FACES AND BASES: DEHN-SOMMERVILLE TYPE RELATIONS

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Abstract. We review several linear algebraic aspects of the Dehn-Sommerville relations and relate redundant analogues of the $f$- and $h$-vectors describing the subsets of a simplex $2^{\{1,\ldots,m\}}$ that satisfy Dehn-Sommerville type relations to integer points contained in some rational polytopes.

1. Introduction and preliminaries

A family $\Delta$ of subsets of a finite set $V$ is called an abstract simplicial complex (or a complex) on the vertex set $V$ if the inclusion $A \subseteq B \in \Delta$ implies $A \in \Delta$, and if $\{v\} \in \Delta$, for any $v \in V$; see, e.g., [3, 7, 9, 10, 15, 21, 24, 30]. The sets from the family $\Delta$ are called the faces. The size of a face $F \in \Delta$ is its cardinality $|F|$. Let $\#$ denote the number of sets in a family. If $\#\Delta > 0$ then the size $d(\Delta)$ of $\Delta$ is defined by $d(\Delta) := \max_{F \in \Delta} |F|$.

The row vector $f(\Delta) := (f_0(\Delta), f_1(\Delta), \ldots, f_{d(\Delta)-1}(\Delta)) \in \mathbb{N}^{d(\Delta)}$, where $f_i(\Delta) := \#\{F \in \Delta : |F| = i+1\}$, is called the $f$-vector of $\Delta$; these vectors are characterized by the Schützenberger-Kruskal-Katona theorem. The row $h$-vector $h(\Delta) := (h_0(\Delta), h_1(\Delta), \ldots, h_{d(\Delta)}) \in \mathbb{Z}^{d(\Delta)+1}$ of $\Delta$ is defined by

$$\sum_{i=0}^{d(\Delta)} h_i(\Delta) \cdot y^{d(\Delta)-i} := \sum_{i=0}^{d(\Delta)} f_{i-1}(\Delta) \cdot (y - 1)^{d(\Delta)-i}.$$ 

For a positive integer $m$, let $[m]$ denote the set $\{1, 2, \ldots, m\}$. We denote by $2^{[m]}$ the simplex $\{F : F \subseteq [m]\}$ and relate to an arbitrary face system $\Phi \subseteq 2^{[m]}$ “long” analogues of the $f$- and $h$-vectors, namely, the row vectors

$$f(\Phi; m) := (f_0(\Phi; m), f_1(\Phi; m), \ldots, f_m(\Phi; m)) \in \mathbb{N}^{m+1},$$

$$h(\Phi; m) := (h_0(\Phi; m), h_1(\Phi; m), \ldots, h_m(\Phi; m)) \in \mathbb{Z}^{m+1},$$

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where \( f_i(\Phi; m) := \# \{ F \in \Phi : |F| = i \} \), for \( 0 \leq i \leq m \), and the vector \( h(\Phi; m) \) is defined by
\[
\sum_{i=0}^{m} h_i(\Phi; m) \cdot y^{m-i} := \sum_{i=0}^{m} f_i(\Phi; m) \cdot (y - 1)^{m-i} ;
\]
see, e.g., [19, 20], where similar combinatorial tools appear. The maps \( \Phi \mapsto f(\Phi; m) \) and \( \Phi \mapsto h(\Phi; m) \) from the Boolean lattice \( D(m) \) of all face systems (ordered by inclusion) to \( \mathbb{Z}^{m+1} \) are valuations on \( D(m) \).

We consider the vectors \( f(\Phi; m) \) and \( h(\Phi; m) \) as elements from the real Euclidean space \( \mathbb{R}^{m+1} \) of row vectors.

Given a complex \( \Delta \subseteq 2^{[m]} \) with \( \# \Delta > 0 \), denote by \( U(d(\Delta)) \) the \((d(\Delta) + 1) \times (d(\Delta) + 1)\) backward identity matrix whose rows and columns are indexed starting with zero and whose \((i, j)\)th entry is the Kronecker delta \( \delta_{i+j,d(\Delta)} \). One says that the \( h \)-vector of a complex \( \Delta \) satisfies the Dehn-Sommerville relations if \( h(\Delta) \) is a left eigenvector of \( U(d(\Delta)) \) corresponding to the eigenvalue 1, that is, it holds
\[
h_l(\Delta) = h_{d(\Delta)-l}(\Delta) , \quad 0 \leq l \leq d(\Delta) \quad (1.1)
\]
(see, e.g., [11 §VI.6], [14 Chapter 5], [19 §II.5], [10 §§I.2, 3.6, 8.6], [15 §III.11], [24 §§II.3, II.6, III.6], [25 §3.14], [30 §8.3],) or
\[
h_l(\Delta; m) = (-1)^{m-d(\Delta)} h_{m-l}(\Delta; m) , \quad 0 \leq l \leq m , \quad (1.2)
\]
see, e.g., [19 p. 171].

In this paper we consider the set of all row vectors \( f \in \mathbb{N}^{m+1} \) such that for each of them there exists a face system \( \Phi \subseteq 2^{[m]} \) satisfying Dehn-Sommerville type relations analogous to (1.2), and \( f(\Phi; m) = f \); we interpret the set of those vectors \( f \) as a set of integer points contained in two rational polytopes.

2. Notation

Throughout the paper, \( m \) denotes an integer, \( m \geq 2 \).

The components of vectors as well as the rows and columns of matrices are indexed starting with zero.

I\((m)\) denotes the \((m+1) \times (m+1)\) identity matrix.

T\((m)\) is the forward shift matrix whose \((i, j)\)th entry is \( \delta_{j-i,1} \).

