Off-shell pions
in Boltzmann-Uehling-Uhlenbeck transport theory *

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(Dated: November 3, 2018)

Abstract

Due to large $\pi N \to \Delta$ cross section a pion acquires a large width in nuclear matter. Therefore, a
$\Delta$-resonance decaying in the nuclear medium can produce far off-shell pions, which will gradually
lose their virtuality while propagating to the vacuum. To describe such off-shell dynamics, we
implemented a pion spectral function for finite nuclear matter density – calculated on the basis
of the $\Delta$-hole model – in the BUU transport approach. The pion off-shell dynamics leads to an
enhancement of the low-$p_t$ pion yields in the collisions of Au+Au at 1 A GeV.

PACS numbers: 25.75.-q; 25.75.Dw; 21.65.+f; 24.10.Jv; 05.60.-k

Keywords: BUU; Au+Au at 1 A GeV; pion production; off-shell pions; $\Delta$-hole model

* Supported by BMBF and GSI Darmstadt
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I. INTRODUCTION

Pions are abundantly produced in heavy-ion collisions at beam energies of 1 A GeV and above. The main pion production channel is via the resonance excitation in nucleon-nucleon collisions: $NN \rightarrow NR$, $R \rightarrow N\pi$, where $R$ stands for $\Delta(1232)$ or higher baryonic resonances. As well established in nuclear transport theory [1, 2], pions propagating through nuclear matter experience the chain of absorption-production processes $\pi_1N_1 \rightarrow R_1 \rightarrow \pi_2N_2 \rightarrow R_2 \rightarrow \pi_3N_3$ etc due to the large resonance $\pi N$ scattering cross section. In other words, a pion with momentum $\sim 0.3$ GeV/c acquires a width in nuclear matter. The pion spectral function in nuclear matter is, therefore, different from the one in vacuum. Actual calculations within the (generalized) $\Delta$-hole model [3, 4, 5, 6, 7] indeed have shown that the pion spectral function is quite broad.

The off-shell nucleon effects in heavy particle ($N^*(1535)$) production have been studied in [8], where it has been shown, that the ground state correlations in colliding nuclei enhance the near-threshold production substantially. Beyond the ground state correlations, the influence of the off-shell nucleon propagation on heavy particle production has been found to be unimportant in [8], while recent BUU calculations [9, 10] have demonstrated an appreciable enhancement of the light meson ($\pi,K^+$) production by off-shell nucleons.

The transport simulations with in-medium pion and $\Delta$ properties have been recently performed by Helgesson and Randrup in a series of works (see [11] and Refs. therein) employing the quasiparticle approximation for pions. The soft pion enhancement due to the in-medium pion modifications has been observed in [11] in agreement with BUU studies in [12, 13]. As has been pointed out in [11], the overall effect of the in-medium modifications on the pion observables is small due to the fact, that the observed pions are emitted from the surface region of the system, where the in-medium modifications are not important. However, by multistep processes, pions influence other particles, e.g. kaons. Thus, the in-medium pion modifications could manifest themselves in the changes of the yields of the other particles.

The present work is the first attempt to include the pion spectral function into the BUU transport theory. The structure of the paper is as follows. In Sect. II we derive the pion spectral function in nuclear matter at finite temperature on the basis of the $\Delta$-hole model. Sect. III contains the equation of motion for off-shell pion propagation. In Sect. IV we
present the results of the BUU calculations for the pion production in Au+Au collisions at 1 A GeV. The summary of the work is given in Sect. V.

II. THE PION SPECTRAL FUNCTION IN NUCLEAR MATTER

In the ∆-hole model the pion polarization function Π(k) is given by the loop shown in Fig. 1, where we neglected the exchange u-diagram. For the evaluation of Π(k) we use a simple Lagrangian of the form:

\[ \mathcal{L}_{\pi N \Delta} = \frac{f_{\pi N \Delta}}{m_\pi} \psi^\mu_\Delta T \psi_N \partial_\mu \pi + h.c. , \]  

(1)

where T is the isospin transition operator \((1/2 \rightarrow 3/2)\) \([4]\), \(\psi^\mu_\Delta\) is the Rarita-Schwinger field of the ∆-resonance, \(\psi_N\) and \(\pi\) are the nucleon and pion fields, \(m_\pi = 0.138\) GeV is the pion mass. The coupling constant \(f_{\pi N \Delta} \approx 2\) will not enter explicitly in final expressions for Π(k) (see Eqs. (17), (18), (20)), therefore, we do not need to specify its value. Using Feynman rules and the Lagrangian (1) the pion polarization function in the case of isospin symmetric nuclear matter may be written as:

\[-i \Pi(k) = -\frac{4f^2_{\pi N \Delta}}{3m^2_\pi} \int \frac{d^4 p}{(2\pi)^4} k_\mu k_\nu \text{Sp} \left( G^\mu_\Delta(p + k)G_0(p) \right) , \]  

(2)

where

\[ G_0(p) = (\not{p} + m_N) \left( \frac{1}{p^2 - m_N^2 + i\epsilon} + 2\pi i\delta(p^2 - m_N^2)\theta(p^0)n(p) \right) \]  

(3)

is the nucleon propagator in noninteracting nuclear matter (c.f. [14]),

\[ n(p) = \frac{1}{\exp((E_N(p) - \mu)/T) + 1} \]  

