ASYMPTOTIC BEHAVIOUR OF BIANCHI VI₀ SOLUTIONS WITH AN EXPONENTIAL-POTENTIAL SCALAR FIELD

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Abstract We obtain some solutions to the Einstein-Klein-Gordon equations without a cosmological constant for an exponential potential scalar field in a Bianchi VI₀ metric and investigate their behaviour.

I. INTRODUCTION

Inflationary theories are expected to explain the observed isotropy of the Universe by assuming an inflationary expansion in the early universe, since as the so-called “Cosmic no-hair” theorems state, all the solutions of the Einstein equations initially expanding and with a positive cosmological constant evolve towards the de Sitter solution [1], [2]. However, only those Bianchi models that have the FRW models as particular solutions isotropize [3]. One of the potentials which has received special attention is that of Liouville form (exponential potential), the one we will use in this work.

Some anisotropic cosmological models have already been studied. Aguirregabiria, Feinstein and Ibáñez [4] analyzed the Bianchi I models with exponential potential by reducing the problem of finding exact solutions to the solution of one third order differential equation. In [5] a non-local transformation was used to linearize this equation and the general solution was found. In this work we shall extend this analysis to the Bianchi VI₀, reducing, again, the problem of finding exact solutions to resolve a nonlinear differential equation and investigating the behaviour of their solutions at early time and its asymptotic stability in the far future.

The line-element for a Bianchi VI₀ cosmological model can be written in the following form:

\[ ds^2 = e^{f(t)} (-dt^2 + dz^2) + G(t) \left(e^z dx^2 + e^{-z} dy^2\right). \]  

and the corresponding Klein-Gordon and Einstein field equations for the metric (1) are as follows:

\[ \ddot{\phi} + \frac{G}{\dot{G}} \dot{\phi} + e^f \frac{\partial V}{\partial \phi} = 0, \]  

\[ \frac{\ddot{G}}{G} = 2e^f V, \]  

\[ \frac{\ddot{G}}{G} - \frac{1}{2} \left(\frac{\dot{G}^2}{G^2} - \frac{\dot{G}}{G} + \frac{1}{2}\phi^2\right) = 0. \]

We will consider a homogeneous self-interacting scalar field (\( \phi = \phi(t) \)) with an exponential potential \( V = \Lambda e^{k\phi} \), which is the only potential that separates the Klein–Gordon equation in two parts, one containing only the scalar field and the other containing only geometrical quantities (see [6]). Making use of this potential we can express the scalar field as follows

\[ \dot{\phi} = \frac{m}{G} - \frac{k}{2} \frac{\dot{G}}{G}. \]
After some algebra, the above equations can be reduced to the following one:

$$\ddot{G}^2 G - K \ddot{G} G^2 - \frac{1}{2} \dot{G} \ddot{G} G + \frac{1}{2} \dddot{G} G^2 + m^2 \dot{G} = 0; \quad (6)$$

where $K = k^2/4 - 1/2$. This is the equation we are going to study in detail. First we will consider the behaviour of the solutions when $t \to 0$ and then we will study their stability.

II. BEHAVIOUR OF THE SOLUTIONS WHEN $t \to 0$

A. Particular case

If we consider Equation $(6)$ when $m = 0$, and we assume that the function $G$ has the form $G(t) = G_0(\Delta t)^n$ when $\Delta t \to 0$ ($\Delta t = t - t_0$), we find two types of solutions, one with $n = 1$, which leads to a vanishing potential, and the other with $n = 1/K$. In the special case of $-\frac{1}{2} < K < 0$ we obtain a power law inflationary solution (see [7]), which can be written in its synchronous form as follows:

$$ds^2 = -dT^2 + \beta T^2 \left[e^{2Z/\sqrt{\alpha}}dX^2 + e^{-2Z/\sqrt{\alpha}}dY^2 + dZ^2\right], \quad (7)$$

where $\beta = \left[(n + 2)/(2\sqrt{\alpha})\right]^{(n+2)/(n+1)}$, $\alpha = \frac{n(n-1)}{2\Lambda|\phi_0|^2}$ and $\phi_0$ a constant.

We have studied the asymptotical behaviour of the scalar curvature, whose expression for the metric $(1)$ is:

$$R = e^{-f(t)} \left[\frac{1}{2} + \frac{\dddot{G}}{2 \dot{G}^2} - 2 \frac{\dddot{G}}{\dot{G}^2} - \dot{f}\right]. \quad (8)$$

In this particular case, we obtain that $R$ diverges at $t = 0$ for $K > 0$, so we can speak about a singularity at that point. In the case of negative $K$ the scalar curvature vanishes when $t \to 0$, due to the fact that $t \to 0$ means $T \to \infty$ in Equation $(7)$.

