Further evidences for a generic Universe

César A. Terrero-Escalante, Alberto A. García
Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN,
Apdo. Postal 14-740, 07000, México D.F., México.

Abstract

Recently it was argued that an inflationary potential yielding power spectra characterized by a scale-invariant tensorial spectral index and a weakly scale-dependent scalar spectral index might account for a generic large-scale structure formation in the multiverses scenario of eternal inflation. Here it is shown that this statement remains true if the tensorial index is allowed to slowly vary in a wide range of angular scales. In the cases analyzed in this paper, the scalar index also slightly depends on the scales. Therefore, the production of closely resembling each other universes seems to be a common feature of inflationary scenarios with weakly scale-dependent spectral indices. With power-law inflation as the extreme case, this is the class of inflationary scenarios which may give the best fit to near future CMB observations.
I. INTRODUCTION

In this paper we would like to address the question of how probable is the existence of a Universe like this that we can observe. Obviously, the answer to this question must integrate the role of a large number of factors but here we shall focus in the generic description of our Universe from the point of view of matter structures we can observe in it. Also, in our analysis we shall argue neither in favor nor against the anthropic principle. Thus, to answer our question it is necessary, first of all, to determine the mechanism generating all the forms of energy densities that are detected today.

Starting from the quark-gluon plasma, the evolution of the micro and macro worlds can be successfully described in the framework of the Standard Cosmological Model (SCM) plus Standard Particle Model (SPM). The main features of such an evolution are in a very good agreement with laboratory and astrophysical observations. Nevertheless, in this framework there are no answers to the questions of where the primordial plasma came from and how were seeded such large structures like voids, galaxies, clusters and superclusters of galaxies. According to a top-bottom scheme, these structures are necessary to explain the formation of stars, planets, heavy chemical elements and life. To find the missing answers it seems to be unavoidable to go beyond the standard paradigms.

The analysis of recent results of the cosmic microwave background (CMB) observations [1] confirm the inflationary scenario as the leading candidate for a theory of the formation of the primordial plasma and, through gravitational instability, of large-scale structures in our Universe. According to this theory, an special kind of matter with a characteristic negative pressure produced a period of rapidly accelerated expansion in the early Universe [2]. The successive decay of this matter, first into non-relativistic particles and then into hot radiation, together with the breaking of fundamental symmetries led to the period known as Hot Big Bang. The fluctuations of matter and space-time taking place in the very early Universe [3] were stretched out of the causal horizon by the accelerated expansion and, after reentering the Universe in a later era, give rise to curvature perturbations, seeding the formation of large-scale structure and imprinting the anisotropies in the CMB radiation. The simplest choice for the inflationary matter is a single real scalar field, coined inflaton, with dynamics dominated by the potential energy.

The primordial fluctuations are characterized in terms of power spectra of scalar and tensor perturbations. Inflaton driven expansion does not give rise to vectorial perturbations. Different theories for the physics of the very early Universe lead to different primordial spectra (different initial conditions for density evolution) and, after evolving, to different distributions of density and temperature anisotropies in the currently observable Universe. The level of accuracy of current experiments measuring the CMB is of the order of 10%, which roughly corresponds to 10% uncertainty for the best determined cosmological parameters. Future experiments [4] will increase this accuracy to the limit of the cosmic variance what must allow to strengthen the constrains upon the cosmological parameters, including the inflationary ones. This way, the number of suitable inflationary potentials could be reduced to a few candidates, and it could be possible to obtain some hints about physics on energies close to the Planck scale if, as we assume throughout this paper, the single scalar field scenario is the dominant feature of the very early Universe dynamics.

Several extensions of the SPM are able to reproduce its features in a equally satisfactory
fashion. Hence, the answer to whether our Universe is generic it is related to the probability for a given SPM extension to be the actual one. But, even if there exist a deterministic way of choosing the appropriate SPM extension, different initial values for the inflaton field dynamics will lead to different kind of universes. The initial values of the scalar field and its derivative with respect to cosmic time determine the amount of inflation produced before the decay into radiation as well as the form of the perturbations spectra. Hence, they will determine the Universe topology, its life time, how isotropic and homogeneous it will be, the temperature of the primordial plasma and the spectra of particles and large-scale structure. It means that, given an inflaton potential (more precisely, given a sector of the potential near a vacuum state of the theory), the fundamental role in the diversification of the cosmological evolution is played by the initial conditions.

In the inflationary scenario the initial conditions for the scalar field classical dynamics are set by the mechanism known as eternal inflation [5]. In most regions of the very early Universe the inflaton rolls down the potential aiming at a stable configuration. However, there is a finite probability that in some causally connected regions the quantum fluctuations dominate upon the classical behavior, displacing the scalar field to higher energies values. These regions inflate more than those where the inflaton is rolling down. This way, large spaces arise which, in turn, can be divided in causally connected regions where either the classical motion or the quantum fluctuations dominate. In principle, this process never ends, producing regions eternally inflating into the future and other where, starting from different initial values, the inflaton decays into hot big bangs, rising different universes. We shall give more details about the mechanism of (future) eternal inflation in the next section.

A plausible outcome of Planck and Map satellites measurements could be a central constant value for the rate between the tensorial and scalar perturbations amplitudes [6] (which, in turn, is related to the tensorial spectral index) as well as the possibility of fitting the CMB observations using a scale-dependent scalar index [7,8]. Recently, a model generating perturbations described by a scale-invariant tensorial spectral index was introduced [9]. This model could be the best fit to near future observations if the scalar index can be fairly approximated by a constant at small angular scales and departs from that behavior at large scales. Taking into account that currently a constant scalar index is typically used to fit the CMB observations [10], these assumptions for its possible scale-dependence are well based. Thus, the physics corresponding to this potential could be close to the actual very early Universe physics. In Ref. [10], assuming a strong similarity between the above mentioned inflationary potential and the actual one, it was concluded that the universes in the eternal inflation picture will be very similar each to the other from the observational point of view. It means that our Universe will be a generic outcome of the cosmological evolution rather than some extraordinary event.

