Generalised Manin transformations and QRT maps

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Abstract

Manin transformations are maps of the plane that preserve a pencil of cubic curves. We generalise to maps that preserve quadratic, and certain quartic and higher degree pencils, and show they are measure preserving. The full 18-parameter QRT map is obtained as a special instance of the quartic case in a limit where two double base points go to infinity. On the other hand, each generalised Manin transformation can be brought into QRT-form by a fractional affine transformation. We also classify the generalised Manin transformations which admit a root.

1 Introduction

We consider pencils of curves of the form \( P_{\alpha,\beta}(u,v) = 0 \), with
\[
P_{\alpha,\beta}(u,v) := \alpha F_a(u,v) + \beta F_b(u,v),
\]
where \( F_a(u,v) \) is a polynomial in two variables \( u,v \) of fixed total degree \( D \) which depends on parameters \( a_1, a_2, \ldots \), and \( F_a \neq F_b \). If the degrees of \( F_a \) and \( F_b \) are not equal then we take \( D \) to be the largest of the two degrees. For all \( (u,v) \) there are \( \alpha, \beta \) such that \( P_{\alpha,\beta}(u,v) = 0 \), i.e. \( \frac{\beta}{\alpha} = -\frac{F_b}{F_a}(u,v) \). For some \( (u,v) \) we have \( P_{\alpha,\beta}(u,v) = 0 \) for all \( \alpha, \beta \). These are called base points, and there are \( D^2 \) of them (considering \( (u,v) \) to be affine coordinates and counting multiplicities), namely the solutions of \( F_a = F_b = 0 \).

For a base point \( p \) of a cubic (\( D = 3 \)) pencil, any line through \( p \) intersects each curve in two points, say \( x \) and \( y \). The map \( \iota_p \) which sends \( x \) to \( y \) (and hence \( y \) to \( x \)) is called a Manin involution. The composition of two Manin involutions is called a Manin transformation. It was first described in [7], see also [5, Section 4.2]. We call \( p \) the involution point of the map \( \iota_p \), and we refer to the map \( \iota_p \) as a \( p \)-switch.

We generalise this construction to maps that preserve pencils of curves which have degree \( D > 1 \). For \( D = 2 \) there are no constraints, neither on the pencil, nor on the positions of the involution points. For \( D = 4 \) we require the pencil to have two base points, \( p \) and \( q \), which are singular points, and which we choose to be the involution points. For \( D > 4 \) the base points \( p \) and \( q \) are required to be singular points of multiplicity \( D - 2 \). This ensures that any line through \( p \) intersects each curve of the degree \( D \) pencil in only two points.

Under these constraints, in section 2, we provide an explicit formula, proven in Appendix A, for the generalised Manin involution \( \iota_p \) that preserves such a pencil of degree \( D \), in terms of the polynomials \( F_a \) and \( F_b \) and their first and second order partial derivatives. In Appendix B we derive a condition which enables one to verify that \( \iota_p \) is anti measure preserving with density \( L^{D-3}/F_a \), where \( L = 0 \) is a line through \( p \).

\footnote{A curve \( C(u,v) = 0 \) has a singular point of multiplicity \( m \) if \( m \geq 1 \) is the smallest number such that all \( k \)-th order partial derivatives with \( k < m \) vanish at \( (c,d) \) [12]. A singular point of multiplicity \( m \) is also called a double point \( (m = 2) \), a triple point \( (m = 3) \), or an \( m \)-ple point.}

\footnote{Recall [11, Section 2.2] that a map \( \phi \) is (anti) measure preserving with density \( \rho \) if its Jacobian \( J \) equals \( (-)\rho/(\rho \circ \phi) \).}
In section 3 we consider the degree $D = 2$ case. Taking two different involution points $p$ and $q$, the 16-parameter map $\tau = \iota_q \circ \iota_p$ is measure preserving with density $1/(L(u,v)F_a(u,v))$, where $L = 0$ is the line through $p$ and $q$. Using Pascal’s hexagrammum mysticum theorem, we show that for any $r$ on $L = 0$ the map $\tau_r$ is a reversing symmetry. This implies that the map $\tau$ has uncountably many symmetries. In section 4 we consider the degree $D = 3$ case. We parameterise the pencil in terms of the coordinates of two distinct base points $p$ and $q$. The 20-parameter map we obtain explicitly, $\tau = \iota_q \circ \iota_p$, is measure preserving with density $1/F_a(u,v)$.

We point out that with $p = (c,0)$ and $q = (0,f)$ in the limit where $c, f \to \infty$, both in the quadratic case and in the cubic case, a special instance of the QRT map \[9,10\] is obtained. In that limit the switch $\iota_p$ becomes the horizontal switch, $\iota_1$, and $\iota_q$ becomes the vertical switch, $\iota_2$, cf. [5 Preface, page viii].

In section 5 pencils of degree $D = 4$ are considered. With $p$ and $q$ double base points, the 22-parameter map $\tau = \iota_q \circ \iota_p$ is measure preserving with density $L(u,v)/F_a(u,v)$. Because the general biquadratic polynomial is a quartic polynomial with double points at $(\infty,0)$ and $(0,\infty)$, the full 18-parameter QRT-map is obtained as a special instance of the degree $D = 4$ generalised Manin transformation.

In section 6 we show that polynomials $F$ of degree $D \geq 5$ with two singular points, $p, q$, of multiplicity $D - 2$ factorise as $F = L^{D-4}Q$ where $L = 0$ is the line through $p$ and $q$, and $Q$ is a quartic polynomial. This allows us to prove that the generalised Manin transformation does not depend on $D \geq 4$.

In section 7 we specify subfamilies of generalised Manin transformations which admit a root, i.e. maps that can be written as $\tau = \rho^2$, such as the 12-parameter symmetric QRT-map.

We complete the picture in section 8, where we show that each generalised Manin transformation can be brought into QRT-form by applying a fractional affine transformation, namely the transformation that sends $p$ to $(\infty,0)$ and $q$ to $(0,\infty)$.

## 2 The generalised Manin involution

The formula $(x,y) = (u + (c - u)z, v + (d - v)z)$ gives a parametrisation of the line going through $(u, v)$, for $z = 0$, and through $(c, d)$, for $z = 1$. Below we provide the value of $z$ such that $(x, y)$ and $(u, v)$ are on the same curve of the given pencil, i.e. such that $F_a(x,y)F_b(u,v) = F_a(u,v)F_b(x,y)$. Denote $F_a(z) := F_a(u + (c - u)z, v + (d - v)z)$, and $F_a^{(z)} := \left. \frac{d}{dz} F_a \right|_{z=0}$. A Taylor expansion, about $z = 0$, gives

$$F_a(z) = F_a(0) + F_a^{(z)}(0)z + \frac{1}{2} F_a^{(z,z)}(0)z^2 + \cdots + \frac{1}{D!} F_a^{(z,\ldots,z)}(0)z^D, \quad (2)$$

where

$$F_a^{(z,\ldots,n,z)}(0) = \sum_{i=0}^n \binom{n}{i} F_a^{(u,\ldots,u,v,\ldots,v)}(u,v)(c-u)^i(d-v)^{n-i}. \quad (3)$$

For $D > 2$ we have the following equations

$$F_a(1) = F_a^{(z)}(1) = \cdots = F_a^{(z,\ldots,3,z)}(1) = 0,$$

and similarly for $F_b$. These equations are used in appendix A to prove the explicit formula for the generalised Manin involution, given in the following theorem.

**Theorem 1.** Let $F_{a,b}(u,v) = 0$ be a pencil of degree $D \geq 2$ and let $p$ be a point which for $D > 2$ is a base point and has multiplicity $D - 2$. Then the generalised Manin involution with involution point $p = (c,d)$ is given by

$$\iota_p : (u,v) \to (u,v) + z(c-u,d-v), \quad (4)$$

where $z$ is given by

$$z = 2 \left( 2(2-D) - \frac{F_a(0)F_b^{(z,z)}(0) - F_a^{(z,z)}(0)F_b(0)}{F_a(0)F_b^{(z)}(0) - F_a^{(z)}(0)F_b(0)} \right)^{-1}. \quad (5)$$


3 The quadratic case

Let

\[ F_a(u, v) := a_1 + a_2 u + a_3 v + a_4 u^2 + a_5 u v + a_6 v^2 \]  

be a polynomial of degree \( D = 2 \) in variables \( u, v \), that is \( a_4, a_5 \) and \( a_6 \) are not all zero. So we have a quadratic pencil of curves (of genus zero), \( P_{\alpha, \beta}(u, v) = 0 \). Any point \( p = (c, d) \) can be taken as involution point. An involution is defined by

\[ \iota_p(u, v) = (u, v) + z(c - u, d - v), \]  

with \( z \) given by (4) (or alternatively by (33)), where, explicitly,

\[
\begin{align*}
F_a(0) &= F_a(u, v), \\
F_a^{(z)}(0) &= F_a^{(u)}(u, v)(c - u) + F_a^{(v)}(u, v)(d - v), \\
F_a^{(z,z)}(0) &= F_a^{(u,u)}(u, v)(c - u)^2 + 2F_a^{(u,v)}(u, v)(c - u)(d - v) + F_a^{(v,v)}(u, v)(d - v)^2,
\end{align*}
\]

and \( F_a^{(u)}(u, v) = a_2 + 2a_4 u + a_5 v, F_a^{(v)} = a_3 + a_5 u + 2a_6 v, F_a^{(u,u)}(u, v) = 2a_4, F_a^{(u,v)}(u, v) = a_5, F_a^{(v,v)}(u, v) = 2a_6. \)

Example 2. Ten curves from the pencil \( P_{\alpha, \beta}(u, v) = 0 \) with

\[ F_a(u, v) = u^2 - u v + v^2 + u - v - 2 \quad \text{and} \quad F_b(u, v) = u v, \]  

are plotted in Figure 1.

