More Anomalies from Fractional Branes

M. Bertolini $^a$, P. Di Vecchia $^a$, M. Frau $^b$, A. Lerda $^{c,b}$, R. Marotta $^d$

$^a$ NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

$^b$ Dipartimento di Fisica Teorica, Università di Torino
and I.N.F.N., Sezione di Torino, Via P. Giuria 1, I-10125 Torino, Italy

$^c$ Dipartimento di Scienze e Tecnologie Avanzate
Università del Piemonte Orientale, I-15100 Alessandria, Italy

$^d$ Dipartimento di Scienze Fisiche, Università di Napoli
Complesso Universitario Monte S. Angelo, Via Cintia, I-80126 Napoli, Italy

Abstract

In this note we show how the anomalies of both pure and matter coupled $\mathcal{N} = 1, 2$ supersymmetric gauge theories describing the low energy dynamics of fractional branes on orbifolds can be derived from supergravity.

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1 Introduction

Recently there has been a lot of activity in trying to generalize the AdS/CFT correspondence and promote it to a more general gauge/gravity duality for theories without conformal invariance or with less supersymmetry. Indeed many different approaches have been proposed to this aim, like for example the study of mass deformations of conformal field theories and of the corresponding RG flows, or the study of D-branes wrapping supersymmetric cycles of K3 and Calabi-Yau spaces, or finally the study of fractional D-branes, in conifold and orbifold backgrounds.

Fractional branes in conifolds backgrounds were first considered in Refs. [1, 2, 3] where a configuration of N regular and M fractional D3 branes on T1,1 has been analyzed and the corresponding supergravity solution has been derived. This solution has been successfully used to describe the dual four dimensional gauge theory, which has N = 1 supersymmetry, a gauge group SU(N + M) × SU(N) and a non trivial matter content [4]. While very interesting IR properties of this gauge theory have been obtained by considering the non-singular solution on a deformed conifold [3], to describe the UV features of the gauge theory it is enough to consider the singular solution of Ref. [2] which indeed accounts for the logarithmic running of the coupling constants and the precise coefficients of the β-functions of the two gauge groups [3]. Very recently, in Ref. [6] it has been shown that also the chiral anomaly and the breaking of the U(1) R-symmetry to Z2M are correctly incorporated in the classical UV supergravity solution of Ref. [2].

Fractional D-branes in orbifold backgrounds have been extensively considered in the recent literature. In Ref. [7] the supergravity solution of M fractional D3 branes in a C2/Z2 orbifold has been explicitly obtained (see also Ref. [8]) and used to derive the exact (perturbative) β-function of the dual N = 2 SU(M) Yang-Mills theory. This analysis has been subsequently generalized by adding fractional D7 branes that yield hypermultiplets in the gauge theory [1, 10], or by considering D-branes in Zk orbifolds [11] (see Ref. [12] for a recent review on fractional branes in N = 2 orbifolds and a complete list of references). More recently, similar results have been also obtained for fractional D-branes in the N = 1 orbifold C3/(Z2 × Z2) [13].
In this paper, prepared while Ref. [6], which contains also a brief discussion of the chiral anomaly for the \( \mathcal{N} = 2 \) orbifold, has appeared, we show how the chiral anomalies of both \( \mathcal{N} = 2 \) and \( \mathcal{N} = 1 \) supersymmetric gauge theories in four dimensions can be very simply obtained from the explicit supergravity solutions describing fractional D3 branes in \( \mathbb{C}^2/\mathbb{Z}_2 \) and \( \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \) orbifolds presented, respectively, in Refs. [7, 10, 12] and Ref. [13]. In particular, we show how to obtain the correct \( \beta \)-function and the chiral anomaly of the pure \( \mathcal{N} = 1 \) super Yang-Mills. It is worth to emphasize that our present analysis does not rely on the probe technique which instead was used in our previous papers. In fact, here we will exploit a simple holographic representation of a gauge theory operator in terms of supergravity bulk quantities and deduce from it how the scale and chiral transformations are realized in the supergravity description.

