From Monte Carlo to neural networks approximations of boundary value problems

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Abstract

In this paper we study probabilistic and neural network approximations for solutions to Poisson equation subject to Hölder or $C^2$ data in general bounded domains of $\mathbb{R}^d$. We aim at two fundamental goals.

The first, and the most important, we show that the solution to Poisson equation can be numerically approximated in the sup-norm by Monte Carlo methods based on a slight change of the walk on spheres algorithm. This provides estimates which are efficient with respect to the prescribed approximation error and without the curse of dimensionality. In addition, the overall number of samples does not depend on the point at which the approximation is performed.

As a second goal, we show that the obtained Monte Carlo solver renders ReLU deep neural network (DNN) solutions to Poisson problem, whose sizes depend at most polynomially in the dimension $d$ and in the desired error. In fact we show that the random DNN provides with high probability a small approximation error and low polynomial complexity in the dimension.

Keywords: Deep Neural Network (DNN); Walk-on-Spheres (WoS); Monte Carlo approximation; high-dimensional approximation; Poisson boundary value problem with Dirichlet boundary condition.

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1 Introduction

From the modelling point of view, the typical mathematical formulation for many physical, technological, biological, economical and financial problems is provided in terms of a corresponding partial differential equation (PDE). The PDE approach has proved to be very useful and numerous simulation schemes have been developed using both deterministic and probabilistic tools. The deterministic methods (e.g. finite differences, finite element methods, etc) are very effective to approximate globally the solutions but their computational effort grows exponentially with respect to the dimension of the space. On the other hand, the probabilistic methods manage to overcome the dimensionality issue, but they are usually employed to obtain approximations at a given point and changing the point requires a different approximation.

In the recent years another powerful method has been developed, namely the (deep) neural network models (in short DNN). These have been able to provide a remarkable number of achievements in the technological realm, such as: image classification, language processing and time series analysis, to name only a few. On the theoretical side the DNNs also provide good approximation properties for continuous functions \[3, 4, 17, 21\]. For a more recent in depth analysis see \[4\] and the references in there.

At the interplay between PDEs and neural networks is the dimensional dependence in the approximations of the solution to a boundary problem. There are attempts to study this for example in \[12, 13, 14\] which propose schemes for solving PDEs using some forms of neural networks. However these approaches depend on a grid discretization of the space and for the case we are interested in, namely PDEs in high dimensions, this is not very promising.

There is already a body of literature which treats the approximation of the solution to PDEs with neural networks as for instance in \[18, 19, 14, 13, 15, 7, 8\]. Typically, these approximations remain at the level of numerical experiments and it is not clear how the estimates depend on the dimension of the ambient space.

Related theoretical results for the analysis of differential equations are provided in \[9\] and \[10\], though the problems and the methods are different from the ones tackled in this paper.

We are primarily interested in studying boundary value problems for PDEs in bounded domains and more quantitatively, we look at the dependence of the estimates and the size of the neural networks in terms of the dimension. We mention two directions in the literature that focus on the the curse of dimensionality. One such attempt is proposed in \[7\] whose approach goes through the stochastic representation of the solution. The second approach is in \[15\] which is rather different and constructs the neural network progressively via a gradient descent method and then calculates the polynomial complexity of neural network approximation.
Our starting point is inspired by [7] which essentially builds on the stochastic representation of the solution to the Poisson equation. In turn, the stochastic representation is then followed by the standard walk on spheres (WoS) method to accelerate the computation of the integrals of the Brownian trajectory. This is then used to construct the neural network. In the present paper we expand, clarify and simplify the results in several ways. For a recent extension of the program developed in [7] to the fractional Laplacian see [20].

Now we descend into the description of our main results. We treat the Poisson boundary value problem

\[
\begin{cases}
\Delta u = f \text{ on } D \\
u|_{\partial D} = g,
\end{cases}
\]

(1.1)

where \(D\) is a bounded domain from \(\mathbb{R}^d\) while \(f : D \to \mathbb{R}\) and \(g : \partial D \to \mathbb{R}\) are given continuous functions. In [7] the domain \(D\) is taken to be convex. We treat several layers of generality for the domain \(D\) which lead to different final results on the complexity of the neural network construction.

The regularity of the boundary is important for the results on the number of steps the walk on spheres takes till it reaches the neighborhood of the boundary. We first get some results on the number of steps the walk on spheres takes for the case of arbitrary \(D\). However, this level of generality is hard to push at the level of the construction of neural networks, so we specialize this to the case of general domains with a Lipschitz boundary.

We also introduce two particular classes of domains for which we provide very explicit results on the size of the neural networks approximations. The first one is provided by domains with finite adiam (some form of annular diameter). The other class is made of defective convex domains, for which we get the best results. We should point out that some of the results here have some overlaps with the results in [2], however the constants in there are not explicit in terms of the dimension, an important aspect which is crucial to us. Also, our proof is based on an appropriate Lyapunov function whose role is to push the WoS process towards the boundary of \(D\), and we expect that this method extends easily to other operators where a WoS algorithm is available.

The next round of results we present here is about the walk on spheres introduced in [16] as a way of acceleration for the computation of integrals along the paths of Brownian motion. The standard walk on spheres uses the following scheme: assume that we are at some point \(x \in D\). Then instead of simulating the whole trajectory of the Brownian motion, it chooses (uniformly) a point on the sphere of maximal radius inside \(D\). We develop here a more general scheme where the maximal radius is replaced by a more general radius which is not necessarily maximal and is compatible with ReLU DNNs. Using this we develop a generalized walk on spheres and the most important result in this context is the estimation of the number of steps the walk on spheres takes until it hits a thin shell near the boundary. It is at this point that different estimates are obtained for general Lipschitz domains, for domains with finite annular diameter and much better results for the case of defective convex domains.

We should point one key aspect here. Our stopping rule for the walk on sphere is deterministic and uniform for all points in the domain, as opposed to the one used in [7] and, in fact, to the ones usually adopted in the literature. This has a number of advantages in the end, but perhaps the most important one is that the neural network construction is explicit.

Using the result on the number of steps of the walk on spheres we get to the principal result of the paper, namely Theorem [2.23]. In here we give the key estimates of the error of the solution to the PDE and its stochastic approximation using the walk on spheres algorithm. We can get results for the case when \(f\) is Hölder regular and in \(g\) has a Hölder extension inside the domain. Better results are obtained in the case \(g\) is \(C^2\) inside \(D\).

The domain is also important, and for all data the best results are obtained in the case of defective convex domains \(D\).

It is appropriate now to make an important remark. In [7], the main estimates are done in term of \(L^2\) distance from the solution to the approximation. However, as pointed out in [15], the estimates depend actually on the volume of the domain \(D\), which provides an exponential dependence on the dimension, for sufficiently large domains. We use here the uniform norm which gives on one hand much better results.
On the other hand, we prove that the uniform norm of the error is small with large probability, whilst the approximation complexity depends on $D$ merely through its (anulus) diameter. This is of extreme practical importance because it shows that no matter how one constructs the approximation, this is guaranteed (with high probability) to be good.

The main estimates in Theorem 2.24 depend on three key parameters, $M$, $N$, and $K$:

- $M$ represents how many steps the walk on spheres is allowed to take,
- $N$ represents the number of simulations used for the Monte-Carlo calculation of the empirical expectation used in the representation of the solution in terms of the walk on spheres,
- $K$ is a technical parameter coming from an auxiliary grid discretization used in conjunction with the regularity of the solution to reduce the error estimates to those at a number of grid points. We emphasize that the grid discretization is just an instrument to prove the main result which remains in fact grid-free.

The interesting choices of these parameters are outlined in Corollary 2.25.

We would like to give here in a very brief fashion the key technical points. The reason for this is to highlight our real contribution in this paper and eventually show a general approach which can be used a general method for other potential problems.

First, let us point out that in order to lighten the notations, all the random variables employed in the sequel are assumed to be on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whilst the expectation is further denoted by $E$. The exact representation of the solution of equation (1.1) is described in detail below in (2.2) and we reproduce it here as

$$ u(x) = E\{g(B^x(\tau_D^x))\} + E\left\{\int_0^{\tau_D^x} f(B^x(t)) \, dt\right\}, \quad x \in D, \quad (1.2) $$

where $(B^x(t))_{t \geq 0}$ is the Brownian motion started at $x$ and run till the first exit time $\tau_D^x := \tau_{D^c}$ from the domain $D$. The next key step in our approach is the generalised walk on sphere process defined as

$$ X_0^x := x, \quad X_n^x := X_n^x + \tilde{r}(X_n^x)U_{n+1}, \quad n \geq 0, \quad (1.3) $$

where $x$ is the starting point in the domain $D$, the function $\tilde{r}$ denotes the replacement of the radius which we use for the walk on spheres algorithm and $U_i$ are drawn independent and identically on the unit sphere in $\mathbb{R}^d$. With this process at hand, we have another exact representation of the solution $u$ (which is discussed in more details in Corollary 2.9) as

$$ u(x) = E\{g(B^x(\tau_{D^c}^x))\} + \frac{1}{d} E\left\{\sum_{k \geq 1} \tilde{r}^2(X_{k-1}^x)f(X_{k-1}^x + \tilde{r}(X_{k-1}^x)Y)\right\}. \quad (1.4) $$

Here $x \in D$ and $Y$ is another random variable, independent of the uniform random variables used in the definition of the walk on spheres from (1.3), defined on the unit ball in $\mathbb{R}^d$ and whose distribution has an explicit density proportional to $|y|^{2-d} - 1$, $|y| < 1$, which is in fact the (normalized) Green kernel of the Laplacian on the unit ball with pole at 0. After these exact representations, the approximation of $u$ is given in two steps.

The first one is the definition of

$$ u_M(x) := E\left\{g(X_M^x) + \frac{1}{d} \sum_{k=1}^M \tilde{r}^2(X_{k-1}^x)f(X_{k-1}^x + \tilde{r}(X_{k-1}^x)Y)\right\} \quad \text{for all} \quad x \in D, \quad (1.5) $$

where the key point is that as, unlike in (1.4), the stopping rule now is a fixed deterministic time $M$ which is independent of the point $x \in D$. This is a crucial choice in all the following analysis.
The second step is the standard Monte-Carlo approximation of $u_M$ defined as

$$u_M^N(x) := \frac{1}{N} \sum_{i=1}^{N} [g(X_{M,i}^{x,j}) + \frac{1}{d} \sum_{k=1}^{M} \tilde{r}^2(X_{k-1}^{x,j}) f(X_{k-1}^{x,j} + \tilde{r}(X_{k-1}^{x,j}) Y_i)], x \in D, \quad (1.6)$$

with the choice of sequences $\{U_{i,n}\}_{n,i \geq 0}$ and $\{Y_i\}_{i \geq 0}$ all being independent.

The second step which is essentially the passage from $u_M$ to $u_M^N$ is done in two steps. The first step is the fact that $u_M^N$ is a.s. Hölder continuous, yet with a Hölder constant which is exponentially large with respect to the number $M$ of WoS steps. The second step is to consider a grid of $D$ and to approximate the solution $u$ in the sup-norm merely on this grid, and then by the Hölder regularity of $u_M^N$ to extrapolate the approximation to the entire domain. The grid needs to be taken extremely refined, and thus leads immediately to an exponential complexity in terms of $M$ and the diameter of $D$. Here is where Höftling's inequality combined with a union bound surprisingly compensates very efficiently the grid exponential complexity. Also, the fact that $f$ and $g$ are bounded is important here. This is how we get the final estimate appearing in Theorem 2.24 which we reproduce here.

**Theorem.** Assume that $f$ is $\alpha$-Hölder on $D$ for some $\alpha \in (0,1]$. Then

$$\mathbb{P}\left( \sup_{x \in D} |u(x) - u_M^N(x)| \geq \gamma \right) \leq 2e^{d(\lceil M/\alpha \rceil \log(2+|\tilde{r}|_1) + \log(K)) - \frac{N}{(\|g\|_\infty + \frac{\beta}{4} \text{diam}(D)^2 |f|_\infty)} \gamma^2(M,K,d,\beta,\varepsilon)} \quad (1.7)$$

for all $N, M, K \geq 1$, and $\varepsilon > 0$ such that $\gamma(M,K,d,\varepsilon) \geq 0$, where $\gamma(M,K,d,\varepsilon)$ is given as follows:

i) If $g$ is $\alpha$-Hölder on $\overline{D}$ then

$$\gamma(M,K,d,\varepsilon) := \gamma - 2 \left( |g|_\alpha + \frac{\text{diam}(D)^2 |f|_\alpha + 2\text{diam}(D) |f|_\infty}{d} \right) \left( \frac{\text{diam}(D)}{K} \right)^\alpha$$

$$- \frac{d^{\alpha/2} |g|_\alpha \cdot |v|_\infty^{\alpha/2}(\varepsilon) - |f|_\infty |v|_\infty(\varepsilon) - (4|g|_\infty + \frac{2}{d} \text{diam}(D)^2 |f|_\infty)}{e^{-\frac{\text{diam}(D)^2}{4 \text{diam}(D)^2} M}}.$$

ii) If $g \in C^2(\overline{D})$ then in $(1.7)$ the term $\lceil M/\alpha \rceil$ can be replaced by $M$ and

$$\gamma(M,K,d,\varepsilon) := \gamma - 2 \left( |g|_\alpha + \frac{\text{diam}(D)^2 |f|_\alpha + 2\text{diam}(D) |f|_\infty}{d} \right) \frac{\text{diam}(D)}{K}$$

$$- \left( \frac{|\Delta g|_\infty}{2} + |f|_\infty(\varepsilon) - (8|g|_\infty + \frac{2}{d} \text{diam}(D)^2 |f|_\infty) \right) e^{-\frac{\text{diam}(D)^2}{4 \text{diam}(D)^2} M}.$$

If in addition $\text{diam}(D) < \infty$ (see (2.26)), then in the above expressions for $\gamma(M,K,d,\beta,\varepsilon)$, $|v|_\infty(\varepsilon)$ may be replaced with $\varepsilon \text{diam}(D)$.

Furthermore, if the domain $D$ is $\delta$-defective convex (see (2.11) for the definition), then the factor $e^{-\frac{\text{diam}(D)^2}{4 \text{diam}(D)^2} M}$ from the above estimates can be replaced by $\left( 1 - \frac{\beta^2 (1-\delta)}{4d} \right)^M \sqrt{\frac{\text{diam}(D)}{\varepsilon}}.$

The term $\gamma(M,K,d,\varepsilon)$ looks ugly and complicated however it is in fact the source of many quantitative conclusions. Thus we will discuss the dependency on each of the terms in there. Notice the final goal is to try to make sure $\gamma(M,K,d,\varepsilon)$ is positive, therefore one needs to consistently count the contribution of each variable to the quantity involved:

We start with the dependency on $\varepsilon$. Here $|v|_\varepsilon$ is defined in (2.3) and represents to some extent some weak form of geometry of the domain. In the case of particular geometries this can be estimated in more concrete terms, as for instance, if the domain has finite annular diameter (this can be potentially done for a much
wider class of domains but we do not focus on this in here). The point is that for small $\varepsilon$, the contribution to the term $\gamma(M,K,d,\varepsilon)$ decreases to 0, thus we can make the corresponding terms very small. Having a rate of decay to 0 gives a concrete way of fitting by choosing $\varepsilon$.

Next, it is the dependency on $d$. If $g$ is $C^2$, the larger the dimension $d$, the smaller the contribution to $\gamma(M,K,d,\varepsilon)$. This is very interesting and unexpected. However in lower regularity cases, the term $d^{\alpha/2} \cdot |v|^{\alpha/2} \varepsilon$ in which the dimension makes things worse it can be compensated by a judicious choice of $\varepsilon$.

Further, there is the dependency on $K$. As we pointed out already, $K$ is a technical parameter which appears from a grid discretization combined with the regularity of the solution $u$. The dependency of $\gamma(M,K,d,\varepsilon)$ on $K$ is clear. The larger the $K$, the smaller the value of $\gamma(M,K,d,\varepsilon)$. As we already pointed out above, usually a grid discretization contributes an exponential term in $M$ and $\text{diam}(D)$, however in here this is compensated by the key usage of regularity of $u_M^N$, the use of union bound and Hölder’s inequality.

Finally, the dependency on $M$ is also obvious, the larger the $M$ the smaller the value of $\gamma(M,K,d,\varepsilon)$.

After the remarks on $\gamma(M,K,d,\varepsilon)$, let us notice that the goal is the estimation from (1.7). The right hand side of (1.7) still depends on, $M$, $K$ and $N$. Notice that as soon as we guarantee that $\gamma(M,K,d,\varepsilon) > 0$, then we can choose $N$ sufficiently large to assure that the right hand side of (1.7) is small. To wrap things up, if we assume the domain has finite annular diameter, and both $f, g$ have $\alpha$-Hölder regularity, then we can take (see more details below in Remark 2.27)

$$
\begin{align*}
\varepsilon &= O(\gamma^2 d^{-1}) \\
K &= O(\gamma^{-2/\alpha}) \\
M &= O\left(\frac{d \log(1/\gamma)}{\gamma^{2/\alpha}}\right), \\
N &= O\left(\frac{d^2 \log(1/\gamma)^2 [d^2 \log(1/\gamma) + \gamma^{2/\alpha} \log(1/\eta)]}{\gamma^{2+4/\alpha}}\right)
\end{align*}
$$

if $D$ is a Lipschitz domain

$$
\begin{align*}
\varepsilon &= O(\gamma^2 d^{-1}) \\
K &= O(\gamma^{-1/\alpha}) \\
M &= O\left(\log(d/\gamma)\right), \\
N &= O\left(\frac{\log^2(d/\gamma)(d \log(d/\gamma) + \log(1/\eta))}{\gamma^2}\right)
\end{align*}
$$

if $D$ is $\delta$ - defective convex.

Here $0 < \eta < 1$ is the confidence level, namely with the above choices, we make sure that the right hand side of (1.7) is less than $\eta$. Inside the $O$ symbols, is the dependency on $f, g, \text{diam}(D)$ and the geometry of $D$. As these expressions make it clear, the dependency on the dimension and $\gamma$ is poly-logarithmic, with better choices in the case the domain $D$ has better geometry. Notice in fact that $K$ does not play any role in the numerical scheme (1.5), it does not depend on the dimension $d$ and it plays only a role in the confidence level of the approximation from (1.7). If we count the number of flops for simulations of the random variables involved in (1.7) we conclude that the total number of flops to compute the approximation is $O(d^2 MN)$, thus in total a polynomial complexity in $d$.

The final part of the paper is dedicated to the construction of the neural network. In fact, in this paper we deal only with the special class of feed-forward neural networks whose activation function is given by ReLU (rectified linear unit), and such a network shall further be referred to as a ReLU DNN. The essential part of this construction is the formula (1.10). The key fact is that we choose $\tilde{r}$ to be already a ReLU DNN. This is the main reason for our modification of the walk on spheres algorithm with $\tilde{r}$ instead of the usual distance to the boundary. In addition, having some ReLU DNN approximations of the data $f$ respectively $g$, we can use these building blocks together with some basic facts about the ReLU DNNs in conjunction with (1.7) to get the result.

Perhaps the most important aspect here is the size (number of non-zero parameters) of the neural network. The main conclusion is that from Theorem 5.10. Informally, it says that for $D$ with a finite annular diameter, we have a theoretical guarantee that given good ReLU DNNs approximations of $f$ and $g$ and $r$, with high probability, at least $1 - \eta$, we can construct a (random) ReLU DNN $U_M^N(x)$ such that

$$
\mathbb{P}\left(\sup_{x \in D} |u(x) - U_M^N(\cdot, x)| \leq \gamma \right) \leq 1 - \eta
$$
Here size denotes the number of non-zero parameters in the neural network, $\phi$. A word is in place here about the distance function $\phi$ width respectively the length of the neural network $r$. Implicitly, the constants involved in $U_M^N$ depend on $|g|/\alpha, |g|/\infty, |f|/\infty, \text{diam}(D), \text{adiam}(D), \delta, \alpha, \log(2+|\phi_r|_1)$.

We outline here that a very important consequence of our results concerns the breaking of the curse of dimensionality in the sense that the size of the neural network approximating the solution $u$ adds at most a (low) polynomial complexity to the overall complexity of the approximating networks for the distance function and the data. Moreover, as typical in machine learning, we also show that despite the fact that the neural network construction is random, it breaks the dimensionality curse with high probability. This should be also put in contrast with the results from [7] where the existence of the neural network is purely existential (and not constructive) and also with the conclusion from [15] where the size of the network is $O(d^{\log(1/\gamma)})$, thus the degree grows with the error.

Before we move to the description of the structure of the paper, we would like to point out that the method outlined here can be extended to a general principle for solving more general PDE in high dimension and without curse of dimensionality. The first stage is to have a probabilistic representation of the solution in terms of a stochastic process. The second stage consists in some fast way of simulating the stochastic process which fits well with the representation of the solution to the PDE. The third stage is a Monte-Carlo method for the approximation.

The paper is organized as follows. In Section 2 we present the main notations, the important quantities and the main results. For the sake of readability of the paper we moved the proofs to Section 4. Section 2 in turn contains several subsections which we discuss now because it shows the main approach. Subsection 2.1 details the first probabilistic representation of the solution and various types of estimates based on the annular diameter of a domain, some one dimensional characterisation of the domain. In Subsection 2.2 we walk on spheres and its modified version enters the scene and the main estimates on the number of steps needed to get in the proximity of the boundary for general domains. Also here we introduce the class of $\delta$-defective convex, a class of domains for which we provide better estimates for the number of steps to the proximity of the boundary. Subsection 2.3 provides the main analysis of the stopped modified walk on spheres at deterministic time (thus uniformly for all points in the domain) as opposed to the one in [7] which is random and very difficult to control. Here we estimate first the sup norm of $u - u_M$ defined in (1.2) and (1.3) which is in terms of the regularity of the data, the geometry of the boundary and the parameter $M$. If the annular diameter is finite, in fact the estimates depend only on the diameter and the annular diameter or the parameter $\delta$ (the convex defectiveness). Furthermore, Subsection 2.4 introduces the Monte Carlo estimator $u_M^N$ (1.6) for $u_M$ and contains the main results, namely Theorem 2.24 and Corollary 2.26. Subsection 2.5 contains the main extensions we need to deal with the walk on sphere algorithm. Fundamentally, in order to construct either (1.5) or (1.6) we need to extend the values of boundary data $g$ inside the domain and we do this under some regularity conditions.

