ON TWO \((p, q)\)-ANALOGUES OF THE LAPLACE TRANSFORM

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Two \((p, q)\)-Laplace transforms are introduced and their relative properties are stated and proved. Applications are made to solve some \((p, q)\)-linear difference equations.

1. INTRODUCTION

The classical Laplace transform of a function \(f(t)\) is given by

\[
\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st}f(t)dt, \quad s = a + ib \in \mathbb{C},
\]

and plays a fundamental role in pure and applied analysis, specially in solving differential equations. If a function of discrete variable \(f(t), t \in \mathbb{Z}\) is considered, then the integral transform \((1.1)\) reads

\[
F(z) = \sum_{j=0}^{\infty} f(j)z^{-j}, \quad z = e^{-p}.
\]

Equation \((1.2)\) is referred to as \(Z\)-transform and plays similar role in difference analysis as Laplace transform in continuous analysis, specially in solving difference equations.

In order to deal with \(q\)-difference equations, \(q\)-versions of the classical Laplace transform have been consecutively introduced in the literature. Studies of \(q\)-versions of Laplace transform go back to Hahn [?]. Abdi [1, 2, 3] published also many results in this domain.

The \(q\)-deformed algebras [15, 16] and their generalizations ((\(p, q\))-deformed algebras) [6, 7, 9] attract much attention these last years. The main reason is that these topics stand for a meeting point of today’s fast developing areas in mathematics and physics like the theory of quantum orthogonal polynomials and special functions, quantum groups, conformal field theories and statistics. From these works, many generalizations of special functions arise. There is a considerable list of references.

In this paper, we introduce two \((p, q)\)-versions of the Laplace transform and provide some of their main properties. Next, some applications are done to solve some \((p, q)\)-difference equations.

The paper is organised as follows:

\[\text{2010 Mathematics Subject Classification. FILL SUBJECT MSCs HERE.}\]
\[\text{Keywords and Phrases. \((p, q)\)-Exponential, \((p, q)\)-Laplace, \((p, q)\)-integral, \((p, q)\)-derivative.}\]
In Section 2, we recall basic notations, definitions and prove some important properties that will help in the next sections. The \( (p, q) \)-number, the \( (p, q) \)-factorial, the \( (p, q) \)-power, the \( (p, q) \)-binomial, the \( (p, q) \)-derivative, the \( (p, q) \)-integral, the \( (p, q) \)-exponentials, the \( (p, q) \)-trigonometric functions are successively introduced and some of their important properties are provided.

In Section 3, we introduce the \( (p, q) \)-Laplace transforms of first and of second kind. Their main properties are studied and the transforms of many fundamental functions are computed.

In Section 4, some applications of the Laplace transform of first kind are made. The same method can be used with the Laplace transform of second kind. The \( (p, q) \)-oscillator is introduced and solved using the Laplace transform of first kind.

In Section 5, we give a conclusion and indicate further possible directions that could be investigated to complete the following work.

### 2. Basic Definitions and Miscellaneous Results

#### 2.1 \( (p, q) \)-number, \( (p, q) \)-factorial, \( (p, q) \)-binomial, \( (p, q) \)-power

Let us introduce the following notation (see \[10,11,13\])

\[
[n]_{p,q} = \frac{p^n - q^n}{p - q},
\]

for any positive integer.

The twin-basic number is a natural generalization of the \( q \)-number, that is

\[
\lim_{p \to 1} [n]_{p,q} = [n]_q.
\]

The \( (p, q) \)-factorial is defined by \([11,13]\)

\[
[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}!, \quad n \geq 1, \quad [0]_{p,q}! = 1.
\]

Let us introduce also the so-called \( (p, q) \)-binomial coefficients

\[
\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n.
\]

Note that as \( p \to 1 \), the \( (p, q) \)-binomial coefficients reduce to the \( q \)-binomial coefficients.

It is clear by definition that

\[
\binom{n}{k}_{p,q} = \binom{n}{n-k}_{p,q}.
\]
Let us introduce also the so-called the \((p, q)\)-powers\(^{13}\)
\[
(x \ominus a)_{p,q}^n = (x - a)(px - aq) \cdots (xp^{n-1} - aq^{n-1}),
\]
\[
(x \oplus a)_{p,q}^n = (x + a)(px + aq) \cdots (xp^{n-1} + aq^{n-1}).
\]
These definitions are extended to
\[
(a \ominus b)_{p,q}^\infty = \prod_{k=0}^{\infty} (ap^k - q^k b)
\]
\[
(a \oplus b)_{p,q}^\infty = \prod_{k=0}^{\infty} (ap^k + q^k b)
\]
where the convergence is required.

2.2 The \((p, q)\)-derivative and the \((p, q)\)-integral

**Definition 2.1.**\(^{13}\) Let \(f\) be an arbitrary function and \(a\) be a real number, then the \((p, q)\)-integral of \(f\) is defined by
\[
\int_0^a f(x) \, dp,q(x) = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} a \right) \quad \text{if} \quad \left| \frac{p}{q} \right| > 1.
\]

**Definition 2.2.**\(^{13}\) The improper \((p, q)\)-integral of \(f(x)\) on \([0; \infty)\) is defined to be
\[
\int_0^\infty f(x) \, dp,q(x) = (p-q) \sum_{j=-\infty}^{\infty} \frac{q^j}{p^{j+1}} f \left( \frac{q^j}{p^{j+1}} \right), \quad 0 < \frac{q}{p} < 1.
\]
Let \(f\) be a function defined on the set of the complex numbers.

**Definition 2.3.** The \((p, q)\)-derivative of the function \(f\) is defined as
\[
D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0,
\]
and \((D_{p,q}f)(0) = f'(0)\), provided that \(f\) is differentiable at 0.

**Proposition 2.1.** The \((p, q)\)-derivative fulfils the following product and quotient rules
\[
D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x),
\]
\[
D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).
\]
Proposition 2.2. Let $n$ be an integer $n \geq 0$, then the following formula applies

$$\tag{2.5} D_{p,q}^n \left[ \frac{1}{x} \right] = (-1)^n \frac{[n]_{p,q}!}{(pq)^{\left(\frac{n+1}{2}\right)}} x^{n+1}.$$  

Proof. The relation is obvious for $n = 0$. Let $n \geq 1$, assume that (2.5) holds true. Then

$$D_{p,q}^{n+1} \left[ \frac{1}{x} \right] = D_{p,q} \left[ (-1)^n \frac{[n]_{p,q}!}{(pq)^{\left(\frac{n+1}{2}\right)}} \right] x^{n+1}$$

$$= \frac{(-1)^n [n]_{p,q}!}{(pq)^{\left(\frac{n+1}{2}\right)}} \times \frac{1}{(p-q)x} \left( \frac{1}{(px)^{n+1}} - \frac{1}{(qx)^{n+1}} \right)$$

$$= \frac{(-1)^n [n]_{p,q}!}{(pq)^{\left(\frac{n+1}{2}\right)}} \times \frac{[n+1]_{p,q}}{(pq)^{n+2}} = (-1)^{n+1} \frac{[n+1]_{p,q}!}{(pq)^{\left(\frac{n+2}{2}\right)}} x^{n+2}.$$  

The proof is then complete. \qed

The next proposition generalizes (2.5).

