Universal nature of replica symmetry breaking in quantum systems with Gaussian disorder

C. Itoi
Department of Physics, GS and CST, Nihon University, Kanda-Surugadai, Chiyoda, Tokyo 101-8308, Japan

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Abstract

We study quantum spin systems with quenched Gaussian disorder. We prove that the variance of all physical quantities in a certain class vanishes in the infinite volume limit. We study also replica symmetry breaking phenomena, where the variance of an overlap operator in the other class does not vanish in the replica symmetric Gibbs state. On the other hand, it vanishes in a spontaneous replica symmetry breaking Gibbs state defined by applying an infinitesimal replica symmetry breaking field. We prove also that the finite variance of the overlap operator in the replica symmetric Gibbs state implies the existence of a spontaneous replica symmetry breaking.

1 Introduction

Coupling constants in a system with quenched disorder are given by i.i.d. random variables. We can regard a given disordered sample as a system obtained by a random sampling of these variables. All physical quantities in such systems are functions of these random variables. In statistical physics, a physical quantity is said to be self-averaging, if its observed value is equal to its expectation value in a disordered sample. In other words, self-averaging quantities obey the law of large numbers. In several disordered spin systems, the free energy density is known to be self-averaging. To be specific, first we consider disordered Ising systems. Let $\Lambda_L := [1, L]^d \cap \mathbb{Z}^d$ be a lattice whose volume is $|\Lambda_L| = L^d$. We define a collection $C_L$ of interaction ranges $X \subset \Lambda_L$ For an arbitrary $X \in C_L$, we denote

$$S_X = \prod_{x \in X} S_x.$$ 

For coupling constants $J_1, J_0, h \in \mathbb{R}$. we define Hamiltonian by

$$H(S, g) := - \sum_{X \in C_L} (J_1 g_X + J_0) S_X - h \sum_{x \in \Lambda_L} S_x$$

as a function of the spin configuration and i.i.d. standard Gaussian random variables $g = (g_X)_{X \in C_L}$. If the Hamiltonian is invariant under the following transformation

$$H(-S, g) = H(S, g).$$
for \( h = 0 \), this symmetry is called \( \mathbb{Z}_2 \) symmetry. In such systems, the magnetization density

\[ m_L := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} S_x, \]

of spin variable \( S_x = \pm 1 \) is an order parameter which is a random variable satisfying the Gibbs distribution depending on the quenched disorder \( g = (g_x)_{x \in \mathcal{C}_L} \). Griffiths’ theorem [17] for the system without disorder \( J_1 = 0 \) shows that the long range order

\[ \lim_{L \to \infty} \lim_{h \to 0} \langle \delta m_L^2 \rangle \neq 0 \]

should imply the spontaneous symmetry breaking

\[ \lim_{h \to 0} \lim_{L \to \infty} \langle m_L \rangle \neq 0, \]

where \( \langle \cdots \rangle \) is expectation in the Gibbs state and a deviation is defined by \( \delta m_L := m_L - \langle m_L \rangle \). This is believed for the spontaneous \( \mathbb{Z}_2 \) symmetry breaking also in disordered systems. Define another deviation \( \Delta m_L := m_L - E\langle m_L \rangle \), and consider the following variance

\[ E\langle \Delta m_L^2 \rangle := E\langle (m_L - E\langle m_L \rangle)^2 \rangle, \]

where \( E \) is the sample expectation with respect to the quenched disorder. By the Chebyshev inequality, the variance gives an upper bound of the probability that a difference between the observed value \( m_L \) and its expectation value \( E\langle m_L \rangle \) becomes larger than a positive number \( c \),

\[ P(|m_L - E\langle m_L \rangle| > c) \leq \frac{E\langle \Delta m_L^2 \rangle}{c^2}. \]

If the variance vanishes in the infinite volume limit

\[ \lim_{L \to \infty} E\langle \Delta m_L^2 \rangle = 0, \]

the observed value of the magnetization density \( m_L \) differs from its expectation value rarely. In this case, \( m_L \) is self-averaging.

Next, we consider replica symmetry breaking phenomena which apparently violate self-averaging of the overlap between two replicated quantities in a replica symmetric expectation. Let \( S^\alpha (\alpha = 1, \cdots, n) \) be \( n \) replicated copies of a spin configuration, and we consider the following Hamiltonian

\[ H(S^1, \cdots, S^n, g) := \sum_{\alpha=1}^n H(S^\alpha, g), \]

where replicated spin configurations share the same quenched disorder \( g \). This Hamiltonian is invariant under an arbitrary permutation \( \sigma \in S_n \).

\[ H(S^1, \cdots, S^n g) = H(S^{\sigma_1}, \cdots, S^{\sigma_n}, g) \]

This permutation symmetry is the replica symmetry. The spin overlap \( R_{1,2} \) between two replicated spin configurations is defined by

\[ R_{1,2} := \frac{1}{|\mathcal{C}|} \sum_{X \in \mathcal{C}_L} S^1_X S^2_X. \]

The covariance of the Hamiltonian is written in terms of the overlap

\[ \mathbb{E}H(S^1, g)H(S^2, g) - \mathbb{E}H(S^1, g)\mathbb{E}H(S^2, g) = |C| J_1^2 R_{1,2}. \]
When the replica symmetry breaking occurs, broadening of the overlap distribution with a finite variance is observed. This phenomenon is well-known in several disordered systems, such as the Sherrington-Kirkpatrick model \cite{24, 27, 28}. In the present paper, we define replica symmetry breaking by the finite variance calculated in the replica symmetric expectation in the infinite volume limit

$$\lim_{L \to \infty} \mathbb{E} \langle \Delta R_{1,2}^2 \rangle = \lim_{L \to \infty} \left[ \mathbb{E} \langle R_{1,2}^2 \rangle - (\mathbb{E} \langle R_{1,2} \rangle)^2 \right] \neq 0,$$

where $\Delta R_{1,2} := R_{1,2} - \mathbb{E} \langle R_{1,2} \rangle$. This definition of the replica symmetry breaking is given by Chatterjee \cite{4}. Although the replica symmetry breaking is believed to be a spontaneous symmetry breaking, there are not many studies of the replica symmetry breaking regarded as spontaneous symmetry breaking \cite{18, 19, 23}. There are lots of important studies for spontaneous symmetry breaking for spin systems without disorder. Griffiths’ theorem shows that the spontaneous symmetry breaking in ferromagnetic systems can be detected by the finite variance of the order parameter calculated in the symmetric Gibbs state \cite{17}. Namely, the long range order implies a finite spontaneous magnetization. In a certain sample with a quenched disorder $g$, we are interested in another variance

$$\lim_{L \to \infty} \mathbb{E} \langle \delta R_{1,2}^2 \rangle,$$

calculated in the replica symmetric Gibbs state, where $\delta R_{1,2} := R_{1,2} - \langle R_{1,2} \rangle$. If this variance does not vanish, this phenomenon can be regarded as a long range order. The following finite variance in a sample expectation

$$\lim_{L \to \infty} \mathbb{E} \langle \delta R_{1,2}^2 \rangle = \lim_{L \to \infty} \left[ \mathbb{E} \langle R_{1,2}^2 \rangle - \mathbb{E} \langle R_{1,2} \rangle^2 \right] \neq 0,$$

implies that the long range order is not a rare event. In this case, one can expect an instability of symmetric Gibbs state because of the strong fluctuation and also spontaneous replica symmetry breaking.

