CONSTANT WEIGHTED MEAN CURVATURE HYPERSURFACES IN SHRINKING RICCI SOLITONS

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ABSTRACT. In this paper, we study constant weighted mean curvature hypersurfaces in shrinking Ricci solitons. First, we show that a constant weighted mean curvature hypersurface with finite weighted volume cannot lie in a region determined by a special level set of the potential function, unless it is the level set. Next, we show that a compact constant weighted mean curvature hypersurface with a certain upper bound or lower bound on the mean curvature is a level set of the potential function. We can apply both results to the cylinder shrinking Ricci soliton ambient space. Finally, we show that a constant weighted mean curvature hypersurface in the Gaussian shrinking Ricci soliton (not necessarily properly immersed) with a certain assumption on the integral of the second fundamental form must be a generalized cylinder.

1. INTRODUCTION

Let \((\bar{M}^{n+1}, \bar{g}, f)\) be a smooth metric measure space and let \(M^n\) be a oriented hypersurface in \(\bar{M}^{n+1}\). The weighted mean curvature vector \(\vec{H}_f\) is defined by
\[
\vec{H}_f = \vec{H} + (\nabla f) \perp
\]
and the weighted mean curvature \(H_f\) is defined by \(\vec{H}_f = -H_f N\), where \(\perp\) is the projection to the normal bundle and \(N\) is the unit normal vector. A hypersurface \(M^n\) is said to have constant weighted mean curvature (CWMC) if \(H_f\) is a constant (see Section 2). These hypersurfaces are also known as \(\lambda\)-hypersurfaces where \(\lambda = H_f\). CWMC hypersurfaces can be seen as critical points of the weighted area functional with respect to weighted volume-preserving variations (see [9], [23]). When \(H_f = 0\) they are called \(f\)-minimal hypersurfaces. For certain choices of \(f\), \(f\)-minimal hypersurfaces are very important singularities of the mean curvature flow in \(\mathbb{R}^{n+1}\), namely self-shrinkers (see p.758 and p.768 in [15]), translating solitons (see p.153 and p.154 in [19]) and self-expanders (see p.9016 and p.9017 in [3]). It turns out that \((\mathbb{R}^{n+1}, \bar{g}_{\text{can}}, f)\) is a Ricci soliton for each of these choices of \(f\) (see Section 2). Thus the study of self-shrinkers, translating solitons and self-expanders in \(\mathbb{R}^{n+1}\) is equivalent to the study of \(f\)-minimal hypersurfaces in the shrinking, steady and expanding Ricci soliton \((\mathbb{R}^{n+1}, \bar{g}, f)\) respectively.

There has been a great interest in CWMC hypersurfaces in the Gaussian shrinking Ricci soliton \((\mathbb{R}^{n+1}, \bar{g}_{\text{can}}, f)\) with \(f(x) = \frac{|x|^2}{4}\). In [23] Mcongale-Ross classified stable CWMC hypersurfaces properly immersed in the Gaussian shrinking Ricci soliton. They also proved that there are no CWMC hypersurfaces properly immersed in this ambient space with index one. In [7] Q.M.Cheng-Ogata-Wei classified complete CWMC hypersurfaces in the Gaussian shrinking Ricci soliton with a certain condition on the norm of the second fundamental form and the mean curvature. In [9] Q.M.Cheng-Wei classified complete CWMC hypersurfaces embedded in the Gaussian shrinking Ricci soliton with polynomial volume growth and \(H - H_f \geq 0\). In [17] Guang classified compact CWMC surfaces embedded in the Gaussian shrinking Ricci soliton with \(H_f \geq 0\) and constant norm of the second fundamental form. He also classified complete CWMC hypersurfaces embedded in the Gaussian shrinking Ricci...
soliton with a certain condition on the norm of the second fundamental form. In [10] Q.M.Cheng-Wei generalized this result to complete CWMC surfaces and removed the condition on $H_{f}$.

There has also been a great interest in f-minimal and CWMC hypersurfaces in the cylinder shrinking Ricci soliton $\left(\mathbb{R}^{n+1-k} \times S^{k}_{\sqrt{2(k-1)}}, \bar{g}, \bar{f}\right)$, where $\bar{g}$ is the product metric and $f(x,y) = \frac{|x|^2}{4}$ (here $x$ is the position vector in $\mathbb{R}^{n-k+1}$ and $y$ is the position vector in $\mathbb{R}^{k+1}$). In [12] X.Cheng-Mejia-Zhou classified compact f-minimal hypersurfaces in a cylinder shrinking Ricci soliton with a certain condition on the norm of the second fundamental form. They also classified compact f-minimal hypersurfaces in this ambient space with index one. In [13] X.Cheng-Zhou generalized these results to complete hypersurfaces. In [22] Barbosa-Santana-Upadhyay obtained theorems for CWMC hypersurfaces in cylinder shrinking Ricci soliton similar to the results of McGonagle-Ross [23] but in the cylinder shrinking Ricci soliton ambient space. They also obtained theorems for CWMC hypersurfaces in cylinder shrinking Ricci soliton similar to the results of X.Cheng-Mejia-Zhou [12] and X.Cheng-Zhou [14] about $L_{f}$-stable (which are those that the second variation of weighted area is non-negative) $f$-minimal hypersurfaces.

In this paper we study geometric properties and classification results for CWMC hypersurfaces in shrinking Ricci solitons. Let $(\bar{M}^{n+1}, \bar{g}, \bar{f})$ be a gradient Ricci soliton with
\[ \bar{Ric} + \bar{\nabla} \bar{f} = \lambda \bar{g}, \]
where $\lambda$ is a constant. We know that
\[ |\nabla f|^2 + \bar{R} - 2\lambda f = C, \]
where $\bar{R}$ is the scalar curvature of $\bar{M}^{n+1}$ and $C$ is a constant (see p.85 in [18] and Lemma 1.1 in [4]). Let us define
\[ D^{\pm}(c_1, c_2) = \{ x \in \bar{M}^{n+1}; f(x) < \Gamma^{\pm}(x) \}, \]
where
\[ \Gamma^{\pm}(x) = \frac{1}{2\kappa} \left\{ \frac{1}{4} \left( \pm c_1 + \sqrt{c_1^2 + 4c_2} \right)^2 + \bar{R}(x) - C \right\}, \]
and $c_1$ and $c_2$ are constants such that $c_1^2 + 4c_2 \geq 0$.

Note that $D^{+}(c_1, c_2)$ and $D^{-}(c_1, c_2)$ are open subsets of $\bar{M}^{n+1}$. In this paper we will always use the above notations and conventions.

In Theorem 3 in [27], Vieira-Zhou proved that if $M^n$ is a complete f-minimal hypersurface properly immersed in the cylinder shrinking Ricci soliton then: (a) $M^n$ cannot lie inside the closed product $\bar{B}^{n+1-k}_{\sqrt{2(n-k)}}(0) \times S^{k}_{\sqrt{2(k-1)}}(0)$, unless $M^n = S^{n-k}_{\sqrt{2(k-1)}}(0) \times S^{k}_{\sqrt{2(k-1)}}(0)$ (see also the earlier work of Cavalcante-Espinar [4] for CWMC hypersurfaces in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$); (b) $M^n$ cannot lie outside the product $\bar{B}^{n+1-k}_{\sqrt{2(n+1-k)}}(0) \times S^{k}_{\sqrt{2(k-1)}}(0)$. We generalize this result in two different ways: we extend the result to CWMC hypersurfaces and we consider an arbitrary shrinking Ricci soliton ambient space.

**Theorem 1.** Let $\bar{M}^{n+1}$ be a shrinking Ricci soliton and let $M^n$ be a complete CWMC hypersurface immersed in $\bar{M}^{n+1}$ with finite weighted volume.

(a) Suppose that $\text{tr}_{\bar{M}^{n}} \bar{\nabla} \bar{f} \geq a$ for some $a > 0$. If $M^n$ lies in $\bar{D}^{-}(\|H_{f}\|, a)$ then $M^n \subseteq \partial \bar{D}^{-}(\|H_{f}\|, a)$.

