ON BODIES IN $\mathbb{R}^5$ WITH DIRECTLY CONGRUENT PROJECTIONS OR SECTIONS

M. ANGELES ALFONSECA, MICHELLE CORDIER, AND DMITRY RYABOGIN

Abstract. Let $K$ and $L$ be two convex bodies in $\mathbb{R}^5$ with countably many diameters, such that their projections onto all 4 dimensional subspaces containing one fixed diameter are directly congruent. We show that if these projections have no rotational symmetries, and the projections of $K, L$ on certain 3 dimensional subspaces have no symmetries, then $K = \pm L$ up to a translation. We also prove the corresponding result for sections of star bodies.

1. Introduction

In this paper we address the following problems (see [Ga, Problem 3.2, page 125 and Problem 7.3, page 289]).

Problem 1. Suppose that $2 \leq k \leq n-1$ and that $K$ and $L$ are convex bodies in $\mathbb{R}^n$ such that the projection $K|H$ is congruent to $L|H$ for all $H \in \mathcal{G}(n,k)$. Is $K$ a translate of $\pm L$?

Problem 2. Suppose that $2 \leq k \leq n-1$ and that $K$ and $L$ are star bodies in $\mathbb{R}^n$ such that the section $K \cap H$ is congruent to $L \cap H$ for all $H \in \mathcal{G}(n,k)$. Is $K$ a translate of $\pm L$?

Here we say that $K|H$, the projection of $K$ onto $H$, is congruent to $L|H$ if there exists an orthogonal transformation $\varphi \in O(k,H)$ in $H$ such that $\varphi(K|H)$ is a translate of $L|H$; $\mathcal{G}(n,k)$ stands for the Grassmann manifold of all $k$ dimensional subspaces in $\mathbb{R}^n$.

Several partial results are known for Problems 1 and 2. For symmetric bodies, the answer is affirmative due to theorems of Aleksandrov (for Problem 1 see [A] and [Ga, Theorem 3.3.6, page 115]) and Funk (for Problem 2 see [Ga, Theorem 7.2.6, page 281]). In the class of convex polytopes, the answer to both problems is also affirmative [MyR]. If the projections are translates of each other, or if the bodies are convex and the corresponding sections are translates of each other, again a positive result is obtained (see [Ga, Theorems 3.1.3 and 7.1.1] and [R1]). For history and additional partial results, we refer the reader to [ACR, My, R, R2].

Key words and phrases. Projections and sections of convex bodies.

The third author is supported in part by U.S. National Science Foundation Grant DMS-1600753.
Hadwiger established that, for \( n \geq 4 \) and \( k = n - 1 \), if the orthogonal transformations between the projections are all translations, it is not necessary to know the the projections onto all subspaces; instead, it is enough to have information about the projections on all subspaces containing a fixed line (see [Ha], and [Ga, pages 126–127]).

In this paper, we obtain several Hadwiger-type results for both Problems when \( k = 4 \) in the case of direct congruence; the fixed line will be given by the direction of one of the diameters of the body \( K \) (see Section 2 for the definitions of direct congruence and diameter). We follow the ideas from [Go], [R] and [ACR], where similar results were obtained in the cases \( k = 2, 3 \).

The case \( k = 4 \) is harder, due to the fact that four dimensional rotations are more difficult to handle than two or three dimensional ones. Nevertheless, here we obtain the expected conclusion of Problems 1 and 2 that \( K = \pm L \) up to a translation, while in [ACR] the conclusion was that \( K = L \) or \( K = OL \) up to a translation, for a certain orthogonal transformation \( O \) of \( \mathbb{R}^n \).

1.1. Results about directly congruent projections. Let \( n \geq 4 \) and \( S^{n-1} \) be the unit sphere in \( \mathbb{R}^n \). Given \( w \in S^{n-1} \), let \( w^\perp \) be the \((n-1)\) dimensional subspace of \( \mathbb{R}^n \) that is orthogonal to \( w \). We denote by \( d_K(\zeta) \) a diameter of the body \( K \) which is parallel to the direction \( \zeta \in S^{n-1} \).

Let \( D \) and \( B \) be two subsets of \( H \in G(n,k) \), \( 3 \leq k \leq n - 1 \). We say that \( D \) and \( B \) are directly congruent if \( \varphi(D) = B + a \) for some vector \( a \in H \) and some rotation \( \varphi \in SO(k,H) \). We also say that \( D \) has an \( SO(k) \) symmetry (respectively, \( O(k) \) symmetry) if \( \varphi(D) = D + a \) for some vector \( a \in H \) and some non-identical rotation \( \varphi \in SO(k,H) \) (respectively, in \( O(k,H) \)).

We prove the following 5 dimensional result.

**Theorem 1.** Let \( K \) and \( L \) be two convex bodies in \( \mathbb{R}^5 \) having countably many diameters. Assume that there exists a diameter \( d_K(\zeta) \), such that the side projections \( K|H, L|H \) onto all four dimensional subspaces \( H \) containing \( \zeta \) are directly congruent, see Figure 1. Assume also that these projections have no \( SO(4) \) symmetries, and that the three dimensional projections \( K|(H \cap \zeta^\perp), L|(H \cap \zeta^\perp) \) have no \( O(3) \) symmetries. Then \( K = L + b \) or \( K = -L + b \) for some \( b \in \mathbb{R}^5 \).

We state a generalization of Theorem 1 to \( n \) dimensions as a Corollary.

**Corollary 1.** Let \( K \) and \( L \) be two convex bodies in \( \mathbb{R}^n \) having countably many diameters. Assume that there exists a diameter \( d_K(\zeta) \), such that the projections \( K|H, L|H \) onto all four dimensional subspaces \( H \) containing \( \zeta \) are directly congruent. Assume also that these projections have no \( SO(4) \) symmetries, and that the three dimensional projections \( K|(H \cap \zeta^\perp) \) and \( L|(H \cap \zeta^\perp) \) have no \( O(3) \) symmetries. Then \( K = L + b \) or \( K = -L + b \) for some \( b \in \mathbb{R}^n \).
1.2. Results about directly congruent sections. We also obtain results related to Problem 2.

**Theorem 2.** Let $K$ and $L$ be two star bodies in $\mathbb{R}^5$ having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$, containing the origin, such that the side sections $K \cap H$, $L \cap H$ by all four dimensional subspaces $H$ containing $\zeta$ are directly congruent. Assume also that these sections have no $SO(4)$ symmetries, and that the three dimensional sections $K \cap (H \cap \zeta^\perp)$, $L \cap (H \cap \zeta^\perp)$ have no $O(3)$ symmetries. Then $K = L + b$ or $K = -L + b$ for some $b \in \mathbb{R}^5$ parallel to $\zeta$.

