FERMION DETERMINANTS: SOME RECENT ANALYTIC RESULTS

M. P. FRY
School of Mathematics, University of Dublin
Dublin 2, Ireland

The use of known analytic results for the continuum fermion determinants in QCD and QED as benchmarks for zero lattice spacing extrapolations of lattice fermion determinants is proposed. Specifically, they can be used as a check on the universality hypothesis relating the continuum limits of the naïve, staggered and Wilson fermion determinants.

1. Introduction

Without internal quark loops such basic phenomena as color charge screening and fast quark fragmentation into hadrons cannot be formulated. Accordingly, the fermion determinant is one of the central quantities in any ab initio lattice QCD calculation that will confront experiment.

It is proposed to use analytic results for fermion determinants in QCD and QED as a check on extrapolated lattice fermion determinants. In particular, the analytic results should be used to verify the universality hypothesis relating the continuum limits of the naïve, staggered and Wilson fermion determinants for a single quark flavor:

\[
\begin{align*}
\left(\det \text{Wilson}\right)_{16}^{a \to 0} &= \det \text{naïve} \\
\left(\det \text{Wilson}\right)_{4}^{a \to 0} &= \det \text{staggered,}
\end{align*}
\]

or stated somewhat differently, verify that

\[
\det \text{Wilson}, \left(\det \text{staggered}\right)^{1/4}, \left(\det \text{naïve}\right)^{1/16} = \det \text{analytic},
\]

It is understood that these are limits computed by taking ratios of the interacting and free determinants. Checks such as these would also be useful in establishing error bounds on determinant extrapolations to the continuum. One important point should be noted: \(\det \text{analytic}\) is renormalization scheme dependent, and so it is necessary to have compatible lattice and continuum renormalization schemes when making the above comparisons.

The analytic results, to be relevant and useful, should be free of ultraviolet and volume cutoffs. Since large quark mass can be dealt with by perturbation theory,
2 M. P. FRY

the results should be valid for all mass values or at least mass values small enough to connect with effective chiral theories. Finally, the background gauge fields should be as general as possible. Obviously these are severe constraints that greatly limit the available benchmark determinants. For lattice theorists such checks will require writing new code and convincing funding agencies that this is a worthwhile project.

2. Defining the Fermion Determinant

By fermion determinant we mean the ratio of determinants of the interacting and free Euclidean Dirac operators, \( \det(\mathcal{P} - gA + m)/\det(\mathcal{P} + m) \), defined by the renormalized determinant on \( \mathbb{R}^d \), namely

\[
\det_{\text{ren}} = \exp(\Pi_2 + \Pi_3 + \Pi_4)\det_{d+1}(1 - gSA),
\]

where

\[
\ln\det_{d+1} = \text{Tr} \left[ \ln(1 - gSA) + \sum_{n=1}^{d} (gSA)^n \right],
\]

and \( S = (\mathcal{P} + m)^{-1} \); \( \Pi_2, \Pi_3, \Pi_4 \) are the second-, third-, and fourth-order contributions to the one-loop effective action defined by some consistent regularization together with a coupling constant subtraction in \( \Pi_2 \) when \( d = 4 \); \( \Pi_4 \) is omitted when \( d < 4 \) as well as \( \Pi_3 \) when \( d = 2 \). The operator \( SA \) is a bounded operator on the Hilbert space \( L^2(\mathbb{R}^d, \sqrt{k^2 + m^2}d^dk) \) for \( A_\mu \in \cap_{n>d} L^n(\mathbb{R}^d) \), in which case it belongs to the trace ideal \( \mathcal{C}_n \) for \( n > d \) \( [\mathcal{C}_n = \{ A | \text{Tr}(A^1A)^{n/2} < \infty \}] \). This means that \( \det_{d+1} \) in 2 can be expanded in terms of the discrete complex eigenvalues \( 1/g_n \) of the non-Hermitian compact operator \( SA \):

\[
\det_{d+1}(1 - gSA) = \prod_n \left[ \left( 1 - \frac{g}{g_n} \right) \exp \left( \sum_{k=1}^{d} \frac{g}{k} \frac{1}{g_n^k} \right) \right].
\]

In QED, C-invariance and the reality of \( \det_{\text{ren}} \) imply that the eigenvalues of \( SA \) appear in quartets \( \pm 1/g_n, \pm 1/g_n^* \); in addition, \( \Pi_3 \) is absent in 1. It is always assumed that \( m \neq 0 \). The \( m = 0 \) case raises nontrivial problems, and most of the above statements, including 3, are false in this case.2 The reader is referred to Refs. 2–4 for a review of these basic results.

Definition 1 of \( \det_{\text{ren}} \) gives back what one gets using the Feynman rules, and so \( \det_{\text{ren}} \) is what physicists mean by a determinant. Closely related to 1 is the Schwinger proper time representation of the determinant provided it has a coupling constant renormalization subtraction included. These are to be contrasted with the zeta function definition whose field-theoretic meaning is not evident. The modified Fredholm determinant 2 in the definition of \( \det_{\text{ren}} \) is necessary because the loop
diagrams representing $\text{Tr}(SA)^n$ for $n = 1, \ldots, d$ are not defined by themselves. We have summarized this basic material because some lattice determinants in the literature, such as $\det(P - gA + m) = \prod_n(\lambda_n + m)$, make no sense when extrapolated to zero lattice spacing.

