A STOCHASTIC MODEL OF MATHEMATICS AND SCIENCE

DAVID H. WOLPERT AND DAVID B. KINNEY

ABSTRACT. We introduce a framework that can be used to model both mathematics and human reasoning about mathematics. This framework involves stochastic mathematical systems (SMSs), which are stochastic processes that generate pairs of questions and associated answers (with no explicit referents). We use the SMS framework to define normative conditions for mathematical reasoning, by defining a "calibration" relation between a pair of SMSs. The first SMS is the human reasoner, and the second is an "oracle" SMS that can be interpreted as deciding whether the question-answer pairs of the reasoner SMS are valid. To ground thinking, we understand the answers to questions given by this oracle to be the answers that would be given by an SMS representing the entire mathematical community in the infinite long run of the process of asking and answering questions. We then introduce a slight extension of SMSs to allow us to model both the physical universe and human reasoning about the physical universe. We then define a slightly different calibration relation appropriate for the case of scientific reasoning. In this case the first SMS represents a human scientist predicting the outcome of future experiments, while the second SMS represents the physical universe in which the scientist is embedded, with the question-answer pairs of that SMS being specifications of the experiments that will occur and the outcome of those experiments, respectively. Next we derive conditions justifying two important patterns of inference in both mathematical and scientific reasoning: i) the practice of increasing one’s degree of belief in a claim as one observes increasingly many lines of evidence for that claim, and ii) abduction, the practice of inferring a claim’s probability of being correct from its explanatory power with respect to some other claim that is already taken to hold for independent reasons.

1. Introduction

In this paper, we introduce a framework that extends both conventional formulations of mathematics and conventional formulations of the structure of the physical universe. Because of how it extends those formulations, this new framework can also be used to represent (and investigate) human reasoning about mathematics, i.e., the reasoning of individual mathematicians, or the collective reasoning of communities of mathematicians. Similarly, our new framework can also be used to represent (and investigate) human reasoning about the structure of the physical universe, i.e., the reasoning of individual scientists, or the collective reasoning of communities of scientists.

The central motivation behind our new framework is simple: One of the defining features of mathematicians and of scientists is that at any given moment, they have what they perceive (rightly or wrongly) to be a set of "(almost) incontrovertibly true" propositions. (Examples of such propositions for mathematicians include accepted proofs, and examples for scientists include observations or the results of past experiments.) They then map those "true" propositions to beliefs concerning other propositions that they are uncertain about. That is, they use a corpus of settled facts to formulate beliefs about propositions that are not settled facts. Our goal is to formalize this map and start to investigate its formal properties. We are particularly interested in presenting a framework that
captures how the inputs to that map change over time as the mathematician / scientist conducts their research, i.e., how their set of established facts and their set of beliefs about non-settled facts evolve over time. In creating this framework this we make the following contributions:

(1) We introduce a very broad formalization of the set of accepted facts as well as the beliefs of mathematicians or scientists concerning propositions that they do not perceive as accepted fact. There is a long tradition within Bayesian epistemology of using probability theory to represent the degrees of belief that epistemic agents (including mathematicians and scientists) assign to propositions that are not in their corpus of accepted facts. Here, we embrace this probabilistic approach to the epistemological modeling of mathematicians and scientists. However, we do not want to limit ourselves by assuming that the settled facts and propositions of math and science are all represented in some single particular formal system — an assumption which would be patently false for any choice of such a formal system, since real mathematicians and scientists formulate the propositions they are investigating using a broad range of formal systems. To avoid that assumption, we formulate both settled facts and unsettled propositions as question-answer pairs, without restricting the formal system used to represent either questions or answers.

(2) In the real world, at any particular moment, any given mathematician or scientist has a set of multiple accepted facts, i.e., multiple question-answer pairs, and a set of multiple non-settled facts that they have beliefs about. Accordingly, we represent agents’ epistemic attitudes as conditional probability distributions over possible answers, conditioned on specific associated questions and a set of possibly multiple (established facts represented as) independent question-answer pairs, where both the questions and answers are elements of some arbitrary (and possibly infinite) associated space. For every question-answer pair that the mathematician / scientist considers to be accepted fact, the conditional probability of that answer given that question equals one. For all other pairs — all other question-answer pairs that they have beliefs about, but do not view as accepted facts — that conditional probability is less than one.

(3) To model how such distributions over sets of question-answer pairs evolve over time, we need to capture the fact that the reasoning of both mathematicians and scientists is stochastic, both in the questions that they choose to investigate and the answers they provide to those questions. Stated formally, such evolution is a stochastic process, which generates successive sets of question-answer pairs. We refer to such a stochastic process as a stochastic mathematical system (SMS). SMSs are the central element of our approach.

(4) Our interrogative approach to the probabilistic modeling of reasoning in mathematics and science allows us to capture several incontrovertible aspects of said reasoning in the real world. (Note that we make no claims about how that reasoning itself should take place.) First, it is a simple fact that for real-world mathematicians (or communities thereof) and real-world scientists (or communities thereof), if the probability distribution over answers to some questions is to be representationally accurate, then it cannot be a delta function. In both mathematics and science, there are questions that amount to open problems, such that even the most advanced practitioners are unsure of the correct answers. If you ask a
computer scientist (for example), “Does $P = NP$?” (one of the most famous open questions in all of mathematics and science), many will refuse to give a definitive answer. If instead you ask them, “What is the probability that $P = NP$?”, now almost all will be happy to provide a response. This is captured in the SMS framework by allowing answers to questions to specify probability distributions over answers to other questions. Thus, the SMS framework allows us to model both the essentially interrogative nature of mathematical and scientific reasoning and the inherent stochasticity of real human reasoners.

(5) We use the SMS framework to address the question of how to formalize the foundational nature of mathematics itself, as opposed to formalizing only the nature of mathematical reasoning. Our conceptual starting point is to take seriously the slogan that “mathematics is what mathematicians do.” This leads us to use the SMS framework to formalize mathematics itself. Using the SMS framework this way allows for the possibility that there is no unique answer to a given mathematical question, but instead a non-degenerate objective probability distribution over possible answers. It allows for the possibility that mathematics itself may inherently and irrevocably contain contradictions. This is in contrast to conventional formulations of mathematics in terms of formal systems, category theory, ZFC set theory, etc., in which one set of hypotheses either does or does not imply another set, with no role for stochasticity, and according to which “mathematical existence is freedom from contradiction” (Hilbert, 1928). In this respect, we build on work by Bueno and Colyvan (2011) and McCullough-Benner (2020); they argue that structuralist approaches in philosophy of mathematics (e.g., Shapiro 1997) are limited by their inability to represent the possibility that mathematics is inherently contradictory. However, our framework departs radically from previous work in the foundations of mathematics by holding that not only can mathematics be fundamentally contradictory, it can also be fundamentally stochastic, such that the answers to mathematical questions are fundamentally generated by sampling from a non-delta-function probability distribution. Thus, we use our framework to put forward a novel position in the foundations of mathematics.

(6) An important normative motivation for mathematicians is almost always that their probability distributions over answers to possible questions be very close to those that would arise if they had access to some oracle. To ground the reader’s thinking, one can follow Peirce (1878), and take the oracle to be the far-future community of all mathematicians, located anywhere in the universe. Many mathematicians would feel they have made a “good” prediction for the answer to a question if it is the same one that would be given by a far-future community. In what follows, we show how the SMS framework is able to formalize this crucial normative feature of mathematical reasoning by considering appropriate expected values of the divergence between the probability distribution of a mathematician SMS and that of mathematics itself. To do this, we first postulate that some “oracle” SMS stipulates what the answers are to arbitrary questions. We then further suppose that the normative goal of any mathematician is for the process that they use to answer to arbitrary mathematical questions to correspond to the greatest degree possible with the process used by that oracle. Specifically, we first postulate that both an individual mathematical reasoner and such an oracle can be represented as a separate SMS. We then define that mathematician-SMS to
be “calibrated” with the SMS representing the oracle if the expected accuracy of the answer that the mathematician gives to a particular question agrees with that of the oracle, up to a given threshold.

(7) Moving to the case of empirical science and the nature of the physical universe, our approach is similar to the one that we use to model mathematical reasoning and the “mathematical universe”, but with some important differences. As is the case with mathematicians and mathematics, we use the SMS framework to formulate humans reasoning stochastically about the physical universe. Similarly, we can also use the SMS framework to formulate stochastic models of the physical universe itself. At one level, this allows us to capture simple truth, that physical facts about the future, governed the present, are random. As an example, conditioned on the data a given climate scientist has (their set of “established facts”), the precise global temperature five years from now is governed by a probability distribution. But modeling the physical universe using the SMS framework allows us to go further than this, to entertain the ontological thesis that the physical universe “is” an SMS, as opposed to the epistemological theses just predictions about some of physical variables conditioned on knowledge of other physical variables are necessarily uncertain. In this respect, our approach has affinities with ontic structural realist approaches in philosophy of science (e.g., Ladyman and French 1998; Ladyman 2007), as well as arguments from theoretical physics that the universe is inherently mathematical (e.g., Barrow 1991, 2011; Schmidhuber 1997; Tegmark 1998, 2008, 2014). However, the SMS framework allows us to represent different ways in which nature might be fundamentally stochastic with a high degree of generality. For example (and as we will show in what follows), our framework allows us represent the possibility that the events of the universe are stochastic (as in some interpretations of quantum mechanics, for example). However, it also allow us to investigate the implications if the laws of the physical universe themselves are stochastic. This includes as a special case the possibility, widely entertained in modern cosmology, that the physical constants in the laws of physics vary randomly across the universe.

(8) As in the mathematical case, we use the SMS framework to formalize the normative aims of physical reasoning, specifically the idea that the answers given by some scientists to some questions are “physically correct.” To do this we first suppose that the human scientist and the physical universe are SMSs. An extra complication in this case, not present in the case of mathematics, is that such scientists are themselves physical systems. So formally, they are answers to certain questions posed to the very physical universe-SMS about which those self-same human scientists are reasoning. To capture this we define a special “embedding” relationship that might hold between the answers of the scientist-SMS and those of the embedding universe-SMS. (This can also be viewed as a way to provide a “physical meaning” to the answers generated by the scientist-SMS.) We then define a scientist-SMS to be “embed-calibrated” with the universe-SMS in which they are embedded if the expected accuracy of the scientist’s answers to some question(s) is the same as that of the embedding physical

1Note that in this regard, formulating the physical universe as having inherent stochasticity might make more sense than formulating mathematics that way.
universe, up to some threshold. This allows us to represent accurate physical scientists in much the same way that we represent accurate mathematicians; the embedding physical universe plays an analogous role for scientists as the far-future community of mathematicians plays for mathematicians. (As elaborated below, ultimately the difference between the definitions of mathematical accuracy and scientific accuracy is due to the fact that scientists are embedded in the SMS about which they are asking questions, whereas mathematicians ask questions about a wholly independent SMS.)

(9) Formalizing these normative constraints on mathematical and scientific reasoning leads to one of our primary goals in this paper, which is to provide conditions that vindicate the use two types of heuristic reasoning commonly used by real world, human researchers. Both of these heuristics have resisted straightforward vindication within the traditional frameworks of Bayesian epistemology:

(a) The first heuristic involves assigning greater degree of belief to a hypothesis when that hypothesis is supported by multiple lines of evidence. That this heuristic is normatively justified is sometimes called the “variety of evidence thesis.” Historically (and particularly in light of results due to Bovens and Hartmann, 2002), this heuristic has been viewed as incompatible with Bayesian epistemology, at least if the heuristic is to be vindicated as a general maxim of epistemic rationality. Indeed, Landes (2020b) declares that “the Bayesian quest for a vindication of the [value of evidence thesis] in full generality has failed” (p. 185). Such claims are made in the context of a traditional Bayesian epistemology. By contrast, we show that on our interrogative approach to modeling both descriptive and normative aspects of reasoning, a highly general version of the variety of evidence thesis holds for both scientific and mathematical reasoning.

(b) The second heuristic is abduction, i.e., the inference from the explanatory power of a claim to its probability of being correct. Here too, there is a widespread perception that this heuristic is not fully consistent with Bayesian epistemology. Indeed, in his recent book-length treatment of abduction, Douven (2022) states that his goal is to convince the reader “that abduction is interestingly different from Bayes’ rule, and that there is nothing intrinsically untoward about it” (p. 25). By ‘Bayes’ rule’ here, Douven implicitly invokes a version of Bayesian epistemology that lacks the interrogative structure of the SMS framework. In what follows, we show that within the SMS framework, one can provide a highly general vindication of abduction that is wholly consistent with the use of probability theory, and only probability theory, to represent the degree to which agents believe a given proposition. Moreover, this defense of abduction does not require that an agent adopt a specific, putatively correct prior as some Bayesian approaches (e.g., Weisberg, 2009) do. Instead, we impose on the weaker constraint that the agent be calibrated with some arbiter of normatively justified reasoning. As discussed above, this crucial notion of calibration is defined in formal detail below.

We take these contributions to demonstrate the fecundity of our approach within philosophy of science and formal epistemology.
1.1. **Roadmap.** In Section 2, we first provide a minimal description of SMSs, which will serve as the basis for the analysis in this paper. (See Appendix A for a more complete, fully formal set of definitions, which may be useful in other contexts.) Next, in Section 3, we introduce a set of distributions concerning a given SMS that play a central role in our analysis. In Section 4 we use these distributions to formally define the notion of calibration between two SMSs. The material in this section is presented in terms of human mathematicians who cohere (or don’t) with a putative far-future community of mathematicians. In the next section, we present similar definitions for embed-calibration, which describes how a scientist can cohere (or not) with the physical universe that they are both embedded in and that they are predicting. Then in Section 5 we describe how calibration can be related to various proposed notions of (mathematical) truth.

Following these formal preliminaries, in Section 7 we present our first application of the SMS framework, to show that if an SMS (set of human mathematical reasoners) is calibrated with a far-future community of mathematicians, then having multiple lines of evidence for a mathematical hypothesis makes that hypothesis more likely. (All proofs can be found in Appendix B). We also show in that section that the analogous result holds for scientists who have multiple lines of evidence for a hypothesis about the outcome of a future experiment.

Next, in Section 8 we present our second application of the SMS framework, to show that if an SMS (set of human mathematical reasoners) is calibrated with a far-future community of mathematicians, then abductive reasoning can be applied to mathematical propositions that support one another. We also show in that section that the analogous result holds for scientists who use abductive reasoning concerning the outcome of an experiment that has already happened but whose result they did not observe. After that, in Section 9 we describe how SMSs are related to other frameworks considered in earlier literature. We end with a discussion of other applications of the SMS framework that we intend to pursue in future work.

As a final comment, it is important to emphasize that we do not claim that our approach can capture every possible vagary of mathematical and scientific reasoning. We do not even claim to capture every aspect of human reasoning that can be formulated as generating sequences of asked and answered questions. Rather, we aim to provide a compelling model that is useful in novel ways.

2. **Stochastic Mathematical Systems**

2.1. **Fundamentals.** To have the SMS framework be as broadly applicable as possible, we impose no *a priori* restrictions on the space of possible “mathematical questions” or the space of “associated answers.” For example, a question could specify a particular formal system as well as a well-formed formula (WFF) in that formal system, and ask whether that formula is a theorem of that system. In this case the answer would be a bit. As another example, a question could specify only the form of a desired solution to a mathematical problem, so that there are many possible answers. For example, the question could be of the form, ‘what is an equation giving the roots of any single-real-variable quartic polynomial?’ Alternatively, a question could be as prosaic as ‘what is $485 + 923874$?’ Our framework can be applied with any of these types of questions.

We start with the following definitions:
Definition 2.1. The elements $q$ of an arbitrary set $Q$ are known as questions. The elements $v$ of an arbitrary set $V$ are known as answers.

As mentioned above, we impose no restrictions on $Q$ or $V$. In particular, we impose no syntactical restrictions; there is no need to specify a set of rules for what constitutes a well-formed question or answer in our formalism. We also do not require either $Q$ or $V$ to be countable. However, to ground thinking and to relate the framework here to the similar framework based on probabilistic Turing machines investigated in Wolpert and Kinney (2020), it may benefit the reader to think of the case where both $Q$ and $V$ are the set of all finite bit strings.

Given a set of questions $Q$ and a set of answers $V$, we make the following associated definitions:

Definition 2.2. A claim $c$ is an arbitrary element of $C = \{(q, v) : q \in Q, v \in V\}$.

Definition 2.3. A claim vector $C$ is a finite sequence of claims.

The set of all claim vectors is written as $C^*$. We make analogous definitions for question vectors, answer vectors, etc.

Definition 2.4. A claim set $\hat{C}$ is a finite (unordered) set of claims. The collection of all claim sets is written as $\hat{C}$.

We suppose that claim vectors are iteratively generated via a discrete stochastic process, which we formalize with the following pair of definitions:

Definition 2.5. A claim vector probability space is a probability space $\mathcal{E} = (\Omega_\mathcal{E}, \Sigma_\mathcal{E}, P_\mathcal{E})$ where $\Omega_\mathcal{E}, \Sigma_\mathcal{E}$ and $P_\mathcal{E}$ are the event space, sigma-field, and probability measure, respectively.

Definition 2.6. A stochastic mathematical system (SMS) $\varphi$ is a pair $(\mathcal{E}, X)$ where $\mathcal{E}$ is a claim vector probability space and the measurable function $X : \mathbb{Z}^+ \times \Omega_\mathcal{E} \to C^*$ specifies a sequence of random variables (i.e., of measurable functions) taking values in $C^*$.

We refer to the integer argument of $X(.)$ as the step of the SMS. We use $P_\mathcal{E}$ to indicate probabilities of events under $(\mathcal{E}, X)$, and use superscripts to indicate the step, e.g., writing $P^n_\mathcal{E}(C)$ for the probability that the SMS $(\mathcal{E}, X)$ produces the claim vector $C$ when the step is $n$. Sometimes we will abuse notation and write for example $X(n)$ for some integer $n$ to mean a sample of $P^n_\mathcal{E}$. Sometimes we will leave the precise specification of the SMS $\varphi$ implicit, and so for example just write $P^n(C)$. For simplicity, we assume that any SMS has zero probability of ever producing the empty set claim vector. We also assume there is zero probability of a claim vector that has multiple copies of the same claim. See Fig. 1 for a visual representation of the process whereby an SMS produces claims.

Whenever we (implicitly) invoke a topology on any of the sets $\mathcal{E}, Q, V, C^*, Q^*, V^*$ (e.g., when discussing measures defined over such sets), that topology is assumed from context. In particular, for countable $\mathcal{E}$ we assume the associated discrete topology.

We can illustrate these definitions with some simple examples:

1. Choose $Q$ to be all well-formed formulas (WFFs) in some formal system, and choose $V = \mathbb{B}$. Then we could interpret the claim $(q, 1)$ to mean that $q$ is a theorem and $(q, 0)$ to mean that $q$ is not a theorem. The SMS could be a function that iteratively produces the decidable
Figure 1. A directed graph showing several of the possible evolutions of the claims that are output by an SMS. Labels on arrows show transition probabilities from each claim to the next, which are determined by the probability space $\mathcal{C}$ of the SMS.

WFFs in the formal system, along with the bit of whether they are (not) a theorem, and adds those claims to the set of all previously decided claims to produce a new, larger claim set.

(2) Fix some universal Turing machine (TM) $T$, and write $T(p)$ for the (perhaps undefined) computation of $T$ starting with bit string $p \in \mathbb{B}^*$. Again choose $Q$ to be all WFFs in some formal system, but now choose $V = \mathbb{B}^*$. Then we could interpret the claim $(q, v)$ to mean that whether $q$ is a theorem or not is given by $T(v)$. In this case a given answer $v$ paired with a question $q$ can be viewed as a “translation” of the WFF $q$ into an input string to the TM. Alternatively, viewing $T$ as a (partial) function, $v$ can be viewed as an encoded version of the bit of whether $q$ is or is not a theorem of the formal system. Whatever interpretation we adopt, as in the previous example the SMS could be a function that iteratively produces the decidable WFFs in the formal system, along with the bit of whether they are theorems, and adds those claims to the set of all previously decided claims to produce a new, larger claim set.

(3) Using some appropriate (and somewhat arbitrary) definition of what it means for a specific question-answer pair to be “commonly accepted” by the global community of human mathematicians, we can view that community as having produced a sequence of sets of commonly accepted claims concerning mathematical questions since 1900. Each of those claims can be formulated as a question-answer pair, and the successive sets of commonly accepted mathematical claims produced by the community is a claim vector. Note that there has been randomness in which precise questions have interested the community of human mathematicians through time. This can be captured in an SMS model of the community of human
mathematicians by introducing stochasticity with respect to which questions occur in each successive claim vector produced by that SMS.\footnote{Of course, the community of mathematicians did much else besides produce sets of claims in those years, e.g., they produced working hypotheses, decided among formalisms to use to address a topic, etc. Those other aspects of the behavior of that community do not concern us here; we are only concerned with the end products of their behavior, so to speak.}

In what follows, we will not need to consider claim vectors directly, instead only directly considering sets of claims. However, distributions over such sets of claims are defined in terms of distributions over claim vectors. Consistently with Definition\footnote{2} we use a hat above a multi-component vector to indicate we are considering the un-ordered rather than ordered version of that vector. To make this precise we define the \textbf{un-ordering} function $U : \mathcal{C}^* \rightarrow \hat{\mathcal{C}}$ for any $C \in \mathcal{C}^*$ by requiring that every element of $U(C)$ is a component of $C$, and vice-versa.

We define a claim vector $C$ to be (non-)repeating if it does (not) contain two claims that have the same question. We say that an SMS is non-repeating after step $k$ if with probability 1 it only produces non-repeating claim vectors after step $k$. As an example, if the SMS of the community of mathematicians is non-repeating after step $k$, then in all such steps, there might be hidden contradictions lurking in the set of all claims currently accepted by mathematicians, but there are not any explicit contradictions, since no single question arises more than once in a claim vector made at any step after $k$, and so cannot be given two different answers in the same step. Though we do not need to assume an SMS is non-repeating in most of what follows, there is no loss of generality if one makes this assumption. In addition, in almost all uses of an SMS to model reasoning, one would expect that SMS to be non-repeating after a certain step $k$.

3. Important Distributions in the SMS Framework

3.1. Notation for distributions over claim sets. Often we will not be concerned with the probability that at some step $n$ an SMS outputs a particular claim vector $C$, but rather with the probability that at some step $n$ the SMS outputs any vector $C$ such that its unordering $U(C)$ contains some particular $\hat{C}$ as a subset. It may even be that the probability of the SMS producing a $C$ such that $U(C) = \hat{C}$ is zero, even though the probability of a $C$ such that $\hat{C} \subset U(C)$ is nonzero.

As an example, suppose that some mathematician has nonzero probability of producing some set $\hat{B}'$ of multiple theorems (i.e., of multiple question-answer pairs) at some specific timestep $n$, where $\hat{B} \subset \hat{B}'$. It may be that nonetheless, the SMS has zero probability of producing $\hat{B}$ at that time step. (Intuitively speaking, the theorems comprising $\hat{B}'$ can only be derived all at once, as a package.) Despite this specific character of that SMS, to analyze that SMS we will sometimes be interested in distributions conditioned only on the theorems in $\hat{B}$, not on the set of all of the theorems in $\hat{B}'$.

Phrased formally (with some abuse of notation), we must distinguish the probability distribution

\begin{equation}
P^n(\hat{C}) := \sum_{C' \in \mathcal{C}^* : \hat{C} \subset U(C')} P^n(C')
\end{equation}

from

\begin{equation}
P^n(\hat{C}) := \sum_{C' \in \mathcal{C}^* : \hat{C} \subset U(C')} P^n(C')
\end{equation}

\footnote{2}
The first distribution takes as its argument the claim set \( \hat{C} \), and returns the probability that the (implicit) SMS produces, at step \( n \), any claim vector whose unordering is the set \( \hat{C} \). By contrast, the second distribution takes as its argument the same claim set \( \hat{C} \), and returns the probability that at step \( n \), the SMS produces any claim vector whose unordering is a superset of the set \( \hat{C} \). So,

\[
P^n(\hat{C}) = \sum_{\hat{C}' : \hat{C} \subseteq \hat{C}'} P^n(\hat{C}') \geq P^n(\hat{C})
\]

The proofs below use the distribution \( P^n(\hat{C}) \), not \( P^n(\hat{C}') \). This reflects the fact that the associated results do not concern whether an SMS produces a particular set of question-answer outputs, but rather whether it produces any set that contains a particular set as a subset. See Appendix A.2 for some further remarks concerning the distribution \( P^n \).

