Practical Approximation Algorithms for $\ell_1$-Regularized Sparse Rank-1 Approximation to Higher-Order Tensors

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Abstract

Two approximation algorithms are proposed for $\ell_1$-regularized sparse rank-1 approximation to higher-order tensors. The algorithms are based on multilinear relaxation and sparsification, which are easily implemented and well scalable. In particular, the second one scales linearly with the size of the input tensor. Based on a careful estimation of the $\ell_1$-regularized sparsification, theoretical approximation lower bounds are derived. Our theoretical results also suggest an explicit way of choosing the regularization parameters. Numerical examples are provided to verify the proposed algorithms.

Key words: tensor; sparse; $\ell_1$ regularization; rank-1 approximation; approximation algorithm; approximation bound

1 Introduction

The goal of sparse rank-1 approximation to higher-order tensors is to find a sparse rank-1 tensor that is as close as possible to the given data tensor in the sense of Euclidean distance. It can be seen as a sparse generalization of the tensor best rank-1 approximation [5,13] and a higher-order extension of the matrix sparse SVD [17]. Sparse rank-1 approximation to tensors is important in higher-order PCA [1], co-clustering [12], sparse tensor decomposition [10,15], and sparse tensor regression [14]; just to name a few.

To encourage sparsity for the tensor rank-1 approximation problem, existing literature employed the $\ell_0$-constraint [14,15] or the $\ell_1$-regularization [1,10,12]. Due to the separable structure of the problem, one usually considers block coordinate descent type algorithms to solve the models [1,10,12,14–16]. However, as the problem is nonconvex, it is hard to measure the solution quality for an iterative algorithm, if we have no a priori information about the data tensor. In the context of sparse matrix PCA and SVD, approximation algorithms with provable lower...
bounds have drawn much attention during the past decade; see, e.g., [3, 4, 6]. Approximation algorithms have also been developed for non-sparse rank-1 approximation to tensors; see, e.g., [7, 8, 18]. Very recently, approximation algorithms were designed for \( \ell_0 \)-constrained sparse rank-1 approximation to tensors [11]. The interest in approximation algorithms is not only on themselves, but also that they can generalize high-quality feasible solutions as initializers for iterative algorithms.

In view of the above, however, in the \( \ell_1 \)-regularized setting, there still lacks approximation algorithms in the literature. This paper attempts to study this point. In the matrix setting, existing work on sparse matrix PCA usually prefers convex relaxations [4, 6]; however, they do not scale well; recent work [2] formulated matrix sparse PCA as a mixed-integer semidefinite program that can deal with large-scale problems, but this approach cannot be generalized to our setting due to the multilinearity of the considered problem and the \( \ell_1 \)-regularization term. In contrast to these approaches, in this paper, we extend the idea of multilinear relaxation [8, 11] to our setting and combine it with the \( \ell_1 \)-regularization induced subproblems. Although the multilinear relaxation is nonconvex, it indeed enables us to design easily implemented and well scalable algorithms, where the second one, in particular, scales linearly with the size of the input tensor. For a \( d \)-th order tensor, \( \prod_{j=1}^{d}(1-\omega_j \sqrt{n_j}) \) and \( \sqrt{\prod_{j=1}^{d} n_j} \)—lower bounds are established when \( \omega_j < 1/\sqrt{n_j} \), where \( \omega_j \)'s are the regularization parameters. Although extending the multilinear relaxation to the \( \ell_1 \)-regularized setting seems to be straightforward, the sparsification needs to be specifically designed and the analysis is somewhat more involved than that of [8] for the non-sparse case and [11] for the \( \ell_0 \)-constrained case. This is because we have to incorporate the \( \ell_1 \)-regularization term into the lower bound analysis, which requires more careful estimation. Besides, our theoretical results also suggest a direct way of choosing the regularization parameters, which is preliminarily confirmed by numerical observations.

The remainder is organized as follows. Sect. 2 introduces the sparse rank-1 approximation models while approximation algorithms and approximation bounds are presented in Sect. 3. Numerical examples are provided in Sect. 4. Sect. 5 draws some conclusions.

Notation. vectors are written as \((x, y, \ldots)\), matrices correspond to \((A, B, \ldots)\), and tensors are written as \((A, B, \ldots)\). \( \mathbb{R}^{n_1 \times \cdots \times n_d} \) denotes the space of \( n_1 \times \cdots \times n_d \) real tensors. The inner product \( \langle A, B \rangle \) between two tensors \( A, B \) of the same size is the sum of entry-wise product. The Frobenius norm of \( A \) is \( \|A\|_F = \langle A, A \rangle^{1/2} \). \( \circ \) denotes the outer product and for \( x_j \in \mathbb{R}^{n_j} \), \( j = 1, \ldots, d \), \( x_1 \circ \cdots \circ x_d \) denotes a rank-1 tensor in \( \mathbb{R}^{n_1 \times \cdots \times n_d} \). For a vector \( x \), \( \|x\| \) is the usual Euclidian norm, \( \|x\|_1 \) is the \( \ell_1 \) norm, and \( \|x\|_0 \) is the \( \ell_0 \) semi-norm; \( |x| \) means that every entry of \( x \) takes its absolute value.

