1) is asymptotically stable in distribution. Then we show that the explicit schemes (3.2) can approximate the invariant measure of SDS (1.1) effectively. From this section as a standing assumption, we always assume $\hat{a} > 0$. For the convenience of invariant measure study we introduce some notations. We write $(x_{i0}, r_l)$ in lieu of $(x_t, r_l)$ to highlight the initial data $(x(0), r(0)) = (x_0, \ell)$. Following [20, p.212], we denote by $\mathcal{P}(\mathbb{R}_+ \times \mathbb{S})$ the space of all probability measures on $\mathbb{R}_+ \times \mathbb{S}$ and for $\bar{\mu}, \bar{\nu} \in \mathcal{P}(\mathbb{R}_+ \times \mathbb{S})$ define

$$d_L(\bar{\mu}, \bar{\nu}) := \sup_{H \in \mathcal{L}} \left| \sum_{i=1}^m \int_{\mathbb{R}_+} H(x, i) \bar{\mu}(dx, i) - \sum_{i=1}^m \int_{\mathbb{R}_+} H(x, i) \bar{\nu}(dx, i) \right|$$

where

$$\mathcal{L} := \left\{ H : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R} : |H(x, i) - H(y, j)| \leq |x - y| + |i - j|, \ |H(\cdot, \cdot)| \leq 1 \right\}.$$

By virtue of Lemma 2.1, SDS (1.1) has a unique continuous positive solution $(x_{i0}, r_l)$, which is a time-homogeneous Markov process. Let $P_t(x_0, \ell; dx \times \{i\})$ denote the transition probability of the process $(x_{i0}, r_l)$.

**Definition 5.1 ([20])** The process $(x(t), r(t))$ is said to be asymptotically stable in distribution if there exists a probability measure $\mu(\cdot \times \cdot)$ on $\mathbb{R}_+ \times \mathbb{S}$ such that the transition probability $P_t(x_0, \ell; \cdot \times \cdot)$ of $(x(t), r(t))$ converges weakly to $\mu(\cdot \times \cdot)$ as $t \to \infty$ for every $(x_0, \ell) \in \mathbb{R}_+ \times \mathbb{S}$. SDS (1.1) is said to be asymptotically stable in distribution if $(x(t), r(t))$ is asymptotically stable in distribution.

It is easy to observe that Theorem 4.1 guarantees that for any $(x_0, \ell) \in \mathbb{R}_+ \times \mathbb{S}$, the family of transition probabilities $\{P_t(x_0, \ell; \cdot \times \cdot) : t \geq 0\}$ is tight. That is, for any $\varepsilon > 0$ there is a compact subset $K_+ = K_+(\varepsilon, x_0, \ell)$ of $\mathbb{R}_+$ such that

$$P_t(x_0, \ell; K_+ \times \mathbb{S}) \geq 1 - \varepsilon \quad \forall t \geq 0.$$  \hspace{1cm} (5.1)

Next we give the existence and uniqueness of the invariant measure for the solution $(x_{i0}, r_l)$ of SDS (1.1).
Lemma 5.1 ([10]) The solutions of SDS (1.1) satisfy
\[ \lim_{t \to \infty} \mathbb{E} \left[ x_t^{x_0, \ell} - x_t^{\bar{x}_0, \ell} \right] = 0 \]
uniformly in \((x_0, \bar{x}_0, \ell) \in \mathbb{K}_+ \times \mathbb{K}_+ \times S\), where \(x_t^{x_0, \ell}, x_t^{\bar{x}_0, \ell}\) are respectively two solutions of SDS (1.1) with initial values \(x_0, \bar{x}_0\) for any compact subset \(\mathbb{K}_+\) of \(\mathbb{R}_+\), and \(\ell \in S\).

Lemma 5.2 ([10]) For any compact subset \(\mathbb{K}_+\) of \(\mathbb{R}_+\) and any \(T \geq 0\),
\[ \sup_{(x_0, \ell) \in \mathbb{K}_+ \times S} \mathbb{E} \left[ \sup_{0 \leq t \leq T} x^2(t) \right] \leq C_T, \]
where \(x(t)\) is the solution of SDS (1.1) with the initial value \((x_0, \ell) \in \mathbb{K}_+ \times S\).

Lemma 5.3 If \(\pi \beta > 0\) hold, for any compact subset \(\mathbb{K}_+\) of \(\mathbb{R}_+\),
\[ \sup_{(x_0, \ell) \in \mathbb{K}_+ \times S} \mathbb{E} \left[ \sup_{0 \leq t \leq T} x^{-\vartheta}(t) \right] \leq C_T, \quad \forall T \geq 0, \]
where \(\vartheta\) is given by Lemma 4.6, \(x(t)\) is the solution of SDS (1.1) with the initial value \((x_0, \ell) \in \mathbb{K}_+ \times S\).

Proof. Borrowing the proof method of [15, Lemma 3.5] we can get the desired result but omit the details to avoid duplication.

By Lemma 5.3 and Jensen’s inequality, \((\mathbb{E} x^\vartheta(t))^{-1} \leq \mathbb{E} [x^{-\vartheta}(t)] \leq C_T\). So
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} x^\vartheta(t) \right] \geq \mathbb{E} x^\vartheta(t) \geq \frac{1}{C_T} > 0. \quad (5.2) \]
Using techniques in the proofs of [20] Lemmas 5.6 and 5.7, we obtain the following lemmas.

Lemma 5.4 Under the condition of Theorem 4.4,
\[ \lim_{t \to \infty} d_{\mathbb{L}}(\mathcal{P}_t(x_0, \ell; \cdot \times \cdot), \mathcal{P}_t(\bar{x}_0, \bar{\ell}; \cdot \times \cdot)) = 0 \]
uniformly in \(x_0, \bar{x}_0 \in \mathbb{K}_+\) and \(\ell, \bar{\ell} \in S\).

Lemma 5.5 Under the condition of Lemma 5.4, for any \((x_0, \ell) \in \mathbb{R}_+ \times S\), \(\{\mathcal{P}_t(x_0, \ell; \cdot \times \cdot)\}_{t \geq 0}\) is Cauchy in the space \(\mathcal{P}(\mathbb{R}_+ \times S)\) with metric \(d_{\mathbb{L}}\).

Theorem 5.1 Under the condition of Lemma 5.4, SDS (1.1) is asymptotically stable in distribution.
Proof. For any $x \in \mathbb{R}_+$, define $v(x) := \log(x)$. For any compact subset $\mathbb{K}_+$ of $\mathbb{R}_+$, and any $(x_0, \ell) \in \mathbb{K}_+ \times \mathbb{S}$, define $\hat{P}(v(x_0), \ell; \mathbb{D} \times \{i\}) := P_t(x_0, \ell; v^{-1}(\mathbb{D}) \times \{i\}) \forall \mathbb{D} \in \mathcal{B}(\mathbb{R}), i \in \mathbb{S}$. Then for any $i \in \mathbb{S}$, $\hat{P}(v(x_0), \ell; \cdot \times i)$ is the transform of $P_t(x_0, \ell; \cdot \times i)$ corresponding to $v(\cdot)$. By the well-known Chebyshev inequality, it is easy to observe that Lemmas 4.3 and 4.6 guarantees that for any $(v(x_0), \ell) \in \mathbb{R} \times \mathbb{S}$, the family of transition probabilities $\{\hat{P}(v(x_0), \ell; \cdot \times \cdot) : t \geq 0\}$ is tight on $\mathbb{R} \times \mathbb{S}$. Since $(\mathbb{R}, |\cdot|)$ is complete and separable, the tightness of $\{\hat{P}(v(x_0), \ell; \cdot \times \cdot) : t \geq 0\}$ on $\mathbb{R} \times \mathbb{S}$ is equivalent to relatively compactness (see [5, Theorems 6.1, 6.2]). Then any sequence $\{P_{t_n}(v(x_0), \ell; \cdot \times \cdot) : n \geq 0\}$ ($t_n \to \infty$ as $n \to \infty$) has a weak convergent subsequence denoted by $\{\hat{P}_{t_n}(v(x_0), \ell; \cdot \times \cdot) : n \geq 0\}$ with some notation abuse. Assume its weak limit is an invariant measure $\hat{\mu}(\cdot \times \cdot)$ on $\mathbb{R} \times \mathbb{S}$. Define $\mu(\mathbb{K}_+ \times \cdot) = \hat{\mu}(v(\mathbb{K}_+) \times \cdot) \forall \mathbb{K} \in \mathcal{B}(\mathbb{R}_+)$. Then $\mu(\cdot \times \cdot)$ is an invariant measure on $\mathbb{R}_+ \times \mathbb{S}$, and the corresponding further subsequence of $\{P_{t_n}(x_0, \ell; \cdot \times \cdot) : n \geq 0\}$ converges weakly to $\mu(\cdot \times \cdot)$ on $\mathbb{R}_+ \times \mathbb{S}$. The following proof is the same way as the proof of [20, Theorem 5.43], so we omit it here.  

5.1 Stability in distribution of numerical solution

We write $(X_{x_0, \ell}, r_k)$ in lieu of $(X_k, r_k)$ to highlight the initial data $(X_0, r_0) = (x_0, \ell)$. By (3.2), we know that $X_{x_0, \ell} = e^{Z_{x_0, \ell}}$ and $x_0 = e^{y_0}$, where $Z_{x_0, \ell}$ denote the numerical solutions defined by (3.1) with initial data $(y_0, \ell)$. Similar to that of [20, Theorem 6.14], we can prove the following result.

Lemma 5.6 $\{(Z_k, r_k)\}_{k \geq 0}$ is a time homogeneous Markov chain.

It is easy to observe that Theorem 4.5 guarantees that for any $(x_0, \ell) \in \mathbb{R}_+ \times \mathbb{S}$, the family of transition probabilities $\{P^\Delta_k(x_0, \ell; \cdot \times \cdot) : k \geq 0\}$ is tight. That is, for any $\varepsilon > 0$ there is a compact subset $\mathbb{K}_+ = \mathbb{K}_+(\varepsilon, x_0, \ell)$ of $\mathbb{R}_+$ such that

$$P^\Delta_k(x_0, \ell; \mathbb{K}_+ \times \mathbb{S}) \geq 1 - \varepsilon \quad \forall k \geq 0. \quad (5.3)$$

To show the numerical solutions $(X_{x_0, \ell}, r_k)$ defined by (3.2) is asymptotically stable in distribution and admit a unique invariant measure $\mu^\Delta(\cdot \times \cdot) \in \mathcal{P}(\mathbb{R}_+ \times \mathbb{S})$, we need the following three lemmas, the proofs of which can be found in Appendix B.

Lemma 5.7 The numerical solutions defined by (3.2) has the property that

$$\lim_{k \to \infty} \mathbb{E}|X_{x_0, \ell} - X_{x_0, \ell}| = \lim_{k \to \infty} \mathbb{E}|e^{Z_{x_0, \ell}} - e^{Z_{x_0, \ell}}| = 0$$

uniformly in $(x_0, \bar{x}_0, \ell) \in \mathbb{K}_+ \times \mathbb{K}_+ \times \mathbb{S}$, for any compact subset $\mathbb{K}_+$ of $\mathbb{R}_+$.  

31
Lemma 5.8  For any compact subset $\mathbb{K}_+$ of $\mathbb{R}_+$,

$$
\sup_{(x_0, \ell) \in \mathbb{K}_+ \times \mathbb{S}} \mathbb{E} \left[ \sup_{0 \leq k \Delta \leq T} X_k^2 \right] \leq C_T, \quad \forall \; T \geq 0,
$$

where $X_k$ is the numerical solutions defined by (3.2) with the initial value $(x_0, \ell) \in \mathbb{K}_+ \times \mathbb{S}$.

Lemma 5.9  Under the condition Lemma 5.3, for any compact subset $\mathbb{K}_+$ of $\mathbb{R}_+$,

$$
\sup_{(x_0, \ell) \in \mathbb{K}_+ \times \mathbb{S}} \mathbb{E} \left[ \sup_{0 \leq k \Delta \leq T} X_k^{-\theta} \right] \leq C_T, \quad \forall \; T \geq 0,
$$

where $X_k$ is the numerical solutions defined by (3.2) with the initial value $(x_0, \ell) \in \mathbb{K}_+ \times \mathbb{S}$.

By Lemma 5.9 and Jensen’s inequality we can get

$$
\mathbb{E} \left[ \sup_{0 \leq k \Delta \leq T} X_k^\theta \right] \geq \mathbb{E} X_k^\theta \geq \frac{1}{C_T} > 0. \quad (5.4)
$$

Using techniques in the proofs of [20, Lemmas 6.11, 6.12 and 6.16], we obtain the following three lemmas, the proofs of which are straightforward, so are omitted.

Lemma 5.10  Under the condition of Theorem 5.1, for any $\Delta \in (0, \Delta^* \wedge \Delta^*_3)$,

$$
\lim_{k \to \infty} d_L(P^\Delta_k(x_0, \ell; \cdot \times \cdot), P^\Delta_k(\bar{x}_0, \bar{\ell}; \cdot \times \cdot)) = 0
$$

uniformly in $x_0, \bar{x}_0 \in \mathbb{K}_+$ and $\ell, \bar{\ell} \in \mathbb{S}$, for any compact subset $\mathbb{K}_+$ of $\mathbb{R}_+$.

Lemma 5.11  Under the condition of Lemma 5.10, $\{P^\Delta_k(x_0, \ell; \cdot \times \cdot)\}_{k \geq 0}$ is Cauchy in the space $\mathcal{P}(\mathbb{R}_+ \times \mathbb{S})$ with metric $d_L$.

Lemma 5.12  Fix any $(x_0, \ell) \in \mathbb{R}_+ \times \mathbb{S}$. Then for any given $T > 0$ and $\varepsilon > 0$, there is a $\Delta^{**} \in (0, 1)$, which is sufficiently small, such that

$$
d_L(P^\Delta_k(x_0, \ell; \cdot \times \cdot), P_k(x_0, \ell; \cdot \times \cdot)) < \varepsilon
$$

provided $\Delta \in (0, \Delta^{**})$ and $k\Delta \leq T$.

