On the electrostatic potential and electric field of a uniformly charged disk

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Abstract
We calculate the electrostatic potential and electric field of a uniformly charged disk everywhere in space. This electrostatic problem was solved long ago, and its gravitational analogue was solved even earlier. However, it seems that physics students are not aware of the solution, because it is not presented in textbooks. The purpose of the present article is to fill this gap in the pedagogical literature.

Keywords: classical electrodynamics, potential theory, electrostatics

1. Introduction
The problem of finding the electric field on the axis of symmetry of a uniformly charged circular disk is considered in many introductory textbooks on classical electrodynamics (see, for example, [1–6]). Naturally, students may wonder how the electric field (or the electrostatic potential) can be found in a more general case for points that do not lie on the axis of symmetry. Surprisingly, students will not find the answer even in more advanced textbooks [7–16]. In [12], a hint is given without any further details that the electric field around a homogeneously charged disk generally requires expression in terms of elliptic integrals.

The desired general answer in terms of elliptic integrals can be found in a comprehensive French treatise [17], in Russian textbooks on potential theory [18, 19], or in special scientific literature [20–22]. However, we are afraid that none of them are especially accessible to the average student.

In an interesting exercise, 2.53 in [6] (marked as a difficult four-star problem), it is proposed to show that the radial component of the electric field of a uniformly charged disk at a point $P$ located at a distance $\eta R$ from the center of the disk (where $0 \leq \eta < 1$ and $R$ is the radius of the disk) and at an infinitesimal distance away from the plane of the disk, has the form...
Figure 1. Dividing the disk into infinitely small wedges centered at point $P$. The black patch at distance $r$ from $P$ is a wedge element containing the charge $dq = \sigma r dr d\theta$.

\[
E_r = \frac{\sigma}{2\pi \varepsilon_0} \int_0^{\pi/2} \ln \left( \frac{\sqrt{1 - \eta^2 \sin^2 \theta + \eta \sin \theta}}{\sqrt{1 - \eta^2 \sin^2 \theta - \eta \sin \theta}} \right) \cos \theta \, d\theta, \quad (1)
\]

where $\sigma$ is surface charge density on the disk.

Instead of using cylindrical coordinates associated with the center of the disk, which might seem natural due to the symmetry of the problem, a hint in the book suggests dividing the disk into infinitely small wedges centered at point $P$, as shown in figure 1, finding the sum of the fields from the two opposite wedges, and finally integrating over the angle $\theta$.

This clever trick can already be found in [23], and it was again discovered in the form of the so-called strip function technique in [24] when studying the electrostatics and magnetostatics of a conducting disk. This method (without much emphasis) was used in [20] when calculating the gravitational potential due to uniform disk.

The purpose of this note is to demonstrate that with a certain level of moderate mathematical complexity (basic knowledge of elliptic integrals), the same technique allows one to find the electrostatic potential and electric field of a uniformly charged disk everywhere in space, and not just on the disk.

2. Electrostatic potential on the disk

An element of the right wedge in figure 1, at distance $r$ from $P$, contains an amount of charge $dq = \sigma r dr d\theta$, and thus creates at $P$ the electrostatic potential $dq/(4\pi \varepsilon_0 r) = \sigma dr d\theta/(4\pi \varepsilon_0)$. The contribution of the entire right wedge is then

\[
d\phi_R(P) = \frac{\sigma d\theta}{4\pi \varepsilon_0} \int_0^{r_1} dr = \frac{\sigma r_1}{4\pi \varepsilon_0} d\theta, \quad (2)
\]

The same consideration applies to the left wedge with the result

\[
d\phi_L(P) = \frac{\sigma r_2}{4\pi \varepsilon_0} d\theta. \quad (3)
\]
When \( \theta \) changes from 0 to \( \pi/2 \), half of the disk’s area is covered. It is evident from the symmetry that another half gives the same contribution to the potential at \( P \). Therefore, finally,

\[
\phi(P) = \frac{\sigma}{2\pi \epsilon_0} \int_0^{\pi/2} [r_1(\theta) + r_2(\theta)] \, d\theta.
\]  

(4)

The functions \( r_1(\theta) \) and \( r_2(\theta) \) can be found from figure 2.