Next, we discuss a possibility of replica symmetry breaking without long range order. Since

$$\lim_{L \to \infty} \mathbb{E} \langle \Delta R_{1,2}^2 \rangle = \lim_{L \to \infty} \mathbb{E} \langle \delta R_{1,2}^2 \rangle + \lim_{L \to \infty} \mathbb{E} \langle \Delta R_{1,2} \rangle^2,$$

the replica symmetry breaking occurs, if $\lim_{L \to \infty} \mathbb{E} \langle \Delta R_{1,2} \rangle^2 > 0$ even in the case $\lim_{L \to \infty} \mathbb{E} \langle \delta R_{1,2}^2 \rangle = 0$. At least in principle, replica symmetry breaking may occur even though long range order is a rare event. For the disordered Ising systems, however, this possibility is ruled out by

$$2\mathbb{E} \langle \Delta R_{1,2}^2 \rangle = 3\mathbb{E} \langle \delta R_{1,2}^2 \rangle = 6\mathbb{E} \langle \Delta R_{1,2} \rangle^2 \quad (1)$$

as follows from the Aizenman-Contucci \cite{1, 7} and the Ghirlanda-Guerra identities \cite{4, 5, 8, 9, 14}. Therefore, the replica symmetry breaking cannot occur without long range order in the disordered Ising systems. To extend this argument to quantum spin systems with quenched disorder, however, the quantum mechanically extended Ghirlanda-Guerra identities do not yield the simple identities \cite{11} because of the non-commutativity of the spin operators \cite{21}. In the present paper, we prove that the finite variance of the overlap in the replica symmetric calculation implies spontaneous replica symmetry breaking. Namely, we show that only spontaneous symmetry breaking Gibbs state causes replica symmetry breaking.
2 Definitions

Let $\Lambda_L$ be a $d$ dimensional cubic lattice with a linear size $L$. A spin operator $S^i_x (i = 1, 2, 3)$ at a site $x \in \Lambda_L$ acting on the Hilbert space $\mathcal{H} := \bigotimes_{x \in \Lambda_L} \mathcal{H}_x$ is defined by a tensor product of the $2S + 1$ dimensional self-adjoint matrix $S^i$ acting on $\mathcal{H}_x \simeq \mathbb{C}^{2S+1}$ and $2S + 1$ dimensional identity matrices. Spin operators $(S^i_x)_{x \in \Lambda_L, i = 1, 2, 3}$ satisfy the following commutation relation

$$[S^i_x, S^j_y] = i \delta_{x,y} \sum_{k=1}^3 \epsilon_{i,j,k} S^k_x.$$

The magnitude of each spin operator $(S^1_x, S^2_x, S^3_x)$ is fixed by

$$\sum_{i=1}^3 (S^i_x)^2 = S(S + 1) \mathbb{1},$$

where $\mathbb{1}$ is the identity operator on the Hilbert space $\mathcal{H}$. We denote a product of the spin operators

$$S_X^i = \prod_{x \in X} S^i_x.$$

To define models, let $A := \{ a \in \mathbb{Z} | 1 \leq a \leq l \}$ be a finite set and $(C^a_L)_{a \in A}$ be a collection of interaction ranges, where each $X \in C^a_L$ is a subset $X \subset \Lambda_L$ and $|X| = n_a$ is a positive integer. We assume that there exists an interaction range $Y \in C^a_L$ such that any interaction range $X \in C^a_L$ can be represented in a translation $X = x + Y$ with a suitable $x \in \Lambda_L$.

The Hamiltonian consists of $m$ terms

$$H(S, g) := \sum_{a \in A} \sum_{X \in C^a_L} \left( J^a_1 g^a_X + J^a_0 \right) S^{i(a)}_X,$$  \hspace{1cm} (2)

where a mapping $i : A \to \{1, 2, 3\}$ defines a model and $(g^a_X)_{X \in C^a_L, a \in A}$ are i.i.d. standard Gaussian random variables or several $g^a$ can be identified to each other.

**Examples**

1. Random field Heisenberg model

   $\Lambda_L = \mathbb{Z}^d \cap [1, L]^d$ is a cubic lattice, $A = \{1, 2\}$, $C^1_L = \Lambda_L$, $C^2_L = \{ \{x, y\} | x, y \in \Lambda_L, |x - y| = 1 \}$ is a collection of bonds and $J^1_0 = 0$, $J^2_1 = 0$. The Hamiltonian is given by

   $$H(S, g) = -J^1_1 \sum_{x \in \Lambda_L} g_x S^z_x - \sum_{X \in C^2_L} \sum_{i=1}^3 J^2_0 S^i_X.$$ \hspace{1cm} (3)

2. Random bond Heisenberg model

   For a cubic lattice $\Lambda_L = \mathbb{Z}^d \cap [1, L]^d$ and a collection of bonds $C_1 = \{ \{x, y\} | x, y \in \Lambda_L, |x - y| = 1 \}$

   $$H(S, g) = - \sum_{X \in C^1_L} \sum_{i=1}^3 (J^1_1 g^i_X + J^1_0) S^i_X.$$ \hspace{1cm} (4)
If \( g^1_X = g^2_X = g^3_X \) and \( J^1_r = J^2_r = J^3_r \) are identified for all \( X \in C^1_L \) and for \( r = 0, 1 \), the Hamiltonian is invariant under SU(2) transformation.

3. Other models

The Hamiltonian (2) contains some other physically interesting models, such as Heisenberg model with random next nearest neighbor interactions, and with random plaquette interactions.

Here, we define Gibbs state for the Hamiltonian. For a positive \( \beta \) and a real number \( J \), the partition function is defined by

\[
Z_L(\beta,J, g) := \text{Tr} e^{-\beta H(S,g)},
\]

where the trace is taken over the Hilbert space \( \mathcal{H} \). The probability \( P_k \) that an energy eigenstate \( \phi_k \) with its eigenvalue \( E_k \) appears is given by

\[
P_k := \frac{e^{-\beta E_k}}{Z_L(\beta, J, g)}.
\]

Let \( f \) be an arbitrary function of spin operators. If \( \psi_i \) is an eigenstate with its eigenvalue \( \lambda_i \) of \( f \), the probability that observed value of \( f \) is the eigenvalue \( \lambda_i \) is given by

\[
P(f = \lambda_i) = \sum_k |(\psi_i, \phi_k)|^2 P_k.
\]

Therefore, the expectation of \( f \) in the Gibbs state is given by

\[
\langle f(S) \rangle = \sum_i \lambda_i P(\lambda_i) = \frac{1}{Z_L(\beta, J, g)} \text{Tr} f(S) e^{-\beta H(S,g)}.
\]

We define the following functions of \((\beta, J) \in [0, \infty) \times \mathbb{R}^{2|A|}\) and randomness \( g = (g^a_X)_{X \in C^a_L, a \in A} \)

\[
\psi_L(\beta, J, g) := \frac{1}{|A_L|} \log Z_L(\beta, J, g),
\]

\[-\frac{E}{\beta} \psi_L(\beta, J, g) \text{ is called free energy in statistical physics. We define a function } p_L : [0, \infty) \times \mathbb{R}^{2|A|} \to \mathbb{R} \text{ by }
\]

\[
p_L(\beta, J) := \mathbb{E} \psi_L(\beta, J, g),
\]

where \( \mathbb{E} \) stands for the expectation of the random variables \((g^a_X)_{X \in C^a_L, a \in A} \). Here, we introduce a fictitious time \( t \in [0, 1] \) and define a time evolution of operators with the Hamiltonian. Let \( O \) be an arbitrary self-adjoint operator, and we define an operator valued function \( O(t) \) of \( t \in [0, 1] \) by

\[
O(t) := e^{-tH} O e^{tH}.
\]

