NOTE ON THE ZERO-FREE REGION OF THE HARD-CORE MODEL

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Abstract. In this paper we prove a new zero-free region for the partition function of the hard-core model, that is, the independence polynomials of graphs with largest degree \( \Delta \). This new domain contains the half disk

\[
D = \left\{ \lambda \in \mathbb{C} \mid \text{Re}(\lambda) \geq 0, |\lambda| \leq \frac{7}{8} \tan \left( \frac{\pi}{2(\Delta - 1)} \right) \right\}.
\]

1. Introduction

The independence polynomial of a graph \( G \) is defined as

\[
Z_G(\lambda) = \sum_{k=0}^{\infty} i_k(G) \lambda^k,
\]

where \( i_k(G) \) be the number of independent sets of the graph \( G \). More generally one can define a multivariate version of the independence polynomial as follows. For each vertex \( v \) let us introduce a new variable \( \lambda_v \), and set

\[
Z_G(\{\lambda_v\}) = \sum_{I \in \mathcal{I}(G)} \prod_{v \in I} \lambda_v,
\]

where \( \mathcal{I}(G) \) is the set of independent sets of \( G \) including the empty one. The polynomial \( Z_G(\{\lambda_v\}) \) is the partition function of the hard-core model in statistical physics. It is an important problem to study its zero-free region. It is related to many things including Lovász local lemma, the computational complexity of the computation of independence polynomial. One of the best-known result is due to Shearer showing that \( Z_G(\lambda) \) does not vanish in a certain disk.

Theorem 1.1 (J. Shearer [7]). Suppose that a graph \( G \) has largest degree at most \( \Delta \), and

\[
|\lambda_v| \leq \frac{(\Delta-1)\Delta^{-1}}{\Delta^2 - 1}
\]

for each vertex \( v \in V(G) \). Then \( Z_G(\{\lambda_v\}) \neq 0 \). In particular, \( Z_G(\lambda) \neq 0 \) if

\[
|\lambda| \leq \frac{(\Delta-1)\Delta^{-1}}{\Delta^2 - 1}.
\]

Very recently H. Peters and G. Regts found other zero-free regions.

Theorem 1.2 (H. Peters and G. Regts [5]). There exists an open domain \( D'_{PR} \) on the complex plane that contains the interval \((0, \frac{(\Delta-1)\Delta^{-1}}{\Delta^2 - 2\Delta + 1})\) such that \( Z_G(\lambda) \neq 0 \) if \( \lambda \in D'_{PR} \) and \( G \) has largest degree at most \( \Delta \).

They also gave a more explicit zero-free region.

2010 Mathematics Subject Classification. Primary: 05C35. Secondary: 05C31, 05C70, 05C80.

Key words and phrases. independence polynomial.

The first author is partially supported by the MTA Rényi Institute Lendület Limits of Structures Research Group. The second author is supported by the Marie Skłodowska-Curie Individual Fellowship grant no. 747430, by the Hungarian National Research, Development and Innovation Office, NKFIH grant K109684 and Slovenian-Hungarian grant NN114614, and by the ERC Consolidator Grant 648017.
The complement of zeros of independence polynomials of graphs of maximum degree \( \Delta \geq 2 \). For a fixed \( \varepsilon > 0 \) et

\[
D_\varepsilon = \left\{ \lambda \in \mathbb{C} \mid |\lambda| \leq \tan \left( \frac{\pi}{(2 + \varepsilon)(\Delta - 1)} \right) \quad \text{and} \quad |\arg(\lambda)| \leq \frac{\varepsilon \pi}{2(2 + \varepsilon)} \right\}.
\]

Set \( D_{PR} = \bigcup \varepsilon D_\varepsilon \). Suppose that for each vertex \( v \in V(G) \) we have \( \lambda_v \in D_\varepsilon \) for a fixed \( \varepsilon \), then \( Z_G(\{\lambda_v\}) \neq 0 \). In particular, if \( \lambda \in D_{PR} \) then \( Z_G(\lambda) \neq 0 \).

At the same time H. Peters and G. Regts \cite{5} also identified a domain \( U_d \) for which the zeros of independence polynomials of graphs of maximum degree \( \Delta = d + 1 \) is dense on the complement of \( U_d \). This domain can be described as follows. Let \( d = \Delta - 1 \), and let

\[
U_d = \left\{ \frac{-\alpha d^d}{(d + \alpha)^{d+1}} \mid |\alpha| \leq 1 \right\}.
\]

In fact, H. Peters and G. Regts showed the following stronger result. Consider the complete \( d \)-ary trees \( T_{k,d} \), where all leaves have distance \( k \) from the root. Then the zeros of independence polynomials of the trees \( T_{k,d} \) are dense outside \( U_d \).

