DEFORMATION OF EXCEPTIONAL COLLECTIONS

Xiaowen Hu

Abstract

We show that in a smooth family of Fano or anti-Fano projective varieties, the existence of full exceptional collection of a fiber preserves in the fibers in a neighborhood. Then we show that the noncommutative deformations of a strong exceptional collection of vector bundles induce the same map on the second Hochschild cohomology as the canonical isomorphism induced by the derived equivalence to the corresponding endomorphism algebra.

1 Introduction

For a smooth complex projective variety $X$, Dubrovin’s conjecture says that $\mathcal{D}b(X)$ has a full exceptional collection if and only if the Frobenius manifold $\mathcal{M}_X$ corresponding to the quantum cohomology of $X$ is generically semisimple, and moreover, the Gram matrix of a certain full exceptional collection of $\mathcal{D}b(X)$ is equal to the so-called Stokes data of $\mathcal{M}_X$. The quantum cohomology is deformation invariant. Thus the above conjecture suggests that the existence of full exceptional collections is preserved in a smooth family. In this paper we show the existence in an open neighborhood.

**Theorem 1.1.** ( = corollary [3.7]) Let $\mathbb{k}$ be a field, $S$ a scheme over $\mathbb{k}$, $X$ a smooth projective scheme over $S$, $s_0$ a point (not necessarily closed) of $S$. Suppose $\omega_{X/S}$ or $\omega_{X/S}^{-1}$ is relatively ample over $S - \{s_0\}$. If $\mathcal{D}b(X_{s_0})$ has a full exceptional collection (resp., a strongly full exceptional collection), then there exists an open subset $V$ containing $s_0$ such that for any geometric point $s$ of $V$, $\mathcal{D}b(X_s)$ has a full exceptional collection (resp., a strongly full exceptional collection).

The deformability of an exceptional collection should have been well-known to the experts. To show the fullness, we use the notion of helix of [Bon89], and an observation that the theorem 4.1 of [Bon89] remains true if we add an assumption that the virtual ranks of the objects in the exceptional collection are coprime. We expect this assumption is true for an exceptional collection whose orthogonal complement is a $K$-phantom, but we cannot prove this. However it preserves under deformations. To make the arguments work we need to generalize the definition of exceptional objects, their mutations, and helices, to a relative setup.

Assume furthermore that the full exceptional collection $(E_i)_{1 \leq i \leq n}$ is strong, and denote $A = \text{End}_{\mathcal{O}_X}(\bigoplus_{i=1}^n E_i)$, then the deformation of $X_{s_0}$ induces a map

$$T_{s_0}S \to \mathcal{H}H^2(A),$$

where $\mathcal{H}H^i(A) := \mathcal{H}H^i(A, A)$ is the Hochschild cohomology of the $\mathbb{k}$-algebra $A$. When the characteristic of $\mathbb{k}$ is 0, there is a natural decomposition (see [Swan96], [Yeku02], [Cîl05])

$$\mathcal{H}H^i(X) = \bigoplus_{p=0}^i \mathcal{H}H^p(X, \wedge^{i-p}T_X),$$
and there is a natural isomorphism (see e.g. \[BH13\])

\[ HH^i(X) \cong HH^i(A). \]  

Therefore it is natural to expect the map \( H^1(X, T_X) \to HH^2(A) \) induced by \( (1) \) coincides with the map induced by \( (2) \) and \( (3) \). It is natural to extend this statement to the noncommutative deformation of Toda [Toda05], such that the image is the whole \( HH^2(A) \). Our second main theorem confirms this under the restriction that the strong full exceptional collection \( (E_i)_{1 \leq i \leq n} \) consists of vector bundles. Let us state it more precisely. First note that the existence of full exceptional collection implies that \( H^2(X, \mathcal{O}_X) = 0 \). Then for \( \beta \in H^1(X, T_X) \) and \( \gamma \in H^0(X, \wedge^2 T_X) \), denote \( u(0, \beta, \gamma) \in HH^2(A) \) the deformation of \( A \) arisen from the deformation of \( X \). Denote \( \Phi^i : \bigoplus_{p=0}^i H^p(X, \wedge^p T_X) \to HH^i(A) \) the composition of the isomorphisms \( (2) \) and \( (3) \).

**Theorem 1.2.** (\( = \) theorem 6.8) Let \( k \) be a field of the characteristic zero, and \( X \) a smooth projective variety over \( k \), with a strong full exceptional collection of vector bundles \( (E_i)_{1 \leq i \leq n} \). Let \( A = \text{End}(\bigoplus_{i=1}^n E_i) \). Then in the notations explained above we have

\[ \Phi^2(0, \beta, \gamma) = u(0, \beta, \gamma). \]  

The keeping of the redundant zero component in the above notations is made to be consistent with the notations in the maintext. In section 5 and 6, we work over characteristic zero, but one can easily check that the theorem \( (1.2) \) and the intermediate statements concerning only Hochschild cohomology of degree \( \leq 2 \) remain valid in characteristics \( > 3 \).

I cannot find a direct conceptual proof of this theorem. Our proof is a bit involved; part of the reason is, I think, that in the definition of noncommutative deformations the Hodge-type decomposition \( (2) \) is used. In fact the reader will see that dealing with the \( \lambda \)-decomposition is the most technical part of the proof. Our basic strategy is to find explicit Hochschild cochains of \( A \) that represent both sides of \( (1) \). In the process we also need to study the deformation of exceptional collections on the first order deformations of \( A \) and the first order noncommutative deformations of \( X \). Notice that in this paper by an explicit construction, we mean explicit in the sense modulo the not-really-explicit construction of inverse images of Čech coboundaries.

I do not the pursue to remove the assumptions in theorem \( (1.2) \) that the full exceptional collection is strong and consists of vector bundles in this paper. I think the construction in the proof of [Toda05, prop. 6.1] will be helpful for the general case. Finally I remark that many existence results of infinitesimal deformations in this paper can also be deduced from the main theorems of [Lowen05].

We organize the paper as follows. In section 2 we introduce the notion of relative exceptional collections and helices. In section 3 we first show the existence of deformations of exceptional collections, then prove theorem \( (1.1) \). In section 4 we show the existence of full strong exceptional collection on a first order deformation of the finite dimensional algebra associated to an acyclic quiver modulo an admissible ideal of paths, and more relevant to the proof of theorem \( (1.2) \) we obtain the formula \( (13) \). In section 5 we recall Toda’s definition of noncommutative deformations associated to \( (\alpha, \beta, \gamma) \in H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(X, \wedge^2 T_X) \), and constructed in an explicit way the deformation of a exceptional collection of vector bundles on such a noncommutative deformation, and by comparing with \( (13) \), we obtain an explicit expression of \( u(\alpha, \beta, \gamma) \). In section 6,
we recall three crucial properties of Hochschild cohomology: the HKR isomorphisms, the Morita equivalence and the $\lambda$-decomposition (also called Hodge-type decomposition). Then we construct a bar-type resolution of the diagonal $\Delta_X$ induced by a strong full exceptional collection, and finally obtain an explicit representation of $\Phi^2(0, \beta, \gamma)$; theorem 1.2 follows by a direct comparison of constructions 5.11 and 6.30. In section 7 we propose some open problems.

Acknowledgement I am indebted to Huazheng Ke, many discussions with whom inspired the question, and to Zhan Li for introducing [AT08] to my scope and for discussions, and to Shizhuo Zhang, who shared to me a lot of knowledge and his interesting ideas in this field. I am also grateful to Yifei Chen, Chenyang Xu, Yuri Prokhorov, Feng Qu, Qizheng Yin, Qingyuan Jiang, Ying Xie, Lei Zhang, Jinxing Xu and Zhiyu Tian for discussions on various geometric problems related to this paper, and to Yining Zhang for answering me a question on Hochschild cohomology.

Notations In this paper, unless otherwise stated, $\delta$ will denotes the differentials of a Čech complex $\check{C}(\cdot)$, $Z^i(\cdot)$, $B_i(\cdot)$ the corresponding $i$-th group of cocyles and coboundaries, $b$ the differentials of a Hochschild cochain complex, and $b'$ the differentials of a bar complex. The symbol $\ll$ will always denotes the contraction of sections of tangent sheaves and cotangent sheaves, or the contraction of sections of their wedge products. The symbols $R$ and $L$ indicate the derived functors. The symbol $\epsilon$ will always denotes a square zero element, e.g., $k[\epsilon] = k[\epsilon]/(\epsilon^2)$. A geometric point means the spectrum of a separably closed field.

2 Relative exceptional collections and helices

In this section we collect some definitions and standard facts on admissible subcategories and semiorthogonal decompositions of a triangulated category. In passing we define the relative exceptional collection and helices. All triangulated subcategories are assumed full.

2.1 Relative exceptional collections

Definition 2.1. Let $\mathcal{A}$ be a triangulated category. A triangulated subcategory $\mathcal{B}$ is called right admissible (resp., left admissible) if the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ has a right adjoint (resp., has a left adjoint). If $\mathcal{B}$ is both right and left admissible, it is called an admissible subcategory.

We will need the following lemma on admissible subcategories.

Lemma 2.2. [Bon89, 3.1] Let $\mathcal{A}$ be a triangulated category, $\mathcal{B}$ a triangulated subcategory, $\mathcal{B}^\perp$ (resp., $\perp \mathcal{B}$) the right orthogonal (resp., the left orthogonal) to $\mathcal{B}$. Then the following are equivalent:

1. $\mathcal{B}$ is right admissible (resp., left admissible).

2. For every $X \in \mathcal{A}$ there is a distinguished triangle $Y \to X \to Z$ with $Y \in \mathcal{B}$ and $Z \in \mathcal{B}^\perp$ (resp., $Y \in \perp \mathcal{B}$ and $Z \in \mathcal{B}$).

3. $\mathcal{B}$ and $\mathcal{B}^\perp$ (resp., $\perp \mathcal{B}$ and $\mathcal{B}$) generate $\mathcal{A}$.

\footnote{This work is supported by 34000-31610265, NSFC 34000-41030364 and 34000-41030338.}
4. The inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ has a left adjoint (resp., the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ has a right adjoint).

For a scheme $S$, denote by $D^b(S)$ the triangulated category of perfect complexes on $S$. For a scheme $X$ over $S$, denote by $D^b(X/S)$ the category of $S$-perfect complexes on $X$ \cite{Lieb06}. If $X$ is smooth over $S$, $D^b(X/S)$ is equivalent to $D^b(X)$.

**Definition 2.3.** \cite{Kuz11} An $S$-linear subcategory of $D^b(X/S)$ is a triangulated subcategory which is closed under the operations of the form $L\pi^* M \otimes L$ where $M \in D^b(S)$, where $\pi: X \to S$ is the structure morphism.

**Lemma 2.4.** \cite[2.7]{Kuz11} A pair of $S$-linear subcategories $A, B$ of $D^b(X/S)$ is semiorthogonal if and only if $R\pi_* R\mathcal{H}om(B, A) = 0$ for any $A \in A, B \in B$.

For a scheme $Y$ a perfect complex $M \in D^b(Y)$, denote by $M^\vee$ the derived dual of $M$, i.e., $M^\vee = R\mathcal{H}om(M, \mathcal{O}_Y)$. In the following of this section we assume $\pi: X \to S$ is smooth and projective.

**Corollary 2.5.** Let $A$ be an $S$-linear triangulated subcategory of $D^b(X/S)$. Then $A$ and $\perp A$ are also $S$-linear triangulated subcategories.

Proof: For $M \in D^b(S)$ and $A, B \in D^b(X/S)$, we have

$$R\pi_* R\mathcal{H}om(L\pi^* M \otimes L B, A) = R\mathcal{H}om(M, R\pi_* R\mathcal{H}om(B, A))$$

and

$$R\pi_* R\mathcal{H}om(B, L\pi^* M \otimes L A) = R\mathcal{H}om(M^\vee, R\pi_* R\mathcal{H}om(B, A)).$$

Thus the conclusion follows from lemma 2.4. \hfill $\Box$

**Definition 2.6.** An ordered set of objects $(E_1, \ldots, E_n)$ of $D^b(X/S)$ is called an exceptional collection of $X/S$ of length $n$ if for $i > j$,

$$R\pi_* R\mathcal{H}om(E_i, E_j) = 0,$$

and the canonical map

$$\mathcal{O}_S \to R\pi_* R\mathcal{H}om(E_i, E_i)$$

is an isomorphism for $1 \leq i \leq n$. It is called a strongly exceptional collection if moreover the cohomology sheaves $\mathcal{H}^k(R\pi_* R\mathcal{H}om(E_i, E_j))$ vanish for $k \neq 0$ and all pairs $i, j$. An exceptional collection of length 2 is called an exceptional pair.

The following fiberwise criterion for exceptionality is immediate from the definition.

**Lemma 2.7.** Let $X$ be a smooth proper variety over $S$, $E = (E_1, \ldots, E_n)$ a sequence of objects of $D^b(X/S)$. Then $E$ is an exceptional collection of $D^b(X/S)$ if and only if $Ls^* E$ is an exceptional collection of $D^b(X_\xi)$ for every geometric point $s: \xi \to S$. \hfill $\Box$
Lemma 2.9. Let \( D \) and \( F \) be closed, i.e., if \( F \) is in \( D \), then \( F \) is in \( D \). Suppose \( L \langle n \rangle \) is an exceptional collection of \( D \), then \( \langle n \rangle \) is an admissible subcategory of \( D \).

Definition 2.8. Let \( (E, F) \) be an exceptional pair. The left mutation \( L_F E \) and the right mutation \( R_F E \) are defined by the distinguished triangles

\[
\begin{align*}
L_F E & \to L \pi^* R \pi_* R \mathcal{H}om(E, F) \otimes^L E \to F \to (L_F E)[1], \\
E & \to (L \pi^* R \pi_* R \mathcal{H}om(E, F))^! \otimes^L F \to R_F E \to E[1].
\end{align*}
\]

For an exceptional collection \( \sigma = (E_1, \cdots, E_n) \) we define the \( i \)-th right and left mutations

\[
R_i \sigma = (E_1, \cdots, E_i-1, E_{i+1}, E_i, E_{i+2}, \cdots, E_n), \\
L_i \sigma = (E_1, \cdots, E_{i-1}, L_i E_i, E_{i+1}, E_i, E_{i+2}, \cdots, E_n).
\]

For a set of object \( G_1, \cdots, G_n \) of \( D^b(X/S) \), denote by \( \langle G_1, \cdots, G_n \rangle \) the smallest \( S \)-linear triangulated subcategory satisfying: (i) it containing \( G_1, \cdots, G_n \), (ii) it is isomorphism closed, i.e., if \( F \in \langle G_1, \cdots, G_n \rangle \) then \( F' \approx F \) then \( F' \in \langle G_1, \cdots, G_n \rangle \).

Lemma 2.9. Let \( \pi : X \to S \) be a smooth and proper morphism, and \( (E_1, \cdots, E_n) \) an exceptional collection of \( D^b(X/S) \). Then \( \langle E_1, \cdots, E_n \rangle \) is an admissible subcategory of \( D^b(X/S) \).

Proof: Using lemma 2.22 the proof goes verbatim as that of [Bon89 theorem 3.2 a]). \( \square \)

Lemma 2.10. 1. \( \langle \sigma \rangle = \langle L_i \sigma \rangle = \langle R_i \sigma \rangle \).

2. There are relations

\[
R_i L_i \cong L_i R_i \cong 1, \quad R_i R_{i+1} R_i \cong R_{i+1} R_i R_{i+1}, \quad L_i L_{i+1} L_i \cong L_{i+1} L_i L_{i+1}, \\
R_i R_j = R_j R_i, \quad L_i L_j = L_j L_i, \quad |i - j| \geq 2.
\]

Proof: The proof goes verbatim as the absolute case. \( \square \)

Lemma 2.11. Let \( (E_1, \cdots, E_n) \) be an exceptional collection of \( D^b(X/S) \). For an object \( F \) of \( D^b(X/S) \), we define inductively \( L^k F \) and \( R^k F \) by \( L^0 F = R^0 F = F \), and the distinguished triangles

\[
\begin{align*}
L^{k+1} F & \to L \pi^* R \pi_* R \mathcal{H}om(E_{n-k}, L^k F) \otimes^L E_{n-k} \to L^k F \to (L^{k+1} F)[1], \\
R^{k+1} F & \to (L \pi^* R \pi_* R \mathcal{H}om(R^k F, E_{k+1}))^! \otimes^L E_{k+1} \to R^k F \to (R^{k+1} F)[1].
\end{align*}
\]

Then \( L^k F \in \langle E_{n-k+1}, \cdots, E_n \rangle^\perp \) and \( R^k F \in \langle E_1, \cdots, E_k \rangle^\perp \) for \( 1 \leq k \leq n \). Moreover, \( \langle E_1, \cdots, E_n \rangle \) is an admissible subcategory of \( D^b(X/S) \).

Proof : We prove the first assertion by induction on \( k \). The claim for \( k = 0 \) is empty. Suppose \( L^k F \in \langle E_{n-k+1}, \cdots, E_n \rangle^\perp \), then for \( j > 0 \), we have

\[
\text{Hom}(E_{n-k+j}, L^k F) = 0.
\]

and for \( j \geq 0 \)

\[
\begin{align*}
R \pi_* \mathcal{H}om(E_{n-k+j}, L \pi^* R \pi_* R \mathcal{H}om(E_{n-k}, L^k F) \otimes^L E_{n-k}) & = R \pi_* (L \pi^* R \pi_* R \mathcal{H}om(E_{n-k}, L^k F) \otimes^L E_{n-k} \otimes^L E_{n-k+j}) \\
& = R \pi_* R \mathcal{H}om(E_{n-k}, L^k F) \otimes^L R \pi_* (E_{n-k} \otimes^L E_{n-k+j}^\vee) \\
& = R \pi_* R \mathcal{H}om(E_{n-k}, L^k F) \otimes^L R \pi_* R \mathcal{H}om(E_{n-k}, E_{n-k}^\vee, E_{n-k+j}).
\end{align*}
\]
which vanishes if \( j > 0 \) by the definition of exceptional collections, and is isomorphic to \( R\pi_*RHom(E_{n-k}, L^k F) \) if \( j = 0 \), and in this case the canonical map to \( R\pi_*RHom(E_{n-k}, L^k F) \) induced by the map

\[
\mathcal{L} \pi^* R\pi_* R\mathcal{H}om(E_{n-k}, L^k F) \otimes \mathcal{L} \to L^k F
\]

is the identity. Thus \( R\pi_* RHom(E_{n-k}, L^{k+1} F) = 0 \) and therefore \( L^{k+1} F \in \langle E_{n-k}, \ldots, E_n \rangle \).

