AN ORTHOGONALITY RELATION FOR A THIN FAMILY OF $GL(3)$ MAASS FORMS

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Abstract. We prove an orthogonality relation for the Fourier-Whittaker coefficients of a thin family of $GL(3)$ Maass forms containing all self-dual forms. This is obtained by analysing the Kuznetsov trace formula on $GL(3)$ for a certain family of test functions. The method also yields Weyl’s law for the same family of Maass forms.

1. Introduction

A well known fact about Dirichlet characters is the following orthogonality relation

$$\sum_{\chi \mod q} \chi(n)\overline{\chi(m)} = \begin{cases} \phi(q), & \text{if } n \equiv m \mod q \\ 0, & \text{otherwise.} \end{cases} \quad (1.1)$$

for integers $m, n$ coprime to $q$, where the sum on the left is over all characters (mod $q$). Since Dirichlet characters can be viewed as automorphic representations of $GL(1, \mathbb{A}_\mathbb{Q})$, this result can be interpreted as the simplest case of the orthogonality relation conjectured by Zhou [19] concerning Fourier-Whittaker coefficients of Maass forms on the space $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$, $n \geq 2$. This orthogonality relation conjectured by Zhou was proved by Bruggeman [2] in the case $n = 2$ and by Goldfeld-Kontorovich [6] and Blomer [1] in the case $n = 3$. Versions of this result have applications to the Sato-Tate problem for Hecke operators, both in the holomorphic [3], [17] and non-holomorphic setting [11], [12], [14], as well as to the problem of determining symmetry types of families of L-functions [6] as introduced in the work of Katz-Sarnak [9]. In this paper, we prove an orthogonality relation for the Fourier-Whittaker coefficients of a thin family of Maass forms on $SL_3(\mathbb{Z})$ which contains all self-dual forms. The main tool used is the Kuznetsov trace formula for $GL(3)$ developed by Blomer [1] and Goldfeld-Kontorovich [6]. The same methods yield a weighted Weyl’s law for the same family of Maass forms.

Weyl’s law was first proved by Selberg [16] for Maass forms on $SL_2(\mathbb{Z})$ using the Selberg trace formula. Miller [13] later showed Weyl’s law for Maass forms on the space $SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO_3(\mathbb{R})$. This result has since been obtained in more general settings in [11], [12], [14]. The version of Weyl’s law presented here tells us that the family of Maass forms being studied has zero density in the set of all Maass forms on $SL_3(\mathbb{Z})$. We also note that a lower bound Weyl’s law for self-dual forms on $SL_3(\mathbb{Z})$ follows from the Gelbart-Jacquet lift [4] and Weyl’s law for Maass forms on $SL_2(\mathbb{Z})$. This work is a step towards a proof of the Gelbart-Jacquet lift by isolating the contribution of symmetric square lifts from $GL(2)$ in the $GL(3)$ Kuznetsov trace formula, in the spirit of Langlands’ “Beyond Endoscopy” [10].

Let $\phi_j$ ($j = 0, 1, 2, \ldots$), where $\phi_0$ is a constant function, be a set of orthogonal $GL(3)$ Hecke-Maass forms with spectral parameters $\nu(j) = (\nu_1^{(j)}, \nu_2^{(j)})$, and define $\nu_3^{(j)} = -\nu_1^{(j)} - \nu_2^{(j)}$. The parameters are normalized such that for tempered forms they are purely imaginary. For a particular non-constant Maass form $\phi$ with spectral parameters $\nu = (\nu_1, \nu_2)$ define $\nu_3 = -\nu_1 - \nu_2$, and define Langlands parameters

$$\alpha_1 = \nu_1 - \nu_3, \quad \alpha_2 = \nu_2 - \nu_1, \quad \alpha_3 = \nu_3 - \nu_2.$$
The Laplace eigenvalue of $\phi$ is given by
\[ \lambda_\phi = 1 - 3(\nu_1^2 + \nu_2^2 + \nu_3^2) = 1 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2). \]
Let
\[ L_j = \text{Res}_{s=1}L(s, \phi_j \times \tilde{\phi}_j). \]
For $T \gg 1$, $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ (with $\nu_3 = -\nu_1 - \nu_2$) and $R > 0$, to be fixed later, we define
\[
T \nu \left( \frac{2+R+3\nu_3}{4} \right) \Gamma \left( \frac{2+R-3\nu_1}{4} \right) \right)^2 \frac{\prod_{j=1}^{3} \Gamma \left( \frac{1+3\nu_j}{2} \right) \Gamma \left( \frac{1-3\nu_j}{2} \right)}{2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{T^2} R^{\nu_j}} \left( T \nu \sqrt{T \nu} R^{\nu_j} \right)^{T^2}.
\] 
This function is essentially supported on the region where $|\nu_1|, |\nu_2|, |\nu_3| \ll T$. Let $\phi$ be tempered and $|\nu_1|, |\nu_2|, |\nu_3| < T$. Note that in this situation, $h_{T,R}(\nu)$ is real valued and positive. We estimate this function in three regions. If one of the $\alpha_i$ is equal to 0, then
\[
|1 + |\nu_1|)(1 + |\nu_2|)(1 + |\nu_3|)|^R \leq h_{T,R}(\nu) \leq |1 + |\nu_1|)(1 + |\nu_2|)(1 + |\nu_3|)|^R,
\]
for some $c_R \geq c_R^{(1)} > 0$. If one of the $|\alpha_i|$ is smaller than $R$, then
\[
h_{T,R}(\nu) \gg R \frac{[(1 + |\nu_1|)(1 + |\nu_2|)(1 + |\nu_3|)]^R}{T^2}.
\]
If $|\alpha_1|, |\alpha_2|, |\alpha_3| > 2R^{3/2}$, then
\[
h_{T,R}(\nu) \ll \frac{1}{T R}.
\]
\textbf{Theorem 1.1.} For $j = 0, 1, 2, \ldots$, let $\phi_j$ be a set of orthogonal $\text{GL}(3)$ Hecke-Maass forms with spectral parameters $\nu^{(j)} = \left( \nu_1^{(j)}, \nu_2^{(j)} \right)$. Here $\phi_0$ is a constant function. Let $h_{T,R}$ be given as in (1.3). We have that for fixed $R > 50$ and some $c_R > 0$,
\[
\sum_{j \geq 1} \frac{|h_{T,R}(\nu^{(j)}), L_j |}{c_R} = \frac{T^{4+3R}}{\sqrt{\log T}} + O(T^{3+3R}), \quad (T \to \infty).
\]
Moreover, for fixed $\epsilon > 0$ and positive integers $m_1, m_2, n_1, n_2$, we have
\[
\sum_j A_j(m_1, m_2)A_j(n_1, n_2) \frac{|h_{T,R}(\nu^{(j)}), L_j |}{c_R} = \begin{cases} \sum_j \frac{|h_{T,R}(\nu^{(j)}), L_j |}{c_R} = O_R, & \text{if } m_1 = m_2 = n_1, \\ O_R, & \text{otherwise.} \end{cases}
\]
\textbf{Remark 1.2.} Note that an analogous result in [2] for all $\text{GL}(3)$ Hecke-Maass forms with spectral parameters $|\nu_1|, |\nu_2|, |\nu_3| \ll T$ yields a main term of size $T^{5+3R}$. This implies that the family of $\text{GL}(3)$ Maass forms picked out by our test function $h_{T,R}$ is indeed significantly thinner (by a factor of $T \sqrt{\log T}$) than the family of all $\text{GL}(3)$ Hecke-Maass forms.
Remark 1.3. The test functions $h_{T,R}$ appearing in this theorem are a product of three terms chosen with the following objectives: the first exponential term contributes with polynomial decay when all $|\alpha_1|, |\alpha_2|, |\alpha_3| \gg 0$ and exponential decay when all $|\alpha_1|, |\alpha_2|, |\alpha_3| \gg T^\epsilon$; the second exponential term contributes with exponential decay when one of $|\nu_i| > T^{1+\epsilon}$; the product of Gamma factors is already partially present in the Kuznetsov trace formula and its particular form is such that it has polynomial growth in $\nu$. We note that the methods presented here have also been carried out for a different family of test functions in [6]. It would be interesting to generalize 1.1 for a broader class of test functions.

