Solutions to Linear Bimatrix Equations with Applications to Pole Assignment of Complex-Valued Linear Systems

Bin Zhou

Abstract

We study in this paper solutions to several kinds of linear bimatrix equations arising from pole assignment and stability analysis of complex-valued linear systems, which have several potential applications in control theory, particularly, can be used to model second-order linear systems in a very dense manner. These linear bimatrix equations include generalized Sylvester bimatrix equations, Sylvester bimatrix equations, Stein bimatrix equations, and Lyapunov bimatrix equations. Complete and explicit solutions are provided in terms of the bimatrices that are coefficients of the equations/systems. The obtained solutions are then used to solve the full state feedback pole assignment problem for complex-valued linear system. For a special case of complex-valued linear systems, the so-called antilinear system, the solutions are also used to solve the so-called anti-preserving (the closed-loop system is still an antilinear system) and normalization (the closed-loop system is a normal linear system) problems. Second-order linear systems, particularly, the spacecraft rendezvous control system, are used to demonstrate the obtained theoretical results.

Keywords: Linear bimatrix equations; Complex-valued linear systems; Pole assignment; Second-order linear systems; Spacecraft rendezvous.

1 Introduction

In this paper we continue to study complex-valued linear systems introduced in [21]. Complex-valued linear systems refer to linear systems whose state evolution depends on both the state and its conjugate (see Subsection 2.1 for a detailed introduction). There are several reasons for studying this class of linear systems [21], for example, they are naturally encountered in linear dynamical quantum systems theory, and can be used to model any real-valued linear systems with lower dimensions (see Subsection 2.2 for a detailed development). Analysis and design of complex-valued linear systems have been studied in our early paper [21], where some fundamental problems such as state response, controllability, observability, stability, pole assignment, linear quadratic regulation, and state observer design, were solved. The conditions and/or methods obtained there are based on bimatrices associated with the complex-valued linear system, which is mathematically appealing.

The pole assignment problem for complex-valued linear system was solved in [21] by using coefficients of the so-called real-representation system, for which any pole assignment algorithms for normal linear systems can be applied. In this paper, we will continue to study the pole assignment problem for complex-valued linear systems by establishing a different method. Our new solution is based on solving the so-called (generalized) Sylvester bimatrix equation whose coefficients are bimatrices associated with the complex-valued linear system. Our study on linear bimatrix equations and their applications in pole assignment has been inspired by early work for normal linear systems. For example, the (normal) Sylvester matrix equation was utilized to solve the pole assignment for normal linear system in [5], and the generalized Sylvester matrix equation was used in [6, 7, 8] and [22] to solve the (parametric) pole assignment problem for normal linear systems, descriptor linear systems, and even high-order linear systems.

We will show that the pole assignment problem for a complex-valued linear system has a solution if and
only if the associated (generalized) Sylvester bimatrix equation has a nonsingular solution. Thus the main task of this paper is to provide complete and explicit solutions to the homogeneous (generalized) Sylvester bimatrix equation. The solutions we provided have quite element expressions that use the original coefficient bimatrices and a right-coprime factorization (in the bimatrix framework) of the system. We also provide solutions to non-homogeneous Sylvester bimatrix equations and Stein bimatrix equations which include the Lyapunov bimatrix equation as a special case.

We are particularly interested in pole assignment for the so-called antilinear system studied recently in [17–18, 19] and [20]. By our approach we first provide closed-form solutions to the associated (generalized) Sylvester bimatrix equations, and then consider two different problems, namely, the anti-preserving problem which ensures that the closed-loop system is still (or equivalent to) a normal linear system, and the normalization problem which guarantees that the closed-loop system is (equivalent to) a normal linear system. The anti-preserving problem was firstly studied in [19]. However, we can provide complete solutions that use full state feedback rather than only normal state feedback used in [19]. We discovered that the anti-preserving problem is meaningful only for discrete-time antilinear systems (as studied in [19]) since any continuous-time antilinear system cannot be asymptotically stable. However, the normalization problem is valid for both continuous-time and discrete-time antilinear systems, and seems more interesting as the closed-loop system is (equivalent to) a normal linear system that is more easy to handle.

**Notation:** For a matrix $A \in \mathbb{C}^{n \times m}$, we use $A^\#, A^T, A^H$, rank $(A)$, $|A|$, $\|A\|$, $\lambda(A)$, $\rho(A)$, $\mu(A)$, Re $(A)$ and Im $(A)$ to denote respectively its conjugate, transpose, conjugate transpose, rank, determinant (when $n = m$), norm, eigenvalue set (when $n = m$), spectral radius (when $n = m$), spectral abscissa (max$_s \in \lambda(A) \{\Re(s)\}$), real part and imaginary part. The notation $0_{n \times m}$ refers to an $n \times m$ zero matrix. For two integers $p, q$ with $p \leq q$, denote $I[p, q] = \{p, p + 1, \ldots, q\}$. Let $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R} = \mathbb{R}^+ \cup \{-\mathbb{R}^+\}$, $\mathbb{Z} = \mathbb{Z}^+ \cup \{-\mathbb{Z}^+\}$, and $j$ the unitary imaginary number. For a series of matrices $A_i, i \in [1, \ell]$, diag$(A_1, A_2, \cdots, A_l)$ denotes a diagonal matrix whose diagonal elements are $A_i, i \in [1, \ell]$.

# Motivation and Preliminaries

## 2.1 Complex-Valued Linear Systems

To introduce complex valued linear systems we recall the bimatrix $\{A_1, A_2\} \in \{\mathbb{C}^{n \times m}, \mathbb{C}^{n \times m}\}$ given in [21], where it was defined in such a manner that, for any $x \in \mathbb{C}^m$,

$$y = \{A_1, A_2\} x \triangleq A_1x + A_2^\# x^\#,$$

which defines a linear mapping over the field of real numbers [21]. Further properties of the bimatrix can be found in [21]. With the notion of bimatrix, we continue to study the following complex-valued linear system [21] (without output equation)

$$x^+ = \{A_1, A_2\} x + \{B_1, B_2\} u,$$

where $A_i \in \mathbb{C}^{n \times n}, B_i \in \mathbb{C}^{n \times m}, i = 1, 2$, are known coefficients, $x = x(t)$ is the state, $u = u(t)$ is the control, and $x^+(t)$ denotes $x(t+1)$ if $t \in \mathbb{Z}^+$ (namely, discrete-time systems) and denotes $\dot{x}(t)$ if $t \in \mathbb{R}^+$ (namely, continuous-time systems). Throughout this paper, the dependence of variables on $t$ will be suppressed unless necessary. The initial condition is set to be $x(0) = x_0 \in \mathbb{C}^n$ [21].

There are several reasons for studying linear systems in the form of (1). Readers are encouraged to refer to [21] for details, while an explicit application of (1) to second-order linear system will be shown in detail in the next subsection. If $A_2$ and $B_2$ are null matrices, then system (1) becomes

$$x^+ = A_1x + B_1u,$$

which is the normal linear system that has been well studied during the past half century [14, 15]. If $A_1$ and $B_1$ are set as zeros, then (1) reduces to the so-called antilinear system

$$x^+ = A_2^\# x^\# + B_2^\# u^\#,$$

which was initially studied in [17] and [20].
In our recent paper [21], we have carried out a comprehensive study on the analysis and design of the complex-valued linear system (1), including state response, controllability, observability, stability, stabilization, pole assignment, linear quadratic regulation, and state observer design. The obtained results will reduce to classical ones when they are applied on the normal linear system (2), and will reduce to and/or improve the existing results when they are applied on the antilinear system (3). Our study is mainly based on the properties of the bimatrix \( \{ P_1, P_2 \} \in \{ \mathbb{C}^{n \times m}, \mathbb{C}^{n \times n} \} \), especially, its real representation
\[
\{ P_1, P_2 \} \triangleq \begin{bmatrix} \Re (P_1 + P_2) & -\Im (P_1 + P_2) \\ \Im (P_1 - P_2) & \Re (P_1 - P_2) \end{bmatrix} \in \mathbb{R}^{2n \times 2m},
\]
and complex-lifting
\[
\{ P_1, P_2 \} \triangleq \begin{bmatrix} P_1 & P_1^# \\ P_2 & P_2^# \end{bmatrix} \in \mathbb{C}^{2n \times 2m}.
\]
Particularly, we shown that, for stabilization and pole assignment of system (1), the so-called full state feedback is necessary [21], and, generally, the well-known normal linear feedback is valid only for the antilinear system (4) when
\[ u = \{ K_1, K_2 \} x, \]
(6)
is necessary [21], and, generally, the well-known normal linear feedback
\[ u = K_1 x, \]
(7)
is valid only for the antilinear system (8) when \( t \in \mathbb{Z}^+ \).

In this paper, we continue to study the complex-valued linear system (1). We are interested in the particular problem of pole assignment of this class of systems by the full state feedback (6). We will show that solutions to the pole assignment problem can be completely characterized by solutions to a class of generalized Sylvester bimatrix equations. Our study is clearly motivated by the existing work on pole assignment of the normal linear system, for which it is well known that solutions to the associated pole assignment can be characterized by solutions to some (generalized) Sylvester matrix equations [3, 5, 7]. We will provide complete solutions to such a type of generalized Sylvester bimatrix equations and will show that the obtained results include the existing ones for both the normal linear system (2) and the antilinear system (3) as special cases. From this point of view, we have built a quite general framework for pole assignment of linear systems.

