GENERAL DECAY FOR A VISCOELASTIC KIRCHHOFF EQUATION WITH BALAKRISHNAN-TAYLOR DAMPING, DYNAMIC BOUNDARY CONDITIONS AND A TIME-VARYING DELAY TERM

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Abstract. In this paper, we consider a viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping, dynamic boundary conditions and a time-varying delay term acting on the boundary. By using the Faedo-Galerkin approximation method, we first prove the well-posedness of the solutions. By introducing suitable energy and perturbed Lyapunov functionals, we then prove the general decay results, from which the usual exponential and polynomial decay rates are only special cases. To achieve these results, we consider the following two cases according to the coefficient $\alpha$ of the strong damping term:

for the presence of the strong damping term ($\alpha > 0$), we use the strong damping term to control the time-varying delay term, under a restriction of the size between the time-varying delay term and the strong damping term;

for the absence of the strong damping term ($\alpha = 0$), we use the viscoelasticity term to control the time-varying delay term, under a restriction of the size between the time-varying delay term and the kernel function.

1. Introduction. In this paper, we are concerned with the following problem:

\[
\begin{align*}
&u_{tt} - M(t) \Delta u + \int_0^t g(t-s) \Delta u(x, s) ds - \alpha \Delta u_t = 0, \quad x \in \Omega, \ t > 0, \\
&u(x, t) = 0, \quad x \in \Gamma_0, \ t > 0, \\
&u_{tt}(x, t) = -M(t) \frac{\partial u}{\partial \nu}(x, t) + \int_0^t g(t-s) \frac{\partial u}{\partial \nu}(x, s) ds - \alpha \frac{\partial u_t}{\partial \nu}(x, t) - \mu_2 u_t(x, t - \tau(t)), \quad x \in \Gamma_1, \ t > 0, \\
&u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \quad x \in \Omega, \\
&u_t(x, t - \tau(0)) = f_0(x, t - \tau(0)), \quad x \in \Gamma_1, \ t \in [0, \tau(0)),
\end{align*}
\]

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where $M(t) = a + b\|\nabla u(t)\|^2 + \sigma \int_\Omega \nabla u(t) \cdot \nabla u_t(t) dx$, $\Omega$ is a regular and bounded domain of $\mathbb{R}^N$ ($N \geq 1$), $\partial \Omega = \Gamma_0 \cup \Gamma_1$, $\text{mes}(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative. Moreover, $a$, $b$, $\sigma$, $\alpha$ and $\mu_2$ are real numbers with $a > 0$, $b > 0$, $\sigma \geq 0$ and $\alpha > 0$, $\tau(t) > 0$ represents the time-varying delay, and the initial datum $u_0$, $u_1$ and $f_0$ are given functions belonging to suitable spaces.

In recent years, wave equation with dynamic boundary conditions have been studied by many authors (see [1], [4], [7], [19], [23] for more details). The papers of [15], [16], [18] and [20] studied problem (1) with $M \equiv 1$ and without delay term. For example, in [15], Gerbi and Said-Houair studied the problem

$$
\begin{aligned}
\begin{cases}
  u_{tt} - \Delta u - \alpha \Delta u_t &= |u|^{p-2}u, & x \in \Omega, t > 0, \\
  u(x, t) &= 0, & x \in \Gamma_0, t > 0, \\
  u_{tt}(x, t) &= -\left(\frac{\partial u}{\partial \nu}(x, t) + \alpha \frac{\partial u_t}{\partial \nu}(x, t) + \gamma |u_t|^{m-2}u_t(x, t)\right), & x \in \Gamma_1, t > 0, \\
  u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x), & x \in \Omega,
\end{cases}
\end{aligned}
$$

(2)

and proved the local existence by using the Faedo-Galerkin approximations combined with a contraction mapping theorem and showed the exponential growth of the energy. Later in [16], they established the global existence and asymptotic stability of solutions starting in a stable set by combining the potential well method and the energy method. A blow-up result for the case $m = 2$ with initial data in the unstable set was also obtained. Recently, when the additional relaxation function $g \neq 0$ is involved in problem (2), they got the existence and exponential growth results in [18].

It is worth mentioning that a wave equation with delay term has become an active area of research, see for instance [5], [10], [29], [30], [36] and the references therein. The delay term may be a source of instability. For example, it was proved in [11, 29, 30] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. In [17], Gerbi and Said-Houair considered the damped wave equation with dynamic boundary conditions and a delay boundary term:

$$
\begin{aligned}
\begin{cases}
  u_{tt} - \Delta u - \alpha \Delta u_t &= 0, & x \in \Omega, t > 0, \\
  u(x, t) &= 0, & x \in \Gamma_0, t > 0, \\
  u_{tt}(x, t) &= -\left(\frac{\partial u}{\partial \nu}(x, t) + \alpha \frac{\partial u_t}{\partial \nu}(x, t) + \mu_1 u_t(x, t - \tau) + \mu_2 u_t(x, t - \tau)\right), & x \in \Gamma_1, t > 0, \\
  u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x), & x \in \Omega, \\
  u_1(x, t - \tau) &= f_0(x, t - \tau), & x \in \Gamma_1, t \in (0, \tau).
\end{cases}
\end{aligned}
$$

(3)

If the weight $\mu_2$ of the delay term is less than the weight $\mu_1$ of the term without delay or if $\mu_2 \geq \mu_1$ and $\alpha > (\mu_2 - \mu_1)B^2$ (with $B$ a constant), they proved the global existence of the solutions and the exponential stability of the system. This results indicated that even when $\mu_2$ is greater than $\mu_1$, the strong damping term still provides exponential stability for the system. However, when $\alpha = \mu_1 = 0$, how can we stabilize the system?
Recently, Ferhat and Hakem in [12] considered the following problem:

\[
\begin{align*}
\left\{
\begin{array}{l}
u_{tt} - \Delta u - \int_0^t g(t-s) \Delta u(x,s) \, ds - \alpha \Delta u_t = |u|^{p-1} u, \\
u(x,t) = 0, \\
u_{tt}(x,t) = -a \left[ \frac{\partial u}{\partial \nu}(x,t) - \int_0^t g(t-s) \frac{\partial u}{\partial \nu}(x,s) \, ds \right] \\
- \alpha \frac{\partial u}{\partial \nu}(x,t) + \mu_1 \psi(u_t(x,t)) + \mu_2 \psi(u_t(x,t - \tau)) \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\
u_t(x,t - \tau) = f_0(x,t - \tau),
\end{array}
\right.
\end{align*}
\]

By using the potential well method and introducing suitable Lyapunov function, they proved the global existence and established general decay estimates depending on the decay rate of the relaxation function \(g\). This result indicated that for a system of nonlinear viscoelastic wave equations with frictional damping, strong damping and nonlinear delay term, the viscoelasticity term also plays an important role in the stability of the whole system. For results of same nature, we refer the reader to [8], [9], [13], [14], [22], [24], [25], [26], [34], [38] and [41]. However, when \(\alpha = \mu_1 = 0\), can the viscoelasticity term showed in problem (4) still cause the similar stability property?

We recall that the stability of PDEs with time-varying delays was studied in [6, 31, 32, 33]. In [33], Nicaise et al. considered the system described by

\[
\begin{align*}
\left\{ \begin{array}{l}
u_{tt} - au_{xx} = 0, \\
u(0,t) = 0, \\
u_x(\pi,t) = -\mu_1 u_t(\pi,t) - \mu_2 u_t(\pi,t - \tau(t)), \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\
u_t(\pi,t - \tau(0)) = f_0(t - \tau(0)),
\end{array} \right.
\end{align*}
\]

and proved the exponential stability result, under the condition

\[
\mu_2 < \sqrt{1 - d\mu_1},
\]

where \(d\) is a constant such that

\[
t'(t) \leq d < 1, \quad \forall \ t > 0.
\]

Later, Nicaise et al. [32] extended the above result to general space dimension.

For equation (1) in the present of Balakrishnan-Taylor damping, it is mainly used to solve the spillover problem. The related problems were considered by Balakrishnan and Taylor in [2], Bass and Zes in [3], Tatar and Zarai in ([35], [39], [40]), Mu in [28] and Wu in [37].

