Property $(FL_p)$ implies property $(FL_q)$ for $1 < q < p < \infty$

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1 Abstract

It is known that for $\sigma$-compact groups Kazhdan’s Property $(T)$ is equivalent to Serre’s Property $(FH)$. Generalized versions of those properties, called properties $(T_B)$ and $(F_B)$, can be defined in terms of the isometric representations of a group on an arbitrary Banach space $B$. Property $(F_B)$ implies $(T_B)$.

It is known that a group with Property $(T_l_p)$ shares some properties with Kazhdan’s groups, for example compact generation and compact abelianization. Moreover in the case of discrete groups, Property $(T_l_p)$ implies Lubotzky’s Property $(\tau)$.

In this paper we prove that in the case of discrete groups and $1 < p < q < \infty$, Property $(F_l_q)$ implies Property $(F_l_p)$.

2 Introduction

Property $(T)$ introduced by Kazhdan \cite{Kaz} in terms of unitary representations, became a fundamental rigidity property of groups, with wide range of applications. It was proved by Delorme \cite{De} and Guichardet \cite{Gu} that Kazhdan’s Property $(T)$ is equivalent to Serre’s Property $(FH)$ for $\sigma$-compact groups. Generalized versions of property $(T)$ and property $(FH)$, called properties $(T_B)$ and $(F_B)$, were introduced in \cite{BFGM} by Bader, Furman, Gelander and Monod, in terms of isometric representations of a group on an arbitrary Banach space $B$. Groups with those properties share some important properties with Kazhdan’s groups, for example groups with property $(T_l_p)$ are compactly generated and have compact abelianization \cite{BO}.

In this article, we study property $(F_l_q)$, a fixed point property for affine actions on $L_p(X,\mu)$ spaces, where $X$ is purely atomic countable space.

Our main result is the following.

**Theorem 1.** For every discrete countable group and $1 < p < q < \infty$, property $(F_l_q)$ implies $(F_l_p)$.
Fixed points properties for groups acting on $L_p$ spaces for $p > 2$ are poorly investigated. Below we present results which summarize our knowledge.

In $[BFGM]$ authors proved that higher rank algebraic groups and theirs lattices have fixed points for every affine action on $L_p$ spaces for $p > 1$. Mimura $[Mi]$ showed that $SL_n(\mathbb{Z}[x_1, x_2, ..., x_d])$ groups have fixed points for every affine isometric action on $L_p$ for $p > 1$ and $d \geq 4$. It is known $[BFGM]$ that for every Kazhdan’s group $G$ there exists a constant $\epsilon(G) > 0$ such that every affine isometric action on $L_p$ spaces has a fixed point for $p \in [2, 2 + \epsilon(G))$. It is also known $[NS]$ that Gromov’s random groups, containing $p$-expanders in theirs Cayley graphs have a fixed point property for affine actions on $L_p$ spaces for $p > 1$. In $[N1]$ Nowak obtained sufficient conditions in terms of $p$-Poincaré constants implying that every affine isometric action of a given group on a reflexive Banach space has a fixed point. We refer to $[N1]$ for recent survey.

It is known $[CDH]$ that fixed point property for affine isometric group actions on $L_p(X, \mu)$ space, where $X$ is a space with measured walls implies property $(T)$. In the paper $[CDH]$ authors stated the following question.

**Question 1.** Is the set of values of $p \in (1, \infty)$ for which group has property $F_{L_p}$ an interval?

Theorem 1 answers this question positively in case of discrete groups and $l_p$ spaces.

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## 3 Preliminaries

We recall the definition $[BFGM]$ of property $(F_B)$ in the special case of discrete groups and $l_p$ spaces. By $l_p = l_p(\mathbb{N})$ we denote the usual Banach space of $p$-summable real sequences. Let $G$ be a discrete group. Denote by $O(B)$ the group of linear bijective isometries of the space Banach $B$.