3.2. Response Distributions. In much of what follows, it will be useful to consider the probability that, in its \( n \)'th step, an SMS produces a claim set that includes both a particular claim \((q, v)\) and a particular set of claims \( \hat{C} \). Abusing notation, we write this probability as follows:

\[
P^n((q, v), \hat{C}) := P^n((q, v) \cup \hat{C}) = \sum_{\{C' \in C' : \hat{C} \cup \{(q, v)\} \subseteq U(C')\}} P^n(C')
\]

Similarly, we again abuse notation to write

\[
P^n(q, \hat{C}) := \sum_{\{C' \in C' : v \in V : \hat{C} \cup \{(q, v)\} \subseteq U(C')\}} P^n(C')
\]

\( P^n(q, \hat{C}) \) and \( P^n((q, v), \hat{C}) \) are not properly normalized distributions over their arguments, in general.\(^3\) We refer to \( P^n(q, \hat{C}) \) as a question semi-distribution for a question \( q \) and claim set \( \hat{C} \) and step \( n \). is the probability that at step \( n \), an (implicit) SMS outputs a set that contains both the claim set \( \hat{C} \) and a claim that contains the question \( q \)\(^4\).

Combining these two definitions, we arrive at the following:

**Definition 3.1.** For an SMS \( \varphi \), step \( n \), question \( q \), and claim set \( \hat{C} \) where \( P^n(q, \hat{C}) \neq 0 \), the associated response distribution is the map from all \( v \in V \) to the value

\[
P^n(v | q, \hat{C}) = \frac{P^n((q, v), \hat{C})}{P^n(q, \hat{C})}.
\]

By comparing Eqs. (3.4) and (3.5), we see that \( P^n(v | q, \hat{C}) \) is a properly normalized distribution over \( v \in V \) (assuming \( P^n(q, \hat{C}) \neq 0 \)). Intuitively, we can interpret it as the distribution over possible answers to a question \( q \) that might be given by a particular SMS at a step \( n \) in response to a question \( q \), given that the SMS also produces a claim set that contains \( \hat{C} \) at step \( n \). We generalize the notation in Definition 3.1 in the obvious way: any probability we write of an indexed set of \( m \) answers conditioned on an indexed set of \( m \) questions (perhaps also conditioned on a claim set) is defined in terms of \( m \) claims that are “broken in two”, with the answer components of those claims on one side of the conditioning bar and the associated question components on the other.

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3This is due to the fact that each \( \hat{C} \) can occur in more than one claim vector \( C' \), and so summing over all \( \hat{C} \) means we are double-counting vectors \( C' \), in general.

4Note that this differs from \( P^n(\{(q, v) \cup \hat{C} : v \in V\}) \), which is the probability that the SMS outputs a claim set that contains \( \hat{C} \) and also contains every claim of the form \( (q, v) \) for some \( v \in V \).
We extend these definitions to the case of multiple claims in addition to \( \hat{C} \) in the obvious way, e.g.,

\[
P^n((q_1, v_1), \ldots, (q_m, v_m), \hat{C}) = \sum_{\{C' \in C^* : \hat{C} \cup \{(q_1, v_1), \ldots, (q_m, v_m)\} \subseteq U(C')\}} P^n(C')
\]

and

\[
P^n(v_1, \ldots, v_m | q_1, \ldots, q_m, \hat{C}) = \frac{P^n((q_1, v_1), \ldots, (q_m, v_m), \hat{C})}{P^n(q_1, \ldots, q_m, \hat{C})}
\]

(Note that in this expression the \( m \) questions and answers are indexed, in order to “match up” with one another, while the questions and answers in \( \hat{C} \) are not.)

Finally, we identify a special class of response distributions of a given question and claim set:

**Definition 3.2.** For an SMS \((C, X)\), step \( n \), question \( q \), and claim set \( \hat{C} \) where \( P^n(q, \hat{C}) \neq 0 \), \( P^n(v | q, \hat{C}) \) is sure if and only if it is a delta function about some answer \( v \).

To illustrate these definitions, consider the abstract model of a real-word human mathematician as an SMS. In a given step the mathematician may carry forward all the question-answer pairs of the claim vector they produced at the previous step, as the first claims in their new claim vector. They may then construct one or more new questions, and then consider those questions, ending the step by (perhaps) producing answers. Prior to such consideration of a new question, the human mathematician may have beliefs about the possible answers to any single such question. We can interpret the response distribution of the mathematician to that question at the current step (conditioned on the previous claim vectors) as precisely such a set of beliefs. For some questions (e.g., ‘what is the value of \( 2 + 2 \) according to the axioms of Peano arithmetic?’), this response distribution may be sure. For other questions, the mathematician’s response distribution will likely not be sure. As an example, consider a human being who aims to evaluate a difficult definite integral using solely pen and paper. That is, they are posed the question, ‘what is your best guess for the value of \( \int_a^b f(x)dx \)?’, where \( f(x) \) is a highly complicated function. Even with significant mathematical training, if the stochastic process in which the question is posed is to represent an actual mathematical reasoner, then the response distribution of possible answers to that question is likely unsure, at least at some step in the SMS process of that reasoner.

3.3. **Backward-Consistent SMSs and the Limit Distribution.** The infinite-step limit of certain types of SMS will play a key role in our analysis. This role is motivated by using an SMS to model the generation of the commonly accepted body of mathematics by the community of human mathematicians. It seems very unlikely that the actual community of human mathematicians would produce a claim that is universally agreed to have been proven, and so included in the commonly accepted body of mathematics, but later on that community determines that that precise claim is wrong, and so removes it from the commonly accepted body of mathematics. To be clear, we do not mean to say that mathematicians do not rely on assumptions that are later found not to hold. Rather, we hold that at an certain level of being well-posed, the answers to mathematical questions are rarely overturned. To illustrate, the parallel line postulate was long thought to be an axiom of geometry in a very general way, until the discovery of non-Euclidean
geometry. However, that discovery did not lead mathematicians to revise the answers to any well-posed questions about whether a given proposition follows or does not follow from the parallel line postulate. Going forward, we will assume that mathematical questions are well-posed in just such a way that retrospective revision is unlikely, even when axioms presupposed in the posing of said questions are ultimately found not to hold in full generality.

We formalize this property by saying that an SMS is **backward-consistent** after step $\kappa$ if for all steps $j > \kappa$, there is zero probability of the SMS producing a sequence of claim vectors $(X(1) = C^1, \ldots, X(n) = C^n)$ such that for some $i$ where $j \geq i > \kappa$, $U(C^i) \notin U(C^j)$. If an SMS is backward-consistent after step $\kappa$, then at any step after $\kappa$, with probability 1 the claim set produced by the SMS includes the claim set of the previous step. When the precise value of $\kappa$ does not matter, we will sometimes say simply that an SMS is “backward-consistent” without specifying $\kappa$.

The following important property of backward-consistent SMSs is proven in the appendix:

**Lemma 3.3.** For any SMS that is backward-consistent and any claim set $\hat{C}$, $\lim_{n \to \infty} P^n(\hat{C})$ is well-defined.

We refer to $\lim_{n \to \infty} P^n(\hat{C})$ as the **limit distribution** of the SMS, and as shorthand write it as $P(\hat{C})$. The limit distribution of an SMS defines associated limit question semi-distributions and limit response distributions of an answer $v$ to a question $q$. For any $m > 1$, we define limit response distributions over $m$-tuples of answers conditioned on $m$-tuples of questions similarly.

### 4. Calibration Between SMSs

Much of our analysis below does not concern a single SMS, but rather the relationship between different SMSs. In particular, it is possible that the answer that one SMS gives to a particular question can be accurately interpreted as a specification of the probability distribution over possible answers to a different question. As a concrete example, consider two human reasoners: Alice and Bob. One can think of both Alice and Bob as SMSs, which we label $\varphi_1$ and $\varphi_2$, respectively. Bob (the SMS $\varphi_2$) might consider the question ‘what is the probability distribution over possible answers that Alice might give to the question $q_1$?’, where $q_1$ is a question in Alice’s question set. Alternatively, instead of representing the human reasoner Alice, the SMS $\varphi_1$ might represent the physical universe itself. If Bob is a scientist, he might ask the question ‘what is the probability distribution over possible outcomes of an experiment specified as $q_1$?’ — in this special case, since the SMS $\varphi_1$ represents the physical universe, the answer that $\varphi_1$ gives to $q_1$ is the result of the experiment.

In both of these kinds of cases, it is natural to begin thinking not just descriptively, but also normatively. For example, we might want to know what it means for Bob to give accurate answers about the probability distribution over possible answers that Alice might output in response to the question $q_1$. Moreover, we may be interested in this kind of accuracy in the relationship between two SMSs when $\varphi_2$ and $\varphi_1$ have both already generated some superset of a claim set $\hat{C}$. We will refer to such relationships by saying that the SMS $\varphi_2$ is “calibrated” at step $n$ with the SMS $\varphi_1$ for

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5Note that it is important that our framework allows $\varphi_2$ to make predictions for distributions over the answers of $\varphi_1$, not just for the answer directly. After all, scientists almost never directly give a single prediction for the outcome of an experiment, but rather give distributions over such outcomes.
a given question \( q \), answer \( v \), and claim set \( \hat{C} \). To illustrate, for some claim set \( \hat{C} \), \( \varphi_2 \) and \( \varphi_1 \) are calibrated for the \( \varphi_1 \) question ‘what is the value of \( 1 + 1 \)’ if, conditioned on both \( \varphi_1 \) and \( \varphi_2 \) having generated the claims in \( \hat{C} \) (perhaps along with other claims), the answer that \( \varphi_2 \) provides at step \( n \) to the question ‘what is the probability that \( \varphi_1 \) answers \( z \) in response to the question, ‘what does \( 1 + 1 \) equal?’’ is close to the actual probability distribution over answers that are outputted by \( \varphi_1 \) in response to the question, ‘what does \( 1 + 1 \) equal?’.

We will define calibration broadly enough to capture the relationship between any two SMSs. However, in the current section we will be concerned with SMSs engaged in explicitly mathematical (rather than physical) reasoning. Moreover, we will typically be concerned with calibration between two specific kinds of SMSs. The first of these SMSs will be the entire far-future community of mathematical reasoners, denoted \( \varphi_1 \). The second SMS will represent a finite (possibly singleton) group of mathematical reasoners, denoted \( \varphi_2 \). We will typically be interested in the specific case where the probability distribution over answers output by \( \varphi_1 \) is defined in terms the limit distribution of \( \varphi_1 \), i.e., the distribution over answers to mathematical questions that will be generated in the infinite long run of mathematical practice. A present-day mathematician \( \varphi_2 \) is calibrated with that far-future community \( \varphi_1 \) (for a particular question and current claim set of the current mathematician) if with high probability, that current mathematician assigns approximately the same probability to the possible answers to the question as does the community of mathematicians under its limit distribution.

4.1. Formal definition of Calibration. To make the concept of calibration fully formal, first let \( \psi : Q_2 \to Q_1^m \) be a partial function from the question set of the SMS \( \varphi_2 \) to the set of all finite vectors of questions that can be asked by the SMS \( \varphi_1 \). Intuitively, we will interpret any question \( q \in \text{Dom } \psi \subseteq Q_2 \) as asking ‘what is the joint distribution assigned to the possible answer(s) to the question(s) specified in \( \psi(q) \), under the distribution of \( \varphi_1 \)?’. We indicate the number of components of \( \psi(q) \) for any specific \( q \in \text{Dom } \psi \) by \( |\psi(q)| \).

Next, let \( \Psi \) be a partial function mapping \( Q_1^m \times V_2 \) for all positive integers \( m \) into the associated space of distributions over \( V_1^m \), i.e., the space of distributions over the set of all vectors of \( m \) answers that might be output by \( \varphi_1 \). Intuitively, we interpret any \( \Psi(\psi(q), v) \) as the specification of a particular joint probability distribution over possible answers to the question(s) specified in \( \psi(q) \). So, when a given SMS \( \varphi_2 \) considers the question ‘what is the probability distribution over possible answer(s) to the question(s) specified in \( \psi(q) \)?’ and outputs the answer \( v \in V_2 \), the function \( \Psi \) translates this answer into a probability distribution over the set of possible answers to the question(s) in \( \psi(q) \) that might be given by the other SMS, \( \varphi_1 \).

Note that in general, for any question \( q \in Q_2 \) that we want to interpret as asking for a distribution over answers to questions \( \psi(q) \in V_1^m \), there may be a non-delta function probability distribution over the possible associated answers of the SMS \( \varphi_2 \). In other words, there is more than one possible distribution over answers to \( \psi(q) \) that the SMS \( \varphi_2 \) might specify. To illustrate, an SMS may be a science journalist, able to generate questions of the form, ‘what is the probability distribution over possible answers by a randomly chosen mathematician from the current community to the question

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6As a technical note, we allow there to be values \( m \) such that the partial function \( \Psi \) is undefined for all arguments that involve \( m \) elements of \( Q_1 \).
can be asked by those of the far-future community of mathematicians. The reason for this requirement is to ensure mathematicians, and similarly for the answers that can be given by the current mathematician and any question a current mathematician might ask can also be asked by the far-future community of mathematicians. Defined, it must be that

Note that there are some implicit requirements in this definition, e.g., for a partial function

Suppose we are given two SMSs

Definition 4.1. Suppose we are given two SMSs \( \varphi_1 \) and \( \varphi_2 \) where \( \hat{C}_2 \subseteq \hat{C}_1 \). Suppose we are also given a partial function \( \psi : Q_2 \rightarrow Q_1^* \), and a partial function \( \Psi \) that maps \( Q_1^m \times V_2 \) into the associated space of distributions over \( V_1^n \) for any positive integer \( m \). Let \( n^1, n^2 \) be two positive integers. A pair of a question \( q \in Q_2 \), and a claim set \( \hat{C} \in \hat{C}_2 \), is a prediction pair (for \( \varphi_1, \varphi_2, \psi, \Psi, n^1 \) and \( n^2 \)) if

1. \( P^n_2(v | q, \hat{C}) \) is well-defined;
2. \( q \in \text{Dom} \psi \);
3. For all \( v \in V_2 \) such that \( P^n_2(v | q, \hat{C}) \) is nonzero, \( (\psi(q), v) \in \text{Dom} \Psi(., ..) \);
4. \( P^n_1(\psi(q) | \psi(q), \hat{C}) \) is a well-defined distribution over \( V_1^{\psi(q)} \),

where we use subscripts to distinguish the distributions of the two SMSs.

Note that there are some implicit requirements in this definition, e.g., for \( P^n_2(v | q, \hat{C}) \) to be well-defined, it must be that \( P^n_2(q, \hat{C}) \neq 0 \).

The requirement in Definition 4.1 that \( \hat{C}_2 \subseteq \hat{C}_1 \) means that any question that can be asked by \( \varphi_2 \) can be asked by \( \varphi_1 \) and every answer that can be given by \( \varphi_2 \) can be given by \( \varphi_1 \). So for example, any question a current mathematician might ask can also be asked by the far-future community of mathematicians, and similarly for the answers that can be given by the current mathematician and those of the far-future community of mathematicians. The reason for this requirement is to ensure
that both \( \varphi_1 \) and \( \varphi_2 \) can be conditioned on the same claim set \( \hat{C} \), e.g., so that the set of claims accepted by the far-future community of mathematicians includes those accepted by the current mathematician.

For any two probability distributions \( p, r \) both defined over the same (sigma algebra with) event set \( Y \), we write \( D[p(Y), r(Y)] \) for some convex divergence measure quantifying the difference between the distributions \( p \) and \( r \). (The canonical example is the Kullback-Leibler divergence; see Cover and Thomas (2012).) Furthermore, for any \( m \in \mathbb{Z}^+ \), we write \( Y^m \) for the event space of all \( m \)-tuples of elements of that set \( Y \).

Given this notation, we define the calibration relation between an SMS \( \varphi_2 \) and an SMS \( \varphi_1 \) as follows:

**Definition 4.2.** Let \( \varphi_1, \varphi_2 \) be two SMSs where \( \hat{C}_2 \subseteq \hat{C}_1 \), with associated partial functions \( \psi \) and \( \Psi \) that have domains and ranges as described in Definition 4.1. Let \( n^2 \) be an integer, \( \hat{C} \) a claim set in \( \hat{C}_2 \), and \( q \) a question in \( Q_2 \). We say that \( \varphi_2 \) is calibrated (with \( \varphi_1 \)) at steps \( n^1, n^2 \) for the pair \((q, \hat{C})\) and for \( \epsilon \geq 0 \) iff

1. \((q, \hat{C})\) is a prediction pair (for \( n^1, n^2 \));
2. \( \sum_{v \in \mathbb{V}_2} \mathbb{P}^2(v(q, \hat{C})D \left[ \Psi(\psi(q), v)(\mathcal{V}^m_{n^1}) \right| P_1^{n^1}(\mathcal{V}^m_{n^1} | \psi(q), \hat{C})] \leq \epsilon \)

where \( m \) is shorthand for \(|\psi(q)|\), and we use subscripts to distinguish whether quantities are defined with respect to SMS \( \varphi_1 \) or \( \varphi_2 \).

In practice, we will usually leave \( \epsilon \) implicit when discussing calibration. Note that calibration is not a symmetric relation; for any \( \epsilon \), if \( \varphi_2 \) is calibrated with \( \varphi_1 \), then it is not necessarily the case that \( \varphi_1 \) is calibrated with \( \varphi_2 \). Note also that for \((q, \hat{C})\) to be a prediction pair, it must be that both \( P_2^{n^2}(q, \hat{C}) \neq 0 \) and \( P_1^{n^1}(\psi(q), \hat{C}) \neq 0 \).

The first condition in Definition 4.2 means that for any answer \( v \) that \( \varphi_2 \) assigns with positive probability to question \( q \) given the claim set \( \hat{C} \), \( \Psi(\psi(q), v) \) is a probability distribution over the set of all vectors of \(|\psi(q)|\) answers that \( \varphi_1 \) can generate. From now on we restrict attention to the special case of Definition 4.2 where \( \varphi_1 \) is backward-consistent and \( n^1 = \infty \). In this case we write \( \mathcal{T}_1 \) instead of \( P_1^{n_1} \) in the fourth requirement of Definition 4.1 and write \( n \) instead of \( n^2 \). Under these conditions, the second condition in Definition 4.2 says that, in expectation conditioned on \( \hat{C} \), the SMS \( \varphi_2 \) generates answers to questions of the form “what is the probability distribution over vectors of \( m \) answers to the associated \( m \) questions in \( \psi(q) \)?” that diverge by less than \( \epsilon \) from \( \varphi_1 \)'s limit distribution over possible vectors of \( m \) answers to the \( m \) questions in \( \psi(q) \), conditioned on the same claim set \( \hat{C} \).

This special case is crucial to several applications of our framework described below. In many instances, we will take \( \varphi_1 \) to be a mathematical “oracle” that acts as the arbiter of what counts as the “correct” response distribution for a given question and claim set. In particular, as described above, one can adopt the interpretation where \( \varphi_1 \) represents the process by which all mathematicians in the universe ask and answer questions, from the past on to the future. Under this interpretation, in

\footnote{Note that \( \varphi_1 \) and / or \( \varphi_2 \) might consider a particular \( \hat{C} \) very unlikely \textit{a priori}. This suggests an alternative definition of calibration to Definition 4.2 under which \( P_1^{n^1} \) and / or \( P_2^{n^2} \) are not conditioned on \( \hat{C} \). We do not investigate this alternative here, for reasons for space.}
keeping with the Peircean idea of truth as what wins out in the long run of intelligent inquiry, we can model an individual mathematician \( \varphi_2 \) as answering questions correctly to the extent that they are calibrated, at some present step \( n \), with the SMS \( \varphi_1 \) in its infinite limit. That is, the mathematician \( \varphi_2 \) answers questions correctly to the extent that it answers them by sampling from a response distribution that is minimally divergent from the limit response distribution of \( \varphi_1 \), conditioned on the same claim set \( \hat{C} \).

One might object to the real world relevance of calibration, arguing that in practice, few human mathematicians ask themselves questions of the form, “what is the probability distribution over answers to the following question (that would be given by some oracle)?” Instead of those kinds of questions, human mathematicians typically just directly ask themselves, “what is the answer to the following question?” (which we can interpret as them asking themselves, “what is the answer that would be given by an oracle to the following question?”). It might seem that the concept of calibration has nothing to say about this second, more common kind of question. However, in our approach, the second kind of question is just a special case of the first.

To see this formally, suppose that the two SMSs in Definition 4.1 are actually identical. In this case we do not use subscripts, and so for example we simply have a single integer \( n \) and have \( \psi : \mathbb{Q} \to \mathbb{Q}^* \). We also do not make the first requirement in Definition 4.1 for such a single-SMS prediction pair. Choose \( \psi(.) \) to be the identity function (so \( |\psi(q)| = m = 1 \)), choose \( \Psi \) so that for all \( v_1, v_2 \in \mathcal{V}_1 = \mathcal{V}_2 \), \( \Psi(q_1, v_2)(v_1) = \delta(v_1, v_2) \) (the Kronecker delta function). For those choices, and for any prediction pair \((q, \hat{C})\), Definition 4.2 reduces to the condition that

\[
D[P^n_2(\mathcal{V}_1 | q, \hat{C}), \overline{P}_1(\mathcal{V}_1 | q, \hat{C})] \leq \epsilon.
\]

So for those choices, a current mathematician-SMS \( \varphi_2 \) is calibrated with the oracle \( \varphi_1 \) if and only if their response distribution is sufficiently non-divergent from that of \( \varphi_1 \), for the question \( q \).

There are several important reasons that the concept of calibration goes further than Eq. (4.1). First, it encompass sets of multiple question-answer pairs that come “as a bundle”, so to speak. Second, it allows a current mathematician to be explicitly uncertain about what they think the oracle’s answer to a question would be, crediting them (in a normative sense) if that uncertainty matches the oracle’s uncertainty.

There is also an important distinction between calibration and simply requiring small divergence between the two response distributions \( P^n_2(\mathcal{V}|q, \hat{C}) \) and \( \overline{P}_1(\mathcal{V}|q, \hat{C}) \). Even if that divergence is identically zero, an answer given by \( \varphi_2 \) may differ greatly from an answer given by \( \varphi_1 \). In other words, the answer \( v \) which the mathematician \( \varphi_2 \) generates in response to the question \( q \) may differ greatly from the answer \( \hat{v} \) which the oracle \( \varphi_1 \) generates in response to that question. Due to this, there is no normative basis to requiring that that divergence be small.

\[8\]To derive Eq. (4.1), note that according to the assumptions made immediately above it, \( P^n_2(v_1|q, \hat{C}) = \sum_{v_2} P^n_2(v_2|q, \hat{C})\delta(v_1, v_2) \) for any \( v_1 \in \mathcal{V}_1 \). With abuse of notation, for any \( v_2 \in \mathcal{V}_2 \), let \( \delta(\mathcal{V}_1, v_2) \) be a vector such that each entry is the value of the delta function \( \delta(v_1, v_2) \) for each \( v_1 \in \mathcal{V}_1 \). Similarly, let \( P^n_2(\mathcal{V}_1|q, \hat{C}) \) be a vector such that each entry is the value of \( P^n_2(\mathcal{V}_1|q, \hat{C}) \) for each \( v_1 \in \mathcal{V}_1 \). Thus, \( P^n_2(\mathcal{V}_1|q, \hat{C}) = \sum_{v_2} P^n_2(v_2|q, \hat{C})\delta(v_1, v_2) \). Since \( D \) is convex in its first argument, we have

\[
D[P^n_2(V_1|q, \hat{C}), \overline{P}_1(V_1|q, \hat{C})] = D[\sum_{v_2} P^n_2(v_2|q, \hat{C})\delta(v_1, v_2), \overline{P}_1(V_1|q, \hat{C})] \leq \sum_{v_2} P^n_2(v_2|q, \hat{C})D[\delta(v_1, v_2), \overline{P}_1(V_1|q, \hat{C})] \leq \epsilon.
\]
On the other hand, there is a normative basis to requiring that \( \varphi_2 \) be calibrated with \( \varphi_1 \); when \( \varphi_2 \) is calibrated with \( \varphi_1 \), the answers of \( \varphi_2 \) to any question in a prediction pair are (on average) very close to the answers of \( \varphi_1 \) (as quantified by the divergence between the two associated distributions specified by those answers). This requirement amounts to requiring that (on average) \( \varphi_2 \) answers the question ‘what is the probability distribution over possible answers to \( \psi(q) \)?’ by outputting a probability distribution with low divergence from the probability distribution that \( \varphi_1 \) outputs in response to the same question. So in this case, the answer \( v \) which the mathematician \( \varphi_2 \) generates in response to the question \( q \) cannot differ greatly from the answer \( v \) which the oracle \( \varphi_1 \) generates in response to that question (on average). So when calibration is small, \( \varphi_2 \) answers mathematical questions about probability distributions over possible mathematical facts in the same way that the oracle will (in the infinite long-run).

