Generalized Gearhart-Koshy acceleration for the Kaczmarz method

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Abstract

The Kaczmarz method is an iterative numerical method for solving large and sparse rectangular systems of linear equations. Gearhart, Koshy and Tam have developed an acceleration technique for the Kaczmarz method that minimizes the distance to the desired solution in the direction of a full Kaczmarz step.

The present paper generalizes this technique to an acceleration scheme that minimizes the Euclidean norm error over an affine subspace spanned by a number of previous iterates and one additional cycle of the Kaczmarz method. The key challenge is to find a formulation in which all parameters of the least-squares problem defining the unique minimizer are known, and to solve this problem efficiently.

A numerical experiment demonstrates that the proposed affine search has the potential to clearly outperform the Kaczmarz and the randomized Kaczmarz methods with and without the Gearhart-Koshy/Tam line-search.

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1 Introduction

The Kaczmarz method for solving systems of linear equations was initially described and analyzed in [11]. It was later rediscovered in the paper [7] in
the context of computerized tomography problems, where it was used with great success. Being a row-action method, it tends to behave well when applied to large and sparse rectangular linear systems, see [4].

The performance of the Kaczmarz method depends on the fixed order in which the method cycles through the rows of the linear system. The randomized Kaczmarz method avoids the selection of a disadvantageous order by selecting the rows at random. It was proved in [16] that this approach yields linear convergence in expectation with a rate corresponding to the geometry of the problem.

Recently, there has been a strong emphasis on the development of acceleration schemes for the randomized Kaczmarz method. Some are based on splittings of the set of rows into a priori known well-conditioned blocks, see [13] and [14], while others are based on Motzkin acceleration see [12]. The latter method selects the next row of the linear system corresponding to the largest component of the current residual instead of cycling through the rows in a given order. Novel probabilistic variants of this approach select the next row with a probability distribution proportional to or otherwise derived from the current residual, see [1], [8], [9], [15] and the references therein. In a sense, these methods are greedy algorithms that aim at decreasing the residual as fast as possible.

The line-search proposed in [6] by Gearhart and Koshy for homogeneous and recently in [17] by Tam for inhomogeneous linear systems in the context of the deterministic Kaczmarz method is pursuing a greedy strategy that is diametrically opposed to Motzkin acceleration: It uses one full cycle of the Kaczmarz method as a search direction and minimizes the Euclidean norm error (instead of the residual) over the corresponding line. This is achieved using only information that is explicitly known at runtime, which means that this acceleration is computationally inexpensive.

Example 3.24 in [2] shows that in pathological situations, the Kaczmarz method with this line-search can be slower than the plain Kaczmarz method, while a straightforward modification of the convergence proof in [3] reveals that it is necessarily convergent.

The aim of this paper is to generalize the Gearhart-Koshy line-search to an acceleration scheme that minimizes the Euclidean norm error over an affine subspace spanned by a number of previous iterates and one additional cycle of the Kaczmarz method. This acceleration strategy is not limited to
the deterministic Kaczmarz method, but can be applied to the randomized Kaczmarz method as well.

The key challenge is to find a formulation in which all parameters of the least-squares problem defining the unique minimizer are known, and to solve this problem efficiently. It turns out that this is possible in linear time because of the particular structure of the problem. A numerical experiment provided in the final section of the paper demonstrates that the proposed affine search has the potential to clearly outperform the Kaczmarz and the randomized Kaczmarz methods with and without the Gearhart-Koshy line-search.

Finally, we would like to point out that the proposed method does not compete with the above-mentioned accelerations based on splittings and Motzkin acceleration, but it can in principle be applied to further enhance these and other methods based on successive projections.

2 Preliminaries

Throughout this paper, we consider a matrix

\[ A = (a_1, \ldots, a_m)^T \in \mathbb{R}^{m \times n} \]

with rows \( a_j \in \mathbb{R}^n \setminus \{0\} \) and a vector \( b \in \mathcal{R}(A) \) in the range of \( A \), and we consider the projectors

\[ P_j : \mathbb{R}^n \to \mathbb{R}^n, \quad P_j(x) := (I - \frac{a_j a_j^T}{\|a_j\|^2})x + \frac{b_j}{\|a_j\|^2}a_j, \quad j = 1, \ldots, m, \]

which project any point \( x \in \mathbb{R}^n \) to the affine subspaces

\[ H_j := \{z \in \mathbb{R}^n : a_j^T z = b_j\}, \quad j = 1, \ldots, m. \]

Their compositions

\[ P(x) := (P_m \circ \ldots \circ P_1)(x) \]

constitute a full cycle of the Kaczmarz method. It is well-known that for any \( x_0 \in \mathbb{R}^n \), we have \( \lim_{k \to \infty} P_k(x_0) \in A^{-1}b \), see e.g. [18].

When determining the computational complexity of the Kaczmarz method and the accelerated variants discussed in this paper, we will denote the number of nonzero elements of the matrix \( A \) by \( \text{nnz}(A) > 0 \). As the matrix \( A \)
is large in typical applications, we will assume that scalar quantities such as the norms \(\|a_j\|^2\) can be stored, but not the normalized rows \(a_j/\|a_j\|\). In this situation, we can carry out one Kaczmarz cycle with \(4 \text{nnz}(A) + m\) flops.

Algorithm 1: Kaczmarz method, originally proposed in [11] (complexity: \(4 \text{nnz}(A) + m\) flops per cycle)

| Input: \(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, x_0 \in \mathbb{R}^n\) |
|---------------------------------------------------------------|
| 1 for \(k = 0\) to \(\infty\) do |
| 2 \(x_{k+1} \leftarrow P(x_k)\); |
| 3 end |

Inspired by the papers [6] and [17], we wish to explore how the residual \(r(x) := \begin{pmatrix} (a_1^T x - b_1)/\|a_1\| \\ (a_2^T P_1(x) - b_2)/\|a_2\| \\ \vdots \\ (a_m^T P_{m-1} \circ \ldots \circ P_1(x) - b_m)/\|a_m\| \end{pmatrix}\) (1) can be used to speed up the Kaczmarz iteration. Note that the quantities required for its computation are explicitly calculated in a cycle of the Kaczmarz method.

We begin by exploring the properties of the residual \(r\), which measures the reduction of the square distance to any solution of the linear system in one Kaczmarz cycle and encodes information on the angle between the vectors \(x^* - x\) and \(P(x) - x\).