The proof of the conclusion for \( R^k F \) is similar. The second assertion follows from the first one by the octahedral axiom. \( \square \)

2.2 Helices

For an exceptional collection \( \sigma = (E_1, \ldots, E_n) \) of \( D^b(X/S) \), define inductively

\[
E_{n+i} = R^n E_i, \quad E_{n-i} = L^n E_i. \tag{7}
\]

**Definition 2.12.** Let \( S \) be a locally noetherian scheme. Suppose \( X/S \) is smooth and projective of pure relative dimension \( d \). We call the sequence \( \{E_i\}_{-\infty \leq i \leq \infty} \) a helix of period \( n \) if \( E_i \cong E_{n+i} \otimes L^\omega_{X/S}[d-n+1] \) for all \( i \). We call an exceptional collection \( \sigma = (E_1, \ldots, E_n) \) a thread of a helix if the infinite sequence (7) generated by \( \sigma \) is a helix of period \( n \).

**Lemma 2.13.** Let \( E_1, \ldots, E_n \) be an exceptional sequence of length \( n \). It is a thread of a helix of period \( n \) if and only if \( E_i = R^n E_i \otimes \omega_{X/S}[d-n+1] \) for \( i = 1, \ldots, n \).

Proof: We compute

\[
(E_{-n}, \ldots, E_{-1}) = (L_1 L_2 \cdots L_n)^n(E_1, \ldots, E_n) \\
= (L_1 L_2 \cdots L_n)^n((E_1, \ldots, E_n) \otimes \omega_{X/S}[-(d-n+1)]) \otimes \omega_{X/S}[d-n+1] \\
= ((L_1 L_2 \cdots L_n)^n(R_n \cdots R_1)^n(E_1, \ldots, E_n)) \otimes \omega_{X/S}[d-n+1] \\
= (E_1, \ldots, E_n) \otimes \omega_{X/S}[d-n+1]. \tag{7}
\]

\( \square \)

2.3 Observations on ranks

Since \( \pi : X \to S \) is smooth, an \( S \)-perfect complex \( E \in D^b(X/S) \) is in fact \( \mathcal{O}_X \)-perfect. Taking a local representative of \( E \) as a bounded complex of locally free sheaves of finite ranks

\[
\cdots \to E^{i-1} \to E^i \to E^{i+1} \to \cdots
\]

and define the rank of \( E \) to be

\[
\text{rank}(E) = \sum_{i=-\infty}^{\infty} (-1)^i \text{rank}(E^i).
\]

Then \( \text{rank}(E) \) is a well-defined locally constant function on \( X \).

**Lemma 2.14.** The following composition of natural maps

\[
\mathcal{O}_X \to E^\vee \otimes \mathcal{L} \to \mathcal{O}_X
\]

is the multiplication by \( \text{rank}(E) \).
Proof: Write $E = E^\bullet$ as a bounded complex of locally free coherent sheaves. Let $f$ be a local section of $\mathcal{O}_X$ and denote by $m_f$ the multiplication by $f$. The first map sends $f$ to $(f_i = m_f)_{i \in \mathbb{Z}}$ where $f_i : E^i \to E^i$. The second map sends $(f_i)_{i \in \mathbb{Z}}$ which represents an element of $\mathcal{H}^0(E^i \otimes^L E)$, to $\sum_i (-1)^i \text{tr}(f_i)$. Composing the two maps we obtain (8). \qed

Lemma 2.15. Let $S$ be a field, $(E_1, \cdots, E_n)$ be a full exceptional collection of $\mathcal{D}^b(X/S) = \mathcal{D}^b(X)$. Then $\gcd(\text{rank}(E_1), \cdots, \text{rank}(E_n)) = 1$.

Proof: Since $(E_1, \cdots, E_n)$ is a full exceptional collection, the classes $[E_i]$ form a basis of $K_0(X)$, thus the conclusion follows. \qed

3 Deformation of full exceptional collections

3.1 Existence of deformations

We need to recall Lieblich’s theorem of the representability of the moduli of objects in $\mathcal{D}^b(X/S)$ [Lieb06].

Let $\pi : X \to S$ be a flat morphism between schemes. An $S$-perfect complex $E$ is called glueable if $R\pi_* R\mathcal{H}om(E, E) \in D^{\geq 0}(S)$, and universally glueable if this remains true for arbitrary base change $T \to S$. Let $\mathcal{D}^b_{\text{pug}}(X/S)$ be the following groupoid fibered over the category of $S$-schemes,

$$T \mapsto \{\text{universally glueable $T$-perfect complexes on } X_T\}.$$

Theorem 3.1. [Lieb06 4.2.1] Let $\pi : X \to S$ be a proper flat morphism of finite presentation. Then $\mathcal{D}^b_{\text{pug}}(X/S)$ is an Artin stack locally of finite presentation, locally quasi-separated with separated diagonal, over $S$.

We also need the following theorem on the deformation and obstruction theory of the perfect complexes, see [Lieb06 3.1.1], [Lowen05] and [HT10].

Theorem 3.2. Let $I \to A \to A_0 \to 0$ be a square zero extension of rings, and $X$ a scheme flat quasi-separated and of finite presentation and over $A$, $E_0 \in \mathcal{D}^b(X_{A_0})$.

1. There is an element $\omega(E_0)$ in $\text{Ext}^2_{X_{A_0}}(E_0, E_0 \otimes_{A_0}^L I)$ which vanishes if and only if there exists $E \in \mathcal{D}^b(X_A)$ such that $L^i \iota^* E \cong E_0$, where $\iota : \text{Spec}(A_0) \hookrightarrow \text{Spec}(A)$ is the embedding.

2. If the deformation $E$ exists, the set of deformations of $E_0$ is a torsor under $\text{Ext}^1_{X_{A_0}}(E_0, E_0 \otimes_{A_0}^L I)$.

From these theorems we can deduce the existence of deformations of exceptional collections in some open neighborhood.

Theorem 3.3. Let $\mathbb{k}$ be a field, $S$ a scheme over $\mathbb{k}$, $X$ a smooth projective scheme over $S$, $s_0$ a point of $S$. Suppose $\mathcal{D}^b(X_{s_0})$ has an exceptional collection (resp., a strongly exceptional collection) $E$, then there exists an étale neighborhood $U$ of $s_0$ in $S$, such that there exists a unique exceptional collection (resp., a strong exceptional collection) $\mathcal{E}$ of $\mathcal{D}^b(X_U/U)$ whose derived restriction to $\mathcal{D}^b(X_{s_0})$ is $E$.  

7
Proof: Let $A = \mathcal{O}_{S,s_0}$, $\mathfrak{m}$ the maximal ideal of $A$, and $A_k = A/\mathfrak{m}^{k+1}$. In particular $A_0$ is the residue field. Let $E_0$ be an exceptional object of $D^b(X_{s_0})$. We will show inductively that there exists uniquely an exceptional object $E_k$ of $D^p(X_{A_k}/A_k)$ such that $L_{t_k}^*E_k = E_0$, where $t_k : \text{Spec}(A_0) \hookrightarrow \text{Spec}(A_k)$ is the closed embedding. Since $E_0$ is exceptional, $\text{Ext}^1_{X_{A_0}}(E_0, E_0) = \text{Ext}^2_{X_{A_0}}(E_0, E_0)$, so there exists a unique deformation $E_1$ in $D^p(X_{A_1}/A_1)$. Suppose we have obtained $E_k$. Then for any $i$,

$$\text{Ext}^i_{X_{A_k}}(E_k, E_k \otimes^L_{A_k} \mathfrak{m}^{k+1}/\mathfrak{m}^{k+2})$$

is acyclic. It is perfect over $\text{Spec}(A_k)$ and its restriction to $\text{Spec}(A_0)$ is acyclic, therefore by the semicontinuity theorem (for perfect complexes, [EGA III, 7.7.5]), it is acyclic over $\text{Spec}(A_k)$. The same argument deduces the existence and uniqueness of the deformation of the exceptional collection onto $X_{A_k}$. The existence of a formal deformation (i.e. a deformation of $E$ over an open subset $U$ containing $s_0$), both follow from theorem 3.1 on the representability of $\mathcal{O}_{pug}^b(X/S)$ as an Artin stack.

\[\square\]

### 3.2 Fullness

**Lemma 3.4.** Let $(E_1, \cdots, E_n)$ be an exceptional collection of $D^p(X/S)$. Then there is a canonical isomorphism

$$R\pi_* R\mathcal{H}om(E_n, F)^! \cong R\pi_* R\mathcal{H}om(F, L^{n-1}E_n[n - 1])$$

for $F \in (E_1, \cdots, E_n)$.

**Proof:** By the proof of [Bon89 4.2] there is a natural homomorphism

$$R\pi_* R\mathcal{H}om(E_n, F)^! \rightarrow R\pi_* R\mathcal{H}om(F, L^{n-1}E_n[n - 1])$$

and lemma 2.7 reduces the conclusion to the absolute case, then use the conclusion of [Bon89 4.2]. \[\square\]

The following theorem is an enhancement of [Bon89 theorem 4.1]; notice that the word *foliation* in the statement of the English version of loc. cit. means *bundle*, according to the russian version.
\textbf{Theorem 3.5.} Let $S$ be a connected locally noetherian scheme over a field $k$, $\pi : X \to S$ a smooth projective morphism. Let $(E_1, \ldots, E_n)$ be an exceptional collection of $D^p(X/S)$. Consider the following two properties:

1) The exceptional collection $(E_1, \ldots, E_n)$ is full;

2) The collection $(E_1, \ldots, E_n)$ is a thread of a helix of period $n$.

Then 1) $\Rightarrow$ 2). If we assume moreover that $\gcd(\text{rank}(E_1), \ldots, \text{rank}(E_n)) = 1$, and $\omega_{X/S}$ or $\omega_{X/S}^{-1}$ is relatively ample, then 2) $\Rightarrow$ 1).

We assume the connectedness of $S$ so that the ranks of $E_i$ are constant.

Proof: 1) $\Rightarrow$ 2): By the Grothendieck duality,

$$R\pi_* R\mathcal{H}om(E_i, F)^\vee \cong R\pi_* R\mathcal{H}om(R\mathcal{H}om(E_i, F), \omega_{X/S}[d])$$

$$\cong R\pi_* R\mathcal{H}om(F, E_i \otimes^L \omega_{X/S}[d]),$$

which by lemma 3.4 induces a map

$$L^{n-1}E_i[n-1] \to E_i \otimes^L \omega_{X/S}[d], \quad (8)$$

which is an isomorphism if $(E_1, \ldots, E_n)$ is a full exceptional collection of $D^p(X/S)$.

2) $\Rightarrow$ 1): We assume $\omega_{X/S}^{-1}$ relatively ample; the case $\omega_{X/S}$ relatively ample is similar. Then there exists $r > 0$ such that $\omega_{X/S}^{-r}$ is relatively very ample, which induces an embedding $\iota : X \hookrightarrow \mathbb{P}^N_S$. Suppose $F \in (E_1, \ldots, E_n)^\perp$. Then for $1 \leq i \leq n$ and any integer $k$,

$$R\pi_* R\mathcal{H}om(E_i \otimes \omega_{X/S}^k, F) = 0. \quad (9)$$

On the other hand, writing $E_i = E_i^* \otimes F = F^*$ as complexes of locally free coherent sheaves on $X$.

$$R\pi_* R\mathcal{H}om(E_i^* \otimes \omega_{X/S}^k, F^*) = R\pi_* t_*(E_i^{\ast \vee} \otimes F^* \otimes \omega_{X/S}^{-rk}),$$

and for $k \gg 0$,

$$R\pi_* R\mathcal{H}om(E_i^p \otimes \omega_{X/S}^k, F^q) = \pi_* t_*(E_i^{p \ast \vee} \otimes F^q \otimes \omega_{X/S}^{-rk}).$$

Taking into account \[\text{[3.4.3] by Serre's theorem EGAII},\] we have

$$t_*(E_i^{\ast \vee} \otimes F^*) = 0$$

in $D^p(\mathbb{P}^N_S/S)$, and thus $E_i^{\vee} \otimes F^* = 0$. Thus the composition

$$F \to E_i \otimes^L E_i^{\ast \vee} \otimes^L F \to F$$

is zero. However, by lemma 2.14, this composition is the multiplication by $\text{rank}(E_i)$. By assumption $\gcd(\text{rank}(E_1), \ldots, \text{rank}(E_n)) = 1$, thus $F = 0$. \hfill \square

\textbf{Theorem 3.6.} Let $k$ be a field, $S$ a locally noetherian scheme over $k$, $X$ a smooth projective scheme over $S$, $s_0$ a point (not necessarily closed) of $S$. Suppose $\omega_{X/S}$ or $\omega_{X/S}^{-1}$ is relatively ample over $S - \{s_0\}$. If $D^p(X_{s_0})$ has a full exceptional collection (resp., a strong full exceptional collection), then there exists an open subset $V$ containing $s_0$ and an étale cover $W$ of $V - s_0$ such that $D^p(X_W/W)$ has a full exceptional collection (resp., a strong full exceptional collection).
Proof: Shrinking $S$ if necessary, we can assume that $S$ is irreducible and $X$ is of pure relative dimension $d$. Let $\sigma_0$ be a full exceptional collection (resp., a strong full exceptional collection) of $D^p(X_{s_0})$. By theorem 3.5, there exists an étale neighborhood $U$ of $s_0$ and an exceptional collection (resp., a strong exceptional collection) $\sigma = (E_1, \cdots, E_n)$ of $D^p(X_U/U)$ extending $\sigma_0$. Since $\sigma_0$ is full, by lemma 2.13 one has $\gcd(\text{rank}(E_1), \cdots, \text{rank}(E_n)) = 1$. Since $S$ is connected, $E_i$ has a constant rank over $U$, thus $\gcd(\text{rank}(E_1), \cdots, \text{rank}(E_n)) = 1$.

By lemma 3.4 there is a natural map

$$L^{n-1}E_i[n-1] \to E_i \otimes \omega_{X/U}[d],$$

as in the proof of theorem 3.5. Since $\sigma_0$ is full, (10) is a quasi-isomorphism after restricting to $X_{s_0}$. By the semicontinuity theorem, there exists an open subset $U'$ of $U$ containing $s_0$ such that the restriction of (10) to $X_U'$ is also a quasi-isomorphism, for $1 \leq i \leq n$. By lemma 2.13 this means that $\sigma$ is a thread of a helix over $X_U'$. Thus by the assumption that $\omega_{X/S}$ or $\omega_{X/S}^{-1}$ is relatively ample over $S - \{s_0\}$. Denote the étale morphism $U \to S$ by $\varphi$, and let $W = U' - \varphi^{-1}(s_0)$. Then by theorem 3.5 $\sigma|_W$ is a full exceptional collection of $D^p(X_W/W)$.

**Corollary 3.7.** Let $k$ be a field, $S$ a locally noetherian scheme over $k$, $X$ a smooth projective scheme over $S$, $s_0$ a point (not necessarily closed) of $S$. Suppose $\omega_{X/S}$ or $\omega_{X/S}^{-1}$ is relatively ample over $S - \{s_0\}$. If $D^p(X_{s_0})$ has a full exceptional collection (resp., a strong full exceptional collection), then there exists an open subset $V$ containing $s_0$ such that for any geometric point $s$ of $V$, $D^p(X_s)$ has a full exceptional collection (resp., a strong full exceptional collection).

Proof: Use theorem 3.5 and 3.6 and that the helicity of a relative exceptional collection can be checked fiberwisely.

**4 First order deformations of full exceptional collections of modules of finite dimensional algebras associated to acyclic quivers with relations**

Fix a base field $k$. We follow the terminology on finite dimensional algebras and quivers of [ASS]. For example, the algebra associated to the following quiver

```
p1  a  p2  b  p3
```

is the $k$-algebra generated by $W = \{p_1, p_2, p_3, a, b\}$ with the relations $p_i^2 = p_i$ for $1 \leq i \leq 3$, $p_1a = a = ap_2$, $p_2b = b = bp_3$, and all the other products of two elements of $W$, except $ab$, are zero. In this section we need to quote several results of [ARS], the reader should notice that our convention of the products is different from that of loc. cit, because this is more convenient for considering right modules. Recall that a quiver is called *acyclic* if it has no oriented nontrivial cycles of arrows; in [Bon89] an acyclic quiver was called *ordered*.

Throughout this section, we assume that $A$ is a finite dimensional algebra of the form $k\Delta/I$, where $\Delta$ is an acyclic quiver and $I$ is an admissible ideal, i.e., $R_{\Delta}^m \subset I \subset R_{\Delta}^1$ for some integer $m \geq 2$, where $R_{\Delta}$ is the ideal of nontrivial arrows of $\Delta$. Let $\{p_1, \ldots, p_n\}$ be the set of vertices of $\Delta$. Then $p_iA$, $1 \leq i \leq n$ form a complete set of indecomposable projective right
A-modules. Denote by $\mathcal{D}^b(A)$ the derived category of finite dimensional right $A$-modules. We arrange the order of $p_1, \ldots, p_n$ such that for $1 \leq i < j \leq n$, there is no path in $\Delta$ that starts from $p_i$ and ends at $p_j$. Thus $p_i x p_j = 0$ for $i < j$ and all $x \in A$. Then by \cite[section 5]{Bon89}, $(p_1 A, \ldots, p_n A)$ is a strong full exceptional collection of $\mathcal{D}^b(A)$, and

$$A \cong \bigoplus_{1 \leq i < j \leq n} \text{Hom}_A(p_i A, p_j A),$$

where $\text{Hom}_A$ is taken in the category of right $A$-modules.

Denote $\mathbb{k}[\epsilon] = \mathbb{k}[\epsilon]/(\epsilon^2)$, and $S = \text{Spec}(\mathbb{k}[\epsilon])$. A deformation of $A$ over $S$ is a flat $\mathbb{k}[\epsilon]$-algebra $A^\dagger$ with an isomorphism $A^\dagger \otimes_{\mathbb{k}[\epsilon]} \mathbb{k} \cong A$. Denote by $HH^n(A) = HH^n(A, A)$ the Hochschild cohomology of the $\mathbb{k}$-algebra $A$. Then the deformation of $A$ over $S$ is parametrized by $HH^2(A)$ due to \cite{Ger64}. For later use let us recall this fact. Explicitly, a Hochschild $n$-cochain is a $\mathbb{k}$-linear map $f : A^\otimes n \rightarrow A$, and the coboundary is given by

$$b(f)(a_1, \ldots, a_{n+1}) = a_1 f(a_2, \ldots, a_{n+1}) + \sum_{1 \leq i \leq n} (-1)^i f(a_1, \ldots, a_ia_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n)a_{n+1}. \quad (11)$$

Given a 2-cocycle $u$, the corresponding deformation of $A_u$ over $S$ is given by the multiplication

$$(a_0 + \epsilon a_1) \cdot_u (b_0 + \epsilon b_1) = a_0 b_0 + \epsilon (a_1 b_0 + a_0 b_1 + u(a_0, b_0)). \quad (12)$$

for $a_i, b_i \in A$, $i = 0, 1$. If $u$ and $u'$ differ by a coboundary $b(v)$, then there is an induced isomorphism $A_u \cong A_{u'}$ over $S$. It is straightforward to see that all deformations of $A$ over $S$ arise in this way.

For a flat finite dimensional algebra $A^\dagger$ over $S$, denote by $\mathcal{D}^p(A^\dagger/S)$ the full $S$-linear triangulated subcategory of $\mathcal{D}^b(A)$ generated by the bounded complexes of projective right $A^\dagger$-modules. The following is the main theorem of this section.

