The Thermodynamics of Quantum Systems and Generalizations of Zamolodchikov’s C-theorem

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Abstract

In this paper we examine the behavior in temperature of the free energy on quantum systems in an arbitrary number of dimensions. We define from the free energy a function $C$ of the coupling constants and the temperature, which in the regimes where quantum fluctuations dominate, is a monotonically increasing function of the temperature. We show that at very low temperatures the system is controlled by the zero-temperature infrared stable fixed point while at intermediate temperatures the behavior is that of the unstable fixed point. The $C$ function displays this crossover explicitly. This behavior is reminiscent of Zamolodchikov’s $C$-theorem of field theories in 1+1 dimensions. Our results are obtained through a thermodynamic renormalization group approach. We find restrictions on the behavior of the entropy of the system for a $C$-theorem-type behavior to hold. We illustrate our ideas in the context of a free massive scalar field theory, the one-dimensional quantum Ising Model and the quantum Non-linear Sigma Model in two space dimensions. In regimes in which the classical fluctuations are important the monotonic behavior is absent.
1 Introduction

Our understanding of the behavior of systems near critical points in 1+1 dimensions has been significantly improved in recent years due to the application of the methods of conformal field theory [1]. One of the aspects of this method which makes it particularly powerful and elegant is that it relates in a simple way the microscopic behavior of such systems with thermodynamics. At the critical point the most important parameter of the theory is the central charge, $c$, which is related with the quantum fluctuations at zero temperature. The central charge can be calculated in many ways. At zero temperature it can be calculated from the correlation function of the energy momentum tensor or from the commutation relations between the generators of the Virasoro algebra [2]. It can be shown that $c$ can also be obtained at finite temperatures from the specific heat of the system and, therefore, it can be measured in an experiment.

Conformal Field Theory (CFT) describes the behavior of systems at fixed points. Away from fixed points, CFT can also describe the “early” and “late” stages of the crossover between fixed points. The behavior at intermediate scales is, in general, non-universal and thus depends on the microscopic description of the system. The universal behavior of this crossovers is described by the famous Zamolodchikov’s $C$-theorem [3]. This theorem states that it is possible to define a function $C$ of the coupling constants of the theory which decreases monotonically under the renormalization group flow, that is, as we change the characteristic scale the function $C$ goes monotonically from a fixed point of larger central charge to one at smaller central charge. The assumptions present in his proof are the euclidian invariance, the existence of a symmetric, conserved, energy momentum tensor and the unitarity of the field theory. The physical interpretation of this theorem is related with loss of information due to the coarsed graining procedure present in the renormalization group equations. Another way to interpret this result asserts that $C$ measures the number of fluctuating degrees of freedom. These must decrease when we integrate over small scales in the renormalization group flow.

All the results of the conformal field theory seem to be very difficult to be applied for higher dimensions due to the proliferation of components in the energy momentum tensor. Nevertheless some importants attemps have been made with the use of the spectral representation for the correlation function of the energy momentum tensor [4], [5]. However, the relation between these results and thermodynamics remains unclear up to now.

The approach of this paper is based on the behavior of the thermodynamic functions under the renormalization group flow. Following this route we can determine the macroscopic conditions which are necessary for any system, in any number of
dimensions, to have a behavior which is analogue to the one described by the $C$-theorem proposed by Zamolodchikov. We can show that the the monotonic behavior implied by the $C$-theorem cannot be derived from the laws of thermodynamics alone and, therefore, this behavior is not universally true. As we will show, the sort of behavior needed for a $C$-theorem to hold is only possible in regimes in which quantum fluctuations dominate. In 1+1 dimensions this is always the case because of the triviality of one-dimensional classical statistical mechanics. But, in higher dimensions, there is a complex quantum-to-classical crossover which spoils the general validity of a $C$-theorem. To put it differently, this is a consequence of the fact that 1+1 dimensions all systems are at or below their lower critical dimension. This fact imposes stringent constraints on the allowed behavior of the entropy of these systems at low temperatures. However, in higher dimensions a variety of complex behaviors are allowed and the constraints on the entropy are much weaker. We believe that this observation is a hint that there is no general $C$-theorem in dimensions higher than 1+1.

We have investigated these issues in the context of several systems. We first consider the case of a free massive real scalar field in arbitrary dimensions. The results also apply to more general bosonic systems as well as to the case of relativistic fermions. Here, the crossover parameter is the mass. We find monotonic behavior in any number of dimensions. We also discuss two more non-trivial models. The first is the quantum Ising model in 1+1 dimensions. We find monotonic behavior in any number of dimensions. We show that the the monotonicity of the $C$-function (calculated without approximations) defined from the thermodynamics is obeyed in the low temperature limit and that near the scaling limit (where the model can be described as a field theory of free massive fermions) we found all the results expected from the point of view of conformal field theory. At temperatures comparable or higher than the fermion “bandwidth”, there is a crossover to classical behavior and monotonicity is lost (this is the “Debye temperature”). Next we discuss the example of the Non-linear Sigma model (the continuum version of the quantum Heisenberg antiferromagnet) in 2+1 dimensions. We show that the $C$-function (calculated from the $1/N$ expansion) is not monotonic for arbitrary temperatures and coupling constant due to a crossover from classical to quantum behavior. This crossover can be observed even near the phase transition. Once again, in the quantum regime we find a term in the free energy which displays the same monotonic behavior expected from a $C$-theorem.

In order to understand these results, we observe that temperature plays two different roles in these problems. At low temperatures, these systems are essentially quantum in $d + 1$ dimensions. Here the temperature mainly affects the geometry in which the system lives since it measures the “perimeter” of Euclidean space-time in the (imaginary) time direction. As the temperature is raised, this perimeter grows
smaller and the temperature drives the system away from criticality, just like a relevant operator. On the other hand, at high temperatures the dimension in the imaginary direction effectively shrinks to zero (compared with all the length scales in the system) and the system behaves like a classical system in $d$ dimensions. In this case the temperature only plays the role of a coupling constant in classical statistical mechanics. Thus, at these high temperatures, the $C$-function of the system is not necessarily monotonic. For instance, one does not expect mononicity if the classical system has a phase transition at a non zero temperature. Even if this is not the case, such as in classical systems at or below their lower critical dimension, non-monotonicity may still occur if the classical system has gapless configurations. We will see below that this is the case of the Non-linear Sigma Model in the “classical regime”. In the case of systems with discrete symmetries (e.g., the Ising Model), non-monotonicity still occurs near what we will refer to as the classical fixed point, but not near the critical zero-temperature fixed point. Thus, we do not expect to have a monotonic behavior of any piece of the free energy for all temperatures but only in regimes in which the quantum fluctuations are dominant. The strategy of this paper is thus to look for the regimes of parameter space (i.e. coupling constants and temperature) in which the physics is dominated by the quantum fluctuations.