2 \( \mathcal{N} = 2 \) gauge theories and fractional branes in \( \mathbb{Z}_2 \) orbifolds

Let us consider a \( \mathcal{N} = 2 \) super Yang-Mills theory in four dimensions with gauge group \( SU(M) \) and \( N \) fundamental hypermultiplets. As is well known (see for example Ref. [14]), this theory has a \( \beta \)-function given by

\[
\beta(g) = \frac{-(2M - N)}{16\pi^2} g^3
\]

where \( g \) is the running coupling constant, and a \( U(1)_R \) anomaly given by

\[
\partial_\alpha J^\alpha_R = 2(2M - N) q(x) \quad ; \quad q(x) = \frac{1}{32\pi^2} F^\alpha a_{\alpha\beta} \tilde{F}^{\alpha\beta} a
\]

where \( J_R \) is the \( R \)-current and \( q(x) \) is the topological charge density. In writing these formulas we have used standard conventions, namely we have normalized the generators of \( SU(M) \) so that the quadratic Casimir invariant is \( M \) in the adjoint representation and \( 1/2 \) in the fundamental representation, and have assigned a \( R \)-charge 1 to the gauginos and \( -1 \) to the chiral fermions in the hypermultiplets.\(^1\)

In supersymmetric theories the scale anomaly, proportional to the \( \beta \)-function, and the chiral anomaly are in the same supersymmetry multiplet. In the theory at hand, this fact can easily be seen by introducing the complex quantity

\[
\tau_{YM} \equiv \frac{\theta_{YM}}{2\pi} + \frac{i}{g^2} \frac{4\pi}{g^2}
\]

where \( \theta_{YM} \) is the Yang-Mills \( \theta \)-angle, and observing that the term reproducing the anomalies has the following form \[\tau_{YM} = i \frac{2M - N}{4\pi} \log \frac{\phi^2}{\Lambda^2} \]

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\(^1\)We note that in Ref. [6] a different \( R \)-charge assignment has been given to the various fields. In particular their charges differ from ours by a factor of 1/3.
where $\Lambda$ is the dynamically generated scale and $\phi$ is the (complex) scalar field of the $\mathcal{N} = 2$ vector multiplet. As is well known, due to $\mathcal{N} = 2$ non-renormalization theorems, Eq. (2.4) is the complete perturbative result which is corrected only by instanton effects. Then the scale and chiral anomalies can be simply obtained by looking at the response of $\tau_{\text{YM}}$ to a rescaling of the energy by a factor of $\mu$ and to a $U(1)_R$ transformation with parameter $\alpha \in [0, 2\pi)$ respectively. Under these transformations the scalar field $\phi$, which has the scale dimension of a mass and a $R$-charge 2 (remember that the gauginos have $R$-charge 1), transforms as follows

$$
\phi \to \mu e^{2i\alpha} \phi ,
$$

and thus from Eq. (2.4) we easily find that

$$
\tau_{\text{YM}} \to \tau_{\text{YM}} + i \frac{2M - N}{2\pi} (\log \mu + 2i\alpha) .
$$

This equation implies that

$$
\frac{1}{g^2} \to \frac{1}{g^2} + \frac{2M - N}{8\pi^2} \log \mu \quad \text{and} \quad \theta_{\text{YM}} \to \theta_{\text{YM}} - 2(2M - N) \alpha ,
$$

which are equivalent to Eqs. (2.1) and (2.2) respectively.

We now show how these results can be obtained in a very simple way by using the supergravity solutions of fractional D-branes that we presented in Refs. [7, 10]. Let us recall that in string theory the simplest way to realize a four dimensional $\mathcal{N} = 2$ Yang-Mills theory with gauge group $SU(M)$ and $N$ hypermultiplets is to consider a stack of $M$ fractional D3 branes and $N$ D7 branes of type IIB in the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ [9, 10]. For definiteness we assume that the $\mathbb{Z}_2$ parity acts as a reflection in the directions 6789 (labeled by indices $\ell, m, ...$), and that the D3 branes extend along the directions 0123 (labeled by indices $\alpha, \beta, ...$) while the D7 branes wrap the directions 01236789. With this arrangement, the directions 4 and 5 are transverse to both types of branes and define a plane where they can move. In this plane it is convenient to use the complex coordinate

$$
z \equiv x^4 + i x^5 = \rho e^{i\theta} .
$$

The supergravity background created by this D3/D7 brane system comprises the dilaton $\varphi$, the (Einstein frame) metric of the form

$$
ds^2 = H^{-1/2} \eta_{\alpha\beta} dx^\alpha dx^\beta + e^{-\varphi} H^{1/2} (d\rho^2 + \rho^2 d\theta^2) + H^{1/2} \delta_{\ell m} dx^\ell dx^m ,
$$
a R-R 0-form $C_{(0)}$, a R-R 4-form $C_{(4)}$, and two 2-forms, namely $B_{(2)}$ from the NS-NS sector and $C_{(2)}$ from the R-R sector, which are given by

