Nested Quantum Dyck Paths and $\nabla(s_{\lambda})$

Nicholas A. Loehr\textsuperscript{1} and Gregory S. Warrington\textsuperscript{2*}

\textsuperscript{1}Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, USA, and \textsuperscript{2}Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109, USA

Correspondence to be sent to: warrings@wfu.edu

We conjecture a combinatorial formula for the monomial expansion of the image of any Schur function under the Bergeron–Garsia nabla operator. The formula involves nested labeled Dyck paths weighted by area and a suitable “diagonal inversion” statistic. Our model includes as special cases many previous conjectures connecting the nabla operator to quantum lattice paths. The combinatorics of the inverse Kostka matrix leads to an elementary proof of our proposed formula when $q = 1$. We also outline a possible approach for proving all the extant nabla conjectures that reduces everything to the construction of sign-reversing involutions on explicit collections of signed, weighted objects.

1 Introduction

The nabla operator introduced by Francois Bergeron and Adriano Garsia [2] plays a fundamental role in the theory of symmetric functions and Macdonald polynomials. To define this operator, let us first introduce some notation. We let $\Lambda$ denote the ring of symmetric functions in the variables $x_1, x_2, \ldots$ with coefficients in the field $\mathbb{Q}(q, t)$. The vector space $\Lambda$ has many well-known bases, all indexed by integer partitions. We will use the following bases of $\Lambda$ in this paper: the monomial symmetric functions $m_{\mu}$, the homogeneous symmetric functions $h_{\mu}$, the elementary symmetric functions $e_{\mu}$, the
power-sum symmetric functions $p_\mu$, the Schur functions $s_\mu$, and the modified Macdonald polynomials $\tilde{H}_\mu$. More details may be found in the encyclopedic reference [26].

The nabla operator is the unique $\mathbb{Q}(q,t)$-linear map on $\Lambda$ such that $\nabla(\tilde{H}_\mu) = q^{n(\mu')^i}t^{n(\mu)}\tilde{H}_\mu$ for all partitions $\mu$, where $n(\mu) = \sum_{i \geq 1} (i-1)\mu_i$ and $\mu'$ is the transpose of $\mu$. Thus, the modified Macdonald polynomials are the eigenfunctions of the nabla operator. From the combinatorial point of view, the nabla operator is important because it encodes a wealth of information about $q,t$-analogues of combinatorial objects such as lattice paths, parking functions, and labeled forests. The connection to combinatorics arises by considering the matrix of the linear operator $\nabla$ relative to various bases for $\Lambda$. Given any two bases $(b_\lambda)$ and $(c_\mu)$ of $\Lambda$ and any linear operator $T$ on $\Lambda$, we write $(c_\mu)[T](b_\lambda)$ to denote the unique matrix of scalars $a_{\mu,\lambda} \in \mathbb{Q}(q,t)$ such that

\[ T(b_\lambda) = \sum_{\mu} a_{\mu,\lambda} c_\mu \]

for all partitions $\lambda$. In particular, if $T = \nabla$ and $(c'_\mu)$ is the dual basis for $c_\mu$ relative to the Hall inner product on $\Lambda$, it follows that

\[ a_{\mu,\lambda} = \langle \nabla(b_\lambda), c'_\mu \rangle. \]

We often restrict consideration to the subspace $\Lambda^n$ of symmetric functions of degree $n$, so that the matrix in question is a finite square matrix with rows and columns indexed by the partitions of $n$.

By definition, $(\tilde{H}_\mu)[\nabla](\tilde{H}_\lambda)$ is a diagonal matrix with diagonal entries $T_\mu = q^{n(\mu')^i}t^{n(\mu)}$. For other choices of the input and output bases, one obtains other $\mathbb{Q}(q,t)$-matrices representing the nabla operator. Remarkably, the entries in these matrices are often polynomials in $q$ and $t$ with integer coefficients all of like sign; i.e. we often have $a_{\mu,\lambda} \in \pm \mathbb{N}[q,t]$. Whenever this occurs, one can seek combinatorial interpretations for various entries $a_{\mu,\lambda}$ as sums of suitable signed, weighted objects. Such interpretations have been sought after, conjectured, and (in some cases) proved by many different authors. Table 1 gives a list (not necessarily exhaustive) of some recent research efforts in this area.

Each of the conjectures mentioned in Table 1 gives only partial information about the nabla operator. For example, the Garsia–Haglund $q,t$-Catalan Theorem establishes a combinatorial interpretation for just one of the coefficients in the matrix $(s_\mu)[\nabla](s_\lambda)$, namely $a_{s_1^n, s_1^n} = \langle \nabla(e_n), s_1^n \rangle$. The main conjecture in [17] extends this result to a combinatorial interpretation for the monomial expansion of $\nabla(e_n)$, but this still only yields information about one column of the matrix $(m_\mu)[\nabla](m_\lambda)$. Our goal in this paper is to present a
new conjecture that gives a combinatorial interpretation for every entry in the matrix \((m_\mu)[\nabla(s_\lambda)]\). We shall see that this conjecture unifies and clarifies the partial conjectures mentioned in Table 1.

In Section 2 of this paper, we describe our conjectured combinatorial model for the monomial expansion of \(\nabla(s_\lambda)\) and explain some connections to the more specialized conjectures in Table 1. In Section 3, we give a proof of our conjecture when \(q = 1\); the proof relies heavily on the combinatorics of the inverse Kostka matrix \(K^{-1} = (s_\mu)[\text{id}](m_\lambda)\). In Section 4, we outline a combinatorial approach that, if implemented, would prove the full conjecture. The proof method suggested in this final section, which relies heavily on the recently discovered combinatorial interpretation for modified Macdonald polynomials, reduces all the extant nabla conjectures to the problem of defining sign-reversing involutions on certain explicit collections of signed, weighted objects.

## 2 The Combinatorial Model

This section presents our conjectured formula for the monomial expansion of \(\nabla(s_\lambda)\). To prepare for this formula, we must first review the known combinatorial interpretation for \(\langle \nabla(e_n), s_1^n \rangle\) and the conjectured interpretation for the monomial expansion of \(\nabla(e_n)\).

### 2.1 Quantum Dyck paths

Fix a positive integer \(n\). A *Dyck sequence* of length \(n\) is a list \(g = (g_0, g_1, \ldots, g_{n-1})\) of non-negative integers such that \(g_0 = 0\) and \(g_{i+1} \leq g_i + 1\) for all \(i < n - 1\). The *area* of a Dyck...
sequence is area(g) = \sum_{i=0}^{n-1} g_i. A diagonal inversion of a Dyck sequence is a pair of indices i < j such that (g_i - g_j) \in \{0, 1\}. We let dinv(g) be the number of diagonal inversions of g. Given any logical statement P, let \chi(P) = 1 if P is true, and \chi(P) = 0 if P is false. Then we can write

\[
\text{dinv}(g) = \sum_{0 \leq i < j < n} \chi(g_i - g_j \in \{0, 1\}).
\]

For example, g = (0, 0, 1, 2, 0, 1, 2, 3, 1) is a Dyck sequence of length 10 with area(g) = 11 and dinv(g) = 15.

Dyck sequences correspond naturally to Dyck paths, which are lattice paths from (0, 0) to (n, n) consisting of n unit north steps and n unit east steps that never go below the line y = x. We convert a Dyck sequence to a Dyck path by drawing \(g_i\) complete lattice squares to the left of the line \(y = x\) in the \(i\)th row from the bottom, and following the north and west boundary of these squares to obtain a lattice path. Then area(g) is the number of squares between the path and the line \(y = x\). The statistic dinv(g) counts pairs of cells lying immediately right of north steps in the path, such that the cells are either on the same diagonal, or such that the lower square lies one diagonal to the left of the upper square. The diagonal inversion statistic was proposed by Haiman in connection with the Garsia–Haiman \(q, t\)-Catalan numbers. Haglund had previously defined another statistic on Dyck paths, called the bounce score [11]. Garsia and Haglund proved that

\[
\langle \nabla(e_n), s_1^n \rangle = \sum_{g \in DS_n} t^{\text{area}(g)} q^{\text{dinv}(g)} = \sum_{\pi \in DP_n} t^{\text{bounce}(\pi)} q^{\text{area}(\pi)},
\]

where \(DS_n\) is the set of Dyck sequences of order \(n\), and \(DP_n\) is the set of Dyck paths of order \(n\) [8, 9].

