Non–commutative Integration Calculus

Edwin Langmann

Theoretical Physics, Royal Institute of Technology, S-10044 Sweden

Abstract

We discuss a non–commutative integration calculus arising in the mathematical description of Schwinger terms of fermion–Yang–Mills systems. We consider the differential complex of forms

$$u_0[\varepsilon, u_1] \cdots [\varepsilon, u_n]$$

with $\varepsilon$ a grading operator on a Hilbert space $\mathcal{H}$ and $u_i$ bounded operators on $\mathcal{H}$ which naturally contains the compactly supported de Rham forms on $\mathbb{R}^d$ (i.e. $\varepsilon$ is the sign of the free Dirac operator on $\mathbb{R}^d$ and $\mathcal{H}$ a $L^2$–space on $\mathbb{R}^d$). We present an elementary proof that the integral of $d$–forms

$$\int_{\mathbb{R}^d} \text{tr}_N(X_0dX_1 \cdots dX_d)$$

for $X_i \in C^\infty_0(\mathbb{R}^d; \text{gl}_N)$, is equal, up to a constant, to the conditional Hilbert space trace of $\Gamma X_0[\varepsilon, X_1] \cdots [\varepsilon, X_d]$ where $\Gamma = 1$ for $d$ odd and $\Gamma = \gamma_d+1$ (‘$\gamma_5$–matrix’) a spin matrix anticommuting with $\varepsilon$ for $d$ even. This result provides a natural generalization of integration of de Rham forms to the setting of Connes’ non–commutative geometry which involves the ordinary Hilbert space trace rather than the Dixmier trace.

1 Introduction

One reason for increasing interest of physicists in non–commutative geometry (NCG) is the hope that it could provide new powerful tools for investigating quantum field theory beyond perturbation theory.

One example strongly supporting this hope is the representation theory of groups $\text{Map}(M^d; G)$ of maps from an odd–$d$ dimensional Riemannian manifold $M^d$ with spin structure to some compact semi–simple Lie group $G$ (e.g. $G = \text{SU}(N)$ in the fundamental representation) and their Lie algebras $\text{Map}(M^d; g)$ ($g$ the Lie algebra of $G$) which are closely related to gauge theories. Indeed, $\text{Map}(M^d; G)$ can be regarded as the gauge group of Yang–Mills theory on $(d + 1)$–dimensional space–time $M^d \times \mathbb{R}$ with structure group $G$ in the Hamiltonian framework. One therefore expects that quantum gauge theories should give rise to non–trivial representations of these groups and Lie algebras satisfying essential physical requirements such as existence of a highest weight vector (ground state).

Both, $\text{Map}(M^d; G)$ and $\text{Map}(M^d; g)$ are subalgebras of $\text{Map}(M^d; \text{gl}_N)$ (the algebra of complex $N \times N$–matrices) for some $N$, and elements $X$ in $\text{Map}(M^d; \text{gl}_N)$ can be naturally identified with a bounded operator on the Hilbert space $\mathcal{H} = L^2(M^d) \otimes V_{\text{spin}} \otimes \mathbb{C}^N$ where $V_{\text{spin}}$ is a vector space carrying the spin structure. Physically $\mathcal{H}$ is the one–particle Hilbert space of fermions on $M^d$, and the Dirac operator $\partial_A$ in an external Yang–Mills field $A$ is a self–adjoint operator on $\mathcal{H}$ playing the role of a one–particle Hamiltonian.

The connection of this with NCG is as follows: $\text{Map}(M^d; \text{gl}_N)$ can be naturally embedded in the algebra $g_p$ of bounded operators $u$ on $\mathcal{H}$ satisfying the Schatten ideal condition that $([\varepsilon, u][\varepsilon, u]^*)^p$ is trace class for $2p = d + 1$ where $\varepsilon = \text{sign}(\partial_0)$ is the sign of the free Dirac operator $\partial_0$ on $M^d$ (see e.g. [MR]), and these very algebras $g_p$ play a fundamental role in understanding the non–commutative spectrum of the Dirac operator.
role in NCG. Moreover, the Yang–Mills field configurations $A$ on $M^d$ can be embedded in the Grassmannian $Gr_p$ of grading operators $F$ on $\mathcal{H}$, $F = F^* = F^{-1}$, also satisfying this Schatten ideal condition, and this embedding is given by $A \mapsto F_A = \text{sign}(\mathcal{P}_A)$ [MR]. Physically the Schatten ideal condition can be regarded as a characterization of the degree of ultra–violet (UV) divergence arising in the fermion sector of $(d+1)$–dimensional Yang–Mills–fermion systems, and $2p = d + 1$ corresponds to the fact that UV divergences are worse in higher dimensions.

It is then natural to develop the representation theory for $g_p$ as a whole and obtain the ones for $\text{Map}(M^d; G)$ by restriction from that as it is the Schatten ideal condition which determines the appropriate regularization procedure required to construct the operators representing $\text{Map}(M^d; gl_N)$. Also the algebra $g_p$ contains other operators of interest to quantum field theory, thus considering $g_p$ as a whole is not only natural mathematically but also very useful from a physical point of view.

This is well–established for $g_1$ which can be represented as current algebra in a fermion Fock space with fermion currents constructed by normal ordering (see e.g. [PS, CR, M2, GL]). From this one obtains as special cases the wedge representations of affine Kac–Moody algebras (central extensions of $\text{Map}(S^1; g)$) and also the Virasoro algebra which played a central role in recent developments in (1+1) dimensional quantum field theory. For $g_{p \geq 2}$ analog representations in a fermion Fock space have been found [MR, M3]. In this case, normal ordering of the fermion currents is not sufficient but an additional multiplicative regularization is required [M4, La].

The regularizations necessary to construct the fermion currents representing $g_p$ lead to non–trivial 2–cocycles of the Lie algebras $g_p$ in the current–current commutator relations. Physically these cocycles are Schwinger terms, and it should be possible to trace back all anomalies of fermion–Yang–Mills systems [J] to these cocycles. At least this is known to be the case for (1+1) dimensions corresponding to $g_1$ (see e.g. a recent construction of QCD_{1+1} with massless quarks exploiting the representation theory of $g_1$ and deriving all anomalies from the $g_1$–cocycle [LS]), and recently it has been shown by explicit calculation that the cocycle of $g_3$ leads to the Gauss law anomaly of (3+1)–dimensional chiral QCD [LM].

The technical difficulty in proving that the 2–cocycle of $g_p$ is indeed equivalent to an anomaly in Yang–Mills gauge theory is that the former is given by a Hilbert space trace of operators — which in general is a highly non–local expression — whereas the latter are local, i.e. integrals of de Rham forms on some manifold $M^d$. For example, the 2–cocycle of $g_1$ (originally found by Lundberg [L]) is $\hat{c}_1(u, v) = \frac{1}{4} \text{Tr} (\varepsilon [u\varepsilon, v])$ where $u, v \in g_1$. As will become clear below, it is actually more natural to write this cocycle as

$$\hat{c}_1(u,v) = \frac{1}{2} \text{Tr}_C (u |\varepsilon, v|) .$$

where $\text{Tr}_C$ is the conditional trace defined by $\text{Tr}_C (a) \equiv \frac{1}{2} \text{Tr} (a + \varepsilon a \varepsilon)$. The corresponding 2–cocycle of $\text{Map}(M^1; g)$ is

$$c_1(X,Y) = \frac{1}{2\pi} \int_{M^1} \text{tr}_N (X dY)$$

where $\text{tr}_N$ is the usual trace of $N \times N$–matrices. At first sight it seems difficult to relate $\hat{c}_1(X,Y)$ to $c_1(X,Y)$. Nevertheless, comparing (1a) with (1b) is very suggestive: in NCG the commutator $[\varepsilon, \cdot]$ is the generalization of the exterior differentiation $d(\cdot)$ of de Rham forms [C], thus (1a) looks exactly like the non–commutative generalization of (1b) if one regards
\(\text{Tr}_C (\cdot)\) as the non–commutative generalization of integration \(\frac{1}{\pi} \int_{M^1} \text{tr} (\cdot)\) of \(\text{gl}_N\)–valued de Rham forms on \(M^1\). Indeed, it is known that (see e.g. [CR])

\[
\text{iTr}_C (X_0[\varepsilon, X_1]) = \frac{1}{\pi} \int_\mathbb{R} \text{tr} (X_0 dX_1) \quad \forall X_0, X_1 \in C^\infty_0 (\mathbb{R}; \text{gl}_N)
\]

(1c)

(and similarly for \(M^1 = S^1\)) which completely justifies this point of view and proves that \(\hat{c}(X, Y) = c(X, Y)\) for \(X, Y \in C^\infty_0 (\mathbb{R}^1; g)\).