For a vector \( w := (w_0, \ldots, w_m) \), \( w^\top \) denotes its transpose; \( \|w\|^2 := \sum_{i=0}^{m} w_i^2 \).

If \( \mathfrak{B} := (b_0, \ldots, b_m) \) is a basis of \( \mathbb{R}^{m+1} \) then, given a vector \( w \in \mathbb{R}^{m+1} \), we denote by \( [w]_{\mathfrak{B}} := (\kappa_0(w, \mathfrak{B}), \ldots, \kappa_m(w, \mathfrak{B})) \in \mathbb{R}^{m+1} \) the \((m+1)\)-tuple satisfying the equality \( \sum_{i=0}^{m} \kappa_i(w, \mathfrak{B}) \cdot b_i = w \). We also set \( \kappa_i(w) := \kappa_i(w, \mathfrak{S}_m) = w_i, 0 \leq i \leq m; \mathfrak{S}_m \) stands for the standard basis of \( \mathbb{R}^{m+1} \).
0 := (0,0,\ldots,0); \iota(m) := (1,1,\ldots,1).

\text{lin}(\cdot), \text{pos}(\cdot) \text{ and } \text{conv}(\cdot) \text{ stand for a linear, conical, and convex hulls, respectively.}

\textit{V} \oplus \textit{W} \text{ denotes the direct sum of subspaces, and } \boxplus \text{ denotes the Minkowski addition.}

We use the notation \( \hat{0} \) to denote the empty set.

If \( A \subseteq C \in 2^{[m]} \) then \([A, C] \) denotes the Boolean interval \( \{B \in 2^{[m]} : A \subseteq B \subseteq C\} \).

If \( \Phi \) is a face system, \( \# \Phi > 0 \), then its size \( d(\Phi) \) is defined by \( d(\Phi) := \max_{F \in \Phi} |F| . \)

3. Dehn-Sommerville type relations for the long \( h \)-vectors

We say, for brevity, that a face system \( \Phi \subset 2^{[m]} \) is a DS-system if the vector \( h(\Phi; m) \) satisfies the Dehn-Sommerville type relations

\[ h_l(\Phi; m) = (-1)^{m-d(\Phi)} h_{m-l}(\Phi; m), \quad 0 \leq l \leq m ; \quad (3.1) \]

see also [18, §7].

If \( \# \Phi > 0 \), then define the integer \( \eta(\Phi) := \begin{cases} |\bigcup_{F \in \Phi} F|, & \text{if } |\bigcup_{F \in \Phi} F| \equiv d(\Phi) \pmod{2}, \\ |\bigcup_{F \in \Phi} F| + 1, & \text{if } |\bigcup_{F \in \Phi} F| \not\equiv d(\Phi) \pmod{2}. \end{cases} \)

A face system \( \Phi \) with \( \# \Phi > 0 \) is a DS-system if and only if for any \( n \in \mathbb{P} \) such that

\[ \eta(\Phi) \leq n, \quad n \equiv d(\Phi) \pmod{2} , \quad (3.2) \]

the vector \( h(\Phi; n) \) is a left eigenvector (corresponding to the eigenvalue 1) of the \((n+1) \times (n+1)\) backward identity matrix \( U(n) \):

\[ h(\Phi; n) = h(\Phi; n) \cdot U(n) . \]

We denote the eigenspace of \( U(m) \) corresponding to the eigenvalue 1 by \( \mathcal{E}^h(m) \).

3.1. The eigenvalues of a backward identity matrix.

Recall that the eigenvalues of the permutation matrix \( U(m) \) are \(-1\) and 1; the matrix is unimodular, see, e.g., [23, Chapter 4], [28, Chapter 8] on unimodular matrices. The algebraic multiplicity of the eigenvalue 1 is \( \left\lceil \frac{m+1}{2} \right\rceil \): the characteristic polynomial \( \varphi(U(m)) \) of the matrix, in the variable \( \lambda \), is

\[ \varphi(U(m)) = \begin{cases} (\lambda - 1)^{\frac{m}{2}} (\lambda + 1)^{\frac{m}{2}}, & m \text{ even,} \\ (\lambda - 1)^{\frac{m-1}{2}} (\lambda + 1)^{\frac{m+1}{2}}, & m \text{ odd.} \end{cases} \]
In other words,

\[
\varphi(U(m)) = \begin{cases}
\sum_{s=0}^{m+1} K_s(m+1, \frac{m+2}{2}) \cdot \lambda^s, & m \text{ even,} \\
\sum_{s=0}^{m+1} K_s(m+1, \frac{m+1}{2}) \cdot \lambda^s, & m \text{ odd,}
\end{cases}
\]

where \( K_s(t, i) \) stands for the Krawtchouk polynomial defined by \( \sum_{s=0}^{t} K_s(t, i) \cdot \lambda^s = (-\lambda + 1)^i (\lambda + 1)^{t-i} \); see, e.g., [29, §1.2] on these polynomials.

The geometric multiplicity of 1 equals its algebraic multiplicity.

3.2. The linear hulls of the long \( h \)-vectors of DS-systems. For a positive integer \( k \), define the complex \( 2^{[k]} := 2^k - \{ [k] \} = [0, [k]] - \{ [k] \} \).

