On the squeezed states for $n$ observables

D. A. Trifonov
Institute of Nuclear Research, 72 Tzarigradsko Chaussée, 1784 Sofia, Bulgaria

Abstract

Three basic properties (eigenstate, orbit and intelligence) of the canonical squeezed states (SS) are extended to the case of arbitrary $n$ observables. The SS for $n$ observables $X_i$ can be constructed as eigenstates of their linear complex combinations or as states which minimize the Robertson uncertainty relation. When $X_i$ close a Lie algebra $L$ the generalized SS could also be introduced as orbit of Aut($L^C$). It is shown that for the nilpotent algebra $h_N$ the three generalizations are equivalent.

For the simple $su(1,1)$ the family of eigenstates of $uK_- + vK_+ (K_\pm$ being lowering and raising operators) is a family of ideal $K_1$-$K_2$ SS, but it cannot be represented as an Aut($su^C(1,1)$) orbit although the $SU(1,1)$ group related coherent states (CS) with symmetry are contained in it.

Eigenstates $|z, u, v, w; k\rangle$ of general combination $uK_- + vK_+ + wK_3$ of the three generators $K_j$ of $SU(1,1)$ in the representations with Bargman index $k = 1/2, 1, \ldots$, and $k = 1/4, 3/4$ are constructed and discussed in greater detail. These are ideal SS for $K_{1,2,3}$. In the case of the one mode realization of $su(1,1)$ the nonclassical properties (sub-Poissonian statistics, quadrature squeezing) of the generalized even CS $|z, u, v; +\rangle$ are demonstrated. The states $|z, u, v, w; k = 1/4, 3/4\rangle$ can exhibit strong both linear and quadratic squeezing.

1 Introduction

In the last decade or so a considerable attention was paid in the literature to the squeezed states (SS), especially to the SS in quantum optics [1]. In the one mode amplitude SS the variance of one of the two quadratures $q, p$ of boson/photon annihilation operator $a, a = (q + ip)/\sqrt{2}$, can be reduced below its value of $1/\sqrt{2}$ in the ground state $|0\rangle$. We shall call these SS the $q$-$p$ SS. Most familiar one mode $q$-$p$ SS are the Stoler $\zeta$-classes $|z, \zeta\rangle$ [2], the Yuen two photon coherent states (CS) $|z, \mu, \nu\rangle$ [2] and the Dodonov et. al. correlated states $|z, u, v\rangle$ [3]. These three types of SS are equivalent [4] and should be called one mode canonical SS (CSS) or standard SS. SS for other pairs of observables are

+ e-mail: dtrif@inrne.acad.bg
also considered in the literature \[5, 6, 7\]. The canonical SS can be defined in the following three equivalent ways \[4\]:

a) as eigenstates of complex combination of \( q \) and \( p \):
\[
|\text{CSS}\rangle = z|\text{CSS}\rangle[^2, 8];
\]

b) as displaced and squeezed vacuum:
\[
|\text{CSS}\rangle = S(\zeta)D(z)|0\rangle, \text{ where } D(z) = \exp(za^\dagger - z^*a),
\]
and \( S(\zeta) = \exp[(\zeta a^2 - \zeta^*a^2)/2] \) is the (canonical or ordinary) squeeze operator \[1, 2\];

c) as states which minimize the Schrödinger inequality (\( \Delta X \) is the variance of \( X \) and \( \Delta XY \) is the covariance of \( X \) and \( Y \))
\[
\Delta^2 X \Delta^2 Y - \Delta^2 XY \geq \frac{1}{4}|\langle [X,Y]\rangle|^2 \tag{1}
\]

for \( X = q \) and \( Y = p \[^3, 4\]. We note that the three equivalent definitions of one mode coherent states (CS) \( |z\rangle[^3] \) are particular cases of the above three definitions of CSS, namely \( \beta_1 = i\beta_2 = i/\sqrt{2} \) in a), \( \zeta = 0 \) in b) and \( \Delta^2qp = 0 \) in c). Here \( q \) and \( p \) are dimensionless quadratures of \( a \). Eigenstates of complex combination of two observables \( X \) and \( Y \) are also called \( X-Y \) SS or \( X-Y \) Schrödinger intelligent states \[5\] (following \[3\] they could also be called Schrödinger correlated states). The term ”intelligent states” was introduced in \[10\] on the example of spin states, which minimize the Heisenberg inequality.

The aim of this paper is to introduce SS for several observables \( X_i, i = 1, ..., n, \) and to construct and analyze SS for the Hermitian generators \( K_1, K_2, K_3 \) of the group \( SU(1,1) \). The idea is to generalize the above three basic properties of the CSS to the case of \( n \) observables \( X_i \). These possibilities stem from the observations that the product of the canonical squeeze and displacement operators \( S(\zeta)D(z) \) belongs \[^11\] to the group of automorphisms \[^12\] of the Heisenberg-Weyl algebra \( h_1 \) (spanned by \( p, q, \) and the identity) and that the Schrödinger inequality for two observables is a particular case of Robertson uncertainty relation for \( n \) observables \[^13\].

The paper is organized as follows. In section 2 we consider the possible extensions of the definitions (a), (b) and (c) to the case on \( n \) arbitrary observables and discuss the problem of equivalence of generalized definitions. The generalizations are based on the Robertson uncertainty relation \[^13\] for \( n \) operators and on the group of automorphisms \( \text{Aut}(L^C) \) of the corresponding complexified Lie algebra \( L^C[^12\). We find that the first way of generalization (the eigenstate way) could be considered as most general one. It is noted that not every continuous family of eigenstates of combinations of \( X_j \) can exhibit squeezing of \( X_j \). A sufficient condition for ideal squeezing (i.e., arbitrarily strong squeezing) in such eigenstates is that of eq. \[^3\].

In section 3 we consider some examples of generalized SS for \( n \) observables which are of current interest in physical literature, especially in quantum optics: the quadratures \( p_\nu, q_\nu \) of \( N \) boson destruction operators \( a_\nu, \nu = 1, 2, ..., N \) (generators of the nilpotent
Heisenberg-Weyl group $H_N$ and the quasi-spin components $K_i$ (generators of the simple $SU(1, 1)$). Here the eigenstates $|z, u, v, w; k\rangle$ of general complex combination of $K_j$ are constructed explicitly (see also [11, 14]) and shown to be ideal SS for the generators $K_{1,2,3}$. The nonclassical properties of $|z, u, v, w; k\rangle$ are analyzed in the quadratic one mode bosonic representation. It is demonstrated that the generalized even CS $|z, u, v; +\rangle$, which are eigenstates of complex combination of $a^2$ and $a^{\dagger 2}$ ($[u a^2/2 + v a^{\dagger 2}/2]|z, u, v; \pm\rangle = z|z, u, v; \pm\rangle$) [13], do exhibit sub-Poissonian photon statistics (Fig. 2a) and strong linear and quadratic quadrature squeezing (Fig. 1a) [The variance of a quadrature component of $a$ or $a^2$ is said to be squeezed if it is less than its value in the ground state $|0\rangle$]. Moreover, there are states from this subfamily, which can exhibit linear and quadratic squeezing simultaneously (joint squeezing). These joint SS, and all $|z, u, v; \pm\rangle$ as well, can be generated using the scheme, which is a modification of the recently proposed scheme of Brif and Mann [16].

We note that the first example of generalized SS (section 3) for the $n$ canonical operators $p_\nu, q_\nu$ is most symmetric: as in the one mode case, here the three generalized definitions are equivalent. The second example of generalized SS (for the generators of $SU(1,1)$) does not possess such symmetry: examples of continuous families of eigenstates of $su(1,1)$ operators (generally called $su(1,1)$ algebraic CS [11]) are pointed out which are not $\text{Aut}(su^C(1,1))$ orbits for any reference state $|\psi_0\rangle$. Such are the sets of the $K_1-K_2$ SS $|z, u, v; k\rangle$ (eigenstates of $u K^- + v K_+ \rangle$ [3], the Barut-Girardello CS (eigenstates of $K_-$) [17] and the even and odd CS (eigenstates of $a^2$) [18].

2 Generalized SS

2.1 Generalization of the eigenvalue property

The generalization of property a) is straightforward: We introduce the shortened notation $\vec{X} = (X_1, X_2, \ldots, X_n)$ and consider the sets of eigenstates of complex combinations of all $X_i$ (summation over repeated indices is adopted),

$$A_\nu(\beta)|\vec{z}, \beta\rangle = z_\nu|\vec{z}, \beta\rangle, \quad A_\nu(\beta) = \beta_\nu X_i,$$

where $\beta$ is an $n_c \times n$ complex matrix and the integer $n_c$ is to be yet specified. The greek indices $\mu, \nu$ run from 1 to $n_c$, and the latin indices $i, j$ run from 1 to $n$. Eigenstates $|\vec{z}, \beta\rangle$ would exhibit arbitrarily strong squeezing (ideal squeezing) of the observable $X_j$ when for a given $\nu$ all but $\beta_{\nu j}$ are let to tend to 0,

$$\rightarrow 0 \quad \text{for all} \quad k \neq j \quad \text{and at least one} \quad \nu.$$
only if $|\psi\rangle$ is an eigenstate of $X$:

$$X|\psi\rangle = x|\psi\rangle \Leftrightarrow \Delta X(\psi) = 0. \quad (4)$$

In the limit (3) the state $|\vec{z},\beta\rangle$ tends to an eigenstate of $X_j$ and this ensure arbitrarily strong squeezing of $X_j$, i.e. $\Delta X_j \to 0$. To observe light squeezing one needs not to take the limit in (3). The variance $\Delta X$ may vanish in mixed states $\rho = \sum p_n|\psi_n\rangle\langle\psi_n|$ if all $|\psi_n\rangle$ are eigenstates of $X$ with the same eigenvalue.

For several observables we need a definition of the family of SS: A family of states $|\psi(l_1,l_2,\ldots)\rangle$ with parameters $l_1, l_2, \ldots$ is called a family of SS for $n$ observables $X_j$ (shortly $\vec{X}$-SS) if for every $j = 1, 2, \ldots n$ one can find in it states, such that $\Delta X_j$ is less than a certain value $\Delta_0$,

$$\Delta X_j < \Delta_0, \quad j = 1, 2, \ldots, n. \quad (5)$$

The reference value $\Delta_0 > 0$ is the variance of some of $X_i$ in some reference state $\psi_0$, selected on certain physical reason, $\Delta_0 = \Delta X_i(\psi_0)$. One natural criterion for $|\psi_0\rangle$ is to provide the equality of two or more $\Delta X_j$ on the as lowest possible level,

$$\Delta_0 = \text{Min}\{\Delta X_1(\psi), \Delta X_2(\psi), \ldots\} \quad \text{provided} \quad \Delta X_1(\psi) = \Delta X_2(\psi) = \ldots = \Delta X_n(\psi_0) \equiv \Delta_0 \quad (6)$$

Note that we work here with dimensionless operators $X_j$. For the electromagnetic field (in quantum optics) one usually takes $|\psi_0\rangle = |0\rangle$, $|0\rangle$ being the vacuum. For the quadratures of any power of the annihilation operators $a_\nu$, $\nu = 1, 2, \ldots N$, the choice $|\psi_0\rangle = |0\rangle$ ensures the equalities $\Delta X_1(\psi_0) = \Delta X_2(\psi_0) = \ldots = \Delta X_n(\psi_0) \equiv \Delta_0$ on the lowest level.

