SINR Diagrams:
Towards Algorithmically Usable SINR Models of Wireless Networks

Chen Avin∗ Yuval Emek† Erez Kantor† Zvi Lotker∗ David Peleg†
Liam Roditty‡

December 10, 2008

Abstract

The rules governing the availability and quality of connections in a wireless network are described by physical models such as the signal-to-interference & noise ratio (SINR) model. For a collection of simultaneously transmitting stations in the plane, it is possible to identify a reception zone for each station, consisting of the points where its transmission is received correctly. The resulting SINR diagram partitions the plane into a reception zone per station and the remaining plane where no station can be heard.

SINR diagrams appear to be fundamental to understanding the behavior of wireless networks, and may play a key role in the development of suitable algorithms for such networks, analogous perhaps to the role played by Voronoi diagrams in the study of proximity queries and related issues in computational geometry. So far, however, the properties of SINR diagrams have not been studied systematically, and most algorithmic studies in wireless networking rely on simplified graph-based models such as the unit disk graph (UDG) model, which conveniently abstract away interference-related complications, and make it easier to handle algorithmic issues, but consequently fail to capture accurately some important aspects of wireless networks.

The current paper focuses on obtaining some basic understanding of SINR diagrams, their properties and their usability in algorithmic applications. Specifically, based on some algebraic properties of the polynomials defining the reception zones we show that assuming uniform power transmissions, the reception zones are convex and relatively well-rounded. These results are then used to develop an efficient approximation algorithm for a fundamental point location problem in wireless networks.

∗Department of Communication Systems Engineering, Ben Gurion University, Beer-Sheva, Israel. E-mail:{avin,zvilo}@cse.bgu.ac.il. Z. Lotker was partially supported by a gift from Cisco research center
†Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot, Israel. E-mail: {yuval.emek,erez.kantor,david.peleg}@weizmann.ac.il. Supported in part by grants from the Minerva Foundation and the Israel Ministry of Science.
‡Department of Computer Science, Bar Ilan University, Ramat-Gan, Israel. E-mail: liam.roditty@gmail.com.
1 Introduction

1.1 Background

It is commonly accepted that traditional (wired, point-to-point) communication networks are satisfactorily represented using a graph based model. The question of whether a station \( s \) is able to transmit a message to another station \( s' \) depends on a single (necessary and sufficient) condition, namely, that there be a wire connecting the two stations. This condition is thus independent of the locations of the two stations, of their other connections and activities, and of the locations, connections or activities of other nearby stations.\(^1\)

In contrast, wireless networks are considerably harder to represent faithfully, due to the fact that deciding whether a transmission by a station \( s \) is successfully received by another station \( s' \) is nontrivial, and depends on the positioning and activities of \( s \) and \( s' \), as well as on the positioning and activities of other nearby stations, which might interfere with the transmission and prevent its successful reception. Thus such a transmission from \( s \) may reach \( s' \) under certain circumstances but fail to reach it under other circumstances. Moreover, the question is not entirely “binary”, in the sense that connections can be of varying quality and capacity.

The rules governing the availability and quality of wireless connections can be described by physical or fading channel models (cf. [14, 4, 15]). Among those, the most commonly studied is the signal-to-interference & noise ratio (SINR) model. In the SINR model, the energy of a signal fades with the distance to the power of the path-loss parameter \( \alpha \). If the signal strength received by a device divided by the interfering strength of other simultaneous transmissions (plus the fixed background noise \( N \)) is above some reception threshold \( \beta \), then the receiver successfully receives the message, otherwise it does not. Formally, denote by \( \text{dist}(p,q) \) the Euclidean distance between \( p \) and \( q \), and assume that each station \( s_i \) transmits with power \( \psi_i \). (A uniform power network is one where all stations transmit with the same power.) At an arbitrary point \( p \), the transmission of station \( s_i \) is correctly received if

\[
\frac{\psi_i \cdot \text{dist}(p,s_i)^{-\alpha}}{N + \sum_{j \neq i} \psi_j \cdot \text{dist}(p,s_j)^{-\alpha}} \geq \beta.
\]

Hence for a collection \( S = \{s_0, \ldots, s_{n-1}\} \) of simultaneously transmitting stations in the plane, it is possible to identify with each station \( s_i \) a reception zone \( \mathcal{H}_i \) consisting of the points where the transmission of \( s_i \) is received correctly. It is believed that the path-loss parameter \( 2 \leq \alpha \leq 4 \), where \( \alpha = 2 \) is the common “textbook” choice, and that the reception threshold \( \beta \approx 6 \) (\( \beta \) is always assumed to be greater than 1).

To illustrate how reception depends on the locations and activities of other stations, consider (the numerically generated) Figure 1. (Throughout, figures are deferred to the end of the Ap-

\(^1\) Broadcast domain wired networks such as LANs are an exception, but even most LANs are collections of point-to-point connections.
Figure 1(A) depicts uniform stations $s_1, s_2, s_3$ and their reception zones. Point $p$ (represented as a solid black square) falls inside $H_2$. Figure 1(B) depicts the same stations except station $s_1$ has moved, so that now $p$ does not receive any transmission. Figure 1(C) depicts the stations in the same positions as Figure 1(B), but now $s_3$ is silent, and as a result, the other two stations have larger reception zones, and $p$ receives the message of $s_1$.

Figure 1 illustrates a concept central to this paper, namely, the SINR diagram. An SINR diagram is a “reception map” characterizing the reception zones of the stations, namely, partitioning the plane into $n$ reception zones $H_i$, $0 \leq i \leq n - 1$, and a zone $H_0$ where no station can be heard. In many scenarios the diagram changes dynamically with time, as the stations may choose to transmit or keep silent, adjust their transmission power level, or even change their location from time to time.

It is our belief that SINR diagrams are fundamental to understanding the dynamics of wireless networks, and will play a key role in the development of suitable algorithms for such networks, analogous perhaps to the role played by Voronoi diagrams in the study of proximity queries and related issues in computational geometry. Yet, to the best of our knowledge, SINR diagrams have not been studied systematically so far, from either geometric, combinatorial, or algorithmic standpoints. In particular, in the SINR model it is not clear what shapes the reception zones may take, and it is not easy to construct an SINR diagram even in a static setting.

Taking a broader perspective, a closely related concern motivating this paper is that while a fair amount of research exists on the SINR model and other variants of the physical model, little has been done in such models in the algorithmic arena. (Some recent exceptions are [8, 9, 10, 11, 12, 13, 16].) The main reason for this is that SINR models are complex and hard to work with. In these models it is even hard to decide some of the most elementary questions on a given setting, and it is definitely more difficult to develop communication or design protocols, prove their correctness and analyze
their efficiency.

Subsequently, most studies of higher-layer concepts in wireless multi-hop networking, including issues such as transmission scheduling, frequency allocation, topology control, connectivity maintenance, routing, and related design and communication tasks, rely on simplified graph-based models rather than on the SINR model. Graph-based models represent the network by a graph $G = (S, E)$ such that a station $s$ will successfully receive a message transmitted by a station $s'$ if and only if $s$ and $s'$ are neighbors in $G$ and $s$ does not have a concurrently transmitting neighbor in $G$. In particular, the model of choice for many protocol designers is the unit disk graph (UDG) model\[6\]. In this model, also known as the protocol model\[9\], the stations are represented as points in the Euclidean plane, and the transmission of a station can be received by every other station within a unit ball around it. The UDG graph is thus a graph whose vertices correspond to the stations, with an edge connecting any two vertices whose corresponding stations are at distance at most one from each other.

Graph-based models are attractive for higher-layer protocol design, as they conveniently abstract away interference-related complications. Issues of topology control, scheduling and allocation are also handled more directly, since notions such as adjacency and overlap are easier to define and test, in turn making it simpler to employ also some useful derived concepts such as domination, independence, clusters, and so on. (Note also that the SINR model in itself is rather simplistic, as it assumes perfectly isotropic antennas and ignores environmental obstructions. These issues can be integrated into the basic SINR model, at the cost of yielding relatively complicated ”SINR+” models, even harder to use by protocol designers. In contrast, graph-based models naturally incorporate both directional antennas and terrain obstructions.) On the down side, it should be realized that graph-based models, and in particular the UDG model, ignore or do not accurately capture a number of important physical aspects of real wireless networks. In particular, such models oversimplified the physical laws of interference; in reality, several nodes slightly outside the reception range of a receiver station $v$ (which consequently are not adjacent to $v$ in the UDG graph) might still generate enough cumulative interference to prevent $v$ from successfully receiving a message from a sender station adjacent to it in the UDG graph; see Figure 2 for an example. Hence the UDG model might yield a “false positive” indication of reception. Conversely, a simultaneous transmission by two or more neighbors should not always end in collision and loss of the message; in reality this depends on other factors, such as the relative distances and the relative strength of the transmissions. We illustrate some of these scenarios in Figures 3-4 where we compare the reception zones of the UDG and SINR models with four transmitting stations $s_1, s_2, s_3, s_4$ and one receiver $p$ (represented as a solid black square). In Figure 3 only station $s_1$ transmits, and all others remain silent, so the diagrams are the same and $p$ can hear $s_1$ in both models. Figure 4 illustrates the next three steps of gradually adding $s_2, s_3$ and $s_4$ to the transmitting set. When both $s_1, s_2$ transmit simultaneously, $p$ cannot hear any station in the UDG model, but it does hear $s_1$ in the SINR model (cases (A) and (B) respectively). Hence in this case the UDG model yields a “false
Figure 2: Cumulative interference in the UDG and SINR models. (A) UDG diagram: \( p \) can hear \( s_1 \). (B) SINR diagram: the cumulative interference of stations \( s_2, s_3, s_4 \) prevents \( p \) from hearing \( s_1 \).

negative” indication. When \( s_3 \) joins the transmitting stations, \( p \) still cannot hear any station in the UDG model, but now it can hear station \( s_3 \) in the SINR model (cases (C) and (D)). In step 4, when \( s_4 \) starts to transmit as well, the effect varies again across the two models (cases (E) and (F)).

Figure 3: Reception zones in the UDG and SINR models. In step 1 only \( s_1 \) transmits, so the reception zones are the same.

In summary, while the existing body of literature on models and algorithms for wireless networks represents a significant base containing a rich collection of tools and techniques, the state of affairs described above leaves us in the unfortunate situation where the more practical graph-based models (such as the UDG model) are not sufficiently accurate, and the more accurate SINR model is not well-understood and therefore difficult for protocol designers. Hence obtaining a better understanding of the SINR model, and consequently bridging the gap between this physical model and the graph based models may have potentially significant (theoretical and practical) implications. This goal is the central motivation behind the current paper.
Figure 4: Reception zones in the UDG and SINR models. Steps 2-4 add stations $s_2, s_3, s_4$, one at a time. (A)-(B): adding $s_2$. (C)-(D): adding $s_3$ (E)-(F): adding $s_4$. 
1.2 Related work

Some recent studies aim at achieving a better understanding of the SINR model. In particular, in their seminal work [9], Gupta and Kumar analyzed the capacity of wireless networks in the physical and protocol models. Moscibroda [11] analyzed the worst-case capacity of wireless networks, making no assumptions on the deployment of nodes in the plane, as opposed to almost all the previous work on wireless network capacity.

Thought provoking experimental results presented in [12] show that even basic wireless stations can achieve communication patterns that are impossible in graph-based models. Moreover, the paper presents certain situations in which it is possible to apply routing / transport schemes that may break the theoretical throughput limits of any protocol which obeys the laws of a graph-based model.

Another line of research, in which known results from the UDG model are analyzed under the SINR model, includes [13], which studies the problem of topology control in the SINR model, and [8], where impossibility results were proven in the SINR model for scheduling.

More elaborate graph-based models may employ two separate graphs, a connectivity graph $G_c = (S, E_c)$ and an interference graph $G_i = (S, E_i)$, such that a station $s$ will successfully receive a message transmitted by a station $s'$ if and only if $s$ and $s'$ are neighbors in the connectivity graph $G_c$ and $s$ does not have a concurrently transmitting neighbor in the interference graph $G_i$. Protocol designers often consider special cases of this more general model. For example, it is sometimes assumed that $G_i$ is $G_c$ augmented with all edges between 2-hop neighbors in $G_c$. Similarly, a variant of the UDG model handling transmissions and interference separately, named the Quasi Unit Disk Graph (Q-UDG) model, was introduced in [10]. In this model, two concentric circles are associated with each station, the smaller representing its reception zone and the larger representing its area of interference. An alternative interference model, also based on the UDG model, is proposed in [16].

