Portfolio Optimization via Quantum Zeno Dynamics on a Quantum Processor

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Portfolio optimization is an important problem in mathematical finance, and a promising target for quantum optimization algorithms. The use cases solved daily in financial institutions are subject to many constraints that arise from business objectives and regulatory requirements, which make these problems challenging to solve on quantum computers. We introduce a technique that uses quantum Zeno dynamics to solve optimization problems with multiple arbitrary constraints, including inequalities. We show that the dynamics of the quantum optimization can be efficiently restricted to the in-constraint subspace via repeated projective measurements, requiring only a small number of auxiliary qubits and no post-selection. Our technique has broad applicability, which we demonstrate by incorporating it into the quantum approximate optimization algorithm (QAOA) and variational quantum circuits for optimization. We analytically show that achieving a constant minimum success probability in QAOA requires a number of measurements that is independent of the problem size for a specific choice of mixer operator. We evaluate our method numerically on the problem of portfolio optimization with multiple realistic constraints, and observe better solution quality and higher in-constraint probability than the state-of-the-art technique of enforcing constraints by introducing a penalty into the objective. We demonstrate the proposed method on the Quantumium H1-2 trapped-ion quantum processor, observing performance improvements from circuits with two-qubit gate depths of up to 148.

I. INTRODUCTION

The daily operation of a large financial institution requires solving many classically-hard optimization problems [1, 2]. Among such problems, one of the most important is portfolio optimization. Modern portfolio theory [3] considers the task of finding a portfolio with a desired trade-off between risk and expected return. This task is typically formulated as an optimization problem, which is hard to solve classically in many settings, such as when the variables are required to only take on a discrete set of values. When designing an algorithm for portfolio optimization, a central consideration is the ability to incorporate a general class of constraints. Such constraints can come from regulatory or business considerations, with examples ranging from portfolio-level constraints (including budget and total number of assets) to asset-level constraints (such as minimum holding size).

A commonly considered class of quantum optimization algorithms uses a parameterized quantum evolution to drive the quantum system towards a state encoding the solution of the optimization problem. This class of algorithms includes the quantum approximate optimization algorithm (QAOA) [4–6] and variational algorithms for optimization [6, 7].

One of the main challenges in applying these quantum algorithms to commercially-relevant optimization problems is the need to enforce the constraints. Concretely, the goal is to prepare a quantum state such that upon measuring it, a high-quality solution that satisfies the constraints is obtained with high probability. Two commonly considered approaches are to encode the constraint into the objective using a penalty term and to directly restrict the parameterized quantum evolution to the in-constraint subspace. In the first approach, a penalty term is added to the objective so that optimizing the objective requires satisfying the constraint. While such approaches are flexible enough to satisfy most constraints, the quality of the result is sensitive to the choice of the penalty strength [8]. As tuning the penalty strength is difficult in general, this approach often leads to sub-optimal performance in practice [9]. This observation motivates the second approach, i.e., restricting the quantum evolution to the in-constraint subspace.

A number of techniques have been proposed to ensure that the parameterized quantum evolution respects the constraints of the problem. Hadfield et al. [10, 11] proposed the quantum alternating operator ansatz algorithm, which applies pairs of alternating operators to an in-constraint initial state. The first alternating operator (phase operator) is diagonal in the computational basis and encodes the objective, and the second operator (mixing operator or mixer) is non-diagonal and restricts the transitions of probability amplitudes to the computational basis states corresponding to the in-constraint solutions. No general rule is known for the construction of a mixing operator preserving arbitrary constraints, though explicit constructions are available for some combinatorial optimization problems [10, 12]. In general, constraint-preserving mixers are difficult to implement, even when constructions are available [13, 14]. The cost of implementing the algorithm on hardware can be reduced for a restricted class of problems by combining the phase and mixing operators [15]. If a uniform superposition of in-constraint states can be prepared efficiently, a Grover operator can be used as the mixer [16–18]. Finally, for problems with an indexable set of feasible

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states (such as those with Hamming-weight constraints), a continuous-time quantum walk in the solution space can be used as a mixer [19–21]. However, none of these techniques are sufficiently flexible to handle the general case of multiple arbitrary constraints directly. Moreover, all of them consider QAOA-like alternating operator circuits, and are not easy to generalize to other variational algorithms.

In this work, we introduce an approach for enforcing multiple arbitrary constraints in quantum optimization. We restrict the quantum evolution to the in-constraint subspace by repeated projective measurements. In each measurement, the value of the constraint is computed onto an auxiliary register, which is then measured. Our technique uses quantum Zeno dynamics, wherein the evolution of the system is restricted to the subspace defined by the repeated projective measurements and transitions outside of this subspace are suppressed. Our approach is applicable to any problem in NPO (the NP optimization complexity class), as the only restriction we impose on the constraints is the existence of an efficient oracle for testing it. We provide explicit constructions for arbitrary combinatorial constraints.

We demonstrate the effectiveness of the proposed technique by applying it to enforcing the constraints in QAOA and the layer variational quantum eigensolver (L-VQE) [22], which is a variational quantum algorithm for optimization. We show analytically that our technique is guaranteed to obtain the optimal in-constraint solution when applied to the digital simulation of the quantum adiabatic algorithm, or equivalently to QAOA in the constrained subspace with sufficiently large depth. We derive an analytical form of the scaling of the number of measurements required to maintain a constant minimum success probability for any given QAOA mixing Hamiltonian. Furthermore, we provide numerical evidence that our technique, applied to QAOA, provides significant performance improvements over the state-of-the-art method of enforcing the constraint by introducing a penalty term. Finally, we implement QAOA with Zeno dynamics on the Quantumum H1-2 trapped-ion quantum processor and observe performance improvements from increased number of measurements up to a two-qubit circuit depth of 148. We make all the data presented in this paper available online at https://doi.org/10.5281/zenodo.7125969.

The rest of the paper is organized as follows. We introduce relevant background on quantum optimization in Section IIA and on quantum Zeno dynamics in Section IIB. We define our approach and provide the scaling of the resource requirements in Section III. We present the numerical experiments with QAOA and L-VQE in Section IV and the experiments on a trapped-ion quantum processor with QAOA in Section V. We conclude with a brief discussion of the results in Section VI.

II. PRELIMINARIES

We begin by briefly introducing the relevant concepts and setting the notation. We undertake the task of minimizing an objective function \( f \) defined on the Boolean cube over the set of feasible solutions \( F \subseteq \mathbb{B}^n \):

\[
\min_{x \in F} f(x).
\]

We consider sets \( F \) of the form \( F = \{ x \in \mathbb{B}^n | \bar{g}_j(x) = 0 \ \forall j \} \), where \( \bar{g}_j(x) \) is an oracle that returns 0 if \( x \) satisfies the \( j \)-th constraint and a value strictly greater-than 0 otherwise. This general definition includes most commonly considered problems such as those with equality and inequality constraints.

This constrained optimization problem can be solved by relaxing the constraints and introducing penalty terms as follows:

\[
\min_{x \in \mathbb{B}^n} f(x) = \min_{x \in \mathbb{B}^n} f(x) + \sum_j \lambda_j \bar{g}_j(x),
\]

where \( \lambda_j \in \mathbb{R}^+ \) are the penalty factors.

