THE PSEUDOSPECTRAL PROPERTIES OF NON-SELF-ADJOINT SCHRÖDINGER OPERATORS IN THE SEMI-CLASSICAL LIMIT

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Abstract

We describe the general qualitative behaviour of the resolvent norm for a very wide class of non-self-adjoint Schrödinger operators in the semi-classical regime, as the spectral parameter $\lambda$ varies over the complex plane.

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1 Introduction

Several authors [4, 9, 11] have demonstrated that the spectrum of certain Schrödinger operators with complex potentials is unstable; this is shown by observing that the resolvent norm $\| (H - \lambda)^{-1} \|$ becomes unbounded as some parameter associated with the operator $H$ varies, even though $\lambda$ may be far from the spectrum of $H$. This phenomena is commonly quantified using the concept of the pseudospectral sets

$$\text{Spec}_\epsilon(H) := \text{Spec}(H) \cup \{ \lambda \in \mathbb{C} : \| (H - \lambda)^{-1} \| \geq \epsilon^{-1} \},$$

where we adopt the convention that if $\lambda \in \text{Spec}(H)$ then $\| (H - \lambda)^{-1} \| := \infty$. For any closed operator $T$ acting on a Hilbert space $\mathcal{H}$ the numerical range, defined by

$$\text{Num}(T) := \{ \langle Tf, f \rangle : f \in \text{Dom}(T), \ \| f \| = 1 \}$$

is a convex subset of $\mathbb{C}$ [1, Theorem 6.1], containing $\text{Spec}(T)$ if $(T+s)$ is maximal quasi-accretive [7, p.279]; equivalently, if

$$\text{Re} \ (\text{Num}(T + s)) \geq 0 \quad \text{and} \quad \text{Ran}(T + t) = \mathcal{H}$$
for some (and hence all) \( t > s \). It is then well-known that, provided \( \mathbb{C} \setminus \text{Num}(T) \) is a connected set
\[
\|(T - \lambda)^{-1}\| \leq \frac{1}{\text{dist}(\lambda, \text{Num}(T))}
\]
for all \( \lambda \notin \text{Num}(T) \) [7, p.268]. We will be concerned with the Schrödinger operator
\[
H_h := -\hbar^2 \Delta + V
\]
with complex-valued \( V \) and \( h > 0 \), acting in \( L^2(\Omega) \) where \( \Omega \) is some region in \( \mathbb{R}^N \). We assume Dirichlet boundary conditions throughout. The \( h \)-independent set
\[
\Phi(V) := \{\text{Ran}(V) + [0, \infty)\}
\]
will be of fundamental importance. In Section 3 we show that for any \( \lambda \in \Phi(V) \), the resolvent norm tends to infinity in the semi-classical limit, \( h \to 0 \). The fact that \( V \) may take complex values means that \( \text{conv}(\Phi(V)) \setminus \Phi(V) \) (\( \text{conv} \) denoting the convex hull) is in general non-empty, and in Section 4 we give new results (Theorems 5 and 6) which show that in the semi-classical limit a bound analogous to (1) holds for \( \lambda \) in this set.

2 Preliminaries

First we discuss the conditions which we shall impose on \( V \) and the question of the domain of the operator \( H_h \). We assume that there exists a closed set \( M \subseteq \Omega \) with Lebesgue measure zero, such that the restriction \( V|_{\Omega \setminus M} \) is continuous and bounded. Since the operator \( H_h \) will not be changed on sets of measure zero, we allow \( V \) to be undefined on \( M \). Defining \( H_{h,0} \) to be the self-adjoint operator \(-\hbar^2 \Delta \) with domain \( W^{1,2}_0(\Omega) \), it is trivial that \( V \) has relative bound zero with respect to \( H_{h,0} \) since \( V \in L^\infty(\Omega) \). Therefore, \( \text{Dom}(H_h) = W^{1,2}_0(\Omega) \), and the fact that
\[
\|V(H_{h,0} + \lambda)^{-1}\| \leq \|V\|_\infty \|(H_{h,0} + \lambda)^{-1}\| = \lambda^{-1}\|V\|_\infty
\]
for all \( \lambda > 0 \), together with an argument similar to the proof of [3, Theorem 1.4.2] shows that \( H_h \) is maximal on \( W^{1,2}_0(\Omega) \). Moreover, since we may integrate by parts
\[
\langle H_h f, f \rangle = \int_\Omega V(x) |f(x)|^2 \, dx + \hbar^2 \int_\Omega |\nabla f|^2 \, dx
\]
for all \( f \in W^{1,2}_0(\Omega) \). Since
\[
\{\text{Re } (V(x)) : x \in \Omega \setminus M \} \geq k
\]
for some \( k \in \mathbb{R} \), it follows that \( H_h \) is maximal quasi-accretive and so \( \text{Spec}(H_h) \subseteq \text{Num}(H_h) \). It also follows from (3) that
Lemma 1 With $H_h$ as defined above
\[ \overline{\text{Num}(H_h)} \subseteq \text{conv}(\Phi(V)) \]
for all $h > 0$.