Lemma 1. Let \(x^* \in A^{-1}b\), and let \(x \in \mathbb{R}^n\) be arbitrary. Then we have

\[
\|r(x)\|^2 + \|P(x) - x^*\|^2 = \|x - x^*\|^2, \tag{2}
\]

\[
\|r(x)\|^2 + \|P(x) - x\|^2 = 2(x - x^*)^T(x - P(x)). \tag{3}
\]

Proof. Since

\[
\langle P_j(x) - x, P_j(x) - x^* \rangle = \frac{b_j - a_j^T x}{\|a_j\|^2} a_j, x + \frac{b_j - a_j^T x}{\|a_j\|^2} a_j - x^*
\]

\[
= \frac{b_j - a_j^T x}{\|a_j\|^2} \left( a_j^T x + (b_j - a_j^T x) - b_j \right) = 0,
\]

4
we may use the Pythagorean theorem to compute
\[ \|x - x^*\|^2 = \|(x - P_j(x)) + (P_j(x) - x^*)\|^2 = \|x - P_j(x)\|^2 + \|P_j(x) - x^*\|^2. \]

Statement (2) follows from the above identity successively applied to \(x, P_1(x), P_2 \circ P_1(x)\) etc. in lieu of \(x\), and from the definitions of \(P\) and \(r\). Now the polarization identity yields
\[
\|r(x)\|^2 + \|x - P(x)\|^2 = \|x - x^*\|^2 - \|P(x) - x^*\|^2 + \|x - P(x)\|^2 = 2(x - x^*)^T(x - P(x)).
\]

\[ \square \]

The mapping \(r\) behaves indeed like a residual.

**Lemma 2.** The following statements are equivalent:

a) We have \(Ax = b\).

b) We have \(P(x) = x\).

c) We have \(r(x) = 0\).

**Proof.** If statement a) holds, then
\[ P_j(x) = (I - \frac{a_ja_j^T}{\|a_j\|^2})x + \frac{b_j}{\|a_j\|^2}a_j = x, \quad j = 1, \ldots, m, \]
which implies \(P(x) = x\).

Assume that statement b) holds, and let \(x^* \in \mathbb{R}^n\) be any point with \(Ax^* = b\). If \(r(x) \neq 0\), then statement (2) gives
\[ \|P(x) - x^*\|^2 < \|x - x^*\|^2, \]
which contradicts \(P(x) = x\). Hence statement c) holds.

If statement c) holds, then statement a) follows by induction. We clearly have \(a_1^Tx = b_1\). If \(a_i^Tx = b_i\) holds for \(i = 1, \ldots, j\), then \(P_i(x) = x\) holds for \(i = 1, \ldots, j\), and
\[ a_{j+1}^T x - b_{j+1} = a_{j+1}^T P_j \circ \ldots \circ P_1(x) - b_{j+1} = r_{j+1}(x) = 0. \]

By induction, we obtain \(Ax = b\). \[ \square \]
Remark 3. A straight-forward modification of the convergence proof in \cite{3} reveals that any sequence \((x_k)_{k \in \mathbb{N}}\) satisfying
\[
\|x_{k+1} - x^*\|^2 \leq \|P(x_k) - x^*\|^2 \quad \forall k \in \mathbb{N}, \ x^* \in A^{-1}b, \tag{4}
\]
converges to a solution \(x^* \in A^{-1}b\), and it is clear that the typical error estimates for cyclic projection-type methods as in Corollary 9.34 in \cite{5} for the sequence \((P^k(x_0))_{k \in \mathbb{N}}\) also hold for the sequence \((x_k)_{k \in \mathbb{N}}\). Hence we will focus on generating a sequence with the above property (4) by minimizing the errors \(\|x_{k+1} - x^*\|^2\) in affine search spaces at a relatively small computational cost.

3 Acceleration by line-search

In this section, we recover the step-size from Theorem 4.1 in \cite{17} with a straight-forward geometric argument. In addition, we quantify the error reduction in terms of the difference between statements (2) and (5). All quantities involved in these formulas are known at runtime.

Theorem 4. Let \(x^* \in A^{-1}b\), and let \(x \in \mathbb{R}^n\) with \(P(x) \neq x\). Then we have
\[
s^* := \arg\min_{s \in \mathbb{R}} \| (1 - s)x + sP(x) - x^* \|^2 = \frac{1}{2} + \frac{\|r(x)\|^2}{2\|P(x) - x\|^2},
\]
\[
\|x - x^*\|^2 - \|(1 - s^*)x + s^*P(x) - x^*\|^2 = \frac{(\|r(x)\|^2 + \|P(x) - x\|^2)^2}{4\|P(x) - x\|^2}. \tag{5}
\]

Proof. Using identity (3), we see that the strictly convex parabola
\[
g(s) = \| (1 - s)x + sP(x) - x^* \|^2
= \|x - x^*\|^2 + 2s(x - x^*)^T(P(x) - x) + s^2\|P(x) - x\|^2
= \|x - x^*\|^2 - s(\|r(x)\|^2 + \|P(x) - x\|^2) + s^2\|P(x) - x\|^2
\]
has the unique minimum
\[
s^* = \frac{\|r(x)\|^2 + \|P(x) - x\|^2}{2\|P(x) - x\|^2} = \frac{1}{2} + \frac{\|r(x)\|^2}{2\|P(x) - x\|^2}.
\]
The second statement follows from
\[
g(s^*) = \|x - x^*\|^2 - \frac{(\|r(x)\|^2 + \|P(x) - x\|^2)^2}{4\|P(x) - x\|^2}. \]

\]
Algorithm 2: Kaczmarz method with line-search, originally proposed in [17]
(complexity: \(4 \text{nnz}(A) + 3m + 5n\) flops per cycle)

**Input:** \(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, x_0 \in \mathbb{R}^n\)

\[
\begin{align*}
\text{for } k = 0 \text{ to } \infty \text{ do} \\
\quad \text{compute } P(x_k) \text{ and } r(x_k) \text{ in Kaczmarz cycle from } x_k; \\
\quad d \leftarrow P(x_k) - x_k; \\
\quad \delta \leftarrow \|d\|^2; \\
\quad \text{if } \delta = 0 \text{ then} \\
\qquad \text{return } x_k; \\
\quad \rho \leftarrow \|r(x_k)\|^2; \\
\quad s \leftarrow \frac{1}{2} + \frac{\rho}{\delta}; \\
\quad x_{k+1} \leftarrow x_k + s d; \\
\end{align*}
\]

It is a straight-forward consequence of Theorem 4 that Algorithm 2 is safe to use.