(4)

is the occupation number written for simplicity in the rest frame (r.f.) of nuclear matter.

\[ E_N(p) = \sqrt{p^2 + m_N^2} \]  

(5)

is the free nucleon energy with \(m_N = 0.938\) GeV being the nucleon mass. For later use we define here also the free ∆ and pion energies:

\[ E_\Delta(p) = \sqrt{p^2 + m_\Delta^2} , \]  

(6)

\[ E_\pi(k) = \sqrt{k^2 + m_\pi^2} \]  

(7)
with the pole delta mass \( m_\Delta = 1.232 \) GeV. The Rarita-Schwinger \( \Delta \)-resonance propagator \( G^\mu_\Delta(p) \) is not unambiguously defined in the literature. We will use an expression from Ref. [15]

\[
G^\mu_\Delta(p) = - \frac{p^\mu + \sqrt{p^2}}{p^2 - m_\Delta^2 + i\sqrt{p^2} \Gamma_\Delta(p^2)} \left( g^{\mu\nu} - \frac{1}{3} \gamma^\mu \gamma^\nu - \frac{2p^\mu p^\nu}{3p^2} + \frac{p^\mu \gamma^\nu - p^\nu \gamma^\mu}{3\sqrt{p^2}} \right),
\]

(8)

which is similar to the one from Ref. [14] with an addition of the term proportional to the total width of the \( \Delta \)-resonance \( \Gamma_\Delta(p^2) \) in the denominator and with a replacement of the pole delta mass \( m_\Delta \) by \( \sqrt{p^2} \) in the numerator and in the tensor part.

The nucleon propagator (3) contains the usual vacuum part \( \propto (p^2 - m_N^2 + i\epsilon)^{-1} \) and the in-medium part \( \propto \delta(p^2 - m_N^2) \). The latter reflects rescattering of a pion on real nucleons. The vacuum part contributes to \( \Im \Pi(k) \) only for \( k^2 > (m_N + m_\Delta)^2 \), which corresponds to the decay of an off-shell pion to the \( \Delta \)-resonance and antinucleon. Obviously, the pions produced in the baryonic resonance decays cannot match this condition. Therefore, we will keep only the in-medium part of the nucleon propagator. Substituting Eqs. (3),(8) into Eq. (2) and keeping only the density dependent \( \propto n(p) \) term in the nucleon propagator we get the following expression for the pion polarization function:

\[
\Pi(k) = \frac{16 f_{\pi N}^2}{9m_\pi^2} \int \frac{d^3p}{(2\pi)^3 E_N(p)} n(p)((pk)^2 - m_N^2 k^2) \frac{p(p+k) + m_N \sqrt{s}}{(s - m_\Delta^2 + i\sqrt{s} \Gamma_\Delta(s))s},
\]

(9)

where \( s = (p+k)^2 \) and the nucleon is on its mass shell, i.e. \( p^2 = m_N^2 \). In the nonrelativistic limit \( |k| \ll m_N, |p| \ll m_N \) and for small \( \Delta \) off-shellness \( |\sqrt{s} - m_\Delta| \ll m_\Delta \) Eq. (9) transforms to well-known formula (c.f. Ref. [4]):

\[
\Pi(k) \approx \frac{16 f_{\pi N}^2}{9m_\pi^2} |k|^2 \int \frac{d^3p}{(2\pi)^3} n(p) \frac{1}{E_N(p) + k^0 - E_\Delta(p + k) + i\Gamma_\Delta(s)/2}. \]

(10)

In the spirit of the transport theory, however, one would like to express the pion polarization function in terms of the \( \pi^+ p \to \Delta^{++} \) cross section

\[
\sigma_{\pi^+ p \to \Delta^{++}}(s, k^2) = \frac{4\pi}{q^2(s, k^2)} \frac{2s \Gamma_{\Delta \to \pi N}(s, k^2) \Gamma_\Delta(s)}{(s - m_\Delta^2)^2 + s \Gamma_\Delta^2(s)},
\]

(11)

generalized here for the case of the off-shell pion with four-momentum \( k \).

\[
q(s, k^2) = \sqrt{(kp)^2 - k^2 m_N^2} / \sqrt{s}
\]

(12)

is the c.m. momentum of the pion and nucleon while \( \Gamma_{\Delta \to \pi N}(s, k^2) \) is the decay width of the \( \Delta \)-resonance. The decay width can, in principle, be easily calculated within the same
Lagrangian of Eq.(11) which produces the expression (c.f. Ref. [14]):

\[ \Gamma^{(0)}_{\Delta \to \pi N}(s, k^2) = \frac{f_{\pi N}^2 q^2(s, k^2)}{12 \pi m_{\pi}^2 s^2} ((p k)^2 - m_N^2 k^2)(p(p + k) + m_N \sqrt{s}) \]

\[ = \frac{f_{\pi N}^2 q^2(s, k^2)}{12 \pi m_{\pi}^2 \sqrt{s}} (\sqrt{q^2(s, k^2) + m_N^2} + m_N) \ . \]  

(13)

