On nonuniqueness and nonregularity for gradient flows of polyconvex functionals

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Abstract
We provide some counterexamples concerning the uniqueness and regularity of weak solutions to the initial-boundary value problem for the gradient flows of certain strongly polyconvex functionals. We show that such a problem can possess a trivial classical solution as well as infinitely many weak solutions that are nowhere smooth. Such polyconvex functions are constructed from some previous examples, and the nonuniqueness and nonregularity results are proved by reformulating the gradient flow as a partial differential relation and then using the convex integration method to construct certain strongly convergent sequences of subsolutions that have a uniform control on local essential oscillations of the spatial gradients.

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1 Introduction
In this paper, we study the energy functionals of the special form

$$\mathcal{E}(u) = \int_\Omega F(Du(x)) \, dx$$ (1.1)

for vector functions $u = (u^1, \ldots, u^m)$ on a bounded domain $\Omega \subset \mathbb{R}^n$, where $m, n \geq 2$ are integers, $F: \mathbb{M}^{m \times n} \to \mathbb{R}$ is a given $C^1$ function, and $Du$ is the Jacobian matrix of $u$ in $\mathbb{M}^{m \times n}$ defined by $Du = (\partial u^i / \partial x_k) (1 \leq i \leq m, 1 \leq k \leq n)$. Here $\mathbb{M}^{m \times n}$ stands for the usual Euclidean space of $m \times n$ matrices with inner product $A : B = \text{tr}(A^T B)$.

Given a number $T > 0$, the $L^2$-gradient flow of energy functional $\mathcal{E}$ on $\Omega_T = \Omega \times (0, T)$ is defined by

$$\partial_t u = \text{div} \, F(Du),$$ (1.2)
where $DF: \mathbb{M}^{m \times n} \rightarrow \mathbb{M}^{m \times n}$ is defined by $DF(A) = (\partial F(A)/\partial A^i_k)$ for all $A = (A^i_k) \in \mathbb{M}^{m \times n}$, $\partial_i u$ and $Du$ denote the time-derivative and spatial Jacobian matrix of $u = u(x, t): \Omega_T \rightarrow \mathbb{R}^m$, respectively.

Note that all classical $C^2$ solutions of the gradient flow (1.2) with given Dirichlet boundary condition $u(x, t) = g(x)$ on $\partial \Omega \times (0, T)$ will always obey the energy decay law

$$\frac{d}{dt} \mathcal{E}(u(\cdot, t)) = -\int_\Omega |\partial_i u|^2 dx \leq 0,$$

and thus the energy always decreases along all such solutions. However, no standard theory ensures the existence of classical solutions of (1.2) even for the general smooth functions $F$ and $g$. We refer to the monograph [2] for further study of general gradient flows. In this paper we focus on the usual weak solutions of the gradient flow (1.2), by which we mean a function $u: \Omega_T \rightarrow \mathbb{R}^m$ in a Sobolev space such that

$$\int_\Omega (u \cdot \phi)|_{t=0}^{t=T} dx = \int_\Omega (u \cdot \partial_i \phi - DF(Du) : D\phi) dx dt$$

holds for all $\phi \in C^1([0, T]; C^\infty_c(\Omega; \mathbb{R}^m))$, where $u(\cdot, 0)$ and $u(\cdot, T)$ are taken in the sense of trace. A function $u: \Omega \times (0, \infty) \rightarrow \mathbb{R}^m$ is called a global weak solution of (1.2) if it is a weak solution of (1.2) on $\Omega_T$ for all $T > 0$.

Recently, for the general energy functional $\mathcal{E}(u) = \int_\Omega f(x, u, Du)dx$, a notion of variational solutions of the gradient flow has been introduced in [5]. It is shown in [5] that under certain growth conditions the existence of variational solutions in a given Dirichlet class for all initial-boundary conditions is equivalent to the convexity of functional $\mathcal{E}(u)$ on the Dirichlet class (this is the so-called integral convexity); moreover, under the integral convexity (and certain growth conditions), variational solutions are unique and coincide with the weak solutions in a suitable Sobolev space. In this regard, integral convexity provides a good theory of existence for the gradient flows in the similar way as quasiconvexity for the minimizers of the time-independent problems (see, e.g., [1, 3, 10, 26]). However, as we shall see below, from the main result of the paper, for functionals of the form (1.1) the integral convexity of $\mathcal{E}$ is generally not guaranteed by the quasiconvexity of function $F$ and the lack of integral convexity can lead to non-uniqueness and non-regularity of weak solutions.

Some existence and regularity results have been known for the weak solutions of the system (1.2), including its stationary (time-independent) solutions; see, e.g., [6, 9, 11, 12, 15, 18, 24, 34]. In most cases, only partial regularity holds. For example, under certain smoothness and growth conditions on $F$, it has been proved in [6, 11, 12] that any weak solution $u$ of (1.2) satisfies $Du \in C^{\alpha,\alpha/2}_{loc}(Q_0)$ for some $\alpha \in (0, 1)$ on some open subset $Q_0$ of $\Omega_T$ of full measure provided that $DF$ is strongly quasimonotone in the sense that for some $\nu > 0$ the inequality

$$\int_\Omega DF(A + D\phi) : D\phi dx \geq \nu \int_\Omega |D\phi|^2 dx$$

holds for all $A \in \mathbb{M}^{m \times n}$ and $\phi \in C^\infty_c(\Omega; \mathbb{R}^m)$. Condition (1.5) implies the strong quasiconvexity of function $F$ (see [14]):

$$\int_\Omega (F(A + D\phi) - F(A))dx \geq \nu \int_\Omega |D\phi|^2 dx$$

for all $A \in \mathbb{M}^{m \times n}$ and $\phi \in C^\infty_c(\Omega; \mathbb{R}^m)$. A specific class of strongly quasiconvex functions is provided by the so-called strongly polyconvex functions. For example, when $m = n = 2$,
a function $F : \mathbb{M}^{2 \times 2} \to \mathbb{R}$ is called strongly polyconvex if

$$F(A) = \frac{\nu}{2} |A|^2 + G(A, \det A),$$

(1.7)

for some $\nu > 0$ and a convex function $G : \mathbb{M}^{2 \times 2} \times \mathbb{R} \to \mathbb{R}$.

We focus on the functions $F$ on $\mathbb{M}^{2 \times 2}$ defined by (1.7) with smooth convex functions $G(A, \delta)$ on $\mathbb{M}^{2 \times 2} \times \mathbb{R}$ and study the initial Dirichlet boundary value problem of the gradient flow:

$$\begin{cases}
\partial_t u = \text{div} F(Du) \text{ in } \Omega_T; \\
u | F(Du)| = 0, \\
\end{cases}
$$

(1.8)

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ and $\partial' \Omega_T = (\Omega \times [0]) \cup (\partial \Omega \times (0, T))$ denotes the parabolic boundary of $\Omega_T$. It is easily seen from the energy decay law (1.3) that $u \equiv 0$ is the unique classical solution to (1.8). Our main result is to provide a counterexample of nonuniqueness and nonregularity for the weak solutions of (1.8) with strongly polyconvex functions $F$.

**Theorem 1.1** There exist smooth strongly polyconvex functions $F$ of the form (1.7) satisfying

$$|DF(A)| \leq C(|A| + 1) \quad \forall A \in \mathbb{M}^{2 \times 2}$$

(1.9)

with the property that for all $\phi \in C^1_0(\Omega; \mathbb{R}^2)$ there is $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$ the initial-boundary value problem (1.8) possesses infinitely many Lipschitz weak solutions $u$ on $\Omega_T$ satisfying that $u(x, T) = \epsilon \phi(x)$ and that $Du$ is nowhere continuous in $\Omega_T$.

We remark that the functions $F$ in the theorem satisfy all the conditions of [5, Theorem 7.5] with $u_* = 0$ and $p = 2$ except the integral convexity of functional $\mathcal{E}$. In fact, from the nonuniqueness of our theorem and by Theorems 7.4, 7.5 and 7.6 of [5], such an integral convexity must fail for the functional $\mathcal{E}$. Furthermore, the nonregularity of the weak solutions in the theorem also implies that the strong quasimonotonicity (1.5) must fail for these functions $F$.

Before outlining the main ideas for Theorem 1.1, we discuss some consequences and related results.

**Proposition 1.2** Let $u$ be any weak solution as given in Theorem 1.1. Then there exists an open dense subset $I$ of $(0, T)$ such that

$$\mathcal{E}(u(\cdot, t)) - \mathcal{E}(u(\cdot, 0)) \geq \frac{\nu}{2} \int_{\Omega} |Du(x, t)|^2 dx > 0 \quad \forall t \in I;$$

(1.10)

thus the energy does not decrease along the solution.

