Quantum algebras in phenomenological description of particle properties

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Quantum and \( q \)-deformed algebras find their application not only in mathematical physics and field theoretical context, but also in phenomenology of particle properties. We describe (i) the use of quantum algebras \( U_q(su_n) \) corresponding to Lie algebras of the groups \( SU_n \), taken for flavor symmetries of hadrons, in deriving new high-accuracy hadron mass sum rules, and (ii) the use of (multimode) \( q \)-oscillator algebras along with \( q \)-Bose gas picture in modelling the properties of the intercept \( \lambda \) of two-pion (two-kaon) correlations in heavy-ion collisions, as \( \lambda \) shows sizable observed deviation from the expected Bose-Einstein type behavior. The deformation parameter \( q \) is in case (i) argued and in case (ii) conjectured to be connected with the Cabibbo angle \( \theta_c \).

1. Introduction

Quantum groups and quantum or \( q \)-deformed algebras \( \mathfrak{s}\mathfrak{l}_q \), whose basic mathematical aspects and diverse quantum physical applications are intensively studied for about a decade and half, will belong to most important and perspective tools of research in the 3rd millenium, too.

In this talk, meant as a mini-review, we concentrate on two examples of applying \( q \)-algebras to phenomenology of hadrons. Within the first one initiated in \( \mathfrak{s}\mathfrak{l}_q \) and developed in subsequent papers, the \( q \)-analogs \( U_q(su_n) \) of the Lie algebras of groups \( SU_n \) are adopted for hadronic flavor symmetries in order to derive new results concerning hadron masses and mass sum rules. Basic tool of this approach is the representation theory of the \( q \)-algebras \( U_q(su_n) \), and in sections 2-8 we discuss a number of results, including unexpected implications: a possibility (gained due to use of \( q \)-algebras) to label different flavors topologically - by knot invariants; a direct link of deformation parameter to the Cabibbo angle \( \theta_c \), etc. In the second part of the talk (section 9) we consider a usage of the algebras of \( q \)-deformed oscillators, within a model of \( q \)-Bose gas, for effective description of unusual (non-Bose type) behavior of two-particle correlations of hadrons (pions or kaons) produced and registered in heavy ion collisions.

2. Vector mesons: \( q \)-deformation vs mixing

We use (see \( \mathfrak{s}\mathfrak{l}_q \)) Gelfand-Tsetlin basis vectors for meson states from \((n^2-1)\)-plet of \('n\)-flavor' \( U_q(u_n) \) embedded into \((n+1)^2-1\)-plet of \('dynamical' \( U_q(u_{n+1}) \); construct mass operator \( \hat{M}_n \), invariant under the 'isospin+hypercharge' \( q \)-algebra \( U_q(u_2) \), from the generators of dynamical algebra \( U_q(u_{n+1}) \) (e.g., \( M_3 = M_0 \mathbf{1} + \gamma_3 A_{34} A_{34} + \delta_3 A_{34} A_{34} \)); calculate expressions for masses \( \omega \( M_4 \) - these involve \( M_0 \), the parameters \( \gamma_3, \delta_3 \), and the \( q \)-parameter. In particular, for \( n = 3 \) one obtains

\[
\begin{align*}
M_\rho &= M_0, \\
M_{K^*} &= M_0 - \gamma_3, \\
m_{\omega_8} &= M_0 - 2[2]_q / [3]_q \gamma_3,
\end{align*}
\]

where \([x]_q \equiv [x] = \frac{q^x - q^{-x}}{q - q^{-1}}\) is the \( q \)-number 'deforming' a number \( x \) and, to have equal masses for particles/antiparticles, \( \delta_3 = \gamma_3 \) was set. \( q \)-Dependence appears only in the mass of \( \omega_8 \) (isosinglet in \( U_q(su_3) \)-octet). Excluding \( M_0, \gamma_3 \), the \( q \)-analogue of Gell–Mann - Okubo (GMO) relation is \( \omega_3 \) :

\[
[3]_q m_{\omega_8} + (2[2]_q - [3]_q) m_\rho = 2[2]_q m_{K^*}.
\]

In the limit \( q = 1 \) (then, \([3]_q / [3]_q = \frac{3}{2}\)), this reduces to usual GMO formula \( 3m_{\omega_8} + m_\rho = 4m_{K^*} \) which
needs singlet mixing [3]. However, it also yields
\[ m_{\omega_s} + m_{\rho} = 2m_K \quad \text{if} \quad q = e^{i\pi/5} \]  
(3)
then, \([3]_q = [2]_q\). With \(m_{\omega_s} \equiv m_{\phi}, \) and no
mixing, eq.(3) coincides with nonet mass formula of
Okubo [3], agreeing ideally with data [3]. The
deformation angle \( \frac{\pi}{5} \), see (3), coincides remarkably
with \( \omega - \phi \) mixing angle (known to be
\( \theta_{\omega \phi} = 36^\circ \)) of traditional \( SU(3) \)-based scheme.
In other words, the \( q \)-deformation of flavor symmetries
supersedes the issue of singlet mixing.