At first, it may seem strange to suppose that a mathematical oracle would ever be unsure of the answer to any mathematical question. However, it is clearly possible under the far-future-community interpretation of such an oracle. Suppose that there is simply eternal disagreement on the answer to a mathematical question \( \psi(q) \): some say that the answer is \( v \), while others say that the answer is \( v' \). Eventually, the mathematical community converges on a stable distribution over possible answers: with probability \( \rho \), \( v \) is output as the answer, while with probability \( (1 - \rho) \), \( v' \) is output as the answer. Under these circumstances, we hold that the best way for a present-day mathematician to respond to the question ‘what is the distribution over possible answers to \( \psi(q) \)?’ is to (in expectation) produce a distribution with low divergence from that which the mathematical community converges on. Thus, while our approach does not require that such answers are the best way to answer mathematical questions, it allows for this possibility. We take this to be a virtue of our account, one that proves all the more fruitful when we turn our attention to scientific reasoning.

4.2. Honesty. Note that the requirement in Definition 4.1 and Definition 4.2 that \( \hat{C}_2 \subseteq \hat{C}_1 \) implies that there can be pairs \( q \in Q_1, v \in V_1 \) such that both \( q \in \text{Dom}(\psi) \) and \( (\psi(q), v) \in \text{Dom}(\Psi) \). Given how we wish to interpret \( \psi \) and \( \Psi \), this means that an SMS \( \varphi \) can ask the question, ‘what is the joint probability distribution that \( I \) assign to the possible answer(s) to the question(s) specified in \( \psi(q) \)?’ — and then provide an answer. This provides what can be seen as a necessary condition for \( \psi \) and \( \Psi \) to have the interpretation we wish to assign them:

**Definition 4.3.** Let \( \varphi \) be an SMS with associated single-SMS partial functions \( \psi \) and \( \Psi \). Let \( q \) be an associated question, \( v \) an associated answer, \( n \) a positive integer, and \( \hat{C} \) an associated claim set. Suppose that \( (q, \hat{C} \cup \{(q, v)\}) \) is a (single SMS) prediction pair. Then we say that \( \varphi \) is honest (for \( (q, v, \hat{C}, n, \psi, \Psi) \)) if for all \( \tilde{v} \in V^{|\psi(q)|} \), \( P^n(\tilde{v} \mid \psi(q), \hat{C} \cup \{(q, v)\}) = \Psi(\psi(q), v)(\tilde{v}) \)

Note that in Definition 4.3 \( \hat{C} \cup \{(q, v)\} \) is the claim set given by the union of \( \hat{C} \) and the single pair \((q, v)\). We make the obvious extension of removing \( n \) from the definition of honesty to allow consideration of the limit distribution of a backward-consistent SMS rather than a step-\( n \) distribution. We will sometimes simply say that \( \varphi \) is honest, without specifying the precise \( q \) and \( v \), if it is honest for all \((q, v)\) such that \( q \in \text{Dom} \psi, (\psi(q), v) \in \text{Dom} \Psi \).

Intuitively, an SMS is honest if it is perfectly calibrated (i.e., calibrated for \( \epsilon = 0 \)) with itself. Viewed alternatively, an SMS is honest if it does indeed "mean" \( \psi(q) \) to be a set of questions,
and “means” $\Psi(\psi(q), v)$ to be an associated joint distribution over $|\psi(q)|$ answers. As an example, suppose that we interpret $\varphi_1$ in Definition 4.2 as the far-future community of mathematicians. Then requiring that the far-future community is an honest SMS (for the provided $\hat{C}$ and $(q, v)$) essentially amounts to requiring that they use what Lewis (1971) calls “immodest” inductive methods. Once they accept a given response distribution $\Psi(\psi(q), v)$ as the correct one for assigning answers to a vector of questions, they make predictions about the answers to those questions by sampling from that distribution $\Psi(\psi(q), v)$.

Strictly speaking, a current mathematician can be perfectly calibrated with the far-future community even if they are not honest. If this were the case, then (the distribution over) their personal answers to the questions in $\psi(q)$ would not be the same as (the distribution over) the answers that they predict the far-future community would provide. For some reason or other, when asked, “do you think $P = \text{NP}$?” they would respond by sampling from a different distribution over answers than the one they would sample if instead asked, “do you predict that the far-future community of mathematicians would think $P = \text{NP}$?”.

While our analysis allows this, typically one would presume that those two distributions must be identical, essentially as a normative principle. Under this presumption, Definition 4.3 would be a definition of $\Psi(\ldots)$, a definition that one would make before defining prediction pairs or calibration. However, many of our results below do not require that $\varphi_2$ be honest, allowing the current mathematician (for whatever reasons) to give a different set of answers to a given set of questions from the set of answers they predict that the far-future community would give. Accordingly, here we do not impose honesty, formulating it as a property that an SMS may or may not have, rather than as a requirement.

5. SMSs, Calibration and Empirically Grounded science

While the details are still unclear, the deepest understanding of the nature of physical reality that modern physics provides us is quantum cosmology, and various possible elaborations of quantum cosmology like string theory (see Barrow, 1991; Greene, 1999; Carroll, 2021). Crucially, the consensus view among a large subset of quantum cosmologists is that quantum cosmology is a complete description of physical reality, in the sense that every experiment that a scientist could ever run (up to and including the “experiment” of posing questions to themselves internally), and every observation they could ever make, is completely described either directly by the laws of quantum cosmology or indirectly via formal implications of those laws. (Examples of those implications being the rules of quantum chemistry, which in turn ultimately provide the statistical regularities of terrestrial biology, etc.) Specifically, the modern view is that the physical universe can be formulated in toto as some appropriate formal system (see Tegmark, 1998; Tegmark, 2014). The outcomes of all experiments ever conducted by scientists, all observations, either in the past or the future, are theorems in that formal system, all following from a particular set of axioms. The crucial point is not that we have all those axioms in hand already — we do not — never mind that we could even imagine calculating those theorems from those axioms. Rather, the crucial point is that physical reality can be formulated as some formal system, in such a way that almost tautologically, there is no conceivable role played by some additional “reality” not captured by that formal system, no sense in which such a reality could be physically accessible to us humans. In modern physics, there
is no unavoidable need for the concept of a “physical reality” outside of the mathematical laws of quantum cosmology. The idea that the physical universe is ultimately a mathematical object and nothing more is explicitly promoted by, among others, Barrow (1991, 2011), Dipert (1997), Schmidhuber (1997), in addition to Tegmark. The view that the universe is fundamentally mathematical is also close to, and arguably entailed by, ontic structural realist views in philosophy of science, such as those put forward by Ladyman (1998), Ladyman (2007), and Wallace (2021). (However, note that some ontic structural realists want to say that, although the physical world is a structure, it is a distinctly physical, rather than mathematical, structure.) For our part, though we offer no argument for it here, we are happy to embrace the literal interpretation of the general claim that the physical universe is a mathematical object. We then extend the perspective of modern physics that the physical universe as a mathematical object is a formal system, in the obvious way, to formulate the physical universe as an SMS.

Note that representing the physical universe as an SMS entails that, as a matter of physical reality, there may be some questions that do not have a single prescribed answer, but are instead only answered by sampling from a non-delta-function distribution over possible answers. This is one way of saying that the physical universe is fundamentally stochastic. There are many ways in which one might think that there are physical questions whose answers are determined in such a fundamentally stochastic way. On the Copenhagen interpretation of quantum mechanics, the result of any measurement of a quantum system is determined stochastically. Alternatively, one might endorse a “many-worlds” interpretation of quantum mechanics (Everett, 1957) in which all possible quantum measurement outcomes occur in some branch of a multiverse structure, but still follow Sebens and Carroll (2018) in holding that it is an objective fact that observers in a given branch of that structure ought to assign specific non-extreme probabilities to their being in that branch, and to their being in some other possible branches. Or, one might adopt the Mentaculus view of cosmology articulated by Albert (2000) and Loewer (2007, 2008, 2009). On their view, which follows in the Humean tradition of Lewis (1986), our actual physical universe is composed of a spatiotemporal array of perfectly intrinsic properties. However, this actual world is generated by sampling from a probability distribution over the set of all possible physical universes defined as such.

All of these theories can be articulated within the SMS framework, where they differ with respect to their specification of the set of questions that are answered by sampling from a non-delta-function probability distribution. For instance, on the Copenhagen interpretation, the universe-SMS might consider a question of the form ‘will tritium atom x decay by time t?’, and arrive at the answer ‘Yes’ or ‘No’ by sampling from an unsure response distribution over those two possible answers. On a Sebens-and-Carroll style multiverse view, the universe-SMS considers questions of the form ‘where am I located in a branching multiverse structure?’, and responds by sampling from an unsure response distribution. Finally, on a Mentaculus view, the universe-SMS considers questions of the form, ‘is the physical universe such that x occurs?’, where x is some event, and answers them by sampling from an unsure response distribution over possible universes. However, representing the universe with an SMS is also perfectly consistent with an entirely deterministic universe; for that to
be the case we simply need to model the universe as an SMS $\varphi_1$ such that the response distribution over all answers to any question $q$ and prior claim set $\hat{C}$ of that universe-SMS is sure.

5.1. WHAT IT MEANS FOR A SCIENTIST TO BE EMBEDDED IN A UNIVERSE. When formulating a model of reasoning in mathematics, we used one SMS $\varphi_2$ to represent a mathematician and a second SMS $\varphi_1$ to represent an arbiter of “correct” answers to mathematical questions, intuitively interpreting that arbiter as the far-future community of mathematicians. Note that $\varphi_1$ and $\varphi_2$ had no a priori relation — if the mathematician $\varphi_1$ was able to give approximately correct answers, then they were calibrated with the far-future community of mathematicians, but nothing more was formalized concerning their relationship.

We can model scientists making predictions about the outcomes of physical experiments similarly — but with one major difference. Choose $\varphi_2$ to be a particular scientist, or more precisely an SMS generating cognitive events of that scientist concerning physical phenomena (i.e., observations, memories, etc., formulated as question-answer pairs), much like the cognitive events of a mathematician that concern mathematical phenomena. Also choose $\varphi_1$ to be the arbiter of correct answers, just as in the case of mathematical reasoning. Of course, just as real human mathematicians do far more than predict how the far-future community of mathematicians will answer specific questions, real scientists do far more than predict what answers will be provided by physical experiments to specific questions. For instance, they often build models of phenomena that contain explicitly unobservable phenomena. Moreover, scientists’ interpretations of the outputs of experiments may be fundamentally shaped by the theories that they have an antecedent commitment to (i.e., their observations may be “theory-laden”). In this paper we are not concerned with those other aspects of scientific reasoning, and so abstract away from them for the sake of building our model, though we are optimistic that these other aspects of scientific reasoning could be formalized in a manner consistent with our broader framework. Similarly, we do not consider here the formal models of scientific (dis)agreement developed in the social epistemology of science (see Grim and Singer, 2022 Sec. 3.2.2; Šešelja, 2022), though here too we believe future work could see overlap between formal modeling of the social dynamics of scientific inquiry and the model we provide here.

In the case of science, we take the arbiter of correct answers to be the physical universe generating outcomes of experiments, not some far-future community of scientists. Moreover, scientists are physically embedded in the very physical universe about which they wish to make predictions. (After all, scientists are physical beings (or so we assume) and so must be accounted for as part of the physical universe-SMS). This means that a scientist $\varphi_2$ is itself a set of (question-answer) pairs, posed to the embedding physical universe $\varphi_1$. There is no analogous property relating a mathematician and the far-future community of mathematicians. In essence, in the case of mathematical reasoning, $\varphi_1$ and $\varphi_2$ are independent SMSs, while in the case of science, $\varphi_2$ is a sub-SMS, embedded in $\varphi_1$.

We formalize what it means for a scientist-SMS to be physically embedded in a universe-SMS by requiring that the distribution of claim sets that a scientist might make, each of which is just some pattern in their physical brain, is given by a distribution over states of the universe that “project down” to those patterns in the brain of the scientist. We formalize this by requiring that there be
a measurable (partial) function from the space of claim sets of the universe into the space of claim sets of the scientist:

**Definition 5.1.** Let \( \varphi_2 \) be an SMS and \( \varphi_1 \) an SMS that is backward-consistent. We say that \( \varphi_1 \) embeds \( \varphi_2 \) for step \( n \) if and only if there is a partial function \( E \) from \( \hat{C}_1 \) into \( \hat{C}_2 \) such that for any non-empty claim set \( \hat{C}_2' \in \hat{C}_2 \), \( P_1(E^{-1}(\hat{C}_2')) = P_2^n(\hat{C}_2') \).

We will sometimes say that “\( \varphi_2 \) is embedded in \( \varphi_1 \)” rather than “\( \varphi_1 \) embeds \( \varphi_2 \)”. We will also sometimes refer to \( E \) as an embedding function from \( \hat{C}_1 \) to \( \hat{C}_2 \). Often below we will leave out specification of the step \( n \), assuming that \( \varphi_2 \) is embedded in \( \varphi_1 \) for all steps after some threshold step \( m \), and that \( n > m \). We refer to \( \varphi_2 \) in Definition 5.1 as the “scientist” SMS, and refer to \( \varphi_1 \) as the “universe” SMS.

In the sequel we will be interested in the case where \( \hat{C}_2' \) can be written as \( \{c_2\} \cup \hat{C}_2 \) for some claim \( c_2 \in C_2 \) and (possibly empty) claim set \( \hat{C}_2 \in \hat{C}_2 \). In these cases the claim \( c_2 \) will be identified with a question by the scientist of the form, “what is the distribution over possible outcomes for this specific experiment / observation?”, together with an answer to that question. We will then identify the associated claim set \( \hat{C}_2 \) as (a subset of) all that the scientist has previously observed that is relevant to their answer given in claim \( c_2 \).

As an example, suppose that \( \hat{C}_2 \) specifies the (brain) state of a scientist in which they remember some parts of a textbook concerning radioactive decay, and \( c_2 \) is the claim (‘what is the probability that tritium atom \( x \) decays by time \( t \)?’, \( .9 \)). In this case, we interpret \( E^{-1}(\{c_2\} \cup \hat{C}_2) \) as the collection of all physical processes that might occur in the universe, within the brain of the scientist and (perhaps) elsewhere in the universe, each of which is sufficient for the scientist to have the (brain state in which) they think they have read some parts of a textbook concerning radioactive decay and also have the thought that the tritium atom \( x \) decays by time \( t \) with probability \( .9 \).

Note that in general there will be many possible facts concerning the universe that are all consistent with some set of fixed claims made by a scientist in a claim set \( \hat{C}_2 \), physical facts that have nothing to do with that scientist or their claims. (As an illustration, there are many possible numbers of stars in the Andromeda galaxy, and all of those numbers are consistent with the claims made by the typical present-day human scientist.) This is captured in Definition 5.1 by the fact that the inverse image of the embedding function will in general be more than just one set of physical facts — one claim set of the universe-SMS — but rather a collection of multiple (perhaps infinite) such sets. Note as well that the scientist’s distribution in Definition 5.1 is of the form of Eq. (3.2), not Eq. (3.1). This reflects the fact that like the state of the universe, the state of the scientist’s brain might have generated other claims besides those specified in \( \{c_2\} \cup \hat{C}_2 \).

As a formal point, suppose that any claim set produced by the scientist-SMS is a subset of the claim set produced by the (infinite step limit of the) universe-SMS, i.e., some of the question-answer pairs of the universe specify the physical properties of the scientist in toto, as discussed above. Then Definition 5.1 holds no matter what the universe-SMS is, i.e., no matter what \( P_1 \) is. In this case, ontologically there is only one SMS, that of the physical universe, with the scientist being in effect a projection the universe-SMS onto a sub-SMS. In our discussion below we will usually be informal and phrase the embedding relationship as though the scientist is indeed a sub-SMS of the physical universe. It’s worth pointing out though that Definition 5.1 can hold even if \( \varphi_1 \) and \( \varphi_2 \) are completely
independent stochastic processes, without the claim set generated by $\varphi_2$ being a subset of the one generated by $\varphi_1$, so long as the probability distributions of those two stochastic processes happen to be related appropriately.

It is important to emphasize that $E^{-1}[[c_2] \cup \hat{C}_2]$ is a collection of claim sets in $\hat{C}_1$, not a union of those claim sets. Formally, $E^{-1}[[c_2] \cup \hat{C}_2]$ is the union of a set of elements of $\hat{C}_1$—the fact that those elements happen to be sets is immaterial. So in terms of the probability measure defining $\varphi_1$, $E^{-1}[[c_2] \cup \hat{C}_2]$ is the event that $\varphi_1$ generates one of the claim sets of $\varphi_1$ whose image under $E$ is $\{c_2\} \cup \hat{C}_2$, not the event that $\varphi_1$ generates the union of all such claim sets. Similarly, the probability $\mathcal{P}_1(E^{-1}[[c_2] \cup \hat{C}_2])$ is the limit probability of $\varphi_1$ generating (a superset of) any of the claim sets in $E^{-1}[[c_2] \cup \hat{C}_2]$. It is not the probability of $\varphi_1$ generating the claim set given by taking the union of all the claim sets whose image is $\{c_2\} \cup \hat{C}_2$. Indeed, due to contradictions among claims in such a union, often the probability that $\varphi_1$ creates that union in its entirety will be zero.

As a final point, we note that many of the results below hold for any partial function $E$ from $\hat{C}_1$ into $\hat{C}_2$, even if it violates the condition on probability distributions in Definition 5.1. For simplicity though, in this paper we always assume that that condition in Definition 5.1 is obeyed.

5.2. Scientists who are calibrated with the universe that embeds them. Given this formalization of what it means for a scientist-SMS to be embedded in a given universe-SMS, we can define what it means for a scientist to accurately predict the outcomes of experiments or observations in the physical universe. This involves a slight modification to the definitions of prediction pair and calibration that were introduced in Section 4.1. In particular, we do not consider whether a scientist is making predictions consistent with the embedding universe conditioned only on a set of their earlier experiments/observations, in analogy to the case with calibration and mathematical reasoning. In addition to conditioning on a claim set, we also condition on the simple physical fact that the scientist is, in fact making a prediction. After all, the fact that they, a part of the universe, are making a prediction, is a restriction on what the precise state of the universe embedding them can be.

**Definition 5.2.** Let $\varphi_1$ be a backward-consistent SMS and $\varphi_2$ an SMS. Let $\psi$ and $\Psi$ be associated partial functions that have domains and ranges as described in Definition 4.1. Let $n$ be a positive integer and let $E$ be an embedding function from $\varphi_1$ to $\varphi_2$. A pair of a question $q \in \mathcal{Q}_2$, and a claim set $\hat{C} \in \hat{C}_2$, is an embedded prediction pair (for $E, \varphi_1, \varphi_2, \psi, \Psi$ and $n$) if

1. $P^n_2(v \mid q, \hat{C})$ is well-defined;
2. $q \in \text{Dom} \psi$;

For all $v \in \mathcal{V}_2$ such that $P^n_2(v \mid q, \hat{C})$ is nonzero:

3. $(\psi(q), v) \in \text{Dom} \Psi(\cdot, \cdot)$;
4. $\mathcal{P}_1\left(\mathcal{C}_1^{\psi(q)} \mid \psi(q), E^{-1}[(q, v) \cup \hat{C}]\right)$ is a well-defined distribution over $\mathcal{C}_1^{\psi(q)}$;

where as usual we use subscripts to distinguish the distributions of the two SMSs.

The definition of an embedded prediction pair in Definition 5.2 is very similar to the definition of a prediction pair in Definition 4.1. One difference is that the requirement in Definition 4.1 that $\hat{C}_2 \subseteq \hat{C}_1$ is replaced in Definition 5.2 with the requirement that there be a specified embedding...
function from \( \varphi_1 \) to \( \varphi_2 \). Note though that there is no requirement that \( \varphi_1 \) actually embeds \( \varphi_2 \) under that embedding function (or any other) in Definition 5.2.

Another difference between the two definitions reflects the fact that the physical act of the scientist asking question \( q \) and giving answer \( v \) is a restriction on the possible state of the physical universe in which they are embedded. This requires us to add \((q, v)\) in the conditioning event in the probability distribution in condition (4) of Definition 5.2. That in turn requires us to stipulate that \( v \) have nonzero probability under SMS \( \varphi_2 \). In contrast, no such stipulation is needed in Definition 4.1. Sometimes we will wish to replace the requirement in Definition 5.2 that conditions (3, 4) hold for all \( v \) such that \( P_2^n(v \mid q, \hat{C}) \) with the requirement that they hold for some specific such \( v \). In such cases we refer to \((q, \hat{C}, v)\) as an embedded prediction triple. So in general every embedded prediction pair corresponds to many embedded prediction triples.

A small modification of the definition of calibration is also needed to define the analogous concept for an empirical scientist:

**Definition 5.3.** Let \( \varphi_1 \) be a backward-consistent SMS and \( \varphi_2 \) an SMS, where \( \varphi_2 \) is embedded in \( \varphi_1 \) with embedding function \( E \) (for step \( n \)). Suppose we are given associated partial functions \( \psi \) and \( \Psi \) that have domains and ranges as described in Definition 4.1. Suppose further that we are given a claim set \( \hat{C} \in \hat{C}_2 \) and a question \( q \in \mathcal{Q}_2 \). We say that \( \varphi_2 \) is embed-calibrated (with \( \varphi_1 \)) at step \( n \) for the pair \((q, \hat{C})\) and for \( \epsilon \geq 0 \) iff

\[
\begin{align*}
(1) \ \ (q, \hat{C}) & \text{ is an embedded prediction pair for embedding function } E \\
(2) \ \ \sum_{v \in \mathcal{V}_2} P_2^n(v \mid q, \hat{C}) D \left[ \Psi(\psi(q), v)(\mathcal{V}_1^m), \overline{P}_1 \left( \mathcal{V}_1^m \mid \psi(q), E^{-1}[\{(q, v)\} \cup \hat{C}] \right) \right] & \leq \epsilon
\end{align*}
\]

where \( m \) is shorthand for \( |\psi(q)| \).

As before, the function \( D \) in Definition 5.3 is some convex divergence measure between probability distributions, and \( \epsilon \) is often implicit.

Intuitively, embed-calibration is the same as calibration simpliciter, just with three coupled changes. First, in Definition 5.3 we drop the assumption that \( \hat{C}_2 \subseteq \hat{C}_1 \). That assumption was needed in condition (3) of Definition 4.2 to specify how conditioning on an already accepted claim set by a present-day mathematician-SMS translates into conditioning on a claim set by the far-future-community SMS. This assumption is replaced in Definition 5.3 by the assumption that the scientist-SMS is embedded in the universe-SMS. Second, calibration requires that \((q, \hat{C})\) be a prediction pair. In contrast, embed-calibration requires that they be an embedded prediction pair.

