Schur-Weyl duality as an instrument of Gauge-String duality

Sanjaye Ramgoolam

Centre for Research in String Theory, Department of Physics,
Queen Mary, University of London,
Mile End Road, London E1 4NS, UK

Abstract. A class of mathematical dualities have played a central role in mapping states in gauge theory to states in the spacetime string theory dual. This includes the classical Schur-Weyl duality between symmetric groups and Unitary groups, as well as generalisations involving Brauer and Hecke algebras. The physical string dualities involved include examples from the AdS/CFT correspondence as well as the string dual of two-dimensional Yang Mills.

Keywords: AdS/CFT, Giant gravitons, Matrix Models, Yang-Mills theory, Schur-Weyl duality

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INTRODUCTION

The AdS/CFT correspondence [1, 2, 3] between string theory on $AdS_5 \times S^5$ and $N = 4$ SYM with $U(N)$ gauge group in four dimensions has provided a remarkably rich setting where gravitons as well as classical brane and spacetime geometries can be seen as emergent phenomena from local operators in the quantum gauge theory. Early work showed how single-particle Kaluza-Klein gravitons in spacetime correspond to gauge invariant single trace operators in the dual gauge theory. The computation of correlation functions in gauge theory and on the gravity side gave evidence for this map [4]. Some qualitatively new features of finite $N$ were highlighted as the stringy exclusion principle [5, 6, 7]. A geometrical realisation of the stringy exclusion principle was discovered in the properties of super-symmetric rotating branes or giant gravitons [8]. These $S^3$-shaped branes in the $S^5$ of spacetime, grow in size as the angular momentum $L$ is increased. This increase in size was related [8] to the growth in transverse size of stringy objects due to the stringy uncertainty principle (see [10] for a review). When the angular momentum $L$ reaches $N$, the size of the worldvolume $S^3$ reaches the size of the background $S^5$. The angular momentum cannot grow any further and the classical SUSY brane solution ceases to exist. Other solutions were found [11] which are also $S^3$-shaped but live in the $AdS_5$ and carry the same angular momentum as the $S^5$ or sphere-giants. These new giant graviton solutions are called dual giants or AdS-giants. These AdS-giants have no upper bound on the size of the $S^3$-worldvolume and no upper bound on the angular momentum.

An attempt to find local gauge invariant duals of the giant gravitons was initiated in [12], where sub-determinant operators were proposed as gauge theory duals of single sphere giants. This lead to the classification of half-BPS multi-trace operators and a more complete map to giant gravitons in [13]. The gauge invariant operators were described in terms of a Young diagram basis, which was shown to have diagonal two-point
functions. This basis lead to a proposal of multi-column Young diagrams as multi-sphere giants, and multi-row Young diagrams as multi-AdS giants. Three-point functions were computed in terms of group-theoretic Littelwood-Richardson coefficients. Subsequently the map of giant gravitons to Young diagrams was given elegant confirmation by constructing open strings attached to multiple-giants (see [14, 15] and refs. therein). The Young diagram picture of half-BPS operators constructed from a complex matrix lead to a system of $N$ free fermions in a harmonic oscillator [13, 16, 17]. Recent work on the classification of half-BPS supergravity solutions discovered the free fermion structure in spacetime, in the guise of black-white colourings of a plane. These colourings provide boundary conditions for non-singular solutions in spacetime and reflect the occupied and unoccupied regions of free fermion phase space [18]. Strings in certain BMN limits of LLM backgrounds have also been studied from the gauge theory point of view [19].

This talk is based mainly on [26, 27], where two generalisations of [13] are considered. In [26] we consider eighth-BPS operators at zero coupling, with a view to finding the gauge invariant operators dual to eighth-BPS giant gravitons. In [27] we consider a non-supersymmetric sector with a view to understanding systems of branes and anti-branes. More complete lists of relevant references are given in these papers. In this talk I would like to highlight an important mathematical equivalence, Schur-Weyl duality, as a recurrent principle in the map between gauge theory states and stringy spacetime states [13, 26, 27]. SW duality had previously played an instrumental role in the string theory of two dimensional Yang Mills [20, 21].

