CHARACTERIZATIONS OF COMPACTLY ALMOST PERIODIC HOMEOMORPHISMS OF METRIZABLE SPACES

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Abstract. For metrizable spaces we replace the notion of almost periodic homeomorphism with a similar notion and verify that the usual characterizations of almost periodic homeomorphisms of compact metric spaces are valid for all metrizable spaces. We include examples and prove some related results.

1. Introduction

A homeomorphism \( h : X \to X \) of a metric space \((X, d)\) is \textit{almost periodic} if given \( \varepsilon > 0 \) there exists an integer \( N \) so that each block of \( N \) consecutive iterates of \( h \) contains a map \( h^n \) such that \( \forall x \in X \ d(x, h^n(x)) < \varepsilon \).

Classifying such maps for various choices of \((X, d)\) is the subject of [1],[5],[8], and [9]. A useful tool is the result of Gottschalk [6], [10].

Theorem 1 (Gottschalk). If \( X \) is a compact metric space then a homeomorphism \( h : X \to X \) is almost periodic if and only if \( \{h^n|n \in \mathbb{Z}\} \) is an equicontinuous family of maps.

Unfortunately the theorem is false if \( X \) is not compact since the property of almost periodicity is metric dependent. For example if \( h \) is an irrational rotation of the plane about the origin then \( h \) is not almost periodic with the usual metric, but \( h \) is almost periodic when \( \mathbb{R}^2 \) is equipped with a metric inherited from the compact 2 sphere \( \mathbb{R}^2 \cup \{\infty\} \). (See Theorem 4, example 6, [1] and [8]).

The main result of this paper, Theorem 2, serves to resurrect Theorem 1 by replacing the notion of almost periodic with the notion of a \textit{compactly almost periodic} homeomorphism defined as follows. If \( X \) is a metrizable space then a homeomorphism \( h : X \to X \) is \textit{compactly almost periodic} if the orbit closure of each compactum \( B \subset X \) under the action of \( h \) is compact, and if for each compact invariant subset \( A \subset X \), \( h_A : A \to A \) is almost periodic.

Though generally distinct, the two notions are often comparable:

1. Almost periodic homeomorphisms of locally complete metric spaces are compactly almost periodic.
2. If \( X \) is compact then \( h \) is compactly almost periodic if and only if \( h \) is almost periodic for each metric on \( X \).
3. If \( X \) is separable and locally compact then \( h \) is compactly almost periodic if and only if \( h \) is almost periodic for some metric on \( X \).

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In particular a homeomorphism \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) is compactly almost periodic if and only if \( h \) is conjugate to rotation or reflection (Corollary 3).

As in the compact case, \( h \) is compactly almost periodic if and only if there exists a metric \( d^* \) on \( X \) such that \( \{ h^n \} \) is a compact abelian group of isometries of \( X \) (the closure is taken in \( C(X, X) \) with the compact open topology). Consequently, the space of orbit closures under \( h \) forms an uppersemicontinuous decomposition of \( X \) (compatible with the Hausdorff metric) into compacta each of which is a topological abelian group (Theorem 5).

One difficulty created by the failure of \( X \) to be compact includes the following: There exists a compactly almost periodic isometry of a locally compact metric space \( X \) such that \( \{ h^n \} \) is a compact isometry group but \( \{ h^n \} \) is not metrizable. (See example 3).

All function spaces are endowed with the compact open topology and all references to metrics \( d \) on metrizable spaces \( X \) are assumed (or proven) to induce a topology compatible with that of \( X \).

2. Main Result

**Theorem 2.** Suppose \( X \) is a metrizable space and \( h : X \to X \) is a homeomorphism such that \( \{ h^n(x) \} \) is compact \( \forall x \in X \). Then the following are equivalent:

1. \( h \) is a compactly almost periodic homeomorphism.
2. For some metric \( d \) on \( X \) \( \{ h^n \} \) is an equicontinuous family of maps.
3. For some metric \( d^* \) on \( X \) \( h : (X, d^*) \to (X, d^*) \) is an isometry.
4. For each metric \( d \) on \( X \) \( \{ h^n \} \) is an equicontinuous family of maps.
5. \( \{ h^n \} \) is a compact subspace of \( C(X, X) \).
6. There exists a metric \( d^* \) on \( X \) such that \( \{ h^n \} \subset C(X, X) \) is a compact abelian topological group of isometries of \( X \).

*Proof.* Trivially 4 \( \Rightarrow \) 2. 2 \( \Rightarrow \) 5 and 5 \( \Rightarrow \) 4 by Theorem 3. Thus 2, 4, and 5 are equivalent. By Lemma 3 2 \( \Rightarrow \) 3. Trivially 3 \( \Rightarrow \) 2. 1 and 4 are equivalent by Lemma 3. Finally, 6 \( \Rightarrow \) 5 is trivial and 5 \( \Rightarrow \) 6 by Theorem 4. \( \square \)

3. Definitions

**Definition 1.** If \( A \) is a metric space then \( 2^A \), the collection of all compact subsets of \( A \) forms a metric space with the Hausdorff metric generated by the following condition. Two compacta \( \{ B, C \} \subset 2^A \) are less than a distance \( \varepsilon \) from each other if each point of \( b \in B \) is within \( \varepsilon \) of some point of \( C \) and if each point \( c \in C \) is within \( \varepsilon \) of some point of \( B \).

