RePair in Compressed Space and Time

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Abstract

Given a string $T$ of length $N$, the goal of grammar compression is to construct a small context-free grammar generating only $T$. Among existing grammar compression methods, RePair (recursive paring) [Larsson and Moffat, 1999] is notable for achieving good compression ratios in practice. Although the original paper already achieved a time-optimal algorithm to compute the RePair grammar $\text{RePair}(T)$ in expected $O(N)$ time, the study to reduce its working space is still active so that it is applicable to large-scale data. In this paper, we propose the first RePair algorithm working in compressed space, i.e., potentially $o(N)$ space for highly compressible texts. The key idea is to give a new way to restructure an arbitrary grammar $S$ for $T$ into $\text{RePair}(T)$ in compressed space and time. Based on the recompression technique, we propose an algorithm for $\text{RePair}(T)$ in $O(\min(N, nm \log N))$ space and expected $O(\min(N, nm \log N \log N))$ time, where $n$ is the size of $S$ and $m$ is the number of variables in $\text{RePair}(T)$. We implemented our algorithm running in $O(\min(N, nm \log N \log N))$ time and show it can actually run in compressed space. We also present a new approach to reduce the peak memory usage of existing RePair algorithms combining with our algorithms, and show that the new approach outperforms, both in computation time and space, the most space efficient linear-time RePair implementation to date.

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1 Introduction

1.1 Motivations and Contributions

Given a string $T$ of length $N$, the goal of grammar compression is to construct a small context-free grammar generating only $T$. Among existing grammar compression methods, RePair (recursive paring) [23] is notable for achieving good compression ratios in practice and in theory [27, 11]. The principle of RePair is quite simple to explain: it chooses one of the most frequent bigrams appearing in $T$ more than once and greedily replaces every occurrence of the bigram with a variable whose righthand side is the bigram, and recursively applies the procedure to the resulting text until there is no bigram with frequency $\geq 2$. This principle successfully captures the regularities frequently appearing in the text, and so it has been shown that RePair (or the essence of RePair) has wide range of applications to, e.g., word-based text compression [34], compression of Web graphs [10], compressed suffix trees [13], compressed wavelet trees [29], tree compression [21], and data mining [32].

In their original paper [23], Larsson and Moffat proposed a time-optimal algorithm to compute the RePair grammar $\text{RePair}(T)$ in expected $O(N)$ time. The space usage is analyzed as $5N + 4\sigma^2 + 4m + \lceil \sqrt{N} \rceil$ words, where $\sigma$ is the alphabet size and $m$ is the number of variables in $\text{RePair}(T)$. However, the space usage is not satisfying since the amount of data becomes larger and larger. Thus, the study to reduce its working space is still active [7, 6].

In this paper, we propose the first RePair algorithm working in compressed space, i.e., potentially $o(N)$ space for highly compressible texts. The key idea is to give a new way to restructure an arbitrary grammar $S$ for $T$ into $\text{RePair}(T)$ in compressed space and time. More precisely, we show how to compute $\text{RePair}(T)$ in $O(\min(N,nm \log N))$ space and $O(\min(N,nm \log N) m)$ time, and improve the expected time complexity to $O(\min(N,nm \log N) \log \log N)$, where $n$ is the size of $S$ and $m$ is the number of variables in $\text{RePair}(T)$. Note that $n$ and $m$ can be exponentially smaller than $N$, while $\log N \leq n$.\footnote{to be precise, the improvement is achieved only when $m = \omega(\log \log N)$, which is likely to hold for compressible texts.}

With our algorithms one can obtain $\text{RePair}(T)$ from $T$ in compressed space as follows: The input string is first processed by an online grammar compression algorithm, such as [33, 25], that works in compressed space, and then its output grammar is recompressed into $\text{RePair}(T)$. This fits well the scenario in which data sources (such as embedded devices with sensors) have weaker computational resources, and thus, the produced data is compressed by a lightweight compression algorithm (to reduce the transmission cost) and sent to server in which further compression can be conducted.

Restructuring a compressed representation of data into another compressed representation \textit{in compressed space} has its own interest and applications, and thus, has been widely studied. In the seminal work [8, 30] in the field of grammar compression, restructuring LZ77 into balanced grammars is the key to obtain a reasonable approximation to the smallest grammar. In [14], a bunch of restructuring algorithms were considered in major lossless compression algorithms including LZ77 [37], LZ78 [38], Bisection [28], and RePair [23]. In [9, 4], the authors gave efficient algorithms to convert any grammar compressed string to LZ78. Recently, compressed space LZ77 parsing was achieved using another compressed scheme of run-length

\footnote{$\log N \leq m$ is not necessarily true since RePair stops producing variables when the input text is compressed into a string $w$ containing no bigram with frequency $\geq 2$. Still, it holds that $\log N \leq m + |w|$.}
compressed Burrows-Wheeler transform [29, 3]. Our contribution in this paper is to draw a missing line from admissible grammars to the RePair grammar in Figure 1 of [14]. As pointed out in [14, 5], restructuring has many applications, e.g., dynamic updates of compressed strings and efficient computation of normalized compression distance (NCD) [9]. As more and more data is available in compressed form, the importance of restructuring algorithms grows.