Furthermore, we define the Duhamel expectation of time depending operators \( O_1(t_1), \cdots, O_k(t_k) \) by

\[
(O_1, O_2, \cdots, O_k)_D := \int_{[0,1]^k} dt_1 \cdots dt_k \langle T[O_1(t_1)O_2(t_2) \cdots O_k(t_k)] \rangle,
\]
where the symbol $T$ is a multilinear mapping of the chronological ordering. If we define a partition function with arbitrary self-adjoint operators $O_0, O_1, \cdots, O_k$ and real numbers $x_1, \cdots, x_k$

$$Z(x_1, \cdots, x_k) := \text{Tr} \exp \beta \left[ O_0 + \sum_{i=1}^k x_i O_i \right],$$

the Duhamel expectation of $k$ operators represents the $k$-th order derivative of the partition function [12, 15, 25]

$$\beta^k(O_1, \cdots, O_k)_D = \frac{1}{Z} \frac{\partial^k Z}{\partial x_1 \cdots \partial x_k}.$$

To study replica symmetry, we define $n$ replicated spin operators $(S_x^i(\alpha))_{\alpha=1,\cdots,n}$ at each site $x \in \Lambda_L$ and a replica symmetric Hamiltonian

$$H(S^1, \cdots, S^n, g) := \sum_{\alpha=1}^n H(S^\alpha, g),$$

which is invariant under an arbitrary permutation $\sigma \in S_n$

$$H(S^1, \cdots, S^n, g) = H(S^\sigma 1, \cdots, S^n \sigma, g),$$

as well as the Ising systems. The covariance of these operators with the expectation in $g$ for $a \in A$

$$\mathbb{E}H_L^a(S^i(\alpha),H_L^a(S^i(\beta)) - \mathbb{E}(H_L^\alpha(S^i(\alpha))\mathbb{E}H_L^\beta(S^i(\beta))) = |C_a^\alpha R_{a,\beta}^a$$

where the overlap $R_{a,\beta}^a$ is defined by

$$R_{a,\beta}^a := \frac{1}{|C_a^\alpha|} \sum_{X \in C_a^\alpha} S^i(\alpha) S^i(\beta).$$

For example, in the random field Heisenberg model, this becomes the site overlap operator

$$R_{a,\beta}^i := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} S_x^i(\alpha) S_x^i(\beta),$$

where we have identified the index $a$ to $i(\alpha)$ for simpler notation. In the random bond Heisenberg model, it becomes the bond overlap operator

$$R_{a,\beta}^i = \frac{1}{|B|} \sum_{X,Y \in B} S_x^i(\alpha) S_x^i(\beta).$$

In short range spin glass models, such as the Edwards-Anderson model [13] the bond overlap is independent of the site overlap unlike the Sherrington-Kirkpatrick (SK) model [26], where the bond overlap is identical to the square of the site overlap. In calculation of thermodynamic quantities of quantum systems, the following self overlap operator appears quite frequently

$$R_{1,1}^a := \frac{1}{|C_a^\alpha|} \sum_{X \in C_a^\alpha} S^i(\alpha)(t_1) S^i(\alpha)(t_2)$$
for \((t_1, t_2) \in [0, 1]^2\). We denote
\[
(R^a_{1,1})_D = \frac{1}{|C^a_L|} \sum_{X \in C^a_L} (S^{a,1}_X, S^{a,1}_X)_D.
\]
Note that \(R^a_{1,1} \to 1\) in the classical limit. As we will see later, the expectation values of overlap operators satisfy the quantum mechanically extended Ghirlanda-Guerra identities and Aizenman-Contucci identities (1) which depend on self-overlap operators.

3 Self-averaging observables

Here, we discuss the self-averaging observables in quantum systems. Let \(O_1, \ldots, O_N\) be a sequence of self-adjoint operators on \(\mathcal{H}\), and we define their mean by
\[
m_N := \frac{1}{N} \sum_{n=1}^N O_n.
\]
Let \(\mu\) be an eigenvalue of \(m_N\) for arbitrary fixed \((g^a_X)_{X \in C^a_L, a \in A}\). The probability that the deviation between the observed value \(\mu\) of \(m_N\) between the expectation \(\langle m_N \rangle\) in the Gibbs state is larger than an arbitrary positive number \(c\) is bounded by the variance of \(m_N\)
\[
P(|\mu - \langle m_N \rangle| > c) \leq \frac{\langle \delta m_N^2 \rangle}{c^2},
\]
where \(\delta m_N := m_N - \langle m_N \rangle\). If the variance \(\langle \delta m_N^2 \rangle\) vanishes in the infinite volume limit, then the observed value differs from the expectation \(\langle m_N \rangle\) rarely. This implies that \(m_N\) obeys the law of large numbers. Next, we consider events in a synthesized quantum spin system with quenched disorder. A disordered sample synthesizing corresponds to a fixing of random variables \((g^a_X)_{X \in C^a_L, a \in A}\) generated by a random sampling. We can evaluate the probability that deviation between the observed value \(\mu\) and the sample and the Gibbs expectation \(\mathbb{E}\langle m_N \rangle\) is larger than \(c > 0\) as follows
\[
P(|\mu - \mathbb{E}\langle m_N \rangle| > c) \leq \frac{\mathbb{E}\langle \Delta m_N^2 \rangle}{c^2},
\]
where \(\Delta m_N := m_N - \mathbb{E}\langle m_N \rangle\). If the variance \(\mathbb{E}\langle \Delta m_N^2 \rangle\) vanishes in the infinite volume limit, then \(m_N\) obeys the law of large numbers. In this case, we can say that \(m_N\) is self-averaging.

Throughout this section, we assume the following general form of the Hamiltonian
\[
H(O, g) := \sum_{a \in A} \sum_{X \in C^a_L} (J^a_1 g^a_X + J^a_0) O^a_X,
\]
where each self-adjoint operators \((O^a_X)_{X \in C^a_L, a \in A}\) are confined in \(X \in C^a_L\) with \(|X| = n_a\) for each \(X \in C^a_L\). If \(X \cap Y = \phi\), the commutator between two operators satisfies
\[
[O^a_X, O^b_Y] = 0.
\]
Each \(O^a_X\) \((a \in A, X \in C^a_L)\) and the commutator among any of them are bounded by constants independent of the system size \(N\). We denote
\[
K^a := \sup_{\phi \in \mathcal{H}} \frac{|(\phi, O^a_X \phi)|}{(\phi, \phi)}.
\]
Here we define a density of a term in the Hamiltonian with the randomness by
\[ h^a_L := \frac{1}{|C^a_L|} \sum_{X \in C^a_L} g^a_X O^a_X, \]  
(12)
and a deterministic term in the Hamiltonian by
\[ m^a_L := \frac{1}{|C^a_L|} \sum_{X \in C^a_L} O^a_X. \]  
(13)

Here, we define two types of deviations of an arbitrary operator \( O \) by
\[ \delta O := O - \langle O \rangle, \quad \Delta O := O - \mathbb{E}\langle O \rangle. \]

We prove the following two lemmas for these deviations of
\[ m^a_L := \frac{1}{|C^a_L|} \sum_{X \in C^a_L} O^a_X. \]

We assume the existence of the following infinite volume limit independent of boundary conditions
\[ p(\beta, J) = \lim_{L \to \infty} p_L(\beta, J), \quad \psi_L(\beta, J) = \lim_{L \to \infty} \psi_L(\beta, J), \]
as proved in [2, 10, 11, 21].

Hereafter, we use a lighter notation \( \psi_L(\beta, J) \) for \( \psi_L(\beta, J, g) \).