**Definition 1.** *(Isometric representation)* An isometric representation of $G$ on a Banach space $B$ is a continuous homomorphism

$$\pi : G \to O(B).$$

The affine group $Aff(B)$ of real affine space (a vector space who forgot its origin) consists of invertible maps satisfying:

$$T(tx + (1 - t)y) = tT(x) + (1 - t)T(y),$$

for $t \in \mathbb{R}$ and $x, y \in B$.

An isometric affine action of a group $G$ on $B$ is a homomorphism $G \to Aff(B)$ of the form:

$$g(x) = \pi_g(x) + b_g,$$
where $\pi$ is an isometric representation and $b : G \to B$ is a $\pi$-cocycle, that is, an element of the Abelian group
\[ Z^1(\pi) = \{ c : G \to B : c_{gh} = \pi_g(c_h) + c_g; g, h \in G \}. \]

**Definition 2.** We say that a group $G$ has property $F_{l_p}$ if any continuous action of $G$ on $l_p$ by affine isometries has a $G$-fixed point.

Group $Z^1(\pi)$ contains the subgroup of $\pi$-coboundaries:
\[ B^1(\pi) = \{ b_g = \pi_g(v) - v : v \in B \}. \]

Observe that $Z^1(\pi)$ describes all affine actions with linear part $\pi$, and $B^1(\pi)$ corresponds to those actions which have $G$-fixed point. This interpretation involves the choice of origin point in the space. Two cocycles differing by a coboundary can be thought of defining the same affine action viewed from different reference points.

To prove that the group $G$ has property $F_{l_p}$ it is sufficient to show that for every isometric representation $\pi : G \to O(l_p)$, we have:
\[ H^1(G, \pi) = Z^1(\pi)/B^1(\pi) = 0. \]

The group $H^1(G, \pi)$ is the first cohomology group of $G$ with $\pi$-coefficients. In this spirit we can reformulate our result as follows:

**Theorem 2.** The set of parameters $p \in (1, \infty)$ for which $H^1(G, \pi) = 0$ for every representation $\pi : G \to O(l_p)$ is an interval.

In our proof we will use the following results:

**Theorem 3.** (Banach-Lamberti) Assume that $(p \neq 2)$. Every isometry $\pi : l_p \to l_p$ is the linear extension of an operator satisfying
\[ \pi(e_n) = \epsilon_n e_{\tilde{\pi}(n)} \]
where $\tilde{\pi} : N \to N$ is some bijection, $\epsilon_n$ is a sequence of signs and $(e_n)_{n=1}^{\infty}$ is the standard basis.

We also use some properties of expander graphs.

**Definition 3.** Let $X = (V, E)$ be a graph. Given $A \subset V$ define its edge boundary $\partial^e(A)$ setting $\partial^e(A) = \{ e \in E : e \text{ has exactly one vertex in } A \}$

Now we the define so called Cheeger constant and expander graphs.

**Definition 4.** The Cheeger constant $h(X)$ of a finite graph $X$ is given by the formula:
\[ h(X) = \min \left\{ \frac{\# \partial^e A}{\# A} : 0 < \# A \leq \frac{\# X}{2} \right\}. \]
Definition 5. A countable collection \( \{X_n\}_{n=1}^{\infty}, X_n = (V_n, E_n) \), of finite graphs of bounded degree, satisfying \( V_n \to \infty \), is said to be a collection of expanders if there exists \( c > 0 \) such that
\[
h(X_n) > c
\]
for every \( n \in \mathbb{N} \).

For expander family of graphs the following Poincaré type inequality holds.

Theorem 4. (Alon-Milman) Let \( X_n = (V_n, E_n) \) be an expander of bounded degree. Then there exists \( c > 0 \) such that for every function \( f : V_n \to \mathbb{R} \) with average 0,
\[
\lambda_1(X_n) = \frac{\sum_{x \in V_n} \sum_{y \in B(v, 1)} |f(x) - f(y)|^2}{\sum_{x \in V} |f(x)|^2} > c.
\]

This theorem is a consequence of the following inequalities:

Theorem 5. Let \( X \) be a \( k \)-regular graph. Then \( \frac{h(X)^2}{2k} \leq \lambda_1(X) \leq 2h(X) \).

