INFINITELY MANY HYPERELLIPTIC CURVES WITH EXACTLY TWO RATIONAL POINTS (2)

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Abstract. In the previous paper, Hirakawa and the author considered a certain infinite family of hyperelliptic curves $C^{(p; i,j)}$ parametrized by a prime number $p$ and integers $i$, $j$, and proved that some of them have exactly two obvious rational points. In this paper, we extend the above work. In the proof, we consider another hyperelliptic curve $C^{(p; i,j)}$ whose Jacobian variety is isogenous to that of $C^{(p; i,j)}$, and prove that the Mordell-Weil rank of the Jacobian variety of $C^{(p; i,j)}$ is 0 by the standard 2-descent argument. Then, we determine the set of rational points of $C^{(p; i,j)}$ by using the Lutz-Nagell type theorem for hyperelliptic curves that was proven by Grant.

Contents

1. Main theorem
2. 2-descent
   2.1. Case $(i, j) = (0, 2), p \equiv 7 \pmod{16}$
   2.2. Case $(i, j) = (2, 2), p \equiv 7 \pmod{16}$
3. Application of the Lutz-Nagell type theorem for hyperelliptic curves
4. Conjectures
5. References

1. Main theorem

In arithmetic geometry, it is one of central problems to determine the sets of rational points on algebraic curves. In this paper, we study the set of rational points on hyperelliptic curves. A hyperelliptic curve defined over $\mathbb{Q}$ with a rational point can be embedded into its Jacobian variety. If $C$ is an elliptic curve, then the 2-descent argument makes it possible to bound the Mordell-Weil rank of $C$ by means of the 2-Selmer group, and the Lutz-Nagell theorem makes it possible to determine the torsion points of $C$ by means of the discriminant. For example, by applying these arguments, we can prove that the only rational points on $C(p)$ defined by $y^2 = x(x + p)(x - p)$ with a prime number $p \equiv 3 \pmod{8}$ are $(x, y) = (0, 0), (p, 0), (-p, 0)$ and $\infty$, i.e., such a prime number $p$ is never a congruent number (cf. [6, D27] and the references therein). It is also known that there exist infinitely many hyperelliptic curves with no or only one rational point for infinitely many genera [11, Theorem 4]. In the genus 2 case, their proof is reduced to the fact that
Theorem 1.1. Let \( p \) be a prime number, \( i, j \in \mathbb{Z} \), and \( C^{(p;i,j)} \) be a hyperelliptic curve defined by
\[
y^2 = x(x^2 + 2^ip^j)(x^2 + 2^{i+1}p^j).
\]
Suppose that one of the following conditions holds.
\begin{itemize}
    \item[(1)] \( p \equiv 7 \pmod{16} \) and \((i, j) = (0, 2)\).
    \item[(2)] \( p \equiv 7 \pmod{16} \) and \((i, j) = (2, 2)\).
\end{itemize}
Then, \( C^{(p;i,j)}(\mathbb{Q}) = \{(0, 0), \infty\} \).

Here, note that, in each case, there exist infinitely many prime numbers satisfying the above congruent condition by Dirichlet’s theorem on arithmetic progressions.

In the previous paper, Hirakawa and the author considered the following conditions.
\begin{itemize}
    \item[(1)] \( p \equiv 3 \pmod{16} \) and \((i, j) = (0, 1)\).
    \item[(2)] \( p \equiv 11 \pmod{16} \) and \((i, j) = (1, 1)\).
    \item[(3)] \( p \equiv 3 \pmod{8} \) and \((i, j) = (0, 2)\).
    \item[(4)] \( p \equiv -3 \pmod{8} \) and \((i, j) = (0, 2)\).
\end{itemize}

Then, we proved that similar results hold (\cite{7} Theorem 1.1). In the proof, we first proved that the \( \mathbb{F}_2 \)-dimension of the 2-Selmer group \( \text{Sel}^{(2)}(\mathbb{Q}, J^{(p;i,j)}) \) is 2 by the 2-descent argument \cite{12}. Since the \( \mathbb{F}_2 \)-dimension of the 2-torsion part \( J^{(p;i,j)}(\mathbb{Q})[2] \) is also 2, we obtain that \( \text{rank}(J^{(p;i,j)}(\mathbb{Q})) = 0 \). Next, we determined the set of rational points on \( C^{(p;i,j)} \) which map to torsion points on \( J^{(p;i,j)} \) via the Abel-Jacobi map by the Lutz-Nagell theorem (\cite{5} Theorem 3).

On the other hand, computation by MAGMA \cite{1} suggests that \( \dim_{\mathbb{F}_2} \text{Sel}^{(2)}(\mathbb{Q}, J^{(p;i,j)}) = 4 \) in Theorem \cite{11}. Thus, we can obtain only an upper bound \( \text{rank}(J^{(p;i,j)}(\mathbb{Q})) \leq 2 \) in the 2-descent calculation for \( J^{(p;i,j)} \) itself. Therefore, in addition to the 2-descent argument and the Lutz-Nagell theorem, we use a Richelot isogeny. A Richelot isogeny is a certain isogeny between the Jacobian varieties of hyperelliptic curves of genus 2 (cf. \cite{2, 8, 9}). The Jacobian variety \( J^{(p;i,j)} \) of the hyperelliptic curve \( C^{(p;i,j)} \) defined by
\[
y^2 = x(x^2 - 2^{i+2}p^j)(x^2 - 2^{i+3}p^j)
\]
is isogenous to the Jacobian variety \( J^{(p;i,j)} \) of \( C^{(p;i,j)} \) by Corollary \cite{8}. Therefore, to prove that the Mordell-Weil rank of \( J^{(p;i,j)} \) is 0, it is sufficient to prove that the Mordell-Weil rank of the Jacobian variety \( J^{(p;i,j)} \) of \( C^{(p;i,j)} \) is 0.

In §2, we carry out this task by the 2-descent argument \cite{12}, and in §3, we determine the set of rational points on \( C^{(p;i,j)} \) which map to torsion points on \( J^{(p;i,j)} \) via the Abel-Jacobi map by the Lutz-Nagell type theorem \cite{5} Theorem 3. Finally, we state some conjectures in §4.

2. 2-descent

Let \( p \) be a prime number, \( i, j \in \mathbb{Z} \), and \( f(x) = x(x^2 + 2^ip^j)(x^2 + 2^{i+1}p^j) \). Let \( C^{(p;i,j)} \) be a hyperelliptic curve defined by \( y^2 = f(x) \) and \( J^{(p;i,j)} \) be its Jacobian variety. In this section, we prove the following theorem.

Theorem 2.1. Suppose that one of the following conditions holds.
\begin{itemize}
    \item[(1)] \( p \equiv 7 \pmod{16} \) and \((i, j) = (0, 2)\).
    \item[(2)] \( p \equiv 7 \pmod{16} \) and \((i, j) = (2, 2)\).
\end{itemize}
Then, we have \( \text{rank}(J^{(p;i,j)}(\mathbb{Q})) = 0 \).
We treat the above two cases separately but in a similar manner in the following two subsections respectively. We prove Theorem 2.1 by considering the Jacobian variety of another hyperelliptic curve \(C^{i,j} \) whose Jacobian variety is isogenous to that of \(C^{i,j} \).

**Theorem 2.2.** ([2, 8, Theorem 5.7.4], [9]) Let \(C \) be a hyperelliptic curve over \(\mathbb{Q} \) defined by \(y^2 = G_1(x)G_2(x)G_3(x) \), where \(G_i(x) = g_{i2}x^2 + g_{i1}x + g_{i0} \) and \(\Delta := \det(g_{ij}) \). Suppose that \(\Delta \neq 0 \), and let \(D \) be a hyperelliptic curve defined by the following equation.

\[
\Delta y^2 = (G_2'G_3 - G_2G_3')(G_3'G_1 - G_3G_1')(G_1'G_2 - G_1G_2').
\]

Here, \(G'_i(x) \) denotes the derivative of \(G_i(x) \). Then, \(J_C := \text{Jac}(C) \) and \(J_D := \text{Jac}(D) \) are isogenous over \(\mathbb{Q} \). In particular,

\[
\text{rank}(J_C(\mathbb{Q})) = \text{rank}(J_D(\mathbb{Q})�).
\]

By this theorem, we obtain a defining equation of our \(C^{i,j} \).