(b) Suppose that $\text{tr}_{\bar{M}^{n}} \bar{\nabla} \bar{f} \leq b$ for some $b > 0$. If $M^n$ lies outside $\bar{D}^{+}(\|H_{f}\|, b)$ then $M^n \subseteq \partial \bar{D}^{+}(\|H_{f}\|, b)$.

We remark that $\bar{\nabla} \bar{\nabla} f$ is a $(0,2)$ tensor in $\bar{M}^{n+1}$, so we can consider its restriction to $M^n$ (in this case the trace $\text{tr}_{M^n} \bar{\nabla} \bar{\nabla} f$ is a function in $M^n$). Note that we can state Theorem 1 in a different way using the fact that
\[ \text{tr}_{M^n} \bar{\nabla} \bar{\nabla} f = n\lambda - \bar{R} + \bar{Ric}(N, N), \]
where $N$ is the unit normal vector of $M^n$. 
Using the above result together with a classification result for the level sets of the potential function (see Theorem 17) we obtain a new result for the cylinder shrinking Ricci soliton ambient space.

**Corollary 2.** Let $M^n$ be a complete CWMC hypersurface properly immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n+1-k} \times S^k/\sqrt{2(k-1)}(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$.

(a) If $M^n$ lies inside the closed product $B_{r}^{n+1-k}(0) \times S^k/\sqrt{2(k-1)}(0)$ with $r = -|H_f| + \sqrt{H_f^2 + 2(n-k)}$, then $H_f \geq 0$ and $M^n = S_{r}^{n-k}(0) \times S^k/\sqrt{2(k-1)}(0)$.

(b) $M^n$ cannot lie outside the product $B_{r}^{n+1-k}(0) \times S^k/\sqrt{2(k-1)}(0)$ with $r = |H_f| + \sqrt{H_f^2 + 2(n+1-k)}$.

In Theorem 3.5 in [17], Guang proved that if $M^n$ is a compact CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$ satisfying $H_f \geq 0$ and

$$|A|^2 \leq \frac{1}{2} + \frac{H_f \left( H_f + \sqrt{H_f^2 + 2n} \right)}{2n},$$

then $M^n = S_{r}^n(0)$, where $r = -H_f + \sqrt{H_f^2 + 2}$. We generalize this result in two different ways: we assume an upper bound on the mean curvature (this assumption is weaker since $H^2 \leq n|A|^2$) and we consider an arbitrary shrinking Ricci soliton ambient space. Here $A$ is the second fundamental form and $H$ is the mean curvature.

**Theorem 3.** Let $M^{n+1}_f$ be a shrinking Ricci soliton and let $M^n$ be a compact CWMC hypersurface immersed in $M^{n+1}_f$. Suppose that $tr_{M^n}\nabla f \geq a$ for some $a > 0$. Assume that $f(p) = sup_{M^n} f$ is a regular value of $f$. If

$$H \leq \frac{2H_f - |H_f| + \sqrt{H_f^2 + 4a}}{2},$$

then $M^n \subseteq \partial D^-(|H_f|, a)$.

**Remark 4.** The theorem above generalizes Theorem 3.5 in [17]. Indeed, when the ambient space is the Gaussian shrinking Ricci soliton (see Example 13) we have $tr_{M^n}\nabla f = \frac{a}{2}$ (so $a = \frac{n}{2}$). Assuming that $H_f \geq 0$ and

$$|A|^2 \leq \frac{1}{2} + \frac{H_f \left( H_f + \sqrt{H_f^2 + 2n} \right)}{2n} = \left( \frac{H_f + \sqrt{H_f^2 + 2n}}{2n} \right)^2$$

and using the fact that $\frac{1}{n}H^2 \leq |A|^2$ we have

$$H \leq \frac{2H_f - |H_f| + \sqrt{H_f^2 + 2n}}{2}.$$ 

Now using Theorem 3 we conclude that $M^n = S_{r}^n(0)$.

In particular, we obtain a new result for the cylinder shrinking Ricci soliton ambient space.

**Corollary 5.** Let $M^n$ be a compact CWMC hypersurface immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n-k+1} \times S^k/\sqrt{2(k-1)}(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$. If $H_f \geq 0$ and

$$H \leq \frac{H_f + \sqrt{H_f^2 + 2(n-k)}}{2},$$

then $M^n = S_{r}^{n-k}(0) \times S^k/\sqrt{2(k-1)}(0)$, where $r = -H_f + \sqrt{H_f^2 + 2(n-k)}$.

Note that the assumption on the upper bound of $H$ is sharp because equality holds for $M^n = S_{r}^{n-k}(0) \times S^k/\sqrt{2(k-1)}(0)$, where $r = -H_f + \sqrt{H_f^2 + 2(n-k)}$.

This is a new result even for $f$-minimal hypersurfaces.
Corollary 6. Let $M^n$ be a compact $f$-minimal hypersurface immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n-k+1} \times S^k(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$. If

$$H \leq \frac{\sqrt{2(n-k)}}{2},$$

then $M^n = S^{n-k}_e(0) \times S^k(0)$, where $r = \sqrt{2(n-k)}$.

In Theorem 1.2 in [7], Q.M.Cheng-Ogata-Wei proved that if $M^n$ is a complete CWMC hypersurface in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$ with polynomial volume growth and satisfying

$$\left( H - \frac{H_f}{2} \right)^2 \geq \frac{H^2_f + n}{4},$$

then $M^n = S^r(0)$, where $r = -H_f + \sqrt{H^2_f + 2n}$. We generalize this result to any ambient space which is a smooth metric measure space.

Theorem 7. Let $(\bar{M}^{n+1}, \bar{g}, f)$ be a smooth measure metric space and let $M^n$ be a complete hypersurface immersed in $\bar{M}^{n+1}$ with finite weighted volume. Suppose that $tr\bar{M} \bar{\nabla} f \leq b$ for some $b > 0$. If

$$H \geq \frac{H_f + \sqrt{H^2_f + 4b}}{2},$$

then $M^n \subseteq f^{-1}(\gamma)$ for some $\gamma \in \mathbb{R}$.

In particular, we recover the result of Q.M.Cheng-Ogata-Wei (see Corollary 26) and we obtain a new result for the cylinder shrinking Ricci soliton ambient space.

Corollary 8. Let $M^n$ be a complete CWMC hypersurface properly immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n+1-k} \times S^k(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$. Then

$$\inf H < \frac{H_f + \sqrt{H^2_f + 2(n+1-k)}}{2}.$$  

This is a new result even for $f$-minimal hypersurfaces.

Corollary 9. Let $M^n$ be a complete $f$-minimal hypersurface properly immersed in the cylinder shrinking Ricci soliton $\mathbb{R}^{n+1-k} \times S^k(0)$, where $n \geq 3$ and $2 \leq k \leq n-1$. Then

$$\inf H < \frac{\sqrt{2(n+1-k)}}{2}.$$  

In the second part of this paper, we obtain some classification theorems for the Gaussian shrinking Ricci soliton ambient space. Le-Sesum [21] proved that if $M^n$ is a complete self-shrinker in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$ with polynomial volume growth and satisfying $|A|^2 < 1/2$, then $M^n$ is a hyperplane passing through the origin. Later, Cao-Li [5] extended this result to arbitrary codimension by showing that if $|A|^2 \leq 1/2$, then $M^n$ is a generalized cylinder. Later, Guang [17] extended this result to CWMC hypersurfaces by assuming polynomial volume growth and a condition on the norm of the second fundamental form. Many of the classification results for CWMC hypersurfaces assume that the hypersurface has polynomial volume growth in order to use integration techniques. Recently, there has been some papers trying to avoid this assumption by using the Omori-Yau maximum principles ([17], [8], [25], etc). We can replace the assumption of polynomial volume growth in Guang’s result by an assumption on the integral of the second fundamental form.
Theorem 10. Let \( M^n \) be a complete embedded CWMC hypersurface in the Gaussian shrinking Ricci soliton \( \mathbb{R}^{n+1} \). If \( A \in L^q_f(M) \) for some \( q \geq 1 \) and 
\[
|A| \leq \sqrt{H_f^2 + 2 - |H_f|},
\]
then either \( M^n \) is a hyperplane or it is a generalized cylinder \( S^k_r(0) \times \mathbb{R}^{n-k} \), where \( 1 \leq k \leq n \).