In the family $\{|\vec{z},\beta\rangle\}$ the SS defining inequality (3) is universally (i.e. for any choice of $\Delta_0 > 0$) satisfied if conditions (3) hold. A set of states in which the inequality (3) can hold for an arbitrarily small $\Delta_0$ (not simultaneously), i.e.,

$$\Delta X_j \to 0, \quad j = 1, 2, \ldots, n, \quad (7)$$

should be called a set of ideal $\vec{X}$-SS. Conditions (3) ensure (5). The known CSS constitute such ideal SS for $q$ and $p$.

It is worth noting that the ideal squeezing conditions (3) (and even the conditions (5)) require enough parameter freedom in the family $\{|\psi(l_1,l_2,\ldots)\rangle\}$. In the case of eigenstates $|\vec{z},\beta\rangle$ of $A_\nu(\beta)$ this means that parameters $\beta_{\nu i}$ should not be fixed, i.e., $|\vec{z},\beta\rangle$ should be eigenstates of a set of complex combinations of $X_j$. The number of free parameters $\beta_{\nu i}$ should evidently be not less than the number $n$ of the observables. Then from this set on can form several (say $n_c$) linearly independent operators $A_\nu(\beta)$.

It is desirable to have the possibility to calculate in $\vec{X}$-SS all second moments of $X_j$ in pure algebraic way as it is the case of CSS. For even $n$ we can perform this for states
if \( n_c = n/2 \). In order to do this we denote \( n/2 = N \) and introduce the \( n \) component vector

\[
\vec{B}(\beta) = (A_1(\beta), \ldots, A_N(\beta), A_1^*(\beta), \ldots, A_N^*(\beta)), \quad A_\nu = \beta_\nu X_i.
\]

Next we express \( \vec{X} \) in terms of \( \vec{B}(\beta) \). Then after some calculations we obtain for the uncertainty matrix \( \sigma \) (an \( n \times n \) matrix with elements \( \sigma_{ij} = \langle X_i X_j + X_j X_i \rangle / 2 - \langle X_i \rangle \langle X_j \rangle \)) in SS \(|\vec{z},\beta\rangle\) the following general expression in terms of first moments of commutators \([X_i, X_j]\)

\[
\sigma(\vec{X}; \beta) = \mathcal{B}^{-1} \left( \begin{array}{cc} 0 & C' \\ C^T & 0 \end{array} \right) \mathcal{B}^{-1T}, \quad \mathcal{B} = \left( \begin{array}{cc} \beta^{(1)} & \beta^{(2)} \\ \beta^{(1)*} & \beta^{(2)*} \end{array} \right),
\]

where \( C' = (C'_{\nu\mu}) \),

\[
C'_{\nu\mu} = \langle [A_\nu(\beta), A^*_\mu(\beta)] \rangle / 2 = \beta_{\nu\mu} \beta_{\mu\nu}^* [X_i, X_j],
\]

\( C^T \) is the transposed \( C \) and the \( N \times N \) matrices \( \beta^{(1,2)} \) are defined as \( \beta_{\nu\mu}^{(1)} = \beta_{\nu\mu}, \beta_{\nu\mu}^{(2)} = \beta_{\nu,N+\mu}. \) Note that \( \beta \) in (2) now \((n_c = N)\) is an \( N \times n \) matrix, while \( \mathcal{B} \) in (8) is \( n \times n \). We suppose that \( \mathcal{B} \) is not singular. This is equivalent to the nonsingularity of the linear real transformation of the observables

\[
\vec{X} \rightarrow \vec{X}' = \Lambda \vec{X}, \quad \vec{X}' = (X'_1, \ldots, X'_n),
\]

where \( X'_i \) and \( X'_{N+i} \) are quadrature components of \( A_\nu(\beta), A_\nu(\beta) = X'_\nu + iX'_{N+i}. \) The \( n \times n \) real matrix \( \Lambda \) is simply composed in terms of matrix elements of \( \beta \): \( \Lambda_{\nu\mu} = \text{Re} \beta_{\nu\mu}, \Lambda_{N+i,n+i} = -\text{Im} \beta_{\nu\mu}. \) Formula (8) could be extended to the case of odd \( n \) if \( n_c = [n/2] \) \([n/2] \) is the integer part of \( n \)\) and we admit that \(|\vec{z},\beta\rangle\) are eigenvectors of one extra Hermitian operator \( X'_{2n_c+1}. \)

Thus \( n_c = [n/2] \) eigenvalue equations (2) and the nonsingularity of the transformation (9) provide a proper generalization of the first definition a) of the CSS to the case of \( n \) arbitrary observables. The family of eigenstates \(|\vec{z},\beta\rangle\) can be qualified as a family of strong (or ideal) SS for \( n \) operators \( X_j \) if the parameters \( \beta_{\nu j} \) could obey the conditions (3) \( \text{or (4)}) \). Some squeezing is not excluded in other parameter ranges. We will below see that, besides the useful formula (8), the requirement \( n_c = [n/2] \) in (4) and (9) provide an efficient generalization of the third property c) of one mode CSS as well.

### 2.2 Generalization of the orbit property of CSS

When \( X_i \) close a Lie algebra \( L \) the second property of CSS can be generalized in the form

\[
|\psi_{\text{GSS}}(g)\rangle = U_\Lambda(g)|\psi_0\rangle,
\]

(10)
where $U_A(g)$ is an unitary representation of the group $G_A \equiv \text{Aut}(L^C)$ of automorphisms of $L^C$ ($L^C$ is the complexified $L$) \cite{12} and $|\psi_0\rangle$ is eigenvector of a fixed element $A_0$ of $L^C$,

$$A_0|\psi_0\rangle = z_0|\psi_0\rangle, \quad A_0 \in L^C.$$ \hfill (11)

The idea of the definition \cite{10} of SS for $n$ Lie group generators $X_j$ came from the observation \cite{4} that the product $D(z)S(\zeta)$ of the displacement $D(z)$ and squeezed $S(\zeta)$ operators, which appears in the second definition b) of CSS, is an element of the semidirect product group $SU(1,1) \ltimes H_1$ of the quasi-unitary group $SU(1,1)$ and the Heisenberg-Weyl group $H_1$. And $SU(1,1) \ltimes H_1$ is the group of automorphisms $\text{Aut}(h^C_1)$ of the complexified algebra $h^C_1$, spanned by the canonical observables $q$, $p$ and the identity.

Not all of the states $|\psi_{GSS}(g)\rangle$ however can exhibit squeezing in the Hermitian operators $X_i \in L$. The standard $G$-group related CS with symmetry $|\psi(g)\rangle$ \cite{18;19} constitute such exceptions ($G$ being the group generated by $L$). Indeed, it is clear that $|\psi(g)\rangle$ are of the form of $|\psi_{GSS}(g)\rangle$, eq. \cite{10}, since $\text{Aut}(L^C)$ contains $G$. More precisely, $G$ is homomorphic to the group $\text{Ad}(L)$ of internal automorphisms of $L$ which is a subgroup of $\text{Aut}(L^C)$:

$$\text{Ad}(L) \subset \text{Ad}(L^C) \subset \text{Aut}(L^C).$$

The $G$-group related CS with symmetry are eigenstates of $U(g)A_0U^{-1}(g)$ which is a very particular combination of the generators $X_j$: if $A_0$ is one of $X_j$, say $X_k$, then condition \cite{10} can holds for $j = k$ only; even if $|\psi_0\rangle$ is eigenvector of several $A_0$ neither the inequalities \cite{3} for $|\psi_0\rangle$ nor the conditions \cite{3} could be satisfied for all $j$. The absence of squeezing (in the sense of definition \cite{3} with \cite{3} in spin ($SU(2)$) and quasi-spin ($SU(1,1)$) group related CS with symmetry was noted in \cite{13}. Thus one has to look for squeezing of elements $X_j$ of $L$ in the states of orbits of the larger group $\text{Aut}(L^C)$ of automorphisms of $L^C$, not in the states of orbits of $G$. However for semisimple Lie algebras even this extension of the group related CS may be insufficient to include ideal SS as we shall see on the example of $su(1,1)$.

Let us examine briefly the relationship between two generalizations \cite{2} and \cite{10}. By the definition of the group of automorphisms $G_A$ the operator $U_A(g)A_0U_A^{-1}(g)$ is a complex combination of $X_j$. Noting that $|\psi_{GSS}(g)\rangle$ is an eigenstate of $U_A(g)A_0U_A^{-1}(g)$ (with eigenvalue $z_0$) we obtain that $\text{Aut}(L^C)$ is equivalent to $\text{Ad}(L^C)$. The inverse however is generally not true: eigenstates $|\vec{z},\beta\rangle$ of some combinations $\beta_jX_j$ may not be represented in the form \cite{10} with unitary $U_A(g(\beta))$ and fixed, $\beta$ and $g$ independent reference vector $|\psi_0\rangle$ (i.e. in the form of $\text{Aut}(L^C)$-group related CS). The standard even and odd CS $|\alpha_\pm\rangle$ \cite{18}, the Barut-Girardello CS (BG CS) $|z; k\rangle$ \cite{17} and their generalizations $|z, u, v; k\rangle$ \cite{3} are examples of such families which are neither $G$ nor $\text{Aut}(L^C)$ group related CS (proof in Appendix A). At the same time the states $|z, u, v; k\rangle$ are ideal SS for $K_1$ or $K_2$ \cite{3}. 


since they do satisfy the conditions \(3\) and can exhibit arbitrarily strong squeezing of \(K_1\) or \(K_2\) [5, 11, 19]. This motivates the necessity to introduce the more general notion of \(L\)-algebraic CS [11] to denote a continuous family of states, which are eigenstates of operators of the complexified \(L^C\). Thus the set of \(SU(1,1)\)-group related CS with symmetry and the three sets of \(|\alpha\rangle_\pm, |z;k\rangle, \) and \(|z,u,v;k\rangle\) are all proper \(su(1,1)\)-algebraic CS. In the next section we construct most general \(su(1,1)\)-algebraic CS and show that they are ideal \(su(1,1)\) SS.

