Energy of test objects on black hole spacetimes: A brief review

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Abstract

In this paper, we review the issue of defining energy for test particles on a background stationary spacetime. We revisit the different notions of energy as defined by different observers. As is well known, the existence of a time-like isometry allows for the notion of total conserved energy to be well defined. We use this well known quantity to show that a gravitational potential energy can be consistently defined. As examples, we study the case of the exterior regions of an asymptotically flat black hole and of the $\Lambda > 0$ Schwarzschild de Sitter case, where an asymptotic region is not available. We then consider the situation in which the test particle is absorbed by the black hole, and analyze the energetics in detail. In particular, we show that the notion of horizon energy as defined by the isolated horizons formalism provides a satisfactory notion of energy compatible with the particle’s total conserved energy. With these choices, there is a global conservation of energy. Finally, we comment on a recent proposal to define energy of the black hole as seen by a nearby observer at rest, for which this feature is lost.

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I. INTRODUCTION

The concept of energy in general relativity is rather subtle. The Newtonian notion that a conserved quantity with the interpretation of energy must be conserved disappears once it is realized that in general relativity there is no “absolute” background on which the theory is defined. Instead, the geometry becomes a dynamical variable. As a consequence, the notion of a gravitational energy density becomes ill defined. Still, there are solutions to the equations of motion that are stationary, in the sense that there is a notion of time invariance. It is then natural to expect that in those situations, there will be conserved quantities associated to the motion of matter fields. That is indeed the case with the so-called Komar quantities, constructed out of the Killing vector fields and the stress energy tensor of matter. Down in the ladder of complexity, one might ask about the existence of a conserved quantity for test particles moving on stationary space-times. Such a quantity has been known for a long time, but its interpretation as a total energy of the particle is not always found in the introductory treatment of the subject.

One of the purposes of this manuscript is to review and revisit this conserved quantity and compare it to other notions of energy that have their origin in the special theory of relativity. As we shall show, the conserved quantity can indeed be interpreted as a total energy, that also contains information about the gravitational field. Thus, in certain circumstances, one can indeed define the notion of a gravitational potential energy satisfying a set of desired properties. The next question pertains to the case of the exterior of a black hole where the spacetime is stationary and for which the horizon represents a boundary. We analyze in detail the behavior of these quantities in the vicinity of the black hole. It is then natural to consider the case when the particle falls into the black hole. Is there a notion of energy associated to the black hole that can capture the intuitive idea that energy should not be lost? Can one keep track of the energy of the in-falling object as being gained by the black hole? As we shall see, the answer is in the affirmative. The natural notion of energy within the isolated horizon formalism satisfies these properties. One can furthermore achieve consistency with the energy balance, for both asymptotically flat and de Sitter black hole spacetimes, provided we associate to the test particle the total (conserved) energy.

Still one might want to consider different notions of energy, such as the one associated to an observer at rest outside the black hole. What would then be the new notion of energy associated to the black hole in order to keep the energy balance? Again, the isolated horizons formalism allows us to consider this possibility and yields a modified notion of black hole energy. As it has been shown, this energy approaches asymptotically the horizon area for large black holes but has some nontrivial dependence in general. We briefly comment on this choice that has been recently advocated in the literature.

The structure of this note is as follows. In Sec. [II] we pose the problem to be considered, namely the energy of test particles on stationary spacetimes. Sec. [III] specializes to the case of a static black holes. In the first part, we consider the Schwarzschild black hole, while in the second part we consider the Schwarzschild-de Sitter spacetime. In Sec. [IV] we consider the case where the particle falls into the black hole and how this affects the energy of the black hole. This will depend on the notion of energy for the horizon. We consider two cases, namely the standard choice within the isolated horizons formalism and, in Sec. [V] a new proposal to describe the situation from the perspective of an observer at rest outside the horizon. We end with a discussion in Sec. [VI].
Since this manuscript is intended to review and revisit some simple notions in general relativity, it should be seen as a complement to the standard texts on the subject. We have tried to make the manuscript self-contained, so the reader is not expected to possess a very deep understanding of general relativity. Throughout the manuscript we use the ‘natural’ units for general relativity, namely geometrical units where $G = c = 1$, and we adopt the abstract index notation as in [1].

II. ENERGY ON STATIONARY SPACETIMES

Let us start by considering an asymptotically flat spacetime $(M, g)$ with a time-like Killing vector field (KVF) $\xi^a$, such that, in the (space-like) asymptotic region $\xi_a \xi^a = -1$. Thus, $\xi$ approaches the four-velocity of an inertial observer at infinity, and can also be interpreted as generator of asymptotic time translations. The existence of the generator of time isometries on the background space-time implies that there are some conserved quantities. In ordinary 'Newtonian' dynamics, time invariance implies conservation of energy, and the situation is no different in general relativity. Here we shall focus on test particles with a four-velocity $u^a$. The corresponding four momentum is $p^a = mu^a$, with $m$ the rest mass of the object (defined by $m^2 := -p_ap^a$). The quantity that is conserved, if the test particle is on free-fall is given by,

$$E = -m \xi_a u^a$$

(1)

(The proof is rather direct: $u^b \nabla_b E = -m(u^b u^a \nabla_b \xi_a + u^a \xi^a \nabla_b u^b) = 0$. The first term vanishes because $\xi$ is a KVF and satisfies the Killing equation $\nabla_{(a} \xi_{b)} = 0$. The second term is equal to zero since $u^a$ satisfies the geodesic equation $u^a \nabla_a u^b = 0$.) What is the physical interpretation of the quantity $E$? Can we say it corresponds to the energy of the particle? The energy as measured by whom? Before answering these questions, let us recall what is the notion of energy in special relativity. In that case one possesses a family of preferred global inertial observers. Their corresponding four velocities $\xi_o$ are Killing vector fields, that are also normalized $\xi_o \xi_o^a = -1$. If we have a particle with four-momentum $p^a$, not necessarily following a geodesic, then the energy of the particle, as measured by the observer $\xi_o$ is given by,

$$E = -\xi_o a p^a$$

(2)