1.3 Our results

As mentioned earlier, a fundamental issue in wireless network modeling involves characterizing the reception zones of the stations and constructing the reception diagram. The current paper aims at gaining a better understanding of this issue in the SINR model, and as a consequence, deriving some algorithmic results. In particular, we consider the structure of reception zones in SINR diagrams corresponding to uniform power networks with path-loss parameter $\alpha = 2$ and examine two specific properties of interest, namely, the convexity and fatness\footnote{The notion of fatness has received a number of non-equivalent technical definitions, all aiming at capturing the same intuition, namely, absence of long, skinny or twisted parts. In this paper we say that the reception zone of station $s_i$ is fat if the ratio between the radii of the smallest ball centered at $s_i$ that completely contains the zone and...} of the reception zones. Apart
from their theoretical interest, these properties are also of considerable practical significance, as obviously, having reception zones that are non-convex, or whose shape is arbitrarily skewed, twisted or skinny, might complicate the development of protocols for various design and communication tasks.

Our first result, which turns out to be surprisingly less trivial than we may have expected, is cast in Theorem 1. This theorem is proved in Section 3 by a complex analysis of the polynomials defining the reception zones, based on combining several observations with Sturm’s condition for counting real roots.

**Theorem 1.** The reception zones in an SINR diagram of a uniform power network with path-loss parameter $\alpha = 2$ and reception threshold $\beta > 1$ are convex.

Note that our convexity proof still holds when $\beta = 1$. In contrast, when $\beta < 1$, the reception zones of a uniform power network are not necessarily convex. This phenomenon is illustrated in (the numerically generated) Figure 5. We then establish an additional attractive property of the reception zones.

**Theorem 2.** The reception zones in an SINR diagram of a uniform power network with path-loss parameter $\alpha = 2$ and reception threshold $\beta > 1$ are fat.

Theorem 2 is proved in Section 4. In a certain sense, this result lends support to the model of Quasi Unit Disk Graphs suggested by Kuhn et al. in [10].

Armed with this characterization of the reception zones, we turn to a basic algorithmic task closely related to SINR diagrams, namely, answering point location queries. We address the following fundamental problem: given a network's reception zones, we want to efficiently determine whether a query point is inside, on the boundary, or outside of the network's reception zones. This problem arises in various applications, such as geographic information systems, sensor networks, and wireless communication.
Figure 6: The reception zones $\mathcal{H}_i$ (enclosed by the bold lines) and the partition of the plane to disjoint zones $\mathcal{H}_i^+$ (dark gray), $\mathcal{H}_i^-$ (light gray), and $\mathcal{H}^-$ (the remaining white).

Following natural question: given a point in the plane, which reception zone contains this point (if any)? For UDG, this problem can be dealt with using known techniques (cf. [1, 2]). For arbitrary (non-unit) disk graphs, the problem is already harder, as the direct reduction to the technique of [2] no longer works. In the SINR model the problem becomes even harder. A naive solution will require computing the signal to interference & noise ratio for each station, yielding time $O(n^2)$. A more efficient ($O(n)$ time) querying algorithm can be based, for example, on the observation that there is a unique candidate $s_i \in S$ whose transmission may be received at $p$, namely, the one whose Voronoi cell contains $p$ in the Voronoi diagram defined for $S$. However, it is not known if a sublinear query time can be obtained. This problem can in fact be thought of as part of a more general one, namely, point location over a general set of objects defined by polynomials and satisfying some “niceness” properties. Previous work on the problem dealt with Tarski cells, namely, objects whose boundaries are defined by a constant number of polynomials of constant degree [5, 1]. In contrast, SINR reception zones are defined by polynomials of degree proportional to $n$.

Consider the SINR diagram of a uniform power network with path-loss parameter $\alpha = 2$ and reception threshold $\beta > 1$ and fix some performance parameter $0 < \epsilon < 1$. The following theorem is proved in Section 5 (refer to Figure 6 for illustration).

**Theorem 3.** A data structure $\text{DS}$ of size $O(n\epsilon^{-1})$ is constructed in $O(n^3\epsilon^{-1})$ preprocessing time. This data structure essentially partitions the Euclidean plane into disjoint zones $\mathbb{R}^2 = \bigcup_{i=0}^{n-1} \mathcal{H}_i^+ \cup \mathcal{H}^- \cup \bigcup_{i=0}^{n-1} \mathcal{H}_i^*$ such that for every $0 \leq i \leq n-1$:

1. $\mathcal{H}_i^+ \subseteq \mathcal{H}_i$;
2. $\mathcal{H}_i^- \cap \mathcal{H}_i = \emptyset$; and
3. $\mathcal{H}_i^*$ is bounded and its area is at most an $\epsilon$-fraction of the area of $\mathcal{H}_i$.

Given a query point $p \in \mathbb{R}^2$, $\text{DS}$ identifies the zone in $\{\mathcal{H}_i^+\}_{i=0}^{n-1} \cup \{\mathcal{H}^-(p)\} \cup \{\mathcal{H}_i^*\}_{i=0}^{n-1}$ to which $p$ belongs, in time $O(\log n)$.
1.4 Open Problems

Various extensions of our original setting may be considered. For instance, it may be of interest to study SINR diagrams in $d > 2$ dimensions, or for path-loss parameter $\alpha > 2$.

Our results concern wireless networks with uniform power transmissions. General wireless networks are harder to deal with. For instance, the point location problem becomes considerably more difficult when different stations are allowed to use different transmission energy, since in this case, the appropriate graph-based model is no longer a unit-disk graph but a (directed) general disk graph, based on disks of arbitrary radii. An even more interesting case is the variable power setting, where the stations can adjust their transmission energy levels from time to time.

The problems discussed above become harder in a dynamic setting, and in particular, if we assume the stations are mobile, and extending our approach to the dynamic and mobile settings are the natural next steps.

2 Preliminaries

2.1 Geometric notions

We consider the Euclidean plane $\mathbb{R}^2$. The distance from point $p$ to point $q$ is denoted by $\text{dist}(p, q) = \|q - p\|$. A ball of radius $r$ centered at point $p$ is the set of all points at distance at most $r$ from $p$, denoted by $B(p, r) = \{q \in \mathbb{R}^2 \mid \text{dist}(p, q) \leq r\}$. We say that point $p \in \mathbb{R}^2$ is internal to the point set $P$ if there exists some $\epsilon > 0$ such that $B(p, \epsilon) \subseteq P$.

Consider some point set $P$. $P$ is said to be an open set if all points $p \in P$ are internal points. $P$ is said to be a closed set if the complement of $P$ is an open set. If there exists some real $r$ such that $\text{dist}(p, q) \leq r$ for every two points $p, q \in P$, then $P$ is said to be bounded. A compact set is a set which is both closed and bounded. The closure of $P$ is the smallest closed set containing $P$. The boundary of $P$, denoted by $\partial P$, is the intersection of the closure of $P$ and the closure of its complement. A connected set is a point set that cannot be partitioned to two non-empty subsets such that each of the subsets has no point in common with the closure of the other. We refer to the closure of an open bounded connected set as a thick set. By definition, every thick set is compact.

A point set $P$ is said to be convex if the segment $\overline{pq}$ is contained in $P$ for every two points $p, q \in P$. The point set $P$ is said to be star-shaped \cite{[7]} with respect to point $p \in P$ if the segment $\overline{pq}$ is contained in $P$ for every point $q \in P$. Clearly, convexity is stronger than the star-shape property in the sense that a convex point set $P$ is star-shaped with respect to any point $p \in P$; the converse is not necessarily true. For thick sets we have the following necessary and sufficient condition for convexity.
Lemma 2.1. A thick set $P$ is convex if and only if every line intersects $\partial P$ at most twice.

We frequently use the term zone to describe a point set with some “niceness” properties. Unless stated otherwise, a zone refers to the union of an open connected set and some subset of its boundary. (A thick set is a special case of a zone.) It may also refer to a single point or to the finite union of zones. Given some bounded zone $Z$, we denote the area and perimeter of $Z$ (assuming that they are well defined) by area($Z$) and per($Z$), respectively. Let $Z$ be a non-empty bounded zone and let $p$ be some internal point of $Z$. Denote

$$\delta(p, Z) = \sup \{ r > 0 \mid Z \supset B(p, r) \}, \quad \Delta(p, Z) = \inf \{ r > 0 \mid Z \subseteq B(p, r) \},$$

and define the fatness parameter of $Z$ with respect to $p$ to be the ratio of $\Delta(p, Z)$ and $\delta(p, Z)$, denoted by $\varphi(p, Z) = \Delta(p, Z)/\delta(p, Z)$. (See Figure 7) The zone $Z$ is said to be fat with respect to $p$ if $\varphi(p, Z)$ is bounded by some constant.

Consider some two points $p_1, p_2$ in the plane. The set of points $q$ that satisfy $\text{dist}(p_1, q) = \text{dist}(p_2, q)$ form a line referred to as the separation line of $p_1$ and $p_2$.

2.2 Wireless networks

We consider a wireless network $\mathcal{A} = \langle S, \psi, N, \beta \rangle$, where $S = \{ s_0, s_1, \ldots, s_{n-1} \}$ is a set of transmitting radio stations embedded in the Euclidean plane, $\psi$ is an assignment of a positive real transmitting power $\psi_i$ to each station $s_i$, $N \geq 0$ is the background noise, and $\beta \geq 1$ is a constant that serves as the reception threshold (will be explained soon). For the sake of notational simplicity, $s_i$ also refers to the point $(a_i, b_i)$ in the plane where the station $s_i$ resides. The network is assumed to contain at least two stations, i.e., $n \geq 2$. We say that $\mathcal{A}$ is a uniform power network if $\psi = 1$, namely, if $\psi_i = 1$ for every $i$.  

Figure 7: The zone $Z$ (enclosed by the solid line) with the ball defining $\delta(p, Z)$ (dotted line) and the ball defining $\Delta(p, Z)$ (dashed line).
The energy of station $s_i$ at point $p \neq s_i$ is defined to be $E_A(s_i, p) = \psi_i \cdot \text{dist}(s_i, p)^{-2}$. The energy of a station set $T \subseteq S$ at point $p$, where $p \neq s_i$ for every $i \in T$, is defined to be $E_A(T, p) = \sum_{i \in T} E_A(s_i, p)$. Fix some station $s_i$ and consider some point $p \notin S$. We define the interference to $s_i$ at point $p$ to be the energies at $p$ of all stations other than $s_i$, denoted $I_A(s_i, p) = E_A(S - \{s_i\}, p)$. The signal to interference & noise ratio (SINR) of $s_i$ at point $p$ is defined as

$$\text{SINR}_A(s_i, p) = \frac{E_A(s_i, p)}{I_A(s_i, p) + \text{N}} = \frac{\psi_i \cdot \text{dist}(s_i, p)^{-2}}{\sum_{j \neq i} \psi_j \cdot \text{dist}(s_j, p)^{-2} + \text{N}}.$$  

Observe that SINR$_A(s_i, p)$ is always positive since the transmitting powers and the distances of the stations from $p$ are always positive and the background noise is non-negative. When the network $A$ is clear from the context, we may omit it and write simply $E(s_i, p), I(s_i, p)$, and SINR$_i(p)$.

The fundamental rule of the SINR model is that the transmission of station $s_i$ is received correctly at point $p \notin S$ if and only if its SINR at $p$ is not smaller than the reception threshold of the network, i.e., SINR$_i(p) \geq \beta$. If this is the case, then we say that $s_i$ is heard at $p$. We refer to the set of points that hear station $s_i$ as the reception zone of $s_i$, denoted as

$$\mathcal{H}_i = \{p \in \mathbb{R}^2 - S \mid \text{SINR}(s_i, p) \geq \beta\} \cup \{s_i\}.$$  

This admittedly tedious definition is necessary as SINR$_i(\cdot)$ is not defined at any point in $S$ and in particular, at $s_i$ itself.

