On Finite Metahamiltonian $p$-Groups *

Lijian An, Qinhai Zhang†
Department of Mathematics, Shanxi Normal University
Linfen, Shanxi, 041004 PR China

Abstract

A group is called metahamiltonian if all non-abelian subgroups of it are normal. This concept is a natural generalization of Hamiltonian groups. In this paper, the properties of finite metahamiltonian $p$-groups are investigated.

Keywords Dedekindian groups, metahamiltonian groups, $A_2$-groups

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1 Introduction

A group is called Dedekindian if every subgroup of it is normal. In 1897, Dedekind classified finite Dedekindian groups in [6]. In 1933, Baer classified infinite Dedekindian groups in [1]. A non-abelian Dedekindian group is also called Hamiltonian.

A non-abelian group is called metahamiltonian if all non-abelian subgroups of it are normal. This concept is a natural generalization of Hamiltonian groups. In the 1960’s and 70’s, many scholars researched metahamilton groups. Romalis and Sesekin [16, 17, 18] investigated some properties on infinite metahamiltonian groups, and Nagrebeckii [11, 12, 13] studied finite metahamiltonian groups. Nagrebeckii [12] proved the following theorem:

**Theorem 1.1.** Suppose that $G$ is a finite non-nilpotent group. Then $G$ is metahamiltonian if and only if $G = SZ(G)$ where $S$ is one of the following groups:

1. $P \times Q$, where $P$ is an elementary $p$-group, $Q$ is cyclic and $(p, |Q|) = 1$;
2. $Q_8 \times Q$, where $Q$ is cyclic and $(|Q|, 2) = 1$;
3. $P \times Q$, where $|P| = p^3$, $p \geq 5$, $Q$ is cyclic and $(p, |Q|) = 1$.

In [12], more detailed information on $S$ is given. Since a nilpotent group is the direct product of its Sylow subgroups, by the above theorem, to study finite metahamiltonian groups, we only need consider finite metahamiltonian $p$-groups, which is more complex than the situation of non-nilpotent.

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†Corresponding author. e-mail: zhangqh@dns.sxnu.edu.cn
Metahamiltonian $p$-groups contain many important classes of $p$-groups. For example, finite $p$-groups all of whose subgroups of index $p^2$ are abelian, are metahamiltonian. All such groups are determined, see [3, 4, 7, 9, 19, 21] for the classification. Another example is finite $p$-groups all of whose non-normal subgroups are cyclic. See [14]. The study of metahamilton $p$-groups is an old problem and many scholar consider it important. In this paper, the properties of finite metahamiltonian $p$-groups are investigated. These properties are useful in the classification of metahamilton $p$-groups [8].

2 Preliminaries

Let $G$ be a finite group. $G$ is said to be minimal non-abelian, if $G$ is non-abelian, but every proper subgroup of $G$ is abelian. A finite $p$-group $G$ is called an $A_1$-group if every subgroup of index $p^t$ of $G$ is abelian, but there is at least one non-abelian subgroup of index $p^{t-1}$. So $A_1$-groups are just the minimal non-abelian $p$-groups.

Let $G$ be a finite $p$-group. We define $\Lambda_1(G) = \{ a \in G \mid a^p = 1 \}$, $V_1(G) = \{ a^p \mid a \in G \}$, $\Omega_1(G) = \langle \Lambda_1(G) \rangle = \langle a \in G \mid a^p = 1 \rangle$, and $U_1(G) = \langle V_1(G) \rangle = \langle a^p \mid a \in G \rangle$; $G$ is called $p$-abelian if $(ab)^p = a^pb^p$ for all $a, b \in G$. We use $c(G)$ and $d(G)$ to denote the nilpotency class and minimal number of generators, respectively.

We use $M_p(m, n)$ to denote groups $\langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{n-1}} \rangle$, where $m \geq 2$, and use $M_p(m, n, 1)$ to denote groups $\langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$, where $m + n \geq 3$ for $p = 2$ and $m \geq n$. We can give a presentation of minimal non-abelian $p$-groups as follows:

Theorem 2.1. ([15])(Rédei) Let $G$ be a minimal non-abelian $p$-group. Then $G$ is $Q_8$, $M_p(m, n)$, or $M_p(m, n, 1)$.

We use $C_n$ and $C_n^m$ to denote the cyclic group and the direct product of $m$ cyclic groups of order $n$, respectively; and use $H * K$ to denote a central product of $H$ and $K$. For undefined notation and terminology the reader is referred to [10].

We have the following information about minimal non-abelian $p$-groups.