Finally, Section 3 contains the neural network consequences of the main probabilistic results and benefits from the very careful preliminary construction of the modified walk on spheres. We should only point out the key fact that from (1.6), once we replaced $g, f$ and $\check{r}$ by a neural networks, the function $u_M^N$ becomes also a neural network. A word is in place here about the distance function $r$, the distance to the boundary. In the original walk on spheres, one is uses $r$ for the construction of the radius of the spheres. We modified this into $\check{r}$. The benefit is that if we have an approximation of $r$ by a neural network, we can easily construct $\check{r}$ which is already a neural network. This avoids complications which arise in [7] from the approximation of $r$ by a neural network after the Monte-Carlo estimator is constructed. The rest of the section here is judicious counting of the size of the neural network obtained for $u_M^N$ replacing $f$ and $g$ by their corresponding approximating networks.
2 The presentation of the main results

This work concerns probabilistic representations and their DNN counterpart for the solution $u$ to problem

$$\begin{cases}
\frac{1}{2} \Delta u = f & \text{in } D \subset \mathbb{R}^d \\
u = g & \text{on } \partial D.
\end{cases} \quad (2.1)$$

Throughout this paper, $D$ is a bounded Lipschitz domain in $\mathbb{R}^d$, $f$ is bounded on $D$ and $g$ is continuous on the boundary $\partial D$. Further regularity shall also be imposed on $D \subset \mathbb{R}^d$, $f$ and $g$, so let us fix some notations:

- We say that a set $D \subset \mathbb{R}^d$ is of class $C^k$ if its boundary $\partial D$ can be locally represented as the graph of a $C^k$ function.

- We write $h \in C(D)$ to say that $h : D \to \mathbb{R}$ is continuous on $D$.

- $L^p(D)$ is the standard Lebesgue space with norm denoted by $| \cdot |_{L^p(D)}$.

- For a bounded function $h : D \to \mathbb{R}$, as well for an element $h \in L^\infty(D)$, we shall denote by $|h|_{\infty}$ both the sup-norm or the essential sup-norm of $h$, respectively.

- For $\alpha \in [0,1]$ and $h : D \to \mathbb{R}$ an $\alpha$-Hölder function, we set $|h|_{\alpha} := \sup_{x,y \in D} \frac{|h(x) - h(y)|}{|x - y|^{\alpha}}$.

Before we proceed, let us remark that one could also consider the anisotropic operator $\nabla \cdot K \nabla$ instead of $\Delta$, where $K$ is a (homogeneous) positive definite symmetric matrix, without altering the forthcoming results. This can be done mainly due to the following straightforward change of variables lemma. For completeness, we also include its short proof in the smooth case.

**Lemma 2.1.** For any given $K$ a $d \times d$ symmetric, positive-definite matrix, assume that $u$ is a classical solution to (2.1) with $\Delta$ replaced by $\nabla \cdot K \nabla$. If we take a $d \times d$ matrix $A$ such that $AA^T = K$, then denoting $D_A := A^{-1}(D)$ and $v(x) := u(Ax)$, $f_A(x) = f(Ax)$, $g_A(x) = g(Ax)$, $x \in D_A$, we have

$$\begin{cases}
\frac{1}{2} \Delta v = f_A & \text{in } D_A \\
v = g_A & \text{on } \partial D_A.
\end{cases}$$

**Proof in Subsection 2.1.**

2.1 Probabilistic representation for Laplace equation and exit time estimates

We fix $B^0(t), t \geq 0$, to be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ which starts from zero, that is $\mathbb{P}(B(0) = 0) = 1$. Then, we set

$$B^x(t) := x + B^0(t), \quad t \geq 0, \quad x \in \mathbb{R}^d,$$

and recall that the law $\mathbb{P}_x := \mathbb{P} \circ (B^x)^{-1}$ on the path-space is precisely the law of a Brownian motion starting from $x \in \mathbb{R}^d$.

By $\tau_{D^c} := \tau_{D^c}^x$ we denote the first hitting time of $D^c := \mathbb{R}^d \setminus D$ by $(B^x(t))_{t \geq 0}$, namely

$$\tau_{D^c}(\omega) := \inf\{t > 0 : B^x(t, \omega) \in D^c\}, \quad \omega \in \Omega.$$

The following result is the fundamental starting point of this work. It is standard for sufficiently regular data, but under the next assumptions we refer to [3, Theorem 6].

**Theorem 2.2** ([3]). Let $g \in C(\partial D)$ and $f \in L^\infty(D)$. Then there exists a unique function $u \in C(D) \cap H^1_{loc}(D)$ such that $u$ is a (weak) solution to problem (2.1). Moreover, it is given by

$$u(x) = \mathbb{E}\{g(B^x(\tau_{D^c}))\} + \mathbb{E} \left\{ \int_0^{\tau_{D^c}} f(B^x(t)) \, dt \right\}, \quad x \in D. \quad (2.2)$$
Let \( v_D : \mathbb{R}^d \to \mathbb{R}_+ \) be given by
\[
v_D(x) := \mathbb{E}\{\tau_D^+\}, \quad x \in \mathbb{R}^d. \tag{2.3}
\]
Most of the main estimates obtained in this paper are expressed in terms of the sup-norm of \( v_D \) in the proximity of the boundary of \( D \) (see e.g. (2.20)). In a following paragraph we shall explore such estimates for \( v_D \) for domains that obey certain "annular" boundary regularity. Before that, let us give the following consequence of Theorem 2.2.

**Corollary 2.3.** The following assertions hold for \( v_D \) given by (2.3):

i) \( |v_D|_\infty \leq \frac{\text{diam}(D)^2}{d} \).

ii) \( v_D \in C^\infty(D) \cap C(\overline{D}) \) is the solution to the Poisson problem
\[
\begin{cases}
-\frac{1}{2} \Delta v_D = 1 & \text{in } D \\
v_D = 0 & \text{on } \partial D.
\end{cases}
\tag{2.4}
\]

**Proof in Subsection 4.1.**

The annulus diameter of a set in \( \mathbb{R}^d \) and exit time estimates. Now we explore more refined bounds for \( v_D(x) := \mathbb{E}\{\tau_D^+\}, \quad x \in D \), aiming at providing a general class of domains \( D \subset \mathbb{R}^d \) for which \( v_D(x) \leq \text{adiam}(D)d(x, \partial D) \), where \( \text{adiam}(D) \) is a sort of annular diameter of \( D \) which in particular is a one dimensional parameter that depends on \( D \). For more details, see (2.3) below for the precise definition, and Proposition 2.5 for the precise result.

The first step here in understanding the exit problem from a domain which is not necessarily convex is driven by our first model, namely the annulus defined by \( 0 < R_0 < R_1 \) as
\[
A(a, R_0, R_1) = \{ x \in \mathbb{R}^d : R_0 < |x - a| < R_1 \}.
\]
We set \( A(R_0, R_1) := A(0, R_0, R_1) \).

The result in this direction is the following.

**Proposition 2.4.** Take \( d \geq 3 \) and \( D := A(R_0, R_1) \). Then, for every point \( x \in D \)
\[
v_D(x) = \mathbb{E}\{\tau_D^+\} \leq d(x, \partial D) \frac{(R_1 - R_0)R_1}{R_0}. \tag{2.5}
\]

**Proof in Subsection 4.1.**

To extend the above result to a larger class of domains, the idea is to define a notion of annulus diameter of a set as follows: For a bounded domain \( D \subset \mathbb{R}^d \), and \( x \in \partial D \), we set
\[
\text{adiam}(D)_x := \inf \left\{ \frac{(R_1 - R_0)R_1}{R_0} : \exists a \in \mathbb{R}^d \text{ such that } x \in \partial A(a, R_0, R_1) \text{ and } D \subset A(a, R_0, R_1) \right\}, \tag{2.6}
\]
with the convention \( \inf \emptyset := \infty \).

Now for a point \( x \), taking \( R_{0,x} \) and \( R_{1,x} \) such that the above infimumum is attained, it is easy to see using the triangle inequality that \( R_{1,x} - R_{0,x} \leq \text{diam}(D) \) for any \( x \in \partial D \) and thus we get that
\[
\text{adiam}(D)_x \leq \text{diam}(D) \left( 1 + \frac{\text{diam}(D)}{R_{0,x}} \right). \tag{2.7}
\]
This is also related to the exterior ball condition at \( x \), since the former holds if and only if \( \text{adiam}(D)_x < \infty \).

If for a point \( x \in \partial D \) there is no pair \( (R_{0,x}, R_{1,x}) \), we set \( R_{0,x} = 0 \) and \( R_{1,x} = +\infty \).
We can now define
\[
\text{adiam}(D) := \sup_{x \in \partial D} \text{adiam}(D)_x = \sup_{x \in \partial D} \frac{(R_{1,x} - R_{0,x})R_{1,x}}{R_{0,x}}. \tag{2.8}
\]
Many domains $D$ in $\mathbb{R}^d$ have finite $\text{adiam}(D)$, including domains which have holes. The above condition gives us that for a bounded domain with smooth boundary and with holes the diameter is finite if the wholes contain balls of a certain positive fixed radius $r > 0$. However, for instance a ball centered at 0 from which we remove a cone with the vertex at the center does not have a finite $\text{adiam}$.

Obviously, a convex bounded domain has finite $\text{adiam}$ and in fact as it turns out this is actually equal to the diameter of the set. Indeed, one can see this by taking a tangent ball of radius $R_0$ and taking $R_1 = R_0 + \text{diam}(D)$. Letting $R_0$ tend to infinity we deduce that for a convex set $D$ we actually have $\text{adiam}(D) = \text{diam}(D)$.

The main result is the following.

**Proposition 2.5.** If $D \subset \mathbb{R}^d$ has $\text{adiam}(D) < \infty$, then for any $x \in D$,

$$v(x) := \mathbb{E}\{\tau_{D^c}\} \leq d(x, \partial D) \text{adiam}(D).$$

In particular, using (2.7) and (2.8), we have

$$v(x) \leq 2d(x, \partial D) \text{diam}(D) \left(1 + \frac{\text{diam}(D)}{R_0}\right), x \in D,$$

where recall that $\tau_{D^c}$ is the first exit time from $D$ and $R_0 := \inf\{R_{0,x} : x \in \partial D\}$.

**Proof in Subsection 4.1.**

### 2.2 Walk-on-Spheres (WoS) and $\varepsilon$-shell estimates revisited

An important benefit of representation (2.2) is that solution $u$ may be numerically approximated by the empirical mean of iid realizations of the random variables under expectation, thanks to the law of large numbers. One way to construct such realizations is by simulating a large number of paths of a Brownian motion that start at $x$ and stopped at the boundary $\partial D$. However, as introduced by Muller in [16], there is a much more (numerically) efficient way of constructing such realisations, based on the idea that (2.2) does not require the entire knowledge of how Brownian motion reaches $\partial D$. This is clearer if one considers the case $f \equiv 0$, when the only information required in (2.2) is the location of Brownian motion at the hitting point of $\partial D$. In this subsection we shall revisit and enhance Muller’s method.

For any $x \in D$ let $r(x) \in [0, \text{diam}(D)]$ denote the distance from $x$ to $\partial D$, or equivalently, the radius of the largest sphere centered at $x$ and contained in $D$, that is

$$r(x) := \inf\{|x - y| : y \in \partial D\} = \sup\{r > 0 : B(x, r) \subseteq D\}. \quad (2.9)$$

Clearly, $r$ is a Lipschitz function.

Recall that the standard WoS algorithm [16] is based on constructing a Markov chain that steps on spheres of radius $r(x)$ where $x \in D$ denotes its current position. However, it is often the case, especially in practice, that merely an approximation $\tilde{r}$ of the distance function is available, and not the exact $r$. This is the case, for example, if for computational reasons, $r$ is simply approximated with the (computationally cheaper) distance function to a polygonal surrogate for the domain $D$. Another situation, which is in fact central to this study, is given in Section 3 below, when $r$ is approximated by a neural network. In both cases, $r$ is approximated by a function $\tilde{r}$ with a certain error. It turns out that considering the chain which walks on spheres of radius $\tilde{r}(x), x \in D$ exhibits certain difficulties regarding the error analysis and also the construction of the chain itself. We do not go into more details at this point, but we refer the reader to Remark (2.21) (iii)-(iv) below for a more technical explanation of these issues.

Our strategy is to solve both of the above difficulties at once, by developing from scratch the entire analysis in terms of (a modification of) $\tilde{r}$, and not in terms of $r$ as it is typically done. This motivates the following concept of distance.

**Definition 2.6.** Let $D \subset \mathbb{R}^d$ be a bounded open set and $r$ the distance function to the boundary. Given $\varepsilon \geq 0$ and $\beta \in (0, 1]$, a Lipschitz function $\tilde{r} : D \rightarrow [0, \text{diam}(D)]$ is called a $(\beta, \varepsilon)$-distance on $D$ if

1) $0 \leq \tilde{r} \leq r$ on $D$, 

2) $\text{adiam}(D) \leq \text{diam}(D)$.

ii) $\tilde{r} \geq \beta r$ on $D_{\varepsilon} := \{ x \in D : r(x) \geq \varepsilon \}$.

When $\varepsilon = 0$ we say that $\tilde{r}$ is a $\beta$-distance. If in addition $\beta = 1$ then it is obvious that $\tilde{r} = r$.

**Remark 2.7.** Suppose that $\phi_r : D \rightarrow [0, \infty)$ is a Lipschitz function such that $|\phi_r - r|_{\infty} \leq \varepsilon$. If $\varepsilon > 2\varepsilon$ and $0 < \beta \leq 1 - \frac{2\varepsilon}{\varepsilon}$, then a simple computation yields that

$$\tilde{r}(x) := (\phi_r(x) - \varepsilon)^+, \quad x \in D$$

is a $(\beta, \varepsilon)$-distance on $D$.

This example already anticipates the amenability of this $\tilde{r}$ to the ReLU neural networks. Indeed, the positive function $x^+$ is precisely the non-linear activation function and from this standpoint, if $\phi_r$ is a neural network, we can argue that $\tilde{r}$ becomes also a ReLU neural network. More on this in Section 3.

**$\tilde{r}$-WoS chain.** Let $U_n : \Omega \rightarrow S(0,1), \ n \geq 1$ be defined (for simplicity) on the same probability space $(\Omega,F,\mathbb{P})$, independent and uniformly distributed, where $S(0,1) \subset \mathbb{R}^d$ is the sphere centered at the origin with radius 1. Let $(\tilde{F}_n)_{n \geq 0}$ be the filtration generated by $(U_n)_{n \geq 0}$, where $U_0 = 0$, namely

$$\tilde{F}_n := \sigma(U_i : i \leq n), \ n \geq 0.$$ 

Also, let $\varepsilon \geq 0, \beta \in (0,1]$, and $\tilde{r}$ be a $(\beta, \varepsilon)$-distance on $D$. For each $x \in D$, we construct the chain $(X^x_n)_{n \geq 0}$ recursively by

$$X^x_0 := x \quad \text{(2.10)}$$

$$X^x_{n+1} := X^x_n + \tilde{r}(X^x_n)U_{n+1}, \ n \geq 0. \quad \text{(2.11)}$$

Clearly, $(X^x_n)_{n \geq 0}$ is a homogeneous Markov chain in $D$ with respect to the filtration $(\tilde{F}_n)_{n \geq 0}$, which starts from $x$ and has transition kernel given by

$$Pf(x) = \int_{S(0,1)} f(x + \tilde{r}(x)z) \sigma(dz), \ x \in D,$$

where $\sigma$ is the normalized surface measure on $S(0,1)$; that is, $\mathbb{E}(f(X^x_n)) = P^n f(x), x \in D, f$ bounded and measurable. We name it an $\tilde{r}$-WoS chain.

We now return now to problem (2.1)

$$\frac{1}{\rho} \Delta u = f \text{ in } D \text{ with } u = g \text{ on } \partial D,$$

which by Theorem 2.2 admits the representations

$$u(x) = \mathbb{E}\{g(B^x(\tau^0_{D^c}))\} + \mathbb{E}\left\{\int_{0}^{\tau^0_{D^c}} f(B^x(t)) \ dt\right\}, \ x \in D.$$

Further, we consider the following sequence of stopping times

$$\tau^0_0 = 0$$

$$\tau^x_{n+1} = \inf\{t > \tau^x_n : |B^x(t) - B^x(\tau^x_n)| \geq \tilde{r}(B^x(\tau^x_n))\}, \ n \geq 1.$$ 

**Remark 2.8.** It is clear that if $\varepsilon = 0$, i.e. $\tilde{r}$ is a $\beta$-distance, then

$$\tau^x_n \uparrow \tau^x_{D^c} \quad \mathbb{P}^0\text{-a.s.}$$

The following result is a generalization of Lemma 3.4 from [7]:
Corollary 2.9. Let $D \subset \mathbb{R}^d$ be a bounded open set, $g \in C(\partial D)$, $f \in L^\infty(D)$, and $u$ be the solution to (2.1). Also, for $\varphi : B(0,1) \to \mathbb{R}$ bounded and measurable set

\[ K_0 \varphi := \mathbb{E} \left\{ \int_0^{\tau^0_{\beta(0,1)}} \varphi(B^0(t)) \ dt \right\} . \]

If $\tilde{r}$ is a $\beta$-distance (i.e. it is a $(\beta,0)$-distance) then the following assertions hold:

i) For all $x \in D$ we have

\[ u(x) = \mathbb{E}\{g(B^x(\tau^x_D))\} + \mathbb{E}\left\{ \sum_{k \geq 1} \tilde{r}^2(X^x_k-1)K_0F_{x,k} \right\} , \]

where $F_{x,k}(y) = f(X^x_{k-1} + \tilde{r}(X^x_{k-1}y)), y \in B(0,1)$, whilst $K_0$ acts on $F_{x,k}$ with respect to the $y$ variable.

ii) The mapping $B(B(0,1)) \ni A \mapsto \mu(A) := dK_01_A \in [0,1]$ renders a probability measure $\mu$ on $B(0,1)$ with density $dG(0,y), y \in D$, where $G(x,y)$ is the Green function associated to $\frac{1}{2}\Delta$ on $B(0,1)$. More explicitly, for $d \geq 3$ we have that $G(0,y)$ is proportional to $|y|^{2-d} - 1, |y| < 1$.

iii) Let $Y$ be a real valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution $\mu$, such that $Y$ is independent of $(U_n)_n$. Then for all $x \in D$ we have

\[ u(x) = \mathbb{E}\{g(B^x(\tau^x_D))\} + \frac{1}{d} \mathbb{E}\left\{ \sum_{k \geq 1} \tilde{r}^2(X^x_k-1)f(X^x_{k-1} + \tilde{r}(X^x_{k-1})Y) \right\} . \]  

(2.12)

Proof in Subsection 4.1.

Consider the $\tilde{r}$-WoS chain described above, and for each $x \in D$ let us define the required number of steps to reach the $\varepsilon$-shell of $\partial D$ by

\[ N^x_\varepsilon := \inf\{n \geq 0 : r(X^n_x) < \varepsilon\} , \]

which is clearly an $(\mathcal{F}_n)$-stopping time.

The estimates to be obtained in the next subsection, and consequently the size of the DNN that we are going to construct in order to approximate the solution to (2.1), depend on how big $N^x_\varepsilon$ is. The goal of this section is to provide upper bounds for $N^x_\varepsilon$, the number of steps the walk on spheres needs to get to the $\varepsilon$-shell. These estimates are first obtained for general domains, and then improved considerably for "defective convex" domains which are introduced in Definition 2.11 below. We provide what are, to our knowledge, the strongest estimates when compared with the currently available literature, as well as rigorous proofs that rely on the general technique of Lyapunov functions. Some of these results have some something in common with the results in [2], though the estimates in there are not clearly determined in terms of the dimension.

It is essentially known that for a general bounded domain in $\mathbb{R}^d$, the average of $N^x_\varepsilon$ grows with respect to $\varepsilon$ at most as $(\text{diam}(D)/\varepsilon)^2$ (see [11] Theorem 5.4] and its subsequent discussion), hence, by Markov inequality, $\mathbb{P}(N^x_\varepsilon \geq R)$ can be bounded by $(\text{diam}(D)/\varepsilon)^2/R$. The next result shows that, in fact, $\mathbb{P}(N^x_\varepsilon \geq R)$ decays exponentially with respect to $R$, and independent of the dimension $d$.

Proposition 2.10. Let $D \subset \mathbb{R}^d$ be a bounded domain, $\varepsilon > 0$, $\beta \in (0,1]$ and $\tilde{r}$ be a $(\beta, \varepsilon)$-distance. Then for any $x \in D$,

\[ \mathbb{E}\left\{ e^{\frac{\beta^2 \varepsilon^2}{\text{diam}(D)^2}} N^x_\varepsilon \right\} \leq 2 , \]

(2.13)

where $\text{diam}(D)$ denotes the diameter of $D$. In particular,

\[ \mathbb{P}(N^x_\varepsilon \geq M) \leq 2e^{-\frac{\beta^2 \varepsilon^2}{\text{diam}(D)^2} M} \] for all $M \in \mathbb{N}$.

