Asymptotic stage of modulation instability for a fourth-order dispersive nonlinear Schrödinger equation with nonzero boundary conditions

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Abstract
In this work, we consider the long-time asymptotics for the Cauchy problem of a fourth-order dispersive nonlinear Schrödinger equation with nonzero boundary conditions at infinity. Firstly, in order to construct the basic Riemann-Hilbert problem associated with nonzero boundary conditions, we analysis direct scattering problem. Then we deform the corresponding matrix Riemann-Hilbert problem to explicitly solving models via using the nonlinear steepest descent method and employing the $g$-function mechanism to eliminate the exponential growths of the jump matrices. Finally, we obtain the asymptotic stage of modulation instability for the fourth-order dispersive nonlinear Schrödinger equation.

Key words: Long-time asymptotics; Fourth-order dispersive nonlinear Schrödinger equation equation; Riemann-Hilbert problem; Nonlinear steepest descent method.

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1 Introduction

The fourth-order dispersive nonlinear Schrödinger (NLS) equation with nonzero boundary conditions (NZBCs) is

\[ iq_t + q_{xx} + 2(|q|^2 - q_0^2)q + \gamma (q_{xxxx} + 8q_{xx}|q|^2 + 2q_x^2q^2 \]
\[ + 6q_x^2q^* + 4q|x|^2 + 6(|q|^4 - q_0^4)q) = 0, \]
\[ \lim_{x \to \pm \infty} q(x, 0) = q_\pm, \tag{1.1} \]

where \( q_\pm \) are complex constants and independent of \( x, t \) with \( |q_\pm| = q_0 > 0 \). Moreover

\[ q(x, 0) - q_\pm \in L^{1,1}(\mathbb{R}^\pm), \quad L^{1,1}(\mathbb{R}^\pm) = \{ f : \mathbb{R} \to \mathbb{C} | \int_{\mathbb{R}^\pm} (1 + |x|)|f(x)|dx < \infty \}. \tag{1.2} \]

With the Gauge transformation

\[ q(x, t) = u(x, t)e^{-2i(3\gamma q_0^2 + 1)q_0^2 t}, \tag{1.3} \]

Eq. (1.1) can be reduced to the following general fourth-order dispersive NLS equation \[1\]

\[ iu_t + u_{xx} + 2|u|^2u + \gamma (u_{xxxx} + 8u_{xx}|u|^2 + 2u_x^2u^2 + 6u_x^2u^* + 4u|x|^2 + 6u|u|^4) = 0, \]
\[ \lim_{x \to \pm \infty} u(x, t) = q_\pm e^{2i(3\gamma q_0^2 + 1)q_0^2 t}, \tag{1.4} \]

where \( u \) is complex function with temporal variable \( t \) and spatial variable \( x \), which denotes the slowly varying envelope of the wave. The parameter \( \gamma \) is a small dimensionless real numbers. In long distance and high speed optical fiber transmission systems, the fourth-order dispersion NLS equation plays a leading role in describing the transmission of ultrashort optical pulses \[2,3,4\]. Moreover, the equation can also depict the nonlinear spin excitation in a one-dimensional isotropic biquadratic Heisenberg ferromagnetic spin with octopole-dipole interaction \[5,6\]. So far, there are some works on the study of the fourth-order dispersive NLS equation. Many methods have been used to derive the exact solutions for the fourth-order dispersive NLS equation, such as Darboux transformation method, Hirota bilinear method and inverse scattering transform (IST) method \[7,8\], and the Lax pair, conservation laws, local wave solutions have also been discussed \[9,10,11\]. Recently, the long-time asymptotic behavior for the fourth-order dispersive NLS equation...
under zero boundary conditions (ZBCs) was investigated \cite{12,13}. To our known of knowledge, the long-time asymptotic behaviors for the fourth-order dispersive NLS equation with NZBCs have not been analyzed yet.

In fact, the asymptotic behavior of solutions for nonlinear integrable systems has a long history and is always a hot topic. Early studies can be traced back to literatures \cite{14, 15, 16, 17, 18, 19}. It is worth mentioning that Deift and Zhou, motivated by the pioneering work of Its \cite{19}, proposed the nonlinear steepest descent method to investigate the long-time asymptotic behavior for the Cauchy problem of the mKdV equation with a oscillatory Riemann-Hilbert (RH) problem \cite{20}. Subsequently, this method was further developed in references \cite{21, 22, 23}. Since the nonlinear steepest descent method been a efficient technique to research the Cauchy problem of integrable equations, the long-time asymptotics for lots of integrable equations as followed have been analyzed \cite{24, 25, 26, 27, 28, 29, 30, 31}. Besides, the method have been extended to the long-time asymptotics of the Cauchy problems for nonlinear integrable systems with a variety of non-decaying initial data, such as the time-periodic boundary conditions \cite{32, 33}, the shock problem \cite{34}, and the step-like initial data \cite{35, 36, 37}. Moreover, as a significant development of RH problem, $\hat{\sigma}$ generalization of the nonlinear steepest descent method was raised to derive the long-time asymptotic expansion of the solution in different fixed space-time regions \cite{38, 39, 40, 41}. Recent years, the researches about NZBCs at infinity have already been become a focal point. Biondini and his cooperators have studied the soliton solutions and the long-time asymptotics for the focusing NLS equation with NZBCs in \cite{42} and \cite{43}, respectively. After that, long-time asymptotics of the focusing Kundu-Eckhaus equation with NZBCs were studied in \cite{44}, long-time dynamics of the Gerdjikov-Ivanov type derivative nonlinear Schrödinger equation with NZBCs were studied in \cite{45}, long-time dynamics of the Hirota equation with NZBCs were studied in \cite{46}, and long-time dynamics of the modified Landau-Lifshitz equation with NZBCs were studied in \cite{47}. Besides, the long-time asymptotic behavior of nonlocal integrable NLS solutions with NZBCs were studied in \cite{48}.

In this work, motivated by the long-time asymptotic analysis presented in \cite{43}, we consider the long-time asymptotics of Eq. (1.1) with the NZBCs at infinity. To the best knowledge of the authors, the long-time asymptotics for the fourth-order dispersive NLS equation under the NZBCs has never been reported up to now.

