Determining the spin Hall conductance via charge transport

Sigurdur I. Erlingsson and Daniel Loss

Department of Physics and Astronomy, University of Basel, Klingelbergstrasse 82, CH-4056, Switzerland

We propose a scheme where transport measurements of charge current and its noise can be used to determine the spin Hall conductance in a four-terminal setup. Starting from the scattering formalism we express the spin current and spin Hall conductance in terms of spin-dependent transmission coefficients. These coefficients are then expressed in terms of charge current and noise. We use the scheme to characterize the spin injection efficiency of a ferromagnetic/semiconductor interface.

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The spin Hall effect (SHE) due to spin-orbit interaction is one of many spintronics phenomena currently under intense study. Initially the SHE was considered as an impurity driven effect referred to as the extrinsic SHE. More recently, theoretical work predicting an intrinsic version of the SHE has sparked renewed interest in this phenomenon. Subsequent papers have dealt with both the effects of impurities and finite size effects within the scattering formalism on the SHE. Recently the extrinsic SHE was observed in electron doped systems, where the spin accumulation at the sample edges was detected via Kerr rotation and from this the spin current was inferred. Also, observation of the intrinsic SHE in two dimensional hole systems has been reported.

The characterization of spin dependent transport via current-current correlations (noise) has received considerable interest recently as well. The charge current noise was proposed as a means to determine spin entanglement, and the full counting statistics of spin entangled electrons was considered. On the other hand, the spin current noise was proposed as a way of probing interactions and the full counting statistics of spin currents were investigated.

In this letter we propose a method of using charge current and charge current noise to determine the spin current and, in particular, the spin Hall conductance in a four-terminal setup. Our proposed scheme is quite flexible and can be applied to many different systems as long as they can be described by the scattering formalism. Quite remarkably, the scheme does not require the detection of individual spin components of the current, and both the spin currents and spin conductances can be expressed in terms of electrically detected quantities, i.e. the charge current and noise.

Spin Hall effect. In extended systems with spin-orbit coupling the spin current density is defined as \( j^\sigma(r) = \frac{1}{\hbar} \langle j_z(r), \hat{\sigma}_\sigma \rangle \), where \( j_z(r) \) is the particle current density operator and \( \hat{\sigma}_\sigma \) is a Pauli matrix. The problem of determining the spin current density is still not clarified.

We consider a four-terminal device where a charge current is driven by an applied voltage between two leads, e.g. lead 1 and 3 in the setup depicted in Fig. and at the same time measuring the voltage, or current, which develops in the two transverse leads 2 and 4. Spin-orbit interaction in the sample plays the role of the magnetic field in the standard Hall setup and gives rise to a spin current flowing perpendicularly to the charge current driven by the applied voltage and thus the spin current is well defined. In the scattering formalism the spin current then becomes

\[
\langle I^\alpha_\alpha \rangle = \frac{e}{h} \sum_\beta \int dE \mathrm{Tr} \{ \hat{\sigma}_z \hat{s}_{\alpha\beta} \hat{s}_{\alpha\beta}^\dagger \} (f_\alpha - f_\beta),
\]

where \( \hat{s}_{\alpha\beta} \) are scattering matrix elements between leads \( \alpha \) and \( \beta \), and \( f_\alpha = f(E - \mu_\alpha) \) is the Fermi function in lead \( \alpha \) at chemical potential \( \mu_\alpha \). The charge current \( \langle I^\alpha_\alpha \rangle \) in lead \( \alpha \) is given by a relation similar to Eq. \( \text{(1)} \), with \( \hat{\sigma}_z \) replaced by the identity. The energy argument of the Fermi function has been suppressed, similarly for the scattering matrices \( \hat{s}_{\alpha\beta} \) which in general also depend on energy. The hat denotes the spin structure of the scattering matrix

\[
\hat{s}_{\alpha\beta} = \begin{pmatrix} T^{\uparrow\uparrow}_{\alpha\beta} & T^{\uparrow\downarrow}_{\alpha\beta} \\ T^{\downarrow\uparrow}_{\alpha\beta} & T^{\downarrow\downarrow}_{\alpha\beta} \end{pmatrix},
\]

