Bose-Einstein condensation in the canonical ensemble

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Abstract

Large-volume, high-temperature Bose-Einstein condensation is illustrated for a relativistic $O(2)$-invariant scalar field with fixed charge using the canonical ensemble. The standard, grand canonical results are reproduced for the infinite-volume limit. Finite-volume corrections are calculated in the canonical ensemble and the results are found to differ from the finite-volume grand canonical approximation in a consistent qualitative way.

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1 Introduction

When dealing with conserved charges in a system in equilibrium the usual procedure is to employ the grand canonical ensemble [1, 2, 3]. The physical picture from which the grand canonical ensemble is derived is that of a system in contact with an infinite reservoir of particles with an associated chemical potential for flow of particles into and out-of the system. Averaging in the grand canonical ensemble is performed over all possible charge states. The average charge is then fixed by a definite choice of chemical potential.

By contrast, in the canonical ensemble, charge cannot move into and out-of the system. There is no chemical potential. The charge is fixed and any averaging must be performed only over states with this definite fixed charge. This provides a more reasonable model for a system which is insulated with respect to charge and a better approximation for experiments involving Bose-Einstein condensation (BEC) of trapped atoms [4].

A clear outline of the path integral formulation of relativistic field theory with exactly conserved energy and charge can be found in Refs.[5, 6]. The central technique is to insert delta-functions into the trace over states in order to select only those states with chosen eigenvalues of the Hamiltonian and charge operators. This is the technique we shall use to exactly conserve charge in the canonical ensemble. An example of the application of this method to BEC in the canonical ensemble can be found in Ref.[7] where fluctuations in the ground state

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occupation are studied in the non-interacting, non-relativistic limit. A further study of the comparison between canonical and grand canonical ensembles with regard to trapped atoms can be found in Ref. [8].

In this article, we outline a fully relativistic calculation of BEC using the canonical ensemble. After mathematically formulating the model in the next section we go on to consider the high-temperature, large-volume limit in section 3 (c.f. Refs. [1, 2, 3]). We calculate the proportion of the total charge that occupies the lowest energy state in section 4, and as expected, the canonical and grand canonical ensembles are found to give the same results in the infinite-volume limit. In section 5 we consider finite-volume corrections where a difference between the two ensembles is observed.

2 Exactly conserved charge

To study a conserved charge \( Q \) using the canonical ensemble we must identify and only average over those states with this specific charge. As noted above, this is taken care of by inserting a delta-function into the trace over states. The partition function is then

\[
Z = \text{Tr} \left\{ \delta_{Q, \hat{Q}} \hat{\rho} \right\},
\]

(1)

The density operator \( \hat{\rho} \) in the canonical ensemble gives a probability to each state in accordance with the temperature of the system \( T = 1/\beta \). It is given by

\[
\hat{\rho} = e^{-\beta \hat{H}},
\]

(2)

where \( \hat{H} \) is the Hamiltonian operator.

The total charge \( Q \) has integer value and so the integral form of the delta-function is

\[
\delta_{Q, Q'} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma \ e^{i(Q-Q')\sigma}.
\]

(3)

The partition function now contains a trace over states where the density operator is multiplied by a charge operator-dependent phase factor

\[
Z = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma \ e^{iQ\sigma} \text{Tr} \left\{ e^{-i\hat{Q}\sigma} \hat{\rho} \right\}.
\]

(4)

This phase factor can be absorbed into the states where its effect is felt through a change in the boundary conditions. However, it is simpler to absorb it as a shift in the Hamiltonian of the system:

\[
\hat{H} \rightarrow \hat{H}' = \hat{H} + \frac{i}{\beta} \hat{Q}.
\]

(5)

The non-hermiticity of this Hamiltonian would concern us if we were to apply it to a dynamical problem. In thermal equilibrium this is nothing more than a mathematical trick.