Theorem 2.2. ([20] Lemma 2.2]) Let $G$ be a finite $p$-group. Then the following conditions are equivalent:

1. $G$ is an inner abelian $p$-group;
2. $d(G) = 2$ and $|G'| = p$;
3. $d(G) = 2$ and $Z(G) = \Phi(G)$.

Lemma 2.3. ([2] p136, Proposition 10.28]) A non-abelian $p$-group is generated by minimal non-abelian subgroups.

Many scholars studied and classified $A_2$-groups, see, for example [3, 4, 7, 9, 19, 21]. We have following Lemma.
Lemma 2.4. Suppose that $G$ is an $A_2$-group. Then $G$ is one of the following groups:

(I) $d(G) = 2$ and $G$ has an abelian maximal subgroup.

1. $\langle a, b \mid a^8 = b^{2m} = 1, a^b = a^{-1} \rangle$, where $m \geq 1$;
2. $\langle a, b \mid a^8 = b^{2m} = 1, a^b = a^3 \rangle$, where $m \geq 1$;
3. $\langle a, b \mid a^8 = 1, b^{2m} = a^4, a^b = a^{-1} \rangle$, where $m \geq 1$;
4. $\langle a_1, b \mid a_1^p = a_2^p = b_1^p = b_2^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_3, b] = 1, [a_1, a_2] = 1 \rangle$, where $p \geq 5$ for $m = 1$, $p \geq 3$ and $1 \leq i, j \leq 3$;
5. $\langle a_1, b \mid a_1^p = a_2^p = b_1^{-m+1} = 1, [a_1, b] = a_2, [a_2, b] = b_1^m, [a_1, a_2] = 1 \rangle$, where $p \geq 3$;
6. $\langle a_1, b \mid a_1^p = b_1^{m+1} = 1, [a_1, b] = a_2, [a_2, b] = b_1^{-p}, [a_1, a_2] = 1 \rangle$, where $p \geq 3$ and $\nu = 1$ or a fixed quadratic non-residue modulo $p$.
7. $\langle a_1, a_2, b \mid a_1^3 = a_2^3 = 1, b_1^3 = a_3, [a_1, b] = a_2, [a_2, b] = a_1^{-3}, [a_2, a_1] = 1 \rangle$.

(II) $d(G) = 3$, $|G'| = p$ and $G$ has an abelian maximal subgroup.

8. $\langle a, b, x \mid a^4 = x^2 = 1, b^2 = a^2 = [a, b], [x, a] = [x, b] = 1 \rangle \cong Q_8 \times C_2$;
9. $\langle a, b, x \mid a^{p^{n+1}} = b^{p^n} = x^p = 1, [a, b] = a^{p^n}, [x, a] = [x, b] = 1 \rangle \cong M_p(n + 1, m) \times C_p$;
10. $\langle a, b, c, x \mid a^{p^n} = b^{p^n} = c^p = x^p = 1, [a, b] = c, [c, a] = [c, b] = [x, a] = [x, b] = 1 \rangle \cong M_p(n, m, 1) \times C_p$, where $n \geq m$, and $n \geq 2$ if $p = 2$;
11. $\langle a, b, x \mid a^4 = 1, b^2 = a^2 = [a, b], [x, a] = [x, b] = 1 \rangle \cong Q_8 \times C_4$;
12. $\langle a, b, x \mid a^{p^n} = b^{p^n} = x^{p^2} = 1, [a, b] = x^p, [x, a] = [x, b] = 1 \rangle \cong M_p(n, m, 1) \times C_{p^2}$, where $n \geq 2$ if $p = 2$ and $n \geq m$.

(III) $d(G) = 3$, $|G'| = p^2$ and $G$ has an abelian maximal subgroup.

13. $\langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = a^2, [c, b] = 1 \rangle$;
14. $\langle a, b, d \mid a^{p^n} = b^{p^n} = d^p = 1, [a, b] = a^{p^n-1}, [d, a] = b^p, [d, b] = 1 \rangle$, where $m \geq 3$ if $p = 2$;
15. $\langle a, b, d \mid a^{p^n} = b^{p^n} = d^{p^2} = 1, [a, b] = d^p, [d, a] = b^{jp}, [d, b] = 1 \rangle$, where $(j, p) = 1, p > 2, j$ is a fixed quadratic non-residue modulo $p$, and $-4j$ is a quadratic non-residue modulo $p$;
16. $\langle a, b, d \mid a^{p^n} = b^{p^n} = d^{p^2} = 1, [a, b] = d^p, [d, a] = b^{jp}d^p, [d, b] = 1 \rangle$, where if $p$ is odd, then $4j = 1 - \rho^{2r+1}$ with $1 \leq r \leq \frac{p-1}{2}$ and $\rho$ the smallest positive integer which is a primitive root (mod $p$); if $p = 2$, then $j = 1$.

