Gauss-Bonnet Dyonic Black Holes: geometry, thermodynamics and test particles’ trajectories

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In this paper, we investigate a class of 5-dimensional black holes in the presence of Gauss-Bonnet gravity with dyonic charges. At first step, thermodynamical quantities of the black holes and their behaviors are explored for different limits. Thermal stability and the possibility of the van der Waals like phase transition are addressed and the effects of different parameters on them are investigated. The second part is devoted to simulation of the trajectory of particles around these black holes and investigation of the angular frequency of particles’ motion. The main goal is understanding the effects of higher curvature gravity (Gauss-Bonnet gravity) and magnetic charge on the structure of black holes and the geodesic paths of particles moving around these black holes.

I. INTRODUCTION

Lovelock gravity is one of the well established/known generalizations of the Einstein gravity [1, 2]. The first three terms of this generalization include three degrees of curvature term including: I) a constant realized as the cosmological constant, II) a first order curvature term known as the Einstein Lagrangian, III) and finally a curvature-squared term called Gauss-Bonnet (GB) Lagrangian. The Gauss-Bonnet term is a topological term in 4 and lower dimensions. Its structure results into the appearance of up to the second order derivations of metric functions in the field equations [3, 6]. It is a higher derivative gravity enjoying the absence of ghost instability [7, 8]. Its Lagrangian could be obtained in the low-energy limit of heterotic string theory [9–13]. Black holes and their properties in the presence of GB gravity have been intensively investigated in literature [14–34]. In addition, the black string solutions with GB generalization and their stabilities have been studied in Refs. [35–39]. In holographical context, the effects of the GB gravity on finite coupling [40], second order transport [41], entanglement entropy [42–45] and superconductivity [46–48] have been addressed. It is worthwhile to mention that inclusion of the GB term in 5 dimensions could results into causality violation unless you have a tower of massive higher-spin particles with a delicately tuned interaction, which happens in string theory [49]. Considering the diverse applications of GB gravity in classical black hole thermodynamics, holography and string theory, we investigate a specific type of GB black holes in the presence of dyonic charge.

The dyonic charge corresponds to the existence of magnetic charge (in addition to electric charge) in the structure of black holes. The dyonic black hole solutions have been vastly used as popular models for investigations in the context of AdS/CFT duality. One of the early applications of the dyonic black holes was studying the Hall conductivity and zero momentum hydrodynamic response functions in the context of AdS/CFT [50]. In addition, it was confirmed that large dyonic black holes in AdS spacetime correspond to stationary solutions of the equations of relativistic magnetohydrodynamics on the conformal boundary of AdS [51]. Of other applications/investigations of on the dyonic black holes in AdS/CFT context, one can name the following ones: I) Inducing external magnetic field on superconductors which results into magnetic dependency similar to the Meissner effect [52], II) Their effects on the holographical properties of solutions such as transport coefficients, Hall conductance and DC longitudinal conductivity [53]. III) The paramagnetism/ferromagnetism phase transitions in case of dyonic Reissner-Nordstrom black holes [54,56] and massive dyonic black holes [57].

In the context of string theory, the exact dyonic black hole solutions were constructed in Refs. [58,59]. In addition, the dyonic black hole solutions were also investigated in effective [60] and heterotic [61] string theories. The extreme solutions [62] and asymptotic degeneracy of dyonic N = 4 string states [63] were studied as well. From supergravity point of view, dyonic black hole solutions were obtained in Refs. [64,65] and their thermodynamics were investigated in Ref. [66]. Furthermore, the existence of dyonic black holes in the Einstein-Yang-Mills theory was investigated in Refs. [67,69] and their gravitating properties [70,71] were studied.

Recently, the dyonic black holes with the Born-Infeld nonlinear electromagnetic field in diverse dimensions were...
constructed \[72\]. Thermodynamic of the dyonic black holes was thoroughly investigated in Ref. \[73\]. Ref. \[74\]

presents a beautiful systematic study regarding computation of conserved quantities of the dyonic black holes in \(AdS_4\). The discreteness of dyonic dilaton black holes \[75\], thermodynamics of dyonic black holes with Thurston horizon geometries \[76\] and dyonic black holes at arbitrary locations \[77\] were also studied. A general scheme for obtaining dyonic solutions in nonlinear electrodynamics was proposed in Ref. \[78\].

Considering the vast applications of both GB gravity and dyonic configuration in the context of black holes, this paper is dedicated to the investigation of dyonic black holes in the presence of GB gravity. To our knowledge, so far the consideration of dyonic configuration with GB gravity was done only for the cases where dyonic contribution is confined to 4-dimensions while GB gravity was contributing from higher dimensions \[79\] \[80\]. In this paper, we intend to construct a 5-dimensional black hole solution in the presence of dyonic matter-field and GB gravity. We first introduce action and corresponding field equations. Then, we extract black hole solutions and investigate the effects of dyonic and GB gravity on geometrical properties of the black holes. We continue our study by obtaining thermodynamical properties of the solutions, investigating the stability and thermal phase transition of the solutions. Later, we focus on trajectories of a test particle around these black holes. We finalize our paper with some closing remarks.

II. LAGRANGIAN ANSATZ

Our line of work in this paper includes investigating dyonic black holes with Gauss-Bonnet gravity generalization. The main goal is to understand the geometrical and thermodynamical properties of the black holes with magnetic and/or electric fields and higher curvature terms generalization (Gauss-Bonnet gravity).

In this section, we first introduce the Lagrangian governing Gauss-Bonnet-dyonic black holes. Then, we obtain the metric function and confirm the existence of black holes. Finally, we investigate the effects of magnetic, electric and GB parameters on geometrical properties of the black holes.

A. Constructing the solutions

The Lagrangian of Einstein gravity in the presence of cosmological constant is given by \(\mathcal{L}_{EA} = R - 2\Lambda\), which contains the first order curvature scalar. Using the variational principle, one finds that its related field equation contains first order or at most second order derivation of the metric functions.

On the other hand, GB Lagrangian includes second order scalar curvature terms which is given by

\[
\mathcal{L}_{GB} = R_{\mu
u\gamma\delta}R^{\mu
u\gamma\delta} - 4R_{\mu
u}R^{\mu
u} + R^2, \tag{1}
\]

where \(R_{\mu
u\gamma\delta}\) and \(R_{\mu\nu}\) are respectively, the Riemann and the Ricci tensors. GB Lagrangian has two important properties: I) Due to the specific factors considered for different terms of Lagrangian, only up to second order derivations of the metric functions could appear in the field equations. II) The Lagrangian is effective for dimensions \(\geq 4\) and it behaves as a topological term for dimensions \(\leq 4\). Considering the latter, we will conduct our study in 5–dimension in this paper. Sure our results can be generalized to higher dimensions.