For \( 3 < n \leq 6 \) the scheme works as well. Again, only
masses of singlets \( \omega_{15}, \omega_{24}, \omega_{35} \) from \((n^2-1)\)
plets of \( U_q(u_n) \) involve \( q \)-dependence. As result, we
get the \( q \)-analog (with isodoublet \( D^* \))
\[
\left[ \frac{4}{3} \right]_q m_{\omega_{35}} + \left( \frac{4}{3} \right)^2_{[2]} - 5 \frac{1}{3} \left[ 2 \right]_{[3]} = \frac{2}{3} + \left[ 2 \right]_{[3]} \right) m_{\rho} = 
\qquad = 2 m_{\rho} + 2 \left( \frac{2}{3} \right)^2_{[2]} - \frac{2}{3} \left[ 2 \right]_{[3]} \right) m_{\rho} = 
\qquad \left( 4 \right)
\end{equation}
and \( q \)-analog for \( n=5,6 \) (see [3]). Fixing \( q \) by
setting \([4]_q = [3]_q \) and \([n]_q = [n-1]_q \) \( n=5,6 \)
simplifies the relations and yields higher analogs of
Okubo’s nonet sum rule (isodoublets in r.h.s.):
\[
\begin{align*}
    m_{\omega_{15}} + (5 - 8/[2]_q) m_{\rho} &= 
    = 2 m_{\rho} + (4 - 8/[2]_q) m_{K^*}, \\
    m_{\omega_{24}} + (9 - 16/[2]_q) m_{\rho} &= 
    = 2 m_{\rho} + (4 - 8/[2]_q) (m_{D^*} + m_{K^*}), \\
    m_{\omega_{35}} + (13 - 24/[2]_q) m_{\rho} &= 
    = 2 m_{\rho} + (4 - 8/[2]_q) (m_{D^*} + m_{K^*}), \\
\end{align*}
\]  
(5)
and (6)
Here the values \( q_n = e^{\pi i/(2n-1)} \) (for which \([2]_q = 2 \cos \frac{\pi}{2n-1} \) )
eq \( q \) by its root, the corresponding MSR from
\( q \)-deformed analog.

Thus, the \( q \)-parameter for each \( n \) is fixed rigidly
as a root \( q_n \) of \( \Delta \{ (2n-1) \} \), contrary to choice
of \( q \) by fitting in other phenomenological
applications [4]. Here, using flavor \( q \)-algebras
along with ‘dynamical’ \( q \)-algebras according to
\( U_q(u_n) \subseteq U_q(u_{n+1}) \), we gain: the torus knots
\( 5_1, 7_1, 9_1, 11_1 \) are put into correspondence [4]
with vector quarkonia \( ss, cb \), and \( tt \) respectively.
The polynomial \( P_n(q) \equiv [n]_q - [n-1]_q \)
by its root \( q_n = q(n) \) determines the value of \( q \-
parameter for each \( n \) and thus serves as defining
polynomial for the MSR/quarkonium/flavor correspon-
ding to \( n \). Hence, the use of \( q \)-algebras suggests
a possibility of topological labeling of flavors:
fixed number \( n \) corresponds to \( 2n-1 \) overcross-
ings of 2-strand braids whose closure gives these
\( (2n-1) \)-torus knots. With the form \( (2n-1, 2) \)
of same torus knots this means the correspondence
\( n \leftrightarrow w \equiv 2n-1 \), \( w \) being the winding number
around one of the two basic cycles on torus.

3. Octet baryon mass sum rules: best can-
didates from \( q \)-deformation
Using \( U_q(su_n) \), the \( q \)-deformed mass relation
\[
[2] M_N + \frac{[2] M_N}{[2] - 1} = [3] M_A + \left( \frac{[2] M_N}{[2] - 1} - [3] \right) M_S \\
\frac{\Delta_M}{[2]} (M_Z + [2] M_N - [2] M_S - M_A)
\]  
(8)
was obtained \([2]\), where \(A_q, B_q\) are certain polynomials of \([2]_q\) with non-overlapping sets of zeros. This \(q\)-analogue yields, as three particular cases, the familiar Gell-Mann - Okubo mass relation (in the ‘classical’ case of \(q = 1\)) and two new MSRs of improved accuracy \([3,4]\):

\[
M_N + M_E = \frac{3}{2}M_A + \frac{1}{2}M_{\Sigma}, \quad (0.58\%) \quad (9)
\]
\[
M_N + \frac{1+\sqrt{3}}{2}M_E = \frac{2M_A}{\sqrt{3}} + \frac{9-\sqrt{3}}{6}M_{\Sigma}, \quad (0.22\%) \quad (10)
\]
\[
M_N + \frac{M_A}{2|q^* - 1|} = \frac{M_A}{|q^* - 1|} + M_{\Sigma}. \quad (0.07\%) \quad (11)
\]