(IV) $d(G) = 2$ and $G$ has no abelian maximal subgroup.
(17) \( \langle a, b \mid a^{p^{r+2}} = 1, b^{p^{r+s+1}} = a^{p^{r+s}}, [a, b] = a^{p^r} \rangle \), where \( r \geq 2 \) for \( p = 2, r \geq 1 \) for \( p \geq 3, t \geq 0, 0 \leq s \leq 2 \) and \( r + s \geq 2 \);

(18) \( \langle a, b \mid a^{p^2} = b^2 = a^p = 1, [a, b] = c, [c, a] = b^{p^p}, [c, b] = a^p \rangle \), where \( p \geq 5, v \) is a fixed square non-residue modulo \( p \);

(19) \( \langle a, b \mid a^{p^2} = b^2 = a^p = 1, [a, b] = c, [c, a] = a^{-p}b^{-lp}, [c, b] = a^{-p} \rangle \), where \( p \geq 5, 4l = p^{2r+1} - 1, r = 1, 2, \ldots, \frac{1}{2}(p-1) \), \( p \) is the smallest positive integer which is a primitive root modulo \( p \);

(20) \( \langle a, b \mid a^q = b^q = c^3 = 1, [a, b] = c, [c, a] = b^{-3}, [c, b] = a^3 \rangle \);

(21) \( \langle a, b \mid a^q = b^q = c^3 = 1, [a, b] = c, [c, a] = b^{-3}, [c, b] = a^{-3} \rangle \).

(V) \( d(G) = 3 \) and \( G \) has no abelian maximal subgroup.

(22) \( \langle a, b, d \mid a^4 = b^4 = d^4 = 1, [a, b] = d^2, [d, a] = b^2d^2, [d, b] = a^2b^2, [a^2, b] = [b^2, a] = 1 \rangle \).

Analyzing the group list in Lemma 2.4, we have following lemma.

**Lemma 2.5.** Suppose that \( G \) is an \( A_2 \)-group with order \( p^n \).

1. \( d(G) \leq 3 \) and \( c(G) \leq 3 \);
2. If \( d(G) = 2 \) and \( \exp(G') = p \), then \( c(G) = 3 \).
3. If \( c(G) > 2 \) and \( \exp(G') = p \), then \( d(G) = 2 \) and \( p \) is odd.

**Theorem 2.6.** ([11] Statz 6.5) If \( [x, y, y] = 1 \) for all \( x, y \in G \), then \( G \) is nilpotent and \( c(G) \leq 3 \). In addition, if \( G \) has no element of order \( 3 \), then \( c(G) \leq 2 \).

A finite \( p \)-group \( G \) is called metacyclic if it has a cyclic normal subgroup \( N \) such that \( G/N \) is also cyclic.

**Lemma 2.7.** ([5]) Suppose that \( G \) is a finite \( p \)-group. Then \( G \) is metacyclic if and only if \( G/\Phi(G')G_3 \) is metacyclic.

## 3 Properties of finite metahamiltonian \( p \)-groups

**Theorem 3.1.** Let \( G \) be a finite metahamiltonian \( p \)-group. Then sections of \( G \) are all metahamiltonian.

**Proof** It is straight forward. \( \square \)

**Theorem 3.2.** Let \( G \) be a finite \( p \)-group. Then \( G \) is metahamiltonian if and only if every minimal non-abelian subgroup is normal in \( G \).

**Proof** If \( G \) is metahamiltonian, then, by the definition of metahamiltonian, every minimal non-abelian subgroup is normal in \( G \). On the other hand, if every minimal non-abelian subgroup is normal in \( G \), then, by Lemma 2.3, every non-abelian subgroup is normal in \( G \). \( \square \)
Theorem 3.3. Let $G$ be a finite metahamiltonian $p$-group. Then, for all $x \in G$, $\langle x \rangle^{G}$ is abelian or minimal non-abelian.

Proof Suppose that $\langle x \rangle^{G}$ is not abelian. Then there exists $g \in G$ such that $[x, x^g] \neq 1$. Let $K = \langle x, x^g \rangle$. Then $K$ is normal in $G$ since $G$ is metahamiltonian. Hence $K = \langle x \rangle^{G}$. Let $y = x^g$ and $L = \langle x, x^y \rangle = \langle x, [x, y] \rangle$. Then $L < K$ and hence $L$ is not normal in $G$. It follows that $L$ is abelian. That is, $[x, y, x] = 1$. Since $\langle y \rangle^{G} = \langle x^y \rangle^{G} = \langle x \rangle^{G}$, similarly we have $[x, y, y] = 1$. Hence $c(K) = 2$.

