On the geometry of the connection with totally skew-symmetric torsion on almost complex manifolds with Norden metric

Dimitar Mekerov

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University of Plovdiv, Faculty of Mathematics and Informatics, 236 Bulgaria Blvd., 4003 Plovdiv, Bulgaria, mircho@uni-plovdiv.bg

Abstract
We consider an almost complex manifold with Norden metric (i.e. a metric with respect to which the almost complex structure is an anti-isometry). On such a manifold we study a linear connection preserving the almost complex structure and the metric and having a totally skew symmetric torsion tensor (i.e. a 3-form). We prove that if a non-Kähler almost complex manifold with Norden metric admits such connection then the manifold is quasi-Kählerian (i.e. has non-integrable almost complex structure). We prove that this connection is unique, determine its form, and construct an example of it on a Lie group. We consider the case when the manifold admits a connection with parallel totally skew-symmetric torsion and the case when such connection has a Kähler curvature tensor. We get necessary and sufficient conditions for an isotropic Kähler manifold with Norden metric.

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Key words: Norden metric, almost complex manifold, indefinite metric, linear connection, Bismut connection, KT connection, skew-symmetric torsion, parallel torsion.

1 Introduction

There is a strong interest in the metric connections with totally skew-symmetric torsion tensor (3-form). These connections arise in a natural way in theoretical and mathematical physics. For example, such a connection is of particular interest in string theory [1]. In mathematics this connection was used by Bismut to prove the local index theorem for non-Kähler Hermitian manifolds [2]. A connection with totally skew-symmetric torsion tensor is called a KT connection.
by physicists, and among mathematicians this connection is known as a Bismut connection.

According to Gauduchon [3], on any non-Kähler Hermitian manifold, there exists a unique Hermitian connection (i.e. one preserving the almost complex structure and the metric), whose torsion tensor is totally skew-symmetric.

The goal of the present work is to solve the analogous problem for almost complex manifolds with Norden metric. We build upon the results in [4], where we have considered a connection with totally skew-symmetric torsion tensor in the case of quasi-Kähler manifolds with Norden metric.

The main results in the present paper are contained in Theorem 3.1. We prove that if a non-Kähler almost complex manifold with Norden metric admits a linear connection \( \nabla' \) preserving the almost complex structure and the metric and having totally skew-symmetric tensor, then the manifold is quasi-Kählerian and the connection \( \nabla' \) is unique. We get the form of \( \nabla' \).

We construct a 4-parametric family of 4-dimensional quasi-Kähler manifolds with Norden metric by a Lie group, and derive the connection \( \nabla' \) for an arbitrary manifold of this family.

We consider the case when \( \nabla' \) has a parallel torsion and obtain a relation between the scalar curvatures of this connection and the Levi-Civita connection. We show that the manifold is isotropic Kählerian iff these curvatures are equal.

We prove that if the manifold admits a parallel connection with a Kähler curvature tensor, then the manifold is isotropic Kählerian.

2 Preliminaries

Let \((M, J, g)\) be a 2\(n\)-dimensional almost complex manifold with Norden metric, i.e.

\[
J^2 x = -x, \quad g(Jx, Jy) = -g(x, y),
\]

for all differentiable vector fields \(x, y\) on \(M\). The associated metric \(\tilde{g}\) of \(g\) on \(M\), given by \(\tilde{g}(x, y) = g(x, Jy)\), is a Norden metric, too. The signature of both metrics is necessarily \((n, n)\).

Further, \(x, y, z, w\) will stand for arbitrary differentiable vector fields on \(M\) (or vectors in the tangent space of \(M\) at an arbitrary point \(p \in M\)).

The Levi-Civita connection of \(g\) is denoted by \(\nabla\). The tensor field \(F\) of type \((0, 3)\) on \(M\) is defined by

\[
F(x, y, z) = g((\nabla_x J)y, z).
\]

It has the following properties [5]:

\[
F(x, y, z) = F(x, z, y) = F(x, Jy, Jz), \quad F(x, Jy, z) = -F(x, y, Jz).
\]

In [6], the considered manifolds are classified into eight classes with respect to \(F\). The class \(W_0\) of the Kähler manifolds with Norden metric is contained in each of the other seven classes. It is determined by the condition \(F(x, y, z) = 0\), which is equivalent to \(\nabla J = 0\).
The condition
\[ S_{x,y,z} F(x, y, z) = 0, \tag{2.2} \]
where \( S_{x,y,z} \) is the cyclic sum over \( x, y, z \), characterizes the class \( W_3 \) of the quasi-Kähler manifolds with Norden metric. This is the only class of manifolds with non-integrable almost complex structure \( J \).

