Limiting curves for the dyadic odometer and the generalized Trollope-Delange formula.*

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Abstract

We study limiting curves resulting from deviations in partial sums in the ergodic theorem for the dyadic odometer and non-cylindric functions. In particular, we generalize the Trollope-Delange formula for the case of the weighted sum-of-binary-digits function and show that the Takagi-Landsberg curve arises.

Key words: limiting curves, weighted sum-of-binary-digits function, Takagi-Landsberg curve, q-analogue of the Trollope-Delange formula

MSC: 11A63, 39B22, 37A30

1 Introduction

Let $T$ be a measure preserving transformation defined on a Lebesgue probability space $(X, B, \mu)$ with an invariant ergodic probability measure $\mu$. Let $g$ denote a function in $L^1(X, \mu)$. In [11] É.Janvresse, T. de la Rue, and Y. Velenik in the process of studying the Pascal adic transformation introduced a new notion of a limiting curve. Following [11] for a point $x \in X$ and a

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†The Pascal adic transformation was invented by A. Vershik (and independently by S. Kakutani), see [30], [15], and intensively studied since then, see e.g. [10], [22], [31], [32]. According to the authors of [11] their research was motivated by the observation by X. Mela.
positive integer $j$ we denote the partial sum $\sum_{k=0}^{j-1} g(T^k x)$ by $S^g_2(j)$. We extend the function $S^g_2(j)$ to a real valued argument by a linear interpolation and denote extended function again by $S^g_2(j)$ or simply $S(j), j \geq 0$.

Let $(l_n)_{n=1}^\infty$ be a sequence of positive integers. We consider continuous on $[0,1]$ functions $\varphi_n(t) = S(t \cdot l_n(x)) - t \cdot S(l_n)$, where the normalizing coefficient $R_n$ is canonically defined to be equal to the maximum in $t \in [0,1]$ of $|S(t \cdot l_n(x)) - t \cdot S(l_n)|$.

**Definition 1** ([11]). If there is a sequence $l^g_n(x) \in \mathbb{N}$ such that functions $\varphi_{x,l^g_n(x)}$ converge to a (continuous) function $\varphi^g_2$ in sup-metric on $[0,1]$, then the graph of the limiting function $\varphi = \varphi^g_2$ is called a **limiting curve**, sequence $l_n = l^g_n(x)$ is called a **stabilizing sequence** and the sequence $R_n = R^g_{x,l^g_n(x)}$ is called a **normalizing sequence**. The quadruple $(x,(l_n)_{n=1}^\infty,(R_n)_{n=1}^\infty,\varphi)$ is called a **limiting bridge**.

Heuristically, the limiting curve describes small fluctuations (of certainly renormalized) ergodic sums $\frac{1}{l} S(l), l \in (l_n)$, along the forward trajectory $x,T(x),T^2(x),\ldots$. More specifically, for $l \in (l_n)$ it holds $S(t \cdot l) = t S(l) + R_l \varphi(t) + o(R_l)$, where $t \in [0,1]$.

For the Pascal adic transformation in [11] and [21] it was shown that for $\mu$-a.e. $x$ a limiting bridge exists and stabilizing sequence $l_n(x)$ can be chosen in such a way that the well-known Takagi curve (and its generalizations) arises in the limit. In [23] results were generalized for a wider class of polynomial adic dynamical systems. In general limiting curves are not well-studied yet for a lot of interesting transformations. In particular, authors of [11] asked (see Sec. 4.3.2.) whether limiting curves can exist for transformations in the rank-one category.

Findings on limiting curves motivated several researches (see e.g. [24]) to attack the problem of the spectrum of the Pascal adic transformation. Despite interesting results were obtained, the problem is still unsolved\footnote{However, there is a general consensus that spectrum should at least comprise a continuous component, see [32].}. A natural starting hypothesis is that for a system with discrete spectrum a (Besicovitch) almost periodic property of trajectories should imply that no limiting curve exists. However, this is not the case and the goal of this paper is to construct a concrete counterexample. We consider the dyadic odometer (as a simplest rank-one system with discrete dyadic spectra) and show that for certain functions limiting curves arise. Surprisingly, limiting curves that we find belong to the so called Takagi class, see below. Technically, we...
generalize the famous Trollope-Delange formula for the case of weighted-sum-of-binary digits function, that does not seem to have been published before.

