BOSONIC FRADKIN-TSEYTLIN EQUATIONS UNFOLDED.

O.V. SHAYNKMAN

I.E. Tamm Theory Department, Lebedev Physics Institute, Leninski prospect 53, 119991, Moscow, Russia

Abstract. We test series of infinite-dimensional algebras as the candidates for higher spin extension of \( \mathfrak{su}(k, k) \). Adjoint and twisted-adjoint representations of \( \mathfrak{su}(k, k) \) on spaces of these algebras are carefully explored. For \( k = 2 \) corresponding unfolded systems are analyzed and they shown to encode Fradkin-Tseytlin equations for some set of integer spins. In each case spectrum of spins is found.

1. Introduction

In this paper we study unfolded formulation of Fradkin-Tseytlin equations \[ \Pi \partial_{\mu} \ldots \partial_{\mu} \phi_{\nu(s)} = C_{\nu(s), \mu(s)}, \] \[ \partial^{\mu} \ldots \partial^{\mu} C_{\nu(s), \mu(s)} = 0, \] which describe free conformal dynamics of spin \( s \) traceless field \( \phi_{\nu(s)} \) in 4-dimensional Minkowski space. Here \( C_{\nu(s), \mu(s)} \) is associated with traceless generalized Weyl tensor. \( \Pi \) is projector that carries out necessary symmetrizations and subtracts traces. Generalized Weyl tensor \( C_{\nu(s), \mu(s)} \) is obviously invariant with respect to gauge transformations
\[ \delta \phi_{\nu(s)} = \partial_{\nu} \epsilon_{\nu(s-1)} - \frac{s - 1}{2s} \eta_{\nu\nu} \partial^\mu \epsilon_{\mu\nu(s-2)} \] with traceless gauge parameter \( \epsilon_{\nu(s-1)} \) and Minkowski metric \( \eta_{\nu\nu} \). If full nonlinear conformal higher spin theory exists, these equations should correspond to its free level.

As \( AdS \) higher spin theory teaches us, the main ingredient needed to construct such kind of theories is higher spin algebra that describes gauge symmetries of the theory. In paper \[ 2 \] Fradkin and Linetsky proposed a number of candidates for the role of infinite-dimensional 4d conformal higher spin gauge symmetry algebra which extends ordinary 4d conformal algebra \( \mathfrak{so(4, 2)} \sim \mathfrak{su(2, 2)} \). Their construction is based on the oscillator realization of \( \mathfrak{su(2, 2)} \) \[ 3 \]. \[ 4 \]. Here we give a straightforward generalization of their results for the case of the algebra \( \mathfrak{su(k, k)} \) with \( k \geq 2 \) and briefly discuss the structure of infinite-dimensional algebras obtained.

Consider star product algebra generated by bosonic oscillators
\[ [b_\beta, a_\alpha]_* = \delta_\beta^\alpha, \quad [\bar{b}_\beta, \bar{a}_\alpha]_* = \delta_\beta^\alpha, \quad \alpha, \bar{\alpha} = 1, \ldots, k, \quad k \geq 2. \] Here \( * \) denotes the star product of Weyl ordered symbols of operators \( f(a, b, \bar{a}, \bar{b}) \) and \( g(a, b, \bar{a}, \bar{b}) \) given by formula \( f * g = f \exp(\Delta)g \), where
\[ \Delta = \frac{1}{2} \left( \frac{\partial}{\partial b} \cdot \frac{\partial}{\partial a} - \frac{\partial}{\partial \bar{a}} \cdot \frac{\partial}{\partial \bar{b}} + \frac{\partial}{\partial \bar{b}} \cdot \frac{\partial}{\partial \bar{a}} - \frac{\partial}{\partial a} \cdot \frac{\partial}{\partial b} \right), \] \( ^{0}\text{e-mail: shayn01pi.ru} \)\(^ {1}\text{Strictly speaking in [2] an extension of the superconformal algebra \( \mathfrak{su(2, 2|N)} \) was considered, but we do not treat super case in the present paper.} \)
\( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \) denote correspondingly left and right derivatives and \( \cdot \) denotes contractions
\[
(1.5) \quad a \cdot b = a^\alpha b_\alpha, \quad \bar{a} \cdot \bar{b} = \bar{a}^\alpha \bar{b}_\alpha, \quad \frac{\partial}{\partial a} \cdot \frac{\partial}{\partial b} = \frac{\partial}{\partial a^\alpha} \frac{\partial}{\partial b_\alpha}, \quad \frac{\partial}{\partial \bar{a}_\alpha} \cdot \frac{\partial}{\partial \bar{b}^\alpha}.
\]

Within this setup bilinears of oscillators (1.3) centralized by helicity operator
\[
(1.6) \quad Z = \frac{i}{2} (a \cdot b - \bar{a} \cdot \bar{b})
\]
furnish algebra \( \mathfrak{gl}(2k, \mathbb{C}) \) with respect to commutator
\[
(1.7) \quad [f, g]_* = f \ast g - g \ast f.
\]
The basis of \( \mathfrak{gl}(2k, \mathbb{C}) \) along with \( Z \) contains elements
\[
(1.8) \quad \mathcal{L}_\beta^\alpha = b_\alpha a^\beta - \frac{1}{k} \delta^\beta_\alpha a \cdot b, \quad \bar{\mathcal{L}}_\beta^\alpha = \bar{b}_\alpha \bar{a}^\beta - \frac{1}{k} \delta^\beta_\alpha \bar{a} \cdot \bar{b},
\]
which satisfy the following commutation relations
\[
(1.9) \quad [\mathcal{L}_\beta^\alpha, \mathcal{L}_\gamma^\delta]_* = \delta^\delta_\alpha \mathcal{L}_\gamma^\beta - \delta^\beta_\alpha \mathcal{L}_\gamma^\delta, \quad [\bar{\mathcal{L}}_\beta^\alpha, \bar{\mathcal{L}}_\gamma^\delta]_* = \delta^\delta_\alpha \bar{\mathcal{L}}_\gamma^\beta - \delta^\beta_\alpha \bar{\mathcal{L}}_\gamma^\delta,
\]
where \( \zeta \) denote complex conjugation (i.e. antilinear involutive map preserving commutators (1.3)) defined on oscillators (1.3) by formulae
\[
(1.10) \quad \zeta(a^\alpha) = \bar{a}^\alpha, \quad \zeta(b_\alpha) = \bar{b}_\alpha.
\]
Real form of \( \mathfrak{gl}(2k, \mathbb{C}) \) singled out by the requirement
\[
(1.11) \quad \zeta(f) = f
\]
is identified with algebra \( \mathfrak{u}(k, k) \). From (1.3) it follows that
\[
(1.12) \quad \zeta(\mathcal{L}_\beta^\alpha) = \bar{\mathcal{L}}_\beta^\alpha, \quad \zeta(\mathcal{P}_\beta^\alpha) = \mathcal{P}_\beta^\alpha, \quad \zeta(\mathcal{K}_\beta^\alpha) = \bar{\mathcal{K}}_\beta^\alpha, \quad \zeta(D) = D, \quad \zeta(Z) = Z
\]
and therefore general element of \( \mathfrak{u}(k, k) \) has a form
\[
(1.13) \quad \mathcal{X}_{\mathfrak{u}(k, k)} = X_\beta^\alpha \mathcal{L}_\beta^\alpha + \bar{X}_\beta^\alpha \bar{\mathcal{L}}_\beta^\alpha + X_\alpha^\beta \mathcal{P}_\beta^\alpha + X_\alpha^\beta \mathcal{K}_\beta^\alpha + XD + X' Z,
\]
where \( X_\alpha^\beta \) and \( \bar{X}_\alpha^\beta \) are mutually complex conjugate \( k \times k \)-matrices, \( X_\alpha^\beta \) and \( X_\alpha^\beta \) are Hermitian \( k \times k \)-matrices and \( X, X' \) are real numbers.

Finally note that algebra \( \mathfrak{u}(k, k) \) decomposes into direct sum
\[
(1.14) \quad \mathfrak{u}(k, k) = \mathfrak{su}(k, k) \oplus \mathfrak{u}(1),
\]
where \( \mathfrak{u}(1) \) is spanned by \( Z \).

To construct an infinite-dimensional extension of \( \mathfrak{su}(k, k) \) let us bring all polynomials (not only bilinear) of oscillators (1.3) into the play still requiring them to be centralized by \( Z \) and to satisfy
Picking out contractions $a \cdot b$ and $\bar{a} \cdot \bar{b}$ from $X_{\text{iu}(k,k)}$ and taking into account that $a \cdot b = D - iZ$, $\bar{a} \cdot \bar{b} = D + iZ$ one gets

\begin{equation}
X_{\text{iu}(k,k)} = \sum_{u,v=0}^{\infty} Z^u \bar{D}^v f_{u,v}(a, b, \bar{a}, \bar{b}),
\end{equation}

where $f_{u,v}$ is centralized by $Z$ \eqref{eq:centralizer}, satisfies reality condition \eqref{eq:reality_condition} and is traceless with respect to $a, b$ and $\bar{a}, \bar{b}$, i.e.

\begin{equation}
\frac{\partial^2}{\partial a \cdot \partial b} f_{u,v} = 0, \quad \frac{\partial^2}{\partial \bar{a} \cdot \partial \bar{b}} f_{u,v} = 0.
\end{equation}

Also note that due to \eqref{eq:centralizer} $f_{u,v}$ is even function $f_{u,v}(a, b, \bar{a}, \bar{b}) = f_{u,v}(-a, -b, -\bar{a}, -\bar{b})$.

Algebra $\text{iu}(k, k)$ admits decomposition analogous to \eqref{eq:decomposition} of $\mathfrak{u}(k, k)$

\begin{equation}
\text{iu}(k, k) = \mathfrak{isu}^\infty(k, k) \oplus \sum_{n=0}^{\infty} \mathfrak{u}(1).
\end{equation}

Here the infinite sum of $\mathfrak{u}(1)$ is spanned by the star product powers of $Z$

\begin{equation}
Z \ast Z \ast \cdots \ast Z = \left(Z - \frac{i}{4} \left(\frac{\partial^2}{\partial a \cdot \partial b} - \frac{\partial^2}{\partial \bar{a} \cdot \partial \bar{b}}\right)\right)^m = Z^m + \cdots, \quad m \geq 0,
\end{equation}

where dots on the right-hand side denote the terms with the lower powers of $Z$. Therefore general element of subalgebra $\mathfrak{isu}^\infty(k, k)$ is given by \eqref{eq:decomposition} with nonconstant $f_{u,0}$.

Unlike the finite-dimensional case algebra $\mathfrak{isu}^\infty(k, k)$ is not semi-simple and contains an infinite chain of ideals $\mathcal{J}^m$

\begin{equation}
\mathfrak{isu}^\infty(k, k) \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \cdots \supset \mathcal{J}^m \supset \cdots.
\end{equation}

Here ideal $\mathcal{J}^m$ is spanned by the elements of the form

\begin{equation}
\mathcal{J}^m : \quad Z \ast \cdots \ast Z \ast \mathcal{D} \ast \cdots \ast \mathcal{D} \ast f_{u,v} = Z^u \mathcal{D}^v f_{u,v} + \cdots, \quad u \geq m,
\end{equation}

where dots on the right-hand side denote the terms of the lower powers of $Z$ and/or $\mathcal{D}$.

Let quotient algebras $\mathfrak{isu}^\infty(k, k)/\mathcal{J}^m$ be denoted as $\mathfrak{isu}^{m-1}(k, k)$

\begin{equation}
\mathfrak{isu}^0(k, k) \subset \mathfrak{isu}^1(k, k) \subset \cdots \subset \mathfrak{isu}^m(k, k) \subset \cdots \subset \mathfrak{isu}^\infty.
\end{equation}

The representatives of the elements of quotient algebra $\mathfrak{isu}^m(k, k)$ can be chosen to have form \eqref{eq:decomposition} with $f_{u,v} \equiv 0$ for $u > m$. Algebra $\mathfrak{isu}^0(k, k)$, spanned by elements independent of $Z$,

\begin{equation}
X_{\mathfrak{isu}(k,k)} = \sum_{v=0}^{\infty} \mathcal{D}^v f_v(a, b, \bar{a}, \bar{b}),
\end{equation}

is semi-simple. In what follows we omit index 0 and denote it $\mathfrak{isu}(k, k)$.

Let us note that in paper \cite{2} algebras $\mathfrak{isu}^\infty(2,2)$ and $\mathfrak{isu}(2,2)$ were denoted as $\mathfrak{hs}(4)$ and $\mathfrak{hs}(4)$ correspondingly, where hsc means higher spin conformal and 4 indicates that they extend 4-dimensional conformal algebra. In the later paper \cite{3} these algebras were denoted as $\mathfrak{c}(1,0|8)$ and
\( \mathfrak{su}_q(1,0|8) \), where 8 indicates the number of oscillators used and pair 1,0 points out that the above algebras have trivial structure in spin 1 Yang-Mills sector.

The rest of the paper is organized as follows. In section 2 we recall some relevant facts about unfolded formulation. The structure of adjoint representations of algebra \( \mathfrak{su}(k,k) \) on the vector spaces of \( \mathfrak{is}u^m(k,k) \) \( m = 0,1,\ldots,\infty \) is discussed in section 3. In section 4 we study twisted-adjoint representation of \( \mathfrak{su}(k,k) \). In section 5 unfolded formulation of higher spin bosonic equations is analyzed for \( k = 2 \). Section 6 contains conclusions. In Appendix A we recall relevant facts concerning finite-dimensional \( \mathfrak{sl}(k) \oplus \mathfrak{sl}(k) \) irreps. In Appendix B we find basis where adjoint and twisted-adjoint modules from sections 3 and 4 are decomposed into submodules. In appendix C \( \sigma_\pm \) and \( \tilde{\sigma}_\pm \)-cohomology corresponding to the gauge sector and Weyl sector of unfolded systems under consideration are found.

2. Unfolded formulation: preliminary remarks

Let \( \mathcal{M}^d \) be some \( d \)-dimensional manifold with coordinates \( x^1,\ldots,x^d \). Any dynamical system on \( \mathcal{M}^d \) can be reformulated in unfolded form of first order differential equations [7] (see [8] for a review)

\[
\text{d}W^\Omega = F^\Omega(W) .
\]

Here \( W^\Omega(x) \) is a collection of differential forms (numered by multiindex \( \Omega \)) of ranks \( \text{deg}(W^\Omega) = p^\Omega \), \( d \) is exterior differential and

\[
F^\Omega(W) = \sum_{n=0}^\infty f^\Omega_{\phi_1\ldots\phi_n} W^{\phi_1} \cdots W^{\phi_n} .
\]

is a form of rank \( p^\Omega + 1 \). It is composed by virtue of exterior product\(^2\) of the elements of \( W^\Omega(x) \), which are contracted with constant functions

\[
f^\Omega_{\ldots\phi\ldots \phi} = (-1)^{p^\phi} f^\Omega_{\ldots\phi\ldots \phi} .
\]

Compatibility conditions of (2.1) require \( F^\Omega(W) \) to satisfy identities

\[
F^\Phi \frac{\partial}{\partial W^\Phi} F^\Omega \equiv 0 ,
\]

where \( \frac{\partial}{\partial W^\Phi} \) is the left derivative. In terms of constants \( f^\Omega_{\phi_1\ldots\phi_n} \), conditions (2.3) have a form of generalized Jacobi identities

\[
\sum_{m=0}^\infty \sum_{n=1}^\infty n f^\Omega_{\psi_1\ldots\psi_m} f^\Omega_{\phi_1\ldots\phi_{n-1}} \equiv 0 ,
\]

where left-hand side of (2.5) is (anti)symmetrised according to (2.3). Any solution of (2.5) defines a free differential algebra (FDA) [9]. In what follows we assume that (2.5) holds independently of the value of space-time dimension \( d \). In this case FDA defined by (2.5) is called universal.

Unfolded system (2.1) corresponding to universal FDA is invariant with respect to gauge transformations

\[
\delta W^\Omega = d\epsilon^\Omega + \epsilon^\Phi \frac{\partial}{\partial W^\Phi} F^\Omega(W) ,
\]

where \( \epsilon^\Omega(x) \) are \( p^\Omega - 1 \)-form gauge parameters.

\(^2\)In this paper all products of differential forms are supposed to be exterior and we omit the designation of exterior product in formulae.
Let us analyze system (2.1) perturbatively assuming that fields of zeroth order form a subclass of 1-forms $W^A(x) \subseteq W^\Omega(x)$. The most general form of $F^A(W)$ in the sector of zero order fields is
\begin{equation}
F^A(W) = -\frac{1}{2} f^A_{BC} W^B W^C,
\end{equation}
where constants $f^A_{BC} = -f^A_{CB}$ due to (2.5) are required to satisfy ordinary Jacobi identities. Therefore $W^A(x)$ can be identified with connection 1-form taking values in some Lie algebra $\mathfrak{g}$ with structure constants $f^A_{BC}$ and system (2.1) reduces to the zero curvature condition
\begin{equation}
dW^A + \frac{1}{2} f^A_{BC} W^B W^C = 0.
\end{equation}
Gauge transformations (2.6) become usual gauge transformation of a connection 1-form in this case.

As one can readily see compatibility conditions (2.5) require matrices $(T_A)^a\_b$ and $(\tilde{T}_A)^i\_j$ to form some representations of Lie algebra $\mathfrak{g}$. Let corresponding modules be denoted as $\mathcal{M}$ and $\tilde{\mathcal{M}}$. Then $D$ and $\tilde{D}$ from (2.10) and (2.11) define $\mathcal{M}$ and $\tilde{\mathcal{M}}$-covariant derivatives respectively. Both derivatives are nilpotent
\begin{equation}
D^2 = 0, \quad \tilde{D}^2 = 0
\end{equation}
as a consequence of the zero curvature condition (2.8).

As it was argued in [10] the term
\begin{equation}
\frac{1}{2} W^A W^B H_{AB}^a\_j
\end{equation}
on the right-hand side of (2.10) should belong to nontrivial class of 2-nd Chevalley-Eilenberg cohomology taking values in $\mathfrak{g}$-module $\mathcal{M} \otimes \tilde{\mathcal{M}}^*$. Indeed, compatibility of (2.10) in the sector of $C^i(x)$ is equivalent to the closedness of (2.13)
\begin{equation}
(\delta_{\text{Che}})^a\_b\_j (W^A W^B H_{AB}^b\_i) = 0
\end{equation}
with respect to Chevalley-Eilenberg differential (see [11])
\begin{equation}
(\delta_{\text{Che}})^a\_b\_j = \frac{1}{2} \delta_b^a \delta_j^i f^A_{BC} W^B W^C \frac{\delta}{\delta W^A} - W^A (T \otimes \tilde{T}^*)^a\_b\_j.
\end{equation}
Here
\begin{equation}
(T \otimes \tilde{T}^*)^a\_b\_j = (T_C)^a\_b\_j - \delta^a_b (\tilde{T}_C)^i\_j
\end{equation}
are matrices acting on module $\mathcal{M} \otimes \tilde{\mathcal{M}}^*$. If (2.13) is $\delta_{\text{Che}}$-exact
\begin{equation}
W^A W^B H_{AB}^a\_i = (\delta_{\text{Che}})^a\_b\_j (W^A \theta^b\_A\_j),
\end{equation}
right-hand side of (2.10) can be removed by field redefinition
\begin{equation}
\omega^a = \omega^a + \frac{1}{2} W^A \theta^a\_A\_i C^i.
\end{equation}
And conversely if some field redefinition removing right-hand side of (2.10) exists, it should necessarily have form (2.13) with $W^A\theta_A^a$, satisfying (2.17).