We define square root interpolating random variables \( G(\vec{u}) = (G^a_X(\vec{u}))_{X \in C^a_L, a \in A} \) for an arbitrary vector \( \vec{u} = (u^a)_{a \in A} \in [0, 1]^{|A|} \) by
\[ G^a_X(\vec{u}) := \sqrt{u^a g^a_X + \sqrt{1 - u^a g^a_X}'}, \]
(14)
where \( (g^a_X')_{a \in C^a_L, a \in A} \) are i.i.d. standard Gaussian random variables. Then, we define a generating function \( \gamma_L(\vec{u}) \) of a parameter \( \vec{u} \in [0, 1]^{|A|} \) by
\[ \gamma_L(\vec{u}) = \mathbb{E}[\mathbb{E}' \psi_L(G(\vec{u}))^2], \]
(15)
where \( \mathbb{E} \) and \( \mathbb{E}' \) denote expectation in \( g \) and \( g' \), respectively. This generating function \( \gamma_L \) is a generalization of a function introduced by Chatterjee [6].

First, we present useful lemmas proved in Ref. [21] as Lemma 3.1-3.6 in the following.

**Lemma 3.1** For any \( (\beta, J) \in [0, \infty) \times \mathbb{R}^{|A|} \), any positive integer \( L \), any positive integer \( k \) and any \( \vec{u}_0 \in [0, 1]^{|A|} \) whose \( a \)-th component is \( u^a = u_0 < 1 \), an upper bound on the \( k \)-th order partial derivative of the function \( \gamma_L \) is given by
\[ \frac{\partial^k \gamma_L(\vec{u}_0)}{\partial u^k} \leq \frac{(k-1)!}{(1-u_0)^{k-1}} \left( \beta J^a \right)^2 |A_L| \]
(16)
The \( k \)-th order derivative of \( \gamma_L \) is represented in the following
\[ \frac{\partial^k \gamma_L(\vec{u})}{\partial u^k} = \sum_{X_1 \in C^a_L} \cdots \sum_{X_k \in C^a_L} \mathbb{E} \left( \mathbb{E}' \frac{\partial^k \psi_L}{\partial g^a_{X_k} \cdots g^a_{X_1}} (G(\vec{u})) \right)^2. \]
(17)
for an arbitrary \( \vec{u} \in [0, 1]^{|A|} \).

If we define a function \( \chi_L : [0, 1] \to \mathbb{R} \) by \( \chi_L(s) := \gamma_L(s, s, \cdots, s) \) for \( s \in [0, 1] \), then Lemma 3.1 gives the following
Lemma 3.2 The variance of $\psi_L$ is bounded from the above as follows

$$\mathbb{E}(\psi_L - p_L)^2 = \chi_L(1) - \chi_L(0) \leq \frac{(\beta J_1^a K^a)^2 n_a |A|}{|\Lambda_L|}. $$

Lemma 3.3 For any $a \in A$ and for $\beta J_1^a \neq 0$, the following quantity is bounded from the above,

$$\mathbb{E}(\delta m_L^a, \delta m_L^a)_D \leq \frac{K^a}{\beta J_1^a} \sqrt{\frac{1}{n_a |\Lambda_L|}}. \tag{18}$$

Lemma 3.4 For any $a \in A$ and any $\beta J_r^a \neq 0$, we have

$$\mathbb{E}(\delta h_L^a, \delta h_L^a)_D \leq \frac{K^a}{\beta J_1^a} \left( \sqrt{\frac{6}{n_a |\Lambda_L|}} + \frac{1}{n_a |\Lambda_L|} \right). \tag{19}$$

Note The functions $p(\beta, J)$, $p_L(\beta, J)$ and $\psi_L(\beta, J, g)$ are convex with respect to $\beta$ and $J_r^a (a \in A, r = 0, 1)$ each. This implies that these functions are differentiable almost everywhere in the coupling constant space $[0, \infty) \times \mathbb{R}^{2|A|}$.

Notation Let $D$ be the measure zero subset of the coupling constant space $[0, \infty) \times \mathbb{R}^{2|A|}$ where $p(\beta, J)$ is not differentiable with respect to $\beta$ or some $J_r^a$.

Lemma 3.5 In the infinite volume limit, the following is valid

$$\lim_{L \to \infty} \mathbb{E} \left( \frac{\partial \psi_L}{\partial J_r^a} - \frac{\partial p}{\partial J_r^a} \right)^2 = 0. \tag{20}$$

on $D^c \subset [0, \infty) \times \mathbb{R}^{2|A|}$.

Lemma 3.6 In $D^c \subset [0, \infty) \times \mathbb{R}^{2|A|}$, we have

$$\frac{\partial p}{\partial J_r^a} = \lim_{L \to \infty} \mathbb{E} \frac{\partial \psi_L}{\partial J_r^a}. \tag{21}$$

Next, we prove upper bound on variances of several quantities.

Lemma 3.7 For any $a \in A$ and for $\beta J_1^a \neq 0$, there exists a positive constant $K'$ independent of $L$, such that

$$\mathbb{E}(\delta m_L^a)^2 \leq \frac{K^a}{\beta J_1^a} \sqrt{\frac{1}{n_a |\Lambda_L|}} + \frac{\beta K'}{12 |\Lambda_L|}. \tag{22}$$

Proof. We use Harris’ Bogolyubov type inequality between the Duhamel product and the Gibbs expectation of the square of arbitrary self-adjoint operator $O$ \cite{20}

$$(O, O)_D \leq \langle O^2 \rangle \leq (O, O)_D + \frac{\beta}{12} \langle [O, [H, O]] \rangle, \tag{23}$$

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for the above inequality. We use an indicator \( I \) defined by \( I[true] = 1 \) and \( I[false] = 0 \). Using the Cauchy-Schwalz inequality and boundedness of operators \( O^a_X \) and their commutators, we have the following

\[
\mathbb{E}(\delta m^a_L)^2 \leq \mathbb{E}(\delta m^a_L, \delta m_L^a) + \frac{\beta}{12|C_L|^2} \sum_{x,y \in C_L^b} \sum_{a \in A} \sum_{z \in C_L^b} \mathbb{E}g_X^a g_Y^b g_Z^b \langle [O^a_X, [O^b_Z, O^a_Y]] \rangle \\
\leq \mathbb{E}(\delta m^a_L, \delta m_L^a) + \frac{\beta}{12|C_L|^2} \sum_{b \in A} \sum_{Z \in C_L^b} \sum_{x,y \in C_L^a} \sqrt{\mathbb{E}(g_X^b)^2 \mathbb{E}(g_Y^a)^2} \mathbb{E}\langle [O^a_X, [O^b_Z, O^a_Y]] \rangle^2 \\
\leq \mathbb{E}(\delta m^a_L, \delta m_L^a) + \frac{\beta}{12|C_L|^2} \sum_{b \in A} \sum_{Z \in C_L^b} \sum_{x,y \in C_L^a} I[X \cap (Y \cup Z) \neq \phi] I[Y \cap Z \neq \phi] K \\
\leq \mathbb{E}(\delta m^a_L, \delta m_L^a) + \frac{\beta K'}{12|L|^2},
\]

where \( K \) and \( K' \) are positive constant independent of \( L \). Therefore, this and (18) complete the proof. □

**Lemma 3.8** For any \( a \in A \) and any \( \beta J^a_1 \neq 0 \), there exists a positive number \( K' \) we have

\[
\mathbb{E}(h_L^a)^2 \leq \frac{K^a}{\beta J^a_1} \left( \sqrt{\frac{6}{n_a|A_L|}} + \frac{1}{n_a|A_L|} \right) + \frac{\beta K'}{12|L|^2}.
\]

