SUPERCONNECTED LEFT QUASIGROUPS AND INVOLUTORY QUANDLES

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Abstract. In this paper we study the classes of superconnected and superfaithful left quasigroups, that are relevant in the study of Mal’cev varieties of left quasigroups [7]. Then we focus on quandles and in particular to the involutory ones. We extend the main result of [13] to the infinite case and we offer a characterization of several classes of involutory quandles in terms of the properties of the canonical generators of the displacement group, improving the main results of [23].

INTRODUCTION

Algebraic structure of interest in many areas of mathematics often have an underlying left quasigroup structure. Examples are quandles that arise in low dimensional topology [19, 22] and the algebraic structure related to the solution of the Yang-Baxter equation [25, 10]. The goal of this paper is to keep developing some tools for understanding left quasigroups as started in [9]. In this paper we study the class of superfaithful and the class of superconnected left quasigroups. Such notions arise naturally in the framework of Mal’cev conditions for left quasigroups that we study in a separate paper [7].

In some sense superfaithful and superconnected left quasigroups are close to quasigroups. Indeed latin left quasigroups (i.e. left quasigroup reducts of quasigroups) are superfaithful and connected and the finite ones are also superconnected (the converse is not true). On the other hand, super-connected left quasigroups have a Mal’cev term [7].

For quandles, the property of being connected is topologically relevant (as connected quandles provide knot invariants). The results of this paper and of [7] suggests that such property is relevant also from an algebraic viewpoint. Indeed, several results on finite latin quandles can be extended to the class of superconnected quandles. For instance, the commutator theory in the sense of [15] is particularly well-behaved in this class (see Proposition 2.14). In some cases, superconnected quandles are indeed latin, as the nilpotent (see Theorem 2.15) and the involutory ones (Theorem 3.6) improves the main result of [13] and partially the main result of [23] that were limited to the finite case.

Involutorial quandles encode the notion of symmetric space as defined in [20] and they are also related to Bruck loops [26, 28]. In the last Section we show that some properties of involutory quandles are determined by the properties of the canonical generators of the displacement group partially inspired by [24] (see Theorem 3.6 and Theorem 3.14). As a byproduct we obtain some group theoretical applications on finite groups generated by a conjugacy class of involutions (see Corollaries 3.11 and 3.15).

The paper is organized as follows: in Section 1.1 we collect all the basic definitions needed in the sequel of the paper, and in 1.2 and 1.3 we collect some basic results on connected and idempotent left quasigroups, respectively (including two characterization of superconnected left quasigroups in Lemma 3.4 and Corollary 1.6). Section 2 is dedicated to racks and quandles. In Section 2.1 we show some construction of (infinite families of) superfaithful quandles and in Sections 2.2 and

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Moreover, the mapping \( (\cdot, \setminus) \) is a well defined surjective homomorphism of groups (see \([1, \text{Lemma 1.8}]\) for racks and \([12]\) for left quasigroups). We denote by \( H(Q), S(Q) \) and \( P(Q) \) respectively the set of isomorphism classes of homomorphic images, subalgebras and powers of the left quasigroup \( Q \). Let \( X \) be a subset of \( Q \), we denote by \( Sg(X) \) the smallest subalgebra of \( Q \) containing \( X \).

The congruences of \( Q \) containing \( a, b, c, d \) hold, i.e. the left multiplications \( L_a : b \mapsto a \cdot b \) are bijective for every \( a \in Q \). The dual notion of right quasigroup is defined analogously. The left multiplication group of \( Q \) is \( LMlt(Q) = \{ (L_a : a \in Q) \} \).

We denote by \( \text{H}(Q), \text{S}(Q) \) and \( \text{P}(Q) \) respectively the set of isomorphism classes of homomorphic images, subalgebras and powers of the left quasigroup \( Q \). Let \( X \) be a subset of \( Q \), we denote by \( \text{Sg}(X) \) the smallest subalgebra of \( Q \) containing \( X \).

1. Preliminary results

1.1. Left quasigroups. A left quasigroup is a binary algebraic structure \((Q, \ast, \setminus)\) such that the identities

\[ x \setminus (x \ast y) = y \setminus x = (x \setminus y) \]

hold, i.e. the left multiplications \( L_a : b \mapsto a \ast b \) are bijective for every \( a \in Q \). The dual notion of right quasigroup is defined analogously. The left multiplication group of \( Q \) is \( LMlt(Q) = \{ (L_a : a \in Q) \} \).

The congruences of \( Q \) form a lattice denoted by \( \text{Con}(Q) \) with minimum \( 0_Q = \{ (a, a) : a \in Q \} \) and maximum \( 1_Q = Q \times Q \) if \( \alpha \) is a congruence of \( Q \), the congruence lattice of \( Q/\alpha \) is given by \( \{ \beta/\alpha : \alpha \leq \beta \in \text{Con}(Q) \} \), where

\[ [a]_\alpha \beta/\alpha [b]_\alpha \text{ if and only if } a \beta b. \]

Moreover, the mapping

\[ \pi_\alpha : LMlt(Q) \longrightarrow LMlt(Q/\alpha), \quad L_a^{k_1} \cdots L_a^{k_n} \mapsto L_{[a_{i_1}]}^{k_{i_1}} \cdots L_{[a_{i_n}]}^{k_{i_n}}, \]

is a well defined surjective homomorphism of groups (see \([1]\) Lemma 1.8) for racks and \([12]\) for left quasigroups). Moreover, \( (2) \)

holds for every \( a \in Q \) and every \( h \in LMlt(Q) \).

The displacement group relative to a congruence \( \alpha \) is the smallest normal subgroup of \( LMlt(Q) \) containing \( \{ L_a L_b^{-1} : a \beta b \} \) (see \([12]\) Section 3.1) i.e.

\[ \text{Dis}_{\alpha} = \{ h L_a L_b^{-1} h^{-1} : a \beta b, h \in LMlt(Q) \}. \]

For \( \alpha = 1_Q \) we denote the relative displacement group as \( \text{Dis}(Q) \) and we call it the displacement group of \( Q \).

**Lemma 1.1.** \([9]\) Lemma 1.4 Let \( Q \) be a left quasigroup. Then

\[ \text{Dis}(Q) = \{ L_{x_1}^{k_1} \cdots L_{x_n}^{k_n} : x_1, \ldots, x_n \in Q, \sum_{i=1}^{n} k_i = 0 \} \]

and in particular \( LMlt(Q) = \text{Dis}(Q)(L_a) \) for every \( a \in Q \).
If \( \alpha, \beta \) are congruences of a left quasigroup \( Q \) and \( \alpha \leq \beta \), the image of \( \text{Dis}_\beta \) under \( \pi_\alpha \) is \( \text{Dis}_{\beta/\alpha} \) and in particular the restriction of \( \pi_\alpha \) to \( \text{Dis}(Q) \) gives a surjective homomorphism \( \text{Dis}(Q) \to \text{Dis}(Q/\alpha) \). The kernels of \( \pi_\alpha \) and of its restriction will be denoted respectively by \( \text{LMlt}^\alpha \) and \( \text{Dis}^\alpha \). The setwise block stabilizers in \( \text{LMlt}(Q) \) is the subgroup \( \text{LMlt}(Q)_{[a]_\alpha} = \{ h \in \text{LMlt}(Q) : h([a]_\alpha) = [a]_\alpha \} \) (and similarly \( \text{Dis}(Q)_{[a]_\alpha} = \{ h \in \text{Dis}(Q) : h([a]_\alpha) = [a]_\alpha \} \)). Note that both \( \text{LMlt}(Q)_{[a]_\alpha} \) and \( \text{LMlt}^\alpha \) are contained in \( \text{LMlt}(Q)_{[a]_\alpha} \) (and the same is true for \( \text{Dis}(Q)_{[a]_\alpha} \)).