In [17], the previous combinatorial formula was extended to a conjectured formula for the monomial expansion of \(\nabla(e_n)\). To describe this extension, we consider pairs \((g, r)\), where \(g\) is a Dyck sequence of length \(n\) and \(r = (r_0, r_1, \ldots, r_{n-1})\) is a list of \(n\) positive integers such that \(g_{i+1} = g_i + 1\) implies \(r_i < r_{i+1}\). Let \(LDP_n\) be the set of all such pairs. We visualize an object \((g, r) \in LDP_n\) by drawing the Dyck path \(\pi\) associated with \(g\) and labeling the \(n\) north steps of \(\pi\) (from bottom to top) with the labels \(r_0, r_1, \ldots, r_{n-1}\). The only restriction on the labels is that the labels in each column must strictly increase.
reading upwards. Now, define \( \text{area}(g, r) = \sum_{i=0}^{n-1} g_i = \text{area}(g) \), and define

\[
dinv(g, r) = \sum_{i < j} \chi(g_i - g_j = 0 \text{ and } r_i < r_j) + \sum_{i < j} \chi(g_i - g_j = 1 \text{ and } r_i > r_j).
\]

Haglund et al. [17] conjectured that

\[
\nabla(e_n) = \sum_{(g, r) \in \text{LD}P_n} t^{\text{area}(g, r)} q^{\text{dinv}(g, r)} \prod_{i=0}^{n-1} x_{r_i},
\]

and they proved that this expression is symmetric in the \( x_j \)'s. Because of this symmetry, the “HHLRU conjecture” is equivalent to the following assertion: for all \( \mu = (\mu_1, \mu_2, \ldots) \models n \), the \((\mu, (1^n))-entry of \langle \nabla(s_{\lambda}), s_1^n \rangle \) is the sum of \( t^{\text{area}(g, r)} q^{\text{dinv}(g, r)} \) over all objects \((g, r) \in \text{LD}P_n\) such that the \( r \)-vector contains \( \mu_i \) copies of \( i \) for all \( i \).

2.2 Combinatorial model for \( \langle \nabla(s_{\lambda}), s_1^n \rangle \)

We are going to conjecture a formula for the monomial expansion of \( \nabla(s_{\lambda}) \). Before doing so, we describe a related conjecture for the “sign character” \( \langle \nabla(s_{\lambda}), s_1^n \rangle \), where \( \lambda \) is an arbitrary partition of \( n \). The sign character conjecture involves \textit{nested quantum Dyck paths}, while the full conjecture involves \textit{nested quantum labeled Dyck paths}.

Our conjecture for the sign character has the form

\[
\langle \nabla(s_{\lambda}), s_1^n \rangle = \text{sgn}(\lambda) \sum_{G \in \text{NDP}_{\lambda}} t^{\text{area}(G)} q^{\text{dinv}(G)},
\]

where \( \text{sgn}(\lambda) \in \{+1, -1\} \), \( \text{NDP}_{\lambda} \) is a certain collection of nested Dyck paths constructed from \( \lambda \), and \( \text{area}, \text{dinv} : \text{NDP}_{\lambda} \to \mathbb{N} \) are suitable weight functions. We will define the quantities \( \text{sgn}(\lambda), \text{NDP}_{\lambda}, \text{area} \) and \( \text{dinv} \) in the context of a specific example.

Suppose \( n = 14 \) and \( \lambda = (5, 3, 2, 2, 2) \). We begin by drawing the Ferrers diagram of the transposed partition \( \lambda' = (5, 5, 2, 1, 1) \). Next, we fill this diagram with “rim hooks” by repeatedly removing the entire northeast border of \( \lambda' \), as shown in Figure 1. For \( 0 \leq i < \lambda_1 = \ell(\lambda') \), let \( n_i \) be the length of the hook that starts in the \( i \)th row from the top of the diagram; let \( n_i \) be zero if there is no such hook. In our example, we have

\[
(n_0, n_1, n_2, n_3, n_4) = (9, 0, 0, 5, 0).
\]
Define the spin of $\lambda'$ to be the total number of times a border hook crosses a horizontal boundary of a unit square in the Ferrers diagram of $\lambda'$, and define the sign $\text{sgn}(\lambda) = (-1)^{\text{spin}(\lambda')}$. In our example, $\text{sgn}(\lambda) = (-1)^5 = -1$. We also define the dinv adjustment by setting

$$\text{adj}(\lambda) = \sum_{i=0}^{\lambda_1-1} (\lambda_1 - 1 - i) \chi(n_i > 0).$$

This adjustment is the sum of the row indices in which the nonzero border hooks start, if we number the rows 0, 1, ... reading from bottom to top. In our example, $\text{adj}(\lambda) = 1 + 4 = 5$.

Next, we describe the collection of objects $NDP_{\lambda}$. Let $l = \lambda_1 = \ell(\lambda')$. We consider $l$-tuples of lattice paths $\Pi = (\pi_0, \pi_1, \ldots, \pi_{l-1})$ such that $\pi_i$ is a lattice path from $(i, i)$ to $(i + n_i, i + n_i)$ consisting of $n_i$ unit north steps and $n_i$ unit east steps that never go strictly below the line $y = x$. If $n_i = 0$ for some $i$, then $\pi_i$ is a degenerate path consisting of a single vertex at $(i, i)$. We say that $\Pi$ is nested iff for all $i \neq j$, no edge or vertex of $\pi_i$ coincides with any edge or vertex of $\pi_j$. By definition, $NDP_{\lambda}$ consists of all such $l$-tuples of nested Dyck paths. Note that degenerate paths are important for determining nesting. Figure 2 shows a typical element of $NDP_{(5,3,2,2,2)}$. 

---

**Fig. 1.** Dissection of $\lambda'$ into border strips

**Fig. 2.** Example of nested Dyck paths
We can represent $\Pi = (\pi_0, \pi_1, \ldots, \pi_{l-1}) \in NDP$ by a “Dyck configuration,” which is the analogue of a Dyck sequence in this setting. A Dyck configuration is an $l$-tuple of words $G = (g^{(0)}, g^{(1)}, \ldots, g^{(l-1)})$, where $g^{(i)}$ is the Dyck sequence for the Dyck path $\pi_i$. We choose the indexing of the letters in these Dyck sequences to match the alignment of paths in the picture. More precisely, for $i \leq a < i + n_i$, let $g^{(i)}_a$ be the number of complete lattice squares in the region bounded below by $y = a$, bounded above by $y = a + 1$, bounded on the right by $y = x$, and bounded on the left by $\pi_i$; for all other values of $a$, $g^{(i)}_a$ is undefined.

For example, the Dyck configuration for the nested paths in Figure 2 is:

$$G = \begin{pmatrix}
g^{(0)} & : & 0 & 1 & 1 & 2 & 3 & 4 & 5 & 5 \\
g^{(1)} & : & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
g^{(2)} & : & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
g^{(3)} & : & \cdot & \cdot & 0 & 1 & 0 & 1 & 1 & . \\
g^{(4)} & : & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}$$

Here, dots indicate positions where $g^{(i)}_a$ is undefined.

It is convenient to identify nested Dyck paths with the associated Dyck configurations. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ with associated border hooks of lengths $n_0, n_1, \ldots, n_{l-1}$ (where $l = \lambda_1$), an $l$-tuple $G = (g^{(i)}_j)$ belongs to $NDP_\lambda$ iff the following requirements are satisfied: (i) for every $i$, $(g^{(i)}_j : i \leq a < i + n_i)$ is a Dyck sequence of length $n_i$; (ii) if $n_k = 0$, then $g^{(i)}_k > 0$ for all $i < k$ such that $g^{(i)}_k$ is defined; (iii) if $n_k > 0$ and $i < k$, then $g^{(i)}_j > g^{(k)}_j$ for all $j$ such that both sides are defined; and (iv) if $n_k > 0$ and $i < k$, then $g^{(i)}_j > g^{(k)}_j + 1$ for all $j$ such that both sides are defined. Conditions (iii) and (iv) express the nesting requirement for two nontrivial paths in terms of the Dyck sequences; condition (ii) expresses the nesting requirement when the inner path has length zero.

Given $G \in NDP_\lambda$, the area of $G$ is the sum of the areas of the Dyck paths comprising $G$:

$$\text{area}(G) = \sum_{i=0}^{l-1} \sum_{i \leq j < i + n_i} g^{(i)}_j.$$

Note that lattice squares in the picture that appear inside multiple Dyck paths are counted multiple times in the area statistic. Next, the diagonal inversion statistic
for $G$ is

$$\text{dinv}(G) = \text{adj}^*(\lambda) + \sum_{a,b,u,v} \chi(g_a^{(u)} - g_b^{(v)} = 1) \chi(a \leq b)$$

$$+ \sum_{a,b,u,v} \chi(g_a^{(u)} - g_b^{(v)} = 0) \chi((a < b) \text{ or } (a = b \text{ and } u < v)).$$

In these sums, we consider all possible choices of $a$, $b$, $u$, $v$ such that $g_a^{(u)}$ and $g_b^{(v)}$ are both defined. For the example shown in Figure 2, we have $\text{area}(G) = 27$ and $\text{dinv}(G) = 28$.

Our conjectured formula

$$\langle \nabla(s_\lambda) , s_1^n \rangle = \text{sgn}(\lambda) \sum_{G \in \text{NDP}_\lambda} t^{\text{area}(G)} q^{\text{dinv}(G)}$$

has been verified by computer for all partitions $\lambda$ of all integers $n \leq 9$.

2.3 Combinatorial model for the monomial expansion of $\nabla(s_\lambda)$

Roughly speaking, our main conjecture for $\nabla(s_\lambda)$ is obtained by “adding labels” to the sign character formula introduced in the previous subsection. This is done in the spirit of the shuffle conjecture for $\nabla(e_n)$ of [17]. For any partition $\lambda$, we associate a collection of labeled nested Dyck paths, which we abbreviate $\text{LNDP}_\lambda$. Note that “nested” in this case will be a slightly weaker notion than that used in the definition of $\text{NDP}_\lambda$.

