Abstract

We suggest a generalization of the dynamical triangulation approach to quantum gravity with both timelike and spacelike edges, which can serve as a toy model for quantum gravity in the Lorentz sector in two dimensions. It is possible to consider the model in a purely Lorentzian sector or to relax this constraint and allow local signature changing moves. We show that, with suitable conventions, the model is equivalent to an Ising model coupled to 2D Euclidean quantum gravity and conduct a preliminary numerical simulation of the Lorentz sector.
1 The Model

Two schools currently exist for the numerical exploration of quantum gravity which are both based on simplicial discretization of spacetime. In the dynamical triangulation approach the edge lengths on the discretized lattice are fixed and connectivities are allowed to vary whereas in the Regge calculus approach at least insofar as practical implementations are concerned, the connectivities are fixed and edge lengths are allowed to vary. In two dimensions dynamical triangulations are a direct implementation of matrix models and the results obtained are in good agreement with the theory, whereas there is currently some uncertainty as to whether the Regge calculus agrees with analytical results. Indeed, some recent results raise the question of whether two dimensions is a suitable testing ground for discretized quantum gravity. In higher dimensions, however, the Regge calculus and dynamical triangulations appear to be in at least qualitative agreement on the phase structure of quantum gravity.

All of the work alluded to above has explored discretized gravity in the Euclidean sector and its relation to Lorentzian gravity is a moot point, as is also the case in the continuum theory. In principle the Regge approach is not shackled to Euclidean space. The volume of a Regge simplex in $n$ dimensions is given by

$$V_n = \frac{1}{n!} \sqrt{\det(e_i \cdot e_j)}$$

where the vectors $e_i$ start at some basis vertex, say 0, and point to vertex $i$ thus spanning the simplex. In the two dimensional case the volume formula gives the the area of a triangle as

$$A = \frac{1}{2} \left| \frac{1}{2}(q_1 + q_2 - q_0) \frac{1}{2}(q_1 + q_2 - q_0) \right|^{1/2}$$

where $q_0, q_1, q_2$ are the squared edge lengths of the triangle. The triangle can be Lorentzian if the determinant is less than zero or Euclidean if it is greater than zero. In simulations to date using Regge calculus the edge lengths have been generally taken to be continuous variables with triangle inequalities being explicitly enforced to ensure that the triangles stay Euclidean. One interesting alternative scheme that has been proposed is the so-called Ising link quantum gravity in which only two squared edge lengths are allowed, $q = 1 + \epsilon \sigma$, where $\sigma$ is an Ising variable and $\epsilon$ is chosen to maintain Euclidean triangles ($\epsilon < 3/5$ in two dimensions).

In the spirit of the Ising link approach, we consider a generalization of dynamical triangulation with both spacelike and timelike edges, $q_{\text{space}} = +1, q_{\text{time}} = -1$. We can accommodate both Euclidean and Lorentzian triangles in this scheme: triangles with $(+1, +1, +1)$ and $(-1, -1, -1)$ edges give Euclidean areas from equ. (2), whereas triangles with either $(+1, -1, -1)$ or $(+1, +1, -1)$ edges give Lorentzian areas. It is possible to take the edge lengths squared themselves as Ising spins, but instead we will take them to be the product of Ising spins at the end of each link, which will allow us to derive some properties of our toy model of quantum gravity in the Lorentz sector from already solved Ising models in the Euclidean sector.

If we allow all four possible sorts of triangle and assign fugacities $g$ to the Euclidean triangles and $z$ to the Lorentzian triangles we can write down the following partition function for our model

$$Z = \sum_{\Delta} \sum_{\text{figs}} g^{(\# \text{ Euclidean } \Delta s)} z^{(\# \text{ Lorentzian } \Delta s)}$$

where the outer sum runs over (Euclidean) triangulations and the inner sum is over the different possible ways of soldering the various sorts of triangles together. This partition function can be retrieved as the free energy of a two-matrix model whose perturbative expansion generates our triangulated universe of mixed Lorentzian and Euclidean patches

$$U = \text{tr} \left( \frac{1}{2} S^2 + \frac{1}{2} D^2 - \frac{2}{3} S^3 - \frac{2}{3} D^3 - z S D^2 - z D S^2 \right)$$

where we use the $M \times M$ hermitian matrix $S$ to represent a spacelike edge and the $M \times M$ hermitian matrix $D$ to represent a timelike edge. The $S^3$, $D^3$ terms thus represents the Euclidean triangles, the $S D^2$, $S^2 D$ terms the Lorentzian triangles and the $S^2$ and $D^2$ terms solder together different triangles.
As it stands this model has not been solved so we make a further bold simplification and restrict ourselves by fiat to one sort of Euclidean triangle, namely (+1,+1,+1), and one sort of Lorentzian triangle, (+1,−1,−1) which leaves us with the following matrix model
\[ U = tr \left( \frac{1}{2} S^2 + \frac{1}{2} D^2 - \frac{g}{3} S^3 - z SD^2 \right). \] (5)