Let \( \{ e_i \} (i = 1, 2, \ldots, 2n) \) be an arbitrary basis of the tangent space of \( M \) at a point \( p \in M \). The components of the inverse matrix of \( g \), with respect to this basis, are denoted by \( g^{ij} \).

Following [7], the square norm \( \| \nabla J \|_2^2 \) of \( \nabla J \) is defined in [8] by
\[ \| \nabla J \|_2^2 = g^{ij} g^{ks} g((\nabla e_i J) e_k, (\nabla e_j J) e_s), \tag{2.3} \]
where it is proven that
\[ \| \nabla J \|_2^2 = -2 g^{ij} g^{ks} g((\nabla e_i J) e_k, (\nabla e_s J) e_j). \tag{2.4} \]

There, the manifold with \( \| \nabla J \|_2^2 = 0 \) is called an isotropic-Kähler manifold with Norden metric. It is clear that every Kähler manifold with Norden metric is isotropic-Kähler, but the inverse implication is not always true.

Let \( R \) be the curvature tensor of \( \nabla \), i.e. \( R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z \).

The corresponding \((0,4)\)-tensor is determined by \( R(x, y, z, w) = g(R(x, y)z, w) \).

The Ricci tensor \( \rho \) and the scalar curvature \( \tau \) with respect to \( \nabla \) are defined by
\[ \rho(y, z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij} \rho(e_i, e_j). \]

A tensor \( L \) of type \((0,4)\) with the properties
\[ L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z), \tag{2.5} \]
\[ S_{x,y,z} L(x, y, z, w) = 0 \quad \text{(the first Bianchi identity)} \tag{2.6} \]
is called a curvature-like tensor. Moreover, if the curvature-like tensor \( L \) has the property
\[ L(x, y, Jz, Jw) = -L(x, y, z, w), \tag{2.7} \]

it is called a Kähler tensor [9].

Let \( \nabla' \) be a linear connection with a tensor \( Q \) of the transformation \( \nabla \rightarrow \nabla' \) and a torsion tensor \( T \), i.e.
\[ \nabla'_x y = \nabla x y + Q(x, y), \quad T(x, y) = \nabla'_y x - \nabla'_y x - [x, y]. \tag{2.8} \]
The corresponding \((0,3)\)-tensors are defined by
\[ Q(x, y, z) = g(Q(x, y), z), \quad T(x, y, z) = g(T(x, y), z). \tag{2.9} \]

The symmetry of the Levi-Civita connection implies
\[ T(x, y) = Q(x, y) - Q(y, x), \quad T(x, y) = -T(y, x). \tag{2.10} \]
A partial decomposition of the space $T$ of the torsion $(0,3)$-tensors $T$ (i.e. $T(x,y,z) = -T(y,x,z)$) is valid on an almost complex manifold with Norden metric $(M,J,g)$: $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4$, where $T_i$ ($i = 1, 2, 3, 4$) are invariant orthogonal subspaces [10]. For the projection operators $p_i$ of $T$ in $T_i$ it is established that:

$$
4p_1(x, y, z) = T(x, y, z) - T(Jx, Jy, z) - T(Jy, Jz, x) - T(y, z, x),
$$
$$
4p_2(x, y, z) = T(x, y, z) - T(Jx, Jy, z) + T(x, Jy, Jz) + T(x, Jy, Jz),
$$
$$
8p_3(x, y, z) = 2T(x, y, z) - T(x, y, z) - T(z, x, y) - T(x, Jy, Jz)
$$
$$
\quad - T(z, Jx, Jy) + 2T(Jx, Jy, z) - T(Jy, Jz, x)
$$
$$
\quad - T(Jz, Jx, y) + T(y, Jz, Jx) + T(Jy, Jz, x),
$$
$$
8p_4(x, y, z) = 2T(x, y, z) + T(x, z, x) + T(z, x, y) + T(Jy, z, Jx)
$$
$$
\quad + T(z, Jx, Jy) + 2T(Jx, Jy, z) + T(Jy, Jz, x)
$$
$$
\quad + T(Jz, Jx, y) - T(y, Jz, Jx) - T(Jz, x, Jy).
$$

A linear connection $\nabla'$ on an almost complex manifold with Norden metric $(M,J,g)$ is called a natural connection if $\nabla'J = \nabla'g = 0$. The last conditions are equivalent to $\nabla'g = \nabla'\tilde{g} = 0$. If $\nabla'$ is a linear connection with a tensor $Q$ of the transformation $\nabla \rightarrow \nabla'$ on an almost complex manifold with Norden metric, then it is a natural connection iff the following conditions are valid:

$$
F(x, y, z) = Q(x, y, Jz) - Q(x, Jy, z),
$$
$$
Q(x, y, z) = -Q(x, z, y).
$$

Since $\nabla'g = 0$, equalities (2.10) and (2.11) imply

$$
Q(x, y, z) = \frac{1}{2}\{T(x, y, z) - T(y, z, x) + T(z, x, y)\},
$$

which is the Hayden theorem ([11]).

