Terao’s conjecture does not extend to weak combinatorics

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Abstract

In this work we study line arrangements consisting in lines passing through three non aligned points. We call them triangular arrangements. We prove that any of this arrangement is associated to another one with the same combinatorics, constructed by removing lines to a Ceva arrangement. We then describe a combinatorics for any possible splitting type of free logarithmic bundles associated to a triangular arrangement. Finally, we give two triangular arrangements having the same weak combinatorics (that means the same number $t_i$ of points with multiplicity $i, i \geq 2$), such that one is free but the other one is not.

1 Introduction

A line arrangement $\mathcal{A} = \{l_1, \ldots, l_n\}$ in $\mathbb{P}^2$ is a finite set of distinct lines. The union of these lines forms a divisor defined by an equation $f = \prod_i f_i = 0$ where $f_i = 0$ is the equation defining $l_i$. The cohomology ring of the complement $\mathbb{P}^2 \setminus \{f = 0\}$ was first studied by Arnold and Brieskorn who proved that it is generated by the logarithmic differential 1-forms $\frac{dl_i}{f_i}$. The sheaf $\Omega_\mathcal{A}$ of logarithmic 1-forms associated to a line arrangement $\mathcal{A}$ (more generally to a hyperplane arrangement), and its dual, the sheaf $\mathcal{T}_\mathcal{A}$ of vector fields tangent to this arrangement become of great interest and many important works appear concerning these objects (see for example [4], [7], [11], [12], [14]). This last sheaf can be heuristically understood as the tangent sheaf of the complement $\mathbb{P}^2 \setminus \{f = 0\}$ and can be defined as the kernel of the Jacobian map, which means (in $\mathbb{P}^2$):

$$0 \rightarrow \mathcal{T}_\mathcal{A} \rightarrow \Omega^3_{\mathbb{P}^2} \xrightarrow{\nabla f} \mathcal{J}_f(n - 1) \rightarrow 0,$$

where $\mathcal{J}_f$ is the ideal sheaf generated by the three partial derivatives $\nabla f = (\partial_x f, \partial_y f, \partial_z f)$. This ideal, called Jacobian ideal, defines the Jacobian scheme supported by the singular points of the arrangement; for instance when $\mathcal{A}$ is generic (i.e. it consists of $n$ lines in general position) then $\mathcal{J}_f$ defines $\binom{n}{2}$ distinct points. These sheaves $\Omega_\mathcal{A}$ and $\mathcal{T}_\mathcal{A}$ are basic tools to study the link between the geometry, the topology and the combinatorics of $\mathcal{A}$. The combinatorics of $\mathcal{A}$ is determined by the intersection lattice $L(\mathcal{A})$ which is, roughly speaking, the set of all intersections of hyperplanes of the arrangement (see [11] for more details).

Notice that the sheaf $\mathcal{T}_\mathcal{A}$ is a reflexive sheaf over $\mathbb{P}^2$ and therefore it is a vector bundle. When $\mathcal{A}$ is generic, one can verify (see [7] for instance) that $\mathcal{T}_\mathcal{A}$ is a Steiner bundle (i.e. its resolution by free $\mathcal{O}_{\mathbb{P}^2}$-modules is given by a matrix of linear forms):

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}^{-3}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}^{n-1} \rightarrow \mathcal{T}_\mathcal{A}(n - 2) \rightarrow 0.$$
When $A$ is not generic, the associated bundle $T_A$ can be of any kind, semi-stable, unstable and even decomposed as a sum of two line bundles. When $T_A$ is a sum of line arrangements, the arrangement $A$ is called free arrangement; these free logarithmic sheaves were studied first by Saito in [12] for any reduced divisor and by Terao [13] for hyperplane arrangements. Let us define freeness precisely in our situation.

**Definition 1.1.** The arrangement $A$ is free with exponents $(a, b)$, where $0 \leq a \leq b$ are integers, if $T_A = O_{\mathbb{P}^2}(-a) \oplus O_{\mathbb{P}^2}(-b)$.

In [11], the main reference about hyperplane arrangements, Terao conjectures that freeness depends only on the combinatorics of $A$, where the combinatorics is described by the set $L(A)$ of all the intersections of lines in $A$. More precisely, if two arrangements $A_0$ and $A_1$ have the same combinatorics (a bijection between $L(A_0)$ and $L(A_1)$) and one of them is free then the other one is also free (of course with the same exponents). This conjecture, despite all the efforts, is proved, for line arrangements, only up to 13 lines (see [2]).

Probably, one of the main difficulty is that few families of free arrangements are known. A weaker problem concerns the weak combinatorics. The weak combinatorics of a given arrangement of $n$ lines is defined the knowledge of the integers $t_i$, $i \geq 2$ of points with multiplicity exactly equal to $i$ of the arrangement. Let us mention the following beautiful formula, found by Hirzebruch in [8], involving these numbers (when $t_n = t_{n-1} = t_{n-2} = 0$):

$$t_2 + t_3 \geq n + \sum_{i \geq 1} it_{i+4}.$$

It is natural to ask if Terao's conjecture can be extended to the assumption of weak combinatorics, i.e. *Do there exist two arrangements with the same weak combinatorics with one free and the other one not?*

In section 6 giving an explicit example, we prove that the answer is yes. To our knowledge, this example is the first known example of two arrangements with the same weak combinatorics (but not the same combinatorics) such that one is free and the other is not.

In [5, Corollary 2.12], Elencjwag and Forster proved that a rank $r$ vector bundle $E$ on $\mathbb{P}^n$ with the same Chern classes than a sum of line bundles $\bigoplus_{i=1}^r O_{\mathbb{P}^n}(-a_i)$ and such that $E_l = \bigoplus_{i=1}^r O_l(-a_i)$ for one line $l \subset \mathbb{P}^n$ is actually $E = \bigoplus_{i=1}^r O_{\mathbb{P}^n}(-a_i)$. In other words, if the Chern classes of $T_A$ are given (they are given by the knowledge of the number of triple points counted with multiplicities, which is weaker than the combinatorics, even weaker than the weak combinatorics), the freeness of $A$ is completely determined by the splitting on one line!

So, the main difficulty is to determine the splitting type of the bundle $T_A$ along a line (of $A$ or not). Wakefield and Yuzvinsky proved (see [14, Theorem 3.1]), using the notion of multiarrangements introduced by Ziegler (see [15]), that except for some special multiplicities, the splitting type of $T_A$ on one line of the arrangement does not depend only on the multiplicities of the restriction, but also on the positions of the restricted points. It means for instance that the splitting type of $T_A$ along a line of $A$ containing 4 multiple points of $A$ will depend on their cross-ratio. Since $\text{PGL}(2, \mathbb{C})$ acts transitively on the set of three distinct points on $\mathbb{P}^1$, this implies that the combinatorics determine the splitting if there are no more than 3 multiple points on a line.
Let us recall that Terao’s conjecture holds for specific configurations of line arrangements, i.e. when:

- All the lines of the arrangement $\mathcal{A}$ pass through two fixed points: such an arrangement is free if and only if the line joining the two fixed points belongs to $\mathcal{A}$;
- One line of the arrangement contains no more than 3 singular points;
- The singular points of $\mathcal{A}$ have multiplicities at most 3.

