Split Instability of a Vortex in an Attractive Bose-Einstein Condensate

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An attractive Bose-Einstein condensate with a vortex splits into two pieces via the quadrupole dynamical instability, which arises at a weaker strength of interaction than the monopole and the dipole instabilities. The split pieces subsequently unite to restore the original vortex or collapse.

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Quantized vortices in gaseous Bose-Einstein condensates (BECs) offer a visible hallmark of superfluidity, where repulsive interatomic interactions play a crucial role in the vortex stabilization and lattice formation. Attractive BECs, on the other hand, cannot hold vortices in any thermodynamically stable state. A fundamental issue of the decay of a many-particle quantum system may be addressed if a vortex is created in an attractive BEC. Such a state has become possible owing to the development of the Feshbach technique by the ENS group. A dynamical instability is also shown to transfer a vortex from one to the other component of an isotropic trap. The eigenvalues become complex at low temperatures. More important than the thermodynamic one at low temperature, experimental situations, such dynamical instabilities are more important than the thermodynamic one at low temperature.

When the complex eigenvalues emerge in the Bogoliubov spectrum, the amplitude of the corresponding mode grows exponentially in time. As noise is inevitable in experimental situations, such dynamical instabilities are more important than the thermodynamic one at low temperature.

Figure 1 shows the real and imaginary parts of the lowest eigenvalues of the $m = -1$ and $3$ excitations in an isotropic trap. The eigenvalues become complex at the critical strength of interaction $g_{Q}^2 = -15.06$, showing the onset of the dynamical instability in the quadrupole mode. The imaginary part of the complex eigenvalue is proportional to $\sqrt{g_{Q}^2 - g}$ as shown in the inset in Fig. 1. The complex eigenvalues emerge also in the dipole modes, i.e., $m = 0$ and 2, for $g < g_{Q}^2 = -18.02$. The eigenvalues with other $m$ are real for $g$ larger than the critical value for the monopole (radial-breathing-mode) instability $g_{M}^2 = -23.7$. 

$\psi_{0}$ is coupled only to $v_{n} \propto e^{i m \phi}$, and we shall refer to $m$ as the angular momentum of the excitation.

We find that there is at least one negative eigenvalue in the $m = 0$ mode for any $g < 0$ and $\lambda$ even in the presence of a rotating drive. The vortex state with attractive interactions is therefore thermodynamically unstable, and eventually decays into the non-vortex ground state by dissipating its energy and angular momentum. At sufficiently low temperatures in a high-vacuum chamber, however, the thermodynamic instability is irrelevant, since the energy and angular momentum are conserved. In fact, recent experiments have demonstrated that the vortex state in a stationary trap has a lifetime of $\sim 1$ s, which is much longer than the characteristic time scales of the dynamics that we shall discuss below.

We first investigate the Bogoliubov spectrum of a single-vortex state. The single-vortex state is determined so as to minimize the Gross-Pitaevskii (GP) energy functional within the axisymmetric functional space $\psi_{0} = f(r, z) e^{i \phi}$ with $r = (x^2 + y^2)^{1/2}$, where we ignore the effect of vortex bending. In the following analysis, we normalize the length, time, and wave functions in units of $d_0 \equiv (\hbar / m \omega_{\perp})^{1/2}$, $\omega_{\perp}^{-1}$, and $(N/d_0^2)^{1/2}$, where $\omega_{\perp}$ is the radial trap frequency, and $N$, the number of BEC atoms. We obtain the Bogoliubov spectrum by numerically diagonalizing the Bogoliubov-de Gennes equations

$$
\begin{align}
(K + 2g|\psi_{0}|^2) u_n + g\psi_{0}^3 v_n &= E_n u_n, \\
(K + 2g|\psi_{0}|^2) v_n + g\psi_{0}^{*3} u_n &= -E_n v_n,
\end{align}
$$

where $K \equiv -\nabla^2/2 + (r^2 + \lambda^2 z^2)/2 - \mu$ with $\lambda \equiv \omega_{\perp}/\omega_{\perp}$, and $n$ is the index of the eigenmode. Here, $g \equiv 4\pi N a/d_0$ characterizes the strength of interaction, where $a$ is the s-wave scattering length. For a vortex state $\psi_{0} \propto e^{i \phi}$, each angular momentum state $u_n \propto e^{i n \phi}$ is coupled only to $v_{n} \propto e^{i (m-2) \phi}$, and we shall refer to $m$ as the angular momentum of the excitation.

