PROFILE FOR THE IMAGINARY PART OF A BLOWUP SOLUTION FOR A COMPLEX-VALUED SEMILINEAR HEAT EQUATION

Giao Ky Duong

Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France.

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Abstract. In this paper, we consider the following complex-valued semilinear heat equation
\[ \partial_t u = \Delta u + u^p, \quad u \in \mathbb{C}, \]
in the whole space \( \mathbb{R}^n \), where \( p \in \mathbb{N} \), \( p \geq 2 \). We aim at constructing for this equation a complex solution
\[ u = u_1 + iu_2, \]
which blows up in finite time \( T \) and only at one blowup point \( a \), with the following estimates for the final profile
\[ u(x, T) \sim \left( \frac{(p - 1)^2|x - a|^2}{8p|\ln|x - a||} \right)^{-\frac{1}{p-1}}, \]
\[ u_2(x, T) \sim \frac{2p}{(p - 1)^2} \left( \frac{(p - 1)^2|x - a|^2}{8p|\ln|x - a||} \right)^{-\frac{1}{p-1}} \frac{1}{|\ln|x - a||}, \]
as \( x \to a \).

1. Introduction

In this work, we are interested in the following complex-valued semilinear heat equation
\[ \left\{ \begin{array}{ll} \partial_t u = \Delta u + F(u), & t \in [0, T), \\ u(0) = u_0 \in L^\infty, \end{array} \right. \tag{1.1} \]
where \( F(u) = u^p \) and \( u(t) : \mathbb{R}^n \to \mathbb{C} \), \( L^\infty := L^\infty(\mathbb{R}^n, \mathbb{C}) \), \( p > 1 \). Though our results hold only when \( p \in \mathbb{N} \) (see Theorem 1.1 below), we keep \( p \in \mathbb{R} \) in the introduction, in order to broaden the discussion.

In particular, when \( p = 2 \), model (1.1) evidently becomes
\[ \left\{ \begin{array}{ll} \partial_t u = \Delta u + u^2, & t \in [0, T), \\ u(0) = u_0 \in L^\infty. \end{array} \right. \tag{1.2} \]

We remark that equation (1.2) is rigidly related to the viscous Constantin-Lax-Majda equation with a viscosity term, which is a one dimensional model for the vorticity equation in fluids. The readers can see more in some of the typical works: Constantin, Lax, Majda [2], Guo, Ninomiya and Yanagida in [7], Okamoto, Sakajo and Wunsch [20], Sakajo in [21] and [22], Schochet [23]. The local Cauchy problem for model (1.1) can be solved (locally in time) in \( L^\infty(\mathbb{R}^n, \mathbb{C}) \) if \( p \) is integer, by using a fixed-point argument. However, when \( p \) is not integer, the local Cauchy problem has not been solved yet, up to our knowledge. This probably

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comes from the discontinuity of \( F(u) \) on \( \{ u \in \mathbb{R}^n \} \). In addition to that, let us remark that equation (1.1) has the following family of space independent solutions:

\[
    u_k(t) = \kappa e^{\frac{2\pi ik}{p-1}} (T-t)^{-\frac{1}{p-1}}, \quad \text{for any } k \in \mathbb{Z},
\]

where \( \kappa = (p-1)^{-\frac{1}{p-1}} \).

If \( p \in \mathbb{Q} \), this makes a finite number of solutions.

If \( p \notin \mathbb{Q} \), then the set

\[
    \left\{ u_k(t) \frac{(T-t)^{\frac{1}{p-1}}}{\kappa} \mid k \in \mathbb{Z} \right\},
\]

is countable and dense in the unit circle of \( \mathbb{C} \).

This latter case \( (p \notin \mathbb{Q}) \), is somehow intermediate between the case \( (p \in \mathbb{Q}) \) and the case of the twin PDE

\[
    \partial_t u = \Delta u + |u|^{p-1}u,
\]

which admits the following family of space independent solutions

\[
    u_\theta(t) = \kappa e^{i\theta} (T-t)^{-\frac{1}{p-1}},
\]

for any \( \theta \in \mathbb{R} \), which turns to be infinite and covers all the unit circle, after rescaling as in (1.4). In fact, equation (1.5) is certainly much easier than equation (1.1). As a matter of fact, it reduces to the scalar case thanks to a modulation technique, as Filippas and Merle did in [5].

Since the Cauchy problem for equation (1.1) is already hard when \( p \notin \mathbb{N} \), and given that we are more interested in the asymptotic blowup behavior, rather than the well-posedness issue, we will focus in our paper on the case \( p \in \mathbb{N} \). In this case, from the Cauchy theory, the solution of equation (1.1) either exists globally or blows up in finite time. Let us recall that the solution \( u(t) = u_1(t) + iu_2(t) \) blows up in finite time \( T < +\infty \) if and only if it exists for all \( t \in [0, T) \) and

\[
    \limsup_{t \to T} \| u_1(t) \|_{L^\infty} + \| u_2(t) \|_{L^\infty} \to +\infty.
\]

If \( u \) blows up in finite time \( T \), a point \( a \in \mathbb{R}^n \) is called a blowup point if and only if there exists a sequence \( \{(a_j, t_j)\} \to (a, T) \) as \( j \to +\infty \) such that

\[
    |u_1(a_j, t_j)| + |u_2(a_j, t_j)| \to +\infty \quad \text{as } j \to +\infty.
\]

The blowup phenomena occur for evolution equations in general, and in semilinear heat equations in particular. Accordingly, an interesting question is to construct for those equations a solution which blows up in finite time and to describe its blowup behavior. These questions are being studied by many authors in the world. Let us recall some blowup results connected to our equation:

(i) **The real case:** Bricmont and Kupiainen [1] constructed a real positive solution to (1.1) for all \( p > 1 \), which blows up in finite time \( T \), only at the origin and they also gave the profile of the solution such that

\[
    \left\| (T-t)^{\frac{1}{p-1}} u(x, t) - f_0 \left( \frac{x}{\sqrt{(T-t) \ln(T-t)}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{1 + \sqrt{\ln(T-t)}},
\]

where the profile \( f_0 \) is defined as follows

\[
    f_0(z) = \left( p - 1 + \frac{(p-1)^2|z|^2}{4p} \right)^{-\frac{1}{p-1}}.
\]

In addition to that, with a different method, Herrero and Velázquez in [12] obtained the same result. Later, in [15] Merle and Zaag simplified the proof of [1] and proposed the following two-step method (see also the note [14]):

- Reduction of the infinite dimensional problem to a finite dimensional one.
- Solution of the finite dimensional problem thanks to a topological argument based on Index theory.
We would like to mention that this method has been successful in various situations such as the work of Tayachi and Zaag [24], and also the works of Ghoul, Nguyen and Zaag in [9], [10], and [8]. In those papers, the considered equations were scale invariant; this property was believed to be essential for the construction. Fortunately, with the work of Ebde and Zaag [4] for the following equation

$$\partial_t u = \Delta u + |u|^{p-1}u + f(u, \nabla u),$$

where

$$|f(u, \nabla u)| \leq C(1 + |u|^q + |\nabla u|^{q'})$$

with $q < p, q' < \frac{2p}{p+1},$

that belief was proved to be wrong.

Going in the same direction as [4], Nguyen and Zaag in [18], have achieved the construction with a stronger perturbation

$$\partial_t u = \Delta u + |u|^{p-1}u + \frac{\mu|u|^{p-1}u}{\ln^{\alpha}(2 + u^2)},$$

where $\mu \in \mathbb{R}, a > 0$. Though the results of [4] and [18] show that the invariance under dilations of the equation is not necessary in the construction method, we might think that the construction of [4] and [18] works because the authors adopt a perturbative method around the pure power case $F(u) = |u|^{p-1}u$. If this is true with [4], it is not the case for [18].

Going in the same direction as [4], Nguyen and Zaag in [18], have arrived at the following equation

$$\partial_t u = \Delta u + |u|^{p-1}u \ln^a(2 + u^2),$$

for some $a \in \mathbb{R}$ and $p > 1$, where we have no invariance under dilation, not even for the main term on the nonlinearity. They were successful in constructing a stable blowup solution for that equation. Following the above-mentioned discussion, that work has to be considered as a breakthrough.

Let us mention that a classification of the blowup behavior of (1.2) was made available by many authors such as Herrero and Velázquez in [12] and Velázquez in [25], [26], [27] (see also Zaag in [30] for some refinement). More precisely and just to stay in one space dimension for simplicity, it is proven in [12] that if $u$ a real solution of (1.1), which blows up in finite time $T$ and $a$ is a given blowup point, then:

A. Either

$$\sup_{|x-a| \leq K\sqrt{(T-t)|\ln(T-t)|}} \left| (T-t)^{\frac{1}{m+1}} u(x,t) - f_0 \left( \frac{x-a}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right| \to 0 \text{ as } t \to T,$$

for any $K > 0$ where $f_0(z)$ is defined in (1.6).

B. Or, there exist $m \geq 2, m \in \mathbb{N}$ and $C_m > 0$ such that

$$\sup_{|x-a| \leq K(T-t)^{\frac{1}{m+1}}} \left| (T-t)^{\frac{1}{m+1}} u(x,t) - f_m \left( \frac{C_m(x-a)}{T-t} \right) \right| \to 0 \text{ as } t \to T,$$

for any $K > 0$, where $f_m(z) = (p-1 + |z|^{2m})^{-\frac{1}{p-1}}$.

(ii) The complex case: The blowup question for the complex-valued parabolic equations has been studied intensively by many authors, in particular for the Complex Ginzburg Landau (CGL) equation

$$\partial_t u = \left(1 + i\beta\right) \Delta u + \left(1 + i\delta\right)|u|^{p-1}u + \gamma u.$$  \hspace{1cm} (1.7)

This is the case of an earlier work of Zaag in [28] for equation (1.7) when $\beta = 0$ and $\delta$ small enough. Later, Masmoudi and Zaag in [16] generalized the result of [28] and constructed a blowup solution for (1.7) with $p - \delta^2 - \beta\delta - \beta\delta p > 0$ such that the solution satisfies the following

$$\left\| (T-t)^{\frac{1}{m+1}} \ln(T-t)|^{-i\mu} u(x,t) - \left( p - 1 + \frac{b_{sub}|x|^2}{(T-t)|\ln(T-t)|} \right)^{\frac{1+i\mu}{p-1}} \right\|_{L^\infty} \leq \frac{C}{1 + \sqrt{|\ln(T-t)|}},$$

where

$$b_{sub} = \frac{(p-1)^2}{4(p-\delta^2-\beta\delta-\beta\delta p)} > 0.$$
Then, Nouaili and Zaag in [19] has constructed for (1.7) (in case the critical where \( \beta = 0 \) and \( p = \delta^2 \)) a blowup solution satisfying

\[
\left\| (T-t)^{\frac{1}{p+1}}(1+\ln(T-t))^{-i\alpha}u(x,t) - \frac{b_{cri}[x]^2}{(T-t)|\ln(T-t)|^{\alpha}} \right\|_{L^\infty} \leq \frac{C}{1+|\ln(T-t)|^{\frac{\alpha}{p}}},
\]

with

\[
b_{cri} = \frac{(p-1)^2}{8\sqrt{p(p+1)}} \mu = \frac{\delta}{8\beta}.
\]

As for equation (1.2), there are many works done in dimension one, such as the work of Guo, Ninomiya, Shimojo and Yanagida, who proved in [7] the following results (see Theorems 1.2, 1.3 and 1.5 in this work):

(i) (A Fourier-based blowup crieterion). We assume that the Fourier transform of initial data of (1.2) is real and positive, then the solution blows up in finite time.

(ii) (A simultaneous blowup criterion in dimension one) If the initial data \( u^0 = u_1^0 + u_2^0 \), satisfies

\[
u_1^0 \text{ is even, } u_2^0 \text{ is odd with } u_2^0 > 0 \text{ for } x > 0.
\]

Then, the fact that the blowup set is compact implies that \( u_1^0, u_2^0 \) blow up simultaneously.

(iii) Assume that \( u_0 = u_0^0 + iu_2^0 \) satisfy

\[
u_1^0, u_2^0 \in C^1(\mathbb{R}^n), 0 \leq u_0^0 \leq M, u_1^0 \neq M, 0 < u_2^0 \leq L,
\]

\[
\lim_{|x| \to +\infty} u_1^0(x) = M \text{ and } \lim_{|x| \to +\infty} u_2^0 = 0,
\]

for some constant \( L, M \). Then, the solution \( u = u_1 + iu_2 \) of (1.2), with initial data \( u^0 \), blows up at time \( T(M) \), with \( u_2(t) \neq 0 \). Moreover, the real part \( u_1(t) \) blows up only at space infinity and \( u_2(t) \) remains bounded.

Still for equation (1.2), Nouaili and Zaag constructed in [17] a complex solution \( u = u_1 + iu_2 \), which blows up in finite time \( T \) only at the origin. Moreover, the solution satisfies the following asymptotic behavior

\[
\left\| (T-t)\tilde{u}(.,t) - f \left( \frac{1}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty} \to 0 \text{ as } t \to T,
\]

where \( f(z) = \frac{1}{8+|z|^2} \) and the imaginary part satisfies the following estimate for all \( K > 0 \)

\[
\sup_{|x| \leq K\sqrt{T-t}} \left| (T-t)\tilde{v}(x,t) - \frac{1}{\ln(T-t)} \sum_{j=1}^n C_j \left( \frac{x_j^2}{T-t} - 2 \right) \right| \leq \frac{C(K)}{|\ln(T-t)|^\alpha}, \tag{1.8}
\]

for some \( (C_j)_1 \neq (0, ..., 0) \) and \( 2 < \alpha < 2 + \eta, \eta \) small enough. Note that the real and the imaginary parts blow up simultaneously at the origin. Note also that [17] leaves unanswered the question of the derivation of the profile of the imaginary part, and this is precisely our aim in this paper, not only for equation (1.2), but also for equation (1.1) with \( p \in \mathbb{N}, p \geq 2 \).

Before stating our result (see Theorem 1.1 below), we would like to mention some classification results by Harada for blowup solutions of (1.2). As a matter of fact, in [11], he classified all blowup solutions of (1.2) in dimension one, under some reasonable assumption (see (1.9), (1.10)), as follows (see Theorems 1.4, 1.5 and 1.6 in that work):

Consider \( u = u_1 + iu_2 \) a blowup solution of (1.2) in one dimension space with blowup time \( T \) and blowup point \( \xi \) which satisfies

\[
\sup_{0<t<T} (T-t)||u(t)||_{L^\infty} < +\infty. \tag{1.9}
\]

Assume in addition that

\[
\lim_{s \to +\infty} ||w_2(s)||_{L^2(\mathbb{R})} = 0, w_2 \neq 0, \tag{1.10}
\]
where $\rho$ is defined as follows

$$\rho(y) = \frac{e^{-\frac{y^2}{4\pi}}}{\sqrt{4\pi}}, \quad (1.11)$$

and $w_2$ is defined by the following change of variables (also called similarity variables):

$$w_1(y,s) = (T-t)u_1(\xi + e^{-\frac{y^2}{2}}y, t) \quad \text{and} \quad w_2(y,s) = (T-t)u_2(\xi + e^{-\frac{y^2}{2}}y, t), \quad \text{where} \quad t = T - e^{-s}.$$  

Then, one of the following cases occurs

$$(C_1) \begin{cases} 
 u_1 &=& 1 - \frac{m}{2}h_2 + O(\ln s) \quad \text{in} \quad L^2(\mathbb{R}), \\
 w_2 &=& c_2s^{-m}e^{-\frac{(m-2)s}{2}}h_m + O\left(s^{-(m+1)}e^{-\frac{(m-2)s}{2}}\ln s\right) \quad \text{in} \quad L^2(\mathbb{R}), \quad m \geq 2.
\end{cases}$$

$$(C_2) \begin{cases} 
 u &=& 1 - c_1e^{-(k-1)s}h_{2k} + O(e^{-\frac{2(k-1)s}{m}}) \quad \text{in} \quad L^2(\mathbb{R}), \\
 v &=& c_2e^{-\frac{(m-2)h_2}{2}}h_m + O\left(e^{-\frac{(m-1)(1+s)}{2}}\right) \quad \text{in} \quad L^2(\mathbb{R}), \quad k \geq 2, m \geq 2k.
\end{cases}$$

where $c_0 = \frac{k}{2}, c_1 > 0, c_2 \neq 0$ and $\rho(y)$ is defined in (1.11) and $h_j(y)$ is a rescaled version of the Hermite polynomial of order $m^{th}$ defined as follows:

$$h_m(y) = \sum_{j=0}^{[\frac{m}{2}]} (-1)^j m! y^{m-2j} \quad j!(m-2j)! \quad (1.12)$$

Besides that, Harada has also given a profile to the solutions in similarity variables:

There exist $\kappa, \sigma, c > 0$ such that

$$(C_1) \Rightarrow \left| u - \frac{1}{1 + c_0s^{-1}h_2} \right| + s^m \frac{e^{\frac{(m-2)s}{2}}}{e^{\frac{(m-2)s}{2}}} v - \frac{c_2s^{-\frac{m}{2}}h_m}{(1 + c_0s^{-1}h_2)^2} < cs^{-\kappa}, \quad (1.13)$$

for $|y| \leq s^{(1+\sigma)}$.

$$(C_2) \Rightarrow \left| u - \frac{1}{1 + c_1e^{-(k-1)s}h_{2k}} \right| + e^{\frac{(m-2k)s}{2}} v - \frac{c_2e^{-\frac{(k-1)h_{2k}}{2}}h_m}{(1 + c_1e^{-(k-1)s}h_{2k})^2} < , \quad (1.14)$$

for $|y| \leq e^{\frac{(k-1)s}{2k}}$.