Namely, using the law of cosines, we get

\[
R^2 = r_1^2 + \eta^2 R^2 + 2\eta r_1 R \cos \theta = r_2^2 + \eta^2 R^2 - 2\eta r_2 R \cos \theta.
\]  

(5)

Solving quadratic equations (5) for \( r_1 \) and \( r_2 \) (note that \( \eta \cos \theta < \sqrt{1 - \eta^2 \sin^2 \theta} \), since \( \eta < 1 \)), we get

\[
r_1 = R \left[ \sqrt{1 - \eta^2 \sin^2 \theta} - \eta \cos \theta \right], \quad r_2 = R \left[ \sqrt{1 - \eta^2 \sin^2 \theta} + \eta \cos \theta \right].
\]  

(6)

Therefore, (4) takes the form

\[
\phi(P) = \frac{\sigma R}{\pi \epsilon_0} \int_0^{\pi/2} \sqrt{1 - \eta^2 \sin^2 \theta} \, d\theta = \frac{\sigma R}{\pi \epsilon_0} E(\eta),
\]  

(7)

where

\[
E(\kappa) = \int_0^{\pi/2} \sqrt{1 - \kappa^2 \sin^2 \theta} \, d\theta
\]  

(8)

is the complete elliptic integral of the second kind [25, 26].

For the radial component of the electric field at the point \( P \), we get from (7) the following expression:

\[
E_r(P) = -\frac{1}{R} \frac{\partial \phi}{\partial \eta} = \frac{\sigma \eta}{\pi \epsilon_0} \int_0^{\pi/2} \frac{\sin^2 \theta}{\sqrt{1 - \eta^2 \sin^2 \theta}} \, d\theta = \frac{\sigma}{\pi \epsilon_0} K(\eta) - E(\eta),
\]  

(9)
where

\[ K(k) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \]  

(10)

is the complete elliptic integral of the first kind, and, at the last step, we have used the relation

\[ \frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k} \]  

(11)

At first sight, (1) and (9) look different. However, we can write (1) in the form

\[ \phi(R, z) = \frac{\sigma R^2}{2\pi \epsilon_0} \left( \int_{0}^{\pi/2} \sqrt{r_1^2 + z^2} \, d\theta - \frac{\pi |z|}{2} \right) \]  

(16)

and this makes clear that (1) and (9) are, in fact, equivalent.

3. Electrostatic potential above or below the disk

Now let us assume that the point \( P \) in figure 1 is just the projection on the disk of the point where we want to find the electrostatic potential. Then, instead of (2) and (3), we will have

\[ d\phi_R = \frac{\sigma d\theta}{4\pi \epsilon_0} \int_{0}^{r_1} \frac{r}{\sqrt{r^2 + z^2}} \, dr = \frac{\sigma d\theta}{4\pi \epsilon_0} \left( \sqrt{r_1^2 + z^2} - |z| \right), \]  

(14)

and

\[ d\phi_L = \frac{\sigma d\theta}{4\pi \epsilon_0} \int_{0}^{r_2} \frac{r}{\sqrt{r^2 + z^2}} \, dr = \frac{\sigma d\theta}{4\pi \epsilon_0} \left( \sqrt{r_2^2 + z^2} - |z| \right), \]  

(15)

Therefore, in this case, the electrostatic potential takes the form

\[ \phi(\eta R, z) = \frac{\sigma}{2\pi \epsilon_0} \int_{0}^{\pi/2} \left( \sqrt{r_1^2 + z^2} + \sqrt{r_2^2 + z^2} - 2|z| \right) \, d\theta, \]  

(16)

where \( r_1(\theta) \) and \( r_2(\theta) \) are given by (6), and it is clear from these expressions that \( r_2(\pi - \theta) = r_1(\theta) \). Then,

\[ \int_{0}^{\pi/2} \sqrt{r_1^2 + z^2} \, d\theta = \int_{\pi/2}^{\pi} \sqrt{r_1^2 + z^2} \, d\theta, \]  

