Spectral Functions and Nuclear Response

Omar Benhar

INFN, Sezione di Roma
Dipartimento di Fisica, Università “La Sapienza”
I-00185 Roma, Italy

Abstract. I discuss the relation between the nuclear response and the Green function describing the propagation of a nucleon in the nuclear medium. Within this formalism, the widely used expressions in terms of spectral functions can be derived in a consistent and rigorous fashion. The results of recent applications to the study of the inclusive electron-nucleus cross section in the impulse approximation regime are briefly analyzed.

Keywords: nuclear response, spectral functions, lepton-nucleus scattering

PACS: 24.10.Cn,25.30.Fj,61.12.Bt

INTRODUCTION

Within non relativistic many-body theory, the nuclear response to a scalar probe delivering momentum \( q \) and energy \( \omega \) can be written in terms of the the imaginary part of the polarization propagator \( \Pi(q,\omega) \) according to [1, 2]

\[
S(q,\omega) = \frac{1}{\pi} \text{Im} \Pi(q,\omega) = \frac{1}{\pi} \text{Im} \frac{1}{\omega - i\eta} \rho_q^{\dagger} H \rho_q \frac{1}{E_0 - i\eta} \rho_0 \Omega ;
\]  

(1)

where \( \eta = 0^+ \), \( \rho_q = \sum_k a_k^{\dagger} q a_k \) is the operator describing the fluctuation of the target density induced by the interaction with the probe, \( a_k^{\dagger} \) and \( a_k \) are nucleon creation and annihilation operators, \( H \) is the nuclear hamiltonian and \( \Omega \) is the target ground state, satisfying the Schrödinger equation \( H \Omega = E_0 \Omega \).

In this short note, I will discuss the relation between \( S(q,\omega) \) and the nucleon Green function, leading to the popular expression of the response in terms of nucleon spectral functions [2, 3]. The main purpose of this work is show that the spectral function formalism, while being often advocated using heuristic arguments, can be derived in a rigorous and fully consistent fashion.

For the sake of simplicity, I will consider uniform nuclear matter with equal numbers of protons and neutrons. In the Fermi gas (FG) model, i.e. neglecting all interactions, such a system reduces to a degenerate Fermi gas of density \( \rho = A/V, A \) and \( V \) being the number of nucleons and the normalization volume, respectively. In the FG ground state the \( A \) nucleons occupy all momentum eigenstates belonging to the eigenvalues \( k \) such that \( k < k_F \), \( k_F = 2\rho=3\pi^2 \) being the Fermi momentum.
FORMALISM

Equation (1) clearly shows that the interaction with the probe leads to a transition of the struck nucleon from a hole state of momentum $k$ to a particle state of momentum $k + q$. To obtain $S(q; \omega)$ one needs to describe the propagation of the resulting particle-hole pair through the nuclear medium.

The fundamental quantity involved in the theoretical treatment of many-body system is the Green function, i.e. the quantum mechanical amplitude associated with the propagation of a particle from $\vec{x}$ ($t_{\text{in}}$) to $\vec{x'}$ ($t_{\text{out}}$) [1]. In nuclear matter, due to translation invariance, the Green function only depends on the difference $\vec{x} - \vec{x'}$, and after Fourier transformation to the conjugate variable $k$ $(k; E)$ can be written in the form

$$G(k; E) = \sum_{H} \frac{1}{E_0 - E + i\eta} a_k^{\dagger} \theta^{j} \frac{1}{E_0 + E - i\eta} a_k^{\dagger} \theta_{i}^{j}$$

(2)

where $G_h$ and $G_p$ correspond to propagation of nucleons sitting in hole and particle states, respectively.