**Proposition 3.1.** (i) If \( m \) is even, then

\[
\mathcal{E}^h(m) = \text{lin} \left( h(2^{[1]}; m), h(2^{[3]}; m), \ldots, h(2^{[m-1]}; m), \iota(m) \right);
\]

if \( m \) is odd, then

\[
\mathcal{E}^h(m) = \text{lin} \left( h(2^{[2]}; m), h(2^{[4]}; m), \ldots, h(2^{[m-1]}; m), \iota(m) \right).
\]

(ii) If \( \Phi \subset 2^{[m]} \) is a DS-system such that \( \Phi \not\ni [m] \) (in particular, if \( \Phi \) is a complex whose \( h \)-vector satisfies (1.1)) then \( h(\Phi; m) \cdot \iota(m)^\top = 0 \).

**Proof.** The proof of assertion (i) is straightforward, with respect to the argument from [31]. Assertion (ii) is a consequence of [18, Eq. (3.8)]. \( \square \)

Define a linear hyperplane \( \mathcal{H}(m) \) in the space \( \mathcal{E}^h(m) \), in the following way: if \( m \) is even, then

\[
\mathcal{H}(m) := \text{lin} \left( h(2^{[1]}; m), h(2^{[3]}; m), \ldots, h(2^{[m-1]}; m) \right),
\]

with \( \dim \mathcal{H}(m) = \frac{m}{2} \); if \( m \) is odd, then

\[
\mathcal{H}(m) := \text{lin} \left( h(2^{[2]}; m), h(2^{[4]}; m), \ldots, h(2^{[m-1]}; m) \right),
\]

\( \dim \mathcal{H}(m) = \frac{m-1}{2} \).

The one-dimensional subspace \( \text{lin}(\iota(m)) \) is the orthogonal complement of the subspace \( \mathcal{H}(m) \) of the space \( \mathcal{E}^h(m) \), with respect to the standard scalar product.
3.3. Some bases of $\mathbb{R}^{m+1}$. Let $\{F_0, \ldots, F_m\} \subset 2^m$ be a face system such that $|F_k| = k$, for $0 \leq k \leq m$; here, $F_0 := \emptyset$ and $F_m := [m]$. In [18, §4] the following six bases of $\mathbb{R}^{m+1}$ were considered:

\[
\mathcal{S}_m = (\sigma(0; m), \ldots, \sigma(m; m)) = (f(\{F_0\}; m), \ldots, f(\{F_m\}; m)),
\]

\[
\mathcal{S}_m^* = (\vartheta^*(0; m), \ldots, \vartheta^*(m; m)) = (h(\{F_0\}; m), \ldots, h(\{F_m\}; m)),
\]

\[
\mathcal{S}_m^\bullet = (\varphi^*(0; m), \ldots, \varphi^*(m; m)) = (f([F_0, F_0]; m), \ldots, f([F_0, F_m]; m)),
\]

\[
\mathcal{S}_m^\circ = (\vartheta^*(0; m), \ldots, \vartheta^*(m; m)) = (h([F_0, F_0]; m), \ldots, h([F_0, F_m]; m)),
\]

\[
\mathcal{S}_m^\circ = (\varphi^*(0; m), \ldots, \varphi^*(m; m)) = (f([F_m, F_m]; m), \ldots, f([F_0, F_m]; m)),
\]

\[
\mathcal{S}_m^\circ = (\vartheta^*(0; m), \ldots, \vartheta^*(m; m)) = (h([F_m, F_m]; m), \ldots, h([F_0, F_m]; m)).
\]

the basis $\mathcal{S}_m^\circ$ is up to rearrangement the standard basis $\mathcal{S}_m$.

Table I collects the representations of the vectors $h(2^{[k]}; m)$ with respect to the above mentioned bases.

We define a matrix $S(m)$ whose $(i, j)$th entry is $(-1)^{j-i}(\frac{m-i}{j-i})$ by

\[
S(m) := \begin{pmatrix}
\vartheta^*(0; m) \\
\vdots \\
\vartheta^*(m; m)
\end{pmatrix},
\]

and denote by $S_m : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ the automorphism $v \mapsto v \cdot S(m)$.

**Table 1.** Representations of $h(2^{[k]}; m)$ and $f(2^{[k]}; m)$, $1 \leq k \leq m$, with respect to various bases

| lth component | Expression |
|---------------|------------|
| $h_i(2^{[k]}; m)$ | $\vartheta^*(k; m) - \vartheta^*(k; m) = (-1)^l \binom{m-k}{l} - (-1)^k \binom{m-k}{l-k}$ |
| $\kappa_i(h(2^{[k]}; m), \mathcal{S}_m^*)$ | $(-1)^l \sum_{s=0}^{m-l} \binom{m-k}{s} - (-1)^k \binom{m-k}{s-k}$ |
| $\kappa_i(h(2^{[k]}; m), \mathcal{S}_m^\circ)$ | $(-1)^{k-l} \binom{m-k}{l}$ |
| $\kappa_i(h(2^{[k]}; m), \mathcal{S}_m^\circ)$ | $(-1)^{m-l} \binom{m-k}{l-k}$ |
| $f_i(2^{[k]}; m)$ | $\varphi^*(k; m) - \sigma_i(k; m) = \varphi^*(k; m) - \delta_{k,l} = \binom{k}{l}$ |
| $\kappa_i(f(2^{[k]}; m), \mathcal{S}_m^*)$ | $\sum_{s=0}^{\min(k, l)} \binom{k}{s} \binom{m-s}{l-s}$ |
| $\kappa_i(f(2^{[k]}; m), \mathcal{S}_m^\circ)$ | $(-1)^{k-l} \binom{m-k}{l}$ |
| $\kappa_i(f(2^{[k]}; m), \mathcal{S}_m^\circ)$ | $(-1)^{m-l} \binom{2k^2l - m - 1}{l}$ |
| $\kappa_i(f(2^{[k]}; m), \mathcal{S}_m^\circ)$ | $\binom{m-k}{m-l} - (-1)^{m-l-k} \binom{m-k}{l}$ |
| $\kappa_i(f(2^{[k]}; m), \mathcal{S}_m^\circ)$ | $\binom{k}{m-l} - \delta_{k,m-l}$ |
4. DEHN-SOMMELVILLE TYPE RELATIONS FOR THE LONG $f$-VECTORS

Define a unimodular matrix $D(m)$ by

$$D(m) := S(m) \cdot U(m) \cdot S(m)^{-1} = S(m) \cdot \begin{pmatrix} \varphi^\vee(0; m) \\ \vdots \\ \varphi^\vee(m; m) \end{pmatrix}^{-1} \cdot S(m)^{-1};$$

its $(i, j)$th entry is $(-1)^{m-i} \binom{i}{j}$.