We implemented a prototype of our recompression algorithm for RePair with complexities of \(O(\min(N, nm \log N))\) space and \(O(\min(N, nm \log N))\) time. While we confirm that it actually has a potential to run in compressed space, the running time is not fast enough to conduct comprehensive experiments over various datasets. Instead of claiming the practicality of the current implementation, we show some evidence that our \(O(\min(N, nm \log N) \log \log N)\) -time algorithm could be practical by further algorithmic engineering work. In particular, our experimental results suggest that the \(nm \log N\) term in the theoretical bounds could be loose, and much smaller, say \(O(n)\), for most of the cases in reality. We also propose a new approach to reduce the peak memory usage of existing RePair algorithms combining with our method. The experimental results show that the approach is promising, outperforming the most space efficient linear-time implementation to date both in time and space.

1.2 Related work.

There have been several attempts to modify the original RePair grammar to improve its performance in terms of working space [35, 31, 25] and compression ratio [12].

For the approximation ratio of RePair grammar to the smallest grammar generating the input string of length \(N\), Charikar et al. proved an upper bound \(O((N/\log N)^{2/3})\) and lower bound \(\Omega(\sqrt{\log N})\). The lower bound was recently improved to \(\Omega(\log N/\log \log N)\) in [15].

Our algorithms simulate the replacements of bigrams on grammars. The technique used here is borrowed from the recompression technique of Jeż, which has been proved to be a powerful tool in problems related to grammar compression [17, 18, 19, 22, 16] and word equations [20, 21]. In particular, the grammar compression method based on recompression [18] considers replacing bigrams in a string with variables level by level like RePair. The difference from RePair lies in the way of choosing bigrams to be replaced. Instead of replacing the most frequent bigram in a single round, recompression chooses several bigrams (which cannot overlap each other) in a way that a given string shrinks by a constant factor after the round. This strategy has lots of merits in theory, e.g., it assures that the number of rounds is \(O(\log N)\) and the approximation ratio to the smallest grammar is \(O(\log N)\), where \(N\) is the length of an input string. Moreover, the procedure is simulated from any grammar of size \(n\) in \(O(n \log^2 (N/n))\) time (or \(O(n \log(N/n))\) time with a slight modification) and \(O(n \log(N/n))\) space (see [16]). The mechanism of replacing bigrams on grammars can also be used for RePair in a somewhat straightforward way. As the way of choosing bigrams is different, we have to thoroughly reanalyze the complexities for RePair, and as a result, unfortunately, we have lost the theoretical cleanliness of recompression. Still, RePair has a strong merit in practical compression ratio and we show that our approach is helpful to overcome its weakness, the peak memory usage in compression.

2 Preliminaries

An alphabet \(\mathcal{A}\) is a finite set of symbols. A string over \(\mathcal{A}\) is an element in \(\mathcal{A}^*\). For any string \(w \in \mathcal{A}^*\), \(|w|\) denotes the length of \(w\). Let \(\varepsilon\) be the empty string, i.e., \(|\varepsilon| = 0\). Let \(\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}\). For any \(1 \leq i \leq |w|\), \(w[i]\) denotes the \(i\)-th symbol of \(w\). For any \(1 \leq i \leq j \leq |w|\), \(w[i..j]\)
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denotes the substring of $w$ beginning at $i$ and ending at $j$. For convenience, let $w[i..j] = \varepsilon$ if $i > j$. For any $0 \leq i \leq |w|$, $w[1..i]$ (resp. $w[|w| - i + 1..|w|]$) is called the prefix (resp. suffix) of $w$ of length $i$. We say that a string $x$ occurs at the interval $[i..i+|x|-1]$ in $w$ iff $w[i..i+|x|-1] = x$. A substring $w[i..j] = c^d$ ($c \in \mathcal{A}, d \geq 1$) of $w$ is called a block iff it is a maximal run of a single symbol, i.e., $(i = 1 \lor w[i-1] \neq c) \land (j = |w| \lor w[j+1] \neq c)$.

An element $c \in \mathcal{A}^2$ is called a \textit{bigram}, and the bigram is said to be \textit{repeating} iff $\hat{c} = c$. When we mention the \textit{frequency} of a bigram $\hat{c}$ in $w \in \mathcal{A}^\ast$, it actually means the \textit{non-overlapping} frequency, which counts the maximum number of occurrences of $\hat{c}$ that do not overlap each other. While the frequency of a non-repeating bigram is identical to the number of occurrences of $\hat{c}$, the frequency of a repeating bigram is counted by summing up $\lfloor d/2 \rfloor$ for every block $c^d$ of $c = \hat{c} = \hat{c}$. Let $\text{freq}(\hat{c}, w)$ denote the frequency of $\hat{c}$ in $w$.