This is the standard expression used in textbooks. If one introduces inertial coordinates such that $\xi_o^a = (\partial/\partial T)^a$, then $E = p_0$, the ‘time component’ of the four-vector $p_a$. This quantity is also helpful to measure the speed $v$ of the particle as measured by $\xi_o$. we have that,

$$E = m \gamma$$

(3)

where $m^2 = -p_ap^a$ and $\gamma = 1/(\sqrt{1 - v^2})$. Then, it is straightforward to see that $E$ is the relativistic generalization of the Kinetic energy. If the object is at rest with respect to the inertial observer, then $E = m$, the rest mass. If the object moves with a small velocity $v \ll 1$ (recall that we are setting $c = 1$) then,

$$E \approx m \left(1 + \frac{1}{2}v^2\right) = m + \frac{1}{2}mv^2$$

That is, $E$ represents the total energy of the particle as measured by the inertial observer, that in the small velocity regime is given by the rest energy plus the Newtonian kinetic
energy. Let us denote the quantity $E$ as the generalized kinetic energy (GKE) of the particle. Clearly, $E$ is conserved if the particle follows a geodesic, namely if it travels at constant speed with respect to the inertial observer $\xi_o$.

Can we generalize the energy $E$ to general relativity? The answer is simple: Yes. The difference is that now, we do not have global inertial observers, so one has to instead consider locally inertial observers. Thus, at a point $p$ of the space-time $M$, a local observer $w^a$, such that $w_aw^a = -1$, will measure the energy of a particle with four-momentum $p^a$ to be,

$$E = -w_a p^a$$

The interpretation is clear. The observer $w^a$ should be thought of as being instantly at rest, and the quantity $E$ is the energy of the moving object as measured by that instantly inertial observer. This quantity can always be defined for any observer and any particle, without any restriction on the background space-time nor the dynamics followed by the observer or the test object. The problem with this quantity is that it gives us no information about the gravitational field. It can not, since one is constructing it by exploiting the fact that locally any point on space-time ‘behaves’ as if it were in special relativity. Thus, the equivalence principle forbids us to draw any information about the gravitational field from this quantity. What one effectively is doing is to go to an instantaneous inertial reference frame where the gravitational field ‘disappears’, to measure the energy of the moving object.

Let us now return to the quantity $E$ defined for stationary spacetimes. Can it represent the energy as measured by some observer? The answer is indeed in the affirmative. Let us now see how that comes about. Suppose that the test object starts ‘at rest’ at spatial infinity, then the observer at infinity that follows the word-lines of the KVF $\xi$ will measure the energy of the particle to be $E = -p_a \xi^a$. But notice that this is precisely the conserved quantity $E$. Then, this quantity corresponds to the generalized kinetic energy as measured by an observer at rest at infinity. But, can we assign to it a further interpretation that ‘knows’ about the gravitational field? That is, can one define a notion of gravitational potential energy and relate it to $E$? In order to answer that question, let us briefly recall the notion of gravitational potential energy in Newtonian theory. For simplicity, consider the case of a central object of mass $M$. The force on an object of mass $m$ at a distance $r$ is $Mm/r^2$ (recall that $G = c = 1$). Thus, the gravitational potential energy is given by $U = -Mm/r + C$ with $C$ a constant. The standard choice is to take $C = 0$ such that $U(r = \infty) = 0$. The total, conserved, energy of the particle is then,

$$E_{\text{Tot}} = KE + U = \frac{1}{2}mv^2 - \frac{Mm}{r} + C'$$

with $C'$ a constant. Again, the standard choice is to take $C' = 0$ but, in order to make contact with general relativity, one can very well take $C' = m$. With this choice, the total Newtonian energy of a particle, at rest at infinity is $E_{\text{Tot}} = m$. Obviously, the conserved quantity in the Newtonian theory has a contribution from the gravitational field. It is then natural to assume that in the general relativistic setting, if there is a conserved quantity representing energy of the particle, it should contain information about the gravitational field. Thus, the object $E$ is the natural quantity to represent the total energy of the particle. And, from our previous discussion, it should have information about the (stationary) gravitational field.

Let us then define $E$ as the total energy of the particle and study some of its properties. The first thing to notice is that there is a simple relation with the generalized kinetic energy
which comes from the normalization of the KVF $\xi^a$. The four-velocity $w^a$ of an observer that follows the word-lines of the KVF $\xi$ is given by,

$$w^a = \frac{\xi^a}{V}$$

with $V^2 = |\xi_a \xi^a|$, the so-called ‘redshift factor’. With this, we have

$$E = -p_\xi \xi^a = -V p_\xi w^a = V E$$

(6)

Note that one can formally define a general relativistic generalization of the potential energy by setting $E := E + U$. Thus, the quantity $U$ becomes,

$$U = E (V - 1)$$

(7)

Or $U = E(1 - 1/V)$. With this choice, a particle that is at rest at infinity has a total energy of $E = m$ equal to the GKE $E$. Thus, in that case the generalized potential energy at infinite vanishes, as in the Newtonian case. For the quantity $U$ to capture the intuitive notion of a negative gravitational potential energy, then the norm of the KVF should satisfy $V < 1$. Note also that, if the particle is moving when it reaches infinity, then $E > m$. If $E < m$ then the object can not reach infinity and is therefore, bounded. A particular case of this situation is when the particle is at rest with respect to the observer following the orbits of $\xi^a$. In this case, the four-momentum is $p_\xi = mw_a$, and the corresponding energies are: $E_{res} = m$, of course, and

$$E_{res} = -m\xi^a w_a = -mV w^a w_a = mV.$$  

(8)

The potential energy becomes then,

$$U_{res} = m(V - 1)$$

Let us now consider the particular case of interest, namely when the spacetime corresponds to the exterior of an eternal black hole.