Below we describe our contribution. Instead of giving immediately the explicit description of the our zero-free domain \( D \) we only give some of its most important properties.

**Theorem 1.4.** (a) Let \( D = \{ \lambda \mid \Re \lambda \geq 0, |\lambda| \leq 1 \} \). If \( \lambda \in D \) then \( Z_G(\lambda) \neq 0 \) for any graph \( G \) with maximum degree 3.

(b) Let \( \Delta \geq 3 \). Let

\[
D_1 = \left\{ \lambda \mid \Re \lambda \geq 0, |\lambda| \leq \frac{7}{8} \tan \left( \frac{\pi}{2(\Delta - 1)} \right) \right\}
\]

and

\[
D_2 = \left\{ \lambda \mid \Re \lambda \geq 0, |\lambda| \leq \tan \left( \frac{\pi}{2(\Delta - 1)} \right) \right\}.
\]

There exists an open domain \( D \subseteq \mathbb{C} \) with the following properties:

1. If \( \lambda \in D \) then \( Z_G(\lambda) \neq 0 \) for any graph \( G \) with maximum degree \( \Delta \),
2. \( D_1 \subseteq D \subseteq D_2 \),
3. \( D_{PR} \subseteq D \),
4. \( \tan \left( \frac{\pi}{2(\Delta - 1)} \right), \pm i \tan \left( \frac{\pi}{2(\Delta - 1)} \right) \in D \).

The zeros of the independence polynomial are strongly connected to the computational complexity of approximating the independence polynomial. A. Sly and N. Sun \cite{8} obtained that there is no efficient algorithm for computing the independence polynomial for positive real numbers outside \( U_d \), unless \( \text{NP} = \text{RP} \). Recently I. Bezáková, A. Galanis, L. A. Goldberg and Štefanković \cite{3} showed that outside \( U_d \) approximating \( Z_G(\lambda) \) on graphs \( G \) with maximum degree at most \( \Delta \) is \( \text{NP} \)-hard. In fact, they also showed that if \( \lambda \) is not a positive real number outside \( U_d \), then approximating the independence polynomial is \#P-hard. For negative \( \lambda \) outside \( U_d \) they also proved that it is \#P-hard to decide whether \( Z_G(\lambda) > 0 \).

Inside \( U_d \) the picture is much more pleasing. First, D. Weitz showed that there is a (deterministic) fully polynomial time approximation algorithm (FPTAS) for computing \( Z_G(\lambda) \) for any \( 0 \leq \lambda < \frac{(\Delta-1)\Delta^{-1}}{(\Delta-2)^2} \) for any graph of maximum degree at most \( \Delta \). V. Patel and G. Regts \cite{4} showed that if \( Z_G(\lambda) \) does not vanish in a domain then there is a polynomial time algorithm for approximating the partition function. Their method is based on the interpolation method introduced by A. Barvinok \cite{1}.
2. Preliminaries

Our proof of Theorem 1.4 is inspired by the proof of Theorem 1.3, but we simplify some arguments and add some extra ideas. One of the simplifications is based on the observation that it is enough to prove Theorem 1.4 for trees as the following lemma shows.

Lemma 2.1 (F. Bencs [2]). For every graph $G$ with maximum degree at most $\Delta$ there exists a tree $T_G$ with maximum degree at most $\Delta$ such that $Z_G(\lambda)$ divides $Z_{T_G}(\lambda)$ as a polynomial.

There are many choices for the tree $T_G$ in Lemma 2.1. This lemma is implicit in the work of D. Weitz [9] and in the work of A. D. Scott and A. D. Sokal [6], but the above algebraic formulation is due to F. Bencs [2].

