CNNs are Globally Optimal Given Multi-Layer Support

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Abstract

Stochastic Gradient Descent (SGD) is the central workhorse for training modern CNNs. Although giving impressive empirical performance it can be slow to converge. In this paper we explore a novel strategy for training a CNN using an alternation strategy that offers substantial speedups during training. We make the following contributions: (i) replace the ReLU non-linearity within a CNN with positive hard-thresholding, (ii) re-interpret this non-linearity as a binary state vector making the entire CNN linear if the multi-layer support is known, and (iii) demonstrate that under certain conditions a global optima to the CNN can be found through local descent. We then employ a novel alternation strategy (between weights and support) for CNN training that leads to substantially faster convergence rates, nice theoretical properties, and achieving state-of-the-art results across large scale datasets (e.g. ImageNet) as well as other standard benchmarks.

1. Introduction

Most modern Convolutional Neural Networks (CNNs) can be expressed simply in terms of the composition of an affine transform followed by a non-linear function\(^1\) such as \( \text{ReLU}(x) = \max(x, 0) \). The output for each layer \( l \) in the network can be expressed recursively as,

\[
 z^{(l)} = \text{ReLU}(W^{(l)}z^{(l-1)} - \beta^{(l)})
\]

where \( \{W^{(l)}, \beta^{(l)}\} \) are the affine parameters and \( z^{(l-1)} \) is the previous layer’s output and \( z^{(0)} = x \) being the input signal. It is well understood that when one applies ReLU to a vector (as in Eq. 1) the resulting output \( z^{(l)} \) will be sparse such that \( m^{(l)} \odot z^{(l)} = z^{(l)} \) where \( \odot \) represents the Hadamard product operator. We shall refer to \( m^{(l)} \) herein as the support of \( z^{(l)} \) where the individual elements are constrained to the binary values \( \{0, 1\} \). Further, we shall refer to the concatenation of these supports across all \( L \) layers within the CNN as the multi-layer support.

In this paper we propose a re-interpretation of the CNN architecture in Eq. 1 such that the network can now be expressed as,

\[
 z^{(l)} = \text{diag}(m^{(l)})(W^{(l)}z^{(l-1)} - \beta^{(l)})
\]

where the ReLU non-linearity at each layer is instead replaced by a binary support. Interestingly, the re-interpretation in Eq. 2 becomes completely linear if one knows the multi-layer support for the input signal a priori.

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1Recent work by [21] has demonstrated that state of the art CNN performance can be attained without max pooling or other non-linearities, resulting in just the use of linear affine operations and ReLU within the network.
Obviously, in practice one cannot determine this multi-layer support without the multi-layer affine weights - so it seems nothing is gained as this modified network remains just as non-linear and problematic to optimize as before.

**A Case for Alternation:** Modern CNNs are typically optimized through stochastic gradient descent (SGD). More efficient optimization strategies are possible such as quasi-Newton methods [4], however, they are problematic due to their high computational cost and the poor positive curvature of CNN objective functions in general. Alternation can be interpreted as a kind of block coordinate descent where the global solution to one factor is found while keeping the other factors fixed [2]. Although alternation is not guaranteed of reaching a local minima, the objective will be reduced after every iteration. As argued by [2], the convergence of alternation iterations is initially very good, and has typically low computational overhead - making it an attractive iteration scheme for many large scale problems in vision and learning.

As discussed by [4], however, alternation is typically dismissed as a viable optimization strategy for modern CNNs. Strategies like alternation (and block coordinate descent) make the most sense when factors in the objective can be separated into groups that play relatively isolated roles. Such a separation of parameters is problematic when applied to CNNs as nearly every parameter in the network heavily influences the other. In this paper we argue that by re-interpreting the ReLU non-linearity as a binary support parameter - one can now orchestrate a parameter separation that makes an alternation strategy attractive over conventional SGD. Specifically, we advocate for a strategy where we hold the multi-layer support fixed, and then solve for the multi-layer affine weights until convergence. We then estimate the support using the updated weights - iterating the whole alternation process until a good solution the CNN is found.