It was proved by Dodziuk [Do], Alon [Al] (left inequality) and Tanner [Tan], Alon and Milman [AM] (right inequality). In the proof of theorem 1 we use the following generalisation of the Alon-Milman inequality [Mat].

Theorem 6. (J. Matoušek) Let \( G_n = (V_n, E_n) \) be an expander of bounded degree. Then there exists \( c > 0 \) such that for every function \( f : V_n \to \mathbb{R} \) with average 0,
\[
\frac{\sum_{x \in V_n} \sum_{y \in B(v, 1)} |f(x) - f(y)|^p}{\sum_{x \in V} |f(x)|^p} > c.
\]

4 Proof of the theorem

It follows from [CDH] that property \( F_{l_p} \) implies property \( (T) \). In the rest of the paper we will assume that \( p, q \neq 2 \).

Suppose that there exist \( p, q \in \mathbb{R} \) satisfying \( 1 < p < q < \infty \), \( p \neq 2 \) such that \( G \) has property \( F(l_q) \) but fails to have property \( F(l_p) \). By [BO], the first assumption implies that \( G \) is finitely generated. The second means that there exists a representation \( \pi^p \) on \( l_p \) which admits a non-trivial cocycle
\[
b^p : G \to l_p,
\]
which means that:

Property 1. For every \( v \) satisfying \( b^p_g = \pi^p_g(v) - v \) where \( g \in G \) we have
\[
v \notin l_p.
\]

By the Banach-Lamperti theorem every isometric representation \( \pi^p \) on \( l_p \) space has the following form (here we use assumption that \( p \neq 2 \)):
\[
\pi^p_g(v_1, v_2, ...) = (\epsilon_1^p v_{g^p(1)}, \epsilon_1^p v_{g^p(1)}, ...)
\]
(1)
where \( \epsilon \) and \( \pi \) are as in Theorem 3. Thus having the representation \( \pi^p \) on \( l_p \) space we can define "the same" representation \( \pi^q \) on \( l_q \) putting \( \pi^q_g = \epsilon_i \pi^p_g \) and \( \tilde{\pi}^q_g(i) = \tilde{\pi}^p_g(i) \). We will denote \( \tilde{\pi}^q_g(i) = \tilde{\pi}_g(i) \) for every \( r \in \mathbb{R} \cup \{ \infty \} \). Let \( i : l_p \to l_q \) be canonical inclusion. It is easy to check that the cocycle \( b^p \) associated to the representation \( \pi^p \) induces the cocycle \( b^q = i(b^p) \) associated to the representation \( \pi^q \) on \( l_q \).

Since \( G \) has property \( F(l_q) \), there exists a vector \( v \in l_q \) such that

\[
b^q_g = \pi_g(v) - v. \tag{2}\]

Note that for every \( g \in G \), we have \( \pi^q_g(v) - v = b^q_g \in l_q \), but Property 1 says that \( v \not\in l_p \). Hence denoting \( v = (v_i)_{i=1}^{\infty} \), and \( \pi^q_g(v) = (\epsilon_i \pi^p_g(v), \epsilon_2 \pi^p_g(v), \ldots) \)\) we have:

\[
\sum_{i=1}^{\infty} |v_i|^p = \infty, \tag{3}\]

and

\[
\sum_{i=1}^{\infty} |\epsilon_i \pi^g v_{\pi_g(i)} - v_i|^p < \infty. \tag{4}\]