We shall consider a relativistic \( O(2) \)-invariant scalar theory. This is one of the simplest theories which contains a conserved charge. It is useful as a
simplified version of the Higgs sector and also for describing atomic gases in its non-relativistic limit. The Hamiltonian and charge are given by

\[ \hat{H} = \int_V d^3x \left( \frac{1}{2} \left[ \pi_1^2 + \pi_2^2 + (\nabla \hat{\phi}_1)^2 + (\nabla \hat{\phi}_2)^2 + m_1^2 \hat{\phi}_1^2 + m_2^2 \hat{\phi}_2^2 \right] \right), \]

\[ \hat{Q} = \int_V d^3x \left( \hat{\phi}_2 \hat{\pi}_1 - \hat{\phi}_1 \hat{\pi}_2 \right). \quad (6) \]

The trace in Eq. (4) can be expressed as a functional integral in terms of the Lagrangian for this system \[9\]

\[ \text{Tr} \left\{ e^{-i\hat{Q}\sigma} \hat{\rho} \right\} = \text{Tr} \left\{ e^{-\beta \hat{H}} \right\} \propto \int [d\phi_a] e^{i \int_0^{-i\beta} dt L'(t)} \quad (7) \]

where the Lagrangian is found to be

\[ L'(t) = \int_V d^3x \left. \frac{1}{2} \left[ \left( \partial_t \phi_1 - i \frac{\sigma}{\beta} \phi_2 \right)^2 + \left( \partial_t \phi_2 + i \frac{\sigma}{\beta} \phi_1 \right)^2 \right. \right. \]

\[ \left. \left. - (\nabla \phi_1)^2 - (\nabla \phi_2)^2 - m_1^2 \phi_1^2 - m_2^2 \phi_2^2 \right] \right. \quad (8) \]

To calculate the functional integral we begin by Fourier decomposing the fields \[9\]. Whilst doing this we shall anticipate the existence of a condensate by giving the fields a spatially constant component \( v_\alpha \) to be determined \[2\]. The constant \( v_\alpha \) acts as a classical replacement for \( \phi_{\alpha,0}(\mathbf{0}) \). We define \( \tau = it \) where \( 0 \leq \tau \leq \beta \). The fields are periodic over this imaginary time interval and we have

\[ \phi_\alpha(x, \tau) = v_\alpha + \frac{1}{\sqrt{\beta}} \sum_n \int_p e^{i \omega_n \tau + i p \cdot x} \phi_{\alpha,n}(\mathbf{p}), \quad (9) \]

where \( \omega_n = 2\pi n / \beta \), and

\[ f = \int \frac{d^3p}{(2\pi)^3}. \quad (10) \]

The action is now given by

\[ i \int_0^{-i\beta} dt L'(t) = -\frac{1}{2} \beta V \left( m_1^2 + \frac{\sigma^2}{\beta^2} \right) |v|^2 \]

\[ + \frac{1}{2} \sum_n \int_p \frac{1}{2} \phi_{\alpha,-n}(-\mathbf{p})K_{ab,n}(\mathbf{p})\phi_{\beta,n}(\mathbf{p}), \quad (11) \]

where, after defining \( \omega^2 = p_1^2 + m_1^2 \), the inverse propagator is

\[ -K_n(p) = \begin{pmatrix} \omega_n^2 + \omega^2 + \frac{\sigma^2}{\beta^2} & 2i \frac{\sigma}{\beta^2} \omega_n \\ -2i \frac{\sigma}{\beta^2} \omega_n & \omega_n + \omega^2 + \frac{\sigma^2}{\beta^2} \end{pmatrix}. \quad (12) \]