A necessary component for our proposed alternation strategy to be feasible is that an approximate global solution to our CNN’s weights can be determined given the known multi-layer support for each training example. This may seem at first glance a tall order, as it has been well documented [8] that modern CNNs with some minor exceptions have no guarantees of converging to a global solution. This brings us to the central insight of our paper - as well as inspiring our paper’s title. Specifically, we demonstrate that we can find the global solution (through local descent) to the weights of our proposed CNN architecture assuming known multi-layer support for all the training examples. This result in isolation is of little practical interest as one never knows the multi-layer support for an input signal a priori. However, the result is of considerable interest when viewed from its applicability for training CNNs within an efficient alternation framework.

**Contributions:** We make the following contributions:-

- Demonstrate how one can linearly separate weights from support within a modern CNN using a modified ReLU non-linearity. Based on this separation we can then demonstrate how the problem of multi-layer affine weight estimation can be represented as determining a rank one tensor. Further, even though this problem is clearly non-convex we characterize under which conditions that a global optima can be found using local descent assuming the multi-layer support of the training examples is known (Section 3.3).
- Present empirical results which show the utility of our proposed alternation strategy for CNN optimization. Specifically, we demonstrate substantially faster convergence results (over traditional SGD) over a number of benchmarks while achieving (and in some circumstances surpassing) state of the art in actual recognition performance (Section 4.1).
- Finally, we discuss computational issues with our approach when attempting to learn a CNN on a large scale data (e.g. ImageNet) - as naively one would be expected to estimate the multi-layer support for every training example. Specifically, we advocate for a novel modification of our alternation strategy where a fixed multi-layer support is assumed across a mini-batch of the training examples. We demonstrate that this strategy can obtain state of the art performance, while still enjoying the rapid convergence properties of our original naive approach (Section 4.2).

**2. Related Work**

Deep neural networks have been revolutionary in many fields of machine learning and artificial intelligence, including computer vision, speed recognition, and natural language processing. Their performance continues to improve with ever deepening network architectures [10].

**Optimization:** Modern CNNs still remain difficult to train due to the nonconvexity of the optimization problem. They are typically optimized through SGD which often cannot return a global minima. More efficient optimization strategies exist such as quasi-Newton methods [4], which however, incur high computational cost. One recent work in [19] views the forward pass of a CNN as a thresholding pursuit serving a Multi-Layered Convolutional Sparse Coding (ML-CSC) model. The authors argued that improved results should be attainable by employing a more sophisticated pursuit algorithm. Promising results were attained, however, due to the computational complexity of the proposed pursuit strategy results were limited to small image datasets.

**Optimality:** Despite the empirical success of CNNs, the theoretical underpinnings for this success and their optimality remains elusive. Of particular note in this regard is the
recent work of Kawaguchi [11] who studied the optimality of simplistic deep linear neural networks. Kawaguchi proved that any local minimum point is also a global minimum, and that any other critical point is a saddle point for deep linear networks. Further, [18, 27] provided some conditions for a critical point of the empirical risk function to be a global minimum for both linear and nonlinear networks.

Recently, Haefele and Vidal [8] derived sufficient conditions to guarantee that local minima for their proposed network are globally optimal, requiring both the network output and the regularization to be positively homogeneous, with the regularization being designed to control the network size. However, the results only apply to networks with one hidden layer, or with multiple subnetworks connected in parallel. Our work does not have such restrictive assumptions about networks.

3. Separating Weights from Supports

Most modern Convolutional Neural Networks (CNNs) are built with a collection of feature layers followed by a small number of fully connected layers. Throughout most of the theoretical treatment of this paper we follow [21] to replace max-pooling by a convolutional layer with increased strides. This makes the convolutional pooling layer linear, leaving activation layers as the sole nonlinearity in a network. We can express such a network simply in terms of the concatenation of an affine transform $(Wx)^2$ followed by a non-linear function $\eta(Wx; \beta)$ such as ReLU($Wx - \beta$). For example a three layer network can be expressed as,

$$F(x; W, \beta) = W^{(3)}\eta(W^{(2)}\eta(W^{(1)}x; \beta^{(1)}); \beta^{(2)}) - \beta^{(3)}.$$  

(3)