The major results of this work is summarized in what follows:

**Theorem 1.1.** As $t \to \infty$, the asymptotic stage of modulation instability for $q(x,t)$ is given by

$$
q(x,t) = \frac{q_0}{q^*_0} (\chi_2 + q_0) \left( \frac{\Omega t + \vartheta + i \ln \left( \frac{q^*_+}{q_0} \right)}{2\pi} - V(\infty) + C \right) \Theta(V(\infty) + C) \\
\left( \frac{\Omega t + \vartheta + i \ln \left( \frac{q^*_+}{q_0} \right)}{2\pi} + V(\infty) + C \right) \Theta(-V(\infty) + C)
$$

$$
e^{2i(g(\infty) + G(\infty)t)} + O(t^{-\frac{1}{2}}),$$

(1.5)

where $V(\infty) = \int_{q_0}^{\infty} d\vartheta$, and $\chi_2, \Omega, G(\infty), \vartheta, g(\infty), C$ are given by Eqs. \cite{3.20, 3.32, 3.33, 3.37, 3.40, 3.56}.
Organization of this work: In Section 2, we perform the spectral analysis for the Cauchy problem of the fourth-order dispersive NLS equation, and construct the basic RH problem, which is the premise to give out the asymptotic behavior of the fourth-order dispersive NLS equation under NZBCs. In Section 3, the asymptotic stage of modulation instability for Eq. (1.1) is analysed in detail.

2 Reconstructing the basic Riemann-Hilbert problem

We recall some notations that will used in our paper. The classical Pauli matrices are defined as follows
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1}
\]
For a $2 \times 2$ matrix $A$ and a scalar variable $\alpha$, we define
\[
e^{\alpha \hat{\sigma}_3} A = e^{\alpha \sigma_3} A e^{-\alpha \sigma_3}. \tag{2.2}
\]

2.1 Direct scattering problem with NZBCs

The Lax pair of Eq. (1.1) is
\[
\psi_x = X \psi, \quad \psi_t = T \psi, \tag{2.3}
\]
with the vector eigenfunction $\psi = (\psi_1, \psi_2)^T$ being a $2 \times 2$ matrix, where the superscript $T$ represents the transpose of the vector,
\[
X = -i k \sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 3i \gamma |q|^4 + i q^2 + i \gamma (q_{xx} q^* + q q_{xx}^*) - 2k \gamma (q_{xx}^* - q_x q^*) \\ -2i k^2 (2 \gamma |q|^2 + 1) + 8i \gamma k^4 - i (3 \gamma q_0^2 + 1) q_0^2 \end{pmatrix} \sigma_3 - 4i \gamma k^2 \sigma_3 Q_x \\
-8 \gamma k^3 Q + 6i \gamma Q^2 Q_x \sigma_3 + i \sigma_3 Q_x + i \gamma \sigma_3 Q_{xxx} + 2k (Q + \gamma Q_{xx} - 2 \gamma Q^3), \tag{2.4}
\]
where $k$ represents the spectrum parameter. For convenience, we set $\gamma = 1$ for the following analysis.

Taking $x \to \pm \infty$ and combining the NZBCs, we turn the Lax pair in Eq. (2.3) into
\[
\psi_{\pm x} = X_{\pm} \psi_{\pm}, \quad \psi_{\pm t} = T_{\pm} \psi_{\pm}, \tag{2.5}
\]
where
\[
X_{\pm} = -i k \sigma_3 + Q_{\pm}, \quad T_{\pm} = (-8k^3 + 2k + 4kq_0^2) X_{\pm}(k), \quad Q_{\pm} = \begin{pmatrix} 0 & q_{\pm} \\ -q_{\pm}^* & 0 \end{pmatrix}, \tag{2.6}
\]
and we have defined $Q_{\pm} = \lim_{x \to \pm \infty} Q$. 

4
It is not hard to obtain the eigenvalues of the matrix $X_{\pm}$ given by $\pm i\lambda$, and $\lambda = \sqrt{k^2 + q_0^2}$. Obviously, the branch cut of $\lambda$ is $\eta = [-iq_0, iq_0]$ which is oriented upward in Figure 1. Here, we should also define $\eta_{+} = [0, iq_0]$ and $\eta_{-} = [-iq_0, 0]$.

\[ \begin{array}{c|c}
 i q_0 & \eta_{+} \\
 \hline
 0 & \mathbb{R} \\
 \eta_{-} & -i q_0
\end{array} \]

**Figure 1.** (Color online) The contour $\Sigma = \mathbb{R} \cup \eta$ of the basic RH problem.

The asymptotic spectral problem (2.5) can be solved by

\[
\psi_{\pm} = E_{\pm} e^{-i\theta(x,t,k)\sigma_3},
\]

where

\[
\theta = \lambda \left[ x + (-8k^3 + 2k + 4kq_0^2)t \right], \quad E_{\pm} = \left( \begin{array}{cc}
1 & \frac{\lambda - k}{iq_\pm} \\
\frac{\lambda - k}{iq_\pm} & 1
\end{array} \right).
\]

Supposing that $\Psi_{\pm}(x,t,k)$ are both the Jost solutions of the Lax pair (2.3), we can define $\Psi_{\pm}(x,t,k) = \psi_{\pm}(x,t,k) + o(1)$ as $x \to \infty$. Furthermore, using variable transforma-

\[
\mu_{\pm}(x,t,k) = \Psi_{\pm}(x,t,k)e^{i\theta(x,t,k)\sigma_3},
\]

we have

\[
\mu_{\pm}(x,t,k) = E_{\pm} + o(1), \quad x \to \pm\infty,
\]

which arrive the following two Volterra integral equations

\[
\mu_{-}(x,t,k) = E_{-} + \int_{-\infty}^{x} E_{-}e^{-i\lambda(x-y)\delta_3} \left[ E_{-}^{-1}(Q_{-})\mu_{-}(y,t,k) \right] dy,
\]

\[
\mu_{+}(x,t,k) = E_{+} - \int_{x}^{+\infty} E_{+}e^{-i\lambda(x-y)\delta_3} \left[ E_{+}^{-1}(Q_{+})\mu_{+}(y,t,k) \right] dy.
\]

**Proposition 2.1.** Suppose $q - q_{\pm} \in L^1(\mathbb{R}_{\pm}^+)$, then $\mu_{\pm}(x,t,k)$ given in Eq. (2.10) uniquely satisfy the Volterra integral equation (2.11) in $\Sigma_0$, and $\mu_{\pm}(x,t,k)$ admit:

- $\mu_{-1}(x,t,k)$ and $\mu_{+2}(x,t,k)$ is analytical in $\mathbb{C}_{+} \setminus \eta_{+}$ and continuous in $\mathbb{C}_{+} \cup \Sigma_0$;
- $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ is analytical in $\mathbb{C}_{-} \setminus \eta_{-}$ and continuous in $\mathbb{C}_{-} \cup \Sigma_0$;
- $\mu_{\pm}(x,t,k) \to I$ as $k \to \infty$;
Therefore, the concrete form of scattering matrix \( s(k) \) of which the scattering matrix \( s(k) \) and \( s(\ast k) \) can be established by scattering matrix \( s(k) \) and \( \ast k \) to be oriented upwards. Now, the fundamental matrix-value function is formulated as
\[
\Psi_-(x, t, k) = \Psi_+(x, t, k)s(k), \quad k \in \Sigma_0,
\]
of which the scattering matrix \( s(k) \) has the following symmetry properties
\[
\Psi_\pm(x, t, k) = \sigma_2 \Psi_\pm(k)\sigma_2, \quad s(\ast k) = \sigma_2 s(k)\sigma_2, \quad k \in \Sigma_0.
\]
Therefore, the concrete form of scattering matrix \( s(k) \) arrives at
\[
s(k) = \begin{pmatrix} a(k) & -b^\ast(k) \\ b(k) & a^\ast(k) \end{pmatrix}, \qquad a(k)a^\ast(k) + b(k)b^\ast(k) = 1,
\]
where \( a^\ast(k) = a^\ast(\ast k) \), \( b^\ast(k) = b^\ast(\ast k) \) means the Schwartz conjugates. Then we get
\[
a(k) = \frac{WR(\Psi_{-1}, \Psi_{+2})}{d(k)}, \quad a^\ast(k) = \frac{WR(\Psi_{+1}, \Psi_{-2})}{d(k)},
\]
\[
b(k) = \frac{WR(\Psi_{-1}, \Psi_{-1})}{d(k)}, \quad b^\ast(k) = \frac{WR(\Psi_{+2}, \Psi_{-2})}{d(k)}.
\]
By taking \( \eta \) to be oriented upwards and defining
\[
\mu_{-1}(k) = \lim_{\varepsilon \to 0^+} \mu_{-1}(k + \varepsilon) = \mu_{-1}(k),
\]
\[
\mu_{+2}(k) = \lim_{\varepsilon \to 0^+} \mu_{+2}(k + \varepsilon) = \mu_{+2}(k), \quad k \in \eta_-
\]
we derive the Jost solutions \( \mu_\pm \) and the scattering data \( a, b \) have the following jump conditions across the branch cut \( \eta \) respectively, given by
\[
\mu_{+1}(x, t, k) = \frac{i(\lambda + k)}{q_+} \mu_{+2}(x, t, k), \quad k \in \eta_+.
\]
\[
\mu_{-2}(x, t, k) = \frac{i(\lambda + k)}{q_-} \mu_{-1}(x, t, k), \quad k \in \eta_+.
\]
\[
\mu_{-1}(x, t, k) = \frac{i(\lambda + k)}{q_-} \mu_{-2}(x, t, k), \quad k \in \eta_-.
\]
\[
\mu_{+2}(x, t, k) = \frac{i(\lambda + k)}{q_+} \mu_{+1}(x, t, k), \quad k \in \eta_-.
\]
and
\[
(a^\ast)^+(k) = \frac{q_-}{q_+} a(k), \quad a^+(k) = \frac{q_+}{q_-} a^\ast(k), \quad b^+(k) = \frac{q^\ast}{q_+} b^\ast(k).
\]

### 2.2 Inverse scattering problem and reconstructing the formula for potential

Now, the fundamental matrix-value function is formulated as
\[
m(x, t, k) = \begin{cases} \frac{(\Psi_{-1}, \Psi_{+2})e^{i\vartheta_3}}{ad}, & k \in \mathbb{C}_+ \setminus \eta_+ \\ \frac{(\Psi_{+1}, \Psi_{-2})e^{i\vartheta_3}}{a^\ast d}, & k \in \mathbb{C}_- \setminus \eta_- \end{cases}
\]
Then the matrix-value function \( m(x, t, k) \) has following jump condition across \( \mathbb{R} \):

\[
m_+(x, t, k) = m_-(x, t, k) \begin{pmatrix}
\frac{1}{2}[1 + \gamma(k)\gamma^*(k)] & \gamma^*(k)e^{-2i\theta(x, t, k)}d(k)
\end{pmatrix}, \quad k \in \mathbb{R}.
\] (2.20)

where \( m_\pm(x, t, k) \) denote the boundary values of \( m(x, t, k) \) as \( k \) approaches the contour from a chosen side, and the reflection coefficient \( \gamma(k) = \frac{b(k)}{a(k)} \). In terms of (2.18), (2.19), (2.20), the jump condition of the matrix-value function \( m(x, t, k) \) across \( \eta_+ \) is given by

\[
m_+(x, t, k) = m_-(x, t, k) \begin{pmatrix}
\frac{1}{2}(-k + \gamma^*(k)) & \gamma^*(k)e^{-2i\theta(x, t, k)}
\end{pmatrix}, \quad k \in \eta_+.
\] (2.21)

Similarly, the matrix-value function \( m^{(0)}(x, t, k) \) has jump condition across \( \eta_- \):

\[
m_+(x, t, k) = m_-(x, t, k) \begin{pmatrix}
\frac{1}{2}(-k + \gamma^*(k)) & \gamma^*(k)e^{-2i\theta(x, t, k)}
\end{pmatrix}, \quad k \in \eta_-.
\] (2.22)

Finally, assuming that the \( a \neq 0 \) for all \( k \in \mathbb{C}_+ \cup \Sigma \), then a matrix RH problem is constructed:

\[
\begin{cases}
m(x, t, k) \text{ is analytic in } \mathbb{C} \setminus \Sigma, \\
m_+(x, t, k) = m_-(x, t, k)J(x, t, k), \quad k \in \Sigma, \\
m(x, t, k) \to I, \quad k \to \infty,
\end{cases}
\] (2.23)

of which the jump matrix \( J(x, t, k) = \{J_i(x, t, k)\}_{i=1}^3 \) is (see Figure 1)

\[
J_1 = \begin{pmatrix}
\frac{1}{2}[1 + \gamma(k)\gamma^*(k)] & \gamma^*(k)e^{-2i\theta(x, t, k)}d(k)
\end{pmatrix},
\]

\[
J_2 = \begin{pmatrix}
\frac{1}{2}(-k + \gamma^*(k)) & \gamma^*(k)e^{-2i\theta(x, t, k)}
\end{pmatrix},
\]

\[
J_3 = \begin{pmatrix}
\frac{1}{2}(-k + \gamma^*(k)) & \gamma^*(k)e^{-2i\theta(x, t, k)}
\end{pmatrix},
\]

where \( f = \lambda \left[ \xi - 8k^3 + 2k + 4kq_0^2 \right], \xi = \frac{t}{\tau} \).

In addition, expanding the \( M^{(0)}(x, t, k) \) at large \( k \) as

\[
m(x, t, k) = I + \frac{m_1(x, t)}{k} + \frac{m_2(x, t)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \to \infty,
\] (2.24)

and combining equations (2.3), (2.19), (2.24), we recover the solution \( q(x, t) \) of the original initial value problem (1.1) in the following form

\[
q(x, t) = 2i \left( m_1(x, t) \right)_1 = 2i \lim_{k \to \infty} km_1(x, t, k).
\] (2.25)
2.3 The sign structure of Re(if)

To find contour deformations, we firstly need to discuss the sign structure of the quantity Re(if). Through taking

\[
\frac{df(\xi, k)}{dk} = \frac{-32k^4 - (16q_0^2 - 4)k^2 + k\xi + 4q_0^4 + 2q_0^2}{\sqrt{k^2 + q_0^2}},
\]

(2.26)

and using Maple symbolic computation, we can get the four stationary phase points (i.e., the points \(k_s\) such that \(f'_k(k_s) = 0\)). Because the expressions are complicated, we’re not going to write it here. We find there are two real stationary phase points and two complex stationary phase points for arbitrary \(\xi\), and the corresponding sign structure of Re(if) is presented at what follows.

\[\begin{align*}
\Re(if) &< 0 & \Re(if) &> 0 \\
\Re(if) &> 0 & \Re(if) &< 0 \\
\Re(if) &> 0 & \Re(if) &< 0
\end{align*}\]

Figure 2. (Color online) The sign structure of Re(if) in the complex \(k\)-plane.