Different dynamical representations, after calculation, produce in (8) differing pairs \(A_q, B_q\). Each \(A_q\) contains the factor \((|2\rangle_q - 2)\) i.e., the ‘classical’ zero \(q = 1\), and some other nontrivial zeros. Eqs. (10), (11) result from two different dynamical representations \(D^{(1)}\) resp. \(D^{(2)}\) producing \(A_q^{(1)}\) resp. \(A_q^{(2)}\) which possess zeros \(q_6 = e^{i\pi/6}\) resp. \(q_7 = e^{i\pi/7}\). The choice (11), i.e. \(q_7 = e^{i\pi/7}\), provides the best mass sum rule.

Sum rule (10) was first derived \([1]\) from a dynamical representation (irrep) \(D^{(1)}\) of \(U_q(4_{1,1})\). However, the ‘compact’ dynamical \(U_q(6)\) is equally well suited. Among the admissible dynamical irreps there exist an entire series of irreps (numbered by integer \(m\), \(6 \leq m < \infty\)) which produce infinite set of MSRs, each given by the first line in (8) with \(q_m\) put for \(q\), where \(q_m = e^{i\pi/m}\) guarantees vanishing of \(\frac{M}{\pi}\). Each of these MSRs shows better agreement with data than the classical GMO one. To illustrate, few cases from the infinite set are shown in the table, the 1st row of which being the classical GMO with \(q_\infty = 1\).

| \(\theta = \frac{\pi}{m}\) | (RHS-LHS), MeV | \(\text{RHS} \times 100\%\) |
|------------------|-----------------|-----------------|
| \(\pi/\infty\)   | 26.2            | 0.58            |
| \(\pi/30\)       | 25.42           | 0.56            |
| \(\pi/12\)       | 20.2            | 0.44            |
| \(\pi/8\)        | 10.39           | 0.23            |
| \(\pi/7\)        | 3.26            | 0.07            |
| \(\pi/6\)        | -10.47          | 0.22            |

We thus gain that a ‘discrete choice’ becomes possible instead of usual fitting; the \(q\)-polynomials \(A_q\)

\[2\] The value \(q_7\) is linked \([1,2]\) to the Cabibbo angle: \(\ln q_7 = 2\theta_C\) (see also sec. 7 below).

due to zeros \(q_m\) serve as \textit{defining} polynomials for the corresponding MSRs.

Quark mass ratio in terms of baryon masses

Since \([2]_q = 2\cos \frac{\pi}{7}\), the MSR (11) takes the equivalent form of “modified average”

\[
\frac{M_\Xi - M_N + M_\Sigma - M_A}{2\cos(\pi/7)} = M_\Sigma - M_N. \quad (12)
\]

From (12) with \(q_7 = 2\theta_C\) (see footnote 2), using the famous relation \([3]\) \(\tan^2 \theta_C = m_d/m_s\) we infer a new formula giving quark mass ratio in terms of (very precisely known) octet baryons:

\[
\frac{m_s}{m_d} = \frac{3M_\Sigma - M_A - 3M_N + M_\Xi}{M_\Sigma + M_A - M_N - M_\Xi} = 18.63 \pm 0.16.
\]

Numerically, the obtained ratio is in nice agreement with the value \(\frac{m_s}{m_d} = 18.9 \pm 0.8\) given in \([4]\).

4. Mass sum rules for decuplet baryons: the \(q\)-analogue matches empirical masses

In the case of \(SU(3)\)-decuplet baryons \(3^+\), 1st order symmetry breaking yields \([5]\) equal spacing rule (ESR) for isospin members in \(10\)-plet. Empirically, \(M_{\Sigma^*} - M_\Delta, M_{\Xi^*} - M_\Sigma^*, M_\Omega - M_{\Xi^*}\) show sensible deviation from ESR: 152.6 MeV \(\leftrightarrow 148.8\) MeV \(\leftrightarrow 139.0\) MeV. The other mass relation known long ago \([6]\): \(M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*}/2 = M_{\Xi^*} - M_{\Sigma^*}\) (13) accounts 1st and 2nd order of \(SU(3)\)-breaking and holds only slightly better than the ESR.

On the contrary, use of the \(q\)-algebras \(U_q(su_n)\) instead of \(SU(n)\) leads to sizable improvement. From evaluations of decuplet masses in two particular irreps of the dynamical algebra \(U_q(4_{1,1})\), the \(q\)-deformed mass relation

\[
\frac{M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*}}{2\cos \theta_{10}} = M_{\Xi^*} - M_{\Sigma^*} \quad (14)
\]

was derived \([7]\). As proven there, this mass relation is \textit{universal} - it results from any admissible irrep (which contains \(U_q(su_3)\)-decuplet embedded in \(20\)-plet of \(U_q(su_4)\)) of the dynamical \(U_q(4_{1,1})\).