**Corollary 2.3.** The Jacobian variety of \(C^{i,j} \) is isogenous to that of \(C^{i,j} \) defined by the following equation.

\[y^2 = x(x^2 - 2^i+2p^j)(x^2 - 2^{i+3}p^j).\]

Let \(J^{i,j}_{C^{i,j}} \) be the Jacobian variety of \(C^{i,j} \). In what follows, we show that \(\text{rank}(J_{C^{i,j}}(\mathbb{Q})) = 0 \). Since \(J_{C^{i,j}}(\mathbb{Q})/2J_{C^{i,j}}(\mathbb{Q}) \) can be embedded into the 2-Selmer group \(\text{Sel}^{(2)}(\mathbb{Q}, J_{C^{i,j}}) \), in order to bound the Mordell-Weil rank from above, it is sufficient to calculate the dimension of the 2-Selmer group. By [12, p. 256], we have the following exact sequence.

\[0 \to J_{C^{i,j}}(\mathbb{Q})/2J_{C^{i,j}}(\mathbb{Q}) \to \text{Sel}^{(2)}(\mathbb{Q}, J_{C^{i,j}}) \to \text{III}(\mathbb{Q}, J_{C^{i,j}})[2] \to 0,\]

where \(\text{III}(\mathbb{Q}, J_{C^{i,j}})[2] \) is the 2-torsion subgroup of the Tate-Shafarevich group of \(J_{C^{i,j}} \). Therefore, we have

\[
\dim \text{Sel}^{(2)}(\mathbb{Q}, J_{C^{i,j}}) = \dim \text{val}^{-1}(G) + \dim (\text{Im}(\delta_2) \times \text{Im}(\delta_p) \times \text{Im}(\delta_\infty))
\]

\[
- \dim ((\text{Im}(\delta_2) \times \text{Im}(\delta_p) \times \text{Im}(\delta_\infty)) + \text{res}_S(\text{val}^{-1}(G))) - \dim \text{III}(\mathbb{Q}, J_{C^{i,j}})[2]
\]

\[
\leq \dim \text{val}^{-1}(G) + \dim (\text{Im}(\delta_2) \times \text{Im}(\delta_p) \times \text{Im}(\delta_\infty))
\]

\[
- \dim ((\text{Im}(\delta_2) \times \text{Im}(\delta_p) \times \text{Im}(\delta_\infty)) + \text{res}_S(\text{val}^{-1}(G))).
\]

In each case, we can prove that the right hand side equals 3. \[\text{Note that } \dim \text{Im}(\delta_2) \text{ (resp. } \dim \text{Im}(\delta_p) \text{) is independent of prime number } p. \text{ For the detail, see Lemmas 2.11, 2.13, 2.19 and 2.21.}\]

**Notation 2.4.** We fix \(p, i \), and \(j \), so we abbreviate \(C^{i,j} \) to \(C \) and \(J^{i,j}_{C^{i,j}} \) to \(J \). Let \(y^2 = f(x) \) be the defining equation of \(C \). Denote

- the \(x \)-coordinate of the point \(P \in C(\mathbb{Q}) \) by \(x_P \),
- every divisor class in \(J(\mathbb{Q}) \) represented by a divisor \(D \) simply by \(D \),
- a fixed algebraic closure of \(\mathbb{Q} \) by \(\overline{\mathbb{Q}} \).

For every place \(v \), we also use a similar notation and fix an embedding \(\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v \). Define

- \(L := \mathbb{Q}[T]/(f(T)) \cong \prod_{i=1}^l L^{(i)} ; T \mapsto (T_1;\cdots;T_l) \), where \(f(T) = f_1(T)\cdots f_l(T) \) is the irreducible decomposition in \(\mathbb{Q}[T] \) and \(L^{(i)} := \mathbb{Q}[T_i]/(f_i(T)) \). We denote the trivial elements in \(L^x \) and \(L^x/L^{x^2} \) by \(1 \).

For every place \(v \), define

- \(L_v := \mathbb{Q}_v[T]/(f(T)) \cong \prod_{i=1}^m L_v^{(i)} ; T \mapsto (T_1;\cdots;T_m) \), where \(f(T) = f_1(T)\cdots f_m(T) \) is the irreducible decomposition in \(\mathbb{Q}_v[T] \) and \(L_v^{(i)} := \mathbb{Q}_v[T_i]/(f_i(T)) \). We denote the trivial elements in \(L_v^x \) and \(L_v^x/L_v^{x^2} \) by \(1_v \).
• \( \delta_v : J(Q_v) \rightarrow L_v^×/L_v^{×2} \); \( D = \sum_{i=1}^n m_i P_i \mapsto \prod_{i=1}^n (x_{P_i} - T)^{m_i} \), where \( D \) is a divisor whose support is disjoint from the support of the divisor \( \text{div}(y) \).

• \( \text{res}_v : L_v^×/L_v^{×2} \rightarrow L_v^×/L_v^{×2} \) as the map induced by \( L \rightarrow L_v ; T \mapsto T \).

• \( I_v(L) := \prod_{i=1}^3 (\text{the group of fractional ideals of } L^{(i)}) \). We often define
  - each element of \( I_v(L) \) and \( I_v(L)/I_v(L)^2 \) by \((a^{(1)}; \ldots; a^{(m)})_v\).
  - the trivial elements of \( I_v(L) \) and \( I_v(L)/I_v(L)^2 \) by \( 1_v \).

• \( \text{val}_v : L_v^× \rightarrow I_v(L) ; a \mapsto (a) \). We denote the induced map \( L_v^×/L_v^{×2} \rightarrow I_v(L)/I_v(L)^2 \) simply by \( \text{val}_v \).

• \( G_v := \text{val}_v(\text{Im}(\delta_v)) \subset I_v(L)/I_v(L)^2 \).

Finally, define

• \( I(L) := \prod_v I_v(L) \cong \prod_{i=1}^3 (\text{the group of fractional ideals of } L^{(i)}) \). We identify \( I_2(L) \times I_p(L) \) with its image in \( I(L) \).

• \( \text{Cl}(L) := \prod_{i=1}^3 \text{Cl}(L^{(i)}) \), where \( \text{Cl}(L^{(i)}) \) are the ideal class groups of \( L^{(i)} \).

• \( S := \{2, p, \infty\} \).

• \( \text{res}_S := \prod_{v \in S \setminus \{\infty\}} \text{res}_v : L^×/L^{×2} \rightarrow \prod_{v \in S \setminus \{\infty\}} L_v^×/L_v^{×2} \).

• \( \text{val} := \prod_v \text{val}_v \circ \text{res}_v : L^× \rightarrow I(L) \). We denote the induced map \( L^×/L^{×2} \rightarrow I(L)/I(L)^2 \) simply by \( \text{val} \).

• \( G := \prod_{v \in S \setminus \{\infty\}} G_v \subset I(L)/I(L)^2 \).

• \( W := \text{Ker}(G \rightarrow \text{Cl}(L)/\text{Cl}(L)^2) \).

2.1. Case \((i,j) = (0,2), p \equiv 7 \pmod{16}\)

Suppose that \( p \equiv 7 \pmod{16} \). Then, we have the following irreducible decompositions:

\[
\begin{align*}
L_2 &= Q_2[T_1]/(T_1^2 - 2p) \times Q_2[T_3]/(T_3 + 2p) \times Q_2[T_4]/(T_4^2 - 8p^2), \\
L_p &= Q_p[T_1]/(T_1^2 - 2p) \times Q_p[T_2]/(T_2 - 2p) \times Q_p[T_3]/(T_3 + 2p) \times Q_p \times Q_p.
\end{align*}
\]

Here, we fix an element \( \alpha \in Q_p \) such that \( \alpha^4 = 2^3 \) and denote \( \alpha^2 \) by \( \sqrt{2} \). Moreover, we also fix an isomorphism \( L_p^{(4)} \cong Q_p \times Q_p \) which sends \( T_4 \) to \((2p\sqrt{2}, -2p\sqrt{2})\). According to this irreducible decomposition, we denote each element in \( L_p \) by the form \((\alpha_1; \alpha_2; \alpha_3; \alpha_4, 1, \alpha_4, 2)\).

Lemma 2.5. The following three elements form an \( F_2 \)-basis of \( J(Q)[2] \) and \( J(Q_2)[2] \):

\[
(0,0) - \infty, \quad (2p,0) - \infty, \quad (-2p,0) - \infty.
\]

On the other hand, the following four elements form an \( F_2 \)-basis of \( J(Q_v)[2] \) for \( v = p, \infty \):

\[
(0,0) - \infty, \quad (2p,0) - \infty, \quad (-2p,0) - \infty, \quad (2p\sqrt{2},0) - \infty.
\]

In particular, we have the following table.

| \( v \) | \( \text{dim } J(Q_v)[2] \) | \( \text{dim } \text{Im}(\delta_v) \) |
|-------|----------------|----------------|
| 2     | 3              | 5              |
| \( p \) | 4              | 4              |
| \( \infty \) | 4              | 2              |

\footnote{Let \( q_v^{(i,1)}, \ldots, q_v^{(i,j_i)} \) be the prime ideals of \( L^{(i)} \) above \( v \), then
\( \text{val}_v(\alpha_1, \ldots, \alpha_m) = (v_{\alpha_1}^{(i,j_1)}, \ldots, v_{\alpha_m}^{(i,j_1)}; \ldots; v_{\alpha_1}^{(i,j_m)}; \ldots, v_{\alpha_m}^{(i,j_m)}) \),
which is well-defined up to the order of \( q_v^{(i,1)}, \ldots, q_v^{(i,j_i)} \) for each \( i \). Here, we identify the prime number \( v \) and the associated valuation map.}

\footnote{If \( p \equiv 7 \pmod{8} \), then \( \mathbb{F}_p^2 = \mathbb{F}_p^2 \). Thus, there exists \( x \in \mathbb{F}_p \) such that \( x^4 = 2 \).}
Proof. The first and second statements follow from \[13, \text{Lemma 5.2}\]. Note that
\begin{itemize}
  \item 2 is not square in \(\mathbb{Q}_2\).
  \item 2 is square in \(\mathbb{Q}_v\) for \(v = p, \infty\).
\end{itemize}
The third statement follows from the following formula (cf. \[4, \text{p. 451, proof of Lemma 3}\]).
\[
\dim_{\mathbb{F}_2} \text{Im}(\delta_v) = \dim_{\mathbb{F}_2} J(\mathbb{Q}_v)[2] \begin{cases} +0 & (v \neq 2, \infty), \\ +2 & (v = 2), \\ -2 & (v = \infty). \end{cases}
\]
\(\square\)

In the calculation of \(\text{Im}(\delta_v)\), we use the following formula.