The Cayley kernel of a left quasigroup \( Q \) is the equivalence relation \( \lambda_Q \) defined as

\[
a \lambda_Q b \quad \text{if and only if} \quad L_a = L_b.
\]

In general, the equivalence \( \lambda_Q \) is not a congruence. If \( \lambda_Q = 0_Q \) then \( Q \) is called faithful and if all subalgebras of \( Q \) are faithful we say that \( Q \) is superfaithful. In particular, if \( Q/\alpha \) is faithful, then \( \lambda_Q \leq \alpha \) (indeed, according to \([2]\) if \( L_a = L_b \) then \( L_{[a]_\alpha} = L_{[b]_\alpha} \)). If \( \lambda_Q = 1_Q \), i.e. \( a \ast b = f(b) \) for every \( a, b \in Q \) where \( f \in \text{Sym}(Q) \), then \( Q \) is called permutation left quasigroup and denoted by \( (Q, f) \).

If \( f \) is the identity mapping then \( a \ast b = b \) for every \( a, b \in Q \) i.e. \( Q \) is a projection left quasigroup. We denote by \( \mathcal{P}_n \) the projection left quasigroup of size \( n \) and we call trivial left quasigroup the one-element projection left quasigroup.

A quasigroup is an algebra \( (Q, \ast, \setminus, /) \) such that \( (Q, \ast, \setminus) \) is a left quasigroup (the left quasigroup reduct of \( Q \)) and \( (Q, \ast, /) \) is a right quasigroup, i.e. also the right multiplications \( R_a : b \mapsto b \ast a \) are bijective for every \( a \in Q \). A left quasigroup is latin if it is the left quasigroup reduct of a quasigroup (in the finite case its multiplication table is a latin square). Note that congruences and subalgebras of a quasigroup and of its left quasigroup reduct might be different since we are considering a different signatures. Nevertheless they coincide in the finite case, since the two algebraic structures are term equivalent. We introduce this rather technical distinction in order to make clear that the results of the paper are tied to the choice of the left quasigroup signature (this detail will be more relevant in the related paper \([2]\)).

Latin left quasigroups are superfaithful. Indeed if \( Q \) is a latin left quasigroup and \( a \ast x = b \ast x \) for some \( a, b, x \in Q \) then \( a = b \).

A left quasigroups \( Q \) is said to be idempotent if \( x \ast x = x \) holds and involutory if \( x \ast (x \ast y) = y \) holds.

Let \( (A, +) \) be an abelian group, \( g \in \text{End}(A), f \in \text{Aut}(A) \) and \( c \in A \). We denote by \( \text{Aff}(A, g, f, c) \) the left quasigroup \( (A, \cdot) \) where \( x \cdot y = g(x) + f(y) + c \) and we call such left quasigroup affine over \( A \). If \( \text{Aff}(A, g, f, c) \) is idempotent, then necessarily \( c = 0 \) and \( g = 1 - f \), so we denote it just by \( \text{Aff}(A, f) \).

1.2. Connected left quasigroup. In this section we introduce the classes of connected and superconnected left quasigroups.

**Definition 1.2.** A left quasigroup \( Q \) is said to be:

(i) connected if \( \text{LMlt}(Q) \) acts transitively on \( Q \).

(ii) Superconnected if every subalgebra of \( Q \) is connected.

The following is a criterion for connectedness for left-quasigroups. The proof of the same criterion for racks stated in \([4]\) Proposition 1.3) can be employed for left quasigroups.

**Lemma 1.3.** Let \( Q \) be left quasigroup and \( \alpha \in \text{Con}(Q) \). Then \( Q \) is connected if and only if \( Q/\alpha \) is connected and \( \text{LMlt}(Q)_{[a]_\alpha} \) is transitive on \([a]_\alpha \) for every \( a \in Q \).

The property of being superconnected is determined by the connectedness of the two-generated subalgebras.

**Lemma 1.4.** Let \( Q \) be a left quasigroup. The following are equivalent:

(i) \( Q \) is superconnected.

(ii) \( \text{Sg}(a,b) \) is connected for every \( a, b \in Q \).

**Proof.** The forward implication is clear. To prove the converse, let \( M \) be a subalgebra of \( Q \) and \( a, b \in M \). The subgroup \( \text{LMlt}(\text{Sg}(a,b)) \) is transitive on \( \text{Sg}(a,b) \) and then in particular there exists \( h \in \langle L_c, c \in \text{Sg}(a,b) \rangle \leq \text{LMlt}(M) \) such that \( h(a) = b \). Therefore \( M \) is connected.
The orbit decomposition $O_Q$ defined by the action of $\text{LMlt}(Q)$ (as $a \in O_Q b$ if and only if $a$ and $b$ are in the same orbit with respect to the action of $\text{LMlt}(Q)$) is a congruence of $Q$ and $Q/O_Q$ is a projection left quasigroup [8, Lemma 1.8].

**Proposition 1.5.** Let $Q$ be a left quasigroup and $\alpha \in \text{Con}(Q)$. Then $Q/\alpha$ is a projection left quasigroup if and only if $O_Q \leq \alpha$. In particular, $Q$ is connected if and only if $\mathcal{P}_2 \notin \text{HS}(Q)$.

**Proof.** If $O_Q \leq \alpha$, then $Q/\alpha \cong (Q/O_Q)/(\alpha/O_Q)$. Therefore, $Q/\alpha$ is a projection left quasigroup. On the other hand, if $Q/\alpha$ is a projection left quasigroup, by virtue of (2), then $[h(a)]_\alpha = \pi_\alpha(h)([a]_\alpha) = [a]_\alpha$ for every $a \in Q$ and $h \in \text{LMlt}(Q)$. Hence, $O_Q \leq \alpha$.

A left quasigroup is connected if and only if $Q/O_Q$ is trivial, i.e. $Q$ has no proper projection factor.

**Corollary 1.6.** A left quasigroup $Q$ is superconnected if and only if $\mathcal{P}_2 \notin \text{HS}(Q)$.

The class of connected left quasigroups is closed under $\text{H}$, but it is not a closed under $\text{S}$ (for instance it is easy to find connected left quasigroups with projection subalgebras). The class of superconnected left quasigroups is closed under $\text{S}$ and $\text{H}$. On the other hand it is not closed under $\text{P}$ (e.g. the permutation left quasigroup $Q = (\mathbb{Z}_m, +1)$ is superconnected, but $Q^2$ is not even connected).

The property of being latin is also related to the properties of 2-generated subalgebras (similarly to what happens for superconnectedness in Lemma 1.4).

