Quantum Deformations of Einstein’s Relativistic Symmetries

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Abstract. We shall outline two ways of introducing the modification of Einstein’s relativistic symmetries of special relativity theory - the Poincaré symmetries. The most complete way of introducing the modifications is via the noncocommutative Hopf-algebraic structure describing quantum symmetries. Two types of quantum relativistic symmetries are described, one with constant commutator of quantum Minkowski space coordinates ($\theta_{\mu\nu}$-deformation) and second with Lie-algebraic structure of quantum space-time, introducing so-called $\kappa$-deformation. The third fundamental constant of Nature - fundamental mass $\kappa$ or length $\lambda$ - appears naturally in proposed quantum relativistic symmetry scheme. The deformed Minkowski space is described as the representation space (Hopf-module) of deformed Poincaré algebra. Some possible perspectives of quantum-deformed relativistic symmetries will be outlined.

Keywords: relativistic symmetries, quantum groups, quantum symmetries

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INTRODUCTION

It is well-known that the nonrelativistic symmetries are described by the Galilei group, with the Galilean boosts relating dynamically equivalent frames which move with relative constant velocity $\vec{v} = (v_1, v_2, v_3)$

$$x'_i = x_i + v_i t', \quad t' = t.$$  (1)

The velocity values $v = |\vec{v}|$ are not bounded and three boost generators $K_i$ ($i = 1, 2, 3$) commute

$$[K_i, K_j] = 0,$$  (2)

Einstein’s equivalence of relativistic frames is described by the following modification of (1) (we choose for simplicity $\vec{v} = (0, 0, v)$,

$$x'_1 = x_1 \quad x'_2 = x_2,$$

$$x'_3 = \cosh \alpha x_3 + \sinh \alpha x_0 \quad x_0 = ct,$$

$$x'_0 = \sinh \alpha x_3 + \cosh \alpha x_0 \quad \tan \alpha = v/c.$$  (3)

The Lorentz boosts $N_i$ generating pseudo-orthogonal rotations in the Lobachevsky planes $(x_i, x_0)$ are described by the deformation of the Abelian algebra (2)

$$[N_i, N_j] = \frac{1}{c^2} M_{ij} = \frac{1}{c^2} \varepsilon_{ijk} M_k,$$  (4)

where $M_{ij} = -M_{ji}$ generate the space rotations in $(i, j)$ plane ($i, j = 1, 2, 3$).

The relativistic transformation (3) should be applied to moving frames if the relative velocity ration $\frac{v}{c}$ is not negligible. Further, from the invariance of Maxwell equations under Poincaré symmetries follows that $c$ should be interpreted as the velocity of electromagnetic waves (light velocity).

The Poincaré symmetries of special relativity theory are described by 10 generators $I_A = (M_{\mu\nu}, P_\mu; \mu, \nu = 0, 1, 2, 3)$ which satisfy the Poincaré Lie algebra (to compare with (4) we should put $M_{00} = c N_0$)

$$[M_{\mu\nu}, M_{\rho\tau}] = \eta_{\mu\rho} M_{\nu\tau} - \eta_{\nu\rho} M_{\mu\tau} + \eta_{\nu\tau} M_{\mu\rho} - \eta_{\mu\tau} M_{\nu\rho},$$
\[ [M_{\mu\nu}, P_{\rho}] = \eta_{\mu\rho} P_{\nu} - \eta_{\nu\rho} P_{\mu}, \]
\[ [P_{\mu}, P_{\nu}] = 0. \]  
\hfill (5)

The Poincaré algebra was considered for almost a century after Einstein’s discovery in 1905 as quite uncontested way of describing the equivalent space-time frames in relativistic elementary particle physics, however under the assumption that the gravitational effects are negligible. Further in consistency with the classical Poincaré group structure (Abelian translation group) it was assumed that the relativistic space-time is described by classical commuting Minkowski space-time coordinates \( x_{\mu} \), i.e.

\[ [x_{\mu}, x_{\nu}] = 0. \]  
\hfill (6)

In last period however, the status of the relations (5) and (6) as describing ultimate geometric description of microworld physics was challenged.