Proposition 2.3. $a$ is a non zero complex number. Then

$$\tag{2.6} D_{p,q}^n \left[ \frac{1}{ax+b} \right] = \frac{(-a)^n [n]_{p,q}!}{\prod_{k=0}^{n} \left( ap^n - k q x + b \right)}$$

$$= \frac{(-a)^n [n]_{p,q}!}{(ap^n x + b)(ap^{n-1} q x + b) \cdots (ap q x + b)(aq^n x + b)}.$$

Proof. The proof follows easily by induction. \qed

Note that for $a = 1$ and $b = 0$, (2.6) reduces to (2.5).

Proposition 2.4. \[13\] If $F(x)$ is a $(p, q)$-antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, we have

$$\int_a^b f(x) d_{p,q}x = F(b) - F(a), \quad 0 \leq a < b \leq \infty.$$  

Corollary 2.4. \[13\] If $f'(x)$ exists in a neighbourhood of $x = 0$ and is continuous at $x = 0$, where $f'(x)$ denotes the ordinary derivative of $f(x)$, we have

$$\int_a^b D_{p,q} f(x) d_{p,q}x = f(b) - f(a).$$  

Proposition 2.5. \[13\] Suppose that $f(x)$ and $g(x)$ are two functions whose ordinary derivatives exist in a neighbourhood of $x = 0$. $a$ and $b$ are two real numbers such that $a < b$, then

$$\tag{2.7} \int_a^b f(px) (D_{p,q} g(x)) d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_b^a g(qx) (D_{p,q} f(x)) d_{p,q}x.$$
2.3 The \((p, q)\)-hypergeometric functions

Here, we give a natural generalization of the \(q\)-hypergeometric series (8)

\[
\sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(q, q)_n (b_1; q)_n \cdots (b_s; q)_n} (-1)^n q^r z^n.
\]

**Definition 2.5.** The \((p, q)\)-hypergeometric series

\[
_{r} \Phi_{s} \left( \begin{array}{c} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{array} \bigg| q; z \right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(q, q)_n (b_1; q)_n \cdots (b_s; q)_n} (-1)^n q^r z^n.
\]

**Theorem 2.6** (Compare to [10]). Let \(a, b\) be two complex numbers, then we have the following

\[
_{1} \Phi_{0} \left( \begin{array}{c} a, b \\ \end{array} \bigg| p, q; z \right) = \sum_{n=0}^{\infty} \frac{(a \otimes b)_p^n (b; p, q)_n}{(p \otimes q)_p^n (a; p, q)_n} z^n.
\]

**Proof.** We first note that

\[
\frac{(a \otimes b)_p^n}{(p \otimes q)_p^n} = \frac{(b; p, q)_n}{(a; p, q)_n} \cdot \frac{a^n}{p^n}.
\]

It follows from the \(q\)-binomial theorem (see [8]) that

\[
\sum_{n=0}^{\infty} \frac{(a \otimes b)_p^n (b; p, q)_n}{(p \otimes q)_p^n (a; p, q)_n} z^n = \frac{(b; q, p)_n}{(a; q, p)_n} \cdot \frac{a^n}{p^n} \cdot \frac{b \otimes q \infty}{(p \otimes q)_{p,q} \infty}.
\]

The following corollary also appears in [10].

**Corollary 2.7.** \(a, b\) and \(c\) are three complex numbers. Then

\[
_{1} \Phi_{0} \left( \begin{array}{c} a, b \\ \end{array} \bigg| p, q; z \right)_{1} \Phi_{0} \left( \begin{array}{c} b, c \\ \end{array} \bigg| p, q; z \right) = _{1} \Phi_{0} \left( \begin{array}{c} a, c \\ \end{array} \bigg| p, q; z \right).
\]
2.4 \((p, q)\)-exponential and \((p, q)\)-trigonometric functions

As in the \(q\)-case, there are many definitions of the \((p, q)\)-exponential function. The following two \((p, q)\)-analogues of the exponential function (see [10]) will be frequently used throughout this paper:

\[
e_{p,q}(z) = {}^1\Phi_0 \left( \frac{(1,0)}{[n]_{p,q}!} ; p, (p-q)z \right) = \sum_{n=0}^{\infty} \frac{p^{(z)}}{[n]_{p,q}!} z^n, \tag{2.10}
\]

\[
E_{p,q}(z) = {}^1\Phi_0 \left( \frac{(0,1)}{[n]_{p,q}!} ; p, (q-p)z \right) = \sum_{n=0}^{\infty} \frac{q^{(z)}}{[n]_{p,q}!} z^n. \tag{2.11}
\]

From the \((p, q)\)-binomial theorem (2.9) and the definitions (2.10) and (2.11) of the \((p, q)\)-exponential functions, it is easy to see that

\[
e_{p,q}(x)E_{p,q}(-x) = 1. \tag{2.12}
\]

The next two propositions give the \(n\)-th derivative of the \((p, q)\)-exponential functions. These formulas are very important for the computations the \((p, q)\)-Laplace transforms of some functions in the next sections.

**Proposition 2.6.** Let \(\lambda\) be a complex number, then the following relations hold

\[
D_{p,q} e_{p,q}(\lambda x) = \lambda e_{p,q}(\lambda px),
\]

\[
D_{p,q} E_{p,q}(\lambda x) = \lambda E_{p,q}(\lambda qx).
\]

**Proof.** The proof follows from the definitions of the \((p, q)\)-exponentials and the \((p, q)\)-derivative.

**Proposition 2.7.** Let \(n\) be a nonnegative integer, then the following equations hold

\[
D_{p,q}^n e_{p,q}(\lambda x) = \lambda^n p^{(z)} e_{p,q}(\lambda p^n x), \tag{2.13}
\]

\[
D_{p,q}^n E_{p,q}(\lambda x) = \lambda^n q^{(z)} \lambda E_{p,q}(\lambda q^n x). \tag{2.14}
\]

**Proof.** The proof follows by induction from the definitions of the \((p, q)\)-exponentials and the \((p, q)\)-derivative.