We remark that for \( M^n = S^n_r(0) \) the assumption on the upper bound is not satisfied unless \( r = \sqrt{2n} \) (that is \( H_f = 0 \)) and for \( S^k_r(0) \times \mathbb{R}^{n-k} \) the assumption on the upper bound is not satisfied unless \( r = \sqrt{2k} \) (that is \( H_f = 0 \)). Therefore the result is only sharp in these cases. It would be interesting to find a sharp result for \( H_f \neq 0 \). This is a new result even for self-shrinkers.

Corollary 11. Let \( M^n \) be a complete embedded self-shrinker in the Gaussian shrinking Ricci soliton \( \mathbb{R}^{n+1} \). If \( A \in L^q_f(M) \) for some \( q \geq 1 \) and 
\[
|A|^2 \leq \frac{1}{2},
\]
then either \( M^n \) is a hyperplane passing through the origin or it is a generalized cylinder \( S^k_r(0) \times \mathbb{R}^{n-k} \), with \( 1 \leq k \leq n \) and \( r = \sqrt{2k} \).

We remark that Ancari-Miranda [1] proved recently it is possible to obtain a similar result if \( |A|^2 \leq \frac{1}{2} \) and \( H_f \in L^q_f(M) \) for some even number \( q \geq 2 \) which is a weaker assumption when \( q \) is even.

For self-shrinker submanifolds \( M^n \) in \( \mathbb{R}^{n+p} \), Q.M.Cheng-Peng [8] proved that if \( \sup |A|^2 < \frac{1}{2} \), then \( M^n \) is a hyperplane in \( \mathbb{R}^{n+1} \). For codimension 1 we generalize Q.M.Cheng-Peng’s result to CWMC hypersurfaces.

Corollary 12. Let \( M^n \) be a complete embedded CWMC hypersurface in \( \mathbb{R}^{n+1} \). If 
\[
\sup |A| < \sqrt{H_f^2 + 2 - |H_f|},
\]
then \( M^n \) is a hyperplane.

This paper is organized as follows. In section 2, we describe our notations and conventions and prove some basic results. In section 3 we prove Theorem 1, Theorem 3, Theorem 7 and related results. In Section 4 we prove Theorem 10 and related results.

2. Preliminaries and basics results

In this section we describe our notations and conventions and prove some basic results.

Smooth metric measure spaces. Let \((M, g)\) be a Riemannian manifold and let \( f \) be a smooth function on \( M \). The triple \((M, g, f)\) is called a smooth metric measure space. The measure \( e^{-f} \) is called weighted volume. If \( u \) and \( v \) are functions on \( M^n \) the \( L^2_f \) inner product \( u \) and \( v \) is defined by
\[
\langle u, v \rangle_{L^2_f(M)} = \int_M u v e^{-f}
\]
and the \( L^p_f \) norm of \( u \) is defined by
\[
|u|_{L^p_f(M)} = \left( \int_M |u|^p e^{-f} \right)^{\frac{1}{p}},
\]
where \( p \geq 1 \). The operator
\[
\Delta_f = \Delta - \langle \nabla f, \nabla \rangle
\]
is called weighted Laplacian. It is well known that the weighted Laplacian is a densely defined self-adjoint operator in \( L^2_f \), that is, if \( u \) and \( v \) are smooth functions on \( M \) with compact support we have
\[
\int_M \langle \Delta_f u, v \rangle e^{-f} = -\int_M \langle \nabla u, \nabla v \rangle e^{-f}.
\]
The Bakry-Émery-Ricci curvature is defined by
\[ \text{Ric}_f = \text{Ric} + \nabla \nabla f. \]
The triple \((M, g, f)\) is called a gradient Ricci soliton if
\[ \text{Ric}_f = \lambda g, \]
where \(\lambda\) is a constant. If \(\lambda\) is positive, zero or negative it is called shrinking, steady or expanding, respectively.

**Hypersurfaces in smooth metric measure spaces.** Let \((\bar{M}^{n+1}, \bar{g}, f)\) be a smooth metric measure space and let \(M^n\) be a oriented hypersurface of \(\bar{M}^{n+1}\). The second fundamental form is defined by
\[ A(u, v) = \langle \bar{\nabla} u v, N \rangle, \]
where \(u\) and \(v\) are vector fields on \(M^n\), \(\bar{\nabla}\) is the Riemannian connection of \(\bar{M}^{n+1}\) and \(N\) is the unit normal vector. The mean curvature vector is defined by
\[ \vec{H} = (\text{tr}_M A)N, \]
The weighted mean curvature vector is defined by
\[ \vec{H}_f = \vec{H} + (\nabla f)^\perp. \]
We remark that \(\vec{H}_f\) does not depend on the choice of the normal vector. The weighted mean curvature is defined by
\[ \vec{H}_f = -H_f N. \]
A hypersurface is said to have constant weighted mean curvature (CWMC) if the weighted mean curvature is constant. Note that \((M^n, g, f)\) is also a smooth metric measure space with weighted measure \(e^{-f} d\text{vol}_M\) and weighted Laplacian
\[ \Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle, \]
where \(g = \bar{g}|M\), \(\nabla\) is the Riemannian connection of \(M^n\) and \(\Delta\) is the Laplacian of \(M^n\).

**Example 13.** Let \(M^{n+1} = \mathbb{R}^{n+1}, \bar{g} = \bar{g}_{\text{can}}\) and \(f(x) = \frac{|x|^2}{4}\). The triple \((\mathbb{R}^{n+1}, \bar{g}, f)\) is a gradient shrinking Ricci soliton with Bakry-Émery-Ricci curvature
\[ \text{Ric}_f = \frac{1}{2} \bar{g}. \]
A hypersurface \(M^n\) of \(\mathbb{R}^{n+1}\) is a CWMC hypersurface if and only if
\[ H = \frac{\langle x, N \rangle}{2} + H_f. \]
Note that \(H_f = 0\) if and only if \(M^n\) is a self-shrinker. The weighted volume of \(M^n\) is \(e^{-\frac{|x|^2}{4}} d\text{vol}_M\) and the weighted Laplacian of \(M^n\) is given by
\[ \Delta_f = \Delta - \frac{1}{2} \langle x, \nabla \cdot \rangle. \]
Note that \(\Delta_f\) is the operator \(\mathcal{L}\) introduced by Colding and Minicozzi \[15\]. We say that \((\mathbb{R}^{n+1}, \bar{g}, f)\) is the Gaussian shrinking Ricci soliton.
**Example 14.** Let $\bar{M}^{n+1} = \mathbb{R}^{n+1}$, $\bar{g} = \bar{g}_{can}$ and $f(x) = -\frac{|x|^2}{4}$. The triple $(\mathbb{R}^{n+1}, \bar{g}, f)$ is a gradient expanding Ricci soliton with Bakry-Émery-Ricci curvature

$$\bar{Ric}_f = -\frac{1}{2} \bar{g}.$$ 

A hypersurface $M^n$ of $\mathbb{R}^{n+1}$ is a CWMC hypersurface if and only if

$$H = \frac{\langle x, N \rangle}{2} + H_f.$$ 

Note that $H_f = 0$ if and only if $M^n$ is a self-expander. The weighted volume of $M^n$ is $e^{\frac{|x|^2}{4}} dvol_M$ and the weighted Laplacian of $M^n$ is given by

