2D binary operadic Lax representation for harmonic oscillator

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Abstract

It is explained how the time evolution of operadic variables may be introduced by using the operadic Lax equation. As an example, a 2-dimensional binary operadic Lax representation for the harmonic oscillator is constructed.

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1 Introduction

In the Hamiltonian formalism, a mechanical system is described by canonical variables $q^i, p_i$ and their time evolution is prescribed by the Hamiltonian system

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \quad (1.1)$$

By a Lax representation [6, 1] of a mechanical system one means such a pair $(L, M)$ of matrices (linear operators) $L, M$ that the above Hamiltonian system may be represented as the Lax equation

$$\frac{dL}{dt} = [M, L] := ML - LM \quad (1.2)$$

Thus, from the algebraic point of view, mechanical systems can be described by linear operators, i.e by linear maps $V \rightarrow V$ of a vector space $V$. As a generalization of this one can pose the following question [7]: how to describe the time evolution of the linear operations (multiplications) $V^\otimes n \rightarrow V$?

The algebraic operations (multiplications) can be seen as an example of the operadic variables [2, 3, 4, 5]. If an operadic system depends on time one can speak about operadic dynamics [7]. The latter may be introduced by simple and natural analogy with the Hamiltonian dynamics. In particular, the time evolution of operadic variables may be given by the operadic Lax equation. In [8] it was shown how the dynamics may be introduced in a 2-dimensional Lie algebra. In the present paper, a 2-dimensional binary operadic Lax representation for the harmonic oscillator is constructed.

2 Operad

Let $K$ be a unital associative commutative ring, and let $C^n$ $(n \in \mathbb{N})$ be unital $K$-modules. For $f \in C^n$, we refer to $n$ as the degree of $f$ and often write (when it does not cause confusion) $f$.
instead of \( \deg f \). For example, \((-1)^f := (-1)^n \), \( C^f := C^n \) and \( \circ_f := \circ_n \). Also, it is convenient to use the reduced degree \( |f| := n - 1 \). Throughout this paper, we assume that \( \otimes := \otimes_K \).

**Definition 2.1** (operad (e.g. [2, 3])). A linear (non-symmetric) operad with coefficients in \( K \) is a sequence \( C := \{C^n\}_{n \in \mathbb{N}} \) of unital \( K \)-modules (an \( \mathbb{N} \)-graded \( K \)-module), such that the following conditions hold:

1. For \( 0 \leq i \leq m - 1 \) there exist the partial compositions
   \[ \circ_i \in \text{Hom}(C^m \otimes C^n, C^{m+n-1}), \quad |\circ_i| = 0 \]

2. For all \( h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g \), the composition (associativity) relations hold,
   \[ (h \circ_i f) \circ_j g = \begin{cases} \((-1)^{|f||g|}(h \circ_{i+|g|} f) \circ_{i+|g|} f & \text{if } 0 \leq j \leq i - 1, \\ h \circ_i (f \circ_{j-i} g) & \text{if } i \leq j \leq i + |f|, \\ (-1)^{|f||g|}(h \circ_{j-|f|} g) \circ_i f & \text{if } i + f \leq j \leq |h| + |f|. \end{cases} \]

3. A unit \( I \in C^1 \) exists such that
   \[ 1 \circ_0 f = f = f \circ_i I, \quad 0 \leq i \leq |f| \]

In the second item, the first and third parts of the defining relations turn out to be equivalent.

**Example 2.2** (endomorphism operad [2]). Let \( V \) be a unital \( K \)-module and \( \mathcal{E}_V^n := \mathcal{E}nd^m_V := \text{Hom}(V^\otimes n, V) \). Define the partial compositions for \( f \otimes g \in \mathcal{E}^f_V \otimes \mathcal{E}^g_V \) as
   \[ f \circ_i g := (-1)^{|g|f} f \circ (\text{id}_V^\otimes \otimes g \otimes \text{id}_V^\otimes (|f|-i)), \quad 0 \leq i \leq |f| \]
Then \( \mathcal{E}_V := \{\mathcal{E}^n_V\}_{n \in \mathbb{N}} \) is an operad (with the unit \( \text{id}_V \in \mathcal{E}^1_V \)) called the endomorphism operad of \( V \).

Therefore, algebraic operations can be seen as elements of the endomorphism operad. Just as elements of a vector space are called vectors, it is natural to call elements of an abstract operad operations.