where \( s_{\alpha\beta}^{ss'} \) denotes scattering of a transverse mode in lead \( \beta \) with spin \( s' \) to a transverse mode in lead \( \alpha \) with spin \( s \). Note that each element \( s_{\alpha\beta}^{ss'} \) in Eq. \( \text{(2)} \) is a \( N_\alpha \times N_\beta \) matrix in transverse mode space. If lead \( \alpha \) has only one single transverse mode, the matrix product in Eq. \( \text{(2)} \) becomes

\[
\hat{s}_{\alpha\beta} \hat{s}_{\alpha\beta}^\dagger = \begin{pmatrix} T^{\uparrow\uparrow}_{\alpha\beta} & T^{\uparrow\downarrow}_{\alpha\beta} \\ (T^{\downarrow\uparrow}_{\alpha\beta})^\dagger & T^{\downarrow\downarrow}_{\alpha\beta} \end{pmatrix},
\]

where the transmission coefficients \( T_{\alpha\beta}^{ss'} \) are numbers, possibly complex for the off-diagonal ones, irrespective of the number of transverse modes in lead \( \beta \). As long as the spin quantization axis of all leads is chosen to be the...
same the off-diagonal matrix elements in Eq. 3 are real [27]. The spin and charge current in lead \( \alpha \) is thus

\[
\langle I^{c/e}_\alpha \rangle = \frac{e}{h} \sum_\beta \int dE (T_{\alpha\beta}^{\uparrow\uparrow} + T_{\alpha\beta}^{\downarrow\downarrow}) (f_\alpha - f_\beta).
\]

In the following we consider the linear transport regime, where the conductances are given by spin dependent transmission coefficients, similar to the ones in Eq. 3. Knowing these allows a complete determination of the charge and spin currents. To illustrate this we consider the SHE in a four-terminal setup, see Fig. 1. We recall that in the infinite system, the (electric field driven) charge current induces a pure spin current (with no accompanying charge current) flowing in perpendicular direction [3, 4, 5, 6, 7, 8, 9, 10, 11, 23, 24, 25]. To achieve such a situation also for the setup in Fig. 1 a positive voltage \( V_1 \) is applied to lead 1 and a negative one \( V_3 \) to lead 3. This results in a charge current flowing from lead 1 to 3 and the values of \( V_1 \) and \( V_3 \) are set by the requirement of zero transverse charge current in lead 4:

\[
\langle I_4^c \rangle = \frac{e^2}{h} \sum_{p=\pm} (T_{41;p}(V_4 - V_1) + T_{43;p}(V_4 - V_3)) = 0.
\]

\( \langle I_2^c \rangle \) is left unspecified, see also below Eq. 3. Here we have introduced the transmission eigenvalues \( T_{41;p} \) and \( T_{43;p} \) of Eq. 3 for \( \beta = 1 \) and 3, resp. In terms of these eigenvalues both the current and noise will take on a simple form [24]. The quantum point contacts (QPC) are used to ensure that only a single transverse mode couples the scattering region to leads 2 and 4. Taking into account the constraint imposed by Eq. 3, the spin current flowing in lead 4 takes the form

\[
\langle I_4^s \rangle = \frac{e^2}{h} \left( (T_{41}^{\uparrow\uparrow} - T_{41}^{\downarrow\downarrow}) - \frac{g_{41}}{g_{43}} (T_{43}^{\uparrow\uparrow} - T_{43}^{\downarrow\downarrow}) \right) (V_4 - V_1),
\]

where \( g_{\alpha\beta} = \sum_p T_{\alpha\beta;p} \geq 0 \) is the dimensionless conductance between leads \( \alpha \) and \( \beta \). To proceed further we make the following assumptions about the spin-orbit interaction. The spin state labeled ‘up’ is predominantly scattered to the right, with respect to its propagation direction. This results in \( (T_{41}^{\uparrow\uparrow} - T_{41}^{\downarrow\downarrow}) > 0 \) and \( (T_{43}^{\uparrow\uparrow} - T_{43}^{\downarrow\downarrow}) < 0 \), since electrons going from lead 3 to 4 ‘turn left’ with respect to their propagation direction. This assumption is not very restrictive and is satisfied by various known spin-orbit scattering mechanisms [3, 4, 5, 6, 7, 8, 9, 10, 11, 23, 24, 25]. With this we write the spin current as