**Proof** Note that $F$ satisfies the strong quasiconvexity (1.6). Let $u$ be a weak solution to the problem (1.8) as given in Theorem 1.1. Then the set $C = \{t \in (0, T) | u(\cdot, t)|_{\Omega} = 0\}$ is closed in $(0, T)$ and has no interior; otherwise, it would follow that $u \equiv 0$ and thus $Du$ is continuous on an open subset of $\Omega_T$. So $I = (0, T) \setminus C$ is open and dense in $(0, T)$, and $\|Du(\cdot, t)\|_{L^2(\Omega)} > 0$ for all $t \in I$. Thus (1.10) follows from (1.6) with $A = 0$ and $\phi(x) = u(x, t)$ for each $t \in I$. \qed

**Proposition 1.3** For the same functions $F$ as in Theorem 1.1, the problem (1.8) possesses infinitely many Lipschitz global weak solutions $u : \Omega \times [0, \infty) \to \mathbb{R}^2$ such that $Du$ is nowhere continuous on $\Omega \times (0, \infty)$. 
Proof Take $T = 1$ and $\phi = 0$ in Theorem 1.1 and let $u_0$ be a Lipschitz weak solution to the problem (1.8) as in the theorem. Then the full Dirichlet boundary condition $u_0|_{\partial \Omega_T} = 0$ is satisfied. Similarly, let $u_i$ be a Lipschitz weak solution of (1.2) with the full zero Dirichlet condition on $\Omega \times \{i + 1\}$ as given in Theorem 1.1 (by simply translating the $t$-variable) for $i = 1, 2, \cdots$. Finally we define function $u$ by gluing the functions $u_i$ together to obtain a global weak solution such that $Du$ is nowhere continuous on $\Omega \times (0, \infty)$. 

We point out that certain polyconvex functions $F$ of the form (1.7) have been constructed in [31] such that the stationary elliptic system

$$\text{div } DF(Du) = 0$$

has Lipschitz weak solutions that are not $C^1$ on any open subset of $\Omega$; this extends the result of [29] proved for the strongly quasiconvex functions $F$. For the time-dependent gradient flow problems, it has been shown in [27] that, given any function $F$ as constructed in [29, 31], for all $\epsilon > 0$ and $\alpha \in (0, 1)$ there exists a function $f \in C^\alpha(\Omega_T; \mathbb{R}^2)$ with $\|f\|_{C^\alpha(\Omega_T)} < \epsilon$ such that the initial-boundary value problem:

$$\begin{align*}
\partial_t u - \text{div } DF(Du) &= f \quad \text{in } \Omega_T; \\
u|_{\partial \Omega_T} &= 0,
\end{align*}$$

(1.11)

has a Lipschitz weak solution $u$ with $Du$ nowhere continuous in $\Omega_T$. The method of [27] relies on constructing a parametrized family of weak solutions $u(\cdot; t)$ of the stationary system $\text{div } DF(Du) = 0$ on $\Omega$ with $Du$ nowhere continuous; in the meantime, the construction ensures that $\|\partial_t u\|_{C^\alpha(\Omega_T)} < \epsilon$; finally, their example is provided by taking $f = \partial_t u$. Our result provides a much simpler counterexample for (1.11) with merely a fixed $f = 0$. We now discuss the main ideas for proving Theorem 1.1. Under the framework of [17, 28, 29], the general gradient flow (1.2) can be reformulated as a partial differential relation:

$$u = \text{div } v, \quad (Du, \partial_t v) \in \mathcal{K},$$

(1.12)

for functions $u : \Omega_T \rightarrow \mathbb{R}^m$ and $v : \Omega_T \rightarrow \mathbb{M}^{m \times n}$, where

$$\mathcal{K} = \{(A, DF(A)) \mid A \in \mathbb{M}^{m \times n}\}$$

is the graph of $DF$. Such a reformulation holds for all dimensions $n, m \geq 2$.

The relation $(Du, \partial_t v) \in \mathcal{K}$ is only a partial relation of the full differential relation for the space-time gradient $\nabla w = ( Dw, \partial_t w )$ of function $w = (u, v) : \Omega_T \rightarrow \mathbb{R}^m \times \mathbb{M}^{m \times n}$. This full differential relation for $\nabla w$ has been analyzed in [33] based on the rank-one hull of its defining matrix set; some useful structures, such as the $T_N$-configurations (see [23, 32]), of the rank-one hull are then projected to the partial relation $(Du, \partial_t v) \in \mathcal{K}$. In the 2-D case (when $m = n = 2$), such projected structures (called the special $\tau_N$-configurations in [33]) become the special $T_N$-configurations in $\mathbb{M}^{2 \times 2} \times \mathbb{M}^{2 \times 2}$. For the nonregularity of 2-D stationary elliptic systems $\text{div } DF(Du) = 0$, the strongly polyconvex functions $F$ constructed in [31] are based on the regular $T_5$-configurations in $\mathbb{M}^{2 \times 2} \times \mathbb{M}^{2 \times 2}$; on the other hand, for the nonuniqueness of the gradient flow problem (1.8), the strongly polyconvex functions $F$ constructed in [33, Section 6] are based on the more restrictive special $T_5$-configurations in $\mathbb{M}^{2 \times 2} \times \mathbb{M}^{2 \times 2}$. We point out that the functions $F$ constructed in [31] do not generally satisfy the special features of the functions $F$ constructed in [33].

For the more restrictive polyconvex functions $F$ constructed in [33], it has been proved using a Baire category method for the relation (1.12) that the initial boundary value problem (1.8) possesses infinitely many Lipschitz weak solutions; for such functions $F$’s, integral
convexity fails for the functional $\mathcal{E}$, as can be seen directly without using the results of [5]. However, the Baire category method used in [33] does not seem to produce a proof of the non-regularity for problem (1.8) as stated in Theorem 1.1. To prove the nonregularity, we follow the convex integration approach of [17, 27, 29, 31] to construct the Cauchy sequences $\{u_j\}$ in $W^{1,1}(\Omega_T; \mathbb{R}^2)$ consisting of certain subsolutions of the relation (1.12) (see Remark 4.1(i) below) and satisfying

$$u_j = \text{div} v_j, \quad \|DF(Du_j) - \partial_t v_j\|_{L^1(\Omega_T)} \to 0 \quad (1.13)$$

such that there is a uniform control on the local (essential) oscillations of $\{Du_j\}$. To accomplish such constructions, we greatly explore and modify the polyconvex functions $F$ constructed in [33] and carefully design an approximation scheme for the convex integration for (1.12) (see Theorem 4.1). To handle the linear constraint $u = \text{div} v$, we apply the antidivergence operator constructed in [4, 20] (see Sect. 3). We finally prove a more general h-principle type of result for (1.12) (see Theorem 5.1), from which our main result (Theorem 1.1) follows directly as a corollary.

To end this introduction, we remark in passing that the convex integration method has recently found remarkable success in the study of many important partial differential equation problems; see, e.g., [7, 8, 13, 19–22, 25, 30].

2 The polyconvex functions and associated properties

By modifying [31, Lemma 3], the function $F_0$ in [33, Proposition 6.2] can be constructed to further satisfy the condition

$$|DF_0(A)| \leq C(|A| + 1) \quad \forall A \in \mathbb{M}^{2\times 2}.$$ 

Therefore, we may assume that the smooth polyconvex function $F : \mathbb{M}^{2\times 2} \to \mathbb{R}$ of the form (1.7) constructed in [33, Definition 6.1] also satisfies the growth condition (1.9) required in Theorem 1.1. Let

$$\mathcal{K} = \{(A, DF(A)) \mid A \in \mathbb{M}^{2\times 2}\} \subset \mathbb{M}^{2\times 2} \times \mathbb{M}^{2\times 2}.$$ 

Rather than discussing the details of the construction of $F$, which we refer to [33], we summarize some properties about $F$ that are crucial for the convex integration of the partial differential relation

$$u = \text{div} v, \quad (Du, \partial_t v) \in \mathcal{K}. \quad (2.1)$$

From the constructions in [33, Definition 6.1], it follows that there exist smooth functions defined on a closed ball $\bar{B}_\beta = \bar{B}_\beta(0) \subset \mathbb{R}^8$,

$$\begin{cases}
\lambda_i : \bar{B}_\beta \to (0, 1), \\
\xi_i : \bar{B}_\beta \to \mathcal{K}, \\
\pi_i : \bar{B}_\beta \to \mathbb{M}^{2\times 2} \times \mathbb{M}^{2\times 2},
\end{cases} \quad (i = 1, \cdots, 5) \quad (2.2)$$

satisfying the following properties:

(P1) Each $\pi_i$ is invertible and thus $\pi_i(B_\beta)$ is open for $1 \leq i \leq 5$. Also $\pi_1(0) = (0, 0) \in \mathbb{M}^{2\times 2} \times \mathbb{M}^{2\times 2}$. Let $\mathbb{P} : \mathbb{M}^{2\times 2} \times \mathbb{M}^{2\times 2} \to \mathbb{M}^{2\times 2}$ be the projection operator: $\mathbb{P}(A, B) = A$. Then

$$\mathbb{P}\xi_i(B_\beta) \cap \mathbb{P}\xi_j(B_\beta) = \mathbb{P}\pi_i(B_\beta) \cap \mathbb{P}\pi_j(B_\beta) = \emptyset \quad \forall i \neq j.$$
Fig. 1 The points $X_i = \xi_i(q)$ and $P_i = \pi_i(q)$ in $\mathbb{M}^{2\times 2} \times \mathbb{M}^{2\times 2}$ for $1 \leq i \leq 5$ with a given $q \in \bar{B}_\beta$. Here $P_{i+1} = \lambda_i X_i + (1 - \lambda_i) P_i$ for $1 \leq i \leq 5$ with $P_6 = P_1$ and $\lambda_i = \lambda_i(q)$.

