Generalized (anti) Yetter-Drinfeld modules as components of a braided T-category

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Abstract

If $H$ is a Hopf algebra with bijective antipode and $\alpha, \beta \in \text{Aut}_{\text{Hopf}}(H)$, we introduce a category $\mathcal{YD}^H(\alpha, \beta)$, generalizing both Yetter-Drinfeld modules and anti-Yetter-Drinfeld modules. We construct a braided T-category $\mathcal{YD}(H)$ having all the categories $\mathcal{YD}^H(\alpha, \beta)$ as components, which if $H$ is finite dimensional coincides with the representations of a certain quasitriangular T-coalgebra $\mathcal{D}T(H)$ that we construct. We also prove that if $(\alpha, \beta)$ admits a so-called pair in involution, then $\mathcal{YD}^H(\alpha, \beta)$ is isomorphic to the category of usual Yetter-Drinfeld modules $\mathcal{YD}^H$.

Introduction

Let $H$ be a Hopf algebra with bijective antipode $S$ and $\alpha, \beta \in \text{Aut}_{\text{Hopf}}(H)$. We introduce the concept of an $(\alpha, \beta)$-Yetter-Drinfeld module, as being a left $H$-module right $H$-comodule $M$ with the following compatibility condition:

$$(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes \beta(h_3)m_{(1)} \alpha(S^{-1}(h_1)).$$

This concept is a generalization of three kinds of objects appeared in the literature. Namely, for $\alpha = \beta = \text{id}_H$, we obtain the usual Yetter-Drinfeld modules; for $\alpha = S^2$, $\beta = \text{id}_H$, we obtain the so-called anti-Yetter-Drinfeld modules, introduced in [7], [8], [10] as coefficients for the cyclic cohomology of Hopf algebras defined by Connes and Moscovici in [5], [6]; finally, an $(\text{id}_H, \beta)$-Yetter-Drinfeld module is a generalization of the object $H_\beta$ defined in [4], which has the property that, if $H$ is finite dimensional, then the map $\beta \mapsto \text{End}(H_\beta)$ gives a group anti-homomorphism from $\text{Aut}_{\text{Hopf}}(H)$ to the Brauer group of $H$.

It is natural to expect that $(\alpha, \beta)$-Yetter-Drinfeld modules have some properties resembling the

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ones of the three kinds of objects we mentioned. We will see some of these properties in this paper
(others will be given in a subsequent one), namely the ones directed to our main aim here, which
is the following: if we denote by $H \mathcal{YD}^H(\alpha, \beta)$ the category of $(\alpha, \beta)$-Yetter-Drinfeld modules and we
define $\mathcal{YD}(H)$ as the disjoint union of all these categories, then we can organize $\mathcal{YD}(H)$ as
a braided T-category (or braided crossed group-category, in the original terminology of Turaev,
see [16]) over the group $G = Aut_{Hopf}(H) \times Aut_{Hopf}(H)$ with multiplication $(\alpha, \beta) \ast (\gamma, \delta) =
(\alpha \gamma, \delta \gamma^{-1} \beta \gamma)$. We also prove that the subcategory $\mathcal{YD}(H)_{fd}$ consisting of finite dimensional
objects has left and right dualities, and that, if $H$ is finite dimensional, then $\mathcal{YD}(H)$ coincides
with the representations of a certain quasitriangular T-coalgebra $DT(H)$ that we construct.
Our second aim is to prove that, if $\alpha, \beta \in Aut_{Hopf}(H)$ such that there exists a so-called pair in
involution $(f, g)$ corresponding to $(\alpha, \beta)$, then $H \mathcal{YD}^H(\alpha, \beta)$ is isomorphic to $H \mathcal{YD}^H$. This result
is independent on the theory concerning $\mathcal{YD}(H)$, but we can give it a very short proof using the
results obtained during the construction of $\mathcal{YD}(H)$.

1 Preliminaries

We work over a ground field $k$. All algebras, linear spaces, etc. will be over $k$; unadorned $\otimes$ means $\otimes_k$. Unless otherwise stated, $H$ will denote a Hopf algebra with bijective antipode $S$.
We will use the versions of Sweedler's sigma notation: $\Delta(h) = h_1 \otimes h_2$ or $\Delta(h) = h^{(1)} \otimes h^{(2)}$.
For unexplained concepts and notation about Hopf algebras we refer to [11], [12], [13], [15]. By
$\alpha, \beta, \gamma...$ we will usually denote Hopf automorphisms of $H$.

Let $A$ be an $H$-bicomodule algebra, with comodule structures $A \rightarrow A \otimes H$, $a \mapsto a_{<0>} \otimes a_{<1>}$ and $A \rightarrow H \otimes A$, $a \mapsto a_{<-1>} \otimes a_{[0]}$, and denote, for $a \in A$,

$$a_{<-1>} \otimes a_{[0]} \otimes a_{(1)} = a_{<0>,-1} \otimes a_{<0>[0]} \otimes a_{<1>} = a_{[-1]} \otimes a_{[0]<0>} \otimes a_{[0]<1>}$$

as an element in $H \otimes A \otimes H$. We can consider the Yetter-Drinfeld datum $(H, A, H)$ as in [2](the second $H$ is regarded as an $H$-bimodule coalgebra), and the Yetter-Drinfeld category $A \mathcal{YD}(H)$,
whose objects are $k$-modules $M$ endowed with a left $A$-action (denoted by $a \otimes m \mapsto a \cdot m$) and a
right $H$-coaction (denoted by $m \mapsto m_{(0)} \otimes m_{(1)}$) satisfying the equivalent compatibility conditions

$$\begin{align*}
(a \cdot m)_{(0)} \otimes (a \cdot m)_{(1)} &= a_{[0]} \cdot m_{(0)} \otimes a_{(1)} m_{(1)} S^{-1}(a_{[-1]}), \\
(a_{<0>} \cdot m)_{(0)} \otimes a_{<1>} m_{(1)} &= (a_{[0]} \cdot m)_{(0)} \otimes (a_{[0]} \cdot m)_{(1)} a_{[-1]},
\end{align*}$$

(1.1)

(1.2)

for all $a \in A$ and $m \in M$.

