1 Introduction

The study of mutual entropy (information) and capacity in classical system was extensively done after Shannon by several authors like Kolmogorov [10] and Gelfand [11]. In quantum systems, there have been several definitions of the mutual entropy for classical input and quantum output [5, 11, 12, 17]. In 1983, the author defined [21] the fully quantum mechanical mutual entropy by means of the relative entropy of Umegaki [32], and it has been used to compute the capacity of quantum channel for quantum communication process; quantum input-quantum output [25].

Recently, a correlated state in quantum systems, so-called quantum entangled state or quantum entanglement, are used to study quantum information, in particular, quantum computation, quantum teleportation, quantum cryptography [6, 7, 8, 9, 14, 15, 29, 30].

In this paper, we mainly discuss three things below: (1) We point out the difference between the capacity of quantum channel and that of classical-quantum-classical channel followed from [28]. (2) So far the entangled state is merely defined as a non-separable state, we give a wider definition of the entangled state and classify the entangled states into three categories. (3) The quantum mutual entropy for an entangled state is discussed. The above (2) and (3) are a joint work with Belavkin [6].

2 Quantum Mutual Entropy

The quantum mutual entropy was introduced in [21] for a quantum input and quantum output, namely, for a purely quantum channel, and it was generalized for a general quantum system described by C*-algebraic terminology [22]. We here review the quantum mutual entropy in usual quantum system described by a Hilbert space.

Let $\mathcal{H}$ be a Hilbert space for an input space, $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ and $S(\mathcal{H})$ be the set of all density operators on $\mathcal{H}$. An output space is described by another Hilbert space $\tilde{\mathcal{H}}$, but often $\mathcal{H} = \tilde{\mathcal{H}}$. A channel from the input system to the output system is a mapping $\Lambda^*$ from $S(\mathcal{H})$ to $S(\tilde{\mathcal{H}})$ [24]. A channel $\Lambda^*$ is said to be completely positive if the dual map $\Lambda$ satisfies the following condition: $\sum_{k,j=1}^n A_k^* \Lambda(B_k^* B_j) A_j \geq 0$ for any $n \in \mathbb{N}$ and any $A_j \in B(\mathcal{H})$, $B_j \in B(\tilde{\mathcal{H}})$.

An input state $\rho \in S(\mathcal{H})$ is sent to the output system through a channel $\Lambda^*$, so that the output state is written as $\tilde{\rho} = \Lambda^* \rho$. Then it is important to ask how much information of $\rho$ is correctly sent to the output state $\Lambda^* \rho$. This amount of information transmitted from input to output is expressed by the mutual entropy in Shannon’s theory.

In order to define the quantum mutual entropy, we first mention the entropy of a quantum state introduced by von Neumann. For a state $\rho$, there exists a unique spectral decomposition $\rho = \sum_k \lambda_k P_k$, where $\lambda_k$ is an eigenvalue of $\rho$ and $P_k$ is the associated projection for each $\lambda_k$. The projection $P_k$ is not one-dimensional when $\lambda_k$ is degenerated, so that the spectral decomposition can be further decomposed into one-dimensional projections. Such a decomposition is called a Schatten decomposition, namely, $\rho = \sum_k \lambda_k E_k$, where $E_k$ is...
the one-dimensional projection associated with \( \lambda_k \) and the degenerated eigenvalue \( \lambda_k \) repeats \( \dim P_k \) times; for instance, if the eigenvalue \( \lambda_1 \) has the degeneracy 3, then \( \lambda_1 = \lambda_2 = \lambda_3 < \lambda_4 \). This Schatten decomposition is not unique unless every eigenvalue is non-degenerated. Then the entropy (von Neumann entropy \( S(\rho) \)) of a state \( \rho \) is defined by

\[
S(\rho) = -\operatorname{tr}\rho \log \rho.
\] (2.1)

The quantum mutual entropy was introduced on the basis of the above von Neumann entropy for purely quantum communication processes. The mutual entropy depends on an input state \( \rho \) and a channel \( \Lambda^* \), so it is denoted by \( I(\rho; \Lambda^*) \), which should satisfy the following conditions:

1. The quantum mutual entropy is well-matched to the von Neumann entropy. Furthermore, if a channel is trivial, i.e., \( \Lambda^* = \) identity map, then the mutual entropy equals to the von Neumann entropy: \( I(\rho; \text{id}) = S(\rho) \).
2. When the system is classical, the quantum mutual entropy reduces to classical one.
3. Shannon’s fundamental inequality \( 0 \leq I(\rho; \Lambda^*) \leq S(\rho) \) is held.

In order to define the quantum mutual entropy followed by the classical one (see\([23]\) for the details), we need the joint state (it is called “compound state” in the sequel) describing the correlation between an input state \( \rho \) and the output state \( \Lambda^* \rho \) and the quantum relative entropy. A finite partition of the classical measurable space corresponds to an orthogonal decomposition \( \{E_k\} \) of the identity operator \( I \) of \( \mathcal{H} \) in quantum case because the set of all orthogonal projections is considered to make an event system for a quantum system. It is known\([24]\) that the following equality holds

\[
\sup \left\{-\sum_k \operatorname{tr}\rho E_k \log \operatorname{tr}\rho E_k; \{E_k\}\right\} = -\operatorname{tr}\rho \log \rho,
\]

and the supremum is attained when \( \{E_k\} \) is a Schatten decomposition of \( \rho \). Therefore the Schatten decomposition is used to define the compound state and the quantum mutual entropy.