**Lemma 1.7.** Let $Q$ be a left quasigroup. If $Sg(a, b)$ is a finite latin left quasigroup for every $a, b \in Q$ then $Q$ is latin.

**Proof.** Assume that $x \ast a = y \ast a$. Then $x \ast a = y \ast a \in U = Sg(a, y) \cap Sg(a, x)$, which is finite and latin and so $R_a(U) = U$. Hence, $x = y$ and right multiplications are surjective. For every $a, b \in Q$ there exists $x \in Sg(a, b)$ for which $x \ast a = b$ and so right multiplications are surjective.

**Example 1.8.**

(i) If a quasigroup $(Q, \ast, \backslash, /)$ and its left quasigroup reduct $(Q, \ast, \backslash)$ are term equivalent then $(Q, \ast, \backslash)$ is superconnected. Hence, any finite latin left quasigroup is superconnected. The converse is not true, as witnessed by the following superconnected non-latin left quasigroup:

$$Q = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & 3 & 4 \\
3 & 2 & 1 & 4 \\
4 & 2 & 3 & 1
\end{array}.$$

(ii) Latin left quasigroup are connected but they might not be superconnected. The left quasigroup $Q = \text{Aff}(\mathbb{Z}, -1)$ is latin. The subalgebra generated by $0, 1$, i.e. $\text{Aff}(\mathbb{Z}, -1)$, is a non-connected subalgebra of $Q$ (and in particular the converse of Lemma 1.4 does not hold).

1.3. **Idempotent left quasigroups.** The blocks of congruences of idempotent left quasigroups are subalgebras, and in particular, the classes of $\lambda_Q$ are projection subalgebras. The blocks of $\text{LMlt}(Q)$ and of $\text{Dis}(Q)$ coincide, because of the structure of $\text{LMlt}(Q)$ given in Lemma 1.4. We extend [8, Proposition 1.4] to the setting of idempotent left quasigroups.

**Lemma 1.9.** Let $Q$ be an idempotent left quasigroup. The following are equivalent:

(i) $Q$ is superfaithful.

(ii) $Sg(a, b)$ is superfaithful for every $a, b \in Q$.

(iii) $\mathcal{P}_2 \notin S(Q)$.

In particular, if $Q$ is superconnected then $Q$ is superfaithful.
Corollary 1.12. The class of (super)faithful (resp. connected) idempotent left quasigroups is blocks of size 3 which are latin, but it is not latin itself: transitive on $L$ then $L$ congruence of $M$.

Proof. (i) $\Rightarrow$ (iii) The subalgebra $P_2$ is not faithful.

(iii) $\Rightarrow$ (i) Let $M$ be a subalgebra of $Q$. The classes of $\lambda_M$ are projection subalgebras, therefore they are trivial.

(i) $\Leftrightarrow$ (ii) The equivalence is clear: indeed $P_2 \in S(Q)$ if and only if $Sg(a,b) \cong P_2$ for some $a,b \in Q$.

Note that the class of superfaithful idempotent left quasigroup is closed under $S$ and $P$.

Example 1.10. Superconnected and latin idempotent left quasigroups are superfaithful wether both the converse implications fails. Indeed the idempotent left quasigroup $\text{Aff}(\mathbb{Z}, -1)$ in Example 1.8(ii) is superfaithful but not connected.

A class of idempotent left quasigroups $K$ is said to be closed under extensions if, whenever $Q/\alpha$ and $[a]_\alpha$ belong to $K$ for every $a \in Q$ then also $Q$ belongs to $K$. It is easy to see that if a class is closed under extensions then it is also closed under finite direct products.

Lemma 1.11. Let $Q$ be a left quasigroup and let $Q/\alpha$ be idempotent. If $Q/\alpha$ and $[a]_\alpha$ are (super)faithful (resp. connected) for every $a \in Q$, then $Q$ is (super)faithful (resp. connected).

Proof. The blocks of $\alpha$ are subalgebras of $Q$ since $Q/\alpha$ is idempotent. Let $M$ be a subalgebra of $Q$. We denote by $M/\alpha$ the image of $M$ under the canonical map $Q \rightarrow Q/\alpha$.

Assume that $Q/\alpha$ and $[a]_\alpha$ are superfaithful for every $a \in Q$. If $L_a[M] = L_b[M]$ for some $a,b \in M$ then $L_a[M/\alpha] = L_b[M/\alpha]$ and so $[a] = [b]$ since the subalgebra $M/\alpha$ of $Q/\alpha$ is faithful. Therefore $L_a[M/\alpha] = L_b[M/\alpha]$ which implies $a = b$ since $[a] \cap M$ is faithful.

Assume that $Q/\alpha$ and $[a]$ are superconnected for every $a \in Q$. The relation $\beta = \alpha \cap M \times M$ is a congruence of $M$. The group 

$$L = \{(L_b : b \in [a]_\beta)\} \leq \text{LMlt}(M)_{[a]_\beta}$$

is transitive over $[a]_\beta$ since $[a]_\beta = [a] \cap M$ is a connected subalgebra of $[a]$. So $\text{LMlt}(M)_{[a]_\beta}$ is transitive on $[a]_\beta$. $M/\beta$ is connected and therefore $M$ is connected by virtue of Lemma 1.8.

For faithfulness and connectedness the same argument applied to the case $M = Q$ will do.

Corollary 1.12. The class of (super)faithful (resp. connected) idempotent left quasigroups is closed under extensions.

The class of idempotent latin left quasigroup is not closed under extensions. For instance the following superconnected idempotent left quasigroup has a congruence with a factor of size 3 and blocks of size 3 which are latin, but it is not latin itself:

$$Q = \begin{pmatrix}
1 & 3 & 2 & 7 & 8 & 9 & 4 & 5 & 6 \\
3 & 2 & 1 & 7 & 8 & 9 & 4 & 5 & 6 \\
2 & 1 & 3 & 7 & 8 & 9 & 4 & 5 & 6 \\
7 & 8 & 9 & 4 & 6 & 5 & 1 & 2 & 3 \\
7 & 8 & 9 & 6 & 5 & 1 & 2 & 3 & . \\
7 & 8 & 9 & 5 & 4 & 6 & 1 & 2 & 3 \\
4 & 5 & 6 & 1 & 2 & 3 & 7 & 9 & 8 \\
4 & 5 & 6 & 1 & 2 & 3 & 9 & 8 & 7 \\
4 & 5 & 6 & 1 & 2 & 3 & 8 & 7 & 9
\end{pmatrix}$$

2. RACKS AND QUANDLES

A rack is a left distributive left quasigroup, i.e. a left quasigroup satisfying the identity

(LD) \[ x * (y * z) = (x * y) * (x * z). \]

An idempotent rack is a quandle. Left-distributivity (LD) implies that for a quandle $Q$, $hL_a h^{-1} = L_{h(a)}$ for every $h \in \text{LMlt}(Q)$ and $a \in Q$. In particular, the displacement group is simply given by

\[ \text{Dis}(Q) = \langle L_a L_b^{-1}, a, b \in Q \rangle. \]

Example 2.1.
(i) Permutation left quasigroups are racks.

(ii) Let $G$ be a group and $H \leq G$ be closed under conjugation. Then $\text{Conj}(H) = (H, *)$ where $x * y = x y x^{-1}$ is a quandle.