Calculations supporting that conclusion rely on several assumptions like a constant tensorial spectral index, the second order precision of the involved expressions, and a semi-classical analysis close to the Planck era. In Ref. [9] it was shown that higher order corrections can introduce relevant features in the form of the inflationary potential derived from information on the functional forms of the spectral indices. For example, to leading order it makes no sense to look for potentials yielding spectra characterized by a constant tensorial index and a scale-dependent scalar index, while this solution can be successfully obtained to next-to-leading order [9]. Consequently, it could be expected that corrections beyond the
next-to-leading order could reveal new important information about the potential form. Nevertheless, even if higher order corrections to primordial spectra have been recently proposed for slow-roll models [11] and some classes of non slow-roll models [12], the given expressions need to be tested for accuracy before the corresponding inverse problem can be introduced. So far, this task has not been accomplished. On the other hand, though the most important from the observational point of view part of inflation occurs at energies substantially lower than Planck scales, we have seen in Ref. [9] that the departure from power-law behavior takes place in the high energies regime, where the initial conditions would be set. Thus, one can wonder whether this departure will occur (and in case it occurs, if it will occur in the same fashion) if the corresponding initial energy scale is close to the Planck scale. Not having a completely satisfactory quantum gravity theory, any answer to this question will imply resorting to additional assumptions. With these regards, in this manuscript we would like to focus on the analysis of the robustness of the conclusion about a generic Universe with respect to the scale-dependence of the tensorial spectral index. Even regarding near future CMB polarization measurements to be carried out by Planck satellite [4], there is not enough motivation for considering a scale-dependent tensorial index while analyzing the cosmological data. The most optimistic expectations are for estimating a very small constant central value for this index. From the theoretical side, it can be shown that the scale dependence of the tensorial index must be very weak. However, in Ref. [1] a weak scale dependence for the scalar index was allowed while keeping constant the tensorial one, and this lead to a significant difference between the obtained solution and power-law inflation for high values of the scalar field. This difference could be a consequence of the high non-linearity of the involved equations. Taking this into account, the question arises of whether a weak scale dependence for the tensorial index could change the inflationary picture in the high energies regime in such a way that the conclusion about a generic Universe would be affected. With this motivation, we introduce in Sec. [1] an ansatz for the tensorial index consisting of a second order polynomial on the first horizon flow function (or first slow-roll parameter) [12]. With this ansatz we solved the Stewart-Lyth inverse problem introduced in Ref. [13] and found a couple of models with slowly varying spectral indices. One of this models can be regarded as a generalization of the scenario described in Ref. [9]. In the second scenario, the spectral indices do not converge to a constant value but still have a very weak scale-dependence in a wide range of scales and met all the conditions for successful inflation. Even if in this scenario different outputs from inflation are possible, we argue that the higher probability is for universes expanded by the same factor than our and with perturbations spectra close to a power-law. In Sec. [14] we summarize our discussion.

**II. FUTURE-ETERNAL INFLATION**

An important effect of the rapidly accelerated primeval expansion is the almost instantaneous flattening of the Universe. Hence, during inflation it can be assumed the space-time geometry being described by flat Friedmann-Robertson-Walker metrics. For a single real scalar field the stress-energy tensor is like that of a perfect fluid with,

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi),$$

(1)
where $\rho$ is the energy density, $p$ is the pressure, $\phi$ is the inflaton, and $V(\phi)$ the inflationary potential. Dot stands for derivatives with respect to cosmic time. Then, looking for a negative pressure we must have $V > \dot{\phi}^2$. Combining the temporal and spatial Einstein equations is obtained,

$$H^2 = \frac{\kappa}{3} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right),$$

while the energy conservation law, yields,

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi).$$

Here, $H = \dot{a}/a$ is the Hubble parameter, $a$ the scale factor, prime stands for derivatives with respect to $\phi$, $\kappa = 8\pi/m_{Pl}^2$ is the Einstein constant and $m_{Pl}$ the Planck mass. Note that for a unique solution of this system it is necessary to set the initial conditions as for $\phi$ as for $\dot{\phi}$.

Since during inflation the coefficient in the friction term of Eq. (4) is almost constant then, for the inflationary period to be long enough, the potential must be rather flat. Inflation ends when the inflaton reaches the true vacuum which, without loss of generality, we shall assume located at the origin. An alternative mechanism for finishing inflation involves several secondary scalar fields frozen in unstable states during the expansion until the inflaton reaches a critical value which triggers the decay of the secondary fields [14]. Thus, inflation must start far enough from the value corresponding to the potential absolute minimum (or from the triggering critical value). That condition imposes some tuning of the initial conditions that it is the kind of problem the inflationary idea is aimed to solve. This paradox is resolved assuming a random initial spatial distribution for the scalar field values. Due to a phenomenon known as (future) eternal inflation [15] which occurs in the early era of the expansion, it is sufficient with a tiny patch of the very early Universe having the required inflaton value. Being the inflaton a quantum object, in a region of linear size $d_H = H^{-1}$ (a region which may be in causal contact within one Hubble time, $\Delta t = H^{-1}$, and we shall call it a Hubble region), the time evolution of the average value of $\phi$ can be described by,

$$\phi(t + \Delta t) = \phi(t) + \Delta_{cl}\phi(\Delta t) + \delta_{qu}\phi(\Delta t),$$

where $\phi(t + \Delta t)$ and $\phi(t)$ are the inflaton values at times $t + \Delta t$ and $t$, $\Delta_{cl}\phi(\Delta t) \simeq \dot{\phi}H^{-1}$ and $\delta_{qu}\phi(\Delta t)$ are changes of the inflaton value during a Hubble time due to the classical motion and quantum fluctuations respectively. The random quantum jump $\delta_{qu}\phi(\Delta t)$ is characterized by a Gaussian distribution with zero mean and standard deviation $\Delta_{qu}\phi \simeq (2\pi)^{-1}H$. Taking into account that $V > \dot{\phi}^2$, then, to leading order, $V \simeq H^2/\kappa$ and larger $|\phi|$ values will allow larger deviations of $\phi$ from the average value. If we assume that in some Hubble regions the inflaton can pick an initial value $\phi_i$ such that,