In (3+1)–dimensions the generalization of (1c) is [LM]

\[
c_3 (X, Y; A) = \frac{i}{24\pi^2} \int_{M^3} \text{tr}_N (A [dX, dY])
\]

(2b)

which was shown in [LM]. Again seems natural to regard (2b) as non–commutative generalization of (2a) if one interprets \(F - \varepsilon\) as the generalization of the 1–form \(A\) (for a more detailed discussion see [LM]). Especially, for \(A\) a pure gauge we have \(A = -i U^{-1} dU\) for some \(U \in \text{Map}(M^3; G)\), hence \(F_A = U^{-1} [\varepsilon, U]\), and (2b) would be indeed the non–commutative generalization of (2a) if we could regard \(\text{Tr}_C (\cdot)\) as extension of \(\frac{1}{3\pi^2} \int_{M^3} \text{tr} (\cdot)\), i.e. (for \(M^3 = \mathbb{R}^3\))

\[
i^3 \text{Tr}_C (X_0[\varepsilon, X_1][\varepsilon, X_2][\varepsilon, X_3]) = \frac{i}{3\pi^2} \int_{\mathbb{R}^3} \text{tr}_N (X_0 dX_1 dX_2 dX_3)
\]

\[
\quad \forall X_0, X_1, X_2, X_3 \in C^\infty_0 (\mathbb{R}^3; \text{gl}(N)).
\]

(2c)

In this paper we give a simple proof of the eqs. (1c) and (2a) and their generalizations to arbitrary (even and odd) dimensions \(d\). (In even dimensions \(d\), the non–commutative integration is \(\propto \text{Tr}_C (\gamma_{d+1})\) involving a spin matrix \(\gamma_{d+1}\) (\(\gamma_5\) for \(d = 4\)) anticommuting with \(\slashed{D}_0\).

The method of proof is very simple and was inspired by the calculation in [LM]: we introduce a regularized trace \(\text{Tr}_\Lambda\) with a ‘momentum cut–off’ \(\Lambda\) such that \(\text{Tr}_\Lambda (a)\) exists and converges to \(\text{Tr}_C (a)\) as \(\Lambda \to \infty\) for all conditionally trace–class operators \(a\) on \(\mathcal{H}\). As was shown in [LM], one can easily calculate expressions of the form \(\text{Tr}_\Lambda ([a, b])\) in the limit \(\Lambda \to \infty\) using symbol calculus of PDOs [H] (see also [CFNW]). If \(a\) is trace–class this is obviously zero, but it is (in general) non–zero if \(ab\) is only conditionally trace–class. In this case it is essentially a ‘surface integral in Fourier space’ which involves only the operators at large momenta (= Fourier variables) and therefore can be calculated using asymptotic expansions of the operator \([a, b]\) in inverse powers of the momenta.

It is worth noting that these ‘surface integral in Fourier space’ (see eqs. (A1) and (2a)) have exactly the form typical for Feynman diagrams giving anomalies (see e.g. [JR]). Moreover, these expressions have also a deep mathematical meaning: as shown in [CFNW], for PDOs \(a, b\) such that \([a, b]\) is conditionally trace class, \(\lim_{\Lambda \to \infty} \text{Tr}_\Lambda ([a, b])\) is equal (up to a constant) to the Wodzicki residue \(\text{Res} (\log (|\slashed{D}_0|), a|b)\) [W] playing an important role in NCG.
We now write (for $d = 3$)

$$\text{Tr}_\Lambda (X_0[\varepsilon, X_1][\varepsilon, X_2][\varepsilon, X_3]) = \text{Tr}_\Lambda (X_0[\varepsilon, X_1][\varepsilon, X_2] \varepsilon, X_3) - \text{Tr}_\Lambda ([X_0[\varepsilon, X_1][\varepsilon, X_2], X_3]\varepsilon)$$

and with the arguments given in [LM] (using the calculus of PDOs) it is not difficult to show that the first term on the r.h.s. in the limit $\Lambda \to \infty$ is equal to the r.h.s. of (2c) (we will, however, give more detailed argument for this in the present paper). The difficult part in the proof of (2c) is to show that second term on the r.h.s. of this eq. actually is zero.

The result of this paper provides a natural generalization of integration of de Rham forms to NCG. It is valid for a differential complex over $g_p$ with differentiation given by $i[\varepsilon, \cdot]$. We note that another such generalization was given in [C2]. In this case, the non–commutative differentiation is defined by $i[D/0, \cdot]$, and the integration is in terms of the Dixmier trace rather than the ordinary Hilbert space trace as in our case. We note that for the differential complex with differentiation $i[\varepsilon, \cdot]$, the algebra product is equal to the product of Hilbert space operators (this follows from $\varepsilon^2 = 1$ — see Section 4). It therefore seem more natural than the one with differentiation $i[D/\cdot, \cdot]$ which does not have this property.

Our proof is restricted to manifolds $M^d = \mathbb{R}^d$. It is natural to conjecture a similar result for arbitrary Riemannian manifolds $M^d$ allowing for a spin structure, but we have not been able to prove this in general. We note that Connes’ non–commutative integration [C2] generalizes integration of de Rham forms for all such manifolds $M^d$.

The plan of this paper is as follows. We introduce the notation and state the result in the next section. The proof is then given in Section 3. It is divided in several Lemmas which are proved in very detail using only elementary arguments. We believe that this is justified because these Lemmas also play a fundamental role in other applications of NCG to quantum field theory and these arguments should therefore be useful, at least for physicists. It is also intended to make explicit that the crucial steps in the proof are very similar to arguments used standard perturbative calculations of anomalies in particle physics. We end with an outline how our result fits into the general framework of NCG and comments on other possible applications to quantum field theory in Section 4.

2 Notation and Result

For $\mathcal{H}$ a separable Hilbert space, we denote as $B(\mathcal{H})$ and $B_1(\mathcal{H})$ the bounded and trace–class operators on $\mathcal{H}$, respectively, and $B_q(\mathcal{H}) = \{ a \in B(\mathcal{H}) \mid (a^*a)^{q/2} \in B_1(\mathcal{H}) \}$ are the Schatten classes ($q \in \mathbb{N}$; $*$ is the Hilbert space adjoint). We note that for $a_0, a_1, \ldots, a_d \in B_q(\mathcal{H})$, $a_0a_1 \cdots a_d$ is trace class for $d + 1 \geq q$.