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that of the negative-norm branch of the

g_{Q} of interaction for the quadrupole mode

is always larger than

|g_{Q}|, and the monopole mode

g_{M}

in axi-symmetric traps, where

Q_{cr} and

1

mode, we obtain

Q_{cr}

and

3

1

g_{Q}/g_{M} = \lambda

= \sqrt{\lambda}/(2\pi)g^{3D}

. In the 2D system, the

dynamical instability arises in the quadrupole, dipole,

and monopole modes at

g_{Q}^{2Dcr} = -7.79, g_{D}^{2Dcr} = -11.48,

and g_{M}^{2Dcr} = -23.4. The dependencies of the complex
eigenvalues on

g are similar to that in Fig. [4].

To understand the dynamical instabilities analytically,

let us consider the GP action integral in 2D

S = \int dt \int dr \psi^{*} \left( -i \frac{\partial}{\partial t} - \frac{\nabla^{2}}{2} + \frac{r^{2}}{2} + \frac{g}{2} |\psi|^{2} \right) \psi. \hspace{1cm} (2)

We assume that the state evolution is described by

\psi = \sum_{m} c_{m}(t) \phi_{m}(r),

where \phi_{m}(r) is the same as the form of \phi_{m}(r) = \psi_{m}^{(m)}(\sqrt{\pi} m \pi^{(m)} 

\exp[-r^{2}/(2d^{2}) + im\phi] and d = [1 + g/(8\pi)]^{1/4} minimizes the GP energy

functional for the m = 1 state. Substituting this \psi into

Eq. (2) and minimizing S with respect to c_{m} yields

i \dot{c}_{m} = \varepsilon_{m} c_{m} + g \sum_{m_{1}m_{2} \neq m_{1}} G_{m_{1}m_{2}}^{m_{1}m_{2}} c_{m_{1}}^{*} c_{m_{2}} c_{m_{3}}, \hspace{1cm} (3)

where \varepsilon_{m} = \int dr (|\nabla \psi_{m}|^{2} + r^{2}|\psi_{m}|^{2})/2 and \sum_{m_{1},m_{2}} G_{m_{1}m_{2},m_{3}}^{m_{1}m_{2},m_{3}} = \int d r \psi_{m_{1}}^{*} \psi_{m_{2}} \psi_{m_{3}} \psi_{m_{4}}. When the BEC exists in the m = 1

mode, we obtain c_{1}(t) = e^{-it} + O(|c_{m} |^{2}) with \mu = 1/d^{2} + d^{2} + g/(4\pi d^{2}). The linear analysis of Eq. (3) for

\tilde{c}_{m} = e^{it} c_{m} (m \neq 1) yields

i \dot{\tilde{c}}_{m} = (\varepsilon_{m} - \mu) \tilde{c}_{m} + 2G^{m,1}_{m_{1},m_{1}} \tilde{c}_{m_{1}} + G^{m,2}_{m_{1},m_{1}} \tilde{c}_{2}. \hspace{1cm} (4)

It follows from this that, for m = 1, the eigenfrequencies are given by A \pm \sqrt{B}, where

A = \left[ g/(8\pi) - 1 \right]/\left[ 1 + g/(8\pi) \right] and

B = 3 + g/(8\pi) + [1 + g/(2\pi) - g^{2}/(2\pi)]/[1 + g/(8\pi)].

We find that B is a monotonically increasing function for g > -8\pi, and B becomes

negative for g < g_{cr} \approx -9.2, which is in reasonable agreement

with g_{Q}^{2Dcr} = -7.79 stated above. We also find that

the imaginary part appearing for g < g_{cr} is proportional to

\sqrt{g_{Q}^{2Dcr}} - g, in agreement with the inset of Fig. [4].

The Bogoliubov analysis described above is valid only

if deviations from a stationary state are small. To follow

further evolution of the wave function, we must solve the
time-dependent GP equation. Since we are studying the

growth of small perturbations, high precision is required in

the numerical integration, and hence we consider the

GP equation in 2D

\frac{\partial \psi}{\partial t} = \left[ -\frac{1}{2} \nabla^{2} + \frac{1}{2} |\psi|^{2} + g^{2D} |\psi|^{2} \right] \psi \hspace{1cm} (5)

to ensure sufficiently small discretization in the Crank-Nicholson scheme. This situation corresponds to an

oblate trap with large \lambda.
with the fact that the $m = 3$ component grows upon the vortex split. The two side vortices cannot be seen in the density plot, and hence they may be called “ghost” vortices that carry very little angular momentum.