From (2.10) we can derive

\[
e_{p,q}(iz) = \sum_{n=0}^{\infty} \frac{p^{(iz)}}{[n]_{p,q}!} (iz)^n = \sum_{n=0}^{\infty} \frac{(-1)^n p^{(2n+1)}}{(2n+1)!} z^{2n+1}.
\]

By (2.10), we define the \((p, q)\)-cosine and the \((p, q)\)-sine functions as follows:

\[
\cos_{p,q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n p^{(2n)}}{[2n]_{p,q}!} z^{2n}, \tag{2.16}
\]

\[
\sin_{p,q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n p^{(2n+1)}}{[2n+1]_{p,q}!} z^{2n+1}. \tag{2.17}
\]
Analogously, from (2.11) we can derive

\[ E_{p,q}(iz) = \sum_{n=0}^{\infty} \frac{q^n(z)}{[n]_{p,q}!} (iz)^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}(z)}{[2n]_{p,q}!} z^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}(z)}{[2n + 1]_{p,q}!} z^{2n+1}. \]

And by (2.14), we define the big \((p,q)\)-cosine and the big \((p,q)\)-sine functions as follows:

\[ \cos_{p,q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}(z)}{[2n]_{p,q}!} z^{2n}, \]

\[ \sin_{p,q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}(z)}{[2n + 1]_{p,q}!} z^{2n+1}. \]

It is easy to see that \( \cos_{p,q}(z) = \cos_{q,p}(z) \) and \( \sin_{p,q}(z) = \sin_{q,p}(z) \).

Clearly,

\[ D_{p,q} \cos_{p,q}(z) = -\sin_{p,q}(qz), \]
\[ D_{p,q} \sin_{p,q}(z) = \cos_{p,q}(pz), \]
\[ D_{p,q} \cos_{p,q}(z) = -\sin_{p,q}(qz), \]
\[ D_{p,q} \sin_{p,q}(z) = \cos_{p,q}(qz). \]

**Proposition 2.8.** The following equations hold

\[ \cos_{p,q}(x) \cos_{p,q}(x) + \sin_{p,q}(x) \sin_{p,q}(x) = 1, \]
\[ \sin_{p,q}(x) \cos_{p,q}(x) - \cos_{p,q}(x) \sin_{p,q}(x) = 0. \]

**Proof.** The proof follows from (2.12). \( \square \)

Let us now define the hyperbolic \((p,q)\)-cosine and the hyperbolic \((p,q)\)-sine functions as follows

\[ \cosh_{p,q}(z) = \frac{e_{p,q}(z) + e_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{p^{(2n)}(z)}{[2n]_{p,q}!} z^{2n}, \]

\[ \sinh_{p,q}(z) = \frac{e_{p,q}(z) - e_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{p^{(2n+1)}(z)}{[2n + 1]_{p,q}!} z^{2n+1}, \]

\[ \cosh_{p,q}(z) = \frac{E_{p,q}(z) + E_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{q^{(2n)}(z)}{[2n]_{p,q}!} z^{2n}, \]

\[ \sinh_{p,q}(z) = \frac{E_{p,q}(z) - E_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{q^{(2n+1)}(z)}{[2n + 1]_{p,q}!} z^{2n+1}. \]
Proposition 2.9. The following equations hold
\[
cosh_{p,q}(z)Cosh_{p,q}(z) - \sinh_{p,q}(z)\sinh_{p,q}(z) = 1,
\]
\[
cosh_{p,q}(z)\sinh_{p,q}(z) - \sinh_{p,q}(z)Cosh_{p,q}(z) = 0.
\]

Proof. The proof follows from (2.12). 

2.5 \((p, q)\)-Gamma function

Definition 2.8. Let \(x\) be a complex number, we define the \((p, q)\)-Gamma function as
\[
\Gamma_{p,q}(x) = \frac{(p \odot q)_{p,q}^\infty}{(p^x \odot q^x)_{p,q}^\infty}(p - q)^{1-x}, \ 0 < q < p.
\]

Proposition 2.10. The \((p, q)\)-Gamma function fulfils the following fundamental relation
\[
\Gamma_{p,q}(x + 1) = [x]_{p,q}\Gamma_{p,q}(x).
\]

Remark 2.9. If \(n\) is a nonnegative integer, it follows from (2.26) that
\[
\Gamma_{p,q}(n + 1) = [n]_{p,q}!.
\]

It can be also easily seen from the definition that
\[
\Gamma_{p,q}(n + 1) = \frac{(p \odot q)_{p,q}^n}{(p - q)^n}.
\]

Very recently, a \((p, q)\)-integral representation of the \((p, q)\)-Gamma function was given in [5] when the argument is a nonnegative integer as follows
\[
\Gamma_{p,q}(n) = \int_0^\infty t^{(n-1)(n-2)}E_{p,q}(-qt)d_{p,q}t.
\]

Note that in this definition, there is maybe a mistake, the factor \(p^{\frac{(n-1)(n-2)}{2}}\) should be replaced by \(p^{\frac{n(n-1)}{2}}\) so we can clearly get \(\Gamma_{p,q}(n + 1) = [n]_{p,q}\Gamma_{p,q}(n)\). Relation (2.27) enables to prove (2.26) again using the formula of the \((p, q)\)-integration by part (2.7).

Now, we propose another definition of the \((p, q)\)-Gamma function which will be frequently use throughout the text.

Definition 2.10. For \(0 < q < p\), we define a \((p, q)\)-Gamma function by
\[
\Gamma_{p,q}(z) = p^{\frac{z(z-1)}{2}}\int_0^\infty t^{z-1}E_{p,q}(-qt)d_{p,q}t.
\]
Proposition 2.11. Let \( z \) be a complex number such that \( \Gamma_{p,q}(z+1) \) and \( \Gamma_{p,q}(z) \) exist, then
\[
\Gamma_{p,q}(z+1) = [z]_{p,q} \Gamma_{p,q}(z).
\]

Proof. Using equation (2.28) and the \((p, q)\)-integration by part (2.7), we have:
\[
\begin{align*}
\Gamma_{p,q}(z+1) &= p \int_{0}^{\infty} t^{z} E_{p,q}(-qt) d_{p,q}t \\
&= -p \int_{0}^{\infty} (pt)^{z} D_{p,q}E_{p,q}(-t) d_{p,q}t \\
&= -p \int_{0}^{\infty} [t^{z} E_{p,q}(-t)]_{0}^{\infty} + p \int_{0}^{\infty} t^{z-1} E_{p,q}(-qt) d_{p,q}t \\
&= [z]_{p,q} \Gamma_{p,q}(z).
\end{align*}
\]

3. TWO \((P, Q)\)-LAPLACE TRANSFORMS

3.6 The \((p, q)\)-Laplace transform of the first kind

Definition 3.11. For a given function \( f(t) \), we define its \((p, q)\)-Laplace transform of the first kind as the function
\[
F(s) = L_{p,q}\{f(t)\}(s) = \int_{0}^{\infty} f(t) E_{p,q}(-qt) d_{p,q}t, \quad s > 0.
\]

Proposition 3.12. For any two complex numbers \( \alpha \) and \( \beta \), we have
\[
L_{p,q}\{\alpha f(t) + \beta g(t)\} = \alpha L_{p,q}\{f(t)\} + \beta L_{p,q}\{g(t)\}.
\]

Proof. The proof follows by (3.30).