\[
\langle I_4^s \rangle = \frac{e^2}{h} \left( (T_{41}^{\uparrow\uparrow} - T_{41}^{\downarrow\downarrow}) + \frac{g_{41}}{g_{43}} (T_{43}^{\uparrow\uparrow} - T_{43}^{\downarrow\downarrow}) \right) (V_4 - V_1).
\]

Unless one knows the details of the spin-orbit scattering one cannot tell whether spin \( \uparrow \) or \( \downarrow \) is scattered more to the right and so the direction (sign) of the spin current is unknown. From the spin current in Eq. 7 we define the spin Hall conductance as \( \langle J^s \rangle \) as a function of the spin dependent transmission coefficients. From this we also see that the maximal possible spin Hall conductance is \( g_{11}^s = e/4\pi \), this bound resulting from the QPC’s single transverse mode. As we will show below, the spin dependent transmission coefficients \( T_{\alpha\beta}^{s,s} \) can be determined using standard transport measurements of charge current and noise. In Eq. 7 no assumption was made about the sample symmetry. In general it is not possible to simultaneously ensure \( \langle J_2^c \rangle = 0 \) and \( \langle J_4^c \rangle = 0 \), due to asymmetries. However, this is not critical since \( G_H^s \) is defined using only one of the two transverse leads. For symmetric structures \( g_{43} = g_{41} \) and \( |T_{41}^{\uparrow\uparrow} - T_{41}^{\downarrow\downarrow}| = |T_{43}^{\uparrow\uparrow} - T_{43}^{\downarrow\downarrow}| \), which reduces the number of measurements required to determine \( T_{\alpha\beta}^{s,s} \). This symmetry will be broken if a perpendicular magnetic field is applied [28].

Let us now determine \( T_{41}^{s,s} \). With all leads at chemical potential \( \mu_0 \), except for lead 1 where a bias \( V_1 \) is applied, the charge current in lead 4 is written in terms of the transmission eigenvalues as

\[
\langle I_4^c \rangle = \frac{e^2}{h} \sum_{p=\pm} T_{41;p} (V_4 - V_1).
\]

Next we introduce the excess noise \( \Delta S_{\alpha\beta} \), which is the total noise \( S_{\alpha\beta} \) minus the Johnson-Nyquist contribution

\[
2kT_e \frac{e}{h} g_{\alpha\beta}.
\]

The excess noise in lead 4 be-
comes (voltage $V_1$ applied to lead 1)

$$\Delta S_{44} = \frac{2e^2}{h} \int dE \text{Tr}\{\hat{s}_{41}\hat{s}_{41}^\dagger(1-\hat{s}_{41}\hat{s}_{41}^\dagger)}(f_1 - f_0)^2$$
$$= \frac{2e^3}{h}V_{1}F\left(\frac{eV_{1}}{k_{B}T}\right) \sum_{p=\pm} T_{41;p}(1 - T_{41;p}), \quad (10)$$

where $f_0 = f(E-\mu_0)$ and $F(x) = \coth(x/2) - 2/x$ [34]. In the following we will assume that the transmission eigenvalues do not change much within the bias window, but apart from that only the unitarity of the scattering matrix was used to derive Eq. (10). From the charge current ($I_{4}^{\uparrow}$) and Eq. (10) we obtain

$$T_{41;\pm} = \frac{1}{2}g_{41} \pm \frac{1}{2}\sqrt{g_{41}(2-g_{41})} - \frac{h\Delta S_{44}}{e^3|V_1|}, \quad (11)$$

The conductance is determined by the current, $g_{41} = \frac{\hbar}{2e^2}$. For simplicity we have assumed $eV_{1}/k_{B}T \gg 1$ but the results are easily generalized to arbitrary temperatures by $|eV_{1}| \rightarrow eV_{1}F(eV_{1}/k_{B}T)$.