Remark 1.4. Blomer [1] shows that the residues $L_j$ are bounded by

$$
\left( (1 + |\nu_1^{(j)}|)(1 + |\nu_2^{(j)}|)(1 + |\nu_3^{(j)}|) \right)^{-1} \ll \epsilon \left( (1 + |\nu_1^{(j)}|)(1 + |\nu_2^{(j)}|)(1 + |\nu_3^{(j)}|) \right)^\epsilon
$$

and, conjecturally, the lower bound is expected to be

$$
\left( 1 + |\nu_1^{(j)}|)(1 + |\nu_2^{(j)}|)(1 + |\nu_3^{(j)}|) \right)^{-\epsilon} \ll \epsilon \ L_j.
$$
2. GL(3) Kuznetsov trace formula

Following ([1], Proposition 4) and [6], the GL(3) Kuznetsov trace formula takes the form

\[ C + \mathcal{E}_{\min} + \mathcal{E}_{\max} = \Sigma_1 + \Sigma_{2\alpha} + \Sigma_{2\beta} + \Sigma_3, \]

(2.1)

where all the quantities are explained below.

**Definition 2.1.** Let \( \nu \in \mathbb{C}^2 \) and \( z \in \mathfrak{h} \) with Iwasawa coordinates

\[ z = xy = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Let

\[ I_{\nu}(z) = y_1^{\nu_1+2\nu_2}y_2^{2\nu_1+\nu_2}, \quad (z = xy \in \mathfrak{h}). \]

Define the Whittaker function

\[ W_{\nu}(z) = \pi^{-\nu_1-\nu_2}\Gamma \left( \frac{1 + 3\nu_1}{2} \right) \Gamma \left( \frac{1 + 3\nu_2}{2} \right) \Gamma \left( \frac{1 + 3\nu_1 + 3\nu_2}{2} \right) \]

\[ \times \int_\mathbb{R}^3 \int_\mathbb{R}^3 \int_\mathbb{R}^3 I_{\nu} \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 & u_3 \\ 0 & u_1 & 0 \end{pmatrix} \right) e(-u_1 - u_2)du_1 du_2 du_3, \]

(2.2)

where \( e(x) = e^{2\pi ix} \).

Let \( F : \mathbb{R}_+^2 \rightarrow \mathbb{C} \) be a bounded function with the following decay

\[ |F(y_1, y_2)| \ll (y_1 y_2)^{2+\epsilon}, \]

(2.3)

as \( y_1 \rightarrow 0 \) or \( y_2 \rightarrow 0 \). For such a function define its Lebedev-Kontorovich transform \( F^\# : D \times D \rightarrow \mathbb{C} \) as

\[ F^\#(\nu) = \int_{\mathbb{R}_+^2} F(y) W_{\nu}(y) \frac{dy_1 dy_2}{y_1 y_2}, \]

(2.4)

where \( W_{\nu}(y) \) is the Whittaker function and \( D \subset \mathbb{C} \) is of the form \((-\delta, \delta) \times i\mathbb{R} \) for some \( \delta > 0 \).

For \( m, n \in \mathbb{Z}, m, n > 0 \), let \( A_\phi(m,n) \) be the Fourier-Whittaker coefficient of a Maass form \( \phi \) as in [6]. We normalize the GL(3) Hecke-Maass forms by choosing the first Fourier-Whittaker coefficient to be \( A_\phi(1, 1) = 1 \). Fix positive integers \( m_1, m_2, n_1, n_2 \). The cuspidal contribution \( C \) is given by

\[ C = \sum_j A_j(m_1, m_2)A_j(n_1, n_2) \frac{|F^\#(\nu_1^{(j)}, \nu_2^{(j)})|^2}{6\mathcal{E}_j \prod_{k=1}^3 \Gamma \left( \frac{1 + 3\nu_k}{2} \right) \Gamma \left( \frac{1 - 3\nu_k}{2} \right)}, \]