Proposition 2 For all $\lambda \notin \text{conv}(\Phi(V))$
\[ \|(H_h - \lambda)^{-1}\| \leq \frac{1}{\text{dist}(\lambda, \text{conv}(\Phi(V)))}. \]

Proof $H_h$ satisfies the conditions for (1) to hold, and then one can apply the lemma.

3 Spectral instability

In a sense our result in this section generalises that of [5, Section 2]; however, our conclusion is not as strong since we do not show super-polynomial growth of the resolvent norm in the semi-classical limit. Note that the specific form of the phase function $x \mapsto \gamma \cdot x$ below has been chosen to ensure that the second derivative disappears, and to exploit the fact that $\lambda := V(c) + |\gamma|^2$. We have not attempted to optimise the choice of phase function, but instead have aimed for a clear demonstration of the underlying processes.

Theorem 3 For $z \in \Phi(V)$, we have
\[ \|(H_h - \lambda)^{-1}\| \to \infty \quad \text{as } h \to 0. \]

Proof Let $\lambda \in \{\text{Ran}(V) + [0, \infty)\}$ so that $\lambda := V(c) + |\gamma|^2$, where $c \in \Omega \setminus M$, and $\gamma \in \mathbb{R}^N$. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be a function whose support is contained within the open ball $B(0; 1)$, and put
\[ \phi_{\lambda,h}(x) := \phi\left(\frac{x - c}{h^p}\right) \]
where $p > 0$ is a constant to be determined, and
\[ f_{\lambda,h}(x) := e^{ih^{-1}\gamma \cdot x} \phi_{\lambda,h}(x). \]

Then supp$(f_{\lambda,h}) \subseteq B(c; h^p)$ and
\[
H_h f_{\lambda,h} = -h^2 \Delta e^{ih^{-1}\gamma \cdot x} \phi_{\lambda,h} + Ve^{ih^{-1}\gamma \cdot x} \phi_{\lambda,h}
= -h^2 e^{ih^{-1}\gamma \cdot x} \Delta \phi_{\lambda,h} - h^2 \phi_{\lambda,h} \Delta e^{ih^{-1}\gamma \cdot x} - 2h^2(\nabla e^{ih^{-1}\gamma \cdot x} \cdot \nabla \phi_{\lambda,h}) + Ve^{ih^{-1}\gamma \cdot x} \phi_{\lambda,h}
= -h^2 e^{ih^{-1}\gamma \cdot x} \Delta \phi_{\lambda,h} + |\gamma|^2 e^{ih^{-1}\gamma \cdot x} \phi_{\lambda,h} - 2ihe^{ih^{-1}\gamma \cdot x} \gamma \cdot \nabla \phi_{\lambda,h} + Ve^{ih^{-1}\gamma \cdot x} \phi_{\lambda,h}.
\]
Therefore, it follows that
\[ \| (H_h - \lambda) f_{\lambda,h} \|_2 \leq h^2 \| \Delta \phi_{\lambda,h} \|_2 + 2h \| \gamma \cdot \nabla \phi_{\lambda,h} \|_2 + \| (V(x) - V(c)) \|_{B(c,h^p)} \| \phi_{\lambda,h} \|_2 \]
and so
\[ \frac{\| (H_h - \lambda) f_{\lambda,h} \|_2}{\| f_{\lambda,h} \|_2} \leq \frac{h^2 \| \Delta \phi_{\lambda,h} \|_2}{\| \phi_{\lambda,h} \|_2} + 2h \| \gamma \cdot \nabla \phi_{\lambda,h} \|_2 + \| (V(x) - V(c)) \|_{B(c,h^p)} \| \phi_{\lambda,h} \|_2. \]

Taking the limit,
\[ \lim_{h \to 0} \frac{\| (H_h - \lambda) f_{\lambda,h} \|_2}{\| f_{\lambda,h} \|_2} \leq \lim_{h \to 0} \left( k_1 h^{2-2p} + k_2 h^{1-p} + \| (V(x) - V(c)) \|_{B(c,h^p)} \right) \]
where \( k_1, k_2 \) are constants dependent upon our choice of \( \phi \) and \( \gamma \).

Thus, by taking \( 0 < p < 1 \), the continuity of \( V \) on \( \Omega \setminus M \) ensures that
\[ \lim_{h \to 0} \frac{\| (H_h - \lambda) f_{\lambda,h} \|_2}{\| f_{\lambda,h} \|_2} = 0, \]
or equivalently, that
\[ \lim_{h \to 0} \| (H_h - \lambda)^{-1} \| = \infty. \]

Now let \( w \in \Phi(V) \) and, aiming for a contradiction, suppose that
\[ \| (H_h - w)^{-1} \| \leq m \quad \text{as } h \to 0. \]

For any \( \delta > 0 \) there exists \( \lambda \in \{ \text{Ran}(V) + [0, \infty) \} \) such that \( |\lambda - w| < \delta \), by the definition of the closure. Hence, using the resolvent identity,
\[ \| (H_h - \lambda)^{-1} \| = \| (H_h - w)^{-1} - (\lambda - w)(H_h - \lambda)^{-1}(H_h - w)^{-1} \| \leq m + |\lambda - w| m \| (H_h - \lambda)^{-1} \| < m + \delta m \| (H_h - \lambda)^{-1} \| \quad \text{as } h \to 0. \]

Since \( \delta \) may be taken arbitrarily small, we have
\[ \| (H_h - \lambda)^{-1} \| \leq m \]
as \( h \to 0 \), contradicting the result just obtained, and completing the proof.