Specifically, for an equality constraint \( g(x) = 0 \), the penalty function may be written as

\[
\bar{g}(x) = [g(x)]^2.
\]

On the other hand, an inequality constraint \( g(x) \geq 0 \) can be converted into an equivalent equality constraint \( g(x) - \bar{s} = 0 \) by introducing a slack variable \( \bar{s} \in [0, \max_g] \), where \( g_{\max} = \max_{x \in F} g(x) \). If we assume \( g(x) \) can be discretized with a spacing of \( \Delta_g \), then \( \bar{s} \) can be implemented using \( n_{\text{slack}} = \lceil \log_2(g_{\max}/\Delta_g) \rceil \) binary variables \( s = (s_1, \ldots, s_{n_{\text{slack}}}) \), and the resultant equality constraint is \( g(x) - \Delta_g \sum_j 2^{j-1}s_j = 0 \). Therefore the penalty function for an inequality constraint can be written as

\[
\bar{g}(x; s) = \left( g(x) - \Delta_g \sum_{j=1}^{n_{\text{slack}}} 2^{j-1}s_j \right)^2.
\]

The magnitudes of the penalty factors \( \lambda_j \) control how much the constraint violations are penalized. Intuitively, a higher value of \( \lambda_j \) should lead to a higher in-constraint probability. However, in practice, the relationship between the penalty factor, the in-constraint probability and the solution quality may be non-monotonic. This makes choosing \( \lambda_j \) harder. We discuss the difficulty of tuning the penalty factors in Section IV C.

A. Quantum algorithms for approximate optimization

In this work, we focus on the class of quantum optimization algorithms that use a parameterized quantum evolution to prepare a state, such that the corresponding measurement outcomes contain a high-quality, valid
solution to the original optimization problem with high probability. This parameterized state is prepared by applying a parameterized evolution \( U(\theta) \) to some initial state \(|s\rangle\):

\[
|\psi(\theta)\rangle = U(\theta) |s\rangle = e^{-i\theta_1 H_1} \ldots e^{-i\theta_n H_n} |s\rangle , \tag{3}
\]

where \( H_j \) is some Hamiltonian, e.g., a tensor product of single-qubit Pauli operators.

Let \( C = \sum_{x \in \mathbb{B}^n} f(x) |x\rangle \langle x| \) be the operator encoding the objective function \( f \) on qubits and \( C_{\text{penalty}} = \sum_{x \in \mathbb{B}^n} f_{\text{penalty}}(x) |x\rangle \langle x| \) be the operator encoding the relaxed objective function \((\ref{penalty})\). The figures of merit used to evaluate the quality of a parameter \( \theta^* \) obtained by algorithms that employ parameterized circuit \((\ref{parameterized})\) are approximation ratios, defined as follows:

\[
r = \frac{\langle \psi(\theta^*) | C_{\text{penalty}} | \psi(\theta^*) \rangle - f_{\text{max}}}{f_{\text{min}} - f_{\text{max}}} \tag{4}
\]

and

\[
r_{\text{penalty}} = \frac{\langle \psi(\theta^*) | C_{\text{penalty}} | \psi(\theta^*) \rangle - f_{\text{max}}}{f_{\text{min}} - f_{\text{max}}} , \tag{5}
\]

where \( C_x = \sum_{x \in \mathbb{F}} f(x) |x\rangle \langle x| \), \( f_{\text{min}} = \min_{x \in \mathbb{F}} f(x) \), \( f_{\text{max}} = \max_{x \in \mathbb{F}} f(x) \), and \( f_{\text{penalty}} = \min_{x \in \mathbb{B}} f_{\text{penalty}}(x) \), and \( f_{\text{penalty}} = \max_{x \in \mathbb{B}} f_{\text{penalty}}(x) \).

This class of algorithms includes QAOA \([4, 5, 23]\) and its generalization, the quantum alternating operator ansatz algorithm \([11]\). In both algorithms, the parameterized quantum evolution is performed by applying pairs of alternating operators:

\[
|\psi(\beta, \gamma)\rangle = \prod_j \left[ U_B(\beta_j) U_C(\gamma_j) \right] |s\rangle , \tag{6}
\]

where \( U_C(\gamma_j) = e^{-i\gamma_j C} \) is the phase operator, and \( U_B(\beta_j) \) is the mixing operator. In the special case of QAOA, the initial state \(|s\rangle\) is the uniform superposition over all computational basis states and the mixing operator \( U_B \) is set to be \( U_B(\beta_j) = e^{-i\beta_j B} \), where \( B = \sum_k x_k \) is a sum of single-qubit Pauli-X operators. In quantum alternating operator ansatz, \( U_B \) and \(|s\rangle\) are allowed to be arbitrary, and are typically set such that the resulting state \(|\psi(\beta, \gamma)\rangle\) preserves the constraints, in the sense that every measurement outcome \( x \) belongs to \( \mathcal{F} \). In this paper, we consider QAOA with an arbitrary mixing Hamiltonian \( B \), defined in Ref. \([11]\) as “Hamiltonian-based QAOA.” In the remainder of the paper, the acronym QAOA will be used to denote this version of the algorithm.

In addition to QAOA, we consider L-VQE \([22]\), which is a version of VQE with the hardware-efficient layered parameterized circuit tailored toward optimization problems. L-VQE uses the parameterized circuit of the form

\[
\prod_{j=1}^{p} \left[ U_{\text{NN}}(\theta_j) \right] V(\theta_0) |0\rangle , \tag{7}
\]

where \( U_{\text{NN}} \) consists of nearest-neighbor CNOT’s and single-qubit Ry’s, and \( V \) is a layer of single-qubit Ry’s. The reader is referred to Ref. \([22]\) for the precise definition of the circuit. While the circuit includes non-parameterized CNOT’s, it is easy to write it equivalently in the form of \((\ref{parameterized})\) by pushing Ry through the control of the CNOT and noting that Ry(\(\theta\)) = \( e^{-i\frac{\theta}{2} y} \) and CNOT\(_{1,2} \) Ry\(_{2}(\theta)\) CNOT\(_{1,2} = e^{-i\frac{\theta}{2} z} y_z \). Here, \( y_j \) and \( z_j \) denote a single-qubit Pauli-Y and Pauli-Z, respectively, acting on the \( j \)-th qubit.

### B. Quantum Zeno dynamics

The quantum Zeno effect (QZE) \([24, 25]\) is named after Zeno’s paradox \([26]\), which regards the continuous observation of a moving arrow. Zeno’s paradox states that an arrow cannot move if no time has elapsed since the point it was last observed. If the time difference between observations is \( \Delta t \), continuous observation occurs in the limit of \( \Delta t \to 0 \). Under continuous observation, no time elapses between observations, and during each observation the arrow is not moving; thus, no overall movement is possible. The analogue in quantum mechanics is a consequence of the Schrödinger equation. We first introduce a simpler one-dimensional version, in which the quantum state is “frozen” by repeated measurements, and then present a more general case in which the dynamics of the system are restricted to a particular “Zeno subspace.”

Suppose a time-dependent quantum state is evolved in a finite-dimensional Hilbert space \( \mathcal{H} \) from some initial state \(|\psi(0)\rangle\) under the action of some Hamiltonian \( H \) for time \( t \). Define a projective measurement \( \mathcal{P} \) given by a pair of complement projections \( P = |\psi_0\rangle\langle\psi_0| \) and \( Q = I - P \), which acts on a density operator \( \rho \) as

\[
\mathcal{P} \rho = P \rho P + Q \rho Q .
\]

If we carry out \( N \) repeated projective measurements \( \mathcal{P} \) at a time interval of \( t/N \), then the probability that the system remains in the initial state is

\[
p(t) = \left| \langle \psi_0 | e^{-iHt/N} |\psi_0\rangle \right|^2 = \left[ \langle \psi_0 | e^{-iHt/N} |\psi_0\rangle \right]^2 .
\]

\[
= \left[ 1 - (t/N \tau_2^z)^2 \right]^N + O(N^{-2}) .
\]

where \( \tau_2^{-2} = \langle \psi_0 | H^2 |\psi_0\rangle - \langle \psi_0 | H |\psi_0\rangle^2 \) is the “Zeno time” quantifying how often the measurements need to be taken. As the frequency at which the measurements are performed increases without bound, the probability of remaining in the initial state approaches one.
projections with ranks all greater than one. Specifically,

$$\mathcal{P} \rho = \sum_{j=1}^{k} P_j \rho P_j,$$

(8)

where \( \sum_{j=1}^{k} P_j = I \), and \( P_j \) is a projection onto some subspace \( \mathcal{H}_j = P_j \mathcal{H} \) of dimensionality \( \text{Tr}(P_j) \geq 1 \). Informally, QZD states that if the evolution starts in \( \mathcal{H}_j \) and the measurement \( \mathcal{P} \) is performed sufficiently often, then the system will remain in \( \mathcal{H}_j \) with high probability.