In what follows, we give some examples. From (3.30), we note that:
\[
\begin{align*}
L_{p,q}\{1\}(s) &= \int_{0}^{\infty} E_{p,q}(-qt) d_{p,q}t = -\frac{1}{s} \int_{0}^{\infty} D_{p,q}E_{p,q}(-st) d_{p,q}t \\
&= -\frac{1}{s} [E_{p,q}(-st)]_{0}^{\infty} = \frac{1}{s}, \quad s > 0.
\end{align*}
\]
\[
\begin{align*}
L_{p,q}\{t\}(s) &= \int_{0}^{\infty} t E_{p,q}(-qt) d_{p,q}t = -\frac{1}{ps} \int_{0}^{\infty} (pt) D_{p,q}E_{p,q}(-st) d_{p,q}t \\
&= -\frac{1}{ps} \left\{ [t E_{p,q}(-st)]_{0}^{\infty} - \int_{0}^{\infty} E_{p,q}(-qt) d_{p,q}t \right\} \\
&= \frac{1}{ps^{2}}, \quad s > 0.
\end{align*}
\]
\[
L_{p,q}\{1 + 5t\}(s) = L_{p,q}\{1\}(s) + 5L_{p,q}\{t\}(s) = \frac{1}{s} + \frac{5}{ps^{2}}, \quad s > 0.
\]
Proposition 3.13. Let $\alpha$ be a non zero complex number, then

\[
(3.31) \quad \int_0^\infty f(at)dp,q_t = \frac{1}{\alpha} \int_0^\infty f(t)dp,q_t.
\]

Theorem 3.12 (Scaling). Let $a$ be a non zero complex number, then the following formula applies

\[
(3.32) \quad L_{p,q}\{f(at)\}(s) = \frac{1}{a}L_{p,q}\{f(t)\}\left(\frac{s}{a}\right).
\]

Proof. Using the definition and Proposition 3.13 we have

\[
L_{p,q}\{f(at)\}(s) = \int_0^\infty f(at)E_{p,q}(-qst)dp,q_t
\]

\[
= \int_0^\infty f(at)E_{p,q}(-aq \frac{s}{a}t)dp,q_t
\]

\[
= \frac{1}{a} \int_0^\infty f(t)E_{p,q}(-q \frac{s}{a}t)dp,q_t = \frac{1}{a}L_{p,q}\{f(t)\}\left(\frac{s}{a}\right).
\]

Theorem 3.13. For $\alpha > -1$, we have the following

\[
(3.33) \quad L_{p,q}(t^{\alpha}) = \frac{\Gamma_{p,q}(\alpha + 1)}{p^{\frac{\alpha(\alpha + 1)}{2}} s^{\alpha + 1}}.
\]

Proof. We have

\[
L_{p,q}\{t^{\alpha}\}(s) = \int_0^\infty t^{\alpha}E_{p,q}(-qst)dp,q_t = \frac{1}{s^{\alpha + 1}} \int_0^\infty E_{p,q}(-qt)t^{\alpha}dp,q_t
\]

\[
= \frac{1}{p^{\frac{\alpha(\alpha + 1)}{2}} s^{\alpha + 1}} \int_0^\infty p^{\frac{\alpha(\alpha + 1)}{2}} t^{(\alpha + 1)-1}E_{p,q}(-qt)dp,q_t
\]

\[
= \frac{\Gamma_{p,q}(\alpha + 1)}{p^{\frac{\alpha(\alpha + 1)}{2}} s^{\alpha + 1}}.
\]

The following theorem is a particular case of Theorem 3.13 when $\alpha = n$ is a nonnegative integer.

Theorem 3.14. Let $n \in \mathbb{N}$, then for $s > 0$, we have

\[
(3.34) \quad L_{p,q}\{t^{n}\}(s) = \frac{[n]_{p,q}!}{p^{\frac{n(\alpha + 1)}{2}} s^{\alpha + 1}}.
\]
Proof. We provide a proof by induction for this result. The result is obvious for $n = 0$. Assume that it holds true for some nonnegative integer $n$, then using the $(p, q)$-integration by part (2.7), we have

$$L_{p,q}\{t^{n+1}\}(s) = \int_0^\infty t^{n+1} E_{p,q}(-qst)dt$$

$$= \frac{1}{p^{n+1}s} \int_0^\infty (pt)^{n+1} D_{p,q}E_{p,q}(-ts)dt$$

$$= \frac{1}{p^{n+1}s} \left\{ [n+1]_{p,q}E_{p,q}(-st)\right\}_0^n - [n+1]_{p,q} \int_0^\infty t^n E_{p,q}(-qts)dt$$

$$= \frac{[n+1]_{p,q}}{p^{n+1}s} L_{p,q}\{t^n\}(s)$$

$$= \frac{[n+1]_{p,q}}{p^{n+1}s} \frac{[n]_{p,q}!}{p^{\binom{n+1}{2}}} s^{n+1}$$

This proves the assertion.

Next, we give explicit formulas for the transform of the $(p, q)$-exponential and the $(p, q)$-trigonometric functions.

**Theorem 3.15.** Let $a$ be a real number, then

$$L_{p,q}\{e_{p,q}(at)\}(s) = \frac{p}{ps - a}, \quad s > \frac{a}{p},$$

$$L_{p,q}\{E_{p,q}(at)\}(s) = \frac{1}{s} \sum_{n=0}^\infty (-1)^n \left( \frac{a}{p} \right)^{\binom{n+1}{2}} \left( \frac{a}{ps} \right)^n.$$

Proof. Using (2.10), (2.11) and (3.34), we have

$$L_{p,q}\{e_{p,q}(at)\}(s) = \int_0^\infty E_{p,q}(-qst) e_{p,q}(at)dt$$

$$= \sum_{n=0}^\infty \frac{a^n p^{\binom{n+1}{2}}}{[n]_{p,q}!} \int_0^\infty E_{p,q}(-qst) t^n dt$$

$$= \sum_{n=0}^\infty \frac{a^n p^{\binom{n+1}{2}}}{[n]_{p,q}! p^{\binom{n+1}{2}}} s^{n+1}$$

$$= \frac{1}{s} \sum_{n=0}^\infty \left( \frac{a}{ps} \right)^n = \frac{p}{ps - a}.$$
Theorem 3.16. The following relations apply

\[ L_{p,q}\{\cos_{p,q}(at)\}(s) = \int_0^\infty E_{p,q}(-qst) \cos_{p,q}(at) d_{p,q}t \]
\[ = \sum_{n=0}^\infty (-1)^n a^2n p^{\binom{2n}{2}} \int_0^\infty E_{p,q}(-qst)t^n d_{p,q}t \]
\[ = \sum_{n=0}^\infty (-1)^n a^2n p^{\binom{2n}{2}} \frac{[n]_{p,q}!}{[2n]_{p,q}!} \frac{[2n+1]_{p,q}!}{p^{\binom{2n+1}{2}} s^{2n+1}} \]
\[ = \frac{1}{s} \sum_{n=0}^\infty (-1)^n \left( \frac{a}{ps} \right)^{2n} \left( \frac{a}{ps} \right)^n. \]