The text subjected to being compressed is denoted by $T \in \Sigma^\ast$ with $N = |T|$ throughout this paper. We assume that $\Sigma$ is an integer alphabet $[1..N^\Theta(1)]$ and the standard word RAM model with word size $\Theta(\lg N)$. The time complexities are expected time as RePair algorithms utilize hash functions to look-up/update frequency tables etc. Also, the space complexities are measured by the number of words (not bits).

In this article, we deal with grammar compressed strings, in which a string is represented by a Context-Free Grammar (CFG) generating the string only. We simply use the term grammars or CFGs to refer to such specific CFGs for string compression. In particular, we consider a normal form of CFGs, called \textit{Straight-Line Programs} (SLPs), in which the right-hand side of every production rule is a bigram.\footnote{Of course, we ignore any trivial input string of length one or zero.} Formally, an SLP that generates a string $T$ is a triple $\mathcal{S} = (\Sigma_S, V_S, D_S)$, where $\Sigma_S$ is the set of terminals (letters), $V_S$ is the set of non-terminals (variables), $D_S$ is the set of deterministic production rules whose right-hand sides are in $(V_S \cup \Sigma_S)^2$, and the last variable derives $T$.

For an SLP $\mathcal{S}$ with $n = |V_S|$, note that $N$ can be as large as $2^n$, and so, SLPs have a potential to achieve exponential compression. Also, $n \geq \lg N$ is always true. We treat variables as integers in $[1..n]$ (which should be distinguishable from $\Sigma_S$ by having one extra bit), and $D_S$ as an injective function that maps a variable to its right-hand side (i.e., $D_S(X)$ represents a bigram for any $X \in V_S$). For any $X \in V_S$, if $D_S(X)[1]$ (resp. $D_S(X)[2]$) is from $V_S$, it is called the left (resp. right) variable of $X$. Let $\mathcal{T}_S$ denote the derivation tree of $\mathcal{S}$. Note that $\mathcal{T}_S$ is implicitly stored by the production rules in $O(n)$ space, which can be seen as a DAG representation of the tree. We assume that variables are in a (reversed) topological sort order, i.e., left/right variable of $X$ is smaller than $X$. Let $\text{vocc}_S(X)$ denote the number of nodes labeled with $X$ in $\mathcal{T}_S$. It is a well-known fact that we can preprocess $\mathcal{S}$ in $O(n)$ time and space to compute $\text{vocc}_S(X)$ for all $X \in V_S$ by a simple dynamic programming (it reduces to the problem of computing the number of paths from the source to nodes in a DAG). We assume that given any variable $X$ we can access in $O(1)$ time the information on $X$, e.g., $D_S(X)$ and $\text{vocc}_S(X)$. For any variable $X \in V$, the string derived from $X$ is denoted by $\text{vals}(X)$, where we omit $\mathcal{S}$ when it is clear from context.

RePair \footnote{We treat the last variable as the starting variable.} is a grammar compression algorithm, which recursively replaces the most frequent bigram (tie-breaking arbitrary) into a variable while there is a bigram with frequency $\geq 2$. Formally, RePair transforms $T_0 := T$ level by level into strings, $T_1, T_2, \ldots, T_m$: at the $h$-th level ($0 \leq h$) we are given $T_h$ and compute $T_{h+1}$ that is obtained by replacing $\text{freq}(\hat{c}, T_h)$ non-overlapping occurrences of the most frequent bigram $\hat{c}$ in $T_h$ with a new variable $\hat{c}$ such that $\hat{c} \rightarrow \hat{c} \hat{c}$. To remove ambiguity in the replacement for a repeating bigram
\( \hat{c} \) with \( \hat{c} = \hat{c} = \hat{c} \), let us conduct a greedy left-to-right parsing on a block \( \hat{c}^d \), namely, \( \hat{c}^d \) is replaced with \( \hat{c}^{(d/2)} \) if \( d \) is even, and otherwise \( \hat{c}^{(d/2)} \). Any appearance \( \hat{c} \) in \( T_h \) is treated as a letter in the later rounds, so we call variable \( \hat{c} \) the letter introduced at level \( h + 1 \). The process shrinks the string monotonically, and finally we get \( T_m \) in which there are no bigram with frequency \( \geq 2 \).

Let \( \text{RePair}(T) \) denote the grammar obtained by RePair with input \( T \). The variables of \( \text{RePair}(T) \) consist of the letters introduced at all levels and the starting variable whose righthand side is \( T_m \). Except the starting variable, the righthands of the rules are bigrams.

3 \( O(\min(N, nm \log N)m) \)-time algorithm

In this section we show how, given an arbitrary SLP \( S \) generating \( T \), we compute \( \text{RePair}(T) \) in \( O(\min(N, nm \log N)m) \) time and \( O(\min(N, nm \log N)) \) space, where \( N \) is the length of \( T \), and \( n \) (resp. \( m \)) is the number of variables in \( S \) (resp. \( \text{RePair}(T) \)).