**Definition 2.** Let \( C(A, Y) \) denote the space of continuous function from the metric space \( A \) to the metric space \( Y \) with the compact open topology. Let \( H(X, X) \subset C(X, X) \) denote the space of homeomorphisms from \( X \) to \( X \).

**Definition 3.** If \( (Y, d_Y) \) is a metric space then a basis for \( C(X, Y) \) consists of sets \( B_A(f, \varepsilon) \) where \( A \subset X \) is compact and \( g \in B_A(f, \varepsilon) \) iff \( d_Y(f(a), g(a)) < \varepsilon \forall a \in A \). For \( \{ f, g \} \subset C(X, Y) \) let \( d_Y(f, g) = \sup_{x \in X} d(f(x), g(x)) \). Note \( d_Y(f, g) \leq \infty \).
Definition 4. Suppose \((X, d)\) is a metric space. Then a homeomorphism \(h : X \to X\) is \textit{almost periodic} if for each \(\varepsilon > 0\) there exists \(N \geq 0\) so that if \(M \in \mathbb{Z}\) then there exists \(i\) such that \(M \leq i \leq N + M\) and such that \(\forall x \in X\) \(d(h^i(x), x) < \varepsilon\). If \(X\) is a metrizable space and \(h : X \to X\) is a homeomorphism such that \(\bigcup_{n \in \mathbb{Z}} h^n(B)\) is compact for each compact \(B \subset X\), and if for each compact invariant set \(A\), \(h_A : A \to A\) is an almost periodic homeomorphism of \(A\) then \(h\) is \textit{compactly almost periodic}.

Definition 5. Suppose \((A, d_A)\) and \((Y, d_Y)\) are metric spaces. A subset \(F \subset C(A, Y)\) is \textit{equicontinuous} if for each \(a \in A\) and each \(\varepsilon > 0\) there exists \(\delta > 0\) such that for each \(f \in F\), if \(d_A(a, x) < \delta\) then \(d_Y(f(a), f(x)) < \varepsilon\).

Definition 6. If \(F \subset C(X, Y)\) and \(A \subset X\) then \(F_A \subset C(A, Y)\) denotes all maps \(f_A\) such that \(f \in F\).

Definition 7. If \((X, d)\) is a metric space then a homeomorphism \(h : X \to X\) is an \textit{isometry} if \(d(x, y) = d(h(x), h(y))\). Let \(\text{ISOM}(X, d) \subset C(X, X)\) denote the collection of isometries from \(X\) to \(X\).

Definition 8. The metric space \((X, d)\) is \textit{locally complete} if for each \(x \in X\) there exists \(\varepsilon > 0\) such that \(\overline{B(x, \varepsilon)}\) is complete with respect to \(d\).

4. Comparing almost periodic and compactly almost periodic homeomorphisms

The notions ‘almost periodic homeomorphism’ and ‘compactly almost periodic homeomorphism’ are distinct but often comparable.

4.1. Positive comparisons.

Lemma 1. Suppose \((X, d)\) is a locally complete metric space and \(h : X \to X\) is an almost periodic isometry. Then \(h\) is compactly almost periodic.

Proof. Since \(h\) is an isometry \(\{h^n\}\) is equicontinuous. By Theorem 3 it suffices to prove \(\{h^n(x)\}\) is compact \(\forall x \in X\). Suppose \(x \in X\). Choose \(\gamma > 0\) so that \(\overline{B(x, \gamma)}\) is complete. To prove \(\{h^n(x)\}\) is complete it suffices to prove that each Cauchy sequence of the form \(\{h^{n_k}(x)\}\) has a limit in \(X\). Suppose \(\{h^{n_k}(x)\}\) is Cauchy. Since \(\{h^{n_k}(x)\}\) is Cauchy we may choose \(K\) so that \(d(h^{n_m}(x), h^{n_l}(x)) < \gamma\) if \(m, l \geq K\). Since \(h^{-N_K}\) is an isometry \(\{h^{n_k-N_K}(x)\}\) is also Cauchy and \(d(h^{n_m-N_K}(x), h^{n_l-N_K}(x)) < \gamma\) if \(m, l \geq K\). Taking \(m = K\) \(d(h^{n_m-N_K}(x), h^{n_l-N_K}(x)) = d(x, h^{n_l-N_K}(x)) < \gamma\). Consequently \(\{h^{n_k-N_K}(x)\}\) has a limit \(y\). Hence \(\{h^{n_k}(x)\}\) has limit \(h^{N_K}(y)\). Thus \(\overline{\{h^n(x)\}}\) is complete. By Lemma 3 \(\{h^n(x)\}\) is totally bounded and therefore \(\overline{\{h^n(x)\}}\) is compact.

