Abstract—A novel method for the fast acquisition and the recovery of ultrasound images disrupts the tradeoff between the image acquisition rate and the image quality. It recovers the spatial compressibility fluctuations in weakly-scattering soft tissue structures from only a few sequential pulse-echo measurements of the scattered sound field. The underlying linear inverse scattering problem uses a realistic d-dimensional physical model for the pulse-echo measurement process, accounting for diffraction, the combined effects of power-law absorption and dispersion, and the specifications of a planar transducer array. Postulating the existence of a nearly-sparse representation of the spatial compressibility fluctuations in a suitable orthonormal basis, the compressed sensing framework ensures its stable recovery by a sparsity-promoting $\ell_1$-minimization method, if the pulse-echo measurements of the individual basis functions are sufficiently incoherent. The novel method meets this condition by leveraging the degrees of freedom in the synthesis of the incident ultrasonic waves. It emits three types of random ultrasonic waves that outperform the widely-used steered quasi-plane waves (QPWs). Their synthesis applies random time delays, apodization weights, or combinations thereof to the voltage signals exciting the individual elements of the planar transducer array. For a numerically-simulated pulse-echo scan configuration in the two-dimensional Euclidean space, the emissions of single pseudorandom incident ultrasonic waves induce less coherent measurements than a QPW for both the canonical and the FOURIER bases. Applying pseudorandom time delays to the excitation voltages, for instance, reduces the full areas at half maximum of the point spread function (PSF) by up to 73.68% in comparison to a QPW. The cumulative distribution function (CDF) of the incoherent artifacts in the PSFs resembles that enforced by the restricted isometry property more closely than the CDF of the coherent side lobes in the PSFs induced by the QPW. The recovery of a tissue-mimicking cosparse object from corrupted pulse-echo measurements, whose signal-to-noise ratios range from 3 dB to 30 dB, significantly increases the mean structural similarity indices by up to $\{43.67\% , 55.1\% \}$, decreases the mean relative root mean-squared errors by up to $\{41.62\% , 59.29\% \}$, and converges up to $\{28.79\% , 44.59\% \}$ faster for $q \in \{1, 0.5\}$. The two other types of pseudorandom incident ultrasonic waves produce similar results.

Index Terms—random sound waves, inverse scattering, compressed sensing, sparse representation, fast imaging, diffraction tomography, fast multipole method, GPU computing, ultrasound imaging, pulse-echo, image recovery

I. INTRODUCTION

A. Review of the Relevant Scientific Literature

In the past decade, various medical imaging technologies, e.g. magnetic resonance imaging (MRI) [1]–[8], x-ray computed tomography (CT) [4]–[6], and photoacoustic tomography (PAT) [7], [8], adopted a novel mathematical concept for data acquisition and recovery that is commonly referred to as “compressed sensing” (CS) [9]–[13], “compressive sensing” [14]–[16], “compressive sampling” [17], [18], “sparse recovery” (SR) [9], [19], or used without any distinct name during its earlier development [20]–[23]. CS enables the imaging of certain objects by these technologies from significantly fewer physical observations than were traditionally thought to be necessary. Although CS reduces the data volume to be acquired and, consequently, the acquisition time, it approximately preserves the image quality. Conversely, for a fixed number of physical observations, CS enhances the image quality compared to traditional methods for image recovery. These enhancements include the reduction of image artifacts and noise as well as the improvement of the spatial resolution [2]–[8]. Graff et al. [24] provide a comprehensive review.

Multiple research groups have recently innovated image recovery methods for fast pulse-echo ultrasound imaging (UI) that combine inverse scattering techniques with CS. These include FOURIER-based plane-wave imaging [25], time-domain compressive beamforming [26], measurement-domain adaptive beamforming [27], and fast CS-based image acquisition [28]–[31]. These methods disrupt the quasi-continuous tradeoff between the image acquisition rate and the image quality inherent in the established methods for fast image acquisition, e.g. coherent plane-wave compounding [32] or limited-diffraction beam imaging [33], [34]. They permit the recovery of high-quality images from only a few sequential pulse-echo measurements of the scattered sound field, if these measurements are sufficiently incoherent. Although the combination of modern UI systems and transducer arrays provides numerous degrees of freedom in the synthesis of the incident ultrasonic wave, these new methods exclusively emit steered plane waves (PWs).

The steered PW is a fundamental and widely-used physical model. It facilitates numerous mathematical derivations and numerical computations, including the times-of-flight in coherent plane-wave compounding [32] and the well-interpretable relation between the d-dimensional spatial FOURIER transform of the heterogeneous object to be imaged over a $(d-1)$-hemisphere and the angular spectrum of the scattered acoustic pressure field in the FOURIER diffraction theorem (FDT) (cf. e.g. [35] Theorem 8.4), [36] Theorem 3.1, [37] Sect. 6.3, or [38]) underlying the FOURIER domain beamforming.

1The initial distinction between the terms “compressed sensing” (CS) and “sparse recovery” (SR), which was based on the usage of either random or deterministic sensing matrices, has been abandoned [20] p. 21, [2].

2The FDT generalizes the FOURIER slice theorem underlying the image recovery in CT from rays to diffracting waves and is also referred to as the “generalized projection-slice theorem” [35] p. 372.

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Methods [25, 33, 34] and ultrasound diffraction tomography (UDT) [38, 39]. Its unlimited spatial extent and the unlimited amount of transmitted energy, however, necessitate its approximation by a steered quasi-plane wave (QPW) in any practical measurement. This steered QPW reflects the anisotropic directivity patterns of the finite number of physical transducer elements and causes a model mismatch. Moreover, it does not induce the incoherent measurements required by CS.

Modern UI systems typically allow the specification of time delays, apodization weights, or combinations thereof for the voltage signals exciting the individual physical elements of the transducer array. The degrees of freedom in this specification permit the synthesis of numerous alternative types of incident ultrasonic waves. Random excitation voltages, for instance, synthesize random incident ultrasonic waves that induce structured random sensing matrices. These types of sensing matrices are central to the CS framework and guarantee the stable image recovery in certain instances. Only a few authors seem to have noticed the significance of these degrees of freedom for CS-based pulse-echo UI.

Liu et al. [40, 41] utilize realizations of uniformly-distributed random variables as the apodization weights in CS-based synthetic transmit aperture UI. They recover the radio frequency (RF) voltage signals provided by a complete synthetic aperture (SA) acquisition scheme and subsequently apply the conventional delay-and-sum (DAS) beamforming protocol. Although they do not leverage inverse scattering techniques for the image recovery, improvements in the image quality occur. The authors, however, require on the order of 30 sequential pulse-echo measurements per image, whereas the method proposed in this paper only requires the minimum number of a single wave emission. Moreover, they misleadingly refer to the resulting incident ultrasonic waves as “diverging wave” [40] and “randomly apodized plane wave” [41] and do not discuss their erratic spatial properties.

Kruizinga et al. [42] present a method for compressive three-dimensional UI using only a single transducer. They equip this single transducer with a custom-made delay mask introducing random time delays into both the emitted and the received ultrasonic wave. The sequential emission of these random ultrasonic waves at numerous angles of rotation permits the recovery of three-dimensional nearly-sparse objects. The fixed delay mask and the absence of multiple physical transducer elements, however, reduce the number of degrees of freedom in the synthesis of the incident ultrasonic wave and increase the required number of sequential pulse-echo measurements per image. The method proposed in this paper enables a significantly faster image acquisition and additional types of random incident ultrasonic waves.

The author published two abstracts outlining the fundamental ideas of this paper in connection with oral presentations at two conferences [43, 44].

B. Specific Contributions of the Paper

The novel method for fast image acquisition proposed in this paper alleviates the tradeoff between the image quality and the image acquisition rate in pulse-echo UI. It formulates the recovery of the spatial compressibility fluctuations in weakly-heterogeneous soft tissue structures from only a few sequential pulse-echo measurements of the scattered sound field as a linear inverse scattering problem (ISP). The ISP bases upon a realistic \(d\)-dimensional, \(d \in \{2, 3\}\), physical model for the pulse-echo measurement process, accounting for diffraction, the combined effects of power-law absorption and dispersion, as well as the specifications of a planar transducer array, including the geometry of its elements, the acoustic lens, and its electromechanical transfer behavior. Postulating the existence of a nearly-sparse representation of the spatial compressibility fluctuations in a suitable orthonormal basis, the CS framework enables its stable recovery by a sparsity-promoting \(\ell_q\)-minimization method, \(q \in (0; 1)\), if the pulse-echo measurements are sufficiently incoherent. The method leverages the degrees of freedom in the synthesis of the incident ultrasonic waves to meet this requirement. It emits three types of random ultrasonic waves that outperform the widely-used steered QPWs in this respect. Their synthesis applies random time delays, apodization weights, or combinations thereof to the voltage signals exciting the individual elements of the planar transducer array. This increases both the image quality and the speed of convergence.

We validated the emissions of single random incident ultrasonic waves in a numerically-simulated pulse-echo scan configuration in the two-dimensional Euclidean space, i.e. \(d = 2\), using the RF voltage signals acquired from a synthetic wire phantom and a tissue-mimicking object, which were sparse in the spatial domain (canonical basis) and the Fourier domain (discrete Fourier basis), respectively, for five signal-to-noise ratios (SNRs). The point spread functions (PSFs) for the canonical basis indicate less coherent measurements than a QPW. The stable recovery was accomplished by a sparsity-promoting \(\ell_q\)-minimization method, \(q \in (0; 1)\), in the convex setting \(q = 1\) and the nonconvex setting \(q \in (0; 1)\).

The paper organizes these contributions as follows. Section II briefly reviews the relevant aspects of the CS framework and Sect. III presents the linear physical model for the pulse-echo measurement process. The synthesis of the incident ultrasonic waves is explained in detail in Section IV. Sections V and VI present the proposed novel method for fast image acquisition based on CS and its matrix-free implementation, respectively. Section VII summarizes the results of the numerical simulations and Sect. VIII concludes this paper with a brief discussion of these results.

The paper uses the following mathematical symbols. The set of consecutive positive integers is \([N] = \{1, 2, \ldots, N\}\) for \(N \in \mathbb{N}\). The set of consecutive nonnegative integers is \([N]_0 = \{0, 1, \ldots, N\}\) for \(N \in \mathbb{N}_0\). For \(q \in (0; 1)\), the \(\ell_q\)-quasinorm of any vector \(\theta = (\theta_1, \ldots, \theta_N)^T \in \mathbb{C}^N\) reads \(\|\theta\|_q = \sum_{n=1}^{N} |\theta_n|^q\). In the limit \(q \to 0\), this expression counts the number of nonzero components, i.e. \(\|\theta\|_0 = \|\theta\|_0 = |\{n \in [N] : \theta_n \neq 0\}|\). For \(q = 1\) and \(q = 2\), this expression yields the \(\ell_1\)-norm and the \(\ell_2\)-norm indicating the length of \(\theta\), respectively. For \(d \in \{2, 3\}\), the vector \(r = (r_{d-1}, r_d)^T = (r_1, \ldots, r_d)^T \in \mathbb{R}^d\) indicates a spatial position in the \(d\)-dimensional Euclidean space \(\mathbb{R}^d\) with the lateral coordinates.
Observations are the image of a nonadaptive linear map \( \Phi \) from the space of vectors to be recovered \( \mathbb{C}^n \) vectors \( \mathbb{C}^d \) transpose matrix and of observations signals in the time domain.

For an underdetermined system of linear algebraic equations

\[
\begin{align*}
\mathbf{y}(\eta) &= \mathbf{A} \mathbf{\theta} + \eta, \\
= & A
\end{align*}
\]

The resulting CS problem reads

\[
\text{Recover nearly-sparse } \mathbf{\theta} \in \mathbb{C}^N \\
\text{subject to } \| \mathbf{y}(\eta) - A \mathbf{\theta} \|_2 \leq \eta
\]

and the methods for its stable solution may be summarized as the sparsity-promoting \( \ell_q \)-minimization method

\[
\hat{\mathbf{\theta}}(\eta, q) = \arg\min_{\mathbf{\theta} \in \mathbb{C}^N} \| \mathbf{\theta} \|_q \\
\text{subject to } \| \mathbf{y}(\eta) - A \hat{\mathbf{\theta}} \|_2 \leq \eta
\]

where the parameter \( q \in [0; 1] \) determines the type of optimization problem. The parameter \( q = 1 \) induces the convex \( \ell_1 \)-minimization method, whose implementation permits computationally-efficient algorithms, whereas the remaining values \( q \in (0; 1) \) induce a nonconvex \( \ell_q \)-minimization method.

Multiple sufficient conditions on the sensing matrix \( M \) have been proven in the scientific literature to guarantee the stable recovery of the nearly-sparse coefficient vector \( \mathbf{\theta} \) in the CS problem \( (5) \) by the sparsity-promoting \( \ell_q \)-minimization method \( \hat{\mathbf{\theta}}(\eta, q) \). These conditions include upper bounds on the coherence \( [14, \text{Chapt. 5}] \), the sparse rank property, the null space property \( [11, 19] \), the restricted isometry ratio \( [48] \), and the restricted isometry property (RIP) \( [12, 17, 49] \). Their direct verification for a given deterministic sensing matrix \( A \), however, is a computationally-intractable combinatorial problem \( [50, 51] \) and, consequently, not practicable for the large sensing matrices \( A \) encountered in medical imaging technologies. Fortunately, the scientific literature additionally has proven that random sensing matrices \( A \), whose entries are realizations of independent and identically distributed (i.i.d.) random variables governed by certain random distributions, e.g. Gaussian or Bernoulli, obey the RIP for a given sparsity \( s \) with high probability, if the number of observations \( M \) is sufficiently large \( [52, \text{Theorem 5.2}] \). These random matrices guarantee the stable recovery of the nearly-sparse coefficient vector \( \hat{\mathbf{\theta}}(\eta, q) \) with \( s \) significant components from \( \mathbb{M} \in \Omega(s \ln(N s^{-1})) \) observations.

The sufficient conditions require nonzero column vectors of similar \( \ell_2 \)-norms or \( \ell_q \)-norms close to unity, i.e. \( \| a_n \|_2 \approx \text{Constant} \) or \( \| a_n \|_2 \approx 1 \) for all \( n \in [N] \). A normalized sensing matrix meets the sufficient conditions for 1-sparse vectors by minimizing both the restricted isometry ratio and the restricted isometry constant, i.e. \( \gamma_1(\mathbf{A}) = 1 \) and \( \delta_1(\mathbf{A}) = 0 \) \( [14, \text{Proposition 6.2}] \). The introduction of suitable diagonal weighting matrices, whose entries equal the \( \ell_q \)-norms of the column vectors or their reciprocals, into the underdetermined system of linear algebraic equations \( \hat{\mathbf{\theta}}(\eta, q) \) accomplishes this normalization and simultaneously preserves the mathematical equivalence.

In medical imaging technologies, the underlying physical models dictate the entries of the observation process \( \Phi \hat{\mathbf{\theta}}(\eta, q) \) and, consequently, the entries of the sensing matrix \( A \). Various degrees of freedom provided by the instrumentation, however,
able the limited manipulation of these entries and the construction of sensing matrices [3] with a random structure that potentially conform better with the aforementioned sufficient conditions. Indeed, randomly and uniformly chosen rows of a discrete FOURIER basis $\Psi$ [53 Theorem 3.3] guarantee similar results. Their suitability for the recovery of the nearly-sparse coefficient vector (2) in the CS problem (5) by the sparsity-promoting $\ell_q$-minimization method (2.4,5) is frequently assessed by the associated transform point spread function (TPSF) [1, 2]. This function equals the mutual correlation coefficient of the column vectors $a_n \in \mathbb{C}^M \setminus \{0\}, n \in [N]$, in the sensing matrix (3) given by

$$TPSF\{(n_1, n_2) = \frac{\langle a_{n_1}, a_{n_2} \rangle}{\|a_{n_1}\|_2 \|a_{n_2}\|_2}, \quad (6)$$

for all $(n_1, n_2) \in [N]^2$ and quantifies their similarity. If the orthonormal basis in the nearly-sparse coefficient vector (4) equals the canonical basis, i.e. $\Psi = I$, the TPSF (6) reduces to the PSF that exclusively describes the properties of the observation process $\Phi$ [1] [2].

For $n_1 = n_2 = n$, the column vectors are identical, and the TPSF (6) attains its maximum absolute value of unity, i.e. $TPSF\{A\}(n, n) = 1$ for all $n \in [N]$. For $n_1 \neq n_2$, however, the column vectors typically differ and the absolute value of the TPSF (6) is desired to be as close to zero as possible, i.e. $|TPSF\{A\}(n_1, n_2)| \approx 0$ for all $(n_1, n_2) \in [N]^2$ with $n_1 \neq n_2$. These small values of the TPSF (6) indicate that the underlying physical model for the observation process distinguishes the associated components in the nearly-sparse coefficient vector (4), which is a major requirement of CS. These deviations from zero quantify the interference between the associated transform coefficients in an approximate linear inversion process using the conjugate transpose sensing matrix (3) [1] [1], i.e. the extent to which the observations originating from $e_{n_2}, i.e. a_{n_2} = A e_{n_2}$, are erroneously attributed to $e_{n_1}$. Ideally, the interference should exhibit noise-like properties [1] [2]. Its absolute values should be close to zero and the nonzero values should be spread uniformly across all vectors $e_{n_1}, n_1 \in [N]$. This type of interference is referred to as incoherent aliasing that is removed by the sparsity-promoting $\ell_q$-minimization method (2.4,5). In contrast, larger values of the interference that are concentrated on only a few vectors $e_{n_1}$ are named coherent aliasing.