### 2.2 Second-Order Linear Systems

In this subsection, we use a second-order linear system model to demonstrate the purpose of studying the complex-valued linear system (1). Consider the second-order linear system
\[
M \ddot{\xi} + D \dot{\xi} + K \xi = G v,
\]
(8)
where \( M, D, K \in \mathbb{R}^{n \times n} \) and \( G \in \mathbb{R}^{n \times q} \) are given matrices, \( \xi \) is the state (often denotes the displacements of the object to be controlled), and \( v \) is the control. Let the initial condition be \( \xi (0) = \xi_{10} \in \mathbb{R}^n \) and \( \dot{\xi} (0) = \xi_{20} \in \mathbb{R}^n \). For simplicity, we only consider the continuous-time case without output equation and assume that \( M \) is nonsingular. We further assume, without loss of generality, that \( q = 2m \), since, otherwise, we set \( G = [G, 0_{n \times 1}] \). Denote \( G = [G_1, G_2] \), \( G_i \in \mathbb{R}^{n \times m} \) and \( v = [v_1^T, v_2^T]^T \), \( v_i \in \mathbb{R}^m \), \( i = 1, 2 \). Second-order linear system can be used to describe many physical systems, for example, the mass-spring system [12], and the spacecraft rendezvous control system [13].

To describe the second-order linear system (9) as a complex-valued linear system, we first write it equivalently as
\[
\begin{bmatrix} \dot{\xi} \\ \xi \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & I_n \\ -M^{-1} K & -M^{-1} D \end{bmatrix} \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix} + \begin{bmatrix} 0_{n \times m} & 0_{n \times m} \\ G_1 & G_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.
\]
(9)
If we choose
\[
x = \xi + j \dot{\xi}, \quad u = v_1 + j v_2,
\]
(10)
then system (9) can be equivalently rewritten as (1) where the initial condition is \( x_0 = \xi_{10} + j \xi_{20} \), and \( A_i \in \mathbb{C}^{n \times n}, B_i \in \mathbb{C}^{n \times m}, i = 1, 2 \), are given by
\[
\begin{align*}
A_1 & = -\frac{1}{2} M^{-1} D - \frac{1}{2} (I_n + M^{-1} K), \\
A_2 & = \frac{1}{2} M^{-1} D - \frac{1}{2} (I_n - M^{-1} K), \\
B_1 & = \frac{1}{2} G_2 + \frac{1}{2} G_1, \\
B_2 & = -\frac{1}{2} G_2 - \frac{1}{2} G_1 = -B_1.
\end{align*}
\]
(11)
Remark 1 We have another method to describe system (8) as (4). Write (8) as
\[
\begin{bmatrix}
\xi \\
\dot{\xi}
\end{bmatrix} = \begin{bmatrix}
0_{n \times n} & I_n \\
-M^{-1}K & -M^{-1}D
\end{bmatrix} \begin{bmatrix}
\xi \\
\dot{\xi}
\end{bmatrix} + \begin{bmatrix}
0_{n \times q} \\
G \\
0_{n \times q}
\end{bmatrix} \begin{bmatrix}
v \\
w
\end{bmatrix},
\] (12)
where \(w \in C^q\) is a temp variable. Then, similar to (4), by setting \(x\) as in (14) and \(u = v + jw\), (14) can be written as (4), where \(A_i, i = 1, 2\), are given by (17) and
\[
B_1 = \frac{j}{2}G, \quad B_2 = -\frac{j}{2}G = -B_1.
\]
This method does not require that \(q\) is an even number, but however leads to higher dimensions of the inputs than (17).

The corresponding complex-valued linear system model (1) for (8) seems more convenient to use than the augmented normal linear system model (9) since it possesses the same dimension as the original system (8). We also mention that, as has been made clear in [21], the full state feedback (6) for the associated complex-valued linear system (1) can be equivalently written as
\[
v = \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = \{K_1, K_2\}_o \begin{bmatrix}
\xi \\
\dot{\xi}
\end{bmatrix},
\] (13)
where \(\{K_1, K_2\}_o\) is a real matrix. Therefore, the full state feedback (6) can be implemented physically.

2.3 Derivation of Linear Bimatrix Equations

The problem of pole assignment for the complex-valued linear system (1) by the full state feedback (6) can be stated as finding the bimatrix \(\{K_1, K_2\} \in \{C^{m \times n}, C^{m \times n}\}\) such that the resulting closed-loop system
\[
x^+ = (\{A_1, A_2\} + \{B_1, B_2\} \{K_1, K_2\}) x,
\] (14)
possesses a desired eigenvalue set \(\Gamma\) that is symmetric with respect to the real axis. Since \(\Gamma\) is symmetric with respect to the real axis, there is a real matrix \(F \in R^{2n \times 2n}\) such that \(\lambda (F) = \Gamma\).

Lemma 1 The pole assignment problem is solvable if and only if there exists a nonsingular bimatrix \(\{X_1, X_2\} \in \{C^{n \times n}, C^{n \times n}\}\) to the following generalized Sylvester bimatrix equation
\[
\{A_1, A_2\} \{X_1, X_2\} + \{B_1, B_2\} \{Y_1, Y_2\} = \{X_1, X_2\} \{F_1, F_2\},
\] (15)
where \(\{F_1, F_2\} \in \{C^{n \times n}, C^{n \times n}\}\) is the unique bimatrix satisfying
\[
F = \{F_1, F_2\}_o.
\] (16)
In this case, the feedback gain bimatrix \(\{K_1, K_2\}\) is determined by
\[
\{K_1, K_2\} = \{Y_1, Y_2\} \{X_1, X_2\}^{-1}.
\] (17)

We give a remark regarding the closed-loop system (14).

Remark 2 By the state transformation \(y = \{X_1, X_2\}^{-1} x\), we have from (15) and (17) that the closed-loop system (14) is equivalent to
\[
y^+ = \{X_1, X_2\}^{-1} x^+
\]
\[
= \{X_1, X_2\}^{-1} (\{A_1, A_2\} + \{B_1, B_2\} \{K_1, K_2\}) x
\]
\[
= \{X_1, X_2\}^{-1} (\{A_1, A_2\} + \{B_1, B_2\} \{K_1, K_2\}) \{X_1, X_2\} y
\]
\[
= \{X_1, X_2\}^{-1} \{A_1, A_2\} \{X_1, X_2\} + \{B_1, B_2\} \{Y_1, Y_2\} y
\]
\[
= \{F_1, F_2\} y.
\] (18)

Clearly, if we want the closed-loop system to be equivalent to a normal linear system, then we should set \(F_2 = 0_{n \times n}\), and to be equivalent to an antilinear system, then we should set \(F_1 = 0_{n \times n}\).
Therefore, to solve the pole assignment problem for system \( \mathbf{A} \), the main task is to finding complete solutions to the generalized Sylvester bimatrix equation \( [15] \), which is one of the main tasks in this paper and will be studied in Sections 5.4.1.

The generalized Sylvester bimatrix equation \( [15] \) is homogeneous and thus its solution is non-unique. As done in pole assignment for normal linear system \( [3] \), sometimes we may first prescribe \( \{Y_1, Y_2\} \) and then seek the (unique) solution for \( \{X_1, X_2\} \) (if exists). In this case, \( [15] \) becomes the following non-homogeneous one:

\[
\{A_1, A_2\} \{X_1, X_2\} - \{X_1, X_2\} \{F_1, F_2\} = \{C_1, C_2\},
\]

where \( \{C_1, C_2\} \triangleq -\{B_1, B_2\} \{Y_1, Y_2\} \) is known. This equation is referred to as the Sylvester bimatrix equation, and will be studied in Section 5. The non-homogeneous equation \( [19] \) also appears in the stability analysis of the complex-valued linear system \( [11] \) with \( t \in \mathbb{R}^+ \), which is asymptotically stable if and only if \( [21] \)

\[
\{A_1, A_2\}^H \{P_1, P_2\} + \{P_1, P_2\} \{A_1, A_2\} = -\{Q_1, Q_2\},
\]

has a (unique) solution \( \{P_1, P_2\} > 0 \) for any given \( \{Q_1, Q_2\} > 0 \). Clearly, \( [20] \) is in the form of \( [19] \), and will be referred to as Lyapunov bimatrix equation.

When we study stability of the complex-valued linear system \( [11] \) with \( t \in \mathbb{Z}^+ \), the discrete-time Lyapunov bimatrix equation

\[
\{P_1, P_2\} = \{A_1, A_2\}^H \{P_1, P_2\} \{A_1, A_2\} + \{Q_1, Q_2\},
\]

is encountered. It is shown in \( [21] \) that stability of \( [11] \) with \( t \in \mathbb{Z}^+ \) is equivalent to the existence of a (unique) solution \( \{P_1, P_2\} > 0 \) for any given \( \{Q_1, Q_2\} > 0 \). Equation \( [21] \) is a special case of

\[
\{X_1, X_2\} = \{A_1, A_2\} \{X_1, X_2\} \{F_1, F_2\} + \{C_1, C_2\},
\]

which is referred to as the Stein bimatrix equation, and will also be studied in Section 5.