System (2.8), (2.10) and (2.11) is locally invariant with respect to gauge transformation (2.9) of connection 1-form $W^A(x)$ and the following gauge transformations of fields $\omega^a(x)$ and $C^a(x)$

\begin{align}
\delta \omega^a &= -\epsilon^A(T_A)^a_b \omega^b + \epsilon^A W^B H_{AB}^a \xi C^i, \\
\delta C^a &= -\epsilon^A (\tilde{T}_A)^i_j C^j.
\end{align}

If some solution

\begin{equation}
W^A = W^A_0
\end{equation}

of zero curvature condition (2.8) is fixed, the above gauge symmetry breaks down to the global symmetry that keeps $W^A_0$ stable. Parameter of this symmetry $\epsilon^A_0(x)$ should obviously satisfy equation

\begin{equation}
\delta W^A_0 = d\epsilon^A_0 + f^A_{BC} W^B_0 \epsilon^C_0 = 0,
\end{equation}

which is consistent due to zero curvature condition (2.8). Equation (2.22) reconstructs $\epsilon^A_0(x)$ in terms of its value $\epsilon^A_0(x_0)$ at any given point $x_0$. So $\epsilon^A_0(x_0)$ plays a role of the moduli space of $W^A_0$ global symmetry algebra, which therefore can be identified with $g$. When substituted to (2.10), (2.11), $W^A$ plays a role of vacuum connection describing $g$-invariant background geometry. The only thing we require is that component of $W^A_0$ corresponding to generator of generalized translation (i.e. generalized coframe) is of maximal possible rank.

Let us consider system (2.10), (2.11) with $W^A = W^A_0$.

\begin{align}
(\tilde{D}_0 \omega)^a &= \frac{1}{2} W^A_0 W^B_0 H_{AB}^a_j C^j, & \text{where } (D_0 \omega)^a &= d\omega^a + W^A_0 (T_A)^a_b \omega^b, \\
(\tilde{D}_0 C)^i &= 0, & \text{where } (\tilde{D}_0 C)^i &= dC^i + W^A_0 (\tilde{T}_A)^i_j C^j.
\end{align}

As follows from the above consideration it is globally $g$-invariant with respect to transformations (2.19), (2.20) with $W^A = W^A_0$ substituted. This system is also gauge invariant with respect to gauge transformations

\begin{equation}
\delta \omega^a = D_0 e^a,
\end{equation}

where $e^a(x)$ is 0-form gauge parameter associated with field $\omega^a(x)$.

To analyze dynamical content of system (2.23), (2.24) let us first consider the case when right-hand side of (2.23) equals zero. In this case equations (2.23), (2.24) are independent and both have a form of covariant constancy conditions. Suppose that modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are graded and this grading is bounded from below. Decompose covariant derivatives (2.23), (2.24) into the summands with definite gradings. We assume that each covariant derivative contains a single operator of negative grading (the case when there are several operators with negative grading was considered in [12])

\begin{equation}
D_0 = D_0 + \sigma_+ + \sum_{\eta} \sigma^\eta_+, \quad \tilde{D}_0 = \tilde{D}_0 + \tilde{\sigma}_- + \sum_{\theta} \tilde{\sigma}^\theta_+.
\end{equation}

Here $D_0$, $\tilde{D}_0$ denote operators of zero grading which include exterior differential, $\sigma_+^\eta$, $\tilde{\sigma}_+^\theta$ denote purely algebraic operators of various positive gradings and $\sigma_-$, $\tilde{\sigma}_-\sigma$ are purely algebraic operators of negative grading. Operators $\sigma_-$ and $\tilde{\sigma}_-$ are nilpotent due to the nilpotency of covariant derivatives (2.12).

Let subspace of $\mathcal{M}$ with fixed grading $n$ be called $n$-th level of $\mathcal{M}$. Analyzing equation (2.23) and its gauge symmetries (2.25) level by level starting from the lowest grading one can see [13] that those fields which are not $\sigma_-$ closed (they are called auxiliary fields) expressed by (2.23) as derivatives of lower level fields, where space-time indices of derivatives are converted into algebraic indices by

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3There are also gauge transformations with the parameter associated with $\omega^a$, which are discussed later (see (2.25)).
virtue of coframe. $\sigma_-$-exact fields can be gauged to zero with the use of Stueckelberg part of gauge symmetry transformations (2.25). Leftover fields (that are called dynamical fields) belong to $H^1_{\sigma_-}$ the 1-st cohomology of $\sigma_-$. We also get that differential gauge parameters (i.e. those that does not correspond to Stueckelberg gauge symmetry) belong to $H^0_{\sigma_-}$.

Let $E_n$ denote the left-hand side of (2.23) on the $n$-th level. Suppose equation $E_m = 0$ is solved up to the $n-1$-st level inclusive, which means that all auxiliary fields up to the $n$-th level properly expressed in terms of derivatives of dynamical fields. Bianchi identities (2.27) $D_0 D_0 \omega \equiv 0$

considered at the $n-1$-st level require $E_n$ to be $\sigma_-$-closed. If $H^2_{\sigma_-}$ the 2-nd cohomology of $\sigma_-$ is trivial on the $n$-th level, equation $E_n = 0$ can be satisfied by appropriate choice of auxiliary field on the $n+1$-st level. In other case $E_n = 0$ also imposes some differential restriction on dynamical fields requiring that $E_n$ belongs to trivial cohomology class. Therefore nontrivial differential equations on dynamical fields are in one-to-one correspondence with $H^2_{\sigma_-}$. Moreover, if $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ that acts horizontally (i.e. keeps levels invariant), differential equations imposed by (2.23) and $H^2_{\sigma_-}$ are isomorphic as $\mathfrak{h}$-modules.

Summarizing, the dynamical content of equation (2.23) with zero right-hand side is described by $H^0_{\sigma_-}, H^1_{\sigma_-}, H^2_{\sigma_-}$ which correspond to differential gauge parameters, dynamical fields and differential equations on dynamical fields respectively. Analogously for equation (2.24) the dynamical fields and differential equations correspond to $\tilde{H}^0_{\sigma_-}$ and $\tilde{H}^1_{\sigma_-}$.

To analyze system (2.23), (2.24) with nonzero right-hand side let us consider operator (2.28) $\hat{D}_0 = \hat{D}_0 + \hat{\sigma}_- + \hat{\sigma}_+$, where

$$\hat{D}_0 = D_0 + \bar{D}_0,$$

$$\hat{\sigma}_- = \sigma_- + \bar{\sigma}_- + \sigma,$$

$$\hat{\sigma}_+ = \sum \sigma^\eta_+ + \sum \sigma^\theta_+.$$ 

System (2.23), (2.24) can be rewritten in the following form (2.30) $\hat{D} \hat{\Psi} = 0,$

where new field $\hat{\Psi}$ is a pair $\hat{\Psi} = (\omega, C)$ incorporating 1-forms $\omega$ and 0-forms $C$ and operators (2.29) are extended by zero on the space where they undefined. Operator $\hat{D}_0$ is nilpotent due to compatibility conditions of system (2.23), (2.24). Gauge transformations (2.25) takes a form (2.31) $\delta \hat{\Psi} = \hat{D}_0 \hat{\Upsilon}$, where $\hat{\Upsilon} = (\epsilon, 0)$.

Let us consider $\hat{\sigma}_-$-cocomplex $\hat{\mathcal{C}} = (\hat{S}, \hat{\sigma}_-)$ with $p$-form element $\hat{\Psi}^p \in \hat{\mathcal{S}}$ defined as a pair $\hat{\Psi}^p = (\omega^p, C^{p-1})$, where $\omega^p$ and $C^{p-1}$ are correspondingly $p$-form taking values in the module $\mathcal{M}$ and $p-1$-form taking values in the module $\tilde{\mathcal{M}}$ ($C^{-1} \equiv 0$). Standard definition of $\hat{\sigma}_-$-closed $p$-forms subspace $\hat{\mathcal{C}}^p = (\omega^p_C, C^{p-1}_C)$ gives in components the following relations

$$\sigma_- \omega^p_C + \sigma C^{p-1}_C = 0,$$

$$\hat{\sigma}_- C^{p-1}_C = 0.$$
Subspace of $\hat{\sigma}_-$-exact $p$-forms $\hat{\mathcal{E}}^p = (\omega^p_{\mathcal{E}}, C^{p-1}_{\mathcal{E}})$ is defined in components by
\[\begin{align*}
\omega^p_{\mathcal{E}} = \sigma_- \omega^{p-1} + \sigma C^{p-2}, \\
C^{p-1}_{\mathcal{E}} = \hat{\sigma}_- C^{p-2}
\end{align*}\]
for some elements $\omega^{p-1}, C^{p-2}$. Let $p$-th $\hat{\sigma}_-$-cohomology be defined as quotient
\[\hat{H}^p_{\hat{\sigma}_-} = \hat{\mathcal{E}}^p / \hat{\mathcal{E}}^{p-1}.
\]
Since the above analysis of equation (2.23) with zero right-hand side is based on Bianchi identities\(^4\) only it is applicable to equation (2.30). We therefore obtain that dynamical content of system (2.33) for some elements $\hat{\omega}$
\[\begin{align*}
(2.33) \quad \hat{\omega} = \hat{\mathcal{E}}^{p-1}
\end{align*}\]
for (2.30).

In section 5 we use the above technic to analyze dynamical content of $\mathfrak{su}(2, 2)$-invariant unfolded system that was originally introduced in [6]. Before it we explore structure of underlying $\mathfrak{su}(k, k)$-modules.

### 3. Structure of Adjoint Module

Consider adjoint action of algebra $\mathfrak{su}(k, k)$ on the vector space of algebra $\mathfrak{isu}^\infty(k, k)$, which is given by brackets $(\text{ad}_{\mathfrak{X}(k, k)}) = [\mathfrak{X}(k, k), \cdot] = 2\mathfrak{X}(k, k) \Delta$. We have
\[\begin{align*}
(\text{ad}^\infty_L)_{\alpha}^{\beta} &= a^\beta \frac{\partial}{\partial a^\alpha} - b^\alpha \frac{\partial}{\partial b^\beta} - \frac{1}{k} b^\alpha (n_a - n_b), \\
(\text{ad}^\infty_L)_{\alpha}^{\beta} &= \bar{a}^\beta \frac{\partial}{\partial \bar{a}^\alpha} - \bar{b}^\alpha \frac{\partial}{\partial \bar{b}^\beta} - \frac{1}{k} \bar{b}^\alpha (n_{\bar{a}} - n_{\bar{b}}), \\
(\text{ad}^\infty_D)_{\alpha}^{\beta} &= \bar{b}^\alpha \frac{\partial}{\partial \bar{b}^\alpha} + \bar{b}^\alpha \frac{\partial}{\partial \bar{a}^\alpha}, \\
(\text{ad}^\infty_Z)_{\alpha}^{\beta} &= -a^\alpha \frac{\partial}{\partial b^\beta} - \bar{a}^\alpha \frac{\partial}{\partial \bar{b}^\beta}, \\
\text{ad}^\infty_D &= \frac{1}{2} (n_a + n_b - n_{\bar{a}} - n_{\bar{b}}), \\
\text{ad}^\infty_Z &= \frac{i}{2} (n_a - n_b + n_{\bar{a}} + n_{\bar{b}}),
\end{align*}\]
where $n_a, n_b, n_{\bar{a}}, n_{\bar{b}}$ denote Euler operators counting the number of corresponding variables. Let $\mathcal{M}^\infty$ denote this $\mathfrak{su}(k, k)$-module.

Obviously, operators
\[\begin{align*}
(3.2) \quad s_1 = n_a + n_b + 1, \quad s_2 = n_b + n_{\bar{a}} + 1
\end{align*}\]
commute with adjoint action of $\mathfrak{su}(k, k)$ (3.1). Moreover due to centralization requirement (1.15)
\[\begin{align*}
(3.3) \quad s_1 f = s_2 f
\end{align*}\]
for any $f \in \mathcal{M}^\infty$. Therefore module $\mathcal{M}^\infty$ decomposes into finite-dimensional submodules $\mathcal{M}^\infty_s$
\[\begin{align*}
(3.4) \quad s_1 \mathcal{M}^\infty_s = s_2 \mathcal{M}^\infty_s = s \mathcal{M}^\infty_s.
\end{align*}\]

Modules $\mathcal{M}^\infty$ and $\mathcal{M}^\infty_s$ are reducible with submodules
\[\begin{align*}
(3.5) \quad \mathcal{M}^\infty \supset \mathcal{J}^1 \supset \cdots \supset \mathcal{J}^m \supset \cdots, \\
\mathcal{M}^\infty_s \supset \mathcal{J}^1_s \supset \cdots \supset \mathcal{J}^{s-1}.
\end{align*}\]
\(^4\)In fact we also required $D$ to have unique $\sigma_-$ and grading to be bounded from below which is obviously also true for (2.30).
where \( \mathfrak{J}^m \) is \( \mathfrak{su}^\infty(k,k) \)-ideal \( (1.22) \) and
\[
(3.6) \quad \mathfrak{J}_s^m = \mathfrak{J}^m \cap \mathcal{M}_s^\infty.
\]
Note that \( \mathfrak{J}_s^m \equiv 0 \) for \( m \geq s \).

Consider quotient modules
\[
(3.7) \quad \mathcal{M}^0 \subset \mathcal{M}^1 \subset \cdots \subset \mathcal{M}^\infty,
\]
\[
\mathcal{M}_s^0 \subset \mathcal{M}_s^1 \subset \cdots \subset \mathcal{M}_s^{s-2} \subset \mathcal{M}_s^\infty,
\]
where
\[
(3.8) \quad \mathcal{M}_s^m = \mathcal{M}_s^\infty / \mathfrak{J}_s^{m+1}, \quad \mathcal{M}_s^m = \mathcal{M}_s^\infty / \mathfrak{J}_s^{m+1}, \quad m = 0, 1, \ldots .
\]
Note that \( \mathcal{M}_s^m \equiv \mathcal{M}_s^\infty \) for \( m \geq s - 1 \). In what follows we omit index 0 and denote \( \mathcal{M}^0, \mathcal{M}_s^0 \) as \( \mathcal{M}, \mathcal{M}_s \), respectively.

As shown in Appendix \[ \ref{app:B} \] module \( \mathcal{M}_s^\infty \) admits the following decomposition
\[
(3.9) \quad \mathcal{M}_s^\infty = \bigoplus_{s' = 1}^s \mathcal{M}_{s'}.
\]
And therefore
\[
(3.10) \quad \mathcal{M}_s^m = \bigoplus_{s' = s-m}^m \mathcal{M}_{s'}, \quad m = 0, 1, \ldots , s - 2
\]
and
\[
(3.11) \quad \mathcal{M}^m = \bigoplus_{s' = 1}^\infty \mathcal{M}_{s'}^m = (m + 1) \bigoplus_{s' = 1}^\infty \mathcal{M}_{s'}, \quad m = 0, 1, \ldots , \infty ,
\]
where the number \( m + 1 \) on the right-hand side of \( (3.11) \) indicates multiplicity of modules \( \mathcal{M}_{s'} \).

The basis where decomposition \( (3.9) \) becomes straightforward has the form
\[
(3.12) \quad Z^{s-s'} g_v^s(Z, \mathcal{D}) f_{s'-v}, \quad s' = 1, \ldots , s, \quad v = 0, \ldots , s' - 1,
\]
where subset with the fixed value of \( s' \) corresponds to the basis of submodule \( \mathcal{M}_{s'} \subset \mathcal{M}_s^\infty \). Here \( g_v^s(Z, \mathcal{D}) \) is homogeneous polynomial of degree \( v \) in two variables \( Z \) and \( \mathcal{D} \), which particular form is found in Appendix \[ \ref{app:B} \], while \( f_{s'-v}(a, b, \bar{a}, \bar{b}) \) is traceless
\[
(3.13) \quad \frac{\partial^2}{\partial a \cdot \partial b} f_{s'-v} = \frac{\partial^2}{\partial \bar{a} \cdot \partial \bar{b}} f_{s'-v} = 0
\]
and forms eigenvector corresponding to eigenvalue \( s' - v \) with respect to operators \( s_1 \) and \( s_2 \) \( (3.2) \).
\[
(3.14) \quad s_1 f_{s'-v} = s_2 f_{s'-v} = (s' - v) f_{s'-v}.
\]
In other words \( f_{s'-v} \) is a sum of monomials of the form
\[
(3.15) \quad m_{s'-v}(n_a, n_b, n_{\bar{a}}, n_{\bar{b}}) = x_{\beta(n_k)}^{\alpha(n_a); \beta(n_k)} a_{\alpha(n_a)}^{\alpha(n_a)} \bar{a}_{\beta(n_k)}^{\beta(n_k)} \bar{b}_{\beta(n_k)}^{\beta(n_k)}.
\]
Here \( x_{\alpha(n_a); \beta(n_k)}^{\alpha(n_a)} \) is a traceless complex tensor symmetric separately with respect to each group of indices \( \alpha(n_a), \bar{a}(n_{\bar{a}}), \beta(n_k), \bar{b}(n_{\bar{b}}) \), where number in parentheses indicates the number of indices in the group, and \( a^{\alpha(n_a)} = a^{\alpha_1} \cdots a^{\alpha_{n_a}} \) denotes \( n_a \)-th power of oscillator \( a \) and analogous notation for oscillators \( \bar{a}, \bar{b} \).

Certainly values of \( n_a, n_{\bar{a}}, n_b, n_{\bar{b}} \) in \( (3.15) \) should be coordinated with \( s' \) and \( v \) through formula \( (3.14) \).

Let \( g_v^s(Z, \mathcal{D}) m_{s'-v}(n_a, n_b, n_{\bar{a}}, n_{\bar{b}}) \) be denoted as \( \mathcal{B}_v^{s'}(n_a, n_b, n_{\bar{a}}, n_{\bar{b}}) \). Due to the above arguments \( \mathcal{B}_v^{s'} \) forms, with respect to generators \( (\text{ad}^\infty_{\mathcal{L}})_\alpha^{\beta}, (\text{ad}^\infty_{\mathcal{L}})_\alpha^{\bar{\beta}}, \) irreducible \( \mathfrak{sl}(k) \oplus \mathfrak{sl}(k) \)-module corresponding...
to the Young tableau

\[
\begin{array}{c|c|c}
\text{undotted} & \text{dotted} \\
\hline
\text{upper} & n_a & n_\bar{a} \\
\text{lower} & n_b & n_{\bar{b}} \\
\end{array}
\]

(see Appendix [A] for more details).

