On direct numerical treatment of hypersingular integral equations arising in mechanics and acoustics

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Abstract

In this paper we present a treatment of hypersingular integral equations, which have relevant applications in many problems of wave dynamics, elasticity and fluid mechanics with mixed boundary conditions. The main goal of the present work is the development of an efficient direct numerical collocation method. The paper is completed with two examples taken from crack theory and acoustics: the study of a single crack in a
linear isotropic elastic medium, and diffraction of a plane acoustic wave by a thin rigid screen.

1 Introduction

Many problems of fluid mechanics, elasticity, and wave dynamics (acoustics) with mixed boundary conditions can be reduced to hypersingular equations of the following type

\[ \int_{-1}^{1} \varphi(t) \left[ \frac{1}{(x-t)^2} + K_0(x,t) \right] dt = f(x), \quad |x| < 1, \quad (1.1) \]

where \( K_0(x,t) \) is a regular (in some sense) part of the kernel. Direct numerical treatment of eq.(1.1) is not easy. Many authors applied the approach, which may be currently considered as classical. If one defines a solution bounded on the both endpoints \( x = \pm 1 \) (that will be shown below to vanish as \( \sqrt{|x \pm 1|} \) at \( x \to \pm 1 \)), then one may search it in the form of the series

\[ \varphi(x) = \sqrt{1-x^2} \sum_{j=0}^{\infty} \varphi_j U_j(x), \quad |x| < 1, \quad (1.2) \]

where \( U_j(x) \) are the Chebyshev’s polynomials of the second kind [1]. Then, using the relation

\[ \int_{-1}^{1} \sqrt{1-t^2} U_j(t) U_i(t) dt = \pi d dx \left[ T_{j+1}(x) \right] = -\pi (j+1) U_j(x) \]

(\( T_j(x) \) is the Chebyshev’s polynomial of the first kind), applying the scalar product of eq.(1.1) with the functions \( \sqrt{1-x^2} U_i(x) \), one can reduce initial integral equation (1.1) to the infinite linear algebraic system of the second kind:

\[ - (i+1) \frac{\pi^2}{2} \varphi_i + \sum_{j=0}^{\infty} k_{ij} \varphi_j = f_i, \quad i = 0, 1, \ldots, \quad (1.3) \]

\[ k_{ij} = \int_{-1}^{1} \int_{-1}^{1} \sqrt{1-x^2} \sqrt{1-t^2} U_i(x) U_j(t) K_0(x,t) dx dt. \quad (1.4) \]
We have used here the well known orthogonality relation [1]

\[ \int_{-1}^{1} \sqrt{1-x^2} \ U_i(x) \ U_j(x) = \frac{\pi}{2} \delta_{ij}, \quad i, j = 0, 1, \ldots, \]

where \( \delta_{ij} \) is the Dirac delta.

The study of qualitative properties of the infinite system (1.3) is not the aim of the present work, and we only note that, when the regular kernel \( K_0(x,t) \) is of a convolution type \( [K_0(x,t) = K_0(x-t)] \), this approach is often applied in a different way.

We rewrite eq. (1.1) in equivalent form

\[ \int_{-1}^{1} K(x-t) \ \phi(t) \ dt = f(x), \quad |x| < 1, \quad K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} L(s) \ e^{-isx} \ ds, \quad (1.5) \]

where \( L(s) \) is the Fourier transform of \( K(x) \). Let \( L(s) \) be integrable for finite \( s \), and for \( s \to \infty \) it has the following asymptotics

\[ L(s) = A \ |s| + L_0(s) \quad \text{and} \quad L_0(s) = O \left( \frac{1}{s^{1+\delta}} \right), \quad (\delta > 0), \quad (1.6) \]

then

\[ K(x) = -\frac{2A}{x^2} + K_0(x), \quad K_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} L_0(s) \ e^{-isx} \ ds, \]

where \( K_0(x) \) is at least continuous, and the generalized value of the integral [2] has been used, that is,

\[ \int_{-\infty}^{\infty} |s| \ e^{-isx} \ ds = 2 \lim_{\epsilon \to +0} \int_{0}^{\infty} e^{-\epsilon s} \cos(sx) \ ds = -\frac{2}{x^2}. \quad (1.7) \]