Though in the above the bimatrix \( \{F_1, F_2\} \) has the same dimension as \( \{A_1, A_2\} \), however, this is not necessary. Hence, without loss of generality, hereafter we assume, if not specified, that \( \{F_1, F_2\} \in \{\mathbb{C}^{p \times p}, \mathbb{C}^{p \times p}\} \), where \( p \) is any positive integer.

### 3 Solutions to Generalized Sylvester Bimatrix Equations

In this section we study solutions to the generalized Sylvester bimatrix equation \( [15] \) and will also consider a special case that its coefficients are determined by the normal linear system \( [2] \). Hereafter we assume that system \( \mathbf{A} \) (or \( \{\{A_1, A_2\}, \{B_1, B_2\}\} \)) is controllable (see \( [21] \) for definition and criterion for the controllability of system \( \mathbf{A} \)).

#### 3.1 General Solutions

Two polynomial bimatrices \( \{N_1(s), N_2(s)\} \in \{\mathbb{C}^{n \times m}, \mathbb{C}^{n \times m}\} \) and \( \{D_1(s), D_2(s)\} \in \{\mathbb{C}^{m \times m}, \mathbb{C}^{m \times m}\} \) are said to be right-coprime if

\[
2m = \text{rank} \left\{ \begin{bmatrix} N_1(s) & N_2(s) \\ D_1(s) & D_2(s) \end{bmatrix} \right\}, \forall s \in \mathbb{C}, \tag{23}
\]

where, and hereafter, \( s \) should be treated as a real parameter when computing its conjugate, namely, \( s = s^\# \). We then can present the following explicit solutions to the generalized Sylvester bimatrix equation \( [15] \).

**Theorem 1** Let \( \{N_1(s), N_2(s)\} \in \{\mathbb{C}^{n \times m}, \mathbb{C}^{n \times m}\} \) and \( \{D_1(s), D_2(s)\} \in \{\mathbb{C}^{m \times m}, \mathbb{C}^{m \times m}\} \) be two polynomial bimatrices such that

\[
\{sI_n - A_1, -A_2\} \{N_1(s), N_2(s)\} = \{B_1, B_2\} \{D_1(s), D_2(s)\}, \tag{24}
\]
and \( \{N_1(s), N_2(s)\} \) and \( \{D_1(s), D_2(s)\} \) are right-coprime. Then complete solutions to \( (27) \) are given by

\[
\begin{align*}
\{X_1, X_2\} &= \sum_{i=0}^{\omega} \{N_1, N_2\} \{Z_1, Z_2\} \{F_1, F_2\}^i, \\
\{Y_1, Y_2\} &= \sum_{i=0}^{\omega} \{D_1, D_2\} \{Z_1, Z_2\} \{F_1, F_2\}^i,
\end{align*}
\]

(25)

where \( \{Z_1, Z_2\} \in \mathbb{C}^{m \times p} \) is an arbitrarily bimatrix and

\[
\begin{bmatrix}
N_j(s) \\
D_j(s)
\end{bmatrix} = \sum_{i=0}^{\omega} \begin{bmatrix}
N_{ji} \\
D_{ji}
\end{bmatrix} s^i, \quad j = 1, 2, \omega \in \mathbb{Z}^+.
\]

(26)

Notice that the polynomial bimatrix equations \( (24) \) can be written as coupled polynomial matrix equations

\[
\begin{align*}
(sI_n - A_1) N_1(s) - A_2^\# N_2(s) &= B_1 D_1(s) + B_2^\# D_2(s), \\
(sI_n - A_1) N_2^\#(s) - A_2 N_1^\#(s) &= B_1 D_2^\#(s) + B_2 D_1^\#(s),
\end{align*}
\]

(27)

and, the generalized Sylvester bimatrix equation \( (15) \) can also be expressed equivalently as coupled matrix equations

\[
\begin{align*}
A_1 X_1 + A_2^\# X_2 + B_1 Y_1 + B_2^\# Y_2 &= X_1 F_1 + X_2^\# F_2, \\
A_1 X_2^\# + A_2 X_1^\# + B_1 Y_2^\# + B_2^\# Y_1^\# &= X_1 F_2^\# + X_2 F_2^\#.
\end{align*}
\]

(28)

In the next subsection, we show how to transform these coupled equations into equivalent decoupled ones.

### 3.2 Decoupling of Coupled Equations

We first show that the coupled polynomial matrix equations in \( (27) \) can be decoupled.

**Lemma 2** The coupled polynomial matrix equations \( (27) \) with unknowns \( (N_1(s), N_2(s), D_1(s), D_2(s)) \) are solvable if and only if the following decoupled matrix equations

\[
\begin{align*}
(sI_n - A_1) N_+(s) - A_2^\# N_-(s) &= B_1 D_+(s) + B_2^\# D_-(s), \\
(sI_n - A_1) N_-(s) + A_2^\# N_+(s) &= B_1 D_-(s) - B_2^\# D_+(s),
\end{align*}
\]

(29)

with unknowns \( (N_+(s), N_-(s), D_+(s), D_-(s)) \) are solvable. Moreover, \( (N_1(s), N_2(s), D_1(s), D_2(s)) \) and \( (N_+(s), N_-(s), D_+(s), D_-(s)) \) are one-to-one according to

\[
\begin{align*}
N_1(s) &= \frac{1}{2} (N_+(s) + N_-(s)), \quad D_1(s) = \frac{1}{2} (D_+(s) + D_-(s)), \\
N_2(s) &= \frac{1}{2} (N_+(s) - N_-(s))^\#, \quad D_2(s) = \frac{1}{2} (D_+(s) - D_-(s))^\#.
\end{align*}
\]

(30)

Furthermore, \( \{N_1(s), N_2(s)\} \) and \( \{D_1(s), D_2(s)\} \) are right-coprime if and only if

\[
\text{rank} \begin{bmatrix}
N_+(s) & -N_-(s) \\
D_+(s) & -D_-(s) \\
N_-(s) & N_+(s) \\
D_-(s) & D_+(s)
\end{bmatrix} = 2m, \quad \forall s \in \mathbb{C}.
\]

(31)

By this lemma, the two matrix pairs \( (N_+(s), D_+(s)) \) and \( (N_-(s), D_-(s)) \) can be solved separately, which is useful in computation. In fact, we need only to solve the first equation since the second one can be solved in a similar way by setting \( A_2 \mapsto -A_2 \) and \( B_2 \mapsto -B_2 \). General solutions to \( (27) \) and \( (31) \) will be investigated in a separate paper.

Similar to Lemma 2, the coupled matrix equations \( (28) \) can also be decoupled in some cases, as shown below.

**Lemma 3** Assume that there exist two real matrices \( F_{ii} \in \mathbb{R}^{p \times p}, i = 1, 2, \) such that

\[
F \triangleq \{F_1, F_2\}_o = \text{diag}\{F_{11}, F_{22}\},
\]

(32)
Then complete solutions to \((34)\) are given by \((33)\) and such that \((31)\). Denote \(F\) such that \((33)\). Let \((33)\). Then the associated coupled matrix equations \((28)\) are solvable with unknowns \((X_1, X_2, Y_1, Y_2)\) if and only if the decoupled matrix equations

\[
\begin{cases}
A_1 X_+ + A_2^\# X_+^\# + B_1 Y_+ + B_2^\# Y_+^\# = X_+ (F_1 + F_2), \\
A_1 X_- - A_2^\# X_-^\# + B_1 Y_- - B_2^\# Y_-^\# = X_- (F_1 - F_2),
\end{cases}
\]

are solvable with unknowns \((X_+, Y_+)\) and \((X_-, Y_-)\). Moreover, \((X_1, X_2, Y_1, Y_2)\) and \((X_+, Y_+, X_-, Y_-)\) are one-to-one according to

\[
\begin{align*}
X_1 &= \frac{1}{2} (X_+ + X_-), & Y_1 &= \frac{1}{2} (Y_+ + Y_-), \\
X_2 &= \frac{1}{2} (X_+ - X_-)^\#, & Y_2 &= \frac{1}{2} (Y_+ - Y_-)^\#.
\end{align*}
\]

The assumption \((32)\) is not restrictive if only asymptotic stability of the closed-loop system is concerned since, in view of \((16)\), we can always find real matrices \(F\) such that \(F\) is asymptotically stable. Combining Lemmas 2 and 3 gives the following result.

