DECOMPOSING FROBENIUS HEISENBERG CATEGORIES

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Abstract. We give two alternate presentations of the Frobenius Heisenberg category, \( \text{Heis}_{F,k} \), defined by Savage, when the Frobenius algebra \( F = F_1 \oplus \cdots \oplus F_n \) decomposes as a direct sum of Frobenius subalgebras. In these alternate presentations, the morphism spaces of \( \text{Heis}_{F,k} \) are given in terms of plane diagrams consisting of strands "colored" by integers \( i = 1, \ldots, n \), where a strand of color \( i \) carries tokens labelled by elements of \( F_i \). In addition, we prove that when \( F \) decomposes this way, the tensor product of Frobenius Heisenberg categories, \( \text{Heis}_{F_1,k} \otimes \cdots \otimes \text{Heis}_{F_n,k} \), is equivalent to a certain subcategory of the Karoubi envelope of \( \text{Heis}_{F,k} \) that we call the partial Karoubi envelope of \( \text{Heis}_{F,k} \).

1. Introduction

In [Kho14], Khovanov developed a graphical calculus for the induction and restriction functors related to the representation theory of the symmetric groups. This led him to defining a monoidal category, \( \mathcal{H} \), which he conjectured is isomorphic to the infinite-dimensional Heisenberg algebra. He called \( \mathcal{H} \) the Heisenberg category. This work has been generalized to q-deformations in [LS13], to categories depending on a graded Frobenius superalgebra in [CL12] and [RS17], and to higher level in [MS18].

In [Bru], Brundan gave two alternate presentations of the higher level Heisenberg categories of [MS18]. One involved an "inversion relation," while the other was similar to the original presentation of the Heisenberg category, given in [Kho14], but with fewer relations.

In [Savb], Savage used the inversion relation approach of Brundan to unify the higher level categories of [MS18] and the categories of [RS17]. He associated to each graded Frobenius superalgebra, \( F \), and integer, \( k \), a graded Heisenberg supercategory, which he called a Frobenius Heisenberg category and denoted \( \text{Heis}_{F,k} \).

In the present paper, we give two alternate presentations of \( \text{Heis}_{F,k} \) in the case where \( F = F_1 \oplus \cdots \oplus F_n \) decomposes as a direct sum of Frobenius subalgebras. Throughout this paper, we assume the trace map of \( F \) is symmetric, we ignore gradings, and we work in the non-super setting. Although we expect analogous presentations of \( \text{Heis}_{F,k} \) to hold without these assumptions, we make these assumptions to simplify exposition. In the original presentation of \( \text{Heis}_{F,k} \) found in [Savb, Def. 1.1], the morphism space of \( \text{Heis}_{F,k} \) is given in terms of planar diagrams consisting of strands carrying tokens labelled by elements of \( F \). In our new presentations of \( \text{Heis}_{F,k} \), the morphism spaces of \( \text{Heis}_{F,k} \) are given in terms of planar diagrams consisting of strands colored by integers \( i = 1, \ldots, n \), where a strand of color \( i \) carries tokens labelled by elements of \( F_i \). Colored morphisms are common in the categorification literature. For example, when \( F \) is a zigzag algebra and \( k = -1 \), \( \text{Heis}_{F,k} \) is the Heisenberg category of Cautis and Licata in [CL12].

The presentation given in [CL12] is in terms of colored planar diagrams. The morphism spaces of Kac-Moody 2-categories are also colored (by elements of the weight lattice). See, for example,

\[ \text{2010 Mathematics Subject Classification.} \quad 18D10. \]
\[ \text{Key words and phrases.} \quad \text{Categorification, Frobenius algebra, Heisenberg algebra, diagrammatic calculus, monoidal category.} \]
[Rou] and [KL10]. Our new presentation of \( \mathcal{H}_{eis,F,k} \) takes advantage of the decomposition of \( F \) to decompose the morphism spaces of \( \mathcal{H}_{eis,F,k} \), which results in a simpler way to view \( \mathcal{H}_{eis,F,k} \). An example of the simplifications that can be made using these new presentations of \( \mathcal{H}_{eis,F,k} \) is given by the proof of Corollary 4.7 in Section 4, where we use one of these new presentations to show that the tensor product of Frobenius Heisenberg categories, \( \mathcal{H}_{eis,F_1,k} \otimes \cdots \otimes \mathcal{H}_{eis,F_n,k} \), is equivalent to a certain subcategory of the Karoubi envelope of \( \mathcal{H}_{eis,F,k} \) that we call the partial Karoubi envelope of \( \mathcal{H}_{eis,F,k} \).

This paper is organized as follows. In Section 2, we review the definition and basic facts about the Frobenius Heisenberg category, \( \mathcal{H}_{eis,F,k} \), introduced in [Savb], and we specialize \( F \) to a Frobenius algebra that decomposes as a direct sum of Frobenius subalgebras, \( F_1 \oplus \cdots \oplus F_n \). In Section 3, we prove two alternate presentations of \( \mathcal{H}_{eis,F,k} \). In Section 4 we recall the definition of the tensor product of categories, and we define the partial Karoubi envelope of \( \mathcal{H}_{eis,F,k} \), denoted \( \text{PK} (\mathcal{H}_{eis,F,k}) \). We then show that \( \text{PK} (\mathcal{H}_{eis,F,k}) \) is equivalent to \( \mathcal{H}_{eis,F_1,k} \otimes \cdots \otimes \mathcal{H}_{eis,F_n,k} \).

**Acknowledgements.** This research was supported by an Undergraduate Student Research Award from the Natural Sciences and Engineering Research Council of Canada and was supervised by Professor Alistair Savage. The author is grateful to Professor Savage for all of his patience and guidance throughout this project.

### 2. The Frobenius Heisenberg category

In this section, we recall the definition and basic properties of the Frobenius Heisenberg category first defined in [Savb]. Fix a commutative ground ring \( k \). Let \( F = F_1 \oplus \cdots \oplus F_n \), be the direct sum of Frobenius \( k \)-algebras, and view \( F \) as a Frobenius algebra with trace map \( \text{tr}(f_1, \ldots, f_n) = \sum_{i=1}^n \text{tr}_i(f_i) \), where \( \text{tr}_i \) denotes the trace of \( F_i \). View \( F_i \) as a subalgebra of \( F \) by identifying an element \( f \in F_i \) with the element \((0, \ldots, 0, f, 0, \ldots, 0) \in F \), where \( f \) appears in the \( i \)-th position, and define \( e_i := 1_{F_i} \). For each \( i \), we fix a basis \( B_i \) of \( F_i \). Thus, \( B := B_1 \cup \cdots \cup B_n \) is a basis of \( F \). It is straightforward to show that for \( b_i \in B_i \) and \( b_j \in B_j \), \( i \neq j \) we have

\[
\text{tr}(b_i b_j) = 0. \tag{2.1}
\]

Fix \( k \in \mathbb{Z} \). We will recall the definition of \( \mathcal{H}_{eis,F,k} \) given in [Savb, Def. 1.1], and impose some additional assumptions. Namely, we assume the trace map of \( F \) is symmetric, we ignore gradings, and we are in the non-super setting.

**Definition 2.1.** The category \( \mathcal{H}_{eis,F,k} \) is the strict \( k \)-linear monoidal category defined as follows. The objects are generated by \( Q_+ \) and \( Q_- \), and we use juxtaposition to denote tensor product. The morphisms of \( \mathcal{H}_{eis,F,k} \) are generated by

\[
\begin{align*}
x &= \uparrow, & s &= \uparrow \downarrow, & c &= \bigcup, & d &= \bigcap, & \beta_f &= \uparrow f, & f \in F.
\end{align*}
\]

We refer to the decoration representing \( x \) as a *dot* and the decorations representing \( \beta_f, f \in F \), as *tokens*. For \( n \geq 1 \), we denote the \( n \)-th power \( x^n \) of \( x \) by labelling the dot with the exponent \( n \):

\[
x^n = \uparrow n
\]

We also define

\[
t : Q_+ Q_- \to Q_- Q_+, \quad t = \bigcup \cup := \bigcup \bigcap.
\]

We impose three sets of relations:

[0x0]KL10

[0x0]Savb

[0x0]4

[0x0]4.7

[0x0]2

[0x0]in Section

[0x0]f

[0x0]we recall the definition of

[0x0]4

[0x0]72x651]

[0x0]An example of the simplifications that can be made using these new presentations of \( \mathcal{H}_{eis,F,k} \) is given by the proof of Corollary 4.7 in Section 4, where we use one of these new presentations to show that the tensor product of Frobenius Heisenberg categories, \( \mathcal{H}_{eis,F_1,k} \otimes \cdots \otimes \mathcal{H}_{eis,F_n,k} \), is equivalent to a certain subcategory of the Karoubi envelope of \( \mathcal{H}_{eis,F,k} \) that we call the partial Karoubi envelope of \( \mathcal{H}_{eis,F,k} \).