Consider an initial state \( \rho_0 \) after \( N \) projective measurements by \( \mathcal{P} \), the state of the system is given by

$$\rho(t) = \mathcal{U}(t) \rho_0 \mathcal{U}(t)^\dagger,$$

(9)

where \( \mathcal{U}(t) = (\mathcal{P} e^{-iHt/N})^N \) and \( p(t) = \text{Tr}(P_j \rho(t)) \) is the probability of the system remaining in \( \mathcal{H}_j \) after evolving for time \( t \). Note that

$$\mathcal{U}(t) = (\mathcal{P} e^{-iHt/N})^N = (\mathcal{P}(1 - ih\Omega + O(N^{-2}))^N = (1 - i\mathcal{P}Ht/N + O(N^{-2}))^N = (1 - i\mathcal{P}Ht/N)^N + O(N^{-1})$$

$$\xrightarrow{N \to \infty} e^{-i\mathcal{P}Ht}\mathcal{P},$$

(11)

and the dynamics of the system are governed by the “Zeno Hamiltonian” \( H_Z = \mathcal{P} H \). Moreover, as \( N \to \infty \) transitions between different subspaces \( \{ \mathcal{H}_1, \ldots, \mathcal{H}_k \} \) of \( \mathcal{H} \) are suppressed. This implies if \( \rho_0 = P_j \rho_0 P_j \) for some \( j \in [k] := \{1, \ldots, k\} \), then in the limit \( N \to \infty \), it follows that \( \rho(t) \to 1 \), and thus the state will remain in \( \mathcal{H}_j \) throughout the evolution. For a more detailed discussion the reader is referred to Refs. [29, 30].

QZE has many applications in algorithms and error mitigation. Childs et al. [31] propose a version of Grover’s search based on QZE that utilizes frequent measurements instead of slow adiabatic evolution. This alternative approach to slow evolution was also observed in [32]. Somma et al. [33] develop a quantum-enhanced version of the simulated annealing algorithm. Their approach makes use of QZE to ensure that the evolution remains in the instantaneous quantum Gibbs state for varying temperature. Boixo et al. [34] show that for Grover’s algorithm and simulated annealing based on QZE, one could use frequent randomized evolutions instead of measurements (“the randomization method”). The randomization method has also been used to implement algorithms for quantum linear systems [35, 36]. Finally, dynamical decoupling, also called “bang-bang” decoupling [37], is a popular error-mitigation technique that uses QZE to suppress decoherence [30] [39] [40].

III. QUANTUM ZENO DYNAMICS FOR CONSTRAINED OPTIMIZATION

We now introduce our approach to enforcing constraints in quantum optimization by repeated non-selective projective measurements. Our method is general, though here we focus on algorithms utilizing parameterized states of the form \([3]. Let P_F \) denote the projector onto the subspace spanned by computational basis states corresponding to feasible solutions in \( F \). We discuss the construction of this operator in Section III C. The measurement \( \mathcal{P} \) is a super-operator as defined by Equation [8]. Without loss of generality, we can assume \( P_1 = P_F \), and define \( P_0 := 1 - P_F = \sum_{j=2}^{k} P_j \).

Assume that the initial state \( |s\rangle \) respects the constraints, that is \( P_F |s\rangle = |s\rangle \). We apply a parameterized unitary \( U(\theta) \) to the initial state following Equation [3]. To enforce the constraints, we insert \( N \) measurements into the parameterized evolution as follows

$$\rho(\theta) = U_Z(\theta) |s\rangle \langle s| U_Z(\theta)^\dagger,$$

(12)

where \( U_Z(\theta) = [PU(\theta/N)]^N \). Following (10)-(11) we get

$$U_Z(\theta) = [\mathcal{P} e^{-i(\theta_1/N)H_1 + \cdots e^{-i(\theta_m/N)H_m}}]^N \approx \left[ P \prod_{j=1}^{m} (1 - i(\theta_j/N)H_j + O(N^{-2})) \right]^N$$

$$\xrightarrow{N \to \infty} e^{-i\sum_{j=1}^{m} \mathcal{P} H_j \theta_j} \mathcal{P} = e^{-i\mathcal{P}H \theta \mathcal{P}}. \quad (13)$$

Thereby the dynamics are described by the Zeno Hamiltonian \( H_Z = \mathcal{P} H \), where \( \mathcal{P} \) acts element-wise on the vector \( H = (H_1, \ldots, H_m)^\dagger \), and the probability of the evolution remaining in the in-constraint subspace tends to one as \( N \) grows without bound. The number of measurements required to maintain a constant success probability depends on \( \{H_j\}_{j=1}^{m} \), and can be hard to estimate in general. However, in the specific case of QAOA, the scaling rule for the number of measurements can be derived explicitly.

A. Constrained QAOA via Zeno dynamics

We now discuss the application of QZE to QAOA. In a QAOA circuit, the phase operator \( U_C(\gamma) \) is diagonal in the computational basis and cannot violate constraints. Therefore the measurements only need to be added to the mixing operator. The full circuit then becomes

$$U_{Z,\text{QAOA}}(\beta, \gamma) = \prod_{j} \left[ U_B(\beta_j, N_j) U_C(\gamma_j) \right], \quad (15)$$
where
\[ U_B(\beta_j, N_j) = \left[ \mathcal{P} e^{-\frac{i \beta_j^2 n^2}{\ln (23-\delta)} B} \right]^{N_j}. \tag{16} \]

As the mixing Hamiltonian \( B \) is known, we can derive explicitly the number of measurements required to maintain a constant success probability. We observe that for any mixer this number of measurements grows linearly with the number of QAOA layers, and for commonly considered mixers, the number of measurements grows no more than quadratically with the number of qubits.

**Corollary 1.** Let \( U_{Z, \text{QAOA}}(\beta, \gamma) \) denote the QAOA circuit on \( n \) qubits with \( N \) measurements added to each mixing operator as defined in \((15)\). Let the initial state \( \rho_0 = |s\rangle\langle s| \) be in-constraint. Then a sufficient number of measurements to maintain a constant probability of at least \( \delta \geq 0.81 \) of obtaining an in-constraint measurement outcome is given by

- if \( B = \sum_{k=1}^{n} x_k \), then \( N_j = \left[ \frac{p \beta_j^2 n^2}{\ln (23-\delta)} \right] \)
- if \( B = \{|\rangle\langle +|\rangle\rangle \), then \( N_j = \left[ \frac{p \beta_j^2 }{\ln (23-\delta)} \right] \)

**Proof.** The proof follows directly from Theorem 1 by noting that for \( B = \sum_{k=1}^{n} x_k \) the range of eigenvalues is integers from \(-n\) to \(n\) and for \( B = \{|\rangle\langle +|\rangle\rangle \) the only eigenvalues are one and zero. For QAOA with \( p \) layers, the number of measurements increases by a factor of \( p \). \( \square \)

Note that the scaling rule of Corollary 1 implies that the number of measurements will change with \( \beta_j \) and thus each mixer layer.

**Remark.** For combinatorial optimization problems, constraint-preserving measurements that correspond to different constraints always commute. Thus \( \mathcal{P}_F \) can be implemented as a composition of measurements corresponding to different constraints.

1. **QAOA with Zeno dynamics in the adiabatic limit**

If the initial state \( |s\rangle \) is the ground state of the mixer Hamiltonian \( B \), QAOA is known to be able to prepare the ground state of the cost Hamiltonian \( C \) and thereby solve the problem exactly in the limit of an infinite number of QAOA layers by approximating adiabatic evolution \[5\].

We now show that this limiting behavior is preserved for constrained QAOA with Zeno dynamics.