Proof in Subsection 4.2.
If the domain $D$ is convex, then the average number of steps required by the WoS to reach the $\varepsilon$-shell is of order $\log(1/\varepsilon)$. This was shown by Muller in [16], and also reconsidered in [2]. Moreover, tracking an explicit constant in front of $\log(1/\varepsilon)$ in terms of the space dimension $d$, is of high importance to the present work. In this subsection we aim to clarify (the proof of) this result and extend it from convex domains to a larger class of domains, resembling the technique of Lyapunov functions from ergodic theory. More importantly, as in the case of Proposition 2.10 we are strongly interested in tail estimates for $N_x^\varepsilon$, not in its expected value. In a nutshell, the idea is to show that the square root of the distance function to the transition kernel of the $\tilde{\varepsilon}$-WoS Markov chain towards the boundary at geometric speed. This is a technique meant to be easily extended for more general operators, in a further work.

The following definition settles the class of domains for which the aforementioned estimate is going to hold.

**Definition 2.11.** We say that a Lipschitz bounded domain $D \subset \mathbb{R}^d$ is "$\delta$-defective convex" if $\delta < 1$ and

$$\Delta r \leq \frac{\delta}{2 \text{rad}(D)} \text{ weakly on } D,$$  \hspace{1cm} (2.14)

where we recall that $r$ is the distance function to the boundary $\partial D$, whilst $\text{rad}(D) := \sup_{x \in D} r(x)$.

**Remark 2.12.** Recall that by [1], $D$ is convex if and only if the signed distance function $r_\ast$ is superharmonic on $\mathbb{R}^d$, where

$$r_\ast := r \text{ on } \mathbb{R}^d \setminus \partial D \text{ and } r_\ast := -r \text{ on } \mathbb{R}^d \setminus D.$$

Hence a defective convex domain as defined by (2.14) is more general than a convex domain; in fact, even if (2.14) holds (on $D$) with $\delta = 0$, it is not necessarily true that $D$ is convex, as explained in [1]. To give an intuition on how a defective convex domain could differ from a convex domain, imagine a ball in $\mathbb{R}^d$. Then there exists a positive thickness $\epsilon$ such that the tubular neighbourhood given by

$$D_\epsilon := \{x \in \mathbb{R}^d \mid \|x\| < \epsilon\},$$

is $\delta$-defective convex. In fact, $\epsilon$ can be chosen explicitly in terms of the principal curvatures of $\Gamma$.

**Example 2.13.** Let $A(R_1, R_2) \subset \mathbb{R}^d$ be an annulus of radii $R_1 < R_2$, namely $A(R_1, R_2) = \{x \in \mathbb{R}^d \mid R_1 < \|x\| < R_2\}$. If $\frac{R_2}{R_1} < 1 + \frac{\delta}{d-1}$ for some $\delta < 1$, then $A(R_1, R_2)$ is a $\delta$-defective convex domain.

**Proof in Subsection 4.2.**

**Example 2.14.** We take $\Gamma$ to be a connected, compact orientable $C^2$ hypersurface in $\mathbb{R}^d$, with $d \geq 2$, endowed with the Riemannian metric $g$ induced by the embedding. We denote by $k_1(x) \leq k_2(x) \leq \ldots \leq k_{d-1}(x)$ the principal curvatures at $x \in \Gamma$. The orientation is specified by a globally defined unit normal vector field $n : \Gamma \to \mathbb{S}^{d-1}$. Then there exists a positive thickness $\epsilon$, such that the tubular neighbourhood given by

$$D_\epsilon := \{x \in \mathbb{R}^d \mid \|x\| < \epsilon\} \quad (2.15) \quad (x,t) \in \Gamma \times (0,1)$$

is $\delta$-defective convex. In fact, $\epsilon$ can be chosen explicitly in terms of the principal curvatures of $\Gamma$.

**Proof in Subsection 4.2.**

**Proposition 2.15.** Let $D \subset \mathbb{R}^d$ be $\delta$-defective convex as in (2.14). Let $\epsilon > 0$, $\beta \in (0,1]$ and $\tilde{r}$ be a $(\beta, \varepsilon)$-distance, and consider $P$ the transition kernel of the $\tilde{r}$-WoS Markov chain $(X_n)_{n \geq 0}$. If we set $V(x) := r(x)^{1/2}, x \in D$, then

$$PV(x) \leq \left( 1 - \frac{\beta^2 (1-\delta)}{4d} \right) V(x), \quad x \in D \text{ a.e.} \quad (2.16)$$

In particular,

$$\mathbb{P}(N_x^\varepsilon > M) \leq \left( 1 - \frac{\beta^2 (1-\delta)}{4d} \right)^M \frac{V(x)}{\sqrt{\varepsilon}}, \quad x \in D, \quad (2.17)$$

and if $\delta_d := 1 - \frac{\beta^2 (1-\delta)}{4d}$, then for any $1 < a < 1/\delta_d$

$$a^{\mathbb{E}(N_x^\varepsilon)} \leq \mathbb{E}\left\{a^{N_x^\varepsilon} \right\} \leq 1 + \frac{a}{1-a\delta_d} \frac{V(x)}{\sqrt{\varepsilon}}, \quad x \in D. \quad (2.18)$$

**Proof in Subsection 4.2.**
Remark 2.16. We have some confidence that Proposition 2.15 still holds if condition (2.14) is satisfied merely in a sufficiently small neighbourhood of the boundary. In particular, in view of Example 2.14, it should hold for domains with smooth boundary. This is going to be considered in more details in a further work.

### 2.3 WoS stopped at deterministic time and error analysis

Throughout this subsection we assume that \( u \) is the solution to problem (2.1), hence Theorem 2.2 and Corollary 2.9 are applicable. Moreover, we keep all the notations from the previous subsections. Before we proceed with the main results of this subsection, let us emphasize several aspects that are essential to this work. To make the explanation simple, assume that \( r = r \) is that the WoS chain is constructed using the \((1,0)\)-distance \( r \). Given \( x \in D \), a usual way to employ WoS Markov chain in order to approximate \( u(x) \) through the representation furnished by Corollary 2.9 is to start the chain from \( x \) and run it until it reaches the \( \varepsilon \)-shell, for some given \( 0 < \varepsilon \ll 1 \). In other words, \( (X^x_k)_{k \geq 0} \) is usually stopped at the (random) stopping time \( N^x_\varepsilon \) and \( u \) represented by (2.2) is then approximated with

\[
u_\varepsilon(x) := \mathbb{E} \left( g(X^x_{N^x_\varepsilon}) \right) + \frac{1}{d} \mathbb{E} \left( \sum_{k=1}^{N^x_\varepsilon} r^2(X^x_{k-1})f(X^x_{k-1} + r(X^x_{k-1})Y) \right).
\]

The intuition behind is that stopping WoS chain at the \( \varepsilon \)-proximity of the boundary should provide a good approximation of how the Brownian motion first hits the boundary \( \partial D \), and estimates that certify this fact are in principle well known. As discussed in the previous subsection, the number of steps required to reach the \( \varepsilon \)-shell is small, especially if the domain is (defective) convex, which eventually leads to a fast numerical algorithm.

From the point of view of this work (also of [7]), the fundamental inconvenience of the above stopping rule is that it depends strongly on the starting point \( x \), mainly through \( N^x_\varepsilon \). In other words, although computationally efficient for estimating a single value \( u(x) \), the above point estimate \( u_\varepsilon \) of \( u \) is expected to fail at overcoming the curse of dimensionality for solving (2.1) globally in \( D \). Moreover, if one aims at constructing a (deep) neural network architecture based on the above representation, as considered in [7] and also in Section 3 below, \( x \) would be the input while \( N^x_\varepsilon \) would give the number of layers; however, the latter should be independent of \( x \), which is obviously not the case. To deal with this architectural impediment, in [7] the authors proposed sup \( N^x_\varepsilon \) as a random time to stop the WoS chain. However, beside the measurability and the stopping time property issues for sup \( N^x_\varepsilon \), it is still unclear, at least to us, that \( \mathbb{E} \left( \sup_{x \in D} N^x_\varepsilon \right) < \infty \) and that this expectation does not depend on the dimension \( d \).

Anyway, our approach is consistently different. Instead of stopping the WoS chain at a random stopping time, be it \( N^x_\varepsilon \) or sup \( N^x_\varepsilon \), the idea is to stop the chain after a deterministic number of steps, say \( M \), independently of the starting point \( x \). Such a choice turns out to be feasible, and it not only avoids the above mentioned issues concerning sup \( N^x_\varepsilon \), but it eventually provides a way to break the curse of dimensionality for solving (2.1), merely using WoS algorithm but in a global fashion. Furthermore, in terms of neural networks, this strategy would also render a way of explicitly constructing a corresponding DNN architecture, that could be easily sampled, and why not, further trained. Also, as already mentioned in Remark 2.7 when we shall deal with neural networks in Section 3 the distance to the boundary \( r \) needs to be replaced by an approximation given by a DNN, with a certain error. Therefore, in light of Definition 2.6 and Remark 2.7 we shall work instead with a \((\beta,\varepsilon)\)-distance \( \tilde{r} \) on \( D \), for some \( \varepsilon > 0 \) and \( \beta \in (0,1] \) properly chosen. Having all these in mind, the aim of this subsection is to estimate the error of approximating the solution \( u \) with

\[
u_M(x) := \mathbb{E} \left( g(X^x_M) + \sum_{k=1}^{M} \tilde{r}^2(X^x_{k-1})K_0f(X^x_{k-1} + \tilde{r}(X^x_{k-1})) \right) \quad \text{for all } x \in D,
\]

for a given (deterministic) number of steps \( M \geq 1 \) that does not depend on \( x \in D \).
To keep the assumption on the regularity of $D$ as general as possible, the forthcoming estimates shall be obtained in terms of the function $v$ defined by (2.3) or (2.4), more precisely in terms of the behavior of $v$ near the boundary measured for each $\varepsilon > 0$ by

\[ |v|_{\infty}(\varepsilon) := \sup \{ v(x) : r(x) := d(x, \partial D) \leq \varepsilon \}. \]  

(2.20)

Let us first consider the case of homogeneous boundary conditions, namely $g \equiv 0$.

**Proposition 2.17.** Let $\varepsilon > 0$, $\beta \in (0,1]$, $\bar{r}$ be a $(\beta, \varepsilon)$-distance, $f \in L^\infty(D)$, $M \in \mathbb{N}^*$ and $u$ be the solution to (2.1) with $g \equiv 0$. If $u_M$ is given by (2.19) then

\[ |u(x) - u_M(x)| \leq |f|_{\infty} \left[ |v|_{\infty}(\varepsilon) + \frac{2}{d} \diam(D)^2 e^{-\frac{\varepsilon^2}{4d\diam(D)^2} M} \right] \quad \text{for all } \varepsilon > 0. \]

**Proof in Subsection 4.3.**

Let us treat now the inhomogeneous Dirichlet problem, this time taking $f \equiv 0$.

**Proposition 2.18.** Let $\varepsilon > 0$, $\beta \in (0,1]$, $\bar{r}$ be a $(\beta, \varepsilon)$-distance, $u$ be the solution to (2.1) with $g \in C(\overline{D})$ and $f \equiv 0$. Further, for each $M \in \mathbb{N}^*$ consider that $u_M$ is given by (2.19) The following assertion hold.

i) If $g$ is $\alpha$-Hölder on $\overline{D}$ for some $\alpha \in [0,1]$ then

\[ \sup_{x \in D} |u(x) - u_M(x)| \leq q^{\alpha/2} |g|_{\alpha} \cdot |v|_{\infty}(\varepsilon) \quad \text{for all } \varepsilon > 0. \]

ii) If $g \in C^2_b(D)$ then

\[ \sup_{x \in D} |u(x) - u_M(x)| \leq \frac{1}{2} |\Delta g|_{\infty} \cdot |v|_{\infty}(\varepsilon) \quad \text{for all } \varepsilon > 0. \]

**Proof in Subsection 4.3.**

We can now superpose Proposition 2.17 and Proposition 2.18 to obtain the following key result:

**Theorem 2.19.** Let $\varepsilon > 0$, $\beta \in (0,1]$, $\bar{r}$ be a $(\beta, \varepsilon)$-distance, and $u$ denote the solution to (2.1) with $g \in C(\overline{D})$ and $f \in L^\infty(D)$. Further, for each $M \in \mathbb{N}^*$ let $u_M$ be given by (2.19). The following assertions hold.

i) If $g$ is $\alpha$-Hölder on $\overline{D}$ for some $\alpha \in [0,1]$, then for all $\varepsilon > 0$ we have

\[ \sup_{x \in D} |u(x) - u_M(x)| \leq q^{\alpha/2} |g|_{\alpha} \cdot |v|_{\infty}(\varepsilon) + |f|_{\infty} |v|_{\infty}(\varepsilon) + (4|g|_{\infty} + \frac{2}{d} \diam(D)^2 |f|_{\infty}) e^{-\frac{\varepsilon^2}{4d\diam(D)^2} M}. \]

ii) If $g \in C^2_b(D)$, then for all $\varepsilon > 0$ we have

\[ \sup_{x \in D} |u(x) - u_M(x)| \leq \frac{1}{2}[\Delta g]_{\infty} \cdot |v|_{\infty}(\varepsilon) + (8 |g|_{\infty} + \frac{2}{d} \diam(D)^2 |f|_{\infty}) e^{-\frac{\varepsilon^2}{4d\diam(D)^2} M}. \]

Let us point out that if $\diam(D) < \infty$ (see (2.19) below), then $|v|_{\infty}(\varepsilon)$ involved above may be replaced with $\varepsilon \diam(D)$. Furthermore, when the domain is defective convex (see subsection 2.2), we can improve considerably the above error estimates with respect to the required number of WoS steps, $M$. This can be done analogously to the proofs of Proposition 2.17 and Proposition 2.18 just by replacing the tail estimate given by Proposition 2.10 with the one provided by Proposition 2.15. Therefore, we give below the precise statement, but we skip its proof.

**Corollary 2.20.** In the context of Theorem 2.19, the following additional assertions hold:

i) If $\diam(D) < \infty$, then by Proposition 2.8 $|v|_{\infty}(\varepsilon)$ involved in the above estimates may be replaced with $\varepsilon \diam(D)$.

ii) If the domain $D$ is $\delta$-defective convex ($\delta < 1$, see (2.14) for the definition) so that the conclusion from Proposition 2.12 is in force, then the factor $e^{-\frac{\varepsilon^2}{4d\diam(D)^2} M}$ from the above estimates can be replaced by

\[ \left( 1 - \frac{\delta^2 (1-d)}{4d} \right)^M \frac{1}{\sqrt{\diam(D)/\varepsilon}}. \]
2.4 Monte-Carlo approximations: mean versus tail estimates

We place ourselves in the same framework as before, namely: $D \subset \mathbb{R}^d$ is a open and bounded Lipschitz domain, $g \in C(\overline{D})$, $f \in L^\infty(D)$, and $u$ is the solution to (2.1).

Further, let $(U_n^i)_{n \geq 1, i \geq 1}$ be a family of independent and uniformly distributed random variables on $S(0,1)$, $	ilde{r}$ be a $(\beta, \varepsilon)$-distance on $D$ for some $\beta \in (0,1]$ and $\varepsilon > 0$ (see Definition 2.6), and set:

$$X_{n+1}^{x,i} := X_n^{x,i} + \tilde{r}(X_n^{x,i}) \cdot U_n^{i}, \quad n \geq 0, i \geq 1.$$

On the same probability space as $(U_n^i)_{n \geq 1, i \geq 1}$, let $(Y^i)_{i \geq 1}$ be iid random variables with distribution $\mu$ given by Corollary 2.9 ii), such that the family $(Y^i)_{i \geq 1}$ is independent of $(U_n^i)_{n \geq 1, i \geq 1}$.

For $N, M \in \mathbb{N}^*$ let $u_M$ be given by (2.19), and consider the Monte Carlo estimator

$$u_M^N(x) := \frac{1}{N} \sum_{i=1}^N g(X_M^{x,i}) + \frac{1}{d} \sum_{k=1}^M \rho_d(X_{k-1}^{x,i}, f(X_k^{x,i} + \tilde{r}(X_k^{x,i})Y^i)),$$

where $x \in D$.

As in Corollary 2.9 iii), we have

$$\mathbb{E}\{u_M^N(x)\} = u_M(x), \quad x \in D, N \geq 1.$$

Remark 2.21. At this point we would like to point out that estimator $u_M^N$ is different than the one employed in [7], Proposition 4.3 in several main aspects:

i) The first aspect was already anticipated in the beginning of Subsection 2.3, namely instead of stopping the WoS chain at $\sup_{x \in D} N_x^\varepsilon$ which is a random time that is difficult to handle both theoretically and practically, we simply stop it at a deterministic time $M$ which is going to be chosen according to the estimates obtained in Theorem 2.24 below and its two subsequent corollaries.

ii) The second aspect is that the estimator used in [7] considers $N$ iid samples drawn from $\mu$, for each of the $N$ iid samples drawn from $(U_n^i)_{n \geq 1, i \geq 1}$, leading to a total of $N^2$ samples. In contrast, $u_M^N$ requires merely $N$ samples, because $Y$ and $(U_n^i)_{n \geq 1, i \geq 1}$ are sampled simultaneously (and independently), on the same probability space.

iii) The third aspect is more subtle: In [7], the Monte Carlo estimator of type (2.21) is constructed based on a given DNN approximation $\overline{r}$ of the distance to the boundary $r$, for any prescribed error, let us say $\eta$. Then, the approximation error of the solution is obtained based on the error of the Monte Carlo estimator constructed with the exact distance $r$, and on how such an estimator varies when $r$ is replaced with $\overline{r}$. However, the latter source of error scales like $2^{N_x^\varepsilon} \eta$, where $N_x^\varepsilon := \sup_{x \in D} N_x^\varepsilon$. To compensate this explosion of error, $\eta$ has to be taken extremely small, and to do so, in [7] it is assumed that $\overline{r}$ can be realized with complexity $O(\log(1/\eta))$; the authors show that such a complexity can indeed be attained for the case of a ball or a hypercube in $\mathbb{R}^d$, and probably can be extended to other domains with a nice geometry. Our approach is different and the key ingredient is to rely on the notion of $(\beta, \varepsilon)$-distance introduced in Definition 2.6. More precisely, using Remark 2.7 we can replace $\overline{r}$ by some $(\beta, \varepsilon)$-distance $r$ at essentially no additional cost, and rely on the herein developed analysis for $r$-WoS. This approach turns out to avoid the additional error of order $2^{N_x^\varepsilon} \eta$ mentioned above, in particular we shall be able to consider domains whose distance function to the boundary may be approximated by a DNN merely at a polynomial complexity with respect to the approximation error.

iv) Another issue regards the construction of the WoS chain itself. Because $\overline{r}$ from iii) may be strictly bigger than $r$, for a given position $x \in D$, the sphere of radius $\overline{r}(x)$ might exceed $D$, so there is a risk that the WoS chain leaves the domain $D$. In particular, if one constructs the WoS chain based on $\overline{r}$, then in order to make the analysis rigorous the boundary data $g$ and the source $f$ should be extended also to the complement of the domain $\overline{D}$. Fortunately, this issue is completely avoided by considering $r$-WoS chains (as it is done in this work), since by definition $\overline{r} \leq r$ on $D$.

Let us begin with the following mean estimate in $L^2(D)$:
Proposition 2.22. Let \( \varepsilon > 0, \beta \in (0,1) \), and \( \bar{r} \) be a \((\beta,\varepsilon)\)-distance. Then for all \( N, M \in \mathbb{N} \), and \( \gamma \geq 0 \)

\[
\mathbb{E} \left\{ |u(\cdot) - u_M^N(\cdot)|^2_{L^2(D)} \right\} \leq 2\lambda(D) \left[ \sup_{x \in D} |u(x) - u_M(x)|^2 + \frac{2(\|g\|_{\infty}^2 + |M|f_\infty|2\text{diam}(D)^4}{N} \right],
\]

where \( \lambda \) is the Lebesgue measure on \( \mathbb{R}^d \), whilst \( u_M \) and \( u_M^N \) are given by (2.19) and (2.21). In particular, the above inequality can be made more explicit by employing the estimates for \( \sup_{x \in D} |u(x) - u_M(x)| \) obtained in Theorem 2.19 and Corollary 2.20, depending on the regularity of \( D \) and \( g \).

Proof in Subsection 4.4.

Remark 2.23. Note that as in [7], Section 4, the above error estimate depends on the volume \( \lambda(D) \). When \( \lambda(D) \) scales well with the dimension (e.g. at most polynomially), then (2.22) can be employed to overcome the curse of dimensionality; in fact, if \( D \) is a subset of a hypercube whose side has length less then some \( \delta < 1 \), then \( \lambda(D) \leq \delta^d \), hence, in this case, the factor \( \lambda(D) \) improves the mean squared error exponentially with respect to \( d \). However, \( \lambda(D) \) may also grow exponentially with respect to \( d \), and the above estimate cannot be used to construct a neural network whose size scales at most polynomially with respect to the dimension. Therefore, our next (and in fact) main goal is to solve this inconvenience, by looking at tail estimates for the Monte-Carlo error; one key idea is to quantify the error using the sup-norm instead of the \( L^2(D) \)-norm.