**Theorem 2** Assume that there exist two real matrices \(F_{ii} \in \mathbb{R}^{p \times p}, i = 1, 2\) satisfying \((32)\) and \((F_1, F_2)\) is given by \((33)\). Let \((N_+(s), D_+(s))\) and \((N_-(s), D_-(s))\) satisfy respectively the first and second equations of \((26)\) and such that \((34)\). Denote

\[
\begin{bmatrix}
N_{\pm i}(s) \\
D_{\pm i}(s)
\end{bmatrix} = \sum_{i=0}^{\omega} \begin{bmatrix}
N_{\pm i} \\
D_{\pm i}
\end{bmatrix} s^i, \quad \omega \in \mathbb{Z}^+. 
\]

Then complete solutions to \((34)\) are given by

\[
\begin{bmatrix}
X_+ \\
Y_+
\end{bmatrix} = \frac{1}{2} \sum_{i=0}^{\omega} \begin{bmatrix}
N_{\pm i} \\
D_{\pm i}
\end{bmatrix} (Z_+ + Z_-^\#) + \begin{bmatrix}
N_{\pm i} \\
D_{\pm i}
\end{bmatrix} (Z_+ - Z_-^\#) (F_1 \pm F_2)^i, 
\]

where \((Z_+, Z_-)\) and \((Z_1, Z_2)\) are one-to-one according to

\[
Z_\pm = Z_1 \pm Z_2^\#.
\]

It follows that, though \((X_+, Y_+)\) and \((X_-, Y_-)\) are decoupled in \((34)\), and \((N_+(s), D_+(s))\) and \((N_-(s), D_-(s))\) are also decoupled in \((26)\), \((X_+, Y_+)\) (and \((X_-, Y_-)\)) depends on both \((N_+(s), D_+(s))\) and \((N_-(s), D_-(s))\). If we let \(Z_\pm \in \mathbb{R}^{m \times p}\), then they are decoupled as

\[
\begin{bmatrix}
X_+ \\
Y_+
\end{bmatrix} = \sum_{i=0}^{\omega} \begin{bmatrix}
N_{\pm i} \\
D_{\pm i}
\end{bmatrix} Z_\pm^i (F_1 \pm F_2)^i, 
\]

which is appealing in mathematics.

### 3.3 Solutions for Normal Linear Systems

When considering the generalized Sylvester bimatrix equation \((15)\) for the normal linear system \((2)\), we may want the closed-loop system to be (similar to) a normal system as well (see Remark 5 later for a different situation). Thus, according to \((15)\), we should choose \(F_2 = 0_{n \times n}\). Consequently, the generalized Sylvester bimatrix equation \((15)\) or the coupled matrix equation \((28)\) becomes

\[
\begin{cases}
A_1 X_1 + B_1 Y_1 = X_1 F_1, \\
A_1 X_2^\# + B_1 Y_2^\# = X_2^\# F_1^\#, 
\end{cases}
\]

where \(F_1\) is such that \(y^+ = F_1 y\) is asymptotically stable. These two equations in \((39)\) are exactly the same one

\[
A_1 X_0 + B_1 Y_0 = X_0 F_0,
\]

where \(F_0\) is such that \(y^+ = F_0 y\) is asymptotically stable.
where \( F_0 = F_1 \) and \( F_1^\# \). Equation \( \text{(10)} \) is known as the generalized Sylvester matrix equation and has been extensively used and studied in the literature for pole assignment of the normal linear system \(|2, 5, 7, 22|\).

In this case, the two polynomial matrix equations \( \text{(29)} \) reduce further to the single one
\[
(sI_n - A_1) N_0(s) = B_1 D_0(s),
\]
that has been investigated in the literature, for example, \( |3, 23| \). We can simply choose
\[
\begin{cases}
N_1(s) = N_0(s), & N_2(s) = 0_{n \times m}, \\
D_1(s) = D_0(s), & D_2(s) = 0_{m \times m},
\end{cases}
\]
to satisfy \( \text{(27)} \). Thus \( \text{(23)} \) is equivalent to, for all \( s \in \mathbb{C} \),
\[
2m = \text{rank} \begin{bmatrix}
N_1(s) & 0_{n \times m} \\
D_1(s) & 0_{m \times m} \\
0_{n \times m} & N_2^\#(s) \\
0_{m \times m} & D_2^\#(s)
\end{bmatrix} = 2 \text{rank} \begin{bmatrix}
N_0(s) \\
D_0(s)
\end{bmatrix},
\]
which implies that \( \{N_1(s), N_2(s)\} \) and \( \{D_1(s), D_2(s)\} \) are right-coprime if and only if
\[
\text{rank} \begin{bmatrix}
N_0(s) \\
D_0(s)
\end{bmatrix} = m, \forall s \in \mathbb{C},
\]
namely, \( N_0(s) \) and \( D_0(s) \) are right-coprime in the normal sense \( |13| \). In this case, by denoting
\[
\begin{bmatrix}
N_0(s) \\
D_0(s)
\end{bmatrix} = \sum_{i=0}^{\omega} \begin{bmatrix}
N_{0i} \\
D_{0i}
\end{bmatrix} s_i,
\]
complete solutions to \( \text{(10)} \) are given by \( |22| \)
\[
\begin{bmatrix}
X_0 \\
Y_0
\end{bmatrix} = \sum_{i=0}^{\omega} \begin{bmatrix}
N_{0i} \\
D_{0i}
\end{bmatrix} Z_0 F_0^i,
\]
where \( Z_0 \in \mathbb{C}^{m \times p} \) is any matrix. On the other hand, in view of \( \text{(12)} \), we can apply solution \( \text{(23)} \) on equation \( \text{(39)} \) (setting \( A_2 = 0_{n \times n}, B_2 = 0_{n \times m}, F_2 = 0_{p \times p} \) and using \( \text{(12)} \)) to obtain
\[
\begin{bmatrix}
X_1 \\
Y_1 \\
X_2 \\
Y_2
\end{bmatrix} = \sum_{i=1}^{\omega} \begin{bmatrix}
N_{0i} \\
D_{0i}
\end{bmatrix} Z_1 F_1^i,
\]
where \( Z_i \in \mathbb{C}^{m \times p}, i = 1, 2, \) are any matrices. These two expressions coincide exactly with \( \text{(13)} \).

**Remark 3** Notice that, if we choose \( Z_2 = 0_{m \times n} \), then we have from \( \text{(44)} \) that \( X_2 = 0_{n \times n} \) and \( Y_2 = 0_{m \times n} \). Then, if \( Z_1 \) is properly chosen such that \( X_1 \) is nonsingular, by \( \text{(17)} \), the feedback gain bimatrix is given by
\[
\{K_1, K_2\} = \{Y_1, 0_{m \times n}\} \{X_1, 0_{n \times n}\}^{-1}
\]
\[
= \{Y_1, 0_{m \times n}\} \{X_1^{-1}, 0_{n \times n}\}
\]
\[
= \{Y_1 X_1^{-1}, 0_{m \times n}\},
\]
which means that the resulting controller is just the normal linear state feedback \( \text{(4)} \). However, if \( Z_2 \neq 0_{m \times n} \), then \( X_2 \neq 0_{n \times n} \) and \( Y_2 \neq 0_{m \times n} \), which in turn implies that the resulting controller is the full state feedback \( \text{(6)} \). Yet in both cases the closed-loop system is equivalent to a normal linear system (see \( \text{(13)} \)).

### 4 Solutions for Antilinear Systems

In this section, we carry out a careful study on pole assignment for the antilinear system \( |3| \) and present explicit solutions to the associated Sylvester bimatrix equations \( |13| \) in different special cases.
4.1 General Solutions

We first provide a method for computing the right-coprime factorization (24) or (29) associated with the antilinear system (3).

Lemma 4 Consider the antilinear system (3). Let \((N_0(s), D_0(s)) \in (C^{n \times m}, C^{m \times m})\) satisfy

\[
sN_0^\#(s) - A_2N_0(s) = B_2D_0(s).
\]

Then the polynomial matrices \(N_+(s), D_+(s), N_-(s)\) and \(D_-(s)\) satisfying (24) can be chosen as

\[
\begin{align*}
N_+(s) &= N_0(s), \quad D_+(s) = D_0(s), \\
N_-(s) &= N_0(-s), \quad D_-(s) = D_0(-s),
\end{align*}
\]

or equivalently, \(\{N_1(s), N_2(s)\}\) and \(\{D_1(s), D_2(s)\}\) satisfying (24) can be chosen as

\[
\begin{align*}
N_1(s) &= \frac{1}{2}(N_0(s) + N_0(-s)), \quad D_1(s) = \frac{1}{2}(D_0(s) + D_0(-s)), \\
N_2(s) &= \frac{1}{2}(N_0(s) - N_0(-s))^\#, \quad D_2(s) = \frac{1}{2}(D_0(s) - D_0(-s))^\#.
\end{align*}
\]

Moreover, \(\{N_1(s), N_2(s)\}\) and \(\{D_1(s), D_2(s)\}\) are right-coprime if and only if

\[
\text{rank}
\begin{bmatrix}
N_0(s) & -N_0(-s) \\
D_0(s) & -D_0(-s) \\
N_0^\#(s) & N_0^\#(-s) \\
D_0^\#(s) & D_0^\#(-s)
\end{bmatrix}
= 2m, \forall s \in \mathbb{C}.
\]

A polynomial matrix pair \((N_0(s), D_0(s))\) satisfying (19) may be called anti-right-coprime, and the equation (24) can be referred to as the anti-right-coprime factorization of the antilinear system (3), which can be studied by using the approach in (24).