Theorem 5.2  Under the condition of Theorem 5.1, for any $\Delta \in (0, \Delta^* \wedge \Delta^*_3)$, the numerical solutions $(X_k^{x_0, \ell}, r_k^\ell)$ is asymptotically stable in distribution and admit a unique invariant measure $\mu^\Delta(\cdot \times \cdot) \in \mathcal{P}(\mathbb{R}_+ \times \mathbb{S})$.  

32
Lemma 6.1 Let \( \sigma \) and any \( \Delta \) \( \in \mathbb{R} \), then for any \( i \in \mathbb{S} \), \( \mathbf{P}_k^\Delta(v(x_0), \ell; \cdot \times i) \) is the transform of \( \mathbf{P}_k^\Delta(x_0, \ell; \cdot \times i) \) corresponding to \( v(\cdot) \). By the well-known Chebyshev inequality, it is easy to observe that Lemmas 4.7 and 4.11 guaranties that for any \( (v(x_0), \ell) \in \mathbb{R} \times \mathbb{S} \), the family of transition probabilities \( \mathbf{P}_k^\Delta(v(x_0), \ell; \cdot \times \cdot) : k \geq 0 \) is tight on \( \mathbb{R} \times \mathbb{S} \). Since \( (\mathbb{R}, |\cdot|) \) is complete and separable, the tightness of \( \mathbf{P}_k^\Delta(v(x_0), \ell; \cdot \times \cdot) : k \geq 0 \) on \( \mathbb{R} \times \mathbb{S} \) is equivalent to relatively compactness. Then any sequence \( \{\mathbf{P}_{k_n}^\Delta(v(x_0), \ell; \cdot \times \cdot) : n \geq 0\} \) \((k_n \to \infty \text{ as } n \to \infty)\) has a weak convergent subsequence denoted by \( \{\mathbf{P}_{\tilde{k}}^\Delta(v(x_0), \ell; \cdot \times \cdot) : n \geq 0\} \) with some notation abuse. Assume its weak limit is an invariant measure \( \bar{\mu}(\cdot \times \cdot) \) on \( \mathbb{R} \times \mathbb{S} \). Define \( \mu^\Delta(\mathbb{K}_+ \times \cdot) = \bar{\mu}^\Delta(v(\mathbb{K}_+) \times \cdot) \forall \mathbb{K}_+ \in \mathcal{B}(\mathbb{K}_+) \). The following proof is the same way as the proof of [20, Theorem 6.19], so we omit it here.

Proof. For any \( x \in \mathbb{R}_+ \), define \( v(x) := \log(x) \). For any compact subset \( \mathbb{K}_+ \) of \( \mathbb{R}_+ \), and any \( (x_0, \ell) \in \mathbb{K}_+ \times \mathbb{S} \), define \( \mathbf{P}_k^\Delta(v(x_0), \ell; \mathbb{D} \times \{i\}) := \mathbf{P}_k^\Delta(x_0, \ell; v^{-1}(\mathbb{D}) \times \{i\}) \forall \mathbb{D} \in \mathcal{B}(\mathbb{R}), i \in \mathbb{S} \). Then for any \( i \in \mathbb{S} \), \( \mathbf{P}_k^\Delta(v(x_0), \ell; \cdot \times i) \) is the transform of \( \mathbf{P}_k^\Delta(x_0, \ell; \cdot \times i) \) corresponding to \( v(\cdot) \). By the well-known Chebyshev inequality, it is easy to observe that Lemmas 4.7 and 4.11 guaranties that for any \( (v(x_0), \ell) \in \mathbb{R} \times \mathbb{S} \), the family of transition probabilities \( \mathbf{P}_k^\Delta(v(x_0), \ell; \cdot \times \cdot) : k \geq 0 \) is tight on \( \mathbb{R} \times \mathbb{S} \). Since \( (\mathbb{R}, |\cdot|) \) is complete and separable, the tightness of \( \mathbf{P}_k^\Delta(v(x_0), \ell; \cdot \times \cdot) : k \geq 0 \) on \( \mathbb{R} \times \mathbb{S} \) is equivalent to relatively compactness. Then any sequence \( \{\mathbf{P}_{k_n}^\Delta(v(x_0), \ell; \cdot \times \cdot) : n \geq 0\} \) \((k_n \to \infty \text{ as } n \to \infty)\) has a weak convergent subsequence denoted by \( \{\mathbf{P}_{\tilde{k}}^\Delta(v(x_0), \ell; \cdot \times \cdot) : n \geq 0\} \) with some notation abuse. Assume its weak limit is an invariant measure \( \bar{\mu}(\cdot \times \cdot) \) on \( \mathbb{R} \times \mathbb{S} \). Define \( \mu^\Delta(\mathbb{K}_+ \times \cdot) = \bar{\mu}^\Delta(v(\mathbb{K}_+) \times \cdot) \forall \mathbb{K}_+ \in \mathcal{B}(\mathbb{K}_+) \). The following proof is the same way as the proof of [20, Theorem 6.19], so we omit it here.

We can now show that the numerical stationary distribution will weakly converge to the stationary distribution of the exact solutions.

Theorem 5.3 Under the condition of Theorem 5.1, 
\[
\lim_{\Delta \to 0} d_{\text{c}}(\mu(\cdot \times \cdot), \mu^\Delta(\cdot \times \cdot)) = 0.
\]

The proof of this theorem is standard (see, e.g. [20, Theorem 6.23]), and hence is omitted to avoid repetition.

6 Numerical examples

In order to illustrate the efficiency of numerical schemes we consider a number of examples and present some simulations. First, we will show that the classical EM method will not be able to reproduce the dynamical properties of the SDS (1.1). To show this, recall that the classical EM method applied to (1.1) produces
\[
\begin{cases}
  X_k^0 = x_0, \\
  X_{k+1}^\Delta = X_k^\Delta + X_k^\Delta \left[ (b(r_k) - a(r_k)X_k^\Delta)\Delta + \sigma(r_k)\Delta B_k \right].
\end{cases}
\]

We choose a number \( \bar{\Delta} \in (0, (0.4e - 1)/|b|] \), the following lemma shows that for any given stepsize \( \Delta \in (0, \bar{\Delta}] \) and any initial value \( (x_0, \ell) \in \mathbb{R}_+ \times \mathbb{S} \), the numerical solutions \( \{|X_k^\Delta|\}_{k \geq 1} \) will tend to infinity super-exponentially with a positive probability.

Lemma 6.1 Let \( \{X_k^\Delta\}_{k \geq 1} \) be defined by (6.1). Suppose for any \( i \in \mathbb{S} \), \( a(i) - 0.5|\sigma(i)| \geq 1.4 \) and \( \sigma(i) \neq 0 \). Then the conditional probability
\[
\mathbb{P}\left( |X_{k+1}^\Delta| \geq \exp(2k) \frac{|X_k^\Delta|}{\Delta}, \forall k \geq 1 \right| |X_1^\Delta| \geq \frac{e}{\Delta}) \geq \exp\left( -\frac{4e^{-2k}}{\exp(2k) - 1} \right).
\]
The proof of this lemma can be found in Appendix C. It then follows from Lemma 6.1 and (C.1) that
\[
P \left( |X_\Delta^k| \geq \frac{\exp(2^{k-1})}{\Delta}, \forall k \geq 1 \right) \\
= P \left( |X_1^\Delta| \geq \frac{e}{\Delta} \right) P \left( |X_{k+1}^\Delta| \geq \frac{\exp(2^k)}{\Delta}, \forall k \geq 1 \middle| |X_1^\Delta| \geq \frac{e}{\Delta} \right) > 0.
\]

In other words, \(|X_\Delta^k|\) will tend to infinity faster than \(\exp(2^{k-1})/\Delta\) with a positive probability. However, our theory established in the previous sections shows that the scheme (3.2) can reproduce the dynamical properties of the SDS (1.1) very well.

To illustrate our theory, as well as to compare to the simulations of the classical EM method, we shall illustrate these conclusions through the following examples.

**Example 6.1** In this example we consider SDS (1.1) with the Markov chain \(r(t)\) is on the state space \(S = \{1, 2\}\) with the generator
\[
\Gamma = \begin{pmatrix} -8 & 8 \\ 2 & -2 \end{pmatrix},
\]
and the coefficients in each state are given in Table 1. By solving the linear equation (4.1) we obtain the unique stationary (probability) distribution \(\pi = (\pi_1, \pi_2) = (0.2, 0.8)\).

| States \(i\)  | Coefficients \(b(i)\) | Coefficients \(a(i)\) | Coefficients \(\sigma(i)\) | Coefficients \(\beta(i) = b(i) - 0.5\sigma^2(i)\) |
|-----------|-----------------|-----------------|-----------------|-----------------|
| \(i = 1\) | 2               | 1.8             | 0.8             | 1.68            |
| \(i = 2\) | 1               | 2.5             | 2               | -1              |

Table 1: Values of the coefficients in Example 6.1

Compute
\[
\pi a > 0, \quad \pi \beta = \pi_1 \beta(1) + \pi_2 \beta(2) = -0.4640.
\]

Therefore, by Theorem 4.2
\[
\limsup_{t \to \infty} \frac{\log x(t)}{t} \leq \pi \beta < 0 \quad a.s.
\]

In other words, this says that the extinction of the population happens. However, by virtue of Lemma 6.1, for any given stepsize \(\Delta \in (0, 0.04]\) and initial value \((x_0, \ell) = (25, 1)\), one observes that \(|X_\Delta^k|\) will tend to infinity super-exponentially with a positive probability, see Figs. 1 and 2. Both simulations show that the classical EM method does not capture the dynamic properties of the underlying SDE (1.1), while the second simulation shows that the classical EM method can blow up very quickly. This contrasts with the extinction of the underlying SDS (1.1).
Figure 1: A sample path of the classical EM solution $X^\Delta_k$ and corresponding state $r_k$. The pink trajectory represents Markov chain while the blue trajectory represents the numerical solution of Scheme (6.1) with $\Delta = 0.02$ and $t \in [0,100]$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Figure 2: A sample path of the classical EM solution $\log(|X^\Delta_k|)$ and corresponding state $r_k$. The pink trajectory represents Markov chain while the blue trajectory represents the numerical solution of Scheme (6.1) with $\Delta = 0.02$ and $t \in [0,100]$. 
In order to represent the simulations by Scheme (3.2) and illustrate its effectiveness, we divide it into five steps.

**Step 1.** MATLAB code. Next we specify the MATLAB code for simulating $X^E_k$ and $X_k$:

```
%MATLAB code for simulating $X^E_k$ and $X_k$
1 clear all;
2 X(1)=25; XE(1)=25; r=1; T=100; Y=log(XE(1)); Z=log(X(1));
3 b=[2,1]; a=[1.8,2.5]; cgm=[0.8,2]; beat=b-0.5*cgm.^2;
4 Gam=[-8 8;2 -2]; dt=0.02; dB=sqrt(dt)*randn(1,T/dt);
5 c=expm(Gam*dt); v=log(10*dt^(-2/5)); % Obviously, v>Z;
6 for n=1:T/dt
7     Y=Y+(beat(r)-a(r)*exp(Y))*dt+cgm(r)*dB(n);
8     Z=Z+(beat(r)-a(r)*exp(Z))*dt+cgm(r)*dB(n);
9     if rand<c(r,1)
10        r=1;
11    else
12        r=2;
13    end
14    XE(n+1)=exp(Y);
15    if Z>v
16        Z=v;
17    end
18    X(n+1)=exp(Z);
19 end
```

**Step 2.** Approximating the error $\mathbb{E}|x(T) - X_\Delta(T)|^p$. To compute the approximation error, we run $M$ independent trajectories where $x^{(j)}(t)$ and $X^{(j)}_\Delta(t)$ represent the $j$th trajectories of exact solution $x(t)$ and the numerical solution $X_\Delta(t)$ respectively. Thus

$$
\mathbb{E}|x(T) - X_\Delta(T)|^p = \frac{1}{M} \sum_{j=1}^{M} |x^{(j)}(T) - X^{(j)}_\Delta(T)|^p.
$$

**Step 3.** The log-log error plot with $M = 2000$. The simulation procedure is carried out by steps 1 and 2. The red dashed line depicts log-log error while the blue solid line is a reference line of slope 1/2 in Fig. 3. Fig. 3 depicts the approximation error $\mathbb{E}|x(32) - X^E_\Delta(32)|$ between the exact solution of the SDS (1.1) and the numerical solution by Scheme (3.15), and the error $\mathbb{E}|x(32) - X_\Delta(32)|$ between the exact solution and that by Scheme (3.2) with $\theta = 0.5$, as the function of stepsize $\Delta \in \{2^{-1}, 2^{-2}, \ldots, 2^{-14}\}$.
Figure 3: The red asterisk trajectory depicts the approximation error $E| x(32) - X^E_{\Delta}(32) |$ of the exact solution of SDS (1.1) and the numerical solution by Scheme (3.15) while the red circle trajectory depicts the approximation error $E| x(32) - X_{\Delta}(32) |$ of the exact solution of SDS (1.1) and the numerical solution by Scheme (3.2) as the functions of stepsize $\Delta \in \{2^{-1}, 2^{-2}, \ldots, 2^{-14}\}$.

**Step 4.** To compare to the simulations of the classical EM method. The simulation procedure is carried out by step 1, and all parameters are same as the classical EM method. The two simulations shown in Figs. 4 and 5 are based on Scheme (3.2). Both figures show clearly that Scheme (3.2) reproduces the dynamic properties of the underlying SDS (1.1).

Figure 4: A sample path of the numerical solutions $X_k$ and corresponding state $r_k$. The pink trajectory represents Markov chain while the blue trajectory represents the numerical solution of Scheme (3.2) with $\Delta = 0.02$, $\theta = 0.4$ and $t \in [0, 100]$. 