Let

$$\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathbb{C}^\nu \otimes \mathbb{C}^N$$

with

$$\nu = 2^{[d/2]}$$

where $[d/2] = (d - 1)/2$ for $d$ odd and $[d/2] = d/2$ for $d$ even. The free Dirac operator $\mathcal{D}_0$ on $\mathbb{R}^d$ is

$$\mathcal{D}_0 = \sum_{i=1}^{d} \gamma_i^{(d)}(-1) \frac{\partial}{\partial x_i}$$
where $\gamma_i^{(d)}$ are self-adjoint $\nu \times \nu$ matrices acting on $\mathbb{C}^\nu$ and obeying
\[ \gamma_i^{(d)} \gamma_j^{(d)} + \gamma_j^{(d)} \gamma_i^{(d)} = 2\delta_{ij} I_{\nu \times \nu} \tag{4b} \]
with $I_{\nu \times \nu}$ the $\nu \times \nu$ unit matrix. This naturally defines a self-adjoint operator on $\mathcal{H}$ which we denote by the same symbol $D_0$.

To be specific we choose the following representation for the $\gamma$–matrices: for $d = 1$ we have $\nu = 1$ and
\[ \gamma_1^{(1)} \equiv 1, \tag{5a} \]
and for all odd $d$ we define inductively ($0_{\nu \times \nu}$ is the $\nu \times \nu$ matrix with all matrix elements $= 0$)
\[ \gamma_i^{(d+2)} \equiv \left( \begin{array}{cc} 0_{\nu \times \nu} & \gamma_i^{(d)} \\ \gamma_i^{(d)} & 0_{\nu \times \nu} \end{array} \right) \text{ for } i = 1, 2, \ldots, d, \]
\[ \gamma_i^{(d+1)} \equiv \left( \begin{array}{cc} 0_{\nu \times \nu} & -i1_{\nu \times \nu} \\ i1_{\nu \times \nu} & 0_{\nu \times \nu} \end{array} \right), \quad \gamma_i^{(d+2)} \equiv \left( \begin{array}{cc} 1_{\nu \times \nu} & 0_{\nu \times \nu} \\ 0_{\nu \times \nu} & -1_{\nu \times \nu} \end{array} \right). \tag{5b} \]

For all even $d$ we choose
\[ \gamma_i^{(d)} \equiv \gamma_i^{(d+1)} \text{ for } i = 1, 2, \ldots, d + 1, \tag{5c} \]
hence there is an additional spin matrix $\gamma_i^{(d)}$ which can be identified with a grading operator in $\mathcal{H}$ (by abuse of notation, we use the same symbol for $\gamma_i^{(d+1)}$, $1_{\nu \times \nu} \in \mathfrak{gl}_N$ and the corresponding operators on $\mathcal{H}$).

Then there is a natural embedding of $C_0^\infty(\mathbb{R}^d; \mathfrak{gl}_N)$ in $B(\mathcal{H})$,
\[ (\hat{X}f)(x) \equiv X(x)f(x) \quad \forall f \in \mathcal{H} \tag{6} \]
(we write $\mathcal{H} \ni f : \mathbb{R}^d \to \mathbb{C}^\nu \otimes \mathbb{C}^N$, $f \mapsto f(x)$, and $\mathfrak{gl}_N$ acts naturally on $\mathbb{C}^N$).

Using the spectral theorem for self–adjoint operators [RS] we define $\varepsilon = \text{sign}(D_0)$ where $\text{sign}(x) = +1(-1)$ for $x \geq 0$ ($x < 0$).

The Schatten ideal discussed in the introduction can be written as
\[ X \in C_0^\infty(\mathbb{R}^d; \mathfrak{gl}_N) \Rightarrow [\varepsilon, \hat{X}] \in B_q(\mathcal{H}) \text{ if } q = d + 1 \tag{7} \]
(see e.g. [MR]).

With $\text{Tr}$ the usual Hilbert space trace, we note that the conditional trace
\[ \text{Tr}_C(a) \equiv \frac{1}{2} \text{Tr} (a + \varepsilon a \varepsilon) \tag{8} \]
is well-defined for all $a$ in the conditional trace class
\[ B_{1,C}(\mathcal{H}) \equiv \{ a \in B(\mathcal{H}) | a + \varepsilon a \varepsilon \in B_1(\mathcal{H}) \}. \tag{9} \]

We can now state our result.

**Theorem:** Let
\[ \Gamma^{(d)} \equiv \begin{cases} 1_{\nu \times \nu} & \text{if } d \text{ is odd} \\ \gamma_i^{(d)} & \text{if } d \text{ is even} \end{cases}. \tag{10} \]

1 to avoid confusion, we distinguish here $X \in C_0^\infty(\mathbb{R}^d; g)$ from the corresponding operator $\hat{X}$ on $\mathcal{H}$.
Then for all $X_0, X_1, \ldots X_d \in C^\infty_0(\mathbb{R}^d; gl_N)$,
\[
\Gamma^{(d)} \hat{X}_0[\varepsilon, \hat{X}_1] \cdots [\varepsilon, \hat{X}_d] \in B_{1,C}(\mathcal{H}),
\]
and
\[
i^d \text{Tr}_C \left( \Gamma^{(d)} \hat{X}_0[\varepsilon, \hat{X}_1] \cdots [\varepsilon, \hat{X}_d] \right) = c_d \int_{\mathbb{R}^d} \text{tr} \left( X_0 dX_1 \cdots dX_d \right) \tag{12a}
\]
with a normalization constant
\[
c_d = (2i)^{d/2} \frac{1}{d(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{12b}
\]
For example
\[
c_1 = \frac{1}{\pi}, \quad c_2 = \frac{i}{2\pi}, \quad c_3 = \frac{i}{3\pi^2}, \quad c_4 = \frac{-1}{2\pi^2} \ldots \tag{13}
\]

3 Proof

To see that (11) is true we observe that — denoting the l.h.s. of (11) as $a —$
\[
a + \varepsilon a \varepsilon = \Gamma^{(d)} \varepsilon [\varepsilon, \hat{X}_0][\varepsilon, \hat{X}_1] \cdots [\varepsilon, \hat{X}_d]
\]
(to see this use repeatedly $\varepsilon [\varepsilon, \hat{X}_i] \varepsilon = -[\varepsilon, \hat{X}_i]$ and $\varepsilon^2 = 1$; for even $d$ one also needs $\varepsilon \Gamma^{(d)} = -\Gamma^{(d)} \varepsilon$) which is trace–class by (7).

For $\Lambda > 0$ and $a \in B_{1,C}(\mathcal{H})$ we define the cutoff trace
\[
\text{Tr}_\Lambda (a) = \text{Tr} (a P_\Lambda), \quad P_\Lambda = \Theta(\Lambda - |\mathcal{D}_0|) \tag{14}
\]
where $\Theta(\Lambda - |x|) = 1$ for $|x| \leq \Lambda$ and 0 otherwise.

Lemma 1: For all $a \in B_{1,C}(\mathcal{H})$, $a P_\Lambda$ is trace class for all $\Lambda < \infty$, and
\[
\text{Tr}_\Lambda (a) = \lim_{\Lambda \to \infty} \text{Tr}_\Lambda (a). \tag{15}
\]

Proof: If $a \in B_1(\mathcal{H})$ then trivially $a P_\Lambda \in B_1(\mathcal{H})$ and $\text{Tr} (a) = \lim_{\Lambda \to \infty} \text{Tr}_\Lambda (a)$ as $P_\Lambda$ converges strongly to the identity for $\Lambda \to \infty$. If $a \in B_{1,C}(\mathcal{H})$ then $\frac{1}{2} \text{Tr}_\Lambda (a + \varepsilon a \varepsilon) = \text{Tr}_\Lambda (a)$ as $\varepsilon$ commutes with $P_\Lambda$. By definition of $\text{Tr}_\Lambda$ and $B_{1,C}(\mathcal{H})$ this implies the Lemma.