As announced in the above remark, we conclude now with the central result of this paper:

Theorem 2.24. Keep the same framework and notations as in the beginning of this subsection, in particular \( \varepsilon > 0, \beta \in (0,1) \), \( \bar{r} \) is a \((\beta,\varepsilon)\)-distance, whilst \( u_M \) and \( u_M^N \) are given by (2.19) and (2.21). Also, assume that \( f \) is \( \alpha \)-Hölder on \( D \) for some \( \alpha \in (0,1] \). Then

\[
P \left( \sup_{x \in D} |u(x) - u_M^N(x)| \geq \gamma \right) \leq 2e^{d \left[ \frac{[M/\alpha]}{\log(2+|\bar{r}|_1) + \log(K)} \right] \gamma^2 (M,K,d,\beta,\varepsilon)}
\]

for all \( N, M, K \geq 1 \) and \( \varepsilon > 0 \) such that \( \gamma(M,K,d,\varepsilon) \geq 0 \), where \( \gamma(M,K,d,\varepsilon) \) is given as follows:

i) If \( g \) is \( \alpha \)-Hölder on \( D \) then

\[
\gamma(M,K,d,\varepsilon) := \gamma - 2 \left( \frac{|g|_\alpha + 2\text{diam}(D)||\alpha| + 2\text{diam}(D)||f|_{\infty}}{d} \right) \left( \frac{\text{diam}(D)}{K} \right)^\alpha
\]

\[
- \frac{d^{\alpha/2} |g|_\alpha \cdot |v|_{\infty}^{\alpha/2} (\varepsilon) - |f|_{\infty} |v|_{\infty} (\varepsilon) - (4|g|_{\infty} + \frac{2}{d}\text{diam}(D)^2 |f|_{\infty})e^{-\frac{\beta^2 \varepsilon^2}{4\text{diam}(D)^2}}}.
\]

ii) If \( g \in C^2(D) \) then in (2.23) the term \( [M/\alpha] \) can be replaced by \( M \) and

\[
\gamma(M,K,d,\varepsilon) := \gamma - 2 \left( \frac{|g|_\alpha + 2\text{diam}(D)||\alpha| + 2\text{diam}(D)||f|_{\infty}}{d} \right) \frac{\text{diam}(D)}{K}
\]

\[
- \frac{\left( |\Delta g|_{\infty} + |f|_{\infty} \right) |v|_{\infty} (\varepsilon) - (8|g|_{\infty} + \frac{2}{d}\text{diam}(D)^2 |f|_{\infty})e^{-\frac{\beta^2 \varepsilon^2}{4\text{diam}(D)^2}}}.
\]

If in addition \( \text{adiam}(D) < \infty \) (see (2.6)), then, using Proposition 2.22 in the above expressions for \( \gamma(M,K,d,\beta,\varepsilon) \), \( |v|_{\infty}(\varepsilon) \) may be replaced with \( \varepsilon \) adiam(D).

Furthermore, if the domain \( D \) is \( \delta \)-defective convex (see (2.14) for the definition) so that the conclusion from Proposition 2.13 is in force, then the factor \( e^{-\frac{\beta^2 \varepsilon^2}{4\text{diam}(D)^2} M} \) from the above estimates can be replaced by

\[
\left( 1 - \frac{\beta^2 (1-\delta)}{4d} \right)^M \sqrt{\frac{\text{diam}(D)}{\varepsilon}}.
\]

Proof in Subsection 4.4.

Remark 2.25. Note that the left hand side of (2.23) does not depend on \( K \), and if \( \bar{r} \) is a \( \beta \)-distance, then it does not depend on \( \varepsilon \) as well. Therefore, in the right hand side of (2.23) one may take the infimum with respect to \( K \geq 1 \), and if \( \bar{r} \) is a \( \beta \)-distance then one may also take the infimum with respect to \( \varepsilon > 0 \); but optimizing the previously obtained bounds in this way may be cumbersome. Anyway, convenient bounds can be easily obtained from particular choices of \( K \) and \( \varepsilon \), so let us do so in the sequel.
Let us conclude this subsection with the following consequence obtained for some convenient choices for \( K \) and \( \varepsilon \): for simplicity we treat only the case when \( g \) is merely Hölder, the \( C^2 \) case being fundamentally similar.

**Corollary 2.26.** Let \( D \subset \mathbb{R}^d \) such that \( \text{adim}(D) < \infty \). Further, let \( f \in L^\infty(D) \), \( g \) be \( \alpha \)-Hölder on \( \bar{D} \) for some \( \alpha \in [0, 1] \), \( \gamma > 0 \) be a prescribed error, \( \eta > 0 \) be a prescribed confidence, and \( \bar{r} \) be a \((\beta, \varepsilon)\)-distance with \( \beta \in (0, 1] \) and \( \varepsilon > 0 \) such that

\[
\varepsilon \leq \varepsilon_0 := [(4|g|_\alpha + |f|_\infty)\text{adim}(D) \vee 1]^{-\frac{2}{\beta}} \frac{\gamma}{2} d^{-1}.
\]  
(2.24)

Also, choose

\[
K := \left\lfloor \text{diam}(D) \left( 8 \left( \frac{|g|_\alpha + \frac{\text{diam}(D)^2|f|_\infty + 2\text{diam}(D)|f|_\infty}}{\gamma} \right) + 1 \right)^{1/\alpha} \right\rfloor.
\]  
(2.25)

Then

\[
\mathbb{P}\left( \sup_{x \in D} |u(x) - u^N_M(x)| \geq \gamma \right) \leq \eta
\]  
(2.26)

whenever we choose

\[
N \geq 16 \left\{ \frac{d \left[ M/\alpha \right] \log(2 + |\bar{r}|_1) + \log(K) + \log(\frac{2}{\eta})}{9\gamma^2} \right\} \left[ |g|_\infty + \frac{M}{d} \text{diam}(D)^2|f|_\infty \right]^2
\]  
(2.27)

and either

i) \( D \) is \( \delta \)-defective convex and

\[
M \geq \frac{\log \left( \frac{4}{\gamma} \sqrt{\frac{\text{diam}(D)}{\varepsilon_0}} \right) + \log(4|g|_\infty + \frac{2}{d} \text{diam}(D)^2|f|_\infty)}{\beta^2(1 - \delta)}
\]  
(2.28)

or

ii) \( D \) is merely a bounded Lipschitz domain and

\[
M \geq \frac{\left[ \log(4/\gamma) + \log(4|g|_\infty + \frac{2}{d} \text{diam}(D)^2|f|_\infty) \right] 4\text{diam}(D)^2}{\beta^2 \varepsilon_0^2}
\]  
(2.29)

**Proof in Subsection 4.4.**

**Remark 2.27.** By some simple computations, \( \varepsilon_0, M, \) and \( N \) from Corollary 2.26 exhibit the following asymptotic behaviors:

\[
\varepsilon_0 \in \mathcal{O}(\gamma^{1/2} d^{-1}),
\]

and if \( D \) is \( \delta \)-defective convex then

\[
M \in \mathcal{O}(\frac{\log(d/\gamma)}{\beta^2(1 - \delta)}),
\]

\[
N \in \mathcal{O} \left( \frac{\log^2(d/\gamma)(d \log(d/\gamma) + \beta^2(1 - \delta) \log(1/\eta))}{\beta^2 \gamma^2(1 - \delta)^2} \right),
\]

whilst if \( D \) is merely a bounded Lipschitz domains then

\[
M \in \mathcal{O}(\frac{d \log(1/\gamma)}{\beta^2 \gamma^2(1 - \delta)}),
\]

\[
N \in \mathcal{O} \left( \frac{d^2 \log(1/\gamma)^2 [d \log(1/\gamma)(d + \beta^2 \gamma^2/\alpha) + \beta^2 \gamma^2/\alpha \log(1/\eta)]}{\beta^2 \gamma^2(1 - \delta)^2} \right).
\]

Here, the Landau symbols tacitly depend on (the regularity of) \( f, g, \text{diam}(D) \), and \( \text{adim}(D) \). In particular, only in terms of the dimension \( d \), if the domain is \( \delta \)-convex then \( M \in \mathcal{O}(\log(d)) \) and \( N \in \mathcal{O}(d \log^3(d)) \), whilst in the case of a bounded Lipschitz domain, \( M \in \mathcal{O}(d) \) and \( N \in \mathcal{O}(d^4) \).
2.5 On regular extensions of the boundary data inside the domain

Recall that one assumption of the main results in the previous subsections (see e.g. Theorem 2.24) is that the boundary data $g$ can be extended as a regular function (Hölder or $C^2$) defined on the entire domain $\overline{\mathcal{D}}$. This is required by the fact that the data needs to be evaluated at the location where the WoS chain is stopped, see (2.21), and such stopped position lies in principle in the interior of the domain $D$. However, usually in practice, $g$ is measured (hence known) merely at the boundary $\partial D$. With this issue in mind, in this subsection we address the problem of extending $g$ regularly from $\partial D$ to $\overline{\mathcal{D}}$, in a constructiveway which is also DNN-compatible.

We take $D \subset \mathbb{R}^d$ to be a set of class $C^k$, $k = 3$ or $k = 2$, hence (see [6] sec. 14.6]) there exists a neighbourhood $D_{\epsilon_0} := \{x \in D; \text{dist}(x, \partial D) < \epsilon_0\}$ of $\partial D$ such that:

- The restriction of the distance function $r : D_{\epsilon_0} \to \mathbb{R}_+$ is of class $C^k$.
- The nearest point projection $\pi_{\partial D} : D_{\epsilon_0} \to \partial D$ is of class $C^{k-1}$.

We have:

**Lemma 2.28.** Let $D \subset \mathbb{R}^d$ to be a set of class $C^2$ hence for any point $x \in \partial D$ there exist a function $\phi_x : \mathbb{R}^{d-1} \to \mathbb{R}$ of class $C^2$ and a radius $r_x > 0$ such that

$$D \cap B(x, r_x) = \{y = (y_1, \ldots, y_d) \in B(x, r_x); y_d < \phi_x(y_1, \ldots, y_{d-1})\}.$$  

We denote by $M := \sup_{x \in \partial D} |\nabla \phi_x|_{\infty}$ the order principal curvatures of $\partial D$ let us take $\epsilon_0 := \min_{x \in \partial D} k_{d-1}^{-1}(x)$. Take $\psi \in C_c^\infty([0, \infty), \mathbb{R})$ to be such that $\psi \equiv 1$ on $[0, 1]$ and $\psi \equiv 0$ on $[3, \infty)$, $|\psi|_{\infty}, |\psi'|_{\infty}, |\psi''|_{\infty}, \leq 1$. We define the extension $G$ in $\overline{\mathcal{D}}$ of the $\alpha$-Hölder function $\psi$ given on the boundary $\partial D$, for $\alpha \in (0, 1]$ as:

$$G : \overline{\mathcal{D}} \to \mathbb{R}, \quad G(x) := \psi \left(\frac{1}{\epsilon_0} r(x)\right) g(\pi_{\partial D}(x)), \quad x \in \overline{\mathcal{D}}. \quad (2.30)$$

i) The extension $G$ is $\alpha$-Hölder on $\overline{\mathcal{D}}$ and we have:

$$|G|_\alpha \leq |\nabla \pi_{\partial D}|_{\infty} g|_\alpha + \frac{|g|_{\infty}}{\epsilon_0} |\text{diam}(\mathcal{D})|^{1-\alpha}.$$  

ii) If, furthermore, the domain is of class $C^3$ and $g \in C^2(\partial D)$ then $G$ is in $C^2(D) \cap C(\overline{\mathcal{D}})$ with $G = g$ on $\partial D$ and furthermore we have:

$$|\nabla G|_{\infty} \leq \frac{1}{\epsilon_0} |g|_{\infty} + 2 |\nabla g|_{\infty}$$

$$|\Delta G|_{\infty} \leq \tilde{C},$$

where $\tilde{C}$ is an explicitly computable constant in terms of $|g|_{\infty}, |\nabla g|_{\infty}, |\Delta g|_{\infty}, M,$ and $\epsilon_0$.

**Proof in Subsection 2.5.**

**Remark 2.29.** One can easily provide a non-constructive $\alpha$-Hölder extension $\tilde{G}$ on $\overline{\mathcal{D}}$ of the $\alpha$-Hölder boundary data $g$ given on $\partial D$ by setting $\tilde{G}(x) := \inf\{g(y) + |g|_\alpha |x - y|^\alpha, y \in \partial D\}, x \in \overline{\mathcal{D}}$.

3 DNN counterpart of the main results

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be the rectified linear unit (ReLU) activation function, that is $\sigma(x) := \max\{0, x\}, x \in \mathbb{R}$. Let $(d_i)^{0, \ldots, d}$ be a sequence of positive integers. Let $A^i \in \mathbb{R}^{d_i \times d_{i-1}}$ and $b^i \in \mathbb{R}^{d_i}$, $i = 1, \ldots, L$, and set $W^i(x) := A^i x + b^i, x \in \mathbb{R}^d$. We define the realization of the DNN $\mathbb{R}^{d_0} \ni x \mapsto \phi(x)$ by

$$\mathbb{R}^{d_0} \ni x \mapsto \phi(x) := W^L \circ \sigma \circ W^{L-1} \circ \cdots \circ \sigma \circ W^1(x) \in \mathbb{R}^{d_L}, \quad x \in \mathbb{R}^{d_0}, \quad (3.1)$$
Lemma 3.1. For every $c > 0$ and $\delta \in (0,1)$, there exists a DNN $\Pi^c_\delta$ such that

$$\sup_{a,b\in[-c,c]} |ab - \Pi^c_\delta(a,b)| \leq \delta$$

and $\text{size}(\Pi^c_\delta) = O\left([\log(\delta^{-1}) + \log(c)]\right)$.

Now, we recall Lemma II.6 from [4]:

**Lemma 3.2.** Let $\phi_i$, $i = 1,\ldots,n$, be ReLU DNNs with the same input dimension $d_0 \in \mathbb{N}$ and the same depth $L := \mathcal{L}(\phi_i), 1 \leq i \leq n$. Let $a_i, i = 1,\ldots,n$, be scalars. Then there exists a ReLU DNN $\phi$ such that

1) $\phi(x) = \sum_{i=1}^{n} a_i \phi_i(x)$ for every $x \in \mathbb{R}^{d_0}$,
2) $\mathcal{L}(\phi) = L$,
3) $\mathcal{W}(\phi) \leq \sum_{1 \leq i \leq n} \mathcal{W}(\phi_i)$,
4) $\text{size}(\phi) \leq \sum_{1 \leq i \leq n} \text{size}(\phi_i)$.

The following lemma is taken from [4, Lemma II.3], with the mention that the last assertion iv) brings some improvement which is relevant to our purpose; it is immediately entailed by the proof of the same [4, Lemma II.3], so we skip its justification.

**Lemma 3.3.** Let $\phi_1: \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ and $\phi_2: \mathbb{R}^{d_3} \to \mathbb{R}^{d_4}$ be two ReLU DNNs. Then there exists a ReLU DNN $\phi: \mathbb{R}^{d_3} \to \mathbb{R}^{d_2}$ such that

1) $\phi(x) = \phi_1(\phi_2(x))$ for every $x \in \mathbb{R}^{d_0}$,
2) $\mathcal{L}(\phi) = \mathcal{L}(\phi_1) + \mathcal{L}(\phi_2)$,
3) $\mathcal{W}(\phi) \leq \max(\mathcal{W}(\phi_1), \mathcal{W}(\phi_2), 2d_1)$,
4) $\text{size}(\phi) \leq \min(\text{size}(\phi_1) + \text{size}(\phi_2) + d_1|\mathcal{W}(\phi_1) + \mathcal{W}(\phi_2)|, 2\text{size}(\phi_1) + 2\text{size}(\phi_2))$.

The next lemma is essentially [4, Lemma II.4]. As in the case of the previous lemma, assertion iv) comes with a slight modification of the original result, which can be immediately deduced from the proof of [4, Lemma II.4].

**Lemma 3.4.** Let $\phi: \mathbb{R}^{d_0} \to \mathbb{R}^{d_4}$ be a ReLU DNN such that $\mathcal{L}(\phi) < L$. Then there exists a second ReLU DNN $\tilde{\phi}: \mathbb{R}^{d_0} \to \mathbb{R}^{d_4}$ such that

1) $\phi(x) = \tilde{\phi}(x)$ for all $x \in \mathbb{R}^{d_0}$,
2) $\mathcal{L}(\tilde{\phi}) = L$,
3) $\mathcal{W}(\tilde{\phi}) = \max(2d_1, \mathcal{W}(\phi))$,
4) $\text{size}(\tilde{\phi}) \leq \min(\text{size}(\phi) + d_1\mathcal{W}(\phi), 2\text{size}(\phi)) + 2d_1(L - \mathcal{L}(\phi))$. 

where $\mathbb{R}^d \ni x \mapsto \sigma(x) := (\sigma(x_1), \ldots, \sigma(x_d))$, $d \in \mathbb{N}$, is defined coordinatewise. The weights of the ReLU DNN $\phi$ are the entries of $(A^i, b^i)_{i=1,\ldots,L}$. The size of $\phi$ denoted by $\text{size}(\phi)$ is the number of non-zero weights. The width of $\phi$ is defined by $\text{width}(\phi^L) = \max\{d_0, \ldots, d_L\}$ and $L$ is the depth of $\phi$ denoted by $\mathcal{L}(\phi)$. In the sequel, we only consider DNNs with ReLU activation function.

For the reader’s convenience, before we proceed to the main result of this section (see Theorem 3.10 below), we present first several technical lemmas following [4] and [21], as well as some of their consequences; all these preparatory results are meant to provide a clear and systematic way of quantifying the size of the DNN which is constructed in the forthcoming main result, namely Theorem 3.10.
As a direct consequence of Lemma 3.1 Lemma 3.3 Lemma 3.4 and [4] Lemma II.5, one gets the following approximation result for products of scalar ReLU DNNs:

**Corollary 3.5.** Let \( \phi_0, \phi_1 : \mathbb{R}^d \to \mathbb{R} \) be two ReLU DNNs, \( D \subset \mathbb{R}^d \) be a bounded subset, and let \( \Pi := \Pi_{\epsilon_{\rho}} \) be given by Lemma 3.4 for \( \epsilon := \max(\sup_{x \in D} \phi_0(x), \sup_{x \in D} \phi_1(x)) \) and \( \epsilon_{\rho} > 0 \). Then there exists a ReLU DNN \( \phi : \mathbb{R}^d \to \mathbb{R} \) such that

1. \( \phi(x) = \Pi(\phi_0(x), \phi_1(x)) \) for every \( x \in \mathbb{R}^d \),
2. \( \sup_{x \in D} |\phi(x) - \phi_0(x)| \leq \epsilon_0 \),
3. \( \sup_{x \in D} |\phi(x) - \phi_1(x)| \leq \epsilon_0 \),
4. \( \sup_{x \in D} |\phi(x) - \phi_0(x)| \leq \epsilon_0 \).

Proof in Subsection 6.5

The following result is easily deduced by employing recursively Lemma 3.6 and Lemma 3.3, so we omit its proof.

**Corollary 3.7.** Let \( \phi : \mathbb{R}^d \to \mathbb{R} \) be a ReLU DNN and \( v \in \mathbb{R}^d \) be a vector. Then there exists a ReLU DNN \( \phi_v : \mathbb{R}^d \to \mathbb{R} \) such that

1. \( \phi_v(x) = x + \phi(x)v \) for all \( x \in \mathbb{R}^d \),
2. \( \mathcal{L}(\phi_v) = \mathcal{L}(\phi) + 1 \),
3. \( \mathcal{W}(\phi_v) \leq 2d + \max(d, \mathcal{W}(\phi)) \),
4. \( \text{size}(\phi_v) \leq 2\text{size}(\phi) + 2d[\mathcal{L}(\phi) + 2] \).

Proof in Subsection 4.5

The following approximation result for products of scalar ReLU DNNs can be given by Lemma 3.1 for \( \epsilon := \max(\sup_{x \in D} \phi_0(x), \sup_{x \in D} \phi_1(x)) \) and \( \epsilon_{\rho} > 0 \). Then there exists a ReLU DNN \( \phi : \mathbb{R}^d \to \mathbb{R} \) such that

1. \( \phi(x) = \Pi(\phi_0(x), \phi_1(x)) \) for every \( x \in \mathbb{R}^d \),
2. \( \sup_{x \in D} |\phi(x) - \phi_0(x)| \leq \epsilon_0 \),
3. \( \sup_{x \in D} |\phi(x) - \phi_1(x)| \leq \epsilon_0 \),
4. \( \sup_{x \in D} |\phi(x) - \phi_0(x)| \leq \epsilon_0 \).

Proof in Subsection 4.5

The following result is easily deduced by employing recursively Lemma 3.6 and Lemma 3.3, so we omit its proof.

**Corollary 3.8.** Let \( \phi_0, \phi_1, \phi_2, \ldots, \phi_k : \mathbb{R}^d \to \mathbb{R} \) be a ReLU DNN and \( v_k \in \mathbb{R}^d, k \geq 1 \) be a sequence of vectors. Then there exist ReLU DNNs \( \theta_k : \mathbb{R}^d \to \mathbb{R}^d, k \geq 0 \), such that for every \( k \geq 0 \)

1. \( \theta_{k+1}(x) = \phi_{v_{k+1}}(\theta_k(x)) \) and \( \theta_0(x) = x \) for all \( x \in \mathbb{R}^d \), where \( \phi_{v_k} \) is the one constructed in Lemma 3.6,
2. \( \mathcal{L}(\theta_{k+1}) = (k + 1)(\mathcal{L}(\phi) + 1) + 1 \),
3. \( \mathcal{W}(\theta_{k+1}) \leq 2d + \max(d, \mathcal{W}(\phi)) \),
4. \( \text{size}(\theta_{k+1}) \leq 2\text{size}(\phi) + 2d[\mathcal{L}(\phi) + 2] + d + 2(k + 1)\text{size}(\phi) \).

We end this first paragraph by a ReLU DNN extension of a boundary data \( g \in C^2(\partial D) \) to the entire domain \( D \).