**Proof.** As well as in the proof of Lemma 3.7, we use the inequality (23) and the Cauchy-Schwalz inequality

\[
\mathbb{E}(h_L^a)^2 \leq \mathbb{E}(h_L^a, h_L^a) + \frac{\beta}{12|L|^2} \sum_{x,y \in C_L^b} \sum_{a \in A} \sum_{z \in C_L^b} \mathbb{E}g_X^a g_Y^b g_Z^b \langle [O^a_X, [O^b_Z, O^a_Y]] \rangle \\
\leq \mathbb{E}(h_L^a, h_L^a) + \frac{\beta}{12|L|^2} \sum_{b \in A} \sum_{Z \in C_L^b} \sum_{x,y \in C_L^a} \sqrt{\mathbb{E}(g_X^b)^2 \mathbb{E}(g_Y^a)^2} \mathbb{E}\langle [O^a_X, [O^b_Z, O^a_Y]] \rangle^2 \\
\leq \mathbb{E}(h_L^a, h_L^a) + \frac{\beta}{12|L|^2} \sum_{b \in A} \sum_{Z \in C_L^b} \sum_{x,y \in C_L^a} I[X \cap (Y \cup Z) \neq \phi] I[Y \cap Z \neq \phi] K \\
\leq \mathbb{E}(h_L^a, h_L^a) + \frac{\beta K'}{12|L|^2},
\]

Therefore, this and (19) complete the proof. □

Aizenman, Greenbatt and Lebowitz proved the rounding effect of disordered interactions which means that \( p \) is differentiable with respect to \( J^a_0 \) for \( J^a_1 \neq 0 \) in \( d \leq 2 \) for a system with discrete symmetry and in \( d \leq 4 \) for a system with a continuous symmetry [2, 16]. Their proof is an extension of the differentiability of the free energy density for classical systems proved by Aizenman and Wehr [3] to the quantum systems. In dimensions \( d \) larger than the critical dimensions, there is a possibility that the function \( p \) is not differentiable on a measure zero subset of the coupling constant space, where the order parameter can be discontinuous and the first order phase transition occurs. In this case,
the Gibbs state is not unique. The discontinuity appears at $J_0^* = 0$ most likely.

Note the relations

$$\beta \langle m_L^o \rangle = \frac{\partial \psi_L}{\partial J_0^o}, \quad \beta \langle h_L^o \rangle = \frac{\partial \psi_L}{\partial J_1^o}, \quad \beta \mathbb{E} \langle m_L^o \rangle = \frac{\partial p_L}{\partial J_0^o}, \quad \beta \mathbb{E} \langle h_L^o \rangle = \frac{\partial p_L}{\partial J_1^o}.$$  

If $\frac{\partial p}{\partial \beta}$ has a discontinuity at $J_0^o$, differentiability of $p$ almost everywhere around $J_0^o$ gives

$$\lim_{x \downarrow J_0^o, p} \frac{\partial p}{\partial J_0^o}(x) = \lim_{x \uparrow J_0^o, p} \beta \mathbb{E} \langle m_L^o \rangle(x), \quad \lim_{x \downarrow J_0^o, p} \frac{\partial p}{\partial J_0^o}(x) = \lim_{x \uparrow J_0^o, p} \beta \mathbb{E} \langle m_L^o \rangle(x).$$

Also integration by parts enables us to calculate the following

$$\lim_{x \downarrow J_0^o, p} \frac{\partial p}{\partial J_1^o}(x) = \lim_{x \downarrow J_0^o, p} \beta \sum_{X \in C_L^o} \mathbb{E} g_X^o \langle O_X^o \rangle(x) = \lim_{x \downarrow J_0^o, p} \beta^2 J_1^o \sum_{X \in C_L^o} \mathbb{E} [(O_X^o, O_X^o)_{\mathbb{D}}(x) - \langle O_X^o \rangle^2],$$

$$\lim_{x \uparrow J_0^o, p} \frac{\partial p}{\partial J_1^o}(x) = \lim_{x \uparrow J_0^o, p} \beta \sum_{X \in C_L^o} \mathbb{E} g_X^o \langle O_X^o \rangle(x) = \lim_{x \uparrow J_0^o, p} \beta^2 J_1^o \sum_{X \in C_L^o} \mathbb{E} [(O_X^o, O_X^o)_{\mathbb{D}}(x) - \langle O_X^o \rangle^2].$$

These identities imply that the order parameter becomes discontinuous at the non-differentiable point of $p$.

Now, we prove that $m_L^o$ and $h_L^o$ are self-averaging on $D^c$.

**Theorem 3.9** For $\beta J_1^o \neq 0$, variances of the density defined by $m_L^o := \frac{1}{|C_L^o|} \sum_{X \in C_L^o} O_X^o$ and that of density of the Hamiltonian with randomness $h_L^o := \frac{1}{|C_L^o|} \sum_{X \in C_L^o} g_X^o O_X^o$ for an arbitrary $a \in A$ vanish in the infinite volume limit

$$\lim_{L \to \infty} \mathbb{E} \langle (m_L^o - \mathbb{E} \langle m_L^o \rangle)^2 \rangle = 0,$$

$$\lim_{L \to \infty} \mathbb{E} \langle (h_L^o - \mathbb{E} \langle h_L^o \rangle)^2 \rangle = 0,$$

on $D^c \subset [0, \infty) \times \mathbb{R}^{2|A|}$.

**Proof** We calculate the following variance of $m_L^o$,

$$\lim_{L \to \infty} \mathbb{E} \langle (m_L^o)^2 \rangle - \langle \mathbb{E} \langle m_L^o \rangle \rangle^2 = \lim_{L \to \infty} \mathbb{E} \langle (m_L^o)^2 \rangle - \langle \mathbb{E} \langle m_L^o \rangle \rangle^2 + \lim_{L \to \infty} \mathbb{E} \langle (m_L^o)^2 \rangle - \langle \mathbb{E} \langle m_L^o \rangle \rangle^2$$

$$= \lim_{L \to \infty} \mathbb{E} \langle (m_L^o - \langle m_L^o \rangle)^2 \rangle + \lim_{L \to \infty} \mathbb{E} \langle (\langle m_L^o \rangle - \mathbb{E} \langle m_L^o \rangle)^2 \rangle = 0.$$  

We have used Lemma 3.4 for the first term and Lemma 3.5 for the second term in the last line. We calculate the following variance of $h_L^o$, as well.

$$\lim_{L \to \infty} \mathbb{E} \langle (h_L^o)^2 \rangle - \langle \mathbb{E} \langle h_L^o \rangle \rangle^2 = \lim_{L \to \infty} \mathbb{E} \langle (h_L^o)^2 \rangle - \langle \mathbb{E} \langle h_L^o \rangle \rangle^2 + \lim_{L \to \infty} \mathbb{E} \langle (h_L^o)^2 \rangle - \langle \mathbb{E} \langle h_L^o \rangle \rangle^2$$

$$= \lim_{L \to \infty} \mathbb{E} \langle (h_L^o - \langle h_L^o \rangle)^2 \rangle + \lim_{L \to \infty} \mathbb{E} \langle (\langle h_L^o \rangle - \mathbb{E} \langle h_L^o \rangle)^2 \rangle = 0.$$  

We have used Lemma 3.8 for the first term and Lemma 3.5 for the second term in the last line. Therefore, $m_L^o$ and $h_L^o$ are self-averaging. □
**Theorem 3.10** The following conditions (I) and (II) are equivalent for almost all sequences of random variables \((g^X_a)\).