**Theorem 4.1.** Let $u \in HH^2(A)$.

(i) For $1 \leq i \leq n$, there exists a unique projective right $A_u$-module $P_i$ that deforms $p_i A$.

(iii) There exists $\lambda_i \in \mathbb{k}$ and $a_i, b_i, c_i \in A$ such that $P_i = (p_i + \epsilon (-\lambda_i p_i + a_i p_i + p_i b_i + c_i))A_u$.

(iii) The sequence $(P_1, \ldots, P_n)$ is a strong full exceptional collection of the $S$-linear triangulated category $\mathcal{D}^p(A_u/S)$, i.e.,

$$\begin{cases} \mathbb{k}[\epsilon] \xrightarrow{\sim} \text{Hom}_{A_u}(P_i, P_i) & \text{for } 1 \leq i \leq n, \\ \text{Hom}_{A_u}(P_i, P_j) = 0 & \text{for } i > j, \\ \text{Ext}^k(P_i, P_j) = 0 & \text{for } k > 0 \text{ and } 1 \leq i, j \leq n. \end{cases}$$

(iv) We have

$$A_u \cong \bigoplus_{1 \leq i \leq j \leq n} \text{Hom}_{A_u}(P_i, P_j). \quad (13)$$

One can find more descriptions on $a_i, b_i, c_i$ in the following lemmas, which we do not spell out in the above theorem. We remark that the choices of $a_i$ and $b_i$ are not unique. We refer the reader to \cite{HSS88} for a result close to ours. The rest of this section is devoted
to an elementary proof of theorem 4.1.

In the following of this section we fix a Hochschild 2-cocycle \( u : A \otimes_k A \rightarrow A \). For \( \forall a, b, c \in A \),
\[
au(b, c) - u(ab, c) + u(a, bc) - u(a, b)c = 0. \tag{14}
\]

**Lemma 4.2.**  
(i) For \( 1 \leq k \leq n \), there exist a unique \( \lambda_k \in k \), and a unique \( c_k \in A \) which modulo \( I \) is a linear combination of paths whose beginnings and ends are not \( p_k \), such that
\[
u(p_k, p_k) = \lambda_k p_k + c_k. \tag{15}
\]

(ii) For \( 1 \leq i, j \leq n \) and \( i \neq j \), there is a unique \( d_{ij} \in A \) which modulo \( I \) is a linear combination of paths connecting \( p_i \) towards \( p_j \), such that
\[
u(p_i, p_j) = d_{ij} - p_i c_j - c_i p_j, \tag{16}
\]
where \( c_i, c_j \in A \) are as in (i). In particular, if \( i < j \), then \( d_{ij} = 0 \).

(iii) For any \( x \in A \), we have \( xu(1, 1) = u(x, 1) \) and \( u(1, 1)x = u(1, x) \).

(iv) Let \( \lambda_k, c_k, 1 \leq k \leq n \), and \( d_{ij}, 1 \leq i \neq j \leq n \) be determined as in (i) and (ii). Let \( 1_u \) be the identity element of \( A_u \), then
\[
u(1) = 1 - \epsilon\left( \sum_{1 \leq k \leq n} \lambda_k p_k - c_k \right) + \sum_{1 \leq i < j \leq n} d_{ij}. \tag{17}
\]

(v) \( A_u \) is a \( k[\epsilon]/(\epsilon^2) \)-algebra via the map
\[
u(k[\epsilon]/(\epsilon^2)) \rightarrow A_u, \mu + \epsilon \nu \mapsto \mu \cdot 1_u + \epsilon \nu = \mu + \epsilon(-\mu u(1, 1) + \nu). \tag{18}
\]

**Proof:** (i) Taking \( a = b = c = p_k \) in (14), we obtain \( p_k u(p_k, p_k) = u(p_k, p_k)p_k \). Thus (15) holds for some \( \lambda_k \in k \), and \( c_k \) satisfies \( p_k c_k = c_k p_k = 0 \), i.e. \( c_k \) is a linear combination of paths whose beginning and ends are not \( p_k \).

(ii) Taking \( a = b = p_i \) and \( c = p_j \) in (14), we obtain
\[
u(p_i, p_j) - u(p_i, p_j) - u(p_i, p_i)p_j = 0.
\]
Taking \( a = p_i \) and \( b = c = p_j \) in (14), we obtain
\[
u(p_i, p_j) + u(p_i, p_j) - u(p_i, p_j)p_j = 0.
\]
Thus
\[
u(p_i, p_j) = p_i u(p_i, p_j) - u(p_i, p_i)p_j = p_i (u(p_i, p_j)p_j - p_i u(p_j, p_j)) - u(p_i, p_i)p_j
\]
\[
u(p_i, p_j) = p_i u(p_i, p_j) - p_i u(p_j, p_j) - u(p_i, p_i)p_j
\]
\[
u(p_i, p_j) = d_{ij} - p_i c_j - c_i p_j,
\]
where \( p_i d_{ij} p_j = d_{ij} \), i.e. \( d_{ij} \) is a linear combination of paths connecting \( p_i \) towards \( p_j \).

(iii) For the first identity, take \( a = x \) and \( b = c = 1 \) in (14). For the second one, take \( a = b = 1 \) and \( c = x \) in (14).
Lemma 4.3. Let \( \lambda_k, c_k, d_{ij} \) be the elements uniquely determined by lemma 4.2.

(i) For \( 1 \leq k \leq n \), the equation

\[
(p_k + \epsilon x) \cdot (p_k + \epsilon x) = p_k + \epsilon x
\]

has solutions, which of the form \( x = -\lambda_k p_k + a_k p_k + p_k b_k + c_k \) such that

\[
p_k a_k = b_k p_k = 0,
\]

i.e., \( a_k \) is a linear combination of paths that do not start at \( p_k \), and \( b_k \) is a linear combination of paths that do not end at \( p_k \).

(ii) The system of idempotents \( \{p_k + \epsilon x_k\}_{1 \leq k \leq n} \), are orthogonal if and only if

\[
p_i a_j p_j + p_i b_j p_j + d_{ij} = 0
\]

for \( 1 \leq i \neq j \leq n \). Such a system of idempotents exist, and they satisfy

\[
\sum_i (p_i + \epsilon (-\lambda_i p_i + a_i p_i + p_i b_i + c_i)) = 1_u.
\]

Proof: The equation (19) is equivalent to

\[
x = p_k x + x p_k + u(p_k, p_k),
\]

i.e.,

\[
x = p_k x + x p_k + \lambda_k p_k + c_k.
\]

From this one easily deduces (i). For \( i \neq j \),

\[
(p_i + \epsilon (-\lambda_i p_i + a_i p_i + p_i b_i + c_i))(p_j - \epsilon (-\lambda_j p_j + a_j p_j + p_j b_j + c_j))
\]

\[
= \epsilon (p_i a_j p_j + p_i b_j p_j + c_j p_j + u(p_i, p_j))
\]

\[
= \epsilon (p_i a_j p_j + p_i b_j p_j + d_{ij}),
\]

where we have used (16). The existence of the solutions for the system of equations (21) follows by induction on \( i \), using the acyclicity of the graph \( \Delta \). Assuming (21), one has, by (17),

\[
\sum_i (p_i + \epsilon (-\lambda_i p_i + a_i p_i + p_i b_i + c_i))
\]

\[
= 1 + \sum_i \epsilon (-\lambda_i p_i + 1 \cdot a_i p_i + p_i b_i \cdot 1 + c_i)
\]

\[
= 1 + \sum_i \epsilon (-\lambda_i p_i + (\sum_j p_j) \cdot a_i p_i + p_i b_i \cdot (\sum_j p_j) + c_i)
\]

\[
= 1 + \epsilon (-\sum_i \lambda_i p_i - \sum_{i \neq j} d_{ij} + \sum_i c_i) = 1_u,
\]
Lemma 4.4. Let \( p_k + \epsilon(-\lambda_k p_k + a_k p_k + p_k b_k + c_k) \) and \( p_k + \epsilon(-\lambda_k p_k + a'_k p_k + p_k b'_k + c_k) \) be two solutions of (19). Then as right \( A_u \)-modules, \( (p_k + \epsilon(-\lambda_k p_k + a_k p_k + p_k b_k + c_k))A_u \) is isomorphic to \( (p_k + \epsilon(-\lambda_k p_k + a'_k p_k + p_k b'_k + c_k))A_u \).

Proof: For \( y, z \in A \) we compute

\[
(1 + ye)(p_k + \epsilon(-\lambda_k p_k + a_k p_k + p_k b_k + c_k))(1 + ze)
= (p_k + \epsilon(-\lambda_k p_k + a_k p_k + p_k b_k + c_k + yp_k + u(1, p_k)))(1 + ze)
= p_k + \epsilon(-\lambda_k p_k + a_k p_k + p_k b_k + c_k + y + u(1,1))p_k + p_k(z + u(1,1)).
\]

Thus we can choose \( y, z \in A \) such that

\[
(1 + ye)(p_k + \epsilon(-\lambda_k p_k + a_k p_k + p_k b_k + c_k))(1 + ze)
= p_k + \epsilon(-\lambda_k p_k + a'_k p_k + p_k b'_k + c_k),
\]

then map for \( \forall w, v \in A, \)

\[
(p_k + \epsilon(-\lambda_k p_k + a'_k p_k + p_k b'_k + c_k))(w + ve)
\mapsto (p_k + \epsilon(-\lambda_k p_k + a'_k p_k + p_k b'_k + c_k))(1 + ze)(w + ve)
\]
gives an isomorphism

\[
(p_k + \epsilon(-\lambda_k p_k + a'_k p_k + p_k b'_k + c_k))A_u \simto (p_k + \epsilon(-\lambda_k p_k + a'_k p_k + p_k b'_k + c_k))A_u.
\]

Lemma 4.5. We have the following identities.

(i) For \( 1 \leq k \leq n, \)

\[
p_k u(p_k, x) - u(p_k, x) + u(p_k, p_k x) - u(p_k, p_k x) = 0,
\]

\[
x u(p_k, p_k) - u(x p_k, p_k) + u(x, p_k) - u(x, p_k) p_k = 0,
\]

\[
p_k u(p_k x, p_k) - u(p_k x, p_k) + u(p_k, p_k x p_k) - u(p_k, p_k x p_k) p_k = 0,
\]

\[
p_k u(x p_k, p_k) - u(p_k x p_k, p_k) + u(p_k, x p_k) - u(p_k, x p_k) p_k = 0,
\]

\[
p_k u(x, p_k) - u(p_k x, p_k) + u(p_k, x p_k) - u(p_k, x) p_k = 0,
\]

(ii) For \( 1 \leq i < j \leq n, \)

\[
p_i u(x p_j, p_j) + u(p_i, x p_j) - u(p_i, x p_j) p_j = 0,
\]

\[
p_i u(x, p_j) + u(p_i, x p_j) - u(p_i, x p_j) p_j = 0,
\]

\[
p_i u(p_i x, p_j) - u(p_i x, p_j) - u(p_i, p_i x) p_j = 0.
\]
(iii) For $1 \leq k \leq n$,

$$p_k u(xp_k, p_k) - p_k u(x, p_k) + p_k u(p_kx, p_k) = u(p_k, xp_k)p_k + u(p_k, x)p_k - u(p_k, x)p_k. \quad (31)$$

(iv) For $1 \leq i < j \leq n$,

$$u(p_i, p_j)p_j + u(p_i, xp_j)p_j - u(p_i, x)p_j = p_i u(p_i, x)p_j - p_i u(x, p_j) + p_i u(xp_j, p_j) = 0. \quad (32)$$

Proof: (i) and (ii) follows easily from (14) and that there is no path starts from $p_i$ and ends at $p_j$. For (iii) we compute

$$u(p_i, p_j)p_j + u(p_i, xp_j)p_j - u(p_i, x)p_j = p_i u(p_i, x)p_j - p_i u(x, p_j) + p_i u(xp_j, p_j),$$

where for the first equality we use (30), for the second we use (29) and for the third we use (28).

For (iv) we compute

$$p_k u(xp_k, p_k) - p_k u(x, p_k) + p_k u(p_kx, p_k) = u(p_kxp_k, p_k) - u(p_k, xp_k)p_k - p_k u(x, p_k) + p_k u(p_k, x)p_k = u(p_kxp_k, p_k) + u(p_k, xp_k)p_k + p_k u(p_kx, p_k) - u(p_kxp_k, p_k) - u(p_kx, p_k) - u(p_k, x)p_k = u(p_kxp_k, p_k) + u(p_k, x)p_k - u(p_k, x)p_k,$$

where for the first equality we use (26), for the second we use (27), for the third we use (25) and $u(p_kxp_k, p_k) = u(p_kxp_k, p_k)$ because $p_kxp_k = \mu p_k$ for some $\mu \in \mathbb{k}$.

Lemma 4.6. For $1 \leq i \leq n$, let $p_i^\dagger = p_i + \epsilon(-\lambda_i p_i + a_ip_i + p_i b_i + c_i)$ be a system of solutions to (19) and (21), and let $P_i = p_i^\dagger A_u$, then

(i) For $1 \leq i < j \leq n$, $\text{Hom}_{A_u}(P_j, P_i) = 0$.

(ii) For $1 \leq i \leq n$, the composition of homomorphisms

$$\mathbb{k}[\epsilon] \to A_u \to \text{Hom}_{A_u}(P_i, P_i)$$

is an isomorphism.

(iii)

$$A_u \cong \bigoplus_{1 \leq i \leq j \leq n} \text{Hom}_{A_u}(P_i, P_j). \quad (33)$$

Proof: By definition $p_i^\dagger$ are idempotents of $A_u$. By [ARS] prop. I.4.9, for $1 \leq i, j \leq n$, $\text{Hom}_{A_u}(P_i, P_j) = p_j^\dagger A_u p_i^\dagger$. 

15
Thus we can show (i) and (ii) by direct computations. For \( i < j \) and \( x, y \in A \), we compute

\[
(p_i + \epsilon(-\lambda_i p_i + c_i + a_i p_i + p_i b_i))(x + \epsilon y) (p_j + \epsilon(-\lambda_j p_j + c_j + a_j p_j + p_j b_j))
\]

\[
= (p_i x + \epsilon(-\lambda_i p_i x + c_i x + a_i p_i x + p_i b_i x + p_i y + u(p_i, x))(p_j + \epsilon(-\lambda_j p_j + c_j + a_j p_j + p_j b_j))
\]

\[
= \epsilon(c_i x p_j + u(p_i, x)p_j + p_i x c_j + u(p_i x, p_j))
\]

\[
= \epsilon(u(p_i, x)p_j + p_i u(x p_j, p_j) - p_i u(x, p_j) + u(p_i, x))
\]

\[
= \epsilon(u(p_i, x)p_j + u(p_i, x p_j) p_j - u(p_i, x) p_j - p_i u(x, p_j) + u(p_i, x))
\]

\[
= \epsilon(u(p_i, x)p_j + u(p_i, x p_j) p_j - u(p_i, x) p_j),
\]

where for the first and second equalities we use the property about directions of \( p_i, p_j \) and \( c_i, c_j \) described in lemma 4.2 (i), for the third equality we use 23, for the fourth we use 24, for the fifth we use 28, and for the sixth we use 29. Thus by 32 we obtain (i).

For \( 1 \leq k \leq n \), and \( x, y \in A \) we compute

\[
(p_k + \epsilon(-\lambda_k p_k + c_k + a_k p_k + p_k b_k))(x + \epsilon y) (p_k + \epsilon(-\lambda_k p_k + c_k + a_k p_k + p_k b_k))
\]

\[
= (p_k x + \epsilon(-\lambda_k p_k x + c_k x + a_k p_k x + p_k b_k x + p_k y + u(p_k, x))(p_k + \epsilon(-\lambda_k p_k + c_k + a_k p_k + p_k b_k))
\]

\[
= p_k x p_k + \epsilon(-\lambda_k p_k x p_k + c_k x p_k + a_k p_k x p_k + p_k b_k x p_k + p_k y p_k + u(p_k, x))
\]

\[
- \lambda_k p_k x p_k + p_k x c_k + p_k x a_k p_k + p_k x p_k b_k + u(p_k, x, p_k)
\]

\[
= p_k x p_k + \epsilon(-2\lambda_k p_k x p_k + c_k x p_k + a_k p_k x p_k + p_k y p_k + u(p_k, x, p_k)
\]

\[
+ p_k x c_k + p_k x p_k b_k + u(p_k, x, p_k),
\]

(34)

where for the first and second equalities we use the property about directions of \( p_i, p_j \) and \( c_i, c_j \) described in lemma 4.2 (i), and for the third equality we use

\[
p_k b_k x p_k = p_k x a_k p_k = 0,
\]

which follows from \( p_k a_k = b_k p_k = 0 \) by lemma 4.3. Then we compute

\[
c_k x p_k + u(p_k, x)p_k + p_k x c_k + u(p_k x, p_k)
\]

\[
= -2\lambda_k p_k x p_k + u(p_k, p_k) x p_k + p_k x u(p_k, p_k) + u(p_k x, p_k)
\]

\[
= -2\lambda_k p_k x p_k + p_k u(p_k, x)p_k + u(p_k, p_k) x p_k + p_k x u(p_k, p_k) + u(p_k x, p_k)
\]

\[
= -2\lambda_k p_k x p_k + p_k u(p_k, x)p_k + u(p_k, p_k) x p_k
\]

\[
+ p_k u(x p_k, p_k) - p_k u(x, p_k) + p_k u(x, p_k) + u(p_k x, p_k)
\]

\[
= -2\lambda_k p_k x p_k + p_k u(p_k, x)p_k + p_k u(x p_k, p_k) - p_k u(x, p_k) + p_k u(x, p_k)
\]

\[
+ p_k u(x p_k, p_k) + u(p_k x, p_k),
\]

(35)

where for the first equality we use 15, for the second we use 23, for the third we use 24 and for the fourth we use 25. Now by 31 we have

\[
p_k u(x p_k, p_k) - p_k u(x, p_k) + p_k u(p_k x, p_k) = (p_k u(x p_k, p_k) - p_k u(x, p_k) + p_k u(p_k x, p_k)) p_k,
\]

thus substitute 35 into the equality of 34, we obtain

\[
(p_k + \epsilon(-\lambda_k p_k + c_k + a_k p_k + p_k b_k))(x + \epsilon y)(p_k + \epsilon(-\lambda_k p_k + c_k + a_k p_k + p_k b_k))
\]

\[
= p_k x p_k + \epsilon(p_k T(x, k)p_k + a_k p_k x p_k + p_k y p_k + p_k x p_k b_k + u(p_k, p_k x p_k)),
\]

(36)
where $T(x, k) \in A$ depends only on $x$ and $k$, and we can ignore its complicated form. There exists $\mu(x, k), \nu(y, k), \tau(x, k) \in k$ which depends on $k$ and $x$ or $y$ as the notations indicate, such that

$$p_kxp_k = \mu(x, k)p_k, \ p_kyp_k = \nu(y, k)p_k, \ p_kT(x, k)p_k = \tau(x, k)p_k.$$  

But for an element $\mu + \epsilon\nu \in k[\epsilon]$, the scalar multiplication on $P_k$ is given by (18), i.e.,

$$(\mu \cdot 1_u + \epsilon\nu)(p_k + \epsilon(-\lambda kp_k + c_k + akp_k + pbk))$$

$$= (\mu + \epsilon(-\mu u(1, 1) + \nu))(p_k + \epsilon(-\lambda kp_k + c_k + akp_k + pbk))$$

$$= \mu p_k + \epsilon(-\mu \lambda kp_k + \mu c_k + \mu akp_k + \mu pbk - \mu u(1, 1)p_k + \nu p_k + u(\mu, p_k))$$

$$= \mu p_k + \epsilon(-\mu \lambda kp_k + \mu c_k + \mu akp_k + \mu pbk + \nu p_k),$$  

where for the third equality we use $\mu u(1, 1)p_k = u(\mu, p_k)$ by lemma 4.2 (iii). Comparing (36) and (38), using (37), we obtain

$$(p_k + \epsilon(-\lambda kp_k + c_k + akp_k + pbk))A_u(p_k + \epsilon(-\lambda kp_k + c_k + akp_k + pbk))$$

$$= k[\epsilon]\cdot(p_k + \epsilon(-\lambda kp_k + c_k + akp_k + pbk)).$$

This completes the proof of (ii). Finally, by (22), $\{P_i\}_{1 \leq i \leq n}$ is a complete list of projective $A_u$-modules. Regard $A_u$ as a $k$-algebra, we obtain $\mathbb{A}$ by [Bon90] section 5.