Recall now from [2] the construction of the (left) diagonal crossed product $H^* \bowtie A$, which is an
associative algebra constructed on $H^* \otimes A$, with multiplication given by

$$\begin{align*}
(p \bowtie a)(q \bowtie b) &= p(a_{<-1>}) q \leftarrow S^{-1}(a_{(1)}) \bowtie a_{[0]} b,
\end{align*}$$

(1.3)

for all $a, b \in A$ and $p, q \in H^*$, and with unit $\varepsilon_H \bowtie 1_A$. Here $\rightarrow$ and $\leftarrow$ are the regular actions
of $H$ on $H^*$ given by $(h \rightarrow p)(l) = p(lh)$ and $(p \leftarrow h)(l) = p(hl)$ for all $h, l \in H$ and $p \in H^*$.

If $H$ is finite dimensional, we can consider the Drinfeld double $D(H)$, which is a quasitriangular
Hopf algebra realized on $H^* \otimes H$; its coalgebra structure is $H^* \text{cop} \otimes H$ and the algebra structure
is just $H^* \bowtie H$, that is

$$\begin{align*}
(p \bowtie h)(q \bowtie l) &= p(h_1 \cdot q \leftarrow S^{-1}(h_3)) \bowtie h_2 l,
\end{align*}$$

(1.4)
for all \( p, q \in H^* \) and \( h, l \in H \).

The diagonal crossed product \( H^* \bowtie A \) becomes a \( D(H) \)-bicomodule algebra, with structures

\[
\begin{align*}
H^* \bowtie A & \rightarrow (H^* \bowtie A) \otimes D(H), \quad p \bowtie a \mapsto (p_2 \bowtie a_{<0>}) \otimes (p_1 \otimes a_{<1>}), \\
H^* \bowtie A & \rightarrow D(H) \otimes (H^* \bowtie A), \quad p \bowtie a \mapsto (p_2 \bowtie a_{[-1]}) \otimes (p_1 \bowtie a_{[0]}),
\end{align*}
\]

for all \( p \in H^* \) and \( a \in A \), see [9].

In the case when \( H \) is finite dimensional, by results in [1], [2] it follows that the category \( \mathcal{YD}(H)^H \) is isomorphic to the category \( H^* \bowtie A \mathcal{M} \) of left modules over \( H^* \bowtie A \).

## 2 \((\alpha, \beta)\)-Yetter-Drinfeld modules

**Definition 2.1** Let \( \alpha, \beta \in \text{Aut}_{\text{Hopf}}(H) \). An \((\alpha, \beta)\)-Yetter-Drinfeld module over \( H \) is a vector space \( M \), such that \( M \) is a left \( H \)-module (with notation \( h \otimes m \mapsto h \cdot m \)) and a right \( H \)-comodule (with notation \( M \mapsto M \otimes H, \ m \mapsto m_0 \otimes m_{(1)} \)) with the following compatibility condition:

\[
(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes \beta(h_3)m_{(1)} \alpha(S^{-1}(h_1)),
\]

(2.1)

for all \( h \in H \) and \( m \in M \). We denote by \( H\mathcal{YD}(\alpha, \beta) \) the category of \((\alpha, \beta)\)-Yetter-Drinfeld modules, morphisms being the \( H \)-linear \( H \)-colinear maps.

**Remark 2.2** As for usual Yetter-Drinfeld modules, one can see that (2.1) is equivalent to

\[
h_1 \cdot m_{(0)} \otimes \beta(h_2)m_{(1)} = (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)} \alpha(h_1).
\]

(2.2)

**Example 2.3** For \( \alpha = \beta = \text{id}_H \), we have \( H\mathcal{YD}(\text{id}, \text{id}) = H\mathcal{YD}^H \), the usual category of (left-right) Yetter-Drinfeld modules.

**Example 2.4** For \( \alpha = S^2 \), \( \beta = \text{id}_H \), the compatibility condition (2.1) becomes

\[
(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes h_3 m_{(1)} S(h_1),
\]

(2.3)

hence \( H\mathcal{YD}(S^2, \text{id}) \) is the category of anti-Yetter-Drinfeld modules defined in [7], [8], [10].

**Example 2.5** For \( \beta \in \text{Aut}_{\text{Hopf}}(H) \), define \( H_\beta \) as in [4], that is \( H_\beta = H \), with regular right \( H \)-comodule structure and left \( H \)-module structure given by \( h \cdot h' = \beta(h_2)h'S^{-1}(h_1) \), for all \( h, h' \in H \). It was noticed in [4] that \( H_\beta \) satisfies a certain compatibility condition, which actually says that \( H_\beta \in H\mathcal{YD}(\text{id}, \beta) \). More generally, if \( \alpha, \beta \in \text{Aut}_{\text{Hopf}}(H) \), define \( H_{\alpha, \beta} \) as follows:

\( H_{\alpha, \beta} = H \), with regular right \( H \)-comodule structure and left \( H \)-module structure given by \( h \cdot h' = \beta(h_2)h'\alpha(S^{-1}(h_1)) \), for \( h, h' \in H \). Then one can check that \( H_{\alpha, \beta} \in H\mathcal{YD}^H(\alpha, \beta) \).

**Example 2.6** Take \( l \) an integer and define \( \alpha_l = S^{2l} \in \text{Aut}_{\text{Hopf}}(H) \). The compatibility in \( H\mathcal{YD}(S^{2l}, \text{id}) \) becomes

\[
(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes h_3 m_{(1)} S^{2l-1}(h_1),
\]

(2.4)

An object in \( H\mathcal{YD}(S^{2l}, \text{id}) \) will be called an \( l - \mathcal{YD} \)-module. Hence, a 0 - \( \mathcal{YD} \)-module is a Yetter-Drinfeld module and a 1 - \( \mathcal{YD} \)-module is an anti-Yetter-Drinfeld module. The right-left version of \( l - \mathcal{YD} \)-modules has been introduced in [14].
Example 2.7 Let \( \alpha, \beta \in \text{Aut}_{H_{opf}}(H) \) and assume that there exist an algebra map \( f : H \to k \) and a group-like element \( g \in H \) such that

\[
\alpha(h) = g^{-1} f(h_1) \beta(h_2) f(S(h_3)) g, \quad \forall \ h \in H.
\]

Then one can check that \( k \in H \mathcal{YD}^H(\alpha, \beta) \), with structures \( h \cdot 1 = f(h) \) and \( 1 \mapsto 1 \otimes g \). More generally, if \( V \) is any vector space, then \( V \in H \mathcal{YD}^H(\alpha, \beta) \), with structures \( h \cdot v = f(h)v \) and \( v \mapsto v \otimes g \), for all \( h \in H \) and \( v \in V \).