2 Designations

2.1 Takagi-Landsberg functions

Let \(|a| < 1\). The Takagi-Landsberg function with parameter \(a\) is defined by the identity

\[
T_a(x) = \sum_{n=0}^{\infty} a^n \tau(2^n x),
\]

where \(\tau(x) = \text{dist}(x, \mathbb{Z})\), the distance from \(x\) to the nearest integer. It is immediately clear that the series converges uniformly and hence defines a continuous function \(T_a\) for \(|a| < 1\). Functions \(\{T_a\}_a\) can be considered as direct generalization of the famous Takagi-Blancmange function \(T\) which is obtained when \(a\) equals to \(1/2\), see [27]. Functions \(T_a\) are 1-periodic and nowhere differentiable for \(|a| \geq \frac{1}{2}\) but differentiable almost everywhere for \(|a| < \frac{1}{2}\), see [19] and more results in [18] and [2]. In [1, 4, 26] the so called approximate midconvexity property of the functions \(T_a, a \in [1/4, 1/2]\), was studied.

It immediately follows from [1] that \(T_a\) satisfies the following de Rham type functional equations for \(x \in [0, 1]\):

\[
\begin{cases}
T_a(x/2) = aT_a(x) + x/2, \\
T_a(x + 1/2) = aT_a(x) + \frac{1-x}{2}.
\end{cases}
\]

Conversely, as shown in [3], using the Banach’s fixed point theorem (see also [8] and [9]) any system of functional equations

\[
\begin{cases}
f(x/2) = a_0 f(x) + g_0(x), \\
f(x + 1/2) = a_1 f(x) + g_1(x),
\end{cases}
\]

\(3\)

More general \(\sum_{n=0}^{\infty} c_n \tau(2^n x)\) with \(\sum_{n=0}^{\infty} |c_n| < \infty\) can be considered; this family of functions also known as the Takagi class.

4For the Pascal adic another class of generalized Takagi functions appeared in [11, 21], the only intersection is the Takagi function \(T\) itself.

5In particular, \(T_{1/4}(x) = x(1 - x)\).
Figure 1: Takagi-Landsberg curves with different parameters: $a = -1/2$ (the alternating sign Takagi curve), $a = 1/2$ (the Takagi-Blancmange curve), $a = 2/3$, $a = 1/4$ (parabola).

with $\max\{|a_0|, |a_1|\} < 1$ and such that consistency condition\(^{[5]}\) holds

$$a_0 \frac{g_1(1)}{1 - a_1} + g_0(1) = a_1 \frac{g_0(0)}{1 - a_0} + g_1(0), \quad (4)$$

defines a unique continuous function on $[0, 1]$.

### 2.2 Dyadic odometer

Consider $X = \mathbb{Z}_2 = \prod_0^\infty \{0, 1\}$, the compact additive dyadic group of dyadic integers with Haar measure $\mu$, and let $T : Tx = x + 1$ be the addition of unity. The dynamical system $(X, T, \mu)$ is called the dyadic odometer. It is one of the simplest transformations in the rank one category. Space $X$ can be identified with a paths space of a simple Bratteli-Vershik diagram with only one vertex at each level $n$, $n = 0, 1, 2\ldots$, see Fig. 3. A cylinder set $C = [c_1c_2\ldots c_n] = \{x \in X | x_1 = c_1, \ldots, x_n = c_n\}$ of a rank $n$ is totally defined by a finite path

\[^{[6]}\text{i.e. both equations gives the same value at } x = 1/2.\]
T(x) = x + 1

Figure 2: The dyadic odometer: the red (solid) path is mapped to the blue (solid) path.

from the origin to the vertex \( n \). Sets \( \pi_n \) of linearly ordered (this order is called adic or colexicographical, see [29] and [30] for the original definition) finite paths are in one to one correspondence with towers \( \tau_n \) made up of corresponding cylinder sets. Towers define approximation of transformation \( T \), see [29]. Cylinder sets \([x_1, x_2, \ldots, x_n], x_i \in \{0, 1\}, \) constituting towers are called rungs; the bottom rung corresponds to the cylinder \([0, 0, \ldots, 0]\). There are a total of \( 2^n \) paths in each \( \pi_n \) (or rungs in \( \tau_n \)).