The Sylvester bimatrix equation (15) or the decoupled matrix equations (34) associated with the antilinear system (3) is equivalent to

\[
\begin{align*}
A_2^\#X_+^\# + B_2^\#Y_+^\# &= X_+(F_1 + F_2), \\
-A_2^\#X_-^\# - B_2^\#Y_-^\# &= X_-(F_1 - F_2).
\end{align*}
\]

These two equations take also the same form while their coefficients are different.

Corollary 1 Assume that there exist two real matrices \(F_{ij} \in \mathbb{R}^{p \times p}, i = 1, 2\) satisfying (49) and \((F_1, F_2)\) is given by (53). Let \((N_0(s), D_0(s)) \in (C^{n \times m}, C^{m \times m})\) satisfy (19) and (49). Denote

\[
\begin{bmatrix}
N_0(s) \\
D_0(s)
\end{bmatrix}
= \sum_{i=0}^{\infty}
\begin{bmatrix}
N_{0,i} \\
D_{0,i}
\end{bmatrix}s^i, \quad s^i \triangleq \sum_{i=0}^{2\infty}
\begin{bmatrix}
N_{0,i} \\
D_{0,i}
\end{bmatrix}s^i.
\]

Then complete solutions to (56) are given by

\[
\begin{bmatrix}
X_\pm \\
Y_\pm
\end{bmatrix}
= \sum_{i=0}^{\infty}
\begin{bmatrix}
N_{0,2i} \\
D_{0,2i}
\end{bmatrix}Z_\pm(F_1 \pm F_2)^{2i} \pm \sum_{i=0}^{\infty-1}
\begin{bmatrix}
N_{0,2i+1} \\
D_{0,2i+1}
\end{bmatrix}Z_\pm^\#(F_1 \pm F_2)^{2i+1},
\]

where \(Z_\pm\) are determined by (38).

4.2 Normalization of Antilinear Systems

We now consider a special case that \(F_2 = 0_{n \times n}\). In this case, by (39), the closed-loop system (14) is equivalent to \(y^+ = F_1y\), which is a normal linear system and is asymptotically stable if and only if \(F_1\) is asymptotically stable \((\mu(F_1) < 0\) when \(t \in \mathbb{R}^+\) and \(\rho(F_1) < 1\) when \(t \in \mathbb{Z}^+)\). In this case, the generalized Sylvester bimatrix equation (15) or the coupled matrix equations (28) become

\[
\begin{align*}
A_2^\#X_2 + B_2^\#Y_2 &= X_1F_1, \\
A_2^\#X_1 + B_2^\#Y_1^\# &= X_2^\#F_1^\#.
\end{align*}
\]
We next consider a special case that where $Z$ is anti-preserving of the antilinear system (3). Hence, in this case, if (53) has a nonsingular solution $\{X_1, X_2\}$, then the full state feedback (6) and (17) will make the system be a normal linear system. We call such a procedure as the normalization of the antilinear system (4).

**Corollary 2** Let $F_2 = 0_{p \times p}$ and $(N_0(s), D_0(s)) \in (\mathbb{C}^{n \times m}, \mathbb{C}^{m \times m})$ satisfy (49), (52) and (55). Then complete solutions to the first equation of (53) are given by

$$
\begin{bmatrix}
X_1 \\
Y_1
\end{bmatrix} = \sum_{i=0}^{\omega} \begin{cases}
N_{0,i} \\
D_{0,i}
\end{cases} \begin{bmatrix}
Z_1 F_1^i, & \text{i is even} \\
Z_2 F_1^i, & \text{i is odd},
\end{cases} \tag{54}
$$

and complete solutions to the second equation of (53) are given by

$$
\begin{bmatrix}
X_2 \\
Y_2
\end{bmatrix} = \sum_{i=0}^{\omega} \begin{cases}
N_{0,i}^\# \\
D_{0,i}^\#
\end{cases} \begin{bmatrix}
Z_2 F_1^i, & \text{i is even} \\
Z_1 F_1^i, & \text{i is odd},
\end{cases} \tag{55}
$$

where $Z_i \in \mathbb{C}^{m \times p}$, $i = 1, 2$ are arbitrary matrices.

### 4.3 Anti-Preserving of Antilinear Systems

We next consider a special case that $F_1 = 0_{n \times n}$. In this case, by (14), the closed-loop system (13) is equivalent to $y^+ = F_2^\# y^\#$, which is still an antilinear system and is asymptotically stable if and only if $t \in \mathbb{Z}^+$ and

$$
\rho \left( F_2 F_2^\# \right) < 1. \tag{56}
$$

In this case, the generalized Sylvester bimatrix equation (15) or the coupled matrix equations (28) become

$$
\begin{align*}
A_2^\# X_1^\# + B_2^\# Y_1^\# &= X_1 F_2^\#, \\
A_2^\# X_2 + B_2^\# Y_2 &= X_2 F_2,
\end{align*} \tag{57}
$$

which are decoupled. Hence, in this case, if (57) has a nonsingular solution $\{X_1, X_2\}$, then the full state feedback (6) and (17) will make the system be (equivalent to) an antilinear system as well. We call such a procedure as the anti-preserving of the antilinear system (4).

**Corollary 3** Let $F_1 = 0_{p \times p}$ and $(N_0(s), D_0(s)) \in (\mathbb{C}^{n \times m}, \mathbb{C}^{m \times m})$ satisfy (49), (52) and (55). Then complete solutions to the first equation of (57) are given by

$$
\begin{bmatrix}
X_1 \\
Y_1
\end{bmatrix} = \sum_{i=0}^{\omega} \begin{cases}
N_{0,i} \\
D_{0,i}
\end{cases} \begin{bmatrix}
Z_1 \left( F_2^\# F_2 \right)^{\frac{i}{2}}, & \text{i is even} \\
Z_2 \left( F_2^\# F_2 \right)^{-\frac{i}{2}}, & \text{i is odd},
\end{cases} \tag{58}
$$

and complete solutions to the second equation of (57) are given by

$$
\begin{bmatrix}
X_2 \\
Y_2
\end{bmatrix} = \sum_{i=0}^{\omega} \begin{cases}
N_{0,i}^\# \\
D_{0,i}^\#
\end{cases} \begin{bmatrix}
Z_2 \left( F_2^\# F_2 \right)^{\frac{i}{2}}, & \text{i is even} \\
Z_1 \left( F_2^\# F_2 \right)^{i/2}, & \text{i is odd},
\end{cases} \tag{59}
$$

where $Z_i \in \mathbb{C}^{m \times p}$, $i = 1, 2$ are arbitrary matrices.
The solution \((X_1, Y_1)\) given in Corollary 3 coincides with those obtained in [19]. We emphasize that the solutions in Corollary 3 can only be used to design discrete-time antilinear systems. In contrast, the coupled equations (60) and (63) can be used to design both continuous-time and discrete-time antilinear systems.

Remark 4 If we choose \(Z_2 = 0_{m \times n}\) and \(Z_1\) such that \(X_1\) is nonsingular, then, by [17], we also have (59), namely, the resulting controller is the normal state feedback (7). This case is just the one studied in [19] and our controller is exactly the one obtained there (yet the rank condition (49) was not available in [19] where a different concept was adopted). However, similar to Remark 3, if we choose \(Z_2 \neq 0_{m \times n}\), such that \(X_2 \neq 0_{n \times n}\) and/or \(Y_2 \neq 0_{m \times n}\), then the resulting controller is the full state feedback (9).

Remark 5 In Subsection 5.3 we have assumed that \(F_2 = 0_{n \times n}\) for the normal linear system (8), namely, the closed-loop system is equivalent to a normal linear system. However, similar to the discussion in this subsection, for the normal linear system (8), we can also set \(F_1 = 0_{n \times n}\) such that the closed-loop system is equivalent to an antilinear system. We may call this procedure as the anti-linearization of the normal linear system (8). The corresponding coupled matrix equations and their complete solutions can be easily stated and are omitted for brevity.

5 Sylvester and Stein Bimatrix Equations

We will study in this section solutions to Sylvester and Stein bimatrix equations. Since we are frequently dealing with the normal linear system (8) and the antilinear system (9), for easy reference, we make the following assumptions.

Assumption 1 \(A_2 = 0_{n \times n}, F_2 = 0_{p \times p}\) and \(C_2 = 0_{n \times p}\), namely, \(\{A_1, A_2\} = \{A_1, 0_{n \times n}\}, \{F_1, F_2\} = \{F_1, 0_{p \times p}\}\) and \(\{C_1, C_2\} = \{C_1, 0_{n \times p}\}\).

Assumption 2 \(A_1 = 0_{n \times n}, F_1 = 0_{p \times p}\) and \(C_1 = 0_{n \times p}\), namely, \(\{A_1, A_2\} = \{0_{n \times n}, A_2\}, \{F_1, F_2\} = \{0_{p \times p}, F_2\}\) and \(\{C_1, C_2\} = \{0_{n \times p}, C_2\}\).