The connection between Green function and spectral functions is established through the Lehman representation [1]

$$G(k; E) = \sum_{E_0} \frac{P_h(k; E^0)}{E^0 + i\eta} dE^0 \frac{P_p(k; E^0)}{E^0 - i\eta}$$

(3)

implying

$$P_h(k; E) = \sum_{N} \sum_{N+1} \frac{\theta}{N+1} \frac{\theta^{j}_{N}}{N} \frac{\theta^{j}_{N+1}}{N+1} \delta(E - E_0) = \frac{1}{\pi} \text{Im} G_h(k; E)$$

(4)

and

$$P_p(k; E) = \sum_{N} \sum_{N+1} \frac{\theta}{N+1} \frac{\theta^{j}_{N}}{N} \frac{\theta^{j}_{N+1}}{N+1} \delta(E + E_0) = \frac{1}{\pi} \text{Im} G_p(k; E)$$

(5)

where $\sum_{N+1} \frac{\theta}{N+1} \frac{\theta^{j}_{N}}{N}$ denotes an eigenstate of the $\langle A \ 1 \rangle$-nucleon system, carrying momentum $k$ and energy $E_n^{(+)}$.

Within the FG model the matrix elements of the creation and annihilation operators reduce to step functions, and the Green function takes a very simple form. For example, for hole states we find

$$G_{FGh}(k; E) = \frac{\theta(k_{F}^{\dagger})}{E + \varepsilon_{k}^{0} + i\eta}$$

(6)

with $\varepsilon_{k}^{0} = k^2 + 2M$, $M$ being the nucleon mass, implying

$$P_{FGh}(k; E) = \theta(k_{F}^{\dagger})$$

(7)

Strong interactions modify the energy of a nucleon carrying momentum $k$ according to $\varepsilon_{k}^{0} = \varepsilon_{k}^{0} + \Sigma(k; E)$, where $\Sigma(k; E)$ is the complex nucleon self-energy, describing
the effect of nuclear dynamics. As a consequence, the Green function for hole states becomes

$$G_h(k;E) = \frac{1}{E + \varepsilon_0 + k \cdot \Sigma(k;E)}$$ (8)

A very convenient decomposition of $G_h(k;E)$ can be obtained inserting a complete set of $(A-1)$-nucleon states (see Eqs. (2)-(5)) and isolating the contributions of one-hole bound states, whose weight is given by [4]

$$Z_k = \sum_{j} k \cdot j \Omega_{1/2} = \theta(k_F - j \cdot k) \Phi_k$$ (9)

Note that in the FG model these are the only nonvanishing terms, and $\Phi_k < 1$. The resulting contribution to the Green function exhibits a pole at $\varepsilon_k$, the quasiparticle energy $\varepsilon_k$ being defined through the equation

$$\varepsilon_k = \varepsilon_0 + k \cdot \Re \Sigma(k;\varepsilon_k)$$ (10)

The full Green function can be rewritten

$$G_h(k;E) = \frac{Z_k}{E + \varepsilon_0 + iZ_k \Im \Sigma(k;\varepsilon_k)} + G_h^B(k;E);$$ (11)

where $G_h^B$ is a smooth contribution, associated with $(A-1)$-nucleon states having at least one nucleon excited to the continuum (two hole-one particle, three hole-two particles . . . ) due to virtual scattering processes induced by nucleon-nucleon (NN) interactions. The corresponding spectral function is

$$P_h(k;E) = \frac{1}{\pi} \frac{Z_k^2 \Im \Sigma(k;\varepsilon_k)}{(E + \varepsilon_0 + iZ_k \Im \Sigma(k;\varepsilon_k))^2 + k \cdot j \Omega_{1/2}} + P_h^B(k;E);$$ (12)

The first term in the right hand side of the above equation yields the spectrum of a system of independent quasiparticles, carrying momenta $j \cdot k < k_F$, moving in a complex mean field whose real and imaginary parts determine the quasiparticle effective mass and lifetime, respectively. The presence of the second term is a consequence of nucleon-nucleon correlations, not taken into account in the mean field picture. Being the only one surviving at $j \cdot k > k_F$, in the FG model this correlation term vanishes.