### III. ENERGY ON THE VICINITY OF A BLACK HOLE

In this section we shall concentrate our attention on the different definitions of energy for test particles in the vicinity of stationary black holes. This section has two parts. In the first one, we discuss the asymptotically flat case, where for simplicity we focus our attention on the Schwarzschild case. In the second part, we consider a non-vanishing and positive cosmological constant, where the notion of an asymptotic inertial observer does not exist.

#### A. Asymptotically flat case ($\Lambda = 0$)

For simplicity and without loss of generality, the spacetime we are going to consider is given by the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

(9)
From this expression we see that the timelike KVF is $\xi^a = (\partial/\partial t)^a$, with norm given by,

$$V = |\xi_a \xi^a|^{1/2} = \left(1 - \frac{2M}{r}\right)^{1/2}$$

Recall that this spacetime is asymptotically flat, and has a horizon at $r_h = 2M$, where $M$ is the Arnowitt-Deser-Misner (ADM) mass of the spacetime and one can also associate it to the mass of the black hole (more about this later). Note that in the exterior of the black hole, namely when $r > 2M$, the red-shift factor satisfies $V < 1$ and $V \to 0$ as one approaches the horizon. Let us now analyse the other quantities we have defined. First, let us write the relation between the GKE $E$, as measured by an observer at rest with respect to the BH at a ‘distance’ $r$, and the total, conserved energy $\mathcal{E}$,

$$E = \frac{\mathcal{E}}{(1 - \frac{2M}{r})^{1/2}} \quad (10)$$

From here we can use the relation $E = m\gamma$ to find the speed of the particle with total energy $\mathcal{E}$, as measured by the observer at rest at a ‘distance’ $r$,

$$v = \left[1 - \left(\frac{m}{\mathcal{E}}\right)^2 \left(1 - \frac{2M}{r}\right)\right]^{1/2} \quad (11)$$

From (10) one can see that, for a particle with given energy $\mathcal{E}$, the GKE as measured by observers at rest will increase and diverge as one approaches the horizon. Eq. (11) is the generalization of the Newtonian equation relating speed with the total energy and distance from the origin. Let us see what information we can extract from it. First, suppose the particle has an energy $\mathcal{E} > m$, then it can reach infinity and there, the speed is given by $v|_\infty = \sqrt{1 - (m/\mathcal{E})^2}$. If the total energy is less than the rest mass, $\mathcal{E} < m$ then the particle cannot reach infinity. For instance if it is ‘thrown’ upward and then follows a radial geodesic, then the maximal radial distance it will attain is,

$$r_{\text{max}} = \frac{2M}{1 - \left(\frac{\mathcal{E}}{m}\right)^2} \quad (12)$$

before falling again in the direction of the black hole. We can compare this quantity with the corresponding Newtonian quantity $r_{\text{max}}^*$ given by: $r_{\text{max}}^* = \frac{M}{1 - \frac{2M}{m}}$, that relates the maximum distance that a test particle would travel away from a point (or spherical) object of mass $M$ (located at $r = 0$), as a function of the energy $E_n$ of the test particle. Finally, note that if we throw the particle in the direction of the black hole, its speed will approach $v \to 1$ as it reaches the horizon, independently of the value of $\mathcal{E}$.

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1 Recall that, in the Newtonian case, for a particle of total energy $E_n$ as defined in the previous section in Eq. (5), the speed $v_n$ is given by $v_n = \sqrt{2\left(\frac{E_n}{m} - 1\right) + \frac{2M}{m}}$. Note that this and the GR expression are rather different, except for the case when $\mathcal{E} = m$ in general relativity and $E_n = m$ in the Newtonian theory, where both theories yield $v = \sqrt{2M/r}$. Incidentally, this ‘coincidence’ is precisely what allowed Michell, with his 1783 calculation within Newtonian mechanics, to predict the “correct” value for the black hole radius. We see that in both GR and Newtonian dynamics, for an object of mass $M$ and radius $r$, the escape velocity is equal to 1, precisely when $r = 2M$. 

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Accelerated motion ‘at rest’. Let us consider the following situation. We start with a test particle of rest mass $m$ following the orbit of the KVF $\xi^a$ at infinity. The total energy is clearly $E = m$. Now, let us assume that we lower the particle in a quasi-static fashion from infinity to a distance $r$, in such a way that it is at rest.\(^2\) We know that in the Newtonian theory, the energy of the particle decreases, since it only possesses potential energy. We expect that in general relativity the total energy $E$ also decreases. In fact, from Eq. (8) we see that the total energy is given by

$$E_{\text{res}} = m \sqrt{1 - \frac{2M}{r}}$$

(13)

There are two points to remark. As expected, the energy decreases as $r$ decreases but as the particle approaches the horizon, the total energy approaches zero. That is, a particle that is held by an observer at rest just outside the black hole horizon has an energy $E = m$, as measured by the observer, but its total energy is much smaller, approaching zero at the horizon. That is, it has lost all its energy due to the negative gravitational potential energy given by,

$$U_{\text{res}} = m \left[ \left( 1 - \frac{2M}{r} \right)^{1/2} - 1 \right],$$

(14)

which takes values in the interval $[-m, 0]$. Note also that there is a “positive energy result” at hand: The conserved energy of a test particle ‘at rest’ is strictly positive in the region where it can be defined (recall that in the interior region, the notion of energy for the conserved quantity is no longer valid, since the Killing vector field $\xi$ becomes spacelike).