In what follows we study the structure of module \( \mathcal{M}_{s'} \) and in particular show that it is irreducible. Elements \( L^\alpha_{\gamma}, \bar{L}^\alpha_{\bar{\gamma}}, D, Z \) of \( \mathfrak{su}(k,k) \) commute with \( Z^{s'-s} g_{s'}^v \) and therefore are represented in \( \mathcal{M}_{s'} \) by the same operators as in module \( \mathcal{M}_\infty \). As shown in Appendix [B.1] elements \( \mathcal{P}_{\alpha\bar{\beta}} \) and \( \mathcal{K}_{\alpha\bar{\beta}} \) are represented up to an overall factor \( Z^{s'-s} \) by the following operators

\[
(\text{ad}_P)_{\alpha\bar{\beta}} \cdot g^v_{s'} m_{s'-v} = \Pi^\perp \left[ v g^v_{s'} b_\alpha \bar{b}_\beta + g^v_{s'} \left( v \varphi(n) + 1 \right) b_\beta \frac{\partial}{\partial a_\alpha} + g^v_{s'} \left( v \varphi(n) + 1 \right) b_\alpha \frac{\partial}{\partial \bar{a}_\beta} \right] m_{s'-v},
\]

\[
(\text{ad}_K)_{\alpha\bar{\beta}} \cdot g^v_{s'} m_{s'-v} = -\Pi^\perp \left[ v g^v_{s'} a^\alpha \bar{a}^\beta + g^v_{s'} \left( v \varphi(n) + 1 \right) \bar{a}^\beta \frac{\partial}{\partial a_\alpha} + g^v_{s'} \left( v \varphi(n) + 1 \right) a^\alpha \frac{\partial}{\partial \bar{a}_\beta} \right] m_{s'-v},
\]

where \( n = n_a + n_b, \bar{n} = n_{\bar{a}} + n_{\bar{b}}, \varphi(n) = 1/(n + k) \) and \( \Pi^\perp = (\Pi^\perp)^2 \) is the projector to the traceless component (3.13)

\[
\Pi^\perp a^\alpha = a^\alpha - \varphi(n - 2) a \cdot \bar{b} \frac{\partial}{\partial b_\alpha}, \quad \Pi^\perp b_\alpha = b_\alpha - \varphi(n - 2) a \cdot \bar{b} \frac{\partial}{\partial a^\alpha},
\]

\[
\Pi^\perp \bar{a}^\bar{\beta} = \bar{a}^\bar{\beta} - \varphi(n - 2) \bar{a} \cdot \bar{b} \frac{\partial}{\partial \bar{b}_\beta}, \quad \Pi^\perp \bar{b}_\beta = \bar{b}_\beta - \varphi(n - 2) \bar{a} \cdot \bar{b} \frac{\partial}{\partial \bar{a}^\bar{\beta}},
\]

\[
\Pi^\perp a^\alpha \bar{a}^\bar{\beta} = \Pi^\perp a^\alpha \Pi^\perp \bar{a}^\bar{\beta}, \quad \Pi^\perp b_\alpha \bar{b}_\beta = \Pi^\perp b_\alpha \Pi^\perp \bar{b}_\beta.
\]

Every element \( \mathcal{B}^v_{s'} \) has a definite conformal weight

\[
\text{ad}_D \mathcal{B}^v_{s'} = \frac{1}{2} (n_a + n_{\bar{a}} - n_b - n_{\bar{b}}) \mathcal{B}^v_{s'} = \Delta \mathcal{B}^v_{s'},
\]

which due to (3.14) ranges for fixed value of \( v = 0, 1, \ldots, s' - 1 \) from

\[
\Delta_{\text{min}}(v) = -s' + v + 1 \text{ for } \Delta_{\text{min}}(v) = Z^{s'-s} g^v_{s'} x^{\alpha(s'-v-1)} \bar{x}^{\beta(s'-v-1)} b_\beta \bar{b}_\beta \bar{b}_\beta \bar{b}_\beta
\]

to

\[
\Delta_{\text{max}}(v) = s' - v - 1 \text{ for } \Delta_{\text{max}}(v) = Z^{s'-s} g^v_{s'} x^{\alpha(s'-v-1)} \bar{x}^{\alpha(s'-v-1)} a^\alpha \bar{a}^\alpha \bar{a}^\alpha \bar{a}^\alpha.
\]

Elements of the form (3.20) and (3.21) with \( v = 0 \) have the lowest conformal weight \(-s' + 1\) and the highest conformal weight \( s' - 1 \) correspondingly.
All elements $B_{s'}^v$ can be arranged on the following diagram

Here every dot (●) indicates some $B_{s'}^v$. All dots in the same row correspond to $B_{s'}^v$-s with the same conformal weight indicated on the left axis. Dots compose the collection of rhombuses, which are distinguished by the value of $v = 0, 1, \ldots, s' - 1$ indicated on the bottom of diagram. The lowest (highest) dot in each rhombus corresponds to the $B_{s'_{\min}}^v$ ($B_{s'_{\max}}^v$) (see (3.20), (3.21)) of the lowest (highest) conformal weight for given $v$. Arrows indicate transformations which change orders of $B_{s'}^v$-s with respect to oscillators $a, b, \bar{a}, \bar{b}$ in such a way that $s'$ is kept constant and $\Delta$ increases by 1. Namely

\[
\begin{align*}
\uparrow: \quad & \{ n_a \to n_a + 1, \quad n_b \to n_b - 1 \}, \\
\downarrow: \quad & \{ n_b \to n_b + 1, \quad n_{\bar{a}} \to n_{\bar{a}} + 1 \}.
\end{align*}
\]

It is convenient to introduce independent "coordinates" on diagram (3.22)

\[
(v, q, t), \quad v = 0, \ldots, s' - 1, \quad q, t = 0, \ldots, s' - v - 1.
\]

Here $v$ numerates the rhombus in (3.22) and $q$ ($t$) indicates the number of upper-right (upper-left) arrows one should pass from the very bottom dot to get to the indicated dot. For instance coordinates $(v, 0, 0), (v, s' - v - 1, 0), (v, 0, s' - v - 1)$ and $(v, s' - v - 1, s' - v - 1)$ indicate on the bottom, right, left, and upper corners of rhombus $v$ correspondingly. In these terms all other variables are expressed as

\[
\begin{align*}
& n_a = q, \quad n_b = s' - v - t - 1, \\
& n_{\bar{a}} = t, \quad n_{\bar{b}} = s' - v - q - 1, \\
& \Delta = -s' + v + q + t - 1.
\end{align*}
\]

Let us note that complex conjugation (1.10) transforms $B_{s'}^v(v, q, t)$ corresponding to the dot $(v, q, t)$ to the $B_{s'}^v(v, t, q)$ corresponding to the dot $(v, t, q)$ symmetric with respect to reflection in a line connecting the top and the bottom of rhombus $v$. Therefore due to reality condition (1.16) coordinate-tensors (i.e. tensors like $x$ in (3.15)) of $B_{s'}^v(v, q, t)$ and $B_{s'}^v(v, t, q)$ are mutually complex conjugate if $q \neq t$ and coordinate-tensor of $B_{s'}^v(v, q, t)$ is self Hermitian conjugate.

Finally, let us show that $\mathcal{M}_{s'}$ is irreducible. Suppose $\mathcal{M}_{s'}$ is reducible then it decomposes into direct sum

\[
\mathcal{M}_{s'} = \mathcal{M}_{s'}' \oplus \mathcal{M}_{s'}''.
\]

Each module in (3.26) has a lowest conformal weight subspace $l'_{s'} \subset \mathcal{M}_{s'}$ and $l''_{s'} \subset \mathcal{M}_{s'}'$ annihilated by $(\text{ad}_{\mathfrak{p}})_{a, \bar{b}}$. Both $l'_{s'}$ and $l''_{s'}$ form $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-modules. Let $m'$ and $m''$ denote some $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-irreducible submodules of $l'_{s'}$ and $l''_{s'}$ correspondingly. Since all $B_{s'}^v$-s in (3.22) with fixed conformal weight (i.e. those contained in fixed row of (3.22)) have different $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-structure, one necessarily concludes that $m'$ and $m''$ coincide with some $B_{s'}^v$ of (3.22). On the other hand as one can

\[\text{Note that variables } n_a, n_b, n_{\bar{a}} \text{ and } n_{\bar{b}} \text{ are not independent on } \mathcal{M}_{s'} \text{ since they obey relations (3.14).} \]
easily see from (3.17) the only $B^{s'}_v$ from (3.22) annihilated by $(ad_{\rho})_{\alpha\beta}$ is $B^{0}_{\min}$ given by formula (3.20) with $v = 0$. We thus conclude that module $M_s$ is irreducible and is generated from $B^{0}_{\min}$ by $(ad_{K})_{\alpha\beta}$.

From (3.17) one finds that quadratic Casimir operator

$$C_{u(k,k)}^2 = \mathcal{L}_{\alpha} \bar{\mathcal{L}}_{\alpha} + \bar{\mathcal{L}}_{\bar{\alpha}} \mathcal{L}_{\bar{\alpha}} + \frac{2}{k}(D^2 + Z^2) - \{P_{\alpha\beta}, K^{\alpha\beta}\}$$

of algebra $u(k,k)$ takes in module $M_{s'}$ the following value

$$C_{u(k,k)}^2 = 2(s' - 1)(s' + 2k - 2).$$

4. Structure of twist-adjoint module

Let us now consider twisted-adjoint $\mathfrak{su}(k,k)$-module $\tilde{M}^\infty$. It is spanned by oscillators $a^\alpha$, $b_\alpha$, $\bar{a}^{\bar{\alpha}}$ and $\bar{b}^\beta$, where oscillator $\bar{b}^\beta$ is obtained from oscillator $\bar{b}_\alpha$ by twist transformation

$$\bar{b}_\alpha \rightarrow \frac{\partial}{\partial \bar{b}^\beta},$$

$$\frac{\partial}{\partial b_\alpha} \rightarrow -\bar{b}^\beta$$

so that commutator

$$[\frac{\partial}{\partial b_\alpha}, \bar{b}_\beta] = [-\bar{b}^\beta, \frac{\partial}{\partial \bar{b}^\beta}]$$

conserves. Due to conservation of commutator (4.2) operators that represent $u(k,k)$ on $\tilde{M}^\infty$ can be obtained from that of $M^\infty$ by simple replacement (4.1). We have (cf. (3.17))

$$(tw_{\tilde{L}}^\infty)_{\alpha\beta} = (ad_{\tilde{L}}^\infty)_{\alpha\beta},$$

$$(tw_{\tilde{L}}^\infty)_{\bar{\alpha}\bar{\beta}} = \bar{a}^{\bar{\beta}} \frac{\partial}{\partial \bar{a}^{\bar{\alpha}}} + \bar{b}^\beta \frac{\partial}{\partial \bar{b}^\beta} - \frac{1}{k} \delta^\beta_\alpha (n_\alpha + n_\beta),$$

$$(tw_{\tilde{P}}^\infty)_{\alpha\beta} = b_\alpha \frac{\partial}{\partial a^\beta} + \frac{\partial^2}{\partial a^\alpha \partial \bar{b}^\beta}, (tw_{\tilde{K}}^\infty)_{\alpha\beta} = a^\alpha \bar{b}^\beta - \bar{a}^{\bar{\beta}} \frac{\partial}{\partial b_\alpha},$$

$$tw_{\tilde{D}}^\infty = \frac{1}{2} (n_\alpha + n_\bar{\alpha} - n_\beta + n_\bar{\beta} + k), \quad tw_{\tilde{X}}^\infty = \frac{i}{2} (n_\alpha - n_\bar{\alpha} - n_\beta + n_\bar{\beta} - k),$$

where $n_\bar{b}$ is the Euler operator for oscillator $\bar{b}$. We require $\tilde{M}^\infty$ to be annihilated by $tw_{\tilde{Z}}^\infty$

$$(n_\alpha - n_\bar{\alpha} - n_\beta - n_\bar{\beta} - k)\tilde{M}^\infty = 0.$$

Analogously to $M^\infty$ module $\tilde{M}^\infty$ can be decomposed into direct sum of submodules $\tilde{M}^\infty_s$ picked out by requirement

$$\tilde{s}_1 \tilde{M}^\infty_s = \tilde{s}_2 \tilde{M}^\infty_s = s \tilde{M}^\infty_s,$$

where

$$\tilde{s}_1 = n_\alpha - n_\bar{\alpha} - k + 1, \quad \tilde{s}_2 = n_\beta + n_\bar{\beta} + 1$$

are operators that commute with (4.3). Note that due to (4.4) $\tilde{s}_1 f = \tilde{s}_2 f$ for any element $f \in \tilde{M}^\infty$.

Twist transformation (4.1) applied to the basis elements of $M^\infty_s$ (3.12) gives rise to the following elements of $\tilde{M}^\infty_s$

$$(\tilde{Z}^{s',s} g_{s'}^v (\tilde{Z}, \tilde{D}) f_{s' - v}), \quad s' = 1, \ldots, s, \quad v = 0, \ldots, s' - 1.$$
Here

\begin{equation}
\tilde{Z} = \frac{i}{2} (a \cdot b - \bar{a} \cdot \frac{\partial}{\partial \bar{b}})
\end{equation}

is annihilated by twist-adjoint action of $\mathfrak{su}(k, k)$ \((4.3)\),

\begin{equation}
\mathcal{D} = \frac{1}{2} (a \cdot b + \bar{a} \cdot \frac{\partial}{\partial \bar{b}})
\end{equation}

and particular form of $\tilde{g}_v^u(\tilde{Z}, \mathcal{D})$ is given in Appendix B.2. Finally, $\tilde{f}_{s'-v}$ is an eigenvector of operators $\tilde{s}_1, \tilde{s}_2$ \((4.6)\) that corresponds to eigenvalue $s' - v$

\begin{equation}
\tilde{s}_1\tilde{f}_{s'-v} = \tilde{s}_2\tilde{f}_{s'-v} = (s' - v)\tilde{f}_{s'-v}
\end{equation}

and satisfies twisted tracelessness relations \((\text{cf. (1.18)})\)

\begin{equation}
\bar{b} \cdot \frac{\partial}{\partial \bar{a}} \tilde{f}_{s'-v} = 0 ,
\end{equation}

\begin{equation}
\frac{\partial^2}{\partial a \cdot \partial \bar{b}} \tilde{f}_{s'-v} = 0 .
\end{equation}

In other words $\tilde{f}_{s'-v}$ can be represented as a sum of monomials

\begin{equation}
\tilde{m}_{s'-v}(n_a, n_{\bar{a}}, n_b, n_{\bar{b}}) = \tilde{z}_{\alpha(n_a); \beta(n_{\bar{a}}); \bar{\alpha}(n_a); \bar{\beta}(n_{\bar{a}})} a^\alpha(n_a) b^\beta(n_{\bar{a}}) \bar{a}^{\bar{\alpha}(n_a)} \bar{\beta}(n_{\bar{a}})
\end{equation}

where $\tilde{z}_{\alpha(n_a); \beta(n_{\bar{a}}); \bar{\alpha}(n_a); \bar{\beta}(n_{\bar{a}})}$ is a complex traceless tensor separately symmetric with respect to upper and lower group of undotted indices and of the symmetry type described by two-row Young tableau with first(second) row of length $n_b(n_{\bar{a}})$ with respect to dotted indices. Certainly values of $n_a, n_b, n_{\bar{a}}, n_{\bar{b}}$ in \((4.12)\) should be coordinated with $s'$ and $v$ through the formula \((4.10)\).

Let $\tilde{g}_v^u(\tilde{Z}, \mathcal{D})\tilde{m}_{s'-v}$ be denoted as $\tilde{B}_v^u$. Due to the above arguments $\tilde{B}_v^u$ forms, with respect to generators $(\text{tw}_L^\infty)_\alpha^\beta$, $(\text{tw}_\bar{L}^\infty)_\bar{\alpha}^{\bar{\beta}}$, irreducible $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-module corresponding to the Young tableau

\begin{equation}
\begin{array}{c}
\text{undotted} \\
\text{dotted}
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
\text{upper} & & & & & & n_a & \\
\text{lower} & & & & & n_b & & \end{array} \\
\begin{array}{cccccccc}
\text{upper} & & & & & & \bar{n}_{\bar{a}} & \\
\text{lower} & & & & & \bar{n}_b & & 
\end{array}
\end{array}
\end{equation}

(see Appendix A for more details).

