Multiple Liquid Bridges with Non-Smooth Interfaces

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Abstract

We consider a coexistence of two axisymmetric liquid bridges $LB_i$ and $LB_m$ of two immiscible liquids $i$ and $m$ which are immersed in a third liquid (or gas) $e$ and trapped between two smooth solid bodies with axisymmetric surfaces $S_1, S_2$ and free contact lines. Evolution of liquid bridges allows two different configurations of $LB_i$ and $LB_m$ with multiple (five or three) interfaces of non-smooth shape. We formulate a variational problem with volume constraints and present its governing equations supplemented by boundary conditions. We find a universal relationship between curvature of the interfaces and discuss the Young relation at the singular curve where all liquids meet together.

Keywords: Isoperimetric problem, multiple liquid bridges, the vectorial Young relation

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1 Introduction

Consider an evolution of two liquid bridges $LB_i$ and $LB_m$ of immiscible liquids, $i$ (inner) and $m$ (intermediate), trapped between two axisymmetric smooth solid bodies with surfaces $S_1, S_2$ in such a way that $LB_i$ is immersed into $LB_m$ and the latter is immersed into the $e$ (external) liquid (or gas) which occupies the rest of the space between the two bodies (see Figure 1a). When liquid $m$ begins to evaporate then $LB_m$ reduces in volume (and width). Depending on the relationships between the contact angles of both liquids on $S_1$ and $S_2$ there are two scenarios for connectivity breakage of the liquid bridge $m$ between the two solids. The first scenario \textit{(five interfaces)} occurs when $LB_m$ splits into two parts each supported by a different solid (see Figure 1b). The second scenario \textit{(three interfaces)} occurs when $LB_m$ is left as a
Figure 1: a) The meridional section of two Und interfaces before $LB_m$ rupture. b) Five Und interfaces of different curvatures for three immiscible liquids trapped between two smooth solid bodies with free BC. The endpoints $C_1, C_2, C_3, C_4$ have one degree of freedom: the upper and lower endpoints are running along $S_1$ and $S_2$, respectively. The endpoints $C_5, C_6$ have two degrees of freedom and are located on two singular curves $L_1, L_2$, respectively, which are passing transversely to the plane of Figure.

The whole but has support only on the upper (see Figure 2b) or lower solid.

Figure 2: a) Two Und interfaces before $LB_m$ rupture. b) Three Und interfaces of different curvatures for three immiscible liquids trapped between two smooth solid bodies with free BCs. The endpoints $C_1, C_2, C_4$ have one degree of freedom while $C_3$ has two degrees and is located on a singular curve $L$ which is passing transversely to the plane of Figure.

Both scenarios lead to a new phenomenon which has not been discussed in literature before, namely, an existence of multiple LBs with non-smooth interfaces. In contrast to the known LBs with fixed and free contact line (CL), here one of CLs appears as a line where three interfaces with different curvatures meet.

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together. From a mathematical standpoint this singular curve is governed by transversality conditions (in physics they are referred to as the Young relations), and coincidence conditions, i.e., three interfaces always intersect at one single curve. We derive a relationship combining the constant mean curvatures of three different interfaces and give the interfaces consistency rules for their coexistence. Another important result is the vectorial Young relation at the triple point which is located on a singular curve.

2 Variational problem for five interfaces

Consider a functional \( E[r, z] \) of surface energy

\[
E[r, z] = \sum_{j=1}^{5} \int_{\phi_j^1}^{\phi_j^5} E_j d\phi_j + \int_{\psi_1^1}^{\psi_1^5} A_{s_1}^i d\psi_1 + \int_{\psi_2^1}^{\psi_2^5} A_{s_2}^m d\psi_1 + \int_{\psi_3^1}^{\psi_3^5} A_{s_3}^0 d\psi_1 + \int_{\psi_4^1}^{\psi_4^5} A_{s_4}^m d\psi_1 + \int_{\psi_5^1}^{\psi_5^5} A_{s_5}^0 d\psi_1 + \int_{\psi_1^1}^{\psi_1^5} A_{r_1}^i d\psi_1 + \int_{\psi_2^1}^{\psi_2^5} A_{r_2}^m d\psi_1 + \int_{\psi_3^1}^{\psi_3^5} A_{r_3}^0 d\psi_1 + \int_{\psi_4^1}^{\psi_4^5} A_{r_4}^m d\psi_1 + \int_{\psi_5^1}^{\psi_5^5} A_{r_5}^0 d\psi_1
\]

(2.1)

where

\[
E_j = \gamma_j r_j \sqrt{r_j^2 + z_j^2}, \quad 1 \leq j \leq 5, \quad A_{s_{\alpha}}^l = \gamma_{s_{\alpha}}^l R_{\alpha} \sqrt{R_{\alpha}^2 + Z_{\alpha}^2}, \quad \alpha = 1, 2, \quad l = i, m, e
\]

(2.2)

Two other functionals \( V_i[r, z] \) and \( V_m[r, z] \) for volumes of liquids \( i \) and \( m \) read

\[
V_i[r, z] = \int_{\phi_i^1}^{\phi_i^5} V_1 d\phi_1 + \int_{\phi_i^2}^{\phi_i^5} V_2 d\phi_2 + \int_{\phi_i^3}^{\phi_i^5} V_3 d\phi_3 + \int_{\phi_i^4}^{\phi_i^5} V_4 d\phi_4 - \int_{\phi_i^1}^{\phi_i^5} B_{s_1} d\psi_1 + \int_{\phi_i^2}^{\phi_i^5} B_{s_2} d\psi_2,
\]

\[
V_m[r, z] = \int_{\phi_m^1}^{\phi_m^5} V_1 d\phi_1 + \int_{\phi_m^2}^{\phi_m^5} V_2 d\phi_2 + \int_{\phi_m^3}^{\phi_m^5} V_3 d\phi_3 + \int_{\phi_m^4}^{\phi_m^5} V_4 d\phi_4 - \int_{\phi_m^1}^{\phi_m^5} B_{s_1} d\psi_1 + \int_{\phi_m^2}^{\phi_m^5} B_{s_2} d\psi_2,
\]