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\[
\begin{align*}
x &= \uparrow, & s &= \uparrow \downarrow, & c &= \bigcup, & d &= \bigcap, & \beta_f &= \uparrow f, & f \in F.
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\]

We refer to the decoration representing \( x \) as a *dot* and the decorations representing \( \beta_f, f \in F \), as *tokens*. For \( n \geq 1 \), we denote the \( n \)-th power \( x^n \) of \( x \) by labelling the dot with the exponent \( n \):

\[
x^n = \uparrow n
\]

We also define

\[
t : Q_+ Q_- \to Q_- Q_+, \quad t = \bigcup \cup := \bigcup \bigcap.
\]

We impose three sets of relations:
(a) **Affine wreath product algebra relations:** We have a homomorphism of algebras

\[
F \to \text{End } Q_+, \quad f \mapsto \beta_f,
\]

so that, in particular,

\[
f \mapsto \beta_f, \quad f, g \in F.
\]

Note also that, for \(a \in k, f_j \in F, j = 1, \ldots, m, m > 0\), we have

\[
(\sum_{i=1}^m f_j) \mapsto \sum_{i=1}^m (f_{i_j}), \quad \quad (\alpha f_1) \mapsto a (f_{i_1}).
\]

Furthermore, we impose the following relations for all \(f \in F\):

\[
(\sum_{i=1}^m f_j) = \sum_{i=1}^m (f_{i_j}), \quad (\alpha f_1) = a (f_{i_1}).
\]

(b) **Right adjunction relations:** We impose the following relations:

\[
(\sum_{i=1}^m f_j) = \sum_{i=1}^m (f_{i_j}), \quad \quad (\alpha f_1) = a (f_{i_1}).
\]

(c) **Inversion relation:** The following matrix of morphisms is an isomorphism in the additive envelope of \(\mathcal{Heis}_{F,k}\):

\[
\begin{bmatrix}
\alpha & r \\
\beta & b^{-}
\end{bmatrix}, \quad 0 \leq r \leq k - 1, \quad b \in B
\]

\[
: Q_+ Q_- \to Q_- Q_+ \oplus 1^{\oplus k \dim F} \quad \text{if } k \geq 0,
\]

\[
\begin{bmatrix}
\alpha & r \\
\beta & b^{-}
\end{bmatrix}, \quad 0 \leq r \leq -k - 1, \quad b \in B
\]

\[
: Q_- Q_+ \oplus 1^{\oplus (-k \dim F)} \to Q_- Q_+ \quad \text{if } k < 0.
\]

(The matrix (2.16) is of size \((1 + k \dim F) \times 1\), while the matrix (2.17) is of size \(1 \times (1 + k \dim F)\).)

Note that these conditions are independent of the choice of basis \(B\) of \(F\).

In the special case \(k = 0\), the inversion relation means that there is another generating morphism

\[
t' = Q_- Q_+ \to Q_+ Q_-,
\]

that is inverse to \(t\). Thus we have

\[
\begin{bmatrix}
Q_- Q_+ & Q_+ Q_-
\end{bmatrix}
\]

The affine wreath product relations correspond to the defining relations of affine wreath product algebras. See [Sava] for further discussion of these algebras. In our case, the category \(\mathcal{Heis}_{F,k}\) is strictly pivotal (see [Savb, page 4]). In particular, there is isotopy invariance of all affine wreath product relations, giving a larger set of relations that hold in \(\mathcal{Heis}_{F,k}\).
For the sake of having notation, we define (as in [Savb, page 11])

\[
t' = \begin{array}{c}
\begin{array}{c}
\bigcirc \\
\bigotimes
\end{array}
\end{array} : \mathbb{Q}_{-} \mathbb{Q}_{+} \to \mathbb{Q}_{+} \mathbb{Q}_{-},
\]

and

\[
(2.19)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} : \mathbb{1} \to \mathbb{Q}_{+} \mathbb{Q}_{-},
\]

\[
(2.20)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} : \mathbb{Q}_{+} \mathbb{Q}_{-} \to \mathbb{1},
\]

for \(0 \leq r < k\) or \(0 \leq r < -k\), respectively, by declaring that

\[
(2.21)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} , \quad 0 \leq r \leq k - 1, \quad b \in B
\]

\[
= \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} , \quad 0 \leq r \leq -k - 1, \quad b \in B
\]

\[
= \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} \mathbf{T}^{-1} \quad \text{if } k \geq 0,
\]

or

\[
(2.22)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} , \quad 0 \leq r \leq k - 1, \quad b \in B
\]

\[
= \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} \mathbf{T}^{-1} \quad \text{if } k \leq 0.
\]

As in [Savb, page 11], we extend the definition of the decorated left cups and caps by linearity in the second argument of the label.

The relations (2.22) and (2.23) hold in \(\mathcal{H}_{\mathbb{F},k}\) for all \(k\):

\[
(2.22)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} + \sum_{r=0}^{k-1} \sum_{b \in B} \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} , \quad (2.23)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} + \sum_{r=0}^{k-1} \sum_{b \in B} \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array}.
\]

When \(k > 0\), the following relations, (2.24) to (2.26), hold in \(\mathcal{H}_{\mathbb{F},k}\). For all \(0 \leq r, s \leq k - 1\) and \(b, c \in B\),

\[
(2.24)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} , \quad (2.25)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} = 0 , \quad (2.26)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} = \delta_{r,s} \delta_{b,c}.
\]

When \(k < 0\), the following relations, (2.27) to (2.29), hold in \(\mathcal{H}_{\mathbb{F},k}\). For all \(0 \leq r, s \leq -k - 1\) and \(b, c \in B\),

\[
(2.27)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} , \quad (2.28)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} = 0 , \quad (2.29)\quad \begin{array}{c}
\begin{array}{c}
\bigotimes \\
(r,b)
\end{array}
\end{array} = \delta_{r,s} \delta_{b,c}.
\]

The inversion relation in \(\mathcal{H}_{\mathbb{F},k}\) is equivalent to the relations (2.22) and (2.24) to (2.26) when \(k > 0\), the relations (2.23) and (2.27) to (2.29) when \(k < 0\), and the relations (2.22) and (2.23) when \(k = 0\).

Note also that the following relations hold in \(\mathcal{H}_{\mathbb{F},k}\), which will be useful in the proof of Lemma 2.2. The relations (2.30) to (2.33) are a result of the relations (2.10) and (2.12) due to isotopy invariance of all affine wreath product relations in \(\mathcal{H}_{\mathbb{F},k}\). The relations (2.34) and (2.35) are equivalent to the relations given by equation number (2.8) in [Savb].
Lemma 2.2. For \( b \in B_i \), the following relations hold in \( \text{Heis}_F \):

\[
\begin{align*}
(2.36) & \quad e_j^\uparrow e_1 = \delta_{i,j} \delta_{i,i}^\uparrow, & k > 0, \\
(2.37) & \quad e_j^\uparrow e_1 = \delta_{i,j} \delta_{i,i}^\uparrow, & k < 0.
\end{align*}
\]

Proof. Attaching the morphism

\[
\begin{array}{c}
e_j^\uparrow \downarrow e_1
\end{array}
\]

below both sides of (2.22) and using (2.9), (2.25), (2.26), (2.30), (2.31), and (2.34) gives (2.36). A similar argument shows that (2.37) holds in \( \text{Heis}_F \). \( \square \)

3. Alternate Presentations of \( \text{Heis}_F \)

In this section, we give two alternate presentations of \( \text{Heis}_F \). First we introduce notation. For \( f \in F \) and \( i = 1, \ldots, n \), define \( i^\uparrow f := i : f^\uparrow \). We say that a strand indexed by \( i \) is of color \( i \). In particular, for \( f = (f_1, \ldots, f_n) \in F \), we can write

\[
f = (f_1, \ldots, f_n) = f_1^\uparrow + \cdots + f_n^\uparrow.
\]

That is, we can write any \( \beta_f \in \text{Heis}_F \) as a sum of uni-colored strands. Additionally, we define the following notation for morphisms in \( \text{Heis}_F \). For \( b \in B \),

\[
\begin{array}{c}
i \quad j := e_i^\uparrow e_j, \\
i := e_1^\uparrow \quad (r,b) := (r,b) e_1^\uparrow, \\
i := e_i^\uparrow \quad (r,b) := (r,b) e_i, \\
i := e_i, \\
i := e_i.
\end{array}
\]

This motivates one alternate presentation of \( \text{Heis}_F \), which we state below and prove in Theorem 3.2.