\[ L_{p,q}\{\sin_{p,q}(at)\}(s) = \int_0^\infty E_{p,q}(-qst) \sin_{p,q}(at) d_{p,q}t \]
\[ = \sum_{n=0}^\infty (-1)^n a^2n p^{\binom{2n+1}{2}} \int_0^\infty E_{p,q}(-qst)t^{2n+1} d_{p,q}t \]
\[ = \sum_{n=0}^\infty (-1)^n a^2n p^{\binom{2n+1}{2}} \frac{[2n+1]_{p,q}!}{[2n+1]_{p,q}!} \frac{[2n+2]_{p,q}!}{p^{\binom{2n+2}{2}} s^{2n+2}} \]
\[ = \frac{1}{s} \sum_{n=0}^\infty (-1)^n \left( \frac{a}{ps} \right)^{2n+1} \left( \frac{a}{ps} \right)^n. \]

\[ \square \]
Remark 3.17. Note that one could also use (3.35), (2.16) and (2.17) to obtain the result.

Theorem 3.18. The following equations apply

\[
L_{p,q}\{\cosh_{p,q}(at)\}(s) = \frac{p^2 s}{(ps)^2 - a^2}, \quad s > \frac{a}{p},
\]
\[
L_{p,q}\{\sinh_{p,q}(at)\}(s) = \frac{pa}{(ps)^2 - a^2}, \quad s > \frac{a}{p}.
\]

Proof. Using (3.30), (2.21) and (2.22) we have

\[
L_{p,q}\{\cosh_{p,q}(at)\}(s) = \frac{1}{2} \left(L_{p,q}\{e_{p,q}(at)\}(s) + L_{p,q}\{e_{p,q}(-at)\}(s)\right)
= \frac{1}{2} \left(\frac{p}{ps - a} + \frac{p}{ps + a}\right)
= \frac{p^2 s}{(ps)^2 - a^2},
\]
\[
L_{p,q}\{\sinh_{p,q}(at)\}(s) = \frac{1}{2} \left(L_{p,q}\{e_{p,q}(at)\}(s) - L_{p,q}\{e_{p,q}(-at)\}(s)\right)
= \frac{1}{2} \left(\frac{p}{ps - a} - \frac{p}{ps + a}\right)
= \frac{pa}{(ps)^2 - a^2}.
\]

\[\square\]

Next, \(f\) being a function, we provide some properties related to the \((p, q)\)-derivative of the \((p, q)\)-Laplace transform of \(f\) and the \((p, q)\)-Laplace transform of the \((p, q)\)-derivative of \(f\). Let us introduce the following notation which makes clear the relative variable on which the \((p, q)\)-derivative is applied:

\[
\frac{\partial_{p,q}}{\partial_{p,q}s} f(x, s) = \frac{f(x, ps) - f(x, qs)}{(p - q)s},
\]

and

\[
\frac{\partial_{p,q}^{n+1}}{\partial_{p,q}s^{n+1}} = \frac{\partial_{p,q}^n}{\partial_{p,q}s^n} \circ \frac{\partial_{p,q}}{\partial_{p,q}s}, \quad n \geq 1, \quad \text{and} \quad \frac{\partial_{p,q}^{0}}{\partial_{p,q}s^0} f = f.
\]

Theorem 3.19 ((\(p, q\))-derivative of transforms). For \(n \in \mathbb{N}\), we have

\[
(3.37) \quad L_{p,q}\{t^n f(t)\}(s) = (-1)^n q^n \frac{\partial_{p,q}^n}{\partial_{p,q}s^n} \left[F \left(q^{-n}s\right)\right].
\]
Proof. The result is obvious for \( n = 0 \). Let \( n \geq 1 \), we have

\[
\frac{\partial^n}{\partial p,q s^n} [F(q^{-n}s)] = \int_0^\infty \frac{\partial^n}{\partial p,q s^n} [E_{p,q}(-q^{-n+1}st)] f(t) dp,q t
\]

Using equation (2.14), it follows that

\[
\frac{\partial^n}{\partial p,q s^n} [E_{p,q}(-q^{-n+1}st)] = (-1)^n q^{\frac{n}{2}} t^n E_{p,q}(-qst).
\]

The proof is then completed.

Note that (3.34) can be obtained using Theorem 3.19. Of course, taking \( f(t) = 1 \) in (3.37) and using (2.5), we have

\[
L_{p,q} \{ t^n \} (s) = (-1)^n q^{\frac{n}{2}} \frac{\partial^n}{\partial p,q s^n} \left[ \frac{q^n}{s} \right] = (-1)^n q^{\frac{n}{2} + 1} \frac{(-1)^n[n]_p,q}{(pq)^{\frac{n+1}{2}} s_{n+1}} = \frac{[n]_p,q}{p^{\frac{n+1}{2}} s_{n+1}}.
\]

Corollary 3.20. The following equation applies:

\[
L_{p,q} \{ t^n e_{p,q}(at) \} (s) = \frac{p^{n+1} q^{\frac{n+1}{2}} [n]_{p,q}}{(p^{n+1} s - aq^n)(p^n q^a - aq^n) \cdots (p^2 q^{n-1} s - aq^n)(p q^n s - aq^n)}
\]

\[
= \frac{n \prod_{k=0}^{n} (p^{n+1-k} q^k s - aq^n)}{n^{n+1} q^{\frac{n+1}{2}} [n]_{p,q}}.
\]

Proof. The proof follows from (2.10) and (3.34).

Theorem 3.21 (Transform of the \((p,q)\)-derivative). The following transform rule applies.

\[
(3.38) \quad L_{p,q} \{ D_{p,q} f(t) \} (s) = \frac{s^n}{p^{\frac{n+1}{2}}} L_{p,q} \{ f(t) \} \left( \frac{s}{p^n} \right) - \sum_{k=0}^{n-1} \frac{s^{n-1-k}}{p^{\frac{n+1}{2}}} (D_{p,q}^k f)(0).
\]

Proof. Let \( f \) be a functions for which the \((p,q)\)-Laplace transform exists. Then,
On two \((p,q)\)-analogues of the Laplace transform

for \(n = 1\),

\[
L_{p,q} \{ D_{p,q} f(t) \} (s) = \int_0^\infty E_{p,q}(-qst)D_{p,q} f(t) dt \\
= [f(t)E_{p,q}(-st)]_0^\infty - \int_0^\infty f(pt)E_{p,q}(-st) dt \\
= -f(0) + s \int_0^\infty f(pt)E_{p,q}(-qst) dt \\
= -f(0) + \frac{s}{p} L_{p,q} \{ f(t) \} \left( \frac{s}{p} \right).
\]