Equation (1.5) together with (1.6) is evidently equivalent to eq.(1.1). If one substitutes the series expansion (1.2) into eqs.(1.5) and then applies a scalar product with functions \( \sqrt{1-x^2} \ U_i(x) \), one finally arrives at the infinite linear algebraic system [1]

\[ \sum_{j=0}^{\infty} l_{ij} \varphi_j = f_i, \quad (i = 0, 1, \ldots), \quad l_{ij} = \int_{-1}^{1} \sqrt{(1-x^2)(1-t^2)} U_i(x)U_j(t) \ dx \ dt \]

\[ \times \int_{-\infty}^{\infty} L(s) e^{-is(x-t)} \ ds \sim \int_{-\infty}^{\infty} \frac{J_{i+1}(s)J_{j+1}(s)}{s^2} L(s) \ ds. \quad (1.8) \]
equivalent to system (1.3). Note that the last integral is finite for all \( i, j = 0, 1, \ldots \), since \( J_n(s) \sim O(s^n), \; s \to 0; \quad J_n(s) \sim O(1/\sqrt{s}), \; s \to \infty \), which, on taking into account the asymptotic property (1.6), is evident.

The integral equation (1.1) with a hypersingular kernel can thus be reduced to an infinite system of linear algebraic equations. This implies numerical treatment of double integrals in (1.4) or single integrals over infinite interval of strongly oscillating functions, like in (1.8). When solving integral equations with more regular kernels, a powerful direct collocation technique, in the framework of the Boundary Element Method [3], can successfully be applied. For the Cauchy-type integral equations, a direct collocation method has been developed by Belotserkovsky and Lifanov [4, 5]. Hence, the main goal of the present paper is to develop an efficient direct numerical collocation method.

### 2 Some properties of hypersingular integrals

First, one should clarify in which sense hypersingular integrals as given by eq.(1.1) may be treated, since they do not exist either as improper integrals of the first kind or as Cauchy-type singular integrals. At least three different definitions of hypersingular integrals are known (see [6]):

1. The integral is a derivative of the Cauchy principal value

\[
\int_a^b \frac{\varphi(t)}{(x-t)^\alpha} dt = -\frac{d}{dx} \int_a^b \frac{\varphi(t)}{x-t} dt. \tag{2.1}
\]

2. The integral is treated as a Hadamard principal value [4,5]

\[
\int_a^b \frac{\varphi(t)}{(x-t)^\alpha} dt = \lim_{\varepsilon \to 0} \left[ \left( \int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \right) \frac{\varphi(t) dt}{(x-t)^\alpha} - \frac{2\varphi(x)}{\varepsilon} \right]. \tag{2.2}
\]

3. The integral is a residue value, in the sense of generalized functions [2], that is is an analytical continuation of the integral

\[
\int_a^b |x-t|^\alpha \varphi(t) dt \tag{2.3}
\]
from domain, where it exists in the classical sense, to the value $\alpha = -2$.

When $\varphi(x) \equiv 1$, the three different approaches give the same result:

\[
\int_{a}^{b} \frac{dt}{(x-t)^2} = - \frac{d}{dx} \int_{a}^{b} \frac{dt}{x-t} = \frac{d}{dx} \ln \left( \frac{b-x}{x-a} \right) = \frac{a-b}{(x-a)(b-x)}. \]

\[
\int_{a}^{b} \frac{dt}{(x-t)^2} = \lim_{\varepsilon \to +0} \left[ \left( - \frac{1}{x-a} + \frac{1}{\varepsilon} + \frac{1}{x-b} \right) - \frac{2}{\varepsilon} \right] = \frac{a-b}{(x-a)(b-x)}. \]

\[
\int_{a}^{b} |x-t|^\alpha dt = \int_{a}^{b} (x-t)^\alpha dt + \int_{x}^{b} (t-x)^\alpha dt = \frac{(x-a)^{\alpha+1}}{\alpha+1} + \frac{(b-x)^{\alpha+1}}{\alpha+1}, \quad (2.4)
\]

when $\Re(\alpha) > -1$. The analytical continuation of (2.4) from the half-plane $\Re(\alpha) > -1$ to the value $\alpha = -2$ gives

\[
\int_{a}^{b} \frac{dt}{(x-t)^2} = \lim_{\alpha \to -2} \left[ \frac{(x-a)^{\alpha+1}}{\alpha+1} + \frac{(b-x)^{\alpha+1}}{\alpha+1} \right] = \frac{a-b}{(x-a)(b-x)}. \]

