On the textures of neutrino mass matrix for maximal atmospheric mixing angle and leptonic CP violation

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Abstract

In this paper, we derive in a novel approach the possible textures of neutrino mass matrix that can naturally lead us to maximal atmospheric mixing angle and leptonic CP violation which are consistent with the current neutrino oscillation data. A total of eleven textures are thus found. Interestingly, the specific texture given by the neutrino $\mu$-$\tau$ reflection symmetry can be reproduced from one of the obtained textures. For these textures, some neutrino mass sum rules which relate the neutrino masses and Majorana CP phases will emerge.

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1 Introduction

Thanks to the enormous neutrino oscillation data, a framework of three-flavor neutrino mixing has been established \cite{ref1}. In the usually used basis of charged lepton fields \( M_l \) being diagonal which is also adopted here, the neutrino mixing matrix \( U \) \cite{ref2} originates from diagonalization of the neutrino mass matrix \( M_\nu \) in a manner as

\[
U^\dagger M_\nu U^* = \text{Diag} \left( m_1, m_2, m_3 \right),
\]

with \( m_i \) (for \( i = 1, 2, 3 \)) being the neutrino masses. In the standard parameterization, \( U \) reads

\[
U = P_l O_{23} U_{13} O_{12} P_\nu,
\]

where \( P_l = \text{Diag} \left( e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau} \right) \) and \( P_\nu = \text{Diag} \left( e^{i\sigma}, e^{i\sigma}, 1 \right) \) are two diagonal phase matrices, and

\[
O_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}, \quad O_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Here we have used the standard notations \( c_{ij} = \cos \theta_{ij} \) and \( s_{ij} = \sin \theta_{ij} \) for the mixing angles \( \theta_{ij} \) (for \( ij = 12, 13, 23 \)). As for the phases, \( \delta \) is known as the Dirac CP phase and responsible for the CP violation effects in neutrino oscillations, while \( \rho \) and \( \sigma \) are known as the Majorana CP phases and control the rates of neutrinoless double beta decays which are used to testify the Majorana nature of neutrinos. \( \phi_{e,\mu,\tau} \) are called unphysical phases since they can be removed by the redefinitions of charged lepton fields. In addition, neutrino oscillations are also dependent on the neutrino mass squared differences \( \Delta m^2_{ij} = m_i^2 - m_j^2 \) (for \( ij = 21, 31 \)).

The experimental results for the neutrino masses are given by \cite{ref3}

\[
\Delta m^2_{21} = \left( 7.50^{+0.19}_{-0.17} \right) \times 10^{-5} \text{ eV}^2, \quad |\Delta m^2_{31}| = \left( 2.524^{+0.039}_{-0.040} \right) \times 10^{-3} \text{ eV}^2.
\]

Note that the sign of \( \Delta m^2_{31} \) has not yet been determined, thereby allowing for two possible neutrino mass orderings: \( m_1 < m_2 < m_3 \) (the normal hierarchy and NH for short) and \( m_3 < m_1 < m_2 \) (the inverted hierarchy and IH for short). And the absolute neutrino mass scale or the lightest neutrino mass \((m_1 \text{ in the NH case or } m_3 \text{ in the IH case})\) remains unknown. On the other hand, the mixing parameters \( \theta_{13}, \theta_{23} \) and \( \delta \) take the values

\[
\sin^2 \theta_{13} = 0.02166 \pm 0.00075, \quad \sin^2 \theta_{23} = 0.441 \pm 0.024, \quad \delta = 261^\circ \pm 55^\circ,
\]

in the NH case, or

\[
\sin^2 \theta_{13} = 0.02179 \pm 0.00076, \quad \sin^2 \theta_{23} = 0.587 \pm 0.022, \quad \delta = 277^\circ \pm 43^\circ,
\]

in the IH case, while \( \theta_{12} \) takes the value \( \sin^2 \theta_{12} = 0.306 \pm 0.012 \) in either case \cite{ref3}. However, information about \( \rho \) and \( \sigma \) is still lacking.

It is interesting to note that the current neutrino oscillation data is consistent with maximal atmospheric mixing angle \( (\theta_{23} = \pi/4) \) and leptonic CP violation \( (\delta = -\pi/2) \). These remarkable relations may point towards some special texture of \( M_\nu \). In this regard, the specific texture given by the neutrino \( \mu-\tau \) reflection symmetry \cite{ref4}-\cite{ref6} serves as a unique example. It is defined by

\[
M_{e\mu} = M_{e\tau}^*, \quad M_{\mu\mu} = M_{\tau\tau}^*, \quad M_{ee} \text{ and } M_{\mu\tau} \text{ being real},
\]

where \( M_{\alpha\beta} \) denotes the \( \alpha\beta \) element of \( M_\nu \) (for \( \alpha, \beta = e, \mu, \tau \)), and can be attributed to the invariance of \( M_\nu \) under a combination of the \( \mu-\tau \) interchange and CP conjugate operations

\[
\nu_e \leftrightarrow \nu_e^c, \quad \nu_\mu \leftrightarrow \nu_\tau^c, \quad \nu_\tau \leftrightarrow \nu_\mu^c.
\]
Such a texture gives $\theta_{23} = \pi/4$ and $\delta = \pm\pi/2$ as well as $\phi_c = \pi/2$, $\phi_\mu = -\phi_\tau$ and trivial Majorana CP phases (i.e., $\rho, \sigma = 0$ or $\pi/2$) [7].

The purpose of this paper is to derive in a novel approach the possible textures of neutrino mass matrix that can naturally generate maximal atmospheric mixing angle and leptonic CP violation [3]. Such a study may help us reveal the underlying flavor symmetries in the lepton sector. A total of eleven textures are thus found. Interestingly, one of the obtained textures can reproduce the matrix that can naturally generate maximal atmospheric mixing angle and leptonic CP violation.

The approach discussed one by one in some detail. Finally, a summary of our main results is given in section 4.

## 2 The approach

A $3 \times 3$ complex symmetric neutrino mass matrix generally contains twelve degrees of freedom (dfs). After the diagonalization process, three dfs will emerge as the unphysical phases $\phi_{e\mu,\tau}$ while nine dfs as the physical parameters $\theta_{ij}$, $\delta$, $\rho$, $\sigma$ and $m_i$. Therefore, one would suffer some uncertainties due to the unphysical phases when retrodicting the textures of $M_\nu$ based on the physical parameters. In comparison, the effective neutrino mass matrix $\bar{M}_\nu = P^T_1 M_\nu P^*_1$ where the unphysical dfs cancel out only consists of nine physical dfs. For this reason, we choose to work on $\bar{M}_\nu$ instead of $M_\nu$ itself so that the uncertainties created by the unphysical phases can be evaded. Two immediate comments are given as follows: (1), One can recover the results for $M_\nu$ from those for $\bar{M}_\nu$ by simply making the replacements $\bar{M}_{\alpha\beta} = M_{\alpha\beta}e^{-i(\alpha+\beta)}$ with $M_{\alpha\beta}$ being the $\alpha\beta$ element of $\bar{M}_\nu$. (2), Since $\bar{M}_\nu$ only has nine dfs, its twelve components $\bar{R}_{\alpha\beta} = \text{Re}(\bar{M}_{\alpha\beta})$ and $\bar{I}_{\alpha\beta} = \text{Im}(\bar{M}_{\alpha\beta})$ are not all independent but subject to three constraint equations.