Lemma 2: Let $\text{tr}_\nu (\cdot)$ be the trace of $\nu \times \nu$ matrices acting on $\mathbb{C}^\nu$ and $i_1, i_2, \ldots i_{d-2n} \in \{1, 2, \ldots d\}$. Then for $n = 0, 1, \ldots [d/2]$,
\[
\text{tr}_\nu \left( \Gamma^{(d)} \gamma^{(d)}_{i_1} \gamma^{(d)}_{i_2} \cdots \gamma^{(d)}_{i_{d-2n}} \right) = \delta_{n,0} (2i)^{[d/2]} \epsilon_{i_1 i_2 \cdots i_n} \tag{16}
\]
where $\epsilon_{i_1 i_2 \cdots i_n}$ is the anti-symmetric tensor (i.e. it is equal to +1 (−1) for $(i_1, i_2, \ldots i_d)$ an even (odd) permutation of $(1, 2, \ldots d)$ and 0 otherwise).

Proof: If $d = 2m$ is even then the eqs. (16) are just special cases of the eqs. for $d = 2m+1$, hence we can restrict ourselves to odd $d$'s. For $d = 1$ eq. (16) is true trivially. For general odd $d$ we prove it by induction. We consider
\[
M = \gamma^{(d+2)}_{i_1} \gamma^{(d+2)}_{i_2} \cdots \gamma^{(d+2)}_{i_{d+2-2n}}
\]
for $n = 0, 1, \ldots, (d + 1)/2$. Without loss of generality we first assume that $1 \leq i_1 < i_2 < \ldots < i_{d+2-2n} \leq d + 2$. (This is because, by using the relation (11), $M$ can be always brought to the form $\pm \gamma_{j_1} (d+2) \cdots \gamma_{j_{d+2-2n}}$ with $1 \leq j_1 < j_2 < \ldots < j_{d+2-2n} \leq d + 2$. If two or more of these indices are equal, $\left( \gamma_{j} (d+2) \right)^2 = 1_{2\nu \times 2\nu}$ implies that $M$ is equal to some matrix $\pm \gamma_{k_1} (d+2) \cdots \gamma_{k_{d+2-2n}}$ with $m > n$ and $1 \leq k_1 < k_2 < \ldots < k_{d+2-2m} \leq d + 2$.)

If $i_{d+2-2n} = d + 2$, $i_{d+1-2n} = d + 1$ then eq. (11) implies
\[
M = \begin{pmatrix} i\gamma_{i_1} (d) & \cdots & \gamma_{i_{d-2n}} (d) & 0_{\nu \times \nu} \\
0_{\nu \times \nu} & i\gamma_{i_1} (d) & \cdots & \gamma_{i_{d-2n}} (d) \end{pmatrix},
\]
hence $\text{tr}_{2\nu} (M) = (2i) \text{tr}_{\nu} \left( \gamma_{i_1} (d) \cdots \gamma_{i_{d-2n}} (d) \right)$, and for $n \geq 1 \text{tr}_{2\nu} (M) = 0$ follows from the induction hypothesis. If $i_{d+2-2n} \neq d + 2$, or $i_{d-2n+2} = d + 2$ and $i_{d-2n+1} \neq d + 1$, then $\text{tr}_{2\nu} (M) = 0$ trivially because $M$ is of the form
\[
\begin{pmatrix} 0_{\nu \times \nu} & (\cdots) \\
\pm (\cdots) & 0_{\nu \times \nu} \end{pmatrix} \text{ or } \begin{pmatrix} (\cdots) & 0_{\nu \times \nu} \\
0_{\nu \times \nu} & -(\cdots) \end{pmatrix}.
\]
This proves (16) for $n \geq 1$. To prove it for $n = 0$ we observe that by the relations above and induction,
\[
\begin{align*}
\text{tr}_{2\nu} \left( \gamma_{1} (d+2) \cdots \gamma_{d+2} (d+2) \right) &= (2i) \text{tr}_{\nu} \left( \gamma_{1} (d) \cdots \gamma_{d} (d) \right) \\
&= \cdots = (2i)^{(d+1)/2} \text{tr}_{1} \left( \gamma_{1} (1) \right) = (2i)^{(d+1)/2}.
\end{align*}
\]
Moreover (as discussed above — we now allow for arbitrary indices $i_1, i_2 \cdots i_{d+2}$),
\[
\text{tr}_{\nu} \left( \gamma_{i_1} (d+2) \gamma_{i_2} (d+2) \cdots \gamma_{i_{d+2}} (d+2) \right)
\]
is non-zero only if $(i_1, i_2, \ldots i_{d+2})$ is a permutation of $(1, 2, \ldots d + 2)$, and if this is the case it is equal to $(+/–) \text{tr}_{\nu} \left( \gamma_{1} (d+2) \gamma_{2} (d+2) \cdots \gamma_{d+2} (d+2) \right)$ depending on whether this permutation is even/odd. This implies (16) for $n = 0$ and completes the proof of Lemma 2.

**Lemma 3:** Let $X_1, \ldots, X_n \in C_0^\infty (\mathbb{R}^d; \text{gl}_N)$. Then for all $n = 0, 1, \ldots [d/2]$ and $\Lambda > 0$,
\[
\text{Tr}_\Lambda \left( \Gamma (d) \hat{X}_1 \hat{X}_2 \cdots \hat{X}_{d-2n} \right) = 0.
\]  

**Proof:** We first introduce some notation. For $L^2$–functions $f$ on $\mathbb{R}^d$ we define the Fourier transform as
\[
\hat{f}(p) = \int_{\mathbb{R}^d} dx \, e^{ipx} f(x)
\]
where $px = \sum_{i=1}^d p_i x_i$ and $dx = d^d x$. The integral kernel $K(a)$ of $a \in B(\mathcal{H})$ is the $\text{gl}_\nu \otimes \text{gl}_N$–valued function on $\mathbb{R}^d \times \mathbb{R}^d$ such that
\[
\overline{(af)}(p) = \int_{\mathbb{R}^d} dq K(a) (p, q) \hat{f}(q) \quad \forall f \in \mathcal{H}
\]
where $dq = d^d q/(2\pi)^d$. Then for all $a, b \in B(\mathcal{H})$,
\[
K(ab)(p, q) = \int_{\mathbb{R}^d} dk K(a)(p, k) K(b)(k, q),
\]
and
and for all \( a \in B_1(\mathcal{H}) \),
\[
\text{Tr}(a) = \int_{\mathbb{R}^d} d\mu(a(p, p))
\]
where \( \text{tr} = \text{tr}_a \text{tr}_N \).

Therefore, as
\[
\begin{align*}
K(\varepsilon)(p, q) &= (2\pi)^d \delta^d(p - q) \varepsilon(p) \\
K(\hat{X})(p, q) &= \hat{X}(p - q) \\
K(P_\Lambda)(p, q) &= (2\pi)^d \delta^d(p - q) \Theta(\Lambda - |p|)
\end{align*}
\]
with
\[
\varepsilon(p) = \frac{1}{|p|}g' = \frac{1}{|p|} \sum_{i=1}^d p_i \gamma_i(d),
\]
we get
\[
\begin{align*}
\text{Tr}_A \left( \Gamma^{(d)} \hat{X}_1 \varepsilon \hat{X}_2 \varepsilon \cdots \hat{X}_n \varepsilon \right) &= \int_{B^d_A} d\mu \int_{\mathbb{R}^d} d\mu q_1 \cdots \int_{\mathbb{R}^d} d\mu q_n \\
\times \text{tr} \left( \Gamma^{(d)} \hat{X}(p - q_1) \varepsilon(q_1) \hat{X}_2(q_1 - q_2) \varepsilon(q_2) \cdots \hat{X}_n(q_n - p) \varepsilon(p) \right)
\end{align*}
\]
with \( B^d_A = \{ p \in \mathbb{R}^d \mid |p| \leq \Lambda \} \), and (17) trivially follows from (16) for \( n < d \).

