Hidden structure in the excitation spectra of strongly correlated electrons

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Abstract

We identify a structure in the spectra of 1d electronic lattice models characterised by the behaviour of a gapped excitation branch. The switching of this branch between symmetry sectors generates a large-scale rearrangement of the spectrum, which is interpreted as a novel form of transition. We argue that this is best characterised by dynamical correlations at high temperature, as opposed static correlations at low temperature. Remarkable similarities with the behaviour of the cuprates are identified, which offers a fresh perspective on the strange metal regime.

1 Introduction

The Fermi liquid is a phenomenally successful theory, which permits the long-wavelength physics of the vast majority of materials to be understood at a single-particle level. The low-lying excitations are electronic in nature, and have a diverging lifetime as the Fermi surface is approached. Interactions play a minor role in this description. The validity of the Fermi liquid does not extend to 1d however, where instead the long-wavelength physics is described by the Luttinger liquid. Here, due to the constrained geometry, the effect of interactions extends right down to the Fermi surface, and low-lying excitations exhibit spin-charge separation.

The challenge to characterise behaviour that lies outside these paradigms is one of central importance. Driven by the discovery of high-temperature superconductivity [1], a major effort has been undertaken to investigate the anomalous behaviour arising from strong electronic correlations [2, 3, 4]. Here we wish to highlight certain features that are of particular relevance to the present work. On the one hand, neutron scattering studies of the optimally doped cuprates [5, 6], see also iron-based and heavy-fermion superconductors [7], reveal the appearance of an upward spin branch, giving the spin excitation spectrum of the cuprates a distinctive ‘hourglass’ form. On the other, recent cuprate measurements of the Hall coefficient in high magnetic fields reveal a sharp jump in the charge carrier density at the pseudogap transition point, unrelated to any charge ordering transition [8]. Taken together, these results suggest that degrees of freedom switch from carrying charge to carrying spin as the doping is reduced towards the Mott insulating state. At high temperature the region between the Fermi liquid and pseudogap regimes has been dubbed a ‘strange metal’, as it defies a quasi-particle description [9]. A hallmark feature is anomalous high-temperature behaviour of dynamical correlation functions, most notably the electrical resistivity [10].

As these features differ sharply from single-particle behaviour, non-perturbative methods are crucial for gaining a concrete understanding. In this paper we focus on interacting electrons in one dimension, which provides access to a family of exactly solvable models. While this comes at the cost of creating distance to direct experimental relevance, remarkably we find behaviour that is highly reminiscent of the experimental observations above.

We identify a structure in the spectra of a family of 1d electronic models with both spin and charge $SU(2)$ symmetry, characterised by whether branches of quasi-particle excitations are (i) gapless, (ii) belong to the spin or charge symmetry sectors. We focus the discussion on the switching of an excitation branch between the spin and charge symmetry sectors without the closure of a gap. This gives rise to a conjecture for a novel transition, associated with a large-scale rearrangement of the spectrum, and it is proposed that this manifests
itself in anomalous behaviour of dynamical correlation functions at high temperatures. Similarities with the physics of the cuprates identified above leads us to suggest that the strange metal is not a consequence of a quantum critical point, as has often been proposed, but instead corresponds to a 2d analogue of the behaviour we consider here.

The paper proceeds as follows. A general model for strongly correlated electrons is introduced, and distinguished parameter regimes are identified. The structure of excitations in the exactly solvable limits is described, with a focus on the switching of a gapped elementary excitation branch between the spin and charge sectors. The effect of breaking the integrability of the model is considered, and it is argued that the structure is not immediately destroyed. Findings are contrasted with the behaviour of the doped cuprates, and the strange metal regime is reinterpreted. Following the conclusion are two appendices which collect some technical details.

## 2 A model of strongly correlated electrons

We focus our study in 1d on a model that combines the Hubbard model \[11, 12, 13\] (here along \(g = 0\)) and the Hubbard-Shastry B-model \[14, 15\] (here along \(U = 0\)), with Hamiltonian

\[
H = \sum_{j=1}^{L} \left[ T_{j,j+1}^0 + \kappa_T T_{j,j+1}^\pm + \kappa_H V_j^H + \kappa_S V_{j,j+1}^{SS} + \kappa_C \left( V_{j,j+1}^{CC} + V_{j,j+1}^{PH} \right) \right],
\]

(1)

on a periodic chain of even length \(L\). A convenient parametrisation of couplings is given by \(\kappa_T = \frac{1}{\cosh g}\)

\(\kappa_H = U + 2 \tanh g\), \(\kappa_S = -\kappa_C = 2 \tanh g\). The correlated hopping terms

\[
T_{j,k}^0 = -\sum_{\sigma = \uparrow, \downarrow} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma})(1 - (n_{j\sigma} - n_{k\sigma})^2), \quad T_{j,k}^\pm = -\sum_{\sigma = \uparrow, \downarrow} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma})(n_{j\sigma} - n_{k\sigma})^2,
\]