5.1 The Sylvester Bimatrix Equation

In this subsection we discuss the Sylvester bimatrix equation (19), which can also be written as coupled matrix equations

\[
\begin{align*}
C_1 &= A_1 X_1 + A_2^\# X_2 - (X_1 F_1 + X_2^\# F_2), \\
C_2 &= A_1^\# X_2 + A_2 X_1 - (X_1^\# F_2 + X_2 F_1).
\end{align*}
\]  

(60)

Proposition 1 The Sylvester bimatrix equation (19) has a unique solution if and only if

\[
\lambda \{A_1, A_2\} \cap \lambda \{F_1, F_2\} = \emptyset.
\]  

(61)

In this case, the unique solution is given by

\[
\{X_1, X_2\} = \left( \sum_{k=0}^{2p} \beta_k \{A_1, A_2\}^k \right)^{-1} \sum_{k=1}^{2p} \beta_k \{D_1(k), D_2(k)\},
\]  

(62)

where \(\beta(s) = s^{2p} + \beta_{2p-1} s^{2p-1} + \cdots + \beta_1 s + \beta_0\) is the characteristic polynomial of \(\{F_1, F_2\}\), and, for \(k \geq 1\),

\[
\{D_1(k), D_2(k)\} = \sum_{i=0}^{k-1} \{A_1, A_2\}^i \{C_1, C_2\} \{F_1, F_2\}^{k-1-i}.
\]  

(63)

Notice that the bimatrix series \(\{D_1(k), D_2(k)\}\) in (63) can also be defined in a recursive way:

\[
\begin{align*}
\{D_1(k+1), D_2(k+1)\} &= \{A_1, A_2\} \{D_1(k), D_2(k)\} + \{C_1, C_2\} \{F_1, F_2\}^k \\
&= \{A_1, A_2\}^{k+1} \{C_1, C_2\} + \{D_1(k), D_2(k)\} \{F_1, F_2\}, \quad k \geq 1,
\end{align*}
\]  

(64)

\[
\{D_1(1), D_2(1)\} = \{C_1, C_2\}.
\]
Corollary 4 Assume that
\[ \mu \{A_1, A_2\} + \mu \{F_1, F_2\} < 0. \] (65)
Then the Sylvester bimatrix equation (14) has a unique solution given by
\[ \{X_1, X_2\} = \int_0^\infty e^{t(A_1+A_2)} \{C_1, C_2\} e^{t(F_1+F_2)} dt. \] (66)
Particularly, if the complex-valued linear system (1) with \( t \in \mathbb{R}^+ \) is asymptotically stable, then the solution to the Lyapunov bimatrix equation (20) is given by
\[ \{P_1, P_2\} = \int_0^\infty e^{t(A_1+A_2)^H} \{Q_1, Q_2\} e^{t(A_1+A_2)} dt. \] (67)

We now take a look at the following well-known Sylvester matrix equation
\[ A_1 X - X F_1 = C_1, \] (68)
which was firstly studied by J. J. Sylvester [16] and then by several authors (for example, [10, 11]). The following corollary reveals the relationship between the Sylvester bimatrix equation (19) and the Sylvester matrix equation (68).

Corollary 5 The Sylvester matrix equation (68) is solvable if and only if the Sylvester bimatrix equation (19) under Assumption 1 is solvable. Particularly,

1. If \( \{X_1, X_2\} \) is a solution to (19), then \( X_1 \) is a solution to (68), and if \( X \) is a solution to (19), then \( \{X_1, X_2\} = \{X, 0_{n \times p}\} \) is a solution to (68).

2. If (19) has a unique solution \( \{X_1, X_2\} \), then \( X_2 = 0_{n \times p} \) and (68) also has a unique solution \( X = X_1 \). If (68) has a unique solution \( X \), then (19) has a unique solution (given by \( \{X, 0_{n \times p}\} \)) if one of the following two conditions is satisfied
\[ s \in \lambda(A_1) \implies s^\# \in \lambda(A_1), \] (69)
\[ s \in \lambda(F_1) \implies s^\# \in \lambda(F_1). \] (70)

Notice that equation (69) (equation (70)) is satisfied if \( A_1 \) (\( F_1 \)) is a real matrix. We can check that, under Assumption 1 the unique solution (62) to (19) coincides exactly with the unique solution to (68) obtained in [11]. We omit the details to save spaces.

Remark 6 Under Assumption 1 it follows from \( \lambda(A_1, 0_{n \times n}) = \lambda(A_1) \cup \lambda(A_1^\#) \) that (68) is equivalent to
\[ \mu(A_1) + \mu(F_1) < 0. \] (71)
On the other hand, we have \( e^{t(A_1+A_2)} = \{e^{At}, 0_{n \times n}\} \) and \( e^{t(F_1+F_2)} = \{e^{F_1 t}, 0_{p \times p}\} \). Then applying Corollary 4 on (68) gives the well-known result [9]: if (71) is satisfied, the Sylvester matrix equation (68) has a unique solution given by \( X = \int_0^\infty e^{tA_1} C_1 e^{tF_1} dt \).

We next discuss the so-called conjugate-Sylvester matrix equation
\[ C_2 = A_1^\# X - X^\# F_2, \] (72)
which was firstly investigated in [1, 2] and then in [19].

Corollary 6 The conjugate-Sylvester matrix equation (72) is solvable if and only if the Sylvester bimatrix equation (17) under Assumption 2 is solvable. Particularly,

1. If \( \{X_1, X_2\} \) is a solution to (17), then \( X_1 \) is a solution to (72), and if \( X \) is a solution to (72), then \( \{X_1, X_2\} = \{X, 0_{n \times p}\} \) is a solution to (17).
2. \([19]\) has a unique solution (denoted by \(\{X_1, X_2\}\)) if and only if \([72]\) has a unique solution (denoted by \(X\)). Moreover, \(X_2 = 0_{n \times p}\) and \(X = X_1\) with

\[
X_1 = \gamma^{-1} \left( A_2^* A_2 \left( \sum_{k=1}^{p} \gamma_k \sum_{i=0}^{k-1} \left( A_2^* A_2 \right)^i \left( C_i^* F_2 + A_2^* C_2 \right) \left( F_2^* F_2 \right)^{k-1-i} \right) \right),
\]

(73)

where \(\gamma(s) = s^p + \gamma_{p-1}s^{p-1} + \cdots + \gamma_1 s + \gamma_0\) is a polynomial with real coefficients, defined by

\[
\gamma(s) = \left| sI_p - F_2^* F_2 \right|.
\]

(74)

We notice that \([73]\) coincides with the solution obtained in \([19]\). Moreover, Item 2 of Corollary \(6\) seems better than that of Corollary \(5\) since it provides an “if and only if” condition.

**Remark 7** An explanation of the solution \([73]\) is given as follows. We have

\[
\{D_1(2), D_2(2)\} = \left\{ A_2^* A_2, 0_{n \times n} \right\} \{X_1, X_2\} - \{X_1, X_2\} \left\{ F_2^* F_2, 0_{p \times p} \right\}
\]

\[
= \left\{ A_2^* A_2 X_1 - X_1 F_2^* F_2, A_2 A_2^* X_2 - X_2 F_2^* F_2 \right\}
\]

\[
= \left\{ C_2^* F_2 + A_2^* C_2, 0_{n \times p} \right\}.
\]

Therefore, if \([19]\) has a unique solution, then \(X_2 = 0_{n \times p}\) and \(X_1\) satisfies

\[
A_2^* A_2 X_1 - X_1 F_2^* F_2 = C_2^* F_2 + A_2^* C_2.
\]

(75)

The closed-form solution to \([73]\) is just \([72]\) by using the result in \([11]\).

### 5.2 The Stein Bimatrix Equation

We now study the Stein bimatrix equation \([22]\), which is equivalent to the coupled matrix equations

\[
\begin{aligned}
X_1 &= A_1 X_1 F_1 + A_2^* X_2 F_1 + A_1 X_1^* F_2 + A_2^* X_2^* F_2 + C_1, \\
X_2 &= A_1^* X_1^* F_2 + A_2 X_2^* F_2 + A_1^* X_2^* F_1 + A_2 X_1 F_1 + C_2.
\end{aligned}
\]

(76)

**Proposition 2** The Stein bimatrix equation \([22]\) has a unique solution if and only if

\[
\lambda_i \{A_1, A_2\} \lambda_j \{F_1, F_2\} \neq 1, \forall i, j.
\]

(77)

In this case, the unique solution is given by

\[
\{X_1, X_2\} = \left( \sum_{k=0}^{2p} \beta_k \{A_1, A_2\}^{2p-k} \{D_1(k), D_2(k)\} \right)^{-1} \\
\sum_{k=1}^{2p} \beta_k \{A_1, A_2\}^{2p-k} \{D_1(k), D_2(k)\},
\]

(78)

where \(\beta(s) = \beta_{2p} + \beta_{2p-1}s^{2p-1} + \cdots + \beta_1 s + \beta_0\) is the characteristic polynomial of \(\{F_1, F_2\}\), and, for \(k \geq 1\),

\[
\{D_1(k), D_2(k)\} = \sum_{i=0}^{k-1} \{A_1, A_2\}^i \{C_1, C_2\} \{F_1, F_2\}^i.
\]

(79)