Furthermore, he also gave the final blowup profiles

The blowup profile of $u = u_1 + iv_2$ is given by

$$(C_1) \Rightarrow \begin{cases} u_1(x,T) = \frac{2}{c_0} \left( \frac{\ln |x|}{x} \right) (1 + o(1)), \\
u_2(x,T) = \frac{c_0^{2m-2}(c_0)^2}{2^{m-2}(c_0)^2} \left( \frac{x^{m-4}}{\ln |x|} \right) (1 + o(1)), \\
u(x,T) = \frac{1 + ic_0}{(c_1 - ic_2)} x^{-2k} (1 + o(1)) \quad \text{if} \quad m = 2k, \\
u_1(x,T) = (c_1)^{-1} x^{-2k} (1 + o(1)) \quad \text{and} \quad u_2(x,T) = \frac{c_0}{(c_1)^2} x^{m-4k} (1 + o(1)), \quad \text{if} \quad m > 2k.
\end{cases}$$

Then, from the work of Nouaili and Zaag in [17] and Harada in [11] for equation (1.2), we derive that the imaginary part $u_2$ also blows up under some conditions, however, none of them was able to give a global profile (i.e. valid uniformly on $\mathbb{R}^n$, and not just on an expanding ball as in (1.13) and (1.14)) for the imaginary part. For that reason, our main motivation in this work is to give a sharp description for the profile of the imaginary part. Our work is considered as an improvement of Nouaili and Zaag in [17] in dimension $n$, which is valid not only for $p = 2$, but also for any $p \geq 3, p \in \mathbb{N}$. In particular, this is the
first time we give the profile for the imaginary part when the solution blows up. More precisely, we have the following theorem:

**Theorem 1.1 (Existence of a blowup solution for (1.1) and a sharp description of its profile).** For each $p \geq 2$, $p \in \mathbb{N}$ and $p_1 \in (0, 1)$, there exists $T_1(p, p_1) > 0$ such that for all $T \leq T_1$, there exist initial data $u_0 = u_1^0 + iu_2^0$, such that equation (1.1) has a unique solution $u(x, t)$ for all $(x, t) \in \mathbb{R}^n \times [0, T)$ satisfying the following:

i) The solution $u$ blows up in finite time $T$ only at the origin. Moreover, it satisfies the following estimates

$$
\left\| (T-t)^{1-\gamma} u(x, t) - f_0 \left( \frac{x}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|\ln(T-t)|^{\frac{1}{2}}},
$$

(1.15)

and

$$
\left\| (T-t)^{-1+\gamma} \ln(T-t)|u_2(x, t) - g_0 \left( \frac{x}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|\ln(T-t)|^{\frac{1}{2}}},
$$

(1.16)

where $f_0$ is defined in (1.6) and $g_0(z)$ is defined as follows

$$
g_0(z) = \frac{|z|^2}{(p-1)\sqrt{|z|^2}}. \quad \text{(1.17)}
$$

ii) There exists a complex function $u^*(x) \in C^2(\mathbb{R}^n \setminus \{0\})$ such that $u(t) \to u^* = u_1^* + iu_2^*$ as $t \to T$ uniformly on compact sets of $\mathbb{R}^n \setminus \{0\}$ and we have the following asymptotic expansions:

$$
u^*(x) \sim \left( \frac{(p-1)^2 |x|^2}{8p|\ln|x||} \right)^{-\frac{1}{p-1}}, \quad \text{as } x \to 0.
$$

(1.18)

and

$$
u_2^*(x) \sim \frac{2p}{(p-1)^2} \left( \frac{(p-1)^2 |x|^2}{8p|\ln|x||} \right)^{-\frac{1}{p-1}} \frac{1}{|\ln|x||}, \quad \text{as } x \to 0.
$$

(1.19)

**Remark 1.2.** The initial data $u_0$ is given exactly as follows

$$
u_0 = u_1^0 + iu_2^0,
$$

where

$$
u_1^0 = T^{-\frac{1}{p-1}} \left( (p-1) + \frac{(p-1)^2 |x|^2}{4p|\ln|T||} \right)^{-\frac{1}{p-1}} + \frac{\kappa}{2p|\ln|T||} T + \frac{A}{|\ln|T||^2} \left( d_{1,0} + d_{1,1} \cdot y \right) (1/2x) \left( \frac{2x}{K|\ln|T||} \right),
$$

$$
u_2^0 = T^{-\frac{1}{p-1}} \left( \frac{|x|^2}{T} + \frac{(p-1)^2 |x|^2}{4p|\ln|T||} \right)^{-\frac{1}{p-1}} - \frac{2\kappa}{(p-1)T} T + \left( \frac{A^2}{|\ln|T||^{p_1+2}} \left( d_{1,0} + d_{1,1} \cdot y \right) (1/2x) + \frac{A^2 |\ln|T||}{|\ln|T||^{p_1+2}} \left( \frac{1}{2}x y^T \cdot d_{2,2} \cdot y - Tr(d_{2,2}) \right) \right) \left( \frac{2x}{K|\ln|T||} \right),
$$

with $\kappa = (p-1)^{-\frac{1}{p-1}}$, $K$, $A$ are positive constants fixed large enough, $d^{(1)} = (d_{1,0}, d_{1,1})$, $d^{(2)} = (d_{2,0}, d_{2,1}, d_{2,2})$ are parameters we fine tune in our proof, and $\chi_0 \in C^0_0[0, +\infty], \|\chi_0\|_{L^\infty} \leq 1, \text{ supp } \chi_0 \subset [0, 2]

**Remark 1.3.** We see below in (2.2) that the equation satisfied by $u_2$ is almost ‘linear’ in $u_2$. Accordingly, we may change a little our proof to construct a solution $u_{c_0}(t) = u_{1,c_0} + iu_{2,c_0}$ with $t \in [0, T], c_0 \neq 0$, which blows up in finite time $T$ only at the origin such that (1.15) and (1.18) hold and also the following

$$
\left\| (T-t)^{1-\gamma} \ln(T-t)|u_2,c_0(x, t) - c_0g_0 \left( \frac{x}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|\ln(T-t)|^{\frac{1}{2}}},
$$

(1.20)

and

$$
u_2^*(x) \sim \frac{2pc_0}{(p-1)^2} \left( \frac{(p-1)^2 |x|^2}{8p|\ln|x||} \right)^{-\frac{1}{p-1}} \frac{1}{|\ln|x||}, \quad \text{as } x \to 0.
$$

(1.21)
Remark 1.4. We deduce from (ii) that \( u \) blows up only at 0. In particular, note both the \( u_1 \) and \( u_2 \) blow up. However, the blowup speed of \( u_2 \) is softer than \( u_1 \) because of the quantity \( \frac{1}{\|x\|^s} \).

Remark 1.5. Nouaili and Zaag constructed a blowup solution of (1.2) with a less explicit behavior for the imaginary part (see (1.8)). Here, we do better and we obtain the profile the the imaginary part in (1.16) and we also describe the asymptotics of the solution in the neighborhood of the blowup point in (1.19). In fact, this refined behavior comes from a more involved formal approach (see Section 2 below), and more parameters to be fine tuned in initial data (see Definition 3.3 where we need more parameters than in Nouaili and Zaag [17], namely \( d_x \in \mathbb{R}^{n(n+1)} \)). Note also that our profile estimates in (1.15) and (1.16) are better than the estimates (1.13) and (1.14) by Harada \((m = 2)\), in the sense that we have a uniform estimate for whole space \( \mathbb{R}^n \), and not just for all \( |y| \leq s^{1+\sigma} \) for some \( \sigma > 0 \). Another point: our result hold in \( n \) space dimensions, unlike the work of Harada in [11], which holds only in one space dimension.

Remark 1.6. As in the case \( p = 2 \) treated by Nouaili and Zaag [17], we suspect this behavior in Theorem 1.1 to be unstable. This is due to the fact that the number of parameters in the initial data we consider below in Definition 3.3 is higher than the dimension of the blowup parameters which is \( n + 1 \) \((n \) for the blowup points and 1 for the blowup time).

Besides that, we can use the technique of Merle [13] to construct a solution which blows up at arbitrary given points. More precisely, we have the following Corollary:

Corollary 1.7 (Blowing up at \( k \) distinct points). For any given points, \( x_1, ..., x_k \), there exists a solution of\( (1.1) \) which blows up exactly at \( x_1, ..., x_k \). Moreover, the local behavior at each blowup point \( x_j \) is also given by (1.15), (1.16), (1.18), (1.19) by replacing \( x \) by \( x_j \) and \( L^\infty(\mathbb{R}^n) \) by \( L^\infty(|x - x_j| \leq \epsilon_0) \), for some \( \epsilon_0 > 0 \).

This paper is organized as follows:
- In Section 2, we adopt a formal approach to show how the profiles we have in Theorem 1.1 appear naturally.
- In Section 3, we give the rigorous proof for Theorem 1.1, assuming some technical estimates.
- In Section 4, we prove the technical estimates assumed in Section 3.

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2. Derivation of the profile (formal approach)

In this section, we aim at giving a formal approach to our problem which helps us to explain how we derive the profile of solution of (1.1) given in Theorem (1.1), as well the asymptotics of the solution.

2.1. Modeling the problem

In this part, we will give definitions and special symbols important for our work and explain how the functions \( f_0, g_0 \) arise as blowup profiles for equation (1.1) as stated in (1.15) and (1.16). Our aim in this section is to give solid (though formal) hints for the existence of a solution \( u(t) = u_1(t) + iu_2(t) \) to equation (1.1) such that

\[
\lim_{t \to T} \|u(t)\|_{L^\infty} = +\infty,
\]

and \( u \) obeys the profiles in (1.15) and (1.16), for some \( T > 0 \). By using equation (1.1), we deduce that \( u_1, u_2 \) solve:

\[
\begin{align*}
\partial_t u_1 &= \Delta u_1 + F_1(u_1, u_2), \\
\partial_t u_2 &= \Delta u_2 + F_2(u_1, u_2).
\end{align*}
\]

where

\[
\begin{align*}
F_1(u_1, u_2) &= \text{Re} \left((u_1 + cu_2)^p\right) = \sum_{j=0}^{\left\lfloor \frac{p}{2} \right\rfloor} \binom{p}{j} j! u_1^{p-2j} u_2^j, \\
F_2(u_1, u_2) &= -\text{Im} \left((u_1 + cu_2)^p\right) = \sum_{j=0}^{\left\lfloor \frac{p-1}{2} \right\rfloor} \binom{p}{j} (p-2j-1)! u_1^{p-2j-1} u_2^{j+1}.
\end{align*}
\]
with $\text{Re}[z]$ and $\text{Im}[z]$ being respectively the real and the imaginary part of $z$ and $C_p^m = \frac{p!}{m!(p-m)!}$, for all $m \leq p$. Let us introduce the similarity-variables:

$$w_1(y,s) = (T-t)^{\frac{1}{p-1}} u_1(x,t), \quad w_2(y,s) = (T-t)^{\frac{1}{p-1}} u_2(x,t), \quad y = \frac{x}{\sqrt{T-t}}, \quad s = -\ln(T-t). \quad (2.4)$$

Thanks to (2.2), we derive the system satisfied by $(w_1, w_2)$, for all $y \in \mathbb{R}^n$ and $s \geq -\ln T$ as follows:

$$\begin{aligned}
\begin{cases}
\partial_s w_1 &= \Delta w_1 - \frac{1}{2} y \cdot \nabla w_1 - \frac{m}{p-1} + F_1(w_1, w_2), \\
\partial_s w_2 &= \Delta w_2 - \frac{1}{2} y \cdot \nabla w_2 - \frac{m}{p-1} + F_2(w_1, w_2).
\end{cases}
\end{aligned} \quad (2.5)$$

Then note that studying the asymptotics of $u_1 + iw_2$ as $t \to T$ is equivalent to studying the asymptotics of $w_1 + iw_2$ in long time. We are first interested in the set of constant solutions of (2.5), denoted by

$$\mathcal{S} = \{(0,0)\} \cup \left\{ \kappa \cos \left( \frac{2k\pi}{p-1} \right), \kappa \sin \left( \frac{2k\pi}{p-1} \right) \right\} \text{ where } \kappa = (p-1)^{-\frac{1}{p-1}}, k = 0, \ldots, p-1.$$ 

With the transformation (2.4), we slightly precise our goal in (2.1) by requiring in addition that

$$(w_1, w_2) \to (\kappa, 0) \text{ as } s \to +\infty.$$ 

Introducing $w_1 = \kappa + \bar{w}_1$, our goal because to get

$$(\bar{w}_1, w_2) \to (0,0) \text{ as } s \to +\infty.$$ 

From (2.5), we deduce that $\bar{w}_1, w_2$ satisfy the following system

$$\begin{aligned}
\begin{cases}
\partial_s \bar{w}_1 &= \mathcal{L}\bar{w}_1 + \bar{B}_1(\bar{w}_1, w_2), \\
\partial_s w_2 &= \mathcal{L}w_2 + \bar{B}_2(\bar{w}_1, w_2).
\end{cases}
\end{aligned} \quad (2.6)$$

where

$$\mathcal{L} = \Delta - \frac{1}{2} y \cdot \nabla + Id, \quad (2.7)$$

$$\bar{B}_1(\bar{w}_1, w_2) = F_1(\kappa + \bar{w}_1, w_2) - \kappa p - \frac{p}{p-1} \bar{w}_1, \quad (2.8)$$

$$\bar{B}_2(\bar{w}_1, w_2) = F_2(\kappa + \bar{w}_1, w_2) - \frac{p}{p-1} w_2. \quad (2.9)$$

It is important to study the linear operator $\mathcal{L}$ and the asymptotics of $\bar{B}_1, \bar{B}_2$ as $(\bar{w}_1, w_2) \to (0,0)$ which will appear as quadratic.

- **The properties of $\mathcal{L}$:**

We observe that the operator $\mathcal{L}$ plays an important role in our analysis. It is not really difficult to find an analysis space such that $\mathcal{L}$ is self-adjoint. Indeed, $\mathcal{L}$ is self-adjoint in $L^2_p(\mathbb{R}^n)$, where $L^2_p$ is the weighted space associated with the weight $\rho$ defined by

$$\rho(y) = e^{-\frac{|y|^2}{4(4\pi)^2}} = \prod_{j=1}^{n} \rho_j(y_j), \text{ with } \rho_j(y_j) = e^{-\frac{|y_j|^2}{4(4\pi)^2}}, \quad (2.10)$$

and the spectrum set of $\mathcal{L}$

$$\text{spec}(\mathcal{L}) = \left\{ 1 - \frac{m}{2}, m \in \mathbb{N} \right\}.$$ 

Moreover, we can find eigenfunctions which correspond to each eigenvalue $1 - \frac{m}{2}, m \in \mathbb{N}$:

- The one space dimensional case: the eigenfunction corresponding to the eigenvalue $1 - \frac{m}{2}$ is $h_m$, the rescaled Hermite polynomial given in (1.12). In particular, we have the following orthogonality property:

$$\int_{\mathbb{R}} h_i h_j \rho dy = i! 2^i \delta_{i,j}, \quad \forall (i, j) \in \mathbb{N}^2.$$ 

- The higher dimensional case: $n \geq 2$, the eigenspace $\mathcal{E}_m$, corresponding to the eigenvalue $1 - \frac{m}{2}$ is defined as follows:

$$\mathcal{E}_m = \{ h_{\beta} = h_{\beta_1} \cdots h_{\beta_n}, \text{ for all } \beta \in \mathbb{N}^n, |\beta| = m, |\beta| = \beta_1 + \cdots + \beta_n \}. \quad (2.11)$$
As a matter of fact, so we can represent an arbitrary function \( r \in L^2_\rho \) as follows

\[
 r = \sum_{\beta, \beta \in \mathbb{N}^n} r_\beta h_\beta(y),
\]

where: \( r_\beta \) is the projection of \( r \) on \( h_\beta \) for any \( \beta \in \mathbb{N}^n \) which is defined as follows:

\[
 r_\beta = \mathbb{P}_\beta(r) = \int r k_\beta \rho dy, \forall \beta \in \mathbb{N}^n,
\]

with

\[
 k_\beta(y) = \frac{h_\beta}{\|h_\beta\|^2_{L^2_\rho}},
\]

- **The asymptotic of \( \tilde{B}_1(\bar{w}_1, w_2), \tilde{B}_2(\bar{w}_1, w_2) \):** The following asymptotics hold:

\[
 \tilde{B}_1(\bar{w}_1, w_2) = \frac{p}{2\kappa} \bar{w}_1^2 + O(|\bar{w}_1|^3 + |w_2|^2),
\]

\[
 \tilde{B}_2(\bar{w}_1, w_2) = \frac{p}{\kappa} \bar{w}_1 w_2 + O (|\bar{w}_1|^2 |w_2|) + O (|w_2|^3),
\]

as \((\bar{w}_1, w_2) \to (0,0)\) (see Lemma A.1 below).