### 2.3 Generalization of the intelligence property of CSS

The third basic property of one mode CSS can be generalized using the Robertson uncertainty relation (RUR) for \(n\) Hermitian operators \(X_i\) [13],

\[
\det \sigma(X) \geq \det C(X), \quad C = (C_{kj}), \quad C_{kj} = (-i/2)\langle [X_k, X_j] \rangle, \tag{12}
\]

where \(\sigma\) is the uncertainty matrix, defined in the subsection 2.1. The inequality (12) is valid for any state, pure or mixed. For two operators it coincides with the Schrödinger relation, eq. (1). The third way of construction of SS for \(n\) observables could be as states \(|\psi\rangle\), which minimize the RUR,

\[
\det \sigma(X; \psi) = \det C(X; \psi). \tag{13}
\]

The properties of the uncertainty matrix \(\sigma(X)\) for \(n\) observables and the minimization of RUR are studied in detail in ref. [20]. It is proven that for any \(n\) the RUR is minimized in a state \(|\psi\rangle\) if \(|\psi\rangle\) is an eigenstate of a real combination \(\beta_iX_i\). For odd \(n\) this condition is also necessary. For even \(n\) the RUR is minimized in eigenstates \(|\vec{z}, \beta\rangle\) of \(n/2\) complex combinations of \(X_i\) (see eq. (2) with \(n_c = n/2\)). By direct calculation one can check that the uncertainty matrix \(\sigma(X; \beta)\), eq. (8), satisfies the equality (13).

In addition to the results of ref. [20] we note here that in case of two arbitrary observables \(X\) and \(Y\) the eq. (2) (with \(n_c = 1\)) is also a necessary condition for a pure state to minimize RUR. For this purpose consider the mean value of nonnegative operators \(F^\dagger(\beta, r)F(\beta, r)\), where \(F = \beta X + irY - (\beta \langle X \rangle + ir \langle Y \rangle)\), \(\beta\) is complex and \(r\) is a real parameter. This mean value is easily expressed as a linear combination of three second moments of \(X\) and \(Y\) and the first moment of their commutator. Then after some simple algebra one gets the result that if a mixed state \(\rho = \sum_m \rho_m |\psi_m\rangle\langle \psi_m|\) minimizes the RUR for \(n = 2\) then \(F(\beta, r) |\psi_m\rangle = 0\) for every \(m\). We anticipate that for any even \(n\) the eqs. (2) with \(n_c = n/2\) are again necessary.

States which minimize \(12\) should be called Robertson intelligent states (RIS), and in case of \(n = 2\) they should be referred to as Schrödinger IS. If the set \(\{X_i\}\) closes an algebra \(L\) then the minimizing states should be also referred to as \(L\)-algebra RIS. It
was proven \(^{20}\) that group related CS with symmetry for semisimple Lie groups are RIS for group generators and CS with maximal symmetry are RIS also for the quadratures of Weyl lowering operators. Thus group related CS with symmetry are subset of the corresponding \(L\)-algebra RIS.

When the covariances (the correlations) of \(X_i, X_j\) are not vanishing the minimizing states should be called also Robertson correlated states, following the ref. \(^{3}\). It was proven however that in any state the correlations can be canceled by means of linear orthogonal or symplectic transformations of the observables \(^{20}\).

RIS could exhibit arbitrarily strong squeezing of the observables if the conditions (3) can be satisfied. Normally this is the case: RIS generally depend on complex parameters \(\beta\nu\) and \(z\nu\). The total number of real parameters (for even \(n\)) is equal to \(3n^2/2\). If \(A\nu(\beta)\) are not subjected to further constrains we have enough parameter freedom to ensure (3). Squeezing may occur also in the cases of eigenstates of \(A\nu(\beta)\) with \(n_c < \lfloor n/2 \rfloor\) which could not minimize the Robertson relation for all \(n\) observables. Note that the large set of RIS contains many well defined subsets, states of which cannot exhibit squeezing - such are the group related CS with symmetry for semisimple (and some nonsemisimple) Lie groups.

The Robertson uncertainty relation provides a natural specification of the reference value \(\Delta_0\) and the reference state \(|\psi_0\rangle\) in the definition (5) and (6) of SS for \(n\) observables. For even \(n\), \(n = 2N\), one can require

\[
\Delta X_j(\psi) < \Delta_0 = \text{Min} \left[ \det C(\vec{X}; \psi) \right]^{1/n},
\]

provided

\[
(X\nu + iX_{N+\nu})|\psi\rangle = z\nu|\psi\rangle, \quad \nu = 1, \ldots, N,
\]

the minimization being with respect to \(z\nu\). The eigenvalue equations (15) ensure the equality in Robertson relation with equal variances \(\Delta^2 X\nu(\vec{z}) = \Delta^2 X_{N+\nu}(\vec{z}) = |\langle \vec{z}|[X\nu, X_{N+\nu}]|\vec{z}\rangle|/2\).

The Eberly-Wodkiewicz definition of relative squeezing \(^{8}\) can be extended as follows: The variances \(\Delta X_j(\psi)\) are squeezed if

\[
\Delta X_j(\psi) < \left[ \det C(\vec{X}; \psi) \right]^{1/n},
\]

This definition fails in the case of \(X_j\) with discrete spectrum: in the eigenstates of \(X_j\) one has absolute squeezing, \(\Delta X_j = 0\), and \(\det C(\vec{X}; \psi) = 0\). For odd number \(n\) both inequalities (14) and (16) fail since \(\det C(\vec{X}; \psi) = 0\) in any state. In these cases one can apply the universal definitions (3) and (8) for ideal and/or strong SS. We shall demonstrate this in the next section on the example of three generators of the \(SU(1, 1)\) group.
3 Examples of SS and RIS for several observables

In this section we consider two explicit constructions of generalized SS and RIS. The first one is related to the nilpotent Lie algebra $h_N$ which has $2N + 1$ basic elements ($2N$ observables) and the second one is related to the simple $su(1,1)$, which has 3 basic elements (3 observables). These algebras and observables are most frequently used in physics, especially in quantum optics.

3.1 $h_N$ SS and RIS

The Heisenberg-Weyl algebra $h_N$ is spanned by $p_\mu$, $q_\mu$ ($\mu = 1, \ldots, N$) and 1. $p_\mu$ and $q_\mu$ are quadrature components of boson operators $a_\mu$, $a_\mu = (q_\mu + ip_\mu)/\sqrt{2}$. Here $N$ independent and mutually commuting linear combinations $A_\mu(\beta)$ of $a_\nu$, $a_\nu^\dagger$ (equivalently of $q_\nu$ and $p_\nu$) always exist, so that $n_c = N$. The solution of eqs. (2) for operators $\vec{A}(\beta) = (A_1(\beta), A_2(\beta), \ldots, A_N(\beta))$

$$\vec{A}(\beta) = \beta^{(1)}\vec{p} + \beta^{(2)}\vec{q}$$

exists for nonsingular $\beta^{(1)}$ and arbitrary $\beta^{(2)}$. In coordinate representation it takes the form of an exponential of a quadratic,

$$\langle \vec{q}|\vec{z},\beta \rangle = \tilde{N} \exp\left[\vec{q}\mathcal{M}\vec{q} + \tilde{N}\vec{q}\right], \quad \mathcal{M} = -(i/2)\beta^{(1)}-1\beta^{(2)}, \quad \tilde{N} = i\beta^{(1)}-1\vec{z},$$

We see a freedom in the set of parameters $\beta_{\nu i}$ which can always be used to subject $A_\nu(\beta)$ to the canonical commutation relations,

$$[A_\nu(\beta), A_\mu^\dagger(\beta)] = \delta_{\nu\mu}.$$  (19)

Eigenstates of $N$ boson operators $A_\nu$, which are linear combinations of $q_\nu$, $p_\nu$ (therefore of $a_\nu$ and $a_\nu^\dagger$) and satisfy (19), were first constructed in [8].

The uncertainty matrix in states $\langle \vec{q}|\vec{z},\beta \rangle$ is given by the general formula (8) with $C' = (1/2)1_N$ and one can check that it satisfies the equality in (13). Therefore these solutions are $h_N$ RIS. Moreover, even if a state $|\psi\rangle$ is an eigenstate of one complex combination of $p_\nu$ and $q_\nu$, then there exist $N$ independent such combinations which have this $|\psi\rangle$ as their common eigenstate, i.e. $|\psi\rangle$ is RIS and takes the form (10) (proof in Appendix A).

The inverse is also true. If a state $|\psi\rangle$ minimizes the Robertson inequality for $p_\nu$ and $q_\nu$ then $|\psi\rangle$ is an eigenstate of $N$ complex combinations $A_\mu$ of $p_\nu$ and $q_\nu$ in the form of new boson operators (proof in [22, 15]). In view of $\text{Aut}(h_N^C) = Mp(N, R) \otimes H_N$ ($Mp(N, R) = Sp(N, R)$) we get that $N$ mode canonical RIS are of the form of (10) with $G_A = Mp(N, R) \otimes H_N$. 

9
In quantum optics eigenstates of complex combinations of $p_\nu, q_\mu$ (i.e. the $h_N$ RIS, given in coordinate representation by (18)) are known as (canonical) multimode SS and their nonclassical properties have been intensively studied (see [21] and references therein). The $h_N$ RIS (18) obey the conditions (3) and can exhibit arbitrarily strong squeezing in $p_\nu$ or $q_\nu$, i.e., they are ideal SS. The limits in (3) however cannot be taken “till the end”, since $p_\nu$ and $q_\nu$ have no normalizable eigenstates and $\Delta q_\nu > 0, \Delta p_\nu > 0$. Moreover no real combination of $p_\nu$ and $q_\nu$ can be diagonalized [20].

Thus we have shown that for canonical observables $p_\nu, q_\nu$ of $N$ mode boson system, i.e., for $h_N$ algebra, all three definitions a), b) and c) of the one mode CSS are equivalently generalized on the basis of the Robertson uncertainty relation and the group of automorphisms of $h_N$. We shall see below that such equivalence does not occur in the case of $su(1,1)$ algebra observables.

3.2 $su(1,1)$ SS and RIS

In this subsection we shall construct the $su(1,1)$ SS and RIS and examine some of their nonclassical properties. We note that eigenstates of complex combinations of the generators of $SU(1,1)$ are discussed in similar ways in the recent papers [11, 14] under the names algebraic CS [11] and algebra eigenstates [14].