Analogously to adjoint case let $\tilde{\mathcal{M}}_s^m$ denote submodule of $\tilde{\mathcal{M}}^\infty$ spanned by the elements \((4.7)\) with the power of $\tilde{Z}$ grater or equal to $m$ and let $\tilde{\mathcal{M}}_s^m = \tilde{\mathcal{M}}_s^m \cap \tilde{\mathcal{M}}_s$ denote corresponding submodule of $\tilde{\mathcal{M}}_s$. Note that $\tilde{\mathcal{M}}_s^m \equiv 0$ for $m > s$. Let us define quotient modules

\begin{equation}
\tilde{\mathcal{M}}^0 \subset \tilde{\mathcal{M}}^1 \subset \cdots \subset \tilde{\mathcal{M}}^\infty ,
\end{equation}

\begin{equation}
\tilde{\mathcal{M}}_s^0 \subset \tilde{\mathcal{M}}_s^1 \subset \cdots \subset \tilde{\mathcal{M}}_s^{s-2} \subset \tilde{\mathcal{M}}_s^\infty ,
\end{equation}

where

\begin{equation}
\tilde{\mathcal{M}}^m = \tilde{\mathcal{M}}^\infty / \tilde{\mathcal{M}}^{m+1} , \quad \tilde{\mathcal{M}}_s^m = \tilde{\mathcal{M}}_s^\infty / \tilde{\mathcal{M}}_s^{m+1} .
\end{equation}

Note that $\tilde{\mathcal{M}}_s^m \equiv \tilde{\mathcal{M}}_s^\infty$ for $m > s - 1$. In what follows we omit index 0 and denote $\tilde{\mathcal{M}}^0$ and $\tilde{\mathcal{M}}_s^0$ as $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_s$, respectively.
To find how $\mathcal{P}_{\alpha\beta}$ and $\mathcal{K}^{\alpha\beta}$ act on (4.7) one should apply twist transformation to (3.17), (3.18). We have

\[
\begin{align*}
&(\text{tw}_P)_{\alpha\beta} \cdot \bar{g}_{s'}^v \bar{m}_{s'-v} = \bar{\Pi}^{\perp} \left[ v \bar{g}_{s'}^{v-1} b_{\alpha} \frac{\partial}{\partial b^2} + \bar{g}_{s'}^v \left( v \varphi(n) + 1 \right) \frac{\partial^2}{\partial a^\alpha \partial \bar{b}^\beta} + \bar{g}_{s'}^v \left( v \varphi(\bar{n}) + 1 \right) b_{\alpha} \frac{\partial}{\partial \bar{a}^\beta} + \right. \\
&\quad \left. + \left( 2s' + 2k - v - 4 \right) \bar{g}_{s'}^{v+1} \varphi(n) \varphi(\bar{n}) \frac{\partial^2}{\partial a^\alpha \partial \bar{a}^\beta} \right] \bar{m}_{s'-v}, \\
&(\text{tw}_K)_{\alpha\beta} \cdot \bar{g}_{s'}^v \bar{m}_{s'-v} = -\bar{\Pi}^{\perp} \left[ v \bar{g}_{s'}^{v-1} a^\alpha a^\beta \bar{b}^\beta + \bar{g}_{s'}^v \left( v \varphi(n) + 1 \right) a^\beta \frac{\partial}{\partial a^\alpha} - \bar{g}_{s'}^v \left( v \varphi(\bar{n}) + 1 \right) a^\alpha \bar{b}^\beta - \right. \\
&\quad \left. - \left( 2s' + 2k - v - 4 \right) \bar{g}_{s'}^{v+1} \varphi(n) \varphi(\bar{n}) \bar{b}^\beta \frac{\partial}{\partial a^\alpha} \right] \bar{m}_{s'-v},
\end{align*}
\]

where $\bar{n} = n_\bar{a} - n_\bar{b} - k$ and $\bar{\Pi}^{\perp}$ is the projector to the component satisfying (4.11)

\[
\begin{align*}
\bar{\Pi}^{\perp} a^\alpha &= a^\alpha - \varphi(n - 2) a \cdot b \frac{\partial}{\partial a^\alpha}, & \bar{\Pi}^{\perp} b_{\alpha} &= b_{\alpha} - \varphi(n - 2) a \cdot b \frac{\partial}{\partial a^\alpha}, \\
\bar{\Pi}^{\perp} a^\beta &= \bar{a}^\beta + \varphi(\bar{n} - 2)(\bar{a}^\beta + \bar{b}^\beta \bar{a} \cdot b \frac{\partial}{\partial \bar{b}^\beta}), & \bar{\Pi}^{\perp} \bar{b}^{\beta} &= \frac{\partial}{\partial \bar{b}^{\beta}} - \varphi(\bar{n} - 2) \bar{a} \cdot \bar{b} \frac{\partial}{\partial \bar{a}^\beta}, \\
\bar{\Pi}^{\perp} a^\alpha \bar{a}^\beta &= \bar{\Pi}^{\perp} a^\alpha \bar{\Pi}^{\perp} a^\beta, & \bar{\Pi}^{\perp} b_{\alpha} \frac{\partial}{\partial \bar{b}^{\beta}} &= \bar{\Pi}^{\perp} b_{\alpha} \bar{\Pi}^{\perp} \frac{\partial}{\partial \bar{a}^\beta}.
\end{align*}
\]

Although all the above formulae were obtained by application of twist transformation (4.11) (which respect commutators (1.2)) to analogous formulae corresponding to adjoint module the structure of twist-adjoint modules and its analysis have some important nuances in comparison with adjoint case.

Firstly, twist-adjoint modules are infinite-dimensional. This is because operator $\bar{s}_1$ contains the difference of $n_a$ and $n_{\bar{b}}$ and, thus, requirement (4.10) does not bound the order of $\bar{m}_{s'-v}$ with respect to $a$ and $\bar{b}$.

Secondly, contrary to adjoint case the elements $\bar{B}^v_{s'}$ of twist-adjoint module are not linearly independent. Indeed, as was discussed above $\bar{m}_{s'-v}$ forms with respect to dotted indices $\mathfrak{sl}(k)$-module corresponding to two-row Young tableaux with first row of length $n_{\bar{b}}$ and second row of length $n_a$ (see (4.13) and Appendix A). Therefore $(a \cdot \bar{b})^v \bar{m}_{s'-v} = (\bar{D} + i \bar{\mathcal{Z}})^v \bar{m}_{s'-v} = 0$ for $u > v_{\text{max}}$, i.e.

\[
\bar{D}^v \bar{m}_{s'-v} = -\sum_{j=1}^{u} C^j_\alpha (i \bar{\mathcal{Z}})^j \bar{D}^{n-j} \bar{m}_{s'-v} \quad \text{for } u > v_{\text{max}},
\]

where

\[
v_{\text{max}} \bar{m}_{s'-v} = (n_{\bar{b}} - n_a) \bar{m}_{s'-v}.
\]

From (4.18) one gets in particular that

\[
\bar{B}_{s'}^v = \min(v_{\text{max}} + 1, s'-1) \sum_{j=1}^{v_{\text{max}} - j + 1} (\cdots) \bar{B}_{s'-j}^v + 1,
\]

where $(\cdots)$ are some coefficients. In what follows let elements $\bar{B}_{s'}^v$ and corresponding monomials $\bar{m}_{s'-v}$ with $v = v_{\text{max}}$ be called terminal and denote them as $\bar{B}_{s'}^v$ and $\bar{m}_{s'-v}$ correspondingly.
Taking the above arguments into account one concludes that for fixed $s'$ all linearly independent $\tilde{\mathcal{B}}^v_{s'}$-s can be arranged in the following diagram

Here every dot (●) indicates some $\tilde{\mathcal{B}}^v_{s'}$. All dots in the same row correspond to $\tilde{\mathcal{B}}^v_{s'} - s$ with the same conformal weight (indicated on the left axis)

$$
tw_D\tilde{\mathcal{B}}^v_{s'} = \frac{1}{2}(n_a + n_{\tilde{a}} - n_b + n_{\tilde{b}} + k)\tilde{\mathcal{B}}^v_{s'} = \tilde{\Delta}\tilde{\mathcal{B}}^v_{s'},
$$

which due to (4.19), (4.6) range for fixed value of $v = 0, 1, \ldots, s - 1$ from

$$
\tilde{\Delta}_{\text{min}}(v) = v + k \text{ for } \tilde{\mathcal{B}}^v_{s'\text{min}} = \tilde{\mathcal{Z}}^{s - s' - v}_{\alpha(s' + k - 1)} \tilde{g}^{\beta(s' - v - 1)}_{\alpha(s' + k - 1)} b^{\alpha(s' + k - 1)} b^{\beta(s' - v - 1)} b^{\beta(v)}
$$
to infinity.

Dots compose the collection of strips of width (number of dots) $s' - v$ and of infinite length, which are distinguished by the value of $v = 0, 1, \ldots, s - 1$ indicated on the bottom of diagram. The lowest dot in each strip corresponds to the $\tilde{\mathcal{B}}^v_{s'\text{min}}$ of the lowest conformal weight for given $v$. Arrows indicate transformations which change orders of $\tilde{\mathcal{B}}^v_{s'}$-s with respect to oscillators $a, b, \tilde{a}, \tilde{b}$ in such a way that $s'$ is kept constant and $\tilde{\Delta}$ increases by 1. Namely

$$
\begin{align*}
\begin{array}{c}
\vdots \quad \begin{cases}
n_{a, s'} & \mapsto n_{a, s'} + 1, \\
n_{b, s'} & \mapsto n_{b, s'} + 1, \\
n_{\tilde{a}, s'} & \mapsto n_{\tilde{a}, s'} + 1,
\end{cases} \\
\vdots
\end{array}
\end{align*}
$$

Introduce independent coordinates on (4.21)

$$
(\nu, q, t), \quad \nu = 0, \ldots, s' - 1, \quad q = 0, \ldots, \infty, \quad t = 0, \ldots, \min(q, s' - v - 1),
$$

where $\nu$ numerates the stripe and $q$ ($t$) indicates the number of upper-right (upper-left) arrows one should pass from the very bottom dot to get to the indicated dot. In these terms all other variables are expressed as

$$
\begin{align*}
n_a &= s' + k + q - 1, \quad n_b = s' - v - t - 1, \\
n_{\tilde{a}} &= t, \quad n_{\tilde{b}} = v + q, \\
\tilde{\Delta} &= k + v + q + t, \quad v_{\text{max}} = v + q - t.
\end{align*}
$$

From the expression for $v_{\text{max}}$ one finds that terminal terms, which are defined by requirement $v_{\text{max}} = v$, correspond to the dots with coordinates $(\nu, t, t)$, $t = 0, \ldots, s' - v - 1$, i.e. the most left dots of each stripe.

Now we are going to show that elements listed in the diagram (4.21) form the basis of $\mathfrak{su}(k, k)$-submodule $\tilde{\mathcal{M}}_s \subset \tilde{\mathcal{M}}^\infty$. Elements $\mathcal{L}_{\alpha}^{\beta}, \tilde{\mathcal{L}}_{\alpha}^{\beta}, \mathcal{D}, \mathcal{Z}$ of $\mathfrak{su}(k, k)$ represented by (4.13) commute with $\tilde{\mathcal{Z}}^{s' - v}_{\alpha(s' + k - 1)} \tilde{g}^{\beta(s' - v - 1)}_{\alpha(s' + k - 1)} b^{\alpha(s' + k - 1)} b^{\beta(s' - v - 1)} b^{\beta(v)}$ and therefore conserve set (4.21). Looking on the right-hand sides of (4.16) one sees that their 1-st, 3-d and 4-s terms also conserve (4.21), but 2-nd terms once act at the terminal term $\tilde{\mathcal{B}}^{s' v}_{s'}$ maps
it to $\tilde{B}_{s'}^{v}$, which due to (4.20) is equivalent to the sum of terms corresponding to $s' - 1, s' - 2, \ldots$. So it looks like $\tilde{M}_s$ is not $\mathfrak{su}(k, k)$-invariant. In Appendix B.2 it is shown that this is not the case, since these problem terms are zero out either (for $v = v_{\text{max}} > 0$) because coefficients in (4.20) vanish or (for $v = v_{\text{max}} = 0$) because projector (4.17) gives zero.

We, thus, show that analogously to adjoint case modules $\tilde{M}_s^\infty$, $\tilde{M}_s^m$ and $\tilde{M}^m$ admit decompositions

$$\tilde{M}_s^\infty = \bigoplus_{s' = 1}^\infty \tilde{M}_{s'},$$

(4.27)

$$\tilde{M}_s^m = \bigoplus_{s' = s - m}^\infty \tilde{M}_{s'}, \quad m = 0, 1, \ldots, s - 2,$$

$$\tilde{M}^m = \bigoplus_{s' = 1}^\infty \tilde{M}_{s'}, \quad m = 0, 1, \ldots, \infty$$

and the basis of $\tilde{M}_s^\infty$ that respects these decompositions is

$$\tilde{Z}^{s-s'}\tilde{B}_{s'}^{v}, \quad s' = 1, \ldots, s, \quad v = 0, \ldots, s' - 1,$$

where $\tilde{B}_{s'}^{v}$-s are listed in diagram (4.21).

In the same manner as for module $\tilde{M}_s$ one can show that $\tilde{M}_s$ is irreducible with quadratic Casimir operator given by formula (3.29).

In what follows we also need complex conjugate modules $\tilde{M}_s^n$, $n = 0, 1, \ldots, \infty$ that are obtained from $\tilde{M}_0^n$ by operation of complex conjugation

$$\tilde{(a, b, \bar{a}, \bar{b}, \bar{D}, \bar{Z})} \rightarrow \tilde{(\bar{a}, \bar{b}, a, b, D, Z)}.$$  

(4.29)

5. Unfolded formulation of Fradkin-Tseytlin equations

In this section we set the number of oscillators $k = 2$. According to procedure described in section 2 consider zero curvature equation (2.8) for 4-d connection 1-form $W(a, b, \bar{a}, \bar{b}) = \xi^\mu W_\mu$ taking values in algebra $\mathfrak{isu}^n(2, 2)$, $n = 0, 1, \ldots, \infty$. Where $W$ is defined on 4-d Minkowski space-time with coordinates $x^\mu$ and basic 1-forms $\xi^\mu$, $\mu = 0, \ldots, 4$.

Let us fix vacuum solution

$$W_0 = \xi^{\alpha \beta} b_\alpha \bar{b}_\beta,$$

(5.1)

where

$$\xi^{\alpha \beta} = \xi^\mu \sigma^\alpha_{\mu \beta}$$

(5.2)

and $\sigma^\alpha_{\mu \beta}$ are 4 Pauli matrices. It obviously satisfies (2.8) and corresponds to Cartesian coordinate choice in 4-d Minkowski space-time.

Let $\omega^\infty(a, b, \bar{a}, \bar{b})$ denote 1-form gauge fields taking values in $\mathcal{M}^\infty$. 2-form curvatures $R^\infty(a, b, \bar{a}, \bar{b})$ of the algebra $\mathfrak{isu}^\infty(2, 2)$ linearized with respect to the vacuum $W_0$ are given by

$$R^\infty = (d + \sigma^\infty)\omega^\infty,$$

(5.3)

where

$$d = \xi^\mu \frac{\partial}{\partial x^\mu} = \xi^{\alpha \beta} \frac{\partial}{\partial x^{\alpha \beta}}$$

(5.4)

and

$$\sigma^\infty = \xi^{\alpha \beta} (ad_\infty^\mu)_{\alpha \beta} = \xi^{\alpha \beta} \left(b_\alpha \frac{\partial}{\partial \bar{a}\beta} + \bar{b}_\beta \frac{\partial}{\partial a\alpha}\right).$$

(5.5)

Consider 0-form fields $C^\infty(a, b, \bar{a}, \bar{b})$ and $\bar{C}^\infty(a, \bar{b}, \bar{a}, \bar{b})$ taking values in twist-adjoint $\mathfrak{isu}(2, 2)$-modules $\tilde{M}^\infty$ and $\tilde{M}^\infty$ correspondingly. The unfolded system of $\mathfrak{su}(2, 2) \sim \mathfrak{so}(4, 2)$-invariant equations (Fradkin-Tseytlin equations) was formulated in paper [6]. In our notation it has the following
From (5.10) one gets that for general $s$

\begin{align}
(\xi + \tilde{\xi}^\infty)C^\infty &= 0, \\
(d + \tilde{\xi}^\infty)C^\infty &= 0,
\end{align}

where

\begin{align}
\sigma^\infty &= \xi^\alpha\beta (\xi^\gamma\delta (\xi^\nu\mu\nu_\alpha\beta + \frac{\partial^2}{\partial a^\alpha \partial b^\beta})), \\
\tilde{\sigma}^\infty &= \xi^\alpha\beta (\tilde{b}_\alpha \partial a^\alpha + \frac{\partial^2}{\partial a^\beta \partial b^\alpha})
\end{align}

and

\begin{align}
\Xi^{\alpha\beta} &= \xi^{\alpha\beta} \xi^{\gamma\delta} \xi^{\epsilon\gamma}, \\
\Xi^{\gamma\delta} &= \xi^{\gamma\delta} \xi^{\epsilon\gamma}.
\end{align}

Here $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ and $\epsilon^{\pi\beta}$, $\epsilon_{\pi\beta}$ are totally antisymmetric rank 2 tensors fixed by relation $\epsilon^{12} = \epsilon_{12} = 1$. They are used to rise and lower indices $x^\alpha\epsilon_{\alpha\beta} = x_\beta$, $x_\beta \epsilon^{\alpha\beta} = x^\alpha$ and analogous formulas for dotted indices.

As was shown above operators $\sigma^\infty$ and $\tilde{\sigma}^\infty$ (\tilde{\sigma}^\infty) commute with $s_1$, $s_2$ and $\tilde{s}_1$, $\tilde{s}_2$ ($\tilde{s}_1$, $\tilde{s}_2$) correspondingly. One can also easily see that operators $\sigma^\infty$ and $\tilde{\sigma}^\infty$ map submodules $\tilde{M}_s^\infty$ and $\tilde{M}_s^\infty$ to $M^\infty$. Therefore system (5.6), (5.7) decomposes into infinite number of subsystems on fields $\omega^\infty$, $C^\infty_s$ and $\tilde{C}^\infty_s$, $s = 1, 2, \ldots$ taking values in submodules $M^\infty_s$, $\tilde{M}^\infty_s$, $\tilde{M}^\infty_s$.

Moreover we argue that the above system is equivalent to the collection of subsystems corresponding to submodules $M_s$, $\tilde{M}_s$, $\tilde{M}_s$, $s' = 1, \ldots, s$. Indeed, since operators $\sigma^\infty$, $\tilde{\sigma}^\infty$ and $\tilde{\sigma}^\infty$ are constructed in terms of $\text{su}(2, 2)$-generators (namely generator of translation in adjoint and twist-adjoint representation) they admit decomposition (3.9), (4.27) into these submodules.

Let us now consider how operator $\sigma^\infty$ acts on the basis elements (1.28) of the module $\tilde{M}_s^\infty$. Suppose first that $s' = s - 1$ and, thus, the power of $Z$ in (1.28) is 1. It is easy to see that

\begin{equation}
\sigma^\infty \tilde{Z}B_{s-1}^v = Z \sigma^\infty \tilde{B}_{s-1}^v + \sigma^\infty \psi^1 \tilde{B}_{s-1}^v + \psi^1 \tilde{\sigma}^\infty \tilde{B}_{s-1}^v,
\end{equation}

where

\begin{equation}
\psi^1 = -i \xi^\alpha\beta \xi^{\gamma\delta} \xi^{\epsilon\beta} \xi^\alpha \partial a^\alpha \bigg|_{b=0}.
\end{equation}

From (5.10) one gets that for general $s'$

\begin{equation}
\sigma^\infty \tilde{Z}^s \tilde{B}_{s'}^v = Z^{s-s'} \sigma^\infty \tilde{B}_{s'}^v + \sigma^\infty \psi^1 \tilde{B}_{s'}^v + \psi^1 \tilde{\sigma}^\infty \tilde{B}_{s'}^v,
\end{equation}

where

\begin{equation}
\psi^1 \tilde{s} \tilde{s}' = \sum_{j=0}^{s-s'} \tilde{Z}^j \psi^1 \tilde{Z}^{s-s'-j}
\end{equation}

and analogous formulae for operator $\tilde{\sigma}^\infty$ acting on basis of $\tilde{M}_s^\infty$.

