HILBERTIAN VERSUS HILBERT W*-MODULES, AND APPLICATIONS TO $L^2$- AND OTHER INVARIANTS

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Abstract. Hilbert(ian) $A$-modules over finite von Neumann algebras with a faithful normal trace state (from global analysis) and Hilbert $W^*$-modules over $A$ (from operator algebra theory) are compared, and a categorical equivalence is established. The correspondence between these two structures sheds new light on basic results in $L^2$-invariant theory providing alternative proofs. We indicate new invariants for finitely generated projective $B$-modules, where $B$ is a unital $C^*$-algebra, (usually the full group $C^*$-algebra $C^*(\pi)$ of the fundamental group $\pi = \pi_1(M)$ of a manifold $M$).

During the last decade W. Lück, A. Carey, V. Mathai, and other authors used the analytical concept of Hilbert(ian) $A$-modules over finite von Neumann algebras $A$ for the study of $L^2$-invariants in global analysis. The technical concept was originally invented by M. F. Atiyah [2] and I. M. Singer [49] in 1976, and further developed by J. Cheeger and M. Gromov [10, 11] in 1985-86. For the detailed history we refer to the handbook [35]. The goal has been to obtain $L^2$-invariants for certain compact closed manifolds, e.g. $L^2$-(co)homology, $L^2$-torsion, $L^2$-Betti numbers and Novikov-Shubin invariants for finitely generated Hilbert $A$-chain complexes. On the other hand Hilbert $C^*$-modules over arbitrary $C^*$-algebras have been used in operator and operator algebra theory, in global analysis, in noncommutative geometry and mathematical physics for about 50 years, [23, 25, 51, 16]. The purpose of the present note is to compare these two categories of $C^*/C^*$-valued inner product modules over finite von Neumann algebras $A$, where for technical purposes the von Neumann algebras $A$ are supposed to admit a faithful normal trace state. We establish a categorical equivalence. Transferring known results on type II$_\infty$ von Neumann algebras and self-dual Hilbert $W^*$-modules through this categorical equivalence to the theory of Hilbertian $A$-modules we obtain more evidence on the background of the theory of $L^2$-invariants from a different viewpoint. We establish new invariants for finitely generated projective $B$-modules over unital $C^*$-algebras $B$ and give a perspective for future research.

1. Introduction

Let $A$ be a finite von Neumann algebra. Since our goal is a comparison of two categories of Hilbert modules over $A$ which are already described in the literature we have to suppose the existence of a faithful normal trace state $tr$ on $A$ to meet the requirements of one of the concepts under consideration. Basically, this assumption only restricts the center of $A$ with respect to its attainable dimension. The restriction is caused by the general type and direct integral decomposition theory of von Neumann algebras, see [50, V. Th. 2.4, 2.6]. Since finite von Neumann algebras always admit a center-valued trace functional we can easily extend the comparison of both the categories under consideration allowing general finite von Neumann algebras. The only loss would be that we would have to treat Hilbert $W^*$-modules over the

1991 Mathematics Subject Classification. Primary 46L08; Secondary 46L35, 46L07, 47L25.
Key words and phrases. finite von Neumann algebra, Hilbertian module, Hilbert $W^*$-module, $L^2$-invariants, modular frames.

The research was partially supported by the INTAS grant INTAS 96-1099.

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center of finite von Neumann algebras $A$ instead of Hilbert spaces carrying an additional proper $A$-module structure as described below. However, for historical reasons we do not invent this more general point of view and consider only the more restrictive setting supposing the existence of a trace state on $A$.

One way to define a sort of Hilbert(ian) modules over $A$ is the following often used one which has been introduced in connection with geometric applications by W. Lück and by A. Carey, V. Mathai in 1991/92. Basic references are the publications by W. Lück and M. Rothenberg [27], A. Carey and V. Mathai [33, 34, 35, 36, 37, 38], M. Rothenberg [11], M. A. Shubin [39] and M. Farber [13, 14, 15], J. Lott and W. Lück [26], D. Burghelea, L. Friedlander, T. Kappeler, P. McDonald [4, 5, 6], W. Lück [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36], T. Schick [17, 18], H. Reich [15, 36], A. Carey, V. Mathai and A. S. Mishchenko [19], and others. Since the authors differ slightly in their denotations we give a comprehensive account at this place adopting the denotations introduced by W. Lück and Th. Schick in the main.

To introduce Hilbertian modules over finite von Neumann algebras $A$ with trace functional we begin with some standard Hilbert modules over $A$ – the completion of the algebraic tensor products $A \otimes H$ of $A$ by Hilbert spaces $H$ with respect to the Hilbert norm induced by the $\mathbb{C}$-valued inner product

$$\langle a \otimes h, b \otimes g \rangle = \text{tr}(ab^*) \cdot \langle h, g \rangle_H,$$

where $a, b \in A$, $h, g \in H$.

The $A$-module action is defined as the (left, w.l.o.g.) multiplication of $A$ by elements of $A$. So all these sets become Hilbert spaces and (left) $A$-modules by construction. Following [27, 6, 37] they are denoted by $\ell^2(A) \otimes H$. A Hilbert module over $A$ is a Hilbert space $\mathcal{M}$ together with a continuous (left) $A$-module structure such that there exists an isometric $A$-linear embedding of $\mathcal{M}$ into one of the standard Hilbert modules $\ell^2(A) \otimes H$ over $A$. A Hilbertian module over $A$ is a topological vector space $\mathcal{M}$ with a continuous (left) $A$-action such that there exists a scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{M}$ generating the topology of $\mathcal{M}$ and turning $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ together with the $A$-action into a Hilbert module over $A$ in the previously defined sense. Any two scalar products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ on a Hilbertian module $\mathcal{M}$ over $A$ that induce one and the same topology on $\mathcal{M}$ and turn $\{\mathcal{M}, \langle \cdot, \cdot \rangle_1\}$ and $\{\mathcal{M}, \langle \cdot, \cdot \rangle_2\}$, respectively, into concrete Hilbert modules over $A$ are linked by a bounded self-adjoint positive $A$-linear map $T : \mathcal{M} \to \mathcal{M}$ fulfilling the identity $\langle \cdot, \cdot \rangle_1 \equiv (T(\cdot), T(\cdot))_2$ on $\mathcal{M} \times \mathcal{M}$. [8, item 1.4]. A Hilbertian module over $A$ is finitely generated if it is embedable as a Hilbertian module over $A$ into some standard Hilbert module $\ell^2(A) \otimes \mathbb{C}^n$ for finite $n \in \mathbb{N}$.