The three generators of $SU(1,1)$ satisfy the relations

$$[K_1, K_2] = -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2. \quad (20)$$

To construct $su(1,1)$ SS according to (2) we have to solve the eigenvalue problem for one operator family

$$A(u, v, w) = \beta_i K_i = uK_- + vK_+ + wK_3, \quad (21)$$

$$u + v = \beta_1, \quad i(v - u) = \beta_2, \quad w = \beta_3.$$ 

We choose the parameters $u, v, w$ and write the eigenvalue equation

$$(uK_- + vK_+ + wK_3)|z, u, v, w; k\rangle = z|z, u, v, w; k\rangle, \quad (22)$$

where $k$ is the Bargman index. Consider first the series $D^{(+)}(k), k = 1/2, 1, ...$. The bosonic realization of $D^{(+)}(k)$ are of current interest in quantum optics [22]. For example in terms of two boson lowering and raising operators $a$ and $b$ one has

$$K_- = ab, \quad K_+ = a^\dagger b^\dagger, \quad K_3 = (a^\dagger a + b^\dagger b + 1)/2. \quad (23)$$
This representation is irreducible in the subspaces with fixed eigenvalue $n_a - n_b$ of $a^\dagger a - b^\dagger b$, the Bargman index being $k = (|n_a - n_b| + 1)/2$. $K_2-K_3$ IS in this representation can be generated and used to improve the accuracy in the interferometric measurements \cite{22}.

To solve eq. (22) it is suitable to use the representation of Barut and Girardello CS (BG representation) \cite{17}. In BG representation $K_\pm$ and $K_3$ are differential operators:

\begin{equation}
K_+ = \eta, \quad K_- = 2k\, d/d\eta + \eta\, d^2/d\eta^2, \quad K_3 = k + \eta\, d/d\eta, \tag{24}
\end{equation}

where $\eta$ is a complex variable. Eq. (22) becomes a second order differential equation, which is easily reduced to the Kummer equation \cite{23}. The solutions ($u \neq 0$) are found in the form \cite{11}

\begin{equation}
\Phi_z(\eta; u, v, w) = N(z, u, v, w) \exp (c(u, w, l)\eta) M(k + z/l, 2k, l\eta/u) \tag{25}
\end{equation}

where $N(z, u, v, w)$ is a normalization constant, $M(a, b, \eta)$ is the Kummer function ($M(a, b, \eta) = _1F_1(a; b; \eta)$) \cite{23, 24} and

\begin{equation}
c(u, w, l) = -\frac{1}{2u}(w + l), \quad l = \sqrt{w^2 - 4uv}. \tag{26}
\end{equation}

Note, $l^2 = (A, A)$, where $(,)$ is the Killing form on the Lie algebra (here $su(1, 1)$) \cite{12}. $\Phi_z(\eta; u, v, w)$ represents normalized states $|z; u, v, w; k\rangle$ if

\begin{equation}
|w - l| < 2|u|, \quad \text{or} \quad |w + l| < 2|u|. \tag{27}
\end{equation}

When these normalizability conditions are broken down the functions $\Phi_z(\eta; u, v, w)$ are still solutions of (22), but represent nonnormalizable states. $|z; u, v, w; k\rangle$ can easily be expressed as series in terms of orthonormalized eigenstates $|n + k, k\rangle$ of $K_3$ using the expansion of $M(a, b, \eta)$ in terms of powers of $\eta$. The orthonormalized eigenstates $|k, k+m\rangle$ of $K_3$ are represented by monomials $\eta^m [\Gamma(2k)/(m!\Gamma(m + 2k))]^{1/2}$.

\begin{equation}
|z, u, v, w; k\rangle = \mathcal{N} \sum_m g_m(z, u, v, w, k) |k, k + m\rangle, \tag{28}
\end{equation}

\begin{equation}
g_m = c^m \sqrt{(2k)_m/m!} \, _2F_1(a, -m; 2k; \zeta), \quad \zeta = \frac{l}{uc} = \frac{2l}{w + l}, \tag{29}
\end{equation}

\begin{equation}
\mathcal{N}^{-2} = (1 + s)^{-2k+a+a^*} \left[(1 + s - s\zeta)^{-a}\right] \, _2F_1 \left(a, a^*; 2k; \frac{-s|\zeta|^2}{|1 + s - s\zeta|}\right) = \mathcal{N}^{-2}(s, \zeta, a), \tag{30}
\end{equation}

\begin{equation}
s = -c^*c = \frac{|w + l|^2}{4|u|^2}, \quad a = k + z/l, \quad l = \sqrt{w^2 - 4uv},
\end{equation}

where $\, _2F_1(a, b; c; z)$ is Gauss hypergeometric function \cite{24}. Closed expressions of the normalization constants of $|z, u, v, w; k\rangle$, eq. (25), were obtained (in different parameters) by Brif \cite{14}.
The above solution can be applied to the cases of \( u = 0 \) or \( l = 0 \) under the appropriate limits \( u \to 0, l \to 0 \) in it (there is also no problem to consider these cases separately). In other parameters solutions of eq. (22) were obtained in [14] using an analytic representation in the unit disk.

The family of \(|z, u, v, w; k\rangle\) contains several known types of states. The BG CS are recovered at \( v = 0 = w \) and the \( SU(1, 1) \) generalized IS (the \( K_1-K_2 \) Schrödinger IS) of ref. [3] are reproduced at \( w = 0 \) (the identification \( u = (\lambda + 1)/2, v = (\lambda - 1)/2 \)). All the \( SU(1, 1) \) group related CS with symmetry are naturally included in \(|z, u, v, w; k\rangle\). It is curious that \( SU(1, 1) \) CS with maximal symmetry \(|\zeta; k\rangle\) are contained also in the subfamily of \( K_1-K_2 \) IS \(|z, u, v; k\rangle \equiv |z, u, v, w = 0; k\rangle: |z = -k\sqrt{-uv}, u, v; k\rangle = |\zeta = \sqrt{-v/u}; k\rangle \) [5].

\(|z, u, v, w; k\rangle\) are eigenstates of linear combination of three observables \( K_1, K_2, K_3 \). According to the discussion in the subsection 2.3 (and the results of [20]) these states should minimize the Robertson inequality for the three observables if and only if they are eigenstates of real combination of \( K_1, K_2, K_3 \), i.e. when \( uK_+ + vK_+ + wK_3 \) is Hermitian. This holds when \( v = u^* \) and \( w \) is real, so the states \(|z, u, u^*, w; \rangle\) with real \( w \) are \( su(1, 1) \) RIS. One can check that \(|z, u, v, w; k\rangle\) and the RIS \(|z, u, u^*, w; \rangle\) as well obey the definition (3) of ideal SS for the three operators \( K_j \).

All \( K_1-K_3 \) Schrödinger IS are naturally contained in \(|z, u, v, w; k\rangle\). For example \( K_2-K_3 \) IS are obtained at \( v = -u^* \) and \( K_1-K_2 \) IS – at \( w = 0 \). The set of IS \(|z, u, v; k\rangle, v \neq 0\) is a set of ideal \( K_1-K_2 \) SS. It is an example of a set of ideal SS which are of the form (2) but cannot be represented in the form (10) of an orbit of the group \( Aut(su^C(1, 1)) \), i.e. in this example the second (the orbit) construction of generalized SS is not equivalent the first and the third ones (see proof in Appendix B). The first (the eigenstate) and the third (the intelligent) constructions of generalized SS are equivalent in this case since \((uK_+ + vK_+)|\psi\rangle = z|\psi\rangle\) is necessary and sufficient for a state to minimize Schrödinger inequality for \( K_1, K_2 \).

In the case of one mode bosonic representation with \( k = 1/4, 3/4 \), when

\[
K_+ = \frac{1}{2}a^2, \quad K_+ = \frac{1}{2}a^\dagger a, \quad K_3 = \frac{1}{2}(a^\dagger a + \frac{1}{2}),
\]  

(31)

it is suitable to use the canonical CS representation in which \( a = d/da, \ a^\dagger = \alpha \). Here we have two independent solutions of eq. (22), represented by the even and odd analytic functions \( \Phi_\pm^z(\alpha; u, v, w) \). The even solution is given by the formula (25) with the replacements \( \eta \to \alpha^2/2, k \to 1/4 \). Its expansion in terms of the eigenstates of \( K_3 = (a^\dagger a/2 + 1/4) \) (the Fock states) is given by formula (28) with \( k = 1/4 \) [The BG representation (of states and operators) can be safely used in the \( su(1, 1) \) irreducible subspaces, such as the subspaces of even and odd states]. The odd solution takes the form
\[
\Phi^{-}_z(\alpha; z, u, v, w) = \alpha N_\alpha \exp \left( c(u, w, l)\alpha^2/2 \right) M \left( a_-, 3/2, l\alpha^2/2u \right),
\]

with parameters
\[
c = -\frac{1}{2u} (w + l), \quad a_\pm = \frac{1}{4} (3 + 2z/\sqrt{-uv}), \quad v' = -\frac{1}{4u} l^2, \quad l = \sqrt{w^2 - 4uv}.
\]

In this \(su(1, 1)\) representation the case \(w = 0\) was solved in [15] where the two independent solutions \(|z, u, v; \pm\rangle\) were called generalized even and odd CS. In fact the even states \(|z, u, v; +\rangle\) were constructed earlier in ref. [3]: in the BG representation \(\Phi_{z'}(z)\) of states \(|z', u, v; k\rangle\) one has to put \(k = 1/4\) and \(z = \alpha^2/2\) in order to get the canonical CS representation of \(|z', u, v; +\rangle\). Since \(|z, u, v; \pm\rangle\) minimize the Schrödinger inequality for the quadratures of \(a^2\) they are also called Schrödinger squared amplitude (even/odd) IS or squared amplitude (even/odd) SS [19]. Other particular cases of eq. (22) for \(k = 1/4, 3/4\) were considered in [24] and in the second and fourth papers of [7]. The general case was solved by Brif [26].

The normalizability conditions on \(u, v, w\) are the same as in the case of \(D^{(+)}(k)\). In the alternative case of \(u = 0\) the solutions can be obtained in a similar manner or by taking the appropriate limit in (23) and (32) [11, 26]. The states \(|z, u, v, w; \pm\rangle\) pertain the property of \(|z, u, v, w; k\rangle\) (noted above) to contain all \(SU(1, 1)\) group related CS with symmetry. The squeezed vacuum states coincides with the \(SU(1, 1)\) CS with maximal symmetry. It is worth to note the following double intelligence property of the squeezed vacuum states: these and only these states minimize the Schrödinger relation for both \(q, p\) and \(K_1, K_2\) pairs of observables. The squeezed one photon states are \(K_1-K_2\) IS only. These properties can easily be derived from the discussion on minimization of Schrödinger inequality. The states \(|z, u, v, w; \pm\rangle\) minimize the Robertson inequality for the three operators \(K_1 = (a^2 + a^4)/4, \quad K_2 = -i(a^2 - a^4)/4\) and \(K_3 = (2a^4 + 1)/4\) when \(w = w^*, \quad v = u^*\).