$$
C_{(2)} = c \omega_{(2)} , \quad B_{(2)} = b \omega_{(2)}
$$

(2.10)
where $\omega^{(2)}$ is the anti-self dual 2-form associated to the vanishing 2-cycle of the orbifold ALE space. The explicit expressions for the various fields have been derived in Ref. [10] (see also Ref. [9]), but for our present purposes it is enough to recall that

\[ e^{\phi} = \frac{1}{1 - \frac{Ng_s}{2\pi} \log \frac{\rho}{\epsilon}} , \]  
\[ C_{(0)} = \frac{Ng_s}{2\pi} \theta , \]  
\[ b = (2\pi^2 \alpha') \frac{1 + \frac{(2M - N)g_s}{\pi} \log \frac{\rho}{\epsilon}}{1 - \frac{Ng_s}{2\pi} \log \frac{\rho}{\epsilon}} , \]  
\[ c = -(2\pi\alpha') g_s \left( 2M - N - \frac{1}{2} - \frac{2Mg_s}{N} \log \frac{\rho}{\epsilon} \right) \theta \]  

where $g_s$ is the string coupling constant and $\epsilon$ is a regulator. Notice that the explicit appearance of the angle $\theta$ in the above expressions of $C_0$ and $c$ implies that this supergravity solution is not invariant under rotations in the transverse plane, a fact that is similar to what has been recently emphasized in Ref. [6] for the fractional D-branes of the conifold.

Let us now consider the world-volume theory of the fractional D3/D7 brane system which, in the limit $\alpha' \to 0$, reduces to $\mathcal{N} = 2$ super Yang-Mills theory with gauge group $SU(M)$ and $N$ fundamental hypermultiplets in four dimensions. In particular, the action $S_{\text{YM}}$ for the bosonic fields in the vector multiplet can be simply obtained by taking the Dirac-Born-Infeld action plus the Wess-Zumino term of a fractional D3 brane and expanding it in the supergravity background previously considered. Taking the limit $\alpha' \to 0$ and keeping fixed the combination

\[ \phi = (2\pi\alpha')^{-1} z \]  

which plays the role of the scalar field of the $\mathcal{N} = 2$ vector multiplet, we easily find [10]

\[ S_{\text{YM}} = -\frac{1}{g^2} \int d^4x \left\{ \frac{1}{4} F_{\alpha\beta}^a F_a^{\alpha\beta} + \frac{1}{2} \partial_{\alpha} \phi \partial^{\alpha} \phi \right\} + \frac{\theta_{\text{YM}}}{32\pi^2} \int d^4x F_{\alpha\beta}^a \tilde{F}^{\alpha\beta}_a \]  

where

\[ \frac{1}{g^2} = \frac{1}{16\pi^3 \alpha' g_s} = \frac{1}{8\pi g_s} + \frac{2M - N}{8\pi^2} \log \frac{\rho}{\epsilon} \]  
\[ \theta_{\text{YM}} = \frac{c + C_{(0)} b}{2\pi\alpha' g_s} = -(2M - N) \theta . \]

Inserting these expressions in Eq.(2.3), we can see that $\tau_{\text{YM}}$ is a simple holomorphic function of $z$, namely

\[ \tau_{\text{YM}} = i \frac{2M - N}{2\pi} \log \frac{z}{\rho_e} \]  

\[ \text{We take this opportunity to correct a sign misprint in Eq.(5.5) of Ref. [10].} \]
where \( \rho_e = e^{\frac{-\pi}{(2M-N)g_s}} \) is the distance in the z-plane where the enhançon phenomenon takes place \[14\]. Finally, using Eqs.(2.4) and (2.13) we deduce that the scale and chiral transformations are realized on the supergravity coordinate \( z \) as follows

\[
z \to \mu e^{2i\alpha} z ,
\]

and hence the field theory results (2.6) and (2.7) are precisely reproduced from the supergravity classical solution.