Definition 2.8 If \( \alpha, \beta \in \text{Aut}_{H_{opf}}(H) \) such that there exist \( f, g \) as in Example 2.7, we will say that \((f, g)\) is a pair in involution corresponding to \((\alpha, \beta)\) (in analogy with the concept of modular pair in involution due to Connes and Moscovici) and the \((\alpha, \beta)\)-Yetter-Drinfeld modules \( k \) and \( V \) constructed in Example 2.7 will be denoted by \( f k^g \) and respectively \( f V^g \).

As an example, if \( \alpha \in \text{Aut}_{H_{opf}}(H) \), then \((\varepsilon, 1)\) is a pair in involution corresponding to \((\alpha, \alpha)\).

Let \( \alpha, \beta \in \text{Aut}_{H_{opf}}(H) \). We define an \( H \)-bicomodule algebra \( H(\alpha, \beta) \) as follows: \( H(\alpha, \beta) = H \) as algebra, with comodule structures

\[
H(\alpha, \beta) \to H \otimes H(\alpha, \beta), \quad h \mapsto h_{[1]} \otimes h_{[0]} = \alpha(h_1) \otimes h_2,
\]

\[
H(\alpha, \beta) \to H(\alpha, \beta) \otimes H, \quad h \mapsto h_{<0>} \otimes h_{<1>} = h_1 \otimes \beta(h_2).
\]

Then we can consider the Yetter-Drinfeld datum \((H, H(\alpha, \beta), \alpha, \beta)\) and the Yetter-Drinfeld modules over it, \( H_{(\alpha, \beta)} \mathcal{YD}(H)^H \).

Proposition 2.9 \( H \mathcal{YD}^H(\alpha, \beta) = H_{(\alpha, \beta)} \mathcal{YD}(H)^H \).

Proof. It is easy to see that the compatibility conditions for the two categories are the same. □

In particular, the category of anti-Yetter-Drinfeld modules coincides with \( H_{(S^2, id)} \mathcal{YD}(H)^H \), which improves the remark in [7] that anti-Yetter-Drinfeld modules are entwined modules.

Consider now the diagonal crossed product \( A(\alpha, \beta) = H^* \bowtie H(\alpha, \beta) \), whose multiplication is

\[
(p \bowtie h)(q \bowtie l) = p(\alpha(h_1) \otimes h_{[1]} \otimes h_{[0]} = \alpha(h_1) \otimes h_2),
\]

\[
(p \bowtie h)(q \bowtie l) = p(S(h_1) \to q \leftarrow S^{-1}(h_3)) \bowtie h_2 l,
\]

for all \( p, q \in H^* \) and \( h, l \in H \). For \( \alpha = \beta = id \) we get \( A(id, id) = D(H) \); for \( \alpha = S^2 \) and \( \beta = id \), the multiplication in \( A(S^2, id) \) is

\[
(p \bowtie h)(q \bowtie l) = p(S^2(h_1) \to q \leftarrow S^{-1}(h_3)) \bowtie h_2 l,
\]

hence \( A(S^2, id) \) coincides with the algebra \( A(H) \) defined in [7].

Assume now that \( H \) is finite dimensional; then \( A(\alpha, \beta) \) becomes a \( D(H) \)-bicomodule algebra, with structures

\[
H^* \bowtie H(\alpha, \beta) \to (H^* \bowtie H(\alpha, \beta)) \otimes D(H), \quad p \bowtie h \mapsto (p_2 \bowtie h_1) \otimes (p_1 \bowtie \beta(h_2)),
\]

\[
H^* \bowtie H(\alpha, \beta) \to D(H) \otimes (H^* \bowtie H(\alpha, \beta)), \quad p \bowtie h \mapsto (p_2 \otimes \alpha(h_1)) \otimes (p_1 \otimes h_2).
\]

In particular, \( A(H) \) becomes a \( D(H) \)-bicomodule algebra, improving the remark in [7] that \( A(H) \) is a right \( D(H) \)-comodule algebra. Since \( H \) is finite dimensional, we have an isomorphism of categories \( H_{(\alpha, \beta)} \mathcal{YD}(H)^H \simeq H^* \bowtie H(\alpha, \beta) \mathcal{M} \), hence \( H \mathcal{YD}^H(\alpha, \beta) \simeq H^* \bowtie H(\alpha, \beta) \mathcal{M} \) (for \( \alpha = S^2 \), \( \beta = id \)).
\[ \beta = \text{id} \] we recover the result in \(^7\) that the category of anti-Yetter-Drinfeld modules is isomorphic to \(A(H)M\). The correspondence is given as follows. If \(M \in \mathcal{Y}D^H(\alpha, \beta)\), then \(M \in \mathcal{Y}D^{H(\alpha, \beta)}M\) with structure
\[
(p \triangleright h) \cdot m = p((h \cdot m)(1))(h \cdot m)(0).
\]
Conversely, if \(M \in \mathcal{Y}D^{H(\alpha, \beta)}M\), then \(M \in \mathcal{Y}D^H(\alpha, \beta)\) with structures
\[
\begin{align*}
\alpha & : \quad h \cdot m = (\varepsilon \triangleright h) \cdot m, \\
m \mapsto m(0) \otimes m(1) = (e^i \triangleright 1) \cdot m \otimes e^i,
\end{align*}
\]
where \(\{e^i\}, \{e^j\}\) are dual bases in \(H\) and \(H^*\).