In terms of $\delta T_{41} = T_{41,+} - T_{41,-}$ the excess noise in Eq. (10) can be written as $\Delta S_{44} = \frac{e^3|V_1|}{h}(g_{41}(2-g_{41}) - \delta T_{41}^2) > 0$. This relation shows that the excess noise $\Delta S_{44}$ decreases with increasing $\delta T_{41}$, ensuring that the argument of the square root in Eq. (10) is positive.

According to Eq. (10) the spin Hall conductance is in part determined by $|T_{41}^{\uparrow} - T_{41}^{\downarrow}|$. This quantity can be written in terms of the transmission eigenvalues as

$$|T_{41}^{\uparrow} - T_{41}^{\downarrow}| = \sqrt{(T_{41,+} - T_{41,-})^2 - 4|T_{41}^{\uparrow}|^2}. \quad (12)$$

If the off-diagonal terms $T_{\alpha\beta}^{\sigma\sigma'}$ vanish then the spin current Eq. (11) is completely determined by the transmission eigenvalues, i.e. $|T_{41}^{\uparrow} - T_{41}^{\downarrow}| = |T_{41,+} - T_{41,-}|$. When $|T_{41}^{\uparrow}| \neq 0$, additional measurements are necessary. With a voltage $V_2$ applied to lead 2 the charge current and excess noise in lead 4 are given by

$$\langle I_{4}^{\uparrow} \rangle = \frac{e^2V_{2}}{h} \sum_{p=\pm} T_{42;p}, \quad (13)$$

$$\Delta S_{44} = \frac{2e^3|V_2|}{h} \sum_{p=\pm} T_{42;p}(1 - T_{42;p}). \quad (14)$$

For non-magnetic leads and forward scattering (from lead 4 to 2) we have $T_{42}^{\uparrow} = T_{42} \pm |T_{42}^{\uparrow}|$. Assuming that $T_{42}^{\uparrow}$ is real we get

$$|T_{42}^{\uparrow}| = \frac{1}{2}\sqrt{g_{42}(2-g_{42}) - \frac{h\Delta S_{44}}{e^3|V_2|}}. \quad (15)$$

Applying a voltage $V_4$ to lead 4, the charge current cross-correlation between leads 2 and 1 gives $T_{41}^{\uparrow}$ via the relation

$$\Delta S_{21} = -\frac{2e^2}{h} \int dE \text{Tr}\{\hat{s}_{42}\hat{s}_{42}^\dagger\hat{s}_{41}\hat{s}_{41}^\dagger\}(f_4 - f_0)^2$$
$$= -\frac{2e^3V_{4}}{h} \frac{1}{2}(g_{42}g_{42} + 2T_{42}^{\uparrow}T_{41}^{\uparrow}). \quad (16)$$

In deriving Eq. (10) time reversal symmetry was assumed, i.e. in the absence of a magnetic field $B$. [We note that this can be generalized to $B \neq 0$ by incorporating the appropriate symmetries of the scattering matrix $s_{\alpha\beta}(+B) = (i\Sigma_{y;\alpha})^{-1}(\hat{s}_{\alpha\beta}(-B))^{*}(i\Sigma_{y;\beta})$ where $i\Sigma_{y;\alpha} = \begin{pmatrix} 0 & \mathbb{I}_G \\ -\mathbb{I}_G & 0 \end{pmatrix}$ comes from the time reversal operation.] The off-diagonal element can thus be expressed in terms of charge transport quantities

$$4|T_{41}^{\uparrow}|^2 = \frac{g_{42}(2-g_{42}) + g_{42}g_{42}}{g_{42}(2-g_{42}) - g_{42}g_{42}} \frac{-\frac{h\Delta S_{44}}{e^3|V_2|}}{g_{42}(2-g_{42}) - g_{42}g_{42}}. \quad (17)$$

Here we have added an argument to each noise measurement to indicate that, e.g., $\Delta S_{44}(V_2)$ corresponds to the excess noise in lead 4 with a finite voltage $V_2$ applied only to lead 2, again assuming $T_{42}^{\uparrow}$ to be constant [34].