In the next section we discuss the relation between the Renormalization Group of quantum systems at finite temperature and Thermodynamics. In section 3 we discuss our results in the context of free scalar and fermi fields. In section 4 we apply our ideas for the case of the quantum Ising model in 1+1 dimensions. In section 5 we show how our results work for the case of the Non-linear Sigma model in 2+1 dimensions. The last section contains our conclusions.

## 2 The Renormalization Group and Thermodynamics

Suppose we are interested in the study of a system in statistical mechanics which can be described in a lattice with some characteristic length $a$ (UV cut-off) and which has a set of dimensionful coupling constants $g$. By naive dimensional analysis we know that the bulk free energy of this system in thermal equilibrium at temperature $T$ ($\beta = 1/T$, $\hbar = k_B = 1$) in $(d + 1)$-dimensions, in the thermodynamic limit, can be written as

$$F(\beta, g, a) = E_0(g, a) - \frac{n(d)VC(\beta, g, a)}{v^d \beta^{d+1}} \quad (2.1)$$

where $E_0$ is the zero temperature energy, $V$ is the volume ($V \to \infty$, but $N/V$ is finite, where $N$ is the number of particles), $v$ is a characteristic velocity in the system.
(for instance, the velocity of the quasi particles), $n(d)$ is a positive real number which depends only on the dimensionality of the system (see next section for details) and $C$ is a dimensionless function of its arguments.

We want to comment here that in (2.1) we are implicitly assuming that the ultra-violet divergences in the field theory can be subtracted in the first term on the r.h.s. of (2.1) since these divergences only occur at zero temperature, that is, since $C$ is a finite size effect (in terms of temperature) it is insensitive to divergences which occur in small scales [8].

Here we are interested in the limit when $a \to 0$, that is, in the limit where the system can be described by a field theory (we do not intend to discuss here how this can be achieved, we just assume that is possible). In this limit the “bare” coupling constants of the original theory are replaced by renormalized dimensionless coupling constants $\alpha$ and since $C$ is dimensionless we should write

$$C(\beta, g, a) = C(\alpha, \beta \Lambda) \tag{2.2}$$

where $\Lambda$ has dimensions of inverse of length.

Since $C$ is an observable it does not change as the scale is changed and therefore it obeys a renormalization group (RG) equation ($t = (\beta \Lambda)^{-1}$)

$$\left( t \frac{\partial}{\partial t} - \tilde{\beta}(\alpha) \frac{\partial}{\partial \alpha} \right) C(\alpha, \beta \Lambda) = 0 \tag{2.3}$$

where

$$\tilde{\beta}(\alpha) = \Lambda \left( \frac{\partial \alpha}{\partial \Lambda} \right)_{g, \beta} \tag{2.4}$$

is the beta function of the system. Observe that the beta function can also be obtained from (2.3).

The solution of (2.3) is obtained with the introduction of the correlation length of the system, $\xi(\alpha, t)$, defined by its renormalization group equation [4]

$$t \frac{\partial \xi(t)}{\partial t} = \tilde{\beta}(\alpha) \frac{\partial \xi}{\partial \alpha} \tag{2.5}$$

It is trivial to show that

$$C(\alpha, t) = C \left( \frac{\xi(\alpha, t)}{\beta} \right) \tag{2.6}$$

Define now a dimensionless correlation length by

$$\gamma = \Lambda \xi \tag{2.7}$$
and therefore from (2.6)  
\[ C = C(t, \gamma(\alpha, t)) \]  \hspace{1cm} (2.8)

and from (2.5) we have  
\[ \frac{\partial(t\gamma)}{\partial t} = \tilde{\beta}(\alpha) \frac{\partial \gamma}{\partial \alpha}. \]  \hspace{1cm} (2.9)

It is also important to define the running coupling constant, \( \alpha(t) \), as the solution of the equation (2.4)  
\[ t \frac{\partial \alpha(t)}{\partial t} = -\tilde{\beta}(\alpha(t)) \]  \hspace{1cm} (2.10)

with the initial condition  
\[ \alpha(t = t_0) = \alpha \]  \hspace{1cm} (2.11)

where \( t_0 \) is an arbitrary renormalization point.

It is well known that at a critical point (where the conformal field theory can be applied \[2\]) the function \( C \) as defined in (2.1) is a constant and, in one dimension, is related with the Virasoro algebra (it is called the central charge of the theory).

Zamolodchikov \[3\] proposed some years ago a function \( C_Z(R, g) \) (where \( R \) is a length scale in the 1+1 dimensional space) which is monotonically decreasing under the RG flow, that is,  
\[ R \left( \frac{\partial C_Z}{\partial R} \right) \leq 0 \]  \hspace{1cm} (2.12)

and which, at the critical point, is the central charge. His proof is based on the euclidian invariance, the existence of a symmetric, conserved, energy-momentum tensor and unitarity of the field theory.

Suppose we can actually calculate the spectrum for the system under consideration. Of course the free energy can always be written in the form (2.1) but, in this case, since \( C \) is dimensionless,  
\[ C = C(\phi(g)\beta) \]  \hspace{1cm} (2.13)

where \( \phi(g) \) is a characteristic energy scale which depends on the bare coupling constants of the theory and possibly on the temperature. In general we are talking about a Hamiltonian with the form  
\[ H = H_0 + gH_1 \]  \hspace{1cm} (2.14)

where \( H_0 \) is the fixed point Hamiltonian and \( H_1 \) is a “pertubation” which drives the system away from criticality. In this case when \( g \) vanishes \( C \) goes to the central charge at the fixed point,  
\[ \text{Lim}_{g \to 0} C(\phi(g)\beta) = c_0 = \text{constant}. \]  \hspace{1cm} (2.15)
Since in thermodynamics, for each system, the coupling constants are fixed, the only energy scale free to be changed is the temperature. Therefore we proposed an analogue of the Zamolodchikov’s equation (2.12) which is valid in any number of dimensions

$$\beta \left( \frac{\partial C}{\partial \beta} \right)_\alpha \leq 0$$  \hspace{1cm} (2.16)

Observe that we are using the fact that in finite temperature field theory we have to impose periodic (or anti-periodic) boundary conditions in the imaginary time direction. These boundary conditions permit us to interpret a quantum system in \( d + 1 \) dimensions as a classical system in \( D(= d + 1) \) dimensions. The manifold under consideration has the form of a cylinder since the other \( d \) dimensions are free. The length of the cylinder in the imaginary time direction is \( \beta \) and, therefore, (2.12) and (2.16) represent the same change in the scale, although, we could say, (2.12) is an isotropic change while (2.16) is anisotropic. Observe that at high temperatures the length of the cylinder goes to zero and therefore the field theory is restricted to \( d \) dimensions.