(17)

and (16) can be rewritten as

\[ \phi(\eta R, z) = \frac{\sigma}{2\pi \epsilon_0} \left( \int_{0}^{\pi/2} \sqrt{r_1^2 + z^2} \, d\theta - \pi |z| \right). \]  

(18)
To calculate the integral in (18), it is convenient to introduce the angle \( \psi \) shown in figure 3 [23] and use \( \phi = \pi/2 - \psi \) as the new integration variable instead of \( \theta \).

Since \( OP = \eta R \), it is clear from figure 3 that

\[
\tan \theta = \frac{R \sin 2\psi}{R \cos 2\psi - \eta R} = \frac{\sin 2\psi}{\cos 2\psi - \eta},
\]

and therefore

\[
d\theta = 2 \frac{1 - \eta \cos 2\psi}{1 + \eta^2 - 2\eta \cos 2\psi} d\psi = -2 \frac{1 + \eta \cos 2\phi}{1 + \eta^2 + 2\eta \cos 2\phi} d\phi.
\]

Figure 3 shows that when \( \theta \) changes from 0 to \( \pi \), \( \psi \) changes from 0 to \( \pi/2 \) and, correspondingly, \( \phi \) changes from \( \pi/2 \) to 0. Therefore, the integral in (18) can be written in the following way:

\[
\int_0^\pi \sqrt{r_1^2 + z^2} d\theta = \frac{2R \sqrt{(1 + \eta^2)^2 + z^2/R^2}}{(1 + \eta^2)} \int_0^{\pi/2} (1 + \eta - 2\eta \sin^2 \phi)(1 - k^2 \sin^2 \phi) \sqrt{1 - n^2 \sin^2 \phi} d\phi.
\]

Here, we have used \( r_1^2 = R^2(1 + \eta^2 - 2\eta \cos 2\psi) = R^2(1 + \eta^2 + 2\eta \cos 2\phi) \) (from figure 3), \( \cos 2\phi = 1 - 2\sin^2 \phi \), and introduced notations [20]

\[
n^2 = \frac{4\eta}{(1 + \eta)^2} < 1, \quad k^2 = \frac{4\eta}{(1 + \eta)^2 + z^2/R^2} < 1,
\]

which allow us to write

\[
1 + \eta^2 + 2\eta \cos 2\phi = (1 + \eta)^2(1 - n^2 \sin^2 \phi),
\]

\[
r_1^2 + z^2 = R^2 \left[ (1 + \eta)^2 + \frac{z^2}{R^2} \right] (1 - k^2 \sin^2 \phi).
\]
To express (21) in terms of complete elliptic integrals, let us decompose
\[
\frac{(1 + \eta - 2\eta \sin^2 \phi)(1 - k^2 \sin^2 \phi)}{1 - n^2 \sin^2 \phi} = A + B(1 - k^2 \sin^2 \phi) + \frac{C}{1 - n^2 \sin^2 \phi}.
\] (24)

For unknown coefficients $A, B, C$ we get a system of equations:
\[
\begin{align*}
Bn^2 &= 2\eta, \\
A + B + C &= 1 + \eta, \\
An^2 + B(n^2 + k^2) &= 2\eta + (1 + \eta)k^2.
\end{align*}
\] (25)

The solution of this system is easily found to be
\[
A = \frac{k^2}{2n^2}(1 - \eta^2), \quad B = \frac{2\eta}{n^2} = \frac{(1 + \eta)^2}{2}, \quad C = \frac{1}{2}(1 - \eta^2) \left[1 - \frac{k^2}{n^2}\right].
\] (26)