Let $S = \langle x, y^p \rangle$. Then $S < K$ and hence $S$ is not normal in $G$. It follows that $S$ is abelian and hence $[x, y^p] = 1$. Since $c(K) = 2$, we get $[x, y]^p = 1$. Thus $K' = \langle [x, y] \rangle$ is of order $p$. By Theorem 2.2, $K$ is minimal non-abelian.

Theorem 3.4. Let $G$ be a metahamiltonian $p$-group. Then $c(G) \leq 3$. In particular, $G$ is metabelian.

Proof By Theorem 3.3, for all $x \in G$, $K = \langle x \rangle^{G}$ is abelian or minimal non-abelian. Then $K' = 1$ or $|K'| = p$. Since $K' \leq G$, we get $K' \leq Z(G)$. Let $\tilde{G} = G/Z(G)$. Then, for all $\bar{x} \in \tilde{G}$, $\langle \bar{x} \rangle^{G}$ is abelian. Hence $\tilde{G}$ satisfies the 2-Engel condition. By Theorem 2.2, $c(\tilde{G}) = 2$ for $p \neq 3$ and $c(\tilde{G}) \leq 3$ for $p = 3$. It follows that $c(G) \leq 3$ for $p \neq 3$ and $c(G) \leq 4$ for $p = 3$.

We claim that $c(G) \leq 3$. If not, then $p = 3$ by the above argument. Let $G$ be a counterexample with minimal order. By Theorem 2.1, $c(G) = 4$, $|G_4| = p$ and the nilpotency class of every proper section of $G$ is at most 3. Hence we may assume that $G_4 = \langle [a, b, c, d] \rangle$, where $a, b, c, d \in G \setminus \Phi(G)$. Let $x = [a, b, c]$. Then $N = \langle x, d \rangle$ is minimal non-abelian by Theorem 2.2. By hypothesis, every subgroup which contains $N$ is normal in $G$. It follows that $G/N$ is Dedekindian. Since $p = 3$, $G/N$ is abelian. It follows that $G' \leq N$. Since $d \notin \Phi(G)$, we have $N \cap \Phi(G) < N$ and hence $G' \leq N \cap \Phi(G) < N$. It follows that $G'$ is abelian. Then $[[c, d], [a, b]] = 1$. Since $[a, b] \in G' < N$ and $d \in N$, $[d, [a, b]] \in N' \leq Z(G)$. It follows that $[d, [a, b], c] = 1$. By Witt’s formula, we have $[[a, b], c, d] = 1$, a contradiction.

Theorem 3.5. Let $G$ be a finite $p$-group. $G$ is metahamiltonian if and only if $G'$ is contained in every non-abelian subgroup of $G$.

Proof If $G'$ is contained in every non-abelian subgroup of $G$, then every non-abelian subgroup of $G$ is normal in $G$. Hence sufficiency holds. In the following we prove the necessity.

Let $G$ be a counterexample with minimal order. Then $G$ is metahamiltonian and there exists a minimal non-abelian subgroup $N = \langle a, b \rangle$ such that $G' \not\leq N$. Since $G$ is metahamiltonian, subgroups containing $N$ are normal in $G$. Hence $G/N$ is Hamiltonian.

By the minimality of $G$, $G/N \cong Q_8$. Let $G/N = \langle x, y \rangle$ and $H = \langle x, y \rangle$. Then $G = HN$, $H/(H \cap N) \cong Q_8$, $z := [x, y] \not\in N$, $H \cap N \leq \Phi(H)$ and $H \cap N$ =
\[ \langle x^4, x^2y^2, x^2[x, y] \rangle^H. \] Since \( z \in \langle x \rangle^H \), it follows from Theorem 3.4 that \( z, x \) is abelian or minimal non-abelian. Hence \( [z, x^2] = [z, x]^2 = 1 \). The same reason gives that \( [z, y^2] = [z, y]^2 = 1 \) and hence \( \exp(H_3) \leq 2 \). Since \( \Phi(H) = \langle x^2, y^2, H' \rangle \) and \( H' \) is abelian (by Theorem 3.4), we have \( \Phi(H), z \) = 1. In particular, \( [H \cap N, z] = 1 \). In the following, we deduce a contradiction on five cases:

Case 1. \( H \cap N = N \).

In this case, \([N, z] = 1 \). Let \( M = \langle za, b \rangle \). Then Theorem 2.2 gives that \( M \) is minimal non-abelian, and hence \( G/M \) is also Dedekindian. Since \( z \not\in M \), \( G/M \) is not abelian. By the minimality of \( G \), \( H/M = G/M \cong Q_8 \). It follows that \( M = \langle x^4, x^2y^2, x^2[x, y] \rangle^H = N = \langle a, b \rangle \), a contradiction.