**Corollary 5.** Either Algorithm 2 terminates in finite time and returns an iterate \(x_k \in A^{-1}b\), or it generates a well-defined sequence \((x_k)_k\) that has the property (4) and, for all \(x^* \in A^{-1}b\), satisfies the identities

\[
\begin{align*}
(x_{k+1} &= \arg\min_{\xi \in \text{aff}(x_k, P(x_k))} \|\xi - x^*\|^2, \tag{6} \\
\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 &= \frac{(\|r(x_k)\|^2 + \|P(x_k) - x_k\|^2)^2}{4\|P(x_k) - x_k\|^2}. \tag{7}
\end{align*}
\]

**Proof.** If Algorithm 2 terminates after \(k \in \mathbb{N}\) steps and returns an iterate \(x_k \in \mathbb{R}^n\), then the stopping criterion implies \(P(x_k) = x_k\), and Lemma 2 yields \(Ax_k = b\). Otherwise, all expressions in Algorithm 2 are well-defined. By Theorem 4 formulas (6) and (7) hold, and formula (6) implies (4). \(\square\)

4 Acceleration by affine search

It is possible to extend the above line-search to a search in an affine subspace spanned by several previous iterates and the latest Kaczmarz cycle, which improves the local error reduction. Again, all required quantities and the exact reduction are computable at runtime.
We begin by proving a simple geometric observation that will give meaning to the stopping criterion of the accelerated iteration.

**Lemma 6.** Let \( x^* \in A^{-1}b \), and let \( x_1, \ldots, x_\ell \in \mathbb{R}^n \) be points such that the condition
\[
x_\ell = \arg\min_{\xi \in \text{aff}(x_1, \ldots, x_\ell)} \|\xi - x^*\|^2
\]
holds. If we have \( P(x_\ell) \in \text{aff}(x_1, \ldots, x_\ell) \), then we also have \( \|r(x_\ell)\|^2 = 0 \), \( P(x_\ell) = x_\ell \) and \( Ax_\ell = b \).

**Proof.** By statement (2), we have
\[
\|P(x_\ell) - x^*\|^2 = \|x_\ell - x^*\|^2 - \|r(x_\ell)\|^2.
\]
If we have \( P(x_\ell) \in \text{aff}(x_1, \ldots, x_\ell) \), then condition (8) yields \( \|r(x_\ell)\|^2 = 0 \), and Lemma 2 implies that \( Ax_\ell = b \).

The following result provides a characterization of the minimizer
\[
\arg\min_{\xi \in \text{aff}(x_1, \ldots, x_\ell, P(x_\ell))} \|\xi - x^*\|^2
\]
that does not use the unknown solution \( x^* \) explicitly. We formulate and prove this theorem for vectors indexed \( x_1, \ldots, x_\ell \) to keep the notation simple, and we will use it later (see Corollary 10) for a varying number of vectors and varying indexation.

**Theorem 7.** Let \( x^* \in A^{-1}b \), let \( x_1, \ldots, x_\ell \in \mathbb{R}^n \) be affinely independent points with (8) and \( P(x_\ell) \notin \text{aff}(x_1, \ldots, x_\ell) \). Consider the matrices
\[
V := (x_1 - x_\ell, \ldots, x_{\ell-1} - x_\ell) \in \mathbb{R}^{n \times (\ell - 1)}, \quad M := (V, P(x_\ell) - x_\ell) \in \mathbb{R}^{n \times \ell}
\]
and define
\[
\gamma := \frac{1}{2}(\|r(x_\ell)\|^2 + \|P(x_\ell) - x_\ell\|^2).
\]
Then the minimizer
\[
s^* := \arg\min_{s \in \mathbb{R}^\ell} \|x_\ell + Ms - x^*\|^2
\]
is the unique solution of the linear system
\[
M^T Ms = \gamma e_\ell^T
\]
where \( e_\ell^T \) is the \( \ell \)-th unit vector, and we have
\[
\|x_\ell - x^*\|^2 - \|x_\ell + Ms^* - x^*\|^2 = \gamma s_\ell^* = \gamma^2 \frac{\det(V^TV)}{\det(M^T M)}.
\]

8
Remark 8. For an interpretation of the identity (11), it is instructive to have a look at the case $\ell = 2$. Elementary computations show that whenever $P(x_2) \notin \text{aff}(x_1, x_2)$, the minimizer
\[ s^* := \arg\min_{s \in \mathbb{R}^2} \|x_2 + s_1(x_1 - x_2) + s_2(P(x_2) - x_2) - x^*\|^2 \]
satisfies
\[ \|x_2 - x^*\|^2 - \|x_2 + s_1^*(x_1 - x_2) + s_2^*(P(x_2) - x_2) - x^*\|^2 \]
\[ = (1 - \cos^2 \angle(x_1 - x_2, P(x_2) - x_2))^{-1}(\|r(x_2)\|^2 + \|P(x_2) - x_2\|^2)^2 \]
\[ = 4\|P(x_2) - x_2\|^2. \]
Comparing with Theorem 3, we see that the planar search outperforms the line-search by a factor $(1 - \cos^2 \angle(x_1 - x_2, P(x_2) - x_2))^{-1}$.

Proof. Since the vectors $x_1, \ldots, x_{\ell}, P(x_{\ell})$ are affinely independent, the Gramian matrices $V^T V$ and $M^T M$ are positive definite. The first derivatives of the strictly convex quadratic function
\[ g(s) := \|x_{\ell} + Ms - x^*\|^2 \]
are given by
\[ \frac{dg}{ds_j}(s) = 2\left(x_{\ell} + Ms - x^*\right)^T(x_j - x_{\ell}), \quad j = 1, \ldots, \ell - 1, \]
\[ \frac{dg}{ds_{\ell}}(s) = 2\left(x_{\ell} + Ms - x^*\right)^T(P(x_{\ell}) - x_{\ell}). \]
Using statements (3) and (8), we see that the unique minimizer $s^*$ of $g$ solves the linear equations
\[ (x_j - x_{\ell})^T Ms = \langle x^* - x_{\ell}, x_j - x_{\ell} \rangle = 0, \quad j = 1, \ldots, \ell - 1, \quad (12) \]
\[ (P(x_{\ell}) - x_{\ell})^T Ms = \langle x^* - x_{\ell}, P(x_{\ell}) - x_{\ell} \rangle = \gamma, \quad (13) \]
which are subsumed in the linear system (10). Using Cramer’s rule, we can express
\[ s^*_\ell = \frac{\det \begin{pmatrix}
\langle x_1 - x_{\ell}, x_1 - x_{\ell} \rangle & \cdots & \langle x_1 - x_{\ell}, x_{\ell - 1} - x_{\ell} \rangle & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\langle x_{\ell - 1} - x_{\ell}, x_1 - x_{\ell} \rangle & \cdots & \langle x_{\ell - 1} - x_{\ell}, x_{\ell - 1} - x_{\ell} \rangle & 0 \\
\langle P(x_{\ell}) - x_{\ell}, x_1 - x_{\ell} \rangle & \cdots & \langle P(x_{\ell}) - x_{\ell}, x_{\ell - 1} - x_{\ell} \rangle & \gamma
\end{pmatrix}}{\det(M^T M)} = \frac{\gamma \det(V^T V)}{\det(M^T M)}. \]
From system (10), we infer
\[ \|Ms^*\|^2 = (s^*)^T M^T Ms^* = \gamma s^*_\ell, \]
and together with statements (12) and (13), we conclude that
\[ g(s^*) = \|x_\ell + Ms^* - x^*\|^2 = \|x_\ell - x^*\|^2 + 2(s^*)^T M^T (x_\ell - x^*) + \|Ms^*\|^2 \]
\[ = \|x_\ell - x^*\|^2 + 2s^*_\ell (P(x_\ell) - x_\ell)^T (x_\ell - x^*) + \gamma s^*_\ell = \|x_\ell - x^*\|^2 - \gamma s^*_\ell. \]