### 2.2. Inner expansion

In this part, we study the asymptotics of the solution in \( L^2_\rho(\mathbb{R}^n) \). Moreover, for simplicity we suppose that \( n = 1 \), and we recall that we aim at constructing a solution of (2.6) such that \((\bar{w}_1, w_2) \to (0,0)\). Note first that the spectrum of \( L \) contains two positive eigenvalues \( 1, \frac{1}{2} \), a neutral eigenvalue 0 and all the other ones are strictly negative. So, in the representation of the solution in \( L^2_\rho \), it is reasonable to think that the part corresponding to the negative spectrum is easily controlled. Imposing a symmetry condition on the solution with respect of \( y \), it is reasonable to look for a solution \( \bar{w}_1, w_2 \) of the form:

\[
 \bar{w}_1 = \bar{w}_{1,0} h_0 + \bar{w}_{1,2} h_2, \\
 w_2 = w_{2,0} h_0 + w_{2,2} h_2.
\]

From the assumption that \((\bar{w}_1, w_2) \to (0,0)\), we see that \( \bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}, w_{2,2} \to 0 \) as \( s \to +\infty \). We see also that we can understand the asymptotics of the solution \( \bar{w}_1, w_2 \) in \( L^2_\rho \) from the study of the asymptotics of \( \bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}, w_{2,2} \). We now project equations (2.6) on \( h_0 \) and \( h_2 \). Using the asymptotics of \( \tilde{B}_1, \tilde{B}_2 \) in (2.14) and (2.15), we get the following ODEs for \( \bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}, w_{2,2} \):

\[
 \partial_s \bar{w}_{1,0} = \bar{w}_{1,0} + \frac{p}{2\kappa} (\bar{w}_{1,0}^2 + 8 \bar{w}_{1,2}^2) + O(|\bar{w}_{1,0}|^3 + |\bar{w}_{1,2}|^3) + O(|w_{2,0}|^2 + |w_{2,2}|^2),
\]

\[
 \partial_s \bar{w}_{1,2} = \frac{p}{\kappa} (\bar{w}_{1,0} \bar{w}_{1,2} + 4 \bar{w}_{1,2}^2) + O(|\bar{w}_{1,0}|^3 + |\bar{w}_{1,2}|^3) + O(|w_{2,0}|^2 + |w_{2,2}|^2),
\]

\[
 \partial_s w_{2,0} = w_{2,0} + \frac{p}{\kappa} (\bar{w}_{1,0} w_{2,0} + 8 \bar{w}_{1,2} w_{2,2}) + O((|\bar{w}_{1,0}|^2 + |\bar{w}_{1,2}|^2)(|w_{2,0}| + |w_{2,2}|)) + O(|w_{2,0}|^3 + |w_{2,2}|^3),
\]

\[
 \partial_s w_{2,2} = \frac{p}{\kappa} (\bar{w}_{1,0} w_{2,2} + \bar{w}_{1,2} w_{2,0} + 8 \bar{w}_{1,2} w_{2,2}) + O((|\bar{w}_{1,0}|^2 + |\bar{w}_{1,2}|^2)(|w_{2,0}| + |w_{2,2}|)) + O(|w_{2,0}|^3 + |w_{2,2}|^3).
\]

Assuming that

\[
 w_{1,0}, w_{2,0}, w_{2,2} \ll \bar{w}_{1,2} \text{ as } s \to +\infty,
\]

we may simplify the ODE system as follows:

- **The asymptotics of \( \bar{w}_{1,2} \):**

  We deduce from (2.17) and (2.20) that

\[
 \partial_s \bar{w}_{1,2} \sim \frac{4p}{\kappa} \bar{w}_{1,2} \text{ as } s \to +\infty,
\]

which yields

\[
 \bar{w}_{1,2} = -\frac{\kappa}{4ps} + o \left( \frac{1}{s} \right), \text{ as } s \to +\infty.
\]
Assuming further that
\[ \bar{w}_{1,0}, w_{2,0}, w_{2,2} \lesssim \frac{1}{s^2}, \]  
we see that
\[ \bar{w}_{1,2} = -\frac{\kappa}{4ps} + O \left( \frac{\ln s}{s^2} \right), \quad \text{as } s \to +\infty. \]  

- **The asymptotics of \( \bar{w}_{1,0} \):** By using (2.16), (2.20) and the asymptotics of \( \bar{w}_{1,2} \) in (2.23), we see that
\[ \bar{w}_{1,0} = O \left( \frac{1}{s^2} \right), \quad \text{as } s \to +\infty. \]  

- **The asymptotics of \( w_{2,0} \) and \( w_{2,2} \):** Besides that, we derive from (2.18), (2.19) and (2.22) that
\[ \partial_s w_{2,2} = \left( -\frac{2}{s} + O \left( \frac{\ln s}{s^2} \right) \right) w_{2,2} + o \left( \frac{1}{s^3} \right), \]  
\[ \partial_s w_{2,0} = w_{2,0} + O \left( \frac{1}{s^3} \right), \]  
which yields
\[ w_{2,2} = o \left( \frac{\ln s}{s^2} \right), \]  
\[ w_{2,0} = O \left( \frac{1}{s^3} \right), \]  
as \( s \to +\infty \). This also yields a new ODE for \( w_{2,2} \):
\[ \partial_s w_{2,2} = -\frac{2}{s} w_{2,2} + o \left( \frac{\ln s}{s^4} \right), \]  
which implies
\[ w_{2,2} = O \left( \frac{1}{s^3} \right). \]  
Using again (2.25), we derive a new ODE for \( w_{2,2} \)
\[ \partial_s w_{2,2} = -\frac{2}{s} w_{2,2} + O \left( \frac{\ln s}{s^4} \right), \]  
which yields
\[ w_{2,2} = \frac{\tilde{c}_0}{s^2} + O \left( \frac{\ln s}{s^4} \right), \quad \text{for some } \tilde{c}_0 \in \mathbb{R}^+. \]  
Noting that our finding (2.23), (2.24), (2.26) and (2.27) are consistent with our hypotheses in (2.20) and (2.22), we get the asymptotics of the solution \( w_1 \) and \( w_2 \) as follows:
\[ w_1 = \kappa - \frac{\kappa}{4ps} (y^2 - 2) + O \left( \frac{1}{s^2} \right), \]  
\[ w_2 = \frac{\tilde{c}_0}{s^2} (y^2 - 2) + O \left( \frac{\ln s}{s^4} \right), \]  
in \( L_p^2(\mathbb{R}) \) for some \( \tilde{c}_0 \) in \( \mathbb{R}^+ \). Using parabolic regularity, we note that the asymptotics (2.28), (2.29) also hold for all \( |y| \leq K \), where \( K \) is an arbitrary positive constant.
2.3. Outer expansion

As Subsection 2.2 above, we assume that \( n = 1 \). We see that asymptotics (2.28) and (2.29) can not give us a shape, since they hold uniformly on compact sets, and not in larger sets. Fortunately, we observe from (2.28) and (2.29) that the profile may be based on the following variable:

\[
    z = \frac{y}{\sqrt{s}}.
\]

This motivates us to look for solutions of the form:

\[
    w_1(y, s) = \sum_{j=0}^{\infty} \frac{R_{1,j}(z)}{s^j},
\]

\[
    w_2(y, s) = \sum_{j=1}^{\infty} \frac{R_{2,j}(z)}{s^j}.
\]

Using system (2.5) and gathering terms of order \( \frac{1}{s^j} \) for \( j = 0, ..., 2 \), we obtain

\[
    0 = -\frac{1}{2} R_{1,0}'(z) \cdot z - \frac{R_{1,0}(z)}{p - 1} + R_{1,0}'(z),
\]

(2.31)

\[
    0 = -\frac{1}{2}z R_{1,1}' - \frac{R_{1,1}}{p - 1} + p R_{1,0}^{p-1} R_{1,1} + R_{1,0}' + \frac{z R_{1,0}'}{2},
\]

(2.32)

\[
    0 = -\frac{1}{2} R_{2,1}'(z) \cdot z - \frac{R_{2,1}}{p - 1} + p R_{1,0}^{p-1} R_{2,1},
\]

(2.33)

\[
    0 = -\frac{1}{2} R_{2,2}'(z) \cdot z - \frac{R_{2,2}}{p - 1} + p R_{1,0}^{p-1} R_{2,2} + R_{2,1} + \frac{1}{2} R_{2,1}' \cdot z + p(p - 1) R_{1,0}^{p-2} R_{1,1} R_{2,1}.
\]

(2.34)

We now solve the above equations:

- **The solution** \( R_{1,0} \): It is easy to solve (2.31)

\[
    R_{1,0}(z) = (p - 1 + b z^2)^{-\frac{1}{p-1}},
\]

(2.35)

where \( b \) is an unknown constant that will be selected accordingly to our purpose.

- **The solution** \( R_{1,1} \): We rewrite (2.32) under the following form:

\[
    \frac{1}{2} z R_{1,1}'(z) = \left( \frac{(p - 1)^2 - b z^2}{(p - 1)(p - 1 + b z^2)} \right) R_{1,1} + F_{1,1}(z),
\]

where

\[
    F_{1,1}(z) = -\frac{2b}{p - 1}(p - 1 + b z^2)^{-\frac{1}{p-1}} + 4 p b^2 z^2 (p - 1 + b z^2)^{-\frac{2p-1}{p-1}} - \frac{b z^2}{p - 1} (p - 1 + b z^2)^{-\frac{1}{p-1}}.
\]

Thanks to the variation of constant method, we see that

\[
    R_{1,1} = H^{-1}(z) \left( \int \frac{2}{z} H(z) F_{1,1}(z) dz + C_1 \right),
\]

(2.36)

where

\[
    H(z) = \frac{(p - 1 + b z^2)^{\frac{1}{p-1}}}{z^2}.
\]

Besides that, we have:

\[
    \frac{2H}{z} F_{1,1} = -\frac{4b}{(p - 1) z^3} + \frac{8 p b^2}{(p - 1)^2} \left( \frac{1}{z(p - 1 + b z^2)} \right) - \frac{2b}{(p - 1) z}
\]

\[
    = -\frac{4b}{(p - 1) z^3} + \frac{1}{z} \left( -\frac{2b}{p - 1} + \frac{8 p b^2}{(p - 1)^3} \right)
\]

\[
    + (p - 1 + b z^2)^{-1} \left( -\frac{8 p b^3 z}{(p - 1)^3} \right).
\]
We can see that if the coefficient of \( \frac{1}{z} \) is non zero, then we will have a \( \ln z \) term in the solution \( R_{1,1} \) and this term would not be analytic, creating a singularity in the solution. In order to avoid this singularity, we impose that

\[
-\frac{2b}{p-1} + \frac{8pb^2}{(p-1)^3} = 0.
\]

which yields

\[
b = \frac{(p-1)^2}{4p}. \tag{2.37}
\]

Besides that, for simplicity, we assume that \( C_1 = 0 \). Using (2.36), we see that

\[
R_{1,1} = \frac{(p-1)}{2p} (p-1 + bz^2)^{-\frac{p-1}{p+1}} - \frac{p-1}{4p} z^2 \ln(p-1 + bz^2)(p-1 + bz^2)^{-\frac{p-1}{p+1}}. \tag{2.38}
\]

- **The solution** \( R_{2,1} \): It is easy to solve (2.33) as follows:

\[
R_{2,1}(z) = \frac{z^2}{(p-1 + bz^2)^{\frac{p-1}{p+1}}}. \tag{2.39}
\]

- **The solution** \( R_{2,2} \): We rewrite (2.34) as follows

\[
\frac{1}{2} z \cdot R_{2,2}^\prime(z) = \left( \frac{(p-1)^2 - bz^2}{(p-1)(p-1 + bz^2)} \right) R_{2,2}(z) + F_{2,2}(z),
\]

where

\[
\begin{align*}
F_{2,2}(z) &= R''_{2,1} + R_{2,1} + \frac{1}{2} R'_{2,1} \cdot z + p(p-1) R^p_{1,0} R_{1,1} R_{2,1} \\
&= 2(pb)^2 (p-1 + bz^2)^{-\frac{2p-1}{p+1}} + 2z^2 (p-1 + bz^2)^{-\frac{p-1}{p+1}} + \frac{(p-1)^2}{2} z^2 (p-1 + bz^2)^{-\frac{3p-2}{p+1}} \\
&+ \frac{4p(2p-1)b^2 z^4}{(p-1)^2} (p-1 + bz^2)^{-\frac{3p-2}{p+1}} - \frac{pb^4}{p-1} (p-1 + bz^2)^{-\frac{2p-1}{p+1}} \\
&- \frac{(p-1)^2}{4} z^4 \ln(p-1 + bz^2)(p-1 + bz^2)^{-\frac{3p-2}{p+1}}.
\end{align*}
\]

By using the variation of constant method, we have

\[
R_{2,2}(z) = \frac{z^2}{(p-1 + bz^2)^{\frac{p-1}{p+1}}} \left( \int \frac{2(p-1 + bz^2)^{-\frac{p}{p+1}}}{z^3} F_{2,2}(z) dz + C_2 \right), \tag{2.40}
\]

where

\[
\frac{2(p-1 + bz^2)^{-\frac{p}{p+1}}}{z^3} F_{2,2}(z) = 4 \frac{z^3}{z^3} + \left[ \frac{5 - 20pb}{(p-1)^2} \right] \frac{z}{z} + \frac{z}{p-1 + bz^2^2} \left[ \frac{20pb}{(p-1)^2} - b - \frac{2pb}{p-1} \right] \\
+ \left[ \frac{8p(2p-1)b^2}{(p-1)^2} - (p-1)p \right] \frac{z}{(p-1 + bz^2^2)^2} \\
- \frac{(p-1)^2}{2} \frac{z}{z} \ln(p-1 + bz^2)(p-1 + bz^2)^{-\frac{3p-2}{p+1}}.
\]

We observe that

\[
5 - \frac{20pb}{(p-1)^2} = 0, \text{ because } b = \frac{(p-1)^2}{4p}.
\]

So, from (2.40) and assuming that \( C_2 = 0 \), we have

\[
R_{2,2}(z) = -2(p-1 + bz^2)^{-\frac{p}{p+1}} + H_{2,2}(z), \tag{2.41}
\]

where

\[
\begin{align*}
H_{2,2}(z) &= C_{2,1}(p) z^2 (p-1 + bz^2)^{-\frac{p}{p+1}} + C_{2,3}(p) z^2 \ln(p-1 + bz^2)(p-1 + bz^2)^{-\frac{p}{p+1}} \\
&+ C_{2,3}(p) z^2 \ln(p-1 + bz^2)(p-1 + bz^2)^{-\frac{p}{p+1}}.
\end{align*}
\]
2.4. Matching asymptotics

Since the outer expansion has to match the inner expansion, this will fix several constant, giving us the following profiles for \( w_1 \) and \( w_2 \):

\[
\begin{align*}
w_1(y, s) & \sim \Phi_1(y, s), \\
w_2(y, s) & \sim \Phi_2(y, s),
\end{align*}
\]  
(2.42)

where

\[
\begin{align*}
\Phi_1(y, s) &= \left(p - 1 + \frac{(p - 1)^2 |y|^2}{4p s} \right)^{-\frac{1}{p+1}} + \frac{n\kappa}{2ps}, \\
\Phi_2(y, s) &= \frac{|y|^2}{s^2} \left(p - 1 + \frac{(p - 1)^2 |y|^2}{4p s} \right)^{-\frac{1}{p+1}} - \frac{2n\kappa}{(p - 1)s^2},
\end{align*}
\]  
(2.43, 2.44)

for all \((y, s) \in \mathbb{R}^n \times (0, +\infty)\).

