Quasi Grand Lebesgue Spaces

Maria Rosaria FORMICA 1, Eugeny OSTROVSKY 2 and Leonid SIROTA 3

1 Università degli Studi di Napoli “Parthenope”, via Generale Parisi 13, Palazzo Pacanowsky, 80132, Napoli, Italy.
e-mail: mara.formica@uniparthenope.it

2, 3 Bar-Ilan University, Department of Mathematics and Statistics, 52900, Ramat Gan, Israel.
e-mail: eugostrovsky@list.ru
e-mail: sirota3@bezeqint.net

Abstract

We introduce a new class of quasi-Banach spaces as an extension of the classical Grand Lebesgue Spaces for “small” values of the parameter, and we investigate some of its properties, in particular, completeness, fundamental function, operators estimates, Boyd indices, contraction principle, tail behavior, dual space, generalized triangle and quadrilateral constants and inequalities.

Keywords: Lebesgue-Riesz spaces, quasi-Banach spaces, quasi Grand Lebesgue Spaces, tail function, slowly varying function, degenerate space, dual space, fundamental function, contraction principle, Hardy and other operators, generalized triangle and quadrilateral inequalities and constants.

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1 Introduction

Let \( X \) be a vector (linear) space. A function \( \| \cdot \| : X \to [0, \infty) \) is said a quasi-norm if

\[
\|x\| \geq 0, \quad \forall x \in X; \quad \|x\| = 0 \iff x = 0; \quad (1.1)
\]

\[
\|\alpha x\| = |\alpha| \cdot \|x\|, \quad \forall x \in X, \ \alpha \in \mathbb{R}. \quad (1.2)
\]

\[
\exists C \in [1, \infty) : \quad \|x + y\| \leq C(\|x\| + \|y\|), \quad \forall x, y \in X \quad (1.3)
\]
The space $X$, equipped with the quasi-norm $\| \cdot \| = \| \cdot \|_X$, is called \textit{quasi-normed space}. Of course, if $C = 1$ in (1.3), then $\| \cdot \|$ is a norm, i.e. $X$ is a normed space.

This definition was introduced by Tosio Aoki in [5]; see also [10, 24, 28, 32, 33, 34, 46, 48].

We highlight the paper of Lech Maligranda [32], which will very useful for our purposes. A modern survey of the theory of these spaces, containing new results and applications, may be found in the recent article [35] (see also reference therein).

The smallest possible constant $C = C(X)$ in (1.3) is called the \textit{quasi-triangle constant} of the space $X = (X, \| \cdot \|)$.

If, in addition, we have for some (constant) value $p \in (0, 1]$,\[ \|x + y\|^p \leq \|x\|^p + \|y\|^p, \tag{1.4}\]
then the functional $x \mapsto \|x\|$ is called a \textit{p-norm} and the value $p$ is named as \textit{power parameter}.

It is known, thanks to Aoki-Rolewicz theorem (see [5, 46]), that if $(X, \| \cdot \|)$ is a complete quasi-normed space, then there exists a constant $p \in (0, 1)$ and a $p$-norm $\|x\|'$ in $X$ equivalent to the source one $\| \cdot \|$.

More precisely, if $C$ is the quasi-triangle constant, then $p$ may be determined by the relation $C = 2^{1/p-1}$ and\[ \|x\|' \leq \|x\| \leq 2^{1/p}\|x\|', \quad \forall x \in X, \tag{1.6}\]
(see also [23]).

Let now and in the sequel $(Q, B, \mu)$ be a non-trivial measure space with sigma-finite measure $\mu$.

An important class of quasi-Banach spaces which are not Banach spaces is the class of $L_p$ spaces, for $0 < p < 1$, with the usual quasi-norm\[ \|f\|_p := \left[ \int_Q |f(x)|^p \mu(dx) \right]^{1/p}. \tag{1.5}\]
In this case\[ \|f + g\|_p \leq 2^{1/p-1}(\|f\|_p + \|g\|_p), \tag{1.6}\]
i.e., the quasi-triangle constant of $L_p$ is $2^{1/p-1}$ (see [32]).

Note, in addition, that in this case\[ \|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p. \tag{1.7}\]
Both the estimates (1.6) and (1.7) follows immediately from the double inequality\[ (A + B)^p \leq A^p + B^p \leq 2^{1-p}(A + B)^p, \quad A, B \geq 0, \quad 0 < p < 1. \tag{1.8}\]

We introduce in this paper a new class of quasi-Banach spaces, as an extension of the known Grand Lebesgue Spaces, but for “small” values of the degree parameter (see Section 3), and we investigate some its properties.
2 Ordinary Grand Lebesgue Spaces

We recall here briefly the definition of (ordinary) Grand Lebesgue Space (GLS).

Let \((Q = \{x\}, B, \mu)\) be a measure space with non-trivial sigma-finite measure \(\mu\) and let \(\psi(p) = \psi_{\alpha, \beta}(p), \ p \in (\alpha, \beta), \) where \(1 \leq \alpha < \beta \leq \infty, \) be a continuous, strictly positive, numerical valued function in the open interval such that
\[
\inf_{p \in (\alpha, \beta)} \psi(p) > 0.
\]

The Grand Lebesgue Space (GLS) \(G\psi = G\psi_{\alpha, \beta}\) is a rearrangement invariant Banach function space in the classical sense ([8, Chapters 1,2]) and consists of all the measurable functions \(f : Q \to \mathbb{R}\) having finite norm
\[
\|f\|_{G\psi} = \|f\|_{G\psi_{\alpha, \beta}} \overset{\text{def}}{=} \sup_{p \in (\alpha, \beta)} \left\{ \frac{\|f\|_p}{\psi(p)} \right\}.
\]