3 A braided T-category \(\mathcal{Y}D(H)\)

Let \(\alpha, \beta \in \text{Aut}_{H_{\text{op}}}(H)\) and consider the objects \(H_\alpha, H_\beta\) as in Example \(^5\). In \(^4\) was considered the object \(M = H_\alpha \otimes H_\beta\), with the following structures:
\[
\begin{align*}
\alpha \cdot (x \otimes y) &= h_1 \cdot x \otimes \alpha(h_2) \cdot y, \\
x \otimes y &\mapsto (x_1 \otimes y_1) \otimes y_2x_2,
\end{align*}
\]
for all \(h, x, y \in H\), where by \(\cdot\) we denoted both the actions of \(H\) on \(H_\alpha\) and \(H_\beta\) given as in Example \(^5\). Then it was noticed in \(^4\) that \(M\) satisfies a compatibility condition which says that \(M \in \mathcal{Y}D^H(\text{id}, \beta \alpha)\).

On the other hand, it was noticed in \(^7\) that the tensor product between an anti-Yetter-Drinfeld module and a Yetter-Drinfeld module becomes an anti-Yetter-Drinfeld module.

The next result can be seen as a generalization of both these facts.

**Proposition 3.1** If \(M \in \mathcal{Y}D^H(\alpha, \beta), N \in \mathcal{Y}D^H(\gamma, \delta),\) then \(M \otimes N \in \mathcal{Y}D^H(\alpha \gamma, \delta \gamma^{-1} \beta \gamma)\), with structures:
\[
\begin{align*}
\alpha \cdot (m \otimes n) &= \gamma(h_1) \cdot m \otimes \gamma^{-1} \beta \gamma(h_2) \cdot n, \\
m \otimes n &\mapsto (m \otimes n)(0) \otimes (m \otimes n)(1) = (m(0) \otimes n(0)) \otimes n(1)m(1).
\end{align*}
\]

**Proof.** Obviously \(M \otimes N\) is a left \(H\)-module and a right \(H\)-comodule. We check now the compatibility condition. We compute:
\[
\begin{align*}
(h \cdot (m \otimes n))(0) \otimes (h \cdot (m \otimes n))(1)
&= (\gamma(h_1) \cdot m \otimes \gamma^{-1} \beta \gamma(h_2) \cdot n)(0) \otimes (\gamma(h_1) \cdot m \otimes \gamma^{-1} \beta \gamma(h_2) \cdot n)(1) \\
&= ((\gamma(h_1) \cdot m)(0) \otimes (\gamma^{-1} \beta \gamma(h_2) \cdot n)(0)) \otimes (\gamma^{-1} \beta \gamma(h_2) \cdot n)(1)(\gamma(h_1) \cdot m)(1) \\
&= (h_1)(2) \cdot m(0) \otimes \gamma^{-1} \beta \gamma(h_2)(2) \cdot n(0) \otimes \\
&\quad \otimes \delta(\gamma^{-1} \beta \gamma(h_3)(2)n(1)\gamma(S^{-1}(\gamma^{-1} \beta \gamma(h_2)(2)))\beta(\gamma(h_1)(3)m(1))\alpha(S^{-1}(\gamma(h_1)(1))) \\
&= (\gamma(h_2) \cdot m(0) \otimes \gamma^{-1} \beta \gamma(h_5) \cdot n(0)) \otimes \delta(\gamma^{-1} \beta \gamma(h_6)n(1)\beta(\gamma(h_4)(4))\beta(\gamma(h_3)m(1))\alpha(S^{-1}(\gamma(h_1))))) \\
&= (h_2 \cdot m(0) \otimes \gamma^{-1} \beta \gamma(h_3) \cdot n(0)) \otimes \delta(\gamma^{-1} \beta \gamma(h_4)n(1)m(1)\alpha(S^{-1}(\gamma(h_1)))) \\
&= h_2 \cdot (m(6) \otimes n(6)) \otimes \delta(\gamma^{-1} \beta \gamma(h_3)n(1)m(1)\alpha(S^{-1}(h_1))) \\
&= h_2 \cdot (m \otimes n)(0) \otimes \delta(\gamma^{-1} \beta \gamma(h_3)(m \otimes n)(1)\alpha(S^{-1}(h_1))).
\end{align*}
\]
that is $M \otimes N \in \mathcal{YD}^H(\alpha\gamma, \delta\gamma^{-1}\beta\gamma)$.

Note that, if $M \in \mathcal{YD}^H(\alpha, \beta)$, $N \in \mathcal{YD}^H(\gamma, \delta)$ and $P \in \mathcal{YD}^H(\mu, \nu)$, then $(M \otimes N) \otimes P = M \otimes (N \otimes P)$ as objects in $\mathcal{YD}^H(\alpha\gamma\mu, \nu\mu^{-1}\delta\gamma^{-1}\beta\gamma\mu)$.

Denote $G = \text{Aut}_{H_{opf}}(H) \times \text{Aut}_{H_{opf}}(H)$, a group with multiplication

$$((\alpha, \beta), (\gamma, \delta)) = (\alpha\gamma, \delta\gamma^{-1}\beta\gamma) \quad (3.1)$$

(the unit is $(id, id)$ and $(\alpha, \beta)^{-1} = (\alpha^{-1}, \alpha^{-1}\beta^{-1})$).

**Proposition 3.2** Let $N \in \mathcal{YD}^H(\gamma, \delta)$ and $(\alpha, \beta) \in G$. Define $(\alpha, \beta)N = N$ as vector space, with structures

$$h \mapsto n = \gamma^{-1}\beta\alpha^{-1}(h) \cdot n,$$

$$n \mapsto n_{<0>} \otimes n_{<1>} = n_{(0)} \otimes \alpha\beta^{-1}(n_{(1)}).$$

Then $(\alpha, \beta)N \in \mathcal{YD}^H(\alpha\gamma\alpha^{-1}, \alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1}) = \mathcal{YD}^H((\alpha, \beta) \ast (\gamma, \delta) \ast (\alpha, \beta)^{-1})$.