Let us recall here some arguments leading to the modification of Einstein’s relativistic symmetries and classical nature of space-time.

i) Due to Einstein’s general relativity theory the space-time manifold with its geometrical structure is a dynamical entity, with its metric (in quantum theory) undergoing the quantum fluctuations. In particular any measurement of position due to Heisenberg uncertainty relations leads to disturbed values of the energy density, what affects through Einstein’s GR equations the values of gravitational field. If we calculate the fluctuations of gravitational field generated by the energy needed to measure the distance with accuracy \( \Delta x_{\mu} \) one arrives at the conclusion, that it is not possible to measure distances which are smaller than the Planck length \( l_{P} \) (\( l_{P} \approx 10^{-33}\text{cm} \)). We see therefore that due to quantum gravity effects the space-time from operational point of view ceases to be a continuous manifold, but becomes a discrete set of small cells with the Planck length size \( l_{P} \). Algebraically the noncontinuous structure of "quantum" Minkowski space can be expressed by the Dopplicher-Haag-Roberts (DHR) relation [1].

\[ [\hat{x}_{\mu}, \hat{x}_{\nu}] = \frac{i}{l_{P}^{2}} \theta_{\mu\nu}, \quad \theta_{\mu\nu} = -\theta_{\nu\mu}, \]  
\hfill (7)

where in simplified version of the model the tensor \( \theta_{\mu\nu} \) is central, i.e. one can postulate that it takes numerical values.

ii) Other argument comes from the brane world scenario, in which we assume that the space-time manifold is described by the \( D \)-brane coordinates [2]. The \( D \)-branes describe the location of the end points of strings (in general of fundamental \( p \)-branes). If we accept the view that the fundamental super-strings describing the most elementary objects in Universe are ten-dimensional, the \( D \)-branes are coupled not only to the gravitational background \( g_{\mu\nu}(x) \) but as well to the nonvanishing antisymmetric tensor field \( B_{\mu\nu}(x) \) (here \( \mu, \nu = 0, 1, \ldots, 9 \)). It was shown using canonical quantization techniques (see e.g. [3, 4]) that the quantized end point \( \hat{x}_{\mu} \) of the open string after first quantization do satisfy the following noncommutativity relation

\[ [\hat{x}_{\mu}, \hat{x}_{\nu}] = \theta_{\mu\nu}(B(\hat{x})) , \]  
\hfill (8)

i.e. \( \theta_{\mu\nu} \) is a local function of \( B_{\mu\nu}(x) \). If we live on \( D \)-brane, the relation (8) describes the noncommutative \( D = 10 \) space-time algebra, which after dimensional reduction \( D = 10 \to D = 4 \) implies the quantum nature of our four-dimensional Minkowski space.

iii) Third source of the space-time noncommutativity is provided by the spin degrees of freedom. Already in seventies it was shown [5] that the space-time coordinates of superparticles satisfy the following equal time (ET) relation

\[ [x_{\mu}, x_{\nu}] = \frac{i}{m^{2}} s_{\mu\nu}, \]  
\hfill (9)

where the relativistic spin operator \( s_{\mu\nu} \) satisfies Lorentz algebra relations and can be expressed in terms of Pauli-Lubanski vector \( W_{\mu} \). Independently, the formula (9) was confirmed in twistor theory, with space-time variables as composites of primary twistor coordinates [6]. It should be added that old Snyder idea of noncommutative space-time (see e.g. [7]) also relates the noncommutativity with nonvanishing angular momentum.

In this short talk we would like to recall the quantum deformation of Einstein’s symmetry scheme which is consistent with simple models of noncommutative space-time. Because the space-time coordinates can be also described by translation sector of Poincaré group, in order to describe noncommutative space-time one should consider new algebraic relativistic symmetries with noncommutative symmetry parameters. We shall consider below in Sect. 2 and Sect. 3 two types of such quantum symmetries: the canonical \( \theta_{\mu\nu} \)-deformation Poincaré symmetries, implying the
relation (7) with the constant values of tensor $\theta_{\mu\nu}$, and the so-called $\kappa$-deformed relativistic symmetries. These two
types of deformations introduce new fundamental mass parameter $\kappa$ which enters the respective noncommutativity
relations of Minkowski space coordinates. In Sect. 4 we shall mention possible prospects of deformed relativistic symmetries.

It should be pointed out that the validity of quantum relativistic symmetries can be also interpreted as the particular
violation of classical Poincaré symmetries [8]. For example if we consider the modification of the classical Poincaré
mass Casimir, which follows from the $\kappa$-deformation of Poincaré algebra [9, 10], we obtain

$$C_2 = p_0^2 - \vec{p}^2 \to C_2^\kappa = 4\kappa^2 \sin^2 \frac{p_0}{2\kappa} - \vec{p}^2 = p_0^2 - \vec{p}^2 - \frac{1}{\kappa} p_0^4 + O(\frac{1}{\kappa^2}), \quad (10)$$

The formula (10) for $C_2^\kappa$ can be interpreted in two ways

i) As a result of a particular violation of classical Lorentz invariance by $\frac{1}{\kappa}$ terms

ii) The indication that the classical Lorentz invariance and the Lorentz transformations should be modified in a way

which leads to the modified mass Casimir $C_2^\kappa$ as a new invariant quantity.