3.1 Overview: Recompress \( S \) into \( \text{RePair}(T) \) in compressed space.

The key idea to compute \( \text{RePair}(T) \) in compressed space is to recompress an arbitrary \( S \) for \( T \) into \( \text{RePair}(T) \) without decompressing \( S \). For a clear description, we add two auxiliary variables that introduce sentinels at the beginning.\( T[0] = \# \notin \Sigma \) and at the end \( T[N + 1] = \$ \notin \Sigma \): we define \( S_0 := (\Sigma_0, \nu_0, D_0) \) such that \( \Sigma_0 := \Sigma \cup \{\#, \$\}, \nu_0 := \nu \cup \{X_\#, X_\$\}, \) and \( D_0 := D \cup \{(X_\# \to \#X), (X_\$ \to X_\$\$)\} \), where \( X_\$ \) is the starting variable of \( S \). Clearly, \( S_0 \) generates \( \#T\$. 

We employ the recompression technique \( [17, 18, 19, 22] \), invented by Jeż, to simulate the transformation from \( T_h \) to \( T_h \) on CFGs. We transform level by level \( S_0 \) into a sequence of \( S_1 = (\Sigma_1, \nu_0, D_1), S_2 = (\Sigma_2, \nu_0, D_2), \ldots, S_m = (\Sigma_m, \nu_0, D_m), \) where each \( S_h \) generates \( \#T_h \$ \in \Sigma_h \). Namely, compression from \( T_h \) to \( T_{h+1} \) is simulated on \( S_h \). We can correctly compute the letters introduced at each level \( h + 1 \) while modifying \( S_h \) into \( S_{h+1} \), and hence, we get all the letters of \( \text{RePair}(T) \) in the end. We note that new variables for \( S_h \) are never introduced and the modification is done by rewriting righthand sides of the original variables in \( \nu_0 \). During the modification, the string represented by a variable \( X \) could be shorten, and \( X \) could be NULL meaning that it represents nothing, i.e., \( \text{val}_{S_h}(X) = \varepsilon \).

Here we introduce the special formation of the CFGs \( S_h \) (it is a generalization of SLPs): For any \( X \in \nu_0 \), \( D_h(X) \) consists of an arbitrary number of letters and at most two non-null variables that are originally in \( D_0(X) \). More precisely, the following condition holds:

For any variable \( X \in \nu_0 \), let \( X \) (resp. \( \hat{X} \)) denote the left (resp. right) variable, where it represents NULL if it does not exist. Then, \( D_h(X) = Xw_X\hat{X} \) with \( w_X \in \Sigma_h \), where null variables are imaginary and actually removed from \( D_h(X) \).

In addition, we compress \( w_X \) by the run-length encoding so that it can be stored in \( O(|w_X|_\text{re}) \) space, where \( |w_X|_\text{re} \) denotes the number of blocks in \( w_X \). We define the size of \( D_h(X) \) by \( |w_X|_\text{re} + |w_X|_\text{re}^\text{null} \) plus the number of non-null variables in \( D_h(X) \), and denote it by \( |D_h(X)|_\text{re} \). The size of \( S_h \), denoted by \( |S_h| \), is defined by \( \sum_{X \in \nu_0} |D_h(X)|_\text{re} \).

In Subsection 3.2 we show how to compute the frequencies of bigrams on \( S_h \) in \( O(|S_h|) \) time and space. In Subsection 3.3 we show, given the most frequent bigram \( \hat{c} \hat{c} \), how to replace \( \hat{c} \hat{c} \) with a new letter \( \hat{c} \) on \( S_h \) to get \( S_{h+1} \) in \( O(|S_h|) \) time and space. In Subsection 3.4 we assign index zero to \( \# \) so that the indexes in \( T \) are persistent with the original ones.
we show that $|S_h| = O(\min(N, nh \log N))$ for any level $h$, and thus, the recompression from $S_h$ to $S_m$ can be done in the claimed time and space complexity.

### 3.2 How to compute frequencies of bigrams on $S_h$.

The goal of this subsection is to show the next lemma:

**Lemma 1.** Given $S_h$ generating $T_h$, we can compute in $O(|S_h|)$ time and space the frequencies of bigrams appearing in $T_h$.

The following fact is useful to compute the frequencies of bigrams in $T_h$ on $S_h$.

**Fact 2.** For any interval $[i..j] \subseteq [0..|T_h| + 1]$ with $j - i > 0$, there is a unique variable $X \in \mathcal{V}_0$ that is the label of the lowest common ancestor of the $i$-th and $j$-th leaf in $T_{S_h}$. We say that such $X$ stabs $[i..j]$.

According to Fact 2 we can detect the occurrences of bigrams by variables that stab the occurrences without duplication or omission. In addition, since each variable $X$ can stab at most $|D_h(X)|_{\text{in}}$ distinct bigrams, it implies that there are at most $\sum_{X \in \mathcal{V}_0} |D_h(X)|_{\text{in}} = |S_h|$ distinct bigrams in total.

In order to compute the frequencies, we use the following auxiliary information for all variables, which can be computed in a bottom-up manner in $O(|S_h|)$ time and stored in $O(n)$ space.