However, we will use Eq.(13) only for the derivation of the pion polarization function in terms of the cross section. In numerical calculations of the pion spectral function we substitute in Eq.(11) the off-shell extension of the vacuum width parameterization from Ref. [16]

\[ \Gamma_{\Delta \to \pi N}(s, k^2) = \Gamma^{(0)}_{\Delta \to \pi N}(s, k^2) \]

with \( q_0 \equiv q(m^2_\Delta, m^2_\pi), \Gamma_0 = 0.118 \) GeV and \( \beta_0 = 0.2 \) GeV. The total width of the \( \Delta \)-resonance in nuclear medium is taken as

\[ \Gamma_\Delta(s) = \Gamma_{\Delta \to \pi N}(s, m^2_\pi) \ . \]  

(15)

Substituting Eqs.(11),(13) into Eq.(9) one gets the expression:

\[ \Pi(k) = \frac{8}{3} \int \frac{d^3p}{(2\pi)^3 E_N(p)} n(p) \frac{q(s - m^2_\Delta - i \sqrt{s} \Gamma_\Delta(s))}{\Gamma_\Delta(s)} \sigma_{\pi p \to \Delta^{++}}(s, k^2) \ . \]  

(16)

Separating now real and imaginary parts we have:

\[ \text{Re} \Pi(k) = \int \frac{2d^3p}{(2\pi)^3 E_N(p)} m_N n(p) \frac{p_{\text{rel}} (s - m^2_\Delta) 4}{\Gamma_\Delta(s) \sqrt{s}} \frac{4}{3} \sigma_{\pi p \to \Delta^{++}}(s, k^2) \ , \]  

(17)

\[ \text{Im} \Pi(k) = -\int \frac{2d^3p}{(2\pi)^3 E_N(p)} m_N n(p) \frac{p_{\text{rel}} (s - m^2_\Delta) 4}{\Gamma_\Delta(s) \sqrt{s}} \frac{4}{3} \sigma_{\pi p \to \Delta^{++}}(s, k^2) \ , \]  

(18)

where \( p_{\text{rel}} = q \sqrt{s}/m_N \) is the pion momentum in the r.f. of a nucleon. By noting that

\[ 4/3 \sigma_{\pi p \to \Delta^{++}} = \sigma_{\pi p \to \Delta} + \sigma_{\pi n \to \Delta} \]  

we see that Eq.(18) can be simply rewritten in a more compact and intuitive form:

\[ \text{Im} \Pi(k) = -\rho \frac{m_N}{2} \left( \frac{1}{E_N(p)} p_{\text{rel}} (\sigma_{\pi p \to \Delta} + \sigma_{\pi n \to \Delta}) \right) \ , \]  

(19)

where the averaging is performed with respect to the nucleon momentum and \( \rho \) is the nuclear matter density. Notice, that Eqs.(17),(18) are explicitly relativistically covariant, since the self-energy of a scalar particle is a Lorentz-scalar.

As pointed out in the beginning of this section, in the derivation of the pion polarization function we have missed the \( u \)-diagram. The particular form (8) of the spin-3/2 propagator...
does not permit to reliably calculate the $u$-diagram contribution, since the tensor part has a pole at $p^2 = 0$ (c.f. [15] for details). However, using the usual Rarita-Schwinger propagator (c.f. [14]) we have checked that the $u$-diagram contributes mostly in the space-like region $k^2 < 0$, which is not important for our study, since in BUU only the time-like pions are propagated.

Finally, we correct the pion polarization function for the repulsive interaction of holes and $\Delta$-resonances at short distances (c.f. [8]):

$$\Pi_c(k) = \frac{\vert k \vert^2 \Pi(k)}{\vert k \vert^2 - g' \Pi(k)}$$

(20)

with $g' = 0.5$ being the Migdal parameter. Here the pion four-momentum $k = (k^0, \mathbf{k})$ has to be taken in the r.f. of nuclear matter. In that frame, due to the isotropy of the nucleon Fermi distribution, the polarization function $\Pi_c$ depends only on the pion energy and on the absolute value of the pion momentum $\Pi_c(k) \equiv \Pi_c(k^0, \vert \mathbf{k} \vert)$.

Fig. 2 shows the real and imaginary parts of the pion optical potential

$$U(|\mathbf{k}|) = \frac{\Pi_c(\varepsilon_{k\mathbf{k}}, \vert \mathbf{k} \vert)}{2E_\pi(|\mathbf{k}|)} ,$$

(21)

where $\varepsilon_k$ is the solution of the dispersion relation (c.f. [7])

$$\varepsilon_k^2 = E_\pi^2(k) + \text{Re} \Pi_c(\varepsilon_k, \vert \mathbf{k} \vert) .$$

(22)