Definition 3.1. The category \( \text{Heis}_F \) is the strict \( k \)-linear monoidal category defined as follows. The objects are generated by \( Q_+ \) and \( Q_- \) and we use juxtaposition to denote tensor product. The morphisms of \( \text{Heis}_F \) are generated by

\[
\begin{align*}
s_{i,j} & := i^\uparrow j : Q_+ Q_+ \to Q_+ Q_+, \\
c_i & := i^\uparrow : 1 \to Q_+ Q_+, \\
d_i & := i^\uparrow : Q_+ Q_- \to 1,
\end{align*}
\]
\[ x_i = \uparrow_i^1 : Q_+ \to Q_+ , \quad \beta_{(f,i)} = \uparrow_i^f : Q_+ \to Q_+ , \quad f \in F_i, \quad i, j = 1, \ldots, n. \]

For \( n \geq 1 \), we denote the \( n \)-th power \( x_i^n \) of \( x_i \), \( i = 1, \ldots, n \), by labelling the dot with the exponent \( n \):

\[ x_i^n = \uparrow_i^n. \]

We also define, for \( i, j = 1, \ldots, n \),

\[ t_{i,j} : Q_+ Q_- \to Q_- Q_+ , \quad t_{i,j} = \begin{array}{cc}
  & j \\
  & i
\end{array}. \]

We impose three sets of relations:

(a) Colored affine wreath product algebra relations: We have a homomorphisms of algebras for \( i = 1, \ldots, n \),

\[ F_i \to \text{End} Q_+ , \quad f \mapsto \beta_{(f,i)}, \]

so that, in particular,

\[ f \uparrow_i g = \uparrow_i f g , \quad f, g \in F_i. \]

Note also that, for \( a \in \mathbb{k} \), \( f_j \in F_i \), \( j = 1, \ldots, m \), \( m > 0 \), we have

\[ \left( \sum_{j=1}^m f_j \right) \uparrow_i = \sum_{j=1}^m \left( f_j \right) \uparrow_i , \quad \left( af_1 \right) \uparrow_i = a \left( f_1 \right) \uparrow_i. \]

Furthermore, we impose the following relations for all \( i, j, l = 1, \ldots, n \):

\[ \begin{array}{ccc}
  & i & j \\
  & l & & \\
\end{array}\left( \begin{array}{c}
  & i & j \\
  & l & & \\
\end{array}\right) = \left( \begin{array}{c}
  & i & j \\
  & l & & \\
\end{array}\right) , \quad \begin{array}{ccc}
  & i & j \\
  & j & & \\
\end{array}\left( \begin{array}{cc}
  & i & j \\
  & i & & \\
\end{array}\right) = \left( \begin{array}{cc}
  & i & j \\
  & i & & \\
\end{array}\right) , \quad \begin{array}{c}
  & i & j \\
  & j & & \\
\end{array}\left( \begin{array}{c}
  & i & j \\
  & i & & \\
\end{array}\right) = \left( \begin{array}{c}
  & i & j \\
  & i & & \\
\end{array}\right). \]

It follows that we also have the relations:

\[ \begin{array}{ccc}
  & i & j \\
  & j & & \\
\end{array}\left( \begin{array}{c}
  & i & j \\
  & j & & \\
\end{array}\right) = \left( \begin{array}{c}
  & i & j \\
  & j & & \\
\end{array}\right) , \quad \begin{array}{c}
  & i & j \\
  & j & & \\
\end{array}\left( \begin{array}{c}
  & i & j \\
  & i & & \\
\end{array}\right) = \delta_{i,j} \sum_{b \in B_i} b \uparrow_i b. \]

(b) Right adjunction relations: We impose the following relations for all \( i = 1, \ldots, n \):

\[ \begin{array}{ccc}
  & i & j \\
  & j & & \\
\end{array}\left( \begin{array}{c}
  & i & j \\
  & j & & \\
\end{array}\right) = \left( \begin{array}{c}
  & i & j \\
  & j & & \\
\end{array}\right) , \quad \begin{array}{c}
  & i & j \\
  & j & & \\
\end{array}\left( \begin{array}{c}
  & i & j \\
  & i & & \\
\end{array}\right) = \delta_{i,j} \sum_{b \in B_i} b \uparrow_i b. \]
(3.13) \[ \begin{array}{c} i \ \\
\end{array} = \begin{array}{c} i \ \\
\end{array} , \quad (3.14) \begin{array}{c} i \ \\
\end{array} = \begin{array}{c} i \ \\
\end{array} . \]

(c) **Inversion relation**: Suppose \( i \neq j, \ i, j = 1, \ldots, n \). Then \( t_{i,j} \) is invertible in \( \mathcal{H}eis_{F,k} \). Additionally, the following matrix of morphisms is an isomorphism in the additive envelope of \( \mathcal{H}eis_{F,k} \) for \( i = 1, \ldots, n \):

\[
\begin{bmatrix}
\begin{array}{c}
\begin{array}{c}
 i \\
\end{array}
\end{array}
\end{bmatrix}^{\top} : Q_+ Q_- \to Q_- Q_+ \oplus 1^\oplus k \dim F_i \quad \text{if } k \geq 0, \\
\begin{bmatrix}
\begin{array}{c}
\begin{array}{c}
 i \\
\end{array}
\end{array}
\end{bmatrix}^{\top} : Q_+ Q_+ \oplus 1^\oplus (-k \dim F_i) \to Q_- Q_- \quad \text{if } k < 0.
\]

(The matrices (3.15) are of size \( (1 + k \dim F_i) \times 1 \), while the matrices (3.16) are of size \( 1 \times (1 + k \dim F_i) \).) Note that these conditions are independent of the choice of basis \( B_i \) of \( F_i \).

In the special case \( k = 0 \), the inversion relation means that there are other generating morphisms

\[
t'_{i,j} = \begin{array}{c}
\begin{array}{c}
 i \\
\end{array}
\end{array} : Q_- Q_+ \to Q_+ Q_-
\]

\( i, j = 1, \ldots, n \), that are inverse, respectively, to \( t_{i,j} \). Thus we have

\[
\begin{array}{c}
\begin{array}{c}
 i \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
 i \\
\end{array}
\end{array} , \quad \begin{array}{c}
\begin{array}{c}
 i \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
 i \\
\end{array}
\end{array}
\]

for all \( i, j = 1, \ldots, n \).

For the sake of having notation, we define

\[
(3.17) \quad t'_{i,j} = \begin{array}{c}
\begin{array}{c}
 i \\
\end{array}
\end{array} : Q_- Q_+ \to Q_+ Q_-
\]

and

\[
(3.18) \quad \begin{array}{c}
\begin{array}{c}
 (r,b) \\
\end{array}
\end{array} : 1 \to Q_+ Q_- , \quad \begin{array}{c}
\begin{array}{c}
 (r,b) \\
\end{array}
\end{array} : Q_+ Q_- \to 1, \quad b \in B_i,
\]
for all $i, j = 1, \ldots, n$, and $0 \leq r < k$ or $0 \leq r < -k$, respectively, by declaring that

\[(3.19)\]

\[
\begin{pmatrix}
\kappa_{i,j}^{i,r} & 0 \leq r \leq k - 1, \ b \in B_i \\
0 & 0 \leq r \leq k - 1, \ b \in B_i
\end{pmatrix}
\]

\[= (\begin{pmatrix}
\kappa_{i,j}^{i,r} & 0 \leq r \leq k - 1, \ b \in B_i \\
0 & 0 \leq r \leq k - 1, \ b \in B_i
\end{pmatrix})^{-1}
\]

when $i = 1, \ldots, n$, and

\[(3.21)\]

\[
\begin{pmatrix}
\kappa_{i,j}^{i,r} & 0 \leq r \leq k - 1, \ b \in B_i \\
0 & 0 \leq r \leq k - 1, \ b \in B_i
\end{pmatrix}
\]

\[= (\begin{pmatrix}
\kappa_{i,j}^{i,r} & 0 \leq r \leq k - 1, \ b \in B_i \\
0 & 0 \leq r \leq k - 1, \ b \in B_i
\end{pmatrix})^{-1}
\]

when $i \neq j, i, j = 1, \ldots, n$. As in [Savb, page 11], we extend the definition of the decorated left cups and caps by linearity in the second argument of the label.