The advantage of second point of view lies in quite strong limitations on the ways in which the classical Einstein
symmetries can be violated. In this talk we assume that the modification of classical symmetries is represented by
a Hopf-algebraic structure. Our hope is that in similar way as hundred years ago the Einstein symmetries replaced
Galilei ones, in future considerations e.g. due to the quantum gravity effects the Einstein symmetries will be modified
and replaced by quantum relativistic symmetries.

**$\theta_{\mu\nu}$-DEFORMATION: AN EXAMPLE OF SOFT QUANTUM DEFORMATION OF EINSTEIN’S RELATIVISTIC SYMMETRIES**

Let us expand the rhs of the general noncommutativity relation

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{1}{\kappa^2} \theta_{\mu\nu} (\kappa \hat{\epsilon}), \quad (11)$$

where $\kappa$ is a masslike parameter, as follows:

$$\theta_{\mu\nu} (\kappa \hat{\epsilon}) = \theta_{\mu\nu}^{(0)} + \kappa \theta_{\mu\nu}^{(1)} + \kappa^2 \theta_{\mu\nu}^{(2)} + \kappa^3 \theta_{\mu\nu}^{(3)} + \ldots . \quad (12a)$$

If in the power serie (11) only first term is nonvanishing ($\theta_{\mu\nu}(\hat{\epsilon}) \equiv \theta_{\mu\nu}^{(0)}$) we shall call the modified Poincaré symmetries preserving the relation (8) the canonically deformed or $\theta_{\mu\nu}$-deformed Poincaré symmetries. Such quantum
symmetries were discovered quite recently [11]–[15] and are obtained by the modification of classical Poincaré-Hopf
algebra only in the Lorentz coalgebra sector by so-called twist function [16]

$$\mathcal{T}_\theta = \exp \frac{i}{2\kappa^2} (\theta_{\mu\nu}^{(0)} P_\mu \wedge P_\nu). \quad (13)$$

The generators ($M_{\mu\nu}, P_\mu$) satisfy the classical Poincaré algebra and the modified Poincaré coalgebra looks as follows

$$\Delta_\theta (P_\mu) = \Delta_0 (P_\mu), \quad \Delta_\theta (M_{\mu\nu}) = \Delta_0 (M_{\mu\nu}) - \frac{1}{\kappa^2} \theta_\mu^\rho \theta_\nu^\sigma \left( [\eta_\rho_\mu P_\nu - \eta_\rho_\nu P_\mu] \otimes P_\sigma \right) + P_\rho \otimes (\eta_\sigma_\mu P_\nu - \eta_\sigma_\nu P_\mu), \quad (14)$$

where ($I_A = (M_{\mu\nu}, P_\mu)$)

$$\Delta_0 (I_A) = I_A \otimes 1 + 1 \otimes I_A, \quad \Delta_\theta (I_A) = \mathcal{T}_\theta \circ \Delta_0 (I_A) \circ \mathcal{T}_\theta^{-1} \quad (15)$$

and $(a \otimes b) \circ (c \otimes d) = ac \otimes bd$.

The dual canonical $\theta_{\mu\nu}$-deformed Poincaré group constructed some years ago [11] and reconstructed recently
[14, 15] looks as follows

$$[\hat{a}_\mu, \hat{a}_\nu] = -\frac{i}{\kappa^2} \theta_\mu^\rho \theta_\rho^\sigma (\hat{\lambda}_\rho \hat{\lambda}_\sigma - \delta_\rho^\mu \delta_\sigma^\nu),$$
\[ \hat{A}_\mu \hat{A}_\nu = [\hat{d}_\mu, \hat{A}_\nu] = 0, \]  
with the coproducts remaining classical.

The twist (13) provides an example of "soft" Abelian deformation, with the classical r-matrix
\[ r = \frac{1}{2\kappa^2} \theta^{\mu \nu}_{(0)} P_\mu \wedge P_\nu, \]  
having the Abelian carrier algebra \([P_\mu, P_\nu] = 0\).