3. Existence of a blowup solution in Theorem 1.1

In Section 2, we adopted a formal approach in order to justify how the profiles \( f_0, g_0 \) arise as blowup profiles for equation (1.1). In this section, we give a rigorous proof to justify the existence of a solution approaching those profiles.

3.1. Formulation of the problem

In this section, we aim at formulating our problem in order to justify the formal approach which is given in the previous section. Introducing

\[
\begin{align*}
w_1 &= \Phi_1 + q_1, \\
w_2 &= \Phi_2 + q_2,
\end{align*}
\]  
(3.1)

where \( \Phi_1, \Phi_2 \) are defined in (2.43) and (2.44) respectively, then using (2.5), we see that \((q_1, q_2)\) satisfy

\[
\begin{align*}
\partial_s \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} L + V \\ 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} V_{1,1} & V_{1,2} \\ V_{2,1} & V_{2,2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} R_1(y, s) \\ R_2(y, s) \end{pmatrix}
\end{align*}
\]  
(3.2)

where linear operator \( L \) is defined in (2.7) and:

- The potential functions \( V, V_{1,1}, V_{1,2}, V_{2,1}, V_{2,2} \) are defined as follows

\[
\begin{align*}
V(y, s) &= p \left( \Phi_1^{p-1} - \frac{1}{p-1} \right), \\
V_{1,1}(y, s) &= \sum_{j=1}^{[\frac{p}{2}]} C_p^{2j} (-1)^j (p - 2j) \Phi_1^{p-2j-1} \Phi_2^{2j}, \\
V_{1,2}(y, s) &= \sum_{j=0}^{[\frac{p}{2}]} C_p^{2j} (-1)^j (2j) \Phi_1^{p-2j-1} \Phi_2^{2j-1}, \\
V_{2,1}(y, s) &= \sum_{j=0}^{[\frac{p-1}{2}]} C_p^{2j+1} (-1)^j (p - 2j - 1) \Phi_1^{p-2j-2} \Phi_2^{2j+1}, \\
V_{2,2}(y, s) &= \sum_{j=1}^{[\frac{p+1}{2}]} C_p^{2j+1} (-1)^j (2j + 1) \Phi_1^{p-2j-1} \Phi_2^{2j}.
\end{align*}
\]  
(3.3, 3.4, 3.5, 3.6, 3.7)

- The quadratic terms \( B_1(q_1, q_2), B_2(q_1, q_2) \) are defined as follows:
Concerning equation (3.8), we recall that we already know the properties of the linear operator \( L \) (see page 8). As for potentials \( V_{j,k} \) where \( j, k \in \{1, 2\} \), they admit the following asymptotics

\[
\sum_{j,k=2} V_{j,k}(y,s) \leq \frac{C}{s}, \forall y \in \mathbb{R}^n, s \geq 1,
\]

(see Lemma A.2). Regarding the terms \( B_1, B_2, R_1, R_2 \), we see that whenever \( |q_1| + |q_2| \leq 2 \), we have

\[
|B_1(q_1, q_2)| \leq C(|q_1|^2 + |q_2|^2),
\]

\[
|B_2(q_1, q_2)| \leq C\left( \frac{|q_1|^2}{s} + |q_1q_2| + |q_2|^2 \right),
\]

\[
\|R_1(y,s)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{s},
\]

\[
\|R_2(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{s^2},
\]

(see Lemmas A.3 and A.4). In fact, the dynamics of equation (3.2) will mainly depend on the main linear operator

\[
\begin{pmatrix}
L + V & 0 \\
0 & L + V
\end{pmatrix},
\]

and the effects of the other terms will be less important. For that reason, we need to understand the dynamics of \( L + V \). Since the spectral properties of \( L \) were already introduced in Section 2.1, we will focus here on the effect of \( V \).

i) Effect of \( V \) inside the blowup region \( \{|y| \leq K\sqrt{s}\} \) with \( K > 0 \) arbitrary, we have

\[
V \to 0 \text{ in } L^2_0(\{|y| \leq K\sqrt{s}\}) \text{ as } s \to +\infty,
\]

which means that the effect of \( V \) will be negligible with respect of the effect of \( L \), except perhaps on the null mode of \( L \) (see item (ii) of Proposition 4.1 below).

ii) Effect of \( V \) outside the blowup region: for each \( \epsilon > 0 \), there exist \( K_\epsilon > 0 \) and \( s_\epsilon > 0 \) such that

\[
\sup_{s \geq K_\epsilon, s \geq s_\epsilon} \left| V(y,s) + \frac{p}{p-1} \right| \leq \epsilon.
\]
Since 1 is the biggest eigenvalue of $\mathcal{L}$, the operator $\mathcal{L} + V$ behaves as one with a fully negative spectrum outside blowup region $\{|y| \geq K \sqrt{s}\}$, which makes the control of the solution in this region easy.

Since the behavior of the potential $V$ inside and outside the blowup region is different, we will consider the dynamics of the solution for $|y| \leq 2K \sqrt{s}$ and for $|y| \geq K \sqrt{s}$ separately for some $K$ to be fixed large. For that purpose, we introduce the following cut-off function

$$\chi(y,s) = \chi_0 \left( \frac{|y|}{K \sqrt{s}} \right),$$  \hspace{1cm} (3.12)

where $\chi_0 \in C^\infty_0[0, +\infty)$, $\|\chi_0\|_{L^\infty} \leq 1$ and

$$\chi_0(x) = \begin{cases} 1 & \text{for } x \leq 1, \\ 0 & \text{for } x \geq 2, \end{cases}$$

and $K$ is a positive constant to be fixed large later. Hence, it is reason able to consider separately the solution in the blowup region $\{|y| \leq 2K \sqrt{s}\}$ and in the regular region $\{|y| \geq K \sqrt{s}\}$. More precisely, let us define the following notation for all functions $q$ in $L^\infty$ as follows

$$q = q_b + q_c \text{ with } q_b = \chi q \text{ and } q_c = (1 - \chi)q.$$  \hspace{1cm} (3.13)

Note in particular that $\text{supp}(q_b) \subset \mathbb{B}(0, 2K \sqrt{s})$ and $\text{supp}(q_c) \subset \mathbb{R}^n \setminus \mathbb{B}(0, K \sqrt{s})$. Besides that, we also expand $q_b$ in $L^2$ as follows; according to the spectrum of $\mathcal{L}$ (see Sention 2.1 above):

$$q_b(y) = q_0 + q_1 \cdot y + \frac{1}{2} y^T \cdot q_2 \cdot y - \text{Tr}(q_2) + q_-(y),$$  \hspace{1cm} (3.14)

where

$$q_0 = \int_{\mathbb{R}^n} q_b \rho(y) \, dy,$$

$$q_1 = \int_{\mathbb{R}^n} q_b \frac{y}{2} \rho(y) \, dy,$$

$$q_2 = \left( \int_{\mathbb{R}^n} q_b \left( \frac{1}{2} y_j y_k - \frac{1}{2} \delta_{j,k} \right) \rho(y) \, dy \right)_{1 \leq j, k \leq n},$$

and $\text{Tr}(q_2)$ is the trace of the matrix $q_2$. The reader should keep in mind that $q_0, q_1, q_2$ are just coordinates of $q_b$, not for $q$. Note that $q_{n_0}$ is the projection of $q_b$, as the eigenspace of $\mathcal{L}$ corresponding to the eigenvalue $\lambda = 1 - \frac{s}{2}$. Accordingly, $q_-$ is the projection of $q_b$ on the negative part of the spectrum of $\mathcal{L}$. As a consequence of (3.13) and (3.14), we see that every $q \in L^\infty(\mathbb{R}^n)$ can be decomposed into 5 components as follows:

$$q = q_b + q_c = q_0 + q_1 \cdot y + \frac{1}{2} y^T \cdot q_2 \cdot y - \text{Tr}(q_2) + q_- + q_c.$$  \hspace{1cm} (3.15)

### 3.2. The shrinking set

In this part, we will construct a shrinking set, such that the control of $(q_1, q_2) \to 0$, will be a consequence of the control of $(q_1, q_2)$ in this shrinking set. This is our definition

**Definition 3.1** (The shrinking set). For all $A \geq 1, p_1 \in (0, 1)$ and $s > 0$, we introduce the set $V_{p_1, A}(s)$ denoted for simplicity by $V_A(s)$ as the set of all $(q_1, q_2) \in (L^\infty(\mathbb{R}^n))^2$ satisfying the following conditions:

$$|q_{1,0}| \leq \frac{A}{s^2} \text{ and } |q_{2,0}| \leq \frac{A^2}{s^{p_1+2}},$$

$$|q_{1,j}| \leq \frac{A}{s^2} \text{ and } |q_{2,j}| \leq \frac{A^2}{s^{p_1+2}}, \forall 1 \leq j \leq n,$$

$$|q_{1,j,k}| \leq \frac{A^2 \ln s}{s^2} \text{ and } |q_{2,j,k}| \leq \frac{A^3 \ln s}{s^{p_1+2}}, \forall 1 \leq j, k \leq n,$$

$$\left\| \frac{q_{1,-}}{1 + |y|^3} \right\|_{L^\infty} \leq \frac{A}{s^2} \text{ and } \left\| \frac{q_{2,-}}{1 + |y|^3} \right\|_{L^\infty} \leq \frac{A^2}{s^{p_1+2}}$$

and

$$\|q_{1,-}\|_{L^\infty} \leq \frac{A^2}{\sqrt{s}} \text{ and } \|q_{2,-}\|_{L^\infty} \leq \frac{A^3}{s^{p_1+2}}.$$
where \( q_1 \) and \( q_2 \) are decomposed as in (3.15).

In the following Lemma, we show that belonging to \( V_A(s) \) implies the convergence to 0. In fact, we have a more precise statement in the following:

**Lemma 3.2.** For all \( A \geq 1, s \geq 1 \), if we have \((q_1, q_2) \in V_A(s)\), then the following estimates hold:

1. \( \|q_1\|_{L^\infty(\mathbb{R}^n)} \leq \frac{CA^2}{s} \) and \( \|q_2\|_{L^\infty(\mathbb{R}^n)} \leq \frac{CA^2}{s} \).

2. \( |q_{1, b}(y)| \leq \frac{CA^2 \ln s}{s^2} (1 + |y|^3) \), \( |q_{1, c}(y)| \leq \frac{CA^2}{s} (1 + |y|^3) \) and \( |q_1| \leq \frac{CA^2 \ln s}{s^2} (1 + |y|^3) \), and

3. \( |q_{2, b}(y)| \leq \frac{CA^3}{s^{1/2}} (1 + |y|^3) \), \( |q_{2, c}(y)| \leq \frac{CA^3}{s^{1/2}} (1 + |y|^3) \) and \( |q_2| \leq \frac{CA^3 \ln s}{s^{1/2}} (1 + |y|^3) \).

(iii) For all \( y \in \mathbb{R}^n \) we have

\[
|q_1| \leq C \left[ \frac{A^2}{s^2} (1 + |y|) + \frac{A^2 \ln s}{s^2} (1 + |y|^2) + \frac{A^2}{s^2} (1 + |y|^3) \right],
\]

and

\[
|q_2| \leq C \left[ \frac{A^2}{s^{1/2}} (1 + |y|) + \frac{A^2 \ln s}{s^{1/2}} (1 + |y|^2) + \frac{A^3}{s^{1/2}} (1 + |y|^3) \right].
\]

where \( C \) will henceforth be an universal constant in our proof which depends only on \( K \).

**Proof.** We only prove the estimate for \( q_2 \) since the estimates for \( q_1 \) follow similarly and has already been proved in previous papers (see for instance Proposition 4.7 in [24]). We now take \( A \geq 1, s \geq 1 \) and \((q_1, q_2) \in V_A(s)\) and \( y \in \mathbb{R}^n \). We also recall from (3.15) that

\[
q_2 = q_{2, b} + q_{2, c},
\]

where \( \text{supp}(q_{2, b}) \subset \mathbb{B}(0, 2K \sqrt{s}) \) and \( \text{supp}(q_{2, c}) \subset \mathbb{R}^n \setminus \mathbb{B}(0, K \sqrt{s}) \).

(i) From (3.14), we have

\[
q_b = q_{2, 0} + q_{2, 1} \cdot y + \frac{1}{2} y^T \cdot q_{2, 2} \cdot y - \text{Tr}(q_{2, 2}) + q_{2, -}.
\]

Therefore,

\[
|q_{2, b}(y)| \leq |q_{2, 0}| + |q_{2, 1}| |y| + \max_{j, k \leq n} |q_{2, j, k}| (1 + |y|^2) + \frac{q_{2, -}}{1 + |y|^3} \|q_{2, b}\|_{L^\infty(\mathbb{R}^n)} (1 + |y|^3).
\]

Then, recalling that \( \text{supp}(q_{2, b}) \subset \mathbb{B}(0, 2K \sqrt{s}) \), using Definition 3.1, we see that

\[
|q_{2, b}(y)| \leq \frac{CA^3}{s^{1/2}}.
\]

Since we also have

\[
|q_{2, c}| \leq \frac{A^3}{s^{1/2}},
\]

We end up with

\[
\|q_2\|_{L^\infty} \leq \|q_{2, b}\|_{L^\infty} + \|q_{2, c}\|_{L^\infty} \leq \frac{CA^3}{s^{1/2}}.
\]

(ii) Using (3.16) and Definition 3.1, we see that

\[
|q_{2, b}(y)| \leq \frac{CA^3}{s^{1/2}} (1 + |y|^3).
\]

We claim that \( q_{2, c} \) satisfies a similar estimate:

\[
|q_{2, c}(y)| \leq \frac{CA^3}{s^{1/2}} (1 + |y|^3).
\]
Indeed, since \( \text{supp}(q_{2,e}) \subset \mathbb{R}^n \setminus \{0, K \sqrt{s}\} \), we may assume that
\[
\frac{|y|}{K \sqrt{s}} \geq 1,
\]
hence, from Definition 3.1, we write
\[
|q_{2,e}(y)| \leq \frac{A^3}{s^{\frac{n+2}{2}}} |y|^3 \leq \frac{|y|^3}{s^{\frac{n+2}{2}}} K^3 s^2 \leq \frac{CA^3}{s^{\frac{n+2}{2}}} (1 + |y|^3),
\]
and (3.18) follows. Using (3.17) and (3.18), we see that
\[
|q_2| \leq |q_{2,e}| + |q_{2,a}| \leq \frac{CA^3}{s^{\frac{n+2}{2}}} (1 + |y|^3).
\]
(iii) It is left to the reader, since this is a direct consequence of Definition 3.1 and the decomposition (3.15).

3.3. Initial data

Here we suggest a class of initial data, depending on some parameters to be fine-tuned in order to get a good solution for our problem. This is initial data:

Definition 3.3 (The initial data). For each \( A \geq 1, s_0 \geq 1, d_1 = (d_{1,0}, d_{1,1}) \in \mathbb{R} \times \mathbb{R}^n, d_2 = (d_{2,0}, d_{2,1}, d_{2,2}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n(n+1)/2} \), we introduce
\[
\phi_{1,A,d_1,s_0}(y) = \frac{A}{s_0} (d_{1,0} + d_{1,1} \cdot y) \chi(2y, s_0),
\]
\[
\phi_{2,A,d_2,s_0}(y) = \left( \frac{A^2}{s_0^{\frac{n+2}{2}}} (d_{2,0} + d_{2,1} \cdot y) + \frac{A^5 \ln s_0}{s_0^{\frac{n+2}{2}}} \left(y^T \cdot d_{2,2} \cdot y - 2 \text{Tr}(d_{2,2})\right) \right) \chi(2y, s_0).
\]

Remark: Note that \( d_{1,0} \) and \( d_{2,0} \) are scalars, \( d_{1,1} \) and \( d_{2,1} \) are vectors, \( d_{2,2} \) is a square matrix of order \( n \). For simplicity, we may drop down the parameters expect \( s_0 \) and write \( \phi_1(y, s_0) \) and \( \phi_2(y, s_0) \).