## 4 Bounded behaviour of the Resolvent norm

We have seen that when \( \lambda \in \Phi(V) \) the resolvent norm becomes infinitely large as \( h \to 0 \) (Theorem 3). Conversely, when \( \lambda \notin \text{conv}(\Phi(V)) \) the resolvent norm is uniformly bounded in \( h > 0 \) (Proposition 2). Therefore, the natural question arises: how does \( \lim_{h \to 0} \| (H_h - \lambda)^{-1} \| \) behave for \( \lambda \in \text{conv}(\Phi(V)) \setminus \Phi(V) \)? On the one hand, Proposition 2 cannot in general be extended to \( \lambda \in \text{conv}(\Phi(V)) \setminus \Phi(V) \), as the following counter-example shows:
Example 4 Consider the operator

\[ H_{\delta,h} f(x) := -h^2 f''(x) + V_{\delta}(x) f(x) \]

acting on \( L^2(-1,1) \)

where

\[ V_{\delta}(x) := \begin{cases} i(x + \delta) & \text{for } x > 0 \\ i(x - \delta) & \text{for } x < 0 \end{cases} \]

For any \( \lambda > 0 \), it follows that \( \lambda \in \text{conv}(\Phi(V_{\delta})) \setminus \Phi(V_{\delta}) \), where

\[ \Phi(V_{\delta}) := \{ \text{Ran}(V_{\delta}) + [0, \infty) \} \]

But, we have shown elsewhere \[ \cite{10} \] that countably many of the eigenvalues \( \{ \lambda_{h,n} \}_{n=1}^{\infty} \) of \( H_{\delta,h} \) are positive real, for every \( h > 0 \), in which case

\[ \| (H_{\delta,h} - \lambda_{h,n})^{-1} \| := \infty. \]

On the other hand, for a wide class of potentials, we have the positive result (initially suggested by numerical simulations of the associated discrete problem using Matlab) that for any given \( \lambda \in \text{conv}(\Phi(V)) \setminus \Phi(V) \) the resolvent norm becomes bounded eventually, as \( h \to 0 \). In the one-dimensional case we have the following result, which we believe to be new.

**Theorem 5** Let \( K_h \) be the non-self-adjoint operator

\[ K_h f(x) := -h^2 \frac{d^2 f}{dx^2} + V(x) f(x) \] (4)

acting in \( L^2(a,b) \), \(-\infty \leq a < b \leq +\infty \) and \( h > 0 \). When \( a \) or \( b \) are finite we impose Dirichlet boundary conditions. We assume that there exists a partition

\[ a = x_0 < x_1 < \cdots < x_n = b \]

of the interval \( (a,b) \), such that the complex-valued \( V \in L^\infty \) satisfies:

(i) \( \{ \text{Re } V(x) : x \in (a,b) \} \geq k \) for some \( k \in \mathbb{R} \).

(ii) \( V \in C^2 \) on each sub-interval \( (x_j, x_{j+1}) \), \( j = 0, \ldots, n-1 \).

(iii)

\[ \int_a^b \left| \frac{q''(x)}{q^{3/2}(x)} - \frac{5}{4} \frac{q''(x)}{q^{5/2}(x)} \right| \, dt < \infty \]

holds, where \( q(x) := V(x) - \lambda \). Then, for any \( \lambda \in \mathbb{C} \setminus \Phi(V) \), we have

\[ \limsup_{h \to 0} \| (K_h - \lambda)^{-1} \| \leq \frac{1}{\text{dist}(\lambda, \Phi(V))}. \]