If $\mathcal{M}, \mathcal{N}$ are two Hilbertian modules over $A$, then a bounded $A$-linear map $T : \mathcal{M} \to \mathcal{N}$ is called an $A$-module morphism. By [12, I.6, Th.2] and [12, Lemma 9.11] the set of all $A$-module morphisms $\text{End}_A(\mathcal{M})$ on a Hilbertian module $\mathcal{M}$ has the structure of a von Neumann subalgebra of the type I von Neumann algebra $\text{End}_\mathbb{C}(\mathcal{M})$ of all bounded linear operators on the Hilbert space $\mathcal{M}$. Its commutant is precisely $A$ itself since the $A$-action on $\mathcal{M}$ was supposed to be continuous. Hilbertian modules over a fixed finite von Neumann algebra $A$ with faithful normal trace state together with the specified $A$-module morphisms form a category. This category is one of the two we are going to compare.

The second kind of Hilbert modules over $C^*$-algebras is widely used in operator algebra theory and its applications. There are also topological applications of this structure, cf. [3, 22, 33, 34, 35, 36, 37] for example. The theory of Hilbert $C^*$-modules goes back to the work of I. Kaplansky [24], W. L. Paschke [40] and M. A. Rieffel [41]. Contemporary explanation are published in E. C. Lance’s lecture notes [25] and in I. Raeburn’s and D. P. Williams’ book [42]. For a detailed bibliography on the subject we refer to [14, Appendix].

A Hilbert $C^*$-module over a $C^*$-algebra $A$ is a (left, w. l. o. g.) $A$-module $\mathcal{M}$ together with a map $\langle \cdot, \cdot \rangle_\mathcal{M} : \mathcal{M} \times \mathcal{M} \to A$ satisfying the following axioms:
(i) \( \langle x, x \rangle_M \geq 0 \) for every \( x \in M \),
(ii) \( \langle x, x \rangle_M = 0 \) if and only if \( x = 0 \),
(iii) \( \langle x, y \rangle_M = \langle y, x \rangle_M \) for every \( x, y \in M \),
(iv) \( \langle ax + by, z \rangle_M = a\langle x, z \rangle_M + \langle y, z \rangle_M \) for every \( a, b \in A, x, y, z \in M \).

In this setting the map \( \| \cdot \| : M \to \mathbb{R}^+ \) defined by \( \| x \| = \| \langle x, x \rangle_M \|^{1/2} \), \( x \in M \), is a norm on \( M \). In the sequel we always assume \( M \) to be complete with respect to this norm. The pair \( \{ M, \langle \cdot, \cdot \rangle_M \} \) is said to be a Hilbert \( A \)-module, and the map \( \langle \cdot, \cdot \rangle_M \) is referred to as to an \( A \)-valued inner product on \( M \).

A morphism of Hilbert \( C^* \)-modules over a certain \( C^* \)-algebra \( A \) is a bounded linear map between them intertwining the given actions of \( A \) on the modules. Hilbert \( C^* \)-modules over a fixed \( C^* \)-algebra \( A \) and bounded \( A \)-linear morphisms form another category to be considered.

One of the unpleasant properties of some Hilbert \( C^* \)-modules is the general failing of the analog of the Riesz representation theorem for bounded module maps from the Hilbert \( C^* \)-module into the \( C^* \)-algebra of its coefficients, where the \( C^* \)-algebra of coefficients is usually assumed to act on itself by multiplication from the left to become an \( A \)-module. For example, consider the finite von Neumann algebra \( A = l_\infty \) of all bounded complex sequences and the Hilbert \( A \)-module \( M = c_0 \) of all sequences converging to zero equipped with the \( A \)-valued inner product \( \langle x, y \rangle_A = ab^* \), \( (a, b) \in A \). Every bounded \( A \)-linear map from \( c_0 \) to \( l_\infty \) is given by a multiplication of \( M = c_0 \) by some element of \( l_\infty \) from the right. However, if this factor belongs to \( l_\infty \setminus c_0 \) then the resulting map is not representable as \( \langle \cdot, \cdot \rangle_A \) for elements \( x \in c_0 \). The next bothering fact is the non-adjointability of some bounded module operators on Hilbert \( C^* \)-modules. To present an example, consider a finite von Neumann algebra \( A \) assumed to act on itself by multiplication from the left to become an \( \mathcal{A} \)-module, where the \( \mathcal{A} \)-algebra of coefficients is usually the fixed \( \mathcal{A} \)-algebra \( A \) and bounded \( \mathcal{A} \)-linear endomorphisms between them intertwining the given actions of \( \mathcal{A} \) on the \( \mathcal{A} \)-modules. Hilbert \( \mathcal{A} \)-modules over a fixed \( \mathcal{A} \)-algebra \( A \) and bounded \( \mathcal{A} \)-linear morphisms form another category to be considered.

For Hilbert \( C^* \)-modules over von Neumann algebras (in short: Hilbert \( W^* \)-modules) there is a manner to overcome these obstacles on the way to a 'better' theory within the category of Hilbert \( W^* \)-modules. The basic idea is to dilate every Hilbert \( W^* \)-module to a larger Hilbert \( W^* \)-module that has properties analogous to those known for Hilbert spaces in any respect, and to select a canonically smallest dilation of that kind. W. L. Paschke described the construction of those envelopes in \( \mathcal{B} \) in 1973. To introduce it we need some further preparation.