Case 2. \( H \cap N < N \) and \( H \cap N \not\leq \Phi(N) \).

In this cases, \( H \cap N \) contains a generator of \( N \). Without losing generality, we assume that \( a \in H \cap N \) and \( b \not\in H \cap N \). Then \([z, a] = 1 \). Since \( H \cap N \) is abelian, we have \([x^2y^2, x^2[x, y]] = 1 \), and hence \([x^2, y^2] = 1 \). By calculation, we have \([x^2, y^2] = [x^2, y^2] = [x, y]^4 = z^4 \). If \( z^2 \not= 1 \), then \( \langle z^2 \rangle = \mathbb{O}_1(H') \) is a minimal normal subgroup of \( H \). Hence we have \( z^2 \in Z(G) \). Particularly, \([z, b]^2 = [z^2, b] = 1 \).

Subcase 2.1. \([z, b] \not= [a, b] \).

Let \( M = \langle za, b \rangle \). By Theorem 2.2, \( M \) is minimal non-abelian and hence \( G/M \) is also Dedekindian. Since \( z \not\in M \), \( G/M \) is not abelian. By the minimality of \( G \), we have \( G/M = HM = H/(H \cap M) \cong Q_8 \). It follows that \( H \cap M = \langle x^4, x^2y^2, x^2[x, y] \rangle^H = H \cap N \), and hence \( a \in H \cap N = H \cap M \leq M \). Thus \( z = (za)a^{-1} \in M \), a contradiction.

Subcase 2.2. \([z, b] = [a, b] \).

Let \( L = \langle z, b \rangle \cap N \). Then \( L \) is normal in \( G \). Let \( K \) be a maximal subgroups of \( N \) which contains \( L \) such that \( K \leq G \). Then \( G/K \) is of order \( 2^4 \), has two generators, and has a quotient group which is isomorphic to \( Q_8 \). By the classification of groups of order \( 2^4 \), \( G/K = \langle xK, yK \rangle := \langle \bar{x}, \bar{y} \rangle \cong M_2(2, 2) \), which has definition relations \( \bar{x}^4 = \bar{y}^4 = 1 \) and \([\bar{x}, \bar{y}] = \bar{x}^2 \). Obviously, \( \langle \bar{y} \rangle \) and \( \langle \bar{x}\bar{y} \rangle \) are not normal in \( G/K \). It follows that their complete inverse images are also not normal in \( G \), hence are abelian. It follows that \( [y, K] = 1 \), \( [xy, K] = 1 \). Thus \([H, K] = 1 \), which is contrary to \([z, b] = [a, b] \not= 1 \).

Case 3. \( H \cap N < \Phi(N) \).

We claim that \( H \cap N \not= 1 \). Otherwise, \( G = H \times N \). Since \( N \cong G/H \) is Dedekind, we have \( N \cong Q_8 \). In this case, \( \langle xa, yb \rangle \cong Q_8 \) is not normal in \( G \), a contradiction.

We claim that \( N' \leq H \cap N \). Otherwise, \( G/(H \cap N) \) is also a counterexample, which is contrary to the minimality of \( G \).

Let \( \overline{G} = G/(H \cap N) \), \( \overline{H} = H/(H \cap N) = \langle \bar{x}, \bar{y} \rangle \) and \( \overline{N} = N/(H \cap N) = \langle \bar{a} \rangle \times \langle \bar{b} \rangle \). Then \( \overline{G} = \overline{H} \times \overline{N} \) and \( \exp(\overline{N}) \geq 4 \). Without loss of generality, we may assume that \( o(\bar{a}) \geq 4 \). Let \( \overline{K} = \langle \bar{x}\bar{a} \rangle \times \langle \bar{b} \rangle \). Then \( \overline{K} \) is not normal in \( \overline{G} \). It follows that its complete
inverse image is not normal in $G$. Hence $[xa, b] = 1$. That is, $[x, b] = [a, b]$. The same reason gives that $[y, b] = [a, b]$ and $[xy, b] = [a, b]$, a contradiction.

Case 4. $H \cap N = \Phi(N) = N'$.

In this case, $|N| = 2^3$, $|H| = 2^4$, $|G| = 2^6$ and $G/N' = H/N' \times \langle aN' \rangle \times \langle bN' \rangle$. Since $\langle aN' \rangle$ and $\langle bN' \rangle$ are normal in $G/N'$, $A := \langle aN' \rangle$ and $B := \langle bN' \rangle$, their complete inverse images, are also normal in $G$. Noting that $A$ and $B$ are of order 4, the NC-Theorem gives that $C_G(A)$ and $C_G(B)$ are maximal in $G$. Let $K = C_G(A) \cap C_G(B)$. Then $|K| \geq 2^4$. Since $K \cap N = Z(N) = N'$, we have $|KN| = (|K||N|)/|K \cap N| \geq 2^6$, and hence $G = K \ast N$. Since $KN/N \cong K/K \cap N \cong \mathbb{Q}_8$, without loss of generality we may assume that $H = K$. By the classification of groups of order $2^4$, $H = \langle x, y \rangle \cong M_2(2, 2)$, which has definition relations $x^4 = y^4 = 1, [x, y] = x^2$ and $N' = H \cap N = \langle x^2y^2 \rangle$.