\section{\( \ell_1 \)-Regularized Sparse Tensor Rank-1 Approximation}

Given \( A \in \mathbb{R}^{n_1 \times \cdots \times n_d} \) with \( d \geq 3 \), the \( \ell_0 \)-constrained sparse tensor rank-1 approximation consists of finding a set of sparse vectors \( x_1, \ldots, x_d \) [11, 14, 15]:

\[
\min_{\lambda \in \mathbb{R}} \|\lambda \cdot x_1 \circ \cdots \circ x_d - A\|_F^2, \\
\text{s.t. } \lambda \in \mathbb{R}, \ x_j \in \mathbb{R}^{n_j}, \ \|x_j\| = 1, \ \|x_j\|_0 \leq r_j, \ 1 \leq j \leq d,
\]
where $r_j$’s are parameters. $\lambda$ above can be eliminated and (2.1) is equivalent to the following maximization problem [11]:

$$\max \langle A, x_1 \circ \cdots \circ x_d \rangle$$

s.t. $\|x_j\| = 1, \|x_j\|_0 \leq r_j, 1 \leq j \leq d.$ \hfill (2.2)

In another thread, the $\ell_1$-regularized version of the problem takes the following form [1] ($d = 3$):

$$\max \langle A, x_1 \circ \cdots \circ x_d \rangle - \sum_{j=1}^d \omega_j \|x_j\|_1$$

s.t. $x_j \in \mathbb{R}^{n_j}, \|x_j\| \leq 1, 1 \leq j \leq d,$ \hfill (2.3)

where $\omega_j \geq 0$ denotes the regularization parameter. $\|x_j\|_1$ can be replaced by $\|Dx_j\|_1$ where $D$ is a structure matrix [10], [16] considered solution methods for the following two models ($d = 3$):

$$\min_{x_j \in \mathbb{R}^{n_j}, 1 \leq j \leq d} \frac{1}{2} \|x_1 \circ \cdots \circ x_d - A\|_F^2 + \sum_{j=1}^d \omega_j \|x_j\|_1;$$

$$\min \frac{1}{2} \|\lambda \cdot x_1 \circ \cdots \circ x_d - A\|_F^2 + \sum_{j=1}^d \omega_j \|x_j\|_1$$

s.t. $\lambda \in \mathbb{R}, x_j \in \mathbb{R}^{n_j}, \|x_j\| = 1, 1 \leq j \leq d.$ \hfill (2.4) \hfill (2.5)

In this paper, we focus on the $\ell_1$-regularized problem of the following form:

$$\max \langle A, x_1 \circ \cdots \circ x_d \rangle - \sum_{j=1}^d \omega_j \|x_j\|_1$$

s.t. $x_j \in \mathbb{R}^{n_j}, \|x_j\| = 1, 1 \leq j \leq d,$ \hfill (2.6)

which is a slight modification of (2.3) with the ball constraints replaced by spherical ones. This is for better designing and analyzing the approximation algorithms, and it can avoid zero solutions.

Note that (2.6) has close relations with the aforementioned problems. Firstly, it is a $\ell_1$-regularized version of (2.2). Secondly, if $\omega_j$’s are chosen small enough, then the ball constrained model (2.3) boils down to (2.6), as the maximizer always lies on the sphere in such a case. Thirdly, considering the spherical constraints, if one minimizes (2.5) with respect to $\lambda$, one obtains $\lambda = \langle A, x_1 \circ \cdots \circ x_d \rangle$. Substituting this into (2.5) gives the objective function $1/2 \langle A, x_1 \circ \cdots \circ x_d \rangle^2 - \sum_{j=1}^d \omega_j \|x_j\|_1$ (if the min is replaced by max), which plays a similar role as that in (2.6).

3 Approximation Algorithms and Approximation Bounds

We need some preparations before presenting the algorithms. For any $a \in \mathbb{R}^n$ and given that $\omega \geq 0$, it is known that the solution to the following problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - a\|^2 + \omega \|x\|_1$$

is given by

$$S(a, \omega) := \text{sgn}(a) \oplus (|a| - \omega e)_+,$$

where $r_j$’s are parameters. $\lambda$ above can be eliminated and (2.1) is equivalent to the following maximization problem [11]:

$$\max \langle A, x_1 \circ \cdots \circ x_d \rangle$$

s.t. $\|x_j\| = 1, \|x_j\|_0 \leq r_j, 1 \leq j \leq d.$ \hfill (2.2)

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$$\min \frac{1}{2} \|\lambda \cdot x_1 \circ \cdots \circ x_d - A\|_F^2 + \sum_{j=1}^d \omega_j \|x_j\|_1$$

s.t. $\lambda \in \mathbb{R}, x_j \in \mathbb{R}^{n_j}, \|x_j\| = 1, 1 \leq j \leq d.$ \hfill (2.4) \hfill (2.5)

In this paper, we focus on the $\ell_1$-regularized problem of the following form:

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where \( \text{sgn}(\cdot) \) is the sign function, \( \odot \) denotes the entry-wise product, \( e \) represents the all-one vector, and \( (\cdot)_+ := \max\{\cdot, 0\} \). The following result holds.