The second observation is that this potential energy (14) satisfies several desired properties. In particular:

i) For $r \gg 2M$, the quantity approaches $U_{\text{res}} \approx -Mm/r$.

ii) It is always negative, and

iii) It serves as a potential for the force felt by the test particle to be held at rest. The magnitude of the force is $F = mg$ with $g$ the norm of the acceleration $\alpha^b = u^a \nabla_a u^b$. That is, $g = \sqrt{\vert \alpha \vert}$. In our case,

$$F = \frac{mM}{r^2} \left( 1 - \frac{2M}{r} \right)^{1/2} = \partial_r U_{\text{res}}$$

(15)

Note that the potential energy $U_{\text{res}}$ that we have here defined improves on the proposal put forward in [2], that does not satisfy the properties i)-iii) above. Thus, to the best of our knowledge, the quantity (14) has not appeared elsewhere.

Circular motion. Let us now consider the test particle of rest mass $m$ following a stable circular orbit. When there is angular momentum, the radial coordinate satisfies a potential equation of the form,

$$(dr/d\tau)^2 + V(r) = (E/m)^2 - 1,$$

with $V(r) = -2m/r + L^2/(m^2 r^2) - 2ML^2/(m^3 r^3)$ and $L, E$ the conserved quantities. Circular orbits are determined by the extrema of the potential. It has two extrema, one maximum

\(^2\) We do not need to perform such a procedure, namely to lower the particle in a quasi-static fashion. We could as well consider, as initial condition, a test particle at rest at a distance $r$. 

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and one local minimum, that represents stable circular orbits. The smallest such orbit is when both extrema coincide, which occurs when \( L^2 = 12M^2m^2 \). Then, the value of this minimum radius \( r_c^- \) is equal to \( r_c^- = 6M = 3r_h \). If one evaluates the potential \( V(r) \) for this value, one finds that,

\[
E_c^- = m\frac{2\sqrt{2}}{3}
\]  

(16)

Let us now consider a test particle that is on circular orbit ‘at infinity’ for which \( E_\infty = m \) (as could be expected \(^3\)). Let us now assume that the particle looses energy as it orbits the black hole in such a way that it follows quasi-circular orbits. It is immediate to see that it will, at most, loose an energy equal to \( \Delta E = E_\infty - E_c = m(1 - 2\sqrt{2}/3) \approx 0.057m \), before plunging into the black hole. This is clearly the energy source of accretion disks around black holes, that loose energy by radiating in the electromagnetic spectrum. One can also find the speed of the particle on the last stable circular orbit to be,

\[
v_c^- = \left[1 - \left(\frac{3}{2\sqrt{2}}\right)^2 \left(1 - \frac{1}{3}\right)\right]^{1/2} = \frac{1}{2}.
\]

That is, the top speed of a test particle on a circular orbit is precisely 50% of the speed of light.

We end here our discussion of the energetics of test particles on an asymptotically flat static spacetime. Clearly, most of the results described here will be qualitatively similar for other asymptotically flat static black holes. As should be clear from our previous discussion, and as we have argued in this part, the right notion of energy to consider is \( E \), if one wants to capture the contribution to the energy coming from the gravitational field. Let us now explore the situation when the cosmological constant is non-vanishing and we have no asymptotic flat regions.

**B. de Sitter Black Holes (\( \Lambda > 0 \))**

So far we have focused on test particles that live on static patches of asymptotically flat spacetimes and have focused on the notion of energy for them. The existence of asymptotic observers has proven crucial for several of our discussions. In particular, it allows us to normalize the KVF \( \xi \) such that its norm becomes unity at infinity. The physical interpretation is that this preferred observer is the one for which the conserved quantity \( E \) coincided with the GKE \( E \), so that we can interpret \( E \) as the (GKE) energy as measured by an inertial observer at infinity. When we depart from the \( \Lambda = 0 \) case, the first observation is that we no longer have an asymptotically flat region. In the patches where the metric looks static, the vector \( \partial/\partial t \) is not canonically normalized, so we immediately loose the interpretation we had in the flat case. Thus, we need to adopt a slightly different strategy. In the remainder of this part, we shall consider the de Sitter case with a positive cosmological constant.

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\(^3\) Note that, just as in the Newtonian theory, the speed \( v \) of circular orbits decreases as the radius of the orbit increases so that, in the \( r \to \infty \) limit, the speed vanishes, and therefore, the total energy \( E \) corresponds to the rest mass of the object.
Let us start by considering the simplest static metric in this case, the so called Schwarzschild-de Sitter (SdS) metric:

\[ ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2 d\Omega^2 \]  

(17)

where \( f^2 = (1 - \frac{2\mu}{r} - \frac{r^2}{\ell^2}) \), with \( \ell = \sqrt{3/\Lambda} \) the natural length scale defined by the cosmological constant, and \( \mu \) the 'mass parameter'\(^4\). The function \( f^2 \) has two zeros, corresponding to the two horizons. The smallest root \( r_b \) corresponds to the black hole horizon, while the larger one \( r_c \) defines the cosmological event horizon. The static patch corresponds to the region \( r_b < r < r_c \). One thing to note is that in no place does \( f^2 = 1 \), which means that \( \bar{\xi}^a = (\partial/\partial t)^a \) can never correspond to the four velocity of an static observer. One needs to normalize it appropriately. Second, note that from all the possible static observers, there is one (sphere worth of) observable(s) that is special. Let us see now which how to select those observers. If a static observer is close to the black hole horizon then, in order to remain static, she needs to accelerate away from the black hole. On the other hand, if the observer is now close to the cosmological horizon, in order to remain static she would need to accelerate \textit{towards} the black hole due to the cosmic expansion. Then, there is an intermediate point \( r_i \) where a static observer is inertial; it is an unstable ‘equilibrium point’. If one particle is following the KVF orbit there and one pushes it slight towards the BH it will fall to it, while if the push is in the outward direction, then the particle ‘falls’ toward the cosmological horizon. The natural normalization for the KFV presents itself before our eyes: chose the KVF \( \xi^a \) such that, for the inertial observer at \( r = r_i \), we have the normalization \( (\xi \cdot \xi)|_{r_i} = -1 \). The value of this preferred radius is given by \( r_i = (\mu/\ell^2)^{1/3} \). Thus, we have,

\[ \xi^a = \frac{\bar{\xi}^a}{f(r_i)} = \frac{1}{\sqrt{1 - 3 \left(\frac{\mu}{\ell^2}\right)^{2/3}}} \left( \frac{\partial}{\partial t} \right)^a. \]  