Therefore after field redefinition

\begin{equation}
\omega^\infty_s \to \omega^\infty_s - \sum_{j=1}^{s-1} \psi^j C^\infty_s - \sum_{j=1}^{s-1} \tilde{\psi}^j C^\infty_s
\end{equation}
system (5.6), (5.7) with s fixed split into subsystems corresponding to modules \( \mathcal{M}_{s'}, \tilde{\mathcal{M}}_{s'}, \tilde{\tilde{\mathcal{M}}}_{s'} \) for \( s' = 1, \ldots, s \)

\[
R_{s'} = (d + \sigma_\omega)\omega_{s'} = \sigma C_{s'} + \sigma \tilde{C}_{s'},
\]

(5.15)

\[
(d + \tilde{\sigma}_-)C_{s'} = 0,
\]

(5.16)

\[
(d + \tilde{\sigma}_-)\tilde{C}_{s'} = 0.
\]

Here \( \omega_{s'}, C_{s'}, \tilde{C}_{s'} \) stand for fields taking values in corresponding modules and

\[
\sigma_- = \xi^{a\beta} (\text{ad}_P)_{a\beta}, \quad \tilde{\sigma}_- = \xi^{a\beta} (\text{tw}_P)_{a\beta}, \quad \tilde{\sigma}_- = \xi^{a\beta} (\text{tw}_P)_{a\beta}.
\]

To find \( \sigma \) let us consider how operator \( \sigma^\infty \) acts on the elements \( \tilde{B}^{\nu}_{\tilde{s}'} = \tilde{g}^{\nu}_{\tilde{s}'} \tilde{m}_{s'-v} \), which form the basis of \( \tilde{\mathcal{M}}_{s'} \). Since \( \sigma^\infty \) sets \( \tilde{b} = 0 \), only \( \tilde{b} \) independent components of \( \tilde{B}^{\nu}_{\tilde{s}'} \) are valid. This means that all oscillators \( \tilde{b} \) in \( \tilde{m}_{s'}^{v} \) should be transformed by \( \tilde{g}^{\nu}_{s'} \) into oscillators \( \tilde{a} \). Therefore to give nonvanishing result \( \tilde{m}_{s'}^{v} \) is required: firstly, to be of the order not higher than \( v \) with respect to \( \tilde{b} \) and, secondly, to be independent of \( \tilde{a} \). From the diagram (4.2140) one finds that the only \( \tilde{B}^{\nu}_{s'} \)'s satisfying to these conditions are those with the lowest conformal weight \( \tilde{B}^{\nu}_{s'}\min \) (see (4.23)), which correspond to the lowest dot of the strip \( v \) in the diagram (4.21). From (4.23) taking into account (B.21) one gets that

\[
\sigma^\infty \tilde{B}^{\nu}_{s'} = \sigma \tilde{B}^{\nu}_{s'}\min = \Xi^{\alpha\beta} (-1)^{v} (2s' - v)! \frac{\partial^2}{\partial\tilde{a}^\alpha \partial\tilde{a}^\beta} \bigg|_{\tilde{a} = \tilde{b}} \tilde{B}^{\nu}_{s'} \min,
\]

(5.17)

\[
\sigma^\infty \tilde{B}^{\nu}_{s'} = \sigma \tilde{B}^{\nu}_{s'}\min = 0 \quad \text{for nonminimal } \tilde{B}^{\nu}_{s'}.
\]

From (5.17) one finds how \( \sigma \) acts on the general element of \( \tilde{\mathcal{M}}_{s'} \)

\[
\sigma C_{s'} = \Xi^{\alpha\beta} (-1)^{\tilde{a}} (n_{\tilde{a}} 1)^{2(2s' - \tilde{a})!} \frac{\partial^2}{\partial\tilde{a}^\alpha \partial\tilde{a}^\beta} \bigg|_{\tilde{a} = 0} C_{s'}.
\]

(5.18)

Here we exploited that requirement \( \tilde{a} = 0 \) in (5.18) sets to zero all nonminimal terms, requirement \( \tilde{b} = \tilde{a} \) along with multiplication by \( v! \) is analogous to \( (\tilde{a} \frac{\partial}{\partial \tilde{b}})^{v} \) in (5.17) and, finally, that \( v = n_{\tilde{a}} \) after we set \( \tilde{a} = \tilde{b} \). The form of operator \( \tilde{\sigma} \) can be obtained from (5.18) by complex conjugation.

As one can see from (5.18) \( \sigma \) maps minimal element \( \tilde{B}^{\nu}_{s'\min} \) of \( \tilde{\mathcal{M}}_{s'} \) corresponding to the stripe \( v \) in the diagram (4.21) into the element of \( \mathcal{M}_{s'} \) of conformal weight \( v \) that corresponds to the upper left edge of the rhombus 0 in the diagram (3.22), while \( \tilde{\sigma} \) maps complex conjugate element \( \tilde{B}^{\nu}_{s'\min} \) of \( \tilde{\mathcal{M}}_{s'} \) into the complex conjugate element of \( \mathcal{M}_{s'} \) corresponding to the upper right edge of rhombus 0.

As was discussed in section 2 the dynamical content of system (5.15) is encoded by cohomology of operators \( \sigma_-, \tilde{\sigma}_- \) and \( \tilde{\sigma}_- \). Let \( \mathcal{H}^{\nu}_{s',\Delta}, \tilde{\mathcal{H}}^{\nu}_{s',\Delta} \) and \( \tilde{\tilde{\mathcal{H}}}_{s',\Delta} \) denote \( p \)-th cohomology of corresponding operator with conformal weight \( \Delta \). These cohomology are found in Appendix C. For cohomology of

\footnote{Recall that \( \tilde{g}^{\nu}_{s'} \) is a polynomial of \( \tilde{Z} \) and \( \tilde{D} \), which contain operators \( \tilde{a} \cdot \frac{\partial}{\partial \tilde{b}} \).}

\footnote{To be precise due to reality conditions imposed on \( \mathcal{M}_{s'} \) operator \( \sigma \) does not map into \( \mathcal{M}_{s'} \), but the sum \( \sigma + \tilde{\sigma} \) does.
\(\sigma\) we have
\[
\mathcal{H}^0_{s';-s'+1} = \xi^{\beta(s'-1)} \beta^{\alpha(s'-1)} \beta^{\gamma(s'-1)} \beta^{\delta(s'-1)},
\]
\[
\mathcal{H}^1_{s';-s'+1} = \xi^{\gamma} \beta^{\alpha(s'-1)} \beta^{\beta(s'-1)} \beta^{\gamma(s'-1)} \beta^{\delta(s'-1)},
\]
\[
\mathcal{H}^2_{s';s'-1} = \Xi^{\gamma \delta(s'-1)} \beta^{\alpha(s'-1)} \beta^{\beta(s'-1)} \beta^{\gamma(s'-1)} \beta^{\delta(s'-1)},
\]
\[
\mathcal{H}^3_{s';s'-1} = i \Xi^{\delta(s'-1)} \beta^{\alpha(s'-1)} \beta^{\beta(s'-1)} \beta^{\gamma(s'-1)} \beta^{\delta(s'-1)},
\]
where
\[
\Xi^{\alpha \beta} = \xi^{\alpha \beta} \xi^{\gamma} \xi^{\delta} \xi^{\epsilon} \epsilon_{\gamma \delta},
\]
\[
\Xi^{\alpha \beta} = \xi^{\alpha \beta} \xi^{\gamma} \xi^{\delta} \epsilon_{\gamma \delta},
\]
\[
\Xi = \xi^{\alpha \beta} \xi^{\gamma} \xi^{\delta} \epsilon_{\gamma \delta},
\]
and tensors \(\varepsilon, \varphi, E\) and \(S\) are traceless. For cohomology of \(\hat{\sigma}\) we have
\[
\mathcal{H}^0_{s';2} = C_{(s'-1)} \beta^{\alpha(s'+1)} \beta^{\beta(s'-1)},
\]
\[
\mathcal{H}^1_{s';s'+1} = \xi^{\delta} \beta^{\alpha(s'+1)} \beta^{\beta(s'-1)} \beta^{\gamma(s'-1)} \beta^{\delta(s'-1)},
\]
where tensors \(\mathcal{C}, \mathcal{E}, \mathcal{S}\) are traceless and symmetry type of \(\mathcal{E} (\mathcal{S})\) with respect to undotted indices corresponds to two row Young tableaux with first row of length \(s'-1\) and second row of length 1 (2) (see Appendix A for more details). Cohomology of \(\hat{\sigma}\) are complex conjugate to (5.21).

As one can easily see operator \(\sigma + \hat{\sigma}\) maps \(\mathcal{H}^0_{s';2} + \mathcal{H}^0_{s';0}\) to \(\mathcal{H}^2_{s';0}\). To speak cohomology \(\mathcal{H}^2_{s';0}\) is ”glued up” by \(\sigma + \hat{\sigma}\). In other words 0-form \(\mathcal{H}^0_{s';2} + \mathcal{H}^0_{s';0}\) is not closed with respect to operator \(\hat{\sigma}\). In other words, 0-forms \(\mathcal{H}^0_{s';2} + \mathcal{H}^0_{s';0}\) is not exact. We, thus, have that 0-th and 1-st cohomology of \(\hat{\sigma}\) (see (2.29) and (2.31)) and \(\mathcal{H}^2_{s';0}\) is \(\hat{\sigma}\)-exact. We, thus, have that 0-th and 1-st cohomology of \(\hat{\sigma}\) (see (2.32) and (2.34)) are
\[
\mathcal{H}^0_{s';-s'+1} = (\mathcal{H}^0_{s';-s'+1}, 0),
\]
\[
\mathcal{H}^1_{s';-s'+1} = (\mathcal{H}^1_{s';-s'+1}, 0),
\]
\[
\mathcal{H}^2_{s';s'+1} = (0, \mathcal{H}^1_{s';s'+1} + \mathcal{H}^1_{s';s'+1}).
\]

From (5.22) one gets that \(\varphi^{\beta(s'-1)} \beta^{\alpha(s'-1)}\) is dynamical field of system (5.19), \(\varepsilon^{\beta(s'-1)} \beta^{(s'-1)}\) is parameter of gauge transformations obeyed by \(\varphi\), \(C^{\beta(s'-1)} \alpha(s'+1)\) is generalized Weyl tensor, which is expressed in terms of \(\varphi\) and \(\mathcal{E}_{\alpha(s'+1)}\). Direct form of these equations can be also easily obtained. We have
\[
C^{\alpha(2s')} = \frac{\partial}{\partial x_\alpha} \cdots \frac{\partial}{\partial x_\alpha} \varphi^{\alpha(s')} \beta^{\alpha(s')}, \quad \mathcal{C}^{\alpha(2s')} = \frac{\partial}{\partial x_\alpha} \cdots \frac{\partial}{\partial x_\alpha} \varphi^{\alpha(s')} \beta^{\alpha(s')},
\]
\[
\frac{\partial}{\partial x_\alpha} \cdots \frac{\partial}{\partial x_\alpha} C^{\alpha(2s')} = 0, \quad \frac{\partial}{\partial x_\alpha} \cdots \frac{\partial}{\partial x_\alpha} \mathcal{C}^{\alpha(2s')} = 0,
\]
where \(\varphi\) obeys gauge transformations
\[
\delta \varphi^{\alpha(s')} \beta^{\alpha(s')} = \frac{\partial}{\partial x_\alpha} \varepsilon^{\alpha(s'-1)} \beta^{\alpha(s'-1)}.
\]
Here symmetrization over the indices denoted by the same latter is implied and to avoid projectors to the traceless and/or Young symmetry components we rose and lowered indices by means of $\epsilon^{\alpha\beta}, \epsilon_{\alpha\beta}, \epsilon^{\dot{\alpha}\dot{\beta}}, \epsilon_{\dot{\alpha}\dot{\beta}}$.

If transformed from spinor indices $\alpha, \dot{\alpha}$ to vector indices $\mu$ (by means of Pauli matrices) equations (5.23), (5.24) coincide with equations (1.1), (1.2) for spin $s'$ field. Here $C_{s'(2s')}$ and $\bar{C}_{s'(2s')}$ correspond to selfdual and antselfdual parts of $C_{\nu(s'),\mu(s')}$. We, thus, showed that system (5.15) realizes unfolded formulation of spin $s'$ Fradkin-Tseytlin equations.

As we showed, any unfolded system corresponding to modules $\mathcal{M}^m_s$, $\bar{\mathcal{M}}^m_s$, $\tilde{\mathcal{M}}^m_s$ for any fixed $s = 1, 2, \ldots$ and $m = 0, \ldots, s - 2, \infty$ or to modules $\mathcal{M}^m_s$, $\tilde{\mathcal{M}}^m_s$ for any fixed $m = 0, 1, \ldots$ decomposes into independent subsystems (5.15). Therefore such system describes the collection of Fradkin-Tseytlin equations for the fields of spins given by decompositions (3.9)-(3.11), (4.27). In particular unfolded system proposed in [6] (see (5.6), (5.7)) describes all integer spin fields $s' = 1, 2, \ldots$, where every spin enters with infinite multiplicity.

6. Conclusion

We have proposed unfolded system (5.15) that describes linear conformal dynamics of spin $s'$ gauge field (spin $s'$ Fradkin-Tseytlin equations). We also have shown that any unfolded system based on $\mathfrak{su}(2,2)$ adjoint and twisted-adjoint representations on space of algebra $\mathfrak{isu}^m(2,2)$, $m = 0, 1, \ldots, \infty$ can be decomposed into independent subsystems of form (5.15) by means of appropriate field redefinition and found spectrum of spins for any $\mathfrak{isu}^m(2,2)$. In particular we have shown that system of equations proposed in [6] (5.6), (5.7) describes linear dynamics of conformal fields of all integer spins greater or equal than 1, where each spin enters with infinite multiplicity.

This work can be considered as a first modest step towards construction of the full nonlinear conformal theory of higher spins. One of the main ingredients of higher spin theories is a higher spin algebra. Our results pretend to be a probe of different candidates to this role. We see that algebras $\mathfrak{isu}^m(2,2)$, $m = 1, 2 \ldots$ mediate between $\mathfrak{isu}^\infty(2,2)$ and $\mathfrak{isu}(2,2)$.

Having in mind that conformal higher spin theory has to be somehow related to $AdS$ higher spin theory one can speculate that algebra $\mathfrak{isu}(2,2)$ is more preferable since its spectrum just literally coincides with the spectrum of some $AdS$ higher spin theory. On the other hand equations proposed in [6], which correspond to $\mathfrak{isu}^\infty(2,2)$, are considerably simpler than (5.15) corresponding to $\mathfrak{isu}(2,2)$. Therefore an interesting question arises wether it is possible to simplify (5.15) maybe by mixing again gauge and Weyl sectors of the theory (5.14). Another important area of investigation is to consider super extensions of $\mathfrak{isu}^m(2,2)$ and, thus, bring fermions into the play.

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Appendix A. Finite-dimensional $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$ irreps

Algebra $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k) \subset \mathfrak{u}(k, k)$ is generated by $\mathcal{L}_\alpha^\beta$ and $\tilde{\mathcal{L}}_{\dot{\alpha}}^\dot{\beta}$ (see (1.9)). Let first summand of $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$ that is generated by $\mathcal{L}_\alpha^\beta$ be referred to as undotted $\mathfrak{sl}(k)$ and second summand that is generated by $\tilde{\mathcal{L}}_{\dot{\alpha}}^\dot{\beta}$ be referred to as dotted $\mathfrak{sl}(k)$. All finite-dimensional irreps of $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$ are given by tensor products of finite-dimensional irreps of undotted and dotted $\mathfrak{sl}(k)$. In what follows we recall some well-known facts about $\mathfrak{sl}(k)$-irreps taking as an example undotted $\mathfrak{sl}(k)$. Needless
to say that the same arguments work for dotted $\mathfrak{s}(k)$ once undotted indices are replaced by dotted indices.

Finite-dimensional irreps of $\mathfrak{s}(k)$ are given by $\mathfrak{s}(k)$-tensors

$$T^{\alpha_1(\lambda_1), \ldots, \alpha_k(\lambda_k)}_{\beta_1(\lambda_1), \ldots, \beta_k(\lambda_k)},$$

(A.1)

$$\lambda_1 \geq \cdots \geq \lambda_k \in \mathbb{Z}_+, \quad \tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k \in \mathbb{Z}_+,$$

written in symmetric basis or equivalently by $\mathfrak{s}(k)$-tensors

$$T^{\alpha_1[\mu_1], \ldots, \alpha_k[\mu_k]}_{\beta_1[\mu_1], \ldots, \beta_k[\mu_k]},$$

(A.2)

$$k \geq \mu_1 \geq \cdots \geq \mu_{\lambda_1} \in \mathbb{Z}_+, \quad k \geq \tilde{\mu}_1 \geq \cdots \geq \tilde{\mu}_{\tilde{\lambda}_1} \in \mathbb{Z}_+,$$

written in antisymmetric basis. Here $\alpha_i(\lambda_i)$ denotes $\lambda_i$ totally symmetrized indices $\alpha_i^1, \ldots, \alpha_i^{\lambda_i}$ and $\alpha_i[\mu_i]$ denotes $\mu_i$ totally antisymmetrized indices $\alpha_i^1, \ldots, \alpha_i^{\mu_i}$. Irreducibility conditions require tensor $T$

1. to be traceless, i.e. such that contraction of any pair of indices $\alpha_i$ and $\beta_j$ gives zero;
2. to satisfy Young symmetry conditions implying for upper indices in (anti)symmetric basis that total (anti)symmetrization of all indices from the set $\alpha_i(\lambda_i) \ (\alpha_i[\mu_i])$ with some index from the set $\alpha_j(\lambda_j) \ (\alpha_j[\mu_j])$ gives zero for $j > i$ and analogous conditions for lower indices.

It is often very useful to visualize structure of tensor $T$ with the help of the pair of Young tableaux

(A.3)

where $i$-th row of upper(lower) tableau corresponds to $i$-th upper(lower) group of totally symmetric indices of tensor $T$ in symmetric basis (A.1) or equivalently $i$-th column of upper(lower) tableau corresponds to $i$-th upper(lower) group of totally antisymmetric indices of tensor $T$ in antisymmetric basis (A.2). Let Young tableau with rows of lengths $\lambda_1 \geq \cdots \geq \lambda_k$ be denoted as $\mathcal{Y}(\lambda_1, \ldots, \lambda_k)$ and Young tableau with columns of heights $\mu_1 \geq \cdots \geq \mu_h$ be denoted as $\mathcal{Y}[\mu_1, \ldots, \mu_h]$.

Using totally antisymmetric tensors $\epsilon^{(k)}$, $\epsilon_{(k)}$ one can rise and lower indices of $T$

(A.4)

$$T_{\ldots, \alpha_i[\mu_i], \ldots} \mapsto \epsilon^{[k-\tilde{\mu}[\tilde{\mu}_i]]} \epsilon_{\ldots, \alpha_i[\mu_i], \ldots} T_{\ldots, \alpha_i[\mu_i], \ldots},$$

(A.5)

$$T_{\ldots, \alpha_i[\mu_i], \ldots} \mapsto \epsilon^{[k-\mu[\mu_i]]} \epsilon_{\ldots, \alpha_i[\mu_i], \ldots} T_{\ldots, \alpha_i[\mu_i], \ldots}.$$

Let us consider composition of (A.4) and (A.5)

(A.6)

$$T_{\ldots, \alpha_i[\mu_i], \ldots} \mapsto \epsilon^{[k-\tilde{\mu}[\tilde{\mu}_i]]} \epsilon_{\gamma[k-\mu[\mu_i]]} \epsilon_{\ldots, \alpha_i[\mu_i], \ldots} T_{\ldots, \alpha_i[\mu_i], \ldots}.$$

Taking into account that tensor product of two $\epsilon$-tensors can be written as alternative sum of Kronecker deltas product

(A.7)

$$\epsilon^{\alpha_1 \cdots \alpha_k} \epsilon_{\beta_1 \cdots \beta_k} = \sum_{\sigma \in S_k} (-1)^{\pi(\sigma)} \delta_{\beta_1^{(1)}}^{\alpha_1} \cdots \delta_{\beta_k^{(k)}}^{\alpha_k},$$

$^8$Note that the number of columns does not limited by $k$ contrarily to the number of rows.
where sum is taken over all permutations of \((1, \ldots, k)\) and \(\pi(\sigma)\) is the oddness of permutation \(\sigma\), one can readily see that (A.6) vanishes if \(\mu_i + \tilde{\mu}_j > k\). Indeed, in this case at least one of the Kronecker deltas in every summand of (A.7) contracts with \(T\) that is considered to be traceless. Therefore, the irreducibility conditions above are consistent only if \(\mu_i + \tilde{\mu}_j \leq k\) for any \(i, j\). Due to the same arguments operations of rising and of lowering indices (A.4), (A.5) result in a tensor satisfying irreducibility conditions only when applied to the highest (i.e. first) column of \(T\).