From the point of view of the renormalization group approach we see that if we follow the flow using the running coupling constant defined by equation (2.10) the function \( C \) is stationary as required. However if we fix the renormalized coupling constant (as required in Zamolodchikov’s theorem) and vary the dimensionless temperature, \( t \), we would get (2.16). Using the result (2.8) and the RG equation (2.9) we find,

$$\frac{\partial C}{\partial t} = \frac{dC}{dt(t\gamma)} \frac{\partial(t\gamma)}{\partial t} = \frac{dC}{dt(t\gamma)} \frac{\partial \gamma}{\partial \alpha} \bar{\beta}(\alpha)$$  \hspace{1cm} (2.17)

This expression will be useful later.

Consider a system initially at temperature zero in a volume \( V \), with ground state energy \( E_0 \) and entropy \( S_0 \). Suppose we perform a thermodynamic transformation at a constant volume which increases the temperature of the system up to a temperature \( T \). At this point the system, in thermal equilibrium, has an internal energy \( U \) and an entropy \( S \). We will show that if,

$$\left( \frac{d + 1}{d} \right) \frac{\delta U}{T} \geq \delta S$$  \hspace{1cm} (2.18)

where

$$\delta U = U - E_0$$  \hspace{1cm} (2.19)

and

$$\delta S = S - S_0$$  \hspace{1cm} (2.20)
then (2.16) follows. We will show at the end of this section that (2.18) can not be obtained from the laws of thermodynamics. Therefore the irreversibility of the RG flow expressed in (2.16) is a consequence of (2.18).

Notice that (2.18) has a very nice interpretation in terms of the number of states. Since in the thermodynamical limit the entropy of a system can be written as

$$S(U) = \ln(\Omega(U))$$  \hspace{1cm} (2.21)

where $\Omega(E)$ is the thermodynamic number of states defined as the number of states with energy between $E$ and $E + dE$, we can reexpress (2.18) as

$$\Omega(E_0)e^{\frac{d+1}{d} \frac{dU}{T}} \geq \Omega(U)$$  \hspace{1cm} (2.22)

which means that the number of states with energy between $U(T)$ and $U(T) + dU(T)$ is bounded from above. This is a highly non trivial statement which, as we shall see below, is not required by the laws of thermodynamics alone.

Before we start to prove that (2.16) is a consequence of (2.18) let us remember that the continuum limit $a \to 0$ is very important in Zamolodchikov's proof. There, he assumed that,

$$R \gg a$$  \hspace{1cm} (2.23)

In this paper we also assume the counterpart of (2.23) in terms of temperature, namely,

$$T \ll \theta$$  \hspace{1cm} (2.24)

where $\theta$ is a characteristic cut-off temperature (for instance, the Debye temperature, $\theta_D$).

Using equation (2.1) the thermodynamical quantities, pressure ($P$), entropy ($S$) and thermal energy ($U$) can be easily calculated,

$$P = \frac{n(d)C}{v^d\beta^{d+1}}$$  \hspace{1cm} (2.25)

$$S = \frac{n(d)V}{v^d\beta^d} \left[(d+1)C - \beta \frac{\partial C}{\partial \beta}\right]$$  \hspace{1cm} (2.26)

$$U = E_0 + \frac{n(d)V}{v^d\beta^{d+1}} \left[dC - \beta \frac{\partial C}{\partial \beta}\right]$$  \hspace{1cm} (2.27)

Notice that as the temperature goes to zero the free energy and the internal energy goes to the ground state energy. This fact means that $C$ and $\partial C/\partial \beta$ go to zero in this limit. From the third law of thermodynamics we know that

$$\lim_{T \to 0} S(T) = S_0$$  \hspace{1cm} (2.28)
So in order to our definition of $C$ be consistent we have to set

$$S_0 = 0$$

(2.29)

which is allowed from the thermodynamic point of view.

From (2.26) and (2.27) we can write

$$\beta \frac{\partial C}{\partial \beta} = \frac{v^d \beta^d}{n(d)V} (d S - (d + 1) \beta \delta U)$$

(2.30)

therefore if (2.18) is obeyed (2.16) follows and we get the same monotonicity property as in the original C-theorem.

We can also rewrite (2.30) in an useful form. Recall that the definition of the specific heat at constant volume is

$$C_v(T) = T \frac{\partial S}{\partial T} = \frac{\partial U}{\partial T}$$

(2.31)

which, when integrated using (2.29), gives

$$S(T) = \int_0^T C_v(T') dT'$$

(2.32)

and

$$\delta U(T) = \int_0^T C_v(T')dT'.$$

(2.33)

Substituting (2.32) and (2.33) in (2.30) we find

$$\beta \frac{\partial C}{\partial \beta} = -\frac{v^d \beta^d}{n(d)V} \int_0^T C_v(T') \left( \frac{1 + d}{T'} - \frac{d}{T'} \right) dT'$$

(2.34)

Expression (2.34) is very suitable for experimental test and can be used to classify the systems which obey assumption (2.18). Moreover, at the fixed point we expect that $\partial C/\partial \beta = 0$ and therefore the specific heat must behave like $T^d$ which is known to be true in one dimension.