**Lemma 3.1.** Given a nonzero vector \( a \in \mathbb{R}^n \) and \( \omega \geq 0 \), it holds that

\[
\max_{x \in \mathbb{R}^n, \|x\|_1 = 1} \langle a, x \rangle - \omega \|x\|_1 = \begin{cases} \|S(a, \omega)\| = \sqrt{\sum_{i=1}^n (|a(i)| - \omega)_+^2}, & \|S(a, \omega)\| \neq 0, \\ \max_{1 \leq i \leq n} |a(i)| - \omega, & \|S(a, \omega)\| = 0, \end{cases} \tag{3.7}
\]

which is solved by

\[
x^* = N(a, \omega) := \begin{cases} \frac{S(a, \omega)}{\|S(a, \omega)\|}, & \|S(a, \omega)\| \neq 0, \\ \text{sgn}(a(i)) \cdot e_i, & \|S(a, \omega)\| = 0, \end{cases} \tag{3.8}
\]

where \( a(i) \) means the \( i \)-th entry of \( a \), \( \hat{i} \) is any index such that \( |a(\hat{i})| = \max_{1 \leq i \leq n} |a(i)| \), \( e_i \) denotes the \( i \)-th vector of the standard basis of \( \mathbb{R}^n \), and we set \( \text{sgn}(a(\hat{i})) = 1 \) if \( a(\hat{i}) = 0 \).

**Proof.** For any maximizer \( x^* \) to the problem \( \max_{x \in \mathbb{R}^n, \|x\|_1 = 1} \langle a, x \rangle - \omega \|x\|_1 \), we first have the observation that \( \text{sgn}(x^*(i)) = \text{sgn}(a(i)) \), otherwise the sign of \( x^*(i) \) can be reversed to obtain a larger objective value. This means that \( \langle a, x^* \rangle = \langle |a|, |x^*| \rangle \), and so

\[
\max_{x \in \mathbb{R}^n, \|x\|_1 = 1} \langle a, x \rangle - \omega \|x\|_1 = \max_{x \in \mathbb{R}^n, \|x\|_1 = 1} \langle |a| - \omega e, |x| \rangle.
\]

Whenever \( S(a, \omega) \neq 0 \), this means that \( |a(i)| > \omega \) for at least one \( i \), and so

\[
\max_{x \in \mathbb{R}^n, \|x\|_1 = 1} \langle a, x \rangle - \omega \|x\|_1 = \max_{x \in \mathbb{R}^n, \|x\|_1 \leq 1} \langle a, x \rangle - \omega \|x\|_1,
\]

whose maximizer is given by \( S(a, \omega)/\|S(a, \omega)\| \) according to [9, Proposition 4.6].

When \( S(a, \omega) = 0 \), we get \( |a| - \omega e \leq 0 \), implying that the maximizer is given by \( |x^*| = e_i \). This together with \( \langle a, x^* \rangle = \langle |a|, |x^*| \rangle \) shows that \( x^* = \text{sgn}(a(\hat{i})) e_i \). Finally, the maximum of the problem can be easily derived. \( \square \)

With the above result at hand, we begin with the general idea on how to design the algorithms. We use multilinear relaxation to obtain a set of non-sparse candidate vectors \( x_1, \ldots, x_d \) from matrix unfoldings and tensor-vector multiplications sequentially, using certain extraction strategies. To sparsify these vectors, in view of the \( \ell_1 \)-regularization, we solve (3.7) with \( a = x_i \) to get the final solutions. That is to say, the sparse solutions are given by \( N(x_i, \omega) \). The designed algorithms are presented in Algorithms V1 and V2, where the former employs SVD to extract the non-sparse vectors, while the latter uses certain maximal energy rule. In the algorithms, \texttt{reshape} is the same as that in Matlab.
Algorithm \((x_1^{\omega}, \ldots, x_d^{\omega}) = \text{approx}_{\text{alg}1}(A, \omega_1, \ldots, \omega_d)\) (V1)

1. Unfold \(A\) to the matrix \(A_1 = \text{reshape}(A, n_1, \prod_{j=2}^d n_j) \in \mathbb{R}^{n_1 \times \prod_{j=2}^d n_j}\) and find the solution to
\[
(x_1^*, y_1^*) \in \arg \max_{(x_1, y_1) \in \mathbb{R}^{n_1} \times \mathbb{R}^{\prod_{j=2}^d n_j}, \|x_1\|_1 = \|y_1\|_1 = 1} x_1^T A_1 y_1;
\]
denote \(x_1^{\omega_1} := N(x_1^*, \omega_1)\), where \(N(\cdot, \cdot)\) was defined in Lemma 3.1.

2. For \(j = 2, \ldots, d - 1\), denote matrices \(A_j := \text{reshape}(A_{j-1}^x x_j^{\omega_{j-1}}, n_j, \prod_{k=j+1}^d n_k) \in \mathbb{R}^{n_j \times \prod_{k=j+1}^d n_k}\) and find the solution to
\[
(x_j^*, y_j^*) \in \arg \max_{(x_j, y_j) \in \mathbb{R}^{n_j} \times \mathbb{R}^{\prod_{k=j+1}^d n_k}, \|x_j\|_1 = \|y_j\|_1 = 1} x_j^T A_j y_j;
\]
denote \(x_j^{\omega_j} := N(x_j^*, \omega_j)\).