Let define rising and lowering Hodge conjugations by formulas

\[(A.8) \quad *_{r}(T^{\alpha_1[\mu_1] \ldots}) = \frac{i^{(k-\tilde{\mu}_1)\mu_1}}{\tilde{\mu}_1!} \epsilon^{[k-\tilde{\mu}_1] \alpha_1[\mu_1]} T^{\alpha_1[\mu_1] \ldots},\]

\[(A.9) \quad *_{l}(T^{\alpha_1[\mu_1] \ldots}) = \frac{i^{(k-\mu_1)\mu_1}}{\mu_1!} \epsilon^{[k-\mu_1] \alpha_1[\mu_1]} T^{\alpha_1[\mu_1] \ldots} \].

They map \(\mathfrak{sl}(k)\)-irrep described by Young tableau (A.3) into equivalent \(\mathfrak{sl}(k)\)-irrep described by the different Young tableau

\[(A.10) \quad \begin{array}{c}
\text{upper} \\
\begin{array}{c}
\mu_1 \mu_2 \mu_3 \ldots
\end{array}
\end{array} \rightarrow \begin{array}{c}
\text{lower} \\
\begin{array}{c}
\hat{\mu}_1 \hat{\mu}_2 \hat{\mu}_3 \ldots
\end{array}
\end{array} \]

Here \(*_{r}\) kills the first column of the lower Young tableau and adds the column of height \(k - \tilde{\mu}_1\) to the left-hand side of the upper Young tableau (since \(\mu_1 + \tilde{\mu}_1 \leq k\) we thus get the proper Young tableau) and \(*_{l}\) acts in the opposite way. Young tableaux that result in consequent application of transformations (A.10) describe one and the same \(\mathfrak{sl}(k)\)-module. Coefficients in (A.8) and (A.9) are chosen such that

\[(A.11) \quad *_{r} *_{l} T = *_{l} *_{r} T = T.\]

Quadratic Casimir operator of algebra \(\mathfrak{sl}(k)\) is given by formula

\[(A.12) \quad C_{\mathfrak{sl}(k)}^{2} = \mathcal{L}_{\alpha}^{\beta} \mathcal{L}_{\beta}^{\alpha} .\]

For the \(\mathfrak{sl}(k)\)-irrep described by Young tableaux (A.3) \(C_{\mathfrak{sl}(k)}^{2}\) is equal to

\[(A.13) \quad C_{\mathfrak{sl}(k)}^{2} = (k+1)(\Sigma + \check{\Sigma}) - \frac{1}{k} (\Sigma - \check{\Sigma})^{2} + \sum_{i} \lambda_{i}(\lambda_{i} - 2i) + \sum_{i} \check{\lambda}_{i}(\check{\lambda}_{i} - 2i) ,\]

where \(\Sigma\) is the total number of upper indices (total number of cells in the upper Young tableau) and \(\check{\Sigma}\) is that of the lower indices. \(C_{\mathfrak{sl}(k)}^{2}\) can be also expressed through the heights of the Young tableaux (A.3)

\[(A.14) \quad C_{\mathfrak{sl}(k)}^{2} = (k-1)(\Sigma + \check{\Sigma}) - \frac{1}{k} (\Sigma - \check{\Sigma})^{2} - \sum_{i} \mu_{i}(\mu_{i} - 2i) - \sum_{i} \check{\mu}_{i}(\check{\mu}_{i} - 2i) .\]
The irreducible representations of $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$ are given by tensor product of two irreps of $\mathfrak{sl}(k)$

\begin{equation}
T^{\beta_1(\lambda_1),\ldots,\beta_k(\lambda_k)}_{\alpha_1(\lambda'_1),\ldots,\alpha_k(\lambda'_k)} \otimes T^{\dot{\beta}_1(\lambda'_1),\ldots,\dot{\beta}_k(\lambda'_k)}_{\dot{\alpha}_1(\lambda'_1),\ldots,\dot{\alpha}_k(\lambda'_k)} = T^{\beta_1(\lambda_1),\ldots,\beta_k(\lambda_k);\dot{\beta}_1(\lambda'_1),\ldots,\dot{\beta}_k(\lambda'_k)}_{\alpha_1(\lambda'_1),\ldots,\alpha_k(\lambda'_k);\dot{\alpha}_1(\lambda'_1),\ldots,\dot{\alpha}_k(\lambda'_k)}, \tag{A.15}
\end{equation}

\begin{equation}
T^{\beta_1[\mu_1],\ldots,\beta_k[\mu_{\lambda_1}]}_{\alpha_1[\mu'_1],\ldots,\alpha_k[\mu'_1]} \otimes T^{\dot{\beta}_1[\mu'_1],\ldots,\dot{\beta}_k[\mu'_1]}_{\dot{\alpha}_1[\mu'_1],\ldots,\dot{\alpha}_k[\mu'_1]} = T^{\beta_1[\mu_1],\ldots,\beta_k[\mu_{\lambda_1}];\dot{\beta}_1[\mu'_1],\ldots,\dot{\beta}_k[\mu'_1]}_{\alpha_1[\mu'_1],\ldots,\alpha_k[\mu'_1];\dot{\alpha}_1[\mu'_1],\ldots,\dot{\alpha}_k[\mu'_1]}, \tag{A.16}
\end{equation}

where first $\mathfrak{sl}(k)$ acts on undotted indices and second $\mathfrak{sl}(k)$ acts on dotted indices. Tensors $T$ are supposed to satisfy irreducibility conditions above separately within undotted and dotted indices. This representation can be described by four Young tableaux

\begin{equation}
\begin{aligned}
\text{undotted} & : \begin{array}{c}
\lambda_1 \\
\vdots \\
\lambda_k
\end{array}
\quad \begin{array}{c}
\mu_1 \mu_2 & \cdots & \mu_{\lambda_1} \\
\vdots & & \vdots \\
\lambda_k & \cdots & \lambda_k
\end{array} \\
\text{dotted} & : \begin{array}{c}
\lambda'_1 \\
\vdots \\
\lambda'_k
\end{array}
\quad \begin{array}{c}
\mu'_1 \mu'_2 & \cdots & \mu'_{\lambda'_1} \\
\vdots & & \vdots \\
\lambda'_k & \cdots & \lambda'_k
\end{array},
\end{aligned} \tag{A.17}
\end{equation}

Quadratic Casimir operator

\begin{equation}
\mathcal{C}_{\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)}^2 = \mathcal{L}_\alpha \mathcal{L}_\beta + \mathcal{L}_{\dot{\alpha}} \mathcal{L}_{\dot{\beta}}, \tag{A.18}
\end{equation}

of $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-irrep described by (A.17) is given by sum of quadratic $\mathfrak{sl}(k)$-Casimirs for undotted and dotted parts of (A.17) (see (A.13) and/or (A.14)).

\section*{Appendix B. The Basis of $\mathcal{M}_s^\infty$ and $\bar{\mathcal{M}}_s^\infty$ Direct Decomposition}

\subsection*{B.1. Module $\mathcal{M}_s^\infty$.} Let us find the basis of the module $\mathcal{M}_s^\infty$ in which it decomposes into submodules $\mathcal{M}_{s'}$ (3.9). Since elements of $\mathfrak{su}(k, k)$ commute with the helicity operator $\mathcal{Z}$ and

\begin{equation}
[s_{1,2}, \mathcal{Z}] = \mathcal{Z}, \quad [s_{1,2}, \mathcal{D}] = \mathcal{D}, \tag{B.1}
\end{equation}

the submodule $\mathcal{M}_{s'}$ enters to $\mathcal{M}_s^\infty$ with the factor $\mathcal{Z}^{s-s'}$. Therefore the natural ansatz for such a basis is

\begin{equation}
\mathcal{Z}^{s-s'}g_{sv}^{s'}(\mathcal{Z}, \mathcal{D})f_{s'-v}, \quad s' = 1, \ldots, s, \quad v = 0, \ldots, s' - 1. \tag{B.2}
\end{equation}

Here $f_{s'-v}(a, b, \bar{a}, \bar{b})$ is traceless (i.e. satisfies (1.18)) and forms eigenvector corresponding to eigenvalue $s' - v$ with respect to operators $s_1$ and $s_2$ and

\begin{equation}
g_{sv}^{s'}(\mathcal{Z}, \mathcal{D}) = \sum_{j=0}^{v} \mathcal{Z}^{v-j}\mathcal{D}^{j}d_{s',j}^{v}, \tag{B.3}
\end{equation}

is homogeneous polynomial of degree $v$ in two variables $\mathcal{Z}$ and $\mathcal{D}$, with coefficients $d_{s',j}^{v}(n_a, n_b, n_{\bar{a}}, n_{\bar{b}})$ to be found from the requirement that elements of (B.2) with fixed value of $s'$ span invariant subspace of module $\mathcal{M}_s^\infty$, which is submodule $\mathcal{M}_{s'}$ in decomposition (3.9).

Operators (3.11) corresponding to the $\mathfrak{su}(k, k)$ elements $\mathcal{L}_\alpha \mathcal{L}_\beta$, $\mathcal{L}_{\dot{\alpha}} \mathcal{L}_{\dot{\beta}}$, $\mathcal{D}$, $\mathcal{Z}$ obviously conserve elements of (B.2) and a fortiori keep $\mathcal{M}_{s'}$ invariant. Suppose that operator corresponding to $\mathcal{P}_{\alpha\dot{\beta}}$ also keeps
\( \mathcal{M}_s \) invariant for some particular choice of \( g_s^v \). Then as one can easily see it has in basis \([B.2]\) the following form

\[
(\text{ad}_P^{\infty})_{\alpha\beta}g_s^v f_{s'v} = \left( g_{s'}^{v-1} P_{s'v}^{v-1} b_{\alpha} \bar{b}_{\beta} + g_s^v P_s^{v+1} \frac{\partial}{\partial a^\alpha} + g_s^v P_s^{v+1} \frac{\partial}{\partial a^\beta} \right) f_{s'v},
\]

(B.4)

where \( P_s^{v+1}, P_{s,0}^v, P_{s,0}^v, P_{s',+}^v \) are some unknown coefficients and \( \Pi^\perp \) is projector to the traceless component \([B.18]\).

The requirement that \( (\text{ad}_P^{\infty})_{\alpha\beta} \) keeps \( \mathcal{M}_s \) invariant reduces to the following system of recurrence equations

\[
(j+1)d_{s',j+1}^v(n_a, n_b, n_{\bar{a}}, n_b) = P_{s',v}^{v-1}(n_a, n_b+1, n_a, n_b+1)d_{s',j}^v(n_a, n_b+1, n_{\bar{a}}, n_b+1),
\]

\[
(j\varphi(n-1) + 1)d_{s',j}^v(n_a, n_b, n_{\bar{a}}, n_b) - i(j+1)\varphi(n-1)d_{s',j+1}^v(n_a, n_b, n_{\bar{a}}, n_b) = P_{s',0}^v(n_a, n_{\bar{a}}, n_b+1, n_{\bar{a}}, n_b+1)d_{s',j}^v(n_a, n_{\bar{a}}-1, n_b, n_{\bar{a}}+1),
\]

(B.5)

with boundary condition \( d_{s',v}^v \equiv 1 \). Here \( n = n_a + n_b, \ \bar{n} = n_{\bar{a}} + n_{\bar{b}} \) and \( \varphi(n) = 1/(n+k) \).

One can show that the solution of system \([B.5]\) is

\[
P_{s',v}^{v-1} = v, \quad P_{s',0}^v = v\varphi(n) + 1, \quad P_{s',0}^v = v\varphi(n) + 1,
\]

(B.6)

\[d_{s',j}^v = \left( \frac{v}{j} \prod_{h=j}^{v-1}(2s' + 2k - v - 3 + h) \right),\]

where \( \delta_{s',j}^v \) satisfies the following recurrence equation

\[
\delta_{s',j}^v = i(n - \bar{n})\delta_{s',j+1}^v + (v - j - 1)(2s' + 2k - v - j - 2)\delta_{s',j+2}^v, \quad j = 0, \ldots, v - 1,
\]

(B.7)

with boundary conditions \( \delta_{s',v}^v \equiv 1, \ \delta_{s',j>v}^v \equiv 0 \).

Consider involution (i.e. involutive antilinear antiautomorphism) of Heisenberg algebra

\[
\tau: \left\{ \begin{array}{c}
a^\alpha \leftrightarrow \frac{\partial}{\partial a^\alpha}, \quad \bar{a}^\alpha \leftrightarrow \frac{\partial}{\partial \bar{a}^\alpha}, \\
 b_\alpha \leftrightarrow \frac{\partial}{\partial b_\alpha}, \quad \bar{b}_\alpha \leftrightarrow \frac{\partial}{\partial \bar{b}_\alpha}.
\end{array} \right.
\]

(B.8)

It induces involution of the algebra \( su(k, k) \)

\[
\tau: \left\{ \begin{array}{c}
\mathcal{L}_\alpha^\beta \leftrightarrow \mathcal{L}_\beta^\alpha, \quad \mathcal{P}_{\alpha\beta} \leftrightarrow -\mathcal{K}^\alpha_{\beta}, \quad \mathcal{D} \leftrightarrow \mathcal{D}, \\
\mathcal{L}_{\bar{\alpha}}^\bar{\beta} \leftrightarrow \mathcal{L}_{\bar{\beta}}^\bar{\alpha}, \quad \mathcal{Z} \leftrightarrow -\mathcal{Z}.
\end{array} \right.
\]

(B.9)

As follows from \([B.6], [B.7]\) Euler operators \( n_a, n_b, n_{\bar{a}}, n_{\bar{b}} \) contribute to coefficients \( d_{s',j}^v \) through the combination \( n - \bar{n} \) only. Therefore elements of \([B.2]\) are invariant up to the factor \(-1\) with respect
to involutive transformation $\tau$ (B.8). Since $P_{\alpha\beta}$ and $K^{\alpha\beta}$ are $\tau$-conjugated one concludes that $K^{\alpha\beta}$ also keeps $M_s$ invariant.

The elements of (B.2) are obviously linearly independent and span the whole $M_s^\infty$. Therefore they form a basis of $M_s^\infty$ under consideration. Substituting values of $P_{s',0}^v$, $P_{s'^{+},0}^v$, $P_{s'^{-},0}^v$ found in (B.6) to (B.4) one gets the representation of $\mathfrak{su}(k,k)$ on $M_s$ (see (3.17) for exact formulas).

Let us note that $Z^{s-s'}g_v^{s'}$ satisfy to reality conditions (1.11), i.e.

$$\zeta(Z^{s-s'}g_v^{s'}) = Z^{s-s'}g_v^{s'}.$$  

It is possible to show that

$$g_v^{s'}(Z,D)|_{Z=\frac{1}{2}} \prod_{\alpha=1/2}^{v} \frac{2s' + 2k - 2h - 4 - (n - \bar{n})}{2s' + 2k - h - 4}. $$

Few lower examples of $g_v^{s'}$ are the following

$$g_v^{s'} = 1,$$

$$g_v^{s'} = Z\frac{i(n - \bar{n})}{2s' + 2k - 4} + D,$$

$$g_v^{s'} = Z^2\frac{2s' + 2k - 4 - (n - \bar{n})^2}{2s' + 2k - 4}(2s' + 2k - 5) + 2ZD\frac{i(n - \bar{n})}{2s' + 2k - 4} + D^2,$$

$$g_v^{s'} = Z^3\frac{6s' + 6k - 14 - (n - \bar{n})^2}{(2s' + 2k - 4)(2s' + 2k - 6)} + 3Z^2D\frac{2s' + 2k - 4 - (n - \bar{n})^2}{(2s' + 2k - 4)(2s' + 2k - 5)} + 3ZD^2\frac{i(n - \bar{n})}{2s' + 2k - 4} + D^3.$$  

B.2. Module $\tilde{M}_s^\infty$. Let us now consider module $\tilde{M}_s^\infty$, which is obtained from $M_s^\infty$ by twist transformation (1.11). Twist transformation of the basis elements (B.2) gives

$$\tilde{Z}^{s-s'}\tilde{g}_v^{s'}(\tilde{Z},\tilde{D})f_{s'-v},$$

where $\tilde{Z}$, $\tilde{D}$ are given by formulas (1.8), (1.9) correspondingly, $f_{s'-v}$ satisfies to (1.10), (1.11) and

$$\tilde{g}_v^{s'}(\tilde{Z},\tilde{D}) = \sum_{j=0}^{v}\tilde{Z}^{v-j}\tilde{D}^j\tilde{d}_{s',j}^{v}.$$  

Here $\tilde{d}_{s',j}^{v}$ are coefficients satisfying to twisted equations (B.5). These equations have the following solution (cf. (B.6), (B.7))

$$\tilde{d}_{s',j}^{v} = \left(\begin{array}{c} v \\ j \end{array}\right)\frac{\delta_{s',j}^{v}}{\prod_{h=j}^{v-1}(2s' + 2k - v - 3 + h)},$$

where $\delta_{s',j}^{v}$ satisfies the recurrence equation

$$\delta_{s',j}^{v} = i(n - \bar{n})\delta_{s',j+1}^{v} + (v - j - 1)(2s' + 2k - v - j - 2)\delta_{s',j+2}^{v},$$

with boundary conditions $\delta_{s',v}^{v} \equiv 1$, $\delta_{s',j>v}^{v} \equiv 0$. Recall that $n = n_a + n_b$ and $\bar{n} = n_a - n_b - k$.

Now according to arguments given in page 16 we need to show that 2-nd terms of right-hand sides of (1.10) vanish when acting on terminal element $\tilde{B}_{s',v}^{t}$, i.e.

$$\tilde{\Pi}^{t}_{s',v}\left(\tilde{g}_v^{t}(\nu (n) + 1)\frac{\partial^2}{\partial a^\alpha \partial b^\beta}m^{t}_{s'-v} = 0, \right.$$  

(B.17)
Suppose first that \( v_{\text{max}} = v = 0 \), i.e. \( \tilde{B}^s_{s'} \) corresponds to the very left dots of the stripe indicated by \( v = 0 \) in the diagram (4.21). As one can easily see the structure of dotted indices of all such \( \tilde{B}^s_{s'} \)'s is described by two-row Young tableau with rows of equal length and, thus, terms (B.17) project it to two-row Young tableau with first row less than second, which is zero.