(2)

allow different amplitudes for processes that either preserve (0) or change (±) the number of doubly occupied sites, and the interactions

\[
V_j^H = (n_j^\uparrow - \frac{1}{2})(n_j^\downarrow - \frac{1}{2}), \quad V_{j,k}^{SS} = S_j^z S_k^z + \frac{S_j^+ S_k^- + S_j^- S_k^+}{2}, \quad V_{j,k}^{CC} = \eta_j^z \eta_k^z, \quad V_{j,k}^{PH} = \frac{\eta_j^+ \eta_k^- + \eta_j^- \eta_k^+}{2},
\]

(3)

expressed in terms of the local operators

\[
S_j^z = \frac{n_j^\uparrow - n_j^\downarrow}{2}, \quad S_j^+ = c_{j\uparrow}^\dagger c_{j\downarrow}, \quad S_j^- = c_{j\downarrow}^\dagger c_{j\uparrow},
\]

\[
\eta_j^z = \frac{n_j^\uparrow + n_j^\downarrow - 1}{2}, \quad \eta_j^+ = (-1)^j c_{j\downarrow}^\dagger c_{j\uparrow}, \quad \eta_j^- = (-1)^j c_{j\uparrow}^\dagger c_{j\downarrow},
\]

(4)

include amplitudes for on-site (H) and nearest-neighbour (CC) charge interaction, spin exchange (SS) and pair hopping (PH). The Hamiltonian possesses spin \(SU(2)\) symmetry with generators \(S^z = \sum_j S_j^z, \quad S^\pm = \sum_j S_j^\pm\)

and and charge \(SU(2)\) symmetry with generators \(\eta^z = \sum_j \eta_j^z, \eta^\pm = \sum_j \eta_j^\pm\). Moreover, along the lines \(U = 0\) and \(g = \pm \infty\) the model has a fermonic symmetry, presented in Appendix A. The model is solvable by Bethe ansatz along the lines \(U = 0, \pm \infty\) and \(g = 0, \pm \infty\), and we restrict attention to the half-filled and zero-magnetised ground state. In addition the model is self-dual under the Shiba transformation \[16\] which interchanges spin and charge and relates \((g, U) \leftrightarrow (-g, -U)\), and so we further concentrate on the most physically interesting region of the parameter space, \(U \geq 0\). A schematic of the parameter space of the model is presented in Fig. 1.

The Hamiltonian has the most general two-site term compatible with translation invariance, inversion symmetry \((H_{j,k} = H_{k,j})\), and invariance under both the spin and charge \(SU(2)\) symmetries. Motivation to study this model comes foremost from the notion that symmetry considerations provide the natural framework
for characterising physics (with the hope that deviations from this model are best understood with respect to it). The Hamiltonian remains a reasonable effective single-band model well beyond the familiar Hubbard line. In the context of Mott insulating behaviour, the suppression of double occupancy of sites and the appearance of antiferromagnetic correlations is often attributed to a large $\kappa_H$, but we wish to stress that it can also be associated with small $\kappa_T$ and $\kappa_S > 0$.

At a more formal level, for an electronic basis of states $\{\downarrow\}, \{\uparrow\}, \{0\}, \{\uparrow\downarrow\}$, which has an inherent graded structure, a natural symmetry to consider is the Lie supergroup $SU(2|2)$[17]. Indeed this symmetry underlies the integrability of the model along the lines $g = \pm \infty$[18]. It is noteworthy that the algebra generating this symmetry admits an exceptional central extension which underlies the integrability of the 1d Hubbard model[19], and the B-model is a distinct integrable model which originates from the same algebraic structure. This motivates a detailed comparison of these two models.

### 3 Structure of excitations: exact results

The foundation of the paper is the combined description of elementary excitations in the exactly solvable limits of the model. The low energy nature of excitations in 1d is well understood in terms of Luttinger liquid theory[20, 21], and its extension taking into account non-linearity of the spectrum[22]. We will however identify a structure beyond the reach of these methods, involving excitation branches that are disconnected from the ground state.

A key feature of interactions in 1d models is the fractionalisation of the underlying degrees of freedom. This was first discovered for the antiferromagnetic Heisenberg model where it was realised that the elementary excitations are not (spin $\pm 1$) magnons with momentum range $2\pi$, but rather (spin $\pm \frac{1}{2}$) spinons with momentum range $\pi$, which are necessarily created in pairs[23].

For the interacting electronic models under consideration here, the electronic, spin and charge degrees of freedom are fractionalised into (spin $\pm \frac{1}{2}$, charge 0) spinons and (spin 0, charge $\pm 1$) holons in an analogous way[24]. Physical excitations must be composed of a combined even number of spinon and holon excitations, with these elementary excitations having momentum range $\pi$.

A subtlety here concerns the number of elementary excitation branches. As the dimensionality of the local Hilbert space $\{\downarrow\}, \{\uparrow\}, \{0\}, \{\uparrow\downarrow\}$ is four, we expect at least three branches$^4$ and indeed there are exactly three in the regimes on which we focus$^5$. We find that the three branches are distributed between the spin and charge sectors, with one of the sectors having an additional branch. It is this additional branch which generates the hidden structure in the spectrum that gives the paper its title.