The compound state \( \theta_E \) (corresponding to joint state (measure) in CS) of \( \rho \) and \( \Lambda^* \rho \) was introduced in\([24, 25]\), which is given by

\[
\theta_E = \sum_k \lambda_k E_k \otimes \Lambda^* E_k,
\] (2.2)

where \( E \) stands for a Schatten decomposition of \( \rho \), so that the compound state depends on how we decompose the state \( \rho \) into basic states (elementary events).

The relative entropy for two states \( \rho \) and \( \sigma \) is defined by Umegaki and Lindblad, which is written as

\[
S(\rho, \sigma) = \begin{cases} \operatorname{tr}\rho (\log \rho - \log \sigma) & (\text{when } \overline{\text{ran } \rho} \subset \overline{\text{ran } \sigma}) \\ \infty & \text{(otherwise)} \end{cases}.
\] (2.3)

Then we can define the quantum mutual entropy by means of the compound state and the relative entropy\([24]\), that is,

\[
I(\rho; \Lambda^*) = \sup \left\{ \sum_k \lambda_k S(\Lambda^* E_k, \Lambda^* \rho); E = \{E_k\}\right\},
\] (2.4)

where the supremum is taken over all Schatten decompositions. Some computations reduce it to the following form:

\[
I(\rho; \Lambda^*) = \sup \left\{ \sum_k \lambda_k S(\Lambda^* \delta_k, \Lambda^* \rho); E = \{E_k\}\right\}
\] (2.5)

This mutual entropy satisfies all conditions (1)∼(3) mentioned above\([23]\).

When the input system is classical, an input state \( \rho \) is given by a probability distribution or a probability measure, in either case, the Schatten decomposition of \( \rho \) is unique, namely, for the case of probability distribution : \( \rho = \{\lambda_k\} \),

\[
\rho = \sum_k \lambda_k \delta_k,
\] (2.6)

where \( \delta_k \) is the delta measure, that is, \( \delta_k(j) = \delta_{k,j} = \{1(k=j) \forall j \}. \) Therefore for any channel \( \Lambda^* \), the mutual entropy becomes

\[
I(\rho; \Lambda^*) = \sum_k \lambda_k S(\Lambda^* \delta_k, \Lambda^* \rho),
\] (2.7)

which equals to the following usual expression of Shannon when it is well-defined:

\[
I(\rho; \Lambda^*) = S(\Lambda^* \rho) - \sum_k \lambda_k S(\Lambda^* \delta_k),
\] (2.8)
which has been taken as the definition of the mutual entropy for a classical-quantum-(classical) channel [6, 11, 12, 17].

Note that the above definition of the mutual entropy (2.5) is also written as

\[ I(\rho; \Lambda^\ast) = \sup \left\{ \sum_k \lambda_k S(\Lambda^\ast \rho_k, \Lambda^\ast \rho) : \rho = \sum_k \lambda_k \rho_k \in F_o(\rho) \right\}, \]

where \( F_o(\rho) \) is the set of all orthogonal finite decompositions of \( \rho \) [23].

More general mutual entropy was defined in [23] based on Araki’s relative entropy [3].

### 3 Communication Processes

The information communication process is mathematically set as follows: \( M \) messages are sent to a receiver and the \( k \)th message \( \omega(k) \) occurs with the probability \( \lambda_k \). Then the occurrence probability of each message in the sequence \( \{\omega(1), \omega(2), \cdots, \omega(M)\} \) of \( M \) messages is denoted by \( \rho = \{\lambda_k\} \), which is a state in a classical system. If \( \xi \) is a classical coding, then \( \xi(\omega) \) is a classical object such as an electric pulse. If \( \xi \) is a quantum coding, then \( \xi(\omega) \) is a quantum object (state) such as a coherent state. Here we consider such a quantum coding, that is, \( \xi(\omega(k)) \) is a quantum state, and we denote \( \xi(\omega(k)) \) by \( \sigma_k \). Thus the coded state for the sequence \( \{\omega(1), \omega(2), \cdots, \omega(M)\} \) is written as \( \sigma = \sum_k \lambda_k \sigma_k \). This state is transmitted through a channel \( \gamma \), which is expressed by a completely positive mapping \( \Gamma^\ast \) from the state space of \( X \) to that of \( \tilde{X} \), hence the output coded quantum state \( \tilde{\sigma} \) is \( \Gamma^\ast \sigma \). Since the information transmission process can be understood as a process of state (probability) change, when \( \Omega \) and \( \tilde{\Omega} \) are classical and \( X \) and \( \tilde{X} \) are quantum, the process is written as

\[ P(\Omega) \xrightarrow{\Xi^\ast} S(\mathcal{H}) \xrightarrow{\Gamma^\ast} S(\tilde{\mathcal{H}}) \xrightarrow{\tilde{\Xi}^\ast} P(\tilde{\Omega}), \quad (3.1) \]

where \( \Xi^\ast \) (resp. \( \Xi^\ast \)) is the channel corresponding to the coding \( \xi \) (resp. decoding \( \tilde{\xi} \)).