(iii) Let $G$ be a group, $f \in \text{Aut}(G)$ and $H \leq \text{Fix}(f) = \{ a \in G : f(a) = a \}$. Let $G/H$ be the set of left cosets of $H$ and the multiplication defined by

$$aH * bH = a f^{-1}(a^{-1}b)H.$$

Then $Q(G, H, f) = (G/H, *, \setminus)$ is a quandle, called a coset quandle. A coset quandle $Q(G, H, f)$ is called principal over $G$ if $H = 1$ and is such case it is denoted by $Q(G, f)$.

(iv) Idempotent affine left quasigroups are quandles.

If the automorphism group of a quandle $Q$ is transitive, we say that $Q$ is homogeneous. The construction in Example 2.1(iii) characterizes homogeneous quandles [19]. For instance, connected quandles are homogeneous and they can be represented as coset quandles over their displacement group.

**Proposition 2.2.** [19, 18] Let $Q$ be a connected quandle $Q$. Then

$$Q \cong Q(\text{Dis}(Q), \text{Dis}(Q)_a, \mathcal{T}_a)$$

for every $a \in Q$.

### 2.1. Superfaithful quandles

In [14] the class of $L$-groupoids have been defined as racks such that the equation $x * a = a$ is uniquely solvable in $x$ for every $a \in Q$. According to [14, Proposition 1] $L$-groupoids are idempotent and so they are exactly quandles with no proper projection subquandles, i.e. $L$-groupoids coincide with superfaithful quandles.

**Lemma 2.3.** [3] Proposition 2.4 Let $Q$ be a quandle. The following are equivalent:

(i) $Q$ is superfaithful.

(ii) $\mathcal{P}_S \notin S(Q)$.

(iii) $\text{Fix}(L_a) = \{ a \}$ for every $a \in Q$.

The coset quandle construction provides a way to construct finite homogeneous superfaithful quandles.

**Lemma 2.4.** Let $Q = Q(G, H, f)$ be a quandle over a finite group $G$. If $|H|$ and $|f|$ are coprime then $|x^f| = |x H^L a|$ for every $a \in G$.

**Proof.** Clearly $n = |x H^L a| = |x H^L f|$. If $|a H^L f|$ divides $|f^a|$ then $f^{na}(x) = xa^g$ for some $a \in H$. Therefore $f^{na}(x) = xa^g$ and so $|x^f| = n a$. Thus $a$ divides both $|H|$ and $|f|$ and so $a = 1$, i.e. $|x^f| = n$. \hfill $\Box$

**Corollary 2.5.** Let $Q = Q(G, \text{Fix}(f), f)$ be a quandle over a finite group $G$. If $|\text{Fix}(f)|$ and $|f|$ are coprime that $Q$ is superfaithful.

**Proof.** Apply Lemma 2.4 to the case $H = \text{Fix}(f)$. Indeed if $\{ H, gH \}$ is a projection subquandle, then $H * gH = f(g)H = gH$. Thus $f(g) = g$, i.e. $g \in H$. Therefore $\text{Fix}(L_H) = \{ H \}$ and since $Q$ is homogeneous, the left multiplications have all the same cycle structure, i.e. $\text{Fix}(L_{aH}) = \{ xH \}$ for every $x \in G$. \hfill $\Box$

The converse of Lemma 2.4 is not true: there exist latin quandles of size $pq$, for $p, q$ primes, with left multiplications of order $2p$ and stabilizers of size $p$ [6] (e.g. SmallQuandle(15,5) from the 21 database of GAP).

Let $G$ be a group, $\theta \in \text{Aut}(G)$ and $t \in \mathbb{N}$. We define

$$\theta_t : G^t \longrightarrow G^t, \quad (x_1, \ldots, x_t) \mapsto (\theta(x_t), x_1, x_2, \ldots, x_{t-1}).$$

It is easy to check that $\theta_t$ is an automorphism of $G^t$ and that $H_t = \text{Fix}(\theta_t) = \{ (a, a, \ldots, a) : a \in \text{Fix}(\theta) \} \subseteq \text{Fix}(\theta)$. We denote by $(G,t,\theta)$ the coset quandle $Q(G^t, H_t, \theta_t)$.

**Lemma 2.6.** Let $Q = (G,t,\theta)$ be a quandle and $\text{Fix}(\theta_t) = H_t$. Then

$$\text{Fix}(L_{H_t}) = \{ (b, ba, ba^2, \ldots, ba^{-2}, ba^{-1})H_t : b \in G, a \in \text{Fix}(\theta), \theta(b) = ba^{-1} \}$$
Proposition 2.9. Let $G$ be a finite group, $\theta \in \text{Aut}(G)$ and $t \in \mathbb{N}$. The following are equivalent:

(i) $(G, 1, \theta)$ is superfaithful and $t$ and $|\text{Fix}(\theta)|$ are coprime.

(ii) $(G, t, \theta)$ is superfaithful.

Proof. According to Lemma 2.6, $H_t \neq xH_t \in \text{Fix}(L_{H_t})$ if and only if $x = (b, ba, ba^2, \ldots, ba^{t-2}, ba^{t-1})$ for some $1 \neq a \in \text{Fix}(\theta)$ and $\theta(b) = ba^{-t}$.

Assume that $t$ and $|\text{Fix}(\theta)|$ are coprime. Under this assumption, the mapping

$$(a) \mapsto (a), \quad x \mapsto x^t$$

is a bijection for every $a \in \text{Fix}(\theta)$ and $x^t = 1$ if and only if $x = 1$. So, if $\theta(b) = ba$ holds for some $b \in G$ and $1 \neq a \in \text{Fix}(\theta)$ then $a = c^t$ for some $c \neq 1$. On the other hand if $\theta(b) = ba^t$ for some $a \neq 1$ then also $a^t \neq 1$. Therefore $(G, t, \theta)$ is superfaithful if and only if $(G, 1, \theta)$ is superfaithful. Thus, the implication (i) $\Rightarrow$ (ii) holds.

To complete the proof of the implication (ii) $\Rightarrow$ (i), we need to show that if $(G, t, \theta)$ is superfaithful then $t$ and $\text{Fix}(\theta)$ are coprime. Assume that $p$ is a prime dividing $|\text{Fix}(\theta)|$ and $t$. Then there exists $a \in \text{Fix}(\theta)$ of order $p$ and $a^t = a^{p t} = 1$. So $\theta(1) = 1 \cdot a^t$ and so $(1, a, a^2, \ldots, a^{t-1})H_t \in \text{Fix}(L_{H_t}) \neq \{H_t\}$. \hfill $\square$

Example 2.8. Let $G$ be a finite group and $Q = (G, t, \theta)$.

(i) If $\text{Core}_G(\text{Fix}(\theta)) = 1$ and $t$ and $|\text{Fix}(\theta)|$ are not coprime, then $Q$ is faithful but not superfaithful.

(ii) Let $\theta = 1$. Then $Q = (G, t, 1)$ is superfaithful if and only if $|G|$ and $t$ are coprime. In particular, if $G$ is simple then $Q$ is a simple quandle (thus simple quandles are faithful but not necessarily superfaithful).