Consider station $s_0$ and an arbitrary point $p = (x, y) \in \mathbb{R}^2$. By rearranging the expression in (1), we correlate the fundamental rule of the SINR model to the 2-variate polynomial $H(x, y)$ so that $s_0$ is heard at $p$ if and only if

$$H(x, y) = \beta \left[ \sum_{i > 0} \psi_i \cdot \prod_{j \neq i} ((a_j - x)^2 + (b_j - y)^2) + N \cdot \prod_i ((a_i - x)^2 + (b_i - y)^2) \right] - \psi_0 \cdot \prod_{i > 0} ((a_i - x)^2 + (b_i - y)^2) \leq 0.$$  

Note that this condition holds even for points $p \in S$. Consequently, we can rewrite $\mathcal{H}_0 = \{(x, y) \in \mathbb{R}^2 \mid H(x, y) \leq 0\}$, where the boundary points of $\mathcal{H}_0$ are exactly the roots of $H(x, y)$. In general, the polynomial $H(x, y)$ has degree $2n$; the degree is $2n - 2$ if the background noise $N = 0$. This polynomial plays a key role in the analysis carried out in Section 3.2.

A uniform power network $A = \langle S, \bar{1}, N, \beta \rangle$ is said to be trivial if $|S| = 2$, $N = 0$, and $\beta = 1$. Note that for $i = 0, 1$, the reception zone $\mathcal{H}_i$ of station $s_i$ in a trivial uniform power network is the half-plane consisting of all points whose distance to $s_i$ is not greater than their distance to $s_{1-i}$. In particular, $\mathcal{H}_i$ is unbounded. For non-trivial networks, we have the following observation that relies on the fact that SINR$_i(\cdot)$ is a continuous function in $\mathbb{R}^2 - S$.

**Observation 2.2.** Let $A = \langle S, \bar{1}, N, \beta \rangle$ be a non-trivial uniform power network. Then the reception zone $\mathcal{H}_i$ is compact for every $s_i \in S$. Moreover, every point in $\mathcal{H}_i$ is closer to $s_i$ than it is to any
other station in $S$ (i.e., $H_i$ is strictly contained in the Voronoi cell of $s_i$ in the Voronoi diagram of $S$).

Next, we state a simple but important lemma that will be useful in our later arguments.

**Lemma 2.3.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping consisting of rotation, translation, and scaling by a factor of $\sigma > 0$. Consider some network $A = \langle S, \psi, N, \beta \rangle$ and let $f(A) = \langle f(S), \psi, N/\sigma^2, \beta \rangle$, where $f(S) = \{ f(s_i) \mid s_i \in S \}$. Then for every station $s_i$ and for all points $p \notin S$, we have $\text{SINR}_A(s_i, p) = \text{SINR}_{f(A)}(f(s_i), f(p))$.

### 3 Convexity of the reception zones

In this section we consider the SINR diagram of a uniform power network $A = \langle S, \bar{1}, N, \beta \rangle$ and establish Theorem 1. As all stations admit the same transmitting power, it is sufficient to focus on $s_0$ and to prove that the reception zone $H_0$ is convex. We shall do so by considering some arbitrary two points $p_1, p_2 \in \mathbb{R}^2$ and arguing that if $s_0$ is heard at $p_i$ for $i = 1, 2$, then $s_0$ is heard at all points in the segment $p_1p_2$. This argument is established in three steps.

First, as a warmup, we prove that $H_0$ is star-shaped with respect to $s_0$. This proof, presented in Section 3.1, establishes our argument for the case that $p_1$ and $p_2$ are colinear with $s_0$. Next, we prove that in the absence of a background noise (i.e., $N = 0$), if $p_i \in H_0$ for $i = 1, 2$, then $\overline{p_1p_2} \subseteq H_0$. This proof, presented in Section 3.3, relies on the analysis of a special case of a network consisting of only three stations which is analyzed in Section 3.2 and in a sense, constitutes the main technical challenge of this paper. Finally, in Section 3.4 we reduce the convexity proof of a uniform power network with $n$ stations and arbitrary background noise, to that of a uniform power network with $n + 1$ stations and no background noise. While the analyses in Sections 3.3 and 3.4 are consistent with some “physical intuition”, the proof presented in Section 3.2 is based purely on algebraic arguments.

#### 3.1 Star-shape

In this section we consider a uniform power network $A = \langle S, \bar{1}, N, \beta \rangle$ and show that the reception zone $H_0$ is star-shaped with respect to the station $s_0$. In fact, we prove a slightly stronger lemma.

**Lemma 3.1.** Consider some point $p \in \mathbb{R}^2$. If $\text{SINR}(s_0, p) \geq 1$, then $\text{SINR}(s_0, q) > \text{SINR}(s_0, p)$ for all internal points $q$ in the segment $s_0p$.

**Proof.** We consider two disjoint cases. First, suppose that there exists some station $s_i$, $i > 0$, such that $E(s_i, p) = E(s_0, p)$. The assumption that $\text{SINR}(s_0, p) \geq 1$ necessitates, by (1), that $N = 0$, $n = 2$ (which means that $i = 1$), and $\text{SINR}(s_0, p) = 1$. Therefore $\text{dist}(s_0, p) = \text{dist}(s_1, p)$ and for all internal points $q$ in the segment $s_0p$, we have $\text{dist}(s_0, q) < \text{dist}(s_1, q)$. Thus $\text{SINR}(s_0, q) > 1$ and the assertion holds.
Now, suppose that $E(s_i, p) < E(s_0, p)$ for every $i > 0$, which means that $\text{dist}(s_i, p) > \text{dist}(s_0, p)$ for every $i > 0$. By Lemma 2.3, we may assume without loss of generality that $s_0 = (0, 0)$ and $p = (-1, 0)$. Consider some station $s_i, i > 0$. Note that if $s_i$ is not located on the positive half of the horizontal axis, then we can relocate it to a new location $s'_i$ on the positive half of the horizontal axis by rotating it around $p$ so that $\text{dist}(s'_i, p) = \text{dist}(s_i, p)$ and $\text{dist}(s'_i, q) \leq \text{dist}(s_i, q)$ for all points $q \in \overline{s_0p}$ (see Figure 8). We can repeat this process with every station $s_i, i > 0$, until all stations are located on the positive half of the horizontal axis without decreasing the interference at any point $q \in \overline{s_0p}$. Therefore it is sufficient to establish the assertion under the assumption that $s_i = (a_i, 0)$, where $a_i > 0$, for every $i > 0$.

Let $q = (-x, 0)$ for some $x \in (0, 1]$. We can express the SINR function of $s_0$ at $q$ as

$$\text{SINR}(s_0, q) = \frac{x^{-2}}{\sum_{i>0}(a_i + x)^{-2} + N}.$$ 

In this context, it will be more convenient to consider the reciprocal of the SINR function,

$$f(x) = \sum_{i>0} \left( \frac{x}{a_i + x} \right)^2 + x^2 \cdot N,$$

so that we have to prove that $f(x) < f(1)$ for all $x \in (0, 1)$. The assertion follows since $\frac{df(x)}{dx} = 2x \cdot \sum_{i>0} \frac{a_i}{(a_i + x)^3} + 2x \cdot N$ is positive when $x \in (0, 1]$.

Consider a non-trivial uniform power network $A = \langle S, \bar{1}, N, \beta \rangle$ and suppose that $s_0 \neq s_j$ for every $j > 0$, that is, the location of $s_0$ is not shared by any other station. Lemma 3.1 implies that the point set $\mathcal{H}'_0 = \{ p \in \mathbb{R}^2 - S \mid \text{SINR}_A(s_0, p) > \beta \} \cup \{s_0\}$ is star-shaped with respect to $s_0$, and in particular, connected. Moreover, since SINR is a continuous function in $\mathbb{R}^2 - S$, it follows that $\mathcal{H}'_0$ is an open set. As $\mathcal{H}_0$ is the closure of $\mathcal{H}'_0$, we have the following corollary.

Figure 8: Relocating stations $s_i, i > 0$. 


Corollary 3.2. In a non-trivial network, if the location of $s_0$ is not shared by any other station, then $\mathcal{H}_0$ is a thick set.

3.2 Three stations with no background noise

In this section we analyze the special case of the uniform power network $\mathcal{A}_3 = \langle S, I, N, \beta \rangle$, where $S = \{s_0, s_1, s_2\}$, $N = 0$, and $\beta = 1$. Our goal is to establish the following lemma, which constitutes the main technical challenge in the course of proving Theorem 1.

Lemma 3.3. The reception zone $\mathcal{H}_0$ of station $s_0$ in $\mathcal{A}_3$ is convex.

Lemma 3.3 clearly holds if $s_j = s_0$ for some $j \in \{1, 2\}$, as this implies that $\mathcal{H}_0 = \{s_0\}$. So, in what follows we assume that no other station shares the location of $s_0$. By Corollary 3.2 we know that $\mathcal{H}_0$ is a thick set. Lemma 2.1 can now be employed to establish Lemma 3.3. To do that, it is required to show that every line intersects $\partial \mathcal{H}_0$ at most twice.

Consider an arbitrary line $L$ in $\mathbb{R}^2$. We claim that $L$ and $\partial \mathcal{H}_0$ have no more than two intersection points. First, note that if $s_0 \in L$, then the claim holds due to Lemma 3.1. Hence in the remainder of this section we assume that $s_0 \notin L$. Recall that point $(x, y) \in \mathbb{R}^2$ is on the boundary of $\mathcal{H}_0$ if and only if it is a root of the polynomial

$$
H(x, y) = \left( (a_0 - x)^2 + (b_0 - y)^2 \right) \left( (a_1 - x)^2 + (b_1 - y)^2 + (a_2 - x)^2 + (b_2 - y)^2 \right) - \left( (a_1 - x)^2 + (b_1 - y)^2 \right) \left( (a_2 - x)^2 + (b_2 - y)^2 \right)
$$

(2)

(see Section 2.2), so it is sufficient to prove that the projection of $H(x, y)$ on the line $L$ has at most two distinct real roots.

Employing Lemma 2.3, we may assume that $s_0$ is located at the origin and that $L$ is the line $y = 1$. By substituting $y = 1$ into (2) and rearranging the resulting expression, we get

$$
H(x) = \left( x^2 + 1 \right) \left( (a_1 - x)^2 + (b_1 - 1)^2 + (a_2 - x)^2 + (b_2 - 1)^2 \right) - \left( (a_1 - x)^2 + (b_1 - 1)^2 \right) \left( (a_2 - x)^2 + (b_2 - 1)^2 \right) = x^4 + (2 - 4a_1a_2)x^2 + (2a_2a_1^2 + 2a_2^2a_1 + 2(1 - b_2)^2a_1 - 2a_1 + 2a_2(1 - b_1)^2 - 2a_2)x + a_1^2 - a_2^2a_2^2 + a_2 - a_2^2(1 - b_1)^2 + (1 - b_1)^2 - a_1^2(1 - b_2)^2 - (1 - b_1)^2(1 - b_2)^2,
$$

so that $(x, 1)$ is on the boundary of $\mathcal{H}_0$ if and only if $x$ is a root of $H(x)$.

Our goal in the remainder of this section is to show that $H(x)$ has at most two distinct real roots, and towards this goal we will first invest some effort in simplifying this polynomial. As a first step we show that we can restrict our attention to the case where both $s_1$ and $s_2$ are in the first quarter above the line $y = 1$, that is, $a_j > 0$ for $j = 1, 2$ and $b_j \geq 1$ for $j = 1, 2$. The latter restriction is trivial due to the symmetry of interference along the line $y = 1$, which implies that
if $b_j < 1$ for some $j \in \{1, 2\}$, then relocating $s_j$ in $(a_j, 1 + |1 - b_j|)$ does not affect the interference at $q$ for all points $q$ on the line $y = 1$, and in particular, does not affect the number of simple real roots of $H(x)$. For the former restriction we prove the following proposition.