It can be checked that this action is real. The trace in Eq. (4) now becomes

\[ \text{Tr} \left\{ e^{-i\hat{Q}\sigma} \hat{\rho} \right\} \propto e^{-\frac{1}{2} \beta V \left( m_1^2 + \frac{\sigma^2}{\beta^2} \right) |v|^2} \int [d\phi_{\alpha,n}] e^{\sum \int \frac{1}{2} \phi_{\alpha,-n}(-\mathbf{p})K_{ab,n}(\mathbf{p})\phi_{\beta,n}(\mathbf{p})} \]

\[ \propto e^{-\frac{1}{2} \beta V \left( m_1^2 + \frac{\sigma^2}{\beta^2} \right) |v|^2 - \ln \sqrt{\det K}}. \quad (13) \]
A few lines of algebra determine the integral form for $\ln \sqrt{\det K}$. We find

$$\ln \sqrt{\det K} = V \int_p [\beta \omega + \ln(1 - e^{-\beta \omega + i\sigma}) + \ln(1 - e^{-\beta \omega - i\sigma})].$$

(14)

The first term on the right-hand side of Eq. (14) is the infinite zero-point contribution which can be factored out since it doesn’t depend on $\sigma$. The other contributions cannot be easily calculated. We are forced to consider their behavior in extreme-temperature limits. Our Eq. (14) is essentially the same as Eq.(2.57) in Ref.[10] with the replacement $\sigma = i\beta \mu$ where $\mu$ is the chemical potential.

3 High-temperature and large-volume

Here we shall demonstrate how the contributions to $\ln \sqrt{\det K}$ can be calculated in the high-temperature and large-volume limit. We first perform a Taylor expansion:

$$\ln(1 - e^{-\beta \omega \pm i\sigma}) = -\sum_{n=1}^{\infty} \frac{e^{-n\beta \omega \pm in\sigma}}{n}.$$  

(15)

The momentum integration can now be taken to act only on the relevant exponential factor

$$\int_p \ln(1 - e^{-\beta \omega \pm i\sigma}) = -\sum_{n=1}^{\infty} \frac{e^{\pm in\sigma}}{n} \int_p e^{-n\beta \omega}.$$  

(16)

To perform this integral in the limit of small $\beta$ we first write out the Mellin-Barnes integral form for the exponential

$$e^{-a} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \ a^{-s} \Gamma(s),$$

(17)

$$\int_p e^{-n\beta \omega} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \ (n\beta)^{-s} \left[ \int_p \omega^{-s} \right] \Gamma(s).$$

(18)

The momentum integral is now straightforward

$$\int_p \omega^{-s} = \frac{m^{3-s} \Gamma \left( \frac{s+3}{2} \right)}{(4\pi)^{3/2} \Gamma \left( \frac{s}{2} \right)}.$$  

(19)

The value of $c$ must be greater than 3 such that the Mellin-Barnes integrand is finite. The contour may then be closed at infinity in the negative real half-plane where the contribution to the integral is zero. The leading pole comes from the factor $\Gamma \left( \frac{s+3}{2} \right)$. Evaluating its residue gives

$$\int_p e^{-n\beta \omega} \sim \frac{1}{n^2} \left( \frac{1}{n\beta} \right)^3.$$  

(20)

This result could also have been found by taking $\omega \sim p + \cdots$. This is a valid approximation when $\beta m \ll 1$. However, the Mellin transform allows us to
consider corrections to this limit in a controlled way (see later). We therefore have
\[ \int p \ln(1 - e^{-\beta \omega + i\sigma}) \sim -\frac{1}{\pi^2 \beta^3} \sum_{n=1}^{\infty} \frac{e^{\pm in\sigma}}{n^4}. \]  
(21)

Finally, summing the two logarithmic contributions to \( \ln \sqrt{\det K} \) we find \[ V \int p \left[ \ln(1 - e^{-\beta \omega + i\sigma}) + \ln(1 - e^{-\beta \omega - i\sigma}) \right] \sim -\frac{2V}{\pi^2 \beta^3} \sum_{n=1}^{\infty} \frac{\cos n\sigma}{n^4} \].
(22)

