COMMUTATOR LENGTH OF POWERS IN FREE PRODUCTS OF GROUPS

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Abstract Given groups $A$ and $B$, what is the minimal commutator length of the 2020th (for instance) power of an element $g \in A * B$ not conjugate to elements of the free factors? The exhaustive answer to this question is still unknown, but we can give an almost answer: this minimum is one of two numbers (simply depending on $A$ and $B$). Other similar problems are also considered.

Keywords: commutator length; stable commutator length; free products of groups

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Introduction

It is well known that, in free groups, non-identity commutators are not proper powers \cite{29}. A product of two commutators in a free group can surely be the square of a non-identity element, and can even be a cube, as Culler noticed \cite{9}: $[a, b]^3 = [a^{-1}ba, a^{-2}bab^{-1}][bab^{-1}, b^2]$. This equality holds in the free group $F(a, b)$ and, therefore, for any elements $a$ and $b$ of any group. Moreover, Culler \cite{9} showed that, in the free group $F(a, b)$, the element $[a, b]^n$ decomposes into a product of $k$ commutators if $n \leq 2k - 1$.

For free groups, Culler’s estimate cannot be improved in any sense:

if, for some elements $x_i, y_i, z$ of a free group, $[x_1, y_1] \ldots [x_k, y_k] = z^n$, where $n \geq 2k$, then $z = 1$.

This remarkable fact was obtained in \cite{7} for $k = 2$ and in \cite{10} in the general case. In the same paper \cite{10}, a similar assertion was proven for free products of \textit{locally indicable} groups (i.e. groups, in which each non-trivial finitely generated subgroup admits an epimorphism onto $\mathbb{Z}$). Later, it was discovered that this assertion remains valid in free products of any torsion-free groups:

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if \([x_1, y_1] \ldots [x_k, y_k] = z^n\) for some elements \(x_i, y_i, z\) of a free product of torsion-free groups, where \(n \geq 2k\), then \(z\) is conjugate to an element of a free factor.

This was shown in [2, 19] (independently). Moreover, both papers mentioned that the arguments remain valid if the torsion-free condition is replaced with a small-torsion-free condition. However, arguments in [2, 19] are different:

- Chen’s proof is based on Calegari’s approach [1],
- while [19] is based on the car-crash lemma [20];

this is why, the results of [2, 19] for groups with torsion are different (and even incomparable — neither one is stronger than the other):

suppose that, for some elements \(x_i, y_i, z\) of a free product of groups without non-identity elements of order less than \(N\), an equality \([x_1, y_1] \ldots [x_k, y_k] = z^n\) holds then \(z\) is conjugate to an element of a free factor

\[
\begin{cases}
  n \geq 2k + \left\lfloor \frac{2n}{N} \right\rfloor & [2] \\
n \geq 2k \text{ and } N > n & [19].
\end{cases}
\]

We show that condition (*) can be replaced with a weaker condition

\[ n \geq 2k + 2 \left\lfloor \frac{n}{N} \right\rfloor. \]

Clearly, this strengthens both results (*). Moreover, estimate (**) is the best possible or almost the best possible. Namely, the situation is as follows.

Let \(G\) be a group with a fixed free-product decomposition: \(G = \bigast_{j \in J} A_j\). Let \(k(G, n)\) be the minimal \(k \in \mathbb{Z}\) such that the \(n\)th power of an element of \(G\) not conjugate to elements of \(\bigcup_{j \in J} A_j\) decomposes into a product of \(k\) commutators and let \(N(G)\) be the minimal order of a non-identity element of \(G\). Thus, according to (**), any free product \(G\) satisfies the inequality \(k(G, n) \geq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{N(G)} \right\rfloor + 1\). This estimate is almost the best possible in the following sense: Theorem 1 (see the following section) asserts, in particular, that

for any free product \(G = \bigast_{j \in J} A_j\), the value \(k(G, n)\) is either \(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{N(G)} \right\rfloor + 1\) or \(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{N(G)} \right\rfloor + 2\).

Putting \(k(G, n) = 1\), we obtain a well-known fact [8]:

in any free product \(G\), a commutator not conjugate to elements of the free factors can be a proper power only if \(N(G) = 2\) or \(N(G) = 3\) in the latter case, this commutator can only be a cube.

For larger \(k(G, n)\), our result is (apparently) new.

Actually, we study equations more general than the equation \([y, z][t, u] \ldots = x^n\) considered above:
the power $x^n$ is replaced with a ‘generalised power’, i.e. the product of conjugate elements;

and the product of commutators is replaced with a product of commutators and elements conjugate to elements of the free factors.

**Main theorem (a simplified form).** Suppose that, in a free product of groups $G = \ast A_j$ without non-identity elements of order less than $N$, an equality

$$c_1 \ldots c_k d_1 \ldots d_l = u_1^{n_1} \ldots u_m^{n_m}$$

holds, where $c_i$ are commutators, $d_i$ are conjugate to elements of $\cup_{j \in J} A_j$, elements $u_i$ are conjugate to each other and not conjugate to elements of $\cup_{j \in J} A_j$, and $n_i$ are positive integers. Then

$$2k + l \geq \sum_{i=1}^{m} (n_i - 1) - 2 \left[ \frac{1}{N} \sum_{i=1}^{m} n_i \right] + 2.$$ 

This result significantly strengthens earlier known facts:

under the hypothesis of the main theorem

$$2k + l \geq \begin{cases} \sum_{i=1}^{m} (n_i - 1) - \left[ \frac{2}{N} \sum_{i=1}^{m} n_i \right] + 2, & \text{if } l = 0 \ [2]; \\ \sum_{i=1}^{m} (n_i - 1) + 2, & \text{if } N > \sum_{i=1}^{m} n_i \ [19]. \end{cases}$$

The main theorem immediately implies what is said above on inequality (**).