**Proposition 3.4.** If $\text{sign}(a_1) \cdot \text{sign}(a_2) \neq 1$, then $H(x)$ has at most two distinct real roots.

**Proof.** We write $H(x) = x^4 + Ax^3 + Bx + C$ for coefficients $A, B, C$ depending on $a_1, b_1, a_2, b_2$, where $A = 2 - 4a_1a_2$. Let $H'(x) = 4x^3 + 2Ax + B$ be the derivative of $H(x)$. The polynomial $H'(x)$ is a cubic polynomial, thus, it has at least one real root. If it has exactly one real root, then $H(x)$ has exactly one extreme point and at most two distinct real roots. A cubic polynomial $c_3x^3 + c_2x^2 + c_1x + c_0$ with real coefficients has one real root when its discriminant

$$\Delta = c_1^2c_2^2 - 4c_0c_2^3 - 4c_1^3c_3 + 18c_0c_1c_2c_3 - 27c_0^2c_3^2$$

is negative. In the case of $H'(x)$ we have $\Delta = -128A^3 - 432B^2$. The assertion follows from the observation that if $\text{sign}(a_1) \neq \text{sign}(a_2)$ or if $a_1 = a_2 = 0$, then $A > 0$ and $\Delta$ is negative. \hfill $\square$

By Proposition 3.4, we may hereafter assume that $\text{sign}(a_1) = \text{sign}(a_2) \neq 0$. The case where $a_j < 0$ for $j = 1, 2$ is redundant, since relocating station $s_j$ in $(-a_j, b_j)$ turns $H(x)$ into $H(-x)$, and in particular, does not affect the number of distinct real roots of the polynomial. Therefore, in what follows we assume that $a_j > 0$ and $b_j \geq 1$ for $j = 1, 2$.

Our next step is to rewrite $H(x)$ as

$$H(x) = (x^2 + 1)^2 - J(x), \quad (3)$$

where

$$J(x) = 4a_2a_1x^2 - (2a_2a_1^2 + 2a_2^3a_1 + 2b_2^3a_1 - 4b_2a_1 + 2a_2b_2^2 - 4a_2b_1)x$$

$$+ a_2^2a_1^2 + b_2^2a_1^2 - 2b_2a_1^2 + a_2^3b_1^2 + b_1^3b_2^2 - 2b_1b_2^2 - 2a_2^3b_1 - 2b_1^2b_2 + 4b_1b_2$$

is a polynomial of degree 2. Under the assumption that $a_1$ and $a_2$ are positive, $J(x)$ has the following (not necessarily distinct) real roots:

$$r_1 = \frac{a_2^2 + (b_1 - 2)b_1}{2a_1} \quad \text{and} \quad r_2 = \frac{a_2^2 + (b_2 - 2)b_2}{2a_2}.$$

Perhaps surprisingly, the root $r_j$ corresponds to the $x$-coordinate of the intersection point of the line $y = 1$ and the separation line $L_j$ of $s_0$ and $s_j$ for $j = 1, 2$ (see Figure 9). To validate this observation, note that the point $(x, y)$ is on $L_j$ if and only if

$$(x - a_j)^2 + (y - b_j)^2 = x^2 + y^2,$$

or equivalently, if and only if

$$a_j^2 + b_j^2 = 2(a_jx + b_jy).$$
Fixing $y = 1$, we get that $x = \frac{a_j^2 + (b_j - 2)b_j}{2a_j} = r_j$.

Moreover, since $x$ has a negative coefficient when $L_j$ is expressed as $y = -\frac{a_j}{b_j}x + a_j^2 + b_j^2$, we conclude that the point $(x', 1)$ is as close to $s_j$ at least as it is to $s_0$ for all $x' \geq r_j$. The following corollary follows since the real roots of $H(x)$ correspond to points on the boundary of $\mathcal{H}_0$, and since $s_0$ is heard at all such points.

**Corollary 3.5.** The real $x'$ is not a root of the polynomial $H(x)$ for any $x' \geq \min\{r_1, r_2\}$.

Incidentally, let us comment without proof that the point $\min\{r_1, r_2\}$ is the intersection point of the line $y = 1$ and the boundary of the Voronoi cell of $s_0$ in the Voronoi diagram of $S$.

Fix $\bar{r} = (r_1 + r_2)/2$ and define the shifted variable $z = x - \bar{r}$. Since $\bar{r}$ is the center of the parabola $J(x)$, it follows that when expressing $J(x)$ in terms of the shifted variable $z$, we get the form $J(z) = \gamma z^2 + \delta$, where $\gamma > 0$ since the leading coefficient of $J(x)$ is positive, and $\delta \leq 0$ since $J(x)$ has at least one real root. We can now express $H(x)$, as represented in (3), in terms of the shifted variable $z$, introducing the polynomial

$$\tilde{H}(z) = ((z + \bar{r})^2 + 1)^2 - \gamma z^2 - \delta,$$

which is obviously much simpler. Clearly, $H(x)$ and $\tilde{H}(z)$ have the same number of distinct real roots.

In what follows, we employ *Sturm’s Theorem*, to be explained next, in order to bound the number of distinct real roots of $\tilde{H}(z)$. Consider some degree $n$ polynomial $P(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0$ over the reals. The *Sturm sequence* of $P(x)$ is a sequence of polynomials denoted by $P_0(x), P_1(x), \ldots, P_m(x)$, where $P_0(x) = P(x)$ and $P_1(x) = P'(x)$, and $P_i(x) = -\text{rem}(P_{i-2}(x)/P_{i-1}(x))$ for $i > 1$. This recursive definition terminates at step $m$ such that $-\text{rem}(P_{m-1}(x)/P_m(x)) = 0$. Since the degree of $P_i(x)$ is at most $n - i$, we conclude that $m \leq n$. Define $\text{SC}_P(t)$ to be the number of sign changes in the sequence $P_0(t), P_1(t), \ldots, P_m(t)$.

![Figure 9: The point $(r_j, 1)$ is on the separation line $L_j$ of $s_0$ and $s_j$.](image)
We are now ready to state the following theorem attributed to Jacques Sturm, 1829 (cf. [3]).

**Theorem 3.6** (Sturm’s condition). Consider two reals \(a, b\), where \(a < b\) and neither of them is a root of \(P(x)\). Then the number of distinct real roots of \(P(x)\) in the interval \((a, b)\) is \(\text{SC}_P(a) - \text{SC}_P(b)\).

We will soon show that \(\text{SC}_\beta(\infty) \leq 3\) and \(\text{SC}_\beta(\infty) \geq 1\), hence \(\text{SC}_\beta(\infty) - \text{SC}_\beta(\infty) \leq 2\). Therefore, by Theorem 3.6, we conclude that \(\beta(x)\) has at most two distinct real roots. It is sufficient for our purposes to consider the first three polynomials in the Sturm sequence of \(\beta(x)\):

\[
\begin{align*}
\beta_0(z) &= z^4 + 4\bar{r}z^3 + (6\bar{r}^2 - \gamma + 2)z^2 + (4\bar{r}^3 + 4\bar{r})z + \bar{r}^4 + 2\bar{r}^2 - \delta + 1 \\
\beta_1(z) &= 4z^3 + 12\bar{r}z^2 + 2(6\bar{r}^2 - \gamma + 2)z + 4\bar{r}^3 + 4\bar{r} \\
\beta_2(z) &= (\gamma/2 - 1)z^2 - \bar{r}(2 + \gamma/2)z - \bar{r}^2 - 1 + \delta.
\end{align*}
\]

**Proposition 3.7.** The polynomial \(\beta(x)\) satisfies \(\text{SC}_\beta(\infty) \geq 1\).

**Proof.** We first argue that \(\beta(x)\) does not have any root in the interval \([0, \infty)\). This argument holds due to Corollary 3.5 since by the definition of \(z = x - \bar{r}\), \(z \geq 0\) implies \(x \geq \bar{r} \geq \min\{r_1, r_2\}\). Therefore Theorem 3.6 guarantees that \(\text{SC}_\beta(\infty) = \text{SC}_\beta(0)\). Now, the sign of \(\beta_0(0)\) is positive while the sign of \(\beta_2(0)\) is negative, so there must be at least one sign change when the Sturm sequence of \(\beta(x)\) is evaluated on 0, hence \(\text{SC}_\beta(\infty) = \text{SC}_\beta(0) \geq 1\).

**Proposition 3.8.** The polynomial \(\beta(x)\) satisfies \(\text{SC}_\beta(-\infty) \leq 3\).

**Proof.** First note that there are at most five polynomials in the Sturm sequence of \(\beta(x)\), hence \(\text{SC}_\beta(-\infty)\) cannot be greater than 4. Suppose, towards deriving contradiction, that \(\text{SC}_\beta(-\infty) = 4\). This implies that there are exactly 5 polynomials in the Sturm sequence of \(\beta(x)\) and the degree of \(\beta_i(z)\) is \(4 - i\) for every \(0 \leq i \leq 4\). Clearly, both \(\text{SC}_\beta(-\infty)\) and \(\text{SC}_\beta(\infty)\) depend solely on the signs of the leading coefficients of the polynomials in the Sturm sequence of \(\beta(x)\). Since \(\text{sign}(\beta_0(-\infty)) = 1\), we must have \(\text{sign}(\beta_i(-\infty)) = -1\) for \(i = 1, 3\) and \(\text{sign}(\beta_i(-\infty)) = 1\) for \(i = 2, 4\). As \(\text{sign}(\beta_i(\infty)) = \text{sign}(\beta_i(-\infty))\) for \(i = 0, 2, 4\) and \(\text{sign}(\beta_i(\infty)) = -\text{sign}(\beta_i(-\infty))\) for \(i = 1, 3\), we conclude that \(\text{sign}(\beta_i(\infty)) = 1\) for every \(0 \leq i \leq 4\). Therefore \(\text{SC}_\beta(\infty) = 0\), in contradiction to Proposition 3.7.

To conclude, combining Propositions 3.7 and 3.8 with Theorem 3.6 we get that \(\beta(x)\) has at most two distinct real roots, and thus \(H(x)\) has at most two distinct real roots. It follows that every line has at most two intersection points with \(\partial H_0\), which completes the proof of Lemma 3.3.

---

\(^3\) We write \(\text{SC}_\beta(\infty)\) and \(\text{SC}_\beta(-\infty)\) as a convenient shorthand for \(\lim_{z \to \infty} \text{SC}_\beta(z)\) and \(\lim_{z \to -\infty} \text{SC}_\beta(z)\), respectively.
3.3 Convexity with \( n \) stations and no background noise

In this section we return to a uniform power network \( A = \{S, \bar{I}, N, \beta\} \) with an arbitrary number of stations and with an arbitrary reception threshold \( \beta \geq 1 \), but still, with no background noise (i.e., \( N = 0 \)), and establish the convexity of \( H_0 \).

**Lemma 3.9.** The reception zone \( H_0 \) of station \( s_0 \) in \( A \) is convex.

Lemma 3.9 is proved by induction on the number of stations in the network, \( n = |S| \). For the base of the induction, we consider the case where \( n = 2 \). The theorem clearly holds if \( s_0 \) and \( s_1 \) share the same location, as this implies that \( H_0 = \{s_0\} \). Furthermore, if \( \beta = 1 \), which means that \( A \) is trivial, then \( H_0 \) is a half-plane and in particular, convex. So, in what follows we assume that \( s_0 \neq s_1 \) and that \( \beta > 1 \).

Corollary 3.2 implies that \( H_0 \) is a thick set, thus, by Lemma 2.1 it is sufficient to argue that every line \( L \) has at most two intersection points with \( \partial H_0 \). If \( s_0 \in L \), then the argument holds due to Lemma 3.1. If \( s_0 \notin L \), then \( E(s_0, q) = \beta \) is essentially a quadratic equation, thus it has at most two real solutions which correspond to at most two intersection points of \( L \) and \( \partial H_0 \).