(18)

With this choice, the red-shift factor \( V \) takes the form:

\[ V = \frac{f(r)}{f(r_i)} = \frac{1}{\sqrt{1 - 3 \left(\frac{\mu}{\ell^2}\right)^{2/3}}} \left( 1 - \frac{2\mu}{r} - \frac{r^2}{\ell^2} \right)^{1/2}. \]

One could argue that these inertial observers play the same role as the asymptotic observers do for the \( \Lambda = 0 \) case, since they are both inertial and ‘far away’ (if \( \ell \gg \mu \)) from the black hole. Thus, one can now define the conserved quantity \( \mathcal{E} \) as before: \( \mathcal{E} = -p_a \xi^a = V E \).

We can again consider the situation in which we have radial geodesics, in which case the relation between speed and ‘radius’ \( r \), for a test particle of total energy \( \mathcal{E} \) is given by,

\[ v = \sqrt{1 - \left(\frac{m}{\mathcal{E}}\right)^2 V^2} \]  

(19)

We can follow much of the analysis that we did in the previous part for the asymptotically flat case. Note that the role that infinity played in the previous case is now played by the ‘point \( r_i \)’. Let us now revisit some of the features that resemble the flat case. The first

\(^4\) while \( \mu \) is sometimes regarded as the mass of the black hole, this assumption is unsupported as was discussed in detail in [3].
observation is that a test particle with $E = m$ is necessarily at rest at $r_i$, since $V(r_i) = 1$. If $E < m$ then the particle is either in a region $r < r_{\text{max}} < r_i$ closer to the black hole, or ‘on the other side of the potential’, namely $r > r_{\text{min}} > r_i$. Here $r_{\text{max}}$ represents the maximum value that $r$ can attain in the region $[r_b, r_i]$, and $r_{\text{min}}$ is the minimum value $r$ can take in the region $[r_i, r_c]$. They are given by the real positive roots of the equation

$$
\left(1 - \frac{2\mu}{r} - \frac{r^2}{l^2}\right) = \left(\frac{E}{m}\right) \left(1 - 3 \left(\frac{\mu}{l}\right)^{2/3}\right).
$$

Clearly, if $E > m$ the particle can climb up the potential away from the black hole and reach infinity. Conversely, if it is originally in the cosmological region ($r > r_i$), and thrown to the center of the spacetime it will overcome the cosmic repulsion and reach the region where the attraction of the black hole dominates, and fall into it.

It is perhaps illustrative to estimate the value of the radius $r_i$ where the transition occurs, for the value of the cosmological constant as recently measured. Its value is currently estimated to be close to $\Lambda \approx 10^{-52} m^2$. Let us now estimate $r_i$ for three different values of $\mu$. First, let us consider a stellar mass black hole with mass of the order of the mass of the sun. In that case $\mu \approx 10^4 m$, and $r_i \approx 6.7 \times 10^{18} m \approx 200$ pc. Note that the Schwarzschild de Sitter metric describes not only a black hole, but the exterior region of a spherically symmetric non-rotating star. Thus, we see that for a system such as the solar system, the cosmic repulsion dominates at scales that are well within the galaxy. Next, let us consider the case of a super-massive black hole at the galactic center with mass of the order of $10^6$ solar masses. In that case, $r_i \approx 20$ kpc, which is again smaller than the galactic radius. Finally, for a mass of the order of the milky way galaxy, we have $r_i \approx 1$ Mpc. Recall that the closest galaxy to ours is Andromeda at about $75$ Mpc, so it is very close to the transition point.

Let us return to the radial geodesic we were considering. Suppose we have a test particle and we throw it radially towards $r_i$, with just enough speed so that $E = m$. From the previous considerations we know that the particle will have zero speed upon arrival to the point $r = r_i$. Let us now calculate the proper time that it would take to reach that point. As usual one starts from the ‘first integral’ of the geodesic equations: $g_{ab}u^a u^b = -1$, and by using the spherical coordinates and considering moving in only the radial direction, one arrives to the expression, for generic $E$,

$$
d\tau = \frac{dr}{\sqrt{\left(1 - 3 \left(\frac{\mu}{l}\right)^{2/3}\right) \left(\frac{E}{m}\right)^2 - 1 + \frac{2\mu}{r} + \frac{r^2}{l^2}}}.
$$

Now, we take the case $E = m$ that we are considering and by redefining the radial coordinate to be $\tilde{r} := r/r_i = r/(\mu l)^{1/3}$, the proper time $\tau$ to go from the point $r_0$ to $r_i$ is given by the

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5 Of course, we do not see this effect inside our galaxy, since the contribution from other stars and the galactic centre out-weights that from the cosmic repulsion.

