Conformally Invariant Braneworld and the Cosmological Constant

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Abstract

A six dimensional braneworld scenario based on a model describing the interaction of gravity, gauge fields and 3+1 branes in a conformally invariant way is described. The action of the model is defined using a measure of integration built of degrees of freedom independent of the metric. There is no need to fine tune any bulk cosmological constant or the tension of the two (in the scenario described here) parallel branes to obtain zero cosmological constant, the only solutions are those with zero 4-D cosmological constant. The two extra dimensions are compactified in a "football" fashion and the branes lie on the two opposite poles of the compact "football-shaped" sphere.

1 The Model

Recently there have been a great number of studies of the possibility that our Universe is built of one or more 3+1 branes and a higher dimensional bulk component $^{1,2,3,4,5}$. Standard model fields would be confined to the branes while gravity can propagate into the bulk also. In this paper, the question of the cosmological constant will be addressed in a model of this kind.

One approach to the cosmological constant problem has been to exploit the possibility of using in the action a measure of integration independent of
the metric $G_{AB}$, that is, different from $\sqrt{-G}d^Dx$, where $G = det(G_{AB})$. See Refs. 6, 7, 8.

Indeed instead of $\sqrt{-G}d^Dx$ one can use $\Phi d^Dx$, where

$$\Phi = \epsilon^{A_1...A_D} \epsilon_{a_1...a_D} \partial_{A_1} \varphi_{a_1} ... \partial_{A_D} \varphi_{a_D}$$  \hspace{1cm} (1)$$

in $D$ - dimensional space-time. Here the $\varphi_a$ fields are scalars and are treated as independent degrees of freedom. Since $\Phi$ has the same transformation properties as $\sqrt{-G}$, it follows that also $\Phi d^Dx$ is a scalar as well as $\sqrt{-G}d^Dx$.

Then we can write for example an invariant action of the form

$$S = \int L \Phi d^Dx$$  \hspace{1cm} (2)$$

where $L$ is a scalar. One can explore the possibility of using also $\sqrt{-G}d^Dx$ in another piece of the action, this "two measures" possibility will not be used here.

Notice that $\Phi$ is a total derivative, indeed we can write

$$\Phi = \partial_{A_1} (\epsilon^{A_1...A_D} \epsilon_{a_1...a_D} \varphi_{a_1} ... \partial_{A_D} \varphi_{a_D})$$  \hspace{1cm} (3)$$

Therefore the shift

$$L \rightarrow L + \text{constant}$$  \hspace{1cm} (4)$$

does not change the equations of motion.

Let us now describe what will be our choice for $L$: it should describe gauge fields, gravity and 3 + 1 branes.

For the (bulk) gauge field contribution to $L$ we will consider a "square root" form $s\sqrt{|F_{CD}F^{CD}|}$ where $F_{CD} = \partial_C A_D - \partial_D A_C$. This is a very interesting possibility: such "square root gauge theory" has been shown to give rise to string solutions 9 and therefore provides a theory which includes strings as special types of excitations, while containing other types of excitations as well. The square root gauge theory was considered also as a model for confinement 8,10.

When including gravity, the bulk dynamics will be then governed by

$$S_{bulk} = \int L_{bulk} \Phi d^6x, L_{bulk} = -\frac{1}{\kappa} R + s\sqrt{|F_{CD}F^{CD}|}$$  \hspace{1cm} (5)$$

where
\[ R = G^{AB} R_{AB}, \quad R_{AB} = R^{C}_{ABC} \]  

and the curvature tensor is defined in terms of the connection coefficients through the relation

\[ R_{ABCD} = \Gamma_{BCD}^{A} - \Gamma_{BD,C}^{A} + \Gamma_{ED}^{A} \Gamma_{BC}^{E} - \Gamma_{EC}^{A} \Gamma_{BD}^{E} \]  

The relation between \( \Gamma_{BC}^{A} \) and \( G_{AB} \) is determined by the variational principle. \( s \) is a number.

Model (5) was studied in Ref. (8) and shown to provide a compactification mechanism without the need of fine tuning a bulk cosmological constant term. The compactification takes place when \( F_{AB} \) takes a monopole expectation value in two extra dimensions.