We see that the real part of the optical potential changes sign and the absolute value of the imaginary part reaches its maximum practically at the same momentum $|\mathbf{k}| \simeq 0.380 \text{ GeV/c}$ for normal nuclear matter case: $\rho = \rho_0$, $T = 5 \text{ MeV}$ (solid lines), where $\rho_0 = 0.16 \text{ fm}^{-3}$ is the nuclear saturation density. This can be qualitatively understood from Eqs. (17),(18) since, neglecting for simplicity the Fermi motion, $\text{Re} \Pi = 0$ and $\sigma_{\pi N \to \Delta}(s)$ is maximal at $s = m_\Delta^2$. The estimated pion momentum is $|\mathbf{k}| = 0.300 \text{ GeV/c}$, which is somewhat different from the numerical value above due to the Fermi motion and the Migdal correction. Increasing temperature (dashed lines) smears out the jump of $\text{Re} U$ and decreases the peak value of $-\text{Im} U$ while shifting the peak to lower values of the pion momentum. Increasing density (dot-dashed lines) acts in the opposite direction. These findings are in an overall agreement with the calculations of Ref. [7] (see Fig. 2 of this Ref.). An essential difference is visible only at high momenta $|\mathbf{k}| > 0.4 \text{ GeV/c}$ where the potentials of Ref. [7] are more repulsive. The pion optical potential is, however, sensitive to the choice of the nucleon and delta
propagators (relativistic in our work and nonrelativistic in Ref. [7]), to the presence of the $u$-diagram contribution missed in our calculations for simplicity and – most important – to the choice of the $\Delta$-resonance width. In this exploratory work we do not intend to study in detail all these effects. Fig. 2 has a purpose to demonstrate only that our simple spectral function calculation is in the reasonable agreement with a state-of-art finite temperature calculation of Ref. [7].

Next we determine the pion spectral function

$$A_\pi(k) \equiv -\frac{1}{\pi} \text{Im} G(k) = -\frac{1}{\pi} \frac{\text{Im} \Pi_c(k)}{(k^2 - m_\pi^2 - \text{Re} \Pi_c(k))^2 + (\text{Im} \Pi_c(k))^2}$$

(23)

with $G(k) = (k^2 - m_\pi^2 - \Pi_c(k))^{-1}$ being the pion propagator. The exact spectral function must satisfy the normalization condition

$$\int_0^\infty d(k^0)^2 A_\pi(k) = 1.$$  

(24)

We checked that the spectral function (23) satisfies the condition (24) with the accuracy of 10 %.

Fig. 3 shows the spectral function vs pion invariant mass squared $M^2_\pi \equiv k^2$ and momentum $|k|$ for normal nuclear matter (density $\rho = \rho_0$ and temperature $T = 5$ MeV). There are two peaks of $A$ as a function of $M^2_\pi$ at fixed momentum $|k|$. At $|k| < 0.4$ GeV/c the lower $M^2_\pi$ sharp peak corresponds to the pion-like collective mode, while the higher $M^2_\pi$ broad peak reflects the $\Delta$-hole mode. With increasing momentum the lower $M^2_\pi$ peak becomes wider and enters the negative $M^2_\pi$ (space-like) region, while the higher $M^2_\pi$ peak gets narrower and asymptotically approaches the free pion limit ($M^2_\pi = m^2_\pi \simeq 0.02$ GeV$^2$). Therefore, at high momenta $|k| > 0.4$ GeV/c the higher $M^2_\pi$ peak corresponds to the pion-like mode and the lower $M^2_\pi$ one describes the $\Delta$-hole excitation. This level crossing is better illustrated in Fig. 4 where along with the peak positions of the spectral function in the $(|k|, \omega)$-plane (full circles) we present also the free pion dispersion relation $\omega = E_\pi(k)$ (solid line), the $\Delta$-hole mode

$$E_{N\Delta}(k) = \sqrt{\omega_\Delta(k)(\omega_\Delta(k) + g'C)}$$

(25)

with $\omega_\Delta(k) = E_\Delta(k) - m_N$ and $C = 8/9(f_{\pi N\Delta}/m_\pi)^2 \rho$ (c.f. [4]) (dashed line), and the solution $\varepsilon_k$ of the dispersion relation (22) (dash-dotted line). The peak positions close to the free pion solution are very well described by the dispersion relation (22), which has a characteristic
jump at $|k| = 0.38\text{ GeV/c}$. We also observe, that broad peaks at higher (lower) energy for small (large) momenta correspond indeed to the $\Delta$-hole excitation.

The high-momentum structure of the pion spectral function ($|k| > 0.4\text{ GeV/c}$) would be modified by taking into account the higher resonances in the pion polarization function. Therefore, the pion peak at high momenta is expected to be smeared out by the higher resonances. Nevertheless, as shown in Ref. [7] for the central heavy-ion collisions at energies about $1\text{ A GeV}$, the higher resonance contribution dominates in the pion yield at the c.m. pion kinetic energy above $0.6\text{ GeV}$ which corresponds to pion c.m. momentum above $0.7\text{ GeV/c}$. Thus, we expect that the inclusion of higher baryonic resonances in the pion spectral function will not strongly modify the results of our work (c.f. Fig. 4 later).

In order to estimate an effect of the in-medium modifications of the $\Delta$-resonance width, we replaced $\Gamma_\Delta(s)$ in Eqs. (1),(7) by $\Gamma_\Delta(s)(1-\bar{n}) + \Gamma_{sp}$, where $\Gamma_{sp} = -2\text{Im}W_{sp} = 80\rho/\rho_0\text{ [MeV]}$ is the phenomenological spreading width given by the imaginary part of the $\Delta$-spreading potential [8]. $\bar{n}$ is the angle averaged nucleon occupation number. The $\Delta$-spreading potential takes into account the quasielastic scattering, two- and three-body absorption contributions [19, 20]. In Fig. 5 we compare the spectral functions at $\rho = 2\rho_0$ and $T = 70\text{ MeV}$ calculated with vacuum (dashed lines) and in-medium (solid lines) $\Delta$-resonance width. We see that the $\Delta$-spreading potential smears-out the peaks of the spectral function. There is a tendency of the strength concentration near $M_\pi^2 = m_\pi^2 = 0.02\text{ GeV}^2$. Correspondingly, the strength at negative and large positive $M_\pi^2$ is reduced somewhat by in-medium $\Delta$-width.