The $n$ dimensional generalization of Theorem 2 is stated as a Corollary.

**Corollary 2.** Let $K$ and $L$ be two star bodies in $\mathbb{R}^n$ having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$, containing the origin, such that the sections $K \cap H$, $L \cap H$ by all four dimensional subspaces $H$ containing $\zeta$ are directly congruent. Assume also that these sections have no $SO(4)$ symmetries, and that the three dimensional sections $K \cap (H \cap \zeta^\perp)$, $L \cap (H \cap \zeta^\perp)$ have no $O(3)$ symmetries. Then $K = L + b$ or $K = -L + b$ for some $b \in \mathbb{R}^n$ parallel to $\zeta$.

The problems still remain open if there is no diameter condition on the bodies. In our proof, this condition allows us to consider only the information on projections or sections on the subspaces containing a diameter, in the spirit of Hadwiger’s result. We remark that the class of convex bodies having a finite number of diameters includes all convex polytopes (which are dense in the class of convex bodies with respect to the Hausdorff metric). In fact, the class of convex polytopes whose three and four dimensional projections have no rigid motion symmetries is also dense in the class of convex bodies with respect to the Hausdorff metric ([Pa], see also [ACR, Proposition 2]). We further observe that the assumption on the countability of the sets of the diameters of $K$ and $L$ can be weakened (for example, the set of...
diameters may be taken to be contained in a countable union of great circles containing $\zeta$). In the proofs below, we only need that the set of unit vectors parallel to diameters be nowhere dense on the sphere.

On the other hand, the restriction of a rotation $\Phi \in SO(4)$ to a two dimensional invariant subspace $\Pi$ containing a diameter is an involution $(\Phi|\Pi)^2 = I$, and it seems to be necessary to exclude involutions from our considerations (in $[Zh]$, a counterexample for Problems 1 and 2 is constructed when the orthogonal transformation is an involution).

The paper is organized as follows. In Section 2 we introduce the needed definitions and notation. In Section 3 we prove the main auxiliary result of the paper, a functional equation similar to Proposition 1 in $[ACR]$. In Section 4 we prove Theorem 1 and Corollary 1, and in Section 5 we prove Theorem 2 and Corollary 2.

2. Notation and auxiliary definitions

We will use the following standard notation. The unit sphere in $\mathbb{R}^n$, $n \geq 2$, is $S^{n-1}$. Given $w \in S^{n-1}$, the hyperplane orthogonal to $w$ and passing through the origin will be denoted by $w^\perp = \{x \in \mathbb{R}^n : x \cdot w = 0\}$, where $x \cdot w = x_1w_1 + \cdots + x_nw_n$ is the usual inner product in $\mathbb{R}^n$. The Grassmann manifold of all $k$ dimensional subspaces in $\mathbb{R}^n$ will be denoted by $G(n, k)$. The orthogonal group in $\mathbb{R}^n$ is denoted by $O(n)$, and the special orthogonal group in $\mathbb{R}^n$ by $SO(n)$. If $U \in O(n)$ is an orthogonal matrix, we will write $U^t$ for its transpose.

We refer to $[Ga]$, Chapter 1, for the next definitions involving convex and star bodies. A body in $\mathbb{R}^n$ is a compact set which is equal to the closure of its non-empty interior. A convex body is a body $K$ such that for every pair of points in $K$, the segment joining them is contained in $K$. For $x \in \mathbb{R}^n$, the support function of a convex body $K$ is defined as $h_K(x) = \max\{x \cdot y : y \in K\}$ (see page 16 in $[Ga]$). The width function $\omega_K(x)$ of $K$ in the direction $x \in S^{n-1}$ is defined as $\omega_K(x) = h_K(x) + h_K(-x)$. A segment $[z, y] \subset K$ is called a diameter of the convex body $K$ if $|z - y| = \max_{\theta \in S^{n-1}} \omega_K(\theta)$. We say that a convex body $K \subset \mathbb{R}^n$ has countably many diameters if the width function $\omega_K$ reaches its maximum on a countable subset of $S^{n-1}$.

Observe that a convex body $K$ has at most one diameter parallel to a given direction $\zeta \in S^{n-1}$ (for, if $K$ had two parallel diameters $d_1$, $d_2$, then $K$ would contain a parallelogram with sides $d_1$ and $d_2$, one of whose diagonals is longer than $d_1$). For this reason, if $K$ has a diameter parallel to $\zeta \in S^{n-1}$, we will denote it by $d_K(\zeta)$.

A set $S \subset \mathbb{R}^n$ is said to be star-shaped with respect to an interior point $p$ if the line segment from $p$ to any point in $S$ is contained in $S$. For $x \in \mathbb{R}^n \setminus \{0\}$, and $K \subset \mathbb{R}^n$ a nonempty, compact, star-shaped set with respect to the origin, the radial function of $K$ is defined as $\rho_K(x) = \max\{c : cx \in K\}$ ($[Ga]$ page 18). We say that a body $K$ is a star body if $K$ is star-shaped with respect to the origin and its radial function $\rho_K$ is continuous.
Given a star body $K$, a segment $[z, y] \subset K$ is called a diameter of $K$ if $|z - y| = \max \{ |a - b| : \{a, b\} \subset K \}$. If a non-convex star body $K$ has a diameter containing the origin that is parallel to $\zeta \in S^{n-1}$, we will also denote it by $d_K(\zeta)$.

Given $\zeta \in S^{n-1}$, the great $(n - 2)$ dimensional subsphere of $S^{n-1}$ that is perpendicular to $\zeta$ will be denoted by $S_{n-2}^{n-2}(\zeta) = \{ \theta \in S^{n-1} : \theta \cdot \zeta = 0 \}$. For $t \in [-1, 1]$, the parallel subsphere to $S_{n-2}^{n-2}(\zeta)$ at height $t$ will be denoted by $S_{n-2}^{n-2}(\zeta) = S^{n-1} \cap \{ x \in \mathbb{R}^n : x \cdot \zeta = t \}$. Observe that when $t = 0$, $S_{n-2}^{n-2}(\zeta) = S^{n-2}(\zeta)$. Figure 2 shows the case $n = 5$.

For $w \in S^4$, we will denote by $O(4, S^3(w))$, $SO(4, S^3(w))$, the orthogonal transformations in the 4 dimensional subspace spanned by the great subsphere $S^3(w)$ of $S^4$. The restriction of a transformation $\varphi \in O(n)$ onto the subspace of smallest dimension containing $W \subset S^{n-1}$ will be denoted by $\varphi|_W$. $I$ stands for the identity transformation.