Finally, in Definition 4.2 we required that, in expectation, the SMS \( \varphi_2 \)'s answer to the question ‘what is the probability distribution over possible answers to the questions in \( \psi(q) \)?', conditional on the claim set \( \hat{C} \), diverges by less than \( \epsilon \) from the SMS \( \varphi_1 \)'s limit response distribution over possible answers to the equations in \( \psi(q) \), conditional on the same claim set \( \hat{C} \). By contrast, in the case of the calibration relation between a scientist-SMS and the universe-SMS in which it is embedded, we require that the scientist-SMS’s answer to the question ‘what is the probability distribution over possible answers to the questions in \( \psi(q) \)?', conditional on the claim set \( \hat{C} \), diverges by less than \( \epsilon \) from the universe-SMS’s limit response distribution over possible answers to the questions in \( \psi(q) \), conditional on the universe-SMS producing one of the claim sets in \( E^{-1}[\{(q, v)\} \cup \hat{C}] \).

The goal of the scientist-SMS, broadly speaking, is to be embed-calibrated with the universe-SMS that embeds it. To illustrate, consider a scientist-SMS who asks themselves the question ‘what is
the probability distribution over possible answers to the question ‘will the specific tritium atom \( x \) decay by specific time \( t \)?’ that would be provided by a future experiment, as determined by the physical universe?’. Suppose that with probability 1 conditional on a body of nuclear theory the scientist knows, which is identified with the claim set \( \hat{C} \), the scientist generates the answer that the probability distribution is ‘Yes’ with probability .35 and ‘No’ with probability .65. If the scientist is embedded in a universe-SMS with a backward-consistent response distribution for the question ‘will tritium atom \( x \) decay by time \( t \)?’, and if conditional on the universe producing a claim set in \( E^{-1}[\{(q, v)\} \cup \hat{C}] \), that universe assigns probability .35 to ‘Yes’ and .65 to ‘No’, then that scientist is calibrated for the question ‘what is the probability distribution over possible answers to the question ‘will tritium atom \( x \) decay by time \( t \)?’, claim set \( \hat{C} \), and \( m = 1 \), for any non-negative value of \( \epsilon \).

Note that real world human scientists always have statistical uncertainty about their predictions. This uncertainty is automatically respected in our approach, due to the stochasticity of the universe-SMS. Note also that in these examples \( E^{-1}[\{(q, v)\} \cup \hat{C}] \) is not a set of physical facts in the embedding universe that comprise the nuclear theory adopted by the scientist-SMS. Rather, it is a set of physical facts that are jointly constitutive of the scientist-SMS having the beliefs that they do about nuclear theory. Concretely, \( E^{-1}[\{(q, v)\} \cup \hat{C}] \) specifies some spatio-temporal biochemical pattern in the brain of the scientist-SMS. On the other hand, the distribution \( \Psi(\psi(q), v) \) is a distribution over physical outcomes of an experiment, not over the state of the brain of the scientist after that experiment.

6. Calibration and Truth

The nature of mathematical and scientific truth is a notoriously thorny philosophical issue that we cannot hope to fully address here. For this reason we are careful not to use the term “true” in our formal definitions, and to make no explicit claims concerning truth. Nevertheless, there are some connections between the two calibration notions developed above and general, informal notions of “truth” which are worth commenting on.

In ordinary human reasoning, we often assert sentences, and moreover, we assert that some of these sentences are true. Dummett (1959) famously investigates why a notion of truth should be part of our apparatus for reasoning, and concludes that the sole function of a notion of truth is to distinguish between those sentences that we can justifiably assert and those that we cannot. Indeed, he concludes that a sentence is true just in case it can be justifiably asserted. While this definition of truth does not yield a procedure for determining whether any given sentence is true (this would require a theory of justifiable assertion), it does provide a theory of the purpose of the concept of truth; namely, truth has the normative function of distinguishing between those sentences that we can justifiably assert and those that we cannot. Indeed, he concludes that a sentence is true just in case it can be justifiably asserted. While this definition of truth does not yield a procedure for determining whether any given sentence is true (this would require a theory of justifiable assertion), it does provide a theory of the purpose of the concept of truth; namely, truth has the normative function of distinguishing between those sentences that it is correct to assert and those that it is not correct to assert. One can easily extend this account to give an analysis of what it means to answer a question \( q \) truthfully; \( v \) is the true answer to \( q \) just in case one is justified in answering \( q \) with \( v \). Note that the idea that truth provides normative governance for assertions extends to more technical notions of truth. For instance, when we define a semantics for a theory written in first-order logic by specifying a model of that theory, we implicitly stipulate which formulas of the theory are assertable and which ones are not.

On its own, calibration can be understood purely mathematically, without any normative significance. However, in our discussion of how to interpret the notion of calibration in the previous two
sections we have interpreted it as also having a normative function. This normative function is similar to the function of truth in Dummett’s account, but is distinct in important ways. In Dummett’s account, truth is a normative constraint on the claims that speakers of a language ought to make. Within the SMS framework, calibration can be interpreted as providing a normative constraint on how an SMS ought to make claims. For a given question $q$ and claim set $\hat{C}$, one can understand an SMS $\varphi_2$ that generates answers to $q$ by sampling from a response distribution that allows $\varphi_2$ to achieve calibration with some other SMS $\varphi_1$ to be generating claims in a “justifiable” way.

Note that defining “justifiable” in terms of calibration does not imply that any particular answer $v$ is the correct or justifiable one for $q$. Defining justifiable (or more generally, “truth”) in terms of calibration allows for the possibility that the answers to some questions are objectively indeterminate or chancy. In the context of calibration and mathematical reasoning this might seem peculiar, if the oracle is some nebulous version of “Platonic truth”. However, it is not so unreasonable if the oracle is interpreted as the far-future community of mathematicians. (In any case, as discussed above, if in fact the infinite-step limit of $\varphi_1$ does output some single answer to a particular question — as in conventional notions of mathematical truth — that simply means that the distribution over the answers they provide is a delta function.)

In more detail, for any present-day real-world mathematician, there is a large set of claims made by earlier mathematicians that they accept as “true.” In the real world, such earlier claims will be part of the claim set of that present-day mathematician. Moreover, since the generation of claim sets is a stochastic process, the full set of claims accepted by the present-day mathematician will modify the distribution of claims made by the far-future community of mathematicians. Indeed, in the extreme case, one particular claim set accepted by the present-day mathematician might completely rule out some claims of the far-future community, claims that would otherwise be quite likely. This is the reason why the definition of calibration involves both conditioning the present-day mathematician’s answers on a given claim set and conditioning the far-future community of mathematicians’ answers on the same claim set. In addition, although we do not exploit the fact here, backward-consistency allows us to require that the full claim set of that far-future community of mathematicians “develops out of” the the claim set of the current community, as the SMS evolves.

The case of empirical science is slightly different. In empirical science, we take the universe-SMS $\varphi_1$ as the arbiter of what counts as a correct or true distribution from which to sample answers to a given question about the outcome of an experiment. A scientist-SMS outputs true or correct answers to questions about an experiment to the extent that they are embed-calibrated (for small $\epsilon$) with the universe-SMS in which they are embedded. As in the case of mathematical reasoning, it may be that there is no unique correct answer to a well-posed question about the outcome of an experiment, but that there is instead a correct probability distribution over possible answers to a well-posed question about the outcome of an experiment. Indeed, as discussed in Sec. 5 our best theories in quantum cosmology imply that there are well-posed questions about the outcomes of experiments that do not admit of a uniquely correct answer, but do admit of a uniquely correct probability distribution over possible answers, (i.e., possible observations of the outcome of a particular experiment). In this sense, our approach is in keeping with Hacking’s (1964) notion of objective statistical facts as grounded in the propensity of a particular “chance set-up” to result in a particular outcome. Indeed, Hacking’s definition of a chance set-up as “a device or part of the world on which might be
conducted one or more trials, experiments, or observations; each particular trial must have a unique result which is a member of a definite class of possible results” is very much in the spirit of what counts as a well-posed question for both a universe-SMS and a scientist-SMS (p. 3).

So in the case of mathematical reasoning calibration involves the relationship between the response distribution of the far-future community of mathematicians and that of the current mathematician to the same set of questions, given that both accept the same claim set $\hat{C}_2$, i.e., given that both accept the same set of chains of proof-like mathematical reasoning. In the case of scientists though, $\hat{C}_2$ is a physically concrete property of the scientist’s own brain, corresponding to their mental perception of the outcomes of chains of experiments or observations they have made. There is no sense in which the universe-SMS “accepts the same claim set”. Rather that claim set is a projection of the full state of the universe down to a set of properties of the brain of the scientist.

Note also that although the universe-SMS can make claims about what happens at particular points in space and time, we do not presume that the universe-SMS successively generates its claims according to any privileged temporal ordering. In other words, for the universe-SMS, the ordering of steps need have nothing to do with the ordering of time in some particular inertial frame of the physical universe generated by that SMS. In this sense, we effectively adopt a “block universe” perspective on quantum cosmology. By contrast, the scientist-SMS does have a privileged temporal ordering to its steps, since it perceives reality to unfold in a particular temporal direction, from past to present. $^9$ This is why we define embed-calibration in terms the limit probability of the universe-SMS; we can only be assured that all cognitive events in the mind of the scientist will be derived as claims of the universe-SMS if we allow that universe-SMS to run arbitrarily many steps, without necessarily interpreting those steps as temporally ordered.

Finally, we note that in the case of empirical science, it is possible for a scientist to be systematically biased in their observations of the outcomes of experimental set-ups (e.g., if they are hallucinating). Indeed, it may be that they are able to make accurate predictions about the outcomes of experiments but are unable to accurately observe the outcomes of such experiments, and so they might falsely conclude that their initial predictions were inaccurate. Even more extremely, it may be that they are unable to discriminate among the possible outcomes of the experiments, due to limitations of their observational apparatus. In such a case there would be no empirical meaning to their predictions, in the sense that the accuracy of those predictions cannot be verified. $^10$

One might object that this is a flaw in the idea of using embed-calibration to assess the quality of a scientist’s predictions, since ultimately any scientist is only interested in the distribution over their perceptions of the outcome of an experiment, not over the “actual” outcomes in some sense independent of their mind. In order to address this objection we must first assure that the perceptions of the scientist can discriminate between the actual outcomes of the experiment. To see how to do this, first note that depending on the precise form of $E$, there may be a set $S \in \text{Dom } E$

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$^9$ One way of accounting for this “psychological arrow of time” is by appealing to the second law of thermodynamics; see Wolpert (1992) and Davies (1977).

$^{10}$ Note that there is not an analogous issue for the case of mathematician-SMSs. That is because we suppose that mathematicians are interested in “mathematical truth”, which they may never directly observe in any sense. For example, this is the case if mathematical truth is interpreted as being the response distribution of some far-future community of mathematicians that the mathematician will not live long enough to encounter.
such that \( S \subset E^{-1}(S) \). In this case, having \( E(S) \) be the state of the scientist’s brain would not fix the state of the universe to be an element in \( S \). So a necessary condition for the observation of the scientist to discriminate between the actual outcomes of the experiment is that if we apply the embedding function \( E \) to each of those possible outcomes of the experiment, we get a state of the scientist’s brain that uniquely distinguishes among those possible outcomes, via the inverse function \( E^{-1} \).

We formalize this as follows:

**Definition 6.1.** Let \( \varphi_1 \) be a backward-consistent SMS and \( \varphi_2 \) an SMS, where \( \psi \) and \( \Psi \) are the associated partial functions that have domains and ranges as described in Definition 4.1. Let \( n \) be a positive integer and let \( E \) be an embedding function from \( \hat{C}_1 \) to \( \hat{C}_2 \). We say that an associated embedded prediction pair \((\hat{C}, q)\) is **discriminating** if for all associated embedded prediction triples \((\hat{C}, q, v)\), for all associated \( v_1^m \in \text{supp} \mathcal{P}_1 \left( \mathcal{V}_1^m \mid \psi(q), E^{-1}[\{(q, v)\} \cup \hat{C}] \right) \),

\[
E^{-1}E \left[ \{(\psi(q), v_1^m)\} \cup E^{-1}[\{(q, v)\} \cup \hat{C}] \right] = \{(\psi(q), v_1^m)\} \cup E^{-1}[\{(q, v)\} \cup \hat{C}].
\]

It is important to be clear about what **Definition 6.1** does not say. It does not say that \( E \) must be an injective function from its domain to its image. Nor does it say that, for any \( v \in \mathcal{V}_2 \) and \( \mathcal{V}_2 \) \( E^{-1}[\{(q, v)\} \cup \hat{C}] = \{(q, v)\} \cup \hat{C} \). Therefore, \((\hat{C}, q)\) being a discriminating pair is still consistent with mental states of the scientist being multiply realizable in terms of outputs of the universe-SMS. Rather, **Definition 6.1** says only that when a scientist poses a question \( q \) of the form “what is the probability distribution over answers to the question(s) in \( \psi(q) \)?”, and provides some answer \( v \in \mathcal{V}_2 \), and that answer is interpreted by the function \( \Psi \) as specifying some probability distribution \( \Psi(q, v)(\mathcal{V}_1^m) \), the scientist-SMS can observationally distinguish between all elements in the support of \( \Psi(q, v)(\mathcal{V}_1^m) \), which are answers given by the universe-SMS. Note that the scientist-SMS need not even “speak universe” for \((\hat{C}, q)\) to be a discriminating pair. That is, elements of the support of \( \Psi(q, v)(\mathcal{V}_1^m) \) need not be in the scientist-SMS’s “cognitive vocabulary” of answers \( \mathcal{V}_2 \). All that is needed is that the scientist-SMS has some way of individuating between elements of the support of \( \Psi(q, v)(\mathcal{V}_1^m) \), even if they do so using different “terminology” (i.e., they use elements of \( \mathcal{V}_2 \) to distinguish between elements of \( \mathcal{V}_1^m \) via a translation process).

This definition of a discriminating pair is in keeping with an informal understanding of the neurobiological process of observation. To illustrate, note that images in the human retina are displayed upside-down relative to how they are observed. So the scientist might be such that their observations systematically lack fidelity with the environment that they aim to represent. Nevertheless, the scientist is able to observe scenes of their environment because each of their mental representations — upside-down though they may be — maps to a distinct set of possible actual scenes. On the other hand, for a scientist to observe whether there is a particular pattern in ultraviolet on a flower does not require that the image be right-side up in their brain. Rather it requires that they be able to discriminate among different patterns in ultraviolet. If they can only observe in the visible part of the spectrum, then they cannot make such an observation. In the same way, if a pair \((\hat{C}, q)\) is discriminating, then it is not necessarily the case that a scientist makes observes possible answers to the questions in \( \psi(q) \) in exactly the same form that the universe
outputs them. But it is necessarily the case that they can make observations that allow them to discriminate between these possible answers.

Combining Definition 6.1 with Definition 5.1 and Definition 5.3 establishes the following:

**Proposition 6.2.** Let $\varphi_1$ be a backward-consistent SMS and $\varphi_2$ an SMS, where $\psi$ and $\Psi$ are the associated partial functions that have domains and ranges as described in Definition 4.1. Let $n$ be a positive integer. Suppose that $\varphi_2$ is embed-calibrated with $\varphi_1$ at step $n$ for the embedding function $E$, the pair $(q, \hat{C})$, and for $\epsilon = 0$. Finally, suppose that $(q, \hat{C})$ is a discriminating pair. Then for all $v_1^m \in V_1^m$,

$$\Psi(\psi(q), v)(v_1^m) = \frac{P^n_2 \left( E \left[ \{ (\psi(q), v_1^m) \} \cup E^{-1}[\{ (q, v) \} \cup \hat{C}] \} \right) }{P^n_2 \left( E \left[ \{ (\psi(q)) \} \cup E^{-1}[\{ (q, v) \} \cup \hat{C}] \} \right) }$$

To understand Proposition 6.2 intuitively, suppose that (like in the analysis of calibration for mathematical reasoning) $C_2$ were a subset of $C_1$. Suppose further that $\psi(q) \in Q_2$, and so the values $v_1^m$ in Proposition 6.2 were all members of the product space $V_2^m$. Then Proposition 6.2 would imply

$$(6.1) \quad \Psi(\psi(q), v)(v_1^m) = P^n_2 \left( v_1^m | \psi(q), E^{-1}[\{ (q, v) \} \cup \hat{C}] \} \right)$$

This would essentially amount to establishing that $\varphi_2$ is honest, in the sense of Definition 4.3.

In the remainder of this paper we turn to two concrete applications of our framework, by showing how it vindicates two heuristics that are common in both mathematics and the empirical sciences. As stated in the introduction, the first of these heuristics is giving more weight to hypotheses that are established via multiple lines of reasoning or pieces of evidence, and the second heuristic is the common practice of abductive inference.

7. **Application I: Multiple Lines of Evidence**

A common heuristic among real world mathematicians is to ascribe greater epistemic value to a proposition if there are multiple lines of evidence that favor it. What is essentially the same heuristic appears among empirical scientists when they greater epistemic value to a prediction about the outcome of an experiment if there are multiple lines of evidence that support that prediction.

In formal epistemology, this heuristic is sometimes called the “variety of evidence thesis,” (see, among others, Wayne (1995), Myrvold (1996), Fitelson (1996), Bovens and Hartmann (2002), Claveau and Grenier (2019), and Landes (2020a, 2020b)). While it has decided intuitive appeal as a principle of reasoning in both empirical science and mathematics, it has resisted straightforward formal vindication. In this section we show how our approach to modeling of mathematical and scientific reasoning using SMSs vindicates a version of this heuristic, assuming that the appropriate calibration relationship holds.

7.1. **Multiple Lines of Evidence in Mathematics**

Consider the probability that a given mathematician-SMS $\varphi_2$ will produce some particular answer $v^*$ in response to a given question $q$. We view some particular set of claims, $B$, as being a “line of evidence” or “evidence path” for that answer to $q$ if the probability that the SMS will in fact produce $v^*$ in response to $q$ (rather than some alternative $v' \neq v^*$) is higher if the probability is conditioned on the SMS having already generated
$B$, as compared to the probability of that answer without conditioning on $B$. In practice, such an evidence path will typically take the form of a “proof” (e.g., as published in the mathematical literature). That is, it will consist of written material that expresses, with varying degrees of formal precision (and varying degrees of confidence by author of the proof), a series of reasoning steps intended to establish a given conclusion. Such a proof $B$ is a line of evidence that $v^*$ is in fact the correct answer to $q$ if the posterior probability that the far-future community of mathematicians will answer $v^*$ in response to question $q$ conditioned on $B$ is larger than the prior, unconditioned probability of that answer by the far-future community of mathematicians.

Here we formally justify this heuristic by proving that if an SMS $\varphi_2$ representing a mathematician is calibrated with an SMS $\varphi_1$ representing an oracle that determines the correct response distribution for each prediction pair $(q, \hat{C})$ (e.g., the far-future community of mathematicians), then the more evidence paths there are for a given claim $(q,v)$ according to $\varphi_2$, the more likely it is that the oracle also responds to $q$ with the answer $v$. This provides a normative justification for the practice of assigning higher probability to a given answer to a mathematical question the more evidence paths there are for that answer.

Define some arbitrary question and answer space $Q, V$, and let $\hat{C}$ be the associated collection of all finite sets of claims ($q \in Q, v \in V$). Let $P$ be any distribution defined over $\hat{C}$. (While we will mostly have in mind the case where $P$ is the distribution of a mathematician-SMS, that is not required below.) Define $B = \{ B(i) : 1, \ldots, n \}$ to be some indexed collection of $n$ claim sets in $\hat{C}$. (Note that we allow the different $B(i)$ to have nonzero intersection, i.e., to share claims.) Let $\beta$ also be a claim set in $\hat{C}$, and let $q \in Q$ be any question. Abusing notation, we will implicitly interpret a list of claim sets as their union, e.g., implicitly equating $P(B(1), \beta) = P(B(1) \cup \beta)$. Note that in the usual way, since any claim $(q,v)$ is a claim set (a claim set having a single element), expressions like $P(v \mid q, \beta, B(1))$ are well-defined, as ratios of probabilities of claim sets. (See Definition 3.1)

We need to formalize the idea that each $B(i)$ is a line of evidence for a particular answer $v^*$ (conditioned on $q$ and $\beta$). We also need to formalize the notion that the separate $B(i)$ do not “confound one another”, i.e., that it is not the case that taken on their own, the separate $B(i)$ each increase the probability of $v^*$, but taken as group, they do not have that effect. Finally, we need to formalize the notion that $\beta$ is a set of “foundational” claims, which underlie the ability of each $B(i)$ to be an evidence path.

There are many ways to translate these desiderata concerning $B, \beta, q$ and $v^*$ into formal requirements. Perhaps most straight-forward is captured in the following definition:

**Definition 7.1.** Let $B = \{ B(i) : 1, \ldots, N \}$ be some indexed collection of $N$ claim sets in $\hat{C}$, $\beta$ another claim set in $\hat{C}$, and $q \in Q$ be any question. Suppose that for all $i : 1 \leq i \leq N$,

(7.1) \[ P(v^* \mid q, \beta, B(i)) > P(v^* \mid q, \beta) \]

Suppose further that for all those $i$,

(7.2) \[ P(v^* \mid q, \beta, B(1), \ldots, B(i)) > P(v^* \mid q, \beta, B(1), \ldots, B(i-1)) \]

(where we interpret the $i = 1$ version of this statement to mean $P(v^* \mid q, \beta, B(1)) \geq P(v^* \mid q, \beta)$).

Finally, suppose that all of the conditional probabilities in those equations are nonzero. Then we
refer to each of the sets $B(i)$ as an evidence path (for the claim $(q,v^*)$ under the distribution $P$ and conditioned on the claim set $\beta$), with $B$ being an evidence collection.\footnote{Below we will assume that a given collection of claim sets is an evidence collection, i.e., that Eq. \eqref{eq:7.2} holds. It turns out that we can derive Eq. \eqref{eq:7.2} instead, from another assumption, one that might seem more innocuous. That alternative is presented, along with the proof that it results in Eq. \eqref{eq:7.2}, in Appendix B.}

(The precise indexing of the $N$ claim sets typically does not matter.)

At this point, despite our language, Definition \ref{def:7.1} does not justify the heuristic of ascribing greater credence to a claim if it is supported by multiple lines of evidence. The problem is that a mathematician-SMS might be delusional, generating arbitrary claims like ‘$1+1=3$’, or even claims which are not well-formed formulas, like ‘$1+\int dx = \sin(d)$’, or other claims which have no connection to “mathematical truth”, however that concept is conceptualized. So it may be that evidence paths combine synergistically as in Eq. \eqref{eq:7.2} for a mathematician, but that this just means their delusional reasoning process has a certain property, with no implication for mathematical truth. To put this another way, even though a mathematician’s reasoning may be such that delusional reasoning process has a certain property, with no implication for mathematical truth. To ground the idea that the mathematician ought to regard $B$ as a set of lines of evidence for that claim.

To address this issue, as discussed in Section \ref{sec:6}, we identify mathematical “truth” with the limit distribution of a mathematical oracle, and suppose that the mathematician is calibrated with that oracle for some small $\epsilon$. We now show that this supposition implies that if a given collection of claim sets are evidence paths of the step-$n$ mathematician for a given question-answer pair, then taken as a whole, they increase the probability of that answer to that question by the oracle.

First, note that calibration is defined only for a specific $q$ and $\hat{C}$. However, to investigate the implications of Eq. \eqref{eq:7.2} for a mathematician-SMS who has multiple evidence paths for a given answer to a question, and is making predictions about the responses of an oracle, we need to consider multiple pairs of $q$ and $\hat{C}$. This requires an additional definition:

\textbf{Definition 7.2.} Suppose we are given a backward-consistent SMS $\varphi_1$, an SMS $\varphi_2$, associated partial functions $\psi, \Psi$ as in Definition \ref{def:4.1} and some $n \in \mathbb{Z}^+$. Assume that $\psi$ is invertible over its domain of definition. Define the associated event space of triples,

$$\Lambda := \{(v^b \in V^b_1, q^b \in Q^b_1, \hat{C}^b_2 \in \hat{C}^b_2) : b \in \mathbb{Z}^+\}.$$

The prediction distribution $F^n_2$ is a distribution over $\Lambda$ defined by

\begin{enumerate}
\item $F^n_2(v^m | \psi(q), \hat{C}_2) := \sum_{v' \in V_2^n} P^n_2(v' | q, \hat{C}_2)[\Psi(\psi(q), v')(v^m)]$ for all $(q, \hat{C}_2)$ that are prediction pairs for $\varphi_2$ and $n$;
\item $F^n_2(\psi(q), \hat{C}_2) \propto P^n_2(q, \hat{C}_2)$ for all $(q, \hat{C}_2)$ that are prediction pairs for $\varphi_2$ and $n$;
\item $F^n_2(\psi(q), \hat{C}_2) = 0$ for all $(q, \hat{C}_2)$ that are not a prediction pair for $\varphi_2$ and $n$;
\end{enumerate}

where as usual $m$ is shorthand for $|\psi(q)|$, and it is assumed that there is at least one $(q, \hat{C}_2)$ that is a prediction pair for $\varphi_2$ and $n$.