Using canonical mapping \( \text{Num} : \mathbb{N} \rightarrow \mathbb{N}_0 \) defined by \( \text{Num}(x) = \sum_{i=1}^{\infty} x_i 2^i \), the set \( N = \{ x \in \mathbb{Z}_2, \text{s.t.} \sum_{i=1}^{\infty} x_i < \infty \} \) can be naturally identified with nonnegative integers \( \mathbb{N}_0 \). We follow the agreement that finite paths can be continued with zeroes, then the image set \( \text{Num}(\pi_n) \) is the discrete interval \([0, 1, 2, \ldots, 2^n - 1]\). Of course, for \( x \in N \) we have \( \text{Num}(Tx) = \text{Num}(x) + 1 \).

The following lemma reads that for a.e. \( x \in X \) we can choose a sequence of levels in the Bratteli diagram such that \( x \) lies \( \varepsilon \)-close to the bottom rung of \( \tau_{n_j}, j = 1, 2, \ldots \). Its proof will follow the ideas of Janvresse and de la Rue from [10].

**Lemma 1.** For \( \mu \text{-a.e.} \ x \) for any \( \varepsilon > 0 \) there is a sequence \( (n_j)_j \) such that index \( \text{Num}(\omega^j) \) of a finite path \( \omega^j = (x_1, x_2, \ldots, x_n) \) satisfies the following inequality \( \text{Num}(\omega^j)/|\pi_{n_j}| < \varepsilon \).

**Proof.** Let \( Z_m = 2 \sum_{i=1}^{m} (x_i - \frac{1}{2}) \). Then \( Z_m \) is a symmetric random walk and thus recurrent. Therefore for \( \mu \text{-a.e.} \ x \) and any \( r \in \mathbb{N} \) we can choose a sequence of moments \( m_j \) such that \( Z_{m_j+1} = Z_{m_j+2} = \cdots = Z_{m_j+r} = -1 \). That means that \( x \) belongs to the one of \( 2^m \) first rungs of the tower \( \tau_{m_j+r} \).
Other way to say this is that \( \text{Num}(x_1, x_2, \ldots, x_m + r) < 2^{m'} \). Having in mind \( |\tau_n| = 2^n \) the former inequality can be rewritten as
\[
\frac{\text{Num}(x_1, x_2, \ldots, x_m + r)}{|\tau_{m + r}|} < 2^{-r}.
\]
Choosing \( r \) such that \( 2^{-r} < \varepsilon \) and setting \( n_j = m_j + r \) we obtain the required statement.

In [14] and [23] it was shown that the necessary condition for a limiting curve to exist is an unbounded growth of the normalizing coefficient \( R_n \). For the stationary odometer (unlike for the Pascal adic) this implies that there are no limiting curves for a cylindric (i.e. depending on a finite number of coordinates) function \( g \) since partial sums \( S^g_x \) are a.s. bounded for such \( g \), see [23] Theorems 1-3. Therefore we need to consider noncylindric functions.

Let \( 0 < |q| < 1 \) we denote by \( s_q : \mathbb{Z}_2 \rightarrow \mathbb{R} \) a weighted-sum-of-digits function defined by the identity \( s_q(x) = \sum_{j=1}^{\infty} x_n q^n \) for \( x = (x_1, x_2, \ldots, x_n, \ldots) \in \mathbb{Z}_2 \). We denote by \( 0 \) the zero path \((0, 0, \ldots)\) and by \( S^q_x(n) \) the partial sums defined by \( S^q_x = \sum_{j=0}^{n-1} s_q(T^j x) \).