$$\Delta_f = \Delta + \frac{1}{2} \langle x, \nabla \cdot \rangle.$$ 

**Example 15.** Let $\bar{M}^{n+1} = \mathbb{R}^{n+1}$, $\bar{g} = \bar{g}_{can}$ and $f(x) = \langle a, x \rangle$, where $a \in \mathbb{R}^{n+1}$. The triple $(\mathbb{R}^{n+1}, \bar{g}, f)$ is a gradient steady Ricci soliton with Bakry-Émery-Ricci curvature

$$\bar{Ric}_f = 0.$$ 

A hypersurface $M^n$ of $\mathbb{R}^{n+1}$ is a CWMC hypersurface if and only if

$$H = \langle a, N \rangle + H_f.$$ 

Note that $H_f = 0$ if and only if $M^n$ is a translating soliton. The weighted volume of $M^n$ is $e^{-\langle a, x \rangle} dvol_M$ and the weighted Laplacian of $M^n$ is given by

$$\Delta_f = \Delta - \frac{1}{2} \langle a, \nabla \cdot \rangle.$$ 

In [22] Lopez classified CWMC surfaces in the steady Ricci soliton $\mathbb{R}^3$ that are invariant by rotations and translations.

**Example 16.** Let $\bar{M}^{n+1} = \mathbb{R}^{n+1-k} \times S^k / \sqrt{2(k-1)}$ with product metric $\bar{g}$ and potential function $f(x, y) = \frac{|x|^2}{4}$. Here $x$ is the position vector in $\mathbb{R}^{n-k+1}$ and $y$ is the position vector in $\mathbb{R}^{k+1}$. The triple $(\mathbb{R}^{n+1-k} \times S^k / \sqrt{2(k-1)}, \bar{g}, f)$ is a gradient shrinking Ricci soliton with Bakry-Émery-Ricci curvature

$$\bar{Ric}_f = \nabla \nabla f + \bar{Ric} = \frac{1}{2} \bar{g}_{\mathbb{R}^{n+1-k}} + \frac{1}{2} \bar{g}_{S^k / \sqrt{2(k-1)}} = \frac{1}{2} \bar{g}.$$ 

A hypersurface $M^n$ of $\mathbb{R}^{n+1-k} \times S^k / \sqrt{2(k-1)}$ is a CWMC hypersurface if and only if

$$H = \frac{\langle x, N \rangle}{2} + H_f.$$ 

The weighted volume of $M^n$ is $e^{-\frac{|x|^2}{4}} dvol_M$ and the weighted Laplacian of $M^n$ is given by

$$\Delta_f = \Delta - \frac{1}{2} \langle x, \nabla \cdot \rangle.$$ 

We say that $(\mathbb{R}^{n+1-k} \times S^k / \sqrt{2(k-1)}, \bar{g}, f)$ is the cylinder shrinking Ricci soliton.

In the rest of the section we prove some results for the level set of the potential function of a shrinking Ricci soliton.
Theorem 17. Let \( \bar{M}^{n+1} \) be a complete gradient shrinking Ricci soliton with constant scalar curvature. Suppose that 
\[
M^n = \{ x \in \bar{M}^{n+1} : f(x) = \gamma \},
\]
where \( \gamma \) is a regular value of \( f \). Then \( M^n \) is a CWMC hypersurface with
\[
H_f = \frac{n\lambda - 2\lambda \gamma - C}{\sqrt{2\lambda \gamma - \bar{R} + C}},
\]
and
\[
\gamma = \frac{1}{2\lambda} \left\{ \frac{1}{4} \left( -H_f + \sqrt{H^2_f + 4(n\lambda - \bar{R})} \right)^2 + \bar{R} - C \right\}.
\]
Moreover \( \bar{R} \in \{0, 2\lambda, ..., n\lambda\} \).

Proof. Using the fact that \( N = |\bar{\nabla}f| \) and \( \bar{\nabla}^2 f = -\bar{Ric} + \lambda \bar{g} \) we have
\[
A(v, w) = -\left\langle \bar{\nabla}_v N, w \right\rangle
= -\left\langle \bar{\nabla}_v \bar{\nabla} f, w \right\rangle
= -\frac{1}{|\bar{\nabla} f|} \bar{\nabla} \bar{\nabla} f(v, w)
= \frac{1}{|\bar{\nabla} f|} \left( \bar{Ric}(v, w) - \lambda \langle v, w \rangle \right).
\]
Then
\[
H_f = H - \left\langle \bar{\nabla} f, N \right\rangle
= \frac{1}{|\bar{\nabla} f|} \left( n\lambda - tr_{M^n} \bar{Ric} \right) - \left\langle \bar{\nabla} f, \frac{\bar{\nabla} f}{|\bar{\nabla} f|} \right\rangle
= \frac{1}{|\bar{\nabla} f|} \left( n\lambda - \bar{R} + \bar{Ric}(N, N) - |\bar{\nabla} f|^2 \right).
\]
It is well known that if \( \bar{M}^{n+1} \) is a gradient Ricci soliton then \( \bar{Ric}(\bar{\nabla} f) = \frac{1}{2} \bar{\nabla} \bar{R} \) (see Equation 1.7 in [1]). Using this and the fact that \( \bar{R} \) is constant and \( N = \frac{\bar{\nabla} f}{|\bar{\nabla} f|} \) we have \( \bar{Ric}(N, N) = 0 \). We find that
\[
H_f = \frac{1}{|\bar{\nabla} f|} \left( n\lambda - \bar{R} - |\bar{\nabla} f|^2 \right).
\]
Using the fact that \( \bar{R} + |\bar{\nabla} f|^2 = 2\lambda f + C \) we get the first part of the result.

Now we prove the second part of the result. Multiplying by \( |\bar{\nabla} f| \) we obtain
\[
|\bar{\nabla} f|^2 + H_f |\bar{\nabla} f| - (n\lambda - \bar{R}) = 0.
\]
Solving the quadratic equation we get
\[
|\bar{\nabla} f| = \frac{1}{2} \left( -H_f + \sqrt{H^2_f + 4(n\lambda - \bar{R})} \right),
\]
or
\[
|\bar{\nabla} f| = \frac{1}{2} \left( -H_f - \sqrt{H^2_f + 4(n\lambda - \bar{R})} \right).
\]
Claim: The second case does not happen. We will prove the claim later. Assuming the claim we have
\[
|\bar{\nabla} f|^2 = \frac{1}{4} \left( -H_f + \sqrt{H^2_f + 4(n\lambda - \bar{R})} \right)^2.
\]
Using the fact that \(|\nabla f|^2 = 2\lambda f - \bar{R} + C\) we get
\[
f = \frac{1}{2\lambda} \left\{ \frac{1}{4} \left( -H_f + \sqrt{H_f^2 + 4(n\lambda - \bar{R})} \right)^2 + \bar{R} - C \right\}.
\]
This proves the second part of the result.

Now we prove the claim. We only need to show that
\[
|\nabla f| = \frac{1}{2} \left( -H_f - \sqrt{H_f^2 + 4(n\lambda - \bar{R})} \right) \leq 0.
\]
Since \(\bar{R}\) is constant on \(M^{n+1}\) we know that \(\bar{R} \in \{0, 2\lambda, \ldots, n\lambda, (n + 1)\lambda\}\) (see Theorem 1 in [16]). We will show that \(\bar{R} \in \{0, 2\lambda, \ldots, n\lambda\}\). Suppose that \(\bar{R} = (n + 1)\lambda\). In this case by Proposition 3.3 in [24] we see that \(M^{n+1}\) is Einstein, which implies that
\[
\text{Ric} = \frac{\bar{R}}{n + 1} \bar{g} = \lambda \bar{g}.
\]
Since \(\nabla \nabla f + \text{Ric} = \lambda \bar{g}\) we have \(\nabla \nabla f = 0\). This implies that \(|\nabla f|\) is constant on \(M^{n+1}\). Since \(2\lambda f = -|\nabla f|^2 - \bar{R} + C\) we see that \(f\) is constant on \(M^{n+1}\) and \(\nabla f = 0\). This contradicts the fact that \(M^{n}\) is a level set of the potential function. Therefore \(\bar{R} \in \{0, 2\lambda, \ldots, n\lambda\}\). Since \(n\lambda - \bar{R} \geq 0\) we have
\[
-H_f - \sqrt{H_f^2 + 4(n\lambda - \bar{R})} \leq 0.
\]
This proves the claim. \(\square\)

Remark 18. Note that when the ambient is a shrinking Ricci soliton with scalar curvature \(\bar{R} = n\lambda\), there is no level set which is \(f\)-minimal. Indeed,
\[
H_f = \frac{1}{|\nabla f|} \left( n\lambda - \bar{R} - |\nabla f|^2 \right)
\]
implies \(|\nabla f| = 0\), a contradiction.