Remark 2.1. Define \(\psi(p) = (\beta - p)^{\theta}, \ p \in (1, \beta), \ 1 < \beta < \infty, \ \theta \geq 0.\) Let \(Q \subset \mathbb{R}^n, \ n \geq 1,\) be a measurable set with finite Lebesgue measure. Replace \(p\) with \(\beta - \varepsilon, \) where \(\varepsilon \in (0, \beta - 1);\) then \(p \in (1, \beta)\) is equivalent to write \(1 < \beta - \varepsilon < \beta.\) Therefore \(\psi(p) = \psi(\beta - \varepsilon) = \varepsilon^{-\frac{\theta}{\beta - \varepsilon}}\) and the \(G\psi\) norm of any measurable function \(f : Q \to \mathbb{R}\) takes the well-known form
\[
\|f\|_{G\psi} = \sup_{0 < \varepsilon < \beta - 1} \varepsilon^{-\frac{\theta}{\beta - \varepsilon}} \|f\|_{\beta - \varepsilon},
\]
introduced in [26] and denoted by \(\|f\|_{L^{\beta, \theta}} (\beta > 1, \ \theta \geq 0).\)

The Grand Lebesgue Spaces spaces have been studied (and applied) in huge numbers of works, see. e.g. [19, 20, 1, 2, 3, 9, 14, 15, 16, 17, 29, 30, 36, 42, 38, 43], etc.

We note that the \(G\psi\) spaces are also interpolation spaces (the so-called \(\Sigma\)-spaces), see [27].

The theory of GLS allows to investigate the exponential decreasing tail behavior of measurable functions, as well as estimates of its Orlicz norms, which is used in Probability theory, Functional Analysis, theory of Partial Differential Equations, etc.

3 Quasi-Grand Lebesgue Spaces. Some properties

Let again \((Q = \{x\}, B, \mu)\) be a measure space with non-trivial sigma-finite measure \(\mu\) and let \(\psi(p) = \psi_{a, b}(p), \ p \in (a, b),\) where now
\[
0 < a < b \leq 1,
\]
(3.1)
be a continuous, strictly positive, numerical valued function in the open interval \((a, b)\), such that
\[
\inf_{p \in (a, b)} \psi(p) > 0.
\]

The Grand Lebesgue Space (GLS) \(G\psi = G\psi_{a,b}\) consists of all the measurable functions \(f : Q \to \mathbb{R}\) such that
\[
\|f\|_{G\psi} = \|f\|_{G\psi_{a,b}} \overset{\text{def}}{=} \sup_{p \in (a, b)} \left\{ \frac{\|f\|_p}{\psi(p)} \right\} < \infty. \tag{3.2}
\]

In contradistinction to the foregoing definition of ordinary GLS, given in Section 2, these spaces are not Banach spaces. They are only quasi-Banach local bounded \(F\) spaces, as proven in next Proposition 3.1.

By definition, the following \(\psi\)-function
\[
\psi_f(p) \overset{\text{def}}{=} \frac{\|f\|_p}{p}, \quad p \in (a, b),
\]
is named **natural function** for the real valued function \(f\).

Evidently \(f \in G\psi_f\) and
\[
\|f\|_{G\psi_f} = 1.
\]

**Example 3.1. Natural function.**

Let \(f(x) = f_{\Delta, \delta, L}(x), \ x \in (0, 1),\) be the following function
\[
f(x) := x^{-\Delta} |\ln x|^{\delta} L(|\ln x|), \ \Delta = \text{const} > 1, \ \delta = \text{const} \geq 0,
\]
where \(L = L(y), \ y \in (0, \infty),\) is a positive continuous **slowly varying** function as \(y \to \infty.\)

The natural function for \(f\) is
\[
\psi_f(p) = \|f\|_p, \quad p \in (0, 1/\Delta).
\]

Obviously \(f \in G\psi_f\) and
\[
\|f\|_{G\psi_f} = 1.
\]

We observe, after some calculations, that the natural function \(\psi_f(p) = \|f\|_p\) for the function \(f\) defined above has the form, for \(p \in (0, 1/\Delta),\)
\[
\psi_f(p) = \|f\|_p \asymp \Gamma^{1/p}(p\delta + 1) \cdot L\left(\frac{1}{1 - p\Delta}\right) \cdot (1 - p\Delta)^{-\Delta - 1/p},
\]
where \(\Gamma(\cdot)\) denotes the Euler’s Gamma function.

It is easy to see that
\[
\lim_{p \to 0^+} \psi_f(p) = \lim_{p \to 0^+} \|f\|_p = L(1) \exp (\Delta - \delta \gamma),
\]
where \(\gamma\) is the Euler-Mascheroni constant
\[
\gamma = - \int_0^\infty \ln z \ e^{-z} \ dz \approx 0.577215667.\]
Remark 3.1. More generally it is known that, for a function \( f : (0, 1) \to \mathbb{R} \),

\[
\lim_{p \to 0^+} \|f\|_p = \exp \left( \int_0^1 \ln|f(x)| \, dx \right).
\]

Namely, when \( f(x) > 0, \ x \in (0, 1) \), then

\[
\int_0^1 f^p(x) \, dx = \int_0^1 \exp(p \ln f(x)) \, dx ^{\frac{p}{p-0^+}} \left[ 1 + p \int_0^1 \ln f(x) \, dx \right],
\]

so

\[
\lim_{p \to 0^+} \|f\|_p = \lim_{p \to 0^+} \left[ 1 + p \int_0^1 \ln f(x) \, dx \right]^{1/p} = \exp \left[ \int_0^1 \ln f(x) \, dx \right].
\]

Remark 3.2. The particular case of these spaces, was introduced in [7], where applications for the study of Hardy operators were given.