The only role of $\varphi_1$ in Definition \ref{def:7.2} is indirect, via the requirement that $(q, \hat{C}_2)$ be a prediction pair. Accordingly we will often shorten $F^n_2$ to simply $F^n$.\footnote{Below we will assume that a given collection of claim sets is an evidence collection, i.e., that Eq. \eqref{eq:7.2} holds. It turns out that we can derive Eq. \eqref{eq:7.2} instead, from another assumption, one that might seem more innocuous. That alternative is presented, along with the proof that it results in Eq. \eqref{eq:7.2}, in Appendix B.}
Note that \( F^m \) is a fully specified distribution. For example, (3) tells us that if \((q, \hat{C}_2)\) is not a prediction pair, then \( F^m(\{(\psi(q), v^m)\} \bigcup \hat{C}_2) = 0 \) for any \( v^m \). As another example, we can define conditional distributions like \( F^n(B(1) \mid \psi(q), \beta) \) for any two claim sets \( B(1), \beta \) such that \((q, \hat{C}_2 = B(1) \cup \beta)\) is a prediction pair. Note also that although defined in terms of an SMS, \( F^n \) is not an SMS, nor a marginal distribution of an SMS. Rather, loosely speaking, \( F^n \) is the average prediction made by the mathematician for the answer of the oracle to the questions specified in \( \psi(q) \), i.e., it is the average prediction of the mathematician for the answers to the questions that will be output by the “true” distribution.\(^{12,13}\)

These definitions allow us to state the following result:

**Proposition 7.3.** Suppose we are given some \( N, n \in \mathbb{Z}^+, \epsilon \in \mathbb{R}^+ \), backward-consistent SMS \( \varphi_1 \) and SMS \( \varphi_2 \) where \( \hat{C}_2 \subseteq \hat{C}_1 \). Suppose further that we are given associated partial functions \( \psi, \Psi \) as in Definition 4.1. Let \( q \) be a question such that \( |\psi(q)| = 1 \). Fix some \( v^* \in V_1 \) and suppose that

1. \( \mathcal{B} = \{B(i)\} \) is a set of \( N \) evidence paths under prediction distribution \( F^n \) for \((\psi(q), v^*)\) conditioned on a claim set \( \beta \subseteq \hat{C}_2 \);
2. \((q, \hat{C}_2)\) is a prediction pair for \( \varphi_2 \) at step \( n \), for any claim set \( \hat{C}_2 \subseteq \beta \bigcup \bigcup_i B(i) \);
3. \( \varphi_2 \) is calibrated with \( \varphi_1 \) at step \( n \) for \( q, \epsilon \) and any \( \hat{C}_2 \subseteq \beta \bigcup \bigcup_i B(i) \), with divergence measure \( D[,] \);
4. \( D[,] \) is a locally Lipschitz continuous function of its arguments.\(^{14}\)

Then for all \( 1 \leq i \leq N \), for small enough \( \epsilon \),

\[
\mathcal{P}_1(v^* \mid \psi(q), \beta, B(1), \ldots, B(i)) > \mathcal{P}_1(v^* \mid \psi(q), \beta, B(1), \ldots, B(i - 1)).
\]

(The restriction that \( |\psi(q)| = 1 \) in Proposition 7.3 is simply for convenience, to avoid introducing extra notation.)

Intuitively, Proposition 7.3 means that if the mathematician is calibrated with an oracle, then the greater the number of evidence paths for the claim \((\psi(q), v^*)\) under the distribution \( F^n \) — the average prediction made by that mathematician for the response distribution of the oracle — the more likely it is that the oracle actually responds with the answer \( v^* \) to the question \( \psi(q) \). In this sense, if a calibrated mathematician-SMS follows the multiple lines of evidence heuristic in their reasoning, assigning greater probability to a proposition if there are more evidence paths supporting it, then the far-future community of mathematicians (or, more generally, any arbiter of mathematical truth) also assigns greater probability to that proposition.

\(^{12}\)One must be careful with this informal interpretation of the prediction distribution though. For example, the proportionality constant in Definition 7.2 does not equal 1 in general, since \( q \) and \( \psi(q) \) live in different dimensional spaces.

\(^{13}\)We require \( \psi \) to be invertible to avoid the problem of how to define \( F^n(\psi(q), \hat{C}_2) \) if there is some \( q' \neq q \) such that \( \psi(q') = \psi(q) \), but while \((q', \hat{C}_2)\) is a prediction pair, \((q', \hat{C}_2)\) is not a prediction pair, e.g., because it has zero probability of being generated by \( \varphi_1 \).

\(^{14}\)Note that depending on the choice of the divergence \( D[,] \), this Lipschitz continuity assumption might require that those distributions have full support.
7.2. Multiple Lines of Evidence in Science. We can analyze the potential benefit of multiple lines of evidence to a scientist using essentially the same approach we used for analyzing the potential benefit of multiple lines of evidence to a mathematician. We simply need to consider a scientist-SMS embedded in a universe-SMS rather than a mathematician-SMS predicting a far-future community of mathematicians SMS.

The formal definition of an evidence path for the case of scientist-SMSs embedded in a universe-SMS is exactly the same as in the case of mathematics, given in Definition 7.1. However, the interpretation of an evidence path is different. In a scientific context, we conceive of evidence paths not as chains of proof-like mathematical reasoning, but instead as chains of (states of the brain of the scientist that correspond to the) outcomes of experiments or observations the scientist has made. Similarly, the formal definitions of prediction pairs and of the distribution $F^n$ are unchanged, but the interpretation of $F^n$ is different. In the case of mathematicians, it is the (expected value of) the prediction of the mathematician for the outcomes generated by the far-future community (or whatever the oracle is) to the mathematical questions specified in $q$. In the case of a scientist, it is instead the (expected value of) the prediction of the scientist for the outcomes generated by the universe to the experiments specified in $q$.

There is also one concrete, non-interpretational distinction between the case of mathematicians predicting the answers of their far-future community to mathematical questions and scientists predicting the outcomes of physical experiments. In the case of mathematicians and calibration, the quantity that we require to have been considered by the far-future community of mathematicians, the claim set, is identical to the chains of proof-like mathematical reasoning considered by the mathematician. In contrast, the associated quantity arising in the assumption of embedding calibration is $E^{-1}([\{(q,v)\} \cup \hat{C}_2])$, the collection of all universe-SMS claim sets that are consistent both with the scientist’s mental perception of the outcomes of chains of experiments or observations they have made, and with their subsequent prediction about the probability distribution over the outcomes of the experiments in $\psi(q)$.

These distinctions yield a result that is similar to Proposition 7.3, but not quite identical. To state this new result, define an embedded prediction distribution exactly as a prediction distribution is defined, in Definition 7.2, with two changes. First, an embedding function $E$ is specified. Second, wherever the pair $(q,\hat{C}_2)$ arises in Definition 7.2, it is now taken to be an embedded prediction pair for embedding function $E$, rather than a prediction pair.

**Proposition 7.4.** Suppose we are given some $N, n \in \mathbb{Z}^+, \epsilon \geq 0$, backward-consistent SMS $\varphi_1$ and SMS $\varphi_2$ with embedding function $E$. Suppose further that we are given associated partial functions $\psi, \Psi$ as in Definition 4.1 and a question $q$ such that $|\psi(q)| = 1$. Fix some $v^* \in V_1$ and suppose that

1. $B = \{B(i)\}$ is an $N$-element evidence collection under embedded prediction distribution $F^n(\psi(q), v^*)$ and the embedding function $E$, conditioned on a claim set $\beta \in \hat{C}_2$;
2. $(q,\hat{C}_2)$ is an embedded prediction pair for $\varphi_2$ at step $n$ for embedding function $E$, for any claim set $\hat{C}_2 \subseteq \beta \cup \bigcup_i B(i)$;
3. Given the $\hat{C}_2$ in condition (2), $\varphi_2$ is embed-calibrated with the SMS $\varphi_1$ at step $n$ for $\epsilon$ and $(q,\hat{C}_2)$, with divergence measure $D[\ldots]$ and embedding function $E$;
4. $D[\ldots]$ is a locally Lipschitz continuous function of its arguments.
Then for all $1 \leq i \leq N$, for small enough $\epsilon$,

\[(7.3) \sum_{v \in V_2} P^2_2(v|q, \beta, B(1), \ldots, B(i)) \overline{P}_1(v^* | \psi(q), E^{-1}[(q, v) \cup \beta, B(1), \ldots, B(i)]) > \sum_{v \in V_2} P^2_2(v|q, \beta, B(1), \ldots, B(i-1)) \overline{P}_1(v^* | \psi(q), E^{-1}[(q, v) \cup \beta, B(1), \ldots, B(i-1)]).
\]

(As with Proposition 7.3, the restriction in Proposition 7.4 that $|\psi(q)| = 1$ is for convenience.)

Suppose that a scientist embedded in a universe-SMS views $B$ as an evidence collection for some claim $(\psi(q), v^*)$, and that this scientist is embed-calibrated with the universe-SMS. Loosely speaking, Proposition 7.4 says that under these conditions, the more evidence paths the scientist observes, the higher (the expected value of) the probability that the universe-SMS itself outputs $v^*$ as the answer to the question $\psi(q)$.

There is an important difference between the conclusion of Proposition 7.3 and the conclusion of Proposition 7.4. Proposition 7.3 establishes that the probability that the mathematical oracle (e.g., the far-future community of mathematicians) $\varphi_1$ outputs $v^*$ as the answer to $\psi(q)$ grows with the number of evidence paths that the present-day mathematician $\varphi_2$ observes for the claim $(\psi(q), v^*)$. By contrast, Proposition 7.4 shows that the expected probability that the universe-SMS $\varphi_1$ outputs $v^*$ as the answer to $\psi(q)$, where that expectation is taken across all possible predictions that the scientist-SMS $\varphi_2$ might make about the probability distribution over answers to $\psi(q)$, grows with the number of evidence paths that the scientist-SMS observes for the claim $(\psi(q), v^*)$.

Ultimately, the reason for this difference is that in the case of empirical science there is only a single SMS (the physical universe) and a projection of that SMS, which can be interpreted as another, dependent SMS (the scientist). In contrast, in the case of mathematics, there are two SMSs which are allowed a priori to be completely independent (e.g., a current mathematician and an oracle, like the far-future community of mathematicians). Concretely, in the case of a scientist, the physical fact that that scientist’s brain is in a state that involves asking question $q$ and giving answer $v$ restricts the set of possible physical universes, over and beyond the restriction given by the fact that the state of the scientist’s brain also involves a claim set $\beta, B(1), \ldots$. (This is why the claim $(q, v)$ is part of the argument of the inverse embedding function in the definition of embed-calibration.) That in turn means that in the definition of embed-calibration, the average of $D(\ldots)$ over all answers $v$ by the scientist affects both of $D(\ldots)$’s arguments. In contrast, in the case of a mathematician-SMS predicting the answers of an oracle-SMS, we don’t assume that the oracle-SMS generates the claim $(q, v)$ made by the mathematician-SMS. So the average of $D(\ldots)$ only affects the first of $D(\ldots)$’s arguments.

Another important point is that one might think that the supposition that $B$ is an evidence collection already says all that’s important, from a normative perspective, with none of the other assumptions needed. Recall though that as discussed above, $B$ could be an evidence collection simply because the scientist is prone to hallucinations about the results of experiments and is self-consistent in their hallucinations. To rule out this possibility of self-consistent hallucination we need to couple what the scientist thinks they are likely to observe in the future with the actual future states of the physical reality that they’re embedded in. The third supposition in Proposition 7.4.
that the scientist is also embed-calibrated, is one way to enforce this coupling. The result of this coupling is made explicit in Eq. (7.3), which involves both the distribution of the scientist and of the universe.

Even given this coupling though, so that the universe-SMS produces answer \( v^\ast \) to the experiments specified in \( \psi(q) \), one might worry that the scientist-SMS observing the universe might not be able to observe the results of such experiments. Or that even if they did observe those results, they might do so in a highly biased manner, being led to believe that the results of the experiments was not \( v^\ast \), even if it was. This possibility can be addressed by adding the requirement that the combination of each of the evidence path with the question \( q \) is a discriminating pair (see Definition 6.1).

There are other ways to couple what the scientist thinks they are likely to observe in the future with the actual future states of the physical reality that they’re embedded in. One way to do this is to flip the roles of the scientist’s distribution and the universe’s distribution in the suppositions in Proposition 7.4, as follows:

**Proposition 7.5.** Suppose we are given some \( N, \epsilon \in \mathbb{Z}^+, \epsilon \geq 0 \), backward-consistent SMS \( \varphi_1 \) and SMS \( \varphi_2 \) with embedding function \( E \). Suppose further that we are given associated partial functions \( \psi, \Psi \) as in Definition 4.1 and a question \( q \) such that \( |\psi(q)| = 1 \). Fix some \( v^\ast \in \mathcal{V}_1 \) and suppose that

1. \( \mathcal{B} = \{ B(i) \} \) is an \( N \)-element evidence collection under distribution \( \overline{P}_1 \) for \( (q, v^\ast) \) conditioned on a claim set \( \beta \);
2. \( (q, E[\hat{C}_1]) \) is an embedding prediction pair for \( \varphi_2 \) at step \( n \) and any claim set \( \hat{C}_1 \subseteq \beta \cup \bigcup_i B(i) \);
3. Given the \( \hat{C}_1 \) in condition (2), \( \varphi_2 \) is embed-calibrated with the universe-SMS \( \varphi_1 \) at step \( n \) for \( (q, E[\hat{C}_1]) \), with divergence measure \( D[.,.] \);
4. \( \overline{D}[.,.] \) is a locally Lipschitz continuous function of its arguments;
5. for all \( 1 \leq i \leq N \) and all \( v \in \mathcal{V}_2 \),

\[
\overline{P}_1(v^\ast|\psi(q), E^{-1}[(\{ q, v \} \cup E[\beta, B(1), \ldots, B(i)])] \propto \overline{P}_1(v^\ast|\psi(q), \beta, B(1), \ldots, B(i));
\]

Then for all \( 1 \leq i \leq N, v \in \mathcal{V}_1 \), for small enough \( \epsilon \), the embedded prediction distribution obeys

\[
F_2^n(v^\ast|q, E[\beta, B(1), \ldots, B(i)]) > F_2^n(v^\ast|q, E[\beta, B(1), \ldots, B(i - 1)]).
\]
8. APPLICATION II: ABDUCTION

8.1. ABDUCTION IN GENERAL. Following modern uses of ‘abduction’ as a synonym for ‘inference to the best explanation’ (see Douven 2021), we take abduction to be the following inference pattern:

1. The question $q^*$ has answer $v^*$ with probability $x^*$.
2. The question $q^\dagger$ has answer $v^\dagger$ with probability $x^\dagger$.
3. If $q^\dagger$ does indeed have answer $v^\dagger$, then $q^*$ has answer $v^*$ with probability $y^* > x^*$.
4. It turns out that $q^*$ does have answer $v^*$.
5. We conclude that $q^\dagger$ has answer $v^\dagger$ with a probability $y^\dagger > x^\dagger$.

In particular, if we take $y^* = 1$, then (3) means that the answer $v^\dagger$ to the question $q^\dagger$ implies that $v^*$ is the answer to the question $q^*$. The conclusion (5) then says that in light of this premise, if in fact $v^*$ is the answer to the question $q^*$, then it is more likely that $v^\dagger$ is the indeed the answer to the question $q^\dagger$.

In what follows we prove that both mathematicians and scientists ought to engage in abductive reasoning, under certain assumptions. This is a partial justification of using abduction as a heuristic in human reasoning.\footnote{See Viteri and DeDeo \citeyear{viteri2022} for a descriptive study of the role of abduction in mathematical reasoning.}

8.2. ABDUCTION IN MATHEMATICS. In this section, we show that if a mathematical oracle treats one claim as abductively supporting another, then an individual mathematician should follow the same inference pattern. To begin, we let $P(\cdot)$ be an arbitrary joint probability distribution defined over $\mathcal{C} \times \mathcal{C} \times \hat{\mathcal{C}}$ for some arbitrary claim space $\mathcal{C}$. (At this point, $P(\cdot)$ has nothing to do with SMSs of any sort.) For simplicity, suppose that under $P$ there is probability 1 that the two claims have questions $q^*$ and $q^\dagger$, respectively. Suppose as well that if the answer $v^\dagger$ is generated in response to the question $q^\dagger$, that would make it more likely that the answer $v^*$ would be generated in response to the question $q^*$, conditioned on a particular claim set $\hat{\mathcal{C}}$. Formally, this condition means that for some $\alpha > 1$,

\begin{equation}
\frac{P(v^* | q^*, (q^\dagger, v^\dagger), \hat{\mathcal{C}})}{P((q^*, v^*), (q^\dagger, v^\dagger), \hat{\mathcal{C}})} = \frac{P(q^*, \hat{\mathcal{C}})}{P(q^*, v^*, \hat{\mathcal{C}})},
\end{equation}

or equivalently,

\begin{equation}
P(v^* | q^*, (q^\dagger, v^\dagger), \hat{\mathcal{C}}) = \alpha P(q^*, v^*, \hat{\mathcal{C}}),
\end{equation}
and so repeatedly using our assumption that both \( q^* \) and \( q^\dagger \) occur with probability 1,

\[
P \left( \left( q^*, v^* \right), \left( q^\dagger, v^\dagger \right), \hat{C} \right) = \alpha \frac{P \left( \left( q^*, v^* \right), \hat{C} \right)}{P(\hat{C})}
\]

(8.3)

\[
P \left( \left( q^*, v^* \right), \left( q^\dagger, v^\dagger \right), \hat{C} \right) = \alpha \frac{P \left( \left( q^\dagger, v^\dagger \right), \hat{C} \right)}{P(\hat{C})}
\]

(8.4)

\[
P \left( \left( q^*, v^* \right), \left( q^\dagger, v^\dagger \right), \hat{C} \right) = \alpha \frac{P \left( \left( q^\dagger, v^\dagger \right), \hat{C} \right)}{P(\hat{C})}
\]

(8.5)

\[
P \left( \left( q^*, v^* \right), \left( q^\dagger, v^\dagger \right), \hat{C} \right) = \alpha \frac{P \left( \left( q^\dagger, v^\dagger \right), \hat{C} \right)}{P(q^\dagger, \hat{C})}
\]

(8.6)

i.e.,

\[
P \left( v^\dagger \mid q^\dagger, \left( q^*, v^* \right), \hat{C} \right) = \alpha P(v^\dagger \mid q^\dagger, \hat{C}).
\]

In the sequel we will refer to Eq. (8.1) as the “abduction premise” and to Eq. (8.6) as the “abduction implication”, and simply say that “abduction holds” when the former implies the latter. The abduction premise says that if the answer \( v^\dagger \) is generated in response to the question \( q^\dagger \), that would make it more likely that the answer \( v^* \) would be generated in response to the question \( q^* \). We have just shown that this premise implies its converse: if the answer \( v^* \) were generated in response to the question \( q^* \), that would make it more likely that the answer \( v^\dagger \) would be generated in response to the question \( q^\dagger \). So we have established that abduction holds for the distribution \( P(\cdot , \cdot) \), for the two question-answer pairs \( (q^*, v^*), (q^\dagger, v^\dagger) \).

In particular, consider the case where the conditional distributions \( P(\cdot | \cdot) \) in Eq. (8.1) are \( P^k_2(\cdot, \cdot | \hat{C}^k_2) \) for the step \( k \) of a mathematician-SMS \( \varphi_2 \), where as usual \( \hat{C}^k_2 \) is the claim set associated with a step-\( k \) claim set that they generate with nonzero probability, \( \hat{C}^k_2 \). Then we have established that abduction holds for that mathematician, at step \( k \), simply by the laws of probability theory. Similarly, we have established that abduction holds for the (limit distribution of the) oracle SMS, e.g., for the far-future community of mathematicians, again simply by the laws of probability theory.

However, as in the case of establishing the evidential value of multiple proof paths, we have made no assumptions at this stage that the output of the mathematician-SMS is in any way connected to mathematical truth. So, while we have stated some conditions under which a mathematician-SMS will reason abductively, we have not yet established that the same mathematician-SMS ought to reason abductively.

To address this issue, recall the prediction distribution \( F^n \) defined in Definition 7.2, which is the structure that captures what an SMS mathematician thinks the distribution of answers of an oracle-universe would be to a given set of questions. Specifically, \( F^n(v^*, v^\dagger | q^*, q^\dagger, \hat{C}_2) \) is a distribution over joint answers \( (v^*, v^\dagger) \) by the oracle-universe to the pair of questions \( (q^*, q^\dagger) \), constructed by averaging the “predictions” for what that distribution is made by the mathematician SMS \( \varphi_2 \) at step \( n \) (via the function \( \psi(q) \)). (All of these distributions are conditioned on the mathematician having also generated the claims listed in \( \hat{C}_2 \) at step \( n \).) Using \( F^n \) and our notational convention Eq. (8.6), we can establish the following sufficient conditions for an SMS \( \varphi_2 \) that is calibrated with an oracle-SMS \( \varphi_1 \) to be justified in abductive reasoning:
Proposition 8.1. Suppose we are given some $N, n \in \mathbb{Z}^+$, backward-consistent SMS $\varphi_1$, SMS $\varphi_2$, and associated partial functions $\psi, \Psi$ as in Definition 4.1 where $\psi$ is an invertible function. Let $v^*, v^1 \in \mathcal{V}_1$, and let $q^*, q^1 \in \mathcal{Q}_1$ be two questions such that all three tuples $q^*, q^1$ and $(q^*, q^1)$ are in the codomain of $\psi$. Suppose as well that

1. $\mathcal{F}^n$ satisfies the abductive premise, Eq. (8.1), for all $\hat{C}_2 \in \hat{C}_2$ and for the specific questions $q^*, q^1$ and answers $v^*, v^1$;
2. $(\psi^{-1}(q^*), \hat{C}_2), (\psi^{-1}(q^1), \hat{C}_2)$, and $(\psi^{-1}(q^*, q^1), \hat{C}_2)$ are all prediction pairs for $\varphi_2$ at step $n$ for any $\hat{C}_2 \in \hat{C}_2$;
3. $\sum v^* \mathcal{F}^n \left( v^*, v^1 | q^*, q^1, \hat{C}_2 \right) = \mathcal{F}^n \left( v^* | q^*, \hat{C}_2 \right)$;
4. $\varphi_2$ is calibrated with $\varphi_1$ at step $n$ for $\hat{C}_2$ for each of the three questions $\psi^{-1}(q^*)$, $\psi^{-1}(q^1)$, and $\psi^{-1}(q^*, q^1)$, and for $m = 1, 2$;
5. $\mathcal{D}[..]$ is a locally Lipschitz continuous function of its probability distribution arguments

Then $\mathcal{P}_1(v^1|q^1, (q^*, v^*), \hat{C}_2) > \mathcal{P}_1(v^*|q^*, \hat{C}_2)$.