### 3 Gerstenhaber brackets and operadic Lax equation

**Definition 3.1** (total composition [2, 3]). The total composition \( \bullet : C^f \otimes C^g \rightarrow C^{f+|g|} \) is defined in an operad \( C \) by
   \[ f \bullet g := \sum_{i=0}^{|f|} f \circ_i g \in C^{f+|g|}, \quad |\bullet| = 0 \]
The pair \( \text{Com} C := \{C, \bullet\} \) is called the composition algebra of \( C \).

**Definition 3.2** (Gerstenhaber brackets [2, 3]). The Gerstenhaber brackets \( [\cdot, \cdot] \) are defined in \( \text{Com} C \) as a graded commutator by
   \[ [f, g] := f \bullet g - (-1)^{|f||g|} g \bullet f = (-1)^{|f||g|}[g, f], \quad |[\cdot, \cdot]| = 0 \]
The commutator algebra of \( \text{Com} C \) is denoted as \( \text{Com}^{-1} C := \{C, [\cdot, \cdot]\} \). One can prove that \( \text{Com}^{-1} C \) is a graded Lie algebra. The Jacobi identity reads
   \[ (-1)^{|f||h|}[f, [g, h]] + (-1)^{|g||f|}[[g, h], f] + (-1)^{|h||g|}[[h, f], g] = 0 \]
Assume that \( K := \mathbb{R} \) and operations are differentiable. Dynamics in operadic systems (operadic dynamics) may be introduced by
Definition 3.3 (operadic Lax pair \[7\]). Allow a classical dynamical system to be described by the Hamiltonian system \[1.1\]. An operadic Lax pair is a pair \((L, M)\) of the homogeneous operations \(L, M \in C\), such that the Hamiltonian system \([1.1]\) may be represented as the operadic Lax equation

\[
\frac{dL}{dt} = [M, L] := M \cdot L - (-1)^{|M||L|} L \cdot M
\]

Evidently, the degree constraints \(|L| = |M| = 0\) give rise to ordinary Lax equation \([1.2]\) \[6, 1\].

4 Operadic harmonic oscillator

Consider the Lax pair for the harmonic oscillator:

\[
L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \frac{\omega}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

Since the Hamiltonian is

\[
H(q, p) = \frac{1}{2}(p^2 + \omega^2 q^2)
\]

it is easy to check that the Lax equation

\[
\dot{L} = [M, L] := ML - LM
\]

is equivalent to the Hamiltonian system

\[
\frac{dq}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q
\]

(4.1)

If \(\mu\) is a homogeneous operadic variable one can use the above Hamilton equations to obtain

\[
\frac{d\mu}{dt} = \frac{\partial \mu}{\partial q} \frac{dq}{dt} + \frac{\partial \mu}{\partial p} \frac{dp}{dt} = p \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = [M, \mu]
\]

Therefore, we get the following linear partial differential equation for the operadic variable \(\mu(q, p)\):

\[
p \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = M \cdot \mu - \mu \cdot M
\]

By integrating one gains sequences of operations called the operadic (Lax representations for) harmonic oscillator.

5 Main example

Let \(A := \{V, \mu\}\) be a binary algebra with an operation \(xy := \mu(x \otimes y)\). We require that \(\mu = \mu(q, p)\) so that \((\mu, M)\) is an operadic Lax pair, i.e the operadic Lax equation

\[
\dot{\mu} = [M, \mu] := M \cdot \mu - \mu \cdot M, \quad |\mu| = 1, \quad |M| = 0
\]

is equivalent to the Hamiltonian system of the harmonic oscillator.

Let \(x, y \in V\). Assuming that \(|M| = 0\) and \(|\mu| = 1\), one has

\[
M \cdot \mu = \sum_{i=0}^{0} (-1)^{|\mu|} M \circ_i \mu = M \circ_0 \mu = M \circ \mu
\]

\[
\mu \cdot M = \sum_{i=0}^{1} (-1)^{|M|} \mu \circ_i M = \mu \circ_0 M + \mu \circ_1 M = \mu \circ (M \otimes \text{id}_V) + \mu \circ (\text{id}_V \otimes M)
\]
Therefore, one has
\[ \frac{d}{dt}(xy) = M(xy) - (Mx)y - x(My) \]