- $\lambda(X)$: the leftmost block in $\text{val}_{S_h}(X)$.
- $\rho(X)$: the rightmost block in $\text{val}_{S_h}(X)$.
- $\text{isSB}(X)$: Boolean that represents if $D_h(X)$ consists of a single block.

For any variable $X \in \mathcal{V}_0$ with $D_h(X) = Xw_X \hat{X}$, we can easily compute $\lambda(X)$, $\rho(X)$ and $\text{isSB}(X)$ in $O(1)$ time, assuming that we have computed those for $\hat{X}$ and $X$: for example, $\lambda(X)$ is identical to $\lambda(\hat{X})$ if the prefix block stops inside $X$, or it is extended if $\lambda(\hat{X})$ can be merged with the first block of $w_X$ (and further with $\lambda(\hat{X})$).

We first focus on the frequencies of non-repeating bigrams $\hat{c}c$. According to Fact 2 we assign any occurrence $[i..i+1]$ of $\hat{c}c$ to the variable that stabs $[i..i+1]$ without duplication or omission. We now intend to count all the occurrences of $\hat{c}c$ assigned to $X$ in $D_h(X) := Xw_X \hat{X}$. Observe that $\hat{c}c$ appears explicitly in $w_X$ or crosses the boundaries of $X$ and/or $\hat{X}$. Thus, it is enough to compute the frequencies in $\rho(\hat{X})w_X \lambda(\hat{X})$. Since each $\hat{c}c$ found in $\rho(\hat{X})w_X \lambda(\hat{X})$ appears every time a node labeled with $X$ appears in $T_{S_h}$, we count each occurrence of $\hat{c}c$ in $\rho(\hat{X})w_X \lambda(\hat{X})$ with the weight $\text{vocc}(X)$. Hence, the frequencies of non-repeating bigrams can be computed in $O(|S_h|)$ time while scanning $\rho(\hat{X})w_X \lambda(\hat{X})$ for all $X \in \mathcal{V}_0$ and incrementing the frequency of $\hat{c}c$ by $\text{vocc}(X)$ whenever we find an occurrence of a non-repeating bigram $\hat{c}c$ in $\rho(\hat{X})w_X \lambda(\hat{X})$.

Next we compute the frequencies of repeating bigrams. To this end, we detect all the blocks with lengths $\geq 2$ without duplication or omission by assigning each block to the smallest variable that “witnesses” the maximality of the block. Formally, we assign a block occurring at $[i..j]$ to the variable $X$ that stabs $[i-1..j+1]$. (Note that $[i-1..j+1]$ is always a valid interval thanks to the sentinels $\#$ and $\$. For any block $X$ in $D_h(X) := Xw_X \hat{X}$, we can find every block assigned to $X$ as a block appearing in $\rho(\hat{X})w_X \lambda(\hat{X})$, where we ignore a block that is a prefix/suffix of $\text{val}_{S_h}(X)$ because $X$ does not witness its maximality. Using the information of $\text{isSB}(\hat{X})$ and $\text{isSB}(X)$, we can easily check if a block is a prefix/suffix of $\text{val}_{S_h}(X)$. The frequencies of repeating bigrams can be computed in $O(|S_h|)$ time while scanning $\rho(\hat{X})w_X \lambda(\hat{X})$ for all $X \in \mathcal{V}_0$ and incrementing the frequency of $c^2$ by $[d/2] \text{vocc}(X)$ whenever we find a block $c^2$ with $d \geq 2$ that is assigned to $X$.

Figure 1 shows an example on how to compute the frequencies on grammars.
we do the following “simultaneously” for all variables of PopOutLet If Lemma 3.

An example of PopInLet If

Figure 1 An example on how the replacements of the first level is done on the grammar in Figure 1.

Figure 2 shows an example on how the replacements of the first level is done on the grammar in Figure 1.

Finally, since $\lambda$ is stabbed by the variables $ab$, we remove all the occurrences of $\lambda$.

Generaing $h$ into $h+1$.

The goal of this subsection is to show the next lemma:

**Lemma 3.** Given $S_h$ generating $T_h$ and the most frequent bigram $\hat{c}\hat{c}$ in $T_h$, we can transform $S_h$ into $S_{h+1}$ in $O(|S_h|)$ time and space.

We first focus on the case where $\hat{c}\hat{c}$ is non-repeating. Some of the occurrences of $\hat{c}\hat{c}$ are explicitly written in $w_X$ and the others are crossing the boundaries of left and/or right variables of $X$ for some $X \in V_0$. While explicit occurrences can be replaced easily, crossing occurrences need additional treatment. To deal with crossing occurrences, we first uncross them by popping out every $\hat{c}$ (resp. $\hat{c}$) occurring at the rightmost (resp. leftmost) position of $valS_h(Y)$ and popping them into the appropriate positions in the other rules. More precisely, we do the following “simultaneously” for all $X \in V_0$:

**PopInLet** If $D_h(X)$ contains a variable $Y \in V_0$ in any position other than the first position and $valS_h(Y)[1] = \hat{c}$, replace the occurrence of $Y$ with $\hat{c}Y$; and if $D_h(X)$ contains a variable $Y \in V_0$ in any position other than the last position and $valS_h(Y)[|valS_h(Y)|] = \hat{c}$, replace the occurrence of $Y$ with $Y\hat{c}$.