Theorem 3. Suppose \((X, d)\) is a metric space and \(h : X \to X\) is an almost periodic homeomorphism. Then \(h\) is compactly almost periodic if and only if \(\forall x \in X\) \(\{h^n(x)\}\) is locally complete with respect to \(d\).

Proof. If \(h\) is compactly almost periodic then \(\overline{\{h^n(x)\}}\) is compact and hence complete and therefore locally complete. Conversely suppose \(\{h^n(x)\}\) is locally complete \(\forall x \in X\). By Lemma 3 \(\{h^n\}\) is equicontinuous. By Lemma 3 \(\overline{\{h^n(x)\}}\) is totally
bounded and hence bounded. Let \( d^*(x, y) = \sup_n d(h^n(x), h^n(y)) \). By Lemma \( d^* \) is a metric equivalent to \( d \) such that \( h \) is an isometry of \((X, d^*)\). Suppose \( x \in X \).

Let \( Y = \{ h^n(x) \} \). To prove \( Y \) is locally \( d^* \) complete suppose \( y \in Y \). Choose \( \gamma > 0 \) so that \( A = \overline{B_\gamma(y, \gamma) \cap \{ h^n(x) \}} \) is \( d \) complete. Choose \( \beta > 0 \) so that \( d(y, z) < \beta \Rightarrow d^*(y, z) < \gamma \). Let \( C = \overline{B_\delta(y, \beta) \cap \{ h^n(x) \}} \). Note \( C \subset A \) and \( C \) and \( A \) are closed. If \( \{ z_n \} \) is \( d^* \) Cauchy in \( C \) then \( \{ z_n \} \) is \( d \) Cauchy in \( A \). Thus \( z = \lim z_n \) exists and \( z \in A \cap C \). Hence \( C \) is \( d^* \) complete and therefore \( Y \) is locally \( d^* \) complete. Now we will prove that \( h_Y \) is an almost periodic isometry of \( Y \). Suppose \( \varepsilon > 0 \). Choose \( \delta < \varepsilon \) so that \( d(x, y) < \delta \Rightarrow d(h^n(x), h^n(y)) < \varepsilon / 2 \). Choose \( N \) so that each block of iterates contains a map \( h^n \) such that \( d(h^n, id) < \delta \). Suppose \( n \in \mathbb{N} \). Choose \( 0 \leq m < N \) such that \( d(h^{n-m}, id) < \delta \). Thus \( d(h^{n-m}(x), x) < \delta \). Hence \( d^*(h^{n-m}(x), x) < \varepsilon / 2 \). However \( h^{n-m} \) is a \( d^* \) isometry. Thus \( d^*(h^{n-m}(h^k(x)), h^k(x)) < \varepsilon / 2 \). Hence, since \( \{ h^k(x) \} \) is dense in \( Y \), \( d^*(h_Y^{n-m}, id_Y) \leq \varepsilon / 2 < \varepsilon \). By Lemma \( h(Y) = Y \). Therefore \( h_Y \) is an almost periodic isometry of the locally complete space \( Y \). Thus by Lemma \( h_Y \) is compactly almost periodic. In particular, since \( \{ h_Y^n(x) \} \) is dense in \( Y \), \( Y \) is compact. Hence by Theorem \( h \) is compactly almost periodic.

**Corollary 1.** If \((X, d)\) is a locally compact metric space then each almost periodic homeomorphism of \( X \) is compactly almost periodic.

**Theorem 4.** Suppose \( X \) is a locally compact separable metrizable space and \( h : X \to X \) is a homeomorphism. Then \( h \) is compactly almost periodic if and only if there exists a metric \( d \) on \( X \) such that \( h \) is an almost periodic homeomorphism of \((X, d)\).

**Proof.** Suppose \( h \) is compactly almost periodic. By Theorem \( h \) choose a metric \( d^* \) on \( X \) such that \( H = \{ h^n \} \subset C(X, X) \) is a compact isometry group of \( X \). Let \( Y = X \cup \{ \infty \} \), the one point compactification of \( X \) with metric \( d \). By Remark \( Y \) is metrizable. By Theorem \( H \) and Remark \( H \) and \( H(Y, Y) \) are topological groups. Define \( \phi : H \to H(Y, Y) \) as \( \phi(g) = G \) where \( G : Y \to Y \) is the unique homeomorphism such that \( G|_{X} = g \) and \( G(\infty) = \infty \). Continuity of \( \phi \) will follow from continuity of \( \phi \) at \( id_X \). Let \( W = B_Y(id_Y, \varepsilon) \subset H(Y, Y) \). Let \( A = Y \setminus B_{d^*}(\infty, 2\varepsilon / 3) \). Let \( C = Y \setminus B_{d^*}(\infty, 2\varepsilon / 3) \). Let \( U = \{ p \in G(p) \subset int(A) \} \). Let \( V_1 \) be an open set containing \( id_X \) such that \( V_1^{-1} \subset U \). Let \( V_2 = \{ g \in G|d^*(id_A, g_A) < \varepsilon \} \). Let \( V = V_1 \cap V_2 \). Suppose \( g \in V \). If \( x \in A \) then \( d^*(g(x), x) \leq \varepsilon \) since \( g \in V_2 \). If \( x \notin A \) then \( g(x) \notin C \) since \( g^{-1} \notin U \). Thus \( d^*(g(x), x) \leq d^*(x, \infty) + d^*(\infty, g(x)) < \varepsilon / 3 + 2\varepsilon / 3 = \varepsilon \).