For any \( C^* \)-algebra \( A \) denote the set of all bounded \( A \)-linear maps from a given Hilbert \( A \)-module \( M \) into \( A \) by \( M' \). This set \( M' \) canonically becomes a Banach \( A \)-module that contains an isometrically embedded copy of the original Hilbert \( A \)-module \( M \) which can be obtained as the image of the map \( x \in M \mapsto \langle \cdot, x \rangle_M \in M' \). In case the \( C^* \)-algebra \( A \) is a von Neumann algebra the \( A \)-valued inner product \( \langle \cdot, \cdot \rangle_M \) on a Hilbert \( A \)-module \( M \rightarrow M' \) can always be continued to an \( A \)-valued inner product on \( M' \) preserving the compatibility with the standard embedding of \( M \) into \( M' \), \( [13, \text{Th. 3.2}] \). As a result \( M' \) turns out to be a self-dual Hilbert \( A \)-module (i.e. \( M' \equiv (M')' \)), and the Banach algebra of all bounded \( A \)-linear endomorphisms \( \text{End}_A(M') \) of \( M' \) becomes a von Neumann algebra, \( [13, \text{Prop. 3.10}] \). If \( A \) is of type I or II or III, respectively, then \( \text{End}_A(M') \) is always exactly of the same type as \( A \), \( [10, \text{Cor. 8.7, Rem.}] \). Moreover, \( M' \) can be constructed from \( M \) in a simple topological way: the unit ball of \( M' \) is the completion of the unit ball of \( M \) with respect to the topology induced by anyone of the two sets of semi-norms:

\[
\{ f(\langle \cdot, \cdot \rangle_M)^{1/2} : f \in A_1^+, \| f \| = 1, x \in M, \| x \| \leq 1 \},
\]

\[
\{ f(\langle \cdot, \cdot \rangle_M)^{1/2} : f \in A_1^+, \| f \| = 1 \},
\]
where $A_+^*$ denotes the set of all normal positive functionals on $A$, [13, 3], [15, Th. 3.2]. In particular, the biorthogonal complement of any Hilbert $A$-submodule $N \subseteq \mathcal{M}$ with respect to the self-dual Hilbert $A$-module $\mathcal{M}'$ can be canonically identified with the $A$-dual Banach $A$-module $\mathcal{N}'$ since these topologies are orthogonality-preserving. Among the various fortunate properties of self-dual Hilbert $W^*$-modules we mention that another $A$-valued inner product $\langle , \rangle_2$ on a self-dual Hilbert $A$-module $\{\mathcal{M}, \langle , \rangle_1\}$ induces the same norm topology on $\mathcal{M}$ if and only if there exists a bounded $A$-linear positive operator $T \in \text{End}_A(\mathcal{M})$ such that the identity $\langle , \rangle_1 \equiv \langle T(\cdot), T(\cdot) \rangle_2$ holds on $\mathcal{M} \times \mathcal{M}$. Furthermore, two self-dual Hilbert $A$-modules are unitarily isomorphic if and only if they are isomorphic as Banach $A$-modules ([15, Prop. 2.2]), a property which sometimes fails to hold for general Hilbert $C^*$-modules (cf. [20]) and which also causes the slight difference between the definition of Hilbert $C^*$-modules and that of Hilbert spaces.

One easily checks that every bounded $A$-linear map defined between two Hilbert $A$-modules over von Neumann algebras $A$ continues to a unique bounded $A$-linear map between their $A$-dual self-dual Hilbert $A$-modules, [13, Cor. 3.7]. These circumstances make 'self-dualisation' of Hilbert $W^*$-modules a suitable operation to overcome the arising technical difficulties of the general theory of Hilbert $C^*$-modules and $W^*$-modules.

Furthermore, self-dual Hilbert $W^*$-modules over von Neumann algebras $A$ can be alternatively described as $w^*$-closures of formal direct sums (i.e. tuples) of *weakly closed left ideals of $A$: every self-dual Hilbert $A$-module $\{\mathcal{M}, \langle , \rangle_\mathcal{M}\}$ is unitarily isomorphic to a Hilbert $A$-module $\mathcal{N} = \{\{a_\alpha\}_{\alpha \in I} : a_\alpha \in Ap_\alpha, 0 \leq p_\alpha = p_\alpha^2 \in A\} =: \sum_{\alpha \in I} \oplus Ap_\alpha$, where $\mathcal{F}$ denotes the net of all finite subsets of the index set $I$, [13, Th. 3.12]. Obviously, $\mathcal{N}$ can be characterized isomorphicly as a direct orthogonal summand of the $A$-dual Hilbert $A$-module $\mathcal{H}'$ of some standard Hilbert $A$-module $\mathcal{H} = A \otimes H$, where the Hilbert space $H$ possesses a Hilbert basis of cardinality $\text{card}(I)$. Here the $A$-valued inner product on $\mathcal{H}$ is usually defined by $\langle a \otimes h, b \otimes g \rangle_H = ab^* \langle h, g \rangle_H$ on elementary tensors.

2. Hilbertian modules versus Hilbert $W^*$-modules

Comparing the direct sum decompositions of (standard) Hilbertian and self-dual Hilbert $W^*$-modules the similarity between these two categories comes to light. In a first step W. Lück was able to identify the appropriate subcategories of finitely generated Hilbertian modules over $A$ with the subcategory of finitely generated projective $W^*$-modules over $A$, [32, Th. 2.1], which are precisely the finitely generated Hilbert $A$-modules by [13, 51]. Our first goal is to establish the hidden categorical identification in full:
Theorem 2.1. Let $A$ be a finite von Neumann algebra that possesses a normal faithful trace state $\text{tr}$. The two categories
(i) Hilbertian modules over $A$, $A$-module morphisms;
(ii) self-dual Hilbert $W^*$-modules over $A$, bounded $A$-linear morphisms;
are equivalent. The involution of morphisms is intertwined by the linking functor $\Phi$.

Proof. We construct a functor $\Phi$ from the first category into the second category and investigate its properties.