Lemma 2.6 (\[10, \text{Lemma 2.2}\]). Let \(C : y^2 = f(x)\) be a hyperelliptic curve over \(\mathbb{Q}\) such that \(\deg f\) is odd. For every place \(v\) of \(\mathbb{Q}\), any point on \(J(\mathbb{Q}_v)\) can be represented by a divisor of degree 0 whose support is disjoint from the support of the divisor \(\text{div}(y)\). Then, we have \(\delta_v(D) = 1\) if \(D\) is supported at \(\infty\). If \(D\) is of the form \(D = \sum_{i=1}^n D_i\) with \(D_i = (\alpha_i, 0)\), where \(\alpha_i\) runs through all roots of a monic irreducible factor \(h(x) \in \mathbb{Q}_v[x]\) of \(f(x)\), then we have
\[
\delta_v(D) = (-1)^{\deg h} \left( h(T) - f(T) \right).
\]

Lemma 2.7. Let \(K = \mathbb{Q}(\sqrt{2})\) and \(u = 1 + \sqrt{2} \in \mathcal{O}_K\). Let \(p \equiv 7 \pmod{16}\) be a prime number, and fix \(a, b \in \mathbb{Z}\) such that \(p = (a + b\sqrt{2})(a - b\sqrt{2})\). Let \(\mathfrak{p}_1 = (a + b\sqrt{2})\), \(\mathfrak{p}_2 = (a - b\sqrt{2})\) and \(\iota_i : \mathcal{O}_K \hookrightarrow \hat{\mathcal{O}}_{K,p_i} \cong \mathbb{Z}_p\) (\(i = 1, 2\)) be embeddings. Then,
\[
\left( \frac{\iota_1(u)}{p} \right) \left( \frac{\iota_2(u)}{p} \right) = -1.
\]

Proof. Let \(\text{id} \neq \sigma \in \text{Gal}(K/\mathbb{Q})\). Then, \(\sigma : \mathcal{O}_K/\mathfrak{p}_1 \to \mathcal{O}_K/\mathfrak{p}_2\) sends \(u \mod \mathfrak{p}_1\) to \(\overline{u} \mod \mathfrak{p}_2\), and \(\overline{u} \mod \mathfrak{p}_1\) to \(u \mod \mathfrak{p}_2\). Therefore, the statement follows. \(\square\)

Lemma 2.8. Let \(K = \mathbb{Q}(\sqrt{2})\) and \(u = 1 + \sqrt{2} \in \mathcal{O}_K\). Let \(p\) be an odd prime number, and fix \(a, b \in \mathbb{Z}\) such that \(p = (a + b\sqrt{2})(a - b\sqrt{2})\). Let \(\mathfrak{p} = (a + b\sqrt{2})\). Then, the following are equivalent.
\begin{enumerate}
  \item \(u\) is square modulo \(p\).
  \item \(\left( \frac{b^2 - ab}{p} \right) = 1\).
\end{enumerate}

Proof. If we write \(\left( \frac{u(x)}{p} \right)\) by \(\left( \frac{\overline{x}}{p} \right)\), then
\[
\left( \frac{u}{p} \right) = \left( \frac{b}{p} \right) \left( \frac{bu}{p} \right)
\]
\[
= \left( \frac{b}{p} \right) \left( \frac{b + b\sqrt{2}}{p} \right)
\]
\[
= \left( \frac{b}{p} \right) \left( \frac{b + b\sqrt{2} - (a + b\sqrt{2})}{p} \right)
\]
\[
= \left( \frac{b}{p} \right) \left( \frac{b - a}{p} \right)
\]
\[
= \left( \frac{b^2 - ab}{p} \right)
\]
since \(\iota_1(a + b\sqrt{2}) \equiv 0 \pmod{p}\). Therefore, the statement holds. \(\square\)
Lemma 2.9. Let $p$ be a prime number such that $p \equiv 7 \pmod{8}$. Then, there exist $a, b \in \mathbb{Z}$ such that $p = a^2 - 2b^2$ and $((b^2 - ab)/p) = -1$.

Proof. Since $p \equiv 7 \pmod{8}$, there exist $a, b \in \mathbb{Z}$ such that $p = a^2 - 2b^2$. If $((b^2 - ab)/p) = -1$, we are done. If $((b^2 - ab)/p) = 1$, let $(a', b') = (a, -b)$. Then,
\[
\left(\frac{b'^2 - a'b'}{p}\right) = \left(\frac{b^2 + ab}{p}\right) = \frac{b}{p}\left(\frac{a + b}{p}\right) = \frac{b}{p}\left(\frac{b - b\sqrt{2} + (a + b\sqrt{2})}{p}\right) = \frac{b}{p}\left(\frac{b - b\sqrt{2}}{p}\right) = \left(\frac{1 - \sqrt{2}}{p}\right) = -1.
\]
The last equality holds by Lemmas 2.7 and 2.8. \hfill \square

Remark 2.10. From now on, we fix $a, b \in \mathbb{Z}$ such that $p = a^2 - 2b^2$ and $((b^2 - ab)/p) = -1$. Then, the isomorphism $L_p^{(4)} = \mathbb{Q}_p \times \mathbb{Q}_p$ sends $1 + T_\delta/2p$ to $(1 + \sqrt{2}, 1 - \sqrt{2}) = (-1, 1)$ by Lemmas 2.7 and 2.8.

By Lemma 2.6 we obtain the following lemma

Lemma 2.11. The following four elements form a basis of $L_p^\times / L_p^{\times 2}$ form an $\mathbb{F}_2$-basis of $\text{Im}(\delta_p)$:
\[
(1; -p; p; -p, p), \quad (p; -1; p; p, -p), \quad (-p; -p; -1; p, -p), \quad (p; -p; -p; 1, p).
\]
In particular, the following four elements of $I_p(L)/I_p(L)^2$ form a basis of $G_p$:
\[
((1); (p); (p); (p)), \quad ((p); (1); (p); (p)), \quad ((p); (1); (p); (p)), \quad ((p); (p); (p); (1), (p)).
\]

Proof. Lemma 2.6 implies that
\[
\delta_p((0, 0) - \infty) = -T + (T - 2p)(T + 2p)(T - 2p\sqrt{2})(T + 2p\sqrt{2}) = (1; -p; p; -p, p),
\]
\[
\delta_p((2p, 0) - \infty) = -(T - 2p) + T(T + 2p)(T - 2p\sqrt{2})(T + 2p\sqrt{2}) = (p; -1; p; -p, p),
\]
\[
\delta_p((-2p, 0) - \infty) = -(T + 2p) + T(T - 2p)(T - 2p\sqrt{2})(T + 2p\sqrt{2}) = (-p; -p; -1; p, -p),
\]
\[
\delta_p((2p\sqrt{2}, 0) - \infty) = -(T - 2p\sqrt{2}) + T(T - 2p)(T + 2p)(T + 2p\sqrt{2}) = (p; -p; -p; 1, p).
\]
Hence, the above four elements lie in $\text{Im}(\delta_p)$. By taking their images in $G_p$ into account, we see that they are linearly independent. By Lemma 2.5 this completes the proof. \hfill \square

Lemma 2.12. The following two elements form a basis of $L_\infty^\times / L_\infty^{\times 2}$ form an $\mathbb{F}_2$-basis of $\text{Im}(\delta_\infty)$:
\[
(1; -1; 1; -1, 1), \quad (-1; -1; -1; -1, 1)
\]
In particular, the following two elements of \( I_\infty(L)/I_\infty(L)^2 \) form a basis of \( G_p \):

\[
(1; -1; 1; -1; 1)_{\infty}, \quad (-1; -1; -1; -1; 1)_{\infty}.
\]

**Proof.** Lemma 2.6 implies that

\[
\begin{align*}
\delta_\infty((0, 0) - \infty) &= -T + (T - 2p)(T + 2p)(T - 2p\sqrt{2})(T + 2p\sqrt{2}) \\
&= (1; -1; 1; -1; 1),
\end{align*}
\]

\[
\begin{align*}
\delta_\infty((-2p, 0) - \infty) &= -(T + 2p) + T(T - 2p)(T - 2p\sqrt{2})(T + 2p\sqrt{2}) \\
&= (-1; -1; -1; -1; 1).
\end{align*}
\]

Hence, the above two elements lie in Im(\( \delta_\infty \)). We see that they are linearly independent. By Lemma 2.5 this completes the proof. \( \square \)

**Lemma 2.13.** The following five elements of \( L_\infty^2/L_2^2 \) form an \( \mathbb{F}_2 \)-basis of \( \text{Im}(\delta_2) \):

\[
(2; 2; -2; -T_4), \quad (-2; -2; -1; -T_4 + 2p), \quad (2; 1; -2; -T_4 - 2p),
\]

\[
(-3; -1; 3; 5 - T_4), \quad (2; -6; 6; 8 - T_4).
\]

In particular, the following three elements of \( I_2(L)/I_2(L)^2 \) form an \( \mathbb{F}_2 \)-basis of \( G_2 \):

\[
((2); (2); (2); (T_4)_{2}), \quad ((2); (2); (1); (1))_{2}, \quad ((2); (1); (2); (1))_{2}.
\]

**Proof.** First, we show that the above four elements actually lie in \( \text{Im}(\delta_2) \). Lemma 2.6 implies that

\[
\begin{align*}
\delta_2((0, 0) - \infty) &= -T + (T - 2p)(T + 2p)(T^2 - 8p^2) = (2; 2; -2; -T_4), \\
\delta_2((2p, 0) - \infty) &= -(T - 2p) + T(T + 2p)(T^2 - 8p^2) = (-2; -2; -1; -T_4 + 2p), \\
\delta_2((-2p, 0) - \infty) &= -(T + 2p) + T(T - 2p)(T^2 - 8p^2) = (2; 1; -2; -T_4 - 2p).
\end{align*}
\]

Hence, the above three elements lie in \( \text{Im}(\delta_2) \).