### 2.3 The Trollope-Delange formula

For \( q = 1 \) function \( s_1 : N \rightarrow \mathbb{R} \) is the well-known sum-of-binary-digits function. Commonly sum-of-binary-digits function is denoted simply by \( s \) instead of \( s_1 \) and defined on \( N_0 \) instead of \( N \), but, as mentioned above, we identify sets \( N \) and \( N_0 \) and we have the identity \( s = s_1 \). For \( S = S^1 \) the following Trollope-Delange formula holds (see [22] and [21]):
\[
\frac{1}{n} S(n) = \frac{1}{2} \log_2(n) + \frac{1}{2} \tilde{F}(\log_2(n)),
\]
where the 1-periodic function \( \tilde{F} \) is given by
\[
\tilde{F}(t) = 1 - t - 2^{1-t} T(2^{-1-t}), \text{ for } 0 \leq t \leq 1.
\]

Several extensions of the Trollope-Delange formula are known. Besides the classical case \( S(n) = \sum_{j=0}^{n-1} s(j) \), its analogues for exponential \( \sum_{j=0}^{n-1} \exp(t s(j)) \), power \( \sum_{j=0}^{n-1} s^{k-1}(j) \) and binomial \( \sum_{j=0}^{n-1} (S(j))_m \) sums were studied, see [17], [3] or [2] for the history and starting links.
The Trollope-Delange formula also appears to be useful for describing functions like \( \varphi_n(t) = \frac{S(t, l_n(x)) - t S(l_n)}{R_n} \). We show below, see Proposition [1] that \( \varphi_n(t_j) = T(t_j) \) for \( t_j = \frac{j}{l_n}, j = 0, 1, 2 \ldots l_n \), and \( l_n = 2^n \). Unfortunately, function \( s_1 \) is not well-defined on \( X = \mathbb{Z}_2 \). So we find generalization of the Trollope-Delange formula for the weighted-sum-of-binary-digits function \( s_q \).

### 3 Main results

The main results of the paper are the following two statements:

**Theorem 1.** Let \( (X, T, \mu) \) be the dyadic odometer and \( \frac{1}{2} < |q| < 1 \). Then for \( \mu \)-a.e. \( x \) there exists a stabilizing sequence \( l_n = l_n(x) \) such that \( \varphi_{x, l_n}^n \) converges in sup-metric on \([0,1]\) to the function \(-T(a)\), where \( a = 1/(2q) \).

For a nonnegative integer \( j \) with binary expansion \( j = \sum_{i \geq 0} \omega_i 2^i \) we denote by \( s_q(j) \) the weighted-sum-of-binary-digits function \( \sum_{i \geq 0} \omega_i q^{i+1} \) and set \( S_q(n) = \sum_{j=0}^{n-1} s_q(j) \). The next proposition does not use the notion of the odometer and its first part generalizes Theorem 5.1 by Krüppel in [16], where author considers the case of “alternating sums”, that are obtained here if \( q = -1 \).

**Proposition 1.** Let \( |q| > 1/2 \) and \( a = 1/(2q) \).

1. The following generalized Trollope-Delange formula holds

\[
\frac{1}{n} S_q(n) = \frac{q}{2} \left( \frac{1 - q \log_2(n)}{1 - q} + q \log_2(n) \hat{F}_q(\log_2(n)) \right),
\]

where the 1-periodic function \( \hat{F}_q \) is given by \( \hat{F}_q(u) = \frac{1 - q^{1-u}}{1-q} - q^{-u} 2^{1-u} T_a(2^{-(1-u)}) \), \( u \in [0,1] \).

2. Let \( l = 2^k \) for any fixed \( k \in \mathbb{N} \), then the following identity holds:

\[
\varphi_l^q(t_j) = -q T_a(t_j),
\]

where \( \varphi_l^q(t) = \frac{S_q(t, l(t)) - t S_q(l(t))}{R_l} \) \( t_j = \frac{j}{l}, j = 0, 1, 2, \ldots, l \). Moreover, the renormalization coefficient \( R_l \) is proportional to \( (2|q|)^{\log_2(l)} \).

**Remark 1.** Conditions \( \frac{1}{2} < |q| < 1 \) and \( |q| > 1/2 \) in Theorem [1] and Proposition [1] respectively are essential. For \( |q| \leq 1/2 \) no continuous limiting curve exists.

**Remark 2.** Another generalization of the Trollope-Delange formula for weighted-sum-of-binary-digits functions was obtained in [20] and [14]. Their
approach is different and result is asymptotic for our choice of weights \((q^i)_{i=1}^{\infty}\), in formula (7) we give a precise expression.

3.1 Proof of Theorem 1

The result of Theorem 1 follows from the second assertion of the Proposition 1 and Lemma 1.