Since the matrices $D(m)$ and $U(m)$ are similar, the properties of $D(m)$ coincide with those of $U(m)$ mentioned in §3.1. Further, for any DS-system $\Phi$ such that $\# \Phi > 0$, and for any integer $n$ satisfying (3.2), it holds

$$f(\Phi; n) = f(\Phi; n) \cdot D(n); \quad (4.1)$$
in other words, we have

$$f_l(\Phi; n) = (-1)^n \sum_{i=l}^{d(\Phi)} (-1)^i \binom{i}{j} f_i(\Phi; n), \quad 0 \leq l \leq d(\Phi);$$
cf., e.g., [30, p. 253].

Let $S_{-1}^{-1} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ be the automorphism $v \mapsto v \cdot S(m)^{-1}$. We define subspaces $F(m)$ of $\mathbb{R}^{m+1}$ by $F(m) := S_{-1}^{-1}(H(m))$ and $E^i(m) := S_{-1}^{-1}(E^h(m))$. Thus, if $m$ is even, then

$$F(m) = \text{lin}(f(2^{[1]}; m), f(2^{[2]}; m), \ldots, f(2^{[m]}; m)); \quad (4.2)$$

if $m$ is odd, then

$$F(m) = \text{lin}(f(2^{[2]}; m), f(2^{[4]}; m), \ldots, f(2^{[m-1]}; m)); \quad (4.3)$$

Define the row vector

$$\pi(m) := ((m+1), (m+1), \ldots, (m+1)) = \nu(m) \cdot S(m)^{-1} \in \mathbb{N}^{m+1}.$$

For any $m$, we have $E^i(m) = F(m) \oplus \text{lin}(\pi(m))$; this is the eigenspace of the matrix $D(m)$ corresponding to its eigenvalue 1.

All the vectors that appear in expressions (4.2) and (4.3), as well as the vector $\pi(m)$, lie in the affine hyperplane $\{z \in \mathbb{R}^{m+1} : z_0 = 1\}$. We also have $\pi(m), 0, 0) = f(2^{[m+1]}; m + 2)$. The representations of the vectors $f(2^{[k]}; m)$ with respect to various bases are collected in Table 1.
5. The long \( f \)-vectors of DS-systems, and integer points in rational polytopes

Studying the long \( f \)-vectors of the DS-systems \( \Phi \subset 2^{[m]} \) such that \( \#\Phi > 0 \) and \( m \equiv d(\Phi) \text{ (mod 2)} \), and considering relation (4.1), we are interested in the solutions \( z \in \mathbb{N}^{m+1}, z \neq 0 \), to the system

\[
z \cdot (I(m) - D(m)) = 0, \quad 0 \leq z \leq \left(\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\right).
\]

The matrix \( I(m) - D(m) \) is lower-triangular, of rank \( \lfloor \frac{m+1}{2} \rfloor \), with the eigenvalues 0 and 2.

Define the polytope

\[
Q_f(m) := \left\{ x \in \mathbb{R}^{m+1} : x \cdot (I(m) - D(m)) = 0, \quad 0 \leq x \leq \left(\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\right) \right\}.
\]

If \( m \) is even, then another description of \( Q_f(m) \) is

\[
Q_f(m) = \text{lin} \left(f(2^{[0]}; m), f(2^{[1]}; m), \ldots, f(2^{[m-1]}; m), \pi(m)\right) \cap \left\{ x \in \mathbb{R}^{m+1} : 0 \leq x \leq \left(\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\right) \right\} ;
\]

if \( m \) is odd, then we have

\[
Q_f(m) = \text{lin} \left(f(2^{[2]}; m), f(2^{[3]}; m), \ldots, f(2^{[m-1]}; m), \pi(m)\right) \cap \left\{ x \in \mathbb{R}^{m+1} : 0 \leq x \leq \left(\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\right) \right\} ;
\]

In other words, for any \( m \), we have

\[
Q_f(m) = E_f(m) \cap \Pi(m),
\]

where

\[
\Pi(m) := \text{conv} \left( \sum_{k=0}^{m} a_k \binom{m}{k} \sigma(k; m) : (a_0, \ldots, a_m) \in \{0,1\}^{m+1} \right) . \tag{5.1}
\]

On the one hand, the points \( \{z \in Q_f(m) \cap \mathbb{N}^{m+1} : z \neq 0\} \) are exactly the vectors \( f(\Phi; m) \) of the DS-systems \( \Phi \subset 2^{[m]} \) such that \( \#\Phi > 0 \) and \( m \equiv d(\Phi) \text{ (mod 2)} \). On the other hand, if \( z' \in Q_f(m) \) then there are \( \prod_{k=0}^{m} \binom{m}{z_k} \) DS-systems \( \Phi \subset 2^{[m]} \) corresponding to the point \( z' \).