Using the normalization \(\lambda = 1/2\) and \(C = \bar{R}\) (which is used in many papers, see for example pag 2 in [3]) we have:

Corollary 19. Let \(M^{n+1}\) be a complete gradient shrinking Ricci soliton with constant scalar curvature. Suppose that
\[
M^n = \{ x \in M^{n+1} : f(x) = \gamma \},
\]
where \(\gamma\) is a regular value of \(f\). Then \(M^n\) is a CWMC hypersurface with
\[
H_f = \frac{n - 2\gamma - 2\bar{R}}{2\sqrt{\gamma}},
\]
and
\[
\gamma = \frac{1}{4} \left( -H_f + \sqrt{H_f^2 + 4\left( \frac{n}{2} - \bar{R} \right)} \right)^2.
\]
Moreover \(\bar{R} \in \{0, 1, \ldots, \frac{n}{2}\}\).

For 3-dimensional shrinking Ricci solitons we have:

Corollary 20. Let \(M^3\) be a complete gradient shrinking Ricci soliton with constant scalar curvature. Suppose that
\[
M^2 = \{ x \in M^3 : f(x) = \gamma \},
\]
where \(\gamma\) is a regular value of \(f\). Then \(M^2\) is a CWMC hypersurface and \(\bar{R} \in \{0, 1\}\). Moreover
\[
H_f = \frac{1 - \gamma}{\sqrt{\gamma}} \text{ in the case } \bar{R} = 0,
\]
and
\[
H_f = -\sqrt{\gamma} \text{ in the case } \bar{R} = 1.
\]
For 4-dimensional shrinking Ricci solitons we have:

**Corollary 21.** Let $M^4$ be a complete gradient shrinking Ricci soliton with constant scalar curvature. Suppose that

$$M^3 = \{x \in M^4 : f(x) = \gamma\},$$

where $\gamma$ is a regular value of $f$. Then $M^3$ is a CWMC hypersurface and $\bar{R} \in \{0, 1, \frac{3}{2}\}$. Moreover

$$H_f = \frac{3 - 2\gamma}{2\sqrt{\gamma}}, \quad \text{in the case} \quad \bar{R} = 0,$$

and

$$H_f = \frac{1 - 2\gamma}{2\sqrt{\gamma}}, \quad \text{in the case} \quad \bar{R} = 1,$$

and

$$H_f = -\sqrt{\gamma}, \quad \text{in the case} \quad \bar{R} = \frac{3}{2}.$$  

Another consequence of Theorem 17 is the following.

**Corollary 22.** Let $r > 0$. Then $S^n_{r^{-k}}(0) \times S^k_{\sqrt{2(k-1)}}(0)$ is a CWMC hypersurface in the cylinder shrinking Ricci soliton $\mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}(0)$ and $r = -H_f + \sqrt{H_f^2 + 2(n-k)}$.

*Proof.* By Example 16 we have $\lambda = \frac{1}{2}$, $\bar{R} = \frac{k}{2}$ and $C = \bar{R}$ (since $C - \bar{R} = |\nabla f|^2 - 2\lambda f$). We have

$$S^n_{r^{-k}}(0) \times S^k_{\sqrt{2(k-1)}}(0) = \{ (x, y) \in \mathbb{R}^{n+1-k} \times S^k_{\sqrt{2(k-1)}}(0); f(x, y) = \gamma \},$$

where $\gamma = \frac{r^2}{4}$. By Theorem 17 we have

$$\gamma = \frac{1}{4} \left( -H_f + \sqrt{H_f^2 + 2(n-k)} \right)^2.$$

This proves the result. \qed

Using similar computations as in the proof of Theorem 17 we show that the level sets cannot be f-minimal when the ambient space is a steady or expanding Ricci soliton with constant scalar curvature.

**Proposition 23.** Let $M^{n+1}$ be a complete gradient Ricci soliton with constant scalar curvature. If there exists a regular level set of the potential function $f$ which is f-minimal, then $M^{n+1}$ is a gradient shrinking Ricci soliton.

*Proof.* Let $M^n = f^{-1}(\gamma)$ be a regular level set of $f$ which is f-minimal. By the proof of Theorem 17 we have

$$H_f = \frac{1}{|\nabla f|} \left( n\lambda - \bar{R} - |\nabla f|^2 \right).$$

From the assumption that $M^n$ is f-minimal we have

$$n\lambda - \bar{R} - |\nabla f|^2 = 0.$$

First suppose that $M^{n+1}$ is a steady soliton ($\lambda = 0$). By Theorem 1.3 in [29] we have $\bar{R} \geq 0$. From the equation above we conclude that $|\nabla f| = 0$, which is a contradiction with the fact that $M^n$ is a level of the potential function.

Now suppose that $M^{n+1}$ is an expanding Ricci soliton ($\lambda < 0$). By Theorem 1 in [16] we have $\bar{R} \in \{(n+1)\lambda, n\lambda, \ldots, 2\lambda, 0\}$. Using the same argument as in the proof of Lemma 17 we have $\bar{R} \geq n\lambda$. From the equation above we conclude that $|\nabla f| = 0$, which is a contradiction with the fact that $M^n$ is a level of the potential function. \qed

In Proposition 5.3 in [5], Cao-Li showed that there are no compact self-expanders in the expanding Ricci soliton $(\mathbb{R}^{n+1}, g_{can}, f = -|x|^2/4)$. Thus, the above result was expected for expanding Ricci solitons with a proper potential function.
3. Geometric results in shrinking Ricci solitons

In this section we prove Theorem 1, Theorem 3, Theorem 7 and related results.

Proof of Theorem 1.

Proof. Fact (a). Let $u$ be a function on $M^n$. If $u$ is bounded from above and $\Delta_fu \geq 0$ then $u$ is constant.

Fact (b). Let $u$ be a function on $M^n$. If $u$ is bounded from below and $\Delta_fu \leq 0$ then $u$ is constant.

Since $M^n$ has finite weighted volume both facts follow from Corollary 1 in [11]. We remark that the extension of these results to smooth metric measure spaces is straightforward. See also Theorem 25 and Remark 26 in [26].