37
Figure 5: A sample path of the numerical solutions $X_k$ and corresponding state $r_k$. The pink trajectory represents Markov chain while the blue trajectory represents the numerical solution of Scheme (3.2) with $\Delta = 0.02$, $\theta = 0.4$ and $t \in [0, 100]$.

**Step 5.** Further show that Scheme (3.2) can reproduce this extinction very well. Fig. 6 depicts 500 sample paths of the numerical solution of Scheme (3.2). This figures show clearly that Scheme (3.2) reproduces positivity and extinction of the underlying SDS (1.1) (see the enlargement in Fig. 6).

Figure 6: 500 trajectories of the numerical solution of Scheme (3.2) with $\Delta = 0.02$, $\theta = 0.4$ and $t \in [0, 200]$.

Before closing this section we carry out some simulations to illustrate the efficiency of Scheme (3.2) in the approximation of invariant measures.
Example 6.2  In this example we consider SDS (1.1) with the Markov chain \( r(t) \) is on the state space \( S = \{1, 2, 3\} \) with the generator
\[
\Gamma = \begin{pmatrix}
-10 & 0 & 10 \\
2 & -2 & 0 \\
0 & 1 & -1
\end{pmatrix},
\]
and the coefficients in each state are given in Table 2. By solving the linear equation (4.1) we obtain the unique stationary (probability) distribution \( \pi = (\pi_1, \pi_2, \pi_3) = \left( \frac{1}{16}, \frac{5}{16}, \frac{10}{16} \right) \).

| States | Coefficients | \( b(i) \) | \( a(i) \) | \( \sigma(i) \) | \( \beta(i) = b(i) - 0.5\sigma^2(i) \) |
|--------|--------------|------------|------------|-------------|-----------------|
| \( i = 1 \) | \( i = 2 \) | \( i = 3 \) | \( i = 1 \) | \( 0.7 \) | \( 0.3 \) | \( \sqrt{3} \) | \( -0.8 \) |
| \( i = 2 \) | \( 0.4 \) | \( 0.8 \) | \( 0.06 \) | \( 0.3982 \) |
| \( i = 3 \) | \( 1 \) | \( 0.5 \) | \( 0.04 \) | \( 0.9992 \) |

Table 2: Values of the coefficients in Example 7.1

Compute
\[
\pi a = 0.5813 > 0, \quad \pi \beta = 0.6989 > 0.
\]
Therefore, by Theorems 4.6 and 5.1 SDS (1.1) is stochastically permanent and asymptotically stable in distribution, namely the probability measure \( P_t(x_0, \ell; \cdot \times \cdot) \) of the solution \( x(t) \) tends to an invariant measure \( \mu(\cdot \times \cdot) \) asymptotically as \( t \to \infty \). On the other hand, by virtue of Theorem 4.5 the numerical solutions \( X_k \) are stochastically permanent. Meanwhile, by Theorems 5.2 and 5.3 the probability measure \( P_k^\Delta(x_0, \ell; \cdot \times \cdot) \) of the solution using Scheme (3.2) with any initial value \( (x_0, \ell) \in \mathbb{R}_+ \times S \) tends to a unique numerical invariant measure \( \mu^\Delta(\cdot \times \cdot) \) asymptotically as \( k \to \infty \), and \( \mu^\Delta(\cdot \times \cdot) \to \mu(\cdot \times \cdot) \) as \( \Delta \to 0 \).

Figure 7: (a) Computer simulation of a sample path of Markov chain \( r(t) \). (b) A sample path of numerical solution of Scheme (3.2) with \( \Delta = 10^{-2} \) and \( \theta = 0.4 \) (the blue solid line).
Next, in order to test the efficiency of the scheme, we carry out numerical experiments by implementing Scheme (3.2) using MATLAB. Let \( (x_0, \ell) = (0.5, 3) \) and take \( \Delta = 10^{-2}, K = 10 \) and \( \theta = 0.4 \). Fig. 7 (a) depicts the path of the Markov chain while Fig. 7 (b) further compares the path of the numerical solution \( X_{\Delta}(t) \). Fig. 8 depicts the empirical density of \( \mu^\Delta \), which predicts the stationary distribution.

7 First order strong convergence

In this section we focus on the numerical approximation of the stochastic logistic population system without regime switching (i.e. \( m = 1 \)), and we will show that construct the explicit schemes with strongly converges with rate one. For \( m = 1 \), we may consider without loss of generality that \( S = \{i\} \), \( b(i) \equiv b, a(i) \equiv a \) and \( \sigma(i) \equiv \sigma \), as a special case, the stochastic logistic population system

\[
dx(t) = x(t) \left[ (b - ax(t))dt + \sigma dB(t) \right]
\]

called the subsystem of the SDS (1.1) is permanent or not depends only on its parameters.

Remark 7.1 For the special case \( S = \{i\} \), the all results on numerical solutions in Sections 3, 4 and 5 still hold.

For the subsystem (7.1), the EM method (3.14) degenerate into the following form

\[
\begin{align*}
Y_0 &= \log x_0, \\
Y_{k+1} &= Y_k + \left( \beta - ae^{Y_k} \right) \Delta + \sigma \Delta B_k,
\end{align*}
\]
for any integer $k \geq 0$ and any $\Delta \in (0, 1]$. To study the rate of convergence of numerical solutions $\{X_k^E\}_{k \geq 0}$ and $\{X_k\}_{k \geq 0}$ of the subsystem $\{i, 1\}$, we need the following lemma, the proof of which can be found in Appendix D.

**Lemma 7.1** For any $q > 0$ there exists a constant $C_T > 0$ such that

$$
\mathbb{E}\left[ \sup_{k=0, \ldots, \lfloor T/\Delta \rfloor} |Y_k - y(t_k)|^q \right] \leq C_T \Delta^q
$$

for any $\Delta \in (0, 1]$ and $T > 0$.

**Theorem 7.1** For any $p > 0$ there exists a constant $C_T > 0$ such that

$$
\sup_{k=0, \ldots, \lfloor T/\Delta \rfloor} \mathbb{E}\left[ |x(t_k) - X_k^E|^p \right] \leq C_T \Delta^p
$$

for any $\Delta \in (0, 1]$ and $T > 0$.

**Proof.** Using Hölder’s inequality, for any $\delta > 1$, we have

$$
\mathbb{E}\left[ |x(t_k) - X_k^E|^p \right] \leq \mathbb{E}\left[ (e^{py(t_k)} + e^{pY_k})^{\delta/(\delta-1)} \right] \left( \mathbb{E}\left[ \sup_{k=0, \ldots, \lfloor T/\Delta \rfloor} |y(t_k) - Y_k|^p \right] \right)^{1/\delta}.
$$

Thus, by applying Lemma 7.1, Corollary D.1 and (2.1), we infer that

$$
\mathbb{E}\left[ |x(t_k) - e^{Y_k}|^p \right] \leq C_T \Delta^p
$$

for any $\Delta \in (0, 1]$. The proof is complete. $\blacksquare$

**Theorem 7.2** If $a > 0$, for any $p > 0$ there exists a constant $C_T > 0$ such that

$$
\mathbb{E}\left[ \sup_{k=0, \ldots, \lfloor T/\Delta \rfloor} |x(t_k) - X_k^E|^p \right] \leq C_T \Delta^p
$$

for any $\Delta \in (0, \Delta^*)$ and $T > 0$.

**Proof.** Now the mean value theorem implies

$$
\mathbb{E}\left[ \sup_{k=0, \ldots, \lfloor T/\Delta \rfloor} |X(t_k) - e^{Y_k}|^p \right] \leq \mathbb{E}\left[ \sup_{k=0, \ldots, \lfloor T/\Delta \rfloor} (e^{py(t_k)} + e^{pY_k}) |y(t_k) - Y_k|^p \right].
$$

Using Hölder’s inequality, for any $\frac{p-2}{p} \vee 0 < \frac{1}{\delta} < 1$, we have

$$
\mathbb{E}\left[ \sup_{k=0, \ldots, \lfloor T/\Delta \rfloor} |x(t_k) - e^{Y_k}|^p \right]
$$
Thus, by applying Lemmas 7.1, 5.2 and 5.8 as well as (2.1), we infer that

\[
\mathbb{E} \left[ \sup_{k=0, \ldots, [T/\Delta]} |x(t_k) - e^{Y_k}|^p \right] \leq C T^p \Delta^p.
\]

The proof is complete.

**Remark 7.2** Using the same method as employed in the proofs of Section 3, we can easily also obtain the first order strong convergence rate for the approximation of the original SDE (7.1) by \(X_k := e^{Z_k}\), and hence is omitted to avoid repetition.

**Remark 7.3** By virtue of Theorem 5.1, we know that SDE (7.1) is asymptotically stable in distribution. On the other hand, under the condition of Theorem 5.1, by solving the Fokker-Planck equation (see details in [26]), the process \(x(t)\) has a unique stationary distribution \(\mu(\cdot)\), and obeys the Gamma distribution with parameter

\[
\alpha_1 = \frac{2b}{\sigma^2} - 1, \quad \alpha_2 = \frac{2a}{\sigma^2},
\]

with a notation abuse slightly, we write \(x \sim \text{Ga}(\alpha_1, \alpha_2)\), with density

\[
p(x) = \left( \frac{\alpha_2}{\Gamma(\alpha_1)} \right)^{\alpha_1} x^{\alpha_1-1} e^{-\alpha_2 x}, \quad x > 0,
\]

where \(\Gamma(\cdot)\) is the Gamma function. By the strong law of large numbers we deduce that

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t x(s) \, ds = \int_0^\infty x p(x) \, dx := \frac{\alpha_1}{\alpha_2} = \frac{2b - \sigma^2}{2a} \quad \text{a.s.}
\]

Let us discuss an example and present some simulations to illustrate our theory before closing this section to highlight the advantages of our new results on the convergence rates.

**Example 7.1** In this example we consider autonomous stochastic logistic model (7.1) of the form

\[
dx(t) = x(t)(0.5 - 0.8x(t)) \, dt + 0.3x(t) \, dB(t) \tag{7.3}
\]

with an initial value \(x_0 = 50\). Then we have

\[
a > 0, \quad \beta = b - 0.5\sigma^2 = 0.455 > 0.
\]

Theorems 4.1 and 5.1 tell us that \(x(t)\) is stochastically permanent (see the red solid line of Fig. 9) and has a unique stationary distribution. Meanwhile, Remark 7.3 also shows that
the distribution of \( x(t) \) weakly converges to the unique invariant probability measure \( \mu(\cdot) \), the Gamma distribution with \( \alpha_1 = \frac{91}{9} \) and \( \alpha_2 = \frac{160}{9} \).

By virtue of Theorem 7.2 and Remark 7.2, the numerical solutions \( X(\Delta, t) \) approximates the exact solution in the mean square sense with error estimate \( \Delta \), respectively. It follows from Theorem 4.5 that given \( a > 0 \) and \( \beta > 0 \), the numerical solution \( X(\Delta, t) \) is stochastically permanent (see the blue dashed line of Fig. 9). Moreover, by Theorems 5.2 and 5.3, the probability measure of the solution using Scheme (3.2) with any initial value \( x_0 \in \mathbb{R}_+ \) tends to a unique numerical invariant measure \( \mu(\cdot) \) asymptotically as \( k \to \infty \), and \( \mu^\Delta(\cdot) \to \mu(\cdot) \) as \( \Delta \to 0 \).

Figure 9: Sample paths of the exact solution (the red solid line) and numerical solution of Scheme (3.2) with \( \Delta = 0.1 \), \( K = 20 \) and \( \theta = 0.4 \) (the blue dashed line). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Figure 10: The red asterisk trajectory depicts the root mean square approximation error \( \left( \mathbb{E}|x(2) - X(\Delta, 2)|^2 \right)^{1/2} \) of the exact solution of SDE (7.3) and the numerical solution by Scheme (3.15), while the red circle trajectory depicts the root mean square approximation error \( \left( \mathbb{E}|x(2) - X(2)|^2 \right)^{1/2} \) of the exact solution of SDE (7.3) and the numerical solution by Scheme (3.15) as the functions of stepsize \( \Delta \in \{2^{-4}, 2^{-5}, \ldots, 2^{-18}\} \).
To test the efficiency of the scheme we carry out numerical experiments by implementing Schemes (3.2) and (3.15) using MATLAB. Fig. 10 plots the root mean square approximation error \( (\mathbb{E}|x(2) - X_E^2|^2)^{1/2} \) between the exact solution of SDE (7.3) and the numerical solution by Scheme (3.15), and the error \( (\mathbb{E}|x(2) - X_\Delta^2|^2)^{1/2} \) between the exact solution and that of Scheme (3.2), as the functions of stepsize \( \Delta \in \{2^{-4}, 2^{-5}, \ldots, 2^{-18}\} \), for \(10^4\) sample points.

One observes that the schemes proposed in [18, 19, 22, 31] are not preserve positivity and therefore are not well defined when directly applied to SDS (1.1), which don’t work for the above stochastic logistic models. However, the performance of Scheme (3.2) is very nice for this case, see Figs. 9 and 11.

![Figure 11](image1.png)

Figure 11: (a) A sample path of the classical EM solution. (b) A sample path of the stopped EM solution. (c) A sample path of the numerical solution of Scheme (3.2) with the same stepsize \( \Delta = 0.025 \) and \( t \in [0, 1000] \).