The inductive step of the proof of Lemma 3.9 is more involved. We consider some arbitrary two points \( p_1, p_2 \in H_0 \) and prove that \( \overline{p_1 p_2} \subseteq H_0 \). Informally, we will show that if there exist at least two stations other than \( s_0 \), then we can discard one station and relocate the rest so that the interference at \( p_i \) remains unchanged for \( i = 1, 2 \) and the interference at \( q \) does not decrease for all points \( q \in \overline{p_1 p_2} \). By the inductive hypothesis, the segment \( \overline{p_1 p_2} \) is contained in \( H_0 \) in the new setting, hence it is also contained in \( H_0 \) in the original setting. This idea relies on the following lemma.

**Lemma 3.10.** Consider the stations \( s_0, s_1, s_2 \) and some distinct two points \( p_1, p_2 \in \mathbb{R}^2 \). If \( E(s_0, p_i) \geq E(\{s_1, s_2\}, p_i) \) for \( i = 1, 2 \), then there exists a location \( s^* \in \mathbb{R}^2 \) such that

1. \( E(s^*, p_i) = E(\{s_1, s_2\}, p_i) \) for \( i = 1, 2 \); and
2. \( E(s^*, q) \geq E(\{s_1, s_2\}, q) \), for all points \( q \) in the segment \( \overline{p_1 p_2} \).

**Proof.** Let \( \rho_i = 1/\sqrt{E(\{s_1, s_2\}, p_i)} \) and let \( B_i \) be a ball of radius \( \rho_i \) centered at \( p_i \) for \( i = 1, 2 \). It is easy to verify that \( B_i \) consists of all station locations \( s \) such that \( E(s, p_i) \geq E(\{s_1, s_2\}, p_i) \). Assume without loss of generality that \( \rho_1 \geq \rho_2 \).

**Proposition 3.11.** \( \partial B_1 \) and \( \partial B_2 \) intersect.

**Proof.** By Lemma 2.3 we may assume that \( p_1 = (0, 0) \) and \( p_2 = (c, 0) \) for some positive \( c \). Since \( s_0 \) must be in both \( B_1 \) and \( B_2 \), it follows that the two balls cannot be disjoint. We establish the claim by showing that \( B_2 \) is not contained in \( B_1 \). Let us define a new uniform power network \( A' \) consisting of the stations \( s_1, s_2 \), and \( s' = (c + \rho_2, 0) \) with no background noise. The points \( p_1 \) and \( p_2 \) are colinear with the station \( s' \), hence Lemma 3.4 may be employed to conclude that

\[
\text{SINR}_{A'}(s', p_1) < \text{SINR}_{A'}(s', p_2). \tag{4}
\]
The construction of $\mathcal{A}'$ guarantees that $\text{SINR}_{\mathcal{A}'}(s', p_2) = E(s', p_2)/E(\{s_1, s_2\}, p_2) = 1$. On the other hand, if $B_2 \subseteq B_1$, then $s'$ is in $B_1$ (see Figure 10), and thus $E(s', p_1) \geq E(\{s_1, s_2\}, p_1)$ which means that $\text{SINR}_{\mathcal{A}'}(s', p_1) \geq 1$, in contradiction to inequality (4). Therefore $\partial B_1$ and $\partial B_2$ must intersect.

Let $s^*$ be an intersection point of $\partial B_1$ and $\partial B_2$ (see Figure 11). We now show that $s^*$ satisfies the assertion of Lemma 3.10. Note that $E(s, p_i) = E(\{s_1, s_2\}, p_i)$ for any station $s$ located at $\partial B_i$, thus $s^*$ produces the desired energy at $p_i$ for $i = 1, 2$, that is, $E(s^*, p_i) = E(\{s_1, s_2\}, p_i)$.

Consider a uniform power network $\mathcal{A}^*$ consisting of the stations $s^*$, $s_1$, and $s_2$ with no background noise. We have $\text{SINR}_{\mathcal{A}^*}(s^*, p_i) = 1$ for $i = 1, 2$. Therefore, Lemma 3.3 guarantees that $\text{SINR}_{\mathcal{A}^*}(s^*, q) \geq 1$ for all points $q \in \overline{p_1 p_2}$ which means that $E(s^*, q) \geq E(\{s_1, s_2\}, q)$. The assertion follows, completing the proof of Lemma 3.10.

We now turn to describe the inductive step in the proof of Lemma 3.9. Assume by induction that the assertion of the theorem holds for $n \geq 2$ stations, i.e., that in a uniform power network with $n \geq 2$ stations and no background noise we have $\overline{p_1 p_2} \subseteq \mathcal{H}_0$ for every $p_1, p_2 \in \mathcal{H}_0$. Now consider a uniform power network $\mathcal{A}$ with $n + 1$ stations $s_0, \ldots, s_n$ and no background noise. Let $p_1, p_2 \in \mathcal{H}_0$. Suppose that $s_1$ is closest to, say, $p_1$ among all stations $s_0, \ldots, s_n$. Since $p_1, p_2 \in \mathcal{H}_0$, we know that $E(s_0, p_1) > E(\{s_1, s_2\}, p_i)$ for $i = 1, 2$. Lemma 3.10 guarantees that there exists a station location $s^* \in \mathbb{R}^2$ such that (1) $E(s^*, p_i) = E(\{s_1, s_2\}, p_i)$ for $i = 1, 2$; and (2) $E(s^*, q) \geq E(\{s_1, s_2\}, q)$ for all points $q$ in the segment $\overline{p_1 p_2}$.
Figure 12: $A^*$ is obtained from $A$ by removing stations $s_1$ and $s_2$ and introducing station $s^*$.

Note that the station location $s^*$ must differ from $s_0$. This is because $E(s^*, p_i) = E(\{s_1, s_2\}, p_i)$ while $E(s_0, p_i) > E(\{s_1, s_2\}, p_i)$ for $i = 1, 2$, thus $\text{dist}(s^*, p_i) > \text{dist}(s_0, p_i)$.

Consider the $n$-station uniform power network $A^*$ obtained from $A$ by replacing $s_1$ and $s_2$ with a single station located at $s^*$ (see Figure 12). Note that $I_{A^*}(s_0, p_i) = I_A(s_0, p_i)$ for $i = 1, 2$ and $I_{A^*}(s_0, q) \geq I_A(s_0, q)$ for all points $q \in \overline{p_1p_2}$, hence $\text{SINR}_{A^*}(s_0, p_i) = \text{SINR}_A(s_0, p_i)$ for $i = 1, 2$ and $\text{SINR}_{A^*}(s_0, q) \leq \text{SINR}_A(s_0, q)$. By the inductive hypothesis, $\text{SINR}_{A^*}(s_0, q) \geq \beta$ for all points $q \in \overline{p_1p_2}$, therefore $\text{SINR}_A(s_0, q) \geq \beta$ and $s_0$ is heard at $q$ in $A$. It follows that every $q \in \overline{p_1p_2}$ belongs to $\mathcal{H}_0$ in $A$, which establishes the assertion and completes the proof of Lemma 3.9.

### 3.4 Adding background noise

Our goal in this section is to show that the reception zones in a uniform power network $A = \langle S, \bar{I}, N, \beta \rangle$, where $N > 0$, are convex, thus establishing Theorem 1. Let $p_1$ and $p_2$ be some points in $\mathbb{R}^2$ and suppose that $s_0$ is heard at $p_1$ and $p_2$ in $A$. Let $B_1$ and $B_2$ be the balls of radius $1/\sqrt{N}$ centered at $p_1$ and $p_2$, respectively. Note that $\text{SINR}_A(s_0, p_i) \geq \beta \geq 1$ implies that $E(s_0, p_i) > N$, thus $\text{dist}(s_0, p_i) < 1/\sqrt{N}$ for $i = 1, 2$. Therefore $\text{dist}(p_1, p_2) < 2/\sqrt{N}$ and $\partial B_1$ and $\partial B_2$ must intersect.

We construct an $(n + 1)$-station uniform power network $A'$ from $A$ by locating a new station $s_n$ (with transmitting power $\psi_n = 1$ like all other stations) in an intersection point of $\partial B_1$ and $\partial B_2$ and omitting the background noise (see Figure 13). Clearly, $E(s_n, p_i) = N$ for $i = 1, 2$. In particular, this means that $s_n \neq s_0$ as $E(s_0, p_i) > N$. Since $\text{dist}(s_n, p_1) = \text{dist}(s_n, p_2) = 1/\sqrt{N}$, it follows that $\text{dist}(s_n, q) \leq 1/\sqrt{N}$ for all points $q \in \overline{p_1p_2}$, hence $E(s_n, q) \geq N$. Therefore $\text{SINR}_{A'}(s_0, p_i) = \text{SINR}_A(s_0, p_i)$ for $i = 1, 2$ and $\text{SINR}_{A'}(s_0, q) \geq \text{SINR}_A(s_0, q)$ for all points $q \in \overline{p_1p_2}$. Since $A'$ has no background noise, we may employ Lemma 3.9 to conclude that $\text{SINR}_{A'}(s_0, q) \geq \beta$ for all points $q \in \overline{p_1p_2}$. The assertion follows. This completes the proof of Theorem 1.
The fatness of the reception zones

In Section 3, we showed that the reception zone of each station in a uniform power network is convex. In this section, we develop a deeper understanding of the "shape" of the reception zones by analyzing their fatness. Consider a uniform power network $A = \langle S, \bar{1}, N, \beta \rangle$, where $S = \{s_0, \ldots, s_{n-1}\}$ and $\beta > 1$ is a constant. We focus on $s_0$ and assume that its location is not shared by any other station (otherwise, the reception zone $H_0 = \{s_0\}$). In Section 4.1, we establish explicit bounds on $\Delta(s_0, H_0)$ and $\delta(s_0, H_0)$. These bounds imply that $\varphi(s_0, H_0) = O(\sqrt{n})$. This is improved in Section 4.2, where we show that $\varphi(s_0, H_0) = O(1)$, thus establishing Theorem 2.

4.1 Explicit bounds

Our goal in this section is to establish an explicit lower bound on $\delta(s_0, H_0)$ and an explicit upper bound on $\Delta(s_0, H_0)$. Since $H_0$ is compact and convex, it follows that there exists some points $q_\delta, q_\Delta \in \partial H_0$ such that $\text{dist}(s_0, q_\delta) = \delta(s_0, H_0)$ and $\text{dist}(s_0, q_\Delta) = \Delta(s_0, H_0)$. In fact, we may redefine $\delta(s_0, H_0)$ as the distance from $s_0$ to a closest point in $\partial H_0$ and $\Delta(s_0, H_0)$ as the distance from $s_0$ to a farthest point in $\partial H_0$.

Fix $\kappa = \min\{\text{dist}(s_0, s_i) \mid i > 0\}$. An extreme scenario for establishing a lower bound on $\delta(s_0, H_0)$ would be to place $s_0$ in $(0, 0)$ and all other $n - 1$ stations in $(\kappa, 0)$. This introduces the uniform power network $A_\delta = \langle\{(0, 0), (\kappa, 0), \ldots, (\kappa, 0)\}, \bar{1}, N, \beta\rangle$. The point $q_\delta$ whose distance to $s_0$ realizes $\delta(s_0, H_0)$ is thus located at $(d, 0)$ for some $0 < d < \kappa$. On the other hand, an extreme scenario for establishing an upper bound on $\Delta(s_0, H_0)$ would be to place $s_0$ in $(0, 0)$, $s_1$ in $(\kappa, 0)$, and all other $n - 2$ stations in $(\infty, 0)$ so that their energy at the vicinity of $s_0$ is ignored. This introduces the uniform power network $A_\Delta = \langle\{(0, 0), (\kappa, 0), (\infty, 0) \ldots, (\infty, 0)\}, \bar{1}, N, \beta\rangle$. The point $q_\Delta$ whose distance to $s_0$ realizes $\Delta(s_0, H_0)$ is thus located at $(-D, 0)$ for some $D > 0$.