(2.5)

with \( j \) ranging over cuspidal Hecke-Maass forms on \( SL(3, \mathbb{Z}) \). The minimal Eisenstein series is

\[ \mathcal{E}_{\min} = \frac{1}{(4\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} A_{\nu}(m_1, m_2)A_\nu(n_1, n_2) \frac{|F^\#(\nu_1, \nu_2)|^2}{\prod_{k=1}^3 \zeta(1 + 3\nu_k) \Gamma \left( \frac{1 + 3\nu_k}{2} \right)} d\nu_1 d\nu_2, \]

(2.6)

where the Fourier coefficients satisfy \( |A_\nu(n_1, n_2)| \ll \epsilon (n_1 n_2)^\epsilon \). The maximal Eisenstein series is
\[ E_{\text{max}} = \frac{c}{2\pi i} \sum_{j=-\infty}^{\infty} \frac{B_{\nu,j}(m_1, m_2) \overline{B_{\nu,j}(n_1, n_2)}}{|L(1, \text{Ad } u_j) L(1 + 3\nu, u_j)|^2} \left| \frac{\nu - \frac{ir_1}{2}}{\frac{3}{2}} \right|^2 \left| \frac{1+3\nu+ir_1}{2} \right|^2 \left( \frac{1+3\nu+ir_1}{2} \right)^2 \right]^2 \nu, \quad (2.7) \]

where \( c \) is an absolute constant and \( \{u_j\} \) is an orthogonal basis of Hecke-Maass forms on \( \text{SL}(2, \mathbb{Z}) \) with eigenvalues \( 1/4 + r_j^2 \). The Fourier coefficients satisfy \( |B_{\nu,j}(n_1, n_2)| \ll (n_1 n_2)^{1/2+\epsilon} \). Set

\[ 1_{A|B} = \begin{cases} 1, & \text{if } A \mid B \\ 0, & \text{otherwise}. \end{cases} \]

On the geometric side we have

\[ \Sigma_1 = 1_{\{m_1 = n_1\}} \int_{\mathbb{R}_+^2} \frac{\sqrt{m_1 n_2}}{(y_1 y_2)^3} |F(y_1, y_2)|^2 \frac{dy_1 dy_2}{(y_1 y_2)^3} = 1_{\{m_2 = n_2\}} \langle F, F \rangle, \quad (2.8) \]

\[ \Sigma_{2a} = \sum_{\delta = \pm 1} \sum_{\frac{D_1 D_2}{m_1 m_2} = n_1 n_2} \tilde{S}(\delta m_1, n_1, n_2, D_1, D_2) \frac{m_1 n_2}{D_1 D_2} \tilde{J}_\delta \left( \frac{\sqrt{m_1 n_2}}{D_1 D_2} \right), \quad (2.9) \]

\[ \Sigma_{2b} = \sum_{\delta = \pm 1} \sum_{\frac{D_1 D_2}{m_1 m_2} = n_1 n_2} \tilde{S}(\delta m_2, n_2, n_1, D_2, D_1) \frac{m_2 n_1}{D_1 D_2} \tilde{J}_\delta \left( \frac{\sqrt{m_2 n_1}}{D_1 D_2} \right), \quad (2.10) \]

\[ \Sigma_3 = \sum_{\delta_1, \delta_2 = \pm 1} \sum_{D_1, D_2} \frac{1}{D_1 D_2} \tilde{S}(\delta_1 m_1, \delta_2 m_2, n_1, n_2, D_1, D_2) \tilde{J}_{\delta_1, \delta_2} \left( \frac{\sqrt{m_1 n_2 D_1}}{D_2}, \frac{\sqrt{m_2 n_1 D_2}}{D_1} \right), \quad (2.11) \]

where

\[ \tilde{J}_\delta(A) = A^{-2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^2} F(A y_1, y_2) e(-\delta A x_1 y_1) F \left( y_2 \frac{\sqrt{1 + x_1^2 + x_2^2}}{1 + x_1^2}, \frac{A y_1 y_2}{1 + x_1^2 + x_2^2} \right) \]

\[ \times e \left( y_2 \frac{x_1 x_2}{1 + x_1^2} + \frac{A x_2}{y_1 y_2} \frac{x_1 y_1}{1 + x_1^2 + x_2^2} \right) \frac{dx_1 dx_2 dy_1 dy_2}{y_1 y_2}, \quad (2.12) \]

\[ \tilde{J}_{\delta_1, \delta_2}(A_1, A_2) = (A_1 A_2)^{-2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^3} F(A_1 y_1, A_2 y_2) e(-\delta_1 A_1 x_1 y_1 - \delta_2 A_2 x_2 y_2) \]

\[ \times F \left( \frac{A_2 \sqrt{(x_1 x_2 - x_3)^2 + x_1^2 + 1}}{y_2 1 + x_3^2}, \frac{A_1 \sqrt{1 + x_2^2 + x_3^2}}{y_1 (x_1 x_2 - x_3)^2 + x_1^2 + 1} \right) \]

\[ \times e \left( -\frac{A_2 x_1 x_3 + x_2}{y_2 1 + x_2^2 + x_3^2} - \frac{A_1 x_2 (x_1 x_2 - x_3) + x_1}{y_1 (x_1 x_2 - x_3)^2 + x_1^2 + 1} \right) \frac{dx_1 dx_2 dx_3 dy_1 dy_2}{y_1 y_2}, \quad (2.13) \]

\[ \tilde{S}(m_1, n_1, n_2, D_1, D_2) = 1_{D_1|D_2} \sum_{C_1 (\text{mod } D_1), \ C_2 (\text{mod } D_2)} e \left( \frac{m_1 C_1 + n_1 \overline{C}_1 C_2}{D_1} \right) e \left( \frac{n_2 \overline{C}_2}{D_2/D_1} \right), \quad (2.14) \]
\[ S(m_1, m_2, n_1, n_2, D_1, D_2) = \sum \sum \sum \sum e \left( \frac{m_1 B_1 + n_1 (Y_1 D_2 - Z_1 B_2)}{D_1} \right) \times e \left( \frac{m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_1)}{D_2} \right), \]  

and \( Y_1, Y_2, Z_1, Z_2 \) are such that

\[ Y_1 B_1 + Z_1 C_1 \equiv 1 \pmod{D_1} \text{ and } Y_2 B_2 + Z_2 C_2 \equiv 1 \pmod{D_2}. \]
3. Bounding the inverse transform