(I) There exists \(b \in A\), such that
\[
\lim_{L \to \infty} \langle (m^b_L - \langle m^b_L \rangle)^2 \rangle \neq 0,
\] (31)
at \((J^b_0, J^b_1) = (0, 0)\).

(II) There exists \(b \in A\), such that
\[
\lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} \langle (m^a_L - \langle m^b_L \rangle)^2 \rangle \neq \lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} \langle (m^b_L - \langle m^b_L \rangle)^2 \rangle.
\] (32)

**Proof.** Lebesgue’s dominated convergence theorem guarantees a commutativity between any limit operations and the sample expectation \(E\), since \(\langle (m^b_L)^k \rangle\) is bounded by a constant independent of \(L\) for any positive \(k\). Therefore, we evaluate the variance of \(m^b_L\)
\[
E \lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} \langle (m^b_L - \langle m^b_L \rangle)^2 \rangle = \lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} E \langle (m^b_L - \langle m^b_L \rangle)^2 \rangle = 0.
\]

The final term vanishes in the coupling constant space \([0, \infty) \times \mathbb{R}^{2|A|}\) by Lemma 3.7. The positive semi-definiteness of \((m^b_L - \langle m^b_L \rangle)^2\) gives that the left hand side in (32) vanishes
\[
\lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} \langle (m^b_L - \langle m^b_L \rangle)^2 \rangle = 0,
\]
for almost all \((g^X_a)\). The continuity of the function of \((J^b_0, J^b_1)\) for an arbitrary finite \(L\) guarantees the equivalence between positive variance and non-commutativity of limits for almost all \((g^X_a)\). This completes the proof. \(\Box\)

Here, we apply Theorem 3.10 to the symmetry breaking phenomena in quantum systems with Gaussian disorder. Assume that the Hamiltonian has a certain symmetry at \((J^a_0, J^a_1) = (0, 0)\), and order parameter \(m^a_L\) breaks this symmetry. Theorem 3.10 claims the equivalence between the finite variance (I) in symmetric Gibbs state and the spontaneous symmetry breaking (II). As in quantum spin systems without disorder, the existence of spontaneous symmetry breaking can be detected by the variance of the corresponding order parameter calculated in the symmetric Gibbs state. The Koma-Tasaki theorem is important for the Heisenberg quantum spin model with SU(2) symmetry. This theorem claims that the finite long range order
\[
\lim_{L \to \infty} \langle m^2_L \rangle \neq 0,
\]
for the order parameter \(m^i_L := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} S^i_x\) in the symmetric Gibbs state guarantees a finite spontaneous magnetization with an infinitesimal symmetry breaking field \([22]\). This is a quantum mechanical extension of Griffiths’ theorem \([17]\). Since \(\langle m^i_L \rangle = 0\) in the symmetric Gibbs state in this case, the finite variance implies the long range order. Although the finite variance of the order parameter seems to violate the law of large numbers, this is just apparent. In this case, one of the symmetry breaking Gibbs states obtained by applying the infinitesimal symmetry breaking field realizes instead of the unstable symmetric one. Theorem 3.10 claims the same conclusion for spontaneous symmetry breaking.
phenomena in disordered quantum spin systems as that of the Koma-Tasaki theorem. If the variance of \( m^b_L \) becomes finite in the symmetric Gibbs state, one of the symmetry breaking Gibbs states should realize actually instead of the unstable symmetric one and the order parameter \( m^b_L \) obeys the law of large numbers.

Next, we present equivalence between non-self-averaging order parameter in symmetric Gibbs state and non-commutativity of the infinite volume limit and the symmetric limit. Therefore, this implies that a violation of the self-averaging in the symmetric calculation should be raised by a spontaneous symmetry breaking.

**Theorem 3.11** The following conditions (III) and (IV) are equivalent for almost all sequence of random variables \( (g^a_X) \) on \( D^c \subset [0, \infty) \times \mathbb{R}^{|A|} \).

(III) There exists \( b \in A \), such that

\[
\lim_{L \to \infty} \langle (m^b_L - \mathbb{E}(m^b_L))^2 \rangle \neq 0, \tag{33}
\]

at \((J^b_0, J^b_1) = (0, 0)\).

(IV) There exists \( b \in A \), such that

\[
\lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} \langle (m^b_L - \mathbb{E}(m^b_L))^2 \rangle \neq \lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} \langle (m^b_L - \mathbb{E}(m^b_L))^2 \rangle. \tag{34}
\]

**Proof.** As in the proof of Theorem 3.10, we have the commutativity between any limit operations and the sample expectation \( \mathbb{E} \). If we use Lemma 3.9, we obtain

\[
\mathbb{E} \lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} \langle (m^b_L - \mathbb{E}(m^b_L))^2 \rangle = \lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} \mathbb{E}\langle (m^b_L - \mathbb{E}(m^b_L))^2 \rangle = 0,
\]
on \( D^c \). The positive semi-definiteness of \((m^b_L - \mathbb{E}(m^b_L))^2\) gives

\[
\lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} \langle (m^b_L - \mathbb{E}(m^b_L))^2 \rangle = 0,
\]
for almost all \((g^a_X)\). Therefore, the violation of the self-averaging

\[
\lim_{L \to \infty} \langle (m^b_L - \mathbb{E}(m^b_L))^2 \rangle \neq 0
\]
at \((J^b_0, J^b_1) = (0, 0)\) is equivalent to the following non-commutativity

\[
\lim_{L \to \infty} \lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \langle (m^b_L - \mathbb{E}(m^b_L))^2 \rangle \neq \lim_{J^b_0 \to 0} \lim_{J^b_1 \to 0} \lim_{L \to \infty} \langle (m^b_L - \mathbb{E}(m^b_L))^2 \rangle,
\]
for almost all \((g^a_X)\) on \( D^c \). This completes the proof. \( \square \)

4 The Ghirlanda-Guerra type identities for quantum systems

The following limit given in Theorem 3.9

\[
\lim_{L \to \infty} \mathbb{E}\langle \Delta h^2_L \rangle = 0,
\]

on \( D^c \).
enables us to derive quantum mechanically extended identities of the Ghirlanda-Guerra type which differ from those obtained in Ref. [21] by the limit

$$\lim_{L \to \infty} E(\Delta h_L^{a}, \Delta h_L^{a})_D = 0.$$  

**Theorem 4.1** For $\beta J_1^a \neq 0$, on $D^c \subset [0, \infty) \times \mathbb{R}^{2|A|}$, for an arbitrary bounded function $f$ of a replicated spin operators, the following identity is valid

$$\lim_{L \to \infty} \left[ \frac{1}{|C_L^a|} \sum_{X \in C_L^a} \sum_{\alpha=1}^{n} E(S_X^{(a),\alpha}, S_X^{(a),1} f)_D - nE(R_{1,n+1}^a f) + E(R_{1,2}^a)E(f) - E(R_{1,1}^a)D E(f) \right] = 0 \quad (35)$$

**Proof.** Theorem 3.9 gives

$$\lim_{L \to \infty} E(\Delta h_L^{a}) = 0. \quad (36)$$

The Cauchy-Schwarz inequality and the boundedness of $f$ imply

$$|E(\Delta h_L^{a} f)| \leq \sqrt{E(\Delta h_L^{a}^2) E(f^2)} \to 0,$$

in the infinite volume limit. The left hand side can be calculated using integration by parts.