**Corollary 1.** Suppose that, in a free product of groups $G = \ast A_j$, an equality $c_1 \ldots c_k = u^n$ holds, where $c_i$ are commutators and $u$ is not conjugate to elements of the free factors. Then $2k \geq n - 2 \left[ \frac{n}{N} \right] + 1$ (or, equivalently, $k \geq \left[ \frac{n}{2} \right] - \left[ \frac{n}{N} \right] + 1$).

The statement of the main theorem above is somewhat simplified. In fact, we prove a stronger estimate under weaker assumptions. The full statement of the main theorem and its proof can be found in the last section. In § 1, we derive Theorem 1 (mentioned above) from the main theorem. Sections 2 and 3 contain necessary information about Howie diagrams and motions on surfaces, i.e. about the car-crash lemma. This lemma from [20] (or its variants) was already applied in [14] and [19] to problems related to the commutator length (and, e.g. in [3–6, 11–13, 15, 20, 22–27], it is applied to other problems). We need a new version of the car-crash lemma, which is discussed in § 3. Surprisingly, a substantial role in that section is played by the fair partition problem, see, e.g. [28].

**Notation**

Our notation is mainly standard. Note only that, if $k \in \mathbb{Z}$, and $x$ and $y$ are elements of a group, then $x^y$, $x^{ky}$, and $x^{-y}$ denote $y^{-1}xy$, $y^{-1}x^ky$, and $y^{-1}x^{-1}y$, respectively. The
commutator \([x, y]\) is \(x^{-1}y^{-1}xy\). The symbol \(\text{cl}(g)\) denotes the commutator length of an element \(g\) of a group, i.e. \(\text{cl}(g)\) is the minimal integer \(k\) such that \(g\) decomposes into a product of \(k\) commutators (and \(\text{cl}(1) = 0\)). The word ‘surface’ always means a closed surface (not necessarily connected). The Euler characteristic of a surface \(S\) is denoted by \(\chi(S)\). The letters \(\mathbb{R}, \mathbb{Z}, \text{and } \mathbb{N}\) denote the set of real, integer, and natural (positive integer) numbers, respectively. The symbol \([x]\) denotes the integer part of a real number \(x\) (i.e. \([x]\) is the maximal integer not exceeding \(x\)).

1. Powers of small commutator length

Culler’s bound mentioned in the very beginning of this paper can be stated as follows.

**Culler’s inequality** [9]. For any elements \(a\) and \(b\) of any group and for any \(n \in \mathbb{N} \cup \{0\}\),
\[
\text{cl}([a, b]^n) \leq \left[ \frac{n}{2} \right] + 1, \quad \text{where } [x]_c \overset{\text{def}}{=} \begin{cases} [x] & \text{if } x \neq 0; \\ -1, & \text{if } x = 0. \end{cases}
\]

**Lemma 1.** If \(a\) and \(b\) are elements of a group and \(m \in \mathbb{N}\), then \((ab)^m\) is conjugate to an element of the form \(a^m b^n c_1 c_2 \ldots c_{[\frac{m}{2}]},\) where \(c_i \in G\) are commutators.

**Proof.**
\[
a^l(ba)^s b^l \cdot [a^{l-2}b^{l-1}, b^{2-l}a^{1-l}] = a^l(ba)^s b^l \cdot b^{l-2}a^{2-l}a^{l-1}b^{2-l}a^{l-2}b^{l-1}b^{2-l}a^{l-1} \\
= a^l(ba)^s bab^{2-l}a^{l-2}b^{2-l}a^{l-2}ba^{1-l} \sim a^{l-2}bab^{2-l}a^{l-2} \\
= a^{l-2}(ba)^s + 2b^{l-2}.
\]

An obvious induction shows that, for some commutators \(c_i\), the element \(a^m b^m c_1 c_2 \ldots c_{[\frac{m}{2}]\}}\) is conjugate to \((ba)^m\) if \(m\) is even, or to \(a(ba)^{m-1}b\) if \(m\) is odd. This completes the proof (because \(a(ba)^{m-1}b = (ab)^m \sim (ba)^m\), where \(\sim\) means conjugation). \(\square\)

**Lemma 2.** If \(a\) and \(b\) are elements of a group, \(m \in \mathbb{N} \ni s\), and \(a^m = b^m = 1\), then \(\text{cl}((ab)^m s) \leq s([m/2] - 1) + [s/2]_c + 1\).

**Proof.** Note that, for any non-identity element \(g\) of the commutator subgroup of any group and for any \(s \in \mathbb{N} \cup \{0\}\),
\[
\text{cl}(g^s) \leq s(\text{cl}(g) - 1) + \left[ \frac{s}{2} \right]_c + 1.
\]

Indeed, representing \(g\) as \(g = ch\), where \(c\) is a commutator, and \(\text{cl}(h) = \text{cl}(g) - 1\), we obtain
\[
\text{cl}(g^s) = \text{cl}((ch)^s) = \text{cl}(c^s h^{c^{s-1}} h^{c^{s-2}} \ldots h^{c^1}) \leq \text{cl}(c^s) + s \cdot \text{cl}(h) \\
\leq \left[ \frac{s}{2} \right]_c + 1 + s(\text{cl}(g) - 1) \quad \text{(by Culler’s inequality)}.
\]

This completes the proof because \(\text{cl}((ab)^m) \leq [m/2]\) by Lemma 1. \(\square\)
Theorem 1. For any free product $G = \bigast_{j \in J} A_j$ and any $n \in \mathbb{N}$,

either $k(G, n) = \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{N(G)} \right\rfloor + 1$ or $k(G, n) = \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{N(G)} \right\rfloor + 2$, where

$$k(G, n) \overset{\text{def}}{=} \min \left\{ \text{cl}(g^n) \mid g \in G, \text{ } g \text{ is not conjugate to elements of } \bigcup_{j \in J} A_j \right\}$$

and

$$N(G) \overset{\text{def}}{=} \min\{|g| \mid g \in G \setminus \{1\}\}.$$ 

Moreover, $k(G, n) = \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{N(G)} \right\rfloor + 1$ if at least one of the following conditions is satisfied:

- a) $n$ is even and $\left\lceil \frac{n}{N(G)} \right\rceil$ is odd; 
- b) $n$ is divisible by $N(G)$; 
- c) $n \leq N(G)$; 
- d) $N(G) = 2$.