Without loss of generality, we may assume that $a \in N$ is of order 4. Then $a^2 = x^2y^2$. By calculations, we have $[x, ay] = x^2$ and $(ay)^2 = x^2$. It follows that $\langle x, ay \rangle$ is neither abelian nor normal in $G$, a contradiction.

Case 5. $H \cap N = \Phi(N) \neq N'$.

Let $\overline{G} = G/K$, $\overline{H} = H/K = \langle \overline{x}, \overline{y} \rangle$ and $\overline{N} = N/K = \langle \overline{a} \rangle \times \langle \overline{b} \rangle$, where $K$ is a maximal subgroup of $H \cap N$ such that $K \leq G$. By the minimality of $G$, $\overline{G}$ is contained in every minimal non-abelian subgroup of $\overline{G}$. Since $G' \leq N$, we have $\overline{G'} \leq \overline{N}$ and hence $\overline{N}$ is abelian. Without loss of generality, we may assume that $o(\overline{a}) = 4$. By the classification of groups of order $2^4$, $\overline{H} \cong M_2(2, 2)$, which has definitions $\overline{x}^4 = \overline{y}^4 = 1$, $[\overline{x}, \overline{y}] = \overline{x}^2$, $\overline{a}^2 = \overline{x}^2\overline{y}^2$, and $\Phi(N)/K = (H \cap N)/K = \langle \overline{x}^2\overline{y}^2 \rangle$. If $\overline{a} \in Z(\overline{G})$, then $\langle \overline{x}, \overline{a}\overline{y} \rangle$ is neither abelian nor normal in $\overline{G}$, a contradiction. Hence $\overline{a} \not\in Z(\overline{G})$. If $[\overline{a}, \overline{x}] = 1$, then $[\overline{a}, \overline{y}] = \overline{x}^2\overline{y}^2$ and hence $\langle \overline{a}\overline{x}, \overline{y} \rangle$ is neither abelian nor normal in $\overline{G}$, a contradiction.

Hence $[\overline{a}, \overline{x}] = \overline{x}^2\overline{y}^2$. The same reason gives that $[\overline{a}\overline{b}, \overline{x}] = \overline{x}^2\overline{y}^2$. It follows that $[\overline{b}, \overline{x}] = 1$. If $[\overline{b}, \overline{y}] \neq 1$, then $[\overline{b}, \overline{y}] = \overline{x}^2\overline{y}^2$. By calculation, $\langle \overline{x}, \overline{b}\overline{y} \rangle$ is neither abelian nor normal in $\overline{G}$, a contradiction. Hence $[\overline{b}, \overline{y}] = 1$.

In this case, it is easy to see that $\langle \overline{x}, \overline{b} \rangle$ and $\langle \overline{a}\overline{x}, \overline{b} \rangle$ are not normal in $\overline{G}$. It follows that their complete inverse images are not normal in $G$, and hence they are abelian. Thus $[x, b] = 1$ and $[ax, b] = 1$, which is contrary to $[a, b] \neq 1$.

$\square$

**Theorem 3.6.** Suppose that $G$ is a finite metahamilton $p$-group. If $d(G) = 2$ and $\exp(G') > p$, then $G$ is metacyclic.

**Proof** Assume that $G = \langle a, b \rangle$ is a counterexample with minimal order. By Lemma 2.7, $\overline{G} := G/\Phi(G')G_3$ is not metacyclic. Since $|\overline{G}| = p$, $\overline{G}$ is minimal non-abelian. By Theorem 2.1, $\overline{G} \cong M_p(n, m, 1)$. That is, we may assume that $\overline{G} = \langle \overline{a}, \overline{b} \mid \overline{a}^{2^n} = \overline{b}^{2^m} = \overline{c}^p = 1, [\overline{a}, \overline{b}] = \overline{c}, [\overline{c}, \overline{a}] = [\overline{c}, \overline{b}] = 1 \rangle$. Since $\langle \overline{a}^p, \overline{b} \rangle$, $\langle \overline{b}^p, \overline{a} \rangle$, $\langle \overline{a}\overline{b}^p, \overline{a} \rangle$ and $\langle \overline{a}\overline{b}^p, \overline{b} \rangle$ are not normal in $\overline{G}$, we have $\langle \overline{a}^p, \overline{b} \rangle = \Phi(G')G_3$, $\langle \overline{b}^p, \overline{a} \rangle = \Phi(G')G_3$, $\langle \overline{a}\overline{b}^p, \overline{a} \rangle = \Phi(G')G_3$ and $\langle \overline{a}\overline{b}^p, \overline{b} \rangle = \Phi(G')G_3$ are not normal in $G$. Hence they are all abelian. Thus we have