3. Denote \(x_d^* := A_{d-1}^x x_{d-1}^{\omega_{d-1}} / \|A_{d-1}^x x_{d-1}^{\omega_{d-1}}\| \in \mathbb{R}^{n_d}\) and compute \(x_d^{\omega_d} := N(x_d^*, \omega_d)\).
4. Return \((x_1^{\omega}, \ldots, x_d^{\omega})\).

Algorithm \((x_1^{\omega}, \ldots, x_d^{\omega}) = \text{approx}_{\text{alg}2}(A, \omega_1, \ldots, \omega_d)\) (V2)

1. Unfold \(A\) to \(A_1 = \text{reshape}(A, n_1, \prod_{j=2}^d n_j)\) and denote \(A_1^k\) as the row of \(A_1\) having the largest magnitude, i.e., \(\|A_1^k\|_1 = \max_{1 \leq k \leq n_1} \|A_1^k\|_1\). Let \(y_1^* = (A_1^k)^T\) and \(x_1^* = A_1^k y_1^* / \|A_1^k y_1^*\| \in \mathbb{R}^{n_1}\); denote \(x_1^{\omega_1} := N(x_1^*, \omega_1)\).

2. For \(j = 2, \ldots, d - 1\), denote \(A_j = \text{reshape}(A_{j-1}^y x_j^{\omega_j}, n_j, \prod_{k=j+1}^d n_k)\) and denote \(A_j^k\) as the row of \(A_j\) having the largest magnitude. Let \(y_j^* = (A_j^k)^T\) and \(x_j^* = A_j^k y_j^* / \|A_j^k y_j^*\| \in \mathbb{R}^{n_j}\); denote \(x_j^{\omega_j} := N(x_j^*, \omega_j)\).

3. Denote \(x_d^* := A_{d-1}^y x_{d-1}^{\omega_{d-1}} / \|A_{d-1}^y x_{d-1}^{\omega_{d-1}}\| \in \mathbb{R}^{n_d}\) and compute \(x_d^{\omega_d} := N(x_d^*, \omega_d)\).
4. Return \((x_1^{\omega}, \ldots, x_d^{\omega})\).

It can be checked that the computational complexity of Algorithm V1 is dominated by \((n_1^2 n_2 \cdots n_d)\) and that of Algorithm V2 is dominated by \(O(n_1 \cdots n_d)\), showing the well-scalability of the proposed algorithms.

In fact, the above two algorithms inherit the ideas of [11, Alg. C and D] for the \(\ell_0\)-constrained problem. The differences lie in the sparsification: For the non-sparse vectors \(x_j\)’s, [11] sparsifies them by finding a normalized sparse vector that is closest to \(x_j\), i.e., solving \(\max_{\|x\|_1 = 1, \|x\|_0 \leq r} \langle x_j, x \rangle\), while here we solve (3.7). Comparing the \(\ell_1\)-regularized problem (2.6) with the \(\ell_0\)-constrained one (2.2), such a modification is natural. However, due to the \(\ell_1\)-regularization term, deriving the approximation bounds is more involved than [11], which needs more refined analysis. These will be detailed in the coming subsection.

### 3.1 Approximation bounds analysis

We first lower bound the sparsification (3.7). Observe that in the algorithms, the vectors to be sparsified are always normalized; this property motivates us to study the following problem with \(\omega \geq 0\):

\[
\min_{x \in \mathbb{R}^n} f(x) := \sum_{i=1}^n (|x(i)| - \omega)_+^2, \text{ s.t. } x^T x = 1.
\]

(3.9)
Denote $\xi(n)$ as the minimum of (3.9). If $\omega \geq 1/\sqrt{n}$, then taking $x(i) = 1/\sqrt{n}$ for each $i$, it is seen that $\xi(n) = 0$. When $\omega < 1/\sqrt{n}$, we have the following estimation.

**Lemma 3.2.** Let $0 < \omega < 1/\sqrt{n}$. Then the minimum $\xi(n)$ of (3.9) satisfies

$$\xi(n) \geq n \left(\frac{1}{\sqrt{n}} - \omega\right)^2.$$

**Proof.** Note that the objective $f(x)$ is differentiable, whose partial derivative with respect to each entry $x(i)$ is given by

$$\frac{\partial f}{\partial x(i)} = 2\text{sgn}(x(i)) \left(|x(i)| - \omega\right)_+.$$

The Lagrangian function of (3.9) is given by $L(x) = f(x) - \lambda^\top x - 1$ with $\lambda \in \mathbb{R}$, and the KKT system of (3.9) reads as

$$\text{sgn}(x(i)) \left(|x(i)| - \omega\right)_+ = \lambda x(i), \; i = 1, \ldots, n, \; x^\top x = 1. \tag{3.10}$$

Multiplying $\text{sgn}(x(i))$ on both sides of the first relation of (3.10) gives

$$\left(|x(i)| - \omega\right)_+ = \lambda |x(i)|, \; i = 1, \ldots, n, \; x^\top x = 1. \tag{3.11}$$