With empirical masses \([5]\), the formula (14) holds remarkably for \(\theta_{10} \approx \frac{\pi}{4}\) (in fact, \(\theta_{10} = \theta_C\), see footnote 2 and sec. 7 below).
5. Highly nonlinear SU(3)-breaking effects in baryon masses

Formula (15) involves highly nonlinear dependence of mass on hypercharge (for decuplet, Y alone causes SU(3)-breaking). Since for the q-number \([N]q\) we have \([N]q = q^{-N-1} + q^{-N-3} + \ldots + q^{-N+3} + q^{-N+1}\) \((N\) terms\), this shows exponential \(Y\)-dependence of masses. Such high nonlinearity makes (14) and (15) radically different from the result (13) of traditional treatment accounting linear and quadratic effects in \(Y\).

For octet baryon masses, nonpolynomiality in SU(3)-breaking effectively accounted for by the model was explicitly shown in [11]. For this, one analyses the expressions for isospet masses with explicit dependence on hypercharge \(Y\) and isospin \(I\), through \(I(I+1)\). Typical matrix element contributing to octet baryon masses contain terms such as \([Y/2+1]q - [I]q[I-1]q\) or \([Y/2-1]q [Y/2-2]q - [I]q[I+1]q\) \((N\) terms\), this shows explicitly the dependence on hypercharge and the factor \([I]q[I+1]q\) q-deforming the SU(2) Casimir. From definition of \(q\)-bracket \([n] = \sin(nh) / n\), \(q = \exp(ih)\), it is clearly seen that baryon masses depend on hypercharge \(Y\) and isospin \(I\) (hence, on SU(3)-breaking effects) in highly nonlinear - nonpolynomial - fashion.

The ability to account highly nonlinear SU(3)-breaking effects by applying quantum analogs \(U_q(su_3)\) of usual flavor symmetries is much alike the fact shown in [17] that, by exploiting appropriate free \(q\)-deformed structure one is able to efficiently study the properties of (undeformed) quantum-mechanical systems with complicated interactions.

6. Using the Hopf-algebra structure

As demonstrated, our approach supplies a plenty of \(q\)-analogues (with different pairs \(A_q, B_q\)) of the form (8). A completely different, as regards (8), version of \(q\)-deformed analog can be derived [1] using for the symmetry breaking term in mass operator a component of \(q\)-tensor operator. This implies usage of the Hopf algebra structure (comultiplication, antipode) of the quantum algebras \(U_q(su_n)\), through the \(q\)-tensor operators \((V_1, V_2, V_3)\) resp. \((V_1, V_2, V_3)\) from elements of \(U_q(su_3)\) and transforming as 3 resp. 3* under the adjoint action of \(U_q(su_3)\). With the Cartan elements \(H_1, H_2\), denoting \([X, Y]q = XY - qYX\), the components \((V_1, V_2, V_3)\) and \((V_1, V_2, V_3)\) can be found explicitly [1], e.g., \(V_2 = E_3^+ E_3^+ q^{H_1/6 - H_2/6}\), \(V_3 = E_3^+ q^{H_1/6 + H_2/3}\), and \(V_3 = q^{-H_1/6 + H_2/3} E_3^-\).

The mass operator is given as

\[
\hat{M} = \hat{M}_0 + \hat{M}_8 = M_0 \mathbf{1} + \alpha V_8^{(1)} + \beta V_8^{(2)} \\
= M_0 \mathbf{1} + \alpha V_3 V_3 + \beta V_3 V_3
\]

where \(\hat{M}_0\) is \(U_q(su_3)\)-invariant and \(\hat{M}_8\) transforms as \(I = 0, Y = 0\) component of tensor operator of 8-irrep of \(U_q(su_3)\), and it is taken into account that the irrep 8 occurs twice in the decomposition of 8 \(\otimes 8\). Besides, the isosinglet operators \(V_0 V_3\) and \(V_0^* V_3\) arise in accordance with \(3 \otimes 3^* = 1 \oplus 8, 3^* \otimes 3 = 1 \oplus 8\).