**Theorem 2.1 ([12]).** For a natural connection with a torsion tensor $T$ on a quasi-Kähler manifold with Norden metric $(M,J,g)$, which is non-Kählerian, the following properties are valid

$$
p_2 \neq 0, \quad p_3 = 0.
$$

\(\square\)

### 3 Connection with totally skew-symmetric torsion

#### 3.1 Main results

The main results in the present work are consist in the following
Theorem 3.1. Let $\nabla'$ be a natural connection with a totally skew-symmetric torsion tensor on a non-Kähler almost complex manifold with Norden metric $(M, J, g)$. Then

(i) $(M, J, g)$ is quasi-Kählerian;

(ii) $\nabla'$ has a torsion tensor $T$ in the class $T_2 \oplus T_4$;

(iii) $\nabla'$ is unique and the tensor $Q$ of the transformation $\nabla \rightarrow \nabla'$ is determined by

$$Q(x, y, z) = \frac{1}{4} \left\{ F(x, Jy, z) - F(Jx, y, z) - 2F(y, Jx, z) \right\}.$$  \hspace{1cm} (3.1)

Proof. Since $\nabla'$ has a totally skew-symmetric torsion tensor $T$, then we have

$$T(x, y, z) = -T(y, x, z) = -T(x, z, y) = -T(z, y, x).$$ \hspace{1cm} (3.2)

From (2.14) and (3.2) it is follows that for the tensor $Q$ of the transformation $\nabla \rightarrow \nabla'$ it is valid

$$Q(x, y, z) = \frac{1}{2} T(x, y, z).$$ \hspace{1cm} (3.3)

Since $\nabla'$ is a natural connection, then (2.12) holds and consequently

$$\mathfrak{S}_{x,y,z} F(x, y, z) = \mathfrak{S}_{x,y,z} \left\{ Q(x, y, Jz) - Q(x, Jy, z) \right\}.$$ \hspace{1cm} (3.4)

According to (3.3), $Q$ is also 3-form and then we have

$$Q(x, y, Jz) = Q(y, Jz, x).$$ \hspace{1cm} (3.5)

Equalities (3.4) and (3.5) imply (2.2), which completes the proposition (i).

According to Theorem (2.1), for the tensor $T$ we have $p_2 \neq 0$ and $p_3 = 0$, i.e. $T \in T_1 \oplus T_2 \oplus T_4$. Let us suppose that $T \in T_1$. Then $T = p_1$ and from (2.11) it is follows

$$3T(x, y, z) + T(Jx, Jy, z) + T(Jx, y, Jz) + T(x, Jy, Jz) = 0.$$ \hspace{1cm} (3.6)

We substitute $Jy$ for $y$ and $Jz$ for $z$ in (3.6) and the obtained equality we add to (3.6). In the result we substitute $Jz$ for $z$ and according to (2.12) we get $Q(x, y, Jz) - Q(x, Jy, z) = 0$. Then, by (2.12) we obtain $F(x, y, z) = 0$, i.e. $(M, J, g)$ is Kählerian, which is a contradiction. Therefore $T \notin T_1$, i.e. $p_1 = 0$.

Combining (2.11), (3.2) and (3.3), we find that

$$p_4(x, y, z) = Q(x, y, z) + Q(Jx, Jy, z).$$ \hspace{1cm} (3.7)

Let us suppose that $p_4 = 0$. Then from (3.7) we have

$$Q(x, y, z) + Q(Jx, Jy, z) = 0.$$ \hspace{1cm} (3.8)
We substitute $y \leftrightarrow z$ in (3.8), and subtract the obtained equality from (3.8). In the result we apply (3.8) and the fact that $Q$ is a 3-form. In such a way we obtain
\[
Q(x, y, z) + Q(J x, y, J z) = 0.
\] (3.9)
The equalities (3.8) and (3.9) imply $Q(x, y, J z) - Q(x, J y, z) = 0$. Thus, according to (2.12), we obtain $F(x, y, z) = 0$, i.e. $(M, J, g)$ is Kählerian, which is impossible. Therefore $p_4 \neq 0$.

Thereby we establish the following conditions for $T$: $p_1 = 0$, $p_2 \neq 0$, $p_3 = 0$, $p_4 \neq 0$. Hence $T \in T_2 \oplus T_4$, i.e. (ii) holds.