Algorithm 3 uses Theorem 7 after every step to reduce the size of the error \(\|x_{k+1} - x^*\|^2\) in statement (4).

Remark 9. a) The parameter \(\ell \in \mathbb{N}_1\) in Algorithm 3 controls how many of the previous iterates are used to span the affine search space. When \(\ell = 1\), then Algorithm 3 reduces to Algorithm 2. When \(\ell > 1\), the algorithm has a startup phase in which it grows the affine basis of the search space, so that
\[ V_0 = [ ], \quad V_1 = (x_0 - x_1), \quad \ldots, \quad V_{\ell-1} = (x_0 - x_{\ell-1}, \ldots, x_{\ell-2} - x_{\ell-1}). \]

After the startup phase, the algorithm keeps the latest \(\ell\) iterates and discards \(x_{k-\ell}\), which gives
\[ V_\ell = (x_1 - x_\ell, \ldots, x_{\ell-1} - x_\ell), \quad V_{\ell+1} = (x_2 - x_{\ell+1}, \ldots, x_{\ell} - x_{\ell+1}), \quad \ldots \]

When \(\ell \geq n\), then the proof of Theorem 10 reveals that Algorithm 3 terminates with an iterate \(x_k \in A^{-1}b, k \leq n\).

b) The computational complexity of one step of Algorithm 3 is composed in the following way:

(i) The Kaczmarz cycle requires \(4 \text{nnz}(A) + m\) flops.

(ii) The computation of \(d_k, \delta_k\) and \(\rho_k\) requires \(3n + 2m\) flops.

(iii) Assembling \(M\) requires \((k - j_k)n\) flops (\(d_k\) is known).

(iv) Computing \(M^T M\) requires \(\frac{1}{2}(k - j_k + 1)(k - j_k + 2)n\) flops.

(v) Solving system (10) via LU factorization and forward and backward substitution requires \(\frac{2}{3}(k - j_k)^3 + \frac{7}{2}(k - j_k)^2 + \frac{5}{6}(k - j_k)\) flops.
Updating $x_k$ requires $2(k - j_k + 1)n$ flops.

After the startup phase, we have $k - j_k + 1 = \ell$, which gives a total computational complexity of roughly

$$4 \text{nnz}(A) + (3 + 3\ell + \frac{1}{2}\ell^2)n + 3m + \ell^3.$$ 

The acceleration comes at a considerable cost, mostly caused by the assembly and by solving system (10), which is not desirable.

**Algorithm 3:** Kaczmarz method with affine search
(for complexity see Remark 9 part b)

| Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $x_0 \in \mathbb{R}^n$, $\ell \in \mathbb{N}$ |
|---|
| for $k = 0$ to $\infty$ do |
| 1. compute $P(x_k)$ and $r(x_k)$ in Kaczmarz cycle from $x_k$; |
| 2. $d_k \leftarrow P(x_k) - x_k$; |
| 3. $\delta_k \leftarrow \|d_k\|^2$; |
| 4. if $\delta = 0$ then |
| 5. return $x_k$; |
| 6. $\rho_k \leftarrow \|r(x_k)\|^2$; |
| 7. $\gamma_k \leftarrow \frac{1}{2}(\rho_k + \delta_k)$; |
| 8. $j_k \leftarrow \max\{k - \ell + 1, 0\}$; |
| 9. $V_k \leftarrow (x_{j_k} - x_k, \ldots, x_{k-1} - x_k) \in \mathbb{R}^{n \times (k-j_k)}$; |
| 10. assemble $M_k^T M_k$ from $M_k = (V_k, d_k) \mathbb{R}^{n \times (k-j_k+1)}$; |
| 11. solve $M_k^T M_k s_k = \gamma_k e^{k-j_k+1} k_{k-j_k+1}$ for $s_k$; |
| 12. $x_{k+1} \leftarrow x_k + M_k s_k$; |
| end |

**Theorem 10.** Let $x^* \in A^{-1}b$. Either Algorithm 3 terminates and returns an iterate $x_k \in A^{-1}b$, or it generates a well-defined sequence $(x_k)_k$ that satisfies

$$x_{k+1} = \arg\min_{\xi \in \text{aff}(x_{j_k}, \ldots, x_k, P(x_k))} \|\xi - x^*\|^2,$$

$$\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 = \gamma_k^2 \frac{\det(V_k^T V_k)}{\det(M_k^T M_k)}$$

for all $k \in \mathbb{N}$. In particular, it has the property (4).
Proof. We prove by induction that Algorithm 3 either returns a solution in finite time or generates a sequence such that identities (14) and (15) and the following statements hold for every $k \in \mathbb{N}$:

a) The vectors $x_{j_k}, \ldots, x_k$ are affinely independent.

b) We have $x_k = \arg\min_{\xi \in \text{aff}(x_{j_k}, \ldots, x_k)} \|\xi - x^*\|^2$.

c) We have $P(x_k) \notin \text{aff}(x_{j_k}, \ldots, x_k)$.

If $k = 0$, then properties a) and b) are trivially satisfied.

Now assume that Algorithm 3 has generated iterates $x_0, \ldots, x_k \in \mathbb{R}^n$ with properties a), b).

If Algorithm 3 terminates and returns $x_k$, then the stopping criterion implies that $P(x_k) = x_k$, and Lemma 2 implies $Ax_k = b$.