**Corollary 3.9.** Let \( D \subset \mathbb{R}^d \) be a set of class \( C^1 \) and \( g \in C^2(\partial D, \mathbb{R}) \) as defined in Lemma 2.28 we assume that for every \( \delta_r, \delta_p, \delta_g, \delta_\psi \in (0, 1) \), there exist ReLU DNNs \( \phi_r, \phi_\psi, \phi_\psi \) and \( \phi_g \) such that

\[
|\phi_r(x) - \phi_\psi(x) - \phi_\psi(x)| \leq \delta_r, \quad |\phi_\psi(x) - \phi_\psi(x) - \phi_\psi(x)| \leq \delta_\psi, \quad |\phi_\psi(x) - \phi_\psi(x) - \phi_\psi(x)| \leq \delta_g.
\]

With \( \epsilon_0 \) the one given in Lemma 2.28 set

\[
\delta := 2 \left( 3\delta_\psi + \delta_p \right) + 2 \left( 3\delta_g + \|\nabla g\|_\infty + \|\nabla g\|_\infty + \delta_r \right) \leq 2 \left( 3\delta_\psi + \delta_p \right) + 2 \left( 3\delta_g + \|\nabla g\|_\infty + \|\nabla g\|_\infty + \delta_r \right) \leq 2 \left( 3\delta_\psi + \delta_p \right) + 2 \left( 3\delta_g + \|\nabla g\|_\infty + \|\nabla g\|_\infty + \delta_r \right).
\]

If \( G \in C^2(\overline{D}) \) is the extension in \( D \) of the boundary data \( g \) given by (2.30), then there exists a ReLU DNN \( \phi_G \) such that

1. \( |G - \phi_G| \leq \delta \),
2. \( \text{size}(\phi_G) \leq 2\text{size}(\phi_r) + 2\text{size}(\phi_\psi) + 2\text{size}(\phi_\psi) + 2\text{size}(\phi_g) + \mathcal{O}(\|\nabla g\|_\infty) \).

Proof in Subsection 5.5
### 3.1 DNN approximations for solutions to problem (2.23)

We are now ready to present the DNN byproduct of Theorem (2.24) in fact of Corollary (2.20). First, let us state that the $\bar{r}$-WoS chain given by (2.10) - (2.11) renders a ReLU DNN as soon as $\bar{r}$ is a ReLU DNN; this follows from a simple corroboration of Lemma 3.6 and Corollary 3.7 so we skip its formal proof.

**Corollary 3.9.** Suppose that $\bar{r}$ is a ReLU DNN on the bounded set $D \subset \mathbb{R}^d$ such that $0 \leq \bar{r} \leq r$, where recall that $r$ specified by (2.9) is the distance function to the boundary of $D$. Further, let $M \geq 0$ and $(X^i_M, x \in D)$ be the $\bar{r}$-WoS chain at step $M$ given by (2.10) - (2.11). Then for each $\omega \in \Omega$ there exists a ReLU DNN defined on $D$ and denoted by $X^i_M(\cdot)$ such that

1. $X^i_M(x) = X^i_M(\omega)$ for all $x \in D$,
2. $\mathcal{L}(X^i_M) = M(\mathcal{L}(\bar{r}) + 1) + 1$,
3. $\mathcal{W}(X^i_M) \leq 2d + \max(d, \mathcal{W}(\bar{r}))$,
4. $\text{size}(X^i_M) \leq 2dM[4d + \mathcal{W}(\bar{r}) + \mathcal{L}(\bar{r}) + 2] + d + 2M\text{size}(\bar{r})$.

The main result of this section is the following, proving that ReLU DNNs can approximate the solution $u$ to problem (2.23) without the curse of dimensions.

**Theorem 3.10.** Because the statement requires a detailed context, let us label the assumption and the conclusion separately.

**Assumption:** Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain with $\text{diam}(D) < \infty$, $f \in L^\infty(D)$, $g$ be an $\alpha$-Hölder function on $\bar{D}$ for some $\alpha \in (0, 1]$, and $u$ be the solution to (2.23), as in Theorem (2.3). Let $\phi_f : D \rightarrow \mathbb{R}, \phi_g : \bar{D} \rightarrow \mathbb{R}, \phi_r : D \rightarrow \mathbb{R}$ be ReLU DNNs such that

$$|f - \phi_f| \leq \epsilon_f \leq |f|, \quad |g - \phi_g| \leq \epsilon_g, \quad |r - \phi_r| \leq \epsilon_r,$$

and set $\bar{r}(x) := (\phi_r(x) - \epsilon_r)^+$, $x \in D$. Also, let $\Pi := \Pi^\infty_\rho$ be the ReLU DNN given by Lemma (3.1).

Further, let $\gamma > 0$ be a prescribed error, $0 < \eta < 1$ be a prescribed confidence, and consider the following assumptions on the parameters $\epsilon_f, \epsilon_g, \epsilon_r, \epsilon_p$ and $c$:

1. $\epsilon_g \leq \gamma/6$,
2. $\epsilon_f \leq \frac{d^r}{4\text{diam}(D)}$,
3. $\epsilon_r < \epsilon_0 := \frac{1}{3} \left[ \left| g \right|_{\infty} + |f|_{\infty} \right] \alpha \text{diam}(D) \vee 1]^{-\frac{2}{\alpha}} \gamma \alpha^{-1}$, so that, by Remark (2.4), $\bar{r}$ is a $(\beta, \epsilon_0)$-distance if we choose $\beta = 1/3$,
4. $\epsilon_p = \frac{\gamma d}{6d(1 + 2|f|_{\infty})^2}$ and $c = \text{max}(\text{diam}(D), 2|f|_{\infty})$.

where $M \geq 1$ is specified below.

Further, consider the iid pairs $((X^i_k, k \geq 1), Y_i), i \geq 1$ on $(\Omega, \mathcal{F}, \mathbb{P})$, as in the beginning of Subsection (2.4) and

$$u_M^N(x) := \frac{1}{N} \sum_{i=1}^N \left[ \phi_g(x^{i, M} + \sum_{k=1}^M \Pi \left( \bar{r}(X^{i, k} - 1), \bar{r}(X^{i, k} - 1) \right) , \phi_f(x^{i, k} + \bar{r}(X^{i, k} - 1)Y_i) \right], x \in D. \quad (3.2)$$

Let us choose

$$N \geq \frac{64 \left( \left[ M/\alpha \right] \log(2 + |\bar{r}|_{\infty} + \log(K)) + \log(\frac{2}{9}) \right) \left[ |g|_{\infty} + \frac{M}{d} \text{diam}(D)|f|_{\infty} \right]^2}{9\alpha^2}, \quad (3.3)$$

where $K := \left[ \text{diam}(D) \left( \frac{16 |g|_{\infty}/\alpha \text{diam}(D)^2 |f|_{\infty} + 2|\text{diam}(D)|f|_{\infty}}{\gamma} \right) + 2 \right]^{1/\alpha}$.
and either a.5) $D$ is $\delta$-defective convex and

$$M \geq 9 \left( \frac{\log \left( \frac{\delta}{\gamma} \sqrt{\frac{\text{diam}(D)}{\varepsilon_0}} \right) + \log(4|g|_\infty + \frac{3}{2}\text{diam}(D)^2|f|_\infty)}{(1-\delta)} \right)$$

or a.5') $D$ is merely a bounded Lipschitz domain and

$$M \geq \frac{36 \left[ \log(8/\gamma) + \log(4|g|_\infty + \frac{3}{2}\text{diam}(D)^2|f|_\infty) \right] \text{diam}(D)^2}{\varepsilon_0^2}.$$

**Conclusion:** Under the above assumption and keeping the same notations, there exits a measurable function $\mathbb{U}_M^N : \Omega \times D \to \mathbb{R}$ such that

c.1) $\mathbb{U}_M^N(\omega, \cdot)$ is a ReLU DNN for each $\omega \in \Omega$, $\mathbb{U}_M^N(\cdot, x) = \mathbb{u}_M^N(x)$, $x \in D$, and

$$\mathbb{P} \left( \sup_{x \in D} |u(x) - \mathbb{U}_M^N(\cdot, x)| \geq \gamma \right) \leq \eta.$$

c.2) For each $\omega \in \Omega$ we have that

$$\text{size}(\mathbb{U}_M^N(\omega, \cdot)) \in \mathcal{O} \left( MN \left[ dk \max(d, W(\phi_r), L(\phi_r)) + k \text{size}(\phi_r) + \text{size}(\phi_f) + \left[ \log \left( \frac{1}{\gamma_d} \right) \right] \right) \right).$$

In particular, if a.5) holds then

$$\text{size}(\mathbb{U}_M^N(\omega, \cdot)) \in \mathcal{O} \left( \frac{d^2 \gamma^2}{\gamma} \log^4 \left( \frac{d}{\gamma} \right) \left[ d \log \left( \frac{d}{\gamma} \right) + \log \left( \frac{1}{\eta} \right) \right] S \right),$$

where

$$S := \left[ \max(d, W(\phi_r), L(\phi_r)) + \text{size}(\phi_r) + \text{size}(\phi_g) + \text{size}(\phi_f) \right]$$

and the tacit constant depends on $|g|_\alpha, |g|_\infty, |f|_\infty, \text{diam}(D), \text{adiam}(D), \delta, \alpha, \log(2 + |\phi_r|_1)$.

If merely a.5') holds then

$$\text{size}(\mathbb{U}_M^N(\omega, \cdot)) \in \mathcal{O} \left( d^8 \gamma^{-16/\alpha - 4} \log^4 \left( \frac{1}{\gamma} \right) \left[ d \log \left( \frac{1}{\gamma} \right) + \log \left( \frac{1}{\eta} \right) \right] S \right),$$

where $S$ and the tacit constant are as above.

**Proof in Subsection 4.5**

We end the exposition of the main results with the remark that it is sufficient to prescribe the Dirichlet data $g$ merely on $\partial D$ (not necessarily extended to $\overline{D}$), as expressed by the following direct consequence of Corollary 3.8.

**Corollary 3.11.** If the domain $D$ is of class $C^2$, and if $g$ is given merely on $\partial D$ and it is $\alpha$-Hölder there, then $g$ can be constructively extended to an $\alpha$-Hölder function on $\overline{D}$. Furthermore, a ReLU DNN approximation $\phi_g$ can be constructed as in Corollary 3.8 so Theorem 2.24 and Theorem 3.10 fully apply.

4 Proofs of the main results

4.1 Proofs for Subsection 2.1

**Proof of Lemma 2.1** Let $v(x) := u(Ax)$. Then $\partial_k v(x) = \sum_{j,l=1}^n \partial_j u(Ax) \partial_k (A_{jl} x_l) = \sum_{j=1}^n \partial_j u(Ax) A_{jk}$ and

$$\sum_{k=1}^n \partial_k \partial_k v(x) = \sum_{i,j,k,l=1}^n \partial_i \partial_j u(Ax) A_{jk} \partial_k (A_{il} x_l) = \sum_{i,j,k=1}^n \partial_i \partial_j u(Ax) A_{jk} A_{ik}.$$
Thus we need to determine $A$ such that $A A^T = K$. We know that there exists a rotation matrix $R \in O(3)$ (hence $R R^T = I_d$) such that $R K R^t = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Since $K$ is positive definite we have $\lambda_i > 0$ for $i \in \{1, \ldots, n\}$. Then we have:

$$RAR^T RAR^T = R K R^T = \text{diag}(\lambda_1, \ldots, \lambda_n).$$

(4.1)

We denote $B := RAR^T$ and observe that (4.1) can be rewritten as $BB^T = \text{diag}(\lambda_1, \ldots, \lambda_n)$. We can now take $B := \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. Thus $A := R^T \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) R$.

**Proof of Corollary 2.3.** The second assertion follows directly from Theorem 2.2 and from classical regularity theory for Poisson equation, so let us prove the first one. To this end, note that without loss of generality we may assume that $x = 0 \in D$, so by Ito’s formula we get that $\langle |B^0(t)|^2 - dt \rangle_{t \geq 0}$ is a martingale and

$$v(0) = E \{ \tau^0_D \} = \frac{1}{d} E \{ |B^0(\tau_D^0)|^2 \} \leq \text{diam}(D)^2/d.$$

**Proof of Corollary 2.4.**

i). Using Remark 2.8 we have that,

$$E \left\{ \int_0^{\tau^0_D} f(B^x(t)) \ dt \right\} = \sum_{n \geq 0} E \left\{ \int_{\tau^0_n}^{\tau^0_{n+1}} f(B^x(t)) \ dt \right\}$$

$$= \sum_{n \geq 1} E \left\{ \tilde{r}^2(B^x_{\tau^0_{n-1}}) K_0 f(B^x_{\tau^0_{n-1}} + \tilde{r}(B^x_{\tau^0_{n-1}})) \right\}$$

$$= E \left\{ \sum_{k \geq 1} \tilde{r}^2(X^{x}_{k-1}) K_0 f(X^{x}_{k-1} + \tilde{r}(X^{x}_{k-1})) \right\}, \ x \in D,$$  

where the second equality follows by the strong Markov and the scaling properties of Brownian motion, whilst the last equality follows from the fact that the law of $(X^n_x)_{n \geq 0}$ under $\mathbb{P}$ and the law of $(B^n_x)_{n \geq 0}$ under $\mathbb{P}^0$ are the same. Therefore, the statement follows by Theorem 2.2.

ii). The claim follows from the fact that $\mu(A) = w(0)$, where $w$ solves

$$\begin{cases}
\frac{1}{d} \Delta w = d1_A & \text{in } B(0, 1) \\
w = 0 & \text{on } \partial B(0, 1)
\end{cases}$$

for all $A \in \mathcal{B}(B(0, 1))$.

iii). We use conditional expectation, namely for every $x \in D$

$$E \left\{ \sum_{k \geq 1} \tilde{r}^2(X^{x}_{k-1}) f(X^{x}_{k-1} + \tilde{r}(X^{x}_{k-1})) \right\} = \sum_{k \geq 1} E \left\{ \tilde{r}^2(X^{x}_{k-1}) f(X^{x}_{k-1} + \tilde{r}(X^{x}_{k-1})) \right\}$$

$$= \sum_{k \geq 1} E \left\{ \tilde{r}^2(X^{x}_{k-1}) f(X^{x}_{k-1} + \tilde{r}(X^{x}_{k-1})) \right\} \right|_{X^{x}_{k-1}} = d \sum_{k \geq 1} E \left\{ \tilde{r}^2(X^{x}_{k-1}) K_0 f(X^{x}_{k-1} + \tilde{r}(X^{x}_{k-1})) \right\},$$

where for the last equality we used that $Y$ has distribution $\mu$ and is independent of $X^{x}_{k-1}$.

**Proof of Proposition 2.4.** Recall that $E\{\tau^0_D\} = v(x), x \in D$, where $v$ is given by (2.3). The idea is to explicitly solve for $v(x) = -2w(|x|)$ in radial form as

$$w''(r) + \frac{n-1}{r} w'(r) = 1 \text{ with } w(R_0) = 0, w(R_1) = 0$$

which is explicitly solved as

$$w(r) = \frac{r^2}{2n} + C_1 r^{2-n} + C_2 \text{ with } C_1 = \frac{R_1^2 - R_0^2}{2n(R_2^{2-n} - R_1^{2-n})}, C_2 = \frac{R_0^n - R_1^n}{2n(R_1^{-n} - R_0^{-n})}.$$
Now, if we start from a point \( x \) we can use the intermediate value theorem to obtain first that

\[
\mathbb{E}_x \{ \tau_{D^c} \} = -2w(|x|) \leq 2d(x, \partial A(R_0, R_1))[w'(r_0)]
\]

for some point \( r_0 \in [R_0, R_1] \). Therefore our task now is to estimate the above derivative, which we can compute explicitly as

\[
w'(r) = \frac{r}{n} + \frac{2 - n}{2n} \frac{R_1^2 - R_0^2}{R_0^2 - R_1^2} r^{1-n}.
\]

To estimate this in a transparent way we set \( R_1 = \rho R_0 \) and \( r = tR_0 \) for \( 1 \leq t \leq \rho \). In these new notations we have

\[
v'(r) = \frac{R_0}{n} \left( t - \frac{(2 - n)(\rho^2 - 1)}{2(\rho^{2-n} - 1)} t^{1-n} \right).
\]

Now, it is an elementary matter that for two functions \( f, g : [a, b] \to \mathbb{R} \) which are differentiable on \( (a, b) \) (and \( g \) non-vanishing), we can find a point \( \xi \in (a, b) \) such that

\[
\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(<\xi)}{g'(<\xi)}.
\]

Using this fact for \( a = 1, b = \rho \), \( f(x) = x^2 \) and \( g(x) = x^{2-n} \) we argue that for some point \( \xi \in (1, \rho) \) we have

\[
w'(r) = \frac{R_0 t}{n} (1 - (\xi/t)^n).
\]

As a function of \( t \in [1, \rho] \) the above function is increasing, thus we have

\[
\frac{R_0}{n} (1 - \xi^n) \leq w'(r) \leq \frac{R_0}{n} (1 - (\xi/\rho)^n). \tag{4.2}
\]

Therefore in order to control \( |w'(r)| \) it suffices to control the absolute values of the two bounds above. The right hand side bound is easy because for any \( x \in (0, 1) \), thus we obtain that

\[
\frac{R_0}{n} \frac{1 - (\xi/\rho)^n}{(1 - \xi^n)} \leq R_0 \rho (1 - \xi/\rho) = R_0 (\rho - \xi) \leq R_1 - R_0. \tag{4.3}
\]

The left hand side of (4.2) in absolute value is bounded by

\[
\left| \frac{R_0 (1 - \xi^n)}{n} \right| = \frac{R_0 (\xi^n - 1)}{n} = \frac{R_0}{n} \left( \frac{(n - 2)(\rho^2 - 1)}{2(1 - \rho^{2-n})} - 1 \right) = \frac{R_0}{n} \left( \frac{(n - 2)(\rho^2 - 1)\rho^{n-2} - 1}{2(\rho^{n-2} - 1)} - 1 \right)
\]

\[
= \frac{R_0}{n} \left( \frac{(n - 2)(\rho^2 - 1)}{2} + \frac{(n - 2)(\rho^2 - 1)}{2(\rho^{n-2} - 1)} - 1 \right)
\]

\[
\leq \begin{cases} 
\frac{R_0}{3} \left( \frac{\rho^2 - 1}{2} + \frac{\rho^2 - 1}{2} \right), & n = 3 \\
\frac{R_0}{2n} (\rho^2 - 1), & n \geq 4
\end{cases} \tag{4.4}
\]

where we go back to the fact that \( \xi^n = \frac{(n-2)(\rho^2-1)}{2(1-\rho)} \) and in the second line we used that \( 2(\rho^{n-2} - 1) \geq (n - 2)(\rho^2 - 1) \) because \( \rho \geq 1 \) for \( n \geq 4 \). Thus combining (4.3) and (4.4) we get that

\[
|w'(r)| \leq \frac{(R_1 - R_0)R_1}{2R_0}
\]

which is our claim.

**Proof of Proposition 2.5** For \( x \in D \), we pick a point \( y \in \partial D \) such that \( d(x, y) = d(x, \partial D) \) and notice that \( \tau_{D^c} \leq \tau_{A(y, R_0, y, R_1, y)^c} \). At the same time, since \( d(x, \partial A(y, R_0, y, R_1, y)) = d(x, \partial D) \), we employ Proposition 2.3 to deduce that

\[
\mathbb{E} \{ \tau_{D^c}^x \} \leq \mathbb{E} \{ \tau_{A(y, R_0, y, R_1, y)^c}^x \} \leq 2d(x, \partial D) \text{diam}(D).
\]

\[\square\]
4.2 Proofs for Subsection 2.2

Proof of Proposition 2.10

The proof goes through several steps. The first step observes the following two basic facts which can be easily checked by direct computations. On the ball of radius \( r \) centered at 0 we have

\[
\text{if } v(x) = 1 - \frac{\gamma}{2d} |x|^2, \text{ for } 0 < \gamma < \frac{2d}{r^2}, \text{ then } \Delta v \leq -\gamma v
\]

and

\[
\text{for } v(x) = 1 - \frac{\gamma}{2d + \gamma r^2} |x|^2 \text{ with } 0 < \gamma, \text{ then } \Delta v \geq -\gamma v.
\]

The second step consists in proving some estimates for the exit time of the Brownian motion from the ball of radius \( r \) and centered at 0. Denote by \( \tau \) this exit time for the Brownian motion started at the origin. Then

\[
\mathbb{E} \{ e^{\gamma \tau} \} \leq \frac{1}{1 - \frac{\gamma^2}{2d}} \text{ for } 0 < \gamma < \frac{2d}{r^2}
\]

and

\[
\mathbb{E} \{ e^{\gamma \tau} \} \geq 1 + \frac{\gamma}{2d} r^2 \text{ for } 0 < \gamma.
\]

The proofs of (4.7) and (4.8) are based on the previous step. For example, using (4.5) we learn that \( \Delta v + \gamma v \leq 0 \) for \( v(x) = 1 - \frac{\gamma}{2d} |x|^2 \) and this combined with Itô’s formula means that \( e^{\gamma t} v(B_t) \) is a supermartingale. In particular, stopping it at time \( \tau \), we obtain that

\[
\mathbb{E} \{ e^{\gamma \tau} v(B_\tau) \} \leq v(0)
\]

from which we deduce (4.7).

In a similar fashion using (4.6) we can deduce (4.8).

With these two steps at hand we can move to proving the actual result. To proceed, we take \( U_1, U_2, \ldots \) the iid sequence of uniform random variables on the unit sphere in \( \mathbb{R}^d \) which drives the walk on spheres. Now set \( N_k^\epsilon \) to denote the number of steps to the \( \epsilon \)-shell for the walk on spheres using the random variables \( U_k, U_{k+1}, \ldots \). Notice that for a fixed point \( x \), in distribution sense, \( N_k^\epsilon \) have the same distribution for all \( k = 1, 2, \ldots \). Also, set \( T_\beta(x, U) = x + \tilde{r}(x) U \) the point on the sphere of radius \( \tilde{r}(x) \) determined by the first step of the walk on spheres determined by \( U \). The key now is the fact that

\[
N_k^\epsilon = 1 + \mathbb{1}_{T_\beta(x, U_1) \in \Omega^\epsilon} N_2^{\beta, T_\beta(x, U_1)}.
\]

The intuitive explanation of this is rather simple, the walk on spheres starts with the first step. If we land in the \( \epsilon \)-shell we stop. Otherwise we have to start again but this time we have already used the random variable \( U_1 \) and thus we have to base our remaining walk on spheres using \( U_2, U_3, \ldots \).