Let \(n \geq 1\), assume (3.38) holds true. Then, applying the result for \(n = 1\) with \(D_{p,q}^n f(t)\), we have

\[
L_{p,q} \{ D_{p,q}^{n+1} f(t) \} (s) = -\left( D_{p,q}^n f(0) + \frac{s}{p} L_{p,q} \{ D_{p,q}^n f(t) \} \left( \frac{s}{p} \right) \right) \\
= -\left( D_{p,q}^n f(0) + \frac{s}{p} \left\{ \frac{s^{n+1}}{p^{(n+1)/2} + n+1} L_{p,q} \{ f(t) \} \left( \frac{s}{p^{n+1}} \right) \\
- \sum_{k=0}^{n-1} \frac{s^{n-k}}{p^{(n-k)/2} + n-k} (D_{p,q}^k f(0)) \right\} \right) \\
= -\left( D_{p,q}^n f(0) + \left\{ \frac{s^{n+1}}{p^{(n+1)/2} + n+1} L_{p,q} \{ f(t) \} \left( \frac{s}{p^{n+1}} \right) \\
- \sum_{k=0}^{n-1} \frac{s^{n-k}}{p^{(n-k)/2} + n-k} (D_{p,q}^k f(0)) \right\} \right) \\
= \frac{s^{n+1}}{p^{(n+1)/2}} L_{p,q} \{ f(t) \} \left( \frac{s}{p^{n+1}} \right) - \sum_{k=0}^{n-1} \frac{s^{n-k}}{p^{(n-k)/2} + n-k} (D_{p,q}^k f(0)) \right) \\
= \frac{s^{n+1}}{p^{(n+1)/2}} L_{p,q} \{ f(t) \} \left( \frac{s}{p^{n+1}} \right) - \sum_{k=0}^{n-1} \frac{s^{n-k}}{p^{(n-k)/2} + n-k} (D_{p,q}^k f(0)) \right).
\]

This completes the proof.

As a direct application, observe that taking \(f(t) = t^n\) in (3.38), we have

\[
L_{p,q} \{ D_{p,q} t^n \} (s) = \frac{s^n}{p^{(n+1)/2}} L_{p,q} \{ t^n \} \left( \frac{s}{p^n} \right).
\]
Taking care that $D^n_{p,q} t^n = [n]_{p,q}!$, and $L_{p,q} \{1\}(s) = \frac{1}{s}$, it follows that 
\[
L_{p,q}\{t^n\} \left( \frac{s}{p^n} \right) = p^{\left( \frac{n+1}{2} \right)} [n]_{p,q}! \frac{L_{p,q}\{1\}(1)}{s^n} = p^{\left( \frac{n+1}{2} \right)} [n]_{p,q}! \frac{1}{s^{n+1}}.
\]
Replacing $s$ by $sp^n$, we then have 
\[
L_{p,q}\{t^n\}(s) = p^{\left( \frac{n+1}{2} \right)} [n]_{p,q}! \frac{L_{p,q}\{1\}(1)}{s^{n+1}p^{n(n+1)}} = \frac{[n]_{p,q}!}{p^{\left( \frac{n+1}{2} \right)} s^{n+1}}.
\]

### 3.7 The $(p, q)$-Laplace transform of second kind

Whereas in the previous sections we introduce the $(p, q)$-Laplace transform of the first kind and prove some of its important properties, in this section, we introduce the $(p, q)$-Laplace transform of the second kind. The main difference is at the level of the $(p, q)$-exponential used in the definition. The motivation of the next definition comes from the fact that when we transform the big $(p, q)$-exponential, the result remains in term of a series which we cannot simplify.

Let first introduce the $(p, q)$-Gamma function of the second kind which will be useful.

**Definition 3.22.** The $(p, q)$-Gamma function of the second kind is defined by

\[
\gamma_{p,q}(z) = q^{\frac{z(z-1)}{2}} \int_0^\infty t^{z-1} e_{p,q}(-pt)d_{p,q}t, \quad \Re(z) > 0.
\]

**Proposition 3.14.** The $(p, q)$-Gamma function fulfils the following fundamental relation

\[
\gamma_{p,q}(z + 1) = [z]_{p,q} \gamma_{p,q}(z),
\]

moreover, for any non negative integer $n > 0$, the following relation holds

\[
\gamma_{p,q}(n + 1) = [n]_{p,q}!.
\]

**Proof.** Let $z$ be a complex number such that $\Re(z) > 0$, then we have

\[
\begin{align*}
\gamma_{p,q}(z + 1) &= q^{\frac{z(z+1)}{2}} \int_0^\infty t^z e_{p,q}(-pt)d_{p,q}t \\
&\quad - q^{\frac{z(z-1)}{2}} \int_0^\infty (qt)^z D_{p,q} e_{p,q}(-t)d_{p,q}t \\
&\quad - q^{\frac{z(z-1)}{2}} \left\{ [t^z e_{p,q}(-t)]_0^\infty - [z]_{p,q} \int_0^\infty t^{z-1} e_{p,q}(-pt)d_{p,q}t \right\} \\
&= [z]_{p,q} \gamma_{p,q}(z).
\end{align*}
\]
Definition 3.23. For a given function \( f(t) \), we define its \((p, q)\)-Laplace transform of the first kind as the function

\[
(3.42) \quad F(s) = \mathcal{L}_{p,q}\{f(t)\}(s) = \int_0^\infty f(t)e_{p,q}(-pts)d_{p,q}t, \quad s > 0.
\]

Proposition 3.15 (Linearity). By (3.42), we have

\[
\mathcal{L}_{p,q}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}_{p,q}\{f(t)\} + \beta \mathcal{L}_{p,q}\{g(t)\}.
\]

Proposition 3.16. For any real number \( \alpha > -1 \), we have

\[
(3.43) \quad \mathcal{L}_{p,q}\{t^{\alpha}\}(s) = \frac{\gamma_{p,q}(\alpha + 1)}{q^{\alpha + \frac{\alpha - 1}{2}}s^{\alpha + 1}}.
\]

Proof. By definition, one has

\[
\mathcal{L}_{p,q}\{t^{\alpha}\}(s) = \int_0^\infty t^\alpha e_{p,q}(-pts)d_{p,q}t
\]

\[
= \frac{1}{s^{\alpha + 1}} \int_0^\infty t^\alpha e_{p,q}(-pt)d_{p,q}t
\]

\[
= \frac{\gamma_{p,q}(\alpha + 1)}{q^{\frac{\alpha - 1}{2}}s^{\alpha + 1}}.
\]

Proposition 3.17. For \( n \in \mathbb{N} \), we have have

\[
(3.44) \quad \mathcal{L}_{p,q}\{t^n\}(s) = \frac{[n]_{p,q}}{q^{\frac{n+1}{2}}s^{n+1}}.
\]

Proof. Clearly, we have for \( n = 0 \)

\[
\mathcal{L}_{p,q}\{1\}(s) = \int_0^\infty e_{p,q}(-pts)d_{p,q}t = -\frac{1}{s} [e_{p,q}(ts)]_0^\infty = \frac{1}{s}.
\]