We have to be careful to study the objects in the above transmission process (3.1). For instance, if we want to know the information capacity of a quantum channel \( \gamma(= \Gamma^\ast) \), then we have to take \( X \) so as to describe a quantum system like a Hilbert space and we need to start the study from a quantum state in quantum space \( X \) not from a classical state associated to a message. If we like to know the capacity of the whole process including a coding and a decoding, which means the capacity of a channel \( \xi \circ \gamma \circ \xi(= \tilde{\Xi}^\ast \circ \Gamma^\ast \circ \Xi^\ast) \), then we have to start from a classical state.

### 4 Channel Capacity

We discuss two types of channel capacity in communication processes, namely, the capacity of a quantum channel \( \Gamma^\ast \) and that of a classical (classical-quantum-classical) channel \( \tilde{\Xi}^\ast \circ \Gamma^\ast \circ \Xi^\ast \).

1. **Capacity of quantum channel**: The capacity of a quantum channel is the ability of information transmission of a quantum channel itself, so that it does not depend on how to code a message being treated as classical object and we have to start from an arbitrary quantum state and find the supremum of the quantum mutual entropy. One often makes a mistake in this point. For example, one starts from the coding of a message and compute the supremum of the mutual entropy and he says that the supremum is the capacity of a quantum channel, which is not correct. Even when his coding is a quantum coding and he sends the coded message to a receiver through a quantum channel, if he starts from a classical state, then his capacity is not the capacity of the quantum channel itself. In his case, usual Shannon’s theory is applied because he can easily compute the conditional probability by a usual (classical) way. His supremum is the capacity of a classical-quantum-classical channel, and it is in the second category discussed below.

The capacity of a quantum channel \( \Gamma^\ast \) is defined as follows: Let \( \mathcal{S}_0(\subset S(\mathcal{H})) \) be the set of all states prepared for expression of information. Then the capacity of the channel \( \Gamma^\ast \) with respect to \( \mathcal{S}_0 \) is defined by

\[ C^{\mathcal{S}_0}(\Gamma^\ast) = \sup \{ I(\rho; \Gamma^\ast) : \rho \in \mathcal{S}_0 \}. \quad (4.1) \]
Here $I(\rho; \Gamma^*)$ is the mutual entropy given in (2.4) or (2.5) with $\Lambda^* = \Gamma^*$. When $S_0 = S(H) \cup C^S(H) (\Gamma^*)$ is denoted by $C(\Gamma^*)$ for simplicity.

In [23], we also considered the pseudo-quantum capacity $C_p(\Gamma^*)$ defined by (4.1) with the pseudo-mutual entropy $I_p(\rho; \Gamma^*)$ where the supremum is taken over all finite decompositions instead of all orthogonal pure decompositions:

$$I_p(\rho; \Gamma^*) = \sup \left\{ \sum_k \lambda_k S(\Gamma^* \rho_k, \Gamma^* \rho) : \rho = \sum_k \lambda_k \rho_k, \right\} . \quad (4.2)$$

However the pseudo-mutual entropy is not well-matched to the conditions explained in Sec.2, and it is difficult to be computed numerically. The relation between $C(\Gamma^*)$ and $C_p(\Gamma^*)$ was discussed in [23]. From the monotonicity of the mutual entropy [23], we have

$$0 \leq C^{S_0}(\Gamma^*) \leq C^{S_0}_p(\Gamma^*) = \sup \left\{ S(\rho) : \rho \in S_0 \right\} .$$

(2) Capacity of classical-quantum-classical channel: The capacity of C-Q-C channel $\Xi^* \circ \Gamma^* \circ \Xi^*$ is the capacity of the information transmission process starting from the coding of messages, therefore it can be considered as the capacity including a coding (and a decoding). As is discussed in Sec.3, an input state $\rho$ is the probability distribution $\{\lambda_k\}$ of messages, and its Schatten decomposition is unique, so the mutual entropy is written by (2.7):

$$I(\rho; \Xi^* \circ \Gamma^* \circ \Xi^*) = \sum_k \lambda_k S(\Xi^* \circ \Gamma^* \delta_k, \Xi^* \circ \Gamma^* \rho) \quad (4.3)$$

If the coding $\Xi^*$ is a quantum coding, then $\Xi^* \delta_k$ is expressed by a quantum state. Let denote the coded quantum state by $\sigma_k$ and put $\sigma = \Xi^* \rho = \sum_k \lambda_k \sigma_k$. Then the above mutual entropy is written as

$$I(\rho; \Xi^* \circ \Gamma^* \circ \Xi^*) = \sum_k \lambda_k S(\Xi^* \circ \Gamma^* \sigma_k, \Xi^* \circ \Gamma^* \rho) \quad (4.4)$$

This is the expression of the mutual entropy of the whole information transmission process starting from a coding of classical messages. Hence the capacity of C-Q-C channel is

$$C^{P_0}(\Xi^* \circ \Gamma^* \circ \Xi^*) = \sup \left\{ I(\rho; \Xi^* \circ \Gamma^* \circ \Xi^*) : \rho \in P_0 \right\} . \quad (4.5)$$

where $P_0(\subset P(\Omega))$ is the set of all probability distributions prepared for input (a-priori) states (distributions or probability measures). Moreover the capacity for coding free is found by taking the supremum of the mutual entropy over all probability distributions and all codings $\Xi^*$:

$$C^{P_0}_c(\Xi^* \circ \Gamma^*) = \sup \left\{ I(\rho; \Xi^* \circ \Gamma^* \circ \Xi^*) : \rho \in P_0, \Xi^* \right\} . \quad (4.6)$$