Using now (4.9) we can write that

\[
\mathbb{E} \{ e^{\lambda N^\epsilon} \} = \mathbb{E} \left\{ \mathbb{E} \{ e^{\lambda N^\epsilon} | U_1 \} \right\} = \mathbb{E} \left\{ e^{\lambda} \mathbb{E} \{ e^{\lambda T_\beta(x, U_1) | U_1} \} \right\},
\]

where we used conditioning with respect to the first random variable \( U_1 \).

Now we are going to use a \( \lambda \) such that

\[
e^{\lambda} \leq \mathbb{E} \{ e^{\gamma \tau_1} \},
\]

where \( \tau_1 \) is the first exit time of the Brownian motion from the ball of radius \( \tilde{r}(x) \geq \beta \epsilon \) starting at \( x \). This is the place where we can use the estimate (4.8) to show that \( \lambda = \log(1 + \gamma \beta^2 e^2 / 2d) \) is sufficient to guarantee the above estimate. Notice the key point here, namely the fact that \( U_1 \) has the same distribution as \( \frac{B_1 - x}{|B_1 - x|} \) where \( B_1 \) is the Brownian motion started at \( x \) and \( \tau_1 \) denotes the exit time of the Brownian motion from the ball of radius \( \tilde{r}(x) \).

Thus now we use this to argue that

\[
\mathbb{E} \{ e^{\lambda N^\epsilon} \} \leq \mathbb{E} \left\{ e^{\gamma \tau_1 + \lambda T_\beta(x, U_1) | U_1} \right\} = \mathbb{E} \left\{ e^{\gamma \tau_1 + \lambda \mathbb{1}_{\tau_1 \leq \epsilon} | U_1} N_2^{\beta, \tau_1} \right\}.
\]
Now repeating this one more step using the new starting point \( B_r \), we will get
\[
\mathbb{E} \left\{ e^{\lambda N^x} \right\} \leq \mathbb{E} \left\{ e^{\gamma (r_1 + r_2) + \lambda (r_1 + r_2) / 2 + \lambda r_1 r_2 / 2} N_2^{T_\beta(x,U_1),U_1} \right\} = \mathbb{E} \left\{ e^{\gamma r_1 + \lambda r_1 r_2 / 2 + \lambda r_2 / 2} N_2^{T_\beta(x,U_1),U_1} \right\}.
\]

Repeating this we finally obtain that
\[
\mathbb{E} \left\{ e^{\lambda N^x} \right\} \leq \mathbb{E} \left\{ e^{\gamma (r_1 + r_2) + \lambda (r_1 + r_2) / 2 + \lambda r_1 r_2 / 2} N_2^{T_\beta(x,U_1),U_1} \right\}.
\]

(4.12)

Here we use \( \gamma > 0 \) and \( \lambda = \log(1 + \frac{\beta^2}{d}) \). To finish the proof, we need to estimate now the right hand side in (4.12). To do this we enclose the domain \( D \) in the ball of radius \( \text{diam}(D) \) centered at \( x \) and now use (4.11) with \( r = \text{diam}(D) \) to get that
\[
\mathbb{E} \left\{ e^{\lambda N^x} \right\} \leq \frac{1}{1 - \frac{2d}{\text{diam}(D)^2}} \text{ for } \lambda = \log(1 + \frac{\beta^2}{d}) \text{ and } \gamma < \frac{2d}{\text{diam}(D)^2}.
\]

Finally, using that \( \log(1 + x) \geq x/2 \) for \( 0 \leq x \leq 1 \) and choosing \( \gamma = \frac{d}{\text{diam}(D)^2} \), we obtain (2.13).

The second inequality of the statement is obtained based on the first estimate and Markov inequality:
\[
\mathbb{P}(N^x \geq R) = \mathbb{P}(e^{\frac{\beta^2}{4d^2} N^x} \geq e^{\frac{\beta^2}{4d^2} R}) \leq \mathbb{E} \left\{ e^{\frac{\beta^2}{4d^2} N^x} \right\} e^{-\frac{\beta^2}{4d^2} R} \leq e^{-\frac{\beta^2}{4d^2} R}.
\]

Proof of Example 2.13

The \( \delta \)-defective convexity condition for this region amounts to the inequality:
\[
\int_{A(R_1, R_2)} \nabla \varphi(x) \nabla r(x) \, dx \leq \frac{\delta}{R_2 - R_1} \int_{A(R_1, R_2)} \varphi(x) \, dx \quad \text{for all } \varphi \in C_c^\infty(A(R_1, R_2)),
\]

where we used the fact that the distance function \( r \) is Lipschitz, hence one can integrate by parts and discard the boundary terms due to the fact that \( \varphi \in C_c^\infty(A(R_1, R_2), \mathbb{R}_+). \)

We use the fact that on \( A \left( \frac{R_1 + R_2}{2}, R_2 \right) \) and \( A \left( R_1, \frac{R_1 + R_2}{2} \right) \) the function \( r \) is in fact smooth and we integrate by parts on each region to obtain that the left hand side above becomes:
\[
\int_{A(R_1, \frac{R_1 + R_2}{2})} \varphi(x) \Delta r(x) \, dx + \int_{S^2} \int_{r_1 + R_2} \varphi(\sigma, s) r'(s) s^{d-1} \, ds \, d\sigma + \int_{A(R_1, \frac{R_1 + R_2}{2})} \varphi(x) \Delta r(x) \, dx - \int_{S^2} \int_{r_1 + R_2} \varphi(\sigma, s) r'(s) s^{d-1} \, ds \, d\sigma
\]
\[
= \int_{A(R_1, \frac{R_1 + R_2}{2})} \varphi(x) \Delta r(x) \, dx,
\]

where we used spherical coordinates on the boundary and denoted by prime the derivative in the radial direction. Also, the \( \Delta r(x) \) is well defined, in a classical sense, on \( A(R_1, R_2) \) except for the set of measure zero that in spherical coordinates is given by \( \{(r, \sigma); r = \frac{R_1 + R_2}{2}, \sigma \in S^2 \} \). Noting that:
\[
r'(\sigma, s) = \begin{cases} s - R_1 & \text{for all } \sigma \in S^2, s \in (R_1, \frac{R_1 + R_2}{2}), \\ R_2 - s & \text{for all } \sigma \in S^2, s \in (\frac{R_1 + R_2}{2}, R_2). \end{cases}
\]

we have \( r' \equiv 1 \) for all \( x \in A \left( R_1, \frac{R_1 + R_2}{2} \right) \) and \( r' \equiv -1 \) for all \( x \in A \left( \frac{R_1 + R_2}{2}, R_2 \right) \), hence the \( \delta \)-defective convexity condition amounts to the inequality:
\[
(d - 1) \left( \int_{S^2} \int_{R_1}^{R_2} \varphi(\sigma, s) s^{d-2} \, ds \, d\sigma \right) - \int_{S^2} \int_{R_1}^{R_2} \varphi(\sigma, s) s^{d-2} \, ds \, d\sigma \leq \frac{\delta}{R_2 - R_1} \int_{S^2} \int_{R_1}^{R_2} \varphi(\sigma, s) s^{d-1} \, ds \, d\sigma,
\]

which (taking into account that \( \varphi \geq 0 \)) is satisfied if for instance \( \frac{R_2}{R_1} < 1 + \frac{\delta}{d-1} \). □
Proof of Example 2.14} Arguing similarly as in the previous example, the $\delta$-defective convexity condition becomes:

$$\int_{D_\varepsilon} \varphi(x) \Delta r(x) \, dx \leq \delta \frac{1}{\varepsilon} \int_{D_\varepsilon} \varphi(x) \, dx$$

for all $0 \leq \varphi \in C^\infty_c(D_\varepsilon)$, \hfill (4.13)

where $\Delta r(x)$ is well-defined, in a classical sense, except on the set of measure zero $\Gamma_\varepsilon := \{ x + \frac{1}{2} \varepsilon \, n(x) \in \mathbb{R}^d \mid x \in \Gamma \}$. We also denote $D_\varepsilon^+ := \{ x + \varepsilon \, t \, n(x) \in \mathbb{R}^d \mid (x, t) \in \Gamma \times \left(0, \frac{1}{2}\right) \}$ respectively $D_\varepsilon^- := \{ x + \varepsilon t \, n(x) \in \mathbb{R}^d \mid (x, t) \in \Gamma \times \left(0, \frac{1}{2}\right) \}$.

We recall (see for instance, [6], Lemma 14.17, p. 355) that

$$\Delta r(x, t) = \begin{cases} \sum_{i=1}^{d-1} \frac{k_i(x)}{1 - k_i(x)(t - r(x, t))} & \text{if } t \in (0, \frac{1}{2}) \\ - \sum_{i=1}^{d-1} \frac{k_i(x)}{1 - k_i(x)(t - r(x, t))} & \text{if } t \in \left(\frac{1}{2}, 1\right) \end{cases}$$

hence we have:

$$\int_{D_\varepsilon} \varphi(x) \Delta r(x) \, dx \leq \int_{D_\varepsilon^+} \varphi(x) \left( \sum_{i=1}^{d-1} \frac{2k_i(x)}{2 - k_i(x)\varepsilon} \right) \, dx - \int_{D_\varepsilon^-} \varphi(x) \left( \sum_{i=1}^{d-1} k_i(x) \right) \, dx$$

thus the condition (4.13) holds for suitably small $\varepsilon > 0$. \hfill \Box

Proof of Proposition 2.15} We split the proof in two steps.

**Step I** (Regularization). Let $D_n := \{ x \in D \mid r(x) > 1/n \}$ for each $n \geq n_0$, where $n_0$ is such that $D_0 \neq \emptyset$. Further, let $\rho \geq 0$ be a (smooth) mollifier on $\mathbb{R}^d$ such that $\text{Supp } \rho \subset B(0, 1)$, and set $\rho_t(\cdot) := t^{-d} \rho(\cdot / t)$, $t > 0$. In particular,

$$\text{Supp } \rho_t + D_n \subset D \quad \text{for all } t < 1/n_0. \hfill (4.14)$$

Now let us consider that we extend $r$ from $\overline{D}$ to $\mathbb{R}^d$ by setting $r \equiv 0$ on $\mathbb{R}^d \setminus \overline{D}$, and set

$$V_t = \sqrt{r_t}, \quad \text{where } r_t := \rho_t * r \in C^2_c(\mathbb{R}^d).$$

We claim that

i) $\Delta r_t \leq \frac{\delta}{2 \text{rad}(D)}$ on $D_n$ for all $n \geq n_0$ and $0 < t < 1/n_0$.

ii) $\Delta V_t \leq -\frac{1}{4} r_t^{-3/2} (|\nabla r_t|^2 - \delta)$ on $D_n$ for all $n \geq n_0$ and $0 < t < 1/n_0$.

To prove the claim, note first that by a simple calculation we get

$$\Delta V_t = -\frac{1}{4} r_t^{-3/2} (|\nabla r_t|^2 - 2r_t \Delta r_t),$$

hence ii) follows from i) and the fact that $r_t \leq \text{rad}(D)$. So, it remains to prove the first assertion of the claim: Let $0 \leq \varphi \in C^\infty_c(D_n), n \geq n_0, t < 1/n_0$, and proceed with integration by parts and Fubini’s theorem as follows

$$\int_{D_n} \varphi \Delta r_t \, dx = - \int_{D_n} \nabla r_t \cdot \nabla \varphi \, dx = - \int_{\mathbb{R}^d} \rho_t(y) \int_{D_n} \nabla r(x - y) \cdot \nabla \varphi(x) \, dx \, dy$$

$$= - \int_{\mathbb{R}^d} \rho_t(y) \int_{D_n} \nabla r(x - y) \cdot \nabla \varphi(x) \, dx \, dy$$

$$= - \int_{\text{Supp } \rho_t} \rho_t(y) \int_{\mathbb{R}^d} \nabla r(x) \cdot \nabla \varphi(\cdot + y)(x) \, dx \, dy,$$

so, by (4.14) and then using (2.14) we can continue with

$$= - \int_{\text{Supp } \rho_t} \rho_t(y) \int_{D} \nabla r(x) \cdot \nabla \varphi(\cdot + y)(x) \, dx \, dy \leq - \int_{\text{Supp } \rho_t} \rho_t(y) \int_{D} \varphi(x + y) \, dx \, dy$$

$$\leq \delta \frac{\delta}{2 \text{rad}(D)} \int_{\text{Supp } \rho_t} \rho_t(y) \int_{\mathbb{R}^d} \varphi(x) \, dx$$

$$= \frac{\delta}{2 \text{rad}(D)} \rho_t(y) \int_{\mathbb{R}^d} \varphi(x) \, dx,$$
which proves i) and hence the entire claim.

Now, by Ito’s formula in corroboration with the claim proved above yield:

For \( x \in D_n \) we have
\[
\mathbb{E} \left\{ V \left( B^x_{\tau(D_n \cap B(x, \bar{r}(x)))} \right) \right\} = V(x) + \mathbb{E} \left\{ \int_0^{\tau(D_n \cap B(x, \bar{r}(x)))} \Delta V_i(B^x_s) \, ds \right\}
\leq V(x) - \frac{1 - \delta}{4} \mathbb{E} \left\{ \int_0^{\tau(D_n \cap B(x, \bar{r}(x)))} r^{-3/2}(B^x_s) \left| \nabla r_s \right|^2 (B^x_s) - \delta \right\} \, ds .
\]
(4.15)

**Step II** (Passing to the limit in (4.15)). The next step is to let \( t \to 0 \) and then \( n \to \infty \) in (4.15). To this end, note that because \( r \in C(\overline{D}) \cap W^{1,\infty}(D) \), we have

iii) \( \lim_{t \to 0} r_t = r \) uniformly on \( D \),

iv) \( \lim_{t \to 0} \nabla r_t = \nabla r \) a.e. and boundedly on \( D \).

In particular, because \( \inf_{x \in D_n} r(x) \geq 1/n \) for large enough \( n_0 \) and \( n \geq n_0 \), we have that

v) \( \lim_{t \to 0} r_t^{-3/2} = r^{-3/2} \) boundedly on \( D_n \), for each \( n \geq n_0 \).

We are now in the position to let \( t \to 0 \) in (4.15) to get that for \( x \in D_n \)
\[
\mathbb{E} \left\{ V \left( B^x_{\tau(D_n \cap B(x, \bar{r}(x)))} \right) \right\} \leq V(x) - \frac{1 - \delta}{4} \mathbb{E} \left\{ \int_0^{\tau(D_n \cap B(x, \bar{r}(x)))} r^{-3/2}(B^x_t) \, dt \right\} ,
\]
where we have used that \( |\nabla r| = 1 \) a.e. The fact that \( |\nabla r| = 1 \), follows from two basic facts. On one hand, \( r \) is 1-Lipschitz so \( |\nabla r| \leq 1 \). On the other hand, from Lipschitz conditions and Rademacher theorem, \( r \) is differentiable almost everywhere. If \( x \) is a point where \( r \) is differentiable, and \( y \in \partial D \) such that \( d(x, y) = r(x) \), and \( v = y - x \), then it is easy to see that the derivative of \( r \) in the direction \( v \) is constant 1, thus the claim.

Now, on the one hand, since \( r \leq 2r(x) \) on \( B(x, \bar{r}(x)) \subset B(x, r(x)) \) we deduce that
\[
\mathbb{E} \left\{ V \left( B^x_{\tau(D_n \cap B(x, \bar{r}(x)))} \right) \right\} \leq V(x) - \frac{1 - \delta}{4r(x)^{3/2}} \mathbb{E} \left\{ \tau(D_n \cap B(x, \bar{r}(x))) \right\} , \quad x \in D_n .
\]
(4.16)

On the other hand, \( D_n \uparrow D \), so \( \tau(D_n \cap B(x, \bar{r}(x))) \downarrow \tau_B(x, \bar{r}(x)) \) a.s., hence for all \( x \in D \)
\[
PV(x) = \mathbb{E} \left\{ V \left( B^x_{\tau_B(x, \bar{r}(x))} \right) \right\} = \lim_n \mathbb{E} \left\{ V \left( B^x_{\tau(D_n \cap B(x, \bar{r}(x)))} \right) \right\} ,
\]
hence letting \( n \to \infty \) in (4.16) we can continue with
\[
PV(x) \leq V(x) - \frac{1 - \delta}{4r(x)^{3/2}} \mathbb{E} \left\{ \tau_B(x, \bar{r}(x)) \right\} = V(x) - \frac{1 - \delta}{4r(x)^{3/2}} \bar{r}(x)^2 \frac{d}{d}
\leq \left( 1 - \frac{\beta^2(1 - \delta)}{4d} \right) V(x),
\]
which proves (2.16). Notice that for the last inequality we used that \( \beta r(x) \leq \bar{r}(x), x \in D \).

The tail estimate (2.17) can be deduced from (2.16) and Markov inequality, as follows:
\[
\mathbb{P}(N^x_\varepsilon > M) \leq \mathbb{P}(r(X^x_M) \geq \varepsilon) = \mathbb{P}(V(X^x_M) \geq \sqrt{\varepsilon}) \leq \frac{1}{\sqrt{\varepsilon}} \mathbb{E} \left\{ V(X^x_M) \right\} = \frac{1}{\sqrt{\varepsilon}} P^M V(x)
\leq \left( 1 - \frac{\beta^2(1 - \delta)}{4d} \right)^M \frac{V(x)}{\sqrt{\varepsilon}}, \quad x \in D .
\]
Let us finally conclude the proof by showing (2.18):
\[
\mathbb{E}\left\{a^{N_x^\epsilon}\right\} = \sum_{k \geq 0} a^k \mathbb{P}(N_x^\epsilon = k) \leq 1 + \sum_{k \geq 1} a^k \mathbb{P}(N_x^\epsilon > k - 1)
\]
\[
\leq 1 + \sum_{k \geq 1} a^k \left(1 - \frac{\beta^2(1 - \delta)}{4d}\right)^{k-1} V(x) = 1 + a \frac{V(x)}{\sqrt{\epsilon}} \sum_{k \geq 0} a^k \left(1 - \frac{\beta^2(1 - \delta)}{4d}\right)^k
\]
\[
= 1 + \frac{a}{1 - a \delta \frac{\beta}{4d}} \frac{V(x)}{\sqrt{\epsilon}}, \quad x \in D.
\]

4.3 Proofs for Subsection 2.3

Proof of Proposition 2.17 First of all, note that by similar arguments to those used in the proof of Corollary 2.8:
\[
\u_M(x) = \mathbb{E}\left\{\sum_{n=0}^{M-1} \int_{\tau_n^x}^{\tau_{n+1}^x} f(B_t^x) \, dt \right\} = \mathbb{E}\left\{\int_{\tau_M^x}^{\tau_1^x} f(B_t^x) \, dt \right\}, \quad x \in D.
\]
Therefore,
\[
|u(x) - \u_M(x)| = \left|\mathbb{E}\left\{\int_{\tau_M^x}^{\tau_1^x} f(B_t^x) \, dt \right\}\right|
\]
\[
\leq |f|_\infty \mathbb{E}\{\tau_{D^c} - \tau_{\tilde{M}^x}\} = |f|_\infty \mathbb{E}\left\{\mathbb{E}\left[B_{\tau_{\tilde{M}^x}}^x \{\tau_{D^c}\}\right]\right\}
\]
\[
= |f|_\infty \mathbb{E}\left\{v(B_{\tau_{\tilde{M}^x}}^x)\right\}
\]
\[
\leq |f|_\infty \left[\mathbb{E}\left\{v(B_{\tau_{\tilde{M}^x \cup N_x^\epsilon}}^x); N_x^\epsilon \leq M\right\} + |v|_\infty \mathbb{P}(N_x^\epsilon > M)\right]
\]
\[
\leq |f|_\infty \left[\mathbb{E}\left\{v(B_{\tau_{\tilde{M}^x \cup N_x^\epsilon}}^x); N_x^\epsilon \leq M\right\} + \frac{1}{d} \text{diam}(D)^2 \epsilon \frac{\beta^2}{4d} \frac{\delta}{\text{diam}(D^2) M}\right], \quad x \in D,
\]
where the last inequality follows by Corollary 2.3 and Proposition 2.10. Now, one can see that \((v(B_{\tau_n^x}))_{n \geq 0}\) is a supermartingale, hence by Doob’s stopping theorem we get that
\[
\mathbb{E}\left\{v(B_{\tau_{\tilde{M}^x \cup N_x^\epsilon}}^x)\right\}; N_x^\epsilon \leq M\right\} = \mathbb{E}\left\{1_{|N_x^\epsilon \leq M|}\mathbb{E}\left[v\left(B_{\tau_{\tilde{M}^x \cup N_x^\epsilon}}^x\right) | \mathcal{F}_{\tau_{\tilde{M}^x \cup N_x^\epsilon}}\right]\right\}
\]
\[
\leq \mathbb{E}\left\{v\left(B_{\tau_{\tilde{M}^x \cup N_x^\epsilon}}^x\right)\right\}
\]
\[
\leq v_\infty(\epsilon),
\]
hence the desired estimate is completely obtained.