Now we can explain why condition (2.24) is important for our results. The perturbation term in (2.14) breaks the scale invariance of the fixed point hamiltonian and it drives the system into the basin of attraction of a more stable fixed point. Once temperature is turned on, this operator generates flows away from the fixed point but which are “cutoff” once the correlation length $\xi$ is of the order of the inverse temperature. Thus, at temperatures higher than $\xi^{-1}$, the system behaves as if it were still at the infrared unstable fixed point but, at lower temperatures
it crosses over to the infrared stable one. It is this crossover which is described by Zamolodchikov’s theorem [8]. Moreover, at high temperatures the specific heat must go to its classical value which is a constant given by the equipartition theorem (Dulong-Petit law), therefore, the entropy diverges logarithmically and we do not expect that assumption (2.18) is obeyed. However, if the spectrum has a width which is much greater than the energy scale \( \phi(g) \) we could imagine a situation where the temperature is much greater than \( \phi(g) \) but much smaller than the bandwidth. In this case, since \( C \) is only a function of \( \phi(g)\beta \), the limit as the temperature goes to infinity is identical (in terms of RG) to the limit as \( g \to 0 \) and therefore, by (2.15), we see that the \( C \) is the central charge in this point. This is a stable high temperature fixed point. Nevertheless, for temperatures larger than the bandwidth we expect that the behaviour of the system is classical and (2.18) will not be obeyed. We will see later that this behaviour can be seen in the Ising model in 1+1 dimensions.

For systems with a phase transition (when \( \phi \) also depends on the temperature) we know that near the critical temperature the correlation length of the system increases exponentially and therefore we do not expect that the thermodynamic number of states be bounded from above. This behaviour near the critical temperature is essentially classical and therefore the monotonicity of \( C \) should be violated. We will show that this is what happen in the Non-linear Sigma Model.

We want to point out that in the assumption that we can describe the system by a quantum field theory we are assuming that there is a zero temperature fixed point which is unstable under the increase of the temperature. This is clearly expected in any quantum system since we always have the crossover from the quantum to the classical system as the temperature increases. Therefore, (2.16) shows that the RG’s flow goes from the zero temperature fixed point to a fixed point at finite temperature (in particular to a high temperature fixed point).

In order to finish this section we will show that we can not obtain (2.18) from the laws of thermodynamics.

Suppose we have initially the system at zero temperature and we put it in contact with a thermal reservoir at temperature \( T \) and wait for the thermal equilibrium. Suppose that this system is isolated from the rest of the universe. The variation of the entropy of the reservoir is \( -Q/T \) where \( Q \) is the heat given by the reservoir to the system. Using the second law we get

\[
S \geq \frac{Q}{T} \tag{2.35}
\]

Using the first law we have

\[
\delta U = Q \tag{2.36}
\]
therefore
\[ S \geq \frac{\delta U}{T} \]  
(2.37)

We see that the laws of thermodynamics can not lead to (2.18), that is, they show that the increase of the entropy is bounded from below. Furthermore, notice that from (2.37), (2.26) and (2.27) we obtain that \( C \) is always positive as required.

3 Free Massive Field Theories

We will now discuss the physical meaning of the results of the last section in the context of specific systems. We will first consider the case of a massive scalar field at non-zero temperature and exhibit the crossover. We will do that by showing that the restriction (2.18) is always obeyed.

The approach is very straightforward. The free energy for a system with one-particle excitation spectrum \( E_{\kappa} \) where \( \kappa \) is a set of quantum numbers is given by

\[
F = E_0 + \frac{\tau}{\beta} \sum_{\kappa} \ln \left( 1 - \tau e^{-\beta E_{\kappa}} \right)
\]

(3.1)

where \( \tau \) is +1 (-1) for bosons (fermions).

Comparing (2.1) and (3.1) we get

\[
C = -\frac{(v\beta)^d}{n(d)V} \tau \sum_{\alpha} \ln \left( 1 - \tau e^{-\beta E_{\alpha}} \right)
\]

(3.2)

Suppose we have the relativistic dispersion relation

\[
E_k = \sqrt{m^2 v^4 + v^2 k^2}
\]

(3.3)

where \( k \) is the wavenumber (in units of \( \hbar \)) which is defined by periodic boundary conditions. In the thermodynamic limit we can replace the sum in (3.2) by an integral as usual. We will also assume \( v = 1 \) and rewrite (3.2) as

\[
C = -\frac{\beta^d}{n(d)(2\pi)^d} \tau \int d^d k \ln \left( 1 - e^{-\beta E(k)} \right)
\]

(3.4)

since the spectrum only depends on the absolute value of \( \vec{k} \), we have

\[
C = -\frac{\beta^d}{n(d)2^{d-1}\pi^{d/2}\Gamma(d/2)} \tau \int_0^{\infty} dk k^{d-1} \ln \left( 1 - e^{-\beta E(k)} \right)
\]

(3.5)

where \( \Gamma(x) \) is the Gamma function.
Using (3.3), changing variables in the integral (3.5), expanding the logarithm in powers and integrating, we easily get

\[ C(m\beta) = \left( m\beta \right)^{d+1\over 2} n(d) \frac{\tau^{l+1}}{l^{d+1}} K_{l+1}(lm\beta) \]  

(3.6)

where \( K_{\nu}(x) \) is the Bessel function of imaginary argument of order \( \nu \).

Observe that, as expected, \( C \) is only function of one variable, namely \( m\beta \), and, as explained before, we find that the correlation length is given by \( m^{-1} \). Furthermore, since \( \alpha = \Lambda^{-1} m \ (\gamma = \alpha^{-1}) \), the beta function is trivial, namely, \( \tilde{\beta}(\alpha) = -\alpha \), and the running coupling constant as defined in (2.10) is just \( \alpha(t) = \alpha t/t_0 \).

We can also obtain the special cases. In the massless limit, when kinetic energy of the system (given by \( m \)) is much smaller than thermal energy (given by \( \beta^{-1} \)) that is, \( m\beta \ll 1 \ (\xi >> 1/T) \), we can approximate the Bessel function for small argument \[ \text{Bessel function} \] and write

\[ C(0) = \frac{\Gamma\left( \frac{d+1}{2} \right)}{n(d) \pi^{d+1}} \sum_{l=1}^{\infty} \frac{\tau^{l+1}}{l^{d+1}} \]  

(3.7)

which is finite. This is the high temperature fixed point.

From (3.6) and using well know relations between Bessel functions \[ \text{Bessel function} \] we find that the rate of change of \( C \) with the temperature is given by,

\[ \frac{dC}{d\beta} = -2m \left( \frac{m\beta}{2\pi} \right)^{d+1\over 2} n(d) \frac{\tau^{l+1}}{l^{d+1}} K_{l+1}(lm\beta) \]  

(3.8)

which is always negative in any number of dimensions. We could also arrive to this result using equation (2.17).