**HALF BPS : ONE COMPLEX MATRIX**

Half-BPS operators, which have scaling dimension $\Delta$ equal to $U(1)$ charge $J$, are constructed from a complex matrix $X = \Phi_1 + i\Phi_2$. $\Phi_1, \Phi_2$ are two of the six hermitian matrices, transforming in the adjoint of the $U(N)$ gauge group of $\mathcal{N} = 4$ supersymmetric gauge theory. The matrix $X$ defines a linear map from an $N$-dimensional vector space $V$ to $V$. It is useful to consider $X = X \otimes X \cdots \otimes X$ as a map from $V^\otimes n$ to $V^\otimes n$. Multi-trace gauge invariant operators can be constructed by composing $X$ with permutations. Take a simple case of $n = 2$. Choose a basis $e_i$ for $V$.

$$tr_{V^\otimes 2}(X \otimes X) \equiv tr_2(X \otimes X) = \langle e^{i_1} \otimes e^{i_2} | (X \otimes X) | e_{i_1} \otimes e_{i_2} \rangle$$

$$= X^{i_1}_{i_1} X^{i_2}_{i_2} = (trX)(trX) \quad (1)$$

Now consider $tr_{V^\otimes 2}((X \otimes X)\sigma)$. The permutation $\sigma$ is an element of the symmetric group of permutations of 2 elements and acts by permuting the two factors of $V \otimes V$. Explicitly,

$$\sigma | e_{i_1} \otimes e_{i_2} \rangle = | e_{i_2} \otimes e_{i_1} \rangle \quad (2)$$

The trace of $(X \otimes X)\sigma$ is

$$tr_2((X \otimes X)\sigma) = \langle e^{i_1} \otimes e^{i_2} | (X \otimes X)\sigma | e_{i_1} \otimes e_{i_2} \rangle$$
\[ = \langle e^{i1} \otimes e^{i2} | X \otimes X | e^{i2} \otimes e^{i1} \rangle \]
\[ = X^{i1}_{i2} X^{i2}_{i1} = \text{tr}(X^2) \quad (3) \]

In general, \( tr_n(X\sigma) \) depends on the cycle structure of the permutation \( \sigma \). Any permutation can be decomposed into a product of cyclic permutations, e.g., the permutation which takes \((1, 2, 3, 4, 5)\) to \((3, 1, 2, 5, 4)\) is the product of cycles \((123)(45)\). Let \( C_i(\sigma) \) be the number of cycles of length \( i \) in \( \sigma \). Then

\[
tr_n(X\sigma) = X^{i_{\sigma(1)}^{1}} \cdots X^{i_{\sigma(n)}^{n}} = \prod_{i=1}^{n} (\text{tr}X^{i_{\sigma(i)}})^{C_i(\sigma)} \quad (4)
\]

In the gauge theory, \( X \) transforms in the adjoint of \( U(N) \). This adjoint action comes from the transformation of \( V \) as the fundamental of \( U(N) \).

\[
U | e_i \rangle = U^j_i | e_j \rangle \quad (5)
\]

The action on \( V^\otimes n \) is given by

\[
U | e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \rangle = U^{j_1}_{i_1} \cdots U^{j_n}_{i_n} | e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n} \rangle \quad (6)
\]

It is an easy exercise to check that the actions of \( U(N) \) and \( S_n \) on \( V^\otimes n \) commute with each other. We can also easily check that the action of the Lie algebra \( u(N) \) generated by the elementary matrices \( E_{ij} \) commuting with \( U(N) \) is given by the action of the group algebra of \( S_n \). Conversely, the commutant of \( S_n \) is the enveloping algebra of \( u(N) \). This has a consequence, via the double commutant theorem, also called the double centraliser theorem (see e.g., [22], [23]), that the decomposition of \( V^\otimes n \) in terms of \( U(N) \times S_n \) is very simple

\[
V^\otimes n = \bigoplus_R V^U_R \otimes V^S_R \quad (7)
\]