**PopOutLet** If $D_h(X)[1] = \hat{c}$, delete it; and if $D_h(X)[|D_h(X)|] = \hat{c}$, delete it. In addition, if $X$ becomes NULL, we remove all the occurrences of $X$ in $D_h$.

**PopOutLet** removes $\hat{c}$ (resp. $\hat{c}$) from the rightmost (resp. leftmost) position of $valS_h(Y)$ (which can be a part of a crossing occurrence of $\hat{c}\hat{c}$), and **PopInLet** introduces the removed letters into appropriate positions in $D_h$ so that the modified $S_h$ keeps to generate $T_h$. The uncrossing can be conducted in $O(|S_h| + n)$ time using the information of $\lambda(\cdot)$ and $\rho(\cdot)$. Since all the occurrences of $\hat{c}\hat{c}$ are now explicitly written in the righthand sides, we can easily replace them with a fresh letter $\hat{c}$ while scanning the righthand sides in $O(|S_h| + n)$ time.

Figure 2 shows an example on how the replacements of the first level is done on the grammar in Figure 1.
Also, the number of occurrences of letters in the righthand sides increases: (1) when letters/blocks are popped in; and (2) when a repeating bigram \( \rho \) is replaced on a run-length encoded block \( \lambda \) with odd \( d \geq 2 \) assigned to \( X \in V^0 \), which can be found in \( \rho(\hat{X})w_X\lambda(\hat{X}) \). In a similar way to the non-repeating case, we first uncross \( c^d \) if it starts in \( \rho(\hat{X}) \) or ends in \( \lambda(\hat{X}) \). The uncrossing for all variables can be done in \( O(n) \) time and space.

### 3.4 Analysis.

The primal goal of this subsection is to prove Lemma 4, which upper bounds the CFG sizes during modification.

**Lemma 4.** For any level \( h \), \( |S_h| = O(\min(N, nh \log N)) \).

**Proof.** When transforming \( S_h \) into \( S_{h+1} \), there are two situations where the size of the righthand sides increases: (1) when letters/blocks are popped in; and (2) when a repeating bigram \( cc \) is replaced on a run-length encoded block \( c^d \) with odd \( d > 2 \). For (1), it is easy to see that for each variable \( X \) the positions where letters/blocks popped in is at most two (the boundaries of left/right variables), and thus, the size of \( S_h \) increases at most \( 2(n+2) = O(n) \) for each level. For (2), we deposit \( \log d \leq \log N \) credit whenever a block \( c^d \) is popped into some position so that the later increase by case (2) can be paid from the credit. Since at most \( O(n \log N) \) credit is issued for each level, we obtain the bound \( |S_h| = O(n \log N) \). Also, the number of occurrences of letters in the righthand sides of \( D_h \) cannot be larger than the uncompressed size \( |T_h| \), and therefore, \( |S_h| \leq |T_h| + 2n = O(N) \) holds.

Our first algorithm running in \( O(\min(N, nm \log N)m) \) time and \( O(\min(N, nm \log N)) \) space is immediately obtained from Lemmas 1, 2 and 4.
The numbers of variables in At each level where \( n \) computing \( \lambda \) along with replacements, but at least we can recollect, for each level beyond boundaries dynamically change. We do not see how we can efficiently maintain it is sometimes problematic as the leftmost/rightmost descendants who possess the contexts left/right variables). Here updating the information for bigrams crossing the boundaries righthand side and pointers to traverse all and only the occurrences of any bigram appearing most \( N \) affected by the replacement in constant time. Since the total number of replacement is at every occurrence of \( \sum \) spending \( O \) \( h=m \) it is analogue to improving a naive \( O(\min(N, nm \log N)) \) time and space. At any level \( h \) \( 0 \leq h < m \), the transform from \( S_{h} \) that generates \( T_{h} \) to \( T_{h+1} \) is simulated on CFGs as follows: Given \( S_{h} \) generating \( T_{h} \), we use Lemma 1 to compute the most frequent bigram in \( T_{h} \), and Lemma 3 to obtain \( S_{h+1} \) that generates \( T_{h+1} \). It can be done in \( O(|S_{h}|) \) time and space. Since \( |S_{h}| = O(\min(N, nh \log N)) \), we can go through from \( S_{0} \) to \( S_{m} \) in \( O(\sum_{h=0}^{m} |S_{h}|) = O(\min(N, nm \log N) m) \) time and \( O(\max\{|S_{h}| \mid 0 \leq h \leq m\}) = O(\min(N, nm \log N)) \) space.