Motivated by these results, in this paper, we intend to study the asymptotic behavior and related decay rates of problem (1), in which no linear damping term is involved (i.e. \(\mu_1 = 0\)). For our purpose, we consider two cases according to the coefficient \(\alpha\) of the strong damping term: the case \(\alpha > 0\) and the case \(\alpha = 0\). As we shall see below, if \(\alpha > 0\), the presence of the strong damping term \(\alpha \Delta u_t\) in (1) plays a decisive role in the stability of the whole system. Thanks to the energy method, by introducing appropriate Lyapunov functionals, we show in this
article that the decay rates of the solution energy are similar to the relaxation function, which are not necessarily decaying like polynomial or exponential function. If $\alpha = 0$, the main difficulty arises since there is no strong damping term to control the time-varying delay term in the estimate of the energy decay. To overcome this difficulty, our basic idea is to control the time-varying delay term by making use of the viscoelasticity term. And to achieve this goal, a restriction of the size between the parameter $\mu_2$ and the kernel $g$ and a new Lyapunov functional is needed. Although the Balakrishnan-Taylor damping is present, we give the remark that it does not change the main result of this paper.

The paper is organized as follows. In Section 2, we present some assumptions needed for our work and state the main result. In Section 3, we prove the well-posedness of the solution. In Section 4, we prove the general decay result in the case of $\alpha > 0$. For the case of $\alpha = 0$, the general decay result is proved in Section 5.

2. Preliminaries and main results. In this section we present some assumptions and state the main results. For the relaxation function $g$, we assume the following:

$(G1)$ $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nonincreasing differentiable function satisfying

$$g(0) > 0, \quad a - \int_0^\infty g(s)ds = l > 0.$$  

$(G2)$ There exists a nonincreasing differentiable function $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$g'(s) \leq -\xi(s)g(s), \quad \forall \ s \in \mathbb{R}_+$$

and

$$\int_0^{+\infty} \xi(t)dt = \infty.$$  

Let $C_p$ and $C_p^*$ be the Poincaré's type constants defined as the smallest positive constants such that

$$\int_{\Gamma_1} |v|^2 d\Gamma \leq C_p \int_{\Omega} |\nabla v|^2 dx, \quad \forall \ v \in H^1_{\Gamma_0}(\Omega)$$

and

$$\int_{\Omega} |v|^2 dx \leq C_p^* \int_{\Omega} |\nabla v|^2 dx, \quad \forall \ v \in H^1(\Omega),$$

where $H^1_{\Gamma_0} = \{u \in H^1(\Omega)| u|_{\Gamma_0} = 0\}$. We denote by $(\cdot,\cdot)$ the scalar product in $L^2(\Omega)$, i.e.,

$$(u, v) = \int_{\Omega} u(x)v(x)dx.$$  

For the coefficient $\mu_2$ of the time-varying delay term, we assume that

$$(7) \begin{cases} |\mu_2| < \frac{\alpha}{C_p} \sqrt{1-d}, & \text{if} \ \alpha > 0, \\
|\mu_2| \leq a_0, & \text{if} \ \alpha = 0, \end{cases}$$

where $d$ is the positive constant of assumption (6) and $a_0$ is a small positive constant defined in (78) below.

Now, we can state the following well-posedness result:
Theorem 2.1. Suppose that (G1), (G2) and (7) hold. Then given \( u_0 \in H^1_0(\Omega), \)
\( u_1 \in L^2(\Omega) \) and \( f_0 \in L^2(\Omega \times (0, 1)) \), there exist \( T > 0 \) and a unique weak solution \((u, u_t)\) of problem (1) on \((0, T)\) satisfying

\[
\begin{align*}
&u \in C \left([0, T], H^1_0(\Omega)\right) \cap C^1 \left([0, T], L^2(\Omega)\right), \\
u_t \in L^2 \left(0, T; H^1_0(\Omega)\right) \cap L^2 \left((0, T) \times \Gamma_1\right).
\end{align*}
\]

We define the energy of problem (1) as

\[
E(t) := \frac{1}{2} \left[\|u(t)\|^2 + \left(a - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{b}{2} \|\nabla u(t)\|^2 \right]
+ (g \circ \nabla u)(t) + \|u_t(t)\|^2 + \frac{\lambda}{2} \int_{\Gamma_1} e^{\lambda(t-s)} u_t^2(x, s)d\Gamma ds, \tag{8}
\]

where \( \zeta \) and \( \lambda \) are suitable positive constants and

\[
(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds \geq 0.
\]

Then, we state the decay result as follows:

Theorem 2.2. Let \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\) be given. Assume that \( g \) and \( \xi \)
are bounded by (G1) and (G2), and the assumption (7) holds. Then, for each \( t_0 > 0 \), there exist two positive constants \( K \) and \( k \) such that, for any solution of problem (1), the energy satisfies

\[
E(t) \leq Ke^{-k t} \int_{t_0}^t \xi(s)ds. \tag{9}
\]

3. Well-posedness of the problem. We now give, with a brief proof of the well-posedness of the problem, which can be established by using the Faedo-Galerkin approximation method (see [15, 31, 33] for the details).

As in [29], we introduce the new variable

\[
z(x, \rho, t) = u_t(x, t - \tau(t)\rho), \quad x \in \Gamma_1, \rho \in (0, 1), t > 0.
\]

Then, we have

\[
\tau(t)\frac{\partial z}{\partial t}(x, \rho, t) + (1 - \tau(t)\rho) z_\rho(x, \rho, t) = 0, \quad \text{in} \quad \Gamma_1 \times (0, 1) \times (0, +\infty).
\]

Then, problem (1) is equivalent to

\[
\begin{cases}
u_{tt} - \xi u_{x} + \int_0^t g(t-s)\Delta u(x, s)ds - \alpha \Delta u_t = 0, & x \in \Omega, t > 0, \\
u(t) \frac{\partial z}{\partial t}(x, \rho, t) + (1 - \tau(t)\rho) z_\rho(x, \rho, t) = 0, & x \in \Gamma_1, \rho \in (0, 1), \quad x \in \Gamma_0, t > 0, \\
u(x, t) = 0, & x \in \Gamma_0, t > 0, \\
u_{tt}(x, t) = -\xi \frac{\partial u}{\partial \nu}(x, t) + \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(x, s)ds
\end{cases}
\]

\[
-\alpha \frac{\partial u_t}{\partial \nu}(x, t) - \mu z(x, 1, t), & x \in \Gamma_1, t > 0,
\]

\[
z(x, 0, t) = u_t(x, t), & x \in \Gamma_1, t > 0,
\]

\[
\frac{\partial z}{\partial t}(x, 0) = u_0(x), u_1(x, 0) = u_1(x), & x \in \Omega,
\]

\[
z(x, \rho, 0) = f_0(x, -\rho \tau(0)), & x \in \Gamma_1, \rho \in (0, 1).
\]
Proof of Theorem 2.1. We divide the proof into two steps: the construction of approximations and then thanks to certain energy estimates, we pass to the limit.

Step 1. Faedo-Galerkin approximation. We construct approximations of the solution \( u, z \) by Faedo-Galerkin method as follows. For every \( n \geq 1 \), let \( W_n = \text{span}\{\omega_1, ..., \omega_n\} \), be a Hilbertian basis of the space \( H^1_0(\Omega) \).

Now, we define for \( 1 \leq j \leq n \) the sequence \( \varphi_j(\rho, x) \) as follows: \( \varphi_j(\rho, 0) = \omega_j(\rho) \). Then, we may extend \( \varphi_j(x, 0) \) by \( \varphi_j(x, \rho) \) over \( L^2(\Omega \times [0,1]) \) and denote \( V_n = \text{span}\{\varphi_1, ..., \varphi_n\} \).

We chose two sequences \( \{u_{0n}\} \) and \( \{u_{1n}\} \) in \( W_n \) and a sequence \( \{z_{0n}\} \) in \( V_n \) such that \( u_{0n} \to u_0 \) strongly in \( H^1_0(\Omega) \), \( u_{1n} \to u_1 \) strongly in \( L^2(\Omega) \) and \( z_{0n} \to f_0 \) strongly in \( L^2(\Omega \times (0,1)) \).