Given a tree $T$ with maximum degree $\Delta \geq 2$ let us pick one of its leaf as a root vertex $v$. Then every vertex has at most $\Delta - 1$ children. If $T'$ is a subtree of $T$ then we can choose a root vertex of $T'$ in a natural way: the vertex $v'$ closest to $v$. Following H. Peters and G. Regts [5] for a fixed $\lambda$ let us introduce the quantity

$$R_{G,v} = \frac{\lambda Z_{G \setminus N[v]}(\lambda)}{Z_{T' - v}(\lambda)}.$$ 

Note that for an arbitrary graph $G$ and vertex $v$ we have

$$Z_G(\lambda) = Z_{G - v}(\lambda) + \lambda Z_{G - N[v]}(\lambda),$$

where $N[v] = N(v) \cup \{v\}$, the closed neighborhood of $v$. Hence $Z_G(\lambda) \neq 0$ if and only if $R_{G,v} \neq -1$. Now if $T'$ is a tree with root vertex $v'$, and $u_1, \ldots, u_k$ are the neighbours of
then a simple computation shows that

\[ R_{T',v'} = \prod_{i=1}^{k} (1 + R_{T_i,u_i}), \]

where \( T_i \) is the subtree of \( T' - v' \) rooted at \( u_i \). Here \( k \leq \Delta - 1 = d \). For a fixed \( \lambda \) let us consider the map

\[ f_{\lambda}(z_1, \ldots, z_d) = \frac{\lambda}{\prod_{i=1}^{d} (1 + z_i)}. \]

It is not a problem that in the above recursion \( k \) may be smaller than \( d \) since we can substitute \( z_i = 0 \) for those variables that we do not need.

Now let \( S_{\lambda} \) be the set of elements of the complex plane that can be obtained by the rules (1) \( 0 \in S_{\lambda} \), (2) \( z_1, \ldots, z_d \in S_{\lambda} \) then \( f_{\lambda}(z_1, \ldots, z_d) \in S_{\lambda} \). The following lemma is now trivial.

**Lemma 2.2.** For a fixed \( \lambda \) let us consider the map

\[ f(z_1, \ldots, z_d) = \frac{\lambda}{\prod_{i=1}^{d} (1 + z_i)}. \]

If there is an open domain \( F_{\lambda} \) such that \( 0 \in F_{\lambda}, -1 \notin F_{\lambda} \) and for \( z_1, \ldots, z_d \in F_{\lambda} \) we have \( f(z_1, \ldots, z_d) \in F_{\lambda} \) then \( Z_G(\lambda) \neq 0 \) for any graph \( G \) with maximum degree \( d + 1 \).

We collected some results on \( S_{\lambda} \).

**Proposition 2.3** (H. Peters, G. Regts [5]). If \( \lambda \notin U_d \) then \( -1 \in \overline{S_{\lambda}} \).

**Proposition 2.4.** If \( -1 \notin \overline{S_{\lambda}} \), and \( z \in S_{\lambda} \) then \( \frac{\lambda}{1+z} \in U_{d-1} \).

**Proof.** For \( z \in S_{\lambda} \) simply consider the map \( f_{\lambda}(z_1, \ldots, z_{d-1}, z) = f_{\lambda/(1+z)}(z_1, \ldots, z_{d-1}) \) and apply the previous proposition. \( \square \)

**Remark 2.5.** Suppose that in a small neighborhood of \( \lambda \), the polynomials of \( Z_G \) do not vanish for each graph \( G \) of largest degree at most \( \Delta \). Equivalently, \( -1 \notin \overline{S_{\lambda}} \). Suppose that \( |\lambda| \geq \frac{d}{(d+1)^{d+1}} \), that is, \( \lambda \) is outside of Shearer’s disk. By the above proposition we get that if \( z \in S_{\lambda} \), then \( \frac{\lambda}{1+z} \in U_{d-1} \). Since \( U_{d-1} \) is contained in a disk of radius \( \frac{(d-1)d^{-1}}{(d-2)d} \) we get that

\[ \left| \frac{\lambda}{1+z} \right| \leq \frac{(d-1)^{d-1}}{(d-2)^d} \]

whence

\[ |1+z| \geq |\lambda| \frac{(d-2)^d}{(d-1)^{d-1}} \geq \frac{(d-2)^d d^d}{(d-1)^{d-1}(d+1)^{d+1}}. \]

So there is an explicit ball around \( -1 \) that is avoided by \( S_{\lambda} \).

**Remark 2.6.** The advantage of Lemma 2.1 is that we can avoid the two-round argument that is typical in the field. In this two-round argument first one proves an auxiliary claim for pairs \((G, v)\), where all neighbors of \( v \) has degree at most \( \Delta - 1 \), and in the second round one can prove the claim in the general case. For a prototype of such an argument see the proof of Theorem 2.3 of [5].
3. Proof of Theorem 1.4

In this section we prove Theorem 1.4. Recall that \( d = \Delta - 1 \). First we specify Lemma 2.2 to a special domain \( F_\lambda \).