**Proof** We first prove the case when \(-\infty < a < b < +\infty \), noting that (iii) is then automatically satisfied. Our proof will involve adding extra points to those in the given partition of the real interval \( (a,b) \). We will then estimate the resolvent norm.
on each of the sub-intervals \((x_j, x_{j+1})\). Without loss of generality therefore, we may initially assume that \(V\) is twice continuously differentiable and bounded on the interval \((-1, 1)\), since the extension to the general case follows easily. Our method uses the so called WKB approximations \([6, 8]\) to the solutions of the differential equation
\[
(K_h - \lambda)f(x) = 0
\]
in the semi-classical limit \(h \to 0\). The operator \(K_h\) is defined formally by (4). By considering only \(\lambda \notin \Phi(V)\) we have \(V(x) - \lambda \neq 0\) for all \(x \in (-1, 1)\). Moreover, since the path \(\gamma\) defined by
\[
\gamma: (-1, 1) \to V(x) - \lambda
\]
does not cross the negative real axis, condition (ii) also ensures that we may choose a twice continuously differentiable branch of \(\sqrt{V(x) - \lambda}\) such that
\[
\text{Re} \sqrt{V(x) - \lambda} > 0
\]
for all \(x \in (-1, 1)\). Denoting the definite integral
\[
\xi(x) := \int_a^x \sqrt{V(t) - \lambda} \, dt
\]
where \(a \in (-1, 1)\) is arbitrary and introduces a constant term which we will omit from our later calculations, it follows from (3) that the function \(x \mapsto \text{Re} \xi(x)\) is increasing on the interval \((-1, 1)\). This fact will be called upon several times in our proof. Property (3) will greatly simplify our application of the WKB approximations, since questions about the Stokes’ phenomenon and valid domains do not then arise.

For fixed \(h > 0\), let the functions \(g_1\) and \(g_2\) be linearly independent, exact classical solutions to (5). For any \(\alpha \in [-1, 1]\), put
\[
g\{\alpha; x\} := g_2(\alpha)g_1(x) - g_2(x)g_1(\alpha) \quad x \in [-1, 1],
\]
so that \(g\{-1; x\}\) and \(g\{1; x\}\) are also independent (classical) solutions satisfying \(g\{-1; -1\} = g\{1; 1\} = 0\). Then by elementary Sturm-Liouville theory, for any \(\lambda \in \mathbb{C}\) which is not an eigenvalue, the Green function is given by
\[
G_\lambda(x, y) = -W^{-1}_\lambda \begin{cases} g\{-1; x\}g\{1; y\} & \text{for } -1 \leq x < y \\ g\{1; x\}g\{-1; y\} & \text{for } y < x \leq 1 \end{cases}
\]
where the Wronskian
\[
W_\lambda := g\{-1; 1\}(g_2(0)g_1'(0) - g_1(0)g_2'(0)).
\]
Now consider the operator \(\tilde{K}_h\), again defined formally by (4), but with the extra ‘boundary condition’ \(f(0) = 0\), so that \(\tilde{K}_h\) effectively acts on the space \(L^2(-1, 0) \oplus L^2(0, 1)\). Then the difference resolvent operator
\[
(\tilde{K}_h - \lambda)^{-1} - (K_h - \lambda)^{-1}
\]
is of rank one, provided \( \lambda \) is not an eigenvalue. Moreover, by Lemma 7 in the Appendix, the operator has a resolvent kernel given by

\[
\psi(x, y) := c \phi(x) \phi(y)
\]

where

\[
\phi(x) := \begin{cases} 
  g\{-1; x\}/g\{-1; 0\} & \text{for } -1 \leq x \leq 0 \\
  g\{1; x\}/g\{1; 0\} & \text{for } 0 \leq x \leq 1
\end{cases}
\]

and

\[
c := W^{-1}_\lambda g\{-1; 0\} g\{1; 0\}.
\]

Here we have chosen the normalising constant \( c \) so that \( \phi(0) = 1 \). (7) is seen to be the rank one operator which acts as

\[
((\tilde{K}_h - \lambda)^{-1} - (K_h - \lambda)^{-1}) f(x) := c \phi(x) \int_{-1}^1 f(y) \phi(y) \, dy = c \phi(f, \overline{\phi})
\]

on \( f \in L^2(-1, 1) \), where

\[
\|((\tilde{K}_h - \lambda)^{-1} - (K_h - \lambda)^{-1}) f\| = |c| \|\phi\|_2 \|\overline{\phi}\|_2 = |c| \|\phi\|^2_2.
\]

We now turn to the semi-classical behaviour as \( h \to 0 \). Under the assumptions of the theorem, asymptotic approximations to solutions of (5) are given by (e.g. [6, p33])

\[
y_1(x) = (V(x) - \lambda)^{-1/4} \exp\{h^{-1} \xi(x)\} (1 + O(h))
\]

and

\[
y_2(x) = (V(x) - \lambda)^{-1/4} \exp\{-h^{-1} \xi(x)\} (1 + O(h))
\]

as \( h \to 0 \). The bound for the remainder term is uniform on \( x \in (-1, 1) \), in the sense that \( |O(h)| \leq mh \) for \( h \leq 1 \), where \( m \) does not depend upon \( x \). These approximations may be differentiated with respect to \( x \), giving

\[
y'_1(x) = -h^{-1}(V(x) - \lambda)^{1/4} \exp\{h^{-1} \xi(x)\} (1 + O(h))
\]

and

\[
y'_2(x) = -h^{-1}(V(x) - \lambda)^{1/4} \exp\{-h^{-1} \xi(x)\} (1 + O(h))
\]