![Figure 12](image2.png)

Figure 12: The red solid line indicates the density function of the Gamma distribution \( Ga(91/9, 160/9) \), the solid blue dots indicates the empirical density function of \( \mu^{\Delta} \).
To further illustrate the result of Theorems 4.6 and 5.3. First, we generate sample paths of the exact solution and numerical solution of Scheme (3.2) in interval \([0, 1000]\), see Fig. 9. It is evident to see that these two sample paths overlap with each other. Secondly, to measure the similarity quantitatively, we use the Kolmogorov-Smirnov test with a significance level of 0.05 to check if the stationary distribution of \(X_\Delta(t)\) is the Gamma distribution. At this level of significance, by MATLAB we do confirm that the stationary distribution of \(X_\Delta(t)\) is the Gamma distribution. So the numerical invariant measure approximates the underlying exact invariant measure very well. Finally, to more intuitively illustrate the result of Theorem 5.3, we plot the empirical density function of \(\mu_\Delta(\cdot)\) and the density function of the Gamma distribution \(Ga(91/9, 160/9)\) in Fig. 12. One observes obviously from the Fig. 12 that the computer simulation results obtained with our method approaches the analytical result which can be obtained by the Fokker-Plank equation. Furthermore, the similarity between the paths as well as the distributions is significant. Thus, this example illustrates the significance of the results of Theorems 4.6 and 5.3.

Appendix A.

Using the technique in the proof Lemma 3.1 yield the following lemma, and hence is omitted to avoid repetition.

**Lemma A.1** For any \(p > 0\), the EM scheme defined by (3.14) has the property that

\[
\sup_{\Delta \in (0,1]} \sup_{0 \leq k \leq [T/\Delta]} \mathbb{E}[e^{pY_k}] \leq C_T, \quad \forall \ T > 0,
\]

where \([T/\Delta]\) represents the integer part of \(T/\Delta\).

**Proof of Lemma 3.16.** By (1.2), we have

\[
y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} (\beta(r(s)) - a(r(s))e^{y(s)})\,ds + \int_{t_k}^{t_{k+1}} \sigma(r(s))\,dB(s). \tag{A.1}
\]

Using (3.14) and (A.1) we have

\[
Y_{k+1} - y(t_{k+1}) =: Y_k - y(t_k) - a(r_k)(e^{Y_k} - e^{y(t_k)})\Delta + J_k,
\]

where \(J_k = J_k^{(1)} + J_k^{(2)} + J_k^{(3)} + J_k^{(4)} + J_k^{(5)}\), \(J_k^{(1)} = -\int_{t_k}^{t_{k+1}} (a(r_k) - a(r(s)))x(t_k)\,ds\), \(J_k^{(2)} = \int_{t_k}^{t_{k+1}} a(r(s))\int_{t_k}^{s} x(u)(b(r(u)) - a(r(u))x(u))\,duds\), \(J_k^{(4)} = \int_{t_k}^{t_{k+1}} [\beta(r_k) - \beta(r(s))]\,ds\), \(J_k^{(3)} = \int_{t_k}^{t_{k+1}} a(r(s))\int_{t_k}^{s} \sigma(r(u))x(u)\,dB(u)\,ds\), \(J_k^{(5)} = \int_{t_k}^{t_{k+1}} [\sigma(r_k) - \sigma(r(s))]\,dB(s)\).
Define \( u_k = Y_k - y(t_k) \). Note that \( u_k(e^{Y_k} - e^{y(t_k)}) \geq 0 \), we get
\[
\begin{align*}
\dot{u}_{k+1}^2 &= u_k^2 + a^2(r_k)(e^{Y_k} - e^{y(t_k)})^2 \Delta^2 + (J_k)^2 + 2u_kJ_k \\
&- 2a(r_k)u_k(e^{Y_k} - e^{y(t_k)})\Delta - 2a(r_k)J_k(e^{Y_k} - e^{y(t_k)})\Delta \\
&\leq u_k^2 + 2a^2(r_k)(e^{Y_k} - e^{y(t_k)})^2 \Delta^2 + 2(J_k)^2 + 2u_kJ_k.
\end{align*}
\]

One further observes that
\[
\mathbb{E}u_{k+1}^2 \leq 2\tilde{a}^2 \Delta^2 \sum_{i=0}^{k} \mathbb{E}\left( (e^{y_i} - e^{y(t_i)})^2 \right) + 2 \sum_{i=0}^{k} \mathbb{E}(u_iJ_i) + 10 \sum_{i=0}^{k} \sum_{j=1}^{5} \mathbb{E}(J_i^{(j)})^2. \tag{A.2}
\]

Now the mean value theorem implies
\[
(e^x - e^y)^2 \leq (e^x + e^y)(e^x - e^y) \leq (e^x + e^y)^2 |x - y| \quad \forall \ x, y \in \mathbb{R}.
\]
The above inequality together with Lemma [A.1] as well as Hölder’s inequality implies
\[
2\tilde{a}^2 \Delta^2 \mathbb{E}\left[ \sum_{i=0}^{k} (e^{Y_i} - e^{y(t_i)})^2 \right] \leq 2\tilde{a}^2 \Delta^2 \sum_{i=0}^{k} \mathbb{E}\left( (e^{Y_i} + e^{y(t_i)})^2 \mid Y_i - y(t_i) \right) \leq \Delta \left[ 2^3 \tilde{a}^4 \Delta^2 \sum_{i=0}^{k} (\mathbb{E}e^{4Y_i} + \mathbb{E}e^{4y(t_i)}) + \sum_{i=0}^{k} \mathbb{E}|u_i|^2 \right] \leq C \Delta \sum_{i=0}^{k} \mathbb{E}|u_i|^2.
\]

This together with (A.2) implies
\[
\mathbb{E}u_{k+1}^2 \leq \Delta \sum_{i=0}^{k} \mathbb{E}|u_i|^2 + C \Delta \sum_{i=0}^{k} \mathbb{E}(u_iJ_i) + 10 \sum_{i=0}^{k} \sum_{j=1}^{5} \mathbb{E}(J_i^{(j)})^2. \tag{A.3}
\]

Then, by the Markov property ( (4.16) in [20 p.116]) and (2.1), we derive that
\[
\mathbb{E}\left[ (J_k^{(1)})^2 \right] \leq \Delta \mathbb{E}\left[ \int_{t_k}^{t_{k+1}} (a(r(s)) - a(r_k))^2 x^2(t_k) ds \right] \leq 4\bar{a}^2 \Delta \int_{t_k}^{t_{k+1}} \mathbb{E}\left[ x^2(t_k) \mathbb{E}\left( I_{r(s) \neq r_k} \right) \right] ds \leq C \Delta^2 \int_{t_k}^{t_{k+1}} \mathbb{E}\left[ x^2(t_k) \right] ds \leq C \Delta^3, \tag{A.4}
\]
and
\[
\mathbb{E}\left[ (J_k^{(2)})^2 \right] \leq \Delta \mathbb{E}\left[ \int_{t_k}^{t_{k+1}} a^2(r(s)) \left( \int_{t_k}^{s} x(u) \left( b(r(u)) - a(r(u))x(u) \right) du \right)^2 ds \right] \leq \bar{a}^2 \Delta^2 \mathbb{E}\left[ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} x^2(u) \left( b(r(u)) - a(r(u))x(u) \right)^2 duds \right]
\]

46
\[
\leq 2\bar{a}^2 \Delta^2 \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \left( \tilde{b}^2 \mathbb{E}[x^2(u)] + \bar{a}^2 \mathbb{E}[x^4(u)] \right) du ds \leq C \Delta^4. \quad (A.5)
\]

By the Itô isometry and (2.1), we have

\[
\mathbb{E}\left[(J_{k}^{(3)})^2\right] \leq \Delta \mathbb{E}\left[\int_{t_k}^{t_{k+1}} a^2(r(s)) \left( \int_{t_k}^{s} \sigma(r(u)) x(u) dB(u) \right)^2 ds \right] \leq \bar{a}^2 \Delta \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \left( \int_{t_k}^{s} \sigma(r(u)) x(u) dB(u) \right)^2 ds \right] \leq |\bar{\sigma}|^2 \bar{a}^2 \Delta \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \mathbb{E}[x^2(u)] du ds \leq C \Delta^3. \quad (A.6)
\]

Furthermore, using Lemma 6.10 in [20, p.251], we yield that

\[
\mathbb{E}\left[(J_{k}^{(4)})^2\right] \leq 4 \bar{\beta}^2 \Delta \int_{t_k}^{t_{k+1}} \mathbb{P}(r(s) \neq r_k) ds \leq C \Delta^3, \quad (A.7)
\]

and

\[
\mathbb{E}\left[(J_{k}^{(5)})^2\right] \leq 4 |\bar{\sigma}|^2 \int_{t_k}^{t_{k+1}} \mathbb{P}(r(s) \neq r_k) ds \leq C \Delta^2. \quad (A.8)
\]

Combining (A.3)–(A.8), we obtain that

\[
\mathbb{E}u_{k+1}^2 \leq \Delta \sum_{i=0}^{k} \mathbb{E}|u_i|^2 + C T \Delta + 2 \sum_{i=0}^{k} \mathbb{E} \left[u_i (J_{i}^{(1)} + J_{i}^{(2)} + J_{i}^{(3)} + J_{i}^{(4)} + J_{i}^{(5)}) \right]. \quad (A.9)
\]

Using the Hölder inequality and (A.4), we obtain that

\[
2 \sum_{i=0}^{k} \mathbb{E}[u_i J_{i}^{(1)}] \leq 2 \sum_{i=0}^{k} \left( \mathbb{E}u_i^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \left[(J_{i}^{(1)})^2\right] \right)^{\frac{1}{2}} \leq 2 \sum_{i=0}^{k} \left( \mathbb{E}u_i^2 \right)^{\frac{1}{2}} \left( C \Delta^3 \right)^{\frac{1}{2}} \leq 2 \Delta^\frac{1}{2} \left( \sum_{i=0}^{k} \mathbb{E}u_i^2 \right)^{\frac{1}{2}} \left( C T \Delta \right)^{\frac{1}{2}} \leq \sum_{i=0}^{k} \mathbb{E}u_i^2 \Delta + C T \Delta.
\]

By (2.1) we derive that

\[
2 \mathbb{E}[u_{k} J_{k}^{(2)}] \leq \mathbb{E}u_{k}^2 \Delta + \mathbb{E}\left[\int_{t_k}^{t_{k+1}} a^2(r(s)) \left( \int_{t_k}^{s} x(u) \left( b(r(u)) - a(r(u)) x(u) \right) du \right)^2 ds\right] \leq \mathbb{E}u_{k}^2 \Delta + \bar{a}^2 \Delta \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} x^2(u) \left( b(r(u)) - a(r(u)) x(u) \right)^2 du ds\right] \leq \mathbb{E}u_{k}^2 \Delta + 2 \bar{a}^2 \Delta \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \left( \bar{b}^2 \mathbb{E}[x^2(u)] + \bar{b}^2 \mathbb{E}[x^4(u)] \right) du ds \leq \mathbb{E}u_{k}^3 \Delta + C \Delta^3.
\]

Since

\[
\mathbb{E}\left[J_{k}^{(3)} | F_{t_k}\right] = \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} a(r(s)) \sigma(r(u)) x(u) dB(u) du ds | F_{t_k}\right] = 0,
\]

47
we have that $E[u_k(J_k^{(3)} + J_k^{(5)})] = 0$. By the Markov property, one observes

$$2E[u_kJ_k^{(4)}] \leq E[u_k^2] + 2\beta^2 \int_{t_k}^{t_{k+1}} E\left[ E(I_{\{r(s) \neq r_k\}} | \mathcal{F}_{t_k}) \right] ds \leq E[u_k^2] + C\Delta^2.$$  

By inserting these four estimates in (A.9) we end up with

$$E[u_k^2] \leq 4\Delta \sum_{i=0}^{k} E|u_i|^2 + C\Delta$$

and Gronwall’s Lemma completes the proof of the assertion. 

**Appendix B.**

**Proof of Lemma 4.8.** Using the well-known Taylor formula we get

$$e^{\tilde{Z}_{k+1}} = e^{Z_k} + e^{Z_k}(\tilde{Z}_{k+1} - Z_k) + \frac{1}{2} e^{\tilde{\xi}_k}(\tilde{Z}_{k+1} - Z_k)^2,$$

where $\tilde{\xi}_k \in (\tilde{Z}_{k+1} \land Z_k, \tilde{Z}_{k+1} \lor Z_k)$. Clear,

$$\frac{1}{2} e^{\tilde{\xi}_k}(\tilde{Z}_{k+1} - Z_k)^2 \leq \frac{1}{2} e^{Z_k} e^{\tilde{Z}_{k+1} - Z_k}(\tilde{Z}_{k+1} - Z_k)^2$$

$$= \frac{1}{2} e^{Z_k} \exp \left( \left( \beta(r_k) - a(r_k)e^{Z_k} \right) \Delta + \sigma(r_k)\Delta B_k \right) \left( \left( \beta(r_k) - a(r_k)e^{Z_k} \right) \Delta + \sigma(r_k)\Delta B_k \right)^2$$

$$\leq e^{\tilde{Z}_k} \tilde{U}_k,$$

where

$$\tilde{U}_k := C \exp \left( |\tilde{\sigma}| |\Delta B_k| \right) \left( \Delta^{2(1-\theta)} + |\Delta B_k|^2 \right).$$

Therefore, we derive from (4.6) and (4.8) that for any integer $k \geq 0$,

$$e^{\tilde{Z}_{k+1}} \leq e^{Z_k} + e^{Z_k} \left[ (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + \sigma(r_k)\Delta B_k + \tilde{U}_k \right].$$

Then using the above inequality, we have

$$(1 + e^{\tilde{Z}_{k+1}})^p \leq (1 + e^{Z_k})^p (1 + \tilde{\varsigma}_k)^p.$$  

(B.1)

where

$$\tilde{\varsigma}_k := (1 + e^{Z_k})^{-1} e^{Z_k} \left[ (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + \sigma(r_k)\Delta B_k + \tilde{U}_k \right],$$

and we can see that $\tilde{\varsigma}_k > -1$. By virtue of [31, Lemma 3.3], without loss the generality we prove (4.20) only for $0 < p \leq 1$. It follows from (B.1) that

$$E\left[ (1 + e^{-\tilde{Z}_{k+1}})^p | \mathcal{F}_{t_k} \right]$$

48
\[ \leq (1 + e^{-Z_k}) \left\{ 1 + p \mathbb{E} [\tilde{\varsigma}_k | \mathcal{F}_{t_k}] + \frac{p(p-1)}{2} \mathbb{E} [\varsigma_k^2 | \mathcal{F}_{t_k}] + \frac{p(p-1)(p-2)}{6} \mathbb{E} [\varsigma_k^3 | \mathcal{F}_{t_k}] \right\}. \]  \tag{B.2}