Now consider QAOA with constraints enforced by measurement \( \mathcal{P} \) as defined in Equation \[6\], in the Zeno limit, the Zeno mixer Hamiltonian is a sum of the original mixer \( B \) projected to the subspaces defined by the projectors constituting \( \mathcal{P} \), i.e.,

\[ H_Z = \mathcal{P} B = \sum_{j=1}^{k} P_j BP_j \]

Concretely, consider the task of using QAOA to approximate the adiabatic evolution under the following time-dependent Hamiltonian:

\[ H_s(t) = s(t)B + (1-s(t))C, \tag{17} \]

where \( s : [0, T] \to [0, 1] \) is the interpolating schedule function. A common schedule function is the linear schedule defined by

\[ s(t) = \frac{t}{T}, \tag{18} \]

where \( T \) is the evolution time scale. Suppose \( T \gg O((\min_n \Delta_n(s))^{-2}) \), where \( \Delta_n(s) \) is the instantaneous minimum difference between the \( n \)-th eigenvalue and any other eigenvalue of \( H(s) \). If \( \forall s \), it holds that \( \Delta_n(s) \neq 0 \), then the quantum adiabatic theorem \[44\] implies:

\[ T \exp \left( \int_0^T H_s(t) dt \right) |\phi_n(0)\rangle = |\phi_n(T)\rangle. \tag{19} \]

In the Zeno case, we consider

\[ H_s(t) = s(t)H_Z + (1-s(t))PC. \tag{20} \]

Consider the QAOA operator with only one measurement per layer, i.e., \( N_j = 1, \forall j \) in \((15)\):

\[ \mathcal{U}(p) = \prod_{j=1}^{p} \mathcal{P} U_B(\beta_j) U_C(\gamma_j). \tag{21} \]

Now it is easy to recover the parameters \( \beta_j, \gamma_j \) giving the limit. From the definition of the product integral \[15\] it follows that

\[ T \exp \left( \int_0^T H_s(t) dt \right) = \lim_{p \to \infty} \prod_{j=1}^{p} \exp \left( i T \frac{p}{T} H_s \left( \frac{jT}{p} \right) \right) \]
\[ = \lim_{p \to \infty} \prod_{j=1}^{p} \exp \left( i T \frac{p}{T} \left( \frac{j}{p} C + (1 - \frac{j}{p}) B \right) \right) \]
\[ = \lim_{p \to \infty} \prod_{j=1}^{p} \mathcal{P} \exp \left( i T \frac{p}{T} B \right) \exp \left( i T \frac{p}{T} \left( 1 - \frac{j}{p} \right) C \right). \tag{22} \]

where the third equality follows from expanding to the first order in \( \frac{T}{p} \) and that \( \frac{j}{p} \) and \( 1 - \frac{j}{p} \) are bounded by 1.

Thus if \( \rho_n(0) = |\psi_n(0)\rangle\langle \psi_n(0)| \) is an \( n \)-th eigenstate of \( H_Z \) then

\[ \rho_n(T) = \lim_{p \to \infty} \mathcal{U}(p) \rho_n(0), \tag{23} \]
2. Mitigating mixer limitations in the Zeno limit

While evolution under $P_FBP_F$ is guaranteed to preserve the in-constraint subspace, it may inhibit transitions between states in $\mathcal{F}$ that were allowed with $B$. This is because states in $\mathcal{F}$ may be connected by a path that passes through states not in $\mathcal{F}$. To see this, consider a simple example of the two-qubit mixer $B_2 = X_1 + X_2$ and the in-constraint space $\mathcal{F} = \{|01\}, |10\}\}$. In the Zeno limit, the mixing operator evolution in the in-constraint subspace is generated by $P_FB_2P_F$, which equals the zero matrix. Thus, the mixing unitary becomes the identity operator and the dynamics become trivial. In general, if there is no path between two computational basis states $|j\rangle, |k\rangle \in \mathcal{F}$ in the graph defined by $B$, the continuous-time quantum walk defined by the mixing operator cannot move probability amplitude from $|k\rangle$ to $|j\rangle$. Whether the transitions between in-constraint states are suppressed in the Zeno limit is in general dependent on the in-constraint space $\mathcal{F}$.

One way to avoid the issue of suppressed transitions is by choosing a mixer $B$ with a complete connectivity graph among computational basis states, i.e., $B = |+\rangle\langle+|$. This mixer is also known as the complete-graph mixer [16]. It has been conjectured [16] that mixers with high connectivity, such as the $B = |+\rangle\langle+|$, can at best produce a Grover-like speedup since they do not make use of the structure of the cost operator. While it is unclear if this conjecture is true, we emphasize that our approach can utilize any mixer and can efficiently enforce constraints as long as the difference between the maximum and minimum eigenvalues of the mixer is polynomial in the number of qubits.

B. Scaling of the number of measurements

In this Section we derive a scaling rule for the number of measurements required to maintain a constant success probability. Our main result, which we use to derive the number of measurements required to enforce constraints in QAOA, is given in Theorem 1. We additionally give an explicit example of a projector and initial state (33) that saturate the lower bound on the success probability in Lemma 2.

Theorem 1. Let $\mathcal{P}$ be the measurement defined in (3). Let the parameterized quantum circuit evolve the system from some initial state $\rho_0 = P_F\rho_0 P_F$ under the action of a Hamiltonian $H$, whose distinct eigenvalues are $\xi_1 < \xi_2 < \cdots < \xi_d = \xi_{\text{max}}$, for time $\theta$. For $\delta \geq 0.81$, if $N$ applications of $\mathcal{P}$ are performed at equally-spaced intervals with

$$N = \left\lceil \frac{|\theta(\xi_{\text{max}} - \xi_{\text{min}})|^2}{\ln(2\delta - 1)^2} \right\rceil,$$

then the probability of measuring a state in $\mathcal{H}_j$ at time $t$ is lower bounded by $\delta$:

$$\text{Tr}[P_j\rho(\theta)] \geq \delta,$$

where

$$\rho(\theta) = U(\theta)\rho_0U(\theta)^\dagger, \quad U(\theta) = [Pe^{-iH\theta/N}]^N$$

Remark. Note that since $2\|H\|_2 \geq |\xi_{\text{max}} - \xi_{\text{min}}|$, the bound can be reformulated in terms of spectral norm of the Hamiltonian. This may be useful as the spectral norm may be easier to bound in practice for complicated mixers.

The following results will be used to prove the above Theorem.

Lemma 2. Let $H$ be a Hermitian matrix. Then

$$\min_{P,|\psi\rangle \in \text{Im}(P)} \|Pe^{-i\theta H} |\psi\rangle\|_2^2 = \cos^2 \left( \frac{\xi_{\text{max}} - \xi_{\text{min}}}{2} \right)$$

$$\forall \theta \in \mathbb{R}, |\theta| \leq \frac{\pi}{\xi_{\text{max}} - \xi_{\text{min}}},$$

where $P$ is an orthogonal projector and $\xi_{\text{max}}$ and $\xi_{\text{min}}$ are the largest and smallest eigenvalues of $H$.