Hence \( g(V) \subset U \). Therefore \( \phi \) is continuous. Hence \( \phi(H) = \{ \phi(h^n) \} \) is compact and therefore by Theorem \( \phi(h) \) is an almost periodic homeomorphism of \((Y, d)\). Thus, since \( \phi(h)(X) = X \) and \( \phi(h)|_X = h|_X \) it follows that \( h \) is an almost periodic homeomorphism of \((X, d)\). The converse follows from Corollary \( \phi(h) \).

**Corollary 2.** Orientation preserving compactly almost periodic homeomorphisms of the plane \( R^2 \) are conjugate to rotation about the origin. Orientation reversing compactly almost periodic homeomorphisms of \( R^2 \) are conjugate to reflection about the \( y \) axis.

**Proof.** If \( h : R^2 \to R^2 \) is compactly almost periodic then, following the proof of Theorem \( h \) extends to an almost periodic homeomorphism of \( S^2 = R^2 \cup \{ \infty \} \) fixing \( \{ \infty \} \). Now apply the results of Ritter \( \phi(h) \).
4.2. Negative comparisons and other counterexamples. In each of the following examples $X$ is a metrizable space and $h : X \to X$ is either a homeomorphism or an isometry. Examples 2 and 3 illustrate two ‘pathological’ isometry groups. In examples 4 and 5, $h$ fails to be compactly almost periodic despite some favorable properties.

**Example 1.** Let $S^1 \subset \mathbb{R}^2$ be the unit circle with the usual metric. Let $h : S^1 \to S^1$ be an irrational rotation. Let $X = \{\bigcup_{n \in \mathbb{Z}} h^n(1)\} \subset S^1$. Then $h_{|X} : X \to X$ is an almost periodic homeomorphism but $h$ is not compactly almost periodic.

**Proof.** Let $J \subset S^1$ be an uncountable set such that for each finite subset $A = \{\alpha_1, .., \alpha_N\} \subset J$, each point $\{z_1, .., z_n\} \in \prod_{i=1}^n S^1$ under the homeomorphism $r_A : \prod_{i=1}^n S^1 \to \prod_{i=1}^n S^1$ defined as $r_A(x_1, .., x_N) = (\alpha_1 x_1, .., \alpha_N x_N)$. Let $X$ denote the disjoint union of uncountably many circles indexed by $J$. Let points of distinct circles in $X$ have distance 1 and let each circle $S_j$ be isometric to $S^1$. For $j \in J$ define $h_j : S_j \to S_j$ to be the rotation $h_j(x) = \alpha_j x$. Define $h : X \to X$ as $h = \bigcup_{j \in J} h_j$. Note $h$ is a compactly periodic isometry of $X$. Hence $\{h^n\}$ is a compact group of isometries of $X$ and $g(S_j) = S_j \forall j \in J \forall g \in \{h^n\}$. Let $Y = \prod_{j \in J} S_j$. Choose a ‘basepoint’ $x_j$ from each $S_j \subset X$. Define $\phi : \{h^n\} \to Y$ as $\phi(g) = \{g(x_j)\}$. Then $\phi$ maps onto a dense subset of $Y$ and hence $\phi$ is surjective. Thus $\{h^n\}$ is not metrizable. Consequently for each equivalent metric $d$ on $X$, $h$ is not an almost periodic homeomorphism of $(X,d)$ since if $h$ were an almost periodic homeomorphism of $(X,d)$ then $\{h^n\}$ would be metrizable via the uniform metric. See 3 for more details.

**Example 3.** Isometry groups need not be closed in $C(X,X)$.

**Proof.** Let $X = \{1, 2, 3, ..\}$ with metric $d(m,n) = 1$ iff $m \neq n$. Let $f_n : X \to X$ be a bijection such that $f_n(k) = 2k$ if $k \leq n$. Define $f : X \to X$ as $f(n) = 2n$. Then $f_n \to f$, $f_n$ is an isometry of $X$ but $f$ is not an isometry of $X$. Hence $ISOM(X,d)$ is not a closed subspace of $C(X,X)$.