The above three layer network can be generalized to $L$ layers by expressing the $\ell$-th network output recursively as,

$$z^{(\ell)} = \eta(W^{(\ell)}z^{(\ell-1)}; \beta^{(\ell)})$$  

(4)

where $z^{(\ell-1)}$ is the previous layer’s output and $z^{(0)} = x$. We can therefore define the $L$ layer network function as,

$$F(x; W, \beta) = W^{(L)}z^{(L-1)} - \beta^{(L)}$$  

(5)

Such a network function can be trained discriminatively on the following objective function,

$$\arg\min_{W,\beta} \sum_{n=1}^{N} L(y_n, F(x_n; W, \beta))$$  

(6)

where $N$ defines the number of training examples, $y_n$ and $x_n$ are the $n$-th output label vector and input vector respectively. Numerous loss functions $L$ can be employed depending on the nature of task ranging from cross-entropy [3] to least-squares error.

As previously discussed, a popular non-linearity within CNN literature at the moment is the Rectified Liner Unit (ReLU) defined as,

$$\eta(q; \beta) = \max(q - \beta, 0)$$  

(7)

where scalar $q$ is the input and $\beta$ is a learnable threshold/bias. With ReLU applied independently across elements of the input vector, the output of the CNN at layer $\ell$ in Equation 4 can be rewritten as

$$z^{(\ell)} = \eta(W^{(\ell)}z^{(\ell-1)}; \beta^{(\ell)}).$$  

(8)

One drawback of ReLU is that the output mixes the input with the threshold/bias complicating the separation of weights from support. To facilitate this separation we advocate to instead use a positive hard thresholding operator to replace ReLU within a CNN. Positive hard-thresholding is defined as

$$\eta^*(q; \beta) = \begin{cases} \eta(q; \beta) & \text{if } q > \beta \\ 0 & \text{otherwise} \end{cases}$$  

(9)

Positive hard-thresholding not only facilitates the decoupling of the weights from support, but we also show in our experiments section that it has no discernable empirical drawbacks compared to ReLU and even improves performance in some circumstances.

There is also a theoretical motivation for this change in non-linearity. Papyan et al. [19] recently demonstrated that the non-linearity ReLU can be viewed as the closed form solution to the following non-non-negative $\ell_1$ minimization:

$$\eta(q; \beta) = \arg\min_p \|q-p\|^2 + 2\beta\|p\|_1, \text{ s.t. } p > 0.$$  

(10)

Taking the view that the $\ell_1$ norm term in Equation 10 is a convex approximation to the sparsity inducing $\ell_0$ norm one could rewrite the objective as

$$\eta^*(q; \beta) = \arg\min_p \|q-p\|^2 + 2\beta\|p\|_0, \text{ s.t. } p > 0.$$  

(11)
Papyan et al. proposed to potentially use the closed form solution to this \( \ell_0 \) problem as an alternative non-linearity to ReLU within a CNN. This closed form solution is the positive hard-thresholding non-linearity defined in Equation 9.

Armed now with the positive hard-thresholding non-linearity, we can now reinterpret a CNN in a new manner. Specifically for each layer of the network we can form the indicator binary support vector

\[
m^{(\ell)} = [(\eta^{(\ell)}_1 > \beta^{(\ell)}_1), \ldots, (\eta^{(\ell)}_M > \beta^{(\ell)}_M)]^T
\]  

(12)

to replace the positive hard-thresholding non-linearity. For example, one could rewrite the three layered network described earlier in Equation 3 as,

\[
\mathcal{F}(x; W, \beta) = W^{(3)} M^{(2)} W^{(2)} M^{(1)} W^{(1)} x - \beta^{(3)}.
\]  

(13)

where \( M^{(\ell)} \) is defined as \( \text{diag}(m^{(\ell)}) \).

3.2. Decoupling Weights and Masks

Let us consider the output of the first layer in a CNN \( z^{(1)} = M^{(1)} W^{(1)} x \). Vectorizing both sides gives,

\[
z^{(1)} = U^{(1)} v^{(1)}.
\]  

(14)

where \( U^{(1)} = x \otimes M^{(1)} \), \( v^{(1)} = \text{vec}(W^{(1)}) \), and the operator \( \otimes \) denotes a kronecker product. This now shows a possible strategy for separating the weights from the binary support in the first layer.