From (2.12), having in mind that $Q$ is a 3-form, we obtain
\[
F(x, J y, z) - F(J x, y, z) - 2F(y, J x, z)
= Q(J x, J y, z) + Q(J x, y, J z) + Q(x, J y, J z) + 3Q(x, y, z).
\] (3.10)

Since $p_1 = 0$, from (2.11) and (3.3) we have
\[
Q(J x, J y, z) + Q(J x, y, J z) + Q(x, J y, J z) = Q(x, y, z).
\] (3.11)
The equalities (3.10) and (3.11) imply (3.1), which completes the proof of (iii).

According to (2.1), equality (3.1) is equivalent to
\[
Q(x, y) = \frac{1}{4} \left\{ (\nabla_x J) y - (\nabla_y J) x - 2 (\nabla_y J) x \right\}.
\] (3.12)

Remark 3.1. In [13] it is found a 2-parametric family of natural connections on almost complex manifolds with Norden metric which contains the connection $\nabla'$ with totally skew-symmetric torsion tensor.

3.2 An example

Let $V$ be a real 4-dimensional vector space with a basis $\{E_i\}$. Let us consider a structure of a Lie algebra determined by the commutators $[E_i, E_j] = C_{ij}^k E_k$, where $C_{ij}^k$ are structure constants satisfying the anti-commutativity condition $C_{ij}^k = -C_{ji}^k$ and the Jacobi identity $C_{ij}^k C_{kl}^m + C_{jk}^l C_{il}^m + C_{ki}^m C_{lj}^m = 0$.

Let $G$ be the associated connected Lie group and $\{X_i\}$ be a global basis for the left invariant vector fields that is induced by the basis $\{E_i\}$ of $V$. Then we have the decomposition
\[
[X_i, X_j] = C_{ij}^k X_k.
\] (3.13)

Let us consider the almost complex manifold with Norden metric $(G, J, g)$, where
\[
J X_1 = X_3, \quad J X_2 = X_4, \quad J X_3 = -X_1, \quad J X_4 = -X_2
\] (3.14)
and
\[
g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1,
g(X_i, X_j) = 0 \text{ for } i \neq j.
\] (3.15)
Connection with skew-symmetric torsion on almost Norden manifolds

Because of (3.15), the following equality is valid
\[ 2g(\nabla X, X_j, X_k) = g([X_i, X_j], X_k) + g([X_k, X_i], X_j) + g([X_j, X_k], X_i). \] (3.16)

We add the condition for the associated metric \( \tilde{g} \) of \( g \) to be a Killing metric [14]. Then
\[ g([X_i, X_j], J X_k) + g([X_k, X_i], J X_j) = 0. \] (3.17)

Combining (2.1), (3.14), (3.15), (3.16) and (3.17), we obtain
\[ 2F(X_i, X_j, X_k) = g([J X_i, X_j], X_k) + g([J X_i, X_k], X_j). \] (3.18)

From (3.15), it follows immediately that \( \tilde{g} F(X_i, X_j, X_k) = 0 \), i.e. \( (G, J, g) \) is a quasi-Kähler manifold with Norden metric.

According to (3.13) and (3.17), we get
\[
\begin{align*}
[X_1, X_2] &= \lambda_1 X_1 + \lambda_2 X_2, & [X_1, X_3] &= \lambda_3 X_2 - \lambda_1 X_4, \\
[X_1, X_4] &= -\lambda_3 X_1 - \lambda_2 X_4, & [X_2, X_3] &= \lambda_4 X_2 + \lambda_1 X_3, \\
[X_2, X_4] &= -\lambda_4 X_1 + \lambda_2 X_3, & [X_3, X_4] &= \lambda_3 X_3 + \lambda_4 X_4,
\end{align*}
\] (3.19)

where
\[
\begin{align*}
\lambda_1 &= C_{12}^1 = C_{23}^2 = -C_{13}^4, & \lambda_2 &= C_{12}^2 = C_{24}^3 = -C_{14}^4, \\
\lambda_3 &= C_{13}^2 = C_{34}^3 = -C_{14}^1, & \lambda_4 &= C_{23}^4 = C_{34}^1 = -C_{24}^2.
\end{align*}
\]

Let equalities (3.19) are valid for an almost complex manifold with Norden metric \( (G, J, g) \), where \( J \) and \( g \) are determined by (3.14) and (3.15). Then we verify directly that the Jacobi identity for the commutators \([X_i, X_j]\) is satisfied and the associated metric \( \tilde{g} \) of \( g \) is a Killing metric.

Therefore, the following theorem is valid.

**Theorem 3.2.** Let \((G, J, g)\) be a 4-dimensional almost complex manifold with Norden metric, where \( G \) is the connected Lie group with an associated Lie algebra determined by a global basis \( \{X_i\} \) of left invariant vector fields, and \( J \) and \( g \) are the almost complex structure and the Norden metric determined by (3.14) and (3.15), respectively. Then \((G, J, g)\) is a quasi-Kähler manifold with a Killing associated metric \( \tilde{g} \) iff \( G \) belongs to the 4-parametric family of Lie groups, defined by (3.19). □

Let \((G, J, g)\) be the quasi-Kähler manifold determined by the conditions of Theorem 3.2.