However the normalization factor \( n(d) \) is still arbitrary. For relativistic systems in 1+1 dimensions, this factor, at a fixed point, equals to the central charge of the Virasoro algebra which can be thought as a measure of the number of fluctuating fields. Its well known \[ \text{Central Charge} \] that in 1+1 dimensions relativistic massless bosons have \( C(0) = 1 \) and relativistic massless spinless fermions (Majorana fermions) have \( C(0) = 1/2 \). Hence,

\[ n(1) = \frac{\pi}{6}. \]  

(3.9)

In higher dimensions we do not know if a generalization of the notion of central charge exists. Still, we may choose the normalization factor in such a way that, for free fields, it counts the number of degrees of freedom. This choice leads to the definition

\[ n(d) = \frac{\Gamma\left( \frac{d+1}{2} \right)}{\pi^{d+1}} \zeta(d+1) \]  

(3.10)
for bosons and
\[ n(d) = \frac{\Gamma \left( \frac{d+1}{2} \right) (2^d - 1)}{\pi^{\frac{d+1}{2}} 2^{d-1}} \zeta(d+1) \]  
(3.11)
for fermions, where \( \zeta(x) \) is the Riemann’s Zeta function. Observe that spinless fermions and bosons only have the same \( n(d) \) at \( d = 1 \). Anyway this arbitrariness in \( n(d) \) does not affect our discussion about the monotonicity properties of the \( C \) function.

Let us remark that the behaviour of the function \( C \) is non analytic as a function of \( m \), that is, if \( m \) is strictly zero the function \( C \) is a constant given by (3.7) in all ranges of temperatures. If \( m \) finite, even very small, the behaviour of \( C \) is completely different (see Fig.1). Indeed, in the massive limit, \( m\beta >> 1 (\xi << 1/T) \), or low temperatures, the Bessel function in (3.6) vanishes exponentially and \( C \) is zero.

Finally, for systems like Fermi liquids we know that independent of the dimensionality the specific heat is linear in the temperature \([12]\). If we substitute this result in (2.34) we will see that \( \partial C/\partial \beta \) is negative in two or three dimensions but it is positive in one dimension. This result is clear from the point of view of relativistic particles since in one dimension we can always expand the energy around the Fermi surface as in (3.3).

4 The Ising Model

Here we will consider the Ising model which can be described by the following Hamiltonian
\[ H = -\Gamma \sum_{n=1}^{N} S_n^3 - J \sum_{n=1}^{N} S_n^1 S_{n+1}^1 \]  
(4.1)
where \( S_n^j \) is the \( j \)th \( (j = 1, 2, 3) \) projection of the spin operator in the site \( n \) \( (n = 1, ..., N) \). We will assume periodic boundary conditions, that is,
\[ S_{N+1}^1 = S_1^1 \]  
(4.2)
and neglect any surface effect which might occur.

This model was solved exactly \([13]\) using the Jordan-Wigner transformation. Since the procedure is somewhat standard we will only quote the results (see \([13]\) for details).

It is possible to show that the problem reduces to a problem of fermions with dispersion relation given by
\[ E_k = \Gamma \sqrt{1 + \varrho^2} - 2 \varrho \cos(k) \]  
(4.3)
where
\[ k = \frac{2\pi m}{N} \]  
with \( m = 0, \pm 1, \pm 2, \ldots, \pm \frac{N}{2} \) \((N \to \infty)\). The constant \( \varrho \) is defined as
\[ \varrho = \frac{J}{2\Gamma} \]
and it controls the physics of the problem. Observe that when \( \varrho = 1 \) the dispersion relation is simply
\[ E_k = 2\Gamma \sin\left(\frac{k}{2}\right) \]
and the fermions are massless. The velocity of the excitations can be easily computed to be \( \Gamma \) near \( k = 0 \). At this point the system is scale invariant and it represents the phase transition.

The free energy is obtained straightforwardly \[14\]
\[ F = E_0 - \frac{N}{\beta\pi} \int_0^\pi dk \ln \left(1 + e^{-\beta E_k}\right) \]  
where
\[ E_0 = -\frac{N}{2\pi} \int_0^\pi dk E_k \]
is the ground state energy.

From (2.1), (3.9) and (4.7) we get
\[ C = \frac{6\beta\Gamma}{\pi^2} \int_0^\pi dk \ln \left(1 + e^{-\beta E_k}\right) \]  
and although we do not have an analytic form for (4.9) we can study its limits.

Observe that at the critical point, \( \varrho = 1 \) \((J = 2\Gamma)\), we can rewrite (4.9) as
\[ C = \frac{6\beta\Gamma}{\pi^2} \sum_{n=1}^\infty \frac{(-1)^n}{n} \int_0^{\pi/2} dk e^{-2n\beta\Gamma\sin(k/2)} \]  
and therefore at the critical point \( C \) depends on the temperature. This occurs because we know that the continuum limit can only be taken at low temperatures. Recall that \( k \) is the lattice momentum. For a system with lattice constant \( a \), the momentum \( q \) in laboratory units is \( q = k/a \). The continuum limit is the limit of \( a \to 0 \), keeping the physical dimensionless temperature \( t^{-1} = \Gamma a/T \) fixed. In this limit, (4.10) becomes
\[ \lim_{a \to 0} C = \lim_{a \to 0} \frac{6}{\pi^2 t} \sum_{n=1}^\infty \frac{(-1)^n}{n} \int_0^{\pi/2} dq e^{-2a\Gamma\sin(\frac{q}{2})} = \frac{1}{2} \]  

(4.11)
as expected from the results of conformal field theory.

We can also understand this result from the point of view of section 3. Observe that at low temperatures and close to the phase transition we can expand (4.3) around \( k = 0 \) as

\[
E_k = \Gamma \sqrt{(1 - \varrho)^2 + \varrho k^2}
\]

which has the same form as (3.3) where the mass term (or the inverse of the correlation length) is proportional to \( 1 - \varrho \) which vanishes at the critical point. If now we let \( k \) varies between 0 and infinity we recover the result of the former section.