For the sake of analysis, we shall replace the background noise $N$ in the above scenarios with a new station $s_n$ located at $(\kappa, 0)$ whose power is $N \cdot \kappa^2$. More formally, the uniform power network

$\footnote{Unlike the convexity proof presented in Section 3, which holds for any $\beta \geq 1$, the analysis presented in the current section is only suitable for $\beta$ being a constant strictly greater than 1. In fact, when $\beta = 1$, the fatness parameter is not necessarily defined (think of a trivial network).}$
\( \mathcal{A}_\delta \) is replaced by the network
\[
\mathcal{A}_\delta' = \langle \{(0,0), (\kappa,0), \ldots, (\kappa,0), (\kappa,0)\}, (1, \ldots, 1, N \cdot \kappa^2), 0, \beta \rangle
\]
and the uniform power network \( \mathcal{A}_\Delta \) is replaced by the network
\[
\mathcal{A}_\Delta' = \langle \{(0,0), (\kappa,0), (\infty,0), \ldots, (\infty,0), (\kappa,0)\}, (1, \ldots, 1, N \cdot \kappa^2), 0, \beta \rangle .
\]
Note that the energy of the new station \( s_n \) at point \((x,0)\) is greater than \( N \) for all \( 0 < x < \kappa \); exactly \( N \) for \( x = 0 \); and smaller than \( N \) for all \( x < 0 \). Therefore the value of \( \delta(s_0, \mathcal{H}_0) \) (respectively, \( \Delta(s_0, \mathcal{H}_0) \)) under \( \mathcal{A}_\delta' \) (resp., \( \mathcal{A}_\Delta' \)) is smaller (resp., greater) than that under \( \mathcal{A}_\delta \) (resp., \( \mathcal{A}_\Delta \)). In the remainder of this section we establish a lower bound (resp., an upper bound) on the former.

In the context of \( \mathcal{A}_\delta' \), we would like to compute the value of \( d > 0 \) that solves the equation
\[
\text{SINR}_{\mathcal{A}_\delta'}(s_0, (d,0)) = \beta ,
\]
which means that
\[
\frac{d^{-2}}{(n-1+N \cdot \kappa^2)(\kappa-d)^{-2}} = \beta ,
\]
or equivalently, \((\kappa-d)^2 = d^2 \beta (n-1+N \cdot \kappa^2)\), or,
\[
d = \frac{\kappa}{\sqrt{\beta(n-1+N \cdot \kappa^2)}+1}.
\]
Hence \( \delta(s_0, \mathcal{H}_0) \geq \frac{\kappa}{\sqrt{\beta(n-1+N \cdot \kappa^2)}+1} \).

In the context of \( \mathcal{A}_\Delta' \), we would like to compute the value of \( D > 0 \) that solves the equation
\[
\text{SINR}_{\mathcal{A}_\Delta'}(s_0, (-D,0)) = \beta ,
\]
which means that
\[
\frac{D^{-2}}{(1+N \cdot \kappa^2)(\kappa+D)^{-2}} = \beta ,
\]
or equivalently, \((\kappa+D)^2 = D^2 \beta (1+N \cdot \kappa^2)\), or
\[
D = \frac{\kappa}{\sqrt{\beta(1+N \cdot \kappa^2)}-1}.
\]
Hence \( \Delta(s_0, \mathcal{H}_0) \leq \frac{\kappa}{\sqrt{\beta(1+N \cdot \kappa^2)}-1} \).

Theorem 4.1 follows from the above bounds and from the following observation.

**Observation.** The inequality \( \frac{a^{n+c+1}}{b^{n+c-1}} \leq \frac{\beta^{n+1}}{\beta^{n-1}} \) holds for any choice of reals \( a \geq b > 1 \) and \( c > 0 \).

**Theorem 4.1.** In a uniform energy network \( \mathcal{A} = \langle S, \bar{I}, N, \beta \rangle \), where \( S = \{s_0, \ldots, s_{n-1}\} \) and \( \beta > 1 \) is a constant, if the minimum distance from \( s_0 \) to any other station is \( \kappa > 0 \), then
\[
\delta(s_0, \mathcal{H}_0) \geq \frac{\kappa}{\sqrt{\beta(n-1+N \cdot \kappa^2)}+1} \quad \text{and} \quad \Delta(s_0, \mathcal{H}_0) \leq \frac{\kappa}{\sqrt{\beta(1+N \cdot \kappa^2)}-1}.
\]
The fatness parameter of \( \mathcal{H}_0 \) with respect to \( s_0 \) thus satisfies
\[
\varphi(s_0, \mathcal{H}_0) \leq \frac{\kappa}{\sqrt{\beta(1+N \cdot \kappa^2)}-1} \left/ \frac{\kappa}{\sqrt{\beta(n-1+N \cdot \kappa^2)}+1} \right. \leq \frac{\sqrt{\beta(n-1)}+1}{\sqrt{\beta}-1} = O(\sqrt{n}).
\]
4.2 An improved bound on the fatness parameter

In this section we prove Theorem 2 by establishing the following theorem.

**Theorem 4.2.** The fatness parameter of $H_0$ with respect to $s_0$ satisfies
\[ \varphi(s_0, H_0) \leq \frac{\sqrt{\beta} + 1}{\sqrt{\beta} - 1} = O(1). \]

Theorem 4.2 is proved in three steps. First, in Section 4.2.1 we bound the ratio $\Delta / \delta$ in a setting of two stations in a one dimensional space. This is used in Section 4.2.2 to establish the desired bound for a special type of uniform power networks called *positive colinear* networks. We conclude in Section 4.2.3 where we reduce the general case to the case of positive colinear networks.

### 4.2.1 Two stations in a one dimensional space

Let $A$ be a network consisting of two stations $s_0, s_1$ with no background noise (i.e., $N = 0$). Consider the embedding of $A$ in the Euclidean one dimensional space $\mathbb{R}$ and assume without loss of generality that $s_0$ is located at $a_0 = 0$ and $s_1$ is located at $a_1 = 1$ (recall that this is made possible due to Lemma 2.3). Suppose that $s_0$ admits a unit transmitting power $\psi_0 = 1$ while the transmitting power of $s_1$ is any $\psi_1 \geq 1$. Let $\mu_r = \max\{p > 0 \mid \text{SINR}_A(s_0, p) \geq \beta\}$ and let $\mu_l = \min\{p < 0 \mid \text{SINR}_A(s_0, p) \geq \beta\}$ (see Figure 14). It is easy to verify that $H_0 = [\mu_l, \mu_r]$ and that $\delta = \mu_r$ and $\Delta = -\mu_l$.

**Lemma 4.3.** The network $A$ satisfies $\Delta / \delta \leq \sqrt{\beta} + 1 \over \sqrt{\beta} - 1$, with equality attained when $\psi_1 = 1$.

**Proof.** The boundary points $\mu_r$ and $\mu_l$ of $H_0$ are the solutions to the quadratic equation
\[ \frac{(x - 1)^2}{\psi_1 x^2} = \beta \iff (\beta \psi_1 - 1)x^2 + 2x - 1 = 0. \]

Solving this equation, we get
\[ \mu_r = \frac{-2 + \sqrt{4 \beta \psi_1}}{2 \beta \psi_1 - 2} = \frac{\sqrt{\beta \psi_1} - 1}{\beta \psi_1 - 1} \]
\[ \mu_l = \frac{-2 - \sqrt{4 \beta \psi_1}}{2 \beta \psi_1 - 2} = \frac{\sqrt{\beta \psi_1} + 1}{\beta \psi_1 - 1}. \]

Therefore the ratio $\Delta / \delta$ satisfies
\[ \frac{\Delta}{\delta} = \frac{\sqrt{\beta \psi_1} + 1}{\sqrt{\beta \psi_1} - 1} \leq \frac{\sqrt{\beta} + 1}{\sqrt{\beta} - 1} \]
as desired. \qed
4.2.2 Positive colinear networks

In this section we switch back to the Euclidean plane $\mathbb{R}^2$ and consider a special type of uniform power networks. A network $A = \langle \{s_0, \ldots, s_{n-1}\}, 1, N, \beta \rangle$ is said to be positive colinear if $s_0 = (0, 0)$ and $s_i = (a_i, 0)$ for some $a_i > 0$ for every $1 \leq i \leq n-1$. Positive colinear networks play an important role in the subsequent analysis due to the following lemma. (Refer to Figure 15 for illustration.)

**Lemma 4.4.** Let $A$ be a positive colinear uniform power network. Fix $\mu_r = \max\{r > 0 \mid \text{SINR}_A(s_0, (r, 0)) \geq \beta\}$ and $\mu_l = \min\{r < 0 \mid \text{SINR}_A(s_0, (r, 0)) \geq \beta\}$. Then

$$
\delta = \mu_r, \\
\Delta = -\mu_l, \\
-\frac{\mu_l}{\mu_r} \leq \frac{\sqrt{\beta} + 1}{\sqrt{\beta} - 1}.
$$

Before we can establish Lemma 4.4, we would like to prove some basic properties of positive colinear networks. First, we argue that the reception zone $\mathcal{H}_0$ of $s_0$ under the positive colinear network $A$ is contained in the halfplanes intersection $\{ (x,y) \mid \mu_l \leq x \leq \mu_r \}$. To see why this is true, suppose towards deriving contradiction that the point $(x, y) \in \mathcal{H}_0$ for some $x > \mu_r$ or $x < \mu_l$. Due to symmetry considerations, we conclude that the point $(x, -y)$ is also in $\mathcal{H}_0$. By the convexity of $\mathcal{H}_0$, it follows that $(x, 0) \in \mathcal{H}_0$, in contradiction to the definitions of $\mu_r$ and $\mu_l$.

**Corollary 4.5.** If $(x, y) \in \mathcal{H}_0$, then $\mu_l \leq x \leq \mu_r$.

We now turn to prove that $\delta = \mu_r$. To do so, we will prove that the ball of radius $\mu_r$ centered at $s_0$ is contained in $\mathcal{H}_0$. In fact, by the convexity of $\mathcal{H}_0$, it is sufficient to show that the point $p(\theta) = (\mu_r \cos \theta, \mu_r \sin \theta)$ is in $\mathcal{H}_0$ for all $0 \leq \theta \leq \pi$. Since the network is positive colinear, it follows that $I_A(s_0, p(\theta))$ attains its maximum for $\theta = 0$. Therefore the fact that $p(0) = (\mu_r, 0) \in \mathcal{H}_0$ implies that $p(\theta) \in \mathcal{H}_0$ for all $0 \leq \theta \leq \pi$ as desired.

**Corollary 4.6.** The positive colinear network $A$ satisfies $\delta = \mu_r$.

Next, we prove that $\Delta$ is realized by the point $(\mu_l, 0)$. Indeed, by the triangle inequality, all points at distance $d$ from $s_0$ are at distance $\leq d + a_i$ from $s_i = (a_i, 0)$, with equality attained for the point $(-d, 0)$. Thus the minimum interference to $s_0$ under $A$ among all points at distance $d$
from $s_0$ is attained at the point $(-d, 0)$. Therefore, by the definition of $\mu_i$, there cannot exist any point $p \in \mathcal{H}_0$ such that $\text{dist}(p, s_0) > -\mu_i$.

**Corollary 4.7.** The positive colinear network $\mathcal{A}$ satisfies $\Delta = -\mu_i$.

It remains to establish the bound on the ratio $-\mu_i/\mu_r = \Delta/\delta$. Fix $d = \min\{a_i \mid 1 \leq i \leq n - 1\}$, that is, the leftmost station other than $s_0$ is located at $(d, 0)$. Clearly, $\mu_r < d$. We denote the energy of station $s_i$ at $(\mu_r, 0)$ by $E_i = (a_i - \mu_r)^{-2}$. We construct a new network $\mathcal{A}' = (S', \psi', 0, \beta)$ consisting of $s_0$ and $n$ new stations $s'_1, \ldots, s'_n$, all located at $(d, 0)$. For $1 \leq i \leq n - 1$, we set the transmitting power $\psi_i'$ of the new station $s'_i$ so that the energy it produces at $(\mu_r, 0)$ is $E_i$. The transmitting power $\psi_n'$ of the new station $s'_n$ is set so that the energy it produces at $(\mu_r, 0)$ is $N$. This accounts to

$$\psi_i' = \begin{cases} E_i \cdot (d - \mu_r)^2 & \text{for } 1 \leq i \leq n - 1; \text{ and} \\ N \cdot (d - \mu_r)^2 & \text{for } i = n. \end{cases}$$

The network $\mathcal{A}'$ falls into the setting of Section 4.2.1. The stations $s'_1, \ldots, s'_n$ share the same location, thus they can be considered as a single station with transmitting power $\sum_{i=1}^n \psi_i'$. We define $\mu'_{r} = \max\{r > 0 \mid \text{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \geq \beta\}$ and $\mu'_{l} = \min\{r < 0 \mid \text{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \geq \beta\}$, so that the restriction of the reception zone of $s_0$ under $\mathcal{A}$ to the $x$-axis is $[\mu'_l, \mu'_r]$. Lemma 4.3 implies that $-\mu'_l/\mu'_r \leq \frac{\sqrt{\pi + 1}}{\sqrt{\pi - 1}}$. The remainder of the proof relies on establishing the following two bounds:

1. $\text{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \leq \text{SINR}_{\mathcal{A}}(s_0, (r, 0))$ for all $\mu_r \leq r < d$; and
2. $\text{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \geq \text{SINR}_{\mathcal{A}}(s_0, (r, 0))$ for all $r \leq \mu_r$, $r \neq 0$.