The black holes that we are interested in, are topological black holes with various horizon topologies. The static metric ansatz is given by

\[
\text{d}s^2 = -f(r)\text{d}t^2 + \frac{\text{d}r^2}{f(r)} + r^2 \text{d}\Omega^2_k, \tag{2}
\]

in which \(f(r)\) is the metric function and \(\text{d}\Omega^2_k\) is the line element of a 3-dimensional hypersurface with the constant curvature \(6k\) and volume \(V_3\), in which its explicit form is

\[
\text{d}\Omega^2_k = \left\{
\begin{array}{ll}
\text{d}\theta^2 + \sin^2\theta\text{d}\psi^2 + \sin^2\theta \sin^2\psi \text{d}\phi^2, & k = 1 \text{ (spherical)} \\
\text{d}\theta^2 + \sinh^2\theta \text{d}\psi^2 + \sinh^2\theta \sinh^2\psi \text{d}\phi^2, & k = -1 \text{ (hyperbolic)} \\
\text{d}\theta^2 + \text{d}\psi^2 + \text{d}\phi^2, & k = 0 \text{ (flat)}
\end{array}
\right.
\]

The matter field of our interest is of dyonic nature. The word ”dyonic” corresponds to the presence of the magnetic charge alongside of the electric charge in the matter field. There are different methods for introducing magnetic charge into the structure of the black holes. In this paper, we follow the method introduced in Ref. \[81\] The
electromagnetic tensor is an antisymmetric second rank tensor with the following components of electric and magnetic field in 5–dimension

$$F_{\mu\nu} = \begin{bmatrix} 0 & F_{t\tau} & F_{t\theta} & F_{t\phi} & F_{t\psi} \\
-F_{t\tau} & 0 & F_{\tau\theta} & F_{\tau\phi} & F_{\tau\psi} \\
-F_{t\theta} & -F_{\tau\theta} & 0 & F_{\theta\phi} & F_{\theta\psi} \\
-F_{t\phi} & -F_{\tau\phi} & -F_{\theta\phi} & 0 & F_{\phi\psi} \\
-F_{t\psi} & -F_{\tau\psi} & -F_{\theta\psi} & -F_{\phi\psi} & 0 \end{bmatrix},$$  \tag{3}

where its nonzero components are functions of all independent coordinates. In this paper, we assume that Eq. (3) satisfies the Maxwell field equations, while the effective components of electromagnetic tensor in the energy-momentum tensor are $F_{t\tau}$ and $F_{\theta\phi}$ with the following forms

$$F_{t\tau} = -F_{\tau t} = \frac{q_E}{r^3} \quad \& \quad F_{\theta\phi} = -F_{\phi\theta} = \frac{q_M}{r} \times \begin{cases} \sin \theta & k = 1 \\ \theta & k = 0 \\ \sinh \theta & k = -1 \end{cases}$$  \tag{4}

in which $q_E$ and $q_M$ are, respectively, electric and magnetic charges related to the total electric and magnetic charges of the solutions. It is notable to mention that although other off-diagonal components of Eq. (3) do not vanish, necessarily, their influences in the energy-momentum tensor are negligible comparing with $F_{t\tau}$ and $F_{\theta\phi}$, $T_{\mu\nu}$. In other words, Eq. (4) denotes the leading terms of $F_{\mu\nu}$ affecting the nonzero components of $T_{\mu\nu}$ dominantly. Thus, we use the gravitational field equation of 5-dimensional GB-dyonic black holes which is given by

$$e_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha H_{\mu\nu} - \frac{1}{2} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} = 0,$$  \tag{5}

in which $\alpha$ is GB parameter, $g_{\mu\nu}$ and $G_{\mu\nu}$ are, respectively, the metric and Einstein tensors, and $H_{\mu\nu}$ is the GB tensor given by

$$H_{\mu\nu} = -\frac{1}{2} \left( 8 R^\sigma_{\mu\nu\rho\sigma} - 4 R^\sigma_{\mu\rho} R_{\nu\sigma\lambda} - 4 R_{\mu\rho} + 8 R_{\mu\lambda} R^\lambda_{\nu\rho} + g_{\mu\nu} L_{GB} \right).$$

Using the metric (2) with the obtained field equation (5), one can extract the nonzero components of field equation. The full form of these components are given in the appendix (see Eqs. (32) and (33)). It is a matter of calculation to obtain the metric function by solving the components of field equation, simultaneously, as

$$f(r) = k + \frac{r^2}{4\alpha} \left\{ 1 - \sqrt{1 + \frac{4 \alpha}{3} \left[ \Lambda + \frac{6m}{2r^4} - \frac{2(q_E^2 + q_M^2)}{r^6} \right]} \right\},$$  \tag{6}

in which $m$ is an integration constant known as the geometrical mass.

**B. Properties of the solution**

Here, we investigate different properties of the obtained metric function (6). Before we start, one point should be noted. Existence of black holes has two conditions: one is the presence of singularity for scalar curvatures and the other one is presence of horizon covering the singularity.

I) Since the electric and magnetic charges are decoupled, in the limit of $q_E \to 0$, one can have purely magnetically charged black holes in the presence of GB gravity. It is also worthwhile to mention that solutions enjoy the electric-magnetic duality and the metric function has the full symmetry of swapping the subscripts "E" and "M".

II) Existence of real metric function is restricted to satisfaction of the following condition

$$3/4 \alpha - \Lambda \geq \frac{2(q_E^2 + q_M^2)}{r^6} - \frac{6m}{r^4}.$$

III) For the case

$$3/4 \alpha - \Lambda = \frac{2(q_E^2 + q_M^2)}{r^6} - \frac{6m}{r^4},$$
the metric function reduces to
\[ f(r) = k + \frac{r^2}{4\alpha}, \] (7)
which yields \( r = \sqrt{-k\alpha} \) as the root of the metric function. Roots of the metric function are horizons of the black holes. Therefore, in this case, only hyperbolic solutions have real positive non-zero root and black holes exist only for hyperbolic horizon (other horizons admit naked singularity not black hole solutions).

