Phase dependent spectrum of scattered light
from two Bose condensates

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Abstract

We calculate the spectrum of the scattered light from quantum degenerate
atomic gases obeying Bose-Einstein statistics. The atoms are assumed to
occupy two ground states which are optically coupled through a common
excited state by two low intensity off-resonant light beams. In the presence
of a Bose condensate in both ground states, the atoms may exhibit light
induced oscillations between the two condensates analogous to the Josephson
effect. The spectrum of the scattered light is calculated in the limit of a
low oscillation frequency. In the spectrum we are able to observe qualitative
features depending on the phase difference between the macroscopic wave
functions of the two condensates. Thus, our optical scheme could possibly
be used as an experimental realization of the spontaneous breakdown of the
U(1) gauge symmetry in the Bose-Einstein condensation.

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The Bose-Einstein condensation (BEC) has finally been observed in a weakly interacting system with well-understood interactions by cooling and trapping alkali metal vapors [1–3]. In these first experiments on BEC the condensates were probed destructively with a short flash of laser light. Recently, Andrews et al. [4] have reported a non-destructive optical detection of a Bose condensate. This detection technique could possibly also be used to measure the spectrum of the scattered light from the condensate, when the driving light is detuned far from the resonance of the optical transitions [3–7].

In the BEC phase transition the Bose gas should acquire nontrivial phase properties. In the traditional reasoning the condensate is given a macroscopic wave function which acts as a complex order parameter with an arbitrary but fixed phase [8]. The selection of the phase in each experiment implicates the spontaneous breaking of the U(1) gauge symmetry. However, even though the condensate is taken to be in a number state with no phase whatsoever, the condensate behaves as if it had a phase [3–7]. The phases of the condensates are expected to exhibit collapses and revivals due to collisions [9–16].

Most of the theoretical studies on optical response of degenerate atomic gases [5–7,18–24] are insensitive to the phase properties of the condensates. Javanainen [25] and Imamoglu and Kennedy [26] have proposed optical schemes to detect spontaneous breakdown of the gauge symmetry in BEC. In Ref. [25] two condensates in different Zeeman states are confined in the same trap, and two phase coherent laser beams are used to drive Raman transitions between the condensates. Amplification of one of the light beams, because of the stimulated Raman scattering, would indicate phase dependent properties. In Ref. [26] nonlocal light scattering between two independent, spatially separated condensates were studied.

In this paper we consider the spectrum of the scattered light from a quantum degenerate Bose-Einstein gas occupying two ground states with a common excited state. Following Ref. [25] all the atoms are confined in the same trap. The possible Bose condensates are in two different Zeeman sublevels $|b\rangle = |g, m\rangle$ and $|c\rangle = |g, m - 2\rangle$. The state $c$ is optically coupled to the electronically excited state $|a\rangle = |e, m - 1\rangle$ by the driving field $E_{d2}$ having a polarization $\sigma_+$ and a dominant frequency $\Omega_2$. Similarly, the state $b$ is coupled to $a$ by
the driving field $E_{d1}$ with a polarization $\sigma_-$ and a dominant frequency $\Omega_1$. We assume that the light fields $E_{d1}$ and $E_{d2}$ propagate in the positive $z$-direction and are detuned far from the resonances of the corresponding atomic transitions. The light fields are also assumed to be in the coherent states. In accordance with our previous work [5, 24] we can immediately write down the Hamiltonian density

$$
H(r) = \psi_b^\dagger(r)H_{\text{c.m.}}\psi_b(r) + \psi_c^\dagger(r)(H_{\text{c.m.}} + \hbar w_{cb})\psi_c(r) + \psi_a^\dagger(r)(H_{\text{c.m.}} + \hbar w_{ab})\psi_a(r) + H_F(r)
- \left( d_{ba} \cdot E_1(r) \psi_b^\dagger(r)\psi_a(r) + \text{h.c.} \right) - \left( d_{ca} \cdot E_2(r) \psi_c^\dagger(r)\psi_a(r) + \text{h.c.} \right),
$$

(1a)

$$
H_F = \int d^3r \, H_F(r) = \hbar \sum_q \omega_q a_q^\dagger a_q.
$$

(1b)