Since every minimizer of (3.9) satisfies (3.11), we analyze the solution property of (3.11). Let $x$ solve (3.11). If $|x(i)| > \omega$, then $\left(|x(i)| - \omega\right)_+ = |x(i)| - \omega > 0$, which together with (3.11) gives

$$\lambda > 0, \; |x(i)| = \frac{\omega}{1 - \lambda}, \; \forall i \in \{i \mid |x(i)| > \omega\}. \tag{3.12}$$

If $|x(i)| \leq \omega$, then $\left(|x(i)| - \omega\right)_+ = 0$, which in connection with $\lambda > 0$ and (3.11) yields that

$$x(i) = 0, \; \forall i \in \{i \mid |x(i)| \leq \omega\}. \tag{3.13}$$

Note that if $x$ is a minimizer to (3.9), then it must obey (3.12) and (3.13). Let $k$ be the number of nonzero entries of a minimizer $x$. Since $\|x\| = 1$, the above analysis implies that $|x(i)| = 1/\sqrt{k} \geq \omega, \forall i \in \{i \mid |x(i)| > \omega\}$, and so

$$\xi(n) = \sum_{i=1}^{n} \left(|x(i)| - \omega\right)^2_+ = k \left(1/\sqrt{k} - \omega\right)^2.$$

We then consider the function $g(y) := y \left(1/\sqrt{y} - \omega\right)^2$, where $1 \leq y \leq n$. Since $\omega < 1/\sqrt{n}$, $g'(y) = \omega(\omega - 1/\sqrt{n}) < 0$, i.e., $g(\cdot)$ is nonincreasing, leading to that

$$k \left(1/\sqrt{k} - \omega\right)^2 \geq n \left(1/\sqrt{n} - \omega\right)^2, \; k \leq n.$$

The proof has been completed. \qed

Let $\lambda_{\text{max}}(\cdot)$ denote the largest singular value of a given matrix. The following lemmas are important.

**Lemma 3.3.** Let $A \in \mathbb{R}^{n \times n}$ be nonzero, where $(x,y)$ is a normalized singular vector pair corresponding to $\lambda_{\text{max}}(A)$. Denote $x^w := N(x,\omega)$ with $N(\cdot,\cdot)$ defined in (3.8). If $\omega < 1/\sqrt{n}$, then

$$\|A^\top x^w\| \geq (\sqrt{\xi(n)} + \omega)\lambda_{\text{max}}(A) \geq (1 - \omega\sqrt{n} + \omega)\lambda_{\text{max}}(A).$$
Proof. Let \( \lambda_1 = \lambda_{\text{max}}(A) > 0 \) provided that \( A \neq 0 \). Since \( \|Ay\| = \lambda_1, A^\top x = \lambda_1 y \) and \( Ay = \lambda_1 x \), we have
\[
\|A^\top x\| = \lambda_1^{-1} \|A^\top x\| \cdot \|A^\top x\| \\
\geq \lambda_1^{-1} \langle A^\top x, A^\top x \rangle = \langle A^\top x, y \rangle \\
= \lambda_1 \langle x, y \rangle \\
= \lambda_1 (\langle x, y \rangle - \omega \|x\|_1) + \lambda_1 \omega \|x\|_1. \tag{3.14}
\]
The assumption \( \omega < 1/\sqrt{n} \) with the fact that \( \|x\| = 1 \) shows that \( \|S(x, \omega)\| \neq 0 \), which together with the definition of \( N(\cdot, \cdot) \) gives that \( x^\omega = S(x, \omega)/\|S(x, \omega)\| \); and Lemma 3.1 tells us that
\[
\langle x, x^\omega \rangle - \omega \|x^\omega\|_1 = \sqrt{\sum_{i=1}^n (|x(i)| - \omega)^2} \\
\geq \sqrt{\min_{z \in \mathbb{R}^n, \|z\| = 1} \sum_{i=1}^n (|z(i)| - \omega)^2} \\
= \sqrt{\xi(n)} \geq (1 - \omega \sqrt{n}), \tag{3.15}
\]
where the last inequality follows from Lemma 3.2. On the other hand,
\[
\|x^\omega\|_1 \geq \min_{z \in \mathbb{R}^n, \|z\| = 1} \|z\|_1 = 1. \tag{3.16}
\]
(3.14), (3.15) and (3.16) then yield \( \|A^\top x^\omega\| \geq (1 - \omega \sqrt{n} + \omega) \lambda_{\text{max}}(A) \), as desired. \( \square \)

Lemma 3.4. Let \( A \in \mathbb{R}^{n \times m} \) be nonzero; let \( A^\omega \) be the row of \( A \) having the largest magnitude. Denote \( y = (A^\omega)^\top \in \mathbb{R}^m, \ x = Ay/\|Ay\|, \) and \( x^\omega := N(x, \omega) \). If \( \omega < 1/\sqrt{n} \), then
\[
\|A^\top x^\omega\| \geq \frac{1 - \omega \sqrt{n} + \omega}{\sqrt{n}} \|A\|_F.
\]