The last capacity is for both coding and decoding free and it is given by

$$C^{P_0}_c(\Gamma^*) = \sup \left\{ I(\rho; \Xi^* \circ \Gamma^* \circ \Xi^*) : \rho \in P_0, \Xi^*, \Xi^* \right\} . \quad (4.7)$$

These capacities $C^{P_0}_c$, $C^{P_0}_c$ do not measure the ability of the quantum channel $\Gamma^*$ itself, but measure the ability of $\Gamma^*$ through the coding and decoding. Remark that $\sum_k \lambda_k S(\Gamma^* \sigma_k)$ is finite, then (4.4) becomes

$$I(\rho; \Xi^* \circ \Gamma^* \circ \Xi^*) = S(\Xi^* \circ \Gamma^* \sigma) - \sum_k \lambda_k S(\Xi^* \circ \Gamma^* \sigma_k) . \quad (4.8)$$

Further, if $\rho$ is a probability measure having a density function $f(\lambda)$ and each $\lambda$ corresponds to a quantum coded state $\sigma(\lambda)$, then $\sigma = \int f(\lambda) \sigma(\lambda) d\lambda$ and

$$I(\rho; \Xi^* \circ \Gamma^* \circ \Xi^*) = S(\Xi^* \circ \Gamma^* \sigma) - \int f(\lambda) S(\Xi^* \circ \Gamma^* \sigma(\lambda)) d\lambda . \quad (4.9)$$
This is bounded by
\[ S(\Gamma^*\sigma) - \int f(\lambda)S(\Gamma^*\sigma(\lambda))d\lambda, \]
which is called the Holevo bound and is computed in several occasions [31, 25].

The above three capacities \( C_{P_0}, C_{\hat{P}_0}, C_{\text{cd}} \) satisfy the following inequalities
\[
0 \leq C_{P_0} \left( \Xi \right) \leq C_{\hat{P}_0} \left( \Xi \right) \leq C_{\text{cd}} \left( \Gamma^* \right) \leq \sup \{ S(\rho); \rho \in P_2 \}
\]
where \( S(\rho) \) is not the von Neumann entropy but the Shannon entropy: \(-\sum \lambda_k \log \lambda_k\).

The capacities (4.1), (4.5), (4.6) and (4.7) are generally different. Some misuses occur due to forgetting which channel is considered. That is, we have to make clear what kind of the ability (capacity) is considered, the capacity of a quantum channel itself or that of a classical-quantum(-classical) channel. The computation of the capacity of a quantum channel was carried in several models in [23, 24, 25, 26].

5 Compound States and Entanglements

Recently the quantum entangled state has been mathematically studied [3, 13, 29], in which the entangled state is defined by a state not written as a form \( \sum k \lambda_k \rho_k \otimes \sigma_k \) with any states \( \rho_k \) and \( \sigma_k \). A state written as above is called a separable state, so that an entangled state is a state not be-

\[ \sigma = \kappa \kappa^\dagger \]
is the (unique) density operator \( \sigma \in \mathcal{A} \) of the state \( \varphi : \varphi(A) = \text{tr} A \sigma, A \in \mathcal{A} \). This \( \kappa \) is called the amplitude operator, and it is called just the amplitude if \( \mathcal{G} \) is one dimensional space \( \mathcal{C} \), corresponding to the pure state \( \varphi(A) = \kappa^\dagger A \kappa \) for a \( \kappa \in \mathcal{K} \) with \( \kappa \kappa^\dagger = \| \kappa \|^2 = 1 \). In general, \( \mathcal{G} \) is not one dimensional, the dimensionality \( \text{dim} \mathcal{G} \) must be not less than rank \( \kappa \), the dimensionality of the range \( \sigma \mathcal{K} \) of the density operator \( \sigma \).

Since \( \mathcal{G} \) is separable, \( \mathcal{G} \) is realized as a subspace of \( l^2(\mathbb{N}) \) of complex sequences (i.e., \( \zeta \in \mathcal{C}, n \in \mathbb{N} \) with \( \sum |\zeta_n|^2 < +\infty \)), so that any vector \( \zeta = (\zeta_n) \) represents a vector \( \zeta = \sum \zeta_n |n\rangle \) in the standard basis \( \{ |n\rangle \} \in \mathcal{G} \) of \( l^2(\mathbb{N}) \).

Given the amplitude operator \( \kappa \), one can define not only the states \( \sigma = \kappa \kappa^\dagger \) and \( \rho = \kappa \kappa^\dagger \kappa \) on the algebra \( \mathcal{A}(=B(\mathcal{K})) \) and \( \mathcal{B}(=B(\mathcal{G})) \) but also an entanglement state \( \Theta \) on the algebra \( \mathcal{B} \otimes \mathcal{A} \) of all bounded operators on the tensor product Hilbert space \( \mathcal{G} \otimes \mathcal{K} \) by
\[ \Theta(B \otimes A) = \text{tr}_\mathcal{G} \kappa \kappa^\dagger A \kappa = \text{tr}_\mathcal{K} A \kappa \kappa^\dagger, \]
for any \( B \in \mathcal{B} \). This state is pure as it is the case of \( \mathcal{F} = \mathcal{C} \) in the theorem below, and it satisfies the marginal conditions: For any \( B \in \mathcal{B}, A \in \mathcal{A} \),
\[ \Theta(B \otimes I) = \text{tr}_\mathcal{G} B \rho, \quad \Theta(I \otimes A) = \text{tr}_\mathcal{K} A \sigma. \]