III. KINETIC EQUATION FOR OFF-SHELL PIIONS

For the description of the off-shell particle dynamics we will use the spectral distribution functions (c.f. [4]) $F_\alpha(r,t,p,M^2)$ with $\alpha$ being the particle type ($\alpha = N, R, \pi$ for nucleons, resonances and pions respectively). The number of particles of type $\alpha$ in the phase space element $d^3rd^3pdM^2/(2\pi)^3$ is $F_\alpha d^3rd^3pdM^2/(2\pi)^3$. The following relation holds:

$$F_\alpha(r,t,p,M^2) = f_\alpha(r,t,p,M^2)A_\alpha(r,t,p,M^2)$$

(26)

where $f_\alpha(r,t,p,M^2)$ is the usual phase space density, and $A_\alpha(r,t,p,M^2)$ is the spectral function. The pion spectral function has been defined earlier by Eq. (23). The spectral
function of a baryonic resonance is taken as the vacuum one:

\[ A_R(M^2) = \frac{1}{\pi} \frac{M \Gamma_R(M^2)}{(M^2 - m_R^2)^2 + M^2 \Gamma_R^2(M^2)} \]  

(27)

with \( m_R \) being the pole mass and \( \Gamma_R^2(M^2) \) the vacuum width of the resonance [16]. The effect of the nucleon off-shellness on the pion and kaon production has been studied in Ref. [9]. In the present work we treat, however, nucleons on-shell in order to underline the genuine pion off-shellness effects on the observables. Thus, we use

\[ A_N(M^2) = \delta(M^2 - m_N^2) \]  

(28)

and everywhere below the nucleon four-momentum is \( p_N = (E_N(p_N), p_N) \).

It is instructive to write down explicitly the coupled transport equations for resonances and pions:

\[
\left( \frac{\partial}{\partial t} + \frac{\partial H_{mf}^R}{\partial \mathbf{p}_R} \frac{\partial}{\partial \mathbf{r}} - \frac{\partial H_{mf}^R}{\partial \mathbf{p}_R} \frac{\partial}{\partial \mathbf{r}} \right) F_R(r, t, \mathbf{p}_R, M_R^2) = I_R ,
\]

(29)

\[
\left( \frac{\partial}{\partial t} + \frac{k}{k^0} \frac{\partial}{\partial \mathbf{r}} \right) F_\pi(r, t, \mathbf{k}, M_\pi^2) + L_{\text{off-shell}}[F_\pi] = I_\pi ,
\]

(30)

where \( H_{mf}^R(r, t, \mathbf{p}_R) \) is the mean field Hamiltonian acting on resonances: in this work we take it the same as for nucleons. \( I_R \) and \( I_\pi \) are the collision integrals. The term \( L_{\text{off-shell}}[F_\pi] \) on the l.h.s. of (30), which will be specified later on, is introduced in order to bring pions back on the mass shell once they reach the vacuum. The collision integrals are expressed in terms of gain (\( < \)) and loss (\( > \)) terms as follows:

\[
I_R = I_R^< - I_R^> = \sum_{N, \pi} \frac{1}{2p_R^0} \int \frac{g_N d^3p_N}{(2\pi)^3 2E_N(p_N)} \int \frac{d^3k dM_\pi^2}{(2\pi)^3 2k^0} 
\]

\[
\times (2\pi)^4 \delta^4(p_R - k - p_N) |\mathcal{M}_{R \rightarrow N\pi}|^2 
\]

\[
\times \{ F_\pi(k, M_\pi^2) f_N(p_N) A_R(M_R^2)(1 - f_R(p_R, M_R^2)) 
\]

\[- F_R(p_R, M_R^2) A_\pi(k, M_\pi^2)(1 + f_\pi(k, M_\pi^2))(1 - f_N(p_N)) \} ,
\]

(31)

\[
I_\pi = I_\pi^< - I_\pi^> = \sum_{N, \pi} \frac{1}{2k^0} \int \frac{g_R d^3p_R dM_R^2}{(2\pi)^3 2p_R^0} \int \frac{g_N d^3p_N}{(2\pi)^3 2E_N(p_N)} 
\]

\[
\times (2\pi)^4 \delta^4(p_R - k - p_N) |\mathcal{M}_{R \rightarrow N\pi}|^2 
\]

\[
\times \{ F_R(p_R, M_R^2) A_\pi(k, M_\pi^2)(1 + f_\pi(k, M_\pi^2))(1 - f_N(p_N)) 
\]

\[- F_\pi(k, M_\pi^2) f_N(p_N) A_R(M_R^2)(1 - f_R(p_R, M_R^2)) \} ,
\]