The first three terms reflect the energies, internal and center-of-mass (c.m.), of the atoms in the absence of electromagnetic fields. The frequencies for the optical transitions $a \rightarrow b$ and $a \rightarrow c$ are $\omega_{ab}$ and $\omega_{ac}$ ($\omega_{cb} = \omega_{ab} - \omega_{ac}$), respectively. The last two terms are for the atom-light dipole interaction. The dipole matrix element for the atomic transition $a \rightarrow b$ is given by $d_{ba}$. The Hamiltonian density for the free electromagnetic field is $H_F$. If the light field is fully treated as a quantum mechanical field with its own dynamics, the Hamiltonian may also contain the polarization self-energy term and the dynamical light field should be the electric displacement instead of the electric field [24, 27]. However, if the detunings ($\Delta_1 = \Omega_1 - \omega_{ab} \simeq \Delta_2 = \Omega_2 - \omega_{ac}$) are sufficiently large [25], multiple scattering is negligible and, at first order in $1/\Delta_1$, these effects may be ignored.

We define slowly varying field operators in the Heisenberg picture by $\tilde{\psi}_a = e^{i\Omega_1 t}\psi_a$, $\tilde{E}_1^+ = e^{i\Omega_1 t}E_1^+$, $\tilde{E}_2^+ = e^{i\Omega_2 t}E_2^+$, and $\tilde{\psi}_c = e^{i(\Omega_1 - \Omega_2) t}\psi_c$. In the limit of large detuning, the excited state field operator $\psi_a$ may be eliminated adiabatically:

$$
\tilde{\psi}_a(r) = \frac{1}{\hbar \Delta_1} \left( d_{ab} \cdot \tilde{E}_1^+(r) \psi_b(r) + d_{ac} \cdot \tilde{E}_2^+(r) \tilde{\psi}_c(r) \right).
$$

(2)

We insert Eq. (2) into the Hamiltonian (3) and keep only the terms of first order in $1/\Delta_1$. If the detuning from the two-photon resonance is given by $\delta_{cb} = \Omega_1 - \Omega_2 - \omega_{cb}$, the Hamiltonian density now reads
\[
\mathcal{H} = \psi_b^\dagger H_{\text{c.m.}} \psi_b + \tilde{\psi}_c^\dagger (H_{\text{c.m.}} - \hbar \delta_{cb}) \tilde{\psi}_c + \mathcal{H}_F - \frac{1}{\hbar \Delta_1} \left\{ d_{ab} \cdot \tilde{\mathbf{E}}_1^- d_{ab} \cdot \tilde{\mathbf{E}}_1^+ \psi_b \psi_b^\dagger + d_{ac} \cdot \tilde{\mathbf{E}}_2^- d_{ca} \cdot \tilde{\mathbf{E}}_2^+ \tilde{\psi}_c \tilde{\psi}_c^\dagger \right\} .
\]

(3)

Then, the electric fields may be solved and, according to Ref. [5], the scattered fields are given by

\[
\tilde{\mathbf{E}}_{s1}^+(r) = \frac{1}{\hbar \Delta_1} \int d^3 r' K(d_{ba}; r - r') \left\{ d_{ab} \cdot \tilde{\mathbf{E}}_{d1}^+(r) \tilde{\psi}_b(r') \psi_b(r') + d_{ac} \cdot \tilde{\mathbf{E}}_{d2}^+(r) \tilde{\psi}_c(r') \psi_c(r') \right\} ,
\]

(4a)

\[
\tilde{\mathbf{E}}_{s2}^+(r) = \frac{1}{\hbar \Delta_1} \int d^3 r' K(d_{ca}; r - r') \left\{ d_{ac} \cdot \tilde{\mathbf{E}}_{d2}^+(r) \tilde{\psi}_c(r') \psi_c(r') + d_{ab} \cdot \tilde{\mathbf{E}}_{d1}^+(r) \tilde{\psi}_b(r') \psi_b(r') \right\} .
\]

(4b)

Here we have used the first Born approximation based on the assumption that the incoming fields dominate inside the sample as the multiple scattering is negligible. The kernel \( K(D; r - r') \) is the familiar expression [28] of the positive-frequency component of the electric field from a monochromatic dipole with the complex amplitude \( D \), given that the dipole resides at \( r' \) and the field is observed at \( r \).