All three definitions are equivalent to each other, if the density $\varphi(x)$ is analytical (i.e. differentiable) on the open interval $(a, b)$. Really, let $\varphi(x)$ be analytic and $\Re(\alpha) > 1$, then

\[
- \frac{d}{dx} \int_{a}^{b} \varphi(t) dt = - \lim_{\varepsilon \to +0} \frac{d}{dx} \left( \int_{a}^{b} \int_{x+\varepsilon}^{b} \varphi(t) dt \right) - \frac{2\varphi(x)}{\varepsilon}, \quad \int_{a}^{b} |x-t|^\alpha \varphi(t) dt = \lim_{\varepsilon \to +0} \left( \int_{a}^{b} \int_{x+\varepsilon}^{b} |x-t|^\alpha \varphi(t) dt \right) =
\]

\[
= \lim_{\varepsilon \to +0} \left[ \int_{a}^{b} (x-t)^\alpha \varphi(t) dt + \int_{x+\varepsilon}^{b} (t-x)^\alpha \varphi(t) dt \right] =
\]

\[
= \frac{d}{dx} \lim_{\varepsilon \to +0} \left[ - \int_{a}^{x+\varepsilon} \frac{(x-t)^{\alpha+1}}{\alpha+1} \varphi(t) dt + \int_{x+\varepsilon}^{b} \frac{(t-x)^{\alpha+1}}{\alpha+1} \varphi(t) dt \right],
\]

so, by applying analytical continuation to the last relation, one can see (with the use of a standard $(\varepsilon, \delta)$ formalism) that the right-hand side results in (2.1).

Therefore, equivalence of 1 to 2 and 3 is evident. Let us prove that if $\varphi(x) \in C_{2}(a, b)$, then a finite value of the limit in expression (2.2) exists, and so for
$x \in (a, b)$ the integral $\int_a^b \varphi(t)/(x - t)^2 \, dt$ is finite in any sense. Really, expression in the square brackets in eq.(2.2) is

$$
\left( \int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \right) \frac{\varphi(t) - \varphi(x) - \varphi'(x)(t-x)}{(x-t)^2} \, dt + \varphi(x) \frac{a-b}{(x-a)(b-x)} + \varphi'(x) \ln \frac{b-x}{x-a},
$$

which has a finite limit at $\varepsilon \to +0$.

3 Integral equation with characteristic hypersingular kernel

Consider hypersingular equation with the characteristic kernel

$$
\int_a^b \frac{g(t) \, dt}{(x-t)^2} = f'(x), \quad x \in (a, b), \quad f(x) \in C^2(a, b).
$$

(3.1)

Let us prove that a bounded solution of Eq.(3.1) is unique and is given as follows

$$
g(x) = \frac{\sqrt{(x-a)(b-x)}}{\pi^2} \int_a^b \frac{f(t) \, dt}{\sqrt{(t-a)(b-t)(x-t)}}.
$$

(3.2)

Really, according to the definition (2.1), equation (3.1) is equivalent to

$$
\frac{d}{dx} \int_a^b \frac{g(t) \, dt}{(x-t)^2} = -f'(x), \quad \sim \quad \int_a^b \frac{g(t) \, dt}{x-t} = -f(x) + C, \quad x \in (a, b),
$$

where $C$ is an arbitrary constant. Now an inversion formula for the Cauchy characteristic integral operator [7] determines the bounded solution as

$$
g(x) = \frac{\sqrt{(x-a)(b-x)}}{\pi^2} \int_a^b \frac{f(t) \, dt}{\sqrt{(t-a)(b-t)(x-t)}}, \quad x \in (a, b),
$$

and the constant $C$ as

$$
C = \frac{1}{\pi^2} \int_a^b \frac{f(t) \, dt}{\sqrt{(t-a)(b-t)}}.
$$

Thus, as indicated in the Introduction, any bounded solution of Eq.(3.1) vanishes at $x \to a, b$. 6
To construct a direct collocation technique to solve eq.(3.1) for arbitrary right-hand side, we divide the interval \((a, b)\) into \(n\) small equal subintervals of the length \(h = (b - a) / n\), by the nodes \(a = t_0, t_1, t_2, \ldots, t_{n-1}, t_n = b\), \(t_j = a + jh\), \(j = 0, 1, \ldots, n\). The central points of each sub-interval \((t_{i-1}, t_i)\) are denoted by \(x_i\), thus \(x_i = a + (i - 1/2)h\), \(i = 1, \ldots, n\).