The only non–trivial case is \( n = d \). Shifting the integration variables to
\[
Q_1 = p - q_1, \quad Q_2 = q_1 - q_2, \quad \ldots \quad Q_{d-1} = q_{d-2} - q_{d-1}
\]
we can write
\[
\begin{align*}
\text{Tr}_A \left( \Gamma^{(d)} \hat{X}_1 \varepsilon \hat{X}_2 \varepsilon \cdots \hat{X}_d \varepsilon \right) &= \int_{\mathbb{R}^d} d\mu Q_1 \cdots \int_{\mathbb{R}^d} d\mu Q_{d-1} \\
\times \text{tr}_N \left( \hat{X}_1(Q_1) \hat{X}_2(Q_2) \cdots \hat{X}_{d-1}(Q_{d-1}) \hat{X}_d(-Q_1 - Q_2 - \cdots - Q_{d-1}) \right) f_\Lambda(Q_1, Q_2, \ldots, Q_{d-1})
\end{align*}
\]
where
\[
f_\Lambda(Q_1, Q_2, \ldots, Q_{d-1}) = \int_{B^d_A} d\mu p \text{tr}_\nu \left( \Gamma^{(d)} \varepsilon(p - q_1) \varepsilon(p - Q_1 - Q_2) \cdots \varepsilon(p - Q_1 - \cdots - Q_{d-1}) \varepsilon(p) \right).
\]
To evaluate this latter function we use the representation
\[
\varepsilon(Q) = \int_{\mathbb{R}} \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2 Q^2} \text{Q}^2
\]
\((Q^2 = \sum_{i=1}^d Q_i^2)\), thus
\[
\begin{align*}
f_\Lambda(Q_1, Q_2, \ldots, Q_{d-1}) &= \int_{\mathbb{R}} \frac{dt_1}{\sqrt{2\pi}} e^{-\frac{1}{2}t_1^2 Q_1^2} \int_{\mathbb{R}} \frac{dt_2}{\sqrt{2\pi}} e^{-\frac{1}{2}t_2^2 (Q_1+Q_2)^2} \cdots \\
\times \int_{\mathbb{R}} \frac{dt_{d-1}}{\sqrt{2\pi}} e^{-\frac{1}{2}t_{d-1}^2 (Q_1+\cdots+Q_{d-1})^2} \int_{\mathbb{R}} \frac{dt_d}{\sqrt{2\pi}} (\ldots)
\end{align*}
\]
where
\[
(\ldots) = \int_{B^d_A} d\mu p e^{-\frac{1}{2}t^2 p^2} e^{\mu} \text{tr}_\nu \left( \Gamma^{(d)} (y' - Q_1)(y' - Q_1 - Q_2) \cdots (y' - Q_1 - \cdots - Q_{d-1})y' \right)
\]
with \( t^2 = \sum_{i=1}^{d} t_i^2 \), \( pv = \sum_{i=1}^{d} p_i v_i \), and
\[
v = t_1^2 Q_1 + t_2^2 (Q_1 + Q_2) + \cdots + t_{d-1}^2 (Q_1 + Q_2 + \cdots + Q_{d-1}).
\]

Using (16) we see that \((\cdots)\) is proportional to
\[
\int_{B_\Lambda^d} \mathcal{P} e^{-\frac{1}{2} \xi^2} \psi \prod_{i=1}^d \epsilon (Q_1) \epsilon Q_2 \cdots \epsilon (p - Q_1 - Q_2 - \cdots - Q_{d-1}) \epsilon = 0 \text{ by Lemma 3.}
\]

We now evaluate the r.h.s. of this using symbol calculus [H]. We recall that every pseudo differential operator (PDO) \( \mathcal{P} \) is a \( \text{PDO} \) on \( \mathcal{H} \) and for \( a \) non-degenerate first-order PDO \( a \) on \( \mathcal{H} \), the symbol is given by
\[
\sigma(a)(p, x) = \mathcal{F}\{ a(x) \} = \int_{\mathbb{R}^d} \mathcal{F}\{ a(x) \} \cdot x \, \mathcal{F}^{-1}\{ x \}.
\]

Obviously the integral \( \int_{B_\Lambda^d} \mathcal{P} e^{-\frac{1}{2} \xi^2} \psi \prod_{i=1}^d \epsilon (Q_1) \epsilon Q_2 \cdots \epsilon (p - Q_1 - Q_2 - \cdots - Q_{d-1}) \epsilon = 0 \text{ by Lemma 3.}
\]

\textbf{Lemma 4:} For all \( X_0, X_1, \ldots, X_d \in C_0^\infty (\mathbb{R}^d; \text{gl}_N) \),
\[
\text{Tr}_\Lambda \left( \Gamma^{(d)} [\dot{X}_0[\epsilon, \dot{X}_1] \cdots [\epsilon, \dot{X}_d] \cdots \epsilon, \dot{X}_{d-1}[\epsilon, \dot{X}_d] \cdots [\epsilon, \dot{X}_d] \cdots \epsilon, \dot{X}_d \right). 
\]

\textbf{Proof:} The difference of the r.h.s. and the l.h.s. of (20) is equal to
\[
\text{Tr}_\Lambda \left( \Gamma^{(d)} [\dot{X}_0[\epsilon, \dot{X}_1] \cdots [\epsilon, \dot{X}_d] \cdots \epsilon, \dot{X}_{d-1}[\epsilon, \dot{X}_d] \cdots [\epsilon, \dot{X}_d] \cdots \epsilon, \dot{X}_d \right).
\]

which is just a linear combination of terms \( \text{Tr}_\Lambda \left( \Gamma^{(d)} [\dot{Y}_1 \epsilon \dot{Y}_2 \cdots \epsilon \dot{Y}_{d-2}[\epsilon, \dot{X}_d] \cdots [\epsilon, \dot{X}_d] \cdots [\epsilon, \dot{X}_d] \cdots [\epsilon, \dot{X}_d] \cdots [\epsilon, \dot{X}_d] \right) \) with \( n \leq d \) and \( Y_i \in C_0^\infty (\mathbb{R}^d; \text{gl}_N) \) (use \( \epsilon^2 = 1 \); for even \( d \) one also needs \( \Gamma^{(d)} = \epsilon \Gamma^{(d)} \) and cyclicity of \( \text{tr}_\nu \) and therefore zero by Lemma 3.

\textbf{Proof of the Theorem:} The Lemmas above imply,
\[
\text{Tr}_\Lambda \left( \Gamma^{(d)} [\dot{X}_0[\epsilon, \dot{X}_1] \cdots [\epsilon, \dot{X}_d] \cdots [\epsilon, \dot{X}_d] \cdots [\epsilon, \dot{X}_d] \cdots [\epsilon, \dot{X}_d] \cdots [\epsilon, \dot{X}_d] \right) \]

We now evaluate the r.h.s. of this using symbol calculus [H]. We recall that every pseudo differential operator (PDO) \( a \) on \( \mathcal{H} \) can be represented by its symbol \( \sigma(a)(p, x) \) which is a \( \text{gl}_v \otimes \text{gl}_N \)-valued function on \( \mathbb{R}^d \times \mathbb{R}^d \) and defined such that for any \( f \in \mathcal{H} \),
\[
(a f)(x) = \int_{\mathbb{R}^d} dp e^{ipx} \sigma(a)(p, x) \tilde{f}(p)
\]
where \( \tilde{f}(p) \) is the Fourier transform of \( f \). It follows then that
\[
\sigma(ab)(p, x) = \int_{\mathbb{R}^d} dq \int_{\mathbb{R}^d} dy e^{i(x-y)(p-q)} \sigma(a)(q, x) \sigma(b)(p, y),
\]
and for a trace–class,
\[
\text{Tr}(a) = \int_{\mathbb{R}^d} dp \int_{\mathbb{R}^d} dx \, \text{tr} (\sigma(a)(p, x)).
\]