We assume the driving electric fields to be plane waves

\[
\tilde{\mathbf{E}}_{d1}^+(r) = \frac{1}{2} \mathcal{E}_1 \hat{\mathbf{e}}_+ e^{i \kappa_1 \cdot r}, \quad \tilde{\mathbf{E}}_{d2}^+(r) = \frac{1}{2} \mathcal{E}_2 \hat{\mathbf{e}}_+ e^{i \kappa_2 \cdot r}.
\]

(5)

We insert these into the Hamiltonian (3) to obtain the Hamiltonian for the matter field dynamics

\[
\mathcal{H}_M = \psi_b^\dagger (H_{\text{c.m.}} - \hbar \delta_1) \psi_b + \tilde{\psi}_c^\dagger (H_{\text{c.m.}} - \hbar \delta_{cb} - \hbar \delta_2) \tilde{\psi}_c + \left( \hbar \kappa \tilde{\psi}_b \tilde{\psi}_c e^{-i \kappa_{12} \cdot r} + \text{h.c.} \right) ,
\]

(6)

where \( \kappa_{12} = \kappa_1 - \kappa_2 \) is the wavevector difference of the incoming light fields. Following the notation in Ref. [23]

\[
\delta_1 = \frac{|\mathcal{E}_1|^2 d_{ab}^2}{4 \hbar^2 \Delta_1}, \quad \delta_2 = \frac{|\mathcal{E}_2|^2 d_{ac}^2}{4 \hbar^2 \Delta_1}, \quad \kappa = \frac{\mathcal{E}_1^* \mathcal{E}_2 d_{ab} d_{ac}}{4 \hbar^2 \Delta_1} .
\]

(7)

The dipole matrix element \( d_{ab} \) contains the reduced dipole matrix element and the corresponding nonvanishing Clebsch-Gordan coefficient. To simplify the algebra, we assume \( \kappa \) to be real.
We assume a translationally invariant and non-interacting Bose gas as in Ref. [25]. The matter field operators are given by the familiar plane wave representations $\psi_b(rt) = V^{-1/2} \sum_k e^{i k \cdot r} b_k(t)$ and $\tilde{\psi}_c(rt) = V^{-1/2} \sum_k e^{i k \cdot r} \tilde{c}_k(t)$. In the absence of light, the c.m. motion in both ground states satisfies the dispersion relation for the frequency $\epsilon_k = \hbar |k|^2 / 2m$.

By assuming that $\kappa_{12}$ is very small we may neglect its contribution to the time evolution. The dynamics of the ground state annihilation operators $\tilde{c}_k$ and $b_k$ may then be easily solved from the Hamiltonian (6). With the definitions $2\tilde{\delta} = \delta_{cb} - \delta_1 + \delta_2$ and $\Omega_R = (\tilde{\delta}^2 + \kappa^2)^{1/2}$ the solutions are given by

$$\tilde{c}_k(t) = e^{i(\tilde{\delta} + \delta_1 - \epsilon_k)t} \left\{ \tilde{c}_k(0) \left( \cos \Omega_R t + \frac{i \tilde{\delta}}{\Omega_R} \sin \Omega_R t \right) - \frac{i \kappa}{\Omega_R} b_{k-k_{12}}(0) \sin \Omega_R t \right\}, \quad (8a)$$

$$b_k(t) = e^{i(\tilde{\delta} + \delta_1 - \epsilon_k)t} \left\{ b_k(0) \left( \cos \Omega_R t - \frac{i \tilde{\delta}}{\Omega_R} \sin \Omega_R t \right) - \frac{i \kappa}{\Omega_R} \tilde{c}_{k+k_{12}}(0) \sin \Omega_R t \right\}. \quad (8b)$$

Before the light is switched on, the atoms in the states $b$ and $c$ are assumed to be uncorrelated. The driving light fields induce a coupling between the two levels. In the presence of Bose condensates in the ground states, the coupling between the two condensates is analogous to the coherent tunneling of Cooper pairs in a Josephson junction [29,25].

According to Ref. [5], the spectrum is obtained by calculating correlation functions for the matter field operators. In particular, in the presence of the condensate the expectation values for single operators are nonvanishing $\langle \psi_c(0) \rangle = (N_c/V)^{1/2} e^{i\varphi_c}$ and $\langle \psi_b(0) \rangle = (N_b/V)^{1/2} e^{i\varphi_b}$. We have assumed that the condensates in the states $c$ and $b$ have the expectation values for the number operators $N_c$ and $N_b$, respectively. The condensate phases $\varphi_c$ and $\varphi_b$ are random variables and are fixed in each experiment.