Let

$$F_{T,R}^\#(\nu_1, \nu_2) = \left( T^{\alpha_1^2/R^2} + T^{\alpha_2^2/R^2} + T^{\alpha_3^2/R^2} \right) e^{(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)/T^2} \left( \prod_{j=1}^{3} \Gamma \left( \frac{2 + R + 3\nu_j}{4} \right) \Gamma \left( \frac{2 + R - 3\nu_j}{4} \right) \right).$$

(3.1)

As $F_{T,R}^\#$ has enough exponential decay on a strip $|\Re(\nu_1)|, |\Re(\nu_2)| < \epsilon$ then by Lebedev-Whittaker inversion as in section 2.2 of [6],

$$F_{T,R}(y) = \frac{1}{(\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{T,R}^\#(\nu) W_\nu(y) \prod_{j=1}^{3} \frac{d\nu_1 d\nu_2}{\Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( \frac{-3\nu_j}{2} \right)}.$$

(3.2)

We also have the Parseval-type identity

$$\langle F_{T,R}, F_{T,R} \rangle = \int_{\mathbb{R}^2_+} |F_{T,R}(y_1, y_2)|^2 \frac{dy_1 dy_2}{(y_1 y_2)^2} = \frac{1}{(\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| F_{T,R}(\nu) \right|^2 d\nu_1 d\nu_2 = \langle F_{T,R}^\#, F_{T,R}^\# \rangle.$$

(3.3)

For proofs of these results refer to [5].

**Proposition 3.1.** Fix $C_1, C_2 > 0$, $R > 3 \max(C_1, C_2) + 6$ and $\epsilon > 0$. For any $y_1, y_2 > 0$, $T \gg 1$, we have

$$|F_{T,R}(y)| \ll_{C_1, C_2, R, \epsilon} y_1 y_2 T^{3R/2+11/2+C_1/2+C_2/2+\epsilon} \left( \frac{y_1}{T} \right)^{C_1} \left( \frac{y_2}{T} \right)^{C_2}.$$

(3.4)

**Proof.** We start by writing out the representation of $W_\nu$ as an inverse Mellin transform [18],

$$W_\nu(y) = \frac{y_1 y_2 \pi^{3/2}}{(2\pi i)^2} \int_{(C_2)} \prod_{j=1}^{3} \frac{\Gamma \left( \frac{s_1+\alpha_j}{2} \right) \Gamma \left( \frac{s_2-\alpha_j}{2} \right)}{4\pi^{s_1+s_2}\Gamma \left( \frac{s_1+s_2}{2} \right)} y_1^{-s_1} y_2^{-s_2} ds_1 ds_2,$$

for $C_1, C_2 > 0$. Combining this with the Lebedev-Whittaker inverse transform of $F_{T,R}$, we observe that

$$F_{T,R}(y) = \int_{(0)} \int_{(0)} \int_{(C_1)} \int_{(C_2)} \frac{F_{T,R}^\#(\nu_1, \nu_2)}{\prod_{j=1}^{3} \frac{3\nu_j}{2}} \Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( \frac{-3\nu_j}{2} \right) 16\pi^{s_1+s_2+5/2} \frac{\Gamma \left( \frac{s_1-\alpha_j}{2} \right) \Gamma \left( \frac{s_2+\alpha_j}{2} \right)}{\Gamma \left( \frac{s_1+s_2}{2} \right)} y_1^{-s_1} y_2^{-s_2} ds_1 ds_2 d\nu_1 d\nu_2.$$

Assume $2k < C_1 < 2k + 2$ for some non-negative integer $k$ and pull $s_1$ from the contour $(C_1)$ to the contour $(-C_1)$. Then

$$F_{T,R}(y) = \mathcal{M} + \sum_{m=0}^{k} \sum_{l=1}^{3} \mathcal{R}_{m,l},$$

(3.5)

where $\mathcal{M}$ is the main term and is given by the same expression as $F_{T,R}$ with the contour $(C_1)$ replaced by $(-C_1)$. Here $\mathcal{R}_{m,l}$ are the residues corresponding to the poles of the integrand at $s_1 = \alpha_l - 2m$ for $m = 0, \ldots, k$ and $l = 1, 2, 3$. These residues are given by
We will now bound each term in (3.6). Let \( \alpha \) for some constant \( C \) the first exponential term in the definition of \( \alpha \) which could possibly show up when two of the \( \alpha_i \) differ by an even integer. In this situation, at least one of \( \pm 3 \nu_1, \pm 3 \nu_2, \pm 3 \nu_3 \) is equal to a non-positive even integer, making the integrand identically zero in that region.

Now assume \( 2k' < C_2 < 2k' + 2 \) for some non-negative integer \( k' \). For each term in (3.5) shift variable \( s_2 \) from the contour \((C_2)\) to \((C_2)\) to get,

\[
F_{T,R}(y) = \bar{M} + \sum_{m' = 0}^{k} \sum_{l' = 1}^{3} M_{m',l'} + \sum_{m,m' = 0}^{k} \sum_{l = 1}^{3} \sum_{l' \neq l} R_{m,l,m',l'}
\]

where \( \bar{M} \) is the main term, given by the same expression as \( M \) with the contour \((C_2)\) replaced by \((-C_2)\) and \( M_{m',l'} \) are the residues corresponding to the poles of the integrand of \( M \) at \( s_2 = -\alpha_l - 2m' \) for \( m' = 0, \cdots, k' \). The terms \( R_{m,l,m',l'} \) are the residues corresponding to the poles of the integrand of \( R_{m,l} \) at \( s_2 = -\alpha_l - 2m' \) for \( m' = 0, \cdots, k' \) and \( l' \neq l \). The residues \( M_{m',l'} \) are given by

\[
M_{m',l'} = C_{m',l'} \int_{(0)(0)(-C_1)} F_{T,R}^{\#}(\nu_1, \nu_2) \prod_{j = 1}^{3} \Gamma \left( \frac{\alpha_j - 2m_\nu - \alpha_j}{2} \right) \prod_{j \neq l'} \Gamma \left( \frac{-\alpha_j - 2m_\nu + \alpha_j}{2} \right) y_1^{1 - s_1} y_2^{1 - s_2} d\nu_1 d\nu_2
\]

and the residues \( R_{m,l,m',l'} \) are given by

\[
R_{m,l,m',l'} = C_{m,l,m',l'} \int_{(0)(0)(-C_1)} F_{T,R}^{\#}(\nu_1, \nu_2) \prod_{j = 1}^{3} \Gamma \left( \frac{\alpha_j - 2m_\nu - \alpha_j}{2} \right) \prod_{j \neq l'} \Gamma \left( \frac{-\alpha_j - 2m_\nu + \alpha_j}{2} \right) y_1^{1 - s_1} y_2^{1 - s_2} d\nu_1 d\nu_2
\]

