DIFFERENTIAL OPTIMIZATION IN FINITE-DIMENSIONAL SPACES

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Abstract. In this paper, a class of optimization problems coupled with differential equations in finite dimensional spaces are introduced and studied. An existence theorem of a Carathéodory weak solution of the differential optimization problem is established. Furthermore, when both the mapping and the constraint set in the optimization problem are perturbed by two different parameters, the stability analysis of the differential optimization problem is considered. Finally, an algorithm for solving the differential optimization problem is established.

1. Introduction. Let $K$ be a subset of $\mathbb{R}^m$, $g : K \rightarrow \mathbb{R}$ be a function. The following optimization problem:

$$\begin{align*}
\text{Minimize} & \quad g(w) \\
\text{subject to} & \quad w \in K
\end{align*}$$

has wide applications in engineering sciences, economics, finance, transportation and so on. There have been many publications devoted to optimization theory and applications [1, 10, 29, 8, 28, 12, 21, 19, 13, 14]. Various kinds of methods have been developed for solving the optimization problem. For example, gradient flow technique [2] can be applied to find the optimal solution. In the approach an optimization problem is formulated as an ordinary differential equation (ODE)
so that the solution of this ODE converges to an equilibrium point of the original problem. The approach has been improved and generalized by many authors in recent years. A unified gradient flow approach to nonlinear constrained optimization problems was presented in [24]. A novel hybrid descent method, consisting of a simulated annealing algorithm and a gradient-based method, was proposed to consider the optimal design of finite precision FIR filters in [26]. Gradient descent methods can be applied to solve integer programming problems in [6]. In addition, many important dynamical systems were modeled by optimization problem coupled with differential equations [5, 18, 23, 22] and gradient flow technique can help us to study the dynamical optimization problem. It is known that the differentiability of objective function is an important assumption in the work. Based on the above researches, we would like to consider the dynamical optimization problem without the differentiability assumption in this paper. Consider the following differential optimization problem (DOP):

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + B(x(t))w(t), \\
\text{Minimize} & \quad g(x(t), w(t)), \\
\text{Subject to} & \quad w(t) \in K, \\
x(0) &= x^0,
\end{align*}
\]

where \(\dot{x}(t) = \frac{dx(t)}{dt}, t \in [0,T], f : \mathbb{R}^n \to \mathbb{R}^n, B : \mathbb{R}^n \to \mathbb{R}^{n \times m}\) and \(g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}\) are given mappings. In this paper, the solution set of the optimization problem (1) is denoted by \(\text{SOL}(K, g)\). \((x(t), w(t))\) defined on \([0,T]\) is called a Carathéodory weak solution of DOP (2) iff \(x(t)\) is an absolutely continuous function on \([0,T]\) and satisfies the differential equation for almost all \(t \in [0,T]\) and \(w \in L^2([0,T], \mathbb{R}^m)\) and \(w(t) \in \text{SOL}(K, g(x(t), \cdot))\) for every \(t \in [0,T]\). The Carathéodory weak solution set of the DOP (2) is denoted by \(\text{SOL}(\text{DOP}(2))\).

Differential optimization problem such as the model (2) is seldom researched. Although it looks like an optimal control problem with \(x\) being the state and \(w\) being the control, there exist some differences between DOP (2) and the optimal control problems. The main difference lies in the following aspect: The control \(w\) in DOP (2) is the solution at time \(t\) for DOP (2), whose objective functions depend on the current state, and it is a pointwise optimization, while the control \(w\) in the optimal control problems is to minimize a performance function that is an integral function. There are many differential optimization problems arising in real worlds. For example, in static portfolio research, we use quadratic programming model to maximize the earning under the assumption that the risk is determined, and the quadratic programming is an optimization problem. In fact, the risk usually varies with time. We believe that the differential optimization problem is a more appropriate model characterizing the portfolio problem in a continuous-time system. Therefore, it is an interesting research to establish the existence result of a Carathéodory weak solution of DOP (2). Furthermore, there are many publications devoted to the stability analysis of static optimization problem. When the objective functions and the constraint sets were perturbed, researchers established various kinds of stability results of the parametric optimization problem (see for example [3, 4, 7, 27]). The results can help us to observe the change of the optimal solution for parametric static optimization problem. However, there is few paper on the stability analysis of DOP (2). The main difference between DOP (2) and the static optimization problem lies in the following aspect: for DOP (2), we need
to consider the state \( x \) and the optimal solution \( u \) at the same time, while for the static optimization problem, we only consider the optimal solution \( u \). Therefore, it is more difficult to study DOP (2) compared with the static problem.

