Highest weight representations of $U_q(\widehat{sl}(2|1))$ and correlation functions of the $q$-deformed supersymmetric $t$-$J$ model

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We re-examine the level-one irreducible highest weight representations of the quantum affine superalgebra $U_q(\widehat{sl}(2|1))$ and derive the characters and supercharacters associated with these representations. We calculate the exchange relations of the vertex operators and find that these vertex operators satisfy the graded Faddeev-Zamolodchikov algebra. We give an integral expression of the correlation functions of the $q$-deformed supersymmetric $t$-$J$ model and derive the difference equations which they satisfy.

I. INTRODUCTION

The algebraic analysis method based on infinite dimensional non-abelian symmetries such as Virasoro and Kac-Moody algebra symmetries has proved eminently successful in both formulating and solving low-dimensional systems on the critical points (see e.g. $^1$ and references therein). The key elements in this approach are the highest weight representation theory and vertex operators which are intertwiners between two irreducible highest weight representations. One advantage over other abelian symmetry methods such as the Bethe ansatz and QISM is that it enables one to compute correlation functions and form factors exactly in the form of integral representations.

Following the success of this approach, one wonders if a similar program can be carried out for massive integrable systems, i.e. integrable systems away from the critical points. After the discovery of quantum groups and quantum affine algebras, one has every reason to believe that the goal is achievable because these quantum algebras are exactly the non-abelian symmetries underlying the off-critical integrable models. One breakthrough is due to Frenkel and Reshetikhin $^2$ who introduced the $q$-deformed vertex operators associated with the quantum affine algebras. They showed that the correlation functions satisfy a set of difference equations, the so-called $q$-KZ equations. Using the $q$-vertex operators and the level one free boson realization of quantum affine algebras of Frenkel and Jing $^3$, the Kyoto group $^4$ (for a nice review, see $^5$) succeeded in diagonalizing the XXZ spin chain directly in the thermodynamic limit. The method is very similar to that used in the critical cases. The idea is to work directly on an infinite lattice and use the full quantum affine algebra symmetry of the model. This way, the highest weight representation theory and $q$-vertex operators of the non-abelian quantum symmetry enter the game in a similar way as in the critical cases. Soon after this ground-breaking work of the Kyoto group, several generalizations have been considered (see e.g. $^6$ for the vertex models with $U_q(\widehat{sl}(n))$-symmetry and $^7$ for the face type statistical mechanics models).

Like in the critical cases, off-critical integrable models with (quantum) superalgebra symmetries have occupied an important place. So it is natural to generalize the above program further to the supersymmetric case. Quantum (affine) superalgebras have much more complicated structures and representation theory than their non-super counterparts $^8$. In general, the knowledge about the representation theory of quantum affine superalgebras is still very much incomplete. For the type I quantum affine superalgebra $U_q(\widehat{sl}(m|n))$ ($m \neq n$), level-one representations and $q$-vertex operators have been studied in $^9$ (see $^10$ for the level-$k$ free boson realization of $U_q(\widehat{sl}(2|1))$). In particular, the level-one irreducible highest weight representations of $U_q(\widehat{sl}(2|1))$ have been investigated in details and the character formulae for these modules have been conjectured.

In this paper, we re-examine the level-one irreducible highest weight representations of $U_q(\widehat{sl}(2|1))$ and calculate their characters by means of the BRST resolution. We also derive the super characters associated with these modules. We find that the conjecture 2.2 proposed in $^{10}$ needs to be slightly modified and we give a modified conjecture (conjecture 1 below). In section 4, we compute the exchange relations of the $q$-vertex operators and show that these vertex operators satisfy the graded Faddeev-Zamolodchikov algebra. A Miki’s construction of $U_q(\widehat{sl}(2|1))$ is also given. In section 5, we consider the application of the irreducible highest weight modules and $q$-vertex operators to the $q$-deformed supersymmetric $t$-$J$ model on an infinite lattice. Generalizing the Kyoto group’s work $^4$, we give a mathematical definition of the space of physical states of the supersymmetric $t$-$J$ model and define its local structure and local operators. We compute the one-point correlation functions of the local operators and give an integral expression of the correlation functions. A set of infinite number of difference equations satisfied by the correlation functions has also been derived.
II. PRELIMINARIES

In this section, we briefly review the bosonization of quantum affine superalgebra $U_q(\hat{sl}(2|1))$ at level one \cite{10}.

A. Quantum affine superalgebra $U_q(\hat{sl}(2|1))$

The symmetric Cartan matrix of the affine Lie superalgebra $\hat{sl}(2|1)$ is

$$ (a_{ij}) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix} $$

where $i,j = 0,1,2$. Quantum affine superalgebra $U_q(\hat{sl}(2|1))$ is a $q$-analogue of the universal enveloping algebra of $\hat{sl}(2|1)$ generated by the Chevalley generators $\{e_i, f_i, t_i^{\pm1}, d | i = 0,1,2\}$, where $d$ is the usual derivation operator. The $Z_2$-grading of the generators are $[e_0] = [f_0] = [e_2] = [f_2] = 1$ and zero otherwise. The defining relations are

$$ t_i t_j = t_j t_i, \quad [d, e_i] = \delta_{i,0} e_i, \quad [d, f_i] = -\delta_{i,0} f_i, $$

$$ t_i e_j t_i^{-1} = q^{\alpha_{ij}} e_j, \quad t_i f_j t_i^{-1} = q^{-\alpha_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij} t_i^{-1} - t_i^{-1}, $$

plus the extra $q$-Serre type relations \cite{13} which we omit. Here and throughout, $[X,Y]_\xi = XY - (-1)^{|X||Y|} \xi YX$ and $[X,Y]_1 = [X,Y]$.

$U_q(\hat{sl}(2|1))$ is a quasi-triangular Hopf superalgebra endowed with the $\mathbb{Z}_2$-graded Hopf algebra structure:

$$ \Delta(t_i) = t_i \otimes t_i, \quad \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, $$

$$ \epsilon(t_i) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0, $$

$$ S(e_i) = -t_i^{-1} e_i, \quad S(f_i) = -f_i t_i, \quad S(t_i^{\pm1}) = t_i^{\mp1}, \quad S(d) = -d. \quad (\text{II.1}) $$

Note the antipode $S$ is a graded algebra anti-automorphism. Namely for homogeneous elements $a,b \in U_q(\hat{sl}(2|1))$, $S(ab) = (-1)^{|a||b|} S(b) S(a)$. The multiplication rule for the tensor product is $\mathbb{Z}_2$ graded and is defined for homogeneous elements $a,b,a',b' \in U_q(\hat{sl}(2|1))$ by $(a \otimes b)(a' \otimes b') = (-1)^{|a'||a|}(aa' \otimes bb')$, which extends to inhomogeneous elements through linearity.

$U_q(\hat{sl}(2|1))$ can also be realized by the Drinfeld generators \cite{12} $\{\xi_d, X_m^{\pm;i}, h_m^{\pm}, K^i, \gamma^{1/2} | i = 1,2, m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}\}$. The $\mathbb{Z}_2$-grading of the Drinfeld generators are $[X_m^{\pm;i}] = 1$ ($m \in \mathbb{Z}$) and zero otherwise. The relations read \cite{13, 14}.

**γ** is central, $[K^i, h_m^{\pm}] = 0$, $[d, K^i] = 0$, $[d, h_m^{\pm}] = mh_m^{\pm}$

$$ [h_m^{\pm}, h_n^{\pm}] = \delta_{m+n,0} \frac{\alpha_{ij} m (\gamma^m - \gamma^{-m})}{m(q-q^{-1})} $$

$$ K^i X_m^{\pm,j} = q^{\pm \alpha_{ij}} X_m^{\pm,j} K^i, \quad [d, X_m^{\pm,j}] = m X_m^{\pm,j} $$

$$ [h_m^{\pm}, X_n^{\pm,j}] = \frac{\alpha_{ij} m}{m} \gamma^{\pm |m|/2} X_n^{\pm,j}, $$

$$ [X_m^{\pm;i}, X_n^{\pm;j}] = \frac{\delta_{ij}}{q-q^{-1}} (\gamma^{(m-n)/2} X_{m+n}^{\pm,j} - \gamma^{-(m-n)/2} X_{m+n}^{\pm,j}) $$

$$ [X_m^0, X_n^0] = 0, \quad [X_m^0, X_n^{\pm}] = 0, \quad [X_n^0, X_m^{\pm}] = 0 $$

where $[m] = \frac{q^m - q^{-m}}{q-q^{-1}}$.