The relations (3.22) and (3.23) hold in $\mathcal{H}(eis)'_{F,k}$ for all $k$. For $i, j = 1, \ldots, n$,

\[(3.22)\]

\[
\begin{pmatrix}
\kappa_{i,j}^{i,r} & 0 \leq r \leq k - 1, \ b \in B_i \\
0 & 0 \leq r \leq k - 1, \ b \in B_i
\end{pmatrix}
\]

\[= (\begin{pmatrix}
\kappa_{i,j}^{i,r} & 0 \leq r \leq k - 1, \ b \in B_i \\
0 & 0 \leq r \leq k - 1, \ b \in B_i
\end{pmatrix})^{-1}
\]

When $k > 0$, the following relations, (3.24) to (3.26), hold in $\mathcal{H}(eis)'_{F,k}$. For all $0 \leq r, s \leq k - 1$, $i, j = 1, \ldots, n$, and $b, c \in B_i$,

\[(3.24)\]

\[
\begin{pmatrix}
\kappa_{i,j}^{i,r} & 0 \leq r \leq k - 1, \ b \in B_i \\
0 & 0 \leq r \leq k - 1, \ b \in B_i
\end{pmatrix}
\]

\[= (\begin{pmatrix}
\kappa_{i,j}^{i,r} & 0 \leq r \leq k - 1, \ b \in B_i \\
0 & 0 \leq r \leq k - 1, \ b \in B_i
\end{pmatrix})^{-1}
\]

When $k < 0$, the following relations, (3.27) to (3.29), hold in $\mathcal{H}(eis)'_{F,k}$. For all $0 \leq r, s \leq -k - 1$, $i = 1, \ldots, n$, and $b, c \in B_i$,

\[(3.27)\]

\[
\begin{pmatrix}
\kappa_{i,j}^{i,r} & 0 \leq r \leq k - 1, \ b \in B_i \\
0 & 0 \leq r \leq k - 1, \ b \in B_i
\end{pmatrix}
\]

\[= (\begin{pmatrix}
\kappa_{i,j}^{i,r} & 0 \leq r \leq k - 1, \ b \in B_i \\
0 & 0 \leq r \leq k - 1, \ b \in B_i
\end{pmatrix})^{-1}
\]

The inversion relation in $\mathcal{H}(eis)'_{F,k}$ is equivalent to the relations (3.22) and (3.24) to (3.26) when $k > 0$, the relations (3.23) and (3.27) to (3.29) when $k < 0$, and the relations (3.22) and (3.23) when $k = 0$. 
**Theorem 3.2.** The categories $\mathcal{Heis}_{F,k}$ and $\mathcal{Heis}'_{F,k}$ are isomorphic.

**Proof.** We define a monoidal functor $F : \mathcal{Heis}'_{F,k} \to \mathcal{Heis}_{F,k}$ as follows. On objects define

$$F(Q_{\pm}) = Q_{\pm},$$

and on morphisms, for $f_i \in F_i$, $i, j = 1, \ldots, n$, define

$$F(s_{l,j}) = s_{l,j}, \quad F(c_i) = c_i, \quad F(d_i) = d_i, \quad F(x_i) = x_i, \quad F(\beta(f_i, i)) = \beta_{f_i}, \quad F(t_{l,j}) = t_{l,j},$$

and extend $F$ so that it is linear and monoidal. We define another monoidal functor $G : \mathcal{Heis}_{F,k} \to \mathcal{Heis}'_{F,k}$ as follows. On objects define

$$G(Q_{\pm}) = Q_{\pm},$$

and on morphisms, define

$$G(s) = \sum_{i,j=1}^{n} s_{l,j}, \quad G(c) = \sum_{i=1}^{n} c_i, \quad G(d) = \sum_{i=1}^{n} d_i, \quad G(\beta(f_1, \ldots, f_n)) = \sum_{i=1}^{n} \beta_i, \quad G(t) = \sum_{i,j=1}^{n} t_{l,j},$$

and extend $G$ so that it is linear and monoidal. We claim that $F$ and $G$ are strict $k$-linear monoidal functors that are two-sided inverses of each other. If $G$ and $F$ exist (i.e. if they are well-defined), it is clear that $G \circ F = \text{id}_{\mathcal{Heis}'_{F,k}}$ and $F \circ G = \text{id}_{\mathcal{Heis}_{F,k}}$. So it is enough to show that $F$ and $G$ are well-defined. To do this, we show that $F$ preserves the defining relations of $\mathcal{Heis}'_{F,k}$ and $G$ preserves the defining relations of $\mathcal{Heis}_{F,k}$. That is, we show that $G$ preserves the relations (2.3), (2.7) to (2.11), (2.14), and (2.15), and the inversion relation in $\mathcal{Heis}_{F,k}$, and that $F$ preserves the relations (3.2), (3.6) to (3.10), (3.13), and (3.14), and the inversion relation in $\mathcal{Heis}'_{F,k}$. We first show that $F$ preserves the defining relations of $\mathcal{Heis}'_{F,k}$.

The functor $F$ preserves the relations (3.2), (3.6) to (3.10), (3.13), and (3.14) in $\mathcal{Heis}'_{F,k}$. This is seen by attaching the appropriate idempotents above or below both sides of the relations (2.4) to (2.11), (2.14), and (2.15) in $\mathcal{Heis}_{F,k}$, and noting that the resulting equations are the images of (3.3) to (3.10), (3.13), and (3.14) under $F$, respectively.

The functor $F$ preserves the inversion relation in $\mathcal{Heis}'_{F,k}$. Showing that $F$ preserves this relation in $\mathcal{Heis}_{F,k}$ is equivalent to showing that $F$ preserves the relations (3.22) to (3.29) in $\mathcal{Heis}'_{F,k}$. To show that $F$ preserves the relations (3.22) to (3.24) and (3.27) in $\mathcal{Heis}'_{F,k}$, we attach the appropriate idempotents above or below both sides of the relations (2.22) to (2.24) and (2.27) in $\mathcal{Heis}_{F,k}$, and note that these equations are the images of (3.22) to (3.24) and (3.27) under $F$, respectively. Now, to show that $F$ preserves the relations (3.25), (3.26), (3.28), and (3.29) in $\mathcal{Heis}'_{F,k}$, we note that the relations (2.25), (2.26), (2.28), and (2.29) in $\mathcal{Heis}_{F,k}$ are the images of (3.25), (3.26), (3.28), and (3.29) under $F$, respectively, since for all $b \in B_i$, $b \in B_i$ for some $i = 1, \ldots, n$, and since Lemma 2.2 holds in $\mathcal{Heis}_{F,k}$. Therefore, $F$ preserves all defining relations in $\mathcal{Heis}'_{F,k}$. 


This shows that $F$ preserves all defining relations in $\mathcal{Heis}_{F,k}'$. Next we show that $G$ preserves all defining relations in $\mathcal{Heis}_{F,k}'$.

The functor $G$ preserves the relation (2.7) in $\mathcal{Heis}_{F,k}'$. This follows by summing both sides of the relation (3.6) over all $i,j,1=1,\ldots,n$ in $\mathcal{Heis}_{F,k}'$, and noting that the resulting equation is the image of (2.7) under $G$.

The functor $G$ preserves the relations (2.8), (2.10), and (2.11) in $\mathcal{Heis}_{F,k}'$. This follows by summing both sides of the relations (3.7), (3.9), and (3.10), over all $i,j,1=1,\ldots,n$ in $\mathcal{Heis}_{F,k}'$, and noting that the resulting equations are the images of (2.8), (2.10), and (2.11) under $G$, respectively.

The functor $G$ preserves the relations (2.3), (2.9), (2.14), and (2.15) in $\mathcal{Heis}_{F,k}'$. This follows by summing both sides of the relations (3.3) to (3.5), (3.8), (3.13), and (3.14) over all $i=1,\ldots,n$ in $\mathcal{Heis}_{F,k}'$, and noting that the resulting equations are the images of (2.4) to (2.6), (2.9), (2.14), and (2.15) under $G$, respectively.

The functor $G$ preserves the inversion relation in $\mathcal{Heis}_{F,k}'$. Showing that $G$ preserves this relation in $\mathcal{Heis}_{F,k}'$ is equivalent to showing that $G$ preserves the relations (2.22) to (2.29) in $\mathcal{Heis}_{F,k}'$. To show that $G$ preserves the relations (2.22) to (2.24) and (2.27) in $\mathcal{Heis}_{F,k}'$, we sum both sides of (2.22) to (2.24) and (2.27), over all $i,j=1,\ldots,n$ in $\mathcal{Heis}_{F,k}'$, and note that these equations are the images of (2.22) to (2.24) and (2.27) under $G$, respectively. Now, to show that $G$ preserves the relations (2.25), (2.26), and (2.29) in $\mathcal{Heis}_{F,k}'$, we note that the relations (3.25), (3.26), and (2.29) in $\mathcal{Heis}_{F,k}'$ are the images of (2.25), (2.26), (2.28), and (2.29) under $G$, respectively, since for all $b \in B$, $b \in B_i$ for some $i=1,\ldots,n$, and since Lemma 2.2 holds in $\mathcal{Heis}_{F,k}'$. Therefore, $G$ preserves all defining relations in $\mathcal{Heis}_{F,k}'$. This completes the proof.

We recall the following alternate presentation of $\mathcal{Heis}_{F,k}'$ introduced in [Savb, Theorem 1.2], and impose some additional assumptions. Namely, we ignore gradings, and we are in the non-super setting.