We have

$$\nabla\nabla f(u, v) = \nabla\nabla f(u, v) + \langle \nabla f, (\nabla u) \rangle.$$

We see that

$$\Delta_f = \text{tr}_{M^n} \nabla\nabla f + \langle \nabla f, \tilde{H} \rangle - \langle \nabla f, \nabla f \rangle$$

$$= \text{tr}_{M^n} \nabla\nabla f + \langle \nabla f, \tilde{H} + (\nabla f) \rangle - \langle \nabla f, \nabla f \rangle$$

$$= \text{tr}_{M^n} \nabla\nabla f + \langle \nabla f, \tilde{H}_f \rangle - |\nabla f|^2.$$

Proof of Item (a). Since $\text{tr}_{M^n} \nabla\nabla f \geq a$ we have

$$\Delta_f f \geq a - |H_f| |\nabla f| - |\nabla f|^2.$$

Claim: We have

$$a - |H_f| |\nabla f| - |\nabla f|^2 \geq 0 \text{ on } M^n,$$

and equality holds if and only if $M^n \subseteq \partial D^- (|H_f|, a)$. We will prove this claim later. By the claim we have

$$\Delta_f f \geq a - |H_f| |\nabla f| - |\nabla f|^2 \geq 0.$$ 

Since $f \leq \Gamma^-$ on $M^n$, by Fact (a) we see that $f$ is constant on $M^n$. We conclude that

$$a - |H_f| |\nabla f| - |\nabla f|^2 = 0 \text{ on } M^n.$$

By the claim we have $M^n \subseteq \partial D^- (|H_f|, a)$. Now we prove the claim. We have

$$a - |H_f| |\nabla f| - |\nabla f|^2 \geq 0 \text{ on } M^n$$

$$\iff \left( |\nabla f| + \frac{|H_f|}{2} \right)^2 \leq \frac{H_f^2}{4} + a \text{ on } M^n$$

$$\iff |\nabla f| \leq \frac{1}{2} \left( -|H_f| + \sqrt{H_f^2 + 4a} \right) \text{ on } M^n.$$ 

Since $|\nabla f|^2 = 2\lambda f - \bar{R} + C$ and

$$2\lambda \Gamma^- - \bar{R} + C = \frac{1}{4} \left( -|H_f| + \sqrt{H_f^2 + 4a} \right)^2,$$

we have

$$M^n \subseteq \overline{D^- (|H_f|, a)} \iff f \leq \Gamma^- \text{ on } M^n$$

$$\iff 2\lambda f - \bar{R} + C \leq 2\lambda \Gamma^- - \bar{R} + C \text{ on } M^n$$

$$\iff |\nabla f|^2 \leq \frac{1}{4} \left( -|H_f| + \sqrt{H_f^2 + 4a} \right)^2 \text{ on } M^n$$

$$\iff |\nabla f| \leq \frac{1}{2} \left( -|H_f| + \sqrt{H_f^2 + 4a} \right) \text{ on } M^n.$$
In the last line we used the fact that \( a \geq 0 \). We conclude that
\[
a - |H_f| |\nabla f| - |\nabla f|^2 \geq 0 \text{ on } M^n \iff M^n \subset D^-(|H_f|, a).
\]
Moreover equality holds in the left hand side if and only if \( M^n \subseteq \partial D^-(|H_f|, a) \). This proves the claim.

Proof of Item (b). Since \( tr_{M^n} \nabla \nabla f \leq b \) we have
\[
\Delta f \leq b + |H_f| |\nabla f| - |\nabla f|^2.
\]

Claim. We have
\[
b + |H_f| |\nabla f| - |\nabla f|^2 \leq 0 \text{ on } M^n,
\]
and equality holds if and only if \( M^n \subseteq \partial D^+(|H_f|, b) \). We will prove this claim later. By the claim we have
\[
\Delta f \leq b + |H_f| |\nabla f| - |\nabla f|^2 \leq 0.
\]
Since \( f \geq \Gamma^+ \) on \( M^n \), by Fact (b) we see that \( f \) is constant on \( M^n \). We conclude that
\[
b + |H_f| |\nabla f| - |\nabla f|^2 = 0 \text{ on } M^n.
\]
By the claim we have \( M^n \subseteq \partial D^+(|H_f|, b) \). Now we prove the claim. We have
\[
b + |H_f| |\nabla f| - |\nabla f|^2 \leq 0
\]
\[
\iff \left( |\nabla f| - \frac{|H_f|}{2} \right)^2 \geq \frac{H_f^2}{4} + b \text{ on } M^n\]
\[
\iff |\nabla f| \geq \frac{1}{2} \left( |H_f| + \sqrt{H_f^2 + 4b} \right) \text{ on } M^n.
\]
Since \( |\nabla f|^2 = 2\lambda f - \bar{R} + C \) and
\[
2\lambda \Gamma^+ - \bar{R} + C = \frac{1}{4} \left( |H_f| + \sqrt{H_f^2 + 4b} \right)^2,
\]
we have
\[
M^n \subset \tilde{M}^{n+1} \setminus D^+(|H_f|, b) \iff f \geq \Gamma^+ \text{ on } M^n\]
\[
\iff 2\lambda f - \bar{R} + C \geq 2\lambda \Gamma^+ - \bar{R} + C \text{ on } M^n\]
\[
\iff |\nabla f|^2 \geq \frac{1}{4} \left( |H_f| + \sqrt{H_f^2 + 4b} \right)^2 \text{ on } M^n\]
\[
\iff |\nabla f| \geq \frac{1}{2} \left( |H_f| + \sqrt{H_f^2 + 4b} \right) \text{ on } M^n.
\]
In the last line we used the fact that \( b \geq 0 \). We conclude that
\[
b + |H_f| |\nabla f| - |\nabla f|^2 \leq 0 \text{ on } M^n \iff M^n \subset \tilde{M}^{n+1} \setminus D^+(|H_f|, b).
\]
Moreover equality holds in the left hand side if and only if \( M^n \subseteq \partial D^+(|H_f|, b) \).\( \square \)

The following lemma will be important throughout this work in order to guarantee the equivalence between properness and finite weighted volume.

Lemma 24. Let \((\tilde{M}^{n+1}, \tilde{g}, f)\) be a complete gradient shrinking Ricci soliton with constant scalar curvature and let \( M^n \) be a complete CWMC hypersurface in \( \tilde{M}^{n+1} \). Suppose that \( \nabla \nabla f(N, N) \geq k_1 \) or \( \text{tr}_{M^n} \nabla \nabla f \leq k_2 \) where \( k_1 \) and \( k_2 \) are constants. Then \( M^n \) is properly immersed if and only if \( M^n \) has finite weighted volume.

Proof. When \( \nabla \nabla f(N, N) \geq k_1 \), the result follows from Theorem 1.3 in [13].

Now suppose that \( \text{tr}_{M^n} \nabla \nabla f \leq k_2 \). Using the fact that \( \bar{R} + \nabla \nabla f = \lambda \tilde{g} \) we have
\[
(n + 1)\lambda = \bar{R} + \Delta f
\]
\[
= \bar{R} + \text{tr}_{M^n} \nabla \nabla f + \nabla \nabla f(N, N).
\]
We see that
\[ \nabla^2 f(N, N) \geq (n + 1)\lambda - R - k. \]

Using Theorem 1.3 in [13] we get the result. \(\square\)

Proof of Corollary 2

**Proof.** By Example 16 we have \(\lambda = \frac{1}{2}\) and \(C = R\) (since \(C - R = |\nabla f|^2 - 2\lambda f\)). We have \(\nabla^2 f = \frac{1}{2}g_{R^{n+1-k}}\) and

\[
tr_{M^n} \nabla^2 f = \Delta f - \nabla^2 f (N, N) \\
= \frac{n+1-k}{2} - \nabla^2 f (N, N).
\]

In particular
\[
\frac{n-k}{2} \leq tr_{M^n} \nabla^2 f \leq \frac{n+1-k}{2}.
\]

For \(D^- (|H_f|, a)\) with \(a = \frac{n-k}{2}\) we have
\[
\Gamma^- = \frac{1}{4} \left( -|H_f| + \sqrt{H_f^2 + 2(n-k)} \right)^2,
\]

and for \(D^+(|H_f|, b)\) with \(b = \frac{n-k+1}{2}\) we have
\[
\Gamma^+ = \frac{1}{4} \left( |H_f| + \sqrt{H_f^2 + 2(n+1-k)} \right)^2.
\]

Since \(M^n\) is properly immersed we know that \(M^n\) has finite weighted volume (see Lemma 24).