This two matrix model is known to represent the Ising model on dynamical triangulations with the spins at the triangle vertices, rather than the more usual formulation with the spins at triangle centres. In this context, \( S \) represents edges with similar spins at each end and \( D \) represents edges with differing spins at each end. In Figures.1–4 we denote the \( S \) edges by bold lines and the \( D \) edges by dotted lines, as well as indicating the corresponding vertex spin arrangements. After suitable scalings the Ising action on dynamical triangulations can be written as
\[ U = tr \left( \frac{1}{2} S^2 + \frac{1}{2} D^2 - g S^3 - z SD^2 \right). \] (6)

so we see that the zero temperature (\( \beta \to \infty \)) limit is the Euclidean sector when the model is regarded as Ising link quantum gravity, whereas an antiferromagnetic limit (\( \beta \to -\infty \), \( g \to 0 \) with \( z = g \exp(-2\beta) \) finite) can be regarded as the purely Lorentzian limit. In this limit the action becomes
\[ U = tr \left( \frac{1}{2} D^2 - \frac{z^2}{2} D^4 \right). \] (7)

and the gaussian integration over \( S \) may be performed, giving
\[ U = tr \left( \frac{1}{2} D^2 - \frac{z^2}{2} D^4 \right). \] (8)

which is again a pure gravity action. In geometrical terms this means that the triangulations generated in the Lorentzian limit that gives equ.(6) are those which admit a perfect matching in which the triangles are paired off into squares. This is clear from equ.(6) where the effect of integrating out the \( S \) is to give a \( D^4 \) term that generates squares.

If we take the opposite convention for the Lorentzian triangles and keep (+1,+1,+1) and (+1,+1,−1) triangles only we have the action
\[ U = tr \left( \frac{1}{2} S^2 + \frac{1}{2} D^2 - g S^3 - z DS^2 \right). \] (9)

which is now gaussian in \( D \), allowing it to be integrated out to give
\[ U = tr \left( \frac{1}{2} S^2 - \frac{g}{3} S^3 - \frac{z^2}{2} S^4 \right). \] (10)

This is an action for a universe built out of both triangles and squares. This action is known to give the Yang-Lee edge singularity on a random surface for suitably tuned \( g \) and \( z \), but generically it will be pure gravity. Again the purely Lorentzian limit is to take \( g \to 0 \) which leaves us with a theory of squares alone (but with spacelike edges, rather than the timelike edges of equ.(6)).

We have glossed over one important respect in which our toy model fails to be realistic: we have implicitly assumed that \( g \) and \( z \) are real to get the standard critical behaviour, ie \( g = \exp(-\mu) \), \( z = g(-2\beta) \), so that for a fixed topology
\[ Z \simeq \sum_n \exp(-\mu n) \cdots \] (11)

where \( n \) is the number of vertices in a triangulation. In the continuum limit the exponential gives the cosmological constant term \( \int \sqrt{g} \) without a factor of \( i \) in front, whether we are in the Euclidean sector or

\[ 1 \mathrm{A possible method for finding a solution in such models has been sketched recently in } 
\] (8). It is also worth noting that making a field redefinition for the Ising model with an external field on \( \phi^3 \) graphs gives an action with identical terms to equ.(6) but the coefficients of \( S^3 \) and \( D^3 \) are not equal, nor are those in front of \( DS^2 \) and \( SD^2 \).
the dense Lorentzian sector \( \mathbb{F} \). However, it is known that the action in equ.(5) may be defined for complex \( z \), so we can approach the standard continuum limit along a complex path, which is precisely what is required for obtaining a factor of \( i \) in the action. The action for the Yang-Lee edge singularity in equ.(10) may also be analytically continued to complex couplings. In any event, these problems are swept under the carpet in a microcanonical simulation at fixed topology as the “factors of \( i \)” only come into play when the number of vertices or the topology are changed. The problem re-emerges when one considers applying a scaling analysis to simulations of differently sized lattices. It is interesting to note in passing that a rather more sophisticated form of Regge calculus with more than two quantized edge lengths devised by Ponzano and Regge for three dimensional gravity \( \mathbb{F} \) also fails to reproduce the correct factors of \( i \).