It is worthwhile to also mention two general facts about the nature of the low-energy excitations here. On the one hand, the generalisation of the Lieb-Schultz-Mattis theorem to electronic systems tells that the ground state is either gapless or degenerate[26]. On the other, bosonisation considerations, taking into account the two $SU(2)$ symmetries of the model, tell that there cannot be linearly dispersing gapless excitations in both the spin and charge sectors[27]. Thus in addition to an upper branch belonging to either the spin or charge sector, the structure of the spectrum is also characterised by whether the single gapless branch belongs to the spin or charge sector.

As systems with Mott insulating ground states at half-filling are of greatest physical relevance, we will focus the presentation on the boundary of the upper-right quadrant of Fig. 1. With the exception of the free fermion point (at the origin), the ground state has gapless spin excitations and gapped charge excitations everywhere here. It is also in this quadrant that we observe an excitation branch switching between carrying spin and carrying charge. To complete the study, we then describe the excitations on the other integrable lines of Fig. 1.

We adopt a convenient notation for characterising the elementary excitations[25]: $C_x S_y$, where $x$ and $y$ represent the number of gapless excitations in respectively the charge and spin sectors, and the overlines

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$^1$A special case is the non-interacting limit, where the Hilbert space splits in two. This results in two identical branches, and the third branch trivialises. It is the fate of these hidden degrees of freedom upon the introduction of interactions that we investigate in this paper.

$^2$Many more branches may appear if there exist bound states in the spectrum[25], which is the case if the ground state is not a singlet of both the spin and charge $SU(2)$ symmetries. These bound states transform under higher-dimensional representations of the symmetries, the generators of which act on more than one site.
Figure 1: A schematic of the parameter space of the Hamiltonian of Eq. 1. The structure of the excitations on the integrable axes of the diagram are labelled by $C_xS_y$, where $x$ and $y$ represent the number of gapless excitation branches in respectively the charge and spin sectors, and the overlines denote the number of branches in each sector. The origin corresponds to free fermions, from which the physics is non-perturbative. The structure of the excitations change between the axes in a manner which either changes the gap structure or does not. The model has a gapless excitation throughout the phase diagram. The ground state remains exactly identical along the outer boundary of the upper-right quadrant, and also along the outer boundary of the lower-left quadrant.

denote the number of branches in each sector (in this example two in the charge sector and one in the spin sector). Explicit expressions for the dispersion curves of the integrable models are collected in Appendix B.

3.1 Boundary of the upper-right quadrant

Here we describe the elementary excitations on the boundary of the upper-right quadrant of Fig. 1 along which the model is integrable, and identify the ‘inverted transition’.

First we take the repulsive Hubbard model, at $g = 0$, $U > 0$. The dispersion curves for the elementary excitations are presented on the left side of Fig. 2. The spinons are gapless, whereas there is a Mott gap for the holons for any $U > 0$ [29]. The $\pi$ range of the spinon momentum indicates a single branch in the spin sector. In contrast, the $2\pi$ range of the holon momentum indicates the existence of two branches of the fractionalised elementary holons. We thus label the excitations as $C_0\overline{S_1}$. (From a microscopic viewpoint, holons are dressed electrons, and spinons are dressed (spin $\pm 1$) spin-flips [24].)

Along the $U = 0$, $g > 0$ line, the B-model’s ground state also has gapless spinon and gapped holon elementary excitations, and their dispersion curves are presented on the right side of Fig. 2. It can clearly be seen that here the extra branch appears in the spin sector, in the form of a distinctive ‘hourglass’ dispersion, and so we label the model as $\overline{C_0S_1}$. (In this case, spinons are dressed electrons, and holons are dressed excitations of (charge $\pm 2$) spin-singlet paired electrons [15].)

We now proceed to examine the upper-right quadrant of Fig. 1 by detailing the behaviour of excitations as one traces clockwise around its boundary. Starting with the repulsive Hubbard model, increasing $U$ causes the charge gap to diverge, and in the limit $U \rightarrow \infty$ the physics is projected onto the spin degrees of freedom $\{|\downarrow\rangle, |\uparrow\rangle\}$. Increasing $g > 0$ here, the model becomes that of the $J = -2 \tanh g$ antiferromagnetic XXX chain, and only the spinon excitations remain in the spectrum.

Next along the $g = \infty$ line the model becomes the Essler-Korepin-Schoutens (EKS) model [18]. This

\[^{3}\text{Taking the limit to } U = g = \infty \text{ at below half-filling, one obtains the } t-J \text{ model at its supersymmetric point } (J/t = 2), \text{ whose} \]
Figure 2: Dispersion curves for the elementary spinon (s) and holon (h) excitations above a half-filled, non-magnetised ground state. H: the repulsive Hubbard model, for $U = 1, 2, 4$. The spinon momentum extends over a range $\pi$ while the holon momentum extends over a range $2\pi$, indicating two branches in the charge sector. For small $U$ the charge gap is beyond the resolution of the plot. B: the ($g > 0$) B-model, for $g = 0.5, 1, 2$. Here the holon momentum extends over a range $\pi$, while the spinon dispersion curves exhibit an ‘hourglass’ shape with two branches over a momentum range $\pi$.