Proof of Item (a). Using the fact that \(f(x, y) = \frac{|x|^2}{4}\) we have \(D^- (|H_f|, a) = B^{n+1-k}_r (0) \times S^k_{2(n-k-1)} (0)\)

and \(\partial D^- (|H_f|, a) = S^{r-k}_r (0) \times S^k_{2(n-k-1)} (0)\) where \(r = -|H_f| + \sqrt{H_f^2 + 2(n-k)}\). Using the fact that \(M^n\) lies in \(B^{n+1-k}_r (0) \times S^k_{2(n-k-1)} (0)\) and Item (a) of Theorem 1 we have \(M^n \subseteq S^{r-k}_r (0) \times S^k_{2(n-k-1)} (0)\).

Since \(M^n\) is complete we have \(M^n = S^{r-k}_r (0) \times S^k_{2(n-k-1)} (0)\). On the other hand, by Corollary 22 we have \(r = -H_f + \sqrt{H_f^2 + 2(n-k)}\), so \(H_f \geq 0\).

Proof of Item (b). Using the fact that \(f(x, y) = \frac{|x|^2}{4}\) we have \(D^+(|H_f|, b) = B^{n-k+1}_r (0) \times S^k_{2(n-k-1)} (0)\)

and \(\partial D^+(|H_f|, b) = S^{r-k}_r (0) \times S^k_{2(n-k-1)} (0)\) where \(r = |H_f| + \sqrt{H_f^2 + 2(n+1-k)}\). Using the fact that \(M^n\) lies outside \(B^{n+1-k}_r (0) \times S^k_{2(n-k-1)} (0)\) and Item (b) of Theorem 1 we have \(M^n \subseteq S^{r-k}_r (0) \times S^k_{2(n-k-1)} (0)\). Since \(M^n\) is complete we have \(M^n = S^{r-k}_r (0) \times S^k_{2(n-k-1)} (0)\). On the other hand, by Corollary 22 we have \(r = -H_f + \sqrt{H_f^2 + 2(n-k)}\). Therefore
\[
-H_f + \sqrt{H_f^2 + 2(n-k)} = |H_f| + \sqrt{H_f^2 + 2(n+1-k)} > |H_f| + \sqrt{H_f^2 + 2(n-k)}.
\]

Hence
\[
-H_f > |H_f|,
\]

which is a contradiction. \(\square\)

For completeness we include an application of Theorem 1 to the Gaussian shrinking soliton ambient space. We omit the proof (it is similar to the proof of Corollary 2).
Corollary 25. Let $M^n$ be a complete CWMC hypersurface properly immersed in the Gaussian shrinking Ricci soliton $\mathbb{R}^{n+1}$.

(a) If $M^n$ lies inside $\bar{B}_r^{n+1}(0)$ with $r = -|H_f| + \sqrt{H_f^2 + 2n}$, then $H_f \geq 0$ and $M^n = S^n_r(0)$.

(b) If $M^n$ lies outside $B_r^{n+1}(0)$ with $r = |H_f| + \sqrt{H_f^2 + 2n}$, then $H_f \leq 0$ and $M^n = S^n_r(0)$.

Proof of Theorem 3.

Proof. Since $M^n$ is a compact hypersurface, the potential function achieves its maximum in some $p \in M$. From the fact that the gradient of $f$ points in the direction of greatest increase and $\nabla f(p) = 0$, we see that $\bar{\nabla} f(p)$ and $N(p)$ have the same direction. Therefore

$$H_f + |\nabla f|(p) = H_f + \langle \nabla f(p), N(p) \rangle = H(p).$$

Using the fact that $|\nabla f|^2 + \bar{R} - 2\lambda f = C$ and the hypothesis on the mean curvature, we get

$$\sqrt{2\lambda f(p) + C - \bar{R}} = |\nabla f|(p) \leq \frac{2H_f - |H_f| + \sqrt{H_f^2 + 4a}}{2} - H_f = \frac{-|H_f| + \sqrt{H_f^2 + 4a}}{2}.$$ 

Therefore

$$f \leq f(p) \leq \frac{1}{2\lambda} \left\{ \left( \frac{-|H_f| + \sqrt{H_f^2 + 4a}}{2} \right)^2 + \bar{R} - C \right\},$$

which implies that $M^n$ lies in $\overline{D^{-}}(|H_f|, a)$. By Theorem 1 we conclude that $M^n \subseteq \partial D^{-}(|H_f|, a)$. \qed

Proof of Corollary 5.

Proof. Take $a = \frac{a - k}{2}$. As in the proof of Corollary 2, Item (a) we have $\text{tr}_{M^n} \nabla \nabla f \geq a$ and $\partial D^{-} (H_f, a) = S^n_{r-k}(0) \times S^k_{\sqrt{2(k-1)}}(0)$ where $r = -H_f + \sqrt{H_f^2 + 2(n - k)}$. The result follows from Theorem 3. \qed

Proof of Theorem 7.

Proof. We compute

$$\Delta f = \text{tr}_M \nabla \nabla f - (\nabla f, N)H$$

$$= \text{tr}_M \nabla \nabla f + (H_f - H)H \leq b + (H_f - H)H$$

$$= \frac{H_f^2 + 4b}{4} - \left( H - \frac{H_f}{2} \right)^2 \leq 0.$$
In the third line we used the assumption on $\nabla^2 f$ and in the fourth line we used the assumption on $H$. Let $\varphi \in C_\infty(M^n)$. Integrating by parts, we have
\[
\int_M \varphi^2 |\nabla f|^2 e^{-f} \leq \int_M \varphi^2 (|\nabla f|^2 - \Delta f) e^{-f}
= -\int_M \varphi^2 \Delta f e^{-f}
= \int_M \langle \nabla \varphi^2, \nabla f \rangle e^{-f}
= 2 \int_M \langle \nabla \varphi, \varphi \nabla f \rangle e^{-f}
\leq \int_M \left[ 2 |\nabla \varphi|^2 + \frac{1}{2} \varphi^2 |\nabla f|^2 \right] e^{-f}.
\]
Therefore
\[
\frac{1}{2} \int_M \varphi^2 |\nabla f|^2 e^{-f} \leq 2 \int_M |\nabla \varphi|^2 e^{-f}.
\]
Fix $p_0 \in M^n$ and consider a sequence $\varphi_j \in C_\infty(M^n)$ such that $\varphi_j = 1$ on $B_{M^n}^j(p_0)$, $\varphi_j = 0$ on $M^n \setminus B_{M^n}^{2j}(p_0)$ and $|\nabla \varphi_j| \leq \frac{1}{j}$. Using the monotone convergence theorem and the fact that the weighted volume is finite we have
\[
\int_M |\nabla f|^2 e^{-f} = 0,
\]
which implies that $f$ is constant on $M^n$.

Using Theorem [7], Lemma [24] and Theorem [17] we obtain a new result for ambient spaces which are shrinking Ricci solitons with constant scalar curvature. Note that taking $b = \frac{n}{2}$, $\lambda = \frac{1}{2}$, $R = 0$ and $C = 0$ in the next result we recover Theorem 1.2 in [7].

**Corollary 26.** Let $M^{n+1}$ be a shrinking Ricci soliton with constant scalar curvature and let $M^n$ be a complete CWMC hypersurface properly immersed in $M^{n+1}$. Suppose that $\text{tr}_M \nabla^2 f \leq b$ for some $b > 0$. If
\[
H \geq \frac{H_f + \sqrt{H_f^2 + 4b}}{2},
\]
then $M^n \subseteq f^{-1}(\gamma)$ for some $\gamma > 0$. Moreover, if $f^{-1}(\gamma)$ is connected and complete then $M^n = f^{-1}(\gamma)$ where
\[
\gamma = \frac{1}{2\lambda} \left\{ \frac{1}{4} \left( -H_f + \sqrt{H_f^2 + 4(n\lambda - R)} \right)^2 + R - C \right\}.
\]

**Proof.** Since $M^n$ is properly immersed we know that $M^n$ has finite weighted volume (see Lemma [24]). Using Theorem [7] and Lemma [24] we have $M^n \subseteq f^{-1}(\gamma)$. Suppose that $f^{-1}(\gamma)$ is connected and complete. Then $M^n = f^{-1}(\gamma)$. Using Theorem [17] we get the conclusion of $\gamma$.