**Proof of theorem 4.1:** Since $p_jA$ is an indecomposable projective right $A$-module, by [ARS] I.4.4 a projective right $A_u$-module $P_i$ deforming $p_jA$ is still indecomposable. Then by [ARS] I.4.5 and I.4.8, one easily sees that $P_j$ is of the form $p_jA_u$ where $p_j$ is an idempotent of $A_u$ such that $p_j \equiv (\epsilon) \equiv p_j$. Thus (i) follows from lemma 4.3 (i) and lemma 4.4, and the remaining statements follow from lemma 4.3 (ii) and lemma 4.6.

5 First order noncommutative deformations of exceptional collections

Let $X$ be a smooth proper variety over $k$, $T_X$ the tangent sheaf of $X$. Given $\beta \in H^1(X, T_X)$, there is a canonically associated smooth projective scheme $X_\beta$ over $k[\epsilon]$ which deforms $X$. Generalizing this classical fact, Toda in [Toda05] introduced the notion of noncommutative deformation $X_{\alpha, \beta, \gamma}$ associated to an element $(\alpha, \beta, \gamma) \in H^2(X, O_X) \otimes H^1(X, T_X) \oplus H^0(X, \wedge^2 T_X)$. Let us recall Toda's definition.

**Definition 5.1.** [Toda05] §3, §4] Choose an affine open covering $U = \{U_i\}_{1 \leq i \leq I}$ of $X$, and choose Čech representatives $\{a_{ijk}\}_{i,j,k \in I} \in \check{C}^2(U, O_X)$, $\{b_{ij}\}_{i,j \in I} \in \check{C}^1(U, T_X)$ of $\alpha$ and $\beta$ respectively. We regard $\gamma \in \Gamma(X, T_X^2)$ as an antisymmetric bi-derivation, i.e., a $k$-linear homomorphism $O_X \otimes kO_X \rightarrow O_X$, which are derivations in both arguments, and is antisymmetric. étale locally, $\gamma$ can be written as $\sum_{i,j=1}^{\dim X} f_{ij} \partial x_i \wedge \partial x_j$, where $x_1, ..., x_{\dim X}$ are étale local coordinates of $X$, and $f_{ij}$ are regular functions on the corresponding chart.

The noncommutative deformation $X_{\alpha, \beta, \gamma}$ consists of the following data.

1. The underlying space is identified to $X$.
2. There is a sheaf of $k[\epsilon]$-algebras $O_{X_{\beta, \gamma}}$ defined as follows. As a sheaf of $k[\epsilon]$-modules, $O_{X_{\beta, \gamma}}$ is the kernel of

$$O_X \oplus \check{C}^0(U, O_X) \ni (a + \epsilon\{b_i\}_{1 \leq i \leq I}) \mapsto \{-\beta_{ij}(a) + \delta\{b_i\}\}_{i,j \leq I} \in \check{C}^1(U, O_X).$$
and the multiplication is given by
\[(a + \epsilon\{b_i\}) \cdot (c + \epsilon\{d_i\}) = ac + \epsilon\{bi + ad_i + \gamma(a, c)\}.
\]

3. An \(\mathcal{O}_{X_{\beta, \gamma}}\)-module twisted by \(\alpha\), is a collection \(\{F_i\}_{i \in I}\), where \(F_i\) is an \(\mathcal{O}_{X_{\beta, \gamma}}|_{U_i}\)-module, and a collection \(\{\phi_{ij}\}_{i, j \in I}\), where \(\phi_{ij} : F_i|_{U_i \cap U_j} \rightarrow F_j|_{U_i \cap U_j}\) is an isomorphism \(\mathcal{O}_{X_{\beta, \gamma}}|_{U_j}\)-modules, such that
\[\phi_{ki} \circ \phi_{jk} \circ \phi_{ij} = \text{id} - \alpha_{ijk}\epsilon.\]

The above definition is independent of the choice of \(U\). For brevity we call an \(\mathcal{O}_{X_{\beta, \gamma}}\)-module twisted by \(\alpha\), an \(\mathcal{O}_{X_{\alpha, \beta, \gamma}}\)-module. Similarly for the \(\mathcal{O}_{X_{\alpha, \beta, \gamma}}\)-linear homomorphisms. We say that an \(\mathcal{O}_{\alpha, \beta, \gamma}\)-module \(F\) is quasi-coherent (resp. coherent, resp. locally free), if \(F|_{U_i}\) is a quasi-coherent (resp. coherent, resp. locally free) \(\mathcal{O}_{X_{\beta, \gamma}}|_{U_i}\)-module. A locally free \(\mathcal{O}_{\alpha, \beta, \gamma}\)-module is also called a vector bundle on \(X_{\alpha, \beta, \gamma}\). The derived category of \(\mathcal{O}_{\alpha, \beta, \gamma}\)-modules (resp. quasi coherent \(\mathcal{O}_{\alpha, \beta, \gamma}\)-modules, resp. coherent \(\mathcal{O}_{\alpha, \beta, \gamma}\)-modules) are denoted by \(\mathcal{D}^*(\mathcal{O}_{X_{\alpha, \beta, \gamma}})\) (resp. \(\mathcal{D}^*_\text{qcoh}(\mathcal{O}_{X_{\alpha, \beta, \gamma}})\), resp. \(\mathcal{D}^*\text{coh}(\mathcal{O}_{X_{\alpha, \beta, \gamma}})\)), where \(* = -, + \text{ or} \ b\). The full subcategory of \(\mathcal{D}^*_\text{coh}(\mathcal{O}_{X_{\alpha, \beta, \gamma}})\) consisting of perfect complexes over \(X_{\alpha, \beta, \gamma}\) is denoted by \(\mathcal{D}^0(X_{\alpha, \beta, \gamma})\).

There is a natural morphism of ringed space \(\pi : X_{\alpha, \beta, \gamma} \rightarrow \text{Spec}(k[\epsilon])\), whose corresponding homomorphism of sheaf of rings \(\pi^{-1}k[\epsilon] \rightarrow \mathcal{O}_{X_{\beta, \gamma}}\) is flat. In particular, on \(X_{0, \beta, \gamma}\), the notion of quasi-coherent sheaves (resp. coherent sheaves, resp. locally free sheaves) reduce to the usual ones on a ringed space.

By [Ioda05 lemma 4.3], the category of \(\mathcal{O}_{X_{\alpha, \beta, \gamma}}\)-modules have enough injectives, thus the derived functor \(\mathcal{R}\mathcal{H\text{om}}\) is defined.

The following corollary 5.5 will be used only in the proof of the strongness statement of theorem 5.14, which is not needed for the proof of theorem 6.8. We outline a proof parallel to the usual one for schemes.

**Lemma 5.2.** Given \(X_{0, \beta, \gamma}\), there exists \(N\) such that for \(q > N\) and any \(\mathcal{O}_{X_{\alpha, \beta, \gamma}}\)-quasi-coherent sheaf \(F\), \(\mathcal{R}^q\pi_* F = 0\).

**Proof:** Let \(F\) be an \(\mathcal{O}_{X_{\alpha, \beta, \gamma}}\)-quasi-coherent sheaf. For any affine open subset \(U\) of \(X\), \(F|_U\) is an \(\mathcal{O}_X\)-quasi-coherent sheaf, thus \(H^i(U, F) = 0\) for \(i > 0\). By Leray’s theorem [Gode58 II, 5.9.2], it follows that for any finite affine open covering \(U\) of \(X\), \(H^i(U, F) \cong H^i(X_{0, \beta, \gamma}, F)\). Thus the conclusion follows from the existence of a finite affine open covering of \(X\) because of the properness of \(X \rightarrow \text{Spec}(k)\).

Now the arguments of [III 3.7 and 3.7.1] carry over verbatim to deduce the following two lemmas. See also [III05 8.3.8].

**Lemma 5.3.** For any \(G \in \mathcal{D}^*_\text{qcoh}(k[\epsilon])\) and \(F \in \mathcal{D}^*_\text{coh}(X_{0, \beta, \gamma})\), there is a canonical isomorphism \(\mathcal{R}\pi_*(\pi^* G \otimes^L F) \cong G \otimes^L_{k[\epsilon]} \mathcal{R}\pi_* F\).

**Lemma 5.4.** Let \(F\) be a perfect complex of \(\mathcal{O}_{X_{\alpha, \beta, \gamma}}\)-modules. Then \(\mathcal{R}\pi_* F\) is perfect over \(\text{Spec}(k[\epsilon])\).

**Corollary 5.5.** Let \(E, F\) be perfect complexes of \(\mathcal{O}_{X_{\alpha, \beta, \gamma}}\)-modules. Then
\[\mathcal{R}\pi_* \mathcal{R}\mathcal{H\text{om}}_{\mathcal{O}_{X_{\alpha, \beta, \gamma}}}(E, F)\]
is perfect over \(\text{Spec}(k[\epsilon])\).
Proof: For open immersions $j : U \rightarrow X$, the extension by zero $j!$ is exact and left adjoint to $j^*$ [Toda05 §4], thus $j^!$ is injective on $U$ for an injective $\mathcal{O}_{X_{\alpha,\beta,\gamma}}$-module. So local properties of $\mathbf{R} \pi_* \mathcal{H}om_{\mathcal{O}_{X_{\alpha,\beta,\gamma}}}(E, F)$ can be computed locally. Then one easily sees that $\mathbf{R} \mathcal{H}om_{\mathcal{O}_{X_{\alpha,\beta,\gamma}}}(E, F)$ is a perfect complex of $\mathcal{O}_{X_{\alpha,\beta,\gamma}}$-modules. The conclusion follows from lemma 5.4.

We define exceptional collections (resp. strong ..., resp. full ...) of $D^b(X_{\alpha,\beta,\gamma})$ relative to $\kappa[\epsilon]$ as the definition 2.6.

From now on in this section we study the deformations of strong exceptional collections consisting of vector bundles, over a noncommutative deformation $X_{\alpha,\beta,\gamma}$.

Let $E$ be a vector bundle over $X$, $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of $X$, and denote $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$, for $i, j, k \in I$. Regarding $E|_{U_i} \oplus E|_{U_j}$ as a vector bundle over $U_i \times_k \kappa[\epsilon]$, we want to glue them to obtain a vector bundle over $X_{\alpha,\beta,\gamma}$. Then we need to specify the isomorphisms

$$ (E|_{U_i} \oplus E|_{U_i})|_{U_{ij}} \xrightarrow{\psi_{ij}} (E|_{U_j} \oplus E|_{U_j})|_{U_{ij}} $$

where $\psi_{ij} \in \text{Hom}_{\kappa[\epsilon]}(E|_{U_i} \oplus E|_{U_i}, E|_{U_j} \oplus E|_{U_j})$, such that

$$ \psi_{ki} \circ \psi_{jk} \circ \psi_{ij} = \text{id} - \alpha_{ijk}\epsilon. \quad (39) $$

 Shrinking $\mathcal{U}$ if necessary, we choose connections $\nabla_i : E|_{U_i} \rightarrow E|_{U_i} \otimes_{\mathcal{O}_{U_i}} \Omega^1_{U_i}$ for $i \in I$.

**Lemma 5.6.** The isomorphisms $\{\psi_{ij}\}_{i,j \in I}$ glue $\{E|_{U_i} \oplus E|_{U_i}\}_{i \in I}$ to be a vector bundle over $X_{\alpha,\beta,\gamma}$ if and only if $\psi_{ij}$ are of the form

$$ \psi_{ij} = \begin{pmatrix} 1 & 0 \\ g_{ij} & 1 \end{pmatrix} $$

where $g_{ij} \in \text{Hom}_k(E|_{U_i}, E|_{U_j})$, and $(g_{ij})_{i,j \in I}$ satisfy

$$ \begin{cases} g_{ij}r - rg_{ij} = -\beta_{ij}(r) + \gamma(r, \cdot) \circ \nabla_j - \gamma(r, \cdot) \circ \nabla_i, & \forall r \in \Gamma(U_{ij}, \mathcal{O}_X), \\
 g_{ij}|_{U_{ijk}} + g_{jk}|_{U_{ijk}} + g_{ki}|_{U_{ijk}} = -\alpha_{ijk}. \end{cases} \quad (40) $$

Proof: Write

$$ \psi_{ij} = \begin{pmatrix} 1 & f_{ij} \\ g_{ij} & h_{ij} \end{pmatrix}, $$

where $f_{ij}, g_{ij}, h_{ij}$ are, a priori, $k$-linear endomorphisms of $E|_{U_{ij}}$. The $\mathcal{O}_{X_{\beta,\gamma}}$-linearity of $\psi_{ij}$ means

$$ \psi_{ij} \begin{pmatrix} r \\ s_i + \gamma(r, \cdot) \circ \nabla_i \end{pmatrix} = \begin{pmatrix} r \\ s_j + \gamma(r, \cdot) \circ \nabla_j \end{pmatrix} \psi_{ij}, $$

for any $r, s_i, s_j \in \Gamma(U_{ij}, \mathcal{O}_X)$ satisfying

$$ s_i - s_j = \beta_{ij}(r). \quad (41) $$

Thus

$$ \begin{align*}
 & \begin{pmatrix}
 r + f_{ij} s_i + f_{ij} \circ \gamma(r, \cdot) \circ \nabla_i \\
 g_{ij} r + h_{ij} s_i + h_{ij} \circ \gamma(r, \cdot) \circ \nabla_i 
\end{pmatrix} \\
 & = \begin{pmatrix}
 r \\
 s_j + \gamma(r, \cdot) \circ \nabla_j + r g_{ij} + r h_{ij} \\
 s_j f_{ij} + \gamma(r, \cdot) \circ \nabla_j \circ f_{ij} + r h_{ij} 
\end{pmatrix}.
\end{align*} $$

19
So \( f_{ij} = 0 \), and \( h_{ij} = rh_{ij} \), i.e. \( h_{ij} \) is \( \mathcal{O}_X \)-linear, and
\[
g_{ij} r + h_{ij} s_i + h_{ij} \circ \gamma(r, \cdot) \circ \nabla_i = s_j + \gamma(r, \cdot) \circ \nabla_j + r g_{ij}. \tag{42}
\]
Since \( s_j = s_i - \beta_{ij}(r) \), \( \tag{42} \) holds for all \( r, s_i, s_j \in \Gamma(U_{ij}, \mathcal{O}_X) \) satisfying \( \tag{41} \) if and only if \( h_{ij} = id \)
and
\[
g_{ij} r - r g_{ij} = -\beta_{ij}(r) + \gamma(r, \cdot) \circ \nabla_j - \gamma(r, \cdot) \circ \nabla_i.
\]
The condition \( \tag{39} \) reduces to the second equation of \( \tag{40} \).

\[\square\]

**Lemma 5.7.** If \( E \) is exceptional, there exists an open covering \( \mathcal{U} \) such that the solution to \( \tag{40} \) exists, and the corresponding vector bundle on \( X_{\alpha,\beta,\gamma} \) is unique up to canonical isomorphisms.

Proof : Shrinking \( \mathcal{U} \) if necessary, we can assume that \( \mathcal{U} \) is an affine covering, and that there exists a solution \((\tilde{g}_{ij})\) of the first equation of \( \tag{40} \). Then for \( r \in \Gamma(U_{ijk}, \mathcal{O}_X) \),
\[
(\tilde{g}_{ij} + \tilde{g}_{jk} + \tilde{g}_{ki}) r - r (\tilde{g}_{ij} + \tilde{g}_{jk} + \tilde{g}_{ki}) = -(\beta_{ij} + \beta_{jk} + \beta_{ki})(r).
\]
Since \( \beta \in H^1(X, T_X) \), the assignment \((i, j, k) \mapsto \tilde{g}_{ij} + \tilde{g}_{jk} + \tilde{g}_{ki} \) lies in \( \tilde{Z}^2(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(E, E)) \); denote it by \( \delta \tilde{g} \), and notice that it does not lie in \( \tilde{B}^2(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(E, E)) \) because \( \tilde{g}_{ij} \) is not \( \mathcal{O}_X \)-linear. It suffices to find \( x = (x_{ij}) \in \tilde{C}^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(E, E)) \), such that
\[
\delta x = -\alpha - \delta \tilde{g},
\]
and thus \( g = x + \tilde{g} \) is a solution to \( \tag{40} \). Since \( \tilde{H}^2(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(E, E)) = Ext^2(E, E) = 0 \), such \( x \) exists.

If \( g' \) is another solution, \( h = g - g' \) is \( \mathcal{O}_X \)-linear and therefore lies in \( \tilde{Z}^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(E, E)) \).
Since \( \tilde{H}^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(E, E)) = Ext^1(E, E) = 0 \), \( h = \delta x \) for some \( x \in \tilde{C}^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(E, E)) \), and therefore it is easy to construct an isomorphism between the vector bundle corresponding to \( g \) and \( g' \).

\[\square\]

**Definition 5.8.** For an exceptional vector bundle \( E \) on \( X \), and \((\alpha, \beta, \gamma) \in H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(X, \wedge^2 T_X) \), denote the unique vector bundle on \( X_{\alpha,\beta,\gamma} \) deforming \( E \) by \( E_{\alpha,\beta,\gamma} \).

Let \( E \) and \( F \) be a strong exceptional pair of vector bundles on \( X \). We want to compute
\[
\text{Hom}_{\mathcal{O}_{X_{\alpha,\beta,\gamma}}}(E_{\alpha,\beta,\gamma}, F_{\alpha,\beta,\gamma}). \tag{43}
\]
Still take an open cover \( \mathcal{U} = (U_i) \) and follow the above notations. First of all, an element of \( \tag{43} \) modulo \( \epsilon \) is an element of \( \text{Hom}_{\mathcal{O}_X}(E, F) \). So we fix \( a \in \text{Hom}_{\mathcal{O}_X}(E, F) \), and denote the restriction \( a|_{U_i} \) still by \( a \).