Practical applications of CS to medical imaging commonly analyze the TPSF (6) for all indices $n_1 \in [N]$ specifying a fixed index $n_2 \in [N]$ (see e.g. [8 Fig. 1], [1 Fig. 6], or [2 Fig. 5]). The fixed index $n_2$ can be selected according to the nearlsparse representation’s expected support, i.e. $n_2 \in \text{supp} (\theta)$. The incomplete evaluation of the TPSF is often necessary owing to the large size of the sensing matrix (3), which renders the complete evaluation impractical. To obtain mathematical guarantees, however, the TPSF has to be calculated for all tuples $(n_1, n_2) \in [N]^2$ to evaluate the maximum sidelobe-to-peak ratio, i.e. the coherence. The latter provides an upper bound on the sensing matrix’s restricted isometry constant.

This paper proposes a linear physical model for the measurement process underlying fast pulse-echo UI that empha-

sizes the degrees of freedom provided by commercial UI systems, e.g. the specification of the voltage signals exciting the individual physical elements of the transducer array. It considers the image recovery as an instance of the CS problem (5) and uses the physical model’s degrees of freedom to introduce random properties into the sensing matrix (3). The resulting sensing matrix (3) is typically large and, consequently, the TPSF (6) will be evaluated to assess its suitability for the recovery of the nearly-sparse coefficient vector (2) in the CS problem (5).

III. LINEAR PHYSICAL MODEL FOR THE PULSE-ECHO MEASUREMENT PROCESS

A. Mechanical Model for Human Soft Tissues

Modern medical UI systems image nearly all soft tissue structures in the human body [55, 56, p. 135], [57], the images exclude the interior of solid structures or gas-filled cavities, e.g. bones, the lung, and the gastrointestinal tract [58, Sects. 1.8.2 and 1.11], [59, pp. 10, 11], [60]. The physical modeling of the interactions between the ultrasonic waves and the soft tissue structures requires a precise specification of the mechanical properties relevant to the ultrasonic waves in the temporal frequency range of interest [61]. This specification constitutes the mechanical model for human soft tissues. Multiple authors review the relevant physical effects, their physical models, and their exploitation in the formation of images [55], [60], [62], [38] (9)], [63]. Owing to the complexity of these interactions, the formulation of a tractable wave equation requires multiple assumptions and approximations [54 pp. 96, 97, 295].

The scientific literature frequently models human soft tissues as heterogeneous, isotropic, quiescent, and lossless fluids [65], [66 Sect. 3.2], [61]. It approximates the longitudinal ultrasonic wave as a small-amplitude disturbance of the tissues’ unperturbed ambient state. The adjective “unperturbed” refers to the values of these material parameters in the absence of any ultrasonic waves and is indicated by the subscript “u”. The relevant acoustic material parameters of the tissues are spatial fluctuations in the unperturbed values of both the compressibility $\kappa_u : \mathbb{R}^d \mapsto \mathbb{R}^+$ and the mass density $\rho_u : \mathbb{R}^d \mapsto \mathbb{R}^+$. These fluctuations are normalized by arbitrary homogeneous reference values for the compressibility $\kappa_0 \in \mathbb{R}^+$ and the mass density $\rho_0 \in \mathbb{R}^+$ of the tissues. Both reference values typically equal the spatial average values of the unperturbed acoustic material parameters [58, p. 354], [67, p. 210], [61]. For the sake of simplicity, this paper considers a homogeneous unperturbed mass density and exclusively accounts for the relative spatial fluctuations in the unperturbed compressibility

$$\gamma (r) = 1 - \frac{\kappa_u (r)}{\kappa_0}, \quad (7)$$

which quantify the relative deviation of the unperturbed compressibility from the homogeneous reference value.
For the limited temporal frequency range of interest in clinical medicine, measurements indicate that the spatial amplitude absorption coefficient $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$ obeys the power law [58 Sect. 4.1.2], [67 (1)], [64 (1.125) and (3.97)], [68 (8)], [69 (1)], [70 Sects. 4.3.5.2 and 4.3.8, Tab. 4.20], [63 (4)]

$\alpha(\omega) = a + b|\omega|^\zeta,$  

(8)

where $\omega \in \mathbb{R}^+$ denotes the angular temporal frequency and the parameters $a, b, \zeta \in \mathbb{R}^+$ depend on both the specific characteristics of the investigated tissue, e.g. type of tissue, age, and pathology, and the ambient conditions, e.g. temperature, static pressure, salinity, and pH. The parameters are usually constant over limited ranges of the temporal frequency. For most soft tissues and biological fluids, the coefficient $a$ equals zero [58 Sect. 4.1.2], [70 p. 112] and the exponent $\zeta$ ranges between 1.0 and 1.5 [67], [71], [73], [70 p. 112].

Given a specific power-law dependence of the spatial amplitude absorption coefficient on the temporal frequency $\zeta$, i.e. a suitable tuple of absorption parameters $(a, b, \zeta)$, in combination with reference values of the angular temporal frequency $\omega_{\text{ref}} \in \mathbb{R}^+$ and the associated phase velocity $c_{\text{ref}} = c(\omega_{\text{ref}}) \in \mathbb{R}$, the KRAMERS-KRONIG relations [67–69] and the time-causal model [73], [74] predict the power-law dispersion relation

$k(\omega) = \frac{\omega}{c_{\text{ref}}}(\omega - j(a + b|\omega|^\zeta),  

= \beta(\omega) + \alpha(\omega)^{-1} - j\omega = \alpha(\omega) \equiv \alpha(\omega) \equiv \alpha(\omega),  

(9a)

expressing the complex-valued wavenumber as a function of the angular temporal frequency, where the phase term $\beta : \mathbb{R}^+ \mapsto \mathbb{R}$ sums the real-valued wavenumber $k_{\text{ref}} = c_{\text{ref}}^{-1}\omega$ and the excess dispersion term

$\beta_{\text{Ex,ref}}(\omega) = \begin{cases} -\frac{2b}{\pi} \omega \zeta \ln\left(\frac{|\omega_{\text{ref}}|}{\omega}\right) & \text{for } \zeta = 1, \\ b \tan(\zeta \pi/2) \omega(\omega^{\zeta - 1} - |\omega_{\text{ref}}|^{\zeta - 1}) & \text{else}. \end{cases}$  

(9b)

The power-law dispersion relation [9] describes the combined effects of power-law absorption and dispersion [28]. The effect of dispersion distorts the shape of broadband pulses.

Let the complex-valued phasor $p : \mathbb{R}^d \mapsto \mathbb{C}$ represent the monofrequent acoustic pressure field $p : \mathbb{R}^{d+1} \mapsto \mathbb{R}$ of the angular temporal frequency $\omega \in \mathbb{R}^+$ using a positive algebraic sign in the argument of the complex exponential function, i.e.

$p(\mathbf{r}, t) = \Re \left\{ p(\mathbf{r}) e^{j\omega t} \right\}.  

(10)$

Under the aforementioned assumptions, the complex-valued phasor obeys the second-order, linear, elliptic, and homogeneous partial differential equation (PDE) (cf. e.g. [65] (2) to (4), [64] (5.30) to (37), Sect. 6.1.2, (25) to (27), [66] (3.1) to (3.3), [75] (9-1.19), [76] (8.1.12)\(^8\))

$(\Delta + k^2)p(\mathbf{r}) = k^2\gamma(\omega,p(\mathbf{r}), \mathbf{r})p(\mathbf{r}),  

(11)$

where $\Delta$ denotes the LAPLACE operator and the power-law dispersion relation [9] substitutes the originally real-valued wavenumber $k_0 = c_0^{-1}\omega \in \mathbb{R}^+$ with respect to the frequency-independent homogeneous reference value of the phase velocity $c_0 = (\kappa_0 \rho_0)^{-1/2}$ to incorporate the combined effects of power-law absorption and dispersion into the mechanical model. The right-hand side of the proposed PDE (11), which is referred to as the induced source, accounts for the interactions of the acoustic pressure field and the scattering potential $k^2\gamma(\mathbf{r})$ [35] (6.1)). In a homogeneous fluid, i.e. $\gamma(\omega, \mathbf{r}) = 0$ for all $\mathbf{r} \in \mathbb{R}^d$, the induced source vanishes and the proposed PDE (11) equals the power-law HELMHOLTZ equation [67] (11)). In a heterogeneous object, i.e. $\gamma(\omega, \mathbf{r}) \neq 0$, the induced source quantifies the physical effect of scattering. The homogeneity of the proposed PDE (11) assumes any independent primary sources of energy to be well-separated from the heterogeneous object of interest.

B. Pulse-Echo Scan Configuration

Figure 11 illustrates the typical scan configuration underlying pulse-echo UI for the two-dimensional Euclidean space, i.e. $d = 2$. A lossy homogeneous fluid with the constant unperturbed values of the compressibility $\kappa_0 \in \mathbb{R}^+$ and the mass density $\rho_0 \in \mathbb{R}^+$ embeds a lossy heterogeneous object represented by the bounded open set $\Omega = \{ \mathbf{r} = (r_1, \ldots, r_d)^T \in \mathbb{R}^d : r_d > 0 \}$ (gray region) with the unperturbed compressibility $\kappa_1 : \Omega \mapsto \mathbb{R}^+$. These material parameters are stationary. The unperturbed compressibility at the position $\mathbf{r} \in \mathbb{R}^d$ thus reads

$\kappa_0(\mathbf{r}) = \begin{cases} \kappa_1(\mathbf{r}) & \text{for } \mathbf{r} \in \Omega, \\ \kappa_0 & \text{for } \mathbf{r} \notin \Omega, \end{cases}$  

(12)

and the associated relative spatial fluctuations in compressibility [7] vanish for $\mathbf{r} \notin \Omega$. The spatial amplitude absorption coefficient exhibits the power-law dependence on the temporal frequency $\zeta$ in the entire Euclidean space with the suitable tuple of absorption parameters $(0, b, \zeta)$.

The planar transducer array emits a broadband ultrasonic wave in the direction of the positive $r_d$-axis. This incident ultrasonic wave penetrates the heterogeneous object and interacts with the relative spatial fluctuations in compressibility [7]. These interactions induce the scattered ultrasonic wave, a portion of which propagates back to the planar transducer array and mechanically excites the faces of its physical elements.

The gray rectangle indicates the specified field of view (FOV) for the image recovery that is represented by the bounded set $\Omega_{\text{FOV}} = \{ \mathbf{r} = (r_1, \ldots, r_d)^T \in \mathbb{R}^d : r_d > 0 \}$.

1) Geometrical Specifications of the Transducer Array: The planar transducer array consists of $N_3 \in \mathbb{N}$ identical discrete

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\(^8\)Setting $K = k_0$, $\gamma_\omega(\mathbf{r}, \omega, \mathbf{r}) = -\gamma(\omega, \mathbf{r})$, and $\gamma_\mathbf{r}(\mathbf{r}, \omega, \mathbf{r}) = \gamma(\omega, \mathbf{r})$. The definition of the relative spatial fluctuation in mass density (3.3) is erroneous.

\(^9\)Setting $k = k_0$, $\gamma_\omega(\mathbf{r}, \omega, \mathbf{r}) = -\gamma(\omega, \mathbf{r})$, and $\gamma_\mathbf{r}(\mathbf{r}, \omega, \mathbf{r}) = \gamma(\omega, \mathbf{r})$. The sign of the divergence term on the right-hand side in (8.1.12)\(^9\) is erroneous and has to be inverted.

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2) Electromechanical Transfer Behavior of the Ultrasound Imaging System: The pulse-echo measurement process typically switches between two modes of operation. The transmission mode synthesizes various types of incident ultrasonic waves by exciting the individual physical elements of the planar transducer array by specified voltage signals. The reception mode records the individual RF voltage signals induced by the mechanical excitation of the physical elements’ vibrating faces by the scattered ultrasonic wave. Both modes of operation utilize specific electromechanical processing chains that share the connecting cables as common parts. This paper models these processing chains as single-input, single-output (SISO) linear time-shift invariant (LTI) systems (cf. [77, Sects. 9.1 and 9.2]) and describes their effect on the signals using the associated transfer functions.

The driving electrical circuits, the connecting cables, and the radiating physical elements of the planar transducer array constitute serial connections of linear two-port systems terminated at both ends [77, Fig. 9.6a]. A single SISO LTI system, whose input and output signals are the specified excitation voltage \( u_{m}^{(tx)} \) and the homogeneous \( r_{d}\)-component of the particle velocity \( v_{d,m} \) on the vibrating face specified by the compact set \( L_{m} \subset \mathbb{R}^{d-1} \), respectively, summarizes this interconnection for each physical transducer element. The associated transmitter electromechanical transfer functions \( h_{m}^{(tx)} \) provide the transfer relations

\[
v_{d,m} = h_{m}^{(tx)} u_{m}^{(tx)}
\]

for all physical transducer elements \( m \in [N_{el} - 1]_{0} \).

Similar to the electromechanical processing chains for the generation of ultrasonic waves, the receiving physical elements of the planar transducer array, the connecting cables, and the receiving electrical circuits constitute serial connections of linear two-port systems terminated at both ends [77, Fig. 9.13a]. A single SISO LTI system, whose input and output signals are the compressive blocked force \( F_{m} \) (physical unit: \( [F_{m}] = N m^{-3} \)) exerted on the blocked face of the physical transducer element specified by the compact set \( L_{m} \subset \mathbb{R}^{d-1} \) and the electrical RF voltage signal \( u_{m}^{(rx)} \) provided by the receiving amplification network, respectively, summarizes this interconnection for each physical transducer element [77, Fig. 9.14a]. The associated receiver electromechanical transfer functions \( h_{m}^{(rx)} \) provide the transfer relations

\[
u_{m}^{(rx)} = h_{m}^{(rx)} F_{m}
\]

for all physical transducer elements \( m \in [N_{el} - 1]_{0} \).

Although the transmitter and the receiver electromechanical transfer functions in the transfer relations (14) and (15) depend on the individual components of the serial connections, these dependencies are not required. The proposed linear physical model for the pulse-echo measurement process combines both types of transfer functions with additional acoustic transfer functions to form the electromechanical pulse-echo responses. Calibration setups with known acoustic transfer functions enable the experimental estimation of these responses [77, Sect. 9.4], [81].
C. Domain Integral Equation Formulation of the Proposed Wave Equation

1) Exact Formulation: The acoustic pressure field in the proposed PDE (11) may be considered as the sum of an incident acoustic pressure field \( p^{\text{inc}} \) and a scattered acoustic pressure field \( p^{\text{sc}} \), i.e. \( p(r) = p^{\text{inc}}(r) + p^{\text{sc}}(r) \) for all \( r \in \mathbb{R}^d \). The consideration of the Helmholtz equation \( \Delta \psi(r) = 0 \) satisfies the Helmholtz equation associated with the operator \( \Delta + \frac{k^2}{c^2} \). The absence of any relative spatial fluctuations in compressibility (7), i.e. \( \gamma^{(\kappa)}(r) = 0 \) for all \( r \in \mathbb{R}^d \), gives rise to the incident acoustic pressure field that satisfies the Helmholtz equation

\[
(\Delta + k^2) p^{\text{inc}}(r) = 0.
\]

This definition assumes that any radiating sources are well separated from the specified FOV [35, p. 232]. The presence of nonzero relative fluctuations in compressibility (7) induces the scattered acoustic pressure field. The insertion of the sum \( p = p^{\text{inc}} + p^{\text{sc}} \) into the proposed PDE (11), the consideration of the Helmholtz equation (16), and the formal treatment of the right-hand side as a source term yield the Lippmann-Schwinger (LS) integral equation [82, (8.13)], [35, (6.7a)], [64, (5.33)]

\[
p^{\text{sc}}(r) = G[p, \gamma^{(\kappa)}](r),
\]

where \( G \) denotes the direct scattering operator

\[
G[p, \gamma^{(\kappa)}](r) = \frac{k^2}{c^2} \int_{\Omega} \gamma^{(\kappa)}(r') p(r') g(r - r') \mathrm{d}r'.
\]

and \( g \) is the outgoing free-space Green’s function [A.57] associated with the operator \( \Delta + k^2 \). The Appendix summarizes the details concerning this Green’s function and states its exact form.

The LS integral equation (17) is a Fredholm equation of the second kind [37, p. 212] that enables the computation of the scattered acoustic pressure field in the d-dimensional Euclidean space from the induced source, i.e. the product of the acoustic pressure field and the scattering potential, inside the heterogeneous object \( \Omega \). As the proposed PDE (11), it defines a linear map from the incident acoustic pressure field to the scattered acoustic pressure field, but a nonlinear map from the relative spatial fluctuations in compressibility (7) to the scattered acoustic pressure field [35, pp. 230, 255]. This nonlinearity of the latter map renders the associated ISP, i.e. the determination of the unknown relative spatial fluctuations in compressibility (7) from the scattered acoustic pressure field, extraordinary difficult [35, pp. 230, 255]. The consideration of a weakly-scattering heterogeneous object, however, permits the linearization of the LS integral equation (17) by the first Born approximation and, consequently, the formulation of a linear ISP. This linearization dominates practical image recovery methods in UI and allows the systematic derivation of analytical inversion schemes. The resulting schemes for the linear ISP can then be generalized to include nonlinear effects [35, p. 256].