The final form of mass operator, with redefined \(M_0, \alpha, \beta\), is

\[
\hat{M} = \hat{M}_0 \mathbf{1} + \alpha E_3^+ E_3^- q^Y + \beta E_3^+ E_3^- q^Y
\]

where the hypercharge \(Y = (H_1 + 2H_2)/3\). Matrix elements with \(\hat{M}\) from (16) are evaluated by embedding 8 in a particular irrepp of \(U_q(su_4)\). Evaluation of baryon masses, say, within the irrep 15 of \(U_q(su_4)\) yields: \(M_\Lambda = M_0 + \beta q, M_\Sigma = M_0, M_\Lambda = M_0 + [2](\alpha + \beta), M_\Sigma = M_0 + \alpha q^{-1}\). Excluding \(M_0, \alpha, \beta\), one finds:

\[
[3] M_\Lambda + M_\Sigma = [2](q^{-1} M_N + q M_\Sigma).
\]

The universality of \(q\)-analog (14) extends also to all admissible irreps of the ‘compact’ dynamical \(U_q(su_3)\). Say, within a dynamical irrep \(\{4000\}\) of \(U_q(su_3)\) calculation yields: \(M_\Delta = M_{10} + \beta, M_{12} = M_{10} + [2]\beta + [2]\alpha, M_0 = M_{10} + [3]\beta + [2]\alpha, M_\Omega = M_{10} + [4]\beta + [3]\alpha\), from which (14) stems. With hypercharge \(Y\), all four masses are comprised by single formula for \(M_{Di} = M(Y(D_i))\):

\[
M_{Di} = M_{10} + \alpha(1 - Y(D_i)) + \beta(2 - Y(D_i)),
\]

with explicit dependence on \(Y\). If \(q = 1\), this reduces to \(M_{Di} = M_{10} + a Y(D_i)\), i.e. linear dependence on hypercharge (or strangeness) where \(a = -\alpha - \beta, M_{10} = M_{10} + \alpha + 2\beta\).
The $q$-parameter now can be fixed by a fitting only and, for each of values $q_{1,2} = \pm 1.035$, $q_{3,4} = \pm 0.903\sqrt{-1}$, the $q$-MR (17) indeed holds within experimental uncertainty.

7. The link: $q$-parameter ↔ Cabibbo angle

For pseudoscalar (PS) mesons, the generalization \([18]\) of GMO-formula

$$f_\pi^2 m_\pi^2 + 3f_\eta^2 m_\eta^2 = 4f_K^2 m_K^2$$

(18)

involves decay constants as coefficients. On imposing the constraint \([12]\) $f_\pi^2 + 3f_\eta^2 = 4f_K^2$ it becomes

$$m_\pi^2 + \left(4 \frac{f_K}{f_\pi} - 1\right) m_\eta^2 = 4\frac{f_K^2}{f_\pi^2} m_K^2,$$

(19)

to be compared with our $q$-analog (2) rewritten for PS mesons (with masses squared):

$$m_\pi^2 + \frac{|3|}{2|2| - |3|} m_\eta^2 = \frac{2|2|}{2|2| - |3|} m_K^2.$$  

(20)

This holds for (the mass of) physical $\eta$-meson put for $\eta_8$ (i.e., no mixing), at properly fixed $q = q_{\text{es}}$.

The two extensions (19) resp. (20) both reduce to standard GMO in the corresponding limits $\frac{f_K}{f_\pi} \to 1$ resp. $q \to 1$. From the identification

$$f_K^2/f_\pi^2 \leftrightarrow \frac{1}{2}|2|/(2|2| - |3|),$$

(21)

using $|3|_q = |2|_q^2 - 1$ and the notation $\xi_{\pi,K} = (4f_K^2/f_\pi^2)^{-1}$, we get

$$|2|_{\pm} = 1 - \xi_{\pi,K} \pm \sqrt{(1 - \xi_{\pi,K})^2 + 1}.$$  

(22)

The ratio $f_K/f_\pi$ is expressible through the Cabibbo angle, e.g., by the formula tan$^2 \theta_C = m_\pi^2 \left(\frac{f_K}{f_\pi} - m_\eta^2/m_K^2\right)^{-1}$ (see \([19]\)). With (21), (22) this implies: the deformation parameter $q_{\text{es}}$ is directly connected with the Cabibbo angle.

One can arrive at similar conclusion in another way. In \([2]\), the $q$-deformed lagrangian for gauge fields of the Weinberg - Salam (WS) model, invariant against quantum-group valued gauge transformations, was constructed. The formula

$$F_\mu^0 = \text{Tr}\left(F_{\mu\nu}\right) \left[2(q^2 + q^{-2})\right]^{-1/2} = B_{\mu\nu} \cos \theta + F_{\mu\nu}^\alpha \sin \theta$$

obtained therein, along with expression for $F_{\mu\nu}^\alpha$ and the relation

$$\tan \theta = (1 - q^2)/(1 + q^2),$$

(23)

exhibits mixing of the $U(1)$-component $B_{\mu}$ with third (nonabelian) component $A_{\mu}^3$. Forming new potentials $\tilde{A}_\mu = B_{\mu} \cos \theta + A_{\mu}^3 \sin \theta$, $Z_\mu = -B_{\mu} \sin \theta + A_{\mu}^3 \cos \theta$ yields physical photon $\tilde{A}_\mu$ and $Z$-boson of WS model, where $\theta = \theta_w$, i.e., the Weinberg angle (at $\theta = 0$ the potentials $B_{\mu}$ and $A_{\mu}^3$ get completely unmixed whereas nonzero $\theta$, i.e., nontrivial $q$-deformation provides proper mixing inherent for the WS model). To summarize: weak mixing is adequately modelled by the $q$-deformation. That is, the $q$-deformation is able to realize proper weak mixing of gauge fields and provides explicit connection of the weak angle and the deformation parameter $q$, see eq.(23).