requires some discussion. The Hamiltonian acts essentially as graded permutations, and so preserves the number of doubly occupied sites. That is, $[H(g = \pm \infty), V^H] = 0$, and so the coefficient of $V^H = \sum_j V^H_j$ in the Hamiltonian can be regarded as a chemical potential for doubly occupied sites. Varying $U$ along this line solely results in shifts of the spectrum, it does not affect the eigenstates in any way. Indeed, the ground state is precisely that of the antiferromagnetic XXX chain for $U > -4 + 4 \log 2 \approx -1.23$ [31]. Spectral functions are solely influenced by the low energy spinon in this regime, in particular the charge dynamical structure factor is identically zero as the operators $\eta^z, \eta^\pm$ annihilate the ground state. Nevertheless, as $U$ decreases along the right-hand side of the quadrant, the degrees of freedom corresponding to the two holon branches of the Hubbard model return to the spectrum, see Fig. 3. They appear separately however, split between the spin and charge sectors. In addition, both are charged under $V^H$ here. That the upper spinon branch shifted its momentum by $\pi$ relative to the upper holon branch of the Hubbard model can be interpreted in terms of the charge operators $\eta^\pm$ having momentum $\pi$ while the operators $S^\pm$ have momentum 0, see Eqs. (4). At $U = 0$, the two spin branches are brought together, with the bottom of the upper branch touching the top of the low energy spin branch. On moving to the B-model line, conservation of $V^H$ gets broken and the two spinon branches cross in the ‘hourglass’ shape of Fig. 2.

Let us emphasise some points here. The ground state of the model remained exactly identical along the outer boundary of the quadrant. (Indeed, one could imagine contracting the outer boundary to a point.) There was no change in the low energy excitations, the spin remained gapless and charge remained gapped throughout. At the same time however, the nature of an elementary excitation branch changed, switching between the spin and the charge sectors. These features are in stark contrast to a widely held perspective that a system’s excitations are intimately linked to its ground state, and cannot change unless an excitation gap closes. Instead the gap for the switching branch diverged, allowing it to completely disconnect from the ground state. This can be interpreted as follows: as elementary excitations have energy of order 1, while the width of the spectrum is of order $L$, when an excitation energy starts to grow with system size it will disconnect from the ground state in the thermodynamic limit, allowing it to mix with the bulk of the spectrum and thereby change its nature. This can be thought of as an ‘inverted transition’.

excitations are studied in [30]. Here there are three states per site, $\{|\downarrow\rangle, |\uparrow\rangle, |0\rangle\}$, and the spinon and holon excitations both have a single branch with momentum range $\pi$. 

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Figure 3: Dispersion curves for the elementary excitations of the EKS model, for $U = 0, 0.5, 1$, above a half-filled, non-magnetised ground state. In addition to the holon $(h)$, there are two spinon excitations $(s)$ and $(s')$. Both $(h)$ and $(s')$ are also charged under $V^H$, and so varying $U$ translates the energy of these branches, while leaving the $(s)$ branch invariant. A horizontal line is drawn in the first two panels to guide the eye to the touching of the two spin branches when $U = 0$, which matches the $g \to \infty$ limit of the B-model.

3.2 Boundaries of the other quadrants

Now we turn to the other integrable lines appearing in Fig. 1. The structure of the excitations for the negative-$U$ Hubbard model and negative-$g$ B-model follow from the self-duality of the model [32], and are labelled respectively as $\mathcal{C} \mathcal{T} \mathcal{S}^{0}$ and $\mathcal{C} \mathcal{T} \mathcal{S}^{0}$. Similarly the behaviour of the excitations on the outer boundary of the lower-left quadrant can be inferred from the above discussion on the upper-right quadrant, and again a gapped branch switches between the charge and spin sectors.

It remains to consider the upper-left and lower-right quadrants of Fig. 1. In these a gap closing transition is guaranteed, as it is the gapless excitation that switches between the spin and charge sectors. Let us trace around the boundary of the upper-left quadrant, this time proceeding anti-clockwise. On decreasing $g < 0$ from the $U \to \infty$ limit of the repulsive Hubbard model the Hamiltonian immediately becomes that of the $J = 2\tanh|g|$ ferromagnetic XXX spin chain, whose excitations are quadratically dispersing spin $\pm 1$ magnons and their bound states. Decreasing $U$ along the $g = -\infty$ line brings back the two additional branches, which are again charged under $V^H$. Both however have non-dispersive flat bands. One is an electronic excitation, allowed as the degenerate ground state breaks spin-charge separation, and the other carries spin 1. When $U$ becomes 4, these flat bands touch the ground state, and induce a sudden change in the nature of excitations.\footnote{This highly singular point also corresponds to a limit of the Hubbard-Shastry $A$-model, which for finite interaction parameter describes itinerant ferromagnetism in 1d [13, 15].}
As $U$ is further lowered the ground state becomes ferrimagnetic\footnote{As we restrict attention to the zero magnetisation sector, the ground state is at the centre of a degenerate multiplet, the size of which decreases along this interval.} down as far as $U = 4 - 4\log 2 \approx 1.23$, and on reaching here the excitations become again spinons and holons, with the extra branch reappearing in the charge sector. Beyond this point the spinons acquire a gap and the holons linearly disperse, and this behaviour continues along the $g < 0$ line of the B-model. This analysis demonstrates the manner in which the ground state becomes degenerate here, as must happen to be compatible with the two general facts on the nature of the low-energy excitations mentioned above.