\textsuperscript{2}all operators of interest to us are PDOs
Especially, $\sigma(\varepsilon)(p, x) = \varepsilon(p)$ and $\sigma(\hat{X})(p, x) = X(x)$ for all $X \in C^\infty_0(\mathbb{R}^d; g)$. Moreover, $\sigma(P_\Lambda)(p, x) = \Theta(\Lambda - |p|)$, hence

$$\text{Tr}_\Lambda (a) = \int_{B_\Lambda} dp \int_{\mathbb{R}^d} dx \text{tr} (\sigma(a)(p, x)).$$

(24)

All operators $a$ of interest to us allow an asymptotic expansion $\sigma(a) \sim \sum_{n=0}^\infty \sigma_{-j}(a)$ where $\sigma_{-j}(a)(p, x)$ is homogeneous of degree $-j$ in $p$ (i.e., $\sigma_{-j}(sp, x) = s^{-j}\sigma_{-j}(a)(p, x)$ for all $s > 0$) and goes to zero like $|p|^{-j}$ for $|p| \to \infty$. We write

$$\sigma(a)(p, x) = \sum_{j=0}^n \sigma_{-j}(a)(p, x) + O(|p|^{-n-1}).$$

(25)

Moreover, eq. 23 has an asymptotic expansion in powers of $|p|^{-1}$,

$$\sigma(ab)(p, x) \sim \sum_{n=0}^\infty \sum_{i_1, \ldots, i_n = 1}^d (-i)^n \frac{\partial^n \sigma(a)(p, x)}{\partial p_{i_1} \cdots \partial p_{i_n}} \frac{\partial^n \sigma(b)(p, x)}{\partial x_{i_1} \cdots \partial x_{i_n}}.$$  

(26)

This allows to determine the asymptotic expansion of $\sigma(ab)$ from the ones of $\sigma(a)$ and $\sigma(b)$. Especially if $\sigma(a)$ is $O(|p|^{-n})$ and $\sigma(b)$ is $O(|p|^{-m})$ then $\sigma(ab)$ is $O(|p|^{-(n+m)})$.

For PDOs $a \in B_1(\mathcal{H})$ and $X \in C^\infty_0(\mathbb{R}^d; g)_N$, $\text{Tr}_\Lambda \left([a, \hat{X}]\right)$ converges to zero for $\Lambda \to \infty$ (as $P_\Lambda$ strongly converges to the identity this trivially follows from the the cyclicity of trace). Especially this is true if $\sigma(a)$ is $O(|p|^{-d-1})$. If $\sigma(a)$ is only $O(|p|^{-d+1})$ then $a$ is not trace class but $\text{Tr}_\Lambda \left([a, \hat{X}]\right)$ still has a well-defined and (in general) non-trivial limit. Indeed, eq. 23 suggests that

$$\text{Tr}_\Lambda \left([a, \hat{X}]\right) = (-i) \int_{B_\Lambda} dp \int_{\mathbb{R}^d} dx \sum_{i=1}^d \text{tr} \left( \frac{\partial \sigma(a)(p, x)}{\partial p_i} \frac{\partial X(x)}{\partial x_i} \right) + O \left(\Lambda^{-1}\right).$$

(27)

This is true as the terms neglected in the asymptotic expansion of the operator products are also total derivatives and, by Stokes theorem (28), can be written as surface integrals over $|p| = \Lambda$ which vanish as $\Lambda \to \infty$ (the details of this argument are given in Appendix A).

Using Stokes’ theorem

$$\int_{B_\Lambda} dp \frac{\partial}{\partial p_i} f(p) = \int_{\mathbb{R}^d} dp \delta(\Lambda - |p|) \frac{p_i}{|p|} f(p)$$

(28)

(which in the present case is just equivalent to

$$\int_{\mathbb{R}^d} dp \frac{\partial}{\partial p_i} \Theta(\Lambda - |p|) f(p) = 0$$

we therefore get

$$\text{Tr}_\Lambda \left([a, \hat{X}]\right) = (-i) \sum_{i=1}^d \int_{\mathbb{R}^d} dp \frac{p_i}{|p|} \delta(\Lambda - |p|) \int_{\mathbb{R}^d} dx \text{tr} \left( \sigma(a)_{-d+1}(p, x) \frac{\partial X(x)}{\partial x_i} \right) + O \left(\Lambda^{-1}\right).$$

(29)
where we replaced $\sigma(a)$ by its $O(|p|^{-d+1})$–piece $\sigma_{-d+1}(a)$ as only this gives a non–zero contribution for $\Lambda \to \infty$.

For $X \in C^\infty_0(\mathbb{R}^d; \mathfrak{gl}_N)$, eq. (26) implies

$$\sigma \left( \left[ \varepsilon, \hat{X} \right] \right) (p, x) = (-i) \sum_{i=1}^d \frac{\partial \varepsilon(p)}{\partial p_i} \frac{\partial X(x)}{\partial x_i} + O(|p|^{-2}),$$

hence $a = \Gamma^{(d)} \hat{X}_0[\varepsilon, \hat{X}_1] \cdots [\varepsilon, \hat{X}_{n-1}] \varepsilon$ is $O(|p|^{-d+1})$ and

$$\sigma(a)_{-d+1}(p, x) = (-i)^{d-1} \Gamma^{(d)} X_0(x) \sum_{i_1 \cdots i_{d-1}=1}^d \frac{\partial \varepsilon(p)}{\partial p_{i_1}} \frac{\partial X_1(x)}{\partial x_{i_1}} \cdots \frac{\partial \varepsilon(p)}{\partial p_{i_{d-1}}} \frac{\partial X_{d-1}(x)}{\partial x_{i_{d-1}}}.$$

Thus with (29),

$$\text{Tr}_\Lambda \left( \left[ \Gamma^{(d)} \hat{X}_0[\varepsilon, \hat{X}_1] \cdots [\varepsilon, \hat{X}_{n-1}] \varepsilon, \hat{X}_n \right] \right) = \sum_{i_1 \cdots i_{d-1}=1}^d (-i)^d J_{i_1 \cdots i_{d-1}}^\Lambda$$

$$\times \int_{\mathbb{R}^d} dx \text{tr}_N \left( X_0(x) \frac{\partial X_1(x)}{\partial x_{i_1}} \cdots \frac{\partial X_{d-1}(x)}{\partial x_{i_{d-1}}} \frac{\partial X_d(x)}{\partial x_i} \right) + O(\Lambda^{-1})$$

where

$$J_{i_1 \cdots i_{d-1}}^\Lambda = \int_{\mathbb{R}^d} dp \left[ \delta(\Lambda - |p|) \left| \frac{p}{|p|} \right| \right]^{d/2} \text{tr}_\nu \left( \Gamma^{(d)} \frac{\partial \varepsilon(p)}{\partial p_{i_1}} \cdots \frac{\partial \varepsilon(p)}{\partial p_{i_{d-1}}} \varepsilon(p) \right).$$