By (3.18) and (3.19) we get the non-trivial components \( F_{ijk} = F(X_i, X_j, X_k) \) of the tensor \( F \):
\[
\begin{align*}
-2F_{114} &= 2F_{123} = -2F_{312} = 2F_{334} = 2F_{411} = 2F_{433} = \lambda_1, \\
2F_{23} &= -2F_{241} = -2F_{322} = -2F_{344} = 2F_{412} = 2F_{434} = \lambda_2, \\
-2F_{112} &= -2F_{134} = F_{211} = F_{233} = 2F_{314} = -2F_{332} = \lambda_3, \\
-F_{122} &= F_{144} = 2F_{221} = 2F_{234} = 2F_{414} = -2F_{432} = \lambda_4.
\end{align*}
\]

This leads to the following
Proposition 3.3. \((G, J, g)\) is a non-Kähler manifold with Norden metric. \(\square\)

Using (3.10) and (3.17), we obtain
\[
2g(\nabla X_i, X_j, X_k) = g([X_i, X_j], X_k) - g(J[X_i, JX_j], X_k) + g(J[X_i, JX_j], X_k).
\]
Then we have
\[
2\nabla X_i, X_j = [X_i, X_j] - J[X_i, JX_j] + J[JX_i, X_j].
\tag{3.20}
\]

The equality (3.20) implies
\[
2\nabla X_i, JX_j = [X_i, JX_j] + [X_i, JX_j] + J[JX_i, JX_j],
2J\nabla X_i, X_j = J[X_i, JX_j] + [X_i, JX_j] - [JX_i, JX_j].
\]

We subtract the last two equalities, apply the formula for covariant derivation and obtain
\[
2(\nabla X_i, J) X_j = J[JX_i, JX_j] + [JX_i, JX_j].
\tag{3.21}
\]

By (3.10) and (3.21) we obtain the components of \(\nabla J:\)
\[
2(\nabla X_1, J) X_1 = 2(\nabla X_3, J) X_3 = -\lambda_3 X_2 + \lambda_1 X_4,
2(\nabla X_2, J) X_2 = 2(\nabla X_4, J) X_4 = \lambda_4 X_1 - \lambda_2 X_3,
2(\nabla X_1, J) X_3 = -2(\nabla X_3, J) X_1 = \lambda_2 X_2 + \lambda_3 X_4,
2(\nabla X_2, J) X_4 = -2(\nabla X_4, J) X_2 = -\lambda_4 X_3 - \lambda_2 X_1,
2(\nabla X_1, J) X_2 = -\lambda_3 X_1 + 2\lambda_4 X_2 - \lambda_1 X_3,
2(\nabla X_1, J) X_4 = -\lambda_1 X_1 + \lambda_3 X_3 + 2\lambda_4 X_4,
2(\nabla X_2, J) X_1 = 2\lambda_3 X_1 + \lambda_4 X_2 + \lambda_2 X_4,
2(\nabla X_2, J) X_3 = \lambda_2 X_2 - 2\lambda_3 X_3 - \lambda_4 X_4,
2(\nabla X_3, J) X_2 = -\lambda_1 X_1 - 2\lambda_2 X_2 + \lambda_3 X_3,
2(\nabla X_3, J) X_4 = \lambda_3 X_1 + \lambda_1 X_3 + 2\lambda_2 X_1,
2(\nabla X_1, J) X_1 = 2\lambda_1 X_1 + \lambda_2 X_2 - \lambda_4 X_4,
2(\nabla X_4, J) X_3 = -\lambda_3 X_2 + 2\lambda_1 X_3 - \lambda_2 X_4.
\tag{3.22}
\]

From (3.3) and (3.22) it follows that \(\|\nabla J\|^2 = -4 (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2)\), and therefore the following is valid.

Proposition 3.4. \((G, J, g)\) is an isotropic Kähler manifold with Norden metric iff \(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0\). \(\square\)

Let \(\nabla^T\) be the connection with totally skew-symmetric torsion tensor \(T\) on \((G, J, g)\).