If Algorithm 3 does not terminate, then we have $P(x_k) \neq x_k$. Because of statement b), Lemma 6 implies statement c). Since statements a), b) and c) hold for $k$, the vectors $x_{j_k}, \ldots, x_k$ satisfy all assumptions of Theorem 7. The linear system (10) with the matrix

$$M = (x_{j_k} - x_k, \ldots, x_{k-1} - x_k, P(x_k) - x_k)$$

possesses a unique solution $s^* \in \mathbb{R}^{k-j_k+1}$, and the iterate $x_{k+1} := x_k + Ms^*$ is well-defined. Because of statements a) and c), we have

$$\det(V^TV) > 0 \text{ and } \det(M^TM) > 0,$$

so by statement (11), we have $s^*_{k-j_k+1} \neq 0$. Combining this fact with statements a) and c) yields statement a) with $k+1$ in lieu of $k$. Statement (9) implies statement (14) and, since

$$\text{aff}(x_{j_k+1}, \ldots, x_k, x_{k+1}) \subset \text{aff}(x_{j_k}, \ldots, x_k, P(x_k)),$$

also statement b) for $k+1$ in lieu of $k$. In addition, statement (11) implies the identity (15).

\qed
5 Efficient updating

The goal of this section is to simplify the solution of the linear system (10), which must be solved after every Kaczmarz cycle to determine $x_{k+1}$ from the previous iterates and the vector $P(x_k)$.

Our first result shows that updating the submatrix $V^T V$ of the matrix $M^T M$ from one iteration to another is straightforward.

**Lemma 11.** In the situation of Theorem 7, and denoting $x_{\ell+1} := x_\ell + Ms^*$, we have

$$\langle x_i - x_\ell, x_j - x_\ell \rangle = (V^T V)_{ij} + \gamma s_{\ell}^*, \quad 1 \leq i, j \leq \ell - 1,$$

$$\langle x_i - x_\ell+1, x_\ell - x_{\ell+1} \rangle = \gamma s_{\ell}^*, \quad 1 \leq i \leq \ell.$$

**Proof.** We can express

$$x_i - x_\ell = Me_i, \quad i = 1, \ldots, \ell - 1.$$

For $1 \leq i, j \leq \ell - 1$, we use the identity (10) to obtain

$$\langle x_i - x_{\ell+1}, x_j - x_{\ell+1} \rangle = \langle x_i - x_\ell - Ms^*, x_j - x_\ell - Ms^* \rangle$$

$$= \langle Me_i - Ms^*, Me_j - Ms^* \rangle$$

$$= e_i^T M^T Me_j - e_i^T M^T Ms^* - e_j^T M^T Ms^* + (s^*)^T M^T Ms^*$$

$$= (M^T M)_{ij} + \gamma s_{\ell}^* = (V^T V)_{ij} + \gamma s_{\ell}^*$$

For $1 \leq i < \ell$, we compute

$$\langle x_i - x_{\ell+1}, x_\ell - x_{\ell+1} \rangle = \langle x_i - x_\ell - Ms^*, -Ms^* \rangle$$

$$= \langle Me_i - Ms^*, -Ms^* \rangle = -e_i^T M^T Ms^* + (s^*)^T M^T Ms^* = \gamma s_{\ell}^*,$$

and we also obtain

$$\langle x_\ell - x_{\ell+1}, x_\ell - x_{\ell+1} \rangle = (s^*)^T M^T Ms^* = \gamma s_{\ell}^*.$$  

□

We will see (in the proof of Theorem 14) that the matrices $V^T V$ generated by Algorithms 3 and 4 have the structure of the matrix $B$ defined below with known coefficients $\alpha_j$. 

13
Lemma 12. Let $\alpha \in \mathbb{R}^n$, and let $f_1^n, \ldots, f_n^n \in \mathbb{R}^n$ be given by $f_j^n = \sum_{i=1}^j e_i^n$, where $e_i^n \in \mathbb{R}^n$ denotes the $i$-th unit vector. Then the matrix

$$B := \sum_{j=1}^n \alpha_j f_j^n (f_j^n)^T$$

has the structure

$$B = \begin{pmatrix}
\sum_{i=1}^n \alpha_i & \sum_{i=2}^n \alpha_i & \cdots & \sum_{i=n} \alpha_i \\
\sum_{i=2}^n \alpha_i & \sum_{i=2}^n \alpha_i & \cdots & \sum_{i=n} \alpha_i \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=n} \alpha_i & \cdots & \cdots & \sum_{i=n} \alpha_i
\end{pmatrix}.$$ 

If $\alpha_j \neq 0$ for $j = 1, \ldots, n$, then

$$C := \begin{pmatrix}
\alpha_1^{-1} & -\alpha_1^{-1} \\
-\alpha_1^{-1} & \alpha_1^{-1} + \alpha_2^{-1} & -\alpha_2^{-1} \\
-\alpha_2^{-1} & \alpha_2^{-1} + \alpha_3^{-1} & -\alpha_3^{-1} \\
& \ddots & \ddots & \ddots \\
& -\alpha_{n-2}^{-1} & \alpha_{n-2}^{-1} + \alpha_{n-1}^{-1} & -\alpha_{n-1}^{-1} \\
& & \alpha_{n-1}^{-1} + \alpha_n^{-1}
\end{pmatrix}$$

is the inverse of the matrix $B$.

Proof. When $\alpha_j \neq 0$ for $j = 1, \ldots, n$, then the matrix $C$ is well-defined. Multiplying the matrices $B$ and $C$ yields the identity. \qed

In conjunction with Lemmas 11 and 12, the next lemma shows that the linear system (10) can be solved in linear time.

Lemma 13. Let $B \in \mathbb{R}^{n \times n}$ be invertible, let $p \in \mathbb{R}^n$ and let $\delta > 0$ and $\gamma \in \mathbb{R}$. If the matrix

$$G := \begin{pmatrix} B & p \\ p^T & \delta \end{pmatrix}$$

is invertible, then we have $p^T B^{-1} p \neq \delta$, and the solution of the linear system

$$G x = \gamma e_n^n$$

is given by

$$G^{-1} \gamma e_n^n = \frac{\gamma}{p^T B^{-1} p - \delta} \begin{pmatrix} B^{-1} p \\ -1 \end{pmatrix}.$$
Proof. Since $G$ is nonsingular, and since
\[
\left( \begin{array}{cc}
B & p \\
\delta & p^T
\end{array} \right) \left( B^{-1}p \right) = \left( \begin{array}{c}
0 \\
p^TB^{-1}p - \delta
\end{array} \right)
\]
holds, we have $p^T Cp \neq \delta$, and the desired result follows.

Lemmas 11, 12 and 13 inspire Algorithm 4. We require $\ell \geq 2$, because for $\ell = 1$, when Algorithm 3 reduces to Algorithm 2, there is no data to be updated. By $C(\alpha)$, we denote the matrix $C$ from Lemma 12 given by the parameter vector $\alpha$ and its dimension.