Let $l = \ell_1 = \ell(\lambda')$. An element of $\text{LNDP}_\lambda$ consists of a pair $(G, R)$. Here, $G$ is an $l$-tuple $(g^{(0)}(0), g^{(1)}(0), \ldots, g^{(l-1)}(0))$ where each $g^{(j)}$ encodes the Dyck sequence of some path $\pi_j$ of length $n_j$ among the entries $(g^{(j)}(j_1), g^{(j)}(j_2), \ldots, g^{(j)}(j_{n_j} - 1))$. The $l$-tuple $R = (r^{(0)}(0), r^{(1)}(0), \ldots, r^{(l-1)}(0))$ is a list of labels. For all $j$, the length of $r^{(j)}$ equals the length of $g^{(j)}$ (i.e. $r^{(j)} = (r^{(j)}_1, r^{(j)}_2, \ldots, r^{(j)}_{n_j - 1})$) and $r^{(j)} \in (\mathbb{Z}_{>0})^{n_j}$. Together, $G$ and $R$ are subject to the following conditions:

(i) If $g^{(j)}_{i+1} = g^{(j)}_i + 1$, then $r^{(j)}_i < r^{(j)}_{i+1}$.

(ii) The value $g^{(j)}_i$ is undefined or greater than zero for all $j < i \leq l - 1$.

(iii) For all $a$ and all $j < k$, either one of $g^{(j)}_a$ or $g^{(k)}_a$ is undefined, or $g^{(j)}_a > g^{(k)}_a$.

(iv) For all $a$ and all $j < k$, if $g^{(j)}_a$ and $g^{(k)}_a$ are defined with $g^{(j)}_a = g^{(k)}_a + 1$, then $r^{(j)}_a \leq r^{(k)}_{a-1}$.

The first condition states that every path $\pi_j$ is matched up with a label vector $r^{(j)}$ that strictly increases up the columns of the path. The remaining conditions imply that no
path encounters the start of any other (even zero-length) path; that the paths are weakly nested with no shared east steps; and that for a given column, no larger label in the row directly above belongs to a lower-indexed path.

Given \((G, R) \in LNDP_\lambda\), the area of \((G, R)\) is (as before) the sum of the areas of the Dyck paths comprising \(G\):

\[
\text{area}(G) = \sum_{i=0}^{l-1} \sum_{i \leq j < i + n_i} g^{(i)}_j.
\]

The diagonal inversion statistic for \((G, R)\) simply incorporates the labels into the two summations:

\[
\text{dinv}(G, R) = \text{adj}(\lambda) + \sum_{u,v,a,b} \chi( g_a^{(u)} - g_b^{(v)} = 1) \chi( r_a^{(u)} > r_b^{(v)}) \chi(a \leq b)
+ \sum_{u,v,a,b} \chi( g_a^{(u)} - g_b^{(v)} = 0) \chi( r_a^{(u)} < r_b^{(v)}) \chi(a < b \text{ or } (a = b \text{ and } u < v)).
\] (3)

In these sums, we consider all possible choices of \(a, b, u, v\) such that both \(g_a^{(u)}\) and \(g_b^{(v)}\) are defined. Set \(x_R = \prod_{u,a} x_a^{(u)}\). For the example shown in Figure 3, we compute the coefficient of \(m_{2,1,1}\) in \(\nabla(s_{2,2})\) as \(-t^2q^2(2 + t + q)\). (Note that in this example, \(\text{adj}(\lambda) = 1\).)

In Figure 4 we illustrate the pair \((G, R)\) where

\[
G = \begin{pmatrix}
g^{(0)} : & 0 & 1 & 2 & 3 & 3 & 3 & 4 & 3 \\
g^{(1)} : & . & 0 & 1 & 1 & 2 & 3 & 2 & . \\
g^{(2)} : & . & . & 0 & 1 & 2 & . & . & .
\end{pmatrix}
\] (4)

and

\[
R = \begin{pmatrix}
r^{(0)} : & 2 & 6 & 10 & 15 & 11 & 3 & 5 & 4 \\
r^{(1)} : & . & 1 & 17 & 7 & 8 & 14 & 12 & . \\
r^{(2)} : & . & . & 9 & 13 & 16 & . & . & .
\end{pmatrix}.
\]
Fig. 4. One term contributing to the coefficient of $m_{1,17}$ in $\nabla (s_{3^2,2})$

We have area$(G, R) = 31$ and dinv$(G, R) = 24$.

**Conjecture 2.1.** For any partition $\lambda$,

$$\nabla (s_{\lambda}) = \text{sgn}(\lambda) \sum_{(G, R) \in \text{LNDP}_{\lambda}} t^{\text{area}(G, R)} q^{\text{dinv}(G, R)} x_R.$$  \hspace{1cm} (5)

**Theorem 2.2.** Our conjectured formula for $\nabla (s_{\lambda})$ (i.e. the right side of (5)), is a symmetric function of the $x_j$'s. \hfill $\Box$

**Proof.** We prove this by expressing our summation in question as a weighted linear combination of the Lascoux–Leclerc–Thibon (LLT) polynomials (introduced in [18]), which are known to be symmetric.

The LLT polynomials can be defined combinatorially as follows. Let $\Gamma$ be a $k$-tuple of skew shapes $(\lambda^1/\nu^1, \lambda^2/\nu^2, \ldots, \lambda^k/\nu^k)$ with $n$ total boxes. For each such $\Gamma$, we write SSYT$^N_{\Gamma}$ to denote the set of $k$-tuples of semistandard Young tableaux $T = (T_1, T_2, \ldots, T_k)$ such that for each $i$,

- $T_i$ has shape $\lambda^i/\nu^i$.
- $T_i$ has entries from $\{x_1, x_2, \ldots, x_N\}$.

We denote the content of $T$ by $x_T = \prod_i \prod_{c \in \lambda^i/\nu^i} x_T(c)$. Given some $T \in \text{SSYT}^N_{\Gamma}$, we define a **diagonal inversion** statistic as follows. For a cell $c = (i, j)$, we define the **diagonal** of $c$ to be $\text{diag}(c) = j - i$. Then set

$$\text{dinv}(T) = \sum_{c \in T, \ d \in T} \left[ \chi(\text{diag}(d) - \text{diag}(c) = 1 \text{ and } T(c) < T(d)) \right. \\
\left. + \chi(\text{diag}(c) - \text{diag}(d) = 0 \text{ and } T(c) > T(d)) \right].$$  \hspace{1cm} (6)
The LLT polynomials are defined as

$$\text{LLT}_N(x_1, x_2, \ldots, x_N) = \sum_{T \in \text{SSYT}_N} q^{\text{dinv}(T)} x_T.$$  

It is proved in [15, 18] that each polynomial $\text{LLT}_N(x_1, x_2, \ldots, x_N)$ is a symmetric polynomial in the $x_i$'s. By taking inverse limits in the usual manner, we obtain symmetric polynomials $\text{LLT}_T \in \Lambda^n$.

We now explore the relationship between elements of $LNDP_\lambda$ and the LLT polynomials. We describe the correspondence via an example. Consider the pair $(G, R)$ illustrated in Figure 4. Reading up the columns from right to left, we encounter five contiguous multisets of north steps of varying lengths. The configuration $G$ will thereby be associated with a tuple of skew shapes $\Gamma(G) = (\lambda_1/\nu_1, \lambda_2/\nu_2, \ldots, \lambda_5/\nu_5)$ of sizes 2, 8, 2, 1, and 4, respectively. The third through fifth groups, each consisting of north steps from a single path, will yield skew shapes that are columns of the appropriate heights. In particular, we get the shapes $(1^2), (1^4)/(1^3), \text{and} (1^4)$, respectively. We have augmented as necessary each $\lambda^i$ and $\nu^i$ in equal amounts to ensure that any north step in $G$ going north from the line $y = x + b$ maps to a cell in $\Gamma(G)$ with diagonal equal to $b$. When we have more than one path contributing to a given multiset of north steps, we proceed in an analogous manner. However, in this case, the cells arising from the $j$th path from the left are additionally shifted to the right and up by $j - 1$ units. (Notice that this is a diagonal-preserving shift.) Doing so yields $\lambda^1/\nu^1 = (2^4)/(2^3)$ and $\lambda^2/\nu^2 = (3^5)/(3^2, 1)$.

The labels in $R$ accompany their respective north steps to give us the element $T \in \text{SSYT}_T^{17}$ illustrated in Figure 5. Note that we have aligned the skew shapes along the diagonals to facilitate computation of $\text{dinv}$. 

**Fig. 5.** $T$ corresponding to $(G, R)$ of Figure 4
This correspondence gives a well-defined correspondence from a Dyck configuration $G$ to a tuple of shapes $\Gamma(G)$. The correspondence can be modified to become invertible by the following two adjustments (which we do not make). First, send all north steps in a given column to a (possibly disconnected) skew shape. Second, include empty skew shapes for columns without any north steps. In any case, by including labels, the correspondence between Dyck configurations and tuples of shapes extends to a map from elements $(G, R) \in LNDP_\lambda$ to tuples of filled shapes $T(G, R)$. As is discussed in the following, these filled shapes are precisely the tuples $T$ appearing in $\text{LLT}_{\Gamma(G)}$.