Now, we demonstrate that this separation also holds for a multi-layer CNN. Suppose the output of \( \ell \)-th layer holds the separation, that is \( z^{(\ell)} = U^{(\ell)} v^{(\ell)} \). The output of the \((\ell + 1)\)-th layer is therefore,

\[
z^{(\ell+1)} = M^{(\ell+1)} W^{(\ell+1)} z^{(\ell)} = M^{(\ell+1)} W^{(\ell+1)} U^{(\ell)} v^{(\ell)}.
\]  

(15)

By applying vectorization sequentially one can derive,

\[
z^{(\ell+1)} = U^{(\ell+1)} v^{(\ell+1)},
\]  

(16)

where

\[
U^{(\ell+1)} = \text{vec}(U^{(\ell)}) \otimes M^{(\ell+1)}
\]  

(17)

\[
v^{(\ell+1)} = v^{(\ell)} \otimes \text{vec}(W^{(\ell+1)}).
\]  

(18)

By mathematical induction, it is implied that a multi-layer CNN can always be separated into a bilinear multiplication of \( U \) and \( v \), where \( U \) depends on the input examples and the binary support while \( v \) only depends on the weights \( W \). Formally, by setting \( M^{(L)} = \text{diag}(1) \) and \( \beta = 0 \), the \( L \) layer network function in Equation 5 can be rewritten as

\[
\mathcal{F}(x; W, M) = U^{(L)} v^{(L)}. \]  

(19)

One can set \( M^{(L)} = \text{diag}(1) \) as the non-linearity is not applied to the last layer, additionally one can set \( \beta^{(L)} = 0 \) as the bias can easily be ignored (for recognition tasks) or estimated independently (e.g. least-squares loss).

From this perspective, the objective function in Equation 6 can be expressed as a bilinear objective, specifically

\[
\sum_{n=1}^{N} \mathcal{L}\{y_n, U^{(L)} v^{(L)}\}. \]  

(20)

3.3. Recovery of Weights Given Support

Given the bilinear form of the objective function, we now characterize the optimality of recovering weights, assuming the masks are given from oracle. To illustrate the basic idea, we first restrict ourselves to a two-layer neural network:

\[
\mathcal{F}^{(2)}(x_n, W) = U_n^{(2)} v^{(2)}
\]  

(21)

using a least-squares loss function \( \mathcal{L}(y, x) = \| y - x \|_2^2 \). Suppose that we have a set of training samples \( \{(x_n, y_n)\}_{n=1,...,N} \). Then the problem of estimating \( W \) given \( M \) is formally defined as the problem (P1):

\[
(P1): \min_{W^{(1)}, W^{(2)}} \sum_{n=1}^{N} \| y_n - \mathcal{F}^{(2)}(x_n; W) \|_2^2.
\]  

(22)

For the sake of brevity, we define:

\[
y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad U = \begin{bmatrix} U_n^{(2)} \end{bmatrix}.
\]  

(23)

Therefore, the problem (P1) can be simplified as

\[
(P2): \min_{W^{(1)}, W^{(2)}} \| y - U v^{(2)} \|_2^2.
\]  

(24)

Further, inspired by the work \[1\], we define a lifting \( \mathfrak{g} \) as a linear operator satisfying

\[
\mathfrak{g}(pq^T) = p \otimes q, \quad \text{for any vectors} \quad p, q.
\]  

(25)

By imposing the lifting function to the problem (P2), one can rewrite the objective function as

\[
\| y - U \mathfrak{g}(\text{vec}(W^{(1)}) \text{vec}(W^{(2)})) \|_2^2
\]  

(26)

By defining \( V = \text{vec}(W^{(1)}) \text{vec}(W^{(2)}) \), we know \( \text{rank}(V) = 1 \). Thus, the problem (P2) can be equivalently rewritten to

\[
(P3): \min_V \| y - U \mathfrak{g}(V) \|_2^2 \quad \text{s.t.} \quad \text{rank}(V) = 1.
\]  

(27)