**Example 4.** $h : X \to X$ is a homeomorphism, $\overline{h^n(x)}$ is compact $\forall x \in X$ and $h_A : A \to A$ is almost periodic for each compact invariant set $A \subset X$, but $h$ is not compactly almost periodic.

**Proof.** Let $\theta \in S^1$. Let $X = ((\{0, 1\} \times S^1) \cup \{0, \theta\} \cup \{1/3, 1/2, 1\} \times S^1)$ and $h(1/n, z) = (1/n, e^{i\pi z})$ for $n \in \mathbb{Z}^+$ and $h(0, \theta) = (0, \theta)$.

**Example 5.** $X$ is compact, $\forall x \in X$ if $A = \overline{h^n(x)}$ then $h_A : A \to A$ is an almost periodic isometry, but $h$ is not compactly almost periodic.

**Proof.** Let $X = D^2$, the closed unit disk in $\mathbb{R}^2$. Let $h(r, \theta) = (r, e^{i\theta})$.

**Example 6.** Let $X = \mathbb{R}^2$. Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ as $h(z) = e^{2\pi i \theta}z$ where $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then $h$ is compactly almost periodic but $h$ is not almost periodic with the usual metric on $\mathbb{R}^2$. If $d$ is any metric on $\mathbb{R}^2 \cup \{\infty\}$ then $h$ is almost periodic with respect to $d$. See Theorem 3.
5. Application: A continuous decomposition of $X$ into compact metrizable abelian groups

A decomposition $X^*$ of a topological space $X$ is a collection of disjoint closed subsets of $X$ whose union is $X$. Endowing $X^*$ with the quotient topology, the natural quotient map $\pi: X \to X^*$ is both open and closed if and only if $X^*$ is both uppersemicontinuous and lowersemicontinuous. Such a decomposition is called continuous.

Theorem 5. Suppose $X$ is a metrizable space, $h: X \to X$ is a compactly almost periodic homeomorphism. Then sets of the form $\{h^n(x)\}$ determine a continuous decomposition of $X$ into compact abelian groups.

Proof. Let $H = \{h^n\}$ and let $G = \overline{H} \subset C(X, X)$. By theorem 6, $G$ is a compact abelian topological group of isometries of $X$ for some metric $d$. By Lemma 8, $G(x) = H(x)$. Thus if $\{x, y\} \subset X$ then $y \in H(x)$ iff there exists $g \in G$ such that $y = g(x)$ iff $G(x) = G(y)$. Thus the collection of orbit closures under the full action of $h$ forms a partition of $X$ into disjoint compacta. Suppose $z^* \in G(x)$ and $w^* \in G(y)$. Then for each $g \in G$ $d(z^*, w^*) = d(g(z^*), g(w^*))$. Thus the Hausdorff distance between $G(x)$ and $G(y)$ is the minimum of $d_{z \in G(x), w \in G(y)} d(z, w)$. Hence, endowing $X/G$ with the Hausdorff topology, the natural map $\pi: X \to X/G$ is both continuous and open since $d(x, y) \geq d(\pi(x), \pi(y)))$. In particular $\pi$ is a quotient map. To show $\pi$ is also a closed map suppose $F \subset X$ is closed and $A = \cup_{g \in G, G \neq \emptyset} G(x)$. Suppose $a \in A \setminus a_n \subset A$ such that $a_n \to a$. Let $a_n = g_n(x_n)$ where $g_n \in G$ and $x_n \in F$. Since $G$ is compact let $g_n \to g \in G$. By Lemma 10, $g_n \to g \in G$. Let $B = a \cup \{a_n\}$. $B$ is compact. Thus $g_n^{-1}(a_n) \to g^{-1}(a)$. Thus $g^{-1}(a) \in F$ since $F$ is closed. Let $x = g^{-1}(a)$. Hence $a = g(x)$. Thus $A = A$. Hence $\pi$ is a closed map. Hence the orbit closures of $H$ determine continuous decomposition of $X$. Now we check the group structure of each orbit $G(x)$. Let $K(x) = \{g \in G| g(x) = x\}$. Note $K(x)$ is a compact abelian normal subgroup of $G$. Hence the compact group $G/K(x)$ acts freely and effectively on $G(x)$. Hence $G(x)$ is a compact topological abelian group under the multiplication $g_1(x) * g_2(x) = g_1g_2(x)$. 

6. Theorems, Lemmas, and Remarks

6.1. Arzela-Ascoli Theorem.

Theorem 6 (Arzela-Ascoli). Suppose each of $X$ and $Y$ is a metric space and $\mathcal{F} \subset C(X, Y)$ in the compact open topology. Suppose $\forall x \in X \{f(x)| f \in \mathcal{F} \}$ has compact closure in $Y$. The following are equivalent

1. $\mathcal{F}$ is equicontinuous
2. $\mathcal{F}$ has compact closure in $C(X, Y)$ with the compact open topology.

Proof. 1 $\Rightarrow$ 2. See [1] p290. Local compactness of $X$ is not required. However the proof makes essential use of the Tychonoff theorem, the fact that the arbitrary product of compacta is compact.