Finally, we define the notion of symmetry for functions, as it will be used throughout the paper.

**Definition 1.** Let $f$ be a continuous function on $S^{n-1}$ and let $\xi \in S^{n-1}$. We say that the restriction of $f$ onto $S^{k-1}(\xi)$ (or just $f$) has an $SO(k)$ symmetry if for some non-identical rotation $\varphi_\xi \in SO(k, S^{k-1}(\xi))$, we have $f \circ \varphi_\xi = f$ on $S^{k-1}(\xi)$. We similarly define the property that $f$ has an $O(k)$ symmetry.

3. A result about a functional equation on $S^4$

**Proposition 1.** Let $f$ and $g$ be two continuous functions on $S^1$. Assume that for some $\zeta \in S^4$ and for every $w \in S^3(\zeta)$ there exists a rotation $\varphi_w \in SO(4, S^3(w))$, verifying that

$$f \circ \varphi_w(\theta) = g(\theta), \quad \forall \theta \in S^3(w).$$
Assume, in addition, that \( \varphi_w(\zeta) = \pm \zeta \) \( \forall w \in S^3(\zeta) \), that the restrictions of \( f \) and \( g \) to each \( S^3(w) \) have no \( SO(4) \) symmetries, and that the restrictions of \( f \) and \( g \) to each \( S^3(w) \cap S^3(\zeta) \) have no \( O(3) \) symmetries.

Then either \( f = g \) on \( S^4 \) or \( f(\theta) = g(-\theta) \forall \theta \in S^4 \).

3.1. Auxiliary Lemmata. We will divide the proof of Proposition 1 in several lemmata. The first Lemma describes the structure of the four dimensional rotations \( \varphi \) that satisfy the condition \( \varphi(\zeta) = \pm \zeta \). Before we state it, let us recall some facts about \( SO(4) \) (see, for example, [St] for proofs).

A rotation \( \varphi \in SO(4) \) such that \( \varphi \neq \pm I \) has (at least) two orthogonal invariant two dimensional subspaces, and the restriction of \( \varphi \) to each of them is a usual two dimensional rotation. If the two angles corresponding to each of these two subspaces are different, then the subspaces are uniquely determined. In this case, we will denote them by \( \Pi \) and \( \Pi^\perp \).

**Lemma 1.** Let \( \varphi \in SO(4) \) and let \( \zeta \in \mathbb{R}^4 \), \( \zeta \neq 0 \), such that \( \varphi(\zeta) = \pm \zeta \). Assume that \( \varphi = \pm I \), then \( \Pi \) and \( \Pi^\perp \) are uniquely determined, without loss of generality \( \zeta \in \Pi^\perp \) and the restriction \( \varphi|\Pi^\perp \) is either a trivial rotation or a rotation by \( \pi \).

**Proof.** Consider first the case in which \( \varphi(\zeta) = \zeta \), i.e. \( \zeta \) is an eigenvector of \( \varphi \) with eigenvalue 1. Then \( \zeta \) must belong to one of the (possibly many) two dimensional invariant subspaces, which we will call \( \Pi^\perp \), and \( \varphi|\Pi^\perp \) is a trivial two dimensional rotation. Since \( \varphi \neq I \), the orthogonal subspace to \( \Pi^\perp \), which we denote by \( \Pi \), cannot be an eigenspace associated to 1. Thus, the subspaces \( \Pi \) and \( \Pi^\perp \) are uniquely determined. The case \( \varphi(\zeta) = -\zeta \) can be treated similarly.

The next Lemma collects some elementary facts about the rotations \( \varphi_w \) given by Proposition 1.

**Lemma 2.** Let \( w \in S^3(\zeta) \) and let \( \varphi_w \in SO(4,S^3(w)) \) be as in the statement of Proposition 1. Then

(a) For each \( w \in S^3(\zeta) \), the rotation \( \varphi_w \) is uniquely determined.
(b) The function mapping \( w \to \varphi_w \) is continuous, and the invariant subspaces \( \Pi_w \) and \( \Pi^\perp_w \) vary continuously with \( w \in S^3(\zeta) \).
(c) \( \varphi_w \) maps \( S^3(\zeta) \cap S^3(w) \) to itself.
(d) Let \( \mathcal{O} \in SO(5) \) be the orthogonal transformation defined by \( \mathcal{O}(\zeta) = \zeta \) and \( \mathcal{O}|_{S^3(\zeta)} = -I \). Then \( \mathcal{O}|_{S^3(w)} \) commutes with \( \varphi_w \in SO(4,S^3(w)) \).

**Proof.** (a) We show that for each \( w \in S^3(\zeta) \), the rotation \( \varphi_w \in SO(4,S^3(w)) \) given by Proposition 1 is unique. On \( \mathbb{R}^5 \) we consider a positively oriented orthonormal basis \( \{u,v,z,\zeta,w\} \), such that \( \{u,v,z,\zeta\} \) is a basis for \( w^\perp \). We think of \( \varphi_w \in SO(4,S^3(w)) \) as restriction to \( w^\perp \) of a rotation \( \Phi_w \in SO(5) \) such that \( \Phi_w(w) = w \).
Assume that there are two different rotations $\varphi_1, \varphi_2 \in SO(4, S^3(w))$, verifying
\[ f \circ \varphi_1(\theta) = g(\theta), \quad \forall \theta \in S^3(w) \]
and
\[ f \circ \varphi_2(\theta) = g(\theta), \quad \forall \theta \in S^3(w). \]
Then $f \circ \varphi_1 \circ (\varphi_2)^{-1}(\theta) = f(\theta)$ for all $\theta \in S^3(w)$, where $\varphi_1 \circ (\varphi_2)^{-1}$ is not the identity. Then $f$ has a rotational symmetry on $S^3(w)$, contradicting the hypotheses of Proposition 1.