3 Asymptotic stage of modulation instability

In order to derive long-time asymptotics of solution for the Eq. (1.1), we carry out similar deformations of the RH problem (2.23) as that in Refs. [34, 43]. As shown in Figure 2, the curves Im\(\theta(k) = 0\) will not intersect the real axis. In order to study the long-time asymptotics of \(q(x, t)\) in this case, we construct a \(g\)-function mechanism and introduce the point \(k_0\).

3.1 First deformation

To achieve the first deformation, we first decompose the jump matrix \(J_1, J_2, J_3\) into following form

\[
\begin{align*}
J_1 &= J_2^{(1)} J_0^{(1)} J_1^{(1)}, & \text{on } & (k_1, k_0) \cup (k_2, \infty), \\
J_1 &= J_4^{(1)} J_3^{(1)}, & \text{on } & (-\infty, k_1) \cup (k_0, k_2), \\
J_2 &= (J_3^{(1)})^\dagger J_0^{(1)} J_3^{(1)}, & \text{on } & \eta_+ \text{ cut}, \\
J_3 &= J_4^{(1)} J_0^{(1)} (J_4^{(1)})^\dagger, & \text{on } & \eta_- \text{ cut},
\end{align*}
\]
where

\[ J_0^{(1)} = \begin{pmatrix} 1 + \gamma \gamma^* & 0 \\ 0 & \frac{1}{1+\gamma^*} \end{pmatrix}, \quad J_1^{(1)} = \begin{pmatrix} d^{-\frac{1}{2}} & \frac{d^{\frac{1}{2}} \gamma^* e^{-2it}}{1+\gamma^*} \\ 0 & \frac{d^{\frac{1}{2}} \gamma^* e^{-2it}}{d^{\frac{1}{2}}} \end{pmatrix}, \quad J_2^{(1)} = \begin{pmatrix} d^{-\frac{1}{2}} & 0 \\ \frac{d^{\frac{1}{2}} \gamma^* e^{-2it}}{1+\gamma^*} & \frac{1}{d^{\frac{1}{2}}} \end{pmatrix}, \]

\[ J_3^{(1)} = \begin{pmatrix} d^{-\frac{1}{2}} & 0 \\ d^{-\frac{1}{2}} \gamma^* e^{2it} & d^{\frac{1}{2}} \end{pmatrix}, \quad J_4^{(1)} = \begin{pmatrix} d^{-\frac{1}{2}} & d^{-\frac{1}{2}} \gamma^* e^{-2it} \\ 0 & d^{\frac{1}{2}} \end{pmatrix}, \quad J_\eta^{(1)} = \begin{pmatrix} 0 & \frac{iq_\eta}{q_0} \\ \frac{iq_\eta}{q_0} & 0 \end{pmatrix}. \] (3.1)

Therefore, we can transform \( m \) into \( m^{(1)} \) by using

\[ m^{(1)} = mB(k), \] (3.2)

where

\[ B(k) = \begin{cases} (J_1^{(1)})^{-1} & \text{on } k \in \Omega_1, \\ J_2^{(1)} & \text{on } k \in \Omega_2, \\ (J_3^{(1)})^{-1} & \text{on } k \in \Omega_3 \cup \Omega_5, \\ J_4^{(1)} & \text{on } k \in \Omega_4 \cup \Omega_6, \\ I & \text{on } k \in \text{others}, \end{cases} \] (3.3)

then the following RH problem about \( m^{(1)} \) can be given

\[ \begin{align*}
& m^{(1)}(x, t, k) \text{ is analytic in } \mathbb{C} \setminus \Sigma^{(1)}, \\
& m^{(1)}(x, t, k) = m^{(1)}_-(x, t, k)J^{(1)}(x, t, k), \quad k \in \Sigma^{(1)}, \\
& m^{(1)}(x, t, k) \rightarrow I, \quad k \rightarrow \infty,
\end{align*} \] (3.4)

where the jump matrix \( J^{(1)} \) has been defined in (3.1). The contour \( \Sigma^{(0)} \) in Figure 3 also become the new contour \( \Sigma^{(1)} \) as shown in Figure 4.

**Figure 3.** (Color online) The initial contour \( \Sigma^{(0)} \).
3.2 Second deformation

Through introducing a scale RH problem

\[
\begin{align*}
\delta(k) & \text{ is analytic in } \mathbb{C} \setminus (k_1, k_0) \cup (k_2, \infty), \\
\delta_+(k) & = \delta_-(k)[1 + \gamma(k)\gamma^*(k)], \quad k \in (k_1, k_0) \cup (k_2, \infty), \\
\delta(k) & \to 1, \quad k \to \infty,
\end{align*}
\] (3.5)

we can delete the jump across the cut \((-\infty, k_0) \cup (k_2, \infty)\). The above RH problem can be solved by Plemelj formula, given by

\[
\delta(k) = \exp \left\{ \frac{1}{2\pi i} \int_{(k_1, k_0) \cup (k_2, \infty)} \frac{\ln[1 + \gamma(y)\gamma^*(y)]}{y - k} \, dy \right\}. \quad (3.6)
\]

To finish the second deformation, we choose the transformation

\[
m^{(2)} = m^{(1)} \delta^{-\sigma_3} \quad (3.7)
\]

to get a new matrix-value function \(m^{(2)}\), which meets the following RH problem with the contour \(\Sigma^{(2)}\) displayed in Fig. 5

\[
\begin{align*}
m^{(2)}(x, t, k) & \text{ is analytic in } \mathbb{C} \setminus \Sigma^{(2)}, \\
m^{(2)}_+ (x, t, k) & = m^{(2)}_-(x, t, k)J^{(2)}(x, t, k), \quad k \in \Sigma^{(2)}, \\
m^{(2)}(x, t, k) & \to I, \quad k \to \infty,
\end{align*}
\] (3.8)

and with the help of \(J^{(2)} = \delta^{\sigma_3} J^{(1)} \delta^{-\sigma_3}\), we have