Let us examine for squeezing the constructed \(su(1, 1)\) SS. According to the general prescription (see eq. (3)) the states \(|z, u, v, w; k\rangle\) would exhibit arbitrarily strong squeezing of the variance of the generators \(K_j\) when all but \(\beta_j\) are let to tend to 0. The calculations confirm this property [11, 19], i.e., \(|z, u, v, w; k\rangle\) are ideal SS. Here the limits \(\beta_1 = 0, \beta_2 = 0\) can be taken: at \(\beta_1 = 0, \beta_2 = 0\) one gets eigenstates of \(K_2\), in which the variance of \(K_2\) vanishes (is absolutely squeezed). The limits \(\beta_3 = 0, \beta_1 = 0\) (i.e., \(w = 0, v = -u\)) or \(\beta_3 = 0, \beta_2 = 0\) (i.e., \(w = 0, v = u\)) however can not be taken - they would violate the normalizability constrains (27). For the sake of simplicity we shall examine in greater detail the case \(w = 0\). The three second moments of \(K_1, K_2\) in \(|z, u, v; k\rangle\) (for \(k = 1/4, 3/4\) and \(k = 1/2, 1, \ldots\)) read [5, 20].
\[ \Delta^2 K_1 = \frac{1}{2} \frac{|u - v|^2}{|u|^2 - |v|^2} \langle K_3 \rangle, \quad \Delta^2 K_2 = \frac{1}{2} \frac{|u + v|^2}{|u|^2 - |v|^2} \langle K_3 \rangle, \quad \Delta K_1 K_2 = \frac{\text{Im}(u^*v)}{|u|^2 - |v|^2} \langle K_3 \rangle. \]  

(33)

These variances can easily be casted in the general matrix form (8) for \( n = 2 \). The mean \( \langle K_3 \rangle \) of \( K_3 \) in \( |z, u, v, w; k \rangle \) can be calculated (for any \( k \), \( k = 1/4, 3/4, 1/2, 1, \ldots \), using the explicit form of the normalization factor \( \mathcal{N}(s, \zeta, a) \), eq. (30), according to the following relation,

\[ \langle K_3 \rangle = k + s \mathcal{N}^2 \frac{\partial \mathcal{N}^{-2}}{\partial s}, \quad s = -|c(u, w, l)|^2. \]  

(34)

For \( k = 1/4, 3/4 \) the generators \( K_1 = (a^2 + a^4)/4, \quad K_2 = -i(a^2 - a^4)/4 \) appear as quadratures of \( a^2 \) and thus here \( K_{1,2} \) squeezing coincides with the "squared amplitude" squeezing (Hillery et. al. [2]). The mean photon number \( \langle a^2 \rangle = 2\langle K_3 \rangle - 1/4 \). The \( K_2 \) squeezing is illustrated (see also [14, 19]) in Fig. 1a on the example of generalized even CS \( |z, u, v; + \rangle \) (\( k = 1/4, w = 0 \)) with \( z = 1, u = \sqrt{1 + x^2}, v = -x, \quad x > 0 \). For convenience we take the quadratures of \( a^2 \) as \( \tilde{K}_1 = 2\sqrt{2}K_1, \quad \tilde{K}_2 = 2\sqrt{2}K_2, \) i.e. \( a^2 = (\tilde{K}_1 + i\tilde{K}_2)/\sqrt{2} \). Then in the ground state \( |0 \rangle \) the variances of the quadratures \( \tilde{K}_1, \tilde{K}_2 \) are both equal to 1. According to [3] with [4] a state \( |\psi \rangle \) exhibits squared amplitude squeezing if \( \Delta \tilde{K}_1(\psi) \) or \( \Delta \tilde{K}_2(\psi) \) is less than 1. Quadratic squeezing is found also in \( |z, \sqrt{1 + v^2}, v; + \rangle \) for \( z = \pm 1/2, \pm 1, \pm 5/2 \) and negative \( v \) [14, 19]. As Fig. 1a shows the generalized even CS \( |1, \sqrt{1 + x^2}, -x; + \rangle \) are \( \tilde{K}_2 \) squeezed when \( x > 1.8 \). States with strong quadratic squeezing have not been pointed out so far (the recent papers [16, 27], where quadratic squeezing is also found, appeared after [14, 19]). We note that states \( |z, u, v; \pm \rangle \) which exhibit quadratic squeezing are not \( SU(1, 1) \) group related CS (even more - they are not of the form [10]).

The states \( |z, u, v; \pm \rangle \) can also exhibit strong (but not arbitrarily strong) ordinary (or linear) squeezing [19]. The quadratures \( q, p \) are squeezed if their squared variance is less than 1/2. In \( |z, u, v; \pm \rangle \) we have

\[ \Delta^2 q = \frac{1}{2} + \langle a^4 \rangle + 2\frac{\text{Re}[(u - v)z^*]}{|u|^2 - |v|^2}, \quad \Delta^2 p = \frac{1}{2} + \langle a^4 \rangle - 2\frac{\text{Re}[(u - v)z^*]}{|u|^2 - |v|^2}. \]  

(35)

On Fig. 1a we show the plot of \( \Delta^2 p(x) \) for the same states \( |1, \sqrt{1 + x^2}, -x; + \rangle, \quad x > 0 \). \( p \) squeezing occurs in the interval \( 0 \leq x \leq 3.8 \). For larger \( |z| \) the \( p \) squeezing is stronger and occurs in wider interval of \( x \). It is worth to underline that in the interval \( 1.8 \leq x \leq 3.8 \) the states \( |1, \sqrt{1 + x^2}, -x; + \rangle \) exhibit \( p \) and \( \tilde{K}_2 \) squeezing simultaneously (joint squeezing of \( p \) and \( \tilde{K}_2 \)). Joint \( q - \tilde{K}_2 \) squeezing occurs in \( |z, \sqrt{1 + v^2}, v; + \rangle \) with negative \( z \) and \( v \) [19] (e.g., for \( z = -1 \) and \( -3.8 \leq v \leq -1.8 \)).

Quadratic and linear squeezing occurs also in SS \( |z, u, v, w; \pm \rangle \) with \( w \neq 0 \). For
example light squeezing of $\tilde{K}_1$ and $\tilde{K}_2$ is found in the squeezed even Schrödinger cats $|\alpha_+\rangle$ [18] [which are of the form (22) for the three observables $K_1, K_2, K_3$]

$$S(\zeta)|\alpha_+\rangle = S(\zeta)|z;+\rangle \equiv |z, \zeta;+\rangle,$$

where $z = a^2/2$ and $K_+|z;\pm\rangle = z|z;\pm\rangle$, $K_- = a^2/2$. Note that $|z, \zeta;\pm\rangle = |z, u, v, w;\pm\rangle$, where

$$u = \cosh^2 r, \quad v = \sinh^2 r e^{2i\theta}, \quad w = \sinh(2r)e^{i\theta}, \quad \zeta = re^{i\theta}.$$ 

In $|z, \zeta;+\rangle$ one has the variances

$$\Delta^2 \tilde{q}_\pm = 1/2 + \langle a^\dagger a \rangle \pm \text{Re}\langle a^2 \rangle,$$

$$\Delta^2 \tilde{X}_\pm = 1 + 2\langle a^\dagger a \rangle + \langle a^\dagger a^2 \rangle \pm \text{Re}\langle a^4 \rangle - \langle \tilde{X}_\pm \rangle^2,$$

where $\tilde{q}_+ = q$, $\tilde{q}_- = p$, $\tilde{X}_+ = \tilde{K}_1$, $\tilde{X}_- = \tilde{K}_2$. The expressions for the means involved are written down in the paper [11].

On Fig. 1b plots of $2\Delta^2 q(d)$ and $\Delta^2 \tilde{K}_1(d)$ are shown for the states $|z, \zeta;+\rangle$ with $z = -d = -|z|$, $\zeta = 0.31$. In the latter the variance of $\tilde{K}_1$ is lightly squeezed for $0.1 < d < 0.31$, and the variance of $q$ is squeezed for $0.17 < d < 0.51$. In the interval $0.17 < d < 0.31$ both variances are squeezed (joint $q$-$\tilde{K}_1$ squeezing). The possibility for joint squeezing of the noncommuting observables $q$, $\tilde{K}_1$ (or $p$, $\tilde{K}_2$) is explained [19] by the fact, that in any even or odd quantum state the Schrödinger inequality reads $\Delta^2 q \Delta^2 \tilde{K}_1 \geq 0$ ($\Delta^2 p \Delta^2 \tilde{K}_2 \geq 0$).

The $su(1, 1)$ SS $|z, u, v, w;\pm\rangle$ can exhibit other nonclassical properties as well, in particular sub- and super-Poissonian photon statistics [19]. In Fig. 2b the super-Poissonian photon number distribution in the $p$ and $\tilde{K}_2$ SS $|z=1, u=\sqrt{10}, v=-3, w=0;+\rangle$ is plotted. Sub-Poissonian statistics occurs in many of the states $|z, u, v;\pm\rangle$, e.g. in $|z, u, v;+\rangle$ with $z = -0.5 - 5i, v = -0.5, u = \sqrt{1.25}$ (see Fig. 2a) and $z = \pm 2.5, u = \sqrt{1+x^2}, v = x$, where $0 < x < 0.5$ [19]. In both cases of the non-Poissonian statistics the photon distributions exhibit well pronounced oscillations. We note that the above pointed nonclassical states with sub-Poissonian statistics are neither $q$ nor $p$ nor $\tilde{K}_1$ or $\tilde{K}_2$ squeezed.