The Chevalley generators are related to the Drinfeld generators by the formulæ:

$$ t_i = K_i, \quad e_i = X_0^{+;i}, \quad t_0 = \gamma (K^2 K^i)^{-1}, \quad f_i = X_0^{-;i} \quad \text{for } i = 1,2, $$

$$ e_0 = -[X_0^{-;0}, X_0^{-;0}] q^{-1} (K^2 K^i)^{-1}, \quad f_0 = K^i K^2 [X_0^{+;1}, X_0^{+;0}] q. \quad (\text{II.2}) $$

$$ e_0 = -[X_0^{-;0}, X_0^{-;0}] q^{-1} (K^2 K^i)^{-1}, \quad f_0 = K^i K^2 [X_0^{+;1}, X_0^{+;0}] q. \quad (\text{II.3}) $$
B. Bosonization of $U_q(\hat{sl}(2|1))$ at level one

Let us introduce the bosonic $q$-oscillators \( \{ a_n^i, a_n^i, b_n, c_n, Q_{a_1}, Q_{a_2}, Q_b, Q_c \mid n \in \mathbb{Z} \} \) which satisfy the commutation relations

\[
[ a_m^i, a_n^j ] = \delta_{i,j} \delta_{m+n,0} \frac{[m]_q^2}{m}, \quad [ a_{0}^i, Q_{a_{a}} ] = \delta_{i,j},
\]
\[
[ b_m, b_n ] = -\delta_{m+n,0} \frac{[m]_q^2}{m}, \quad [ b_{0}, Q_{b} ] = -1
\]
\[
[ c_m, c_n ] = \delta_{m+n,0} \frac{[m]_q^2}{m}, \quad [ c_{0}, Q_{c} ] = 1.
\]

The remaining commutators vanish. Define the Drinfeld currents or generating functions $X_i^\pm (z) = \sum_{m \in \mathbb{Z}} X_i^\pm(z^{-m-1})$, and introduce $h_i^k$ by setting $K^i = q^{h_i^k}$. Let $Q_{h^1} = Q_{a_1} - Q_{a_2}$, $Q_{h^2} = Q_{a_2} + Q_b$. Define

\[
h_i(z; \kappa) = -\sum_{n \neq 0} \frac{h_i^n}{[n]_q^{-\kappa n}} z^{-n} + Q_{h_i} + h_i^0 \ln z. \quad (\text{II.4})
\]

Other bosonic fields $b(z; \kappa)$ and $c(z; \kappa)$ are defined similarly. Also introduce the $q$-differential operator defined by $\partial_z f(z) = \frac{(f(qz) - f(q^{-1}z))}{(q - q^{-1})z}$. We have [10]

**Theorem 1**: The Drinfeld generators at level-one are realized by the free boson fields as

\[
\gamma = q, \quad h_m^1 = a_m q^{-|m|/2} - a_m q^{-|m|/2}, \quad h_m^2 = a_m q^{-|m|/2} + b_m q^{-|m|/2}, \quad m \in \mathbb{Z},
\]
\[
X_1^\pm (z) = \pm e^{\pm h_1(z; \frac{1}{2})} : e^{\pm \pi n a_0} : \quad X_2^\pm (z) = : e^{h_2(z; \frac{1}{2})} e^{\pm q z} e^{\mp \pi n a_0} :.
\]

C. Level-one vertex operators

Let $V$ be the 3-dimensional evaluation representation of $U_q(\hat{sl}(2|1))$ and $E_{ij}$ be the $3 \times 3$ matrix whose $(i, j)$-element is unity and zero otherwise. The grading of the basis vectors $v_1, v_2, v_3$ of $V$ is chosen to be $[v_1] = [v_2] = 1, [v_3] = 0$. Then the 3-dimensional level-0 representation $V_0$ of $U_q(\hat{sl}(2|1))$ is given by

\[
e_1 = E_{12}, \quad e_2 = E_{23}, \quad e_0 = -z E_{31}, \quad t_1 = q E_{11} - E_{22}, \quad t_2 = q E_{22} + E_{33},
\]
\[
f_1 = E_{21}, \quad f_2 = E_{32}, \quad f_0 = z^{-1} E_{13}, \quad t_0 = q^{-1} E_{11} - E_{33}, \quad d = z \frac{d}{dz}.
\]

We define the dual modules $V^*_V$ of $V$ by $\pi V \cdot s(a) = \pi V \cdot (S(a))^{st}$, $\forall a \in U_q(\hat{sl}(2|1))$, where $st$ is the supertransposition operation.

Throughout, we denote by $V(\lambda)$ an irreducible highest weight $U_q(\hat{sl}(2|1))$-module with highest weight $\lambda$. Consider the following intertwiners of $U_q(\hat{sl}(2|1))$-modules:

\[
\Phi(\lambda) : V(\lambda) \rightarrow V(\mu) \otimes V, \quad \Phi(\mu) : V(\mu) \otimes V \rightarrow V(\lambda),
\]
\[
\Psi(\lambda) : V(\lambda) \rightarrow V \otimes V(\mu), \quad \Psi(\mu) : V \otimes V(\mu) \rightarrow V(\lambda).
\]

They are intertwiners in the sense that for any $x \in U_q(\hat{sl}(2|1))$,

\[
\Theta(x) \cdot x = \Delta(x) \cdot \Theta(x), \quad \Theta(z) = \Phi(z), \Phi^*(z), \Psi(z), \Psi^*(z). \quad (\text{II.5})
\]

The intertwiners are even operators, that is their grading is $[\Theta(z)] = 0$. $\Phi(z)$ ($\Phi^*(z)$) is called type I (dual) vertex operator and $\Psi(z)$ ($\Psi^*(z)$) type II (dual) vertex operator.

We expand the vertex operators as
Define the operators $\Phi_j(z)$, $\phi_j^\ast(z)$, $\psi_j(z)$ and $\psi_j^\ast(z)$ ($j = 1, 2, 3$) by

$$
\Phi(z) = \sum_{j=1,2,3} \Phi_j(z) \otimes v_j, \quad \Phi^\ast(z) = \sum_{j=1,2,3} \Phi_j^\ast(z) \otimes v_j^\ast,
$$

$$
\Psi(z) = \sum_{j=1,2,3} v_j \otimes \Psi_j(z), \quad \Psi^\ast(z) = \sum_{j=1,2,3} v_j^\ast \otimes \Psi_j^\ast(z).
$$

(II.6)

(II.7)

Then we have

**Proposition 1** (II.14): The operators $\phi_j(z)$, $\phi_j^\ast(z)$, $\psi_j(z)$ and $\psi_j^\ast(z)$ satisfy the same commutation relations as $\Phi(z)$, $\Phi^\ast(z)$, $\Psi(z)$ and $\Psi^\ast(z)$, respectively.

### III. LEVEL-ONE HIGHEST WEIGHT $U_q(\widehat{sl}(2|1))$-MODULES REVISITED

In this section, we re-examine the level-one irreducible highest weight $U_q(\widehat{sl}(2|1))$-modules in the Fock space defined by the bosonic $q$-oscillators.

Denote by $F_{\lambda_1, \lambda_2, \lambda_b, \lambda_c}$ the bosonic Fock spaces generated by $a_{-m}, b_{-m}, c_{-m}$ ($m > 0$) over the vector $|\lambda_1, \lambda_2, \lambda_b, \lambda_c >$

$$
F_{\lambda_1, \lambda_2, \lambda_b, \lambda_c} = C[a_{-1, -2, \ldots, -b_{-1, -2, \ldots, c_{-1, \ldots}}] |\lambda_1, \lambda_2, \lambda_b, \lambda_c >.
$$

(III.11)

where

$$
|\lambda_1, \lambda_2, \lambda_b, \lambda_c > = e^{\lambda_1 Q_a + \lambda_2 Q_b + \lambda_3 Q_c} |0 >.
$$

(III.12)

The vacuum vector $|0 >$ is defined by $a_m^i |0 > = b_m |0 > = c_m |0 > = 0$ for $m > 0$. Obviously,

$$
\begin{align*}
a_m^i |\lambda_1, \lambda_2, \lambda_b, \lambda_c > &= 0, \quad \text{for } m > 0, \\
b_m |\lambda_1, \lambda_2, \lambda_b, \lambda_c > &= 0, \quad \text{for } m > 0, \\
c_m |\lambda_1, \lambda_2, \lambda_b, \lambda_c > &= 0, \quad \text{for } m > 0.
\end{align*}
$$

(III.13)

$|\lambda_1, \lambda_2, \lambda_b, \lambda_c >$ is said to be a $U_q(\widehat{sl}(2|1))$ highest weight vector of the weight $\lambda = \lambda_0 \Lambda_0 + \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2$, if it satisfies

$$
\begin{align*}
\epsilon_i |\lambda_1, \lambda_2, \lambda_b, \lambda_c > &= 0, \\
h_i |\lambda_1, \lambda_2, \lambda_b, \lambda_c > &= \lambda^i |\lambda_1, \lambda_2, \lambda_b, \lambda_c >, \quad i = 0, 1, 2.
\end{align*}
$$

(III.13)
Here $\Lambda_i$ ($i = 0, 1, 2$) are the fundamental weights of $U_q(\hat{sl}(2|1))$. In order to classify the highest weight $U_q(\hat{sl}(2|1))$-modules in the Fock space, we introduce the spaces

$$F_{(\alpha;\beta)} = \bigoplus_{i,j \in \mathbb{Z}} F_{\beta+i,\beta-i+j,\alpha+j}$$

where $\alpha$ and $\beta$ are arbitrary parameters. In the following we restrict ourselves to $\alpha \in \mathbb{Z}$ case.

Remark. In [10], the following two extra spaces

$$F_{((1,0);\beta)} = \bigoplus_{i,j \in \mathbb{Z}} F_{\beta+1+i,\beta-i+j,j},$$

$$F_{((0,1);\beta)} = \bigoplus_{i,j \in \mathbb{Z}} F_{\beta+i,\beta+1-i+j,j}$$

were also introduced. However, it is easily seen that $F_{((1,0);\beta)}$, $F_{((0,1);\beta)} \subset F_{(\alpha;\beta)}$. In fact, $F_{((1,0);\beta)} = F_{(1;\beta)}$ and $F_{((0,1);\beta)} = F_{(2;\beta)}$.

It can be shown that the bosonized action of $U_q(\hat{sl}(2|1))$ on $F_{(\alpha;\beta)}$ is closed, i.e. $U_q(\hat{sl}(2|1))F_{(\alpha;\beta)} = F_{(\alpha;\beta)}$. Hence each Fock space $F_{(\alpha;\beta)}$ constitutes a $U_q(\hat{sl}(2|1))$-module. However these modules are not irreducible in general. To obtain the irreducible subspaces in $F_{(\alpha;\beta)}$, we introduce a pair of fermionic currents [10, 15].