We define now the approximations:

\[
\begin{align*}
\{ u_n(t, x) = \sum_{j=1}^n g_{jn}(t) \omega_j(x) \} \quad \text{and} \quad \{ z_n(t, x, \rho) = \sum_{j=1}^n h_{jn}(t) \varphi_j(x, \rho) \},
\end{align*}
\]

where \( (u_n(t), z_n(t)) \) are solutions to the finite dimensional Cauchy problem (written in normal form):

\[
\begin{cases}
\int_{\Omega} u_{tt} \omega_j dx + \int_{\Omega} M(t) \nabla u_n \cdot \nabla \omega_j dx \\
+ \int_0^t g(t-s) \int_{\Omega} \nabla u_n \cdot \nabla \omega_j dx ds + \alpha \int_{\Omega} \nabla u_n \cdot \nabla \omega_j dx \\
+ \int_{\Gamma_1} u_{tt} \omega_j d\Gamma + \mu_2 \int_{\Gamma_1} z_n(x, 1, t) \omega_j d\Gamma = 0, \quad x \in \Omega, t > 0, \\
u(x, t) = 0, \quad x \in \Gamma_0, t > 0,
\end{cases}
\]

\[
\begin{cases}
\int_{\Gamma_1} u_{tt} \omega_j d\Gamma = - \int_{\Gamma_1} M(t) \frac{\partial u_n}{\partial \nu} \omega_j d\Gamma \\
+ \int_0^t g(t-s) \int_{\Gamma_1} \frac{\partial u_n}{\partial \nu} \omega_j d\Gamma ds - \alpha \int_{\Gamma_1} \frac{\partial u_n}{\partial \nu} \omega_j d\Gamma \\
- \mu_2 \int_{\Gamma_1} z_n(x, 1, t) \omega_j d\Gamma, \quad x \in \Gamma_1, t > 0, \\
z_n(x, 0, t) = u_{tt}(x, t), \quad x \in \Gamma_1, t > 0, \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{cases}
\]

and

\[
\begin{cases}
\int_{\Gamma_1} [\tau(t) z_{nt}(x, \rho, t) + (1 - \tau(t)) \rho \\
\times z_{n\rho}(x, \rho, t) \varphi_j] d\Gamma = 0, \quad x \in \Gamma_1, \rho \in (0,1), t > 0, \\
z_n(x, \rho, 0) = z_{0n}(x, -\rho \tau(0)), \quad x \in \Gamma_1, \rho \in (0,1).
\end{cases}
\]

According to the standard theory of ordinary differential equations, the finite dimensional of problem (12), (13) has solution \( (g_{jn}(t), h_{jn}(t))_{j=1,...,n} \) defined on \( [0, t_n) \). The a priori estimates that follow imply that in fact \( t_n = T \).
Step 2. Energy estimates. Multiplying (12) by \( g'_n \) integrating over \((0,t)\) and multiplying (13) by \( \zeta h_j \) integrating over \((0,t) \times (0,1)\), we can obtain

\[
E_n(t) = \frac{1}{2} (g' \circ \nabla u_n)(t) - \frac{1}{2} g(t) \| \nabla u_n(t) \|_2^2 - \sigma \left( \frac{1}{2} \frac{d}{dt} \| \nabla u_n(t) \|_2^2 \right)^2 \leq E_n(0). \tag{14}
\]

where

\[
E_n(t) = \frac{1}{2} \left[ \| u_{tn}(t) \|_2^2 + \left( a - \int_0^t g(s) ds \right) \| \nabla u_n(t) \|_2^2 + b \| \nabla u_n(t) \|_2^2 + (g \circ \nabla u_n)(t) \right.
\]

\[
\left. + \| u_{tn}(t) \|_2 \| \nabla u_n(t) \|_2 + \frac{\zeta}{2} \int_{t-\tau(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)} u_{tn}(x,s) d\Gamma d\sigma, \right] \tag{15}
\]

Since the sequences \( \{u_{n0}\}_{n \in \mathbb{N}}, \{u_{n1}\}_{n \in \mathbb{N}} \) and \( \{z_{n0}\}_{n \in \mathbb{N}} \) converge, using (G1) and (G2), we can find a positive constant \( C \) independent of \( n \) such that

\[
E_n(t) \leq C. \tag{16}
\]

Therefore, using the fact that \( 1 - \int_0^t g(s) ds \geq 1 \), by estimates (16) and (15), for all \( n \in \mathbb{N}, t_n = T \), we have

\[
\{u_n\}_{n \in \mathbb{N}} \text{ is bounded in } L^\infty(0,T; H^1_0(\Omega)), \tag{17}
\]

\[
\{u_{tn}\}_{n \in \mathbb{N}} \text{ is bounded in } L^\infty(0,T; L^2(\Omega)) \tag{18}
\]

and

\[
\{z_n\}_{n \in \mathbb{N}} \text{ is bounded in } L^\infty(0,T; L^2(\Omega \times (0,1))). \tag{19}
\]

Consequently, we may conclude that:

\[
\begin{align*}
    u_n \rightarrow u & \quad \text{weak}^* \quad \text{in } L^\infty(0,T; H^1_0(\Omega)), \\
    u_{tn} \rightarrow u & \quad \text{weak}^* \quad \text{in } L^\infty(0,T; L^2(\Omega)), \\
    z_n \rightarrow z & \quad \text{weak}^* \quad \text{in } L^\infty(0,T; L^2(\Omega)).
\end{align*}
\]

From (17), (18) and (19), we have \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( L^\infty(0,T; H^1_0(\Omega)) \). Then, \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( L^2(0,T; H^1_0(\Omega)) \). Since \( \{u_{tn}\}_{n \in \mathbb{N}} \) is bounded in \( L^\infty(0,T; L^2(\Omega)) \), \( \{u_{tn}\}_{n \in \mathbb{N}} \) is bounded in \( L^2(0,T; L^2(\Omega)) \). Consequently \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( H^1(0,T; H^1_0(\Omega)) \).

Since the embedding \( H^1(0,T; H^1_0(\Omega)) \hookrightarrow L^2(0,T; L^2(\Omega)) \) is compact, using Aubin-Lions theorem [27], we can extract a subsequence \( \{u_\mu\}_{\mu \in \mathbb{N}} \) of \( \{u_n\}_{n \in \mathbb{N}} \) such that

\[
u_\mu \rightarrow u \quad \text{strongly in } L^2(0,T; L^2(\Omega)). \]

Therefore,

\[
u_\mu \rightarrow u \quad \text{strongly and a.e on } (0,T) \times \Omega. \]

The proof now can be completed arguing as in ([27], Theorem 3.1). \( \square \)
4. Decay of solutions for $\alpha > 0$. As mentioned earlier, in this section, we prove the general decay result for problem (1) under the first inequality of assumption (7). For our purpose, we use the idea of Nicuase and Pignotti in [31]. We fix $\zeta$ such that

$$\frac{|\mu_2|}{\sqrt{1-d}} <\zeta <\frac{2\alpha}{C_\rho} - \frac{|\mu_2|}{\sqrt{1-d}}$$

which is possible by assumption (7). Moreover, the parameter $\lambda$ is fixed satisfying

$$\lambda < \frac{1}{\tau} \left| \log \frac{|\mu_2|}{\zeta \sqrt{1-d}} \right|,$$

where $\tau$ is a positive constant such that $\tau \geq \tau(t) \geq \tau_0 > 0, \forall \ t > 0$.

**Proposition 1.** For any regular solution of problem (1), under the conditions of Theorem 2.2, the energy is non-increasing and for a suitable positive constant $C$, we have

$$E'(t) \leq \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) ||\nabla u(t)||_2^2 - C \left( ||\nabla u(t)||_2^2 + ||u(t) - \tau(t)||_{2,1}^2 \right)$$

$$- \sigma \left( \frac{1}{2} \int_0^t ||\nabla u(t)||_2^2 \right)^2 - \frac{\lambda \zeta}{2} \int_{t-\tau(t)}^t \int_{\Gamma_1} e^{\lambda(t-s)} u_t^2(x,t) d\Gamma ds + \frac{\zeta}{2} ||u_t||_{2,1}^2,$$

where $u_t$ is a positive constant such that $\tau \geq \tau(t) \geq \tau_0 > 0, \forall \ t > 0$.