Let \((Z_1,d_1)\) and \((Z_2,d_2)\) be two metric spaces. Assume that a nonempty closed and convex set \( K \subset \mathbb{R}^n \) is perturbed by a parameter \( u \), which varies over \((Z_1,d_1)\), that is, \( K : Z_1 \rightrightarrows \mathbb{R}^n \) is a set-valued mapping with nonempty closed convex values. Let the objective function \( g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) be perturbed by a parameter \( v \), which varies over \((Z_2,d_2)\), that is, \( g : \mathbb{R}^n \times \mathbb{R}^m \times Z_2 \to \mathbb{R} \). We consider the parametric DOP:

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + B(x(t))u(t), \\
\text{Minimize} & \quad g(x(t), w(t), v), \\
\text{Subject to} & \quad u(t) \in K(u), \\
x(0) &= x^0.
\end{align*}
\]

The Carathéodory weak solution set of the parametric DOP (3) is denoted by SOL(DOP\((u,v)\)).

The remainder of this paper is organized as follows. In section 2, we introduce some preliminary results. In section 3, we establish the existence result of a Carathéodory weak solution of DOP\((2)\). In section 4, we study the stability analysis of DOP\((3)\). In section 5, we give an algorithm for solving the differential optimization problem. In section 6, we give some numerical experiments to verify the validity of the proposed algorithm.

2. Preliminaries. In this section, we will introduce some preliminary results.

**Definition 2.1.** Let \( Y, Z \) be topological spaces and \( G : Y \rightrightarrows Z \) be a set-valued mapping with nonempty values. We say that \( G \) is

(i) upper semicontinuous at \( x_0 \in Y \) iff for any neighborhood \( \mathcal{N}(G(x_0)) \) of \( G(x_0) \), there exists a neighborhood \( \mathcal{N}(x_0) \) of \( x_0 \) such that

\[ G(x) \subseteq \mathcal{N}(G(x_0)), \quad \text{for all } x \in \mathcal{N}(x_0); \]

(ii) lower semicontinuous at \( x_0 \in Y \) iff for any \( y_0 \in G(x_0) \) and any neighborhood \( \mathcal{N}(y_0) \) of \( y_0 \), there exists a neighborhood \( \mathcal{N}(x_0) \) of \( x_0 \) such that

\[ G(x) \cap \mathcal{N}(y_0) \neq \emptyset, \quad \text{for all } x \in \mathcal{N}(x_0). \]

We say that \( G \) is continuous at \( x_0 \) iff it is both upper and lower semicontinuous at \( x_0 \). \( G \) is said to be continuous on \( Y \) iff it is both upper and lower semicontinuous at every point of \( Y \).

**Lemma 2.2.** \[17\] Let \( \Omega \equiv [0,T] \times \mathbb{R}^n, F : \Omega \rightrightarrows \mathbb{R}^n \) be an upper semicontinuous set-valued map with nonempty closed convex values. Suppose that there exists a scalar \( \rho_F > 0 \) satisfying

\[ \sup \{ \| y \| : y \in F(t,x) \} \leq \rho_F(1 + \| x \|), \quad \forall (t,x) \in \Omega. \]  

(4)

For every \( x^0 \in \mathbb{R}^n \), the differential inclusion \( \dot{x} \in F(t,x), x(0) = x^0 \) has a weak solution in the sense of Carathéodory.

**Lemma 2.3.** \[17\] Let \( h : \Omega \times \mathbb{R}^m \to \mathbb{R}^n \) be a continuous function and \( U : \Omega \rightrightarrows \mathbb{R}^m \) be a closed set-valued map such that for some constant \( \eta_U > 0 \),

\[ \sup_{u \in U(t,x)} \| u \| \leq \eta_U(1 + \| x \|), \quad \forall (t,x) \in \Omega. \]
Let \( v : [0, T] \to R^n \) be a measurable function and \( x : [0, T] \to R^n \) be a continuous function satisfying \( v(t) \in h(t, x(t), U(t, x(t))) \) for almost all \( t \in [0, T] \). There exists a measurable function \( u : [0, T] \to R^m \) such that \( u(t) \in U(t, x(t)) \) and \( v(t) = h(t, x(t), u(t)) \) for almost all \( t \in [0, T] \).