(P2) For all $1 \leq i \leq 5$ and $q \in \bar{B}_\beta$,

$$\pi_{i+1}(q) = \lambda_i(q) \xi_i(q) + (1 - \lambda_i(q)) \pi_i(q),$$

where we set $\pi_6(q) = \pi_1(q)$. There exist numbers $0 < \nu_0 < \nu_1 < 1$ such that

$$\nu_0 < \lambda_i(q) < \nu_1 \ \forall \ 1 \leq i \leq 5, \ q \in \bar{B}_\beta.$$

(P3) For $0 \leq \lambda \leq 1$ and $1 \leq i \leq 5$, define the sets in $\mathbb{M}^{2\times 2} \times \mathbb{M}^{2\times 2}$:

$$
\begin{align*}
S_i(\lambda) &= \{\lambda \xi_i(q) + (1 - \lambda) \pi_i(q) : q \in \bar{B}_\beta\}, \\
K(\lambda) &= \bigcup_{i=1}^5 S_i(\lambda), \quad \Sigma(0) = K(0), \\
\Sigma(\lambda) &= \bigcup_{0 < \lambda' < \lambda} K(\lambda') \ \forall \ 0 < \lambda < 1.
\end{align*}
$$

Then the following properties hold:

(a) $S_i(0) = \pi_i(\bar{B}_\beta)$ is open for each $1 \leq i \leq 5$ and the family $\{S_i(0)\}_{i=1}^5$ is disjoint (see Property (P1));

(b) $\Sigma(\lambda)$ is open for all $0 \leq \lambda \leq 1$ (see [33, Proof of Theorem 6.11]);

(c) there is a number $0 < \delta_1 < 1$ such that $\{S_i(\lambda)\}_{i=1}^5$ is a family of disjoint open sets for all $\delta_1 \leq \lambda < 1$ (see [33, Remark 6.3]).

(See Fig. 1 for a typical special $T_5$-configuration of points $X_i = \xi_i(q)$ and $P_i = \pi_i(q)$ in $\mathbb{M}^{2\times 2} \times \mathbb{M}^{2\times 2}$ for $1 \leq i \leq 5$ with a given $q \in \bar{B}_\beta$.)

The following elementary lemma, which follows by direct iteration, clarifies some computations below; we omit its proof.

**Lemma 2.1** If quantities $P_k$ and $X_k$, $1 \leq k \leq 5$, satisfy the relation $P_{k+1} = t_k X_k + (1 - t_k) P_k$ with $0 < t_k < 1$ for all $k$ modulo 5, then $P_i = \sum_{j=1}^5 v_{ij} X_j$ for all $1 \leq i \leq 5$, with coefficients
$v_i^j$ being determined for $1 \leq i, j \leq 5$ by
\[
v_k^{k+1} = \frac{t_k}{t}, \quad v_{k+1}^{k-1} = \frac{(1 - t_k)t_{k-1}}{t}, \quad v_{k+1}^{k-2} = \frac{(1 - t_k)(1 - t_{k-1})t_{k-2}}{t}, \quad v_{k+1}^{k-3} = \frac{(1 - t_k)(1 - t_{k-1})(1 - t_{k-2})t_{k-3}}{t}, \quad v_{k+1}^{k-4} = \frac{(1 - t_k)(1 - t_{k-1})(1 - t_{k-2})(1 - t_{k-3})t_{k-4}}{t},
\]
for all $k$ modulo 5, where $t = 1 - (1 - t_1) \cdots (1 - t_5)$.

A crucial property of the function $F$ is the following result, which is the building block for the convex integration of relation (2.1).

**Theorem 2.2** Let $Y = (A, B) \in S_i(\lambda)$ for some $1 \leq i \leq 5$ and $0 \leq \lambda < 1$. Let $1 > \mu' > \mu > \max\{\lambda, \nu_1, \delta_1\}$ be given numbers, where $\nu_1$ and $\delta_1$ are the numbers in (P2) and (P3). Assume $Y = \lambda \xi_i(q) + (1 - \lambda)\pi_i(q)$ for some $q \in B_\beta$. Let $X_k = \mu \xi_k(q) + (1 - \mu)\pi_k(q) \in S_k(\mu)$ for all $1 \leq k \leq 5$. We have
\[
\pi_{k+1}(q) = \frac{\lambda(k)}{\mu} X_k + \left(1 - \frac{\lambda(k)}{\mu}\right) \pi_k(q) \quad \forall k.
\]
By the lemma above, we have $\pi_k(q) = \sum_{j=1}^5 \tilde{v}_k^j(\mu, q) X_j$ with coefficients $\tilde{v}_k^j(\mu, q) \in (0, 1)$ satisfying
\[
\sum_{j=1}^5 \tilde{v}_k^j(\mu, q) = 1, \quad \tilde{v}_k^j(\mu, q) > (\mu - \nu_1)^4 v_0 \quad \forall 1 \leq k, j \leq 5,
\]
where $v_0$ is the number in (P2), and also
\[
Y = \left(\frac{\lambda}{\mu} + (1 - \frac{\lambda}{\mu}) \tilde{v}_i^j(\mu, q)\right) X_i + \sum_{1 \leq k \leq 5, k \neq i} \left(1 - \frac{\lambda}{\mu}\right) \tilde{v}_k^j(\mu, q) X_k.
\]
Then, for any bounded open set $G \subset \mathbb{R}^2 \times \mathbb{R}$ and $0 < \epsilon < 1$, there exists $(\varphi, \psi) \in C^\infty_c(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{M}^{2 \times 2})$ with support $\subset \subset G$ such that
(a) $\text{div} \, \varphi = 0$ and $(A + D\varphi, B + \partial_t \psi) \in \Sigma(\mu')$ on $G$;
(b) $\int_{\mathbb{R}^2} \varphi(x, t) \, dx = 0$ for all $t \in \mathbb{R}$;
(c) $\|\varphi\|_{L^\infty(G)} + \|\partial_t \varphi\|_{L^\infty(G)} < \epsilon$;
(d) there exist disjoint open subsets $G_1, \cdots, G_5$ of $G$ such that
\[
\begin{cases}
(A + D\varphi, B + \partial_t \psi)|_{G_k} = X_k, \\
|G_i| > (1 - \epsilon) \left(\frac{\lambda}{\mu} + (1 - \frac{\lambda}{\mu}) \tilde{v}_i^j(\mu, q)\right) |G|, \\
|G_k| > (1 - \epsilon)(1 - \frac{\lambda}{\mu}) \tilde{v}_k^j(\mu, q) |G| \quad \forall k \neq i;
\end{cases}
\]
in particular, $\sum_{j=1}^5 |G_j| > (1 - \epsilon)|G|$.

**Proof** The theorem follows from further special features of $F$ in terms of functions $\xi_i$ and $\pi_i$, in the similar way as in the Steps 1 and 2 of the proof of [33, Theorem 5.1] using Theorem 2.4 of the paper [33]; we refer the reader to that paper and omit the details here. \hfill $\Box$
3 The antidivergence operator

Throughout the rest of the paper, given a bounded open set \( G \subseteq \mathbb{R}^d \) and a function \( f \in W^{1,\infty}(G; \mathbb{R}^q) \), we define the Dirichlet class

\[
W_f^{1,\infty}(G; \mathbb{R}^q) = \{ g \in W^{1,\infty}(G; \mathbb{R}^q) \mid (g - f)|_{\partial G} = 0 \}.
\]

In this section, we construct a simple antidivergence operator on the rectangles in \( \mathbb{R}^2 \) to handle the linear constraint \( u = \text{div} \, v \) in the relation (2.1). See [20, Section 2.4] for the case of general dimensions and [4] for some deeper results.

Let \( J = (a_1, b_1) \times (a_2, b_2) \) be a rectangle in \( \mathbb{R}^2 \). For any function \( u = (u^1, u^2) : J \to \mathbb{R}^2 \), we define \( v = \mathcal{R}^J u : J \to \mathbb{M}^{2 \times 2} \) by setting \( v = (v^i_j(x)) \) for all \( x = (x_1, x_2) \in J \), with

\[
\begin{align*}
v^1_1(x) &= \rho(x_2) \int_{a_1}^{x_1} \left( \int_{a_2}^{b_2} u^1(r, s) ds \right) dr, \\
v^2_2(x) &= \int_{a_2}^{b_2} u^2(x_1, s) ds - \int_{a_2}^{b_2} \rho(s) ds \int_{a_2}^{b_2} u^1(x_1, s) ds
\end{align*}
\]

where \( \rho \in C^\infty_c(a_2, b_2) \) is a fixed function with \( \int_{a_2}^{b_2} \rho(s) ds = 1 \). We easily verify the following result.