**Example 5.1.** We have \( Q_f(2) = \text{conv}(0, (1,0,0), (0,1,1), (1,1,1)) \), see Figure 1(a).

One more description of \( Q_f(m) \) is

\[
Q_f(m) = \left\{ x \in C_f(m) : x \leq \left(\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\right) \right\} ,
\]
Figure 1.
where a convex pointed polyhedral cone $\mathcal{C}^f(m)$ is defined by

$$\mathcal{C}^f(m) := \mathcal{E}^f(m) \cap \text{pos}(\sigma(0; m), \sigma(1; m), \ldots, \sigma(m; m)) .$$

**Proposition 5.2.**

(i) If $m$ is even, then

$$\mathcal{C}^f(m) \supseteq \text{pos} \left( \varphi^i(i; m) \cdot T(m)^i : 0 \leq i \leq \frac{m}{2} \right) .$$

For any $i \in \mathbb{N}$, $i \leq \frac{m}{2}$, the ray $\text{pos} \left( \varphi^i(i; m) \cdot T(m)^i \right)$ is an extreme ray of the $\frac{m+2}{2}$-dimensional unimodular cone $\text{pos} (\varphi^i(i; m) \cdot T(m)^i : 0 \leq i \leq \frac{m}{2}) . $ \hspace{1cm} (5.2)

(ii) If $m$ is odd, then

$$\mathcal{C}^f(m) \supseteq \text{pos} \left( \varphi^i(i + 1; m) \cdot T(m)^i + \varphi^i(i; m) \cdot T(m)^{i+1} : 0 \leq i \leq \frac{m-1}{2} \right) .$$

For any $i \in \mathbb{N}$, $i \leq \frac{m-1}{2}$, the ray $\text{pos} (\varphi^i(i + 1; m) \cdot T(m)^i + \varphi^i(i; m) \cdot T(m)^{i+1})$ is an extreme ray of the $\frac{m+1}{2}$-dimensional unimodular cone $\text{pos} (\varphi^i(i + 1; m) \cdot T(m)^i + \varphi^i(i; m) \cdot T(m)^{i+1} : 0 \leq i \leq \frac{m-1}{2})$.

**Proof.** (i) The vectors from the sequence

$$\left( \varphi^i(i; m) \cdot T(m)^i : 0 \leq i \leq \frac{m}{2} \right) \hspace{1cm} (5.3)$$

are linearly independent, and for $i, j \in \mathbb{N}$ such that $i \leq \frac{m}{2}$ and $j \leq m$, we have

$$\kappa_j \left( \varphi^i(i; m) \cdot T(m)^i \cdot S(m) \right) = (-1)^{j-i} \binom{m-2}{j-i} , \hspace{1cm} (5.4)$$

hence sequence (5.3) is a basis of the space $\mathcal{E}^f(m)$, and the cone generated by it is simplicial. The matrix

$$\begin{pmatrix}
\kappa_0 (\varphi^0(0; m) \cdot T(m)^0) & \cdots & \kappa_{\frac{m}{2}} (\varphi^0(0; m) \cdot T(m)^0) \\
\cdots & \cdots & \cdots \\
\kappa_0 (\varphi^{\frac{m}{2}}(\frac{m}{2}; m) \cdot T(m)^{\frac{m}{2}}) & \cdots & \kappa_{\frac{m}{2}} (\varphi^{\frac{m}{2}}(\frac{m}{2}; m) \cdot T(m)^{\frac{m}{2}})
\end{pmatrix}$$

is upper-triangular whose diagonal entries are 1. This implies that (5.3) is an integral basis of the intersection of the linear hull of (5.3) with $\mathbb{Z}^{m+1}$. In other words, cone (5.2) is unimodular.

Pick a vector $\mathbf{v} \in \mathcal{E}^f(m)$ such that $\mathbf{v} \geq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. If $\mathbf{v} = \sum_{i=0}^{m/2} a_i \cdot \varphi^i(i; m) \cdot T(m)^i$, for some $a_0, \ldots, a_{\frac{m}{2}} \in \mathbb{R}$, then the equalities $\kappa_0 (\mathbf{v}) = a_0$, $\kappa_1 (\mathbf{v}) = a_1$ and $\kappa_{\frac{m}{2}} (\mathbf{v}) = a_{\frac{m}{2}}$ imply that $a_0, a_1, a_{\frac{m}{2}} \geq 0$. As a consequence, if $m \in \{2, 4\}$ then

$$\mathcal{C}^f(m) = \text{pos} \left( \varphi^i(i; m) \cdot T(m)^i : 0 \leq i \leq \frac{m}{2} \right) .$$
The proof of assertion (ii) is analogous to that of (i). In particular, for \( i, j \in \mathbb{N} \) such that \( i \leq \frac{m-1}{2} \) and \( j \leq m \), we have

\[
\kappa_j \left( (\varphi^* (i + 1; m) \cdot T(m)^i + \varphi^* (i; m) \cdot T(m)^{i+1}) \cdot S(m) \right)
\]

or \( \kappa_{m-j} \)

\[
= (-1)^{j-i} (\binom{m-2i-1}{j-i} - \binom{m-2i-1}{j-i-1}) ; \quad (5.5)
\]

the sequence

\[
(\varphi^* (i + 1; m) \cdot T(m)^i + \varphi^* (i; m) \cdot T(m)^{i+1} : 0 \leq i \leq \frac{m-1}{2}) \quad (5.6)
\]

is a basis of \( \mathcal{E}^f (m) \).