More precisely, let \( Q = (0, 1), \ a = \frac{b}{2}, 0 < b \leq 1 \). Define \( \psi(p) = (b - p)^{-\frac{1}{p}}, \ p \in \left( \frac{b}{2}, b \right) \).

Let \( \varepsilon \in (0, b/2) \) and replace \( p \) with \( b - \varepsilon \) in the function \( \psi(p) \). Therefore we have \( \psi(p) = \psi(b - \varepsilon) = \varepsilon^{-\frac{1}{p-1}}, \ \varepsilon \in \left( 0, \frac{b}{2} \right), \ 0 < b \leq 1 \), and the \( G\psi \) norm of any measurable function \( f : (0, 1) \to \mathbb{R} \) is given by

\[
\|f\|_{G\psi} = \sup_{0<\varepsilon<\frac{b}{2}} \varepsilon^{\frac{1}{p-1}} \|f\|_{b-\varepsilon}, \quad (3.3)
\]

so we recover the quasi-norm introduced in [7], denoted by \( L^b(0,1), \ (0 < b \leq 1) \).

Proposition 3.1. The Grand Lebesgue space \( G\psi = G\psi_{a,b}, \ 0 < a < b \leq 1 \), is a quasi-normed space, having quasi-triangle constant

\[
C = C[\psi_{a,b}] = 2^{1/a-1}; \quad (3.4)
\]

i.e., let \( f, g \in G\psi \), then

\[
\|f + g\|_{G\psi} \leq 2^{1/a-1} (\|f\|_{G\psi} + \|g\|_{G\psi}).
\]

Proof.

Let \( f, g \in G\psi \). The first two properties of the quasi-norm (1.1) and (1.2) trivially holds. Now we have

\[
\|f + g\|_p^p = \int_Q |f(x) + g(x)|^p \, \mu(dx).
\]

Applying inequality (1.8) we have

\[
\|f + g\|_p^p \leq \int_Q |f(x)|^p \, \mu(dx) + \int_Q |g(x)|^p \, \mu(dx) = \|f\|_p^p + \|g\|_p^p,
\]
so we get inequality (1.7). Furthermore, using once again (1.8), we deduce
\[ \|f + g\|_p \leq 2^{1/p-1} (\|f\|_p + \|g\|_p). \] (3.5)

Since \( f, g \in G\psi \) then \( \|f\|_{G\psi} < \infty \) and \( \|g\|_{G\psi} < \infty \). Therefore
\[ \|f\|_p \leq \|f\|_{G\psi} \cdot \psi(p), \quad \|g\|_p \leq \|g\|_{G\psi} \cdot \psi(p), \quad p \in (a, b). \] (3.6)

By (3.5) and (3.6) it follows
\[ \|f + g\|_p \leq 2^{1/p-1} (\|f\|_{G\psi} + \|g\|_{G\psi}) \psi(p), \quad p \in (a, b). \]

Taking into account that \( p > a > 0 \), we get
\[ \|f + g\|_{G\psi} \leq 2^{1/a-1} (\|f\|_{G\psi} + \|g\|_{G\psi}), \quad p \in (a, b). \]

as desired. \( \square \)

The proof of completeness of the \( G\psi_{a,b} \) spaces, \( 0 < a < b \leq 1 \), is given in the next Section. We also observe that \( G\psi_{a,b} \) spaces are rearrangement invariant.

Let us discuss now the fundamental function for these spaces. We suppose here that the measure \( \mu \) is not trivial (non-zero) and atomless.

Recall that the fundamental function of an arbitrary \( G\psi_{a,b} \) space has the form
\[ \phi_{G\psi_{a,b}}(\delta) = \sup_{p \in (a,b)} \left\{ \frac{\delta^{1/p}}{\psi(p)} \right\}, \quad \delta \in [0, \mu(Q)]. \] (3.7)

This function plays a very important role in functional analysis, theory of Fourier series, etc. It was investigated in detail especially for ordinary GLS in [43].

We extrapolate this notion (3.7) one-to-one also in the case of quasi-Grand Lebesgue Spaces, i.e. when \( 0 < a < b \leq 1 \).

Denote
\[ \bar{\psi} := \sup_{p \in (a,b)} \psi(p), \quad \underline{\psi} := \inf_{p \in (a,b)} \psi(p). \]

Recall that by definition \( \bar{\psi} > 0 \).

We state the following simple bilateral estimate for the fundamental function for the values \( \delta \in [0,1] \).

**Proposition 3.2.** The fundamental function \( \phi_{G\psi_{a,b}}(\delta) \) of the GLS space \( G\psi_{a,b} \), with \( 0 < a < b \leq 1 \), satisfies, for the values \( \delta \in [0,1] \),
\[ \frac{\delta^{1/a}}{\bar{\psi}} \leq \phi_{G\psi_{a,b}}(\delta) \leq \frac{\delta^{1/b}}{\underline{\psi}}. \] (3.8)

Of course, the left-hand side of the estimate (3.8) has a sense only if \( \bar{\psi} < \infty \).
4 Completeness.

In this Section we prove that the quasi Grand Lebesgue spaces introduced in Section 3 are complete.

We will use a well known criterion of completeness of quasi-normed spaces, proved by Lech Maligranda, which we recall here for convenience of the reader.