Our first claim is that the shapes are, in fact, skew shapes. This follows from Conditions (ii) and (iii) in the definition of $LNDP_\lambda$. Our second claim is that the fillings of the skew shapes are semistandard. That the entries increase up columns is the content of Condition (i) while rows are forced to weakly increase by Condition (iv). Furthermore, it is easily checked that any element $T$ contributing to $\text{LLT}_{\Gamma(G)}$ appears as the image of $(G, R)$ for some labeling $R$. It follows that

$$
\sum_{R: (G, R) \in LNDP_\lambda} x_R = \sum_{T \in \text{SSYT}_{\Gamma(G)}} x_T.
$$

We now consider the relationship between $\text{dinv}(G, R)$ and $\text{dinv}(T(G, R))$. Fix $(G, R) \in LNDP_\lambda$ corresponding to a $T(G, R) \in \text{SSYT}_{\Gamma(G)}$. Suppose we have $u, v, a$, and $b$ such that $g_a^{(u)} - g_b^{(v)} = 1$, $r_a^{(u)} > r_b^{(v)}$ and $a \leq b$. Under our correspondence, $g_a^{(u)}$ and $g_b^{(v)}$ will correspond to cells $c$ and $d$ (labeled $r_a^{(u)}$ and $r_b^{(v)}$, respectively) in $T(G, R) = (T_1, T_2, \ldots, T_k)$. We may write $c \in T_i$ and $d \in T_j$ for some $i, j$. Since $a \leq b$ but $g_a^{(u)} > g_b^{(v)}$, we must have $i > j$. Since diagonals are preserved, we see that $c$ and $d$ will contribute to the first summand in (6). A similar analysis of terms with $a < b$ in the second summation of (3) will yield the second summand of (6).

We are left to consider those quadruples $a, b, u, v$ for which $g_a^{(u)} = g_b^{(v)}$, $a = b$ and $u < v$. Such a quadruple will contribute to (3) exactly when $r_a^{(u)} < r_b^{(v)}$. However, the corresponding pair of cells in $T(G, R)$ will never contribute to $\text{dinv}(T(G, R))$ because they will lie in the same skew shape. Fortunately, for $(G, R) \in LNDP_\lambda$ and such a quadruple, we will always have $r_a^{(u)} < r_b^{(v)}$. To see this, note that by Conditions (ii) and (iii), $g_a^{(v)}$ is defined. By Condition (iv), $r_a^{(u)} \leq r_{a-1}^{(v)}$. Finally, $r_{a-1}^{(v)} < r_a^{(v)}$ by Condition (i). For brevity, write

$$
\delta(\Gamma(G)) = \sum_{i=1}^k \delta(\lambda^i / \nu^i).
$$
where we define $\delta$ for a skew shape by

$$
\delta(\lambda/\nu) = \sum_{(i,j) \in \lambda/\nu, a > 0} \chi((i + a, j + a) \in \lambda/\nu).
$$

Then

$$
dinv(G, R) = \text{adj}(\lambda) + \text{dinv}(T(G, R)) + \delta(\Gamma(G)). \tag{8}
$$

Combining (7) and (8), we conclude

$$
\sum_{R : (G, R) \in \text{LNDP}_\lambda} q^{\text{dinv}(G, R)} x_R = q^{\text{adj}(\lambda) + \delta(\Gamma(G))} \sum_{T \in \text{SSYT}_{\Gamma(G)}} q^{\text{dinv}(T(G, R))} x_T. \tag{9}
$$

Of course, the summation on the right side of (9) is $\text{LLT}_{\Gamma(G)}$. So, if we allow $G$ to vary as well and sum over all such $G$, we get

$$
\sum_{(G, R) \in \text{LNDP}_\lambda} q^{\text{area}(G, R)} q^{\text{dinv}(G, R)} x_R = \sum_{G} q^{\text{area}(G)} q^{\text{adj}(\lambda) + \delta(\Gamma(G))} \text{LLT}_{\Gamma(G)}. \tag{10}
$$

Since, as mentioned, each $\text{LLT}_{\Gamma(G)}$ is a symmetric function in the $x_i$’s, it follows that the left side of (10) is as well.

\[\blacksquare\]

### 2.4 Hook shapes

Let $\lambda = (a, 1^{n-a})$ for some $a \geq 1$; so $\lambda' = (n - a + 1, 1^{a-1})$. When we fill the Ferrers diagram of $\lambda'$ with rim hooks as described in Section 2.2, we find that there is only one nonzero rim hook. Thus, $(n_0, n_1, \ldots, n_{a-1}) = (n, 0, \ldots, 0)$, so $\text{sgn}(\lambda) = (-1)^{a-1}$ and $\text{adj}(\lambda) = a - 1$. Furthermore, objects in $\text{LNDP}_\lambda$ can be identified with those elements $(g, r)$ of $\text{LDP}_n$ for which
\(g_i > 0\) for \(0 < i < a\). So when \(\lambda\) is a hook, we obtain the following simplifications of the main conjecture.

**Conjecture 2.3.**

\[
\langle \nabla(s_{a,1^{n-a}}), s_{1^n} \rangle = (-q)^{a-1} \sum_{g \in DP_1; g_i > 0 \text{ for } 0 < i < a} t^{\text{area}(g)} q^{\text{dinv}(g)}
\]

\[
\nabla(s_{2,1^{n-a}}) = (-q)^{a-1} \sum_{(g,r) \in LDP_1; g_i > 0 \text{ for } 0 < i < a} t^{\text{area}(g,r)} q^{\text{dinv}(g,r)} x_r.
\]

\[\square\]

### 2.5 Trapezoidal paths

Conjecture 2.1 neatly explains some of the formulas conjectured in [20, 23] regarding lattice paths in trapezoids. Consider the trapezoid bounded by the vertices \((0, k), (0, k+n), (k,k), \) and \((k+n, k+n)\) for some \(k, n\) with \(k \geq 0\) and \(n \geq 1\). A *trapezoidal lattice path* of type \((n,k)\) is a path from \((0, k)\) to \((k+n, k+n)\) consisting of \(n\) north steps and \(k+n\) east steps (all of length one) such that no vertex of the path lies strictly below the line \(y = x\). The case of \(k = 0\) is that of Dyck paths. Write \(T_{n,k}\) for the set of trapezoidal lattice paths of type \((n,k)\).

Given a path \(P \in T_{n,k}\), define the sequence \(g(P) = (g_0, g_1, \ldots, g_{n-1})\) by taking \(g_i\) to be the number of unit squares in the strip bounded below by \(y = k + i\), above by \(y = k + i + 1\), on the left by \(P\), and on the right by \(y = x\). Define \(\text{area}(P) = \sum_{i=0}^{n-1} g_i\) and

\[
\text{dinv}(P) = \sum_{i<j} \chi(g_i - g_j \in \{0, 1\}) + \sum_{i=0}^{n-1} \max(k - g_i, 0).
\]

It is conjectured in [20, 23] that \(\sum_{P \in T_{n,k}} t^{\text{area}(P)} q^{\text{dinv}(P)}\) is symmetric in \(q\) and \(t\).

We now show that this conjecture follows from Conjecture 2.1. To this end, for \(n, k \geq 0\) and \(n + k > 0\), define

\[
\lambda(n, k) = \left(\left\lceil \frac{k + 1}{2} \right\rceil, 1^n \right).
\]

The partition \(\lambda(n, k)\) has been defined so that when the special rim hooks are placed in \(\lambda(n, k)'\), the hooks are of lengths \(n_0 = k + n\) and \(n_i = k - 2i\) for \(1 \leq i \leq \lfloor k/2 \rfloor\). Since the
successive length differences are only 2 and the shortest path is of length 0 or 1, this forces the path of length $n_i$ for $i > 0$ to consist of $n_i$ north steps followed by $n_i$ east steps. The outermost path, of length $k + n$ has the single restriction that it must begin with $k$ north steps. It follows that the elements $G \in NDP_{\lambda(n,k)}$ are in natural bijection with the $g(P)$ for $P \in T_{n,k}$ by sending $G = (g^{(0)}, g^{(1)}, \ldots, g^{(\lfloor k/2 \rfloor)})$ to $g' = (g^{(0)}_k, g^{(0)}_{k+1}, \ldots, g^{(0)}_{k+n-1})$.

To complete the proof, we need only examine how $\text{area}(G)$ compares to $\text{area}(g')$ and how $\text{dinv}(G)$ compares to $\text{dinv}(g')$. It is a simple computation to show that

$$\text{area}(G) = \text{area}(g') + \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k - 2i}{2}.$$ 

As for the diagonal inversion statistic, first note that by throwing away all but the outermost path, we have lost $\text{adj}(\lambda(n, k)) = \binom{\lfloor k/2 \rfloor + 1}{2}$. We have also lost the contributions to $\text{dinv}(G)$ arising from interactions between any two of the paths. However, we gain $\sum_{i=k}^{k+n-1} \max(k - g^{(0)}_i, 0)$. We leave it to the reader to show, in fact, that these collectively give the appropriate difference.