Here \( R \) runs over Young diagrams with first column of length no greater than \( N \). The vector space \( V^U_R \) is the irreducible representation (irrep.) of \( U(N) \) corresponding to Young diagram \( R \) and \( V^S_R \) is the irrep. of \( S_n \) corresponding to Young diagram \( R \).

Several powerful consequences follow from (7), when it is combined with standard group theory properties, such as orthogonality of characters. These are used along with the free field correlators

\[
\langle X^i_j (X^{\dagger})^k_l \rangle = \delta^i_j \delta^k_l \quad (8)
\]

to show [13] that the Schur-basis of matrix operators

\[
\chi_R(X) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) tr_n(\sigma X) \quad (9)
\]

satisfies the orthogonality

\[
\langle \chi_R(X) \chi_S(X^{\dagger}) \rangle = \delta_{RS} f_R \quad (10)
\]
The normalisation factor $f_R$ is

$$f_R = \frac{n! \text{Dim} R}{d_R}$$  \hspace{1cm} (11)

where $\text{Dim} R$ is the dimension of $V_{R}^{U(N)}$ and $d_R$ is the dimension of $V_{R}^{\mathcal{S}_n}$. We have dropped the trivial spacetime dependence of these correlators. Local operators in CFT correspond to states in CFT. Duality maps states in the CFT to states in spacetime. The column lengths of the Young diagrams map to angular momenta of sphere giants. The row lengths map to AdS giants. These descriptions are valid in different regimes, where the semiclassical descriptions in terms of S and AdS giants respectively are appropriate. The fact that the S-giants have angular momentum cutoff at $N$ whereas the AdS-giants have unbounded angular momentum is naturally explained by the Young diagram description [13]. This is compatible with the backreacted geometries of LLM and with the description of open strings attached to multi-giants. The normalisation factor (11) encodes the stringy exclusion principle simply in that it vanishes for Young diagrams with column lengths greater than $N$. Three-point functions are

$$\langle \chi_{R_1}(X) \chi_{R_2}(X) \chi_{R_3}(X^\dagger) \rangle = g(R_1, R_2; R_3) f_{R_3}$$  \hspace{1cm} (12)

where $g(R_1, R_2; R_3)$ is the Littlewood-Richardson coefficient for the coupling of Young diagrams $R_1, R_2$ with $n_1, n_2$ boxes into Young diagram $R_3$ of $n_3 = n_1 + n_2$ boxes.

For the application of Schur-Weyl duality (7) in the string theory of two dimensional $U(N)$ Yang Mills see section 4 of [21]. In that context cycle structures of permutations correspond to winding numbers of multi-string states, whereas free fermions are related to the irreducible representation basis of observables in the $U(N)$ gauge theory. The counting of string worldsheet maps to spacetime, described according to classic theorems on branched covers in mathematics, by appropriate sums over permutations, is recovered from the large $N$ expansion of the gauge theory using the fundamental equation (7). In [24] the projectors for the different Young diagrams in (7), along with simple diagrammatic techniques for manipulations in tensor spaces (for any rank $n$), are used to study the factorisation equations of CFT in the half-BPS context. The ref. [25] contains a review of the underlying operator-state correspondence of CFT and applications of the factorisation equations to the probabilistic interpretation of correlators. Useful mathematical references on Schur-Weyl duality are [29, 28]. Schur-Weyl duality is an application of the double-commutant theorem. For a review of this and several useful related theorems see for example section 1 of [30] and [22, 23].