Recall that for a quandle $Q$, the equivalence relation $\sigma_Q$ is defined by

$$(a) \sim_Q (b) \text{ if and only if } \text{Dis}(Q)_a = \text{Dis}(Q)_b.$$

The blocks of $\sigma_Q$ are subquandles and they are also blocks with respect to the action of $\text{LMLt}(Q)$ [S Section 2.3].

Proposition 2.9. Let $Q$ be a finite superfaithful quandle. Then $[a]_{\sigma_Q}$ is a principal latin quandle over $N_{\text{Dis}(Q)}(\text{Dis}(Q)_a)/\text{Dis}(Q)_a$ and $N_{\text{Dis}(Q)}(\text{Dis}(Q)_a)/\text{Dis}(Q)_a$ is solvable for every $a \in Q$.

Proof. The block $S = [a]_{\sigma_Q}$ is a finite superfaithful semiregular quandle. Then $S$ is latin and in particular it is connected [S Corollary 2.9]. Hence $[a]_{\sigma_Q}$ is contained in the orbit of $a$ with respect to $\text{Dis}(Q)$. Hence, according to [S Theorem 3.4], $[a]_{\sigma_Q}$ is principal over $\text{Dis}(Q) \cong N_{\text{Dis}(Q)}(\text{Dis}(Q)_a)/\text{Dis}(Q)_a$ that is solvable (the displacement group of a finite latin quandle is solvable [27]). \hfill $\square$

According to Proposition 2.9 finite superfaithful quandles are the disjoint union of principal latin quandles. Note that such a partition can be trivial.
2.2. Superconnected racks. Finite Latin quandles are superconnected, but the converse implication fails, although examples seem to be rare. Examples of superconnected non-Latin quandles are provided by the family \( \text{SmallQuandle}(i) \) with \( i = 3,4,5,6 \) in the [27] library of GAP (such quandles have a congruence with Latin blocks and Latin factor, but they are not Latin).

Example 2.10.

(i) A permutation rack \( (Q,f) \) is connected if and only if \( Q = \{ f^j(a) : j \in \mathbb{Z} \} \) for every \( a \in Q \). Let \( C \) be a cyclic group generated by \( c \) and \( |C| = |Q| \). The map

\[
(Q,f) \mapsto \text{Aff}(C,0,1,c), \quad f^j(a) \mapsto jc
\]

is an isomorphism of racks. The rack \( \text{Aff}(C,0,1,c) \) is generated by any of its elements and then it is superconnected (note that every monogenerated rack arise in this way).

(ii) Let \( C \) be a conjugacy class in a group \( G \). Then \( C \) is superconnected if and only if every pair of elements \( a,b \in C \) are conjugate in the subgroup \( \langle a,b \rangle \) (see Lemma 1.4).

Let \( Q \) be a rack, \( A \) an abelian group, \( \psi \in \text{Aut}(A) \) and a map \( \theta : Q \times Q \to A \). We define the left quasigroup \( E = Q \times \psi,\theta A = (Q \times A, *) \) where

\[
(a,s) \ast (b,t) = (a \ast b, (1 - \psi)(s) + \psi(t) + \theta(a,b))
\]

for every \( a,b \in Q \) and \( s,t \in A \). Under suitable conditions on \( \theta \) and \( \psi \), \( E \) is a rack [12] Section 7 and we say that \( E \) is a central extension of \( Q \) by \( A \). The projection onto \( Q \) is a rack morphism and if \( \psi = 1 \), then its kernel is contained in the congruence \( \lambda_E \). In this case, following [13] and [11], we say that \( E \) is an abelian cover of \( Q \).

Recall that for a rack \( Q \) the equivalence relation \( \mathfrak{ip}_Q \) which blocks are \( [a]_{\mathfrak{ip}_Q} = Sg(a) \) is a congruence of \( Q \) contained in \( Q \) [11] Proposition 7.1.

Proposition 2.11. Let \( Q \) be a rack. The following are equivalent:

(i) \( Q \) is superconnected.

(ii) \( Q/\mathfrak{ip}_Q \) is superconnected.

If particular, if \( Q \) is superconnected then \( \lambda_Q = \mathfrak{ip}_Q \) and \( Q \) is an abelian cover of \( Q/\mathfrak{ip}_Q \).

Proof. The blocks of \( Q/\mathfrak{ip}_Q \) are subracks since \( Q/\mathfrak{ip}_Q \) is idempotent. The block \( [a]_{\mathfrak{ip}_Q} \) is the subrack generated by \( a \) and so it superconnected according to Example 2.10(i). So, the equivalence between (i) and (ii) follows by Lemma 1.11.

If \( Q \) is superconnected, then \( Q/\mathfrak{ip}_Q \) is superconnected and so \( Q \) is faithful. Therefore, \( \lambda_Q \leq \mathfrak{ip}_Q \leq \lambda_Q \) and so equality holds. Finally, \( Q \) is superconnected and then homogeneous, so we can apply [11] Corollary 7.1(5), i.e. \( Q \) is an abelian cover of \( Q/\mathfrak{ip}_Q \).

Some of the contents of Section 2.4 of [8] on principal Latin quandles extend to principal superconnected quandles.

Proposition 2.12. Let \( Q = Q(G,f) \) be a superconnected quandle.

(i) The subquandles of \( Q \) are coset with respect to \( f \)-invariant subgroups of \( G \) and they are principal.

(ii) \( \text{Dis}_\alpha = \text{Dis}^\alpha = \text{Dis}(Q)_{[\alpha]} \) for every \( \alpha \in \text{Con}(Q) \).

(iii) \( \text{Con}(Q) \equiv \{ N \unlhd G : f(N) = N \} \) and \( Q/\alpha \) is principal for every \( \alpha \in \text{Con}(Q) \).

Proof. All subquandles of \( Q \) are connected, then we can apply [8] Lemma 2.7 for (i), and [8] Corollary 2.11 for (ii) and (iii).

2.3. Commutator theory for superconnected quandles. According to the commutator theory developed in [15] we can define abelianess and centrality for congruences of arbitrary algebraic structures (e.g. for left quasigroups) using the commutator between congruences (we omit the general definition and we denote the commutator between two congruences \( \alpha, \beta \) by \([\alpha, \beta]\)). Consequently, nilpotence and solvability are defined by using a special chain of congruences defined in
Proposition 2.14. Let \( Q \) be a superconnected quandle. Then:

(i) If \( \alpha, \beta \in \text{Con}(Q) \), then by virtue of Proposition 2.12(ii) \( \text{Dis}^n = \text{Dis}_\alpha \). Thus, we can apply [12, Proposition 3.7] we have that for every \( \alpha \in \text{Con}(Q) \).

(ii) The mapping \( \text{Dis} \) is injective and the mapping \( \text{con} \) is surjective.

Proof. Every superconnected quandle is faithful. If \( Q \) is a principal superconnected quandle then \( Q = \text{Dis}_\alpha \). Thus, we can apply [12, Proposition 3.7] we have that for every \( \alpha \in \text{Con}(Q) \).