We illustrate an example in Figure 6 for $k = 6$ and $n = 5$. On the left is shown $\lambda(5,6)'$ along with its special rim hooks of lengths 11, 4, 2, and 0. On the right we show a typical element of $NDP_{\lambda(5,6)}$. Notice that by considering the portion of the outermost path weakly above the dotted line, we have an element of $T_{5,6}$. For this example,

$$G = \begin{pmatrix}
g^{(0)} : 0 & 1 & 2 & 3 & 4 & 5 & 4 & 2 & 0 & 1 & 2 
g^{(1)} : 0 & 1 & 2 & 3 & \ldots & \ldots 
g^{(2)} : \ldots & 0 & 1 & \ldots & \ldots 
g^{(3)} : \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}.$$
So \( \text{area}(G) = 31 \), \( \text{dinv}(G) = 45 \) and \( \text{area}(g') = 9 \). We now check that \( \text{dinv}(G) - \text{dinv}(g') \) is indeed 22 = 31 - 9. In going from \( G \) to \( g' \), we lose the adjustment \( \text{adj}(\lambda(5, 6)) = 6 \). We also lose all of the interactions involving \( g_j^{[0]} \) for \( j < 6 \) and \( g_i^{[i]} \) for \( i = 1, 2, 3 \) that contribute to \( \text{dinv}(G) \). These account for a loss of 37 more. However, we gain the sum \( \sum_{i=0}^{4} \max(6 - g'_i, 0) = 21 \). This gives us a difference of 22 as desired.

2.6 Higher powers of nabla

Following [17], there is a natural conjecture to make regarding a combinatorial framework for \( \nabla^m(s_\lambda) \) for \( m > 1 \). To start, define an \( m \)-Dyck path of length \( n \) to be a lattice path with \( n \) north steps and \( mn \) east steps that never go below the line \( my = x \). These reduce to Dyck paths when \( m = 1 \). An \( m \)-Dyck sequence, \( g(P) = (g_0, g_1, \ldots, g_{n-1}) \), is defined by setting \( g_i \) to be the number of unit squares lying between the lines \( y = i \) and \( y = i + 1 \), to the right of \( P \), and to the left of \( my = x \). An \( n \)-tuple of non-negative integers is an \( m \)-Dyck sequence for an \( m \)-Dyck path if and only if \( g_0 = 0 \) and \( g_i \leq g_{i-1} + m \) for all \( 1 \leq i < n \).

For any given \( \lambda \), we define a set \( LNDP_m^{\lambda} \) in a manner entirely analogous to how we defined \( LNDP_\lambda \). The primary difference is that for a pair \( (G, R) \in LNDP_m^{\lambda} \), \( G \) is an \( m \)-Dyck configuration; i.e. each element of \( G \) is an \( m \)-Dyck sequence. While \( \text{area}(G, R) \) is defined by summing the entries in \( G \) as usual, we generalize the definition of \( \text{dinv}(G, R) \) as follows:

\[
\text{dinv}(G) = \text{adj}(\lambda) + \sum_{u,v,a,b} \sum_{d=0}^{m-1} \chi(g_a^{[u]} - g_b^{[v]} + d = m)\chi(r_a^{[u]} > r_b^{[v]})\chi(a \leq b) \\
+ \sum_{u,v,a,b} \sum_{d=0}^{m-1} \chi(1 \leq g_a^{[u]} - g_b^{[v]} + d < m)\chi(a \leq b) \\
+ \sum_{u,v,a,b} \sum_{d=0}^{m-1} \chi(g_a^{[u]} - g_b^{[v]} + d = 0)\chi(r_a^{[u]} < r_b^{[v]})\chi(a < b \text{ or } (a = b \text{ and } u < v)).
\]

(11)

**Conjecture 2.4.** For any partition \( \lambda \) and \( m \geq 1 \),

\[
\nabla^m(s_\lambda) = \text{sgn}(\lambda) \sum_{(G, R) \in LNDP_m^{\lambda}} t^{\text{area}(G, R)} q^{\text{dinv}(G, R)} x_R.
\]

(12)

We illustrate a typical element of \( LNDP_{(2,2)}^3 \) in Figure 7. As an exercise, the reader can explicitly write down the elements of \( LNDP_{(2,2)}^2 \) labeled with three 1’s and a 2 to
compute that
\[ \langle \nabla^2(s_{2,2}), h_{3,1} \rangle = -(tq)^3(1 + t + t^2 + q + qt + q^2). \]

We remark that trapezoidal lattice paths and $m$-Dyck paths can be treated at the same time by replacing the line $y = x$ with $my = x$ in the definition of trapezoidal paths. Of course, the dinv statistic needs to be adjusted accordingly. See [20, 23] for details.

3 Proof when $q = 1$

In this section, we will prove the following specialization of the main conjectures.

**Theorem 3.1.** For all partitions $\lambda$,
\[
\langle \nabla(s_{\lambda}), s_{1^n} \rangle|_{q=1} = \text{sgn}(\lambda) \sum_{G \in \text{NDP}_\lambda} t^{\text{area}(G)};
\]
\[
\nabla(s_{\lambda})|_{q=1} = \text{sgn}(\lambda) \sum_{(G,R) \in \text{LNDP}_\lambda} t^{\text{area}(G)} x_R.
\]

We remark that Lenart [19] proved a closely related result that establishes the Schur positivity of $\nabla|_{q=1}(s_{\lambda/\nu})$ by expanding the latter polynomials in terms of skew Schur functions. Lenart’s proof relies heavily on Jacobi–Trudi determinantal formulas. We adopt a more combinatorial approach that makes heavy use of the inverse Kostka matrix. One benefit of the present method is that the combinatorial significance of the global sign $\text{sgn}(\lambda)$ is more readily apparent.
3.1 Specialized nabla operator

The first step is to replace nabla by a more convenient operator. Let $\tilde{H}_\mu |_{q=1}$ denote the image of $\tilde{H}_\mu$ under the specialization sending $q$ to $1$; these specialized Macdonald polynomials form a basis for the $\mathbb{Q}(t)$-vector space $\Lambda_{\mathbb{Q}(t)}$ of symmetric functions with coefficients in $\mathbb{Q}(t)$. Define $\nabla_q = 1$ to be the unique $\mathbb{Q}(t)$-linear map such that $\nabla_q(\tilde{H}_\mu |_{q=1}) = t^{n(\mu)} \tilde{H}_\mu |_{q=1}$ for all partitions $\mu$. It is known that $\nabla_q$ is a ring homomorphism (this follows easily from the combinatorial interpretation of $\tilde{H}_\mu$ given in Section 4). Furthermore, it can be shown that

$$\nabla(s_\lambda) |_{q=1} = \nabla_q(s_\lambda).$$

Henceforth we will study the ring homomorphism $\nabla_q$.

3.2 Elementary symmetric function expansions

The second step is to study the matrix $(e_\mu) |_{\nabla_q = 1}(e_\nu)$. Given a Dyck path $\pi$, let $\alpha(\pi) = (\alpha_1(\pi), \alpha_2(\pi), \ldots)$ be the lengths of the vertical columns formed by consecutive north steps of $\pi$. Garsia and Haiman proved that

$$\nabla_q(e_n) = \sum_{\pi \in DP_n} t^{\text{area}(\pi)} e_{\alpha(\pi)}$$ (15)

(see Theorem 1.2 in [10]). Keeping in mind the combinatorial interpretation of $e_k = s_1^k$ in terms of semistandard tableaux, it is clear that this formula is equivalent to (1) when $q = 1$. Since the ring homomorphism $\nabla_q$ preserves multiplication, we immediately deduce from (15) that

$$\nabla_q(e_{\nu}) = \sum_{(\pi_1, \pi_2, \ldots) \in DP_\nu} t^{\sum_{i} \text{area}(\pi_i)} \prod_i e_{\alpha(\pi_i)}$$ (16)

where $DP_\nu$ is the set of all lists of paths $(\pi_1, \pi_2, \ldots)$ such that $\pi_i$ is a Dyck path of order $\nu_i$. Furthermore, since $\langle e_\xi, s_{1^n} \rangle = 1$ for all $\xi \vdash n$, we also deduce that

$$\langle \nabla_q(e_{\nu}), s_{1^n} \rangle = \sum_{(\pi_1, \pi_2, \ldots) \in DP_\nu} t^{\sum_{i} \text{area}(\pi_i)} (\nu \vdash n).$$ (17)
3.3 Transition matrices

The third step is to multiply by suitable transition matrices to change the input and output bases in the matrix $| e_j \rangle \langle q = 1 | e_i \rangle$. The relevant transition matrix turns out to be the inverse Kostka matrix, which we now review. (For more background on transition matrices, see Section I.6 of [26] or the references [1, 6].) Recall that the Kostka number $K_{\lambda, \mu}$ is the number of semistandard tableaux of shape $\lambda$ and content $\mu$. We have the identities

\begin{align*}
s_\lambda &= \sum_{\mu} K_{\lambda, \mu} m_\mu, \quad (18) \\
h_\mu &= \sum_{\lambda} K_{\lambda, \mu} s_\lambda, \quad (19) \\
e_\mu &= \sum_{\lambda} K_{\lambda', \mu} s_\lambda. \quad (20)
\end{align*}