We next claim that we can find a domain for \((d_1, d_2)\) so that initial data belongs to \( V_A(s_0) \):

Lemma 3.4 (Control of initial data in \( V_A(s_0) \)). There exists \( A_1 \geq 1 \) such that for all \( A \geq A_1 \), there exists \( s_1(A) \geq 1 \) such that for all \( s_0 \geq s_1(A) \), if \( (q_1, q_2)(s_0) = (\phi_1, \phi_2)(s_0) \) where \((\phi_1, \phi_2)(s_0)\) are defined in Definition 3.3, then, the following properties hold:

i) There exists a set \( \mathcal{D}_{A, s_0} \in [-2, 2]^{2^{\frac{n(n+1)}{2}}} \) such that the mapping
\[
\Psi_1 : \mathbb{R}^{2^{\frac{n(n+1)}{2}}} \to \mathbb{R}^{2^{\frac{n(n+1)}{2}}},
\]
\[
(d_1, d_2) \mapsto (q_{1,0}, (q_{1,j})_{j=1}^n, q_{2,0}, (q_{2,j})_{j=1}^n, (q_{2,j,k})_{j,k=1}^n) (s_0)
\]
is linear, one to one from \( \mathcal{D}_{A, s_0} \) to \( V_A(s_0) \), where
\[
\hat{V}_A(s) = \left. -\frac{A}{s^{\frac{n+2}{2}}} \left[ A + A^2 \left( \frac{A^2}{s_0^{n+2}} \right)^{1+n} \left[ \frac{A^2}{s_0^{n+2}} \right]^{1+n} \left[ \frac{A^5 \ln s_0}{s_0^{n+2}} \right]^{\frac{n(n+1)}{2}} \right] \right|.
\]
Moreover,
\[
\Psi_1(\partial \mathcal{D}_{A, s_0}) \subset \partial \hat{V}_A(s_0) \quad \text{and} \quad \text{deg} (\Psi_1 |_{\partial \mathcal{D}_{A, s_0}}) \neq 0.
\]
ii) In particular, we have \((q_1, q_2)(s_0) \in V_A(s_0)\), and
\[
|q_{1,j,k}(s_0)| \leq \frac{A^2 \ln s_0}{2 s_0^3}, \forall j, k \leq n,
\]
\[
\left\| q_{1,\cdot}(s_0) \right\|_\infty \leq \frac{A}{2 s_0^3} \quad \text{and} \quad \left\| q_{2,\cdot}(s_0) \right\|_\infty \leq \frac{A^2}{2 s_0^3}.
\]

Proof. The proof is straightforward and a bit length. For that reason, the proof is omitted, and we friendly refer the reader to Proposition 4.5 in [24] for a quite similar case. \( \square \)
Now, we give a key-proposition for our argument. More precisely, in the following proposition, we prove an existence of a solution of equation (3.2) trapped in the shrinking set:

**Proposition 3.5** (Existence of a solution trapped in \( V_A(s) \)). There exists \( A_2 \geq 1 \) such that for all \( A \geq A_2 \) there exists \( s_2(A) \geq 1 \) such that for all \( s_0 \geq s_2(A) \), there exists \((d_1, d_2) \in \mathbb{R}^{\frac{d+1}{3}}\) such that the solution \((q_1, q_2)\) of equation (3.2) with the initial data at the time \( s_0 \), given by \((q_1, q_2)(s_0) = (\phi_1, \phi_2)(s_0)\), where \((\phi_1, \phi_2)(s_0)\) is defined in Definition 3.3, we have

\[
(q_1, q_2) \in V_A(s), \quad \forall s \in [s_0, +\infty).
\]

The proof is divided into 2 steps:

1. The first step: In this step, we reduce our problem to a finite dimensional one. In other words, we aim at proving that the control of \((q_1, q_2)(s)\) in the shrinking set \( V_A(s) \) reduces to the control of the components

\[
(q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s)
\]

in \( \hat{V}_A(s) \).

2. The second step: We get the conclusion of Proposition 3.5 by using a topological argument in finite dimension.

**Proof.** We here give proof of Proposition 3.5:

- **Step 1:** Reduction to a finite dimensional problem: Using a priori estimates, our problem will be reduced to the control of a finite number of components.

**Proposition 3.6** (Reduction to a finite dimensional problem). There exists \( A_3 \geq 1 \) such that for all \( A \geq A_3 \), there exists \( s_3(A) \geq 1 \) such that for all \( s_0 \geq s_3(A) \), the following holds:

(a) If \((q_1, q_2)(s)\) a solution of equation (3.2) with initial data at the time \( s_0 \) given by \((q_1, q_2)(s_0) = (\phi_1, \phi_2)(s_0)\) defined as in Definition 3.3 with \((d_1, d_2) \in D_{A,s_0}\) defined in Lemma 3.4.

(b) If we furthermore assume that \((q_1, q_2)(s) \in V_A(s)\) for all \( s \in [s_0, s_1] \) for some \( s_1 \geq s_0 \) and \((q_1, q_2)(s_1) \in \partial V_A(s_1)\).

Then, we have the following conclusions:

(i) (Reduction to finite dimensions): We have \((q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s) \in \partial \hat{V}_A(s_1)\).

(ii) (Transverse outgoing crossing) There exists \( \delta_0 > 0 \) such that

\[
\forall \delta \in (0, \delta_0), (q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s_1 + \delta) \notin \hat{V}_A(s_1 + \delta),
\]

which implies that \((q_1, q_2)(s_1 + \delta) \notin V_A(s_1 + \delta)\) for all \( \delta \in (0, \delta_0) \).

This proposition makes the heart of the paper and needs many steps to be proved. For that reason, we dedicate a whole section to its proof (Section 4 below). Let us admit it here, and get to the conclusion of Proposition 3.5 in the second step.

- **Step 2:** Conclusion of Proposition 3.5 by a topological argument. In this step, we finish the proof of Proposition 3.5. In fact, we aim at proving the existence of a parameter \((d_1, d_2) \in D_{A,s_0}\) such that the solution \((q_1, q_2)(s)\) of equation (3.2) with initial data \((q_1, q_2)(s_0) = (\phi_1, \phi_2)(s_0)\), exists globally for all \( s \in [s_0, +\infty) \) and satisfies

\[
(q_1, q_2)(s) \in V_A(s).
\]

Our argument is analogous to the argument of Merle and Zaag in [15]. For that reason, we only give a brief proof. Let us fix \( K, A, s_0 \) such that Lemma 3.4 and Proposition 3.6 hold. We first consider \((q_1, q_2)_{d_1, d_2}(s)\), \( s \geq s_0 \) a solution of equation (3.2) with initial data at \( s_0 \) is \((q_1, q_2)(s_0)\), which depend on \((d_1, d_2)\) as follows

\[
(q_1, q_2)_{d_1, d_2}(s_0) = (\phi_1, \phi_2)(s_0).
\]

From Lemma 3.4 and by construction of the set \( D_{A,s_0}\), we know that

\[
(q_1, q_2)(s_0) \in V_A(s_0).
\]

By contradiction, we assume that for all \((d_1, d_2) \in D_{A,s_0}\) there exists \( s_1 \in [s_0, +\infty) \) such that

\[
(q_1, q_2)_{d_1, d_2}(s_1) \notin V_A(s_1).
\]
Then, for each \((d_1, d_2) \in \mathcal{D}_{A,s_0}\), we can define
\[
s^*(d_1, d_2) = \inf \{s_1 \geq s_0 \text{ such that } (q_1, q_2)_{d_1,d_2}(s_1) \notin V_A(s_1)\}.
\]
Since there exists \(s_1\) such that \((q_1, q_2)(s_1) \notin V_A(s_1)\) we deduce that \(s^*(d_1, d_2) < +\infty\) for all \((d_1, d_2) \in \mathcal{D}_{A,s_0}\).
Besides that, using (3.22), and the minimality of \(s^*(d_1, d_2)\), the continuity of \((q_1, q_2)\) in \(s\) and the closeness of \(V_A(s)\) we derive that \((q_1, q_2)(s^*(d_1, d_2)) \in \partial V_A(s^*(d_1, d_2))\) and for all \(s \in [s_0, s^*(d_1, d_2)]\),
\[
(q_1, q_2)(s) \in V_A(s).
\]
Therefore, from item (i) of Proposition 3.6 we see that
\[
(q_1, 0, (q_{1,j})_{j \leq n}, q_2, 0, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s^*(d_1, d_2)) \in \hat{V}_A(s^*(d_1, d_2)).
\]
This means that following mapping \(\Gamma\) is well-defined:
\[
\Gamma: \mathcal{D}_{A,s_0} \rightarrow \partial \left([-1, 1]^{1+n} \times [-1, 1]^{1+n} \times [-1, 1]^{n(n+1)/2}\right)
\]
\[
(d_1, d_1) \mapsto \left(\frac{s_2^2(d_1, d_2)}{A}(q_1, 0, (q_{1,j})_{j \leq n}, \frac{s_{p_1+2}}{A^2}(q_2, 0, (q_{2,j})_{j \leq n}), \frac{s_{p_1+2}}{A^2}(d_1, d_2) (q_{2,j,k})_{j,k \leq n}\right)(s^*(d_1, d_2)).
\]
Moreover, it satisfies the two following properties:
(i) \(\Gamma\) is continuous from \(\mathcal{D}_{A,s_0}\) to \(\partial \left([-1, 1]^{2+2n+1}\right)\). This is a consequence of item (ii) in Proposition (3.6).
(ii) The degree of the restriction \(\Gamma|_{\partial \mathcal{D}_{A,s_0}}\) is non zero. Indeed, again by item (ii) in Proposition 3.6, we have
\[
s^*(d_1, d_2) = s_0,
\]
in this case. Applying (3.20), we get the conclusion. In fact, such a mapping \(\Gamma\) cannot exist by Index theorem, this is a contradiction. Thus, Proposition 3.5 follows, assuming that Proposition 3.6 (see Section 4 for the proof of latter)

3.4. The proof of Theorem 1.1

In this section, we aim at giving the proof of Theorem 1.1.

Proof. Proof of Theorem 1.1 assuming that Proposition 3.6

+ The proof of item (i) of Theorem 1.1: Using Proposition 3.5, there exists initial data \((q_1, q_2)_{d_1,d_2}(s_0) = (\phi_1, \phi_2)(s_0)\) such that the solution of equation of (3.2) exists globally on \([s_0, +\infty)\) and satisfies:
\[
(q_1, q_2)(s) \in V_A(s), \forall s \in [s_0, +\infty),
\]
Thanks to similarity variables (2.4), (3.1) and item (i) in Lemma 3.2, we conclude that there exist initial data \(u^0\) of the form given in Remark 1.2 with \((d_1, d_2)\) given in Proposition 3.5 such that the solution \(u(t)\) of equation (1.1) exists on \([0, T]\), where \(T = e^{-\sigma_0}\) and satisfies (1.15) and (1.16). Using these two estimates, we see that
\[
u(0, t) \sim \kappa(T - t)^{-\frac{p}{p-1}}\text{ as } t \rightarrow T,
\]
which means that \(u\) blows up at time \(T\) and the origin is a blowup point. It remains to prove that for all \(x \neq 0\), \(x\) is not a blowup point of \(u\). The following Lemma allows us to conclude.

Lemma 3.7 (No blow up under some threshold). For all \(C_0 > 0, 0 \leq T_1 < T\) and \(\sigma > 0\) small enough, there exists \(c_0(C_0, T, \sigma) > 0\) such that \(u(\xi, \tau)\) satisfies the following estimates for all \(|\xi| \leq \sigma, \tau \in [T_1, T]\):
\[
|\partial_\tau u - \Delta u| \leq C_0 |u|^p,
\]
and
\[
|u(\xi, \tau)| \leq c_0(1 - \tau)^{-\frac{p}{p-1}}.
\]
Then, \(u\) does not blow up at \(\xi = 0, \tau = T\).

Proof. The proof of this Lemma is processed similarly to Theorem 2.1 in [6]. Although the proof of [6] was given in the real case, it extends naturally to the complex valued case. \(\square\)
We next use Lemma 3.7 to conclude that \( u \) does not blow up at \( x_0 \neq 0 \). Indeed, if \( x_0 \neq 0 \) we use (1.15) to deduce the following:

\[
\sup_{|x-x_0| \leq \frac{|x_0|}{3}} (T-t)^{\frac{1}{p-1}} |u(x,t)| \leq |f_0 \left( \frac{|x_0|}{\sqrt{(T-t)|\ln(T-t)|}} \right) | + \frac{C}{\sqrt{|\ln(T-t)|}} \to 0, \text{ as } t \to T. \tag{3.23}
\]

Applying Lemma 3.7 to \( u(x-x_0, t) \), with some \( \sigma \) small enough such that \( \sigma \leq \frac{|x_0|}{2} \), and \( T_1 \) close enough to \( T \), we see that \( u(x-x_0, t) \) does not blow up at time \( T \) and \( x = 0 \). Hence \( x_0 \) is not a blow-up point of \( u \). This concludes the proof of item (i) in Theorem 1.1.

+ The proof of item (ii) of Theorem 1.1: Here, we use the argument of Merle in [13] to deduce the existence of \( u^* = u_1^* + iu_2^* \) such that \( u(t) \to u^* \) as \( t \to T \) uniformly on compact sets of \( \mathbb{R}^n \setminus \{0\} \). In addition to that, we use the techniques in Zaag [29], Masmoudi and Zaag [16], Tayachi and Zaag [24] for the proofs of (1.18) and (1.19).

Indeed, for all \( x_0 \in \mathbb{R}^n, x_0 \neq 0 \), we deduce from (1.15), (1.16) that not only (3.23) holds but also the following satisfied:

\[
\sup_{|x-x_0| \leq \frac{|x_0|}{3}} (T-t)^{\frac{1}{p-1}} |\ln(T-t)||u_2(x,t)| \leq \left| f_0 \left( \frac{|x_0|}{2\sqrt{(T-t)|\ln(T-t)|}} \right) \right| \leq \frac{3|x_0|^2}{2(T-t)|\ln(T-t)|} f_0^p \left( \frac{|x_0|}{\sqrt{(T-t)|\ln(T-t)|}} \right) \tag{3.24}
\]

where \( u_2(x_0, \xi, \tau) \) is defined by

\[
U(x_0, \xi, \tau) = (T-t_0(x_0))^\frac{1}{p-1} u(x,t).
\]

and

\[
V_2(x_0, \xi, \tau) = |\ln(T-t_0(x_0))|U_2(x_0, \xi, \tau),
\]

where \( U_2(x_0, \xi, \tau) \) is defined by

\[
U(x_0, \xi, \tau) = U_1(x_0, \xi, \tau) + iU_2(x_0, \xi, \tau),
\]

and

\[
(x, t) = \left( x_0 + \xi \sqrt{T-t_0(x_0)} - t_0(x_0) + \tau(T-t_0(x_0)) \right), \text{ and } (\xi, \tau) \in \mathbb{R}^n \times \left[ -\frac{t_0(x_0)}{T-t_0(x_0)}, 1 \right].
\]

We can see that with these notations, we derive from item (i) in Theorem 1.1 the following estimates for initial data at \( \sigma = 0 \) of \( U \) and \( V_2 \)

\[
\sup_{|\xi| \leq |\ln(T-t_0(x_0))|^{\frac{1}{2}}} |U(x_0, \xi, 0) - f_0(K_0)| \leq \frac{C}{1 + (|\ln(T-t_0(x_0))|^{\frac{1}{2}})} \to 0 \text{ as } x_0 \to 0, \tag{3.29}
\]

\[
\sup_{|\xi| \leq |\ln(T-t_0(x_0))|^{\frac{1}{2}}} |V_2(x_0, \xi, 0) - g_0(K_0)| \leq \frac{C}{1 + (|\ln(T-t_0(x_0))|^{\frac{1}{2}})} \to 0 \text{ as } x_0 \to 0. \tag{3.30}
\]

where \( f_0(x), g_0(x) \) are defined as in (1.6) and (1.17) respectively, and \( \gamma_1 = \min \left( \frac{1}{2}, \frac{n}{2p} \right) \). Moreover, using equations (2.2), we derive the following equations for \( U, V_2 \) for all \( \xi \in \mathbb{R}^n, \tau \in [0, 1) \)

\[
\partial_\tau U = \Delta \xi U + U^p,
\]

\[
\partial_\tau V_2 = \Delta \xi V_2 + V_2G_2(U_1, U_2),
\]

where \( G \) is defined by

\[
G(U_1, U_2)U_2 = F_2(U_1, U_2),
\]

and \( F_2 \) is defined in (2.3). We note that \( G_2, F_2 \) are polynomials of \( U_1, U_2 \).
Besides that, from (3.24) and (3.31), we can apply Lemma 3.7 to $U$ when $|\xi| \leq |\ln(T - t_0(x_0))|^{\frac{1}{2}}$ and obtain:

$$\sup_{|\xi| \leq \frac{1}{2} |\ln(T - t_0(x_0))|^{\frac{1}{2}}, \tau \in [0, 1]} |U(x_0, \xi, \tau)| \leq C. \quad (3.34)$$

and we aim at proving for $V_2(x_0, \xi, \tau)$ that

$$\sup_{|\xi| \leq \frac{1}{2} |\ln(T - t_0(x_0))|^{\frac{1}{2}}, \tau \in [0, 1]} |V_2(x_0, \xi, \tau)| \leq C. \quad (3.35)$$