4 On breaking integrability

A striking feature emerged from the above study of the integrable regimes of the model: the switching of an elementary excitation branch between the spin and charge sectors without a change in the ground state. Specifically, the upper holon branch of the Hubbard model transformed into the upper spinon branch of the B-model. Elementary excitations constitute the building blocks which generate the entire spectrum of each of
these models, and thus the change of the branch reflects a large-scale rearrangement of the spectrum as one tunes between these two models. We now switch attention to the interior of the upper-right quadrant. Our analysis focuses on the region of parameter space, extending in from the boundary of the quadrant, for which the ground state remains non-degenerate and has a charge gap\(^6\).

It is not straightforward to proceed. Away from the integrable lines the excitations no longer correspond to exact eigenstates of the system. Unlike Landau’s theory of the Fermi liquid where the single-particle nature of a free electron gas facilitates the characterisation of quasi-particle excitations in the presence of interactions\(^33\), for an interacting integrable model the excitations are inherently fractionalised. As a result, spectral functions are broadened already at the integrable lines, even though elementary excitations can be regarded as having infinite lifetimes.

The characterisation of quasi-particle excitations in 1d in the vicinity of integrable models is an important and long-standing open problem. Despite the absence of a firm theoretical framework, the perspective that appropriate quasi-particles connect directly to elementary excitations at some nearby integrable point is a common one, see e.g.\(^34\)\(^35\). It is well substantiated, both theoretically and experimentally. The anomalously slow thermalisation\(^36\)\(^37\) of systems which are not integrable (but close to it) is most naturally interpreted in this way, as the local conserved operators of the integrable model are directly related to its particle content through string-charge duality\(^38\). A key feature of interacting electronic systems in 1d is the separation of spin and charge, which is necessarily understood in terms of excitations distinct from those of a non-interacting system. This is well described in the low-energy long-wavelength limit by Luttinger liquid theory\(^21\). Experimentally, there exists a growing body of evidence that effectively 1d compounds have quasi-particle excitations which exhibit spin-charge separation over a wide range of energies\(^39\)\(^40\)\(^41\)\(^42\)\(^43\)\(^44\)\(^45\). Also noteworthy are predicted and observed violations of the Wiedemann-Franz law in one dimension\(^46\)\(^47\).

Moreover, although integrability is lost in the interior of the quadrant, the model retains both the spin and charge SU(2) symmetries. This has a number of consequences. The ground state, being non-degenerate, is then a singlet of both SU(2) symmetries\(^7\). It is thus vacuum-like, in that it has only particle-like excitations and no hole-like excitations, as can be seen for the integrable models in Figs. 2\(^3\)\(^6\). This enhances the stability of the elementary excitations under perturbation, as collisions are suppressed. That is, there is no quadratic term in the spinon dispersion\(^22\), and so the energy window of excitations with energy \(\omega\) scales as \(\omega^3\) at low \(\omega\), significantly constraining the decay of excitations in comparison to the \(\omega^2\) scaling which arises upon breaking spin-inversion symmetry. With regards to a low energy field theory, the symmetries imply that the spin and charge degrees of freedom cannot be coupled by any relevant or marginal operators\(^48\). Thus spin-charge separation is robust at the lowest energies and, with the interactions preserving both the spin and charge symmetries, there is no reason not to expect it to be at higher energies also.

We thus adopt the perspective that on departing from the integrable lines the quasi-particle excitations directly connect to the elementary excitations at the lines, and in particular remain spin-charge separated. This leads to an immediate conflict: coming from the Hubbard model line there are two holon branches and a single spinon branch, while coming from the B-model there is a single holon branch and two spinon branches. Something must happen in between.

Unlike the boundary of the quadrant, where symmetries allow for the exact characterisation of the entire spectrum, there is no guarantee of well defined quasi-particle excitations throughout the whole interior. Indeed, the necessity of a large-scale rearrangement of the spectrum is most naturally compatible with a breakdown of a quasi-particle description of the degrees of freedom characterised by the switching branch as one traces through the interior of the quadrant.

5 On strange metals

Quantum phase transitions are associated with an excitation gap closing, which leads to an abrupt change of the ground state and its excitations, and exhibits signatures in static correlation functions at low to zero temperature\(^49\). In contrast, what we have observed here is that excitations can change their nature without

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\(^6\)An exact diagonalisation study on small system sizes indicates this region covers the whole interior.