![Figure 1: Ten curves from the quadratic pencil defined by (1) and (8), labeled by the value of \(-\beta/\alpha\).](image)

Taking \( p = (2, -2) \) yields the involution

\[ \iota_{2,-2}(u, v) = -\frac{2}{u - v - 2}(v, u), \]  

and taking \( p = (-1, 1) \) yields the involution

\[ \iota_{-1,1}(u, v) = \frac{-v(2u + v + 1), u(u + 2v - 1)}{u^2 + uv + v^2 - 1}. \]
In general, the involution \[\iota\] has the form
\[
\iota_p(u,v) = \frac{(N_1(u,v), N_2(u,v))}{D(u,v)},
\]
where \(N_1\) and \(D\) are generically of degree \(t = 3\). If \(t = 3\) the point \(p = (c,d)\) is a double point on \(N_1 = 0\), \(N_2 = 0\) and on \(D = 0\), and all points on the curve \(C\) defined by \(F_a(c,d)F_b(u,v) = F_b(c,d)F_a(u,v)\) are mapped to \((c,d)\). When \(p\) is a point on one line through two base points, the degree is lowered to \(t = 2\), and \(p\) is a simple point on \(N_1 = 0, N_2 = 0\), and on \(D = 0\). An example is given by (10). Here the map \(\iota_p\) is an involution on the line that contains \(p\), but the other line of the union \(C\) is mapped to \(p\). When \(p\) is the intersection of two straight lines through two base points, the degree is lowered to \(t = 1\) and \(p\) is a not on \(N_1 = 0, N_2 = 0\) or on \(D = 0\). The involution is an involution on both lines, (9) provides an example. For base points \(p\) the degree is \(t = 0\), i.e. we have \(\iota_p = \text{id}\), the identity.

The involution \([6]\) is anti measure preserving with density
\[
\frac{1}{(p(u - c) + q(v - d))F_a(u,v)},
\]
where the first factor represents any straight line through \((c,d)\). Taking the composition of two such involutions \([6]\) we construct a map
\[
\tau_{p,q} = \iota_q \circ \iota_p.
\]
The following holds.

**Theorem 3.** The map \(\tau_{p,q}\) defined by (12) is an integrable map of the plane. It preserves each curve of the quadratic pencil \(P_{\alpha,\beta}(u,v) = 0\) with \([3]\), and it is measure preserving with density
\[
\frac{1}{L(u,v)F_a(u,v)},
\]
where
\[
L(u,v) = (d - f)(u - e) - (c - e)(v - f),
\]
so that \(L = 0\) is the line through the involution points \(p = (c,d)\) and \(q = (e,f)\).

Let us now define two special involutions,
\[
\iota_1 = \lim_{c \to \infty} \iota_{c,0}, \quad \iota_2 = \lim_{d \to \infty} \iota_{0,f}.
\]
These are called the horizontal, respectively vertical, switch in [3] page viii]. Considering the involution \([3]\), it is clear that \(z\) is of the form \(z = N/D\) where \(N\) is linear in \(c,d\), and \(D\) quadratic. Hence, the involutions have the form \(\iota_1(u,v) = (u + cz, v)\), and \(\iota_2(u,v) = (u, v + fz)\). In the respective limits we find
\[
cz = -2\frac{F_a(u,v)F_b^{(u)}(u,v) - F_b(u,v)F_a^{(u)}(u,v)}{F_a(u,v)F_b^{(u,u)}(u,v) - F_b(u,v)F_a^{(u,u)}(u,v)},
\]
and
\[
fz = -2\frac{F_a(u,v)F_b^{(v)}(u,v) - F_b(u,v)F_a^{(v)}(u,v)}{F_a(u,v)F_b^{(u,v)}(u,v) - F_b(u,v)F_a^{(u,v)}(u,v)}.
\]
The map \(\tau = \iota_2 \circ \iota_1\) is a special case of the asymmetric QRT-map \([9][10]\), with matrices
\[
A^0 = \begin{pmatrix} 0 & 0 & a_4 \\ 0 & a_5 & a_2 \\ a_6 & a_3 & a_1 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & b_4 \\ 0 & b_5 & b_2 \\ b_6 & b_3 & b_1 \end{pmatrix},
\]

The involutions $\iota_1$ and $\iota_2$ are anti measure preserving with densities
\[ \frac{1}{F_a(u,v)(r_1v + r_2)}, \quad \frac{1}{F_a(u,v)(s_1u + s_2)} \]
respectively, for arbitrary $r_i, s_i$. This implies in particular that $\tau$ is measure preserving with density $1/F_a(u,v)$, and $\iota_{c,d} \circ \iota_1$ is measure preserving with density $1/((v - d) F_a(u,v))$.

**Symmetries**

The following theorem follows from Pascal’s theorem, which is illustrated by Figure 2.

Figure 2: Lines through opposite sides of a hexagon on a conic meet in three points which lie on a straight line, called the Pascal line.

**Theorem 4.** The map $\tau_{p,q}$ defined by (13) has uncountably many symmetries.

**Proof.** We first show that the map $\tau_{p,q}$ has uncountably many reversing symmetries. Let $r$ be on the line through $p$ and $q$, and let
\[ B = \iota_p(A), \quad C = \iota_q(B), \quad D = \iota_r(C), \quad E = \iota_p(D), \quad F = \iota_q(E), \]
as in Figure 2. By construction $A, B, C, D, E, F$ lie on a conic. The lines $AB$ and $DE$ meet in $p$, the lines $BC$ and $EF$ meet in $q$. According to Pascal’s theorem the lines $CD$ and $AF$ meet in a point $s$ on the Pascal line $pq$. But $r$ is on $CD$ and on $pq$, so we have $s = r$ and hence $A = \iota_r(F)$. It follows that $\iota_r \circ \iota_q \circ \iota_p$ is an involution. Thus, we have $\iota_r \tau_{p,q} = \tau_{p,q}^{-1} \iota_r$, showing that $\iota_r$ is a reversing symmetry. Uncountably many symmetries are obtained by composition of reversing symmetries (and more reversing symmetries by composition of symmetries and reversing symmetries).

4 The cubic case

An irreducible plane curve of degree three with no singular points has genus one. Two such curves generically intersect in nine points. To find these intersection points, in general one needs to find the roots of a ninth order polynomial. However, we use the coordinates of two distinguished, and distinct, intersection points, $p = (c,d)$ and $q = (e,f)$, to parameterise the cubic curves. We require the cubics
\[ F_a(u,v) := a_1 + a_2u + a_3v + a_4u^2 + a_5uv + a_6v^2 + a_7u^3 + a_8u^2v + a_9uv^2 + a_{10}v^3 \]
(15)
to vanish at \( p \) and \( q \). Assuming that \( K := c^3 f^3 - d^3 e^3 \) does not vanish\(^3\), we can solve the constraints for the parameters \( a_7 \) and \( a_{10} \). We find \( a_7 = G_a/K \), \( a_{10} = H_a/K \) with

\[
G_a = (d^3 - f^3) a_1 + (d^3 c - c f^3) a_2 + d f (d^2 - f^2) a_3 + (d^3 c e^2 - c^2 f^3) a_4 + d f (d^2 e - c f^2) a_5 + d^2 f^2 (d - f) a_6 + d^2 f (d c^2 - e^2 f^2) a_9,
\]

\[
H_a = (e^3 - c^3) a_1 + c e (e^3 - c^2) a_2 + (d e^3 - f^3) a_3 + c^2 e^2 (e - c) a_4 + c e (d e^2 - f^2 c) a_5 + d^2 c e^2 (d - f) a_8 + c e (d^2 e^2 - f^2 c^2) a_9.
\]

We have chosen this parametrization so we can easily set \( d = e = 0 \) and take a limit where \( c \) or \( f \) goes to infinity, which yields \( a_7 = 0 \), \( a_{10} = 0 \) respectively. If both limits are taken we are left with a biquadratic

\[
F_a(u, v) = u^2 v a_8 + u v^2 a_9 + u^2 a_4 + u v a_5 + v^2 a_6 + u a_2 + v a_3 + a_1. \tag{16}
\]

For finite involution points \( p \) and \( q \) we obtain the following general form

\[
F_a(u, v) = G_a u^3 + H_a v^3 + K (u^2 v a_8 + u v^2 a_9 + u^2 a_4 + u v a_5 + v^2 a_6 + u a_2 + v a_3 + a_1) \tag{17}
\]

We have two Manin involutions,

\[
\iota_p(u, v) := (u, v) + z (c - u, d - v), \quad \iota_q(u, v) := (u, v) + z (e - u, f - v), \tag{18}
\]

with \( z \) given by \(^4\) and \(^7\), where for the latter involution \( (c, d) \) should be replaced by \( (e, f) \), and

\[
F_{a(u)}(u, v) = 3G_a u^2 + K (2u v a_8 + v^2 a_9 + 2u a_4 + v a_5 + a_2),
\]

\[
F_{a(v)}(u, v) = 3H_a v^2 + K (u^2 a_8 + 2u v a_9 + a_5 + 2v a_6 + a_3),
\]

\[
F_{a[u,u]}(u, v) = 6G_a u + 2K (v a_8 + a_4),
\]

\[
F_{a[v,v]}(u, v) = 6H_a v + 2K (u a_9 + a_6),
\]

\[
F_{a[u,v]}(u, v) = K (2u a_8 + 2v a_9 + a_5).
\]

The involutions \(^{18}\) are anti measure preserving with density \( 1/F_a(u, v) \).

**Theorem 5.** The composition of the involutions \(^{18}\) is an integrable map of the plane. It preserves each curve of the cubic pencil \( P_{a,b}(u, v) = 0 \) with \(^{17}\) (or \(^{16}\)) and it is measure preserving with density \( 1/F_a(u, v) \).