as \( h \to 0 \), again the remainder term being uniform on \( x \in (-1, 1) \). The functions (8), (9), (10) and (11) will be used to estimate the norm of the rank one difference operator (7) as \( h \to 0 \). Indeed, substituting the approximate solutions \( y_1, y_2, y'_1 \) and \( y'_2 \) for the exact solutions \( g_1, g_2, g'_1 \) and \( g'_2 \), one obtains for \( \alpha, \beta \in (-1, 1), \alpha < \beta \),

\[
g\{\alpha; \beta\} := g_2(\alpha) g_1(\beta) - g_2(\beta) g_1(\alpha) = y_2(\alpha) y_1(\beta) - y_2(\beta) y_1(\alpha) = 2(V(\alpha) - \lambda)^{-1/4}(V(\beta) - \lambda)^{-1/4} \sinh(h^{-1}(\xi(\beta) - \xi(\alpha))) (1 + O(h)) = (V(\alpha) - \lambda)^{-1/4}(V(\beta) - \lambda)^{-1/4} \exp\{h^{-1}(\xi(\beta) - \xi(\alpha))\} (1 + O(h))
\]
as $h \to 0$. The last line uses the fact that the function $x \mapsto \text{Re } \xi(x)$ is increasing on the interval $(-1, 1)$, and that the vanishing term in the sinh function decreases (much) more rapidly than $O(h)$. The Wronskian simplifies to

$$
W_λ := g\{-1; 1\}(g_2(0)g_1'(0) - g_1(0)g_2'(0)) = g\{-1; 1\}(y_2(0)y_1'(0) - y_1(0)y_2'(0))(1 + O(h)) = g\{-1; 1\}2h^{-1}(1 + O(h))
$$

as $h \to 0$, enabling us to estimate

$$
c := W_λ^{-1}g\{-1; 0\}g\{1; 0\} = h g\{-1; 0\}g\{1; 0\}/2 g\{-1; 1\}(1 + O(h)) = \frac{h(V(0) - λ)^{-1/2} \exp\{h^{-1}(ξ(1) - ξ(-1))\}(1 + O(h))}{2 \exp\{h^{-1}(ξ(1) - ξ(-1))\}(1 + O(h))} = O(h)
$$

as $h \to 0$. It also follows that for $0 < x ≤ 1$

$$
φ(x) := g\{1; x\}/g\{1; 0\} = \frac{(V(1) - λ)^{-1/4}(V(x) - λ)^{-1/4} \exp\{h^{-1}(ξ(1) - ξ(x))\}(1 + O(h))}{(V(1) - λ)^{-1/4}(V(0) - λ)^{-1/4} \exp\{h^{-1}(ξ(1) - ξ(0))\}(1 + O(h))} = \frac{(V(x) - λ)^{-1/4}}{(V(0) - λ)^{-1/4}} \exp\{h^{-1}(ξ(0) - ξ(x))\}(1 + O(h))
$$

as $h \to 0$. We note that

$$\text{Re } (ξ(0) - ξ(x)) < 0$$

for $0 < x ≤ 1$; and $φ(0) = 1 + O(h)$. Then, using the fact that $φ$ is even, and applying the method of steepest descents

$$
\|φ\|^2_{L^2(-1,1)} = 2 \int_0^1 |φ(x)|^2 \, dx ≤ c_1 \int_0^1 \left|\exp\{h^{-1}(ξ(0) - ξ(x))\}(1 + O(h))\right|^2 \, dx = c_1 \int_0^1 \exp\{2h^{-1}\text{Re } (ξ(0) - ξ(x))\} \, dx (1 + O(h)) = c_1 \int_0^1 \exp\{-2h^{-1}\text{Re } (ξ'(0)x)\} \, dx (1 + O(h)) = \frac{c_1 h}{2\text{Re } (ξ'(0))}(1 + O(h)) = O(h)
$$
as \( h \to 0 \), since \( \text{Re} (\zeta'(0)) = \text{Re} \sqrt{V(0) - \lambda} > 0 \). Therefore

\[
\|(\tilde{K}_h - \lambda)^{-1} - (K_h - \lambda)^{-1}\| = |c| \|\phi\|^2
\]
\[
= O(h) \|\phi\|^2
\]
\[
= O(h^2)
\]
as \( h \to 0 \).

Now defining the operator \( \tilde{K}_h \) formally by (4) but with the finite number of boundary conditions

\[
f(x_j) = 0 \quad j = 0, \ldots, n,
\]
\( \tilde{K}_h \) then effectively acts on

\[
L^2(-1, x_1) \oplus L^2(x_1, x_2) \oplus \cdots \oplus L^2(x_{n-1}, 1).
\]

By induction, our argument so far shows that the operator

\[
(\tilde{K}_h - \lambda)^{-1} - (K_h - \lambda)^{-1}
\]
is of at most rank \( n \), and we have the norm resolvent convergence

\[
\lim_{h \to 0} \|(\tilde{K}_h - \lambda)^{-1} - (K_h - \lambda)^{-1}\| = 0.
\]

(12)