If we try to arrange an approximation of integral in (3.1) by using finite sum, like in regular cases, then for \(x = x_i\) we have

\[
\int_a^b \frac{g(t)}{(x_i - t)^2} \, dt \approx \sum_{j=0}^{n} g(t_j) \int_{t_{j-1}}^{t_j} \frac{dt}{(x_i - t)^2} = g(t_i) \int_{-h/2}^{h/2} \frac{dt}{t^2} + \sum_{j \neq i} g(t_j) \left( \frac{1}{x_i - t_j} - \frac{1}{x_i - t_{j-1}} \right),
\]

where a value of the above hypersingular integral of the function \(1/t^2\) has been used. So, we try to approximate eq.(3.1) by the linear algebraic system

\[
\sum_{j=1}^{n} g(t_j) \left( \frac{1}{x_i - t_j} - \frac{1}{x_i - t_{j-1}} \right) = f'(x_i), \quad i = 1, \ldots, n. \tag{3.4}
\]

Further considerations are similar to those in [4,5]. Let us prove that, by assuming \(x = x_l \in (a, b)\) is fixed, the difference between solution \(g(x_l)\) of the system (3.4) and analytical solution (3.2) tends to zero, when \(n \to \infty\). Obviously, the system (3.4) can be rewritten as

\[
\sum_{j=1}^{n} g(t_j) \left( \frac{1}{x_j - t_i} - \frac{1}{x_j - t_{i-1}} \right) = f'(x_i), \quad i = 1, \ldots, n, \tag{3.5}
\]

and its principal determinant is

\[
\Delta = \begin{vmatrix}
\left( \frac{1}{x_1 - t_1} - \frac{1}{x_1 - t_0} \right) & \ldots & \left( \frac{1}{x_1 - t_{n-1}} - \frac{1}{x_1 - t_{n-2}} \right) \\
\ldots & \ddots & \ldots \\
\left( \frac{1}{x_n - t_n} - \frac{1}{x_n - t_{n-1}} \right) & \ldots & \left( \frac{1}{x_n - t_{n-1}} - \frac{1}{x_n - t_{n-2}} \right)
\end{vmatrix}. \tag{3.6}
\]

Let us rewrite the \(i - th\) row \((i = 2, \ldots, n)\) of \(\Delta\) as a sum of all other rows,
from $k = 1$ to $k = i$:

$$\Delta = \begin{vmatrix}
\frac{t_1 - t_0}{(x_1 - t_1)(x_1 - t_0)} & \cdots & \frac{t_1 - t_0}{(x_1 - t_1)(x_1 - t_0)} \\
\frac{t_n - t_0}{(x_1 - t_n)(x_1 - t_0)} & \cdots & \frac{t_n - t_0}{(x_1 - t_n)(x_1 - t_0)}
\end{vmatrix} = \prod_{i=1}^{n} \frac{(t_j - t_0)}{\prod_{i=1}^{n} (x_i - t_0)}$$

(3.7)

\[ \times \begin{vmatrix}
\frac{1}{x_1 - t_1} & \cdots & \frac{1}{x_1 - t_1} \\
\frac{1}{x_1 - t_1} & \cdots & \frac{1}{x_1 - t_1} \\
\frac{1}{x_1 - t_n} & \cdots & \frac{1}{x_1 - t_n}
\end{vmatrix} = \frac{\prod (t_j - t_0) \prod (t_q - t_p) \prod (x_p - x_q)}{\prod (x_i - t_0) \prod (x_q - t_p)} \prod_{q,p} (x_q - t_p), \]

where the lower limit in all products is 1 and the upper is $n$. A known value of the last determinant has been taken from [4,5].

According to the Cramer’s rule, one needs to calculate the determinant $\Delta_l$ where a column with the right-hand side (3.5) is substituted instead of the $l-th$ column of $\Delta$ (3.6):