$$\frac{\Delta_{qu}\phi(\phi_i)}{\Delta_{cl}\phi(\phi_i)} \simeq \frac{1}{2\pi} \frac{H^2}{\dot{\phi}} = \frac{1}{2\pi} \frac{H}{\frac{d\phi}{dN}} \geq 1,$$

then we have a finite probability (of about a 16%) for the negative (positive) quantum contribution to be larger than the positive (negative) classical displacement. In Eq. (3)
$N \equiv \ln(a/a_i)$ is the number of efolds by which the size of the patch is increased since some initial time $t_i$. (Note that usually the efolds number is counted backward in time, we count it forward, i.e., $N(t_i) = 0$.) The conclusion is that there is a finite probability for the existence of some Hubble regions in the very early Universe where the overall scalar field motion will be in the direction of larger $|\phi|$ values, i.e., higher energies. During inflation, larger values of $|\phi|$ correspond to a larger factor $e^N$. Then, regions where the scalar field rolls upward inflate more and after some Hubble times we will have large spaces which can be divided again in Hubble regions, some of them with a finite probability for the inflaton to diffuse up the potential, and so on. In regions where the overall scalar field motion is in the direction of smaller $|\phi|$ values, i.e., lower energies, the inflaton eventually will reach the potential minimum (or the triggering critical value) and will give rise to big bangs leading to different universes. Thus, the general picture involves some regions eternally inflating and a large number of universes arising through big bangs, with their large-scale structures seeded by primordial curvature perturbations and growing due to gravitational instability.

For each one of these universes the inflaton final roll-down starts with different values of $\phi$ and $\dot{\phi}$ and, according with Eq. (4), it means that the inflationary period will set different initial conditions for the big bang processes. Notice, however, that the mechanism of future eternal inflation provides that most of the initial conditions are set at the higher absolute values of the scalar field. Even if we assume that the whole set of initial conditions leads to a unique vacuum state, after inflation there will be universes with different degree of flatness, and/or different degree of homogeneity and isotropy, and/or different rate of expansion, and/or different number of topological defects, and/or different number and spectrum of particles, and/or different large scale-distribution of matter. Summarizing, from the observational point of view, most of these universes are different each from the other. Therefore, the question is which is the probability for a universe like our to exist in this multiverses scenario. It can be naively concluded that with an infinite number of universes this probability must be high, but it is not so simple. The point is that, in the definition

$$\text{Probability} = \frac{\text{number of universes like our}}{\text{total number of universes}}$$

we are dealing with infinite quantities, $\infty/\infty$, i.e., it is an ill-defined probability.

III. WEAKLY SCALE-DEPENDENT SPECTRAL INDICES

In order to assess the influence that different initial conditions around a given vacuum state exert upon the results of the inflationary era it is necessary to know which is the inflationary potential. However, this is perhaps the ultimate quest in the very early Universe research. No particle physics based theory has been completely successful in explaining inflation \[15\]. An alternative to solve the problem we are concerned about it is to look for common features of those potentials likely to fit the results of CMB measurements. In Ref. \[13\] a procedure coined Stewart-Lyth inverse problem (SLIP) was introduced to determine the potential corresponding to given functional forms for the spectral indices. In the current analysis of observations, a constant scalar spectral index close to unity and zero primordial gravitational waves contribution have been used to fit the CMB spectrum.
with a rather remarkable success \cite{1}. It implies that in the inflaton range corresponding to the horizon-crossing out of scales currently observed, the potential must look like a very flat exponential typical of power-law inflation \cite{10}. Obviously, a large number of functional expressions for the potential can have the appropriate form.

On the other hand, from upcoming measurements of CMB polarization to be done by satellite Planck it could be possible to determine a central constant value for the rate between tensorial and scalar inflationary amplitudes. From this value it is possible to estimate a constant leading order value for the tensorial index \cite{6}. Furthermore, it was already found currently available data to be compatible with scale-dependent scalar index \cite{7}, though this dependence seems to be highly constrained \cite{8}. The increased range and resolution of the CMB observations must allow to discern the scale-dependence of the scalar index. If the leading order of tensor-scalar ratio is measured and the scale-dependence of the scalar index is hinted, then stronger constraints in the potential functional form will be set.

The SLIP procedure was used to find the inflationary scenario, other than power-law, with constant tensorial spectral index \cite{9}. The corresponding scalar index was found to be almost constant at small angular scales but diverges from that behavior at large angular scales. Therefore, the obtained potential could be a good approximation to the actual inflaton potential. The theoretical potential is given as a parametric function described by two branches. In Ref. \cite{10} it was shown that the form of these branches can be fitted by some hybrid inflation potentials with running exponential couplings (see report \cite{15} for suitable patterns and related physics). If the similarity between the parametric potential and the actual one is realistic, then it can be shown that different initial conditions do not lead to different outcomes from the inflationary era. Therefore, our universe would be a generic result of the cosmological evolution rather than an accident \cite{10}. Nevertheless, the assumption of a constant tensorial index is justified by measurements limitation but not by theoretical judgment. From the theoretical point of view some scale-dependence for this index must be expected, even if it is very weak to be observed. Moreover, the nonlinearity of the SLIP related equations may amplify any differences in the input of the inverse problem. That amplification could happen in such a way that the SLIP solutions corresponding to closely related functional forms of the spectral indices could be very different each from other. It means that a conclusion drawn from the analysis of one SLIP solution may not be true for other SLIP solution, even if the input of the corresponding inverse problems are slightly different. Taking this into account, is of interest to test the robustness of the prediction of a generic Universe drawn in Ref. \cite{10} by allowing a weak scale dependence of the tensorial index.