To proceed, we diagonalize $\bar{M}_\nu$ to give the expressions for the physical parameters in terms of $\bar{R}_{\alpha\beta} = \text{Re}(\bar{M}_{\alpha\beta})$ and $\bar{I}_{\alpha\beta} = \text{Im}(\bar{M}_{\alpha\beta})$. From Eqs. (12), one finds

$$O^T_{12} U_{13}^T \bar{M}_\nu O_{23} U_{13}^* O_{12} = \text{Diag} (m_1 e^{2i\rho}, m_2 e^{2i\sigma}, m_3) .$$

In order to simplify the related expressions, we define the following matrices after the rotations $O_{23}$, $U_{13}$ and $O_{12}$ are implemented in succession

$$M_1 = O^T_{23} \bar{M}_\nu O_{23} , \quad M_2 = O_{13}^T \bar{M}_1 O_{13}^* , \quad M_3 = O_{12}^T \bar{M}_2 O_{12} .$$

By taking $\theta_{23} = \pi/4$ and $\delta = -\pi/2$, the elements of these matrices are obtained as

$$M_1^{11} = \bar{M}_{ee} , \quad M_1^{12} = \frac{\bar{M}_{e\mu} - \bar{M}_{e\tau}}{\sqrt{2}} , \quad M_1^{13} = \frac{\bar{M}_{e\mu} + \bar{M}_{e\tau}}{\sqrt{2}} ,$$

$$M_1^{22} = \frac{\bar{M}_{\mu\mu} + \bar{M}_{\tau\tau} - \bar{M}_{\mu\tau}}{2} , \quad M_1^{23} = \frac{\bar{M}_{\mu\mu} - \bar{M}_{\tau\tau}}{2} , \quad M_1^{33} = \frac{\bar{M}_{\mu\mu} + \bar{M}_{\tau\tau} + \bar{M}_{\mu\tau}}{2} ,$$

and

$$M_2^{11} = c_{13} M_1^{11} - i \sin 2\theta_{13} M_1^{13} - s_{13} M_1^{33} , \quad M_2^{12} = c_{13} M_1^{12} - i s_{13} M_1^{23} ,$$

$$M_2^{13} = \cos 2\theta_{13} M_1^{13} - \frac{i}{2} \sin 2\theta_{13} (M_1^{11} + M_1^{33}) , \quad M_2^{22} = M_1^{22} ,$$

$$M_2^{23} = c_{13} M_1^{23} - i s_{13} M_1^{12} , \quad M_2^{33} = c_{13} M_1^{33} - i \sin 2\theta_{13} M_1^{13} - s_{13}^2 M_1^{11} ,$$

and

$$M_3^{11} = c_{12} M_2^{11} - \sin 2\theta_{12} M_2^{12} + s_{12} M_2^{22} ,$$

$$M_3^{12} = \cos 2\theta_{12} M_2^{12} + \frac{1}{2} \sin 2\theta_{12} (M_2^{11} + M_2^{22}) , \quad M_3^{13} = c_{12} M_2^{13} - s_{12} M_2^{33} ,$$

$$M_3^{22} = s_{12} M_2^{11} + \sin 2\theta_{12} M_2^{12} + c_{12} M_2^{22} , \quad M_3^{23} = s_{12} M_2^{13} + c_{12} M_2^{23} , \quad M_3^{33} = M_2^{33} ,$$

Finally, a summary of our main results is given in section 4.
where $M_{ij}^k$ stands for the $ij$ element of $M_k$ (for $i, j, k = 1, 2, 3$). By comparing two sides of Eq. (9), one obtains

$$ \text{Re/Im} \left( M_2^{13} \right) = \text{Re/Im} \left( M_2^{23} \right) = \text{Re/Im} \left( M_3^{12} \right) = \text{Im} \left( M_3^{33} \right) = 0 , \quad (14) $$

and

$$ m_1 e^{2i\nu} = M_3^{11} , \quad m_2 e^{2i\sigma} = M_3^{22} , \quad m_3 = M_3^{33} . \quad (15) $$

Explicitly, the seven equations in Eq. (14) which are labelled as A-G in order appear as

$$ \begin{align*}
A : & \quad \cos 2\theta_{13} R_{13}^{13} = -\frac{1}{2} \sin 2\theta_{13} (I_{11}^{11} + I_{11}^{33}) , \\
B : & \quad \cos 2\theta_{13} R_{13}^{13} = \frac{1}{2} \sin 2\theta_{13} (R_{11}^{11} + R_{11}^{33}) , \\
C : & \quad c_{13} R_{13}^{23} = -s_{13} I_{11}^{12} , \\
D : & \quad c_{13} R_{13}^{23} = s_{13} I_{11}^{12} , \\
E : & \quad \cos 2\theta_{12} R_{12}^{12} = -\frac{1}{2} \sin 2\theta_{12} (R_{11}^{11} - R_{11}^{22}) , \\
F : & \quad \cos 2\theta_{12} R_{12}^{12} = -\frac{1}{2} \sin 2\theta_{12} (I_{11}^{11} - I_{11}^{22}) , \\
G : & \quad \sin 2\theta_{13} R_{13}^{13} = c_{13} I_{11}^{33} - s_{13} I_{11}^{11} ,
\end{align*} \quad (16) $$

with $R_{ij}^k = \text{Re} \left( M_{ij}^k \right)$ and $r_{ij}^k = \text{Im} \left( M_{ij}^k \right)$.

The expressions for $\theta_{12}$ and $\theta_{13}$ in terms of the components of $M_\nu$ can be directly read from Eq. (16). For example, equation E gives $\theta_{12}$ as

$$ \tan 2\theta_{12} = \frac{-2 R_{12}^{12}}{R_{11}^{11} - R_{22}^{22}} , \quad (17) $$

while equation F gives $\theta_{12}$ as

$$ \tan 2\theta_{12} = \frac{-2 I_{12}^{12}}{I_{11}^{11} - I_{22}^{22}} . \quad (18) $$

By relating these two expressions for $\theta_{12}$, a constraint equation for the components of $\tilde{M}_\nu$ arises as

$$ R_{12}^{12} (I_{11}^{11} - I_{22}^{22}) = I_{12}^{12} (R_{11}^{11} - R_{22}^{22}) . \quad (19) $$

It can be expressed in terms of $\tilde{R}_{\alpha\beta}$ and $\tilde{I}_{\alpha\beta}$ by taking the expressions

$$ \begin{align*}
R_{12}^{12} & = \text{sgn} \left( \tilde{R}_{\mu\mu} - \tilde{R}_{\eta\eta} \right) \sqrt{\frac{1}{2} (\tilde{R}_{\mu\mu} - \tilde{R}_{\eta\eta})^2 + \frac{1}{4} (\tilde{I}_{\mu\mu} - \tilde{I}_{\eta\eta})^2} , \\
I_{11}^{11} - I_{22}^{22} & = \frac{\tilde{I}_{\eta\eta} + \tilde{I}_{\mu\mu}}{2} - \frac{3}{4} (\tilde{I}_{\mu\mu} + \tilde{I}_{\eta\eta}) - \text{sgn} \left( \tilde{R}_{\mu\mu} - \tilde{R}_{\eta\eta} \right) \\
& \times \sqrt{\frac{1}{2} (\tilde{R}_{\mu\mu} + \tilde{R}_{\eta\eta})^2 + \frac{1}{4} \left( \tilde{I}_{\eta\eta} + \tilde{I}_{\mu\mu} + \frac{\tilde{I}_{\mu\mu} + \tilde{I}_{\eta\eta}}{2} \right)^2} , \\
I_{12}^{12} & = \text{sgn} \left( \tilde{I}_{\mu\mu} - \tilde{I}_{\eta\eta} \right) \sqrt{\frac{1}{2} (\tilde{I}_{\mu\mu} - \tilde{I}_{\eta\eta})^2 + \frac{1}{4} (\tilde{R}_{\mu\mu} - \tilde{R}_{\eta\eta})^2} , \\
R_{11}^{11} - R_{22}^{22} & = \frac{\tilde{R}_{\mu\mu} + \tilde{R}_{\eta\eta}}{2} - \frac{3}{4} (\tilde{R}_{\mu\mu} + \tilde{R}_{\eta\eta}) + \text{sgn} \left( \tilde{I}_{\mu\mu} - \tilde{I}_{\eta\eta} \right) \\
& \times \sqrt{\frac{1}{2} (\tilde{I}_{\mu\mu} + \tilde{I}_{\eta\eta})^2 + \frac{1}{4} \left( \tilde{R}_{\mu\mu} + \tilde{R}_{\eta\eta} + \frac{\tilde{R}_{\mu\mu} + \tilde{R}_{\eta\eta}}{2} \right)^2} .
\end{align*} \quad (20) $$
In a similar way, one will arrive at the following constraint equations by relating the expressions
for $\theta_{13}$ derived from equations A-D