From the equations (4) and (8) we immediately see that the spectrum of the scattered light is time-dependent. However, with the measurement times short enough compared to the characteristic time scale of the changes in the spectrum, the spectrum may be considered steady. Evidently, we do not want too short measurement times which would wipe out qualitative features from the spectrum. According to Ref. [5], the most relevant choices for the characteristic frequency scales in the spectrum are the effective recoil frequencies
\[ \omega_{Ri} = \hbar (\Delta \kappa_i)^2 / 2m, \] where \[ \Delta \kappa_i = \Omega_i \hat{n} / c - \kappa_i \] is the change of the wave vector of the light field \( \mathbf{E}_i \) upon scattering and \( \hat{n} \) is a unit vector pointing into the direction of the scattered light.

We present a set of assumptions based on the limit of a weak coupling between the ground states to obtain a stable spectrum during short measurement times. We assume that the oscillation frequency \( \Omega_R \) is small, \( \omega_{Ri} \gg \Omega_R \), or \( \omega_{Ri} \gg \bar{\delta} \) and \( \omega_{Ri} \gg \kappa \). The latter indicates \( \Delta_1 \omega_{Ri} \gg R_1 \) \( R_2 \), where \( R_1 = d_{ab} \mathcal{E}_1 / 2\hbar \) and \( R_2 = d_{ac} \mathcal{E}_2 / 2\hbar \) are the Rabi flopping frequencies for the electronic transitions \( b \to a \) and \( c \to a \), respectively. The quantities \( \omega_{Ri} \) are expected to be in the neighborhood of atomic recoil frequencies, \( \omega_{Ri} \sim \epsilon_R \sim 100 \) kHz as a rule of thumb [5]. The detuning \( \Delta_1 \) may be chosen large keeping in mind that the intensity of the scattered light scales as \( 1 / \Delta_1^2 \). If the detuning from the two photon resonance becomes small, the effective linewidth \( \bar{\gamma} \) of the transition \( c \to b \) may have an effect. However, it may be shown to be proportional to \( \Delta_1^{-2} \) or smaller. Thus, we can also safely assume \( \omega_{Ri} \gg \bar{\gamma} \).

The requirement, for the measurement times \( T \) to be short enough, sets the conditions \( \kappa T \ll 1 \) or \( \Delta_1 \gg T R_1 R_2 \) and \( \bar{\delta} T \ll 1 \). With these assumptions the time evolution for the atom operators from Eq. (6) with the short measurement times becomes very simple:

\[ \tilde{c}_k(t) = \tilde{c}_k(0) e^{i(\delta_1 - \epsilon_k)t} \quad \text{and} \quad \tilde{b}_k(t) = \tilde{b}_k(0) e^{i(\delta_1 - \epsilon_k)t}. \]

The two components of the scattered light \( \mathbf{E}_{s1} \) Eq. (4a), and \( \mathbf{E}_{s2} \) Eq. (4b), have polarizations \( \sigma_- \) and \( \sigma_+ \), respectively. By selecting the polarization we may detect these fields separately. Let us first consider the spectrum of the scattered field \( \mathbf{E}_{s1} \). To detect only \( \mathbf{E}_{s1} \), we need to select the polarization \( \mathbf{e} = (\sqrt{2} \hat{e}_{-1} - \tan \theta e^{-i \phi} \hat{e}_z) / \sqrt{\tan^2 \theta + 2} \) from the scattered light, where \( \theta \) is the scattering angle in the conventional spherical representation (the angle of the scattered light with respect to the positive \( z \)-axis). The polar angle \( \phi \) is the angle with respect to the \( x \)-axis in the \( xy \)-plane. This angle has an influence on scattering processes in which the phases of the two circularly polarized light beams are important. The calculation of the angular distribution of the scattered light with the given polarizations and atomic transitions is explained in Ref. [5]. If we neglect the forward scattering, immediately after switching on the light the spectrum is given by
\[ S_1(r; \omega) = C_1(r) [S_1(\omega) + S_{1b}(\omega)], \quad C_1(r) = \frac{\Omega_1^4 \rho_{ab}^2}{16(\tan^2 \theta + 2)\pi^2 \Delta_1 \epsilon_0 c^3 r^2}, \] \hfill (9a)