**EIGHTH-BPS**

The eighth-BPS operators at zero coupling are constructed from holomorphic combinations of three complex matrices $X_1, X_2, X_3$. These are charged respectively \((1,0,0), (0,1,0), (0,0,1)\) under the $U(1) \times U(1) \times U(1)$ subgroup of $SO(6)$ global symmetry group of $\mathcal{N} = 4$ SYM theory. The holomorphic subspace is preserved by the $U(3)$ subgroup of $SO(6)$. The general holomorphic covariant operators are

$$X_{a_1 j_1}^{i_1} X_{a_2 j_2}^{i_2} \cdots X_{a_n j_n}^{i_n}$$  \hspace{1cm} (13)
The indices $a_1..a_n$ each take values from 1 to 3 in the fundamental of $U(3)$. The discussion generalizes to $U(M)$ where $a_i$ are in the fundamental of $U(M)$. The $i, j$ indices take values from 1 to $N$. The covariant operators can be written as

$$X_{a_1} \otimes X_{a_2} \otimes \cdots \otimes X_{a_n}$$

and viewed as maps from $V^\otimes n$ to $V^\otimes n$. The operators in (14) transform as the $V^\otimes n$ of $U(M)$ where $V_M$ is the fundamental $M$-dimensional representation of $U(M)$.

In solving the problem of finding a diagonal basis of gauge invariant operators constructed from (14) we encounter two roles for Schur-Weyl duality. As in the half-BPS case of section 2, we use

$$V^\otimes n = \bigoplus_R V^{U(N)}_R \otimes V^{S_n}_R$$

(15)

$R$ runs over Young diagrams with no more than $N$ rows. Since we also want to organise the operators according to their $U(M)$ transformation properties we also have

$$V^\otimes n_M = \bigoplus_\Lambda V^{U(M)}_\Lambda \otimes V^{S_n}_\Lambda$$

(16)

$\Lambda$ runs over Young diagrams with no more than $M$ rows.

The main result

Employing the decompositions (15) (16) the gauge invariant operators in a diagonal basis are labelled by $R$, a Young diagram with $n$ boxes associated to the $U(N)$ gauge symmetry ; by $\Lambda$ another Young diagram with $n$ boxes associated to the $U(M)$ global symmetry, along with some additional labels explained below. Consider the class of operators with $\mu_1$ copies of $X_1$, $\mu_2$ copies of $X_2$, $\mu_3$ copies of $X_3$. Clearly at zero coupling, using the free field correlator

$$\langle (X_a)^i_j (X_b)^k_l \rangle = \delta_{ai} \delta_{bj} \delta_{kl}$$

(17)

operators characterised by different choices of $\mu_1, \mu_2, \mu_3$ are orthogonal to each other. Hence the diagonalisation problem in the eighth-BPS sector can be solved for each fixed $\mu = (\mu_1, \mu_2, \mu_3)$ with $\mu_1 + \mu_2 + \mu_3 = n$. Let

$$X^\mu = X_1^{\otimes \mu_1} \otimes \cdots \otimes X_M^{\otimes \mu_M}$$

(18)

The following operators provide a diagonal basis for two-point functions, hence for the metric on operators defined using two-point functions

$$\mathcal{O}^{\Lambda, R}_{\beta, \tau} = \frac{1}{n!} \sum_\alpha B_{j\beta} S_{j \otimes R}^{\tau, \Lambda} D_{p q}^{R} (\alpha) \ tr(\alpha X^\mu)$$