\[
\begin{align*}
AB & : \quad R_{11}^{13} (R_{11}^{11} + R_{11}^{33}) = -I_{11}^{13} (I_{11}^{11} + I_{11}^{33}) , \\
AC & : \quad I_{12}^{12} R_{11}^{23} (I_{11}^{11} + I_{11}^{33}) = R_{11}^{13} \left[ (I_{12}^{12})^2 - (R_{23}^{12})^2 \right] , \\
AD & : \quad I_{23}^{12} R_{11}^{12} (I_{11}^{11} + I_{11}^{33}) = -R_{11}^{13} \left[ (R_{23}^{12})^2 - (I_{12}^{23})^2 \right] , \\
BC & : \quad -R_{11}^{12} R_{11}^{23} (R_{11}^{11} + R_{11}^{33}) = R_{11}^{13} \left[ (I_{12}^{12})^2 - (R_{23}^{12})^2 \right] , \\
BD & : \quad R_{23}^{12} R_{11}^{12} (I_{11}^{11} + I_{11}^{33}) = I_{11}^{13} \left[ (R_{23}^{12})^2 - (I_{12}^{23})^2 \right] , \\
CD & : \quad R_{23}^{12} R_{11}^{12} = -I_{11}^{23} R_{11}^{12} ,
\end{align*}
\]

(21)

where the symbol AB (and so on) is used to denote the constraint equation resulting from equations
A and B (and so on). In terms of $\bar{R}_{\alpha\beta}$ and $\bar{I}_{\alpha\beta}$, they are expressed as

\[
\begin{align*}
(\bar{R}_{\mu
u} + \bar{R}_{\tau\tau}) & \left( \bar{R}_{ee} + \bar{R}_{\mu\mu} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2} \right) = - (I_{\mu
u} + I_{\tau\tau}) \left( I_{ee} + I_{\mu\mu} + \frac{I_{\mu\mu} + I_{\tau\tau}}{2} \right) , \\
(I_{\mu
u} - I_{\tau\tau}) & \left( \bar{R}_{\mu\mu} - \bar{R}_{\tau\tau} \right) \left( I_{ee} + I_{\mu\mu} + \frac{I_{\mu\mu} + I_{\tau\tau}}{2} \right) = - (\bar{R}_{\mu\mu} - \bar{R}_{\tau\tau}) \left[ (I_{\mu\mu} - I_{\tau\tau})^2 \right] , \\
(I_{\mu
u} - I_{\tau\tau}) & \left( \bar{R}_{\mu\mu} - \bar{R}_{\tau\tau} \right) \left( \bar{R}_{ee} + \bar{R}_{\mu\mu} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2} \right) = - (\bar{R}_{\mu\mu} - \bar{R}_{\tau\tau}) \left[ (I_{\mu\mu} - I_{\tau\tau})^2 \right] , \\
(I_{\mu
u} - I_{\tau\tau}) & \left( \bar{R}_{\mu\mu} - \bar{R}_{\tau\tau} \right) \left( \bar{R}_{ee} + \bar{R}_{\mu\mu} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2} \right) = - (I_{\mu\nu} - I_{\tau\tau}) \left[ (I_{\mu\nu} - I_{\tau\tau})^2 \right] , \\
(I_{\mu
u} - I_{\tau\tau}) & \left( \bar{R}_{\mu\mu} - \bar{R}_{\tau\tau} \right) \left( \bar{R}_{ee} + \bar{R}_{\mu\mu} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2} \right) = - (I_{\mu\nu} - I_{\tau\tau}) \left[ (I_{\mu\nu} - I_{\tau\tau})^2 \right] .
\end{align*}
\]

(22)

But not all of these six constraint equations are independent. For example, equation BC can be
derived from equations AB and AC. In fact, at most three of them can be independent. A set of
three independent constraint equations (e.g., AB, AC and AD) can be chosen in such a way that
each of equations A-D has been used at least once in deriving them. Finally, we obtain a constraint
equation as

\[
-I_{11}^{11} + I_{11}^{33} = \text{sng} (R_{11}^{13}) \sqrt{4 (R_{11}^{13})^2 + (I_{11}^{11} + I_{11}^{33})^2} ,
\]

(23)

by relating the expressions for $\theta_{13}$ derived from equations A and G. Its expressions in terms of $\bar{R}_{\alpha\beta}$
and $\bar{I}_{\alpha\beta}$ appears as

\[
-I_{ee} + I_{\mu\mu} + \frac{I_{\mu\mu} + I_{\tau\tau}}{2} = \text{sng} (\bar{R}_{\mu\nu} + \bar{R}_{\tau\tau}) \sqrt{4 (\bar{R}_{\mu\nu} + \bar{R}_{\tau\tau})^2 + \left( I_{ee} + I_{\mu\mu} + \frac{I_{\mu\mu} + I_{\tau\tau}}{2} \right)^2} .
\]

(24)
To sum up, a total of five independent constraint equations for the components of \( \bar{M}_\nu \) (i.e., Eqs. \( \text{[20]} \)) and three independent ones from Eq. \( \text{[22]} \) will arise from the eliminations of \( \theta_{12} \) and \( \theta_{13} \) in Eq. \( \text{[16]} \). This can be understood from the fact that two more conditions (i.e., \( \theta_{23} = \pi/4 \) and \( \delta = -\pi/2 \)) have been imposed on the basis of three inherent constraint equations for the components of \( \bar{M}_\nu \). At last, one can say that an \( M_\nu \) with its components satisfying these constraint equations will necessarily produce \( \theta_{23} = \pi/4 \) and \( \delta = -\pi/2 \).

By taking the expressions for \( \theta_{12} \) and \( \theta_{13} \) derived from Eq. \( \text{[16]} \) in Eq. \( \text{[15]} \), the neutrino masses in combination with the Majorana CP phases are obtained as

\[
\text{Re} \left( m_1 e^{i\phi} \right) = \frac{R_{11}^2 + R_{22}^2}{2} - \text{sgn} \left( R_{2}^{12} \right) \sqrt{\left( R_{2}^{12} \right)^2 + \frac{1}{4} \left( R_{2}^{11} - R_{2}^{22} \right)^2},
\]

\[
\text{Im} \left( m_1 e^{i\phi} \right) = \frac{I_{11}^2 + I_{22}^2}{2} - \text{sgn} \left( I_{2}^{12} \right) \sqrt{\left( I_{2}^{12} \right)^2 + \frac{1}{4} \left( I_{2}^{11} - I_{2}^{22} \right)^2},
\]

\[
\text{Re} \left( m_2 e^{i\phi} \right) = \frac{R_{11}^2 + R_{22}^2}{2} + \text{sgn} \left( R_{2}^{12} \right) \sqrt{\left( R_{2}^{12} \right)^2 + \frac{1}{4} \left( R_{2}^{11} - R_{2}^{22} \right)^2},
\]

\[
\text{Im} \left( m_2 e^{i\phi} \right) = \frac{I_{11}^2 + I_{22}^2}{2} + \text{sgn} \left( I_{2}^{12} \right) \sqrt{\left( I_{2}^{12} \right)^2 + \frac{1}{4} \left( I_{2}^{11} - I_{2}^{22} \right)^2},
\]

\[
m_3 = -\frac{\bar{R}_{ee} - \bar{R}_{\mu\tau}}{2} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{4} + \text{sgn} \left( I_{em} + I_{et} \right)
\times \sqrt{\left( I_{em} + I_{et} \right)^2 + \frac{1}{4} \left( \bar{R}_{ee} + \bar{R}_{\mu\tau} + \bar{R}_{\mu\mu} + \bar{R}_{\tau\tau} \right)^2},
\]