\[ S_1(\omega) = \begin{align*}
\delta_1 \delta(\omega + \omega_{R1}) N_b (1 + \bar{n}_{b_{\Delta k_1}}^l) + \delta_1 \delta(\omega - \omega_{R1}) N_b \bar{n}_{b_{\Delta k_1}}^l + \\
\delta_2 \delta(\omega + \omega_{R2}) N_c (1 + \bar{n}_{b_{\Delta k_2}}^l) + \delta_2 \delta(\omega - \omega_{R2}) N_c \bar{n}_{b_{\Delta k_2}}^l + 2\kappa \cos(\varphi) \delta(\omega + \omega_{R1}) \delta(\Delta \kappa_1 - \Delta \kappa_2) \sqrt{N_c N_b (1 + \bar{n}_{b_{\Delta k_1}}^l)},
\end{align*} \hfill (9b)

\[ S_{1b}(\omega) = \begin{align*}
\sum_k \delta_1 \delta(\omega - \epsilon_k + \epsilon_{k - \Delta k_1}) \bar{n}_{bk}^l (1 + \bar{n}_{b_{k - \Delta k_1}}^l) + \\
\sum_k \delta_2 \delta(\omega - \epsilon_k + \epsilon_{k - \Delta k_2}) \bar{n}_{ck}^l (1 + \bar{n}_{b_{k - \Delta k_2}}^l). \hfill (9c)
\end{align*} \]

We have explicitly separated the contribution of the Bose condensates into \( S_1(\omega) \). This term is nonvanishing only, if there is a condensate present at least in one of the states \( b \) or \( c \). The term \( S_{1b}(\omega) \) arises from scattering processes between the noncondensate atoms. The expectation values for the number operators for the states \( c \) and \( b \) in the c.m. state \( k \) are
\[ \bar{n}_{ck} = (z^{-1} e^{\beta \hbar (\epsilon_k + \omega_{cb})} - 1)^{-1} \quad \text{and} \quad \bar{n}_{bk} = (z^{-1} e^{\beta \hbar \epsilon_k} - 1)^{-1}. \]

The primes in these terms indicate that the condensate states are excluded to avoid double counting. For example, the notation in the last term in Eq. \((9b)\) \( \bar{n}_{b_{\Delta k_1}}^l \) should be interpreted as \( \Delta \kappa_1 \neq 0 \). The difference between the condensate phases is written as \( \Delta \varphi = \varphi_b - \varphi_c \). We have used the same normalization in Eq. \((9)\) as in Ref. \cite{5}.

The first term in Eq. \((9c)\) describes a scattering process in which an atom in the ground state \( b \) with the c.m. state \( k \) scatters to the c.m. state \( k - \Delta \kappa_1 \) still remaining in the state \( b \). The delta function dictates the energy conservation, which coincides with the theory for Doppler velocimetry of atoms \cite{30} shifted by the effective recoil frequency \( \omega_{R1} \) \cite{5}. The product of the occupation numbers indicates that the scattering is enhanced if the final state is already occupied. The first term in Eq. \((9c)\) and the first two terms in Eq. \((9b)\) coincides with the results given in Ref. \cite{3}, apart from the angular distribution of the scattered light and possibly the Clebsch-Gordan coefficients of the atomic transitions. In the second term in Eq. \((9c)\) an atom scatters from the ground state \( c \) to the ground state \( b \), while the c.m. state undergoes the change \( k \rightarrow k - \Delta \kappa_2 \).

The last term in Eq. \((9b)\) is purely a consequence of the broken gauge symmetry in BEC and it depends on the phase difference \( \Delta \varphi \) between the two condensates in the states
$b$ and $c$. It varies from measurement to measurement, because $\Delta \varphi$ is essentially fixed in each experiment as a random number. The delta function for momenta indicates the exact conservation of the momentum of the two scattered photons for an ideal gas in a spatially homogeneous system. For a weakly interacting gas in a magnetic trap, the momenta of the scattered photons do not need to be exactly the same.

The last term in Eq. (9b) is strongly peaked at $\omega = -\omega_R$. With $\Delta \kappa_1 = \Delta \kappa_2$ and by setting $N_b = N_c$ and $\delta_1 = \delta_2 = \kappa$ the condensate part of the spectrum reduces to an especially simple form:

$$S_1(\omega) = 2\delta_1 \delta(\omega + \omega_R) N_b (1 + \bar{n}_{b\Delta \kappa}) [1 + \cos (\Delta \varphi)] + \delta_1 \delta (\omega - \omega_R) N_b (\bar{n}_{b\Delta \kappa} + \bar{n}_{c\Delta \kappa}) .$$

The effect of the phase difference between the condensates is clearly observed. The peak at $\omega = -\omega_R$ reaches its maximum at $\Delta \varphi = 0$ and completely vanishes at $\Delta \varphi = \pi$.