6 But in that case, since Andromeda has a mass comparable to that of our galaxy, the test mass approximation clearly fails and one would have to treat the system as a two body problem, in which case the effect of the cosmological constant is less transparent.
expression
\[ \tau = \ell \int_{\tilde{r}_0}^{1} \frac{d\tilde{r}}{\sqrt{-3 + \frac{2}{\ell^2} + \tilde{r}^2}}, \] 
that diverges. Also, note that if the conserved energy is slightly different from \( m \), say \( 0 < E - m \ll m \), then the corresponding integral converges, so one can transition from the black hole region to the cosmological region in a finite proper time\(^7\).

**Accelerated motion at rest.** Let us consider the following situation. We start with a test particle of rest mass \( m \) following the orbit of the KVF \( \xi^a \) at the point \( r_i \). The total energy is clearly \( E = m \). Now, let us assume that we lower the particle in a quasi-static fashion in such a way that it is at rest at a distance \( r \). We know from the previous part that the energy \( E \) of the particle decreases, since it only possesses potential energy. From Eq. (8) we see that the total energy is given by

\[
E_{\text{res}} = \frac{m}{\sqrt{1 - \left(\frac{2\mu}{r} - \frac{r^2}{\ell^2}\right)^{2/3}}} \left(1 - \frac{2\mu}{r} - \frac{r^2}{\ell^2}\right)^{1/2} \tag{22}
\]

There are two points to remark. As expected, the energy decreases as \( r \) decreases but as the particle approaches the horizon, the total energy approaches zero. That is, a particle that is held by an observer at rest just outside the black hole horizon has an energy \( E = m \), as measured by the observer, but its total energy is much smaller, approaching zero at the horizon. That is, it has lost all its energy due to the negative gravitational energy given by,

\[
U_{\text{res}} = m \left[ \left(1 - \frac{2\mu}{r} - \frac{r^2}{\ell^2}\right)^{1/2} - 1 \right], \tag{23}
\]

which, as in the previous case, takes values in the interval \([-m, 0] \). Note that in this case we also have a positive energy result, given that the conserved energy of a test particle is strictly positive in the region where it can be defined (recall that in the interior regions, the notion of energy for the conserved quantity is no longer valid, since the Killing vector ceases to be timelike). The potential energy takes its minimum value \(-m\) at both horizons and is zero at point \( r_i \).

\( \Lambda < 0 \) Case. Let us now briefly comment on the other possibility when there is a cosmological constant present, namely the case of asymptotically AdS spacetimes for \( \Lambda < 0 \). This case has drawn a lot of attention due to the conjectured AdS/CFT dualities. The first observation is that in static coordinates for which the metric takes the form,

\[
ds^2 = - \left(1 - \frac{2\mu}{r} + \frac{r^2}{\ell^2}\right) dt^2 + \left(1 - \frac{2\mu}{r} + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{24}
\]

We do not have again, as in the de-Sitter case, a preferred asymptotic observer, since the norm \( V \) of the Killing field \( \xi \) tends to infinity as \( r \) grows. There is however a preferred

\(^7\) And if the energy is smaller than \( m \), one reaches the turning point at a finite proper time.
observer located at the point $r_m = (2\mu l^2)^{1/3}$, where the norm of the KVF is unit $V = 1$. Thus, the total energy $\mathcal{E}$ as defined by $\xi$ has the interpretation of being the energy as measured by an static observer at $r = r_m$. The main difference with the previous cases is that this observer is not invariantly defined (it depends on the choice of coordinate $t$), in contrast to the previous two cases where the preferred static observers follow geodesics. As is well known for the AdS case, there is a very large ‘potential wall’ for large values of $r$, such that one has to imprint and increasing amount of energy to a radial particle to climb the wall. In order to send it to infinity, the energy would also have to be unbounded. Thus, no physical (massive) object can reach infinity $\mathcal{I}$. Note also that $\mu$ can not be the energy of the black hole as measured at infinity.

So far, we have considered test particles in a static black hole background and studied the different types of energy one can consider for the particle. What we have not considered is what happens when the particle crosses the horizon and falls into the black hole. That will be the subject of the next section.

IV. ENERGY OF THE BLACK HOLE

In this section we shall consider the issue of the energy of the particle and how it relates to the energy of the black hole. In the previous section we considered only test particles, where the test object does not affect the background black hole. In this part we shall relax that assumption a bit. We will still neglect any back-reaction on the background geometry as long as the particle remains outside the horizon, but shall consider the change in area and mass of the black hole as the objects fall into the horizon.

Let us start with the simplest case, namely an object of rest mass $m$ at rest at infinity in an asymptotically flat spacetime of a black hole of mass $M$. Here, the total energy of the spacetime, as captured by the ADM mass is then $M_{ADM} = M + m$. We are assuming that $m \ll M$ and therefore, there is little ‘interaction’ between the black hole and the test object. Here, the mass $M$ of the black hole, refers to the horizon mass $M_h$ as defined by the isolated horizon formalism [4].

Let us now assume that we drop the object radially towards the BH, following a radial geodesic. Then the total energy of the particle is $\mathcal{E} = m$, and that quantity shall be conserved along the geodesic. The object will cross the horizon at a finite proper time (assuming we did not actually drop it from infinity but rather from a sufficiently distant asymptotic region). The horizon will react to the crossing of the object and will quickly settle down to a new isolated horizon. The question now is: what is the new horizon mass $M'_h$? To answer that question is rather simple. After the object falls into the black hole and it settles, to the future the spacetime will be vacuum and approximately static (in our approximation we are assuming the energy radiated is negligible with respect to $m$), so it is very well approximated by a Schwarzschild metric with mass $M'_h$. In the isolated horizon formalism it has been established that for static spacetimes with an isolated horizon, as is our case, the ADM mass and the horizon mass coincide (More generally, when there is