Proof. Clearly we have the identities \(S_q(n) = S_0(n)\) and \(\varphi_0^{r_j} = \varphi_q^{r_j}\). Lemma 1 implies that for \(r = r_j\) arbitrary large (we skip index \(j \in \mathbb{N}\) below) we can assume that \(x = (x_1, x_2, \ldots, x_m, 0, 0, \ldots, 0, \ast, \ast, \ast, \ldots)\), where \(m = m(r)\) and symbol \(*\) means either 0 or 1 entry. Let \(y = (x_1, x_2, \ldots, x_m, 0, 0, \ldots, 0, \ldots)\).

We have

\[
\frac{|S_{y,q}(i) - S_{0,q}(i)|}{R_n} \leq \frac{2^{m+1} \sum_{i=1}^{2^m} q^i}{2^{m+r}} \rightarrow 0, \quad i = 0, \ldots, 2^{m+r} - 2^m - 1.
\]

and, therefore, \(\|\varphi_s^{x,q} - \varphi_s^{0,q}\|_{\infty} \rightarrow 0, \quad l_j = 2^{m(r_j)+r_j}\). Next note, that tail coordinates of \(x\) denoted by \((\ast, \ast, \ast, \ldots, \ast)\) above do not change under \(T^i, i = 0, \ldots, 2^{m+r} - 2^m - 1\). Namely, we have \(s_q(T^i x) = s_q(T^i y) + b\), where \(b = b(r)\) is constant in \(i, i = 0, \ldots, 2^{m+r} - 2^m - 1\). Due to the renormalization\(^7\), we have \(\varphi_x^{q+y} = \varphi_x^{q}\) for any constant \(C\), so the value of \(b\) does not affect\(^8\) \(\varphi_s^{x,q}\). Finally, we conclude that our choice of the stabilizing sequence \(l_j(x)\) implies \(\|\varphi_s^{x,q} - \varphi_s^{0,q}\|_{\infty} \rightarrow 0\).

3.2 Proof of Proposition 1

Several approaches can be used to prove (7). We adopt approach suggested by Girgensohn in [9]. The idea of the approach is to start with the sequence \(S(n)\) itself, discover the functional equations within this sequence and then identify the limiting functions from the obtained functional equations. Advantage of this approach is that it does not require any advance knowledge of the functions appearing in the answer.

\(^7\)Specifically, subtraction of the linear part \(t \cdot S_q(x_n)\).

\(^8\)Equivalently, we may assume that \(b = b_j = 0\) for the given \(l_j\).
Proof. First we note that for any \( p = 2^{k-1} \)
\[
\begin{align*}
    s_q(2j) &= q \cdot s_q(j), \\
    s_q(2j + 1) &= q \cdot s_q(j) + q, \\
    s_q(j + p) &= s_q(j) + q^k, \quad j = 0, 1, \ldots, p - 1, \\
    s_q(j + p) &= s_q(j) - q^k(1 - q), \quad j = p, p + 1, \ldots, 2p - 1.
\end{align*}
\]

Let \( k_n = [\log_2(n)] \) and \( u_n = \{\log_2(n)\} \), where \([\cdot]\) and \(\{\cdot\}\) stand for the integer and fractional parts respectively. Following \([9]\) we denote by \( p(n) = 2^{k_n} \) the largest power of 2 less than or equal to \( n \) and by \( r_n \) we denote \( q^{\log_2(p_n)} = q^{k_n} \).

For any \( n \in \mathbb{N} \) we have
\[
\begin{align*}
    S_q(n + 2p_n) &= S_q(n) + S_q(2p_n) + nq^{k_n+1}, \\
    S_q(n + p_n) &= S_q(n) + (2q - 1)S_q(p_n) - (n - p_n)q^{k_n+1}(1 - q) + qp_n, \\
    S_q(2n) &= 2qS_q(n) + nq.
\end{align*}
\]

It is straightforward to obtain from (15) that for any \( p = 2^k \)
\[
S_q(p) = q \frac{1 - q^{k-1}2^{k-1}}{1 - q} = q \frac{1 - q^{k-1}}{2}. \tag{16}
\]

We define function \( G_q(n) \) by the identity
\[
G_q(n) = \frac{1}{p(n)r_n}
\left(S(n) - \frac{n}{p(n)}S(p(n))\right). \tag{17}
\]