Now lets \( v_{\text{max}} = v > 0 \), i.e. \( \tilde{B}^s_{s'} \) corresponds to the very left dots of the stripe indicated by \( v \) in the diagram (4.21). As one can easily see from (1.23), (1.24) the monomials \( \tilde{m}^v_{s'-v} \), corresponding to such terms have value

\[
\tilde{n} - \tilde{n} = (s' + k - 1) \tilde{m}^v_{s'-v} = 2(s' + k - 1) \tilde{m}^v_{s'-v}
\]

and operators \( \alpha^{2} \partial_{\alpha} \) from (B.17) decrease it by 2. Let us find the form of function \( \tilde{g}^v_{s'} \) in (B.17). Substituting the value of \( n - \tilde{n} \) to (B.16) one finds that

\[
\tilde{g}^v_{s';j} = i^{v-j} \prod_{h=j}^{v-1} (2s' - 2k - v - 3 + h)
\]

and, thus,

\[
\tilde{d}^v_{s';j} = \left( \begin{array}{c} v \\ j \end{array} \right) i^{v-j} \quad \tilde{g}^v_{s'} = (\tilde{D} + i\tilde{Z})^v.
\]

Taking into account that operators \( \alpha^{2} \partial_{\alpha} \) from (B.17) decrease the value of \( v_{\text{max}} = v \) by 1 one gets (B.17).

Finally let us note that formula (1.11) in twist-adjoint case is

\[
\tilde{g}^v_{s'}(\tilde{Z}, \tilde{D}) |^{\tilde{D} = 1/2}_{\tilde{Z} = 1/2} = \frac{1}{2^v} \prod_{h=0}^{v-1} \frac{2s' + 2k - h - 4 - (n - \tilde{n})}{2s' + 2k - h - 4}.
\]

**Appendix C. \( \sigma_- \)-cohomology**

Let \( \mathcal{C} = \{ \mathcal{C}, \partial \} \) be some co-chain complex. Here \( \mathcal{C} = \oplus_{p=0}^{\infty} C^p \) is graded space and

\[
\partial : \ C^p \mapsto C^{p+1}, \quad \partial^2 = 0
\]

is differential. The powerful tool to calculate cohomology of \( \mathcal{C} \) consists in consideration of homotopy operator \( \partial^* \)

\[
\partial^* : \ C^{p+1} \mapsto C^p, \quad (\partial^*)^2 = 0.
\]

According to standard result of cohomological algebra (see e.g. [14]) every element of \( \mathcal{C} \)-cohomology group \( \mathcal{H} \) has representative belonging to kernel of the anticommutator \( \Theta = \{ \partial, \partial^* \} \) provided that \( \Theta \) is diagonalizable on \( \mathcal{C} \).

Indeed, \( \Theta \) obviously commutes with \( \partial \) and, thus, they have the common set of eigenvectors. Suppose \( \psi \in \mathcal{C} \) is \( \partial \)-closed vector from this set, such that \( \Theta \psi = q\psi, \quad q \neq 0 \). Then acting on \( \psi \) by operator \( \frac{1}{q} \Theta \) one gets that \( \psi = \frac{1}{q} \partial \partial^* \psi \) is \( \partial \)-exact.

In paper [15] it was observed that if along with the above assumptions operator \( \Theta \) is positive or negative semi-definite, representatives of \( \mathcal{H} \) are in one-to-one correspondence with the elements from the kernel of \( \Theta \).

Let us recall the arguments of [15]. Suppose for definiteness that \( \Theta \) is nonnegative. Then one can define on \( \mathcal{C} \) positive scalar product \( \langle | \rangle \) with respect to which operators \( \partial \) and \( \partial^* \) are mutually Hermitian conjugate \( \partial^* = \partial^* \). One then has that:

1. \( \langle \psi | \Theta | \psi \rangle = \langle \psi | \partial | \partial^* \psi \rangle + \langle \psi | \partial^* | \partial^\dagger \psi \rangle \) and, thus, elements from the kernel of \( \Theta \) a necessarily \( \partial \) and \( \partial^* \)-closed;
2. those \( \partial \)-closed \( \psi \) that are not annulled by \( \Theta \) are \( \partial \)-exact due to the above arguments;
such that any of its rows are not longer than $M$.

Consider subspace $\Xi^p \subset \Xi$ of the $p$-th order monomials in $\xi$. Algebra $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$ is represented on $\Xi^p$ by operators

$$\mathcal{L}_\alpha^\beta = \xi^{\beta\gamma} \frac{\partial}{\partial \xi^{\alpha\gamma}} - \frac{1}{k} \delta_{\alpha}^\beta n_\xi,$$

with quadratic Casimir operator equal to

$$\mathcal{C}^2_{\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)} = \mathcal{L}_\alpha^\beta \mathcal{L}_\beta^\alpha + \mathcal{L}_\alpha^\beta \mathcal{L}_\beta^\alpha = 2k n_\xi - \frac{2}{k} k^2 = 2kp - \frac{2}{k} p^2.$$

In these notation operator $\Theta$ has the following form

$$\Theta = \{\partial, \partial^*\} = -\frac{1}{2} \mathcal{C}^2_{\mathfrak{u}(k,k)} + \frac{1}{2} \mathcal{C}^2_{\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)} + \frac{1}{k} (D+p)(D+p-k^2) + \frac{1}{k} Z^2,$$

where $p$ is the rank of differential form, $D$ and $Z$ are dilatation and helicity operators represented in $M$, $\mathcal{C}^2_{\mathfrak{u}(k,k)}$ is quadratic $\mathfrak{u}(k,k)$-Casimir operator \[^3\] calculated on the module $M$ and $\mathcal{C}^2_{\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)}$ is quadratic $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-Casimir operator \[^{18}\] calculated on the module $\mathfrak{C} = M \otimes \Xi$, where both $M$ and $\Xi$ are considered as $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-modules, i.e.

$$\mathcal{C}^2_{\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)} = ((\text{ad}_\xi)_{\alpha}^\beta + \hat{\mathcal{L}}_\alpha^\beta)((\text{ad}_\xi)_{\beta}^\alpha + \hat{\mathcal{L}}_\beta^\alpha) + ((\text{ad}_\xi)_{\alpha}^\beta + \hat{\mathcal{L}}_\alpha^\beta)((\text{ad}_\xi)_{\beta}^\alpha + \hat{\mathcal{L}}_\beta^\alpha)$$

Note that all the ingredients of (C.6) commute and their common eigenvectors form basis diagonalizing $\Theta$.

For subsequent analysis we also need to study $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-tensorial structure of $\Xi^p$. To this end consider the basis element of $\Xi^p$

$$\Xi(p)^{\alpha_1 \dot{\alpha}_1 \ldots \alpha_p \dot{\alpha}_p} = \xi^{\alpha_1 \dot{\alpha}_1} \ldots \xi^{\alpha_p \dot{\alpha}_p}.$$

Since $\xi$-s anticommute one can easily see that symmetrization of any group of undotted indices of $\Xi(p)$ imply antisymmetrization of corresponding group of dotted indices and conversely antisymmetrization of any group of dotted indices imply symmetrization of corresponding group of undotted indices. Therefore, if undotted indices are projected to obey some symmetry conditions\[^4\] corresponding to Young tableau $\mathcal{Y}$ with rows of lengths $\lambda_1, \ldots, \lambda_k$, $\lambda_1 + \cdots + \lambda_k = p$, dotted indices are automatically projected to obey symmetry conditions corresponding to Young tableau $\mathcal{Y}^T$ with columns of heights $\lambda_1, \ldots, \lambda_k$. Note that all rows of $\mathcal{Y}$ are required to be not greater than $k$ since in opposite case antisymmetrization of more than $k$ dotted indices implied. On the other hand projection of undotted indices of $\Xi(p)$ to symmetry conditions corresponding to any Young tableau not longer than $k$ (i.e. such that any of its rows are not longer than $k$) leads to nonzero result.

\[^3\] For instance that can be done by contracting undotted indices of $\Xi(p)$ with the $\mathfrak{sl}(k)$-tensor of symmetry type under consideration.
Let us define operation of transposition $T$ that maps Young tableau $Y$ with rows $\lambda_1, \ldots, \lambda_k$ to Young tableau $Y^T$ with columns $\lambda_1, \ldots, \lambda_k$ and let $\mathcal{Y}^T$ be called the transpose of $\mathcal{Y}$. In these notation decomposition of $\Xi(p)$ into the $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-irreducible components can be written as

\begin{equation}
\Xi(p) : \bigoplus_{\mathcal{Y}^{p,k}} \text{upper} Y^{p,k} \bigoplus_{\mathcal{Y}^{p,k}} \text{lower} (\mathcal{Y}^{p,k})^T,
\end{equation}

where $\mathcal{Y}^{p,k}$ denote any Young tableau of length not longer than $k$ and with the total number of cells equal to $p$.

Using formulas (A.13) and (A.14) one can calculate Casimir operator $\xi_{\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)}$ of the $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-representations corresponding to Young tableaux listed in decomposition (C.9). It can be easily seen that terms of (A.13) depending on lengths of rows of $\mathcal{Y}^{p,k}$ are cancelled out by terms of (A.14) depending on heights of columns of $(\mathcal{Y}^{p,k})^T$ and one finally gets the same result as in (C.5).

### C.1. Gauge sector

Consider co-chain complex $\mathcal{C}_s = (\mathcal{C}_s, \sigma_-)$, where $\mathcal{C}_s = \Xi \otimes \mathcal{M}_s$ is the space of differential forms taking values in $\mathfrak{su}(k,k)$-adjoint module $\mathcal{M}_s$ and operator $\sigma_- = \xi_{\mathfrak{ad}_P,\mathfrak{a}_\beta}$ is differential. $\mathcal{C}_s$ is obtained from the above consideration if one sets $M$, $P_{a\beta}$ and $K_{a\beta}$ equal to $\mathcal{M}_s$, $(\mathfrak{ad}_P)_{a\beta}$ and $(\mathfrak{ad}_K)_{a\beta}$ correspondingly. To coordinate notation let homotopy $\partial^* = \frac{\partial}{\partial \xi_{a\beta}}(\mathfrak{ad}_K)_{a\beta}$ be denoted as $\sigma^*_-$ in this case.

Note that we should also require reality of $\mathcal{C}_s$, i.e.

\begin{equation}
\zeta(\omega_s) = \omega_s,
\end{equation}

where $\omega_s \in \mathcal{C}_s$ and $\zeta$ is given by (1.10) and

\begin{equation}
\zeta(\xi_{a\beta}) = \xi_{\beta a}.
\end{equation}

However one can ignore (C.10) until all $\sigma_-\text{-cohomology}$ are found. Indeed, suppose (C.10) disregarded. If $\mathcal{H}_s^p$ is some $\sigma_-\text{-cohomology}$, $\zeta(\mathcal{H}_s^p)$ is also $\sigma_-\text{-cohomology}$ since differential $\sigma_-$ is real. So combinations $\mathcal{H}_s^p + \zeta(\mathcal{H}_s^p)$ give us all real $\sigma_-\text{-cohomology}$.

Let $\mathcal{C}_s^p$ denote subspace of $p$-forms in $\mathcal{C}_s$. Consider such a scalar product on $\mathcal{C}_s$ with respect to which involution $\tau$ (A.8) redefined on $\xi_{a\beta}$ by

\begin{equation}
\tau : \xi_{a\beta} \leftrightarrow \frac{\partial}{\partial \xi_{a\beta}}
\end{equation}

plays a role of Hermitian conjugation. Such scalar product is obviously positive definite.$^{10}$

Due to (B.9) and (C.12) operators $\sigma_-$ and $-\sigma^*_-$ are mutually $\tau$-conjugate and, thus, $\Theta = \{\sigma_-, \sigma^*_-\}$ is negative semidefinite. Therefore $\sigma_-\text{-cohomology}$ $\mathcal{H}_s$ coincide with the kernel of operator $\Theta$, which can be found by analyzing those elements of $\mathcal{C}_s$ that correspond to maximal eigenvalues of $\Theta$.

According to (C.3) with the value of $\mathcal{C}_s^2$ (see (3.28)) substituted one has

\begin{equation}
\Theta = \frac{1}{2} \xi_{\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)} - (s - 1)(s + 2k - 2) + \frac{1}{k}(\Delta + p)(\Delta + p - k^2).
\end{equation}

If conformal weight $\Delta$ and differential form order $p$ are fixed the maximal value of $\Theta$ corresponds to maximum of $\mathcal{C}_s^2$.

The general element of $\mathcal{C}_s^p$ has the following form

\begin{equation}
\omega_s^p = \omega_s^p \gamma_1 \cdots \gamma_p ; \xi_1 \cdots \xi_n ; a_{\alpha(n_s)} a_{\alpha(n_s)} b_{\beta(n_s)} \bar{b}_{\beta(n_s)},
\end{equation}

$^{10}$Note that it is not $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-invariant. However it does not effect our consideration.
Its $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-tensorial structure is described by Young tableaux found in tensor product of those describing $\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)$-structure of $\Xi(p)$ (see (C.14)) and of $\mathcal{M}_s$ (see (3.16)), i.e.

\[
\omega^p_s : \bigoplus_{\mathcal{Y}^p,k} \begin{array}{c} \text{upper} \\ \bigotimes (\mathcal{Y}^p,k)^T (n_a) \\ \bigotimes (n_b) \end{array},
\]

where $(n)$ denotes one-row Young tableau of length $n$.

So now we need to find such an irreducible component of the tensor product (C.15) which maximizes $\mathcal{C}^2_{\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)}$. Let us first consider undotted part of (C.15). Suppose $\mathcal{Y}^p,k = (\lambda_1, \ldots, \lambda_h)$, i.e. consists of $h \leq k$ rows of lengths $k \geq \lambda_1 \geq \cdots \geq \lambda_h > 0$, $\lambda_1 + \cdots + \lambda_h = p$. As one can readily see from the formula (A.13) for the value of $\mathfrak{sl}(k)$-Casimir operator depending of the rows of Young tableau, the maximal value of $\mathfrak{sl}(k)$-Casimir corresponds to undotted component of (C.15) with the upper row $(n_a)$ symmetrized to the first row of $\mathcal{Y}^p,k$ (i.e. located in the mostly upper manner) and without any contractions done between $\mathcal{Y}^p,k$ and the lower row $(n_b)$. The same arguments true for the dotted part of (C.15) also.

Let such component be denoted as

\[
\mathcal{C}^2_{\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)} = n_a(n_a + 2\lambda_1 - 1) + n_a(n_a + 2h - 1) + n_b(n_b - 1) + n_b(n_b - 1) + 2k(p + s - v - 1) - \frac{1}{k}\left[(p + n_a - n_b)^2 + (p + n_a - n_b)^2\right].
\]

As seeing from (C.17), $\mathcal{C}^2_{\mathfrak{sl}(k) \oplus \mathfrak{sl}(k)}$ does not depend on the shape of Young tableau $\mathcal{Y}^p,k$ except the length of first row $\lambda_1$, the number of rows $h$ and the total number of cells $p$.

Substituting (C.17) to (C.13) and expressing all variables in terms of independent coordinates on $\mathcal{M}_s$ (3.24), (3.25) one gets

\[
\Theta = q(q + \lambda_1 - k - s + 1) + t(t + h - k - s + 1) + v(v + q + t - 2s - 2k + 3).
\]

Due to inequalities $q, t \leq s - v - 1$ and $k \geq 2$ one sees that the last term in (C.18) is non positive and, thus, maximization of $\Theta$ requires $v = 0$. We finally arrive at

\[
\Theta = q(q + \lambda_1 - k - s + 1) + t(t + h - k - s + 1).
\]

Let $\mathcal{H}_s^p \Delta$ denote $p$-th $\sigma_-$-cohomology corresponding to the module $\mathcal{M}_s$ with conformal weight $\Delta$. First consider degenerate case $s = 1$. Since $\mathcal{M}_1$ is trivial and $\sigma_- \equiv 0$ one gets that cohomology $\mathcal{H}_{1,0}^p$ are all real $p$-forms

\[
\mathcal{H}_{1,0}^p = \{\mathcal{Y}^p,k, (\mathcal{Y}^p,k)^T\} + \text{comp. conj.}, \quad p = 0, \ldots, k^2,
\]

where complex conjugated of $\{\mathcal{Y}^p,k, (\mathcal{Y}^p,k)^T\}$ is $\{(\mathcal{Y}^p,k)^T, \mathcal{Y}^p,k\}$. 

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Suppose now that \( s > 1 \). All zeros of \((C.19)\) and corresponding \( \sigma_- \)-cohomology are listed below.

(1) \( v = q = t = 0 \), i.e. \( n_a = n_b = 0 \), \( n_b = n_b = s - 1 \) and \( \Delta = -s + 1 \). Formally \((C.19)\) does not impose any additional limitations on \( \lambda_1 \) and \( h \) (recall that we always require \( \lambda_1, h \leq k \)), but according to argument given in Appendix A (see page 22) traceless tensor identically vanishes if corresponds to Young tableau with total height of some upper and lower columns greater than \( k \). Since due to \((C.16)\) cohomology in this case has form \( \{ Y^{p,k} \otimes b Y, (Y^{p,k})^T \otimes b Y \} \) and \( n_b, n_b > 0 \) for \( s > 1 \) one should require \( \lambda_1, h \leq k - 1 \). So we have

\[
H_{s; s+1}^p = \{ Y^{p,k} Y^{p,k}_\lambda \leq k, h \leq k-1 \} \otimes b (s-1), (Y^{p,k}_\lambda \leq k, h \leq k-1)^T \otimes b (s-1) \} \} + c.c., \quad p = 0, \ldots, (k - 1)^2.
\]

(2) \( v = t = 0, q = s - 1, \lambda_1 = k \), i.e. \( n_a = n_b = 0, n_b = n_b = s - 1, \Delta = 0 \) and complex conjugate \( v = q = 0, t = s - 1, h = k \), i.e. \( n_a = n_b = 0, n_a = n_b = s - 1, \Delta = 0 \). Analogously to item 1 to get nonzero result we additionally require \( h \leq k - 1 \) in first case and \( \lambda_1 \leq k - 1 \) in complex conjugate case. We, thus, have

\[
H_{s; 0}^p = \{ Y^{p,k} Y^{p,k}_\lambda \leq k, h \leq k-1 \} \otimes b (s-1), (Y^{p,k}_\lambda \leq k, h \leq k-1)^T \otimes b (s-1) \} + c.c., \quad p = k, \ldots, k(k - 1).
\]

(3) \( v = 0, q = t = s - 1, \lambda_1 = h = k \), i.e. \( n_a = n_b = s - 1, n_b = n_b = 0, \Delta = s - 1 \). We have

\[
H_{s+1; s}^p = \{ Y^{p,k} Y^{p,k}_\lambda \leq k, h \leq k-1 \} \otimes b (s-1), (Y^{p,k}_\lambda \leq k, h \leq k-1)^T \otimes b (s-1) \} + c.c., \quad p = 2k - 1, \ldots, k^2.
\]

For the case \( k = 2 \) one gets \((C.19)\).