**Lemma 3.1** The operator \( \mathcal{R}^J : C(\bar{J}; \mathbb{R}^2) \to C(\bar{J}; \mathbb{M}^{2 \times 2}) \) is a bounded linear operator and satisfies \( \text{div} \, v = u \) in \( J \). Moreover, if \( u \in W^{1,\infty}(\bar{J}; \mathbb{R}^2) \) then \( v = \mathcal{R}^J u \in W^{1,\infty}(\bar{J}; \mathbb{M}^{2 \times 2}) \); if \( u \in W^{1,\infty}_0(J; \mathbb{R}^2) \) satisfies \( \int_J u(x) dx = 0 \), then \( v = \mathcal{R}^J u \in W^{1,\infty}_0(\bar{J}; \mathbb{M}^{2 \times 2}) \). Furthermore, if \( u \in C^1(\bar{J}; \mathbb{R}^2) \) then \( v = \mathcal{R}^J u \) is in \( C^1(\bar{J}; \mathbb{M}^{2 \times 2}) \).

**Remark 3.1** Other antidivergence operators \( \mathcal{A} \) may be defined through certain unique solutions of Poisson’s equation \( \Delta f = \text{div} \, u \) on \( J \) by setting \( v = \mathcal{A} u := Df \). Although they may have the \( L^\infty-L^\infty \) bound on \( W^{1,\infty} \) functions, they don’t satisfy the boundary property that \( v = \mathcal{A} u \in W^{1,\infty}_0(J; \mathbb{M}^{2 \times 2}) \) for all \( u \in W^{1,\infty}_0(J; \mathbb{R}^2) \) satisfying \( \int_J u(x) dx = 0 \). Let \( Q_0 = (-1, 1)^2 \) be the cube in \( \mathbb{R}^2 \) and \( Q = Q_0 \times (-1, 1) \) be the cube in \( \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3 \).

**Lemma 3.2** Let \( u \in W^{1,\infty}_0(Q; \mathbb{R}^2) \) satisfy \( \int_{Q_0} u(x, t) dx = 0 \) for all \( t \in (-1, 1) \). Then there exists a function \( v = \mathcal{R} u \in W^{1,\infty}_0(Q; \mathbb{M}^{2 \times 2}) \) such that \( \text{div} \, v = u \) in \( Q \) and

\[
\| \partial_t v \|_{L^\infty(Q)} \leq C_0 \| \partial_t u \|_{L^\infty(Q)}.
\]

where \( C_0 \) is a constant. Moreover, if in addition \( u \in C^1(\bar{Q}; \mathbb{R}^2) \) then \( v = \mathcal{R} u \in C^1(\bar{Q}; \mathbb{M}^{2 \times 2}) \).

**Proof** Here we simply take \( \mathcal{R} u = \mathcal{R}^{Q_0} u(\cdot, t) \). The estimate (3.2) follows from \( \partial_t (\mathcal{R} u) = \mathcal{R}^{Q_0} (\partial_t u(\cdot, t)) \).

For \( y = (x, t) \in \mathbb{R}^2 \times \mathbb{R} \) and \( l > 0 \), let \( Q_{y,l} = y + lQ \) be the cube of center \( y \) and side length \( 2l \). For convenience, we denote the number \( l = \text{rad}(Q_{y,l}) \) as a radius of \( Q_{y,l} \), with a possible abuse of notation.

Define the rescaling operator \( \mathcal{L}_{y,l} : W^{1,\infty}(Q; \mathbb{R}^d) \to W^{1,\infty}(Q_{y,l}; \mathbb{R}^d) \) by setting

\[
(\mathcal{L}_{y,l} f)(z) = lf\left( \frac{z - y}{l} \right) \quad (z \in Q_{y,l}) \quad \forall f \in W^{1,\infty}(Q; \mathbb{R}^d).
\]

Note that \( \nabla (\mathcal{L}_{y,l} f)(z) = (\nabla f)(\frac{z - y}{l}) \) for \( z \in Q_{y,l} \), where \( \nabla = (D_x, \partial_t) \) if \( z = (x, t) \in \mathbb{R}^2 \times \mathbb{R} \). From the previous lemma, the following result is immediate.
Corollary 3.3 Let \( \varphi \in W^{1,\infty}_0(Q; \mathbb{R}^2) \) satisfy \( \int Q_\varphi(x, t)\, dx = 0 \) for all \( t \in (-1, 1) \). Let \( \bar{\varphi} = L_{y,t} \varphi \) and \( \bar{g} = R_{y,t} \varphi := L_{y,t}(R \varphi) \) in \( W^{1,\infty}_0(Q_{\bar{y},\bar{t}}; \mathbb{R}^{2 \times 2}) \). Then

\[
\text{div } \bar{g} = \bar{\varphi} \text{ in } Q_{\bar{y},\bar{t}}, \quad \| \partial_t \bar{g} \|_{L^\infty(Q_{\bar{y},\bar{t}})} \leq C_0 \| \partial_t \varphi \|_{L^\infty(Q_{y,t})}.
\]

Moreover, if in addition \( \varphi \in C^1(\bar{Q}; \mathbb{R}^2) \) then \( \bar{g} = R_{y,t} \varphi \in C^1(\bar{Q}_{\bar{y},\bar{t}}; \mathbb{R}^{2 \times 2}) \).

### 4 Convex integration and the main approximations

In this section, we prove the main approximations needed for the convex integration of relation (2.1).

**Theorem 4.1** Let \( G \subset \mathbb{R}^2 \times \mathbb{R} \) be a bounded open set and let \((u, v) \in C^1(\bar{G}; \mathbb{R}^2 \times \mathbb{R}^{2 \times 2})\) satisfy

\[
u = \text{div } v, \quad (Du, \partial_t v) \in \Sigma(\lambda') \text{ on } \bar{G}
\]

for some \( 0 < \lambda' < 1 \). Let \( 0 \leq \lambda < \lambda' < \mu < \mu' < 1 \) with \( \mu \) be given numbers such that

\[
\{S_k(\lambda)\}_{k=1}^5 \text{ is a family of disjoint open sets.}
\]

Then, for all \( 0 < \varepsilon < 1 \), there exists a function \((\bar{u}, \bar{v}) \in W^{1,\infty}_{(u,v)}(G; \mathbb{R}^2 \times \mathbb{R}^{2 \times 2})\) that is piecewise \( C^1 \) on at most countable partition of \( G \) by disjoint closed cubes \( \{Q_j\}_{j=1}^M \) \((M \leq \infty)\) such that

(a) \( 0 < \text{rad}(Q_j) < \varepsilon \) for all \( j \);

(b) \( \bar{u} = \text{div } \bar{v}, \ (D\bar{u}, \partial_t \bar{v}) \in \Sigma(\mu') \) on each \( \bar{Q}_j \);

(c) \( \| \bar{u} - u \|_{L^\infty(G)} + \| \partial_t \bar{u} - \partial_t u \|_{L^\infty(G)} < \varepsilon \);

(d) \( |Q_j \Delta \{(D\bar{u}, \partial_t \bar{v}) \in K(\mu)\}| > (1 - \varepsilon) |Q_j| \) for all \( j \);

(e) for all \( 1 \leq k \leq 5 \),

\[
\frac{1}{2} |(D\bar{u}, \partial_t \bar{v}) \in S_k(\mu')| > \frac{1}{2} |(D\bar{u}, \partial_t \bar{v}) \in S_k(\mu)|,
\]

(f) if \( G_3 = G \cap \{(Du, \partial_t v) \notin K(\lambda)\}, \) then

\[
|\|D\bar{u} - Du\|_{L^1(G)}| \leq C[|G_0| + (\varepsilon + (\mu - \lambda)) |G|],
\]

where \( C > 0 \) depends only on the polyconvex function \( F \).

**Remark 4.1** (i) A function \((u, v)\) satisfying (4.1) with \( 0 < \lambda' < 1 \) is called a (strict) *subsolution* of the relation (2.1) on \( G \).

(ii) The condition (4.2) seems necessary for the theorem; nevertheless, by Property (P3), it is always satisfied when \( \lambda = 0 \) or \( \lambda \geq \delta_1 \). This will be sufficient for the proof in the next section.

**Proof of Theorem 4.1** Let \( G_k = G \cap \{(Du, \partial_t v) \in S_k(\lambda)\} \) for \( 1 \leq k \leq 5 \). Then \( G_k \) is open for each \( 1 \leq k \leq 5 \) and the family \( \{G_k\}_{k=0}^5 \) is disjoint and covers \( G \), but some \( G_k \) may be empty. For each \( 1 \leq k \leq 5 \), we select another open set \( G'_k \subset G_k \) such that

\[
|\partial G'_k| = 0, \quad |G_k \setminus G'_k| \leq \varepsilon' |G_k|,
\]

where \( \varepsilon' > 0 \) is a constant.
where $0 < \epsilon' < \epsilon$ is a sufficiently small number to be determined later; thus $G'_k \neq \emptyset$ if and only if $G_k \neq \emptyset$. Define the open set

$$G'_0 = G \setminus \bigcup_{k=1}^{\infty} G'_k = G \setminus \bigcup_{k=1}^{\infty} \overline{G'_k}.$$  