Next, for \( n > 0 \), we have

\[
\mathcal{L}_{p,q}\{t^n\}(s) = \int_0^\infty t^ne_{p,q}(-pts)d_{p,q}t
\]

\[
= -\frac{1}{q^n s} \int_0^\infty (qt)^n D_{p,q} e_{p,q}(-ts)d_{p,q}t
\]

\[
= -\frac{1}{q^n s} \left\{ [t^n e_{p,q}(-ts)]_0^\infty - [n]_{p,q} \int_0^\infty t^{n-1} e_{p,q}(-pts)d_{p,q}t \right\}
\]

\[
= \frac{[n]_{p,q}}{q^n s} \mathcal{L}_{p,q}\{t^{n-1}\}(s).
\]

The proof then follows by induction. \( \square \)
Proposition 3.18. The following equation holds

\[ \mathcal{L}_{p,q}\{E_{p,q}(at)\}(s) = \frac{q}{qs - a}, \quad s > \left| \frac{a}{q} \right|. \]  

**(Proof.** We have

\[ \mathcal{L}_{p,q}\{E_{p,q}(at)\}(s) = \sum_{n=0}^{\infty} \frac{q^n}{[n]_{p,q}} \int_0^s t^n e_{p,q}\left(-ptst\right)dt \]

\[ = \sum_{n=0}^{\infty} \frac{q^n}{[n]_{p,q}} \times \frac{[n]_{p,q}!}{q^{(n+1)}s^{n+1}} \]

\[ = \frac{1}{s} \sum_{n=0}^{\infty} \left( \frac{a}{qs} \right)^n = \frac{q}{qs - a}. \]

\[ \square \]

**Corollary 3.24.** The following equations hold

\[ \mathcal{L}_{p,q}\{Cos_{p,q}(at)\}(s) = \frac{q^2s}{(qs)^2 + a^2}, \quad s > \left| \frac{a}{q} \right|, \]

\[ \mathcal{L}_{p,q}\{Sin_{p,q}(at)\}(s) = \frac{qa}{(qs)^2 + a^2}, \quad s > \left| \frac{a}{q} \right|. \]

**(Proof.** The proof follows from the definitions (2.19), (2.20) and equation (3.45). \[ \square \]

**Corollary 3.25.** The following equations hold

\[ \mathcal{L}_{p,q}\{Cosh_{p,q}(at)\}(s) = \frac{q^2s}{(qs)^2 - a^2}, \quad s > \left| \frac{a}{q} \right|, \]

\[ \mathcal{L}_{p,q}\{Sinh_{p,q}(as)\}(s) = \frac{qa}{(qs)^2 - a^2}, \quad s > \left| \frac{a}{q} \right|. \]

**(Proof.** The proof is similar to the proof of Corollary 3.24. \[ \square \]

Next, \( f \) being a function, we provide some properties related to the \((p, q)\)-derivative of the \((p, q)\)-Laplace transform of \( f \) and the \((p, q)\)-Laplace transform of the \((p, q)\)-derivative of \( f \).

**Theorem 3.26 \((p, q)\)-derivative of transforms.** For \( n \in \mathbb{N} \), we have

\[ \mathcal{L}_{p,q}\{t^n f(t)\}(s) = (-1)^n p_2^{(n)} \frac{\partial^n}{\partial p^n} F\left(p^{-n}s\right) \]

where \( F(s) = \mathcal{L}_{p,q}\{f(t)\}(s) \).
Proof. The result is obvious for \( n = 0 \). Let \( n \geq 1 \), we have
\[
\frac{\partial^n_{p,q}}{\partial p_q s^n} [F(p^{-n}s)] = \int_0^\infty \frac{\partial^n_{p,q}}{\partial p_q s^n} [e_{p,q}(-p^{-n+1}st)] f(t) d_p_q t
\]
Using equation (2.13), it follows that
\[
\frac{\partial^n_{p,q}}{\partial p_q s^n} [e_{p,q}(-p^{-n+1}st)] = \prod_{j=0}^{n-1} (-p^{n-1-j}t) e_{p,q}(-pst)
\]
\[
= (-1)^n p^{-\frac{q}{2}} t^n e_{p,q}(-pst).
\]
The proof is then completed.

Note that (3.34) can be obtained using Theorem 3.26. Of course, taking \( f(t) = 1 \) in (3.46) and using (2.5), we have \( F(s) = \frac{1}{s} \) and
\[
L_{p,q}\{ t\} (s) = \frac{1}{s} L_{p,q}\{ f(t) \} (s) = (-1)^n p^{-\frac{q}{2}} \left( \frac{1}{s} \right) = \frac{[n]_{p,q}!}{q^{\frac{n+1}{2}} s^{n+1}}.
\]

Corollary 3.27. The following equation applies:
\[
L_{p,q}\{ t^n E_{p,q}(at) \} (s) = \frac{q^{n+1} p^{\frac{n+1}{2}} [n]_{p,q}!}{(q^{n+1}s - ap^n)(q^n ps - ap^n) \cdots (q^2 p^{n-1}s - ap^n)(pq^n s - ap^n)}
\]
\[
= \frac{q^{n+1} p^{\frac{n+1}{2}} [n]_{p,q}!}{\prod_{k=0}^{n} (q^{n+1-k}p^k s - ap^n)}.
\]

Proof. The proof follows from (2.6) and (3.46).

Theorem 3.28. (Transform of the \((p,q)\)-derivative). For any nonnegative integer \( n \), we have
\[
L_{p,q}\{ D_{p,q}^n f(t) \} = \frac{s^n}{q^{\frac{n+1}{2}}} L_{p,q}\{ f(t) \} \left( \frac{s}{q^n} \right) - \sum_{k=0}^{n-1} \frac{s^{n-1-k}}{q^{\frac{n-1-k}{2}}} \left( D_{p,q}^k f \right) (0).
\]

Proof. For \( n = 1 \), we have
\[
L_{p,q}\{ f(t) \} (s) = \int_0^\infty D_{p,q} f(t) e_{p,q}(-pst) d_p_q t
\]
\[
= [f(t) e_{p,q}(-st)] + s \int_0^\infty f(qt) e_{p,q}(-pst) d_p_q t
\]
\[
= -f(0) + \frac{s}{q} \int_0^\infty f(t) e_{p,q} \left( -p\frac{s}{q} t \right) d_p_q t
\]
\[
= -f(0) + \frac{s}{q} L_{p,q}\{ f(t) \} \left( \frac{s}{q} \right).
\]
So the relation is true for \( n = 1 \). Let \( n \geq 1 \), assume that (3.47) holds true, then using the case \( n = 1 \), we can write

\[
\mathcal{L}_{p,q} \{ D_{p,q}^{n+1} f(t) \} = -(D_{p,q}^{n} f)(0) + \frac{s}{q} \mathcal{L}_{p,q} \{ D_{p,q}^{n} f(t) \} \left( \frac{s}{q} \right) - \sum_{k=0}^{n-1} \frac{s^{n-1-k}}{q^{n-1-k}} \left( D_{p,q}^{k} f \right)(0)
\]

\[
= -(D_{p,q}^{n} f)(0) + \frac{s^{n+1}}{q^{n+1}} \mathcal{L}_{p,q} \{ f(t) \} \left( \frac{s}{q^{n+1}} \right) - \sum_{k=0}^{n-1} \frac{s^{n-k}}{q^{n-k}} \left( D_{p,q}^{k} f \right)(0)
\]

\[
= \frac{s^{n+1}}{q^{n+1}} \mathcal{L}_{p,q} \{ f(t) \} \left( \frac{s}{q^{n+1}} \right) - \sum_{k=0}^{n-1} \frac{s^{n-k}}{q^{n-k}} \left( D_{p,q}^{k} f \right)(0).
\]