Function \( G_q(n) \) satisfies the following three identities:
\[
\begin{align*}
    G_q(2n) &= G_q(n), \\
    G_q(n + p_n) &= \frac{1}{2q}G_q(n) + \frac{p_n - n}{4p_n}(3 - 2q), \tag{19}
\end{align*}
\]
\[
G_q(n + 2p_n) = \frac{1}{2q}G_q(n) + \frac{n}{4p_n}(2q - 1). \tag{20}
\]

Identities (18)–(20) follow from (13)–(15); here we deduce the first one, other two are obtained in the same way.
\[
\begin{align*}
G_q(n + p_n) &= \frac{1}{2qp_n}r_n\left(S(n + p_n) - \frac{n + p_n}{p(n + p_n)}S(p(n + p_n))\right) = \\
&= \frac{1}{2q}G_q(n) + \frac{1}{2qp_n}r_n\left(p_n - n\right)\left(\frac{n}{p_n} - 1\right)S(p_n) + (n - p_n)q^{k_n} = \\
&= \frac{1}{2q}G_q(n) + \frac{p_n - n}{4p_n}(3 - 2q).
\end{align*}
\]

\(\Box\)

*Identities (9) and (10) can be generalized for \( \omega \in \mathbb{Z}_2 \) and \( s_q \) function using the shift operator but we do not use that.*
Figure 3: Graph of the function $F_{2/3}$.

Following [9], we set $x_n = x(n) = 2^{n_1} - 1 = \frac{n-p_n}{p_n} \in [0, 1]$, then the following simple lemma proved in [9] holds:

Lemma 2. Let $G : \mathbb{N} \rightarrow \mathbb{R}$ be a function on the integers. For $n \in \mathbb{N}$, set

$$x := \frac{n-p_n}{p_n} \in [0, 1) \quad \text{and} \quad F(x) = F\left(\frac{n-p_n}{p_n}\right) := G(n).$$

Then $F$ is a well-defined function on the dyadic rationals in $[0, 1)$ iff $G(2n) = G(n)$ for all $n \in \mathbb{N}$.

Also we have

$$\frac{x(n)}{2} = \frac{n-p_n}{2p_n} = x(n+p_n), \quad (21)$$

$$\frac{x(n)+1}{2} = \frac{n}{2p_n} = x(n+2p_n). \quad (22)$$

By Lemma 2 the function $F_q$ given by $F_q(x_n) = G_q(n)$ is well-defined on the dyadic rationals in $[0, 1)$. Identities (19)–(20) for $x = x_n$ rewrites as follows

$$\begin{cases}
F_q(x/2) = aF_q(x) + (2q-3)\frac{x}{4}, \\
F_q(x+1/2) = aF_q(x) + (2q-1)\frac{x+1}{4}.
\end{cases} \quad (23)$$

From the system (23) we obtain $F_q(x) = qx - \frac{1}{2}T_q(x)$. Using $S_q(n) = r_n p_n G_q(n) + \frac{n}{p_n} S_q(p_n)$ we see that

$$\frac{1}{n} S_q(n) = q^{k_n} \frac{F_q(x_n)}{x_n+1} + \frac{q}{2} \frac{1}{1-q} \frac{1 - q^{k_n}}{2}. \quad (24)$$
Using \( x_n = 2^n - 1 \) and rewriting \( F_q \) as \( F_q(x) = q^{x+1} - q^T_a(x^{x+1}) \) we obtain \([5]\). This finishes part 1 of the proof.

Now we prove part 2. We fix some \( l = 2^j, j \in \mathbb{N} \). We set \( j_m = j - m, m = 1, 2, \ldots, j \) and \( l_m = l/2^m \). We split the whole discrete interval \( I = [1, l] \) into \( j \) disjoint intervals \( (I_m)_{m=1}^j, I_m = [l_m, l_{m+1}], \text{ thus } I = \cup_m I_m. \)

Let \( r = q^{j-1} \) and \( p = l/2 \), importantly \( r, p, \) and \( j_m \) do not change in \( n \) now, they depend only on \( l \) and \( m \). We set \( \tilde{G}(n) = \frac{1}{r^p} \left(S(n) - \frac{n}{r^p} S(2p)\right) \equiv \varphi\left(\frac{n}{r}\right). \) We prove \([\ref{5}]\) at each \( I_m, m = 1, 2, \ldots, j \), by starting with \( I_1 \) and then going to the general case \( I_m, m \geq 1 \).