Taking \( d = e = 0 \), with \( \iota_1 = \lim_{c \to \infty} \iota_{c, 0} \) and \( \iota_2 = \lim_{f \to \infty} \iota_{0, f} \), the map \( \tau = \iota_2 \circ \iota_1 \) is a special case of the QRT-map with

\[
A^0 = \begin{pmatrix} 0 & a_8 & a_4 \\ a_9 & a_5 & a_2 \\ a_6 & a_3 & a_1 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & b_8 & b_4 \\ b_9 & b_5 & b_2 \\ b_6 & b_3 & b_1 \end{pmatrix}.
\]

**Example 6.** We choose particular values for the constants in \( F_a, F_b \) \(^{17}\).

\[
a_1 = a_9 = 1, a_2 = a_3 = a_4 = -1, a_5 = a_6 = a_8 = 0,
\]

\[
b_1 = b_9 = 0, b_2 = b_3 = b_4 = -1, b_5 = b_6 = b_8 = 1, c = 2, d = e = 0, f = 1.
\]

This gives

\[
F_a(u, v) = 5u^3 + 8(2v^2 - u^2 - u - v + 1), \quad F_b(u, v) = 6u^3 + 8(u^2 v - u^2 + uv + v^2 - u - v). \tag{19}
\]

\(^3\)One can also consider the case where \( K = 0 \): if \( c \neq e \) one can solve for \( a_1 \) and \( a_2 \), or when \( d \neq f \) one can solve for \( a_1 \) and \( a_3 \).
In Figure 3 we have drawn 10 curves of this cubic pencil. In addition to the involution points (2, 0) and (0, 1) there is one other finite real base point, near $(-1.140, 0.782)$. We have $\iota_{2,0}(u,v) = (u,v) - \frac{4}{h}(u-2,v)$, with
\[
g = u^5 + 3u^4v + 21u^3v^2 + 24u^2v^3 + 8uv^4 - 2u^3v - 46u^2v^2 - 16uv^3 - 16v^4 - 22u^3 - 22u^2v \\
+ 16uv^2 + 24v^3 + 52u^2 + 16uv - 16v^2 - 24u + 24v - 16,
\]
\[
h = (u-2) \left( u^4 + 3u^3v + 21u^2v^2 + 24uv^3 + 8v^4 - 3u^3 - 7u^2v - 44uv^2 - 8v^3 - 6u^2 - 8uv \right) \\
+ 4v^2 + 28u + 20v - 24,
\]
and $\iota_{0,1}(u,v) = (u,v) - \frac{l}{h}(u,v-1)$, where
\[
k = 6u^5 + u^4v - 11u^3v^2 + 8u^2v^3 + 8uv^4 - u^4 + 44u^3v - 8u^2v^2 - 16uv^3 - 33u^3 - 16u^2v \\
+ 24uv^2 + 8v^3 + 16u^2 - 32uv - 24v^2 + 16u + 24v - 8
\]
\[
l = u \left( 6u^4 + u^3v - 11u^2v^2 + 8uv^3 + 8v^4 - u^3 + 33u^2v - 16uv^2 - 24v^3 - 22u^2 + 8uv \\
+ 24v^2 - 8v \right).
\]
As indicated in the figure, the image of the point $(\sqrt{2}, 0)$ under the involution $\iota_{2,0}$ is $(-\sqrt{2}, 0)$, and the image of $(-\sqrt{2}, 0)$ under $\iota_{0,1}$ is $\left( \frac{9}{7} + \frac{1}{7}\sqrt{2}, \frac{10}{7} + \frac{2}{7}\sqrt{2} \right)$. The image of the curve labeled -1 is the point (0,1) as this is a singular point of that curve.

5 The quartic case

Let the quartic curve $F_a(u,v) = 0$, with
\[
F_a(u,v) := a_0 + a_1u + a_2u^2 + a_3u^3v + a_4u^2v + a_5uv + a_6v^2 + a_7u^3 + a_8u^2v + a_9uv^2 + a_{10}v^3 \\
+ a_{11}u^4 + a_{12}u^3v + a_{13}u^2v^2 + a_{14}uv^3 + a_{15}v^4
\]
have double points at $p = (c,d)$ and $q = (e,f)$, i.e. at these points we require the function $F_a$ as well as its first partial derivatives $F_a^{(u)}$, $F_a^{(v)}$ to vanish. Generically the genus of such a curve is
one, the same as in the cubic case. Assuming that
\[ V := c^3 f^3 - d^3 e^3 \neq 0, \quad W := (cf - de)^2((cf + de)^2 + 2cdef) \neq 0, \]
we can solve for
\[ a_7 = \frac{P}{V} \cdot a_{10} = \frac{Q}{V} \cdot a_9 = \frac{R}{W} \cdot a_{12} = \frac{S}{W} \cdot a_{14} = \frac{T}{W} \cdot a_{15} = \frac{U}{W}, \]
where the functions \( P, Q, R, S, T, U \) can be found in Appendix D. If \( V \) or \( W \) vanishes one has to solve for other parameters. If \( c \neq e \) one can solve for \( a_1, a_2, a_3, a_4, a_5, a_7 \) and if \( d \neq f \) one can solve for \( a_1, a_2, a_3, a_5, a_6, a_{10} \). The parameters \( a_7, a_{10}, a_{11}, a_{12}, a_{14}, a_{15} \) vanish when \( d = e = 0 \) in the limit where both \( c \) and \( f \) go to infinity, leaving us with the most general biquadratic. For finite \( p \)
and \( q \), we obtain
\[
F_a(u, v) = \left( u^2 v^2 a_{13} + u^2 v a_8 + u v^2 a_9 + u^2 a_4 + u v a_5 + v^2 a_6 + u a_2 + v a_3 + a_1 \right) W V
+ (Pu^3 + Qv^3) W + u^4 R + v^3 S + uv^3 T + v^4 U. \tag{20}
\]
As in the previous section, we have two involutions,
\[ \iota_p(u, v) := (u, v) + z(c - u, d - v), \quad \iota_q(u, v) := (u, v) + z(e - u, f - v). \tag{21} \]
Here \( z \) is again given by (44) and (7), where for the second involution \((c, d)\) should be replaced by \((e, f)\), but now
\[
F_a^{(u)}(u, v) = (2u^2 v^2 a_{13} + 2 u v a_8 + v^2 a_9 + 2 u a_4 + v a_5 + a_2) W V
+ 3 u^2 P W + 4 u^3 R + 3 u^2 S + v^3 T,
F_a^{(v)}(u, v) = \left( 2 u^2 v a_{13} + u^2 a_8 + 2 u v a_9 + u a_5 + 2 v a_6 + a_3 \right) W V
+ v^3 Q W + u^3 S + 3 u v^2 T + 4 v^3 U,
F_a^{(u, u)}(u, v) = (2v^2 a_{13} + 2 v a_8 + 2 a_4) W V + 6 u P W + 12 u^2 R + 6 u v S,
F_a^{(v, v)}(u, v) = (2 u^2 a_{13} + 2 u a_9 + 2 a_6) W V + 3 u^2 Q W + 6 u v T + 12 v^2 U,
F_a^{(u, v)}(u, v) = \left( 4 u v a_{13} + 2 u a_8 + 2 v a_9 + a_3 \right) W V + 3 u^2 S + 3 v^2 T.
\]
Both involutions are anti measure preserving, \( \iota_p \) with density \((s_1(u - c) + s_2(v - d)) / F_a(u, v)\), and \( \iota_q \) with density \((t_1(u - e) + t_2(v - f)) / F_a(u, v)\), for arbitrary \( s_i, t_i \).

**Theorem 7.** The composition of the Manin involutions (21) is an integrable map of the plane. It preserves the quartic pencil \( P_{\alpha, \beta}(u, v) = 0 \) with (20), and it is measure preserving with density \( L(u, v) / F_a(u, v) \), where \( L = 0 \) is the line through the involution points, \( p \) and \( q \), see (42).

**Example 8.** Consider the quartic pencil where
\[ F_a(u, v) = u^2 (7(2u - 1)(2u + 1) - 4(3v - 2)(2u - v)) \]  \tag{22}
is a product of a double line and an ellipse, and
\[ F_b(u, v) = (u - 3v) (2u + v - 1)(3u + v)(u + 5v - 5). \]  \tag{23}
is a product of four lines. All 10 base points are finite, the involution points are the singular base points (0, 0) and (0, 1). Some curves of the pencil are plotted in Figure 4.
Figure 4: Ten curves from the quartic pencil defined by (1), (22) and (23), labeled by the value of $-\beta/\alpha$.

The curve of the pencil which contains the point $(-\frac{3}{2}, \frac{3}{10})$ and some of its iterates are plotted in Figure 5.

Figure 5: Six iterations of the point $(-\frac{3}{2}, \frac{3}{10})$ under the composition of Manin involutions (24), $\iota_{0,1} \circ \iota_{0,0}$.

The involutions are explicitly given by:

\[ \iota_{0,0}(u, v) = (u, v)A, \quad \iota_{0,1}(u, v) = (0, 1) - 3(u, v - 1)B \] (24)
with
\[ A = \frac{154 u^2 - 43 u v + 95 v^2 + 3 u - 110 v}{340 u^4 + 176 u^2 v - 116 u v^2 + 80 v^3 - 154 u^2 + 43 u v - 95 v^2}, \]
and
\[ B = \frac{25 u^2 - 16 u v + 15 u^2 + 16 u - 8 v - 7}{200 u^3 + 88 u^2 v - 152 u v^2 + 24 v^3 - 13 u^2 + 256 u v - 27 v^2 - 104 u - 18 v + 21}. \]

The set of base points is the disjoint union of points at which \( A \) is undefined and points where \( B \) is undefined. This is made clear in Figure 6.

We have
\[ \iota_{0,1}(b_2) = b_4, \quad \iota_{0,0}(b_7) = b_9 \]
\[ \iota_{0,1}(b_3) = b_5, \quad \iota_{0,0}(b_8) = b_{10}. \]

To define the action of \( \iota_{0,0} \) at \( b_2, b_3, b_4 \) and \( b_5 \), one needs to blow up at these points. Similarly, for \( \iota_{0,1} \) blowups are required at \( b_7, b_8, b_9 \) and \( b_{10} \).

The special involutions with base points at infinity, with \( d = e = 0 \),
\[ \iota_1 = \lim_{c \to \infty} \iota_{c,0}, \quad \iota_2 = \lim_{f \to \infty} \iota_{0,f} \]
are anti measure preserving. The horizontal switch \( \iota_1 \) has density \((s_1 v + s_2) / F_a(u, v)\), and the vertical switch \( \iota_2 \) has density \((t_1 u + t_2) / F_a(u, v)\), for arbitrary \( s_1, t_1 \). This implies in particular that \( \iota_{c,d} \circ \iota_1 \) is measure preserving with density \((v - d) / F_a(u, v)\) and, that \( \tau = \iota_2 \circ \iota_1 \) is measure preserving with density \( 1 / F_a(u, v) \). This map \( \tau \) is the QRT-map.

6 Higher degree pencils with singular points

Theorem 9. Higher degree \( D > 4 \) curves with two distinct points of multiplicity \( D - 2 \) are products of the form \( C = L^{D-4} Q \), where \( L \) is the line through the two points, and \( Q \) a quartic.
This implies that we obtain the same involutions as in the case $D = 4$, see Theorem 10.