B. General Case

In the most general case the equation we have to deal with is:

$$\frac{1}{(\Delta t)^4} \left[n^2 - (1 + K)n^3 + Kn^4\right] - \left(\frac{1}{2} \frac{m^2}{(\Delta t)^2}\right) \frac{1}{(\Delta t)^2} \left[n^2 - n\right] = 0. \quad (9)$$

When $n \leq 1$ we obtain the same results that we had in the particular case. Solutions with $n > 1$, however, are not permitted. This means that solutions with $0 > K > 1$ do not appear in the general case.

As we did for the general case, we can calculate the behaviour of the scalar curvature, given by equation $(8)$. The relevant solutions (those with $-\frac{1}{2} < K < 0$ or $1 \leq K$) behave as they did in the $m = 0$ case, and therefore, the conclusions about the divergence of $R$ remain unchanged.

III. STABILITY OF THE ASYMPTOTIC SOLUTION

A. Particular Case

To investigate the behaviour and stability of the solutions, we can rewrite Equation $(6)$ in a quite different way, by introducing a new function $h = \frac{\dot{G}}{G}$ and redefining the time variable as follows:

$$d\eta = hdt = d\ln G. \quad (10)$$

After some transformations Equation $(6)$ reads

$$\frac{d}{d\eta} \left[\frac{h^2}{2} + K \frac{h^2}{2} - \frac{1}{2} \ln h\right] = - \left[-\frac{1}{2h^2} + (1 + K)\right] h^2, \quad (11)$$
where primes are derivatives with respect to the new variable $\eta$. This is the equation of motion for a dissipative or antidissipative system, with the potential $V(h) = K h^2 - \frac{1}{2} \ln h$.

Equation (11) presents local minima when $h^2_0 = \frac{1}{2K}$, for $K > 0$ (i.e. $k^2 > 2$). As the dissipative term given by the right-hand side of Equation (11) is negative definite in the asymptotic regime, the bracket on the l.h.s. of Equation (11) define a Liapunov Function [8], [9], [10]. Then, the corresponding exact solution, expressed by

$$G = G_0 e^{\sqrt{\frac{1}{2K} t}}$$

(with $G_0$ a constant) is stable for $t \to \infty$ and for any initial condition. This result allows us to study the behaviour of the solutions around these equilibrium points. To first order in perturbations and for $K > \frac{1}{2}$, there is a two-parameter family of stable solutions that behaves as Equation (12). The trajectories in the phase plane $(h, \dot{h})$ can be divided in two different groups: for $K > \frac{1}{8}$ the solutions cut the axis in the phase plane an infinite number of times, so they spiral around the constant solution $h_0$ (Equation (12)). For $-\frac{1}{2} < K \leq \frac{1}{8}$ the solutions do not cut the $h$ axis or they cut it once.

B. General Case

Equation (6) when $m = 0$ can be written using $h$ and $\eta$ variables, as follows:

$$\frac{d}{d\eta} \left[ \frac{h'^2}{2} + V(h) \right] = 2m^2 e^{-2\eta} \ln h - \left( (1 + K) - \frac{1 + 2m^2 e^{-2\eta}}{2h^2} \right) h'^2,$$

(13)

where now the “potential” is $V(h) = \frac{K h^2}{2} - \left( \frac{1}{2} + m^2 e^{-2\eta} \right) \ln h$, and the local minima will be given by $h^2_0 = \frac{1 + 2m^2 e^{-2\eta}}{2K}$. One has to be more careful in this case because the equilibrium points are stable only if $K > \frac{1}{8}$, as a consequence of Liapunov’s theorem. In this case, the equilibrium points are not fixed, they depend on $\eta$, and the equation that rules their behaviour gives us the following solution:

$$G_{\text{min}} = \pm \left\{ \frac{m^2}{2} e^{\sqrt{\frac{1}{2K} (t-t_0)}} + e^{-\sqrt{\frac{1}{2K} (t-t_0)}} \right\},$$

(14)

so the final asymptotically stable solution behaves again as Equation (12), which represents an anisotropic solution. This result had been obtained numerically in [11] and, in our case, it corresponds to the simplest solution we can obtain from Equation (6) when $m = 0$.

IV. CONCLUSIONS

We have studied the general behaviour of Bianchi VI sub solutions with a self-interacting scalar field and an exponential potential, by reducing the system of field equations to one differential equation of third order, Equation (6).

We have analyzed the behaviour of these solutions when $t \to 0$ and have found several anisotropic solutions which appear to have a singularity at $t = 0$, since the scalar curvature diverges there. A power-law inflationary solution corresponding to $k^2 < 2$ has also been found.

We have studied the asymptotical behaviour of the solutions when $t \to \infty$, finding that there is a simple asymptotically stable anisotropic solution for all cases. It can be easily shown (see [5]) that this solution remains anisotropic for large $t$.

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