Let \( \dim V = n \). In a basis \( \{e_1, \ldots, e_n\} \) of \( V \), the structure constants \( \mu_{jk}^i \) of \( A \) are defined by
\[ \mu(e_j \otimes e_k) := \mu_{jk}^i e_i, \quad j, k = 1, \ldots, n \]

In particular,
\[ \frac{d}{dt}(e_j e_k) = M(e_j e_k) - (Me_j)e_k - e_j(Me_k) \]

By denoting \( Me_i := M_i^s e_s \), it follows that
\[ \dot{\mu}_{jk}^i = \mu_{jk}^i M_s^i - M_j^s \mu_{sk}^i - M_k^s \mu_{sj}^i, \quad i, j, k = 1, \ldots, n \]

In particular, one has

**Lemma 5.1.** Let \( \dim V = 2 \) and \( M := (M_j^i) := \frac{\omega}{2} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \). Then the 2-dimensional binary operadic Lax equations read
\[ \begin{align*}
\dot{\mu}_{11}^1 &= -\frac{\omega}{2} (\mu_{11}^2 + \mu_{12}^1 + \mu_{21}^1), \\
\dot{\mu}_{12}^1 &= -\frac{\omega}{2} (\mu_{12}^2 - \mu_{11}^1 + \mu_{22}^1), \\
\dot{\mu}_{21}^1 &= -\frac{\omega}{2} (\mu_{21}^2 - \mu_{11}^1 + \mu_{22}^1), \\
\dot{\mu}_{22}^1 &= -\frac{\omega}{2} (\mu_{22}^2 - \mu_{12}^1 + \mu_{21}^1)
\end{align*} \]

For the harmonic oscillator, define its auxiliary functions \( A_\pm \) and \( D_\pm \) by
\[ \begin{align*}
A_+^2 + A_-^2 &= 2\sqrt{2}H \\
A_+^2 - A_-^2 &= 2p \\
A_+ A_+ + A_- A_- &= \omega q
\end{align*} \]

Differentiating the defining relations (5.1) of \( A_\pm \) with respect to \( t \) one gets
\[ \begin{align*}
A_+ \dot{A}_+ + A_- \dot{A}_- &= \frac{1}{\sqrt{2}H} (pp + \omega^2 qq) \\
A_+ \dot{A}_+ - A_- \dot{A}_- &= \dot{p} \\
A_- \dot{A}_+ + A_+ \dot{A}_- &= \omega \dot{q}
\end{align*} \]

Now one can propose

**Theorem 5.2.** Let \( C_\nu \in \mathbb{R} \) (\( \nu = 1, \ldots, 8 \)) be arbitrary real-valued parameters, \( M := \frac{\omega}{2} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \) and
\[ \begin{align*}
\mu_{11}^1(q, p) &= C_5 A_- + C_6 A_+ + C_7 D_- + C_8 D_+ \\
\mu_{12}^1(q, p) &= C_1 A_- + C_2 A_+ - C_7 D_+ + C_8 D_- \\
\mu_{21}^1(q, p) &= -C_1 A_- - C_2 A_- - C_3 A_+ - C_4 A_- - C_5 A_+ + C_6 A_+ - C_7 D_+ + C_8 D_- \\
\mu_{22}^1(q, p) &= -C_3 A_- + C_4 A_+ - C_7 D_- - C_8 D_+ \\
\mu_{11}^2(q, p) &= C_3 A_- + C_4 A_- - C_7 D_+ + C_8 D_- \\
\mu_{12}^2(q, p) &= C_1 A_- - C_2 A_+ + C_3 A_- - C_4 A_+ + C_5 A_- + C_6 A_+ - C_7 D_- - C_8 D_+ \\
\mu_{21}^2(q, p) &= -C_1 A_- - C_2 A_+ - C_7 D_- - C_8 D_+ \\
\mu_{22}^2(q, p) &= -C_5 A_+ + C_6 A_- + C_7 D_+ - C_8 D_- \\
\end{align*} \]

Then \((\mu, M)\) is a 2-dimensional binary operadic Lax pair of the harmonic oscillator.
Proof. Denote

\[
\begin{aligned}
C_{±}^\omega/2 & := \dot{A}_± \pm \frac{ω}{2} A_± \\
C_{±}^{3ω/2} & := D_± \pm \frac{3ω}{2} D_±
\end{aligned}
\]