Constant spectral indices correspond to a power-law scenario of inflation where they are given by \cite{16,9},

\[ \delta = \Delta = \frac{\epsilon_1}{\epsilon_1 - 1} = -\left(\epsilon_1 + \epsilon_1^2 + \epsilon_1^3 + \cdots\right) \leq 0, \]

where \( \delta = n_T/2 \), \( \Delta = (n_S - 1)/2 \), \( n_T \) and \( n_S \) are the tensorial and scalar spectral indices respectively, and the first horizon flow function \cite{12}

\[ \epsilon_1 \equiv \frac{d \ln \delta}{dN} = 3 \frac{\dot{\phi}^2}{2} + V(\phi), \]

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measures, in general, the logarithmic change of the Hubble distance per e-fold of expansion and, particularly in the case of inflaton driven expansion, the contribution of the scalar field kinetic energy to the total field energy. Inflation proceeds for $\epsilon_1 < 1$ (equivalent to $p < 0$ and $\dot{a} > 0$) and, on the other hand, $\epsilon_1 > 0$ from the weak energy condition (for a spatially flat universe). For power-law inflation, $\epsilon_1 = \text{const.}$

With this regard, we propose the following ansatz for the tensorial index,

$$\delta(\epsilon_1) = -(1 + a)\epsilon_1^2 + (1 + b)\epsilon_1 + c,$$

where $\epsilon_1$ is now a scale-dependent function and $a, b, c$ are real numbers. The second order of $\delta$ is for compatibility with the order of the algebraic equations for the spectral indices \cite{7} from which the SLIP differential equations are derived. Taking into account that $\delta(\epsilon_1) < 0$ \cite{9}, the extremum of the parabola given by Eq. (9) must be a maximum. Combining this with condition $\epsilon_1 > 0$ it is obtained that $1 + a > 0$ and $b + 1 < 0$. On the other hand, combined with condition $\epsilon_1 < 1$ it give us that $1 + b > -2 - 2a$. Finally, requiring $\delta \leq 0$ while evaluated at this maximum yields $c \geq (1 + b)^2/4(1 + a) \geq 0$. Summarizing we have,

$$a > -1, -1 > b > -3 - 2a \text{ (or } b \geq -2\sqrt{(1 + a)c - 1}), c > 0.$$

We shall see that two different cases arise corresponding to $b^2 > 4ac$ and $b^2 < 4ac$.

**A. Case 1, $b^2 > 4ac$**

We start by analyzing case $b^2 > 4ac$. As we shall see, the result for this case is close to that obtained in Ref. \cite{1} for a constant tensorial index. In fact, $0 \geq a > -1$ is just a generalization of that SLIP solution. Thus, the conclusion about a generic Universe drawn in Ref. \cite{10} can be trivially extended to this model when $0 \geq a > -1$. With this regard, we shall focus here in the SLIP solution for $a > 0$, where the corresponding expression for the potential will be slightly different. For $a > 0$ the strongest condition on $b$ is $b \geq -2\sqrt{(1 + a)c - 1}$. Values of $a, b, c$ can be additionally constrained if more conditions for successful inflation are taken into account. Firstly, as it will be shown below, the values of these coefficients determine the central values for $n_T$ and $n_S$. A range for the last can be estimated from the CMB spectrum and averaging the results in Ref. \cite{1} we obtain $n_S \in [0.87, 1.07]$. Without loss of generality in our analysis, for all of the plots and numerical estimates in this subsection, we have chosen $a = 0.35$, $b = -1.1$ and $c = 0.01$.

To determine $n_S(\epsilon_1)$ we use the SLIP equations \cite{13},

$$2C\epsilon_1\dot{\epsilon}_1 - (2C + 3)\epsilon_1\dot{\dot{\epsilon}_1} - \dot{\epsilon}_1 + \epsilon_1^2 + \epsilon_1 + \Delta = 0,$$

$$2(C + 1)\dot{\epsilon}_1\dot{\epsilon}_1 - \epsilon_1^2 - \epsilon_1 - \delta = 0,$$

where $C = -0.7296$ and a hat denotes derivatives with respect to $\tau \equiv \ln H^2$. Substituting the ansatz (8) in Eq. (12) we can separate $\dot{\epsilon}_1$. Deriving with respect to $\tau$ we obtain an expression for $\dot{\dot{\epsilon}_1}$. Then, $\epsilon_1$ and $\dot{\epsilon}_1$ can be substituted in Eq. (11) and $\Delta$ as function of $\epsilon_1$ will be given by,
\[\Delta(\epsilon_1) = -\frac{1}{2(C+1)^2} \left\{ [Ca^2 + (2C + 3)(C + 1)a + 2(C+1)^2] \epsilon_1^2 \\
+ [(C + 1)a + Cab + (2C + 3)(C + 1)b + 2(C+1)^2] \epsilon_1 \\
+ (C + 1)b + (2C + 3)(C + 1)c + (C + 1)c \frac{1}{\epsilon_1} - Cc^2 \frac{1}{\epsilon_1} \right\} . \quad (13)\]

On the other hand, the wavenumber \( k \) at horizon crossing, i.e., \( k = aH \), as function of \( \epsilon_1 \) can be obtained as solution of equation,

\[(C + 1)(\epsilon_1 - 1)\tilde{\epsilon}_1 - \epsilon_1^2 - \epsilon_1 - \delta = 0, \quad \text{(14)}\]

where \( \tilde{\epsilon}_1 \equiv d\epsilon_1/d \ln k \) [18]. Using ansatz (9) and after integration,

\[k = k_0 \left| a\epsilon_1^2 + b\epsilon_1 + c \right|^\frac{C+1}{2} \frac{2a\epsilon_1 + \sqrt{b^2 - 4ac}}{2a\epsilon_1 - \sqrt{b^2 - 4ac}} \frac{(C+1)(2a+b)}{2a\sqrt{b^2 - 4ac}}, \quad \text{(15)}\]

where \( k_0 \) is the integration constant. Hereafter, the integration constants will be denoted by the subscript 0 in the corresponding variable. The plots of \( n_T(k) \) and \( n_S(k) \), using \( \epsilon_1 \) as parameter, are presented in Fig. [1] where the scale range was chosen to allow detailed observation at large scales (small \( k \)) but it can be straightforwardly extended to infinity. As