Let us mention also the infinite family of reflection arrangements $\mathcal{A}_3^0(n)$, $\mathcal{A}_3^1(n)$, $\mathcal{A}_3^2(n)$ and $\mathcal{A}_3^3(n)$ defined respectively by the equations $f_n = 0$, $xf_n = 0$, $yf_n = 0$ and $z^2f_n = 0$ where $f_n = (x^n - y^n)(y^n - z^n)(x^n - z^n)$. They are free with exponents respectively $(n + 1, 2n - 2)$, $(n + 1, 2n - 1)$ and $(n + 1, 2n + 1)$ (see [8] or Corollaries 2.8 and 2.9 of this text). Their number of multiple points $t_{2n}, t_{3n}, t_{4n}, t_{5n}$ (their weak combinatorics) are given in [11] and again in [8]. The last one, $\mathcal{A}_3^3(n)$, is often called Ceva arrangement.

Let us introduce now the family of triangular arrangements. They are line arrangements consisting in lines passing through three non aligned fixed points. Reflection arrangements belong to this family. Our goal in this paper is to study these triangular arrangements. After describing some preliminary results (see the next section), the paper is organized as follows:

- First of all, we prove in Theorem 3.1 that we can associate, to any triangular arrangement, a further one having the same combinatorics and obtained by deleting particular lines from a reflection arrangement.
- We construct, for any possible exponent, a free arrangement and we describe its combinatorics. These free arrangements will be obtained from the reflection ones by deletion of lines well chosen in order to keep freeness at each step (see Section 4).
- We characterize, in Section 5, every free triangular arrangement constructed deleting at least one side line of the triangle. Observe that, because of the previous point, any triangular arrangement can be related to this situation.
- Then we propose two arrangements $\mathcal{A}$ and $\mathcal{B}$ having the same weak combinatorics (even quite the same combinatorics!) such that $\mathcal{A}$ is free and $\mathcal{B}$ is not (see Section 6).

## 2 The inner triple points of a triangular arrangement

In this section we will explicit the importance of the set $T$ of the triple points, defined by the triangular arrangement, which are not the vertices of the triangle. We will describe in particular the case where $T$ is either empty or a complete intersection.

Let $A, B, C$ be three points not aligned. Consider the union of $a + 1$ lines through $A$, $b + 1$ lines through $B$ and $c + 1$ through $C$. The set of corresponding arrangements of $a + b + c$ distinct lines through $A,B$ and $C$ possessing $a + b + c$ is denoted $\text{Tr}(a, b, c)$. The subset
of arrangements in $\text{Tr}(a, b, c)$ that contain the three lines joining the vertices is denoted by $\text{Tr}(a, b, c)$.

**Proposition 2.1.** Let $A \in \text{Tr}(a, b, c)$ then, there is an exact sequence

$$0 \longrightarrow \mathcal{T}_A \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b) \oplus \mathcal{O}_{\mathbb{P}^2}(-c) \longrightarrow \mathcal{J}_T(-1) \longrightarrow 0,$$

where $T$ is the smooth finite set of inner triple points (i.e. $A, B, C \notin T$).

**Proof.** Let $Z \subset \tilde{\mathbb{P}}^2$ the finite set of points corresponding by projective duality to the lines of $\mathcal{A}$. Since $Z$ is contained in a triangle $\Delta$, this induces the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow \mathcal{J}_Z(1) \longrightarrow \mathcal{J}_{Z/\Delta}(1) \longrightarrow 0.$$

The hypothesis says that the vertices of $\Delta$ belong to $Z$, then it implies that $\mathcal{J}_{Z/\Delta}(1) = \mathcal{O}_L(-a) \oplus \mathcal{O}_L(-b) \oplus \mathcal{O}_L(-c)$. Let us consider the incidence variety

$$\mathcal{F} = \{(x, l) \in \mathbb{P}^2 \times \tilde{\mathbb{P}}^2 \mid x \in l\}$$

and the projection maps $p : \mathcal{F} \to \mathbb{P}^2$ and $q : \mathcal{F} \to \tilde{\mathbb{P}}^2$. According to [6, Theorem 1.3] $\mathcal{T}_A = p_* q^*(\mathcal{J}_Z(1))$ and the Fourier-Mukai transform $p_* q^*$ applied to the above exact sequence gives:

$$0 \longrightarrow \mathcal{T}_A \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b) \oplus \mathcal{O}_{\mathbb{P}^2}(-c) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \downarrow$$

$$0 \longleftarrow \mathcal{R}^1 p_* q^* \mathcal{O}_L(-a) \oplus \mathcal{R}^1 p_* q^* \mathcal{O}_L(-b) \oplus \mathcal{R}^1 p_* q^* \mathcal{O}_L(-c) \longleftarrow \mathcal{R}^1 p_* q^* \mathcal{J}_Z(1),$$

The sheaf $\mathcal{R}^1 p_* q^* \mathcal{J}_Z(1)$ is supported on the scheme of triple points defined by $\mathcal{A}$, while, the last sheaf of the sequence is supported on the vertices of the triangle $(ABC)$. Therefore the kernel of the last map is the structural sheaf of the set of triple inner points $T$. This implies that we have the following exact sequence

$$0 \longrightarrow \mathcal{T}_A \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b) \oplus \mathcal{O}_{\mathbb{P}^2}(-c) \longrightarrow \mathcal{J}_T(-1) \longrightarrow 0.$$

**Remark 2.2.** By the hypothesis on $\mathcal{A}$, the set $T$ is smooth. Its length is related to the second Chern class of $\mathcal{T}_A$, more precisely we have that $c_1(\mathcal{T}_A) = 1 - a - b - c$ and

$$c_2(\mathcal{T}_A) = \binom{a + b + c - 1}{2} - \binom{a}{2} - \binom{b}{2} - \binom{c}{2} - |T| = (ab + bc + ac - a - b - c + 1) - |T|.$$

First of all we prove that these arrangements, under conditions over $a, b$ and $c$, lead to stable bundles.

**Proposition 2.3.** Let us assume that $a \leq b \leq c$. Then, when $T = \emptyset$ we have

$$\text{H}^0(\mathcal{T}_A(a + b - 2)) = 0 \text{ and } \text{H}^0(\mathcal{T}_A(a + b - 1)) \neq 0.$$
Proof. If $T = \emptyset$, then we have a short exact sequence

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2}(c) \longrightarrow \mathcal{T}_A' \longrightarrow 0.
$$

Being $\mathcal{T}_A' = \mathcal{T}_A(a + b + c - 1)$, if we tensor by $\mathcal{O}_{\mathbb{P}^2}(-c)$ we obtain

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1 - c) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(a - c) \oplus \mathcal{O}_{\mathbb{P}^2}(b - c) \oplus \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{T}_A(a + b - 1) \longrightarrow 0,
$$

which proves the proposition. \hfill \Box

Corollary 2.4. Let us assume that $a \leq b \leq c$ and that $T = \emptyset$. Then, $\mathcal{T}_A$ is stable if and only if $a + b > c + 1$.

Proof. Under the hypothesis $T = \emptyset$, we have $H^0(\mathcal{T}_A(a + b - 1)) \neq 0$ and $H^0(\mathcal{T}_A(a + b - 2)) = 0$. Then $\mathcal{T}_A$ is stable if and only if $c_1(\mathcal{T}_A(a + b - 1)) > 0$. Since $c_1(\mathcal{T}_A(a + b - 1)) = a + b - c - 1$, this proves that $\mathcal{T}_A$ is stable if and only if $a + b > c + 1$. \hfill \Box

Theorem 2.5. The bundle $\mathcal{T}_A$ is free with exponents $(a + b - 1, c)$ if and only if $T$ is a complete intersection $(a - 1, b - 1)$.