Now (13) implies $\partial \varepsilon(p)/\partial p_i = \sum_{j=1}^d P_{ij} \varepsilon^{(d)} / |p|$ with $P_{ij} = (\delta_{ij} - p_i p_j / |p|^2)$, hence with Lemma 2 we get

$$\text{tr}_\nu \left( \Gamma^{(d)} \frac{\partial \varepsilon(p)}{\partial p_{i_1}} \cdots \frac{\partial \varepsilon(p)}{\partial p_{i_{d-1}}} \varepsilon(p) \right) = \sum_{j_1 \cdots j_d=1}^d (2i)^{d/2} \epsilon_{j_1 \cdots j_d} \frac{1}{|p|^{d}} P_{i_1 j_1} \cdots P_{i_{d-1} j_{d-1}} P_{i_d j_d}$$

$$= (2i)^{d/2} \sum_{j=1}^d \epsilon_{i_1 \cdots i_{d-1} j} P_{i_d j_d}$$

where we used the antisymmetry of the $\varepsilon$–symbol. Thus

$$I_{i_1 \cdots i_{d-1}}^\Lambda = (2i)^{d/2} \sum_{j=1}^d \epsilon_{i_1 \cdots i_{d-1} j} \cdots$$

with

$$\cdots_{ij} = \int_{\mathbb{R}^d} dp \delta(\Lambda - |p|) \frac{p_i p_j}{|p|^2} = \delta_{ij} \frac{2\pi^{d/2}}{d(2\pi)^d \Gamma(d/2)}$$

we rescaled $\xi = p/\Lambda$ and used $\int_{\mathbb{R}^d} d\xi \delta(1 - |\xi|) = 2\pi^{d/2} / \Gamma(d/2)$ (volume of the unit sphere $S^{d-1}$). This implies

$$I_{i_1 \cdots i_{d-1}}^\Lambda = c_d \epsilon_{i_1 \cdots i_{d-1} i}$$

with $c_d$ (12b). Putting these equations together we end up with

$$\text{Tr}_\Lambda \left( \left[ \Gamma^{(d)} \hat{X}_0[\varepsilon, \hat{X}_1] \cdots [\varepsilon, \hat{X}_{d-1}] \varepsilon, \hat{X}_d \right] \right)$$

$$= (-i)^d c_d \int_{\mathbb{R}^d} dx \sum_{i_1 \cdots i_d=1}^d \epsilon_{i_1 \cdots i_d} \text{tr}_N \left( X_0(x) \frac{\partial X_1(x)}{\partial x_{i_1}} \cdots \frac{\partial X_d(x)}{\partial x_{i_d}} \right) + O(\Lambda^{-1}),$$

thus eq. (21) implies (12d) which completes our proof.
4 Final Comments

Let \( A \) be an associative algebra over \( \mathbb{C} \). The basic object of NCG is the differential complex \((\Omega, d)\) over \( A \), which is a \( \mathbb{N}_0 \)-graded complex algebra,

\[
\Omega = \bigoplus_{n=0}^{\infty} \Omega^{(n)}
\]

where \( \Omega^{(n)} \) are \( A \)-bimodules and \( \Omega^{(0)} = A \). Moreover, there is a linear operator

\[
d : \Omega^{(n)} \to \Omega^{(n+1)}
\]

satisfying \( d^2 = 0 \) and \( d(\omega \omega') = (d\omega)\omega' + (-)^n \omega (d\omega') \) for all \( \omega \in \Omega^{(n)}, \omega' \in \Omega \). Here we restrict ourselves to the case where for all \( n \), \( \Omega^{(n)} \) is equal to the linear span of forms \( u_0 d u_1 \cdots d u_n \) with \( u_i \in A \), hence \( \Omega \) is determined by \( A \) and \( d \).

The most prominent example is the de Rham complex \((\Omega_d, d)\) of forms on \( \mathbb{R}^d \) which is the differential complex over \( A_d = C_0^\infty(\mathbb{R}^d; gl_N) \) with \( d \) the exterior differentiation of forms.

In this case \( \Omega^{(n)} = \emptyset \) for \( n > d \).

Another important example is the differential complex \((\hat{\Omega}_p, \hat{d})\) over \( A_p = \{ \hat{X} \in B(H) \mid [\varepsilon, \hat{X}] \in B \} \) where \( 2p \in \mathbb{N}, \) \( H \) a separable Hilbert space, \( \varepsilon \) a grading operator on \( H \) (\( \varepsilon = \varepsilon^* = \varepsilon^{-1} \)), and

\[
\hat{d} u = i[\varepsilon, u] \quad \forall u \in g_p.
\]

Then \( \hat{d}\hat{\omega} = i(\varepsilon \hat{\omega} - (-)^n \hat{\omega} \varepsilon) \) for all \( \hat{\omega} \in \hat{\Omega}_p^{(n)} \) showing that the algebra product in \( \hat{\Omega}_p \) is equal to the product as operators in \( B(H) \). Moreover, \( \hat{\omega} \in \hat{\Omega}_p^{(n)} \) is conditionally trace class for \( n \geq 2p \) (as before, \( \text{Tr}_C(a) = \frac{1}{2} \text{Tr}(a + \varepsilon a \varepsilon) \)).

The Schatten ideal condition shows that \( \hat{\omega} \in \hat{\Omega}_p^{(n)} \) for all \( \hat{\omega} \in \hat{\Omega}_p \) showing that the de Rham complex to NCG is the linear mapping \( \hat{\int} : \hat{\Omega}_p^{(n)} \to \mathbb{C} \), defined by

\[
\hat{\int} \hat{\omega} = \begin{cases} 
\int \omega & \text{for } \omega \in \Omega_d^{(n)} \\
0 & \text{otherwise}
\end{cases}
\]

and Stokes theorem holds, \( \int d\omega = 0 \) for all \( \omega \in \Omega_d \).

The theorem of the present paper shows that the natural generalization of this integration of de Rham forms to NCG is the linear mapping \( f : \hat{\Omega}_p \to \mathbb{C} \), defined by

\[
f \hat{\omega} = \begin{cases} 
\frac{1}{c_{2p}^{-1}} \text{Tr}_C(\Gamma^{2p} \hat{\omega}) & \text{for } \hat{\omega} \in \hat{\Omega}_p^{(n)}, n \geq 2p \\
0 & \text{otherwise}
\end{cases}
\]
where $\Gamma$ is a grading operator on $\mathcal{H}$ such that $\varepsilon \Gamma = -\Gamma \varepsilon$. For this also Stokes’ theorem holds (due to the cyclicity of trace), and under the homomorphism $\tilde{\omega} = \int c(\tilde{\omega})$ for all $\tilde{\omega} \in \hat{\Omega}'_p$.

As mentioned, it has been suggested that NCG should provide appropriate mathematical tools for formulating and studying quantum field theory models. A universal Yang–Mills theory in this spirit has been proposed and studied in $[R, FR]$. In these cases, the models were designed such that a formulation solely in terms of notions from NCG was possible. We believe that it would be extremely interesting to extend such an approach to Yang–Mills theories of the usual kind.

The theorem of the present paper allows such a formulation for topological Yang–Mills field theories (whose actions involve only integrals of de Rham forms and no Riemannian structure, e.g. Chern–Simons theories) by using the embedding of Yang–Mills field configurations $A \in \Omega^{(1)}$ in $\hat{\Omega}^{(1)}_p$ as discussed above. This should provide the first step to implement such a program for this class of models. To extend it to other Yang–Mills gauge theories would require a generalization of the Hodge–$\star$ operation to the non–commutative setting.

Appendix A: $\text{Tr}_\Lambda \left( [a, \hat{X}] \right)$

In this appendix we prove eq. (27). Using the notation from the proof of Lemma 3 (eq. (13) ff) we have

$$\text{Tr}_\Lambda \left( [a, \hat{X}] \right) = \int_{\mathcal{B}_A^d} dp \int_{\mathbb{R}^d} dq \text{tr} \left( K(a)(p, q) \hat{X}(q - p) - \hat{X}(p - q) K(a)(q, p) \right).$$

Using cyclicity of tr and changing variables to $Q = (-/+)(p - q)$ in the first/second term, we get

$$\text{Tr}_\Lambda \left( [a, \hat{X}] \right) = \int_{\mathcal{B}_A^d} dp \int_{\mathbb{R}^d} dq \text{tr} \left( (K(a)(p| - Q) - K(a)(p - Q| - Q)) \hat{X}(Q) \right) \quad (A1)$$

where we introduced

$$K(a)(p|Q) \equiv K(a)(p, p - Q).$$

As mentioned in the introduction, this integral of the same kind as the ones from Feynman diagrams giving anomalies (i.e. it would be zero for $\Lambda \to \infty$ if one could shift the integration variables, see e.g. $[J]$).