Let $\mathcal{N}$ be a Hilbertian module over $A$. Suppose, $\mathcal{N}$ is already $A$-linearly and continuously embedded into a certain standard Hilbertian module $l^2(A) \otimes H$ for some Hilbert space $H$. Consider the intersection $\Phi(\mathcal{N})$ of $\mathcal{N}$ with the norm-dense subset $(A \otimes H)^I \subseteq l^2(A) \otimes H$. This intersection $\Phi(\mathcal{N})$ is non-empty and $A$-invariant by construction. Moreover, for topological reasons it is complete with respect to the norm $\|f\| = \|\langle \cdot, f \rangle_{(A \otimes H)^I}\|_{A^2}$, and the unit ball of $\Phi(\mathcal{N})$ is complete with respect to the topology induced by the semi-norms $\{f(\langle \cdot, \cdot \rangle_{A \otimes H})^{1/2} : f \in A^\perp, \|f\| = 1\}$. By [23, Th. 3.2] $\Phi(\mathcal{N})$ is a self-dual Hilbert $W^*$-module over $A$, and the completion of it with respect to the norm $\|\cdot\| = tr(\langle \cdot, \cdot \rangle_{\Phi(\mathcal{N})})^{1/2}$ recovers $\mathcal{N}$. Note, that $\Phi(\mathcal{N})$ is a direct orthogonal summand of the Hilbert $W^*$-module $(A \otimes H)^{\perp}$ ([35, Th. 2.8]), and that the $A$-linear projection to it extends to the $A$-linear projection from $l^2(A) \otimes H$ to $\mathcal{N}$.

Consider two $A$-linear continuous embeddings of $\mathcal{N}$ into $l^2(A) \otimes H_1$ and $l^2(A) \otimes H_2$, respectively. The images of $\mathcal{N}$ in $l^2(A) \otimes H_1$ and in $l^2(A) \otimes H_2$ are linked by an isometric $A$-linear operator with carrier projections equal to the projections to the embedded copies of $\mathcal{N}$ in $l^2(A) \otimes H_1$ and $l^2(A) \otimes H_2$, respectively. Repeating our construction we obtain two self-dual Hilbert $W^*$-modules $\Phi_1(\mathcal{N})$ and $\Phi_2(\mathcal{N})$ which are $A$-linearly and isometrically isomorphic. By E. C. Lance’s theorem [23, Th. 3.5] they are unitarily isomorphic, too. Consequently, the functor $\Phi$ does not depend on the embedding of $\mathcal{N}$ as a Hilbertian module over $A$.

Let $H$ be any Hilbert space. Then $l^2(A) \otimes H$ is the Hilbert norm closure of the self-dual Hilbert $A$-module $H = (A \otimes H)^{\perp}, \langle \cdot, \cdot \rangle_{l^2}$ with respect to the Hilbert space norm $\|\cdot\| = tr(\langle \cdot, \cdot \rangle_H)^{1/2}$. For general self-dual Hilbert $A$-modules $H, \langle \cdot, \cdot \rangle_H$ consider their canonical decomposition as $H = \sum_{\alpha \in I} \oplus A p_\alpha$ with $p_\alpha = p_{\alpha^*} > 0$ of $A$. Obviously, $H$ is an orthogonal direct summand of the self-dual Hilbert $A$-module $(A \otimes H)^{\perp}$ for a certain Hilbert space $H$ with $\dim(H) = \text{card}(I)$, where the orthogonal complement is the self-dual Hilbert $A$-module $(H)^{\perp} = \sum_{\alpha \in I} \oplus A(1_A - p_\alpha)$.

Hence, the completion of $H$ with respect to the norm $tr(\langle \cdot, \cdot \rangle_H)^{1/2}$ is a Hilbertian module over $A$ by definition. This shows the functor $\Phi$ to be surjective. At the same time we constructed the inverse functor $\Phi^{-1}$.

The respective sets of bounded $A$-linear morphisms of both these categories can be seen to coincide looking at the present construction. Since every bounded module map on a self-dual Hilbert $A$-module possesses an adjoint and since every bounded module map on a Hilbertian module $\mathcal{N}$ over $A$ preserves the subset $\Phi(\mathcal{N})$ invariant, the coincidence of both the involutions comes to light.

Corollary 2.2. The Hilbert space orthogonal complement of a Hilbertian module $\mathcal{N}$ over a finite von Neumann algebra $A$ with faithful normal trace state inside a standard Hilbertian module $l^2(A) \otimes H$ is $A$-invariant and, hence, a Hilbertian module, too.

Corollary 2.3. (W. Lück’s theorem, [32, Th. 2.1], [33, Th. 1.8])

The functor $\Phi$ identifies the subcategory of finitely generated Hilbertian modules over $A$ with the subcategory of finitely generated projective $A$-modules (i.e. finitely generated Hilbert $A$-modules).

For a proof we have only to recall that $\Phi(l^2(A) \otimes \mathbb{C}^N) = (A \otimes \mathbb{C}^N)^{\perp} = A \otimes \mathbb{C}^N$ for every finite $N \in \mathbb{N}$.
Corollary 2.4. Let $\mathcal{N}$ be a Hilbertian module over a certain finite von Neumann algebra $A$ that admits a normal trace functional $\text{tr}$. Suppose, $\mathcal{N}$ can be identified with a direct summand of the standard Hilbertian module $l^2(A) \otimes H$ for a certain Hilbert space $H$. Then the von Neumann algebra $\text{End}_A(l^2(A) \otimes H)$ of all bounded $A$-linear module maps on $l^2(A) \otimes H$ is *-isomorphic to the W*-tensor product $A \overline{\otimes} \text{End}_C(H)$ and, hence, a type $\Pi_\infty$ von Neumann algebra. The von Neumann algebra $\text{End}_A(\mathcal{N})$ can be identified with a full corner of the von Neumann algebra $\text{End}_A(l^2(A) \otimes H)$, i.e. $P : \text{End}_A(l^2(A) \otimes H) \cdot P \cong \text{End}_A(\mathcal{N})$ for the orthogonal projection $P : l^2(A) \otimes H \to \mathcal{N}$.