Proof. Assume that $T$ is a complete intersection $(a - 1, b - 1)$. Since $T$ is the locus of inner triple points, the curve defined by $(a - 1)$ lines passing through $A$ contains $T$ and the curve defined by $(b - 1)$ lines passing through $B$ contains also $T$. These two curves generate the ideal defining $T$, which implies that the kernel of the last map of the exact sequence

$$
0 \longrightarrow \mathcal{T}_A \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b) \oplus \mathcal{O}_{\mathbb{P}^2}(-c) \longrightarrow \mathcal{J}_T(1) \longrightarrow 0,
$$

is $\mathcal{O}_{\mathbb{P}^2}(-a - b + 1) \oplus \mathcal{O}_{\mathbb{P}^2}(-c)$. This proves that $\mathcal{T}_A$ is free with exponents $(a + b - 1, c)$. Reciprocally, if $\mathcal{T}_A = \mathcal{O}_{\mathbb{P}^2}(-a - b + 1) \oplus \mathcal{O}_{\mathbb{P}^2}(-c)$ then we have $H^0(\mathcal{J}_T(a - 1)) \neq 0$ and $H^0(\mathcal{J}_T(a - 2)) = 0$. Moreover, the length of $T$, given by the numerical invariant of the above exact sequence, is $(a - 1)(b - 1)$ and this proves that $T$ is a complete intersection $(a - 1, b - 1)$. \hfill \Box

Remark 2.6. If $c \geq a + b - 1$ the splitting type of $\mathcal{T}_A$ along the lines joining $A$ to $C$ or $B$ to $C$ is fixed and it is $\mathcal{O}_I(1 - a - b) \oplus \mathcal{O}_I(-c)$; this is a consequence of [[14, Theorem 3.1]]. Therefore, under the condition $c \geq a + b - 1$ the arrangement is free if and only if

$$
\mathcal{T}_A = \mathcal{O}_{\mathbb{P}^2}(1 - a - b) \oplus \mathcal{O}_{\mathbb{P}^2}(-c).
$$

That’s why, if we want to describe all the possible splitting types of free triangular arrangements with $a + b + c$ lines $(a + 1$ by $A$, $b + 1$ by $B$ and $c + 1$ by $C$), we can assume that $c \leq a + b - 2$. Then the biggest possible gap $|a + b - 1 - c|$ is realized by the complete intersection $(a - 1)(b - 1)$, in particular it could be described with a roots-of-unity-arrangement (see the definition below): let $\rho$ be a primitive $(c - 1)$-root of unity, then the arrangement

$$
xyz \prod_{i=0}^{a-2} (x - \rho^i y) \prod_{j=0}^{b-2} (y - \rho^j z) \prod_{k=0}^{c-2} (x - \rho^k z) = 0
$$

is such an example of free triangular arrangement with exponents $(a + b - 1, c)$.  

5
Definition 2.7. A triangular arrangement $\mathcal{A}$ of $a + b + c$ lines, defined by an equation

$$xyz \prod_{i=1}^{a-1} (x - \alpha_i y) \prod_{j=1}^{b-1} (y - \beta_j z) \prod_{k=1}^{c-1} (x - \gamma_k z) = 0,$$

is called a \textit{Roots-of-Unity-Arrangement} (RUA for short) if the coefficients $\alpha_i, \beta_j$ and $\gamma_k$ can all be expressed as powers of a $n$-root of unity $\rho$.

The following two results are well known (they are described in particular in [11]).

Corollary 2.8. The arrangements $A_3^3(n)$ (sometimes called Ceva arrangements), defined by the equation $xyz(x^n - y^n)(y^n - z^n)(x^n - z^n) = 0$, are free with exponents $(n + 1, 2n + 1)$.

Proof. The set of inner triple points $T$ is a complete intersection of length $n^2$ defined by the ideal $(x^n - y^n, y^n - z^n)$. \hfill \square

Corollary 2.9. The arrangements $A_3^2(n), A_3^1(n)$ and $A_3^0(n)$ are obtained respectively from $A_3^3(n), A_3^2(n)$ and $A_3^1(n)$ by deleting one line between two vertices of the triangle. They are free with exponents respectively equal to $(n + 1, 2n - 1)$ and $(n + 1, 2n - 2)$.

Proof. Starting with $A_3^3(n)$ we remove the line $l = \{x = 0\}$. This gives the arrangement $A_3^2(n)$. The line $l$ contains $2n$ triple points ($n$ at each vertex, 0 elsewhere). Then we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_{P^2}(-n - 1) \oplus \mathcal{O}_{P^2}(-2n - 1) \longrightarrow \mathcal{T}_{A_3^2(n)} \longrightarrow \mathcal{O}_l(-2n) \longrightarrow 0.$$

By the Addition-Deletion theorem (see [11, Theorem 4.51]) we obtain

$$\mathcal{T}_{A_3^2(n)} = \mathcal{O}_{P^2}(-n - 1) \oplus \mathcal{O}_{P^2}(-2n).$$

With the same arguments, removing the line $y = 0$ containing $2n - 1$ triple points from $A_3^3(n)$ we obtain

$$\mathcal{T}_{A_3^1(n)} = \mathcal{O}_{P^2}(-n - 1) \oplus \mathcal{O}_{P^2}(-2n + 1),$$

and removing the line $z = 0$ containing $2n - 2$ triple points from $A_3^2(n)$ we obtain

$$\mathcal{T}_{A_3^1(n)} = \mathcal{O}_{P^2}(-n - 1) \oplus \mathcal{O}_{P^2}(-2n + 2).$$

\hfill \square

3 Roots-of-Unity-Arrangement

This section is dedicated to prove the following result.

Theorem 3.1. Given a triangular arrangement, it is always possible to find a RUA with the same combinatorics.
Let us consider the triangular arrangement defined by the following equations
\[
\begin{align*}
x &= 0 \\
x &= \alpha_i y \\
y &= 0 \\
y &= \beta_j z \\
z &= 0 \\
z &= \gamma_k x
\end{align*}
\]
where \(x = y = z = 0\) are the lines which compose the triangle, \(\alpha_i \neq 0, i = 1, \ldots, a - 1\), and \(\alpha_{i_1} \neq \alpha_{i_2}\) for \(i_1 \neq i_2\), and the same properties hold for the \(\beta_j\)'s and the \(\gamma_k\)'s, \(j = 1, \ldots, b - 1\) and \(k = 1, \ldots, c - 1\).

Observe that the existence of an inner triple point, defined by three lines \(x = \alpha_i y, y = \beta_j z\) and \(x = \gamma_k z\) is given by a relation of the following type
\[\alpha_i \beta_j \gamma_k = 1.\]

Therefore, we can translate the combinatorics of the arrangement in a family of equalities
\[
\alpha_{i_1} \beta_{j_1} \gamma_{k_1} = 1
\] (1)
for each \(i_1, j_1, k_1\) whose associated lines define an inner triple point of the arrangement, a family of inequalities
\[
\alpha_{i_2} \beta_{j_2} \gamma_{k_2} \neq 1
\] (2)
for each \(i_2, j_2, k_2\) whose associated lines do not define an inner triple point of the arrangement, and finally, the inequalities
\[
\alpha_{i_1} \neq \alpha_{i_2}, \alpha_{i_1} \neq 0, \beta_{j_1} \neq \beta_{j_2}, \beta_{j_1} \neq 0, \gamma_{k_1} \neq \gamma_{k_2}, \gamma_{k_1} \neq 0,
\] (3)
for each \(i_1, i_2 = 1, \ldots, a - 1\), with \(i_1 \neq i_2\), \(j_1, j_2 = 1, \ldots, b - 1\), with \(j_1 \neq j_2\), and \(k_1, k_2 = 1, \ldots, c - 1\), with \(k_1 \neq k_2\).