![Case 1 plots](image)

**Fig. 1.** Case 1. Plots of the tensorial index (left) and scalar index (right) as functions of the horizon crossing wavenumber.

it can be observed, at small angular scales the indices behave roughly as constants. Two branches can be noted at large angular scales departing from the power-law like behavior. To understand the origin of these branches we must analyze the dynamics of \( \epsilon_1 \). The solution of Eq. (12) with \( \delta \) given by Eq. (8) is

\[\tau = \ln \left| a\epsilon_1^2 + b\epsilon_1 + c \right|^\frac{C+1}{a} \frac{2a\epsilon_1 + \sqrt{b^2 - 4ac}}{2a\epsilon_1 - \sqrt{b^2 - 4ac}} \frac{(C+1)b}{a\sqrt{b^2 - 4ac}} + \tau_0. \quad \text{(16)}\]

This solution diverges when \( \epsilon_1 \rightarrow \epsilon_1^\pm \) where, \( \epsilon_1^\pm \) are the roots of \( a\epsilon_1^2 + b\epsilon_1 + c \). Both these roots are positive for \( a > 0, b < 0 \) and \( c > 0 \) but for most suitable values of \( a, b \) and \( c \) only
\( \epsilon_1^- \) will be less than 1. In Fig. 2, \( \epsilon_1(\tau) \) as given by solution (10) is plotted. We observe that the branches in Fig. 1 correspond to initial values for \( \epsilon_1 \) which are greater or less than \( \epsilon_1^- \). This value is shown in all the figures of this subsection by mean of a dashed line. Recalling that \( d\tau/dt < 0 \), independently of the initial conditions, with cosmic time the solutions asymptotically converge to the solution given by \( \epsilon_1 = \epsilon_1^- \). This can be observed in Fig. 2 where the time flow is represented by small arrows near the corresponding branch (also used in the remaining figures of this subsection). Substituting the value for \( \epsilon_1^- \) in Eqs. (9) and (13) it is obtained that,

\[
\Delta(\epsilon^-_1) = \delta(\epsilon^-_1) = \frac{c}{a} + \left( \frac{b}{a} - 1 \right) \epsilon^-_1
\]

confirming that the solution converges to a power-law solution. This asymptotic behavior is consistent with the result by Hoffmann and Turner [19] of power-law being an attractor of the inflationary dynamics. However, we shall show in the next subsection that it is not generally true. Expression (17) can be used to fit the values of \( a, b \) and \( c \) for \( \Delta \) to match the power-law value required to fit the CMB spectrum.

Recall that as functions of \( \epsilon_1 \), the scalar field and its potential are given by [4],

\[
V(\epsilon_1) = \frac{1}{\kappa} (3 - \epsilon_1) \exp[\tau(\epsilon_1)],
\]

\[
\phi(\epsilon_1) = -\frac{2(C + 1)}{\sqrt{2\kappa}} \int \frac{\sqrt{\epsilon_1} d\epsilon_1}{\epsilon_1^2 + \epsilon_1 + \delta} + \phi_0.
\]

Thus, for our case, the parametric potential is
\[
V(\phi) = \begin{cases}
\phi(\epsilon_1) = \frac{(C+1)}{\sqrt{\kappa}} \frac{\sqrt{\kappa}}{\sqrt{b^2-4ac}} \left[ -\sqrt{-b + \sqrt{b^2-4ac}} \ln \frac{\sqrt{2a\epsilon_1 + \sqrt{-b + \sqrt{b^2-4ac}}}}{\sqrt{2a\epsilon_1 - \sqrt{-b + \sqrt{b^2-4ac}}}} + \phi_0 \right] \\
+ \sqrt{-b - \sqrt{b^2-4ac}} \ln \frac{\sqrt{2a\epsilon_1 + \sqrt{-b - \sqrt{b^2-4ac}}}}{\sqrt{2a\epsilon_1 - \sqrt{-b - \sqrt{b^2-4ac}}}} 
\end{cases} 
\]

Plotting \(\phi(\epsilon_1)\) and \(V(\epsilon_1)\) as in Fig. 3 and comparing with Fig. 2, one can see that

\[
V(\epsilon_1) = V_0(3 - \epsilon_1) |ae_1^2 + b\epsilon_1 + c|^{-\frac{C+1}{a} \left( \frac{2ae_1 + b + \sqrt{b^2-4ac}}{2ae_1 + b - \sqrt{b^2-4ac}} \right) - \frac{b(C+1)}{a \sqrt{b^2-4ac}}}.
\]

FIG. 3. Case 1. Plots of the inflaton (left) and the potential (right) as functions of \(\epsilon_1\).

The asymptotic behavior \(\epsilon_1 \to \epsilon_1^-\) should ensure that the inflaton roll-down along any of these two branches will produce enough inflation. For inflation to make the Universe as flat and homogeneous as our, it is required that \(N > 60\). To see clearly that this is the case, the efolds number as function of \(\epsilon_1\) can be determined using definition (8). Changing variables it is obtained,

\[
\dot{N} = -\frac{1}{2\epsilon_1},
\]

and after using \(\epsilon_1\) as given by Eq. (12),
FIG. 4. Case 1. Plots of the potential as function of $\phi$ (left) and the efolds number as function of $\epsilon_1$ (right).

$$N = -(C + 1) \int \frac{d\epsilon_1}{\epsilon_1^2 + \epsilon_1 + \delta} + N_0.$$ \hspace{1cm} (23)

Note that for a given functional form of the tensorial index it can be assessed whether the corresponding inflationary scenario will produce the required amount of expansion without previous knowledge on the corresponding potential. In our case, the integration of Eq. (23) yields,

$$N = -\frac{(C + 1)}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2a\epsilon_1 + b + \sqrt{b^2 - 4ac}}{2a\epsilon_1 + b - \sqrt{b^2 - 4ac}} \right| + N_0,$$ \hspace{1cm} (24)

with the plot presented in the right part of Fig. 4. As was expected, the closer $\epsilon_1$ to $\epsilon_1^c$, the larger the increase of expansion of the Universe.