Observe that here the fact that the energy cuttoff (i.e. the bandwidth of the spectrum) is finite is very important. We can not excite particles in the system with an energy larger than the bandwidth and therefore we have a temperature cut-off exactly as discussed in second section for the case of constraint (2.24). This fact will result that the \( C \) function defined in (4.9) will decrease with temperature for temperatures larger than the bandwidth. If we take the limit that \( \beta \Gamma << 1 \) in (4.10) we easily see that \( C \) goes to zero in this limit.

We will pay attention to two limits, the limit where the we have unbroken symmetry (the vaccum is invariant under the symmetry of the Hamiltonian) and the limit of broken symmetry [12]. When \( \varrho << 1 \) \( (J << \Gamma) \) the vaccum is an eigenstate of \( S^3 \) where all spins are pointing up. From (4.9) is easy to see that

\[
C = \frac{6\beta\Gamma}{\pi} \ln \left( 1 + e^{-\beta\Gamma} \right)
\]

and get a behaviour where \( C \) increases monotonically in the low temperature region and when reaches temperatures of order of the bandwidth, namely \( \Gamma \), \( C \) goes monotonically to zero (see Fig.2).

When \( \varrho >> 1 \) \( (J >> \Gamma) \) the vaccum is an eigenstate of \( S^1 \) and it is degenerate (all the spins can be up or down), that is, the symmetry is broken. From (4.9) is trivial to show that

\[
C = \frac{6\beta\Gamma}{\pi} \ln \left( 1 + e^{-\frac{\beta J}{\Gamma}} \right)
\]

and therefore we see that \( C \) increases monotonically from \( T = 0 \) \( (C = 0) \) to temperatures of order \( J \) and then decreases monotonically to zero again. We see that the function \( C \) in the low temperature regime increases monotonically with the temperature as expected by Zamolodchikov’s theorem and the cut-off temperature, as explained in (2.24), is \( J \). This result can also be obtained from the free energy of a classical one-dimensional Ising model. It is well known that this trivial classical model has an infrared unstable fixed point at \( T = 0 \). We call this the classical zero temperature fixed point. In this classical problem it is obvious that the free energy is not a monotonic function of the temperature even though the correlation length.
is indeed monotonic. Monotonicity is only present near the fixed point, i.e., near \( T = 0 \).

Anyway, even away from the critical point the monotonicity properties of the \( C \) function are consistent with Zamolodchikov’s theorem if we are in the region where the temperature is low compared with the temperature cut-off of the problem, exactly as considered in (2.24). Again the behaviour of the \( C \) function is non-analytic in \( g \), this is a clear sign of problems in one dimension.

5 The Non-Linear Sigma Model

The non-linear sigma model is described by the following Lagrangian density

\[
L = \frac{1}{2g} (\partial_\mu \vec{n}(\vec{x}, t))^2
\]

with \( \mu = 1, ..., d + 1 \) and the constraint

\[
(\vec{n})^2 = 1
\]

where \( \vec{n} \) is a vector with \( N \) components and \( g \) is the coupling constant.

It can be shown that the Heisenberg antiferromagnet in the continuum limit can be described by the non-linear sigma model (see [16] and references therein). This model has been studied in the context of the High-Temperature materials with the use the renormalization group method [17]. Here we will use the expansion \( 1/N \) in order to study its properties and we will see that our results are equivalent to those found in [17] and [18].

The thermodynamical partition function is written as

\[
Z = \int D\vec{n}(\vec{x}, \tau) \delta(\vec{n}^2 - 1) e^{-\frac{S_0[\vec{n}]}{g}}
\]

where \( \tau \) is the imaginary time and

\[
S_0 = \int_0^\beta d\tau \int_{-\infty}^{+\infty} dx^d L(\vec{n}(\vec{x}, \tau))
\]

is the euclidean action with \( \beta = 1/T \) is the inverse of the temperature.

Observe that in the low temperature limit we can extend the integral (5.4) over the whole imaginary time axis and we find a non-linear sigma model in \( d + 1 \) dimensions with the coupling constant \( g \). In the high temperature limit the upper limit in the integral in (5.4) is small and we can rewrite the action as the action for the
classical non-linear sigma model in \( d \) dimensions with a coupling constant given by \( g T \) \cite{17}.

We now decompose the field \( \vec{n} \) in its components as

\[
\vec{n} = (\sigma, \vec{\pi}) \tag{5.5}
\]

where \( \vec{\pi} \) is a vector with \( N-1 \) components. Substituting (5.5) in (5.3) changing variables in the functional integral as \( \vec{\pi} = \sqrt{g} \vec{\phi} \) and tracing over \( \vec{\phi} \) we get

\[
Z = \int D\sigma(\vec{x}, \tau) \int D\lambda(\vec{x}, \tau) e^{-\frac{S_{\text{eff}}[\vec{\sigma}, \lambda]}{g}} \tag{5.6}
\]

where we have introduced a representation for the delta function and

\[
S = \int_0^\beta d\tau \int_{-\infty}^{+\infty} d^d x \frac{1}{2} \left[ (\partial_\mu \sigma)^2 + \lambda \left( \sigma^2 - 1 \right) \right] + \frac{N}{2} tr \ln \left( -\partial_\mu^2 + \lambda \right) \tag{5.7}
\]

where \( tr \) is the trace.

Now rescale the parameters as

\[
\tilde{g} = (N-1) g \tag{5.8}
\]

\[
\sigma = \sqrt{(N-1)g} \tilde{\sigma} \tag{5.9}
\]

and rewrite (5.6) as

\[
Z = \int D\tilde{\sigma} \int D\lambda e^{-(N-1)S_{\text{eff}}[\tilde{\sigma}, \lambda]} \tag{5.10}
\]

where the effective action is given by

\[
S_{\text{eff}} = \int_0^\beta d\tau \int_{-\infty}^{+\infty} d^d x \frac{1}{2} \left[ (\partial_\mu \tilde{\sigma})^2 + \lambda \tilde{\sigma}^2 - \frac{\lambda}{\tilde{g}} \right] + tr \ln \left( -\partial_\mu^2 + \lambda \right) \tag{5.11}
\]