(2.3)

where

\[
V_j = \frac{1}{2} z_j r_j^2, \quad 1 \leq j \leq 5, \quad B_{s_{\alpha}} = \frac{1}{2} Z_{\alpha}^2 R_{\alpha}, \quad \alpha = 1, 2.
\]

The isoperimetric problem requires to find a set of functions \( \phi_j, \), \( \psi_j \), providing a local minimum of \( E[r, z] \) with two constraints \( V_i[r, z] = V_i \) and \( V_m[r, z] = V_m \) imposed on the volumes of liquids \( i \) and \( m \). Consider a composite functional

\[
W[r, z] = E[r, z] - \lambda_1 V_m[r, z] - \lambda_3 V_i[r, z],
\]

(2.4)
with two Lagrange multipliers $\lambda_j$ and represent it in the following form,

$$W[r, z] = \sum_{j=1}^{5} \int_{\phi_j}^{\phi_j^1} F_j d\phi_j + \int_{0}^{\psi_j^2} G_1^1 d\psi_j + \int_{\psi_1^2}^{\psi_j^2} G_1^m d\psi_j + \int_{\psi_1^2}^{\infty} G_1^s d\psi_j - \int_{0}^{\psi_j^2} G_2^1 d\psi_j - \int_{\psi_1^2}^{\psi_j^2} G_2^m d\psi_j - \int_{\psi_1^2}^{\infty} G_2^s d\psi_j,$$

(2.5)

where $F_j = F(r_j, z_j, r_j', z_j')$ and $G_1^\alpha(R_\alpha, Z_\alpha, R_\alpha', Z_\alpha')$ are given as follows,

$$F_1 = E_1 - \lambda_1 V_1, \quad F_2 = E_2 - \lambda_2 V_2, \quad F_3 = E_3 - \lambda_3 V_3, \quad F_4 = E_4 - \lambda_4 V_4, \quad F_5 - \lambda_5 V_5,$$

$$G_1^i = \lambda_3 B_{s_1} + A_{s_1}^i, \quad G_1^m = \lambda_1 B_{s_1} + A_{s_1}^m, \quad \lambda_2 = \lambda_3 - \lambda_1, \quad \lambda_5 = \lambda_1 \quad \lambda_4 = \lambda_2,$$

$$G_2^i = \lambda_3 B_{s_2} - A_{s_2}^i, \quad G_2^m = \lambda_1 B_{s_2} - A_{s_2}^m, \quad G_2^e = -A_{s_2}^e, \quad G_1^e = A_{s_1}^e.$$

Calculate first variation of $W$ when the functions $\bar{r}_j(\phi_j)$ and $\bar{z}_j(\phi_j)$ are perturbed by $u_j(\phi_j)$ and $v_j(\phi_j)$, respectively,

$$\delta W = \sum_{j=1}^{5} \int_{\phi_j}^{\phi_j^1} \Delta F_j d\phi_j + (G_1^i - G_1^m) \delta \psi_j^2 + (G_1^m - G_1^e) \delta \psi_j^1 - (G_2^i - G_2^m) \delta \psi_j^4 - (G_2^m - G_2^e) \delta \psi_j^5,$$

(2.7)

$$G_1^i - G_1^m = A_{s_1}^i - A_{s_1}^m + \lambda_2 B_{s_1}, \quad G_1^m - G_1^e = A_{s_1}^m - A_{s_1}^e + \lambda_1 B_{s_1},$$

$$G_2^i - G_2^m = A_{s_2}^i - A_{s_2}^m + \lambda_2 B_{s_2}, \quad G_2^m - G_2^e = A_{s_2}^m - A_{s_2}^e + \lambda_1 B_{s_1},$$

$$\Delta F_j = \frac{\partial F_j}{\partial r_j} u_j + \frac{\partial F_j}{\partial v_j} v_j + \frac{\partial F_j}{\partial z_j} v_j,$$

(2.8)

The functions $u_j(\phi_j)$ and $v_j(\phi_j)$ may be derived using a requirement that the upper free endpoints of the first and second interfaces at Figure[1], are running along $S_1$ and the lower free endpoints of the forth and fifth interfaces - along $S_2$ ,

$$\bar{r}_j(\phi_j^0) = R_\alpha(\psi_j^0), \quad \bar{r}_j(\phi_j^+ \bar{r}_j(\phi_j^0) + u_j(\phi_j^0) = R_\alpha(\psi_j^+ + \delta \psi_j^0), \quad u_j(\phi_j^0) = \frac{dR_\alpha}{d\psi_j^0},$$

$$\bar{z}_j(\phi_j^0) = Z_\alpha(\psi_j^0), \quad \bar{z}_j(\phi_j^0 + v_j(\phi_j^0) = Z_\alpha(\psi_j^+ + \delta \psi_j^0), \quad v_j(\phi_j^0) = \frac{dZ_\alpha}{d\psi_j^0},$$

(2.9)

where $\alpha = 1$ for $j = 1, 2$ and $\alpha = 2$ for $j = 4, 5$. Substitute (2.8) into (2.7) and integrate by parts

$$\delta W = \sum_{j=1}^{5} \left[ \int_{\phi_j}^{\phi_j^1} \left( u_j \frac{\partial F_j}{\partial r_j} + v_j \frac{\partial F_j}{\partial z_j} \right) d\phi_j + \left( u_j \frac{\partial F_j}{\partial r_j} + v_j \frac{\partial F_j}{\partial z_j} \right) \delta \phi_j \right] +$$