By (4.13) and Lemma 3.2, we derive that
\[
\mathbb{E} [\tilde{U}_k | \mathcal{F}_{t_k}] \leq C\mathbb{E} \left[ \Delta^{2(1-\theta)} e^{\frac{\Delta B_k^2}{4\Delta}} + e^{-\frac{\Delta B_k^2}{4\Delta}} \right] \leq C\Delta,
\]
and
\[
\mathbb{E} [\tilde{\varsigma}_k | \mathcal{F}_{t_k}] \leq (1 + e^{Z_k})^{-1} e^{Z_k} \left[ (\beta(r_k) - a(r_k)e^{Z_k}) \Delta + C\Delta \right]
\leq -a(r_k)(1 + e^{Z_k})^{-1} e^{2Z_k} \Delta + C(1 + e^{Z_k})^{-1} e^{Z_k} \Delta
= -a(r_k)(1 + e^{Z_k}) \Delta + 2a(r_k)\Delta - a(r_k)(1 + e^{Z_k})^{-1} \Delta
+ C\Delta - C(1 + e^{Z_k})^{-1} \Delta
\leq -a(r_k)(1 + e^{Z_k}) \Delta + C\Delta - C(1 + e^{Z_k})^{-1} \Delta.
\]
Similarly, we can also prove that
\[
\mathbb{E} [\tilde{\varsigma}_k^2 | \mathcal{F}_{t_k}] \geq -C\Delta, \quad \mathbb{E} [\tilde{\varsigma}_k^3 | \mathcal{F}_{t_k}] \leq C\Delta^2.
\]
Thus, we obtain that
\[
\mathbb{E} \left[ (1 + e^{-Z_{k+1}})^p \right] \leq \mathbb{E} \left[ (1 + e^{-Z_k})^p \left( 1 - pa(r_k)(1 + e^{Z_k}) \Delta + C\Delta - C(1 + e^{Z_k})^{-1} \Delta \right) \right]
\leq -p\hat{a} \mathbb{E} \left[ (1 + e^{-Z_k})^{p+1} \right] \Delta + (1 + C\Delta) \mathbb{E} \left[ (1 + e^{-Z_k})^p \right] - C\mathbb{E} \left[ (1 + e^{Z_k})^{p-1} \right] \Delta \leq C.
\]
The proof is therefore complete.

**Proof of Lemma 4.9.** Since \( \pi a = 0 \), we have
\[ Z_{k+1} = Z_{k+1} = Z_k + \beta(r_k) \Delta + \sigma(r_k) \Delta B_k. \]
It is easy to see that
\[ e^{Z_{k+1}} \leq e^{Z_k} e^{Z_k (Z_{k+1} - Z_k) + \frac{1}{2} e^{2Z_k} (Z_{k+1} - Z_k)^2 + e^{Z_k} \tilde{\nu}_k}, \]
implies that
\[ e^{Z_{k+1}} \leq e^{Z_k} (1 + \varsigma_k), \]  \tag{B.3}
where
\[ \varsigma_k = \beta(r_k) \Delta + \sigma(r_k) \Delta B_k + \frac{\sigma^2(r_k)}{2} (\Delta B_k)^2 + \sigma(r_k) \Delta B_k \beta(r_k) \Delta + \frac{1}{2} \beta^2(r_k) \Delta^2 + \tilde{\nu}_k, \]
and we can see that $c_k > -1$. Note that

$$\pi[\beta - (\pi\beta)\mathbb{1}_m] = 0, \quad \sum_{i=1}^{m} \pi_i = 1.$$  

It follows from Lemma 4.1 (1) that the equation

$$\Gamma v = \beta - (\pi\beta)\mathbb{1}_m$$

has a solution $v = (v_1, \ldots, v_m)^T \in \mathbb{R}^m$. Thus we have

$$-\beta(i) + \sum_{j=1}^{m} \gamma_{ij} v_j = -\pi \beta > 0, \quad i \in \mathcal{S}. \quad (B.4)$$

Choose a constant $0 < \rho_0 \leq 1$ such that for each $0 < \rho \leq \rho_0$,

$$\xi_i^{\rho,v} := 1 - v_i \rho > 0, \quad i = 1, \ldots, m.$$  

Using the techniques in the proof of Lemma 4.7, it follows from (B.3) that

$$\mathbb{E}\left[e^{\rho Z_{k+1}} \xi_i^{\rho,v} \bigg| \mathcal{F}_{t_k}\right] \leq e^{\rho Z_k} \left\{ \mathbb{E}\left[\xi_i^{\rho,v} \bigg| \mathcal{G}_{t_k}\right] + \rho \mathbb{E}\left[\xi_i^{\rho,v} \bigg| \mathcal{F}_{t_k}\right] + \frac{\rho(\rho - 1)}{2} \mathbb{E}\left[\xi_i^{2 \rho,v} \bigg| \mathcal{F}_{t_k}\right] \right\}.$$  

By (4.10), (4.13) and Lemma 3.2, we derive that

$$\mathbb{E}\left[\xi_i^{\rho,v} \bigg| \mathcal{F}_{t_k}\right] = \mathbb{E}\left[\xi_i^{\rho,v} \mathbb{E}\left(\xi_i^{\rho,v} \bigg| \mathcal{G}_{t_k}\right) \bigg| \mathcal{F}_{t_k}\right] \leq \mathbb{E}\left[\xi_i^{\rho,v} \bigg| \mathcal{F}_{t_k}\right] \leq \left(\beta(r_k) + \frac{\sigma^2(r_k)}{2}\right) \xi_i^{\rho,v} \Delta + C \Delta^2,$$

and

$$\mathbb{E}\left[\xi_i^{2 \rho,v} \bigg| \mathcal{F}_{t_k}\right] \geq \mathbb{E}\left\{ \left[ \beta(r_k) + \frac{\sigma^2(r_k)}{2}(\Delta B_k)^2 \right. \right.$$  

$$\left. + \sigma(r_k) \Delta B_k \beta(r_k) \Delta + \frac{1}{2} \beta^2(r_k) \Delta^2 + \bar{U}_k \right]^2 \bigg| \mathcal{G}_{t_k}\right\}$$

$$\geq \mathbb{E}\left\{ \sigma^2(r_k)(\Delta B_k)^2 + 2 \sigma(r_k) \Delta B_k \left[ \beta(r_k) \Delta + \frac{\sigma^2(r_k)}{2}(\Delta B_k)^2 \right. \right.$$  

$$\left. + \sigma(r_k) \Delta B_k \beta(r_k) \Delta + \frac{1}{2} \beta^2(r_k) \Delta^2 + \bar{U}_k \right] \bigg| \mathcal{G}_{t_k}\right\}$$

$$= \sigma^2(r_k) \Delta + 2 \sigma^2(r_k) \beta(r_k) \Delta^2 + 2 \sigma(r_k) \mathbb{E}[\Delta B_k \bar{U}_k] \geq \sigma^2(r_k) \Delta - C \Delta^2,$$

implies that

$$\mathbb{E}\left[\xi_i^{2 \rho,v} \bigg| \mathcal{F}_{t_k}\right] \geq \mathbb{E}\left[\xi_i^{\rho,v} \mathbb{E}\left(\xi_i^{2 \rho,v} \bigg| \mathcal{G}_{t_k}\right) \bigg| \mathcal{F}_{t_k}\right] \geq \sigma^2(r_k) \xi_i^{\rho,v} \Delta - C \Delta^2.$$
Similarly, we can also prove that \( \mathbb{E}[\xi^3_{k}\xi^v_{r_{k+1}} | \mathcal{F}_k] \leq C\Delta^2 \). Thus, we obtain that

\[
\begin{align*}
\mathbb{E}\left[e^{\rho Z_{k+1} \xi^v_{r_{k+1}} | \mathcal{F}_k} \right] & \leq e^{\rho Z_k \xi^v_{r_k}} \left\{ \xi^v_{r_k} + \sum_{j \in S} \xi^v_{j} \left( \gamma_{r_j} \Delta + o(\Delta) \right) + \rho \left( \beta(r_k) + \frac{\sigma^2(r_k)}{2} \right) \xi^v_{r_k} \Delta + C \Delta^2 \right\} \\
& \leq e^{\rho Z_k \xi^v_{r_k}} \left\{ 1 + \frac{1}{\xi^v_{r_k}} \sum_{j \in S} \xi^v_{j} \gamma_{r_j} \Delta + \rho \beta(r_k) \Delta + \frac{\rho^2}{2} \sigma^2(r_k) \Delta + o(\Delta) \right\}. \tag{B.5}
\end{align*}
\]

By the properties of the generator, we have

\[
\frac{1}{\xi^v_{i}} \sum_{j \in S} \xi^v_{j} \gamma_{ij} = \frac{1}{1 - v_i \rho} \sum_{j=1}^{m} (1 - v_j \rho) \gamma_{ij}
\]

\[
= - \rho \left( \sum_{j=1}^{m} \gamma_{ij} v_j + \frac{v_i \rho}{1 - v_i \rho} \sum_{j=1}^{m} \gamma_{ij} v_j \right). \tag{B.6}
\]

It follows from (B.4), (B.5) and (B.6) that

\[
\begin{align*}
\mathbb{E}\left[e^{\rho Z_{k+1} \xi^v_{r_{k+1}} | \mathcal{F}_k} \right] & \leq e^{\rho Z_k \xi^v_{r_k}} \left\{ 1 - \rho \left( \sum_{j=1}^{m} \gamma_{r_j} v_j + \frac{v_r \rho}{1 - v_r \rho} \sum_{j=1}^{m} \gamma_{r_j} v_j \right) \Delta + \rho \beta(r_k) \Delta + \frac{\rho^2}{2} \sigma^2(r_k) \Delta + o(\Delta) \right\} \\
& = e^{\rho Z_k \xi^v_{r_k}} \left\{ 1 - \rho \left( \beta(r_k) - \pi \beta + \frac{v_r \rho}{1 - v_r \rho} \sum_{j=1}^{m} \gamma_{r_j} v_j - \frac{\rho}{2} \sigma^2(r_k) \Delta \right) \Delta + \rho \beta(r_k) \Delta + o(\Delta) \right\} \\
& = e^{\rho Z_k \xi^v_{r_k}} \left\{ 1 - \rho \left( - \pi \beta + \frac{v_r \rho}{1 - v_r \rho} \sum_{j=1}^{m} \gamma_{r_j} v_j - \frac{\rho}{2} \sigma^2(r_k) \Delta \right) \Delta + o(\Delta) \right\}.
\end{align*}
\]

Choose a constant \( 0 < \rho_1 \leq \rho_0 \) such that for any \( 0 < \rho \leq \rho_1 \),

\[
\lambda_1 := - \pi \beta + \rho \left( \frac{v_i}{1 - v_i \rho} \sum_{j=1}^{m} \gamma_{ij} v_j - \frac{\rho}{2} \sigma^2(i) \right) > 0.
\]

Choose \( \Delta_2^* \in (0, 1) \) sufficiently small such that \( \Delta_2^* < 2 / \rho \lambda, \ o(\Delta_2^*) \leq \rho \Delta_2^*/2 \). Then, for any \( \Delta \in (0, \Delta_2^*] \) yields

\[
\begin{align*}
\mathbb{E}\left[e^{\rho Z_{k+1} \xi^v_{r_{k+1}} | \mathcal{F}_k} \right] & \leq (1 - \rho \lambda \Delta) e^{\rho Z_k \xi^v_{r_k}} \\
& \leq (1 - \rho \lambda \Delta) e^{\rho Z_k \xi^v_{r_k}}.
\end{align*}
\]

Since the proof method is as that of Lemma 4.7, we omit the details. \( \blacksquare \)
Proof of Lemma 5.7. Obviously,

\[ Z_{k+1}^{y_0,\ell} = Z_k^{y_0,\ell} + (\beta(r_k^\ell) - a(r_k^\ell)e^{\theta Z_k^y}) \Delta + \sigma(r_k^\ell) DB_k, \]

and

\[ \tilde{Z}_{k+1}^{\bar{y}_0,\ell} = Z_k^{\bar{y}_0,\ell} + (\beta(r_k^\ell) - a(r_k^\ell)e^{\theta Z_k^y}) \Delta + \sigma(r_k^\ell) DB_k. \]

It is easy to see that

\[ \tilde{Z}_{k+1}^{y_0,\ell} - \tilde{Z}_{k+1}^{\bar{y}_0,\ell} = Z_k^{y_0,\ell} - Z_k^{\bar{y}_0,\ell} - a(r_k^\ell)(e^{Z_k^y} - e^{\theta Z_k^y}) \Delta = Z_k^{y_0,\ell} - Z_k^{\bar{y}_0,\ell} - a(r_k^\ell)(X_{k_0,\ell}^x - X_{k_0,\ell}^x) \Delta, \]

which implies that

\[ |\tilde{Z}_{k+1}^{y_0,\ell} - \tilde{Z}_{k+1}^{\bar{y}_0,\ell}| = |Z_k^{y_0,\ell} - Z_k^{\bar{y}_0,\ell}| - a(r_k^\ell)|(X_{k_0,\ell}^x - X_{k_0,\ell}^x)| \Delta. \]  \hspace{1cm} (B.7)

Note that

\[ |Z_k^{y_0,\ell} - Z_k^{\bar{y}_0,\ell}| \leq |\tilde{Z}_k^{y_0,\ell} - \tilde{Z}_k^{\bar{y}_0,\ell}|. \]  \hspace{1cm} (B.8)