On the other hand, the relation found in \([21]\)

$$\theta_w = 2(\theta_{12} + \theta_{23} + \theta_{13})$$

(24)

connects $\theta_w$ with the Cabibbo angle $\theta_{12} \equiv \theta_C$ (and the angles $\theta_{13}, \theta_{23}$ of mixing with 3rd family, that will be neglected). The eqn. (24) is important as it links apparently different mixings: in the bosonic (interaction) versus fermionic (matter) sectors of the electroweak model.

Combining (23) and (24) we conclude: the Cabibbo angle can be linked with $q$-parameter of a quantum-group (or quantum-algebra) based structure applied in the fermion sector. Hence, there must exist a direct connection of the $q$-parameter in (12), (14) with the fermion mixing angle. Setting $\theta_{10} = g(\theta_C)$ and $\theta_8 = h(\theta_C)$ we find for $g(\theta_C)$ and $h(\theta_C)$ the following:

$$\theta_{10} = \theta_C, \quad \theta_8 = 2 \theta_C.$$  

(25)

With $\theta_8 = \frac{\pi}{2}$, see (12), this suggests for Cabibbo angle the exact value $\frac{\pi}{14}$.

**Cabibbo mixing from noncommutative extra dimensions?**

Quantum groups and the related quantum algebras provide necessary tools in constructing covariant differential calculi and particular noncommutative geometry on quantum spaces \([2]\), e.g.

\(^3\) It leads to the single dimensionless quantity $\frac{m_\eta}{m_K}$ involved in the multipliers of masses.
quantum vector spaces, quantum homogeneous spaces.

The direct link found between the Cabibbo angle \( \theta_c = \frac{\pi}{4} \) and the \( q \)-parameter which measures strength of \( q \)-deformation for the \( q \)-algebras \( U_q(\mathfrak{su}_n) \) used for flavor symmetry, can be viewed [12] as indicating towards noncommutative-geometric origin of fermion mixing. The exact value \( \theta_c = \frac{\pi}{4} \) of the Cabibbo angle would then serve as the noncommutativity measure of relevant quantum space, responsible for the mixing and explicitly as yet unknown, in extra dimensions whose number should be not less than 2.

8. Mass relations from anyonic realization of \( U_q(\mathfrak{su}_N) \)

Necessary setting adopted from [22] includes lattice angle functions \( \theta_i(x, y) \) and \( \delta_i(x, y) \) for the two opposite (\( \gamma \)- and \( \delta \)-) types of cuts and the related definition of ordering of sites on the lattice (\( x > y \) or \( y > x \)). Accordingly, the two types of statistical operator, \( K_i(x) \) and \( K_i(y) \), are formed using \( N \) sorts of lattice fermions \( c_i(x) \), \( c_i^\dagger(y) \), \( i = 1, ..., N \), obeying usual (lattice) anticommutation relations (ARs), as

\[
K_i(x) = \exp(i \nu \sum_{y \neq x} \theta_i(x, y)c_j^\dagger(y)c_j(y)) \tag{26}
\]

and similarly for \( K_i(x) \). In terms of them, the two types of anyonic oscillators are given as [22]

\[
a_i(x) = K_i(x)c_i(x), \quad a_i^\dagger(x) = K_i(x)c_i^\dagger(x). \]

The relations of permutation (PRs) obtained for anyonic oscillators include simple ARs, and also nontrivial PRs involving the deformation parameter \( q \) (the latter is connected with the statistics parameter \( \nu \) in eq. (26) as: \( q = \exp(i \pi \nu) \)). For instance, the braiding properties are described by the following nontrivial PRs (\( x \neq y \)):

\[
a_i(x) a_i(y) + q^{-\text{sgn}(x-y)} a_i(y) a_i(x) = 0, \quad a_i^\dagger(x) a_i^\dagger(y) + q^{\text{sgn}(x-y)} a_i(x) a_i(x) = 0.
\]