(b) We prove the continuity of the map $w \to \varphi_w$. Let $(w_l)_{l=1}^\infty$ be a sequence of elements of $S^3(\zeta)$ converging to $w \in S^3(\zeta)$ as $l \to \infty$, and let $\theta$ be any point on $S^3(w)$. Consider a sequence $(\theta_l)_{l=1}^\infty$ of points $\theta_l \in S^3(w_l)$ converging to $\theta$ as $l \to \infty$ (to see why such a sequence exists, see [ACR, Lemma 3], where an analogous statement is proved). Let $\Phi_{w_l}$ be the rotation in $\mathbb{R}^5$ such that $\Phi_{w_l}(w_l) = w_l$ and $\Phi_{w_l}|w_l^\perp = \varphi_{w_l}$. By compactness, the sequence $\{\Phi_{w_l}\} \subseteq SO(5)$ has a convergent subsequence. Suppose that $(\Phi_{w_l})_{l=1}^\infty$ has two subsequences that converge to different rotations in $SO(5)$, $(\Phi_{w_j}^1) \to \Phi_{w^1}$ and $(\Phi_{w_j}^2) \to \Phi_{w^2}$ as $j \to \infty$, where $\Phi_{w^1} \neq \Phi_{w^2}$. Since $w_l$ converges to $w$, and $\Phi_{w_l}(w_l) = w_l$, we have that $\Phi_{w^1}(w) = w$, $\Phi_{w^2}(w) = w$. Let $\varphi_{w^j}$ be the restriction of $\Phi_{w^j}$ to the subspace $w^\perp$, and similarly $\varphi_{w^2} = \Phi_{w^2}|w^\perp$. We know that $f \circ \varphi_{w^j}(\theta_j) = g(\theta_j)$ and by passing to the limit as $j \to \infty$ we obtain that $f \circ \varphi_{w^1}(\theta) = g(\theta)$. Similarly, we have $f \circ \varphi_{w^2}(\theta) = g(\theta)$. This implies that $f \circ \varphi_{w^1}(\theta) = f \circ \varphi_{w^2}(\theta)$.

Assume that there are two different rotations $\varphi_1, \varphi_2 \in SO(4, S^3(w))$, verifying
\[ f \circ \varphi_1(\theta) = g(\theta), \quad \forall \theta \in S^3(w) \]
and
\[ f \circ \varphi_2(\theta) = g(\theta), \quad \forall \theta \in S^3(w). \]
Then $f \circ \varphi_1 \circ (\varphi_2)^{-1}(\theta) = f(\theta)$ for all $\theta \in S^3(w)$, where $\varphi_1 \circ (\varphi_2)^{-1}$ is not the identity. Then $f$ has a rotational symmetry on $S^3(w)$, which is a contradiction. Therefore, all convergent subsequences of $\{\Phi_{w_l}\}$ must have the same limit, which we will denote by $\Phi_w$. Since $\Phi_{w_l}(w_l) = w_l$, it follows that $\Phi_w(w) = w$, and the restriction $\varphi_w = \Phi_w|w^\perp$ is in $SO(4, S^3(w))$.

If $\Phi_w|w^\perp \neq \pm I$, let $\Pi_w$ and $\Pi_w^\perp$ be the unique invariant two dimensional subspaces guaranteed by Lemma 1 and assume that $\Phi_w|\Pi_w^\perp = I$ (the case $\Phi_w|\Pi_w^\perp = -I$ can be treated in a similar way). Consider a sequence $\{\Phi_{w_l}\}$ converging to $\Phi_w$. Then, we must have (for a subsequence) that $\Phi_{w_l}|w_l^\perp \neq \pm I$, and letting $\{u_l, v_l, \zeta, z_l, w_l\}$ be the orthonormal basis of $\mathbb{R}^5$ such that $\Pi_{w_l} = \text{span}(u_l, v_l)$ and $\Pi_{w_l}^\perp = \text{span}(\zeta, z_l)$, $\Phi_{w_l}|\Pi_{w_l}^\perp = I$. Consider the case where $\Phi_{w_l}|\Pi_{w_l} = I$ (the other case is analogous). Given that $\Phi_{w_l}(\zeta) = \zeta$ for all $l$, we obtain that $\Phi_w(\zeta) = \zeta$. We also know that $\Phi_{w_l}(z_l) = z_l$ for all $l$, hence there exists $z \in S^3(w)$ such that a subsequence of $\{z_l\}$ converges to $z$, and $\Phi_w(z) = z$. Therefore, $\text{span}(\zeta, z)$ must coincide with $\Pi_w^\perp$, and it follows that $\Pi_{w_l}$ converges to $\Pi_w$. 

Directly Congruent Projections in $\mathbb{R}^3$
(c) Given \( \varphi_w \in SO(4, S^3(w)) \) such that \( \varphi_w(\zeta) = \pm \zeta \), by Lemma 3 there are two invariant two dimensional subspaces, \( \Pi_w \) and \( \Pi_w^\perp \), where \( \zeta \in \Pi_w^\perp \), and \( \varphi_w|\Pi_w^\perp = \pm I \). If \( z \in S^3(w) \) is such that \( \{\zeta, z\} \) is an orthogonal basis for \( \Pi_w \), we have that \( \varphi_w(z) = \pm z \) and hence \( \text{span}(z) \) is an invariant subspace for \( \varphi_w \).

Since \( \zeta \) is orthogonal to \( z \) and \( \Pi_w \), we have that \( S^3(\zeta) \cap S^3(w) = S^3(w) \cap (\Pi_w \oplus \text{span}(z)) \). Given that both \( \Pi_w \) and \( \text{span}(z) \) are invariant subspaces for \( \varphi_w \), it follows that \( \varphi_w \) maps \( S^3(\zeta) \cap S^3(w) \) to itself.

(d) By the definition of \( O \), both \( \Pi_w \) and \( \text{span}(z) \) are eigenspaces for \( O \), with eigenvalue \(-1\), while \( \text{span}(\zeta) \) is an eigenspace with eigenvalue \(1\). Hence, \( \varphi_w \) and \( O \) commute on each of them. Since \( S^3(w) = \Pi_w \oplus \text{span}(\zeta) \oplus \text{span}(z) \), the result follows.

\( \square \)

The next Lemma is an observation about the geometry of the sphere.

**Lemma 3.** Let \( \zeta \) and \( x \) be in \( S^k \), \( k \geq 3 \). Then,

\[
\bigcup_{w \in S^{k-1}(\zeta) \cap S^{k-1}(x)} S^{k-1}(w) = S^k.
\]

**Proof.** Let \( y \) be any point on \( S^k \). Then \( S^{k-1}(\zeta) \cap S^{k-1}(x) \cap S^{k-1}(y) \) is nonempty, since \( k \geq 3 \). Taking any \( w \in S^{k-1}(\zeta) \cap S^{k-1}(x) \cap S^{k-1}(y) \subset S^{k-1}(\zeta) \cap S^{k-1}(x) \), it follows that \( y \in S^{k-1}(w) \). \( \square \)

**Lemma 4.** (cf. Lemma 1, [R].) Let \( \zeta \in S^k \), \( k \geq 4 \). If for every \( w \in S^{k-1}(\zeta) \) we have either \( f(\theta) = g(\theta) \) for all \( \theta \in S^{k-1}(w) \) or \( f(-\theta) = g(\theta) \) for all \( \theta \in S^{k-1}(w) \), then either \( f = g \) on \( S^k \) or \( f(-\theta) = g(\theta) \) for all \( \theta \in S^k \).