3. Existence of the solution for DOP. In Theorem 3.1, we establish the existence conclusion of a Carathéodory weak solution.

**Theorem 3.1.** Let \( f : R^n \to R^n, B : R^n \to R^{n \times m} \) be two Lipschitz continuous mappings, \( g : R^n \times R^m \to R \) be a convex function, \( K \) be a compact and convex set in \( R^n \). Then for every \( x_0 \in R^n \), the following differential inclusion:

\[
\begin{align*}
\dot{x}(t) &\in F(t, x(t)) = \{ f(x(t)) + B(x(t))w(t), \ w(t) \in SOL(K, g(x(t), \cdot)) \}, \\
x(0) &\in x_0.
\end{align*}
\]

has a weak solution in the sense of Carathéodory. Furthermore, DOP(2) has a weak solution in the sense of Carathéodory.

**Proof.** The convexity of \( g \) on \( R^n \times R^m \) implies that \( g \) is continuous on \( R^n \times R^m \). By the conditions that \( g \) is continuous on \( R^n \times R^m \) and \( K \) is a compact subset of \( R^m \), it implies that for any \( q \in R^m \), the following optimization problem:

\[
\begin{align*}
\text{Minimize} & \quad g(q, w) \\
\text{subject to} & \quad w \in K,
\end{align*}
\]

has a solution. The continuity of \( g \) implies that for every \( q \in R^m \), \( SOL(K, g(q, \cdot)) \) is closed. Now we prove that \( SOL(K, g(q, \cdot)) \) is convex. Let \( g_{\min} \) denote the minimum of the objective function \( g(q, \cdot) \) on \( K \). For every \( w_1, w_2 \in SOL(K, g(q, \cdot)) \) and \( \lambda \in [0, 1] \), the convexity of \( g \) implies that

\[
\begin{align*}
g_{\min} &\leq g(q, \lambda w_1 + (1 - \lambda)w_2) \\
&\leq \lambda g(q, w_1) + (1 - \lambda)g(q, w_2) \\
&= \lambda g_{\min} + (1 - \lambda)g_{\min} \\
&= g_{\min}.
\end{align*}
\]

It follows that \( g(q, \lambda w_1 + (1 - \lambda)w_2) = g_{\min} \) and so \( \lambda w_1 + (1 - \lambda)w_2 \in SOL(K, g(q, \cdot)) \).

Next we prove that \( F \) is upper semicontinuous, it suffices to prove that \( F \) is closed from the compactness of \( K \). Take a sequence \((t_k, x_k)\) \( \in [0, T] \times R^n \). Let \( w_k \in SOL(K, g(q, \cdot)) \) for every \( k = 1, 2 \cdots \). Assume that

\[
(t_k, x_k) \to (t_0, x_0) \in [0, T] \times R^n
\]

and

\[
f(x_k) + B(x_k)w_k \to z_0 \in R^n
\]
as \( k \to \infty \). The compactness of \( K \) implies that there exists a subsequence of \( \{w_k\} \), which is denoted again by \( \{w_k\} \), such that \( w_k \to w_0 \). The continuity of \( g \) implies that \( w_0 \in SOL(K, g(x_0, \cdot)) \). From the uniqueness of limitation and the continuity of \( f \) and \( B \), it follows that

\[
z_0 = f(x_0) + B(x_0)w_0 \in F(t_0, x_0).
\]

Therefore, \( F \) is closed and so upper semicontinuous.

Then we prove that the linear growth property of \( F \). Since \( f \) and \( B \) are Lipschitz continuous, it implies that there exist \( \rho_f > 0 \) and \( \rho_B > 0 \) such that

\[
\|f(x)\| \leq \rho_f (1 + \|x\|)
\]
and
\[ \|B(x)\| \leq \rho_B(1 + \|x\|). \]

The compactness of \( K \) implies that there exists \( M > 0 \) such that \( \|u\| \leq M \). Then
\[ \|f(x(t)) + B(x(t))w(t)\| \leq (\rho_f + \rho_B M)(1 + \|x\|). \]

So \( F \) satisfies the linear growth property.