Letting $K = (K_{\lambda, \mu})$ be the matrix of Kostka numbers, these identities assert that $K^t = (m_\mu) [\text{id}]_{(s_\lambda)}$; $K = (s_\lambda) [\text{id}]_{(h_\mu)}$; and $K = (s_\lambda) [\text{id}]_{(e_\mu)}$. Now, let $K^{-1} = (K_{\lambda, \mu})$ be the inverse of the matrix $K$. The previous identities now read

\begin{align*}
m_\mu &= \sum_{\lambda} K^{-1}_{\mu, \lambda} s_\lambda, \quad (21) \\
s_\lambda &= \sum_{\mu} K^{-1}_{\lambda, \mu} h_\mu, \quad (22) \\
s_\lambda &= \sum_{\mu} K^{-1}_{\lambda', \mu} e_\mu. \quad (23)
\end{align*}

Remmel and Eğecioğlu discovered the following important combinatorial interpretation for the entries $K^{-1}_{\mu, \lambda}$ of the inverse Kostka matrix [6]. A special rim hook tabloid of shape $\lambda$ and type $\mu$ is a filling of the Ferrers diagram $\mathcal{F}(\lambda)$ with rim hooks of length $\mu_i$ that all start in the leftmost column. For example, Figure 1 displays one special rim hook tabloid of shape $\lambda' = (5, 5, 2, 1, 1)$ and type $(9, 5)$. A rim hook spanning $r$ rows has sign $(-1)^{r-1}$. The sign of a rim hook tabloid is the product of the signs of all the rim hooks in the tabloid. Let $\text{SRHT}(\mu, \lambda)$ be the set of all special rim hook tabloids of shape $\lambda$ and type $\mu$. Remmel and Eğecioğlu showed that

\[ K^{-1}_{\mu, \lambda} = \sum_{T \in \text{SRHT}(\mu, \lambda)} \text{sgn}(T). \]
For example, we see from Figure 8 that $K^{-1}_{(4,3,1),(3,2,2,1)} = +3$. It will be convenient to introduce a modified notion of the “type” of a special rim hook tabloid. Given a rim hook tabloid $T$, let $\alpha_i(T)$ be the length of the rim hook that starts $i$ rows from the top of the diagram for $T$ (for $i = 0, 1, \ldots$). If no rim hook starts in row $i$, we let $\alpha_i(T) = 0$. The ordered sequence $\alpha(T) = (\alpha_0(T), \alpha_1(T), \ldots)$ will be called the total type of $T$. By dropping zero entries in $\alpha(T)$ and arranging into decreasing order, we obtain the type of $T$ (which is a partition). The total type of the tabloid in Figure 1 is $(9,0,0,5,0)$.

3.4 Intersecting path model

Combining (23) with (16), we immediately obtain a combinatorial interpretation for the entries in the matrix $[\nabla_{q=1}(s_\lambda)]_{\lambda \vdash n}$. Using linearity of $\nabla_{q=1}$, we calculate

$$\nabla_{q=1}(s_\lambda) = \sum_{\mu} K^{-1}_{\mu,\lambda} \nabla_{q=1}(e_\mu)$$

$$= \sum_{\mu} K^{-1}_{\mu,\lambda} \sum_{(\pi_i) \in DP_\mu} t^{\sum_{i} \text{area}(\pi_i)} \prod_{i} \epsilon_{\alpha(\pi_i)}.$$

Here is an explicit combinatorial interpretation of the right side. Given $\lambda \vdash n$, we consider all pairs $(T, \Pi)$ where $T$ is a special rim hook tabloid of shape $\lambda'$ and $\Pi = (\pi_i : i \geq 0)$ is a sequence of labeled Dyck paths such that $\pi_i$ has order $\alpha_i(T)$ for $0 \leq i < \lambda_1 = \ell(\lambda')$. The sign of such a pair is $\text{sgn}(T)$; the $t$-weight of the pair is $\sum_{i} \text{area}(\pi_i)$; and the monomial weight is obtained as usual from the labels of $\pi_i$. The sum of all such signed, weighted objects gives us $\nabla_{q=1}(s_\lambda) \in \mathbb{Q}(t)[x_1, x_2, \ldots]$. Using the fact that $\langle e_\xi, s_1^n \rangle = 1$ for all $\xi$, we obtain an analogous combinatorial interpretation for the quantity $\langle \nabla_{q=1}(s_\lambda), s_1^n \rangle$. The only difference is that $\Pi$ now consists of unlabeled Dyck paths.

For example, Figure 9 depicts a typical object contributing to the coefficient $\langle \nabla_{q=1}(s_{6,5,4,2,1,1,1}), s_1^{20} \rangle$. The sign of this object is $(-1)^7 = -1$, and the $t$-weight is $33 + 0 + \ldots$
2 + 0 + 4 + 0 = 39. As in Section 2, it is convenient to display the paths in \( \Pi \) by letting \( \pi_i \) start at \((i, i)\) and end at \((i + \alpha_i(T), i + \alpha_i(T))\).

3.5 Cancellation for unlabeled paths

To complete the proof of the theorem, we define sign-reversing involutions that cancel objects of opposite sign, and then show that the fixed points are enumerated by the formulas (13) and (14). For ease of exposition, we consider the unlabeled case first. So far, we have shown that

\[
\langle \nabla_{q=1}(s_k), s_1^n \rangle = \sum_{(T, \Pi)} \text{sgn}(T)t^{\text{area}(\Pi)},
\]

where \( T \) is any special rim hook tabloid of shape \( \lambda' \), and \( \Pi = (\pi_i : 0 \leq i < \lambda) \) is a collection of unlabeled Dyck paths such that \( \pi_i \) has order \( \alpha_i(T) \) for all \( i \).

We can cancel pairs of objects with the same \( t \)-weight and opposite signs as follows. Suppose that \((T, \Pi)\) is an object such that there exist two paths in \( \Pi \) that intersect at some vertex. More precisely, there exist \( i < j \) and \((x, y)\) such that \( \pi_i \) and \( \pi_j \) both reach \((x, y)\). Among all such choices of \( i, j, x, y \), choose one such that \( x, y \), then \( i \), then \( j \) is minimized. Write \( \pi_i = \beta \gamma \) and \( \pi_j = \delta \epsilon \), where \( \beta \) is a path from \((i, i)\) to \((x, y)\), \( \gamma \) is a path from \((x, y)\) to \((i + \alpha_i(T), i + \alpha_i(T))\), \( \delta \) is a path from \((j, j)\) to \((x, y)\), and \( \epsilon \) is a path from \((x, y)\) to \((j + \alpha_j(T), j + \alpha_j(T))\). Replace \( \pi_i \) by \( \beta \epsilon \) and \( \pi_j \) by \( \delta \gamma \) to get a new list of paths \( \Pi' \) with the same earliest intersection and the same total \( t \)-weight. Next, consider the special...
Fig. 10. Example of tail-switching for paths and tabloids

rim hook tabloid $T$. It is easy to see that there is a unique way to “switch the tails” of the special rim hooks starting in rows $i$ and $j$ so that the new special rim hooks in these rows have lengths $j + \alpha_j(T) - i$ and $i + \alpha_i(T) - j$, respectively. Furthermore, the sign of the new tabloid $T'$ is opposite to the sign of $T$. Figure 10 displays the object that is matched to the object in Figure 9 by this process. It is clear from the description that the map $(T, \Pi) \mapsto (T', \Pi')$ is an involution.

What do the fixed points of the involution look like? Clearly, $(T, \Pi)$ is a fixed point iff no two Dyck paths in $\Pi$ intersect. Because of the way the starting points of the Dyck paths are arranged along the line $y = x$, this can only occur if the lengths of the nontrivial Dyck paths in the list $(\pi_1, \pi_2, \ldots)$ form a strictly decreasing sequence. In other words, the lengths of the nonzero rim hooks in $T$ must strictly decrease reading from top to bottom. One sees easily that this condition forces $T$ to consist of a succession of “border hooks” as described earlier in connection with $\text{sgn}(\lambda)$. Indeed, we now see that $\text{sgn}(\lambda)$ is simply the sign of the unique special rim hook tabloid $T$ that occurs in the objects that are fixed points. Taking the cancellation and fixed points into account, we see that the desired result (13) follows from (24).

3.6 Cancellation for labeled paths

We sketch the cancellation for labeled paths while focusing on the differences with respect to the unlabeled case. In general, paths can be rerouted at intersections as in the unlabeled case. In particular, any two paths that intersect at the beginning of a common
Nested Quantum Dyck Paths and $\nabla(s_i)$

Fig. 11. Cancellation for labeled paths along north steps

...east step can be cancelled with an object of equal $t$-weight and opposite sign. Similarly, if we have an object where one path intersects the beginning of another (possibly zero-length) path, we can cancel with an object of equal $t$-weight and opposite sign. The label of each step should be envisioned to remain with the individual step rather than with a particular path.