**Theorem 5.1.** Let \( \Theta : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{C} \) be a state
\[ \Theta(B \otimes A) = \text{tr}_\mathcal{F} \psi^\dagger (B \otimes A) \psi, \]
defined by an amplitude operator \( \psi \) on a separable Hilbert space \( \mathcal{E} \) into the tensor product Hilbert space \( \mathcal{G} \otimes \mathcal{K} \); \( \psi : \mathcal{E} \rightarrow \mathcal{G} \otimes \mathcal{K} \) with \( \text{tr}_\mathcal{E} \psi^\dagger \psi = 1 \). Then there exist a Hilbert space \( \mathcal{F} \) and an amplitude operator \( \kappa : \mathcal{G} \rightarrow \mathcal{F} \otimes \mathcal{K} \) with
\[ \kappa^\dagger (I \otimes \mathcal{A}) \kappa \subset \mathcal{B}, \quad \text{tr}_\mathcal{F} \kappa \kappa^\dagger \subset \mathcal{A} \quad (5.1) \]
such that the state \( \Theta \) can be achieved by an entanglement
\[
\Theta(B \otimes A) = \text{tr}_\mathcal{G} B \kappa^\dagger (I \otimes \mathcal{A}) \kappa \kappa \kappa^\dagger \kappa^\dagger = \text{tr}_\mathcal{K} (I \otimes \mathcal{A}) \kappa B \kappa^\dagger \quad (5.2)
\]
The entangling operator \( \kappa \) is uniquely defined up to a unitary transformation of the minimal space \( \mathcal{F} \).
Note that the entangled state (5.2) is written as
\[ \Theta(B \otimes A) = \text{tr}_G B \phi(A) = \text{tr}_K A \phi_*(B), \] (5.3)
where \( \phi(A) = \kappa^\dagger (I \otimes A) \kappa \) is in the predual space \( B_* \subset B \) of all trace-class operators in \( \mathcal{G} \), and \( \phi_*(B) = \text{tr}_FB \kappa^\dagger \kappa \) is in \( A_* \subset A \). The map \( \phi \) is the Steinspring form of the general completely positive map \( A \rightarrow B \), written in the eigen-basis \( \{|n\} \} \) of \( \mathcal{G} \subseteq \ell^2(N) \) of the density operator \( \rho = \phi(I) \) as
\[ \phi(A) = \sum_{m,n} |m\rangle \kappa_m^\dagger (I \otimes A) \kappa_n |n\rangle, \quad A \in \mathcal{A} \] (5.4)
where \( \kappa_n \) is the vector in \( \mathcal{F} \otimes \mathcal{K} \) such that \( \kappa = \sum_n \kappa_n |n\rangle \). The dual operation \( \phi_* \) is the Kraus form of the general completely positive map \( \mathcal{B} \rightarrow \mathcal{A}_* \), given in this basis as
\[ \phi_*(B) = \sum_{n,m} \langle n | B | m\rangle \text{tr}_F \kappa_n \kappa_m^\dagger, \quad B \in \mathcal{B}. \] (5.5)
It corresponds to the general form of the density operator
\[ \theta_\phi = \sum_{m,n} |n\rangle \langle m| \otimes \text{tr}_F \kappa_n \kappa_m^\dagger \] (5.6)
for the entangled state \( \Theta \), characterized by the weak orthogonality property
\[ \text{tr}_K \kappa_n \kappa_m^\dagger = p_n \delta_n^m = \kappa_m^\dagger \kappa_n. \] (5.7)

**Definition 5.2.** The dual map \( \phi_* : \mathcal{B} \rightarrow \mathcal{A}_* \) to a completely positive map \( \phi : \mathcal{A} \rightarrow \mathcal{B}_* \), normalized as \( \text{tr}_G \phi(I) = 1 \), is called the quantum entanglement of the state \( \rho = \phi(I) \) on \( \mathcal{B} \) to the state \( \sigma = \phi_*(I) \) on \( \mathcal{A} \). The entanglement by \( \phi(A) = \sigma^{1/2} A \sigma^{1/2} \) of the state \( \rho = \sigma \) on the algebra \( \mathcal{B} = \mathcal{A} \) given by the standard entangling operator \( \kappa = \sigma^{1/2} \) is called standard.

### 6 d-Entanglements and Correspondences

A compound state, playing the similar role as the joint input-output probability measures in classical systems, was introduced in [21] as explained in Sec.2. It corresponds to a particular diagonal type
\[ \theta_\phi = \sum_n |n\rangle \langle n| \otimes \kappa_n \kappa_n^\dagger \]
of the entangling map (5.6) in the eigen-basis of the density operator \( \rho = \sum p_n |n\rangle \langle n| \), and is discussed in this section. Therefore the entangled states, generalizing the compound state, also play the role of the joint probability measures.