**Proof.** Assume at first that there exists an \( x \in S^k \) such that for all \( w \in S^{k-1}(\zeta) \cap S^{k-1}(x) \) we have \( f(\theta) = g(\theta) \) for all \( \theta \in S^{k-1}(w) \). Then, using the previous lemma, we obtain \( f = g \) on \( S^k \).

Assume now that there exists an \( x \in S^k \) such that for all \( w \in S^{k-1}(\zeta) \cap S^{k-1}(x) \) we have \( f(-\theta) = g(\theta) \) for all \( \theta \in S^{k-1}(w) \). Then, using the previous lemma, we obtain \( f(-\theta) = g(\theta) \) for all \( \theta \in S^k \).

Finally, assume that for every \( x \in S^k \) there exist two directions \( w_1 \) and \( w_2 \) in \( S^{k-1}(\zeta) \cap S^{k-1}(x) \) such that \( f(\theta) = g(\theta) \) for all \( \theta \in S^{k-1}(w_1) \) and \( f(-\theta) = g(\theta) \) for all \( \theta \in S^{k-1}(w_2) \). Then \( f(-x) = f(x) = g(x) \), and since \( x \) was chosen arbitrarily, we obtain \( f = g \) on \( S^k \). \( \square \)

Let \( O \in SO(5) \) be the orthogonal transformation defined in Lemma 2 (d) by \( O(\zeta) = \zeta \) and \( O|S^3(\zeta) = -I \). A function \( f \) defined on \( S^k \) can be decomposed in the form

\[
(2) \quad f(\theta) = \frac{f(\theta) + f(O\theta)}{2} + \frac{f(\theta) - f(O\theta)}{2} = f_{O^{\circ}}(\theta) + f_{O^+}(\theta), \quad \theta \in S^4,
\]
and we will call $f_{O,e}$, $f_{O,o}$, the even and odd parts of $f$ with respect to $O$. Since $O^2 = I$, we have

$$f_{O,e}(\theta) = f_{O,e}(O\theta), \quad f_{O,o}(\theta) = -f_{O,o}(O\theta).$$

Given $y \in S^4$, we have that $y \in S^3(\zeta)$ for some $t \in [-1,1]$, i.e. we can write

$$y = \sqrt{1 - t^2}x + t\zeta,$$

for some $t \in [-1,1]$ and $x \in S^3(\zeta)$ (see Figure 2). For any function $f$ on $S^4$, we define the function $F_t$ on $S^3(\zeta)$,

$$F_t(x) = F_{t,\zeta}(x) = f(\sqrt{1 - t^2}x + t\zeta), \quad x \in S^3(\zeta),$$

which is the restriction of $f$ to $S^3(\zeta)$. Observe that the even part of $F_t$, $(F_t)_e$ equals

$$(F_t)_e(x) = \frac{f(\sqrt{1 - t^2}x + t\zeta) + f(-\sqrt{1 - t^2}x + t\zeta)}{2} = \frac{f(y) + f(Oy)}{2},$$

where $y$ is as in (3), i.e.,

$$(F_t)_e(x) = f_{O,e}(y), \quad (F_t)_o(x) = f_{O,o}(y).$$

Note that $(F_t)_e(x) = (F_t)_e(-x)$ for every $x \in S^3(\zeta)$. We similarly define $G_t$ from the function $g$.

Every two dimensional great subshpere of $S^3(\zeta)$ is of the form $E_w := S^3(w) \cap S^3(\zeta)$ for some $w \in S^3(\zeta)$. For $\varphi_w \in SO(4, S^3(w))$ as in Proposition 1 we have that $\varphi_w(E_w) = E_w$ by Lemma 2(c). Denote by $\phi_{E_w} = \varphi_w|_{E_w}$ the restriction of $\varphi_w$ to $E_w$, and define $G_t$ from $g$ similarly to $F_t$ in (4). Observe that if $y$ is as in (3) with $x \in E_w$, then $\varphi_w(y) = \sqrt{1 - t^2}\phi_{E_w}(x) \pm t\zeta$, and it follows from (6) that, for every $t \in [-1,1],$

$$(F_t)_e(x) = G_t(x) \quad \forall x \in E_w.$$

Lemma 5. Assume that $f, g$ satisfy equation (1) for all $w \in S^3(\zeta)$. Then $f_{O,e}(y) = g_{O,e}(y)$ for every $y \in S^4$.

Proof. For $w \in S^3(\zeta)$, we consider the spherical Radon transform

$$Rf(w) = \int_{E_w} f(x)dx$$

(see [Ga, pg. 429]). Since Lebesgue measure is invariant under orthogonal transformations on $E_w$, by (6) we have

$$\int_{E_w} F_t(x)dx = \int_{E_w} F_{t,\zeta}(x)dx = \int_{E_w} G_{t}(x)dx,$$

for each $t \in [-1,1]$. Hence, $RF_t(w) = RG_t(w)$ for every $w \in S^3(\zeta)$ and $t \in [-1,1]$, and it follows from Proposition 3.4.12 [Gr, pg. 108] that the even parts of $F_t$ and $G_t$ coincide. By equations (4) and (5) and Lemma 2(d), this means that $f_{O,e}(y) = g_{O,e}(y)$ for every $y \in S^4$. \qed
Note: Because of Lemma 5 from now on we will assume that $f, g$ are odd with respect to $O$.

By Lemmas 1 and 2 given $w \in S^3(\zeta)$ and $\varphi_w \neq \pm I$ verifying the hypotheses of Proposition 1 we have that $w^\perp = \Pi_w \oplus \Pi_w^\perp$, where $\zeta \in \Pi_w^\perp$. $\varphi_w|\Pi_w^\perp = \pm I$, and $\varphi_w|\Pi_w$ is a 2 dimensional rotation. Given $t \in \Pi_w \cap S^3$, if the angle between the vectors $t$ and $\varphi_w(t) \in \Pi_w \cap S^3$ is $\pi$, for $\alpha \in (0, 2)$, $\alpha \neq 1$, and $\{t, \varphi_w(t), \zeta, z, w\}$ forms a positively oriented basis of $\mathbb{R}^5$, then we will denote $\varphi_w|\Pi_w$ by $\varphi_{w}^{\alpha \pi}$ when we want to specify the angle of rotation.