(2.11)

$$(G_1^i - G_1^m) \delta \psi_j^2 + (G_1^m - G_1^e) \delta \psi_j^1 - (G_2^i - G_2^m) \delta \psi_j^4 - (G_2^m - G_2^e) \delta \psi_j^5,$$

where $\delta F / \delta y_j = \partial F / \partial y_j - d / dx \left( \partial F / \partial y_j \right)$, denotes the variational derivative for the functional $\int F(x, y_j, y_j') dx$. Since $u_j(\phi)$ and $v_j(\phi)$ are independent functions, vanishing of the integral part of $\delta W$ in (2.9) gives rise to the Young-Laplace equations (YLE) [1],

$$\frac{\delta F_j}{\delta r_j} = 0, \quad \frac{\delta F_j}{\delta z_j} = 0 \quad \rightarrow \quad \frac{z_j'}{r_j} + z_j'' r_j' - z_j' r_j'' = \frac{\lambda_j}{\gamma_j}, \quad 1 \leq j \leq 5.$$
Setting the remaining terms (2.11) to zero gives rise to the four transversality relations,

\[
\frac{dR_1}{d\psi_1} (\psi_1^1) + \frac{dZ_1}{d\psi_1} (\psi_1^1) \frac{\partial F_1}{\partial r_1} (\phi_1^1) + \frac{G_i m_i}{\partial z_1^r} (\psi_1^1) - G_i^c (\psi_1^1) = 0 \\
\frac{dR_1}{d\psi_2} (\psi_2^1) + \frac{dZ_1}{d\psi_2} (\psi_2^1) \frac{\partial F_2}{\partial r_2} (\phi_2^1) + G_i^c (\psi_2^1) - G_i^m (\psi_2^1) = 0, \\
\frac{dR_2}{d\psi_2} (\psi_2^2) + \frac{dZ_2}{d\psi_2} (\psi_2^2) \frac{\partial F_2}{\partial r_2} (\phi_2^2) + G_i^c (\psi_2^2) - G_i^m (\psi_2^2) = 0, \\
\frac{dR_2}{d\psi_2} (\psi_2^3) + \frac{dZ_2}{d\psi_2} (\psi_2^3) \frac{\partial F_3}{\partial r_3} (\phi_3^3) + G_i^c (\psi_2^3) - G_i^m (\psi_2^3) = 0, \\
\frac{dR_2}{d\psi_2} (\psi_2^4) + \frac{dZ_2}{d\psi_2} (\psi_2^4) \frac{\partial F_3}{\partial r_4} (\phi_4^4) + G_i^c (\psi_2^4) - G_i^m (\psi_2^4) = 0,
\]

(2.13)

and one more transversality relation

\[
u_1 (\phi_1^2) \frac{\partial F_1}{\partial r_1} + v_1 (\phi_2^2) \frac{\partial F_2}{\partial r_2} + v_2 (\phi_2^3) \frac{\partial F_3}{\partial r_3} - u_3 (\phi_3^3) \frac{\partial F_3}{\partial z_3} - v_3 (\phi_3^3) \frac{\partial F_3}{\partial z_3} = 0, \\
u_3 (\phi_3^3) \frac{\partial F_3}{\partial r_3} + v_3 (\phi_3^3) \frac{\partial F_3}{\partial r_3} - u_4 (\phi_4^4) \frac{\partial F_4}{\partial r_4} - v_4 (\phi_4^4) \frac{\partial F_4}{\partial r_4} - u_5 (\phi_5^5) \frac{\partial F_5}{\partial r_5} - v_5 (\phi_5^5) \frac{\partial F_5}{\partial r_5} = 0.
\]

(2.14)

In case of one liquid bridge \(LB_m\) and two immiscible liquids \(m\) and \(e\) between two smooth solids \(S_1, S_2\) the first and forth relations in (2.13) coincide with those derived in [1], formula (2.15), while the rest of relations disappear. Regarding condition (2.14), the perturbations \(u_j (\phi_j^k)\) and \(v_j (\phi_j^k)\) are related in such a way that the three disturbed interfaces 1, 2, 3 (and other three 3, 4, 5) always intersect at one point,

\[
u_1 (\phi_1^2) = u_2 (\phi_2^2) = u_3 (\phi_3^2), \quad v_1 (\phi_1^2) = v_2 (\phi_2^2) = v_3 (\phi_3^2), \\
u_3 (\phi_3^3) = u_4 (\phi_4^4) = u_5 (\phi_5^5), \quad v_3 (\phi_3^3) = v_4 (\phi_4^4) = v_5 (\phi_5^5).
\]

(2.15)

Combine (2.14), (2.15) and use independence of \(u_1 (\phi_1^2), v_1 (\phi_1^2), u_3 (\phi_3^2), v_3 (\phi_3^2)\) and obtain four relations,

\[
\frac{\partial F_1}{\partial r_1} (\phi_1^2) + \frac{\partial F_2}{\partial r_2} (\phi_2^2) - \frac{\partial F_3}{\partial r_3} (\phi_3^2) = 0, \quad \frac{\partial F_1}{\partial z_1} (\phi_1^2) + \frac{\partial F_2}{\partial z_2} (\phi_2^2) - \frac{\partial F_3}{\partial z_3} (\phi_3^2) = 0, \\
\frac{\partial F_3}{\partial r_3} (\phi_3^3) - \frac{\partial F_4}{\partial r_4} (\phi_4^4) - \frac{\partial F_5}{\partial r_5} (\phi_5^5) = 0, \quad \frac{\partial F_3}{\partial z_3} (\phi_3^3) - \frac{\partial F_4}{\partial z_4} (\phi_4^4) - \frac{\partial F_5}{\partial z_5} (\phi_5^5) = 0.
\]