To detect only $E_{s2}$, we need to select from the scattered light the polarization $\hat{e} = (\sqrt{2}\hat{e}_+ + \tan \theta e^{i\phi} \hat{e}_z)/\sqrt{\tan^2 \theta + 2}$. The scattering spectrum $S_2(r; \omega)$ obtained in this case is the same as $S_1(r; \omega)$ given by Eq. (9) with the following obvious changes: $c \leftrightarrow b$, $\Delta \kappa_1 \leftrightarrow \Delta \kappa_2$, and $\Omega_2 \rightarrow \Omega_1$.

If the detector is insensitive to polarization of the scattered light, the spectrum has contributions from $S_1(r; \omega)$, $S_2(r; \omega)$, and from the cross terms $S_{12}(r; \omega) \propto \langle \tilde{E}_{s2} \tilde{E}_{s1}^+ \rangle + \langle \tilde{E}_{s1} \tilde{E}_{s2}^+ \rangle$. These cross terms depend on the phases of the two circular polarizations and the scattered light does not have a rotational symmetry around the $z$-axis. Neglecting the forward scattering, the resulting spectrum, immediately after switching on the light, reads

$$S_1(r; \omega) = C'_1(r) [S_1(\omega) + S_{1b}(\omega)] + C'_2(r) [S_2(\omega) + S_{2b}(\omega)] + S_{12}(r; \omega) ,$$

$$C'_1(r) = \frac{\Omega_1^4 d_{ab}^2 (2 - \sin^2 \theta)}{64\pi^2 \Delta_1 \epsilon_0 c^3 r^2} , \quad C'_2(r) = \frac{\Omega_2^4 d_{ac}^2 (2 - \sin^2 \theta)}{64\pi^2 \Delta_1 \epsilon_0 c^3 r^2} ,$$

$$S_{12}(r; \omega) = \frac{\Omega_1^2 \Omega_2^2 d_{ab} d_{ac} \sin^2 \theta}{32\pi^2 \Delta_1 \epsilon_0 c^3 r^2} \cos (2\phi + \Delta \varphi) \sqrt{N_b N_c} (\bar{n}_{b\Delta \kappa} + \bar{n}_{c\Delta \kappa}) \bigl( \delta \delta (\omega - \omega_{R1}) n_{b\Delta \kappa} + \delta \delta (\omega - \omega_{R2}) n_{c\Delta \kappa} \bigr) .$$
The spectra $S_1(\omega)$ and $S_{1b}(\omega)$ are given by Eq. (9) and the spectra $S_2(\omega)$ and $S_{2b}(\omega)$ may be obtained from the same expressions with the previously explained changes. The new term (11c) also is purely a consequence of the phase difference between the two condensates. By setting $\Delta \kappa_1 = \Delta \kappa_2$, $N_b = N_c$, $d_{ab} = d_{ac}$, $\delta_1 = \delta_2 = \kappa$, and the scattering angle $\theta = \pi/2$ the total scattered spectrum may be expressed in the simple form: $S(r; \omega) = C'_1(r)[S(\omega) + S_b(\omega)]$, where

$$S(\omega) = 2 \delta_1 \delta(\omega + \omega_R) N_b[2 + \bar{n}'_{b,\Delta \kappa} + \bar{n}'_{c,\Delta \kappa}][1 + \cos(\Delta \varphi)]$$

$$+ 2 \delta_1 \delta(\omega - \omega_R) N_b(\bar{n}'_{b,\Delta \kappa} + \bar{n}'_{c,\Delta \kappa})[1 + \cos(2 \phi + \Delta \varphi)], \quad (12a)$$

$$S_b(\omega) = \sum_{\alpha, \beta} \sum_{k} \delta_1 \delta(\omega - \epsilon_k + \epsilon_{k-\Delta \kappa}) \bar{n}'_{\alpha k}(1 + \bar{n}'_{\beta, k-\Delta \kappa}). \quad (12b)$$

Here the summation over $\alpha$ and $\beta$ denotes the summation over the ground states $b$ and $c$. The phase difference between the condensates again has clearly observable effects on the spectrum. The two condensate peaks in Eq. (12a) oscillate as a function of the phase difference between the two condensates. However, these oscillations depend on the observation point of the scattered light (the angle $\phi$ in the $xy$-plane). The detection of the scattered light at several different points in space (with different values of $\phi$) may provide a scheme to determine the phase difference between the condensates very accurately.