IV) In general, the metric function could admit up to three roots. The roots are given in the appendix (see Eq. 34). These roots contain square root functions. These square root functions put certain conditions for having real positive non-zero roots for the metric function. If these conditions are satisfied, the metric function could have one to three roots. Otherwise, the solutions will admit naked singularity. In order to elaborate these issues, we have plotted Fig. 1 which confirms our discussion. Existence of roots (horizons) for metric function indicates that the second condition for having black hole solutions is satisfied.

\[ \Lambda = -1 \]

\[ \alpha = 0.1 \]
\[ \alpha \sim 0.42 \]
\[ \alpha = 1 \]

\[ 0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \quad 1.2 \quad 1.4 \]
\[ -1.0 \quad -0.5 \quad 0.0 \quad 0.5 \quad 1.0 \]
\[ r \quad f(r) \]

\[ \Lambda = 1 \]

\[ \alpha = 0.1 \]
\[ \alpha = 0.42 \]
\[ \alpha = 1 \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \]
\[ -1.0 \quad -0.5 \quad 0.0 \quad 0.5 \quad 1.0 \]
\[ r \quad f(r) \]

FIG. 1: Metric function versus \( r \) for \( q_M = q_E = 0.1 \) and \( k = m = 1 \).

V) The presence of singularity could be detected by studying the divergencies of curvature scalars. In this regard, we study a curvature scalar known as Kretschmann scalar. In general, for 5-dimensional topological black holes, the Kretschmann scalar is given by
\[ R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \left( \frac{d^2 f(r)}{dr^2} \right) + \frac{4}{r^2} \left( \frac{df(r)}{dr} \right)^2 + \frac{12}{r^4} (f(r) - k)^2. \] (8)
Inserting metric function (6) in the Kretschmann scalar (8) leads to
\[ R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \propto O\left( \frac{1}{r^6} \right). \] (9)
The full form of the Kretschmann scalar is given in appendix (see Eq. 35). Eq. (9) shows that
\[ \lim_{r \to 0} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \to \infty, \]
which confirms that there is a curvature singularity at \( r = 0 \). By series expanding the obtained Kretschmann scalar for small values of \( r \), one can find the following relation
\[ \lim_{r \to 0} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \propto -\frac{3}{\alpha^6} \left( q_E^2 + q_M^2 \right) + \frac{3m}{\alpha r^4} + O\left( \frac{1}{\alpha r^3} \right). \]
This limit confirms that the singularity and its nature (being temporal or spatial singularity) is affected by GB gravity, electric and magnetic fields.

VI) The asymptotic behavior of Kretschmann scalar is a measure that hints us to find whether solutions have asymptotic AdS/dS behavior or not. Using the obtained Kretschmann scalar (Eq. 35) and its series expansion for large \( r \) result into
\[ \lim_{r \to \infty} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \propto \frac{(-\alpha - 9)\sqrt{1 + \frac{4\alpha\Lambda}{3}}}{2\alpha^2} + \frac{(1 + 6\Lambda)\alpha + 9}{2\alpha^2} + O\left( \frac{1}{r} \right). \]
This indicates that the asymptotic behavior of the solutions is not anymore the regular AdS/dS, but in fact an effective relation is obtained for it.
III. THERMODYNAMICS

In this section, we focus on thermodynamical aspects of the GB-dyonic black holes. Our goal is to understand the effects of magnetic charge alongside the GB gravity generalization. We conduct our study in three subsections, each focusing on a specific thermodynamical property of these black holes. First, we extract thermodynamical quantities. Then, we investigate the possible van der Waals like phase transition. Finally, we study thermal stability of solutions through canonical ensemble.

A. Thermodynamical quantities

In this subsection, we obtain the thermodynamical quantities of the solutions and check the validity of first law of black hole thermodynamics.

The first item of our interest is the entropy. In Einsteinian black holes, without higher curvature terms, the entropy could be obtained by using the area law \[S = \frac{1}{4} \int d^3x \sqrt{\gamma} (1 + 2a \tilde{R}),\] (10)
in which $\tilde{R}$ is the Ricci scalar of the induced metric $\gamma_{ab}$ on the 3-dimensional boundary. It is a matter of calculation to show that in general, for topological GB black holes, the entropy per volume $V_3$ is obtained as

\[S = \frac{1}{4} r^3_+ \left(1 + \frac{12\alpha k}{r^2_+}\right).\] (11)

This expression for the entropy confirms the following important points:

I) The entropy is not directly affected by electric and magnetic charges. This quantity is affected by topological structure of the black holes and GB parameter (metric and left hand side of gravitational field equation).

II) The GB parameter and topological factor are coupled. Since GB parameter is positive valued, the effects of GB parameter and topological factor on entropy is determined by the topological factor, $k$. Therefore, entropy is:

1) a decreasing function of GB parameter for the hyperbolic black holes. 2) not affected by GB gravity for the flat horizon. 3) and an increasing function of GB parameter for the spherical horizon.

III) For hyperbolic black holes, the entropy has a root at $r_+ = 2\sqrt{3}\alpha$. Whereas, for flat and spherical horizon black holes, entropy is a smooth function of the horizon radius without any root. It is worthwhile to mention that for flat horizon, the effects of the higher curvature terms on entropy disappears and the area law is valid.

Next, we calculate the temperature. To do so, one should first obtain the surface gravity through \[\kappa = \sqrt{-\frac{1}{2} (\nabla_\mu \chi_\nu) (\nabla^\mu \chi^\nu)},\]
in which $\chi^\nu$ is the Killing vector. Considering the metric employed in this paper (2), the Killing vector will be temporal in the form of $\chi = \partial_t$. It is a matter of calculation to show that surface gravity is obtained as

\[\kappa = \frac{1}{2} \frac{df(r)}{dr}.\] (12)

Using the surface gravity (12) with metric function (6), the temperature is calculated as

\[T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{df(r_+)}{dr} = \frac{3kr^3_+ - 8Ar^2_+ - q^2_\xi - q^2_M}{24\pi\alpha kr^4_+ + 6\pi r^5_+}.\] (13)

As one can see, obtained temperature gives us the following details:

I) For spherical and flat horizon black holes, the temperature is divergence free. Whereas, for hyperbolic black holes, the temperature has a divergence at $r_+ = 2\sqrt{3}\alpha$.