We define the sets
\[ \Xi_+ = \left\{ w \in S^3(\zeta) : f(\theta) = g(\theta) \forall \theta \in S^3(w) \right\}, \]
\[ \Xi_- = \left\{ w \in S^3(\zeta) : f(\theta) = g(-\theta) \forall \theta \in S^3(w) \right\}, \]
\[ \Xi_0 = \left\{ w \in S^3(\zeta) : f|\Pi_w = g|\Pi_w, \text{ and } f(\theta) = g(-\theta) \forall \theta \in \Pi_w^\perp \right\}, \]
\[ \Xi_1 = \left\{ w \in S^3(\zeta) : f|\Pi_w^\perp = g|\Pi_w^\perp, \text{ and } f(\theta) = g(-\theta) \forall \theta \in \Pi_w \right\}, \]
and, for $\alpha \in (0, 1) \cup (1, 2)$,
\[ \Xi_\alpha = \left\{ w \in S^3(\zeta) : \varphi_w|\Pi_w = \pm I, \varphi_w|\Pi_w = \varphi_w^{\alpha \pi} ight\}. \]

With this notation, the hypothesis of Proposition 1 is that
\[ S^3(\zeta) = \Xi_+ \cup \Xi_- \cup \bigcup_{\alpha \in [0, 2]} \Xi_\alpha, \]
and we want to show that, under the condition on the lack of symmetries, we have either $S^3(\zeta) = \Xi_+$ or $S^3(\zeta) = \Xi_-$. By Lemma 4 this will imply that either $f = g$ on $S^4$ or $f(\theta) = g(-\theta)$ for all $\theta \in S^4$.

We observe that the sets on the right hand side of (11) are disjoint. Indeed, assume that there exists $z \in \Xi_\alpha \cap \Xi_\beta$, where $\alpha$ and $\beta$ are either $+, -$ or a rational number in $[0, 2]$, and $\alpha \neq \beta$. Then, there are two rotations $\varphi_{z, \alpha}, \varphi_{z, \beta} \in SO(4, S^3(z))$ such that $f \circ \varphi_{z, \alpha}(\theta) = g(\theta)$ and $f \circ \varphi_{z, \beta}(\theta) = g(\theta)$ for all $\theta \in S^3(z)$. But then $f \circ \varphi_{z, \beta}(\theta) \circ \varphi_{z, \alpha}^{-1}(\theta) = g(\theta)$ for all $\theta \in S^3(z)$, where $\varphi_{z, \beta}(\theta) \circ \varphi_{z, \alpha}^{-1}$ is not the identity since $\alpha \neq \beta$. Thus, if $\Xi_\alpha \cap \Xi_\beta \neq \emptyset$, $f$ has a symmetry on $S^3(z)$, contradicting the hypothesis of Proposition 1.

Lemma 6. The sets $\Xi_+, \Xi_-, \Xi_\alpha$ are closed.

Proof. Since the empty set is closed, we can assume that the sets $\Xi_+, \Xi_-$ and $\Xi_\alpha$ are not empty. First we prove that $\Xi_+$ is closed. Let $(w_l)_{l=1}^{\infty}$ be a sequence of elements of $\Xi_+$ converging to $w \in S^3(\zeta)$ as $l \to \infty$, and let $\theta$ be any point on $S^3(w)$. Consider a sequence $(\theta_l)_{l=1}^{\infty}$ of points $\theta_l \in S^3(w_l)$ converging to $\theta$ as $l \to \infty$. By definition of $\Xi_+$ we have the following,
\[ f(\theta_l) = g(\theta_l), \quad \theta_l \in S^3(w_l), \quad l \in \mathbb{N}. \]
Since $f$ and $g$ are continuous, we may pass to the limit and obtain $f(\theta) = g(\theta)$. Thus $w \in \Xi$, since the choice of $\theta \in S^3(w) \subset S^3(\z)$ was arbitrary, and hence $\Xi_+$ is closed. A similar proof shows that $\Xi_-$ is closed, replacing $g(\theta_l)$ with $g(-\theta_l)$ and $g(\theta)$ with $g(-\theta)$. Now we prove that $\Xi_{\alpha}$ is closed, where $\alpha \in [0, 2)$. As above, let $(w_{l})_{l=1}^{\infty}$ be a sequence of elements of $\Xi_{\alpha}$ converging to $w \in S^3(\z)$ as $l \to \infty$. Let $\varphi_{w_l}$ be the rotation associated to $w_l^{\perp}$ by Proposition [1]. By Lemma [2](b), the limit of $\varphi_{w_l}$ is the rotation $\varphi_w$, and if $\Pi_{w_l}$ and $\Pi_{w_l}^{\perp}$ are the two dimensional invariant subspaces of $\varphi_{w_l}$, and $\Pi_w$ and $\Pi_{w}^{\perp}$ are those of $\varphi_w$, we have $(\Pi_{w_l}) \to \Pi_w$ and $(\Pi_{w_l}^{\perp}) \to \Pi_w^{\perp}$ as $l \to \infty$. Hence, there is either a subsequence $\varphi_{w_j}|\Pi_{w_j}^{\perp} = I$ (which would imply $\varphi_{w}|\Pi_{w}^{\perp} = I$), or a subsequence $\varphi_{w_j}|\Pi_{w_j}^{\perp} = -I$ (which implies that $\varphi_{w}|\Pi_{w}^{\perp} = -I$). Furthermore, since all $\varphi_{w_j}|\Pi_{w_j}$ are rotations by the angle $\alpha \pi$, for the limit we obtain that $\varphi_{w}|\Pi_{w}^{\perp}$ is also a rotation by the same angle, and since

\[ (9) \quad f \circ \varphi_{w_l}^{\alpha \pi}(\theta_l) = g(\theta_l) \quad \theta_l \in \Pi_{w_l}, \quad l \in \mathbb{N}, \]

we conclude by continuity that $f \circ \varphi_{w}(\theta) = g(\theta)$. This shows that $\Xi_{\alpha}$ is closed.

\[ \Box \]

The next Lemma shows that rotations by an irrational multiple of $\pi$ do not occur because of the lack of symmetries of $f$ and $g$.

**Lemma 7.** Under the hypotheses of Proposition [1], we have $\Xi_{\alpha} = \emptyset$ for $\alpha \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 2)$.

**Proof.** Let $\alpha \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 2)$, and take $w \in \Xi_{\alpha}$. Following the ideas of Schneider [Sch], we claim at first that $f^2 = g^2$ on $S^3(w)$. Indeed, since $f$ and $g$ are odd with respect to $O$, $f^2$ and $g^2$ are even with respect to $O$, and [1] holds with $f^2$, $g^2$ instead of $f$, $g$. Thus, by Lemma [3], we obtain that $f^2 = g^2$ on $S^3(w)$.