Proof. Suppose $H$ has the following eigendecomposition

$$H = \sum_{k=1}^d \xi_k Q_k,$$

where $\xi_k$ are the unique eigenvalues of $H$ (including 0 if $H$ is not full rank) and $\{Q_k\}_{k=1}^d$ is the complete set of projectors onto the corresponding eigenspaces. Therefore

$$p(\theta) = \|Pe^{-i\theta H} |\psi\rangle\|_2^2$$

$$\geq \|\langle\psi| e^{-i\theta H} |\psi\rangle\|_2^2 = \|\langle\psi| e^{-i\theta H} |\psi\rangle\|_2^2$$

$$= \sum_{j,k=1}^d e^{i\theta(\xi_j - \xi_k)} \langle\psi| Q_j |\psi\rangle \langle\psi| Q_k |\psi\rangle$$

$$= \sum_{j,k=1}^d \cos(\theta(\xi_j - \xi_k)) \langle\psi| Q_j |\psi\rangle \langle\psi| Q_k |\psi\rangle$$

$$= \sum_{j,k=1}^d c_{jk} x_j x_k,$$

where $c_{jk} = \cos(\theta(\xi_j - \xi_k))$, $x_j = \langle\psi| Q_j |\psi\rangle \geq 0$. Note that the second to the last equality follows from

$$e^{i\theta(\xi_j - \xi_k)} x_j x_k + e^{i\theta(\xi_j - \xi_k)} x_k x_j$$

$$= \cos(\theta(\xi_j - \xi_k)) x_j x_k + \cos(\theta(\xi_j - \xi_k)) x_k x_j.$$

Let $C$ be the matrix with elements $c_{jk}$ at the $j$-th row and $k$-th column. Then using simple trigonometric identities, it can be shown that

$$C = v(\theta) v(\theta)^T + v(\frac{\pi}{2} - \theta) v(\frac{\pi}{2} - \theta)^T$$

(28)
where
\[ \psi(\theta) = (\cos(\xi_1 \theta), \ldots, \cos(\xi_d \theta))^T. \]  
(29)

Since \( C \) is the sum of positive semi-definite matrices, it too is positive semi-definite.

Therefore, minimizing \( p(\theta) \) is equivalent to solving the following convex constrained minimization problem
\[ \min_{x \in S} x^T C x, \quad \text{where} \quad S := \{x \in \mathbb{R}^d_+ \mid x \|_1 = 1\}, \]  
(30)
\[ x = (x_1, \ldots, x_d)^T \] and thus a sufficient condition \[\text{Theorem 2.2.5}\] for \( x^* \) to be the optimum is
\[ x^* \in S. \]  
(31)

Consider the following trial solution
\[ x_{\min}^* = x_{\max}^* = \frac{1}{2} \]
\[ x_j^* = 0 \quad \forall j \notin \{\min, \max\}. \]  
(32)

We have that \( \forall x \in S \)
\[ 2x^* \in S \quad \forall \theta \leq \frac{\pi}{\xi_{\max} - \xi_{\min}}, \] where \( c_{j,k} \geq c_{\max, \min} \), and thus
\[ 1 + c_{\min, \max} = 2 \cos^2 \left( \frac{\xi_{\max} - \xi_{\min}}{2} \right) \]
\[ \leq 2 \cos \left( \frac{\xi_{\max} - \xi_{\min}}{2} \right) \cos \left( \frac{\xi_{\max} + \xi_{\min} - 2 \xi_j \theta}{2} \right) \]
\[ = c_{\max, j} + c_{\min, j}. \]

Combining the above results, we obtain that \( 2x^* \in S \) for all steps are equalities in \[\text{Lemma 2}\] when \( P = |\psi\rangle \langle \psi| \), we obtain:
\[ \min_{P, |\psi\rangle \in \text{Im}(P)} \left\| P e^{-i \theta H} |\psi\rangle \right\|_2^2 \]
\[ = x^* \in S \]
\[ = \cos^2 \left( \frac{\xi_{\max} - \xi_{\min}}{2} \right) \quad \text{in} \] \[\text{Eq. (27)}\]

Additionally, the result implies that minimization occurs when
\[ |\psi\rangle = |\pm H\rangle := \frac{1}{\sqrt{2}} (|\xi_{\max}\rangle \pm \frac{1}{\sqrt{2}} |\xi_{\min}\rangle) \]  
(33)
for any \( |\xi_{\max}\rangle \in \text{Im}(Q_{\max}) \) and \( |\xi_{\min}\rangle \in \text{Im}(Q_{\min}) \).

**Corollary 2.** Let \( H \) be a Hermitian matrix. Then
\[ \min_{P, |\psi\rangle \in \text{Im}(P)} \left\| P e^{-i \theta H} |\psi\rangle \right\|_2^2 \]
\[ = \frac{1}{2} + \frac{1}{2} \left[ \frac{2 p^* (\theta \theta^*)}{N} - 1 \right]^N \]
\[ \forall \theta \in \mathbb{R}, \quad \theta \leq \frac{\pi N}{\xi_{\max} - \xi_{\min}}, \] where \( P \) is a projective measurement as defined in Equation \[\text{S}\] with projectors \( P \) and \( I - P \), \( p^* \) and \( \xi_{\max} \) and \( \xi_{\min} \) are the largest and smallest eigenvalues of \( H \).

**Proof.** Consider a fixed \( \theta \) and some \( N \) that satisfies the hypothesis. The stochastic process formed by random variables indicating whether the system is in \( \text{Im}(P) \) or its complement after each evolution segment \( P e^{-i \theta H} \) is a two-state Markov chain. According to Lemma \[\text{2}\] the probability of remaining in a “state” on the chain at any point in time is at least
\[ p^* \left( \frac{\theta}{N} \right) := \cos \left( \frac{\xi_{\max} - \xi_{\min}}{2 \pi N} \theta \right), \]  
(34)
and this minimum probability is attained at each segment when \( |\psi\rangle \) is \[\text{Eq. (27)}\] and \( P = |\psi\rangle \langle \psi| \). Because, in this case, the evolution lies in the two-dimensional space spanned by \( |\pm H\rangle \), the result is a Markov chain with transition matrix
\[ A(k) = \hat{A} = \left( \begin{array}{cc} p^* & 1 - p^* \\ 1 - p^* & p^* \end{array} \right) \quad \forall k \in [N], \]  
(35)
and \( \forall k > N, A(k) = I \).

Therefore the probability of the state remaining in \( \text{Im}(P) \) after \( N \) steps of the chain is \( \hat{A}_{0,0}^N \), or the first diagonal element of the matrix \( \hat{A} \) after raising it to the \( N \)-th power. Applying diagonalization on \( \hat{A} \), we obtain
\[ \hat{A}_{0,0}^N = \frac{1 + (2 p^* - 1)^N}{2}. \]  
(36)

**Proof of Theorem 4.** For all \( \theta \in \mathbb{R} \), such that
\[ |\theta| < \frac{N}{\xi_{\max} - \xi_{\min}}, \]  
(37)
it follows that
\[
\cos^2 \left( \frac{\xi_{\text{max}} - \xi_{\text{min}}}{2N} \theta \right) \geq \left( 1 - \frac{1}{2} \left[ \frac{\theta(\xi_{\text{max}} - \xi_{\text{min}})}{2N} \right]^2 \right)^2 \\
\geq 1 - \frac{[\theta(\xi_{\text{max}} - \xi_{\text{min}})]^2}{4N^2}.
\]

If we combine this result with Corollary \[2\] then we obtain
\[
\frac{1}{2} + \frac{1}{2} \left[ 2 p^* \left( \frac{\theta}{N} \right) - 1 \right]^N \\
\geq \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{[\theta(\xi_{\text{max}} - \xi_{\text{min}})]^2}{2N^2} \right)^N \\
\geq \frac{1}{2} + \frac{1}{2} \exp \left( - \frac{N - 1}{N} \right).
\]

To lower bound this by \(\delta\), we can choose \(N\) as stated in Theorem \[1\] Note that to ensure Equation (37) we must have
\[
\frac{[\theta(\xi_{\text{max}} - \xi_{\text{min}})]^2}{N} < N,
\]
and thus
\[
\frac{1}{2} + \frac{1}{2} \exp \left( - \frac{N - 1}{N} \right) \\
> \frac{1}{2} + \frac{1}{2} \exp \left( - \frac{N}{2} \right).
\]

At the minimum of value of \(N\), we have
\[
\frac{1}{2} + \frac{1}{2} \exp \left( - \frac{1}{2} \right) \approx 0.81.
\]

Figure \[1\] visualizes how the number of measurements required to maintain in-constraint probability of at least \(\delta_{\text{min}}\) for the \(B = \sum_j x_j \) (\(\bullet\) marker) and \(B = |+\rangle \langle +| \) (\(\star\) marker) mixers with 3 qubits. Many more measurements are required for \(B' = \sum_j x_j\) as the number of measurements grows quadratically with number of qubits. Note that due to periodicity, \(|\beta| \leq \pi/2\) for \(B = \sum_j x_j\) and \(|\beta| \leq \pi\) for \(B = |+\rangle \langle +|\).

practice the initial state is unlikely to align perfectly with the eigenvectors of the mixer. Thus one could consider a relaxed version of the rules provided in \[\text{Figure 1}\] as follows:
\[
N = \left\lceil \frac{[\beta(\xi_{\text{max}} - \xi_{\text{min}})]^2}{\eta} \right\rceil,
\]
where \(\eta\) is some hyperparameter to be fine tuned. One could always efficiently estimate the the in-constraint probability of a QAOA circuit with a fixed \(\eta\) by measuring a single auxiliary qubit indicating whether the final state output by the circuit is in constraint. In the portfolio optimization experiments, we successfully use an \(\eta\) for the \(B = \sum_j x_j\) mixer that is orders of magnitude larger than predicted by Corollary \[\text{Figure 1}\] requiring a correspondingly smaller number of measurements.