If \( \nu := \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} a_i \cdot (\varphi^* (i + 1; m) \cdot T(m)^i + \varphi^* (i; m) \cdot T(m)^{i+1}) \in \mathcal{E}^f (m) \) and \( \nu \geq 0 \) and \( \nu \neq 0 \), for some \( a_0, \ldots, a_{m-1} \in \mathbb{R} \), then the equalities \( \kappa_0 (\nu) = a_0 \) and \( \kappa_{m-1} (\nu) = a_{\frac{m-1}{2}} \) imply that \( a_0, a_{\frac{m-1}{2}} \geq 0 \) and, as a consequence, it holds

\[
\mathcal{C}^f (3) = \text{pos} \left( \varphi^* (i + 1; 3) \cdot T(3)^i + \varphi^* (i; 3) \cdot T(3)^{i+1} : 0 \leq i \leq 1 \right).
\]

The structure of bases \( (5.3) \) and \( (5.6) \) of \( \mathcal{E}^f (m) \) allows us to come to the following conclusion:

**Corollary 5.3.**

(i) Let \( \Phi \subset 2^{|m|} \) be a DS-system.

If \( m \) is even, then either \( f_{m-1} (\Phi; m) = \frac{m}{2} \) and \( f_m (\Phi; m) = 1 \), or \( f_{m-1} (\Phi; m) = f_m (\Phi; m) = 0 \); if \( m \) is odd, then it holds \( f_{m-1} (\Phi; m) = f_m (\Phi; m) = 0 \).

(ii) If \( m \) is even, then for any \( t \in \mathbb{N} \), \( t \leq \frac{m-2}{2} \), it holds

\[
\text{lin} \left( \left\{ f(2^{[i+1]}; m) : 0 \leq i \leq t \right\} \right) = \text{lin} \left( \left\{ \varphi^* (i; m) \cdot T(m)^i : 0 \leq i \leq t \right\} \right);
\]

if \( m \) is odd, then for any \( t \in \mathbb{N} \), \( t \leq \frac{m-3}{2} \), it holds

\[
\text{lin} \left( \left\{ f(2^{[i+1]}; m) : 0 \leq i \leq t \right\} \right) = \text{lin} \left( \left\{ \varphi^* (i + 1; m) \cdot T(m)^i + \varphi^* (i; m) \cdot T(m)^{i+1} : 0 \leq i \leq t \right\} \right).
\]

(iii)

\[
\text{lin} (f(\Phi; m) : \Phi \subset 2^{|m|}, m \equiv d(\Phi) \pmod{2}, \Phi \text{ is DS}) = \begin{cases} \mathcal{E}^f (m), & m \text{ even}, \\ \mathcal{F} (m), & m \text{ odd}. \end{cases}
\]
Table 2. Representations of \( \mathbf{w} := \varphi^* (i; m) \cdot \mathbf{T} (m)^i \), where \( m \) is even and \( 0 \leq i \leq \frac{m}{2} \), with respect to various bases

| \( i \)th component | Expression |
|---------------------|------------|
| \( \kappa_i (\mathbf{w}, \mathcal{S}_m) \) | \( (\left( \begin{array}{c} m \end{array} \right) ) \) |
| \( \kappa_i (\mathbf{w}, \mathcal{S}^*_m) \) | \( \sum_{s=i}^{\min (2i, i)} \left( \begin{array}{c} i-s \end{array} \right) (m-s) \) |
| \( \kappa_i (\mathbf{w}, \mathcal{S}^*_m) \) | \( (-1)^i \left( \begin{array}{c} i \end{array} \right) \) |
| \( \kappa_i (\mathbf{w}, \mathcal{S}^*_m) \) | \( (-1)^i \sum_{s=\max (i, m-i) \frac{1}{m}} (s-i) \left( \begin{array}{c} m-i \end{array} \right) \) |
| \( \kappa_i (\mathbf{w}, \mathcal{S}^*_m) \) | \( (-1)^i \left( \begin{array}{c} m-2i \end{array} \right) \) |
| \( \kappa_i (\mathbf{w}, \mathcal{S}^*_m) \) | \( (-1)^i \left( \begin{array}{c} m-2i \end{array} \right) \) |

Table 3. Representations of \( \mathbf{w} := \varphi^* (i+1; m) \cdot \mathbf{T} (m)^i + \varphi^* (i; m) \cdot \mathbf{T} (m)^{i+1} \), where \( m \) is odd and \( 0 \leq i \leq \frac{m-1}{2} \), with respect to various bases

| \( i \)th component | Expression |
|---------------------|------------|
| \( \kappa_i (\mathbf{w}, \mathcal{S}_m) \) | \( (\left( \begin{array}{c} m \end{array} \right) ) + \sum_{s=i}^{\min (2i+1, i)} \left( \begin{array}{c} i+s \end{array} \right) \left( \begin{array}{c} m-s \end{array} \right) \) |
| \( \kappa_i (\mathbf{w}, \mathcal{S}^*_m) \) | \( (-1)^{i-1} \left( \begin{array}{c} i-1 \end{array} \right) + \left( \begin{array}{c} i \end{array} \right) \) |
| \( \kappa_i (\mathbf{w}, \mathcal{S}^*_m) \) | \( (-1)^{i-1} \sum_{s=\max (i, m-i) \frac{1}{m}} (s-i) \left( \begin{array}{c} m-s \end{array} \right) \) |
| \( \kappa_i (\mathbf{w}, \mathcal{S}^*_m) \) | \( (-1)^{i-1} \left( \begin{array}{c} m-2i \end{array} \right) \) |
| \( \kappa_i (\mathbf{w}, \mathcal{S}^*_m) \) | \( (-1)^{i-1} \left( \begin{array}{c} m-2i \end{array} \right) \) |