Define the matrix

\[
\Gamma = (\Gamma_β^α) :=
\begin{pmatrix}
0 & G_+^{\omega/2} & -G_-^{\omega/2} & 0 & 0 & G_-^{\omega/2} & -G_+^{\omega/2} & 0 \\
0 & G_-^{\omega/2} & -G_+^{\omega/2} & 0 & 0 & G_+^{\omega/2} & -G_-^{\omega/2} & 0 \\
0 & 0 & -G_-^{\omega/2} & -G_+^{\omega/2} & G_-^{\omega/2} & G_+^{\omega/2} & 0 & 0 \\
0 & 0 & -G_+^{\omega/2} & G_-^{\omega/2} & G_+^{\omega/2} & -G_-^{\omega/2} & 0 & 0 \\
G_-^{3ω/2} & 0 & -G_+^{3ω/2} & -G_-^{3ω/2} & -G_+^{3ω/2} & G_-^{3ω/2} & G_+^{3ω/2} & G_-^{3ω/2} \\
G_+^{3ω/2} & G_-^{3ω/2} & G_+^{3ω/2} & -G_-^{3ω/2} & -G_+^{3ω/2} & -G_-^{3ω/2} & G_+^{3ω/2} & G_-^{3ω/2}
\end{pmatrix}
\]

Then, by using Lemma 5.1, it follows that the 2-dimensional binary operadic Lax equations read

\[
C_β^ω/2 = 0, \quad α = 1, \ldots, 8
\]

Since parameters \( C_β \) are arbitrary, the latter constraints imply \( Γ = 0 \). Thus one has to consider the differential equations

\[
G_±^{\omega/2} = 0 = G_±^{3ω/2}
\]

We show that

\[
\begin{aligned}
\dot{p} &= -ω^2 q \quad \iff \quad G_±^{\omega/2} = 0 \quad \iff \quad G_±^{3ω/2} = 0
\end{aligned}
\]

First prove (I). \( \Longrightarrow \): Assume that the Hamilton equations (4.1) for the harmonic oscillator hold. Then it follows from (5.2) that

\[
\begin{aligned}
A_± \dot{A}_± + A_- \dot{A}_- &= 0 \\
A_+ \dot{A}_+ - A_- \dot{A}_- &= -ω^2 q \\
A_- \dot{A}_+ + A_+ \dot{A}_- &= ωp
\end{aligned}
\]

\[
\begin{aligned}
2A_- \dot{A}_- &= ω^2 q \\
2A_+ \dot{A}_+ &= -ω^2 q \\
A_- \dot{A}_+ + A_+ \dot{A}_- &= ωp
\end{aligned}
\]

\[
\begin{aligned}
\dot{A}_- &= \frac{ω^2 q}{2A_+} = \frac{ω^2 q A_±}{2A_+ A_-} = \frac{ω}{2} A_+ \\
A_+ &= \frac{ω^2 q}{2A_+} = \frac{ω^2 q A_±}{2A_+ A_-} = -\frac{ω}{2} A_- \\
A_+^2 - A_-^2 &= 2p
\end{aligned}
\]

\[
\iff \quad G_±^{\omega/2} = 0
\]

and the latter the required system for \( A_± \).

\( \iff \): Assume that the differential equations \( G_±^{\omega/2} = 0 \) hold. Then it follows from (5.2) that

\[
\begin{aligned}
A_- A_+ - A_+ A_- &= \frac{2(\dot{p} + ω^2 q \dot{q})}{\sqrt{2H}} \\
A_± A_± + A_- A_+ &= -\frac{ω^2 q}{\dot{p}} \\
A_+^2 - A_-^2 &= 2q
\end{aligned}
\]

\[
\begin{aligned}
\dot{p} + ω^2 q \dot{q} &= 0 \\
A_± A_± &= \frac{1}{2} \dot{p} \\
A_+^2 - A_-^2 &= 2q
\end{aligned}
\]

\[
\iff \quad \dot{p} = -ω A_± A_- = -ω^2 q \\
\dot{q} = \frac{1}{2} (A_+^2 - A_-^2) = p
\]

\[\text{5}\]
where the first relation easily follows from the Hamiltonian system (4.1).