Proof. Surely, $N(G)$ is either prime or infinite. For $N(G) = 2$, the assertion holds, because the group $G$ in this case contains an infinite dihedral subgroup, whose commutator subgroup coincides with the set of commutators (and trivially intersects conjugates of free factors). For $N(G) = \infty$, the argument below is essentially valid, but we leave it to readers, because the assertion of the theorem in this case follows immediately from the results of [2, 19] mentioned in the introduction. Thus, we assume that $N(G)$ is odd.

If $z^m = 1$, then we have two bounds:

$$\text{cl}([x, y]^m) \leq \left\lceil \frac{m}{2} \right\rceil + 1 \text{ and } \text{cl}([z, u]^m) \leq s \left( \left\lceil \frac{m}{2} \right\rceil - 1 \right) + \left\lceil \frac{s}{2} \right\rceil + 1, \quad (***)$$

The first one is Culler’s inequality, and the second one is Lemma 2.

Consider in $G$ a commutator $[z, u]$, where $z^{N(G)} = 1$ and $u$ does not lie in the same free factors as $z$. Let us divide $n$ by $N = N(G)$ with remainder: $n = rN + t$, where $0 \leq t < N$ (and $r = \left\lceil \frac{N}{t} \right\rceil$). Let the symbols $\Delta_0(a, b, \ldots)$ and $\Delta_{odd}(a, b, \ldots)$ denote the number of zeros and the number of odd numbers in the tuple $(a, b, \ldots)$. Then for odd $N$, we obtain

$$k(G, n) \overset{(***)}{=} \text{cl}([z, u]^{rN + t}) \leq \text{cl}([z, u]^{rN}) + \text{cl}([z, u]^t)$$

$$\leq \left( r \left( \left\lceil \frac{N}{2} \right\rceil - 1 \right) + \left\lceil \frac{r}{2} \right\rceil + 1 \right) + \left( \left\lceil \frac{t}{2} \right\rceil + 1 \right)$$

$$= \left( r \left( \frac{N - 1}{2} - 1 \right) + \left\lceil \frac{r}{2} \right\rceil + 1 \right) + \left( \left\lceil \frac{n - rN}{2} \right\rceil + 1 \right)$$

$$= r \left( \frac{N - 1}{2} - 1 \right) + \frac{r}{2} + 1 + \frac{n - rN}{2} + 1 - \Delta_0(r, n - rN) - \frac{1}{2} \Delta_{odd}(r, n - rN)$$

$$= \frac{n}{2} - r + 2 - \Delta_0(r, n - rN) - \frac{1}{2} \Delta_{odd}(r, n - rN)$$

$$= \left\lceil \frac{n}{2} \right\rceil - r + 2 - \Delta_0(r, n - rN) - \frac{1}{2} (\Delta_{odd}(r, n - rN) - \Delta_{odd}(n)).$$
Commutator length of powers in free products of groups

\[
\frac{n}{2} - r + 2 - \Delta_0(r, n - rN) - \Delta_{\text{odd}}(r)(1 - \Delta_{\text{odd}}(n))
\]

\[
= \left\{ \begin{array}{ll}
\left\lfloor \frac{n}{2} \right\rfloor - r + 1 & \text{if a), b), or c) holds; } \\
\left\lfloor \frac{n}{2} \right\rfloor - r + 2 & \text{otherwise. }
\end{array} \right.
\]

Comparing this with Corollary 1, we conclude that Theorem 1 is proven (modulo the main theorem). \qed

2. Howie diagrams

Suppose that \( S \) is a closed oriented surface (possibly non-connected), and \( \Gamma \) is a finite (undirected) graph embedded into \( S \) and dividing it into simply connected domains. Such a graph determines a cell decomposition of \( S \), i.e. a mapping \( M \) called a map on \( S \):

\[
M : \bigsqcup_{i=1}^{m} D_i \to S, \quad \text{where } D_i \text{ are two-dimensional disks,}
\]

such, that

- the mapping \( M \) is continuous surjective, injective on the interior (i.e. on \( \bigsqcup_{i=1}^{m} (D_i \setminus \partial D_i) \));
- the preimage of each point is finite, and the preimage of the graph \( \Gamma \) is the union of the boundaries of the faces: \( M^{-1}(\Gamma) = \bigsqcup_{i=1}^{m} \partial D_i \).

The preimages of the vertices of \( \Gamma \) are called corners of the map; we say that a corner \( c \) is at a vertex \( v \) if \( M(c) = v \). The vertices and edges of \( \Gamma \) are referred to as vertices and edges of the map \( M \). The disks \( D_i \) are called faces or cells of the map. Such a map is called a diagram over a free product \( A \ast B \) if

- the graph \( \Gamma \) is bipartite, i.e. there are two types of vertices: \( A \)-vertices and \( B \)-vertices, and each edge joins an \( A \)-vertex with a \( B \)-vertex;
- the corners at \( A \)-vertices are labeled by elements of the group \( A \), and the corners at \( B \)-vertices are labelled by elements of \( B \);
- some vertices are distinguished and called exterior, the other vertices are called interior;
- the label of each interior \( A \)-vertex equals 1 in the group \( A \), and the label of each interior \( B \)-vertex equals 1 in \( B \), where the label of a vertex is the product of labels of corners at this vertex in clockwise order (thus, the label of a vertex is defined up to conjugation in \( A \) or \( B \)).

Similar diagrams were considered in \([15–17, 20]\) and many other works, but our definitions slightly differ and correspond to the definitions from \([19]\) (except that exterior and interior vertices are called irregular and regular in \([19]\)).
The label of a face of a diagram is the product of labels of all corners of this face in counterclockwise order. The label of a face is an element of the free product $A \ast B$ defined up to conjugation.