We can now bring everything together in order to write out the canonical partition function with a conserved charge \( Q \). We may factor out any \( \sigma \)-independent terms as an overall normalization \( N \). We also write in terms of a new variable \( x \) where \( 2\pi x = \sigma \). We thus have
\[ Z \sim N(\beta) \int \frac{dx}{2\pi} e^{2\pi iQx} e^{-\frac{1}{2\beta} V \left( m^2 + \frac{4\pi^2 x^2}{\beta^2} \right) |v|^2} e^{-\frac{2V x^2}{3\pi^2} x^2 (|x|-1)^2}. \]
(23)

Within this range of integration, in the limit of large \( V \) and small \( \beta \) we may make the approximation
\[ e^{-\frac{2V x^2}{3\pi^2} x^2 (|x|-1)^2} \sim e^{-\frac{2V x^2}{3\pi^2} x^2}. \]
(24)

Since this exponential factor suppresses any potential contribution for larger values of \( x \) we may further extend the integration range to infinity with negligible effect giving
\[ Z \sim N(\beta) \int \frac{dx}{2\pi} \cos(2\pi Qx) e^{-\frac{1}{2\beta} V \left( m^2 + \frac{4\pi^2 x^2}{\beta^2} \right) |v|^2} e^{-\frac{2V x^2}{3\pi^2} x^2}. \]
(25)

This is the canonical partition function for an \( O(2) \)-invariant scalar field theory with total conserved charge \( Q \). The integral can be performed analytically. In the next section we shall use \( Z \) to demonstrate Bose-Einstein condensation in the canonical ensemble.

4 Bose-Einstein condensation

The classical variable \( v_a \) represents the constant expectation value of the field and is to be determined. We first identify \( \ln Z(v_a) \) with the effective potential for the field. The correct solution \( v_a \) must minimize the effective potential:
\[ \frac{d}{dv_a} \ln Z = \frac{1}{Z} \frac{dZ}{dv_a} = 0. \]
(26)

Applying this condition to \( Z \) in Eq. (25) we find
\[ 0 = v_a \left( m^2 + \frac{4\pi^2}{\beta^2} \int_0^\infty dx \cos(2\pi Qx) x^2 e^{-\frac{2V x^2}{3\pi^2} \left( \frac{1}{3\pi^2} + |v|^2 \right) x^2} \right). \]
(27)
This can be evaluated with the following standard integrals [11]

\[
\int_0^\infty dx \cos(ax) e^{-bx^2} = \frac{1}{2} \sqrt{\frac{\pi}{b}} e^{-\frac{a^2}{4b}},
\]

(28)

\[
\int_0^\infty dx \cos(ax) x^2 e^{-bx^2} = \frac{1}{2} \sqrt{\frac{\pi}{b}} \left( \frac{1}{2b} - \frac{a^2}{4b^2} \right) e^{-\frac{a^2}{4b}}.
\]

(29)

Given these integrals we find

\[
0 = v_a \left( m^2 - \frac{Q^2}{V^2 \left( \frac{1}{3\beta^2} + |v|^2 \right)^2} \right).
\]

(30)

This constraint equation looks very much like the grand canonical constraint (see Eq.(1.42) in [2]) once we define the chemical potential by

\[
\mu = \frac{Q}{V \left( \frac{1}{3\beta^2} + |v|^2 \right)}.
\]

(31)

There are two solutions:

\[
v_a = 0 \quad (32)
\]

\[
|v|^2 = \frac{Q}{V m} - \frac{1}{3\beta^2} \quad (33)
\]

The first solution is trivial. The second solution exists when \( \frac{Q}{V m} > \frac{1}{3\beta^2} \). We may therefore define a critical temperature to mark the point at which the field develops a non-zero expectation value

\[
T_c = \frac{1}{\beta_c} = \sqrt{\frac{3Q}{Vm}}.
\]

(34)

To express the charge stored in the ground state we write out the total charge in integral form:

\[
Q = \langle Q \rangle = \frac{1}{2\pi Z} \int_{-\pi}^\pi d\sigma e^{iQ\sigma} \frac{i}{d\sigma} \text{Tr} \left\{ e^{-i\hat{Q}\sigma} \hat{\rho} \right\}
\]