**Proof.** Differentiating (8) and by Cauchy-Schwarz’s inequality, a trace estimate, Poincaré’s inequality, we get

$$E'(t) = \int_\Omega u_t u t dx + \left( a - \int_0^t g(s) ds \right) \int_\Omega \nabla u \cdot \nabla u_t dx - \frac{1}{2} g(t) ||\nabla u(t)||_2^2$$

$$+ \frac{1}{2} (g' \circ \nabla u)(t) + \int_0^t g(t-s) \int_\Omega \nabla u_t(t) \cdot [\nabla u(t) - \nabla u(t)] dx ds$$

$$+ \int_{\Gamma_1} u_t u_t d\Gamma + \frac{b}{4} \int_\Omega ||\nabla u(t)||_2^2 - \frac{\zeta}{2} \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(x,t) (1 - \tau(t)) d\Gamma ds$$

$$- \frac{\lambda \zeta}{2} \int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(x,t) d\Gamma ds + \frac{\zeta}{2} ||u_t||_{2,1}^2,$$

where $u_t$ is a positive constant such that $\tau \geq \tau(t) \geq \tau_0 > 0, \forall \ t > 0$.

$$\leq - \alpha ||u_t||_2^2 + \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) ||\nabla u(t)||_2^2 - \sigma \left( \frac{1}{2} \int_0^t ||\nabla u(t)||_2^2 \right)^2$$

$$+ \frac{\zeta}{2} ||u_t||_{2,1} + \frac{|\mu_2|}{2\sqrt{1-d}} ||u_t(t)||_{2,1}^2 + \frac{|\mu_2|}{2} ||u_t(t) - \tau(t)||_{2,1}^2$$

$$- \frac{\zeta}{2} (1-d) e^{-\lambda t} \int_{\Gamma_1} u_t^2(x,t) d\Gamma \geq \frac{\lambda \zeta}{2} \int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(x,t) d\Gamma ds$$

$$\leq \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) ||\nabla u(t)||_2^2 - \sigma \left( \frac{1}{2} \int_0^t ||\nabla u(t)||_2^2 \right)^2$$

$$- \frac{\zeta}{2} (1-d) e^{-\lambda t} - \frac{|\mu_2|}{2\sqrt{1-d}} ||u_t(t) - \tau(t)||_{2,1}^2$$

$$- \left( \alpha - \frac{|\mu_2| C_\rho}{2\sqrt{1-d}} - \frac{\zeta}{2} C_\rho \right) ||u_t||_{2,1}^2 - \frac{\lambda \zeta}{2} \int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(x,t) d\Gamma ds.$$

Therefore, by (20) and (21) we can easily get (22). \qed
Now, we use the following modified functional, for positive constants $N$, $\epsilon_1$ and $\epsilon_2$, we have
\[ L(t) = NE(t) + \epsilon_1 \psi(t) + \epsilon_2 \phi(t), \]
where
\[ \psi(t) = \int_{\Omega} u_t(t)u(t)dx + \int_{\Gamma_1} u_t(t)u(t)d\Gamma + \frac{\alpha}{2}\|\nabla u(t)\|^2 + \frac{\sigma}{4}\|\nabla u(t)\|^4 \]
(25) and
\[ \phi(t) = -\int_{\Omega} u_t(t) \int_0^t g(t-s) (u(t) - u(s)) ds dx \\
- \int_{\Gamma_1} u_t(t) \int_0^t g(t-s) (u(t) - u(s)) ds d\Gamma. \]
(26)

It is easy to check that, by using Poincare’s inequality, trace inequality, the functional $L$ is equivalent to the energy $E$, that is, for $\epsilon_1$ and $\epsilon_2$ small enough, choosing $N$ large enough, there exist two constants $\alpha_1$ and $\alpha_2$ such that
\[ \alpha_1 L(t) \leq E(t) \leq \alpha_2 L(t). \]
(27)

Next, we estimate the derivative of $L(t)$ according to the following lemmas.

**Lemma 4.1.** Under the conditions of Theorem 2.2, the functional $\psi(t)$ defined in (25) satisfies
\[ \psi'(t) \leq \|u_t(t)\|^2 + \|u_t(t)\|^2_{\Gamma_1} - (l - \delta_1 - C_\rho \delta_1 + b\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2 \\
+ \frac{\mu_1^2}{4\delta_1} \|u_t(t - \tau(t))\|^2_{\Gamma_1} + \frac{a-l}{4\delta_1} (g \circ \nabla u)(t), \]
(28)
forsome $\delta_1 > 0$.

**Proof.** By using the equation in (1), we get
\[ \psi'(t) = \|u_t(t)\|^2 + \int_{\Omega} u_t(t)u(t)dx + \|u_t(t)\|^2_{\Gamma_1} + \int_{\Gamma_1} u_t(t)u(t)d\Gamma \\
+ \alpha \int_{\Omega} \nabla u(t) \cdot \nabla u(t)dx + \frac{\sigma}{2} \left( \frac{d}{dt}\|\nabla u(t)\|_2^2 \right) \|\nabla u(t)\|_2^2 \\
- \int_0^t g(t-s) \int_{\Gamma_1} \frac{\partial u(s)}{\partial \nu} u(s) d\Gamma ds + \int_0^t g(t-s) \int_{\Omega} \nabla u(s) \cdot \nabla u(t)dx ds \\
+ M(t) \int_{\Gamma_1} \frac{\partial u(t)}{\partial \nu} u(t)d\Gamma + \alpha \int_{\Gamma_1} \frac{\partial u(t)}{\partial \nu} u(t)d\Gamma \\
= \|u_t(t)\|^2 + \|u_t(t)\|^2_{\Gamma_1} - (a + b\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2 \\
+ \int_{\Omega} \nabla u(t) \cdot \int_0^t g(t-s)\nabla u(s)ds dx - \mu_2 \int_{\Gamma_1} u_t(t - \tau(t))u(t)d\Gamma. \]
(29)

We now estimate the right-hand side of (29). For a positive constant $\delta_1$, we have the estimates as follows
\[ \int_{\Omega} \nabla u(t) \cdot \int_0^t g(t-s)\nabla u(s)ds dx \leq (\delta_1 + a - l)\|\nabla u(t)\|_2^2 + \frac{a-l}{4\delta_1}(g \circ \nabla u)(t). \]
(30)
Lemma 4.2. Under the conditions of Theorem 2.2, the functional $\phi(t)$ defined in (26) satisfies

$$
\phi'(t) \leq \left( \delta_2 - \int_0^t g(s) \, ds \right) \| u_t(t) \|^2_2 + \left( \delta_2 - \int_0^t g(s) \, ds \right) \| u_t(t) \|^2_{2,\Gamma_1} \\
+ \left[ \frac{1}{4\delta_2} \left( a + \frac{2\beta E(0)}{l} \right)^2 + \left( 2\delta_2 + \frac{1}{4\delta_2} + \alpha \right) + \frac{1}{4l} + \frac{C_p \mu_2}{4\delta_2} \right] (a - l)(g \circ \nabla u)(t) \\
+ \frac{\alpha}{2} \| \nabla u_t(t) \|^2_2 + \sigma^2 E(0) \left( \frac{1}{2l} \| \nabla u(t) \|^2_2 \right)^2 + \frac{g(0)(C_p + C_p^*)}{4\delta_2} (-g' \circ \nabla u)(t) \\
+ [\delta_2 + 2\delta_2(a - l)^2] \| \nabla u(t) \|^2_2 + \delta_2 \mu_2 \| u_t(t - \tau(t)) \|^2_{2,\Gamma_1},
$$

for some $\delta_2 > 0$.