The mode expansion of $\eta(z)$ is well defined on $F_{(\alpha;\beta)}$ for $\alpha \in \mathbb{Z}$, and the modes satisfy the relations

$$\xi_m\xi_n + \xi_n\xi_m = \eta_m\eta_n + \eta_n\eta_m = 0, \quad \xi_m\eta_n + \eta_n\xi_m = \delta_{m,n}.$$ 

Obviously, $\eta_0\xi_0$ and $\xi_0\eta_0$ qualify as projectors and so we use them to decompose $F_{(\alpha;\beta)}$ into a direct sum $F_{(\alpha;\beta)} = \eta_0\xi_0 F_{(\alpha;\beta)} \oplus \xi_0\eta_0 F_{(\alpha;\beta)}$. Following [10], $\eta_0\xi_0 F_{(\alpha;\beta)}$ is referred to as Ker$_{\eta_0}$ and $\xi_0\eta_0 F_{(\alpha;\beta)} = F_{(\alpha;\beta)}/\eta_0\xi_0 F_{(\alpha;\beta)}$ as Coker$_{\eta_0}$.

Since $\eta_0$ commutes (or anticommutes) with $U_q(\hat{sl}(2|1))$, Ker$_{\eta_0}$ and Coker$_{\eta_0}$ are both $U_q(\hat{sl}(2|1))$-modules.

A. Characters and supercharacters

Now we study the characters and supercharacters of these $U_q(\hat{sl}(2|1))$-modules Ker$_{\eta_0}$ and Coker$_{\eta_0}$. We first of all bosonize the derivation operator $d$ as

$$d = \sum_{m=1}^{\infty} \sum_{[n]_2} m^2 \left[ h_{-m} h_m + h_{-m}^2 h_m + \frac{1}{n} (q^m + q^{-m}) h_{-m} h_m^2 \right] - \frac{m^2}{[n]_2^2} c_{-m} c_m,$$

$$+ \left\{ h_{m}^3 h_0^2 + h_0^4 h_{-2} - \frac{1}{2} c_0 (c_0 + 1) \right\}.$$  

One can easily check that this $d$ obeys the commutation relations,

$$[d, K^+] = 0, \quad [d, h_m^i] = m h_m^i, \quad [d, X_m^{\pm,j}] = m X_m^{\pm,j},$$

as required. Moreover, we have $[d, \xi_0] = [d, \eta_0] = 0$.

Remark. In [10], the authors also gave a bosonized expression for the operator $d$. However, their $d$ does not satisfy the derivation properties (III.17).

The character and supercharacter of a $U_q(\hat{sl}(2|1))$-module $V$ are defined by

$$ch_V(q, x, y) = tr_V (q^{-2d} x^{h_0^1} y^{h_0^2}),$$

$$Sch_V(q, x, y) = St (q^{-2d} x^{h_0^1} y^{h_0^2}) = tr_V ((-1)^{N_f} q^{-2d} x^{h_0^1} y^{h_0^2}),$$

respectively. The Fermi-number operator $(-1)^{N_f}$ can also be bosonized and we derive $(-1)^{N_f} = (-1)^{b_0}$. Indeed such a bosonized operator satisfies
Proposition 2:

Proof. We have the following BRST complexes:

\[
(1)^{b_0} \Theta(z) = (-1)^{[\Theta(z)]} \Theta(z)(1)^{b_0}, \quad \text{for} \quad \Theta(z) = X^{\pm}_i(z), \Phi(z), \Phi^*(z), \Psi(z), \Psi^*(z),
\]
as required. The characters of \( \text{Ker}_{\eta_0} \) and \( \text{Coker}_{\eta_0} \) were calculated in [10] by inserting the projectors \( \eta_0 \xi_0 \) and \( \xi_0 \eta_0 \) into the trace over the Fock space \( F_{(\alpha;\beta)} \) and then computing the resultant traces. Here we rederive these character formulae by using the BRST resolution. We also derive the supercharacters of \( \text{Ker}_{\eta_0} \) and \( \text{Coker}_{\eta_0} \).

Let us define the Fock spaces, for \( l \in \mathbb{Z} \),

\[
F^{(l)}_{(\alpha;\beta)} = \bigoplus_{i,j \in \mathbb{Z}} F^{\beta+i,\beta-i+j,\beta+\alpha+j,\alpha-j+l}_{i,j}. 
\]

We have \( F^{(0)}_{(\alpha;\beta)} = F_{(\alpha;\beta)} \). It can be shown that \( \eta_0, \xi_0 \) intertwine various Fock spaces

\[
\eta_0 : F^{(l)}_{(\alpha;\beta)} \rightarrow F^{(l+1)}_{(\alpha;\beta)}, \quad \xi_0 : F^{(l)}_{(\alpha;\beta)} \rightarrow F^{(l-1)}_{(\alpha;\beta)},
\]

We have the following BRST complexes:

\[
\begin{align*}
\cdots \xrightarrow{Q_{l-1}=\eta_0} F^{(l)}_{(\alpha;\beta)} \xrightarrow{Q_l=\eta_0} F^{(l+1)}_{(\alpha;\beta)} \xrightarrow{Q_{l+1}=\eta_0} \cdots \\
\cdots \xrightarrow{Q_{l-1}=\xi_0} F^{(l)}_{(\alpha;\beta)} \xrightarrow{Q_l=\xi_0} F^{(l+1)}_{(\alpha;\beta)} \xrightarrow{Q_{l+1}=\xi_0} \cdots,
\end{align*}
\]

where \( \mathbf{O} \) is an operator such that \( F^{(l)}_{(\alpha;\beta)} \rightarrow F^{(l)}_{(\alpha;\beta)} \), and \( \mathbf{O} \) commutes with the BRST charges \( Q_l \). We have

Proposition 2:

\[
\begin{align*}
\text{Ker}_{Q_l} = \text{Im}_{Q_{l-1}}, \quad \text{for any} \quad l \in \mathbb{Z} \quad \text{and} \\
\text{tr}(\mathbf{O})|_{\text{Ker}_{Q_l}} = \text{tr}(\mathbf{O})|_{\text{Im}_{Q_{l-1}}} = \text{tr}(\mathbf{O})|_{\text{Coker}_{Q_{l-1}}},
\end{align*}
\]

Proof. It follows from the fact that \( \eta_0\xi_0 + \xi_0\eta_0 = 1 \), \( \xi_0^2 = \eta_0^2 = 0 \) and \( \eta_0\xi_0 \) (\( \xi_0\eta_0 \)) is the projection operator from \( F_{(\alpha;\beta)} \) to \( \text{Ker}_{\eta_0} \) (\( \text{Coker}_{\eta_0} \)).

In the following we simply write \( \text{Ker}_{\eta_0} \) and \( \text{Coker}_{\eta_0} \) of \( F_{(\alpha;\beta)} \) as \( \text{Ker}F_{(\alpha;\beta)} \) and \( \text{Coker}F_{(\alpha;\beta)} \), respectively. Then

Proposition 3: The characters of \( \text{Ker}F_{(\alpha;\beta)} \) and \( \text{Coker}F_{(\alpha;\beta)} \) for \( \alpha \in \mathbb{Z} \) are given by

\[
\begin{align*}
\text{ch}_{\text{Ker}F_{(\alpha;\beta)}} &= \text{ch}_{F^{(0)}_{(\alpha;\beta)}} = \sum_{l=1}^{\infty} (-1)^{l+1} \text{ch}^{(l)}_{F_{(\alpha;\beta)}} = \frac{q^{-\alpha(\alpha+1)}}{\prod_{n=1}^{\infty}(1-q^{2n})^3} \\
&\times \sum_{i,j \in \mathbb{Z}} \sum_{l=1}^{\infty} (-1)^{l+1} q^{l(l-1)+2l(a-j)+(2i^2-2ij+j^2)+x^{2i-j}y^{a-i}},
\end{align*}
\]

\[
\begin{align*}
\text{ch}_{\text{Coker}F_{(\alpha;\beta)}} &= \text{ch}_{\text{Coker}F^{(0)}_{(\alpha;\beta)}} = \sum_{l=1}^{\infty} (-1)^{l+1} \text{ch}^{(l)}_{F_{(\alpha;\beta)}} = \frac{q^{-\alpha(\alpha+1)}}{\prod_{n=1}^{\infty}(1-q^{2n})^3} \\
&\times \sum_{i,j \in \mathbb{Z}} \sum_{l=1}^{\infty} (-1)^{l+1} q^{l(l-1)-2l(a-j)+(2i^2-2ij+j^2)+x^{2i-j}y^{a-i}},
\end{align*}
\]

respectively.