C. Realizing oracles for combinatorial constraints

In this Section, we review the constructions of quantum oracles for implementing polynomial inequality and equality constraints. We use the constructions provided in this Section in the experiments on a trapped-ion quantum computer described in Section \[\text{V}\]. Since any function on the Boolean cube can be expressed as a polynomial it suffices to only demonstrate constructions for polynomial constraints \[\text{XIV}\]. In addition, since we are considering problems in NPO we can assume the existence of a polynomially-sized classical circuit for evaluating any constraints to sufficient precision. Given that all classical basis gates can be represented as polynomials, we can represent our constraint as the composition of polyno-
mially many polynomial functions. Of course, one could also directly implement the classical circuit in a reversible fashion on a quantum device efficiently. For the remainder of this Section, we consider a polynomial function \( g \):
\[
g(b) = \sum_{k=1}^{K} d_k \prod_{l \in S_k} b_l,
\]
where \( S_k \subseteq [n] \) and \( d_k \in \mathbb{R} \). In addition for \( S_k = \emptyset \), \( \prod_{l \in S_k} b_l := 1 \).

Without loss of generality we can assume that equality constraints are of the form \( g(b) = 0 \) and inequality constraints are of the form \( g(b) \geq 0 \). We assume that there exists an oracle that computes the value of \( g(b) \) into a quantum register (constructions of such oracles are briefly reviewed in Sections III.C.1, III.C.2). For an equality constraint, we implement the constraint-enforcing measurement by simply measuring the entire register. A projection onto the in-constraint subspace implies that we have observed a 0 in the register. For an inequality constraint, we measure the qubit corresponding to the sign, a 0 corresponds to a successful projection, and apply the inverse of the oracle post measurement.

In the near term and in the absence of error correction, a more resource efficient approach is to switch to the Fourier basis using the quantum Fourier transform (QFT) and perform the arithmetic in the Fourier basis. This approach has worse asymptotic complexity in \( T \) counts, but requires fewer qubits and CNOT gates. We use this approach in the hardware experiments discussed in Section VI. The discussion in this Section is based on Ref. [17], though the idea of using the QFT for quantum arithmetic is well-known, see e.g. [55–57].

For \( s \in [2^m] \), the QFT on \( \mathbb{Z}_{2^m} \) is defined as follows:
\[
\text{QFT}_{2^m} : |s\rangle \mapsto \sum_{k \in [2^m]} e^{-i2\pi ks/2^m} |k\rangle.
\]
It can be shown [58] that the right-hand side of (44) is a product state and can be expressed in the following form:
\[
\bigotimes_{k=1}^{m} \frac{|0\rangle + e^{-i\pi m/2}}{\sqrt{2}} |1\rangle = F_m \left( \frac{s}{2^m} \right) |+\rangle^\otimes m,
\]
where
\[
F_m(\theta) := \bigotimes_{k=1}^{m} R(\pi 2^k \theta)
\]
implies the desired operation. In addition, \( R(\alpha) \) denotes the phase gate \(|0\rangle |0\rangle + e^{i\alpha} |1\rangle |1\rangle \). The angle \( \theta \) is restricted to \([-\frac{1}{2}, \frac{1}{2}] \) to avoid overflow and allow for representing negative numbers. Thus, when implementing a polynomial \( g \), we require that its range match the range of \( \theta \), i.e., \( \|g\|_{\infty} \leq \frac{1}{2} \). This can always be satisfied by scaling \( g \) accordingly.

As an example, we can add two integers \( a \) and \( b \), with the conditions \( a, b, a + b \in \{-2^{m-1}, \ldots, 0, \ldots, 2^{m-1} - 1\} \), as follows:
\[
\text{QFT}_{2^m}^\dagger F_m \left( \frac{a}{2^m} \right) F_m \left( \frac{b}{2^m} \right) |+\rangle^\otimes m = |a + b\rangle.
\]

Note, the value in the quantum register is really the two’s complement of \( a + b \). We define the following controlled operation:
\[
F_m(b, \theta) := |b\rangle \langle b| \otimes F_m(\theta) + (I - |b\rangle \langle b|) \otimes I,
\]
where \( b \in \mathbb{B}^n \). For \( S_k \subseteq [n] \), let \( \mathbf{1}_{S_k} \in \mathbb{B}^n \) denote the indicator vector of \( S_k \). The process for (approximately)
loading the value of the polynomial $\tilde{g}$ into a quantum register is:

$$(I \otimes \text{QFT}_{2m}^{1}) \prod_{k=1}^{K} F_{m}(1_{S_{k}}, d_{k}) |b\rangle |+\rangle^{\otimes m} = |b\rangle |\tilde{g}(b)\rangle,$$

(49)

where by the assumption on the range of $g$, $|\tilde{g}(b) - g(b)| \leq 2^{-m}$. The result is stored in an auxiliary quantum register of size $O(m)$. The operation $F_{m}(b, \theta)$ requires $m$ $n$-controlled rotation gates. Thus overall it requires $Kmn$ $n$-controlled rotation gates. An $O(n)$-controlled Toffoli can be implemented with $O(n)$ $T$'s \cite{53,54} and each controlled rotation can be $\epsilon$-approximately implemented with $O(\log(1/\epsilon))$ $T$'s \cite{59,60}. Thus, assuming a fixed rotation-gate approximation error the total cost is $O(Kmn)$.

The operation $\text{QFT}_{2m}$ requires $O(m^{2})$ gates to be implemented exactly \cite{55} and can be implemented approximately, for a fixed approximation error, on a fault-tolerant device with $O(m \log(m))$ $T$ gates \cite{60}. For equality constraints, since we will be measuring the entire register containing the value $\tilde{g}(b)$, we swap the coherent implementation of the inverse QFT for the semiclassical variant \cite{61,62}. This semiclassical version of the QFT replaces all two-qubit gates with classically-controlled single qubit gates and requires only a single auxiliary qubit that is repeatedly measured and reset to compute the bits of $\tilde{g}(b)$. Thus, this approach benefits from both mid-circuit measurements and QCL. A fault-tolerant version of this circuit can be approximately implemented with $O(m \log(m))$ $T$ gates \cite{55}. Thus in a fault-tolerant setting the overall $T$ count of the QFT-based approach is $O(Kmn + m \log(m))$.

D. Initial state construction

Our proposed approach is flexible with regards to the choice of the initial state, any initial state that is in constraint suffices. Thus, unlike \cite{10}, when using the complete-graph mixer our approach does not require repeated applications of a unitary and its inverse for preparing the uniform superposition of in-constraint states. However, the initial state we use in experiments discussed in Sections \ref{sec:4}, \ref{sec:5} is the uniform superposition over all computational basis states encoding in-constraint solutions. In general, this superposition is hard to prepare. However, there exist constructions for a wide range of practically relevant cases. If the set of feasible solutions is efficiently indexable, \cite[Section IIIB]{20} gives an efficient procedure for the initial state preparation. In the specific case of a Hamming-weight equality or inequality constraint, the uniform superposition over feasible states is a superposition of Dicke states with corresponding Hamming weights, which can be constructed efficiently \cite{64}. Since, our technique does not require the state preparation method be reversible, we can make use of repeat-until-success schemes.