The basic fact proven in [22] states that generating elements \( A_{j,j+1}, A_{j+1,j} \) and \( H_j \) realized bilinearly in terms of anyonic oscillators \( a_i(x) \), \( a_i^\dagger(y) \) satisfy the defining relations [13] of the quantum algebra \( U_q(\mathfrak{su}_N) \). Similarly, dual realization in terms of \( a_i(x) \), \( a_i^\dagger(x) \) does also exist. On this basis, within anyonic realization of \( U_q(\mathfrak{su}_N) \), one can explicitly construct both basis vectors for hadronic irreps and hadron mass operator. Starting point is the highest weight vector (HWV) of the irrep \{4000\} of ‘dynamical’ \( U_q(\mathfrak{su}_5) \) which is of the form \{1111\} in the notation \{\( n_1 n_2 n_3 n_4 \)\} for the state vector, that means \( a_1^\dagger(x_1) a_2^\dagger(x_2) a_3^\dagger(x_3) a_4^\dagger(x_4) |0\rangle \). All basis state vectors of baryons \( \frac{1}{2}^+ \) are constructed, by acting with lowering generators, in accordance with the chain of \( q \)-algebras \( U_q(\mathfrak{su}_3) \subset U_q(\mathfrak{su}_4) \subset U_q(\mathfrak{su}_5) \) and respective chain of irreps [30] \{300 \subset \{4000\} \}. For isoquartet baryon \( |\Delta^{++}\rangle \) one finds \( \sqrt{\frac{1}{4}}(|10111\rangle + q^{-1}|1511\rangle + q^{-2}|1151\rangle + q^{-3}|1115\rangle) \), and similarly for \( |\Sigma^+\rangle, |\Xi^+\rangle, |\Omega^-\rangle \).

The dual basis \( |\Delta^{++}\rangle \), etc., obtained by acting on the HWV with lowering operators in dual anyonic realization, is also needed. Masses \( M_{D_i} \) of baryons \( D_i \) are calculated within the dynamical \( U_q(\mathfrak{su}_5) \)-irrep \{4000\} as \( M_{D_i} = \langle D_i | M | D_i \rangle \) with mass operator formed from anyonic operators) to yield:

\[
M_{\Delta} = M_{10} + \beta, \quad M_{\Sigma^+} = M_{10} + [2]_q \alpha + [2]_q \beta, \quad M_{\Sigma^-} = M_{10} + [2]_q \alpha + [3]_q \beta, \quad M_{\Omega^-} = M_{10} + [2]_q \beta + [4]_q \beta. \]

One easily checks that these masses satisfy the relation (14). This proves applicability [23] of quantum algebras and their irreps for treating hadron mass relations within anyonic realization.

9. Algebras of \( q \)-oscillators, \( q \)-Bose gas and two-pion (two-kaon) correlations

The model of ideal gas of \( q \)-bosons based on the algebra of \( q \)-deformed oscillators either of Biedenharn-Macfarlane (BM) type [24] or Arik-Coon (AC) type [25] was recently utilized within the approach aimed to describe [26,27] unusual properties of 2-particle correlations of identical pions or kaons produced in heavy ion collisions. The approach yields clear predictions based on explicit expressions for the intercept \( \lambda \) (dependent on temperature, particle mass, pair mean momentum, and the deformation parameter \( q \)).

To obtain needed observables, one evaluates thermal averages \( \langle A \rangle = \text{Sp}(A \rho)/\text{Sp}(\rho) \), \( \rho = \sum_{i < j} |i,j \rangle \langle i,j| \) for the two-pion (two-kaon) correlations. The model of ideal gas of \( q \)-bosons based on the algebra of \( q \)-deformed oscillators either of Biedenharn-Macfarlane (BM) type [24] or Arik-Coon (AC) type [25] was recently utilized within the approach aimed to describe [26,27] unusual properties of 2-particle correlations of identical pions or kaons produced in heavy ion collisions. The approach yields clear predictions based on explicit expressions for the intercept \( \lambda \) (dependent on temperature, particle mass, pair mean momentum, and the deformation parameter \( q \)).
$e^{-\beta H}$, where the Hamiltonian $H = \sum \omega_i N_i$ and $\beta = 1/T$. With $b_i^\dagger b_i = [N_i]_q$ and $q + q^{-1} = 2 \cos \theta$, the $q$-deformed distribution function results for BM-type $q$-bosons as

$$\langle b_i^\dagger b_i \rangle = \frac{e^{\beta \omega_i} - 1}{e^{2\beta \omega_i} - 2 \cos \theta e^{\beta \omega_i} + 1}.$$  \hspace{1cm} (27)

At $\theta = 0$ (or $q = 1$), it yields Bose-Einstein (B-E) distribution, since $q = 1$ recovers usual bosonic commutation relations. As seen, deviation of $q$-distribution (27) from the quantum B-E distribution tends towards the Maxwell-Boltzmann one. For AC-type $q$-bosons, the $q$-distribution is especially simple: $\langle b_i^\dagger b_i \rangle = \frac{1}{1 - q^4}$.