(35)

giving

\[
Q = \frac{2V\pi}{\beta} \left( \frac{1}{3\beta^2} + |v|^2 \right) \frac{\int_0^\infty dx \sin(2\pi Qx) x e^{-2\frac{V\pi^2}{\beta} (\frac{1}{3\beta^2} + |v|^2) x^2}}{\int_0^\infty dx \cos(2\pi Qx) e^{-2\frac{V\pi^2}{\beta} (\frac{1}{3\beta^2} + |v|^2) x^2}} \quad (36)
\]

\[
= Q_{\text{fluct}} + Q_{\text{cond}}.
\]

(37)

Here we can divide the contributions to the charge into two types. The first term in brackets corresponds to the charge contained in the fluctuations or excited modes. The second term is the charge which is contained in the ground state, the Bose-Einstein condensate. We find

\[
Q_{\text{cond}} = |v|^2 \left( \frac{Q}{\left( \frac{1}{3\beta^2} + |v|^2 \right)} \right) = \begin{cases} 
0 & \text{when } \beta^2 < \frac{Vm}{3\beta^2} \\
Q - \frac{Vm}{3\beta^2} & \text{when } \beta^2 > \frac{Vm}{3\beta^2}
\end{cases}.
\]

(38)
At $T = 1/\beta = 0$ all the charge resides in the condensate. As the temperature increases charge is excited out. Eventually, when $T > T_c$, the condensate melts and all the charge resides in the fluctuation modes. Of course, these equations are only valid in the high-temperature limit and we cannot rely on them when the condensate becomes big.

It has been demonstrated how we can understand BEC without using a chemical potential. We achieve the same result as for the grand canonical ensemble in the high-temperature, large-volume limit.

5 Finite-volume

Up until now, the canonical ensemble has proved to be more complicated than the grand canonical ensemble without offering any new results. However, now we stand to benefit from the fact that the canonical ensemble provides a more realistic picture of a finite-volume system, closed to the movement of charge into and out-of that volume. To continue from the previous section we work in the high-temperature region and the bosons are not subjected to any external potential within their volume.

It is well known that there is no phase transition involved in Bose-Einstein condensation for finite volumes. However, we may still characterize the process by studying the proportion of the total charge stored in the lowest energy state. In the grand canonical ensemble the charge in the lowest energy state is given by

$$Q_0 = \frac{2T\mu}{m^2 - \mu^2}. \quad (39)$$

For $T < T_c$ when the charge occupying the lowest energy state compares with the charge stored in fluctuations, we may think of $(m - \mu)^{-1} \sim \mathcal{O}(VM)$ \[12\]. The chemical potential is determined by the constraint that the total energy is fixed. To leading order in the large-volume expansion we have \[13\]

$$Q = \frac{2T\mu}{m^2 - \mu^2} + V \left[ \frac{\mu T^2}{3} + \frac{\mu T(m^2 - \mu^2)\frac{\xi}{\sqrt{2\pi}}}{2\pi} + \frac{\mu(3m^2 - 2\mu^2)}{12\pi^2} + \cdots \right] + \cdots \quad (40)$$

For any given $T$, $V$ and $Q$, this equation can be solved for $\mu$ enabling $Q_0$ to be calculated. (For a more detailed finite-volume calculation see \[14\].)

Returning to the canonical ensemble and the expression for $\ln \sqrt{\det K}$ given in equation \[15\], for finite volumes, the integral over momentum states should be replaced with an infinite sum over those momentum states compatible with the boundary conditions. We again expand as follows

$$\sum_p \ln(1 - e^{-\beta \omega \pm i\sigma}) = -\sum_{n=1}^{\infty} \frac{e^{\pm i\sigma}}{n} \sum_p e^{-n\beta \omega}, \quad (41)$$

$$\sum_p e^{-n\beta \omega} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \left( n\beta \right)^{-s} \left[ \sum_p \omega^{-s} \right] \Gamma(s). \quad (42)$$