Given (21), (22), (24), and (26), the electrostatic potential (18) can be expressed in terms of complete elliptic integrals as follows (in full agreement with the results of [20]):
\[
\phi(\eta R, z) = \frac{\sigma R}{2\pi \varepsilon_0} \left[\frac{1 - \eta^2}{\sqrt{(1 + \eta^2 + z^2/R^2) K(k)}} + \frac{\eta}{1 + \eta} \sqrt{1 + \eta^2 + z^2/R^2} E(k) \right] + \frac{1}{\sqrt{(1 + \eta^2 + z^2/R^2)}} \Pi(n^2, k) - \pi \frac{|z|}{R}. \] (27)

Here,
\[
\Pi(n^2, k) = \int_0^{\pi/2} \frac{d\theta}{(1 - n^2 \sin^2 \phi)\sqrt{1 - k^2 \sin^2 \theta}}
\] (28)

is the complete elliptic integral of the third kind [25, 26].

If the observation point $P$ is situated on the $z$-axis, then $\eta = 0$, $k^2 = n^2 = 0$, and using $K(0) = E(0) = \Pi(0, 0) = \pi/2$, (27) simplifies to the well-known expression
\[
\phi(0, z) = \frac{\sigma}{2\pi \varepsilon_0} \left(\sqrt{z^2 + R^2} - |z|\right). \] (29)

4. Electric field above or below the disk

Due to the rather complicated character of the expression (27) for the electrostatic potential, it will be easier to calculate the components of the corresponding electric field using (18) instead of the integrated form (27). Namely, for the radial component of the electric field, it follows from (18) that
\[
E_r(\eta R, z) = -\frac{1}{R} \frac{\partial}{\partial \eta} \phi(\eta R, z) = -\frac{\sigma}{2\pi \varepsilon_0 R} \int_0^\pi \frac{r_1}{r_1^2 + z^2} \frac{\partial}{\partial \phi} \phi(\eta R, z) \, d\theta.
\] (30)
In this expression, the partial derivative \( \left( \frac{\partial r}{\partial \eta} \right)_\theta \) should be calculated with constant \( \theta \). On the other hand, it is clear from the figure 3 that

\[ r_1^2 = R^2 (1 + \eta^2 - 2\eta \cos 2\psi), \]  

(31)

where \( \psi \), as equation (19) indicates, is a function of both \( \eta \) and \( \theta \). Therefore,

\[ r_1 \left( \frac{\partial r_1}{\partial \eta} \right)_\psi = r_1 \left[ \left( \frac{\partial r_1}{\partial \eta} \right)_\psi + \left( \frac{\partial r_1}{\partial \psi} \right)_\eta \left( \frac{\partial \psi}{\partial \eta} \right)_\theta \right]. \]  

(32)

Differentiating (31), we get for partial derivatives

\[ r_1 \left( \frac{\partial r_1}{\partial \eta} \right)_\psi = R^2 (\eta - \cos 2\psi), \]

\[ r_1 \left( \frac{\partial r_1}{\partial \psi} \right)_\eta = 2\eta R^2 \sin 2\psi. \]  

(33)

Meanwhile, it follows from (19) that

\[ \tan \theta \left[ -2 \sin 2\psi \left( \frac{\partial \psi}{\partial \eta} \right)_\theta - 1 \right] = 2 \cos 2\psi \left( \frac{\partial \psi}{\partial \eta} \right)_\theta, \]  

(34)

and, therefore,

\[ \left( \frac{\partial \psi}{\partial \eta} \right)_\theta = -\tan \theta \frac{\tan \theta}{2 \cos 2\psi + \tan \theta \sin 2\psi} = -\frac{\sin 2\psi}{2(1 - \eta \cos 2\psi)}. \]  

(35)

Substituting (33) and (35) into (32), we get

\[ r_1 \left( \frac{\partial r_1}{\partial \eta} \right)_\theta = -\frac{R^2 \cos 2\psi (1 + \eta^2 - 2\eta \cos 2\psi)}{1 - \eta \cos 2\psi}. \]  

(36)