IV. NUMERICAL EXPERIMENTS

We now present the numerical experiments showing the power of the proposed method. The technique we propose is general, though in this section we consider only the problem of portfolio optimization (with both equality and inequality constraints) and only the QAOA and L-VQE algorithms. We compare the results to the state-of-the-art method of encoding constraints by introducing a penalty into the objective, and observe significant improvements in both approximation ratio and in-constraint probability. In addition to better performance, the proposed method does not require complicated tuning of the penalty factor. The data used to generate the figures in this Section is available online at https://doi.org/10.5281/zenodo.7125969

A. Benchmark: portfolio optimization

The particular constrained optimization problems we study numerically arise from the discrete mean-variance Markowitz model \cite{39} and have the following objective function

$$\min_{x \in \mathcal{F}} q x^{T} \Sigma x - \mu^{T} x,$$

(50)

where $\mathcal{F}$ is defined by some set of constraints on the portfolio. We consider two sets of problems. In the first set, we impose an inequality constraint on the total size of the portfolio ($\sum_{j} x_{j} \leq C$). In the second set of problems, in addition to the inequality constraint on portfolio size, we include a constraint on the total expected return ($\sum_{j} \mu_{j} x_{j} \geq R$). For each of the two sets of constraints, we consider seven instances with between four and ten assets, for a total of fourteen instances.

B. Zeno dynamics improves quantum optimization performance

Figure \ref{fig:2} presents the comparison between QAOA with Zeno dynamics and QAOA with constraints enforced using penalty factor on fourteen problem instances described in Sec. \ref{sec:4A}. We consider QAOA with mixers $B = \sum_{j} x_{j} (\bigotimes \text{marker})$ and $B = |+\rangle \langle +| (\bigotimes \text{marker})$, and optimize the QAOA parameters exhaustively. To improve the performance of parameter optimization, we follow Ref. \cite{65} and rescale the cost function so that the gradients with respect to $\beta$ and $\gamma$ are roughly of the same magnitude.

For instances with a single constraint, we perform extensive tuning of the penalty factor $\lambda$. For multi-constraint problems, the tuning becomes prohibitively expensive. Therefore, we exclude QAOA with constraints enforced through penalties from the comparison for problems with multiple constraints. The choice of the penalty
Figure 2. Approximation ratio $r$ and in-constraint probability $\delta$ achieved by QAOA with Zeno dynamics (solid lines) and QAOA with constraints enforced using penalty terms (dotted lines) on problems with single (a,b,c,d) and multiple (e,f) constraints. For all single constraint problems QAOA with Zeno dynamics produces superior approximation ratio and in-constraint probability (solid line is above dotted with the same color). As penalty factor tuning is prohibitively difficult for problems with multiple constraints (see Sec. IV C), for these problems only Zeno dynamics results are presented.

We observe that Zeno dynamics (solid lines) enables consistently better solution quality and in-constraint probability as compared to QAOA with constraints enforced using a penalty (dotted lines) for all problems considered. Furthermore, Figure 2(b) shows that for 6 and 10 assets the in-constraint probability drops off rapidly with the number of QAOA layers if the penalty factor is kept constant. This highlights an important limitation of enforcing the constraints via penalties, namely that the penalty factor must be tuned independently for each QAOA depth. In contrast, for QAOA with Zeno dynamics we obtain an explicit rule for how $\eta$, from (42), should change with the QAOA depth (see Corollary 1). Moreover, in the experiments shown in Figure 2, good performance is obtained while keep $\eta$ a depth-independent constant for each instance.

We note that the in-constraint probability can be improved arbitrarily for the Zeno dynamics approach by decreasing $\eta$, without the need to re-optimize the QAOA parameters. This is due to the objective function landscape becoming independent of $\eta$ as the Zeno limit is approached. In fact, we observe that transferring parameters from a smaller to a larger number of measurements (larger to smaller $\eta$) works well even for practically relevant values of $\eta$. Figure 4 shows the approximation ratio $r$ and in-constraint probability with directly optimized QAOA parameters and with pre-optimized parameters transferred from a fixed value of $\eta = 1.6$ (marked with a star in the plot). We observe that for sufficiently small $\eta$, transfer works well and the difference in approximation ratio is negligible. Specifically, parameter transfer using the $B = \sum_j x_j$ mixer and a total of 33, 75, and 200 measurements results in in-constraint probabilities of at least 85%, 89%, and 96%, respectively for the nine-
assets, single-constraint problem at \( p = 5 \). At the same time, if the number of measurements is very small (\( \eta \) large), the objective function landscape is very different from the landscape in the Zeno limit, and the parameter transfer does not work well. We remark that while the in-constraint probability increases monotonically as \( \eta \) decreases, no such guarantee is given for approximation ratio \( r \). In fact, in Figure 4 we observe that depending on the problem and the circuit depth, \( r \) can either increase or decrease with \( \eta \).

Note that the same approach of boosting the in-constraint probability without re-optimizing the QAOA parameters does not work if the constraints are enforced using penalties. Figure 3 shows that transferring parameters from a fixed value of penalty factor (marked with a star) leads to the approximation ratio rapidly dropping off to random guess. It is however possible that better performance may be achieved by leveraging more sophisticated parameter transfer strategies, such as the rescaling rule proposed for the weighted MaxCut problem [66] or machine learning methods [67].

While for QAOA with Zeno dynamics the approximation ratio \( r \) given in Equation 4 increases monotonically with the number of QAOA layers, this is not guaranteed for QAOA with constraints enforced through penalties. This is because the QAOA parameters are chosen with respect to the objective with penalties and the increased expressivity of the higher-depth circuit is only guaranteed to improve the performance with respect to that objective. Figure 8 shows that this is indeed the case and the approximation ratio \( r_{\text{penalty}} \) given in 5 increases with the number of QAOA layers as expected.

Finally, we include the results for Zeno-dynamics L-VQE. Unlike QAOA, for L-VQE we do not have analytical lower bounds on the number of measurements required to maintain a fixed in-constraint probability. Therefore, we heuristically set the number of measurements to 100. Table 1 presents the results. As expected, L-VQE achieves high approximation ratio, while Zeno dynamics enables high in-constrain probability. As the total number of measurements is kept fixed for all problems and parameter values, slightly lower in-constraint probability is observed for higher qubit counts. As is the case for QAOA, the in-constraint probability can be increased by increasing the number of measurements.

| \# assets | Single \( r \) | Multiple \( r \) |
|-----------|----------------|----------------|
| 4         | 0.995          | 0.9996         | 0.980          |
| 5         | 0.995          | 0.977          | 0.909          |
| 6         | 0.972          | 0.964          | 0.963          |
| 7         | 0.979          | 0.917          | 0.936          |
| 8         | 0.956          | 0.948          | 0.944          |
| 9         | 0.967          | 0.961          | 0.974          |
| 10        | 0.914          | 0.910          | 0.960          |

Table 1. Performance of L-VQE on the benchmark problems. L-VQE obtains high approximation ratio and high in-constraint probability.
C. Penalty factor tuning is difficult

An important advantage of our method is the simplicity of hyperparameter tuning, as only \( \eta \) needs to be chosen. This choice is made easy by the theoretical bounds derived in Section III and the monotonic increase of in-constraint probability with decrease in \( \eta \). This is in sharp contrast with the penalty approach, where the performance crucially depends on the penalty strength, which is hard to tune in general. We now present how the penalty strength was chosen for the experiments above, and discuss the challenges that arose in doing so.