Moreover, letting \( V_j \) denote the potential \( V \) restricted to the interval \((x_j, x_{j+1})\), condition (i) ensures that one can apply Proposition 2 to \( \tilde{K}_h \) separately on each interval \((x_j, x_{j+1})\). Therefore, taking

\[
\lambda \in \mathbb{C} \setminus \bigcup_{j=0}^{n-1} \text{conv}(\Phi(V_j))
\]
we have

\[
\|(\tilde{K}_h - \lambda)^{-1}\| \leq \max_{j} \left\{ \frac{1}{\text{dist}(z, \text{conv}(\Phi(V))))} \right\} < \infty
\]

(13)
uniformly on \( h > 0 \). As the partition of \((-1, 1)\) becomes increasingly fine, it is clear that for every \( \lambda \in \mathbb{C} \setminus \Phi(V) \) one has

\[
\min_{j} \{\text{dist}(\lambda, \text{conv}(\Phi(V_j)))\} \longrightarrow \text{dist}(\lambda, \Phi(V)).
\]

Then, by (12) and (13) it follows that

\[
\limsup_{h \to 0} \|(K_h - \lambda)^{-1}\| \leq \frac{1}{\text{dist}(\lambda, \Phi(V))}
\]
to complete the proof in the finite interval case.
On the interval \((a, +\infty)\), the WKB approximations to the classical solutions of (5) take the form
\[
\tilde{y}_1(x) = (V(x) - \lambda)^{-1/4} \exp\{h^{-1}\xi(x)\}(1 + o(1))
\]
and
\[
\tilde{y}_2(x) = (V(x) - \lambda)^{-1/4} \exp\{-h^{-1}\xi(x)\}(1 + o(1))
\]
as \(x \to \infty\), for all \(h > 0\), provided \((iii)\) holds (see [6, p50]). Comparing \(\tilde{y}_{1,2}\) with \(y_{1,2}\) in the proof above, and noting that \(\tilde{y}_2(x) \to 0\) exponentially as \(x \to +\infty\), one can check that the proof just given still carries through. The interval \((-\infty, b)\) is dealt with similarly, by a change of signs, completing the proof for the general case.

5 A Different Approach

In Theorem 5 we relied upon the theory of ODEs to prove the norm resolvent convergence (12). The proof of the next theorem uses a powerful but technically simple construction, the so-called ‘Twisting Trick’ [3, Section 8.6] or [2].

Theorem 6 Let \(K_h\) be defined by (4) on \(L^2(\Omega)\), where \(\Omega\) is a bounded region in \(\mathbb{R}^N\) and \(V : \bar{\Omega} \to \mathbb{C}\) is continuous. Then, for \(\lambda \notin \Phi(V)\)
\[
\limsup_{h \to 0} \| (K_h - \lambda)^{-1} \| \leq \frac{1}{\text{dist}(\lambda, \Phi(V))}.
\]

Proof If \(\lambda \notin \text{conv}(\Phi(V))\) then Proposition 4 applies and we are done. So, assume that \(\lambda \in \text{conv}(\Phi(V))\) \(\cap \Phi(V)\) is given. For any \(\delta > 0\) we define \(\{S_j\}\) to comprise \(N\)-dimensional cubes of the form \(\{(x_1, \ldots, x_N) : \delta r_i < x_i < \delta(r_i + 1)\}\), where the \(r_1, \ldots, r_N\) take integer values. Then by the uniform continuity of \(V\) on \(\bar{\Omega}\), there exists a covering of \(\bar{\Omega}\) by disjoint cubes
\[
\bar{\Omega} \subseteq \bigcup_{j=1}^{M} \bar{S}_j
\]
each of side length \(\delta\), such that
\[
\lambda \notin \bigcup_{j=1}^{M} \text{conv} \left\{ \Phi \left( V|_{\bar{\Omega} \cap \bar{S}_j} \right) \right\}.
\]
In addition, for any given \(0 < \alpha < 1\) we can always take \(\delta > 0\) small enough so that
\[
\text{dist} \left( \lambda, \bigcup_{j=1}^{M} \text{conv} \left\{ \Phi \left( V|_{\bar{\Omega} \cap \bar{S}_j} \right) \right\} \right) \geq \alpha \text{dist}(\lambda, \Phi(V)). \tag{14}
\]
The proof proceeds by a series of bisections in each of the \(N\) dimensions of \(\Omega\). Choose a point \(c = (c_1, \ldots, c_N)\) such that each \(c_i = \delta r_i\) for some integer \(r_i\). Then the
hyperplane $\{x : x_i = c_i\}$ splits the cubes into two families; one covering the region $\Omega_1 := \{x : x_i - c_i > 0\}$, and the other covering the region $\Omega_2 := \{x : x_i - c_i < 0\}$. Define

$$V_1(x) := \begin{cases} V(x) & \text{if } x \in \Omega_1 \\ m & \text{if } x \in \Omega_2 \end{cases}$$

and

$$V_2(x) := \begin{cases} V(x) & \text{if } x \in \Omega_2 \\ m & \text{if } x \in \Omega_1 \end{cases}$$

where $m$ is some sufficiently large real number. Then consider the two operators