Our goal is to find solutions, or at least prove their existence, which satisfy all the previous relations and that can be expressed as various powers of a \(n\)-th root of the unity, for a given \(n\).

Let us consider a prime number \(p\) and one of its primitive roots \(\omega\), hence, working modulo \(p\), we can translate all the relations of type (1) as
\[
\omega^{v_{i_1}} \omega^{w_{j_1}} \omega^{t_{k_1}} \equiv 1 \pmod{p}
\]
or equivalently, as a family of linear equations
\[
v_{i_1} + w_{j_1} + t_{k_1} \equiv 0 \pmod{p - 1}.
\] (4)

We claim that we always have solutions, for any choice of \(p\), of the linear system defined by the family (4); indeed we have the following result.

**Lemma 3.2.** If we consider the linear system given by the equations
\[
v_{i_1} + w_{j_1} + t_{k_1} = 0,
\] (5)
i.e. considering all the linear forms of (4) in \(\mathbb{C}[v_{i_1}, w_{j_1}, t_{k_1}]\), we always have infinite solutions.
Proof. Let us consider the arrangement $A_3^3(n)$. It leads to the maximal system of equations $v_{i_1} + w_{j_1} + t_{k_1} = 0$, in the sense that it contains all the possible equations involving the variables. Writing down the corresponding square matrix we verify that its determinant vanishes. The system of equations (5) can be seen as a subsystem of a maximal one associated to $A_3^3(n)$, hence it will also have infinite solutions. 

Since the coefficients of the equations of (5) are integers we get in particular infinite integer solutions. Therefore, choosing well $p >> 0$, we get as many solutions of the system of linear congruences as we want.

Our next goal is to prove the following fact: consider the inequality

$$v + w + t \not\equiv 0 \pmod{p-1}$$

and add to our previous linear system, module $p$, the associated equality

$$v + w + t \equiv 0 \pmod{p-1}.$$  

Then, for an infinite number of $p$'s, either we have less solutions than before or the added condition is a consequence of the others.

Indeed, suppose that for a fixed $p$ we have exactly the same solutions, this means that the added condition is a linear combination of the previous ones, i.e.

$$v + w + t \equiv \sum_s \lambda_{s,p}(v_{i_s} + w_{j_s} + t_{k_s}) \pmod{p-1}$$

which is equivalent to have

$$\omega^v \omega^w \omega^t \equiv \prod_s (\omega^{v_{i_s}} \omega^{w_{j_s}} \omega^{t_{k_s}})^{\lambda_{s,p}} \pmod{p}$$

and therefore

$$\alpha \beta \gamma \equiv \prod_s (\alpha_{i_s} \beta_{j_s} \gamma_{k_s})^{\lambda_{s,p}} \pmod{p}.$$  

We conclude noticing that if we have the previous relation for an infinite set of prime numbers, then we must also have

$$\alpha \beta \gamma \equiv \prod_s (\alpha_{i_s} \beta_{j_s} \gamma_{k_s})^{\lambda_s},$$

which implies that the added condition is a consequence to the other equalities, which is a contradiction because of our hypothesis on the triple points.

Following the same reasoning as before for the relations expressed in the family (3), it is possible to find solutions that do also satisfy those inequalities.

Therefore, by what we have said, we can find a prime number $p$ (big if necessary) such that we can find $\alpha_i = \rho^{\alpha_i}$, $\beta_j = \rho^{\beta_j}$ and $\gamma_k = \rho^{\gamma_k}$, powers of the $(p-1)$-th unity root $\rho$, which satisfies all the conditions expressed in (1), (2) and (3).

Example 3.3. Consider the line arrangement in $\mathbb{P}^2$ defined by the curve

$$xyz \prod_{i=-1}^{2} (x - \alpha^i y) \prod_{i=0}^{3} ((y - \alpha^j z) (x - \alpha^j z)) = 0$$
with $\alpha$ a generic complex number, which has 12 inner triple points, i.e. excluding the three vertices of the triangle.

Following the proof of the previous result, in order to find a root arrangement with the same combinatorics as the given one, we have to solve the following linear system of equalities

\[
\begin{align*}
  x_1 + y_1 + z_1 &= 0 \\
  x_1 + y_2 + z_2 &= 0 \\
  x_1 + y_3 + z_3 &= 0 \\
  x_1 + y_4 + z_4 &= 0 \\
  x_2 + y_1 + z_2 &= 0 \\
  x_2 + y_2 + z_3 &= 0 \\
  x_2 + y_3 + z_4 &= 0 \\
  x_3 + y_1 + z_3 &= 0 \\
  x_3 + y_2 + z_4 &= 0 \\
  x_4 + y_1 + z_1 &= 0 \\
  x_4 + y_3 + z_2 &= 0 \\
  x_4 + y_4 + z_3 &= 0 \\
\end{align*}
\]

and which, moreover, does not satisfy any other relation $x_i + y_j + z_k = 0$ which is not present in the previous system, and, finally, such that

\[
x_p \neq x_q, \quad y_p \neq y_q, \quad z_p \neq z_q, \quad \text{for} \quad p, q = 1, 2, 3, 4, \quad p \neq q.
\]

The set of integers

\[
\begin{align*}
  x_1 &= 0 & y_1 &= 0 & z_1 &= 0 \\
  x_2 &= 1 & y_2 &= 1 & z_2 &= 5 \\
  x_3 &= 2 & y_3 &= 2 & z_3 &= 4 \\
  x_4 &= 5 & y_4 &= 3 & z_4 &= 3 \\
\end{align*}
\]

satisfy all the required conditions, and therefore, we can consider $\rho$ a 6-root of unity and the root arrangement with the same combinatorics as the starting one is given by

\[
xyz \prod_{i=2}^{5} ((x - \rho^iy)(y - \rho^iz)) \prod_{j=0}^{3} (x - \rho^jz) = 0
\]

4 Free arrangements obtained by deletion from the Ceva’s

4.1 General case: arrangements in $\overline{\text{Tr}(a,b,c)}$

Let us consider an arrangement $\mathcal{A} \in \overline{\text{Tr}(a,b,c)}$. It possesses $N = a + b + c$ lines. We will construct an explicit free arrangement for each possible splitting. According to Remark 2.6 we assume that $c \leq a + b - 2$.

Let us begin by describing the case of the maximal number of inner triple points which corresponds to the more unbalanced splitting. The maximal number of inner triple points is evidently $|T| = (a - 1)(b - 1)$ and we already know that this complete intersection corresponds to $\mathcal{T}_A = \mathcal{O}_{\mathbb{P}^2}(-c) \oplus \mathcal{O}_{\mathbb{P}^2}(1 - a - b)$ (see Theorem 2.5). As shown in the remark 2.6 such an arrangement can be obtained by considering a $c-1$ root of unity $\zeta$ generating
the multiplicative group of \((c - 1)\)-roots of unity, then choose \(\alpha_i = \zeta^i\) for \(i = 1, \ldots, a - 1\), \(\beta_j = \zeta^j\) for \(j = 1, \ldots, b - 1\) and \(\gamma_k = \zeta^k\) for \(k = 1, \ldots, c - 1\).

If \(c = a + b - 2\) there is no other splitting, since with these values, the most unbalanced possible splitting type is actually balanced. In this case an arrangement is free if and only if \(|T|\) is a complete intersection \((a - 1, b - 1)\).