The two clusters in Fig. 3 (d) may be regarded as revolving “solitons” whose phases differ by $\pi$. In fact, at an energy only slightly below that of Fig. 3 (d), there is a low-lying state in which two solitons revolve without changing their shapes. It is interesting to note that this situation is similar to the soliton–train formation observed by the Rice group [22], where the modulation (dynamical) instability causes a quasi-1D condensate to split into solitons when the interaction is changed from repulsive to attractive using the Feshbach resonance. We note that similar instabilities split an optical vortex propagating in a nonlinear medium into spiraling solitons [23–25]. This similarity between attractive BECs and optical solitons [20] implies that other nonlinear phenomena, such as pattern formation, which has been predicted in attractive BECs [27], may also be realized in optical systems.

When the system becomes too small to be observed by the in situ imaging method due to the attractive interaction, the condensate must be expanded before imaging. Figure 3 (g) shows the expanded image at $t = 17$, where the interaction is switched from $g^{2D} = -9$ to $g_{\text{expand}} = 50$, and the trapping potential is switched off at $t = 16$. (h) An expanded image at $t = 18$ with $g_{\text{expand}} = 0$. The sizes of the images are $7 \times 7$ in (a)-(f) and $18 \times 18$ in (g) and (h) in units of $(\hbar/m\omega_L)^{1/2}$. The sensitivity of imaging in (g) and (h) is 20 times higher than that in (a)-(f).

Figures 3 (a)-(f) depict the time evolution of the density and phase profiles with $g^{2D} = -9$, which is smaller than the critical value for the quadrupole mode $g_{Q}^{2D_{\text{cr}}} = -7.79$ but larger than that for the dipole mode $g_{D}^{2D_{\text{cr}}} = -11.48$. A small symmetry-breaking perturbation is added to the initial state to imitate noise in realistic situations. Due to the quadrupole instability, the vortex is first stretched [Fig. 3 (b)], and then splits into two clusters that revolve around the center of the trap [Fig. 3 (d)] with angular velocity $\simeq 0.73\omega_L$. In the first deformation process the $m = -1$ and 3 components grow exponentially, and their Lyapunov exponents agree with the imaginary part of the complex eigenvalues. Interestingly, the split process is reversible: the two clusters subsequently unite to restore the ring shape [Fig. 3 (f)], and this split-merge process repeats. We numerically checked that no split-merge phenomenon occurs for $g^{2D} > g_{2D_{\text{cr}}}$, where the system is metastable. The insets in Fig. 3 illustrate the phase plots. At $t = 16$, there are three topological defects: the central one exists from the outset, and the other two enter as the vortex splits, in accordance...
and with $g$ from 0 to $-\frac{12}{\pi} (h)$, and 2 to yield a vortex-antivortex pair 3-4 [inset of (c)]. Vortex 5 note that the end part of the central branch cut is separated topological defects and branch cuts in the left one. Here we and antivortex 4 are subsequently combined [inset of (f)]. The sizes of the images are 5 in (a)-(f), 8 × 8 in (g), 4 × 4 in (h), and 2 × 2 in (i) in units of $(h/m\omega_{\perp})^{1/2}$. The sensitivity of imaging of the density plots is 1/4 in (a)-(f), 1 in (g), 1/5 in (h), and 1/20 in (i) in units of those in Figs. 3 (a)-(f).

$g_{\mathrm{nl}}(t) = 0$. The vortex first shrinks due to the monopole instability [(g) → (h)], then splits into two clusters due to the quadrupole instability [(h) → (i)], and both clusters collapse. When the interaction is switched to an even greater attractive one, a shell structure is formed [27], which splits into several parts due to multipole instabilities, and each fragment collapses and explodes [8], producing very complicated collapsing dynamics.

In conclusion, we have studied the dynamical instabilities of a quantized vortex in an attractive BEC. The dynamical quadrupole instability spontaneously breaks the axisymmetry and splits a vortex into clusters that revolve around the center of the trap, which then unite to restore the vortex or eventually collapse. The dynamical instabilities presented here play a much larger role than the thermodynamic one at low temperature, and serve as a dominant mechanism for the collapsing dynamics of a rotating condensate: vortices collapse via the dynamical instabilities around the topological defects.

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