Remark. By using the following identity:

\[
\sum_{l \in \mathbb{Z}} (-1)^{l+1} q^{l(l-1)+2lt} = 0, \quad \text{for any} \quad t \in \mathbb{Z},
\]

one can show that the above character formulae coincide with those given by Kimura et al in [10].
Proposition 4: The supercharacters of $\text{Ker} F_{(\alpha, \beta)}$ and $\text{CoKer} F_{(\alpha, \beta)}$ for $\alpha \in \mathbb{Z}$ are given by

$$\text{Sch}_{\text{Ker} F_{(\alpha, \beta)}} = \text{Sch}_{F^{(0)}_{(\alpha, \beta)}} = \sum_{l=1}^{\infty} (-1)^{l+1} \text{Sch}_{F^{(l)}_{(\alpha, \beta)}} = \frac{(-1)^{\beta-a} q^{-\alpha(\alpha+1)}}{\prod_{n=1}^{\infty} (1-q^{2n})} \times \sum_{i,j \in \mathbb{Z}} (-1)^i \sum_{l=1}^{\infty} (-1)^l q^{(l-1)+2l(\alpha-j)+(2i-j)^2} x^{2i-j} y^\alpha, \quad (\text{III}.23)$$

$$\text{Sch}_{\text{CoKer} F_{(\alpha, \beta)}} = \text{Sch}_{F^{(0)}_{(\alpha, \beta)}} = \sum_{l=1}^{\infty} (-1)^{l+1} \text{Sch}_{F^{(l)}_{(\alpha, \beta)}} = \frac{(-1)^{\beta-a} q^{-\alpha(\alpha+1)}}{\prod_{n=1}^{\infty} (1-q^{2n})} \times \sum_{i,j \in \mathbb{Z}} (-1)^i \sum_{l=1}^{\infty} (-1)^l q^{(l+1)-2l(\alpha-j)+(2i-j)^2} x^{2i-j} y^\alpha. \quad (\text{III}.23)$$

Proof. We sketch the proof of these two propositions. Because $q^{-2d}x^h y^b$ and $(-1)^{N_f} q^{-2d}x^h y^b$ commute with the BRST charges $Q_i$, the trace over Ker and Coker can be written as the sum of trace over $F^{(l)}_{(\alpha, \beta)}$. The latter can be computed by the technique introduced in [10], which is given in appendix C.

Since $F_{(\alpha, \beta)} = F_{(\alpha-1; \beta+1)}$, we have

Corollary 1: The following relations hold for any $\alpha$ and $\beta$,

$$\text{ch}_{\text{Coker} F_{(\alpha, \beta)}} = \text{ch}_{\text{Coker} F_{(\alpha+1; \beta-1)}} = \text{ch}_{\text{Coker} F_{(\alpha+1; \beta-1)}},$$

$$\text{Sch}_{\text{Coker} F_{(\alpha, \beta)}} = \text{Sch}_{\text{Coker} F_{(\alpha+1; \beta-1)}}.$$  

B. $U_q(\hat{\mathfrak{sl}}(2|1))$-module structure of $F_{(\alpha; \beta; \alpha)}$

Set $\lambda_\alpha = (1-\alpha)\Lambda_0 + \alpha \Lambda_2$, and

$$|\lambda_\alpha| = |\beta - \alpha, \beta - \alpha, \beta - 2\alpha, -\alpha| \in F_{(\alpha, \beta; \alpha)}, \quad \alpha \in \mathbb{Z},$$

$$|A_1| = |\beta, \beta - 1, \beta - 1, 0| \in F_{(1; \beta-1)},$$

$$|A_2| = |\beta - 1, \beta - 1, \beta - 2, 0| \in F_{(2; \beta-2)}.$$  

The above vectors play the role of the highest weight vectors of $U_q(\hat{\mathfrak{sl}}(2|1))$-modules [11]. One can check

$$\eta_0 |\lambda_\alpha| = 0, \quad \text{for} \quad \alpha = 0, -1, -2, \cdots,$$

$$\eta_0 |A_1| = 0, \quad \eta_0 |A_2| = 0,$$

$$\eta_0 |\lambda_\alpha| \neq 0, \quad \text{for} \quad \alpha = 1, 2, \cdots. \quad (\text{III}.24)$$

It follows that the modules $\text{Coker} F_{(\alpha; \beta; \alpha)}$ ($\alpha = 1, 2, 3, \ldots$), $\text{Ker} F_{(\alpha; \beta; \alpha)}$ ($\alpha = 0, -1, -2, \ldots$), $\text{Ker} F_{(1; \beta-1)}$ and $\text{Ker} F_{(2; \beta-2)} = \text{Coker} F_{(1; \beta-1)}$ are highest weight $U_q(\hat{\mathfrak{sl}}(2|1))$-modules. Denote by $\hat{V}(\lambda_\alpha)$ and $\hat{V}(A_1)$ the $U_q(\hat{\mathfrak{sl}}(2|1))$-modules with highest weights $\lambda_\alpha$ and $A_1$, respectively. From (\text{III}.23) and corollary 1, we find

Theorem 2: We have the following identifications of the highest weight $U_q(\hat{\mathfrak{sl}}(2|1))$-modules:

$$\hat{V}(\lambda_\alpha) \cong \text{Ker} F_{(\alpha; \beta; \alpha)} \cong \text{Coker} F_{(\alpha-1; \beta-\alpha+1)}, \quad \text{for} \quad \alpha = 0, -1, -2, \cdots,$$

$$\cong \text{Coker} F_{(\alpha; \beta; \alpha)} \cong \text{Ker} F_{(\alpha+1; \beta-\alpha-1)}, \quad \text{for} \quad \alpha = 1, 2, \cdots,$$

$$\hat{V}(A_1) \cong \text{Ker} F_{(1; \beta-1)} \cong \text{Coker} F_{(0; \beta)}. \quad (\text{III}.25)$$

Therefore we have
Theorem 3: For \( \alpha \in \mathbb{Z} \), each Fock space \( F(\alpha; \beta - \alpha) \) can be decomposed explicitly into a direct sum of the highest weight \( U_q(\hat{sl}(2|1)) \)-modules

\[
\begin{array}{ccc}
& Ker & Coker \\
F(-2; \beta + 2) = & \bar{V}(\lambda_{-2}) & \bigoplus & \bar{V}(\lambda_{-1}) \\
& \phi(z) \uparrow \downarrow \phi^*(z) & \phi(z) \uparrow \downarrow \phi^*(z) \\
F(-1; \beta + 1) = & \bar{V}(\lambda_{-1}) & \bigoplus & \bar{V}(\Lambda_0) \\
& \phi(z) \uparrow \downarrow \phi^*(z) & \phi(z) \uparrow \downarrow \phi^*(z) \\
F(0; \beta) = & \bar{V}(\Lambda_0) & \bigoplus & \bar{V}(\Lambda_1) \\
& \phi(z) \uparrow \downarrow \phi^*(z) & \phi(z) \uparrow \downarrow \phi^*(z) \\
F(1; \beta - 1) = & \bar{V}(\Lambda_1) & \bigoplus & \bar{V}(\Lambda_2) \\
& \phi(z) \uparrow \downarrow \phi^*(z) & \phi(z) \uparrow \downarrow \phi^*(z) \\
F(2; \beta - 2) = & \bar{V}(\Lambda_2) & \bigoplus & \bar{V}(\lambda_2) \\
& \phi(z) \uparrow \downarrow \phi^*(z) & \phi(z) \uparrow \downarrow \phi^*(z) \\
F(3; \beta - 3) = & \bar{V}(\lambda_2) & \bigoplus & \bar{V}(\lambda_3) \\
& \phi(z) \uparrow \downarrow \phi^*(z) & \phi(z) \uparrow \downarrow \phi^*(z) \\
& \ldots & \ldots & \ldots \\
\end{array}
\]

It is expected that the modules \( \bar{V}(\lambda_\alpha) \) and \( \bar{V}(\Lambda_1) \) are also irreducible with respect to the action of \( U_q(\hat{sl}(2|1)) \). From theorem 2, we are led to the following conjecture which is a modified version of the conjecture 2.2 proposed in [10].

**Conjecture 1**: \( \bar{V}(\lambda_\alpha) \) and \( \bar{V}(\Lambda_1) \) are irreducible highest weight \( U_q(\hat{sl}(2|1)) \)-modules of the weights \( \lambda_\alpha \) and \( \Lambda_1 \), respectively, i.e.

\[
\bar{V}(\lambda_\alpha) = V(\lambda_\alpha), \quad \alpha \in \mathbb{Z}, \\
\bar{V}(\Lambda_1) = V(\Lambda_1).
\]

(III.26)

IV. EXCHANGE RELATIONS OF VERTEX OPERATORS

In this section, we derive the exchange relations of the type I and type II vertex operators of \( U_q(\hat{sl}(2|1)) \). As expected, these vertex operators satisfy the graded Faddeev-Zamolodchikov algebra.

A. The R-matrix

Let \( R(z) \in \text{End}(V \otimes V) \) be the R-matrix of \( U_q(\hat{sl}(2|1)) \):

\[
R(z)(v_i \otimes v_j) = \sum_{kl} R_{kl}^{ij}(z)v_k \otimes v_l, \quad \forall v_i, v_j, v_k, v_l \in V,
\]

(IV.27)
where the matrix elements are given by

\[ R^{31}_{23}(z) = \frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R^{21}_{23}(z) = \frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R^{32}_{23}(z) = \frac{(q - q^{-1})z_2}{z_1 q - z_2 q^{-1}}, \quad R^{33}_{23}(z) = \frac{(q - q^{-1})z_2}{z_1 q - z_2 q^{-1}}. \]

The R-matrix satisfies the graded Yang-Baxter equation (YBE) on \( V \otimes V \otimes V \)

\[ R_{12}(z)R_{13}(zu)R_{23}(w) = R_{23}(w)R_{13}(zu)R_{12}(z), \]

and moreover enjoys: (i) initial condition, \( R(1) = P \) with \( P \) being the graded permutation operator; (ii) unitarity condition, \( R_{12}(z)R_{21}(w) = 1 \), where \( R_{21}(z) = PR_{12}(z)P; \) and (iii) crossing-unitarity,

\[ R^{-1, st_1}(z) ((q^{-1} q^{1/2} \otimes 1) R(zq^{-2})(q^{1/2} \otimes 1))^{st_1} = 1 \otimes 1, \]

where

\[ q^{2g} \triangleq \begin{pmatrix} q^{2g_1} & q^{2g_2} \\ q^{2g_3} & q^{2g_3} \end{pmatrix} = \begin{pmatrix} 1 & q^{-2} \\ q^{-2} & q^{-2} \end{pmatrix}. \]  