\[
\begin{align*}
J^{(2)}_1 & = \left( \begin{array}{cc} d^{-\frac{1}{2}} & \delta^2 d^2 \gamma e^{-2ift} \\ 0 & d^2 \end{array} \right), & J^{(2)}_2 & = \left( \begin{array}{cc} d^{-\frac{1}{2}} & 0 \\ \delta^2 d^2 \gamma e^{2ift} & d^2 \end{array} \right), & J^{(2)}_3 & = \left( \begin{array}{cc} \delta^2 d^2 \gamma e^{-2ift} & 0 \\ d^2 & 0 \end{array} \right), \\
J^{(2)}_4 & = \left( \begin{array}{cc} d^{-\frac{1}{2}} & 0 \\ \delta^2 d^2 \gamma e^{2ift} & d^2 \end{array} \right), & J^{(2)}_5 & = \left( \begin{array}{cc} 0 & \delta^2 d^2 \gamma e^{-2ift} \\ 0 & 0 \end{array} \right), & J^{(2)}_6 & = \left( \begin{array}{cc} 0 & \delta^2 d^2 \gamma e^{2ift} \\ 0 & 0 \end{array} \right), & J^{(2)}_7 & = \left( \begin{array}{cc} 0 & \delta^2 d^2 \gamma e^{-2ift} \\ 0 & 0 \end{array} \right).
\end{align*}
\] (3.9)
3.3 Third deformation

For the third deformation, we select the following transformation

\[ m^{(3)} = m^{(2)} \hat{B}(k), \] (3.10)

with

\[ \hat{B}(k) = \begin{cases} 
  \frac{d^+}{d \tau} & \text{on } k \in \hat{\Omega}_1, \\
  \frac{d^-}{d \tau} & \text{on } k \in \hat{\Omega}_2, \\
  I & \text{on } k \in \hat{\Omega}_3 \cup \hat{\Omega}_4,
\end{cases} \] (3.11)

The goal of this transformation is to wipe out the term \( \Delta(k) \). Then the following RH problem about \( m^{(3)} \) is obtained

\[ \begin{cases} 
  m^{(3)}(x, t, k) \text{ is analytic in } \mathbb{C} \setminus \Sigma^{(3)}, \\
  m^{(3)}(x, t, k) = m_-(^{(3)})(x, t, k) J^{(3)}(x, t, k), & k \in \Sigma^{(3)}, \\
  m^{(3)}(x, t, k) \to I, & k \to \infty,
\end{cases} \] (3.12)

of which the contour \( \Sigma^{(3)} = \Sigma^{(2)} \) is shown in Figure 5, and \( J^{(3)} \) is

\[ J_1^{(3)} = \begin{pmatrix} 1 & \frac{\delta^2 \gamma^* e^{-2i\tau}}{1+\gamma \gamma^*} \\ 0 & \frac{\delta^{-2} \gamma e^{2i\tau}}{1+\gamma \gamma^*} \end{pmatrix}, \quad J_2^{(3)} = \begin{pmatrix} 1 & 0 \\ \delta^{-2} \gamma^* e^{-2i\tau} & 1 \end{pmatrix}, \quad J_3^{(3)} = \begin{pmatrix} 1 & 0 \\ \delta^{-2} \gamma e^{2i\tau} & 1 \end{pmatrix}, \quad J_4^{(3)} = J_7^{(2)}. \] (3.13)

3.4 Eliminating of the exponential growth

According to the sign structure of \( \text{Re}(if) \) shown in Figure 2, we see that the jump matrices \( J_3^{(3)} \) and \( J_4^{(3)} \) in Eq. (3.13) should be grow exponentially in the segment \([k_0, \chi]\) and \([k_0, \chi^*]\), respectively. Thus, the matrices \( J_3^{(3)} \) and \( J_4^{(3)} \) must be decomposed(see Figure 6)

\[ J_3^{(3)} = J_5^{(3)} J_7^{(3)} J_5^{(3)}, \quad J_4^{(3)} = J_6^{(3)} J_8^{(3)} J_6^{(3)}, \] (3.14)
where
\[
J^{(3)}_5 = \begin{pmatrix} 1 & \delta^2 \gamma^{-1} e^{-2i f t} \\ 0 & 1 \end{pmatrix}, \quad J^{(3)}_6 = \begin{pmatrix} 1 & \delta^{-2} (\gamma^*)_1 e^{2i f t} \\ 0 & 1 \end{pmatrix}, \\
J^{(3)}_7 = \begin{pmatrix} 0 & -\delta^2 \gamma^{-1} e^{-2i f t} \\ 1 & 0 \end{pmatrix}, \quad J^{(3)}_8 = \begin{pmatrix} 0 & \delta^2 \gamma^* e^{-2i f t} \\ 0 & 1 \end{pmatrix}.
\]
(3.15)

Next, we set the transformation \( m^{(4)} = m^{(3)} e^{i G(k) \sigma_3} \) via using a time-dependent \( G \) function which is analytic off the cuts \( \eta \cup \varpi \). Of which \( \varpi = \varpi_+ \cup \varpi_- \) with \( \varpi_+ = [k_0, \chi] \) and \( \varpi_- = [\chi^*, k_0] \). Then the new jump matrices \( J^{(4)} \) is calculated as follows
\[
J^{(4)}_1 = \begin{pmatrix} 1 & \delta^2 \gamma^* e^{-2i (f + G) t} \\ 0 & 1 \end{pmatrix}, \quad J^{(4)}_2 = \begin{pmatrix} \delta^{-2} \gamma e^{2i (f + G) t} & 0 \\ 1 & \end{pmatrix}, \\
J^{(4)}_3 = \begin{pmatrix} 0 & \delta^2 \gamma^* e^{-2i (f + G) t} \\ 1 & 0 \end{pmatrix}, \quad J^{(4)}_4 = \begin{pmatrix} 1 & \delta^2 \gamma^* e^{-2i (f + G) t} \\ 0 & 1 \end{pmatrix},
\]
(3.16)

Furthermore, we introduce a new function \( \omega \):
\[
\omega(k) = f(k) + G(k),
\]
(3.17)
whose properties need to be investigated up to find the parameters \( k_0 \) and \( \chi \). First of all, a function \( z \) should be defined as

\[
z(k) = \sqrt{(k^2 + q_0^2)(k - \chi)(k - \chi^*)}, \quad (3.18)
\]

which has branch cuts \( \eta \cup \varpi \) and satisfies \( z(k) = -z_+(k) = z_-(k) \). We implement this algebraic curve as two Riemann surfaces, and the basis \( \{ L_1, L_2 \} \) cycles of this Riemann surface can be described as: the \( L_1 \)-cycle is a simple counterclockwise closed ring around the bifurcation incision \( \eta \), which lies on the lower sheet. The \( L_2 \)-cycle starts from the point \( \chi \) on the upper sheet, then accesses \(-iq_0\) and gets back to the starting point on the lower sheet.