\(^7\)The situation is different in higher dimensions where the ground state on a bipartite lattice (necessary for the charge SU(2) symmetry) will typically be Néel ordered at zero temperature.
the closure of a gap. Not only is there no dramatic change in the ground state, we have demonstrated the existence of a path in parameter space along which the ground state remains exactly the same as an excitation branch switches between symmetry sectors.

We thus do not expect any discernible feature of this behaviour at low temperatures. Instead, given that a switching branch implies a large-scale rearrangement of the spectrum, we propose that a signature may appear in the high to infinite temperature regime. Specifically, we advocate a detailed investigation of the high temperature behaviour of dynamical correlation functions, which are quantities that remain non-trivial right up to infinite temperature.

This proposal is motivated in part by the behaviour of the cuprate materials, to which our analysis bears some striking resemblances. As highlighted in the introduction, these exhibit a reduction in charge carrier density and the appearance of an upward spin branch (with notable similarity to the hourglass dispersion in the right panel of Fig. 2) as doping is reduced and the system moves from the Fermi liquid regime to the pseudogap regime. This can be compared with the transition of degrees of freedom from the charge sector to the spin sector as one goes from the half-filled Hubbard model to the half-filled B-model. At high temperatures, the strange metal lies between the Fermi liquid and pseudogap regimes in the phase diagram of the cuprates. It is worthwhile to remark that the anomalous behaviour exhibited by the electrical resistivity here is observed up to the highest temperatures measured, to the extent that an inferred mean free path for a potential quasi-particle would be smaller than the electron’s de Broglie wavelength $\lambda$. This provides clear support for our proposal, as the electrical resistivity is related to the zero frequency component of the current-current correlation function.

These observations lead us to suggest that the 1d analogue of a strange metal can be found in the interior of the upper-right and lower-left quadrants of Fig. 1 and moreover that the strange metal itself corresponds to a 2d analogue of what we describe here. Let us be clear that our analysis is restricted to half-filling, while the strange metal appears in the cuprates when the system is doped away from half-filling. The spectra of the integrable lines is not just exactly characterised at half-filling however, but at all fillings and magnetisation densities. We anticipate that it is possible to identify a region in the interior of the quadrants where related behaviour can be induced by varying the filling.

6 Conclusion

Focusing on the solvable limits of a general nearest-neighbour Hamiltonian with spin and charge $SU(2)$ symmetry, we have revealed a hidden structure in the spectra of 1d electronic lattice models. This gives rise to a four-way classification of the excitations of critical phases, as summarised in Fig. 1.

A standout feature of the analysis is the switching of an excitation branch between the spin and charge sectors without the closure of a gap. This is explicitly observed on a special path in parameter space along which the ground state remains precisely that of the antiferromagnetic Heisenberg spin chain. The switching of the branch corresponds to a large-scale rearrangement of the spectrum, which reflects a novel form of transition. Unlike quantum phase transitions which influence behaviour at low temperatures, we instead conjecture that signatures for this transition appear at high temperature through dynamical correlation functions.

We have highlighted remarkable similarities with behaviour observed in the cuprates, materials generally regarded to admit an effective 2d description. In particular, this leads to the suggestion that the strange metal results from a 2d analogue of the novel transition proposed here. This is in contrast to the widespread, yet controversial, perspective that strange metal behaviour is the consequence of some hidden quantum critical point $^50$.

Going forward, it would be most interesting to further map out the phase diagram of Fig. 1 both in 1d and above, and also away from half-filling. The Hubbard model and the B-model appear to capture two distinct forms of Mott insulating behaviour, and remain distinguished models beyond 1d. As a system’s degrees of freedom are characterised by its symmetries, it is hoped that this will lead to a more complete understanding of strongly correlated electrons.
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A Additional symmetries

The Hamiltonian of Eq. [1] has many additional symmetries along its exactly solvable lines in 1d, see [51] [52]. In this appendix we describe a fermionic symmetry that the model possesses along the lines $U = 0$ and $g = \pm \infty$ when put on any bipartite lattice with a unique coordination number.

Let us first discuss a generalised notion of symmetry, which can be called dynamical symmetry. We call a symmetry dynamical if its generators, $A_\gamma$, say, do not commute with the Hamiltonian, but rather obey $[H, A_\gamma] = \lambda_\gamma A_\gamma$ for some constant scalars $\lambda_\gamma$. The generators are not conserved under Heisenberg’s equation of motion $\frac{dA_\gamma}{dt} = \frac{\partial A_\gamma}{\partial t} + \frac{i}{\hbar}[H, A_\gamma]$, but instead a dynamical set of generators $\hat{A}_\gamma(t) = e^{-i\lambda_\gamma t/\hbar}A_\gamma$ are, and these still satisfy the same algebra. Relevant examples of dynamical symmetry are given by the charge $SU(2)$ and spin $SU(2)$ symmetries of the model when respectively a chemical potential, coupling to $\eta^2$, or a magnetic field, coupling to $S^z$, are added to the Hamiltonian.