Further, we need the following lemma:

Lemma 4.1. Let \(\epsilon > 0, \beta \in (0, 1], \tilde{r} \) be a \((\beta, \epsilon)-\)distance, \(x \in D\), and \((X_n^\epsilon)_{n \geq 0}\) be the corresponding \(\tilde{r}\)-WoS chain. If \(\tau' \geq \tau\) is a finite stopping times such that \(\tau \geq N_x^\epsilon\), then the following estimates hold:

i) If \(g\) is \(\alpha\)-Hölder on \(\overline{D}\) for some \(\alpha \in [0, 1]\) then
\[
\mathbb{E}\{|g(X_{\tau'}^x) - g(X_{\tau}^x)|\} \leq d^\alpha/2|g|_\alpha \cdot |v|_\infty^{\alpha/2}(\epsilon).
\]

ii) If \(g \in C^2_b(\overline{D})\) then
\[
|\mathbb{E}\{g(X_{\tau'}^x)\} - \mathbb{E}\{g(X_{\tau}^x)\}| \leq \frac{\Delta g|_\infty}{2} \cdot |v|_\infty(\epsilon).
\]
Proof. i). It is easy to see that for $z \in \mathbb{R}^d$, $(\langle X^n_t, z \rangle - \langle x, z \rangle)_{n \geq 1}$ is a bounded martingale, hence

$$\mathbb{E}\{\langle X^n_t, X^n_t \rangle \} = \mathbb{E}\{|X^n_t|^2\}$$

and thus

$$\mathbb{E}\{|X^0_t - X^n_t|^2\} = \mathbb{E}\{|X^n_t|^2\} - \mathbb{E}\{|X^0_t|^2\}.$$

By employing the martingale problem for the Markov chain $(X^n_t)_{n \geq 0}$, we get that for any finite stopping time $T$, $\mathbb{E}\{|X^n_T|^2\} = |x|^2 + \mathbb{E}\left\{\sum_{i=0}^{T-1} \hat{r}^2(X^n_i)\right\}$, hence $\mathbb{E}\{X^n_t - X^0_t|^2\} = \mathbb{E}\left\{\sum_{i=0}^\infty \hat{r}^2(X^n_i)\right\}$ and therefore

$$|\mathbb{E}\{g(X^n_t)\} - \mathbb{E}\{g(X^0_t)\}| \leq |g|_{\alpha}\mathbb{E}\left\{\sum_{i=0}^\infty \hat{r}^2(X^n_i)\right\} \leq d^\alpha/2|g|_{\alpha}\mathbb{E}\left\{\sum_{i=N^\infty_t}^\infty \tau_i^x - \tau_i^x\right\} \leq d\alpha/2d|g|_{\alpha}\cdot \mathbb{E}\{v(B^n_{\tau^x})\} \alpha/2$$

$$\leq d\alpha/2|g|_{\alpha}\cdot |v|\alpha/2(\epsilon).$$

ii). Suppose now that $g \in C^2(\overline{D})$. Then, by the martingale problem we deduce

$$|\mathbb{E}\{g(X^n_t)\} - \mathbb{E}\{g(X^0_t)\}| = \mathbb{E}\{\sum_{i=0}^{\sum^{T-1}} (Pg - g)(X^n_i)\} \leq \mathbb{E}\left\{\sum_{i=0}^\infty |Pg - g|(X^n_i)\right\}.$$

On the other hand, by Itô’s formula

$$|Pg(z) - g(z)| = \left|\mathbb{E}\left\{\int_0^{\tau_{\partial D}^{1,z}} \frac{1}{2} \Delta g(B_t) \, dt\right\}\right| \leq \frac{1}{2} |\Delta g|_\infty \cdot \mathbb{E}\left\{\tau_{\partial D}^{1,z}\right\} = |\Delta g|_\infty \cdot \frac{\hat{r}(z)^2}{2d},$$

hence

$$|\mathbb{E}\{g(X^n_t)\} - \mathbb{E}\{g(X^0_t)\}| \leq \frac{|\Delta g|_\infty}{2d} \cdot \mathbb{E}\left\{\sum_{i=0}^{\sum^{T-1}} \hat{r}^2(X^n_i)\right\} \leq \frac{|\Delta g|_\infty}{2} \cdot \mathbb{E}\left\{\sum_{i=0}^{\infty} \tau_i^x\right\} \leq \frac{|\Delta g|_\infty}{2} \cdot \mathbb{E}\left\{v(B^n_{\tau^x})\right\} \leq \frac{|\Delta g|_\infty}{2} \cdot |v|_\infty(\epsilon).$$

Proof of Proposition 2.18. Recall that $(B^x_{\tau^x})_n$ and $(X^x)_n$ are equal in law and hence $B^x_{\tau^x_{N^n}}$ and $X^x_{N^n}$ are also equal in law. In particular,

$$\mathbb{E}\{g(B^x_{\tau^x})\} = \lim_n \mathbb{E}\{g(B^x_{\tau^x})\} = \lim_n \mathbb{E}\{g(X^x_n)\} = \mathbb{E}\{g(B^x_{\tau^x})\}, \quad x \in D.$$

Also, if $T$ is a finite random time then

$$\lim_n \mathbb{E}\{g(X^x_{n\wedge \tau})\} = \lim_n \{\mathbb{E}\{g(X^x_n)\}; \ T < n \} + \mathbb{E}\{g(X^x_T)\}; \ T \geq n\} = \mathbb{E}\{g(B^x_{\tau^x_{\tau^x}})\}, \quad x \in D.$$

Now let us fix $\epsilon > 0$ and argue as follows:

$$|u(x) - u_M(x)| \leq \lim_n \mathbb{E}\{g(X^x_n) - g(X^x_M)\}$$

$$= \lim_n \mathbb{E}\{g(X^x_n) - g(X^x_M); \ N^n \leq M\} + \lim_n \mathbb{E}\{g(X^x_n) - g(X^x_M); \ N^n > M\}$$

$$\leq \lim_n \mathbb{E}\{g(X^x_{n\wedge M \vee N^n}) - g\left(\frac{N^n}{X^x_{n\wedge M \vee N^n}}\right); \ N^n \leq M\} + 2|g|_\infty \mathbb{P}(N^n > M).$$

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Now we consider the two cases separately:

i). Assume that \( g \) is \( \alpha \)-Hölder. Then for \( x \in D \),
\[
|u(x) - u_M(x)| \leq \lim_{n} \mathbb{E} \left\{ g(X_{n,M,N}^x) - g(X_{M,N}^x) \right\} + 2|g|_{\infty} \mathbb{P}(N^x_\varepsilon > M),
\]
and by employing Lemma 4.1, i), we can continue with
\[
\leq d^{3/2}|g|_\alpha \cdot |v|_{\infty}^{3/2}(\varepsilon) + 2|g|_{\infty} \mathbb{P}(N^x_\varepsilon > M)
\leq d^{3/2}|g|_\alpha \cdot |v|_{\infty}^{3/2}(\varepsilon) + 4|g|_{\infty} e^{-\frac{\alpha^2}{4diam(D)^2} M},
\]
where the last inequality is due to Proposition 2.10

ii). Suppose now that \( g \in C^2(D) \). Then
\[
|u(x) - u_M(x)| \leq \lim_{n} \mathbb{E} \left\{ g(X_{n,M,N}^x) - g(X_{M,N}^x) \right\} + 4|g|_{\infty} \mathbb{P}(N^x_\varepsilon > M);
\]
\[
\leq \frac{\Delta g}{2} \cdot |v|_{\infty}(\varepsilon) + 8|g|_{\infty} e^{-\frac{\alpha^2}{4diam(D)^2} M},
\]
where the last inequality follows by employing Lemma 11, ii).

\[\square\]

4.4 Proofs for Subsection 2.4

Proof of Proposition 2.22 We have
\[
\mathbb{E} \left\{ \left| u(\cdot) - u^N_M(\cdot) \right|^2_{L^2(D)} \right\}
\leq 2|u(\cdot) - u_M(\cdot)|^2_{L^2(D)} + 2\mathbb{E} \left\{ |u_M(\cdot) - u^N_M(\cdot)|^2_{L^2(D)} \right\}
\leq 2\lambda(D) \sup_{x \in D} |u(x) - u_M(x)| + 2 \int_{D} \mathbb{E} \left\{ |u_M(\cdot) - u^N_M(\cdot)|^2 \right\} \, dx
= 2\lambda(D) \sup_{x \in D} |u(x) - u_M(x)|^2 + \frac{2}{N} \int_{D} \mathbb{E} \left\{ |u_M(\cdot) - u^1_M(\cdot)|^2 \right\} \, dx
\leq 2\lambda(D) \sup_{x \in D} |u(x) - u_M(x)|^2 + \frac{4}{N} \left( \int_{D} |g|_{\infty}^2 + \frac{1}{d^2} M|f|_{\infty}^2 \operatorname{diam}(D)^2 \mathbb{E} \left\{ \sum_{k=1}^{M} r^2(X_{k-1}^1) \right\} \right) \, dx
\leq 2\lambda(D) \sup_{x \in D} |u(x) - u_M(x)|^2 + \frac{4}{N} \left( \int_{D} |g|_{\infty}^2 + \frac{1}{d^2} M|f|_{\infty}^2 \operatorname{diam}(D)^2 \mathbb{E} \{ r_{D^x} \} \right) \, dx
\]
and by Corollary 2.3
\[
\leq 2\lambda(D) \left[ \sup_{x \in D} |u(x) - u_M(x)|^2 + \frac{2(|g|_{\infty}^2 + \frac{1}{d^2} M|f|_{\infty}^2 \operatorname{diam}(D)^4)}{N} \right].
\]

Before proving the main result, Theorem 2.24, we need several preliminary Lemmas.
Lemma 4.2. Let $\varepsilon > 0$ and $\beta \in (0, 1]$. If $f$ and $g$ are $\alpha$-Hölder for some $\alpha \in [0, 1]$, and $M, N \geq 1$, then

$$|u_M^n(x) - u_M^N(y)| \leq \left( |g|_\alpha + \frac{\text{diam}(D)^2|f|_\alpha + 2\text{diam}(D)|f|_\infty}{d} \right) (2 + |\bar{r}|)_1^M \left( |x-y|^\alpha \vee |x-y| \right)$$

for all $x, y \in D$ almost surely.

In particular,

$$|u_M(x) - u_M(y)| \leq \left( |g|_\alpha + \frac{\text{diam}(D)^2|f|_\alpha + 2\text{diam}(D)|f|_\infty}{d} \right) (2 + |\bar{r}|)_1^M \left( |x-y|^\alpha \vee |x-y| \right)$$

for all $x, y \in D$.

Proof. Clearly, it is sufficient to prove the estimate for $|u_M^i(x) - u_M^i(y)| \leq 1$, independently of $i$, $1 \leq i \leq N$. To this end, since $\bar{r}$ is Lipschitz

$$|X_{M}^{x,i} - X_{M}^{y,i}| \leq |X_{M-1}^{x,i} - X_{M-1}^{y,i}| + |\bar{r}(X_{M-1}^{x,i}) - \bar{r}(X_{M-1}^{y,i})|$$

$$\leq (1 + |\bar{r}|)_1 |X_{M-1}^{x,i} - X_{M-1}^{y,i}|$$

$$\leq (1 + |\bar{r}|)_1 M |x-y|$$

for all $x, y \in D, M \geq 0$.

Therefore,

$$|u_M^i(x) - u_M^i(y)| \leq |g|_\alpha |X_{M}^{x,i} - X_{M}^{y,i}|^\alpha$$

$$+ \frac{1}{d} \sum_{k=1}^{M} 2|f|_\infty \text{diam}(D)|\bar{r}(X_{k-1}^{x,i}) - \bar{r}(X_{k-1}^{y,i})|$$

$$+ \frac{1}{d} \sum_{k=1}^{M} \text{diam}(D)^2|f|_\alpha |X_{k-1}^{x,i} - X_{k-1}^{y,i}| + |\bar{r}(X_{k-1}^{x,i}) - \bar{r}(X_{k-1}^{y,i})|^\alpha$$

$$\leq |g|_\alpha (1 + |\bar{r}|)_1 M |x-y|^\alpha$$

$$+ \frac{2\text{diam}(D)|f|_\infty |x-y|}{d} \sum_{k=1}^{M} (1 + |\bar{r}|)^k_1$$

$$+ \frac{\text{diam}(D)^2|f|_\alpha |x-y|^\alpha}{d} \sum_{k=1}^{M} (1 + |\bar{r}|)^{\alpha(k-1)}_1$$

$$\leq \left( |g|_\alpha + \frac{\text{diam}(D)^2|f|_\alpha + 2\text{diam}(D)|f|_\infty}{d} \right) (2 + |\bar{r}|)_1^M \left( |x-y|^\alpha \vee |x-y| \right).$$

The next lemma is the well-known Hoeffding’s inequality:

Lemma 4.3. Suppose that $(Z_i)_{i \geq 1}$ are iid real random variables such that $a_i \leq Z_i \leq b_i$ for all $i$. Then for all $N \in \mathbb{R}$ and $\gamma \geq 0$

$$\mathbb{P} \left( \left| \mathbb{E}(Z_1) - \frac{1}{N} \sum_{i=1}^{N} Z_i \right| \geq \gamma \right) \leq 2e^{-\frac{2N\gamma^2}{\sum_{i=1}^{N}(b_i-a_i)^2}}.$$

Using Hoeffding’s inequality we immediately get the following estimate:

Corollary 4.4. For all $N, M \in \mathbb{N}$, and $\gamma \geq 0$ we have

$$\mathbb{P} \left( \left| u_M^i(x) - u_M^N(x) \right| \geq \gamma \right) \leq 2e^{-\frac{N\gamma^2}{(|g|_\infty + M\text{diam}(D)^2|f|_\infty/d)^2}}, \quad x \in D.$$

Proof. The result follows directly from Lemma 4.3 since

$$\left| g(X_{M}^{x,i}) + \frac{1}{d} \sum_{k=1}^{M} \bar{r}^2(X_{k-1}^{x,i}) f \left( X_{k-1}^{x,i} + \bar{r}(X_{k-1}^{x,i}) Y_i \right) \right|_\infty \leq |g|_\infty + M\text{diam}(D)^2|f|_\infty/d.$$

Finally, we are in the position to prove the main theorem.
Proof of Theorem 2.24. First of all, assume without loss of generality that $D \subset [0, \text{diam}(D)]^d$, and for each $M, K \geq 1$ consider the grid

$$F = F(K, M, \alpha, |\bar{F}|_1) := \left\{ \frac{i \text{diam}(D)}{K(1 + |\bar{F}|_1)^{1/M/\alpha}} : 1 \leq i \leq K (2 + |\bar{F}|_1)^{1/M/\alpha} \right\} \cap D.$$

For $x \in D$ such that $x \in \left\{ \frac{i_1 \text{diam}(D)}{K(2 + |\bar{F}|_1)^{1/M/\alpha}}, \frac{(i_1 + 1) \text{diam}(D)}{K(2 + |\bar{F}|_1)^{1/M/\alpha}} \right\} \times \cdots \times \left\{ \frac{i_d \text{diam}(D)}{K(2 + |\bar{F}|_1)^{1/M/\alpha}}, \frac{(i_d + 1) \text{diam}(D)}{K(2 + |\bar{F}|_1)^{1/M/\alpha}} \right\}$ we set

$$x^F := \left( \frac{i_1 \text{diam}(D)}{K(2 + |\bar{F}|_1)^{1/M/\alpha}}, \cdots, \frac{i_d \text{diam}(D)}{K(2 + |\bar{F}|_1)^{1/M/\alpha}} \right).$$

Note that

$$\sup_{x \in D} |u(x) - u^N_M(x)| \leq \sup_{x \in D} |u(x) - u_M(x)| + \sup_{x \in D} |u_M(x) - u^N_M(x)|$$

$$\leq \sup_{x \in F} |u(x) - u_M(x)| + \sup_{x \in F} |u_M(x) - u^N_M(x)|$$

$$+ 2 \left( |g|_\alpha + \frac{\text{diam}(D)^2 |f|_\alpha + 2 \text{diam}(D) |f|_\infty}{d} \right) \left( \frac{\text{diam}(D)}{K} \right)^\alpha,$$

where the last inequality follows from Lemma 4.2 and by the fact that

$$|x - x_F| \leq \frac{\text{diam}(D)}{K(2 + |\bar{F}|_1)^{1/M/\alpha}}$$

for all $x \in D$.

Consequently, by setting

$$\bar{\gamma} := \gamma - \sup_{x \in F} |u(x) - u_M(x)| - 2 \left( |g|_\alpha + \frac{\text{diam}(D)^2 |f|_\alpha + 2 \text{diam}(D) |f|_\infty}{d} \right) \left( \frac{\text{diam}(D)}{K} \right)^\alpha$$

and using union bound inequality we have

$$\mathbb{P} \left( \sup_{x \in F} |u(x) - u^N_M(x)| \geq \bar{\gamma} \right) \leq \mathbb{P} \left( \sup_{x \in F} |u_M(x) - u^N_M(x)| \geq \bar{\gamma} \right) \leq \sum_{x \in F} \mathbb{P} \left( |u_M(x) - u^N_M(x)| \geq \bar{\gamma} \right),$$

so, the two desired estimates now follow by Theorem 2.19 and Corollary 4.4.

The last two assertions are clear. 

□

Proof of Corollary 2.26. Note that since $\bar{r}$ is a $(\beta, \varepsilon)$-distance, it is also a $(\beta, \varepsilon_0)$-distance since $\varepsilon \leq \varepsilon_0$.

Now, using (2.25) we get that

$$2 \left( |g|_\alpha + \frac{\text{diam}(D)^2 |f|_\alpha + 2 \text{diam}(D) |f|_\infty}{d} \right) \left( \frac{\text{diam}(D)}{K} \right)^\alpha \leq \frac{\gamma}{4}.$$

Also, because $\text{adiam}(D) < \infty$, by Proposition 2.16 we have that $|v|_\infty(\varepsilon_0) \leq \varepsilon_0 \text{adiam}(D)$. Therefore, by choosing $\varepsilon_0$ as in (2.24), we get that

$$d\alpha/2 |g|_\alpha \cdot |v|_\infty^{\alpha/2}(\varepsilon) + |f|_\infty |v|_\infty(\varepsilon) \leq \frac{\gamma}{4}.$$

Further, if $D$ is $\delta$-defective convex and $M$ is chosen as in (2.28), we have that

$$4 |g|_\infty + \frac{2}{d} \text{diam}(D)^2 |f|_\infty a_M \sqrt{\frac{\text{diam}(D)}{\varepsilon_0}} \leq \frac{\gamma}{4},$$

whilst if $D$ is merely a bounded Lipschitz domain and $M$ is as in (2.29), then

$$4 |g|_\infty + \frac{2}{d} \text{diam}(D)^2 |f|_\infty e^{-\text{adiam}(D)^2 M} \leq \frac{\gamma}{4}.$$

All the choices above ensure that $\gamma(M, K, d, \beta, \varepsilon)$ from Theorem 2.24 satisfies

$$\gamma(M, K, d, \beta, \varepsilon) \geq \frac{3\gamma}{4}.$$

Taking into account the above inequality and the estimate (2.23), it is simple to see that the right hand side of (2.23) is less than $\eta$ if $N$ satisfies (3.3), which concludes the proof. 

□
4.5 Proofs for Subsection 2.5

Proof of Lemma 2.28 i). We have:

\[
\frac{|G(x) - G(y)|}{|x-y|^\alpha} \leq \frac{|\psi \left( \frac{1}{\varepsilon_0} r(x) \right) (g(\pi_{\partial D}(x)) - g(\pi_{\partial D}(y)))|}{|x-y|^\alpha} + \frac{|((\psi \left( \frac{1}{\varepsilon_0} r(y) \right))')g(\pi_{\partial D}(y))|}{|x-y|^\alpha} \\
\leq \frac{|\pi_{\partial D}(x) - \pi_{\partial D}(y)|^\alpha |g|_\alpha^\prime}{|x-y|^\alpha} + |g|_\infty |r(x) - r(y)| \\
\leq |\nabla \pi_{\partial D}|^\alpha |g|_\alpha + \frac{|g|_L \infty |x-y|^\alpha |\text{diam}(D)|^{1-\alpha}}{|x-y|^\alpha}.
\]

ii). Using the definition of \( G \) we have

\[
\nabla G = \frac{1}{\varepsilon_0} \psi' \left( \frac{1}{\varepsilon_0} r \right) \nabla r g(\pi_{\partial D}) + \psi \left( \frac{1}{\varepsilon_0} r \right) \nabla g(\pi_{\partial D}) \nabla \pi_{\partial D},
\]

respectively

\[
\Delta G = \left( \psi'' |\nabla r|^2 + \psi' \Delta r \right) g(\pi_{\partial D}) + 2 \varepsilon_0 \frac{\psi'}{\varepsilon_0} |\nabla r| \nabla g(\pi_{\partial D}) \nabla \pi_{\partial D} \\
+ \psi(\Delta g(\pi_{\partial D}) |\nabla \pi_{\partial D}|^2 + |\nabla g(\pi_{\partial D}) \Delta \pi_{\partial D}|).
\]

Taking into account that \( |\psi|_\infty, |\psi'|_\infty, |\psi''|_\infty \leq 1 \) and \( |\nabla r| \leq 1 \) we have:

\[
|\nabla G|_\infty \leq \frac{1}{\varepsilon_0} |g|_\infty + |\nabla g|_\infty |\nabla \pi_{\partial D}|_\infty, \\
|\Delta G|_\infty \leq (1 + |\Delta r|_\infty) |g|_\infty + \frac{2}{\varepsilon_0} |\nabla g|_\infty |\nabla \pi_{\partial D}|_\infty + |\Delta g|_\infty |\nabla \pi_{\partial D}|^2_\infty + |\nabla g|_\infty |\Delta \pi_{\partial D}|_\infty.
\]

For any point \( P \in \partial D \) let \( \nu(P) \) and \( T_P \) denote respectively the unit exterior normal to \( \partial D \) at the point \( P \) and the tangent hyperspace to \( \partial D \) at \( P \). By a rotation of coordinates we can assume that the \( P_d \) coordinate lies in the direction \( \nu(P) \). In some neighbourhood \( N = N(P) \) of \( P \), \( \partial D \) is given by \( P_d = \varphi(P') \) where \( P' = (P_1, \ldots, P_{d-1}) \), \( \varphi \in C^3(T_P \cap N) \) and \( D\varphi(P') = 0 \). The eigenvalues of the matrix \( [\nabla^2 \varphi(P')] \) denoted \( \{k_1, \ldots, k_{d-1}\} \) are then the principal curvatures at \( \partial D \) at \( P \). By a further rotation of coordinates we can assume that the \( P_1, \ldots, P_d-1 \) axes lie along principal directions corresponding to \( k_1, \ldots, k_{d-1} \) at \( P \).