Proof. By using the equation in (1), we get

$$
\phi'(t) = -\int_\Omega u_t(t) \int_0^t g(t - s) (u(t) - u(s)) \, ds \, dx \\
- \int_\Omega u_t(t) \int_0^t g'(t - s) (u(t) - u(s)) \, ds \, dx \\
- \left( \int_0^t g(s) \, ds \right) \int_\Omega |u_t(t)|^2 \, dx - \int_{\Gamma_1}^t u_t(t) \int_0^t g(t - s) (u(t) - u(s)) \, ds \, d\Gamma \\
- \int_{\Gamma_1}^t u_t(t) \int_0^t g'(t - s) (u(t) - u(s)) \, ds \, d\Gamma - \left( \int_0^t g(s) \, ds \right) \int_{\Gamma_1}^t |u_t(t)|^2 \, d\Gamma \\
= \int_\Omega (a + b \| \nabla u(t) \|^2_2) \nabla u(t) \cdot \int_0^t g(t - s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \\
+ \int_\Omega \sigma \left( \int_\Omega \nabla u(t) \cdot \nabla u_t(t) \, dx \right) \nabla u(t) \cdot \int_0^t g(t - s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \\
- \int_\Omega \left( \int_0^t g(t - s) \nabla u(s) \, ds \right) \cdot \left( \int_0^t g(t - s) (\nabla u(t) - \nabla u(s)) \, ds \right) \, dx \\
+ \alpha \int_\Omega \nabla u_t(t) \cdot \int_0^t g(t - s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \\
- \int_\Omega u_t(t) \int_0^t g'(t - s) (u(t) - u(s)) \, ds \, dx \\
- \int_{\Gamma_1}^t u_t(t) \int_0^t g'(t - s) (u(t) - u(s)) \, ds \, d\Gamma \\
+ \mu_2 \int_{\Gamma_1}^t u_t(t - \tau(t)) \int_0^t g(t - s) (u(t) - u(s)) \, ds \, d\Gamma \\
- \left( \int_0^t g(s) \, ds \right) \int_\Omega |u_t(t)|^2 \, dx - \left( \int_0^t g(s) \, ds \right) \int_{\Gamma_1}^t |u_t(t)|^2 \, d\Gamma.
$$
Then, from (34), we have
\[
\sum_{i=1}^{7} I_i - \left( \int_{0}^{t} g(s) ds \right) \int_{\Omega} |u_t(t)|^2 dx - \left( \int_{0}^{t} g(s) ds \right) \int_{\Gamma_1} |u_t(t)|^2 d\Gamma.
\]
We now estimate the right-hand side of (33). By using Young’s inequality, Hölder’s inequality and Cauchy-Schwarz’s inequality, we get
\[
I_1 \leq \left| \int_{\Omega} \left( a + \frac{2b}{l} E(0) \right) \nabla u(t) \cdot \int_{0}^{t} g(t-s) (\nabla u(t) - \nabla u(s)) \, dxds \right|.
\]
As \( a - \int_{0}^{t} g(s) ds \| \nabla u(t) \|^2 \leq 2E(t) \leq 2E(0) \) and by (G1), (8), (22), we can obtain
\[
\| \nabla u(t) \|^2 \leq \frac{2}{l} E(0).
\]
Then, from (34), we have
\[
I_1 \leq \left| \int_{\Omega} \left( a + \frac{2b}{l} E(0) \right) \nabla u(t) \cdot \int_{0}^{t} g(t-s) (\nabla u(t) - \nabla u(s)) \, dxds \right|
\leq \delta_2 \| \nabla u(t) \|_2^2 + \frac{1}{4\delta_2} \left( a + \frac{2b}{l} E(0) \right)^2 (a - l) (g \circ \nabla u(t)).
\]
\[
I_2 \leq \sigma \left| \int_{\Omega} \nabla u(t) \cdot \nabla u(t) dx \right| \left| \int_{\Omega} \nabla u(t) \cdot \int_{0}^{t} g(t-s) (\nabla u(t) - \nabla u(s)) \, dxd\right|
\leq \sigma^2 \left( \int_{\Omega} \nabla u(t) \cdot \nabla u(t) dx \right)^2 \frac{l}{2} \| \nabla u(t) \|_2^2
\]
\[
+ \frac{1}{2l} \int_{\Omega} \left( \int_{0}^{t} g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \right)^2 dx
\leq \sigma^2 E(0) \left( \frac{1}{2} \frac{dl}{dt} \| \nabla u(t) \|_2^2 \right)^2 + \frac{a - l}{2l} (g \circ \nabla u(t)),
\]
\[
I_3 \leq \left| \int_{\Omega} \left( \int_{0}^{t} g(t-s) \nabla u(s) ds \right) \cdot \left( \int_{0}^{t} g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \right) \right| dx
\leq 2\delta_2 (a - l)^2 \| \nabla u(t) \|_2^2 + \left( 2\delta_2 + \frac{1}{4\delta_2} \right) (a - l) (g \circ \nabla u(t)),
\]
\[
I_4 \leq \left| \alpha \int_{\Omega} \nabla u(t) \cdot \int_{0}^{t} g(t-s) (\nabla u(t) - \nabla u(s)) \, dsdx \right|
\leq \frac{\alpha}{2} \| \nabla u(t) \|_2^2 + \frac{\alpha(a - l)}{2} (g \circ \nabla u(t)),
\]
\[
I_5 \leq \left| \int_{\Omega} u(t) \int_{0}^{t} g'(t-s) (u(t) - u(s)) \, dsdx \right|
\leq \delta_2 \| u_t(t) \|_2^2 + C_g^2 g(0) \frac{1}{4\delta_2} (-g' \circ \nabla u)(t),
\]
\[
I_6 \leq \left| \int_{\Gamma_1} u_t(t) \int_{0}^{t} g'(t-s) (u(t) - u(s)) d\Gamma \right|
\]
Proof of Theorem 2.2 (case $\alpha > 250$)

Once $\delta$ is fixed, we then pick $\delta_2 < \frac{l_0}{l + 1 + 2(a - l)^2}$ such that $g_0 - \delta_2 > 0$.

Once $\delta_2$ is fixed, we then pick $\delta_1 < \frac{l}{1 + C_p} + \frac{\delta_2 + 2\delta_2(a - l)^2}{(\delta_2 - g_0)(1 + C_p)}$ to get $l - \delta_1 - C_p\delta_1 > 0$

and

$$\frac{1}{g_0 - \delta_2} < \frac{l - \delta_1 - C_p\delta_1}{\delta_2 + 2\delta_2(a - l)^2}.$$
\[ k_2 = \varepsilon_1 (l - \delta_1 - C_p \delta_1) - \varepsilon_2 (\delta_2 + 2\delta_2(a - l)^2) > 0, \]
\[ k_5 = (a - l) \left\{ \frac{\varepsilon_1}{4\delta_1} + \varepsilon_2 \left[ \frac{1}{4\delta_2} \left( a + \frac{2bE(0)}{t} \right)^2 \right] + \left( 2\delta_2 + \frac{1}{4\delta_2} \right) + \frac{\alpha}{2} + \frac{1}{2l} + \frac{C_p|\mu_2|}{4\delta_2} \right\} > 0. \]

When \( \delta_1, \delta_2, \varepsilon_1, \) and \( \varepsilon_2 \) are fixed, we choose \( N > 0 \) large enough such that
\[ k_3 = NC - \frac{\varepsilon_2\alpha}{2} > 0, \]
\[ k_4 = \frac{N}{2} - \frac{\varepsilon_2 g(0)(C_p + C_p^*)}{4\delta_2} > 0, \]
\[ k_6 = NC - \frac{\varepsilon_1\mu_2^2}{4\delta_1} - \varepsilon_2\delta_2|\mu_2| > 0, \]
\[ k_7 = N - \varepsilon_2\sigma E(0) > 0. \]

We have
\[ L'(t) \leq -k_1 (\|u(t)\|^2 + \|u(t)\|^2_{H^1(I)}) - k_2\|\nabla u(t)\|^2 - k_3\|\nabla u_t(t)\|^2 + k_4(g' \circ \nabla u)(t) + k_5 (g \circ \nabla u)(t) - C \int_{t-\tau(t)}^t e^{\lambda(s-t)}u_1^2(x,s)d\sigma \int_{\Gamma_1} e^{\lambda(s-t)}u_1^2(x,s)d\sigma \Omega, \tag{44} \]
where \( C = \frac{N\lambda\zeta}{2} > 0. \) By \((8), (G2)\) and \((44), \) there exist two positive constants \( M \) and \( \kappa_8 \) such that
\[ L'(t) \leq -ME(t) + k_8 (g \circ \nabla u)(t), \forall \ t \geq t_0. \tag{45} \]