Then $\{G'_j\}_{j=0}^{5}$ is a family of disjoint open sets in $G$ and, since $|\partial G'_k| = 0$,

$$G = \bigcup_{k=0}^{5} G'_k \cup N, \quad |N| = 0.$$

For each $\tilde{y} \in G$, the point $Y = (A, B) = (Du(\tilde{y}), \partial_t v(\tilde{y}))$ belongs to $\Sigma(\lambda')$; thus $Y \in S_1(\lambda'')$ for some $1 \leq i' \leq 5$ and $0 < \lambda'' < \lambda'$. In particular, if $\tilde{y} \in G_k$ for some $1 \leq k \leq 5$, then we set $i' = k$ and $\lambda'' = \lambda$. With this notation, we write $Y = \lambda'' \xi_{i'}(q') + (1 - \lambda'')\pi_{i'}(q')$ for some $q' \in B_\beta$. Let $X_j = \mu \xi_j(q') + (1 - \mu)\pi_j(q') \in S_j(\mu)$ for all $1 \leq j \leq 5$. We apply Theorem 2.2 to $Y$ and $\{X_j\}$ with $G$ being the cube $Q$ to obtain $(\varphi, \psi) \in C_c^\infty(Q; \mathbb{R}^2 \times \mathbb{M}^{2 \times 2})$ such that

(i) $\text{div } \psi = 0$ and $(A + D\varphi, B + \partial_t \psi) \in \Sigma(\mu')$ on $\tilde{Q}$;

(ii) $\int_{Q_0} \varphi(x, t) \, dx = 0$ for all $t \in (-1, 1)$;

(iii) $\|\varphi\|_{L^\infty(Q)} + \|\partial_t \psi\|_{L^\infty(Q)} < \epsilon'$;

(iv) there exist disjoint open subsets $P_1, \ldots, P_5$ of $Q$ such that

$$\begin{aligned}
(A + D\varphi, B + \partial_t \psi)|_{P_j} &= X_j \quad \forall 1 \leq j \leq 5, \\
|P_i| &= (1 - \epsilon') \left( \frac{\lambda''}{\mu} + (1 - \frac{\lambda''}{\mu})\nu_i'(\mu, q) \right) |Q|, \\
|P_j| &= (1 - \epsilon')(1 - \frac{\lambda''}{\mu})\nu_i'(\mu, q) |Q| \quad \forall j \neq i'.
\end{aligned} \quad (4.4)$$

Note that $\sum_{j=1}^{5} |P_j| > (1 - \epsilon') |Q|$. Moreover, since $\lambda'' < \lambda'$, by (2.3),

$$\begin{aligned}
|P_i| &= (1 - \epsilon') \left( \frac{\lambda''}{\mu} + (\mu - \lambda')(\mu - v_1)^4 v_0 \right) |Q|, \\
|P_j| &= (1 - \epsilon')(\mu - \lambda')(\mu - v_1)^4 v_0 |Q| \quad \forall 1 \leq j \leq 5.
\end{aligned} \quad (4.5)$$

Given any $0 < l < \epsilon$, let $(\tilde{\varphi}, \tilde{\psi}) = L_{\tilde{y}, l}(\varphi, \psi)$ and $\tilde{g} = R_{\tilde{y}, l}\psi$ be the rescaled functions on $\tilde{Q}_{\tilde{y}, l}$ as defined in Sect.3. Define

$$\tilde{u} = u_{\tilde{y}, l} = u + \tilde{\varphi}, \quad \tilde{v} = v_{\tilde{y}, l} = v + \tilde{\psi} + \tilde{g} \quad \text{on } \tilde{Q}_{\tilde{y}, l}. \quad (4.6)$$

Then, $\tilde{u} \in u + C_c^\infty(Q_{\tilde{y}, l}; \mathbb{R}^2)$, $\tilde{v} \in W_1^{1,\infty}(Q_{\tilde{y}, l}; \mathbb{M}^{2 \times 2}) \cap C^1(\tilde{Q}_{\tilde{y}, l})$, and

$$\begin{aligned}
\tilde{u} &= \text{div } \tilde{v}, \\
\|\tilde{u} - u\|_{L^\infty(Q_{\tilde{y}, l})} + \|\partial_t \tilde{u} - \partial_t u\|_{L^\infty(Q_{\tilde{y}, l})} &< \epsilon', \\
\|\partial_t \tilde{g}\|_{L^\infty(Q_{\tilde{y}, l})} &\leq C_0 l \|\partial_t \varphi\|_{L^\infty(Q)} < C_0 l.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left| (D\tilde{u}(y), \partial_t \tilde{v}(y)) - (A + D\varphi(\frac{y - \tilde{y}}{l}), B + \partial_t \psi(\frac{y - \tilde{y}}{l})) \right| \\
&\leq |Du(y) - Du(\tilde{y})| + |\partial_t v(y) - \partial_t v(\tilde{y})| + |\partial_t \tilde{g}(y)| \quad \text{on } \tilde{Q}_{\tilde{y}, l}.
\end{aligned}$$

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By the continuity of \((Du, \partial_t v)\) and the openness of sets \(G'_k\), \(\Sigma(\mu')\) and \(S_j(\mu)\), we select a sufficiently small \(l_y \in (0, \varepsilon)\) such that for all \(0 < l < l_y\),

\[
\left\{ \begin{aligned}
\tilde{Q}_{y,l} &\subset G'_k \text{ if } \tilde{y} \in G'_k \text{ for some } 0 \leq k \leq 5, \\
(Du, \partial_t v)|_{\tilde{Q}_{y,l}} &\in B_\varepsilon(Y), \quad (D\tilde{u}, \partial_t \tilde{v})|_{\tilde{Q}_{y,l}} \in \Sigma(\mu'), \\
(D\tilde{u}, \partial_t \tilde{v})|_{\tilde{P}_j} &\in S_j(\mu) \cap B_\varepsilon(X_j) \quad \forall \ 1 \leq j \leq 5,
\end{aligned} \right.
\tag{4.7}
\]

where \(\tilde{P}_j\)'s are the disjoint open sets of \(\tilde{Q}_{y,l}\) obtained by rescaling \(P_j\). Hence,

\[
\begin{align*}
|Q_{y,l}| &\geq \left| \bigcup_{j=1}^S \tilde{P}_j \right| > (1 - \varepsilon') |Q_{y,l}|, \\
|\tilde{P}_j| &> (1 - \varepsilon') \left( \frac{\varepsilon'}{\mu} + (\mu - \lambda')(\mu - \nu_1)^4 v_0 \right) |Q_{y,l}|, \\
|\tilde{P}_j| &> (1 - \varepsilon')(\mu - \lambda')(\mu - \nu_1)^4 v_0 |Q_{y,l}| \quad \forall \ 1 \leq j \leq 5.
\end{align*}
\tag{4.8}
\]

For each \(0 \leq k \leq 5\) with \(G_k \neq \emptyset\), the nonempty open set \(G'_k\) is covered by the family of closed cubes \(\{\tilde{Q}_{y,l} \mid \tilde{y} \in G'_k, \ 0 < l < l_y\}\) in the sense of Vitali covering (see [10]); thus, we have

\[G'_k = \left( \bigcup_{n=1}^{m_k} \tilde{Q}^k_n \right) \cup N_k, \quad 1 \leq m_k \leq \infty, \quad |N_k| = 0,\]

where the cubes \(\tilde{Q}^k_n = \tilde{Q}^k_{y^k_n,l^k_n} \subset G'_k\) are such that \(y^k_n \in G'_k\), \(0 < l^k_n = \text{rad}(Q^k_n) < \varepsilon\) and \(\tilde{Q}^k_n \cap \tilde{Q}^j_m = \emptyset\) for all \(n \neq m\). Clearly if \(G_j \neq \emptyset\) and \(j \neq k\) then \(\tilde{Q}^k_n \cap \tilde{Q}^j_m = \emptyset\) for all \(1 \leq n \leq m_k\) and \(1 \leq m \leq m_j\). We thus achieve an at most countable partition of \(G\) by disjoint closed cubes:

\[G = \left( \bigcup_{k=0}^5 \bigcup_{n=1}^{m_k} \tilde{Q}^k_n \right) \cup N, \quad |N| = 0,\]

where we set \(m_k = 0\) if \(G_k = \emptyset\) and drop the \(k\)-terms from the union.

For each \(0 \leq k \leq 5\) with \(G_k \neq \emptyset\), let \(\tilde{u}^k_n = u_{y^k_n,l^k_n}\) and \(\tilde{v}^k_n = v_{y^k_n,l^k_n}\) be the functions defined by (4.6) on \(\tilde{Q}^k_n = \tilde{Q}^k_{y^k_n,l^k_n}\) with center \(y^k_n \in G'_k\) and \(\text{rad}(Q^k_n) < \varepsilon\). Define

\[(\tilde{u}, \tilde{v}) = \sum_{k=0}^5 \sum_{n=1}^{m_k} (\tilde{u}^k_n, \tilde{v}^k_n) \chi_{\tilde{Q}^k_n} + (u, v) \chi_N,\]

where again no \(k\)-terms are in the summation if \(m_k = 0\). Then \((\tilde{u}, \tilde{v}) \in W^{1,\infty}_{(u,v)}(G; \mathbb{R}^2 \times \mathbb{M}^{2 \times 2})\) and \((\tilde{u}, \tilde{v})|_{\tilde{Q}^k_n} \in C^1(\tilde{Q}^k_n; \mathbb{R}^2 \times \mathbb{M}^{2 \times 2})\) for all \(0 \leq k \leq 5\) and \(1 \leq n \leq m_k\). We define the cubes \(\{\tilde{Q}^k_j\}_{j=1}^M\) by re-indexing:

\[\{\tilde{Q}^k_n \mid 0 \leq k \leq 5, \ 1 \leq n \leq m_k\} = \{\tilde{Q}^k_j \mid 1 \leq j \leq M\}.\]

It is easily seen that the requirements (a)-(c) are satisfied; the requirement (d) also follows readily from the first line of (4.8).