We note that the bound \( |S_{h}| = O(nh \log N) \) of Lemma 4 could be quite rough because the analysis considers the following (probably too pessimistic) scenario: there are \( \Omega(n) \) run-compressed letters are popped in and each of them produces \( \Omega(\log N) \) remainders during replacing repeating bigrams on it. In addition, the analysis does not take into account the fact that each replacement on non-repeating bigrams reduces the grammar size by one. It is open if there is an example to achieve the upper bound. In our preliminary experiments, we observed that \( |S_{h}| \) is just a few times larger than \( n \) in highly repetitive datasets.

4 \( O(\min(N, nm \log N) \log \log N) \)-time algorithm

In this section, we improve the time complexity \( O(\min(N, nm \log N) m) \) of Theorem 5 to \( O(\min(N, nm \log N) \log \log N) \). It is analogue to improving a naive \( O(Nm) \)-time RePair algorithm that works on plain text \( T \) to an \( O(N) \)-time algorithm. At level \( h \) \( 0 \leq h < m \), the naive algorithm simply scans text \( T_{h} \) to compute the most frequent bigram and replace its non-overlapping occurrence with a fresh letter spending \( O(|T_{h}|) \) time, and thus, it takes \( O(\sum_{h=0}^{m} |T_{h}|) = O(Nm) \) time in total. The essential idea of 23 to obtain \( O(N) \)-time algorithm is to:

1. represent \( T_{h} \) by a linked list so that replacements can be done locally without breaking adjacent letters apart,
2. maintain, for every bigram in \( T_{h} \), pointers to traverse all and only the occurrences of the bigram,
3. maintain the frequencies of all bigrams in a priority queue.

At each level \( h \), we obtain the most frequent bigram \( \hat{c} \) from the priority queue and replace every occurrence of \( \hat{c} \) using the pointers to visit the occurrences of \( \hat{c} \). While replacing each occurrence, we can easily update the linked-list, pointers and frequencies of bigrams that are affected by the replacement in constant time. Since the total number of replacement is at most \( N \), the algorithm runs in \( O(N) \) time.

We apply this idea to our algorithm in Section 3, we maintain the linked-list for each righthand side and pointers to traverse all and only the occurrences of any bigram appearing in the grammar (it is explicitly written in the grammar rules or crossing the boundaries of left/right variables). Here updating the information for bigrams crossing the boundaries is sometimes problematic as the leftmost/rightmost descendants who possess the contexts beyond boundaries dynamically change. We do not see how we can efficiently maintain it along with replacements, but at least we can recollect, for each level \( h \), the information by computing \( \lambda(\cdot), \rho(\cdot) \) and \( isSB(\cdot) \) in \( O(n_{h}) \) time (as we did in the algorithm in Section 3), where \( n_{h} \) is the number of non-null variables in \( S_{h} \).
RePair in Compressed Space and Time

\[
\sum_{h=0}^{m} |S_h| \leq n \\
\sum_{h=0}^{m} n_h \\
R
\]

| time | space | n \(2^{10}\times\) | m \(2^{10}\times\) | Max \(2^{10}\times\) | \(\sum_{h=0}^{m} |S_h| \) \(2^{10}\times\) | \(\sum_{h=0}^{m} n_h \) \(2^{10}\times\) | R \(2^{10}\times\) |
|------|-------|----------------|----------------|---------------|----------------|----------------|--------|
| einstein.en.txt | 5,626 | 27.36 | 413 | 98 | 1,157 | 38,408,764 | 11,149,315 | 5,241 |
| world_leaders | 19,872 | 33.07 | 807 | 204 | 1,920 | 139,854,080 | 36,212,346 | 14,249 |
| fib41 | 20 | 24.01 | 0.4 | 0.04 | 0.2 | 3 | 3 | 6,495 |

Table 1: Table showing time and working space for our algorithm to compute RePair from each dataset (including the time and space of SOLCA). SOLCA takes 53, 9 and 19 seconds for each dataset. The peak memory usage of fib41 is from the constant-size hash table used in SOLCA. For other columns, \(n\) is the number of variables in the output grammar \(S\) of SOLCA, \(m\) is the number of variables in the RePair grammar, \(\text{Max} := \max\{|S_h| | 0 \leq h \leq m\}\) and \(R\) is the total number of replacements executed on the grammars in the algorithm.

Note that in the algorithm working on uncompressed texts, the priority queue can be implemented by a simple linked-list because every single replacement increases/decreases the frequency of a bigram by one, and we can afford to spend the cost of maintaining the list to run in \(O(N)\) time. However, this is not satisfactory for our “compressed-time” algorithm, which potentially runs in \(o(N)\) time. Thus, we use dynamic data structure for predecessor queries to implement the priority queue. For example, using the y-fast trie we can update the frequency of a bigram in \(O(\log \log N)\) expected time while supporting the function of the priority queue in \(O(\log \log N)\) time as well. Then the algorithm runs in \(O(\sum_{h=0}^{m} n_h + R \log \log N)\) time and \(O(n + R)\) space, where \(R\) is the total number of replacements executed on the grammars in our algorithm. Since \(R = O(\min(N, nm \log N))\) by Lemma 4, we can get the following theorem:

**Theorem 6.** Given an SLP \(S\) generating \(T\) of length \(N\), we can compute RePair(\(T\)) in expected \(O(\min(N, nm \log N) \log \log N)\) time and \(O(\min(N, nm \log N))\) space, where \(n\) and \(m\) are the numbers of variables in \(S\) and RePair(\(T\)), respectively.