Figure 1 illustrates the energy dependence of the hole spectral function of nuclear matter, calculated in Ref. [3] with a realistic nuclear hamiltonian yielding an accurate description of NN scattering data up to pion production threshold. Comparison with the FG model clearly shows that the effects of nuclear dynamics and NN correlations are large, resulting in a shift of the quasiparticle peaks, whose finite width becomes large for deeply-bound states with $j \cdot k > k_F$. In addition, NN correlations are responsible for the appearance of strength at $j \cdot k > k_F$. The energy integral

$$n(k) = \int E P_h(k;E)$$ (13)

yields the occupation probability of the state of momentum $k$. The results of Fig. 1 clearly show that in presence of correlations $n(j \cdot k > k_F) \neq 0$. 
In general, the calculation of the response requires the knowledge of $P_h$ and $P_p$, as well as of the particle-hole effective interaction \cite{2, 5}. The spectral functions are mostly affected by short range NN correlations (see Fig. 1), while the inclusion of the effective interaction, e.g. within the framework of the Random Phase Approximation (RPA) \cite{5}, is needed to account for collective excitations induced by long range correlations, involving more than two nucleons.

At large momentum transfer, as the space resolution of the probe becomes small compared to the average NN separation distance, $S(q;\omega)$ is no longer significantly affected by long range correlations. In this kinematical regime the zero-th order approximation in the effective interaction is expected to be applicable, and the response can be written in the simple form

$$S(q;\omega) = \frac{Z}{d^3k dE \ P_h(k\ , E) P_p(k + q\ , \omega\ , E)} :$$

(14)

The widely employed impulse approximation (IA) can be readily obtained from the above definition replacing $P_p$ with the FG result, which amounts to disregarding final state interactions (FSI) between the struck nucleon and the spectator particles:

$$S_{IA}(q;\omega) = \frac{Z}{d^3k dE \ P_h(k\ , E) \theta(k - q\ , \kappa) \delta(\omega\ , E\ , q\ , k)} :$$

(15)

At moderate momentum transfer, both the full response and the particle and hole spectral functions can be obtained using non relativistic many-body theory. The results of Ref.\cite{2} suggest that the zero-th order approximations of Eqs.(14) and (15) are fairly accurate at $q > 500$ MeV. However, it has to be pointed out that in this kinematical regime the motion of the struck nucleon in the final state can no longer be described using the non relativistic formalism. While at IA level this problem can be easily circumvented, replacing the non relativistic kinetic energy with its relativistic counterpart,
obtaining the response at large \( q \) from Eq. (14) involves further approximations, needed to calculate the particle spectral function.

A systematic scheme to include corrections to Eq. (15) and take into account FSI effects, originally proposed in Ref. [6], is discussed in Ref. [7]. In the simplest implementation of this approach the response is obtained from the IA result according to

\[
S(q; \omega) = \int d\omega' S_{\text{IA}}(q; \omega') F_q(\omega, \omega') \;
\]

the folding function \( F_q \) being related to the particle spectral function through

\[
F_q(\omega, E, \frac{q}{Q}) = P_p(q; \omega, E) \;
\]

with \( \epsilon_0^q = \frac{p}{q^2 + M^2} \). Obviously, at large \( q \) the calculation of \( P_p(q; \omega, E) \) cannot be carried out using a nuclear potential model. However, \( F_q \) can be obtained form the measured NN scattering amplitude within the eikonal approximation [7]. It has to be pointed out that NN correlation, whose effect on \( P_p \) is illustrated in Fig. 1, also affect the particle spectral function and, as a consequence, the folding function of Eq. (17). In the absence of FSI \( F_q \) shrinks to a \( \delta \)-function and the IA result of Eq. (15) is recovered.

**APPLICATIONS TO LEPTON-NUCLEUS SCATTERING**

The formalism outlined in the previous section can be readily generalized to describe lepton-nucleus scattering, replacing the density fluctuation operator \( \rho_q \) with the appropriate vector and axial-vector currents. The large body of theoretical and experimental work on inclusive electron-nucleus scattering has been recently reviewed in Ref. [8].