In fact, if \( \tilde{Z}_k^{y_0,\ell} \lor \tilde{Z}_k^{\bar{y}_0,\ell} \leq \log(K\Delta^{-\theta}) \), (B.8) holds obviously. If \( \tilde{Z}_k^{y_0,\ell} \land \tilde{Z}_k^{\bar{y}_0,\ell} > \log(K\Delta^{-\theta}) \), (B.8) holds obviously. If \( \tilde{Z}_k^{y_0,\ell} \leq \log(K\Delta^{-\theta}) < \tilde{Z}_k^{\bar{y}_0,\ell} \), we have

\[ Z_k^{y_0,\ell} = \tilde{Z}_k^{y_0,\ell} \leq \tilde{Z}_k^{\bar{y}_0,\ell} = \log(K\Delta^{-\theta}) < \tilde{Z}_k^{\bar{y}_0,\ell}, \]

and

\[ |Z_k^{y_0,\ell} - Z_k^{\bar{y}_0,\ell}|^2 - |\tilde{Z}_k^{y_0,\ell} - \tilde{Z}_k^{\bar{y}_0,\ell}|^2 = (Z_k^{y_0,\ell} + \tilde{Z}_k^{\bar{y}_0,\ell})(Z_k^{\bar{y}_0,\ell} - \tilde{Z}_k^{\bar{y}_0,\ell}) - 2Z_k^{y_0,\ell}(Z_k^{y_0,\ell} - \tilde{Z}_k^{\bar{y}_0,\ell}) \]

\[ = (Z_k^{y_0,\ell} + \tilde{Z}_k^{\bar{y}_0,\ell} - 2Z_k^{y_0,\ell})(Z_k^{y_0,\ell} - \tilde{Z}_k^{\bar{y}_0,\ell}) \leq 0. \]

Then (B.8) follows immediately. If \( \tilde{Z}_k^{y_0,\ell} \leq \log(K\Delta^{-\theta}) < \tilde{Z}_k^{\bar{y}_0,\ell} \), (B.8) holds also by symmetry on \( \tilde{Z}_k^{y_0,\ell} \) and \( \tilde{Z}_k^{\bar{y}_0,\ell} \). Thus, the desired inequality (B.8) holds for all cases. It follows from (B.7) and (B.8) that for any integer \( k \geq 0 \),

\[ |Z_k^{y_0,\ell} - Z_k^{\bar{y}_0,\ell}| \leq |Z_k^{y_0,\ell} - Z_k^{\bar{y}_0,\ell}| - a(r_k^\ell)|(X_{k_0,\ell}^x - X_{k_0,\ell}^x)| \Delta \]

\[ \leq |y_0 - \bar{y}_0| - \sum_{i=0}^{k} a(r_i^\ell)|(X_i^x - X_i^x)| \Delta. \]

Then we have \( \mathbb{E}|Z_k^{y_0,\ell} - Z_k^{\bar{y}_0,\ell}| \leq |y_0 - \bar{y}_0| - \bar{a} \sum_{i=0}^{k} \mathbb{E}|X_i^x - X_i^x| \Delta. \) Due to \( \bar{a} > 0 \),

\[ \sum_{i=0}^{\infty} \mathbb{E}|X_i^x - X_i^x| \Delta \leq \frac{|y_0 - \bar{y}_0|}{\bar{a}} < \infty. \]

The proof is therefore complete. \( \blacksquare \)
Proof of Lemma 5.8. By (4.9), we see that
\[ e^{Z_{k+1}} \leq x_0 + \sum_{i=0}^{k} e^{Z_i} \left[ (\beta(r_i) - a(r_i)e^{Z_i}) \Delta + \sigma(r_i) \Delta B_i + \sigma(r_i) \Delta B_i (\beta(r_i) - a(r_i)e^{Z_i}) \Delta \\
+ \frac{\sigma^2(r_i)}{2} (\Delta B_i)^2 + \frac{1}{2} \left( \beta(r_i) - a(r_i)e^{Z_i} \right)^2 \Delta^2 + \tilde{C} e^{\frac{\Delta B_i^2}{2\Delta}} \left( \Delta^3 + |\Delta B_i|^3 \right) \right], \]
where \( \tilde{C} = 2/3 \exp \left( (\tilde{\beta} + 2|\tilde{\sigma}|^2) \Delta \right) (|\tilde{\beta}| \lor |\tilde{\sigma}|)^3 \). Then we have
\[ e^{Z_{k+1}} \leq 8 \left\{ x_0^2 + \left[ \sum_{i=0}^{k} e^{Z_i} (\beta(r_i) - a(r_i)e^{Z_i}) \Delta \right]^2 + \left[ \sum_{i=0}^{k} e^{Z_i} \sigma(r_i) \Delta B_i \right]^2 + \frac{|\tilde{\sigma}|^4}{4} \left[ \sum_{i=0}^{k} e^{Z_i} (\Delta B_i)^2 \right]^2 \right. \\
+ \Delta^2 \left[ \sum_{i=0}^{k} e^{Z_i} (\beta(r_i) - a(r_i)e^{Z_i}) \sigma(r_i) \Delta B_i \right]^2 + \frac{\Delta^4}{4} \left[ \sum_{i=0}^{k} e^{Z_i} (\beta(r_i) - a(r_i)e^{Z_i})^2 \right]^2 \\
+ \tilde{C}^2 \Delta^6 \left[ \sum_{i=0}^{k} e^{Z_i} \exp \left( \frac{|\Delta B_i|^2}{8\Delta} \right) \right]^2 \left. + \tilde{C}^2 \left[ \sum_{i=0}^{k} e^{Z_i} \exp \left( \frac{|\Delta B_i|^2}{8\Delta} \right) |\Delta B_i|^3 \right]^2 \right\}, \]
which implies that
\[ \mathbb{E} \left[ \sup_{0 \leq k \Delta \leq T} X_{k+1}^2 \right] \]
\[ \leq 8 \left\{ x_0^2 + 2T|\tilde{\beta}|^2 \Delta \sum_{i=0}^{T/\Delta} \mathbb{E} X_i^2 + 2T \tilde{\alpha}^2 \Delta \sum_{i=0}^{T/\Delta} \mathbb{E} X_i^4 + \mathbb{E} \sum_{i=0}^{T/\Delta} \sigma^2(r_i) X_i^2 \Delta + 2|\tilde{\beta}|^4 T \Delta^3 \sum_{i=0}^{T/\Delta} \mathbb{E} X_i^2 \\
+ \frac{|\tilde{\sigma}|^4}{4} \left[ T/\Delta \right] \mathbb{E} \sum_{i=0}^{T/\Delta} X_i^2 \mathbb{E} \left( (\Delta B_i)^4 |\mathcal{F}_t \right) \Delta + \Delta^2 \mathbb{E} \sum_{i=0}^{T/\Delta} X_i^2 (\beta(r_i) - a(r_i)X_i)^2 \sigma^2(r_i) \Delta \\
+ 2\tilde{\alpha}^4 T \Delta^3 \sum_{i=0}^{T/\Delta} \mathbb{E} X_i^6 + \sqrt{2} \tilde{C}^2 T \Delta^5 \sum_{i=0}^{T/\Delta} \mathbb{E} X_i^2 + C[T/\Delta] \sum_{i=0}^{T/\Delta} \mathbb{E} X_i^2 \Delta^3 \right\} \]
\[ \leq C \left( x_0^2 + \Delta \sum_{i=0}^{T/\Delta} \mathbb{E} X_i^2 + \Delta \sum_{i=0}^{T/\Delta} \mathbb{E} X_i^4 + \Delta^3 \sum_{i=0}^{T/\Delta} \mathbb{E} X_i^6 \right). \]
Using Lemma 4.8, we obtain \( \mathbb{E} \left[ \sup_{0 \leq k \Delta \leq T} X_k^2 \right] \leq C \). The proof is therefore complete. \[ \blacksquare \]
Proof of Lemma 5.9. Due to $\vartheta < 1$, by (4.21) we have

$$\left(1 + e^{-Z_{k+1}}\right)^{\vartheta} \leq \left(1 + e^{-Z_k}\right)^{\vartheta} \left[1 + \frac{\vartheta}{2} \mathbf{s}_k + \frac{\vartheta(\vartheta - 2)(\vartheta - 4)}{48} \mathbf{s}_k^3\right]. \quad (B.9)$$

Obviously,

$$\left(1 + e^{-Z_k}\right)^{\vartheta} = \left(1 + e^{-Z_k}\right)^{\vartheta} I_{\Omega_k} + \left(1 + e^{-Z_k}\right)^{\vartheta} I_{\Omega_k} \leq \left(1 + e^{-Z_k}\right)^{\vartheta} + (1 + K^{-1}\Delta)^{\vartheta} I_{\Omega_k}, \quad (B.10)$$

where $\Omega_k$ is defined by (4.29). Using (B.9) and (B.10) yields

$$\left(1 + e^{-Z_{k+1}}\right)^{\vartheta} \leq \left(1 + e^{-Z_k}\right)^{\vartheta} \left[1 + \frac{\vartheta}{2} \mathbf{s}_k + \frac{\vartheta(\vartheta - 2)(\vartheta - 4)}{48} \mathbf{s}_k^3\right] + 2I_{\Omega_k} \leq \left(1 + x_0^{-1}\right)^{\vartheta} + \frac{\vartheta}{2} \sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta} \left(\mathbf{s}_k + \frac{(\vartheta - 2)(\vartheta - 4)}{24} \mathbf{s}_k^3\right) + 2\sum_{i=0}^{k} I_{\Omega_k},$$

which implies that

$$\left(1 + e^{-Z_{k+1}}\right)^{\vartheta} \leq 4\left(1 + x_0^{-1}\right)^{\vartheta} + \frac{\vartheta^2(\vartheta - 2)^2(\vartheta - 4)^2}{24^2} \left[\sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta} \mathbf{s}_k^3\right]^2$$

$$+ \vartheta^2 \left[\sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta} \mathbf{s}_k\right]^2 + 16\left[\sum_{i=0}^{k} I_{\Omega_k}\right]^2.$$  

Then we have

$$\mathbb{E} \sup_{0 \leq k \Delta \leq T} \left(1 + e^{-Z_{k+1}}\right)^{\vartheta} \leq 4\mathbb{E} \sup_{0 \leq k \Delta \leq T} \left[\sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta} \mathbf{s}_k\right]^2 + \mathbb{E} \sup_{0 \leq k \Delta \leq T} \left[\sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta} \mathbf{s}_k^3\right]^2$$

$$+ 4\left(1 + x_0^{-1}\right)^{\vartheta} + 16\mathbb{E} \sup_{0 \leq k \Delta \leq T} \left[\sum_{i=0}^{k} I_{\Omega_k}\right]^2,$$  

(B.11)

we deduce that

$$\mathbb{E} \sup_{0 \leq k \Delta \leq T} \left[\sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta} \mathbf{s}_k\right]^2$$

$$\leq 7\vartheta^2 \Delta^2 \mathbb{E} \sup_{0 \leq k \Delta \leq T} \left(\sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta - 1}\right)^2 + 7|\hat{\vartheta}|^2 \Delta^2 \mathbb{E} \sup_{0 \leq k \Delta \leq T} \left(\sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta}\right)^2$$

$$+ C \Delta^4 (1 - \vartheta) \mathbb{E} \sup_{0 \leq k \Delta \leq T} \left(\sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta}\right)^2 + 4|\hat{\sigma}|^2 \mathbb{E} \sup_{0 \leq k \Delta \leq T} \left(\sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta} (\Delta B_i)^2\right)^2$$

$$+ 7\mathbb{E} \sup_{0 \leq k \Delta \leq T} \left|\sum_{i=0}^{k} \sigma(r_i) \left(1 + e^{-Z_i}\right)^{\vartheta - 1} e^{-Z_k} \Delta B_i\right|^2 + 7|\hat{\sigma}|^2 \mathbb{E} \sup_{0 \leq k \Delta \leq T} \left(\sum_{i=0}^{k} \left(1 + e^{-Z_i}\right)^{\vartheta} \mathbf{U}_k\right)^2$$

$$+ 7\Delta^2 \mathbb{E} \sup_{0 \leq k \Delta \leq T} \left|\sum_{i=0}^{k} \sigma(r_i) \left(1 + e^{-Z_i}\right)^{\vartheta - 1} (a(r_i) - \beta(r_i) e^{-Z_i}) \Delta B_i\right|^2$$

54
\[ \leq 7T^2\tilde{a}^2 + 7T|\tilde{\beta}|^2 \Delta \sum_{i=0}^{k} \mathbb{E}(1 + e^{-Z_i})^\vartheta + CT\Delta^{3-4\vartheta} \sum_{i=0}^{[T/\Delta]} \mathbb{E}(1 + e^{-Z_i})^\vartheta \]
\[ + 12T|\tilde{\sigma}|^2 \Delta \sum_{i=0}^{[T/\Delta]} \mathbb{E}(1 + e^{-Z_i})^\vartheta + 7|\tilde{\sigma}|^2 \Delta \sum_{i=0}^{[T/\Delta]} \mathbb{E}(1 + e^{-Z_i})^\vartheta + 14T\tilde{a}^2|\tilde{\sigma}|^2 \Delta^2 \]
\[ + 14|\tilde{\sigma}|^2|\tilde{\beta}|^2 \Delta^3 \sum_{i=0}^{[T/\Delta]} \mathbb{E}(1 + e^{-Z_i})^\vartheta + CT\Delta^2 \sum_{i=0}^{[T/\Delta]} \mathbb{E}(1 + e^{-Z_i})^\vartheta, \quad (B.12) \]

and
\[ \mathbb{E} \sup_{0 \leq k \Delta \leq T} \left[ \sum_{i=0}^{k} \left( 1 + e^{-Z_i} \right)^\vartheta \bar{\varsigma}_k^3 \right]^2 \leq [T/\Delta] \mathbb{E} \left[ \sum_{i=0}^{[T/\Delta]} \left( 1 + e^{-Z_i} \right)^\vartheta \mathbb{E}(\bar{\varsigma}_k^6|\mathcal{G}_\Delta) \right] \leq CT \Delta \sum_{i=0}^{[T/\Delta]} \mathbb{E}(1 + e^{-Z_i})^\vartheta. \quad (B.13) \]

By Chebyshev’s inequality and Lemma 4.8
\[ \mathbb{E} \sup_{0 \leq k \Delta \leq T} \left[ \sum_{i=0}^{k} I_{\Omega_k} \right]^2 \leq [T/\Delta] \sum_{i=0}^{[T/\Delta]} \mathbb{P}\left\{ \bar{Z}_{i+1} > \log(K\Delta^{-\vartheta}) \right\} \]
\[ \leq [T/\Delta] \sum_{i=0}^{[T/\Delta]} \mathbb{E} \exp\left( \frac{2\vartheta^{-1}\bar{Z}_{i+1}}{K^{2\vartheta^{-1}} \Delta^{-2}} \right) \leq \frac{CT^2}{K^{2\vartheta^{-1}}}, \quad (B.14) \]

Inserting (B.12), (B.13) and (B.14) into (B.11), and using Lemma 4.11 we obtain
\[ \mathbb{E} \sup_{0 \leq k \Delta \leq T} (1 + e^{-Z_k})^\vartheta \leq C_T. \]

The proof is therefore complete. \[ \blacksquare \]
Appendix C.