The twist (17) determines the noncommutative structure of \(\theta_{\mu \nu}\)-deformed Minkowski space \(\mathcal{M}_4^{(\theta)}\). In accordance with general framework for twisted quantum symmetries [16, 17, 12, 13] the coproduct formulae (15) enter into the definition of noncommutative associative \(*\)-product for the functions \(f, g \in \mathcal{M}_4^{(\theta)}\)
\[ f(\hat{x}) \ast g(\hat{x}) := \omega_\theta \left( f(\hat{x}) \otimes g(\hat{x}) \right) = \omega \left( F_{\theta}^{-1} \circ f(\hat{x}) \otimes g(\hat{x}) \right). \]  
In such a description the quantum Minkowski space \(\mathcal{M}_4^{(\theta)}\) is a quantum representation (a Hopf module) of canonical \(\theta_{\mu \nu}\)-deformed Poincaré algebra. The relation (18) applied to \(f(\hat{x}) = \hat{x}_\mu\) and \(g(\hat{x}) = \hat{x}_\nu\) provides the formula
\[ \hat{x}_\mu \ast \hat{x}_\nu = \frac{i}{2} \theta^{(0)}_{\mu \nu}, \]  
and leads to the relation (8). Interestingly enough, the relation
\[ [x_\mu, x_\nu]_s \equiv x_\mu \ast x_\nu - x_\nu \ast x_\mu = i \theta^{(0)}_{\mu \nu}, \]  
is covariant under the Hopf-algebraic action of twisted Poincaré algebra generators. Using the general formula for the action of canonical quantum symmetry generators
\[ I_A \triangleright \omega_\theta(f(\hat{x}) \otimes g(\hat{x})) = \omega_\theta(\Delta_\theta(I_A) \circ (f(\hat{x}) \otimes g(\hat{x}))), \]  
and choosing \(I_A \equiv M_{\mu \nu}\), one obtains
\[ M_{\mu \nu} \triangleright ([x_\mu, x_\nu]_s) - i \theta_{\mu \nu} = 0. \]  
The algebraic multiplication formula (18) after the use of classical realizations
\[ (P_\mu \circ f(x)) = i \partial_\mu f(x), \quad (M_{\mu \nu} \circ f(x)) = i (x_\mu \partial_\nu - x_\nu \partial_\mu) f(x). \]  
can be represented on commutative Minkowski space as the Moyal-Weyl star product
\[ f(x) \ast g(x) = \omega(e^{2\pi i \theta^{(0)}_{\rho \mu} \partial^\rho \wedge \partial^\nu} \circ (f(x) \otimes g(x))). \]  
The product (24) has been used recently quite often in order to describe the effects of the noncommutativity of space-time coordinates in classical and quantum field theory (see e.g. [18, 19]).

**STANDARD \(\kappa\)-DEFORMATION: THE MODIFICATION OF HIGH ENERGY LORENTZ BOOSTS AND QUANTUM TIME**

The next class of quantum deformations is obtained by assuming that the linear term in (11) is nonvanishing. In order to obtain the classical nonrelativistic physics we postulate
\[ [\hat{\xi}_i, \hat{\xi}_j] = 0. \]  
The only remaining \(O(3)\)-covariant deformed relation is the one proposed firstly in the \(\kappa\)-deformed framework of Einstein’s symmetries [20, 21, 10]
\[ [\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{\xi}_i. \]  

We see from the relations (26)–(27) that the space coordinates remain classical, but the time variable becomes quantum. The $\kappa$-deformed relativistic symmetries preserving the relations (25)–(26) are generated by the following $\kappa$-deformed Poincaré-Hopf algebra

\[ [N_i, P_j] = i\delta_{ij} \frac{\kappa}{2} \left( 1 - e^{-\frac{2\pi}{\kappa}} \right) + \frac{1}{2\kappa} \bar{P}^2 + \frac{1}{\kappa} P_j P_i, \]  

(27)

a) algebraic sector

The only modified classical relation is between boosts (we put $c = 1$ i.e. $N_i = M_{i0}$) and three momenta $P_j$:

\[ [N_i, P_j] = i\delta_{ij} \frac{\kappa}{2} \left( 1 - e^{-\frac{2\pi}{\kappa}} \right) + \frac{1}{2\kappa} \bar{P}^2 + \frac{1}{\kappa} P_j P_i, \]