The diagonal entanglements are quantum correspondences of classical symbols to quantum, in general not orthogonal and pure, states. The general entangled states \( \Theta \) are described by the density operators \( \theta_\phi \) of the form (5.6) which is not necessarily diagonal in the eigen-representation of the density operator \( \rho = \sum p_n |n\rangle \langle n| \). Such non-diagonal entangled states were called in [21] the quasicompound (q-compound) states, so we can call also the non-diagonal entanglement the quantum quasi-correspondence (q-correspondence) in contrast to the d-correspondences, described by the diagonal entanglements, giving rise to the d-compound states.

Let us consider a finite or infinite input system indexed by the natural numbers \( n \in \mathbb{N} \). The associated space \( \mathcal{G} \subseteq \ell^2(N) \) is the Hilbert space of the input system described by a quantum projection-valued measure \( n \mapsto |n\rangle \langle n| \) on \( \mathbb{N} \) giving an orthogonal partition of unity \( I = \sum |n\rangle \langle n| \in \mathcal{B} \) of the finite or infinite dimensional input Hilbert space \( \mathcal{G} \). Each input pure state, identified with the one-dimensional density operator \( |n\rangle \langle n| \in \mathcal{B} \) corresponding to the elementary symbol \( n \in \mathbb{N} \), defines the elementary output state \( \omega_n \) on \( \mathcal{A} \). If the elementary states \( \omega_n \) are pure, they are described by pure output amplitudes \( v_n \in \mathcal{K} \) satisfying \( v_n^* v_n = 1 = \text{tr} \omega_n \), where \( \omega_n = v_n v_n^* \) are the corresponding output one-dimensional density operators. If these amplitudes are non-orthogonal \( v_m^* v_n \neq \delta_m^m \), they cannot be identified with the input amplitudes \( |n\rangle \).

The elementally joint input-output states are given by the density operators \( |n\rangle \langle n| \otimes \omega_n \) in \( \mathcal{G} \otimes \mathcal{K} \), and their mixtures
\[ \theta = \sum_n |n\rangle \langle n| \otimes \sigma_n, \quad \sigma_n = p_n v_n v_n^* \] (6.1)
define the compound states on \( \mathcal{B} \otimes \mathcal{A} \), giving the quantum correspondences \( n \mapsto |n\rangle \langle n| \) with the probabilities \( p_n \). Here we note that the quantum correspondence is described by a classical-quantum channel, and the general d-compound state for a quantum-quantum channel in quantum communication can be obtained in this way due to the or-
thonogonality of the decomposition (6.1), corresponding to the orthogonality of the Schatten decomposition $\rho = \sum_n p_n |n\rangle \langle n| \propto \rho = tr_k \theta$. The comparison of the general compound state (5.6) with (6.1) suggests that the quantum correspondences are described as the diagonal entanglements

$$\phi_s(B) = \sum_n p_n \langle n| B|n\rangle v_n \psi_n^\dagger$$ (6.2)

which are dual to the orthogonal decompositions

$$\phi(A) = \sum_n p_n |n\rangle \langle n| \psi_n\psi_n^\dagger$$ (6.3)

These are the entanglements with the stronger orthogonality

$$tr_{\mathcal{F}}\kappa_m^\dagger = p_n \omega_n \delta^m_n$$ (6.4)

for the amplitudes $\kappa_n \in \mathcal{F} \otimes \mathcal{K}$ of the decomposition $\kappa = \sum_n \kappa_n |n\rangle$ in comparison with the weak orthogonality of $\kappa_n$ in (5.7). The orthogonality (6.4) can be achieved in the following manner: Take $\kappa_n = |n\rangle \otimes \psi_n$ with $\psi_n = p_n^{1/2} \nu_n$ so that

$$\kappa_m^\dagger (I \otimes A) \kappa_n = \langle m | n\rangle \psi_m^\dagger A\psi_n = p_n \psi_m^\dagger A\psi_n \delta^m_n$$

for any $A \in \mathcal{A}$. Then, we have the following theorem.

**Theorem 6.1.** Let $\mathcal{F} = \oplus_n \mathcal{F}_n$ and let $\psi_n$ be the operators, defining a compound state of the diagonal form

$$\Theta(B \otimes A) = \sum_n \langle n| B|n\rangle tr_{\mathcal{F}_n} \psi_n^\dagger A\psi_n$$ (6.5)

Then it corresponds to the entanglement by the orthogonal decomposition

$$\phi(A) = \sum_n |n\rangle \kappa_n^\dagger (I \otimes A) \kappa_n |n\rangle$$ (6.6)

mapping from the algebra $\mathcal{A}$ into a diagonal subalgebra of $\mathcal{B}$.

Thus the entanglement (5.5) corresponding to (6.5) is given by the dual to (6.6) diagonal map

$$\phi_s(B) = \sum_n \langle n| B|n\rangle \psi_n \psi_n^\dagger$$ (6.7)

with the density operators $\sigma_n = \psi_n \psi_n^\dagger$ normalized to the probabilities $p_n = tr_K \psi_n \psi_n^\dagger$.

**Definition 6.2.** The positive diagonal map

$$\phi_s(B) = \sum_n \langle n| B|n\rangle \sigma_n$$ (6.8)

into the subspace of trace-class operation $\mathcal{K}$ with $tr_{\mathcal{G}} \phi_s(I) = 1$, is called quantum $d$-entanglement with the input probabilities $p_n = tr_K \sigma_n$ and the output states $\omega_n = p_n^{1/2} \sigma_n$, and the corresponding compound state (2.2) is called $d$-compound state. The $d$-entanglement is called c-entanglement and compound state is called c-compound if all density operators $\sigma_n$ commute: $\sigma_n \sigma_m = \sigma_m \sigma_n$ for all $m$ and $n$.