We are left to consider the scenario that two paths intersect as in Figure 11, left panel or right panel ($z$ is allowed to be one in either case). We claim that each object containing an intersection of type 1 can be paired with an object containing a region of type 2.

Assume we have an intersection such as that of Figure 11, left panel. Set

$$C = \{ z \} \cup \{ 1 \leq i < z : c_i > d_{i-1} \} \quad \text{and} \quad D = \{ 1 \} \cup \{ 1 < i \leq z : d_i > c_{i-1} \}.$$  

Suppose that $j \in C \cap D$; choose the smallest such $j$. Then we can set $d'_i = d_i$ for $0 \leq i < j$, $c'_i = c_i$ for $1 \leq i < j$, $d'_i = c_i$ for $j \leq i < z$ and $c'_i = d_i$ for $j \leq i \leq z$. This yields a paired object as in Figure 11, right panel, with the same $t$-weight, but with opposite sign. This process is invertible. So it only remains to show that $C \cap D \neq \emptyset$.

Assume in fact that $C \cap D = \emptyset$. Hence $1 \notin C$ and $z \notin D$. Let $j$ be as small as possible such that $j \in C$ (such a $j$ must exist since $z \in C$). It is immediate that $j > 1$ and $j - 1 \notin C$. Therefore, $c_{j-1} \leq d_{j-2} < d_{j-1} < d_j$. From this we conclude that $j \in D$. This is a contradiction. So, as desired, $C \cap D \neq \emptyset$.

The foregoing argument shows that all configurations with an intersection such as in Figure 11, left panel, cancel with an equal-weight object of the opposite sign. It
remains to characterize the configurations that do not cancel. We have just seen that they must be weakly nested. That is, while any two paths are allowed to share some north steps, the path that started earlier can never pass under a path that started later. In fact, the fixed points are described by Conditions (i), (ii), and (iii) along with

(iv') For all \(a\) and \(j < k\), if \(g_a^{(j)}\) and \(g_{a-1}^{(k)}\) are defined with \(g_a^{(j)} = g_{a-1}^{(k)} + 1\), then either

(a) \(r_a^{(j)} \leq r_{a-1}^{(k)}\), or
(b) \(g_a^{(k)}\) is not defined, or
(c) \(g_{a-1}^{(j)}\) is not defined, or
(d) \(g_a^{(k)}\) and \(g_{a-1}^{(j)}\) are both defined with \(r_a^{(k)} \leq r_{a-1}^{(j)}\).

We now show that any fixed point (which must satisfy Condition (iv')) must actually satisfy Condition (iv) as well. So suppose on the contrary we have a fixed point such as in Figure 11, right panel, with \(a\) and \(j < k\) for which \(r_a^{(j)} > r_{a-1}^{(k)}\). It is easy to check that having \(g_a^{(k)}\) or \(g_{a-1}^{(j)}\) not being defined would contradict the fact that we have a fixed point. So assume they are, in fact, both defined. The only possibility is that \(r_a^{(k)} \leq r_{a-1}^{(j)}\). But then we give a similar argument to the one in the foregoing to obtain a contradiction.

4 Proof Strategy for the Full Conjecture

Most of the known facts about combinatorial interpretations of the nabla operator (cf. Table 1) were proved via long, laborious algebraic manipulations making heavy use of the machinery of plethystic calculus \([3, 4, 8, 9, 13]\). In the past, one barrier to finding purely combinatorial proofs has been the absence of combinatorial conjectures that fully characterize the action of the nabla operator. Of course, this barrier has been overcome by the conjecture in Section 2. Another obstacle to a combinatorial analysis of nabla was the lack of combinatorial information about the Macdonald polynomials, which appear in the definition of nabla. However, recent breakthroughs by Haglund et al. \([12, 15, 16]\) have resolved this difficulty as well. We review Haglund’s combinatorial description of Macdonald polynomials in the next subsection. Combining this material with more inverse Kostka combinatorics, we then show how to reduce the proof of conjecture (5) to the problem of finding sign-reversing involutions on certain explicit collections of objects. If this proof strategy can be completed, it would yield a fully combinatorial proof of all the conjectured facts about the nabla operator.
Nested Quantum Dyck Paths and $\nabla(s_i)$

4.1 Combinatorial Macdonald polynomials

Let $\mu$ be a fixed integer partition. A Haglund filling of $\mu$ is a function $T: F(\mu) \to \mathbb{N}^+$. Informally, $T$ is a labeling of the cells in the diagram of $\mu$ with arbitrary positive integers. The content monomial of $T$ is

$$x_T = \prod_{c \in F(\mu)} x_{T(c)}.$$

Recall that the major index of a word $w = w_1 w_2 \cdots w_k$ is the sum of the positions of the descents of $w$, i.e., $\text{maj}(w) = \sum_{i=1}^{k-1} i \chi(w_i > w_{i+1})$. We define $\text{maj}(T)$ to be the sum of the major indices of the words obtained by reading the labels from top to bottom in each column of $T$.

Next, we define the notion of an inversion triple. Consider three cells $c, d, e \in F(\mu)$ such that $c$ and $e$ are in the same row (with $c$ to the left of $e$), and $d$ is the cell immediately below $c$. We call $(c, d, e)$ an inversion triple for $T$ iff

$$T(c) \leq T(d) < T(e) \text{ or } T(d) < T(e) < T(c) \text{ or } T(e) < T(c) \leq T(d).$$

By convention, if $c$ is to the left of $e$ in the bottom row of $F(\mu)$, we regard $(c, e)$ as an inversion triple iff $T(c) > T(e)$. Finally, let $\text{inv}(T)$ be the total number of inversion triples in $T$.

Haglund [12] discovered the formula

$$\hat{H}_\mu = \sum_{T: F(\mu) \to \mathbb{N}^+} q^{\text{inv}(T)} t^{\text{maj}(T)} x_T.$$

This formula was later proved by Haglund et al. [15, 16]. This formula essentially gives the transition matrix $(m_{\lambda})_{(\mu \rightarrow \lambda)}$. More precisely, writing $x_\lambda = \prod_{i \geq 1} x_i^{\lambda_i}$, we have

$$\hat{H}_\mu = \sum_\lambda A(\lambda, \mu) m_\lambda \quad \text{where } A(\lambda, \mu) = \sum_{T: F(\mu) \to \mathbb{N}^+ \atop x_T = x_\lambda} q^{\text{inv}(T)} t^{\text{maj}(T)}.$$  \hspace{1cm} (25)

Note that the $\mathbb{Q}(q, t)$-valued coefficient matrix $A = (A_{\lambda, \mu})$ is invertible, since the modified Macdonald polynomials form a basis of $\Lambda$. It follows that the $m_\lambda$’s are the unique vectors $v_\lambda$ solving the system of equations $\hat{H}_\mu = \sum_\lambda A(\lambda, \mu) v_\lambda$. Now, apply nabla
to both sides of (25). We obtain

\[ T_\mu \tilde{H}_\mu = \sum_\lambda A(\lambda, \mu)(\nabla (m_\lambda)). \quad (26) \]

Reasoning just as before, it follows that the vectors \( \nabla (m_\lambda) \) constitute the unique solution to the system of equations

\[ T_\mu \tilde{H}_\mu = \sum_\lambda A(\lambda, \mu)v_\lambda. \]

Extracting the coefficient of \( m_\nu \) on both sides, we see that a given indexed family of vectors \( (v_\lambda) \) is equal to the indexed family \( (\nabla (m_\lambda)) \) iff

\[ T_\mu A(v, \mu) = \sum_\lambda A(\lambda, \mu)(v_\lambda|m_\nu) \text{ for all partitions } \mu, \nu. \quad (27) \]

### 4.2 Combinatorial formulation of the problem

To proceed, we need to have a combinatorial interpretation of the quantities \( \nabla (m_\lambda) \). One such interpretation follows immediately from our conjecture for \( \nabla (s_\lambda) \) by using the inverse Kostka matrix again. More precisely, note that linearity of nabla gives

\[ \nabla (m_\lambda) = \sum_\rho K_{\lambda, \rho}^{-1} \nabla (s_\rho) \]

\[ = \sum_\rho \sum_{T \in \text{SRHT}(\lambda, \rho)} \text{sgn}(T) \sum_{(G, R) \in \text{LNDP}_\rho} \text{sgn}(\rho) t^{\text{area}(G, R)} q^{\text{dinv}(G, R)} x_{R, G}. \]

Extracting the coefficient of \( m_\nu \), this formula says that the \( \nu, \lambda \)-entry of the matrix \( (m_\nu)[\nabla (m_\lambda)] \) is

\[ \sum_\rho \sum_{T \in \text{SRHT}(\lambda, \rho)} \sum_{(G, R) \in \text{LNDP}_\rho: X_R = x_\nu} \text{sgn}(T) \text{sgn}(\rho) t^{\text{area}(G, R)} q^{\text{dinv}(G, R)}. \]

Computer calculations indicate that every entry of \( (m_\nu)[\nabla (m_\lambda)] \) is a polynomial in \( q \) and \( t \) with coefficients all of like sign. Yet the formula just written is a sum of both positive and negative objects. This indicates that there should be a sign-reversing, weight-preserving involution on the objects just described whose fixed points all have
the same sign. If such an involution could be found, we would have a better description of the entries of the matrix under consideration.