+ The proof for (3.35): We first use (3.34) to derive the following rough estimate:

$$\sup_{|\xi| \leq \frac{1}{2} |\ln(T - t_0(x_0))|^{\frac{1}{2}}, \tau \in [0, 1]} |V_2(x_0, \xi, \tau)| \leq C |\ln(T - t_0(x_0))|. \quad (3.36)$$

We first introduce $\psi(x)$ a cut-off function $\psi \in C_0^\infty(\mathbb{R}^n), 0 \leq \psi \leq 1, \text{supp}(\psi) \subset B(0, 1), \psi = 1$ on $B(0, \frac{1}{2})$. We introduce

$$\psi_1(\xi) = \psi \left( \frac{2\xi}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}} \right) \quad \text{and} \quad V_{2,1}(x_0, \xi, \tau) = \psi_1(\xi)V_2(x_0, \xi, \tau). \quad (3.37)$$

Then, we deduce from (3.32) an equation satisfied by $V_{2,1}$

$$\partial_\tau V_{2,1} = \Delta_\xi V_{2,1} - 2 \text{div} (V_2\nabla \psi_1) + V_2 \Delta \psi_1 + V_{2,1} G_1(U_1, U_2). \quad (3.38)$$

Hence, we can write $V_{2,1}$ with a integral equation as follows

$$V_{2,1}(\tau) = e^{\Delta(\tau)}(V_{2,1}(0)) + \int_0^\tau e^{(\tau - \tau')\Delta} (-2 \text{div} (V_2\nabla \psi_1) + V_2 \Delta \psi_1 + V_{2,1} G_1(U_1, U_2)(\tau')) d\tau'. \quad (3.39)$$

Besides that, using (3.34) and (3.36) and the fact that

$$|\nabla \psi_1| \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}}, \quad |\Delta \psi_1| \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}},$$

we deduce that

$$\left| \int_0^\tau e^{(\tau - \tau')\Delta} (-2 \text{div} (V_2\nabla \psi_1)) d\tau' \right| \leq C \int_0^\tau \frac{||V_2\nabla \psi_1||_{L^\infty}(\tau')}{\sqrt{\tau - \tau'}} d\tau' \leq C |\ln(T - t_0(x_0))|^{\frac{1}{2}},$$

$$\left| \int_0^\tau e^{(\tau - \tau')\Delta} (V_2(\tau')\Delta \psi_1) d\tau' \right| \leq C \int_0^\tau ||V_2 \Delta \psi_1||_{L^\infty}(\tau') d\tau' \leq C |\ln(T - t_0(x_0))|^{\frac{1}{2}},$$

$$\left| \int_0^\tau e^{(\tau - \tau')\Delta} (V_{2,1} G_1(U_1, U_2)(\tau')) d\tau' \right| \leq C \int_0^\tau ||V_{2,1} G_2(U_1, U_2)||_{L^\infty}(\tau') d\tau'.$$

Note that $G_2(U_1, U_2)$ in the last line is bounded on $|\xi| \leq |\ln(T - t_0)|^{\frac{1}{2}}, \tau \in [0, 1]$ because it is a polynomial in $U_1, U_2$ and (3.34) holds, then, we derive

$$||V_{2,1} G_2(U_1, U_2)||_{L^\infty}(\tau') \leq C ||V_{2,1}||_{L^\infty}(\tau').$$

Hence, from (3.39) and the above estimates, we derive

$$||V_{2,1}(\tau)||_{L^\infty} \leq C |\ln(T - t_0(x_0))|^{\frac{1}{2}} + C \int_0^\tau ||V_{2,1}(\tau')||_{L^\infty} d\tau'.$$

Thanks to Gronwall Lemma, we deduce that

$$||V_{2,1}(\tau)||_{L^\infty} \leq C |\ln(T - t_0(x_0))|^{\frac{1}{2}}, \forall \tau \in [0, 1),$$

which yields

$$\sup_{|\xi| \leq \frac{1}{2} |\ln(T - t_0(x_0))|^{\frac{1}{2}}, \tau \in [0, 1]} |V_2(x_0, \xi, \tau)| \leq C |\ln(T - t_0(x_0))|^{\frac{1}{2}}. \quad (3.40)$$

We apply iteratively for

$$V_{2,2}(x_0, \xi, \tau) = \psi_2(\xi)V_2(x_0, \xi, \tau) \quad \text{where} \quad \psi_2(\xi) = \psi \left( \frac{4\xi}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}} \right).$$
Similarly, we deduce that
\[ \sup_{|\xi| \leq \frac{1}{4} \ln(T-t_0(x_0))} |V_2(x_0, \xi, \tau)| \leq C |\ln(T-t_0(x_0))|^{\frac{1}{2}}. \]
We apply this process a finite number of steps to obtain (3.35). We now come back to our problem, and aim at proving that:
\[ \sup_{|\xi| \leq \frac{1}{4} \ln(T-t_0(x_0))} \left| U(x_0, \xi, \tau) - \hat{U}_{K_0}(\tau) \right| \leq \frac{C}{1 + |\ln(T-t_0(x_0))|^{\gamma_2}}, \tag{3.41} \]
\[ \sup_{|\xi| \leq \frac{1}{4} \ln(T-t_0(x_0))} \left| V_2(x_0, \xi, \tau) - \hat{V}_{2,K_0}(\tau) \right| \leq \frac{C}{1 + |\ln(T-t_0(x_0))|^{\gamma_3}}, \tag{3.42} \]
where \( \gamma_2, \gamma_3 \) are positive small enough and \((\hat{U}_{K_0}, \hat{V}_{2,K_0})(\tau)\) is the solution of the following system:
\[ \partial_\tau \hat{U}_{K_0} = \hat{U}_{K_0}^p, \tag{3.43} \]
\[ \partial_\tau \hat{V}_{2,K_0} = p\hat{U}_{K_0}^{p-1} \hat{V}_{2,K_0}. \tag{3.44} \]
with initial data at \( \tau = 0 \)
\[ \hat{U}_{K_0}(0) = f_0(K_0), \]
\[ \hat{V}_{2,K_0}(0) = g_0(K_0). \]
given by
\[ \hat{U}_{K_0}(\tau) = \left( (p-1)(1-\tau) + \frac{(p-1)^2 K_0^2}{4p} \right)^{-\frac{1}{p-1}}, \tag{3.45} \]
\[ \hat{V}_{2,K_0}(\tau) = K_0^2 \left( (p-1)(1-\tau) + \frac{(p-1)^2 K_0^2}{4p} \right)^{-\frac{p}{p-1}}. \tag{3.46} \]
for all \( \tau \in [0, 1) \). The proof of (3.41) is cited to Section 5 of Tayachi and Zaag [24] and the proof of (3.42) is similar. For the reader’s convenience, we give it here. Let us consider
\[ V_2 = V_2 - \hat{V}_{2,K_0}(\tau). \tag{3.47} \]
Then, \( V_2 \) satisfies
\[ \sup_{|\xi| \leq \frac{1}{4} \ln(T-t_0(x_0))} |V_2| \leq C. \tag{3.48} \]
We use (3.32) to derive an equation on \( V_2 \) as follows:
\[ \partial_\tau V_2 = \Delta V_2 + p\hat{U}_{K_0}^{p-1} V_2 + p(U_1^{p-1} - \hat{U}_{K_0}^{p-1}) V_2 + G_2(x_0, \xi, \tau), \tag{3.49} \]
where
\[ G_2(x_0, \xi, \tau) = V_2[G_2(U_1, U_2) - pU_1^{p-1}]. \]
Note that, from definition of \( G_2 \) and (3.34) we deduce that
\[ \sup_{|\xi| \leq \frac{1}{4} \ln(T-t_0(x_0))} |G_2(U_1, U_2) - pU_1^{p-1}| \leq C|U_2|, \]
Hence, using (3.27) and (3.35) and we derive
\[ \sup_{|\xi| \leq \frac{1}{4} \ln(T-t_0(x_0))} |G_2(x_0, \xi, \tau)| \leq \frac{C}{|\ln(T-t_0(x_0))|}. \tag{3.50} \]
We also define
\[ \tilde{V}_2 = \psi_* (\xi) V_2, \]
where
\[ \psi_* = \psi \left( \frac{16\xi}{|\ln(T-t_0(x_0))|} \right), \]
and $\psi$ is the cut-off function which has been introduced above. We also note that $\nabla \psi_*, \Delta \psi_*$ satisfy the following estimates

$$\|\nabla \xi \psi_*\|_{L^\infty} \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}}, \quad \|\Delta \xi \psi_*\|_{L^\infty} \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}},$$

(3.51)

In particular, $\tilde{V}_2$ satisfies

$$\partial_t \tilde{V}_2 = \Delta \tilde{V}_2 + p\tilde{U}_K^{p-1}(\tau)\tilde{V}_2 - 2 \text{div} \left( \nabla \tilde{V}_2 \right) + \nabla \tilde{V}_2 \Delta \psi_* + p(U_1^{p-1} - \tilde{U}_K^{p-1})\psi_* V_2 + \psi_* \mathcal{G}_2,$$

(3.52)

By Duhamel principal, we derive the following integral equation

$$\tilde{V}_2(\tau) = e^{\tau \Delta} \tilde{V}_2(\tau) + \int_0^\tau e^{(\tau - \tau') \Delta} \left( p\tilde{U}_K^{p-1} \tilde{V}_2 - 2 \text{div} \left( \nabla \tilde{V}_2 \right) + \nabla \tilde{V}_2 \Delta \psi_* + p(U_1^{p-1} - \tilde{U}_K^{p-1})\psi_* V_2 + \psi_* \mathcal{G}_2 \right) (\tau') d\tau'.

(3.53)

Besides that, we use (3.41), (3.48), (3.51), (3.50) to derive the following estimates: for all $\tau \in [0,1)$

$$\left| \tilde{U}_K(\tau) \right| \leq C,$$

$$\|\nabla \tilde{V}_2\|_{L^\infty}(\tau) \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{4}}},$$

$$\|\nabla \Delta \psi_*\|_{L^\infty}(\tau) \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{4}}},$$

$$\left\| \left( U_1^{p-1} - \tilde{U}_K^{p-1} \right) \psi_* \right\|_{L^\infty}(\tau) \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}},$$

$$\|\mathcal{G}_2\psi_*\|_{L^\infty} \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}}.$$

where $\gamma_2$ given in (3.41). Hence, we derive from the above estimates that: for all $\tau \in [0,1)$

$$|e^{(\tau - \tau') \Delta} p\tilde{U}_K^{p-1} \tilde{V}_2(\tau')| \leq C \|\tilde{V}_2(\tau')\|,$$

$$|e^{(\tau - \tau') \Delta} \left( \text{div} \left( \nabla \tilde{V}_2 \right) \right)| \leq C \frac{1}{\sqrt{\tau - \tau'}} \frac{1}{|\ln(T - t_0(x_0))|^{\frac{1}{4}}},$$

$$|e^{(\tau - \tau') \Delta} (\nabla \tilde{V}_2 \Delta \psi_*)| \leq C \frac{1}{|\ln(T - t_0(x_0))|^{\frac{1}{4}}},$$

$$|e^{(\tau - \tau') \Delta} (p(U_1^{p-1} - \tilde{U}_K^{p-1})\psi_* V_2)(\tau')| \leq C \frac{1}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}},$$

$$|e^{(\tau - \tau') \Delta} (\psi_* \mathcal{G}_2)(\tau')| \leq C \frac{1}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}}.$$

Pluggin into (3.53), we obtain

$$\|\tilde{V}_2(\tau)\|_{L^\infty} \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{3}}} + C \int_0^\tau \|\tilde{V}_2(\tau')\|_{L^\infty} d\tau',$$

where $\gamma_3 = \min\left( \frac{1}{4}, \gamma_2 \right)$. Then, thanks to Gronwall inequality, we get

$$\|\tilde{V}_2\|_{L^\infty} \leq \frac{C}{|\ln(T - t_0(x_0))|^{\gamma_3}}.$$

Hence, (3.42) follows. Finally, we easily find the asymptotics of $u^*$ and $u_2^*$ as follows, thanks to the definition of $U$ and $\tilde{V}_2$ and to estimates (3.41) and (3.42):

$$u^*(x_0) = \lim_{t \to T} u(x_0, t) = (T - t_0(x_0))^{-\frac{1}{p-1}} \lim_{\tau \to 1} U(x_0, 0, \tau) \sim (T - t_0(x_0))^{-\frac{1}{p-1}} \left( \frac{(p - 1)^2}{4p} K_0^2 \right)^{-\frac{1}{p-1}},$$

(3.54)

and

$$u_2^* = \lim_{t \to T} u_2(x_0, t) = \frac{(T - t_0(x_0))^{-\frac{1}{p-1}}}{|\ln(T - t_0(x_0))|} \lim_{\tau \to 1} \tilde{V}_2(x_0, 0, \tau) \sim \frac{(T - t_0(x_0))^{-\frac{1}{p-1}}}{|\ln(T - t_0(x_0))|} \left( \frac{(p - 1)^2}{4p} \right)^{-\frac{1}{p-1}} K_0^{-\frac{p-1}{2}}.$$
Using the relation (3.25), we find that
\[ T - t_0 \sim \frac{|x_0|^2}{2K_0^n |\ln|x_0||} \quad \text{and} \quad \ln(T - t_0(x_0)) \sim 2 \ln(|x_0|), \quad \text{as} \ x_0 \to 0. \tag{3.56} \]

Plugging (3.56) into (3.54) and (3.55), we get the conclusion of item (ii) of Theorem 1.1.

This concludes the proof of Theorem 1.1 assuming that Proposition 3.6 holds. Naturally, we need to prove this proposition on order to finish the argument. This will be done in the next section. \(\square\)

4. The proof of Proposition 3.6

This section is devoted to the proof of Proposition 3.6, which is the heart of our analysis. We proceed into two parts. In the first part, we derive a priori estimates on \(q(s)\) in \(V_A(s)\). In the second part, we show that the new bounds are better than those defined in \(V_A(s)\), except for the first components \((q_1, q_1, q_2, q_2)\). This means that the problem is reduced to the control of these components, which is the conclusion of item (i) of Proposition 3.6. Item (ii) of Proposition 3.6 is just a direct consequence of the dynamics of these modes. Let us start the first part.

4.1. A priori estimates on \((q_1, q_2)\) in \(V_A(s)\).

In this subsection, we aim at proving the following proposition:

Proposition 4.1. There exists \(A_4 \geq 1\), such that for all \(A \geq A_4\) there exists \(s_4(A) \geq 1\), such that the following holds for all \(s_0 \geq s_4(A)\): we assume that for all \(s \in [\sigma, s_1], (q_1, q_2) \in V_A(s)\) for some \(s_1 \geq s_0\). Then, the following holds for all \(s \in [s_0, s_1]\):

(i) (ODE satisfied by the positive modes) For all \(j \in \{1, n\}\) we have
\[ |q'_{1,j}(s) - q_{1,j}(s)| + \frac{1}{2} q_{1,j}(s) \leq \frac{C}{s^2}, \forall j \leq n. \tag{4.1} \]
\[ |q'_{2,j}(s) - q_{2,j}(s)| + \frac{1}{2} q_{2,j}(s) \leq \frac{C}{s^2}, \forall j \leq n. \tag{4.2} \]

(ii) (ODE satisfied by the null modes) For all \(j, k \leq n\)
\[ \left| q'_{1,j,k}(s) + \frac{2}{s} q_{1,j,k}(s) \right| \leq \frac{CA}{s^3}, \tag{4.3} \]
\[ \left| q'_{2,j,k}(s) + \frac{2}{s} q_{2,j,k}(s) \right| \leq \frac{CA^2 \ln s}{s^{p_1+3}}. \tag{4.4} \]

(iii) (Control the negative part)
\[ \frac{|q_{1,-}(\cdot, s)|}{1 + |\cdot|^3} \leq C e^{-\frac{s-\tau}{2}} \frac{|q_{1,-}(\cdot, \tau)|}{1 + |\cdot|^3} + C e^{-\frac{(s-\tau)}{2}} \frac{|q_{1,-}(\cdot, \tau)|}{1 + |\cdot|^3} \leq \frac{C(1 + s - \tau)}{s^2}, \tag{4.5} \]
\[ \frac{|q_{2,-}(\cdot, s)|}{1 + |\cdot|^3} \leq C e^{-\frac{s-\tau}{2}} \frac{|q_{2,-}(\cdot, \tau)|}{1 + |\cdot|^3} + C e^{-\frac{(s-\tau)}{2}} \frac{|q_{2,-}(\cdot, \tau)|}{1 + |\cdot|^3} \leq \frac{C(1 + s - \tau)}{s^{p_1+3}}. \tag{4.6} \]

(v) (Outer part)
\[ \frac{|q_{1,-}(\cdot, s)|}{1 + |\cdot|^3} \leq C e^{-\frac{s-\tau}{2}} \frac{|q_{1,-}(\cdot, \tau)|}{1 + |\cdot|^3} + C e^{-\frac{(s-\tau)}{2}} \frac{|q_{1,-}(\cdot, \tau)|}{1 + |\cdot|^3} \leq \frac{C(1 + s - \tau)e^{s-\tau}}{s^{\frac{p_1+3}{2}}}, \tag{4.7} \]
\[ \frac{|q_{2,-}(\cdot, s)|}{1 + |\cdot|^3} \leq C e^{-\frac{s-\tau}{2}} \frac{|q_{2,-}(\cdot, \tau)|}{1 + |\cdot|^3} + C e^{-\frac{(s-\tau)}{2}} \frac{|q_{2,-}(\cdot, \tau)|}{1 + |\cdot|^3} \leq \frac{C(1 + s - \tau)e^{s-\tau}}{s^{\frac{p_1+3}{2}}}. \tag{4.8} \]

Proof. The proof of this Proposition is given in two steps:

+ **Step 1:** We will give a proof to items (i) and (ii) by using the projection the equations which are satisfied by \(q_1\) and \(q_2\).