**Proof.** Obviously $(\alpha, \beta)N$ is a left $H$-module and right $H$-comodule, so we check the compatibility condition. We compute:

$$(h \mapsto n)<_{<0>} \otimes (h \mapsto n)<_{<1>}
\begin{align*}
&\quad = (\gamma^{-1}\beta\alpha^{-1}(h) \cdot n)_{(0)} \otimes \alpha\beta^{-1}((\gamma^{-1}\beta\alpha^{-1}(h) \cdot n)_{(1)}) \\
&\quad = \gamma^{-1}\beta\alpha^{-1}(h_2) \cdot n_{(0)} \otimes \alpha\beta^{-1}(\delta\gamma^{-1}\beta\alpha^{-1}(h_3) n_{(1)} \cdot \gamma^{-1}\beta\alpha^{-1}(S^{-1}(h_1))) \\
&\quad = \gamma^{-1}\beta\alpha^{-1}(h_2) \cdot n_{(0)} \otimes \alpha\beta^{-1}\delta\gamma^{-1}\beta\alpha^{-1}(h_3) \cdot n_{(1)} \cdot \alpha\gamma\alpha^{-1}(S^{-1}(h_1)) \\
&\quad = h_2 \mapsto n_{(0)} \otimes \alpha\beta^{-1}\delta\gamma^{-1}\beta\alpha^{-1}(h_3) n_{<0>} \otimes \alpha\gamma\alpha^{-1}(S^{-1}(h_1)),
\end{align*}$$

that is $(\alpha, \beta)N \in \mathcal{YD}^H(\alpha\gamma\alpha^{-1}, \alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1})$.

**Remark 3.3** Let $M \in \mathcal{YD}^H(\alpha, \beta)$, $N \in \mathcal{YD}^H(\gamma, \delta)$ and $(\mu, \nu) \in G$. Then we have

$$(\alpha, \beta) \ast (\mu, \nu) N = (\alpha, \beta)(\mu, \nu) N$$

as objects in $\mathcal{YD}^H(\alpha\mu\gamma\mu^{-1}\alpha^{-1}, \alpha\beta^{-1}\mu\nu^{-1}\delta\gamma^{-1}\nu\mu^{-1}\beta\gamma\mu^{-1}\alpha^{-1})$, and

$$(\mu, \nu)(M \otimes N) = (\mu, \nu) M \otimes (\mu, \nu) N$$

as objects in $\mathcal{YD}^H(\mu\alpha\gamma\mu^{-1}, \mu\nu^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1} \nu\alpha\gamma\mu^{-1})$.

**Proposition 3.4** Let $M \in \mathcal{YD}^H(\alpha, \beta)$ and $N \in \mathcal{YD}^H(\gamma, \delta)$. Define $MN = (\alpha, \beta)N$ as object in $\mathcal{YD}^H((\alpha, \beta) \ast (\gamma, \delta) \ast (\alpha, \beta)^{-1})$. Define the map

$$c_{M,N} : M \otimes N \to MN \otimes M, \quad c_{M,N}(m \otimes n) = n_{(0)} \otimes \beta^{-1}(n_{(1)}) \cdot m.$$

Then $c_{M,N}$ is $H$-linear $H$-colinear and satisfies the conditions (for $P \in \mathcal{YD}^H(\mu, \nu)$):

$$c_{M \otimes N, P} = (c_{M,N} \otimes id_P) \circ (id_M \otimes c_{N,P}), \quad (3.2)$$

$$c_{M,N \otimes P} = (id_{MN} \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P). \quad (3.3)$$

Moreover, if $M \in \mathcal{YD}^H(\alpha, \beta)$, $N \in \mathcal{YD}^H(\gamma, \delta)$ and $(\mu, \nu) \in G$, then $c_{(\mu, \nu)M,(\mu, \nu)N} = c_{M,N}$. 

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Proof. We prove that \(c_{M,N}\) is \(H\)-linear. We compute:
\[
c_{M,N}(h \cdot (m \otimes n)) = c_{M,N}((\gamma h_1) \cdot m \otimes \gamma^{-1}\beta \gamma h_2 \cdot n)
\]
\[
= ((\gamma^{-1}\beta \gamma h_2) \cdot n)(0) \otimes \beta^{-1}((\gamma^{-1}\beta \gamma h_2) \cdot n)(1) \gamma h_1 \cdot m
\]
\[
= \gamma^{-1}\beta \gamma h_2 \cdot n(0) \otimes \beta^{-1}(\delta \gamma^{-1}\beta \gamma h_2) n(1) \gamma (S^{-1}((\gamma^{-1}\beta \gamma h_2)1)) \gamma h_1 \cdot m
\]
\[
= \gamma^{-1}\beta \gamma h_3 \cdot n(0) \otimes \beta^{-1}\delta \gamma^{-1}\beta \gamma h_4 \beta^{-1}(n(1)) \gamma (S^{-1}(h_2)) \gamma h_1 \cdot m
\]
\[
= \gamma^{-1}\beta \gamma h_1 \cdot n(0) \otimes \beta^{-1}\delta \gamma^{-1}\beta \gamma h_2 \beta^{-1}(n(1)) \cdot m,
\]
so the two terms are equal. The fact that \(c_{M,N}\) is \(H\)-colinear is similar and left to the reader.

We prove now (3.3). First note that, due to Remark 3.3, we have \(M \otimes N P = M \otimes N P\) and \(M(N \otimes P) = M \otimes N \otimes M P\). We compute:
\[
(c_{M,N} \otimes \text{id}_N) \circ (\text{id}_M \otimes c_{N,P})(m \otimes n \otimes p) = c_{M,N} \otimes \text{id}_N(m \otimes p(0)) \otimes \delta^{-1}(p(1)) \cdot n
\]
\[
= p(0,0) \otimes \beta^{-1}(p(0,1)) \cdot m \otimes \delta^{-1}(p(1)) \cdot n
\]
\[
= p(0,0) \otimes \beta^{-1}\gamma \delta^{-1}(p(0,1)) \cdot m \otimes \delta^{-1}(p(1)) \cdot n
\]
\[
= p(0) \otimes \beta^{-1}\gamma \delta^{-1}(p(1)) \cdot m \otimes \delta^{-1}(p(1)) \cdot n
\]
and we are done. The proof of (3.3) is easier and left to the reader, and similarly the last statement of the Proposition.