Proof of Lemma 6.1. We first note that
\[
|X_1^\Delta| \geq x_0 \min_{i \in S} \{|\sigma(i)|\} |\Delta B_0| - x_0 \left( 1 + \bar{b} \Delta + \bar{a} x_0 \Delta \right) \geq \frac{e}{\Delta}
\]
if
\[
|\Delta B_0| \geq \frac{e/\Delta + x_0 (1 + \bar{b} \Delta + \bar{a} x_0 \Delta)}{x_0 \min_{i \in S} \{|\sigma(i)|\}}
\]
In other words, we have
\[
\mathbb{P} \left( |X_1^\Delta| \geq \frac{e}{\Delta} \right) \geq \mathbb{P} \left( |\Delta B_0| \geq \frac{e/\Delta + x_0 (1 + \bar{b} \Delta + \bar{a} x_0 \Delta)}{x_0 \min_{i \in S} \{|\sigma(i)|\}} \right) > 0 \tag{C.1}
\]
due to $\Delta B_0 \sim \mathcal{N}(0, \Delta)$. Let $M_i = (a(i) - 1.4)/|\sigma(i)|$, we observe that, for $k \geq 1$, if $|X_k^\Delta| \geq \exp(2^{k-1})/\Delta$ and $|\Delta B_k| \leq \bar{M} \exp(2^{k-1})$ hold, then
\[
|X_{k+1}^\Delta| \geq \frac{\exp(2^k)}{\Delta}.
\]
In fact,
\[
|X_{k+1}^\Delta| = |X_k^\Delta| \left| 1 + b(r_k) \Delta - a(r_k) X_k^\Delta + \sigma(r_k) \Delta B_k \right|
\geq |X_k^\Delta| \left| a(r_k) |X_k^\Delta| \Delta - 1 - b(r_k) \Delta - |\sigma(r_k)||\Delta B_k| \right|
\geq \frac{\exp(2^{k-1})}{\Delta} \left| a(r_k) \exp(2^{k-1}) - 1 - b(r_k) \Delta - |\sigma(r_k)|\bar{M} \exp(2^{k-1}) \right|
= \frac{\exp(2^k)}{\Delta} \left( a(r_k) - \exp(-2^{k-1}) - b(r_k) \exp(-2^{k-1}) \Delta - |\sigma(r_k)|\bar{M} \right)
\geq \frac{\exp(2^k)}{\Delta} \left( a(r_k) - e^{-1} - b(r_k) e^{-1} \Delta - |\sigma(r_k)|\bar{M} \right) \geq \frac{\exp(2^k)}{\Delta}.
\]
We therefore have
\[
\left\{ |X_k^\Delta| \geq \frac{e}{\Delta} \text{ and } |\Delta B_k| \leq \bar{M} \exp(2^{k-1}), \ \forall \ k \geq 1 \right\} \subset \left\{ |X_k^\Delta| \geq \frac{\exp(2^{k-1})}{\Delta}, \ \forall \ k \geq 1 \right\}.
\]
Since $X_k^\Delta$ and $\Delta B_k$ for $k \geq 1$ are all independent,
\[
\mathbb{P} \left( |X_k^\Delta| \geq \frac{\exp(2^{k-1})}{\Delta}, \ \forall \ k \geq 1 \right) \geq \mathbb{P} \left( |X_k^\Delta| \geq \frac{e}{\Delta} \text{ and } |\Delta B_k| \leq \bar{M} \exp(2^{k-1}), \ \forall \ k \geq 1 \right)
= \mathbb{P} \left( |X_k^\Delta| \geq \frac{e}{\Delta} \right) \mathbb{P} \left( |\Delta B_k| \leq \bar{M} \exp(2^{k-1}), \ \forall \ k \geq 1 \right).
\]
By the conditional probability formula implies
\[
\mathbb{P} \left( |X_{k+1}^\Delta| \geq \frac{\exp(2^k)}{\Delta}, \ \forall \ k \geq 1 \left| |X_k^\Delta| \geq \frac{e}{\Delta} \right) = \begin{cases} \mathbb{P} \left( |X_k^\Delta| \geq \frac{\exp(2^{k-1})}{\Delta}, \ \forall \ k \geq 1 \right) \mathbb{P} \left( |X_k^\Delta| \geq \frac{e}{\Delta} \right) 
\end{cases}
\]
56
\[ \geq \mathbb{P}\left( |\Delta B_k| \leq \hat{M} \exp(2^{k-1}), \forall k \geq 1 \right) \]
\[ = \prod_{k=1}^{\infty} \mathbb{P}\left( |\Delta B_k| \leq \hat{M} \exp(2^{k-1}) \right), \quad (C.2) \]

Now, because \( \Delta B_k \sim \mathcal{N}(0, \Delta) \), we have
\[
\mathbb{P}\left( |\Delta B_k| > \hat{M} \exp(2^{k-1}) \right) = \mathbb{P}\left( \frac{|\Delta B_k|}{\sqrt{\Delta}} > \frac{\hat{M} \exp(2^{k-1})}{\sqrt{\Delta}} \right) \leq \mathbb{P}\left( \frac{|\Delta B_k|}{\sqrt{\Delta}} > \hat{M} \exp(2^{k-1}) / \sqrt{\Delta} \right)
\]
\[
= \frac{2}{\sqrt{2\pi}} \int_{\hat{M} \exp(2^{k-1}) / \sqrt{\Delta}}^{\infty} e^{-x^2/2} \, dx \leq \int_{\hat{M} \exp(2^{k-1})}^{\infty} \frac{x e^{-x^2/2}}{\hat{M} \exp(2^{k-1})} \, dx
\]
\[
\leq \frac{2}{\exp(2^{k-1})} \int_{\hat{M} \exp(2^{k-1})}^{\infty} x e^{-x^2/2} \, dx
\]
\[
= 2 \exp \left( -2^{k-1} - \hat{M}^2 \exp(2^k) / 2 \right)
\]
\[
\leq 2 \exp \left( -2^{k-1} - 2^{-3} \exp(2^k) \right).
\]

But, by the elementary inequality \( \log(1 - u) \geq -2u \) for \( 0 \leq u < 0.5 \), we derive
\[
\log \left( \prod_{k=1}^{\infty} \left[ 1 - 2 \exp \left( -2^{k-1} - 2^{-3} \exp(2^k) \right) \right] \right)
\]
\[
= \sum_{k=1}^{\infty} \log \left( 1 - 2 \exp \left( -2^{k-1} - 2^{-3} \exp(2^k) \right) \right) \geq -4 \sum_{k=1}^{\infty} \exp \left( -2^{k-1} - 2^{-3} \exp(2^k) \right).
\]

Noting that \( e^x \geq 1 + x \) and \( 2^{k-1} \geq 2(k-1) \), we then get
\[
-4 \sum_{k=1}^{\infty} \exp \left( -2^{k-1} - 2^{-3} \exp(2^k) \right)
\]
\[
\geq -4 \sum_{k=1}^{\infty} \exp \left( -2^{k-1} - 2^{-3} - 2^{-3} \right) \geq -4 \sum_{k=1}^{\infty} \exp \left( -2(k-1) - 2^{-3} - 2(k-3) \right)
\]
\[
= -4 \exp \left( 8 - 2^{-3} \right) \sum_{k=1}^{\infty} e^{-4k} = -4 \exp \left( 8 - 2^{-3} \right) \lim_{k \to \infty} \frac{e^{-4(1 - e^{-4k})}}{1 - e^{-4}} = -4 e^{4 - 2^{-3}} - 1 / e^{-4}.
\]

Hence, in \( (C.2) \),
\[
\log \left( \mathbb{P}\left( |X_{k+1}^\Delta| \geq \frac{\exp(2^k)}{\Delta}, \forall k \geq 1 \left| |X_1^\Delta| \geq \frac{e}{\Delta} \right. \right) \right) \geq -\frac{4 e^{4 - 2^{-3}}}{1 - e^{-4}}
\]
and the result follows. \( \blacksquare \)
Appendix D.

We can easily obtain from Lemma A.1 the following corollary.

**Corollary D.1** For any \( p > 0 \), the EM scheme defined by (7.2) has the property that

\[
\sup_{\Delta \in (0,1]} \sup_{0 \leq k \leq [T/\Delta]} \mathbb{E}[e^{pY_k}] \leq C_T, \quad \forall \ T > 0,
\]

where \([T/\Delta]\) represents the integer part of \( T/\Delta \).

**Proof of Lemma 7.1.** By (1.2), we have

\[
y(t_{k+1}) = y(t_k) + \beta \Delta - a \int_{t_k}^{t_{k+1}} e^{y(s)} ds + \sigma \Delta B_k.
\]  

(D.1)

Using (7.2) and (D.1) we have

\[
Y_{k+1} - y(t_{k+1}) = Y_k - y(t_k) - a(e^{Y_k} - e^{y(t_k)}) \Delta + a \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} d(u) ds
\]

\[
= Y_k - y(t_k) - a(e^{Y_k} - e^{y(t_k)}) \Delta + \Xi_k,
\]

where

\[
\Xi_k = a \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} x(u)(b - ax(u))dud + a \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} x(u)dB(u)ds =: \Xi^{(1)}_k + \Xi^{(2)}_k.
\]  

(D.2)

Let us define \( u_k = Y_k - y(t_k) \). Note that \( u_k (e^{Y_k} - e^{y(t_k)}) \geq 0 \), we get

\[
u_{k+1}^2 \leq u_k^2 - 2au_k(e^{Y_k} - e^{y(t_k)}) \Delta + 2a^2 (e^{Y_k} - e^{y(t_k)})^2 \Delta^2 + 2\Xi^2_k + 2u_k\Xi_k
\]

\[
\leq u_k^2 + 2a^2 (e^{Y_k} - e^{y(t_k)})^2 \Delta^2 + 2\Xi^2_k + 2u_k\Xi_k
\]

\[
\leq 2a^2 \Delta^2 \sum_{i=0}^{k} (e^{Y_i} - e^{y(t_i)})^2 + 2 \sum_{i=0}^{k} \Xi^2_i + 2 \sum_{i=0}^{k} u_i \Xi_i.
\]  

(D.3)

Let \( \mathfrak{M}_0 = 0 \), and \( \mathfrak{M}_k = \sum_{i=0}^{k-1} u_i \Xi^{(2)}_i \) for any \( k \geq 1 \), since

\[
\mathbb{E}[\Xi^{(2)}_k | \mathcal{F}_{t_k}] = a\sigma \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} x(u)dB(u)ds | \mathcal{F}_{t_k} \right] = 0.
\]

It is then easy to show that

\[
\mathbb{E}[\mathfrak{M}_{k+1} | \mathcal{F}_{t_k}] = \mathbb{E}[\mathfrak{M}_k + u_k \Xi^{(2)}_k | \mathcal{F}_{t_k}] = \mathfrak{M}_k + u_k \mathbb{E}[\Xi^{(2)}_k | \mathcal{F}_{t_k}] = \mathfrak{M}_k,
\]