Note that due to the commutativity of the operators $B \otimes I$ with $I \otimes A$ on $\mathcal{G} \otimes \mathcal{K}$, one can treat the correspondences as the nondemolition measurements in $\mathcal{B}$ with respect to $\mathcal{A}$. So, the compound state is the state prepared for such measurements on the input $\mathcal{G}$. It coincides with the mixture of the states, corresponding to those after the measurement without reading the sent message. The set of all $d$-entanglements corresponding to a given Schatten decomposition of the input state $\rho$ on $\mathcal{A}$ is obviously convex with the extreme points given by the pure elementary output states $\omega_n$ on $\mathcal{A}$, corresponding to a not necessarily orthogonal decompositions $\sigma = \sum_n \sigma_n$ into one-dimensional density operators $\sigma_n = \rho_n \omega_n$.

The orthogonal Schatten decompositions $\sigma = \sum_n p_n \omega_n$ correspond to the extreme points of $c$-entanglements which also form a convex set with mixed commuting $\omega_n$ for a given Schatten decomposition of $\sigma$. The orthogonal $c$-entanglements were used in [2] to construct a particular type of Accardi’s transition expectations [3] and to define the entropy in a quantum dynamical system via such transition expectations [4].

Thus we classified the entangled states into three categories, namely, q-entangled state, d-entangled state and c-entangled state, and their rigorous expressions were given.
7 Quantum Mutual Entropy via Entanglements

Let us consider the entangled mutual entropy by means of the above three types compound states. We denote the quantum mutual entropy of the compound state \( \Theta \) achieved by an entanglement \( \phi : B \to A \), with the marginals

\[
\Theta (B \otimes I) = \text{tr}_B \rho, \quad \Theta (I \otimes A) = \text{tr}_K A \sigma
\]

by \( I_\phi (\rho, \sigma) \) or \( I_\phi (A, B) \) and it is given as

\[
I_\phi (\rho, \sigma) = \text{tr} \theta_\phi (\log \theta_\phi - \log (\rho \otimes \sigma)).
\]  

(7.2)

Besides this quantity describes an information gain in a quantum system \((A, \sigma)\) via an entanglement \( \phi \) with another system \((B, \rho)\), it is naturally treated as a measure of the strength of an entanglement, having zero the value only for completely disentangled states (7.1), corresponding to \( \theta_\phi = \rho \otimes \sigma \).

**Definition 7.1.** The maximal quantum mutual entropy for a fixed state \( \sigma \)

\[
H_\sigma (A) = \sup \{ I_\phi (A, B) : \phi (I) = \sigma \}
\]

is called q-entropy of the state \( \sigma \). The differences

\[
H_\phi (B|A) = H_\sigma (A) - I_\phi (A, B),
\]

\[
D_\phi (B|A) = S (\sigma) - I_\phi (A, B)
\]

are respectively called the q-conditional entropy on \( B \) with respect to \( A \) and the degree of disentanglement for the compound state \( \phi \).

\( H_\phi (B|A) \) is obviously positive, however \( D_\phi (B|A) \) has the positive maximal value \( S (\sigma) = \sup \{ D_\phi (B|A) : \phi (I) = \sigma \} \) and can achieve also a negative value

\[
\inf \{ D_\phi (B|A) : \phi (I) = \sigma \} = S (\sigma) - H_\sigma (A)
\]

(7.4)

for the entangled states \( \mathbb{1} \).

**Theorem 7.2.** Let \( \mathcal{A} \) be a discrete decomposable algebra \( \oplus B(K_i) \) with a normal state \( \sigma = \oplus \sigma_i \), and \( \mathcal{C} \subseteq \mathcal{A} \) be its center with probability distribution \( \mu = \oplus \mu_i \) induced by \( \sigma \). Then the q-entropy is given by

\[
H_\sigma (A) = \sum_i (\mu_i \ln \mu_i - 2 \text{tr}_K \sigma_i \ln \sigma_i),
\]

(7.5)

It is positive, \( H_\sigma (A) \in [0, \infty) \), and if \( \mathcal{A} \) is finite dimensional, it is bounded, \( H_\sigma (A) \leq \dim \mathcal{A} \).

Let us consider \( \mathcal{G} \) as a Hilbert space describing a quantum input system and \( \mathcal{K} \) as its output Hilbert space. A quantum channel \( \Lambda^\dagger \) sending each input state defined on \( \mathcal{G} \) to an output state defined on \( \mathcal{K} \). A deterministic quantum channel is given by a linear isometry \( \Upsilon : \mathcal{G} \to \mathcal{K} \) with \( \Upsilon^\dagger \Upsilon = I_0 \) (\( I_0 \) is the identify operator in \( \mathcal{G} \)) such that each input state vector \( \eta \in \mathcal{G}, \| \eta \| = 1 \) is transmitted into an output state vector \( \Upsilon \eta \in \mathcal{K}, \| \Upsilon \eta \| = 1 \). The mixtures \( \rho = \sum_n p_n \omega_n \) of the pure input states \( \omega_n = \eta_n \eta_n^\dagger \) are sent into the mixtures \( \sigma = \sum_n p_n \sigma_n \) with pure states \( \sigma_n = \Upsilon \omega_n \Upsilon^\dagger \). A noisy quantum channel sends pure input states \( \omega \) into mixed ones \( \sigma = \Lambda^\dagger \omega \) given by the dual of the following completely positive map \( \Lambda \)

\[
\Lambda (A) = \Upsilon^\dagger (I_1 \otimes A) \Upsilon, \quad A \in \mathcal{A}
\]