**Lemma 3.1.** Let \( \lambda \) be fixed, and argue \( \lambda = \alpha \in (-\pi/2, \pi/2) \). Let \( 0 \leq \beta, \gamma \leq \pi/2 \). Set

\[
\beta' = \arctan \frac{|\lambda| \sin \beta}{1 + |\lambda| \cos \beta} \quad \text{and} \quad \gamma' = \arctan \frac{|\lambda| \sin \gamma}{1 + |\lambda| \cos \gamma}.
\]

Suppose that \( d\gamma' - \beta \leq \alpha \leq \gamma - d\beta' \). Then \( Z_G(\lambda) \neq 0 \) for any graph \( G \) with maximum degree \( d + 1 \).

**Proof.** We will show that the domain

\[
F_\lambda = \{ z \mid |z| \leq |\lambda|, -\beta \leq \arg z \leq \gamma \}
\]

satisfies the requirements above. Then Lemma 2.2 implies the claim. It is clear that \( 0 \in F_\lambda \) and \( -1 \notin F_\lambda \). Now suppose that \( z_1, \ldots, z_d \in F_\lambda \). Then

\[
|f(z_1, \ldots, z_d)| = \frac{|\lambda|}{\prod_{i=1}^{d} |1 + z_i|} \leq |\lambda|
\]

since \( |1 + z_i| \geq 1 \) if \( z_i \in F_\lambda \). Furthermore,

\[
\arg f(z_1, \ldots, z_d) = \arg \lambda - \sum_{i=1}^{d} \arg(1 + z_i).
\]

Note that for \( z_i \in F_\lambda \) we have \(-\beta' \leq \arg(1 + z_i) \leq \gamma'\) since we have \( \{1 + z \mid z \in F_\lambda\} \subseteq \{z' \mid -\beta' \leq \arg z' \leq \gamma'\} \). Hence

\[
-\beta \leq \alpha - d\gamma' \leq \arg f(z_1, \ldots, z_d) \leq \alpha + d\beta' \leq \gamma
\]

implying that \( f(z_1, \ldots, z_d) \in F_\lambda \).

Now we reverse our argument. We first fix \( \beta \) and \( \gamma \) and we choose the largest possible \( |\lambda| \) and the corresponding \( \alpha \). The only condition is that \( d\gamma' - \beta \leq \gamma - d\beta' \), that is, \( d(\beta' + \gamma') \leq \beta + \gamma \). As we increase \( |\lambda| \) the angles \( \beta' \), \( \gamma' \) will increase too. So for the largest possible \( |\lambda| \) we have \( d(\beta' + \gamma') = \beta + \gamma \). We need that

\[
d \left( \arctan \frac{|\lambda| \sin \beta}{1 + |\lambda| \cos \beta} + \arctan \frac{|\lambda| \sin \gamma}{1 + |\lambda| \cos \gamma} \right) = \beta + \gamma.
\]

Since

\[
\arctan x + \arctan y = \arctan \left( \frac{x + y}{1 - xy} \right),
\]

and \( \tan \) is a monotone increasing function on \((0, \pi/2)\) we get that

\[
\frac{|\lambda| \sin \beta}{1 + |\lambda| \cos \beta} + \frac{|\lambda| \sin \gamma}{1 + |\lambda| \cos \gamma} = \tan \left( \frac{\beta + \gamma}{d} \right).
\]

This simplifies to

\[
\frac{|\lambda|(\sin \beta + \sin \gamma) + |\lambda|^2 \sin(\beta + \gamma)}{1 + |\lambda|(\cos \beta + \cos \gamma) + |\lambda|^2 \cos(\beta + \gamma)} = \tan \left( \frac{\beta + \gamma}{d} \right).
\]

After multiplying with the denominator on the left hand side we get a quadratic equation. Let

\[
A = \sin(\beta + \gamma) - \tan \left( \frac{\beta + \gamma}{d} \right) \cos(\beta + \gamma),
\]
By Lemma 3.1 we have $B = \sin \beta + \sin \gamma - \tan \left(\frac{\beta + \gamma}{d}\right) (\cos \beta + \cos \gamma)$, 

$$C = -\tan \left(\frac{\beta + \gamma}{d}\right),$$

and $0 = A|\lambda|^2 + B|\lambda| + C$. Note that 

$$\frac{\sin \beta + \sin \gamma}{\cos \beta + \cos \gamma} = \tan \left(\frac{\beta + \gamma}{2}\right).$$