(25)

where

\[
R_{11}^2 + R_{22}^2 = \frac{\bar{R}_{ee} - 3\bar{R}_{\mu\tau}}{2} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{4} + \text{sgn} \left( I_{em} + I_{et} \right)
\times \sqrt{\frac{1}{2} \left( I_{em} + I_{et} \right)^2 + \frac{1}{4} \left( \bar{R}_{ee} + \bar{R}_{\mu\tau} + \bar{R}_{\mu\mu} + \bar{R}_{\tau\tau} \right)^2},
\]

\[
I_{11}^2 + I_{22}^2 = \frac{\bar{I}_{ee} - 3\bar{I}_{\mu\tau}}{2} + \frac{\bar{I}_{\mu\mu} + \bar{I}_{\tau\tau}}{4} - \text{sgn} \left( \bar{R}_{em} + \bar{R}_{et} \right)
\times \sqrt{\frac{1}{2} \left( \bar{R}_{em} + \bar{R}_{et} \right)^2 + \frac{1}{4} \left( \bar{I}_{ee} + \bar{I}_{\mu\tau} + \bar{I}_{\mu\mu} + \bar{I}_{\tau\tau} \right)^2},
\]

(26)

while the expressions for \( R_{12}, I_{12}, R_{11}^2 - R_{22}^2 \) and \( R_{21}^2 - R_{22}^2 \) have been given in Eq. \( \text{[20]} \).

However, it should be pointed out that the above results are derived in the general case where none of equations A-G has its two sides vanish. When an equation has its two sides vanish, it fails to give an expression for \( \theta_{12} \) or \( \theta_{13} \) and thus the constraint equation(s) resulting from it will become ineffective. In such kind of case, the number of independent constraint equations will get increased compared to in the general case. (For example, in the case of two sides of equation E being vanishing, the expression for \( \theta_{12} \) in Eq. \( \text{[17]} \) and thus the constraint equation in Eq. \( \text{[19]} \) become ineffective. But there are two new constraint equations \( R_{21}^2 = R_{21}^2 - R_{22}^2 = 0 \). So, in effect, the number of independent constraint equations in this case gets increased by one compared to in the general case.) When this number gets increased by one (and so on), there will correspondingly be one (and so on) neutrino mass sum rules as we will see. In the next section, all the possible cases where one or more equations in Eq. \( \text{[16]} \) have their two sides vanish will be examined.

Before doing that, we make several observations: (1), In the case of two sides of equation A being vanishing (i.e., \( R_{11}^2 = I_{11}^1 + I_{33}^3 = 0 \), two sides of equation G are necessarily also vanishing (i.e.,
$R_{13}^{13} = c_{13}^2 I_{33}^{33} - s_{13}^2 I_{11}^{11} = 0$. The reverse is also true. It is easy to see that a combination of $c_{13}^2 I_{33}^{33} - s_{13}^2 I_{11}^{11} = 0$ and $I_{11}^{11} + I_{33}^{33} = 0$ leads us to $I_{11}^{11} = I_{33}^{33} = 0$. So equations A and G always have their two sides vanish simultaneously. And in such a case one has

$$R_{13}^{13} = I_{11}^{11} = I_{33}^{33} = 0.$$  \hspace{1cm} (27)

(2), In the case of two sides of equation C or D being vanishing (i.e., $R_{23}^{23} = I_{12}^{12} = 0$ or $R_{12}^{12} = I_{23}^{23} = 0$), as a result of the relation $I_{22}^{22} = c_{13} I_{12}^{12} - s_{13} R_{23}^{23}$ or $R_{12}^{12} = c_{13} R_{11}^{11} + s_{13} I_{23}^{23}$, two sides of equation F or E are necessarily also vanishing (i.e., $I_{22}^{22} = I_{11}^{11} - I_{22}^{22} = 0$ or $R_{22}^{22} = R_{11}^{11} - R_{22}^{22} = 0$). The reverse is also true. So equations C and F or D and E always have their two sides vanish simultaneously. And in such a case one has

$$I_{22}^{22} - I_{22}^{22} = R_{23}^{23} = I_{12}^{12} = 0 , \hspace{1cm} \text{(28)}$$

or

$$R_{22}^{22} - R_{22}^{22} = I_{23}^{23} = R_{12}^{12} = 0 . \hspace{1cm} \text{(29)}$$

(3), Equations E and F (or A, B, C, D and G) are not allowed to have their two sides vanish simultaneously. Otherwise, $\theta_{12}$ (or $\theta_{13}$) would be free of any constraint and have no reason to take the measured value. For the above observations, we just need to consider the cases where equations A&G, B, C&F, D&E, A&B&G, A&C&F&G, A&D&E&G, B&C&F, B&D&E, A&B&C&F&G or A&B&D&E&G have their two sides vanish.

3 Various textures

3.1 A&G

In the case of two sides of equations A and G being vanishing, equations AB, AC, AD and Eq. $R_{23}^{23}$ become ineffective. We are thus left with three independent constraint equations (i.e., Eq. $R_{11}^{11}$ and two of equations BC, BD and CD). But, as discussed at the end of section 2, there are three new constraint equations given by Eq. (27) which lead to the following relations for $\bar{R}_{\alpha\beta}$ and $\bar{I}_{\alpha\beta}$

$$\bar{R}_{\mu\mu} = -\bar{R}_{\tau\tau} , \quad \bar{I}_{ee} = 0 , \quad -2\bar{I}_{\mu\tau} = \bar{I}_{\mu\mu} + \bar{I}_{\tau\tau} . \hspace{1cm} \text{(30)}$$

By taking these relations, the expressions for the surviving constraint equations in Eqs. (20, 22) can be simplified to some extent. In total, the number of independent constraint equations in the case under study gets increased by one compared to in the general case. So one neutrino mass sum rule will arise.

The neutrino masses in combination with the Majorana CP phases in Eq. (25) can also be simplified to some extent by taking the relations in Eq. (30). Thereinto, $\text{Im} \left( m_1 e^{i\rho} \right)$ and $\text{Im} \left( m_2 e^{i\sigma} \right)$ can be written as

$$\text{Im} \left( m_1 e^{i\rho} \right) = \frac{1}{2} \left( \bar{I}_{\mu\mu} + \bar{I}_{\tau\tau} - \bar{I}_{\mu\tau} \right) \left( 1 - \frac{1}{\cos 2\theta_{12}} \right) , \hspace{1cm} \text{(31)}$$

$$\text{Im} \left( m_2 e^{i\sigma} \right) = \frac{1}{2} \left( \bar{I}_{\mu\mu} + \bar{I}_{\tau\tau} - \bar{I}_{\mu\tau} \right) \left( 1 + \frac{1}{\cos 2\theta_{12}} \right) ,$$

from which it is easy to get

$$m_1 c_{12}^2 \sin 2\rho + m_2 s_{12}^2 \sin 2\sigma = 0 . \hspace{1cm} \text{(32)}$$
In the particular case of \( \sin 2\rho = \sin 2\sigma = 0 \) (i.e., \( \rho, \sigma = 0 \) or \( \pi/2 \)), the neutrino masses will become independent of the Majorana CP phases. For this case, an additional constraint equation

\[
\bar{I}_{\mu\mu} + \bar{I}_{\tau\tau} = 2\bar{I}_{\mu\tau},
\]

results from Eq. (31). It is found that Eq. (33) together with Eq. (30) would lead us to a situation where two sides of equations C and F are also vanishing. This happens to be the case of two sides of equations A, C, F and G being vanishing which will be discussed in subsection 3.6. For the case of \( \sin 2\rho, \sin 2\sigma \neq 0 \), we present \( \sin 2\sigma/\sin 2\rho \) as a function of the lightest neutrino mass in the NH and IH cases in Fig. 1. In the IH case, it takes a value close to \(-c_{12}^2/s_{12}^2 \simeq -2.27\) in the whole mass range as a result of \( m_1 \simeq m_2 \). In the NH case, its value is very small in the range of \( m_1 \) being vanishingly small but approaches \(-2.27\) in the range of \( m_1 \simeq m_2 \simeq 0.1 \text{ eV} \).