Proof. Using the relation \( \|w\| = \max_{\|z\| = 1} \langle w, z \rangle \) for any vector \( w \), we have
\[
\|A^\top x^\omega\| = \max_{\|z\| = 1} \langle A^\top x^\omega, z \rangle \\
\geq \langle A^\top x^\omega, y/\|y\| \rangle = \langle x^\omega, Ay/\|y\| \rangle = \frac{\|Ay\|}{\|y\|} \langle x^\omega, x \rangle \\
= \frac{\|Ay\|}{\|y\|} (\langle x^\omega, x/\|x\| \rangle - \omega \|x\|_1) + \omega \frac{\|Ay\|}{\|y\|} \|x^\omega\|_1 \\
\geq \frac{\|Ay\|}{\|y\|} (1 - \omega \sqrt{n} + \omega),
\]
where the last inequality follows from \( x^\omega = S(x, \omega) \) due to that \( \omega < 1/\sqrt{n} \) and \( \|x\| = 1 \), and from (3.15) and (3.16). On the other hand,
\[
\|Ay\|^2/\|y\|^2 = \sum_{k=1}^n \|A_k y\|^2/\|y\|^2 \geq \left( \frac{\|A_k y\|^2}{\|y\|^2} \right) \geq \|A_k\|^2 \geq \frac{1}{n} \|A\|_F^2,
\]
where the last inequality comes from the definition of \( A^\omega \). The required result follows. \( \square \)

Lemma 3.5. If \( A \neq 0 \) and \( (x_1^\omega, \ldots, x_d^\omega) \) is generated by Algorithms \( V1 \) or \( V2 \), then \( A_j \neq 0 \) and \( x_j^\omega \neq 0, \ 1 \leq j \leq d \).
Proof. We only prove the results for Algorithm V1 while that for Algorithm V2 is analogous. It follows from $\mathcal{A} \neq 0$ that $x_i \neq 0$ and $y_i \neq 0$. Lemma 3.1 shows that $x_i^{\omega_i} \neq 0$. Then, $\langle A_1^T x_i^{\omega_i}, y_j \rangle = \lambda_{\max}(A_1) (x_i, x_i^{\omega_i}) > 0$, where the strict inequality also comes from Lemma 3.1. Thus $A_1^T x_i^{\omega_i} \neq 0$, and so $A_2 \neq 0$. Similar argument can be applied to show that $A_j \neq 0$ and $x_j^{\omega_j} \neq 0$ for each $j$. \hfill \Box

Now we are in the position to analyze the approximation bound. To simplify notations, we denote

$$Ax_1 \cdots x_d := \langle A, x_1 \circ \cdots \circ x_d \rangle$$

in the sequel. In fact, it is more reasonable to derive the approximation bound on $Ax_1 \cdots x_d$ than on $Ax_1 \cdots x_d - \sum_{j=1}^d \omega_j \|x_j\|_1$, because the goal of the studied problem is to approximate $\|A - \lambda x_1 \circ \cdots \circ x_d\|_F$ essentially, while it is clear that the larger the $Ax_1 \cdots x_d$ is, the better the approximation will be. In the following, let $\mathbf{x}_1, \ldots, \mathbf{x}_d$ be any feasible point to (2.6) or (2.2), i.e., $(\mathbf{x}_1, \ldots, \mathbf{x}_d) \in \{ (x_1, \ldots, x_d) | \|x_j\|_1 = 1, 1 \leq j \leq d \}$. \hfill \Box

**Theorem 3.1.** Let $d \geq 3$ and let $(x_1^{\omega_1}, \ldots, x_d^{\omega_d})$ be generated by Algorithm V1. If $\omega_j < 1/\sqrt{n_j}$, $1 \leq j \leq d$, then

$$Ax_1^{\omega_1} \cdots x_d^{\omega_d} \geq \frac{\prod_{j=1}^d (1 - \omega_j \sqrt{n_j} + \omega_j)}{\sqrt{\prod_{j=2}^d n_j}} \lambda_{\max}(A_1)$$

$$= \frac{\prod_{j=1}^d (1 - \omega_j \sqrt{n_j} + \omega_j)}{\sqrt{\prod_{j=2}^d n_j}} Ax_1 \cdots \hat{x}_j \cdots x_d.$$

**Remark 3.1.** To prove the theorem, by noticing the definitions of $A_j$’s and using the Kronecker representation, we will use the following fact

$$Ax_1^{\omega_1} \cdots x_d^{\omega_d} = \langle A_1^T x_1^{\omega_1}, x_d^{\omega_d} \otimes \cdots \otimes x_2^{\omega_2} \rangle = \langle A_2^T x_2^{\omega_2}, x_d^{\omega_d} \otimes \cdots \otimes x_3^{\omega_3} \rangle$$

$$= \cdots = \langle A_{d-1}^T x_{d-1}^{\omega_{d-1}}, x_d^{\omega_d} \rangle,$$

where $\otimes$ denotes the Kronecker product. \hfill \Box

Proof. Using the above representation and recalling the definitions of $x_j^{\omega_j}$ and $x_j^*$ in the algorithm, we first rewrite $Ax_1^{\omega_1} \cdots x_d^{\omega_d}$ as $\langle A_{d-1}^T x_{d-1}^{\omega_{d-1}}, x_d^{\omega_d} \rangle$. It follows from $\omega_d < 1/\sqrt{n_d}$ and $\|x_d^\ast\|_1 = 1$ that $S(x_d^\ast, \omega_d) \neq 0$, and so Lemma 3.1 gives $(x_d^\ast, x_d^{\omega_d}) - \omega_d \|x_d^{\omega_d}\|_1 = \sqrt{\sum_{k=1}^d (|x_d^\ast(i)| - \omega_d)^2}$. Then we have