- Since \( f(5) \equiv 1 \pmod{8} \), there exists \( P \in C(\mathbb{Q}_2) \) such that \( x_P = 5 \). In particular, \((3; -1; 3; 5 - T_4) = \delta_2(P - \infty) \) lies in \( \text{Im}(\delta_2) \).
- Since \( f(8)/2^8 \equiv 1 \pmod{8} \), there exists \( Q \in C(\mathbb{Q}_2) \) such that \( x_Q = 8 \). In particular, \((2; -6; 6; 8 - T_4) = \delta_2(Q - \infty) \) lies in \( \text{Im}(\delta_2) \).

Since \( \nu_2(2) = \nu_2(-2) = \nu_2(-6) = 1 \), the first three elements and the last element are non-trivial in \( L_\infty^2/L_2^2 \). Since \(-3 \) is non-trivial in \( \mathbb{Q}_2^2/\mathbb{Q}_2^2 \), the fourth element is non-trivial in \( L_\infty^2/L_2^2 \).

Finally, by taking the first and the second components into account, we see that they are linearly independent. By Lemma 2.5 this completes the proof. \( \square \)

**Corollary 2.14.** The following five elements of \( L^\times/L^\times 2 \) form an \( \mathbb{F}_2 \)-basis of \( \text{Ker}(\text{val}) \):

\[
(-1; 1; 1; 1), \quad (1; -1; 1; 1), \quad (1; 1; -1; 1), \quad (1; 1; 1; -1), \quad (1; 1; 1 + T_4/2p).
\]

**Proof.** By the exactly same manner as in [7, Corollary 2.10], we obtain a short exact sequence

\[
1 \to O_L^\times/O_L^{\times 2} \to \text{Ker}(\text{val}) \to \text{Cl}(L)[2] \to 0.
\]

The lemma follows from the facts that the above five elements form an \( \mathbb{F}_2 \)-basis of \( O_L^\times/O_L^{\times 2} \), and the class number of \( L^{(1)} \) is 1 for every \( i = 0, 1, 2, 3 \). \( \square \)

**Lemma 2.15.** The following seven elements of \( I(L)/I(L)^2 \) form an \( \mathbb{F}_2 \)-basis of \( W = G \):

\[
((2); (2); (2); (T_4))_2 \times 1_p \times 1_{\infty}, \quad ((2); (2); (1); (1))_2 \times 1_p \times 1_{\infty}, \quad ((2); (1); (2); (1))_2 \times 1_p \times 1_{\infty},
\]

\[
1_2 \times ((1); (p); (p); (p))_p \times 1_{\infty}, \quad 1_2 \times ((p); (1); (p); (p))_p \times 1_{\infty}, \quad 1_2 \times ((p); (p); (1); (p))_p \times 1_{\infty}.
\]
Lemma 2.17. Fix \(a, b \in \mathbb{Z}\) such that \(p = a^2 - 2b^2\) and \((\frac{b^2 - ab}{p}) = -1\). Then, the following twelve elements of \(L^\times/L^\times_2\) form an \(\mathbb{F}_2\)-basis of \(\text{val}^{-1}(G)\):
\[
(-1; 1; 1; 1), \quad (1; -1; 1; 1), \quad (1; 1; -1; 1), \quad (1; 1; 1; 1 + T_4/2p), \quad (2; 2; 2; T_4),
\]
\[
(2; 2; 1; 1), \quad (2; 1; 2; 1), \quad (1; p; p; p), \quad (p; 1; p; p), \quad (p; p; 1; p), \quad (p; p; p; a + b \cdot T_4/2p).
\]

Lemma 2.17. Fix \(a, b \in \mathbb{Z}\) such that \(p = a^2 - 2b^2\) and \((\frac{b^2 - ab}{p}) = -1\). Set elements of \(L^\times_2/L^\times_2 \times L^\times_2/p \times L^\times_2 \times L^\times_2\) as follows:
\[
d_1 := (2; 2; -2; -T_4)_2 \times 1_p \times 1_\infty,
\]
\[
d_2 := (-2; -2; -1; -T_4 + 2p)_2 \times 1_p \times 1_\infty,
\]
\[
d_3 := (2; 1; -2; -T_4 - 2p)_2 \times 1_p \times 1_\infty,
\]
\[
d_4 := (-3; -1; 3; 5 - T_4)_2 \times 1_p \times 1_\infty,
\]
\[
d_5 := (2; -6; 6; 8 - T_4)_2 \times 1_p \times 1_\infty,
\]
\[
d_6 := 1_2 \times (1; -p; p; -p)_p \times 1_\infty,
\]
\[
d_7 := 1_2 \times (p; -1; p; -p)_p \times 1_\infty,
\]
\[
d_8 := 1_2 \times (p; p; -1; p)_p \times 1_\infty,
\]
\[
d_9 := 1_2 \times (p; -p; -1; p)_p \times 1_\infty,
\]
\[
d_{10} := 1_2 \times 1_p \times (1; -1; 1; -1, 1)_\infty,
\]
\[
d_{11} := 1_2 \times 1_p \times (-1; -1; -1; -1, 1)_\infty,
\]
\[
h_1 := \text{res}_S(-1; 1; 1; 1) = (-1; 1; 1; 1)_2 \times (-1; 1; 1; 1)_p \times (-1; 1; 1; 1)_\infty,
\]
\[
h_2 := \text{res}_S(1; -1; 1; 1) = (1; -1; 1; 1)_2 \times (1; -1; 1; 1)_p \times (1; -1; 1; 1)_\infty,
\]
\[
h_3 := \text{res}_S(1; 1; -1; 1) = (1; 1; -1; 1)_2 \times (1; -1; 1; 1)_p \times (1; -1; 1; 1)_\infty,
\]
\[
h_4 := \text{res}_S(1; 1; 1; -1) = (1; 1; 1; 1)_2 \times (1; 1; 1; 1)_p \times (1; 1; 1; 1)_\infty,
\]
\[
h_5 := \text{res}_S(1; 1; 1; 1 + T_4/2p) = (1; 1; 1; 1 + T_4/2p)_2 \times (1; -1; 1; -1)_p \times (1; 1; 1; 1)_\infty,
\]
\[
h_6 := \text{res}_S(2; 2; 2; T_4) = (2; 2; 2; T_4)_2 \times (1; 1; 1; 1)_p \times (1; 1; 1; 1)_\infty,
\]
\[
h_7 := \text{res}_S(2; 2; 1; 1) = (2; 2; 1; 1)_2 \times 1_p \times 1_\infty,
\]
\[
h_8 := \text{res}_S(2; 2; 1; 1) = (2; 2; 1; 1)_2 \times 1_p \times 1_\infty,
\]
\[
h_9 := \text{res}_S(1; p; p; p) = (1; -1; -1)_2 \times (1; p; p; p)_p \times 1_\infty,
\]
\[
h_{10} := \text{res}_S(p; 1; p; p) = (-1; 1; -1; 1)_2 \times (p; 1; p; p)_p \times 1_\infty,
\]
\[
h_{11} := \text{res}_S(p; p; 1; p) = (-1; -1; 1; -1)_2 \times (p; p; 1; p)_p \times 1_\infty,
\]
\[
h_{12} := \text{res}_S(p; p; p; a + b \cdot T_4/2p) = (1; -1; -1; -1)_2 \times (p; p; p; a + b \cdot T_4/2p)_p \times 1_\infty.
\]

Proof. (1) We can check it by direct calculation.

\[\text{(2) The twenty elements } d_1, d_4, d_5, d_7, \ldots, d_{11}, h_1, \ldots, h_{12} \text{ form an } \mathbb{F}_2\text{-basis of } \text{Im}(\delta_2) \times \text{Im}(\delta_\infty) + \text{res}_S(\text{val}^{-1}(G)).\]