For instance, Figure 1 shows a diagram on a torus (which is drawn as a rectangle with identified opposite sides) containing two vertices, three edges, one face, and six corners with labels $a \in A$ and $b \in B$. If both vertices are interior, then $a^3$ must be equal to 1 in $A$, and $b^3$ must be equal to 1 in $B$. The label of the face is $(ab)^3$. Actually, this diagram shows that the cube of the product of two elements of order three is always a commutator (in any group).

3. Motions

This section is very similar to corresponding sections of [14] and [19] and contains definitions and statements from [22] with some simplifications.

Let $M$ be a map on a closed oriented surface $S$ and let $\Gamma \subset S$ be the corresponding graph. A car moving around a face $D$ is an orientation preserving homeomorphism from an oriented circle $R$ (the circle of time) to the boundary $\partial D$ of $D$.

If the number of cars being at a moment of time $t$ at a point $p$ of $\Gamma$ equals the degree $d$ of this point, then we say that a complete collision (of degree $d$) occurs at $p$ at the moment $t$; this point $p$ is called a point of complete collision. Here, the degree of a point $p \in \Gamma$ is the number of edges incident to $p$ if $p$ is a vertex; and $\deg p \overset{\text{def}}{=} 2$ if $p$ is not a vertex (i.e. if $p$ is an interior point of an edge).

Note that, according to the definition, when a car arrives to the vertex of degree one (a dead end), a complete collision occurs.

A multiple motion of period $T$ on a map $M$ is a tuple of cars $\alpha_{D,j} : R \to \partial D$, where $j = 1, \ldots, d_D$, such that

1. $d_D \geq 1$ for any face $D$ (i.e. each face is moved around by at least one car);
2. $\alpha_{D,j}(t + T) = \alpha_{D,j+1}(t)$ for any $t \in R$ and $j = \{1, \ldots, d_D\}$ (here indices are modulo $d_D$, and the addition of points of the circle $R$ is defined naturally: $R = \mathbb{R}/P\mathbb{Z}$);
3. for every face $D$, there exists a partition of $\partial D$ into $d_D$ consecutive arcs with disjoint interiors such that, during the time interval $0 \leq t \leq T$, each car $\alpha_{D,j}$ is moving along the $j$th arc of the partition.
Informally, several \((d_D)\) cars are moved around each face \(D\) in counterclockwise direction (the interior of \(D\) remains on the left) without U-turns and stops; and the motion is periodic in the sense that the boundary of \(D\) is partitioned into \(d_D\) segments, and, during the period (of \(T\) minutes), each car is moving along its segment (thus, after \(T\) minutes, the cars’ positions interchange cyclically).

**Car-crash lemma** (for multiple motions) \([21, 22]\). For any multiple motion on a map on a closed oriented surface \(S\), the number of points of complete collision is at least \(\chi(S) + \sum_D (d_D - 1)\), where the summation runs over all faces of the map.

In \([21, 22]\), this lemma was stated and proven for connected surfaces, but it remains valid in non-connected cases because the both sides of the inequality are additive with respect to the disjoint union.

Consider, for instance, the following motion on the one-cell map on a torus shown in Figure 1: three cars move around the unique face with constant speed one edge per minute; at zero moment of time, these three cars are at three different angles with label \(a\). Figure 1 shows the location of cars at the moment \(t = 1/3\). This is a periodic motion with a period of two minutes. Complete collisions occur at both vertices; while outside the vertices (i.e. at interior points of edges) there are no collisions. The car-crash lemma says that the following inequality must hold:

\[
\left(\text{the number of points of complete collision,}\right) \\
\quad \text{i.e. } 2 \\
\geq \left(\text{the Euler characteristic of the torus,}\right) \\
\quad \text{i.e. } 0 \\
+ \left(\text{\(d_D\), i.e. the number of cars moving around the unique face } D,\right) \\
\quad \text{i.e. } 3 - 1,
\]

which appears to be an equality in this example.

4. Clusters

The idea of clusters is that collisions that occur near each other can be treated as one collision; the modified car-crash lemma (the cluster lemma below) says that not only the number of points of collision is large but also the number of points of collision that are far from each other is large.

Suppose that we have a multiple motion with period \(T\) on some map on a surface \(S\) and all cars move with the same constant speed one edge per minute. A set \(K\) of points of complete collision is called a cluster centred at \(v \in K\) if, during less than \(T/2\) minutes after the collision at \(v\), each point \(w \in K\) is visited by at least one car having collided at \(v\). The cars colliding at the centre \(v\) of a cluster \(K\) are referred to as the connecting cars of \(K\); the connecting paths of \(K\) are the paths (of length \(< T/2\)) the connecting cars move along on the way from the centre of \(K\) to other points of the cluster. A set \(C\) of clusters is called independent if the centre of each cluster from \(C\) does not lie on any connecting path of another cluster from \(C\).
The statement of the cluster lemma (see below) uses the *fair partition function* \( \text{fp}(\mathcal{M}) \) of a multiset \( \mathcal{M} \) consisting of positive integers:

\[
\text{fp}(\mathcal{M}) \overset{\text{def}}{=} \min \left\{ \max \left( \sum_{i \in A} i, \sum_{i \in \mathcal{M} \setminus A} i \right) \mid A \subseteq \mathcal{M} \right\}.
\]

For example, \( \text{fp}(10, 4, 4, 3, 2) = \max(10 + 2, 4 + 4 + 3) = 12 \).