We will now bound each term in (3.6). Let \( \nu_j = it_j \) and \( s_j = C_j + iu_j \). Note that, for \( \Re(\nu_j) = 0 \), the first exponential term in the definition of \( F_{T,R}^{\#} \) is bounded and the second one has exponential decay for \( |t_j| > T^{1+\delta} \). Using Stirling’s formula to estimate the Gamma factors we get

\[
|\bar{M}| \ll y_1^{1 + C_1} y_2^{1 + C_2} \iint_{|t_1|,|t_2| < T^{1+\delta}} \iint_{\mathbb{R}^2} P \cdot \exp(\mathcal{E}) \, du_1 du_2 dt_1 dt_2,
\]

where the exponential factor is given by

\[
\frac{4\mathcal{E}}{\pi} = 3 \sum_{j = 1}^{3} |t_j| + |u_1 + u_2| - \sum_{j = 1}^{3} |iu_1 - \alpha_j| - \sum_{j = 1}^{3} |iu_2 + \alpha_j|
\]

and the polynomial term is given by
\[ P = \left( \prod_{j=1}^{3} (1 + |t_j|) \right)^{(R+2)/2} (1 + |u_1 + u_2|)^{(1+C_1+C_2)/2} \]
\[ \times \left( \prod_{j=1}^{3} (1 + |iu_1 - \alpha_j|) \right)^{(-C_1-1)/2} \left( \prod_{j=1}^{3} (1 + |iu_2 + \alpha_j|) \right)^{(-C_2-1)/2}. \] (3.7)

We now show that the exponential factor is non-positive. As the exponential factor is invariant under cyclic permutations of \((t_1, t_2, t_3)\), we may assume, without loss of generality, that \(t_1\) and \(t_2\) have the same sign. Then \(|\alpha_1| + |\alpha_3| = 3|t_1| + 3|t_2|\). As \(|u_1 + u_2| \leq |iu_1 - \alpha_2| + |iu_2 + \alpha_2|\), we get

\[ \frac{4\mathcal{E}}{\pi} \leq 3 \sum_{j=1}^{3} |t_j| - |u_1 - \alpha_1| - |u_1 - \alpha_3| - |u_2 + \alpha_1| - |u_2 + \alpha_3| \]
\[ \leq 6|t_1| + 6|t_2| - 2(|\alpha_1| + |\alpha_3|) = 0. \]

For either \(|u_1| > 5T^{1+\delta}\) or \(|u_2| > 5T^{1+\delta}\), the exponential factor is bounded above by \(-T^{1+\delta}\) giving exponential decay to the integral. Integrating first over \(u_1, u_2\) we get

\[ |\mathcal{M}| \ll y_1^{1+C_1} y_2^{1+C_2} \int \int \int \mathcal{P} du_1 du_2 dt_1 dt_2 \ll y_1^{1+C_1} y_2^{1+C_2} T^{(3R+11-C_1-C_2)/2+\epsilon} \]

by choosing \(\delta\) appropriately. To bound the residues \(\mathcal{M}_{m',l'}\) we start by shifting variables \(\nu_1\) and \(\nu_2\) to contours \((B_1)\) and \((B_2)\), respectively, where \(|B_1|, |B_2| < R/3\) and \(B_j' < C_1\) for \(j \neq l\), defining \(B_1' = 2B_1 + B_2, B_2' = B_2 - B_1, B_3' = -B_1 - 2B_2\), in a manner similar to the \(\nu_j\). Note that the first exponential term is now bounded by \(3T^{(\max(B_1',B_2',B_3')^3/R^2) < T}\). It follows that

\[ |\mathcal{M}_{m',l'}| \ll y_1^{1+C_1} y_2^{1+B_1'+2m'} T \int \int \mathcal{P} \cdot \exp(\mathcal{E}) du_1 dt_1 dt_2 \]

where the exponential factor is given by

\[ \frac{4\mathcal{E}}{\pi} = 3 \sum_{j=1}^{3} |t_j| + |u_1 - \Im(\alpha_{l'})| - \sum_{j=1}^{3} |u_1 - \Im(\alpha_j)| - \sum_{j \neq l'} |\Im(\alpha_j - \alpha_{l'})|, \]

and the polynomial term is given by

\[ P = \left( \prod_{j=1}^{3} (1 + |t_j|) \right)^{(R+2)/2} (1 + |u_1 - \Im(\alpha_{l'})|)^{(1+C_1+B_1'+2m')/2} \]
\[ \times \left( \prod_{j=1}^{3} (1 + |u_1 - \Im(\alpha_j)|) \right)^{(-C_1-1)/2} \left( \prod_{j \neq l'} (1 + |\Im(-\alpha_{l'} + \alpha_j)|) \right)^{(B_j' - B_{l'} - 2m' - 1)/2}. \] (3.8)

The exponential factor is again non-positive as

\[ \frac{4\mathcal{E}}{\pi} \leq 3 \sum_{j=1}^{3} |t_j| - \sum_{j \neq l'} |u_1 - \Im(\alpha_j)| - \sum_{j \neq l'} |\Im(\alpha_j - \alpha_{l'})| \leq 0. \]
by using the triangle inequality on the second sum to get a difference of \(\Im(\alpha_j)\). We now pick 
\[ B'_j = C_2 - 2m' \]  
and 
\[ B'_j < 0 \]  
for \( j \neq l' \) to get 
\[ |M_{m',l'}| \ll y_1^{1+C_1} y_2^{1+C_2}T \int_{|t_1|,|t_2|<T^{1+\delta}} \mathcal{P} dt_1 dt_2 \ll y_1^{1+C_1} y_2^{1+C_2} T^{(3R+11-C_1-C_2)/2+\epsilon}. \]