Combined with (20) and (30), this result implies

\[ E_r = \frac{\sigma R}{\pi \varepsilon_0} \int_0^{\pi/2} \cos 2\psi \frac{d\psi}{\sqrt{r_1^2 + z^2}} = -\frac{\sigma}{\pi \varepsilon_0 \sqrt{(1 + \eta^2 + z^2)/R^2}} \int_0^{\pi/2} \frac{1 - 2 \sin^2 \phi}{\sqrt{1 - k^2 \sin^2 \phi}} \frac{d\phi}{R}. \]  

(37)

But \( \sin^2 \phi = [1 - (1 - k^2 \sin^2 \phi)]/k^2 \) and, respectively, (37) can be immediately expressed in terms of complete elliptic integrals of the first and second kinds (in agreement with [21]):

\[ E_r = -\frac{\sigma}{\pi \varepsilon_0 \sqrt{(1 + \eta^2 + z^2)/R^2}} \left[ \left( 1 - \frac{2}{k^2} \right) K(k) + \frac{2}{k^2} E(k) \right]. \]  

(38)

The combination of complete elliptic integrals that appear in (38) can be expressed through the Legendre function of the second kind and half-integral order \( Q_{1/2} \) (the so called toroidal function of zeroth order; see [27] and references therein). Indeed, we have [22, 28]

\[ Q_{1/2} \left( 2 \frac{k^2}{k^2} - 1 \right) = 2 - k^2 \frac{K(k) - 2 k E(k)}{k}. \]  

(39)

Therefore, (38) can be rewritten as follows:

\[ E_r = -\frac{\sigma}{2 \pi \varepsilon_0 \sqrt{\eta}} Q_{1/2} \left( \frac{1 + \eta^2 + z^2}{2 \eta} \right). \]  

(40)
It is not immediately clear that (9) is the $z \to 0$ limit of (38). However, this follows from the identities

$$K(n) = (1 + \eta)K(\eta), \quad E(n) = \frac{2}{1 + \eta} E(\eta) - (1 - \eta)K(\eta),$$

(41)

that by themselves can be obtained from Gauss’ transformation formulas for complete elliptic integrals (transformations 164.02 in [25]).

Similarly, for the vertical component of the electric field, we obtain after some simple manipulations,

$$E_z = -\frac{\partial \phi(\eta R, z)}{\partial z} = \frac{\sigma}{2\pi \epsilon_0} \left[ \int_0^\pi \frac{z}{\sqrt{R^2 + z^2}} d\theta - \pi \text{sign}(z) \right],$$

(42)

Here,

$$\text{sign}(z) = \frac{d|z|}{dz} = \begin{cases} 1 & \text{if } z > 0, \\ 0 & \text{if } z = 0, \\ -1 & \text{if } z < 0. \end{cases}$$

(43)

Using \( \frac{2}{1 + \eta} - n^2 \sin^2 \phi = \frac{1 - \eta}{1 + \eta} + 1 - \sin^2 \phi \), the integral in (42) can be immediately expressed in terms of the complete elliptic integrals, and the final answer is

$$E_z = \frac{\sigma}{2\pi \epsilon_0} \left[ \frac{z/R}{\sqrt{(1 + \eta)^2 + z^2/R^2}} \left( K(k) + \frac{1 - \eta}{1 + \eta} \Pi(n^2, k) \right) - \pi \text{sign}(z) \right].$$

(44)

This answer does not look identical to the expression given in [21]. However, it coincides with the result obtained in [29], and it was shown in [29] that the results of [21, 29] for $E_z$ are actually equivalent.

It follows from the solid angle formula (see figure 4),

$$\Omega = -\int \hat{n} \cdot d\vec{S} = \int \frac{\cos \alpha}{r^2} dS,$$

(45)

that the $z$-component of the gravitational attraction at $P$ of any plane homogeneous lamina with constant surface density is equal to the negative of surface density times the Newton’s constant times the solid angle which the lamina subtends at $P$ [30]. In the case of our charged disk, this translates into

$$E_z = \frac{\sigma}{4\pi \epsilon_0} \Omega,$$

(46)

where $\Omega$ is the solid angle which the disk subtends at $P$. There exist independent calculations of $\Omega$ [31, 32]. It can be verified that (for $z > 0$) the solid angle $\Omega$ that follows from (44) and (46) is consistent with the results of [31, 32].
5. Electrostatic potential and electric field at points outside the disk boundary

If the projection point $P$ lies outside the disk boundary ($\eta > 1$), we can arrange only one wedge crossing the disk, as shown in figure 5.