The above analysis can be easily generalized by discussing a more general bound state in which both types of fractional D3 branes present in the \( \mathbb{Z}_2 \) orbifold are considered \[12\]. The most general configuration one can have is made of \( N_1 \) branes of type 1 and \( N_2 \) branes of type 2. In this case the world-volume theory is a \( \mathcal{N} = 2 \) Yang-Mills theory with gauge group \( SU(N_1) \times SU(N_2) \) \[8\] and two hypermultiplets in the bifundamental representations \((N_1, \overline{N}_2)\) and \((\overline{N}_1, N_2)\) respectively. In this case our previous analysis can be generalized in a straightforward way and the correct results are simply obtained by replacing in all formulas the quantity \( 2(M-N) \) with \( 2(N_1-N_2) \) if we refer to the first factor of the gauge group, or with \( 2(N_2-N_1) \) if instead we refer to the second factor of the gauge group.

Finally, we would like to point out that the method of obtaining the \( \beta \)-function and the chiral anomaly from the fractional brane solutions as we have described it now, does not rely on the use of the probe technique which instead was employed in Refs. \[7, 10, 11\], and in particular does not require that the analysis be made in the Coulomb branch of the \( \mathcal{N} = 2 \) theory. The only necessary ingredients are the holographic identification of a world-volume field in terms of bulk supergravity quantities, as we did for example in Eq.(2.15), and its behavior under scale and chiral transformations. Everything else then follows from the explicit expressions Eqs.(2.17) and (2.18) of the gauge coupling constant and the \( \theta \)-angle in terms of the supergravity fields, expressions that are dictated by the low-energy limit of the world-volume action of the fractional branes. Thus this method can be in principle applied also to \( \mathcal{N} = 1 \) models as we will discuss in the next section.

### 3 \( \mathcal{N} = 1 \) gauge theories and fractional branes in \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifolds

Let us now consider type IIB string theory in the orbifold \( \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \) which provides the simplest set up to realize supersymmetric gauge theories with \( \mathcal{N} = 1 \) supersymmetry in four dimensions by means of fractional D3 branes, whose supergravity solution was obtained in Ref. \[13\]. For definiteness we take the orbifold directions to be \( x^4, ..., x^9 \), introduce three complex coordinates defined by

\[
z_1 \equiv x^4 + i x^5 = \rho_1 e^{i\theta_1} , \quad z_2 \equiv x^6 + i x^7 = \rho_2 e^{i\theta_2} , \quad z_3 \equiv x^8 + i x^9 = \rho_3 e^{i\theta_3} ,
\]

\[3\]

\[We neglect a diagonal \( U(1) \) factor which is decoupled, and the relative \( U(1) \) which are subleading in the large \( N \)-limit.
and consider fractional D3 branes that are completely transverse to the orbifold, \textit{i.e.} that are extended along \(x^\alpha\) with \(\alpha = 0, 1, 2, 3\). As explained in Refs. [17, 13], there are four types of such fractional D3 branes corresponding to the four irreducible representations of \(Z_2 \times Z_2\), none of which is free to move in the transverse space. The most general configuration we can consider is therefore a stack of \(N_1\) branes of type 1, \(N_2\) of type 2, \(N_3\) of type 3 and \(N_4\) of type 4, all located at the orbifold fixed point. On the world volume of this bound state there is a four dimensional \(\mathcal{N} = 1\) Yang-Mills theory with gauge group \(SU(N_1) \times SU(N_2) \times SU(N_3) \times SU(N_4)\) and bifundamental matter. In particular for each factor \(SU(N_I)\) of the gauge group one finds 6 chiral multiplets, 3 of them transforming in the bifundamental representation \((N_I, N_J)\) and 3 in the conjugate representation \((N_I, \bar{N}_J)\) with \(J \neq I\), for a total of 12 chiral multiplets. In the explicit string realization these fields are equipped with a suitable 4 \(\times\) 4 Chan-Paton matrix that specifies on which of the four types of branes the two end-points of the open string are attached. Taking this fact into account and picking the same complex structure as in Eq.(3.1), under the same conventions as those of Ref. [13], one can rearrange the 12 chiral multiplets into three 4 \(\times\) 4 matrices given by

\[
\Phi_1 = \begin{pmatrix}
0 & A_1 & 0 & 0 \\
B_1 & 0 & 0 & 0 \\
0 & 0 & C_3 & 0 \\
0 & 0 & D_1 & 0
\end{pmatrix}, \quad \Phi_2 = \begin{pmatrix}
0 & 0 & A_2 & 0 \\
0 & 0 & 0 & B_2 \\
C_2 & 0 & 0 & 0 \\
0 & D_2 & 0 & 0
\end{pmatrix}, \quad \Phi_3 = \begin{pmatrix}
0 & 0 & 0 & A_3 \\
0 & 0 & B_3 & 0 \\
0 & C_3 & 0 & 0 \\
D_3 & 0 & 0 & 0
\end{pmatrix}
\] (3.2)

where \(A_i, \cdots, D_i\) are each a chiral multiplet. The position of these multiplets inside the matrices indicates from which types of open strings they originate, for example \(A_1\) arises from strings stretched between branes of type 1 and branes of type 2, whereas \(C_3\) from strings stretched between branes of type 3 and branes of type 2 and so on. Each field matrix \(\Phi_i\) encodes those chiral superfields having dynamics in the \(z_i\) plane.