The B-model, which resides along the $U = 0$ line, has a richer dynamical symmetry. Together with the charge and spin $SU(2)$ symmetry generators, the super-generators

$$Q_{\omega \uparrow} = \sum_{j=1}^{L} Q_{j, \omega \downarrow} = \alpha_+ c_{j, \uparrow}^\dagger - \beta n_{j, \downarrow} c_{j, \uparrow}^\dagger, \quad Q_{\omega \downarrow} = \sum_{j=1}^{L} Q_{j, \omega \uparrow} = (-1)^{j+1}(\alpha_- c_{j, \downarrow} - \beta n_{j, \uparrow} c_{j, \downarrow}),$$

with $\alpha_+ = (\beta \pm 1/\beta)/2$ and $\beta = \sqrt{\tanh(g/2)}$, obey the centrally extended $su(2|2)$ symmetry algebra:

$$\{Q_{j, \omega \downarrow}, Q_{j', \omega' \uparrow}\} = \{Q_{j, \omega \uparrow}, Q_{j', \omega' \downarrow}\} = S_{j}^+, \quad \{Q_{j, \omega \uparrow}, Q_{j', \omega' \downarrow}\} = \{Q_{j, \omega \downarrow}, Q_{j', \omega' \uparrow}\} = S_{j}^-, \quad \{Q_{j, \omega \uparrow}, Q_{j', \omega' \uparrow}\} = \frac{\coth g}{2} - S_{j}^- - \eta_{j}^\dagger, \quad \{Q_{j, \omega \downarrow}, Q_{j', \omega' \downarrow}\} = \frac{\coth g}{2} + S_{j}^+ - \eta_{j}^\dagger.$$

The fermionic generators are not symmetries of the B-model for any value of the coupling $g$, but rather they are dynamical symmetries obeying (where here we replace $V_j^H$ by $V_{j,k}^H = \frac{1}{2}(V_j^H + V_k^H)$ in the bond Hamiltonian)

$$[H, Q_{\omega \uparrow}^\dagger] = z Q_{\omega \sigma}^\dagger, \quad [H, Q_{\omega \downarrow}] = z Q_{\omega' \sigma}^\dagger, \quad [H, Q_{\omega \sigma}] = -z Q_{\omega' \sigma}, \quad [H, Q_{\omega' \sigma}] = -z Q_{\omega \sigma},$$

where $z$ is the coordination number of the lattice. At the free fermion point, $g = 0$, the central charges of the algebra diverge.

The model also possesses a dynamical fermionic symmetry along the lines $g = \pm \infty$ [18]. The fermionic charges correspond to the $g \to \infty$ limit of those of the B-model, and in addition $V^H$ also commutes with the Hamiltonian, which enhances the symmetry to $U(2|2)$. At $U = -4$ on the $g = \infty$ line and $U = 4$ on the $g = -\infty$, this dynamical fermionic symmetry becomes a full symmetry of the model.

\[\text{In 1d, these symmetries becoming dynamical is associated with the appearance of towers of bound states in the spectrum.}\]
B Displacement curves

In this appendix we present the displacement relations for the excitations of the model along its exactly solvable lines, which can be found, up to conventions, in \cite{12,15,53}. These are sufficient to reproduce the plots in Figs. 2 and 3. While we restrict attention to excitations above a half-filled, zero-magnetised ground state, we express them for a more general model \( H - 2 \mu q^2 - BS^2 \), so that the dressed spin and charge of the excitations can be read off from \(-\frac{\partial E}{\partial B}\) and \(-\frac{\partial E}{\partial \mu}\).

Before proceeding let us first introduce some useful conventions. We denote convolutions between kernels \( K(v, t) \) and functions \( f(v) \) as

\[
(K * f)(v) = \int_{-\infty}^{\infty} dt \ K(v, t) f(t), \quad (K^* f)(v) = \int_{-1}^{1} dt \ K(v, t) f(t), \quad K^* f = K * f - K * f.
\]

Some useful functions are

\[
s(v) = \frac{1}{4c \cosh \frac{\pi v}{c}}, \quad K_M(v) = \frac{M}{\pi c(M^2 + v^2/c^2)}, \quad \Theta_M(v) = 2 \arctan \left( \frac{v}{cM} \right), \quad \Upsilon(v) = i \log \left[ \frac{\Gamma\left(1 + \frac{iv}{c}\right) \Gamma\left(1 - \frac{iv}{c}\right)}{\Gamma\left(1 - \frac{iv}{c}\right) \Gamma\left(1 + \frac{iv}{c}\right)} \right], \quad \Psi(v) = \frac{\pi}{2} - 2 \arctan \left( \exp \left( \frac{\pi v}{2c} \right) \right).
\]

A function appearing on the left side of a convolution is regarded as a two-variable function \( K(v, t) \equiv K(v-t) \).