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8 It has been shown that for asymptotically flat stationary electro-vac spacetimes the horizon mass and the ADM mass coincide. Here we are assuming that the presence of the test object does not modify the horizon mass $M$, so our approximation is valid.
radiation, the Bondi energy at future null infinity and the ‘mass at $i^+$’, coincide \[4\]). That is, the new horizon mass will simply be,

$$M'_h = M_h + \mathcal{E}$$

(25)

which means that the change in horizon mass is equal to the total energy of the test object, $\delta M_h = \mathcal{E}$. The next scenario is to consider a particle that has a total energy smaller that its rest mass, $\mathcal{E} < m$. This could be the case if initially the test object is at rest at some distance from the black hole and is then released to fall into the black hole following a radial geodesic. Just as in the previous case, the horizon mass $M_h$ of the black hole will change by an amount equal to the total energy of the test object $\mathcal{E}$. For our case here,

$$\delta M_h = m \sqrt{1 - \frac{2M}{r_{\text{res}}}}$$

(26)

where $r_{\text{res}}$ is the distance from which the object is released. The extreme case is when the object of mass $m$ is held at rest just outside the horizon. As we saw in the previous section, the total energy of the object can be arbitrarily small as one approaches the horizon, so when released into the black hole, the change in BH mass will be arbitrarily small, approaching zero in the limit.

Let us now recall the first law of black hole mechanics. For the non-rotating case we are considering, there is a relation between the change of mass of the black hole and how much it ‘grows’ as measured by a change in area. The relation is given by,

$$\delta M_h = \frac{\kappa}{8\pi} \delta A_h$$

(27)

where $\kappa$ is the so-called surface gravity and $A_h$ is the horizon area $A_h = 4\pi r_h^2$. The surface gravity can be found from the equation $\ell^a \nabla_a \ell^b = \kappa \ell^b$ with $\ell$ the null generator of the isolated horizon. For static spacetimes it is standard to set $\ell^a = \xi^a$ but, in principle, other choices are allowed. What the Eq. (27) is telling us is that the horizon mass can only be a function of the area $A$ (or radius $r$, of course). Thus $\kappa$ could be in principle an arbitrary function of $r_h$. For any asymptotically flat spacetime in vacuum, with a non-rotating isolated horizon, which includes also the Schwarzschild spacetime, we have that $\kappa_o(r_h) = 1/(2r_h)$, from which one can integrate the first law to obtain \[4,\]

$$M_h(r_h) = \frac{1}{2} \int_0^{r_h} \beta(r) \, dr$$

(28)

with $\beta(r) = 2r \kappa(r)$. This formula is valid for any non-rotating static horizon. Note that, for the standard choice for $\kappa_o$, $\beta_o = 1$, from which we obtain a horizon mass $M_h(r_h) = r_h/2$. Note also that when integrating the 1st law, one is setting $M_h(r_h)|_{r_h=0} = 0^9$.

From our previous discussion regarding energetics, we see that the standard choice for surface gravity leading to an equality between horizon and ADM masses (for electro-vac spacetimes) is fully justified. The reason is that, as we have seen, the change in mass of the horizon has to be equal to the total energy $\mathcal{E}$ of the in-falling object. This energy is ‘measured’ by the KVF $\xi$, and then its natural extension to the horizon is the now null vector $\ell = \xi$. Energy is conserved in a “global sense”.

\[9\] this choice is modified when there are nontrivial static solitonic solutions as in EYM.
Let us now discuss the case of a test particle falling into a de-Sitter black hole. Some of these observations have been previously discussed in [3], so we refer the interested reader to that paper for a deeper discussion of black hole mass and energy in the de-Sitter context. The important point to remember here is that the role of the asymptotic observer in AF gravity gets replaced by the static observer located at \( r_i \). The Killing vector field \( \bar{\xi} = \partial/\partial t \) was rescaled to be of unit norm at the sphere corresponding to \( r_i \), in order to define the total energy. The same strategy was followed in [3] to define the BH horizon mass and energy. With that choice for the Killing vector field \( \xi \), we recover the same global conservation of energy we had in the asymptotically flat case, as the test particle falls into the black hole horizon. The change in horizon mass \( \delta M_h \) corresponds exactly to the total energy \( E \) of the test particle, as measured by \( \xi \). The Bondi energy is now replaced by the energy associated to the cosmological horizon [3]. Energy is conserved in a global sense.

In this part we have seen that the notion of total energy \( E \) for a test particle is consistent with the standard definition of horizon energy for the black hole in order to have (global) conservation of energy. Note that any other choice will not benefit from this feature. Let us now comment on one such possibility.

V. ANOTHER PROPOSAL: ENERGY FOR OBSERVERS AT REST

Any other choice of vector \( \ell \) of the type \( \alpha \xi \), when extended to the horizon will give different values of the surface gravity \( \kappa_\ell \) and therefore, of the measure of energy falling into the black hole, and of its horizon mass. One possibility is that one takes, for instance, the vector \( w^a = \xi/V \) that, as we have seen, corresponds to the four-velocity of an observer at rest. In that case, we can not really extend it to the horizon since \( V = 0 \) there. What one could do is to stay a little bit outside the horizon and define the various quantities there. This is precisely the proposal we shall comment on now.

