Combustion of Fractal Distributions

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Abstract:
The advantages of introducing a fractal viewpoint in the field of combustion is emphasized. It is shown that the condition for perfect combustion of a collection of drops is the self-similarity of the distribution.
1 Introduction

The advantages of introducing geometrical viewpoints, specially fractal geometry in the analysis of complex phenomena is already unquestionable. Particularly, such complex processes as brittle fracture have been studied experimentally \cite{1} and theoretically\cite{2,3} in order to reveal their multifractal behaviour and the scaling laws present in the size distribution of the resulting fragments.

Scaling is present in many fragmentation processes i.e., the Korcak’s law for the distribution of islands, the distribution of icefields, and also the distribution of lunar craters, which at the same time reveals the distribution of meteorites (see \cite{2} for more details). More recently, we have shown that scaling is also present in the breaking of a fluid drop\cite{4} and in some regimes of atomization \cite{5}.

The process of fracture is important for combustion, since many combustion chambers burn a collection of fuel drops rather than a massive jet of fuel. Besides, the use of water-oil emulsions in combustion devices leads to the study of liquid fragmentation since it has been shown\cite{6} that drops of water-oil emulsions during combustion show a disruptive behavior giving rise to a collection of secondary droplets which increases the surface of the fluid for the reaction with air in the process of combustion. In all these above mentioned distributions the common characteristic is that the cumulative number of particles with radius \( r \) (i.e, the number of particles with radius larger than \( r \)) \( N(r) \) varies as

\[
N(r) \sim r^{-x}, \quad (1)
\]

\( x \) being the scale exponent of the distribution. Though the geometrical model proposed by Matsushita in \cite{2} for fracture does not give any specific value of \( x \), his viewpoint permits a simple description of this process. To obtain \( x \) one has to know details of the dynamics of the fracture, which is a very difficult problem, yet unsolved. The main goal of this paper is to show the advantage of this fractal viewpoint in the analysis of some combustion processes, specially in his application for the analysis of the burning of water-oil emulsions.

2 Combustion of a drop of water-oil emulsion

The presence of small droplets of water inside a drop of fuel gives to the drops of water-oil emulsion a disruptive character since they explode when burning, transforming the original drop in a collection of small droplets that improve the process of combustion as we pointed above. As it was already shown\cite{4} the break of a liquid drop by an explosive process produces a collection of fragments in which their cumulative number is given by (1). From this equation the number of drops with radius between \( r \) and \( r + dr \) can be found differentiating (1), so that if we denote such distribution by \( n(r)dr \) we have

\[
n(r)dr \sim xr^{-x-1}dr, \quad (2)
\]
We may suppose that the original drop, of radius $A$, breaks into fragments the largest of which is of radius $R = \frac{A}{\beta}$ where $\beta > 1$ is a constant. Normalizing (2):

$$\int_{0}^{R} n(r)r^3 dr = A^3,$$

with this we are assuming that the process of drop fracture is fast enough as to neglect fuel consumption during fragmentation and apply mass conservation. The normalization condition leads to the expression for $n(r)$:

$$n(r) = (3 - x)\beta^3 R^x r^{-x - 1}$$

To consider the combustion of this system of fragments we may adopt a simple model to describe the variation of the radius of an isolated drop according to the law[^7]:

$$r^2_f = r^2_i - kt$$

where $r_i$ is the radius of the drop at the initial time, $r_f$ the radius once elapsed the time $t$ and $k$ a constant characteristic of the fuel. We may introduce the combustion time $\tau$, so that such drops with radius larger than $r_0 = \sqrt{k\tau}$ lead to a given quantity of unburned matter, leading to soot production and waste of fuel. This quantity can be calculated as:

$$I = \int_{r_0}^{R} i(r)n(r)dr,$$

where $i(r)$ is the quantity of unburned matter given by a fragment of radius $r$, $r_0$ is the already introduced "critical radius". The integral starts at $r_0$ since all drops with radius $r < r_0$ will be consumed during the time $\tau$. With (5) it is easy to evaluate the final volume of the fragment once elapsed $\tau$. If we introduce the variable $\xi = \frac{r}{R}$, the quantity of unburned matter for one drop of radius $A$, expressed in units of the volume of the original drop $\frac{4}{3}\pi A^3$ is

$$i(A) = (3 - x)\int_{\xi_0}^{1} \xi^{-x - 1}(\xi^2 - \xi^2_0)^{\frac{3}{2}}d\xi$$

As it can be seen, when $x$ approaches 3, $i(A)$ goes to zero for any value of $\xi_0$, i.e, for any time of combustion. As $x$ is near to 3 the combustion is improved and for small values of $\xi_0$ the fuel is consumed. The case $x=3$ is the "ideal case" and corresponds to some kind of "ideal" or "complete" combustion of the drop. To interpret the case $x=3$ we can imagine a cube of unit length containing a scaled distribution of drops given by (1). Let us take from this cube a sub-cube of length $\lambda^{-1}$, ($\lambda < 1$) then the cumulative number of drops in this sub-cube is given by

$$N_{\lambda^{-1}}(r) \sim \lambda^{-3} r^{-x},$$

now we look this sub-cube with a microscope of magnification $\lambda$ such that the resolution of our observation will go from $r$ to $\lambda^{-1}r$, then the cumulative number observed is

$$N_{\lambda^{-1}}(\lambda^{-1}r) \sim \lambda^{x-3} r^{-x}.$$
Comparing (9) with (1) we may conclude that $x=3$ means that we can observe the same distribution of drops in any scale, i.e., in that case the distribution is scale invariant. As the operation here performed is essentially a renormalization transformation it can be said that a distribution of fragments with the scale exponent equal to the dimension of the space is the fixed point of the RG transformation\cite{8}.