Squeezing and statistical properties of other subsets of algebraic CS $|z, u, v, w; k = \frac{1}{4}, \frac{3}{4}\rangle$ are studied in the recent papers [14, 19, 16, 27] and in the second paper of [7]. In [27] it was wrongly concluded that all $|z, u, v;+\rangle$ with real $u, v$ are super-Poissonian.
\[ \Delta^2 K_2 - 2 \Delta^2 p \]

\[ \Delta^2 K_1 - 2 \Delta^2 q \]

Fig. 1. Squeezing of quadratures of \( a \) and \( a^2 \) in the \( su(1,1) \) even SS \(|z, u, v; +\rangle\).

a) Variances of \( p = i(a^\dagger - a)/\sqrt{2} \) and \( \tilde{K}_2 = i(a^{12} - a^2)/\sqrt{2} \) in generalized even CS \(|1, \sqrt{1 + x^2}, -x, 0; +\rangle, x > 0 \). Joint \( p, \tilde{K}_2 \) squeezing occurs in \( 1.8 < x < 3.8 \).

b) Variances of \( q = (a^\dagger + a)/\sqrt{2} \) and \( \tilde{K}_1 = (a^{12} + a^2)/\sqrt{2} \) in ordinary squeezed even CS \(|z, \zeta; +\rangle\), eq. (3), for \( \zeta = 0.31, z = -d, d > 0 \). Joint \( q, \tilde{K}_1 \) squeezing in \( 0.17 < d < 0.31 \).

Fig. 2. Non-Poissonian photon distributions in generalized even CS \(|z, u, v; +\rangle\):

a) \(|-0.5-5i, \sqrt{1.25}, -0.5; +\rangle, Q < 0 (Q = -0.21), \langle a^\dagger a \rangle = 7.06;\)

b) \(|1, \sqrt{10}, -3; +\rangle, Q > 0, \langle a^\dagger a \rangle = 6.88 \) (this is \( p, \tilde{K}_2 \) joint squeezed state as in Fig. 1a).

Poissonian distributions with mean photon numbers 7.06 and 6.88 are also shown.

For physical realization of a new set of states it is important to know the general form of Hamiltonian operator which preserves the set stable in the time evolution. It was shown [5] that the \( su(1,1) \) states \(|z, u, v; k\rangle\), which minimize the Schrödinger uncertainty relation for \( K_{1,2} \), are stable under the action of \( H \sim \tilde{K}_3 \) only. For the case of \( k = 1/4, 3/4 \) it is the free field Hamiltonian only which keeps the whole set of \(|z, u, v; k\rangle\) stable [19]. However this Hamiltonian can not change the value of \(|v|\). Therefore there is no unitary evolution process in which the squared amplitude SS \(|z, u, v; k\rangle\) can be generated from BG CS \(|z; k\rangle\) or from eigenstates of \( a^2 \) (in particular from Glauber CS \(|\alpha\rangle\)).

For a large subset of \(|z, u, v, w; k\rangle\) there is an other possibility for generation from states which are finite superpositions of orthonormalized eigenstates \(|n + k, k\rangle\) (in particular of Fock states). Indeed, \(|z, u, v, w; k\rangle\) can be represented as \( S(\zeta)|\psi_0(u, v, w)\rangle\), where in accordance with the discussion after eq. (10) the "reference" vector \(|\psi_0\rangle\) is not independent
of the parameters $u, v, w$ (see \[1\]). When in (25) $a = -n$ (i.e., the quantization condition $z = -(k+n)\sqrt{\lambda^2} \equiv z_n$ is imposed) the "reference" state $|\psi_0(u, v, w)\rangle$ is a finite superposition of $|n+k, k\rangle$ which for $k=1/4, 3/4$ are number states. Fine superposition of number states in principle can be experimentally constructed [28]. Then the one mode $su(1, 1)$ SS which are squeezed finite superpositions of number states can be generated using the latter as input in the degenerate amplifier scheme. The $su(1, 1)$ SS in the form of eq. (30) can be generated in the same scheme, using as input the ordinary even/odd CS.

A scheme for generation of two mode $su(1, 1)$ SS in the form of $K_2-K_3$ IS was presented recently by Luis and Perina [22]. A generation scheme for several subsets of one mode $su(1, 1)$ algebraic CS, in particular, for subsets of Schrödinger $K_1-K_2$ and $K_2-K_3$ IS is presented very recently [16]. In both schemes a photon number measurement in the process of generation is involved, i.e. the total evolution process is not unitary, which is in accordance of the stable evolution results [5, 20, 19].

In the scheme of Brif and Mann [16] two light beams of modes $a$ and $b$ are mixed in the non-degenerate parametric amplifier, the mode $a$ being beforehand squeezed in the degenerate parametric amplifier. These processes are described by interaction Hamiltonians

$$H_1 = \frac{1}{2}(g_1a^2 + g_1^*a^2), \quad H_2 = \frac{1}{2}(g_2a^\dagger b^\dagger + g_2^*ab),$$

where parameters $g_{1,2}$ depend on the pump and the media nonlinear characteristics. In [16] the input state was taken as a two mode vacuum, $|\psi_{in}\rangle = |0\rangle_a |0\rangle_b$. If a measurement of $a$-mode photon number is performed in the output beam (with the result $n$) and an $SU(1, 1)$ transformation $\exp(i\omega K_2) \exp(i\varphi K_3)$ is performed on the mode $b$, then for $\omega, \varphi$ suitably chosen, the final state $|\psi_{fin}\rangle \equiv |\lambda, n\rangle$ is an eigenstate of $\lambda K_1 - iK_2$:

$$\langle \lambda K_1 - iK_2 |\psi_{fin}\rangle = (k + |n|)\sqrt{1 - \lambda^2} |\psi_{fin}\rangle,$$

where $|n|$ is the integer part of $n$, $k = 1/4 (3/4)$ for even (odd) $n$, $\lambda = 1/\cosh\omega = \sqrt{|\chi|^2 - 1/|\chi|}$, $|\chi| > 1$ ($0 < \lambda < 1$). The parameter $\chi$ is related to $g_{1,2} = |g_{1,2}|e^{i\theta_{1,2}}$ and the interaction times $t_{1,2}$ according to $\chi = (\tanh |g_1t_1|/\sinh^2 |g_2t_2|) \exp[i(\theta_1 - 2\theta_2)]$ [16]. The choice of the $SU(1, 1)$ parameters is $\varphi = \arg \chi$ and $\tanh \omega = 1/|\chi|$.

We see that for a given $\lambda$ and $n$ this scheme produces only one eigenstate $|\lambda, n\rangle$ of the operator $\lambda K_1 - iK_2 = \frac{1}{2}[(\lambda + 1)K_- + (\lambda - 1)K_+]$. In particular, the even joint squeezed states $|z, u, v; +\rangle$, $z = 1, v = -x, u = \sqrt{1 + x^2}$, the statistical properties of which are presented on Figures 1a and 2b, are beyond the family of final states $|\lambda, n\rangle$.

We produce the whole family $|z, \lambda, k\rangle$, $z \in C$, of the eigenstates of the operator $\lambda K_1 - iK_2$ one has to modify the Brif and Mann scheme in order to introduce extra free parameters. A suitable modification is that which includes two displacements $D(\gamma_1)$ and $D(\gamma_2)$ on the mode $b$.
prior to and after the above described $SU(1,1)$ transformation. Then an input state of the form $|\alpha\rangle_a |0\rangle = D(\alpha)|0\rangle_a |0\rangle_b$, under a certain relation among $\gamma_{1,2}, \varphi$ and $\alpha$, would be finally transformed into $|\psi_{\text{fin}}\rangle$,

$$(\lambda K_1 - iK_2)|\psi_{\text{fin}}\rangle = \frac{1}{2}[\frac{n}{2} + n + \zeta(\gamma_1, \omega, \varphi)]\sqrt{1 - \lambda^2}|\psi_{\text{fin}}\rangle,$$

where

$$\zeta = |\gamma_1|^2 + \gamma_2^2 \coth \omega e^{i\varphi} - \gamma_1 \bar{\alpha} + \frac{1}{2}[\gamma_2^2 \coth \frac{\omega}{2} - \gamma_2^2 \tanh \frac{\omega}{2}],$$

$$\gamma_2 = \tanh \frac{\omega}{2} \left[ \gamma_1 \sinh \frac{\omega}{2} e^{i\varphi/2} - (\gamma_1^* + 2\gamma_1 \chi) \cosh \frac{\omega}{2} e^{-i\varphi/2} \right],$$

$$\bar{\alpha} = (\cosh \frac{\omega}{2} + \sinh \frac{\omega}{2}) e^{i\varphi/2} \text{Re} \tilde{\beta} - i(\cosh \frac{\omega}{2} - \sinh \frac{\omega}{2}) e^{i\varphi/2} \text{Im} \tilde{\beta},$$

$$\tilde{\beta} = (\gamma_1 + 2\gamma_1^* \chi^*) \cosh \frac{\omega}{2} e^{i\varphi/2} + (\gamma_1^* + 2\gamma_1 \chi) \sinh \frac{\omega}{2} e^{-i\varphi/2} +$$

$$\gamma_1 \sinh \frac{\omega}{2} \tanh \frac{\omega}{2} e^{i\varphi/2} - \gamma_1^* \cosh \frac{\omega}{2} e^{-i\varphi/2}.$$  

This $|\psi_{\text{fin}}\rangle$ is of the form of $|z, u, v; k\rangle$, $|u|^2 - |v|^2 = 1$, with $u = (\lambda + 1)/2\sqrt{\lambda}$, $v = (\lambda - 1)/2\sqrt{\lambda}$ and $z = (1/2)(1/2 + n + \zeta)\sqrt{(1 - \lambda^2)/\lambda}$. The $p\bar{K}_2$ joint SS shown on Figure 2b is produced by this generation scheme when $|\chi| = \coth \omega = 1.0004$, $\varphi = 0$, $n = 0$ and $\gamma_1 = 0.91$. The sub-Poissonian even SS shown on Figure 2a is produced when $|\chi| = \coth \omega = 1.08$, $\varphi = 2.374$, $n = 0$ and $\gamma_1 = 1.01 + i0.36$.

4 Conclusion

We have examined three possible ways of generalization of the one mode canonical squeezed states (CSS) to the case of $n$ arbitrary observables $X_j$ on the basis of the Robertson uncertainty relation and the group of automorphisms of the corresponding complexified Lie algebra. The eigenstate way of generalization appeared as the most general one. The cases of $N$ pairs of canonical observables $p_\nu$, $q_\nu$ (which span the algebra $h_N$ and generates the group $H_N$) and the three quasi-spin observables $K_j$ (the generators of $SU(1,1)$) are considered as examples. The case of nilpotent algebra $h_N$ is most symmetric in the sense that the three constructions of $\vec{X}$-SS by means of eqs. (2), (10) and (13) are equivalent to each other. For the simple $su(1,1)$ algebra such equivalence is lacking. In this context it is shown that the overcomplete family of eigenstates $|z; k\rangle$ of $su(1,1)$ lowering operator $K_-$ (the Barut-Girardello coherent states (CS)) and the continuous family of ideal SS $|z, u, v; k\rangle$ (eigenstates of $uK_- + vK_+$) are not orbits of the group $\text{Aut}(su_C(1,1)) \supset SU(1,1)$. Such continuous sets are examples of proper algebraic coherent states [11].