In the initial step of Algorithm 4, the matrices $V_0$ and $C_0$ as well as the vectors $p_0$ and $q_0$ are empty and have to be ignored where they occur. We split the solution $s_k$ of the linear system $M_k^T M_k s_k = \gamma_k e_{k-j+1}$ into the vector $s_k \in \mathbb{R}^{k-j}$ of the first several components and the last component $s_k \in \mathbb{R}$ to exploit the structure of system (10).

Algorithm 4: Kaczmarz method with enhanced affine search
(for complexity see Remark 15)

\begin{verbatim}
Input: A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, x_0 \in \mathbb{R}^n, \ell \in \mathbb{N}_2
for k = 0 to \infty do
    compute P(x_k) and r(x_k) in Kaczmarz cycle from x_k;
    d_k \Leftarrow P(x_k) - x_k;
    \delta_k \Leftarrow ||d_k||^2;
    if \delta_k = 0 then
        return x_k;
    end
    \rho_k \Leftarrow ||r(x_k)||^2;
    \gamma_k \Leftarrow \frac{1}{2}(\rho_k + \delta_k);
    j_k \Leftarrow \max\{k - \ell + 1, 0\};
    V_k \Leftarrow (x_{j_k} - x_k, \ldots, x_{k-1} - x_k) \in \mathbb{R}^{n \times (k-j_k)};
    p_k \Leftarrow V_k^T d_k \in \mathbb{R}^{k-j_k};
    C_k \Leftarrow C(\gamma_{j_k} s_{j_k}, \ldots, \gamma_{k-1} s_{k-1}) \in \mathbb{R}^{(k-j_k) \times (k-j_k)};
    q_k \Leftarrow C_k p_k \in \mathbb{R}^{k-j_k};
    s_k \Leftarrow -\frac{\gamma_k}{\delta - p_k^T q_k} \in \mathbb{R};
    s_k \Leftarrow - s_k q_k \in \mathbb{R}^{k-j_k};
    x_{k+1} \Leftarrow x_k + V_k s_k + s_k d_k;
end
\end{verbatim}
Theorem 14. Algorithms 3 and 4 generate identical iterations.

Proof. We prove by induction that one of the following alternatives holds:

i) Algorithms 3 and 4 both terminate in step $k$.

ii) We have

$$V_k^T V_k = \begin{cases} \sum_{i=1}^{k-j} \gamma_{jk+i-1} s_{jk+i-1} f_i^{k-j} (f_i^{k-j})^T, & k > 0 \\ \text{the empty matrix } [\ ], & k = 0, \end{cases}$$

where $f_1^{k-j}, \ldots, f_{k-j}^{k-j} \in \mathbb{R}^{k-j}$ are given by $f_i^{k-j} = \sum_{h=1}^{i} e_h^{k-j}$, and both algorithms compute identical $\gamma_k$, $s_k$ and $x_{k+1}$.

When $k = 0$, both algorithms terminate if and only if $d_0 = P(x_0) - x_0 = 0$. Otherwise, both algorithms compute identical $\delta_0$, $\rho_0$ and $\gamma_0$, and they both have $j_0 = 0$ and $V_0 = [\ ]$. Algorithm 3 solves

$$\delta_0 s_0 = \|d_0\|^2 s_0 = M_0^T M_0 s_0 = \gamma_0,$$

while Algorithm 4 has $p_0 = [\ ]$, $C_0 = [\ ]$ and $q_0 = [\ ]$, computes $s_0 = \delta_0^{-1} \gamma_0$ and sets $s_0 = [\ ]$. Hence both algorithms generate identical $s_0 \in \mathbb{R}$ and the same next iterate

$$x_1 = x_0 + s_0 d_0 \in \mathbb{R}^n.$$

Now assume that alternative ii) holds for $0, \ldots, k$. Then both algorithms compute identical $\delta_{k+1}$ and $\delta_k$, and both terminate and return $x_{k+1}$ if and only if $\delta_{k+1} = 0$. Otherwise, they compute identical $\rho_{k+1}$ and $\gamma_{k+1}$. For the update $V_k \mapsto V_{k+1}$, we distinguish the following cases:

a) When $k = 0$, we have $j_0 = j_1 = 0$ and $V_1 = (x_0 - x_1)$, so using statements (16) and (17), we find

$$V_1^T V_1 = \|x_0 - x_1\|^2 = s_0^2 \delta_0 = \gamma_0 s_0.$$

b) When $k > 0$ and $j_{k+1} = j_k$, then we have $j_{k+1} = j_k = 0$ as well as $V_k = (x_0 - x_k, \ldots, x_{k-1} - x_k)$ and $V_{k+1} = (x_0 - x_{k+1}, \ldots, x_k - x_{k+1})$. Lemma 11 tells us that

$$\langle x_i - x_{k+1}, x_j - x_{k+1} \rangle = (V_k^T V_k)_{ij} + \gamma_k s_k, \quad 0 \leq i, j \leq k - 1,$$

$$\langle x_i - x_{k+1}, x_k - x_{k+1} \rangle = \gamma_k s_k, \quad 0 \leq i \leq k.$$
The induction hypothesis implies that

\[
V_{k+1}^T V_k = \begin{pmatrix} V_k^T V_k & 0 \\ 0 & 0 \end{pmatrix} + \gamma_k s_k \mathbb{1}_{\mathbb{R}^{k+1}}^T \mathbb{1}_{\mathbb{R}^{k+1}} \\
= \sum_{i=1}^{k-j_k} \gamma_{j_k+i-1} s_{j_k+i-1} \begin{pmatrix} f_i^{k-j_k} \\ 0 \end{pmatrix} (f_i^{k-j_k})^T, 0) + \gamma_k s_k \mathbb{1}_{\mathbb{R}^{k+1}}^T \mathbb{1}_{\mathbb{R}^{k+1}} \\
= \sum_{i=1}^{k+1-j_k} \gamma_{j_k+i-1} s_{j_k+i-1} f_i^{k+1-j_k+1} (f_i^{k+1-j_k+1})^T,
\]

where we have used that \( \gamma_k = \gamma_{k+1} = 0 \) and \( f_k^{k+1} = f_k^{k+1} \).