(32)

where we dropped space and time variables for brevity. In Eqs.(31),(32) \(|\mathcal{M}_{R \rightarrow N\pi}|^2 \) is the invariant matrix element squared averaged over spins of initial and final particles, \( g_\alpha = \)}
\(2s_\alpha + 1 (\alpha = N, R)\) is the spin degeneracy factor. For simplicity, we did not display the contributions of \(NN \leftrightarrow NR\) processes to \(I_R\) and of the direct processes \(NN \leftrightarrow NN\pi\) to \(I_\pi\). However, these processes are always included in BUU. Sums in Eqs. (31), (32) run over all possible isospin states of the particles. Noting that the resonance decay width to a nucleon and an off-shell pion of mass \(M_\pi\) in the resonance r.f. is given by the expression

\[
\Gamma_{R \rightarrow N\pi}(M_R^2, M_\pi^2) = \frac{1}{2M_R} \int \frac{g_Nd^3p_N}{(2\pi)^32E_N(p_N)} \int \frac{d^3k}{(2\pi)^32k^0} \times (2\pi)^4\delta(4)(p_R - k - p_N)|M_{R \rightarrow N\pi}|^2 = \int d\Omega_{c.m.} \frac{1}{32\pi^2M_R^2}|M_{R \rightarrow N\pi}|^2 qg_N
\] (33)

and using the fact that the spin-averaged matrix element does not depend on the center-of-mass scattering angles, one can rewrite Eq. (31) as follows:

\[
I_R = \sum_{N,\pi} \int dM_\pi^2 M_R^2 P_R^0 \Gamma_{R \rightarrow N\pi}(M_R^2, M_\pi^2)
\]

\[
\times \int d\Omega_{c.m.} \frac{1}{4\pi} \left\{ F_\pi(k, M_\pi^2)f_N(p_N)A_R(M_R^2)(1 - f_R(p_R, M_R^2)) - F_R(p_R, M_R^2)A_\pi(k, M_\pi^2)(1 + f_\pi(k, M_\pi^2))(1 - f_N(p_N)) \right\} .
\] (34)

Introducing the cross section

\[
\sigma_{\pi N \rightarrow R}(M_R^2, M_\pi^2) = \frac{4\pi^2}{q^2} \Gamma_{R \rightarrow N\pi}(M_R^2, M_\pi^2) M_R A_R(M_R^2) \frac{g_R}{g_N}
\] (35)

the pion collision integral can be also simplified as

\[
I_\pi = \sum_{R, N} \int \frac{g_Nd^3p_N}{(2\pi)^3} v_{\pi N}\sigma_{\pi N \rightarrow R}(M_R^2, M_\pi^2)
\]

\[
\times \left\{ F_\pi(p_R, M_R^2)A_\pi(k, M_\pi^2)(1 + f_\pi(k, M_\pi^2))(1 - f_N(p_N)) - F_R(p_R, M_R^2)A_N(p_N)(1 - f_R(p_R, M_R^2)) \right\}
\] (36)

with \(v_{\pi N} = \sqrt{(kp_N)^2 - M_\pi^2m_N^2/(k^0E_N(p_N))}\) being the relative velocity of the pion and nucleon.

In the following we neglect for simplicity the second-order effect of the pion off-shellness on the resonance width. Thus, we replace in Eq. (34) and, to maintain detailed balance, also in Eq. (35)

\[
\Gamma_{R \rightarrow N\pi}(M_R^2, M_\pi^2) \rightarrow \Gamma_{R \rightarrow N\pi}(M_R^2, m_\pi^2),
\] (37)

i.e. the off-shellness of the pion in the resonance decay vertex is neglected. We have to comment that this simplification has been done only in the BUU implementation.
pion polarization function calculation the full off-shell decay width (14) has been applied.)
Correspondingly, the mass squared of the outgoing pion in the resonance decay has been
sampled in the interval $[0; (M_R - m_N)^2]$ according to the distribution $\propto A_\pi(k, M_\pi^2)$. In
the case of the direct (background) pion production $NN \to NN\pi$ the value of $M_\pi^2$ has been
sampled in the interval $[0; (\sqrt{s} - 2m_N)^2]$ using the same distribution.

The propagation of the off-shell pion between its production in the resonance decay (or
in the direct process $NN \to NN\pi$) and absorption in the collision with a nucleon $\pi N \to R$
(or with two nucleons $\pi NN \to NN$) is described by the l.h.s. of Eq.(30). In BUU the
test particle method is used to solve the transport equations. The pion spectral distribution
function is represented by test particles in the following way:

$$F_\pi(r, t, k, M_\pi^2) = \sum_i \delta(r - r_i(t))\delta(k - k_i(t))\delta(M_\pi^2 - M_\pi^2, i(t)) . \quad (38)$$

Following Refs. [9, 16] we introduce a scalar off-shell potential $s_{\pi, i}$ acting on the i-th pion
test particle:

$$s_{\pi, i}(r_i(t), t) = \frac{M_{\pi, i}(t_{cr}) - m_\pi}{\rho_N(r_i(t_{cr}), t_{cr})}\rho_N(r_i(t), t) , \quad (39)$$

where $\rho_N(r, t)$ is the density of nucleons in the local r.f. of nuclear matter. $t_{cr}$ is the
production time of the test particle. The potential (39) drives the off-shell pion back to its
mass shell when it leaves the nucleus, i.e. we assume that

$$M_{\pi, i}(t) = m_\pi + s_{\pi, i}(r_i(t), t) . \quad (40)$$

Therefore, Eqs.(39),40) provide the correct boundary condition for large times, i.e. when
pions reach the vacuum:

$$\lim_{t \to \infty} M_{\pi, i}(t) = m_\pi . \quad (41)$$

The operator $L_{off-shell}[F_\pi]$ in the l.h.s. of Eq.(30) is defined by its action on the distribution
function in the test particle representation (38) as follows:

$$L_{off-shell}[F_\pi] \equiv \sum_i \left(2M_{\pi, i}(t)s_{\pi, i}\frac{\partial}{\partial M_\pi^2} - \frac{\partial H_{off-shell, i}}{\partial r_i} \frac{\partial}{\partial k}\right) \times \delta(r - r_i(t))\delta(k - k_i(t))\delta(M_\pi^2 - M_{\pi, i}(t)) , \quad (42)$$

where

$$H_{off-shell, i} = \sqrt{k_i^2 + (m_\pi + s_{\pi, i})^2} . \quad (43)$$
is the energy of the pion test particle $i$. From Eqs. (30), (42) we get equations of motion of test particles in coordinate and momentum space:

$$\dot{r}_i = \frac{\partial H_{\text{off-shell}},i}{\partial k_i},$$  \hspace{1cm} (44)

$$\dot{k}_i = -\frac{\partial H_{\text{off-shell}},i}{\partial r_i}.$$  \hspace{1cm} (45)

These two Hamiltonian equations together with Eq. (40) determine the evolution of the spectral distribution function (38) completely.

Due to the explicit time dependence of the potential (39), which acts as an external field, the energy of the whole system is not conserved. However, in practice, the violation of the energy conservation due to the off-shell potential turns out to be very small ($\sim 0.2$ MeV/nucleon for the central Au+Au collision at 1 A GeV, i.e. $\sim 0.1\%$ of the total c.m. kinetic energy per nucleon). This is caused by a relatively small number of pions with respect to the baryon number.

Some comments are in order on the relation of the phenomenological approach of Refs. [9, 16] applied in the present work with the more general test particle equations of motion derived in Refs. [10, 21] on the basis of the Kadanoff-Baym equations. Indeed, Eqs. (39), (40) which determine the evolution of the pion mass and the Hamiltonian equations (44), (45) can be obtained from the formalism of Refs. [10, 21] by assuming that $\text{Im} \Pi \propto \rho_N$ and neglecting $\text{Re} \Pi_c$ in the propagation of the pions. Earlier BUU studies [11, 12, 13] within the on-shell approximation have shown that the inclusion of the real part of the pion optical potential results in an enhanced soft pion yield. However, our attempted treatment of the off-shell pion propagation within the formalism of Refs. [10, 21] with the $\Delta$-hole pion polarization function resulted in the appearance of the superluminous pions. We have, therefore, chosen the simplified Hamiltonian approach of Refs. [9, 16] for the propagation of the off-shell pions which has no this problem, since the off-shell potential $s_{\pi,i}$ is momentum-independent.

IV. NUMERICAL RESULTS

The pion spectral function has been implemented in the BUU model in the version of Ref. [16]. The temperature $T$ and the baryon density $\rho$ have been calculated locally in BUU and their values have been used to interpolate $A_\pi$ which was stored on the 4-dimensional grid in the space ($M_\pi^2, |k|, \rho, T$). This allows one to save CPU time by avoiding on-line calculation
of $A_\pi$. Masses of outgoing pions from resonance decays or from NN collisions have been sampled according to $A_\pi$. The pions have then been propagated by using Eqs.(44),(45) until they have been reabsorbed in $\pi N \rightarrow R$ or $\pi NN \rightarrow NN$ processes or emitted to vacuum. The BUU calculations of the present work are performed using the soft momentum-dependent mean field (SM) with the incompressibility $K = 220$ MeV and employing the quenching of the resonance production in order to get the correct pion multiplicity $[22]$. Below, if the opposite is not stated explicitly, the vacuum $\Delta$-width $[14]$ is used in the pion spectral function.

Fig. 6 shows the time evolution of the pion mass distribution in the central collision Au+Au at 1.06 A GeV. At early times ($t = 10 \div 20$ fm/c) a rather broad range in $M_\pi$ is populated by the resonance decays in dense nuclear matter, where the pion spectral function is broad. In order to separate the effect of the off-shell transport by Eqs.(44),(45) we performed also the calculation putting $L_{off-shell} = 0$ in Eq.(30) (dash-dotted lines). In this case pions emitted to vacuum do not lose their off-shellness and the $M_\pi$-spectrum stays quite broad even at $t = 40$ fm/c when all violent processes are already finished and the system is already quite dilute (central baryon density $\simeq 0.02$ fm$^{-3}$). The full calculation (solid lines) results in a strong narrowing of the $M_\pi$-spectrum at the late stage of the reaction: at $t = 40$ fm/c the most part of pions is already on the mass shell. Continuing the calculation until 80 fm/c (dotted line on lower right panel) produces a rather moderate variation of the $M_\pi$-spectrum towards precise $\delta(M_\pi - m_\pi)$. Thus, for the comparison with experimental data below we have stopped calculations at $t = 40$ fm/c.