**Proof.** Consider a degree $D = 5$ curve $C$ with two distinct points of multiplicity 3. Let $L$ be the line through these points. Near each triple point there is a line which intersects the curve in at least three points, see Figures 7 and 8. Note that while we have drawn the generic case where 3 tangents intersect at each triple point, the statement is still true when some of these tangents are imaginary, e.g. when the curve contains a cusp. If $C$ does not contain $L$ there is a line close to $L$ which intersects $C$ in 6 points, which contradicts $D = 5$.

![Figure 7: A degree 5 curve does not intersect a line in 6 points.](image)

![Figure 8: Any degree 5 curve with two triple points contains the line through the triple points.](image)

Next, let $m$ be the multiplicity of $L$ in a degree $D$ curve $C$. We need $2(D - 2) - 2 = D$, which implies $m = D - 4$. □

**Theorem 10.** The value of $z$, (3), does not depend on $D$, for $D \geq 4$.

**Proof.** Consider the degree $D+1$ pencil $\alpha \hat{F}_a(u,v) + \beta \hat{F}_b(u,v) = 0$ where $\hat{F}_a(u,v) = F_a(u,v)L(u,v)$, where $F_a$ has degree $D$ and two singular points of multiplicity $D - 2$, $(c, d)$ and $(e, f)$, and $L(u,v) = (d-f)(u-e)-(c-e)(v-f)$. We evaluate the functions in (4) at $u + (\tilde{c}-u)z, v + (d-v)z$, we let $'$ denote differentiation with respect to $z$ and we evaluate at $z = 0$. We have $\hat{F}_a' = F_a'L + L'\hat{F}_a$ and $\hat{F}_a'' = F_a''L + 2L'\hat{F}_a'$, as $L'' = 0$. Let

$$K = \frac{F_aF_b'' - F_a''F_b}{F_aF_b' - F_a'F_b}.$$ 

Then

$$\hat{K} = \frac{\hat{F}_a\hat{F}_b'' - \hat{F}_a''\hat{F}_b}{\hat{F}_a\hat{F}_b' - \hat{F}_a'\hat{F}_b} = K + \frac{L'}{L},$$

and

$$\frac{L'}{L} = -1 + \frac{\tilde{c}(d-f) + \tilde{d}(e-c) + cf - de}{L} = -1$$

when $(\tilde{c}, \tilde{d})$ equals $(c, d)$ or $(e, f)$. Therefore, from (4),

$$z_{D+1} = 2 \left(2 - 2 - (K - 2)\right)^{-1} = 2 (2 - (K - 2))^{-1} = z_D.$$ □
7 Roots

Recall that the QRT map is obtained by considering a $D = 4$ pencil with double base points at $(0, z)$ and $(z, 0)$ as involution points, and taking the limit where $z \to \infty$. In that limit the quartic polynomials $F_a(u, v)$ and $F_b(u, v)$ become biquadratic polynomials. A special case of the QRT map, the so called symmetric QRT map, arises when the biquadratic polynomials are symmetric in $u, v$, i.e. they are invariant under what Duistermaat calls the symmetry switch \[ \text{[Section 10.1]} \]

\[
\sigma(u, v) = (v, u). \tag{25}
\]

The symmetric QRT map $\tau = \iota_2 \circ \iota_1$ equals $\tau = \rho^2$, where $\rho = \sigma \circ \iota_1 = \iota_2 \circ \sigma$ is called the QRT-root.

We note that $\sigma$ may arise as a Manin involution corresponding to the base point $(z, -z)$ in the limit where $z \to \infty$, and we provide an example of a map which can be written as a Manin transformation in various different ways.

Example 11. The Lyness map

\[ \lambda : (u, v) \to \left(v, \frac{v + a}{u}\right) \]

leaves invariant the pencil of cubic curves

\[ \alpha(u + 1)(v + 1)(u + v + a) + \beta uv = 0. \]

The pencil has finite base points $p_1 = (-1, 0), p_2 = (0, -1), p_3 = (-a, 0), p_4 = (0, -a)$, which gives rise to involutions

\[
\iota_{p_1}(u, v) = \left(\frac{a(u + 1) + v}{uv}, \frac{a + v}{u}\right), \quad \iota_{p_2}(u, v) = \left(\frac{a + u}{v}, \frac{u + a(v + 1)}{uv}\right),
\]

\[
\iota_{p_3}(u, v) = \left(\frac{u + a(v + 1)}{uv}, \frac{a(uw + v + 1) + u}{u(u + a)}\right), \quad \iota_{p_4}(u, v) = \left(\frac{a(uw + u + 1) + v}{v(v + a)}, \frac{v + a(u + 1)}{uv}\right),
\]

as well as base points at infinity $p_5 = \lim_{x \to \infty}(0, x), p_6 = \lim_{x \to \infty}(x, 0)$ (these have multiplicity two), and $p_7 = \lim_{x \to \infty}(x, -x)$, which yield the involutions

\[
\iota_{p_5}(u, v) = \left(u, \frac{a + u}{v}\right), \quad \iota_{p_6}(u, v) = \left(\frac{a + v}{u}, v\right), \quad \iota_{p_7}(u, v) = (v, u).
\]

The latter Manin involution, $\iota_{p_7} = \sigma$, is the symmetry switch of the pencil of curves, it is a reversing symmetry for the Lyness map, and it corresponds to negation in the group law of the pencil $\iota_{p_5}$. The other involutions are also reversing symmetries, generated by $\lambda$ and $\sigma$:

\[
\iota_{p_1} = \sigma \circ \lambda^2, \quad \iota_{p_2} = \lambda^2 \circ \sigma, \quad \iota_{p_3} = \lambda^3 \circ \sigma, \quad \iota_{p_4} = \sigma \circ \lambda^3, \quad \iota_{p_5} = \lambda \circ \sigma, \quad \iota_{p_6} = \sigma \circ \lambda.
\]

Thus the Lyness map is a QRT root: we have $\iota_{p_5} = \iota_2$ and $\iota_{p_6} = \iota_1$, see \[ [14] \], and hence

\[
\lambda = \sigma \circ \iota_1 = \iota_2 \circ \sigma.
\]

On the other hand, it can also be written as the composition of two Manin involutions which correspond to finite involution points

\[
\lambda = \iota_{p_1} \circ \iota_{p_4} = \iota_{p_3} \circ \iota_{p_2},
\]

or as the composition of an Manin involution which corresponds to a finite involution point and a horizontal or vertical switch

\[
\lambda = \iota_{p_2} \circ \iota_2 = \iota_1 \circ \iota_{p_1}.
\]

In the sequel we call a transformation $\sigma$ a symmetry switch of the pencil $P = 0$ if $\sigma$ is a symmetry of $P$ and it is an involution.
Theorem 12. Let $\sigma$ be a symmetry switch of the pencil $P_{\alpha,\beta}(u,v) = 0$ which maps lines to lines. Then
\[ \tau_p = \iota_{\sigma(p)} \circ \iota_p = \rho_p^2, \quad \text{with } \rho_p = \sigma \circ \iota_p = \iota_{\sigma(p)} \circ \sigma. \]

We call $\rho_p$ the root of $\tau_p$.

**Proof.** Let $q$ be a point on a curve $C$ in a pencil of degree $D$, and let the involution point $p$ be a singular point of multiplicity $D - 2$. Note that $\sigma(p)$ has the same multiplicity as $p$. Defining $r = \iota_p(q) \in C$, the points $p, q, r$ are collinear. Because $\sigma$ maps lines to lines the points $\sigma(p), \sigma(q), \sigma(r)$ are also collinear. Because $\sigma$ is a symmetry, both $\sigma(q), \sigma(r)$ are on the curve $C$. Therefore we must have $\sigma(r) = \iota_{\sigma(p)}(\sigma(q))$, cf. Figure 9. And hence $\tau_p = \iota_{\sigma(p)} \circ \iota_p = \iota_{\sigma(p)} \circ \sigma \circ \iota_p = \rho_p^2$.

\[ \square \]

### 7.1 Symmetric generalised Manin transformations

We require that the symmetric quartic polynomials $F_a$ and $F_b$, where
\[ F_a = a_1 + a_2 (u + v) + u a_3 + (u^2 + v^2) a_4 + (u^2 v + u v^2) a_5 + u^2 v^2 a_6 + (u^3 + v^3) a_7 + (u^3 v + u v^3) a_8 + (u^4 + v^4) a_9, \]
have a singular point at $p = (c, d)$. Solving the constraints for $F_a$ for $a_7, a_8, a_9$ gives
\[ a_7 = -\frac{4 a_1 + (3 c + 3 d) a_2 + 2 c d a_3 + (2 c^2 + 2 d^2) a_4 + (c^2 d + c d^2) a_5}{(c + d) (c^2 - c d + d^2)}, \]
\[ a_8 = -\frac{1}{(c^2 - c d + d^2) (c^4 + 4 c^2 d^2 + d^4) (c + d)^2} \left( -12 c^2 d^2 a_1 + (c^5 + d c^4 - 8 d^2 c^3 - 8 d^3 c^2 + d^4 c + d^5) a_3 + (2 d^5 - 4 d^2 c^4 - 4 d^4 c^2 - 2 c^5 d) a_4 + (c^7 + 3 d c^6 - 6 d^2 c^5 + 4 d^3 c^4 + 4 d^4 c^3 + 3 d^5 c^2 + 3 d^6 c + d^7) a_5 + (2 c^7 d + 2 c^5 d^2 + 2 c^3 d^3 + 2 c^2 d^4 + 2 c d^5) a_6 \right), \]
\[ a_9 = \frac{1}{(c^2 - c d + d^2) (c^4 + 4 c^2 d^2 + d^4) (c + d)^2} \left( (3 c^4 + 3 c^3 d + 12 c^2 d^2 + 3 c d^3 + 3 d^4) a_1 + (2 c^5 + 5 d c^4 + 11 d^2 c^3 + 11 d^3 c^2 + 5 d^4 c + 2 d^5) a_2 + (2 d^5 + 2 d^2 c^4 + 6 d^3 c^3 + 2 c^5 d) a_3 + (c^7 + d c^6 + 7 d^2 c^5 + 2 c^3 d^4 + 7 d^3 c^2 + d^4 c + d^5) a_4 + (d c^6 + 3 d^2 c^5 + 4 d^4 c^3 + 3 d^5 c^2 + 3 d^6 c + d^7) a_5 + (c^5 d^2 + 4 c^3 d^3 + 4 d c^5 + c^2 d^6) a_6 \right) \]
and similar expressions are obtained for $b_7, b_8, b_9$. Taking $\sigma(u, v) = (v, u)$, one defines $\rho_p = \sigma \circ \iota_p$ and verifies that $\rho_p = \iota_{\sigma(p)} \circ \sigma$. The symmetric QRT-root is obtained by considering the limit $d \to \infty$ (in which $a_7, a_8, a_9, b_7, b_8, b_9 \to 0$), or by performing a fractional affine transformation explained in section 8.