Next, we let \( \omega(k) \) meets

\[
\omega(k) = \frac{1}{2} \left( \int_{iq_0}^{k} + \int_{-iq_0}^{k} \right) d\omega(y), \quad (3.19)
\]

which is a Abelian integral and \( d\omega \) is given by

\[
d\omega(k) = -32 \frac{(k - k_0)(k - k_1)(k - k_2)(k - \chi)(k - \chi^*)}{z(k)} dk. \quad (3.20)
\]

Moreover, the sign signatures of \( \text{Im}\omega(k) \) must be same as that ones of \( \text{Im}f(k) \) for large \( k \), given by

\[
\text{Im}\omega = \text{Im}f + O\left(\frac{1}{k}\right), \quad k \to \infty. \quad (3.21)
\]

Therefore, we have

\[
\omega(k) = -8k^4 + 2k^2 + \xi k + \omega_0 + O\left(\frac{1}{k}\right), \quad k \to \infty. \quad (3.22)
\]

When \( k \to \infty \), the large \( k \) expression of \( z(k) \) become

\[
z(k) = k^2 \left[ 1 - \frac{\chi + \chi^*}{2k} + \frac{4q_0^2 - (\chi - \chi^*)^2}{8k^2} + O\left(\frac{1}{k^3}\right) \right], \quad k \to \infty. \quad (3.23)
\]

Taking \( \chi = \chi_1 + \chi_2i \), from (3.20), we easily obtain

\[
\frac{d\omega}{dk} = -32k^3 + 32(\chi_1 + k_0 + k_1 + k_2)k^2 - [32(k_0 + k_1 + k_2)\chi_1 + 32(k_1k_0 + k_2k_0 + k_2k_1) \\
- 16q_0^2 + 16\chi_2^2]k + (32(k_1k_0 + k_2k_0 + k_2k_1) - 16q_0^2 + 16\chi_2^2)\chi_1 + 32k_0k_1k_2 \\
+ 16(\chi_2^2 - q_0^2)(k_0 + k_1 + k_2) + O\left(\frac{1}{k}\right), \quad k \to \infty. \quad (3.24)
\]

Meanwhile, since \( f = \lambda \left[ \xi - 8k^3 + 2k + 4kq_0^2 \right] \), one has

\[
\frac{df(k)}{dk} = -32k^3 + 4k + \xi + O\left(\frac{1}{k}\right), \quad k \to \infty. \quad (3.25)
\]
Since (3.21) is allowed, and from Eqs. (3.24), (3.25), we can derive
\[
\chi_1 = -k_0 - k_1 - k_2,
\]
\[
\chi_2 = \frac{1}{2} \sqrt{8(k_0^2 + k_1^2 + k_2^2) + 8(k_1k_0 + k_2k_0 + k_2k_1) + 4q_0^2 - 1},
\]
(3.26)
where parameter \( k_0 \) is still need to be derived later. Observing that
\[
-16 \left( \int_{i\rho_0}^{k} + \int_{-i\rho_0}^{k} \right) \left( y^3 - \frac{y}{8} - \frac{\xi}{32} \right) dy = -8k^4 + 2k^2 + \xi k + 8q_0^4 + 2q_0^2,
\]
(3.27)
then the expression of \( \omega(k) \) in Eq. (3.19) can be reconstructed as
\[
\omega(k) = -16 \left( \int_{i\rho_0}^{k} + \int_{-i\rho_0}^{k} \right) \left[ \frac{(y-k_0)(y-k_1)(y-k_2)(y-\chi)(y-\chi^*)}{z(y)} \right. \\
- (y^3 - \frac{y}{8} - \frac{\xi}{32}) \left] dy - 8k^4 + 2k^2 + \xi k + 8q_0^4 + 2q_0^2. \]
(3.28)
As \( k \to \infty \) in (3.28), we have
\[
\omega_0 = -16 \left( \int_{i\rho_0}^{\infty} + \int_{-i\rho_0}^{\infty} \right) \left[ \frac{(y-k_0)(y-k_1)(y-k_2)(y-\chi)(y-\chi^*)}{z(y)} \right. \\
- (y^3 - \frac{y}{8} - \frac{\xi}{32}) \left] dy + 8q_0^4 + 2q_0^2. \]
(3.29)
Next, we will devote to reveal the parameter \( k_0 \) on the real line by presenting the asymptotic expansions of \( \omega(k) \) near point \( \chi \). Similar to reference [43], it is not hard to obtain that
\[
\int_{i\rho_0}^{i\rho_0} \left[ \frac{(y-k_0)(y-k_1)(y-k_2)(y-\chi)(y-\chi^*)}{z(y)} \right] dy \\
= \int_{-i\rho_0}^{i\rho_0} \sqrt{\frac{(y-\chi_1)^2 + \chi^2_2}{y^2 + q_0^2}}(y-k_0)(y-k_1)(y-k_2)dy = 0,
\]
(3.30)
which uniquely gives us the point \( k_0 \) is uniquely expressed.

Now, the function \( \omega(k) \) satisfies the following jump condition:
\[
\omega_+(k) + \omega_-(k) = 0, \quad k \in \eta, \\
\omega_+(k) + \omega_-(k) = \Omega, \quad k \in \infty,
\]
(3.31)
where \( \Omega \) is real constant given by
\[
\Omega = -32 \left( \int_{i\rho_0}^{\chi} + \int_{-i\rho_0}^{\chi} \right) \frac{(k-k_0)(k-k_1)(k-k_2)(k-\chi)(k-\chi^*)}{z(k)} dk.
\]
(3.32)
Besides, since function \( \omega(k) \) is defined in Eq. (3.17), one easily obtains
\[
G(\infty) = \omega_0 - 3q_0^4 - q_0^2, \quad k \to \infty,
\]
(3.33)
and we also have $m^{(4)} \rightarrow e^{G(\infty)\sigma_3}$ as $k \rightarrow \infty$. Finally, we can derive the RH problem for $m^{(4)}$, whose jump matrices $J^{(4)}$ are

$$J_1^{(4)} = \begin{pmatrix} 1 & \delta^2 \gamma^* e^{-2i\omega t} \\ 0 & 1 + \gamma \gamma \end{pmatrix}, \quad J_2^{(4)} = \begin{pmatrix} 1 & 0 \\ \delta^2 \gamma e^{2i\omega t} & 1 \end{pmatrix},$$

$$J_3^{(4)} = \begin{pmatrix} 1 & 0 \\ \delta^2 e^{2i\omega t} & 1 \end{pmatrix}, \quad J_4^{(4)} = \begin{pmatrix} 1 & \delta^2 \gamma^* e^{-2i\omega t} \\ 0 & 1 \end{pmatrix},$$

$$J_5^{(4)} = \begin{pmatrix} 1 & 0 \\ \delta^2 \gamma e^{2i\omega t} & 1 \end{pmatrix}, \quad J_6^{(4)} = \begin{pmatrix} 1 & 0 \\ \delta^2 \gamma^* e^{-2i\omega t} & 1 \end{pmatrix},$$

$$J_7^{(4)} = \begin{pmatrix} 0 & -\delta^2 \gamma \gamma e^{\Omega t} \\ \delta^2 \gamma e^{\Omega t} & 0 \end{pmatrix}, \quad J_8^{(4)} = \begin{pmatrix} 0 & \delta^2 \gamma^* e^{-i\Omega t} \\ -\delta^2 \gamma^* e^{-i\Omega t} & 0 \end{pmatrix}. \quad (3.34)$$

The sign signature of $\text{Im}(\omega)(k)$ ensures that the jump matrices $J_i^{(4)} (i = 1, 2, 3, 4, 5, 6)$ are all exponentially decaying in the associated branch cuts.