(2.16)

Boundary conditions (BC) (2.13, 2.15) have to be supplemented by condition of coincidence of interfaces in \(C_5, C_6\) located on singular curves \(L_1, L_2\), respectively,

\[
r_1 (\phi_1^2) = r_2 (\phi_2^2) = r_3 (\phi_3^2), \quad r_4 (\phi_4^4) = r_5 (\phi_5^4) = r_3 (\phi_3^2), \\
z_1 (\phi_1^2) = z_2 (\phi_2^2) = z_3 (\phi_3^2), \quad z_4 (\phi_4^4) = z_5 (\phi_5^4) = z_3 (\phi_3^2),
\]

(2.17)

while the angular coordinates \(\phi_j^k\) and \(\psi_j^k\) are related by

\[
z_1 (\phi_1^1) = Z_1 (\psi_1^1), \quad r_1 (\phi_1^1) = R_1 (\psi_1^1), \quad z_2 (\phi_2^1) = Z_1 (\psi_2^1), \quad r_2 (\phi_2^1) = R_1 (\psi_2^1), \\
z_4 (\phi_4^1) = Z_2 (\psi_4^2), \quad r_4 (\phi_4^1) = R_2 (\psi_4^2), \quad z_5 (\phi_5^2) = Z_2 (\psi_5^2), \quad r_5 (\phi_5^2) = R_2 (\psi_5^2).
\]

(2.18)
Thus, we have 24 BC for the ten YLE (2.12) of the second order. Let us arrange them as follows,

\[
\begin{align*}
\frac{\delta F_1}{\delta r_1} &= \frac{\delta F_1}{\delta z_1} = 0, \\
\frac{\delta F_2}{\delta r_2} &= \frac{\delta F_2}{\delta z_2} = 0, \\
\frac{\delta F_3}{\delta r_3} &= \frac{\delta F_3}{\delta z_3} = 0, \\
\frac{\delta F_4}{\delta r_4} &= \frac{\delta F_4}{\delta z_4} = 0, \\
\frac{\delta F_5}{\delta r_5} &= \frac{\delta F_5}{\delta z_5} = 0,
\end{align*}
\]

\[
\begin{align*}
&\left\{ \frac{d\psi_1}{d\psi_1} (\phi_1^1) + \frac{d\psi_2}{d\psi_2} (\phi_2^1) + \frac{d\psi_3}{d\psi_3} (\phi_3^1) + G_{1me}^m (\psi_1^1) = 0, \\
&\quad r_1 (\phi_1^1) = r_3 (\phi_3^1), \quad z_1 (\phi_1^1) = z_3 (\phi_3^1), \\
&\quad z_1 (\phi_1^1) = Z_1 (\psi_1^1), \quad r_1 (\phi_1^1) = R_1 (\psi_1^1), \\
&\quad r_2 (\phi_2^1) = r_3 (\phi_3^1), \quad z_2 (\phi_2^1) = z_3 (\phi_3^1), \\
&\quad z_2 (\phi_2^1) = Z_1 (\psi_2^1), \quad r_2 (\phi_2^1) = R_1 (\psi_2^1), \\
&\quad r_3 (\phi_3^1) = r_3 (\phi_3^2), \quad z_3 (\phi_3^1) = z_3 (\phi_3^2), \\
&\quad z_3 (\phi_3^1) = Z_2 (\psi_3^1), \quad r_3 (\phi_3^1) = R_2 (\psi_3^1), \\
&\quad r_4 (\phi_4^1) = r_3 (\phi_3^2), \quad r_4 (\phi_4^1) = r_3 (\phi_3^2), \\
&\quad z_4 (\phi_4^1) = Z_2 (\psi_4^1), \quad r_4 (\phi_4^1) = R_2 (\psi_4^1), \\
&\quad r_5 (\phi_5^1) = r_3 (\phi_3^2), \quad z_5 (\phi_5^1) = z_3 (\phi_3^2), \\
&\quad z_5 (\phi_5^1) = Z_2 (\psi_5^1), \quad r_5 (\phi_5^1) = R_2 (\psi_5^1),
\end{align*}
\]

where

\[
\begin{align*}
G_{1me}^m (\psi_1^1) &= G_{1m}^m (\psi_1^1) - G_{1e}^m (\psi_1^1), \quad G_{1m}^m (\psi_2^2) = G_{1e}^m (\psi_2^2),
\end{align*}
\]

2.1 Curvature law and interface consistency rules

Analysis of YLE (2.12) yields an important conclusion about the curvatures $H_j$ of five interfaces. Consider (2.12) and recall that according to [1], $\lambda_j = 2\gamma_j H_j$. Combining this scaling with (2.6) we arrive at relationships between the curvatures of three interfaces,

\[
\gamma_1 H_1 + \gamma_2 H_2 = \gamma_3 H_3, \quad H_1 = H_5, \quad H_2 = H_4.
\]