As we have already noted, if we choose the vector \( w^a = \xi/V \) and try to extend it to the horizon as a generator of the horizon, we do not succeed since the redshift factor \( V \) vanishes there. A recent proposal is to consider instead an observer that is at a small distance, of the order of the Planck length, outside the horizon [3]. Next, the generalized kinetic energy is taken as the “right” notion of energy and then the authors consider a modified 1st law that reflects this observer at rest. Let us describe this procedure and offer some comments. The first step is to look at the first law and see if one can find a new \( \bar{\kappa} \) that captures the fact that the description of the particle falling will be done by the observer at rest. The first attempt, namely to generalize the equation \( \ell^a \nabla_a \ell^b = \kappa \ell^b \) (valid at the horizon) and use the acceleration \( g \) of the observer at rest given by \( g = M r^2 (1 - \frac{2M}{r})^{-\frac{1}{2}} \). Note that, at the horizon, the surface gravity \( \kappa \) satisfies \( \kappa = V g \), namely it is equal to the red-shifted value of the acceleration (that diverges on the horizon). Thus, one could propose that the “effective” surface gravity just outside the horizon is given by,

\[
\bar{\kappa} = \frac{\kappa}{V} = \kappa \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}
\]

(29)

Note that this is precisely the proposal of [3]. Their argument is the following: As seen by the observer at rest, the energy that the observer assigns to the particle is \( \mathcal{E} = \mathcal{E}/V \). Thus,\(^{10}\) which, incidentally, do not coincide for the cosmological horizon, in the \( \Lambda \not= 0 \) case [3].

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\(^{10}\) which, incidentally, do not coincide for the cosmological horizon, in the \( \Lambda \not= 0 \) case [3].
according to the observer, the black hole mass $\tilde{M}$ must have changed by $E$ and not $\mathcal{E}$. Then

$$\delta \tilde{M} = \frac{\mathcal{E}}{V} = \frac{\delta M_h}{V}$$

(30)

Then, using the 1st law, they conclude that $\delta \tilde{M} = \frac{\kappa}{8\pi V} \delta A$. Thus, the corresponding surface gravity for the observers mass $\tilde{M}$ is $\bar{\kappa} = \kappa/V$.

With this choice, there is a local sense of conservation, which means that the energy that the observer sees falling into the black hole is recovered in the mass of the black hole when it grows. However, if we decide to consider this as the notion of energy of the black hole horizon, we loose any contact with the canonical definition of total energy at infinity, so the energy balance that we had before with the energy $\mathcal{E}$ and the ADM energy for the whole spacetime, is lost. This, in particular implies that we have to forgo the notion of ‘global’ conservation of energy, even in the restricted context of stationary backgrounds. It is our belief that these considerations render the proposal of [5] difficult to justify, from a physical viewpoint.\(^{11}\)

VI. DISCUSSION

In this note we have considered test particles on stationary spacetimes and, as concrete examples, in the context of an asymptotically flat, and de Sitter, black hole static spacetimes. As we have argued in detail, there is an important difference between the two main notions of energy. The direct generalization to general relativity, of the energy as measured by an observer, even when well defined independently of the spacetime geometry and observer, does not provide us with information about the gravitational contribution to the energy. On the other hand, when the spacetime is stationary, there is a conserved quantity that can be interpreted as the total energy of the particle, including the gravitational (potential energy) contribution. We have also isolated the gravitational potential energy for certain cases and shown to satisfy a set of physically motivated properties.

We have explored some of the consequences of these notions of energy for the simplest cases of a Schwarzschild and a de Sitter black hole. We considered also the different notions of energy for the black hole itself. For that the formalism that has proved useful is that of isolated horizons. In that case there is a preferred notion of horizon energy that is consistent with energy balance at infinity. If we use this notion of energy for the horizon, then we get a full consistent description of the energetics if we associate to the in-falling object the notion of total energy $\mathcal{E}$. In the de Sitter case, one has to appropriately modify the notion of asymptotic infinity, but the qualitative picture remains intact. We have also considered an alternative description where the energy associated to the particle is the one given by an observer at rest. If we modify the notion of horizon energy to retain local energy conservation, we are lead to a very different mass for the horizon that is however, not tailored

\(^{11}\) Note also that in the proposal of [5] and in subsequent papers, it is argued that the ‘mass’ $\tilde{M}$ of the black hole is proportional to its area. This is only partially true after certain approximations have been made that, in particular, involve the large area limit. One should, nevertheless, not forget that the actual relation between $\tilde{M}$ and area is much more complicated. Making them equal and, furthermore, equating them as operators is then an unjustified extrapolation.
to the global notion of energy at infinity. One should note that, even when we have explicitly considered only static vacuum solutions, one expects these results to be qualitatively similar on other static black hole spacetimes, including matter, and for stationary solutions.

One could also wonder what would be the situation of a test matter field as defined by a stress energy tensor $T_{ab}$. In this case, there is also a distinction between the special relativistic notion of energy momentum flux as seen by an observer $w^a$, given by $j^a = T_{ab}w^b$, and the conserved current when a KVF $\xi^a$ is present, given by $\tilde{j}^a := T_{ab}\xi^b$. Note that since $\nabla_a j^a = 0$, the integral,

$$\mathcal{E}_{\text{matt}} := \int_{\Sigma} d^3x \sqrt{q} \tilde{j}^a n_a = \int_{\Sigma} d^3x \sqrt{q} T_{ab} \xi^a n^b$$

over a Cauchy surface $\Sigma$, is conserved. Again, the difference between the two integrands depends on the redshift factor $V = |\xi^a \xi^a|^{1/2} < 1$ that accounts for the gravitational (potential) energy. In this case, it affects the local notion of energy flux density and distinguishes it from the (non-conserved) generalization of the SR notion of energy defined by $E_{\text{matt}} := \int_{\Sigma} d^3x \sqrt{|q|} T_{ab} n^a n^b$. It should be noted that the conserved energy (31) is closely related to the quantity defined by Komar. If we consider the issue of matter flowing into the black hole, then the relevant quantity representing flux across the horizon will be $T_{ab} \xi^a$, and not the quantity $T_{ab} w^a$. The relation between energy flux and the change in mass of the black hole will again depend on the ‘observer’ we choose to describe the process. Compatibility with the total notion of energy will select the quantity $\mathcal{E}$, while the ‘local’ viewpoint will select the quantity $E$ as in [5]. This discussion also point to the challenges that the proposal of [5] faces in order to be physically compelling.

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