The requirement that $\psi$ be invertible is for convenience. Note that since by condition (2) there are questions $q_2 \in \mathcal{Q}_2$ such that $(q_2, \hat{C}_2)$ is a prediction pair, $\hat{C}_2 \subseteq \hat{C}_1$. Intuitively, condition (3) means that the extra information that $\varphi_1$ produces the question $q^1$ doesn’t affect the expectation of the conditional probability that the mathematician assigns to the event that the oracle-SMS produces the answer $v^*$ to the question $q^*$ (given the information already contained in $\hat{C}_2$). Similarly, condition (5) means that the extra information that the limit distribution of $\varphi_1$ generates the question $q^1$ doesn’t affect the conditional probability that it produces the answer $v^*$ to the question $q^*$ (given the information already contained in $\hat{C}_2$).

Proposition 8.1 says that if the distribution $\mathcal{F}^n$ representing the beliefs of a mathematician-SMS $\varphi_2$ about the output of an oracle-SMS $\varphi_1$ satisfies the abductive premise for $q^*, q^1, v^*$ and $v^1$, and if $\varphi_2$ is appropriately calibrated with $\varphi_1$, then the oracle-SMS must obey the associated abduction implication. More concretely, note that we have specifically assumed that the mathematician-SMS is calibrated with the oracle-SMS for the questions $\psi^{-1}(q^*)$, $\psi^{-1}(q^1)$, and $\psi^{-1}(q^*, q^1)$. These are questions for the mathematician, which can be interpreted as asking them, ‘what is the probability distribution over possible answer(s) by the oracle-SMS to the questions $q^*/q^1/(q^*, q^1)$ (resp.)?’ So a mathematician who is calibrated with the oracle with respect to the probabilities that the oracle assigns to the answers to the questions $q^*$ and $q^1$ will also follow the abductive inference pattern when answering these questions. To the extent that we take it to be a normative goal of a mathematician or finite group of mathematicians that their predictions about the answers to questions match those of the far-future mathematical community, it stands to reason that said mathematicians ought to make use of abduction in at least some circumstances.

8.3. Abduction in Science. As when we considered the benefit of multiple evidence paths for empirical scientific reasoning rather than for mathematical reasoning, we begin by re-defining $\mathcal{F}^n$ as an embedded prediction distribution, rather than a prediction distribution simpliciter. This allows
Proposition 8.2. Suppose we are given some $N,n \in \mathbb{Z}^+$, backward-consistent SMS $\varphi_1$, SMS $\varphi_2$, and associated partial functions $\psi, \Psi$ as in Definition 4.1. Let $v^*, v^\dagger \in V_1$, and let $q^*, q^\dagger \in Q_1$ be two questions such that all three tuples $q^*, q^\dagger$ and $(q^*, q^\dagger)$ are in the codomain of $\psi$. Let $E$ be an embedding function. Suppose as well that:

1. $F^n$ satisfies the abductive premise, Eq. (8.1), for all $\hat{C}_2 \in \hat{C}_2$ and for the specific questions $q^*, q^\dagger$ and answers $v^*, v^\dagger$;
2. $(\psi^{-1}(q^*), \hat{C}_2), (\psi^{-1}(q^\dagger), \hat{C}_2)$, and $(\psi^{-1}(q^*, q^\dagger), \hat{C}_2)$ are all embedded prediction pairs for $\varphi_2$ at step $n$ for any $\hat{C}_2 \in \hat{C}_2$;
3. $\sum_{v^\dagger} F^n \left( v^*, v^\dagger | q^*, q^\dagger, \hat{C}_2 \right) = F^n \left( v^* | q^*, \hat{C}_2 \right)$;
4. $\varphi_2$ is embed-calibrated with $\varphi_1$ at step $n$ for $\hat{C}_2$ for each of the three questions $\psi^{-1}(q^*), \psi^{-1}(q^\dagger)$, and $\psi^{-1}(q^*, q^\dagger)$, and for $m = 1, 2$;
5. $D[,,]$ is a locally Lipschitz continuous function of its probability distribution arguments (where those distributions are considered as vectors in a Euclidean metric space);
6. For any $v \in V_2$,

$$\sum_{v^\dagger} P_1 \left( v^*, v^\dagger | q^*, q^\dagger, E^{-1} \left[ \{ (\psi^{-1}(q^*, q^\dagger), v \} \} \cup \hat{C}_2 \right] \right) = P_1 \left( v^* | q^*, E^{-1} \left[ \{ (\psi^{-1}(q^*), v \} \} \cup \hat{C}_2 \right] \right)$$

Then

$$\sum_{v \in V_2} P^n_2(v | \psi^{-1}(q^\dagger), (q^*, v^*) , \hat{C}_2) P_1(v^\dagger | q^\dagger, (q^*, v^*) , E^{-1} \left[ \{ (\psi^{-1}(q^\dagger), v \} \} \cup \hat{C}_2 \right]) > \sum_{v \in V_2} P^n_2(v | \psi^{-1}(q^\dagger), \hat{C}_2) P_1(v^\dagger | q^\dagger, E^{-1} \left[ \{ (\psi^{-1}(q^\dagger), v \} \} \cup \hat{C}_2 \right]).$$

This proposition shows that if a scientist-SMS believes that one claim $(q^*, v^*)$ abductively supports a second claim $(q^\dagger, v^\dagger)$, and they are embed-calibrated with the universe-SMS, then in expectation the universe-SMS itself is such that $(q^*, v^*)$ abductively supports a second claim $(q^\dagger, v^\dagger)$.

Next, as we did in the case of multiple evidence paths, we use the assumption that $\varphi_2$ is embed-calibrated with $\varphi_1$ to “project down” from the fact that the outcomes of experiments in the universe conform with the abductive inference pattern, to establish the same property for the observations by the scientists of the outcomes of those experiments:

Proposition 8.3. Suppose we are given some $N,n \in \mathbb{Z}^+$, backward-consistent SMS $\varphi_1$, SMS $\varphi_2$, and associated partial functions $\psi, \Psi$ as in Definition 4.1. Let $v^*, v^\dagger \in V_1$, and let $q^*, q^\dagger \in Q_1$ be two questions such that all three tuples $q^*, q^\dagger$ and $(q^*, q^\dagger)$ are in the codomain of $\psi$. Let $E$ be an embedding function. Suppose as well that:

1. $F^n$ satisfies the abductive premise, Eq. (8.1), for all $\hat{C}_2 \in \hat{C}_2$ and for the specific questions $q^*, q^\dagger$ and answers $v^*, v^\dagger$;
2. $(\psi^{-1}(q^*), \hat{C}_2), (\psi^{-1}(q^\dagger), \hat{C}_2)$, and $(\psi^{-1}(q^*, q^\dagger), \hat{C}_2)$ are all embedded prediction pairs for $\varphi_2$ at step $n$ for any $\hat{C}_2 \in \hat{C}_2$;
(3) For any $v \in V_2$,
\[
\sum_{v^1} \mathcal{P}_1 \left( v^*, v^1 \mid q^*, q^1, E^{-1} \left[ \{ \psi^{-1}(q^*, q^1), v \} \cup \hat{C}_2 \} \right] \right) = \mathcal{P}_1 \left( v^*, E^{-1} \left[ \{ \psi^{-1}(q^*), v \} \cup \hat{C}_2 \} \right] \right);
\]

(4) $\varphi_2$ is embed-calibrated with $\varphi_1$ at step $n$ for $E[\hat{C}_1]$ for each of the three questions $\psi^{-1}(q^*)$, $\psi^{-1}(q^1)$, and $\psi^{-1}(q^*, q^1)$, and for $m = 1, 2$;

(5) $D[., .]$ is a locally Lipschitz continuous function of its probability distribution arguments (where those distributions are considered as vectors in a Euclidean metric space);

(6) $\sum_{v^1} F^n \left( v^*, v^1 \mid q^*, q^1, E[\hat{C}_1] \right) = F^n \left( v^* \mid q^*, E[\hat{C}_1] \right)$;

Then $F^n(v^1 | q^1, (q^*, v^*), E[\hat{C}_1]) > F^n(v^1 | q^1, E[\hat{C}_1])$.

Intuitively, Proposition 8.3 shows that if the occurrence of the experimental outcome $(q^*, v^*)$ increases the probability that the universe-SMS assigns the question $q^1$ the answer $v^1$, then for a scientist who is embed-calibrated with the universe that embeds them, conditioning on $(q^1, v^1)$ ought to increase their expected degree of belief that the question $q^*$ has the answer $v^*$, in keeping with the abductive inference pattern. To the extent that being embed-calibrated with their physical universe is an appropriate normative goal for an empirical scientist, this result shows that such a scientist should engage in abductive reasoning, in expectation.

9. Relation with previous work

There are several points worth elaborating about how the SMS framework differs from frameworks previously investigated in the literature. First, it is important to contrast the SMS framework with concepts that have been studied in probability logic (e.g., Carnap 1950, Burgess 1969, Hoover 1978, Leblanc 1979, Hailperin et al. 1984, Nilsson 1986, Fagin, Halpern, and Meggido 1990, Leitgeb 2008, Haenni et al. 2010, Christiano et al. 2013, and Campbell-Moore 2015). When building a system of probability logic, one seeks “to start with a classical (propositional/modal/etc.) system of logic and to ‘probabilify’ it in one way or another, by adding probabilistic features to it” (Demey, Kooi, and Sack 2019). That is, whereas ordinary system of logic aim to axiomatize truth-preserving inferences between sentences that do not contain probabilistic language, probability logics seek either to axiomatize the process of making probability-preserving inferences from one sentence to another, or to axiomatize the process of making truth-preserving inferences between sentences that contain probabilistic language. (See also work on Markov logic networks due to Richardson and Domingos, 2006.) In contrast, the SMS framework does not require the axiomatization of either truth-preserving or probability-preserving inferences. Rather, we aim to provide a framework in which any pattern of inferences can be understood as the output of a stochastic process.

Although his proposal is less technical than the work in probability logic discussed immediately above, Franklin (1987) argues for the descriptive claim that a non-deductive logic, formulated in the language of probability theory, plays a crucial role in the development of mathematical results. His proposal contains a germ of an idea that we have developed here, namely that mathematical reasoning in human agents can be represented as a stochastic process. However, he holds that non-extreme probabilities assigned to mathematical statements should be treated as solely representing the subjective degrees of belief of individual mathematicians, and that such degrees of belief are only
justified prior to a statement’s being proven. By contrast, we hold that in principle, a mathematician
may always be justified in assigning non-extreme degree of belief to a mathematical claim.

Second, our framework shares some features with "erotetic logic," which aims to provide a for-
mal logic of questions and answers. Belnap and Steel (1976) provide a textbook introduction to
the subject, and subsequent developments include Hintikka’s Interrogative Model of Inquiry (see
Hintikka and Bachman, 1991) and Wiśniewski’s Inferential Erotetic Logic (IEL) (see Wisniewski,
2013). Our framework and erotetic logic are both primarily concerned with asking and answering
of questions. However, research in erotetic logic follows the general methodology of formal logic,
defining a syntax, semantics, and proof theory that allows one to prove that certain questions have
certain answers. By contrast, we use a probabilistic formalism to descriptively represent the asking
and answering of questions. We also extend this probabilistic formalism to model conditions under
which this process of question answering can be held to certain normative standards. In addition,
insofar as it provides a model for inquisitive reasoning under uncertainty, our framework shares
some similarities with recent work by Hoek (2022). However, the focus of Hoek’s work is on the
connections between an inquisitive model of belief and decision theory, rather than mathematical
or scientific reasoning.

Third, our approach has significant similarities to work from Garrabrant et al. (2016) and
Garrabrant et al. (2017), who present a computable algorithm that assigns probabilities to well-
formed formulas in a given logical language, and is able to update those probabilities as it obtains
more evidence about possible theorems of that language. We submit that their algorithm is a par-
ticular instance of the more general class of SMSs. The type of SMS Garrabrant et al. define is one
that considers questions of the form ‘what is the probability distribution over possible answers to
whether \( \phi \) a theorem of the logical language?’ and returns a probability distribution over the pos-
sible answers ‘Yes’ and ‘No.’ As a point of contrast between their approach and ours, we note that
the normative standards that they impose on their algorithm, where that algorithm is understood
as an SMS, are entirely internal to the SMS. For their algorithm to reason “correctly,” they require
only that it is impossible for a Turing machine to construct a series of bets that leads the algorithm
to sure loss in time polynomial in the size of the input to that Turing machine. In other words, they
require that their algorithm not be internally inconsistent in a way that can be efficiently exploited.
By contrast, in our approach an SMS is subject to normative standards that are external to the
SMS itself, in virtue of its calibration with a second SMS. We note that one can draw a similar
comparison between our approach and work by Lample and Charton (2019), in which they use
deep neural networks to approximate integrals and solve differential equations. These deep neural
networks can be thought of a special case of an SMS, in which questions are considered and answers
are output depending on the pattern of activation within the neural network, with the distribution
over possible outputs updated via supervised learning.

We note also that the approaches due to Garrabrant et al. do not provide a mechanism for
considering questions that are about the probability distribution over the answers produced by a
different SMS, made by that SMS in response to other questions. This flexibility is particularly
important when using SMSs to model human scientists making predictions about the outcome of a
future experiment concerning the embedding SMS of the physical universe, since such predictions
invariably take the form of probability distributions. For example, when asked to predict what
the weather will be tomorrow over London, commonly a scientist would respond by providing a probability distribution over the possible states of the weather, rather than a point prediction.

Fourth, we note the connections between our work and work in computational cognitive science on methods for defining probabilistic generative models, which are models that enable agents to generate random samples from a distribution. These include the programming language Church, first introduced by Goodman et al. (2012), and the related probabilistic Turing machine QUERY, which is introduced by Freer, Roy, and Tenenbaum (2014), and uses conditional simulation to implement hierarchical Bayesian inference. As in the case of Garrabrant et al.’s algorithm, we understand QUERY as a particular type of SMS that answers questions about its environment by randomly sampling a probability distribution over possible answers to a question, randomly sampling an answer from the sampled distribution, and comparing its predicted answer to observed answers, updating both of the distributions that it samples from over time. Notably, QUERY is the sort of SMS that can answer questions about the probability distributions over answers to some question, since it uses this information as part of its hierarchical Bayesian architecture. However, the category of SMSs is more general that the category of inference techniques implementable in QUERY. Unlike QUERY, the distributions the define an SMS are not required to be computable, nor do Freer, Roy, and Tenenbaum provide any explicit notion of semantics or normative constraints for QUERY that could be analogized to the notion of calibration for an SMS. That said, more carefully considering the connections between SMSs, QUERY, hierarchical Bayesian inference, and calibration is an intriguing avenue for future work.

Fifth, Icard (2020) considered a probabilistic version of the Chomsky-Schützenberger (1959) hierarchy of linguistic grammars. Each grammar in Icard’s hierarchy is defined by a set of restrictions on the rules for generating finite strings from other finite strings by sampling from other finite strings. Icard derives results showing that as we move to less-restricted probabilistic grammars, we are able to use those grammars to define increasingly general probability distributions. (For example, the most restricted class of probabilistic grammars can define finite-support, rational-valued probability distributions, while the least restricted can define any probability distribution that can be implemented by a probabilistic Turing machine.) Any SMS in which the claim set is restricted to pairs of finite strings can be placed in Icard’s hierarchy, with implications for the probability distribution(s) that define each SMS. However, SMSs also allow for even more general probabilistic grammars, if we allow for infinite strings in the claim set. An intriguing avenue for future work would be to consider how calibration between SMSs is effected by where those two SMSs stand in Icard’s hierarchy. (See also work by Lin and Tegmark (2017) on criticality in probability distributions definable at different levels of a probabilistic Chomsky hierarchy.)

Sixth and finally, we note recent work in the foundations of physics by Gisin (2019, 2021). He argues that even classical mechanics should be understood as indeterministic, on the grounds that real-valued quantities are physically unrealistic, and should be replaced with intuitionistic numbers, which are finite descriptions of stochastic processes that produce real numbers in the infinite limit. By representing physical systems (even classical physical systems) as stochastic processes, Gisin’s framework bears a significant resemblance to ours. However, there are also crucial differences. In particular, Gisin argues that indeterminism in physical systems is driven by the fact that physical quantities only exist in finite time, and so their precise value is not allowed to tend towards a limit.
This emphasis on the role of time in explaining key aspects of physical reality leads him to explicitly reject the block universe hypothesis in favor of an ontologically meaningful notion of time (2021, p. 13364). By contrast, within the SMS framework time plays no explicit role in conceptualizing indeterminism in physical systems. As explicitly noted above, we intend for our framework to be consistent with a block-universe view.

10. Discussion

There are several aspects of the stochastic mathematical systems framework we introduced and explored above that are worth exploring further. First, note that we needed to introduce additional structure when formalizing scientific reasoning, beyond the formal structure needed to formalize mathematical reasoning. In particular, we had to introduce additional formal structure (namely, the concept of an embedding) to capture our normative constraints on scientific reasoning, beyond the formal structure needed to capture normative constraints on mathematical reasoning. This reflected the fact that we take abstract mathematical reasoning by a human reasoner not to be normatively constrained by the physical universe in which the mathematician finds themselves, whereas we do take scientific reasoning to be normatively constrained in this way. This amounts to a sharp distinction between the norms of mathematics and the norms of science that we hope to explore in future work.

Focusing just on the case of scientific reasoning, there has been previous work that considers the implications of a reasoner being embedded in the system they are reasoning about. In particular, see the “inference devices” framework of Wolpert (2017). Within that framework a “monotheism” theorem is proven, showing that no two reasoners embedded in the same physical system can both be perfectly accurate predictors. An intriguing avenue for future work on the SMS framework is to see whether similar results hold for two scientist-SMSs both embedded within the same universe-SMS.

Another potentially fruitful line of research might be consider whether there are analogs of Gödel’s second incompleteness theorems for the SMS framework. Specifically, potential future work would investigate whether, if an SMS assigns non-zero probability to both answers to all true/false questions, then it would in particular assign non-zero probability to the answer of ‘false’ to self-referential questions about that same SMS’s probabilistic pattern of answers to questions.

In a similar vein, we believe that our approach may be fruitful for extending the “no free lunch theorems” of Wolpert (1996), Wolpert and Macready (1997), and Wolpert (2021), which provide a set of formal bounds on how well any machine learning algorithm or search algorithm can be guaranteed to perform if one does not make assumptions for the prior probability distribution of the underlying stochastic process, to mathematical and scientific learning and reasoning more generally. Fourth, we hope to study the relationships between multiple mathematician-SMSs or scientist-SMSs that are calibrated with the same community-SMS and/or embedded in the same universe-SMS. Our hope is that such an inquiry will be fruitful for understanding, in a general way, the properties of reasoners that, despite differing in crucial respects, share the same normative goals.

Another potential line of future research involves the “problem of (logical) omniscience” in epistemic logic. Briefly, the problem is that many forms of epistemic logic suppose that if a reasoner knows any proposition \( A \), and knows that \( A \) implies \( B \) according to some canonical set of axioms, then they know \( B \). For example, according to many epistemic logics a reasoner who knows the
axioms of number theory automatically must also know all theorems of number theory, which is clearly absurd. However, consider defining what an SMS at a step \( n \) “knows” to be the contents of a claim set it produces at that step. In general, an SMS can with nonzero probability produce a claim \( A \), a claim \( A \rightarrow B \), and also a claim \( \neg B \). Thus, the problem of logical omniscience disappears in the SMS framework, if we adopt its natural definition of what it means to “know”. These considerations can also be generalized into a Bayesian setting. There the problem of omniscience is that, because the sample space over which probabilities are defined is typically assumed to be closed under the entailment relation of some logic, a reasoner whose degrees of belief are modeled by that probability function must be represented as maximally confident in any theorems of that logic. See Skipper and Bjerring (2020) for a recent and more detailed discussion of this problem in a Bayesian context. The SMS framework makes no assumptions about the closure of the sample space under any logic, and so might therefore be seen as immune to the problem of logical omniscience, though more work is needed to defend this thesis rigorously.

As another possible research direction, we note that in both the mathematical universe approach of Tegmark (1998, 2008, 2014), and the work of the ontic structural realists, physical reality is given a structural representation as a collection of functions between sets. While this may seem to be a very different approach to our representation of the physical universe as an SMS, we hope in future work to show how one can model structure as being generated through the SMS’s process of asking and answering questions. We suspect that in so doing, we can shed new light on the relationships between computability, indeterminism, and the structuralist approach to the representation of physical reality.

More speculatively, we are also interested in whether the framework presented here may be helpful in quantifying the extent to which foundational results in mathematics, like Gödel’s incompleteness theorem or the undecidability of the halting problem are robust to small perturbations in the oracle-SMS that is used to set the standards of mathematical correctness. This could in turn lead to improved understanding of the modal status of these results across alternative possibilities with respect to the norms of mathematical reasoning.

Having said all this, we hope that there are many more applications, both practical and theoretical, of the approach presented here that we have not yet anticipated. Indeed, the two applications of our SMS framework presented in this paper—establishing the benefit of abduction and establishing the benefit of multiple lines of evidence—do not exploit the fact that the relevant response distributions arise via an infinitely iterated stochastic process. The analysis would still go through if that were not the case. However, we expect that some of these future directions of research on other applications of the SMS framework would exploit that fact.

**Appendix A. Additional Facets of the SMS Framework**

A.1. **Claim Trajectories.** Although it was not needed for the results or discussion in this paper, we can also use the SMS framework to assign probabilities to trajectories of claims produced at different steps of an SMS.

**Definition A.1.** A *claim vector trajectory* \( \vec{C}^n \) is a map taking any \( i \in 1, \ldots, n \) to a claim vector if \( n \) is finite, and taking any \( i \in \mathbb{Z}^+ \) to a claim vector if \( n = \infty \).
As shorthand, if a claim vector trajectory contains only a single claim vector, then we will sometimes write that trajectory as that single claim vector. In the sequel the spaces $Q$ and $V$ will often be implicit. We will also cavalierly use $|.|$ to indicate cardinality. So for example, $|C|$ is the number of components of a claim vector $C$, $|\hat{C}|$ is the number of elements in the claim set $\hat{C}$, etc.

It will occasionally be useful to consider probability distributions concerning the event that the claim vector generated at the $n$-th step of a particular SMS contains the answer $v$ in response to the question $q$. We present some definitions that will be useful when working with such distributions for the case where $C$ is countable; their extensions to other spaces is straightforward.

**Definition A.2.** Suppose we are given an SMS $\varphi = (\mathcal{C}, X)$ and a claim vector trajectory $\vec{C}^{n-1} = (C_1, C_2, \ldots, C_{n-1})$ for some finite $n > 1$. The associated question semi-distribution is the function mapping all $q \in Q$ to the value

$$P^n(q, \vec{C}^{n-1}) := \sum_{v' \in V} P_C\left(X(1) = C_1, \ldots, X(n-1) = C_{n-1}, (q, v') \in \hat{X}(n)\right).$$

$P^n(q, \vec{C}^{n-1})$ is the joint probability that the question $q$ occurs in a claim in the step-$n$ claim vector and that the earlier claim vectors were $C_1, C_2, \ldots, C_{n-1}$. Intuitively, one can think of this as the probability that a reasoner considers a particular question at step $n$ and also outputs a particular set of claims at previous steps. Note that in general, for any fixed $\vec{C}^{n-1}$, the associated question semi-distribution is not normalized when considered as a function of questions $q$.

We define claim semi-distributions $P^n((q, v), \vec{C}^{n-1})$ analogously to question semi-distributions. We combine these definitions as follows:

**Definition A.3.** For an SMS $\varphi = (\mathcal{C}, X)$, question $q$, and claim vector trajectory $\vec{C}^{n-1}$ where $P^n(q, \vec{C}^{n-1}) \neq 0$, the associated response distribution is the map from all $v \in V$ to the value

$$P^n(v|q, \vec{C}^{n-1}) = \frac{P^n((q, v), \vec{C}^{n-1})}{P^n(q, \vec{C}^{n-1})}.$$ 

The response distribution for a given SMS $\varphi$, question $q$, and claim vector trajectory $\vec{C}^{n-1}$ specifies the probability that $q$ is given an answer $v$ in the $n$-th step of $\varphi$, conditional on $\varphi$ having produced the earlier claims specified in the partial claim vector trajectory $\vec{C}^{n-1}$ and having produced some claim that includes the question $q$ in the $n$-th claim vector. This conditional character of the response probability distribution will be exploited below to represent path-dependent mathematical reasoning, in which the answers given by an SMS to questions earlier in the process of generating mathematical claims affect the answers given to questions posed later in the process.