The relation holds then true for each integer \( n \geq 1 \). \( \square \)

We now have another possibility to compute \( \mathcal{L}_{p,q} \{ t^n \} (s) \) using (3.47). Of course, applying (3.47) to \( f(t) = t^n \), we have

\[
\mathcal{L}_{p,q} \{ D_{p,q}^{n} t^n \} (s) = \frac{s^n}{q^{n+1}} \mathcal{L}_{p,q} \{ t^n \} \left( \frac{s}{q^n} \right).
\]

Taking care that \( D_{p,q}^{n} t^n = [n]_{p,q}! \), it follows that

\[
\mathcal{L}_{p,q} \{ t^n \} \left( \frac{s}{q^n} \right) = q^{n+1} \frac{[n]_{p,q}!}{s^n} \mathcal{L}_{p,q} \{ 1 \} (s) = \frac{[n]_{p,q}! q^{n+1}}{s^{n+1}}.
\]

Replacing \( s \) by \( sq^n \), it follows that

\[
\mathcal{L}_{p,q} \{ t^n \} (s) = \frac{[n]_{p,q}! q^{n+1}}{s^{n+1} q^n (n+1)} = \frac{[n]_{p,q}! q^{n+1}}{q^{n+1} s^{n+1}}.
\]

4. APPLICATION OF \((P,Q)\)-LAPLACE TRANSFORM TO CERTAIN \((P,Q)\)-DIFFERENCE EQUATIONS

As Laplace transform and \( Z \)-transform are largely applied in solving differential and difference equations respectively, and the \( q \)-Laplace transforms are applied to solve \( q \)-difference equations, the \((p,q)\)-Laplace transforms are expected to play
similar role but now in \((p, q)\)-difference equations. The idea lying behind is always the same. In this section, we show on few examples how the Laplace transforms introduced before can be used to solve some \((p, q)\)-differential equations.

Consider the problem of finding \(f(t)\), where \(f(t)\) satisfies \((p, q)\)-Cauchy problem

\[ D_{p,q}f(t) + cf(pt) = 0, \quad f(0) = 1, \]

where \(c\) stands for a complex constant.

Applying the Laplace transform of the first kind to (4.48), we obtain

\[ -f(0) + \frac{s}{p} L_{p,q}\{f(t)\} \left( \frac{s}{p} \right) + c L_{p,q}\{f(pt)\}(s) = 0. \]

Next, using equation (3.32), and the initial condition \(f(0) = 0\), we get

\[ -1 + \frac{s}{p} L_{p,q}\{f(t)\} \left( \frac{s}{p} \right) + \frac{c}{p} L_{p,q}\{f(t)\} \left( \frac{s}{p} \right) = 0. \]

Hence,

\[ L_{p,q}\{f(t)\} \left( \frac{s}{p} \right) = \frac{ps + c}{s + c}, \]

and so

\[ L_{p,q}\{f(t)\}(s) = \frac{p}{ps + c}. \]

It follows that \(f(t) = e_{p,q}(-ct)\).

Now, consider the \((p, q)\)-differential equation

\[ D_{p,q}h(t) - \lambda h(pt) = e_{p,q}(\lambda qt), \quad h(0) = 0. \]

Applying the \((p, q)\)-Laplace transform of first kind to (4.49), it follows that

\[ -h(0) + \frac{s}{p} L_{p,q}\{h(t)\} \left( \frac{s}{p} \right) - \frac{\lambda}{p} L_{p,q}\{h(t)\} \left( \frac{s}{p} \right) = \frac{p}{ps - \lambda q}. \]

Simplifications give

\[ L_{p,q}\{h(t)\} \left( \frac{s}{p} \right) = \frac{p^2}{(s - \lambda)(ps - \lambda q)}, \]

and finally, replacing \(s\) by \(ps\), we have

\[ L_{p,q}\{h(t)\}(s) = \frac{p^2}{(ps - \lambda)(p^2 s - \lambda q)}. \]

So, clear \(h(t) = te_{p,q}(\lambda t)\).
For the last example, we consider the classical \((p, q)\)-oscillator

\[
D_{p,q}^2 f(t) + \omega^2 f(p^2 t) = 0, \quad D_{p,q} f(0) = A, \quad f(0) = B.
\]

Applying the \((p, q)\)-Laplace transform of first kind to (4.50), it follows that

\[
-A - \frac{Bs}{p} + \frac{s^2}{p^2} L_{p,q} \{ f(t) \} \left( \frac{s}{p^2} \right) + \frac{\omega^2}{p^2} L_{p,q} \{ f(t) \} \left( \frac{s}{p^2} \right) = 0.
\]

By an easy simplification, we get

\[
L_{p,q} \{ f(t) \} \left( \frac{s}{p^2} \right) = \frac{Bs + Ap}{p} \times \frac{p^3}{s^2 + p\omega^2}.
\]

It happens that

\[
L_{p,q} \{ f(t) \} (s) = \frac{Bp^2 s}{(ps)^2 + (\frac{\omega}{\sqrt{p}})^2} + A \frac{\sqrt{p}}{\omega} \frac{p^3 \omega}{(ps)^2 + (\frac{\omega}{\sqrt{p}})^2}.
\]

Hence, the solutions of the \((p, q)\)-oscillators are

\[
f(t) = B \cos_{p,q} \left( \frac{\omega}{\sqrt{p}} t \right) + A \frac{\sqrt{p}}{\omega} \sin_{p,q} \left( \frac{\omega}{\sqrt{p}} t \right).
\]

5. CONCLUSION AND PERSPECTIVES

In this work, we have introduced two Laplace transforms. Many properties of these new transforms have been proved. This work is certainly not complete and should be a starting point of many other works. For example, in future works, one could define the \((p, q)\)-convolution product and compute its \((p, q)\)-Laplace transform. This will of course enable to solve some \((p, q)\)-convolution equations. Also, another work will be to find the inversion formula for these transform, so we could be able to solve many more \((p, q)\)-differential equations.

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