This model lacks a graceful exit into the SCM, i.e., there is no “natural” way out of inflation here. To solve this problem, $\phi$ can be regarded as the dominant scalar field in a hybrid scenario with $\epsilon_1^c$ being the value corresponding to the critical value of $\phi$ near which the false vacuum becomes unstable and the multiple scalar fields roll to the true potential minimum. It means that for any initial condition with $\epsilon_1(t_i)$ slightly different from $\epsilon_1^c$ the Universes will get sufficiently flat and homogeneous and the observable spectra of perturbations will be almost of power-law type.

Let us now see which are the conditions for eternal inflation to take place in this model. First of all, we note that in general,

$$\frac{d\phi}{dN} = \sqrt{\frac{2}{K}} \sqrt{\epsilon_1}.$$ \hspace{1cm} (25)

Then, condition (6) will read

$$\frac{H^2(\epsilon_{1i})}{\pi m_{P_i}^2 \epsilon_{1i}} \geq 1,$$ \hspace{1cm} (26)
for some value $\epsilon_{1i}$. For the particular model we are analyzing in this subsection, and using $H^2$ from Eq. (13), we obtain,

$$\frac{H_i^2}{\pi m_p^2 \epsilon_{i_1}}\left[ac_{i_1} + b \epsilon_{i_1} + c\right] - \frac{(C+1)}{a} \frac{2ae_{i_1} + b + \sqrt{b^2 - 4ac}}{2ae_{i_1} + b - \sqrt{b^2 - 4ac}} \geq 1. \quad (27)$$

It can be checked (for example, by plotting the left hand side of this relation) that a suitable selection of the scale given by $H_0$ must ensure that condition (27) will be met for any value of $\epsilon_{1i}$ far enough from $\epsilon_{1i}$. $H_0$ must be chosen taking into account that the damping of inflaton evolution due to the back reaction of the particles produced during the expansion must take place far enough from the quantum dominated regime [20]. Then, for any proper inflationary energy scale $V_0$, the scalar field starts to roll down the potential in Hubble regions with values of $\epsilon_1$ very close to zero or to $\epsilon_{1i}^*$, i.e., regions with larger values of $|\phi|$. On the other hand, there is no reason to assume a stochastic initial distribution for $\phi$ and not for $\dot{\phi}$. According to Eq. (8), starting with random $\phi$ and $\dot{\phi}$ means starting with random $\epsilon_1$ then, in the beginning, in some Hubble regions the inflaton is located in branch $V_I$, and in some other Hubble regions, in branch $V_{II}$. Therefore, in the subset of these regions where the roll-up the potential will eventually dominate, different high energies physics will be set up. Now, while rolling down, $\Delta_{\text{qu}} \phi$ decreases and, due to the uncertainty principle, the equal-time standard deviation for the canonical conjugate of $\phi$, i.e., $\Delta_{\text{qu}} \dot{\phi}$ must increase. Since, for $\epsilon_1 \in I$ as well as for $\epsilon_1 \in II$, at these energies the evolution of $\epsilon_1$ is characterized by the asymptotic convergence $\epsilon_1 \rightarrow \epsilon_{1i}^*$, any quantum perturbation of the canonical momentum $\dot{\phi}$ (translated into quantum perturbation of $\epsilon_1$) makes possible that the inflaton “jumps” from branch $V_I$ to branch $V_{II}$ and vice versa. This way, memories from the corresponding high energies physics will be smoothly erased. Moreover, this asymptotic behavior of $\epsilon_1$ ensures that most of the inflaton perturbations were produced close to the logarithmic end of inflation. These are the perturbations which play the most important role from the point of view of large-scale structure formation. As we have already seen, at these scales the perturbations spectra produced while rolling down $V_I$ or $V_{II}$ are almost identical to power-law spectra. Therefore, according with potential (20), no matter which the initial conditions were, there is a big chance for all the big bang universes (including our) being very similar each to the other.

### B. Case 2. $b^2 < 4ac$

In this subsection we shall analyze case $b^2 < 4ac$ which, as it will be shown, it is a step further in the relaxation of the scale-dependence of the spectral indices while still satisfying the observational bounds for such a dependence. With respect to the condition labeling this case, new restrictions for the coefficients values arise, i.e., $a > 0$, $-1 > b > -1/2 - 2ac$ and $c > 1/4a$. On the other hand, we shall see that, for a sufficient efolds number we must require $4ac - b^2 \ll 1$. Once more, $c$ will be chosen for $n_S$ to be in the center of $[0.87, 1.07]$. In this subsection the coefficients numerical values are, $a = 50.01$, $b = -1.0002$ and $c = 0.005004$. For this case, the scalar index as function of $\epsilon_1$ is also given by expression (13), but the wavenumber $k$ at the horizon crossing, as function of $\epsilon_1$ is now,
\[ k = k_0 \left| a\epsilon_1^2 + b\epsilon_1 + c \right|^\frac{C+1}{2\alpha} \exp \left[ \frac{(C + 1)(2a + b)}{a\sqrt{4ac - b^2}} \arctan \left( \frac{2ae_1 + b}{\sqrt{4ac - b^2}} \right) \right]. \tag{28} \]