Assume that \(c = a + b - 3\), then another splitting is possible, that is

\[
T_A = O_{\mathbb{P}^2}(-c - 1) \oplus O_{\mathbb{P}^2}(2 - a - b).
\]

In order to obtain it we begin with an arrangement \(A_3^n(c)\). It is free with exponents \((c + 1, 2(c + 1) - 1)\). We remove one inner line from each vertex such that the inner triple points removed are \(c + 1\), then \(c - 1\). By the Addition-Deletion theorem, the new arrangement is free with exponents \((c + 1, 2(c + 1) - 4) = (c + 1, 2(c - 1))\). Then we remove \(c - b\) lines from \(B\) and \(c - a\) lines from \(A\). Since we have removed the appropriate number of triple points, thanks to the Addition-Deletion theorem, the arrangement that we obtain is still free with exponents \((c + 1, a + b - 2)\).

The same process gives the splitting \(O_{\mathbb{P}^2}(-c - i) \oplus O_{\mathbb{P}^2}(1 + i - a - b)\) for any \(i \geq 1\): begin with the arrangement \(A_3^n(c + i - 1)\), remove \(i\) lines from each vertex in order to keep a free arrangement then remove \(c - b\) and \(c - a\) lines from \(B\) and \(A\).

4.2 Special case: arrangements in \(\text{Tr}(n, n, n)\)

Let us explain the above construction with more details for triangular arrangements of \(\text{Tr}(n, n, n)\). First of all we establish the link between the number of triple inner points and the splitting type of free arrangements.

**Lemma 4.1.** Let \(A\) be the arrangement of \(3n\) lines defined by an equation

\[
xyz \prod_{i=1}^{n-1} (x - \alpha_i y) \prod_{j=1}^{n-1} (y - \beta_j z) \prod_{k=1}^{n-1} (x - \gamma_k z) = 0.
\]

Let \(s\) be an integer verifying \(0 \leq s \leq \left\lceil \frac{n}{2} \right\rceil\) and \(A\) be a free arrangement with exponents \((s + n, 2n - s - 1)\). Then the number of inner triple points is \(|T| = (n - 1)^2 - s(n - 1 - s)\).

**Proof.** Consider the exact sequence defining \(T\) for a free triangular arrangement with exponents \((s + n, 2n - s - 1)\), which is given by

\[
0 \longrightarrow O_{\mathbb{P}^2}(-s - n) \oplus O_{\mathbb{P}^2}(s + 1 - 2n) \longrightarrow O_{\mathbb{P}^2}^3(-n) \longrightarrow J_T(-1) \longrightarrow 0.
\]

The second Chern class of \(J_T(-1)\) is exactly the length of \(T\).

\[\square\]

We propose now a construction of examples of free arrangements in \(\text{Tr}(n, n, n)\), with all the possible splitting types.

To construct these examples we delete the same number of inner lines from each vertex of a Ceva arrangement. Let us begin with the Ceva arrangement \(A_3^n(n - 1)\). The logarithmic bundle associated to this arrangement is \(T_0 = O_{\mathbb{P}^2}(-n) \oplus O_{\mathbb{P}^2}(1 - 2n)\) and its number \(|T_0|\) of triple inner points is \((n - 1)^2\) which is the maximal possible number.
To find an arrangement with the next exponents, which are \((n + 1, 2n - 2)\), and with \(|T_1| = (n - 1)^2 - (n - 2)\) triple inner points we remove one inner line from each vertex \(A, B, C\) from the arrangement \(A^3_2(n)\) which is free with exponents \((n + 1, 2n + 1)\). The inner line removed through the vertex \((1,0,0)\) contains \(n\) triple inner points of \(A^3_2(n)\) which implies, by the Addition-Deletion theorem, that the new arrangement \(B\) is free with exponents \((n + 1, 2n)\); the removed line through the vertex \((0,1,0)\) contains \(n - 1\) triple inner points of \(B\) which implies, by the Addition-Deletion theorem, that the new arrangement \(C\) is free with exponents \((n + 1, 2n - 1)\), as wanted. Then we choose to remove a line through the vertex \((0,0,1)\) that contains \(n - 2\) triple inner points of \(C\) (and not \((n - 1)\) which is possible if the line passes through the intersection point of the two previous removed lines) which implies by the addition-deletion theorem that the new arrangement \(T_1\) is free with exponents \((n + 1, 2n - 2)\).

This process can be applied to find also the other splittings. More precisely, to obtain \(T_s\) we begin with \(A^3_2(n - 1 + s)\) and we remove the lines \(x = \zeta^i z, y = \zeta^{n-1+s-i} z\) and \(x = \zeta^i y\) for \(0 \leq i \leq s\), with \(s \leq \left\lfloor \frac{n}{2} \right\rfloor\). The reduction is done by choosing alternatively lines passing through \(A, B, C\) such that the the number of triple points has the expected value in order to find a new free bundle. The computation gives the number of remaining inner triple points:

\[
(n - 1 + s)^2 - \sum_{i=1}^{s} [(n - (2i - 1) + s) + (n - 2i + s) + (n - (2i + 1) + s)]
\]

and this number is equal to

\[
|T_s| = (n - 1)^2 - s(n - 1 - s).
\]

**Example 4.2.** Let us consider the case of 15 lines (6 by each vertex). Then the splitting types allowed are \(T_A = \mathcal{O}_{p^2}(-5) \oplus \mathcal{O}_{p^2}(-9)\), which correspond to 16 inner triple points, \(T_A = \mathcal{O}_{p^2}(-6) \oplus \mathcal{O}_{p^2}(-8)\), which correspond to 13 inner triple points, and \(T_A = \mathcal{O}_{p^2}(-7) \oplus \mathcal{O}_{p^2}(-7)\), which correspond to 12 inner triple points. Since we have an exact sequence

\[
0 \longrightarrow T_A \longrightarrow \mathcal{O}_{p^2}^3(-5) \longrightarrow J_T(-1) \longrightarrow 0
\]

these three splitting are the only possible ones (\(H^0(T_A(4)) = 0\)).

— To obtain the case \(|T| = 16\) triple inner points and \(T_A = \mathcal{O}_{p^2}(-5) \oplus \mathcal{O}_{p^2}(-9)\) there is only one possibility. Indeed a syzygy of degree 0 means that the curve \(\prod_{0 \leq i \leq 3}(x - \alpha_i y) = 0\) belongs to the pencil generated by \(\prod_{0 \leq i \leq 3}(x - \gamma_i z) = 0\) and \(\prod_{0 \leq i \leq 3}(y - \beta_i z) = 0\). This arises only when this system is equivalent by a linear change of coordinates to

\[
[x^4 - y^4, x^4 - z^4, y^4 - z^4].
\]

— To obtain the case \(|T| = 13\) triple inner points and \(T_A = \mathcal{O}_{p^2}(-6) \oplus \mathcal{O}_{p^2}(-8)\) we begin with the roots of unity of order 5. Then it is possible to remove three lines with the required number of triple points. The three associated partitions along the three directions are always \(13 = 3 + 3 + 3 + 4\).

Another method to prove the freeness is also to remark that this inductive process gives a syzygy. In this case we find a syzygy \((P, Q, R)\) of degree 1 verifying

\[
(x - z)P - (y - z)Q + (x - \zeta^4 y).R = 0
\]
where \( P = \frac{y^5 - z^5}{(x-z)} \), \( Q = \frac{y^5 - z^5}{(y-z)} \) and \( R = \frac{y^5 - z^5}{(x-z^5)} \).