**Lemma 5.9.** Assume \( b_i \in \text{Hom}_k(F|_{U_i}, E|_{U_i}) \), \( c_i \in \text{Hom}_k(E|_{U_i}, F|_{U_i}) \), \( d_i \in \text{Hom}_k(F|_{U_i}, F|_{U_i}) \) for \( i \in I \). Then
\[
\begin{pmatrix}
  a & b_i \\
  c_i & d_i
\end{pmatrix} : E|_{U_i} \oplus E|_{U_i} \to F|_{U_i} \oplus F|_{U_i}
\]
glue to be an $O_{X_{\alpha,\beta,\gamma}}$-linear homomorphism from $E_{\alpha,\beta,\gamma}$ to $F_{\alpha,\beta,\gamma}$ if and only if $b_i = 0$, $d_i = 0$ and

\[
\begin{align*}
  c_ir - rc_i &= \gamma(r, \cdot) \circ \nabla_i \circ a - a \circ \gamma(r, \cdot) \circ \nabla_i, \quad \forall r \in \Gamma(U_i, O_X) \\
  c_i - c_j &= ag^E_{ij} - g^F_{ij}a.
\end{align*}
\] (44)

Proof: Suppose

\[
\begin{pmatrix}
  a & b_i \\
  c_i & d_i
\end{pmatrix}
: E|_{U_i} \oplus E|_{U_i} \to F|_{U_i} \oplus F|_{U_i}
\]
is an $O_{X_{\alpha,\beta,\gamma}}$-linear homomorphism. Then

\[
\begin{pmatrix}
  r \\
  s_i + \gamma(r, \cdot) \circ \nabla_i
\end{pmatrix}
\begin{pmatrix}
  a & b_i \\
  c_i & d_i
\end{pmatrix}
= \begin{pmatrix}
  a & b_i \\
  c_i & d_i
\end{pmatrix}
\begin{pmatrix}
  r \\
  s_i + \gamma(r, \cdot) \circ \nabla_i
\end{pmatrix}
\]
for any $r, s_i \in \Gamma(U_i, O_X)$. This is equivalent to

\[
b_i = 0, \quad rd_i = d_ir,
\]
and

\[
s_i a + \gamma(r, \cdot) \circ \nabla_i \circ a + rc_i = c_i r + d_i s_i + d_i \circ \gamma(r, \cdot) \circ \nabla_i.
\] (45)

These holds for all $r, s_i$ if and only if $d_i = a$ and

\[
c_i r - rc_i = \gamma(r, \cdot) \circ \nabla_i \circ a - a \circ \gamma(r, \cdot) \circ \nabla_i.
\] (46)

Moreover, a system of homomorphisms

\[
\left\{ \begin{pmatrix}
  a & 0 \\
  c_i & a
\end{pmatrix} \right\}_{i \in I}
\]
glue to be an element of $\text{Hom}_{O_{X_{\alpha,\beta,\gamma}}}(E_{\alpha,\beta,\gamma}, F_{\alpha,\beta,\gamma})$ if and only if

\[
\begin{pmatrix}
  1 \\
  g^F_{ij}
\end{pmatrix}
\begin{pmatrix}
  a & 0 \\
  c_i & a
\end{pmatrix}
= \begin{pmatrix}
  a & 0 \\
  c_j & a
\end{pmatrix}
\begin{pmatrix}
  1 \\
  g^E_{ij}
\end{pmatrix}
\]
which is equivalent to

\[
g^F_{ij}a_i + c_i = a_j g^E_{ij} + c_j.
\]

Lemma 5.10. Let $E, F$ be an strong exceptional pair of vector bundles on $X$. Then there exists an open covering $U$ such that there exists a solution $\{c_i\}_{i \in I}$ to the system of equations (44). And two different solutions differ by $\{c'_i\}_{i \in I}$, where $c'_i = c'|_{U_i}$, $i \in I$, for some $c' \in \text{Hom}_{O_X}(E, F)$.

Proof: Shrinking $U$ if necessary, we can assume that $U$ is an affine covering, and that there exists a solution $\{\tilde{c}_i\}_{i \in I}$ of the first equation. Thus

\[
(\tilde{c}_i - \tilde{c}_j)r - r(\tilde{c}_i - \tilde{c}_j) = \gamma(r, \cdot) \circ (\nabla_i - \nabla_j) \circ a - a \circ \gamma(r, \cdot) \circ (\nabla_i - \nabla_j),
\]
while, by the first equation of (44),

\[
(ag^E_{ij} - g^F_{ij}a)r - r(ag^E_{ij} - g^F_{ij}a) = \gamma(r, \cdot) \circ (\nabla_i - \nabla_j) \circ a - a \circ \gamma(r, \cdot) \circ (\nabla_i - \nabla_j).
\]
So the assignment \((i, j) \mapsto -(\tilde{c}_i - \tilde{c}_j) + (ag_{ij}^E - g_{ij}^F a)\) lies in \(\check{C}^1(U, \mathcal{H}om_{\mathcal{O}_X}(E, F))\). Moreover, by \(40\),
\[
(ag_{ij}^E - g_{ij}^F a) + (ag_{jk}^E - g_{jk}^F a) + (ag_{ki}^E - g_{ki}^F a) = 0,
\]
thus the assignment \((i, j) \mapsto -(\tilde{c}_i - \tilde{c}_j) + (ag_{ij}^E - g_{ij}^F a)\) lies in \(\check{H}^1(U, \mathcal{H}om_{\mathcal{O}_X}(E, F))\). Since \(\check{H}^1(U, \mathcal{H}om_{\mathcal{O}_X}(E, F)) = \text{Ext}^1(E, F) = 0\), there exists \(x \in \check{C}^0(U, \text{Hom}(E, F))\) such that \(\delta x = \{-(\tilde{c}_i - \tilde{c}_j) + (ag_{ij}^E - g_{ij}^F a)\}_{i, j \in I}\), thus \(x + \{\tilde{c}_i\}_{i \in I}\) gives a solution to \(41\). The second statement is obvious.

Now let \((E_i)_{1 \leq j \leq n}\) be a strong exceptional collection of vector bundles. Let \(E = F = \bigoplus_{j=1}^n E_i\), and \(A = \text{Hom}_{\mathcal{O}_X}(E, E)\).

**Construction 5.11.** Choosing a \(k\)-basis of \(A\), by lemma \(5.7\) and lemma \(5.10\) there exists an affine open covering \(U = \{U_i\}_{i \in I}\) of \(X\) such that for any \(a\) in the chosen basis, the system of equations for \(g_{ij} \in \text{Hom}_k(E|_{U_{ij}}, E|_{U_{ij}})\) for \(i, j \in I\) and \(i \neq j\), and \(c_i \in \text{Hom}_k(E|_{U_i}, E|_{U_i})\) for \(i \in I\)
\[
\begin{align*}
g_{ij}r - rg_{ij} &= -\beta_{ij}(r) + \gamma(r, \cdot) \circ \nabla_j - \gamma(r, \cdot) \circ \nabla_i, \quad \forall r \in \Gamma(U_{ij}, \mathcal{O}_X),
g_{ij} + gjk + gki &= -\alpha_{ijk},
c_i r - rc_i &= \gamma(r, \cdot) \circ \nabla_i \circ a - a \circ \gamma(r, \cdot) \circ \nabla_i, \quad \forall r \in \Gamma(U_i, \mathcal{O}_X)
\end{align*}
\]
has a solution. Thus we can assign a solution \(c(a)_i\) for each \(a \in A\), such that \(c(\lambda a)_i = \lambda c(a)_i\) for \(\lambda \in k\). On each \(U_i\) we define
\[
u_{\alpha, \beta, \gamma}(a', a)_i = -c(a')_i a - a' c(a)_i + c(a')_i,
\]
which glue to be an \(\mathcal{O}_X\)-endomorphism of \(E\) by the following lemma \(5.12\) thus we obtain an element \(u_{\alpha, \beta, \gamma}(a', a) \in A\).

**Lemma 5.12.** The elements \(-c(a')_i a - a' c(a)_i + c(a')_i\) constructed above are independent of \(i\), and are \(\mathcal{O}_X\)-linear.

**Proof:** First we check the independence of \(i\).
\[
\begin{align*}
\left(-c(a')_i a - a' c(a)_i + c(a')_i\right) - \left(-c(a')_j a - a' c(a)_j + c(a')_j\right)
&= \left(-c(a')_i a + c(a')_j a\right) + \left(-a c(a)_i + a' c(a)_j\right) + \left(c(a')_i - c(a')_j\right)
&= \left(-a g_{ij} a + g_{ij} a' a\right) + \left(-a' g_{ij} + a' g_{ij} a\right) + (a' g_{ij} - g_{ij} a') = 0,
\end{align*}
\]
where for the second equality we use the fourth equation of \(47\). Then we check the \(\mathcal{O}_X\)-linearity.
\[
\begin{align*}
\left(-c(a')_i a - a' c(a)_i + c(a')_i\right)r - r\left(-c(a')_j a - a' c(a)_j + c(a')_j\right)
&= \left(c(a')_i r - rc(a')_i a - a' (c(a)_i r - rc(a)_i)\right) + \left(c(a')_i r - rc(a')_i\right)
&= -\left(\gamma(r, \cdot) \circ \nabla_i \circ a - a' \circ \gamma(r, \cdot) \circ \nabla_i\right) a - a' \left(\gamma(r, \cdot) \circ \nabla_i \circ a - a \circ \gamma(r, \cdot) \circ \nabla_i\right)
&+ \gamma(r, \cdot) \circ \nabla_i \circ a d - a d' \circ \gamma(r, \cdot) \circ \nabla_i = 0,
\end{align*}
\]
where for the second equality we use the third equation of \(47\). \(\square\)
Lemma 5.13. The assignment \((a', a) \mapsto u_{a, \beta, \gamma}(a', a)\) gives an element \(u_{a, \beta, \gamma} \in Z^2(A, A)\), and the choices of \(c(a)\) and the open covering \(U\) do not affect the class of \(u_{a, \beta, \gamma}\) in \(HH^2(A)\). Moreover, \(u_{a, \beta, \gamma}\) depends \(k\)-linearly on \(\alpha, \beta\) and \(\gamma\).

Proof : By construction, \(u_{a, \beta, \gamma}(a', a)\) is \(k\)-linear in \(a\) and \(a'\), and it is straightforward to verify that \(u_{a, \beta, \gamma}(\cdot, \cdot)\) is a cocycle. Given a solution \(\{g_{ij}\}_{i,j \in I}\) to the first and second equations of \((47)\), by the last statement of lemma \([5.10]\) different choices of \(c(a)\) do not change the class of \(u_{a, \beta, \gamma}\) in \(HH^2(A)\). If \(\{g'_{ij}\}_{i,j \in I}\) is another solution to the first and second equations of \((47)\), then \(\{g'_{ij}\}_{i,j \in I} - \{g_{ij}\}_{i,j \in I} = \{x_{ij}\}_{i,j \in I}\), where \(\{x_{ij}\}_{i,j \in I} \in \check{Z}^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(E, E))\). Since \(\check{H}^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(E, E)) = \text{Ext}^1(E, E) = 0\), there exists \(\{y_{ij}\}_{i \in I} \in \check{C}^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(E, E))\) such that \(\tilde{\delta}(\{y_{ij}\}_{i \in I}) = \{x_{ij}\}_{i,j \in I}\). Then we can solve the third and the fourth equation by

\[
\tilde{c}(a) = c(a) + ay_i - y_i a,
\]

for \(i \in I\) and \(a \in A\). Then the corresponding \(\tilde{u}_{a, \beta, \gamma}\) is given by

\[
\tilde{u}_{a, \beta, \gamma}(a', a)_i = -\tilde{c}(a')_i a - a'\tilde{c}(a)_i + \tilde{c}(a' a)_i = u_{a, \beta, \gamma}(a', a)_i - (a' y_i - y_i a') a - a' (ay_i - y_i a) + (a' a y_i - y_i a') a = u_{a, \beta, \gamma}(a', a)_i.
\]

The remaining statements are also obvious from the construction. \(\Box\)

Now we are ready to come to the main theorem of this section.

Theorem 5.14. Let \(\{E_j\}_{1 \leq j \leq n}\) be a strong exceptional collection of vector bundles on \(X\), and denote \(E = \bigoplus_{j=1}^n E_j\) and \(A = \text{Hom}_{\mathcal{O}_X}(E, E)\). For \((\alpha, \beta, \gamma) \in H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(X, \wedge^2 T_X)\), there exists a unique strong exceptional collection of vector bundles \(\{(E_j)_{\alpha, \beta, \gamma}\}_{1 \leq j \leq n}\) on \(X_{\alpha, \beta, \gamma}\) such that \((E_j)_{\alpha, \beta, \gamma}\) is the unique deformation of \(E_j\), and

\[
\text{Hom}_{\mathcal{O}_X(\alpha, \beta, \gamma)}(\bigoplus_i (E_i)_{\alpha, \beta, \gamma}, \bigoplus_i (E_i)_{\alpha, \beta, \gamma}) \cong A_{u_{a, \beta, \gamma}}, \tag{49}
\]

Proof : By corollary \([5.5]\) the complexes \(R\pi_* R\mathcal{H}om((E_i)_{\alpha, \beta, \gamma}, (E_j)_{\alpha, \beta, \gamma})\) are perfect \(k[\varepsilon]\)-complexes. Thus the strong exceptionality follows from the semicontinuity and base change theorem on \(\text{Spec}(k[\varepsilon])\). To show \((49)\), it suffices to notice that, by \([12]\) and \([18]\), the product of \(a + \varepsilon c(a)_i\) and \(a' + \varepsilon c(a')_i\) in \(A_{u_{a, \beta, \gamma}}\) is \(aa' + \varepsilon c(aa')_i\), as wanted. \(\Box\)

Remark 5.15. I do not address the problem of fullness of the exceptional collections \(\{(E_j)_{\alpha, \beta, \gamma}\}\) in this paper. For Fano varieties, I expect that a theory of noncommutative Grothendieck duality will show the fullness along the line of the proof of theorem \([5.6]\). In the general cases, it might be possible to show the fullness by adapting the method of \([\text{Toda}05]\) to finite dimensional algebras.

6 A comparison theorem

In this section we assume that \(k\) is a field of characteristic zero, and \(X\) a smooth projective variety over \(k\), \((E_1, \cdots, E_m)\) an strong full exceptional collection of vector bundles on \(X\), and denote

\[
E = \bigoplus_{i=1}^m E_i.
\]
Thus $E$ is a tilting object of $\mathcal{D}^b(X)$. Denote
\[ A = \text{Hom}_{\mathcal{O}_X}(E, E). \]

Our goal is to show that the assignment
\[ (\alpha, \beta, \gamma) \in H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(\wedge^2 T_X) \mapsto u_{\alpha, \beta, \gamma} \in HH^2(A) \]
coincides with the composition
\[ H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \oplus H^0(\wedge^2 T_X) \cong HH^2(X) \cong HH^2(A). \]

First recall that we have $\alpha = 0$, by the following well-known fact.

**Lemma 6.1.** For a smooth projective variety $X$ over a field of characteristic zero, if $\mathcal{D}^b(X)$ has a full exceptional collection, then $H^2(X, \mathcal{O}_X) = 0$.

**Proof:** Since the characteristic is zero, one has the HKR isomorphism ([Swan96], [Yeku02] or [C˘ al05])
\[ HH_i(X) \cong \bigoplus_{q-p=i} H^p(X, \Omega^q). \]

It suffices to show that under the assumption of existence of a full exceptional collection, one has $HH_i(X) = 0$ for $i > 0$. This is well-known. One way (in the spirit of this paper) to see this, at least in the case that a strong full exceptional collection exist, is via the isomorphism $HH_i(X) = HH_i(A)$, and use the theorem of [Cib86] which says that the higher Hochschild homology of, an algebra associated to an acyclic quiver with relations, is zero. For the general case (there exists a full exceptional collection which is not necessarily strong), one notices that Cibils’ theorem can be easily generalized to the case of acyclic $dg$-quivers with relations, so we can apply the main theorem of [Bod15] to conclude. $\square$

To state our comparison theorem, we need to recall the definition of the canonical isomorphisms
\[ \bigoplus_{i=0}^{n} H^i(X, \wedge^{n-i} T_X) \cong HH^n(X) \cong HH^n(A). \]

### 6.1 HKR isomorphisms

By [Swan96] section 1] there is a spectral sequence
\[ E_2^{p,q} = H^p(X, \mathcal{E}xt^q_{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)) = H^p(X \times X, \mathcal{E}xt^q_{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)) \]
\[ \Rightarrow \text{Ext}^{p+q}_{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta). \] (50)

By using a theorem of [GS87], Swan showed that [Swan96] cor. 2.6] this spectral sequence degenerates, and there is moreover a Hodge-type decomposition. See also [Yeku02] and [C˘ al05]. Some details of the isomorphism $\Upsilon^n$ will be reviewed in section 6.5.

**Theorem 6.2.** The spectral sequence (50) degenerates at $E_2$, and there is a canonical decomposition
\[ \Upsilon^n : \bigoplus_{i=0}^{n} H^i(X, \mathcal{E}xt^{n-i}_{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)) \xrightarrow{\sim} \text{Ext}_{\mathcal{O}_{X \times X}}^{n}(\mathcal{O}_\Delta, \mathcal{O}_\Delta). \] (51)
We need also the HKR isomorphism for smooth affine algebras, due to [HKR62]. Our presentation follows [Loday, section 3.4]. Let \( R \) be a commutative algebra over \( k \), and
\[
T^1_{R/k} = \text{Hom}_R(\Omega^1_{R/k}, R) = \text{Der}_k(R, R),
\]
and let
\[
T^n_{R/k} := \wedge^n T^1_{R/k}
\]
be the \( n \)-th exterior product of \( T^1_{R/k} \). If \( R \) is smooth over \( k \), \( T^n_{R/k} \cong \text{Hom}_R(\Omega^n_{R/k}, R) \). For \( f_1, \ldots, f_n \in \text{Der}_k(R, R) \), define the antisymmetrization map
\[
\epsilon_n : \text{Der}_k(R, R)^{\otimes n} \to \text{Hom}_k(R^{\otimes n}, R)
\]
to be
\[
\epsilon_n(f_1 \otimes \ldots \otimes f_n)(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma)(f_1(a_{\sigma(1)}), \ldots, f_n(a_{\sigma(n)})). \tag{52}
\]
Then \( \epsilon_n \) induces a map, still denoted by \( \epsilon_n \),
\[
\epsilon_n : T^n_{R/k} \to \text{Hom}_k(R^{\otimes n}, R) = \text{Hom}_R(\text{C}^\text{bar}_n(R), R).
\]

Lemma 6.3. The image of \( \epsilon_n \) lies in the kernel of \( b \). Thus there is an induced map
\[
\epsilon_n : T^n_{R/k} \to HH^n(R). \tag{53}
\]

Theorem 6.4. If \( R \) is a smooth \( k \)-algebra, the map (53) is an isomorphism.