(27)

b) coalgebra sector

We obtain the following set of coproducts for $N_i$ and $P_j$, satisfying the relation (27)

\[ \Delta P_i = P_i \otimes 1 + e^{-\frac{\pi}{\kappa}} P_i \otimes P_i, \]

\[ \Delta N_i = N_i \otimes 1 + e^{-\frac{\pi}{\kappa}} N_i \otimes N_i + \frac{1}{\kappa} \epsilon_{ijk} P_j \otimes M_k. \]

(28)

Remaining coproducts for $P_0$ and $M_i$ are primitive, i.e.

\[ \Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \]

\[ \Delta M_i = M_i \otimes 1 + 1 \otimes M_i, \]

(29)

c) antipodes

The classical value $S(I_A) = -I_A$ is modified for the generators $(P_i, N_i)$

\[ S(P_i) = -e^{-\frac{\pi}{\kappa}} P_i \]

\[ S(N_i) = -e^{-\frac{\pi}{\kappa}} N_i + \frac{1}{\kappa} \epsilon_{ijk} e^{-\frac{\pi}{\kappa}} P_j \otimes M_k. \]

(30)

Using the formulae (27)–(30) one can calculate the $\kappa$-deformation of finite Lorentz transformations of the fourmomenta. The classical Lorentz boost transformations

\[ P_\rho(\alpha_i) = e^{\alpha_i N_i} P_\rho e^{-\alpha_i N_i}, \]

(31)

in deformed case are generalized as follows

\[ P_\rho(\alpha) = \text{ad}_{\alpha_i N_i} P_\rho = \sum_{n=0}^{\infty} \frac{\alpha_{i_1} \cdots \alpha_{i_n}}{n!} \left( \text{ad}_{N_{i1}} \cdots \text{ad}_{N_{in}} P_\rho \right), \]

(32)

where the quantum adjoint action is defined by the formula

\[ \text{ad}_Y X = Y_1 X S(Y_2). \]

(33)

where $\Delta(Y) = Y_1 \otimes Y_2$. Choosing in (33) $Y = N_i$ and $X = F(P_i, P_0)$ one obtains after using (27) and (30) (see e.g. [22])

\[ \text{ad}_{N_i} F(P_i, P_0) = [N_i, F(P_i, P_0)]. \]

(34)

If we use (34) in (32) we see that the formula (32) takes the standard form (31). Choosing $\alpha = (\alpha, 0, 0)$ one obtains the differential equation for $\kappa$-deformed Lorentz transformations of $P_\rho$ in $(p_1, p_0)$ plane

\[ \frac{dP_\rho(\alpha)}{d\alpha} = [N_i, P_\rho]. \]

(35)

\[ \text{We provide the formulae in so-called bicrossproduct basis, with classical Lorentz algebra} \]
Substituting in (35) the rhs of (27) one gets the nonlinear equation for $P_\mu$, which has been solved in bicrossproduct basis firstly in [23] (the solution for arbitrary vector $\vec{a} = \alpha \vec{n}$ ($\vec{n}^2 = 1$) has been given in [24]).

The $\kappa$-deformed Minkowski space (25)–(26) as the Hopf-algebra module of the $\kappa$-deformed Poincaré algebra has been firstly considered in [21]. Using

$$I_A \triangleright \omega(f \otimes g) = \omega(I_A(1) \triangleright f \otimes I_A(2) \triangleright g),$$

and the classical form of the actions

$$M_i \triangleright \hat{x}_j = \epsilon_{ijk} \hat{x}_k, \quad M_j \triangleright \hat{x}_0 = 0,$$

$$N_i \triangleright \hat{x}_j = -\delta_{ij} \hat{x}_0, \quad N_i \triangleright \hat{x}_0 = -x_i,$$

one gets

$$N_i \triangleright \hat{\xi}_0 \hat{x}_j = \delta_{ij} \hat{\xi}_0^2 - \hat{\xi}_j \hat{x}_j + \frac{1}{\kappa} \hat{\xi}_j \hat{x}_0,$$

$$N_i \triangleright \hat{x}_j \hat{\xi}_0 = -\delta_{ij} \hat{x}_0^2 - \hat{x}_j \hat{\xi}_j,$$

and subsequently

$$N_i \triangleright (\hat{\xi}_0 \hat{x}_0 - \frac{1}{\kappa^2} x_i) = 0.$$

Similarly one can show that $N_i \triangleright ([\hat{x}_i, \hat{x}_j]) = 0$.