In the rest of the paper we show that (3) and (4) lead to a contradiction. We can assume that \( v_i \geq 0 \) for all \( i \) and \( \epsilon_i = 1 \). Indeed, since:

\[
\sum_{i=1}^{\infty} |\epsilon_i \pi^g v_{\pi_g(i)} - v_i|^p \geq \sum_{i=1}^{\infty} \left| |\epsilon_i \pi^g v_{\pi_g(i)}| - |v_i| \right|^p,
\]

our assumption does not change (3) and (4). From now on we assume that

\[
\pi^p_g(v) = (v_{\pi_g(1)}, v_{\pi_g(2)}, \ldots) \tag{5}\]

We will also denote by \( \pi^r : G \to O(l_r) \) the representation satisfying \( \tilde{\pi}^r_g = \tilde{\pi}_g \), and \( \epsilon_i^r = 1 \) for every \( r \in \mathbb{R} \cup \infty \). Put \( w = (w_i)_{i=1}^{\infty} \) where \( w_i = (v_i)^{1/2} \). Obviously \( w \not\in l_q(G) \), but \( w \in l_r \) where \( r = \frac{1}{2} \frac{2}{p} \) and \( \pi^r_g(w) - w \in l_q \), for every \( g \in G \). Indeed, since \(|x^{\alpha} + y^{\alpha}| < 2|x + y|^{\alpha}, \) for \( \alpha < 1 \) we get:

\[
||\pi_g(w) - w||_q^q = \sum_{i=1}^{\infty} |w_{\pi_g(i)} - w_i|^q
\]

\[
\sum_{i=1}^{\infty} |w_{\pi_g(i)} - w_i|^q \leq 2^q \sum_{i=1}^{\infty} (|v_{\pi_g(i)} - v_i|^{1/2})^q = 2^q \sum_{i=1}^{\infty} |v_{\pi_g(i)} - v_i|^p < \infty.
\]

Now we use the assumption that \( G \) has property \( F(l_q) \). Then it follows that there exists \( u \in l_q(G) \) such that

\[
w - \pi^r_g(w) = u - \pi^r_g(u), \tag{6}\]

or equivalently

\[
\pi^r_g(w - u) = w - u. \tag{7}\]
i.e. $z = w - u$ is a fixed point of representation. Therefore $z$ belongs to $l_r(G)$, but $z \not\in l_q(G)$, because $w \not\in l_q(G)$.

Let $z = (z_i)_{i=1}^\infty$. Put $J = \{i : z_i = 0\}$ and let $N - J = \bigcup_{I \in P} I$ be the decomposition onto disjoint orbits of the representation $\pi^r$, where $P$ is the (countable) family of components. Here by component we mean any smallest set closed under permutations induced by the representation. Note that all components $I \in P$ are finite. Indeed, since $z$ is a fixed point of $\pi^r$ it follows that $z_i$ is constant on every $I \in P$. On the other hand $P$ is infinite because otherwise $z$ would have finite support, which contradicts $z \not\in l_q(G)$. Note that we can assume that $J$ is an empty set. Indeed, $\sum_{i \in J} (w_i)^q \leq \infty$ i.e. $\sum_{i \in J} (v_i)^p \leq \infty$ which means that the coordinates from $J$ does not disturb the assumption.

We have to consider three cases.

### 4.1 Components of unbounded length with expander property

In this section we show that if the representation $\pi^p$ has the expander property, then every cocycle $b^p$ with respect to this representation is coboundary for each $p \in (1, \infty)$. We define the expander property as follows. Suppose that the length of components is not uniformly bounded. Let $S$ be any symmetric set of generators of $G$. We consider the family of graphs $X_I = (V_I, E_I)$ for $I \in P$, where $V_I = I$ and vertices $a, b \in V_n$ are connected by an edge if there exists a generator $g \in S$ such that $\pi^r g(1_a) = 1_b$ i.e., for some $g \in S$ the permutation $\tilde{\pi}_g^r$, associated to the representation $\pi^r$ satisfies $\tilde{\pi}_g^r(a) = b$. Note that two vertices may be connected by more than one edge.