---

To obtain the case \(|T| = 12\) triple inner points and \( T_A = \mathcal{O}_{P^2}(-7) \oplus \mathcal{O}_{P^2}(-7) \) we begin with the roots of unity of order 6. Then we remove six lines with the required number of triple points. The three associated partitions along the three directions are always \( 12 = 2 + 3 + 4 + 3 \). Observe that by [14, Theorem 3.1] the splitting type on the line \( l \) containing only 2 inner triple points is \( \mathcal{O}_l(-7) \oplus \mathcal{O}_l(-7) \). Since its second Chern class is 49, \( T_A \) is free by [5, Corollary 2.12]. Another method to prove the freeness is to remark that this inductive process gives a syzygy. In this case we find a syzygy \((P,Q,R)\) of degree 2 verifying

\[
(x - z)(x - \zeta z)P - (y - z)(y - \zeta z)Q + (x - \zeta^5 y)(x - \zeta^4 y)R = 0
\]

where \( P = \frac{x^6 - z^6}{(x-z)(x-\zeta z)} \), \( Q = \frac{y^6 - z^6}{(y-z)(y-\zeta z)} \) and \( R = \frac{x^6 - z^6}{(x-\zeta^5 y)(x-\zeta^4 y)} \).
4.3 Free arrangements obtained by removing a “big” number of lines from a Ceva

In the examples described above we have removed at most \( i \leq \left\lceil \frac{n}{2} \right\rceil \) lines from each vertex of a Ceva arrangement \( \mathcal{A}_3^3(n-1+i) \). One can ask what kind of free arrangement could be obtained by removing more lines?

Removing inner lines from a Ceva arrangement we find two triangular arrangements, \( \mathcal{A} \) consisting in the remaining lines and \( \mathcal{B} \) consisting in the deleted lines. More precisely, when we remove \( k \) lines from each three vertices with equations \( f_A = 0, f_B = 0, f_C = 0 \) from a Ceva arrangement \( \mathcal{A}_3^3(N-1) \), these lines form a subarrangement \( \mathcal{B} \) defined by the equation \( f_A f_B f_C = 0 \) and this arrangement possesses, outside the vertices, a number \( T_{\text{rem}} \) of triple inner points. This operation induces a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{T} & \to & \mathcal{O}_{\mathbb{P}^2}(-N+k) & \phi & \mathcal{J}_T(-1) & \to & 0 \\
& & & & & \downarrow & & & \\
0 & \to & \mathcal{J}_T(4k-3N+1) & \to & \mathcal{F} & \to & \mathcal{J}_T(-1) & \to & 0
\end{array}
\]

where \( \mathcal{T}_A \) is the logarithmic bundle associated to the triangular arrangement \( \mathcal{A} \) consisting in \( 3(N-k) \) lines such that \( N+1-k \) of them pass through each vertex, \( \Gamma \) is the zero locus of the section of \( \mathcal{T}_A \) induced by the syzygies \( (f_A, f_B, f_C) \), \( T \) is the set of triple inner points of \( \mathcal{A} \) and \( \text{Sing}(\mathcal{F}) = T_{\text{rem}} \). Since \( \mathcal{T}_A \) comes from the Ceva arrangement \( \mathcal{A}_3^3(N-1) \) we have of course

\[
\Phi = \left\{ \frac{x^{N-1} - y^{N-1}}{f_A}, \frac{y^{N-1} - z^{N-1}}{f_B}, \frac{x^{N-1} - z^{N-1}}{f_C} \right\}.
\]

Moreover, since \( T_{\text{rem}} \) and \( T \) are disjoint sets of simple points, we conclude that \( \Gamma = T_{\text{rem}} \). Emptyness of \( T_{\text{rem}} \) implies that \( \Gamma \) is empty. Then \( H_1(\mathcal{T}_A) = 0 \) which implies by Horrocks’ criterion \([9]\) that \( \mathcal{T}_A \) is free with exponents \( (k, 4k-3N+1) \). Let us emphasize this fact:

**Theorem 4.3.** Let \( \mathcal{A} \) a triangular arrangement, obtained from a Ceva arrangement by removing a certain number of lines which define the complementary triangular arrangement \( \mathcal{B} \). If \( \mathcal{B} \) does not have inner triple point, then \( \mathcal{A} \) is free.

**Remark 4.4.** If \( T_{\text{rem}} = \emptyset \), the arrangement \( \mathcal{A} \) is free with exponents \( (k, 4k-3N+1) \). This arises only if we remove a “small” number of lines from each vertex. If we remove too many lines then, necessarily \( T_{\text{rem}} \neq \emptyset \) and we cannot conclude directly that \( \mathcal{T} \) will be free. However we will be able to produce a free bundle, even if we remove a “big” number of lines if we remove lines in order to minimize \( T_{\text{rem}} \). Let us precise what we mean by “small” and “big” number of removed lines:

1. First of all we remove a “small” number of lines \( i \leq k \) to each vertex of a Ceva arrangement \( \mathcal{A}_3^3(n-1+i) = \mathcal{A}_3^3(2k+i) \). Then we can remove \( i \) lines from each vertex without passing through \( T_{\text{rem}} \). In other words the syzygy of degree \( i \) does
not vanish and the logarithmic bundle associated to this arrangement obtained by deletion from $\mathbb{A}_3^2(n-1)$ is free with exponents $(n+i, 2n-1-i)$ for $i = 0, \ldots, \frac{n-1}{2} = k$. Notice that in this case, freeness is obtained if and only if the diagonal lines do not contain points of $T_{\text{rem}}$; indeed if such a point is on a removed diagonal then it is not in $T$, but the syzygy (corresponding to the removed lines) vanishes at this point and it arises as the zero locus of a global section of the bundle. Considering its first Chern class, it cannot be free.

2. In the second case we remove a “big” number of lines $k+i$ with $i \geq 1$ to each vertex of a Ceva arrangement $\mathbb{A}_3^2(n-1+k+i) = \mathbb{A}_3^2(3k+i)$. We have a square with $(3k+i)^2$ intersection points. We remove $k+i$ vertical lines and $k+i$ horizontal lines. They meet in $(k+i)^2$ points, we denote these points by $T_{\text{rem}}$. Of course the other $2k$ horizontal and vertical lines meet in $4k^2$ points. Now we remove one by one the $k+i$ diagonals in order to minimize the number of points in $T_{\text{rem}}$ in the removed diagonals. It implies that, if a diagonal contains $j$ points among $T_{\text{rem}}$, we have removed $(3k+i-2(k+i)+j)$ triple points (among the $4k^2$). Then when we remove $k+i$ diagonal lines we have removed a number of triple points from $4k^2$ that is:

$$\sum_{s=1,\ldots,k+i} (k+j_s-i) = k^2 - i^2 + \sum_{s=1,\ldots,k+i} j_s, \text{ with } 0 \leq j_1 \leq \cdots \leq j_{k+i}.$$ 

It remains $|T| = 3k^2 + i^2 - \sum_{s=1,\ldots,k+i} j_s$ triple points. If $i = k - t - 1$ then

$$j_1 = \cdots = j_{t-1} = 0, j_s = j_{s+1} = 1, j_{s+2} = j_{s+3} = 2, \ldots$$

Then we obtain

$$|T| = 3k^2 \text{ because } i^2 = 2(1 + 2 + \cdots + i) - i.$$ 

The arrangements obtained in this way are free, again because of the Addition-Deletion theorem. Notice that in the second case we will always get balanced exponents, i.e. exponents with the minimum possible gap.

Figure 1: With $T_{\text{rem}} = 1$
5 Free arrangements in the non complete triangle

All the free triangular arrangements coming from Ceva’s ones by deletion in the above construction contain the three sides of the triangle, i.e. the three lines joining the three vertices. Indeed, if the arrangement does not contain one of these three sides, then it is free if and only if it is $A_{1}\n\(n\), $A_{2}\n\(n\) or $A_{3}\n\(n\).