(7.6)

where \( \Upsilon \) is a linear isometry from \( \mathcal{G} \) to \( \mathcal{F}_1 \otimes \mathcal{K} \), \( \Upsilon^\dagger (I_1 \otimes I) = I_0 \), and \( I_1 \) is the identity operator in a separable Hilbert space \( \mathcal{F}_1 \) representing the quantum noise. Each input mixed state \( \rho \in B (\mathcal{G}) \) is transmitted into the output state \( \sigma = \Lambda^\dagger \rho \) on \( \mathcal{A} \subseteq B (\mathcal{K}), \) which is given by the density operator

\[
\sigma = \text{tr}_K \Upsilon \rho \Upsilon^\dagger = \Lambda^\dagger \rho \in \mathcal{A}.
\]

(7.7)

We apply the proceeding discussion of the entanglement to the above situation containing a channel \( \Lambda^\dagger \). For a given Schatten decomposition \( \rho = \sum_n p_n | n \rangle \langle n | \) and the state \( \sigma = \Lambda^\dagger \rho \), we can construct three entangled states of the preceding section:

(1) q-entanglement \( \phi^q \) and q-compound state \( \theta^q \) are given as

\[
\phi^q (B) = \sum_{n,m} | n \rangle \langle m | \text{tr}_K \kappa_n \kappa_m^\dagger
\]

\[
\theta^q = \sum_{m,n} | n \rangle \langle m | \otimes \text{tr}_K \kappa_n \kappa_m^\dagger
\]

with the marginals \( \rho = \sum_n p_n | n \rangle \langle n |, \sigma = \Lambda^\dagger \rho = \text{tr}_\phi \theta^q \phi^q \) and \( \text{tr}_K \kappa_n \kappa_m^\dagger = p_n \omega_n \kappa_m^\dagger \kappa_n \) for \( \omega_n = \Lambda^\dagger \kappa_n | n \rangle \langle n | \). Let \( \mathcal{E}_q \) be the convex set of all completely positive maps \( \phi^q \).

(2) d-entanglement \( \phi^d \) and d-compound state \( \theta^d \) are given as
\[
\phi^d(B) = \sum_n \langle n \mid B \mid n \rangle \text{tr}_K \kappa_n\kappa_n^\dagger
\]

\[
\theta^d_\phi = \sum_n |n\rangle\langle n| \otimes \text{tr}_K \kappa_n\kappa_n^\dagger
\]

with the same marginal conditions as (1). Let \( \mathcal{E}_d \) be the convex set of all completely positive maps \( \phi^d \).

(3) c-entanglement \( \phi^c_\phi \) and c-compound state \( \theta^c_\phi \) are same as those of (2) with commuting \( \{ \omega_n \} \). Let \( \mathcal{E}_c \) be the convex set of all completely positive maps \( \phi^c \).

Now, let us consider the entangled mutual entropy and the capacity of quantum channel by means of the above three types of compound states.

**Definition 7.3.** The mutual entropy \( I_q(\rho, \Lambda^*) \) and q-capacity \( C_q(\Lambda^*) \) for a quantum channel \( \Lambda^* \) are defined by the supremums

\[
I_q(\rho, \Lambda^*) = \sup \left\{ S(\phi^q_\phi, \rho \otimes \Lambda^* \rho) ; \phi^q \in \mathcal{E}_q \right\},
\]

\[
C_q(\Lambda^*) = \sup \left\{ I_q(\rho, \Lambda^*) \mid \rho \right\}.
\]

The d-mutual entropy, d-capacity and c-mutual entropy, c-capacity are defined as above using \( \theta^d_\phi \) and \( \theta^c_\phi \), respectively.

Note that due to \( \mathcal{E}_c \subseteq \mathcal{E}_d \subseteq \mathcal{E}_q \), we have the inequalities

\[
I_q(\rho, \Lambda^*) \geq I_d(\rho, \Lambda^*) \geq I_c(\rho, \Lambda^*),
\]

\[
C_q(\Lambda^*) \geq C_d(\Lambda^*) \geq C_c(\Lambda^*)
\]

for a deterministic channel \( (\Lambda^* = id) \), the two lower mutual entropies coincide with the von Neumann entropy:

\[
I_d(\rho, id) = -\text{tr} \rho \log \rho = I_c(\rho, id).
\]

The capacity for such a channel is finite if \( \mathcal{A} \) has a finite rank, \( C_d(\Lambda^*) \leq \dim \mathcal{K} \). On the other hand, the q-mutual entropy can achieve the q-entropy

\[
I_q(\rho, id) = -2\text{tr} \rho \log \rho
\]

and its capacity is bounded by the dimension of the algebra \( \mathcal{A} \), \( C_q(\Lambda^*) \leq \dim \mathcal{A} \) which doubles the d-capacity \( \dim \mathcal{K} \) when \( \mathcal{A} = B(\mathcal{K}) \).

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