Figure 1: \( \sin 2\sigma/\sin 2\rho \) as a function of the lightest neutrino mass (\( m_1 \) in the NH case or \( m_3 \) in the IH case) in the case of two sides of equations A and G being vanishing.

### 3.2 B

In the case of two sides of equation B being vanishing, equations AB, BC and BD become ineffective. We are thus left with four independent constraint equations (i.e., Eqs. (19, 23) and two of equations AC, AD and CD). But there are two new constraint equations

\[
I_1^{13} = R_{11}^{11} + R_{13}^{33} = 0,
\]

which lead to the following relations for \( \bar{R}_{\alpha\beta} \) and \( \bar{I}_{\alpha\beta} \)

\[
\bar{I}_{\mu\mu} = -\bar{I}_{\tau\tau}, \quad -2(\bar{R}_{ee} + \bar{R}_{\mu\tau}) = \bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}.
\]

By taking these relations, the expressions for the surviving constraint equations in Eqs. (20, 22, 24) can be simplified to some extent. In total, the number of independent constraint equations in the case under study gets increased by one compared to in the general case. So one neutrino mass sum rule will arise.
The neutrino masses in combination with the Majorana CP phases in Eq. (25) can also be simplified to some extent by taking the relations in Eq. (35). Thereinto, \( \text{Re}(m_1 e^{2i\rho}) \), \( \text{Re}(m_2 e^{2i\sigma}) \) and \( m_3 \) can be written as

\[
\text{Re}(m_1 e^{2i\rho}) = -\bar{R}_{\mu\tau} + (\bar{R}_{ee} + \bar{R}_{\mu\tau}) \frac{1}{\cos 2\theta_{12}}, \\
\text{Re}(m_2 e^{2i\sigma}) = -\bar{R}_{\mu\tau} - (\bar{R}_{ee} + \bar{R}_{\mu\tau}) \frac{1}{\cos 2\theta_{12}}, \\
m_3 = -\bar{R}_{ee},
\]

from which it is easy to get

\[
m_1 c_{12}^2 \cos 2\rho + m_2 s_{12}^2 \cos 2\sigma + m_3 = 0.
\]

With the help of the inequality

\[
m_1 c_{12}^2 \cos 2\rho + m_2 s_{12}^2 \cos 2\sigma \geq -m_1 c_{12}^2 - m_2 s_{12}^2 > -m_2,
\]

one can see that this sum rule can never be fulfilled in the NH case. Hence we discuss the implications of this sum rule in the IH case. In Fig. 2, we show the correlation between \( \rho \) and \( \sigma \) for some representative values of \( m_3 \). For the particular value of \( m_3 = 0 \), an interesting solution to Eq. (37) is \( \cos 2\rho = \cos 2\sigma = 0 \) (i.e., \( \rho, \sigma = \pi/4 \) or \( 3\pi/4 \)) which gives two additional constraint equations

\[
\bar{R}_{ee} = \bar{R}_{\mu\tau} = 0,
\]

from Eq. (36). In this case, the effective neutrino mass

\[
|\langle m \rangle_{ee}| = \left| m_1 e^{2i\rho} c_{12}^2 c_{13}^2 + m_2 e^{2i\sigma} s_{12}^2 c_{13}^2 + m_3 s_{12}^2 e^{-2i\delta} \right|,
\]

which controls the rates of neutrinoless double beta decays takes a value of \( m_1 c_{12}^2 c_{13}^2 + m_2 s_{12}^2 c_{13}^2 \simeq 0.049 \text{ eV} \) for \( \rho = \sigma \) or \( m_1 c_{12}^2 c_{13}^2 - m_2 s_{12}^2 c_{13}^2 \simeq 0.019 \text{ eV} \) for \( \rho \neq \sigma \). In the case of only \( \cos 2\rho = 0 \) or \( \cos 2\sigma = 0 \) which results in a constraint equation as

\[
-\bar{R}_{\mu\tau} \text{ or } \bar{R}_{\mu\tau} = \text{Sign}(\bar{R}_{e\mu} - \bar{R}_{e\tau}) \sqrt{\frac{1}{2} (\bar{R}_{e\mu} - \bar{R}_{e\tau})^2 + \frac{1}{4} (\bar{I}_{\mu\mu} - \bar{I}_{\tau\tau})^2 + (\bar{R}_{ee} + \bar{R}_{\mu\tau})^2},
\]

\( \sigma \) or \( \rho \) as well as \( |\langle m \rangle_{ee}| \) can be presented as a function of \( m_3 \) as shown in Fig. 3. In the range of \( m_3 \) being vanishingly small, \( \sigma \) and \( \rho \) take a value close to \( \pi/4 \) or \( 3\pi/4 \) while \( |\langle m \rangle_{ee}| \) takes a value close to 0.049 eV or 0.019 eV. This is consistent with the results in the case of \( m_3 = \cos 2\rho = \cos 2\sigma = 0 \). When \( m_3 \) takes its upper value of \( m_2 s_{12}^2 \simeq 0.016 \text{ eV} \) or \( m_1 c_{12}^2 \simeq 0.048 \text{ eV} \), \( \sigma \) or \( \rho \) becomes \( \pi/2 \) while \( |\langle m \rangle_{ee}| \) becomes \( \sqrt{(m_1 c_{12}^2 c_{13}^2)^2 + (m_2 s_{12}^2 c_{13}^2 + m_3)^2} \simeq 0.039 \text{ eV} \) or \( \sqrt{(m_2 s_{12}^2 c_{13}^2)^2 + (m_1 c_{12}^2 c_{13}^2 + m_3)^2} \simeq 0.052 \text{ eV} \).

### 3.3 C&F

In the case of two sides of equations C and F being vanishing, equations AC, BC, CD and Eq. (19) become ineffective. We are thus left with three independent constraint equations (i.e., Eq. (23) and two of equations AB, AD and BD). But, as discussed at the end of section 2, there are three new constraint equations given by Eq. (28) which lead to the following relations for \( \bar{R}_{\alpha\beta} \) and \( \bar{I}_{\alpha\beta} \)

\[
\bar{I}_{ee} = \bar{I}_{\mu\mu} + \bar{I}_{\tau\tau}, \quad \bar{R}_{\mu\mu} = \bar{R}_{\tau\tau}, \quad \bar{I}_{e\mu} = \bar{I}_{e\tau}.
\]

By taking these relations, the expressions for the surviving constraint equations in Eqs. (22, 24) can be simplified to some extent. In total, the number of independent constraint equations in the
Figure 2: The correlation between $\rho$ and $\sigma$ for some representative values of $m_3$ in the case of two sides of equation B being vanishing.

Figure 3: Left: $\rho$ (or $\sigma$) as a function of $m_3$ for $\cos 2\sigma = 0$ (or $\cos 2\rho = 0$) in the case of two sides of equation B being vanishing. Right: $|\langle m \rangle_{ee}|$ as a function of $m_3$ for $\cos 2\sigma = 0$ (or $\cos 2\rho = 0$) in the case of two sides of equation B being vanishing.