**Proposition 5.1.**
- The only free triangular arrangement of $3n−1$ lines passing through three points with $n−1$ inner lines through each vertex plus two sides is $A_{3}\n\(n\).
- The only free triangular arrangement of $3n−2$ lines passing through three points with $n−1$ inner lines through each vertex plus one side is $A_{2}\n\(n\).
- The only free arrangement of $3n−3$ lines passing through three points with $n−1$ inner lines through each vertex and no side is $A_{3}\n\(n\).

**Proof.** Let us consider an arrangement $A∪L∈\mathbb{R}(n−1,n−1,n−1)$ of $3n$ lines $(n+1$ from each vertex) where $L$ is a line joining two vertices. Let us denote by $|t_\A|$ and $|t_\A∪L|$ the number of triple points of $A$ and $A∪L$ counted with multiplicities. Then we have

- $|t_\A| = |T| + \binom{n}{2} + \binom{n−3}{2} + \binom{n−1}{2},$
- $|t_\A∪L| = |T| + \binom{n}{2} + \binom{n−2}{2} + \binom{n−1}{2}.$

Indeed the number of inner triple points $|T|$ is the same. Then their difference is $2n−2$ and we have an exact sequence (see [7, Proposition 5.1])

$$0 → \mathcal{T}_{A∪L} → \mathcal{T}_{A} → \mathcal{O}_{L}(2−2n) → 0.$$ 

So the arrangement $A$ is free if and only if $\mathcal{T}_{A} = \mathcal{O}_{p^2}(−2n) ⊕ \mathcal{O}_{p^2}(2−2n);$ this implies that $\mathcal{T}_{A∪L} = \mathcal{O}_{p^2}(−n) ⊕ \mathcal{O}_{p^2}(1−2n)$ in other words $A∪L$ is the Ceva arrangement $A_{3}\n\(n−1\)$ and $A$ is defined by the equation $yz(x^{n−1}−y^{n−1})(y^{n−1}−z^{n−1})(x^{n−1}−z^{n−1}) = 0.$

Assume now that $A′∪L$ is an arrangement of $3n−1$ lines passing through three points with $n−1$ inner lines through each vertex plus two sides, where $L$ is one of these sides. Let us denote by $|t_\A′|$ and $|t_\A′∪L|$ the number of triple points of $A′$ and $A′∪L$ counted with multiplicities. Then we have

- $|t_\A′| = |T| + \binom{n−1}{2} + \binom{n−3}{2} + \binom{n−2}{2},$
- $|t_\A′∪L| = |T| + \binom{n}{2} + \binom{n−3}{2} + \binom{n−1}{2}.$

Indeed the number of inner triple points $|T|$ is the same. Then their difference is $2n−3$ and we have an exact sequence (see [7, Proposition 5.1])

$$0 → \mathcal{T}_{A′∪L} → \mathcal{T}_{A′} → \mathcal{O}_{L}(3−2n) → 0.$$ 

The arrangement $A′$ is free if and only if $\mathcal{T}_{A′} = \mathcal{O}_{p^2}(−n) ⊕ \mathcal{O}_{p^2}(3−2n)$ and this implies that $\mathcal{T}_{A′∪L} = \mathcal{O}_{p^2}(−n) ⊕ \mathcal{O}_{p^2}(2−2n);$ in other words $A′∪L$ is the previous arrangement defined by the equation $yz(x^{n−1}−y^{n−1})(y^{n−1}−z^{n−1})(x^{n−1}−z^{n−1}) = 0$ and $A′$ is defined by an equation $z(x^{n−1}−y^{n−1})(y^{n−1}−z^{n−1})(x^{n−1}−z^{n−1}) = 0.$

15
Assume now that \( A'' \cup L \) is an arrangement of \( 3n - 2 \) lines passing through three points with \( n - 1 \) inner lines through each vertex plus one side \( L \). Let us denote by \( |t_{A''}| \) and \( |t_{A'' \cup L}| \) the number of triple points of \( A'' \) and \( A'' \cup L \) counted with multiplicities. Then we have

- \( |t_{A''}| = |T| + (n-2) + (n-2) + (n-2) \),
- \( |t_{A'' \cup L}| = |T| + (n-2) + (n-1) + (n-1) \).

Indeed the number of inner triple points \(|T|\) is the same. Then their difference is \( 2n - 4 \) and we have an exact sequence (see [7, Proposition 5.1])

\[
0 \longrightarrow \mathcal{T}_{A'' \cup L} \longrightarrow \mathcal{T}_{A''} \longrightarrow \mathcal{O}_L(4-2n) \longrightarrow 0.
\]

Then \( A'' \) is free if and only if \( \mathcal{T}_{A''} = \mathcal{O}_{\mathbb{P}^2}(-n) \oplus \mathcal{O}_{\mathbb{P}^2}(4-2n) \) and this implies that \( \mathcal{T}_{A'' \cup L} = \mathcal{O}_{\mathbb{P}^2}(-n) \oplus \mathcal{O}_{\mathbb{P}^2}(3-2n) \) in other words \( A'' \cup L \) is the previous arrangement defined by the equation \( z(x^{n-1} - y^{n-1})(y^{n-1} - z^{n-1})(x^{n-1} - z^{n-1}) = 0 \) and \( A'' \) is defined by an equation \( (x^{n-1} - y^{n-1})(y^{n-1} - z^{n-1})(x^{n-1} - z^{n-1}) = 0 \).

\[ \square \]

6 Weak combinatorics

The combinatorics of \( A \) is determined by the set \( L(A) \) of all the intersections of lines in \( A \). There is a partial order on this set corresponding to the inclusion of points \((L_1 \cap L_2 \subset L_1 \text{ for instance})\) in lines. Two line arrangements \( A_0 \) and \( A_1 \) have the same combinatorics if and only if there is a bijection between \( L(A_0) \) and \( L(A_1) \) preserving the partial order. In [11] Terao conjectures that if two arrangements have the same combinatorics and one of them is free then the other one is also free. This problem posed in any dimension and on any field is still open even on the projective plane and seems far from being proved, probably because few free arrangements are known.

In this section we will show that if we only suppose the weak combinatorics hypothesis, the conjecture does not hold. Indeed, we get the following result.

**Theorem 6.1.** There exist pairs of arrangements possessing the same weak combinatorics such that one is free and the other is not.

**Proof.** We prove it by describing an example. We will explain next how to produce a family of examples of the same kind.

We will construct two triangular arrangements of 15 lines \( A_0 \), which will be free with exponents \((7, 7)\), and \( A_1 \), which won’t be free, in \( \text{Tr}(5, 5, 5) \) with the same following numbers of multiple points \( t_3 = 12, t_4 = t_5 = 0, t_6 = 3 \) and \( t_i = 0 \) for \( i > 6 \) (the number of double points is a given by the combinatorial formula \((15 \choose 2) = t_2 + 3t_3 + (6 \choose 2)t_6 \).