\text{Note that } 2 = 1 \text{ in } L_2^{(4)} \times L_2^{(4)} \times 2.
(2) By (1) and Lemmas 2.11 to 2.13 and 2.16, \((\text{Im}(\delta_2) \times \text{Im}(\delta_p)) + \text{res}_S(\text{val}^{-1}(G))\) is generated by \(d_4, \ldots, d_{11}, h_1, \ldots, h_{12}\). Set
\[
\begin{align*}
d_4 &:= (-3; -1; 3; 5 - T_4)_2 \times 1_p \times 1_\infty, \\
d'_4 &:= d_5 h_8 = (1; -6; 3; 8 - T_4)_2 \times 1_p \times 1_\infty, \\
d'_6 &:= d_6 d_7 h_{11} = (-1; -1; 1; -1)_2 \times (1; 1; 1; -p, -p)_p \times 1_\infty, \\
d'_7 &:= d_7 h_{10} = (-1; -1; -1; -1)_2 \times (1; -1; 1; -1)_p \times 1_\infty, \\
d'_8 &:= d_8 d_{11} h_2 h_3 h_4 h_5 h_{11} = (1; 1; -1; 1 + T_4/2p)_2 \times 1_p \times 1_\infty, \\
d'_9 &:= d_9 d_{11} h_1 h_2 h_3 h_4 h_5 h_{11} = (1; -1; -1; 1)_2 \times (1; -1; 1; -p)_p \times 1_\infty, \\
d'_{10} &:= d_{10} d_7 h_2 h_3 h_4 h_5 h_{10} = (-1; -1; -1; -1 + T_4/2p)_2 \times 1_p \times 1_\infty, \\
d'_{11} &:= d_{11} h_1 h_2 h_3 h_4 h_5 = (-1; -1; -1; -1 - T_4/2p)_2 \times (-1; -1; -1; -1)_p \times 1_\infty, \\
h_1 &:= (-1; 1; 1; 1)_2 \times (-1; 1; 1; 1)_p \times (-1; 1; 1; 1)_\infty, \\
h_2 &:= (1; -1; 1; 1)_2 \times (1; -1; 1; 1)_p \times (-1; 1; 1; 1)_\infty, \\
h_3 &:= (1; 1; -1; 1)_2 \times (1; 1; -1; 1)_p \times (1; 1; -1; 1)_\infty, \\
h_4 &:= (1; 1; 1; -1)_2 \times (1; 1; 1; -1)_p \times (1; 1; -1; 1)_\infty, \\
h_5 &:= (1; 1; 1; 1 + T_4/2p)_2 \times (1; 1; 1; -1)_p \times (1; 1; 1; -1)_\infty, \\
h'_5 &:= h_6 d_6 d_{10} h_2 h_4 h_5 h_{11} = (1; 1; -2; -T_4)_2 \times 1_p \times 1_\infty, \\
h'_7 &:= h_7 h_8 = (1; 2; 2; 1)_2 \times 1_p \times 1_\infty, \\
h_8 &:= (2; 1; 2; 1)_2 \times 1_p \times 1_\infty, \\
h'_9 &:= h_9 h_{10} h_{11} = (1; 1; -1; 1)_2 \times (1; 1; 1; p, p)_p \times 1_\infty, \\
h_{10} &:= (-1; -1; -1; -1)_2 \times (p; 1; 1; p, p)_p \times 1_\infty, \\
h'_{11} &:= h_{11} h_{10} = (1; -1; -1; 1)_2 \times (1; p; p; 1, 1)_p \times 1_\infty, \\
h'_{12} &:= h_{12} h_{11} = (1; 1; -1; -a - b \cdot T_4/2p)_2 \times (1; 1; p; p(a + b\sqrt{2}), p(a - b\sqrt{2}))_p \times 1_\infty.
\end{align*}

It is sufficient to prove that the above twenty elements are linearly independent.

(a) By taking the first components at \(v = \infty\) into account, we see that there is no relation containing \(h_1\).

(b) By taking the second components at \(v = \infty\) into account, we see that there is no relation containing \(h_2\).

(c) By taking the third components at \(v = \infty\) into account, we see that there is no relation containing \(h_3\).

(d) By taking the fourth and fifth components at \(v = \infty\) into account, we see that there is no relation containing \(h_4\) and \(h_5\).

(e) By taking the first components at \(v = p\) into account, we see that there is no relation containing \(d'_4\) and \(h_{10}\).

(f) By taking the second components at \(v = p\) into account, we see that there is no relation containing \(d'_7\) and \(h'_{11}\).

(g) By taking the third components at \(v = p\) into account, we see that there is no relation containing \(d'_9\) and \(h'_{12}\).

(h) By taking the forth and fifth components at \(v = p\) into account, we see that there is no relation containing \(d'_9\) and \(h'_9\).

(i) By taking the first components at \(v = 2\) into account, we see that there is no relation containing \(d_4, d'_{10}\) and \(h_8\).

(j) By taking the second components at \(v = 2\) into account, we see that there is no relation containing \(d'_5\) and \(h'_7\).
(k) By taking the third components at \( v = 2 \) into account, we see that there is no relation containing \( d'_8 \) and \( h_6' \).

Recall that
\[
\dim \text{Sel}^2(\mathbb{Q}, J^{(p;i,j)}) = \dim \text{val}^{-1}(G) + \dim(\text{Im}(\delta_2) \times \text{Im}(\delta_p) \times \text{Im}(\delta_\infty)) \\
- \dim((\text{Im}(\delta_2) \times \text{Im}(\delta_p) \times \text{Im}(\delta_\infty)) + \text{res}_S(\text{val}^{-1}(G)))
\]
and
\[
J(\mathbb{Q})/2J(\mathbb{Q}) \cong \mathbb{F}_2^{\text{rank}(J(\mathbb{Q}))} \oplus J(\mathbb{Q})[2].
\]
Therefore, by Lemmas 2.5, 2.11 to 2.16 and 2.17 we obtain
\[
\text{rank}(J(\mathbb{Q})) \leq \dim \text{Sel}^2(\mathbb{Q}, J) - \dim J(\mathbb{Q})[2] = 12 + 11 - 20 - 3 = 0.
\]
This completes the proof of Theorem 2.1 (1).

2.2. Case \((i, j) = (2, 2), p \equiv 7 \pmod{16}\)

Let \((i, j) = (2, 2), \) and suppose that \( p \equiv 7 \pmod{16}\).

By Corollary 2.3, the Jacobian variety of \( C'(p;i,j) \) is isomorphic to that of the hyperelliptic curve defined by
\[
y^2 = x(x - 4p)(x + 4p)(x^2 - 32p^2).
\]
The above curve is isomorphic to a hyperelliptic curve defined by
\[
y^2 = x(x - p)(x + p)(x^2 - 2p^2),
\]
which is also denoted by \( C'(p;i,j) \). Let \( J := \text{Jac}(C'(p;i,j)) \), \( L := \mathbb{Q}[T]/(x(x - p)(x + p)(x^2 - 2p^2)) \) and \( L_v := \mathbb{Q}_v[T]/(x(x - p)(x + p)(x^2 - 2p^2)) \) for \( v \in S \). Denote the \( \delta_v \) map by
\[
\delta_v : J(\mathbb{Q}) \rightarrow L_v^\times/L_v^\times^2.
\]
Then, we have the following irreducible decompositions:
\[
\begin{align*}
L_2 &= \mathbb{Q}_2[T_1]/(T_1) \times \mathbb{Q}_2[T_2]/(T_2 - p) \times \mathbb{Q}_2[T_3]/(T_3 + p) \times \mathbb{Q}_2[T_4]/(T_4^2 - 2p^2), \\
L_p &= \mathbb{Q}_p[T_1]/(T_1) \times \mathbb{Q}_p[T_2]/(T_2 - p) \times \mathbb{Q}_p[T_3]/(T_3 + p) \times \mathbb{Q}_p \times \mathbb{Q}_p.
\end{align*}
\]
Here, we fix an element \( \alpha \in \mathbb{Q}_p \) such that \( \alpha^4 = 2 \), and denote \( \alpha^2 \) by \( \sqrt{2} \). Moreover, we also fix an isomorphism \( L_p^{(i)} \cong \mathbb{Q}_p \times \mathbb{Q}_p \) which sends \( T_4 \) to \((p\sqrt{2}, -p\sqrt{2})\). According to this irreducible decomposition, we denote each element in \( L_p \) by the form \((\alpha_1; \alpha_2; \alpha_3; \alpha_4, 1, \alpha_4, 2)\).