There are several important points that we would like to emphasize. First of all, the Lagrangian of this \(\mathcal{N} = 1\) theory contains a cubic superpotential of the form \(W = \text{Tr}(\Phi_1[\Phi_2, \Phi_3])\), which is renormalizable in the UV. This is to be contrasted to what happens in the conifold theory where the matter fields have a quartic unrenormalizable superpotential [13]. Secondly, the Lagrangian of the orbifold theory is classically invariant under scale transformations of the energy and \(U(1)_R\) transformations. Due to the presence of the cubic superpotential, the superfields \(\Phi_i\) have \(R\)-charge 2/3, and hence their scalar components \(\phi_i\) have charge 2/3 while the chiral fermions have charge \(-1/3\). Therefore, under a scale transformation with parameter \(\mu\) and a \(U(1)_R\) transformation with parameter \(\alpha\) we have

\[
\phi_i \rightarrow \mu e^{\frac{4i}{3}\alpha} \phi_i
\] (3.3)

\[\footnote{\text{Again we do not consider irrelevant } U(1) \text{ factors.}}\]
for $i = 1, 2, 3$. As is well known, these transformations become anomalous in the quantum theory. Indeed, focusing for simplicity on the first factor of the gauge group (similar considerations hold for any gauge groups), we find a scale anomaly proportional to the (Wilsonian) $\beta$-function

$$
\beta(g) = -\frac{(3N_1 - N_2 - N_3 - N_4)}{16\pi^2} g^3
$$

(3.4)

where $g$ is the running coupling constant of $SU(N_1)$, and a $U(1)_R$ anomaly given by

$$
\partial_a J^a_R = 2 \left( N_1 - \frac{1}{3} (N_2 + N_3 + N_4) \right) q(x)
$$

(3.5)

where $q(x)$ is the topological charge density for $SU(N_1)$ (see Eq.(2.2)). Just like in the $\mathcal{N} = 2$ theories, also in this case we can combine the effect of the scale and chiral anomalies together by writing

$$
\tau_{YM} \rightarrow \tau_{YM} + i \frac{(3N_1 - N_2 - N_3 - N_4)}{2\pi} \left( \log \mu + \frac{2}{3} i \alpha \right)
$$

(3.6)

where $\tau_{YM}$ is defined as in Eq.(2.3) in terms of the coupling constant and $\theta$-angle of the first factor of the gauge group. Of course similar expressions hold for the other factors and can be obtained from the previous formulas in a straightforward way. Notice that while the chiral anomaly is a one-loop effect, in $\mathcal{N} = 1$ gauge theories the $\beta$-function receives corrections at all loops. The reason why it is possible to construct the complex combination (3.6) is because here we are discussing the Wilsonian $\beta$-function which is perturbatively exact at one loop [18].

Let us now consider the supergravity background corresponding to our bound state of fractional D3 branes [13]. This is characterized by a metric of the form

$$
ds^2 = H^{-1/2} \eta_{\alpha\beta} dx^\alpha dx^\beta + H^{1/2} \delta_{\ell m} dx^\ell dx^m,
$$

(3.7)

a R-R 4-form $C(4)$ and three pairs of scalars $b_i$ and $c_i$ ($i = 1, 2, 3$) which correspond to the components of the 2-forms $B(2)$ and $C(2)$ along the anti-self dual forms $\omega^i_{(2)}$ associated to the three exceptional vanishing cycles of the orbifold, namely

$$
C_{(2)} = c_i \ \omega^i_{(2)} \ , \ \ B_{(2)} = b_i \ \omega^i_{(2)}
$$

(3.8)

The explicit form of the solution can be found in Ref. [13]; here we simply recall that

$$
b_i = (2\pi^2 \alpha') \left( 1 + \frac{2g_s}{\pi} f_i(N_I) \log \frac{\rho_i}{\epsilon} \right)
$$