**Theorem 3.3.** There are unique morphisms $c': \mathbb{1} \to Q_+Q_-$ and $d': Q_-Q_+ \to \mathbb{1}$ in $\mathcal{Heis}_{F,k}'$, drawn as

$$c' = \begin{array}{c} \circlearrowright \\ + \sum_{r,s \geq 0} \sum_{a,b \in B} a'sb\end{array}$$

such that the following relations hold:

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = \sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1,$$

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1,$$

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1,$$

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1,$$

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1,$$

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1,$$

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1,$$

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1,$$

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1,$$

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1,$$

$$\sum_{r,s \geq 0} \sum_{a,b \in B} a'sb = 1.$$
Moreover, $\mathcal{Heis}_{F,k}$ can be presented equivalently as the strict $\mathbb{k}$-linear monoidal supercategory generated by the objects $Q_+, Q_-$, and morphisms $s, x, c, d, c', d'$, and $\beta_f$, $f \in F$, subject only to the relations (2.3), (2.7) to (2.11), (2.14), (2.15), and (3.30) to (3.35). In the above relations, in addition to the rightward crossing $t$ defined by (2.2), we have used the left crossing $t'$: $Q_-Q_+ \to Q_+Q_-$ defined by

(3.36) 

$\text{t}' = \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture}$

and the negatively dotted bubbles defined, for $f \in F$, by

(3.37) 

$f \circ \circ_{r-k-1} = \sum_{b_1, \ldots, b_{r-1} \in B} \det \left( \begin{smallmatrix} i-j+k & b_{j-1} \ b_{j} \\ i_j & \end{smallmatrix} \right)^r_{i,j=1}$, if $r \leq k$,

(3.38) 

$r+k-1 \circ f = (-1)^{r+1} \sum_{b_1, \ldots, b_{r-1} \in B} \det \left( \begin{smallmatrix} b_{j-1} \ b_{j} & b_{i-j} \ b_{j} \\ i_{j} \ b_{j} & \end{smallmatrix} \right)^r_{i,j=1}$, if $r \leq -k$,

where we adopt the convention that $b_0^c = f$ and $b_1 = 1$, and we interpret the determinants as $\text{tr}(f)$ if $r = 0$ and as 0 if $r < 0$.

The morphisms $c'$ and $d'$ of Theorem 3.3 are defined as (see [Savb, page 12]):

(3.39) 

$c' = \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture}$ if $k > 0$,

(3.40) 

d' = \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture}$ if $k < 0$.

The following theorem, Theorem 3.4, gives another alternate presentation of $\mathcal{Heis}_{F,k}$.

**Theorem 3.4.** There are unique morphisms $c'_i: \mathbb{1} \to Q_+Q_-$ and $d'_i: Q_-Q_+ \to \mathbb{1}$, $i = 1, \ldots, n$, in $\mathcal{Heis}_{F,k}$, drawn as

\[ c'_i = \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture}, \quad d'_i = \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture}, \]

such that the following relations hold for all $i, j = 1, \ldots, n$:

(3.41) 

\[ \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture} = \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture} + \delta_{i,j} \sum_{r,s \geq 0} \sum_{a, b \in B_i} a^s b = \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture} + \delta_{i,j} \delta_{k,1} \sum_{b \in B_i} b^i \quad \text{if } k \leq 1 \),

(3.42) 

\[ \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture} = \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture} + \delta_{i,j} \sum_{r,s \geq 0} \sum_{a, b \in B_i} a^s b = \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0.5) -- (0,0);
\draw[thick] (0.5,0) -- (1,0.5);
\end{tikzpicture} + \delta_{i,j} \delta_{k,-1} \sum_{b \in B_i} b^i \quad \text{if } k \geq -1 \),
The objects and morphisms, for $i, j = 1, \ldots, n$, subject only to the relations (3.2), (3.6) to (3.10), (3.13), (3.14), and (3.41) to (3.46). In the above relations, in addition to the rightward crossing defined by (3.1), we have used the left crossing, for $i, j = 1, \ldots, n$, defined by

$$t'_{i,j} = \bigotimes_j^i : = \bigotimes_j^i , \quad Q_+ Q_+ \mapsto Q_+ Q_-, $$

and the negatively dotted bubbles defined, for $i = 1, \ldots, n$, by

$$f_{l} = \bigotimes_{l}^{r-k-1} \sum_{b_1, \ldots, b_{r-1} \in B_l} \det \left( \bigotimes_{i,j=1}^{i,j+k} b_{l-1}^{b_j} \right)^{r} , \quad \text{if } r \leq k,$$

$$f_{l} = \bigotimes_{l}^{r+k-1} (-1)^{r+1} \sum_{b_1, \ldots, b_{r-1} \in B_l} \det \left( b_{l-1}^{b_j} \right)^{r} , \quad \text{if } r \leq -k,$$

where we adopt the convention that $b_0^f = f$ and $b_{r} = 1$, and we interpret the determinants as $\text{tr}_l(f)$ if $r = 0$ and as 0 if $r < 0$.

Proof. Let $\mathcal{Heis}_{F,k}$ be the strict $k$-linear monoidal category generated by the objects $Q_+$, $Q_-$ and morphisms $s_{i,j}, x_i, c_i, d_i, c'_i, d'_i$, $i, j = 1, \ldots, n$, subject only to the relations (3.2), (3.6) to (3.10), (3.13), (3.14), and (3.41) to (3.46). The rightward crossing, $t_{i,j}$, and leftward crossing, $t'_{i,j}$, are given by (3.1) and (3.47), respectively, and the negatively dotted bubbles are defined by (3.48) and (3.49).

View $\mathcal{Heis}_{F,k}$ in the presentation given in Theorem 3.3. That is, $\mathcal{Heis}_{F,k}$ is generated by the objects $Q_+$, $Q_-$ and morphisms $s, x, c, d, c', d'$, subject only to the relations (2.3), (2.7) to (2.11), (2.14), (2.15), and (3.30) to (3.35), where the rightward crossing, $t$, and leftward crossing, $t'$, are given by (2.2) and (3.36), respectively, and the negatively dotted bubbles are defined by (3.37) and (3.38).

We first prove that $\mathcal{Heis}_{F,k}$ is isomorphic to $\mathcal{Heis}_{F,k}$.

We define a monoidal functor $A: \mathcal{Heis}_{F,k} \rightarrow \mathcal{Heis}_{F,k}$ as follows. On objects define

$$A(Q_+) = Q_+,$$

and on morphisms, for $f_i \in F_i$, $i, j = 1, \ldots, n$, define

$$A(s_{i,j}) = s_{i,j}, \quad A(c_i) = c_i, \quad A(d_i) = d_i, \quad A(x_i) = x_i, \quad A(\beta_{f_i, i}) = \beta_{f_i}.$$

We then have

$$A(c'_i) = c'_i, \quad A(d'_i) = d'_i.$$
and extend \( A \) so that it is linear and monoidal. We define another monoidal functor \( B : \mathcal{H}(\text{Heis}_{F,k}) \rightarrow \mathcal{H}(\text{Heis}'_{F,k}) \) as follows. On objects define

\[
B(\mathbb{Q}_\pm) = \mathbb{Q}_\pm,
\]

and on morphisms, define

\[
B(s) = \sum_{i,j=1}^{n} s_{i,j} r \quad B(c) = \sum_{i=1}^{n} c_i r \quad B(d) = \sum_{i=1}^{n} d_i r \quad B\left(\beta(f_1,\ldots,f_n)\right) = \sum_{i=1}^{n} \beta(f_i,1),
\]

and extend \( B \) so that it is linear and monoidal.

We claim that \( A \) and \( B \) are strict \( k \)-linear monoidal functors that are two-sided inverses of each other. If \( A \) and \( B \) exist (i.e. if they are well-defined), it is clear that \( B \circ A = \text{id}_{\mathcal{H}(\text{Heis}'_{F,k})} \) and \( A \circ B = \text{id}_{\mathcal{H}(\text{Heis}_{F,k})} \). So it is enough to show that \( A \) and \( B \) are well-defined. To do this, we show that \( A \) preserves the defining relations of \( \mathcal{H}(\text{Heis}_{F,k}) \) and \( B \) preserves the defining relations of \( \mathcal{H}(\text{Heis}'_{F,k}) \). That is, we show that \( B \) preserves the relations (2.3), (2.7) to (2.11), (2.14), (2.15), and (3.30) to (3.35) in \( \mathcal{H}(\text{Heis}_{F,k}) \), and that \( A \) preserves the relations (3.2), (3.6) to (3.10), (3.13), (3.14), and (3.41) to (3.46) in \( \mathcal{H}(\text{Heis}'_{F,k}) \). We first show that \( A \) preserves the defining relations of \( \mathcal{H}(\text{Heis}'_{F,k}) \).