The various supertranspositions of the R-matrix are given by

\[ (R^{st_1}(z))^{kl}_{ij} = R(z)^{kl}_{ij}(-1)^{[i][j]+[k]}, \quad (R^{st_2}(z))^{kl}_{ij} = R(z)^{kj}_{il}(-1)^{[i][j]+[l]}, \]

\[ (R^{st_12}(z))^{kl}_{ij} = R(z)^{lj}_{ik}(-1)^{[i][j]+[l]+[j]+[k]} = R(z)^{ij}_{kl}. \]

B. The graded Faddeev-Zamolodchikov algebra

Now, we are in the position to calculate the exchange relations of the type I and type II vertex operators of \( U_q(\mathfrak{gl}(2|1)) \). Define

\[ \oint dz f(z) = \text{Res}(f) = f_{-1}, \quad \text{for formal series function } f(z) = \sum_{n \in \mathbb{Z}} f_n z^n. \]

Then the Chevalley generators of \( U_q(\mathfrak{gl}(2|1)) \) can be expressed by the integrals,

\[ e_1 = \oint dz X_1^+(z), \quad e_2 = \oint dz X_2^+(z), \quad f_1 = \oint dz X_1^-(z), \quad f_2 = \oint dz X_2^-(z), \]

\[ e_0 = -\oint \oint dz dw z [X_2^-(w), X_1^-(z)] q^{-b_0^{1} - b_0^3}, \]

\[ f_0 = \oint \oint dz dw z^{-1} q^{b_0^{1} + b_0^3} [X_1^+(z), X_2^+(w)]_q. \]

One can also obtain the integral expression of the vertex operators defined in [11.9]-[11.10]
\[ \phi_2(z) = \int dw : \left\{ \frac{e^{-c(wq,0)}}{wq(1 - \frac{wq}{w})} + \frac{e^{-c(wq^{-1},0)}}{zq^2\left(1 - \frac{wq}{zq}\right)} \right\} e^{-h_2^z(q^2; \frac{1}{2}) - h_2(w; \frac{1}{2}) + c(q^2; 0)} e^{i\pi h_0^z}, \]

\[ \phi_1(z) = \int dw_1 \int dw \frac{q^2 - 1}{w(1 - \frac{wq}{w})(1 - \frac{wq}{w^2})} \times \left\{ \frac{e^{-c(wq,0)}}{wq(1 - \frac{wq}{w})} + \frac{e^{-c(wq^{-1},0)}}{zq^2\left(1 - \frac{wq}{zq}\right)} \right\} e^{-h_2^z(q^2; \frac{1}{2}) - h_2(w; \frac{1}{2}) + h_1(w; \frac{1}{2}) + c(q^2; 0)} e^{i\pi a_0^z}, \]

\[ \phi_2^z(z) = \int dw \frac{1 - q^{-2}}{w(1 - \frac{wq}{w})(1 - \frac{wq}{w^2})} : e^{h_1^z(q^2; \frac{1}{2}) - h_1(w; \frac{1}{2})} e^{i\pi h_0^z}, \]

\[ \phi_3^z(z) = \int dw_1 \int dw \frac{1 - q^{-2}}{w(1 - \frac{wq}{w})(1 - \frac{wq}{w^2})} \times \left\{ \frac{e^{-c(wq,0)}}{wq(1 - \frac{wq}{w})} - \frac{e^{-c(wq^{-1},0)}}{zq^2\left(1 - \frac{wq}{zq}\right)} \right\} e^{-h_1^z(q^2; \frac{1}{2}) - h_1(w; \frac{1}{2}) - h_2(w; \frac{1}{2})} e^{i\pi a_0^z}, \]

\[ \psi_2(z) = \int dw \frac{1 - q^{-2}}{wq(1 - \frac{wq}{w})(1 - \frac{wq}{w^2})} : e^{-h_1^z(q^2; \frac{1}{2}) + h_1(w; \frac{1}{2})} e^{i\pi h_0^z}, \]

\[ \psi_3(z) = \int dw_1 \int dw \frac{(q^2 - 1)(q - q^{-1})}{wq(1 - \frac{wq}{w})(1 - \frac{wq}{w^2})} \times \left\{ e^{-h_1^z(q^2; \frac{1}{2}) + h_1(w; \frac{1}{2}) + h_2(w; \frac{1}{2}) + c(w; 0)} e^{i\pi a_0^z} \right\}, \]

\[ \psi_2^z(z) = \int dw \left\{ \frac{e^{-c(zq,0)}}{w(1 - \frac{wq}{w})} + \frac{e^{-c(zq^{-1},0)}}{zq^{-1}\left(1 - \frac{wq}{zq}\right)} \right\} : e^{h_2^z(z; \frac{1}{2}) + h_2(w; \frac{1}{2}) + h_1(w; \frac{1}{2}) + c(w; 0)} e^{i\pi h_0^z}, \]

\[ \psi_3^z(z) = \int dw_1 \int dw \frac{1 - q^{-2}}{w(1 - \frac{wq}{w})(1 - \frac{wq}{w^2})} \times \left\{ \frac{e^{-c(zq,0)}}{w(1 - \frac{wq}{w})} + \frac{e^{-c(zq^{-1},0)}}{zq^{-1}\left(1 - \frac{wq}{zq}\right)} \right\} e^{h_2^z(z; \frac{1}{2}) + h_2(w; \frac{1}{2}) + h_1(w; \frac{1}{2}) + c(w; 0)} e^{i\pi a_0^z}. \]

(IV.29)

Using these integral expressions and the relations given in appendix A and appendix B, we derive

**Proposition 5**: The bosonic vertex operators defined in (II.9)–(II.10) satisfy the graded Faddeev-Zamolodchikov algebra

\[ \phi_j(z_2)\phi_i(z_1) = \sum_{kl} \tilde{R}(\frac{z_1}{z_2})_{ij}^{kl} \phi_k(z_1)\phi_l(z_2)(-1)^{|i||j|}, \]

(IV.30)

\[ \phi_j^z(z_2)\phi_i^z(z_1) = \sum_{kl} \tilde{R}(\frac{z_1}{z_2})_{ij}^{kl} \phi_k^z(z_1)\phi_l^z(z_2)(-1)^{|i||j|}, \]

(IV.31)

\[ \psi_i(z_1)\psi_j(z_2) = \sum_{kl} \tilde{R}(\frac{z_1}{z_2})_{ij}^{kl} \psi_k(z_1)\psi_l(z_2)(-1)^{|i||j|}, \]

(IV.32)

\[ \psi_i^z(z_1)\psi_j^z(z_2) = \sum_{kl} \tilde{R}(\frac{z_1}{z_2})_{ij}^{kl} \psi_k^z(z_1)\psi_l^z(z_2)(-1)^{|i||j|}, \]

(IV.33)

\[ \psi_i^z(z_1)\phi_j(z_2) = -(-1)^{|i||j|}\phi_j(z_2)\psi_i^z(z_1), \]

(IV.34)

\[ \phi_j(z_2)\phi_i^z(z_1) = \sum_{kl} \tilde{R}(\frac{z_1}{z_2})_{ij}^{kl} \phi_k(z_1)\phi_l^z(z_2)(-1)^{|k||l|}, \]

(IV.35)

where \( \tilde{R}(z) = R^{-1,\text{st}}(z) \).

In the derivation of this proposition the fact that \( R(z)_{ij}^{kl}(-1)^{|k||l|} = R(z)_{ij}^{kl}(-1)^{|i||j|} \) is helpful.

**Proposition 6**: We have the first invertibility relations,
\[ \phi_i(z)\phi_j^*(z) = (-1)^{[j]_\delta} \delta_{ij}, \]
\[- \sum_k (-1)^{[k]} \phi_k^*(z) \phi_k(z) = 1, \quad (IV.36)\]

and the second invertibility relations,
\[ \phi_i^*(z q^2) \phi_j(z) = \delta_{ij} q^{2p_i}, \]
\[ \sum_k q^{-2p_k} \phi_k(z) \phi_k^*(z q^2) = 1. \quad (IV.37)\]

Using the fact that \( \eta \xi_0 \) is a projection operator, we can make the following identifications:
\[ \Phi_1(z) = \eta \xi_0 \phi_1(z) \eta \xi_0, \quad \Phi_1^*(z) = \eta \xi_0 \phi_1^*(z) \eta \xi_0, \quad (IV.38) \]
\[ \Psi(z) = \eta \xi_0 \psi(z) \eta \xi_0, \quad \Psi^*(z) = \eta \xi_0 \psi^*(z) \eta \xi_0. \quad (IV.39) \]

Since the vertex operators \( \phi, \phi^*, \psi, \psi^* \) commute (or anti-commute) with the BRST charge \( \eta \), we have

**Theorem 4**: Set
\[ \mu_\alpha = \begin{cases} \Lambda_\alpha, & \alpha = 0, 1, 2, \\ \Lambda_{\alpha - 1}, & \text{for } \alpha > 2, \\ \Lambda_\alpha, & \text{for } \alpha < 0. \end{cases} \quad (IV.40) \]