Squaring [1], we have

\[ f^2 \circ \varphi_{w}(\theta) = g^2(\theta) = f^2(\theta) \quad \forall \theta \in S^3(w). \]

Iterating, for any $k \in \mathbb{Z}$,

\[ (10) \quad f^2 \circ \varphi_{w}^{k}(\theta) = f^2 \circ \varphi_{w}^{k-1}(\theta) = \cdots = f^2(\theta) \quad \forall \theta \in S^3(w). \]

Let $\{\zeta, z\}$ be an orthonormal basis for $\Pi_{w}^{\perp}$, and consider the three dimensional subspace generated by $\Pi_w$ and $z$, and its unit sphere $S^3(w) \subset S^3(\z)$. If $\varphi_{w}|\Pi_{w}^{\perp} = I$, then $\varphi_{w}|S^3(w) \cap S^3(\z)$ is a rotation of angle $\alpha \pi$ around the vector $z$. For each $\theta \in S^3(w) \cap S^3(\z)$, equation [10] holds for any $k \in \mathbb{Z}$, and since the orbit of $(\varphi_{w}^{k}(\theta))_{k \in \mathbb{Z}}$ is dense, we conclude that $f^2$ and $g^2$ are constant on each parallel of $S^3(w) \cap S^3(\z)$ perpendicular to $z$. By continuity, $f$ and $g$ must also be constant on each parallel, and thus $f \circ \varphi_{w}(\theta) = f(\theta)$ for every $\theta \in S^3(w) \cap S^3(\z)$. But then $f$ has an $SO(3)$ symmetry on $S^3(w) \cap S^3(\z)$, contradicting the hypothesis of Proposition [1].
On the other hand, if \( \varphi_i (\zeta) = -\zeta \), then \( \varphi_i | S^3(w) \cap S^3(\zeta) \) is a rotation around \( z \) by the angle \( 2\alpha \pi \), and similarly to the previous case, \( f \) must be constant on every parallel of \( S^3(w) \cap S^3(\zeta) \), and thus has a rotational symmetry on \( S^3(w) \cap S^3(\zeta) \). This is a contradiction, and thus we have proven that \( \Xi_\alpha = \emptyset \) for irrational \( \alpha \). \( \square \)

**Proof of Proposition 1.** By (8) and Lemma 7, we have written \( S^3(\zeta) \) as a countable union of disjoint closed sets,

\[
S^3(\zeta) = \Xi_+ \cup \Xi_- \cup \bigcup_{\alpha \in [0,2) \cap \mathbb{Q}} \Xi_\alpha.
\]

By a well-known result of Sierpiński’s [Si], \( S^3(\zeta) \) must equal just one of the sets. We will now assume that \( S^3(\zeta) = \Xi_\alpha \) for some \( \alpha \in [0,2) \cap \mathbb{Q} \), and derive a contradiction. This will leave us only with the possibilities \( S^3(\zeta) = \Xi_+ \) or \( S^3(\zeta) = \Xi_- \), and Proposition 1 will be proven.

Let \( S^3(\zeta) = \Xi_\alpha \). Choose \( w \in S^3(\zeta) \) and \( \xi \in S^3(w) \cap S^3(\zeta) \). By Lemma 1 the subspace \( \xi^\perp \) is equal to \( \Pi_\xi \oplus \Pi_\xi^\perp \), with \( \zeta \in \Pi_\xi^\perp \). Let \( \zeta \in \Pi_\xi^\perp \) be such that \( \{\zeta, \zeta\} \) is an orthonormal basis of \( \Pi_\xi^\perp \).

We will write the set \( S^3(w) \cap S^3(\zeta) \) as the union of two closed sets, \( \Theta_{\text{good}} \) and \( \Theta_{\text{bad}} \), where

\[
\Theta_{\text{good}} = \{ \xi \in S^3(w) \cap S^3(\zeta) : \Pi_\xi = \xi^\perp \cap w^\perp \cap \zeta^\perp \},
\]

\[
\Theta_{\text{bad}} = \{ \xi \in S^3(w) \cap S^3(\zeta) : \dim(\Pi_\xi \cap w^\perp \cap \zeta^\perp) = \dim(\Pi_\xi^\perp \cap w^\perp \cap \zeta^\perp) = 1 \}.
\]

Observe that \( \Theta_{\text{good}} \) and \( \Theta_{\text{bad}} \) are closed. Indeed, for any \( \xi \in \Theta_{\text{good}} \), \( \Pi_\xi^\perp \) is spanned by \( \{\zeta, w\} \), and the result follows from Lemma 2 (b). If \( \xi \in \Theta_{\text{bad}} \), then \( \Pi_\xi^\perp \) is spanned by \( \{\zeta, z\} \) and \( \Pi_\xi \) is spanned by \( \{w, v\} \) for some \( v \). Let \( \{\xi_j\} \) be a sequence from \( \Theta_{\text{bad}} \) converging to \( \xi \in S^3(w) \cap S^3(\zeta) \). Applying Lemma 2 (b) once again, we obtain that \( \Pi_{\xi_j} \) converges to \( \Pi_\xi \) and \( \Pi_{\xi_j}^\perp \) converges to \( \Pi_\xi^\perp \) as \( j \to \infty \). In particular, \( w \in \Pi_\xi \), while \( \zeta \in \Pi_\xi^\perp \), and thus \( \xi \in \Theta_{\text{bad}} \).

The sets \( \Theta_{\text{good}} \) and \( \Theta_{\text{bad}} \) are **disjoint** and their union equals \( S^3(w) \cap S^3(\zeta) \). It follows that either \( S^3(w) \cap S^3(\zeta) = \Theta_{\text{good}} \) or \( S^3(w) \cap S^3(\zeta) = \Theta_{\text{bad}} \).