Choosing a cubic container for our system with side $L$ and Neumann boundary conditions we have

$$\sum_p \omega^{-s} = \sum_p \left[ p^2 + m^2 \right]^{-\frac{s}{2}}.$$
\[ e^{\sum_{n_i=0}^{\infty} \left[ \left( \frac{\pi n_1}{L} \right)^2 + \left( \frac{\pi n_2}{L} \right)^2 + \left( \frac{\pi n_1}{L} \right)^2 + m^2 \right]^{-\frac{1}{2}}} \]

\[ = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty dt \ t^{s-1} e^{-m^2 t} \sum_p e^{-p^2 t}. \quad (43) \]

The sum over \( p \) can be performed as a large-volume expansion \[14\]

\[ \sum_p e^{-p^2 t} = \left[ \sum_{n=0}^{\infty} e^{-\left(\frac{\pi n}{L}\right)^2} \right]^3 \]

\[ = \frac{V}{8\pi t} + \frac{3V^{\frac{3}{2}}}{8\pi t} + \frac{3V^{\frac{5}{2}}}{8\pi t} + 1 + \cdots \quad (44) \]

This gives

\[ \sum_p \omega^{-s} = V^{m^3 - s} \Gamma\left(\frac{s-1}{2}\right) + V^{\frac{3}{2}} m^{2-s} \Gamma\left(\frac{s-2}{2}\right) \]

\[ + V^{\frac{3}{2}} m^{l-s} \Gamma\left(\frac{2l-1}{2}\right) + \frac{m^s}{8} + \cdots, \quad (45) \]

and eventually

\[ \ln \sqrt{\text{det} K} \sim -V \frac{2}{\pi^2 \beta^3} \sum_{n=1}^{\infty} \cos(n \sigma) \frac{n^4}{n^2} + \frac{V m^2}{2 \pi^2 \beta} \sum_{n=1}^{\infty} \cos(n \sigma) \frac{n^2}{n^2} \]

\[ - V^{\frac{4}{2}} \frac{3}{\pi \beta^2} \sum_{n=1}^{\infty} \frac{\cos(n \sigma)}{n^3} + V^{\frac{3}{2}} m^{2} \frac{3}{4 \pi} \sum_{n=1}^{\infty} \frac{\cos(n \sigma)}{n} \]

\[ - V^{\frac{3}{2}} \frac{3}{\pi \beta^2} \sum_{n=1}^{\infty} \frac{\cos(n \sigma)}{n^2} - 2 \sum_{n=1}^{\infty} \frac{e^{-n \beta m \cos(n \sigma)}}{n} + \cdots. \quad (46) \]

We have kept all terms in the \( 1/L \) expansion up to \( O(L^0) \) and in the high temperature (small mass) expansion up to \( O(m^2) \). All but one of these infinite sums have analytic solutions for the given integration range of \( \sigma \). Referring to \[11\] we have

\[ \sum_{n=1}^{\infty} \frac{\cos(n \sigma)}{n^4} = \frac{\pi^4}{90} - \frac{\pi^2 \sigma^2}{12} + \frac{\pi |\sigma|^3}{12} - \frac{\sigma^4}{48} \quad (47) \]

\[ \sum_{n=1}^{\infty} \frac{\cos(n \sigma)}{n^2} = \frac{\pi^2}{6} - \frac{\pi |\sigma|}{2} + \frac{\sigma^2}{4} \quad (48) \]

\[ \sum_{n=1}^{\infty} \frac{\cos(n \sigma)}{n} = -\frac{1}{2} \ln(2 - 2 \cos \sigma), \quad (49) \]

\[ \sum_{n=1}^{\infty} \frac{e^{-n \beta m \cos(n \sigma)}}{n} = -\frac{1}{2} \left[ \ln(1 - e^{-\beta m + i \sigma}) + \ln(1 - e^{-\beta m - i \sigma}) \right]. \quad (50) \]

The remaining sum is convergent and can easily be evaluated numerically.
There is no guarantee that the expansion of Eq. (46) is a good one. The best we can expect is that subsequent terms in the expansion quickly become negligibly small in comparison with the leading order terms over the whole integration range of $\sigma$. This is clearly not the case for the fourth term in the expansion when close to $\sigma = 0$. The term diverges leading to a zero in the integrand. However, by appropriate choice of parameters, the domination of this term is made sufficiently brief that its effect is negligible.