The neutrino masses in combination with the Majorana CP phases in Eq. (25) can also be simplified to some extent by taking the relations in Eq. (42). Thereinto, Im($m_1 e^{2i\rho}$) and Im($m_2 e^{2i\sigma}$)
become
\[
\text{Im}(m_1 e^{2i\rho}) = \frac{1}{2} (I_{\mu\mu} + I_{\tau\tau}) - I_{\mu\tau}, \quad \text{Im}(m_2 e^{2i\sigma}) = \frac{1}{2} (I_{\mu\mu} + I_{\tau\tau}) - I_{\mu\tau},
\]
from which it is easy to get
\[
m_1 \sin 2\rho - m_2 \sin 2\sigma = 0.
\]
In the particular case of \(\sin 2\rho = \sin 2\sigma = 0\) (i.e., \(\rho, \sigma = 0\) or \(\pi/2\)), the neutrino masses will become independent of the Majorana CP phases. In this case, an additional constraint equation same as that in Eq. (33) results from Eq. (43). It is found that Eq. (33) together with Eq. (42) would lead us to a situation where two sides of equations A and G are also vanishing. As mentioned in subsection 3.1, this happens to be the case of two sides of equations A, C, F and G being vanishing which will be discussed in subsection 3.6. For the case of \(\sin 2\rho, \sin 2\sigma \neq 0\), we present \(\sin 2\sigma/\sin 2\rho\) as a function of the lightest neutrino mass in the NH and IH cases in Fig. 4. In the IH case, it takes a value close to 1 in the whole mass range as a result of \(m_1 \approx m_2\). In the NH case, its value is very small in the range of \(m_1\) being vanishingly small but approaches 1 in the range of \(m_1 \approx m_2 \approx 0.1\) eV.

\[\text{Figure 4: } \sin 2\sigma/\sin 2\rho \text{ as a function of the lightest neutrino mass (} m_1 \text{ in the NH case or } m_3 \text{ in the IH case) in the case of two sides of equations C and F being vanishing.}\]

### 3.4 D&E

In the case of two sides of equations D and E being vanishing, equations AD, BD, CD and Eq. (19) become ineffective. We are thus left with three independent constraint equations (i.e., Eq. (23) and two of equations AB, AC and BC). But, as discussed at the end of section 2, there are three new constraint equations given by Eq. (29) which lead to the following relations for \(\bar{R}_{\alpha\beta}\) and \(\bar{I}_{\alpha\beta}\):

\[
\bar{R}_{ee} + \bar{R}_{\mu\tau} - \frac{3}{2} (\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}) = -\text{sgn} (\bar{I}_{e\mu} + \bar{I}_{e\tau}) \sqrt{2(\bar{I}_{e\mu} + \bar{I}_{e\tau})^2 + \left(\bar{R}_{ee} + \bar{R}_{\mu\tau} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}\right)^2},
\]
\[
\bar{I}_{\mu\mu} = \bar{I}_{\tau\tau}, \quad \bar{R}_{e\mu} = \bar{R}_{e\tau}.
\]

(45)
By taking these relations, the expressions for the surviving constraint equations in Eqs. (22, 24) can be simplified to some extent. In total, the number of independent constraint equations in the case under study gets increased by one compared to in the general case. So one neutrino mass sum rule will arise.

The neutrino masses in combination with the Majorana CP phases in Eq. (25) can also be simplified to some extent by taking the relations in Eq. (45). Thereinto, \( \text{Re} \left( m_1 e^{2i\rho} \right) \) and \( \text{Re} \left( m_2 e^{2i\sigma} \right) \) become

\[
\text{Re} \left( m_1 e^{2i\rho} \right) = \frac{1}{2} \left( \bar{R}_{\mu\mu} + \bar{R}_{\tau\tau} \right) - \bar{R}_{\mu\tau}, \quad \text{Re} \left( m_2 e^{2i\sigma} \right) = \frac{1}{2} \left( \bar{R}_{\mu\mu} + \bar{R}_{\tau\tau} \right) - \bar{R}_{\mu\tau},
\]

from which it is easy to get

\[
m_1 \cos 2\rho - m_2 \cos 2\sigma = 0.
\]

(47)

In the particular case of \( \cos 2\rho = \cos 2\sigma = 0 \) (i.e., \( \rho, \sigma = \pi/4 \) or \( 3\pi/4 \)), the neutrino masses will become independent of the Majorana CP phases. In this case, an additional constraint equation

\[
\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau} = 2 \bar{R}_{\mu\tau},
\]

(48)

results from Eq. (46). For the case of \( \cos 2\rho, \cos 2\sigma \neq 0 \), one can also present \( \cos 2\sigma/\cos 2\rho \) as a function of the lightest neutrino mass. The result is the same as that for \( \sin 2\sigma/\sin 2\rho \) in subsection 3.3.

3.5 A&B&G

In the case of two sides of equations A, B and G being vanishing, equations AB, AC, AD, BC, BD and Eq. (23) become ineffective. We are thus left with two constraint equations (i.e., Eq. (19) and equation CD). But there are five new constraint equations given by Eqs. (27, 34) which lead to the relations for \( \bar{R}_{\alpha\beta} \) and \( \bar{I}_{\alpha\beta} \) in Eqs. (30, 35). By taking these relations, the expressions for the surviving constraint equations in Eqs. (20, 22) can be simplified to some extent. In total, the number of independent constraint equations in the case under study gets increased by two compared to in the general case. So two neutrino mass sum rules will arise.

By taking the relations in Eqs. (30, 35), the neutrino masses in combination with the Majorana CP phases in Eq. (25) can be written as in Eqs. (31, 36). So the desired neutrino mass sum rules are the same as those in Eqs. (32, 37). As discussed in subsection 3.2, these sum rules can only be fulfilled in the IH case. In Fig. 5, we present \( \rho, \sigma \) and \( |\langle m_{ee} \rangle| \) as functions of the lightest neutrino mass \( m_3 \). There is a lower value 0.021 eV of \( m_3 \) at which \( \rho \) and \( \sigma \) respectively take the values 0 and \( \pi/2 \) (or \( \pi/2 \) and 0). As discussed in subsection 3.1, \( \sin 2\rho = \sin 2\sigma = 0 \) (which gives an additional constraint equation given by Eq. (33)) would lead us to a situation where two sides of equations C and F are also vanishing. This happens to be the case of two sides of equations A, B, C, F and G being vanishing which will be discussed in subsection 3.10. On the other hand, \( |\langle m_{ee} \rangle| \) is found to be equal to \( m_3 \). The sum rules in Eqs. (32, 37) can be reorganized into a single complex equation

\[
m_1 c_{12}^2 e^{2i\rho} + m_2 s_{12}^2 e^{2i\sigma} + m_3 = 0,
\]

(49)

which immediately tells us that \( |\langle m_{ee} \rangle| = m_3 \).

3.6 A&C&F&G

In the case of two sides of equations A, C, F and G being vanishing, equations AB, AC, AD, BC, CD and Eqs. (19, 23) become ineffective. We are thus left with only one constraint equation (i.e., equation BD). But there are six new constraint equations given by Eqs. (27, 28) which lead to
Figure 5: $\rho$, $\sigma$ and $|\langle m \rangle_{ee}|$ as functions of the lightest neutrino mass $m_3$ in the case of two sides of equations A, B and G being vanishing.