A.2. An Alternative Distribution Over Claim Sets. In the body of this paper, we considered the following distribution over claim sets:

$$(A.1) \quad P^n(\hat{C}) := \sum_{\{C \in C^* : \hat{C} \subseteq U(C)\}} P_C(X(n) = C).$$

---

The terms ‘earlier’ and ‘later’ implicitly presuppose a temporal interpretation of the ordering of the SMS. While such an interpretation is not required, it is a natural one, especially when our framework is taken to represent mathematical reasoning by actual humans.
This is the probability that an SMS outputs, at step $n$, a claim vector whose unordering is a superset of $\hat{C}$. But we may also want to consider the probability that an SMS outputs, at a step $n$, a vector whose unordering simply is the claim set $\hat{C}$. Such a distribution, which we denote $P^n(\hat{C})$, can be defined in the obvious way:

(A.2) \[ P^n(\hat{C}) := \sum_{\{C \in C^*: \hat{C} = U(C)\}} P\epsilon(X(n) = C). \]

We can then use this distribution to define a related question semi-distribution and response distribution:

(A.3) \[ P^n((q, v), \hat{C}) := \sum_{\{C \in C^*: \hat{C} \cup \{(q, v)\} = U(C)\}} P\epsilon(X(n) = C) \]

(A.4) \[ P^n(q, \hat{C}) := \sum_{v \in V} P^n((q, v), \hat{C}) \]

(A.5) \[ P^n(v|q, \hat{C}) := P^n((q, v), \hat{C}) P^n(q, \hat{C}) \]

Finally, we can extend this distribution to apply to trajectories of claim sets, rather than claim vectors, e.g.:

(A.6) \[ P^n(q, \tilde{C}^{n-1}) := \sum_{\tilde{C}^{n-1} \cup \{C'\} = \hat{C}^i \forall 1 \leq i \leq n-1} P^n(q, \tilde{C}^{n-1}), \]

where $\tilde{C}^{n-1}$ is a trajectory of unordered claim sets $(\hat{C}^1, \ldots, \hat{C}^{n-1})$.

**Appendix B. Proofs of Propositions**

**B.1. Proof of Lemma 3.3**

*Proof.* Choose some SMS that is backward-consistent after step $\kappa$, and some claim set $\hat{C}$. For all steps $j > \kappa$ and $j > i > \kappa$,

\[ P^j(\hat{C}) = \sum_{\hat{C}^r : \hat{C}^r \subseteq \hat{C}^j} P^j(\hat{C}^r) \]

\[ = \sum_{C' : \hat{C}^r \subseteq U(C')} P\epsilon(X(j, .) = C') \]

\[ = \sum_{C' : \hat{C}^r \subseteq U(C')} \left[ \sum_{C'' : C'' \subseteq U(C')} P\epsilon(X(j, .) = C', X(i, .) = C'') + \sum_{C'' : \hat{C} \subseteq U(C'')} P\epsilon(X(j, .) = C', X(i, .) = C'') \right] \]

\[ = \sum_{C' : \hat{C}^r \subseteq U(C')} \left[ \sum_{C'' : \hat{C} \subseteq U(C'')} P\epsilon(X(j, .) = C' | X(i, .) = C'') P\epsilon(X(j, .) = C'') \right] \]

\[ + \sum_{C'' : \hat{C} \subseteq U(C'')} P\epsilon(X(j, .) = C', X(i, .) = C'') \]
Due to backward-consistency, \( U(C') \subseteq U(C'') \) and so if \( \hat{C} \subseteq U(C'') \), then \( \hat{C} \subseteq U(C') \). This means that for all \( C'' \) such that \( \hat{C} \subseteq U(C'') \),

(B.2) \[ P_e(X(j, .) \in \{ C' : \hat{C} \subseteq U(C') \}) | X(i, .) = U(C'') \) = 1.

Thus, the sum over \( C' \) of the conditional distribution in the summand in the last line equals 1, for all \( C'' \) in the associated sum. Therefore we get

\[
P_\hat{C}(C') = \sum_{C'' : \hat{C} \subseteq U(C'')} P_e(X(i, .) = C'') + \sum_{C'' : \hat{C} \subseteq U(C'')} \sum_{C'' : \hat{C} \cap U(C'')} P_e(X(j, .) = C', X(i, .) = C'')
\]

(B.3) \[ = P_\hat{C}(C') + \sum_{C'' : \hat{C} \subseteq U(C'')} \sum_{C'' : \hat{C} \cap U(C'')} P_e(X(j, .) = C', X(i, .) = C'') \geq P_\hat{C}(C')
\]

So for any claim set \( \hat{C} \), \( P_i(\hat{C}) \) is a monotonically increasing function of \( j \). In addition, that probability is upper-bounded as one varies over all \( j \), by 1. By the completeness axiom of the reals, that means it has a least upper bound. Therefore the limit of that probability goes to infinity is well-defined. Replacing \( j \) with \( m \) establishes the claim. \( \square \)

### B.2. Proof of Proposition 6.2

**Proof.** The divergence function \( D \) in the definition of embed-calibrated can only equal 0 if its two arguments are identical. Therefore for all embedded prediction triples \( (v, q, \hat{C}) \), and \( v^m_1 \),

(B.4) \[ \Psi(\psi(q), v)(v^m_1) = \overline{P}_1 \left( v^m_1 | \psi(q), E^{-1}[\{(q, v)\} \cup \hat{C}] \right). \]

Define

(B.5) \[ \hat{C}_2(v^m_1) := E \left[ \left( \psi(q), v^m_1 \right) \cup E^{-1}[\{(q, v)\} \cup \hat{C}] \right]. \]

Since \( (q, \hat{C}) \) is a discriminating pair,

(B.6) \[ E^{-1}(\hat{C}_2(v^m_1)) = \{(q, v), v^m_1 \} \cup E^{-1}[\{(q, v)\} \cup \hat{C}]. \]

By the definition of embedding function, this means that

(B.7) \[ \overline{P}_1 \left( \left\{(q, v), v^m_1 \right\} \cup E^{-1}[\{(q, v)\} \cup \hat{C}] \right) = \overline{P}_2 \left( \hat{C}_2(v^m_1) \right). \]

Since this is true for all \( v^m_1 \), it also is true when we sum over such \( v^m_1 \). Therefore

(B.8) \[ \overline{P}_1 \left( v^m_1 | \psi(q), E^{-1}[\{(q, v)\} \cup \hat{C}] \right) = \frac{P_2(\hat{C}_2(v^m_1))}{\sum_{v^m_1} P_2(\hat{C}_2(v^m_1))}. \]

(See discussion just below Definition 3.1) This completes the proof. \( \square \)
B.3. Derivation of Definition 7.1. We begin with an alternative definition, that a collection of evidence paths \( \mathcal{B} \) is a set of claim sets \( \{B(i)\} \) such that the premise:

\[
\forall i \in \{2, \ldots, n\} \quad \frac{P(B(1), \ldots, B(i) \mid (q, v^*), \beta)}{P(B(1), \ldots, B(i-1) \mid \beta, (q, v^*)) \times P(B(i) \mid (q, v^*))} \geq \frac{P(B(1), \ldots, B(i) \mid q, \beta)}{P(B(1), \ldots, B(i-1) \mid q, \beta) \times P(B(i) \mid q, \beta)}
\]

Intuitively speaking, Eq. (7.1) tells us that each \( B(i) \) is a line of evidence for the claim \( (q, v^*) \), when considered in isolation of all the others. Eq. (B.9) goes on to tell us that those different lines of evidence do not “work at cross-purposes”, thwarting one another. (The precise indexing of the \( n \) sets in \( \mathcal{B} \) is not important, so long as we can find such an indexing for which Eq. (B.9) holds.)

To illustrate Eq. (B.9), suppose that \( B(1) \) and \( B(2) \) are two lines of reasoning that each establish for a given mathematician-SMS that (conditional on \( \beta \)) the answer to \( q \) is \( v^* \). However, suppose that both lines of reasoning depend on steps such that, if the answer to \( q \) did indeed turn out to be \( v^* \) rather than something else, it would imply to the mathematician-SMS that \( B(1) \) and \( B(2) \) are very unlikely to both be true. Thus, from the perspective of the mathematician-SMS, \( P(B(1), B(2) \mid (q, v^*), \beta) \ll P(B(1), B(2) \mid q, \beta) \). This is one way of formalizing the notion that \( B(1) \) and \( B(2) \) thwart one another, conditioned on the answer \( v^* \) in response to the question \( q \) (in addition to the foundational claim set \( \beta \)), but not if that answer is unspecified.

We still need to quantify the strength of that thwarting though. To see how to do this, suppose that although the supposition that \( q \) has the answer \( v^* \) might increase the mathematician-SMS’s degree of belief that \( B(1) \) is true, the overall effect is mild: \( P(B(1) \mid (q, v^*), \beta) \) is not much greater than \( P(B(1) \mid q, \beta) \). Suppose that similarly, \( P(B(2) \mid (q, v^*), \beta) \) is not much greater than \( P(B(2) \mid q, \beta) \). Under these conditions, Eq. (B.9) may not hold for \( B(1) \) and \( B(2) \), so that they are not both evidence paths for the claim \( (q, v^*) \). Intuitively, they thwart each other too much. Thus, enforcing Eq. (B.9) ensures that such thwarting does not occur.

If \( \mathcal{B} \) is a collection of evidence paths, then not only is each \( B(i) \) a line of evidence for the answer \( v^* \) to the question \( q \) when considered in isolation, but furthermore, as we iteratively combine more and more of those evidence paths, we strictly increase the probability of the response \( v^* \) to the question \( q \):

**Proposition B.1.** Let \( \mathcal{B} \) be a collection of \( N \) evidence paths for the claim \( (q, v^*) \). Then for all \( i : N \geq i \geq 1, \)

\[
P(v^* \mid q, \beta, B(1), \ldots, B(i)) > P(v^* \mid q, \beta, B(1), \ldots, B(i-1))
\]

(where we interpret the \( i = 1 \) version of this statement to mean \( P(v^* \mid q, \beta, B(1)) \geq P(v^* \mid q, \beta) \)).

**Proof.** Evaluating Eq. (7.1) for \( i = 1 \) directly establishes the \( i = 1 \) version of Eq. (7.2):

\[
P(v^* \mid q, \beta, B(1)) > P(v^* \mid q, \beta)
\]
Next, for any $i : 2 \leq i \leq n$, we can use Bayes’ theorem to expand

\begin{equation}
\frac{P(v^* \mid q, \beta, B(1), \ldots, B(i))}{P(v^* \mid q, \beta, B(1), \ldots, B(i-1))} = \frac{P(v^* \mid q, \beta, B(i))}{P(v^* \mid q, \beta)} \times \frac{F^n(B(i) \mid q, \beta) \times P(B(1), \ldots, B(i-1) \mid q, \beta)}{P(B(1), \ldots, B(i) \mid q, \beta)} \times \frac{P(B(1), \ldots, B(i) \mid (q, v^*, \beta))}{P(B(i) \mid (q, v^*, \beta)) \times P(B(1), \ldots, B(i-1) \mid (q, v^*), \beta)}
\end{equation}

Using Eq. (7.1) again establishes that the first term on the RHS of Eq. (B.11) is $>1$. Eq. (B.9) establishes that the product of the second and third terms on the RHS is also $>1$. Therefore the entire RHS of Eq. (B.11) is $>1$. This establishes the claim. \hfill \Box

So this alternative definition of an evidence path implies the one we adopted in the main text.

\subsection*{B.4. Proof of Prop. 7.3}

\textit{Proof.} By hypothesis, $\varphi_2$ is calibrated with the community-SMS $\varphi_1$ at step $n$ for any $\hat{C}_2 \subseteq \beta \cup \bigcup_i B(i)$. So for all $1 \leq i \leq n$,

\begin{equation}
\sum_{v \in V_2} P^n_2(v \mid q, \beta, B(1), \ldots, B(i))D \left[ \Psi(\psi(q), v)(V_1), \mathcal{P}_1(V_1 \mid \psi(q), \beta, B(1), \ldots, B(i)) \right] \leq \epsilon.
\end{equation}

The convexity of $D[., .]$ over its first argument then establishes

\begin{equation}
D \left[ \sum_{v \in V_2} P^n_2(v \mid q, \beta, B(1), \ldots, B(i))\Psi(\psi(q), v)(V_1), \mathcal{P}_1(V_1 \mid \psi(q), \beta, B(1), \ldots, B(i)) \right] \leq \epsilon.
\end{equation}

i.e.,

\begin{equation}
D \left[ F^n(V_1 \mid \psi(q), \beta, B(1), \ldots, B(i)), \mathcal{P}_1(V_1 \mid \psi(q), \beta, B(1), \ldots, B(i)) \right] \leq \epsilon.
\end{equation}

Next, since all of the $B(i)$ are evidence paths for $(\psi(q), v^*)$ under the distribution $F^n$ and conditioned on $\beta$, by Eq. (7.2), for all $1 \leq i \leq N$,

\begin{equation}
F^n(v^* \mid \psi(q), \beta, B(1), \ldots, B(i)) > F^n(v^* \mid \psi(q), \beta, B(1), \ldots, B(i-1))
\end{equation}

Finally, by hypothesis $D[., .]$ is a locally Lipschitz continuous function of its probability distribution arguments (where those distributions are considered as vectors in a Euclidean metric space) when evaluated for the distributions specified in Eqs. (B.14) and (B.15). Then since a divergence equals zero only if its arguments are identical, for all $1 \leq i \leq n$, for small enough $\epsilon$,

\begin{equation}
\mathcal{P}_1(v^* \mid \psi(q), \beta, B(1), \ldots, B(i)) > \mathcal{P}_1(v^* \mid \psi(q), \beta, B(1), \ldots, B(i-1)).
\end{equation}

as claimed. \hfill \Box
B.5. Proof of Prop. 7.4

Proof. By hypothesis $\varphi_2$ is embed-calibrated with the universe-SMS $\varphi_1$ at step $n$ for any $\hat{C}_2 \subseteq \beta \cup \bigcup_i B(i)$. So for all $1 \leq i \leq n$,

$$\sum_{v \in \mathcal{V}_2} P_2^n(v|q, \beta, B(1), \ldots, B(i)) D \left[ \Psi(q), v(\mathcal{V}_1) \right],$$

$$\mathcal{P}_1 \left( \mathcal{V}_1 \mid \psi(q), E^{-1}[\{(q, v) \cup \beta, B(1), \ldots, B(i)\}] \right] \leq \epsilon.$$

The convexity of $D[.,.]$ then establishes

$$\sum_{v \in \mathcal{V}_2} P_2^n(v|q, \beta, B(1), \ldots, B(i)) \mathcal{P}_1 \left( \mathcal{V}_1 \mid \psi(q), E^{-1}[\{(q, v) \cup \beta, B(1), \ldots, B(i)\}] \right] \leq \epsilon.$$

i.e.,

$$D \left[ \sum_{v \in \mathcal{V}_2} P_2^n(v|q, \beta, B(1), \ldots, B(i)) \Psi(q), v(\mathcal{V}_1) \right],$$

$$\sum_{v \in \mathcal{V}_2} P_2^n(v|q, \beta, B(1), \ldots, B(i)) \mathcal{P}_1 \left( \mathcal{V}_1 \mid \psi(q), E^{-1}[\{(q, v) \cup \beta, B(1), \ldots, B(i)\}] \right] \leq \epsilon.$$

Next, since all of the $B(i)$ are evidence paths for $(\psi(q), v^*)$ under the distribution $F^n$ and conditioned on $\beta$, by Eq. (7.2), for all $1 \leq i \leq N$,

$$F^n(v^* \mid \psi(q), \beta, B(1), \ldots, B(i)) > F^n(v^* \mid \psi(q), \beta, B(1), \ldots, B(i-1))$$

Finally, by hypothesis $D[.,.]$ is a locally Lipschitz continuous function of its probability distribution arguments (where those distributions are considered as vectors in a Euclidean metric space) when evaluated for the distributions specified in Eqs. (B.19) and (B.20). Then since a divergence equals zero only if its arguments are identical, for all $1 \leq i \leq n$, for small enough $\epsilon$,

$$\sum_{v \in \mathcal{V}_2} P_2^n(v|q, \beta, B(1), \ldots, B(i)) \mathcal{P}_1(v^* \mid \psi(q), E^{-1}[\{(q, v) \cup \beta, B(1), \ldots, B(i)\}]$$

$$\sum_{v \in \mathcal{V}_2} P_2^n(v|q, \beta, B(1), \ldots, B(i-1)) \mathcal{P}_1(v^* \mid \psi(q), E^{-1}[\{(q, v) \cup \beta, B(1), \ldots, B(i-1)\}]$$

as claimed. $\Box$

B.6. Proof of Prop. 7.5

Proof. By hypothesis, $(q, E[\hat{C}_1])$ is an embedding prediction pair for step $n$ for any claim set $\hat{C}_1 \subseteq \beta \cup \bigcup_i B(i)$. By hypothesis it is also true that $\varphi_2$ is embed-calibrated with SMS $\varphi_1$ at step $n$ for the prediction pair $(q, E[\hat{C}_1])$ for any such $\hat{C}_1$. So for all $1 \leq i \leq n$,

$$\sum_{v \in \mathcal{V}_2} P_2^n(v|q, E[\beta, B(1), \ldots, B(i)]) D \left[ \Psi(q), v(\mathcal{V}_1) \right],$$

$$\mathcal{P}_1 \left( \mathcal{V}_1 \mid \psi(q), E^{-1}[\{(q, v) \cup E[\beta, B(1), \ldots, B(i)]}] \right] \leq \epsilon.$$
Due to the convexity of $D[.,.]$, this means that
\begin{equation}
D \left[ \sum_{v \in V} P^n_2(v|q, E[\beta, B(1), \ldots, B(i)]) \Psi(q, v)(V_1), \right. \\
\left. \sum_{v \in V} P^n_2(v|q, E[\beta, B(1), \ldots, B(i)]) \mathbf{P}_1(V_1 | \psi(q), E^{-1}[(q, v)] \cup E[\beta, B(1), \ldots, B(i)]) \right] \leq \epsilon.
\end{equation}

or
\begin{equation}
D \left[ F^n(V_1 | \psi(q), E[\beta, B(1), \ldots, B(i)]), \right. \\
\left. \sum_{v \in V} P^n_2(v|q, E[\beta, B(1), \ldots, B(i)]) \mathbf{P}_1(V_1 | \psi(q), E^{-1}[(q, v)] \cup E[\beta, B(1), \ldots, B(i)]) \right] \leq \epsilon.
\end{equation}

The same line of reasoning for $E[\beta, B(1), \ldots, B(i-1)]$ yields:
\begin{equation}
D \left[ F^n(V_1 | \psi(q), E[\beta, B(1), \ldots, B(i-1)]), \right. \\
\left. \sum_{v \in V} P^n_2(v|q, E[\beta, B(1), \ldots, B(i-1)]) \mathbf{P}_1(V_1 | \psi(q), E^{-1}[(q, v)] \cup E[\beta, B(1), \ldots, B(i-1)]) \right] \leq \epsilon.
\end{equation}

Since, by hypothesis, all of the $B(i)$ are evidence paths for $(\psi(q), v^*)$ under the distribution $\mathbf{P}_1$ and conditioned on $\beta$, for all $1 \leq i \leq N$, Eq. (7.2) allows us to write
\begin{equation}
\mathbf{P}_1(v^* | \psi(q), \beta, B(1), \ldots, B(i)) > \mathbf{P}_1(v^* | \psi(q), \beta, B(1), \ldots, B(i-1)).
\end{equation}

Via the fifth condition of the proposition, this implies that
\begin{equation}
\mathbf{P}_1(v^* | \psi(q), E^{-1}[(q, v)] \cup E[\beta, B(1), \ldots, B(i)]) > \mathbf{P}_1(v^* | \psi(q), E^{-1}[(q, v)] \cup E[\beta, B(1), \ldots, B(i-1)])
\end{equation}
for all $v \in V_2$. By hypothesis $D[.,.]$ is a locally Lipschitz continuous function of its probability distribution arguments (where those distributions are considered as vectors in a Euclidean metric space) when evaluated for the distributions specified in Eqs. (B.24) to (B.26). Then since a divergence equals zero only if its arguments are identical, for all $1 \leq i \leq n$, for small enough $\epsilon$,
\begin{equation}
F^n(V_1 | \psi(q), E[\beta, B(1), \ldots, B(i)]) > F^n(V_1 | \psi(q), E[\beta, B(1), \ldots, B(i-1)]),
\end{equation}
as claimed.

\begin{flushright}
\Box
\end{flushright}

\textbf{B.7. Proof of Prop. 8.1}

\textit{Proof.} By hypothesis, $F^n$ satisfies the abductive premise,
\begin{equation}
F^n(v^* | q^*, (q^1, v^1), \hat{C}_2) = \alpha F^n(v^* | q^*, \hat{C}_2),
\end{equation}
which means the abduction implication must hold,
\begin{equation}
F^n(v^1 | q^1, (q^*, v^*), \hat{C}_2) = \alpha F^n(v^1 | q^1, \hat{C}_2).
\end{equation}

Since hypothesis $F^n_2(v^* | q^*, q^1, \hat{C}_2) := \sum_{v^1} F^n_2(v^*, v^1 | q^*, q^1, \hat{C}_2) = F^n_2(v^* | q^*, \hat{C}_2)$, Eq. (B.30) implies
\begin{equation}
F^n(v^*, v^1 | q^*, q^1, \hat{C}_2) = \alpha F^n(v^1 | q^1, \hat{C}_2) F^n(v^* | q^*, \hat{C}_2),
\end{equation}
Also by hypothesis, \( \varphi_2 \) is calibrated with \( \varphi_1 \) at step \( n \) for \( \hat{C}_2 \), for each of the three questions \( \psi^{-1}(q^*) \), \( \psi^{-1}(q^\dagger) \), and \( \psi^{-1}(q^*, q^\dagger) \), and for \( m = 1, 2 \). Then choosing \( m = 2 \), we get

\[
(B.32) \quad \sum_{v \in V_2} P^m_2(v|\psi^{-1}(q^*, q^\dagger), \hat{C}_2)D \left[ \Psi(\psi^{-1}(q^*, q^\dagger)), v)(V_1, V_1), \bar{P}_1 \left( V_1, V_1 | q^*, q^\dagger, \hat{C}_2 \right) \right] \leq \epsilon.
\]

The convexity of \( D[.,.] \) establishes

\[
(B.33) \quad D \left[ \sum_{v \in V_2} P^m_2(v|\psi^{-1}(q^*, q^\dagger), \hat{C}) \Psi(\psi^{-1}(q^*, q^\dagger)), v)(V_1, V_1), \bar{P}_1 \left( V_1, V_1 | q^*, q^\dagger, \hat{C}_2 \right) \right] \leq \epsilon,
\]
or

\[
(B.34) \quad D \left[ F^m(V_1, V_1 | q^*, q^\dagger, \hat{C}_2), \bar{P}_1 \left( V_1, V_1 | q^*, q^\dagger, \hat{C}_2 \right) \right] \leq \epsilon,
\]
where with abuse of notation we write \( \bar{P}_1 \left( V_1, V_1 | q^*, q^\dagger, \hat{C}_2 \right) \) for a fixed value of the triple \( (q^*, q^\dagger, \hat{C}_2) \) for the associated distribution over an event space \( V_1 \times V_1 \).

Go through the analogous reasoning for \( m = 1 \) twice, once for each of the two distinct questions \( q' \neq q, q'' \neq q \) that we assume exist, questions which (via \( \psi(.) \)) specify the single question \( q^* \) and the single question \( q^\dagger \), respectively. In these two cases calibration means that:

\[
(B.35) \quad D \left[ F^m(V_1 | q^*, \hat{C}_2), \bar{P}_1 \left( V_1 | q^*, \hat{C}_2 \right) \right] \leq \epsilon
\]

\[
(B.36) \quad D \left[ F^m(V_1 | q^\dagger, \hat{C}_2), \bar{P}_1 \left( V_1 | q^\dagger, \hat{C}_2 \right) \right] \leq \epsilon
\]
(where we have extended the definition of \( F^m \) in the obvious way to the case where it has one claim as an argument rather than two).