Multiplying \((45)\) by \( \xi(t), \) we have
\[ \xi(t)L'(t) \leq -M\xi(t)E(t) + k_8 \xi(t)(g \circ \nabla u)(t), \forall \ t \geq t_0. \tag{46} \]

Because \( \xi \) and \( g \) are nonincreasing, we get
\[ \xi(t)\int_0^t (g(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds \]
\[ \leq -\int_0^t g'(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds \leq -2E'(t) \tag{47} \]

Inserting the last inequality in \((46), \) we obtain
\[ \xi(t)L'(t) + 2k_8E'(t) \leq -M\xi(t)E(t), \forall \ t \geq t_0. \tag{48} \]

Now, we define
\[ H(t) = \xi(t)L(t) + 2k_8E(t). \]

Since \( \xi(t) \) is a non-increasing positive function, we can easily get that \( H \sim E. \) Thus \((48)\) implies that
\[ H'(t) \leq -k\xi(t)H(t), \forall \ t \geq t_0, \]
for some \( k > 0. \) Then, by direct integration over \((t_0, t), \) we have
\[ H(t) \leq H(t_0) e^{-k \int_{t_0}^t \xi(s)ds}, \forall \ t \geq t_0. \]

Consequently, using the equivalent relations of \( H(t) \) and \( E(t), \) we can conclude
\[ E(t) \leq k_9 H(t_0) e^{-k \int_{t_0}^t \xi(s)ds} = Ke^{-k \int_{t_0}^t \xi(s)ds}, \forall \ t \geq t_0, \]
where \( k_0 \) is a positive constant and \( K = k_0 H(t_0) \). This completes the proof.

\[ \square \]

5. Decay of solutions for \( \alpha = 0 \). In this section, we prove the general decay result of problem (1) in the absence of the strong damping \(-\Delta u(t)\) (i.e. \( \alpha = 0 \)). For our purpose, we use the idea of Dai and Yang in [10]. Before proving the result, we need the following proposition and lemmas.

**Proposition 2.** For any regular solution of problem (1), under the conditions of Theorem 2.2, we have

\[
E'(t) \leq \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \| \nabla u(t) \|^2 - \sigma \left( \frac{1}{2} \frac{d}{dt} \| \nabla u(t) \|^2 \right)^2 + \left( \frac{\zeta}{2} + \frac{|\mu_2|}{2} \right) \| u(t) \|^2_{2, \Gamma_1} - \left( \frac{\zeta}{2} e^{-\lambda \tau} - \frac{|\mu_2|}{2} \right) \| u(t - \tau(t)) \|^2_{2, \Gamma_1} - \frac{\lambda \zeta}{2} \int_{t - \tau(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)} u^2(x, s) d\Gamma ds.
\]

**Proof:** Differentiating (8) and by Cauchy-Schwarz’s inequality, Poincaré’s inequality, we get

\[
E'(t) = \int_{\Omega} u_t u_t dx + \left( a - \int_0^t g(s) ds \right) \int_{\Omega} \nabla u \cdot \nabla u_t dx - \frac{1}{2} g(t) \| \nabla u \|^2 \]

\[
+ \frac{1}{2} (g' \circ \nabla u)(t) + \int_0^t g(t - s) \int_{\Omega} \nabla u_t(t) \cdot (\nabla u(s) - \nabla u(t)) dxdz ds + \frac{b}{4} d \int_{\Gamma_1} \| \nabla u(t) \|^2 - \frac{\zeta}{2} \int_{\Gamma_1} e^{-\lambda \tau(t)} u^2(x, t - \tau(t))(1 - \tau'(t)) d\Gamma \]

\[
- \frac{\lambda \zeta}{2} \int_{t - \tau(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)} u^2(x, s) d\Gamma ds + \frac{\zeta}{2} \| u \|^2_{2, \Gamma_1} \]

\[
\leq \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \| \nabla u(t) \|^2 - \sigma \left( \frac{1}{2} \frac{d}{dt} \| \nabla u(t) \|^2 \right)^2 + \frac{\zeta}{2} \| u \|^2_{2, \Gamma_1} - \mu_2 \int_{\Gamma_1} u_t(t) u_t(t - \tau(t)) d\Gamma + \frac{\zeta}{2} e^{-\lambda \tau} \int_{\Gamma_1} u^2(x, t - \tau(t)) d\Gamma \]

\[
- \frac{\lambda \zeta}{2} \int_{t - \tau(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)} u^2(x, s) d\Gamma ds \]

\[
\leq \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \| \nabla u(t) \|^2 - \sigma \left( \frac{1}{2} \frac{d}{dt} \| \nabla u(t) \|^2 \right)^2 + \left( \frac{\zeta}{2} + \frac{|\mu_2|}{2} \right) \| u(t) \|^2_{2, \Gamma_1} + \left( \frac{|\mu_2|}{2} - \frac{\zeta}{2} e^{-\lambda \tau} \right) \| u(t - \tau(t)) \|^2_{2, \Gamma_1} - \frac{\lambda \zeta}{2} \int_{t - \tau(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)} u^2(x, s) d\Gamma ds.
\]

This completes the proof. \( \square \)

**Remark 1.** In Proposition 1, we proved that the energy functional is non-increasing. However, since \( \left( \frac{\zeta}{2} + \frac{|\mu_2|}{2} \right) \| u(t) \|^2_{2, \Gamma_1} \geq 0 \), \( E(t) \) may not be non-increasing here.
Now, we define the Lyapunov functional, for positive constants $\varepsilon_3$ and $\varepsilon_4$, we have
\[
\dot{L}(t) = N_1 E(t) + \varepsilon_3 G(t) + \varepsilon_4 \phi(t),
\]
where
\[
G(t) = \int_{\Omega} u_t(t)u(t)dx + \int_{\Gamma_1} u_t(t)u(t)d\Gamma + \frac{\sigma}{4} \|\nabla u(t)\|_2^4.
\]

It is easy to check that, by using Poincaré’s inequality, trace inequality, the functional $L$ is equivalent to the energy $E$, that is, for $\varepsilon_3$ and $\varepsilon_4$ small enough, choosing $N_1$ large enough, there exist two constants $\beta_1$ and $\beta_2$ such that
\[
\beta_1 E(t) \leq \dot{L}(t) \leq \beta_2 E(t).
\]

Next, we estimate the derivative of $\dot{L}(t)$ according to the following lemmas.

**Lemma 5.1.** Under the conditions of Theorem 2.2, the functional $G(t)$ defined in (51) satisfies
\[
G'(t) \leq \|u_t(t)\|_2^2 + \|u_t(t)\|_{2,G_1}^2 - b\|\nabla u(t)\|_2^2 + (\delta_3 + C_\rho \delta_3|\mu_2 - l|) \|\nabla u(t)\|_2^2
\]
\[\quad + \frac{|\mu_2|}{4\delta_3}\|u_t(t - \tau(t))\|_{2,G_1}^2 + \frac{a - l}{4\delta_3} (g \circ \nabla u)(t),
\]
for some positive constant $\delta_3$.