II) Temperature is a decreasing function of electric and magnetic charges. The effects of cosmological constant and topological factor depend on the AdS/dS nature and horizon of the solutions. The effect of GB parameter depends on the sign of $k$ (due to coupling with topological factor).

III) Considering the structure of numerator of the temperature, it is possible to obtain a root for the temperature. The full form of the root(s) is given in the appendix (see Eq. 36).
IV) By series expanding the temperature for small horizon radius, one can obtain the following relation
\[
\lim_{r_+ \to 0} T \propto -\frac{q_E^2 + q_M^2}{24\pi\alpha kr_+^3} + O\left(\frac{1}{r_+}\right),
\]
which confirms that: 1) Temperature diverges at \(r_+ \to 0\). 2) The only quantity which does not have direct effect on the high energy limit of the temperature is cosmological constant. 3) For small black holes, the temperature is highly affected and governed by the topological factor, GB parameter, electric and magnetic charges.

V) The asymptotic behavior of the temperature is given by
\[
\lim_{r_+ \to \infty} T \propto -\frac{\Lambda r_+}{6\pi} + O\left(\frac{1}{r_+}\right),
\]
which shows that for large black holes, the cosmological constant is the governing factor in temperature. The other details about the temperature will be discussed in the following subsections.

Now, we focus on the conserved quantities of the matter field. The first items of interest are the total electric and magnetic charges. For calculating these quantities, one can employ the Gauss law. Using this method results into the following total electric and magnetic charges per unit volume \(V_3\)
\[
Q_E = \frac{q_E}{4\pi},
\]
\[
Q_M = \frac{q_M}{4\pi}.
\]

For calculating the total mass, one can use the ADM (Arnowitt-Deser-Misner) method [86], finding the following total mass per unit volume \(V_3\) for the solutions
\[
M = 3\frac{1}{16}m.
\]

In order to obtain the general form of the mass, one should obtain geometrical mass, \(m\). To do so, one can evaluate the metric function on its outer horizon \((f(r = r_+) = 0)\). It is a matter of calculation to show that the total mass per unit volume \(V_3\) for these black holes is
\[
M = \frac{4\alpha k^2 + 3kr_+^2}{16} + \frac{q_E^2 + q_M^3}{16r_+^2} - \frac{\Lambda r_+^4}{32}.
\]

The obtained total mass gives us the following information:

I) Electromagnetic part of the solutions and GB generalization have positive contributions on the value of mass. As for the topological factor, it depends on the horizon of black hole under consideration. The contribution of cosmological constant depends on the AdS/dS nature of our solutions.

II) Due to structure of the expression for mass, it is possible to obtain root(s) and a region of negativity for the mass. The full form of the mass’s root is given in the appendix (see Eq. [37]). In order to elaborate the possible cases, we have plotted the variation of mass for different parameters in Fig. 2. It is worthwhile to mention that in the classical black hole thermodynamics, mass would be interpreted as the internal energy and it should be positive in order for the solutions to be physical. Plotted diagrams confirm that depending on different choices, one can observe regions of negative mass, hence non-physical region for these black holes.

III) Series expanding the mass for small values of the horizon radius results into
\[
\lim_{r_+ \to 0} M \propto \frac{q_E^2 + q_M^2}{16r_+^2} + \frac{3\alpha k^2}{8} + O\left(\frac{r_+^2}{r_+}\right),
\]
which confirms three important results: 1) Due to presence of the electric and magnetic charges, the total mass diverges for vanishing \(r_+\). 2) In the absence of electric and magnetic charges, the mass is non-zero and finite at \(r_+ = 0\). It is in fact a function of GB parameter coupled with topological factor. This is a very important consequence of generalization to GB gravity which is absent in the Einstein theory of the gravity. 3) In the presence of electric and/or magnetic charges, the high energy limit of the mass is governed by them. In their absence, the high energy limit of mass is determined by GB gravity and horizon of the solutions.
IV) The asymptotic behavior of the mass, similar to the temperature, is governed by cosmological constant and it is given by

$$\lim_{r_+ \to \infty} M \propto -\frac{\Lambda r_+^4}{32} + O \left( r_+^4 \right).$$

It is a well established fact that black holes admit the first law of black hole thermodynamics. In other words, the following relation is valid for black holes

$$dM = TdS + \Phi_E dQ_E + \Phi_M dQ_M,$$  \hspace{1cm} (18)

in which $\Phi_E$ and $\Phi_M$ are electric and magnetic potentials. Using the obtained mass [17] with total electric [14] and magnetic [15] charges, one may calculate the electric and magnetic potentials as

$$\Phi_E = \frac{\pi q_E^2}{2r_+^3},$$  \hspace{1cm} (19)

$$\Phi_M = \frac{\pi q_M^2}{2r_+^3},$$  \hspace{1cm} (20)

which confirm the independency of all conserved quantities of the matter from GB gravity generalization and horizon of the solutions.

**B. van der Waals like behavior**

In this section, we investigate the possibility of van der Waals like phase transition for the solutions. To do so, we employ the proposal in which cosmological constant is considered as a thermodynamical variable known as pressure [87, 88]. The relation between cosmological constant and pressure is given by

$$P = -\frac{\Lambda}{8\pi}. $$

Replacing the cosmological constant with pressure in total mass results into changing the role of the total mass from internal energy to enthalpy. Using this enthalpy, it is a matter of calculation to obtain the volume of these black holes (per $V_3$) as

$$V = \left( \frac{\partial H}{\partial P} \right)_{q_M, q_E} = \frac{\pi r_+^4}{4},$$  \hspace{1cm} (21)

Evidently, there is a direct relation between total volume of the black holes and horizon radius. This indicates that instead of using volume in calculations, one can use the horizon radius (which is related to specific volume, linearly [87]). For the sake of uniformity, we use the horizon radius instead of total volume in our calculations.
By replacing the cosmological constant in the temperature and using horizon radius instead of volume, one can obtain the equation of state as

$$P = T \left( \frac{12\alpha k + 3q_E^2}{4r_+^3} - \frac{3k}{8\pi r_+^2} + \frac{q_E^2 + q_M^2}{8\pi r_+^6} \right). \quad (22)$$

Obtained equation of state gives us the following information regarding the pressure:

I) The pressure is an increasing function of the electric charge and magnetic charges. The contribution of temperature term (which includes GB gravity as well) depends on the horizon under consideration; 1) Flat horizon: the GB gravity’s effects on the pressure do not appear and pressure is an increasing function of temperature. 2) Spherical horizon: both temperature and GB gravity contribute positively to the pressure. 3) Hyperbolic horizon: the GB gravity has negative affect on the pressure, while for the temperature, if $r_+ = 2\sqrt{\alpha}$, the effects of temperature term is vanished. For $r_+ < 2\sqrt{\alpha}$, the temperature term has negative effect on the pressure. The opposite is true for the case $r_+ > 2\sqrt{\alpha}$.