By virtue of (3.3), (3.12) and (3.21), we have
\[
4T(X_i, X_j) = 3[JX_i, JX_j] - J[X_i, JX_j] - J[JX_i, X_j] + [X_i, X_j].
\tag{3.23}
\]
which implies
\[
T(X_1, X_2) = \lambda_3 X_3 + \lambda_4 X_4, \quad T(X_1, X_3) = \lambda_3 X_2 - \lambda_1 X_4, \\
T(X_1, X_4) = \lambda_4 X_2 + \lambda_3 X_3, \quad T(X_2, X_3) = -\lambda_3 X_1 - \lambda_2 X_4, \\
T(X_2, X_4) = -\lambda_4 X_1 + \lambda_2 X_3, \quad T(X_3, X_4) = \lambda_1 X_1 + \lambda_2 X_2. \tag{3.24}
\]

Then the non-trivial components $T_{ijk} = T(X_i, X_j, X_k)$ of the corresponding $(0,3)$-tensor $T$ are:
\[
T_{134} = \lambda_1, \quad T_{234} = \lambda_2, \quad T_{123} = -\lambda_3, \quad T_{124} = -\lambda_4. \tag{3.25}
\]

According to (3.19) and (3.24), we have
\[
T(X_i, X_j) = [JX_i, JX_j]
\]
and then, from (3.23) it follows that
\[
[JX_i, JX_j] + J[JX_i, X_j] = [X_i, X_j] - J[X_i, JX_j]. \tag{3.26}
\]

Since we have $2Q = T$ for $\nabla'$, from (2.8), (3.23) and (3.24), we obtain
\[
\nabla'_{X_i} X_j = [X_i, X_j] - J[X_i, JX_j]. \tag{3.27}
\]
Combining (3.19) and (3.27), we get the components of $\nabla'$:
\[
\begin{align*}
\nabla'_{X_1} X_1 &= \nabla'_{X_3} X_3 = -\lambda_1 X_2 - \lambda_3 X_4, \\
\nabla'_{X_2} X_2 &= \nabla'_{X_4} X_4 = \lambda_2 X_1 + \lambda_4 X_3, \\
\nabla'_{X_1} X_2 &= \nabla'_{X_3} X_4 = \lambda_1 X_1 + \lambda_3 X_3, \\
\nabla'_{X_1} X_3 &= -\nabla'_{X_3} X_1 = \lambda_3 X_2 - \lambda_1 X_4, \\
\nabla'_{X_1} X_4 &= -\nabla'_{X_3} X_2 = -\lambda_3 X_1 + \lambda_1 X_3, \\
\nabla'_{X_2} X_1 &= \nabla'_{X_4} X_3 = -\lambda_2 X_2 - \lambda_4 X_4, \\
\nabla'_{X_2} X_3 &= -\nabla'_{X_4} X_1 = \lambda_4 X_2 - \lambda_2 X_4, \\
\nabla'_{X_2} X_4 &= -\nabla'_{X_4} X_2 = -\lambda_4 X_1 + \lambda_2 X_3. \tag{3.28}
\end{align*}
\]

Thus we arrive at the following

**Proposition 3.5.** Let $\nabla'$ be the connection with totally skew-symmetric torsion tensor $T$ on $(G, J, g)$. Then the components of $\nabla'$ and $T$ with respect to the basis $\{X_i\}$ are (3.28) and (3.25), respectively. \(\square\)

In [12] and [15] we have considered the canonical connection $\nabla^C$ and the $B$-connection $\nabla^B$ on quasi-Kähler manifolds with Norden metric. Now we will show how these two connections relate to the connection $\nabla'$ with totally skew-symmetric torsion tensor on $(G, J, g)$.

The connection $\nabla^B$ is defined in [15] by
\[
\nabla^B_{X_i} X_j = \nabla_{X_i} X_j + \frac{1}{2} (\nabla_{X_i} J) J X_j.
\]
Hence, applying (3.14) and (3.22), we obtain directly \( \nabla' = \frac{4}{3} \nabla' \). On the other hand, according to [12], we have \( \nabla' = \frac{1}{2} (\nabla C + \nabla') \). Thus we proved the following

**Proposition 3.6.** Let \( \nabla', \nabla B \) and \( \nabla C \) be the connection with totally skew-symmetric torsion, the B-connection and the canonical connection on \((G, J, g)\), respectively. Then

\[
\nabla' = \frac{4}{3} \nabla B = 2 \nabla C.
\]

□

In [4] we have proved that \( \nabla' \) has a Kähler curvature tensor on any quasi-Kähler manifolds with Norden metric iff the following identity holds

\[
\mathcal{R}(x, y, z) = 0. \tag{3.29}
\]

Equality (3.21) implies

\[
2 (\nabla_x J) J x + (\nabla J x) J x = [J x, J x] - [x, x].
\]

We add the above, and according to (3.20) obtain

\[
(\nabla_x J) J x + (\nabla J x) J x = [J x, J x] - [x, x].
\]

Then, the condition (3.29) takes the following form on \((G, J, g)\):

\[
\mathcal{R}(x, y, z) = 0. \tag{3.20}
\]

The last equality and equalities (3.14), (3.15) and (3.19) imply

**Proposition 3.7.** The connection \( \nabla' \) with totally skew-symmetric torsion on \((G, J, g)\) has a Kähler curvature tensor iff \( \lambda_1^2 + \lambda_2^2 = \lambda_3^2 + \lambda_4^2 \).