Then the vertex operators defined by (IV.38) and (IV.39) interweave the level-one irreducible highest weight \( U_q(\hat{sl}(2|1)) \)-modules \( V(\mu_\alpha) \) \( (\alpha \in \mathbb{Z}) \)
\[ \Phi(z) : V(\mu_\alpha) \rightarrow V(\mu_{\alpha - 1}) \otimes V_z, \quad \Phi^*(z) : V(\mu_\alpha) \rightarrow V(\mu_{\alpha + 1}) \otimes V_{z}^\ast, \]
\[ \Psi(z) : V(\mu_\alpha) \rightarrow V_z \otimes V(\mu_{\alpha - 1}), \quad \Psi^*(z) : V(\mu_\alpha) \rightarrow V_z^\ast \otimes V(\mu_{\alpha + 1}), \]

and satisfy the graded Faddeev-Zamolodchikov algebra. Moreover, the type I vertex operators \( \Phi(z), \Phi^*(z) \) obey the following invertibility relations:
\[ \Phi_1(z)\Phi_j^*(z)\big|_{V(\mu_\alpha)} = (-1)^{[j]_\delta} id_{V(\mu_\alpha)}, \quad (IV.41) \]
\[ - \sum_k (-1)^{[k]} \Phi_k^*(z)\Phi_k(z)\big|_{V(\mu_\alpha)} = id_{V(\mu_\alpha)}, \]
\[ \Phi_j^*(z q^2)\Phi_j(z)\big|_{V(\mu_\alpha)} = \delta_{ij} q^{2p_i} id_{V(\mu_\alpha)}, \quad (IV.42) \]
\[ \sum_k q^{-2p_k} \Phi_k(z)\Phi_k^*(z q^2)\big|_{V(\mu_\alpha)} = id_{V(\mu_\alpha)}. \]

C. Miki's construction of \( U_q(\hat{sl}(2|1)) \)

We generalize the Miki’s construction to the supersymmetric case. Define
\[ (L^+(z))^i_j = \Phi_i(z q^\pm)\Psi_j^*(z q^{-\pm}), \]
\[ (L^-(z))^i_j = \Phi_i(z q^{\pm})\Psi_j^*(z q^{-\pm}). \]

**Proposition 7**: The \( L^\pm(z) \) defined above give a realization of the super Reshetikhin-Semenov-Tian-Shansky algebra at level one in the quantum affine superalgebra \( U_q(\hat{sl}(2|1)) \)
\[ R(z \frac{\mu}{w})L^+_1(z) L^+_1(w) = L^+_1(w) L^+_1(z) R(z \frac{\mu}{w}), \]
\[ R(z \frac{\mu}{w})L^-_1(z) L^-_2(w) = L^-_2(w) L^+_1(z) R(z \frac{\mu}{w}), \]
where \( L^\pm_1(z) = L^\pm(z) \otimes 1, L^\pm_2(z) = 1 \otimes L^\pm(z) \) and \( z^\pm = q^\pm \).

**Proof**. Straightforward by means of the graded Faddeev-Zamolodchikov algebra.
V. Q-DEFORMED SUPERSYMMETRIC T-J MODEL

In this section, we give a mathematical definition of the q-deformed supersymmetric t-J model on an infinite lattice.

A. Space of states

By means of the R-matrix \( \text{R}^{(21)} \) of \( U_q(\widehat{sl}(2|1)) \), one can define the q-deformed supersymmetric t-J model on the infinite lattice \( \cdots \otimes V \otimes V \otimes V \cdots \). Let \( h \) be the operator on \( V \otimes V \) such that

\[
PR(\frac{z_1}{z_2}) = 1 + uh + \cdots, \quad u \rightarrow 0,
\]

\( P \) : the graded permutation operator, \( e^u \equiv \frac{z_1}{z_2} \).

The Hamiltonian \( H \) of the q-deformed supersymmetric t-J model is defined by

\[
H = \sum_{l \in \mathbb{Z}} h_{l+1,l}. \tag{V.43}
\]

\( H \) acts formally on the infinite tensor product,

\[
\cdots V \otimes V \otimes V \cdots. \tag{V.44}
\]

It can be easily checked that

\[
[U'_q(\widehat{sl}(2|1)), H] = 0,
\]

where \( U'_q(\widehat{sl}(2|1)) \) is the subalgebra of \( U_q(\widehat{sl}(2|1)) \) with the derivation operator \( d \) being dropped. So \( U'_q(\widehat{sl}(2|1)) \) plays the role of infinite dimensional non-abelian symmetries of the q-deformed supersymmetric t-J model on the infinite lattice. Following [4], we replace the infinite tensor product (V.44) by the level-0 \( U_q(\widehat{sl}(2|1)) \)-module,

\[
F_{\alpha\alpha'} = \text{Hom}(V(\mu_\alpha), V(\mu_{\alpha'})) \cong V(\mu_\alpha) \otimes V(\mu_{\alpha'})^*,
\]

where \( V(\mu_\alpha) \) is level-one irreducible highest weight \( U_q(\widehat{sl}(2|1)) \)-module and \( V(\mu_{\alpha'})^* \) is the dual module of \( V(\mu_{\alpha'}) \). By theorem 4, this homomorphism can be realized by applying the type I vertex operators repeatedly. So we shall make the (hypothetical) identification:

\[
"\text{the space of physical states"} = \bigoplus_{\alpha,\alpha' \in \mathbb{Z}} V(\mu_\alpha) \otimes V(\mu_{\alpha'})^*.
\]

Namely, we take

\[
F \equiv \text{End}(\bigoplus_{\alpha \in \mathbb{Z}} V(\mu_\alpha)) \cong \bigoplus_{\alpha,\alpha' \in \mathbb{Z}} F_{\alpha\alpha'}
\]

as the space of states of the q-deformed supersymmetric t-J model on the infinite lattice. The left action of \( U_q(\widehat{sl}(2|1)) \) on \( F \) is defined by

\[
x.f = \sum x_{(1)} \circ f \circ S(x_{(2)})(-1)^{|f||x_{(2)}|}, \quad \forall x \in U_q(\widehat{sl}(2|1)), \quad f \in F,
\]

where we have used notation \( \Delta(x) = \sum x_{(1)} \otimes x_{(2)} \). Note that \( F_{\alpha\alpha} \) has the unique canonical element \( id_{V(\mu_\alpha)} \). We call it the vacuum \[ \] and denote it by \( |\text{vac} >_{\alpha} \in F_{\alpha\alpha} \).
B. Local structure and local operators

Following Jimbo et al [5], we use the type I vertex operators and their variants to incorporate the local structure into the space of physical states $F$, that is to formulate the action of local operators of the $q$-deformed supersymmetric $t$-$J$ model on the infinite tensor product (V.44) in terms of their actions on $F_{\alpha \alpha'}$.

Using the isomorphisms (c.f. theorem 4)

$$
\Phi(1) : V(\mu_\alpha) \rightarrow V(\mu_{\alpha-1}) \otimes V,
\Phi^{st}(q^2) : V \otimes V(\mu_\alpha)^* \rightarrow V(\mu_{\alpha-1})^*,
$$

were $st$ is the supertransposition on the quantum space, we have the following identification:

$$
V(\mu_\alpha) \otimes V(\mu_{\alpha'})^* \sim V(\mu_{\alpha-1}) \otimes V (\mu_{\alpha'-1})^*.
$$

The resulting isomorphism can be identified with the super translation (or shift) operator defined by

$$
T = \sum_i \Phi_i(1) \otimes \Phi_i^{st}(q^2)(-1)^{|i|}q^{-2\overrightarrow{i}}.
$$

Its inverse is given by

$$
T^{-1} = \sum_i \Phi_i^*(1) \otimes \Phi_i^{st}(1).
$$

Thus we can define the local operators on $V$ as operators on $F_{\alpha \alpha'}$ [5]. Let us label the tensor components from the middle as $1, 2, \cdots$ for the left half and as $0, -1, -2, \cdots$ for the right half. The operators acting on the site 1 are defined by

$$
E_{ij}^{(1)} = \Phi^*_i(1) \Phi_j(1)(-1)^{|i|} \otimes \text{id}.
$$

More generally we set

$$
E_{ij}^{(n)} = T^{-(n-1)}E_{ij}T^{n-1} \quad (n \in \mathbb{Z}).
$$

Then, from the invertibility relations of the type I vertex operators of $U_q(\hat{sl}(2|1))$, we have

**Theorem 5**: The local operators $E_{ij}^{(n)}$ acting on $F_{\alpha \alpha'}$ satisfy the following relations:

$$
E_{ij}^{(m)} E_{kl}^{(n)} = \begin{cases} 
\delta_{jk} E_{il}^{(n)} & \text{if } m = n \\
(-1)^{|i|+|j|+|k|+|l|} E_{il}^{(n)} E_{ij}^{(m)} & \text{if } m \neq n.
\end{cases}
$$

As is expected from the physical point of view, we also have

**Proposition 8**: The vacuum vectors $|\text{vac}_\alpha>$ are super-translationally invariant and singlets (i.e. belong to the trivial representation of $U_q(\hat{sl}(2|1))$)

$$
T|\text{vac}_\alpha> = |\text{vac}_{\alpha-1}>,
$$

$$
\tilde{x}|\text{vac}_\alpha> = \epsilon(x)|\text{vac}_\alpha>.
$$

**Proof.** Let $u_i^{(\alpha)}$ ($u^*_i^{(\alpha)}$) be a basis vectors of $V(\mu_\alpha)$ ($V(\mu_\alpha)^*$) and

$$
|\text{vac}_\alpha> \overset{\text{def}}{=} \text{id}_{V(\mu_\alpha)} = \sum_i u_i^{(\alpha)} \otimes u^*_i^{(\alpha)}.
$$