$$H_1 := \begin{pmatrix} -h^2\Delta + V & 0 \\ 0 & -h^2\Delta + m \end{pmatrix}, \quad \quad H_2 := \begin{pmatrix} -h^2\Delta + V_1 & 0 \\ 0 & -h^2\Delta + V_2 \end{pmatrix},$$

both acting in the Hilbert space $H := L^2(\Omega) \oplus L^2(\Omega)$. We will show that

$$\|(H_1 - \lambda)^{-1}\| - \|(H_2 - \lambda)^{-1}\| \to 0$$
as $h \to 0$. Defining $\theta : \mathbb{R} \to [0, \pi/2]$ by

$$\theta(s) := \begin{cases} \pi/2 & \text{if } s \leq -1/3 \\ \pi(1-3s)/4 & \text{if } -1/3 \leq s \leq 1/3 \\ 0 & \text{if } s \geq 1/3 \end{cases}$$

we define the unitary operator $U_h : H \to H$ by

$$U_h \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} := \begin{pmatrix} \cos(\theta((x_i - c_i)/h^\gamma)) & \sin(\theta((x_i - c_i)/h^\gamma)) \\ -\sin(\theta((x_i - c_i)/h^\gamma)) & \cos(\theta((x_i - c_i)/h^\gamma)) \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$$

where $\gamma > 0$ is to be determined. Thus

$$U_h(x) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x_i \geq c_i + h^\gamma/3 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{if } x_i \leq c_i - h^\gamma/3 . \end{cases}$$

To ease notation we denote the following functions, which are to be regarded as multiplication operators on $L^2(\Omega)$,

$$C_h(x) := \cos(\theta((x_i - c_i)/h^\gamma))$$

$$S_h(x) := \sin(\theta((x_i - c_i)/h^\gamma))$$

together with the partial differentiation operator $D_i := \partial/\partial x_i$. Then, one can show using elementary matrix calculations that

$$U_h H_1 U_h^* = H_2 + P_h D_i + Q_h + G_h$$

(15)

where $P_h$, $Q_h$ and $G_h$ are the matrix-valued functions on $\Omega$ given by

$$P_h := \frac{3\pi h^{2-\gamma}}{2} \chi_c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
\[ Q_h := \frac{9\pi^2 h^{2-2\gamma}}{16} \chi_c \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \]

and

\[ G_h := (m - V) \left( \begin{array}{cc} \chi_{\Omega_1} S_h^2 \chi_{\Omega_2} C_h^2 & C_h S_h \\ C_h S_h & -\chi_{\Omega_1} S_h^2 + \chi_{\Omega_2} C_h^2 \end{array} \right), \]

where

\[ C := \{ x \in \Omega : -h^7/3 < x_i - c_i < h^7/3 \}. \]

Thus \( \|P_h\| = O(h^{2-\gamma}) \), \( \|Q_h\| = O(h^{2-2\gamma}) \) and since \( \chi_{\Omega_1} S_h \) and \( \chi_{\Omega_2} C_h \) are \( O(h^\gamma) \), \( \|G_h\| = O(h^\gamma) \), as \( h \to 0 \). We therefore take the optimal value \( \gamma = 2/3 \). Now, for our given \( \lambda \), one may write

\[(H_2 - \lambda)^{-1} - (U_h H_1 U_h^* - \lambda)^{-1} = (H_2 - \lambda)^{-1}(P_h D_i + Q_h + G_h)(U_h H_1 U_h^* - \lambda)^{-1},\]

so that

\[ \|(H_2 - \lambda)^{-1} - (U_h H_1 U_h^* - \lambda)^{-1}\| \leq \|(H_2 - \lambda)^{-1}\| \|G_h\| + \|P_h\| \|D_i(U_h H_1 U_h^* - \lambda)^{-1}\| + \|Q_h\|. \]

(16)

Now \( (H_1 - \lambda)^{-1} \) and \( (H_2 - \lambda)^{-1} \) are bounded from \( H \) to \( W_0^{1,2} \), \( D_i \) is bounded from \( W_0^{1,2} \) to \( H \), and \( U_h \) is uniformly bounded from \( W_0^{1,2} \) to \( W_0^{1,2} \) for all \( h \leq 1 \). Thus

\[ \|D_i(U_h H_1 U_h^* - \lambda)^{-1}\| = \|D_i U_h(H_1 - \lambda)^{-1} U_h^*\| \leq \beta \]

for some \( \beta < \infty \) and all \( h \leq 1 \). Therefore, from (16), we obtain

\[ \|(U_h H_1 U_h^* - \lambda)^{-1} - (H_2 - \lambda)^{-1}\| = O(h^{1/2}) \]

as \( h \to 0 \), so that

\[ \|(H_1 - \lambda)^{-1}\| = \|(U_h H_1 U_h^* - \lambda)^{-1}\| = \|(H_2 - \lambda)^{-1}\| + O(h^{1/2}) \]

where, applying Proposition 2, one has

\[ \|(H_2 - \lambda)^{-1}\| \leq \max \left\{ \frac{1}{\text{dist}(\lambda, \text{conv}(\Phi(V_1)))}, \frac{1}{\text{dist}(\lambda, \text{conv}(\Phi(V_2)))} \right\}. \]