II) Regarding the expression of the pressure, one can obtain its real valued root(s). Later, in phase transition diagrams, we confirm the presence of roots for the pressure (see Fig. 3).

III) Series expanding the pressure for small values of the horizon radius results into

$$\lim_{r_+ \to 0} P \propto \frac{q_E^2 + q_M^2}{8\pi r_+^6} + \frac{3k\alpha T}{r_+^3} + O\left(\frac{1}{r_+^2}\right),$$

which confirms the following points: 1) The high energy limit of pressure (also the behavior of pressure for large black holes) is governed only by the electric and magnetic charges. 2) In the absence of the electric and magnetic fields, the high energy limit of the pressure is governed by the temperature, topological factor and GB parameter with the same order of magnitude. 3) Evidently, for vanishing $r_+$, pressure diverges.

IV) The asymptotic behavior of the pressure (also the behavior of pressure for small black holes) is only governed by the temperature in the following form

$$\lim_{r_+ \to \infty} P \propto \frac{3T}{4r_+} + O\left(\frac{1}{r_+^2}\right).$$

In order to investigate the existence of van der Waals like phase transition, one can use the properties of inflection point of isotherm $P - V$ diagrams, as

$$\left( \frac{\partial P}{\partial r_+} \right)_{T,q_E,q_M} = \left( \frac{\partial^2 P}{\partial r_+^2} \right)_{T,q_E,q_M} = 0. \quad (23)$$

which leads to the following equation for obtaining the critical horizon radius (specific volume)

$$r_+^2 (kr_+^4 - 5q_E^2 - 5q_M^2) - 12\alpha kr_+^4 + 3q_E^2 + 3q_M^2 = 0. \quad (24)$$

Evidently, only for the flat horizon, the effects of the GB gravity disappear in critical horizon radius. It is easy to obtain the critical horizon radius. The full form is given in the appendix (see Eq. 24). The obtained form contains square root functions indicating that certain condition must be satisfied in order to have real valued positive critical horizon radius, hence critical behavior. Using obtained critical horizon radius (Eq. 24), one can obtain the critical temperature and pressure as well. (the full forms are given in the appendix (see Eqs. 39 and 40)). In order to elaborate the possibility of van der Waals like phase transition, we have plotted Fig. 4.

Evidently, by suitable choices of different parameters, one can obtain van der Waals like phase transition for these black holes. The presence of subcritical isobars in $T - r_+$ and isothermal diagrams in $P - r_+$ confirm the existence of van der Waals like phase transition. In order to study the effects of magnetic charge and GB parameter on critical values, we have plotted another set of diagrams (see Fig. 4).

As one can see, the critical horizon radius is an increasing function of the magnetic charge and GB parameter. Whereas, the critical temperature and pressure are decreasing functions of the magnetic charge and GB parameter. The effects of variation of the GB parameter on these critical values are more significant compared to magnetic charge. The blank space in critical temperature and pressure diagrams correspond to having non-real value for the critical temperature and pressure (for more details see Eqs. (39) and (40) in the appendix). In other words, these blank spaces correspond to the absence of van der Waals like phase transition for these black holes.
By suitable tuning, this expression could vanish. Such a case could only be achieved for hyperbolic black holes. This

The GB parameter is present in the denominator of the heat capacity as a factor of the expression

It is possible to extract the root of heat capacity analytically from this expression. The full form is given in appendix

In general, the heat capacity of these black holes could be obtained as

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The obtained heat capacity provides the following information:

I) The GB parameter is coupled with the cosmological factor. This indicates that for the horizon flat black holes, heat capacity is free of the GB gravity, so it is the same as that in Einstein gravity.

II) Regarding the roots of heat capacity, we have two distinctive terms from which we can extract the roots. 1) One of these expressions is

Evidently, for this expression, the root of heat capacity could be extracted in form of

This shows that only for hyperbolic black holes, GB parameter could contribute to the root of heat capacity. Otherwise, the root of heat capacity is independent of $\alpha$. 2) The other expression is

It is possible to extract the root of heat capacity analytically from this expression. The full form is given in appendix (see Eq. 25). The obtained root is independent of the GB parameter. In order to understand the effects of magnetic charge and pressure on the root, we have plotted Fig. 5. Evidently, the root of heat capacity is an increasing (a decreasing) function of the magnetic charge (pressure). It is worthwhile to mention that both the heat capacity and temperature share the same root.

III) In general, divergencies of the heat capacity are GB parameter dependent. But, it is possible to omit such effect. The GB parameter is present in denominator of the heat capacity as a factor of the expression

By suitable tuning, this expression could vanish. Such a case could only be achieved for hyperbolic black holes. This

FIG. 3: Van der Waals like phase diagrams for $q_M = q_E = 0.1$, $\alpha = 0.01$ and $k = 1$. The dot corresponds to phase transition point between small and large black holes. **Left Panel:** Bold line corresponds to $T = T_c$. **Middle Panel:** Bold line corresponds to $P = P_c$.

FIG. 4: Variation of critical horizon radius, temperature and pressure as functions of Gauss-Bonnet parameter and magnetic charge for: $q_E = 0.1$ and $k = 1$.

C. Stability of the solutions

Now, we focus on thermal stability of the solutions. In the canonical ensemble, we consider the heat capacity of the solutions. The heat capacity could be used to extract two important properties of the solutions: I) Phase transition points which could be detected as discontinuity in the heat capacity. II) Thermal stability of the solutions. If the heat capacity is positive valued, solutions are thermally stable, while the opposite stands for the unstable black holes.

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By suitable tuning, this expression could vanish. Such a case could only be achieved for hyperbolic black holes. This
is due to the fact that only the topological term with hyperbolic horizon could be negative, otherwise the mentioned expression is positive valued and non-zero.