One can also solve the constraints for other variables, depending on what variables one chooses to be non-zero.

**Example 13.** Setting $a_4 = 1, a_5 = a_6 = a_7 = a_8 = 0$ and $b_3 = 1, b_1 = b_5 = b_6 = b_7 = b_8 = 0$, both polynomials $F_a$ and $F_b$ have singular points at both $(0, 1)$ and $(1, 0)$ if
\[ a_1 = a_9 = -\frac{1}{2}, a_2 = 0, b_1 = \frac{3}{4}, b_2 = -1, b_3 = \frac{1}{4}. \]

Thus we obtain the map
\[ (u, v) \to (v, u) - 2 \frac{u^4 + v^4 - 2 u^3 + 2 u - 1}{u^4 + v^4 - 4 u^3 + 6 u^2 - 4 u + 1} (v, u - 1), \]
which preserves the pencil
\[ \alpha (u^4 + v^4 - 2(u^2 + v^2) + 1) + \beta (u^4 + v^4 + 4(u v - u - v) + 3) = 0. \]
7.2 Linear symmetry switches

We introduce a symmetry switch that is more general than (25), but which is still linear. In terms of

\[ U = (u, v), \quad V = (b, -a), \quad W = (ad - bc, ae - bd), \quad E = V \cdot W, \quad G = G(U) = U \cdot W \]

we define

\[ \sigma_{a,b,c,d,e} : U \rightarrow U - \frac{2G}{E} V. \] (26)

The ‘symmetric switch’ given by (25) is a special case of (26), we have \( \sigma = \sigma_{a,a,c,d,e} \) and the matrices of \( \sigma \) and \( \sigma_{a,b,c,d,e} \) are conjugate. In the sequel we will omit the index \( a,b,c,d,e \). The linear transformation \( \sigma \) given by (26) is a reflection in the line through \((0,0)\) perpendicular to \( W \) along a line with direction \( V \), i.e. we have

\[ \sigma(V) = -V, \quad \sigma(JW) = JW, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Importantly, \( \sigma \) (26) leaves the linear respectively quadratic forms

\[ L = L(U) = au + bv, \quad Q = Q(U) = cu^2 + 2duv + ev^2 \]

invariant (and it also negates the linear form \( G \)), that is

\[ L(\sigma(U)) = L(U), \quad Q(\sigma(U)) = Q(U), \quad G(\sigma(U)) = -G(U). \]

For \( D = 2 \) the most general pencil which admits \( \sigma \) (26) as a symmetry is given by

\[ F_a = a_1 + a_2 L + a_3 L^2 + a_4 Q, \quad F_b = b_1 + L + L^2 + Q. \] (27)

Note that the constants \( b_2, b_3, b_4 \) can be absorbed by the other constants,

\[ (a, b) \rightarrow \frac{1}{b_2} (a, b), \quad (c, d, e) \rightarrow \frac{1}{b_4} (c, d, e) + \left( 1 - \frac{b_3}{b_2^2} \right) \frac{1}{b_4} (a^2, ab, b^2). \]

We are still free to choose the coordinates of \( p \), so in total the degree \( D = 2 \) family of maps which admit a root has 12 parameters.

**Theorem 14.** The root \( \rho_p = \sigma \circ \iota_p \), where \( \sigma \) is given by (26) and \( \iota_p \) by (4), is an integrable map of the plane. It preserves each curve of the quadratic pencil \( P_{a,b}(u,v) = 0 \) with (7) and (27), and it is measure preserving with density \( (F_a(U)(L(U) - L(p))^{-1}). \)

**Example 15.** Let \((a, b, c, d, e) = (1, 2, -3, 4, 5)\), \((a_1, a_2, a_3, a_4) = (1, -2, -3, 4)\), and \( b_1 = 1 \). Then

\[ \sigma(u, v) = \frac{1}{23} \begin{pmatrix} -17 & 12 \\ 20 & 17 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \] (28)

and

\[ F_a = -15 u^2 + 20 u v + 8 v^2 - 2 u - 4 v + 1, \quad F_b = -2 u^2 + 12 u v + 9 v^2 + u + 2 v + 1. \]

The point \( q = (1/2, -1) \) is on the curve

\[ 0 = P_{k,7}(u,v) = -134 u^2 + 244 u v + 127 v^2 - 9 u - 18 v + 15. \] (29)

Choosing \( p = (2, -1) \) we find \( r = \iota_p(q) = (-160/67, -1) \). The points

\[ \sigma(p) = (-2, 1), \quad \sigma(q) = \left( \frac{41}{46}, -\frac{7}{23} \right), \quad \sigma(r) = \left( \frac{1916}{1541}, -\frac{4339}{1541} \right). \]
are collinear, and

\[ \iota_p(\sigma(r)) = \left( \frac{259627}{86963}, \frac{118690}{86963} \right), \quad \iota_{\sigma(p)}(r) = \left( -\frac{5651}{3781}, \frac{13630}{3781} \right). \]

It can be seen, see Figure 9, that \( \sigma(\iota_p(\sigma(r))) = \iota_{\sigma(p)}(r) \).

Figure 9: A degree 2 curve, given by (29), which admits the symmetry switch (28). The symmetry switch is a reflection in the line through \((0, 0)\) perpendicular to \(W = (10, -3)\) (purple), in the direction \((2, -1)\) (dotted).

For \( D = 3 \) the most general pencil left invariant by \( \sigma \) is

\[ F_a = a_1 + a_2L + a_3L^2 + a_4Q + a_5L^3 + a_6LQ, \quad F_b = b_1 + L + L^2 + Q + b_3L^3 + b_6LQ. \]

We require that the involution point \( p \) is a point on both \( F_a = 0 \) and \( F_b = 0 \) and thus we have a 14 parameter family of maps which admit a root. In the cubic case the root is measure preserving with density \( 1/F_a(u, v) \).

For \( D = 4 \) the most general pencil invariant under \( \sigma \) is defined by

\[ F_a = a_1 + a_2L + a_3L^2 + a_4Q + a_5L^3 + a_6LQ + a_7L^4 + a_8L^2Q + a_9Q^2, \]
\[ F_b = b_1 + L + L^2 + Q + b_3L^3 + b_6LQ + b_7L^4 + b_8L^2Q + b_9Q^2. \]

Here we require that the involution point \( p \) is a double point of \( F_a = 0 \) and \( F_b = 0 \), which gives 6 constraints. Thus we are left with a 16-parameter family whose square root can be taken. In the quartic case the root is measure preserving with density \( (L(U) - L(p))/F_a(U) \).

In \[4\] it was shown that the Kahan map obtained from discretizing the ODE \( U_t = J \nabla H(U) \), where \( H(U) \) is a symmetric cubic Hamiltonian is (the root of) a Manin transformation. We note that the Kahan map for the system in \[3\] Section 2] is the root of a generalised Manin transformation preserving a quadratic pencil, as in Theorem 14, cf. \[13\].
8 Transforming a generalised Manin transformation into QRT-form

The QRT-map preserves a pencil of curves defined by polynomials that are quadratic in two variables. In this paper we have considered maps which preserve more general polynomials of degree \(D = 2, 3, 4, \ldots\). For \(D > 2\), in order to define the maps, as composition of generalised Manin involutions, the pencil needs to have two base points which for \(D > 3\) are singular with multiplicity \(D - 2\). We have shown that no new maps are obtained for \(D > 5\). We will now show that for \(D = 2, 3, 4\) every map can be brought into QRT form (which can be regarded as a normal form for generalised Manin transformation) by a fractional affine transformation which transform the line through the involution points to infinity.

Consider the fractional affine transformation

\[
\psi : (u, v) \rightarrow (U, V) = \left( \frac{au + bv + c}{gu + hv + i}, \frac{du + ev + f}{gu + hv + i} \right).
\]  

Such a transformation maps lines to lines, which can be seen as follows. The coordinates \((u, v)\) can be taken as affine coordinates of a projective space and then \(\psi\) is induced by a linear transformation of the vector space it is derived from. Indeed, we can write \(\psi = \kappa \phi \kappa^{-1}\) where \(\phi\) is a linear map and \(\kappa : (u, v, w) \rightarrow (u/w, v/w)\). Since \(\kappa(p + t(q - p)) = \kappa(p) + s(\kappa(q) - \kappa(p))\), with \(sp_3 - tq_3 = ts(p_3 - q_3)\) the maps \(\kappa, \kappa^{-1}\) and hence \(\psi\) maps lines to lines. Such map is called a homography, or, a projective collineation. The fundamental theorem of projective geometry states that every map which sends lines to lines (in a projective space of dimension at least two) is a projective collineation [1, Thm 2.26].

If \(p = (c, d)\) and \(q = (e, f)\) are points in the plane and \(L = (d - f)(u - e) - (c - e)(v - f)\), so that \(L = 0\) is the line through \(p\) and \(q\), then any projective collineation of the form,

\[
(u, v) \rightarrow \left( \frac{A(u - e) + B(v - f)}{L}, \frac{C(u - c) + D(v - d)}{L} \right),
\]

with \(AC \neq BD\) and neither \((A, B)\) nor \((C, D)\) perpendicular to \(L\), sends \(p\) to \((\infty, 0)\), and \(q\) to \((0, \infty)\). Thus we have the following result.

**Theorem 16.** Let \(p = (c, d)\) and \(q = (e, f)\) be the involution points of a pencil of curves \(P_{p,q}(u, v) = 0\) of degree \(2 \leq D \leq 5\), so that if \(D > 2\) then \(p, q\) are base points of multiplicity \(D - 2\). The projective collineation \((31)\) brings the generalised Manin transformation \(\tau_{p,q} = \iota_q \circ \iota_p\) into QRT form.