The label $\beta$ runs over the number of times the trivial representation of $H_\mu \equiv S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_M}$ is contained in $S_n$, where $n = \mu_1 + \cdots + \mu_M$. The label $\tau$ runs over the number
of times \( \Lambda \) appears in the \( S_n \) Clebsch-Gordan decomposition of \( R \otimes R \). \( D^R_{pq}(\alpha) \) are the matrix elements of the permutation \( \alpha \) in an orthogonal basis. \( S^\Lambda_{pq} \) is a Clebsch-Gordan coefficient for the coupling of \( R \otimes R \) to \( \Lambda \). The \( B \)-factor \( B_{j\beta} \) is a branching coefficient, giving a change of basis in the subspace of the irrep. \( \Lambda \) of \( S_n \) invariant under the subgroup \( H_\mu \). The counting of the operators is easily written down in terms of Clebsch-Gordan multiplicities \( C(R, R, \Lambda) \) and Littlewood-Richardson multiplicities for coupling horizontal Young diagrams of lengths \( \mu_1, \mu_2, \ldots, \mu_M \) into the Young diagram \( \Lambda \). This agrees with counting derived by expanding partition functions using properties of characters [31].

The \( O^{\Lambda}_{\beta\tau} \) can be proved to have orthogonal 2-point functions [26]

\[
\langle O^{\Lambda_1(1),R_1}_{\beta_1,\tau_1} O^{\Lambda_2(2),R_2}_{\beta_2,\tau_2} \rangle = \delta^{(1)} \delta^{(2)} \delta^{\Lambda_1 \Lambda_2} \delta^{R_1 R_2} \delta^{\beta_1 \beta_2} \delta^{\tau_1 \tau_2} \frac{|H_{\mu}| \text{Dim}R_1}{d_{R_1}^2}
\]

In this way, the problem of computing a diagonal basis for the multi-matrix operators has been reduced to the computation of standard \( S_n \) group theory quantities which are available for example in [32]. Likewise 3-point (and higher-point) functions can be computed in terms of this type of group theory data ([26]).

**Weak coupling and the chiral ring**

The above basis (19) solves the problem of diagonalising the two-point functions at zero coupling. In order to compare to eighth-BPS giant gravitons, we need to consider the strong coupling spectrum. The spectrum of eighth-BPS operators at weak coupling can be obtained by finding the subspace of the zero coupling operators which are orthogonal to operators such as \( tr([X_1,X_2][X_1,X_2]) \) which are descendants. The zero-coupling diagonalisation allows us to write some neat formulae for the inverse metric in the trace basis. This gives some useful information [26] on the problem of finding a diagonal basis of eighth-BPS operators at weak coupling, but it remains an open problem. Further work will be interesting as it will allow comparison with strong coupling discussions [33, 34].

**THE \( X, X^\dagger \) SECTOR: BRANES AND ANTI-BRANES**

The operators \( \chi_R(X) \) of section 2 correspond to states with \( \Delta = J \) (scaling dimension equals angular momentum) and are giant gravitons. The \( \chi_R(X^\dagger) \) have \( L_0 = -J \). They correspond to branes moving in the opposite direction. The time reversed brane solutions are solutions of the anti-brane action. Hence \( \chi_R(X^\dagger) \) correspond to anti-giant gravitons. Operators constructed from both \( X, X^\dagger \) correspond to composite systems built from branes and anti-branes. The consideration of diagonal bases for these composite operators can be done using a generalisation of Schur Weyl duality involving Brauer
algebras [27], for the case where $U(N)$ is acting on tensor products of the fundamental and anti-fundamental, i.e $V^\otimes m \otimes \bar{V}^\otimes n$

We have seen that $S_n$ is the commutant of $U(N)$ (or $GL(N)$) acting on $V^\otimes n$ leading to (7). $S_m \times S_n$ is contained in commutant of $U(N)$ (or $GL(N)$) acting in $V^\otimes m \otimes \bar{V}^\otimes n$. But we also need contractions, which along with the permutations, generate the Brauer algebra $B_N(m, n)$. Hence Schur-Weyl duality, in this case (see for example [30]), states that

$$V^\otimes m \otimes \bar{V}^\otimes n = \bigoplus_\gamma V^U(N) \otimes V^B_N(m, n)$$

(19)