Over the past few years, significant effort has been devoted to the study of the kinematical region corresponding to beam energies around 1 GeV, whose understanding is relevant to the analysis of many neutrino oscillation experiments [9].

In Fig. 2 the results of Ref. [10], obtained using the realistic hole spectral functions of Ref. [11] and the particle spectral functions resulting from the approach of Ref. [6], are compared to the measured electron scattering cross sections off Carbon and Oxygen of Refs. [12] and [13], respectively. It appears that, while for the kinematics corresponding to the higher value of \( Q^2 = \frac{q^2}{2} \) the peaks corresponding to quasi-elastic scattering and delta resonance production are both very well described, at the lower \( Q^2 \) the delta peak is somewhat underestimated. In both cases, a sizable deficit of strength is observed in the region of the dip between the two peaks. The possibility that these problems may be ascribed to deficiencies in the description of the elementary electron-nucleon cross section above pion production threshold is being actively investigated [14].

**ACKNOWLEDGMENTS**

This paper is dedicated to the memory of Adelchi Fabrocini and Vijay Pandharipande, whose work led to important and lasting progress in nuclear response theory.
FIGURE 2. Upper panel: inclusive electron scattering cross section off carbon at beam energy 1.3 GeV and scattering angle 37.5°, as a function of the electron energy loss $\omega$. The shaded area shows the results of Ref. [10]. Data from Ref. [12]. Lower panel: same as in the upper panel, but for oxygen target, beam energy 1.2 GeV and scattering angle 32°. Data from Ref. [13].

REFERENCES

1. A. Fetter, and J. Walecka, *Quantum Theory of Many Particle Systems*, McGraw-Hill, New York, NY, 1971.
2. O. Benhar, A. Fabrocini, and S. Fantoni, *Nucl. Phys. A* 550, 201 (1992).
3. O. Benhar, A. Fabrocini, and S. Fantoni, *Nucl. Phys. A* 505, 267 (1989).
4. O. Benhar, A. Fabrocini, and S. Fantoni, *Phys. Rev. C* 41, R24 (1990).
5. W. Dickhoff, and C. Barbieri, *Prog. Part. Nucl. Phys.* 52, 377 (2004).
6. O. Benhar, A. Fabrocini, S. Fantoni, G. Miller, V. Pandharipande, and I. Sick, *Phys. Rev. C* 44, 2328 (1991).
7. M. Petraki, E. Mavrommatis, O. Benhar, J. Clark, A. Fabrocini, and S. Fantoni, *Phys. Rev. C* 67, 014605 (2001).
8. O. Benhar, D. Day, and I. Sick, *Rev. Mod. Phys.* (2007), in press [nucl-ex/0603029].
9. O. Benhar, N. Farina, H. Nakamura, M. Sakuda, and R. Seki, *Phys. Rev. D* 72, 053005 (2005).
10. O. Benhar, and D. Meloni, *Phys. Rev. Lett.* 97, 192301 (2006).
11. O. Benhar, A. Fabrocini, S. Fantoni, and I. Sick, *Nucl. Phys. A* 579, 493 (1994).
12. R. Sealeck, K. Giovanetti, S. Thornton, Z. Meziani, O. Rondon-Aramayo, S. Auffret, J.-P. Chen, D. Christian, D. Day, J. McCarthy, R. Minehard, L. Dennis, K. Kemper, B. Mecking, and J. Morgenstern, *Phys. Rev. Lett.* 62, 1350 (1989).
13. M. Anghinolfi, M. Battaglieri, N. Bianchi, R. Cenni, P. Corvisiero, A. Fantoni, P. Levi Sandri, A. Longhi, V. Lucherini, V. Mokeev, V. Muccifora, E. Polli, A. Reolon, G. Ricco, M. Ripani, P. Rossi, S. Simula, M. Taitti, A. Teglia, and A. Zucchatti, *Nucl. Phys. A* 602, 405 (1996).
14. H. Nakamura, M. Sakuda, T. Nasu, and O. Benhar (2007), these Proceedings.