Under the change

\[ \varphi_{a} \rightarrow \varphi'_{a} (\varphi_{b}) \]  

which means that \( \Phi \rightarrow |\frac{\partial \varphi'}{\partial \varphi}| \Phi \) we can achieve invariance of the action defined by (2), (5), (6), (7) if \( G_{AB} \) is also transformed according to

\[ G^{AB} \rightarrow |\frac{\partial \varphi_{a}}{\partial \varphi_{b}}| G^{AB} \]  

In this case we can recover the General Relativity form if by means of the conformal invariance displayed before, we choose the gauge \( \Phi = \sqrt{-G} \).

Notice that the conformal invariance (8), (9) is possible because both terms in \( L \) in (5) have the same homogeneity in \( G_{AB} \). They are both homogeneous of degree one in this variable, so their transformation can be simultaneously compensated by the transformation of the measure \( \Phi \).

To construct a brane scenario we must of course study the introduction of a brane term to \( L \). We will see that the action of a 3 + 1 brane embedded in a six dimensional space is consistent with the conformal symmetry (8)-(9).

Before considering the action of a 3 + 1 brane in the context of an action of the form (1) and (2), let us review what is the action in the context of the standard formulation. In this case

\[ S_{4}^{\text{standard}} = \int d^{4} \sqrt{-g} l_{4} = \int d^{6} x \sqrt{-G} L_{4} \]  

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here \( g = \text{det}(g_{\mu \nu}) \) and \( g_{\mu \nu} \) being the metric pulled back to the brane world volume.

\[
g_{\mu \nu} = G_{AB} x_A^\mu x_B^\nu \tag{11}
\]

and

\[
L_4 = \int d^4 \sigma \sqrt{-G} \delta^{(D)}(x - x(\sigma)) l_4 \tag{12}
\]

and we will consider the case where \( D = 6 \) and \( l_4 = T = \text{constant} \). This constant \( T \) has the interpretation of '3 + 1 surface tension'.

We can incorporate (12) into the general form (1), (2) by changing the measure of integration: \( d^6 x \sqrt{-G} \rightarrow d^6 x \Phi \), so that the brane contribution will be now

\[
S_4 = \int d^6 x \Phi L_4 \tag{13}
\]

where \( L_4 \) is given by (12) for \( l_4 = T = \text{constant} \).

It is very important to see the very special role of the dimensionalities 3 + 1 of the brane and 6 = 5 + 1 of the embedding space: indeed \( L_4 \) as defined by (12) is of degree one in \( G^{AB} \) only for the very special choice \( D = 6 \). So that

\[
S = S_{\text{bulk}} + S_4 \tag{14}
\]

with \( S_{\text{bulk}} \) given by (5) and \( S_4 \) given by (12), (13), has the symmetry (8), (9). Notice that by using the symmetry (8), (9) we can set the gauge \( \Phi = \sqrt{-G} \) and in this case \( S_4 = S_4^{\text{standard}} \). This once again, only for a 3 + 1 brane when embedded in a six dimensional space.

### 2 A zero four dimensional cosmological constant without fine tuning

The fact that \( L = L_{\text{bulk}} + L_4 \), where \( L_{\text{bulk}} \) and \( L_4 \) are given by (5) and (12), is homogeneous in \( G^{AB} \) with homogeneity one implies,

\[
G^{AB} \frac{\partial L}{\partial G^{AB}} = L \tag{15}
\]
(and a similar equation holds also separately for $L_{\text{bulk}}$ and $L_4$) The variation of the action with respect to the $\varphi_a$ fields gives the equation

$$A^A_a \partial_A L = 0$$  \hspace{1cm} (16)

where

$$A^A_a = \epsilon^{AA_2..A_a} \epsilon_{aa_2..a_6} \partial_{A_2} \varphi_{a_2} \ldots \partial_{A_6} \varphi_{a_6}$$  \hspace{1cm} (17)

since one can easily see that $\det(A^A_a) = 6^{-6} \Phi^6$, we have that if $\Phi \neq 0$, then $\partial_A L = 0$, which means that

$$L = M = \text{constant}$$  \hspace{1cm} (18)

Taking a variation of the action (2) with respect to a conformal transformation, i.e. to a transformation of the form $G^{AB} \rightarrow \Omega^2(x)G^{AB}$, we obtain, for the case that $L$ is an homogeneous function of $G^{AB}$ (of non trivial homogeneity), that $L = 0$, so that the constant $M$ in (18) equals zero.