All factors of superconnected quandles are superconnected and then faithful. So, according to [12, Proposition 3.7] we have that \( \alpha \leq \text{con}_{\text{Dis}_\alpha} \leq \text{con}_{\text{Dis}^\alpha} = \alpha \) and so \( \alpha = \text{con}_{\text{Dis}_\alpha} \) for every \( \alpha \in \text{Con}(Q) \). For the other statements, we can apply directly [12, Propositions 3.8 and 5.5] since all the factor of \( Q \) are faithful.

Nilpotent superconnected quandles are indeed latin.

Theorem 2.15. Nilpotent superconnected quandles are latin.

Proof. If \( Q \) is abelian and superconnected, then it is faithful and connected and so latin [8, Corollary 2.6]. Let \( Q \) be nilpotent of length \( n+1 \), i.e. \( \gamma_n(Q) \) is central. The group

\[
D = \langle L_{a}L_{a}^{-1}, b \in [a]_{\gamma_n(Q)} \rangle \leq \text{Dis}_{\gamma_n(Q)}
\]

is transitive on the block of \([a]_{\gamma_n(Q)}\) and \( Q \) is connected. Then we can apply [12, Proposition 7.8] and we have that \( Q \) is a central extension of \( Q/\gamma_n(Q) \), i.e. the quandle operation of \( Q \) is defined as in [5] by

\[
R_{(a,s)}(b,t) = (b,t) \ast (a,s) = (R_{a}(b), ((1-\psi)(t) + \psi(s) + \theta_{a,b})).
\]

By induction on the nilpotency length, \( Q/\gamma_n(Q) \) is latin and the blocks of \( \gamma_n(Q) \) are abelian and therefore latin, i.e. \( 1 - \psi \) is bijective. Therefore the right multiplication \( R_{(a,s)} \) has inverse \( R_{(a,s)}^{-1}(b,t) = (R_{a}^{-1}(b), (1-\psi)^{-1}(t - \psi(s) - \theta_{a,b})) \) and so \( Q \) is latin.

The converse of Theorem 2.15 does not hold. Indeed there exist infinite affine latin quandles which are not superconnected (see Example 1.8(ii)).
The biggest central congruence of a quandle $Q$ is called the center of $Q$ (the analog of the center of a group) and denoted by $\zeta_Q$. According to [12, Proposition 5.8] (and [9, Proposition 3.14]) we have
\[ a \zeta_Q b \text{ if and only if } L_a L_b^{-1} \in Z(\text{Dis}(Q)) \text{ and } a \sigma_Q b, \]
where $\sigma_Q$ is defined in [11].

**Lemma 2.16.** Let $Q$ be a finite connected quandle and $\alpha \leq \zeta_Q$. If $Q$ is superfaithful then $Q/\alpha$ is superfaithful.

**Proof.** Assume that $[a] * [b] = [b]$. Then $L_a L_b^{-1} \in \text{Dis}(Q)_b$. According to [9, Corollary 3.2] the block stabilizer is the direct product of $\text{Dis}_{\alpha} = \{ L_a L_b^{-1} : c \alpha b \}$ and the stabilizer of $b$ in $\text{Dis}(Q)$. Thus, there exists $c \alpha b$ and $h \in \text{Dis}(Q)_h = \text{Dis}(Q)_c$ such that $L_a L_b^{-1} = h L_c L_b^{-1}$. Then $L_a = h L_c \in \text{LMlt}(Q)_c$ and accordingly $a * c = c$. Then, $a = c \alpha b$ and so $\text{Fix}(L_{[a]}) = \{ [a] \}$ and Lemma 2.15 applies. \qed

**Proposition 2.17.** Let $Q$ be a finite nilpotent quandle. The following are equivalent:

(i) $Q$ is connected and superfaithful.

(ii) $Q$ is latin.

**Proof.** (i) $\Rightarrow$ (ii) Let us proceed by induction on the nilpotency length. If $Q$ is abelian then it follows by [8, Corollary 2.6], since $Q$ is connected and faithful. Let $Q$ be nilpotent of length $n$. By Lemma 2.16 $Q/\zeta_Q$ is superfaithful and then by induction $Q/\zeta_Q$ is latin. So we can apply [8, Lemma 3.4] and conclude that $Q$ is latin.

(ii) $\Rightarrow$ (i) True in general. \qed

Note that the superconnected quandles of size 28 mentioned earlier in the previous section are solvable but not latin, so Theorem 2.15 does not extend to the solvable case. Nevertheless finite solvable superconnected quandles have the Lagrange property, i.e. the size of every subalgebra divides the size of the quandle (extending a known result for left distributive quasigroups [16]).

In the proof of the following Proposition we are using that the blocks of a congruence of a connected left quasigroup have all the same size [12, Lemma 2.5].

**Proposition 2.18.** Finite solvable superconnected quandles have the Lagrange property.

**Proof.** If $Q$ is abelian, the statement is true because subquandles correspond to submodules with respect to the structure given by the affine representation [8, Proposition 2.18]. Let $Q$ be solvable of length $n + 1$ i.e. $\gamma^n(Q)$ is an abelian congruence and let $M$ be a subquandle. Then $|M| = |M/\gamma^n(Q)||[a]| \cap M$. Since $[a]$ is affine, $|M \cap [a]|$ divides $|[a]|$ and since $Q/\gamma^n(Q)$ is solvable of length $n$, by induction we have that $|M/\gamma^n(Q)|$ divides $|Q/\gamma^n(Q)|$. Therefore, $|M|$ divides $|Q| = |Q/\gamma^n(Q)||[a]|$. \qed

### 3. Involutory quandles

#### 3.1. Two generated involutory quandles

Recall that a quandle $Q$ satisfying the identity $x * (x * y) = y$ is called involutory.

The local properties of 2-generated subquandles determine the global properties such as superconnectedness and latinity (see Lemmas 1.14 and 1.7). A description of the free involutory quandle on 2 generators is given in [19, Corollary 10.4], namely such quandle is isomorphic to Aff($\mathbb{Z}$, -1).

We investigate the properties of involutory quandles according to the properties of the canonical generators of the displacement group. A similar approach was take in [24] using the concept of cycle. The main original contribution is to partially extend the main result of [24] to the infinite case.

Let $Q$ be an involutory quandle and $a, b \in Q$. Following [24] we define the cycle generated by $a$ with base $b$ as

\[ C(a, b) = \{ a^k : k \in \mathbb{Z} \}, \text{ where } a^k = \begin{cases} (L_a L_b)^i(b), & \text{if } k = 2i, \\ (L_a L_b)^i(a), & \text{if } k = 2i + 1. \end{cases} \]
According to [19] Corollary 5.4, \( Sg(a, b) = a^{L_a L_b} \cup b^{L_a L_b} \) and so we have that \( C(a, b) \) is the sub-quandle generated by \( a, b \). If \([L_a L_b]\) is finite we can define \( \text{ord}_{a, b} = \min_{k \geq 0} \{ a^k = b \} \).

**Proposition 3.1.** Let \( Q \) be an involutory quandle generated by \( a, b \in Q \). Then \( \text{Dis}(Q) \) is the cyclic group generated by \( L_a L_b \).