— Let us construct \( A_0 \): it is obtained by removing the six lines \( x = z, x = \zeta z, y = z, y = \zeta z, x = \zeta^2 y \) and \( x = \zeta^4 y \) from the Ceva arrangement \( xyz(x^6 - y^6)(y^6 - z^6)(x^6 - z^6) = 0 \) as represented in the following picture:
This arrangement is free because the syzygy of degree 2, that is
\[
\psi = [(x - z)(x - \zeta z), (y - z)(y - \zeta z), (x - \zeta^2 y)(x - \zeta^4 y)],
\]
has no zero. Indeed this syzygy gives for
\[
\begin{array}{cccc}
0 & \longrightarrow & T_{A_0} & \longrightarrow & O_{P^2}^3(-5) & \phi & J_T(-1) & \longrightarrow & 0 \\
\end{array}
\]
a non zero section \(O_{P^2}(-7) \longrightarrow T_{A_0}\), being
\[
\phi = \left[\frac{x^6 - z^6}{(x - z)(x - \zeta z)}, \frac{y^6 - z^6}{(y - z)(y - \zeta z)}, \frac{x^6 - y^6}{(x - \zeta^2 y)(x - \zeta^4 y)}\right].
\]
This induces a commutative diagram:
\[
\begin{array}{cccc}
O_{P^2}(-7) & \approx & O_{P^2}(-7) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & T_{A_0} & \longrightarrow & O_{P^2}^3(-5) & \phi & J_T(-1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & J_T(-7) & \longrightarrow & F & \longrightarrow & J_T(-1) & \longrightarrow & 0,
\end{array}
\]
where the singular locus of the rank two sheaf \(F\) is the zero set of \(\phi\). Since this zero set is empty, \(F\) is a vector bundle. That proves \(\text{Ext}^1(F, O_{P^2}) = 0\) and then \(\Gamma = \emptyset\).

Another argument can be used to establish the freeness: the 12 inner triple points are distributed as a partition \(3 + 3 + 3 + 3\) along the vertical lines, \(3 + 3 + 3 + 3\) along the horizontal lines but \(2 + 3 + 4 + 3\) along the diagonal (this means that this example is very closed to have the same combinatorics than the non free case: indeed in the following example we’ll see that the partition along vertical, horizontal and diagonal lines is always \(3 + 3 + 3 + 3\)); thanks to [13, Theorem 3.1], the bundle restricted to the line containing only 2 inner triple point has the splitting \((7, 7)\) which proves the freeness according to [5, Corollary 2.12].

— Let us construct now \(A_1\): it is obtained by removing the three lines \(x = z, y = z, x = y\) from the Ceva arrangement \(xyz(x^5 - y^5)(y^5 - z^5)(x^5 - z^5) = 0\) as it appears on the picture:
Beginning with $A_3^3(5)$ which is free with exponents $(6,11)$ and removing the first line we obtain, by Addition-Deletion theorem, a free bundle with exponents $(6,10)$. Removing the second line we find again a free arrangement with exponents $(6,9)$. Removing the third line, we don’t find a free bundle (with splitting $(7,7)$) but a nearly free bundle with generic splitting $(6,8)$. The jumping point is the intersection point of the three removed lines. The three partitions appearing along the horizontal, vertical and diagonal lines are $12 = 3 + 3 + 3 + 3$.

Let us make it more explicit. We found a syzygy of degree 1, which is

$$\psi = [x - z, y - z, x - y],$$

and which induces a non zero section $O_{\mathbb{P}^2}(-6) \to T$ where

$$0 \to \mathcal{T}_{A_1} \to O_{\mathbb{P}^2}^3(-5) \xrightarrow{\phi} J_T(-1) \to 0$$

and

$$\phi = \left[ \frac{x^5 - z^5}{x - z}, \frac{y^5 - z^5}{y - z}, \frac{x^5 - y^5}{x - y} \right].$$

This syzygy admits a common zero $p = (1,1,1)$ and induces a commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{O}_{\mathbb{P}^2}(-6) & \cong & \mathcal{O}_{\mathbb{P}^2}(-6) \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{T}_{A_1} & \to & O_{\mathbb{P}^2}^3(-5) & \xrightarrow{\phi} & J_T(-1) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & J_T(-8) & \to & \mathcal{F} & \to & J_T(-1) & \to & 0,
\end{array}$$

where the singular locus of the rank two sheaf $\mathcal{F}$ is the zero set of $p$. Since $p \notin T$ then $p \in \Gamma$ (actually $p = \Gamma$) and $\mathcal{T}_{A_1}$ cannot be free.

This example proves that the arrangement consisting in these 15 lines is Nearly free (defined in [3] and studied by the authors in [10]) with the same weak-combinatorics (same
numbers \( t_2, t_3, \ldots \) and quite the same combinatorics (only one partition along the diagonals differs) than the one described just before. This shows that we cannot replace the term combinatorics by weak-combinatorics in the hypothesis of Terao's conjecture.

\[ \textbf{Remark 6.2.} \] In the famous Ziegler's example of two arrangements (9 lines with 6 triple points) with the same combinatorics but with different free resolutions, the situation was explained by the existence of a smooth conic containing the 6 triple points. Here the situation can be geometrically explained by the existence of a cubic containing the 12 inner triple points. Indeed, since the bundle \( T_{A_1} \) described in the previous example is the kernel of the following exact sequence

\[
0 \longrightarrow T_{A_1} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-5) \longrightarrow J_T(-1) \longrightarrow 0,
\]

(where \( |T| = 12 \)) it gives \( H^0(J_T(3)) = H^1(T_{A_1}(4)) \). Moreover, the following non zero global section

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-6) \longrightarrow T_{A_1} \longrightarrow J_p(-8) \longrightarrow 0,
\]

where \( p \) is the jumping point associated to the Nearly Free arrangement, proves that

\[
h^1(T_{A_1}(4)) = h^1(J_p(-4)) = h^0(\mathcal{O}_p) = 1.
\]

\[ \textbf{Remark 6.3.} \] It is possible to generalize the described examples, and find a family of them in the following way: consider triangular arrangements in the family \( \text{Tr}(n, n, n) \). The multiplicity of each vertex is \( n + 1 \). Assume that \( n = 2k + 1 \). For arrangements of this family, the maximal possible number for the inner triple points is \( T = 4k^2 \) (then the arrangement is \( A_3^2(n-1) \)). The minimal number \( T \) is of course 0 but if we want to consider free arrangements, this minimal number is \( T = 3k^2 \). This minimal number corresponds to the balanced free arrangement with exponents \( (3k+1, 3k+1) \). We construct a nearly free arrangement with generic splitting \( O_l(-3k) \oplus O_l(-2-3k) \) and \( 3k^2 \) triple inner points in the same family \( \text{Tr}(n, n, n) \). This generic splitting is also the one of the free arrangement with \( 3k^2 + 1 \) triple inner points constructed by removing \( 3(k-1) \) lines to the arrangement \[ xyz(x^{3k-1} - y^{3k-1})(y^{3k-1} - z^{3k-1})(x^{3k-1} - z^{3k-1}) = 0 \]. In this example we do the same construction except for the last removed line. Indeed, instead of removing a line with \( k-1 \) triple inner points we remove a line with \( k \) triple points. It is always possible by choosing a line of the third direction passing through a intersection point of the previous removing lines in the two other directions.

\section{About Terao’s conjecture}

We described examples of free triangular arrangements by deleting lines in a Ceva arrangement which pass in every step through the minimal number of triple inner points. If all combinatorics of free triangular arrangements could be described in this way then Terao’s conjecture would be proved for this family. Unfortunately, it is not possible yet to state that any free arrangement is constructed in this way, i.e. always removing in each step a line the minimum possible number of triple points.

We conclude proposing the following conjecture, which would imply Terao’s conjecture for triangular arrangement.
Conjecture 1. Let \( A \) be a roots of unity arrangement, then \( A \) is free if and only if is obtained from a Ceva one, deleting in each step a line with the minimum possible triple points on it.

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