+ **Step 2:** We will control the other components by studying the dynamics of the linear operator \(L + V\).
a) **Step 1:** We observe that the techniques of the proof for (4.1), (4.2), (4.3) and (4.4) are the same. So, we only deal with the proof of (4.3). For each \( j, k \leq n \) by using the equation in (3.2) and the definition of \( q_{1,j,k} \) we deduce that

\[
|q'_{i,j}(s) - \int [\mathcal{L}q_1 + Vq_1 + B_1(q_1, q_2) + R_1(y, s)] \chi(y, s) \left( \frac{y_j y_j}{4} - \frac{\delta_{i,j}}{2} \right) \rho dy | \leq C e^{-s},
\]

if \( K \) is large enough. In addition to that, using the fact \( (q_1, q_2) \in V_A(s) \) and Lemma 3.2, Lemma A.2, Lemma A.3, Lemma A.4 that

\[
\left| \int \mathcal{L}(q) \chi \left( \frac{y_j y_j}{4} - \frac{\delta_{i,j}}{2} \right) \rho dy \right| \leq \frac{C}{s^3},
\]

\[
\left| \int Vq_1 \chi \left( \frac{y_j y_j}{4} - \frac{\delta_{i,j}}{2} \right) \rho dy + \frac{2}{s} q_{1,i,j}(s) \right| \leq \frac{CA}{s^3},
\]

\[
\left| \int B_1(q_1, q_2) \chi \left( \frac{y_j y_j}{4} - \frac{\delta_{i,j}}{2} \right) \rho dy \right| \leq \frac{C}{s^3},
\]

\[
\left| \int R_1(y, s) \chi \left( \frac{y_j y_j}{4} - \frac{\delta_{i,j}}{2} \right) \rho dy \right| \leq \frac{C}{s^3},
\]

if \( s \geq s_A(A) \). Then, (4.3) is derived by adding all the above estimates.

**Step 2:** In this part, we will concentrate on the proof of items (iii) and (iv). We now rewrite (3.2) in its integral form: for each \( s \geq \tau \)

\[
\begin{align*}
q_1(s) & = K(s, \tau) q_1(\tau) + \int_{\tau}^{s} K(s, \sigma) [(V_{1,1} q_1)(\sigma) + (V_{1,2} q_2)(\sigma) + B_1(q_1, q_2)(\sigma) + R_1(\sigma)] d\sigma \\
q_2(s) & = K(s, \tau) q_2(\tau) + \int_{\tau}^{s} K(s, \sigma) [(V_{2,1} q_1)(\sigma) + (V_{2,2} q_2)(\sigma) + B_2(q_1, q_2)(\sigma) + R_2(\sigma)] d\sigma
\end{align*}
\]

where \( \{K(s, \tau)\}_{s \geq \tau} \) is the fundamental solution associated to the linear operator \( \mathcal{L} + V \) and defined by

\[
\begin{align*}
\partial_s K(s, \tau) & = (\mathcal{L} + V) K(s, \tau), \quad \forall s > \tau, \\
K(\tau, \tau) & = Id.
\end{align*}
\]

Let us now introduce some notations:

\[
\begin{align*}
\vartheta_{1,1}(s, \tau) & = K(s, \tau) q_1(\tau), \quad \vartheta_{1,2}(s, \tau) = \int_{\tau}^{s} K(s, \sigma) (V_{1,1} q_1)(\sigma) d\sigma, \quad \vartheta_{1,3}(s, \tau) = \int_{\tau}^{s} K(s, \sigma) (V_{1,2} q_2)(\sigma) d\sigma, \\
\vartheta_{1,4}(s, \tau) & = \int_{\tau}^{s} K(s, \sigma) (B_1(q_1, q_2))(\sigma) d\sigma, \quad \vartheta_{1,5} = \int_{\tau}^{s} K(s, \sigma) (R_1(\cdot, \sigma)) d\sigma,
\end{align*}
\]

and

\[
\begin{align*}
\vartheta_{2,1}(s, \tau) & = K(s, \tau) q_2(\tau), \quad \vartheta_{2,2}(s, \tau) = \int_{\tau}^{s} K(s, \sigma) (V_{2,1} q_1)(\sigma) d\sigma, \quad \vartheta_{2,3}(s, \tau) = \int_{\tau}^{s} K(s, \sigma) (V_{2,2} q_2)(\sigma) d\sigma, \\
\vartheta_{2,4}(s, \tau) & = \int_{\tau}^{s} K(s, \sigma) (B_2(q_1, q_2))(\sigma) d\sigma, \quad \vartheta_{2,5} = \int_{\tau}^{s} K(s, \sigma) (R_2(\cdot, \sigma)) d\sigma.
\end{align*}
\]

From (4.10), we can see the strong influence of the kernel \( K \). For that reason, we will study the dynamics of that operator:

**Lemma 4.2** (A-priori estimates of the linearized operator). For all \( \rho^* \geq 0 \), there exists \( s_5(\rho^*) \geq 1 \), such that if \( s \geq s_5(\rho^*) \) and \( v \in L^2_\rho \) satisfies

\[
\sum_{m=0}^{2} |v_m| + \left\| \frac{v_n}{1 + |y|^i} \right\|_{L^\infty} + \|v\|_{L^\infty} < \infty,
\]

then, for all \( s \in [\sigma, \sigma + \rho^*] \), the function \( \theta(s) = K(s, \sigma) v \) satisfies

\[
\left\| \frac{\vartheta_{i,(y,s)}}{1 + |y|^i} \right\|_{L^\infty} \leq \frac{C e^{-(\sigma - \rho^*)^2 + 1}}{s} \left( |v_0| + |v_1| + \sqrt{s} |v_2| \right) + Ce^{-(\sigma - \rho^*)^2} \|v\|_{L^\infty} + C e^{-(\sigma - \rho^*)^2} \|v\|_{L^\infty},
\]

\( i = 1, 2 \).
and
\[ \| \theta_c(y,s) \|_{L^\infty} \leq C e^{s-\sigma} \left( \sum_{i=0}^{2} s^{\frac{i}{2}} |v_i| + s^{\frac{3}{2}} \left\| \frac{v - \|y\|^3}{1 + \|y\|^3} \right\|_{L^\infty} \right) + C e^{-\frac{s-\sigma}{\rho}} \| v_c \|_{L^\infty}. \] (4.14)

**Proof.** The proof of this result was given by Bricmont and Kupiainen [1] in the one dimensional case. Later, it was extended to the higher dimensional case by Nguyen and Zaag [18]. We kindly refer interested readers to Lemma 2.9 in [18] for details of the proof.

We now use Lemmas 4.2, 3.2, A.2, A.3 and A.4 to deduce the following Lemma which implies Proposition 4.1.

**Lemma 4.3.** For all $A \geq 1, \rho^* \geq 0$, there exists $s_6(A, \rho^*) \geq 1$ such that $\forall s_0 \geq s_6(A, \rho^*)$ and $q(s) \in S_A(s), \forall s \in [\tau, \tau + \rho^*]$ where $\tau \geq s_0$. Then, we have the following properties: for all $s \in [\tau, \tau + \rho^*],$

i) (The linear term $\vartheta_{1,1}(s, \tau)$ and $\vartheta_{2,1}(s, \tau)$)
\[ \left\| \frac{(\vartheta_{1,1}(s, \tau))_e}{1 + \|y\|^3} \right\|_{L^\infty} \leq C e^{-\frac{s-\tau}{\rho}} \left\| \frac{q_{1,1}(\cdot, \tau)}{1 + \|y\|^3} \right\|_{L^\infty} + \frac{C e^{s-\tau}}{s^2}, \]
\[ \| (\vartheta_{2,1}(s, \tau))_e \|_{L^\infty} \leq C e^{-\frac{s-\tau}{\rho}} \| q_{1,1}(\cdot) \|_{L^\infty} + C e^{s-\tau} \frac{s^2}{1 + \|y\|^3} \| q_{1,1}(\cdot) \|_{L^\infty} + \frac{C}{\sqrt{s}}, \]
\[ \left\| \frac{(\vartheta_{2,1}(s, \tau))_e}{1 + \|y\|^3} \right\|_{L^\infty} \leq C e^{-\frac{s-\tau}{\rho}} \left\| \frac{q_{2,1}(\cdot, \tau)}{1 + \|y\|^3} \right\|_{L^\infty} + \frac{C e^{s-\tau}}{s^2}, \]
\[ \| (\vartheta_{2,1}(s, \tau))_e \|_{L^\infty} \leq C e^{-\frac{s-\tau}{\rho}} \| q_{2,1}(\cdot) \|_{L^\infty} + C e^{s-\tau} \frac{s^2}{1 + \|y\|^3} \| q_{2,1}(\cdot) \|_{L^\infty} + \frac{C}{s^{\frac{3}{2}}}. \]

ii) The quadratic term $\vartheta_{1,2}(s, \tau)$ and $\vartheta_{2,2}(s, \tau)$
\[ \left\| \frac{(\vartheta_{1,2}(s, \tau))_e}{1 + \|y\|^3} \right\|_{L^\infty} \leq \frac{C(s - \tau)}{s^2}, \quad \| (\vartheta_{1,2}(s, \tau))_e \|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}, \]
\[ \left\| \frac{(\vartheta_{2,2}(s, \tau))_e}{1 + \|y\|^3} \right\|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}, \quad \| (\vartheta_{2,2}(s, \tau))_e \|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}. \]

iii) The correction terms $\vartheta_{1,3}(s, \tau)$ and $\vartheta_{2,3}(s, \tau)$
\[ \left\| \frac{(\vartheta_{1,3}(s, \tau))_e}{1 + \|y\|^3} \right\|_{L^\infty} \leq \frac{C(s - \tau)}{s^2}, \quad \| (\vartheta_{1,3}(s, \tau))_e \|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}, \]
\[ \left\| \frac{(\vartheta_{2,3}(s, \tau))_e}{1 + \|y\|^3} \right\|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}, \quad \| (\vartheta_{2,3}(s, \tau))_e \|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}. \]

iv) The correction terms $\vartheta_{1,4}(s, \tau)$ and $\vartheta_{2,4}(s, \tau)$
\[ \left\| \frac{(\vartheta_{1,4}(s, \tau))_e}{1 + \|y\|^3} \right\|_{L^\infty} \leq \frac{C(s - \tau)}{s^2}, \quad \| (\vartheta_{1,4}(s, \tau))_e \|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}, \]
\[ \left\| \frac{(\vartheta_{2,4}(s, \tau))_e}{1 + \|y\|^3} \right\|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}, \quad \| (\vartheta_{2,4}(s, \tau))_e \|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}. \]

v) The correction terms $\vartheta_{1,5}(s, \tau)$ and $\vartheta_{2,5}(s, \tau)$
\[ \left\| \frac{(\vartheta_{1,5}(s, \tau))_e}{1 + \|y\|^3} \right\|_{L^\infty} \leq \frac{C(s - \tau)}{s^2}, \quad \| (\vartheta_{1,5}(s, \tau))_e \|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}, \]
\[ \left\| \frac{(\vartheta_{2,5}(s, \tau))_e}{1 + \|y\|^3} \right\|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}, \quad \| (\vartheta_{2,5}(s, \tau))_e \|_{L^\infty} \leq \frac{C(s - \tau)}{s^{\frac{3}{2}}}. \]

**Proof.** The result is implied from the definition of the shrinking set $V_A(s)$ and Lemma 3.2 and the bounds for $V, V_{j,k}, B_1, B_2, R_1, R_2$ with $j, k \in \{1, 2\}$ which are shown in Lemmas A.2, A.3 and A.4. For details in a quite similar case, see Lemma 4.20 in Tayachi and Zaag [24].

Finally, the conclusion of (iii) and (iv) of Proposition 4.1 follows by using formular (4.10) and Lemma (4.3). This concludes the proof of Proposition 4.1.
4.2. Conclusion of the proof of Proposition 3.6

In this subsection, we will give prove a Proposition which implies Proposition 3.6 directly. More precisely, this is our statement:

**Proposition 4.4.** There exists $A_7 \geq 1$ such that for all $A \geq A_7$, there exists $s_7(A) \geq 1$ such that for all $s_0 \geq s_7(A)$, we have the following properties: If the following conditions hold:

a) $(q_1, q_2)(s_0) = (\phi_1, \phi_2)$ with $(d_0, d_1) \in D_{A,s_0}$,
b) For all $s \in [s_0, s_1]$ we have $(q_1, q_2)(s) \in V_A(s)$.

Then for all $s \in [s_0, s_1]$, we have

$$\forall i, j \in \{1, \cdots, n\}, \ |q_{2,i,j}(s)| \leq \frac{A^2 \ln s}{2s^2},$$

$$\left\| \frac{q_{1,-}(y,s)}{1+|y|^2} \right\|_{L^\infty} \leq \frac{A}{2s^2}, \ \left\| q_{1,e}(s) \right\|_{L^\infty} \leq \frac{A^2}{2\sqrt{s}},$$

$$\left\| \frac{q_{2,-}(y,s)}{1+|y|^2} \right\|_{L^\infty} \leq \frac{A^2}{2s^2}, \ \left\| q_{2,e}(s) \right\|_{L^\infty} \leq \frac{A^3}{2s^{\frac{3}{2}}},$$

where $D_{A,s_0}$ is introduced in Lemma 3.4 and $(\phi_1, \phi_2)$ is defined as in Definition (3.3).

**Proof.** The proof relies on Proposition 4.1 and details are similar to Proposition 4.7 of Merle and Zaag [15]. For that reason, we only give a short proof to (4.15). We use (4.3) to deduce that

$$\int_{s_0}^s (\tau^2 q_{1,k}(\tau)) d\tau \leq C(A \ln(s) - \ln(s_0)),$$

which yields

$$|q_{1,j,k}(s)| \leq CAs^{-2} \ln s \leq \frac{A^2 \ln s}{2s^2},$$

if $A \geq A_7$ large enough and $s \geq s_7(A)$. Then, (4.15) follows.

We here give the conclusion of the proof of Proposition 3.6:

**Proof.** From Proposition 4.4, if $(q_1, q_2)(s_1) \in \partial V_A(s_1)$ then:

$$(q_{1,0} (q_{1,j})_{1 \leq j \leq n}, q_{2,0}, (q_{2,j})_{1 \leq j \leq n}, (q_{2,j,k})_{1 \leq j,k \leq n}) (s_1) \in \partial \tilde{V}_A(s_1).$$

(4.18)

This concludes item (i) of Proposition 3.6.

The proof of item (ii) of Proposition 3.6. Thanks to (4.18), we derive two the following cases:

+ The first case: There exists $j_0 \in \{1, \cdots, n\}$ and $\epsilon_0 \in (-1, 1)$ such that either $q_{1,0}(s_1) = \epsilon_0 \frac{A}{s_1^2}$ or $q_{1,j_0} = \epsilon_0 \frac{A}{s_1^2}$ or $q_{2,0} = \epsilon_0 \frac{A^2}{s_1^{\frac{3}{2}}}$ or $q_{2,j_0}(s_1) = \epsilon_0 \frac{A^2}{s_1^{\frac{3}{2}}}$. Without loss of generality, we can suppose that $q_{1,0} = \epsilon_0 \frac{A}{s_1^2}$ (the other cases are similar). Then, by using (4.1), we can prove that the sign of $q_{1,0}'(s_1)$ is opposite to the sign of $\left(\epsilon_0 \frac{A}{s_1^2}\right)'$. In other words,

$$\epsilon_0 \left( q_{1,0} - \epsilon_0 \frac{A}{s_1^2} \right)'(s_1) > 0.$$

+ The second case: There exists $j_0, k_0, \epsilon_0 \in (-1, 1)$ such that $q_{2,j_0,k_0}(s_1) = \epsilon_0 \frac{A^2}{s_1^{\frac{3}{2}}}$, by using (4.4) we can prove that

$$\epsilon_0 \left( q_{2,j_0,k_0} - \epsilon_0 \frac{A^2}{s_1^{\frac{3}{2}}} \right)'(s_1) > 0.$$

Finally, we deduce that there exists $\delta_0 \geq 0$ such that for all $\delta \in (0, \delta_0)$ we have

$$(q_{1,0} (q_{1,j})_{1 \leq j \leq n}, q_{2,0}, (q_{2,j})_{1 \leq j \leq n}, (q_{2,j,k})_{1 \leq j,k \leq n}) (s_1 + \delta) \notin \tilde{V}_A(s_1 + \delta).$$

if $A \geq A_3$ and $s_0 \geq s_3(A)$ large enough. Then, the item (ii) of Proposition follows. Hence, we also derive the conclusion of Proposition 3.6.
A. Appendix

In this appendix, we state and prove several technical and straightforward results needed in our paper.