**Property 2.** We say that the family $P$ of components has the expander property if there exists an enumeration of $P$, say $(I_n)_{n=1}^\infty$, such that the sequence of graphs $X_{I_n}$ is an expander.

Consider the vector $\hat{v} = (\hat{v})_{i=1}^\infty$, defined by $\hat{v}_i = v_i - \frac{1}{|S|} \sum_{i \in I} v_i$ for $i \in I$. This vector satisfies

$$\pi^p g(v) - v = \pi^p g(\hat{v}) - \hat{v}.$$ 

Thus, considering the sum of differences:

$$\sum_{g \in S} \pi^p g(v) - v = \sum_{g \in S} \pi^p g(\hat{v}) - \hat{v},$$

we get, by Theorem 6

$$\infty > \sum_{g \in S} ||\pi_g(v) - v||^p_p = \sum_{g \in S} ||\pi_g(\hat{v}) - \hat{v}||^p_p \geq c ||\hat{v}||^p_p.$$ 

Thus $\hat{v} \in l_p$ - which contradicts Property 1.
4.2 Components of unbounded length without expander property

In this section we show that if there exists an infinite sequence of components without expander property, then the representation $\pi^q$ admits a non-trivial co-cycle on $l_q$ for $1 \leq q < \infty$.

Suppose that the sequence $X_{I_n}$ constructed in 3.1 is not an expander, i.e., there exists an infinite family $P' \subset P$ such that $\sum_{I \in P'} \frac{\# A_I}{\# A_I} < \infty$.

Consider $v = (v_i)_{i=1}^\infty \in l_\infty$ where $v_i = (\frac{1}{\# A_I})^{\frac{1}{q}}$ for $i \in I$ and $v_i = 0$ for $i \notin \bigcup_{I \in P} I$. For every generator $g \in S$ and $q > 2$ we have:

$$||\pi_g(v) - v||_q^q = \sum_{i=1}^\infty |v_i - v_{\pi_g(i)}|^q$$

$$= \sum_{I \in P'} \sum_{i \in I} \left( \sum_{i \in A_I, \pi(i) \notin A_I} \frac{1}{\# A_I} + \sum_{i \notin A_I, \pi(i) \in A_I} \frac{1}{\# A_I} \right)$$

$$\leq 2 \sum_{I \in P} \frac{\partial^1 A_I}{\# A_I} < \infty$$

Thus $\pi^\infty_g(v) - v \in l_q$ for $g \in S$. Put $v_{\emptyset(i)} = v_i$. For any $g = s_1 s_2 ... s_l$ where $s_i \in S$ we have:

$$||\pi_g(v) - v||_q^q \leq \sum_{i=1}^\infty |v_{\pi_{s_1 s_2 ... s_l}(i)} - v_i|^q$$

$$\leq \sum_{i=1}^\infty \sum_{j=1}^l |v_{\pi_{s_1 s_2 ... s_j}(i)} - v_{\pi_{s_1 s_2 ... s_j-1}(i)}|^q$$

$$\leq \sum_{j=1}^l \sum_{I \in P'} \left( \sum_{\pi_{s_1 s_2 ... s_j}(i) \in A_I} \frac{1}{\# A_I} + \sum_{\pi_{s_1 s_2 ... s_j-1}(i) \notin A_I} \frac{1}{\# A_I} \right)$$

$$\leq 2 \sum_{I} \frac{\partial^1 A_I}{\# A_I} < \infty.$$

Hence $\pi^\infty_g(v) - v \in l_q(G)$ for $g \in G$. It is easy to check that $b^q(g) = \pi^\infty_g(v) - v$ is a cocycle with respect to the representation $\pi^q$. We have to show that the vector $\pi^\infty_g(v) - v$ is a non-trivial cocycle on $l_q$.