IV) The high energy limit of heat capacity is governed by topological factor and GB gravity in the following form

\[ \lim_{r_+ \to 0} C \propto -\alpha k r_+ - \frac{r_+^3}{12} + O \left( r_+^5 \right). \]

As one can see, the next leading term in the high energy limit is only a function of the horizon radius. This comes from the contribution of the Einstein gravity. This shows that the effects of GB gravity are more dominant in the high energy limit of the heat capacity comparing to the Einstein gravity. In the limit of \( r_+ = 0 \), the heat capacity completely vanishes which is in contrast to the same cases of enthalpy, temperature and pressure where they diverge.

V) The asymptotic behavior is given by

\[ \lim_{r_+ \to \infty} C \propto \frac{3r_+^3}{4} + \left( \frac{9k}{16\pi P} - 3\alpha k \right) r_+ + O \left( \frac{1}{r_+^2} \right). \]

Evidently, the dominant term in asymptotic behavior includes only horizon radius. This comes from the contribution of Einstein gravity. Therefore, the dominant factor in asymptotic behavior of the heat capacity is Einstein gravity. The second dominant term includes pressure, GB parameter and topological factor. As we see, the magnetic and electric charge effects are not significant in both high energy limit and asymptotic behavior of the heat capacity.

Thermal stability in canonical ensemble is determined by the sign of heat capacity. Positivity (negativity) shows that black holes are thermally stable (unstable). Considering this issue, we have plotted Fig. 6 to study the effects of different parameters and possible cases of stability/instability for these black holes.

In general, there are three possibilities for the heat capacity:

I) Existence of two divergencies and one root: in this case, the stable regions are between root and smaller divergency, and after larger divergency.

II) One root and one divergency: in which the regions of the stability are between root and divergency, and after the divergency.

III) Only one root: where the stable black holes are after the root.

Plotted diagrams shows that there is a critical value for the GB parameter where for values smaller than it, two divergencies exist for the heat capacity. For the GB parameters larger than this critical value, the heat capacity enjoys the absence of divergency in its structure. The same applies to magnetic charge and pressure. Since we have plotted diagrams for spherical black holes, the root of heat capacity is independent of the GB parameter. Whereas, for the magnetic charge, the root is a sensitive function of it.

IV. TRAJECTORY OF THE PARTICLE

Investigation of particle geodesics is one of the interesting subjects in the context of black hole physics. This is due to the fact that analyzing the accretion disk and out flows of black holes have a direct relation to the trajectory of
Our final study in this paper is devoted to the trajectory of a particle around GB-dyonic black hole. Our main motivation is to investigate the effects of different parameters on such a trajectory. In addition, we would like to address the possible scenarios for the trajectory of the particle to some extent. In order to obtain the trajectory of the particle, we use Euler-Lagrange equation in the following form \[ \frac{d}{d\tau} \frac{\partial}{\partial \dot{x}^\alpha} \mathcal{L} - \frac{\partial}{\partial x^\alpha} \mathcal{L} = 0, \] \[ \text{(26)} \]

where \( \mathcal{L} \) is the particle Lagrangian, \( x^\alpha \) is coordinate, \( \dot{x}^\alpha = dx^\alpha/d\tau \) and \( \tau \) is the affine parameter. The Lagrangian of the particle can be obtained as

\[ 2\mathcal{L} = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -f(r) \left( \frac{dt}{d\tau} \right)^2 + \frac{1}{f(r)} \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left( \frac{d\psi}{d\tau} \right)^2 + \sin^2 \theta \sin^2 \psi \left( \frac{d\phi}{d\tau} \right)^2. \] \[ \text{(27)} \]

We choose \( \theta = \psi = \pi/2 \) to confine the motion of the particle to equatorial plane \( \theta = \psi = \pi/2 \) at all times. Considering the structure of the metric \[ \text{(2)}, \] one can confirm that the Lagrangian is independent of \( t \) and \( \phi \) coordinates. This indicates that \( t \) and \( \phi \) components yield conserved quantities as

\[ E = g_{tt} \frac{dt}{d\tau} = f(r) \frac{dt}{d\tau}, \] \[ \text{(28)} \]

\[ L = g_{\phi\phi} \frac{d\phi}{d\tau} = r^2 \frac{d\phi}{d\tau}, \] \[ \text{(29)} \]

in which \( E \) is the energy (per unit mass for massive particles) of the particle and \( L \) is the orbital angular momentum. Using the obtained relations and replacing them in Eq. \[ \text{(27)} \] result into

\[ \mathcal{L} = - \frac{E^2}{2f(r)} + \frac{1}{2f(r)} \left( \frac{dr}{d\tau} \right)^2 + \frac{L^2}{2r^2}. \] \[ \text{(30)} \]

Here, we focus only on the null orbits, requiring \( \mathcal{L} = 0 \). For the timelike orbits, one should consider \( \mathcal{L} = 1 \). In order to obtain the motion of the particle around black holes, one should replace Eqs. \[ \text{(28)-(30)} \] in the Euler-Lagrange equation \[ \text{(26)} \] and solve it. Due to the complexity of the obtained relations, we employed numerical computation to obtain expressions governing the motion of particle around these black holes. We fixed the initial conditions and study the effects of different parameters on the trajectory. The results are presented in Figs. \[ \text{7 and 8} \] and \[ \text{9} \] and \[ \text{10} \].

Evidently, depending on the choices of different parameters, the null orbits of particle around the black holes would be different. The shape of orbital movement of particle is highly sensitive to the variation of GB parameter, energy and orbital angular momentum. Whereas, the effects of the variation of magnetic charge on the shape is not as significant as other parameters. The effects of this parameter could be better seen in the context of the speed of

![Graphs showing thermally stable and/or unstable regions of the black holes for different parameters.](image-url)
(a) $q_M = 1$, $E = 3$ and $L = 5$.

(b) $\alpha = 0.5$, $E = 3$ and $L = 5$.

(c) $\alpha = 0.5$, $q_M = 1$ and $L = 5$.

(d) $\alpha = 0.5$, $q_M = 1$ and $E = 3$.

FIG. 7: Trajectory of the particle around black holes for $m = k = q_E = 1$, $\Lambda = -1$, $r(\tau = 0) = 10$ and $\tau$ running from 0 to 200.
\( q_M = 1, E = 3, L = 5, \alpha = 0.7 \) \( \alpha = 0.5, E = 3, L = 5 \) and \( q_M = 0.5 \)
\( q_M = 1, L = 5, \alpha = 0.5 \) and \( E = 3.2 \) \( q_M = 1, E = 3, \alpha = 0.5 \) and \( L = 6 \)

FIG. 8: Trajectory of a particle around black holes for \( m = k = q_E = 1 \), \( \Lambda = -1 \), \( r(\tau = 0) = 10 \) and \( \tau \) running from 0 to 200.