The constant of integration $M$, if it were different from zero would have spontaneously broken the conformal invariance, since $L$ changes under a conformal transformation (as mentioned $L$ is of homogeneity one in $G^{AB}$ ) and $M$ does not (is fixed by the boundary conditions). If we work with theories with global scale invariance, there is the possibility of spontaneous breaking of global scale invariance by the appearance of non zero constant of integration $M$. See Refs. (7)

Separating Gravity and matter pieces of the action, we define

$$L = -\frac{1}{\kappa} R + L_{\text{matter}}$$  \hspace{1cm} (19)

and we obtain, from the variation with respect to $G^{AB}$ (using the fact that the fields in the measure $\Phi$ are independent of $G^{AB}$),

$$R_{AB} = \kappa \frac{\partial L_{\text{matter}}}{\partial G^{AB}}$$  \hspace{1cm} (20)

Alternatively, one can perform the variation with respect to $G^{AB}$ after setting the ”Einstein” gauge $\Phi = \sqrt{-G}$. In this case one obtains the Einstein’s equations corresponding to the matter lagrangian $L_{\text{matter}}$. These equations are identical to (20) if use is made of the fact that $L_{\text{matter}}$ is homogeneous of degree one in $G^{AB}$.
In any case, even before we start to solve in detail the equations of motion, we see something remarkable from eq. (20): Let us consider a product metric of the form

\[ ds^2 = g_{\mu\nu}(x^\alpha)dx^\mu dx^\nu + \gamma_{ij}(x^k)dx^i dx^j \]  

(21)

where the ordinary dimensions are labeled with greek indices, \( \mu, \nu, \alpha = 0, 1, 2, 3 \) and the extra dimensions with small latin indices, \( i, j, k = 4, 5 \) and consider the case when \( F_{AB} \) takes non zero values only in the extra dimensions \( i, j, k = 4, 5 \) and let us consider the branes to be oriented in the \( 3 + 1 \) hyperplanes orthogonal to those of the two extra dimensions, i.e. hyperplanes \( x^i = constant \).

In this case we see that in \( s\sqrt{|F_{CD}F^{CD}|} \) only the extra dimensional metric appears and likewise the same is true for \( S_4 \), because there, in the ratio \( \frac{g}{G} \) the four dimensional metric is cancelled.

Therefore (20) implies that both \( s\sqrt{|F_{CD}F^{CD}|} \) and \( S_4 \) curve only the extra dimensions. A solution containing four dimensional flat Minkowskii space is possible without fine tuning whatsoever!, i.e. we can take

\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \gamma_{ij}(x^k)dx^i dx^j \]  

(22)

3 The explicit braneworld solutions

Given the metric (22), choosing the Einstein form \( \Phi = \sqrt{-G} \), we can see that, if we want an extra dimensional space of constant curvature (at the points where there are no branes), we must consider a field strength which has expectation value in the extra dimensions of the form, as has been done in many previous studies of spontaneous compactification\(^\text{11}\)

\[ F_{ij} = B_0 \sqrt{\gamma} \epsilon_{ij} \]  

(23)

where \( B_0 \) is a constant. This automatically gives \( F^{AB}F_{AB} = 2B_0^2 = constant \) and from (20) and (23) a constant extradimensional curvature appears.

This configuration satisfies also the field equations for the gauge fields

\[ \partial_A(\sqrt{-G} \frac{F^{AB}}{\sqrt{|F^{CD}F_{CD}|}}) = 0 \]  

(24)
If there are no branes \((T = 0, S_4 = 0)\), we take the extra dimensions to have a spherical shape,

\[
\gamma_{ij}(x^k)dx^i dx^j = b^2(d\theta^2 + \sin^2\theta d\phi^2), \quad b = \text{const.}
\]  
(25)

and then eq. (20) implies that there is the following relation:

\[
b^2 = \frac{\sqrt{2}}{s\kappa B_0}
\]  
(26)

When branes are present, we can keep still a gauge field configuration satisfying (23), but then the extra dimensional part of the metric has to be changed, to a "foot balllike configuration" in the language of Carroll and Guica\textsuperscript{12}, see also Ref. 13.

We consider two branes located at opposite poles of the spherical extra dimensions. Following the analysis of Carroll and Guica\textsuperscript{12}, we represent the extra dimensions as

\[
\gamma_{ij}(x^k)dx^i dx^j = \psi(r)(dr^2 + r^2 d\phi^2)
\]  
(27)

To describe brane sources we need to introduce singularities at the north pole \(r = 0\) and at the south pole \(r = \infty\), but a new coordinate system should be used there). The standard two dimensional delta function with respect to integration measure \(rdrd\phi\) is

\[
\delta^{(2)}(r) = \frac{1}{2\pi} \nabla^2 \ln r
\]  
(28)

where \(\nabla^2 f = f'' + \frac{1}{r}f'\), \('\) being derivation with respect to \(r\).