$$\Delta_l = \begin{vmatrix}
\frac{1}{x_1 - t_1} - \frac{1}{x_1 - t_0} & \cdots & \frac{1}{x_1 - t_1} - \frac{1}{x_1 - t_0} \\
\frac{1}{x_1 - t_2} - \frac{1}{x_1 - t_1} & \cdots & \frac{1}{x_1 - t_2} - \frac{1}{x_1 - t_1} \\
\cdots & \cdots & \cdots \\
\frac{1}{x_1 - t_n} - \frac{1}{x_1 - t_{n-1}} & \cdots & \frac{1}{x_1 - t_n} - \frac{1}{x_1 - t_{n-1}}
\end{vmatrix} = \begin{vmatrix}
\frac{t_1 - t_0}{(x_1 - t_1)(x_1 - t_0)} & \cdots & \frac{t_1 - t_0}{(x_1 - t_1)(x_1 - t_0)} \\
\frac{t_2 - t_0}{(x_1 - t_2)(x_1 - t_0)} & \cdots & \frac{t_2 - t_0}{(x_1 - t_2)(x_1 - t_0)} \\
\cdots & \cdots & \cdots \\
\frac{t_n - t_0}{(x_1 - t_n)(x_1 - t_0)} & \cdots & \frac{t_n - t_0}{(x_1 - t_n)(x_1 - t_0)}
\end{vmatrix},$$

where we have used the same summation of rows as in the case of $\Delta$.

The last determinant may be calculated arranging expansion by elements of the $l-th$ column as follows (see also [4,5])

$$\Delta_l = \frac{\prod (t_j - t_0)}{\prod (x_i - t_0)} \sum_{k=1}^{m} \frac{f'(x_k)}{t_m - t_0} (-1)^{m+l} q < p : q, p \neq m \prod_{q < p : q, p \neq l} \frac{(t_q - t_p)(x_p - x_q)}{\prod_{q \neq l, p \neq m} (x_q - t_p)}. $$
so

$$g(x_l) = \frac{\Delta_l}{\Delta} = (x_l - t_0) \sum_{m=1}^{n} \frac{\sum_{k=1}^{m} f'(x_k) \prod_{p} (x_l - t_p) \prod_{q} (x_q - t_m)}{(t_m - t_0)(x_l - t_m) \prod_{p \neq m} (t_m - t_p) \prod_{q \neq l} (x_q - x_l)}. \quad (3.8)$$

The last expression at \( h \to 0 \sim n \to \infty \), under the condition \( x_l \in (a, b) \) is fixed, can be estimated as follows:

$$\prod_{p=1}^{n} (x_l - t_p) \prod_{q \neq l} (x_q - x_l) = (x_l - t_l) \prod_{p=1}^{l-1} (x_l - t_p) \prod_{q=1}^{l-1} (x_q - x_l) \prod_{g=l+1}^{n} (x_q - x_l) =$$

$$= \frac{(-1)^n h^{l-1} \prod_{p=1}^{l-1} (1 - \frac{1}{2p}) \prod_{q=1}^{l-1} (1 + \frac{1}{2q})}{\prod_{g=l+1}^{n} (x_q - x_l)} \sim (-1)^n \frac{h \sqrt{b - x_l}}{\pi \sqrt{x_l - a}}, \quad n \to \infty ,$$

if \( x \) belongs to the open interval \((a, b)\) (i.e. \( l \neq n, \ l \neq 1 \)). Here we have used the asymptotic estimate \([4,5]\)

$$\prod_{m=1}^{n} \left(1 + \frac{\beta}{m}\right) = \frac{n^\beta}{\Gamma(1 + \beta)} + O\left(n^{\beta-1}\right), \quad n \to \infty ,$$

Further, by analogy

$$\prod_{q} (x_q - t_m) \prod_{p \neq m} (t_m - t_p) = (-1)^n (t_m - x_m) \prod_{q=1}^{m-1} \left(1 + \frac{1}{2q}\right) \prod_{q=1}^{n-m} \left(1 - \frac{1}{2q}\right) \sim (-1)^n \frac{h \sqrt{t_m - a}}{\pi \sqrt{b - t_m}}.$$

Other terms in eq. (3.8) can be simplified as follows \((h \to 0)\):

\((x_l - t_0) \to (x_l - a), \ (t_m - t_0) \to (t_m - a), \ h \sum_{k=1}^{m} f'(t_k) \to \int_{a}^{t_m} f'(t) dt = f(t_m),\)

hence expression (3.8) with \( h \to 0 \) tends to

$$g(x_l) \sim \frac{h}{\pi^2} \sqrt{(x_l - a) (b - x_l)} \sum_{m=1}^{n} \frac{f(t_m)}{\sqrt{(t_m - a) (b - t_m) (x_l - t_m)}} \sim$$

$$\sim \frac{\sqrt{(x_l - a) (b - x_l)}}{\pi^2} \int_{a}^{b} \frac{f(t) dt}{\sqrt{(t - a) (b - t) (x_l - t)}}.$$