Eigenstates $|z, u, v, w; k\rangle$ of general complex combinations of $K_j$ are explicitly constructed using the analytic Barut-Girardello CS representation. These are ideal SS for
the three $SU(1,1)$ generators. Their nonclassical properties are analyzed in the case of $k = 1/4, 3/4$, when $K_j$ are quadratic combinations of boson/photon operators $a$ and $a^\dagger$. Intelligent even states $|z, u, v, 0; +\rangle$ are pointed out which exhibit sub-Poissonian photon statistics and joint linear and quadratic amplitude squeezing. These sub-Poissonian and joint SS could be produced from the canonical CS by an appropriate modification of the recently proposed Brif and Mann generation scheme [16]. Joint SS of the field could be useful in improvement of the sensitivity of the interferometric measurements.

**Acknowledgments.** The author thanks prof. V.I. Man’ko for valuable comments. The work is partially supported by the Bulgarian Science Foundation under Contracts No. F-559, F-644.
Appendix

A. Equivalence of the three definitions of SS for $2N$ canonical observables

The three generalized definitions of SS for $h_N$ algebra elements $p_\nu$, $q_\nu$ are given by the eqs. (2), (10) and (13) with $X_j = Q_j$, $Q_\nu = p_\nu$ and $Q_{N+\nu} = q_\nu$. In ref. [20] it was proven that in case of $X_j = Q_j$ the equality (13) entails (2) with $n_c = N$ and (10) as well. What remains to be proven in order to establish the equivalence of the three definitions (2), (10) and (13) is that eq. (2) with any $n_c \geq 1$ also entails (13).

Proposition 1. If a state $|\psi\rangle$ is an eigenstate of an operator $A(\vec{\beta}, \vec{\gamma}) = \beta_\nu p_\nu + \gamma_\nu q_\nu,

$$A(\vec{\beta}, \vec{\gamma})|\psi\rangle = z|\psi\rangle,$$

then $|\psi\rangle$ is an eigenstate of $N$ new boson operators $A_\nu$, which are linear in $p_\nu$ and $q_\nu$.

Proof. Let $|\psi\rangle$ satisfy eq. (45). In coordinate representation the solution to (45) is given by an exponent of a quadratic (see eq. (18)),

$$\psi_z(\vec{q}, \vec{\beta}, \vec{\gamma}) = \tilde{N} \exp \left[ -\vec{q}\mathcal{M}\vec{q} + \vec{N}\vec{q} \right],$$

where

$$\mathcal{M}_{\nu\mu} = -(i/2)\beta_\nu^{-1}\gamma_\mu, \quad \vec{N}_\nu = i\beta_\nu^{-1}z,$$

The matrix $\mathcal{M}$ is symmetric and the state $\psi_z(\vec{q}, \vec{\beta}, \vec{\gamma})$ is normalizable iff $\mathcal{M}^* + \mathcal{M}$ is positive definite.

Let us now treat $\mathcal{M}$ and $\vec{N}$ in eq. (47) as given, $z$ as arbitrary and consider (47) as algebraic equations for $\gamma$ and $\beta$. The solution is easily seen to be not unique. For an arbitrary $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_N)$ the vector $\vec{\gamma}$ is uniquely determined as $\vec{\gamma} = -2i\mathcal{M}\vec{\beta}$. Thus for every state $\psi(\vec{q}, \mathcal{M}, \vec{N})$ of the form (46), i.e. for every given $\mathcal{M}$ and $\vec{N}$, we have an $N$ complex parameter family of linear in $p_\nu$, $q_\nu$ operators $A(\mathcal{M}, \vec{N}; \vec{\beta})$ which have the same state $\psi(\vec{q}, \mathcal{M}, \vec{N})$ as their common eigenstate. Since parameters $\beta_\nu$ are free we may choose $N$ vectors $\vec{\beta}(\nu)$ (i.e. a matrix $\beta$), consider then $N$ operators from this family,

$$A_\mu(\mathcal{M}, \vec{N}; \beta) = \beta_{\mu\nu} p_\nu + \gamma_{\mu\sigma}(\beta) q_\sigma, \quad \gamma_{\mu\sigma}(\beta) = -2i(\beta\mathcal{M}^T)_{\mu\sigma},$$

and try to subject $A_\mu(\mathcal{M}, \vec{N}; \beta)$ to the canonical boson commutation relations (19). The latter require $\beta\gamma^T - \gamma\beta^T = i$ and $\beta\gamma^T - \gamma\beta^T = 0$. Substituting here $\gamma = -2i\beta\mathcal{M}^T$ we get equations for $\beta$,

$$\beta(\mathcal{M}^* + \mathcal{M}^T)\beta^T = \frac{1}{2}, \quad \beta\mathcal{M}\beta^T - \beta\mathcal{M}^T\beta^T = 0.$$
is congruent to a multiple of unity by the matrix \( \beta \) which is symplectic or is a product of one orthogonal and one diagonal positive matrix \([29]\). Thus eq. \((19)\) always has a solution (not unique) for \( \beta \) which ensures the canonical boson relations \((19)\) for \( A_\mu(\mathcal{M}, \mathcal{N}; \beta) \). End of proof.

**B. The sets of \( K_1-K_2 \) SS \( |z, u, v; k\rangle \) and the Barut-Girardello CS are not orbits of \( \text{Aut}(su^C(1,1)) \)**

Unlike the case of \( h_N \) (canonical observables) the eigenstates of linear combinations of \( su(1,1) \) operators \( K_j \) (quasi-spin observables) and of other semisimple Lie algebras can not always be presented in the form \([10]\), i.e. as orbit of unitary operators from \( \text{Aut}(su^c(1,1)) \). In subsection 2.2 we have noted that any \( \text{Aut}(L^C) \equiv G_A \) group related CS with symmetry is of the form of \([2]\), i.e. is an eigenstate of a complex linear combination of algebra operators \( X_i \), while the inverse is generally not true. In this Appendix we provide such a ”negative” examples of generalized SS: we shall prove that the \( su(1,1) \) Schrödinger intelligent states (IS) \( |z, u, v; k\rangle \) (constructed first in \([3]\)) can’t be represented in the form of eq. \((10)\)). These states can exhibit arbitrary strong squeezing in \( SU(1,1) \) generators \( K_1 \) or \( K_2 \) when \( v \to \pm u \ (|v| < |u|) \), i.e. they are ideal SS of the form \([2]\). But the family of states \( |z, u, v; k\rangle \) is not neither \( SU(1,1) \) nor \( \text{Aut}(su^C(1,1)) \) unitary orbit of any fixed reference vector \( |\psi_0\rangle \), as we are going to prove below.

The \( K_1-K_2 \) SS \( |z, u, v; k\rangle \) are defined as eigenstates of \( A_-(u, v) = uK_- + vK_+ \),

\[
A_-(u, v)|z, u, v; k\rangle = z|z, u, v; k\rangle, \tag{50}
\]

and for \( k = 1/2, 1, \ldots \) and \( k = 1/4, 3/4 \) are explicitly constructed in subsection 3.2.

**Proposition 2.** There is no Hilbert space vector \( |\psi_0\rangle \) such that the family \( \{|z, u, v; k\rangle\} \) be an \( \text{Aut}(su^C(1,1)) \) unitary orbit of \( |\psi_0\rangle \), i.e.

\[
|z, u, v; k\rangle = U(u, v, z)|\psi_0\rangle, \quad U \in \text{Aut}(su^C(1,1)), \quad U^\dagger = U^{-1}. \tag{51}
\]

**Proof.** Let us suppose the inverse, i.e. let \((51)\) holds for some state vector \( |\psi_0\rangle \), which is independent of the parameters \( u, v, z \). We shall show that this leads to a contradiction (and therefore \((51)\) is impossible).

In view of \( U \in \text{Aut}(su^C(1,1)) \) the transformation \( K_i \to U^\dagger(z, u, v)K_iU(z, u, v) \equiv K'_i \) is linear in \( K_j \). One has

\[
U^\dagger(z, u, v)A_-(u, v)U(z, u, v) = \mu(z, u, v)K_- + \nu(z, u, v)K_+ + \sigma(z, u, v)K_3, \tag{52}
\]

Eqs. \((52)\), \((51)\) and \((50)\) imply that

\[
(\mu(z, u, v)K_- + \nu(z, u, v)K_+ + \sigma(z, u, v)K_3)|\psi_0\rangle = z|\psi_0\rangle \tag{53}
\]
It is worth noting that the invariance of the Killing form $B(X, Y)$ for $su^C(1, 1)$ \[ \tag{12} \sigma^2(z, u, v) - 4\mu(z, u, v)\nu(z, u, v) = 4uv = B(A_-, A_-), \] requires that neither $\sigma$ and $\mu$ nor $\sigma$ and $\nu$ can vanish simultaneously. At $v = 0$ (then $A_-(u, 0) = uK_-$ and we put $u = 1$) \[ \tag{54} \] reads $\sigma^2(z) - \mu(z)\nu(z) = 0$. It is interesting to note that using only the invariance of the Killing form one can easily derive that the orthonormalized eigenstates $|k, k+n\rangle$ of $K_3$ can’t satisfy eq. \[ \tag{53}, \] i.e. $|\psi_0\rangle \neq |k, k+n\rangle$, $n = 0, 1, \ldots$ [For $k = 1/4, 3/4 |k, k+n\rangle$ coincide with the Fock states $|n\rangle$]. For the sake of brevity henceforth we write $|n\rangle$ instead of $|k, k+n\rangle$.