As already was shown in [2] this fixed point is impossible to reach.

3 Combustion of a spray of emulsified fuel

The analysis of the quantity of unburned matter given by a spray of common fuel is not different from the preceding one for the drop, just minor changes in notation must be made. In this respect, we will denote as $n(A)$ the number of atomized drops with radius $A$, the total atomized volume will be denoted by $V$ and $R$ denotes the largest value of $A$. We also denote the scale exponent for the atomization as $y$. Thus the quantity of unburned matter for this case is (in units of $V$)

$$I_S = (3 - y) \int_0^1 \frac{1}{\zeta} (\xi^2 - \xi^2_0) \frac{1}{2} d\zeta, \quad (10)$$

where $\zeta = \frac{A}{R}$ and $\zeta_0 = \frac{\sqrt{\kappa\tau}}{R}$. The calculation of the unburned matter when the atomized fluid is an emulsified fuel is straightforward if we consider that this process can be divided in two steps:

- A first one in which a volume $V$ of emulsified fuel enters the combustion chamber with a distribution of drops characterized by a scale exponent $y$.
- A second step when each drop of initial radius $A$ "explodes" giving a distribution of fragments with scale exponent $x$.

The quantity of unburned matter can be calculated in this case, summing up all the quantities provided by each of the drops:

$$I_T = \int_{\sqrt{\kappa\tau}}^R i(A)n(A) dA, \quad (11)$$

here $i(A)$ is given by (7). If we now express:

$$n(A) = (3 - y) \gamma R^y A^{-y-1}, \quad (12)$$

where $\gamma = \frac{3V}{4\pi R^3}$, as the distribution of atomized drops and use (7) and (12) in (11), we obtain for the total quantity of unburned matter in units of $V$:

$$I_T = (3 - x)(3 - y) \int_0^1 \frac{1}{\zeta} (\xi^2 - \xi^2_0) \frac{1}{2} d\zeta \int_0^{\beta \xi_0} \xi^1 (\xi^2 - \beta^2 \xi^2_0) \frac{1}{2} d\xi. \quad (13)$$

This expression gives us the possibility of analyzing the process of combustion as a function of two main factors: the scale exponents of each distribution. It is evident that the self-similarity of the distribution plays a major role in the improvement of combustion, leading to the fastest consumption rate.
4 Additional remarks

There are some curious facts emerging from this viewpoint that we believe important to notice:

-When evaluating the surface presented by the distribution of drops to combustion, we must sum the surface of each small drop and, as the drops are distributed according to (4) it is necessary to integrate on this distribution. If $\Sigma$ represents the total surface of the drop distribution, then

$$\Sigma = \int_{r_0}^{R} 4\pi r^2 n(r) dr,$$

the integral starts at $r_0$ since we are interested in the area of the drops with size larger than $r_0$. Let us denote $R = \alpha r_0$ with $\alpha > 1$. The total surface expressed in units of the area of the drop is

$$\Sigma = 4\pi \left( \frac{3-x}{2-x} \right) (1 - \alpha^{x-2})$$

and has a finite limit when $x = 2$:

$$\Sigma(x = 2) \sim \log \alpha,$$

as can be noted, for $x = 3$, $\Sigma = 0$. This means that when the distribution is completely self-similar no drops with radius larger than $r_0$ exist. but as we have not fixed it, this occurs for any value of $r_0$ no matter how small could it be. This corresponds with an infinite subdivision of the drop, which leads to obvious self-similarity. The impossibility of this value of $x$ was obtained in [2] by another way.

-For a distribution like (4) it is always possible to choose a value of $x$ such that the quantity of unburned matter given is less than that given by a collection of small equal drops slightly larger than $r_0$. Indeed, if we represent that kind of distribution as $n_1(r) \sim \delta(r_1 - r_0)$, assuming that $r_1 = r_0 + \epsilon$, with $\epsilon$ small, the quantity of unburned matter produced by this kind of distribution is (neglecting higher order infinitesimals):

$$i_2 = \frac{2\epsilon}{k\tau}.$$

Comparing (7) and (17) it is obvious that for a fixed $\epsilon$ and $\tau$ it is always possible to choose a value of $x$ for which $i(A) < i_2$. This condition may seem shocking at first sight, but is a logical consequence of the scaling property of the distribution.
5 Conclusions

The behavior of water-oil emulsions in the combustion process is qualitatively different from that of conventional fuel. This is expressed in higher consumption rates. The viewpoint here presented permits the introduction of the scaling exponent as one of the important parameters to analyze combustion processes. The fractal viewpoint seems to be natural in the analysis of drop microexplosion and jet atomization.

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