Theorem 4.1. [32, Theorem 1.1]
A quasi-normed space $X = (X, \| \cdot \|)$ with a quasi-triangle constant $C \geq 1$ is complete (quasi-Banach space) if and only if, for every series such that

$$\sum_{k=1}^{\infty} C^k \|x_k\| < \infty,$$

we have $\sum_{k=1}^{\infty} x_k \in X$ and

$$\left\| \sum_{k=1}^{\infty} x_k \right\| \leq C \sum_{k=1}^{\infty} C^k \|x_k\|.$$

Now we state our result.

Theorem 4.2. Let $(Q = \{z\}, B, \mu)$ be a measure space with non-trivial sigma-finite measure $\mu$. The quasi-Grand Lebesgue Space $G_{\psi_{a,b}}$, $0 < a < b \leq 1$, builded on $(Q, B, \mu)$, is complete with respect to the quasi-distance $\rho(x,y) := \|x-y\|_{G_{\psi_{a,b}}}$.

Proof.

Let us consider first the space $L_p$, $0 < p < 1$; the case $p = 1$ is trivial. We use Theorem 4.1, which we transform as follows.

Note that for $L_p$ spaces, with $0 < p < 1$, the metric used is $\rho(x,y) = \|x-y\|_p^p$, i.e. without the $p$-th root of the integral, since in this case we have, for $x, y, w \in L_p$,

$$\rho(x,y) = \|x-y\|_p^p = \|(x-w) + (w-y)\|_p^p$$

$$\leq \int_Q |x-w|^p d\mu(z) + \int_Q |w-y|^p d\mu(z)$$

$$= \rho(x,w) + \rho(w,y)$$

with constant $C = 1$.

Assume that for an arbitrary sequence $\{x_k\}$, $k = 1, 2, \ldots$, $x_k \in L_p$ such that

$$S := \sum_{k=1}^{\infty} \|x_k\|_p^p < \infty, \quad (4.1)$$

the sum

$$x := \sum_{k=1}^{\infty} x_k \quad (4.2)$$
belongs to the space $L_p$ and moreover
\[ \|x\|_p^p \leq \sum_{k=1}^{\infty} \|x_k\|_p^p. \] (4.3)

Then actually the quasi-metric space $(L_p, \rho)$ is complete.

So, let $S < \infty$ or equally
\[ \sum_{k=1}^{\infty} \int_Q |x_k(z)|^p \mu(dz) < \infty. \]

It follows immediately from the theorem of Beppo-Levi that the sum $\sum_{k=1}^{\infty} |x_k(z)|^p$ is integrable. Therefore
\[ \|x\|_p^p \leq \sum_{k=1}^{\infty} \|x_k\|_p^p = S < \infty. \]

Let us consider now the case of the space $G\psi = G\psi_{a,b}$, $0 < a < b < 1$. Let $C = 2^{1/a-1} \in (1, \infty)$ the quasi-triangle constant (as seen in Proposition 3.1) and let \{x_k\}, $k = 1, 2, \ldots$, be any sequence of the elements of the space $G\psi_{a,b}$ such that
\[ \Theta := \sum_{k=1}^{\infty} C^k \|x_k\|_{G\psi_{a,b}} < \infty. \]

We have
\[ \Theta = \sum_{k=1}^{\infty} C^k \sup_{p \in (a,b)} \left[ \frac{\|x_k\|_p}{\psi(p)} \right] < \infty, \]
\[ \sum_{k=1}^{\infty} C^k \|x_k\|_p \leq \Theta \psi(p). \]

As we yet know, the value $x$ there exists as element of the space $L_p(Q)$ for all the values $p \in (a, b)$ and moreover
\[ \|x\|_p \leq C\Theta \psi(p). \]

Consequently, $x \in G\psi$ and $\|x\|_{G\psi} \leq C\Theta$.

This completes the proof. \(\Box\)

## 5 Dual space

We suppose in this Section that the measure $\mu$ is atomless and that the function $\psi(p) = \psi_{a,b}(p)$, $p \in (a, b)$, $0 < a < b < 1$, is described as before: it is continuous and strictly positive in $(a, b)$ and $\inf_{p \in (a, b)} \psi(p) > 0$. 
We intend here to investigate the dual (conjugate) space of the quasi-GLS space $G\psi = G\psi_{a,b}$; more precisely, we will prove its essential absence.

It is known that the dual space $L^*_p$ of the $L_p$ space, $p \in (0, 1)$, is degenerate: it consists only of the null element (see [47, pp. 36-37]). It is natural to expect that the same conclusion is also true for the quasi-Grand Lebesgue Spaces.

**Theorem 5.1.** The dual space $[G\psi_{a,b}]^*$ of the quasi-Grand Lebesgue Space $G\psi_{a,b}$, $0 < a < b < 1$, builted on a measure space $(Q, B, \mu)$, with atomless sigma-finite measure $\mu$, is degenerate, that is

$$ [G\psi_{a,b}]^* = \{0\}. \quad (5.1) $$

As a consequence, the associate space $[G\psi_{a,b}]'$ is also degenerate.

**Proof.**

We will follow mainly arguments like the ones described in the book of W. Rudin [47, §1.47, pp. 36-37]. Denote, for brevity, by $\phi(\delta)$, $\delta \in (0, 1)$, the fundamental function $\phi_{G\psi_{a,b}}(\delta)$ for the space $G\psi = G\psi_{a,b}$, defined in (3.7).

We claim that $G\psi$ contains no convex open sets, other than $\emptyset$ and $G\psi$.