In the limit as \( N \to \infty \) the solution of (5.10) can be obtained by the saddle point equations

\[
\left( -\partial_\mu^2 + \lambda \right) \tilde{\sigma} = 0 \tag{5.12}
\]

\[
\tilde{\sigma}^2 = \frac{1}{\tilde{g}} - G_\lambda \left( \vec{x}, \tau; \vec{x}, \tau \right) \tag{5.13}
\]

and

\[
\left( -\partial_\mu^2 + \lambda \right) G_\lambda \left( \vec{x}, \tau; \vec{x'}, \tau' \right) = \delta^d \left( \vec{x} - \vec{x'} \right) \delta \left( \tau - \tau' \right) \tag{5.14}
\]

Assuming that \( \tilde{\sigma} \) and \( \lambda \) are constants we can replace the equations above by

\[
\lambda \sigma = 0 \tag{5.15}
\]
and
\[
\sigma^2 = 1 - \frac{\tilde{g}}{\beta} \sum_n \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + \omega_n^2 + \lambda} \tag{5.16}
\]
where we have used the periodic boundary conditions for the fields in the imaginary time direction
\[
\phi(\vec{x}, \tau) = \phi(\vec{x}, \tau + \beta) \tag{5.17}
\]
and therefore \(\omega_n = \frac{2\pi n}{\beta}\) with \(n = 0, \pm 1, \pm 2, \ldots\).

We can proceed further and perform the sum in (5.16)
\[
\sigma^2 = 1 - \frac{\tilde{g} S_d}{2} \int_0^\infty dk \frac{k^{d-1} \coth \left( \frac{\beta}{2} \sqrt{k^2 + \lambda} \right)}{\sqrt{k^2 + \lambda}} \tag{5.18}
\]
where \(S_d^{-1} = 2^{d-1} \pi^{\frac{d}{2}} \Gamma \left( \frac{d}{2} \right) \).

From (5.15) is easy to see that we have two phases in the problem, one ordered phase \((\lambda = 0 \text{ and } \sigma \neq 0)\) and one disordered phase \((\lambda \neq 0 \text{ and } \sigma = 0)\). In the ordered phase we have \(N - 1\) Goldstone bosons and one massive boson and in the disordered phase the full symmetry of the system is recovered and the modes are massive.

Notice that expression (5.18) contains a ultra-violet divergence \((k \to \infty)\) at zero temperature in one and two dimensions which means that we have to renormalize the theory. In order to do so define the renormalization group transformations
\[
\sigma = \sqrt{Z_0} M \tag{5.19}
\]
\[
\lambda = \frac{Z_1}{Z_0} \Lambda^2 m^2 \tag{5.20}
\]
\[
\tilde{g} = Z_1 \Lambda^{1-d} \alpha \tag{5.21}
\]
\[
T = \Lambda t \tag{5.22}
\]
\[
U = \tilde{g} T = Z_1 \Lambda^{2-d} u \tag{5.23}
\]
where
\[
u = \alpha t \tag{5.24}
\]
Here \(M, m, \alpha, t\) and \(u\) are dimensionless quantities and \(\Lambda\) is a scale with dimensions of inverse of length \((\Lambda = 1/a, \text{ where } a \text{ is the lattice spacing})\). Observe that the temperature \(T\) is not renormalized since it fixes the length of the manifold (the cylinder). In order to achieve the renormalization in two dimensions we have to introduce a new variable \(U\), which is the high temperature coupling constant as explained before.
It is easy to see that the theory is renormalized in any number of dimensions if we choose

\[ Z_0 = Z_1 \]  

and

\[
\frac{1}{Z_0} = 1 + \frac{\alpha K_d}{2} \int_0^\infty dx x^{d-1} \coth \left( \frac{\alpha}{2u} \sqrt{x^2 + 1} \right)
\]  

(5.26)

From (5.26) is possible to calculate the $\tilde{\beta}$-functions of the model. Using (5.21), (5.22) and (5.23) we have

\[
\tilde{\beta}_\alpha = \Lambda \left( \frac{\partial \alpha}{\partial \Lambda} \right)_{g,T} = (d - 1)\alpha - \alpha \Omega(\alpha, u)
\]  

(5.27)

\[
\tilde{\beta}_u = \Lambda \left( \frac{\partial u}{\partial \Lambda} \right)_{g,T} = (d - 2)u - u \Omega(\alpha, u)
\]  

(5.28)

and

\[
\tilde{\beta}_t = \Lambda \left( \frac{\partial t}{\partial \Lambda} \right)_{g,T} = -t
\]  

(5.29)

where

\[
\Omega(\alpha, u) = \Lambda \left( \frac{\partial \ln Z_0}{\partial \Lambda} \right)_{g,T} = \tilde{\beta}_\alpha \left( \frac{\partial \ln Z_0}{\partial \alpha} \right) + \tilde{\beta}_u \left( \frac{\partial \ln Z_0}{\partial u} \right)
\]  

(5.30)

We can now solve the system formed by (5.27) and (5.28) using expression (5.26). Although we can do it in general we will choose $d = 2$ since all integrals can be done explicitly. For $d = 2$ we find (the zero temperature integrals are done as in [19] in the context of $\epsilon$ expansion)

\[
\tilde{\beta}_\alpha = \alpha - \frac{\alpha^2}{4\pi} \coth \left( \frac{\alpha}{2u} \right)
\]  

(5.31)

and

\[
\tilde{\beta}_u = -\frac{\alpha u}{4\pi} \coth \left( \frac{\alpha}{2u} \right)
\]  

(5.32)

which agrees with the one loop result found in [17]. Furthermore, we can solve (5.18) explicitly and one finds (use definition (5.24))

\[
m = 2t \arcsinh \left( e^{2\pi(M^2\alpha - 1)} \frac{1}{\alpha^t} \sinh \left( \frac{1}{2t} \right) \right)
\]  

(5.33)

which, together with

\[
M m = 0
\]  

(5.34)
completes the solution of the saddle point equations.

Notice that the fixed points of the model (where the the \( \tilde{\beta} \)-functions vanish) are
\[ u = 0, \alpha = 0 \] and \( u = 0, \alpha = 4\pi \) (or simply, \( (0,0) \) and \( (0,4\pi) \)). \( (0,0) \) is a stable fixed point and \( (0,4\pi) \) is unstable.