By the remark at the end of the last subsection, we see that all the combinatorial formulas for nabla will be proved if the following “master identities” can be verified for all partitions \(\mu, \nu\):

\[
T_\mu A(\nu, \mu) = \sum_\lambda A(\lambda, \mu) \sum_\rho \sum_{T \in SRHT(\lambda, \rho)} \sum_{(G, R) \in LNDP_\rho} \text{sgn}(T)\text{sgn}(\rho) t^{\text{area}(G, R)} q^{\text{dinv}(G, R)}.
\] (28)

Recalling the combinatorial interpretation for the entries of \(A\), we can reformulate the master identities as follows. Fix \(\mu, \nu \vdash n\). On one hand, let \(X\) be the set of all tuples \(z = (U, \lambda, T, \rho, (G, R))\) such that:

(i) \(\lambda\) and \(\rho\) are partitions of \(n\);
(ii) \(U\) is a Haglund filling of shape \(\mu\) and content \(\lambda\);
(iii) \(T\) is a special rim hook tabloid of shape \(\rho\) and content \(\lambda\);
(iv) \((G, R) \in LNDP_\rho\) has content \(x_R = x_\nu\).

The sign of \(z\) is \(\text{sgn}(T)\text{sgn}(\rho) \in \{+1, -1\}\); the \(t\)-weight of \(z\) is \(\text{maj}(U) + \text{area}(\Pi)\); and the \(q\)-weight of \(z\) is \(\text{inv}(U) + \text{dinv}(\Pi)\).

On the other hand, let \(X'\) be the set of all Haglund fillings \(U'\) of shape \(\mu\) and content \(x_\nu\). The sign of \(U'\) is always positive; the \(t\)-weight is \(n(\mu) + \text{maj}(U')\); and the \(q\)-weight is \(n(\mu') + \text{inv}(U')\). The master identity (28) holds for \(\mu\) and \(\nu\) iff there exists a sign-reversing, weight-preserving involution on \(X\) with positive fixed points, and a weight-preserving bijection between these fixed points and \(X'\).

As a closing remark, we recast the preceding discussion in terms of matrices. We are essentially trying to prove a matrix identity of the form \(TA = ALB\), where \(T\) is the diagonal matrix of scalars \(T_\mu\), \(A\) is the matrix describing the monomial expansion of modified Macdonald polynomials, \(L\) is the inverse Kostka matrix, and \(B\) is the matrix conjectured to give the monomial expansion of \(\nabla(s_\lambda)\). Now, \(ALB = A(LB) = (AL)B\). The matrix \(LB\) gives the monomial expansion of \(\nabla(m_\lambda)\), as discussed in the foregoing, while the matrix \(AL\) gives the Schur expansion of modified Macdonald polynomials. Using special rim hook tabloids, we can write down collections of signed, weighted objects to interpret either of the matrix products \(LB\) or \(AL\). In each case, computer evidence (and the known Schur-positivity of Macdonald polynomials) tells us that we should be able to cancel objects in these collections to obtain smaller collections of objects all of like sign. Finding an interpretation for the entries of \(AL\) that involves only positive objects is
a well-known open problem. It is likely that this problem (or the analogous problem for $LB$) will need to be solved first before further progress can be made in understanding the triple product $ALB$.

Acknowledgements

The authors were partially supported by a National Science Foundations Postdoctoral Fellowship and a Sterge Faculty Fellowship.

References

[1] Beck, Desiree A., Jeffrey B. Remmel, and Tamsen Whitehead. “The combinatorics of transition matrices between the bases of the symmetric functions and the $B_n$ analogues.” In *Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics*, edited by Peter L. Hammer, M. Delest, R. Pinzani, 3–27. Vol. 153. New York: Elsevier Science, 1996.

[2] Bergeron, F., and A. M. Garsia. “Science fiction and Macdonald’s polynomials.” In *Algebraic Methods and q-Special Functions*, 1–52. Centre de Recherches Mathématiques Proceedings Lecture Notes 22. Providence, RI: American Mathematical Society, 1999.

[3] Bergeron, F., A. M. Garsia, M. Haiman, and G. Tesler. “Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions.” *Methods of Applications and Analysis* 6, no. 3 (1999): 363–420.

[4] Bergeron, François, Nantel Bergeron, Adriano M. Garsia, Mark Haiman, and Glenn Tesler. “Lattice diagram polynomials and extended Pieri rules.” *Advances in Mathematics* 142, no. 2 (1999): 244–334.

[5] Can, Mahir, and Nicholas Loehr. “A proof of the $q, t$-square conjecture.” *Journal of Combinatorial Theory, Series A* 113, no. 7 (2006): 1419–34.

[6] Eğecioğlu, Ömer, and Jeffrey B. Remmel. “A combinatorial interpretation of the inverse Kostka matrix.” *Linear and Multilinear Algebra* 26, no. 1-2 (1990): 59–84.

[7] Egge, E. S., J. Haglund, K. Killpatrick, and D. Kremer. “A Schröder generalization of Haglund’s statistic on Catalan paths.” *Electronics Journal of Combinatorics* 10, R16 (2003): 1–21. http://www.combinatorics.org/Volume_10/PDF/v10i1r16.pdf.

[8] Garsia, A. M., and J. Haglund. “A proof of the $q, t$-Catalan positivity conjecture.” *Discrete Mathematics* 256, no. 3 (2002): 677–717.

[9] Garsia, A. M., and J. Haglund. “A positivity result in the theory of Macdonald polynomials.” *Proceedings of the National Academy of Sciences, USA* 98, no. 8 (2001): 4313–6.

[10] Garsia, A. M., and M. Haiman. “A remarkable $q, t$-Catalan sequence and $q$-Lagrange inversion.” *Journal of Algebraic Combinatorics* 5, no. 3 (1996): 191–244.

[11] Haglund, J. “Conjectured statistics for the $q, t$-Catalan numbers.” *Advances in Mathematics* 175, no. 2 (2003): 319–34.
Nested Quantum Dyck Paths and $\nabla(s)$

[12] Haglund, J. “A combinatorial model for the Macdonald polynomials.” *Proceedings of the National Academy of Sciences, USA* 101, no. 46 (2004): 16127–31.

[13] Haglund, J. “A proof of the $q, t$-Schröder conjecture.” *International Mathematical Research Notices* 2004, no. 11 (2004): 525–60.

[14] Haglund, J., and N. Loehr. “A conjectured combinatorial formula for the Hilbert series for diagonal harmonics.” *Discrete Mathematics* 298, nos. 1–3 (2005): 189–204.

[15] Haglund, J., M. Haiman, and N. Loehr. “A combinatorial formula for Macdonald polynomials.” *Journal of the American Mathematical Society* 18, no. 3 (2005): 735–61.

[16] Haglund, J., M. Haiman, and N. Loehr. “Combinatorial theory of Macdonald polynomials 1: Proof of Haglund’s formula.” *Proceedings of the National Academy of Sciences, USA* 102, no. 8 (2005): 2690–6.

[17] Haglund, J., M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. “A combinatorial formula for the character of the diagonal coinvariants.” *Duke Mathematical Journal* 126, no. 2 (2005): 195–232.

[18] Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. “Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties.” *Journal of Mathematical Physics* 38, no. 2 (1997): 1041–68.

[19] Lenart, Cristian. “Lagrange inversion and Schur functions.” *Journal of Algebraic Combinatorics*, 11, no. 1 (2000): 69–78.

[20] Loehr, N. “Trapezoidal lattice paths and multivariate analogues.” *Advances in Applied Mathematics* 31, no. 4 (2003): 597–629.

[21] Loehr, Nicholas A. “Conjectured statistics for the higher $q, t$-Catalan sequences.” *Electronic Journal of Combinatorics* 12, R9 (2005): 1–54. http://www.combinatorics.org/Volume_12/PDF/v12i1r9.pdf.

[22] Loehr, Nicholas A. “Combinatorics of $q, t$-parking functions.” *Advances in Applied Mathematics* 34, no. 2 (2005): 408–25.

[23] Loehr, Nicholas A. “Multivariate analogues of Catalan numbers, parking functions, and their extensions.” PhD thesis, University of California at San Diego, 2003.

[24] Loehr, Nicholas A., and Jeffrey B. Remmel. “Conjectured combinatorial models for the Hilbert series of generalized diagonal harmonics modules.” *Electronic Journal of Combinatorics* 11, no. 1, R68 (2004): 1–64. http://www.combinatorics.org/Volume_11/PDF/v11i1r68.pdf.

[25] Loehr, Nicholas A., and Gregory S. Warrington. “Square $q, t$-lattice paths and $\nabla(p_n)$.” *Transactions of the American Mathematical Society* 359, no. 2 (2007): 649–69.

[26] Macdonald, I. G. *Symmetric Functions and Hall Polynomials*. 2nd ed. Oxford Mathematical Monographs. New York: Oxford University Press, 1995.