To prove this, suppose $V \neq \emptyset$ is an arbitrary open and convex set in $G\psi$. Assume, without loss of generality, that $0 \in V$. Then, for some positive finite number $r$, $B_r \subset V$, where $B_r$ is the centered open ball in $G\psi$:

$$ B_r = \{ g \in G\psi : \|g\|_{G\psi} < r \}. $$

Let us pick an arbitrary non-zero function $f \in G\psi$ such that $\|f\| = \|f\|_{G\psi} \in (0, \infty)$.

Let $n \in \mathbb{N}$, $n \geq 2$, and introduce the following measurable partition $P = P_n$ of the whole set $Q$:

$$ P_n = \{ A_n(i) \}, \quad i = 1, 2, \ldots, n; \quad i \neq j : A_n(i) \cap A_n(j) = \emptyset, \quad (5.2) $$

$$ \bigcup_{i=1}^{n} A_n(i) = Q, \quad (5.3) $$

and a corresponding double sequence of (measurable) functions

$$ g_{i,n} = n \cdot f \cdot I_{A_n(i)}, $$

where $I_{A_n(i)}$ denotes the indicator function of the sets $A_n(i)$, such that

$$ \|g_{i,n}\| = n \cdot \phi(1/n) \cdot \|f\|. $$

By the right-hand side of (3.8) in Proposition 3.2 we have

$$ \phi(1/n) \leq \frac{n^{-1/b}}{\psi}, \quad (5.4) $$
and we get
\[ \forall i = 1,2,\ldots,n \Rightarrow \| g_{i,n} \| \leq \frac{n^{1-1/b}}{\psi} \cdot \| f \| \leq r, \quad n > 2, \]
since \( n^{1-1/b} \to 0 \) as \( n \to \infty \). On the other words, for all the large values \( n, \quad n > 2 \),
\[ \forall i = 1,2,\ldots,n \Rightarrow g_{i,n} \in B_r \subset V. \]
On the other hand,
\[ f = \frac{1}{n} \sum_{i=1}^{n} g_{i,n}. \]
As long as the set \( V \) is convex, we deduce from the last equality that \( f \in V \).

Thus, an arbitrary non empty convex set in the space \( G_\psi \) coincides with the whole space \( G_\psi \). The lack of convex open sets implies, as consequence, that the dual space of \( G_\psi \) is degenerate, that is \( [G_\psi]^* = \{0\} \) (see [47, pp. 36-37]).

\[ \square \]

Theorem 5.1 may be generalized as follows.

**Theorem 5.2.** Let \( Y = (Y, \| \cdot \|_Y) \) be an arbitrary quasi-Banach rearrangement invariant function space builded over an atomless sigma-finite measure space \((Q,B,\mu)\). Let \( f \in Y \) be any non-zero function such that \( 0 < \| f \|_Y < \infty \) and define the following generalization of the notion of the fundamental function (weighted version):
\[ \phi_{Y,f}(\delta) \overset{\text{def}}{=} \sup_{A \in B, \mu(A) \leq \delta} \| f \cdot I_A \|_Y, \quad \delta \in (0,\infty). \]
Evidently,
\[ \phi_{Y,f}(\delta) \leq \text{ess sup}_{v \in Q} |f(v)| \cdot \phi_Y(\delta), \]
where \( \phi_Y \) is the ordinary fundamental function.

Consider some partition of the whole space \( Q \) containing \( n \) elements
\[ P_n = P_n(\{A_n(i)\}, \quad i = 1,2,\ldots,n; \quad n = 2,3,\ldots, \]
and denote the collection of such partitions with \( S(n) := \{P_n\} \). Define the following variables:
\[ H[P_n,f](Y) \overset{\text{def}}{=} \inf_{\{A_n(i)\} \in P_n} \max_{i=1,2,\ldots,n} \phi_{Y,f}(\mu(A_n(i))), \quad (5.5) \]
\[ \overline{H}_n[Y](f) \overset{\text{def}}{=} \inf_{P_n \in S(n)} H[P_n,f](Y). \quad (5.6) \]

We assert that, if for an arbitrary non-zero function \( f \in Y \), we have
\[ \lim_{n \to \infty} n \overline{H}_n[Y](f) = 0, \quad (5.7) \]
then the dual space \( Y^* \) of \( Y \) is degenerate, i.e. \( Y^* = \{0\} \).
Proof.
Suppose $V \subset Y$, $V \neq Y$, is an arbitrary centered (i.e. $0 \in V$) non-trivial convex set and suppose that $\exists r \in (0, \infty)$ such that $B_r \subset V$, where
$$B_r = \{ g \in Y : \|g\|_Y < r \}.$$ 
Let $f \in Y$ be any non-zero function such that $0 < \|f\|_Y < \infty$.

As in Theorem 5.1, let $n \geq 2$ be a certain "great" natural number and let $P_n = \{ A_n(i) \}$ be an optimal partition in the sense of (5.5), (5.6):
$$H[P_n, f](Y) = \Pi_n[Y](f).$$

We pick
$$g_{i,n} = n \cdot f \cdot I_{A_n(i)},$$
so that, as $n \to \infty$,
$$\|g_{i,n}\|_Y = n \phi_{Y,f}(\mu(A_n(i))) \leq n \Pi_n[Y](f) \to 0,$$
by virtue of the condition (5.7).
Therefore, for $n$ sufficiently "large", $g_{i,n} \in B_r \subset V$, $\forall i = 1, \ldots , n$.