3. Analytic Results

Because $\det_{\text{ren}}$ has no zeros on the real $g$-axis and $\det_{\text{ren}}(g = 0) = 1$, it is positive for real $g$. By inspection it is an entire function of $g$, and because $SA \in \mathbb{C}_n$, $n > d$, it is of order $d$. This means that for suitable constants $A(\epsilon)$, $K(\epsilon)$ and any complex value of $g$, $|\det_{\text{ren}}| < A(\epsilon) \exp(K(\epsilon)^{d+\epsilon})$ for any $\epsilon > 0$. For $d = 2, 3$ both the QCD and QED determinants satisfy $\det_{\text{ren}} < 1$ for real $g$ and all values of $m > 0$.

This is known as the "diamagnetic" bound\textsuperscript{3–6} which can be interpreted in QED as a reflection of the paramagnetism of charged fermions in a magnetic field. Therefore, in these cases $\det_{\text{ren}}$ definitely does not achieve its maximal growth on the real $g$-axis. Already at this point we have exhausted all that can generally be said so far about fermion determinants in QCD and QED.

Keeping to the criteria we have stated at the end of Sec. 1, we have only two cases to report. The first deals with QCD’s fermion determinant in the instanton background:

$$A_\mu(x) = A_\mu^a\tau_a = \frac{\tau_a}{g} \frac{\eta_{\mu\nu}x^\nu}{x^2 + \rho^2}. \quad (4)$$

The $\eta$ symbols are defined in the Appendix of Ref. 7; $\rho$ is a scale parameter. For $m\rho << 1$,

$$\ln\det_{\text{ren}} = \ln(m\rho) - \frac{2}{3} \ln(\mu\rho) + 2\alpha \left(\frac{1}{2}\right) + [\ln(m\rho) + \gamma - \ln 2] (m\rho)^2 + O(m\rho)^4, \quad (5)$$

where

$$\alpha \left(\frac{1}{2}\right) = \frac{1}{6}(\gamma + \ln \pi) - \frac{\zeta'(2)}{\pi^2} - \frac{17}{72} = 0.145873, \quad (6)$$

$\gamma$ is Euler’s constant and $\mu$ is the subtraction point of the coupling constant $g$. The first three terms in (5) were obtained by ’t Hooft.\textsuperscript{7} Later, Carlitz and Creamer\textsuperscript{8} obtained the $(m\rho)^2 \ln(m\rho)$ term. Then Kwon, Lee and Min\textsuperscript{9} verified the calculation in Ref. 8 and also obtained the remaining terms of order $(m\rho)^2$. Note that the coefficient of the leading $\ln(m\rho)$ term, 1, is the chiral anomaly in this case, which

\textsuperscript{a}This bound is initially derived on a lattice. There remains the technical problem of proving that the zero lattice spacing limit of the lattice fermion determinant coincides with $\det_{\text{ren}}$ defined by (1) and (2). This is partially dealt with in Refs 3, 4. A continuum proof of $\det_{\text{ren}} \leq 1$ with $g$ real for $d = 2, 3$ is needed.
should be of special interest in extrapolations of lattice calculations of the instanton
determinant.

Recently, Dunne, Hur, Lee and Min\textsuperscript{10} have proposed a partial-wave WKB phase-
shift method for calculating the instanton determinant for all values of $m$. Although
their predicted value for $\alpha(1/2)$ is 0.137827 to third order in the WKB approxima-
tion, inclusion of higher order terms may improve this result. If so this would raise
the question of why the WKB analysis should work for other than large values of
$m$.

The inclusion of the subscript ren on the determinant in (5) is provisional
because it was not calculated following (1) and (2). Instead, the $2\alpha(1/2)$ term in
(5) was obtained using Pauli-Villars regulator fields\textsuperscript{b} with space-time dependent masses followed by a change in the subtraction point to the constant mass $\mu$ in (5).

It would be reassuring if the instanton determinant were recalculated for
$m \neq 0$ using (1) and (2) or the Schwinger proper time representation without space-time
dependant masses and verify that the $2\alpha(1/2)$ term emerges when $m\rho << 1$.