Notice that, similar to \([61]\), the bimatrix \(\{D_1(k), D_2(k)\}\) in \([79]\) can also be defined in a recursive way:

\[
\begin{aligned}
\{D_1(k + 1), D_2(k + 1)\} &= \{A_1, A_2\} \{D_1(k), D_2(k)\} \{F_1, F_2\}, k \geq 1, \\
\{D_1(1), D_2(1)\} &= \{C_1, C_2\}.
\end{aligned}
\]

(80)

We can also state the following corollary that parallels to corollary \(4\).
Corollary 7 Assume that
\[ \rho \{ A_1, A_2 \} \rho \{ F_1, F_2 \} < 1, \quad (81) \]
and \( \{ D_1(k), D_2(k) \} \) is defined by (79). Then the Stein bimatrix equation (22) has a unique solution given by
\[ \{ X_1, X_2 \} = \sum_{k=0}^{\infty} \{ A_1, A_2 \}^k \{ C_1, C_2 \} \{ F_1, F_2 \}^k. \quad (82) \]

Particularly, if the complex-valued linear system (1) with \( t \in \mathbb{Z}^+ \) is asymptotically stable, then the solution to the discrete-time Lyapunov bimatrix equation (21) is given by
\[ \{ P_1, P_2 \} = \sum_{k=0}^{\infty} \left( \{ A_1, A_2 \}^H \right)^k \{ Q_1, Q_2 \} \{ A_1, A_2 \}^k. \quad (83) \]

We now check the following well-known Stein matrix equation
\[ X = A_1 XF_1 + C_1, \quad (84) \]
which has been widely studied in the literature (see, for example, [12] and [24]). The relationship between (22) and (84) can be made clear as follows.

Corollary 8 The Stein matrix equation (84) is solvable if and only if the Stein bimatrix equation (22) under Assumption 1 is solvable. Particularly,

1. If \( \{ X_1, X_2 \} \) is a solution to (22), then \( X = X_1 \) is a solution to (84), and if \( X \) is a solution to (84), then \( \{ X_1, X_2 \} = \{ X, 0_{n \times p} \} \) is a solution to (22).

2. If (22) has a unique solution \( \{ X_1, X_2 \} \), then \( X_2 = 0_{n \times p} \) and (84) also has a unique solution \( X = X_1 \). If (84) has a unique solution \( X \), then (22) has a unique solution (given by \( \{ X, 0_{n \times p} \} \)) if one of the two conditions (69)-(70) is satisfied.

Under Assumption 1 one may check that the unique solution (78) to (22) coincides with the unique solution to (84) obtained in [12]. The details are omitted.

Remark 8 Similar to Remark 6 under Assumption 1, we can show that (87) is equivalent to
\[ \rho(A_1) \rho(F_1) < 1. \quad (85) \]

On the other hand, we have from (76) that \( D_2(k) = 0_{n \times p} \) and \( D_1(k+1) = A_1 D_1(k) F_1 \) with \( D_1(1) = C_1 \). Then applying Corollary 7 on (84) gives the well-known result (see, for example, [24]): if (80) is satisfied, the Stein matrix equation (84) has a unique solution given by \( X = \sum_{k=0}^{\infty} A_1^k C_1 F_1^k \).

We next discuss the so-called conjugate-Stein matrix equation
\[ X = A_2 X^\# F_2 + C_2, \quad (86) \]
which was firstly investigated in [12] and was also studied in several other papers, for example, [19] and [24], where it was equivalently transformed into a normal Stein matrix equation in the form of (84). The following corollary reveals the relationship between (86) and (22).

Corollary 9 The conjugate-Stein matrix equation (86) is solvable if and only if the Stein bimatrix equation (22) under Assumption 3 is solvable. Particularly,

1. If \( \{ X_1, X_2 \} \) is a solution to (22), then \( X = X_2 \) is a solution to (86), and if \( X \) is a solution to (86), then \( \{ X_1, X_2 \} = \{ 0_{n \times p}, X \} \) is a solution to (22).
According to the development in Subsection 2.2, this system can be written as the complex-valued linear system (90) is exactly in the form of (8) with dynamics. Notice that in this case the system is still controllable.

\[ X_2 = \left( \sum_{k=0}^{p} \gamma_k(A_2 A_2^\#)^{p-k} \right)^{-1} \left( \sum_{k=1}^{p} \gamma_k(A_2 A_2^\#)^{p-1} \left( C_2 + A_2 C_2^\# F_2 \right) \left( F_2^\# F_2 \right)^{k-1} \right). \]  

(87)

Similar to the situation in Subsection 5.1, Item 2 of Corollary 8 is better than that of Corollary 5 since it reveals an “if and only if” relation. Notice that (87) coincides with the solution obtained in [12].

Remark 9: Parallel to Remark 4, we give an explanation on (87). We have

\[ \{X_1, X_2\} = \{A_1, A_2\}^2 \{X_1, X_2\} \{F_1, F_2\}^2 + \{D_1(2), D_2(2)\} \]

\[ = \left\{ A_2^\# A_2, 0_{n \times p} \right\} \{X_1, X_2\} \left\{ F_2^\# F_2, 0_{p \times p} \right\} + \left\{ 0_{n \times p}, C_2 + A_2 C_2^\# F_2 \right\} \]

\[ = \left\{ A_2^\# A_2 X_2 F_2^\# F_2, A_2 A_2^\# X_2 F_2^\# F_2 + C_2 + A_2 C_2^\# F_2 \right\}. \]

Therefore, if (87) has a unique solution, we have \( X_1 = 0_{n \times p} \) and \( X_2 \) satisfying

\[ X_2 = A_2 A_2^\# X_2 F_2^\# F_2 + C_2 + A_2 C_2^\# F_2, \]

(88)

whose unique solution is exactly (87) by using the result in [12]. Moreover, (87) is equivalent to

\[ \rho \left( A_2 A_2^\# \right) \rho \left( F_2^\# F_2 \right) < 1. \]

(89)

Hence, under this condition, the unique solution to (85) can also be expressed by (see (72) in [24]):

\[ X_2 = \sum_{k=0}^{\infty} \left( A_2 A_2^\# \right)^k \left( C_2 + A_2 C_2^\# F_2 \right) \left( F_2^\# F_2 \right)^k. \]

6 Example: Design of the Spacecraft Rendezvous System

We use the spacecraft rendezvous system model to illustrate the effectiveness of the proposed methods. The linearized equation of the spacecraft rendezvous control system is known as the C-W equation [4]

\[ \begin{align*}
\dot{\xi}_1 &= 2\omega \dot{\xi}_2 + 3\omega^2 \xi_1 + a_1, \\
\dot{\xi}_2 &= -2\omega \dot{\xi}_1 + a_2, \\
\dot{\xi}_3 &= -\omega^2 \xi_3 + a_3,
\end{align*} \]

(90)

where \( \xi_i, i = 1, 2, 3 \), are relative positives of the chase spacecraft with respect to the target, \( a_1, a_2 \) and \( a_3 \) are the control accelerations that the thrusts generate in the three directions, and \( \omega \) is the orbit rate of the target orbit, which is a known constant. For more information about this model, see [4] and the references therein. Notice that (90) is a full-actuated system. To demonstrate the design in a general case, we assume that \( a_1 = 0 \), since otherwise, the design is trivial as \( a_1 \) can be easily designed to cancel all of the open-loop dynamics. Notice that in this case the system is still controllable.

System (90) is exactly in the form of (8) with \( \xi = [\xi_1, \xi_2, \xi_3]^T, v = [a_2, a_3]^T, n = 3, q = 2 \) and \( m = 1 \). Then, according to the development in Subsection 2.2, this system can be written as the complex-valued linear system (1), where \( A_i, B_i, i = 1, 2 \), are computed according to (11) as

\[ A_1 = \begin{bmatrix}
\frac{3\omega^2}{2} + \frac{i}{2} & \omega & 0 \\
-\omega & -\frac{1}{2} & 0 \\
0 & 0 & -\omega^2 - \frac{i}{2}
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 \\
\frac{i}{2}
\end{bmatrix}, \]