One can now bisect each of \( \Omega_1 \) and \( \Omega_2 \) in the same manner, and carry out the above process on four copies of \( L^2(\Omega) \). Repeating until \( \Omega \) has been divided into ‘strips’ of thickness \( \delta \), one changes to another coordinate direction and repeats the process until all \( N \) dimensions have been decomposed. Then, recalling (14) we have

\[ \|(H_h - \lambda)^{-1}\| \leq \frac{\alpha^{-1}}{\text{dist}(\lambda, \Phi(V))} + O(h^{1/2}) \]

as \( h \to 0 \), where \( 0 < \alpha < 1 \) was arbitrarily chosen. Taking \( \alpha \) as close as one likes to 1 will then complete the proof.
Appendix

We give a proof of the following well-known result.

Lemma 7 Let \( L \) be the Sturm-Liouville operator

\[
(L - \lambda)f(x) := -\frac{d^2 f(x)}{dx^2} + V(x)f(x) - \lambda f(x) = 0 \tag{17}
\]

acting in \( L^2(a, b) \), together with boundary conditions

\[
f(a) = f(b) = 0. \tag{18}
\]

Here \( V \) is a complex-valued continuous function on \([a, b]\), and \( \lambda \) is a complex constant. We may assume that \( a < 0 < b \), and let \( \tilde{L} \) denote the operator given formally by (17) but subject to the additional condition \( f(0) = 0 \). Then

\[
(\tilde{L} - \lambda)^{-1} - (L - \lambda)^{-1}
\]

is a rank one operator for all \( \lambda \notin \{\text{Spec}(\tilde{L}) \cup \text{Spec}(L)\} \).

Proof. Let \( u, v \) be a pair of independent classical solutions to the differential equation (17), with \( u, v \) satisfying the boundary conditions at \( a, b \) respectively. Multiplying \( u \) by the constant \( v(0)/u(0) \), we may further assume that \( u(0) = v(0) \). Then the Green function is given by

\[
G_{\lambda}(x, y) := -W_{\lambda}^{-1} \begin{cases} u(x)v(y) & \text{if } a \leq x \leq y \leq b \\ v(x)u(y) & \text{if } b \geq x \geq y \geq a. \end{cases}
\]

For any \( x \in (a, b) \), the Wronskian is given by

\[
W_{\lambda} := u(x)v'(x) - u'(x)v(x).
\]

Imposing the boundary condition \( f(0) = 0 \), \( \tilde{L} \) effectively acts on the Hilbert space \( L^2(a, 0) \oplus L^2(0, b) \). Then, putting

\[
w(x) := u(x) - v(x)
\]

we see that \( w(0) = 0 \), and the new Wronskian on \( L^2(a, 0) \):

\[
W^- := u(0)w'(0) - u'(0)w(0) = u(0)(u'(0) - v'(0)) - u'(0)(u(0) - v(0)) = -u(0)v'(0) + u'(0)v(0) = -W.
\]

By a similar calculation, the Wronskian on \( L^2(0, b) \) is given by \( W^+ = W \). Hence we construct the new ‘Green’ function

\[
\tilde{k}(x, y) := -W^{-1} \begin{cases} 0 & \text{if } a \leq x \leq 0 \text{ and } b \geq y \geq 0 \\ 0 & \text{if } b \geq x \geq 0 \text{ and } a \leq y \leq 0 \\ w(y)v(x) & \text{if } 0 \leq y \leq x \leq b \\ v(y)w(x) & \text{if } 0 \leq x \leq y \\ -w(x)u(y) & \text{if } a \leq y \leq x \leq 0 \\ -u(x)w(y) & \text{if } a \leq x \leq y \leq 0. \end{cases}
\]
It is then straightforward to check that putting
\[ \sigma(x) := \begin{cases} v(x) & \text{if } b \geq x \geq 0 \\ u(x) & \text{if } a \leq x \leq 0 \end{cases} \]
the ‘Green’ function of
\[ (\tilde{L} - \lambda)^{-1} - (L - \lambda)^{-1} \]
is given by
\[ W^{-1}\sigma(x)\sigma(y) \]
for all \( x, y \) in \((a, b)\). Clearly \( \sigma \in L^2(a, b) \), and for all \( f \in L^2(a, b) \)
\[ \left( (\tilde{L} - \lambda)^{-1} - (L - \lambda)^{-1} \right) f(x) = W^{-1} \int_a^b \sigma(x)\sigma(y)f(y) \, dy \]
\[ = W^{-1}\sigma(x)\int_a^b \sigma(y)f(y) \, dy. \]
\[ (\tilde{L} - \lambda)^{-1} - (L - \lambda)^{-1} \]
is a compact rank one operator on \( L^2(a, b) \), which completes the proof.

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