We conclude this paper with two remarks, and we provide for all generalised Manin transformations in the examples given their QRT-form.

**Remark 1.** As Theorem 12 concerns symmetry switches which map lines to lines, it is worthwhile to determine which projective collineations are symmetry switches and to study the corresponding pencils. In Appendix C we show that the highest dimensional solution yields pencils comprising singular curves only.

**Remark 2.** Theorem 4 implies that QRT-maps which preserve a pencil of quadratic curves admit uncountably many reversing symmetries, namely all generalised Manin involutions with involution point at infinity.

**Example 17.** Consider the pencil of Example 3 with involution points \(p = (-2, 2)\) and \(q = (-1, 1)\). The line through \(p\) and \(q\) is \(u + v = 0\). Introducing new coordinates

\[
(x, y) = \left( \frac{u + 1}{u + v}, \frac{v + 2}{u + v} \right)
\]
the involution \( \iota_p \) becomes
\[
\iota_1 : (x, y) \rightarrow (y - x + \frac{1}{2}, y)
\]
and the involution \( \iota_q \) becomes
\[
\iota_2 : (x, y) \rightarrow \left( x, \frac{x + 2 - xy}{x - y} \right).
\]

The ratio \( F_a/F_b \) becomes
\[
\frac{y^2 + 14x(y - x) + 7x - 8y - 2}{(2x - y + 1)(2x - y - 2)}.
\]

The QRT mapping \( \tau = \iota_2 \circ \iota_1 \) has matrices
\[
A^0 = \begin{pmatrix}
0 & 0 & -14 \\
0 & 14 & 7 \\
1 & -8 & -2
\end{pmatrix},
A^1 = \begin{pmatrix}
0 & 0 & 4 \\
0 & -4 & -2 \\
1 & 1 & -2
\end{pmatrix}.
\]

Other involutions in the \((u, v)\)-plane whose involution point is on the line \( u + v = 0 \) give rise to mappings that are reversing symmetries of \( \tau \). Examples are \( \iota_{0,0} \) which gives rise to
\[
(x, y) \rightarrow (x, y) - (y - 1)(1, 2)
\]
and \( \iota_{1,-1} \) which gives rise to
\[
(x, y) \rightarrow (x, y) - \frac{4x^2y - 10xy^2 + 4y^3 - 2x^2 + 6xy - y^2 + 13x - 10y - 2}{2x^2 - 2xy - 4y^2 - x + 5y + 8} (2, 1).
\]

**Example 18.** Consider the pencil of Example \([6]\) with involution points \( p = (2, 0) \) and \( q = (0, 1) \). The line through \((2, 0)\) and \((0, 1)\) is given by \( L(u, v) = 2 - u - 2v = 0 \). In terms of variables \((x, y) = (u, v)/L(u, v)\) the involutions \( \iota_{2,0} \) and \( \iota_{0,1} \) become the horizontal and vertical switches of the QRT-map with matrices
\[
A^0 = \begin{pmatrix}
0 & 0 & \frac{5}{2} \\
0 & -2 & 0 \\
-2 & 0 & -1
\end{pmatrix},
A^1 = \begin{pmatrix}
0 & 3 & 4 \\
2 & 4 & 1 \\
2 & 1 & 0
\end{pmatrix},
\]
i.e. we have
\[
\iota_{2,0} \mapsto \iota_1 : (x, y) \rightarrow \left( -\frac{(18xy^2 + 16xy + 10y^2 + 4x + 5y + 1)(2y + 1)}{36xy^3 + 74xy^2 + 36y^3 + 35xy + 50y^2 + 9x + 24y + 4}, y \right),
\]
\[
\iota_{0,1} \mapsto \iota_2 : (x, y) \rightarrow \left( x, \frac{33x^4 - 26x^3y - 5x^3 - 28x^2y - 14x^2 + x + 2y + 1}{2(18x^2y + 13x^2 + 8xy + x - 2y - 1)(x + 1)} \right),
\]

preserving the ratio of biquadratics
\[
\frac{12x^2y - 4xy^2 + 5x^2 - 4y^2 - x - 4y - 1}{2(3x^2y + 2xy^2 + 4x^2 + 4xy + 2y^2 + x + y)}.
\]

**Example 19.** Consider the pencil of Example \([\text{3}]\) with involution points \( p = (0, 0) \) and \( q = (0, 1) \). Performing a change of variables, \((x, y) = (1 - u - v, -v)/u\), the involutions become
\[
(x, y) \rightarrow \left( \frac{15y^2x - 150xy - 3y^2 - 157x - 58y + 29}{110y^2 - 15y^2 + 3x + 150y + 157}, y \right),
\]
\[
(x, y) \rightarrow \left( x, -\frac{21x^2y - 56x^2 + 6xy - 50x - 102y + 206}{21x^2 - 66y^2 - 6x - 66y - 102} \right).
\]
preserving the ratio of biquadratics

\[
\frac{7x^2 - 22yx + 3y^2 - 2x - 30y - 37}{15x^2y^2 - 40x^2y - 3y^2x - 8x^2 - 14yx - 9y^2 + x + 2y - 25},
\]

i.e. we obtain the QRT map with matrices

\[
A^0 = \begin{pmatrix} -15 & 40 & 1 \\ 3 & 36 & 1 \\ 6 & 28 & 62 \end{pmatrix}, \quad A^1 = \begin{pmatrix} -15 & 40 & 8 \\ 3 & 14 & -1 \\ 9 & -2 & 25 \end{pmatrix}.
\]

Example 20. In Example 13 the involution points are (0,1) and (1,0). After a transformation, 
\((x, y) = (u, v)/(1 - u - v),\) the map becomes the composition of \((x, y) \to (y, x)\) and the horizontal switch which preserves the ratio of biquadratics

\[
2x^2y^2 + 8xy^2 + 4x^2 + 12xy + 4y^2 + 4x + 4y + 1
\]
\[
2x^2y^2 + 8xy^2 + 6x^2 + 16xy + 6y^2 + 8x + 8y + 3.
\]

Example 21. In Example 14 the involution points are \(p = (2, -1)\) and \(\sigma(p) = (-2, 1)\). After a transformation, with new coordinates

\[
(x, y) = \left(\frac{3u - 23 - 29v}{2u + 4v}, \frac{-23u - 1 + v}{2u + 4v}\right),
\]

we have that \(\sigma\) switches \(x\) and \(y\), we have \(\iota_p \to \iota_1 : (x, y) \to (F(x, y), y), \iota_{\sigma(p)} \to \iota_2 : (x, y) \to (x, F(y, x)),\)

where

\[
F(x, y) = \frac{12y^3 - 213xy + 651y^2 - 5966x + 12084y - 3268}{12xy + 213x + 213y + 5966},
\]

and the preserved ratio is

\[
91x^2 - 186xy + 91y^2 + 20x + 20y - 836
\]
\[
22x^2 - 48xy + 22y^2 - 49x - 49y - 1710.
\]

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Appendix A

It is convenient to use abbreviated notation \(F_a^{(i)} := F_a^{(z, \ldots, z)}(0).\) We start with the Taylor expansion about \(z = 0,\) equation [2], and Taylor expand it about \(z = 1:\)

\[
F_a(z) = \sum_{i=0}^{D} c_i (z - 1)^i, \quad \text{with} \quad c_i = \sum_{j=i}^{D} \frac{F_a^{(j)}}{j!(j-i)!}.
\]

As \(c_i = 0\) for \(i < D - 2\) we have

\[
F_a(z) = \frac{(z - 1)^{D-2}}{D!} \left( D(D - 1)(F_a^{(D-2)} + F_a^{(D-1)} + \frac{1}{2}F_a^{(D)}) + D(F_a^{(D-1)} + F_a^{(D)})(z - 1) + F_a^{(D)}(z - 1)^2 \right).
\]
Due to $\sum_{i=0}^{D-3} (-1)^i c_i = 0$ we have
\[
\frac{1}{2} (D-1)(D-2) F_a^{(D)} + D(D-2) F_a^{(D-1)} + D(D-1) F_a^{(D-2)} = (-1)^D D! F_a^{(0)}
\]
and hence
\[
F_a(z) = \frac{(z-1)^{D-2}}{D!} \left( F_a^{(D)} z^2 + (DF_a^{(D-1)} - (D-2)F_a^{(D)}) z + (-1)^D D! F_a^{(0)} \right),
\]
and similarly for $F_b(z)$. Substituting these into the equation $F_a(z) F_b(0) = F_b(z) F_a(0)$, after dividing out $z(z-1)^{D-2}$ the constant term vanishes, and we are left with a linear equation
\[
(F_a^{(D)}(z + D - 2) + D F_a^{(D-1)}) F_b^{(0)} = (F_a^{(D)}(z + D - 2) + D F_b^{(D-1)}) F_a^{(0)},
\]
which provides
\[
z = 2 - D \left( 1 + \frac{F_a(0) F_b(z, D-1, 0) - F_a(z, D-1, 0) F_b(0)}{F_a(0) F_b(z, D, 0) - F_a(z, D, 0) F_b(0)} \right).
\]
To get the expression (4) we solve the system $c_i = 0$, $0 \leq i \leq D - 3$. This can be done as follows. Define $x_{0,j} = (D - j)! c_j$ and $x_{i+1,j} = \frac{x_i,j - x_i,j+1}{i+1}$. Explicitly we have, for $0 \leq i \leq D - 3$,
\[
x_{i,0} = \sum_{j=0}^{D-i} \prod_{k=0}^{D-i-j-1} \frac{(D - i - k)(i + k + 1) F_a^{(j)}}{(D - i - j)!},
\]
and the linear combination
\[
\sum_{h=3}^{k} (-1)^h k! \frac{(D - h)!}{(D-k)!} \binom{k}{h} x_{D-h,0} =
\]
\[
F_a^{(k)} - (-1)^k \frac{k!}{2} \left\{ \left( \frac{D-2}{k-2} \right) F_a^{(2)} + 2 \left( \frac{D-1}{k-1} \right) (k-2) F_a^{(1)} + \left( \frac{D}{k} \right) (k-1)(k-2) F_a^{(0)} \right\},
\]
and similar for $F_b^{(*)}$. In terms of
\[
G_n = F_a^{(0)} F_b^{(n)} - F_a^{(n)} F_b^{(0)}
\]
\[
= (-1)^k \frac{k!}{2} \left\{ \left( \frac{D-2}{k-2} \right) G_2 + 2 \left( \frac{D-1}{k-1} \right) (k-2) G_1 \right\}
\]
one can show that
\[
\frac{G_{D-1}}{G_D} + 1 = \frac{2 (2D-3) G_1 + G_2}{D (2D-4) G_1 + G_2}.
\]

**Appendix B**

We provide a condition that is equivalent to the generalised Manin involution $i_p$ given by (5) being anti measure preserving with density
\[
\rho = \frac{L^{D-3}}{F_a},
\]
where $L = 0$ is a line through $p$.