Figure 6 presents the performance of QAOA on a single-constraint problem enforced using a penalty term with varying penalty factors \( \lambda \). In the plot, the in-constraint probability \( \delta \) monotonically increases with \( \lambda \), while the approximation ratio \( r \) decreases. This indicates a trade-off between \( r \) and \( \delta \), and hence hyperparameter tuning on \( \lambda \) must be performed in order to obtain a good approximation ratio while meeting requirements on the minimum in-constraint probability. We also observe that for QAOA with small \( \rho \), \( \delta \) tends to levels off at a value far below what is achievable by using Zeno dynamics. For example, the top figure in Figure 6 shows that the highest in-constraint probability achievable with \( \rho = 1 \) is around 80% for the problem tested. Given that the approximation ratio with the penalty term \( r_{\text{penalty}} \) is above 0.9 for the high \( \lambda \) regime, it indicates that the maximum achievable in-constraint probability may be limited by the expressivity of the variational circuit. On the other hand, constraints enforced by Zeno dynamics do not suffer from such problems, as the in-constraint probability can be arbitrarily boosted regardless of the expressivity of the variational circuit (see Figure 4). In the numerical experiments, we choose the value of \( \lambda \) independently.
for each problem instance with the goal of obtaining a high in-constraint probability $\delta$. Since we show that the factor $\lambda$ trades off $r$ and $\delta$, both cannot be improved at the same time. This suggests that no choice of $\lambda$ would lead to a QAOA with constraints enforced by penalties outperforming QAOA with Zeno dynamics.

In Figure 7, we show how hyperparameter tuning works with two penalty factors $\lambda_1$ and $\lambda_2$ corresponding to penalty terms enforcing the budget constraint and the return constraint respectively. The figure shows the in-constraint probability of the optimal solution obtained with varying $\lambda_1$ and $\lambda_2$. Similar to the single-constraint case, maximal ap-
proximation ratio $r$ and maximal in-constraint probability $\delta$ cannot be simultaneously achieved. Specifically, the solutions with the maximal $r$ and maximal $\delta$ have very different values in $\lambda_1$ and $\lambda_2$. Moreover, unlike Figure 4, Figure 7 clearly shows the non-monotonic behavior of $\delta$ in both $\lambda_1$ and $\lambda_2$. In fact, we observe a similar behavior across many of the single- and multi-constraint problems that we have tested, and for both the $B = \sum_j x_j$ and $B = |+\rangle\langle+|$ mixers. This indicates that tuning the penalty factors is indeed difficult in the general case.

V. HARDWARE EXPERIMENTS

To demonstrate the near-term feasibility of our approach, we execute QAOA with Zeno dynamics on the Quantinuum H1-2 trapped-ion quantum processor. Our implementation uses the constraint-checking oracles that perform quantum arithmetic in the Fourier domain, following directly the construction in Sec. III C. We observe that increasing the number of measurements improves the in-constraint probability $\delta$, as expected. The improvement from additional measurements continues up to a two-qubit gate depth of 148, at which point the hardware noise prevents further improvements. The circuits executed on hardware are available online at [https://doi.org/10.5281/zenodo.7125969](https://doi.org/10.5281/zenodo.7125969).

The experiments presented in this Section utilize $p = 1$ QAOA and the $B = \sum_j x_j$ mixer. We use the cost function of the four-assets portfolio optimization problem used in the numerics described in Sec. IV but apply different constraints. We consider two instances with linear constraints, one with an equality constraint and one with an inequality constraint. Figure 10 shows a high-level circuit diagram. For each problem, the QAOA parameters are first optimized using a noiseless simulator. All circuit executions use 2000 shots and no error mitigation.

The first portfolio optimization instance we consider has an equality constraint on the four binary variables $x_1, x_2, x_3, x_4$: $2x_1 - x_2 - x_3 = 0$. As discussed in Sec. III C, the semiclassical QFT can be utilized for equality constraints. The semiclassical QFT makes use of QCL and midcircuit measurements, which are features supported by the H1-2 device. This results in an oracle that uses only one auxiliary qubit, and thus the circuit uses five qubits in total. The circuit for the oracle is shown in Figure 11.

As a comparison, we also implement the coherent QFT on three qubits, resulting in seven qubits in total. After applying the oracle and measuring, all auxiliary qubits are reset to the ground state for the uncompute step. Figure 8 shows the in-constraint probability as a func-
tion of the number of projective measurements. Figure 8a shows the distributions of measurement outcomes of QAOA for varying numbers of measurements ($N_x$), with the outcomes (computational basis states) ordered by the objective function value. For both implementations, the in-constraint probability improves with the number of measurements up to $N_x \approx 15$. For a higher number of measurements, the hardware noise arising from high circuit depth prevents further improvements in the in-constraints probability $\delta$.

While the QCL and non-QCL implementations both perform similarly, we do note a reduction in the number of two-qubit gates and auxiliary qubits. For QCL and $N_x = 15$, the two-qubit gate depth was 122 and the count was 123. Without QCL, for $N_x = 15$, the two-qubit gate depth was 148 and the count was 165. The similar performance between QCL and non-QCL versions despite the difference in gate count may be due to the higher impact of measurement error on the QCL implementation.

Note that the performance deteriorates at a significantly lower $N_x$ for the inequality constraint problem than equality. This occurs even though the two-qubit circuit depth is lower for the inequality case and the two-qubit gate count is not significantly higher. Besides the inclusion of an additional qubit, one potential reason for this is that for the inequality constraint, only one of the auxiliary qubits is measured and then the inverse oracle is applied. This allows for errors to accumulate more and propagate to the rest of the circuit. However, in the equality constraint case, after applying the oracles, all auxiliary qubits are measured and then reset to the ground state. In addition, the total gate count happens to be significantly higher for the inequality constraint case.

VI. DISCUSSION

In this work, we propose an approach for enforcing constraints in quantum optimization and demonstrate its effectiveness by applying it to constrained instances of portfolio optimization in simulation and on a trapped-ion quantum processor. Our technique has two major

The second portfolio optimization instance we consider has a cardinality (Hamming-weight) inequality constraint $\sum_{j=1}^{n} x_j \leq 2$. For this problem, it is necessary to utilize the coherent QFT, and thus QCL does not lead to a resource-reduction technique. The QFT adder is used to compute $\sum_j x_j - 3$, which requires four qubits to accommodate the range. In addition, unlike the equality-constraint case, the inverse oracle is necessary for uncomputation. The system is in-constraint when the most-significant qubit, i.e., the sign bit, is a one. The circuit for the oracle is shown in Figure 12. Similar to the previous run, we plot the in-constraint probability for varying numbers of measurements (Figure 9a), as well as, the measurement distributions obtained from QAOA (Figure 9b). For $N_x = 3$, the two-qubit gate depth is 112 and the count is 186. Similarly to the experiments with the equality constraint, the in-constraint probability $\delta$ improves until $N_x = 3$. For a higher number of measurements, the hardware noise prevents further improvements.

Advantages: the ability to enforce a very general class of constraints and the simplicity of hyperparameter tuning. Two important downsides of our approach are the complexity of implementing the measurement and the possibility of the measurements resulting in trivial dynamics.

Implementing the oracle for a constraint in general requires quantum arithmetic and may lead to high gate count for more complex constraints. At the same time, our hardware experiments show that the overhead is low enough to make our approach viable in the near-term. Reductions in the cost of implementing quantum arithmetic, such as techniques utilizing quantum conditional logic, can further reduce the overhead of the proposed method. Moreover, additional performance improvements can be obtained by leveraging advanced algorithm-specific error mitigation techniques such as the ones recently proposed for QAOA.

As discussed in Sec. III A 2, restricting the evolution to the Zeno subspace may result in trivial dynamics for certain mixers. Therefore an important consideration when applying the proposed technique is evaluating whether the particular choice of mixer has this behavior. As this effect would apply generally to all instances with a given
class of constraints, the mixer may be analyzed only once for a class of problems.

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