B.1 Hubbard model with \( U > 0 \)

The energy and momenta of the spinon and holon excitations of the Hubbard model are given parametrically by

\[
E_s = s^* (e_+ - e_-) - B/2, \quad P_s = -\frac{1}{2\pi} \Psi^* \left( \frac{dp_+}{dv} - \frac{dp_-}{dv} \right) + \pi/2,
\]

\[
E_h^\pm = -e_\pm + s^* e_\mp - \mu, \quad P_h^\pm = p_\pm - \frac{1}{2\pi} \Upsilon^* \left( \frac{dp_+}{dv} - \frac{dp_-}{dv} \right) - \pi/2,
\]

where \( E_h^\pm \) and \( P_h^\pm \) take values on the interval \((-1, 1)\), and \( p_\pm(v) = \frac{1}{2} \log(\pm \sqrt{1 - v^2}) \), \( e_\pm(v) = -2 \cos p_\pm(v) - 2c \), \( e_\mp(v) = e_+(v + ic) + e_+(v - ic) \), and \( c = U/4 \).

B.2 B-model with \( g > 0 \)

The energy and momenta of the spinon and holon excitations of the B-model are given parametrically by (correcting a shift by \( \pi/2 \) typo in Eq. (3.14) of \cite{15})

\[
E_s^\pm = e_\pm - s^* e_\mp - B/2, \quad P_s^\pm = p_\pm - \frac{1}{2\pi} \Psi^* \left( \frac{dp_+}{dv} - \frac{dp_-}{dv} \right) - \pi/2,
\]

\[
E_h = -s^* (e_+ - e_-) - \mu, \quad P_h = -\frac{1}{2\pi} \Psi^* \left( \frac{dp_+}{dv} - \frac{dp_-}{dv} \right) + \pi/2,
\]

where \( E_s^\pm \) and \( P_s^\pm \) take values on the interval \((-\infty, -1) \cup (1, \infty)\), and

\[
p_\pm(v) = \frac{1}{2} \log \left( \frac{-\sqrt{1 + c^2 \pm iv \sqrt{1 - 1/v^2}}}{c + iv} \right),
\]

\[
p_\mp(v) = p_+(v + ic) + p_+(v - ic), \quad e_\pm(v) = -2 \cos p_\pm(v), \quad e_\mp(v) = e_+(v + ic) + e_+(v - ic), \quad c = \sinh g.
\]

B.3 EKS model

We will break up the description of excitations of the EKS model, to focus on the physically most interesting regime \( U > 0 \) for the Hamiltonian of Eq. (1).
First focusing on the region $U > -4 + 4 \log 2 \approx -1.23$ of the model on the $g = \infty$ line, the energy and momenta of the spinons ($s$) and ($s'$), and holon ($h$) excitations are given parametrically by
\begin{align}
E_s &= 4 \pi s - B/2, \\
E_{s'} &= 4 - 4 \pi K_2 \ast s - B/2 + U, \\
E_h &= 2 - 4 \pi K_1 \ast s - \mu + U/2, \\
P_s &= \Psi + \frac{\pi}{2}, \\
P_{s'} &= \Psi + \Theta_1 + \pi/2, \\
P_h &= -\Upsilon + \pi/2,
\end{align}
(14)
with $c = 1$. In the limit $U \to \infty$ the ($s'$) and ($h$) excitations disappear from the spectrum, and just the spinon excitation ($s$) of the $J = 2$ antiferromagnetic XXX chain remains.

Next we turn to the $U > 4$ region of the $g = -\infty$ line. Here the dispersive excitations are the magnons of the $J = 2$ ferromagnetic XXX chain, and their bound states ($M = 1, 2, 3, \ldots$), and their dispersion is given parametrically by
\begin{align}
E_M &= 4 \pi K_M - MB, \\
P_M &= \pi - \Theta_M,
\end{align}
(15)
with $c = 1$. In addition there are electronic (e) and magnonic (1') non-dispersive flat bands whose energy depends on $U$ as
\begin{align}
E_e &= U/2 - 2 - \mu - B/2, \\
E_{1'} &= U - 4 - B.
\end{align}
(16)

Below $U = 4 - 4 \log 2 \approx 1.23$ on the $g = -\infty$ line, the excitations can be obtained from Eqs. (12) through the self-duality transformation [5].

Along the region $4 - 4 \log 2 < U < 4$ on the $g = -\infty$ line, the nature of the excitations varies. Their energies correspond to the solution of the coupled system of linear integral equations
\begin{align}
E_1 &= 4 \pi s - \mu + s \ast Q E_2, \\
E_{1'} &= 4 - U - \mu - 4 \pi K_2 \ast s - s \ast Q E_2, \\
E_{M+1} &= -4 + U - MB + 4 \pi K_{M+1} - K_{M+1} \ast Q E_2 - K_{M-1} \ast Q E_2, \\
\end{align}
(17)
with $c = 1$, and $M$ takes values on the positive integers. The subscript on the convolution indicates the interval over which the integral is taken, with $Q = (-\infty, -q) \cup (q, \infty)$, and $Q$ is its complement in $\mathbb{R}$. The parameter $q$ is determined self-consistently by $E_2(q) = 0$. The spectra should be evaluated at $\mu = B = 0$, but keeping these explicit allows one to track the spin and charge of the excitations in this interval. At $U = 4 - 4 \log 2$, the holon bands are (1) and (1'), the spinon band is (2), and the higher ($M$) all have zero energy, and represent the degeneracy arising from the ground state becoming symmetric under the spin $SU(2)$.

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