2) Linearization Based on the First Born Approximation: A Liouville-Neumann perturbation expansion, which is commonly known as the Born series, formally solves the LS integral equation (17) (cf. e.g. [35, p. 255], [37, p. 213], [83, p. 710], [88]). Its first two summands constitute the first Born approximation [35, (6.53)], [36, Sects. 3.3, 3.8.3, and 7.4.1], [37, Sect. 6.2.1], [83, p. 700], [76, pp. 413–414] given by

\[
\begin{align*}
p^{(\text{sc})}(r) &\approx p^{(B)}(r) = G[p^{\text{inc}}, \gamma^{(\kappa)}](r). \tag{18}
\end{align*}
\]

The first Born approximation (18) is a Fredholm integral equation of the first kind that substitutes the acoustic pressure field in the LS integral equation (17) by the incident acoustic pressure field. This substitution is valid if \( p(r) \approx p^{\text{inc}}(r) \), i.e. \( |p^{(\text{sc})}(r)| \ll |p^{\text{inc}}(r)| \) for all \( r \in \Omega \) [35, pp. 268, 287], [36, p. 47], [27, p. 212], [83, pp. 699, 700, 708], [84, (4)], [76, p. 413]. The validity of this condition requires the absolute value of the relative spatial fluctuations in compressibility (7) and the spatial extent of the heterogeneous object \( \Omega \) to be small [88]. If both aforementioned conditions are valid, the heterogeneous object is weakly-scattering [83, p. 708]. The described mechanism is referred to as single scattering [83, p. 709], i.e. the interaction of the incident acoustic pressure field with the relative spatial fluctuations in compressibility (7) exclusively induces all sources of the scattered acoustic pressure field. In contrast to the LS integral equation (17), however, the induced scattered acoustic pressure field does not interact with the relative spatial fluctuations in compressibility (7). Given the incident acoustic pressure field, the first Born approximation (18) thus defines a linear map from the relative spatial fluctuations in compressibility (7) to the scattered acoustic pressure field [35, p. 256].

D. First Born Approximation of the Received Radio Frequency Voltage Signals

The SISO LTI system models of the receiver electromechanical processing chains (cf. Subsect. III-B2) convert the compressive blocked forces exerted on the blocked faces of the physical transducer elements specified by the compact sets \( L_m \subset \mathbb{R}^{d-1} \) into the received RF voltage signals using the transfer relations (15). The first Born approximation of the scattered acoustic pressure field (18) enables the approximation of these compressive blocked forces as [71, (9.13)], [86, Sect. 5.2], [64, p. 299], [65, (12)], [80, (5a)]

\[
F_m \approx F_m^{(B)} = 2 \int_{L_m} \chi^{(m)}(r_{d-1}) p^{(B)}(r_{d-1}, 0) \mathrm{d}r_{d-1}, \tag{19}
\]

where the receiver apodization functions \( \chi^{(m)} : L_m \to \mathbb{C} \) account for the heterogeneous sensitivities as well as the time delays introduced by the acoustic lens and the outgoing free-space Green’s function [A.57] in combination with the factor of 2 models a rigid baffle. The successive insertions of the first Born approximations of the compressive blocked forces (19) and the scattered acoustic pressure field (18) into the transfer relations (15) approximate the received RF voltage signals as

\[
u_{m}^{(B)} \approx \nu_{m}^{(B)} = h_{m}^{(m)} F_m^{(B)} = \Phi_m \left[ p^{\text{inc}}, \gamma^{(\kappa)} \right]. \tag{20a}
\]
E. Pulse-Echo Measurement Process

The pulse-echo measurement process occurs in the time domain. The UI system sequentially performs $N_m \in \mathbb{N}$ pulse-echo measurements, which are identified by the superscript $n \in [N_m - 1]_0$. Given the incident acoustic pressure field $p_{\text{in}}^{(in,n)} : \mathbb{R}^d \rightarrow \mathbb{C}$ satisfying the Helmholtz equation (16), the observation operators (20b) map the relative spatial fluctuations in compressibility (7) to the first Born approximation of the received RF voltage signals (20) for all physical transducer elements $m \in [N_d - 1]_0$. The first stage applies the direct scattering operator (17b) to the relative spatial fluctuations in compressibility (7) to obtain the first Born approximation of the scattered acoustic pressure field (18). The second stage integrates this field over the face of the physical transducer element specified by the compact set $L_m$ to approximate the compressive blocked force (19). The third stage represents the transfer relation (15) and applies the receiver electromechanical transfer function $h_m^{(rx)}$ to the approximate compressive blocked force (19) to obtain the first Born approximation of the received RF voltage signals (20).

Fig. 2: Schematic representation of the proposed linear physical model for the pulse-echo measurement process by a linear signal processing chain in the temporal frequency domain. The UI system sequentially performs $N_m \in \mathbb{N}$ pulse-echo measurements identified by the superscript $n \in [N_m - 1]_0$. Given the incident acoustic pressure field $p_{\text{in}}^{(in,n)} : \mathbb{R}^d \rightarrow \mathbb{C}$ satisfying the Helmholtz equation (16), the observation operators (20b) map the relative spatial fluctuations in compressibility (7) to the first Born approximation of the received RF voltage signals (20) for all physical transducer elements $m \in [N_d - 1]_0$. The first stage applies the direct scattering operator (17b) to the relative spatial fluctuations in compressibility (7) to obtain the first Born approximation of the scattered acoustic pressure field (18). The second stage integrates this field over the face of the physical transducer element specified by the compact set $L_m$ to approximate the compressive blocked force (19). The third stage represents the transfer relation (15) and applies the receiver electromechanical transfer function $h_m^{(rx)}$ to the approximate compressive blocked force (19) to obtain the first Born approximation of the received RF voltage signals (20).
length of this interval, i.e. \( B_u = \| B_u \| = f_{ab} - f_{ub} \in \mathbb{R}^+ \), is referred to as the effective bandwidth. The presence of measurement noise additionally degrades the SNR of the received RF voltage signals \( (20) \) and prohibits their usage for the image recovery outside of the compact interval of temporal frequencies \( (22) \). This interval consequently identifies a limited number of discrete FOURIER coefficients relevant to the image recovery among the theoretically unlimited total number of discrete FOURIER coefficients. In the following, the bounds on the temporal frequency in the compact interval of temporal frequencies \( (22) \) are selected such that the corrupted RF voltage signals exhibit a sufficient SNR.

Multiple time-domain methods for the analog-to-digital conversion (ADC) of the received RF voltage signals permit the determination of the relevant FOURIER coefficients. These methods differ in their efficiency, i.e. the quotient of the data volume occupied by the quantized relevant FOURIER coefficients and the data volume digitized during the pulse-echo measurement. Both the conventional sampling and the in-phase and quadrature (IQ) sampling of the received RF voltage signals provide regular temporal samples that enable the computation of the relevant FOURIER coefficients via the discrete FOURIER transform (DFT). The sampling rate in the former method \( f_s = \frac{1}{T_s} \in \mathbb{R}^+ \) typically exceeds twice the signals’ upper frequency bound, i.e. \( f_s > 2f_{ub} \), whereas that in the latter method \( f_{s, IQ} \in \mathbb{R}^+ \) approximately equals the effective bandwidth \( B_u \), i.e. \( B_u < f_{s, IQ} \ll f_s \) at the expense of complex-valued samples \( (87) \). Alternatively, the combination of the multichannel sampling scheme \( (88) \) and the special hardware \( (89) \) enables the direct ADC of the relevant FOURIER coefficients from the received RF voltage signals. Chernenkova et al. \( (90) \) Sect. IV-B] present a thorough discussion of suitable acquisition methods.

IV. SYNTHESIS OF THE INCIDENT ULTRASONIC WAVE

The UI system synthesizes various types of incident ultrasonic waves by exciting the individual physical elements of the planar transducer array by specified voltage signals. The linearity of the HELMHOLTZ equation \( (16) \) in the incident acoustic pressure field validates this superposition of incident ultrasonic waves emitted by the individual elements if the respective boundary conditions add up consistently. The geometrical properties of the elements’ planar vibrating faces on the hyperplane \( r_d = 0 \) (cf. Subsect. III-B1) and the acoustic lens cause anisotropic directivity patterns that differ from the isotropic directivity pattern of a point source. This paper consequently refers to the ultrasonic wave emitted by an individual physical transducer element as a quasi-\(-(d - 1)\)-spherical wave to emphasize this difference with respect to the outgoing \((d - 1)\)-spherical wave induced by the point source. In the two-dimensional Euclidean space, i.e. \( d = 2 \), the paper specifically refers to the quasi-1-spherical wave as a quasi-cylindrical wave (QCW). The UI system typically provides multiple degrees of freedom in the specification of the exciting voltage signals. This paper proposes leveraging their significant effect on the synthesized incident ultrasonic wave and, consequently, the properties of the observation operator \( (20) \) to implement the incoherent observations required by CS (cf. Sect. III). The highly-probable satisfaction of the sufficient conditions for the instance optimal and the robust recovery of nearly-sparse representations from relatively few observations by the classes of random sensing matrices motivates the synthesis of random incident ultrasonic waves by the UI system. Although the underlying linear physical model for the pulse-echo measurement process always imposes structural constraints, the resulting structured random sensing matrices might be advantageous for image recovery.

The characterization of the observations necessitates solving the HELMHOLTZ equation \( (16) \) for the incident acoustic pressure field subject to the boundary conditions imposed by the scan configuration. These conditions include the planar transducer array consisting of the physical elements’ vibrating faces, which are excited by the specified voltage signals, and the embedding materials, e.g. the kerfs and the acoustic lens. The extraordinary complexity of their exact specification combined with a lack of boundary data, however, encourages their simplifying approximation by the half-space \( r_d > 0 \) in combination with idealized boundary conditions on the hyperplane \( r_d = 0 \). This simple geometry enables an explicit analytical representation of the incident acoustic pressure field by integrals over the specified boundary values. Assuming the planar vibrating face of each physical transducer element to be embedded in a rigid (acoustically hard) or soft baffle of infinite extent coinciding with the hyperplane \( r_d = 0 \), the RAYLEIGH-SOMMERFELD (RS) diffraction equations (cf. e.g. \( (3.2-78) \) and \( (3.2-84) \)) predict the velocity potentials representing the incident acoustic pressure fields. These predictions are used widely in UI to model the ultrasonic waves radiated from transducers of various shapes \( (92, (3.2-78) \) and \( (3.2-84) \)) predict the velocity potentials representing the incident acoustic pressure fields.

The transmitter electromechanical processing chains, which convert the specified voltage signals exciting the individual physical elements of the planar transducer array into the normal components of the vibrating faces’ velocities (cf. Subsect. III-B2), provide the required boundary waveforms. The associated transmitter electromechanical transfer functions account for the effective bandwidth of the UI system and limit the spatial frequencies of the incident acoustic pressure field. The resulting bandpass character enables its efficient sampling on a regular lattice. The first subsection Subsect. IV-A introduces the spatial velocity transfer function to compute the incident acoustic pressure field associated with the superposition of independent quasi-\(-(d - 1)\)-spherical waves.
The second subsection Subsect. [IV-B] introduces the four types of incident ultrasonic waves investigated in this paper and states the associated excitation voltages. The types include the deterministic steered quasi-plane wave (QPW), which approximates a conventional steered PW while accounting for the finite number of physical transducer elements and their anisotropic directivity patterns, as well as three types of random ultrasonic waves. In contrast to the deterministic steered QPW, whose observations serve as a reference, these random incident ultrasonic waves will be demonstrated to generate less coherent observations and provide advantages for CS-based fast image acquisition.

A. Superposition of Quasi-\((d-1)\)-Spherical Waves

The consideration of a lossless fluid, i.e. \(\alpha = 0\) and \(\beta = k_0 \in \mathbb{R}^+\), in the three-dimensional Euclidean space, i.e. \(d = 3\), enables the simplified representations of the RS diffraction equations for an individual physical transducer element as time-domain convolutions of the specified boundary waveform, i.e. either the acoustic pressure or the normal component of the vibrating face’s velocity, and the associated spatial impulse response [64, (2.36)], [56, Sect. 5.2], [94]. The spatial impulse response characterizes the anisotropic directivity pattern of the individual physical transducer element by relating the geometrical properties of its planar vibrating face (compact support: \(L_m \subset \mathbb{R}^{d-1}\), center coordinates: \(\mathbf{r}_{el,m} \in \mathcal{M}\)) to the time-dependent velocity potential observed at the position \(\mathbf{r}\) and induced by a temporally-impulsive boundary waveform [56, Sect. 5], [57, 24]. It exclusively depends on the relative position of both the observation position \(\mathbf{r}\) and the physical transducer element’s planar vibrating face [56, pp. 150, 152]. [94]. The efficient evaluation, which hinges on the time-domain equality of the outgoing free-space GREEN’s function [A.57] (cf. Sect. A) and the DIRAC delta distribution, however, fails for media exhibiting frequency-dependent absorption or the two-dimensional Euclidean space, i.e. \(d = 2\) [58, p. 247]. In addition, sharp discontinuities in the spatial impulse responses associated with bounded and nonapodized planar vibrating faces require special treatment by excessive temporal sampling rates, time-adapted evaluation, or integrated responses [56, 94].

This paper leverages the spatial transfer function, which equals the temporal FOURIER transform of the spatial impulse response, to circumvent the aforementioned shortcomings. This function reads

\[
\mathcal{Y}^{(tx)}_m(\mathbf{r}) = -2\int_{L_m} \chi^{(tx)}_m(\mathbf{r'})g(\mathbf{r}_{d-1} - \mathbf{r'}, r_d) d\mathbf{r'}
\]  

for a rigid baffle and \(r_d > 0\), where the transmitter apodization function \(\chi^{(tx)}_m : L_m \mapsto \mathbb{C}\) accounts for a nonuniform normal velocity and the outgoing free-space GREEN’s function [A.57] (cf. Sect. A) in combination with the factor of 2 models the rigid baffle. The spatial transfer function [23] is valid for both the two- and three-dimensional Euclidean spaces, i.e. \(d \in \{2, 3\}\). It exactly includes the combined effects of frequency-dependent absorption and dispersion described by the complex-valued wavenumber [9]. Its exclusive evaluation in the bounded interval of temporal frequencies associated with the effective bandwidth of the UI system additionally avoids the excessive temporal sampling rates required to approximate the discontinuities in the spatial impulse response. The angular spectrum approach [58 Sect. 6.7], [35 Sects. 3.2.3 and 4.3], [64 Sect. 2.3] permits its efficient evaluation.

Identifying the spatial transfer function [24] in the RS diffraction equation for the rigid baffle (cf. e.g. [35, (2.48)], [91 (3-38)], [83 (13) of §8.11], and ), the quasi-\((d-1)\)-spherical wave emitted by the individual physical transducer element with the index \(m \in [N_{el} - 1]_0\), the quasi-\((d-1)\)-spherical wave emitted by the individual physical transducer element with the index \(m \in [N_{el} - 1]_0\) is represented by the incident acoustic pressure field

\[
p^{(in)}(\mathbf{r}, L_m) = j\omega \rho_0 v_{m,d} \mathcal{Y}^{(tx)}_m(\mathbf{r}),
\]

where \(\rho_0\) denotes the homogeneous unperturbed mass density and \(v_{m,d}\) indicates the uniform \(r_d\)-component of the particle velocity \(\mathbf{v}_m\).

Inserting the uniform \(r_d\)-component of the particle velocity [14] and assuming identical transmitter electromechanical transfer functions for each physical element, i.e. \(h^{(tx)}_m = h^{(tx)}_1\) for all \(m \in [N_{el} - 1]_0\), the superposition of the \(N_{el}\) independent quasi-\((d-1)\)-spherical waves represented by the individual incident acoustic pressure fields [25] results in the incident acoustic pressure field generated by the entire planar transducer array

\[
p^{(in)}(\mathbf{r}) = j\omega \rho_0 \sum_{m=0}^{N_{el}-1} u^{(tx)}_m \mathcal{Y}^{(tx)}_m(\mathbf{r}).
\]

The neglect of variations in the transmitter electromechanical transfer functions for each physical element, which is demonstrated to be accurate for three commercial linear arrays [77, Sect. 10.2], [81].

B. Types of Incident Ultrasonic Waves

The specification of the excitation voltages in the incident acoustic pressure field [25] permits the synthesis of various types of incident ultrasonic waves. Modern digital UI systems typically weight and delay a common reference voltage signal to produce the individual excitation voltages. The digital representation of the associated weights and time delays, however, necessitates their quantization and prohibits arbitrary values. This paper thus specifies excitation voltages of the general form

\[
u^{(tx)}_m = u^{(tx)}_m a_m e^{-j\omega Q (\Delta t_m)},
\]

for all \(m \in [N_{el} - 1]_0\), where \(u^{(tx)}_m \in \mathbb{C}\) is the amplitude at the angular temporal frequency \(\omega\) and \(Q\) denotes an operator accounting for the quantization of the nonnegative time delays \(\Delta t_m \in \mathbb{R}^+_\). The time delays correspond to the multiplications by the complex exponential functions and induce phase shifts with a linear frequency dependence. For a clock signal with the temporal frequency \(f_{clk} = T_{clk}^{-1} \in \mathbb{R}^+\), the admissible time delays read

\[
Q (\Delta t_m) = \lceil \Delta t_m f_{clk} \rceil T_{clk},
\]

where the asymmetric brackets indicate the rounding to the nearest integer.
The following subsections introduce the four types of investigated incident ultrasonic waves and specify the associated combinations of apodization weights and time delays in the excitation voltages (26). The four types of investigated incident ultrasonic waves are the deterministic steered QPW and three random superpositions of quasi-(d - 1)-spherical waves using random apodization weights, random time delays, and combinations of both random apodization weights and random time delays.

1) Steered Quasi-Plane Wave: Practical UI approximates steered PWs to leverage their simplifying properties introduced in Subsect. 1A. The approximation, which will be referred to as steered quasi-plane wave (QPW), combines apodization weights of unity with time delays \( \Delta t_m \) depending affinely on the center coordinates of the physical transducer elements \( r_{el,m} \) on each coordinate axis. The apodization weights and the time delays in the excitation voltages (26) read

\[
a_m = 1 \quad \text{and} \quad \Delta t_m (e_\theta) = \frac{(r_{el,m} - r_{ref, e_\theta})}{e_{ref}} \quad (27a)
\]

for all \( m \in [N_{el} - 1] \), where \( e_\theta = (e_{\theta,1}, \ldots, e_{\theta,d})^T \in S^d \) denotes the unit vector indicating the preferred direction of propagation and \( r_{ref} = (r_{ref,1}, \ldots, r_{ref,d-1}, 0)^T \) is the reference position with the components

\[
r_{ref,l} = \begin{cases} -N_{el}^{-1} \Delta r_{el,l} & \text{for } e_{\theta,l} \geq 0, \\ N_{el}^{-1} \Delta r_{el,l} & \text{for } e_{\theta,l} < 0, \end{cases} \quad (27b)
\]

for all \( l \in [d - 1] \). The nonzero reference position ensures the nonnegativity of the time delays in (27a) and, consequently, the causality of the excitation voltages in the time domain.