The multiplication leading to the formulae (25)–(26) can be represented by a particular CBH *-product (CBH≡Cambell-Baker-Haussdorf) on commutative Minkowski space $x_\mu$. From (25)–(26) follows that (see [25])

$$e^{ip_\mu \hat{x}_\mu} \cdot e^{ik_\mu \hat{\xi}_\mu} = e^{ip_\mu \hat{x}_\mu + k_\mu \hat{\xi}_\mu},$$

where

$$r^0 = p^0 + k^0, \quad r^j = \frac{f_\kappa(p^0) e^{ip^0 \hat{x}^0 + f_\kappa(k^0) \hat{\xi}^j}}{f_\kappa(p^0 + k^0)},$$

and $f_\kappa(\alpha) = \frac{\alpha}{\alpha} (1 - e^{-\frac{\alpha}{\kappa}}) = 1 - \frac{1}{\kappa} \alpha + O(\frac{1}{\kappa^2})$.

The CBS star product representing the multiplication rule (40) can be described by the formula

$$f(x) \ast g(x) = f\left( \frac{1}{i} \frac{\partial}{\partial p^\mu} \right) g\left( \frac{1}{i} \frac{\partial}{\partial k^\mu} \right) e^{ip_\mu \hat{x}_\mu},$$

In particular we obtain from (42)

$$x_0 \ast x_i = x_0 x_i + \frac{i}{2\kappa} x_i,$$

$$x_i \ast x_0 = x_0 x_i - \frac{i}{2\kappa} x_i,$$

i.e. we reproduce the relation (26).

After the discovery of $\kappa$-deformation of the Poincaré symmetries in 1991 there was found in 1995 the light-cone deformation of Poincaré algebra [26], with one of the light-cone directions quantized, and the relations (25)–(26) replaced by the following ones ($r, s = 1, 2$)

$$[\hat{x}_-, \hat{x}_r] = \frac{1}{\kappa} \hat{x}_r, \quad [\hat{x}_-, \hat{x}_s] = \frac{1}{\kappa} \hat{x}_-, \quad [\hat{x}_r, \hat{x}_s] = [\hat{x}_+, \hat{x}_r] = 0,$$

where $x_\pm = x_0 \pm x_3$. In 1996 both relations (25)–(26) and (44) were obtained as special cases of generalized $\kappa$-deformation of Poincaré symmetries [27, 18] with the quantized noncommutative direction $a_\mu \hat{x}_\mu$ in deformed Minkowski space. We get the following $a_\mu$-dependent noncommutativity structure of quantum space-time

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa} (a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu).$$
It follows from [27]–[29] that only if \( a_\mu^2 = 0 \), i.e. when the light-cone direction in Minkowski space is quantized, the corresponding deformation of Poincaré symmetries can be represented by the twist factor, which modifies by the similarity transformation the classical coproduct (see (15)). In such a case only the coalgebraic sector is modified and we deal with twisted quantum relativistic symmetries.

It should be added that \( \kappa \)-deformations were further employed from 2000 as a mathematical structure describing so-called doubly special relativistic (DSR) theories (see e.g. [30]–[32]). Basic aspects of the relation between the DSR framework and the \( \kappa \)-deformation of Poincaré symmetries were discussed by the present author in [24, 33].

**OUTLOOK**

The idea of noncommutative space-time can be traced back to Heisenberg who first indicated that a space-time uncertainty relation might be useful for the removal of infinities in renormalization procedure. Unfortunately till present time the idea of compensating the divergent terms (see e.g. [34]) in noncommutative field theory has not been successful. Also other ways of justifying the modifications of classical relativistic symmetries, based on the observation of modified kinematics in cosmic ray physics (see e.g. [35]) after a period of hopeful uncertainties in the interpretation of experimental data at present are rather not confirmed.

Important theoretical arguments for noncommutativity of space-time are based on quantum gravity and quantized string theory. They indicate that the implementation of quantum symmetries and quantum modification of relativistic symmetries should be a necessary step for the description of the distances comparable with the Planck length \( l_P \) (\( l_P \simeq 10^{-33} \) cm). Also the role of noncommutative geometry in the ultimate unification of all interactions - the \( M \)-theory - appears to be important and should be further explored.

Now however the main challenge is to find among experimental data in high energy physics and astrophysics the ones which indicate the violation of the postulates of Einstein special relativity theory (e.g. constant velocity of light, classical conservation law of fourmomenta, classical energy momentum dispersion relation, isotropy of "empty" space-time etc.). One can conjecture that these possible classical symmetry breaking effects can be recast into a new quantum relativistic symmetry invariance.

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