**Lemma 2.18.** The following three elements form an \( \mathbb{F}_2 \)-basis of \( J(\mathbb{Q})[2] \) and \( J(\mathbb{Q}_v)[2] \):
\[
(0, 0) - \infty, \quad (p, 0) - \infty, \quad (-p, 0) - \infty.
\]
On the other hand, the following four elements form an \( \mathbb{F}_2 \)-basis of \( J(\mathbb{Q}_v)[2] \) for \( v = p, \infty \):
\[
(0, 0) - \infty, \quad (p, 0) - \infty, \quad (-p, 0) - \infty, \quad (p\sqrt{2}, 0) - \infty.
\]
In particular, we have the following table.

| \( v \) | \( \dim J(\mathbb{Q}_v)[2] \) | \( \dim \text{Im}(\delta_v) \) |
|---|---|---|
| 2 | 3 | 5 |
| \( p \) | 4 | 4 |
| \( \infty \) | 4 | 2 |

**Proof.** The first and second statements follow from [13, Lemma 5.2]. Note that
- 2 is not square in \( \mathbb{Q}_2 \).
- 2 is square in \( \mathbb{Q}_v \) for \( v = p, \infty \).
The third statement follows from the following formula (cf. \cite{1} p. 451, proof of Lemma 3). 

\[
\dim_{\mathbb{F}_2} \text{Im}(\delta_v) = \dim_{\mathbb{F}_2} J(Q_v)[2] \begin{cases} 
+0 & (v \neq 2, \infty), \\
+2 & (v = 2), \\
-2 & (v = \infty).
\end{cases}
\]

**Lemma 2.19.** The following four elements form a basis of \(L_p^\times / L_p^{\times 2}\) form an \(\mathbb{F}_2\)-basis of \(\text{Im}(\delta_p)\):

\[
(1; -p; p; -p, p), \quad (p; -1; p; p, -p), \quad (-p; -p; -1; p, -p), \quad (p; -p; -p; 1, p).
\]

In particular, the following four elements of \(I_p(L)/I_p(L)^2\) form a basis of \(G_p\):

\[
((1); (p); (p); (p), p), \quad ((p); (1); (p); (p), p), \quad ((p); (p); (1); (p), p), \quad ((p); (p); (p); (1), (p), p).
\]

**Proof.** Lemma 2.6 implies that

\[
\delta_p((0, 0) - \infty) = -T + (T - p)(T + p)(T - p\sqrt{2})(T + p\sqrt{2}) \\
= (1; -p; p; -p, p),
\]

\[
\delta_p((p, 0) - \infty) = -(T - p) + T(T + p)(T - p\sqrt{2})(T + p\sqrt{2}) \\
= (p; -1; p; p, -p),
\]

\[
\delta_p((-p, 0) - \infty) = -(T + p) + T(T - p)(T - p\sqrt{2})(T + p\sqrt{2}) \\
= (-p; -p; -1; p, -p),
\]

\[
\delta_p((p\sqrt{2}, 0) - \infty) = -(T - p\sqrt{2}) + T(T - p)(T + p\sqrt{2}) \\
= (p; -p; -p; 1, p).
\]

Hence, the above four elements lie in \(\text{Im}(\delta_p)\). By taking their images in \(G_p\) into account, we see that they are linearly independent. By Lemma 2.18 this completes the proof. \(\square\)

**Lemma 2.20.** The following two elements form a basis of \(L_{\infty}^\times / L_{\infty}^{\times 2}\) form an \(\mathbb{F}_2\)-basis of \(\text{Im}(\delta_{\infty})\):

\[
(1; -1; 1; -1, 1), \quad (-1; -1; -1; -1, 1)
\]

In particular, the following two elements of \(I_{\infty}(L)/I_{\infty}(L)^2\) form a basis of \(G_{\infty}\):

\[
(1; -1; 1; -1, 1), \quad (-1; -1; -1; -1, 1)
\]

**Proof.** Lemma 2.6 implies that

\[
\delta_{\infty}((0, 0) - \infty) = -T + (T - p)(T + p)(T - p\sqrt{2})(T + p\sqrt{2}) \\
= (1; -1; 1; -1, 1),
\]

\[
\delta_{\infty}((-p, 0) - \infty) = -(T + p) + T(T - p)(T - p\sqrt{2})(T + p\sqrt{2}) \\
= (-1; -1; -1; -1, 1).
\]

Hence, the above two elements lie in \(\text{Im}(\delta_{\infty})\). We see that they are linearly independent. By Lemma 2.18 this completes the proof. \(\square\)

**Lemma 2.21.** The following five elements of \(L_2^\times / L_2^{\times 2}\) form an \(\mathbb{F}_2\)-basis of \(\text{Im}(\delta_2)\):

\[
(2; 1; -1; -T_4), \quad (-1; -2; -2; -1 + p), \quad (1; 2; -2; -1 + p), \quad (6; -1; -3; 6 - T_4), \quad (1; -3; -3; 1).
\]

In particular, the following three elements of \(I_2(L)/I_2(L)^2\) forms an \(\mathbb{F}_2\)-basis of \(G_2\):

\[
((1); (2); (2); (1)), \quad ((2); (2); (1)), \quad ((2); (1); (1); (T_4)).
\]
Proof. First, we show that the above four elements actually lie in \( \text{Im}(\delta_2) \). Lemma 2.18 implies that
\[
\delta_2((0, 0) - \infty) = -T + (T - p)(T + p)(T^2 - 2p^2) = (2; 1; -1; -T_4),
\]
\[
\delta_2((p, 0) - \infty) = -(T - p) + T(T + p)(T^2 - 2p^2) = (-1; -2; -2; -T_4 + p),
\]
\[
\delta_2((-p, 0) - \infty) = -(T + p) + T(T - p)(T^2 - 2p^2) = (1; 2; -2; -T_4 - p).
\]
Hence, the above three elements lie in \( \text{Im}(\delta_2) \).

- Since \( f(6)/4 \equiv 1 \pmod{8} \), there exists \( P \in \mathcal{C}(\mathbb{Q}_2) \) such that \( x_P = 6 \). In particular, \((6; -1; -3; 6 - T_4) = \delta_2(P - \infty) \) lies in \( \text{Im}(\delta_2) \).
- Since \( 2^{10}f(1/4) \equiv 1 \pmod{8} \), there exists \( Q \in \mathcal{C}(\mathbb{Q}_2) \) such that \( x_Q = 1/4 \). In particular, \((1; -3; -3; 1) = \delta_2(Q - \infty) \) lies in \( \text{Im}(\delta_2) \).

Since \( v_2(\pm 2) = v_2(6) = 1 \), the first four elements are non-trivial in \( L_2^x/L_2^x \). Since \(-3\) is non-trivial in \( \mathcal{Q}_2^x/\mathcal{Q}_2^x \), the fifth element is non-trivial in \( L_2^x/L_2^x \).

Finally, by taking the first and the second components into account, we see that they are linearly independent. By Lemma 2.18 this completes the proof. \( \square \)

Corollary 2.22. The following five elements of \( L^x/L^x \) form an \( \mathbb{F}_2 \)-basis of \( \ker(\text{val}) \):
\[
(-1; 1; 1; 1), \quad (1; -1; 1; 1), \quad (1; 1; -1; 1), \quad (1; 1; 1; -1), \quad (1; 1; 1; 1 + T_4/p).
\]

Proof. By the exactly same manner as in [7, Corollary 2.10], we obtain a short exact sequence
\[
1 \to \mathcal{O}_L^x/\mathcal{O}_L^{x^2} \to \ker(\text{val}) \to \text{Cl}(L)[2] \to 0.
\]
The lemma follows from the facts that the above five elements form an \( \mathbb{F}_2 \)-basis of \( \mathcal{O}_L^x/\mathcal{O}_L^{x^2} \), and the class number of \( L^{(i)} \) is 1 for every \( i = 0, 1, 2, 3 \). \( \square \)

Lemma 2.23. The following seven elements of \( I(L)/I(L)^2 \) form an \( \mathbb{F}_2 \)-basis of \( W = G \):
\[
((1); (2); (2); (1))_2 \times 1_p \times 1_{\infty}, \quad ((2); (1); (1); (T_4))_2 \times 1_p \times 1_{\infty},
\]
\[
1_2 \times ((1); (p); (p); (p); (p))_p \times 1_{\infty}, \quad 1_2 \times ((p); (1); (p); (p); (p))_p \times 1_{\infty},
\]
\[
1_2 \times ((p); (p); (1); (p); (p))_p \times 1_{\infty}, \quad 1_2 \times ((p); (p); (p); (1); (p))_p \times 1_{\infty}.
\]

Proof. Since the class number of \( L^{(i)} \) is 1 for every \( i = 0, 1, 2, \) we have \( W = G \). Thus, the lemma follows from Lemmas 2.19 and 2.21. \( \square \)

Recall that there exists an exact sequence \( 1 \to \ker(\text{val}) \to \text{val}^{-1}(G) \to W \to 1 \). By Corollary 2.22 and Lemma 2.23 we have the following lemma.

Lemma 2.24. Fix \( a, b \in \mathbb{Z} \) such that \( p = a^2 - 2b^2 \) and \( (\frac{b^2 - ab}{p}) = -1 \). Then, the following twelve elements of \( L^x/L^x \) form an \( \mathbb{F}_2 \)-basis of \( \text{val}^{-1}(G) \):
\[
(-1; 1; 1; 1), \quad (1; -1; 1; 1), \quad (1; 1; -1; 1), \quad (1; 1; 1; -1), \quad (1; 1; 1; 1 + T_4/p), \quad (1; 2; 2; 1),
\]
\[
(2; 1; 1; T_4), \quad (1; p; p; p), \quad (p; 1; p; p), \quad (p; p; 1; p), \quad (p; p; p; a + b \cdot T_4/p).
\]
Lemma 2.25. Fix $a$, $b \in \mathbb{Z}$ such that $p = a^2 - 2b^2$ and $(\frac{b^2 - ab}{p}) = -1$. Set elements of $L_p^\times / L_p^\times \times 2 \times \mathbb{L}^\times / \mathbb{L}^\times \times 2$ as follows:

\[
\begin{align*}
d_1 &:= (2; 1; -1; -T_4)2 \times 1_p \times 1_\infty, \\
d_2 &:= (-1; -2; -2; -T_4 + p)2 \times 1_p \times 1_\infty, \\
d_3 &:= (1; 2; -2; -T_4 - p)2 \times 1_p \times 1_\infty, \\
d_4 &:= (6; -1; -3; 6 - T_4)2 \times 1_p \times 1_\infty, \\
d_5 &:= (1; -3; -3; 1)2 \times 1_p \times 1_\infty, \\
d_6 &:= 1_p \times (-1; -p; p, -p, p) \times 1_\infty, \\
d_7 &:= 1_p \times (p; -1; p; p, -p) \times 1_\infty, \\
d_8 &:= 1_p \times (-p; -p; -1; p, -p) \times 1_\infty, \\
d_9 &:= 1_p \times (p; -p; -p; 1, p) \times 1_\infty, \\
d_{10} &:= 1_p \times (1; -1; 1; -1, 1)_\infty, \\
d_{11} &:= 1_p \times (-1; -1; -1; -1, 1)_\infty, \\
h_1 &:= \text{res}_S(-1; 1; 1; 1) = (-1; 1; 1; 1)_2 \times (-1; 1; 1; 1)_p \times (-1; 1; 1; 1)_\infty, \\
h_2 &:= \text{res}_S(1; -1; 1; 1) = (1; -1; 1; 1)_2 \times (1; -1; 1; 1)_p \times (1; -1; 1; 1)_\infty, \\
h_3 &:= \text{res}_S(1; 1; -1; 1) = (1; 1; -1; 1)_2 \times (1; -1; 1; 1)_p \times (1; -1; 1; 1)_\infty, \\
h_4 &:= \text{res}_S(1; 1; 1; -1) = (1; 1; 1; -1)_2 \times (1; 1; 1; -1)_p \times (1; 1; 1; -1)_\infty, \\
h_5 &:= \text{res}_S(1; 1; 1; 1 + T_4/p) = (1; 1; 1; 1 + T_4/p)_2 \times (1; 1; 1; -1, 1)_p \times (1; 1; 1; -1, 1)_\infty, \\
h_6 &:= \text{res}_S(1; 2; 2; 1) = (1; 2; 2; 1)_2 \times 1_p \times 1_\infty, \\
h_7 &:= \text{res}_S(2; 1; 1; T_4) = (2; 1; 1; T_4)_2 \times (1; 1; 1; p, -p)_p \times (1; 1; 1; 1, -1)_\infty, \\
h_8 &:= \text{res}_S(1; p; p; p) = (1; -1; -1; -1)_2 \times (1; p; p; p)_p \times 1_\infty, \\
h_9 &:= \text{res}_S(p; 1; p; p) = (-1; 1; -1; 1)_2 \times (p; 1; p; p)_p \times 1_\infty, \\
h_{10} &:= \text{res}_S(p; p; 1; p) = (-1; -1; -1; 1)_2 \times (p; p; 1; p)_p \times 1_\infty, \\
h_{11} &:= \text{res}_S(p; p; p; a + b \cdot T_4/p) = (-1; -1; -1; a + b \cdot T_4/p)_2 \times (p; p; p; a + b\sqrt{2}, a - b\sqrt{2})p \times 1_\infty.
\end{align*}
\]

(1) We have $d_1 d_2 d_3 d_4 d_5 h_1 h_2 h_3 h_4 h_5 h_6 h_7 h_8 h_9 h_{10} = d_2 d_3 d_4 d_5 d_1 d_1 h_1 h_2 h_3 h_4 h_5 h_6 h_7 h_8 h_9 = 1_2 \times 1_p \times 1_\infty.

(2) The nineteen elements $d_1, \ldots, d_{11}, h_1, \ldots, h_{11}$ form an $\mathbb{F}_2$-basis of $(\text{Im}(\delta_2) \times \text{Im}(\delta_3) \times \text{Im}(\delta_{\infty})) + \text{res}_S(\text{val}^{-1}(G))$.

Proof. (1) We can check it by direct calculation.
(2) By (1) and Lemmas 2.19 to 2.21 and 2.24, \((\text{Im}(\delta_2) \times \text{Im}(\delta_p)) + \text{res}_S(\text{val}^{-1}(G))\) is generated by \(d_4, \ldots, d_{11}, h_1, \ldots, h_{11}\). Set

\[ d_4 := (6; -1; -3; 6 - T_4)2 \times 1_p \times 1_\infty, \]
\[ d_5 := (1; -3; 3; -1)2 \times 1_p \times 1_\infty, \]
\[ d_6 := d_5d_7h_{10} = (-1; -1; -1; 2) \times (1; -1; 1; -p, -p)p \times 1_\infty, \]
\[ d_7 := 1_2 \times (p; -1; p; -p)p \times 1_\infty, \]
\[ d_8 := d_8d_1h_1h_2h_3h_4h_5h_{10} = (1; 1; -1; 1 + T_4/2)2 \times 1_p \times 1_\infty, \]
\[ d_9 := d_9d_7 = 1_2 \times (1; p; -1; p; -1)p \times 1_\infty, \]
\[ d_{10} := d_{10}h_2h_4h_5 = (1; -1; 1; -1 - T_4/p)2 \times (1; -1; 1; -1)p \times 1_\infty, \]
\[ d_{11} := d_{11}h_1h_2h_3h_4h_5 = (-1; -1; -1; -1 - T_4/p)2 \times (1; -1; 1; -1)p \times 1_\infty, \]
\[ h_1 := (-1; 1; 1; 1)2 \times (-1; 1; 1; 1)p \times (1; -1; 1; 1)_\infty, \]
\[ h_2 := (1; -1; 1; 1)2 \times (1; -1; 1, 1)_\infty, \]
\[ h_3 := (1; 1; 1; -1)_2 \times (1; 1; -1, 1)_\infty, \]
\[ h_4 := (1; 1; 1; -1)_2 \times (1; 1; 1; -1)_\infty, \]
\[ h_5 := (1; 1; 1 + T_4/2)2 \times (1; 1; -1, 1)_\infty, \]
\[ h_6 := (1; 2; 2; 1)_2 \times 1_p \times 1_\infty, \]
\[ h'_7 := h_7d_6d_7h_{10} = (-2; -1; -1 - T_4(1 + T_4/p)2) \times 1_p \times 1_\infty, \]
\[ h'_8 := h_8d_6d_7h_4h_5h_{10} = (-1; -1; -1 - T_4/p)2 \times (1; 1; 1; p, p)_\infty, \]
\[ h'_9 := h_9d_6d_7h_4h_5 = (-1; -1; 1 + T_4/p)2 \times 1_p \times 1_\infty, \]
\[ h'_{10} := h_{10}d_6d_7h_4h_5 = (-1; 1; 1 + T_4/p)2 \times (1; 1; -p, -p)_\infty, \]
\[ h'_{11} := h_{11}d_6d_7h_4h_5 = (-1; 1; -1; -a + b \cdot T_4/p)(1 + T_4/p)2 \times (1; 1; a + b\sqrt{2}, -p(a - b\sqrt{2})) \times 1_\infty. \]

It is sufficient to prove that the above nineteen elements are linearly independent.

(a) By taking the first components at \(v = \infty\) into account, we see that there is no relation containing \(h_1\).

(b) By taking the second components at \(v = \infty\) into account, we see that there is no relation containing \(h_2\).

(c) By taking the third components at \(v = \infty\) into account, we see that there is no relation containing \(h_3\).

(d) By taking the fourth and fifth components at \(v = \infty\) into account, we see that there is no relation containing \(h_4\) and \(h_5\).

(e) By taking the first components at \(v = p\) into account, we see that there is no relation containing \(d_7\), \(d'_{11}\).

(f) By taking the second components at \(v = p\) into account, we see that there is no relation containing \(d_9\) and \(d'_{10}\).

(g) By taking the third components at \(v = p\) into account, we see that there is no relation containing \(h'_{10}\) and \(h'_{11}\).

(h) By taking the forth and fifth components at \(v = p\) into account, we see that there is no relation containing \(d'_8\) and \(h'_8\).

(i) By taking the first components at \(v = 2\) into account, we see that there is no relation containing \(d_4\), \(h'_7\) and \(h'_9\).

(j) By taking the second components at \(v = 2\) into account, we see that there is no relation containing \(d_5\) and \(h_6\).

(k) By taking the third components at \(v = 2\) into account, we see that there is no relation containing \(d'_9\).

\[ \Box \]
Recall that
\[
\dim \text{Sel}^2(Q, J^{(p; i, j)}) = \dim \text{val}^{-1}(G) + \dim(\text{Im}(\delta_2) \times \text{Im}(\delta_p) \times \text{Im}(\delta_{\infty})) \\
- \dim((\text{Im}(\delta_2) \times \text{Im}(\delta_p) \times \text{Im}(\delta_{\infty})) + \text{res}_{\delta}(\text{val}^{-1}(G)))
\]
and
\[
J(Q)/2J(Q) \simeq \mathbb{F}_2^{\text{rank}(J(Q))} \oplus J(Q)[2].
\]
Therefore, by Lemmas 2.18 to 2.21 and 2.25, we obtain
\[
\text{rank}(J(Q)) \leq \dim \text{Sel}^2(Q, J) - \dim J(Q)[2] = 11 + 10 - 19 - 3 = 0.
\]
This completes the proof of Theorem 2.1 (2).

In this section, we applied the 2-descent argument to all prime numbers \( p \equiv 7 \pmod{16} \). If we can reduce the calculation to \( p = 7 \), the proof of Theorem 2.1 will be short, but we couldn’t.