To bound the residues \( R_{m,l',m'} \) we again shift variables \( \nu_1 \) and \( \nu_2 \) to contours \((B_1)\) and \((B_2)\), respectively, where \( |B_1|, |B_2| < R/3 \). To simplify notation, we will assume without loss of generality that \( l = 1 \) and \( l' = 2 \). We obtain 
\[ |R_{m,1,m',2}| \ll y_1^{1-B'_1+2m} y_2^{1+B'_2+2m'} T \int_{|t_1|,|t_2|<T^{1+\delta}} \mathcal{P} \cdot \exp(\mathcal{E}) dt_1 dt_2 \]
where the exponential factor \( \mathcal{E} \) is given by 
\[ \frac{4\mathcal{E}}{\pi} = 3 \sum_{j=1}^{3} |t_j| - \sum_{j \neq 1} |\Im(\alpha_j - \alpha_1)| - \sum_{j \neq 2} |\Im(\alpha_j - \alpha_2)| + |\Im(\alpha_1 - \alpha_2)| = 0 \]
and the polynomial term is given by 
\[ \mathcal{P} = \left( \prod_{j=1}^{3} (1 + |t_j|)^{(R+2)/2} \right) \left( 1 + |t_1| \right)^{(3B_1-1)/2} \left( 1 + |t_2| \right)^{(3B_2-2m-1)/2} \left( 1 + |t_3| \right)^{(3B_3-2m-1)/2}, \]
where \( B_3 = B_1 + B_2 \). Then pick \( B'_1 = -C_1 + 2m \) and \( B'_2 = C_2 - 2m' \) to get 
\[ |R_{m,1,m',2}| \ll y_1^{1+C_1} y_2^{1+C_2} T \int_{|t_1|,|t_2|<T^{1+\delta}} \mathcal{P} dt_1 dt_2 \ll y_1^{1+C_1} y_2^{1+C_2} T^{(3R+9-2C_1-2C_2+2m+2m')/2+\epsilon} \]
\[ \ll y_1^{1+C_1} y_2^{1+C_2} T^{(3R+9-C_1-C_2)/2+\epsilon}. \]
Combining all the bounds we get the desired result. 

\[ \square \]

**Remark 3.2.** It follows from this proposition that \( F_{T,R} \) satisfies the decay condition \( 2.3 \).
4. Bounds for the Kloosterman terms

We start by getting bounds for the Kloosterman integrals \( \tilde{J}_6(A) \) and \( J_{6_1, 6_2}(A_1, A_2) \). Break the integral \( \tilde{J}_3(A) = \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 \) depending on whether \( y_1 \) and \( y_2 \) are smaller or greater than 1. To estimate \( \tilde{J}_1 \), start by taking absolute values inside the integral

\[
|\tilde{J}_1| \ll A^{-2} \int_0^1 \int_0^1 \int_{\mathbb{R}^2} |F_{T,R}(A_1 y_1, A_2 y_2)| \left| F_{T,R} \left( \frac{1 + x_1^2 + x_2^2}{1 + x_1^2}, \frac{A}{y_1 y_2} \right) \right| \, dx_1 \, dx_2 \, dy_1 \, dy_2.
\]

Use (3.4) with \( C_1 = C_2 = 6 - \epsilon/2 \) to bound the first instance of \( F_{T,R} \) and \( C_1 = \epsilon, C_2 = 6 - \epsilon \) to bound the second instance of \( F_{T,R} \) in order to get

\[
|\tilde{J}_1| \ll A^{12 - 3\epsilon/2} T^{3R+2+3\epsilon/2} \int_0^1 \int_0^1 \int_{\mathbb{R}^2} \frac{y_1^{-1+\epsilon/2} - y_2^{-1+3\epsilon/2}}{(1 + x_1^2)^{3-3\epsilon/2}} \left( 1 + x_1^2 + x_2^2 \right)^{6-3\epsilon/2} \, dx_1 \, dx_2 \, dy_1 \, dy_2.
\]

It is clear that the integral in \( y \) converges and the integral in \( x \) also converges for small values of \( \epsilon \). After a suitable redefinition of \( \epsilon \) we get

\[
|\tilde{J}_1| \ll R_\epsilon A^{12-\epsilon} T^{3R+2+\epsilon}.
\]

For the remaining three integrals the argument is almost identical. The only necessary changes are to pick \( C_1 = 6 - 3\epsilon/2 \) when \( y_1 > 1 \) and \( C_2 = 6 - 5\epsilon/2 \) when \( y_2 > 1 \), while bounding the first instance of \( F_{T,R} \). Putting all the bounds together, we obtain

\[
|\tilde{J}_6(A)| \ll R_\epsilon A^{12-\epsilon} T^{3R+2+\epsilon}.
\] (4.1)

To bound \( J_{6_1, 6_2}(A_1, A_2) \), we also break it into \( J_{6_1, 6_2}(A_1, A_2) = J_1 + J_2 + J_3 + J_4 \) depending on whether \( y_1 \) is smaller or greater than 1. We will first bound \( J_1 \) by taking absolute values and doing the change of variables that sends \( x_3 \) to \( x_3 + x_1 x_2 \). We have

\[
|J_1| \ll (A_1 A_2)^2 \int_0^1 \int_0^1 \int_{\mathbb{R}^3} \left| F_{T,R}(A_1 y_1, A_2 y_2) \right| \left| F_{T,R} \left( \frac{A_1 \sqrt{1 + x_1^2 + x_2^2}}{y_1 y_2}, \frac{A_2 \sqrt{1 + x_1^2 + x_3^2} + 1}{y_2}, \frac{A_1 \sqrt{1 + x_2^2 + x_3^2} + 1}{x_3 + x_1 + 1} \right) \right| \, dx_1 \, dx_2 \, dx_3 \, dy_1 \, dy_2.
\]