Based on the above discussion, it implies that \( F \) is upper semicontinuous with nonempty closed and convex values, and \( F \) satisfies the linear growth property. From Lemma 2.2, it follows that differential inclusion (5) has a weak solution in the sense of Carathéodory. Furthermore, Lemma 2.3 implies that DOP(2) has a weak solution in the sense of Carathéodory. This completes the proof. \( \square \)

4. Stability analysis for DOP(3).

**Theorem 4.1.** Let \((Z_1, d_1), (Z_2, d_2)\) be two metric spaces. Let \( K : Z_1 \Rightarrow R^m \) be a continuous set-valued mapping with nonempty closed, convex and compact values, \( g : R^n \times R^m \times Z_2 \rightarrow R \) be a continuous and convex function. Let \( f : R^n \rightarrow R^n, B : R^n \rightarrow R^{n \times m} \) be two Lipschitz continuous mappings. Then SOL(DOP(u, v)) is closed on \( Z_1 \times Z_2 \).

**Proof.** Take \((u_0, v_0) \in Z_1 \times Z_2\). Assume that \( U \times V \subset Z_1 \times Z_2 \) is a neighborhood of \((u_0, v_0)\). Then from Theorem 3.1, it follows that for every \((u, v) \in U \times V\), SOL(DOP(u, v)) is nonempty. Take a subsequence \( \{(u_n, v_n)\} \subset U \times V \) with \((u_n, v_n) \rightarrow (u_0, v_0)\), and \((x_n, w_n) \in SOL(DOP(u_n, v_n)) \) with \((x_n, w_n) \rightarrow (x_0, w_0)\).

To prove the closeness of SOL(DOP(u, v)), it suffices to prove that \((x_0, w_0) \in SOL(DOP(u_0, v_0))\). By the assumption that \((x_n, w_n) \in SOL(DOP(u_n, v_n))\), we have

(i)
\[ \frac{dx_n(t)}{dt} = f(x_n(t)) + B(x_n(t))w_n(t), \] (6)

(ii) for every \( t \in [0, T] \) and \( \tilde{w}_n \in K(u_n) \),
\[ g(x_n(t), \tilde{w}_n, v_n) \geq g(x_n(t), w_n(t), v_n), \] (7)

(iii) the initial condition
\[ x_n(0) = x^0. \] (8)

The absolute continuity of \( x_n(t) \) implies that the equation (6) is equivalent to the following relation: for any \( 0 \leq s \leq t \leq T \),
\[ x_n(t) - x_n(s) = \int_s^t [f(x_n(\tau)) + B(x_n(\tau))w_n(\tau)]d\tau. \]

As \((x_n, w_n) \rightarrow (x_0, w_0)\), it follows from the continuity of \( f \) and \( B \) that for any \( 0 \leq s \leq t \leq T \),
\[ x_0(t) - x_0(s) = \int_s^t [f(x_0(\tau)) + B(x_0(\tau))w_0(\tau)]d\tau. \] (9)

Next we prove that \( w_0(t) \in SOL(K(u_0), g(x_0(t), \cdot, v_0)) \) for every \( t \in [0, T] \). Since \( w_n \rightarrow w_0 \), it follows that for every \( t \in [0, T] \), \( w_n(t) \rightarrow w_0(t) \). The upper semicontinuity of \( K \) and \( w_n(t) \in K(u_n) \) imply that \( w_0(t) \in K(u_0) \) for every \( t \in [0, T] \). Assume for the sake of contradiction that there exists \( t \in [0, T] \) such that \( w_0(t) \notin SOL(K(u_0), g(x_0(t), \cdot, v_0)) \). Then there exists \( \tilde{w}_0 \in K(u_0) \) such that
\[ g(x_0(t), \tilde{w}_0, v_0) < g(x_0(t), w_0(t), v_0), \]
and
\[ g(x_0(t), \tilde{w}_n, v_n) < g(x_0(t), w_0(t), v_0) - \frac{g(x_0(t), w_0(t), v_0) - g(x_0(t), \tilde{w}_n, v_0)}{2}. \] (10)