Corollary 6.5. There are canonical isomorphisms
\[
\epsilon_i : \wedge^i T_X \cong \text{Ext}^i_{O_{X \times X}}(O_{\Delta}, O_{\Delta})
\]
and
\[
H^p(X, \wedge^q T_X) \cong H^p(X, \text{Ext}^q_{O_{X \times X}}(O_{\Delta}, O_{\Delta})). \tag{54}
\]

Denote the isomorphism (54) by \( \mathcal{E}^{p,q} \), and
\[
\mathcal{E}^n = \bigoplus_{p=0}^n \mathcal{E}^{p,n-p} : \bigoplus_{p=0}^n H^p(X, \wedge^{n-p} T_X) \cong \bigoplus_{p=0}^n H^p(X, \text{Ext}^{n-p}_{O_{X \times X}}(O_{\Delta}, O_{\Delta})).
\]

6.2 Statement of the theorem

The part 1 of the following theorem is [Bon89, 6.2], and the part 2 is [BH13, 3.4, 3.5]. Recall that \( F \boxtimes G := q_1^* F \otimes q_2^* G \), where \( q_1 \) and \( q_2 \) are the two projections from \( X \times X \) to \( X \).

Theorem 6.6. Let \( Y \) be a smooth projective variety over \( k \), and \( E \) a tilting object of \( \text{D}^b(Y) \), and \( A = \text{Hom}_{O_X}(E, E) \), \( A^c = A^{\text{op}} \otimes_k A \).

1. The functor
\[
\Psi = \text{RHom}_{O_X}(E, \cdot) : \text{D}^b(X) \to \text{D}^b(A) \tag{55}
\]
is an equivalence. Moreover, \( \Psi(E) = A \).
2. The functor
\[ \Psi^e : \text{RHom}_{O_X \times X}(E^\vee \boxtimes L \cdot, \cdot) : D^b(X \times X) \to D^b(A^e) \] (56)
is an equivalence. Moreover, \[ \Psi^e(E^\vee \boxtimes L E) = A^e, \Psi^e(O_\Delta) = A. \]

Thus \( \Psi^e \) induces an isomorphism
\[ \Xi^n_E : \text{Ext}^n_{O_X \times X}(O_\Delta, O_\Delta) \xrightarrow{\sim} \text{Ext}^n_{A^e}(A, A) = HH^n(A). \] (57)

**Definition 6.7.** We denote the composition of the isomorphisms (54), (51) and (57) by
\[ \Phi^2 = \Xi^n_E \circ \Upsilon^n \circ \mathcal{E}^n : \bigoplus_{p=0}^n H^p(X, \wedge^{n-p} T_X) \xrightarrow{\sim} HH^n(A). \] (58)

Now we are ready to state our theorem.

**Theorem 6.8.** For \( \beta \in H^1(X, T_X) \), \( \gamma \in H^0(X, \wedge^2 T_X) \),
\[ \Phi^2(\beta, \gamma) = u_{0, \beta, \gamma}, \] (59)
where \( u_{\alpha, \beta, \gamma} \) is given by the construction 5.11.

The proof of this theorem occupies the rest of this section. The following corollary is a direct consequence of theorem 6.8.

**Corollary 6.9.** A first order noncommutative deformation of \( X \) is trivial, if it induces a trivial deformation of \( A \).

**Remark 6.10.** This corollary is also a consequence of [AT08, prop. 5.1] in our special case (smooth projective varieties with a strong full exceptional collection of vector bundles).

### 6.3 Morita equivalence and \( \lambda \)-decomposition

In this subsection we review the Morita equivalence and the \( \lambda \)-decomposition of Hochschild cohomology, and make some observations that we will need later. Our references are [Loday], [GS87]. Let \( B \) be a \( k \)-algebra, \( M_r(B) \) the \( k \)-algebra of matrices of rank \( r \) with coefficients in \( B \). The \((i,j)\)-entry of a matrix \( G \) is denoted by \( G_{ij} \).

**Definition 6.11.** For \( f \in C^0(B, B) = B \), define \( \text{cotr}(f) = f \cdot \text{id} \in M_r(B) = C^0(M_r(B), M_r(B)) \).
For \( n \geq 1 \) and \( f \in C^n(M_r(B), B) \), define \( \text{cotr}(f) \) to be the element of \( C^n(M_r(B), M_r(B)) \) such that for \( \alpha^1, \ldots, \alpha^n \in M_r(B) \),
\[ \text{cotr}(f)(\alpha^1, \ldots, \alpha^n)_{ij} = \sum_{i_2, \ldots, i_n} f(\alpha^1_{i_2}, \ldots, \alpha^n_{i_n}) \] (60)
where the sum is over all possible indices \( 1 \leq i_2, \ldots, i_n \leq r \).

For a given positive integer \( r \), let \( E_{i,j}(a) \) be the \( r \times r \) matrix whose entry at \((i,j)\) is \( a \), and all the other entry is zero. The inclusion map
\[ \text{inc}^* : C^n(M_r(B), M_r(B)) \to C^n(B, B) \]
is defined by

$$\text{inc}^*(F)(a_1, \ldots, a_n) = F(E_{11}(a_1), \ldots, E_{11}(a_n))$$  \tag{61}

for a $k$-linear map $F : M_r(B)^{\otimes n} \to M_r(B)$. It is easily seen that cotr and inc* are chain maps. The following theorem is given in [Loday, 1.5.6] without a proof. For the readers' convenience I write a proof by mimicking the proof of the homological version [Loday, 1.2.4].

**Theorem 6.12.** For positive integers $n$, cotr and inc* induce isomorphisms of Hochschild cohomology

$$\text{cotr} : HH^n(B) \xrightarrow{\sim} HH^n(M_r(B)), \quad \text{inc}^* : HH^n(M_r(B)) \xrightarrow{\sim} HH^n(B),$$

and which are inverse to each other.

**Proof:** It is obvious that $\text{inc}^* \circ \text{cotr} = \text{id}$. It suffices to show that $\text{cotr} \circ \text{inc}^*$ is homotopic to id. By definition,

$$(\text{cotr} \circ \text{inc}^*)(F)(\alpha^1, \ldots, \alpha^n)_{ij} = \sum_{i_2, \ldots, i_n} F(E_{11}(\alpha^n_{i_1}), \ldots, E_{11}(\alpha^n_{i_n})))), \tag{62}$$

define

$$(\text{cotr} \circ \text{inc}^*)(F)(\alpha^1, \ldots, \alpha^n) = \sum_{i, i_2, \ldots, i_n, j} E_{i, 1} F(E_{11}(\alpha^1_{i_2}), \ldots, E_{11}(\alpha^n_{i_n})) E_{1, j}. \tag{63}$$

For $i = 1, \ldots, n - 1$, define

$$h_i : \text{Hom}_k(M_r(B)^{\otimes n}, M_r(B)) \to \text{Hom}_k(M_r(B)^{\otimes n-1}, M_r(B))$$

by

$$h_i(F)(\alpha^1, \ldots, \alpha^{n-1}) = \sum_{k, m, \ldots, p, q} E_{k1}(1) F(E_{11}(\alpha^1_{i_2}) \otimes \ldots \otimes E_{i q}(1) \otimes \alpha^{i+1} \otimes \ldots \otimes \alpha^{n-1}). \tag{64}$$

Set

$$h_0(F)(\alpha^1, \ldots, \alpha^{n-1}) = \sum_k E_{k1}(1) F(E_{1k}(1) \otimes \alpha^1 \otimes \ldots \otimes \alpha^{n-1}). \tag{65}$$

Set temporarily (in this proof),

$$b_0(F)(\alpha^1, \ldots, \alpha^{n+1}) := \alpha^1 F(\alpha^2, \ldots, \alpha^{n+1}),$$

$$b_i(F)(\alpha^1, \ldots, \alpha^{n+1}) := F(\alpha^1, \ldots, \alpha^i \alpha^{i+1} \ldots \alpha^{n+1}), \text{ for } 1 \leq i \leq n,$$

$$b_n(F)(\alpha^1, \ldots, \alpha^{n+1}) := F(\alpha^1, \ldots, \alpha^n) \alpha^{n+1}$$

such that

$$b(F) = \sum_{i=0}^{n+1} (-1)^i b_i(F).$$

Thus

$$h_0 b_0 = \text{id}, \quad h_n b_{n+1} = \text{cotr} \circ \text{inc}^*.$$
One can verify by some tedious computations the \textit{pre-cosimplicial homotopy} relations

\[
\begin{align*}
\begin{cases}
h_i b_j = b_j h_{i-1}, & 0 \leq j < i \leq n, \\
h_i b_i = h_{i-1} b_i, & 0 < i \leq n, \\
h_i b_j = b_{j-1} h_i, & 1 \leq i + 1 < j \leq n + 1,
\end{cases}
\end{align*}
\]

which imply

\[
\left( \sum_{i=0}^{n} (-1)^i h_i \right) \circ \left( \sum_{j=0}^{n+1} (-1)^j b_j \right) + \left( \sum_{j=0}^{n} (-1)^j b_j \right) \circ \left( \sum_{i=0}^{n-1} (-1)^i h_i \right) = h_0 b_0 - h_n b_{n+1},
\]

and therefore give the homotopy from \text{id} to \text{cotr} \circ \text{inc}^*.

Now let \(L\) be free \(B\) module of rank \(r\), and \(M = \text{End}_B(L)\). Choosing a \(B\)-basis of \(L\), we obtain an isomorphism \(M \cong M_r(B)\), and thus the isomorphisms of Hochschild cohomology.

\textbf{Lemma 6.13.} The induced isomorphisms

\[
\text{cotr} : HH^n(B) \xrightarrow{\sim} HH^n(M), \quad \text{inc}^* : HH^n(M) \xrightarrow{\sim} HH^n(B)
\]

are independent of the choice of \(B\)-basis of \(L\).

\textbf{Proof :} The conclusion is a direct consequence of a more general Morita equivalence, see e.g. \cite{Loday} 1.2.5. Recall that two \(k\)-algebras \(R\) and \(S\) are Morita equivalent if there are \(R\)-\(S\)-bimodule \(P\) and \(S\)-\(R\)-bimodule \(Q\) and an isomorphism of \(R\)-bimodules \(u : P \otimes_S Q \cong R\), and an isomorphism of \(S\)-bimodules \(v : Q \otimes_R P \cong S\). Moreover, such \(u\) and \(v\) induce a natural isomorphism

\[
HH^n(R, R) \cong HH^*(S, Q \otimes_R P).
\]

Consider \(R = B, S = M = \text{End}_B(L)\), and take \(P = L, Q = L^\vee = \text{Hom}_B(L, B)\). Then there are an obvious isomorphism of \(B\)-bimodules \(u : L \otimes_M L^\vee \cong B\) given by the pairing, and an obvious isomorphism of \(M\)-bimodules \(L^\vee \otimes_B L \cong M\), and notice that \(u\) and \(v\) do not depend on the choice of basis of \(L\).

A proof of Hochschild homology version of (68) is given in \cite{Loday} 1.2.7, and one easily checks the construction of the isomorphism coincides with the isomorphism of \(\text{inc}^* : H^*_s(B, B) \cong H^*_s(M_r(B), M_r(B))\) after choosing a basis of \(M\), which implies the independence of basis for Hochschild homology. The case for Hochschild cohomology is similar, as the proof of theorem 6.12, and we omit it.

Next we recall the Hodge-type decomposition \cite{GSS7}, which is called \(\lambda\)-decomposition in \cite{Loday} §4.5. Denote by \(S_n\) the symmetric group of \(n\) elements. For the definition of the elements \(e_{n}^{(i)}\) of \(\mathbb{Q}(S_n)\), and the proof of the following proposition, see e.g. \cite{Loday} 4.5.2, 4.5.3, 4.5.7.

\textbf{Proposition 6.14.} The elements \(e_{n}^{(1)}, \ldots, e_{n}^{(n)}\) satisfy

\(\begin{align*}
(i) & \quad \text{id} = e_{n}^{(1)} + \ldots + e_{n}^{(n)}. \\
(ii) & \quad e_{n}^{(i)} e_{n}^{(j)} = 0 \quad \text{for} \quad 1 \leq i \neq j \leq n, \quad \text{and} \quad e_{n}^{(i)} e_{n}^{(i)} = e_{n}^{(i)} \quad \text{for} \quad 1 \leq i \leq n.
\end{align*}\)
(iii) In particular,
\[ e_2^{(1)} = \frac{1}{2}(\text{id} + (12)), \]
and
\[ e_n^{(n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn(\sigma)} \sigma = \frac{1}{n!} \epsilon_n. \]

**Definition 6.15.** Let \( k \) be a field of characteristic zero, \( B \) a \( k \)-algebra. For \( \sigma \in S_n \), and \( f \in C^n(A, A) \), define
\[ \sigma(f)(a_1, ..., a_n) = f(a_{\sigma(1)}, ..., a_{\sigma(n)}), \]
and extend the action linearly to \( Q(S_n) \).

**Theorem 6.16.** [Loday, 4.5.10, 4.5.12] Let \( k \) be a field of characteristic zero, and \( B \) a commutative \( k \)-algebra.

(i) \[ b \circ e_n^{(i)} = e_{n+1}^{(i)} \circ b. \] (69)

(ii) The idempotents \( e_n^{(i)} \) split the Hochschild cochain complex \( C^*(B, B) \) into a direct sum
\[ C^*(B, B) = \bigoplus_{i \geq 0} C^*_i(B, B), \] (70)
where \( C^*_i(B, B) = 0 \) for \( i > n \). This induces a direct decomposition of Hochschild cohomology
\[ HH^n(B) = \sum_{i=1}^n HH^i_n(B). \] (71)

(iii) If \( B \) is smooth, then \( HH^i_n(B) = 0 \) for \( i < n \) and the isomorphism \([74]\) reduces to
\[ e_n : T_{B/k}^n \cong HH^i_n(B). \]

Now let \( B \) be a commutative \( k \)-algebra, and \( L \) a free \( B \)-module of rank \( r \), \( M = \text{End}_B(L) \). Then the Morita equivalence and the \( \lambda \)-decomposition induce a decomposition
\[ HH^n(M) \cong \bigoplus_{i=1}^n HH^i_n(M) \] (72)
such that \( HH^i_n(M) = HH^i_n(B) \) via the isomorphism \([77]\). However, to my knowledge, we do not have an \( \lambda \)-decomposition on the cochain level \( C^*(M, M) \). Fortunately, the following naive characterization is enough for our use.

**Lemma 6.17.** Let \( F \in Z^n(M, M) \), i.e., \( F \) a Hochschild \( n \)-cocycle of \( M \). Then the class of \( F \) lies in \( HH^i_n(M) \) if, after choosing a basis of \( L \) and identify \( M \) to \( M_r(B) \),
\[ e_n^{(i)}(F)(E_{11}(b_1), ..., E_{11}(b_n))_{11} = F(E_{11}(b_1), ..., E_{11}(b_n))_{11} \] (73)
for all \( b_1, ..., b_n \in B \).

Proof: By the definition \([61]\) of \( inc^* \), \([73]\) implies \( e_n^{(i)} inc^*(F) = inc^*(F) \). \(\square\)
6.4 A bar resolution

For \( i = 1, 2 \), the homomorphisms of \( \mathcal{O}_X \)-modules (regarding \( A \) as a constant sheaf)

\[
E^\vee \otimes_k A = E^\vee \otimes_k \text{Hom}_{\mathcal{O}_X}(E, E) \rightarrow E^\vee
\]
and

\[
E \otimes_k A = \text{Hom}_{\mathcal{O}_X}(E, E) \otimes_k E \rightarrow E
\]
induce homomorphisms of \( \mathcal{O}_{X \times X} \)-modules, respectively,

\[
\sigma : q_1^*E^\vee \otimes_k A \rightarrow q_1^*E^\vee
\]
and

\[
\tau : A \otimes_k q_1^*E \rightarrow q_1^*E.
\]

Set

\[
\mathcal{C}^{\text{bar}}_i = E^\vee \otimes E \otimes_k A^{\otimes i},
\]
and define \( b_i' : \mathcal{C}^{\text{bar}}_i \rightarrow \mathcal{C}^{\text{bar}}_{i-1} \) by

\[
b_i'(x, y, a_1, ..., a_i) = (\sigma(x \otimes a_1), y, a_2, ..., a_i) \\
+ \sum_{j=1}^{i-1} (-1)^j (x, y, a_1, ..., a_{j-1}, a_ja_{j+1}, a_{j+2}, ..., a_i) + (-1)^i (x, \tau(a_i \otimes y), a_1, ..., a_{i-1}). \tag{74}
\]

We define an augmentation map \( \mu : E^\vee \otimes E \rightarrow \mathcal{O}_\Delta \) by adjointness, via \( q_2^*E \rightarrow q_1^*E \otimes \mathcal{O}_\Delta \cong q_2^*E \big|_\Delta \), or equivalently, via \( q_1^*E^\vee \rightarrow q_2^*E^\vee \otimes \mathcal{O}_\Delta \cong q_1^*E^\vee \big|_\Delta \).

**Lemma 6.18.** There is an quasi-isomorphisms of complex of coherent sheaves

\[
\mathcal{C}^{\text{bar}}_*(E) \rightarrow \mathcal{O}_\Delta \tag{75}
\]
on \( X \times X \), which is transformed by \( \Psi^e \) to the bar resolution \( \mathcal{C}^{\text{bar}}_*(A) \) of \( A \).

**Proof:** It is easy to check that \( b_i' \circ b_i'_{i+1} = 0 \) and \( \mu \circ b_i' = 0 \). By the definition of the bar resolution of a \( k \)-algebra [Loday 1.1.11], one easily sees that \( \Psi^e(\mathcal{C}^{\text{bar}}_*(E)) = \mathcal{C}^{\text{bar}}_*(A) \). By [Loday 1.1.12] and theorem 6.6, \( \mathcal{C}^{\text{bar}}_*(E) \) is a resolution of \( \mathcal{O}_\Delta \). \( \square \)

For an open subset \( U_i \) of \( X \), regarded as an open subset of the diagonal \( \Delta_X \subset X \times X \), by theorem 6.3 and lemma 6.18 we have

\[
\bigwedge^q T_{U_i} \cong \delta^q_{\mathcal{O}_{\Delta U_i} \otimes \mathcal{O}_{U_i}} (\mathcal{O}_{\Delta U_i}, \mathcal{O}_{U_i}) \cong \mathcal{H}^q(\mathcal{H}om_{\mathcal{O}_{\Delta U_i} \otimes \mathcal{O}_{U_i}}(\mathcal{C}^{\text{bar}}_*(E)|_{U_i \times U_i}, \mathcal{O}_{\Delta U_i})). \tag{76}
\]

It will turn out to be more convenient to work with a Hochschild cochain complex rather than the bar resolution. Let us introduce first the Hochschild cochain complex for a module over a sheaf of algebras.