Figure 3 illustrates the emission of a steered QPW by a linear transducer array consisting of \( N_{el} \) discrete physical elements in the two-dimensional Euclidean space, i.e. \( d = 2 \). The gray semicircles represent the individual QCWs emitted by each physical transducer element. These QCWs propagate into the half-plane \( r_2 > 0 \) and reflect the anisotropic directivity patterns of the elements. The dashed gray line indicates the planar wavefront approximated by the superposition of the QCWs. This approximation is only accurate for a bounded area in front of the array. The black arrow shows the preferred direction of propagation \( e_\theta = (\cos(\vartheta), \sin(\vartheta))^T \in S^1_+ \) with \( \vartheta = 3^{-1} \pi \). Its positive \( r_1 \)-component \( e_{\theta,1} = \cos(\vartheta) > 0 \) requires the reference position with the components (27b) to equal \( r_{ref} = r_{el,0,1} e_1 \). Figure 5(a) displays the incident acoustic pressure field (25) obtained from numerical calculations for the excitation voltages (26) with the apodization weights and the time delays (27). The preferred direction of propagation equals \( e_2 \).

2) Superposition of Randomly-Apodized Quasi-(d - 1)-Spherical Waves: The first type of random incident ultrasonic wave leverages realizations of i.i.d. random variables \( \Phi_m \) as the apodization weights in the excitation voltages (26), whereas all time delays equal zero. These specifications yield

\[
a_m = \Phi_m \quad \text{and} \quad \Delta t_m = 0 \quad (28a)
\]

for all \( m \in [N_{el} - 1] \). The choice of the Bernoulli distribution

\[
\Phi_m = \begin{cases} 1 & \text{with probability 0.5}, \\ -1 & \text{with probability 0.5}, \end{cases} \quad (28b)
\]

exclusively requires inverting buffer amplifiers to generate the excitation voltages and, consequently, enables a relatively simple hardware realization within the digital logic of the UI system. It additionally ensures a constant transmitted acoustic energy, which equals that of a steered QPW. Figure 5(b) displays the incident acoustic pressure field (25) obtained from numerical calculations for the excitation voltages (26) with the apodization weights and the time delays (28).

Contrasting the proposed choice of the Bernoulli distribution, the SA method devised by LIU et al. [40, 41] relies on the continuous uniform distribution for the apodization weights in (28a), i.e. \( \Phi_m \sim \mathcal{U}[0; 1] \). This choice, however, increases the complexity of the transmit hardware, requires approximations within the digital logic of the UI system, and does not ensure wave emissions of constant acoustic energy. Moreover, the authors misleadingly refer to the resulting superposition of randomly-apodized quasi-(d - 1)-spherical waves as “diverging wave” [40] and “randomly apodized plane wave” [41]. The deviation of the resulting incident ultrasonic wave from these conventional waves, which becomes manifest in erratic properties of the associated incident acoustic pressure field, however, will be identified as a crucial advantage of the random apodization for CS in Subsect. VII-B1.

10Assuming that the random variables \( \Phi_m \) are i.i.d., which is not explicitly specified by the authors.
3) Superposition of Randomly-Delayed Quasi-$(d - 1)$-Spherical Waves: The second type of random incident ultrasonic wave leverages realizations of random permutations for the time delays in the excitation voltages \((26)\), whereas all apodization weights equal unity. These specifications yield \(a_m = 1\) and \(\Delta t_m = \Pi_m([N_d - 1]_0)T_{inc}\) \((29a)\) for all \(m \in [N_d - 1]_0\), where \(\Pi_m([N_d - 1]_0)\) denotes the element of index \(m\) in a random permutation of the set \([N_d - 1]_0\) and \(T_{inc} \in \mathbb{R}^+\) is a fixed time interval. The assignment rule for the time delays in \((29a)\) first specifies \(N_d\) distinct time instants by uniformly sampling the time axis based on the fixed time interval \(T_{inc}\) and starting at \(t = 0\). The uniform spacing and the distinctness of the specified time instants ensure the sequential emission of \(N_d\) quasi-\((d - 1)\)-spherical waves by the planar transducer array in uniform time intervals. The random permutation then uniquely assigns each distinct time instant to a transmitting physical transducer element. This assignment resembles the multiplexing in a shared-medium network based on a synchronous time-division multiple access (TDMA) scheme allocating time slots to multiple users.

The specification of the fixed time interval \(T_{inc}\) influences the properties of the resulting superposition and, consequently, the sensing matrix. In the limiting case \(T_{inc} \to 0^+\), the superposition of randomly-delayed quasi-\((d - 1)\)-spherical waves approximates the steered QPW with the preferred direction of propagation \(e_\theta\), because the time delays in \((29a)\) converge to zero and equal those in \((27)\). A fixed time interval \(T_{inc}\) shorter than the maximum time-of-flight of the incident and scattered ultrasonic waves induces range ambiguities that are undesired in conventional UI methods but can be resolved by the proposed CS-based method for fast image acquisition. The fixed time interval \(T_{inc}(e_\theta) = \frac{1}{N_d - 1} \max_{m \in [N_d - 1]_0} \left\{ \frac{(r_{el,m} - r_{ref}, e_\theta)}{c_{ref}} \right\}\) \((29b)\) ensures a total time duration of the pulse-echo measurement process that is identical to that of the steered QPW with the time delays specified in \((27)\). In the two-dimensional Euclidean space, i.e. \(d = 2\), the fixed time interval \((29b)\) simplifies to \(T_{inc}(e_\theta) = c_{ref}^{-1}\Delta r_{el,1}|e_{\theta,1}|\) and induces a random permutation of the time delays specified for the steered QPW in \((27)\). The superposition of randomly-delayed QCWs thus equals a steered QPW with randomly-permuted time delays. Any UI system able to synthesize a steered QPW possesses the capabilities to synthesize this superposition of randomly-delayed QCWs. Since the choice of a time interval \(T_{inc}\) equal to or exceeding the duration of the recorded RF voltage signals results in the SA acquisition scheme \((101)\), the superposition of randomly-delayed quasi-\((d - 1)\)-spherical waves may be interpreted as an accelerated and, consequently, time-compressed SA imaging method. Figure \(5(c)\) displays the incident acoustic pressure field \((25)\) obtained from numerical calculations for the excitation voltages \((26)\) with the apodization weights and the time delays \((29)\).

4) Superposition of Randomly-Apodized and Randomly-Delayed Quasi-$(d - 1)$-Spherical Waves: The third type of random incident ultrasonic wave combines realizations of i.i.d. random variables \(\Phi_m\) as the apodization weights with realizations of random permutations for the time delays in the excitation voltages \((26)\). These specifications yield \(a_m = \Phi_m\) and \(\Delta t_m = \Pi_m([N_d - 1]_0)T_{inc}\) \((30)\) for all \(m \in [N_d - 1]_0\), where the Bernoulli distribution \((28b)\) is combined with the fixed time interval \((29b)\).

Figure \(4\) illustrates the emission of a superposition of both randomly-apodized and randomly-delayed quasi-cylindrical waves (QCWs) by a linear transducer array consisting of \(N_d \in \mathbb{N}\) discrete physical elements in the two-dimensional Euclidean space, i.e. \(d = 2\). The gray semicircles represent the individual QCWs emitted by each physical transducer element. These QCWs propagate into the half-plane \(r_2 > 0\) and reflect the anisotropic directivity patterns of the elements. The dashed semicircles indicate negatively-apodized QCWs, i.e. \(a_m = -1\) in \((30)\), whereas the solid semicircles indicate positively-apodized QCWs, i.e. \(a_m = 1\) in \((30)\). Their distinct radii reflect the random time delays in \((30)\). The fixed time interval \((29b)\) simplifies to \(T_{inc}(e_\theta) = c_{ref}^{-1}\Delta r_{el,1}\cos(\vartheta)\) for \(e_\theta = (\cos(\vartheta), \sin(\vartheta))^T\) with \(\vartheta = 3^{-1}\pi\) and induces a random permutation of the time delays specified for the steered QPW in \((27)\), which are illustrated in Fig. \(3\). The maximum time delay is \(\Delta t_{max} = \max_{m \in [N_d - 1]_0} \{\Delta t_m\} = (N_d - 1)T_{inc}(e_\theta)\). The superposition
of these QCWs results in a random wavefront that does not resemble any commonly used model. Figure S(d) displays the incident acoustic pressure field $P_{\text{inc}}$ obtained from numerical calculations for the excitation voltages $V_{\text{mod}}$ with the apodisation weights and the time delays $\tau_{\text{th}}$.

V. NOVEL METHOD FOR FAST IMAGE ACQUISITION BASED ON COMPRESSED SENSING

A. Discretization of the Linear Physical Model

The numerical computations underlying this paper require a suitable discretization of the linear physical model for the pulse-echo measurement process proposed in Sect. III. Specifically, the surface integrals in the spatial transfer functions (20c) and (23), the volume integral in the first BORN approximation of the observation operator (20b), as well as the temporal frequency axis have to be discretized. The discretization proposed in the following exploits the finite effective bandwidth of the received RF voltage signals to reduce the number of discrete points and, consequently, the computational costs.

1) Spatial Discretization of the Transducer Array: A set of point-like mathematical elements emulates the FOV containing the lossy heterogeneous object. For $l \in [d-1]$, let $N_{\text{lat},l} \in \mathbb{N}$ be the number of point-like mathematical elements per vibrating face along the $l$-th coordinate axis and $\Delta r_l \in \mathbb{R}^+$ be the constant spacing between the adjacent point-like mathematical elements along this axis. Generalizing the spatial discretization proposed in [1] for the two-dimensional Euclidean space, i.e., $d = 2$, to its $d$-dimensional version, the set of discrete positions

$$V_m = \left\{ r_{\text{mat},\nu}^{(m)} \in \mathbb{R}^d : \right\},$$

$$r_{\text{mat},\nu}^{(m)} = r_{\text{el},m} + \sum_{l=1}^{d-1} \left( \nu_l - \frac{N_{\text{int},l} - 1}{2} \right) \Delta r_{\text{mat},l} e_l,$$

$$\nu_l \in [N_{\text{lat},l} - 1],$$

where $r_{\text{el},m} \in \mathcal{M}$ are the center coordinates specified in (13), represents each vibrating face $L_m$, $m \in [N_{\text{el}} - 1]$. The total number of point-like mathematical elements per vibrating face equals the product $N_{\text{int}} = \prod_{l=1}^{d-1} N_{\text{int},l}$.

Defining the $(d-1)$-dimensional surface element $\Delta A = \prod_{l=1}^{d-1} \Delta r_{\text{mat},l}$, the receiver and the transmitter apodization functions in the spatial transfer functions (20c) and (23) are discretized as

$$\chi^{(m)}_{\text{m},r}(r_{d-1}) = \Delta A \sum_{\nu = 0}^{N_{\text{int},d-1} - 1} \chi^{(m)}_{\text{m},\nu}(r_{d-1} - r_{\text{mat},\nu,d-1}),$$

$$\chi^{(m)}_{\text{m},r}(r_{d-1}) = \Delta A \sum_{\nu = 0}^{N_{\text{int},d-1} - 1} \chi^{(m)}_{\text{m},\nu}(r_{d-1} - r_{\text{mat},\nu,d-1}),$$

where $\chi^{(m)}_{\text{m},\nu} = \chi_{\text{m},\nu}^{(m)}[F_{\text{mat},\nu,d-1}]$ and $\chi^{(m)}_{\text{m},\nu} = \chi_{\text{m},\nu}^{(m)}[F_{\text{mat},\nu,d-1}]$ denote their regular samples for all $m \in [N_{\text{el}} - 1]$ and all $\nu \in [N_{\text{int}} - 1]$.  

2) Spatial Discretization of the Field of View: A set of point-like scatterers emulates the FOV containing the lossy heterogeneous object. For $l \in [d-1]$, let $N_{\text{lat},l} \in \mathbb{N}$ be the number of lattice points along the $l$-th coordinate axis and $\Delta r_l \in \mathbb{R}^+$ be the constant spacing between the adjacent lattice points along this axis. In addition, let $r_{\text{lat}} = (r_{0,1}, \ldots, r_{0,d})^T \in \mathbb{R}^{d-1} \times \mathbb{R}^+$ denote an arbitrary offset vector. The set of discrete positions

$$\mathcal{L} = \left\{ r_i \in \mathbb{R}^d : r_i = r_0 + \sum_{l=1}^{d-1} i_l \Delta r_l e_l, \right\}$$

$$i_l \in [N_{\text{lat},l} - 1], i = \sum_{l=1}^{d-1} i_l \prod_{k=l+1}^{d} N_{\text{lat},k} + i_d$$

represents the FOV containing the lossy heterogeneous object. The total number of lattice points equals the product

$$N_{\text{lat}} = |\mathcal{L}| = \prod_{l=1}^{d-1} N_{\text{lat},l}.$$

Defining the $d$-dimensional volume element $\Delta V = \prod_{l=1}^{d} \Delta r_l$, the relative spatial fluctuations in compressibility (7) are discretized as

$$\gamma^{(\nu)}(r) = \Delta V \sum_{i=0}^{N_{\text{lat}}-1} \gamma^{(\nu)}(r_i) \delta(r - r_i),$$

where $\gamma^{(\nu)}(r_i)$ denote the regular samples for all $i \in [N_{\text{lat}} - 1]$.  

3) Discretization of the Temporal Frequency Axis: The boundedness of the common observation time intervals (21) enables the representation of the received RF voltage signals in the time domain by the FOURIER series (102) (2.2.1)]

$$\tilde{u}_{m,n}(t) = \sum_{l=-\infty}^{\infty} \tilde{u}_{m,l} e^{j\omega_l t}$$

$$= u_{m,0} + 2\text{Re}\left\{ \sum_{l=1}^{\infty} \tilde{u}_{m,l} e^{j\omega_l t} \right\}$$

with the discrete angular temporal frequencies $\omega_l = 2\pi f_l = 2\pi f \in \mathbb{Z}$ and the coefficients (102) (2.2.2)]

$$u_{m,n} = \frac{1}{T_{\text{obs}}} \int_{-T_{\text{obs}}}^{T_{\text{obs}}} \tilde{u}_{m,n}(t) e^{-j\omega_l t} dt.$$  

The second identity in the sum (35a) exploits the conjugate symmetry of the FOURIER coefficients (35b) for real-valued signals and, consequently, requires only nonnegative frequency indices.

The finite effective bandwidth of the received RF voltage signals additionally enables the truncation of the FOURIER series (35) with respect to the compact interval of temporal frequencies (22). The resulting sets of relevant discrete temporal frequencies read

$$F_{\text{h}}^{(n)} = \left\{ f_l \in \mathbb{R}^+ : f_l = \frac{l}{T_{\text{obs}}} \in \mathbb{L}_{\text{h}}^{(n)} \right\}$$

for all $n \in [N_{\text{h}} - 1]$, where the sets of admissible frequency
indices are
\[ I_{n,\text{BP}} = \left\{ l \in \mathbb{Z} : l_{\text{lh}}^{(n)} \leq l \leq l_{\text{ub}}^{(n)} \right\} \]  
(36b)
with the lower and the upper bounds
\[ l_{\text{lh}}^{(n)} = \left\lfloor T_{\text{obs}}^{(n)} f_{\text{lh}} \right\rfloor \quad \text{and} \quad l_{\text{ub}}^{(n)} = \left\lceil T_{\text{obs}}^{(n)} f_{\text{ub}} \right\rceil, \]  
(36c)
respectively. The numbers of relevant discrete temporal frequencies equal \( N_{\text{f, BP}}^{(n)} = |I_{n,\text{BP}}| = l_{\text{ub}}^{(n)} - l_{\text{lh}}^{(n)} + 1 \) for all \( n \in [N_{l, \text{lat}} - 1]_0 \). The effective time-bandwidth products approximate these numbers, i.e., \( N_{\text{f, BP}}^{(n)} \approx T_{\text{obs}}^{(n)} B_u \). For each sequential wave emission, the time-domain representation of the received RF voltage signals by the FOURIER series requires \( N_{\text{el}} N_{\text{f, BP}}^{(n)} \) complex-valued FOURIER coefficients.

**B. Systems of Linear Algebraic Equations Obtained from the First BORN Approximation**

Let the subscript \( l \in I_{n,\text{BP}} \) indicate the admissible frequency indices in the sets of relevant discrete temporal frequencies \( I_{n,\text{BP}} \) in the following. Inserting the discretized receiver apodization functions \( (32a) \) and the discretized relative spatial fluctuations in compressibility \( (34) \) into the first BORN approximation of the received RF voltage signals \( (25) \) yields
\[ u_{m, l}^{(B, n)} \approx -k_b^2 \Delta V h_{m, l}^{(n)} \sum_{i=0}^{N_{l, \text{el}}} \gamma_i^{(\nu)} p_i^{(n, m)}(r_i) \gamma_i^{(n, l)}(r_i) \]
\[ \approx 2k_b^2 \Delta A \Delta V h_{m, l}^{(n)} \sum_{i=0}^{N_{l, \text{el}}} \gamma_i^{(\nu)} p_i^{(n, m)}(r_i) \]
\[ \times \sum_{\nu=0}^{N_{l, \text{el}} - 1} \chi_{m, \nu, l}(r_i) \left[ f_{\text{mat}, \nu} \right] (r_i) \]  
(37)
for all sequential wave emissions \( n \in [N_{l, \text{el}} - 1]_0 \), all physical transducer elements \( m \in [N_{l, \text{el}} - 1]_0 \), and all relevant discrete temporal frequencies \( l \in I_{n,\text{BP}} \).