The problem of finding a fair partition is sometimes called ‘the easiest NP-hard problem’ [28]. We need a simple example of such calculation:

\[
\text{fp} \left( \underbrace{1, 1, 1, 1, \ldots, 1}_{\text{\# numbers}}, N, N, \ldots, N \right) = \begin{cases} 
\left[ \frac{\kappa + 1}{2} \right], & \text{if } \kappa \leq l; \\
\left[ \frac{l + 1}{2} \right] + N \cdot \left( \frac{\kappa - l}{2} \right), & \text{if } \kappa > l \text{ and } \kappa - l \text{ is even}; \\
\left[ \frac{l + 1 - \min(l, N)}{2} \right] + N \cdot \left( \frac{\kappa - l + 1}{2} \right), & \text{if } \kappa > l \text{ and } \kappa - l \text{ is odd.}
\end{cases}
\]

This is true for all \( \kappa, N \in \mathbb{N} \) and \( l \in \mathbb{N} \cup \{0\} \). The point is that the following algorithm gives a fair partition \( \mathcal{M} = A \sqcup B \):

- divide large items (i.e. \( N \)s) fairly, i.e. give \( \left( \frac{\kappa - l}{2} \right) \) of these items to \( A \) (if \( \kappa \geq l \));
- use small items (i.e. ones) to compensate the difference between \( A \) and \( B \) (which arises for odd \( \kappa - l \));
- when (and if) the difference is compensated, divide the remaining 1s fairly.

We leave the proof to the reader as an easy exercise; Figures 2 and 3 show all possible cases (\( f \) denotes \( \text{fp}(\underbrace{1, 1, 1, 1, \ldots, 1}_{\text{\# things}}, N, N, \ldots, N) \) in these figures).

We call a multiple motion *uniform* if all cars move with same constant speed one edge per minute and are at some vertices at the moment \( t = 0 \).

**Cluster lemma.** Suppose that, for a multiple uniform motion on a map on a closed oriented surface \( S \), the set \( \Pi \) of points of complete collision is partitioned into the minimal possible number \( \kappa \) of independent clusters: \( \Pi = \bigsqcup_{i=1}^{\kappa} v_a K_i \). Then

(a) \( \kappa \geq \chi(S) + \sum_D (d_D - 1), \) where the summation runs over all faces

(b) if there are precisely \( n \) points of complete collision, and their degrees are \( N_1 \leq \ldots \leq N_n \), then the number of cars of this motion (i.e. \( \sum_D d_D \), where the sum runs over
all faces) satisfies the inequality $\sum_{D} d_D \geq \max(fp(N_1, \ldots, N_\kappa), N_n)$; in particular, if all points of complete collision have degree at least $N$, then $\sum_{D} d_D \geq \left\lfloor \frac{\kappa+1}{2} \right\rfloor \cdot N$.

**Proof.** We assume that all collisions occur at vertices. This can be achieved by the subdivision of each edge of the initial map into two equal parts by new vertices of degree two (and slowing down all cars).

Let us prove the first assertion. For each cluster $K = K_i$ centred at $v = v_i$, consider a minimal set of connecting paths $\pi_j = \pi_{i,j}$ such that these paths contain all points of $K$. Due to minimality, for each connecting car, we have at most one corresponding connecting
path $\pi_j$ lying in the boundary of a cell $D_j = D_{ij}$, which is moved around by this car. By the definition of cluster, the length $\tau_j$ of the path $\pi_j$ is less than $T/2$.

Let us connect the starting and the ending points of the path $\pi_j$ by a new path $\pi'_j$ of the same length lying inside the cell $D_j$ (so, we duplicate the path $\pi_j$). (Note that these duplications for all clusters under consideration may produce several chords inside a cell, but these chords never intersect, because the clusters are independent.)

The cell $D_j$ turns into two cells (see Figure 4, on the left): the large cell $D'_j$ of the same perimeter as the initial cell $D_j$ and the small cell $\Gamma_j$ of perimeter $2\tau_j$.

We want to define car motion on this modified map. Cars moving around the large cells imitate the cars moving around the initial cells $D_j$, except that they use the new road $\pi'_j$ instead of $\pi_j$.

To define a car moving around the small cell, let us consider the motion of already defined cars on the boundary of a small cell. The boundary of each small cell $\Gamma = \Gamma_j$ has length $2\tau_j$ and consists of three segments (listed counterclockwise, see Figure 4, on the right):

- part $a = \pi'_j$ of length $\tau = \tau_j$; in this segment, the connecting car is moving during time interval $0 \leq t \leq \tau$ (to simplify notation, we assume that the complete collision at $v_i$ occurs at zero moment, other cases can, of course, be considered similarly); part $a$ ends at the corner $c$ at the centre $v_i$ of the cluster $K_i$;

- part $b$ of length one (this is the first edge of the path $\pi_j$), starting at the corner $c$; in this segment, another connecting car of the cluster $K_i$ is moving during time $-1 \leq t \leq 0$;

- part $c$ of length $|c| = \tau - 1$; not much is known about cars moving here; however, we know that $\tau < T/2$, therefore, $|c| = \tau - 1 < T - \tau - 1$; this means that there are no cars in $c$ (including its ends) at some moment of time $\tau < t < T - 1$ and even during some subinterval $\Delta_\Gamma$ (of positive duration) of the time interval $\tau < t < T - 1$ (because the time interval $\tau < t < T - 1$ has duration $T - 1 - \tau > |c|$, and all cars move with the unit speed). Note also that we can choose time intervals $\Delta_\Gamma$ disjoint for different small cells $\Gamma$:

$$\Delta_\Gamma \cap \Delta_{\Gamma'} = \emptyset \quad \text{for} \quad \Gamma \neq \Gamma'.$$

Now, let us define a new car $\alpha_\Gamma$ moving around the small cell $\Gamma = \Gamma_j$:
– at moment zero, \( \alpha_\Gamma \) is at the corner \( c \) (and participates in the complete collision at \( v_i \));

– then \( \alpha_\Gamma \) moves (slowly) along the segment \( b \) without any collisions, because the (connecting) car moving in the opposite direction has left segment \( b \) having met our car \( \alpha_\Gamma \) at \( v_i \); so segment \( b \) is safe until the moment \( T - 1 \);

– during the time interval \( \Delta_\Gamma \) (which starts earlier than \( T - 1 \) by definition of \( \Delta_\Gamma \)), our new car \( \alpha_\Gamma \) (rapidly) moves through the segment \( c \); no collisions occur, because \( c \) is safe during the time interval \( \Delta_\Gamma \) by definition of \( \Delta_\Gamma \) (and because \( \Delta_\Gamma \cap \Delta_{\Gamma'} = \emptyset \) for \( \Gamma \neq \Gamma' \));

– thus, our car \( \alpha_\Gamma \) arrives to segment \( a \) later than moment \( \tau \) (again, by definition of \( \Delta_\Gamma \)); this means that the connecting car already left \( a \), and our car \( \alpha_\Gamma \) safely without any collisions arrives to corner \( c \) at the end of the period.