For \( I_1 \) we have \( \frac{l}{2} \leq n < l \) and at this interval \( p = p_n, r = r_n \) and \( \frac{x+1}{2^m} = \frac{n}{r} \), where \( p_n, r_n \) and \( x = x_n \) are defined in the proof of part 1. Using \([\ref{13}]\) \(-\([\ref{14}]\) we rewrite \( G(n) \) as follows:

\[
\tilde{G}_q(n) = \frac{1}{p} \left(S_q(n) - \frac{nq}{2p}(2S_q(p) + p)\right) = \frac{1}{pr} \left(S_q(n) - \frac{nq}{p}S_q(p) + (1-q)\frac{n}{p}S_q(p) - q\frac{n}{2}\right) = \\
= \frac{1}{2pr} \left(S_q(n) - \frac{nq}{p}S_q(p)\right) = \frac{1}{2pr} \left(S_q(n) - \frac{nq}{p}(1 - q^{-1})\right) = G_q(n) - \frac{qn}{2p} = \\
= qx - \frac{1}{2} T_a(x) - \frac{x}{2} = q \left(\frac{x}{2} - \frac{1}{2} T_a(x)\right) = -q T_a\left(\frac{x}{2}\right) = -q T_a\left(\frac{n}{r}\right),
\]

Generally, we define \( l_m \leq n < l_{m-1} \) and set \( t = \frac{x+1}{2^m} \), note that \( \frac{t}{2^m} = \frac{n}{r} \).

From \([\ref{15}]\) \(\text{and } \[\ref{3}\] \) we see that

\[
S_q(l) = (2q)^m S_q\left(\frac{l}{2^m}\right) + 2^{m-1} l_m q (1 + q + \cdots + q^{m-1}) \quad \text{(25)}
\]

and

\[
T_a(t) = (2q)^m T_a\left(\frac{l}{2^m}\right) - t q (1 + q + \cdots + q^{m-1}). \quad \text{(26)}
\]

Using \([\ref{13}]\) \(\text{and } \[\ref{14}\] \) \(\text{on the interval } I_m \) we have \( l_m \leq n < l_{m-1} \) and \( \tilde{G}_q(n) = \frac{2}{l r_{j_1}} \left(S_q(n) - \frac{n}{l} S_q(l)\right) = \frac{2}{l r_{j_1}} \left(S_q(n) - \frac{n}{l_{m-1}} S_q(l_{m-1}) + \frac{n}{l_{m-1}} S_q(l_{m}) - \frac{n}{2^{m-1} l_{m-1}} S_q(l)\right) = \\
= \frac{2}{l r_{j_1}} \left(r_{l-m+1} l_{m+1} \tilde{G}_q(n) + \frac{n}{l_{m}} S_q(l_{m}) - \frac{n}{2^{m-1} l_{m-1}} \left((2q)^m S_q\left(\frac{l}{2^m}\right) + \right) + \\
+ 2^{m-2} l_{m-1} q (1 + q + \cdots + q^{m-1})\right) = \frac{2}{l r_{j_1}} \left(1 - q^{-1}\right) S_q\left(\frac{l}{2^m}\right) + \\
2^{m-2} l_{m-1} q (1 + q + \cdots + q^{m-1}) = -q T_a(t) + \frac{n}{l r_{j_1}} \left(\frac{l}{2^m}\right) \left((2q)^{m-1} S_q\left(\frac{l}{2^m}\right) + \\
2^{m-2} l_{m-1} q (1 + q + \cdots + q^{m-1})\right) = -q T_a(t) + \frac{n}{l r_{j_1}} q^{m-1} (q + \cdots + q^{m-1}) \quad \text{and \( T_a(t/2^{m-1}) = q T_a(t) \). }
Question. One can consider a more general function $\tilde{s}_2(\omega)$ defined by $\tilde{s}_2(\omega) = \sum_{i=1}^{\infty} c_i \omega_i$ with $\sum_{i=1}^{\infty} |c_i| < \infty$. What are the limiting curves in this case?

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