□

4 Connection with parallel totally skew-symmetric torsion

Let \( \nabla' \) be the connection with totally skew-symmetric torsion tensor \( T \) on the quasi-Kähler manifold with Norden metric \((M, J, g)\).

Now we consider the case when \( \nabla' \) has a parallel torsion, i.e. \( \nabla' T = 0 \).

It is known that the curvature tensors \( R' \) and \( R \) of \( \nabla' \) and \( \nabla \), respectively, satisfy:

\[
R'(x, y, z, w) = R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) + Q(x, Q(y, z), w) - Q(y, Q(x, z), w). \tag{4.1}
\]
Equality (3.3) implies $\nabla'Q = 0$ in the considered case. Then from the formula for covariant derivation with respect to $\nabla'$ it follows that

$$xQ(y, z, w) - Q(\nabla'_x y, z, w) - Q(y, \nabla'_x z, w) - Q(y, z, \nabla'_x w) = 0. \quad (4.2)$$

According to the first equality of (2.8) we have

$$Q(\nabla'_x y, z, w) = Q(\nabla x y, z, w) + Q(Q(x, y), z, w),$$
$$Q(y, \nabla'_x z, w) = Q(y, \nabla x z, w) + Q(y, Q(x, z), w),$$
$$Q(y, z, \nabla'_x w) = Q(y, z, \nabla x w) + Q(y, z, Q(x, w)). \quad (4.3)$$

Combining (4.2), (4.3), the first equality of (2.9) and having in mind the formula for covariant derivation with respect to $\nabla$, we obtain

$$(\nabla x Q)(y, z, w) = Q(Q(x, y), z, w)$$
$$- g(Q(x, z), Q(y, w)) - g(Q(y, z), Q(x, w)). \quad (4.4)$$

From (4.4) and the first equality of (2.10) we have

$$(\nabla x Q)(y, z, w) - (\nabla y Q)(x, z, w) = Q(T(x, y), z, w)$$
$$- 2g(Q(x, z), Q(y, w)) + 2g(Q(y, z), Q(x, w)). \quad (4.5)$$

Because of (4.5), equality (4.1) can be rewritten as

$$R'(x, y, z, w) = R(x, y, z, w) + Q(T(x, y), z, w)$$
$$- g(Q(x, z), Q(y, w)) + g(Q(y, z), Q(x, w)). \quad (4.6)$$

Since $Q(e_i, e_j) = -Q(e_j, e_i)$ it follows that $g^{ij}Q(e_i, e_j) = 0$. Then, from (4.6) after contraction by $x = e_i, w = e_j$, we obtain the following equality for the Ricci tensor $\rho'$ of $\nabla'$:

$$\rho'(y, z) = \rho(y, z) + 2g^{ij}g(Q(e_i, y), Q(z, e_j))$$
$$- g^{ij}g(Q(e_i, z), Q(y, e_j)). \quad (4.7)$$

Contracting by $y = e_k, z = e_s$ in (4.7), we get

$$\tau' = \tau + g^{ij}g^{ks}g(Q(e_i, e_k), Q(e_s, e_j)), \quad (4.8)$$

where $\tau'$ is the scalar curvature of $\nabla'$.

By virtue of (4.8), (3.12), (2.3) and (2.4) we have

$$\tau' = \tau - \frac{1}{8} \left\| \nabla J \right\|^2. \quad (4.9)$$

Thus we arrive at the following...
**Theorem 4.1.** Let $\nabla'$ be the connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric $(M, J, g)$. Then for the Ricci tensor $\rho'$ and the scalar curvature $\tau'$ of $\nabla'$ are valid (4.7) and (4.9), respectively.

Equality (4.9) leads to the following

**Corollary 4.2.** Let $\nabla'$ be the connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric $(M, J, g)$. Then the manifold $(M, J, g)$ is isotropic Kählerian iff $\nabla'$ and $\nabla$ have equal scalar curvatures.

**Remark 4.1.** The 4-parametric family of 4-dimensional quasi-Kähler manifolds $(G, J, g)$ considered in subsection 3.2 does not admit any connection with parallel totally skew-symmetric torsion.

### 5 Connection with parallel totally skew-symmetric torsion and Kähler curvature tensor

Let $\nabla'$ be a connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric $(M, J, g)$.

We will find conditions for the curvature tensor $R'$ of $\nabla'$ to be Kählerian.