2 $\Rightarrow$ 1. Suppose in order to obtain a contradiction that $\mathcal{F}$ is not equicontinuous at $x$. Choose $\varepsilon > 0$ and $x_n \to x$ and $f_n \in \mathcal{F}$ such that $d(f_n(x), f_n(x_n)) \geq \varepsilon$. Let $A = \{x \cup \{x_1, x_2, \ldots \}\}$. By compactness of $\mathcal{F}$ choose a convergent subsequence such that $f_{n_k} \to f$. In particular $f_{n_k} A \to f A$ uniformly and hence $f_{n_k} A$ is equicontinuous. Thus $d(f_{n_k}(x), f_{n_k}(x_{n_k})) \to 0$ and we have our contradiction. 


6.2. Remarks.

Remark 1. Suppose \((X,d)\) is a metric space and \(f : X \to X\) is a surjective function such that \(d(x,y) = d(f(x),f(y))\) for all \(x,y \in X\). Then \(f\) is an isometry.

Remark 2. Suppose \(X\) and \(Y\) are metric spaces \(\mathcal{F} \subset C(X,Y)\) and \(\mathcal{F}_A\) is equicontinuous for each compact subset \(A\). Then \(\mathcal{F}\) is equicontinuous.

Remark 3. Suppose \(h : X \to X\) is a homeomorphism, \(B \subset X\) and \(\bigcup_{n \in \mathbb{Z}} h^n(B) = \bigcup_{n \in \mathbb{Z}} h^n(B)\). Then \(h(A) = A\).

Remark 4. If \((A,d)\) is a compact metric space then ISOM\((A,d)\) is a compact topological group.

Remark 5. If \(X\) is a locally compact separable metrizable space then the one point compactification of \(X\) is metrizable. See Theorem 8.6 p.247 of Dugundji \(^3\).

Remark 6. Suppose \(Y\) is a compact metric space. Then \(H(Y,Y)\) is a topological group under function composition in the uniform topology.

6.3. Lemmas.

Lemma 2. Suppose \(X\) and \(Y\) are metric spaces, \(\mathcal{F} \subset C(X,Y)\) is compact and \(B \subset X\). Then \(A = \bigcup_{f \in \mathcal{F}} f(B)\) is compact.

Proof. Let \(f_n(x_n) \in A\) with \(f_n \in \mathcal{F}\) and \(x_n \in B\). Since \(\mathcal{F}\) is sequentially compact, and \(B\) is compact. Choose convergent subsequences \(f_{n_k} \in \mathcal{F}\) and \(x_{n_k} \in B\) such that \(f_{n_k} \to f \in \mathcal{F}\) and \(x_{n_k} \to x \in B\). Hence \(f_{n_k}(x_{n_k}) \to f(x)\). Thus \(A\) is sequentially compact and hence compact.

Lemma 3. Suppose \((X,d)\) is a metric space, \(h : X \to X\) is a homeomorphism, and \(\overline{\{h^n(x)\}}\) is compact \(\forall x \in X\). Then \(h\) is compactly almost periodic iff \(\{h^n\}\) is equicontinuous.

Proof. Suppose \(h\) is compactly almost periodic and \(B \subset X\) is compact. Let \(A = \bigcup_{n \in \mathbb{Z}} h^n(B)\). By Remark 1 \(h(A) = A\). Thus by Theorem 1 \(\{h^n\}\) is equicontinuous. Since \(B \subset A\) \(\{h^n\}\) is equicontinuous. Thus

by Remark \(\{h^n\}\) is equicontinuous. Conversely suppose \(\{h^n\}\) is equicontinuous and \(B \subset X\) is compact. By Lemma \(A = \bigcup_{n \in \mathbb{Z}} h^n(B)\) is compact. Suppose \(A \subset X\) is compact and \(h(A) = A\). Then \(\{h^n\}\) is equicontinuous and hence by Theorem 1 \(h_A\) is almost periodic. Thus \(h\) is compactly almost periodic.

Lemma 4. Suppose \((X,d)\) is a metric space and \(h : X \to X\) is an almost periodic homeomorphism. Then \(\{h^n\}\) is equicontinuous.

Proof. Suppose \(\varepsilon > 0\) and \(x \in X\). Choose \(N > 0\) so that each block of \(N\) iterates contains a map \(h^m\) such that \(d(id,h^m) < \varepsilon/3\). Choose \(\delta > 0\) such that \(d(x,y) < \delta \Rightarrow d(h^i(x),h^i(y)) < \varepsilon\) whenever \(|i| \leq N\). Suppose \(n \in \mathbb{Z}\). Choose \(m\) such that \(0 \leq m \leq N\) and \(d(h^{n-m},id) < \varepsilon/3\). Note \(d(h^n,h^m) < \varepsilon/3\). If \(d(x,y) < \delta\) then

\[
\begin{align*}
&d(h^n(x),h^m(y)) \leq d(h^n(x),h^m(x)) + d(h^m(x),h^m(y)) + d(h^m(y),h^n(y)) \\
&< \varepsilon.
\end{align*}
\]

Thus \(\{h^n\}\) is equicontinuous.