By the lower semicontinuity of \( K \), it implies that there exist \( \hat{w}_n, v_n \in K(u_n) \) such that \( \hat{w}_n \to \tilde{w}_n \) and so \( (x_n, \hat{w}_n, v_n) \to (x_0, \tilde{w}_0, v_0) \). The continuity of \( g \) on \( R^n \times R^n \times Z_2 \) implies that
\[ g(x_n(t), \hat{w}_n, v_n) \to g(x_0(t), \tilde{w}_0, v_0). \]
Then there exists \( N_1 > 0 \) with \( n > N_1 \) such that
\[ g(x_n(t), \hat{w}_n, v_n) < g(x_0(t), w_0(t), v_0) + \frac{g(x_0(t), w_0(t), v_0) - g(x_0(t), \tilde{w}_0, v_0)}{4}. \] (11)

Since \( (x_n, w_n, v_n) \to (x_0, \tilde{w}_0, v_0) \), it implies that
\[ g(x_n(t), w_n(t), v_n) \to g(x_0(t), w_0(t), v_0). \]
Then there exists \( N_2 > 0 \) with \( n > N_2 \) such that
\[ g(x_n(t), w_n(t), v_n) < g(x_0(t), w_0(t), v_0) + \frac{g(x_0(t), w_0(t), v_0) - g(x_0(t), \tilde{w}_0, v_0)}{4}. \] (12)

It follows from (10), (11) and (12) that for any \( n > \max\{N_1, N_2\} \),
\[ g(x_n(t), w_n(t), v_n) > g(x_n(t), \hat{w}_n(t), v_n) \]
and so \( w_n(t) \notin \text{SOL}(\text{DOP}(u_n, v_n)) \), this is a contradiction to (7). Then for every \( t \in [0, T] \) and \( \tilde{w}_0 \in K(u_0) \),
\[ g(x_0(t), \tilde{w}_0, v_0) \geq g(x_0(t), w_0(t), v_0). \] (13)
Since \( x_n \to x_0 \), it follows that
\[ x_0(0) = x^0. \] (14)

Therefore, (9), (13) and (14) imply that \( (x_0, w_0) \in \text{SOL}(\text{DOP}(u_0, v_0)) \). As \( (u_0, v_0) \) is taken arbitrarily on \( Z_1 \times Z_2 \), it deduces that \( \text{SOL}(\text{DOP}(u, v)) \) is closed on \( Z_1 \times Z_2 \). This completes the proof. \( \square \)

5. **An algorithm for DOP.** Based on the numerical method to derive the solution for differential variational inequality [17], we consider the algorithm for DOP (2).

It begins with the division of the time interval \([0, T]\) into \( N_l + 1 \) subintervals:
\[ 0 = t_{l,0} < t_{l,1} < \cdots < t_{l,N_l} < t_{l,N_l+1} = T, \]
where \( l > 0 \) and \( (N_l + 1)l = T \) and \( t_{l,i+1} = t_{l,i} + l \), for all \( i = 0, 1, \cdots, N_l \).

Starting from \( x^0 \in R \), it computes \( w^0 \), which satisfies the following optimization problem:
\[
\text{Minimize} \quad g(x^0, w) \\
\text{subject to} \quad w \in K, \] (15)

and two finite families of vectors:
\[ \{x^{l,1}, x^{l,2}, \cdots, x^{l,N_l+1}\} \subset R^n, \]
\[ \{w^{l,1}, w^{l,2}, \cdots, w^{l,N_l+1}\} \subset K, \]
by the recursion: for \( i = 1, 2, \cdots, N_l + 1, \)
\[
\begin{align*}
    x^{l,i+1} &= x^{l,i} + l(f(x^{l,i}) + B(x^{l,i})w^{l,i}), \\
    w^{l,i+1} &\in \text{SOL}(K, g(x^{l,i+1}, \cdot)), \\
    x^{l,0} &= x^0.
\end{align*}
\] (16)
Let
\[
x^l(t) \equiv x^{l,i} + \frac{t-t_{i,i}}{t_{i,i+1}-t_{i,i}}(x^{l,i+1} - x^{l,i}), \quad \forall t \in [t_{i,i}, t_{i,i+1}].
\]
(17)

We study the convergence of the time-stepping scheme in Theorem 5.1.