It gives the decomposition of the tensor product $V^\otimes m \otimes \bar{V}^\otimes n$ in terms of irreps of $U(N)$ and $B_N(m, n)$. $\gamma$ runs over sets of integers $(\gamma_1, \gamma_2, \cdots, \gamma_N)$ obeying $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_N$. The set of positive integers defines $\gamma_+$ which is a partition of $m - k$ while the negative integers define a partition $\gamma_-$. Here $k$ is an integer lying between 0 and $\min(m, n)$. Equivalently $\gamma_+$ determines a Young diagram with $m - k$ boxes, $\gamma_-$ one of $n - k$ boxes. A choice of $\gamma$ is equivalent to a choice of $(k, \gamma_+, \gamma_-)$. If we write $\gamma_+$ as a Young diagram, with row lengths equal to the parts in the partition, $c_1(\gamma_+)$ is defined as the length of the first column. It follows from the above definitions that $c_1(\gamma_+) + c_1(\gamma_-) \leq N$.

The Brauer algebra contains a class of elements,

$$Q^\gamma_{A,ij}$$

(20)

called symmetric branching operators in [27], which have the properties

$$Q^\gamma_{A,ij}Q^\delta_{B,kl} = \delta_{\gamma_1 \gamma_2} \delta_{AB} \delta_{jk} Q^\gamma_{A,il}$$

(21)

and

$$h Q^\gamma_{A,ij} h^{-1} = Q^\gamma_{A,ij}$$

(22)

for $h \in S_m \times S_n$. The index $A$ consists of a pair of labels $(\alpha, \beta)$ for irreps. of $S_m \times S_n$. The indices $i, j$ each run over the multiplicity of the irrep. $A$ in the $S_m \times S_n$ decomposition of the irrep. $\gamma$ of $B_N(m, n)$.

The operators $tr_{m,n}(\Sigma(Q^\gamma_{A,ij})(X \otimes X^\dagger))$ diagonalise the two-point functions.

$$\langle tr_{m,n}(\Sigma(Q^\gamma_{A_2, j_2})(X \otimes X^\dagger))tr_{m,n}(\Sigma(Q^\gamma_{A_1, i_1})(X \otimes X^\dagger)) \rangle = m! n! \delta_{\gamma_1 \gamma_2} \delta_{A_1 A_2} \delta_{i_1 i_2} \delta_{j_1 j_2} d_{A_1} \text{ Dim } \gamma_1$$

(23)

The trace is in $V^\otimes m \otimes \bar{V}^\otimes n$, $X$ denotes the $m$-fold tensor product $X \otimes \cdots \otimes X$, and $X^\dagger$ is the $n$-fold tensor product $X^\dagger \otimes \cdots \otimes X^\dagger$. $\Sigma$ is a map from $B_N(m, n)$ to $S_{m+n}$ defined in [27]. Dim $\gamma_1$ is the dimension of the $U(N)$ irrep. labelled by $\gamma_1$.

A special class of operators are those corresponding to $k = 0$. In these cases, $\alpha = \gamma_+, \beta = \gamma_-$, the reduction from Brauer to $S_m \times S_n$ gives a unique irrep with multiplicity 1. In this $k = 0$ case the $Q$-operators become projection operators, denoted as $P_{RS}$ (using the relabelling $\alpha = R, \beta = S$ to match the notation of two dimensional Yang Mills [20]) which are related to the coupled characters $\chi_{RS}(U)$ in two dimensional Yang Mills.
The operators $tr_{m,n}(\Sigma(P_{RS}(X \otimes X^\dagger)))$ are proposed as ground states of a composite system made from a giant graviton corresponding to Young diagram $R$ and an anti-giant corresponding to Young diagram $S$. Higher $k$ operators are interpreted as excited states of brane-anti-branes. The stringy exclusion principle for individual giant gravitons imposes the condition $c_1(R) \leq N$ and $c_1(S) \leq N$, which says that Young diagrams for $U(N)$ cannot have more than $N$ boxes in the first column. The surprising lesson from the Brauer algebra description of brane-anti-branes and their excited states is that there is no ground state corresponding to a giant-anti-giant if the condition $c_1(R) + c_1(S) \leq N$ is violated.