Then
We want to show $T|\text{vac} >_\alpha = |\text{vac} >_{\alpha -1}$. This is equivalent to proving

$$\sum_{m,l} q^{-2\rho_m} \Phi_m(1) u_l^{(\alpha)} \Phi_m^*(q^2) u_l^{*(\alpha)} (-1)^{|m|+|l|} = v, \quad \text{for any } v \in V(\mu_{\alpha-1}).$$

Now

$$l.h.s = \sum_{m,l} q^{-2\rho_m} \Phi_m(1) u_l^{(\alpha)} u_l^{*(\alpha)} (\Phi_m(q^2))^{st} v (-1)^{|m|}$$

$$= \sum_{m,l} q^{-2\rho_m} \Phi_m(1) u_l^{(\alpha)} u_l^{*(\alpha)} (\Phi_m(q^2)) v$$

$$= \sum_{m} q^{-2\rho_m} \Phi_m(1) \Phi_m^*(q^2) v = v,$$

where we have used $(\Phi_m^*(z))^{st} = \Phi_m(z)(-1)^{|m|}$ and the second invertibility of the type I vertex operators. As to the second equation, we have

$$x \cdot |\text{vac} >_\alpha = \sum x(1) u_l^{(\alpha)} \otimes x(2) u_l^{*(\alpha)} (-1)^{|l||x(2)|}$$

$$= \sum x(1) u_l^{(\alpha)} \otimes \pi_{V(m\alpha)}(x(2))_{tm} u_m^{*(\alpha)} (-1)^{|l||x(2)|}$$

$$= \sum x(1) u_l^{(\alpha)} \otimes \pi_{V(m\alpha)}(S(x(2)))_{ml} u_m^{*(\alpha)}$$

$$= \sum x(1) \pi_{V(m\alpha)}(S(x(2)))_{ml} u_l^{(\alpha)} \otimes u_m^{*(\alpha)}$$

$$= \sum x(1) S(x(2)) u_m^{*(\alpha)} \otimes u_m^{*(\alpha)} = \epsilon(x)|\text{vac} >_\alpha.$$

This completes the proof.

For any local operator $O \in F$, its vacuum expectation value is given by

$$a < \langle \text{vac} | O | \text{vac} >_\alpha = \frac{\text{tr}_{V(m\alpha)}(q^{-2d+2\lambda_0}O)}{\text{tr}_{V(m\alpha)}(q^{-2d+2\lambda_0})},$$

where we have chosen the normalization $a < \langle \text{vac} | \text{vac} >_\alpha = 1$. We shall denote the correlator $a < \langle \text{vac} | O | \text{vac} >_\alpha$ by $< O >_\alpha$.

**VI. Correlation Functions**

The aim of this section is to calculate $< E_{mn} >_\alpha$. The generalization to the calculation of the multi-point functions is straightforward.

Throughout we use the abbreviation

$$(z;x)_\infty = \prod_{n=0}^{\infty} (1 - zx^n).$$

Set

$$P^m_{n}(z_1,z_2|q(\alpha)) = \frac{\text{tr}_{V(m\alpha)}(q^{-2d+2\lambda_0} \hat{\Phi}_m(z_1) \hat{\Phi}_n(z_2))}{\text{tr}_{V(m\alpha)}(q^{-2d+2\lambda_0})},$$

then $< E_{mn} >_\alpha = P^m_{n}(z,z|q(\alpha))$. By [IV.38]–[IV.39], conjecture 1 and theorem 2, it is sufficient to calculate

$$F^{(\alpha)}_{mn}(z_1,z_2) = \frac{\text{tr}_{F(\alpha,\beta,-\alpha)}(q^{-2d+2\lambda_0} \phi_m(z_1) \phi_n(z_2) \eta_0 \xi_0)}{\text{tr}_{F(\alpha,\beta,-\alpha)}(q^{-2d+2\lambda_0} \eta_0 \xi_0)},$$

(VI.50)
By the procedure similar to the derivation of the (super)characters, we get

\[ F_{mn}^{(\alpha)}(z_1, z_2) = \delta_{mn} \sum_{l=1}^{\infty} (-1)^{l+1} F_{m,-l}, \]

where

\[ \chi_\alpha = \text{tr} F_{(\alpha;\beta-\alpha)}(q^{-2d+2\alpha} \phi_0^* \xi_0) = \prod_{n=1}^{\infty} (1 - q^{2n}) \frac{q^{-\alpha(\alpha+1)}}{\prod_{n=1}^{\infty} (1 - q^{2n})^3} \times \sum_{i,j \in \mathbb{Z}} (-1)^{l+i} q^{(l(\alpha+1)+2l(\alpha-\beta)+2i^2-2i+j^2+j)+2(\alpha-i)}, \]

\[ F_{1,l} = \frac{q^{-\alpha(\alpha+1)}(\frac{2zq}{1-z}\frac{q^2}{q}; q^2)^{\infty} (\frac{2zq}{1-z}\frac{q^2}{q}; q^2)^{\infty}}{(q^2; q^2)^{\infty}} \sum_{i,j \in \mathbb{Z}} q^{(l(\alpha+1)+2l(\alpha-\beta)+2i^2-2i+j^2+j)} \times \left( \int dw_1 \int dw_2 \frac{1 - q^2}{w_1 q} (\frac{w_2 q}{w_1 q}; q^2)^{\infty} (\frac{w_1 q}{w_2 q}; q^2)^{\infty} \frac{q^{-\alpha-i}(\frac{2zq}{1-z}\frac{q^2}{q})^{\infty}}{(w_1 q; q^2)^{\infty} (w_2 q; q^2)^{\infty}} \right) \]

\[ F_{2,l} = -\frac{q^{-\alpha(\alpha+1)}(\frac{2zq}{1-z}\frac{q^2}{q}; q^2)^{\infty} (\frac{2zq}{1-z}\frac{q^2}{q}; q^2)^{\infty}}{(q^2; q^2)^{\infty}} \sum_{i,j \in \mathbb{Z}} q^{(l(\alpha+1)+2l(\alpha-\beta)+2i^2-2i+j^2+j)} \times \left( \int dw_1 \int dw_2 \frac{q - 1}{w_1 w_2 q} (\frac{w_2 q}{w_1 q}; q^2)^{\infty} (\frac{w_1 q}{w_2 q}; q^2)^{\infty} \frac{q^{-\alpha-i}(\frac{2zq}{1-z}\frac{q^2}{q})^{\infty}}{(w_1 q; q^2)^{\infty} (w_2 q; q^2)^{\infty}} \right) \]

\[ F_{3,l} = \frac{q^{-2d}}{(w_1 q; q^2)^{\infty} (w_2 q; q^2)^{\infty} (\frac{w_1 q}{w_2 q}; q^2)^{\infty}} \sum_{i,j \in \mathbb{Z}} q^{(l(\alpha+1)+2l(\alpha-\beta)+2i^2-2i+j^2+j)} \times \left( \int dw_1 \int dw_2 \frac{q - 1}{w_1 w_2 q} (\frac{w_2 q}{w_1 q}; q^2)^{\infty} (\frac{w_1 q}{w_2 q}; q^2)^{\infty} \frac{q^{-\alpha-i}(\frac{2zq}{1-z}\frac{q^2}{q})^{\infty}}{(w_1 q; q^2)^{\infty} (w_2 q; q^2)^{\infty}} \right) \]

\[ (VI.51) \]

We now derive the difference equations satisfied by these one-point functions. Let

\[ \overline{F}_{mn}^{(\alpha)}(z_1, z_2) = \text{tr} F_{(\alpha;\beta-\alpha)}(q^{-2d+2\alpha} \phi_0^* \phi_0 \phi_0^* \xi_0) \overset{\text{def}}{=} \delta_{mn} F_{m}^{(\alpha)}(z_1, z_2). \]

Noticing that

\begin{align*}
  x^d \phi(z) x^{-d} &= \phi(z x^{-1}), & x^d \phi^*_z(z) x^{-d} &= \phi^*_z(z x^{-1}), \\
  x^d \psi(z) x^{-d} &= \psi(z x^{-1}), & x^d \psi^*_z(z) x^{-d} &= \psi^*_z(z x^{-1}), \\
  x^d \eta_0 x^{-d} &= \eta_0, & x^d \xi_0 x^{-d} &= \xi_0,
\end{align*}

we get the difference equations

\[ \overline{F}_m^{(\alpha)}(z_1, z_2 q^2) = q^{-2p_m} \sum_k R(z_2, z_1)^{km} k_m \overline{F}_k^{(\alpha-1)}(z_1, z_2) (-1)^{[m]+[k]+[m][k]}. \]

Since \( \alpha \in \mathbb{Z} \), it is easily seen that this is a set of infinite number of difference equations.
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APPENDIX A.

In this appendix, we give the normal order relations of fundamental bosonic fields:

\[
\begin{align*}
& e^{h_1(z;\beta_1)} e^{h_2(z;\beta_2)} = \frac{1}{z_1 - q^{-(\beta_1 + \beta_2)} z_2} : e^{h_1(z;\beta_1)} e^{h_2(z;\beta_2)} : , \\
& e^{h_2(z;\beta_1)} e^{h_1(z;\beta_2)} = \frac{1}{z_1 - q^{-(\beta_1 + \beta_2)} z_2} : e^{h_2(z;\beta_1)} e^{h_1(z;\beta_2)} : , \\
& e^{h_1(z;\beta_1)} e^{h_2(z;\beta_2)} = \frac{1}{z_1 - q^{-(\beta_1 + \beta_2)} z_2} : e^{h_1(z;\beta_1)} e^{h_2(z;\beta_2)} : , \\
& e^{h_1(z;\beta_1)} e^{h_1(z;\beta_2)} = : e^{h_1(z;\beta_1)} e^{h_1(z;\beta_2)} : ,
\end{align*}
\]

APPENDIX B.