Now prove (II). Differentiate the auxiliary functions \( D_\pm \) to get

\[
\begin{align*}
\dot{D}_+ &= \frac{1}{2} \dot{A}_+(A_+^2 - 3A_+^2) + A_+(A_+ \dot{A}_+ - 3A_- \dot{A}_-) \\
\dot{D}_- &= \frac{1}{2} \dot{A}_-(3A_-^2 - A_-^2) + A_-(3A_+ \dot{A}_+ - A_- \dot{A}_-)
\end{align*}
\]

\[\implies\text{ Assume that functions } A_\pm \text{ satisfy the differential equations } G_{\pm}^{\omega/2} = 0. \text{ Then} \]

\[
\begin{align*}
\dot{D}_+ &= -\frac{\omega}{2} A_-(A_+^2 - 3A_+^2) - \frac{A_+ \omega}{2} (A_+ A_- + 3A_- A_+) \\
\dot{D}_- &= \frac{\omega}{2} A_+(3A_+^2 - A_-^2) - \frac{A_- \omega}{2} (3A_+ A_+ + A_- A_-)
\end{align*}
\]

and

\[
\begin{align*}
\dot{D}_+ &= -\frac{\omega}{2} A_-(A_+^2 - 3A_+^2) = -\frac{\omega}{2} D_- \\
\dot{D}_- &= \frac{\omega}{2} A_+(3A_+^2 - A_-^2) = \frac{\omega}{2} D_+ 
\end{align*}
\]

\[\iff G_{\pm}^{3\omega/2} = 0 \]

\[\iff \text{ Assume that functions } D_\pm \text{ satisfy the differential equations } G_{\pm}^{3\omega/2} = 0. \text{ Then} \]

\[
\begin{align*}
\begin{cases}
\dot{A}_+ (3A_+^2 - 3A_+^2) + \dot{A}_- (-6A_+ A_+) = -3\omega D_- \\
\dot{A}_+ (6A_+ A_-) + \dot{A}_- (3A_+^2 - 3A_+^2) = 3\omega D_+
\end{cases}
\end{align*}
\]

To use the Cramer formulae, calculate

\[
\Delta = \begin{vmatrix}
p & -\omega q \\
\omega q & p
\end{vmatrix} = p^2 + \omega^2 q^2 = 2H
\]

\[
\Delta \dot{A}_+ = \begin{vmatrix}
-\frac{\omega}{2} D_- & -\omega q \\
\frac{\omega}{2} D_+ & p
\end{vmatrix} = -\frac{\omega}{2}(D_- p - D_+ \omega q)
\]

\[
\Delta \dot{A}_- = \begin{vmatrix}
p & -\frac{\omega}{2} D_- \\
\omega q & \frac{\omega}{2} D_+
\end{vmatrix} = \frac{\omega}{2}(D_+ p + D_- \omega q)
\]

Note that

\[
\begin{align*}
D_- p - D_+ \omega q &= \frac{A_-}{2} p(3A_+^2 - A_-^2) - \frac{A_+}{2} \omega q (A_+^2 - 3A_+^2) \\
&= \frac{A_-}{2} \frac{1}{2} (A_+^2 - A_-^2)(3A_+^2 - A_-^2) - \frac{A_+}{2} A_- A_- (A_+^2 - 3A_+^2) \\
&= \frac{A_-}{2} (A_+^2 + A_-^2)^2 = 2A_- H
\end{align*}
\]

\[
\begin{align*}
D_+ p + D_- \omega q &= \frac{A_+}{2} p(3A_+^2 - 3A_+^2) + \frac{A_-}{2} \omega q (3A_+^2 - A_-^2) \\
&= \frac{A_+}{2} \frac{1}{2} (A_+^2 - A_-^2)(3A_+^2 - A_-^2) + \frac{A_-}{2} A_+ A_- (3A_+^2 - A_-^2) \\
&= \frac{A_+}{2} (A_+^2 + A_-^2)^2 = 2A_+ H
\end{align*}
\]

Thus,

\[
\begin{align*}
\dot{A}_+ &= \frac{\Delta \dot{A}_+}{\Delta} = -\frac{\omega}{2} \frac{2HA_-}{2H} = -\frac{\omega}{2} A_- \iff G_{\pm}^{\omega/2} = 0 \\
\dot{A}_- &= \frac{\Delta \dot{A}_-}{\Delta} = \frac{\omega}{2} \frac{2HA_+}{2H} = \frac{\omega}{2} A_+ \end{align*}
\]

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