We observe that
$$f = \frac{1}{n} \sum_{i=1}^{n} g_{i,n},$$
i.e. the (arbitrary) function $f$ belongs to the convex linear shell of the elements $g_{i,n}$ from the set $V$. As long as the set $V$ is convex, we conclude $f \in V$, therefore $V = Y$, in contradiction.

Thus, an arbitrary non empty convex set in the space $Y$ coincides with the whole space $Y$. For the same arguments at the end of the previous Theorem, we conclude that the dual space of $Y$ is degenerate, that is $Y^* = \{0\}$.

\[\square\]

Remark 5.1. As a particular case of $G\psi$ spaces, thanks to Remark 3.2, we have that the dual of the grand Lebesgue space $L^b(0, 1)$, $0 < b < 1$, described in [7], is $[L^b(0, 1)]^* = \{0\}$.

For the interested reader, the dual of the grand Lebesgue space $L^b(0, 1)$, $b > 1$, has been described in [16] (see also [12, 13, 4]), while for the dual of the general (ordinary) GLS spaces $G\psi = G\psi_{\alpha,\beta}$, $1 \leq \alpha < \beta \leq \infty$, see [39, 44].

6 Other results.

6.1 Connections between tail behavior and quasi Grand Lebesgue Norm

Let $(Q, B, \mu)$ be a non-trivial measure space with sigma-finite measure $\mu$. We define, for an arbitrary measurable function $f : Q \to \mathbb{R}$, its so-called tail function
\[ T_f(u) \overset{\text{def}}{=} \mu \{ z : |f(z)| \geq u \}, \quad u > 0. \]

It follows from Tchebychev inequality, analogously as done in [38],

\[ T_f(u) \leq \frac{\|f\|_p^p}{u^p} \leq \frac{\|f\|_{G_\psi}^p}{u^p} \psi^p(p), \]

therefore

\[ T_f(u) \leq \inf_{p \in (a,b)} \left\{ \frac{\|f\|_p^p}{u^p} \right\} \leq \inf_{p \in (a,b)} \left\{ \frac{\|f\|_{G_\psi}^p}{u^p} \psi^p(p) \right\}, \quad (6.1) \]

where 0 < a < b ≤ 1.

Conversely, introduce the following \( \psi \) function

\[ \psi_0(p) := \left[ p \int_0^\infty u^{p-1} T_f(u) \, du \right]^{1/p} = \|f\|_p, \]

If the last integral converges in some interval \( (a,b) \), 0 < a < b ≤ 1, then

\[ f \in G_{\psi_0}, \quad \|f\|_{G_{\psi_0}} = 1. \quad (6.2) \]

Let us show an example.

**Example 6.1.** Define the following tail function

\[ T^{b,\gamma,L}(x) = x^{-b} (\ln x)^\gamma L(\ln x), \quad x \geq e, \]

where \( b \in (0,1), \ \gamma = \text{const} \geq 0, \ \mu(Q) = 1, \ L = L(y), \ y \geq 1, \) is a positive continuous *slowly varying* function as \( y \to \infty \). We deduce after some calculations, alike ones in [30], that if a measurable function (random variable) \( \xi \geq 1 \), defined on some probability space, is such that

\[ T_\xi(x) \leq C T^{b,\gamma,L}(x), \quad x \geq e, \quad (6.3) \]
then, for \( p \in (0, b) \), \( b \in (0, 1) \), \( x \geq e \)

\[
\|\xi\|_p = \left[ p \int_0^\infty x^{p-1} T_\xi(x) \, dx \right]^{1/p} \\
\leq C \left[ p \int_0^\infty x^{p-1} x^{-b} (\ln x)^{\gamma} L(\ln x) \, dx \right]^{1/p} \\
\leq C \left[ p \int_0^\infty x^{p-1} x^{-b} (\ln x)^{\gamma} L(\ln x) \, dx \right]^{1/p} \\
= C \left[ p \int_0^\infty e^{-y(b-p)} y^{\gamma} L(y) \, dy \right]^{1/p} \\
= C \left[ p \int_0^\infty e^{-z} (b-p)^{-\gamma} z^{\gamma} L\left(\frac{z}{b-p}\right) (b-p)^{-1} \, dz \right]^{1/p} \\
\sim C \left[ p(b-p)^{-\gamma} (b-p)^{-\gamma} L\left(\frac{1}{b-p}\right) \int_0^\infty e^{-z} z^{\gamma} \, dz \right]^{1/p} \quad \text{(as } p \to b^-) \\
= C \left[ p (b-p)^{-(\gamma+1)/p} L\left(\frac{1}{b-p}\right) \Gamma(\gamma + 1) \right]^{1/p} \\
= C_1(b, \gamma, L) (b-p)^{-(\gamma+1)/p} L^{1/p}(1/(b-p)) \\
\asymp C_2(b, \gamma, L) (b-p)^{-(\gamma+1)/b} L^{1/b}(1/(b-p)) \\

In conclusion

\[
\|\xi\|_p \leq C_2(b, \gamma, L) (b-p)^{-(\gamma+1)/b} L^{1/b}(1/(b-p)) =: C_2 \psi(p) \quad (6.4)
\]

We state that the random variable \( \xi \) belongs to the quasi-Grand Lebesgue Space \( G\psi \), where

\[
\psi(p) = (b-p)^{-(\gamma+1)/b} L^{1/b}(1/(b-p)), \quad p \in (0, b), \quad b \in (0, 1).
\]

Inversely, suppose that the tail estimate

\[
\|\xi\|_p \leq C_1(b, \gamma, L) (b-p)^{-(\gamma+1)/p} L^{1/p}(1/(b-p))
\]

holds for some random variable \( \xi \), under the restrictions \( p \in (0, b), \ b \in (0, 1) \). We obtain, by Tchebychev’s inequality,

\[
T_\xi(x) \leq \frac{\|\xi\|_p}{x^p} \leq C_1^p x^{-p} (b-p)^{-(\gamma+1)} L(1/(b-p)),
\]

and, choosing \( p := b - \frac{c}{\ln x} \), \( x \geq e \), we get the following tail estimate

\[
T_\xi(x) \leq C_3 x^{-p} (\ln x)^{\gamma+1} L(\ln x), \quad x \geq e. \quad (6.5)
\]
It is very important to notice that there is a "gap" \((\ln x)^1\) between the estimates (6.5) and (6.3). This "gap" is essential, see correspondent examples in [38, 30].