**Definition 6.19.** For a sheaf \( \mathcal{A} \) of \( k \)-algebras over a topological space \( Y \), let \( \mathcal{A}^{\otimes i} \) be the sheaf associated to the presheaf \( U \mapsto \Gamma(U, A)^{\otimes i} \), which is still a sheaf of \( k \)-algebras. For a sheaf \( \mathcal{M} \) of \( \mathcal{A} \)-bimodules, we define the Hochschild cochain complex \( \mathcal{C}^*(\mathcal{A}, \mathcal{M}) \) of sheaves of \( k \)-vector spaces on \( X \) by

\[
\mathcal{C}^k(\mathcal{A}, \mathcal{M}) = \mathcal{H}om_k(\mathcal{A}^{\otimes k}, \mathcal{M})
\]

30
with the differentials given by

\[ b(f)(a_1, \cdots, a_{k+1}) = a_1 f(a_2, \cdots, a_{k+1}) + \sum_{1 \leq i \leq k} (-1)^i f(a_1, \cdots, a_i a_{i+1}, \cdots, a_{k+1}) + (-1)^{k+1} f(a_1, \cdots, a_k) a_{k+1}. \]  

When \( Y \) is a point, \( C^\bullet(A, \mathcal{M}) \) is the ordinary Hochschild cochain complex which computes the Hochschild cohomology \( HH^\bullet(A, \mathcal{M}) \) [Loday, 1.5.1].

Return to the setup at the beginning of this section. We denote the constant sheaf of \( k \)-algebras associated to \( A = \text{Hom}_{\mathcal{O}_X}(E, E) \) by \( A \). Then \( E^\vee \otimes_{\mathcal{O}_X} E \cong \mathcal{H} \text{om}_{\mathcal{O}_X}(E, E) \) is a sheaf of \( A \)-bimodules in an obvious way. The corresponding Hochschild cochain complex is denoted by \( C^\bullet(A, E^\vee \otimes E) \). There is an obvious homomorphism between two Hochschild cochain complex

\[ C^\bullet(E^\vee \otimes E, E^\vee \otimes E) \rightarrow C^\bullet(A, E^\vee \otimes E) \]  

induced by the homomorphism of sheaves of algebras \( A \rightarrow E^\vee \otimes E \) given by restrictions of global endomorphisms of \( E \).

Let us recall the Čech complex associated to a complex of sheaves. Let \( \mathcal{U} = \{U_i\}_{i \in I} \) be an affine open covering of \( X \). For a complex of sheaves \((L^\bullet, \partial_L)\), the associated Čech double complex is \( \check{C}^p(\mathcal{U}, L^q) \) with \( \delta : \check{C}^p(\mathcal{U}, L^q) \rightarrow \check{C}^{p+1}(\mathcal{U}, L^q) \) the Čech coboundary map, \( \partial_L : \check{C}^p(\mathcal{U}, L^q) \rightarrow \check{C}^p(\mathcal{U}, L^{q+1}) \) the map induced by \( \partial_L \). The differential of the associated simple complex is

\[ d = \delta + (-1)^p \partial_L. \]  

**Lemma 6.20.** The cohomology of (the simple complex associated to) the double complex

\[ \check{C}^\bullet(\mathcal{U}, C^\bullet(A, E^\vee \otimes E)) \]  

computes the Hochschild cohomology \( HH^\bullet(X) \).

**Proof:** For every integer \( m \geq 0 \), there is an identity of sheaves on \( U_i \)

\[ \mathcal{H} \text{om}_{\mathcal{O}_{U_i \times U_j}}(\mathcal{O}_{U_i \times U_j}^\bar{\otimes}(E)|_{U_i \times U_j}, \mathcal{O}_{\Delta_{U_i}}) = \mathcal{H} \text{om}_{k}(A^\otimes_m, (E^\vee \otimes E)|_{U_i}). \]  

One easily checks, by comparing (74) and (77), that (80) induces an isomorphism

\[ \mathcal{H} \text{om}_{\mathcal{O}_{U_i \times U_j}}(\mathcal{O}_{\Delta_{U_i}})(\mathcal{O}_{U_i \times U_j}^\bar{\otimes}(E)|_{U_i \times U_j}) \cong C^\bullet(A, E^\vee \otimes E)|_{U_i}. \]

By the isomorphisms (76) and (81), \( HH^\bullet(X) \) is isomorphic to the hypercohomology of \( C^\bullet(A, E^\vee \otimes E) \). Since \( \mathcal{H} \text{om}_{k}(A^\otimes_k, E^\vee \otimes E) \) is coherent, the conclusion follows from e.g. [ET, theorem 2.8.1].

We denote the resulting isomorphism by

\[ \mathcal{O}^n : H^n(\check{C}^\bullet(\mathcal{U}, C^\bullet(A, E^\vee \otimes E))) \xrightarrow{\sim} \text{Ext}^n_{\mathcal{O}_{X \times X}}(\Delta_X, \Delta_X). \]

By (76) and (81), there are also isomorphisms

\[ \mathcal{B}^{p, q} : H^p(\mathcal{U}, \mathcal{E}xt^q_{\mathcal{O}_{X \times X}}(\Delta_X, \Delta_X)) \xrightarrow{\sim} \check{H}^p(\mathcal{U}, \mathcal{H}^q(C(A, E^\vee \otimes E))), \]

and we denote \( \mathcal{B}^n = \bigoplus_{p+q=n} \mathcal{B}^{p, q} \).
6.5 Some canonical isomorphisms

In this subsection we prove some canonical isomorphism together commutativity, for preparing the explicit construction of $\Phi^n$.

According to definition 6.19, let $C^\bullet(O_X, O_X)$ be the Hochschild cochain complex associated to the sheaf of $k$-algebras $O_X$.

**Lemma 6.21.** The cohomology sheaf $H^q(C^\bullet(O_X, O_X))$ is canonically isomorphic to $T^q_X = \wedge^q T_X$.

Proof: This follows from theorem 6.4, see also [Swan96, lemma 2.4 (3)].

**Corollary 6.22.** Let $U$ be an affine open covering of $X$, then

$$H^p(X, \mathcal{H}^q(C^\bullet(O_X, O_X))) \cong \check{H}^p(U, \mathcal{H}^q(C^\bullet(O_X, O_X))).$$

Proof: By lemma 6.21, $\mathcal{H}^q(C^\bullet(O_X, O_X))$ is a coherent sheaf, thus the conclusion follows.

**Lemma 6.23.** There are quasi-isomorphisms $C^\bullet(O_X, O_X) \rightarrow C^\bullet(E^\vee \otimes E, E^\vee \otimes E) \rightarrow C^\bullet(A, E^\vee \otimes E)$.

Proof: The first map is induced by the natural maps $E^\vee \otimes E \rightarrow O_X$ and $O_X \rightarrow E^\vee \otimes E$. By theorem 6.12 and lemma 6.13, the first map is a quasi-isomorphism. The second map is (78). Then by (76), (81) and lemma 6.21, the second map is also a quasi-isomorphism.

**Lemma 6.24.** There is a canonical isomorphism

$$\bigoplus_{p+q=n} \check{H}^p(U, \mathcal{H}^q(C^\bullet(O_X, O_X))) \cong H^n(X, C^\bullet(O_X, O_X)).$$

Proof: There is a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(C^\bullet(O_X, O_X))) \Rightarrow H^{p+q}(X, C^\bullet(O_X, O_X)).$$

The $\lambda$-decomposition $C^q(O_X, O_X) = \bigoplus_{i=0}^q C^q_i(O_X, O_X)$ induces the degeneration of the spectral sequence, and moreover the decomposition (84), see the argument of [Swan96, cor. 2.6].

**Notations 6.25.** For a given affine open covering $U$ of $X$, denote by $\eta$ the canonical isomorphism

$$\eta: \bigoplus_{p+q=n} \check{H}^p(U, \mathcal{H}^q(C^\bullet(O_X, O_X))) \cong H^n(X, C^\bullet(O_X, O_X))$$

induced by (82) and (84), and denote $\xi$ and $\zeta$ the isomorphisms

$$\xi: \bigoplus_{p+q=n} \check{H}^p(U, \mathcal{H}^q(C^\bullet(O_X, O_X))) \cong \bigoplus_{p+q=n} \check{H}^p(U, \mathcal{H}^q(C^\bullet(A, E^\vee \otimes E))).$$

and

$$\zeta: H^{p+q}(X, C^\bullet(O_X, O_X)) \cong H^n(\check{C}^\bullet(U, C^\bullet(A, E^\vee \otimes E))).$$

the isomorphisms induced by (83).
Lemma 6.26. There are natural isomorphisms $\rho$ and $\sigma$ such that the following diagrams commute.

\[
\bigoplus_{p+q=n} \check{H}^p(U, \mathcal{E}xt^q_{O_X \times X}(\Delta_X, \Delta_X)) \xrightarrow{\rho} \bigoplus_{p+q=n} \check{H}^p(U, \mathcal{H}^q(\mathcal{C}^\bullet(O_X, O_X)))
\]

and

\[
\bigoplus_{p+q=n} \check{H}^p(U, \mathcal{E}xt^q_{O_X \times X}(\Delta_X, \Delta_X)) \xrightarrow{\sigma} \check{H}^n(X, \mathcal{C}^\bullet(O_X, O_X))
\]

\[
\bigoplus_{p+q=n} \check{H}^p(U, \mathcal{E}xt^q_{O_X \times X}(\Delta_X, \Delta_X)) \xrightarrow{\eta} \check{H}^n(X, \mathcal{C}^\bullet(U, E^\vee \otimes E))
\]

Proof: The quasi-isomorphisms (73), (75) and the isomorphism (71) induce canonical isomorphisms $\rho$ and $\sigma$, and the commutativity of (76). In addition, they induce an isomorphism of $E_2$-spectral sequences

\[
E_2^{p,q} = \check{H}^p(U, \mathcal{E}xt^q_{O_X \times X}(\Delta_X, \Delta_X)) \Rightarrow \check{H}^n(X, \mathcal{C}^\bullet(O_X, O_X))
\]

and

\[
E_2^{p,q} = \check{H}^p(U, \mathcal{H}^q(\mathcal{C}^\bullet(O_X, O_X))) \Rightarrow \check{H}^{p+q}(X, \mathcal{C}^\bullet(O_X, O_X)).
\]

Thus the decomposition (71) induces a decomposition $\check{Y}^n$ and a commutative diagram

\[
\bigoplus_{p+q=n} \check{H}^p(U, \mathcal{E}xt^q_{O_X \times X}(\Delta_X, \Delta_X)) \xrightarrow{\rho} \bigoplus_{p+q=n} \check{H}^p(U, \mathcal{H}^q(\mathcal{C}^\bullet(O_X, O_X)))
\]

\[
\bigoplus_{p+q=n} \check{H}^p(U, \mathcal{E}xt^q_{O_X \times X}(\Delta_X, \Delta_X)) \xrightarrow{\sigma} \check{H}^n(X, \mathcal{C}^\bullet(O_X, O_X)).
\]

It remains to show $\check{Y}^n = Y^n$. Following [Swan96, §2], let $\mathcal{C}_i$ be the sheaf associated to the presheaf $U \mapsto C_i(\Gamma(U, O_X)) = \Gamma(U, O_X)^{\otimes i+1}$, and together with the usual Hochschild boundary map $b$, we obtain a complex of sheaves of $O_X$-modules, denoted by $\mathcal{C}^\bullet$. Then by [Swan96, theorem 2.1 and 2.5], there is the following commutative diagram of isomorphisms

\[
\bigoplus_{p+q=n} \check{H}^p(U, \mathcal{E}xt^q_{O_X \times X}(\Delta_X, \Delta_X)) \xrightarrow{\rho} \bigoplus_{p+q=n} \check{H}^p(U, \mathcal{H}^q(\mathcal{C}^\bullet(O_X, O_X)))
\]

\[
\bigoplus_{p+q=n} \check{H}^p(U, \mathcal{E}xt^q_{O_X \times X}(\Delta_X, \Delta_X)) \xrightarrow{\sigma} \check{H}^n(O_X, (\mathcal{C}_i(O_X), O_X)).
\]

In fact, [Swan96, theorem 2.5] says that there is an $E_2$-spectral sequence

\[
E_2^{p,q} = \check{H}^p(U, \mathcal{H}^q(\mathcal{C}^\bullet(O_X, O_X))) \Rightarrow \check{H}^{p+q}(O_X, (\mathcal{C}_i(O_X), O_X))
\]

which is isomorphic to the spectral sequence (77), and then the decomposition $Y^n$ follows from the right one $\eta'$, which is also deduced from the $\lambda$-decomposition of $\mathcal{C}^\bullet(O_X, O_X)$.
Therefore the spectral sequences (88) and (91) are isomorphic, thus induce an isomorphism \( \chi : \mathcal{H}^n(X, \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X)) \xrightarrow{\sim} \mathcal{H}^n \mathcal{O}_X (\mathcal{C}_\bullet(\mathcal{O}_X), \mathcal{O}_X) \). Then \( \tilde{\mathcal{Y}}^n = \mathcal{Y}^n \) is equivalent to the commutativity of the decompositions \( \eta \) and \( \eta' \):

\[
\begin{array}{ccc}
\bigoplus_{p+q=n} \mathcal{H}^n(\mathcal{U}, \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X)) & \xrightarrow{\eta} & \bigoplus_{p+q=n} \mathcal{H}^n(\mathcal{U}, \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X)) \\
\mathcal{H}^n(X, \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X)) & \xrightarrow{\chi} & \mathcal{H}^n(X, \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X)) \\
\mathcal{H}^n \mathcal{O}_X (\mathcal{C}_\bullet(\mathcal{O}_X), \mathcal{O}_X) & \xrightarrow{\eta'} & \mathcal{H}^n \mathcal{O}_X (\mathcal{C}_\bullet(\mathcal{O}_X), \mathcal{O}_X)
\end{array}
\]

But both decomposition \( \eta \) and \( \eta' \) follows from the same decomposition of

\[
E_2^{p,q} = E''_{p,q} = \mathcal{H}^p(\mathcal{U}, \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X))
\]

which in turn is induced by the \( \lambda \)-decomposition of \( \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X) \), the commutativity of \( \eta \) follows.

**Corollary 6.27.** Given an affine open covering \( \mathcal{U} \) of \( X \), there is a canonical isomorphism

\[
\begin{aligned}
\mathbf{L}^n : \bigoplus_{p+q=n} \mathcal{H}^p(\mathcal{U}, \mathcal{C}^q(\mathcal{C}(A, E^\vee \otimes E))) & \xrightarrow{\sim} H^n(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{C}^\bullet(A, E^\vee \otimes E))) \\
\mathcal{H}^n(X, \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X)) & \xrightarrow{\mathbf{L}^n} H^n(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{C}^\bullet(A, E^\vee \otimes E)))
\end{aligned}
\]

such that the following diagram

\[
\begin{array}{ccc}
\bigoplus_{p+q=n} \mathcal{H}^n(\mathcal{U}, \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X)) & \xrightarrow{\mathbf{L}^n} & \bigoplus_{p+q=n} \mathcal{H}^n(\mathcal{U}, \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X)) \\
\mathcal{H}^n(X, \mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{O}_X)) & \xrightarrow{\mathbf{L}^n} & H^n(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{C}^\bullet(A, E^\vee \otimes E)))
\end{array}
\]

commutes.

Proof: The commutativity of the upper triangle follows from naturality. The commutativity of the left trapezoid and the lower triangle is lemma \( \ref{lemma:6.26} \). The isomorphism \( \mathbf{L}^n \) is induced by demanding the commutativity of the right trapezoid.

**6.6 An explicit description of \( \Phi^2 \)**

In this subsection we give an explicit description of \( \Phi^2 \), and compare it to \( u(\cdot, \cdot, \cdot) \) of section 5, and thus complete the proof of theorem \( \ref{theorem:6.8} \).

**Lemma 6.28.** Let \( l \geq 0 \) be an integer, and \( v \in \text{Hom}_k(A^\otimes l, A) \) such that \( b(v) = 0 \). Let \( v_i \in \text{Hom}_k(A^\otimes l, E^\vee \otimes E) \) be the restriction of \( v \) to \( U_i \). Thus \( \{v_i\}_{i \in I} \in \mathcal{C}^0(\mathcal{U}, \mathcal{C}^l(A, E^\vee \otimes E)) \) induces a class in \( H^n(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{C}^\bullet(A, E^\vee \otimes E))) \), denoted by \( \tilde{v} \). Then

\[
\Xi_E \circ \Omega^n(\tilde{v}) = v.
\]
Proof: This follows directly from the second statement of lemma 6.18 and the identification (81).

Now we are ready to give an explicit description of $\Psi^2$. Consider the following commutative diagram, keeping in mind the diagram (94) which microscopes the following right square:

\[
\begin{array}{c}
\bigoplus_{p+q=n} \check{H}^p(U, \wedge^q T_X) \cong \bigoplus_{p+q=n} \check{H}^p(U, \mathcal{S}^0_{\mathcal{O}_{\Delta X}}(\Delta X, \Delta X)) \cong \bigoplus_{p+q=n} \check{H}^p(U, \mathcal{H}^0(C^\bullet(A, E))) \\
\check{H}^n(A) \\
\end{array}
\]

(96)

**Definition 6.29.** Define

\[ Z^i(A, \mathcal{H}om_{\mathcal{O}_X}(E, E)) = \ker \left( C^i(A, \mathcal{H}om_{\mathcal{O}_X}(E, E)) \to C^{i+1}(A, \mathcal{H}om_{\mathcal{O}_X}(E, E)) \right), \]

and define $Z^i(p)(A, \mathcal{H}om_{\mathcal{O}_X}(E, E))$ to be the subsheaf of $Z^i(A, \mathcal{H}om_{\mathcal{O}_X}(E, E))$ consists of the local sections cohomological to local sections of $C^i_{(p)}(\mathcal{O}_X, \mathcal{O}_X)$ via the quasi-isomorphism (83).

Given a local section of $Z^i(A, \mathcal{H}om_{\mathcal{O}_X}(E, E))$, one can choose a local basis of $E$ to show that it lies in $Z^i(p)(A, \mathcal{H}om_{\mathcal{O}_X}(E, E))$, by checking the criterion in lemma 6.17.

Our general strategy to find an explicit description of $\Phi^n(\tau)$ for $\tau \in \check{H}^n(U, \wedge^l T_X)$ consists of the following steps:

1. Find an explicit expression for $\mathfrak{B}^n \circ \mathfrak{C}^n(\tau)$.
2. Find an explicit expression for $\mathfrak{L}^n \circ \mathfrak{B}^n \circ \mathfrak{C}^n(\tau)$.
3. By a zigzag in a double complex, find an element $v$ in $\check{C}^0(U, C^{m+1}(A, E^\vee \otimes E))$ that differs by a coboundary from $\mathfrak{L}^n \circ \mathfrak{B}^n \circ \mathfrak{C}^n(\tau)$, and observe that $v$ in fact is of the form in the lemma 6.28.