The functor \( A \) preserves the relations (3.2), (3.6) to (3.10), (3.13), (3.14), and (3.41) in \( \mathcal{H}(\text{Heis}'_{F,k}) \). This follows by the same arguments used in Theorem 3.2, where we showed that the functor \( F \) preserves these relations in \( \mathcal{H}(\text{Heis}'_{F,k}) \).

The functor \( A \) preserves the relations (3.41) to (3.43) and (3.45) in \( \mathcal{H}(\text{Heis}'_{F,k}) \). This follows directly from Theorem 3.3 by attaching the appropriate idempotents above and below both sides of the equations (3.30) to (3.32) and (3.34), and noting that the resulting are the images of (3.41) to (3.43) and (3.45) under \( A \), respectively.

The functor \( A \) preserves the relations (3.44) and (3.46) in \( \mathcal{H}(\text{Heis}'_{F,k}) \). This follows by letting \( f \in F_i, \ i = 1,\ldots,n, \) in equations (3.33) and (3.35), and noting that the resulting equations are the images of (3.44) and (3.46) under \( A \), respectively. Therefore, we have shown that \( A \) preserves the defining relations of \( \mathcal{H}(\text{Heis}'_{F,k}) \). Now we show that \( B \) preserves the defining relations of \( \mathcal{H}(\text{Heis}_{F,k}) \).

The functor \( B \) preserves the relations (2.3), (2.7) to (2.11), (2.14), and (2.15) in \( \mathcal{H}(\text{Heis}_{F,k}) \). This follows by the same arguments used in Theorem 3.2, where we showed that the functor \( G \) preserves these relations in \( \mathcal{H}(\text{Heis}_{F,k}) \).

The functor \( B \) preserves the relations (3.30) and (3.31) in \( \mathcal{H}(\text{Heis}_{F,k}) \). This follows by summing both sides of the relations (3.41) and (3.42) over all \( i,j = 1,\ldots,n, \) and noting that the resulting equations are the images of (3.30) and (3.31) under \( B \), respectively.

The functor \( B \) preserves the relations (3.32) and (3.34) in \( \mathcal{H}(\text{Heis}_{F,k}) \). This follows by summing both sides of the relations (3.43) and (3.45) over all \( i = 1,\ldots,n, \) and noting that the resulting equations are the images of (3.32) and (3.34) under \( B \), respectively.

The functor \( B \) preserves the relations (3.33) and (3.35) in \( \mathcal{H}(\text{Heis}_{F,k}) \). Let \( f = (f_1,\ldots,f_n) \), and note that the relations (3.33) and (3.35) can be written as

\[
\sum_{i=1}^{n} r \quad f_i = -\delta_{r,k-1} \sum_{i=1}^{n} \text{tr}(f_i) \quad \text{if} \quad 0 \leq r < k, \quad \text{and}
\]

\[
\sum_{i=1}^{n} f_i \quad r = \delta_{r,-k-1} \sum_{i=1}^{n} \text{tr}(f_i) \quad \text{if} \quad 0 \leq r < -k, \quad f_i \in F_i,
\]
respectively. Since the relations (3.44) and (3.46) hold in $\mathcal{Heis}_{F,k}$, $B$ preserves the relations (3.33) and (3.35) in $\mathcal{Heis}_{F,k}$. Therefore, we have shown that $B$ preserves the defining relations of $\mathcal{Heis}_{F,k}$. Hence, $\mathcal{Heis}_{F,k}$ is isomorphic to $\mathcal{Heis}_{F,k}$.

Now we show that there are unique morphisms, $c'_i$ and $d'_i$, $i = 1, \ldots, n$, in $\mathcal{Heis}_{F,k}$ satisfying (3.41) to (3.46). Define

$$c'_i := \begin{cases} i & \text{if } k > 0, \\ i \circ (-k-1,1) & \text{if } k \leq 0, \\ i \circ (k,1,1) & \text{if } k < 0. \end{cases}$$

(3.50)

$$d'_i := \begin{cases} i & \text{if } k > 0, \\ i \circ (-k-1,1) & \text{if } k \leq 0, \\ i \circ (k,1,1) & \text{if } k < 0. \end{cases}$$

(3.51)

First we show that the morphisms $c'_i$ and $d'_i$, $i = 1, \ldots, n$, satisfy the relations (3.41) to (3.46) in $\mathcal{Heis}_{F,k}$. Since (3.30) to (3.32) and (3.34) hold in $\mathcal{Heis}_{F,k}$, attaching the appropriate idempotents above or below both sides of these relations gives (3.41) to (3.43) and (3.45), respectively, in $\mathcal{Heis}_{F,k}$. Since (3.33) and (3.35) hold in $\mathcal{Heis}_{F,k}$, we can let the label of the tokens of these relations be $f \in F_i$. This gives the relations (3.44) and (3.46), respectively, in $\mathcal{Heis}_{F,k}$. So we have shown that $c'_i$ and $d'_i$ are morphisms in $\mathcal{Heis}_{F,k}$ that satisfy the relations (3.41) to (3.46).

Now we show that $c'_i$ and $d'_i$, $i = 1, \ldots, n$, are unique morphisms in $\mathcal{Heis}_{F,k}$ that satisfy (3.41) to (3.46). Summing both sides of (3.41) and (3.42) over all $i, j = 1, \ldots, n$ gives the relations (3.30) and (3.31), respectively. Summing both sides of (3.43) and (3.45) over all $i = 1, \ldots, n$, gives the relations (3.32) and (3.34), respectively. Letting $f = (f_1, \ldots, f_n) \in F_i$ and noting that the relations (3.33) and (3.35) in $\mathcal{Heis}_{F,k}$ are equivalent to the equations

$$\sum_{i=1}^{n} f_i \circ r \circ i = -\delta_{r,k-1} \sum_{i=1}^{n} \text{tr}_i(f_i) \text{ if } 0 \leq r < k, \quad f_i \in F_i, \quad \text{and} \quad$$

$$\sum_{i=1}^{n} f_i \circ r \circ i = \delta_{r,-k-1} \sum_{i=1}^{n} \text{tr}_i(f_i) \text{ if } 0 \leq r < -k, \quad f_i \in F_i,$$

respectively, shows that the relations (3.44) and (3.46) in $\mathcal{Heis}_{F,k}$ imply the relations (3.33) and (3.35) in $\mathcal{Heis}_{F,k}$. So the existence of the morphisms $c'_i$ and $d'_i$, $i = 1, \ldots, n$, in $\mathcal{Heis}_{F,k}$ satisfying (3.41) to (3.46) implies the existence of morphisms $c'$ and $d'$ in $\mathcal{Heis}_{F,k}$ satisfying (3.30) to (3.35). Attaching the appropriate idempotent to the bottom or top of $c'$ and $d'$ gives the morphisms $c'_i$ and $d'_i$ in $\mathcal{Heis}_{F,k}$, respectively. Therefore, since $c'$ and $d'$ are unique morphisms in $\mathcal{Heis}_{F,k}$ satisfying (3.30) to (3.35), we must have that $c'_i$ and $d'_i$, $i = 1, \ldots, n$, are unique morphisms in $\mathcal{Heis}_{F,k}$ satisfying (3.41) to (3.46).

**Theorem 3.5.** Using the notation from Theorem 3.4, the following relations are consequences of the defining relations.

(a) Infinite grassmannian relations: For $i = 1, \ldots, n$ and $f, g \in F_i$ we have

$$\sum_{i=1}^{n} f_i \circ r \circ i = -\delta_{r,k-1} \text{tr}_i(f) \text{ if } r \leq k-1, \quad \text{and} \quad$$

(3.52)

$$\sum_{i=1}^{n} g_i \circ r \circ i = \delta_{r,-k-1} \text{tr}_i(f) \text{ if } r \leq -k-1,$$

(3.53)
(3.54) \[
\sum_{r,s\in\mathbb{Z}} \sum_{b\in B_i} r\cdot g \cdot s \cdot i \cdot f b = \sum_{r,s\geq 0} \sum_{b\in B_i} r+k-1 \cdot g \cdot s-k-1 \cdot i \cdot f b = -\delta_{t,0} tr_i(fg).
\]

(b) Left adjunction: For all \( i = 1, \ldots, n \)

(3.55) \[
i = i
\]

(3.56) \[
i = i
\]

(c) Rotation relations: For all \( f \in F_i, i = 1, \ldots, n \),

(3.57) \[
f := f = f, \quad (3.58) \quad := :=
\]

(3.59) \[
:=
\]

(d) Curl relations: For all \( r \geq 0 \) and \( i = 1, \ldots, n \),

(3.60) \[
\sigma^r_i = \sum_{s \geq 0} \sum_{b \in B_i} r-s-1 \cdot b \cdot g \cdot s \cdot i \cdot f b, \quad (3.61) \quad = -\sum_{s \geq 0} \sum_{b \in B_i} s \cdot b \cdot g \cdot r-s-1 \cdot i \cdot f b.
\]