When $\theta$ changes from 0 to $\theta_{\text{max}}$, the wedge covers half the disk. The limit angle $\theta_{\text{max}}$ corresponds to the wedge touching the disk. Therefore, $\sin \theta_{\text{max}} = 1/\eta$. The other symmetric half of the disk makes an equal contribution to the electrostatic potential, and we can write

$$\phi(\eta R, z) = \frac{\sigma}{2\pi \varepsilon_0} \int_0^{\theta_{\text{max}}} d\theta \int_{r_1}^{r_2} \frac{r}{\sqrt{r^2 + z^2}} dr = \frac{\sigma}{2\pi \varepsilon_0} \int_0^{\theta_{\text{max}}} \left( \sqrt{r_2^2 + z^2} - \sqrt{r_1^2 + z^2} \right) d\theta,$$

(47)

where $r_1 = PA$ and $r_2 = PB$ (see figure 5). Both $r_1$ and $r_2$ obey the same quadratic equation $R^2 = \eta^2 R^2 + r^2 - 2\eta R \cos \theta$ with solutions

$$r_1 = R \left[ \eta \cos \theta - \sqrt{1 - \eta^2 \sin^2 \theta} \right], \quad r_2 = R \left[ \eta \cos \theta + \sqrt{1 - \eta^2 \sin^2 \theta} \right].$$

(48)

In the first integral, let us again introduce the angle $\psi$ as defined in figure 5. Then, $r_2^2 = R^2(1 + \eta^2 - 2\eta \cos 2\psi)$ and

$$\sqrt{r_2^2 + z^2} = R \sqrt{(1 + \eta^2) + z^2/R^2} \sqrt{1 - k^2 \sin^2 \phi},$$

(49)

where $\phi = \pi/2 - \psi$ and $k$ was defined in (22). In addition, figure 5 indicates that

$$\tan \theta = \frac{R \sin(\pi - 2\psi)}{\eta R + R \cos(\pi - 2\psi)} = \frac{\sin 2\psi}{\eta - \cos 2\psi},$$

(50)

and, correspondingly,

$$d\theta = 2 \frac{\eta \cos 2\psi - 1}{1 + \eta^2 - 2\eta \cos 2\psi} d\psi = 2 \frac{1 + \eta \cos 2\phi}{1 + \eta^2 + 2\eta \cos 2\phi} d\phi.$$
Figure 5. Definition of variables when the projection point $P$ lies outside the disk boundary.

Therefore,

$$
\int_0^{\theta_{\text{max}}} \sqrt{r^2 + z^2} \, d\theta = \frac{2R}{(1 + \eta)^2} \sqrt{(1 + \eta)^2 + \frac{z^2}{R^2}} \int_{\psi_m}^{\pi/2} \frac{(1 + \eta - 2\eta \sin^2 \phi)(1 - k^2 \sin^2 \phi)}{(1 - n^2 \sin^2 \phi)\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi.
$$

(52)

where $\psi_m$ corresponds to $\theta_{\text{max}}$ and thus satisfies $\cos^2 \psi_m = \sin \theta_{\text{max}} = 1/\eta$.

In the second integral of (47), we use an alternative definition of the angle $\psi$ shown in figure 6.