To obtain explicitly the intercept $\lambda$ of two-particle correlations one starts with the defining ratio $\lambda + 1 = \langle b_i^\dagger b_i b_i^\dagger b_i \rangle / (\langle b_i^\dagger b_i \rangle)^2$, calculates the two-particle distribution $\langle b_i^\dagger b_i b_i^\dagger b_i \rangle$ and takes into account the $\langle b_i^\dagger b_i \rangle$. The result for AC-type $q$-bosons reads $\lambda = q - \frac{q(1-q^2)}{2 \cos \theta(q)}$, $-1 < q < 1$, and for BM-type $q$-bosons, with $F(\beta \omega) \equiv \cosh(\beta \omega)$, it is

$$\lambda = -1 + \frac{2 \cos \theta (F(\beta \omega) - \cos \theta)^2}{(F(\beta \omega) - 1)(F(\beta \omega) - 2 \cos^2 \theta + 1)}.$$ \hspace{1cm} (28)

Both (27), (28) are real owing to the sum $q + q^{-1}$.

The intercept $\lambda$ with $\omega = (m^2 + \mathbf{K}^2)^{1/2}$, shows a remarkable feature: asymptotically, at large mean momentum of pion (kaon) pairs and fixed temperature, $\lambda$ tends to a constant given by the $q$-parameter: $\lambda_{\text{as}} = q$ ($q$ real) for the AC-type $q$-bosons, and

$$\lambda_{\text{as}} = 2 \cos \theta - 1, \quad \theta = \frac{1}{i} \ln q,$$ \hspace{1cm} (29)

for the BM-type $q$-bosons.

As conjectured in [27], correlations of pions and kaons are determined by the same value of $q$ (a kind of universality). Then, experimentally one should observe the tendency of merging $\lambda(\pi)$ and $\lambda(K)$ at large enough mean momenta, i.e., $\lambda_{\text{as}}(\pi) = \lambda_{\text{as}}(K)$. Preliminary results of recent RHIC/STAR experiment give three values [28] $\lambda_i(\pi^-), \lambda_2(\pi^-)$ and $\lambda_3(\pi^-)$ for the $\pi^-$-intercept, obtained by averaging over three intervals of transverse momenta (in MeV/c): $\langle K \rangle = \frac{125 \pm 225, (225 \pm 325), (325 \pm 450), and by integrating over rapidity in the range $-0.5 \leq y \leq 0.5$.

In Fig.1 the three values $\lambda_j(\pi^-), j=1,2,3$, with error bars, are shown along with five curves for the intercept $\lambda$ which correspond to fixing in (28) different values of the deformation angle $\theta$, all curves being at the temperature $T = 180$ MeV.

One can see remarkable agreement between the data of curve E obtained at $\theta = 28.5^\circ$. The other interesting curve D corresponds to $\theta = \pi/7 \simeq 25.7^\circ$ (twice the Cabibbo angle, see footnote 2 and eq.(25)). At the same temperature $T = 180$ MeV, the curve D agrees (within error bar) with the points $\lambda_2(\pi)$ and $\lambda_3(\pi)$. However, suffice it to take slightly higher effective temperature $T \approx 198$ MeV, and the resulting curve marked by $\theta = \pi/7 \approx 2\theta_c$, respects all the three error bars. Among different mixing angles known for hadrons, see [8], only the angle $2\theta_c$ seems to be relevant to the discussed topic of intercept $\lambda(\pi)$. It is tempting to suggest that just this angle $2\theta_c$ can be the benchmark of assumed universality (to be) seen in 2-particle correlations since, then, $\lambda_{\text{as}}(\pi) |_{\theta = \pi/7} = \lambda_{\text{as}}(K) |_{\theta = \pi/7} = 2 \cos \frac{\pi}{7} - 1 \approx 0.80194$. Insisting on the asymptotical coincidence $\lambda_{\text{as}}(\pi) = \lambda_{\text{as}}(K)$ we may predict for kaon intercepts: at any transverse momentum, the intercept $\lambda(K)$ of 2-kaon correlations should not exceed 0.80194.
10. Outlook

A question naturally arises: does there exist more intimate connection between the two discussed applications - of the quantum algebras $U_q(su_n)$ taken as flavor symmetries, on one hand, and of the algebras of $q$-deformed oscillators corresponding to discretized momenta of (correlated) pairs of pions or kaons as produced in relativistic heavy ion collisions, on the other hand? The value of $q$-parameter (given by $2\theta_C$) shared by the two applications in case of octet hadrons gives a guess for possible physical reason for such a connection (recall also the well-known fact that generating elements of $U_q(su_n)$ admit realization in terms of $q$-deformed oscillators \[24\]). Future research possibly involving noncommutative geometry in extra dimensions should give ultimate answer.

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