An example on application of the proposed numerical method is shown in Fig.1, where \((a, b) = (-1, 1)\).
4 Full hypersingular equation

Consider the full equation

\[ \int_a^b \left[ \frac{1}{(x-t)^2} + K_0(x,t) \right] g(t) \, dt = f'(x), \quad x \in (a,b), \]  

(4.1a)

where

\[ K_0(x,t) = \frac{\partial K_1(x,t)}{\partial x}. \]  

(4.1b)

Its bounded solution can be constructed by applying inversion of the characteristic part, that reduces eq.(4.1a) to a second-kind Fredholm integral equation

\[ g(x) + \int_a^b N_1(x,t) g(t) \, dt = f_1(x), \quad x \in (a,b), \]  

(4.2)

where

\[ N_1(x,t) = \frac{\sqrt{(x-a)(b-x)}}{\pi^2} \int_a^b \frac{K_1(\tau,t) \, d\tau}{\sqrt{(\tau-a)(b-\tau)(x-\tau)}}, \]

\[ f_1(x) = \frac{\sqrt{(x-a)(b-x)}}{\pi^2} \int_a^b \frac{f(\tau) \, d\tau}{\sqrt{(\tau-a)(b-\tau)(x-\tau)}}. \]

Let us prove that, if \( f(x) \in C_1(a,b); \ K_1(x,t) \in C_1[(a,b) \times (a,b)] \), then for any \( x \in (a,b) \) the difference between solution \( g(x) \) of the linear algebraic system

\[ \sum_{j=1}^n \left[ \frac{1}{x_i-t_j} - \frac{1}{x_i-t_{j-1}} + hK_0(x_i,t_j) \right] g(t_j) = f'(x_i), \quad i = 1,\ldots,n \]

and the bounded solution of eq.(4.1) tends to zero when \( h \to 0 \) (i.e. \( n \to \infty \)). Really, if one transfers the terms, related to the regular kernel, to the right-hand side and solves so written linear algebraic system with the characteristic matrix

\[ \frac{1}{x_i-t_j} - \frac{1}{x_i-t_{j-1}}, \]

then one arrives at a finite-difference approximation of the eq.(4.2). The proof is finally completed, if one applies classical results on numerical solution of the second-kind Fredholm integral equation.

An example for \( K_0(x,t) = A(x-t), \ f'(x) = -\pi \sim f(x) = -\pi x \) is shown in Fig.2, where numerical solution is compared with the exact one \( g(x) = 8\sqrt{1-x^2}(4+Ax)/(32+A^2) \) in the case \( A = 3, \ (a,b) = (-1,1) \).
It should also be noted that some interesting results on the theory and numerical treatment of hypersingular integral equations have been recently obtained in [8-10].

5 Some examples from mechanics and acoustics

5.1 Single crack in linear isotropic elastic medium

Consider the two-dimensional in-plane problem. Then the displacement field is \( \mathbf{u} = (u_x, u_y, 0) \), where \( u_x = u_x(x, y) \), \( u_y = u_y(x, y) \) can be represented in the Papkovich-Neuber form

\[
\begin{align*}
  u_x &= \frac{\partial \varphi}{\partial x} + y \frac{\partial \psi}{\partial x}, \\
  u_y &= \frac{\partial \varphi}{\partial y} + y \frac{\partial \psi}{\partial y} - \chi \psi, \\
  \chi &= 3 - 4\nu,
\end{align*}
\]

where the potentials \( \varphi \) and \( \psi \) are harmonic

\[
\Delta \varphi = 0, \quad \Delta \psi = 0,
\]

and \( \nu = \lambda / 2(\lambda + \mu) \) is the Poisson ratio (\( \lambda \) and \( \mu \) are elastic constants).

Let a constant normal load \( \sigma_{yy} = \sigma_0 \) (\( \tau_{xy} = 0 \)) be applied symmetrically to internal faces of the straight-line crack \( y = 0 \), \( x \in (-a, a) \). Then, due to a symmetry, the problem can be reduced to the upper half-plane \( y \geq 0 \) with the boundary conditions on the boundary line \( y = 0 \)

\[
\tau_{xy} = 0, \quad -\infty < x < \infty; \quad u_y = 0, \quad |x| > a; \quad \sigma_{yy} = -\sigma_0, \quad |x| < a.
\]

Solution of eqs.(5.1) can be constructed with the use of Fourier transform (all capital letters are Fourier transforms of corresponding functions):

\[
\Phi (s, y) = A(s) e^{-|s|y}, \quad \Psi (s, y) = B(s) e^{-|s|y},
\]

where the well known properties

\[
f (x) \implies F (s), \quad f'' (x) \implies -s^2 F (s), \quad \text{so} \quad \Delta (\varphi, \psi) \implies \frac{d^2}{dy^2} (\Phi, \Psi) - s^2 (\Phi, \Psi)
\]
have been used.