Then $r^2 = R^2(1 + \eta^2 - 2\eta \cos 2\psi)$; equations (50) and (51) are still valid, but now $\psi$ changes from 0 to $\psi_m$ when $\theta$ changes from 0 to $\theta_{\text{max}}$, and correspondingly, $\phi$ changes from $\pi/2$ to $\pi/2 - \psi_m$. Therefore,

$$
\int_0^{\theta_{\text{max}}} \sqrt{r_1^2 + z^2} \, d\theta = \frac{2R}{(1 + \eta)^2} \sqrt{(1 + \eta)^2 + \frac{z^2}{R^2}} \int_{\psi_m}^{\pi/2} \frac{(1 + \eta - 2\eta \sin^2 \phi)(1 - k^2 \sin^2 \phi)}{(1 - n^2 \sin^2 \phi)\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi.
$$

(53)

As we see, the difference of two integrals, (52) and (53), just gives the right-hand side of (21), and therefore the electrostatic potential when $\eta > 1$ is still given by (27), but without the last, proportional to $|z|$ term. Introducing the Heaviside step function $H(x) = (1 + \text{sign}(x))/2$, the final expression for the electrostatic potential, valid for all values of $\eta$, takes the form

$$
\phi(\eta R, z) = \frac{\sigma R}{2\pi \varepsilon_0} \left[ \frac{1 - \eta^2}{\sqrt{(1 + \eta)^2 + z^2/R^2}} K(k) + \sqrt{(1 + \eta)^2 + z^2/R^2} E(k) \right.
$$

$$
+ \frac{1 - \eta z^2/R^2}{1 + \eta \sqrt{(1 + \eta)^2 + z^2/R^2}} \Pi(n^2, k) - \pi H(1 - \eta) \frac{|z|}{R} \right].
$$

(54)
The radial electric field is still given by (40), except that the derivative of Heaviside step function will bring an additional term proportional to the Dirac delta function:

$$E_r = \frac{\sigma}{2\pi\epsilon_0\sqrt{\eta}} Q_{1/2} \left( \frac{1 + \eta^2 + z^2/R^2}{2\eta} \right) - \frac{\sigma}{2\epsilon_0} \delta(1 - \eta) \frac{|z|}{R}. \quad (55)$$

Finally, the vertical component of the electric field, valid for all values of $\eta$, has the form

$$E_z = -\frac{\sigma}{2\pi\epsilon_0} \left[ \frac{z/R}{\sqrt{(1 + \eta^2) + z^2/R^2}} \left( K(k) + \frac{1 - \eta}{1 + \eta} \Pi(n^2, k) \right) - \pi H(1 - \eta) \text{sign}(z) \right]. \quad (56)$$

To use these formulas near the rim of the disk ($\eta \approx 1$), the following asymptotic expansions [25], valid when $n^2 \approx 1$, are useful:

$$K(n) \approx \ln 4 - \ln(1 - n^2), \quad \Pi(n^2, k) \approx K(k) - \frac{E(k)}{1 - k^2} + \frac{\pi(2 - k^2 - n^2k^2)}{4\sqrt{(1 - n^2)(1 - k^2)}}. \quad (57)$$

Then it can be seen from (9) that the radial electric field has a logarithmic singularity at the rim of the disk. This singularity is related to the discontinuity in the surface charge density at $\eta = 1$ and can be avoided by using a distribution of charge with surface density that falls to zero at the rim [22].

6. Concluding remarks

Disk-shaped structures are widespread in astrophysics, their formation being associated with the conservation of angular momentum in gas flow around compact gravitating objects [33]. It is not surprising, therefore, that the gravitational counterpart of the considered problem and its generalization to the case of non-uniform disks has attracted much more attention (see, for example, [18–22, 29, 34–38] and references cited therein).
We were able to find Plana’s publication [54], in which Plana indeed gives the gravitational pull from the ring and from the circular disk. All the modern articles that we checked cite [17, 20, 21] as primary sources where the contribution of Cayley [55] is expressed in terms of complete elliptic integrals. The gravitational potential was given much later by Arthur Cayley. 

The footnote in [19] gives a hint that according to Todhunter [53], Giovanni Antonio Amedeo Plana calculated gravitational pulls from the ring and from the circular disk. We were able to find Plana’s publication [54], in which Plana indeed gives the gravitational attraction of a homogeneous circular disk in terms of complete elliptic integrals. The corresponding expression for the gravitational potential was given much later by Arthur Cayley [55].