The corresponding parametric plots of \( n_T(k) \) and \( n_S(k) \) for a given range of scales are presented in Fig. 5. It is observed that, within the observational error, both indices can be approximated by constant values in a wide range of scales but neither of them asymptotically converges to a constant. Again, for an understanding of this behavior we must analyze the dynamics of \( \epsilon_1 \). Here, the solution of Eq. (12) is,

\[
\tau = \ln \left| a\epsilon_1^2 + b\epsilon_1 + c \right| + \frac{2(C + 1)b}{a\sqrt{4ac - b^2}} \arctan \left( \frac{2ae_1 + b}{\sqrt{4ac - b^2}} \right) + \tau_0, \tag{29} \]

and the plot is presented in Fig. 6. A period of almost constant \( \epsilon_1 \) (\( L_2 \equiv (\epsilon_1^b, \epsilon_1^t) \)), corresponding to near power-law inflation, is bounded by periods where \( \epsilon_1 \) abruptly increases with cosmic time. This result disagrees with those in Ref. [19]. We believe that this disagreement is explained by the precision of the expressions involved in each analysis. The analysis by Hoffmann and Turner was done using the leading order of the relevant quantities while we are using here the next-to-leading order precision. The dynamical analysis presented in Ref. [13] shows that to next-to-leading order the expressions are even more nonlinear that to leading order, and this nonlinearity introduces further complexity in the inflationary dynamics. As can be observed in the figures of the reduced phase spaces for the evolution of \( \epsilon_1 \) presented Ref. [13], for \( \Delta < 0 \) we have a saddle point located in the interesting range of values of \( \epsilon_1 \). It means that an asymptotic behavior will be characteristic only of those trajectories very close to the unstable separatrices (recall that \( d\tau/dt < 0 \)). On the other hand, some trajectories must be expected that come near the saddle point and then blow up. This could be the case for the model we are analyzing in this subsection.

The corresponding parametric potential is,
FIG. 6. Case 2. Plot of $\epsilon_1$ as function of $\tau$.

$$V(\phi) = \left\{ \phi(\epsilon_1) = \frac{(C+1)}{\sqrt{\epsilon_1}} \sqrt[4]{ac-b^2} \left\{ \sqrt{2\sqrt{ac-b}} - b \right\} - \arctan \left( \frac{2\sqrt{ac+b}}{2\sqrt{ac-b}} \right) \right\} + \phi_0,$$

$$V(\epsilon_1) = V_0(3 - \epsilon_1) \left| a\epsilon_1^2 + b\epsilon_1 + c \right|^{\frac{C+1}{2}} \exp \left[ \frac{2(C+1)b}{a\sqrt{4ac-b^2}} \arctan \left( \frac{2a\epsilon_1+b}{\sqrt{4ac-b^2}} \right) \right].$$

Analyzing the behaviors of $\phi(\epsilon_1)$ and $V(\epsilon_1)$ plotted in Fig. 7, and the dynamics of $\epsilon_1$ (Fig. 6), it is observed that potential (30) plotted in the left part of Fig. 8 fulfills criterion (21) for any $\epsilon_1 \in [0, 1)$. Moreover, the integration of Eq. (23) for this case yields,
FIG. 8. Case 2. Plots of the potential as function of \( \phi \) (left) and the efolds number as function of \( \epsilon_1 \) (right).

\[
N = 2 \frac{(C + 1)}{\sqrt{4ac - b^2}} \arctan \left( \frac{2a \epsilon_1}{\sqrt{4ac - b^2}} \right) + N_0 ,
\]

(31)

and from the plot presented in the right part of Fig. 8 one can see that, with an appropriate selection of \( a, b, \) and \( c \), this scenario will produce the required amount of inflation.

Even if in this model the inflationary period can be finished by \( \epsilon_1 \) reaching unity (in fact \( \epsilon_1 = 1 \) is reached shortly after the power-law like period), since this potential does not have a minimum, we still need to appeal to the hybrid inflation mechanism in order to account for the big bang relativistic matter. The triggering value could be any value of \( \phi \) corresponding to \( 1 \geq \epsilon_1 > \epsilon_1' \). The scales crossing the horizon after \( \epsilon_1 = \epsilon_1' \) will be extraordinarily small and reenter it back immediately with not relevant effect.

In this scenario, the values of possible initial conditions for the inflaton roll-down can be divided in three intervals, namely, \( L_1 \equiv [0, \epsilon_1'] \), \( L_2 \) and \( L_3 \equiv [\epsilon_1', 1) \). Starting from \( L_3 \) negligible expansion will be produced and the universes will collapse almost immediately. Since in \( L_2 \) \( \epsilon_1 \) is almost constant, \( L_2 \) is a very narrow interval and the initial conditions are only slightly different. Nevertheless, even that tiny difference could account for very different amount of inflation and, correspondingly, to universes with different degree of flatness, homogeneity and so on. Finally, any inflationary dynamics starting from \( L_1 \) will face the huge expansion produced in \( L_2 \) and the resulting perturbations spectra will be very similar each other, differing from a power-law spectrum only at very large and very small scales.

Putting \( H^2 \) as given by Eq. (29) in condition (26) yields,

\[
\frac{H_0^2}{\pi m^2_{pl} \epsilon_{1i}} \left| a \epsilon_{1i}^2 + b \epsilon_{1i} + c \right| \frac{a^{\epsilon_{1i} + 1}}{a} \exp \left[ \frac{2(C + 1)b}{a \sqrt{4ac - b^2}} \arctan \left( \frac{2a \epsilon_{1i} + b}{\sqrt{4ac - b^2}} \right) \right] \geq 1 .
\]

(32)

For any \( H_0 \) there is a sufficiently small \( \epsilon_{1i} \) such that condition (26) will be met, meaning that there is a larger number of Hubble regions where the initial conditions for the scalar field roll-down toward the big bang will belong to \( L_1 \). It implies that, for inflation driven by
potential \(^{30}\), even if there are finite probabilities for several different inflationary outputs, the most likely one is that of universes expanded by the same factor given by \(L_2\) and with power-law like spectra of perturbations in a wide range of scales.