When the continuum limit is taken in the Lorentzian sector such a theory gives no \( i \) in front of the action \( \mathbb{F} \), whereas taking the continuum limit in the Euclidean sector does give a factor of \( i \).

### 2. A Numerical Simulation

In a numerical simulation of the model the allowed moves for various sorts of edges can be deduced by looking at the Ising interpretation of the model. One new feature compared with an undecorated Euclidean triangulation is that the standard “flip” move used in simulations can have the effect of changing the sign of an edge, depending on the local vertex spin configuration, which in turn has the effect of changing the signature of one of the participating triangles. The model thus incorporates local signature changing moves. The choice of only \((+1,+1,+1)\) triangles is completely trivial in a purely Euclidean theory - it is simply a matter of convention. In a purely Lorentzian sector we can see that the choice of \((+1,-1,-1)\) triangles is compatible with the dynamical triangulation moves shown in Figure 1, neither of which change the sign of the flipped edge or the nature of the participating triangles. The further two flip moves shown in Figure 2 do change the sign of the flipped edge and exchange Lorentzian and Euclidean triangles. These signature changing moves will be present if we choose a non-zero fugacity \( g \) for the Euclidean triangles.

As we have defined the signature of an edge in terms of the Ising spins living at either end it is also possible for a changing spin to influence the local geometry. An example of this in the purely Lorentzian sector is shown in Figure 3 where flipping the central spin still keeps only \(SD^2\) triangles but rotates the bold \(S\) edges around the central point. In general any vertex with an equal number of alternating \(S\) and \(D\) edges emanating from it allows a similar sort of move, whilst still keeping the restriction to \(SD^2\) triangles only. In all cases, as we might expect given the underlying matrix model of equ.(5), \((+1,+1,-1)\) and \((-1,-1,-1)\) triangles are consistently excluded, if the starting configuration does not contain them.

For a lattice of toroidal topology, for instance, a section of a suitable starting configuration might look like Figure 4 in which horizontal spacelike lines are joined by the inclined timelike edges. Such a starting configuration gives a foliation of the spacetime manifold by spacelike lines. A lattice of spherical topology would require in addition “caps” of purely timelike edges radiating from the poles.

We conducted a simulation of the model with the action in equ.(7) (“perfect matching”) which is in a purely Lorentzian sector \( \mathbb{F} \) according to our definitions and measured the probability \( \omega(n) \) for the occurrence of vertices with coordination number \( n \). These are listed in the following table for \( n = 3, 4, \ldots 10 \) for a toroidal lattice with 1024 vertices and a statistics of 40k independent configurations obtained via 1M sweeps. The statistical error is about 0.2 percent. The columns \( \omega_L \) and \( \omega_E \) contain the numerical values for Lorentz and Euclidean sector respectively, while \( \omega_{\text{exact}} \) shows the exact result for the Euclidean sector.

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2 The curvature term, \( N^X \), that has been suppressed at fixed topology in the above also fails to display a factor of \( i \) in the continuum limit, giving \( \int \sqrt{g} R \).

3 In this sector there are no signature changing moves, only those in Fig.1 being admissible.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$n$ & $\omega_L$ & $\omega_E$ & $\omega_{exact}$ \\
\hline
3 & 0.193 & 0.210 & 0.210937... \\
4 & 0.212 & 0.197 & 0.197753... \\
5 & 0.156 & 0.155 & 0.155731... \\
6 & 0.119 & 0.116 & 0.116798... \\
7 & 0.086 & 0.086 & 0.086034... \\
8 & 0.063 & 0.062 & 0.062912... \\
9 & 0.045 & 0.045 & 0.045873... \\
10 & 0.033 & 0.033 & 0.033422... \\
\hline
\end{tabular}
\caption{Measured values of the probability $\omega(n)$ for the occurrence of vertices with coordination number $n$.}
\end{table}

We can see that the main effect in the Lorentz sector is to increase $\omega(4)$ (and to a much lesser extent $\omega(6)$) at the expense of $\omega(3)$, the $\omega$'s for larger $n$ being essentially identical. This ties in with the intuition that a perfect matching is going to be easier to achieve when there is a larger proportion of vertices with an even number of neighbours. The simulation indicates that local properties need not be identical to those of pure Euclidean gravity, even though we are still dealing with a summation over (in this case a restricted set of) triangulations.