(e) Bubble slides: For all \( f \in F_i, i = 1, \ldots, n \), and \( r \geq 0 \),

(3.62) \[
\sigma^r_i = \sum_{t \geq 0} \sum_{s=0}^t \sum_{a,b \in B_i} a \cdot f \cdot r-t-2 \cdot i \cdot f b, \quad (3.63) \quad = \sum_{t \geq 0} \sum_{s=0}^t \sum_{a,b \in B_i} a \cdot f \cdot r-t-2 \cdot i \cdot f b.
\]
(f) Alternating braid relation: for all \( i, j, l = 1, \ldots, n \)

\[(3.64)\]

\[
\begin{cases}
\delta_{ij} \sum_{r,s,t \geq 0} \sum_{a,b,c \in \mathcal{B}_l} \delta_{L,l} \sum_{r,s,t \geq 0} \sum_{a,b,c \in \mathcal{B}_l} & 
\text{if } k \geq 2, \\
0 & 
\text{if } -1 \leq k \leq 1, \\
\delta_{ij} \sum_{r,s,t \geq 0} \sum_{a,b,c \in \mathcal{B}_l} \delta_{L,l} \sum_{r,s,t \geq 0} \sum_{a,b,c \in \mathcal{B}_l} & 
\text{if } k \leq -2.
\end{cases}
\]

Proof. The infinite Grassmanian relations, \((3.52)\) to \((3.54)\), follow from the infinite Grassmanian relations, \((1.27)\) to \((1.29)\), in [Savb, Theorem 1.3], respectively, by letting \( f \in F_l, i = 1, \ldots, n \). The remaining relations, \((3.55)\) to \((3.64)\), follow from [Savb, Theorem 1.3] by attaching the appropriate idempotents to the bottoms or tops of both sides of the relations \((1.30)\) to \((1.39)\) in [Savb, Theorem 1.3], respectively.

\[\square\]

4. An Equivalence of Categories

Let \( \mathcal{C} \) and \( \mathcal{D} \) be strict \( k \)-linear monoidal categories.

**Definition 4.1.** Following [GK14, Def. 1.1], the tensor product of \( \mathcal{C} \) and \( \mathcal{D} \), denoted \( \mathcal{C} \otimes_k \mathcal{D} \), is defined as follows. For all \( U, W, Y \in \text{Ob} (\mathcal{C}) \) and \( V, X, Z \in \text{Ob} (\mathcal{D}) \), we have

- \( \text{Ob} (\mathcal{C} \otimes_k \mathcal{D}) = \{(X,Y) \mid X \in \text{Ob}(\mathcal{C}) \text{ and } Y \in \text{Ob}(\mathcal{D})\} \),
- \( \text{Hom}_{\mathcal{C} \otimes_k \mathcal{D}} ((W,X), (Y,Z)) = \text{Hom}_{\mathcal{C}} (W,Y) \otimes_k \text{Hom}_{\mathcal{D}} (X,Z) \), and
- For all morphisms of the form \( g = g_c \otimes g_D : (U,V) \to (W,X) \) and \( f = f_c \otimes f_D : (W,X) \to (Y,Z) \) in \( \mathcal{C} \otimes_k \mathcal{D} \), the composition \( f \circ g \) is defined as

\[
f \circ g = (f_c \otimes f_D) \circ (g_c \otimes g_D) = (f_c \circ g_c) \otimes (f_D \circ g_D) : (U,V) \to (Y,Z).
\]

Define composition between any two morphisms of \( \mathcal{C} \otimes_k \mathcal{D} \) by extending the definition on simple tensors by linearity.

It is easy to see that the tensor product, \( \mathcal{C} \otimes \mathcal{D} \), is a category. We add additional structure to \( \mathcal{C} \otimes_k \mathcal{D} \) to turn it into a strict \( k \)-linear monoidal category, and we call resulting category \( \mathcal{C} \otimes_k \mathcal{D} \) (i.e.: we do not change the name):

- Equip \( \mathcal{C} \otimes_k \mathcal{D} \) with a bifunctor \( \otimes : (\mathcal{C} \otimes_k \mathcal{D}) \times (\mathcal{C} \otimes_k \mathcal{D}) \to \mathcal{C} \otimes_k \mathcal{D} \) sending \((A_c, A_D), (B_c, B_D)) \rightarrow (A_c \otimes B_c, A_D \otimes B_D)\), and
- Let \( a \in k \), and \( f = f_c \otimes f_D \). We define \( k \)-action by the following equation:

\[
a \cdot f = a \cdot (f_c \otimes f_D) = (a \cdot f_c) \otimes (a \cdot f_D).
\]

It is straightforward to verify that \( \mathcal{C} \otimes \mathcal{D} \) is a strict \( k \)-linear monoidal category, with \( 1 = (1_c, 1_D) \) being the unit object of \( \mathcal{C} \otimes_k \mathcal{D} \), where \( 1_c \) is the unit object of \( \mathcal{C} \) and \( 1_D \) is the unit object of \( \mathcal{D} \).
Now, we are interested in a particular subcategory of the Karoubi envelope, $\text{Kar}(\text{Heis}_{F,k})$, which we call the partial Karoubi envelope of $\text{Heis}_{F,k}$ and denote $PK(\text{Heis}_{F,k})$. Below, we will abuse notation: in $\text{Heis}_{F,k}$, $\beta(e_i)$, $i = 1, \ldots, n$, is a morphism in $Q_+$ and a downward strand carrying a token labelled by $e_i$, $i = 1, \ldots, n$, is a morphism in $Q_-$, but below we simply refer to both morphisms as $e_i$ (note that there will be no ambiguity to this notation).

**Definition 4.2.** The partial Karoubi envelope of $\text{Heis}_{F,k}$, denoted $PK(\text{Heis}_{F,k})$, is defined as follows:

- **Ob** $PK(\text{Heis}_{F,k}) = \{(Q,e,e) \mid (e,\lambda) \in \{+,-\}^m \times \{1, \ldots, n\}^m, m \geq 0\}$, where for $m > 0$, we define $(Q,e,e) := (Q_{e_1} \otimes \cdots \otimes Q_{e_m}, e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_m})$, where $e = (e_1, \ldots, e_m)$ and $\gamma = (\gamma_1, \ldots, \gamma_m)$. For $m = 0$, we define $(Q,e,e) := (1, \text{id}_1)$, where $1$ is the unit object of $\text{Heis}_{F,k}$ and $\text{id}_1$ is the identity morphism on $1$ in $\text{Heis}_{F,k}$.
- $\text{Hom}_{PK(\text{Heis}_{F,k})}(\{(Q,e_1,e_{\gamma_1}),(Q,e_2,e_{\gamma_2})\}) = \{f \in \text{Hom}_{\text{Heis}_{F,k}}(Q,e_1,Q,e_2) \mid f \circ e_{\gamma_1} = e_{\gamma_2} \circ f\}$.
- For all morphisms $g : (Q,e_1,e_{\gamma_1}) \to (Q,e_2,e_{\gamma_2})$ and $f' : (Q,e_2,e_{\gamma_2}) \to (Q,e_3,e_{\gamma_3})$ in $PK(\text{Heis}_{F,k})$, there are morphisms $g : Q,e_1 \to Q,e_2$ and $f : Q,e_2 \to Q,e_3$ in $\text{Heis}_{F,k}$ such that $g' = f'$, respectively. The composition $f' \circ g' : (Q,e_1,e_{\gamma_1}) \to (Q,e_3,e_{\gamma_3})$ in $PK(\text{Heis}_{F,k})$ is then defined as the composition $f \circ g : Q,e_1 \to Q,e_3$ in $\text{Heis}_{F,k}$. Indeed, we have that

$$ (f \circ g) \circ e_1 = f \circ (g \circ e_1) = f \circ g = (e_3 \circ f) \circ g = e_3 \circ (f \circ g), $$

and

- Equip $PK(\text{Heis}_{F,k})$ with a bifunctor $\otimes : PK(\text{Heis}_{F,k}) \times PK(\text{Heis}_{F,k}) \to PK(\text{Heis}_{F,k})$ sending $((Q,e,e),(Q,e,e)) \mapsto (Q,e_1 \otimes Q,e_2,e_{\gamma_1} \otimes e_{\gamma_2})$.

The partial Karoubi envelope, $PK(\text{Heis}_{F,k})$, is a strict monoidal subcategory of the Karoubi envelope $\text{Kar}(\text{Heis}_{F,k})$, with the unit object of $PK(\text{Heis}_{F,k})$ being $(1, \text{id}_1)$, where $1$ is the unit object of $\text{Heis}_{F,k}$, and $\text{id}_1$ is the identity morphism of $\text{Heis}_{F,k}$.