Fixing the trace functionals $\text{tr}$ and $\text{Tr}_{B(H)}$ on $A$ and $B(H)$, respectively, the canonical semifinite trace functional $\text{Tr}$ on $\text{End}_A(l^2(A) \otimes H)$ defined by $\text{Tr}(a \otimes T) = \text{tr}(a) \cdot \text{Tr}_{B(H)}(T)$ on elementary tensors assigns either a real number $\text{Tr}(P) \in [0, +\infty)$ or the symbol $+\infty$ to $\mathcal{N}$. The assigned value does not depend on the choice of the embedding of $\mathcal{N}$, and the assigned value is a finite number if and only if $\mathcal{N}$ is finitely generated.

These elementary conclusions can be derived from standard von Neumann algebra theory identifying the Hilbertian module $\mathcal{N}$ with its image $\Phi(\mathcal{N})$ in the category of Hilbert $\text{W}^*$-modules over $A$, cf. [43]. If we have two embeddings of $\mathcal{N}$ into standard Hilbertian modules $l^2(A) \otimes H_1$ and $l^2(A) \otimes H_2$, respectively, we may always assume that both $H_1$ and $H_2$ are isomorphic Hilbert spaces enlarging the smaller one appropriately. Since there exists a partial isometry $U$ between the orthogonal projections $P_1$ on $H_1$ and $P_2$ on $H_2$ we obtain $\text{Tr}(P_2) = \text{Tr}(UP_2U^*) = \text{Tr}(P_1)$, and so the value is independent of the concrete representation of $\mathcal{N}$.

3. $L^2$- and other invariants of $C^*$-modules – revisited

The purpose of the present section is to investigate the freedom of choice for (center-valued) semifinite trace functionals on the set of all bounded module operators on some Hilbertian modules, and to translate the obtained results into the language of $L^2$-invariants. We rely on the categorical equivalence obtained in the previous section. So the results are much easier to obtain for self-dual Hilbert $\text{W}^*$-modules because of their well-known composition structure. As a result we obtain that the consideration of finitely generated Hilbert $B$-modules and their invariants can replace the investigation of the corresponding finitely generated Hilbert $B^{**}$-modules which arise as categorically equivalent objects of the w*-completion of the former with respect to the respective $C^*$-valued inner product $\text{tr}(\langle \cdot, \cdot \rangle)$.

In applications of Hilbertian modules $\mathcal{M} \subseteq l^2(A) \otimes H$ over finite von Neumann algebras $A$ with faithful normal trace state $\text{tr}$ in the theory of $L^2$-invariants the properties of the canonical projection $P^H_{\mathcal{M}} : l^2(A) \otimes H \to \mathcal{M}$ and the stability of some of these properties with respect to changes of the representations of $\mathcal{M}$ are of major interest, [32]. One of these invariant properties of the family $\{P^H_{\mathcal{M}} : H - \text{Hilbert space}\}$ is the (non-)existence of a finite $Z(A)$-valued trace value for it, where $Z(A)$ denotes the center of $A$ which can be canonically identified with the center of $\text{End}_A(l^2(A) \otimes H)$. Another property is the existence of a Fuglede-Kadison determinant for positive invertible module operators on a Hilbertian $A$-module. If a finite trace value exists then generalized Betti-numbers $b^H_p(C)$ of chain complexes $C$ of Hilbertian modules over $A$ and Novikov-Shubin invariants $\alpha_p(C)$ can be defined, [32]. However, the value of the trace in $Z(A)$ is not uniquely determined, cf. [43] V. Th. 2.34 for a description of the variety of faithful semifinite normal extended center-valued traces on type $\Pi_\infty$ von Neumann algebras. To fix a standard $Z(A)$-valued trace $\tau_o$ on a $C^*$-subalgebra of $\text{End}_A(l^2(A) \otimes H)$ we use a standard *-isomorphism

$$\text{End}_A(l^2(A) \otimes H) \cong A \overline{\otimes} \text{End}_C(H)$$

(where $\overline{\otimes}$ denotes the W*-tensor product of W*-algebras), and we set

$$\tau_o(a \otimes S) = \tau(a) \cdot \text{Tr}_{B(H)}(S) = \tau \otimes \text{Tr}_{B(H)}(a \otimes S)$$
for \(a \in A\) and trace class operators \(S\) on \(H\). Here \(\tau\) denotes the unique faithful normal center-valued trace on \(A\) (cf. \([50, \text{V. Th. 2.6}]\), and \(\text{Tr}_{B(H)}\) is a normal trace on \(B(H)\) normalized by the requirement that it takes the value one on projections to one-dimensional subspaces of \(H\).

The following theorem gives a precise description of the situation. It was already partially stated by W. Lück in \([52, \text{Cor. 3.2}]\). In difference to his result we do neither require the existence of a finite faithful trace functional on the finite von Neumann algebra \(A\) nor have any preference for any specific center-valued faithful trace on the \(\text{C}^*\)-algebra of all bounded module operators. The standard trace \(\tau_o\) is only a tool to prove the assertions.

**Theorem 3.1.** Let \(A\) be a finite von Neumann algebra and \(M\) be a self-dual Hilbert \(A\)-module. Then \(M\) is finitely generated if and only if for some/every isometric embedding of \(M\) into a standard self-dual Hilbert \(A\)-module \((A \otimes H)\) the canonical projection \(P^H_M\) from \((A \otimes H)\) to \(M\) possesses a finite center-valued trace value \(\tau_1(P^H_M)\), where \(\tau_1\) denotes an arbitrary faithful semifinite normal extended center-valued trace given on the type \(\text{II}_\infty\) von Neumann algebra \(\text{End}_A((A \otimes H))\).

In other words, the value \(\tau_1(P)\) of a center-valued trace \(\tau_1\) applied to an orthogonal projection \(P \in \text{End}_A((A \otimes H))\) is finite if and only if \(P((A \otimes H))\) is a finitely generated Hilbert \(A\)-module, despite the concrete value depends on the choice of \(\tau_1\).