Aggregating the regular samples in the discretized relative spatial fluctuations in compressibility \( (34) \) for all lattice points \( i \in [N_{l, \text{lat}} - 1]_0 \) into the complex-valued \( N_{\text{lat}} \times 1 \) vector
\[ \gamma^{(n)} = \left[ \gamma_0^{(n)} \ldots \gamma_{N_{l, \text{el}} - 1}^{(n)} \right]^T \]  
(38)
and the received RF voltage signals \( (37) \) for all physical transducer elements \( m \in [N_{l, \text{el}} - 1]_0 \) into the complex-valued \( N_{\text{el}} \times 1 \) vector
\[ u_{l}^{(B, n)} = \left[ u_{0, l}^{(B, n)} \ldots u_{N_{l, \text{el}} - 1, l}^{(B, n)} \right]^T, \]  
(39a)
the complex-valued \( N_{\text{el}} \times N_{\text{lat}} \) matrix \( \Phi_l^{(\nu, n)} \) with the entries
\[ \phi_{m, i, j}^{(\nu, n)} = 2k_b^2 \Delta A \Delta V h_{m, l}^{(n)} \sum_{\nu=0}^{N_{l, \text{el}} - 1} \chi_{m, \nu, l}(r_i) \left[ f_{\text{mat}, \nu} \right] (r_i) \]  
(39b)
for all \( m \in [N_{l, \text{el}} - 1]_0 \) and all \( i \in [N_{l, \text{lat}} - 1]_0 \) defines the partial system of linear algebraic equations
\[ u_{l}^{(B, n)} = \Phi_l^{(\nu, n)} \gamma^{(n)} \]  
(39c)
for all sequential wave emissions \( n \in [N_{l, \text{el}} - 1]_0 \) and all relevant discrete temporal frequencies \( l \in I_{n,\text{BP}} \). The individual rows in the matrix \( (39b) \) discretize the observation operator \( (20) \), whose schematic representation as a linear signal processing chain is illustrated in Fig. 2 for each physical transducer element. The slow spatial decay of the outgoing free-space GREEN’s function \( (A.57) \) induces a dense population of the matrix \( (39b) \) \( (103), (104) \).

The vertical stacking of the partial systems of linear algebraic equations \( (39) \) for all relevant discrete temporal frequencies \( l \in I_{n,\text{BP}} \) and all sequential wave emissions \( n \in [N_{l, \text{el}} - 1]_0 \) results in the complete system of linear algebraic equations.

The total number of observations in this system equals
\[ N_{\text{obs}} = N_{\text{el}} \sum_{n=0}^{N_{l, \text{el}} - 1} N_{\text{f, BP}}^{(n)}, \]  
(40)
the vertical stacking yields the complex-valued \( N_{\text{obs}} \times 1 \) vector of received RF voltage signals
\[ u^{(B)} = \left[ u_{0, l}^{(B, 0)} \ldots u_{N_{l, \text{el}} - 1, l}^{(B, N_{l, \text{el}} - 1)} \right] \]  
(41a)
the complex-valued \( N_{\text{obs}} \times N_{\text{lat}} \) matrix representing the complete observation process
\[ \Phi^{(\nu, n)} = \left[ \phi_{0, l}^{(\nu, n)} \ldots \phi_{N_{l, \text{el}} - 1, l}^{(\nu, n)} \right] \]  
(41b)
and the complete system of linear algebraic equations
\[ u^{(B)} = \Phi^{(\nu, n)} \gamma^{(n)}. \]  
(41c)

The complete system of linear algebraic equations \( (41) \) resembles the linear system relating the observations to the unknown vector to be recovered \( (??) \) in the idealized CS problem (cf. ??). The observation process \( (41b) \) includes the entirety of physical transducer elements and the relevant discrete temporal frequencies for all sequential wave emissions. It accounts for the physical laws of ultrasonic wave propagation and scattering within the first BORN approximation, the instrumentation, i.e., the geometrical specifications of the planar transducer array and the electromechanical transfer behavior of the UI system, as well as the imaging parameters. The dense population of the matrix \( (39b) \) transfers to the observation process \( (41b) \). The total number of observations \( (40) \) exclusively depends on the total number of physical transducer elements \( N_{\text{el}} \), the effective time-bandwidth products \( N_{\text{f, BP}}^{(n)} \approx T_{\text{obs}}^{(n)} B_u \), and the number of sequential wave emissions \( N_{l, \text{el}} \). Note that other quantities such as the temporal sampling rate \( f_s \) or the total number of point-like mathematical elements per vibrating face \( N_{\text{lat}} \) do not affect this number. In a fixed scan configuration, the specifications of the incident ultrasonic waves are the only degrees of freedom to influence the properties of the observation process \( (41b) \) and implement the incoherent observations required by CS (cf. Sect. II). The significant effect of the four types of
incident ultrasonic waves introduced in Subsect. [V-B] on the observation process (41b) will be investigated in Sect. [VI].

The linear physical model for the pulse-echo measurement process, however, will also be shown to impose limitations on the implementation of the requirements of CS.

C. Computation of the Incident Acoustic Pressure Field

Inserting the discretized transmitter apodization functions (32b) into the incident acoustic pressure field generated by the entire planar transducer array (25) yields

\[ p_i^{(in,n)}(r_i) = j\omega_0 p_0 h_i^{(tx)} \sum_{m=0}^{N_l-1} u_{m,l}^{(tx,n)} \gamma_{m,l}^{(tx)}(r_i) \approx -j2\omega_0 p_0 A h_i^{(tx)} \sum_{m=0}^{N_l-1} u_{m,l}^{(tx,n)} \chi_{m,v} \delta_{r_i,r_{lat,m,v}} \]

for all lattice points \( i \in [N_{lat} - 1]_0 \), all relevant discrete temporal frequencies \( l \in \mathbb{L}_{BP} \), and all sequential wave emissions \( n \in [N_l - 1]_0 \), where the general form of the excitation voltages (26a) was adapted to

\[ u_{m,l}^{(tx,n)} = u_{m,l}^{(tx,n)} a_{m,n} e^{-j\omega_0 Q(\Delta t_m^{(n)})} \]

for all physical transducer elements \( m \in [N_{el} - 1]_0 \).

The validity of this discretization requires the discrete positions \( r_i \in \mathcal{L} \) in the regular lattice (33) to exceed a minimum distance from the positions of the point-like mathematical elements \( r_{lat,m,v} \in \mathcal{V}_{lat} \) specified in the sets (31) representing the vibrating faces \( L_m \), where the outgoing free-space Green’s function (A.57) becomes singular (77 p. 27). The instances of the regular lattice (33) investigated in this paper, however, circumvent this problem.

In a fixed scan configuration, the synthesis of the incident ultrasonic wave by the UI system exclusively provides the degrees of freedom to influence the properties of the observation process (41b) and implement the incoherent observations required by CS (cf. Sect. [II]).

D. Regularization of the Discretized Linear Inverse Scattering Problem

The Fredholm integral equation of the first kind representing the first Born approximation of the scattered acoustic pressure field (13) causes the complete system of linear algebraic equations (41) to be ill-conditioned. Moreover, the sequential emission of only a few ultrasonic waves per image in fast pulse-echo UI typically renders this system underdetermined (cf. Subsect. [VII-A]). Both properties prevent the direct image recovery, i.e. the recovery of the unknown vector associated with the relative spatial fluctuations in compressibility (38) from the vector of received RF voltage signals (41a). The reformulation of the image recovery as a CS problem (5) will, however, circumvent these difficulties. The sparsity-promoting \( \ell_q \)-minimization method (\( P_{\kappa,\eta} \)) guarantees the stable recovery of a nearly-sparse representation of the unknown vector associated with the relative spatial fluctuations in compressibility (38) provided that the sufficient conditions on the sensing matrix (3) are met.

Postulating the existence of a suitable orthonormal basis of \( \mathbb{C}^{N_{lat}} \), which is represented by the complex-valued unitary \( N_{lat} \times N_{lat} \) matrix \( \Phi \), the complex-valued \( N_{lat} \times 1 \) vector of transform coefficients

\[ \theta^{(\kappa)} = \Phi^{H} \gamma^{(\kappa)} \]

is a nearly-sparse representation of the vector associated with the relative spatial fluctuations in compressibility (38). Inserting the nearly-sparse coefficient vector (43), defining the complex-valued \( N_{obs} \times N_{lat} \) sensing matrix

\[ \Lambda [P^{(in)}] = \Phi [P^{(in)}] \Psi, \]

and adding the complex-valued \( N_{obs} \times 1 \) vector of observation errors with \( ||\gamma||_2 < \eta \), the complete system of linear algebraic equations (21) becomes

\[ u^{(B,n)} = \Phi [P^{(in)}] \theta^{(\kappa)} + \eta = \Lambda [P^{(in)}] \theta^{(\kappa)} + \eta. \]

Depending on the choice of the orthonormal basis (43), the dense population of the observation process (41b) transfers to the sensing matrix (44). The additive observation errors reflect the limited precision, e.g. effects of thermal noise, and the limited resolution, e.g. quantization errors, of the measurement devices providing the voltage measurements in the UI system. They further account for the inaccuracy of the linear physical model underlying the observation process (41b) and predicting the observations.

The column vectors \( a_i \in \mathbb{C}^{N_{lat}}, i \in [N_{lat}] \), equal the received RF voltage signals (41a) induced by the individual components in the nearly-sparse coefficient vector (43). The squared \( \ell_2 \)-norms of the column vectors reflect the energy in the received RF voltage signals. They characterize the transfer behavior of the sensing matrix (44) for the individual components in the nearly-sparse coefficient vector (43) induced by the underlying linear physical model for the pulse-echo measurement process. These \( \ell_2 \)-norms vary significantly and their dynamic range depends on the exact choice of the orthonormal basis. The sensing matrix (44) thus potentially violates the aforementioned sufficient conditions. The sparsity-promoting recovery methods preferentially recover those components in the nearly-sparse coefficient vector (43) that have the strongest impact on the \( \ell_2 \)-error in the specified data fidelity constraint.

In the canonical basis, i.e. \( \Psi = I \), for example, the samples of the relative spatial fluctuations in compressibility located on the lattice points near the physical transducer elements induce observations of relatively high energy, owing to the increasing effects of diffraction and absorption on the ultrasonic waves with propagation distance. The promotion of a sparse solution may thus erroneously neglect significant samples located on distant lattice points.

The normalization of each column vector in the sensing matrix (44) by its \( \ell_2 \)-norm generates the normalized sensing matrix \( \bar{A} \) whose column vectors \( \bar{a}_i = \| a_i \|_2^{-1} a_i, i \in [N_{lat}] \), exhibit \( \ell_2 \)-norms of unity, i.e. \( \| \bar{a}_i \|_2 = 1 \) for all \( i \in [N_{lat}] \), and
eliminate this erroneous neglect in the absence of observation errors. The inverse weighting matrices may be interpreted as inverse filters that compensate the loss of energy in the transfer behavior of the sensing matrix \( A \) for the individual components in the nearly-sparse coefficient vector \( A \). This compensation, however, amplifies the additive observation errors and increases the recovery errors if the dynamic range of the column vectors’ \( \ell_2 \)-norms is high. The imposition of a hard threshold on the \( \ell_2 \)-norms of the column vectors mitigates this amplification, similar to a pseudo-inverse filter. The SNR of the received RF voltage signals dictates the permissible threshold. The simulation study in Sect. VII demonstrates that the thresholded normalized sensing matrix reduces the recovery errors and increases the speed of convergence of the recovery algorithms using numerical calculations for the Fourier basis.

Let the factor \( \xi \in (0; 1] \) specify the lower bound on the \( \ell_2 \)-norms of the sensing matrix’s column vectors

\[
a_{i,\text{lb}}(\xi) = \xi \max_{i \in [N_{\text{lat}}]} \{ ||a_i||_2 \}.
\]

(46)

The thresholded \( \ell_2 \)-norms of the column vectors

\[
a_{i,\xi} = \max\{ ||a_i||_2, a_{i,\text{lb}}(\xi) \}
\]

constitute the entries of the real-valued positive definite diagonal \( N_{\text{lat}} \times N_{\text{lat}} \) weighting matrix

\[
W_\xi = \text{diag}\{ a_{1,\xi}, a_{2,\xi}, \ldots, a_{N_{\text{lat}},\xi} \}
\]

(47a)

and its inverse matrix

\[
W_\xi^{-1} = \text{diag}\{ 1/a_{1,\xi}, 1/a_{2,\xi}, \ldots, 1/a_{N_{\text{lat}},\xi} \}.
\]

(47b)

The right-hand side multiplication of the sensing matrix \( A \) by the inverse weighting matrix \( W_\xi^{-1} \) yields the complex-valued normalized \( N_{\text{obs}} \times N_{\text{lat}} \) sensing matrix

\[
\tilde{A}_\xi[p^{(n)}] = A[p^{(n)}] W_\xi^{-1} = \Phi[p^{(n)}] \Psi W_\xi^{-1},
\]

(48)

whose column vectors \( \tilde{a}_{i,\xi} \in \mathbb{C}^{N_{\text{obs}}, i \in [N_{\text{lat}}]} \), exhibit \( \ell_2 \)-norms of unity, if the \( \ell_2 \)-norm of the associated column vector in the original sensing matrix \( A \) is not smaller than the specified lower bound \( \xi \). This multiplication does not affect the transfer of a potential dense population of the sensing matrix \( A \) to the normalized sensing matrix \( \tilde{A}_\xi \).

Using the weighting matrices \( W_\xi \), the normalized sensing matrix \( \tilde{A}_\xi \), and the normalized vector of corrupted RF voltage signals \( \tilde{u}^{(b,\eta)} = ||u^{(b,\eta)}||_2^{-1} u^{(b,\eta)} \), the complete system of linear algebraic equations \( 45 \) and \( 48 \) can equivalently be written as

\[
\tilde{u}^{(b,\eta)} = \frac{1}{||u^{(b,\eta)}||_2} \left[ \tilde{A}_\xi[p^{(n)}] W_\xi^{-1} W_\xi \vartheta^{(\kappa)} + \eta \right]
\]

\[
= \tilde{A}_\xi[p^{(n)}] \hat{\theta}_\xi^{(\kappa)} + \tilde{\eta},
\]

where \( \tilde{\eta} = ||u^{(b,\eta)}||_2^{-1} \eta \) are the normalized observation errors and the complex-valued nearly-sparse normalized \( N_{\text{lat}} \times 1 \) coefficient vector

\[
\hat{\theta}_\xi^{(\kappa)} = \frac{1}{||u^{(b,\eta)}||_2} W_\xi \vartheta^{(\kappa)}
\]

(50)

was introduced.

The CS problem \( 5 \) associated with the normalized system of linear algebraic equations \( 48 \) reads

\[
\text{Recover nearly-sparse } \hat{\theta}_\xi^{(\kappa)} \in \mathbb{C}^{N_{\text{obs}}}
\]

subject to \( ||u^{(b,\eta)} - \tilde{A}_\xi[p^{(n)}] \hat{\theta}_\xi^{(\kappa)}||_2 \leq \tilde{\eta} \),

(51)

and the sparsity-promoting \( \ell_q \)-minimization method for its stable solution \( \hat{\theta}_\xi^{(\kappa,\eta)} \) becomes

\[
\hat{\theta}_\xi^{(\kappa,\eta)} = \arg \min_{\theta \in \mathbb{C}^{N_{\text{obs}}}} ||\theta||_q
\]

subject to \( ||u^{(b,\eta)} - \tilde{A}_\xi[p^{(n)}] \theta||_2 \leq \tilde{\eta} \),

where the parameter \( q \in (0; 1] \) determines the type of optimization problem and \( \hat{\theta}_\xi^{(\kappa,\eta)} \) denotes the complex-valued \( N_{\text{lat}} \times 1 \) vector of recovered normalized transform coefficients.

Given the unique solution to the \( \ell_q \)-minimization problem \( \hat{\theta}_\xi^{(\kappa,\eta)} \), the inversion of both the weighting \( W_\xi \) and the basis transform in \( 43 \) estimates the unknown vector associated with the relative spatial fluctuations in compressibility \( 38 \) as

\[
\hat{\gamma}_\xi^{(\kappa,\eta)} = ||u^{(b,\eta)}||_2 \Psi W_\xi^{-1} \hat{\theta}_\xi^{(\kappa,\eta)}.
\]

(52)

Physically-meaningful estimates of the real-valued relative spatial fluctuations in compressibility can be obtained from their real parts multiplied by a factor of two.