This example remains true still for the values \(p \in (0, 1)\).

### 6.2 Boyd indices of the quasi-Grand Lebesgue spaces \(G\psi_{a,b}\)

Let now \(Q = \mathbb{R}_+ = (0, \infty)\) and \(d\mu(z) = dz\). We provide the values of the Boyd indices for the quasi-GLS space \(G\psi = G\psi_{a,b}\) on \((0, \infty)\), having parameters \(a, b\) such that \(0 < a < b \leq 1\).

Boyd indices play an important role in the theory of interpolation of operators and in Fourier Analysis.

We recall the definition (see, e.g., [8]). Denote \(\sigma_s : G\psi_{a,b} \to G\psi_{a,b}\) the dilatation operators given by

\[
\sigma_s f(x) = f(x/s), \quad s > 0.
\]

The Boyd indices \(\gamma_1\) and \(\gamma_2\) of the quasi-GLS space \(G\psi\), with \(\gamma_1 \leq \gamma_2\), are defined by

\[
\gamma_1 = \gamma_1[G\psi] = \lim_{s \to 0^+} \frac{\log \|\sigma_s\|_{G\psi \to G\psi}}{\log s}, \quad \gamma_2 = \gamma_2[G\psi] = \lim_{s \to \infty} \frac{\log \|\sigma_s\|_{G\psi \to G\psi}}{\log s} \quad (6.6)
\]

Following the same arguments concerning the classical GLS spaces (see [31]), we state that

\[
\gamma_1 = \gamma_1[G\psi] = \frac{1}{b}, \quad \gamma_2 = \gamma_2[G\psi] = \frac{1}{a}.
\]

We recall that the Boyd indices for the particular case of the classical GLS spaces, namely for the space \(L^b)(0, 1)\), \(b > 1\), were computed in [18].

### 6.3 Contraction principle in quasi metric spaces

Let \((X, d)\) be a complete, closed, non-trivial quasi-metric space, for instance, a quasi-Banach space, equipped with the quasi-distance function \(d = d(x, y), \; x, y \in X\), such that

\[
\exists K = \text{const} \in (1, \infty) : d(x, y) \leq K[d(x, z) + d(z, y)], \; x, y, z \in X. \quad (6.7)
\]

The map \(f : X \to X\) is said to be a \textit{contraction} with parameter \(\alpha = \text{const} \in (0, 1)\), iff the Lipschitz condition is satisfied:

\[
d(f(x), f(y)) \leq \alpha d(x, y). \quad (6.8)
\]
Theorem 6.1. Let \((X,d)\) be a complete, closed, non-trivial quasi-metric space and \(f : X \to X\) a contraction, with constant \(\alpha\) in (6.8) satisfying
\[
\alpha < \frac{1}{K^2},
\]
where \(K\) is the constant in (6.7).

Then \(f\) admits a unique fixed point \(x^* \in X\), i.e. \(x^*\) is the unique solution of the equation
\[
f(x) = x,
\]
and \(x^*\) may be obtained as limit, as \(n \to \infty\), of the iterations (recursion)
\[
x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots,
\]
where \(x_0\) is an arbitrary point in \(X\). Herewith
\[
d(x^*, x_n) \leq \frac{K \alpha^n}{1 - \alpha K^2} d(x_0, x_1).
\]

Proof.
The proof is a simple generalization of the one presented by R. S. Palais in [45].
First of all note that, for \(x, y, z, v \in X\), from (6.7) we have
\[
d(x, y) \leq K d(x, z) + K^2 d(z, v) + K^2 d(v, y),
\]
therefore, for arbitrary elements \(y_1, y_2 \in X\), by (6.8) follows
\[
d(y_1, y_2) \leq K d(y_1, f(y_1)) + K^2 d(f(y_1), f(y_2)) + K^2 d(y_2, f(y_2))
\]
\[
\leq K d(y_1, f(y_1)) + \alpha K^2 d(y_1, y_2) + K^2 d(y_2, f(y_2)).
\]
Consequently,
\[
d(y_1, y_2) \leq \frac{K d(y_1, f(y_1)) + K^2 d(y_2, f(y_2))}{1 - \alpha K^2},
\]
since \(\alpha K^2 < 1\).