**Proof.** Every isometric copy of \(M\) as an \(A\)-submodule of another Hilbert \(A\)-module \(N\) is an orthogonal summand of \(N\) since \(M\) is self-dual by assumption, cf. \([15]\). In case \(M = (A \otimes H)\) for some Hilbert space \(H\) we have the isometric algebraic embedding \(A \otimes K_C(H) \subseteq \text{End}_A((A \otimes H))\), where \(\otimes\) denotes the algebraic tensor product. Let us fix the standard center-valued trace \(\tau_o\) on \(A \otimes K_C(H)\). The bounded module operators on \(M = (A \otimes H)\) that admit a finite center-valued trace are contained in the \(\text{C}^*\)-algebra of all ‘compact’ operators on \(M\) which is defined as the norm-closure of the linear hull of the operators \(\{\theta_{x,y} : \theta_{x,y}(z) = \langle z, x\rangle y \text{ for } x, y, z \in M\}\), since the trace class operators \(S \in B(H)\) are all compact operators. Therefore, projections onto a Hilbert \(\text{C}^*\)-module \(M = (A \otimes H)\) admit a finite value with respect to the fixed standard semifinite center-valued trace \(\tau_o\) if and only if they are ‘compact’ operators, i.e. if and only if their image is a finitely generated Hilbert \(\text{C}^*\)-module, \([51, \text{Th. 15.4.2, Remark 15.4.3}]\).

By \([51, \text{Th. 2.34}]\) the type \(\text{II}_\infty\) von Neumann algebra \(\text{End}_A((A \otimes H))\) may admit more then just one faithful semifinite normal extended center-valued trace, and an easy classification is available. However, if a projection \(P \in \text{End}_A((A \otimes H))\) admits a finite center-valued trace value with respect to a certain such trace on \(\text{End}_A((A \otimes H))\), then the von Neumann algebra \(P \cdot \text{End}_A((A \otimes H)) \cdot P\) is of type \(\text{II}_1\), i.e. also the restriction of the standard faithful center-valued trace \(\tau_o\) to it would give only finite values in \(Z(A)\). The converse obviously also holds. This implies our statement.

The theorem has some consequences for the point of view taken for the investigations in the field of \(L^2\)- and other invariants. The usual approach is to consider the (full) group \(\text{C}^*\)-algebra \(\text{C}^*(\pi)\) of the fundamental group \(\pi = \pi_1(M)\) of a certain compact manifold \(M\) or, more specifically, of a finite connected CW-complex \(M\). The \(\text{C}^*\)-algebra \(\text{C}^*(\pi)\) admits a canonical finite trace functional, so its bidual von Neumann algebra \(\mathcal{N}(\pi)\) has a finite normal trace functional and, hence, is the block-diagonal sum of \(\sigma\)-finite type \(\text{II}_1\) and finite type \(\text{I}\) components. The theorem above suggests that it might be more perspective to consider the category of finitely generated projective \(\text{C}^*(\pi)\)-modules instead of the category of finitely generated projective \(\mathcal{N}(\pi)\)-modules.

This thought is supported by a number of results of A. Carey, V. Mathai and A. S. Mishchenko \([4]\) and of D. Burghelea, L. Friedlander and T. Kappeler \([8]\). They consider the analytic torsion of cone complexes that arise from finite-dimensional non-simply connected Riemannian manifolds \(M\) and their de Rham complexes \(\Omega^*(M; \text{C}^*(\pi))\). Their investigations rely merely on the properties of \(\text{C}^*(\pi)\) and of finitely generated projective \(\text{C}^*(\pi)\)-modules, and they avoid any weak* completions of appearing structural elements of the basic constructions. So one might get the idea that Betti-numbers and Novikov-Shubin invariants could be already derived in case
$C^*(\pi)$ admits a finite center-valued trace. However, unitarily non-isomorphic finitely generated projective $C^*(\pi)$-modules can have equal dimension values in case $C^*(\pi)$ is infinite-dimensional. Furthermore, the monoid of all finitely generated projective $C^*(\pi)$-modules can fail to have the cancellation property. Also, the extension of the center-valued trace on $C^*(\pi)$ to the standard semifinite center-valued trace on $\text{End}_A(l^2(A) \otimes l^2)$ may assign a finite value to some orthogonal projection $P$ even if the image $P(l^2(A) \otimes l^2)$ is not finitely generated (cf. G. G. Kasparov’s theorem [25, Th. 6.2] applied to countably generated ideals of $C^*(\pi)$). So the entire approach does not work for the situation of general finitely generated projective $C^*(\pi)$-modules without completing them with respect to the standard weak topology.

However, there is another way to classify finitely generated $B$-modules over arbitrary unital $C^*$-algebras $B$. It is suggested by the modular frame theory of countably generated Hilbert $B$-modules that has been recently worked out by D. R. Larson and the author in [18, 22]. Let $B = C^*(\pi)$ and consider a finitely generated projective $A$-module $M$. The module $M$ can be equipped with an $B$-valued inner product $\langle \cdot, \cdot \rangle$ and becomes a finitely generated Hilbert $B$-module that way. Conversely, every finitely generated Hilbert $B$-module is projective as an $B$-module, [51, Cor. 15.4.8]. The choice of the $B$-valued inner product on $M$ is unique up to a positive invertible bounded module operator $T$ on $M$ linking any other inner product structure to a fixed one.

By [18, 22] every finitely generated Hilbert $B$-module $\{M, \langle \cdot, \cdot \rangle_M \}$ admits at least one finite normalized tight (modular) frame, i.e. a $k$-tuple $\{x_1, ..., x_k\}$ of elements of $M$ such that the equality $\langle x, x \rangle = \sum_{i=1}^{k} \langle x, x_i \rangle \langle x_i, x \rangle$ holds for any $x \in M$. Furthermore, a reconstruction formula $x = \sum_{i=1}^{k} \langle x, x_i \rangle x_i$ is valid for any $x \in M$, so the knowledge of the values $\{\langle x_i, x_j \rangle : i, j = 1, ..., k\}$ turns out to be sufficient to describe the $B$-module $M$ up to uniqueness. Note that the elements $\{x_1, ..., x_k\}$ need not to be ($B$-)linearly independent, in general.