\(\square\)
Lemma 5. Suppose \((X, d)\) is a metric space and \(h : X \to X\) is an almost periodic homeomorphism. Then \(\{h^n(x)\}\) is totally bounded \(\forall x \in X\).

Proof. Suppose \(x \in X\) and \(\varepsilon > 0\). Choose \(N > 0\) so that each block of \(N\) iterates contains a map \(h^k\) such that \(d(id, h^k) < \varepsilon/2\). Let \(A = \{x, h(x), \ldots, h^N(x)\}\). Suppose \(y \in \{h^n(x)\}\). Let \(d(y, h^n(x)) < \varepsilon/2\). Choose \(m\) such that \(0 \leq m \leq N\) and \(d(h^{n-m}, id) < \varepsilon/2\). Note \(d(h^n, h^m) < \varepsilon/2\). Thus \(d(y, h^n(x)) \leq d(y, h^m(x)) + d(h^n(x), h^m(x)) < \varepsilon\) and hence \(\{h^n(x)\}\) is totally bounded.

Lemma 6. Suppose \((X, d)\) is a metric space and \(h : X \to X\) is a homeomorphism such that \(\{h^n\}\) is equicontinuous over \((X, d)\) and \(\{h^n(x)\}\) is bounded \(\forall x \in X\). Then \(d^*(x, y) = \sup_{n \in N} d(h^n(x), h^n(y))\) is an equivalent metric and \(h\) is an isometry of \((X, d^*)\).

Proof. Note \(d^*\) is well defined since \(d\) is bounded on \(\{h^n(x)\} \cup \{h^n(y)\}\).
1) By definition \(d^* \geq 0\) since \(d \geq 0\).
2) If \(x = y\) then \(h^n(x) = h^n(y) \forall n \in Z\) and hence \(d^*(x, x) = 0\).
3) If \(x \neq y\) then \(d^*(x, y) \geq d(id(x), id(y)) = d(x, y) > 0\).
4) \(d^*\) is symmetric since \(d\) is symmetric.
5) Suppose \(\varepsilon > 0\). Choose \(n \in Z\) so that \(d^*(x, z) < d(h^n(x), h^n(z)) + \varepsilon\).

Note \(d(h^n(x), h^n(z)) \leq d(h^n(x), h^n(y)) + d(h^n(y), h^n(z)) \leq d^*(x, y) + d^*(y, z)\).

Thus \(d^*(x, z) \leq d^*(x, y) + d^*(y, z) + \varepsilon\) and hence, since \(\varepsilon\) is arbitrary, \(d^*(x, z) \leq d^*(x, y) + d^*(y, z)\).

6) To show \(d\) and \(d^*\) are comparable it suffices to prove \(ID : (X, d^*) \to (X, d)\) is a homeomorphism. Suppose \(x_n \to x\). Then \(ID(x_n) \to ID(x)\) since \(d(x, x_n) \leq d^*(x, x_n)\). Thus \(ID\) is continuous. To prove \(ID^{-1}\) is continuous at \(x\) suppose \(\varepsilon > 0\). By equicontinuity of \(\{h^n\}\) at \(x\) choose \(\delta > 0\) so that \(d(y, x) < \delta \Rightarrow d(h^n(y), h^n(x)) < \varepsilon/2\). Suppose \(d(x, y) < \delta\). Then \(d^*(x, y) < \varepsilon/2\). Thus \(d^*(x, y) < \varepsilon/2 < \varepsilon\). Hence \(ID^{-1}\) is continuous.

7) To prove \(h\) is an isometry of \((X, d^*)\) it suffices by Remark [3] since \(h\) is surjective to prove \(d^*(x, y) = d^*(h^*(x), h^*(y))\). Note \(d^*(h(x), h(y)) = \sup_{n \in Z} d(h^n(x), h^n(y)) = \sup_{n \in Z} d(h^{n+1}(x), h^{n+1}(y)) = d^*(x, y)\).

Thus \(h\) is an isometry of \((X, d^*)\).

6.4. Theorem [7]. We assume throughout this section that \(G\) is a countable abelian group of isometries of the metric space \((X, d)\) and that \(\overline{G}\) is a compact subspace of \(C(X, X)\). In general the isometry group of a metric space \(X\) is not a closed subspace of \(C(X, X)\) and compact isometry groups are generally not metrizable [See examples [8] and [9]]. Hence some care is required to establish Theorem [7]. The countability of \(G\) is not essential for these results but enables us to detect limit points of \(G\) with sequences rather than nets.

Theorem 7. Suppose \((X, d)\) is a metric space and \(G \subset C(X, X)\) is a countable abelian group of isometries of \(X\) such that \(\overline{G}\) is compact in \((X, X)\) with the compact open topology. Then \(\overline{G}\) is a compact abelian topological group of isometries of \(X\).