Then apply (3.4) with \( C_1 = C_2 = 6 - \epsilon \) to bound the first instance of \( F_{T,R} \) and \( C_1 = C_2 = 6 - 2\epsilon \) to bound the second instance of \( F_{T,R} \) in order to get

\[
|J_1| \ll (A_1 A_2)^{10-3\epsilon} T^{3R-1+2\epsilon} \int_0^1 \int_0^1 \int_{\mathbb{R}^3} \frac{(y_1 y_2)^{-1+\epsilon} \, dx_1 \, dx_2 \, dx_3 \, dy_1 \, dy_2}{(1 + x_2^2 + x_3^2)(1 + x_1^2 + x_3^2)^{3-3\epsilon}}.
\]

As the integral in \( y \) is convergent and the \( x \) integral is also convergent for small values of \( \epsilon \) then, after redefining \( \epsilon \)

\[
|J_1| \ll R_\epsilon (A_1 A_2)^{10-\epsilon} T^{3R-1+\epsilon}.
\]

For the remaining three integrals the same argument works by choosing \( C_1 = 6 - \epsilon/2 \) when \( y_1 > 1 \) and \( C_2 = 6 - \epsilon/2 \) when \( y_2 > 1 \), while bounding the second instance of \( F_{T,R} \). Combining all four bounds, one obtains

\[
|J_{6_1, 6_2}(A_1, A_2)| \ll R_\epsilon (A_1 A_2)^{10-\epsilon} T^{3R-1+\epsilon}.
\] (4.3)
We may now bound the Kloosterman terms $\Sigma_{2a}$, $\Sigma_{2b}$ and $\Sigma_3$. The only necessary bounds for the Kloosterman sums will be

$$\tilde{S}(m_1, n_1, n_2, D_1, D_2) \ll \epsilon \left( D_1 D_2 \right)^{1+\epsilon}$$

$$S(m_1, m_2, n_1, n_2, D_1, D_2) \ll \epsilon \left( D_1 D_2 \right)^{1+\epsilon}.$$

To bound $\Sigma_{2a}$ we use (4.1) together with the first bound for Kloosterman sums, to obtain

$$|\Sigma_{2a}| \ll_{R, \epsilon} \sum_{D_2 = 1}^{\infty} \sum_{D_1 \mid D_2, \atop m_2 D_1^2 \equiv n_1 D_2} \left| \tilde{S}(\delta m_1, n_1, n_2, D_1, D_2) \right| \left| \tilde{J}_\delta \left( \sqrt{\frac{m_1 n_1 n_2}{D_1 D_2}} \right) \right|$$

$$\ll T^{3R+2+\epsilon} \sum_{D_1, D_2} \left( \frac{m_1 n_1 n_2}{D_1 D_2} \right)^6 \ll (m_1 n_1 n_2)^6 T^{3R+2+\epsilon}.$$

The bound for $\Sigma_{2b}$ is obtained in the same manner. In order to bound $\Sigma_3$ we use (4.3) together with the second bound for Kloosterman sums, to obtain

$$|\Sigma_3| \ll_{R, \epsilon} \sum_{D_1, D_2} \left| S(m_1, m_2, n_1, n_2, D_1, D_2) \right| \left| J_{\delta_1, \delta_2} \left( \sqrt{\frac{m_1 n_1 D_1}{D_2}}, \sqrt{\frac{m_2 n_1 D_2}{D_1}} \right) \right|$$

$$\ll (m_1 m_2 n_1 n_2)^5 T^{3R-1+\epsilon} \sum_{\delta_1, \delta_2 = \pm 1} \sum_{D_1, D_2} \frac{1}{(D_1 D_2)^{5-3\epsilon/2}} \ll (m_1 m_2 n_1 n_2)^5 T^{3R-1+\epsilon}.$$

For future reference, we write down these bounds in the following proposition.

**Proposition 4.1.** Fix $R > 50$, $T \gg 1$ and $\epsilon > 0$. We have the following bounds for the Kloosterman terms:

$$|\Sigma_{2a}| \ll_{R, \epsilon} (m_1 n_1 n_2)^6 T^{3R+2+\epsilon},$$

$$|\Sigma_{2b}| \ll_{R, \epsilon} (m_2 n_1 n_2)^6 T^{3R+2+\epsilon},$$

$$|\Sigma_3| \ll_{R, \epsilon} (m_1 m_2 n_1 n_2)^5 T^{3R-1+\epsilon}.$$
5. Bounds for the Eisenstein terms

We start by obtaining a bound for the contribution to the Kuznetsov trace formula of the minimal Eisenstein series $E_{\text{min}}$. We will require the de la Vallée Poussin bound for the Riemann zeta function,

$$|\zeta(1 + it)| \gg \frac{1}{\log(2 + |t|)}.$$  

Using the previous bound and Stirling’s formula for the Gamma factors, we get

$$|E_{\text{min}}| \ll \int_{-iT^{1+\epsilon}}^{iT^{1+\epsilon}} \int_{-iT^{1+\epsilon}}^{iT^{1+\epsilon}} (m_1 m_2 n_1 n_2)^\epsilon \sum_{k=1}^{3} \left| \frac{\Gamma \left( \frac{2+R+3\nu_k}{4} \right) \Gamma \left( \frac{2+R-3\nu_k}{4} \right)}{\zeta(1+3\nu_k) \Gamma \left( \frac{1+3\nu_k}{2} \right)} \right|^2 |dv_1 dv_2|$$

$$\ll (m_1 m_2 n_1 n_2)^\epsilon \int_{-iT^{1+\epsilon}}^{iT^{1+\epsilon}} \int_{-iT^{1+\epsilon}}^{iT^{1+\epsilon}} \prod_{k=1}^{3} \left( (1 + |\nu_k|)^R \log(2 + |\nu_k|)^2 \right) |dv_1 dv_2| \ll_{R,\epsilon} (m_1 m_2 n_1 n_2)^\epsilon T^{3R+2+\epsilon}.$$  