c) When \( k > 0 \) and \( j_{k+1} \neq j_k \), then we have \( j_{k+1} = k - \ell + 2 \) and \( j_k = k - \ell + 1 \), and we consider \( V_k = (x_{k-\ell+1} - x_k, \ldots, x_{k-1} - x_k) \) and \( V_{k+1} = (x_{k-\ell+2} - x_{k+1}, \ldots, x_k - x_{k+1}) \). Again by Lemma 11 and defining \( \mu \in \mathbb{R} \) and \( v \in \mathbb{R}^{\ell-1} \) by:

\[
\mu := \langle x_{k-\ell+1} - x_{k+1}, x_{k-\ell+1} - x_{k+1} \rangle \\
v_i := \langle x_{k-\ell+1} - x_{k+1}, x_{k-\ell+1} - x_{k+1} \rangle, \quad i = 1, \ldots, \ell - 1,
\]

we find for similar reasons that

\[
\begin{pmatrix} \mu \\ v \end{pmatrix}^T V_{k+1}^T V_{k+1} = \begin{pmatrix} V_k^T V_k & 0 \\ 0 & 0 \end{pmatrix} + \gamma_k s_k \mathbb{1}_{\mathbb{R}^{\ell}}^T \mathbb{1}_{\mathbb{R}^{\ell}} \\
= \sum_{i=1}^{k-j_k} \gamma_{j_k+i-1} s_{j_k+i-1} \begin{pmatrix} f_i^{k-j_k} \\ 0 \end{pmatrix} (f_i^{k-j_k})^T, 0) + \gamma_k s_k \mathbb{1}_{\mathbb{R}^{\ell}}^T \mathbb{1}_{\mathbb{R}^{\ell}} \\
= \sum_{i=1}^{k+1-j_k} \gamma_{j_k+i-1} s_{j_k+i-1} f_i^{k+1-j_k+1} (f_i^{k+1-j_k+1})^T,
\]

The desired statement

\[
V_{k+1}^T V_{k+1} = \sum_{i=1}^{k+1-j_{k+1}} \gamma_{j_{k+1}+i-1} s_{j_{k+1}+i-1} f_i^{k+1-j_{k+1}} (f_i^{k+1-j_{k+1}})^T
\]

can be verified by a component-wise comparison of equations (18) and (19).
Hence in all three cases, the matrix $V_{k+1}^T V_{k+1}$ has the desired representation, and Lemma 12 yields that
\[
(V_{k+1}^T V_{k+1})^{-1} = C(\gamma_{j_{k+1}} s_{j_{k+1}}, \ldots, \gamma_k s_k) = C_{k+1}.
\]
Since we have
\[
M_{k+1}^T M_{k+1} = \begin{pmatrix} V_{k+1}^T V_{k+1} & V_{k+1}^T d_{k+1} \\ d_{k+1} V_{k+1} & \delta \end{pmatrix},
\]
Lemma 13 tells us that the remaining steps of Algorithm 4 compute the solution $s_k = (s_k^T, s_k)^T$ of the linear system
\[
M_{k+1}^T M_{k+1} s_{k+1} = \gamma_{k+1} e_{k+1} - \gamma_{j_{k+1}} e_{j_{k+1}}
\]
and hence the same iterate $x_{k+1}$ as Algorithm 3.

Remark 15. The computational complexity of one step of Algorithm 4 is composed in the following way:

(i) The Kaczmarz cycle requires $4 \text{nnz}(A) + m$ flops.

(ii) The computation of $d_k$, $\delta_k$ and $\rho_k$ requires $3n + 2m$ flops.

(iii) Assembling $V_k$ requires $(k - j_k)n$ flops.

(iv) Computing $p_k$ requires $2(k - j_k)n$ flops.

(v) Computing $q_k$ requires $4(k - j_k)$ flops.

(vi) Computing $s_k^T$ requires $(k - j_k)$ flops.

(vii) Updating $x_k$ requires $2(k - j_k + 1)n$ flops.

After the startup phase, when $k - j_k + 1 = \ell$, we have a total computational complexity of roughly
\[
4 \text{nnz}(A) + (3 + 5\ell)n + 3m + 5\ell.
\]
Hence Algorithm 3 arrives at the same numerical results as Algorithm 4, but it replaces the most expensive operations (assembly of $M^T M$ at roughly $\frac{1}{2} \ell^2 n$ flops and solution of system (10) at roughly $\ell^3$ flops) with cheap ones (matrix-vector product with tridiagonal matrix at roughly $4 \ell$ flops and inner product at roughly $\ell$ flops).
6 Application to the random Kaczmarz method

The random Kaczmarz method is displayed in Algorithm 5. We organize the iterations in epochs of \( m \) projection steps. Please refer to Algorithm 5 for the random indices used in this section.

**Algorithm 5:** Random Kaczmarz method, originally proposed in [16]
(complexity: \( 4 \text{nnz}(A) + m \) flops per epoch with \( m \) projections)

| Input: \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), \( x_0 \in \mathbb{R}^n \) |
|---|
| 1 for \( k = 0 \) to \( \infty \) do |
| 2 \( x_{k,0} \leftarrow x_k \); |
| 3 for \( j = 0 \) to \( m - 1 \) do |
| 4 draw \( i_{k,j} \) uniformly at random from \( \{1, \ldots, m\} \); |
| 5 \( x_{k,j+1} \leftarrow P_{i_{k,j}}(x_{k,j}) \); |
| 6 end |
| 7 \( x_{k+1} \leftarrow x_{x,m} \); |
| 8 end |

The random Kaczmarz method is known to converge in expectation when \( A \) has full rank and \( x^* := A^{-1}b \) is unique, see Theorem 2 in [16]. The final statement of the induction step in its proof shows that

\[
E\|x_{k+1} - x^*\|^2 \leq (1 - \kappa(A)^{-2})^m E\|x_k - x^*\|^2 \quad \forall k \in \mathbb{N},
\]

where \( \kappa(A) > 0 \) is a specific condition number, see Section 1 of [16] for details. Statement (20) reveals that an acceleration of the sequence \( (x_k)_k \) that reduces the error \( \|x_k - x^*\|^2 \) maintains the convergence properties of the sequence as well as the error estimate. This motivates us to transfer the acceleration techniques from the previous sections to the random Kaczmarz method, which gives Algorithm 6.

**Remark 16.** Algorithm 6 is justified by the following reasoning: The \( k \)-th epoch of the random Kaczmarz method can be regarded as one cycle

\[
\bar{P}^{(k)}(x) := (P_{i_{k,m-1}} \circ \ldots \circ P_{i_{k,0}})(x)
\]

of the deterministic Kaczmarz method applied to the matrix \( \tilde{A}^{(k)} \in \mathbb{R}^{m \times n} \) and a vector \( \tilde{b}^{(k)} \in \mathbb{R}^m \) given by

\[
\tilde{A}^{(k)} := (a_{ik,0}, \ldots, a_{ik,m-1})^T, \quad \tilde{b}^{(k)} := (b_{ik,0}, \ldots, b_{ik,m-1})^T,
\]
Algorithm 6: Random Kaczmarz method with affine search
(for an upper bound on the complexity see Remark 15)

Input: \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), \( x_0 \in \mathbb{R}^n \), \( \ell \in \mathbb{N}_2 \)