Proof. This is a direct consequence of Lemmas [3] and [10].

Lemma 7. If \((X, d)\) is a metric space then isometry composition is continuous in the sense that the function \(\phi : ISOM(X, d) \times ISOM(X, d) \to ISOM(X, d)\) defined as \(\phi(f, g) = fg\) is continuous.
such that if \( df \) is a metric space it suffices to prove \( \overline{G(A)} \subset \overline{G(A)} \) is compact.

**Lemma 8.** If \( A \subset X \) is compact then \( \overline{G(A)} = \overline{\overline{G(A)}} \) and \( \overline{G(A)} \) is compact.

**Proof.** Suppose \( y \in \overline{G(A)} \). If \( y \in G(A) \) then \( y \in \overline{G(A)} \). Suppose \( y \in \overline{G(A)} \backslash G(A) \). Let \( y = \lim n_a(a_n) \) where \( a_n \subset A \) and \( g_n \to g \in \overline{G} \). Let \( B = \{a\} \cup \{a_n\} \). Since \( B \) is compact \( g_nB \to gB \) uniformly. Thus \( y = \lim n_a(a_n) = \lim g_n(a_n) = g(a) \). Hence \( \overline{G(A)} \subset \overline{G(A)} \). Conversely, suppose \( y \in \overline{G(A)} \). Let \( y = g(a) \) with \( a \in A \) and \( g \in \overline{G} \). Since \( G \) is countable we may choose \( g_n \in G \) such that \( g_n \to g \). Hence \( g_n(a) \to g(a) = y \). Thus \( y \in \overline{G(A)} \). Hence \( \overline{G(A)} = \overline{\overline{G(A)}} \) is compact.

**Lemma 9.** \( \overline{G} \) is a collection of isometries of \( X \).

**Proof.** Suppose \( g \in \overline{G} \) and \( y \in X \). Since \( G \) is countable choose \( g_n \in G \) such that \( g_n \to g \). Let \( y = g_n(a_n) \) where \( a_n \subset A \). Choose a convergent subsequence \( a_{n_i} \to a \in A \). Let \( B = \{a\} \cup \{a_{n_i}\} \). Then \( y = \lim g_n(a_{n_i}) = g(a) \). Thus \( g \) is surjective. Suppose \( x, y \in X \). By continuity of \( d \) at \( (g(x), g(y)) \) choose \( \delta > 0 \) such that if \( d(w, g(x)) < \delta \) and \( d(z, g(y)) < \delta \) then \( d(w, g(x)) - d(z, g(y)) < \epsilon \). Choose \( h \in ISOM(X, d) \) so that \( d(h(x, y), g(x, y)) < \delta \). Let \( z = h(x) \) and \( w = h(y) \) and observe that \( d(g(x), g(y)) = d(x, y) \) and \( d(h(x), h(y)) < \epsilon \). Thus \( d(h(x, y)) = d(g(x), g(y)) \). Letting \( Y = im(g) \) and \( \delta = \epsilon \) shows that \( g \) is a homeomorphism of \( X \) onto \( Y \).

**Lemma 10.** Suppose \( f, g \in \overline{G} \). Then \( fg \in \overline{G} \), \( g^{-1} \in \overline{G} \), \( \overline{G} \) is abelian, and the map \( \phi : \overline{G} \to \overline{G} \) is continuous where \( \phi(g) = g^{-1} \).

**Proof.** Since \( G \) is countable let \( f_n \to f \) and \( g_n \to g \) with \( f_n, g_n \in G \). Then \( f_n g_n \to fg \) by Lemma 8 and hence \( fg \in \overline{G} \) since \( f_n g_n \in G \) and \( \overline{G} \) is closed. By Lemma 8 \( g^{-1} \) is an isometry and hence \( g^{-1} \) exists. We will prove \( g_n^{-1} \to g^{-1} \) and conclude that \( g^{-1} \in \overline{G} \) since \( \overline{G} \) is closed. Suppose \( B \subset X \) is compact. Let \( A = \overline{B}(B) \). By Lemma 8 \( A \) is compact and invariant under \( G \). Hence \( g_n[A] \to g[A] \) and thus by Remark 8 \( (g_n[A])^{-1} \to (g[A])^{-1} = (g^{-1})[A] \). Thus \( g^{-1} \in \overline{G} \). Hence \( \phi \) is well defined. To check continuity let \( U = B_{\overline{B}(g^{-1}, \epsilon)} \subset \overline{G} \). Let \( V = B_{\overline{B}(g, \epsilon)} \cap \overline{G} \). Note \( \phi(V) \subset U \). Thus \( \phi \) is continuous at \( g \) and hence continuous. By Lemma 8 \( fg(x) = \lim f_n g_n(x) = \lim g_n f_n(x) = gf(x) \). Thus \( \overline{G} \) is abelian.
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