VI. IMPLEMENTATION

A. Determination of the Relevant Fourier Coefficients

The simultaneous ADC of all received RF voltage signals based on their regular temporal sampling at the rate \( f_s > 2 f_{\text{lb}} \) in combination with the subsequent computation of the DFT provides the relevant Fourier coefficients. This combination necessitates the quantization of the lower and the upper bounds in the specified common observation time intervals \( 21 \) to \( \ell^{(n)}_b = q^{(n)}_b T_s \) and \( \ell^{(n)}_u = q^{(n)}_u T_s \), respectively, where \( q^{(n)}_b, q^{(n)}_u \in \mathbb{N}_0 \) and \( q^{(n)}_b > q^{(n)}_u \). The quantized common observation time intervals read \( T_{\text{obs}}^{(n)} = \{ q^{(n)}_b T_s, q^{(n)}_u T_s \} \) and the regular temporal sampling provides \( N^{(n)}_t = q^{(n)}_u - q^{(n)}_b \) real-valued samples per received RF voltage signal. Each ultrasonic wave emission thus triggers the acquisition of \( N_{\text{el}} N^{(n)}_t \) real-valued samples from the \( N_{\text{el}} \) physical transducer elements. The \( N^{(n)}_t \)-point DFTs (cf. e.g. \( 105 \) Sec. 3.3.2 or \( 102 \) Sect. 2.2.3) provide the Fourier coefficients \( 35 \) associated with the sets of relevant discrete temporal frequencies \( 36 \) for the quantized common observation times \( T_{\text{obs}}^{(n)} = N^{(n)}_t T_s \). The total number of observations \( 40 \), the quantized common observation times \( T_{\text{obs}}^{(n)} = N^{(n)}_t T_s \), the effective time-bandwidth products \( N_{\text{el}} f_{\text{up}} \approx N^{(n)}_t B_u \), the requirement on the sampling rate \( f_s > 2 f_{\text{lb}} \), and the effective bandwidth \( B_u = f_{\text{lb}} - f_{\text{ub}} \) permit the approximation of this combination’s efficiency as

\[
\text{Efficiency} = \frac{2 N_{\text{obs}}}{N_{\text{el}} \sum_{n=0}^{N^{(n)}_t - 1} N^{(n)}_t} \approx 2 B_u T_s < 1 - \frac{f_{\text{ub}}}{f_{\text{sub}}}
\]

(53)
Its upper bound demonstrates that the data volume digitized during the pulse-echo measurement exceeds the data volume occupied by the quantized FOURIER coefficients.

B. Observation Errors

Additive observation errors, which are statistically modeled as white noise with zero mean and the variance $\sigma_n^2$, corrupt the regular temporal samples of the received RF voltage signals. The expected total energy of the corrupted RF voltage signals resulting from this model amounts to

$$E\left(\|u^{(B,n)}\|_2^2\right) = \|u^{(B)}\|_2^2 + \sigma_n^2 N_{\text{obs}}$$

and permits the $\ell_2$-norm of the normalized observation errors in both the normalized CS problem \((51)\) and the sparsity-promoting $\ell_q$-minimization method \((\mathbf{R}_{q,\xi,n})\) to be estimated as

$$\hat{\eta} = \sqrt{\frac{\sigma_n^2 N_{\text{obs}}}{\|u^{(B)}\|_2^2 + \sigma_n^2 N_{\text{obs}}}} \left[1 + \frac{\|u^{(B)}\|_2^2}{\sigma_n^2 N_{\text{obs}}}\right]^{-\frac{1}{2}}. \quad (54)$$

C. Sparsity-Promoting $\ell_q$-Minimization Method

Spectral projected gradient for $\ell_1$-minimization (SPG\(_{\ell_1}\)) \((106)\) implements the convex sparsity-promoting $\ell_1$-minimization method \((\mathbf{R}_{q,\xi,n})\) induced by the parameter $q = 1$. FOUCART’s algorithm \((48)\) Sect. 4] iteratively applies this convex $\ell_1$-minimization method based on SPG\(_{\ell_1}\) to a sequence of renormalized CS problems to implement the nonconvex sparsity-promoting $\ell_q$-minimization method \((\mathbf{R}_{q,\xi,n})\) induced by the open parameter interval $q \in (0; 1)$. Both algorithms are iterative and left multiply a sequence of recursively-generated vectors by the potentially densely-populated normalized sensing matrix \((48)\) or its conjugate transpose. Their matrix-free implementations numerically evaluate both types of matrix-vector products in two efficient auxiliary functions. These functions pursue the two main objectives of circumventing the matrices’ complete storage in the fast but limited random-access memory (RAM) and accelerating the numerical computations. This paper proposes the fast multipole method (FMM) for the complete observation process \((41b)\) in combination with a fast basis transform for the nearly-sparse coefficient vector \((43)\) to accomplish both objectives. The diagonal weighting matrices \((47)\) directly permit an efficient numerical evaluation of the associated matrix-vector products.

D. Fast Multipole Method for the Complete Observation Process

The fast multipole method (FMM) (cf. e.g. \((104), (107), (108)\) or \((109)\)) approximates the action of the complete observation process \((41b)\) or its conjugate transpose on a suitable vector. An error-controlled approximate representation of the spatially-shifted outgoing free-space GREEN’s function \((A.57)\) in the entries of the partial observation process \((39b)\), which is valid under a specific geometrical relationship between the positions of the lattice point $r_i \in \mathcal{L}$ and the point-like mathematical element $r_{\text{mat},m}^{(n)} \in \mathcal{V}_m$, constitutes its foundation \((107)\).

This representation decomposes the complete observation process \((41b)\) into the sum $\Phi[p^{(in)}] = \Phi^{(near)}[p^{(in)}] + \Phi^{(far)}[p^{(in)}]$, where $\Phi^{(near)}[p^{(in)}]$ accounts for the lattice points close to the planar transducer array and $\Phi^{(far)}[p^{(in)}]$ accounts for those exceeding a specified distance from the planar transducer array \((110)\) Sect. 9.1]. The sparse population of the summand $\Phi^{(near)}[p^{(in)}]$ enables both its complete storage in the fast RAM and a fast numerical evaluation of the associated matrix-vector product. The additional decomposition of the summand $\Phi^{(far)}[p^{(in)}]$ into the products of only a few unique diagonal translation matrices as well as densely-populated aggregation and disaggregation matrices provides similar benefits. The small number of unique matrices enables their complete storage in the fast RAM. The diagonality of the translation matrices may simultaneously reduce the computational costs raised by the numerical evaluation of the associated matrix-vector product.

Two C programs based on CUDA (NVIDIA Corp., Santa Clara, CA, USA) implement parallelized versions of the FMM for the complete observation process \((41b)\) and its conjugate transpose. A Tesla K40c (NVIDIA Corp.) graphics processing unit (GPU) performs all computations with 32 bit single precision, which was found to be sufficiently accurate. A MATLAB (The MathWorks, Inc., Natick, MA, USA) interface based on the MEX framework simplifies the data transfers and the analysis of the results.

E. Fast Basis Transform for the Nearly-Sparse Coefficient Vector

Fast basis transforms implement the matrix-vector products between the unitary matrix $\Psi$, which represents the orthonormal basis of $\mathbb{C}^{N_u}$ specified in the nearly-sparse coefficient vector \((43)\), or its conjugate transpose and a suitable vector. The fast FOURIER transform (FFT), for example, efficiently implements the matrix-vector products including the $N_u$-dimensional DFT matrix $\Psi$, which represents the discrete FOURIER basis of $\mathbb{C}^{N_u}$, or its conjugate transpose. Various wavelet or wave atom bases provide comparably fast transforms that circumvent the matrices’ complete storage in the limited RAM and minimize the computational costs.

VII. SIMULATION STUDY

A. Parameters of the Simulations

1) Pulse-Echo Scan Configuration: The following study numerically simulated a pulse-echo scan configuration in the two-dimensional Euclidean space, i.e. $d = 2$ (cf. Sect. III-B). It included a linear transducer array whose geometrical parameters equaled those of a commercial off-the-shelf product (number of physical elements: $N_{\text{el}} = 128$, element width: $w_{\text{el},1} = 279.8 \mu \text{m}$, kerf width: $k_{\text{el},1} = 25 \mu \text{m}$, element pitch: $\Delta w_{\text{el},1} = 304.8 \mu \text{m}$). The lack of the third spatial dimension, however, prevented the simulation of the vibrating face’s height $w_{\text{el},2}$ and the elevational focus. The receiver and transmitter apodization functions in the spatial transfer functions \((20c)\) and \((25)\), respectively, were set to unity, i.e. $\chi_m^{(n)}(r_1) = 1$ and $\chi_m^{(t)}(r_1) = 1$ for all $r_1 \in L_m$ and all $m \in [127]_0$. These specifications resulted in a piston...
transducer array with a homogeneous spatial sensitivity. The product of the identical transmitter electromechanical transfer functions defined in the transfer relations (14) and $-j\omega l/p_0$, i.e. $-j\omega l_0(\xi)$, in the discretized incident acoustic pressure field generated by the entire linear transducer array (42) corresponded to a modulated Gaussian envelope in the time domain (temporal center frequency: $f_c = 4$ MHz, fractional bandwidth: $B_T/\Delta f_{\text{inc}} = 0.7$, reference attenuation level defining the fractional bandwidth: $\alpha_{\text{f}}/\Delta f_{\text{db}} = 60$ dB, reference attenuation level defining the time duration: $\alpha_{t}/\Delta f_{\text{db}} = 60$ dB). For the sake of simplicity, the receiver electromechanical transfer functions defined in the transfer relations (13) were set to unity, i.e. $h_n^t(\omega) = 1$ V m$^{-1}$ for all physical transducer elements $m \in [127]_0$.

2) Pulse-Echo Measurement Process: The simulated UI system insomniated two complementary lossy heterogeneous objects, which will be specified in Subsections VII-AS and VII-AD by single realizations, i.e. $N_{\text{in}} = 1$, of each type of incident ultrasonic wave (cf. Subsect. IV-B). Each object required a distinct common observation time interval (21), which was identical for each type of incident ultrasonic wave. The lower and upper bounds on the temporal frequency in the compact interval of temporal frequencies (22) equaled $f_b = 2.6$ MHz and $f_a = 5.4$ MHz, resulting in the effective bandwidth $B_a = 2.8$ MHz. The implemented method for the determination of the relevant FOURIER coefficients used the temporal sampling rate $f_s = 20$ MHz ($T_s = 50$ ns) and its approximate efficiency (53) amounted to Efficiency $\approx 28$%.

3) Incident Ultrasonic Waves: The amplitude of the excitation voltage identically exciting all physical transducer elements $u^t(\omega)$ in the adapted general form of the excitation voltages (42) was a sinusoidal signal in the time domain (temporal frequency: $f_\text{tx} = T_\text{tx}^{-1} = f_c = 4$ MHz, time duration: $T_\text{tx} = 250$ ns). The frequency-independent apodization weights and the nonnegative time delays in the adapted general form of the excitation voltages (42b) depended on the specific type of the incident ultrasonic wave. In accordance with the specifications of a commercial UI system, the clock signal used for the quantization of the nonnegative time delays in (26b) had the temporal frequency $f_{\text{clk}} = 80$ MHz ($T_{\text{clk}} = 12.5$ ns). The steered QPW had the preferred direction of propagation indicated by the unit vector $e_\theta = e_2$. The superposition of randomly-delayed QCWs and the superposition of both randomly-apodized and randomly-delayed QCWs used the fixed time interval $T_{\text{inc}} = 43.637$ ns in the computations of the time delays in (22b) and (30). This fixed time interval approximately comprised a random permutation of the time delays specified for the steered QPW in (27b) for a preferred direction of propagation $e_\phi = (\cos(\phi), \sin(\phi))^T$ with $\phi = 72^\circ - 131\pi$.

4) Spatial Discretization: The set of discrete positions (31) represented each vibrating face $L_m$, $m \in [127]_0$, of the linear transducer array (number of point-like mathematical elements per vibrating face along the $r_1$-axis: $N_{\text{lat},1} = 4$). This resulted in the constant spacing between the adjacent point-like mathematical elements along the $r_1$-axis $\Delta r_{\text{lat},1} = 4^{-1}\Delta r_{\text{el},1} = 76.2$ $\mu$m. The regular lattice (33) was square-shaped and laterally-centered in front of the linear transducer array (numbers of lattice points along both coordinate axes: $N_{\text{lat},1} = N_{\text{lat},2} = 512$, constant spacings: $\Delta r_{1} = \Delta r_{2} = \Delta r_{\text{mat},1} = 76.2$ $\mu$m, offset vector: $r_0 = 2^{-1}(-511, 1)^T \Delta r_{\text{mat},1} \approx (-19.47$ $\text{mm}, 38.1$ $\mu$m)$^T$). The total number of lattice points amounted to $N_{\text{lat}} = |\mathcal{L}| = 512^2 = 262444$.

5) Object A: Synthetic Wire Phantom: Object A mimicked a typical wire phantom, i.e. an ensemble of isolated thin wire targets immersed in a homogeneous fluid. The wire targets were represented by $N_{\text{cof}} = 21$ identical nonzero components in the vector associated with the relative spatial fluctuations in compressibility (38). Their axial distances from the linear transducer array ranged from 5 mm to 37 mm and their axial and lateral spacings amounted to approximately 5 mm. The canonical basis of $\mathbb{C}^{N_{\text{in}}}$, which is represented by the identity matrix $\Psi = I$, induced an $N_{\text{cof}}$-sparse coefficient vector (43). The absorption parameters in the complex-valued wavenumber (70) equaled those of pure water at an ambient temperature of $T_{\text{amb}} = 20^\circ$ C, i.e. $a = 0$, $b = 2.17 \times 10^{-3}$ dB MHz$^{-1}$ cm$^{-1}$, and $\zeta = 2$ (70 Tab. 4.8). The exponent $\zeta = 2$ prevented dispersion and the phase velocity was constant, i.e. $c(f) = c_0 = c_{\text{ref}} = 1500$ m s$^{-1}$ for all $f \in \mathbb{R}^+$. The quantized lower and upper bounds in the specified common observation time interval for the received RF voltage signals (21) equaled $T_{\text{lb}}^{(0)} = 50$ ns and $T_{\text{ub}}^{(0)} = 16487T_s = 82.4$ $\mu$s, respectively, resulting in the quantized common observation time $T_{\text{obs}}^{(0)} = 1647T_s$ and the acquisition of $N_{\text{off}}^{(0)} = 1647$ real-valued samples per received RF voltage signal. The lower and the upper bounds defining the set of admissible frequency indices (36c) equaled $l_{\text{lb}}^{(0)} = 215$ and $l_{\text{ub}}^{(0)} = 444$, respectively, truncating the FOURIER series representing the received RF voltage signals in the time domain (35) to $N_{\text{f},\text{lb}}^{(0)} = 230$ relevant discrete temporal frequencies. The total number of observations (40) in the observation processes induced by each of the four types of incident ultrasonic waves (41b) was $N_{\text{obs}} = 29 440 \ll N_{\text{lat}} = 262444$.

6) Object B: Tissue-Mimicking Cosparse Object: Object B approximately mimicked the structure and the properties of human soft tissues. It was represented by $N_{\text{cof}} = 10$ randomly-specifed nonzero components in the vector of transform coefficients (43). The discrete FOURIER basis of $\mathbb{C}^{N_{\text{in}}}$, which is represented by the $N_{\text{lat}}$-dimensional DFT matrix $\Psi$, induced the vector associated with the relative spatial fluctuations in compressibility (38). The absorption parameters in the complex-valued wavenumber (9) were typical for biologic soft tissues, i.e. $a = 0$, $b = 0.5$ dB MHz$^{-1}$ cm$^{-1}$, and $\zeta = 1$ (70 Tab. 4.20). The reference values of the temporal frequency and the associated phase velocity were $f_{\text{ref}} = f_c$ and $c_{\text{ref}} = c(f_{\text{ref}}) = 1540$ m s$^{-1}$. In contrast to object A, the exponent $\zeta = 1$ induced dispersive absorption. The quantized lower and upper bounds in the specified common observation time interval for the received RF voltage signals (21) equaled $T_{\text{lb}}^{(0)} = 0$ and $T_{\text{ub}}^{(0)} = 1607T_s = 80.35$ $\mu$s, respectively, resulting in the quantized common observation time $T_{\text{obs}}^{(0)} = 1607T_s$ and the acquisition of $N_{\text{off}}^{(0)} = 1607$ real-valued samples per received RF voltage signal. The lower and the upper bounds defining the set of admissible frequency indices (36c) equaled $l_{\text{lb}}^{(0)} = 209$ and $l_{\text{ub}}^{(0)} = 433$, respectively, truncating the
FOURIER series representing the received RF voltage signals in the time domain \([35]\) to \(N_f^{(0)} = 225\) relevant discrete temporal frequencies. The total number of observations \([40]\) in the observation processes induced by each of the four types of incident ultrasonic waves \([41b]\) was \(N_{\text{obs}} = 28,800 \ll N_{\text{lat}}\).

7) Observation Errors: Additive observation errors of five distinct energy levels corrupted the first BORN approximations of the received RF voltage signals \([41a]\) induced by the insonifications of each heterogeneous object by the four investigated types of incident ultrasonic waves. Their regular temporal samples were realizations of i.i.d. Gaussian random variables with zero mean and the variances

\[
\sigma^2_{\eta} = \frac{2\|u^{(\text{QPW})}\|^2_{L_2}}{N_{\text{d}}N_{\text{t}}^{(0)}}, \quad (55)
\]

where \(\|u^{(\text{QPW})}\|^2_{L_2}\) denotes the energy of the first BORN approximation of the received RF voltage signal \([41a]\) induced by the QPW with the preferred direction of propagation \(e_2\) and \(SNR_{dB} \in \{3\text{dB}, 6\text{dB}, 10\text{dB}, 20\text{dB}, 30\text{dB}\}\) specifies the reference SNR in dB. This study created \(N_{\text{lat}} = 10\) realizations of the observation errors for each admissible reference SNR and solved the associated instances of the normalized CS problem \([51]\) by the sparsity-promoting \(\ell_2\)-minimization method \([R_{\ell_2,\xi,n}]\).

8) Regularization Parameters: The variances \([55]\), the total number of observations \([40]\), the quantized common observation time \(T_i^{(0)} = N_i^{(0)}T_s\), and the effective time-bandwidth product \(N_f^{(0)}B_u\) permit the approximation of the estimated \(\ell_2\)-norm of the observation errors \([54]\) as

\[
\hat{\eta} \approx \left[1 + \frac{\|u^{(\text{BP})}\|^2_{L_2}T_s}{\|u^{(\text{QPW})}\|^2_{L_2}2B_u} - \frac{\text{SNR}_{\text{dB}}}{10}\right]^{-\frac{1}{2}}.
\]

For each admissible reference SNR, the empirical factor

\[
\xi = 10^{-\frac{\text{SNR}_{\text{dB}}}{10}}
\]

specified the lower bounds on the \(\ell_2\)-norms of the sensing matrices’ column vectors \([49]\). The admissible maximum number of iterations in SPG\(\ell_1\) amounted to \(N_{\text{iter}} = 1000\). The normalization parameters \(\epsilon_n = (2 + n)^{-1}\), \(n \in [4]_0\), generated a sequence of five renormalized CS problems in FOUCART’s algorithm and the result provided by SPG\(\ell_1\) served as the initial guess.