1. \( k = 0 \) to \( \infty \) do
   
   repeat
     \( \rho_k \leftarrow 0; \)
     \( x_{k,0} \leftarrow x_k; \)
     
     for \( j = 0 \) to \( m - 1 \) do
       draw \( i_{k,j} \) uniformly at random from \( \{1, \ldots, m\} \);
       \( \rho_k \leftarrow \rho_k + (a_{i_{k,j}}^T x_{k,j} - b_{i_{k,j}})^2 / \|a_{i_{k,j}}\|^2; \)
       \( x_{k,j+1} \leftarrow P_{i_{k,j}}(x_{k,j}); \)
     end
     
   until \( x_{k,m} \neq x_k; \)

2. \( d_k \leftarrow x_{k,m} - x_k; \)
3. \( \delta_k \leftarrow \|d_k\|^2; \)
4. \( \gamma_k \leftarrow \frac{1}{2}(\rho_k + \delta_k); \)
5. \( j_k \leftarrow \max\{k - \ell + 1, 0\}; \)
6. \( V_k \leftarrow (x_{j_k} - x_k, \ldots, x_{k-1} - x_k) \in \mathbb{R}^{n \times (k-j_k)}; \)
7. \( p_k \leftarrow V_k^T d_k \in \mathbb{R}^{k-j_k}; \)
8. \( C_k \leftarrow C(\gamma_{j_k} s_{j_k}, \ldots, \gamma_{k-1} s_{k-1}) \in \mathbb{R}^{(k-j_k) \times (k-j_k)}; \)
9. \( q_k \leftarrow C_k p_k \in \mathbb{R}^{k-j_k}; \)
10. \( s_k \leftarrow \frac{\gamma_k}{p_k^T d_k} \in \mathbb{R}; \)
11. \( \overline{s_k} \leftarrow -s_k q_k \in \mathbb{R}^{k-j_k}; \)
12. \( x_{k+1} \leftarrow x_k + V_k \overline{s_k} + s_k d_k; \)

end
which gives rise to the residual

\[ \tilde{r}^{(k)}(x) := \begin{pmatrix}
    (a^T_{ik,0} x - b_{ik,0})/\|a_{ik,0}\| \\
    (a^T_{ik,1} P_{ik,0}(x) - b_{ik,1})/\|a_{ik,1}\| \\
    \vdots \\
    (a^T_{ik,m-1} P_{ik,m-2} \circ \ldots \circ P_{ik,0}(x) - b_{ik,m-1})/\|a_{ik,m-1}\| 
\end{pmatrix}. \]

The solution \( x^* = A^{-1}b \) also solves \( \tilde{A}^{(k)} x^* = \tilde{b}^{(k)} \), and all statements on errors given in the previous sections remain valid.

However, it is no longer true that \( \tilde{P}^{(k)}(x_k) \in \text{aff}(x_{j_k}, \ldots, x_k) \) if and only if \( Ax_k = b \), because \( \tilde{A}^{(k)} \) and \( \tilde{b}^{(k)} \) are only subsamples of \( A \) and \( b \), which invalidates the stopping criteria we previously used. On the other hand, the inclusion \( \tilde{P}^{(k)}(x_k) \in \text{aff}(x_{j_k}, \ldots, x_k) \) still implies that \( \tilde{P}^{(k)}(x_k) = x_k \) and \( \tilde{r}^{(k)}(x_k) = 0 \). Hence, in this situation, we can ignore the last epoch in the acceleration scheme (see line 10 of Algorithm 6), and accelerate only when the random Kaczmarz method made progress. This guarantees that whenever Theorem 7 is invoked, its assumptions are satisfied.

### 7 Numerical results

We test the algorithms presented in this paper in the context of the computerized tomography problem, which is one of the most important applications of the Kaczmarz method. To generate benchmark problems, we apply the parallel_tomo function of the AIR Tools library [10] with default parameters to the Shepp-Logan medical phantom.

In all simulations, we choose the initial guess \( x_0 = 0 \) and apply an initial random shuffling to the rows of \( A \), because the canonical ordering tends to induce very slow and hence atypical convergence. To see how the method behaves under scaling, we investigate the following scenarios, where

\[ \text{oncost}(\ell) := \frac{\text{cost(acceleration(\ell))}}{\text{cost(Kaczmarz cycle)}} \approx \frac{(3 + 5\ell)n + 2m + 5\ell}{4\text{nnz}(A) + m} \]

measures the cost of an acceleration step relative in terms of the cost of a Kaczmarz cycle.
We see that in all three scenarios, the cost of an acceleration step relative to
the cost of a Kaczmarz cycle is small, though the matrices are moderately sparse.

It is well-known that the normal equations (10) are prone to become ill-conditioned. We observed unstable behavior of Algorithm 3 e.g. in scenario a) at $\ell = 20$. Though Algorithm 4 essentially solves the same problem, it remained stable, which is probably a benefit of applying an explicitly known inverse over solving the linear system numerically. Only when the approximation error $\|x_k - x^*\|$ was very small (roughly $10^{-13}$), we observed instability in the form of oscillating errors.

Algorithm 4 always performs better than Algorithm 3 and the outperformance increases with the problem dimension $n$ and the dimension $\ell$ of the
affine search space, see Remark 9b and Remark 15. As it is also more stable, we only display the numerical errors of Algorithm 4 in Figure 1.

The wobble that is most pronounced in the error plot of $K_1$ for $N = 40$ is neither caused by an unstable algorithm nor an artefact. It is typical across a range of acceleration schemes (not presented in this paper, but investigated by the author numerically) and seems to be caused by going back and forth between the Euclidean geometry and the geometry of the Kaczmarz map $P$.

Roughly speaking, the error curves of the accelerated Kaczmarz methods cluster at the error curve of $K_\infty$, which is the variant of Algorithm 4 that spans the affine search space using all previously computed iterates in every step. This seems to suggest that the benefit of working with many or all previous iterates outweighs the additional cost incurred by processing them.

Figure 2 shows the performance of Algorithm 6 and it demonstrates that our acceleration technique can be successfully applied to the random Kaczmarz method. Comparing figures 1 and 2 in terms of absolute values is not meaningful, because the performance of the deterministic Kaczmarz method depends on the chosen order of the rows of $A$, and the performance of the random Kaczmarz method depends to some degree on the particular random numbers drawn.

However, there seems to be a slight qualitative difference in the performance of the accelerated methods between the deterministic and the random setting. In the random setting, a larger $\ell$ seems to be needed to achieve a similar level of outperformance of the plain Kaczmarz method as in the deterministic setting, which is particularly noticeable in the error plot of the methods $K_1$ and $rK_1$ for $N = 40$. On the other hand, in both settings, the method $rK_\infty$ always clearly outperforms the Kaczmarz method.

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