### 3.5 Further deformation

In order to delete the variable $k$ from the jump matrices $J_7^{(4)}, J_8^{(4)}$, we need to introduce the $g$-function mechanism again. In the same way, we select following transformation

$$m^{(5)} = m^{(4)} g(k) \sigma_3, \quad (3.35)$$

where the function $g(k)$, which is analytic in $\mathbb{C} \setminus (\eta \cup \varpi)$, satisfies

$$g_+(k)g_-(k) = \begin{cases} \delta^2 & \text{on } k \in \eta, \\ e^{i\vartheta \delta^2} & \text{on } k \in \varpi_+, \\ e^{i\vartheta \delta^2 \gamma^*} & \text{on } k \in \varpi_-, \end{cases} \quad (3.36)$$

of which $\vartheta$ is a real constant and given by

$$\vartheta = i \frac{\int_{\eta} \frac{2\ln \delta}{z} ds + \int_{[k_0, \chi]} \ln \frac{\delta^2}{z} ds + \int_{[k_0, \chi^*]} \ln \frac{\delta^2 \gamma^*}{z} ds}{\int_{[k_0, \chi] \cup [k_0, \chi^*]} \frac{1}{z} ds}. \quad (3.37)$$

Applying the Plemelj’s formula, the $g(k)$ function can be solved by the following integral representation

$$g(k) = \exp \left\{ -\frac{z}{2\pi i} \left( \int_{[\eta]} \frac{2\ln \delta}{z(s-k)} ds + \int_{[k_0, \chi]} \frac{i\vartheta + \ln \frac{\delta^2}{z}}{z(s-k)} ds + \int_{[k_0, \chi^*]} \frac{i\vartheta + \ln \frac{\delta^2 \gamma^*}{z}}{z(s-k)} ds \right) \right\}, \quad (3.38)$$

which implies that $g(k)$ has the following behavior for the large $k$:

$$g(k) = e^{ig(\infty)} + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty, \quad (3.39)$$
where the \( g(\infty) \) is a real constant, given by

\[
g(\infty) = \frac{1}{2\pi} \int_\eta \frac{2\ln \delta - 2}{z} ds + \int_{[k_0, \chi]} \frac{i\gamma + \ln \delta^2}{z} ds + \int_{[k_0, \chi]} \frac{i\gamma + \ln \delta^2 \gamma^*}{z} ds. \tag{3.40}
\]

Finally, we get the following RH problem for \( m^{(5)} \)

\[
\begin{cases}
m^{(5)}(x, t, k) \text{ is analytic in } \mathbb{C} \setminus \Sigma^{(5)}, \\
m^{(5)}_\Sigma(x, t, k) = m^{(5)}_\Sigma(x, t, k), \\
m^{(5)}(x, t, k) \rightarrow e^{i(g(\infty) + G(\infty))\sigma_3}, & k \rightarrow \infty,
\end{cases}
\]

where the contour \( \Sigma^{(5)} = \tilde{\Sigma}^{(3)} \) and the jump matrices \( J^{(5)} \) become

\[
\begin{align*}
J^{(5)}_1 &= \begin{pmatrix} 1 & \delta^2 \gamma e^{-2i\omega t} g^{-2} \\ 0 & 1 \end{pmatrix}, & J^{(5)}_2 &= \begin{pmatrix} 1 & 0 \\ \delta - 2 \gamma e^{2i\omega t} g^2 & 1 \end{pmatrix}, \\
J^{(5)}_3 &= \begin{pmatrix} 1 & 0 \\ \delta^2 \gamma e^{-2i\omega t} g^2 & 1 \end{pmatrix}, & J^{(5)}_4 &= \begin{pmatrix} 1 & 0 \\ \delta^2 \gamma e^{-2i\omega t} g^{-2} & 1 \end{pmatrix}, \\
J^{(5)}_5 &= \begin{pmatrix} 1 & \delta^2 \gamma^{-1} e^{-2i\omega t} g^{-2} \\ 0 & 1 \end{pmatrix}, & J^{(5)}_6 &= \begin{pmatrix} 1 & 0 \\ \delta - 2 \gamma e^{2i\omega t} g^2 & 1 \end{pmatrix}, \\
J^{(5)}_7 &= \begin{pmatrix} 0 & -e^{-i(\Omega t + \vartheta)} \\ e^{i(\Omega t + \vartheta)} & 0 \end{pmatrix}, & J^{(5)}_8 &= \begin{pmatrix} 0 & e^{-i(\Omega t + \vartheta)} \\ -e^{i(\Omega t + \vartheta)} & 0 \end{pmatrix}. \tag{3.41}
\end{align*}
\]

### 3.6 Model problem and the results

Since the jump matrices \( J^{(5)}_i (i = 1, 2, 3, 4, 5, 6) \) are all exponentially decaying to the identity away from the points \( k_0, k_1, k_2 \), \( \chi \) and \( \chi' \) as \( t \to \infty \), we can obtain a model problem to determine the leading term of the solution, given by

\[
\begin{cases}
m^{mod}(x, t, k) \text{ is analytic in } \mathbb{C} \setminus (\eta \cup \varpi_+ \cup (-\varpi_-)), \\
m^{mod}_+ (x, t, k) = m^{mod}_+ (x, t, k), \\
m^{mod}(x, t, k) \rightarrow e^{i(g(\infty) + G(\infty))\sigma_3}, & k \rightarrow \infty,
\end{cases}
\]

where

\[
\begin{align*}
J^{mod}_\eta &= J^{(5)}_\eta = \begin{pmatrix} 0 & \frac{iq_+}{q_0} \\ \frac{iq_+}{q_0} & 0 \end{pmatrix}, \\
J^{mod}_{\varpi_+ \cup (-\varpi_-)} &= \begin{pmatrix} 0 & -e^{-i(\Omega t + \vartheta)} \\ e^{i(\Omega t + \vartheta)} & 0 \end{pmatrix}. \tag{3.43}
\end{align*}
\]

and \( -\varpi_- \) means the negative direction of cut \( \varpi_- \).
For large $k$, introducing the factorization $m^{(5)} = m^{err} m^{mod}$ and taking the Laurent series for matrices $m^{err}, m^{mod}$ as

$$m^{err} = I + \frac{m^{err}_1(x,t)}{k} + \frac{m^{err}_2(x,t)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \to \infty,$$

$$m^{mod} = e^{i(g(\infty)+G(\infty)t)}\sigma_3 + \frac{m^{mod}_1(x,t)}{k} + \frac{m^{mod}_2(x,t)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \to \infty,$$

we can represent the solution $q(x,t)$ for the Eq. (1.1) via the solution of model problem

$$q(x,t) = 2i \left( m^{mod}_1(x,t)e^{i(g(\infty)+G(\infty)t)} + m^{err}_1(x,t) \right)_{12}. \quad (3.46)$$

Similar to reference [43], we get $|m^{err}_1| = O(t^{-\frac{1}{2}})$.