Note that \(c_{M,N}\) is bijective with inverse \(c_{M,N}^{-1}(m \otimes n) = \beta^{-1}(S(n(1))) \cdot m \otimes n(0)\).

We are ready now to introduce the desired braided T-category (we use terminology as in [18]; for the subject of Turaev categories, see also the original paper of Turaev [16] and [2], [17]).

Define \(\mathcal{YD}(H)\) as the disjoint union of all \(H\mathcal{YD}^H(\alpha, \beta)\), with \((\alpha, \beta) \in G\) (hence the component of the unit is just \(H\mathcal{YD}^H\)). If we endow \(\mathcal{YD}(H)\) with tensor product as in Proposition 3.1, then it becomes a strict monoidal category with unit \(k\) as object in \(H\mathcal{YD}^H\) (with trivial structures). The group homomorphism \(\varphi : G \rightarrow aut(\mathcal{YD}(H))\), \((\alpha, \beta) \mapsto \varphi_{(\alpha, \beta)}\), is given on components as
\[
\varphi_{(\alpha, \beta)} : H\mathcal{YD}^H(\gamma, \delta) \rightarrow H\mathcal{YD}^H((\alpha, \beta) * (\gamma, \delta) * (\alpha, \beta)^{-1}), \quad \varphi_{(\alpha, \beta)}(N) = (\alpha, \beta) N
\]
and the functor \(\varphi_{(\alpha, \beta)}\) acts as identity on morphisms. The braiding in \(\mathcal{YD}(H)\) is given by the family \(\{c_{M,N}\}\). As a consequence of the above results, we obtain:

**Theorem 3.5** \(\mathcal{YD}(H)\) is a braided T-category over \(G\).
We consider now the problem of existence of left and right dualities.

**Proposition 3.6** Let \( M \in H\mathcal{YD}^H(\alpha, \beta) \) and assume that \( M \) is finite dimensional. Then \( M^* = \text{Hom}(M, k) \) becomes an object in \( H\mathcal{YD}^H(\alpha^{-1}, \alpha\beta^{-1}\alpha^{-1}) \), with \( (h \cdot f)(m) = f((\beta^{-1}\alpha^{-1}S(h)) \cdot m) \) and \( f(0)(m) \otimes f(1) = f(m(0)) \otimes S^{-1}(m(1)) \). Moreover, the maps \( b_M : k \rightarrow M \otimes M^* \), \( b_M(1) = \sum_i e_i \otimes e^i \) (where \( \{e_i\} \) and \( \{e^i\} \) are dual bases in \( M \) and \( M^* \)) and \( d_M : M^* \otimes M \rightarrow k \), \( d_M(f \otimes m) = f(m) \), are morphisms in \( H\mathcal{YD}^H \) and we have \( \text{id}_M \otimes \text{id}_M \) and \( (d_M \otimes d_M)(\text{id}_M \otimes \text{id}_M) = \text{id}_M \) and \( (d_M \otimes d_M)(\text{id}_M \otimes b_M) = \text{id}_{M^*} \).

**Proof.** We first prove that \( M^* \) is indeed an object in \( H\mathcal{YD}^H(\alpha^{-1}, \alpha\beta^{-1}\alpha^{-1}) \). We compute:

\[
(h \cdot f)(0)(m) \otimes (h \cdot f)(1) = (h \cdot f)(m(0)) \otimes S^{-1}(m(1))
\]
\[
= f((\beta^{-1}\alpha^{-1}S(h) \cdot m)) \otimes (\alpha\beta^{-1}\alpha^{-1})(h(3)) \cdot (\alpha^{-1}S^{-1})(h(1))
\]
\[
= f((\beta^{-1}\alpha^{-1}S(h(2)) \cdot m(0)) \otimes (\alpha\beta^{-1}\alpha^{-1})(h(3)))
\]
\[
S^{-1}((\beta^{-1}\alpha^{-1}S(h(2)) \cdot m(1)))(\alpha^{-1}S^{-1}(h(1)))
\]
\[
= f((\beta^{-1}\alpha^{-1}S)(h(3)) \cdot m(0)) \otimes (\alpha\beta^{-1}\alpha^{-1})(h(5))
\]
\[
S^{-1}((\alpha^{-1}S)(h(2)) \cdot m(1))(\alpha\beta^{-1}\alpha^{-1})(h(4))) \cdot (\alpha^{-1}S^{-1}(h(1)))
\]
\[
= f((\beta^{-1}\alpha^{-1}S)(h(3)) \cdot m(0)) \otimes (\alpha\beta^{-1}\alpha^{-1})(h(5)) \cdot S^{-1}(m(1)) \cdot \alpha^{-1}(h(2)) \cdot S^{-1}(h(1))
\]
\[
= f((\beta^{-1}\alpha^{-1}S)(h) \cdot m(0)) \otimes S^{-1}(m(1)),
\]

which means that

\[
(h \cdot f)(0) \otimes (h \cdot f)(1) = (h(2) \cdot f(0)) \otimes (\alpha\beta^{-1}\alpha^{-1})(h(3)) \cdot f(1)(\alpha^{-1}S^{-1})(h(1)), \quad \text{q.e.d.}
\]