$$Ax_1^{\omega_1} \cdots x_d^{\omega_d} \overset{\text{by (3.17)}}{=} \langle A_{d-1}^T x_{d-1}^{\omega_{d-1}}, x_d^{\omega_d} \rangle = \|A_{d-1}^T x_{d-1}^{\omega_{d-1}}\| (x_d^*, x_d^{\omega_d})$$

$$\geq (1 - \omega_d \sqrt{n_d} + \omega_d) \|A_{d-1}^T x_{d-1}^{\omega_{d-1}}\|,$$

where the second equality follows from the definition of $x_d^*$, and Lemma 3.2 together with $\|x_d^\ast\|_1 = 1$ yields the inequality.

We then build the relation between $\|A_j^T x_j^{\omega_j}\|$ and $\|A_{j-1}^T x_j^{\omega_{j-1}}\|$ for $j = d-1, \ldots, 2$. Lemma 3.5 shows that $A_j \neq 0$ . It then follows from the range of $\omega_j$, the definition of $x_j^{\omega_j}$, and Lemma 3.3 that $\|A_j^T x_j^{\omega_j}\| \geq \lambda_{\max}(A_j)(1 - \omega_j \sqrt{n_j} + \omega_j)$, and so

$$\|A_j^T x_j^{\omega_j}\| \geq \lambda_{\max}(A_j)(1 - \omega_j \sqrt{n_j} + \omega_j) \geq \frac{1 - \omega_j \sqrt{n_j} + \omega_j}{\sqrt{n_j}} \|A_j\|_F$$

$$= \frac{1 - \omega_j \sqrt{n_j} + \omega_j}{\sqrt{n_j}} \|A_{j-1}^T x_{j-1}^{\omega_{j-1}}\|,$$

(3.19)
where the second inequality uses that $\lambda_{\max}(A_j)/\|A_j\|_F \geq 1/\sqrt{n_j}$ and the equality follows from that the reshape operation does not change the norm size. Finally, it follows from the range of $\omega_1$ and Lemma 3.3 that $\|A_1^T x_1^{\omega_1}\| \geq (1 - \omega_1 \sqrt{n_1} + \omega_1)\lambda_{\max}(A_1)$. The above analysis together with the simple fact that $\lambda_{\max}(A_1) \geq A \tilde{x}_1 \cdots \tilde{x}_d$ gives the desired bound. 

In particular, if $\omega_j = O(1/\sqrt{n_j})$ where the constant behind the big O is strictly less than 1, then the order of the ratio is $O\left(\left(\prod_{j=2}^{d-1} n_j\right)^{-1/2}\right)$, which is of the same order as [8, Alg. 1] in the non-sparse rank-1 approximation setting.

The approximation bound for Algorithm V2 is given as follows.

**Theorem 3.2.** Let $(x_1^{\omega_1}, \ldots, x_d^{\omega_d})$ be generated by Algorithm V2. If $\omega_j < 1/\sqrt{n_j}$, $1 \leq j \leq d$, then

$$\|A\|_F \geq \prod_{j=1}^d \frac{(1 - \omega_j \sqrt{n_j} + \omega_j)}{\sqrt{\prod_{j=1}^{d-1} n_j}} \lambda_{\max}(A_1) \geq \prod_{j=1}^d \frac{(1 - \omega_j \sqrt{n_j} + \omega_j)}{\sqrt{\prod_{j=1}^{d-1} n_j}} A \tilde{x}_1 \cdots \tilde{x}_d.$$

**Proof.** Similar to the proof of Theorem 3.1, the definition of $x_d^{\omega_d}$ shows that

$$\|A\|_F \geq \prod_{j=1}^d \frac{(1 - \omega_j \sqrt{n_j} + \omega_j)}{\sqrt{\prod_{j=1}^{d-1} n_j}} \lambda_{\max}(A_1) \geq \prod_{j=1}^d \frac{(1 - \omega_j \sqrt{n_j} + \omega_j)}{\sqrt{\prod_{j=1}^{d-1} n_j}} A \tilde{x}_1 \cdots \tilde{x}_d.$$

For $j = d-1, \ldots, 2$, since $A_j$'s are nonzero by Lemma 3.5, according to Lemma 3.4, the definition of $x_j^{\omega_j}$, and the range of $\omega_j$, we have

$$\|A_j^T x_j^{\omega_j}\| \geq \frac{1 - \omega_j \sqrt{n_j} + \omega_j}{\sqrt{n_j}} \|A_j\|_F = \frac{1 - \omega_j \sqrt{n_j} + \omega_j}{\sqrt{n_j}} \|A_j^{-1} x_j^{\omega_j-1}\|.$$

Finally, again by Lemma 3.4, the definition of $x_1^{\omega_1}$, and the range of $\omega_1$, we obtain $\|A_1^T x_1^{\omega_1}\| \geq \frac{1 - \omega_1 \sqrt{n_1} + \omega_1}{\sqrt{n_1}} \|A\|_F$. Combining the above analysis gives the result. 