This observation is backed up by considering the dual viewpoint of equ.(5), where one regards the $S^3$ and $SD^2$ terms in equ.(5) as representing cubic vertices rather than triangles. This recasts the model as an $N = 1$ version of the $O(N)$ loop gas on a random surface, whose critical behaviour has been investigated in some detail \[16\]. Viewed in this way the model generates $D$ loops floating in a background of $S$ edges. In this model, as well as the pure Euclidean gravity phase observed on increasing $g$ with $z$ fixed, a dense critical phase is encountered on increasing $z$ with $g$ fixed. In this the entire lattice is densely filled with $D$ loops, which translates back to the Lorentzian sector of perfectly matched triangulations in our original, non-dual, picture \[16\].

The $N$ appearing in the $O(N)$ model can conveniently be parametrized as

$$ N = 2 \cos(\theta \pi), \quad 0 \leq \theta \leq 1 $$

Looking at the scaling of various operators and demanding consistency with the continuum KPZ/DDK \[17\] results then gives (for spherical topology on the underlying Euclidean lattice) that

$$ \gamma_{\text{string}} = -\theta \quad (\text{Critical Point}) $$

$$ \gamma_{\text{string}} = -\frac{\theta}{1 - \theta} \quad (\text{Dense Phase}) $$

so in our case $N = 1$ gives $\theta = 1/3$. We recover the Ising model value of $\gamma_{\text{string}} = -1/3$ at the critical point and we find the (Euclidean) pure gravity value of $\gamma_{\text{string}} = -1/2$ throughout the dense phase that we have identified as Lorentzian.

Although $\gamma_{\text{string}} = -1/2$ in the dense/Lorentzian phase at least some of the local geometrical properties are not the same as the Euclidean theory. Watermelon operators on the form

$$ \chi_L = \langle (\text{tr}(D)^L)^2 \rangle, $$

measure the expectation values of $L$ $D$ lines pinned at two points. Their conformal dimensions $\Delta_L$ are known to satisfy

$$ \Delta_L = \frac{L}{4}(1 + \theta) - \frac{\theta}{2} \quad (\text{Critical Point}) $$

$$ \Delta_L = \frac{L}{4} - \frac{1}{2} \frac{\theta}{1 - \theta} \quad (\text{Dense Phase}) $$

As pure gravity is retrieved at the critical point for $\theta = 1/2$ and Ising link gravity still has $\theta = 1/3$ in the dense phase these scaling dimensions are not the same.

\footnote{The coalescence of the two singularities is the standard Ising model critical point.}
3 Conclusions

We have suggested that a model combining dynamical triangulations and Ising link quantum gravity might capture some of the features of the discretized microgeometry of a 2D gravity theory admitting both Lorentzian and Euclidean signatures. With suitable restrictions on the allowed triangles this model is precisely the Ising model coupled to 2D gravity. The purely Lorentzian sector appears as an antiferromagnetic limit in the Ising model, giving either a summation over perfectly matched triangulations or a mixed triangulation/quadrangulation depending on the choice of convention for the Lorentzian triangles. As the model is being used away from the Ising critical point some properties are essentially those of discretized pure Euclidean gravity based on triangulations (i.e. $\gamma_{\text{string}} = -1/2$), but non-universal features such as curvature distributions may be different, as witnessed by the simulation and the watermelon dimensions.

We have not really resolved the issue of “factors of $i$” in this paper. As noted, this does not play a role in microcanonical simulations but would have to be faced in extracting scaling information from the results. However, it is reassuring that the models of eqn. (8) and eqn. (10) representing the extreme Lorentzian limit for both possible choices of convention can be defined for complex couplings, so the standard scaling limit can be approached through complex coupling values, which is precisely what is required in our context. Modulo this caveat, the results here appear to be rather reassuring for Euclidean practitioners, as we are still dealing with a theory of triangulations, albeit decorated, in our toy Lorentzian model.

Perhaps the most important question we have left unanswered in the current work is whether the microscopic causality structure in our model, which looks very chaotic, becomes tamer at larger scales. This would be a prerequisite for extracting a sensible continuum theory. A numerical investigation of the role of signature changing would also be possible within the context of the model. Other ways of grafting a Lorentzian structure onto Euclidean lattices also suggest themselves, such as using $\phi^4$ graphs and taking the edges emanating from a vertex as the lightcone \[8\]. These might also be amenable to numerical investigation. In any event the current work suggests that the Ising antiferromagnet, or more generally the complex temperature Ising model, coupled to 2D gravity might be worthy of further investigation.

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Figure 1: The allowed flip moves in the purely Lorentzian sector. The timelike edges are shown dotted and spacelike edges solid. The spins in the underlying Ising model are also shown.
Figure 2: The two signature changing flip moves.
Figure 3: A geometry changing spin flip in the purely Lorentzian sector.
Figure 4: A possible starting configuration in the purely Lorentzian sector.