It can be verified that the determinant of the Jacobian of the map $i_p$ equals
\[
\frac{(2 (D - 1) G_1 + G_2) X}{(2 (D - 2) G_1 + G_2)^3},
\]
with
\[ X = 2 \left( (c - u) G_2^{(u)} + (d - v) G_2^{(v)} \right) G_1 + 4 \left( (D - 1) \left( (D - 2) G_1 + G_2 \right) G_1 - G_2^2 \right). \]

On the other hand, by substituting the expressions for \( F^{(D)} \) and \( F^{(D-1)} \) as given by (34) into (32) with \( z \) given by (33) we find
\[ \frac{-\rho(u, v)}{\rho(u, v)} = \frac{(2 (D - 1) G_1 + G_2) Y}{(2 (D - 2) G_1 + G_2)^3}, \]

with
\[ Y = 2 G_1 \left( \frac{F_a^{(1)}}{F_a} G_2 - \frac{F_a^{(2)}}{F_a} G_1 \right) - 2 (D - 2) G_1 (\alpha - 1) G_1 + G_2) - G_2^2. \]

We have \( Y = X \) if
\[ (c - u) G_2^{(u)} + (d - v) G_2^{(v)} + F_a^{(1)} F_b^{(2)} - F_a^{(2)} F_b^{(1)} = 2 (D - 1) ((D - 2) G_1 + G_2) + (D - 2) ((D - 1) G_1 + G_2). \]  \( (36) \)

It is easy, using Maple [8], to verify that condition (36) is satisfied for pencils of degree \( D = 2, 3, 4 \).

Appendix C

The projective collineation (30) is an involution for solutions of
\[ \begin{align*}
ab + be + ch &= 0, & ac + bf + ci &= 0, & ad + de + fg &= 0, & ag + dh + gi &= 0 \\
bg + eh + hi &= 0, & cd + ef + fi &= 0, & a^2 + bd &= fh + i^2, & c^2 + bd &= cg + i^2.
\end{align*} \]  \( (37) \)

Assuming that \( b \neq 0 \), the highest dimensional family of solutions to (37) can be parameterised in terms of \( b, c, e, h, i \) by
\[ \begin{align*}
a &= -e - \frac{ch}{b}, & g &= -\frac{h (e + i)}{b}, & f &= \frac{c (be - hi + ch)}{b^2}, & d &= -\frac{(e + i) (be - bi + ch)}{b^2}.
\end{align*} \]  \( (38) \)

We reparameterise the solution (38) in terms of parameters \( \alpha, \beta, \gamma, P, Q \)
\[ \begin{align*}
b &= h P, & c &= \frac{h (\beta PQ - (\alpha + \gamma) Q - 2 \alpha)}{\alpha + \gamma}, \\
e &= \frac{h (\alpha Q + \gamma Q + \alpha)}{\alpha + \gamma}, & i &= \frac{h (\beta PQ - (\alpha + \gamma) Q - \alpha)}{\alpha + \gamma}.
\end{align*} \]

The parameters \( P, Q \) play a special role; defining \( Y = (P, Q) \) the projective collineation takes the form
\[ \sigma : U \to U + z(U - Y), \quad z = -1 + \frac{\alpha}{(\alpha + \gamma) v - \beta Q u + \delta}, \]  \( (39) \)

with
\[ \delta = (\beta P - \gamma Q - \alpha Q + 1). \]  \( (40) \)

The constraint (40) ensures that \( \sigma \) is an involution. The form of (39) shows that \( \sigma \) preserves lines through \( Y \). We take
\[ P = \frac{BF - CE}{AE - BD}, \quad Q = \frac{CD - AF}{AE - BD}. \]

\(^4\)Lower dimensional solutions can be obtained by taking \( b = 0 \) and either \( c = 0 \) or \( h = 0 \).

\(^5\)This 'reparameterisation' is invertible outside the set \( bh(bh + ch - bi)(be + ch - bi) \). In particular, this means that the linear switch from the previous section (where \( h = 0 \)) is not included.
so that \( Y = (P, Q) \) is the intersection point of the lines \( S = 0 \) and \( T = 0 \), where

\[
S = Au + Bv + C, \quad T = Du + Ev + F,
\]

Having fixed \( Y \), the four parameter family of projective collineations \( [39] \) leaves the ratio \( S/T \) invariant, and a three parameter subfamily, defined by \( [40] \), consists of involutions.

Using \( S, T \) we can build pencils of fixed degree which are invariant under \( \sigma \) \( [39] \). For \( D = 2 \) we have \( P_{\alpha, \beta}(u, v) = 0 \) where

\[
F_a = a_1S^2 + a_2ST + a_3T^2, \quad F_b = b_1S^2 + b_2ST + b_3T^2.
\]

The point \( Y \) is a double point of the pencil \( P_{\alpha, \beta}(u, v) = 0 \). Because the degree of the pencil is two, all curves are singular, i.e. each curve factorises into a product of lines. If \( \alpha, \beta \) are such that \( P_{\alpha, \beta}(u, v) = 0 \), then \( P_{\alpha, \beta}(u, v) = LK \). If \( L = 0 \) is the line through \( Y \) and \( U = (\hat{u}, \hat{v}) \), then \( K = 0 \) is the line through \( Y \) with direction

\[
(a_1b_2 - a_2b_1 \quad a_2b_3 - a_3b_2 \quad a_1b_3 - a_3b_1) \begin{pmatrix}
-B\hat{S} & A\hat{S} \\
-E\hat{T} & D\hat{T} \\
-(B\hat{T} + E\hat{S}) & A\hat{T} + D\hat{S}
\end{pmatrix}.
\]

Choosing involution points \( p \) and \( \sigma(p) \), for some \( \alpha, \beta, \gamma, \) and \( \delta \) given by \( [40] \), the map \( \iota_{\sigma(p)} \circ \iota_p \) admits a root.

**Theorem 22.** The root \( \rho_p = \sigma \circ \iota_p \), where \( \sigma \) is given by \( [39] \), with \( [40] \), and \( \iota_p \) by \( [4] \), is an integrable map of the plane. It preserves each curve of the quadratic pencil \( P_{\alpha, \beta}(u, v) = 0 \) with \( [7] \) and \( [41] \), and it is measure preserving with density \((LF_a)^{-1}\) where, with \( p = (c, d) \),

\[
L = (d - Q)(u - P) - (c - P)(v - Q), \tag{42}
\]

so that \( L = 0 \) is the line through \( p \) and \( Y \).

For \( D = 3 \) we have that

\[
P_{\alpha, \beta}(u, v) = \alpha(a_1S^3 + a_2S^2T + a_3ST^2 + a_4T^3) + \beta(b_1S^3 + b_2S^2T + b_3ST^2 + b_4T^3)
\]

admits the symmetry switch \( [39] \). We require that the involution point \( p = (c, d) \) is a base point of the pencil. Because \( Y \) is a triple point, each curve is a product of three lines, with common intersection point \( Y \). Thus the line \( L \) through \( p \) and \( Y \) \( [42] \) is contained in each curve, we have \( P_{\alpha, \beta}(u, v) = LZ \), where \( Z \) is a quadratic polynomial with a double point at \( Y \). No new maps which admit a root are obtained, other than the ones already obtained in the \( D = 2 \) case. Similarly no other maps are obtained in the \( D > 3 \) case where the requirement of the involution point \( p \) being a singular point of multiplicity \( D - 2 \) leads to the factorisation \( P_{\alpha, \beta}(u, v) = L^{D-2}Z \).

**Example 23.** We take \( S = u + 12v + 2 \) and \( T = 2u - 4v - 3 \), so the lines \( S = 0 \) and \( T = 0 \) intersect in \( Y = (1, -1/4) \). Taking \( D = 2 \), \( a_i = i + 1, b_i = 4 - i \), the point \( q = (1, 3/7) \) lies on the curve \( P_{34, -31}(u, v) = 35(u - 1)(9u - 88v - 31) = 0 \). Choosing \( \alpha = -6, \beta = 20, \gamma = 6 \) gives \( d = 1 \) and

\[
\sigma : (u, v) \rightarrow \left(7 - u \quad \frac{15u + 24v + 7}{5u + 1}\right).
\]

One verifies that

\[
P_{34, -31}(\sigma(u, v)) = \frac{1260(u - 1)(9u - 88v - 31)}{(5u + 1)^2}.
\]

We choose the point \( p = (2, 1) \) as involution point, and we find \( r = \iota_p(q) = -(129, 115)/289 \). The points

\[
\sigma(p) = (5/11, -41/44), \quad \sigma(q) = (1, -13/14), \quad \sigma(r) = (-538/89, -691/712)
\]
lie on a straight line, see Figure 10. It can also be checked that
\[
\sigma(t_p(\sigma(r))) = \sigma(1, 4325/5728) = (1, -7189/5728) = t_{\sigma(p)}(r).
\]

Figure 10: A product of lines admitting fractional linear symmetries.