From this it follows that $A > 0, B \geq 0, C < 0$ since $0 \leq \beta + \gamma \leq \pi$, and $d \geq 2$. Hence exactly one of the solutions of the quadratic equation $Ax^2 + Bx + C = 0$ is positive. Set 

$$s(\beta, \gamma) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

For $d = 2$ it turns out that $s(\beta, \gamma) \equiv 1$. For $d \geq 2$ it is always true that 

$$s(\beta, \pi/2) > \frac{2\beta + \pi}{\pi(1 + \sin \beta)} \tan \left(\frac{\pi}{2d}\right) > \frac{7}{8} \tan \left(\frac{\pi}{2d}\right),$$

(Some comment about these facts. The function $g(d) = \frac{s_d(\beta, \pi/2)}{\tan \left(\frac{\pi}{2d}\right)}$ is monotone decreasing for fixed $\beta$, and its limit is $\frac{2\beta + \pi}{\pi|1 + \sin \beta|}$. The justifications of these facts are standard but somewhat tedious computations.) It turns out that $s(\beta, \pi/2)$ is a convex function on $[0, \pi/2]$ and at the end points, $s(0, \pi/2) = s(\pi/2, \pi/2) = \tan \left(\frac{\pi}{2d}\right)$.

Now let us turn our attention to the angle $\alpha = t(\beta, \gamma) = \gamma - d\beta'$, where the length $|\lambda|$ is fixed to be $s(\beta, \gamma)$. Here we only study the angle $t(\beta, \pi/2)$. Since $s(\beta, \pi/2)$ is continuous we get that $\beta'$ is continuous, and so $t(\beta, \pi/2)$ is continuous too. For $\beta = 0$ we get $\beta' = 0$ independently of $|\lambda|$, and so $t(0, \pi/2) = \pi/2$. For $\beta = \pi/2$ the equation $d(\beta' + \gamma') = \beta + \gamma$ implies that $t(\pi/2, \pi/2) = 0$. Hence $t(\beta, \pi/2)$ takes all values in the interval $[0, \pi/2]$.

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** Consider the domain 

$$D = \{\lambda \in \mathbb{C} \mid \exists \beta \in [0, \pi/2] : |\arg \lambda| = t(\beta, \pi/2), |\lambda| \leq s(\beta, \pi/2)\}.$$

Then by the bounds on $s(\beta, \pi/2)$ we have $D_1 \subseteq D \subseteq D_2$, and $\tan \left(\frac{\pi}{2(D - 1)}\right), \pm i \tan \left(\frac{\pi}{2(D - 1)}\right) \in D$. Furthermore, for $\Delta = 3$ we have $s(\beta, \pi/2) \equiv 1$.

Now suppose that $\lambda \in D$, we can assume that $\arg \lambda = \alpha \geq 0$. Then there exists a $\beta \in [0, \pi/2]$ such that $|\arg \lambda| = t(\beta, \pi/2)$ and $|\lambda| \leq s(\beta, \pi/2)$. Set $\gamma = \pi/2$ and 

$$\beta_1' = \arctan \frac{|\lambda|\sin \beta}{1 + |\lambda| \cos \beta} \quad \text{and} \quad \gamma_1' = \arctan \frac{|\lambda|\sin \gamma}{1 + |\lambda| \cos \gamma};$$

and 

$$\beta_2' = \arctan \frac{s(\beta, \pi/2) \sin \beta}{1 + s(\beta, \pi/2) \cos \beta} \quad \text{and} \quad \gamma_2' = \arctan \frac{s(\beta, \pi/2) \sin \gamma}{1 + s(\beta, \pi/2) \cos \gamma}.$$ 

Since $|\lambda| \leq s(\beta, \pi/2)$ we get that $\beta_1' \leq \beta_2'$ and $\gamma_1' \leq \gamma_2'$. Furthermore, 

$$\alpha = \gamma - d\beta_2' = d\gamma_2' - \beta$$

by the definition of $s(\beta, \gamma)$. Hence 

$$d\gamma_1' - \beta \leq \alpha \leq \gamma - d\beta_1'.$$

By Lemma 3.1 we have $Z_G(\lambda) \neq 0$ for any graph $G$ with maximum degree $\Delta$. 


The fact that $D_{PR}$ is contained in $D$ follows from the fact that their argument of zero-freeness is a variation of the special case of Lemma 3.1 with $\beta = 0, \gamma = \pi/2$.

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