**Proof of Corollary 26**

**Proof.** Assuming that the conclusion of the result is false we will show that this leads to a contradiction. Take $b = \frac{n\lambda - R}{2}$. If the conclusion is false we have
\[
H \geq \frac{H_f + \sqrt{H_f^2 + 4b}}{2}.
\]
As in the proof Corollary [2] we have $\text{tr}_M \nabla^2 f \leq b$, $\lambda = \frac{1}{2}$ and $C = R = \frac{k}{2}$. Using Corollary [26] we have $M^n = f^{-1}(\gamma)$ where
\[
\gamma = \frac{1}{4} \left( -H_f + \sqrt{H_f^2 + 2(n - k)} \right)^2.
\]
Using the fact that \( f(x, y) = \frac{|x|^2}{4} \) we see that \( M^n = S^{n-k}_p(0) \times S^k_{\sqrt{2(k-1)}}(0) \) with \( r = -H_f + \sqrt{H_f^2 + 2(n-k)} \). However for \( M^n \) we have (see Corollary 5)

\[
H = \frac{H_f + \sqrt{H_f^2 + 2(n-k)}}{2} < \frac{H_f + \sqrt{H_f^2 + 2(n+1-k)}}{2} = \frac{H_f + \sqrt{H_f^2 + 4b}}{2},
\]
a contradiction. \( \square \)

4. Constant weighted mean curvature hypersurfaces in \( \mathbb{R}^{n+1} \)

In this section we prove Theorem 10 and Corollary 12.

Proof of Theorem 10

Proof. By [17] we have

\[
\Delta_f |A|^2 = 2 \left( \frac{1}{2} - |A|^2 \right) |A|^2 - 2H_f \text{tr}A^3 + 2|\nabla A|^2.
\]

Considering \( q \geq 1 \) as in the hypothesis, we have

\[
|A|^q \Delta_f |A|^2 = 2 \left( \frac{1}{2} - |A|^2 \right) |A|^{q+2} - 2|A|^q H_f \text{tr}A^3 + 2|A|^q |\nabla A|^2 \\
\geq 2 \left( \frac{1}{2} - |A|^2 \right) |A|^{q+2} - 2H_f |A|^{q+3} + 2|A|^q |\nabla A|^2 \\
= |A|^{q+2} (1 - 2|A|^2 - 2H_f |A|) + 2|A|^q |\nabla A|^2.
\]

Note that since

\[
|A| \leq \sqrt{\frac{H_f^2 + 2 - |H_f|}{2}},
\]

we have that \( 1 - 2|A|^2 - 2H_f |A| \geq 0 \). Therefore

\[
|A|^q \Delta_f |A|^2 \geq 2|A|^q |\nabla A|^2.
\]

Let \( \varphi \in C^\infty_c(M^n) \). Integrating by parts we have

\[
2 \int_M \varphi^2 |A|^q |\nabla A|^2 e^{-f} \leq \int_M \varphi^2 |A|^q \Delta_f |A|^2 e^{-f} \\
= -\int_M \langle \nabla \varphi^2 |A|^q, \nabla |A|^2 \rangle e^{-f} \\
= -2 \int_M q |A|^q |A|^{q+2} |\nabla |A||^2 e^{-f} - 4 \int_M \langle |A|^{q+1} \nabla \varphi, |A|^{\frac{q}{2}} |\nabla |A|| \rangle e^{-f} \\
\leq -2 \int_M q |A|^q |\nabla |A||^2 e^{-f} + 4 \int_M |A|^{q+1} |\nabla \varphi| |A|^{\frac{q}{2}} |\nabla |A|| e^{-f}.
\]

Here in the last line we use the Cauchy-Schwarz inequality. From the identity \( 2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2 \), choosing \( a = |A|^{q+1} |\nabla \varphi| \) and \( b = |A|^{\frac{q}{2}} |\nabla |A|| \) we have

\[
2 \int_M \varphi^2 |A|^q |\nabla A|^2 e^{-f} \leq 2\varepsilon \int_M |A|^{q+2} |\nabla \varphi|^2 e^{-f} + \left( \frac{2}{\varepsilon} - 2q \right) \int_M \varphi^2 |A|^q |\nabla A|^2 e^{-f}.
\]

Choosing \( \varepsilon = \frac{1}{q} \), we get

\[
\int_M \varphi^2 |A|^q |\nabla A|^2 e^{-f} \leq \frac{1}{q} \int_M |A|^{q+2} |\nabla \varphi|^2 e^{-f}.
\]

Since \( |A|^2 \leq C \), we have

\[
\int_M \varphi^2 |A|^q |\nabla A|^2 e^{-f} \leq \frac{C}{q} \int_M |A|^q |\nabla \varphi|^2 e^{-f}.
\]
Let us fix \( p_0 \in M \) and consider the sequence \( \phi_j \) such that \( \phi_j = 1 \) on \( B_j^M(p_0) \), \( \phi_j = 0 \) on \( M \setminus B_j^M(p_0) \) and \( |\nabla \phi_j| \leq \frac{1}{j} \). Using the monotone convergence theorem and the assumption that \( A \in L^q_f(M) \), we conclude that

\[
|A| |\nabla A| = 0.
\]

Let \( \mathcal{A} = \{ x \in M : |A|(x) = 0 \} \). The set \( M \setminus \mathcal{A} \) is open and since

\[
|A| |\nabla A| \leq |A| |\nabla A| = 0,
\]

we see that \( |A| \) is constant on \( M \setminus \mathcal{A} \). If \( \mathcal{A} \neq \emptyset \), then using continuity we conclude that \( |A| = 0 \), which implies that \( M^n \) is a hyperplane. If \( \mathcal{A} = \emptyset \), then \( |\nabla A| = 0 \). In this case using Theorem 4 in Lawson \[20\] and the fact that \( M^n \) is complete we conclude that \( M^n \) is a generalized cylinder. \( \square \)

Proof of Corollary \[12\].

Proof. We estimate the Bakry-Emery-Ricci tensor \( R_{icf} \) of a CWMC hypersurface. Let \( p \in M \) and choose an orthonormal basis of \( T_p M \) such that \( \nabla e_i e_j(p) = 0 \). Let \( h_{ij} = A(e_i, e_j) \). We have

\[
(R_{icf})_{ij} = R_{ij} + \nabla_{e_i} \nabla_{e_j} f = R_{ij} + \nabla_{e_i} \nabla_{e_j} f + \langle \nabla f, A(e_i, e_j) N \rangle = R_{ij} + \frac{1}{2} \delta_{ij} + \frac{\langle x, N \rangle}{2} h_{ij} = -H h_{ij} - \sum_{l=1}^{n} h_{il} h_{lj} + \frac{1}{2} \delta_{ij} h_{ij}
\]

\[
\geq -|H_f| |h_{ij}| - \sum_{l=1}^{n} h_{il} h_{lj} + \frac{1}{2} \delta_{ij} h_{ij}
\]

\[
\geq -|H_f| |A| - \sum_{l=1}^{n} h_{il} h_{lj} + \frac{1}{2} \delta_{ij} h_{ij}.
\]

In the fourth line we used the Gauss equation. Since \( \sup |A| < \frac{\sqrt{H_f^2 + 2 - |H_f|}}{2} \) we have

\[
R_{icf} \geq -|A|^2 - |H_f| |A| + \frac{1}{2} \geq -\sup |A|^2 - |H_f| \sup |A| + \frac{1}{2} > 0.
\]

By Theorem 4.1 in \[25\], the condition above implies that \( M^n \) has finite weighted volume. Since \( |A|^q \) is bounded and \( M^n \) has finite weighted volume, we see that \( |A| \in L^q_f(M) \). Applying Theorem \[10\] we conclude that \( M^n \) is a hyperplane. \( \square \)

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