We obtain a periodic motion on a map on surface \( S \), the number of complete collisions is precisely \( \kappa \), and the sum \( \sum_D (d_D - 1) \) over all faces remains the same as for the initial map (because each small face \( \Gamma_j \) is moved around by one car, i.e. \( d_{\Gamma_j} = 1 \)). Thus, applying the car-crash lemma to this motion, we obtain assertion a).

To prove assertion b), we divide the period of time \( I = \{ t \mid 0 \leq t < T \} \) into two half-periods: \( I = I_1 \sqcup I_2 \), where \( I_1 = \{ t \mid 0 \leq t < T/2 \} \) and \( I_2 = \{ t \mid T/2 \leq t < T \} \).

The periodicity of the motion implies that, at each point, not more than one complete collision occurs during period \( I \). Therefore, the set of points of complete collision \( \Pi = \{ p_1, \ldots, p_n \} \) is partitioned into two subsets: \( \Pi = \Pi_1 \sqcup \Pi_2 \), and the multiset \( N = (N_1, \ldots, N_n) \) of the degrees of these points is partitioned into two submultisets:

\[ N = N_1 \sqcup N_2, \quad \text{where } N_i = (\text{degrees of points of complete collision occurring during } I_i). \]

Suppose that \( \Pi_i \) can be partitioned into \( \kappa_i \) independent clusters and cannot be partitioned into a fewer number of independent clusters. Then \( \kappa_1 + \kappa_2 \geq \kappa \) (because \( \Pi \) can be partitioned into \( \kappa_1 + \kappa_2 \) independent clusters, which is impossible if \( \kappa_1 + \kappa_2 < \kappa \)).

We say that a set of points of complete collision is independent if the sets of colliding cars at these points during the period \( I \) are disjoint.

Let us concentrate on \( \Pi_1 \) now. Suppose that

– \( v_1 \in \Pi_1 \) is a point at which the first (timewise) collision occurs (if such \( v_1 \) exists);

– \( v_2 \in \Pi_1 \) is a point at which the first (timewise) collision occurs such that the set \( \{v_1, v_2\} \) is independent (if such \( v_2 \) exists);

– \( v_3 \in \Pi_1 \) is a point at which the first (timewise) collision occurs such that the set \( \{v_1, v_2, v_3\} \) is independent (if such \( v_3 \) exists);

– \( \ldots \)

The number of points \( v_i \) is at least \( \kappa_1 \), because otherwise \( \Pi_1 \) would admit a partition on a less than \( \kappa_1 \) number of independent clusters (e.g. if \( v_1 \) and \( v_2 \) do exist, and \( v_3 \) does
not, then each point from \( \Pi_1 \) is either in a cluster centred at \( v_1 \) or in a cluster centred at \( v_2 \).

Thus, \( \Pi_1 \) contains independent points \( v_1, \ldots, v_{\kappa_1} \), and \( \Pi_2 \) contains independent points \( w_1, \ldots, w_{\kappa_2} \) (by similar reasons). Therefore, the number of all existing cars is at least

\[
\max \left( \sum \deg v_i, \sum \deg w_i \right) \geq \text{fp}(\deg v_1, \ldots, \deg v_{\kappa_1}, \deg w_1, \ldots, \deg w_{\kappa_2}) \\
\geq \text{fp}(N_1, \ldots, N_\kappa)
\]

(where the last estimate follows immediately from the inequalities \( \kappa_1 + \kappa_2 \geq \kappa \) and \( N_1 \leq \ldots \leq N_n \)).

This completes the proof of assertion b), because the bound \( \sum_D d_D \geq N_n \) is obvious (if, at some point, \( N_n \) cars collide, then \( N_n \) cars do exist). \( \square \)

The cluster lemma implies the following fact (not mentioning clusters at all).

**Corollary of the cluster lemma.** Suppose that a multiple uniform motion on a map on an oriented closed surface \( S \) has precisely \( n \) points of complete collision, and their degrees are \( N_1 \leq \ldots \leq N_n \). Then the number of cars of this motion (i.e. \( \sum_D d_D \), where the summation runs over all faces) satisfies the inequality

\[
\sum_D d_D \geq \max(\text{fp}(N_1, \ldots, N_\kappa), N_n),
\]

where

\[
\kappa = \chi(S) + \sum_D (d_D - 1) \quad (\text{this value never exceeds } n).
\]

Moreover, for all \( l \in \mathbb{N} \cup \{0\} \),

\[
\chi(S) - l + \sum_D (d_D - 1) \\
\leq \begin{cases} 
2 \left[ \frac{1}{N_{l+1}} \left( \sum_D d_D - \left[ \frac{l+1}{2} \right] \right) \right] , & \text{if } \sum_D (d_D - 1) - l \text{ is even} \\
2 \left[ \frac{1}{N_{l+1}} \left( \sum_D d_D - \left[ \frac{l+1-N_{l+1}}{2} \right] \right) \right] - 1, & \text{if } \sum_D (d_D - 1) - l \text{ is odd}
\end{cases}
\]  

(3)

where \( \lfloor x \rfloor_+ = \max(\lfloor x \rfloor, 0) \) and \( N_i = \infty \) for \( i > n \) (in particular, for \( N_{l+1} = \infty \), the right-hand side is 0 or \(-1\)).