To satisfy the first line of the requirement (e), note that by the third line of (4.8) we have

\[|(Du, \partial_t v)\in S_j(\mu)| \geq (1 - \varepsilon')(\mu - \lambda')(\mu - \nu_1)^4 v_0 \sum_{k=0}^5 \sum_{n=1}^{m_k} |Q^k_n|.
\]

\[= (1 - \varepsilon')(\mu - \lambda')(\mu - \nu_1)^4 v_0 |G|\]
which holds if the number \( \epsilon' \in (0, \epsilon) \) is chosen to satisfy \( \epsilon' < 1/2 \).

The second line of the requirement (e) is automatically satisfied if \( G_k = \emptyset \). Now assume \( 1 \leq k \leq 5 \) with \( G_k \neq \emptyset \). In this case, we have \( i' = k \) and \( \lambda'' = \lambda \) in (4.5) and thus, by (4.3) and the second line of (4.8)

\[
|\{(D\tilde{u}, \partial_t \tilde{v}) \in S_k(\mu)\}| \geq \sum_{n=1}^{m_k} |\tilde{Q}_n^k \cap \{(D\tilde{u}_n^k, \partial_t \tilde{v}_n^k) \in S_k(\mu)\}|
\]

\[
> (1 - \epsilon') \left( \frac{\lambda}{\mu} + (\mu - \lambda')(\mu - v_1)^4 v_0 \right) \sum_{n=1}^{m_k} |Q_n^k|
\]

\[
= (1 - \epsilon') \left( \frac{\lambda}{\mu} + (\mu - \lambda')(\mu - v_1)^4 v_0 \right) |G_k'|
\]

\[
\geq (1 - \epsilon')^2 \left( \frac{\lambda}{\mu} + (\mu - \lambda')(\mu - v_1)^4 v_0 \right) |G_k|.
\]

Thus the second line of the requirement (e) is ensured if the number \( \epsilon' \in (0, \epsilon) \) is chosen to further satisfy

\[
(1 - \epsilon')^2 \left( \frac{\lambda}{\mu} + (\mu - \lambda')(\mu - v_1)^4 v_0 \right) > \frac{\lambda}{\mu}, \quad (4.11)
\]

which is possible because \( (\mu - \lambda')(\mu - v_1)^4 v_0 > 0 \). Therefore, both requirements in (e) are ensured.

Finally, to verify the requirement (f), note that

\[
\|D\tilde{u} - Du\|_{L^1(G)} = \sum_{k=0}^{5} \|D\tilde{u} - Du\|_{L^1(G_k)}
\]

\[
\leq C|G_0| + \sum_{k=1}^{5} \|D\tilde{u} - Du\|_{L^1(G_k)}
\]

\[
\leq C|G_0| + \sum_{k=1}^{5} C|G_k \setminus G_k'| + \sum_{k=1}^{5} \|D\tilde{u} - Du\|_{L^1(G_k')}
\]

\[
\leq C(|G_0| + \epsilon'|G|) + \sum_{k=1}^{5} \sum_{n=1}^{m_k} \int_{Q_n^k} |(D\tilde{u}, \partial_t \tilde{v}) - (Du, \partial_t v)|
\]

\[
\leq C(|G_0| + \epsilon|G|) + \sum_{k=1}^{5} \sum_{n=1}^{m_k} \int_{\tilde{P}_n^k} |(D\tilde{u}, \partial_t \tilde{v}) - (Du, \partial_t v)|
\]

\[
+ \sum_{k=1}^{5} \sum_{n=1}^{m_k} \int_{\tilde{Q}_n^k \setminus \tilde{P}_n^k} |(D\tilde{u}, \partial_t \tilde{v}) - (Du, \partial_t v)|,
\]

where \( \tilde{P}_n^k \subset Q_n^k \) for all \( 1 \leq k \leq 5 \) and \( 1 \leq n \leq m_k \) are the sets defined as in (4.7) with \( Y = Y_n = (Du(\tilde{y}_n), \partial_t v(\tilde{y}_n)) \in S_k(\lambda) \) and with \( i' = k \) and \( \lambda'' = \lambda \) in (4.8). By (4.11),

\[
(1 - \epsilon') \left( \frac{\lambda}{\mu} + (\mu - \lambda')(\mu - v_1)^4 v_0 \right) > \frac{\lambda}{\mu},
\]
and thus $|\tilde{P}_n^k| > \frac{1}{\mu} |Q_n^k|$. Hence
\[
\int_{Q_n^k \setminus \tilde{P}_n^k} |(Du, \partial_t \tilde{v}) - (Du, \partial_t v)| \leq C |Q_n^k \setminus \tilde{P}_n^k| < C(\mu - \lambda) |Q_n^k|.
\] (4.13)

Let $X_k = X_n^k \in S_k(\mu)$ be defined as in (4.7) with $Y = Y_n$ by some $q_n \in B_\beta$. Then $|X_n^k - Y_n| = (\mu - \lambda)|\xi_k(q_n) - \pi_k(q_n)| \leq C(\mu - \lambda)$; thus, by (4.7) and (4.8), we have
\[
\int_{\tilde{P}_n^k} |(Du, \partial_t \tilde{v}) - (Du, \partial_t v)|
\leq \int_{\tilde{P}_n^k} \left( |(Du, \partial_t \tilde{v}) - X_n^k| + |X_n^k - Y_n| + |(Du, \partial_t v) - Y_n| \right)
\leq C(\epsilon + (\mu - \lambda)) |Q_n^k|.
\] (4.14)

Plugging (4.13) and (4.14) into (4.12) proves the requirement (f).

The proof is completed.

5 Proof of the main theorem

Given a function $f \in L^\infty(G; \mathbb{R}^q)$ on an open set $G \subset \mathbb{R}^d$, for any $x_0 \in G$, we define the essential oscillation of $f$ at $x_0$ by
\[
\omega_f(x_0) = \inf \{ \| f(x) - f(y) \|_{L^\infty(U \times U)} \mid U \subset G, \ U \text{ open}, \ x_0 \in U \}.
\]
We say that $f$ is essentially continuous at $x_0$ if $\omega_f(x_0) = 0$. Clearly, if $f$ is continuous at $x_0$ then it is essentially continuous at $x_0$, but the converse is false, as easily seen by the Dirichlet function on $\mathbb{R}$.

In this final section, we prove the following more general theorem.

Theorem 5.1 Assume the polyconvex function $F$ is defined as above. Let $(\bar{u}, \bar{v}) \in C^1(\bar{\Omega}_T; \mathbb{R}^2 \times \mathbb{M}^{2 \times 2})$ satisfy
\[
\bar{u} = \text{div} \, \bar{v}, \quad (D\bar{u}, \partial_t \bar{v}) \in \Sigma(\tilde{\lambda}) \quad \text{on} \ \bar{\Omega}_T
\] (5.1)
for some $0 < \tilde{\lambda} < 1$. Then, for each $\rho > 0$, the problem
\[
\begin{cases}
\partial_t u = \text{div} \, D\bar{u} \quad \text{on} \ \Omega_T; \\
u|_{\partial \Omega_T} = \bar{u}
\end{cases}
\] (5.2)
possesses a Lipschitz weak solution $u$ such that $\| u - \bar{u} \|_{L^\infty} + \| \partial_t u - \partial_t \bar{u} \|_{L^\infty} < \rho$ and that $Du$ is not essentially continuous at any point; in fact, $\omega_{Du}(y_0) \geq \delta$ for all $y_0 \in \Omega_T$, where $\delta > 0$ is a uniform constant.

Remark 5.1 (i) This result is a type of $h$-principle for the gradient flow (1.2) in the sense of [17].

(ii) The theorem asserts that the weak solutions exist in every $L^\infty$ neighborhood of $\bar{u}$; therefore, such solutions must be infinitely many.
5.1 Proof of Theorem 1.1

Before proving Theorem 5.1, we show that it implies our main theorem as a corollary. For example, let \( \phi \in C_0^1(\Omega; \mathbb{R}^2) \) be given as in Theorem 1.1. Let \( J \subset \mathbb{R}^2 \) be a rectangle containing \( \Omega \) and extend \( \phi \) to \( J \) by zero. Let \( h = R^j \phi \in C^1(\tilde{J}; \mathbb{M}^{2\times 2}) \) and define

\[
\tilde{u}(x, t) = \frac{\epsilon}{T} \phi(x)t, \quad \tilde{v}(x, t) = \frac{\epsilon}{T} h(x)t. \tag{5.3}
\]

Then, \( (\tilde{u}, \tilde{v}) \in C^1(\tilde{\Omega}_T; \mathbb{R}^2 \times \mathbb{M}^{2\times 2}) \), \( \tilde{u} = \text{div} \, \tilde{v} \) on \( \tilde{\Omega}_T \), \( \tilde{u}|_{|\tilde{v}|\Omega_T} = 0 \) and

\[
\|(D\tilde{u}, \partial_t \tilde{v}) - (0, 0)\|_{L^\infty} \leq |\epsilon|\|(D\phi)\|_{L^\infty} + \frac{1}{T}\|h\|_{L^\infty}).
\]

Since \( (0, 0) = \pi_1(0) \in \pi_1(B_\beta) = S_1(0) \), which is open in \( \mathbb{M}^{2\times 2} \times \mathbb{M}^{2\times 2} \), we select a number \( \epsilon_0 > 0 \) depending only on \( \phi \) such that

\[
(D\tilde{u}, \partial_t \tilde{v}) \in S_1(0) \subset \Sigma(\tilde{\lambda}) \quad \forall |\epsilon| < \epsilon_0, \tag{5.4}
\]

where \( 0 < \tilde{\lambda} < 1 \) is any fixed number. Then, for all \( |\epsilon| < \epsilon_0 \), with the strict subsolution \( (\tilde{u}, \tilde{v}) \) defined by (5.3), Theorem 1.1 follows from Theorem 5.1.