By means of the bosonic realization of \( U_q(\widehat{sl}(2|1)) \), the integral expressions of the vertex operators and the technique given in Ref. 8, one can check the following relations

- For the type I vertex operators

\[
\begin{align*}
& [\phi_1(z), f_{1,q}^{-1}] = 0, \quad [\phi_1(z), f_1] = 0, \\
& [\phi_1(z), f_2] = 0, \quad [\phi_2(z), f_2] = 0, \\
& [\phi_1(z), e_1] = t_1 \phi_2(z), \quad [\phi_2(z), e_1] = 0, \\
& [\phi_1(z), e_2] = 0, \quad [\phi_2(z), e_2] = t_2 \phi_3(z), \\
& \phi_1(z) t_1 = q t_1 \phi_1(z), \quad \phi_2(z) t_1 = q^{-1} t_1 \phi_2(z), \\
& \phi_1(z) t_2 = t_2 \phi_1(z), \quad \phi_2(z) t_2 = q t_2 \phi_2(z), \\
& \phi_3(z) t_1 = q^{-1} t_1 \phi_3(z), \quad \phi_3(z) t_2 = q^{-1} t_2 \phi_3(z).
\end{align*}
\]

\[
\begin{align*}
& [\phi_1(z), f_{2,q}^{-1}] = 0, \quad [\phi_2(z), f_2] = 0, \\
& [\phi_1(z), f_1] = 0, \quad [\phi_2(z), f_1] = 0, \\
& [\phi_1(z), e_1] = 0, \quad [\phi_2(z), e_1] = 0, \\
& [\phi_1(z), e_2] = 0, \quad [\phi_2(z), e_2] = 0, \\
& [\phi_1(z), f_{1,q}^{-1}] = 0, \quad [\phi_2(z), f_{1,q}^{-1}] = 0, \\
& [\phi_1(z), f_{2,q}^{-1}] = 0, \quad [\phi_2(z), f_{2,q}^{-1}] = 0, \\
& [\phi_1(z), e_1] = 0, \quad [\phi_2(z), e_1] = 0, \\
& [\phi_1(z), e_2] = 0, \quad [\phi_2(z), e_2] = 0.
\end{align*}
\]
• For the type II vertex operators

\[ \psi_2(z) = [\psi_1(z), e_1]_g, \quad [\psi_2(z), e_1]_{q^{-1}} = 0, \quad [\psi_3(z), e_1] = 0, \]
\[ [\psi_1(z), e_2] = 0, \quad [\psi_3(z), e_2]_{q^{-1}} = 0, \]
\[ [\psi_2(z), f_1] = 0, \quad [\psi_3(z), f_1] = t_1^{-1} \psi_1(z), \quad [\psi_4(z), f_1] = 0, \]
\[ [\psi_2(z), f_2] = 0, \quad [\psi_3(z), f_2] = t_2^{-1} \psi_2(z), \quad [\psi_4(z), f_2] = 0, \]
\[ \psi_1(z) t_1 = qt_1 \psi_1(z), \quad \psi_2(z) t_1 = q^{-1} t_1 \psi_2(z), \quad \psi_3(z) t_1 = t_1 \psi_3(z), \quad \psi_4(z) t_1 = t_1 \psi_4(z), \]
\[ \psi_1(z) t_2 = t_2 \psi_1(z), \quad \psi_2(z) t_2 = q t_2 \psi_2(z), \quad \psi_3(z) t_2 = q^{-1} t_2 \psi_3(z), \]
\[ \psi_4(z) t_2 = t_2 \psi_4(z). \]

\[ [\psi_1^*(z), e_1]_{q^{-1}} = 0, \quad [\psi_2^*(z), e_1] = -q [\psi_2^*(z), e_1]_{q^{-1}}, \quad [\psi_3^*(z), e_1] = 0, \]
\[ [\psi_2^*(z), e_2] = 0, \quad [\psi_3^*(z), e_2]_{q^{-1}}, \quad [\psi_4^*(z), f_1] = -q t_1^{-1} \psi_3^*(z), \quad [\psi_5^*(z), f_1] = 0, \]
\[ [\psi_2^*(z), f_2] = 0, \quad [\psi_3^*(z), f_2] = -q t_2^{-1} \psi_4^*(z), \quad [\psi_5^*(z), f_2] = 0, \]
\[ \psi_1^*(z) t_1 = q^{-1} t_1 \psi_1^*(z), \quad \psi_2^*(z) t_1 = q t_1 \psi_2^*(z), \quad \psi_3^*(z) t_1 = t_1 \psi_3^*(z), \quad \psi_4^*(z) t_1 = t_1 \psi_4^*(z), \]
\[ \psi_1^*(z) t_2 = t_2 \psi_1^*(z), \quad \psi_2^*(z) t_2 = q t_2 \psi_2^*(z), \quad \psi_3^*(z) t_2 = q^{-1} t_2 \psi_3^*(z), \]
\[ \psi_4^*(z) t_2 = t_2 \psi_4^*(z). \]

\textbf{APPENDIX C.}

In computing the characters/supercharacters and the correlation functions, one encounters the trace of the form

\[ tr(x^{-d} e^{\sum_{m=1}^{\infty} (\sum_{i=1,2} A_m h_i^{m} + B_m c_m) e^{\sum_{m=1}^{\infty} (\sum_{i=1,2} C_m h_i^{m})} + D_m c_m + f_1 t_1^m f_2 t_2 f_3 f_4}), \]

where \( A_m, B_m, C_m, t, D_m \) and \( f_4 \) are all some coefficients. We can calculate the contributions from the oscillators modes and the zero modes separately. The trace over the oscillator modes can be carried out as follows by using the Clavelli-Shapiro technique [1]. Let us introduce the extra oscillators \( h_i^{m}, c_m \) which commute with \( h_i^{m}, c_m \). \( h_i^{m}, c_m \) satisfy the same commutation relations as those satisfied by \( h_i^{m}, c_m \). Introduce the operators

\[ H_i^{m} = h_i^{m} \otimes 1 + 1 \otimes h_i^{m}, \quad C_m = c_m \otimes 1 + 1 \otimes c_m, \quad m \geq 0, \]
\[ H_i^{m} = h_i^{m} \otimes 1 + \frac{1 \otimes h_i^{m}}{x^{m} - 1}, \quad C_m = c_m \otimes 1 + \frac{1 \otimes c_m}{x^{m} - 1}, \quad m < 0. \]

Then for any bosonic operator \( O(h_i^{m}, c_m) \), one can show

\[ tr(x^{-d} O(h_i^{m}, c_m)) = \frac{<0|O(H_i^{m}, C_m)|0>}{\prod_{n=1}^{\infty} (1 - x^n)^3} \]

providing that \( d \) satisfies the derivation properties [III.17]. We write \( <0|O(H_i^{m}, C_m)|0> \equiv <O(H_i^{m}, C_m)> \). Then by the Wick theorem, one obtains

\[ << e^{h_1(z_1 \rightarrow \frac{1}{2})} e^{h_2(z_2 \rightarrow \frac{1}{2})} >> = C_1 \left( -\frac{1}{2} \right) C_2 \left( -\frac{1}{2} \right) g_1 \left( \frac{z_2}{z_1} \right), \]
\[ << e^{h_1(z_1 \rightarrow \frac{1}{2})} e^{h_2(z_2 \rightarrow \frac{1}{2})} >> = C_1 \left( \frac{1}{2} \right) C_2 \left( \frac{1}{2} \right) g_2 \left( \frac{z_2}{z_1} \right), \]
\[ << e^{h_2(z_1 \rightarrow \frac{1}{2})} e^{h_1(z_2 \rightarrow \frac{1}{2})} >> = C_1 \left( \frac{1}{2} \right) C_2 \left( -\frac{1}{2} \right) g_2 \left( \frac{z_2}{z_1} \right), \]
\[ << e^{h_2(z_1 \rightarrow \frac{1}{2})} e^{h_2(z_2 \rightarrow \frac{1}{2})} >> = C_2 \left( -\frac{1}{2} \right) C_2 \left( -\frac{1}{2} \right), \]
\[ << e^{h_1(z_1 \rightarrow \frac{1}{2})} e^{h_4(z_2 \rightarrow \frac{1}{2})} >> = C_1 \left( \frac{1}{2} \right) C_4 \left( -\frac{1}{2} \right) \left( g_2 \left( \frac{z_2}{z_1} \right) \right)^{-\delta_{ij}}, \]
\[ << e^{h_1(z_1 \rightarrow \frac{1}{2})} e^{h_4(z_2 \rightarrow \frac{1}{2})} >> = C_1 \left( -\frac{1}{2} \right) C_4 \left( -\frac{1}{2} \right) \left( g_2 \left( \frac{z_2}{z_1} \right) \right)^{-\delta_{ij}}, \]

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\[ g_t(z) = (xzq^2; x)_\infty (xz^{-1}q^2; x)_\infty (xz^{-1}; x)_\infty, \quad g_2(z) = \frac{1}{(xzq; x)_\infty (xz^{-1}q; x)_\infty}, \]
\[ g_0(z) = (xz; x)_\infty (xz^{-1}; x)_\infty, \quad C_0(0) = (x; x)_\infty, \]
\[ C_1(-1/2) = (xz^2; x)_\infty (x; x)_\infty = \frac{1}{C_2(-1/2)}, \quad C_2(-1/2) = \frac{1}{C_1(-1/2)} = 1. \]