It is easy to generalise the first two terms in (5) to any square-integrable self-dual
background gauge field with range $\rho$ interacting with a massive fermion. Referring
to Eqs. (3.5) and (3.10) in Ref. 12 one sees by inspection that for QCD$_4$ and QED$_4$

\begin{equation}
\lim_{m^2 \to 0} m^2 \frac{\partial^2}{\partial m^2} \ln \det_{\text{ren}} = \frac{1}{32\pi^2} \int d^4 x \text{tr} F_{\mu\nu}^2 - \frac{1}{48\pi^2} \int d^4 x \text{tr} F_{\mu\nu}^2 + \frac{1}{16\pi^2} \int d^4 x \text{tr} F_{\mu\nu}^2 + R(m^2, \mu^2),
\end{equation}

The last term in (7) arises when renormalization is carried out by a momentum space subtraction at $p^2 = 0$. Subtracting instead at $p^2 = \mu^2$ gives for $m\rho << 1$

\begin{equation}
\ln \det_{\text{ren}} = \frac{1}{16\pi^2} \left[ \ln(m\rho) - \frac{2}{3} \ln(\mu\rho) \right] \int d^4 x \text{tr} F_{\mu\nu}^2 + R(m^2, \mu^2),
\end{equation}

where $R(0, \mu^2)$ is a $\mu$ and $\rho-$ independent constant that depends of $F_{\mu\nu}$.

The second analytic result available is the fermion determinant in QED$_2$ for the
background field $A_\mu = \epsilon_{\mu\nu} \partial_\nu \phi$, where $\epsilon_{12} = -\epsilon_{21} = 1$ and $B = - \partial^2 \phi$. The magnetic
field $B$ is arbitrary except that $B(r) \geq 0$ with continuous first and second derivatives and $B(r) = 0$ for $r \geq a$ with $\int_0^a dr r B^2(r) < \infty$. In polar coordinates, $A_\phi = 0$, $A_\theta = \Phi(r)/2\pi r$, where $\Phi(r)$ is the total flux of $B$ out to $r$. Hence $A_\theta = \Phi/2\pi r$, $r > a$, where $\Phi$ is $B$’s total flux. Then for the sequence of limits indicated we obtain

\textsuperscript{b} Different regulator schemes can affect the $2\alpha(1/2)$ term in (5). If dimensional renormalization is used instead then

$$2\alpha \left( \frac{1}{2} \right) \to 2\alpha \left( \frac{1}{2} \right) - \frac{1}{3} \ln(4\pi - \gamma) = -0.35952.$$
the following result:

\[
\lim_{|e\Phi| > 1} \lim_{ma \ll 1} \ln det_{\text{ren}} = -\frac{|e\Phi|}{4\pi} \ln \left( \frac{|e\Phi|}{(ma)^2} \right) + O(|e\Phi|, (ma)^2|e\Phi| \ln(|e\Phi|)). \tag{9}
\]

The minus sign is in accord with the diamagnetic bound cited in Sec. 2, and the coefficient of the \(\ln(ma)\) term, \(e\Phi/2\pi\), is the two-dimensional chiral anomaly, assuming \(e > 0\). The analysis in Ref. 13 is not sufficient to rule out the \((ma)^2|e\Phi| \ln |e\Phi|\) term in the remainder in (9); it may only be \((ma)^2|e\Phi|\).

This result is therefore valid for a whole class of background gauge fields in the nonperturbative region of small mass and strong coupling. Note that the \(\Pi_2\) term in (1) is cancelled by \(det_3\) in (1) and (2). We believe this is a general result and that \(\Pi_4\) will be cancelled by \(det_5\) in \(d = 4\).

Finally, although result (9) is for an Abelian background field in \(d = 2\) it remains a challenge for lattice QCD fermion determinant algorithms – which we take to include the discretization of the Dirac operator adopted – to reproduce it.

**Added Note:** The authors cited in Ref. 10 have substantially improved their result in hep-th/0410190 (“Precise Quark Mass Dependence of the Instanton Determinant”).

**References**

1. D.H. Adams, Phys. Rev. Lett. 92 (2004) 162002.
2. E. Seiler, Phys. Rev. D 22 (1980) 2412.
3. E. Seiler, Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics, Lecture Notes in Physics, Vol. 159 (Springer, Berlin, 1982)
4. E. Seiler, in Gauge Theories: Fundamental Interactions and Rigorous Results, Proceedings of the International Summer School of Theoretical Physics, Poiana Brasov, Romania, 1981, edited by P. Dita, V. Georgescu, and R. Purice, Progress in Physics Vol. 5 (Birkhäuser, Boston, 1982), p. 263.
5. D. Brydges, J. Fröhlich, and E. Seiler, Ann. Phys. (N.Y.) 121 (1979) 227.
6. D. Weingarten, Ann. Phys. (N.Y.) 126 (1980) 154.
7. G. ’t Hooft, Phys. Rev. D 14 (1976) 3432; ibid. 18 (1978) 2199E.
8. R.D. Carlitz and D.B. Creamer, Ann. Phys. (N.Y.) 118 (1979) 429.
9. O-K. Kwon, C. Lee and H. Min, Phys. Rev. D 62 (2000) 114022.
10. G.V. Dunne, J. Hur, C. Lee and H. Min, hep-th/0407222
11. G. ’t Hooft, in Under the Spell of the Gauge Principle, Advanced Series in Mathematical Physics 19 (1994), edited by H. Araki et al. (World Scientific, Singapore).
12. M.P. Fry, Phys. Rev. D 55 (1997) 968.
13. M.P. Fry, Phys. Rev. D 67 (2003) 065017.