$$\frac{1}{|C_L^a|} \sum_{X \in C_L^a} \sum_{\alpha=1}^{n} E g_X^a (S_X^{(a)} f) = \frac{1}{|C_L^a|} \sum_{X \in C_L^a} \sum_{\alpha=1}^{n} E \frac{\partial}{\partial g_X^a} (S_X^{(a)} f)$$

$$= \beta J_1^a \left[ \frac{1}{|C_L^a|} \sum_{X \in C_L^a} \sum_{\alpha=1}^{n} E(S_X^{(a),1}, S_X^{(a),\alpha} f)_D - nE(S_X^{(a)})(S_X^{(a),1} f) \right]$$

$$= \beta J_1^a \sum_{X \in C_L^a} \sum_{\alpha=1}^{n} E(S_X^{(a),\alpha}, S_X^{(a),1} f)_D - nE(R_{1,n+1}^a f) \quad (37)$$

Substituting $f = 1$ to the above, we have

$$\frac{1}{|C_L^a|} \sum_{X \in C_L^a} E g_X^a (S_X^{(a)})$$

$$= \beta J_1^a \left[ \frac{1}{|C_L^a|} \sum_{X \in C_L^a} E(S_X^{(a),1}, S_X^{(a),1})_D + \sum_{\alpha=2}^{n} E(R_{1,\alpha}) - nE(R_{1,n+1}^a) \right]$$

$$= \beta J_1^a [E(R_{1,1}^a)_D - E(R_{1,2}^a)] \quad (38)$$

From the above two identities, we have

$$E(\Delta h_L^{a} f) = \beta J_1^a \left[ \frac{1}{|C_L^a|} \sum_{X \in C_L^a} \sum_{\alpha=1}^{n} E(S_X^{(a),\alpha}, S_X^{(a),1} f)_D - nE(R_{1,n+1}^a f)$$

$$- (E(R_{1,1}^a)_D - E(R_{1,2}^a))E(f) \right], \quad (39)$$

Therefore, we obtain the given identity $\square$
Now, we derive the relation between two kinds of variance $\mathbb{E}\langle \Delta R_{1,2}^a \rangle$ and $\mathbb{E}\langle \delta R_{1,2}^a \rangle$, where $\Delta R_{1,2}^a := R_{1,2}^a - \mathbb{E}\langle R_{1,2}^a \rangle$ and $\delta R_{1,2}^a := R_{1,2}^a - \langle R_{1,2}^a \rangle$.

For $n = 2$, $f = R_{1,2}^a$, Theorem 4.1 gives

$$\lim_{L \to \infty} \left[ \frac{1}{|C_L^a|} \sum_{X \in C_L^a} \mathbb{E}(S_X^{i(a),2}, S_X^{i(a),1} R_{1,2}^a)_D - 2 \mathbb{E}(R_{1,3}^a, R_{1,2}^a) + (\mathbb{E}(R_{1,2}^a))^2 \
+ \frac{1}{|C_L^a|} \sum_{X \in C_L^a} \mathbb{E}(S_X^{i(a),1}, S_X^{i(a),1} \Delta R_{1,2}^a)_D \right] = 0. \quad (40)$$

For $n = 3$, $f = R_{2,3}^a$, Theorem 4.1 gives

$$\lim_{L \to \infty} \left[ 2 \mathbb{E}(R_{1,2}^a R_{2,3}^a)_D - 3 \mathbb{E}(R_{1,4}^a R_{2,3}^a) + \mathbb{E}(R_{1,2}^a) \mathbb{E}(R_{2,3}^a) \
+ \mathbb{E}(R_{1,1}^a) \mathbb{E}(R_{2,3}^a) \right] = 0. \quad (41)$$

We have used the replica symmetry, since we calculate above terms in the replica symmetric expectation. The identities (40) and (41) give

$$\lim_{L \to \infty} \left[ 2(\mathbb{E}(R_{1,2}^a))^2 - 3 \mathbb{E}(R_{1,2}^a)^2 + \frac{1}{|C_L^a|} \sum_{X \in C_L^a} \mathbb{E}(S_X^{i(a),2}, S_X^{i(a),1} R_{1,2}^a)_D \
+ \frac{1}{|C_L^a|} \sum_{X \in C_L^a} \mathbb{E}(S_X^{i(a),1}, S_X^{i(a),1} \Delta R_{1,2}^a)_D + \mathbb{E}(R_{1,1}^a) \mathbb{E}(R_{1,2}^a) + 2 \mathbb{E}(R_{1,2}^a R_{1,3}^a)_D - 2 \mathbb{E}(R_{1,2}^a R_{1,3}^a)_D \right] = 0. \quad (42)$$

This identity has not been very useful so far. In the classical limit, four terms in the second line vanish, and we obtain

$$\lim_{L \to \infty} \left[ 2(\mathbb{E}(R_{1,2}^a))^2 - 3 \mathbb{E}(R_{1,2}^a)^2 + \mathbb{E}(R_{1,2}^a)^2 \right] = 0,$$

which yields

$$2 \lim_{L \to \infty} \mathbb{E}\langle \Delta R_{1,2}^a \rangle^2 = 3 \lim_{L \to \infty} \mathbb{E}\langle \delta R_{1,2}^a \rangle^2.$$

If the right hand side vanishes, then the left hand side vanishes. In quantum systems, however, we cannot judge whether or not $\lim_{L \to \infty} \mathbb{E}\langle \Delta R_{1,2}^a \rangle^2$ vanishes even if $\lim_{L \to \infty} \mathbb{E}\langle \delta R_{1,2}^a \rangle^2 = 0$.

Note that another relation (27) in Theorem 3.9 yields the following identity for an arbitrary bounded function $f$ of spin operators

$$\mathbb{E}(\Delta m_{1,2}^a f) = 0.$$

This should be useful some times.

5 Spontaneous replica symmetry breaking

Here, we study replica symmetry breaking phenomena applying Theorem 3.10 and 3.11 to the overlap operator

$$R_{1,2}^c := \frac{1}{|C_L^c|} \sum_{X \in C_L^c} S_X^{i(c),1} S_X^{i(c),2},$$
for an arbitrary fixed \( c \in A \). We extend the coupling constant space \([0, \infty) \times \mathbb{R}^{2|A|}\) to \([0, \infty) \times \mathbb{R}^{2|A|+1}\) with new coupling constants \((J^0_1, J^0_0)\) to introduce inter replica coupling. 

We define a total Hamiltonian by

\[
H_{\text{tot}}(S^1, \cdots, S^n, g, g^0) := H(S^1, \cdots, S^n, g) + H_{\text{int}}(S^1, S^2, g^0),
\]

where

\[
H(S^1, \cdots, S^n, g) = \sum_{a=1}^{n} \sum_{a \in A} \sum_{X \in C^0_L} (J^a_1 g_X + J^a_0)S^{i(a),a}_X,
\]

\[
H_{\text{int}}(S^1, S^2, g^0) = \sum_{X \in C^0_L} (J^0_1 g_X + J^0_0)S^{i(0),1}_X S^{i(0),2}_X,
\]

with \( C^0_L = C^c_L \) and \( i(c) = i(0) \). Note that the inter replica coupling breaks the replica symmetry of the Hamiltonian. We can apply all obtained theorems to this system, since the operators

\[
O^a_X := S^{i(a)}_X
\]

for \( a \in A \) and

\[
O^0_X := S^{i(c),1}_X S^{i(c),2}_X
\]

for \( c \in A \) and \( X \in C^0_L \) are bounded by

\[
\sup_{\phi \in \mathcal{H}} \left| \frac{\langle \phi, S^{i(a)}_X \phi \rangle}{\langle \phi, \phi \rangle} \right| = S^{n_a}, \quad \sup_{\phi \in \mathcal{H} \otimes \mathcal{H}} \left| \frac{\langle \phi, S^{i(b),1}_X S^{i(b),2}_X \phi \rangle}{\langle \phi, \phi \rangle} \right| = S^{2n_b}.
\]

**Note**: The density \( m^0_L \) is identical to the overlap

\[
m^0_L = R^0_{1,2},
\]

and the replica symmetry at \((J^0_1, J^0_0) = (0,0)\) does not imply \( m^0_L = 0 \), but implies \( \langle R^0_{1,2} \rangle = \langle R^c_{k,l} \rangle \) for \( 1 \leq k < l \leq n \).