With the same procedure we obtain $T_{43}^{\uparrow}$: (i) apply voltage $V_3$ to lead 3 and measure $\Delta S_{44}$, which gives $T_{43;\pm}$, (ii) apply voltage $V_4$ to lead 4 and measure $\Delta S_{23}$, which gives $|T_{43}^{\uparrow}|$. All these measured quantities taken together yield the spin Hall conductance from Eq. (9). The spin Hall conductance $G_{H}^{s}$ is determined by

$$G_{H}^{s} = \frac{e}{4\pi} \sum_{\beta=1,3} \frac{1}{g_{44}^{-1} - g_{44}} \left\{ g_{44}(2-g_{44}) + g_{44}g_{44} \right\} \frac{h\Delta S_{44}(V_{\beta})}{e^3|V_{\beta}|} \left( \frac{g_{42}(2-g_{42}) - g_{42}g_{42}}{g_{42}(2-g_{42}) - g_{42}g_{42}} \right)^{\frac{1}{2}}. \quad (18)$$

This equation is the central result of our paper from which we see that the spin Hall conductance $G_{H}^{s}$ can be solely obtained by charge transport measurements without the need of any spin dependent detection. This result applies equally to the extrinsic and intrinsic spin Hall effect, with no size restriction on the SO scattering region. Our scheme is restricted to only a single transverse mode in the QPCs. In case of more transverse modes, but still few, numerical analysis can be used to extract information about the transmission eigenvalues [32]. However, since the spin polarization only comes from the ‘active region’ we expect that opening up the QPC (for larger current) should not greatly change the amount of spin polarization determined for the single mode case.

**Spin injection.** Here we consider a hetero-junction, consisting of a semiconductor and a magnetic material (ferromagnet or magnetic semiconductor) which serves as a spin injector, see Fig. (2). This setup allows the characterization of the spin injection from lead $L$ to lead
C. The QPC, which is assumed to have known scattering properties, ensures a single transverse mode in the detection lead \(R\). The number of transverse modes in leads \(C\) and \(L\) is arbitrary, so the area of the interface cross-section is not constrained. The spin quantization axis is chosen along the magnetization direction of the ferromagnet and it is assumed that \(T_{LR}^{\uparrow\downarrow} \ll T_{LR}^{\uparrow\uparrow}, T_{LR}^{\downarrow\downarrow}\), which is a good approximation for ferromagnets. \(^{27}\) In this case only a single charge current and noise measurement is needed to obtain \(|T_{LR}^{\uparrow\downarrow} - T_{LR}^{\downarrow\uparrow}|\), from which the polarization \(p\) of the injected current is obtained from Eqs. \(^{8}, 11\) and \(^{12}\) (replacing the subscripts 4, 1 with \(L, R\))

\[
p = \frac{\langle I_{R} \rangle}{\langle I_{L} \rangle} = \frac{1}{g_{LR}} \sqrt{\frac{2 - g_{LR}}{g_{LR}}} \frac{\hbar \Delta S_{RR} V_L}{e^2 |V_L|},
\]

where \(V_L\) is the bias applied at lead \(L\). Since the scattering from \(C\) to \(R\) is assumed to be non-spin-selective, it is a reasonable assumption that the polarization obtained via Eq. \(^{14}\) characterizes the spin polarization of the interface between leads \(L, C\), even for a QPC with many transverse modes.

In summary, we have introduced a scheme using charge current and noise measurements to extract spin polarization, without the need of spin-resolved measurements. The theory is based on the scattering formalism which makes this scheme quite flexible and applicable to many different systems.

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\(^{[34]}\) Applying the voltage symmetrically around \(\mu\) the component of the transmission coefficient which is linear in energy vanishes upon integration, and deviations \(\delta T_{\alpha\beta}^{\pm}(E)\) are at least quadratic in energy away from \(\mu\). To resolve the spin dependence of the transmission coefficients we require that \(\delta T_{\alpha\beta}^{ss} \approx \delta T_{\alpha\beta}^{\pm}\), where \(\delta T_{\alpha\beta}^{ss} \approx \delta T_{\alpha\beta}^{\pm}\).
is the maximum deviation within the bias window.