We define a $k$-action in $PK(\text{Heis}_{F,k})$ to turn it into a strict $k$-linear monoidal category, and we call resulting category $PK(\text{Heis}_{F,k})$ (i.e.: we do not change the name). Let $a \in k$, and $f' : (Q,e_1,e_{\gamma_1}) \to (Q,e_2,e_{\gamma_2}) \in \text{Mor}(PK(\text{Heis}_{F,k}))$. Then there is a morphism $f : Q,e_1 \to Q,e_2 \in \text{Mor}(\text{Heis}_{F,k})$ equalling $f'$, with the property $f \circ e_{\gamma_1} = e_{\gamma_2} \circ f$. We define $k$-action as follows:

- $a \cdot f' : (Q,e_1,e_{\gamma_1}) \to (Q,e_2,e_{\gamma_2}) \in \text{Mor}(PK(\text{Heis}_{F,k}))$ equals the morphism $a \cdot f : Q,e_1 \to Q,e_2 \in \text{Mor}(\text{Heis}_{F,k})$, so that

$$ (a \cdot f) \circ e_1 = a \cdot (f \circ e_1) = a \cdot f = a \cdot (e_2 \circ f) = (a \cdot e_2) \circ f = e_2 \circ (a \cdot f). $$

**Definition 4.3.** Suppose $\mathcal{C}$ is a strict $k$-linear monoidal category and $X = X_1 \cup \cdots \cup X_n$, $X_i \subseteq \text{Ob}(\mathcal{C})$, is a subclass of the objects of $\mathcal{C}$ such that the objects of $\mathcal{C}$ are generated by the objects in $X$, and suppose $\mathcal{C}_i$ is a subcategory of $\mathcal{C}$ such that $\text{Ob}(\mathcal{C}_i) = X_i$. We define a functor $F : \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n \to \mathcal{C}$ as follows. On objects, we define

$$ F(c_1, \ldots, c_n) = c_1 \otimes \cdots \otimes c_n, $$

and on morphisms that are simple tensors, we define

$$ F(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_n, $$

and extend $F$ by linearity.

**Theorem 4.4.** Suppose $\mathcal{C}$ is a strict $k$-linear monoidal category and $X = X_1 \cup \cdots \cup X_n$, $X_i \subseteq \text{Ob}(\mathcal{C})$, is a subclass of the objects of $\mathcal{C}$ such that the objects of $\mathcal{C}$ are generated by the objects in $X$, and such that there is no nonzero morphism between objects in $X_i$ and objects in $X_j$ for $i \neq j$, and such that $X_i \otimes X_i$ is isomorphic to $X_i \otimes X_j$ whenever $x_i \in X_i$ and $x_j \in X_j$ for $i \neq j$. Suppose $\mathcal{C}_i$ is a subcategory of $\mathcal{C}$ such that
\( \text{Ob} (\mathcal{C}_1) = X_i \) and such that the induced functor \( F \) from Definition 4.3 is full on \( \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n \). Then \( \mathcal{C} \) is equivalent to \( \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n \).

**Proof.** We claim that the functor \( F \) from Definition 4.3 induces an equivalence of categories between \( \mathcal{C} \) and \( \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n \).

\( F \) is essentially surjective on objects. Since \( x_i \otimes x_j \) and \( x_j \otimes x_i \) are isomorphic whenever \( x_i \in X_i \) and \( x_j \in X_j \) for \( i \neq j \), we can construct an invertible morphism from an object in \( \mathcal{C} \) to an object in \( \text{im} F \) by composing the invertible morphisms between the objects \( x_i \otimes x_j \) and \( x_j \otimes x_i \) when \( i \neq j \). This morphism acts on an object in \( \mathcal{C} \) by shifting its components one by one in such a way that the resulting object is in \( \text{im} F \).

\( F \) is full and faithful on morphisms. \( F \) is full by assumption. Moreover, if \( F(g) = F(f) \), then by the definition of \( F \), we have \( f = g \), since \( F(g) = g \) and \( F(f) = f \). Therefore, \( F \) is full and faithful. \( \square \)

**Definition 4.5.** Define \( PK (\text{Heis}_{F,k}) \) as the subclass of the objects of \( PK (\text{Heis}_{F,k}) \) generated by the objects \( (Q_+, e_i) \) and \( (Q_-, e_i) \) and the morphisms of \( PK (\text{Heis}_{F,k}) \) of color \( i \).

**Lemma 4.6.** The categories \( PK (\text{Heis}_{F,k})_1 \otimes \cdots \otimes PK (\text{Heis}_{F,k})_n \) and \( \text{Heis}_{F,1,k} \otimes \cdots \otimes \text{Heis}_{F,n,k} \) are isomorphic.

**Proof.** For \( (Q_\varepsilon, e_\gamma) \), denote the number of components of \( \varepsilon \) by \( l(\varepsilon) \). Define a functor \( G : \text{Heis}_{F,k} \otimes \cdots \otimes \text{Heis}_{F,n,k} \to PK (\text{Heis}_{F,k})_1 \otimes \cdots \otimes PK (\text{Heis}_{F,k})_n \) as follows. On objects, define

\[
G (Q_1, \ldots, Q_n) = \left( \left( Q_1^{(e_1^{(1)})}, \ldots, Q_n^{(e_n^{(1)})} \right) \right),
\]

and on morphisms that are simple tensors, define

\[
G (f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_n,
\]

and extend \( G \) by linearity.

Now define another functor \( H : PK (\text{Heis}_{F,k})_1 \otimes \cdots \otimes PK (\text{Heis}_{F,k})_n \to \text{Heis}_{F,1,k} \otimes \cdots \otimes \text{Heis}_{F,n,k} \) as follows. On objects, define

\[
H \left( \left( Q_1^{(e_1^{(1)})}, \ldots, Q_n^{(e_n^{(1)})} \right) \right) = \left( Q_1, \ldots, Q_n \right),
\]

and on morphisms that are simple tensors, define

\[
H (f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_n,
\]

and extend \( H \) by linearity. Clearly, \( H \circ G = \text{id}_{\text{Heis}_{F,1,k} \otimes \cdots \otimes \text{Heis}_{F,n,k}} \) and \( G \circ H = \text{id}_{PK (\text{Heis}_{F,k})_1 \otimes \cdots \otimes PK (\text{Heis}_{F,k})_n} \), so the categories in question are isomorphic. \( \square \)

**Corollary 4.7.** \( PK (\text{Heis}_{F,k}) \) is equivalent to \( \text{Heis}_{F,1,k} \otimes \cdots \otimes \text{Heis}_{F,n,k} \).

**Proof.** \( PK (\text{Heis}_{F,k}) \) is a strict \( \mathbb{K} \)-linear monoidal category, and

\[
X = \text{Ob} (PK (\text{Heis}_{F,k})_1) \cup \cdots \cup \text{Ob} (PK (\text{Heis}_{F,k})_n)
\]

is a subclass of the objects of \( PK (\text{Heis}_{F,k}) \) such that all objects of \( PK (\text{Heis}_{F,k}) \) are generated by the objects in \( X \), and such that there is no nonzero morphism between objects in \( PK (\text{Heis}_{F,k})_i \) and objects in \( PK (\text{Heis}_{F,k})_j \) for \( i \neq j \), and there is an isomorphism between the objects \( q_i \otimes q_j \) and \( q_j \otimes q_i \) in \( PK (\text{Heis}_{F,k}) \) when \( q_i \in PK (\text{Heis}_{F,k})_i \) and \( q_j \in PK (\text{Heis}_{F,k})_j \) for \( i \neq j \), given by compositions of the diagrams:
Following the techniques of [RS17, Prop. 8.14], there is a spanning set of $\text{Mor}(PK(\text{Heis}_{F,k}))$ over $k$, denoted $S$, which is given by diagrams satisfying:

- Any two strands of the diagram intersect at most once,
- No strands of the diagram self-intersect,
- Each strand consists of at most one cup or cap,
- Dots appear above all crossings of strands, and
- All bubbles are to the right of the strands of the diagram.

We have that for all diagrams in $S$, all bubbles of color $i$ slide through morphisms of color $j$ when $i \neq j$. This property arises from the relations (3.62) and (3.63) in Theorem 3.5. Therefore, the induced functor $F$ from Definition 4.3 is full on morphisms of $PK(\text{Heis}_{F,k})_1 \otimes \cdots \otimes PK(\text{Heis}_{F,k})_n$. In light of Lemma 4.6, we identify $PK(\text{Heis}_{F,k})_1 \otimes \cdots \otimes PK(\text{Heis}_{F,k})_n$ with $\text{Heis}_{F,k}^1 \otimes \cdots \otimes \text{Heis}_{F,k}^n$. Therefore, by Theorem 4.4, there is an equivalence of categories between $PK(\text{Heis}_{F,k})$ and $\text{Heis}_{F,k}^1 \otimes \cdots \otimes \text{Heis}_{F,k}^n$. □

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