**Theorem 5.1.** Let \( f : R^n \to R^n, B : R^n \to R^{n \times m} \) be two Lipschitz continuous mappings, let \( f(R^n) \) and \( B(R^n) \) be bounded, and let \( K \) be a bounded, closed and convex subset of \( R^n \). Then there is a sequence \( \{l_v\} \downarrow 0 \) such that the following two limits exist: \( x^{l_v} \to x \) and \( w^{l_v} \to w \) in \( L^2([0,T],[K]) \), where \( x^{l_v} \) and \( w^{l_v} \) are defined by (17), and \( \rightarrow \) denotes the weak convergence.

Furthermore, assume that the following conditions hold: (i) for any \( w \in K \), \( g(\cdot,w) \) is continuous on \( R^n \), (ii) for any \( x \in R^n \), \( g(x,\cdot) \) is strongly continuous on \( L^2([0,T],[K]) \), then all such limits \( (x,w) \) are weak solutions of DOP(2).

**Proof.** Since \( x^{l,v,i+1} = x^{l,i} + I(f(x^{l,i}) + B(x^{l,i})w^{l,i}) \) and the boundedness of \( f, B \) and \( K \), it follows that there exists \( M > 0 \) such that
\[
\|x^{l,v,i+1} - x^{l,i}\| \leq Ml_v.
\]
This implies that \( \{x^l(t)\} \) is equicontinuous and uniform bounded. By Arzela-Ascoli theorem, there exists a sequence \( \{l_v\} \downarrow 0 \) such that \( \{x^{l,v}\} \) converges to a function \( x \) with respect to the norm \( \|x\|_1 = \sup_{t \in [0,T]} \|x(t)\| \). Following a similar way to the proof of Theorem 7.1 in [17], we have
\[
x^{l,v,i+1} - x^{l,v,i} = l_v(f(x^{l,v,i}) + B(x^{l,v,i})w^{l,v,i})
\]
\[
= \int_{t^{l,v,i}}^{t^{l,v,i+1}} (f(x^{l,v,i}) + B(x^{l,v,i})w^{l,v,i})d\tau + O(l_v),
\]
where \( O(t) \) denotes a function satisfying \( \lim_{t \downarrow 0} \frac{O(t)}{t} < \infty \). Then for any \( 0 \leq s \leq t \leq T \), we have
\[
x^{l,v}(t) - x^{l,v}(s) = \int_{s}^{t} (f(x^{l,v}(\tau)) + B(x^{l,v}(\tau))w^{l,v}(\tau))d\tau + O(l_v).
\]
As \( l_v \to 0 \), it follows that,
\[
x(t) - x(s) = \int_{s}^{t} (f(x(\tau)) + B(x(\tau))w(\tau))d\tau.
\]
(18)

The boundedness of \( K \) implies that \( \{w^{l,v}\} \) is uniform bounded. By Alaoglu’s theorem, it follows that the sequence \( \{w^{l,v}\} \) has a weak* limit \( w \). The reflexive Banach space \( L^2([0,T],[K]) \) implies that weak* convergent sequences are also weakly convergent sequences. In addition, we have
\[
\|g(x^{l,v}(t),\hat{w}) - g(x^{l,v}(t),w^{l,v}(t))\| \leq \|g(x^{l,v}(t),\hat{w}) - g(x^{l,v}(t),w^{l,v}(t))\| + \|g(x(t),\hat{w}) - g(x(t),w^{l,v}(t))\| + \|g(x(t),\hat{w}) - g(x(t),w^{l,v}(t))\|,
\]
and for every \( t \in [0,T] \),
\[
g(x^{l,v}(t),\hat{w}) - g(x^{l,v}(t),w^{l,v}(t)) \geq 0, \quad \forall \hat{w} \in K.
\]
Then the strongly continuity of \( g \) implies that
\[
g(x(t),\hat{w}) - g(x(t),w(t)) \geq 0, \quad \forall \hat{w} \in K.
\]
(19)
Therefore, it follows from (18) and (19) that \((x, w)\) is a weak solutions of the DOP(2).

6. **Numerical experiments.** In this section, we provide some examples to verify the validity of algorithm, which has been introduced in Section 5. We firstly study a differential optimization problem without a perturbed parameter in Example 6.1.