From (5.3), having in mind that $Q$ is a 3-form, we have

$$Q(T(x, y), z, w) = g(Q(z, w), T(x, y)) = g(T(x, y), Q(z, w)) = 2g(Q(x, y), Q(z, w)).$$

Then (4.6) obtains the form

$$R'(x, y, z, w) = R(x, y, z, w) + 2g(Q(x, y), Q(z, w)) - g(Q(x, z), Q(y, w)) + g(Q(y, z), Q(x, w)).$$

From (5.1), identities (2.5) and (2.7) for $R'$ follow immediately. Therefore $R'$ is a Kähler tensor iff the first Bianchi identity (2.6) for $R'$ is satisfied. Since this identity is valid for $R$, then (5.1) implies that $R'$ is Kählerian iff

$$\mathcal{S}_{x, y, z} \{2g(Q(x, y), Q(z, w)) - g(Q(x, z), Q(y, w)) + g(Q(y, z), Q(x, w))\} = 0.$$

Thus, using that $Q$ is a skew-symmetric tensor, we arrive the following

**Theorem 5.1.** Let $\nabla'$ be the connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric $(M, J, g)$. Then the curvature tensor for $\nabla'$ is a Kähler tensor iff

$$\mathcal{S}_{x, y, z} g(Q(x, y), Q(z, w)) = 0.$$ (5.2)
Because of the skew-symmetry of $Q$, (5.2) implies
\[ g(Q(y, z), Q(x, w)) - g(Q(x, z), Q(y, w)) = -g(Q(x, y), Q(z, w)). \]
The last equality and (5.1) lead to the following

**Corollary 5.2.** Let $\nabla'$ be the connection with parallel totally skew-symmetric torsion and Kähler curvature tensor on the quasi-Kähler manifold with Norden metric $(M, J, g)$. Then
\[ R'(x, y, z, w) = R(x, y, z, w) + g(Q(x, y), Q(z, w)). \] (5.3)

If $R'$ is a Kähler tensor then $R'(x, y, Jz, Jw) = -R'(x, y, z, w)$, and because of (5.3) we have
\[ R(x, y, Jz, Jw) + R(x, y, z, w) = -g(Q(x, y), Q(Jz, Jw)) - g(Q(x, y), Q(z, w)). \] (5.4)

From (5.12) we get
\[ Q(x, Jy) = JQ(x, y) - (\nabla_x J) y. \]
Then we have
\[ Q(Jx, Jy) = -Q(x, y) - (\nabla_x J) y - (\nabla_y J) Jx \]
and consequently
\[ g(Q(x, y), Q(Jz, Jw)) = -g(Q(x, y), Q(z, w)) - g(Q(x, y), (\nabla_J J) Jz). \]

The last equality and (5.1) imply the following

**Corollary 5.3.** Let $\nabla'$ be the connection with parallel totally skew-symmetric torsion and Kähler curvature tensor on the quasi-Kähler manifold with Norden metric $(M, J, g)$. Then
\[ R(x, y, Jz, Jw) + R(x, y, z, w) = g(Q(x, y), (\nabla_J J) Jz). \] (5.5)

Contracting by $x = e_i$, $w = e_j$ in (5.5), we obtain
\[ g^{ij} R(e_i, y, Jz, Je_j) + \rho(y, z) = g^{ij} g(Q(e_i, y), (\nabla_J J) e_j + (\nabla_e J) Jz). \]
Then, after a contraction by $y = e_k$, $z = e_s$, it follows
\[ \tau^{**} + \tau = g^{ij} g^{ks} g(Q(e_i, e_k), (\nabla_J e_s, J) e_j + (\nabla_e J) J e_s), \] (5.6)
where $\tau^{**} = g^{ij} g^{ks} R(e_i, e_k, Je_s, Je_j)$.

From (3.12), (2.3) and (2.4) we have

$$g^{ij} g^{ks} g(Q(e_i, e_k), (\nabla_{Je_s} J) e_j + (\nabla_{ej} J) Je_s) = -\frac{1}{8} \|\nabla J\|^2.$$ 

Then (5.6) can be rewritten as

$$\tau^{**} + \tau = -\frac{1}{8} \|\nabla J\|^2.$$ 

On the other hand, according to [8], we have

$$\tau^{**} + \tau = -\frac{1}{2} \|\nabla J\|^2.$$ 

Then $\|\nabla J\|^2 = 0$ and therefore the following is valid.

**Theorem 5.4.** Let $(M, J, g)$ be a quasi-Kähler manifold with Norden metric which admit a connection with parallel totally skew-symmetric torsion and Kähler curvature tensor. Then $(M, J, g)$ is a isotropic Kähler manifold with Norden metric. 

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Connection with skew-symmetric torsion on almost Norden manifolds

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