IV. CONCLUSIONS

Two inflationary scenarios were introduced with weakly scale-dependent spectral indices yielding perturbations amplitudes which, within current observational limits, can be fairly approximated by power-law spectra in a wide range of angular scales.

If, as it could be expected, any of these potentials strongly resembles the actual inflaton potential, then the big bang universes arising in the eternal inflation picture will be very similar each to the other from the observational point of view. In that case, our observable Universe will be a generic outcome of the cosmological evolution rather than an extraordinary entity in the multiverses space.

This conclusion is limited by the assumptions underlying the calculations. First of all, the precision of the expressions for the spectral indices upon which the SLIP is based might determine the functional forms of the spectral indices and, consequently, of the corresponding inflaton potential. Since the perturbations spectra is expected to be nearly power-law, the information added by higher order corrections may be irrelevant though that must be proved. Work in this direction is in progress.

On the other hand, our predictions are valid for inflation taking place in the neighborhood of a given vacuum state. The observable size of this neighborhood is determined by the range of scales probed by the CMB experiments and, obviously, will always be finite. Nevertheless, a theory may have several equivalent vacua arising through different fundamental symmetries breaking. In this case, inflation can happen around every vacuum and, if the vacua are not symmetric, it leads to different big bang processes, this way diversifying the cosmological outcomes. The observation of our Universe will not yield direct information on the inflationary dynamics taking place near other vacua. However, it could be expected that our conclusion holds for all of the universes related with each vacuum. Therefore, the universes would be grouped in a number of classes corresponding to the number of actually different vacuum states. Whether universes like our are a generic cosmological product will depend on how large could this number be.

Finally, our conclusion heavily relies on the fact that most initial conditions are set in the higher energies regime. If the characteristic energy scale is too close or beyond the Planck scale, the validity of the semi-classical analysis can break down.

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REFERENCES

[1] C. B. Netterfield et al., astro-ph/0104460; A. T. Lee et al., astro-ph/0104459; R. Stompor et al., astro-ph/0105062; N. W. Halverson et al., astro-ph/0104489; C. Pryke et al., astro-ph/0104490.

[2] A. Linde, Particle Physics and Inflationary Cosmology (Harwood, Chur, Switzerland, 1990); A. R. Liddle and D. H. Lyth, Cosmological Inflation and Large-Scale Structure (Cambridge University Press, Cambridge, UK, 2000).

[3] A. A. Starobinsky, Pis’ma Zh. Eksp. Teor. Fiz. 30, 719 (1979) [JETP Lett. 30, 682 (1979)]; V. Mukhanov and G. Chibisov, Pis’ma Zh. Eksp. Teor. Fiz.33, 549 (1981) [JETP Lett. 33, 532 (1981)]; A. H. Guth and S. Y. Pi, Phys. Rev. Lett. 49, 1110 (1982); S. Hawking, Phys. Lett. 115B, 295 (1982); A. A. Starobinsky, Phys. Lett. 117B, 175 (1982).

[4] Microwave Anisotropy Probe (MAP) http://map.gsfc.nasa.gov/; Planck http://astro.estec.esa.nl/SA-general/Projects/Planck/.

[5] P. J. Steinhardt, in The Very Early Universe, Proc. Nuffield Workshop, 21 June - 9 July, 1982, eds: G. W. Gibbons, S. W. Hawking, and S. T. Siklos (Cambridge Univ. Press, 1983), p. 251; A. Vilenkin, Phys. Rev. D 27, 2848 (1983); A. D. Linde, Mod. Phys. Lett. A1 81 (1986); A. D. Linde, Phys. Lett. 175B, 395 (1986); A. H. Guth, Phys. Rep. 333 555 (2000).

[6] M. Kamionkowski and A. Kosowsky, Ann. Rev. Nucl. Part. Sci. 49, 77 (1999).

[7] L. Covi and D.H. Lyth, astro-ph/0008165; S. Hannestad, S. H. Hansen and F. L. Villante, Astropart. Phys. 16 (2001) 137.

[8] S. Hannestad, S.H. Hansen, F.L. Villante and A. J. S. Hamilton astro-ph/0103047.

[9] C. A. Terrero-Escalante, E. Ayón-Beato, and A. A. García, Phys. Rev. D 64, 023503 (2001).

[10] C. A. Terrero-Escalante, and A. A. García, Proc. IV Mexican School on Gravitation and Mathematical Physics, December 2001. Eds: N. Bretón, and J. Socorro. astro-ph/0107325

[11] E. D. Stewart and J. O. Gong, Phys. Lett. B 510 1, (2001).

[12] D. J. Schwarz, C. A. Terrero-Escalante, and A. A. García, astro-ph/0106020.

[13] E. Ayón-Beato, A. García, R. Mansilla and C. A. Terrero-Escalante, Phys. Rev. D 62, 103513 (2000).

[14] A. D. Linde, Phys. Lett. 249B, 18 (1990); A. D. Linde, Phys. Lett. 259B, 38 (1991).

[15] D. H. Lyth, A. Riotto, Phys. Rep. 314 1 (1998).

[16] F. Lucchin and S. Matarrese, Phys. Rev. D 32, 1316 (1985).

[17] E. D. Stewart and D. H. Lyth, Phys. Lett. 302B, 171 (1993).

[18] E. Ayón-Beato, A. García, R. Mansilla and C. A. Terrero-Escalante, in Proceedings of III DGFM-SMF Workshop on Gravitation and Mathematical-Physics, León, México (2000). Eds: N. Bretón, O. Pimentel and J. Socorro. astro-ph/0009358

[19] M. B. Hoffmann and M. S. Turner, Phys. Rev. D 64, 023506 (2001).

[20] A. D. Dolgov and S. H. Hansen, Nucl. Phys. B 548, 408 (1999).