**Example 6.1.**

\[
\begin{align*}
\dot{x}(t) &= |x(t) - 0.3| + 3x(t)w(t) + 5 \\
w(t) &\in \text{SOL}([-3, 3], w^2(t) + x(t)w(t)) \\
x(0) &= 0.1 \\
t &\in [0, 3].
\end{align*}
\]

**Algorithm 6.1.**

**Step 0.** It begins with the division of the time interval \([0, 3]\) into 30 subintervals:

\[0 < 0.1 < 0.2 < \ldots < 2.8 < 2.9 < 3,\]

with each of length \(l = 0.1\).

**Step 1.** Let \(x^{l,0} = 0.1\). Compute \(w^{l,0}\) which satisfies the following optimization problem,

\[
\begin{align*}
\text{Minimize} & \quad w^2 + x^{l,0}w \\
\text{subject to} & \quad w \in [-3, 3].
\end{align*}
\]

**Step 2.** Let

\[x^{l,i+1} = x^{l,i} + l(|x^{l,i} - 0.3| + 3x^{l,i}w^{l,i} + 5),\]

and let \(w^{l,i+1}\) be the solution of the following optimization problem,

\[
\begin{align*}
\text{Minimize} & \quad w^2 + x^{l,i}w \\
\text{subject to} & \quad w \in [-3, 3].
\end{align*}
\]

By the recursion, for \(i = 0, 1, 2, \ldots, 30\), the numerical results are shown in Figure 1.

**Figure 1.** The trajectory of \(x(t)\) and \(w(t)\)

In Example 6.2, we consider the differential optimization problem when the objective function is perturbed by a parameter \(v\).
Example 6.2.

\[
\begin{align*}
\dot{x}(t) &= |x(t) - 0.3| + 3x(t)w(t) + 5 \\
w(t) &\in \text{SOL}([-3, 3], w^2(t) + vx(t)w(t)) \\
x(0) &= 0.1 \\
t &\in [0, 3].
\end{align*}
\]

Following a similar algorithm to Example 6.1, the numerical results of Example 6.2 are shown in Figure 2. Observe the trajectories of \(x(t)\) and \(w(t)\) when the parameters \(v_1 = 1, v_2 = 0.9, v_3 = 0.7, v_4 = 0.4\), we can find that \((x_v, w_v) \rightarrow (x_1, w_1)\) as \(v \rightarrow 1\), where \((x_v, w_v)\) denotes a Carathéodory weak solution of DOP (22).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The trajectory of \(x(t)\) and \(w(t)\)}
\end{figure}

Furthermore, in Example 6.3, we consider the differential optimization problem when the constraint set is perturbed by a parameter \(u\).

Example 6.3.

\[
\begin{align*}
\dot{x}(t) &= |x(t) - 0.3| + 3x(t)w(t) + 5 \\
w(t) &\in \text{SOL}([-3u, 3u], w^2(t) + x(t)w(t)) \\
x(0) &= 0.1 \\
t &\in [0, 3].
\end{align*}
\]

Following a similar algorithm to Example 6.1, the numerical results of Example 6.3 are shown in Figure 3.

Observe the trajectories of \(x(t)\) and \(w(t)\) when the parameters \(u_1 = 1, u_2 = 0.33, u_3 = 0.3, u_4 = 0.1\), we can find that \((x_u, w_u) \rightarrow (x_1, w_1)\) as \(u \rightarrow 1\), where \((x_u, w_u)\) denotes a Carathéodory weak solution of DOP (23). The conclusions of Example 6.2 and 6.3 are consistent with the conclusion of Theorem 4.1 that the Carathéodory weak solution set mapping is closed.

7. Conclusions. In this paper, a class of differential optimization problems have been introduced and studied. The main contributions in this paper include establishing the existence theorem of a Carathéodory weak solution of the differential optimization problem, studying the stability analysis of differential optimization problem, and establishing an algorithm for solving the differential optimization problem. As future extensions, the research on how realizable this approach is for dynamical portfolio problem and other real world large scale problems should be given. In addition, differential vector optimization problem, consisting of a system
Figure 3. The trajectory of \( x(t) \) and \( w(t) \)

of differential equation and vector optimization problem, can also be studied, since
the model can be used to study the fermentation dynamics problem [20], human
migration networks [16] and so on.

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