Choosing in (6.14)
\[
y_1 := x_n, \quad y_2 := x_m, \quad m > n \geq 1
\]
and taking into account the inequalities
\[
d(x_n, x_{n+1}) \leq d_0 \alpha^n, \quad d(x_m, x_{m+1}) \leq d_0 \alpha^m,
\]
where \(d_0 = d(x_1, x_0)\), we get
\[
d(x_n, x_m) \leq \frac{K d(x_n, x_{n+1}) + K^2 d(x_m, x_{m+1})}{1 - \alpha K^2}
\]
\[
\leq \frac{K d_0 \alpha^n + K^2 d_0 \alpha^m}{1 - \alpha K^2}, \quad 1 \leq n < m.
\]
The last estimate in (6.15) denotes that the sequence \( \{x_n\} \) is fundamental (Cauchy sequence). Since the space \((X,d)\) is complete, there exists the limit
\[
x^* = \lim_{n \to \infty} x_n \in X.
\]
Finally, passing to the limit as \( m \to \infty \) in (6.15), since \( \alpha \in (0,1) \), we conclude
\[
d(x_n,x^*) \leq \frac{Kd_0}{1 - \alpha K^2} \alpha^n.
\]
This completes the proof.

\[\square\]

**Remark 6.1.** As we know, the condition (6.7) is satisfied for the quasi Grand Lebesgue Spaces \( G_{\psi_a,b} \), \( 0 < a < b \leq 1 \) with respect to the quasi-distance function
\[
d(f,g) = \|f - g\|_{G_{\psi_a,b}},
\]
with constant \( K = 2^{1/a-1} \).

Assume now that the quadrilateral inequality
\[
\exists K \in [1, \infty) : d(x,y) \leq K[d(x,z) + d(z,v) + d(v,y)], \tag{6.16}
\]
holds instead of the one in (6.13). Then Theorem 6.1 may be reformulated as follows.

**Theorem 6.2.** Let \((X,d)\) be a complete, closed, non-trivial quasi-metric space and \( f : X \to X \) a contraction, with constant \( \alpha \) in (6.8) such that
\[
\alpha < \frac{1}{K}, \tag{6.17}
\]
where \( K \) is the constant in (6.16).

Then \( f \) admits a unique fixed point \( x^* \in X \), i.e. \( x^* \) is the unique solution of the equation
\[
f(x) = x, \tag{6.18}
\]
and \( x^* \) may be obtained as limit, as \( n \to \infty \), of the iterations (recursion)
\[
x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots, \tag{6.19}
\]
where \( x_0 \) is an arbitrary point in \( X \). Herewith
\[
d(x^*,x_n) \leq \frac{K\alpha^n}{1 - \alpha K} d(x_0,x_1). \tag{6.20}
\]

**Remark 6.2.** Note that the quadrilateral inequality (6.16) is satisfied for the quasi-Banach (Lebesgue-Riesz) spaces \( L_p \), \( p \in (0,1) \), with constant \( K = 3^{1/p-1} \) as well as for the quasi-GLS spaces \( G_{\psi_a,b} \), \( 0 < a < b \leq 1 \); in this case
\[
K = K[G_{\psi_a,b}] = 3^{1/a-1}.
\]
The last constant \( 3^{1/a-1} \) is lesser than its old value \( K^2 = [2^{1/a-1}]^2 \).

Some results concerning fixed point theorem in quasi-Banach spaces, with applications to integral equations, was given in [25].
7 Estimates for operators in quasi-Grand Lebesgue Spaces

Let \( U : L_p \to L_p, \ p \in (a, b), \ 0 < a < b \leq 1, \) be a bounded operator, not necessarily linear or sublinear, acting from the quasi-Lebesgue-Riesz space \( L_p \) into itself: \( \forall f \in L_p \exists U[f] \in L_p \) and
\[
\|U[f]\|_p \leq \Theta(p) \|f\|_p, \quad p \in (a, b),
\]
(7.1)
\[ \forall p \in (a, b) \Rightarrow \Theta(p) < \infty. \]

We can and will understood as the value of \( \Theta(p) \) its minimal value, indeed
\[
\Theta(p) \overset{\text{def}}{=} \sup_{0 \neq f \in L_p} \left\{ \frac{\|U[f]\|_p}{\|f\|_p} \right\}.
\]

Let also \( G\psi = G\psi_{a,b}, \ 0 < a < b \leq 1 \) be again the quasi-Grand Lebesgue Space and let \( g \in G\psi_{a,b}, \) then
\[
\|g\|_p \leq \|g\|_{G\psi} \cdot \psi(p), \quad p \in (a, b).
\]

We conclude, using (7.1),
\[
\|U[g]\|_p \leq \Theta(p) \psi(p) \|g\|_{G\psi}. \tag{7.2}
\]

On the other words, if we introduce a new \( \psi \) function \( \Psi[\Theta] = \Theta(p) \psi(p), \) we have
\[
\|U[g]\|_{G\psi[\Theta]} \leq \|g\|_{G\psi}. \tag{7.3}
\]

**Proposition 7.1.**
\[
\|U[\cdot]\|_{G\psi \to G\psi[\Theta]} = 1. \tag{7.4}
\]

**Proof.**

The upper estimate in (7.4), i.e. \( \|U[\cdot]\|_{G\psi \to G\psi[\Theta]} \leq 1, \) follows immediately from (7.3) and from the direct definition of the norm in the quasi Grand Lebesgue Spaces.

The lower estimate may be proved quite analogously to the one for the ordinary GLS, see [37], [40, Theorem 2.1.].

Note, in addition, that the weighted estimates for Hardy and Hausdorff operators in quasi Banach spaces \( L^p \) and \( L^p \) was obtained in [7, 10, 11, 6].
8 Concluding remarks.

A. Note that the case $K = 1$ in Theorems 6.1 and 6.2 corresponds to the classical contraction principle.

B. Perhaps, the results obtained in this paper may be applied, for instance, in the investigation of the sums of random variables having heavy tails of distributions, in the spirit of articles [41, 42], etc., and, as a consequence, in the Monte-Carlo method ([21, 22]).

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