If \( m \) is even, define the \( \frac{m}{2} \) dimensional polytope

\[ \mathcal{P}^i (m) := \mathcal{E}^i (m) \cap \{ \mathbf{z} \in \mathbb{R}^{m+1} : z_0 = 0, \ 0 \leq (z_1, \ldots, z_m) \leq (\left( \begin{array}{c} m \end{array} \right), \left( \begin{array}{c} m \end{array} \right), \ldots, (\left( \begin{array}{c} m \end{array} \right)) \} \],
and note that the polytope $Q^f(m)$ is a prism whose basis is $P^f(m)$:

$$Q^f(m) = P^f(m) \boxplus \varphi^A(0; m) \cdot T(m)^0 = P^f(m) \boxplus \sigma(0; m) ;$$

as a consequence, we have

$$\sum_{\alpha \in Q^f(m) \cap \mathbb{N}^{m+1}} x^\alpha = (1 + x_0) \sum_{\alpha \in P^f(m) \cap \mathbb{N}^{m+1}} x^\alpha ,$$

where $x^\alpha := x_0^{a_0} \cdots x_m^{a_m}$.

Since for any $m$, the generating function for the long $f$-vectors of DS-systems contained in $2^m$ is

$$\sum_{f \in \mathbb{N}^{m+1} \colon \exists \Phi \subset 2^m, \Phi \text{ is DS, } f(\Phi; m) = f} x^f = -1 + \sum_{\alpha \in Q^f(m) \cap \mathbb{N}^{m+1}} x^\alpha + \sum_{(\alpha, 0) \in P^f(m+1) \cap \mathbb{N}^{m+2}: \alpha \leq (m, \ldots, m)} x^\alpha ,$$

we come to the conclusion:

**Proposition 5.4.** (i) If $m$ is even, then

$$\sum_{f \in \mathbb{N}^{m+1} \colon \exists \Phi \subset 2^m, \Phi \text{ is DS, } f(\Phi; m) = f} x^f = -1 + (1 + x_0) \sum_{(\beta, 0) \in P^f(m) \cap \mathbb{N}^{m+1}} x^\beta$$

$$+ \sum_{(\gamma, 0, 0) \in Q^f(m+1) \cap \mathbb{N}^{m+2}: \gamma \leq (m, \ldots, m)} x^\gamma .$$

(ii) If $m$ is odd, then

$$\sum_{f \in \mathbb{N}^{m+1} \colon \exists \Phi \subset 2^m, \Phi \text{ is DS, } f(\Phi; m) = f} x^f = -1 + \sum_{(\beta, 0, 0) \in Q^f(m) \cap \mathbb{N}^{m+1}} x^\beta$$

$$+ (1 + x_0) \sum_{(0, \gamma, 0, 0) \in P^f(m+1) \cap \mathbb{N}^{m+2}: \gamma \leq (m, \ldots, m)} x^\gamma .$$

See, e.g., [1, Chapter VIII], [2], [3], [4], [5], [8], [17], [21, Chapter 12] on lattice-point counting in polytopes. Effective tools for manipulating rational generating functions are presented in [3].

Table 4 collects some illustrative information obtained with the help of the software LattE [11], [12], [13].
Table 4. The number of the long $f$-vectors of the DS-systems contained in $2^{[m]}$, $2 \leq m \leq 10$

| $m$ | $|\{ f \in \mathbb{N}^{m+1} : \exists \Phi \subset 2^{[m]}, \# \Phi > 0, f(\Phi; m) = \mathbf{f} \}|$ | $|\{ f \in \mathbb{N}^{m+1} : \exists \Phi \subset 2^{[m]}, \# \Phi > 0, f(\Phi; m) = \mathbf{f} \}|$ | $|\{ f \in \mathbb{N}^{m+1} : \exists \Phi \subset 2^{[m]}, \Phi \text{ is DS}, f(\Phi; m) = \mathbf{f} \}|$ |
|-----|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| 2   | 3                                               | 1                                               | 5                                               |
| 3   | 1                                               | 7                                               | 9                                               |
| 4   | 19                                              | 5                                               | 25                                              |
| 5   | 7                                               | 71                                              | 79                                              |
| 6   | 291                                             | 41                                              | 333                                             |
| 7   | 103                                             | 2223                                            | 2327                                            |
| 8   | 17465                                           | 1107                                            | 18573                                           |
| 9   | 4905                                            | 271619                                          | 276525                                          |
| 10  | 3959091                                         | 103057                                          | 4062149                                         |

6. Appendices

6.1. The long $h$-vectors of DS-systems, and integer points in rational polytopes. We now consider the polytope $\mathcal{Q}^h(m) := S_m(\mathcal{Q}^f(m)) := \{ z \cdot S(m) : z \in \mathcal{Q}^f(m) \}$.

The relation $h(\Phi; m) = h(\Phi; m) \cdot U(m)$, $h(\Phi; m) \neq 0$, describes the long $h$-vectors of the DS-systems $\Phi \subset 2^{[m]}$ such that $\# \Phi > 0$ and $m \equiv d(\Phi)$ (mod 2), and we are interested in the solutions $z \in \mathbb{Z}^{m+1}$, $z \neq 0$, to the system

$$z \cdot (I(m) - U(m)) = 0,$$

$$z \in S_m(\Pi(m)) = \text{conv}\left(\sum_{k=0}^{m} a_k \binom{m}{k} \vartheta^*(k; m) : (a_0, \ldots, a_m) \in \{0, 1\}^{m+1}\right).$$

(6.1)

The matrix $I(m) - U(m)$ of rank $\lfloor \frac{m+1}{2} \rfloor$ is totally unimodular (see, e.g., [23, §19] on totally unimodular matrices,) that is, any its minor is either $-1$ or 0 or 1; this matrix can be substituted in system (6.1) by the submatrix composed of the first $\lfloor \frac{m+1}{2} \rfloor$ columns.