By combining bounds (1) and (2), we conclude that $\mu'_r \leq \mu_r$ and $\mu'_l \leq \mu_l$, which completes the proof of Lemma 4.4.

To establish bounds (1) and (2), consider some point $p = (r, 0)$, where $r < d$, $r \neq 0$. For every $1 \leq i \leq n - 1$, we have

$$E(s_i, p) = \frac{1}{(a_i - r)^2}, \text{ while } E(s'_i, p) = \frac{\psi'_i}{(d - r)^2} = \frac{(d - \mu_r)^2}{(d - r)^2(a_i - \mu_r)^2}.$$  

Comparing between the former expression and the latter, we get

$$E(s_i, p) \geq E(s'_i, p),$$

or equivalently,

$$(d - r)(a_i - \mu_r) \geq (d - \mu_r)(a_i - r).$$

Rearranging, we get

$$da_i - d\mu_r - a_i r + r\mu_r \geq da_i - dr - a_i \mu_r + r \mu_r,$$

or

$$\mu_r(a_i - d) \geq r(a_i - d),$$

where the last inequality holds if and only $a_i = d$, which, by definition, implies that $E(s_i, p) = E(s'_i, p)$, or $\mu_r \geq r$. Therefore the contribution of $s'_i$ to the interference to $s_0$ at $p = (0, r)$ is not
larger than that of \( s_i \) as long as \( r \leq \mu_r \) and not smaller than that of \( s_i \) as long as \( \mu_r \leq r < d \). On the other hand, the energy of \( s_n' \) at \( p = (r,0) \) is not larger than the background noise \( N \) for all \( d \leq \mu_r \) and not smaller than \( N \) for all \( \mu_r \leq r < d \). Bounds (1) and (2) follow.

### 4.2.3 A general uniform power network

We are now ready to prove the main theorem of Section 4.

**Proof of Theorem 4.2.** Let \( \mathcal{A} = \langle S, \bar{I}, N, \beta \rangle \), where \( S = \{s_0, \ldots, s_{n-1}\} \) and \( \beta > 1 \) is a constant, be an arbitrary uniform power network. We employ Lemma 2.3 to assume that \( s_0 \) is located at \((0,0)\) and that \( \max \{ \text{dist}(s_0, q) \mid q \in \mathcal{H}_0 \} \) is realized by a point \( q \) on the negative \( x \)-axis. By definition, we have \( q = (-\Delta, 0) \).

We construct a new uniform power network \( \mathcal{A}' = \langle \{s_0, s_1', \ldots, s_{n-1}'\}, \bar{I}, N, \beta \rangle \), obtained from \( \mathcal{A} \) by rotating each station \( s_i \) around the point \( q \) until it reaches the positive \( x \)-axis (see Figure 16). More formally, the location of \( s_0 \) remains unchanged and \( s_1' = (a_i', 0) \), where \( a_i' = \text{dist}(s_i, q) - \Delta \) for every \( 1 \leq i \leq n - 1 \). Since \( s_0 \) is heard at \( q \) under \( \mathcal{A} \), it follows that \( \Delta = \text{dist}(s_0, q) < \text{dist}(s_i, q) \) for every \( 1 \leq i \leq n - 1 \), hence \( a_i' > 0 \) and \( \mathcal{A}' \) is a positive colinear network. Clearly, \( \text{dist}(s_i', q) = \text{dist}(s_i, q) \) for every \( 1 \leq i \leq n - 1 \).

Let \( \mathcal{H}_0 \) denote the reception zone of \( s_0 \) under \( \mathcal{A}' \). Fix \( \delta' = \max \{ r > 0 \mid B(s_0, r) \subseteq \mathcal{H}_0 \} \) and \( \Delta' = \min \{ r > 0 \mid B(s_0, r) \supseteq \mathcal{H}_0 \} \). Let \( \mu_r' = \max \{ r > 0 \mid \text{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \geq \beta \} \) and let \( \mu_0' = \min \{ r < 0 \mid \text{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \geq \beta \} \). Lemma 4.4 guarantees that \( \delta' = \mu_r', \Delta' = -\mu_0', \) and \( \frac{\delta'}{\mu_r'} \leq \frac{\sqrt{\beta + 1}}{\sqrt{\beta - 1}} \). We shall establish the proof of Theorem 4.2 by showing that \( \Delta' = \Delta \) and \( \delta' \leq \delta \). The former is a direct consequence of Lemma 4.4 since \( \text{SINR}_{\mathcal{A}'}(s_0, q) = \text{SINR}_{\mathcal{A}}(s_0, q) = \beta \), it follows that \( \max \{ \text{dist}(s_0, p) \mid p \in \mathcal{H}_0 \} \) is realized at \( p = q \).

It remains to prove that \( \delta' \leq \delta \). We shall do so by showing that \( B(s_0, \delta') \subseteq \mathcal{H}_0 \). Fix \( \rho_i = \text{dist}(s_i, q) \) for every \( 1 \leq i \leq n - 1 \). We argue that the ball \( B(s_0, \delta') \) is strictly contained in the ball \( B(q, \rho_i) \) for every \( 1 \leq i \leq n - 1 \). To see why this is true, observe that \( -\Delta < 0 < \delta' = \mu_r' < a_i' \), hence the ball centered at \( q = (-\Delta, 0) \) of radius \( \rho_i = \Delta + a_i' \) strictly contains the ball of radius \( \delta' \) centered at \( s_0 = (0, 0) \).

Consider an arbitrary point \( p \in B(s_0, \delta') \). We can now rewrite

\[
\text{dist}(s_i', (\delta', 0)) = a_i' - \delta' = \min \{ \text{dist}(t, t') \mid t \in B(s_0, \delta'), t' \in \partial B(q, \rho_i) \}
\]

for every \( 1 \leq i \leq n - 1 \). Recall that \( s_i \in \partial B(q, \rho_i) \), thus \( \text{dist}(s_i, p) \geq \text{dist}(s_i', (\delta', 0)) \). Therefore \( \text{I}_{\mathcal{A}}(s_0, p) \leq \text{I}_{\mathcal{A}'}(s_0, (\delta', 0)) \) and \( \text{SINR}_{\mathcal{A}}(s_0, p) \geq \text{SINR}_{\mathcal{A}'}(s_0, (\delta', 0)) = \beta \). It follows that \( p \in \mathcal{H}_0 \), which completes the proof. \( \square \)
Handling approximate point location queries

Our goal in this section is to prove Theorem 3. In fact, our technique for approximate point location queries is suitable for a more general framework of zones (and diagrams). Let $Q(x, y)$ be a 2-variate polynomial of degree $m$ and suppose that the zone $Q = \{(x, y) \in \mathbb{R}^2 | Q(x, y) \leq 0\}$ is a thick set and that $Q(x, y)$ is strictly negative for all internal points $(x, y)$ of $Q$. Moreover, suppose that we are given an internal point $s$ of $Q$, a lower bound $\tilde{\delta}$ on $\delta(s, Q)$, and an upper bound $\tilde{\Delta}$ on $\Delta(s, Q)$. Let $0 < \epsilon < 1$ be a predetermined performance parameter. We construct in $O(m^2(\tilde{\Delta}/\tilde{\delta})^2\epsilon^{-1})$ preprocessing time a data structure $QDS$ of size $O((\tilde{\Delta}/\tilde{\delta})^2\epsilon^{-1})$. $QDS$ essentially partitions the Euclidean plane into three disjoint zones $\mathbb{R}^2 = Q^+ \cup Q^- \cup Q'$ such that
1. $Q^+ \subseteq Q$;
2. $Q^- \cap Q = \emptyset$; and
3. $Q'$ is bounded and its area is at most an $\epsilon$-fraction of the area of $Q$.

Given a query point $p \in \mathbb{R}^2$, $QDS$ answers in constant time whether $p$ is in $Q^+$, $Q^-$, or $Q'$.

In Section 5.1 we describe the construction of $QDS$. In Section 5.2 we explain how the reception zones and the SINR diagram fall into the above framework and establish Theorem 3.

Figure 16: $A'$ is obtained from $A$ by relocating each station $s_i$ on the $x$-axis.
5.1 The construction of QDS

In this section we describe the construction of QDS. Let $\gamma$ be a positive real to be determined later on. The data structure QDS is based upon imposing a $\gamma$-spaced grid, denoted by $G_\gamma$, on the Euclidean plane. The grid is aligned so that the point $s$ is a grid vertex. The Euclidean plane is partitioned to grid cells with respect to $G_\gamma$ in the natural manner, where ties are broken such that each cell contains all points on its south edge except its south east corner and all points on its west edge except its north west corner (the cell does contain its south west corner). Given some cell $C$, we define its 9-cell, denoted by $\#C$, as the collection of $3 \times 3$ cells containing $C$ and the eight cells surrounding it.

The grid cells will be classified to three types corresponding to the zones $Q^+$, $Q^-$, and $Q^?$: cells of type $T^+$ are fully contained in $Q$; cells of type $T^-$ do not intersect $Q$; and cells of type $T^?$ are suspect of partially overlapping $Q$, i.e., having some points in $Q$ and some points not in $Q$. A query on point $p \in \mathbb{R}^2$ is handled merely by computing the cell to which $p$ belongs and returning its type. Our analysis relies on bounding the number (and thus the total area) of $T^?$ cells.

By definition, the zone $Q$ contains a ball of radius $\tilde{\delta}$ and it is contained in a ball of radius $\tilde{\Delta}$, both centered at $s$. Clearly, the area of $Q$ is lower bounded by the area of any ball it contains. Since $Q$ is convex, it follows that its perimeter is upper bounded by the perimeter of any ball that contains it. Therefore the zone $Q$ satisfies

$$\text{area}(Q) \geq \pi \tilde{\delta}^2 \quad \text{and} \quad \text{per}(Q) \leq 2\pi \tilde{\Delta} . \quad (5)$$

We will soon present an iterative process, referred to as the Boundary Reconstruction Process (BRP), which identifies the $T^?$ cells. The union of the $T^?$ cells form the zone $Q^?$ that contains $Q$’s boundary $\partial Q = \{(x,y) \in \mathbb{R}^2 \mid Q(x,y) = 0\}$. In fact, the zone $Q^?$ is isomorphic to a ring and in particular, it partitions $\mathbb{R}^2 - Q^?$ to a zone enclosed by $Q^?$ and a zone outside $Q^?$. The cells in the former zone (respectively, latter zone) are subsequently classified as $T^+\text{ cells}$ (resp., $T^-\text{ cells}$). We shall conclude by bounding the area of $Q^?$, showing that it is at most an $\epsilon$-fraction of the area of $Q$.

The main ingredient of BRP is a procedure referred to as the segment test. On input segment $\sigma$ (which may be open or closed in each endpoint), the segment test returns the number of distinct intersection points of $\partial Q$ and $\sigma$. (Since $Q$ is convex, this number is either 0, 1, or 2.) The segment test is implemented to run in time $O(m^2)$ by employing Sturm’s condition of the projection of the polynomial $Q(x,y)$ on $\sigma$ and by direct calculations of the SINR function in the endpoints of $\sigma$. Typically, the segment test will be invoked on segments consisting of edges of the grid $G_\gamma$.