\( q_M = 1, E = 3 \) and \( L = 5 \) \( \alpha = 0.5, E = 3 \) and \( L = 5 \)
\( q_M = 1, \alpha = 0.5 \) and \( L = 5 \) \( q_M = 1, \alpha = 0.5 \) and \( E = 3 \)

FIG. 9: Angular frequency of the orbit for \( m = k = q_E = 1 \) and \( \Lambda = -1 \).

In order to have a better picture regarding the effects of different parameters on the motion of particle, we calculate angular frequency as measured at infinity. This is given by

\[
\Omega_c = \frac{d\phi}{dt} = \left( \frac{d\phi}{d\tau} \right)/\left( \frac{dt}{d\tau} \right) = \frac{f(r)L}{r^2E}. \tag{31}
\]

Using obtained metric function (6) and considering plotted diagrams in Fig. 7, we plot the following diagrams for the angular frequency (Fig. 9). The blank space corresponds to the non-real angular frequency. The diagrams confirm that angular frequency is a decreasing function of magnetic charge, GB parameter, energy and angular momentum. The plotted diagrams for angular frequency could also be used to determine the regions where there is no physical geodesics for the particle moving around the black holes. These regions are where no real valued angular frequency exists (hence blank spaces in Fig. 9).
V. CONCLUSION

In this paper, we have investigated dyonic black holes in the presence of GB gravity. The paper was focused on three aspects of these black holes including: solutions and geometrical properties, thermodynamic behavior and possible phase transition, and finally the trajectory of a particle around these black holes.

The metric function was obtained and it was shown that: I) These solutions could enjoy from zero up to three roots in their structures. II) The Kretschmann scalar admitted the existence of singularity and the fact that this singularity is affected by both magnetic charge and GB gravity. III) The asymptotic behavior of the Kretschmann admitted a modified version of the AdS/dS behavior.

The investigations regarding the thermodynamics of these black holes confirmed: I) Decoupling between electric and magnetic parts of the solutions enabling possibility of studying purely magnetically charged cases. II) Existence of the root for entropy and divergency for temperature due to GB gravity contributions. III) Dependency of the high energy limit of temperature and mass on both magnetic charge and GB gravity. IV) Dependency of the high energy limit of the pressure on magnetic charge and, heat capacity on GB gravity. V) Existence of van der Waals like phase transition for these black holes restricted to satisfaction of certain conditions. VI) Independency of heat capacity’s root on GB gravity (except for hyperbolic black holes) and its sensitivity on magnetic charge. VI) Dependency of the heat capacity’s divergencies on both GB gravity and magnetic charges which results into dependency of the thermal stability/instability of the black holes on these quantities.

Studying the trajectory of a particle around these black holes resulted into: I) High sensitivity of the shape of orbit on the GB gravity, energy and the orbital angular momentum whereas significantly less sensitivity to variation of magnetic charge and GB gravity. II) The asymptotic behavior of the Kretschmann admitted a existence of singularity and the fact that this singularity is affected by both magnetic charge and GB gravity. III) The Kretschmann scalar admitted the existence of singularity and the fact that this singularity is affected by both magnetic charge and GB gravity. IV) Dependency of the high energy limit of temperature and mass on both magnetic charge and GB gravity. V) Existence of van der Waals like phase transition for these black holes restricted to satisfaction of certain conditions. VI) Independency of heat capacity’s root on GB gravity (except for hyperbolic black holes) and its sensitivity on magnetic charge. VI) Dependency of the heat capacity’s divergencies on both GB gravity and magnetic charges which results into dependency of the thermal stability/instability of the black holes on these quantities.

The obtained results of this paper could be employed in the context of AdS/CFT. Especially, it would be interesting to study the effects of magnetic charge and GB gravity on the magnetization and susceptibility of the boundary theory, and the conditions for having diamagnetic and paramagnetic behaviors. The dynamical stability, quasinormal modes and gravitational waves of these black holes are other subjects of interest.

VI. APPENDIX

A. components of the field equations

Considering the obtained field equation (5) with the given metric (2), one can obtain the following non-zero components for the field equation

\[ e_{tt} = e_{rr} = 12r^3 \alpha f(r) f'(r) - 2q_E^2 - 2q_M^2 - 6r^4 f(r) - 2r^6 \Lambda - 3r^5 f'^4 k \left( \frac{2\alpha f'(r)}{r} - 1 \right) = 0, \]  

(32)

\[ e_{\theta\theta} = e_{\phi\phi} = 4r^4 \alpha f'^2 - r^6 f''(r) f(r) + 2q_E^2 + 2q_M^2 - 2r^6 \Lambda - 4r^5 f'^4 f(r) - 2r^4 k(2\alpha f''(r) - 1) = 0. \]  

(33)

Solving these two set of field equations, one can obtain the metric function that is given in Eq. (6).

B. roots of the metric function

Using the obtained metric function (6), it is a matter of calculation to obtain the following roots

\[ r_f(r) = \begin{cases} \sqrt{\frac{A_1^{2/3} + 2 \sqrt{3} i k + 4 \alpha k^2 \Lambda + 4 k^2 - 2 \Lambda m}{\sqrt{A_1^2 \Lambda}}} & \text{if} \quad (\sqrt{A_1^2 + 2(1+i \sqrt{3}) (\Lambda m - 2k^2 (\alpha \Lambda + 1))} \\ \sqrt{2 A_1^2 \Lambda}} \\ \sqrt{\frac{i(\sqrt{3} + 1) A_1^{2/3} + 4 \sqrt{3} i k + 2(1-i \sqrt{3}) (\Lambda m - 2k^2 (\alpha \Lambda + 1))}{2 \sqrt{A_1^2 \Lambda}}} & \text{if} \quad (\sqrt{A_1^2 + 2(1-i \sqrt{3}) (2k^2 (\alpha \Lambda + 1) - \Lambda m)} \\ \sqrt{2 A_1^2 \Lambda}} \end{cases}, \]  

(34)

where

\[ A_1 = 8k^3 + 6k \Lambda (2\alpha k^2 - m) + \Lambda^2 (q_E^2 + q_M^2) + \sqrt{(4k^3 (3\alpha \Lambda + 2) - 6k \Lambda m + \Lambda^2 (q_E^2 + q_M^2))^2 + 8(\Lambda m - 2k^2 (\alpha \Lambda + 1))^3}. \]