The second step will make use of the $\lambda$-decomposition. The following construction illustrates an attempt to carry out this strategy, but it is not completely fulfilled. The problem arises in the second step: I don’t know how to find $\mathfrak{t}^{m,l}$ that satisfies (99). In the final proof of theorem 6.3 I will show that for $\Phi^2$, the construction 5.11 indeed provides a construction of $\mathfrak{t}^{1,1}$ and $\mathfrak{t}^{0,2}$.

**Construction 6.30.** Let $U = \{U_i\}_{i \in I}$ be an affine open covering of $X$, such that $E|_{U_i}$ is free for any $i \in I$. For any ordered set of indices $I = (i_0, ..., i_k) \subset I$, denote $U_I = U_{i_0} \cap ... \cap U_{i_k}$. For each $U_I$, choose a connection $\nabla_I : E \to E \otimes_{\mathcal{O}_X} \Omega^1_X$. For $a \in \mathcal{H}om_{\mathcal{O}_U}(E, E)$, define $\nabla_I(a) = \nabla_I \circ a - a \circ \nabla_I \in \mathcal{H}om_{\mathcal{O}_{U_I}}(E, E \otimes_{\mathcal{O}_{U_I}} \Omega^1)$.
Let \( \theta \in \Gamma(U_I, \wedge^l T_X) \). For \( a_1 \otimes ... \otimes a_l \in \mathcal{H}om_{\mathcal{O}_{U_I}}(E, E)^{\otimes l} \), define

\[
\text{cotr}(\theta)(a_1 \otimes ... \otimes a_l) = \theta_\perp(\nabla_I(a_1) \circ ... \circ \nabla_I(a_l)) \in \mathcal{H}om_{\mathcal{O}_{U_I}}(E, E).
\]  

(97)

For example, if \( \theta = \theta_1 \wedge ... \wedge \theta_l \), where \( \theta_j \in \Gamma(U_I, T_X) \) for \( 1 \leq j \leq l \), and set \( (\nabla_I)_{\theta_j}(s) = \theta_j \cdot \nabla_I(s) \) to be the covariant derivative, then

\[
\text{cotr}(\theta)(a_1 \otimes ... \otimes a_l) = \sum_{\sigma \in S_l} \text{sgn}(\sigma)(\nabla_I)_{\theta_{\sigma_1}}(a_{\sigma_1}) \circ ... \circ (\nabla_I)_{\theta_{\sigma_l}}(a_{\sigma_l}).
\]

(98)

Let \( \tau \) be an element of \( H^m(X, \wedge^l T_X) \) for certain integers \( m, l \geq 0 \). Let \( \{\theta_I\}_{|I|=m+1} \) be a Čech representative of \( \tau \), where \( \theta_I \in \Gamma(U_I, \wedge^l T_X) \). Thus

\[
\{\text{cotr}(\theta_I)\}_{|I|=m+1} \in \check{C}^m(U, C^I(A, \mathcal{H}om_{\mathcal{O}_X}(E, E))).
\]

Denote \( t^{m,l} = \{\text{cotr}(\theta_I)\}_{|I|=m+1} \). Look at the following commutative diagram, where \( C^I = C^I(A, \mathcal{H}om_{\mathcal{O}_X}(E, E)) \).

\[
\begin{array}{ccc}
\check{C}^{m-1}(U, C^I) & \xrightarrow{b} & \check{C}^{m-1}(U, C^{I+1}) \\
\delta \downarrow & & \delta \downarrow \\
\check{C}^{m}(U, C^{I-1}) & \xrightarrow{b} & \check{C}^{m}(U, C^{I+1}) \\
\delta \downarrow & & \delta \downarrow \\
\check{C}^{m+1}(U, C^{I-1}) & \xrightarrow{b} & \check{C}^{m+1}(U, C^I) \\
\end{array}
\]

Since \( \{\theta_I\}_{|I|=m+1} \in Z^m(U, \wedge^l T_X), \ b(t^{m,l}) = 0 \). Moreover, by the definition (97), and lemma 6.17 and trivializing \( E \) by the connections chosen, one easily sees \( t^{m,l} \in \check{C}^m(U, Z_l^I) \). But \( \delta t^{m,l} \) is not necessarily zero. Suppose we can find \( \tilde{t}^{m,l} \) such that

\[
\begin{cases}
\tilde{t}^{m,l} \in \check{C}^m(U, Z_l^I), \\
\tilde{t}^{m,l} - t^{m,l} \in b(\check{C}^m(U, C^{I-1})), \\
\delta \tilde{t}^{m,l} = b(t^{m,l}) = 0.
\end{cases}
\]

(99)

Then since

\[
\check{H}^m(U, C^I) = \text{Hom}_k\left(A^{\otimes l}, \check{H}^m(U, \mathcal{H}om_{\mathcal{O}_X}(E, E))\right)
\]

and \( \check{H}^m(U, \mathcal{H}om_{\mathcal{O}_X}(E, E)) = \text{Ext}^{m}(E, E) = 0 \) for \( m \geq 1 \), there exists \( t^{m-1,l} \in \check{C}^{m-1}(U, C^I) \) such that \( \delta t^{m-1,l} = t^{m,l} \). Put \( t^{m-1,l+1} = b(t^{m-1,l}) \). Then \( t^{m-1,l+1} \in \check{C}^{m-1}(U, C^{I+1}) \) and \( \delta t^{m-1,l+1} = 0 \). We can continue this process, until we obtain \( 0^{m,l} \in \check{C}^0(U, C^{m+l}) \). Moreover, because \( b(0^{m,l}) = 0 \) and \( \delta 0^{m,l} = 0 \), \( 0^{m,l} \) lies in \( \text{Hom}_k(A^{\otimes m+l}, A) \) and produces a Hochschild cocycle, and we denote the resulting class in \( HH^{m+l}(A) \) by \( v(\tau) \).

\[\square\]

**Theorem 6.31.** Given \( \tilde{t}^{m,l} \) satisfying (99), then

\[\Phi^n(\tau) = (-1)^m v(\tau).\]

(100)

36
Proof: By the definition (60) of cotrace map, the definition of affine HKR isomorphism (52) - (53), and the construction of the quasi-isomorphism (53), $t^{m,l}$ represents $B^n \circ \mathcal{E}^n(\tau) \in H^n(\mathcal{U}, \mathcal{K}^l(\mathcal{C}^*(A, E)))$. So does $\tilde{t}^{m,l}$. Moreover, $\tilde{t}^{m,l}$ represents a class in $H^n(\mathcal{C}^*(\mathcal{U}, \mathcal{C}^*))$, and by the first condition of (99),

$$\tilde{t}^{m,l} = \zeta \circ \eta \circ \rho^{-1} \circ \mathcal{E}^n(\tau) = \mathcal{L}^n([t^{m,l}]).$$

By the construction (6.30) and the sign convention (79), $(-1)^m t^{0,m+l}$ and $\tilde{t}^{m,l}$ represents the same class in $H^n(\mathcal{C}^*(\mathcal{U}, \mathcal{C}^*))$. Thus (100) follows from lemma (6.28). □

**Proof of theorem 6.8**

Since $\Phi^2(0, \beta, \gamma) = \Phi^2(0, \beta, 0) + \Phi^2(0, 0, \gamma)$, and $u_{0,\beta,\gamma} = u_{0,\beta,0} + u_{0,0,\gamma}$ by lemma 5.13, we can prove theorem 6.8 in the case $\beta = 0$ and the case $\gamma = 0$ separately.

**6.6.1 The case $\gamma = 0$**

This corresponds to the case $m = l = 1$ in the construction 6.30. Let $\tilde{t}^{1,1} = - \{g_{ij}\}_{i,j \in I}$, where $g_{ij}$ is defined in construction 5.11. Then by the construction 5.11 $\tilde{t}^{1,1}$ satisfies (99) by lemma 6.17 and proposition 6.14 (iii); in fact, this is automatic for $l = 1$. Thus again by the construction 5.11 we can take $t^{0,1} = \{c_i\}_{i \in I}$. Then $t^{0,2} = -u_{0,0,0}$. So by theorem 6.31 $\Phi^2(0, \beta, 0) = u_{0,\beta,0}$.

**6.6.2 The case $\beta = 0$**

It suffices to show that $\tilde{t}^{0,2} := \{u_{0,0,0}(\cdot, \cdot, \cdot)\}_{i \in I}$ satisfies (6.31). The third condition of (6.31) follows by the construction of $u_{0,0,0}$, see lemma 5.12 and 5.13. The second condition of (6.31) is a local property, so we can check this locally on each sufficiently small $U_i$. Thus suppose $\gamma = \partial_1 \wedge \partial_2$, where $\partial_k = \partial_{x_k}$ for $i = 1, 2$, and $\{x_k\}_{1 \leq k \leq \dim X}$ are (étale) local coordinates of $X$. In addition we trivialize $E$ by choose a local basis, on $U_i$, and obtain a corresponding connection $\nabla$. Take

$$c_i(a) = \nabla_1 \otimes \nabla_2(a) - \nabla_1(a) \otimes \nabla_2,$$

where $\nabla_k = \partial_k \cdot \nabla$, $k = 1, 2$. For $a, a' \in A$, write $a$ and $a'$ as $(a_{rs})$ and $(a'_{rs})$ in the chosen local basis of $E$. Then $c_i(a') = (C'_{rs}), c_i(a) = (C'_{rs})$, and $c_i(a'a) = (C''_{rs})$ where

$$C'_{rs} = \partial_1 \otimes \partial_2(a_{rs}) - \partial_1(a'_{rs}) \otimes \partial_2,$$
$$C_{rs} = \partial_1 \otimes \partial_2(a_{rs}) - \partial_1(a_{rs}) \otimes \partial_2,$$
$$C''_{rs} = \partial_1 \otimes \partial_2(\sum_{p} a'_{rp}a_{ps}) - \partial_1(\sum_{p} a'_{rp}a_{ps}) \otimes \partial_2.$$
Then
\[
- \sum_s C'_r a_{st} - \sum_s a'_r s C_{st} + C''_r t
= - \sum_s (\partial_1 (a_{st}) \partial_2 (a'_r) + a_{st} \partial_2 (a'_r)) \partial_1 - \partial_1 (a'_r) \partial_2 (a_{st}) - a_{st} \partial_1 (a'_r) \partial_2
-
\sum_s (a'_r s \partial_2 (a_{st}) \partial_1 - a'_r s \partial_1 (a_{st}) \partial_2)
+ \sum_s (a'_r s \partial_2 (a_{st}) \partial_1 + a_{st} \partial_2 (a'_r) \partial_1 - a'_r s \partial_1 (a_{st}) \partial_2 - a_{st} \partial_1 (a'_r) \partial_2)
= \sum_s (\partial_1 (a'_r) \partial_2 (a_{st}) - \partial_1 (a_{st}) \partial_2 (a'_r))
\]
(101)
Comparing to (98) one sees
\[u_{0,0,\gamma}(a', a) = \cotr(\gamma)(a', a).\]
Thus the second condition of (99) is shown.
The first condition of (99) is also local. By the expression (101), \(u_{0,0,\gamma}\) is anti-symmetric in \(\partial_1\) and \(\partial_2\). By lemma 6.17 and proposition 6.14 (iii), \(u_{0,0,\gamma} \in C^0(U, Z^2(2))\).

7 Open problems

I propose two problems partly inspired by theorem 1.1.

1. Bernardara and Bolognesi proposed a notion of \textit{categorical representability dimension}. By [BB12, definition 2.4], one says that a smooth projective variety \(X\) over \(k\) is \textit{categorically representable in dimension} \(n\) if \(D^b(X)\) has a semiorthogonal decomposition
\[D^b(X) = \langle B_1, ..., B_l \rangle\]
such that each \(B_j\) is an admissible subcategory of \(D^b(Y_j)\) where \(Y_j\) is a smooth projective variety over \(k\) of dimension \(\leq n\). By [AB15, lemma 1.19], if \(k\) is separably closed, \(X\) is categorically representable in dimension zero if and only if \(X\) has a full exceptional collection. So according to theorem 1.1 the following question seems natural.

\textbf{Question 7.1.} For a family of smooth projective varieties, is the categorical representability dimension of the geometric fibers upper semicontinuous over the base scheme ?

2. It seems a folklore conjecture that smooth projective varieties possessing full exceptional collections are rational. In dimension \(\leq 2\) this is a conjecture attributed to Orlov. By [Kawa06], a smooth projective toric variety has a full exceptional collection. This together with theorem 1.1 motivate the following question.

\textbf{Question 7.2.} Does small smooth deformation of a smooth toric Fano variety remain rational ?

If the answer is negative, there exist nonrational smooth projective varieties that possess full exceptional collections. We can also ask similar questions for all the varieties possessing full exceptional collections, but among them toric varieties seem the ones that most probably have nonrational deformations.
References

[AT08] Anel, Mathieu; Toën, Bertrand. Dénombrabilité des classes d’équivalences dérivées de variétés algébriques. J. Algebraic Geom. 18 (2009), no. 2, 257–277.

[AZ94] Artin, M.; Zhang, J. J. Noncommutative projective schemes. Adv. Math. 109 (1994), no. 2, 228–287.

[AB15] Auel, Asher; Bernardara, Marcello. Semiorthogonal decompositions and birational geometry of del Pezzo surfaces over arbitrary fields. arXiv:1511.07576

[ARS] Auslander, Maurice; Reiten, Idun; Smalø, Sverre O. Representation theory of Artin algebras. Corrected reprint of the 1995 original. Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1997.

[ASS] Assem, Ibrahim; Simson, Daniel; Skowroński, Andrzej. Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory. London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006.

[Barr68] Barr, Michael. Harrison homology, Hochschild homology and triples. J. Algebra 8 (1968), 314–323.

[BBR] Bartocci, Claudio; Ugo Bruzzo, and Daniel Hernández Ruipérez. Fourier-Mukai and Nahm transforms in geometry and mathematical physics. Vol. 276. Springer Science & Business Media, 2009.

[BB12] Bernardara, Marcello; Bolognesi, Michele. Categorical representability and intermediate Jacobians of Fano threefolds. Derived categories in algebraic geometry, 1–25, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012.

[Bod15] Bodzenta, Agnieszka. DG categories and exceptional collections. Proc. Amer. Math. Soc. 143 (2015), no. 5, 1909–1923.

[Bon89] Bondal, Alexei I. Representations of associative algebras and coherent sheaves. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 1, 25–44; translation in Math. USSR-Izv. 34 (1990), no. 1, 23–42

[BH13] Buchweitz, Ragnar-Olaf; Hille, Lutz. Hochschild (co-)homology of schemes with tilting object. Trans. Amer. Math. Soc. 365 (2013), no. 6, 2823–2844.

[Căl05] Căldăraru, Andrei. The Mukai pairing. II. The Hochschild-Kostant-Rosenberg isomorphism. Adv. Math. 194 (2005), no. 1, 34–66.

[Cib86] Cibils, Claude. Hochschild homology of an algebra whose quiver has no oriented cycles. Representation theory, I (Ottawa, Ont., 1984), 55-59, Lecture Notes in Math., 1177, Springer, Berlin, 1986.

[ET] El Zein, Fouad; Tu, Loring W. From sheaf cohomology to the algebraic de Rham theorem. Hodge theory, 70–122, Math. Notes, 49, Princeton Univ. Press, Princeton, NJ, 2014.
[EGAII] Grothendieck, A. Eléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): II. Étude globale élémentaire de quelques classes de morphismes. Publications mathématiques de l’IHES., tome 8 (1961), p. 5-222.

[EGAIII] Grothendieck A. Eléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): III. Étude cohomologique des faisceaux cohérents, Premiere partie. Publications Mathématiques de l’IHES, 1961, 11: 5-167.

[Ger64] Gerstenhaber, Murray. On the deformation of rings and algebras. Ann. of Math. (2) 79 (1964), 59–103.

[GS87] Gerstenhaber, Murray; Schack, S. D. A Hodge-type decomposition for commutative algebra cohomology. J. Pure Appl. Algebra 48 (1987), no. 3, 229–247.

[Gode58] Godement, Roger. Topologie algébrique et théorie des faisceaux. Actualit’es Sci. Ind. No. 1252. Publ. Math. Univ. Strasbourg. No. 13 Hermann, Paris 1958

[HS88] Happel, Dieter; Schaps, Mary. Deformations of tilting modules. Perspectives in ring theory (Antwerp, 1987), 1–20, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 233, Kluwer Acad. Publ., Dordrecht, 1988.

[HPT16] Hassett B; Pirutka A; Tschinkel Y. Stable rationality of quadric surface bundles over surfaces. arXiv preprint arXiv:1603.09262.

[HKR62] Hochschild, G.; Kostant, Bertram; Rosenberg, Alex. Differential forms on regular affine algebras. Trans. Amer. Math. Soc. 102 1962 383–408.

[HT10] Huybrechts, Daniel; Thomas, Richard P. Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes. Math. Ann. 346 (2010), no. 3, 545–569.

[Ill] Illusie, Luc. Conditions de finitude relatives. Théorie des Intersections et Théorème de Riemann-Roch. Springer, Berlin, Heidelberg, 1971. 222–273.

[Ill05] Illusie, Luc. Grothendieck’s existence theorem in formal geometry. With a letter of Jean-Pierre Serre. Math. Surveys Monogr., 123, Fundamental algebraic geometry, 179–233, Amer. Math. Soc., Providence, RI, 2005.

[Jrg97] Jørgensen, Peter. Serre-duality for Tails(A). Proc. Amer. Math. Soc. 125 (1997), no. 3, 709–716.

[Kawa06] Kawamata, Yujiro. Derived categories of toric varieties. Michigan Math. J. 54 (2006), no. 3, 517–535.

[Kuz11] Kuznetsov, Alexander. Base change for semiorthogonal decompositions. Compos. Math. 147 (2011), no. 3, 852–876.

[Lieb06] Lieblich, Max. Moduli of complexes on a proper morphism. J. Algebraic Geom. 15 (2006), no. 1, 175–206.

[Loday] Loday, Jean-Louis. Cyclic homology. Appendix E by María O. Ronco. Second edition. Chapter 13 by the author in collaboration with Teimuraz Pirashvili. Grundlehren der Mathematischen Wissenschaften, 301. Springer-Verlag, Berlin, 1998.
[Lowen05] Lowen, Wendy. Obstruction theory for objects in abelian and derived categories. Comm. Algebra 33 (2005), no. 9, 3195–3223.

[Swan96] Swan, Richard G. Hochschild cohomology of quasiprojective schemes. J. Pure Appl. Algebra 110 (1996), no. 1, 57–80.

[Toda05] Toda, Yukinobu. Deformations and Fourier-Mukai transforms. J. Differential Geom. 81 (2009), no. 1, 197–224.

[Yeku02] Yekutieli, Amnon. The continuous Hochschild cochain complex of a scheme. Canad. J. Math. 54 (2002), no. 6, 1319–1337.

[YZ97] Yekutieli, Amnon; Zhang, James J. Serre duality for noncommutative projective schemes. Proc. Amer. Math. Soc. 125 (1997), no. 3, 697–707.

School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P.R. China

Email address: luxw06@gmail.com