We now use Taylor’s expansion

$$K(a)(p| - Q) - K(a)(q - Q| - Q) = \sum_{i=1}^d Q_i \frac{\partial}{\partial p_i} K(a)(p| - Q) - \frac{1}{2} \sum_{i,j=1}^d Q_i Q_j \frac{\partial^2}{\partial p_i \partial p_j} K(a)(p - t| - Q)$$

where $t = t(p, Q)$ with $0 \leq t_i \leq Q_i$. Thus

$$\text{Tr}_\Lambda \left( [a, \hat{X}] \right) = \int_{\mathcal{B}_A^d} dp \int_{\mathbb{R}^d} dq \sum_{i=1}^d \text{tr} \left( \frac{\partial K(a)(p| - Q)}{\partial p_i} Q_i \hat{X}(Q) \right) + R_\Lambda.$$

---

3I am grateful to J. Mickelsson for pointing this out to me
Comparing (18) with (22) it is easy to see that the first term on the r.h.s. of this eq. is equal to the first term on the r.h.s. of (27). Moreover, using (28) we get

\[ R_\Lambda = -\frac{1}{2} \int_{B^d_A} d\rho \int_{\mathbb{R}^d} dQ \delta(\Lambda - |p|) \sum_{i,j=1}^d \text{tr} \left( \frac{p_i}{|p|} \frac{\partial K(a)(p - t - Q)}{\partial p_j} Q_j \tilde{X}(Q) \right), \]

hence \( R_\Lambda = O(\Lambda^{-1}) \) follows from \( K(a)(p - Q) = O(|p|^{-d+1}) \) and \( \tilde{X}(Q) = O(|Q|^{-\infty}) \).

**Appendix B: Volume of \( S^{d-1} \)**

To make this paper self-contained we give an elementary proof of

\[ \Omega_d = \int_{\mathbb{R}^d} d\xi \delta(1 - |\xi|) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad \text{(B1)} \]

(volume of \( S^{d-1} \)). We have

\[ \Omega_{d+1} = \int_{\mathbb{R}^d} d\xi \int_{\mathbb{R}} dy \delta \left( 1 - \sqrt{\xi^2 + y^2} \right) = \]
\[ \Omega_d \frac{1}{2} \int_{\mathbb{R}} d\rho |\rho|^{d-1} \int_{\mathbb{R}} dy \delta \left( 1 - \sqrt{\rho^2 + y^2} \right) = \]
\[ \Omega_d \frac{2}{\pi^{d/2}} \int_{0}^{\pi} d\varphi (\cos(\varphi))^{d-1} = \Omega_d \frac{\Gamma(1/2)\Gamma(d/2)}{\Gamma((d+1)/2)}, \]

(in the second line we introduced spherical coordinates with \( \rho = |\xi| \)), thus

\[ \Gamma \left( \frac{d+1}{2} \right) \Omega_{d+1} = \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{d}{2} \right) \Omega_d = \ldots = \Gamma \left( \frac{1}{2} \right)^{d/2} \Gamma \left( \frac{1}{2} \right) \Omega_1 \]

and (B1) follows with \( \Omega_1 = 2 \) and \( \Gamma \left( \frac{1}{2} \right) = \pi^{1/2} \).

**Acknowledgments**

I am grateful to G. Ferretti, H. Grosse, and S.G. Rajeev and especially J. Mickelsson for helpful discussions. I would like to thank the Erwin Schrödinger International Institute in Vienna for hospitality where part of this work was done. This work was supported in part by the "Österreichische Forschungsgemeinschaft" under contract Nr. 09/0019.

**References**

[CR] Carey A. L. and Ruijsenaars S. N. M.: On fermion gauge groups, current algebras and Kac-Moody algebras. *Acta Appl. Mat.* 10, 1 (1987).

[C1] Connes A.: Non–commutative differential geometry. *Publ. Math. IHES* 63, 257 (1985).

[C2] Connes A.: The action functional in non–commutative geometry. *Commun. Math. Phys.* 117, 117 (1988).

[CFNW] Cederwall M., Ferretti G., Nilsson B. E. W. and Westerberg A.: Schwinger Terms and Cohomology of Pseudodifferential Operators. [hep-th/9410016](http://arxiv.org/abs/hep-th/9410016).
[FS] Faddeev L. D.: Operator anomaly for the Gauss law. *Phys. Lett.* 145B, 81 (1984); Faddeev L. D. and Shatashvili S. L.: Algebraic and Hamiltonian methods in the theory of non-Abelian anomalies. *Theor. Math. Phys.* 60, 206 (1984).

[FR] Ferretti G. and Rajeev S. G.: Universal Dirac–Yang–Mills theory. *Phys. Lett.* 244B, 265 (1990).

[H] Hörmander L.: *The Analysis of Linear Partial Differential Operators III.* Springer-Verlag, Berlin (1985).

[J] Jackiw R.: Topological investigations of quantized gauge theories. in: *Relativity, Groups and Topology II*, Les Houghes 1983, DeWitt B. S. and Stora R (eds.), North Holland, Amsterdam (1984).

[JJ] Jackiw R. and Johnson K.: Anomalies of axial–vector current. *Phys. Rev.* 182, 1459 (1969).

[GL] Grosse H. and Langmann E.: The geometric phase and the Schwinger term in some models. *Int. Jour. of Mod. Phys.* 21, 5045 (1992).

[La] Langmann E.: Fermion current algebras and Schwinger terms in (3+1)–Dimensions. *Commun. Math. Phys.* 162, 1 (1994).

[LM] Langmann E., and Mickelsson J.: (3+1)-dimensional Schwinger terms and non-commutative geometry. *Phys. Lett.* B (in press), hep-th/9407193.

[LS] Langmann E. and Semenoff G. W.: QCD_{1+1} with massless quarks and gauge covariant Sugawara construction. *Phys. Lett.* B (in press), hep-th/9404153.

[Lu] Lundberg L.-E.: Quasi-free “second quantization”. *Commun. Math. Phys.* 50, 103 (1976).

[M1] Mickelsson J.: On a relation between massive Yang–Mills theories and dual string models. *Lett. Math. Phys.* 7, 45 (1983); Chiral anomalies in even and odd dimensions. *Commun. Math. Phys.* 97, 361 (1985).

[M2] Mickelsson J.: *Current Algebras and Groups.* Plenum Monographs in Nonlinear Physics, Plenum Press (1989).

[M3] Mickelsson J.: Current algebra representations for 3+1 dimensional Dirac-Yang-Mills theory. *Commun. Math. Phys.* 117, 261 (1988).

[M4] Mickelsson J.: Commutator anomalies and the Fock bundle. *Commun. Math. Phys.* 127, 285 (1990).

[MR] Mickelsson J., and Rajeev S. G.: Current algebras in d + 1-dimensions and determinant bundles over infinite dimensional Grassmannians. *Commun. Math. Phys.* 116, 400 (1988).

[PS] Pressly A., Segal G.: *Loop Groups.* Oxford Math. Monographs, Oxford (1986).

[RS] Reed R., and Simon B.: *Methods of Modern Mathematical Physics I. Functional Analysis,* Academic Press, New York (1968).
[R] Rajeev S. G.: Universal gauge theory. *Phys. Rev.* D42, 2779 (1990); *Phys. Lett.* 209B, 53 (1988); Embedding Yang–Mills theory into universal Yang–Mills theory. *Phys. Rev.* D44, 1836 (1991).

[S] Simon B.: *Trace Ideals and Their Applications.* Cambridge University Press, Cambridge, UK (1979).

[W] Wodzicki M.: Noncommutative Residue. in: *K–theory, arithmetic and geometry,* Yu. I. Manin (ed.), Lecture notes in Mathematics 1289, Springer–Verlag, Berlin (1985).