If a solution to (P3) can be found, we then can utilize Singular Value Decomposition to extract \( W^{(1)} \), \( W^{(2)} \) from \( V \):

\[
\text{vec}(W^{(1)}) = \sqrt{\sigma^*} u^*, \quad \text{vec}(W^{(2)}) = \sqrt{\sigma^*} v^*
\]  

(28)

where \( V = \sigma^* u^*(v^*)^T \).
Theorem 1: The globally optimal solution to $V$ in the problem (P3) can be achieved by stochastic gradient descent with high possibility if there exists a positive number $0 < \epsilon < 1/10$, such that the matrix $U$ satisfies
\[ \frac{9}{10} + \epsilon \|x\|^2 \leq \|Ub\|^2 \leq \frac{11}{10} - \epsilon \|x\|^2. \]  
where $\lambda_{\min}, \lambda_{\max}$ are the smallest and largest eigenvalue of $U$, and $m$ is the size of $y$.

Proof: From the property of eigenvalues, it is implied that, under certain conditions, for any vector $b$, it is true that
\[ \lambda_{\min}^2 \|b\|^2 \leq \|Ub\|^2 \leq \lambda_{\max}^2 \|b\|^2. \]  
By applying the bounds in Equation 29, it is implied that
\[ \frac{9}{10} + \epsilon \|b\|^2 \leq \frac{1}{m} \|Ub\|^2 \leq \frac{11}{10} - \epsilon \|b\|^2. \]  
Since $\hat{S}(V)$ is a vector, the following must holds:
\[ \frac{9}{10} + \epsilon \|\hat{S}(V)\|^2 \leq \frac{1}{m} \|U\hat{S}(V)\|^2 \leq \frac{11}{10} - \epsilon \|\hat{S}(V)\|^2. \]  
Because one possible construction of $\hat{S}$ is
\[ \hat{S}(V) = \text{vec}(V^T), \]  
one can see $\|\hat{S}(V)\|^2 = \|V\|_{F}^2$. Then
\[ \frac{9}{10} + \epsilon \|V\|^2 \leq \frac{1}{m} \|U\hat{S}(V)\|^2 \leq \frac{11}{10} - \epsilon \|V\|^2. \]  
By defining a linear operation $A(V) = U\hat{S}(V)$, we have
\[ \frac{9}{10} + \epsilon \|V\|^2 \leq \frac{1}{m} \|A(V)\|^2 \leq \frac{11}{10} - \epsilon \|V\|^2. \]  
This condition characterizes that the operator $A$ satisfies the RIP condition with $\epsilon = \frac{\lambda_{\min}}{\lambda_{\max}}$. By the Theorem 3.1 in [1], a global optimal solution $V^*$ will be discovered by stochastic gradient descent with high possibility. 

4. Experimental Results
We start with evaluating our alternation strategy on four standard image classification benchmarks: CIFAR-10 [12], CIFAR-100 [12], MNIST [14] and SVHN [17]. Two network architectures are employed - the All Convolutional Network (ALL-CNN) [21] where ReLU is the only nonlinearity, and Network in Network (NIN) [16] with nonlinearities beyond ReLU (i.e., max pooling). We derive a method to estimate the multi-layer support for every training example from the considered small datasets. The resulting alternating training algorithm is found to often converge to a better local minima, even if ReLU is not the sole nonlinearity. Our better results also come independent of network architectures and capacities, as well as different datasets. More importantly, we demonstrate much faster convergence properties of our algorithm. Lastly, we propose a batch-based modification of our alternation algorithm to scale up to the large ImageNet dataset [13]. We show that by sharing a fixed multi-layer support across a batch of examples, our alternation algorithm is computationally more friendly than the naive per-example approach. We experimented with several popular deep and highly nonlinear networks (e.g., GoogLeNet [25]), achieving state-of-the-art performance for all networks. We performed all experiments using the Caffe framework.