**Proof.** According to [19] Corollary 10.4, the free 2-generated involutory quandle \( F \) is isomorphic to \( \text{Aff}(\mathbb{Z}, -1) \) that is generated by 0 and 1. Since we have that

\[
L_a L_b(c) = 2(a - b) + c
\]

for every \( a, b, c \in F \), the displacement group of \( F \) is \( (L_1 L_0) = 2\mathbb{Z} \equiv \mathbb{Z} \). The canonical surjective quandle homomorphism \( F \rightarrow Q \), induces a surjective group homomorphism \( \mathbb{Z} \rightarrow \text{Dis}(Q) \) and so \( \text{Dis}(Q) \) is cyclic and it is generated by \( L_a L_b \).

**Lemma 3.2.** Let \( Q \) be an involutory quandle, \( a, b \in Q \) and \( n \in \mathbb{N} \). Then:

(i) \( (L_a L_b)^{2n+1} = L_a L_{(L_b L_a)^n(b)} \).

(ii) \( (L_a L_b)^{2n} = L_a L_{(L_b L_a)^n(a)} \).

**Proof.** Let \( a, b \in Q \). For (i) we have

\[
(L_a L_b)^{2n+1} = L_a (L_b L_a)^n L_b (L_a L_b)^n = L_a (L_b L_a)^n (L_b L_a)^{-n} = L_a L_{(L_b L_a)^n(b)}
\]

For (ii):

\[
(L_a L_b)^{2n} = (L_a L_b)^n (L_a L_b)^n = L_a (L_b L_a)^n L_a (L_b L_a)^{-n} = L_a L_{(L_b L_a)^n(a)}
\]

**Corollary 3.3.** Let \( Q \) be an involutory quandle and \( a, b \in Q \). The following are equivalent:

(i) \( |L_a L_b| \) is finite.

(ii) \( Sg(a, b) \) is finite

(iii) \( \text{ord}_{a, b} \) is finite.

**Proof.** Let \( S = Sg(a, b) \). According to Lemma 3.1 \( \text{Dis}(S) \) is generated by \( L_a L_b \) and so, by [19] Corollary 5.4, \( S = a^{L_a L_b} \cup b^{L_a L_b} \)

(i) \( \Rightarrow \) (ii) Clearly if \( |L_a L_b| \) is finite then \( S \) is finite.

(ii) \( \Rightarrow \) (iii) If \( S \) is finite then \( \text{ord}_{a, b} \leq 2|a^{L_a L_b}| \leq |S| \) is finite.

(i) \( \Leftrightarrow \) (iii) It follows by Lemma 3.2. Indeed, if \( s = \text{ord}_{a, b} \) is even then \( (L_a L_b)^{2s} = L_a L_{(L_b L_a)^s(a)} = L_a^2 = 1 \). If \( s \) is odd then \( (L_a L_b)^{2s+1} = L_a L_{(L_b L_a)^s(b)} = L_a^2 = 1 \).

**Proposition 3.4.** Let \( Q \) be an involutory quandle and \( a, b \in Q \). The following are equivalent:

(i) \( Sg(a, b) \) is connected.

(ii) \( Sg(a, b) \) is a finite latin quandle of odd order.

(iii) \( \text{ord}_{a, b} \) is finite and odd.

**Proof.** According to Lemma 3.1 \( S = Sg(a, b) \) has cyclic displacement group generated by \( L_a L_b \) and every orbit of \( S \) is isomorphic to \( \text{Aff}(C, -1) \) where \( C \) is a cyclic group.

(i) \( \Rightarrow \) (ii) If \( Q \) is connected, then \( Q \equiv \text{Aff}(C, -1) \). If \( C \) is infinite then \( Q \) is not connected. Hence \( Q \) is a finite connected affine quandle, and then \( Q \) is latin (see [5] [8]). In particular, \( R_0 : x \mapsto 2x \) is bijective and so \( |C| \) is odd.

(ii) \( \Rightarrow \) (iii) If \( Q \) is a finite latin quandle, then \( Q \equiv \text{Aff}(\mathbb{Z}_{2^n+1}, -1) \) for some \( n \). The condition \( (L_1 L_0)^n(0) = 2n = 0 \) is satisfied just for \( n = 0 \).

(iii) \( \Rightarrow \) (i) If \( \text{ord}_{a, b} \) is odd, then there exists \( n = 2k + 1 \) such that \( b = (L_a L_b)^k(a) \) and so \( Q \) is connected.
3.2. Superconnected and latin involutory quandles. Let us first note that in one direction
Proposition 6] works also for infinite superfaithful quandle (indeed the proof indeed just
requires that the order of the canonical generators of the displacement group have finite order and
that the two-generated subquandles are faithful). We provide an alternative proof using Lemma

**Lemma 3.5.** Let $Q$ be an involutory quandle and $a,b \in Q$ such that $|L_aL_b|$ is finite. If $Q$ is
superfaithful then $|L_aL_b| = \text{ord}_{a,b}$ is odd.

**Proof.** According to [24] Proposition 5] $|L_aL_b| = \text{ord}_{a,b}$ since $Q$ is faithful.
The group $C = (L_a L_b)$ is finite and so it is $S = Sg(a,b)$ Assume that $S$ is not connected, i.e.
$S = a^C \cup b^C = O_a \cup O_b$. If $|O_a|$ is even then $O_a \cong \text{Aff}(\mathbb{Z}_{2m},-1)$ has projection subquandle. Then
$|O_a|$ is odd and $L_b$ acts on $O_a$. Since $L_b$ has order 2 then $L_b|O_a$ has fixed points. According to
Lemma 2.3, $P_2 \in S(Q)$, contradiction. Hence $S$ is connected and so Proposition 3.3 applies. □

The following theorem characterizes superconnected involutory quandles.

**Theorem 3.6.** Let $Q$ be an involutory quandle. The following are equivalent:
(i) $Q$ is superconnected.
(ii) $Q$ is latin and $|L_aL_b|$ is finite for every $a, b \in Q$.
(iii) $\text{ord}_{a,b} = |L_aL_b|$ is finite and odd for every $a, b \in Q$.

**Proof.** (i) $\Rightarrow$ (ii) According to Proposition 3.3] if the subquandle $Sg(a,b)$ is connected then it is
finite and latin. Then we can conclude that $Q$ is latin by Lemma 1.4 and that $|L_aL_b|$ is finite by
Corollary 3.3.
(ii) $\Rightarrow$ (iii) Follows by Lemma 3.5 since $Q$ is superfaithful.
(iii) $\Rightarrow$ (i) By virtue of Proposition 3.3 every pair of elements of $Q$ generates a finite connected
quandle. Thus we can conclude by Lemma 1.4.

**Example 3.7.** Let $A$ be a torsion abelian group. If $A$ is 2-divisible (i.e. $2A = A$) and $A$ has no
2-torsion (i.e. $A$ has no elements of order 2) then $Q = \text{Aff}(A,-1)$ is a superconnected involutory
quandle. Indeed $Q$ is latin and the order of the generators of $\text{Dis}(Q) \cong A$ is finite. For instance
take $A$ to be the Prüfer group $\mathbb{Z}_p^\infty$ for $p > 2$.

In the (locally) finite case we also recover the main result of [14] by using Proposition 3.3] and
Theorem 3.6. Theorem also extends the main result of [24] to infinite involutory quandles such that
the order of the canonical generators of the displacement group is finite.