5.2 Proof of Theorem 5.1

First we select sequences \( \{\lambda_n\}, \{\lambda'_n\} \) and \( \{\epsilon_n\} \) such that

\[
\begin{align*}
\lambda_0 &= 0, \quad \lambda_1 = \max\{\tilde{\lambda}, \delta_1, \nu_1\}, \\
\lambda_{n+1} &= \lambda_n, \quad \lim_{n \to \infty} \lambda_n = 1; \\
\lambda'_0 &= \tilde{\lambda}, \quad \lambda'_n = \frac{1}{2}(\lambda_n + \lambda_{n+1}) \quad \forall n = 1, 2, \ldots; \\
\epsilon_n &= \rho/3^n.
\end{align*}
\]

(5.5)

Since \( \{S_i(0)\}_{i=1}^5 \) is a family of disjoint open sets, we can apply Theorem 4.1 to \( (u, v) = (\tilde{u}, \tilde{v}) \) on the set \( G = \Omega_T \) with

\[
\lambda = \lambda_0, \quad \lambda' = \lambda'_0, \quad \mu = \lambda_1, \quad \mu' = \lambda'_1, \quad \epsilon = \epsilon_1
\]

to obtain a function \( (u_1, v_1) = (\tilde{u}, \tilde{v}) \in W^{1,\infty}(\Omega_T; \mathbb{R}^2 \times \mathbb{M}^{2\times 2}) \) that is piece-wise \( C^1 \) on at most countable partition of \( \Omega_T \) by disjoint closed cubes \( \{Q_j^1\}_{j=1}^{M_1} (M_1 \leq \infty) \) such that

\[
\begin{align*}
0 &< \text{rad}(Q_j^1) < \epsilon_1; \\
u_1 &= \text{div} \, v_1, \quad (Du_1, \partial_t v_1) \in \Sigma(\lambda'_j) \quad \text{on each } Q_j^1; \\
\|u_1 - \tilde{u}\|_{L^\infty(\Omega_T)} + \|\partial_t u_1 - \partial_t \tilde{u}\|_{L^\infty(\Omega_T)} &< \epsilon_1; \\
|Q_j^1| \cap \{|(Du_1, \partial_t v_1) \in K(\lambda_1)|\} &> (1 - \epsilon_1)|Q_j^1| \quad \forall 1 \leq j \leq M_1. 
\end{align*}
\]

(5.6)

We now construct \( (u_n, v_n) \) for all \( n \geq 2 \) by induction. Suppose that for a fixed \( n \geq 1 \) we have constructed \( (u_n, v_n) \in W^{1,\infty}(\Omega_T; \mathbb{R}^2 \times \mathbb{M}^{2\times 2}) \) that is piece-wise \( C^1 \) on at most countable partition of \( \Omega_T \) by disjoint closed cubes \( \{Q_j^n\}_{j=1}^{M_n} (M_n \leq \infty) \).

Since \( \lambda_n > \delta_1 \), by Property (P3)(c), \( \{S_i(\lambda_n)\}_{i=1}^5 \) is a family of disjoint open sets, and thus we can apply Theorem 4.1 to \( (u, v) = (u_n, v_n) \) on the set \( G = Q_j^n \) for each \( 1 \leq j \leq M_n \) with

\[
\lambda = \lambda_n, \quad \lambda' = \lambda'_n, \quad \mu = \lambda_{n+1}, \quad \mu' = \lambda'_{n+1}, \quad \epsilon = \epsilon_{n+1},
\]

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to obtain a function \((\tilde{u}^{n,j}, \tilde{v}^{n,j}) \in W^{1,\infty}(Q^n_j; \mathbb{R}^2 \times \mathbb{M}^{2 \times 2})\) that is piece-wise \(C^1\) on at most countable partition of \(Q^n_j\) by disjoint closed cubes \(\{Q^n_{j,M_{n,j}}\}_{i=1}^{M_{n,j}}\) \((M_{n,j} \leq \infty)\) such that

\[
\begin{align*}
0 &< \text{rad}(Q^n_{j,i}) < \epsilon_{n+1} \quad \forall i; \\
\tilde{u}^{n,j} &\in \partial v^{n,j} \in \Sigma(\lambda'_{n+1}) \quad \text{on } Q^n_{j,i}; \\
||\tilde{u}^{n,j} - u_n||_{L^\infty(Q^n_j)} + ||\partial_t \tilde{u}^{n,j} - \partial_t u_n||_{L^\infty(Q^n_j)} &< \epsilon_{n+1}; \\
|Q^n_{j,i} \cap ((D\tilde{u}^{n,j}, \partial_t \tilde{v}^{n,j}) \in K(\lambda_{n+1}))| &> (1 - \epsilon_{n+1}) |Q^n_{j,i}| \quad \forall i; \\
|D\tilde{u}^{n,j} - Du_n||_{L^1(Q^n_j)} &\leq C [\epsilon_n + \epsilon_{n+1} + (\lambda_{n+1} - \lambda_n)] |Q^n_j|; \\
\end{align*}
\]

(5.7)

\[
\begin{align*}
|Q^n_{j,i} \cap ((D\tilde{u}^{n,j}, \partial_t \tilde{v}^{n,j}) \in S_k(\lambda_{n+1}))| \\
> \frac{1}{4} (\lambda_{n+1} - \lambda_n)(\lambda_{n+1} - v_1)^d v_0 |Q^n_j| \quad \forall 1 \leq k \leq 5; \\
|Q^n_{j,i} \cap ((D\tilde{u}^{n,j}, \partial_t \tilde{v}^{n,j}) \in S_k(\lambda_{n+1}))| \\
> \frac{1}{4} (\lambda_{n+1} - \lambda_n)(\lambda_{n+1} - v_1)^d v_0 |Q^n_j| \quad \forall 1 \leq k \leq 5.
\end{align*}
\]

Here the fifth line of (5.7) follows from the part (f) of Theorem 4.1 with the set \(G_0 = Q^n_j \cap \{(Du_n, \partial_t v_n) \notin K(\lambda_{n+1})\}\), which satisfies \(|G_0| < \epsilon_n |Q^n_j|\) by the induction assumption:

\[
|Q^n_{j,i} \cap ((Du_n, \partial_t v_n) \in K(\lambda_{n+1}))| > (1 - \epsilon_n) |Q^n_j|
\]

(see the last line of (5.6) for \(n = 1\)).

We glue all cubes \(\{Q^n_{j,M_{n,j}}\}_{j=1}^{M_n}\) together to have a partition of \(\Omega_T\) with

\[
\Omega_T = \left( \bigcup_{j=1}^{M_n} \tilde{Q}^n_j \right) \cup N_n = \left( \bigcup_{j=1}^{M_n} Q^n_j \right) \cup \tilde{N}_n, \quad \tilde{N}_n = \left( \bigcup_{j=1}^{M_n} \partial Q^n_j \right) \cup N_n,
\]

where \(|N_n| = |\tilde{N}_n| = 0\). Define function \((u_{n+1}, v_{n+1}) \in W^{1,\infty}(\Omega_T; \mathbb{R}^2 \times \mathbb{M}^{2 \times 2})\) by setting

\[
(u_{n+1}, v_{n+1}) = \sum_{j=1}^{M_n} (\tilde{u}^{n,j}, \tilde{v}^{n,j}) \chi_{\tilde{Q}^n_j} + (u_n, v_n) \chi_{\tilde{N}_n}
\]

(5.8)

and family of closed cubes \(\{\tilde{Q}^{n+1}_h\}_{h=1}^{h_{n+1}}\) by simply re-indexing:

\[
\{\tilde{Q}^{n+1}_h \mid 1 \leq j \leq M_n, 1 \leq i \leq M_{n,j} \} = \{\tilde{Q}^{n+1}_h \mid 1 \leq h \leq M_{n+1}\}.
\]

It is easily seen that \((u_{n+1}, v_{n+1}) \in W^{1,\infty}(\Omega_T; \mathbb{R}^2 \times \mathbb{M}^{2 \times 2})\), is piece-wise \(C^1\) on at most countable partition of \(\Omega_T\) by disjoint closed cubes \(\{\tilde{Q}^{n+1}_h\}_{h=1}^{h_{n+1}}\) \((M_{n+1} \leq \infty)\), and satisfies the following properties:

\[
0 < \text{rad}(\tilde{Q}^{n+1}_h) < \epsilon_{n+1} \quad \forall h;
\]