3. Application of the Lutz-Nagell type theorem for hyperelliptic curves

Let \( p \) be a prime number, \( i, j \in \mathbb{Z}_{>0} \), and \( f(x) = x(x^2 + 2^i p^j)(x^2 + 2^{i+1} p^j) \). Let \( C \) be a hyperelliptic curve defined by \( y^2 = f(x) \) and \( J \) be its Jacobian variety. Let \( \phi : C \to J \) be the Abel-Jacobi map defined by \( \phi(P) = [P - \infty] \).

By taking Theorem 2.1 into account, Theorem 1.1 is an immediate consequence of the following proposition that we proved in [7].

**Proposition 3.1** (Proposition 3.1). Let \( P \in C(Q) \setminus \{ \infty \} \) such that \( \phi(P) \in J(Q)_{\text{tors}} \).

1. Suppose that \( p \neq 3 \). Then, \( P = (0, 0) \).
2. Suppose that \( p = 3 \) and \( (i, j) \neq (2, 2), (3, 2) \pmod{4} \). Then, \( P = (0, 0) \).

This proposition follows from the following Lutz-Nagell type theorem.

**Theorem 3.2** (cf. [5] Theorem 3). If \( P = (a, b) \in C(Q) \setminus \{ \infty \} \) and \( \phi(P) \in J(Q)_{\text{tors}} \), then \( a, b \in \mathbb{Z} \) and either \( b = 0 \) or \( b^2 \mid \text{disc}(f) \).

Note that Grant [5] proved the above Lutz-Nagell type theorem in more general settings.

4. Conjectures

In this section, we state some conjectures on the set of rational points of \( C^{(p; i, j)} \) for each pair \( (i, j) \).

**Conjecture 4.1.** Let \( p \) be an odd prime number, \( i, j \in \mathbb{Z} \), and \( C^{(p; i, j)} \) be a hyperelliptic curve defined by the following equation.
\[
y^2 = x(x^2 + 2^i p^j)(x^2 + 2^{i+1} p^j).
\]

1. Suppose that \( (i, j) = (0, 1) \). Then,
\[
C^{(p; i, j)}(Q) = \begin{cases} 
\{(0, 0), (8, \pm 252), \infty\} & (p = 17), \\
\{(0, 0), \infty\} & (p \neq 17).
\end{cases}
\]

2. Suppose that \( (i, j) = (2, 2) \). Then,
\[
C^{(p; i, j)}(Q) = \begin{cases} 
\{(0, 0), (6, \pm 216), \infty\} & (p = 3), \\
\{(0, 0), (5, \pm 375), \infty\} & (p = 5), \\
\{(0, 0), (136, \pm 235824), \infty\} & (p = 17), \\
\{(0, 0), \infty\} & (p \neq 3, 5, 17).
\end{cases}
\]
Remark 4.2. six infinite families.

(3) Suppose that \((i, j) = (2, 3)\). Then,

\[
C^{(p; i, j)}(\mathbb{Q}) = \begin{cases} 
\{(0, 0), (72, \pm 45360), \infty\} & (p = 3), \\
\{(0, 0), (98, \pm 115248), \infty\} & (p = 7), \\
\{(0, 0), \infty\} & (p \neq 3, 7).
\end{cases}
\]

(4) Suppose that \((i, j) = (1, 1)\) or \((0, 2)\) or \((2, 1)\). Then, \(C^{(p; i, j)}(\mathbb{Q}) = \{(0, 0), \infty\}\).

For each pair \((i, j)\), we have checked that there exist no additional rational points whose height of the \(x\)-coordinate is less than \(10^5\) for \(p < 1000\).\footnote{We checked it by MAMGA. The commands are as follows.}

\begin{itemize}
\item \(P<x>:=PolynomialRing(Rationals());\)
\item \(for\ \text{p}\ \in\ [3..1000]\ \text{do};\)
\item \(\text{if}\ \text{IsPrime(p)}\ \text{then};\)
\item \(C:=HyperellipticCurve(x*(x^2+p)*(x^2+2*p)); // (i,j)=(0,1).\)
\item \(\text{Points(C; Bound:=10^5);}\)
\item \(end;\)
\item \(end\ for;\)
\end{itemize}

\{(0, 0), \infty\}\ for \(p = 3, 7\) and \(p \neq 3, 7\).

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\footnote{We checked it by MAMGA. The commands are as follows.}

\begin{itemize}
\item \(P<x>:=PolynomialRing(Rationals());\)
\item \(for\ \text{p}\ \in\ [3..1000]\ \text{do};\)
\item \(\text{if}\ \text{IsPrime(p)}\ \text{then};\)
\item \(C:=HyperellipticCurve(x*(x^2+p)*(x^2+2*p)); // (i,j)=(0,1).\)
\item \(\text{Points(C; Bound:=10^5);}\)
\item \(end;\)
\item \(end\ for;\)
\end{itemize}

\{(0, 0), \infty\}\ for \(p = 17\).

\section*{Remark 4.2. (1) \(C^{(p; i+j)} (\text{resp.} \ C^{(p; i-j)})\) is isomorphic to \(C^{(p; i, j)}\) via a map which maps \((x, y)\) to \((x/4, y/32)\) (resp. \((x/p^2, y/p^3)\)). Moreover, we have isomorphisms of the following curves over \(\mathbb{Q}\):\footnote{The isomorphisms are given by the following:}

\begin{itemize}
\item \(C^{(p; 0,1)} \cong C^{(p;0,3)}; (x, y) \mapsto (2p^2/x, 2p^2/y/x^3).\)
\item \(C^{(p;1,1)} \cong C^{(p;0,3)}; (x, y) \mapsto (2p^2/x, 2p^4/y/x^3).\)
\item \(C^{(p;1,2)} \cong C^{(p;1,6)} = C^{(p;1,3)} \cong C^{(p;0,2)}; (x, y) \mapsto (p^2x, p^5y) \mapsto (2p^2/x, 2p^3y/x^3).\)
\item \(C^{(p;2,1)} \cong C^{(p;3,3)}; (x, y) \mapsto (8p^2/x, 32p^4y/x^3).\)
\item \(C^{(p;2,2)} \cong C^{(p;3,1)}; (x, y) \mapsto (8p^2/x, 32p^2y/x^3).\)
\item \(C^{(p;2,6)} \cong C^{(p;3,3)}; (x, y) \mapsto (p^2x, p^5y) \mapsto (8p^2/x, 32p^3y/x^3).\)
\end{itemize}
References

[1] Wieb Bosma, John Cannon, and Catherine Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), no. 3-4, 235–265, DOI 10.1006/jsco.1996.0125. Computational algebra and number theory (London, 1993). MR1484478

[2] Jean-Benoît Bost and Jean-François Mestre, Moyenne arithmético-géométrique et périodes des courbes de genre 1 et 2, Gaz. Math. 38 (1988), 36–64 (French). MR970659

[3] Benedict H. Gross and David E. Rohrlich, Some results on the Mordell-Weil group of the Jacobian of the Fermat curve, Invent. Math. 44 (1978), no. 3, 201–224, DOI 10.1007/BF01403161. MR491708

[4] E. V. Flynn, Bjorn Poonen, and Edward F. Schaefer, Cycles of quadratic polynomials and rational points on a genus-2 curve, Duke Math. J. 90 (1997), no. 3, 435–463, DOI 10.1215/S0012-7094-97-09011-6. MR1480542

[5] David Grant, On an analogue of the Lutz-Nagell theorem for hyperelliptic curves, J. Number Theory 133 (2013), no. 3, 963–969, DOI 10.1016/j.jnt.2012.02.023. MR2997779

[6] Richard K. Guy, Unsolved problems in number theory, 3rd ed., Problem Books in Mathematics, Springer-Verlag, New York, 2004. MR2076335

[7] Yoshinosuke Hirakawa and Hideki Matsumura, Infinitely many hyperelliptic curves with exactly two rational points (preprint), arXiv:1904.00215v2 (2019).

[8] Chris Nicholls, Descent methods and torsion on Jacobians of higher genus curves, University of Oxford (Ph.D. thesis) (2018).

[9] Fried. Jul. Richelot, De transformatione integralium Abelianorum primi ordinis commentatio, J. Reine Angew. Math. 16 (1837), 221–284, DOI 10.1515/crll.1837.16.221 (Latin). MR1578134

[10] Edward F. Schaefer, 2-descent on the Jacobians of hyperelliptic curves, J. Number Theory 51 (1995), no. 2, 219–232, DOI 10.1006/jnth.1995.1044. MR1326746

[11] Jin Nakagawa and Kuniaki Horie, Elliptic curves with no rational points, Proc. Amer. Math. Soc. 104 (1988), no. 1, 20–24, DOI 10.2307/2047452. MR958035

[12] Michael Stoll, Implementing 2-descent for Jacobians of hyperelliptic curves, Acta Arith. 98 (2001), no. 3, 245–277, DOI 10.4064/aa98-3-4. MR1829626

[13] _______, Arithmetic of hyperelliptic curves (unpublished lecture note), available from http://www.mathe2.uni-bayreuth.de/stoll/teaching/ArithHypKurven-SS2014/Skript-ArithHypCurves-pub-screen.pdf.

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