The map $\Sigma$ appearing in the formula for the operators in (23) is also used in the explicit construction of Brauer algebra projectors in [27]. As a corollary of that construction, there is a new formula for the coupled dimension

$$
\frac{d_R^2 d_S^2}{\text{Dim}RS} = \frac{m!^2 n!^2}{(m+n)!^2} \sum_T \frac{d_T^2}{\text{Dim}T} g(R;S;T) \tag{24}
$$

where $R, S$ are Young diagrams with $m, n$ boxes respectively, $T$ has $m + n$ boxes and $g(R;S;T)$ are Littlewood-Richardson coefficients. This formula is developed further to derive a new formulation of the nonchiral large $N$ expansion of 2d Yang Mills [35].

**SUMMARY AND OUTLOOK**

Schur-Weyl duality has provided a powerful set of tools for the mapping of gauge theory states to strings or branes in spacetime or large deformations of spacetime. The simplest and most thoroughly understood cases are those of half-BPS operators in $\mathcal{N} = 4$ super Yang Mills theory in four dimensions. Recent work has been undertaken on the quarter and eighth BPS sectors [26], as well as a non-supersymmetric sector [27], providing bases of multi-matrix operators which diagonalise the free field two-point functions in the field theory. Clebsch-Gordan coefficients of symmetric groups and Branching coefficients of Brauer algebras have appeared in the solutions of these diagonalisation problems.

Some other recent work related to this theme is described here. The computation of the one-loop dilatation operator in the $SU(2)$ sector [36] using the basis of [26] was undertaken [37]. A closely related diagonalisation method, inspired by earlier work on the construction of strings attached to branes, has been proposed [38]. The Brauer algebras used in the diagonalisation problem of the $X, X^\dagger$ sector have been used to simplify the complete large $N$ expansion of two dimensional Yang Mills, giving it a new holomorphic string interpretation [35]. The application of Schur-Weyl duality in the case of $q$-deformed two dimensional Yang Mills was done in [39] where Hecke algebras, which are $q$-deformations of symmetric group algebras and Schur-Weyl duals of $q$-deformed $U(N)$, provide $q$-deformations of branched cover counting problems. The extension of work on diagonalising correlators in four dimensions from $U(N)$ to $SU(N)$ has produced interesting results (section 10 of [24] and [40, 41]).

Interesting open questions include the understanding of finite $N$ cutoffs (stringy exclusion principle) on gauge invariant operators in less supersymmetric contexts, and in
non-supersymmetric contexts in terms of spacetime physics of branes and black holes. We would like all the cutoffs on matrix operators in the gauge theory to have simple spacetime stringy interpretations like the ones for sphere giant gravitons. It has been suggested recently that the stringy exclusion principle is related to black hole complementarity [42]. It is an interesting problem to develop this idea using detailed information about the counting and construction of operators in gauge theory at finite $N$. Since the diagonal bases naturally encode the finite $N$ cutoffs in the normalisations (e.g. eq. (11)) of two-point functions (even at zero coupling) they should be useful tools in this direction. A clear interpretation of three-point functions such as (12) in terms of giant graviton moduli spaces in spacetime would also be very desirable, perhaps using known connections between Littlewood-Richardson coefficients and the cohomology of Grassmanians [43].

Schur-Weyl duality has been a suprisingly effective technical tool in gauge-string duality, capturing crucial aspects of the map between gauge theory states and spacetime string theory states, both for two dimensional and four dimensional gauge theory. It is undoubtedly going to continue to play this role and provide valuable information on many interesting physical questions on gauge theory, especially in relation to its stringy spacetime dual. It is natural to wonder if an appropriately enriched version of Schur-Weyl duality might actually give a complete mathematical expression of the background independent content of gauge string duality.

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