Generally $|\psi_0\rangle$ is a superposition of $|n\rangle$, \[ |\psi_0\rangle = \sum_{n=0} C_n |n\rangle, \sum_n |C_n|^2 = 1. \] (55)

Substituting \[ \tag{54} \] into \[ \tag{53} \] we obtain the recurrence relations for the coefficients $C_n$, \[ \mu(z, u, v)\sqrt{n+1}C_{n+1} + \nu(z, u, v)\sqrt{n}C_{n-1} + \sigma(z, u, v)(n+k)C_n = zC_n. \] (56)

It is sufficient to prove that \[ \tag{51} \] is impossible for some subset of states $|z, u; v; k\rangle$. We shall carry out the proof for the subsets \{|0, u; v; k\rangle\} and \{|z, 1, 0; k\rangle\}. Note that the states $|z, 1, 0; k\rangle$ are the BG $|z\rangle$, $|z, 1, 0; k\rangle = |z; k\rangle$. Let us first choose the subset $|0, u; v; k\rangle$. We easily see that if $C_0 = 0 = C_1$ then all $C_n = 0$. Moreover both $C_0$ and $C_1$ are nonvanishing. For $n = 0$ and $n = 1$ \[ \tag{53} \] produces \[ C_1\mu = k\sigma C_0, \quad \mu C_2\sqrt{2} = -\nu C_0 - (k+1)\sigma C_1. \] (57)

From $C_1\mu = k\sigma C_0$ we derive that $C_1 = 0 \iff C_0 = 0$. Indeed, if e.g. $C_0 = 0$ but $C_1 \neq 0$, then $\mu = 0$ and the second equation in \[ \tag{57} \] yields $\sigma = 0$ which contradicts to \[ \tag{54} \]. Thus $C_0 \neq 0 \neq C_1$. The two eqs. \[ \tag{57} \] tell us also that the ratios $\mu/\sigma$ and $\nu/\sigma$ must be $u$ and $v$ independent. Denoting $\mu/\sigma = kC_0/C_1 \equiv a_1, \nu/\sigma = a_2$ we rewrite \[ \tag{54} \] as $\sigma^2(0, u, v)(1 - a_1a_2) = 4uv$, which at $v = 0$ produces $\sigma(0, 1, 0) = 0$ [since $1 - a_1a_2 = 0$ would lead to $0 = 4uv$]. On the other hand is eq. \[ \tag{52} \], which now reads $U^\dagger(0, u, v)A(u, v)U(0, u, v) = \sigma(0, u, v)[a_1K_- + a_2K_+ + K_3]$. Herefrom at $v = 0$ we obtain $U^\dagger(0, 1, 0)K_-U(0, 1, 0) = 0$ which is impossible for the unitary operator $U \neq 0 [U(0, 1, 0) = 0$ would lead to $|k, k\rangle = 0$ since $|0\rangle \equiv |k, k\rangle = U(0, 1, 0)|\psi_0\rangle]$. This contradiction proves that the continuous set of states $|0, u; v; k\rangle$ (which are annihilated by $uK_- + vK_+$) is not an $\text{Aut}(su^C(1, 1))$ orbit of any reference vector. End of proof.

**Proposition 3.** The set of Barut-Girardello CS $|z; k\rangle$ is not an $\text{Aut}(su^C(1, 1))$ unitary orbit.

**Proof.** The BG CS constitute a subset of $|z, u; v; k\rangle$, $|z; k\rangle = |z, 1, 0; k\rangle$. We follow the scheme of the proof of Proposition 2. Let us first note that if the unitary operator
$U(z) = U(z,1,0)$ (obeying (51) with $u = 1$, $v = 0$) exists, it cannot commute with $K_-$, otherwise $K_-|z\rangle = U(z)K_-|0\rangle = 0$. The recurrence relations for the coefficients $C_n$ in (53) are given by eq. (56) and this time the Killing form vanishes identically with respect to $z$:

$$\sigma^2(z) - 4\mu(z)\nu(z) = 0. \quad (58)$$

Again $C_0 \neq 0 \neq C_1$, and instead of (57) now we have

$$C_1\mu = (z - k\sigma)C_0, \quad \mu C_2\sqrt{2} = -\nu C_1 - [(k+1)\sigma - z]C_0. \quad (59)$$

After some consideration we get from (59) that the ratios $\sigma/\mu \equiv a_2 \neq 0$, $\nu/\mu \equiv a_1 \neq 0$ and $z/\mu \equiv a_3$ are independent of $z$. Then we rewrite (52) in the form

$$U^\dagger(z)K_- U(z) = za_3[K_- + a_1K_+ + a_2K_3], \quad (60)$$

which at $z = 0$ yields the contradiction $U^\dagger(0)K_- U(0) = 0$. For $z \neq 0$ another contradictions arise: Let us apply both sides of (60) to $U^\dagger(z)|0\rangle$. This gives $0 = [K_- + a_1K_+ + a_2K_3]U^\dagger(z)|0\rangle$. In view of $U(z) \in \text{Aut}(su(1,1))$ we have $U(z)[K_- + a_1K_+ + a_2K_3]U^\dagger(z) = \mu'(z)K_- + \nu'(z)K_+ + \sigma'(z)K_3 \neq 0$ and thus $0 = [\mu'(z)K_- + \nu'(z)K_+ + \sigma'(z)K_3]|0\rangle = [\nu'(z)K_+ + \sigma'(z)k]|0\rangle$. One sees that the last equality is possible if and only if $\nu' = 0 = \sigma'$. Then, we have $(1 - \mu'(z))U(z)K_- U^\dagger(z) = U(z)[a_1K_+ + a_2K_3]U^\dagger(z)$. Applying this to $U(z)|0\rangle$ (and noting that $\mu' \neq 1$, otherwise the commutator $[K_-, U(z)]$ vanishes) we have $0 = [a_1K_+ + a_2k]|0\rangle$, which is impossible since $a_1 \neq 0 \neq a_2$. End of the proof.

Remark: The Propositions 2 and 3 are valid for any Hermitian representation of $su(1,1)$ for which the BG CS $|z\rangle$ and the eigenstates $|z,u,v\rangle$ of $uK_- + vK_+$ exist. These results can be extended to semisimple Lie groups.
References

[1] Loudon R. and Knight P.L., J. Mod. Opt. 34 709 (1987).
[2] Stoler D.A., Phys. Rev. D 1 3217 (1970); Yuen H., Phys. Rev. A 13 2226 (1976).
[3] Dodonov V.V., Kurmyshev E. and Man’ko V.I., Phys. Lett. A 79 150 (1980).
[4] Trifonov D.A., J. Math. Phys. 34 100 (1993).
[5] Trifonov D.A., J. Math. Phys. 35 2297 (1994); Trifonov D.A., Generalized intelligent states and SU(1,1) and SU(2) squeezing, Preprint INRNE-TH-93/4, May 1993.
[6] Wodkiewicz K. and Eberly J., J. Opt. Soc. Am. B 2 458 (1985).
[7] Vaccaro J. and Pegg D., J. Mod. Opt. 37 17 (1990); Bergou J.A., Hillery M. and Yu D., Phys. Rev. A 43 515 (1991); Kitagawa M. and Ueda M., Phys. Rev. A 47 5138 (1993); Nieto M.M. and Truax D.R., Phys. Rev. Lett. 71 2843 (1993).
[8] Malkin I.A., Man’ko V.I. and Trifonov D.A., Phys. Rev. D 2 1371 (1970); J. Math. Phys. 14 576 (1973).
[9] Zhang W.M., Feng D.H. and Gilmore R., Rev. Mod. Phys. 62 (1990) 867.
[10] Aragone C., Chalband E. and Salamo S., J. Math. Phys. 17 (1976) 1963.
[11] Trifonov D.A., Algebraic coherent states and squeezing, E-print quant-ph/9609001.
[12] Barut A.O. and Raczka R. Theory of Group Representations and Applications (Polish Publishers, Warszawa, 1977).
[13] Robertson H.R., Phys. Rev. 46 (1934) 794.
[14] Brif C., Int. J. Theor. Phys. 36 (1997) 1677 [E-print quant-ph/9701003].
[15] Trifonov D.A., Uncertainty matrix, multimode squeezed states and generalized even and odd coherent states, Preprint INRNE-TH-95/5 (1995) (1995 Annual Report of INRNE, p. 57).
[16] Brif C. and Mann A., Quant. Semiclass. Opt. 9 (1997) 879 [E-print quant-ph/9707003].
[17] Barut A.O. and Girardello L., Commun. Math. Phys. 21 (1971) 41.
[18] Dodonov V.V., Malkin I.A. and Man’ko V.I., Physica 72 (1974) 597.
[19] Trifonov D.A., Schrödinger intelligent states and linear and quadratic amplitude squeezing, E-print quant-ph/9609017.
[20] Trifonov D.A., J. Phys. A 30 (1997) 5941 [E-print quant-ph/9701018].
[21] Ma X. and Rhodes W., Phys. Rev. A 41 (1990) 4624; Dodonov V.V., Man’ko O.V. and Man’ko V.I., Phys. Rev. A 50 (1994) 813.

[22] Brif C. and Mann A., Phys. Rev. A, 54 (1996) 4505; Luis A. and Perina J., Phys. Rev. A 53 (1996) 1886.

[23] Handbook of mathematical functions, edited by M. Abramowitz M. and Stegun I.A. (National bureau of standards, 1964) (Russian translation, Nauka, Moskva, 1979).

[24] H. Bateman and A. Erdélyi, Higher transcendentlal functions (McGraw-Hill, New York, 1953), v. 1 (Russian translation, Nauka, Moskva, 1973).

[25] B. Nagel, in Modern Group Theoretical Methods in Physics, (J. Bertrand et al., eds, Kluwer Academic Publishers 1995), p. 211 [E-print quant-ph/9711018]; Wünsche A., Acta Phys. Slovaca 45 (1995) 413.

[26] Brif C., Ann. Phys. 251 (1996) 180.

[27] Marian P., Phys. Rev. A, 55 3051 (1997).

[28] Janszky J., Domokos P., Szabo S. and Adam P., Phys. Rev. A 51 (1995) 4191.

[29] Gantmaher, F., Teoria Matrits (Nauka, Moskva, 1975).
Fig. 1. Squeezing of quadratures of a and $a^2$ in the $su(1, 1)$ even SS $|z, u, v, w; +\rangle$.

a) Variances of $p = i(a^\dagger - a)/\sqrt{2}$ and $\tilde{K}_2 = i(a^{12} - a^2)/\sqrt{2}$ in generalized even CS $|1, \sqrt{1 + x^2}, -x, 0; +\rangle$, $x > 0$. Joint $p$, $\tilde{K}_2$ squeezing occurs in $1.8 < x < 3.8$.

b) Variances of $q = (a^\dagger + a)/\sqrt{2}$ and $\tilde{K}_1 = (a^{12} + a^2)/\sqrt{2}$ in ordinary squeezed even CS $|z, \zeta; +\rangle$, eq. (10), for $\zeta = 0.31$, $z = -d$, $d > 0$. Joint $q$, $\tilde{K}_1$ squeezing in $0.17 < d < 0.31$. 
Fig. 2. Nonpoissonian photon distributions in generalized even CS $|z, u, v; +\rangle$:
a) $|{-1, 10i, -0.5}; +\rangle$, $Q < 0$ ($Q = -0.21$), $\langle a^\dagger a \rangle = 7.06$;
b) $|1, \sqrt{10}, -3; +\rangle$, $Q > 0$, $\langle a^\dagger a \rangle = 6.88$ (this is $p, \tilde{K}_2$ joint squeezed state as in Fig. 1a). Poissonian distributions with mean photon numbers 7.06 and 6.88 are also shown.