**Corollary 3.8.** Let $Q$ be an involutory quandle such that $|L_aL_b|$ is finite for every $a, b \in Q$. The
following are equivalent:
(i) $Q$ is superfaithful.
(ii) $Q$ is latin.
(iii) $\text{ord}_{a,b} = |L_aL_b|$ is odd for every $a, b \in Q$.

Note that the classical result [17] Theorem 1.2] is exactly Corollary 3.8] for quandles given by
conjugacy classes of involutions in finite groups.

The quandle $Q = \text{Aff}(\mathbb{Q},-1)$ is an infinite involutory latin quandle such that $|L_aL_b|$ is infinite
for every $a, b \in Q$, so Corollary 3.8] cannot be pushed any further. Finite simple superfaithful
involutory quandles are isomorphic to $Q \cong \text{Aff}(\mathbb{Z}_p,-1)$ where $p$ is a prime (the unique simple latin
involutory quandles). Simple involutory non-latin quandles exist, e.g. the smallest example is
SmallQuandle(10,1) from the 21] database of GAP.

**Corollary 3.9.** Let $Q$ be an involutory quandle such that $|L_aL_b|$ is odd for every $a, b \in Q$. Then
(i) $Q/\lambda_Q$ is latin.
(ii) If $Q/\lambda_Q$ is finite then $Q$ is solvable.

**Proof.** (i) Assume that $|L_aL_b| = 2n + 1$. According to Lemma 3.2 we have $L_aL_{(L_aL_a)^n(b)} = (L_aL_b)^{2n+1} = 1$, i.e.
$[a]_{\lambda_Q} = [(L_bL_a)^n(b)]_{\lambda_Q} = (L_{[b]_{\lambda_Q}} L_{[a]_{\lambda_Q}})^n([b]_{\lambda_Q})$. 

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Thus \(Q/\lambda_Q\) is superconnected and then latin by Theorem 3.10.

(ii) If \(Q/\lambda_Q\) is finite, the group \(\text{Dis}(Q/\lambda_Q)\) is solvable according to the main result of [27] and \(\text{Dis}^{\lambda_Q}\) is central in \(\text{Dis}(Q)\) [11 Corollary 2.3]. Therefore \(\text{Dis}(Q)\) is also solvable and we can conclude that \(Q\) is solvable by using [12 Lemma 6.1].

**Theorem 3.10.** Let \(Q\) be a finite nilpotent involutory quandle. Then \(Q\) is latin if and only if \(Q\) is connected and faithful.

**Proof.** The forward implication is clear. Let \(Q\) be faithful and connected. According to Corollary 3.9 (i) we just need to prove that \([L_aL_b]\) is odd for every \(a, b \in Q\). According to [12 Theorem 1.4], \(Q\) decomposes as the direct product of connected quandles of prime power order. Such quandles are constructed over the \(p\)-Sylow subgroups of the nilpotent group \(\text{Dis}(Q)\). According to [12 Theorem 8.1] there are no connected involutory quandle of size a power of 2. Therefore the 2-Sylow of \(\text{Dis}(Q)\) is trivial and so the order of \(L_aL_b\) is odd.

Let us include the following group theoretical application in the same direction of the main result of [2].

**Corollary 3.11.** Let \(G\) be a finite group generated by a conjugacy class of involutions \(C\).

(i) If \(C\) contains no commuting elements then \(G\) is solvable.

(ii) If \(|abZ(G)|\) is odd for every \(a, b \in C\) then \(G\) is solvable.

**Proof.** The quandle \(Q = \text{Conj}(C)\) is involutory and \(\text{LMlt}(Q) \cong G/Z(G)\). If \(Q\) is solvable, e.g. if \(Q\) is latin, then \(G\) is also solvable [12 Lemma 6.1].

(i) If \(C\) has no commuting elements, then \(Q\) is superfaithful and so latin.

(ii) If \([L_aL_b] = |abZ(G)|\) is odd for every \(a, b \in Q\), \(Q\) is solvable by Corollary 3.9.

### 3.3. Locally reductive involutory quandles.

In [5] we investigate several classes of quandles, including quandles with no connected subquandles (in some sense the dual class with respect to superconnected quandles). In the finite case, such class is defined by a set of identities.

Let us define

\[
u_0(a, b) = a, \quad u_{n+1}(a, b) = u_n(a, b) * b.
\]

A quandle is said \(n\)-locally reductive if \(u_n(a, b) = b\) for every \(a, b \in Q\) (see [5 Section 3.2]). According to the theory developed in [5], for a finite quandle \(Q\) the following properties are equivalent:

(i) \(Q\) is locally reductive.

(ii) \(Q/\lambda_Q\) is locally reductive.

(iii) \(Q\) has no (proper) connected subquandles.

(iv) \(\text{LMlt}(Q)\) is nilpotent.

In this section we offer a characterization of involutory quandles satisfying one of this conditions in terms of the properties of the canonical generators of the displacement group.

**Lemma 3.12.** Let \(Q\) be an involutory quandle, \(a, b \in Q\) and \(n \in \mathbb{N}\). Then \((L_aL_b)^{2^n} = L_{u_n(a, b)L_b}\).

**Proof.** For \(n = 0\), the statement is trivial. By induction

\[
L_{u_{n+1}(a, b)L_b} = L_{u_n(a, b)L_bL_b} = L_{u_n(a, b)L_b}L_{u_{n+1}(a, b)L_b} = (L_aL_b)^{2^n}(L_aL_b)^{2^{n+1}} = (L_aL_b)^{2^{n+1}}.
\]

For involutory quandles, the property of being locally reductive is also determined by the properties of the canonical generators of the displacement group.

**Proposition 3.13.** Let \(Q\) be an involutory quandle. Then \(Q/\lambda_Q\) is \(n\)-locally reductive if and only if \((L_aL_b)^{2^n} = 1\) every \(a, b \in Q\).

**Proof.** The quandle \(Q/\lambda_Q\) is \(n\)-locally reductive if and only if

\[
L_{u_n(a, b)} = [u_n(a, b)]_{\lambda_Q} = u_n([a]_{\lambda_Q}, [b]_{\lambda_Q}) = [b]_{\lambda_Q} = L_b
\]
for every $a, b \in Q$. By Lemma 3.12 we have that
\[ L_{a,b} = (L_a L_b)^{2^n} \]
Therefore $L_{a,b} = L_b$ if and only if $(L_a L_b)^{2^n} = 1$. □

**Corollary 3.14.** An involutory quandle $Q$ is locally reductive if and only if there exists $n \in \mathbb{N}$ such that $(L_a L_b)^{2^n} = 1$ for every $a, b \in Q$.

**Corollary 3.15.** Let $G$ be a finite group generated by a conjugacy class of involutions $C$. Then $G$ is nilpotent if and only if there exists $n \in \mathbb{N}$ such that $|abZ(G)| = 2^n$ for every $a, b \in C$.

**Proof.** The quandle $Q = \text{Conj}(C)$ is involutory and $\text{LMlt}(Q) = G/Z(G)$. Therefore $G$ is nilpotent if and only if $Q$ is locally reductive. Hence, we can conclude by Corollary 3.14 using that $|L_a L_b| = |abZ(G)|$ for every $a, b \in Q$. □

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