Evidently, under certain circumstances, it is possible to have from zero up to three roots for the metric function. This issue is confirmed in Fig. 1.
C. Kretschmann

The general formula for calculating the Kretschmann scalar of 5−dimensional topological black holes is given in Eq. (8). Using the obtained metric function (6), it is a matter of calculation to obtain Kretschmann scalar as

\[
K = \frac{1}{12A_3 r^{12}} \left\{ -2\alpha(A_2 - 3)A_3^3 r^{12} + (A_2 - 3)^2 A_3^2 r^{12} + 72\alpha^2 A_3^7 r^{12} (7 (q_E^2 + q_M^2) - 10mr^2) \\
-288\alpha^2 A_3^6 r^{12} \left(-2mr^2 + q_E^2 + q_M^2\right) + 2A_2 \left((A_2 - 3)A_3 r^6 + 36\alpha \left(-2mr^2 + q_E^2 + q_M^2\right)\right)^2 \\
+5184\alpha^3 \left(-2mr^2 + q_E^2 + q_M^2\right)^2 \right\},
\]

in which

\[
A_2 = \sqrt{\frac{6m}{r^4} - \frac{2(q_E^2 + q_M^2)}{r^6}} + 9.
\]

D. roots and critical values of thermodynamical quantities

Using the obtained temperature (13), one is able to extract the its roots in the following form

\[
r_{T=0} = \begin{cases} 
\sqrt{\frac{2^{2/3}A_3^2 + 6A_3 k + 18 \sqrt{2}k^2}{6A_3 k}}, \\
\sqrt{\frac{2^{2/3}(-1 - i\sqrt{3})A_3^2 + 12A_3 k + 18i \sqrt{3}(\sqrt{3}+i)k^2}{12A_3 k}}, \\
\sqrt{\frac{2^{2/3}(\sqrt{3}+i)A_3^2 + 12A_3 k + 18i \sqrt{3}(\sqrt{3}-i)k^2}{12A_3 k}},
\end{cases}
\]

where

\[
A_3 = \sqrt{\frac{54k^3 - 27\Lambda^2 (q_E^2 + q_M^2)}{2}} + \sqrt{(54k^3 - 27\Lambda^2 (q_E^2 + q_M^2))^2 - 2916k^6}.
\]

In general, three roots exist for temperature that under satisfaction of certain conditions, all three cases might yield positive real valued roots for the temperature.

As for the total mass (17), it is a matter of calculation to obtain the following roots

\[
r_{M=0} = \begin{cases} 
\sqrt{\frac{2^{2/3}A_3^2 + 12A_3 k + 72 \sqrt{2}k^2}{6A_3 k}}, \\
\sqrt{\frac{2^{2/3}(-1 - i\sqrt{3})A_3^2 + 24A_3 k + 72i \sqrt{3}(\sqrt{3}+i)k^2(a\Lambda+1)}{12A_3 k}}, \\
\sqrt{\frac{2^{2/3}(\sqrt{3}+i)A_3^2 + 24A_3 k - 72i \sqrt{3}(\sqrt{3}-i)k^2(a\Lambda+1)}{12A_3 k}},
\end{cases}
\]

in which

\[
A_4 = \sqrt{648\alpha k^3 \Lambda + 432k^3 + 54\Lambda^2 (q_E^2 + q_M^2)} + \sqrt{(648\alpha k^3 \Lambda + 432k^3 + 54\Lambda^2 (q_E^2 + q_M^2))^2 + 4(-36\alpha k^2 \Lambda - 36k^2)^3}.
\]

Evidently, the mass, similar to the temperature and metric function could enjoy three roots in its structure.

Our next items of the interest are critical horizon radius, temperature and pressure. By solving Eq. (24), one can obtain the critical horizon radius as

\[
r_c = \sqrt{\frac{\sqrt{3}A_5^2 + 12\alpha A_5 k^2 + 48 3^{2/3}\alpha^2 k^4 + 5 3^{2/3}k (q_E^2 + q_M^2)}{3A_5 k}},
\]

in which

\[
A_5 = \sqrt{576\alpha^3 k^6 + \sqrt{3} \sqrt{k^3 (q_E^2 + q_M^2) (-144\alpha^2 k^4 + q_E^2 + q_M^2) (432\alpha^2 k^3 + 125 (q_E^2 + q_M^2)) + 252 \alpha k^3 (q_E^2 + q_M^2)}}.
\]
Using obtained critical horizon, it is a matter of calculation to obtain critical temperature and pressure in the following forms

\[ T_c = \frac{-kr^4 + q^2_M}{12\pi kr^3 + \pi r^5}, \]  

\[ P_c = \frac{-3kr^4 + 24\pi kr^3 T_c + q^2_E + q^2_M + 6\pi r^5 T_c}{8\pi r^6}, \]  

(39)  

(40)

Finally, considering the calculated heat capacity \( (25) \), one can extract its roots as

\[
 r_{C=0} = \begin{cases} 
 \sqrt{\frac{18\sqrt{2}\sqrt{2}\pi (\sqrt{3} + i) k^2 \left( -1 - i \sqrt{3} \right) A_6^2}{18 \sqrt{2} (\sqrt{3} + i) k^2 - 12 k A_6 + i 2^{3/2}}}, \\
 \sqrt{\frac{18\sqrt{2}\sqrt{2}\pi (\sqrt{3} + i) k^2 \left( -1 - i \sqrt{3} \right) A_6^2}{96 \pi A_6 P}}, \\
 \sqrt{\frac{18\sqrt{2}\sqrt{2}\pi (\sqrt{3} + i) k^2 \left( -1 - i \sqrt{3} \right) A_6^2}{96 \pi A_6 P}}, \\
 \sqrt{\frac{18\sqrt{2}\sqrt{2}\pi (\sqrt{3} + i) k^2 \left( -1 - i \sqrt{3} \right) A_6^2}{96 \pi A_6 P}}, \\
 \end{cases}
\]  

(41)

in which

\[
 A_6 = \sqrt{-54k^3 + \sqrt{(-54k^3 + 1728\pi^2 P^2(q^2_E + q^2_M))^2 - 2916k^6 + 1728\pi^2 P^2(q^2_E + q^2_M)}}.
\]

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