It seems that the contributions of Plana and Cayley have now been forgotten, because all the modern articles that we checked cite [17, 20, 21] as primary sources where the gravitational attraction of a homogeneous ring [39–41], and of the electric field of a homogeneous ring [27, 42–45]. However, we were able to find only one paper [46] discussing the gravitational field of a homogeneous massive disk.

In [46], the gravitational field of a hypothetical flat Earth was considered in the approximation $\eta \ll 1$. To get the radial electric field in the $\eta \ll 1$ approximation from (40), we use the expansion [27, 50]

$$Q_{1/2}(\beta) = \frac{\pi}{2(\beta)^{3/2}} \sum_{n=0}^{\infty} \frac{(4n + 1)!!}{2^{2n}(n + 1)!n!} \frac{1}{(\beta)^{2n}} = \frac{\pi}{2(\beta)^{3/2}} \left( 1 + \frac{15}{8} \frac{1}{(\beta)^2} \ldots \right), \quad (58)$$

with $2\beta = (1 + \eta^2 + z^2/R^2)/\eta$. As a result, we get

$$E_\eta \approx \frac{\sigma \eta}{4\epsilon_0(1 + z^2/R^2)^{3/2}} \left[ 1 + \frac{3(1 - 4z^2/R^2)}{8(1 + z^2/R^2)^2} \eta^2 \right]. \quad (59)$$

For the vertical component of the electric field, we use expansions [25]

$$K(k) = \frac{\pi}{2} \left[ 1 + \frac{1}{4} k^2 + \frac{9}{64} k^4 + \ldots \right], \quad \Pi(n^2, k) = \frac{\pi}{2} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{(2m)!}{4^m (m!)^2 (j!)^2} \frac{k(j) \rho^{2m-j}}{\beta^{2(n^j)}} \quad (60)$$

and get in the $\eta \ll 1$ approximation (for $z > 0$):

$$E_z \approx \frac{\sigma}{4\epsilon_0} \left[ 2 \left( 1 - \frac{z/R}{\sqrt{1 + z^2/R^2}} \right) - \frac{3z/R}{2(1 + z^2/R^2)^{5/2}} \eta^2 \right]. \quad (61)$$

Under substitution $1/(4\pi\epsilon_0) \rightarrow -G$, $G$ being the Newton gravitational constant, (59) and (61) reproduce the results of [46].

In fact, the problem addressed in this note was resolved a long time ago, much earlier than is commonly thought. Conway notes in [51] that the fact that the potential of a homogeneous disk is expressible in terms of elliptic integrals was already known to Weber in 1873 [52]. However, Weber does not provide any reference.

The footnote in [19] gives a hint that according to Todhunter [53], Giovanni Antonio Amedeo Plana calculated gravitational pulls from the ring and from the circular disk. We were able to find Plana’s publication [54], in which Plana indeed gives the gravitational attraction of a homogeneous circular disk in terms of complete elliptic integrals. The corresponding expression for the gravitational potential was given much later by Arthur Cayley [55].

It seems that the contributions of Plana and Cayley have now been forgotten, because all the modern articles that we checked cite [17, 20, 21] as primary sources where the contributions of Plana and Cayley have now been forgotten, because all the modern articles that we checked cite [17, 20, 21] as primary sources where the contributions of Plana and Cayley have now been forgotten, because all the modern articles that we checked cite [17, 20, 21] as primary sources where the scientific method in modern society [47]. Alas, calculations, as in [46], cannot change their mind—they do not believe in gravity. For every educated person, starting from the ancient times of Greek and Roman writers, the sphericity of the Earth was an indisputable fact based on solid empirical data [48, 49]. The fact that flat Earth proponents frantically and ignorantly attack this ancient knowledge is an alarming sign of the growing irrationality and decline of faith in the scientific method in modern society [47].
problem of a homogeneous disk was solved. We think that the upcoming bicentennial anniversary of Giovanni Plana’s paper [54] is a good occasion both to restore the legacy of Plana and Cayley, and to make this venerable problem accessible to a wide audience of physics students.

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