\begin{align*}
\end{align*}
\[ \Phi(G')G_3 \leq Z(G) \text{ and} \]
\[ [a^p, b] = [b^p, a] = [(ab)^p, a] = [(ab)^p, b] = 1 \quad (\ast) \]

If \( p = 2 \), then \((ab)^2 = a^2b^2[a, b]. \) By \((\ast)\), \([a, b] \in Z(G). \) Hence \( G' = \langle [a, b] \rangle. \) By \((\ast)\), \([a, b]^2 = [a^2, b] = 1 \), which is contrary to \( \exp(G') > 2. \)

If \( p > 2 \), then, by calculation, we have \([a, b]^p = [a^p, b, a] = 1 \) and \([a, b]^p = [a^p, b, b] = 1 \). It follows that \( \exp(G_3) \leq p. \) By \((\ast)\), \([a, b]^p = [a^p, b] = 1 \), which is contrary to \( \exp(G') > p. \)

**Lemma 3.7.** Suppose that \( G \) is a finite metahamiltonian \( p \)-group which has elementary abelian derived group. If \( G \) is not an \( A_2 \)-group, then \( A_2 \)-subgroups of \( G \) have nilpotency class 2.

**Proof** Assume the contrary. Then there exists \( K < G \) such that \( K \in A_2 \), \( \exp(K') = p \) and \( c(K) \geq 3. \) Hence \( p > 2 \) and \( K \) is a group of Type (4)–(7) or (18)–(21) in Lemma 2.3.

Case 1: \( K = \langle a_1, b \rangle \) is a group of Type (4)–(7) in Lemma 2.3.

Let \( H \leq G \) such that \( K < H \). Since \( K/Z(K) \) is minimal non-abelian, non-metacyclic and of order \( p^3 \), \( H/Z(K) \) is not metacyclic and of order \( p^4. \) If \( d(H/Z(K)) = 2 \), then, by the classification of groups of order \( p^4 \), \( H/Z(K) \) is of maximal class. It follows that \( K'Z(K)/Z(K) = H_3Z(K)/Z(K). \) Hence \([a_1, b] \in H_3Z(K) \) and \([a_1, b, b] \in H_4 \). Since \([a_1, b, b] \notin 1 \), we have \( H_4 \neq 1 \) and \( c(H) \geq 4 \), which is contrary to Theorem 3.3. If \( d(H/Z(K)) = 3 \), then, by the classification of groups of order \( p^4 \), there exists \( d \in H \) such that \( H/Z(K) = K/Z(K) \times \langle dZ(K) \rangle \) or \( H/Z(K) = K/Z(K) \times \langle dZ(K) \rangle. \) By calculation, \([d^p, k] = [d, k]^p = 1 \) for all \( k \in K. \) It follows that \( d^p \in Z(K) \) and \( H/Z(K) = K/Z(K) \times \langle dZ(K) \rangle. \) Since \( a_2 = [a_1, b] \notin \langle a_1, d \rangle \), by Theorem 3.5 we have \([a_1, d] = 1. \) The same reason gives that \([b, d] = 1. \) Hence \( d \in Z(H). \) In this case \( \langle a_2d, b \rangle \) is neither abelian nor normal, a contradiction.

Case 2: \( K = \langle a_1, b \rangle \) is a group of Type (18)–(21) in lemma 2.4.

Let \( H \leq G \) such that \( K < H \). By Theorem 3.5, \( H' \leq \langle c, a \rangle \cap \langle c, b \rangle = \langle c, a^p, b^p \rangle. \) It follows that \( H' = K' \) and \( H_3 = K_3 = \langle a^p, b^p \rangle. \) By the classification of groups of order \( p^4 \), there exists \( d \in H \setminus K \) such that \([a, d] \equiv [b, d] \equiv 1 \) (mod \( K_3). \) By calculation, \([a, d^p] = [a, d]^p = 1 \) and \([b, d^p] = [b, d]^p = 1. \) It follows that \( d^p \in Z(K) = K_3. \) Since \( c \notin \langle a, d \rangle\), by Theorem 3.5 we have \([a, d] = 1. \) The same reason gives that \([ac, d] = 1. \) In this case \( \langle a, cd \rangle \) is neither abelian nor normal, a contradiction.