In Fig. 7 we present the inclusive transverse momentum spectra at midrapidity for the same system Au+Au at 1.06 A GeV in comparison with the experimental data from Refs. $[23, 24]$. The calculation with on-shell pions (dashed lines) describes reasonably well the high-$p_t$ part of the spectra, but underpredicts the data at low $p_t$. Taking into account the pion off-shellness (dotted lines) increases the low-$p_t$ pion yields improving, thus, the agreement with the data. The low-$p_t$ pion enhancement is caused by the off-shell pion propagation. To check this, we switched off again the $L_{off-shell}$ term in Eq.(30) (dash-dotted lines). We see that, indeed without this term the low-$p_t$ enhancement disappears. The effect is caused by the low mass pions ($M_\pi < m_\pi$) produced by the resonance decays at early stage of the collision (Fig. 3). These pions experience the attractive off-shell potential $[33]$, and, therefore, they are decelerated propagating out of the dense region of the nuclear system.

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One may notice also in Fig. 7 a slightly enhanced yield of high-$p_t$ pions by the off-shell calculation with vacuum $\Delta$-width. The agreement with data at large transverse momenta can be improved by using the $\Delta$-spreading width (solid lines). This is caused by the reduced strength of the pion spectral function at large pion masses (c.f. Fig. 5).

V. SUMMARY AND OUTLOOK

In this work we have generalized the BUU approach for the pion off-shellness in the nuclear medium. The $\Delta$-hole model has been applied to calculate the pion spectral function which has been used, then, in BUU to choose randomly the outgoing pion mass in the resonance decay. The propagation of the produced off-shell pions has been described in a Hamiltonian approach – developed earlier in Refs. [9, 16] – using the baryon density dependent off-shell scalar potential which acts on pions. This guarantees that a pion emitted to vacuum is always on the mass shell.

As an observable effect in heavy-ion collisions at 1 A GeV, we have demonstrated the low $p_t$ enhancement of the pion yield due to the off-shell pion propagation, which is in agreement with earlier BUU calculations [11, 12, 13] using the in-medium pion dispersion relation. We expect, that the pion off-shellness will also influence other observables, which are not covered in this work, e.g. the $K^+, K^-$ and $\rho$ production via reactions $\pi N \rightarrow K^+ \Lambda$, $\pi \Lambda \rightarrow K^- N$ and $\pi\pi \rightarrow \rho$ respectively (study in progress).

Another important aspect is the modification of the $\Delta$-width due to coupling to an off-shell pion. We have studied this effect assuming the in-medium broadening of the $\Delta$-resonance given by the phenomenological $\Delta$-spreading potential [18]. Overall, a rather weak influence of the in-medium $\Delta$-width on pion observables is found. We have to remark, however, that we included the in-medium $\Delta$-width in the pion spectral function only, not in BUU. The explicit in-medium modification of the $\Delta$-width in BUU has been studied earlier in Refs. [11, 25], where a quite weak effect on pion observables in heavy-ion collisions and on $\gamma$-absorption cross section on nuclei was found. Finally, it is straightforward to generalize the $\Delta$-hole model including higher baryonic resonances which will then also contribute to the pion spectral function.
Acknowledgments

Stimulating discussions with W. Cassing, S. Leupold, M. Post and V. Shklyar are gratefully acknowledged. The authors are especially grateful to Stefan Leupold for the careful reading of the manuscript, useful advices and comments.
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FIG. 1: Feynman graph representing $-i\Pi(k)$ with $\Pi(k)$ being the pion polarization function. $k$ and $p$ are the pion and nucleon four-momenta respectively.
FIG. 2: Real (upper panel) and imaginary (lower panel) parts of the pion optical potential as functions of the pion momentum at various densities and temperatures.
FIG. 3: Pion spectral function in nuclear matter at the density $\rho = \rho_0$ and temperature $T = 5$ MeV shown vs mass squared and momentum.
FIG. 4: Peak positions of the pion spectral function $A_\pi$ in the $(|k|, \omega)$-plane for nuclear matter at the density $\rho = \rho_0$ and temperature $T = 5$ MeV – full circles. Free pion dispersion relation $E_\pi(k)$ is shown by the solid line. Dashed line shows the $\Delta$-hole energy $E_{N\Delta}(k)$. Solution $\varepsilon_k$ of the dispersion relation (22) is depicted by dash-dotted line.
FIG. 5: Pion spectral function vs pion invariant mass squared at fixed pion momentum $|k| = 0.25$ GeV/c (upper panel) and $|k| = 0.55$ GeV/c (lower panel) in nuclear matter at $\rho = 2\rho_0$ and $T = 70$ MeV. Dashed and solid lines present the calculations with vacuum and in-medium $\Delta$-width respectively.
FIG. 6: Time evolution of the pion mass distribution in the central $Au + Au$ collision at 1.06 A GeV. Solid and dash-dotted lines show the results of calculations with and without the off-shell term $L_{\text{off-shell}}[F_\pi]$ in pion propagation respectively. Dotted line in the lower right panel shows the mass distribution at $t = 80$ fm/c for the calculation with off-shell term.
FIG. 7: Pion transverse momentum spectra at midrapidity for Au+Au collisions at 1.06 A GeV in comparison with experimental data from Refs. [23, 24]. Dashed lines show the calculation with on-shell pions. Dotted and dash-dotted lines present the off-shell calculations using vacuum ∆-width with and without $L_{\text{off-shell}}[F_\pi]$ term respectively. Solid lines show the off-shell calculation with in-medium ∆-width.