The relations for $\bar{R}_{\alpha\beta}$ and $\bar{I}_{\alpha\beta}$ in Eqs. (30, 42). By taking these relations, the expression for the surviving constraint equation in Eq. (22) can be simplified to some extent. It is found that the texture thus obtained can reproduce the specific texture of $M_\nu$ given by the neutrino $\mu$-$\tau$ reflection symmetry: The relations in Eqs. (30, 42) can be reorganized into

$$ M_{\mu\mu} = M_{\tau\tau}^*, \quad M_{e\mu} = M_{e\tau}^*, \quad M_{ee} \text{ and } M_{e\tau} \text{ being real.} \quad (50) $$

In view of the definition $\bar{M}_{\alpha\beta} = M_{\alpha\beta}e^{-i(\phi_\alpha + \phi_\beta)}$, the relations in Eq. (50) will become those in Eq. (7) by taking $\phi_e = \pi/2$ and $\phi_\mu = -\phi_\tau$. In total, the number of independent constraint equations in the case under study gets increased by two compared to in the general case. So two neutrino mass sum rules will arise.

The neutrino masses in combination with the Majorana CP phases in Eq. (25) can also be simplified to some extent by taking the relations in Eqs. (30, 42). Thereinto, $\text{Im}(m_1e^{2i\rho})$ and $\text{Im}(m_2e^{2i\sigma})$ become vanishing, implying that the Majorana CP phases take trivial values (i.e., $\rho, \sigma = 0$ or $\pi/2$). For the possible combinations $[\rho, \sigma] = [0, 0], [0, \pi/2], [\pi/2, 0]$ and $[\pi/2, \pi/2]$, we present $|\langle m \rangle_{ee}|$ as a function of the lightest neutrino mass in the NH and IH cases in Fig. 6. In the NH case, three terms of $|\langle m \rangle_{ee}|$ add constructively to a maximal level for $[\rho, \sigma] = [\pi/2, \pi/2]$ but will cancel out (i.e., $|\langle m \rangle_{ee}| \approx 0$) at $m_1 \approx 0.002$ eV (or 0.007 eV) for $[\rho, \sigma] = [\pi/2, 0]$ (or $[0, \pi/2]$).

In the IH case, $|\langle m \rangle_{ee}|$ is dominated by the first two terms since the third one is highly suppressed. For $\rho = \sigma$ (or $\rho \neq \sigma$), $|\langle m \rangle_{ee}|$ approximates to $m_1c_1^2 + m_2s_1^2$ (or $m_1c_1^2 - m_2s_1^2$).

### 3.7 A&D&E&G

In the case of two sides of equations A, D, E and G being vanishing, equations AB, AC, AD, BD, CD and Eqs. (19, 23) become ineffective. We are thus left with only one constraint equation (i.e., equation BC). But there are six new constraint equations given by Eqs. (27, 29) which lead to the relations for $\bar{R}_{\alpha\beta}$ and $\bar{I}_{\alpha\beta}$ in Eqs. (30, 45). By taking these relations, the expression for the surviving constraint equation in Eq. (22) can be simplified to some extent. In total, the number
3.8 B&C&F

In the case of two sides of equations B, C and F being vanishing, equations AB, AC, BC, BD, CD and Eq. (19) become ineffective. We are thus left with two constraint equations (i.e., Eq. (23) and equation BC). But there are five new constraint equations given by Eqs. (34, 28) which lead to the relations for $\bar{R}_{\alpha\beta}$ and $\bar{I}_{\alpha\beta}$ in Eqs. (35, 42). By taking these relations, the expressions for the surviving constraint equations in Eqs. (22, 24) can be simplified to some extent. In total, the number of independent constraint equations in the case under study gets increased by two compared to in the general case. So two neutrino mass sum rules will arise.

By taking the relations in Eqs. (35, 42), the neutrino masses in combination with the Majorana CP phases in Eq. (25) can be written as in Eqs. (36, 43). So the desired neutrino mass sum rules are the same as those in Eqs. (32, 47). In Figs. 7-8, we present $\rho, \sigma$ and $|\langle m_{ee} \rangle|$ as functions of the lightest neutrino mass in the NH and IH cases. In the NH case, there is a lower value 0.004 eV of $m_1$ at which $\rho$ and $\sigma$ respectively take the values $\pi/4$ and $3\pi/4$ (or $3\pi/4$ and $\pi/4$). As discussed in subsection 3.4, $\cos 2\rho = \cos 2\sigma = 0$ results in an additional constraint equation given by Eq. (48). At the lower value of $m_1$, $|\langle m_{ee} \rangle|$ takes a value close to $m_3 s_{13}^2$. In the mass range of $m_1 \simeq m_2 \simeq 0.1$ eV, Eqs. (32, 47) give

$$\frac{\sin 2\sigma}{\sin 2\rho} \simeq -\frac{c_{12}^2}{s_{12}^2}, \quad \cos 2\rho \simeq \cos 2\sigma.$$

Only for $\rho \simeq \pi - \sigma \simeq 0$ or $\pi/2$, can these two relations be fulfilled simultaneously. Consequently, $|\langle m_{ee} \rangle|$ approximates to $m_1 c_{12}^2 + m_2 s_{12}^2$ in this mass range. In the IH case, we have $m_1 \simeq m_2$ and thus $\rho \simeq \pi - \sigma \simeq 0$ or $\pi/2$ and $|\langle m_{ee} \rangle| \simeq m_1 c_{12}^2 + m_2 s_{12}^2$ in the whole mass range.
Figure 7: $\rho$ and $\sigma$ as functions of the lightest neutrino mass ($m_1$ in the NH case or $m_3$ in the IH case) in the case of two sides of equations A, D, E and G being vanishing.

Figure 8: $|\langle m_{ee}\rangle|$ as a function of the lightest neutrino mass ($m_1$ in the NH case or $m_3$ in the IH case) in the case of two sides of equations A, D, E and G being vanishing.

are the same as those in Eqs. (37, 44). As discussed in subsection 3.2, these sum rules can only be fulfilled in the IH case. In Fig. 9, we present $\rho$, $\sigma$ and $|\langle m_{ee}\rangle|$ as functions of the lightest neutrino mass $m_3$. As a result of $m_1 \simeq m_2$ in the IH case, one finds $\rho \simeq \sigma$ or $\pi/2 - \sigma$ from Eq. (44). Eq. (37) further leads us to $\rho \simeq \sigma \simeq \pi/4$ or $3\pi/4$ for vanishingly small $m_3$ and $\rho \simeq \sigma \simeq \pi/2$ for $m_3 \simeq m_1 \simeq m_2 \simeq 0.1$ eV. Consequently, $|\langle m_{ee}\rangle|$ takes a value close to $m_1 c_{12}^2 + m_2 s_{12}^2$ for these two mass ranges. Finally, for $m_3 \simeq 0.02$ eV, it is found that one of the allowed solutions to Eqs. (37, 44).
Figure 9: $\rho$, $\sigma$ and $|\langle m \rangle_{ee}|$ as functions of the lightest neutrino mass $m_3$ in the case of two sides of equations B, C and F being vanishing.

is $\rho \simeq \pi/2$ and $\sigma \simeq 0$ in which case $|\langle m \rangle_{ee}|$ becomes $m_1 c_{12}^2 - m_2 s_{12}^2 \simeq 0.02$ eV.

3.9 B&D&E

In the case of two sides of equations B, D and E being vanishing, equations AB, BC, BD, AD, CD and Eq. (19) become ineffective. We are thus left with two constraint equations (i.e., Eq. (23) and equation AC). But there are five new constraint equations given by Eqs. (34) which lead to the relations for $\hat{R}_{\alpha\beta}$ and $\hat{I}_{\alpha\beta}$ in Eqs. (35). By taking these relations, the expressions for the surviving constraint equations in Eqs. (22) can be simplified to some extent. In total, the number of independent constraint equations in the case under study gets increased by two compared to in the general case. So two neutrino mass sum rules will arise.