By hypothesis, \( D[.,.] \) is a locally Lipschitz continuous function of its probability distribution arguments (where those distributions are considered as vectors in a Euclidean metric space) when evaluated for the distributions specified in Eq. \( (B.34) \) to Eq. \( (B.36) \). Then since a divergence equals zero only if its arguments are identical, for small \( \epsilon \) Eq. \( (B.34) \) implies:

\[
(B.37) \quad \bar{P}_1 \left( v^*, v^\dagger | q^*, q^\dagger, \hat{C}_2 \right) \simeq F^m(v^*, v^\dagger | q^*, q^\dagger, \hat{C}_2) = \alpha F^m(v^\dagger | q^\dagger, \hat{C}_2)P(v^* | q^*, \hat{C}_2).
\]

Eq. \( (B.35) \) implies:

\[
(B.38) \quad \bar{P}_1 \left( v^* | q^*, \hat{C}_2 \right) \simeq F^m(v^* | q^*, \hat{C}_2),
\]
and Eq. \( (B.36) \) implies:

\[
(B.39) \quad \bar{P}_1 \left( v^\dagger | q^\dagger, \hat{C}_2 \right) \simeq F^m(v^\dagger | q^\dagger, \hat{C}_2).
\]

Together, Eq. \( (B.37) \) through Eq. \( (B.39) \) imply:

\[
(B.40) \quad \bar{P}_1 \left( v^*, v^\dagger | q^*, q^\dagger, \hat{C}_2 \right) \simeq \alpha \bar{P}_1 \left( v^* | q^*, \hat{C}_2 \right) \bar{P}_1 \left( v^\dagger | q^\dagger, \hat{C}_2 \right)
\]

By hypothesis,

\[
(B.41) \quad \bar{P}_1 \left( v^* | q^*, q^\dagger, \hat{C}_2 \right) = \bar{P}_1 \left( v^* | q^*, \hat{C}_2 \right)
\]
and so Eq. \( (B.40) \) implies that

\[
(B.42) \quad \bar{P}_1(v^\dagger | q^\dagger, (q^*, v^*), \hat{C}_2) \simeq \alpha \bar{P}_1(v^\dagger | q^\dagger, \hat{C}_2) > \bar{P}_1(v^\dagger | q^\dagger, \hat{C}_2),
\]
as claimed. □

B.8. Proof of Prop. 8.2

Proof. By hypothesis, \( F^n \) satisfies the abductive premise. Following the same steps as at the beginning of the proof of Proposition 8.1, we get

\[
F^n(v^*, v^\dagger | q^*, q^\dagger, \hat{C}_2) = \alpha F^n(v^\dagger | q^\dagger, \hat{C}_2) F^n(v^* | q^*, \hat{C}_2),
\]

By hypothesis, \( \varphi_2 \) is embed-calibrated with \( \varphi_1 \) for \( \hat{C}_2 \) for each of the three questions \( \psi^{-1}(q^*) \), \( \psi^{-1}(q^\dagger) \), and \( \psi^{-1}(q^*, q^\dagger) \), and for \( m = 1, 2 \). Then choosing \( m = 2 \), we get

\[
\sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^*, q^\dagger), \hat{C}_2) D \left[ \Psi(\psi^{-1}(q^*, q^\dagger)), v \right](\mathcal{V}_1, \mathcal{V}_1), \mathcal{P}_1 \left( \mathcal{V}_1, \mathcal{V}_1 | q^*, q^\dagger, E^{-1}[\{(\psi^{-1}(q^*, q^\dagger), v) \} \cup \hat{C}_2] \right) \leq \epsilon.
\]

The convexity of \( D[., .] \) establishes

\[
D \left[ F^n(\mathcal{V}_1 | \psi^{-1}(q^*, q^\dagger), \hat{C}_2) \right],
\]

\[
\sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^*, q^\dagger), \hat{C}_2) \mathcal{P}_1 \left( \mathcal{V}_1, \mathcal{V}_1 | q^*, q^\dagger, E^{-1}[\{(\psi^{-1}(q^*, q^\dagger), v) \} \cup \hat{C}_2] \right) \leq \epsilon.
\]

Go through the analogous reasoning for \( m = 1 \) twice, once for each of the two distinct questions \( q' \neq q, q'' \neq q \) that we assume exist, questions which (via \( \psi(.) \)) specify the single question \( q^* \) and the single question \( q^\dagger \), respectively. In these two cases calibration means that:

\[
D \left[ F^n(\mathcal{V}_1 | q^*, \hat{C}_2) \right],
\]

\[
\sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^*, q^\dagger), \hat{C}_2) \mathcal{P}_1 \left( \mathcal{V}_1, \mathcal{V}_1 | q^*, q^\dagger, E^{-1}[\{(\psi^{-1}(q^*, q^\dagger), v) \} \cup \hat{C}_2] \right) \leq \epsilon,
\]

\[
D \left[ F^n(\mathcal{V}_1 | q^\dagger, \hat{C}_2) \right],
\]

\[
\sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^*, q^\dagger), \hat{C}_2) \mathcal{P}_1 \left( \mathcal{V}_1, \mathcal{V}_1 | q^\dagger, q^*, E^{-1}[\{(\psi^{-1}(q^*, q^\dagger), v) \} \cup \hat{C}_2] \right) \leq \epsilon.
\]

(where we have extended the definition of \( F^n \) in the obvious way to the case where it has one claim as an argument rather than two).

By hypothesis, \( D[., .] \) is a locally Lipschitz continuous function of its probability distribution arguments (where those distributions are considered as vectors in a Euclidean metric space) when evaluated for the distributions specified in Eq. (B.45) to Eq. (B.47). Then since a divergence equals zero only if its arguments are identical, for small \( \epsilon \) Eq. (B.45) implies:

\[
F^n(v^*, v^\dagger | \psi^{-1}(q^*, q^\dagger), \hat{C}_2) \approx F^n(v^\dagger | q^\dagger, \hat{C}_2) F^n(v^* | q^*, \hat{C}_2)
\]

\[
\approx \sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^*, q^\dagger), \hat{C}_2) \mathcal{P}_1 \left( v^*, v^\dagger | q^*, q^\dagger, E^{-1}[\{(\psi^{-1}(q^*, q^\dagger), v) \} \cup \hat{C}_2] \right).
\]

Eq. (B.46) implies:

\[
F^n(v^* | q^*, \hat{C}_2) \approx \sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^*), \hat{C}_2) \mathcal{P}_1 \left( v^* | q^*, E^{-1}[\{(\psi^{-1}(q^*), v) \} \cup \hat{C}_2] \right),
\]
and Eq. (B.47) implies:

\[
F^n(v^t | q^t, \hat{C}_2) \simeq \sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^t), \hat{C}_2) \overline{P}_1 \left( v^t | q^t, E^{-1}[\{\psi^{-1}(q^t), v\} \cup \hat{C}_2] \right).
\]

Together, Eq. (B.48) through Eq. (B.50) imply:

\[
\sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^t), \hat{C}_2) \overline{P}_1 \left( v^t, v^t | q^*, q^t, E^{-1}[\{\psi^{-1}(q^*, q^t), v\} \cup \hat{C}_2] \right) \\
\quad \simeq \alpha \sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^t), \hat{C}_2) \overline{P}_1 \left( v^t | q^*, E^{-1}[\{\psi^{-1}(q^t), v\} \cup \hat{C}_2] \right) \\
\quad \times \sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^t), \hat{C}_2) \overline{P}_1 \left( v^t | q^t, E^{-1}[\{\psi^{-1}(q^t), v\} \cup \hat{C}_2] \right).
\]

By hypothesis, for all \( v \in V_2, \)

\[
\overline{P}_1 \left( v^t | q^*, q^t, E^{-1}[\{\psi^{-1}(q^*, q^t), v\} \cup \hat{C}_2] \right) = \overline{P}_1 \left( v^t | q^*, E^{-1}[\{\psi^{-1}(q^*), v\} \cup \hat{C}_2] \right)
\]

and so Eq. (B.51) implies that

\[
\sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^t), \hat{C}_2) \overline{P}_1(v^t|q^t, (q^*, v^*), E^{-1}[\{\psi^{-1}(q^t), v\} \cup \hat{C}_2]) \\
\quad > \sum_{v \in V_2} P^n_2(v|\psi^{-1}(q^t), \hat{C}_2) \overline{P}_1(v^t|q^t, E^{-1}[\{\psi^{-1}(q^t), v\} \cup \hat{C}_2])
\]

as claimed. \(\square\)

B.9. PROOF OF PROP. 8.3

Proof. By hypothesis, \( \overline{P}_1 \) satisfies the abductive premise for any \( \hat{C}_1 \in \hat{C}_1, \) meaning that for any \( v \in V_2, \)

\[
\overline{P}_1 \left( v^t | q^*, (q^t, v^t), E^{-1}[\{\psi^{-1}(q^*, q^t), v\} \cup E[\hat{C}_1]] \right) = \alpha \overline{P}_1 \left( v^t | q^*, E^{-1}[\{\psi^{-1}(q^*), v\} \cup E[\hat{C}_1]] \right),
\]

which allows us to derive

\[
\overline{P}_1 \left( v^t | q^t, (q^*, v^*), E^{-1}[\{\psi^{-1}(q^*, q^t), v\} \cup E[\hat{C}_1]] \right) = \alpha \overline{P}_1 \left( v^t | q^t, E^{-1}[\{\psi^{-1}(q^t), v\} \cup E[\hat{C}_1]] \right).
\]

Since, also by hypothesis,

\[
\overline{P}_1 \left( v^t | q^t, v^t, E^{-1}[\{\psi^{-1}(q^*, q^t), v\} \cup E[\hat{C}_1]] \right) = \overline{P}_1 \left( v^t | q^t, E^{-1}[\{\psi^{-1}(q^t), v\} \cup E[\hat{C}_1]] \right),
\]

we are able to rewrite Eq. (B.55) as

\[
\overline{P}_1 \left( v^t, v^t | q^t, q^*, E^{-1}[\{\psi^{-1}(q^*, q^t), v\} \cup E[\hat{C}_1]] \right) \\
\quad = \alpha \overline{P}_1 \left( v^t | q^t, E^{-1}[\{\psi^{-1}(q^t), v\} \cup E[\hat{C}_1]] \right) \overline{P}_1 \left( v^t | q^*, E^{-1}[\{\psi^{-1}(q^*), v\} \cup E[\hat{C}_1]] \right).
\]
By hypothesis, $\varphi_2$ is embed-calibrated with $\varphi_1$ for $E[\hat{C}_1]$ for each of the three questions $\psi^{-1}(q^*)$, $\psi^{-1}(q^\dagger)$, and $\psi^{-1}(q^*, q^\dagger)$, and for $m = 1, 2$. Then choosing $m = 2$ and applying Eq. (B.57), we get

(B.58) \[ \sum_{v \in V_2} P_2^n(v|\psi^{-1}(q^*, q^\dagger), E[\hat{C}_1]) D\left[ \Psi(\psi^{-1}(q^*, q^\dagger)), v\right| (V_1, V_1), \right] \]

Together, Eq. (B.62) through Eq. (B.64) imply:

(B.63) \[ F^n(v^*, q^*|\psi^{-1}(q^*, q^\dagger), E[\hat{C}_1]) \simeq \sum_{v \in V_2} P_2^n(v|\psi^{-1}(q^*), E[\hat{C}_1]) \mathcal{P}_1 \left( v^*, q^*|\psi^{-1}(q^*, q^\dagger), E[\hat{C}_1] \right), \]

and Eq. (B.61) implies:

(B.64) \[ F^n(q^\dagger|q^\dagger, E[\hat{C}_1]) \simeq \sum_{v \in V_2} P_2^n(v|\psi^{-1}(q^\dagger), E[\hat{C}_1]) \mathcal{P}_1 \left( q^\dagger|q^\dagger, E[\hat{C}_1] \right). \]

Together, Eq. (B.62) through Eq. (B.64) imply:

(B.65) \[ F^n(v^*, q^\dagger|\psi^{-1}(q^*, q^\dagger), E[\hat{C}_1]) \simeq \alpha F^n(q^*, q^\dagger, E[\hat{C}_1]) F^n(q^\dagger, E[\hat{C}_1]). \]
By hypothesis,

\[(B.66) \quad F_n^m(v^* | q^*, q^\dagger, E[\hat{C}_1]) = F_n^m(v^* | q^*, E[\hat{C}_1]) \]

and so Eq. \[(B.65)\] implies that

\[(B.67) \quad F_n^m(v^\dagger | q^\dagger, (q^*, v^*), E[\hat{C}_1]) > F_n^m(v^\dagger | q^\dagger, E[\hat{C}_1]),\]

as claimed.

\[\square\]

**References**

Albert, David Z. *Time and Chance*. Harvard University Press, 2000.

Barrow, John D. “Godel and Physics”. In: *Kurt Gödel and the Foundations of Mathematics: Horizons of Truth* (2011), p. 255.

— *Theories of everything: The quest for ultimate explanation*. Clarendon Press Oxford, 1991.

Belnap, Nuel D. and Thomas B. Steel. *The Logic of Questions and Answers*. New Haven/London: Yale University Press, 1976.

Bovens, Luc and Stephan Hartmann. “Bayesian Networks and the Problem of Unreliable Instruments”. In: *Philosophy of Science* 69.1 (2002), pp. 29–72. DOI: [10.1086/338940](https://doi.org/10.1086/338940).

Bueno, Otávio and Mark Colyvan. “An inferential conception of the application of mathematics”. In: *Noûs* 45.2 (2011), pp. 345–374.

Burgess, John P. “Probability logic”. In: *The Journal of Symbolic Logic* 34.2 (1969), pp. 264–274.

Campbell-Moore, Catrin. “How to express self-referential probability”. In: *The Review of Symbolic Logic* 4 (2015), pp. 680–704.

Carnap, Rudolf. *Logical foundations of probability*. University of Chicago Press, 1950.

Carroll, Sean M. “The Quantum Field Theory on Which the Everyday World Supervenes”. In: *arXiv preprint arXiv:2101.07884* (2021).

Chomsky, Noam and Marcel P Schützenberger. “The algebraic theory of context-free languages”. In: *Studies in Logic and the Foundations of Mathematics*. Vol. 26. Elsevier, 1959, pp. 118–161.

Christiano, Paul F et al. “Definability of truth in probabilistic logic”. In: *Unpublished Manuscript* (2013).
Claveau, François and Olivier Grenier. “The Variety-of-Evidence Thesis: A Bayesian Exploration of its Surprising Failures”. In: *Synthese* 196.8 (2019), pp. 3001–3028. DOI: [10.1007/s11229-017-1607-5](10.1007/s11229-017-1607-5)

Cover, Thomas M. and Joy A. Thomas. *Elements of information theory*. John Wiley & Sons, 2012.

Davies, Paul Charles William. *The physics of time asymmetry*. Univ of California Press, 1977.

Demey, Lorenz, Barteld Kooi, and Joshua Sack. “Logic and Probability”. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Summer 2019. Metaphysics Research Lab, Stanford University, 2019.

Dipert, Randall R. “The mathematical structure of the world: The world as graph”. In: *The Journal of Philosophy* 94.7 (1997), pp. 329–358.

Douven, Igor. “Abduction”. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Summer 2021. Metaphysics Research Lab, Stanford University, 2021.

– *The art of abduction*. MIT press, 2022.

Dummett, Michael. “Truth”. In: *Proceedings of the aristotelian society*. Vol. 59. 1. 1959.

Everett, Hugh. "" Relative state" formulation of quantum mechanics". In: *Reviews of modern physics* 29.3 (1957), p. 454.

Fagin, Ronald, Joseph Y Halpern, and Nimrod Megiddo. “A logic for reasoning about probabilities”. In: *Information and computation* 87.1-2 (1990), pp. 78–128.

Fitelson, Branden. “Wayne, Horwich, and Evidential Diversity”. In: *Philosophy of Science* 63.4 (1996), pp. 652–660. DOI: [10.1086/289982](10.1086/289982)

Franklin, James. “Non-deductive logic in mathematics”. In: *The British journal for the philosophy of science* 38.1 (1987), pp. 1–18.

Freer, Cameron E, Daniel M Roy, and Joshua B Tenenbaum. “Towards common-sense reasoning via conditional simulation: legacies of Turing in Artificial Intelligence.” In: *Turing’s Legacy* 42 (2014), pp. 195–252.

Garrabrant, Scott et al. “A formal approach to the problem of logical non-omniscience”. In: *arXiv preprint arXiv:1707.08747* (2017).

– “Logical induction”. In: *arXiv preprint arXiv:1609.03543* (2016).
Gisin, Nicolas. “Indeterminism in physics and intuitionistic mathematics”. In: Synthese 199.5 (2021), pp. 13345–13371.

- “Indeterminism in Physics, Classical Chaos and Bohmian Mechanics: Are Real Numbers Really Real?” In: Erkenntnis (2019), pp. 1–13.

Goodman, Noah et al. “Church: a language for generative models”. In: arXiv preprint arXiv:1206.3255 (2012).

Greene, Brian. The Elegant Universe. Vintage Books, 1999.

Grim, Patrick and Daniel Singer. “Computational Philosophy”. In: The Stanford Encyclopedia of Philosophy. Ed. by Edward N. Zalta and Uri Nodelman. Fall 2022. Metaphysics Research Lab, Stanford University, 2022.

Hacking, Ian. “On the foundations of statistics”. In: The British Journal for the Philosophy of Science 15.57 (1964), pp. 1–26.

Haenni, Rolf et al. Probabilistic logics and probabilistic networks. Vol. 350. Springer Science & Business Media, 2010.

Hailperin, Theodore et al. “Probability logic.” In: Notre Dame Journal of Formal Logic 25.3 (1984), pp. 198–212.

Hilbert, David. “Die grundlagen der mathematik”. In: Die Grundlagen der Mathematik. Springer, 1928, pp. 1–21.

Hintikka, Jaakko and James Bachman. What If...?: Toward Excellence in Reasoning. Mayfield, 1991.

Hoek, Daniel. “Questions in Action”. In: The Journal of Philosophy 119.3 (2022), pp. 113–143.

Hoover, Douglas N. “Probability logic”. In: Annals of mathematical logic 14.3 (1978), pp. 287–313.

Icard, Thomas F. “Calibrating generative models: The probabilistic Chomsky–Schützenberger hierarchy”. In: Journal of Mathematical Psychology 95 (2020), p. 102308.

Ladyman, James. “Scientific structuralism: On the identity and diversity of objects in a structure”. In: Aristotelian Society Supplementary Volume. Vol. 81. 1. Wiley Online Library. 2007, pp. 23–43.

- “What is Structural Realism?” In: Studies in History and Philosophy of Science Part A 29.3 (1998), pp. 409–424. DOI: 10.1016/s0039-3681(98)80129-5

Lample, Guillaume and François Charton. “Deep learning for symbolic mathematics”. In: arXiv preprint arXiv:1912.01412 (2019).
Landes, Jürgen. “The Variety of Evidence Thesis and its Independence of Degrees of Independence”. In: *Synthese* (2020a), pp. 1–31.

– “Variety of Evidence”. In: *Erkenntnis* 85.1 (2020b), pp. 183–223.

Leblanc, Hugues. “Probabilistic semantics for first-order logic”. In: *Mathematical Logic Quarterly* 25.32 (1979), pp. 497–509.

Leitgeb, Hannes. “On the Probabilistic Convention T.” In: *Rev. Symb. Log.* 1.2 (2008), pp. 218–224.

Lewis, David. “Immodest inductive methods”. In: *Philosophy of Science* 38.1 (1971), pp. 54–63.

– *On the plurality of worlds*. Oxford Blackwell, 1986.

Lin, Henry W and Max Tegmark. “Critical behavior in physics and probabilistic formal languages”. In: *Entropy* 19.7 (2017), p. 299.

Loewer, Barry. “Counterfactuals and the Second Law”. In: *Causation, Physics, and the Constitution of Reality: Russell’s Republic Revisited*. Ed. by Huw Price and Richard Corry. Oxford University Press, 2007.

– “Why is There Anything Except Physics?” In: *Synthese* 170.2 (2009), pp. 217–233.

– “Why There is Anything Except Physics?” In: *Being Reduced: New Essays on Reduction, Explanation, and Causation*. Ed. by Jakob Hohwy and Jesper Kallestrup. Oxford University Press, 2008.

McCullough-Benner, Colin. “Representing the World with Inconsistent Mathematics”. In: *The British Journal for the Philosophy of Science* 71.4 (2020), pp. 1331–1358.

Myrvold, Wayne C. “Bayesianism and Diverse Evidence: A Reply to Andrew Wayne”. In: *Philosophy of Science* 63.4 (1996), pp. 661–665.

Nilsson, Nils J. “Probabilistic logic”. In: *Artificial intelligence* 28.1 (1986), pp. 71–87.

Peirce, C. S. “How to Make Our Ideas Clear”. In: *Popular Science Monthly* 12.Jan. (1878), pp. 286–302.

Richardson, Matthew and Pedro Domingos. “Markov logic networks”. In: *Machine learning* 62.1-2 (2006), pp. 107–136.

Schmidhuber, Jürgen. “A computer scientist’s view of life, the universe, and everything”. In: *Foundations of computer science*. Springer. 1997, pp. 201–208.
Sebens, Charles T and Sean M Carroll. “Self-locating uncertainty and the origin of probability in Everettian quantum mechanics”. In: The British Journal for the Philosophy of Science 69.1 (2018), pp. 25–74.

Šešelja, Dunja. “Agent-Based Models of Scientific Interaction”. In: Philosophy Compass 17.7 (2022), e12855. DOI: 10.1111/phc3.12855.

Shapiro, Stewart. Philosophy of mathematics: Structure and ontology. Oxford University Press on Demand, 1997.

Skipper, Mattias and Jens Christian Bjerring. “Bayesianism for non-ideal agents”. In: Erkenntnis (2020), pp. 1–23.

Tegmark, Max. “Is “the theory of everything” merely the ultimate ensemble theory?” In: Annals of Physics 270.1 (1998), pp. 1–51.

– Our mathematical universe: My quest for the ultimate nature of reality. Vintage, 2014.

– “The mathematical universe”. In: Foundations of physics 38.2 (2008), pp. 101–150.

Viteri, Scott and Simon DeDeo. “Epistemic phase transitions in mathematical proofs”. In: Cognition 225 (2022), p. 105120.

Wallace, David. Stating structural realism: mathematics-first approaches to physics and metaphysics. 2021. URL: http://philsci-archive.pitt.edu/20048/

Wayne, Andrew. “Bayesianism and Diverse Evidence”. In: Philosophy of Science 62.1 (1995), pp. 111–121.

Weisberg, Jonathan. “Locating IBE in the Bayesian framework”. In: Synthese 167.1 (2009), pp. 125–143.

Wisniewski, Andrzej. Questions, Inferences, and Scenarios. College Publications, 2013.

Wolpert, David H. “Constraints on physical reality arising from a formalization of knowledge”. In: arXiv preprint arXiv:1711.03499 (2017).

– “Memory systems, computation, and the second law of thermodynamics”. In: International Journal of Theoretical Physics 31.4 (1992), pp. 743–785.

– “The Implications of the No-Free-Lunch Theorems for Meta-induction”. In: Journal of General Philosophy of Science (2021).
Wolpert, David H. “The lack of a priori distinctions between learning algorithms”. In: *Neural Computation* 8.7 (1996), pp. 1341–1390.

Wolpert, David H and David Kinney. “Noisy Deductive Reasoning: How Humans Construct Math, and how Math Constructs Universes”. In: *Undecidability, Uncomputability, and Unpredictability*. Springer Nature, 2020.

Wolpert, David H and W. G. Macready. “No Free Lunch Theorems for Optimization”. In: *IEEE Transactions on Evolutionary Computation* 1.1 (1997), pp. 67–82.

_Santa Fe Institute, Santa Fe, New Mexico, Complexity Science Hub, Vienna, Arizona State University, Tempe, Arizona, http://davidwolpert.weebly.com_

_Princeton University, Princeton, New Jersey, http://davidbkinney.com_