The integration of Eq. (12b) is straightforward in the continuum limit and the results are given in Ref. [5]. For simplicity, we choose $\omega_{cb} = 0$, so that the statistical distributions of the states $b$ and $c$ are identical. We give representative spectra of Eq. (12) in Fig. 1 with different values of $\Delta \varphi$. We choose the detection point at $\theta = \pi/2$ and $\phi = \pi/2$. To simplify further, we set $\omega_R = \omega_D = [k_B T(\Delta \kappa)^2/m]^{1/2}$, where $\omega_D$ is the effective Doppler width corresponding to the change of the wave vector $\Delta \kappa$. For $^{87}$Rb atoms this corresponds approximately to the temperature 270 nK. We remove the integrable singularities of Eqs. (12a) and (12b) by convoluting the spectra with a Gaussian whose variance is 0.01 $\omega_R$. It is assumed that a fraction 0.8 of the atoms in both ground states are in the Bose condensate.

At $\Delta \varphi = 0$ the peak at $\omega = -\omega_R$ has its maximum height. The contribution of the scattering processes into the condensates completely vanishes in the spectrum. The peak at
\[ \omega = \omega_R \] has its minimum height. At \( \Delta \varphi = \pi/2 \) the cosine terms in Eq. (12a) vanish. At \( \Delta \varphi = \pi \) the contribution of the scattering processes out of the condensates vanishes in the spectrum. The peak at \( \omega = -\omega_R \) reaches its minimum, while the peak at \( \omega = \omega_R \) is in its maximum.

The spectra (9) and (11) were calculated immediately after switching the light on. The expectation value of the atom number in the two ground states oscillates with the frequency \( 2\Omega_R \), see Eq. (8). The atom numbers of the condensates oscillate even if the number of atoms in each condensate is initially equal, analogous to the Josephson effect [29,25]. With \( \delta \simeq 0 \) and \( \kappa_{12} \simeq 0 \) the expectation values for the number operators in the two condensates would approximately interchange in the time period of \( \pi/2\kappa \).

We have calculated the spectrum for an ideal, noninteracting gas. In a magnetic trap and in the case of a weakly interacting gas the momentum distribution is broadened, due to both the uncertainty principle and the interparticle interactions. However, the qualitative features in the spectrum are still well recognisable [6,7]. If we wanted to consider Bose condensates in arbitrary momentum states, we would evidently have to abandon the assumption of translational invariance in space and use some other eigenfunction basis than the plane wave basis for matter field operators.

In conclusion, the spectrum of the scattered light from two Bose condensates in a trap may show an unambiguous signature of the broken gauge symmetry in BEC. The phase difference of the macroscopic wave functions for the condensates has clearly observable effects on the spectrum and its value may possibly be determined very accurately from the oscillating heights of the peaks. Recently, Myatt et al. [31] have managed to produce two overlapping Bose condensates in two different angular momentum states of \( ^{87}\text{Rb} \). The condensates were created using nearly lossless sympathetic cooling of one state via thermal contact with the other evaporatively cooled state. The two states \( |F = 1, m = -1\rangle \) and \( |F = 2, m = 2\rangle \) cannot be coupled by two photon transitions, as was assumed in our calculations, but the generalization for the three photon case should be rather straightforward.
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FIGURES

FIG. 1. Spectra of light scattered from a Bose gas occupying two ground states with various values of the phase difference $\Delta \varphi$ between the two condensates. The scattered light is measured at $\theta = \pi/2$ and $\phi = \pi/2$. A fraction 0.8 of the atoms in both ground states are in the Bose condensate. For the solid line $\Delta \varphi = 0$, for the dashed line $\Delta \varphi = \pi/2$, and for the dotted line $\Delta \varphi = \pi$. The origin corresponds to the common laser frequency $\Omega_1 = \Omega_2$. The spectra are expressed in units of the effective recoil frequency $\omega_R$, and the effective Doppler width is chosen as $\omega_D = \omega_R$. To regularize integrable divergences, all computed spectra are convolved with a Gaussian with the variance $0.01 \omega_R$. 
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\[ \Delta \phi = \pi \]