On \( k \) we have the trivial module and comodule structure, and with these \( k \in H\mathcal{YD}^H \). We want to prove that \( b_M \) and \( d_M \) are \( H \)-module maps. We compute:

\[
(h \cdot b_M(1))(m) = (h \cdot (\sum_i e_i \otimes e^i))(m)
\]
\[
= \sum_i \alpha^{-1}(h(1)) \cdot e_i \otimes ((\alpha\beta\alpha^{-1})(h(2)) \cdot e^i)(m)
\]
\[
= \alpha^{-1}(h(1)) \cdot e_i \otimes e^i((\beta^{-1}\alpha^{-1}S\alpha\beta\alpha^{-1})(h(2)) \cdot m)
\]
\[
= \sum_i \alpha^{-1}(h(1)) \cdot e_i \otimes e^i((\alpha^{-1}S)(h(2)) \cdot m)
\]
\[
= \alpha^{-1}(h(1))S(h(2)) \cdot m
\]
\[
= \varepsilon(h) \sum_i e_i \otimes e^i(m)
\]
\[
= (\varepsilon(h)b_M(1))(m),
\]

\[
d_M(h \cdot (f \otimes m)) = d_M(\alpha(h(1)) \cdot f \otimes \beta^{-1}(h(2)) \cdot m)
\]
\[
= (\alpha(h(1)) \cdot f)(\beta^{-1}(h(2)) \cdot m)
\]
\[
= f((\beta^{-1}\alpha^{-1}S\alpha(h(1)))\beta^{-1}(h(2)) \cdot m)
\]
\[
= f(\beta^{-1}(S(h(1))h(2)) \cdot m)
\]
\[
= \varepsilon(h)d_M(f \otimes m).
\]
Also they are $H$-comodule maps:

\[
((b_M(1))(0) \otimes (b_M(1))(1))(m) = \sum_i (e_i)(0) \otimes (e^i)(0)(m) \otimes (e_i)(1)(e_i)(1)
\]

\[
= \sum_i (e_i)(0) \otimes (e^i)(m(0)) \otimes S^{-1}(m(1))(e_i)(1)
\]

\[
= m(0) \otimes S^{-1}(m(1)_2)m(1)_1
\]

\[
= (b_M(1) \otimes 1)(m),
\]

\[
d_M((f \otimes m)(0)) \otimes (f \otimes m)(1) = f(0)(m(0)) \otimes m(1)f(1)
\]

\[
= f(m(0)) \otimes m(1)_2S^{-1}(m(1)_1)
\]

\[
= d_M(f \otimes m) \otimes 1.
\]

Finally, the last two identities $(id_M \otimes d_M)(b_M \otimes id_M) = id_M$ and $(d_M \otimes id_M^*)(id_{M^*} \otimes b_M) = id_{M^*}$ are trivial. □

Similarly, one can prove:

**Proposition 3.7** Let $M \in H\mathcal{YD}^H(\alpha, \beta)$ and assume that $M$ is finite dimensional. Then $^*M = \text{Hom}(M, k)$ becomes an object in $H\mathcal{YD}^H(\alpha^{-1}, \alpha\beta^{-1}\alpha^{-1})$, with $(h \cdot f)(m) = f((\alpha^{-1}\beta^{-1}S^{-1}(h)) \cdot m)$ and $f(0)(m) \otimes f(1) = f(m(0)) \otimes S(m(1))$. Moreover, the maps $b_M : k \rightarrow ^*M \otimes M$, $b_M(1) = \sum_i e_i^i \otimes e_i$ and $d_M : M \otimes ^*M \rightarrow k$, $d_M(m \otimes f) = f(m)$, are morphisms in $H\mathcal{YD}^H$ and we have $(d_M \otimes id_M)(id_M \otimes b_M) = id_M$ and $(id_{M^*} \otimes d_M)(b_M \otimes id_{M^*}) = id_{M^*}$.

Consequently, if we consider $\mathcal{YD}(H)_{fd}$, the subcategory of $\mathcal{YD}(H)$ consisting of finite dimensional objects, we obtain:

**Theorem 3.8** $\mathcal{YD}(H)_{fd}$ is a braided T-category with left and right dualities over $G$, the left (respectively right) duals being given as in Proposition 3.6 (respectively Proposition 3.7).

Assume now that $H$ is finite dimensional. We will construct a quasitriangular T-coalgebra over $G$, denoted by $DT(H)$, with the property that the T-category $\text{Rep}(DT(H))$ of representations of $DT(H)$ is isomorphic to $\mathcal{YD}(H)$ as braided T-categories.

For $(\alpha, \beta) \in G$, the $(\alpha, \beta)$-component $DT(H)(\alpha, \beta)$ will be the diagonal crossed product algebra $H^* \bowtie H(\alpha, \beta)$. Define

\[
\Delta_{(\alpha, \beta), (\gamma, \delta)} : H^* \bowtie H((\alpha, \beta) * (\gamma, \delta)) \rightarrow (H^* \bowtie H(\alpha, \beta)) \otimes (H^* \bowtie H(\gamma, \delta)),
\]

\[
\Delta_{(\alpha, \beta), (\gamma, \delta)}(p \bowtie h) = (p_2 \bowtie \gamma(h_1)) \otimes (p_1 \bowtie \gamma^{-1}\beta\gamma(h_2)).
\]

One can check, by direct computation, that these maps are algebra maps, satisfying the necessary coassociativity conditions.

The counit $\varepsilon$ is just the counit of $DT(H)_{(id, id)} = D(H)$, the Drinfeld double of $H$.

For $(\alpha, \beta), (\gamma, \delta) \in G$, define now

\[
\varphi_{(\alpha, \beta), (\gamma, \delta)} : H^* \bowtie H(\gamma, \delta) \rightarrow H^* \bowtie H((\alpha, \beta) * (\gamma, \delta) * (\alpha, \beta)^{-1}),
\]

\[
\varphi_{(\alpha, \beta), (\gamma, \delta)}(p \bowtie h) = p \circ \beta^{-1} \bowtie \alpha \gamma^{-1}\beta\gamma(h).
\]
Moreover, the structure of \( DT(H) \) for all \((\alpha, \beta) \in G\), by

\[
S_{(\alpha, \beta)} : H^* \bowtie H(\alpha, \beta) \to H^* \bowtie H((\alpha, \beta)^{-1}),
\]

\[
S_{(\alpha, \beta)}(p \bowtie h) = (\varepsilon \bowtie \alpha \beta(S(h))) \cdot (S^{*^{-1}}(p) \bowtie 1),
\]

where the multiplication \( \cdot \) in the right hand side is made in \( H^* \bowtie H((\alpha, \beta)^{-1}) \).