5 Experiments

In this section, we show the results of our preliminary experiments. We implemented in C++ our algorithm to compute RePair(\(T\)) from an arbitrary grammar \(S\) for \(T\) running in \(O(\min(N, nm \log N)m)\) expected time and \(O(\min(N, nm \log N))\) space.

We choose the following three highly repetitive texts in repcorpus, einstein.en.txt (446 MB), world_leaders (45 MB) and fib41 (255 MB). We first compress each dataset by SOLCA, a space-optimal online grammar compression, to obtain \(S\), and feed \(S\) to our algorithm. In theory, SOLCA runs in \(O(N \log \log n)\) time and \(O(n)\) space.

Table 1 summarizes the results, where we also collected some data during the execution, which are useful for understanding the performance. The running time and working space of our algorithm deeply depend on the compressibility of each dataset. We confirmed that our algorithm potentially runs in compressed space for repetitive texts. We see that the recompression part for the extremely compressible text fib41 is done in a second. Unfortunately, for less compressible datasets our implementation does not scale well as \(n\) and \(m\) become larger. More precisely, the running time of our algorithm depends on

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7 See [http://pizzachili.dcc.uchile.cl/repcorpus/statistics.pdf](http://pizzachili.dcc.uchile.cl/repcorpus/statistics.pdf) for statistics of the datasets.
Figure 3 Comparisons in textsize [MB] / time [s] (larger one is faster) and space [MB] / textsize [MB] (smaller one is better).

\[ \sum_{h=0}^{m} |S_h| \], i.e., our algorithm runs in \( \Theta(\sum_{h=0}^{m} |S_h|) \) time. As the value \( \sum_{h=0}^{m} |S_h| \) is large even for relatively compressible datasets we tested, it may be hopeless to make the algorithm practical.

As mentioned in Section 4, our second algorithm runs in \( O(\sum_{h=0}^{m} n_h + R \log \log N) \) time and \( O(n + \max\{|S_h| \mid 0 \leq h \leq m\}) \) space, where \( n_h \) is the number of non-null variables in \( S_h \) and \( R \) is the total number of replacements executed on the grammars in the algorithm. Because \( R \) is upper bounded by \( N \), the term \( R \log \log N \) is almost linear in the worst-case. As we see Table 1, \( R \) is actually much smaller than \( N \). Also, Table 1 shows that \( \sum_{h=0}^{m} n_h \) is not so big compared to \( \sum_{h=0}^{m} |S_h| \), and thus, we expect that our second algorithm runs in a reasonable time.

Next we propose a new approach to reduce the peak memory usage of existing algorithms by combining with our algorithms. Since the peak memory usage is achieved at the very beginning of RePair, we can avoid it as follows: introducing parameter \( t \), we first use our algorithms until the input text \( T \) becomes sufficiently small, i.e., \( |T_h| < |T|/t \), and then, switch to a linear time algorithm that works in \( O(|T_h|) \) time and space. In our experiments, we combine our implementation described above with a well-tuned implementation of linear-time RePair by Maruyama [11] (denote it by RP). Setting \( t \in \{2, 3, 4, 5\} \), we compare our method with RP and the most space efficient linear-time algorithm [6, 2] to date (denote it by SERP). In theory, SERP runs in \( O(N/\epsilon) \) time using at most \( (1.5 + \epsilon)N \) words of space for arbitrary small \( \epsilon \leq 1 \), but \( \epsilon \) is fixed to 1 in their implementation. The results for some datasets from repcorpus are shown in Figure 3. We can see that our approach successfully slashes the peak memory usage of RP. Also, the time-space tradeoff is controlled by parameter \( t \) and our method with \( t = 3 \) outperforms SERP both in time and space.

6 Conclusions and Future Work

We have proposed the first recompression algorithm for obtaining an output of the RePair algorithm via other space-saving grammar compression without decompressing it. As a consequence, depending on the size of preliminarily compressed input text, our recompression algorithm can simulate the RePair algorithm in compressed space. We showed that our algorithm runs in reasonable time for several benchmarks consisting of highly compressible texts. Moreover, we showed that our algorithms can be used to reduce the peak memory
usage of existing RePair algorithms, and the approach outperforms the most space efficient linear-time algorithm to date. A future work is to implement the improved version of the recompression algorithm achieving the smaller time complexity and examine the performance of running time compared with other implementations of RePair and its variants. Another important future work is to prove preciser upper bound and/or lower bound of the recompression for RePair. An acquisition of new knowledge about the complexity would further reduce the running time and space of the proposed algorithm. These improvements lead us to the final goal: a faster recompression of RePair than the original one working in uncompressed space.

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