C.2. Weyl sector. Consider co-chain complex \( \tilde{\mathcal{C}}_s = (\tilde{C}_s, \tilde{\sigma}_-), \) where \( \tilde{C}_s = \Xi \otimes \tilde{\mathcal{M}}_s \) is the space of differential forms taking values in \( \mathfrak{su}(k, k) \) twist-adjoint module \( \tilde{\mathcal{M}}_s \) and \( \tilde{\sigma}_- = \xi^{\alpha\beta}(tw_\mathcal{P})_{\alpha\beta} \). Unfortunately the powerful homotopy technic described at the beginning of this section is not applicable in this case. This is because of the sign change in twist transformation \((4.11)\), which breaks mutual Hermitian conjugacy of \( (tw_\mathcal{P})_{\alpha\beta} \) and \( (tw_\mathcal{K})^{\alpha\beta} \) with respect to any positive definite scalar product. Therefore anticommutator of \( \tilde{\sigma}_- \) with homotopy \( \tilde{\sigma}_-^* = \frac{\partial}{\partial \kappa^{\alpha\beta}} (tw_\mathcal{K})^{\alpha\beta} \) is indefinite.\(^{11}\)

For the purposes of the present paper one need to know 0-th and 1-st \( \tilde{\sigma}_- \)-cohomology only and also can fix \( k = 2 \). Let us focus on this case leaving general situation for the future investigation.

Zeroth \( \tilde{\sigma}_- \)-cohomology \( \tilde{\mathcal{H}}_{s; \tilde{\Delta}}^0 \) coincide with the elements of \( \tilde{\mathcal{M}}_s \) annihilated by \( (tw_\mathcal{P})_{\alpha\beta} \). Since module \( \tilde{\mathcal{M}}_s \) is irreducible there is a single element of such kind the one with the lowest conformal weight \((4.23)\)

\[
\tilde{\mathcal{H}}_{s; 2}^0 = C_{\alpha\beta(s+1)} a^{\alpha(s+1)} b^{\beta(s-1)}.
\]

Recall that the basis in \( \tilde{\mathcal{M}}_s \) is \( \tilde{B}_{s}^v = \tilde{g}_{s}^v \tilde{m}_{s-v}(n_a, n_b, n_b) \), \( v = 0, \ldots, s - 1 \), where \( \tilde{m}_{s-v} \) is a monomial of form \((4.12)\) which can be fixed by independent coordinates \((v, q, t)\) \((4.25)\). Since \( k = 2 \) one can rewrite \( \tilde{m}_{s-v} \) as follows

\[
\tilde{m}_{s-v}(n_a, n_b, n_b) = \tilde{x}(n_a) \varepsilon_a(n_a) \varepsilon_b(n_b) \varepsilon_{\tilde{\alpha}}(n_a) \varepsilon_{\tilde{\beta}}(n_b) \varepsilon_{\tilde{\gamma}}(n_a),
\]

\(^{11}\)Note that anticommutator of \( \tilde{\sigma}_- \) and \( \frac{\partial}{\partial \kappa^{\alpha\beta}} (tw_\mathcal{P})^{\alpha\beta} \), where \( (tw_\mathcal{P})^{\alpha\beta} \) is Hermitian conjugate to \( (tw_\mathcal{P})_{\alpha\beta} \), is semi-definite but non-diagonalizable.
where \( b^\alpha = b_\beta \epsilon^{\alpha\beta} \) and \( \epsilon^{\alpha\beta} \) is totally antisymmetric tensor. Such monomial forms \( \mathfrak{sl}(k) \oplus \mathfrak{sl}(k) \) irrep corresponding to the Young tableau \[ \begin{array}{c|c} \text{undotted} & \text{dotted} \\ \hline n_a+n_b & n_b-n_a \end{array} \] (C.26)

In what follows we denote diagrams like (C.26) as \((l_1, l_2)\), where \( l_1 \) and \( l_2 \) are the number of cells of undotted and dotted rows correspondingly.

Decompose operator \( \tilde{\sigma}_- \) into the sum of three operators in accordance with their action on basis element \( \tilde{B}_s^{\mathfrak{c}} \). Namely operator \( \tilde{\sigma}_- (\tilde{\sigma}_+^\top) \) decrease (increase) \( v \) by 1 and \( \tilde{\sigma}_0^- \) does not change it

\[
\tilde{\sigma}_-(\tilde{g}_s v \tilde{m}_{s-v}) = \tilde{g}_s v \xi^{\alpha\beta} \epsilon_{\gamma\alpha} \tilde{\Pi} b^\gamma \frac{\partial}{\partial b^\beta} \tilde{m}_{s-v},
\]

(C.27)

\[
\tilde{\sigma}_0^-(\tilde{g}_s v \tilde{m}_{s-v}) = \tilde{g}_s v \xi^{\alpha\beta} \tilde{\Pi} (v \varphi(n) + 1) \frac{\partial^2}{\partial \alpha^\alpha \partial \beta^\beta} \tilde{m}_{s-v},
\]

\[
\tilde{\sigma}_+(\tilde{g}_s v \tilde{m}_{s-v}) = \tilde{g}_s v (2s - v) \varphi(n) \varphi(\tilde{n}) \xi^{\alpha\beta} \frac{\partial^2}{\partial \alpha^\alpha \partial \beta^\beta} \tilde{m}_{s-v},
\]

where \( \varphi(n) = 1/(n+2) \) and \( \tilde{\Pi}^\top \) are projectors\(^\text{13} \) given by (4.17). From the nilpotency of \( \tilde{\sigma}_- \) it follows that

\[
(\tilde{\sigma}_-)^2 = (\tilde{\sigma}_+^\top)^2 = 0, \quad (\tilde{\sigma}_0^-)^2 + \{\tilde{\sigma}_-, \tilde{\sigma}_+^\top\} = 0.
\]

Representative of \( \tilde{\sigma}_- \)-cohomology can always be chosen to have definite conformal weight and definite irreducible \( \mathfrak{sl}(k) \oplus \mathfrak{sl}(k) \)-structure. General element of \( \Xi^p \otimes \tilde{\mathcal{M}}_{v, \Delta} \) with fixed conformal weight \( \tilde{\Delta} \) can be decomposed as

\[
\tilde{\mathcal{F}}_p_{s, \Delta} = \sum_{v=v_{\text{min}}}^{v=v_{\text{max}}} \tilde{\mathcal{F}}_{p,v}^{s, \Delta},
\]

where summand \( \tilde{\mathcal{F}}_{p,v}^{s, \Delta} = \Xi^p \otimes \tilde{g}_s (\tilde{m}_{s-v} + \tilde{m}'_{s-v} + \cdots) \) is a linear combination of basis elements with fixed \( v \) and \( \tilde{\Delta} \) tensored by \( \Xi^p \) and \( v \) vary from \( v_{\text{min}} \) to \( v_{\text{max}} \).

Within this decomposition \( \tilde{\sigma}_- \)-closedness condition for \( \tilde{\mathcal{F}}_{p,v}^{s, \Delta} \) split into the system

\[
\begin{align*}
\tilde{\sigma}_- \tilde{\mathcal{F}}_{p,v}^{s, \Delta} &= 0, \\
\tilde{\sigma}_- \tilde{\mathcal{F}}_{p,v+1}^{s, \Delta} + \tilde{\sigma}_0^- \tilde{\mathcal{F}}_{p,v}^{s, \Delta} &= 0, \\
\tilde{\sigma}_- \tilde{\mathcal{F}}_{p,v+2}^{s, \Delta} + \tilde{\sigma}_0^- \tilde{\mathcal{F}}_{p,v+1}^{s, \Delta} + \tilde{\sigma}_- \tilde{\mathcal{F}}_{p,v}^{s, \Delta} &= 0, \\
&\vdots \\
\tilde{\sigma}_- \tilde{\mathcal{F}}_{p,v}^{s, \Delta} + \tilde{\sigma}_0^- \tilde{\mathcal{F}}_{p,v+1}^{s, \Delta} + \tilde{\sigma}_- \tilde{\mathcal{F}}_{p,v+2}^{s, \Delta} &= 0, \\
\tilde{\sigma}_- \tilde{\mathcal{F}}_{p,v}^{s, \Delta} + \tilde{\sigma}_0^- \tilde{\mathcal{F}}_{p,v+1}^{s, \Delta} + \tilde{\sigma}_- \tilde{\mathcal{F}}_{p,v+2}^{s, \Delta} &= 0.
\end{align*}
\]

\( ^\text{12} \)This Young tableau is obtained from (A.10) by Hodge conjugation \( \{(C.26)\} \).

\( ^\text{13} \)Projector \( \tilde{\Pi}^\top b^\alpha \tilde{m}_{s-v} \) which acts on the oscillator \( b^\alpha \) with risen index carries out symmetrization of \( b^\alpha \) with all \( a \)-s and \( b \)-s in \( \tilde{m}_{s-v} \)

\[
\tilde{\Pi}^\top b^\alpha = \frac{1}{n_b} \left( n_b b^\alpha + a^\alpha b^\beta \frac{\partial}{\partial a^\beta} \right).
\]
According to the first equation of (C.30) \( \tilde{F}_{s:\Delta}^{p;v_{\min}} \) is required to be \( \tilde{\sigma}_- \)-closed. Suppose it is \( \tilde{\sigma}_- \)-exact, i.e. such \( \varepsilon_{s:\Delta+1}^{p-1;v_{\min}+1} \) exists that \( \tilde{F}_{s:\Delta}^{p;v_{\min}} = \tilde{\sigma}_- \varepsilon_{s:\Delta+1}^{p-1;v_{\min}+1} \). Then \( \tilde{\sigma}_- \)-exact shift \( \tilde{F}_{s:\Delta}^p - \tilde{\sigma}_- \varepsilon_{s:\Delta+1}^{p-1} \) zeros out term \( \tilde{F}_{s:\Delta}^{p;v_{\min}} \). So in order for \( \tilde{F}_{s:\Delta}^p \) to be \( p \)-th \( \tilde{\sigma}_- \)-cohomology one can require its term with the lowest value of \( v \) to be \( p \)-th \( \tilde{\sigma}_- \)-cohomology.

Let us find 1-st \( \tilde{\sigma}_- \)-cohomology \( \tilde{h}^{-1} \). Note that unlike the whole operator \( \tilde{\sigma}_- \) operators \( \tilde{\sigma}_-^{\pm,0} \) acting separately map monomials into monomials and, thus, one can look for cohomology \( \tilde{h}^{-1} \) among components of tensor product \( \xi \otimes \tilde{m}_{s'-v} \). These components are described by Young tableaux obtained in tensor product of (C.26) with one undotted and one dotted cells (C.31)

\[
\left( n_a + n_b - 1, n_b - n_a + 1 \right), \left( n_a + n_b - 1, n_b - n_a - 1 \right), \left( n_a + n_b + 1, n_b - n_a + 1 \right), \left( n_a + n_b + 1, n_b - n_a - 1 \right), \left( n_a + n_b - 1, n_b - n_a + 1 \right), \left( n_a + n_b - 1, n_b - n_a - 1 \right),
\]

\[
n_a + n_b \geq 0, \quad n_a + n_b \geq 1, \quad n_a + n_b \geq 0, \quad n_a + n_b \geq 1,
\]

\[
n_b - n_a \geq 0, \quad n_b - n_a \geq 1, \quad n_b - n_a \geq 1, \quad n_b - n_a \geq 0,
\]

Closed if \( n_a = n_b = 0 \) nonclosed, nonclosed, closed,

exact if \( n_b > 0 \).

Closedness and exactness of each component can be easily checked by direct computation. In (C.31) the results are collected.

We thus have two series of \( \tilde{F}_{s:\Delta}^{1;v_{\min}} \) that pretend to contribute to \( \tilde{H}_{s:\Delta}^p \). Let us consider both series separately.

(1) \( n_a = n_b = 0 \). As one can see from \( \tilde{M}_s \) diagram (1.21) the only element of \( \tilde{M}_s \) with \( n_a = n_b = 0 \) is that with coordinates \( (0, 0, 0) \). It has \( n_a = s + 1, n_b = s - 1 \) and the lowest conformal weight \( \Delta = 2 \). One can also see that there are no more elements in \( \tilde{M}_s \) with the same conformal weight. So decomposition (C.29) reduces to one term (C.32)

\[
\tilde{F}_{s:2}^1 = \xi^{\alpha \beta} x_{\alpha(2s+1)}; \beta^{\alpha(s+1)} y^{\alpha(s-1)}.
\]

As one can readily see (C.32) is \( \tilde{\sigma}_- \)-exact (C.33)

\[
\tilde{F}_{s:2}^1 = \tilde{\sigma}_- \varepsilon_{s:3}^0, \quad \varepsilon_{s:3}^0 = x_{\alpha(2s+1)}; \beta^{\alpha(s+2)} y^{\alpha(s-1)} \tilde{\beta}.
\]

Indeed, \( \varepsilon_{s:3}^0 \) is \( \tilde{\sigma}_-^{\pm,0} \)-closed and mapped by \( \tilde{\sigma}_-^0 \) to \( \tilde{F}_{s:2}^1 \).

(2) \( n_b = 0 \). In this case we get monomial \( \tilde{m}_{v_{\min}} \) with coordinates (C.34)

\[
\left( v_{\min}, s - v_{\min} - 1, s - v_{\min} + m - 1 \right), \quad \text{for some fixed} \quad v_{\min} = 0, \ldots, s - 1, \quad \text{and} \quad m = 0, 1, \ldots, \infty
\]

which corresponds to some dot at the upper north-east boundary of the stripe \( v_{\min} \) (see diagram (1.21)). Conformal weight and orders with respect to oscillators of \( \tilde{m}_{v_{\min}} \) are

\[
n_a = 2s - v_{\min} + m, \quad n_a = s - v_{\min} - 1,
\]

\[
n_b = 0, \quad n_b = s + m - 1,
\]

\[
\tilde{\Delta} = 2s - v_{\min} + m.
\]

Therefore \( \tilde{F}_{s:\Delta}^{p;v_{\min}} \) from decomposition (C.29) has the following \( \mathfrak{sl}(k) \oplus \mathfrak{sl}(k) \)-structure (C.36)

\[
(2s - v_{\min} + m - 1, v_{\min} + m + 1).
\]

Suppose \( v_{\min} < s - 1 \). To construct other possible terms from decomposition (C.29) on should find such elements of \( \tilde{M}_s \) that

(1) have the same conformal weight;

(2) their coordinate \( v \) is greater than \( v_{\min} \);
(3) contribute to component (C.36) when tensored by ξ.

Suppose the element ˆm′ we are looking for has coordinates (v′, q′, t′). Let the orders of ˆm′ with respect to oscillators (which are expressed through coordinates via formula (4.26)) be denoted as n′, n′, n′, n′. Since tensoring by ξ either adds or subtracts one cell to/from Young tableau we require

\[(n'_a + n'_b, n'_c - n'_d) = (n_a + n_b ( - 2), n_b - n_a [ + 2]),\]

where n, n, n, n are given in (C.35) and numbers in parenthesis (brackets) could be either skipped or taken into account. Condition (C.37) guarantees that ξ ⊗ ˆm′ contains component (C.36). In terms of coordinates the above requirements give the following system

\[
\begin{align*}
v' + q' + t' + 2 = 2s - v_{\text{min}} + m, \\
v' - q' + t' = v_{\text{min}} - m ( + 2), \\
v' + q' - t' = v_{\text{min}} + m [ + 2].
\end{align*}
\]

The solution is

\[
\begin{align*}
v' = v_{\text{min}} ( + 1) [ + 1], \\
q' = s - v_{\text{min}} + m - 1 ( - 1) = q_{\text{min}} ( - 1), \\
t' = s - v_{\text{min}} - 1 [ - 1] = t_{\text{min}} [ - 1],
\end{align*}
\]

where qmin = s − vmin + m − 1 and tmin = s − vmin − 1 are corresponding coordinates of ˆmmin.

Looking at the diagram (4.21) one sees that if we take into account both numbers (in parenthesis and brackets), vmin increases by 2 and, thus, the width of the stripe decreases by 2 while tmin decreases by 1 only. Since monomial ˆmmin is at the boundary of the stripe vmin monomial ˆm′ is out of the boundary of stripe vmin + 2, i.e. does not belong to M. The same situation occurs if one takes into account the number in parenthesis only. So the only possibility is

\[
\begin{align*}
v' = v_{\text{min}} + 1, \\
q' = q_{\text{min}}, \\
t' = t_{\text{min}} - 1 \\
v_{\text{min}} < s - 1,
\end{align*}
\]

\[
\begin{align*}
n'_a = n_a, \\
n'_b = 0, \\
n'_c = n_c - 1, \\
n'_d = n_d + 1.
\end{align*}
\]

In this case ˆF1s,Δ is

\[
\begin{align*}
\hat{F}^1_{s,\Delta} = &\tilde{g}_s v_{\text{min}} \xi^\gamma \beta \tilde{\alpha} (n_a - 1); \beta (n_b - n_c + 1) \epsilon \alpha \gamma \epsilon \alpha \beta \cdot \cdot \cdot \epsilon \alpha \beta a^\alpha (n_d) \tilde{\beta} (n_c) \tilde{\alpha} (n_d) + \\
&+ \tilde{g}_s v_{\text{min}} + 1 \xi^\gamma \beta \tilde{\alpha} (n_a - 1); \beta (n_b - n_c + 1) \epsilon \alpha \gamma \epsilon \beta \cdot \cdot \cdot \epsilon \alpha \beta a^\alpha (n_d) \tilde{\beta} (n_c + 1) \tilde{\alpha} (n_d - 1),
\end{align*}
\]

where n, n, n are fixed by (C.35). As one can directly see such ˆF1s,Δ is not ˆσ_{−}-closed.

So we are finally left with the case vmin = s − 1, when

\[
\begin{align*}
n_a = s + m + 1, \\
n_c = 0, \\
m = 0, 1, \ldots, \infty
\end{align*}
\]

\[
\Delta = s + m + 1
\]

and

\[
\hat{F}^1_{s,s+m+1} = \tilde{g}_s^{s-1} \xi^\gamma \beta \tilde{\alpha} (s+m); \beta (s+m) \epsilon \alpha \gamma a^\alpha (s+m+1) \tilde{\beta} (s+m+1).
\]

\[
\hat{F}^1_{s,s+m+1}
\]

is obviously not ˆσ0_{−}-closed for m > 0. For m = 0 this is the case since ˆF1s,s+1 is composed of terminal monomial which zero out by ˆσ0_{−} due to the arguments explained in Appendix B.2 (see formula (B.17)).
By means of analogous analysis checking $\varepsilon^0_{s;s+2} s$ that could have contributed to $\tilde{\mathcal{F}}^1_{s;s+1}$ one can show that $\tilde{\mathcal{F}}^1_{s;s+1}$ is not $\tilde{\sigma}_-$-exact. Therefore

$$\tilde{\mathcal{H}}^1_{s;s+1} = \tilde{g}^{s-1}\xi_{\gamma\beta} (s) \varepsilon_{\alpha(s)} \beta(s) \varepsilon_{\alpha(s)} \varepsilon^{\alpha(s+1)} \beta(s-1).$$

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