(3.9)

$$
c_i = -(4\pi \alpha') g_s f_i(N_I) \theta_i
$$

(3.10)
where
\[
\begin{align*}
f_1(N_I) &= N_1 + N_2 - N_3 - N_4, \\
f_2(N_I) &= N_1 - N_2 + N_3 - N_4, \\
f_3(N_I) &= N_1 - N_2 - N_3 + N_4.
\end{align*}
\] (3.11)

As we mentioned before, the world-volume action of our bound state of fractional D3 branes in the limit $\alpha' \to 0$ reduces to a $\mathcal{N} = 1$ super Yang-Mills theory in four dimensions with gauge group $SU(N_1) \times SU(N_2) \times SU(N_3) \times SU(N_4)$ and bifundamental matter. The bosonic part of this action can be obtained by expanding the Dirac-Born-Infeld action plus the Wess-Zumino term for fractional D3 branes in the corresponding supergravity background, and then observing that the complex coordinates $z_i$ of Eq. (3.1) can be traded for the scalar components $\phi_i$ of the chiral superfields $\Phi_i$ of Eq. (3.2), similarly to what we have done in Eq. (2.15) with the scalar component of the $\mathcal{N} = 2$ vector multiplet. This correspondence can also be understood by looking at the explicit open-string realization of the scalars $\phi_i$, each of which is indeed related to a position in the $z_i$ plane. Using this identification between $\phi_i$ and $z_i$, and Eq. (3.3), we can find the realization of the scale and chiral transformations on the supergravity coordinates, namely
\[
z_i \to \mu e^{\frac{\pi}{2} i \alpha} z_i
\] (3.12)

We now focus for simplicity on those terms of world-volume action that, in the low-energy limit, depend only on the gauge fields of the first factor $SU(N_1)$ of the gauge group. These terms are simply
\[
S_{YM} = -\frac{1}{4g^2} \int d^4x \, F_{a\alpha\beta}^a F_a^{a\alpha\beta} + \frac{\theta_{YM}}{32\pi^2} \int d^4x \, F_{a\alpha\beta}^a \tilde{F}^{a\alpha\beta}
\] (3.13)

where
\[
\frac{1}{g^2} = \frac{1}{8\pi g_s} \left( \frac{1}{4\pi^2 \alpha'} \sum_i b_i - 1 \right) = \frac{1}{16\pi g_s} + \frac{1}{8\pi^2} \sum_i f_i(N_I) \log \frac{\rho_i}{\epsilon}
\]
\[
\theta_{YM} = \frac{1}{4\pi \alpha' g_s} \sum_i c_i = -\sum_i f_i(N_I) \theta_i
\] (3.14)

Using these formulas, we see that the supergravity realization of the complex coupling $\tau_{YM}$ is
\[
\tau_{YM} = i \left[ \frac{1}{4g_s} + \frac{1}{2\pi} \sum_i f_i(N_I) \log \frac{z_i}{\epsilon} \right]
\] (3.15)

From this equation it is now immediate to see that the field theory result (3.6) is correctly reproduced if we use the transformation (3.12) and the explicit definitions of the functions $f_i(N_I)$ given in Eq. (3.11).
We conclude by observing that a pure $\mathcal{N} = 1$ super Yang-Mills theory in four dimensions can be realized on a stack of $M$ fractional D3 branes of just one type, for example by putting $N_1 = M$ and $N_2 = N_3 = N_4 = 0$ in our previous analysis. Therefore, the method we have described allows to obtain the correct $\beta$-function and chiral anomaly of this theory from the supergravity solution, namely

$$\left. \frac{1}{g^2} \to \frac{1}{g^2} + \frac{3M}{8\pi^2} \log \mu \quad \text{and} \quad \theta_{\text{YM}} \to \theta_{\text{YM}} - 2M \alpha \ . \right. \quad (3.16)$$

Notice that the equation above correctly accounts for the breaking of the $U(1)$ R-symmetry to $\mathbb{Z}_{2M}$. To our knowledge this is the first quantitative derivation of these results for pure $\mathcal{N} = 1$ super Yang-Mills theory using a supergravity dual background. Since these are UV results, the naked singularity of the supergravity solution does not play any role. It would be very interesting to explore the possibility of resolving this singularity, for example by suitably deforming the orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, and see whether in this way one can get some information on the IR behavior of the dual gauge theory.

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