**Proof.** To prove (2), it suffices to substitute bound a) of the cluster lemma to bound b) of the same lemma (as the fair partition function \( \text{fp}(N_1, \ldots, N_\kappa) \) is surely non-decreasing as a function of \( \kappa \)).
Let us prove (3). Using the monotonicity of the function $\mathbf{f}_p$ with respect to each argument and formulae (2) and (1), for $\kappa = \chi(S) + \sum D (d_D - 1)$, we obtain

$$\sum_{D} d_D \geq \max_{\kappa \text{ numbers}} \left( \mathbf{f}_p(N_1, \ldots, N_{\kappa}), N_n \right) \geq \mathbf{f}_p \left( 1, 1, 1, 1, \ldots, 1, N_{l+1}, N_{l+1}, \ldots, N_{l+1} \right) \overset{\substack{\kappa \text{ numbers} \\ \min(l, \kappa) \text{ ones}}}{=} \overset{(1)}{=} \left[ \frac{\kappa + 1}{2} \right]$$

if $\kappa \leq l$;

$$\left[ \frac{l + 1}{2} \right] + N_{l+1} \cdot \frac{\kappa - l}{2}$$

if $\kappa > l$ and $\kappa - l \in 2\mathbb{Z}$;

$$\left[ \frac{l + 1 - \min(l, N_{l+1})}{2} \right] + N_{l+1} \cdot \frac{\kappa - l + 1}{2}$$

if $\kappa > l$ and $\kappa - l \notin 2\mathbb{Z}$.

**Case 0: $\kappa \leq l$.**

- If $\kappa - l$ is even, then $\lfloor (\kappa + 1)/2 \rfloor \geq \lfloor (l + 1)/2 \rfloor + N_{l+1} \cdot (\kappa - l)/2$ (since $N_{l+1} \geq 1$).

- If $\kappa - l$ is odd, then $\lfloor (\kappa + 1)/2 \rfloor \geq \lfloor l/2 \rfloor + N_{l+1} \cdot (\kappa - l + 1)/2 \geq \lfloor (l + 1 - \min(l, N_{l+1}))/2 \rfloor + N_{l+1} \cdot (\kappa - l + 1)/2$.

Thus, for all $\kappa$ and $l$,

$$\sum_{D} d_D \geq \left\{ \begin{array}{ll}
\left[ \frac{l + 1}{2} \right] + N_{l+1} \cdot \frac{\kappa - l}{2} & \text{if } \kappa - l \in 2\mathbb{Z}; \\
\left[ \frac{l + 1 - \min(l, N_{l+1})}{2} \right] + N_{l+1} \cdot \frac{\kappa - l + 1}{2} & \text{if } \kappa - l \notin 2\mathbb{Z}.
\end{array} \right.$$  

**Case 1: $\kappa - l$ is even.**

$$\sum_{D} d_D \geq \left[ \frac{l + 1}{2} \right] + N_{l+1} \cdot \frac{\kappa - l}{2} \implies \kappa - l \leq \frac{2}{N_{l+1}} \left( \sum_{D} d_D - \left[ \frac{l + 1}{2} \right] \right) \implies$$

$$\kappa - l \leq 2 \left[ \frac{1}{N_{l+1}} \left( \sum_{D} d_D - \left[ \frac{l + 1}{2} \right] \right) \right],$$

where the last implication is valid since $\kappa - l \in 2\mathbb{Z}$. The obtained bound coincides with (3), because $\kappa = \chi(S) + \sum D (d_D - 1)$.

**Case 2: $\kappa - l$ is odd.**

$$\sum_{D} d_D \geq \left[ \frac{l + 1 - N_{l+1}}{2} \right] + N_{l+1} \cdot \frac{\kappa - l + 1}{2} \implies$$

$$\kappa - l + 1 \leq 2 \left( \sum_{D} d_D - \left[ \frac{l + 1 - N_{l+1}}{2} \right] \right);$$
\(\kappa - l + 1\) is even, hence, \(\kappa - l + 1 \leq 2 \left[ \frac{1}{N_{l+1}} \left( \sum_{D} d_{D} - \left[ \frac{l+1-N_{l+1}}{2} \right]_{+} \right) \right] \), as required. This completes the proof. \(\square\)

5. Main theorem

Main theorem. Suppose that, in a free product of groups \(G = \ast A_{j}\), an equality

\[c_{1} \ldots c_{k} d_{1} \ldots d_{l} = u_{1}^{n_{1}} \ldots u_{m}^{n_{m}}\]  

(4)

holds, where \(c_{i}\) are commutators, \(d_{i}\) are conjugate to elements of \(\cup_{j \in J} A_{j}\), elements \(u_{i}\) are conjugate to each other and not conjugate to elements of \(\cup_{j \in J} A_{j}\), and \(n_{i}\) are positive integers. Then

\[2 - 2k - l + \sum_{i=1}^{m} (n_{i} - 1) \leq \begin{cases} 2 \left[ \frac{1}{N} \left( \sum_{i=1}^{m} n_{i} - \left[ \frac{l+1}{2} \right] \right) \right], & \text{if } \sum_{i=1}^{m} (n_{i} - 1) - l \text{ is even} \\ 2 \left[ \frac{1}{N} \left( \sum_{i=1}^{m} n_{i} - \left[ \frac{l+1-N}{2} \right]_{+} \right) \right] - 1, & \text{if } \sum_{i=1}^{m} (n_{i} - 1) - l \text{ is odd,} \end{cases}\]

where \([x]_{+} \overset{\text{def}}{=} \max([x], 0)\), and \(N\) is the minimal order of a letter (from \(\cup_{j \in J} A_{j}\)) of a cyclically reduced word \(u\) conjugate to \(u_{1}\) (in particular, for \(N = \infty\), the right-hand side is 0 or \(-1\)).