Here we employ the definition of replica symmetry breaking for disordered Ising systems given by Chatterjee [4] also for quantum systems. For an arbitrary sequence of \((g^0_X)\) of random variables, we say that a replica symmetry breaking occurs, if

\[
\lim_{L \to \infty} \langle (R^a_{1,2} - \mathbb{E}(R^a_{1,2}))^2 \rangle \neq 0,
\]

for \( \exists a \in A \) and for \((J^0_1, J^0_0) = (0,0) \).

Next, we remark the nature of spontaneous replica symmetry breaking. For an arbitrary sequence \((g^0_X)\) of random variables we can say that a spontaneous replica symmetry breaking occurs, if

\[
\lim_{J^0_1 \to 0} \lim_{J^0_0 \to 0} \lim_{L \to \infty} \langle R^a_{1,2} \rangle \neq \lim_{J^0_1 \to 0} \lim_{J^0_0 \to 0} \lim_{L \to \infty} \langle R^0_{1,2} \rangle,
\]

for \( \exists a \in \{0, 1\} \), for \( \exists a \in A \) and for \( \exists k > 0 \).

Now, we apply Theorem 3.10 and 3.11 to \( m^0_L = R^0_{1,2} \). If (I) in Theorem 3.10 occurs in some samples, namely

\[
\lim_{L \to \infty} \langle (R^c_{1,2} - \langle R^c_{1,2} \rangle)^2 \rangle \neq 0,
\]
at \((J_0^0, J_1^0) = (0, 0)\), then (II) the limit operations are non-commutative and the spontaneous replica symmetry breaking occurs in such samples. This result corresponds to the Koma-Tasaki theorem which shows the detection of the spontaneous symmetry breaking by the long range order in symmetric Gibbs state in deterministic quantum systems.

Theorem 3.11 for \(m_0^2 = R_{1,2}^c\) implies that the replica symmetry breaking should be spontaneous replica symmetry breaking. If (III) in Theorem 3.11 occurs in some samples, namely

\[
\lim_{L \to \infty} \langle (R_{1,2}^c - \mathbb{E}(R_{1,2}^c))^2 \rangle \neq 0,
\]

in the replica symmetric calculation, then (IV) and the spontaneous replica symmetry breaking occurs in such samples for almost everywhere in the coupling constant space. In each sample, one of the replica symmetry breaking Gibbs states realizes, and the overlap operator becomes self-averaging. Theorem 3.11 implies also that if the spontaneous replica symmetry breaking is a rare event, then the replica symmetry breaking rarely occurs, namely \(R_{1,2}^c\) is self-averaging even in the replica symmetric expectation.

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References

[1] Aizenman, M., Contucci, P.: On the stability of quenched state in mean-field spin glass models. J. Stat. Phys. 92, 765-783(1997)

[2] Aizenman, M., Greenbatt,R.L., Lebowitz, J. L.:Proof of rounding by quenched disorder of first order transitions in low-dimensional quantum systems J. Math. Phys. 53 10.1063, (2012)

[3] Aizenman, M. Wehr, J.:Rounding effects of quenched randomness on first-order phase transitions. Commun. Math. Phys. 130, 489-528(1990)

[4] Chatterjee, S.: Absence of replica symmetry breaking in the random field Ising model. Commun. Math. Phys. 337, 93-102(2015)

[5] Chatterjee, S.: The Ghirlanda-Guerra identities without averaging, preprint, arXiv:0911.4520 (2009).

[6] Chatterjee, S.: Disorder chaos and multiple valleys in spin glasses. preprint, arXiv:0907.3381 (2009).

[7] Contucci, P., Giardinà, C.: Spin-glass stochastic stability: A rigorous proof. Annales Henri Poincare, 6, 915-923, (2005)

[8] Contucci, P., Giardinà, C.: The Ghirlanda-Guerra identities. J. Stat. Phys. 126, 917-931,(2007)
[9] Contucci, P., Giardinà, C.: Perspectives on spin glasses. Cambridge university press, 2012.

[10] Contucci, P., Giardinà, C., Pulé, J.: The infinite volume limit for finite dimensional classical and quantum disordered systems. Rev. Math. Phys. 16, 629-638, (2004)

[11] Contucci, P., Lebowitz, J. L.: Correlation inequalities for quantum spin systems with quenched centered disorder. J. Math. Phys. 51, 023302-1-6 (2010)

[12] Crawford, N.: Thermodynamics and universality for mean field quantum spin glasses. Commun. Math. Phys. 274, 821-839(2007)

[13] Edwards, S. F., Anderson, P. W.: Theory of spin glasses J. Phys. F: Metal Phys. 5, 965-974(1975)

[14] Ghirlanda, S., Guerra, F.: General properties of overlap probability distributions in disordered spin systems. Towards Parisi ultrametricity. J. Phys. A31, 9149-9155(1998)

[15] Goldschmidt, C., Ueltschi, D., Windridge, P: Quantum Heisenberg models and their probabilistic representations Entropy and the quantum II, Contemp. Math. 562 177-224, (2011)

[16] Greenbatt, R.L., Aizenman, M. Lebowitz, J. L.: Rounding first order transitions in low-dimensional quantum systems with quenched disorder Phys. Rev.Lett. 10319721(2009)

[17] Griffiths, R. B.: Spontaneous magnetization in idealized ferromagnets. Phys.Rev. 152 , 240-246, (1964)

[18] Guerra, F.: The phenomenon of spontaneous replica symmetry breaking in complex statistical mechanics systems J. Phys: Conf. Series 442 012013(2013)

[19] Guerra, F.: Spontaneous replica symmetry breaking and interpolation methods for complex statistical mechanical systems Lecture Notes in Mathematics 2143, 45-70, Springer (2013)

[20] Harris, A.B.: Bounds for certain thermodynamic averages J. Math. Phys. 8 1044-1045.(1967)

[21] Itoi, C.: General properties of overlap operators in disordered quantum spin systems J. Stat. phys. 163 1339-1349 (2016)

[22] Koma, T., Tasaki, H. Symmetry breaking in Heisenberg antiferromagnets Commun. Math. Phys. 158, 198-214 (1993).

[23] Mukaida, H.: Non-differentiability of the effective potential and the replica symmetry breaking in the random energy model J. Phys. A 49 45002,1-15 (2016)

[24] Parisi, G.: A sequence of approximate solutions to the S-K model for spin glasses. J. Phys. A 13, L-115 (1980)
[25] Seiler, E., Simon, B.: Nelson’s symmetry and all that in Yukawa and $(\phi^4)_3$ theories. Ann. Phys. 97, 470-518, (1976)

[26] Sherrington, S., Kirkpatrick, S.: Solvable model of spin glass. Phys. Rev. Lett. 35, 1792-1796, (1975).

[27] Talagrand, M.: The Parisi formula. Ann. Math. 163, 221-263 (2006).

[28] Talagrand, M.: Mean field models for spin glasses. Springer, Berlin (2011).