**Proof.** By using the differential equation in (1) and (G1), we get
\[
G'(t) = \|u_t(t)\|_2^2 + \int_{\Omega} u_t(t)u(t)dx + \|u_t(t)\|_{2,G_1}^2 + \int_{\Gamma_1} u_t(t)u(t)d\Gamma
\]
\[\quad + \frac{\sigma}{2} \left( \frac{d}{dt} \|\nabla u(t)\|_2^2 \right) \|\nabla u(t)\|_2^2
\]
\[\quad = \|u_t(t)\|_2^2 + \|u_t(t)\|_{2,G_1}^2 + \int_{\Omega} \nabla u(s) \cdot \int_0^t g(t - s) (\nabla u(s) - \nabla u(t)) dsdx
\]
\[\quad - b\|\nabla u(t)\|_2^2 - \left( a - \int_0^t g(s)ds \right) \|\nabla u(t)\|_2^2 - \mu_2 \int_{\Gamma_1} u_t(t - \tau(t))u(t)d\Gamma
\]
\[\quad \leq \|u_t(t)\|_2^2 + \|u_t(t)\|_{2,G_1}^2 + \int_{\Omega} \nabla u(s) \cdot \int_0^t g(t - s) (\nabla u(s) - \nabla u(t)) dsdx
\]
\[\quad - b\|\nabla u(t)\|_2^2 - l\|\nabla u(t)\|_2^2 - \mu_2 \int_{\Gamma_1} u_t(t - \tau(t))u(t)d\Gamma.
\]

We now estimate the right-hand side of (54). For a positive constant $\delta_3$, by Young’s inequality, we get (see [21])
\[
\int_{\Omega} \nabla u(s) \cdot \int_0^t g(t - s) (\nabla u(s) - \nabla u(t)) dsdx
\]
\[\leq \delta_3 \|\nabla u(t)\|_2^2 + \frac{1}{4\delta_3} \int_0^t \left( \int_0^s g(t - s) \nabla u(s) - \nabla u(t)ds \right)^2 dx
\]
\[\leq \delta_3 \|\nabla u(t)\|_2^2 + \frac{1}{4\delta_3} (g \circ \nabla u)(t).
\]

By Young’s inequality and Poincaré’s inequality, we have
\[
-\mu_2 \int_{\Gamma_1} u_t(t - \tau(t))u(t)d\Gamma \leq C_\rho \delta_3 |\mu_2| \|\nabla u(t)\|_2^2 + \frac{|\mu_2|}{4\delta_3} \|u_t(t - \tau(t))\|_{2,G_1}^2.
\]
Combining (54)-(56), we arrive at (53).

**Lemma 5.2.** Under the conditions of Theorem 2.2, the functional \( \phi(t) \) defined in (26) satisfies

\[
\phi'(t) \leq \left( \delta_4 - \int_0^t g(s)ds \right) \|u_t(t)\|_{2,\Gamma_1}^2 + \left( \delta_4 - \int_0^t g(s)ds \right) \|u_t(t)\|_{2,\Gamma_1}^2
\]

\[
+ [\delta_4 + 2\delta_4(a - l)^2] \|\nabla u(t)\|_2^2 + \sigma^2 E(0) \left( \frac{1}{2} \frac{d}{dt}\|\nabla u(t)\|_2^2 \right)^2
\]

\[
+ \left[ \frac{1}{4\delta_4} \left( a + \frac{2bE(0)}{l} \right)^2 + \left( \frac{\delta_4 + 1}{4\delta_4} \right) + \frac{C_p\mu_2}{4\delta_4} \right] (a - l)(g \circ \nabla u)(t)
\]

\[
+ \frac{g(0)}{4\delta_4} (C_p + C_p^\ast)(-g' \circ \nabla u)(t) + \delta_4\|u_t(t) - \tau(t)\|_{2,\Gamma_1}^2,
\]

for some positive constant \( \delta_4 \).

**Proof.** By using the differential equation in (1), we get

\[
\phi'(t) = -\int_\Omega u_t(t) \int_0^t g(t-s) (u(t) - u(s)) \, ds \, dx
\]

\[
- \int_\Omega u_t(t) \int_0^t g'(t-s) (u(t) - u(s)) \, ds \, dx
\]

\[
- \left( \int_0^t g(s)ds \right) \int_\Omega |u_t(t)|^2 \, dx - \int_{\Gamma_1} u_t(t) \int_0^t g(t-s) (u(t) - u(s)) \, ds \, d\Gamma
\]

\[
- \int_{\Gamma_1} u_t(t) \int_0^t g'(t-s) (u(t) - u(s)) \, ds \, d\Gamma - \left( \int_0^t g(s)ds \right) \int_{\Gamma_1} |u_t(t)|^2 \, d\Gamma
\]

\[
= \int_\Omega (a + b\|\nabla u(t)\|_2^2) \nabla u(t) \cdot \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx
\]

\[
+ \int_\Omega \sigma \left( \int_\Omega \nabla u(t) \cdot \nabla u_t(t) \right) \nabla u(t) \cdot \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx
\]

\[
- \int_\Omega \left( \int_0^t g(t-s) \nabla u(s) \, ds \right) \cdot \left( \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \right) \, dx
\]

\[
- \int_{\Gamma_1} u_t(t) \int_0^t g'(t-s) (u(t) - u(s)) \, ds \, d\Gamma
\]

\[
- \int_{\Gamma_1} u_t(t) \int_0^t g'(t-s) (u(t) - u(s)) \, ds \, d\Gamma
\]

\[
+ \mu_2 \int_{\Gamma_1} u_t(t - \tau(t)) \int_0^t g(t-s) (u(t) - u(s)) \, ds \, d\Gamma
\]

\[
- \left( \int_0^t g(s)ds \right) \int_\Omega |u_t(t)|^2 \, dx - \left( \int_0^t g(s)ds \right) \int_{\Gamma_1} |u_t(t)|^2 \, d\Gamma
\]

\[
= \sum_{i=1}^6 I_i - \left( \int_0^t g(s)ds \right) \int_\Omega u_t^2(t) \, dx - \left( \int_0^t g(s)ds \right) \int_{\Gamma_1} u_t(t)^2 \, d\Gamma.
\]
We now estimate the right-hand side of (58). By using Young’s inequality, Hölder’s inequality and Cauchy-Schwarz’s inequality, for a positive constant \( \delta \), we get

\[
d_1 \leq \left| \int_{\Omega} \left( a + b\|\nabla u(t)\|^2 \right) \nabla u(t) \cdot \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \right|.
\]

As \( a - \int_0^t g(s) \, ds \) \( \|\nabla u(t)\|^2 \leq 2E(t) \leq 2E(0) \) and by (G1), (8), (22), we can obtain

\[
\|\nabla u(t)\|^2 \leq \frac{2}{t} E(0).
\]

Then, from (59), we have

\[
d_1 \leq \left| \int_{\Omega} \left( a + \frac{2b}{t} E(0) \right) \nabla u(t) \cdot \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \right|
\leq \delta_4 \|\nabla u(t)\|^2 + \frac{1}{4\delta_4} \left( a + \frac{2b}{t} E(0) \right)^2 (a - l) (g \circ \nabla u)(t).
\]

\[
d_2 \leq \sigma \left| \int_{\Omega} \nabla u(t) \cdot \nabla u(t) \, dx \int_{\Omega} \nabla u(t) \cdot \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \right|
\leq \sigma^2 \left( \int_{\Omega} \nabla u(t) \cdot \nabla u(t) \, dx \right)^2 \frac{l}{2} \|\nabla u(t)\|^2
+ \frac{1}{2l} \int_{\Omega} \left( \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \right)^2 \, dx
\leq \sigma^2 E(0) \left( \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|^2 \right)^2 + \frac{a - l}{2l} (g \circ \nabla u)(t),
\]

\[
d_3 \leq \left| \int_{\Omega} \left( \int_0^t g(t-s) \nabla u(s) \, ds \right) \cdot \left( \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \right) \, dx \right|
\leq 2\delta_4 (a - l)^2 \|\nabla u(t)\|^2 + \left( 2\delta_4 + \frac{1}{4\delta_4} \right) (a - l) (g \circ \nabla u)(t),
\]

\[
d_4 \leq \left| \int_{\Omega} u_t(t) \int_0^t g'(t-s) (u(t) - u(s)) \, ds \, dx \right|
\leq \delta_4 \|u_t(t)\|^2 - \frac{C_p g(0)}{4\delta_4} (g' \circ \nabla u)(t),
\]

\[
d_5 \leq \left| \int_{\Gamma} u_t(t) \int_0^t g'(t-s) (u(t) - u(s)) \, ds \, ds \right|
\leq \delta_4 \|u_t(t)\|^2_{\Gamma, \Gamma} - \frac{C_p g(0)}{4\delta_4} (g' \circ \nabla u)(t),
\]

\[
d_6 \leq \left| \mu_2 \int_{\Gamma} u_t(t - \tau(t)) \int_0^t g(t-s) (u(t) - u(s)) \, ds \, ds \right|
\leq \delta_4 \mu_2 \|u_t(t - \tau(t))\|^2_{\Gamma, \Gamma} + \frac{\mu_2 C_p (a - l)}{4\delta_4} (g \circ \nabla u)(t).
\]