Finally, the universal \( R \)-matrix is given by

\[
R_{(\alpha, \beta), (\gamma, \delta)} = \sum_i (\varepsilon \bowtie \beta^{-1}(e_i)) \otimes (e_i \bowtie 1) \in (H^* \bowtie H(\alpha, \beta)) \otimes (H^* \bowtie H(\gamma, \delta)),
\]

for all \((\alpha, \beta), (\gamma, \delta) \in G\), where \( \{e_i\} \) are dual bases in \( H \) and \( H^* \).

Thus, we have obtained:

**Theorem 3.9** \( DT(H) \) is a quasitriangular \( T \)-coalgebra over \( G \), with structure as above.

Moreover, the structure of \( DT(H) \) was constructed in such a way that, via the isomorphisms 

\[
H^* \bowtie H(\alpha, \beta) \mathcal{M} \simeq H \mathcal{YD}^H(\alpha, \beta)
\]

from Section 2, we obtain:

**Theorem 3.10** \( \text{Rep}(DT(H)) \) and \( \mathcal{YD}(H) \) are isomorphic as braided \( T \)-categories over \( G \).

**Remark 3.11** From \( \mathcal{YD}(H) \) (respectively \( DT(H) \)) we can obtain, by pull-back along the group morphism 

\[
\text{Aut}_{\text{Hopf}}(H) \to G, \alpha \mapsto (\alpha, \text{id})
\]

a braided \( T \)-category (respectively a quasitriangular \( T \)-coalgebra) over \( \text{Aut}_{\text{Hopf}}(H) \). If \( \pi \) is a group together with a group morphism \( \pi \to \text{Aut}_{\text{Hopf}}(H) \), by pull-back along it we obtain a braided \( T \)-category (respectively a quasitriangular \( T \)-coalgebra) over \( \pi \). Quasitriangular \( T \)-coalgebras over such \( \pi \) have been studied by Virelizier in \cite{17}.

### 4 An isomorphism of categories \( H \mathcal{YD}^H(\alpha, \beta) \simeq H \mathcal{YD}^H \)

in the presence of a pair in involution

The aim of this section is to prove the following result.

**Theorem 4.1** Let \( \alpha, \beta \in \text{Aut}_{\text{Hopf}}(H) \) and assume that there exists \((f, g)\) a pair in involution corresponding to \((\alpha, \beta)\). Then the categories \( H \mathcal{YD}^H(\alpha, \beta) \) and \( H \mathcal{YD}^H \) are isomorphic.

A pair of inverse functors \((F, G)\) is given as follows. If \( M \in H \mathcal{YD}^H(\alpha, \beta) \), then \( F(M) \in H \mathcal{YD}^H \), where \( F(M) = M \) as vector space, with structures

\[
h \to m = f(\beta^{-1}(S(h_1)))\beta^{-1}(h_2) \cdot m,
\]

\[
m \mapsto m_{<0>} \otimes m_{<1>} = m_{(0)} \otimes m_{(1)} g^{-1}.
\]

If \( N \in H \mathcal{YD}^H \), then \( G(N) \in H \mathcal{YD}^H(\alpha, \beta) \), where \( G(N) = N \) as vector space, with structures

\[
h \to n = f(h_1)\beta(h_2) \cdot n,
\]

\[
n \mapsto n^{(0)} \otimes n^{(1)} = n_{(0)} \otimes n_{(1)} g.
\]

Both \( F \) and \( G \) act as identities on morphisms.
Proof. One checks, by direct computation, that $F$ and $G$ are functors, inverse to each other. Alternatively, we can give a very short proof using results from the previous section. By Example 2.7 we have $f^g \in H\mathcal{YD}^H(\alpha, \beta)$. By Proposition 3.6 we get $(f^g)^\ast \in H\mathcal{YD}^H((\alpha, \beta)^{-1})$. Then, one can check that actually $F(M) = (f^g)^\ast \otimes M \in H\mathcal{YD}^H$ and $G(N) = (f^g) \otimes N \in H\mathcal{YD}^H(\alpha, \beta)$. Also, one can see that $(f^g)^\ast \otimes (f^g) = f^g \otimes (f^g)^\ast = k$ as objects in $H\mathcal{YD}^H$, hence $F \circ G = G \circ F = id$, using the associativity of the tensor product. □

As we have noticed before, for any $\alpha \in Aut_{Hopf}(H)$ we have that $(\varepsilon, 1)$ is a pair in involution corresponding to $(\alpha, \alpha)$, hence we obtain:

**Corollary 4.2** $H\mathcal{YD}^H(\alpha, \alpha) \simeq H\mathcal{YD}^H$.

Also, as a consequence of the theorem, we obtain the following result (a right-left version was given in [13]), which might be useful for the area of applicability of anti-Yetter-Drinfeld modules:

**Corollary 4.3** Assume that there exists a pair in involution $(f, g)$ corresponding to $(S^2, id)$. Then the category $H\mathcal{YD}^H(S^2, id)$ of anti-Yetter-Drinfeld modules is isomorphic to $H\mathcal{YD}^H$, and any anti-Yetter-Drinfeld module can be written as a tensor product $f^g \otimes N$, with $N \in H\mathcal{YD}^H$.

Let again $\alpha, \beta \in Aut_{Hopf}(H)$ such that there exists $(f, g)$ a pair in involution corresponding to $(\alpha, \beta)$, and assume that $H$ is finite dimensional. Then we know that $H\mathcal{YD}^H(\alpha, \beta) \simeq H \bowtie H(\alpha, \beta)\mathcal{M}$, $H\mathcal{YD}^H \simeq D(H)\mathcal{M}$, and the isomorphism $H\mathcal{YD}^H(\alpha, \beta) \simeq H\mathcal{YD}^H$ constructed in the theorem is induced by an algebra isomorphism between $H^* \bowtie H(\alpha, \beta)$ and $D(H)$, given by

$D(H) \rightarrow H^* \bowtie H(\alpha, \beta), \quad p \otimes h \mapsto g^{-1} \rightarrow p \bowtie f(\beta^{-1}(S(h_1)))\beta^{-1}(h_2)$,

$H^* \bowtie H(\alpha, \beta) \rightarrow D(H), \quad p \bowtie h \mapsto g \rightarrow p \otimes f(h_1)\beta(h_2)$.

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