System (6.1) describes the intersection

$$\mathcal{Q}^h(m) := \bigcap_{1 \leq k \leq \lfloor (m+1)/2 \rfloor} \{ x \in \mathbb{R}^{m+1} : x_{k-1} = x_{m-k+1} \} \cap S_m(\Pi(m))$$

(6.2)

of the $\lfloor \frac{m+1}{2} \rfloor$-dimensional center of a graphical hyperplane arrangement (see, e.g., [22, §2.4], [25, Lecture 2] on graphical arrangements) with an $(m+1)$-dimensional parallelepiped. The intersection poset of the arrangement $\{ x \in \mathbb{R}^{m+1} : x_{k-1} = x_{m-k+1} : 1 \leq k \leq \lfloor \frac{m+1}{2} \rfloor \}$ is a Boolean lattice of rank $\lfloor \frac{m+1}{2} \rfloor$. 
Example 6.1. $Q^h(2) = \text{conv}(0, (1, -1, 1), (0, 1, 0), (1, -1, 1))$, see Figure 1(b).

Another description of polytope $Q^h(m)$ is

$$Q^h(m) = \text{lin}(h(2|1]; m), h(2|3]; m), \ldots, h(2|m-1]; m), \iota(m)) \cap S_m(\Pi(m))$$

in the case of $m$ even, and

$$Q^h(m) = \text{lin}(h(2|2]; m), h(2|4]; m), \ldots, h(2|m-1]; m), \iota(m)) \cap S_m(\Pi(m)),$$

if $m$ is odd. In other words, for any $m$, we have

$$Q^h(m) = \mathcal{E}^h(m) \cap S_m(\Pi(m)).$$

6.2. Biorthogonality. By the principle of biorthogonality [16, Theorem 1.4.7], we have

$$h(2|s]; m) \cdot h(2|t]; m)^\top = 0,$$

for all positive integers $s$ and $t$ less than or equal to $m$ such that $s \neq t$ (mod 2). Indeed, one of the vectors $h(2|s]; m)$ and $h(2|t]; m)$ is a left eigenvector of the backward identity matrix $U(m)$ corresponding to an eigenvalue $\lambda \in \{-1, 1\}$, while the other vector is a right eigenvector corresponding to the other eigenvalue $\mu \in \{-1, 1\}$.

6.3. The norms. For a positive integer $k$ such that $k \leq m$, we have

$$\|h(2|k]; m)\|^2 = 2\left(\binom{2(m-k)}{m-k} - (-1)^k \binom{2(m-k)}{m}\right)$$

and

$$\|f(2|k]; m)\|^2 = \binom{2k}{k} - 1.$$

6.4. Orthogonal projectors.

- If $m$ is even, then the matrix of the orthogonal projector (see, e.g., [27, Chapter 3] on projectors) into $\mathcal{H}(m)$, relative either to the standard basis $\mathcal{S}_m$ or to the basis $\mathcal{F}_m^\top$, is

$$
\begin{pmatrix}
\vdots \\
(h(2|1]; m)^\top & h(2|3]; m)^\top & \ldots & h(2|m-1]; m)^\top \\
(h(2|1]; m) & h(2|3]; m) & \ldots & h(2|m-1]; m) \\
\vdots \\
(h(2|1]; m)^\top & h(2|3]; m)^\top & \ldots & h(2|m-1]; m)^\top \\
\end{pmatrix}^{-1}
\begin{pmatrix}
(h(2|1]; m) \\
(h(2|3]; m) \\
\vdots \\
(h(2|m-1]; m) \\
\end{pmatrix}.
$$
The matrix of the orthogonal projector (relative to the standard basis $S_m$) into the subspace $F(m)$ is

$$
\begin{pmatrix}
    f(2^{[1]}; m)^	op & f(2^{[3]}; m)^	op & \cdots & f(2^{[m-1]}; m)^	op \\
    \vdots & \vdots & \ddots & \vdots \\
    f(2^{[m-1]}; m)^	op & f(2^{[m-3]}; m)^	op & \cdots & f(2^{[1]}; m)^	op \\
\end{pmatrix}^{-1}
\begin{pmatrix}
    f(2^{[1]}; m) \\
    f(2^{[3]}; m) \\
    \vdots \\
    f(2^{[m-1]}; m) \\
\end{pmatrix}.
$$

The matrices of the orthogonal projectors in the case of $m$ odd are built in an analogous way.

- Let $k$ be a positive integer such that $k \leq m$. The $(i, j)$th entry of the matrix of the orthogonal projector into the one-dimensional subspace $\operatorname{lin}(h(2^k; m))$ of the space $\mathbb{R}^{m+1}$, relative to either of the bases $S_m$ and $H_m$, is

$$
\frac{(-1)^{i+j}((m-k)_i - (-1)^k(m-k)_j)((m-k)_j - (-1)^k(m-k)_i)}{2((2^{(m-k)}_{m-k}) - (-1)^k(2^{(m-k)}_m))};
$$

the $(i, j)$th entry of the matrix of the orthogonal projector into the one-dimensional subspace $\operatorname{lin}(f(2^k; m))$ of $\mathbb{R}^{m+1}$, relative to $S_m$, is

$$
\frac{\binom{k}{i}}{\binom{k}{j} - 1}.
$$

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