By taking the relations in Eqs. (35), the neutrino masses in combination with the Majorana CP phases in Eq. (25) can be written as in Eqs. (36). So the desired neutrino mass sum rules are the same as those in Eqs. (37). A combination of these sum rules further yields

$$m_1 \cos 2\rho = m_2 \cos 2\sigma = -m_3$$

(52)

Apparently, these relations can only be fulfilled in the IH case. In Fig. 10, we present $\rho$, $\sigma$ and $|\langle m \rangle_{ee}|$ as functions of the lightest neutrino mass $m_3$. In consideration of $m_1 \simeq m_2$ in the IH case, the results for $\rho$ and $\sigma$ are presented by the same lines. One has $\rho, \sigma \simeq \pi/4$ or $3\pi/4$ for vanishingly small $m_3$ and $\rho \simeq \sigma \simeq \pi/2$ for $m_3 \simeq m_1 \simeq m_2 \simeq 0.1$ eV. Consequently, $|\langle m \rangle_{ee}|$ takes a value close to $m_1 c_{12}^2 + m_2 s_{12}^2 \simeq 0.05$ eV in the case of $\rho \simeq \sigma$ (or $m_1 c_{12}^2 - m_2 s_{12}^2 \simeq 0.02$ eV in the case of $\rho \simeq \pi/2 + \sigma$) for vanishingly small $m_3$ and $m_1 c_{12}^2 + m_2 s_{12}^2$ for $m_3 \simeq m_1 \simeq m_2 \simeq 0.1$ eV.

3.10 A&B&C&F&G

In the case of two sides of equations A, B, C, F and G being vanishing, Eqs. (19) become ineffective. We are thus left with no constraint equations. But there are eight new constraint equations given by Eqs. (27) which lead to the relations for $\hat{R}_{\alpha\beta}$ and $\hat{I}_{\alpha\beta}$ in Eqs. (30).
Figure 10: $\rho$, $\sigma$ and $|\langle m_{ee} \rangle|$ as functions of the lightest neutrino mass $m_3$ in the case of two sides of equations B, D and E being vanishing.

In total, the number of independent constraint equations in the case under study gets increased by three compared to in the general case. So three neutrino mass sum rules will arise.

The neutrino masses in combination with the Majorana CP phases in Eq. (25) can be simplified to some extent by taking the relations in Eqs. (30, 42, 35). Thereinto, $\text{Im} \left( m_1 e^{2i\rho} \right)$ and $\text{Im} \left( m_2 e^{2i\sigma} \right)$ become vanishing while $\text{Re} \left( m_1 e^{2i\rho} \right)$, $\text{Re} \left( m_2 e^{2i\sigma} \right)$ and $m_3$ can be written as in Eq. (36). So the desired neutrino mass sum rules are given by Eq. (37) and $\rho, \sigma = 0$ or $\pi/2$. As discussed in subsection 3.2, these sum rules can only be fulfilled in the IH case. It turns out that only for the combination $[\rho, \sigma] = [\pi/2, 0]$ can Eq. (37) have a realistic solution ($m_3 \simeq 0.02$ eV) in which case $|\langle m_{ee} \rangle|$ takes a value of 0.02 eV.

### 3.11 A&B&D&E&G

In the case of two sides of equations A, B, D, E and G being vanishing, Eqs. (19, 21, 23) become ineffective. We are thus left with no constraint equations. But there are eight new constraint equations given by Eqs. (27, 29, 34) which lead to the relations for $\bar{R}_{\alpha\beta}$ and $\bar{I}_{\alpha\beta}$ in Eqs. (30, 45, 35). In total, the number of independent constraint equations in the case under study gets increased by three compared to in the general case. So three neutrino mass sum rules will arise.

The neutrino masses in combination with the Majorana CP phases in Eq. (25) can be simplified to some extent by taking the relations in Eqs. (30, 42, 35). Thereinto, $\text{Re} \left( m_1 e^{2i\rho} \right)$ and $\text{Re} \left( m_2 e^{2i\sigma} \right)$ become $\bar{R}_{ee}$ while $\text{Im} \left( m_1 e^{2i\rho} \right)$, $\text{Im} \left( m_2 e^{2i\sigma} \right)$ and $m_3$ can be written as in Eqs. (31, 36). So the desired neutrino mass sum rules are given by Eqs. (32, 52). By taking the relations given by Eq. (52) in Eq. (32), one arrives at a neutrino mass sum rule as

$$ (m_1^2 - m_3^2) c_{12}^4 = (m_2^2 - m_3^2) s_{12}^4. \quad (53) $$

Unfortunately, this sum rule has no chance to be in agreement with the experimental results.
4 Summary

Motivated by the fact that the current neutrino oscillation data is consistent with maximal atmospheric mixing angle and leptonic CP violation, we derive in a novel approach the possible textures of neutrino mass matrix that can naturally lead us to $\theta_{23} = \pi/4$ and $\delta = -\pi/2$. In order to evade the uncertainties created by the unphysical phases, we work on the effective neutrino mass matrix $\bar{M}_\nu$ instead of $M_\nu$ itself. Since the unphysical phases have cancelled out in $\bar{M}_\nu$, its twelve components $\bar{R}_{\alpha\beta}$ and $\bar{I}_{\alpha\beta}$ are not all independent but subject to three constraint equations. After imposing the conditions $\theta_{23} = \pi/4$ and $\delta = -\pi/2$, there will be five independent constraint equations for $\bar{R}_{\alpha\beta}$ and $\bar{I}_{\alpha\beta}$. We derive these constraint equations (i.e., Eqs. (20, 24) and three independent ones from Eq. (22)) by eliminating $\theta_{12}$ and $\theta_{13}$ in Eq. (16) in the general case where none of equations A-G has its two sides vanish. On the basis of this, we further study the possible textures of $\bar{M}_\nu$ by considering that one or more of equations A-G may have their two sides vanish. When an equation has its two sides vanish, the constraint equation(s) resulting from it will become ineffective. But the fact that its two sides vanish itself brings about two new constraint equations. So the number of independent constraint equations gets increased compared to in the general case. When this number gets increased by one (and so on), there will correspondingly be one (and so on) neutrino mass sum rules relating the neutrino masses and Majorana CP phases.

Thanks to the observations that equations A and G (or C and F or D and E) always have their two sides vanish simultaneously and equations E and F (or A, B, C, D and G) are not allowed to have their two sides vanish simultaneously, one just needs to consider the cases where equations A&G, B, C&F, D&E, A&B&G, A&C&F&G, A&D&E&G, B&C&F, B&D&E, A&B&C&F&G or A&B&D&E&G have their two sides vanish. In the case of two sides of equations A&G, B, C&F or D&E being vanishing, there is one neutrino mass sum rule. In the case of two sides of equations A&B&G, A&C&F&G, A&D&E&G, B&C&F or B&D&E being vanishing, there are two neutrino mass sum rules. In the case of two sides of A&B&C&F&G or A&B&D&E&G being vanishing, there are three neutrino mass sum rules. The neutrino mass sum rule Eq. (37) arising from the vanishing of two sides of equation B can only be fulfilled in the IH case. By taking $\phi_e = \pi/2$ and $\phi_\mu = -\phi_\tau$, the texture of $\bar{M}_\nu$ obtained in the case of two sides of equations A&C&F&G being vanishing can reproduce the specific texture given by the neutrino $\mu$-$\tau$ reflection symmetry. In the case of two sides of equations A&B&C&F&G, the unknown neutrino parameters can be completely determined: the neutrino masses are of the inverted hierarchy with $m_3 \simeq 0.02$ eV while the Majorana CP phases are $[\rho, \sigma] = [\pi/2, 0]$. But in the case of two sides of equations A&B&D&E&G, the resulting neutrino mass sum rule has no chance to be in agreement with the experimental results.

Finally, we note that the results obtained in this work can be further studied from two aspects: On the one hand, one can study the origins of these special textures from some underlying flavor symmetries in the lepton sector [10]. On the other hand, one can study the breaking effects of these special textures so as to accommodate the deviations of $\theta_{23}$ and $\delta$ from $\pi/4$ and $-\pi/2$ [11].

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