Appendix D

Here we give the constants that appear in our formula for quartic polynomials with two double points \((c, d)\) and \((e, f)\). \[20\]:

\[
P = (4d^3 - 4f^3) a_1 + (-3c f^3 + 3d^3 e) a_2 + (3d^5 f - 3d f^3) a_3 + (-2c^2 f^6 + 2c^2 d^3) a_4 \\
+ (-2 c d f^3 + 2d^3 e) a_5 + (2d^5 f^2 - 2d^3 f^3) a_6 + (-2 c d f + 3d^2 e f) a_8 + (-c d f^3 + d^3 e f^2) a_9
\]

\[
Q = (-4c^6 + 4e^3) a_1 + (-3c^3 e + 3ce^3) a_2 + (-3c^3 f + 3de^3) a_3 + (-2c^2 e^6 + 2c^2 e^3) a_4 \\
+ (-2 c^3 e f + 2d e^3) a_5 + (-2 c^3 f^2 + 2e^3 d^2) a_6 + (-c^3 e^2 f + c^2 d e^3) a_8 + (-c^3 e f^2 + c d^2 e^3) a_9
\]

\[
R = (9c^3 d^4 f^3 - 12c^3 d^3 f^4 + 3c^3 f^7 + 18c^2 d^6 e f^2 - 24c^2 d^4 e f^3 + 6c^2 d e^6 f - 6c d^6 e f + 24c d^2 e^6 f^4 \\
- 18c d^3 e f^5 - 3d^4 c f^3 + 12 d^4 e f^3 - 9 d^4 e^3 f) a_1 + (-2c d^4 f^4 + 2e^4 f^7 + 5 c d^4 e f^3 - 9 c d^4 e f^3 \\
+ 4c d e f^3 + 12 c^2 d^2 c^2 f^2 - 18 c^2 d^2 e^2 f^3 + 18 c^2 d^2 e^2 f^4 - 12 c^2 d^2 e^2 f^5 - 4 e d e^3 f + 9 c d e^3 f^3 \\
- 5c d^3 e^3 f^4 - 2d^3 e^3 f^4 + 2 e d e^4 f) a_4 + (6c^4 d^4 f^4 - 9 c^3 f^5 + 3 d^3 f^6 + 12 c^2 d^2 e f^3 - 18 c^2 d^2 e f^4 \\
+ 6 c^2 d e^3 f^6 - 6c^2 d e^2 f^7 + 18 c d^2 e^2 f^8 - 12 c d e^2 f^9 - 3 d c f^9 + 9 d c^3 e^3 f^3 - 6 d^3 e^3 f^4) a_3 \\
+ (c^4 f^5 - 4 c^3 d^3 c^2 e f^4 + 2 c d e^4 f^6 + c^3 d^3 e^3 f^3 + 6 c^2 d^3 c^2 e^3 f^4 - 6 c^2 d^3 c^2 e^3 f^5 \\
- 2c^2 d^3 c^2 e^4 f^2 + 2d^3 c^2 e^4 f^2 + 4 c d e^3 f^6 + c^2 d^3 c^2 e^3 f^3 + 6 c^2 d^3 c^2 e^3 f^4 - 6 c^2 d^3 c^2 e^3 f^5 \\
+ 3 c d e^4 f^6) a_5 + (3 c d^4 f^5 - 6 c d^4 f^6 + 3 c^3 d^4 f^6 + 6 c^2 d^4 e f^6 - 12 c^2 d^4 e f^7 + 6 c d^3 e^2 f^6 - 6 c d^3 e^2 f^7 + 12 c d^2 e^2 f^4 \\
- 6 c d^2 e^2 f^5 - 3 d e c^3 f^5 + 6 d e c^3 f^6 - 3 d e c^3 f^7) a_6 + (c^4 d^2 f^7 - 4 c^3 d^2 e^3 f^4 + 2 c d^3 e^4 f^6 + 4 c^2 d^4 e^4 f^4 \\
- 3 c^2 d^4 e^3 f^5 + 3 c^2 d^4 e^3 f^6 - 4 c^2 d^3 e^3 f^4 - 2 c d^4 e^4 f^3 + 4 c d e^4 f^3 - d e^5 f) a_8 + (-2c d^3 f^6 \\
+ 2c d^2 e f^7 - c^2 d e^5 f + c^2 d^3 e^5 f - 6 c d^2 e^2 f^5 - 6 c d^3 e^2 f^4 - 4 c d e^4 f^3 + 2d^2 e^4 f^2 \\
+ 2d^4 e f) a_9 + (c^2 d^2 f^7 - 2c^2 d^2 e^3 f^6 + c^2 d^4 e^4 f^5 - c^2 d^4 e^3 f^5 - 2 c d^4 e^4 f^3 + 2 d^6 e^4 f - 3 d^6 e^3 f^3) a_{113}
\]
\[ S = (-12c^d f^3 + 12c^d e^f^2 - 24c^d e f^3 + 24c^d e f^3 + 36c^d e^2 f^2 - 36c^d e^2 f^4 - 24c^d e f^3 + 12d^e f^3 + 12d^e f^3) \alpha_1 + (3c^d f^3 - 6c^d e f^3 + 9c^d e f^3) \alpha_2 \]

\[ -15c^d d^e f^3 + 18c^d d^e f^3 - 27c^d d^e f^3 + 27c^d d^e f^3 + 18c^d d^e f^3 + 15c^d d^e f^3 - 9c^d f^3 d^3 + 6c^d e f^3 - 3d^e e f^3) \alpha_2 + (-8c^d d^e f^3 + 9c^d d^e f^3 - c^f^3 - 16c^d e f^3 + 18c^d e f^3 - 2c^d e f^3 \]

\[ +3c^d d^e f^3 + 27c^d d^e f^3 - 27c^d d^e f^3 - 3c^d d^e f^3 + 2c^d e f^3 - 18c^d e f^3 + 16c^d d^e f^3 + d^e f^3 \]

\[ -9d^e f^3 + 8d^e f^3) \alpha_3 + (6d^e d^e f^3 - 12d^e d^e f^3 - 6d^e d^e f^3 + 6d^e d^e f^3 + 12d^e d^e f^3 \]

\[ -6d^e d^e f^3) \alpha_3 + (3c^d d^e f^3 - c^f^3 - 2c^d e f^3 + 6c^d e f^3 - 2c^d e f^3 - 7c^d d^e f^3 - 6c^d e f^3 \]

\[ -3c^d d^e f^3 + 3c^d d^e f^3 + 6c^d d^e f^3 + 7c^d d^e f^3 + 2c^d e f^3 - 6c^d e f^3 + 2c^d e f^3 + d^e f^3 \]

\[ -3c^d d^e f^3) \alpha_5 + (-4c^d d^e f^3 + 6c^d d^e f^3 - 2c^d d^e f^3 - 8c^d d^e f^3 + 12c^d d^e f^3 - 4c^d d^e f^3 + 6c^d d^e f^3 \]

\[ -6c^d d^e f^3 + 4c^d d^e f^3 - 12c^d d^e f^3 + 8c^d d^e f^3 + 2c^d e f^3 - 6c^d e f^3 + 4d^e f^3) \alpha_6 + (-c^f^3 \]

\[ +6c^d d^e f^3 - 2c^d e f^3 - 5d^e d^e f^3 - 4c^d d^e f^3 + 4c^d d^e f^3 + 5c^d d^e f^3 + 2c^d e f^3 - 6c^d e f^3 \]

\[ +d^e f^3) \alpha_8 + (3c^d d^e f^3 - 2c^d d^e f^3 + 2c^d d^e f^3 - 8c^d d^e f^3 + 8c^d d^e f^3 + cd^e f^3 - 2c^d e f^3 \]

\[ +2d^e f^3 - 3d^e f^3) \alpha_9 + (-2c^d f^3 + 2c^d d^e f^3 - 2c^d d^e f^3 - 2c^d d^e f^3 \]

\[ T = (24c^d d^e f^3 - 36c^d d^e f^3 + 12c^d d^e f^3 - 24c^d d^e f^3 + 2c^d d^e f^3 - 12c^d f^3 - 12c^d d^e f^3 + 36c^d d^e f^3 - 24c^d d^e f^3 + 18c^d d^e f^3 - 8c^d e f^4 + 8c^d e f^4 - 18c^d d^e f^3 + 27c^d d^e f^3 - 18c^d d^e f^3 - 9c^d f^3 d^3 + 2c^d e f^4 + 15c^d d^e f^3 - 27c^d d^e f^3 + 9c^d e f^4 + 3c^d d^e f^3 + 6c^d d^e f^3 - 18c^d d^e f^3 - 15c^d d^e f^3) \alpha_3 \]

\[ +6c^d d^e f^3 - 4c^d d^e f^3 + 8c^d d^e f^3 - 8c^d d^e f^3 + 4c^d d^e f^3 - 12c^d d^e f^3 + 6c^d d^e f^3 - 6c^d d^e f^3 + 2c^d e f^3 + 3c^d d^e f^3 - 2c^d e f^3 - 3c^d e d^e f^3 + 6c^d d^e f^3 \]

\[ +3c^d d^e f^3 - 7c^d d^e f^3 - 6c^d d^e f^3 + 2c^d d^e f^3 + 6c^d d^e f^3 - 7c^d d^e f^3 - 3c^d e d^e f^3 - 6c^d d^e f^3 + 3c^d d^e f^3 + 2c^d e f^3 + 6c^d d^e f^3 - 6c^d d^e f^3 \]

\[ +12c^d e f^3 - 6c^d e f^3 - 6c^d e f^3 + 6c^d e f^3 - 6c^d e f^3 - 6c^d e f^3 + c^f^3 \]

\[ -2c^d e f^3 - 6c^d e f^3 + 3c^d e f^3 + 3c^d e f^3 - 3c^d e d^e f^3 + 6c^d e f^3 \]

\[ +4c^f^3 + 6c^d e f^3 - 12c^d e f^3 + 5c^d e f^3 + 18c^d e f^3 - 9c^d e f^3 - 2c^d e f^3 + 2c^d e f^3 + 9c^d e f^3 - 18c^d e f^3 - 12c^d e f^3 + 6c^d e f^3 - 6c^d e f^3 \]

\[ +12c^d e f^3 - 6c^d e f^3 - 6c^d e f^3 + 2c^d e f^3 + 3c^d e f^3 + 2c^d e f^3 + 8c^d e f^3 - 8c^d e f^3 \]

\[ -2c^d e f^3 - 6c^d e f^3 + 3c^d e f^3 + 6c^d e f^3 + 4c^d e f^3 - 5c^d e f^3 + 5c^d e f^3 \]

\[ +5c^d e f^3 - 4c^d e f^3 + 6c^d e f^3 + 6c^d e f^3 + 6c^d e f^3 - 6c^d e f^3 \]

\[ -2c^d e f^3 - 2c^d d^e f^3 + 2c^d d^e f^3 \]
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