9) Reference Observation Processes: Two reference observation processes, which did not originate from the proposed linear physical model for the pulse-echo measurement process, facilitated the assessment of the four distinct sensing matrices \([44]\) induced by the four types of incident ultrasonic waves for each heterogeneous object. Their concatenations with the canonical or the FOURIER bases resulted in fictitious reference sensing matrices whose TPSFs \([6]\) served as benchmarks and emphasized the limitations imposed by the proposed linear physical model for the pulse-echo measurement process on the image recovery. The entries in the real-valued \(N_{\text{obs}} \times N_{\text{lat}}\) matrix representing the first reference observation process \(\Phi^{(\text{RIP})}\) were realizations of i.i.d. Gaussian random variables, i.e. \(\phi_{m,i}^{(\text{RIP})} \sim \mathcal{N}(0; \mathbb{1})\) for all \(m \in [N_{\text{obs}}]\) and all \(i \in [N_{\text{lat}}]\). This observation process and the associated sensing matrix \(\Phi^{(\text{RIP})} \approx \Phi^{(\text{GWN})}\) provably satisfy the RIP with very high probability, if the total number of observations \([40]\) is sufficiently large (cf. Sect. II). The entries in the complex-valued \(N_{\text{obs}} \times N_{\text{lat}}\) matrix representing the second reference observation process \(\Phi^{(\text{GWN})}\) equaled those of the observation process \([41b]\) with the discretized incident acoustic pressure field generated by the entire linear transducer array \([42]\) substituted by realizations of i.i.d. complex-valued Gaussian random variables, i.e. \(\text{Re}\{p_i^{(in,0)}(r_i)\} \sim \mathcal{N}(0; 1)\) and \(\text{Im}\{p_i^{(in,0)}(r_i)\} \sim \mathcal{N}(0; 1)\) for all lattice points \(i \in [N_{\text{lat}} - 1]_0\) and all relevant discrete temporal frequencies \(l \in \mathbb{N}_{\leq 0}\). This substitution introduces random properties into the observation process \([41b]\) and simultaneously accounts for the scattering and the reception of the ultrasonic waves. The violation of the HELMHOLTZ equation \([16]\) governing the incident acoustic pressure field by the Gaussian white noise (GWN), however, renders the induced observation process \([41b]\) fictitious. The three types of random incident ultrasonic waves introduced in Subsect. IV-B attempt to approximate this type of incident acoustic pressure field in a physically valid manner.

B. Results of the Numerical Simulations for Object A: Synthetic Wire Phantom

1) Incident Acoustic Pressure Fields: Figure 5 illustrates the incident acoustic pressure fields \([42]\) associated with the QPW with the preferred direction of propagation \(e_2\) (cf. (a) and (e)), the superposition of randomly-apodized QCWs (cf. (b) and (f)), the superposition of randomly-delayed QCWs (cf. (c) and (g)), and the superposition of both randomly-apodized and randomly-delayed QCWs (cf. (d) and (h)) inside the specified FOV. The top row (cf. (a), (b), (c), and (d)) displays these fields at the temporal center frequency \(f_t\) as functions of the spatial position. For each type of incident ultrasonic wave, the large images show the normalized absolute value in decibel (dB) [right colorbar], whereas the inset images show the phase in radian (rad) [top colorbar] inside the region indicated by the red square. The reference value for normalization is the maximum absolute value inside the specified FOV, which is attained by the superposition of both randomly-apodized and randomly-delayed QCWs at the position \(r = (-6.1341\text{ mm}; 6.8199\text{ mm})^T\) (cf. (d)). The bottom row (cf. (e), (f), (g) and (h)) displays the normalized absolute values in decibel (dB) (solid lines) and the normalized unwrapped phases (dashed lines) at the three distinct reference positions indicated by the crosses in the top row as functions of the normalized temporal frequency. The reference values for the normalizations are the maximum absolute value and the minimum unwrapped phase at the three distinct reference positions, which are attained by the superposition of randomly-delayed QCWs (cf. (g)).

The incident acoustic pressure field \([42]\) associated with the QPW with the preferred direction of propagation \(e_2\) exhibits a relatively homogeneous normalized absolute value (cf. (a)), which is promoted by the small spatial amplitude absorption coefficient \([8]\). The linear transducer array of finite spatial...
Fig. 5: Incident acoustic pressure fields \((42)\) associated with the quasi-plane wave (QPW) with the preferred direction of propagation \(e_2\) (cf. \((a)\) and \((e)\)), the superposition of randomly-apodized quasi-cylindrical waves (QCWs) (cf. \((b)\) and \((f)\)), the superposition of randomly-delayed QCWs (cf. \((c)\) and \((g)\)), and the superposition of both randomly-apodized and randomly-delayed QCWs (cf. \((d)\) and \((h)\)). The top row (cf. \((a)\), \((b)\), \((c)\), and \((d)\)) displays these fields at the temporal center frequency \(f_t\) as functions of the spatial position. For each type of incident ultrasonic wave, the large images show the normalized absolute value in decibel (dB) [right colorbar], whereas the inset images show the phase in radian (rad) [top colorbar] inside the region indicated by the red square. The bottom row (cf. \((e)\), \((f)\), \((g)\), and \((h)\)) displays the normalized absolute values in decibel (dB) (solid lines) and the normalized unwrapped phases (dashed lines) at the three distinct reference positions indicated by the crosses in the top row as functions of the normalized temporal frequency.

The properties of the three types of random incident ultrasonic waves introduced in Subsect. IV-B significantly differ from those of the QPW with the preferred direction of propagation \(e_2\). The associated incident acoustic pressure fields \((42)\) exhibit pronounced beamlike fluctuations of up to 9 dB in the normalized absolute value. The approximately constant phase on lines parallel to the vibrating faces of the linear transducer array on the \(r_1\)-axis (cf. inset image) indicates plane wavefronts. At the three distinct reference positions, the normalized absolute values accurately reflect the Gaussian shape of the pulse-echo response and the normalized unwrapped phases depend approximately linearly on the normalized temporal frequency (cf. \((e)\)). The normalized absolute values are very similar for each reference position. The three distinct slopes of these linear functions indicate the times-of-flight of the QPW from the linear transducer array to the three distinct reference positions.

The illustrated instances of the discretized incident acoustic pressure field \((42)\) induce observation processes \((41)\) of very different properties. The strong dependence of the discretized incident acoustic pressure field \((42)\) on the reference position is an additional crucial difference to the QPW with the preferred direction of propagation \(e_2\).
Fig. 6: Absolute values of the point spread functions (PSFs) \( \Phi_{(6)} \) associated with both reference observation processes (cf. \( \Phi_{(a)} \) and \( \Phi_{(b)} \)) as well as the observation processes \( \Phi_{(41b)} \) induced by the four investigated types of incident ultrasonic waves (cf. \( \Phi_{(c)} \) to \( \Phi_{(f)} \)) in decibel (dB). The position of the point-like relative spatial fluctuation in compressibility \( r_{12} \) is fixed and indicated by the green crosshairs, whereas the lattice position \( r_{ij}, i_j \in [N_{lat} - 1]_0 \), varies. The inset images magnify the regions indicated by the white squares.

in the vector associated with the relative spatial fluctuations in compressibility \( \Phi_{(6)} \). The transfer of this low correlation to the column vectors of the normalized sensing matrix \( \Phi_{(6)} \) potentially aids in the stable recovery of the nearly-sparse normalized coefficient vector \( \Phi_{(50)} \) in the normalized CS problem \( \Phi_{(51)} \) by \( \ell_q \)-minimization \( \Phi_{(\ell_q,\xi,\eta)} \). The following subsection characterizes the observation processes \( \Phi_{(41b)} \) induced by the four types of incident ultrasonic waves and both reference observation processes using the PSF \( \Phi_{(6)} \).

2) Point Spread Functions: Figure 6 illustrates the absolute values of the PSFs \( \Phi_{(6)} \) associated with both reference observation processes (cf. \( \Phi_{(a)} \) and \( \Phi_{(b)} \)) as well as the observation processes \( \Phi_{(41b)} \) induced by the four investigated types of incident ultrasonic waves (cf. \( \Phi_{(c)} \) to \( \Phi_{(f)} \)) in decibel (dB). The position of the point-like relative spatial fluctuation in compressibility \( r_{12} \) is fixed and indicated by the green crosshairs, whereas the lattice position \( r_{ij}, i_j \in [N_{lat} - 1]_0 \), varies. The inset images magnify the regions indicated by the white squares. Although all illustrated PSFs \( \Phi_{(6)} \) attain the maximum absolute value of unity at the position of the point-like relative spatial fluctuation in compressibility \( r_{12} \), i.e. \( i_1 = i_2 \), they significantly differ in their behavior for the remaining positions, i.e. \( i_1 \neq i_2 \).

The PSF \( \Phi_{(6)} \) associated with the first reference observation process \( \Phi_{(41)} \) (cf. \( \Phi_{(a)} \)) exhibits an isolated sharp maximum at the position of the point-like relative spatial fluctuation in compressibility \( r_{12} \), i.e. \( i_1 = i_2 \), and random absolute values close to zero, which appear to be uniformly distributed over the FOV, for the remaining positions, i.e. \( i_1 \neq i_2 \). The sharp maximum implies an excellent spatial resolution that equals the specified constant spacing between the adjacent lattice points in the set of discrete positions \( \Phi_{(25)} \). The noise-like properties indicate that the observations are uncorrelated and that the type of aliasing induced by the relatively small number of observations \( N_{obs} \ll N_{lat} \) is incoherent (cf. Sect. \( \Pi \)).

The PSF \( \Phi_{(6)} \) associated with the second reference observation process \( \Phi_{(42)} \) (cf. \( \Phi_{(b)} \)) deviates only slightly from that associated with the first reference observation process \( \Phi_{(41)} \). Although the maximum absolute value at the position of the point-like relative spatial fluctuation in compressibility \( r_{12} \), i.e. \( i_1 = i_2 \), is embedded in an hourglass-shaped region of nonzero values (cf. inset image), it is still sharp and isolated. As before, this feature implies an excellent spatial resolution that equals the specified constant spacing between the adjacent lattice points in the set of discrete positions \( \Phi_{(33)} \). The nonzero values approximately maintain the uniform spatial distribution for the remaining positions, i.e. \( i_1 \neq i_2 \), with the exception of two gaps that are laterally adjacent to the maximum absolute value and shape the aforementioned hourglass. These gaps originate from the lack of reception angles in the pulse-echo measurement process caused by the fixed position of the linear transducer array on a single edge of the FOV. They are an inevitable consequence of the typical scan configuration used in pulse-echo UI. The desirable features of the PSF \( \Phi_{(6)} \) associated with the second reference observation process \( \Phi_{(42)} \), however, motivate the synthesis of physically valid incident acoustic pressure fields that approximate the GWN.

The remaining PSFs \( \Phi_{(6)} \) associated with the observation processes \( \Phi_{(41b)} \) induced by the four investigated types of incident ultrasonic waves (cf. \( \Phi_{(c)} \) to \( \Phi_{(f)} \)) exhibit more significant deviations from those associated with both reference observation processes. First, relatively large absolute values close to unity concentrate in elliptical-shaped regions centered on the maximum absolute value at the position of the point-like relative spatial fluctuation in compressibility \( r_{12} \), i.e. \( i_1 = i_2 \), with minor and major axes of approximately 0.3 mm and 0.7 mm, respectively (cf. inset images). The indicated high correlation of the observations associated with adjacent lattice points in the elliptical-shaped regions deteriorates the spatial resolution to distances exceeding the specified constant spacing between the adjacent lattice points. Second, the nonzero values are distributed less uniformly for the remaining positions, i.e. \( i_1 \neq i_2 \), and form sidelobes of various shapes, spatial extents, and absolute values. Both the dimensions of the elliptical-shaped regions and the sidelobes vary significantly with the
The three types of random incident ultrasonic waves, however, achieve FAHMs that are smaller than or equal to those of the QPW with the preferred direction of propagation \( \mathbf{e}_2 \) (cf. [3]) that were approximately uniformly distributed along the diagonal line from \((-17.5 \text{ mm}, 2 \text{ mm})^T\) to \((17.5 \text{ mm}, 37 \text{ mm})^T\) and numbered from 1 to 9 with increasing axial position, i.e. \( i_2 = 513 \left[ 25.55 + (s - 1)57.4875 \right] \) for all \( s \in [9] \). Both reference observation processes (cf. Subsect. VII-A9) consistently achieve the minimum FAHM of a single two-dimensional volume element \( \Delta V \approx 5.81 \times 10^{-3} \text{ mm}^2 \) for all reference positions. Figure 6 illustrates the absolute values of the FAHMs for reference position \( i = 3 \).

| Incident ultrasonic wave | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Mean & std. dev. |
|-------------------------|---|---|---|---|---|---|---|---|---|-----------------|
| QPW                     | 0.19 | 0.18 | 0.21 | 0.24 | 0.28 | 0.32 | 0.38 | 0.42 | 0.51 | 0.30 ± 0.11 |
| Rnd. apo.               | 0.07 | 0.09 | 0.14 | 0.17 | 0.16 | 0.18 | 0.44 | 0.36 | 0.51 | 0.24 ± 0.16 |
| Rnd. del.               | 0.05 | 0.10 | 0.09 | 0.11 | 0.14 | 0.33 | 0.20 | 0.34 | 0.46 | 0.20 ± 0.14 |
| Rnd. apo. del.          | 0.11 | 0.15 | 0.12 | 0.19 | 0.23 | 0.19 | 0.33 | 0.28 | 0.39 | 0.22 ± 0.10 |

The PSFs (6) associated with the observation processes (41b) induced by the three investigated types of random incident ultrasonic waves (cf. Subsect. VII-A3). The FAHMs were evaluated for nine distinct fixed reference positions of the point-like relative spatial fluctuation in compressibility \( r_{12} \) that were approximately uniformly distributed along the diagonal line from \((-17.5 \text{ mm}, 2 \text{ mm})^T\) to \((17.5 \text{ mm}, 37 \text{ mm})^T\) and numbered from 1 to 9 with increasing axial position, i.e. \( i_2 = 513 \left[ 25.55 + (s - 1)57.4875 \right] \) for all \( s \in [9] \). The nonzero values strongly concentrate around the maximum and in sidelobes. The absolute values in these sidelobes do not fluctuate and lack noise-like properties.

The PSFs (6) associated with the observation processes (41b) induced by the three investigated types of random incident ultrasonic waves (cf. [3]) resemble those associated with the second reference observation process \( \Phi^{(\text{GWN})} \). The sizes of the elliptical-shaped regions reduce in comparison to that achieved by the QPW. The sidelobes diffuse and fluctuate in their absolute values, similar to a speckle pattern, resulting in significantly more uniform distributions. The PSFs (6) associated with the superposition of randomly-delayed QCWs (cf. [6]) and the superposition of both randomly-apodized and randomly-delayed QCWs (cf. [4]) exhibit a slightly wider spread distribution of the nonzero values than that associated with the superposition of randomly-apodized QCWs (cf. [2]). This pattern seems to be less random for the superposition of randomly-apodized QCWs (cf. [2]) and the superposition of both randomly-apodized and randomly-delayed QCWs (cf. [4]), whereas it becomes random for the delays (cf. [6]) and [4]). This effect suggests that the introduction of random time delays is beneficial.

Table 1 summarizes the full areas at half maximum (FAHMs) of the point spread functions (PSFs) (6) associated with the observation processes (41b) induced by the four investigated types of incident ultrasonic waves (cf. Subsect. VII-A3). The FAHMs were evaluated for nine distinct fixed reference positions of the point-like relative spatial fluctuation in compressibility \( r_{12} \) that were approximately uniformly distributed along the diagonal line from \((-17.5 \text{ mm}, 2 \text{ mm})^T\) to \((17.5 \text{ mm}, 37 \text{ mm})^T\) and numbered from 1 to 9 with increasing axial position, i.e. \( i_2 = 513 \left[ 25.55 + (s - 1)57.4875 \right] \) for all \( s \in [9] \). They generally increase with the axial distance of the point-like relative spatial fluctuation in compressibility from the linear transducer array for all types of incident ultrasonic waves. The three types of random incident ultrasonic waves, however, achieve FAHMs that are smaller than or equal to those of the QPW for all reference positions, except those numbered \( s \in [6, 7] \). The maximum normalized differences in the FAHMs range from 23.53% for the superposition of both randomly-apodized and randomly-delayed QCWs at the reference position \( s = 9 \) to 73.68% for the superposition of randomly-delayed QCWs at the reference position \( s = 1 \). The mean FAHMs reflect these reductions relative to the QPW. Both reference observation processes (cf. Subsect. VII-A9) consistently achieve the minimum FAHM of a single two-dimensional volume element \( \Delta V \approx 5.81 \times 10^{-3} \text{ mm}^2 \) for all reference positions. Figure 6 illustrates the absolute values of the PSFs (6) for the third reference position, i.e. \( i = 3 \).

Figure 7 characterizes the statistical properties of the PSFs for the regions outside of the FAHMs by QQ plots with respect to the \( \chi^2 \)-distribution. The statistic investigated the PSFs for all nine reference positions of the point-like relative spatial fluctuation in compressibility \( r_{12} \). The first reference observation process \( \Phi^{(\text{RIP})} \) achieves an approximate linear dependence of the quantiles that suggests a \( \chi^2 \)-distribution of the associated PSFs outside of the FAHMs. The second reference observation process \( \Phi^{(\text{GWN})} \) in contrast, exhibits modest deviations from this linear dependence. These deviations reflect the nonuniformity of the nonzero values in the associated PSFs and the two gaps that are laterally adjacent to the FAHMs (cf. Fig. 6(b)). The observation processes induced by the three types of random incident ultrasonic waves introduced in Subsect. IV-B cause further deviations from the linear dependence. Clearly, the observation process induced by the quasi-plane wave (QPW) with the preferred direction of propagation \( \mathbf{e}_2 \) exhibits the most pronounced deviations.