The two constants $A(s)$ and $B(s)$ can be determined from the boundary conditions (5.2) using
\[\tau_{xy} = \mu \left[ 2 \frac{\partial^2 \varphi}{\partial x \partial y} + (1 - \chi) \frac{\partial \psi}{\partial x} \right], \quad \sigma_{yy} = 2\mu \frac{\partial^2 \varphi}{\partial y^2} + (1 - \chi)(\lambda + 2\mu) \frac{\partial \psi}{\partial y}. \quad (5.3)\]

It follows from (5.2) that
\[2(-is)(-|s|) A(s) + (1 - \chi)(-is) B(s) = 0 \implies A(s) = \frac{1 - \chi}{2|s|} B(s). \quad (5.4)\]

Introducing a new function
\[g(x) = \begin{cases} u_y, & |x| < a \\ u_y = 0, & |x| > a \end{cases}, \quad U_y(s,0) = \int_{-a}^{a} g(t) e^{ist} dt \quad (5.5)\]
using eq.(5.4), we obtain the constants $A(s)$ and $B(s)$ in terms of $g(x)$
\[A(s) = \frac{\chi - 1}{|s| (\chi + 1)} \int_{-a}^{a} g(t) e^{ist} dt, \quad B(s) = -\frac{2}{\chi + 1} \int_{-a}^{a} g(t) e^{ist} dt. \]

From (5.3) one obtains the Fourier transform of the normal stress $\sigma_{yy}$:
\[\Sigma_{yy}(s,0) = 2\mu s^2 A(s) + (1 - \chi)(\lambda + 2\mu) (-|s|) B(s) = -\frac{\mu}{1 - \nu} |s| \int_{-a}^{a} g(t) e^{ist} dt, \quad (5.6)\]
and the boundary condition (5.2)$_3$ leads thus to the integral equation ($|x| < a$)
\[\int_{-a}^{a} g(t) K(t - x) dt = \frac{2\pi(1 - \nu)}{\mu} \sigma_0, \quad K(x) = \int_{-\infty}^{\infty} |s| e^{-isx} ds = -\frac{2}{x^2}. \quad (5.7)\]

Equation (5.6) is the characteristic hypersingular integral equation
\[\int_{-a}^{a} \frac{g(t)}{(x-t)^2} dt = -\frac{\pi(1 - \nu)}{\mu} \sigma_0, \quad |x| < a \quad (5.7)\]
with respect to the function $g(x)$ which is, due to eq.(5.5), the relative opening of the crack’s faces. Solution of eq.(5.7) is (see Section 3)
\[g(x) = u_y(x,0) = \frac{\sigma_0}{\mu}(1 - \nu)\sqrt{a^2 - x^2}, \quad |x| < a, \quad (5.8)\]
which can also be obtained by the proposed direct numerical collocation method (see Fig.1). Note that (5.8) coincides with a classical solution of the considered problem.
5.2 Diffraction by a thin rigid screen

Acoustically hard thin screen is placed horizontally on the interval $-a < x < a, \ y = 0$. A plane acoustic wave is normally incident on the screen: $p^\text{inc} = e^{iky}, \ k = \omega/c$, where $\omega$ is the angular wave frequency, $c$ is the wave speed.

The total pressure is then a sum of the incident and the scattered wave fields $p(x, y) = p^\text{inc} + p^\text{sc}$, where all three components of the wave field satisfy Helmholtz equation

$$\Delta p + k^2 p = 0 \ . \quad (5.9)$$

By analogy to the previous problem, the Fourier images in the lower ($x < 0$) and upper ($x > 0$) half-plane are, respectively ($\gamma(s) = \sqrt{s^2 - k^2}$):

$$P_-^\text{sc}(s, y) = A(s) e^{\gamma(s)y}, \ y < 0 \ ; \ P_+^\text{sc}(s, y) = B(s) e^{-\gamma(s)y}, \ y > 0 \ . \quad (5.10)$$