The $O(V^0)$ term of Eq. (46) may be rewritten as

$$\frac{1}{2} \sum_n \ln \left[ (\omega_n^2 + m^2 + T^2 \sigma^2)^2 - T^2 \sigma^2 \omega_n^2 \right]$$

from which we may extract the $n = 0$ contribution and identify this as the contribution to $\ln \sqrt{\det K}$ of the ground state mode $\phi_{a;0}(0)$:

$$\ln(m^2 + T^2 \sigma^2).$$

The expectation of charge in the lowest energy state is given by

$$\langle Q_0 \rangle = \frac{1}{2\pi Z} \int_{-\pi}^{\pi} d\sigma e^{iQ\sigma} \text{Tr} \left\{ e^{-i\hat{Q}\sigma} \hat{Q}_0 \right\}. \tag{53}$$

This can be derived by differentiating with respect to $\sigma$ only those terms in $Z$ where $\sigma$ is coupled to the ground state mode (expression (52)). We find

$$\langle Q_0 \rangle = \frac{1}{2\pi Z} \int_{-\pi}^{\pi} d\sigma e^{iQ\sigma} \left( \frac{-2i\sigma}{\beta^2 m^2 + \sigma^2} \right) e^{-\ln \sqrt{\det K}} \tag{54}$$

where $\ln \sqrt{\det K}$ is given above in Eq. (46).

Having defined the condensate by the expectation of charge in the lowest energy mode, we may directly calculate the condensate fraction by numerical integration. The results can be seen for different values of the total charge in Fig.1 and Fig.2. A comparison is made with the infinite-volume limit and with the finite-volume approximation in the grand canonical ensemble given by Eq. (40). The parameters must be chosen such that $\beta/V^{1/3}, \beta m \ll 1$ in order to justify the expansion of $\ln \sqrt{\det K}$. We chose $Q \sim 10^3$, $V = 10^9/T_c^3$ and $m = 3Q/(VT_c^3)$, working in units where $T_c = 1$.

For $T > T_c$ we consistently observe a lower occupation of the ground state in the canonical ensemble as compared with the grand canonical ensemble. This makes the infinite-volume limit a better approximation for the canonical than the grand canonical case. As the temperature approaches the critical temperature we see the ground state occupation increase as expected. At low temperatures the expansion of $\ln \sqrt{\det K}$ breaks down and canonical results deviate from the other approximations.

6 Conclusions

It has been shown how we can understand BEC for a system of fixed charge in the canonical ensemble. Though the chemical potential plays a crucial role in our understanding of BEC in the grand canonical ensemble, there is no need for a chemical potential in the canonical ensemble.
Figure 1: Expectations of charge in the lowest energy state against temperature. The solid line represents the infinite-volume approximation in the high-temperature limit; the dotted line is a finite-volume approximation using the grand canonical ensemble; the dashed line uses the canonical ensemble. Parameters are $Q = 20000$, $V = 10^9$, mass is chosen to give $T_c = 1$ ($m = 0.00006$).

We have seen that the two ensembles produce identical results in the infinite-volume limit. For finite volumes, the canonical method produces sensible results which differ from the grand canonical approximation in a consistent qualitative way.

For future work, a low-temperature, non-relativistic version of this calculation could feasibly be tested against experimental data from BEC in atomic fluids and other theoretical results $[7, 8]$. 

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Figure 2: $Q = 40000$, $V = 10^9$, $m = 0.00012$ in units of $T_c$. Canonical (dashed line), grand canonical (dotted line), infinite volume approximation (solid line).

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