Figure 8 illustrates the absolute values of the recovered relative spatial fluctuations in compressibility (62) for the QPW with the preferred direction of propagation \( \mathbf{e}_2 \) (cf. [4] and [6]), the superposition of randomly-apodized QCWs (cf. [2] and [4]), the superposition of randomly-delayed QCWs (cf. [3] and [5]), and the superposition of both randomly-apodized and randomly-delayed QCWs (cf. [3] and [5]) in decibel (dB). The top row (cf. [4] to [7]) shows the relative spatial fluctuations
All four investigated types of random incident ultrasonic waves achieve mean SSIM indices close to unity and comparable trends in the relative RMSEs for all investigated values of the reference SNR and the convex $\ell_1$-minimization method $(R_{q,\xi,\eta})$ induced by the parameter $q = 1$. The former metric confirms the excellent recovery of the object’s structure exemplified in Fig. 8. The superposition of randomly-apodized QCWs (cf. [b]) performs slightly worse than the other types of incident ultrasonic waves at low reference SNRs. In combination with the nonconvex $\ell_{0.5}$-minimization method $(R_{q,\xi,\eta})$ induced by the parameter $q = 0.5$, which significantly increases the numbers of iterations, all four investigated types of incident ultrasonic waves achieve even better mean SSIM indices for all investigated values of the reference SNR. The three investigated types of random incident ultrasonic waves (cf. [f] to [h]), however, produce significantly higher relative RMSEs than the QPW with the preferred direction of propagation $e_2$ (cf. [e]) for the reference SNRs of $\text{SNR}_{\text{ref}} \in \{3\,\text{dB}, 6\,\text{dB}, 10\,\text{dB}\}$. The spatial variations in the associated random incident acoustic pressure fields illustrated in Fig. 8 across the positions of the $N_{\text{coef}} = 21$ wire targets induce this behavior.

Figure 10 illustrates the normalized energies in the discretized incident acoustic pressure fields (42) at the positions of the $N_{\text{coef}} = 21$ wire targets (cf. [a]) and the first BORN approximations of the received RF voltage signals induced by the individual wire targets, i.e., the column vectors in the complete observation processes (41b) (cf. [b]), for the four investigated types of incident ultrasonic waves. The reference values for the normalizations are the maximum energies achieved by the superposition of both randomly-apodized and randomly-delayed QCWs (gray bars) at the position of the wire target with the index 3. The $r_2$-coordinate of each wire target increases monotonically with its index. The QPW with the preferred direction of propagation $e_2$ (black bars) provides approximately constant incident acoustic energies, whereas those provided by the three types of random incident ultrasonic waves (blue, green, and gray bars) vary significantly. The energies in the received RF voltage signals, which strongly depend on the incident acoustic energies, generally decrease with increasing spatial distances of the wire targets from the linear transducer array. They reflect the SNRs of the corrupted RF voltage signals (45) induced by the individual wire targets for additive observation errors of fixed energy. The individual wire targets insonified by relatively low incident acoustic energy, e.g., the wire targets with the indices 6, 8, and 16 for the superposition of randomly-apodized QCWs (cf. blue bars in [a]), induce corrupted RF voltage signals of worse SNR (cf. blue bars in [b]) than those insonified by relatively high incident acoustic energy, e.g., the wire targets with the indices 3, 11, and 16 for the superposition of both randomly-apodized and randomly-delayed QCWs (cf. gray bars in [a]). These variations in the SNRs induce variations in the mean relative recovery errors achieved by the nonconvex $\ell_{0.5}$-minimization method $(R_{q,\xi,\eta})$ with the parameter $q = 0.5$ for the reference SNR of $\text{SNR}_{\text{ref}} = 3\,\text{dB}$ (cf. [c]). The latter variations explain the residual relative RMSEs discovered in Fig. 9 for the three variations of iterations are normalized by their admissible maximum of $N_{\text{iter}} = 10^3$ (cf. Subsect. VII-A9).
investigated types of random incident ultrasonic waves and the low reference SNRs of $\text{SNR}_{\text{ref}} \in \{3\,\text{dB}, 6\,\text{dB}, 10\,\text{dB}\}$.  

### C. Results of the Numerical Simulations for Object B: Tissue-Mimicking Cosparse Object

1) Relative Spatial Fluctuations in Compressibility Recovered by $\ell_q$-Minimization, $q \in \{0.5; 1\}$: Figure 11 illustrates the absolute values of the recovered relative spatial fluctuations in compressibility (52) for the QPW with the preferred direction of propagation $e_2$ (cf. (a) and (e)), the superposition of randomly-apodized QCWs (cf. (b) and (f)), the superposition of randomly-delayed QCWs (cf. (c) and (g)), and the superposition of both randomly-apodized and randomly-delayed QCWs (cf. (d) and (h)) in decibel (dB). The top row (cf. (a) to (d)) shows the relative spatial fluctuations in compressibility (52) recovered by the convex $\ell_1$-minimization method $\mathbf{R}_{q,\xi,n}^{0.5}$ induced by the parameter $q = 1$, whereas the bottom row (cf. (e) to (h)) shows those recovered by the nonconvex $\ell_{0.5}$-minimization method $\mathbf{R}_{q,\xi,n}^{0.5}$ induced by the parameter $q = 0.5$. The inset images magnify the regions indicated by the white squares. The reference SNR amounted to $\text{SNR}_{\text{ref}} = 20\,\text{dB}$.

Fig. 9: Sample means and sample standard deviations of the mean structural similarity (SSIM) indices (solid blue), the relative root mean-squared errors (RMSEs) (dashed green), and the normalized numbers of iterations (solid gray) obtained by the sparsity-promoting $\ell_q$-minimization method $\mathbf{R}_{q,\xi,n}^{0.5}$ induced by the parameters $q = 1$ (top row) and $q = 0.5$ (bottom row) for the four investigated types of incident ultrasonic waves in percent ($\%$). The assignment of the four types of incident ultrasonic waves to the columns in this figure equals that in Fig. 8. For each reference SNR, $N_{\text{rcn}} = 10$ recovery experiments were conducted. The dashed red line indicates the reference SNR for Fig. 8.

Fig. 8: Absolute values of the recovered relative spatial fluctuations in compressibility (52) for the QPW with the preferred direction of propagation $e_2$ (cf. (a) and (e)), the superposition of randomly-apodized QCWs (cf. (b) and (f)), the superposition of randomly-delayed QCWs (cf. (c) and (g)), and the superposition of both randomly-apodized and randomly-delayed QCWs (cf. (d) and (h)) in decibel (dB). The top row (cf. (a) to (d)) shows the relative spatial fluctuations in compressibility (52) recovered by the convex $\ell_1$-minimization method $\mathbf{R}_{q,\xi,n}^{0.5}$ induced by the parameter $q = 1$, whereas the bottom row (cf. (e) to (h)) shows those recovered by the nonconvex $\ell_{0.5}$-minimization method $\mathbf{R}_{q,\xi,n}^{0.5}$ induced by the parameter $q = 0.5$. The inset images magnify the regions indicated by the white squares. The reference SNR amounted to $\text{SNR}_{\text{ref}} = 20\,\text{dB}$. 

![Fig. 8](image1.png)

![Fig. 9](image2.png)
fluctuations in compressibility \( \tilde{\mathbf{R}}_{q,\xi,\eta}^{\ell_{0.5}} \) for the QPW with the preferred direction of propagation \( \mathbf{e}_2 \) (cf. (a) and (c)), the superposition of randomly-apodized QCWs (cf. (b) and (e)), and the superposition of randomly-delayed QCWs (cf. (c) and (g)), and the superposition of both randomly-apodized and randomly-delayed QCWs (cf. (d) and (h)). The top row (cf. (a) to (d)) shows the relative spatial fluctuations in compressibility \( \tilde{\mathbf{R}}_{q,\xi,\eta}^{\ell_{0.5}} \) recovered by the convex \( \ell_1 \)-minimization method \( \tilde{\mathbf{R}}_{q,\xi,\eta}^{\ell_1} \) induced by the parameter \( q = 1 \), whereas the bottom row (cf. (e) to (h)) shows those recovered by the nonconvex \( \ell_{0.5} \)-minimization method \( \tilde{\mathbf{R}}_{q,\xi,\eta}^{\ell_{0.5}} \) induced by the parameter \( q = 0.5 \). The inset images magnify the regions indicated by the white squares. The reference SNR amounted to \( \text{SNR}_{\text{ref}} = 10 \text{ dB} \).

Figure 12 illustrates the sample means and the sample standard deviations of the mean SSIM indices (solid blue), the relative RMSEs (dashed green), and the normalized numbers of iterations (solid gray) obtained by the sparsity-promoting \( \ell_q \)-minimization method \( \tilde{\mathbf{R}}_{q,\xi,\eta}^{\ell_q} \) induced by the parameters \( q = 1 \) (top row) and \( q = 0.5 \) (bottom row) for the four investigated types of incident ultrasonic waves in percent (%). The assignment of the four types of incident ultrasonic waves to the columns in this figure equals that in Fig. 11. The numbers of iterations are normalized by their admissible maximum of \( N_{\text{iter}} = 10^3 \) (cf. Subsect. VII-A8).

### VIII. DISCUSSION

#### A. Simulation Study

1) **Object A: Synthetic Wire Phantom:** The PSFs (6) demonstrate the theoretical advantages of the complete observation processes (41b) induced by the three investigated types of random incident ultrasonic waves over that induced by the conventional QPW. The FAHMs summarized in Tab. II confirm smaller regions of highly-correlated observations and the QQ plots in Fig. 7 attest statistical properties more similar to both reference observation processes. Despite these theoretical advantages, the complete observation processes (41b) induced by the three investigated types of random incident ultrasonic waves also increase the sensitivity of the recovered relative spatial fluctuations in compressibility \( \tilde{\mathbf{R}}_{q,\xi,\eta} \) to the observation errors. The nonconvex \( \ell_{0.5} \)-minimization method \( \tilde{\mathbf{R}}_{q,\xi,\eta}^{\ell_{0.5}} \) induced by the parameter \( q = 0.5 \), for example, produces residual relative RMSEs for the low reference SNRs of \( \text{SNR}_{\text{ref}} \in \{3 \text{ dB}, 6 \text{ dB}, 10 \text{ dB} \} \) in Fig. 9(f) to Fig. 9(h), whereas the conventional QPW minimizes those in Fig. 9(e).

The sparse vector associated with the relative spatial fluctuations in compressibility \( \tilde{\mathbf{R}}_{q,\xi,\eta}^{\ell_{0.5}} \) exclusively samples the discretized incident acoustic pressure fields (42) at the \( N_{\text{coef}} = 21 \) positions of the wire targets. The random variations in the incident acoustic energy at these positions, which are illustrated in Fig. 10, cause variations in the SNRs of the corrupted RF voltage signals \( \text{SNR}_{\text{ref}} \) induced by the individual wire targets. The quantitative recovery of the wire targets associated with the observations of relatively low SNR is impeded. These findings suggest that the relocation of the wire targets to positions of high incident acoustic energy, different realizations of the random incident ultrasonic waves, or the sequential emission of multiple ultrasonic waves improve the quantitative recovery. The findings further suggest that cosparse objects with spatially-extended basis functions are less sensitive to the random variations in the incident acoustic energy and potentially permit lower relative RMSEs. The mean SSIM indices, which reflect the visual impression more accurately than the relative RMSEs, however, indicate a comparably excellent structural image quality for all four investigated types of incident ultrasonic waves and all reference SNRs in Fig. 9.

2) **Object B: Tissue-Mimicking Cosparse Object:** The \( \mathbf{N}_{\text{iter}} \)-dimensional DFT matrix \( \Psi \), which relates the vector associated with the relative spatial fluctuations in compressibility \( \tilde{\mathbf{R}}_{q,\xi,\eta}^{\ell_{0.5}} \) to the sparse coefficient vector \( \mathbf{a} \), exhibits completely-populated column vectors. These indicate the presence of nonzero samples of the relative spatial fluctuations in compressibility at all discrete positions in the regular lattice \( \mathbf{a} \).
Fig. 11: Absolute values of the recovered relative spatial fluctuations in compressibility (52) for the QPW with the preferred direction of propagation $e_2$ (cf. (a) and (e)), the superposition of randomly-apodized QCWs (cf. (b) and (f)), the superposition of randomly-delayed QCWs (cf. (c) and (g)), and the superposition of both randomly-apodized and randomly-delayed QCWs (cf. (d) and (h)) in decibel (dB). The top row (cf. (a) to (d)) shows the relative spatial fluctuations in compressibility (52) recovered by the convex $\ell_1$-minimization method $R_{q,\xi,\eta}$ induced by the parameter $q = 1$, whereas the bottom row (cf. (e) to (h)) shows those recovered by the nonconvex $\ell_{0.5}$-minimization method $R_{q,\xi,\eta}$ induced by the parameter $q = 0.5$. The inset images magnify the regions indicated by the white squares. The reference SNR amounted to $\text{SNR}_{\text{ref}} = 10 \, \text{dB}$.

In contrast to the isolated wire targets defining object A, these column vectors completely sample the discretized incident acoustic pressure fields (42) and capture the entire incident acoustic energy. The random variations in the incident acoustic energy thus do not increase the sensitivity of the recovered relative spatial fluctuations in compressibility (52) to the observation errors. In fact, they facilitate the distinction of the FOURIER coefficients in the sparse coefficient vector (42) by the normalized sensing matrices (48) induced by the three investigated types of random incident ultrasonic waves. The
sparsity-promoting $\ell_q$-minimization method $\|x\|_q \in C$ induced by the parameters $q = 1$ and $q = 0.5$ produces smaller relative RMSEs for the low reference SNRs of $\text{SNR}_{ab} \in \{3 \text{ dB}, 6 \text{ dB}, 10 \text{ dB}\}$ in Fig. 12(b) to Fig. 12(d) and Fig. 12(f) to Fig. 12(h) than for the conventional QPW in Figures 12(a) and 12(e). These findings confirm the predicted relevance of the heterogeneous object’s spatial extent.

B. Experimental Validation

The author presents an experimental validation of the proposed novel method for fast image acquisition based on CS (cf. Sect. V) using the plane-wave imaging challenge in medical ultrasound (PICMUS) data and $N_m \in \{1, 3\}$ sequential emissions of deterministic steered QPWs (cf. Subsect. [V-B] in [11]). Additional experimental validations of the proposed method at increasing stages of evolution and comparisons to conventional image recovery methods, including SA, DAS, filtered backpropagation (FBP), and focused beam sweeping, are given in [28–31].

Appendix

For $d \in \mathbb{N}$, $r \in \mathbb{R}^d$, and considering the positive sign convention [10], the outgoing free-space GREEN’S function $g : \mathbb{R}^d \rightarrow \mathbb{C}$ is the unique solution to the fundamental $d$-dimensional inhomogeneous HELMHOLTZ equation

$$(\Delta + k^2) g(r) = \delta(r)$$

subject to the SOMMERFELD radiation condition (SRC) (cf. e.g. [15] (1.48) or (2.8)), [35] (7.61), [11] 12

$$\lim_{r \rightarrow \infty} \|r\|^d_{r=\tau} \left( \mathbf{e}_r(r), \nabla g(r) \right)_r + j k g(r) = 0,$$

(A.56)

where the complex-valued wavenumber $\bar{k}$ satisfies $\bar{k} \in \mathbb{C} \setminus \{0\}$, $\text{Im}\{\bar{k}\} \leq 0$, $\delta$ denotes the DIRAC delta distribution, and $\mathbf{e}_r(r) = \|r\|^{-1}_{r=\tau} \mathbf{r}$ for all $r \in \mathbb{R}^d \setminus \{0\}$ indicates the unit vector pointing in the radial direction.

The SRC (A.56) eliminates converging waves incoming from sources distributed across a hypothetical boundary surrounding $\Omega$ and located at infinite distance as well as standing waves from the set of valid solutions and ensures that the scattered acoustic pressure field $p^{(sc)}$ can be written as an integral (superposition) over diverging, outgoing (radiating) waves that vanish when $\|r\|_{r=\tau}$ becomes infinitely large.

With the SRC (A.56), the outgoing free-space GREEN’S function reads [11], [14]

$$g(r) = \frac{i}{4} \left( \frac{k}{2\pi \|r\|_{r=\tau}} \right)^{\frac{d-1}{2}} H^{(2)}_{\frac{d-1}{2}} (k \|r\|_{r=\tau}),$$

(A.57)

where $H^{(2)}_{\nu} (Z)$ denotes the HANKEL function of second kind of order $\nu \in \mathbb{C}$, which is also referred to as BESSEL function of the third kind [11] §10.2(ii), [116] 9.1.1. The HANKEL function is a weighted sum of a BESSEL function of the first kind $J_\nu$ and a BESSEL function of the second kind $Y_\nu$, i.e. for $Z \in \mathbb{C}$ [115] 10.4.3, [116] 9.1.4.

$$H^{(2)}_{\nu} (Z) = J_\nu (Z) - j Y_\nu (Z).$$

For $d \in \{2, 3\}$, the outgoing free-space GREEN’S function (A.57) becomes

$$g(r) = \left\{ \begin{array}{ll}
\frac{\pi}{2} H^{(2)} (k \|r\|_{2}) & \text{for } d = 2,
\frac{1}{4\pi} e^{-j k |r|_{2}} & \text{for } d = 3.
\end{array} \right.$$ (A.58)

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