58
This implies immediately that $\mathcal{M}_k$ is a martingale and the Burkholder-Davis-Gundy inequality implies that

$$
\mathbb{E}
\left[
\sup_{k=0,\ldots,l} |\mathcal{M}_k|^q
\right]
\leq
C_T \mathbb{E}
\left[
\sum_{i=0}^{l-1} \left( u_i \Xi_i^{(2)} \right)^2
\right]^{\frac{q}{2}}
\leq
C_T \mathbb{E}
\left[
\sum_{i=0}^{l-1} u_i^2 |\Xi_i^{(2)}|^2
\right]^{\frac{q}{2}}
$$

for any $q \geq 2$ and $l = 0, \ldots, \lfloor T/\Delta \rfloor$. Using this and Jensen’s inequality in (D.3) we now arrive at

$$
\mathbb{E}
\left[
\sup_{k=0,\ldots,l} |u_{k+1}|^2
\right]
\leq
2^q \mathbb{E}
\left[
\sup_{k=0,\ldots,l} \left| a^2 \Delta^2 \sum_{i=0}^{k} (e^{y(t_i)} - e^{y(t_i)})^2 + \sum_{i=0}^{k} |\Xi_i|^2 + \sum_{i=0}^{k} u_i |\Xi_i|^q \right|
\right]
\leq 6^q \mathbb{E}
\left[
(a\Delta)^2 \sum_{i=0}^{l} (e^{y(t_i)} - e^{y(t_i)})^2 + \left( \sum_{i=0}^{l} |\Xi_i|^2 \right)^q + \sup_{k=0,\ldots,l} \left| \sum_{i=0}^{k} u_i |\Xi_i|^q \right|
\right]
\leq 6^q \mathbb{E}
\left[
(a\Delta)^2 \left( \frac{T}{\Delta} \right)^{q-1} \sum_{i=0}^{l} (e^{y(t_i)} - e^{y(t_i)})^2 + \left( \sum_{i=0}^{l} |\Xi_i|^2 \right)^q + \sum_{i=0}^{l} |u_i|^q |\Xi_i^{(1)}|^q
\right]
$$

$$
+ 2^q \left( \frac{T}{\Delta} \right)^{q-1} \sum_{i=0}^{l} |u_i|^q |\Xi_i^{(1)}|^q + 2^q \sup_{k=0,\ldots,l} \left| \mathcal{M}_{k+1} \right|^q
\right]
\leq C_T \mathbb{E}
\left[
\Delta^{q-1} \sum_{i=0}^{l} (e^{y(t_i)} - e^{y(t_i)})^2 + \left( \frac{T}{\Delta} \right)^{q-1} \sum_{i=0}^{l} |\Xi_i^{(1)}|^q + \left( \sum_{i=0}^{l} u_i^2 |\Xi_i^{(2)}|^2 \right)^{\frac{q}{2}}
\right]
\leq C_T \mathbb{E}
\left[
\Delta^{q-1} \sum_{i=0}^{l} (e^{y(t_i)} - e^{y(t_i)})^2 + \left( \frac{T}{\Delta} \right)^{q-1} \sum_{i=0}^{l} |\Xi_i^{(1)}|^q + \left( \sum_{i=0}^{l} u_i^2 |\Xi_i^{(2)}|^2 \right)^{\frac{q}{2}}
\right]
$$

for any $q \geq 2$ and $l = 0, \ldots, \lfloor T/\Delta \rfloor$. It is easy to see that

$$
\mathbb{E}
\left[
\Delta^{q-1} \sum_{i=0}^{l} (e^{y(t_i)} - e^{y(t_i)})^2
\right]
\leq
\Delta^{q-1} T^{q-1} \sum_{i=0}^{l} \mathbb{E}
\left[
(e^{y(t_i)} - e^{y(t_i)})^2 |Y_i - y(t_i)|^q
\right].
$$

On the other hand, by applying Corollary (D.1) and (2.1), we infer that

$$
\sup_{0 \leq k \leq \lfloor T/\Delta \rfloor} \mathbb{E}
\left[
2^q |y_k|
\right]
\leq C_T,
\sup_{0 \leq t \leq T} \mathbb{E}
\left[
2^q |e^{2q(t)}|
\right]
= \sup_{0 \leq t \leq T} \mathbb{E}
\left[
x^{2q(t)}
\right]
\leq C_T.
$$

Now note that

$$
\mathbb{E}
\left[
\Delta^{q-1} \sum_{i=0}^{l} (e^{y(t_i)} - e^{y(t_i)})^2
\right]
\leq
\Delta^{q-1} T^{q-1} \sum_{i=0}^{l} \mathbb{E}
\left[
(e^{y(t_i)} - e^{y(t_i)})^2 |Y_i - y(t_i)|^q
\right]
$$

59
\[ \leq C_T \Delta^{2q} + C_T \Delta \sum_{i=0}^{l} \mathbb{E}|u_i|^{2q}. \]  

(D.6)

By (D.2) and (D.5), for any \( q \geq 2 \),

\[ \mathbb{E}|\Xi_{i}^{(1)}|^{2q} \leq a^{2q}\mathbb{E}\left[ \left( \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} x(u)(b + ax(u))duds \right)^{2q} \right] \]
\[ \leq a^{2q}\mathbb{E}\left[ \left( \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} x(u)(b + ax(u))du \right)^{2q} \right] \]
\[ \leq (a \vee b)^{2q}(2a)^{2q}\Delta^{2q-2} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \left( \mathbb{E}[x^{2q}(u)] + \mathbb{E}[x^{4q}(u)] \right) duds \leq C_T \Delta^{4q}, \]  

(D.7)

and

\[ \mathbb{E}|\Xi_{i}^{(2)}|^{2q} \leq (a \sigma)^{2q}\Delta^{2q-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[ \left( \int_{t_k}^{s} x(u)dB(u) \right)^{2q} \right] ds \]
\[ \leq C_T \Delta^{3q-2} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \mathbb{E}[x^{2q}(u)] duds \leq C_T \Delta^{3q}. \]  

(D.8)

Thus, the Cauchy-Schwarz inequality give that

\[ \mathbb{E}[|u_i|^q|\Xi_{i}^{(1)}|^q] \leq \left( \mathbb{E}[|u_i|^{2q}] \right)^{1/2} \left( \mathbb{E}[|\Xi_{i}^{(1)}|^{2q}] \right)^{1/2} \leq C_T \left( \mathbb{E}[|u_i|^{2q}] \right)^{1/2} \Delta^{2q}. \]  

(D.9)

Similar we also obtain

\[ \mathbb{E}[|u_i|^q|\Xi_{i}^{(2)}|^q] \leq \left( \mathbb{E}[|u_i|^{2q}] \right)^{1/2} \left( \mathbb{E}[|\Xi_{i}^{(2)}|^{2q}] \right)^{1/2} \leq C_T \left( \mathbb{E}[|u_i|^{2q}] \right)^{1/2} \Delta^{3q/2}. \]  

(D.10)

Thus, for any integer \( k \geq 0 \), substituting (D.6)-(D.10) into (D.4), we know that

\[ \mathbb{E}\left[ \sup_{k=0, \ldots, l} |u_{k+1}|^{2q} \right] \leq C_T \left\{ T^q \Delta^{2q} + T^{q-1} \Delta \sum_{i=0}^{l} \mathbb{E}|u_i|^{2q} + (|T/\Delta|)^q \sum_{i=0}^{l} \mathbb{E}\left[ |\Xi_{i}^{(1)}|^{2q} \right] + (|T/\Delta|)^{q-1} \sum_{i=0}^{l} \mathbb{E}\left[ |\Xi_{i}^{(2)}|^{2q} \right] \right\} \]
\[ + (|T/\Delta|)^{q-2} \sum_{i=0}^{l} \mathbb{E}\left[ |u_i|^q|\Xi_{i}^{(1)}|^q \right] + (|T/\Delta|)^{q/2-1} \sum_{i=0}^{l} \mathbb{E}\left[ |u_i|^q|\Xi_{i}^{(2)}|^q \right] \}
\[ \leq C_T \left\{ T^q \Delta^{2q} + T^{q-1} \Delta \sum_{i=0}^{l} \mathbb{E}|u_i|^{2q} + (|T/\Delta|)^q \Delta^{4q} + (|T/\Delta|)^q \Delta^{3q} \right. \]
\[ + (|T/\Delta|)^{q-1} \sum_{i=0}^{l} \mathbb{E}|u_i|^{2q} \right\}^{1/2} \Delta^{2q} + (|T/\Delta|)^{q/2-1} \sum_{i=0}^{l} \left( \mathbb{E}|u_i|^{2q} \right)^{1/2} \Delta^{3q/2} \}
\[ \leq C \left\{ 2T^q \Delta^{2q} + T^{q-1} \Delta \sum_{i=0}^{l} \mathbb{E}|u_i|^{2q} + (T^{q-1} + T^{q/2-1}) \sum_{i=0}^{l} \left( \mathbb{E}|u_i|^{2q} \right)^{1/2} \Delta^{q+1} \right\} \]

60
\[ \leq C_T \left\{ 2T^q \Delta_2^q + T^{q-1} \Delta \sum_{i=0}^{l} \mathbb{E}|u_i|^{2q} + (T^{q-1} + T^{q/2-1}) \Delta \sum_{i=0}^{l} \left( \mathbb{E}|u_i|^{2q} + \Delta_2^q \right) \right\} \]

\[ \leq C_T \left( 3T^q \Delta_2^q + 2T^{q-1} \Delta \sum_{i=0}^{l} \mathbb{E}|u_i|^{2q} + T^{q/2-1} \Delta \sum_{i=0}^{l} \mathbb{E}|u_i|^{2q} + T^{q/2} \Delta_2^q \right) \]

\[ \leq C_T \left( T^q \Delta_2^q + T^{q-1} \Delta \sum_{i=0}^{l} \mathbb{E}|u_i|^{2q} \right) \]

for any \( q \geq 2 \) and \( l = 0, \ldots, \lfloor T/\Delta \rfloor \). By Gronwall’s Lemma

\[ \mathbb{E} \left[ \sup_{k=0,\ldots,l} |u_{k+1}|^{2q} \right] \leq C_T \Delta_2^q \exp(C_T) \]

for any \( q \geq 2 \) and \( l = 0, \ldots, \lfloor T/\Delta \rfloor \). This completes now the proof of the assertion for \( q \geq 2 \). The case \( q \in (0,2) \) follows now by Lyapunov’s inequality.

References

[1] A. Alfonsi, On the discretization schemes for the CIR (and Bessel squared) processes, Monte Carlo Methods Appl. 11 (2005) 355-384.
[2] A. Alfonsi, Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process, Statist. Probab. Lett. 83 (2013) 602-607.
[3] W. J. Anderson, Continuous-Time Markov Chains, Springer, New York, 1991.
[4] A. Berkaoui, M. Bossy, A. Diop, Euler scheme for SDEs with non-Lipschitz diffusion coefficient: strong convergence, ESAIM Probab. Stat. 12 (2008) 1-11.
[5] P. Billingsley, Convergence of Probability Measures, New York: Wiley, 1968.
[6] J. F. Chassagneux, A. Jacquier, I. Mihaylov, An explicit Euler scheme with strong rate of convergence for financial SDEs with non-Lipschitz diffusion coefficients, SIAM J. Financial Math. 7 (2016) 993-1021.
[7] L. Chen, S. Gan, X. Wang, First order strong convergence of an explicit scheme for the stochastic SIS epidemic model, J. Comput. Appl. Math. 392 (2021) 113482.
[8] S. Dereich, A. Neuenkirch, L. Szpruch, An Euler-type method for the strong approximation of the Cox-Ingersoll-Ross process, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science 468 (2012) 1105-1115.
[9] D. Higham, X. Mao, A. Stuart, Strong convergence of Euler-type methods for nonlinear stochastic differential equations, SIAM J. Numer. Anal. 40 (2002) 1041-1063.
[10] G. Hu, K. Wang, Stability in distribution of competitive Lotka-Volterra system with Markovian switching, Appl. Math. Model. 35 (2011) 3189-3200.
[11] M. Hutzenthaler, A. Jentzen, P. E. Kloeden, Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients, Ann. Appl. Probab. 22 (2012) 1611-1641.
[12] P. E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations. Springer-Verlag, Berlin, 1992.
[13] P. E. Kloeden, A. Neuenkirch, Convergence of numerical methods for stochastic differential equations in mathematical finance. Recent developments in computational finance, 49-80, Interdiscip. Math. Sci., 14, World Sci. Publ., Hackensack, NJ, 2013.
[14] X. Li, D. Jiang, X. Mao, Population dynamical behavior of Lotka-Volterra system under regime switching, J. Comput. Appl. Math. 232 (2009) 427-448.
[15] X. Li, X. Mao, Population dynamical behavior of non-autonomous Lotka-Volterra competitive system with random perturbation, Discrete Contin. Dyn. Syst. 24 (2009) 523-545.
[16] X. Li, A. Gray, D. Jiang, X. Mao, Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching, J. Math. Anal. Appl. 376 (2011) 11-28
[17] X. Li, G. Yin, Logistic models with regime switching: Permanence and ergodicity, J. Math. Anal. Appl. 441 (2016) 593-611.
[18] X. Li, X. Mao, G. Yin, Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: truncation methods, convergence in pth moment and stability, IMA J. Numer. Anal. 39 (2018) 847-892.
[19] W. Liu, X. Mao, Strong convergence of the stopped Euler-Maruyama method for nonlinear stochastic differential equations, Appl. Math. Comput. 223 (2013) 389-400.
[20] X. Mao, C. Yuan, Stochastic Differential Equations with Markovian Switching, Imperial College Press, 2006.
[21] X. Mao, Y. Shen, A. Gray, Almost sure exponential stability of backward Euler-Maruyama discretizations for hybrid stochastic differential equations, J. Comput. Appl. Math. 235 (2011) 1213-1226.
[22] X. Mao, The truncated Euler-Maruyama method for stochastic differential equations, J. Comput. Appl. Math. 290 (2015) 370-384.
[23] X. Mao, F. Wei, T. Wiriyakraikul, Positivity preserving truncated Euler-Maruyama Method for stochastic Lotka-Volterra competition model, J. Comput. Appl. Math. 394 (2021) 113566.
[24] A. Neuenkirch and L. Szpruch, First order strong approximations of scalar SDEs defined in a domain, Numer. Math. 128 (2014) 103-136.
[25] D. T. Nguyen, S. L. Nguyen, T. A. Hoang, G. Yin, Tamed-Euler method for hybrid stochastic differential equations with Markovian switching, Nonlinear Anal. Hybrid Syst. 30 (2018) 14-30.
[26] S. Pasqual, The stochastic logistic equation: stationary solutions and their stability, Rendiconti del Seminario Matematico della Università di Padova 106 (2001) 165-183.
[27] S. Sabanis, A note on tamed Euler approximations, Electron. Commun. Probab. 18 (2013) 1-10.
[28] S. Sabanis, Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients, Ann. Appl. Probab. 26 (2016) 2083-2105.
[29] R. Wang, X. Li, D. S. Mukama, On stochastic multi-group Lotka-Volterra ecosystems with regime switching, Discrete Contin. Dyn. Syst. Ser. B 22 (2017) 3499-3528.
[30] X. Wang, S. Gan, The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients, J. Difference Equ. Appl. 19 (2013) 466-490.
[31] H. Yang, X. Li, Explicit approximations for nonlinear switching diffusion systems in finite and infinite horizons, J. Differential Equations 265 (2018) 2921-2967.
[32] G. Yin, C. Zhu, Hybrid Switching Diffusions: Properties and Applications, New York: Springer-Verlag, 2010.