Assume that $\pi^\infty_g(v) - v$ is a coboundary. Then there exists a vector $w \in l_q(G)$ such that

$$w - \pi^q_g(w) = v - \pi^\infty_g(v),$$

for every $g \in G$ i.e. $w - v$ is fixed point of representation $\pi^\infty$. This implies that
for $I \in P'$ there exists $c_I \in \mathbb{R}$ such that

$$w_i = \begin{cases} 
  \left(\frac{1}{|A_I|}\right)^\frac{1}{q} + c_I & \text{for } i \in A_I \\
  c_I & \text{for } i \in V_I - A_I \\
  d_I & \text{for } i \notin V_I, I \in P'
\end{cases}$$

(8)

It is clear that

$$||w_i||_q^q = \sum_{I \in P} ||w_i||_q^q.$$  

For every $I \in P'$ either $c_I \geq -(\frac{1}{|A_I|})^{\frac{1}{q}}/2$ or $c_I \leq -(\frac{1}{|A_I|})^{\frac{1}{q}}/2$. In the first case we have

$$||w_i||_q^q \geq \frac{1}{2q}||v||_q^q.$$  

In the second case, since $\#A_I \leq \frac{1}{2\#V_I}$, we get the same conclusion. Summing over $I \in P'$ we obtain $||w||_q \geq \frac{1}{2}||v||_q = \infty$, which ends the proof.

4.3 $\#I < D$ for $I \in P$

In this case we prove that every cocycle on $l_p$ with respect to the representation $\pi^p$ is a coboundary. Denote by $S(A)$ the permutation group of set $A$. Suppose that $\#I \leq D$ for every $I \in P$. Let $\Lambda_I : I \rightarrow \{1, 2, ..., D\}$ be an injection. For every permutation $\sigma$ of $I$ there is a permutation $H_I(\sigma)$ satisfying formula $H_I(\sigma) \circ \Lambda_I = \Lambda_I \circ \sigma$. We say that two components $I, I'$ are equivalent if and only if $H_I(\tilde{\pi}^p_g|I) = H_I(\tilde{\pi}^p_g'|I')$ for every $g \in G$. Since $G$ is finitely generated and every representation is determined by permutations corresponding to generators, there are only finitely many of equivalence classes. Moreover the permutations $H_I(\tilde{\pi}^p_g|I)$ form a subgroup of $S(\{1, 2, ..., D\})$. Thus there exists a finite set $Q \subset G$ such that for every $g \in G$ and $I \in P$ there exists $g' \in Q$ satisfying $\tilde{\pi}^p_g|I = \tilde{\pi}^p_{g'}|I$.

For every $I \in P$ fix $i_I \in I$. By transitivity, for every $j \in I$ there exists $g_j \in G$ such that $\tilde{\pi}^p_{g_j}(j) = i_I$. Thus we get

$$\sum_{g \in Q} ||\pi^p_g(v) - v||_p^p = \sum_{I \in P} \sum_{g \in Q} ||(\pi^p_g(v) - v)|_I||_p^p \geq \sum_{I \in P} \sum_{j \in I} \sum_{g \in Q} |v_{\pi^p_g(j)} - v_j|^p$$

(9)

$$\geq \sum_{I \in P} \sum_{j \in I} \sum_{g \in Q} |a_i - a_{i_I}|^p$$

By (4), the left hand side is finite. We put $\bar{w} = (c_i)_{i=1}^\infty$ by $c_i = a_i - a_{i_I}$ for $i \in I$. By (3) $\bar{w} \in l_p$. Since $\bar{w} = v - \bar{w}$ is constant on every $I \in P$, it is a fixed point of $\pi^\infty$, i.e.

$$\pi^\infty_g(v - \bar{w}) = v - \bar{w}$$
or equivalently
\[ \pi_\varphi^\infty(v) - v = \pi_\varphi^\infty(\bar{w}) - \bar{w} = \pi_\varphi^g(\bar{w}) - \bar{w}. \]

This contradicts Property 1.

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