Simple verification of (2.24) can be done in special cases. Indeed, if the liquids $i$ and $m$ are indistinguishable, i.e., $\gamma_1 = \gamma_3$ and $\gamma_2 = 0$, then $H_1 = H_3$. On the other hand, if the liquids $m$ and $e$ are indistinguishable, i.e., $\gamma_2 = \gamma_3$ and $\gamma_1 = 0$, then $H_2 = H_3$. In the case $\gamma_1 = \gamma_2 = \gamma_3 \neq 0$, we arrive at relation known in theory of double bubble [4] when three spherical soap surfaces meet at a contact line.
We can formulate strong statements on consistency of five interfaces based on relations (2.24). Recall [1] that there exists only one type, $\text{Nod}^-$, of interfaces with negative $H$, while the other interfaces have positive $H$: nodoid $\text{Nod}^+$, cylinder $\text{Cyl}$, unduloid $\text{Und}$, sphere $\text{Sph}$, or zero curvature, catenoid $\text{Cat}$. Denote by $\text{Mns}^+ = \{\text{Nod}^+, \text{Cyl}, \text{Und}, \text{Sph}\}$ a set of interfaces with $H > 0$ and by $\text{Mns}^\pm = \{\text{Mns}^+, \text{Cat}, \text{Nod}^-\}$ a set of all admissible interfaces. The rules of interfaces consistency with curvatures $H_1, H_2, H_3$ are given in Table, e.g., if the first and second interfaces are $\text{Cat}$ and $\text{Nod}^-$ then the third interface has to be also $\text{Nod}^-$, but if the first and second interfaces are $\text{Und}$ and $\text{Nod}^-$ then the third interface may be any of six interfaces.

| Interfaces | $\text{Mns}^+$ | $\text{Cat}$ | $\text{Nod}^-$ |
|------------|----------------|--------------|---------------|
| $\text{Mns}^+$ | $\text{Mns}^+$ | $\text{Mns}^+$ | $\text{Mns}^\pm$ |
| $\text{Cat}$ | $\text{Mns}^+$ | $\text{Cat}$ | $\text{Nod}^-$ |
| $\text{Nod}^-$ | $\text{Mns}^\pm$ | $\text{Nod}^-$ | $\text{Nod}^-$ |

### 2.2 Standard parameterization and symmetric setup

Consider non-zero curvature interfaces $r_j(\phi_j), z_j(\phi_j), 1 \leq j \leq 5,$ between two solid bodies, $\{R_\alpha(\psi_\alpha), Z_\alpha(\psi_\alpha)\}, \alpha = 1, 2,$ and choose interfaces parameterization in such a way that the lower $\phi^1_j$ and the upper $\phi^2_j$ coordinates of endpoints $C_1, C_2, C_3, C_4$ are located on the solid surfaces $S_1, S_2$ and governed by BC while the other two points $C_5, C_6$ denote the triple points located on singular curves $L_1, L_2$ where three different interfaces meet together.

Following [1] write the parametric expressions for the shape of such interfaces $z_j(\phi_j)$ and $r_j(\phi_j)$,

\[
\begin{align*}
   z_j(\phi_j) &= M(\phi_j, B_j) + d_j, \\
   r_j(\phi_j) &= \frac{1}{2|H_j|} \sqrt{1 + B^2_j + 2B_j \cos \phi_j}, \\
   M(\phi, B) &= (1 + B)E\left(\frac{\phi}{2}, m\right) + (1 - B)F\left(\frac{\phi}{2}, m\right), \quad m^2 = \frac{4B}{(1 + B)^2}, \\
   r_j' &= -\frac{B_j \sin \phi_j}{2|H_j|r_j}, \quad z_j' = \frac{1 + B_j \cos \phi_j}{2|H_j|r_j}, \quad \frac{z_j'}{r_j'} = -\frac{1 + B_j \cos \phi_j}{B_j \sin \phi_j}, \quad r_j'^2 + z_j'^2 = 1. \quad (2.26)
\end{align*}
\]

For all interfaces we have to find 24 unknowns: 15-1=14 interfaces parameters $H_j, B_j, d_j$ (due to (2.24)) and 10 endpoint values $\phi^1_j, \phi^2_j$. These unknowns have to satisfy 24 BCs in (2.19-2.23).

When both surfaces $S_1$ and $S_2$ are similar and the picture in Figure 7b is symmetric w.r.t. midline between $S_1$ and $S_2$, then such setup reduces the problem above to six YLE (2.19-2.21) for the first, second and third interfaces with twelve unknowns:

\[
\phi^1_1, \phi^2_1, \phi^1_3, \phi^2_2, d_1, d_2, B_1, B_2, B_3, H_1, H_2,
\]
and \( \phi_2^2 = \pi, \ 2d_3 = -M(\pi, B_3)/|H_3| \) and \( H_3 = (\gamma_1 H_1 + \gamma_2 H_2)/\gamma_3 \). This number coincides with twelve BCs which comprise ten BCs in (2.20, 2.21) and two first BCs in (2.22). Calculate the partial derivatives \( \partial F_j/\partial r_j, \partial F_j/\partial z_j \) and write these twelve BCs,

\[
\gamma_1 r_1 \left( r_1' R_1' + z_1' Z_1' \right) + \left( \gamma_{s_1}^m - \gamma_{s_1}^e \right) R_1 \sqrt{R_1'^2 + Z_1'^2} + \lambda_1 Z_1 \frac{R_1^2 - r_1^2}{2} = 0, \quad \phi_1 = \phi_1^1, \ \psi_1 = \psi_1^1, \\
\gamma_2 r_2 \left( r_2' R_2' + z_2' Z_2' \right) + \left( \gamma_{s_1}^m - \gamma_{s_1}^e \right) R_1 \sqrt{R_1'^2 + Z_1'^2} + \lambda_2 Z_1 \frac{R_1^2 - r_2^2}{2} = 0, \quad \phi_2 = \phi_2^1, \ \psi_2 = \psi_2^1, \\
\gamma_3 r_3 \left( r_3' R_3' + z_3' Z_3' \right) + \left( \gamma_{s_1}^m - \gamma_{s_1}^e \right) R_1 \sqrt{R_1'^2 + Z_1'^2} + \lambda_3 Z_1 \frac{R_1^2 - r_3^2}{2} = 0, \quad \phi_3 = \phi_3^1, \ \psi_3 = \psi_3^1. 
\]