To solve the model RH problem (3.43), we first define the Abelian differential in what follows

$$d\vartheta = \frac{\vartheta_0}{z(k)} \frac{dz}{dk}, \quad \vartheta_0 = \left( \oint_{L_1} \frac{1}{z(k)} \frac{1}{dk} \right)^{-1}, \quad (3.47)$$

which is normalized at the case of $\oint_{L_1} d\vartheta = 1$. At the same time, the above Abelian differential (3.47) admits following Riemann period $\tau$

$$\tau = \oint_{L_2} d\vartheta, \quad (3.48)$$

which is purely imaginary when $i\tau < 0[19]$. Therefore, the theta function can be written into

$$\Theta(k) = \sum_{\vartheta \in \mathbb{Z}} e^{2\pi i k + \pi i \tau \vartheta^2}, \quad (3.49)$$

which yields the properties

$$\Theta(k + n) = \Theta(k), \quad \Theta(k + n\tau) = e^{-(2\pi i nk + \pi i n^2)\theta} \Theta(k), \quad n \in \mathbb{Z}. \quad (3.50)$$

According to the Abelian map

$$V(k) = \int_{i\vartheta_0}^{k} d\vartheta, \quad (3.51)$$

it arrives at

$$V_+(k) + V_-(k) = n - \tau, \quad n \in \mathbb{Z}, \quad k \in \varpi_+ \cup (-\varpi_-),$$

$$V_+(k) + V_-(k) = n, \quad n \in \mathbb{Z}, \quad k \in \eta. \quad (3.52)$$
Finally, a $2 \times 2$ matrix-valued function $M(k) = M(x, t, k)$ are constructed to solve the mod problem (3.43), whose elements are

$$M_{11}(k) = \frac{1}{2} [r(k) + r^{-1}(k)] \frac{\Theta \left( \frac{\Omega t + \theta + i \ln \left( \frac{\chi_1 q_0 + q_0 \chi_2}{q_0} \right)}{2 \pi} + V(k) + C \right)}{\sqrt{\frac{q_0}{q_0^*}} \Theta (V(k) + C)},$$

$$M_{12}(k) = \frac{i}{2} [r(k) - r^{-1}(k)] \frac{\Theta \left( \frac{\Omega t + \theta + i \ln \left( \frac{\chi_1 q_0 + q_0 \chi_2}{q_0} \right)}{2 \pi} - V(k) + C \right)}{\sqrt{\frac{q_0}{q_0^*}} \Theta (-V(k) + C)},$$

$$M_{21}(k) = -\frac{i}{2} [r(k) - r^{-1}(k)] \frac{\Theta \left( \frac{\Omega t + \theta + i \ln \left( \frac{\chi_1 q_0 + q_0 \chi_2}{q_0} \right)}{2 \pi} + V(k) - C \right)}{\sqrt{\frac{q_0}{q_0^*}} \Theta (V(k) - C)},$$

$$M_{22}(k) = \frac{1}{2} [r(k) + r^{-1}(k)] \frac{\Theta \left( \frac{\Omega t + \theta + i \ln \left( \frac{\chi_1 q_0 + q_0 \chi_2}{q_0} \right)}{2 \pi} - V(k) - C \right)}{\sqrt{\frac{q_0}{q_0^*}} \Theta (-V(k) - C)},$$

(3.53)

of which the function $r(k)$ is

$$r(k) = \left( \frac{(k - \chi)(k - iq_0)}{(k - \chi^*)(k + iq_0)} \right)^{\frac{i}{4}}$$

(3.54)

which has the identical jump discontinuity across $\eta$ and $\varpi_+ \cup (-\varpi_-)$, as well as $r_+(k) = ir_-(k)$, and it’s large-$k$ asymptotic is

$$r(k) = 1 - \frac{i(\chi_2 + q_0)}{2k} + O \left( \frac{1}{k^2} \right), \quad k \to \infty,$$

$$r(k) - r^{-1}(k) = -\frac{i(\chi_2 + q_0)}{k} + O \left( \frac{1}{k^2} \right), \quad k \to \infty.$$  

(3.55)

Besides, we also have

$$C = V(\hat{k}) + \frac{1}{2} (1 + \tau), \quad \hat{k} = \frac{q_0 \chi_1}{q_0 + \chi_2}.$$  

(3.56)

Then the model RH problem (3.43) is solved as

$$m^{\text{mod}}(x, t, k) = e^{i(\varphi(\infty) + G(\infty) t) \sigma_3} M^{-1}(\infty, C) M(k, C),$$

(3.57)
further we obtain

\[(m_1^{\text{mod}})_2 = \frac{q_0(\chi_2 + q_0)\Theta\left(\frac{\Omega t + \vartheta + i \ln q^*_{0/0}}{2\pi} - V(\infty) + C\right)\Theta(V(\infty) + C)}{2iq^*_{0/0}\Theta\left(\frac{\Omega t + \vartheta + i \ln q^*_{0/0}}{2\pi} + V(\infty) + C\right)\Theta(-V(\infty) + C)e^{-i(g(\infty) + G(\infty)t)}}\]

which implies the long-time asymptotics of solution \(q(x, t)\) for the fourth-order dispersive NLS equation (1.1) is

\[q(x, t) = \frac{q_0}{q^*_{\text{e}}} (\chi_2 + q_0) \frac{\Theta\left(\frac{\Omega t + \vartheta + i \ln q^*_{0/0}}{2\pi} - V(\infty) + C\right)\Theta(V(\infty) + C)}{\Theta\left(\frac{\Omega t + \vartheta + i \ln q^*_{0/0}}{2\pi} + V(\infty) + C\right)\Theta(-V(\infty) + C)e^{-i(g(\infty) + G(\infty)t)}} e^{-2i(g(\infty) + G(\infty)t)} + O(t^{-\frac{1}{2}})\]

where \(V(\infty) = \int_{i q_0}^{\infty} d\vartheta\), and \(\chi_2, \Omega, G(\infty), \vartheta, g(\infty), C\) are presented in Eqs. (3.26), (3.32), (3.33), (3.37), (3.40), (3.56), respectively.

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