**Remark 3.2.** The analysis in this section suggests that for the regularization parameters, we can choose $\omega_j < 1/\sqrt{n_j}$. Empirically, we find that $\omega_j$ closing to $1/\sqrt{n_j}$ gives better results, and in the experiments we always set $\omega_j = 1/\sqrt{n_j} - 10^{-5}$.

### 4 Preliminary Numerical Examples

All the computations are conducted on an Intel i7 CPU desktop computer with 32 GB of RAM. The supporting software is Matlab R2019b. The tensors are given by $A = \sum_{i=1}^d x_{1,i} \cdots x_{d,i} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$, where the vectors are first randomly drawn from the normal distribution, and then some of the entries are randomly set to be zero. The sparsity ratio of the tensor is denoted as sr. We set $d = 4$. For each case, we randomly generated 50 instances, and the averaged results are presented. To be more stable, we actually use $A/\|A\|_{\infty}$ as the data tensor, and multiply the results by $\|A\|_{\infty}$ after the computation; here $\| \cdot \|_{\infty}$ denotes the largest entry in magnitude. We always set $\omega_j = 1/\sqrt{n_j} - 10^{-5}$ for the algorithms. The value $v^{ub} :=$
The tensor sparsity ratio $sr$ varies from 10% to 90%. Left panel: $\mathbf{A}x^{\omega_1} \cdots x^{\omega_d}$ versus the tensor sparsity ratio $sr$; right panel: the sparsity ratio of the output vectors versus the true tensor sparsity ratio $sr$.

We first fix $n_j = 50$ for each $j$ and let the sparsity ratio of the tensor $sr$ vary from 10% to 90%. The results are plotted in Fig. 1. The results of Algorithm V1 is in blue with square markers while those of Algorithm V2 is in red with star markers. The values $\mathbf{A}x^{\omega_1} \cdots x^{\omega_d}$ are plotted in the left panel, from which we observe that both algorithms perform well, as the results are close to the upper bound, which is in black with diamond markers. Algorithm V1 is better because it is based on SVDs, which retain more information in the non-sparse candidate vectors $x^*_j$. The sparsity ratios of $x^{\omega_j}$'s are plotted in the right panel, from which we observe that both algorithms are quite close to the true sparsity ratios, which is in black with diamond markers. This confirms the theoretical suggestion on choosing the regularization parameters in the previous section.

We then fix $sr = 70\%$ and let $n_j$'s vary from 20 to 100. The results are plotted in Fig. 2, where the left panel still shows $\mathbf{A}x^{\omega_1} \cdots x^{\omega_d}$ while the right one shows the CPU time. Considering $\mathbf{A}x^{\omega_1} \cdots x^{\omega_d}$, both algorithms still perform well when $n$ changes, and considering the CPU time, Algorithm V2 is significantly better because it does not need to compute SVDs.

Finally, we compare Algorithms V1 and V2 as initialization procedures for the alternating maximization method (AMM) for solving (2.6). AMM is stopped if the distance between the
successive two iterative points is smaller than $10^{-6}$. AMM initialized by the random initialization is used as a baseline, which is generated as $N(x/\|x\|, \omega)$ where $x$ is randomly drawn from the normal distribution. We plot the value $\mathcal{A}x_{1}^{\text{iter}} \cdots x_{d}^{\text{iter}} (x_{j}^{\text{iter}}$s are outputted by AMM), the CPU time (counting both that of the approximation algorithm and AMM), and the sparsity ratio of the output vectors in Fig. 3. From the first subfigure, we see that the output value of AMM initialized by Algorithm V1 is the best one, followed by Algorithm V2, and both two are far more better than that initialized by the random initialization. From the second subfigure, we see that AMM initialized by Algorithm V2 is the most efficient one, followed by that initialized by the random initialization and that initialized by Algorithm V1. We do not plot the number of iterations as all the cases take about 4 to 6 iterations to converge. From the third subfigure, we observe that the sparsity ratios of the output vectors of AMM initialized by Algorithm V1 and V2 are close to the true sparsity ratio 70%, which are far more better than AMM initialized by the random initialization.

In summary, the above experiments preliminarily demonstrate the effectiveness and efficiency of the proposed algorithms, and confirm the theoretical suggestion on choosing the regularization parameters.

5 Conclusions

By extending the idea of multilinear relaxation [8, 11], we devised two approximation algorithms for $\ell_1$-regularized rank-1 approximation to tensors. The algorithms are easily implemented.
and well scalable. For a $d$-th order tensor, \( \prod_{j=1}^{d} (1 - \omega_j \sqrt{n_j} + \omega_j) \)–lower bounds were established when \( \omega_j < 1/\sqrt{n_j} \), where \( \omega_j \)'s are the regularization parameters. Numerical examples were provided to verify the algorithms and the effectiveness of setting \( \omega_j = 1/\sqrt{n_j} - 10^{-5} \). We also remark that similar ideas of approximation algorithms can be designed for the \( \ell_0 \)-regularized and \( \ell_1 \)-constrained cases, while the key is how to analyze their lower bounds.

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