After simplification we obtain

\[
\gamma_1 \cos \theta_1^1 + \gamma_{s_1}^m - \gamma_{s_1}^e = 0, \quad \gamma_2 \cos \theta_1^1 + \gamma_{s_1}^i - \gamma_{s_1}^m = 0, \\
\gamma_1 r_1' \left( r_1' R_1' + z_1' Z_1' \right) + \left( \gamma_{s_1}^m - \gamma_{s_1}^e \right) R_1 \sqrt{R_1'^2 + Z_1'^2} + \lambda_1 Z_1 \frac{R_1^2 - r_1^2}{2} = 0, \quad \phi_1 = \phi_1^1, \ \psi_1 = \psi_1^1, \\
\gamma_1 r_3' \left( r_3' R_3' + z_3' Z_3' \right) + \left( \gamma_{s_1}^m - \gamma_{s_1}^e \right) R_1 \sqrt{R_1'^2 + Z_1'^2} + \lambda_3 Z_1 \frac{R_1^2 - r_3^2}{2} = 0, \quad \phi_3 = \phi_3^1, \ \psi_3 = \psi_3^1. 
\]

where \( \cos \theta_1^j = \left( r_j' R_1' + z_j' Z_1' \right) / \sqrt{R_1'^2 + Z_1'^2} \) determines the contact angle \( \theta_1^j \) of the \( j \)-th interface and \( S_1 \). Two equalities in (2.30) give the Young relations at the points \( C_1, C_2 \) on \( S_1 \) while two equalities in (2.31) represent the vectorial Young relation at the triple point \( C_5 \) located on a singular curve. Indeed, the latter equalities are the \( r \) and \( z \) projections of the vectorial equality for capillary forces \( f_j \) at \( C_5 \) in outward directions w.r.t. \( C_5 \) and tangential to meridional section of menisci,

\[
f_1(C_5) + f_2(C_5) + f_3(C_5) = 0, \quad f_j(C_5) = \gamma_j \left( r_j'(C_5), z_j'(C_5) \right). 
\]

Finish this section with one more observation related the surface tensions \( \gamma_j \) and contact angles of three interfaces on solid surface. Bearing in mind that \( \gamma_3 \cos \theta_3^3 + \gamma_{s_1}^i - \gamma_{s_1}^e = 0 \), combine the last equality with two others in (2.30) and obtain,

\[
\gamma_1 \cos \theta_1^1 + \gamma_2 \cos \theta_1^2 = \gamma_3 \cos \theta_3^3. 
\]

### 2.3 Solving the BC equations (liquid bridges between two parallel plates)

Making use of standard parametrization (2.25) we present below twelve BCs (2.30, 2.31) for twelve unknowns \( \phi_1^1, \phi_2^1, \phi_3^1, \phi_1^2, \phi_2^2, d_1, d_2, B_1, B_2, B_3, H_1, H_2 \), in a way convenient for numerical calculations,

\[
\gamma_1 B_1 \sin \phi_1^2 / |H_1| + \gamma_2 B_2 \sin \phi_2^2 / |H_2| = \gamma_3 B_3 \sin \phi_3^1 / |H_3|, \quad \gamma_1 B_1 \cos \phi_1^2 / |H_1| + \gamma_2 B_2 \cos \phi_2^2 / |H_2| = \gamma_3 B_3 \cos \phi_3^1 / |H_3|. 
\]
\[
\sqrt{1 + 2B_1 \cos \phi_1^1 + B_1^2} |H_1| = \sqrt{1 + 2B_2 \cos \phi_2^2 + B_2^2} |H_2| = \sqrt{1 + 2B_3 \cos \phi_3^3 + B_3^2} |H_3|,
\]

\[
M(\phi_1^1, B_1) + d_1 = \frac{M(\phi_2^2, B_2)}{2|H_1|} + d_2 = \frac{M(\phi_3^3, B_3)}{2|H_2|} + d_3, \quad d_3 = -\frac{M(\pi, B_3)}{2|H_3|},
\]

\[
M(\phi_j^j, B_j) - M(\phi_j^j, B_j) = Z_1 \left( \psi_1^j \right) - \frac{M(\phi_3^3, B_3)}{2|H_3|} - d_3,
\]

\[
|H_j| = \sqrt{1 + 2B_j \cos \phi_j^j + B_j^2} \frac{1}{2R_1 \left( \psi_1^j \right)}, \quad B_j = \left[ \cos \phi_j^j + \sin \phi_j^j \tan \theta_1^j \right]^{-1}, \quad j = 1, 2,
\]

where \( H_3 = H_1 \gamma_1 / \gamma_3 + H_2 \gamma_2 / \gamma_3 \).

The numerical optimization of the solution was done by a standard gradient descent algorithm. The cost function for the optimization problem was chosen to be the weighted sum of absolute values of the differences between the right and the left hand sides of the six first equations in (2.34).

Figure 3: a) Five Imm interfaces for three immiscible media \((i – \text{water}, m – \text{octane}, e – \text{air})\) trapped between two similar solid plates \((Z_1 – Z_2 = 2)\) with free BC. b) Enlarged view of the vicinity of the triple point \(C_5\) on the singular curve where three phases coexist. The angles between the adjacent interfaces read: \(\Phi_{12} = 36.76^\circ, \Phi_{23} = 159.43^\circ, \Phi_{31} = 163.81^\circ\).

In Figure we present the shapes of five interfaces of different curvatures for three immiscible media: \(i – \text{water}, m – \text{octane} (C_{8}H_{18}, \text{a component of petrol}), e – \text{air},\) trapped between two similar glass plates with free BCs and capillary parameters taken from [5]. The interfaces have the following parameters,
The volumes of liquids confined inside interfaces read $V_m = 4.009$, $V_i = 2.674$.