\[ A_2 = \begin{bmatrix}
-\frac{3\omega^2}{2} - \frac{i}{2} & -\omega & 0 \\
\omega & -\frac{1}{2} & 0 \\
0 & 0 & -\omega^2 - \frac{i}{2}
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 \\
-\frac{i}{2}
\end{bmatrix}. \]
The eigenvalue set of the open-loop system is known as \{0, 0, \pm \omega j, \pm \omega i\} \cite{1}. We are going to design a full state feedback \cite{1} such that the closed-loop system possesses an eigenvalue set that is a shift of the open-loop system along the real axis, say, \( \Gamma = \{-\gamma, -\gamma, -\gamma \pm \omega j, -\gamma \pm \omega i\} \), where \( \gamma > 0 \) is a real constant. Thus we can choose
\[
F = \text{diag} \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \right\} - \gamma I_6,
\]
and the unique bimatrix \( \{F_1, F_2\} \) satisfying \cite{16} can be obtained as
\[
F_1 = \begin{bmatrix} -\gamma & \frac{1}{2} \omega & \frac{1}{2} \\ \frac{1}{2} \omega & -\gamma & \frac{1}{2} \\ -\frac{1}{2} \omega & -\frac{1}{2} \omega & -\gamma \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & \frac{1}{2} \omega & \frac{1}{2} \\ -\frac{1}{2} \omega & 0 & -\frac{1}{2} \omega \\ -\frac{1}{2} \omega & \frac{1}{2} \omega & 0 \end{bmatrix}.
\]
The right-coprime bimatrix polynomials \( \{N_1(s), N_2(s)\} \) and \( \{D_1(s), D_2(s)\} \) satisfying \cite{24} can be computed as
\[
\begin{align*}
N_1(s) &= \begin{bmatrix} \omega s (1 + js) \\ \frac{s}{2} - \frac{1}{2} \\ -\omega s (-1 + js) \end{bmatrix} \sum_{i=0}^{3} N_{1i}s^i, \\
N_2(s) &= \begin{bmatrix} \frac{s}{2} - \frac{1}{2} \\ j(\omega^2 - s^2)(s + i) \end{bmatrix} \sum_{i=0}^{3} N_{2i}s^i,
\end{align*}
\]
and
\[
\begin{align*}
D_1(s) &= \begin{bmatrix} \frac{s}{2} - \frac{1}{2} \\ j(\omega^2 - s^2)(s + i) \end{bmatrix} \sum_{i=0}^{4} D_{1i}s^i, \\
D_2(s) &= \begin{bmatrix} \frac{s}{2} - \frac{1}{2} \\ j(\omega^2 - s^2)(s + i) \end{bmatrix} \sum_{i=0}^{4} D_{2i}s^i.
\end{align*}
\]
Then, according to Theorem \cite{1} complete solutions to the associated generalized Sylvester bimatrix equation \cite{15} are given by
\[
\begin{align*}
\{X_1, X_2\} &= \sum_{i=0}^{3} \{N_{1i}, N_{2i}\} \{Z_1, Z_2\} \{F_1, F_2\}^i, \\
\{Y_1, Y_2\} &= \sum_{i=0}^{4} \{D_{1i}, D_{2i}\} \{Z_1, Z_2\} \{F_1, F_2\}^i,
\end{align*}
\]
where \( Z_i \in \mathbb{C}^{1 \times 3}, i = 1, 2, \) are arbitrary matrices. Particularly, if we choose
\[
Z_1 = [1 + j \quad 0 \quad 0], \quad Z_2 = [0 \quad 0 \quad 1],
\]
the state feedback gain bimatrix \( \{K_1, K_2\} \) can be obtained according to \cite{17} as (we omit to display the variables \( \{X_1, X_2\} \) and \( \{Y_1, Y_2\} \))
\[
\begin{align*}
K_1 &= \begin{bmatrix} k_{11} + \frac{k_{11}}{12\omega^2} & k_{12} + \frac{k_{12}}{6\omega^2} & -\frac{\gamma(2 + j\gamma)}{2} \\ -k_{11} & k_{12} & \frac{\gamma(2 + j\gamma)}{2} \end{bmatrix}, \\
K_2 &= \begin{bmatrix} k_{21} + \frac{k_{21}}{12\omega^2} & k_{22} + \frac{k_{22}}{6\omega^2} & -\frac{\gamma(2 + j\gamma)}{2} \\ -k_{21} & k_{22} & \frac{\gamma(2 + j\gamma)}{2} \end{bmatrix},
\end{align*}
\]
where
\[
\begin{align*}
k_{11} &= \gamma^4 j - 12\gamma^3 \omega^2 + 19\gamma^2 \omega^2 j + \gamma^2 - 42\gamma \omega^4 + 6\gamma^2 \omega^2 j + 4\omega^2, \\
k_{21} &= -\gamma^4 j + 12\gamma^3 \omega^2 - 19\gamma^2 \omega^2 j + \gamma^2 + 42\gamma \omega^4 + 6\gamma^2 \omega^2 j + 4\omega^2, \\
k_{12} &= \gamma^4 j + \gamma^3 \omega^2 - \gamma^2 j + 12\gamma \omega^4 - \omega^2 j, \\
k_{22} &= \gamma^4 j + \gamma^3 \omega^2 + \gamma^2 j + 12\gamma \omega^4 + \omega^2 j.
\end{align*}
\]
Finally, the resulting controller can be implemented according to \cite{13}, which is physically realizable.

### 7 Conclusion

This paper has studied several kinds of linear bimatrix equations, whose coefficients are bimatrices that were introduced in our early studies. These equations arise from full state feedback pole assignment and
stability analysis of complex-valued linear systems. Explicit solutions to these linear bimatrix equations are established. Particularly, explicit solutions are provided for the case that the coefficients of the bimatrix equations are determined by the so-called antilinear systems. Explicit solutions are then used to solve the pole assignment problem for complex-valued linear systems, particularly, for second-order linear systems that can be easily converted into complex-valued linear systems. The spacecraft rendezvous control system is then used to demonstrate the obtained theoretical results. The results in this paper can be readily extended to linear bimatrix equations associated with complex-valued descriptor linear systems and high-order complex-valued linear systems.

References

[1] Bevis J H, Hall F J, Hartwing R E. Consimilarity and the matrix equation $AX - XB = C$. Current Trends in Matrix Theory, 1987, 1: 51-64.

[2] Bevis J H, Hall F J, Hartwig R E. The matrix equation $AX - XB = C$ and its special cases. SIAM Journal on Matrix Analysis and Applications, 1988, 9(3): 348-359.

[3] Bhattacharyya S P, De Souza E. Pole assignment via Sylvester’s equation. Systems & Control Letters, 1982, 1(4): 261-263.

[4] Clohessy W H, Wiltshire R. Terminal guidance system for satellite rendezvous. Journal of the Aeronautical Sciences, 27(9) (1960) 653–658.

[5] Duan G R. Solutions of the equation $AV + BW = VF$ and their application to eigenstructure assignment in linear systems. IEEE Transactions on Automatic Control, 1993, 38(2): 276-280.

[6] Duan G R. On the solution to the Sylvester matrix equation $AV + BW = EVF$. IEEE Transactions on Automatic Control, 1996, 41(4): 612-614.

[7] Duan G R. Generalized Sylvester Equations: Unified Parametric Solutions. CRC Press, 2015.

[8] Duan G R, Zhou B. Solution to the second-order Sylvester matrix equation $MVF^2 + DVF + KV = BW$. IEEE Transactions on Automatic Control, 2006, 51(5): 805-809.

[9] Lancaster P. Explicit solutions of linear matrix equations. SIAM Review, 1970, 12(4): 544-566.

[10] Hartwig R E. Resultants and the solution of $AX - XB = -C$. SIAM Journal on Applied Mathematics, 1972, 23(1): 104-117.

[11] Jameson A. Solution of the equation $AX + XB = C$ by inversion of an $M \times M$ or $N \times N$ matrix. SIAM Journal on Applied Mathematics, 1968, 16(5): 1020-1023.

[12] Jiang T, Wei M. On solutions of the matrix equations $X - AXB = C$ and $X - AXB = C$. Linear Algebra and its Applications, 2003, 367: 225-233.

[13] Kailath T. Linear Systems, Englewood Cliffs, NJ: Prentice-Hall, 1980.

[14] Kalman R E, Falb P L, Arbib M A. Topics in Mathematical System Theory, New York: McGraw-Hill, 1969.

[15] Rugh W J. Linear System Theory, Upper Saddle River, NJ: prentice hall, 1996.

[16] Sylvester J J. Sur la solution du cas le plus général des équations linéaires en quantités binaires, c'est-a-dire en quaternions ou en matrices du second ordre. CR Acad. Sci. Paris, 1884, 99: 117-118.

[17] Wu A G, Duan G R, Liu W, Sreeram V. Controllability and stability of discrete-time antilinear systems. 2013 3rd Australian Control Conference (AUCC), 2013: 403-408.

[18] Wu A G, Qian Y Y, Liu W, Sreeram V. Linear quadratic regulation for discrete-time antilinear systems: An anti-Riccati matrix equation approach. Journal of the Franklin Institute, 2016, 353(5): 1041-1060.

[19] Wu A G, Zhang Y. Complex Conjugate Matrix Equations for Systems and Control. Springer, 2017.
[20] Wu A G, Zhang Y, Liu W, Sreeram V. State response for continuous-time antilinear systems. *IET Control Theory & Applications*, 2015, 9(8): 1238-1244.

[21] Zhou B. Analysis and design of complex-valued linear systems, arXiv:1708.05120. Available at: https://arxiv.org/abs/1708.05120.

[22] Zhou B, Duan G R. A new solution to the generalized Sylvester matrix equation $AV - EVF = BW$. *Systems & Control Letters*, 2006, 55(3): 193-198.

[23] Zhou B, Duan G R, Li Z Y. A Stein matrix equation approach for computing coprime matrix fraction description. *IET Control Theory & Applications*, 2009, 3(6): 691-700.

[24] Zhou B, Lam J, Duan G R. Toward solution of matrix equation $X = Af(X)B + C$. *Linear Algebra and its Applications*, 2011, 435(6): 1370-1398.