A Combination of (58)-(65) yields (57).
Proof of Theorem 2.2 (case $\alpha = 0$). Since $g$ is continuous and $g(0) > 0$, then for any $t_0 > 0$, we have
\[
\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0, \quad \forall \ t \geq t_0.
\]
(66)

From (49), (53), (57) and (66), then from (50), we get
\[
\dot{L}'(t) = N_1 E'(t) + \varepsilon_3 G'(t) + \varepsilon_4 \phi'(t)
\]
\[
\leq - \left[ \varepsilon_4 (g_0 - \delta_4) - \varepsilon_3 \|u_t(t)\|^2_\Gamma \right. \\
\left. - \left[ \varepsilon_3 (l - \delta_3 - C_p \delta_3 |\mu_2|) - \varepsilon_4 \delta_4 \left( 1 + 2(a - l)^2 \right) \right] \|\nabla u(t)\|^2_\Gamma - b \varepsilon_3 \|\nabla u(t)\|^4_\Gamma \\
+ (a - l) \left\{ \varepsilon_3 \frac{c_4}{4 \delta_4} + \varepsilon_4 \left[ 4 \delta_4 \left( a + \frac{2 b E(0)}{l} \right)^2 \right] + \left( 2 \delta_4 + \frac{1}{4 \delta_4} \right) \\
+ \frac{1}{2l} + \frac{C_p |\mu_2|}{4 \delta_4} \right\} \right] (g \circ \nabla u)(t) \\
- \left[ N_1 \left( \frac{\zeta}{2e^{\lambda \tau}} - \frac{|\mu_2|}{2} \right) - \varepsilon_3 \frac{|\mu_2|}{4 \delta_3} - \varepsilon_4 \delta_4 |\mu_2| \right] \|u_t(t - \tau(t))\|^2_\Gamma \\
+ \left[ \frac{1}{2} - \frac{\varepsilon_4 g(0) (C_p + C_p^*)}{4 \delta_4} \left( g' \circ \nabla u \right)(t) \right] \\
- \left[ N_1 - \varepsilon_4 \sigma E(0) \right] \sigma \left( \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|^2_\Gamma \right)^2 \\
- \frac{N_1 \zeta}{2} \int_{t - \tau(t)}^t \int_{\Gamma_1} e^{\lambda(s - t)} u_t^2(x, s)dsd\Gamma.
\]
(67)

Now, we should find that, for the positive constants $\zeta$, $\delta_3$, $\delta_4$, $\varepsilon_3$, $\varepsilon_4$ and $N_1$, by adding some suitable conditions to $|\mu_2|$, the following system of inequalities:
\[
\begin{cases}
\varepsilon_4 (g_0 - \delta_4) - \varepsilon_3 > 0,
\varepsilon_4 (g_0 - \delta_4) - \varepsilon_3 - N_1 \left( \frac{\zeta}{2} + \frac{|\mu_2|}{2} \right) > 0,
\varepsilon_3 (l - \delta_3 - C_p \delta_3 |\mu_2|) - \varepsilon_4 \delta_4 \left( 1 + 2(a - l)^2 \right) > 0,
N_1 \left( \frac{\zeta}{2e^{\lambda \tau}} - \frac{|\mu_2|}{2} \right) - \varepsilon_3 \frac{|\mu_2|}{4 \delta_3} - \varepsilon_4 \delta_4 |\mu_2| > 0,
\frac{1}{2} - \frac{\varepsilon_4 g(0) (C_p + C_p^*)}{4 \delta_4} > 0,
N_1 - \varepsilon_4 \sigma E(0) > 0,
\end{cases}
\]
(68)
is solvable.

In fact, we first pick $\delta_4 < \frac{l g_0}{l + 1 + 2(a - l)^2}$ and $\delta_3 < l - \frac{\delta_4 \left( 1 + 2(a - l)^2 \right)}{g_0 - \delta_4}$ to get
\[
g_0 - \delta_4 > 0
\]
(69)
and
\[
\frac{\delta_4 \left( 1 + 2(a - l)^2 \right)}{l - \delta_3} < g_0 - \delta_4.
\]
(70)
Then, by (70), we select \(\varepsilon_3\) and \(\varepsilon_4\) satisfying the relation
\[
\varepsilon_4 \delta_4 \left[ 1 + 2(a - l)^2 \right] < \varepsilon_3 < \varepsilon_4 (g_0 - \delta_4)
\] (71)
and
\[
\varepsilon_3 (l - \delta_3) - \varepsilon_4 \delta_4 \left[ 1 + 2(a - l)^2 \right] > 0.
\] (72)
From (71), we have
\[
\varepsilon_4 (g_0 - \delta_4) - \varepsilon_3 > 0.
\] (73)
We choose \(N_1\) large enough that (52) remain valid and
\[
\frac{N_1}{2} - \frac{\varepsilon_4 g(0)(C_p + C_p^*)}{4\delta_4} > 0,
\] (74)
and
\[
N_1 - \varepsilon_4 \sigma E(0) > 0.
\] (75)
Now, we must ensure that the second, third and the fourth inequalities in (68) hold. We have to prove the following system of inequalities:
\[
\begin{cases}
|\mu_2| + \zeta - K_1 < 0, \\
\varepsilon_3 C_p \delta_3 |\mu_2| - K_2 < 0, \\
-|\mu_2| + K_3 > 0,
\end{cases}
\] (76)
where
\[
K_1 = \frac{2[\varepsilon_4 (g_0 - \delta_4) - \varepsilon_3]}{N_1}, \quad K_2 = \varepsilon_3 (l - \delta_3) - \varepsilon_4 \delta_4 \left[ 1 + 2(a - l)^2 \right], \\
K_3 = \frac{2N_1 \delta_3 \zeta}{(2N_1 \delta_3 + \varepsilon_3 + 4\varepsilon_4 \delta_3 \delta_4) e^{\lambda \tau}}.
\]
By (73), we have \(\varepsilon_4 (g_0 - \delta_4) - \varepsilon_3 > 0\). Thus, we choose a positive constant \(\zeta\) such that
\[
\zeta < K_1
\] (77)
to get \(K_1 - \zeta > 0\). If we choose
\[
|\mu_2| < \min \left\{ K_1 - \zeta, \frac{K_2}{\varepsilon_3 C_p \delta_3}, K_3 \right\} := a_0,
\] (78)
then the system of inequalities (76) is solvable. Therefore, by (73), (74), (75) and (78), we can get (68) is solvable.
Consequently, from (8), (G2) and (67), there exist two positive constants \(M_1\) and \(K_4\) such that
\[
\hat{L}'(t) \leq -M_1 E(t) + K_4 (g \circ \nabla u)(t), \quad \forall \ t \geq t_0.
\] (79)
By the same method as in the proof of last section, we omit the details. Then, we can conclude
\[
E(t) \leq Ke^{-k \int_{t_0}^{t} \xi(s)ds}, \quad \forall \ t \geq t_0.
\]
This completes the proof.

Remark 2. We note from the proof of Theorem 2.2 (Sections 3 and 4) that, in the case of \(\sigma = 0\), the general decay result is still valid.
Remark 3. We note that when $\alpha > 0$, we have both the strong damping term $\alpha \Delta u_t$ and the viscoelastic term $\int_0^t g(t-s)\Delta u(x,s)ds$ to stabilize the whole system. However, when $\alpha > 0$, we observe from (45), (44) and (23) that, the assumption condition of $|\mu_2| \lesssim \alpha$ (appearing in (7) for the case $\alpha > 0$) is not optimal. In fact, since

$$k \sim M \sim k_3 \leq N \left( \alpha - \frac{|\mu_2| C_p}{2\sqrt{1 - \varepsilon}} - \frac{\zeta C_p}{2} \right) - \varepsilon_2 \alpha$$

the constant $k$ in (9) will approach 0 as $\alpha \to 0$. But when $\alpha = 0$, only $|\mu_2| \leq a_0$ ($a_0 > 0$) is sufficient to obtain the decay estimate.

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