4.1. Image Classification on Small Benchmarks
On small datasets, one simple way to implement our alternation algorithm is to estimate the support mask for every training instance during alternation. We call this approach as “I-alternation” where the instance-wise mask can be stored or even cached online for small data. The closed-form solution to the mask can be obtained via solving the reconstruction problem:
\[ \min_{m} \|x - m \odot x\|^2 + \beta \|m\|_0, \quad m_i \in \{0, 1\}, \]  
where the least-square error is used (the solution can generalize to the cross-entropy or other loss). Obviously this objective function is separable with respect to each element $m_i$ in the mask $m$. Given $m_i$ as a binary mask, the objective function of each subproblem is
\[ (x_i - m_i x_i)^2 + \beta m_i = \begin{cases} \beta & \text{if } m_i = 1 \\ x_i^2 & \text{if } m_i = 0 \end{cases} \]  
Therefore, the optimal mask solution is:
\[ m_i^* = \begin{cases} 1 & \text{if } x_i^2 > \beta \\ 0 & \text{otherwise}. \end{cases} \]  
We first evaluate this naive approach on the small CIFAR-10 dataset. The ALL-CNN network [21] is adopted,
which only consists of convolutional layers with ReLU non-linearities and softmax layer. Such a linearized architecture provides a perfect testbed to specially evaluate the capability of our ReLU-induced alternation algorithm. We follow the detailed training settings of [21], and similarly use the models A, B and C with increasing network capacities. For testing, as is common practice we apply ReLU to each testing example. To predict image classes, we follow [21] to produce 10 outputs for different positions at the top 1 × 1 convolutional layer, and simply average them over the whole image before computing softmax probabilities.

Table 2 conducts an in-depth study of our various baselines without data augmentation. The classification error as well as the number of network parameters are compared. The following observations can be made from the table:

- Our simple “I-alternation” approach already shows successful convergence with competitive performance with respect to the baseline [21]. Such empirical results are consistent for various network capacities. On the other hand, our approach converges much faster than the SGD baseline (shown later).
- The “I-alternation+MaskT” baseline only applies the binary mask rather than ReLU activations for testing. Not surprisingly the performance drops dramatically, but is not too unacceptable. This suggests the learned mask preserves some discrimination ability for classification, while also confirming the value of standard testing procedure.

CIFAR-10 and CIFAR-100: Table 3 and Table 4 further compare our I-alternation approach using the ALL-CNN-C network [21] with other state-of-the-art methods on CIFAR-10 and CIFAR-100. On CIFAR-10, for cases with and without data augmentation, our method performs on par with or surpasses prior arts. On CIFAR-100, our approach achieves competitive performance as well. Note we use a much smaller ALL-CNN-C network (1.4M parameters) than the top performing Fractional Pooling network [7] (50M parameters) thus we train much faster.

MNIST and SVHN: On the two datasets, we adopt the NIN architecture [16] with nonlinearities of both ReLU and max pooling. Therefore, the experiments here can be regarded as a generalization test of our I-alternation algorithm to different network architectures and their nonlinearities beyond ReLU. We follow the training hyper-parameters of [16], and use the same data preprocessing and splitting for both datasets. Table 5 and Table 6 compare our results with previous methods that do not augment data. Our method performs comparable to the state of the arts again, even though they either use more complicated training schemes or net-
work architectures. This shows the efficacy of our alternation algorithm which is consistent across networks and datasets. We claim that our algorithm can be applied to more complex networks to compare with more recent methods that have even stronger performance on these datasets. The experiments here mainly act as a validation test of our alternation algorithm’s efficacy.

**Analysis of convergence speed:** One good property of our alternation strategy is that its strong performance comes with a consistent boost in convergence speed. Fig. 3 compares the training curves (in terms of loss and error rate) of traditional SGD and our I-alternation algorithm on two example datasets of CIFAR-10 and MNIST (using different networks).

Note the alternation iteration is identified at the corresponding epoch number, whose gap ranges from 4 to 50 epochs. This gap is approximately determined by the convergence speed of SGD while keeping the mask fixed - when the loss no longer decreases on the considered dataset, we use the passed epochs as the gap and turn to estimate a new mask from the updated weights. After seeing one whole epoch of training examples, we store their individual masks which are then used for the following SGD training. For networks with randomly initialized weights, the generated masks will be meaningless at the beginning of alternation. We run regular SGD for few epochs to obtain reasonable masks to start the whole alternation process. The bias \( \beta \) is also initialized this way, and we found good performance can be generally achieved.