**Theorem 3.2.** Let $B$ be a unital $C^*$-algebra and let $\{M, \langle \cdot, \cdot \rangle_M \}$ and $\{N, \langle \cdot, \cdot \rangle_N \}$ be two finitely generated Hilbert $B$-modules. Then the following conditions are equivalent:

(i) $M$ and $N$ are algebraically isomorphic as projective $B$-modules.

(ii) $\{M, \| \cdot \|_M \}$ and $\{N, \| \cdot \|_N \}$ are isometrically isomorphic as Banach $B$-modules.

(iii) $\{M, \langle \cdot, \cdot \rangle_M \}$ and $\{N, \langle \cdot, \cdot \rangle_N \}$ are unitarily isomorphic as Hilbert $B$-modules.

(iv) There are finite normalized tight (modular) frames $\{x_1, ..., x_k\}$ and $\{y_1, ..., y_l\}$ of $M$ and $N$, respectively, such that $k = l$, $x_i \neq 0$ and $y_i \neq 0$ for any $i = 1, ..., k$, and $\langle x_i, x_j \rangle_M = \langle y_i, y_j \rangle_N$ for any $i, j = 1, ..., k$.

**Proof.** The equivalence of the conditions (i), (ii) and (iii) has been shown for countably generated Hilbert $B$-modules in [23, Th. 4.1]. The implication (iii)$\Rightarrow$(iv) can be shown to hold setting $y_i = U(x_i)$ for the existing unitary operator $U : M \to N$ and for $i = 1, ..., k$. The demonstration of the inverse implication requires slightly more work. For the given normalized tight (modular) frames $\{x_1, ..., x_k\}$ and $\{y_1, ..., y_l\}$ of $M$ and $N$, respectively, we define a $B$-linear operator $V$ by the rule $V(x_i) = y_i$, $i = 1, ..., k$. For this operator $V$ we have

$$\langle V(x), y_j \rangle_N = \sum_{i=1}^{k} \langle V(x), y_i \rangle_N \langle y_i, y_j \rangle_N$$

$$= \sum_{i=1}^{k} \left( \sum_{m=1}^{k} \langle x, x_m \rangle_M V(x_m), y_i \right) \langle x_i, x_j \rangle_M$$
holds for certain coefficients 

\[ \sum_{i=1}^{k} \sum_{m=1}^{k} \langle x, x_m \rangle_M \langle x_m, x_i \rangle_M \langle x_i, x_j \rangle_M \]

\[ = \sum_{m=1}^{k} \langle x, x_m \rangle_M \left( \sum_{i=1}^{k} \sum_{j=1}^{k} \langle x_j, x_i \rangle_M x_i \right) \]

\[ = \left( \langle x, \sum_{m=1}^{k} \langle x_j, x_m \rangle_M x_m \rangle_M \right) \]

\[ = \langle x, x_j \rangle_M \]

for every \( x \in M \). Consequently,

\[ \langle x, x \rangle_M = \sum_{i=1}^{k} \langle x, x_i \rangle_M \langle x_i, x \rangle_M = \sum_{i=1}^{k} \langle V(x), y_i \rangle_N \langle y_i, V(x) \rangle_N = \langle V(x), V(x) \rangle_N \]

for any \( x \in M \), and the operator \( V \) is unitary.

**Corollary 3.3.** Every finitely generated projective \( B \)-module \( M \) over a unital \( C^* \)-algebra \( B \) can be reconstructed up to isomorphism from the following data:

(i) A finite set of algebraic non-zero modular generators \( \{ x_1, ..., x_k \} \) of \( M \).

(ii) A symmetric \( k \times k \) matrix \( (a_{ij}) \) of elements from \( B \), where \( a_{ij} \) is supposed to be equal to \( \langle x_i, x_j \rangle_0 \) for \( i, j = 1, ..., k \) and for the (existing and unique) \( B \)-valued inner product \( \langle ..., \rangle_0 \) on \( M \) that turns the set of algebraic modular generators \( \{ x_1, ..., x_k \} \) into a normalized tight modular frame of the Hilbert \( B \)-module \( \{ M, \langle ..., \rangle_0 \} \).

The number of elements in sets of algebraic modular generators of \( M \) has a minimum, and it suffices to consider sets of generators of minimal length. Then the modular invariants can be easier compared permuting the elements of the generating sets if necessary.

**Proof.** The set of algebraic generators \( \{ x_1, ..., x_k \} \) of \( M \) is a frame with respect to any \( B \)-valued inner product on \( M \) which turns \( M \) into a Hilbert \( B \)-module. That is the inequality

\[ C \cdot \langle x, x \rangle \leq \sum_{i=1}^{k} \langle x, x_i \rangle \langle x_i, x \rangle \leq D \cdot \langle x, x \rangle \]

is satisfied for two finite positive real constants \( C, D \) and any \( x \in M \), see [18, Th. 5.9]. What is more, for any frame of \( M \) there exists another \( B \)-valued inner product \( \langle ..., \rangle_0 \) on \( M \) with respect to which it becomes normalized tight, that is \( \langle x, x \rangle_0 = \sum_{i=1}^{k} \langle x, x_i \rangle \langle x_i, x \rangle_0 \) holds for any \( x \in M \). This inner product is unique, cf. [18, Cor. 4.3, Th. 6.1], [22, Th. 4.4]. So assertion (iv) of the previous theorem gives the complete statement.

We can say more in case the finitely generated Hilbert \( B \)-module contains a modular Riesz basis, i.e. a finite set of modular generators \( \{ x_1, ..., x_k \} \) such that the equality \( 0 = b_1 x_1 + \ldots + b_k x_k \) holds for certain coefficients \( \{ b_1, ..., b_k \} \subset B \) if and only if \( b_i x_i = 0 \) for any \( i = 1, ..., k \). Obviously, a modular Riesz basis is minimal as a set of modular generators, i.e. we cannot drop any of its elements preserving the generating property. However, there can exist totally different Riesz bases for the same module that consist of less elements, cf. [18, Ex. 1.1]. Note that the coefficients \( \{ b_1, ..., b_k \} \) can be non-trivial even if \( b_i x_i = 0 \) for any index \( i \) since every non-trivial \( C^* \)-algebra \( B \) contains zero-divisors. Not every Hilbert \( C^* \)-module containing a normalized tight modular frame does possess a modular Riesz basis. An example can be found in [22, Ex. 2.4].