3 Variational problem for three interfaces

Consider a functional $E[r, z]$ of surface energy

$$E[r, z] = \sum_{j=1}^{3} \int_{\phi_j^1}^{\phi_j^3} E_j d\phi_j + \int_0^{\psi_1^3} A_{s1}^i d\psi_1 + \int_0^{\psi_1^3} A_{s1}^m d\psi_1 + \int_0^{\psi_1^3} A_{s1}^e d\psi_1 + \int_0^{\psi_2^3} A_{s2}^i d\psi_2 + \int_0^{\psi_2^3} A_{s2}^e d\psi_2,$$

and two functionals $V_i[r, z]$ and $V_m[r, z]$ of volumes of the $i$ and $m$ liquids

$$V_i[r, z] = \int_{\phi_i^1}^{\phi_i^3} V_1 d\phi_1 - \int_{\phi_i^2}^{\phi_i^3} V_2 d\phi_2 - \int_{\psi_i^2}^{\psi_i^3} B_{s1} d\psi_1,$$

$$V_m[r, z] = \int_{\phi_m^1}^{\phi_m^3} V_2 d\phi_2 + \int_{\phi_m^2}^{\phi_m^3} V_3 d\phi_3 - \int_{\psi_m^2}^{\psi_m^3} B_{s1} d\psi_1 + \int_{\psi_m^3}^{\psi_m^3} B_{s2} d\psi_2,$$

where all integrands $E_j$, $A_{s1}^{i,m,e}$, $V_j$ and $B_{s1}$ are defined in (2.2, 2.4).

Consider the composed functional $W[r, z] = E[r, z] - \lambda_1 V_{m0}[r, z] - \lambda_3 V_i[r, z]$ and represent it in the following form,

$$W[r, z] = \sum_{j=1}^{3} \int_{\phi_j^1}^{\phi_j^3} F_j d\phi_1 + \int_0^{\psi_1^3} G_1^i d\psi_1 + \int_0^{\psi_1^3} G_1^m d\psi_1 + \int_0^{\psi_1^3} G_1^e d\psi_1 - \int_0^{\psi_2^3} G_2^i d\psi_2 - \int_0^{\psi_2^3} G_2^e d\psi_2,$$

where the integrands are given in (2.6).

Applying a similar technique as in section 2, we arrive at the first variation,

$$\delta W = \sum_{j=1}^{5} \left[ \int_{\phi_j^1}^{\phi_j^3} \left( u_j \frac{\delta F_j}{\delta r_j} + v_j \frac{\delta F_j}{\delta z_j} \right) d\phi_j + \left( \frac{u_j}{\phi_j^0} \frac{\delta F_j}{\delta r_j} + \frac{v_j}{\phi_j^0} \frac{\delta F_j}{\delta z_j} \right) \phi_j^1 \right] +$$

$$(G_1^i - G_1^m) \delta \psi_1^2 + (G_1^m - G_1^e) \delta \psi_1^1 - (G_2^i - G_2^e) \delta \psi_2^3.$$

This case does not allow a symmetric version and therefore is less reducible compared to the 5 interface case w.r.t. the number of unknowns and BC equations. This number equal fifteen: nine interface...
parameters $H_j, B_j, d_j$, and six endpoint values $\phi_j^1, \phi_j^2$. They satisfy fifteen BC equations

$$
\begin{align*}
\delta F_1 &= \delta F_1 \bigg|_{\delta z_1 = 0}, \\
\delta F_2 &= \delta F_2 \bigg|_{\delta z_2 = 0}, \\
\delta F_3 &= \delta F_3 \bigg|_{\delta z_3 = 0},
\end{align*}
$$

that gives

$$
\begin{align*}
\gamma_1 \cos \theta_1^1 + \gamma_1^m - \gamma_1^e &= 0, \\
\gamma_2 \cos \theta_1^2 + \gamma_2^i - \gamma_2^m &= 0, \\
\gamma_3 \cos \theta_1^3 + \gamma_3^i - \gamma_3^e &= 0, \\
\gamma_1 \rho_1^1 (\phi_1^1) + \gamma_2 \rho_1^2 (\phi_1^2) - \gamma_3 \rho_1^3 (\phi_1^3) &= 0, \\
\rho_1 (\phi_1^1) &= \rho_2 (\phi_2^1), \\
\rho_1 (\phi_1^2) &= \rho_2 (\phi_2^2), \\
\rho_1 (\phi_1^3) &= \rho_2 (\phi_2^3), \\
\gamma_1 \rho_3^1 (\phi_3^1) + \gamma_2 \rho_3^2 (\phi_3^2) - \gamma_3 \rho_3^3 (\phi_3^3) &= 0, \\
\rho_3 (\phi_3^1) &= \rho_3 (\phi_3^2), \\
\rho_3 (\phi_3^3) &= \rho_3 (\phi_3^3), \\
\end{align*}
$$

Three equalities in (3.6) cannot be reduced to a single equality similar to (2.25) if the upper and lower solid bodies have different capillary properties, namely, $\gamma_{i_2} - \gamma_{i_1} \neq \gamma_{i_2} - \gamma_{i_1}$, i.e.,

$$
\gamma_1 \cos \theta_1^1 + \gamma_2 \cos \theta_1^2 \neq \gamma_3 \cos \theta_1^3.
$$