It can be observed from Fig. 3 that our I-alternation algorithm converges much faster than conventional SGD. The training loss sometimes increases abruptly at the time of alternation due to the mismatch between outdated mask and updated weights. But overall, our I-alternation algorithm reaches similar performance to SGD in 30%-40% fewer iterations. Note when masks are known for all training examples, SGD have been proved to be able to find a global optima under certain conditions, but we can still choose more efficient solvers (e.g. advanced Augmented Lagrange methods or SVD) in the future.

**4.2. Generalization to Large-Scale Dataset**

We finally discuss computational issues with our I-alternation approach when scaling up to large-scale data (e.g. ImageNet [13]). Specifically, we claim that estimating instance-wise support is storage and computationally demanding for many large scale problems in vision and learning. To this end, we propose a batch-based modification of our “I-alternation” algorithm, called “B-alternation” where a fixed multi-layer support is shared across a mini-batch of training examples. The closed-form solution to such batch-wise mask can be obtained via solving the problem:

\[
\min_m \sum_{n=1}^{N} \| x^{(n)} - m \odot x^{(n)} \|_2^2 + \beta \| m \|_0, \quad m_i \in \{0, 1\},
\]

where \( x^{(n)} \) is one example from the \( N \)-sized batch. Given \( m_i \) as a binary mask in \( m \), the objective function of the corresponding subproblem is

\[
\sum_{n=1}^{N} (x_i^{(n)} - m_i x_i^{(n)})^2 + \beta m_i = \begin{cases} \beta & \text{if } m_i = 1 \\ \frac{1}{\sum_{n=1}^{N} (x_i^{(n)})^2} & \text{if } m_i = 0. \end{cases}
\]

Therefore, the optimal mask solution is:

\[
m_i^* = \begin{cases} 1 & \text{if } \sum_{n=1}^{N} (x_i^{(n)})^2 > \beta \\ 0 & \text{otherwise}. \end{cases}
\]

In this way, we generate and apply batch-wise masks on-the-fly during training. Table 7 first examines this B-alternation algorithm on small datasets, and finds no accuracy loss compared to the original I-alternation algorithm. The B-alternation algorithm is computationally more friendly as well, which plays a more important role for large datasets. It is worth noting that, we also tried to do the...
majority vote among individual masks in one batch to produce the batch-wise mask. Although this is not theoretically grounded, we still achieve comparable results to the B-alternation algorithm. We do not report the numbers here to avoid cluttering the experiments.

Next, we apply the novel algorithm to the large ImageNet dataset. We choose the deep GoogLeNet [25], VGG-16 [20], ResNet-18 [10], which have higher levels of non-linearities than just ReLU and more practical values due to their wide use. We followed the DSD (Dense-Sparse-Dense) [9] training flow. We first train a Dense network to learn full weight connections. In the Sparse step of [9], the network is regularized by pruning the connections with small weights and finetuning the rest. A final Dense network is obtained by retraining all connections with the previously pruned weights re-initialized from zero. Our B-alternation algorithm is integrated in the Sparse step due to the sparse nature of the generated masks. By running our alternation algorithm in this step, we simply prune the connections with zero mask, eliminating the need to set the sparsity degree as in [9]. We train for the same number of epochs, and do not change any other training settings or hyper-parameters.

Table 8 summarizes the Top-1 error rates of the base network, DSD training and its variants using our alternation algorithms. Surprisingly, our method is able to achieve the best results for all types of network architectures on ImageNet. This suggests that our found support offers better guidance than DSD on reducing the weights redundancy. When compared to the original I-alternation algorithm, the B-alternation variant is found to achieve stronger results. One hypothesis for this is that such batch-wise “regularization” can help avoid overfitting and escape noisy local minima by instance-wise optimization. Also, the batch-wise operation reflects the current good practices of CNN training and thus enables parallelization in the future. Furthermore, the B-alternation algorithm has typically low computational overhead, while still enjoying the rapid convergence properties of the I-alternation algorithm (about 30% fewer iterations than SGD).

5. Conclusion

This paper reinterprets CNNs as being composed of weights and support using a modified ReLU-nonlinearity.
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