Note that if $\sigma$ is tangent to $\partial Q$, then the segment test reports a single intersection point. To distinguish this (extreme) case from the (common) case where $\partial Q$ crosses $\sigma$ in a single point, we can append three other segments to $\sigma$, thus closing a virtual square, and apply the segment test to these three new segments. Since $\partial Q$ is a closed curve, if $\partial Q$ crosses $\sigma$ and enters the virtual square,
then it must exit it at some point. On the other hand, by the convexity of $Q$, we know that if $\sigma$ is tangent to $\partial Q$, then such a virtual square cannot intersect $\partial Q$ at any other point. (Of course, one should consider two possible such squares, one on each side of $\sigma$.)

We now turn to describe BRP. Informally, the process traverses the boundary of $Q$ in the clockwise direction and identifies the grid cells that intersect it (with some slack). Let $C_s$ be the grid cell that contains the point $s$. (We will choose the parameter $\gamma$ to ensure that $\gamma < \tilde{\delta}/\sqrt{2}$ so that $C_s$ and the three other cells that share the vertex $s$ are fully contained in $Q$.) BRP begins by identifying the unique cell $C_1$ north to $C_s$ ($C_1$ and $C_s$ are in the same grid column) which contains a point of $\partial Q$ along its west edge. Note that all grid vertices between $s$ and the south west corner of $C_1$ are in $Q$, while the north west corner of $C_1$ and all the grid vertices to its north are not in $Q$. The computation of $C_1$ is performed by direct calculations of the SINR function at grid vertices north of $s$ in a binary search fashion, starting with a grid vertex at distance at most $\tilde{\Delta}$ from $s$, and ending with a grid vertex at distance at least $\tilde{\delta}$ from $s$, so that the total number of SINR calculations is $O(\log(\tilde{\Delta}/\tilde{\delta}))$.

Let $q$ denote the (unique) intersection point of $\partial Q$ and the west edge of $C_1$. Consider some continuous (injective) curve function $\phi : [0, 2\pi) \to \partial Q$ that traverses $\partial Q$ in the clockwise direction, aligned so that $\phi(0) = q$. For the sake of formality, we extend the domain of $\phi$ to $[0, \infty)$ by setting $\phi(z) = \phi(z - \lfloor z/(2\pi) \rfloor \cdot 2\pi)$ for every $z > 2\pi$. Let $z_1 = 0$. Given the cell $C_{j-1}$ and the real $z_{j-1} \in [0, 2\pi)$, we define $z_j = \inf\{z > z_{j-1} \mid \phi(z) \notin \#C_{j-1}\}$. (Informally, $\phi(z_j)$ is the first point out of $\#C_{j-1}$ encountered along a clockwise traversal of $\partial Q$ that begins at $\phi(z_{j-1})$.)

If $z_j \geq 2\pi$ (which means that the process have completed a full encirclement of $\partial Q$), then we fix $m = j$ and BRP is over. Assume that $z_j < 2\pi$. If $\phi(z_j) \notin \#C_{j-1}$, then the cell $C_j$ is defined to be the cell containing $\phi(z_j)$. Otherwise, the cell $C_j$ is defined to be the cell containing the point $\phi(z_j + \delta)$ for sufficiently small $\delta > 0$. BRP then continues, gradually constructing the collection of $T^7$ cells, consisting of all cells in the $9$-cell of $C_j$ for every $1 \leq j < m$. The choice of cells $C_1, \ldots, C_{m-1}$ is illustrated in Figure [17].

It should be clarified that from an algorithmic perspective, we do not explicitly compute the real sequence $z_1, \ldots, z_m$, but rather the cell sequence $C_1, \ldots, C_{m-1}$. This is done by $O(1)$ applications of the segment test for every 9-cell involved in the process. Since $\partial Q$ is a closed curve, and since $Q$ is convex, these applications are sufficient to identify the grid edges (and vertices) crossed by (or tangent to) $\partial Q$, and hence to compute the desired cell sequence $C_1, \ldots, C_{m-1}$.

Next, we bound the number of $T^7$ cells. In every iteration of BRP, we introduce at most 9 new $T^7$ cells, hence the total number of $T^7$ cells is at most $9(m - 1)$. Recall our choice of reals $z_1, \ldots, z_m$. As $\phi(z_{j-1})$ lies on the boundary of $C_{j-1}$ and $\phi(z_j)$ lies on the boundary of $\#C_{j-1}$, we conclude that $\text{dist}(\phi(z_j), \phi(z_{j-1})) \geq \gamma$ for every $1 < j \leq m$. Therefore in each iteration of BRP, at least $\gamma$ units of length are “consumed” from $\text{per}(Q)$. By inequality [5], we have $m \leq \lceil \text{per}(Q)/\gamma \rceil \leq \lceil 2\pi \tilde{\Delta}/\gamma \rceil$, thus the number of $T^7$ cells is at most $9(m - 1) < 18\pi \tilde{\Delta}/\gamma$. 

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Figure 17: The cell collection $C_1, \ldots, C_{m-1}$ (dark gray) on top of the boundary of $Q$ (bold curve). The $T^?$ cells are the union of $C_1, \ldots, C_{m-1}$ and the 8 cells surrounding each one of them (in light gray).

Since the area of each grid cell is $\gamma^2$, it follows that the total area of $Q^?$ (which is the union of the $T^?$ cells) is smaller than $18\pi\tilde{\Delta}\gamma$. In order to guarantee that $\text{area}(Q^?) \leq \epsilon \cdot \text{area}(Q)$, we employ inequality (5) once more and demand that $18\pi\tilde{\Delta}\gamma \leq \epsilon\pi\tilde{\delta}^2$. Therefore it is sufficient to fix

$$\gamma = \frac{\epsilon\tilde{\delta}^2}{18\Delta},$$

which means that the number of $T^?$ cells is $O((\tilde{\Delta}/\tilde{\delta})^2\epsilon^{-1})$.

Let $Q$ be the collection of grid columns with at least one $T^?$ cell. Clearly, $|Q| = O((\tilde{\Delta}/\tilde{\delta})^2\epsilon^{-1})$. Each column $\chi$ in $Q$ contains at most 6 $T^?$ cells. Consider some cell $C$ in $\chi$ which is not a $T^?$ cell. If there is a $T^?$ cell to the north of $C$ and a $T^?$ cell to its south, then $C$ is a $T^+$ cell; otherwise, $C$ is a $T^-$ cell. It follows that the data structure $QDS$ can be represented as a vector with one entry per each grid column in $Q$ ($O((\tilde{\Delta}/\tilde{\delta})^2\epsilon^{-1})$ entries altogether), where the entry corresponding to the grid column $\chi \in Q$ stores the $T^?$ cells of $\chi$ (at most 6 of them). On input point $p \in \mathbb{R}^2$, we merely have to compute the grid cell to which $p$ belongs and (possibly) look up at the relevant entry of $QDS$.

5.2 Approximate point location queries in the SINR diagram

In this section we explain the relevance of the construction presented in Section 5.1 to $\epsilon$-approximate point location queries in the SINR diagram and establish Theorem 3. Consider some uniform power
network \( (S, I, N, \beta) \), where \( S = \{s_0, \ldots, s_{n-1}\} \) and \( \beta > 1 \) is a constant. Recall that the reception zone \( \mathcal{H}_i = \{(x, y) \in \mathbb{R}^2 \mid H_i(x, y) \leq 0\} \) for every \( 0 \leq i \leq n-1 \), where \( H_i(x, y) \) is a 2-variate polynomial of degree at most 2 that is strictly negative for all internal points \((x, y)\) of \( \mathcal{H}_i \) (see Section 2.2). Assuming that the location of \( s_i \) is not shared by any other station (if it is, then \( \mathcal{H}_i = \{s_i\} \) and point location queries are answered trivially), we know that \( s_i \) is an internal point of \( \mathcal{H}_i \). Furthermore, Theorem 4.1 provides us with a lower bound \( \tilde{\delta} \) on \( \delta(s_i, \mathcal{H}_i) \), and an upper bound \( \tilde{\Delta} \) on \( \Delta(s_i, \mathcal{H}_i) \) such that \( \tilde{\Delta}/\tilde{\delta} = O(\sqrt{n}) \).

In fact, we can obtain much tighter bounds on \( \delta(s_i, \mathcal{H}_i) \) and \( \Delta(s_i, \mathcal{H}_i) \). Let \( r \) be some positive real and assume that we are promised that \( \delta(s_i, \mathcal{H}_i) = O(r) \) and that \( \Delta(s_i, \mathcal{H}_i) = \Omega(r) \). Theorem 4.2 guarantees that \( \Delta(s_i, \mathcal{H}_i)/\delta(s_i, \mathcal{H}_i) = O(1) \), hence both \( \delta(s_i, \mathcal{H}_i) \) and \( \Delta(s_i, \mathcal{H}_i) \) are \( \Theta(r) \). Such a positive real \( r \) is found via an iterative binary-search-like process that directly computes the values of the SINR function of \( s_i \) at points to the, say, north of \( s_i \), starting with a point at distance \( \tilde{\Delta} \) form \( s_i \), and ending with a point at distance at least \( \tilde{\delta} \) from \( s_i \). Since \( \tilde{\Delta}/\tilde{\delta} = O(\sqrt{n}) \), it follows that this process is bound to end within \( O(\log n) \) iterations. Each iteration takes \( O(n) \) time, thus the improved bounds for \( \delta(s_i, \mathcal{H}_i) \) and \( \Delta(s_i, \mathcal{H}_i) \) are computed in time \( O(n \log n) \).

Given a performance parameter \( 0 < \epsilon < 1 \), we apply the technique presented in Section 5.1 to \( \mathcal{H}_i \) and its corresponding polynomial \( H_i \) with the improved bounds on \( \delta(s_i, \mathcal{H}_i) \) and \( \Delta(s_i, \mathcal{H}_i) \) and construct in time \( O(n^2 \epsilon^{-1}) \) a data structure \( \mathcal{QDS}_i \) of size \( O(\epsilon^{-1}) \) that partitions the Euclidean plane to disjoint zones \( \mathbb{R}^2 = \mathcal{H}_i^+ \cup \mathcal{H}_i^- \cup \mathcal{H}_i^j \) such that (1) \( \mathcal{H}_i^+ \subseteq \mathcal{H}_i \); (2) \( \mathcal{H}_i^- \cap \mathcal{H}_i = \emptyset \); and (3) \( \mathcal{H}_i^j \) is bounded and its area is at most an \( \epsilon \)-fraction of \( \mathcal{H}_i \). Given a query point \( p \in \mathbb{R}^2 \), \( \mathcal{QDS}_i \) answers in constant time whether \( p \) is in \( \mathcal{H}_i^+ \), \( \mathcal{H}_i^- \), or \( \mathcal{H}_i^j \). (We construct a separate data structure \( \mathcal{QDS}_i \) for every \( 0 \leq i \leq n-1 \).

Recall that by Observation 2.2 point \( p \) cannot be in \( \mathcal{H}_i \) unless it is closer to \( s_i \) than it is to any other station in \( S \). Thus for such a point \( p \) there is no need to query the data structure \( \mathcal{QDS}_j \) for any \( j \neq i \). A Voronoi diagram of linear size for the \( n \) stations is constructed in \( O(n \log n) \) preprocessing time, so that given a query point \( p \in \mathbb{R}^2 \), we can identify the closest station \( s_i \) in time \( O(\log n) \) and invoke the appropriate data structure \( \mathcal{QDS}_i \).

Combining the Voronoi diagram with the data structures \( \mathcal{QDS}_i \) for all \( 0 \leq i \leq n-1 \), we obtain a data structure \( \mathcal{DS} \) of size \( O(n^2 \epsilon^{-1}) \), constructed in \( O(n^3 \epsilon^{-1}) \) preprocessing time, that decides in time \( O(\log n) \) whether the query point \( p \) is in \( \mathcal{H}_i^+ \) for some \( i \), in \( \mathcal{H}_i^- \) for some \( i \), or neither, which means that \( p \in \mathcal{H}^- = \bigcap_{i=0}^{n-1} \mathcal{H}_i^- \). Theorem 3 follows.
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