In case of finitely generated projective \( W^* \)-modules (and therefore, in the case of Hilbertian modules over finite \( W^* \)-algebras) we are in the pleasant situation that they always contain a modular Riesz basis by [13, Th. 3.12]. Moreover, by spectral decomposition every element \( x \)
of a Hilbert $W^*$-module $\mathcal{M}$ has a carrier projection of $\langle x, x \rangle$ contained in the $W^*$-algebra of coefficients $B$. So we can ascertain the following fact:

**Proposition 3.4.** Let $\mathcal{M}$ be a finitely generated projective $B$-module over a $W^*$-algebra $B$ that possesses two finite modular Riesz bases $\{x_1, ..., x_k\}$ and $\{y_1, ..., y_l\}$. Then there exists an $l \times k$ matrix $C = (c_{ij})$, $i = 1, ..., l$, $j = 1, ..., k$, with entries from $B$ such that $y_i = \sum_{j=1}^{k} c_{ij} x_j$ for any $i = 1, ..., l$, and analogously, there exists a $k \times l$ matrix $D = (d_{ji})$ with entries from $B$ such that $x_j = \sum_{i=1}^{l} d_{ji} y_i$ for any $j = 1, ..., k$.

Suppose the left carrier projections of $c_{ij}$ and $d_{ji}$ equal the carrier projections of $\langle y_i, y_i \rangle$ and $\langle x_j, x_j \rangle$, respectively, and the right carrier projection of $c_{ij}$ and $d_{ji}$ equal the carrier projections of $\langle x_j, x_j \rangle$ and $\langle y_i, y_i \rangle$, respectively. Then the matrices $C$ and $D$ are Moore-Penrose invertible in $M_{kl}(B)$ and $M_{lk}(B)$, respectively. The matrix $C$ is the Moore-Penrose inverse of $D$, and vice versa.

**Proof.** Since both the modular Riesz bases are sets of modular generators of $\mathcal{M}$ we obtain two $B$-valued (rectangular, w.l.o.g.) matrices $C = (c_{ij})$ and $D = (d_{ji})$ with $i = 1, ..., l$ and $j = 1, ..., k$ such that

$$y_i = \sum_{m=1}^{k} c_{im} x_m \quad , \quad x_j = \sum_{n=1}^{l} d_{jn} y_n .$$

Combining these two sets of equalities in both the possible ways we obtain

$$y_i = \sum_{n=1}^{l} \left( \sum_{m=1}^{k} c_{im} d_{mn} \right) y_n \quad , \quad x_j = \sum_{m=1}^{k} \left( \sum_{n=1}^{l} d_{jn} c_{nm} \right) x_m$$

for $i = 1, ..., l$, $j = 1, ..., k$. Now, since we deal with sets of coefficients $\{c_{ij}\}$ and $\{d_{ji}\}$ that are supposed to admit special carrier projections, the coefficients in front of the elements $\{y_n\}$ and $\{x_m\}$ at the right side can only take very specific values:

$$\sum_{m=1}^{k} c_{im} d_{mn} = \delta_{in} \cdot q_n \quad , \quad \sum_{n=1}^{l} d_{jn} c_{nm} = \delta_{jm} \cdot p_m ,$$

where $\delta_{ij}$ is the Kronecker symbol, $p_m \in B$ is the carrier projection of $\langle x_m, x_m \rangle$ and $q_n \in B$ is the carrier projection of $\langle y_n, y_n \rangle$. So $C \cdot D$ and $D \cdot C$ are positive idempotent diagonal matrices with entries from $B$. The Moore-Penrose relations $C \cdot D \cdot C = C$, $D \cdot C \cdot D = D$, $(C \cdot D)^* = C \cdot D$ and $(D \cdot C)^* = D \cdot C$ turn out to be fulfilled. 

Summing up, we can replace the single center-valued trace value that characterizes a finitely generated projective $A$-module over a finite von Neumann algebra $A$ up to isomorphism by another set of data of a finitely generated projective $B$-module over a unital $C^*$-algebra $B$. It consists of a set of non-zero modular generators $\{x_1, ..., x_k\}$ together with a $k \times k$ matrix with entries $\langle x_i, x_j \rangle_0$, where $\langle ., . \rangle_0$ is supposed to be the $B$-valued inner product on the module with respect to which the generating set becomes a normalized tight frame. In the von Neumann case the generators can be described as $k$-tuples $x_i = (0, ..., 0, p_i, 0, ..., 0)$ for orthogonal projections $p_i \in A$, $i = 1, ..., k$, and the $k \times k$ matrix can be chosen to be a diagonal one with the elements $p_i \in A$ on the diagonal. This follows from the general structure of self-dual Hilbert $W^*$-modules as described by W. L. Paschke at [43, Th. 3.11]. For general finitely generated Hilbert $B$-modules a diagonal structure of the $k \times k$ matrix may not exist for any $k \in \mathbb{N}$ since cancellation may not hold in the monoid of all finitely generated projective $B$-modules. We would like to formulate the problem whether Moore-Penrose type transfer matrices between modular Riesz bases appear for more general $C^*$-algebras of coefficients then $W^*$-algebras or monotone complete $C^*$-algebras, or not.
To derive appropriate invariants for finitely generated Hilbert $B$-chain complexes in case $B$ is the full group $C^*$-algebra of the fundamental group of a certain compact manifold $M$ has to await another time since the considerations would go beyond the set limits of the present paper.

Acknowledgement: The author is grateful to V. M. Manuilov, A. S. Mishchenko, G. K. Pedersen and E. V. Troitsky for the fruitful discussions, the exchange of ideas and the continuous support during the recent years of collaboration.

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