We first give a Taylor expansion of the quadratic terms defined in (2.8) and (2.9).

**Lemma A.1** (Asymptotics of $\tilde{B}_1$ and $\tilde{B}_2$). We consider $\tilde{B}_1(\tilde{w}_1, w_2)$ and $\tilde{B}_2(\tilde{w}_1, w_2)$ as defined in (2.8) and (2.9). Then, the following holds

\[
\tilde{B}_1(\tilde{w}_1, w_2) = \frac{p}{2\kappa} \tilde{w}_1^2 + O(|\tilde{w}_1|^3 + |w_2|^2),
\]

\[
\tilde{B}_2(\tilde{w}_1, w_2) = \frac{p}{\kappa} \tilde{w}_1 w_2 + O(|\tilde{w}_1|^2|w_2|) + O(|w_2|^3),
\]

as $(\tilde{w}_1, w_2) \to (0, 0)$.

**Proof.** Using the Newton binomial formula (remember that $\tilde{w}$), we derive that:

\[(\tilde{w}_1 + \kappa + iw_2)^p = (\tilde{w}_1 + \kappa)^p + ip(\tilde{w}_1 + \kappa)^{p-1}w_2 + p(p-1)(\tilde{w}_1 + \kappa)^{p-2}w_2^2 + G(\tilde{w}_1, w_2),\]

with

\[|G(\tilde{w}_1, w_2)| \leq C|w_2|^3, \quad \forall |\tilde{w}_1| + |w_2| \leq 1.\]

Then,

\[
\Re ((\tilde{w}_1 + \kappa + iw_2)^p) = (\tilde{w}_1 + \kappa)^p + p(p-1)(\tilde{w}_1 + \kappa)^{p-1}w_2 + \Re (G),
\]

\[
\Im ((\tilde{w}_1 + \kappa + iw_2)^p) = p(\tilde{w}_1 + \kappa)^{p-1}w_2 + \Im (G).
\]

Moreover, we apply again the Newton binomial formula to $(\kappa + \tilde{w}_1)^p, (\kappa + \tilde{w}_1)^{p-1}$ around $\tilde{w}_1 = 0$ and we get

\[(\kappa + \tilde{w}_1)^p = \kappa^p + \frac{p}{p-1} \tilde{w}_1 + \frac{p}{2\kappa} \tilde{w}_1^2 + O(|\tilde{w}_1|^3),\]

\[(\kappa + \tilde{w}_1)^{p-1} = \frac{1}{p-1} + \frac{1}{\kappa} \tilde{w}_1 + O(|\tilde{w}_1|^2).\]

Then, (A.1) follows by (A.3) and (A.5) and (A.2) follows by (A.4) and (A.6). \(\square\)

Now, we give an expansion of the potentials defined in (3.3) and (3.4) - (3.7). The following is our statement:

**Lemma A.2** (The potential functions $V$ and $V_{j,k}$ with $j, k \in \{1, n\}$). We consider $V, V_{1,1}, V_{1,2}, V_{2,1}$ and $V_{2,2}$ as defined in (3.3) and (3.4) - (3.7). Then, the following holds:

(i) For all $s \geq 1$ and $y \in \mathbb{R}^n$, we have $|V(y, s)| \leq C,$

\[|V(y, s)| \leq \frac{C(1 + |y|^2)}{s},\]

and

\[V(y, s) = -\left(\frac{|y|^2 - 2n}{4s}\right) + \tilde{V}(y, s),\]

where $\tilde{V}$ satisfies

\[|\tilde{V}(y, s)| \leq \frac{C(1 + |y|^4)}{s^2}, \forall s \geq 1, |y| \leq 2K \sqrt{s}.\]

(ii) For all $s \geq 1$ and $y \in \mathbb{R}^n$, the potential functions $V_{j,k}$ with $j, k \in \{1, 2\}$ satisfy

\[\|V_{1,1}\|_{L^\infty} + \|V_{2,2}\|_{L^\infty} \leq \frac{C}{s^2},\]

\[\|V_{1,2}\|_{L^\infty} + \|V_{2,1}\|_{L^\infty} \leq \frac{C}{s},\]

\[|V_{1,1}(y, s)| + |V_{2,2}(y, s)| \leq \frac{C(1 + |y|^4)}{s^4},\]

\[|V_{1,2}(y, s)| + |V_{2,1}(y, s)| \leq \frac{C(1 + |y|^2)}{s^2}.\]

**Proof.** We see that item (ii) is derived directly from the definition of $V_{j,k}$. In addition to that, the proof of (i) is quite similar to Lemma B.1, page 1270 in [18]. \(\square\)
Now, we give a Taylor expansion of the quadratics terms $B_1$ and $B_2$ given in (3.8) and (3.9).

**Lemma A.3** (The quadratic terms $B_1(q_1, q_2)$ and $B_2(q_1, q_2)$). We consider $B_1(q_1, q_2)$ and $B_2(q_1, q_2)$ as defined in (3.8) and (3.9) respectively. For all $A \geq 1$, there exists $s_8(A) \geq 1$ such that for all $s \geq s_8(A)$, if $(q_1, q_2)(s) \in V_A(s)$, then

\[
|B_1(q_1, q_2)| \leq C \left( |q_1|^2 + |q_2|^2 \right), \quad (A.10)
\]

\[
|B_2(q_1, q_2)| \leq C \left( \frac{|q_1|^2}{s} + |q_1 q_2| + |q_2|^2 \right). \quad (A.11)
\]

**Proof.** We first recall the two functions $F_1(u_1, u_2)$ and $F_2(u_1, u_2)$ which are defined in (2.3). As a matter of facts, they belong to $C^\infty(\mathbb{R}^2)$. Then, by applying a Taylor expansion to $F_1, F_2$, we obtain

\[
F_1(\Phi + q_1, \Phi + q_2) = \sum_{j,k \leq p} \frac{1}{j!k!} \partial^{j+k} u_1^{j} u_2^k F_1(\Phi_1, \Phi_2)q_1^j q_2^k,
\]

\[
F_2(\Phi + q_1, \Phi + q_2) = \sum_{j,k \leq p} \frac{1}{j!k!} \partial^{j+k} u_1^{j} u_2^k F_2(\Phi_1, \Phi_2)q_1^j q_2^k.
\]

Then, (A.10) and (A.11) follow by definition of $B_1$, $B_2$ and also the definition of the shrinking set $V_A(s)$. □

In the following lemma, we give various estimates involving the rest terms $R_1$ and $R_2$ defined in (3.10) and (3.11).

**Lemma A.4** (The rest terms $R_1, R_2$). For all $s \geq 1$, we consider $R_1, R_2$ defined in (3.10) and (3.11). Then,

(i) For all $s \geq 1$ and $y \in \mathbb{R}^n$

\[
R_1(y,s) = \frac{c_1}{s^2} + \tilde{R}_1(y,s),
\]

\[
R_2(y,s) = \frac{c_2}{s^3} + \tilde{R}_2(y,s),
\]

where $c_1, c_2$ are constants depended on $p$ and $\tilde{R}_1, \tilde{R}_2$ satisfy: for all $|y| \leq 2K \sqrt{s}$

\[
|\tilde{R}_1(y,s)| \leq \frac{C(1 + |y|^4)}{s^3},
\]

\[
|\tilde{R}_2(y,s)| \leq \frac{C(1 + |y|^6)}{s^4}.
\]

(ii) Moreover, we have for all $s \geq 1$

\[
\|R_1(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{s},
\]

\[
\|R_2(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{s^2},
\]

**Proof.** The proofs for $R_1$ and $R_2$ are quite similar. For that reason, we only give the proof of the estimates on $R_2$. This means that we need to prove the following estimates:

\[
R_2(y,s) = -\frac{n(n+4)\kappa}{(p-1)s^3} + \tilde{R}_2(y,s), \quad (A.12)
\]

with

\[
|\tilde{R}_2(y,s)| \leq \frac{C(1 + |y|^6)}{s^4}, \forall |y| \leq 2K \sqrt{s}
\]

and

\[
\|R_2(\cdot, s)\|_{L^\infty} \leq \frac{C}{s^2}. \quad (A.13)
\]

We first from (3.11), recall the definition of $R_2(y,s)$

\[
R_2(y,s) = \Delta \Phi_2 - \frac{1}{2} y \cdot \nabla \Phi_2 - \frac{\Phi_2}{p-1} + F_2(\Phi_1, \Phi_2) - \partial_y \Phi_2,
\]

Then, we can rewrite $R_2$ as follows

\[
R_2(y,s) = \Delta \Phi_2 - \frac{1}{2} y \cdot \nabla \Phi_2 - \frac{\Phi_2}{p-1} + p \Phi_1^{p-1} \Phi_2 - \partial_y \Phi_2 + R'_2(y,s),
\]
where

\[ R_2^*(y, s) = F_2(\Phi_1, \Phi_2) - p\Phi_1^{p-1}\Phi_2. \]

Using the definition of \( F_2 \) in (2.3), and the definitions of \( \Phi_1, \Phi_2 \) in (2.43) and (2.44), we derive that

\[ |R_2^*(y, s)| \leq \frac{C(1 + |y|^6)}{s^3}, \quad \forall |y| \leq 2K\sqrt{s}, \]

and

\[ \|R_2^*(y, s)\|_{L^\infty} \leq \frac{C}{s^2}. \]

In addition to that, we introduce \( \tilde{R}_2 \) as follows:

\[ \tilde{R}_2(y, s) = \Delta \Phi_2 - \frac{1}{2} y \cdot \nabla \Phi_2 - \frac{\Phi_2}{p-1} + p\Phi_1^{p-1}\Phi_2 - \partial_s \Phi_2. \]

Then, we may obtain the conclusion if the following two estimates hold:

\[ |\tilde{R}_2(y, s) + \frac{n(n+4)\kappa}{(p-1)s^2}| \leq \frac{C(1 + |y|^6)}{s^4}, \quad \forall |y| \leq 2K\sqrt{s}, \tag{A.14} \]

\[ \|\tilde{R}_2(., s)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{s^3}. \tag{A.15} \]

**+ The proof of (A.14):** We first aim at expanding \( \Delta \Phi_2 \) in a polynomial in \( y \) of order less than 4 via the Taylor expansion. Indeed, \( \Delta \Phi_2 \) is given by

\[
\Delta \Phi_2 = \frac{2n}{s^2} \left( p-1 + \frac{(p-1)^2|y|^2}{4ps} \right)^{-\frac{n\kappa}{s}} - (p-1)\frac{|y|^2}{s^3} \left( p-1 + \frac{(p-1)^2|y|^2}{4ps} \right)^{-\frac{2n\kappa}{s}} \]

\[ - \frac{(n+2)(p-1)|y|^2}{2s^3} \left( p-1 + \frac{(p-1)^2|y|^2}{4ps} \right)^{-\frac{2n\kappa}{s}} - \frac{(2p-1)(p-1)^2|y|^4}{4s^{4p}} \left( p-1 + \frac{(p-1)^2|y|^2}{4ps} \right)^{-\frac{2n\kappa}{s}}. \]

Besides that, we make a Taylor expansion in the variable \( z = \frac{|y|}{\sqrt{s}} \) for \( \left( p-1 + \frac{(p-1)^2|y|^2}{4ps} \right)^{-\frac{n\kappa}{s}} \) when \( |z| \leq 2K \), and we get

\[
\left( p-1 + \frac{(p-1)^2|y|^2}{4ps} \right)^{-\frac{n\kappa}{s}} - \frac{\kappa}{p-1} + \frac{\kappa}{4(p-1)} |y|^2 \leq \frac{C(1 + |y|^4)}{s^2}, \quad \forall |y| \leq 2K\sqrt{s}.
\]

which yields

\[
\left| \frac{2n}{s^2} \left( p-1 + \frac{(p-1)^2|y|^2}{4ps} \right)^{-\frac{n\kappa}{s}} - \frac{2n\kappa}{(p-1)s^2} + \frac{n\kappa|y|^2}{2(p-1)s^3} \right| \leq \frac{C(1 + |y|^4)}{s^4} \leq \frac{C(1 + |y|^6)}{s^4}, \quad \forall |y| \leq 2K\sqrt{s}.
\]

It is similar to estimate the other terms in \( \Delta \Phi_2 \) as the above. Finally, we obtain

\[
\left| \Delta \Phi_2 - \frac{2n\kappa}{(p-1)s^2} + \frac{n\kappa|y|^2}{(p-1)s^3} + 2\kappa \frac{|y|^2}{(p-1)s} \right| \leq \frac{C(1 + |y|^6)}{s^4}, \quad \forall |y| \leq 2K\sqrt{s}. \tag{A.16}
\]

As we did for \( \Delta \Phi_2 \), we estimate similarly the other terms in \( \tilde{R}_2 \): for all \( |y| \leq 2K\sqrt{s} \)

\[
\left| -\frac{1}{2} y \cdot \nabla \Phi_2 + \frac{\kappa|y|^2}{(p-1)s^2} - \frac{\kappa|y|^4}{4(p-1)s^3} \right| \leq \frac{C(1 + |y|^6)}{s^4}, \tag{A.17}
\]

\[
\left| \frac{-\Phi_2}{p-1} + \frac{\kappa|y|^2}{(p-1)s^2} - \frac{\kappa|y|^4}{4(p-1)^2s^3} \right| \leq \frac{C(1 + |y|^6)}{s^4}, \tag{A.18}
\]

\[
\left| p\Phi_1^{p-1}\Phi_2 - \frac{p\kappa|y|^2}{(p-1)^2s^2} + \frac{(2p-1)p\kappa|y|^4}{4(p-1)^2s^3} - \frac{n\kappa|y|^2}{(p-1)s^3} + \frac{2n\kappa}{(p-1)^2s^2} + \frac{n^2\kappa}{(p-1)s^3} \right| \leq \frac{C(1 + |y|^6)}{s^4}. \tag{A.19}
\]

\[
\left| -\partial_s \Phi_2 - \frac{2n\kappa|y|^2}{(p-1)s^3} + \frac{4n\kappa}{(p-1)s^3} \right| \leq \frac{C(1 + |y|^6)}{s^4}. \tag{A.20}
\]

Thus, we use (A.16), (A.17), (A.18), (A.19) and (A.20) to deduce the following

\[
\left| \tilde{R}_2(y, s) + \frac{n(n+4)\kappa}{(p-1)s^2} \right| \leq \frac{C(1 + |y|^6)}{s^4}, \quad \forall |y| \leq 2K\sqrt{s},
\]
and (A.14) follows

+ The proof (A.15): We rewrite $\Phi_1, \Phi_2$ as follows

$$
\Phi_1(y,s) = R_{1,0}(z) + \frac{nk}{2ps} \quad \text{and} \quad \Phi_2(y,s) = \frac{1}{s} R_{2,1}(z) - \frac{2nk}{(p-1)s^2} \quad \text{where} \quad z = \frac{y}{\sqrt{s}},
$$

where $R_{1,0}$ and $R_{2,1}$ are defined in (2.35) and (2.39), respectively. In addition to that, we rewrite $\bar{R}_2$ in terms of $R_{1,0}$ and $R_{2,1}$, and we note that $R_{1,0}$ and $R_{2,1}$ satisfy (2.31) and (2.33). Then, it follows that

$$
|\bar{R}_2(y,s)| \leq \frac{C}{s^2}, \forall y \in \mathbb{R}^n.
$$

Hence, (A.15) follows. This concludes the proof of this Lemma. \qed

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**Address:** Paris 13 University, Institute Galilée, Laboratory of Analysis, Geometry and Applications, CNRS UMR 7539, 95302, 99 avenue J.B Clément, 93430 Villetaneuse, France

e-mail: duong@math.univ-paris13.fr