For the metric defined by equations (22) and (27) we have,

\[
R_{\mu\nu} = 0, \quad R_{rr} = -\frac{1}{2}\nabla^2 \psi, \quad R_{\phi\phi} = -\frac{r^2}{2} \nabla^2 \psi, \quad R = -\frac{1}{\psi} \nabla^2 \psi
\]  
(29)

From (20) and from the fact that \(G^{AB} \frac{\partial L_{\text{matter}}}{\partial G^{AB}} = L_{\text{matter}}\), we get that

\[
R = \kappa L_{\text{matter}}
\]  
(30)

equation that gives (using the representation (28) for the two dimensional delta function and eqs. (5) and (12) for the matter lagrangians of the gauge fields and 3 + 1 brane)
\[-\frac{1}{\psi} \nabla^2 \psi = s \kappa \sqrt{2} B_0 + \frac{\kappa T}{2 \pi \psi} \nabla^2 (\ln r) \tag{31}\]

This equation, which appears also in the context of 2 + 1 dimensional gravity, has the solution\(^1\) (the different constants in (31) having a different meaning in the corresponding eq. in Ref. 14 of course)

\[
\psi = \frac{4 \alpha^2 b^2}{r^2\left[(\frac{r}{r_0})^\alpha + (\frac{r}{r_0})^{-\alpha}\right]^2} \tag{32}
\]

where \(r_0\) is an arbitrary parameter and

\[
\alpha = 1 - \frac{\kappa T}{4 \pi}, \quad b^2 = \frac{\sqrt{2}}{s \kappa B_0} \tag{33}\]

Such a metric can be transformed into the form (where \(r_0\) goes away),

\[
\gamma_{ij}(x^k) dx^i dx^j = b^2 (d \theta^2 + \alpha^2 \sin^2 \theta d \phi^2) \tag{34}
\]

where \(\phi\) ranges from 0 to 2\(\pi\), or equivalently

\[
\gamma_{ij}(x^k) dx^i dx^j = b^2 (d \theta^2 + \sin^2 \theta d \tilde{\phi}^2) \tag{35}
\]

where \(\tilde{\phi}\) now ranges from 0 to 2\(\pi\alpha < 2\pi\). The effect of the branes (at the opposite poles of the extra dimensional sphere and having the same tension) is just to produce a deficit angle and changing the shape of the extra dimensional into a football like sphere.

It should be pointed out that the solutions with \(F^{AB} F_{AB} = \text{constant}\) (used here) are not the most general solutions. In four dimensions the square root gauge theory allows string like solutions\(^9\). The generalizations of the string solutions in the square root gauge theory in six dimensions are 3 + 1 brane solutions, as we will see elsewhere, so that in fact the 3 + 1 branes do not have to be added to \(S_{\text{bulk}}\) defined in eq.(5).

A complete study of the solutions of the square root gauge theory plus gravity will be done elsewhere, but still, as we have seen in section 2, which does not depend on the details of the solutions, we have that as long as \(F_{AB}\) takes expectation values in the extra dimensions only, does not matter in which precise way, only the extra dimensions curve and the four dimensional space remains flat.
4 Discussion and Conclusions

We have seen that it is possible to construct a totally conformally invariant six dimensional brane world scenario by means of the introduction of a new measure of integration in the action. This new measure of integration depends of degrees of freedom independent of the metric. The model includes gravity, gauge fields and 3 + 1 branes.

We have also seen that if the gauge fields take expectation values in the extra dimensions and if the 3 + 1 branes are orthogonal to the extra dimensions, we get that the four dimensional subspace can remain flat without need of any fine tuning, only the extra dimensions are curved. For this it is essential that $\sqrt{-G}$ did not enter in the measure of integration of the action ($\Phi = \sqrt{-G}$ can be choosen as a particular ”gauge” however).

An explicit solution with the ”football ” like compactification for the extra dimensions $^{12,13}$ and flat four dimensional space which exemplifies the above has been displayed. More general solutions will be studied elsewhere.

Of course to describe the real Universe one has to go beyond this model and introduce spontaneous symmetry breaking of scale invariance, a small vacuum energy to describe the presently accelerated Universe, as has been done in the context of alternative measure theories in four dimensions $^7$. This six dimensional model represents nevertheless significant progress, since it allows us to see the applicability of the ”modified measure” approach to braneworld scenarios and therefore for a starting point of a new direction in the research of these kind of theories.

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