The coefficients $A$ and $B$ are defined from boundary conditions at $y = 0$

$$\frac{\partial p_+(x, 0)}{\partial y} = \frac{\partial p_-(x, 0)}{\partial y} = 0, \quad \frac{\partial p_-^\text{sc}(x, 0)}{\partial y} = \frac{\partial p_+^\text{sc}(x, 0)}{\partial y} = -ik, \ |x| < a \ , \quad (5.11)$$

$$\frac{\partial p_+(x, 0)}{\partial y} = \frac{\partial p_-(x, 0)}{\partial y}, \quad \frac{\partial p_-^\text{sc}(x, 0)}{\partial y} = \frac{\partial p_+^\text{sc}(x, 0)}{\partial y}, \ |x| > a \quad (5.12)$$

$$p_+(x, 0) = p_-(x, 0), \quad p_-^\text{sc}(x, 0) = p_+^\text{sc}(x, 0), \ |x| > a \ . \quad (5.13)$$

The conditions (5.11), (5.12) imply that $\partial p_-^\text{sc}/\partial y = \partial p_+^\text{sc}/\partial y$ for $y = 0$ and all $-\infty < x < \infty$, thus $\gamma A = -\gamma B \Longrightarrow B(s) = -A(s)$. Now it is obvious from (5.10) that $p_-^\text{sc}(x, 0) = -p_+^\text{sc}(x, 0)$. It thus implies, together with (5.13), that $p_-^\text{sc}(x, 0) = -p_+^\text{sc}(x, 0) = 0, \ |x| > a$.

Introducing a new function $g(x)$

$$p_-^\text{sc}(x, 0) = \begin{cases} 
0, & |x| > a \\
g(x), & |x| < a 
\end{cases} ,$$

it follows from (5.10) that

$$A(s) = \int_{-a}^{a} g(t) e^{ist} dt \ ,$$

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and the boundary condition (5.11) leads to the integral equation involving function \( g(x) \)

\[
\int_{-a}^{a} g(t) K(x-t) \, dt = -ik, \quad |x| < a , \quad K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(s) e^{-isx} \, ds , \quad (5.14)
\]

where the kernel

\[
K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |s| e^{-isx} \, ds + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \sqrt{s^2 - k^2} - |s| \right) e^{-isx} \, ds = \quad (5.15)
\]

\[
= -\frac{1}{\pi x^2} + K_0(x) , \quad K_0(x) = \frac{1}{\pi} \int_{0}^{\infty} \left( \sqrt{s^2 - k^2} - s \right) \cos(sx) \, ds ,
\]

and the regular kernel \( K_0(x) \) is differentiable for all \( x \neq 0 \), with a weak singularity in origin: \( K_0(x) = O(\ln|x|) \), \( x \to 0 \), that permits application of the proposed numerical method. An example for some values of \( k \) is shown in Fig.3.

### 6 Conclusions

1. The proposed method is an efficient alternative to a standard reduction to infinite systems of linear algebraic equations. This is based on direct numerical treatment of hypersingular integrals, and reduces the problem to a finite system of linear algebraic equations. Its principal merit is that there is no need in numerical computations when calculating elements of respective matrix.

2. In some cases the proposed method permits explicit analytical solution of respective hypersingular integral equation. Otherwise, the proposed method provides an efficient numerical treatment, and convergence of the algorithm has been proved when the step of the mesh tends to zero.

3. The considered examples show a good precision of the method, since with a few decades of nodes on the mesh the obtained numerical results almost coincide with respective analytical solution, in the cases when the latter is known.

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Legends to Figures

Figure 1. Comparison between exact and numerical solutions of the characteristic equation with \( f'(x) = -\pi \sim f(x) = -\pi x \): — exact solution \( g(x) = \sqrt{1 - x^2} \), - - - numerical solution; \( n = 40 \).

Figure 2. Comparison between exact and numerical solutions of the full equation with \( f'(x) = -\pi \sim f(x) = -\pi x \), \( K_0(x,t) = A(x-t), \ A = 3 \): — exact solution, - - - numerical solution; \( n = 40 \).

Figure 3. Comparison between exact and numerical solutions of the equation (5.14)-(5.15), \( (a = 1) \): 1 - \( k = 0.5 \), 2 - \( k = 1.5 \), 3 - \( k = 2.5 \). The factor \(-\pi ik\) is omitted.
Fig. 1
Fig. 2
Fig. 3