Observe that at zero temperature in the ordered phase \( (m = 0, M \neq 0) \) we get
\[
M^2 = 1 - \frac{\alpha}{\alpha_c} \tag{5.35}
\]
where
\[
\alpha_c = 4\pi \tag{5.36}
\]
is the critical dimensionless coupling constant which was expected from the above analysis of the \( \tilde{\beta} \)-functions. Observe that the expression (5.35) is only valid as \( \alpha \leq \alpha_c \). In the disordered phase \( (m \neq 0, M = 0, \alpha \geq \alpha_c) \) we get
\[
m = 1 - \frac{\alpha_c}{\alpha} \tag{5.37}
\]

We can also solve exactly the equation (2.9) for the running coupling constant
\[
\alpha(t) = \alpha t_0 \left( t + \frac{\alpha t_0 t}{2\pi} \ln \left( \frac{\sinh \left( \frac{1}{2} t \right)}{\sinh \left( \frac{1}{2} t_0 \right)} \right) \right)^{-1} \tag{5.38}
\]

We can easily see that the dimensionless correlation length as defined by (2.7) is \( m^{-1} \) (use (5.33) in (2.9)). Another possible way to achieve this result is to expand the action (5.11) around the saddle point solutions and recall the definition of the free energy
\[
F = -\frac{1}{\beta} \ln Z \tag{5.39}
\]
we get that the finite term (which does not need to be regularized) in the leading order term in \( 1/N \) is given by
\[
\frac{F - E_0}{(N - 1)V} = -\frac{S_d 2^{d-1} \Gamma \left( \frac{d}{2} \right)}{\pi^{1/2} \beta^{d+1}} \left( \beta \lambda^{1/2} \right)^{d+1} \sum_{n=1}^{\infty} \frac{1}{n^{d+1}} K_{d+1} \left( n \beta \lambda^{1/2} \right) \tag{5.40}
\]
The free energy only depends on the mass of the \( N - 1 \) modes, which is represented by \( \lambda \). Notice the resemblance with expression (3.6), the main difference is that here the correlation length depends on the temperature. Moreover, comparing (5.40) with (2.1) we find the \( C \) function,
\[
C = \frac{S_d 2^{d-1} \Gamma \left( \frac{d}{2} \right)}{n(d) \pi^{1/2}} \left( \frac{m(\alpha,t)}{t} \right)^{d+1} \sum_{n=1}^{\infty} \frac{1}{n^{d+1}} K_{d+1} \left( n \frac{m(\alpha,t)}{t} \right) \tag{5.41}
\]
20
Comparing (5.41) with (2.8) we conclude that $\gamma = m^{-1}$, as expected.

Now we want to know how $C$ changes with the temperature. From (3.8) is easy to see that
\[ \frac{dC}{d(\gamma t)} \geq 0 \quad (5.42) \]
and from (5.33) one finds
\[ \frac{\partial \gamma}{\partial \alpha} \leq 0. \quad (5.43) \]
Using our result (2.17) we arrive to the conclusion that the sign of the derivative in (2.17) equals to minus the sign of the beta function.

We can clearly distinguish two regions in the parameter space. One of them has a positive beta function and is called [17] the classical renormalized region because the correlation length diverges exponentially near $T = 0$ as it does in the classical version of the theory with the bare coupling constant replaced by the renormalized one (see (5.33)); the second region has a negative beta function and is called quantum region. The border between these regions is given by the zeros of $\tilde{\beta}_{\alpha}$, namely,
\[ \alpha = \alpha_c \tanh \left( \frac{1}{2t} \right) \quad (5.44) \]
The meaning of this border line is clear, we see that at low temperatures the correlation length at this line equals the thermal wavelength (which in our units is $1/T$). Therefore, in the classical region the correlation length is always greater than the thermal wavelength which gives the classical character of the region, in the quantum region the correlation length is smaller than the thermal wavelength, as expected.

We finally conclude that if we start the flow from the classical region our $C$ function will decrease with increase the temperature (since $\tilde{\beta}_{\alpha}$ is positive) as far as it finds the border line (5.44) and then, in the quantum region, it will start to increase monotonically with the increase of the temperature (see Fig.3).

We see that the non monotonicity of the $C$ is again related with the crossover between quantum and classical behaviour. In one dimension the crossover is simply dictated by the temperature in a trivial way, that is, any lattice leads to a bandwidth in the spectrum of excitations which is a natural temperature cut-off. This behaviour is implicit in the original assumptions of the Zamolodchikov's theorem. In higher dimensions the existence of phase transitions (even in the presence of a infinite bandwidth as in the case of the Non-linear Sigma model) produces a region in the parameter space where the behaviour is clearly classical, that is, where the correlation length is much greater than the thermal wavelength.

Now we can understand why the Zamolodchikov's theorem is so powerful in one dimension in the continuum limit. In $d = 1$ we have only a fixed point at
(0, 0) and therefore there is no classical region in the parameter space. This is also true for systems with relativistic dispersion relation and although we can arbitrarily choose a region in the parameter space where the correlation length (independent of temperature) is smaller than the thermal wavelength we do not have a phase transition, that is, the correlation length does not blow up dynamically.

6 Conclusions

We have shown in this article that, in regions of the parameter space (coupling constants and temperature) where quantum fluctuations dominate, it is possible to define a \( C \) function (in any number of dimensions) which has the same properties as the \( C \) function defined in Zamolodchikov’s \( C \) theorem. We showed that in the regions where the system behaves classically (where the correlation length is greater than the thermal wavelength) monotonic properties are not to be expected.

We illustrate these issues in the context of several models. We showed that free massive field theories have the mononicity properties proposed in Zamolodchikov’s theorem in any number of dimensions. This result is due to the lack of a classical region in the parameter space of the theory, exactly as in one-dimensional field theories. Applying our ideas for the case of the Ising model in 1+1 dimensions we showed that the Zamolodchikov’s theorem applies in the region of low temperatures (as expected) and the fact that we have a bandwidth (or temperature cut-off) implies a crossover between classical and quantum behavior.

In the case of the Non-linear Sigma model in 2+1 dimensions we found that in regions where the correlation length is greater than the thermal wavelength (that is, close to a phase transition) the thermodynamic functions exhibit classical behavior and consequently the function \( C \) is not a monotonic function of the temperature. Nevertheless, in regions where the system is essentially quantum mechanical our function \( C \) does increase monotonically with the temperature, a result similar to the Zamolodchikov’s \( C \)-theorem. In this way we see that \( C \) can be used to describe the crossover between classical and quantum behaviour.

Since our approach is thermodynamical, it is possible to extract these behavior from experiments which measure the specific heat of the system under consideration.

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