Proof. Without loss of generality, we can assume that the number \(|J|\) of free factors is two. Indeed, suppose that the cyclically irreducible form \(u\) of the elements \(u_{i}\) contains a letter \(a_{j} \in A_{j} \setminus \{1\}\) for some \(j \in J\). Then \(G\) decomposes into the free product \(G = A \ast B\), where \(A = A_{j}\) and \(B = \ast A_{j^{'}}\), and the conditions of the theorem remain fulfilled for this decomposition. Thus, we assume that the number of factors is two: \(G = A \ast B\).

Equality (4) allows us to draw a Howie diagram on an oriented closed (possibly non-connected) surface \(S\) of genus \(k' \overset{\text{def}}{=} \frac{1}{2}(2 - \chi(S))\) with \(l'\) exterior vertices and \(m\) cells, whose labels are \(u_{1}^{n_{1}}, \ldots, u_{m}^{n_{m}}\), where

\[k' \leq k \text{ and } 2k' + l' \leq 2k + l.\]

In detail, this construction was explained in [19]. Let us restrict ourselves to just one example. If \(k = m = 1\) and \(l = 0\), then, in most cases, we naturally obtain a torus without exterior vertices (with several interior vertices), and with one cell, whose label is \(u_{1}^{n_{1}}\) (see Figure 1, for instance); but if, e.g. commutator \(c_{1}\) has the form \(c_{1} = [a, v]\), where \(a \in A \setminus \{1\}\) and \(v \in (A \ast B) \setminus A\), then we obtain a sphere with two exterior vertices (whose labels are \(a\) and \(a^{-1}\)) and one cell, whose label is \(c_{1}\). For \(m > 1\), we can even obtain a non-connected surface (if equality (4) decomposes into a product of two equalities of the same type).
On the obtained diagram, a multiple uniform motion is naturally defined: a cell with label $u_n$ are moved around by $n_i$ cars with unit speed (one edge per minute); at moment $s \in \mathbb{Z}$, each car is at the corner whose label is the $s$th letter of the word $u$ (where $s$ is counted modulo the length of $u$).

Thus, collisions outside vertices (i.e. inside edges) cannot occur, because at each moment of time either

- all cars are at $A$-vertices,
- or all cars are at $B$-vertices,
- or each car is moving along an edge from an $A$-vertex to a $B$-vertex,
- or each car is moving along an edge from a $B$-vertex to an $A$-vertex.

A complete collision at a vertex $v$ means that all corners at this vertex have the same label equal to a letter of the word $u$. If vertex $v$ is interior, then the product of these labels must be 1, i.e. $\deg v \geq N$. Applying the corollary of the cluster lemma to this motion, we obtain the inequality

$$
\Phi(k', l') \defeq 2 - 2k' - l' + \sum_{i=1}^{m} (n_i - 1) \leq \Psi(l') \defeq \\
\begin{cases} 
2 \left[ \frac{1}{N} \left( \sum_{i=1}^{m} n_i - \left\lfloor \frac{l' + 1}{2} \right\rfloor \right) \right], & \text{if } \sum_{i=1}^{m} (n_i - 1) - l' \in 2\mathbb{Z} \\
2 \left[ \frac{1}{N} \left( \sum_{i=1}^{m} n_i - \left\lfloor \frac{l' + 1 - N}{2} \right\rfloor \right) + 1 \right], & \text{if } \sum_{i=1}^{m} (n_i - 1) - l' \notin 2\mathbb{Z}.
\end{cases}
$$

Note that

$$\Psi(l + 2) \leq \Psi(l) \text{ and } \Psi(l + 1) \leq \Psi(l) + 1 \text{ for all } l. \quad (6)$$

The first inequality is obvious; to explain the second one, we put $n \defeq \sum n_i$. Now, if $\sum(n_i - 1) - l$ is even, then

$$
\Psi(l \pm 1) \leq \Psi(l - 1) = 2 \left\lfloor \frac{1}{N} \left( n - \left\lfloor \frac{l - N}{2} \right\rfloor \right) \right\rfloor - 1 \leq 2 \left\lfloor \frac{1}{N} \left( n - \left\lfloor \frac{l - N}{2} \right\rfloor \right) \right\rfloor - 1 \\
\leq 2 \left[ \frac{1}{N} \left( n - \left\lfloor \frac{l + 1 - 2N}{2} \right\rfloor \right) \right] - 1 = 2 \left[ \frac{1}{N} \left( n - \left\lfloor \frac{l + 1}{2} \right\rfloor + N \right) \right] - 1 \\
= 2 \left[ \frac{1}{N} \left( n - \left\lfloor \frac{l + 1}{2} \right\rfloor \right) \right] + 2 - 1 = \Psi(l) + 1;
$$

if $\sum(n_i - 1) - l$ is odd, then

$$\Psi(l \pm 1) \leq \Psi(l - 1) = \left\lfloor \frac{1}{N} \left( n - \left\lfloor \frac{l - N}{2} \right\rfloor \right) \right\rfloor \leq 2 \left[ \frac{1}{N} \left( n - \left\lfloor \frac{l + 1 - N}{2} \right\rfloor \right) \right] = \Psi(l) + 1. \text{ This proves (6).}
$$

Now, if $l' \geq l$, then $\Phi(k, l) \leq \Phi(k', l')$, because $2k' + l' \leq 2k + l$ (as was noted above), and $\Psi(l') \leq \Psi(l) + 1$. Therefore, $\Phi(k, l) \leq \Phi(k', l') \leq \Psi(l') \leq \Psi(l) + 1$ and, hence,
Φ(k, l) ≤ Ψ(l), because Φ(k, l) and Ψ(l) have the same parity [see (5)]. This completes the proof in the case where l′ ≥ l.

Now, suppose that l′ < l. Note that (6) implies that the function l ↦ l + Ψ(l) is non-decreasing. Thus, the assertion of the theorem follows immediately from (5), because k′ ≤ k (as is noted above).

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