**Theorem 3.8.** Suppose that \( G \) is a finite metahamiltonian \( p \)-group having an elementary abelian derived group. If \( c(G) = 3 \), then \( G \) is an \( A_2 \)-group.

**Proof** Assume the contrary and \( G \) is a counterexample with minimal order. Then \( c(G) = 3 \) and \( G \in A_3. \)
We claim that $G$ does not satisfy the 2-Engel condition. If not, then, by Theorem 2.6, $G$ is a 3-group. In this case, there exist $x, y, z \in G$ such that $[x, y, z] \neq 1$. Since $G$ has minimal order, we have $G = \langle x, y, z \rangle$ and $[x, y, z]^3 = [x^3, y, z] = 1$. Since $[x, yz, yz] = 1$, by calculation, we get $[x, y, z] = [z, x, y]$. Similar reasons give that $[x, y, z] = [y, z, x] = [z, x, y] = 1$. Then $G' = \langle a, b, c, d \rangle$. Since $[b, y] = d \neq 1$, we have $\langle b, y \rangle \leq G$. It follows that $c = [x, y] \in \langle b, y \rangle$. Since $[c, b] = [c, y] = 1$, we may assume that $c = y^3d$. Hence $d = [c, z] = [y^3d^w, z] = [y^3t, z] = 1$, a contradiction.

Since $G$ does not satisfy the 2-Engel condition, there exist $x, y, z \in G$ such that $[x, y, z] \neq 1$. Since $G$ has minimal order, we have $G = \langle x, y, z \rangle$, $[x, y, z]^p = 1$ and $[x, y, z]^p = 1$. Let $[x, y] = c, [c, y] = b$ and $[c, x] = a$. Then $G_3 = \langle b, a \rangle$ and $G' = \langle c, G_3 \rangle$. If $[c, x] \in \langle b \rangle$, then, by suitable replacement, we may assume that $[c, x] = a = 1$. Hence we may assume that $\langle a \rangle \cap \langle b \rangle = 1$.

The maximal subgroups of $G$ are $M = \langle x^iy, \Phi(G) \rangle$ and $K = \langle x, \Phi(G) \rangle$, where $i = 0, 1, \ldots, p$. It is easy to see that $\Phi(G) = \langle x^p, y^p, c, a, b \rangle$ is abelian. Since $[c, x^iy] = a^i b \neq 1$, by Lemma 2.5 (2), we have that $N = \langle c, x^iy \rangle \in A_1$. By Theorem 3.3, $G' \leq N$. Since $[cx^p, x^iy] = ba^{i+1} \neq 1$, by Lemma 2.5 (2), we have $\langle cx^p, x^iy \rangle \in A_1$. By Lemma 3.3, $\langle cx^p, x^iy \rangle = \langle x^iy \rangle^G = N$. It follows that $x^p \in N$. Since $(x^iy)^p \equiv x^{ip}y^p \pmod{G'}$, we have $x^{ip}y^p \in N$ and hence $y^p \in N$. Thus $\Phi(G) \leq N$ and $M = N \in A_1$.

If $[c, x] = a \neq 1$, then, by Lemma 2.5 (2), $\langle c, x \rangle \in A_1$. By Theorem 3.3, $G' \leq L$. Since $[cy^p, x] \neq 1$, by Lemma 2.5 (2), $\langle cy^p, x \rangle \in A_1$. By Lemma 3.3, $\langle cy^p, x \rangle = \langle x \rangle^G = \langle c, x \rangle$. It follows that $y^p \in \langle c, x \rangle$ and hence $\Phi(G) \leq \langle c, x \rangle$. Thus $K = \langle c, x \rangle \in A_1$.

If $[c, x] = a = 1$ and $p > 2$, then $[x, y^p] = 1$. Hence $\Phi(G), x = 1$ and $K$ is abelian. If $[c, x] = a = 1$ and $p = 2$, then $[x, y^2] = b \neq 1$. By Lemma 2.5 (2), $\langle x, y^2 \rangle \in A_1$. By 3.3, $G' \leq \langle x, y^2 \rangle$. Hence $K = \langle x, y^2 \rangle \in A_1$.

By the above argument, all maximal subgroup of $G$ are abelian or minimal non-abelian. Hence $G \in A_2$, a contradiction. □

**Corollary 3.9.** Suppose that $G$ is a finite metahamiltonian $p$-group having an elementary abelian derived group. If $c(G) = 3$, then $d(G) = 2$ and $p$ is odd.

**Proof** By Theorem 3.8, $G \in A_2$. Then the results follow from Corollary 2.5 (3). □

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