### 3.1 Solving the BC equations (liquid bridges between two parallel plates)

Using a standard parametrization (2.25) and relation (2.24) for $H_3$ we present below fourteen BCs (3.6.3.7) for fourteen unknowns: $\phi_j^1, \phi_j^2, \phi_j^3, \phi_j^4, \phi_j^5, \phi_j^6, d_1, d_2, d_3, B_1, B_2, B_3, H_1, H_2$ in a way convenient for numerical calculations,

$$
\begin{align*}
B_j &= \left[ \cos \phi_j^1 + \sin \phi_j^1 \tan \theta_j^i \right]^{-1}, \\
B_3 &= \left[ \cos \phi_3^2 + \sin \phi_3^2 \tan \theta_3^2 \right]^{-1}, \\
|H_1| &= \sqrt{1 + 2B_1 \cos \phi_1^1 + B_1^2}, \\
|H_2| &= \sqrt{1 + 2B_2 \cos \phi_2^1 + B_2^2}, \\
M(\phi_j^1, B_j) &= d_j = Z_1 \left( \psi_j^1 \right), \\
M(\phi_3^2, B_3) &= d_3 = Z_2 \left( \psi_2^3 \right), \\
\end{align*}
$$

(3.8)
\[
\frac{\gamma_1 B_1 \sin \phi_1^2 + \gamma_2 B_2 \sin \phi_2^2}{|H_1|} + \sqrt{1 + 2B_1 \cos \phi_1^2 + B_1^2} = \frac{\gamma_3 B_3 \sin \phi_3^2}{|H_3|},
\]
\[
\frac{\gamma_1 B_1 \cos \phi_1^2}{|H_1|} + \sqrt{1 + 2B_1 \cos \phi_1^2 + B_1^2} = \frac{\gamma_3 B_3 \cos \phi_3^2}{|H_3|},
\]
\[
\frac{M(\phi_1^1, B_1)}{2|H_1|} + d_1 = \frac{M(\phi_2^2, B_2)}{2|H_2|} + d_2 = \frac{M(\phi_3^3, B_3)}{2|H_3|} + d_3,
\]

where \( H_3 = (H_1 \gamma_1 + H_2 \gamma_2) / \gamma_3 \). In Figure 4 we present the shapes of three interfaces of different

![Figure 4](image_url)

Figure 4: a) Two Und and one Nod interfaces for three immiscible media (\( i \) – water, \( m \) – hexane, \( e \) – air) trapped between two (not similar) solid plates (\( Z_1 - Z_2 = 1 \)) with free BC. b) Enlarged view of the vicinity of the triple point \( C_5 \) on singular curve \( L \) where three phases coexist. The angles between the adjacent interfaces reach the following values: \( \Phi_{12} = 46.04^\circ \), \( \Phi_{23} = 173.28^\circ \), \( \Phi_{31} = 140.69^\circ \).

Curvatures for three immiscible media: \( i \) – water, \( m \) – hexane (\( C_6H_{14} \), a component of petrol), \( e \) – air, trapped between two plates composed of different materials (glass and glass coated with polymer film) with free BCs and capillary parameters taken from [5]. The interfaces have the following parameters,

| Interfaces | \( \gamma_j \) mN/m | \( \theta_j \) | \( B_j \) | \( H_j \) | \( d_j \) | \( \phi_j^2 \) | \( \phi_j^3 \) |
|------------|----------------------|-----------|---------|---------|---------|------------|------------|
| 1(\( e - m \)) | 18.4 | 19^\circ | 1.091 | 0.229 | -2.521 | 199.51^\circ | 228.89^\circ |
| 2(\( m - i \)) | 51.1 | 40^\circ | 0.775 | 0.379 | -3.257 | 217.19^\circ | 228.50^\circ |
| 3(\( e - i \)) | 72.8 | 49^\circ | 0.841 | 0.324 | -3.574 | 169.79^\circ | 211.11^\circ |

The volumes of liquids confined inside interfaces read \( V_m = 0.4377, V_i = 0.8940 \).
4 Conclusion

We formulate a variational problem for coexistence of axisymmetric interfaces of three immiscible liquids: two of them, \( i \) and \( m \), immersed in a third liquid (or gas) \( e \) and trapped between two smooth solid bodies with axisymmetric surfaces \( S_1, S_2 \) and free contact lines. Assuming the volume constraints of two liquids \( i \) and \( m \), we find the governing (Young-Laplace) equations \( (2.12) \) supplemented by boundary conditions and Young relation \( (2.13) \) on \( S_1, S_2 \) and transversality relations \( (2.16) \) on singular curve where all liquids meet together.

We consider two different cases when the problem allows the coexistence of five (section 2) or three (section 3) interfaces. In the first case the problem is reduced solving 16 boundary conditions, 4 Young relations and 4 transversality relations \( (2.19, 2.23) \), i.e., 24 equations for 24 variables. In the second case this number is reduced substantially, namely, 15 equations with 15 variables \( (3.5) \) including 10 boundary conditions, 3 Young relations and 2 transversality relations.

We derive the relationship \( (2.24) \) combining the constant mean curvatures of three different interfaces, \( e - m, m - i, e - i \), and give consistency rules for interface coexistence (section 2.1).

Another result is the vectorial Young relation \( (2.32) \) at the triple point which is located on a singular curve. It has a clear physical interpretation as the balance equation of capillary forces. More importantly, it gives a new insight on an old assertion about the usual Young relations \( (2.30, 3.6) \) at a solid/liquid/gas interface refered by R. Finn \( [2] \) to T. Young: the contact angle at a solid/liquid/gas interface is a physical constant depending only on the materials, and in no other way on the particular conditions of problem, and a well known contradiction with uncompensated normal force reaction of solid (see \( [2] \) and references therein). Indeed, being applied to the contact line of three continuous media, liquid-gas-solid, vectorial relation \( (2.32) \) assumes a singular deformation of solid surface if its elastic modules take finite values.

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