The Hessian matrix \( [D^2 \varphi(P')] \) with respect to the principal coordinate system at \( P \) described above is given by

\[
[D^2 \varphi(P')] = \text{diag}[k_1, \ldots, k_{d-1}].
\]

As noted in the proof of Lemma 14.16 in [10] the maximal radius of the interior ball that can be associated to each point on the boundary, is bounded from below by a certain \( \mu > 0 \) and we have that \( \mu^{-1} \) bounds the principal curvatures, hence our choice of \( \mu \) as \( \varepsilon_0 \).

The unit exterior normal vector \( \hat{\nu}(P') := \nu(P) \) at a point \( P = (P', \varphi(P')) \in N \cap \partial D \) is given by

\[
\nu_i(P) = \frac{\partial \varphi_i(P')}{\partial x_j(P')} \sqrt{1 + |D\varphi(P')]^2}, i = 1, \ldots, d-1, \nu_d(P) = \frac{1}{\sqrt{1 + |D\varphi(P')]^2}}.
\]

Hence with respect to the principal coordinate system at \( P \) we have:

\[
\frac{\partial \nu_i}{\partial x_j}(P') = k_i \delta_{ij}, i, j = 1, \ldots, d-1. \quad (4.17)
\]

We note that for each \( x \in D_{\varepsilon_0} \) there exists a unique \( \pi_{\partial D}(x) \in \partial D \) such that \( |\pi_{\partial D}(x) - x| = r(x) \). We have:

\[
x = \pi_{\partial D}(x) + \nu(\pi_{\partial D}(x)) r(x). \quad (4.18)
\]
As pointed in the proof of Lemma 14.6 in [5] we have $\pi_{\partial D} \in C^2(D_{\varepsilon_0}), r \in C^3(D_{\varepsilon_0})$. Differentiating the i-th coordinate of (4.18) with respect to $x_j$ we get:

$$\delta_{ij} = \frac{\partial(\pi_{\partial D})_i}{\partial x_j} + \sum_l \frac{\partial \nu_i}{\partial y_l} \frac{\partial(\pi_{\partial D})_l}{\partial x_j} r + \nu_i \frac{\partial r}{\partial x_j}$$  \hspace{1cm} (4.19)

Furthermore, differentiating (4.19) with respect to $x_k$ we get:

$$0 = \frac{\partial^2(\pi_{\partial D})_i}{\partial x_j \partial x_k} + \sum_{l,m} \frac{\partial^2 \nu_i}{\partial y_l \partial y_m} \frac{\partial(\pi_{\partial D})_l}{\partial x_k} \frac{\partial(\pi_{\partial D})_m}{\partial x_j} r + \sum_l \frac{\partial \nu_i}{\partial y_l} \frac{\partial^2(\pi_{\partial D})_l}{\partial x_j \partial x_k} r + \sum_l \frac{\partial \nu_i}{\partial y_l} \frac{\partial r}{\partial x_j} \frac{\partial(\pi_{\partial D})_l}{\partial x_k} + \sum_l \frac{\partial \nu_i}{\partial y_l} \frac{\partial^2 r}{\partial x_j \partial x_k}$$  \hspace{1cm} (4.20)

As noted in the proof of Lemma 14.17 in [6] we have that in terms of a principal coordinate system at $\pi_{\partial D}(x)$ as chosen before

$$\nabla r(x) = \nu(\pi_{\partial D}(x))$$  \hspace{1cm} (4.21)

and,

$$\nabla^2 r = \text{diag}(\frac{k_1}{1+k_1 r}, \ldots, \frac{k_{d-1}}{1+k_{d-1} r}, 0),$$  \hspace{1cm} (4.22)

hence

$$|\nabla r|_\infty \leq 1, |\nabla^2 r|_\infty \leq \max_{x \in \partial D} k_{d-1}(x) = \varepsilon_0^{-1}.$$ 

Using (4.21) in (4.17) implies:

$$\frac{\partial(\pi_{\partial D})_i}{\partial x_j} = (\delta_{ij} - \nu_i \nu_j) \frac{1}{1+k_i r}$$  \hspace{1cm} (4.23)

hence

$$|\nabla \pi_{\partial D}|_\infty \leq 2.$$ 

Furthermore using (4.21) and (4.22) in (4.20) we can bound:

$$|\nabla^2 \pi_{\partial D}|_\infty \leq |\nabla^2 \nu|_\infty 4\varepsilon_0 + 4 + \varepsilon_0^{-1},$$ 

where we estimated $|r|_\infty \leq \varepsilon_0$ because due to the cut-off function $\psi$ we only need to estimate $r, \pi_{\partial D}$ in the $\varepsilon_0$ neighbourhood of the boundary.

**Proof of Lemma 3.6.** First note that if $\phi = W^L \circ \sigma \cdots \sigma W^1$, where $W^i$ is of the form $W^i(z) = A^i z + b^i$, then using the relation $x = \sigma(x) - \sigma(-x)$ we have

$$\phi(x) = (v \cdots v \sigma \circ \left(\begin{array}{c} W^L \\ -W^L \end{array}\right) \circ \sigma \cdots \sigma \circ W^1 x), \quad x \in \mathbb{R}^d.$$ 

The assertions follow now from Lemma 3.4 and Lemma 3.2. \hfill \Box

**Proof of Corollary 3.8.** We first note that

$$\left| \frac{1}{\varepsilon_0} r - \frac{1}{\varepsilon_0} \phi_r \right|_\infty \leq \frac{\delta_r}{\varepsilon_0} \quad \text{and} \quad \left| \psi \left(\frac{1}{\varepsilon_0} r \right) - \phi_{\psi} \left(\frac{1}{\varepsilon_0} r \right) \right|_\infty \leq \delta_{\phi}.$$ 

Therefore, by the triangle inequality we get

$$\left| \phi_{\psi} \left(\frac{r}{\varepsilon_0} \right)(x) - \phi_{\psi} \left(\frac{\phi_r}{\varepsilon_0} \right)(x) \right| \leq 2\delta_{\phi} + |\psi'|_\infty \left| \frac{1}{\varepsilon_0} r(x) - \frac{1}{\varepsilon_0} \phi_r(x) \right| \leq 2\delta_{\phi} + |\psi'|_\infty \frac{\delta_r}{\varepsilon_0},$$

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hence
\[ \left| \psi \left( \frac{1}{\varepsilon_0} r \right) - \phi \left( \frac{\phi_x}{\varepsilon_0} \right) \right|_\infty \leq 3\delta_\psi + \frac{\delta_d}{\varepsilon_0}. \]

Reasoning analogously we get
\[ |g(\pi_{\partial D}) - \phi_g \circ \phi_x|_\infty \leq 3\delta_g + |\nabla g|_\infty \delta_x, \]

so again by the triangle inequality
\[ \left| \psi \left( \frac{1}{\varepsilon_0} r \right) g(\pi_{\partial D}) - \phi \left( \frac{\phi_x}{\varepsilon_0} \right) \phi_g \circ \phi_x \right|_\infty \leq \left( 3\delta_\psi + \frac{\delta_d}{\varepsilon_0} \right) |g|_\infty + \left( 3\delta_g + |\nabla g|_\infty \delta_x \right) \left( \delta_\psi + 1 \right). \]

Let now \( \Pi \) be the ReLU DNN given in Corollary 3.5 with \( U \) denoted in the statement by \( M \), namely
\[ u \leq \hat{u}_M, \]

Next, let \( \omega \) be given by (2.21) and consider the modification of \( u_N^\omega \) by replacing \( g \) and \( f \) with \( \phi_g \) and \( \phi_f \), namely
\[ \hat{u}_M(x) := \frac{1}{N} \sum_{i=1}^N \left[ \phi_g(X^\omega_{M,i}) + \frac{1}{d} \sum_{k=1}^M \bar{r}^2(X^\omega_{M,k-1}) \phi_f \left( X^\omega_{M,k-1} + \bar{r}(X^\omega_{M,k-1}) Y^\omega \right) \right], \quad x \in D. \]

Then using assumption a.4) we can easily deduce that
\[ |\hat{u}_M - \bar{u}_M|_\infty \leq \frac{M \text{diam}(D)^2 \epsilon_f}{d} (1 + 2|f|_\infty) \leq \gamma/6. \]

Therefore,
\[ \text{sup}_{x \in D} \left| u(x) - \bar{u}_M(x, x) \right| = |u - \bar{u}_M|_\infty \leq |u - \hat{u}_M|_\infty + |u_N^\omega|_\infty + |\hat{u}_M - \bar{u}_M| + |\hat{u}_M - \bar{u}_M| \]
\[ \leq |u - \hat{u}_M|_\infty + \epsilon_g + \frac{M \text{diam}(D)^2 \epsilon_f}{d} + \gamma/6 \]
\[ \leq |u - \hat{u}_M|_\infty + \gamma/2, \]

where the last inequality follows from assumptions a.1), a.2). Consequently,
\[ \mathbb{P} \left( \text{sup}_{x \in D} \left| u(x) - \bar{u}_M(x, x) \right| \geq \gamma \right) \leq \mathbb{P} \left( |u - \hat{u}_M|_\infty \geq \gamma/2 \right), \]

so, we can employ Corollary 3.4 to conclude the first assertion.

Let us proceed to proving ii). First of all, note that
\[ \mathcal{L}(\bar{r}) = \mathcal{L}(\phi_r) + 1, \quad \mathcal{W}(\bar{r}) = \mathcal{W}(\phi_r), \quad \text{size}(\bar{r}) \leq \text{size}(\phi_r) + 2. \]

Then, by Lemma 5.3 and Corollary 3.3 we have
\[ \mathcal{L}(\phi_g(X^\omega_M)) = \mathcal{L}(\phi_g) + \mathcal{L}(X^\omega_M) = \mathcal{L}(\phi_g) + M(\mathcal{L}(\bar{r}) + 1) + 1 = \mathcal{L}(\phi_g) + M(\mathcal{L}(\phi_r) + 2) + 1, \]
\[ \mathcal{W}(\phi_g(X^\omega_M)) \leq \max(\mathcal{W}(\phi_g), \mathcal{W}(X^\omega_M), 2d) \leq \max(\mathcal{W}(\phi_g), 2d + \text{max}(d, \mathcal{W}(\phi_r))), \]
\[ \text{size}(\phi_g(X^\omega_M)) \leq 2\text{size}(\phi_g) + 2\text{size}(X^\omega_M) \leq 2\text{size}(\phi_g) + 4dM[4d + \mathcal{W}(\bar{r}) + \mathcal{L}(\bar{r}) + 2] + 2d + 4M\text{size}(\phi_g) + 2 \]
\[ \leq 2\text{size}(\phi_g) + 4dM[4d + \mathcal{W}(\phi_r) + \mathcal{L}(\phi_r) + 3] + 2d + 4M\text{size}(\phi_g) + 2 \]
\[ \in O(\text{size}(\phi_g) + M\text{size}(\phi_r) + Md \max(d, \mathcal{W}(\phi_r)), \mathcal{L}(\phi_r))). \]

(4.24)
Further, for each $k \geq 0$, by Lemma 3.3 we get
\[\text{size} \left( \phi_f \left( X_{k+1}^{i+1} + \bar{r}(X_{k+1}^{i+1})Y^i \right) \right) \leq 2 \text{size} (\phi_f) + 2 \text{size} \left( X_{k+1}^{i+1} + \bar{r}(X_{k+1}^{i+1})Y^i \right)\]
and since $X_k^{i+1} + \bar{r}(X_k^{i+1})Y^i$ has the same size as $X_{k+1}^{i+1}$, we can continue with
\[= 2 \text{size} (\phi_f) + 2 \text{size} (X_{k+1}^{i+1}) \leq 2 \text{size} (\phi_f) + 4d(k+1)[4d + W(\phi_r) + L(\phi_r) + 3] + 2d + 4(k+1)[\text{size}(\phi_r) + 2]. \tag{4.25}\]

The next step is to use Corollary 3.5 to get
\[\text{size} \left( \Pi \left( \bar{r}(X_k^{i+1}), \bar{r}(X_k^{i+1}) \right) \right) \leq 8 \text{size}(\bar{r}(X_k^{i+1})) + O(\log(\epsilon_p^{-1}) + \log(c)) \leq 16 \text{size}(\bar{r}) + 16 \text{size}(X_k^{i+1}) + O(\log(\epsilon_p^{-1}) + \log(c)) \leq 16 \text{size}(\phi_r) + 32dk[4d + W(\phi_r) + L(\phi_r) + 3] + 16d + 32[k \text{size}(\phi_r) + 2] + O(\log(\epsilon_p^{-1}) + \log(c)), \tag{4.26}\]
so that by (4.25) and (4.26) together with Corollary 3.3 we obtain
\[\text{size} \left( \Pi \left( \bar{r}(X_k^{i+1}), \bar{r}(X_k^{i+1}) \right), \phi_f \left( X_k^{i+1} + \bar{r}(X_k^{i+1})Y^i \right) \right) \leq 4 \text{size} \left( \Pi \left( \bar{r}(X_k^{i+1}), \bar{r}(X_k^{i+1}) \right) \right) + 4 \text{size} \left( \phi_f \left( X_k^{i+1} + \bar{r}(X_k^{i+1})Y^i \right) \right) + O(\log(\epsilon_p^{-1}) + \log(c)) \leq 24 \text{size}(\phi_r) + 128dk[4d + W(\phi_r) + L(\phi_r) + 3] + 64d + 128k[\text{size}(\phi_r) + 2] + 8 \text{size}(\phi_f) + 16d(k+1)[4d + W(\phi_r) + L(\phi_r) + 3] + 8d + 16k(1)[\text{size}(\phi_r) + 2] + 32\phi_r + 32[k \text{size}(\phi_r) + 2] + O(\log(\epsilon_p^{-1}) + \log(c)) \in O \left( dk \max(d, W(\phi_r), L(\phi_r)) + k \text{size}(\phi_r) + \text{size}(\phi_f) + \text{size}(\phi_f) + [\log(\epsilon_p^{-1}) + \log(c)] \right). \tag{4.27} \]

Finally, corroborating (4.27) with (4.28) and by applying Lemma 3.2 we obtain that for each $\omega \in \Omega$
\[\text{size}(U_M^N(\omega, \cdot)) \in O \left( MN \left[ dM \max(d, W(\phi_r), L(\phi_r)) + M \text{size}(\phi_r) + \text{size}(\phi_f) + \text{size}(\phi_g) + [\log(\epsilon_p^{-1}) + \log(c)] \right] \right), \]
and by assumption a.4) we deduce that
\[\text{size}(U_M^N(\omega, \cdot)) \in O \left( MN \left[ dM \max(d, W(\phi_r), L(\phi_r)) + M \text{size}(\phi_r) + \text{size}(\phi_f) + \log \left( \frac{1}{\gamma d} \right) \right] \right), \]
where the tacit constant depends on $\max(\text{diam}(D), |f|_{\infty})$.

Now, if assumption a.5) is in force in the case that $D$ is $\delta$-defective convex, then
\[M \in O \left( \log \left( \frac{d}{\gamma} \right) \right), \]
\[N \in O \left( \frac{1}{\gamma} \log^2 \left( \frac{d}{\gamma} \right) \left[ d \log \left( \frac{d}{\gamma} \right) \log(2 + |\phi_r|_1) + \log \left( \frac{1}{\eta} \right) \right] \right), \]
\[\text{size}(U_M^N(\omega, \cdot)) \in O \left( \frac{d^2}{\gamma^2} \log^4 \left( \frac{d}{\gamma} \right) \left[ d \log \left( \frac{d}{\gamma} \right) + \log \left( \frac{1}{\eta} \right) \right] S \right), \]
where
\[S := \left[ \max(d, W(\phi_r), L(\phi_r)) + \text{size}(\phi_r) + \text{size}(\phi_g) + \text{size}(\phi_f) \right], \]
and the tacit constant depends on $|g|_{\alpha}, |g|_{\infty}, |f|_{\infty}, \text{diam}(D), \text{adiam}(D), \delta, \alpha, \log(2 + |\phi_r|_1)$. 

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Finally, if assumption a.5′ holds, namely $D$ is nearly a Lipschitz domain, then

$$M \in O \left( d^2 \gamma^{-4/\alpha} \log \left( \frac{1}{\eta} \right) \right),$$

$$N \in O \left( d^4 \gamma^{-8/\alpha - 2} \log^2 \left( \frac{1}{\eta} \right) \left[ d^2 \gamma^{-4/\alpha} \log \left( \frac{1}{\eta} \right) + \log \left( \frac{1}{\eta} \right) \right] \right),$$

$$\text{size}(U^N_M(\omega, \cdot)) \in O \left( d^8 \gamma^{-16/\alpha - 4} \log^4 \left( \frac{1}{\eta} \right) \left[ d^2 \gamma^{-4/\alpha} \log \left( \frac{1}{\eta} \right) + \log \left( \frac{1}{\eta} \right) \right] S \right),$$

where $S$ is as above and the tacit constant depends on $|g|_{\alpha}, |g|_{\infty}, |f|_{\infty}, \text{diam}(D), \text{adiam}(D), \delta, \alpha, \log(2 + |\phi_r|_1)$.

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**References**

[1] D.H. Armitage and Ü. Kuran. The convexity of a domain and the superharmonicity of the signed distance function. *Proceedings of the Amer. Math. Soc.*, 93(4):598–600, 1985.

[2] I. Binder and M. Braverman. The rate of convergence of the walk on spheres algorithm. *Geometric and Functional Analysis*, 22(3):558–587, 2012.

[3] G. Cybenko. Approximation by superpositions of a sigmoidal function. *Mathematics of Control, Signals and Systems*, 2(4):303–314, 1989.

[4] D. Elbrächter, D. Perekrestenko, P. Grohs, and H. Bölcskei. Deep neural network approximation theory. *IEEE Transactions on Information Theory*, 67(5):2581–2623, 2021.

[5] W.D. Gerhard. The probabilistic solution of the Dirichlet problem for $1/2 \Delta + \langle a, \nabla \rangle + b$ with singular coefficients. *Journal of Theoretical Probability*, 5(3):503–520, 1992.

[6] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer, 2001.

[7] P. Grohs and L. Herrmann. Deep neural network approximation for high-dimensional elliptic PDEs with boundary conditions. *IMA Journal of Numerical Analysis*, 42(3):2055–2082, 2021.

[8] J. Han, A. Jentzen, and E. Weinan. Solving high-dimensional partial differential equations using deep learning. *Proceedings of the National Academy of Sciences*, 115(34):8505–8510, 2018.

[9] A. Jentzen, D. Salimova, and T. Welti. A proof that deep artificial neural networks overcome the curse of dimensionality in the numerical approximation of kolmogorov partial differential equations with constant diffusion and nonlinear drift coefficients. *arXiv preprint arXiv:1809.07321*, 2018.

[10] G. Kutyniok, P. Petersen, M. Raslan, and R. Schneider. A theoretical analysis of deep neural networks and parametric pdes. *Constructive Approximation*, 55(1):73–125, 2022.

[11] A.E. Kyprianou, A. Osojnik, and T. Shardlow. Unbiased ‘walk-on-spheres’ monte carlo methods for the fractional laplacian. *IMA Journal of Numerical Analysis*, 38, 09 2017.

[12] I.E. Lagaris, A. Likas, and D.I. Fotiadis. Artificial neural networks for solving ordinary and partial differential equations. *IEEE transactions on neural networks*, 9(5):987–1000, 1998.
[13] I.E Lagaris, A.C. Likas, and D.G. Papageorgiou. Neural-network methods for boundary value problems with irregular boundaries. *IEEE Transactions on Neural Networks*, 11(5):1041–1049, 2000.

[14] A. Malek and R. Shekari Beidokhti. Numerical solution for high order differential equations using a hybrid neural network—optimization method. *Applied Mathematics and Computation*, 183(1):260–271, 2006.

[15] T. Marwah, Z.C. Lipton, and A. Risteski. Parametric complexity bounds for approximating pdes with neural networks. In M. Ranzato, A. Beygelzimer, Y. Dauphin, P.S. Liang, and J. Wortman Vaughan, editors, *Advances in Neural Information Processing Systems*, volume 34, pages 15044–15055. Curran Associates, Inc., 2021.

[16] M.E. Muller. Some continuous monte carlo methods for the dirichlet problem. *The Annals of Mathematical Statistics*, pages 569–589, 1956.

[17] A. Pinkus. Approximation theory of the mlp model in neural networks. *Acta Numerica*, 8:143–195, 1999.

[18] M. Raissi, P. Perdikaris, and G. Karniadakis. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*, 378:686–707, 2019.

[19] J. Sirignano and K. Spiliopoulos. DGM: A deep learning algorithm for solving partial differential equations. *Journal of Computational Physics*, 375:1339–1364, 2018.

[20] N. Valenzuela. A new approach for the fractional laplacian via deep neural networks. *arXiv preprint arXiv:2205.05229*, 2022.

[21] D. Yarotsky. Error bounds for approximations with deep relu networks. *Neural Networks*, 94:103–114, 2017.