To bound the maximal Eisenstein series contribution $E_{\text{max}}$ we require the following lower bounds for $L$-functions

$$L(1, \text{Ad } u_j) \gg (1 + |r_j|)^{-\epsilon}$$

$$|L(1 + 3\nu, u_j)| \gg (1 + |\nu| + |r_j|)^{-\epsilon}$$

where the eigenvalue of $u_j$ is $1/4 + r_j^2$. These lower bounds can be found in [7] and [8]. It follows that

$$|E_{\text{max}}| \ll \sum_{r_j < T^{1+\epsilon} - iT^{1+\epsilon}} (m_1 m_2 n_1 n_2)^{1/2+\epsilon} \left| \frac{\Gamma \left( \frac{2+R+3\nu+ir_j}{4} \right)}{\Gamma \left( \frac{1+3\nu+ir_j}{2} \right)} \right|^8 \left| \frac{\Gamma \left( \frac{2+R+2ir_j}{4} \right)}{\Gamma \left( \frac{1+2ir_j}{2} \right)} \right|^4 (1 + |r_j|)^\epsilon (1 + |\nu| + |r_j|)^{2\epsilon} |dv|$$

$$\ll (m_1 m_2 n_1 n_2)^{1/2+\epsilon} \sum_{r_j < T^{1+\epsilon} - iT^{1+\epsilon}} (1 + |r_j|)^{R+\epsilon} (1 + |\nu| + |r_j|)^{2R+2\epsilon} |dv|$$

$$\ll_{R,\epsilon} (m_1 m_2 n_1 n_2)^{1/2+\epsilon} T^{3R+3+\epsilon}.$$  

For the last inequality, we use Weyl’s law for $GL(2)$ Maass forms which tells us that

$$|\{\phi_j : r_j < T\}| \sim cT^2$$

for some constant $c > 0$. In summary, we obtain the following proposition.

**Proposition 5.1.** Fix $R > 50$, $T \gg 1$ and $\epsilon > 0$. We have the following bounds for the Eisenstein terms in the Kuznetsov trace formula:

$$|E_{\text{min}}| \ll_{R,\epsilon} (m_1 m_2 n_1 n_2)^\epsilon T^{3R+2+\epsilon};$$

$$|E_{\text{max}}| \ll_{R,\epsilon} (m_1 m_2 n_1 n_2)^{1/2+\epsilon} T^{3R+3+\epsilon}.$$
6. Estimating the main term on the geometric side

To estimate the main term on the geometric side, $\Sigma_1 = \langle F_{T,R}, F_{T,R} \rangle$, we use the Parseval-type identity (3.3) which says that $\Sigma_1 = \langle F^\#_{T,R}, F^\#_{T,R} \rangle$. Hence

$$\langle F^\#_{T,R}, F^\#_{T,R} \rangle = \frac{1}{(\pi i)^2} \int_{-\infty}^{i\infty} \int_{-\infty}^{i\infty} \frac{|F_{T,R}(\nu)|^2}{\prod_{j=1}^3 \Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( -\frac{3\nu_j}{2} \right)} d\nu_1 d\nu_2$$

$$= \frac{27^{R+1} \pi}{64^R} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \sum_{j=1}^3 T^{-\beta_j^2/R^2} \right)^2 \exp \left( -2 \sum_{j=1}^3 \frac{\beta_j^2}{T^2} \right) \prod_{j \neq k} \left( |\beta_j - \beta_k|^{R+1} + O(|\beta_j - \beta_k|^{R+1}) \right) d\beta_1 d\beta_2$$

where $\nu_j = i t_j$ and $\alpha_j = i \beta_j$. The second equality follows from Stirling’s approximation of the Gamma factors. Making a linear change of variables of integration from $t_1, t_2$ to $\beta_1 = t_3 - t_1 = 2t_1 + t_2$, $\beta_2 = t_2 - t_1$ and using the symmetry of the integral in the $\beta_j$, one gets

$$= \frac{9^{R+14/3}}{64^R} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \sum_{j=1}^3 T^{-\beta_j^2/R^2} \right)^2 \exp \left( -2 \sum_{j=1}^3 \frac{\beta_j^2}{T^2} \right) \prod_{j \neq k} \left( |\beta_j - \beta_k|^{R+1} + O(|\beta_j - \beta_k|^{R+1}) \right) d\beta_1 d\beta_2$$

$$= \frac{9^{R+18/3}}{32^R} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \sum_{j=1}^3 T^{-\beta_j^2/R^2} \right)^2 \exp \left( -2 \sum_{j=1}^3 \frac{\beta_j^2}{T^2} \right) \left( \beta_2^{3R+3} + O(\beta_2^{3R+2}\beta_1 + \beta_2^{3R+2}) \right) d\beta_1 d\beta_2.$$
Performing a change of variables on the innermost integral, we get

\[
M = \frac{9R+18\pi}{32Rc_1} \int_0^\infty \int_{\beta_2 \frac{c_2^2}{c_1}}^{\beta_1} \exp \left( -\beta_1^2 - \left( c_2^2 - \frac{c^4}{4c_1^2} \right) \beta_2 \right) \beta_2^{3R+3} d\beta_1 d\beta_2.
\]

We then use the following bounds

\[
\int_0^x e^{-t^2} dt \ll x
\]
\[
\int_x^\infty e^{-t^2} dt \ll \frac{e^{-x^2}}{x}
\]

to obtain

\[
M = \frac{9R+14\pi}{32Rc_1} \int_0^\infty \exp \left( - \left( c_2^2 - \frac{c^4}{4c_1^2} \right) \beta_2 \right) \beta_2^{3R+3} \left( 1 + O \left( \frac{c_2^2}{2c_1} \right) \right) d\beta_2
\]
\[
+ \int_0^\infty \exp \left( - \left( c_2^2 + \frac{1}{c_1} \right) \beta_2 \right) \beta_2^{3R+2} O \left( \frac{1}{(1 + c_2^2)} \right) d\beta_2
\]
\[
= \frac{9R+12\pi^2}{32R} \left( \frac{c_2^2 - \frac{c^4}{4c_1^2}}{c_1} \right)^{-3R+4/2} + O \left( \frac{c_2^2 \left( c_2 - \frac{c^4}{4c_1^2} \right)^{-3R+5/2}}{c_1^2} \right) + O \left( \frac{\left( c_2^2 + \frac{1}{c_1} \right)^{-3R+3/2}}{1 + c_2^2} \right)
\]
\[
= \frac{3^{3R+2} \pi^2}{2^{8R+7/2} \sqrt{\log T}} + O(T^{3R+3}).
\]

Combining the bounds for the Kloosterman terms (4.1), the bounds for the Eisenstein terms (5.1) and the computation of the main term we get (1.1).
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