(5.9)

\[
u_{n+1} = \text{div}(v_n, (Du_{n+1}, \partial_t v_{n+1}) \in N(\lambda_{n+1}) \quad \text{on } \tilde{Q}^{n+1}_h;
\]

(5.10)

\[
||u_{n+1} - u_n||_{L^\infty(\Omega_T)} + ||\partial_t u_{n+1} - \partial_t u_n||_{L^\infty(\Omega_T)} < \epsilon_{n+1};
\]

(5.11)

\[
\{(Du_{n+1}, \partial_t v_{n+1}) \in K(\lambda_{n+1})\} > (1 - \epsilon_{n+1}) |\Omega_T|;
\]

(5.12)

\[
\{(Du_{n+1}, \partial_t v_{n+1}) \in S_k(\lambda_{n+1})\} > (1 - \epsilon_{n+1}) |\Omega_T|;
\]

(5.13)

\[
|\tilde{Q}^{n+1}_h \cap (\tilde{Q}^{n+1}_h, \partial_t v_{n+1}) \in S_k(\lambda_{n+1})|\]

(5.14)
\[
> \frac{\lambda_n}{\lambda_{n+1}} |Q_j^n \cap \{(Du_n, \partial_tv_n) \in S_k(\lambda_n)\}| \forall 1 \leq j \leq M_n, 1 \leq k \leq 5. \quad (5.15)
\]

**Lemma 5.2** For all \( n > m \geq 1 \) and \( 1 \leq j \leq M_m \), we have

\[
|Q_j^m \cap \{(Du_{n+1}, \partial_tv_{n+1}) \in S_k(\lambda_{n+1})\}| > \frac{1}{4} \frac{\lambda_{m+1}}{\lambda_{n+1}} (\lambda_{m+1} - \lambda_m)(\lambda_{m+1} - v_1)^4 v_0 |Q_j^m|,
\]

for all \( 1 \leq k \leq 5 \).

**Proof** By the construction of sequence \( (u_n, v_n)_{n=1}^{\infty} \) and the cubes \( \{Q_j^n\} \) above, we see that if \( n > m \geq 1 \) and \( 1 \leq j \leq M_m \) then \( Q_j^m = (\bigcup_{k \in I} Q_k^n) \cup N \) for some index set \( I \) and a null-set \( N \). Thus, for all \( n > m \geq 1 \), \( 1 \leq j \leq M_m \) and \( 1 \leq k \leq 5 \), by (5.14) and (5.15) we have

\[
|Q_j^m \cap \{(Du_{n+1}, \partial_tv_{n+1}) \in S_k(\lambda_{n+1})\}| > \frac{1}{4} \frac{\lambda_{m+1}}{\lambda_{n+1}} (\lambda_{m+1} - \lambda_m)(\lambda_{m+1} - v_1)^4 v_0 |Q_j^m|.
\]

\[\Box\]

**Lemma 5.3** Let \( f(X) = DF(A) - B \) for \( X = (A, B) \in \mathbb{M}^{2 \times 2} \times \mathbb{M}^{2 \times 2} \). Then, there is a constant \( C > 0 \) such that

\[
|f(X)| \leq C (1 - \lambda) \quad \forall 0 \leq \lambda \leq 1, \ X \in K(\lambda).
\]

**Proof** Since \( F \) is smooth, \( f \) is locally Lipschitz and thus \( |f(X_1) - f(X_2)| \leq L |X_1 - X_2| \) for all \( X_1, X_2 \in \Sigma(1) \), where \( L > 0 \) is a constant. Let \( X \in K(\lambda) \). Then \( X = \lambda \xi_i(q) + (1 - \lambda)\pi_i(q) \) for some \( 1 \leq i \leq 5 \) and \( q \in B_\beta \). Note that \( \xi_i(q) \in \mathcal{K} \) and thus \( f(\xi_i(q)) = 0 \); hence \( |f(X)| \leq L |X - \xi_i(q)| = L (1 - \lambda) |\xi_i(q) - \pi_i(q)| \leq C (1 - \lambda) \). \[\Box\]

We are now in a position to complete the proof of Theorem 5.1.

By (5.11) and (5.13), \( (u_n) \) is a Cauchy sequence in \( W^{1,1}(\Omega_T; \mathbb{R}^2) \). Let \( u_n \to u \) in \( W^{1,1}(\Omega_T; \mathbb{R}^2) \) as \( n \to \infty \). Clearly, \( u \) also belongs to \( W^{1,\infty}_u(\Omega_T; \mathbb{R}^2) \). Moreover, by (5.11), we have

\[
\|u - \bar{u}\|_{L^\infty(\Omega_T)} + \|\partial_tv - \partial_tv\|_{L^\infty(\Omega_T)} \leq \sum_{n=0}^{\infty} \|u_{n+1} - u_n\|_{L^\infty(\Omega_T)} + \|\partial_tv_{n+1} - \partial_tv_n\|_{L^\infty(\Omega_T)} \leq \sum_{n=0}^{\infty} \epsilon_n + \frac{1}{2} \rho < \rho.
\]
By (5.12) and Lemma 5.3, we have
\[ \int_{\Omega_T} |f(Du_{n+1}, \partial_t v_{n+1})| \leq C[(1 - \lambda_{n+1}) + \epsilon_{n+1}]|\Omega_T| ; \]
thus \( \| f(Du_{n+1}, \partial_t v_{n+1}) \|_{L^1(\Omega_T)} \to 0 \) as \( n \to \infty \). From \( u_{n+1} = \text{div} v_{n+1} \) a.e. on \( \Omega_T \), it follows easily that
\[ \int_{\Omega} (u_{n+1} \cdot \varphi) \bigg|_{t=0}^{t=T} - \int_{\Omega_T} (u_{n+1} \cdot \partial_t \varphi - DF(Du_{n+1}) : D\varphi) \]
\[ = \int_{\Omega_T} (DF(Du_{n+1}) - \partial_t v_{n+1}) : D\varphi \quad \forall \varphi \in C^1([0, T]; C_c^\infty(\Omega; \mathbb{R}^2)). \] (5.16)

Since \( u_{n+1} \to u \) in \( W^{1,1}(\Omega_T) \), \( f(Du_{n+1}, \partial_t v_{n+1}) \to 0 \) in \( L^1(\Omega_T) \) both strongly as \( n \to \infty \) and \( \{ \| Du_{n+1} \|_{L^\infty(\Omega_T)} \} \) is bounded, taking the limits in (5.16), we have that
\[ \int_{\Omega} (u \cdot \varphi) \bigg|_{t=0}^{t=T} - \int_{\Omega_T} (u \cdot \partial_t \varphi - DF(Du) : D\varphi) = 0 \]
holds for all \( \varphi \in C^1([0, T]; C_c^\infty(\Omega; \mathbb{R}^2)) \). This proves that \( u \) is a weak solution of (5.2).

Finally, to show \( Du \) is nowhere essentially continuous on \( \Omega_T \), let \( y_0 \in \Omega_T \) and \( U \) be any open subset of \( \Omega_T \) containing \( y_0 \). From
\[ U \cap \bigcap_{n=1}^\infty \cup_{j=1}^{M_n} \tilde{Q}_j^n \neq \emptyset \]
and \( \text{rad}(Q_j^n) < \epsilon_n \to 0 \), it follows that there exist some \( m \geq 1 \) and \( 1 \leq j \leq M_m \) such that \( \tilde{Q}_j^m \subset U \).

Recall the projection operator: \( P(A, B) = A \). By Lemma 5.2, we have for all \( n > m \) and \( 1 \leq k \leq 5 \),
\[ |Q_j^m \cap \{ Du_{n+1} \in P(S_k(\lambda_{n+1})) \}| \]
\[ \geq |Q_j^m \cap \{ (Du_{n+1}, \partial_t v_{n+1}) \in S_k(\lambda_{n+1}) \}| \]
\[ \geq \frac{1}{4} \frac{\lambda_{m+1}}{\lambda_{n+1}} (\lambda_{m+1} - \lambda_m)(\lambda_{m+1} - v_1)^4 v_0 |Q_j^m| \]
Taking the limits as \( n \to \infty \), since \( u_{n+1} \to u \) in \( W^{1,1}(\Omega_T; \mathbb{R}^2) \), we have
\[ |Q_j^m \cap \{ Du \in P(S_k(1)) \}| \geq \frac{1}{4} \frac{\lambda_{m+1}}{\lambda_{n+1}} (\lambda_{m+1} - \lambda_m)(\lambda_{m+1} - v_1)^4 v_0 |Q_j^m| > 0 \]
for all \( 1 \leq k \leq 5 \). Let
\[ \delta = \min_{k \neq l} \text{dist}(P(S_k(1)); P(S_l(1))). \]
By Property (P1), we have \( \delta > 0 \). Thus \( \| Du(y) - Du(z) \|_{L^\infty(U \times U)} \geq \delta > 0 \) for all open sets \( U \subset \Omega_T \) containing \( y_0 \); this proves
\[ \omega_{Du}(y_0) \geq \delta > 0. \]

The proof of Theorem 5.1 is now completed.
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