Assume that \( S^3(w) \cap S^3(\zeta) = \Theta_{\text{bad}} \). We claim that, in this case, the map

\[
\xi \to \ell(\xi) = \Pi_\xi^\perp \cap w^\perp \cap \zeta^\perp
\]

defines a non-vanishing continuous tangent line field on the two dimensional sphere \( S^3(w) \cap S^3(\zeta) \). If this map were not continuous, then there would exist two subsequences \( \{\xi_j^1\} \) and \( \{\xi_j^2\} \), both with limit \( \xi_0 \), such that

\[
\lim_{j \to \infty} \ell(\xi_j^1) \neq \lim_{j \to \infty} \ell(\xi_j^2).
\]

Denote by \( z_0^1 \) a unit vector in the direction of the line \( \lim_{j \to \infty} \ell(\xi_j^1) \) and by \( z_0^2 \) a unit vector in the direction of the line \( \lim_{j \to \infty} \ell(\xi_j^2) \). We have \( z_0^1 \neq \pm z_0^2 \). Let \( \varphi_{\xi_0}^i = \lim_{j \to \infty} \varphi_{\xi_j^i} \), for \( i = 1, 2 \), and let \( \Pi_{\xi_0}^i = (\Pi_{\xi_0})^\perp \) the corresponding
decompositions of $\xi_0^\perp$. Since all the rotations $\varphi_{\xi_0} |_{\Pi_{\xi_0}}$ are by the angle $\alpha \pi$, the limiting rotations $\varphi_{\xi_0}^1 |_{\Pi_{\xi_0}^1}$ and $\varphi_{\xi_0}^2 |_{\Pi_{\xi_0}^2}$ are by the angle $\alpha \pi$ as well (this can be shown by a reasoning similar to the one in the proof of Lemma 6). But given that $z_{\xi_0}^1 = \pm z_{\xi_0}^2$, the corresponding limiting subspaces $(\Pi_{\xi_0}^1)^\perp$ and $(\Pi_{\xi_0}^2)^\perp$ must be different. This means that $\varphi_{\xi_0}^1$ and $\varphi_{\xi_0}^2$ are two different rotations on $\xi_0^\perp$. From equation (1), it follows that $f \circ \varphi_{\xi_0}^1 \circ (\varphi_{\xi_0}^2)^{-1} = f$ on $S^3(\xi_0)$, where $\varphi_{\xi_0}^1 \circ (\varphi_{\xi_0}^2)^{-1} \neq I$. Therefore, we conclude that $f$ has a rotational symmetry on $\xi_0^\perp$. Thus, the line field spanned by $\ell(\xi)$ must be continuous on the two dimensional sphere $S^3(w) \cap S^3(\xi)$. This is impossible by a well known result of Hopf (see [Sa]).

We now consider the case in which $S^3(w) \cap S^3(\xi) = \Theta_{\text{good}}$, i.e. the two dimensional space $\Pi_{\xi}$ is equal to $\xi^\perp \cap w^\perp \cap \xi^\perp$, and the restriction of $\varphi_{\xi}$ to this subspace, which we will denote by $\psi_{\xi}$, is a rotation by the angle $\alpha \pi$. We have that the restrictions of $f$ and $g$ to $S^3(w) \cap S^3(\xi)$ satisfy

\begin{equation}
(12) \quad f \circ \psi_{\xi}(\theta) = g(\theta), \ \forall \theta \in \Pi_{\xi},
\end{equation}

for every $\xi \in S^3(w) \cap S^3(\xi)$. But every one dimensional great circle on the two dimensional sphere $S^3(w) \cap S^3(\xi)$ is of the form $S^3(\xi) \cap S^3(w) \cap S^3(\xi)$ for some $\xi \in S^3(w) \cap S^3(\xi)$. We are thus under the hypothesis of the continuous Rubik’s cube [H], and therefore we can conclude that either $f = g$ on $S^3(w) \cap S^3(\xi)$, or $f(\theta) = g(-\theta)$ for every $\theta \in S^3(w) \cap S^3(\xi)$.

Therefore, we have

\[ f \circ \varphi_w(\theta) = g(\theta) \quad \text{and} \quad f(\theta) = g(\theta), \quad \forall \theta \in S^3(w) \cap S^3(\xi), \]

or

\[ f \circ \varphi_w(\theta) = g(\theta) \quad \text{and} \quad f(\theta) = g(-\theta), \quad \forall \theta \in S^3(w) \cap S^3(\xi). \]

This implies that either

\[ f \circ \varphi_w(\theta) = f(\theta), \quad \forall \theta \in S^3(w) \cap S^3(\xi), \]

or

\[ f \circ \varphi_w(\theta) = f(-\theta), \quad \forall \theta \in S^3(w) \cap S^3(\xi), \]

where the restriction of $\varphi_w$ to $S^3(w) \cap S^3(\xi)$ is not the identity, since we are assuming that $w \in S^3(\xi) = \Xi_\alpha$. Thus, the restriction of $f$ to the 3 dimensional subspace spanned by $S^3(w) \cap S^3(\xi)$ has an $O(3)$ symmetry, contradicting the hypothesis of Proposition 1.

Since the case $S^3(\xi) = \Xi_\alpha$ leads to a contradiction, we conclude that either $S^3(\xi) = \Xi_+$, or $S^3(\xi) = \Xi_-$. Proposition 1 is proven.
4. Proof of Theorems 1 and 2

As in [ACR], the key ingredient in the proof of Theorem 1 is the existence of a diameter $d_K(\zeta)$ such that the side projections of $K$ and $L$ are directly congruent. We will first show that this implies that $L$ must also have a diameter in the $\zeta$ direction, which necessarily has the same length as $d_K(\zeta)$. We can thus translate the bodies $K$ and $L$ so that their diameters $d_K(\zeta)$ and $d_L(\zeta)$ coincide and are centered around the origin. Since the translated bodies, $K$ and $L$, have countably many diameters, almost all side projections contain only this particular diameter, which must be fixed by the rotation. Therefore, we have reduced Theorem 1 to Proposition 1 with $f = h_K$ and $g = h_L$.

4.1. Theorem 1 and Corollary 1

Let $\zeta \in S^4$ be the direction of the diameter $d_K(\zeta)$ given in Theorem 1. By hypothesis, for every $w \in S^3(\zeta)$, the projections $K|w^\perp$ and $L|w^\perp$ are directly congruent. Hence, for every $w \in S^3(\zeta)$ there exists $\chi_w \in SO(4, S^3(w))$ and $a_w \in w^\perp$ such that
\[
\chi_w(K|w^\perp) = L|w^\perp + a_w.
\]

Let $A_K \subset S^4$ be the set of directions parallel to the diameters of $K$, and $A_L \subset S^4$ be the set of directions parallel to the diameters of $L$. We define
\[
\Omega = \{w \in S^3(\zeta) : (A_K \cup A_L) \cap S^3(w) = \{\pm \zeta\}\}.
\]

The following two Lemmata are proven by the same arguments used in [ACR]. Lemma 8 shows that for most of the directions $w \in S^3(\zeta)$ the projections $K|w^\perp$ and $L|w^\perp$ have exactly one diameter, $d_K(\zeta)$ and $d_L(\zeta)$, respectively. We can thus translate the bodies $K$ and $L$ by vectors $a_K$, $a_L \in \mathbb{R}^5$, to obtain $\tilde{K} = K + a_K$ and $\tilde{L} = L + a_L$ such that their diameters $d_{\tilde{K}}(\zeta)$ and $d_{\tilde{L}}(\zeta)$ coincide and are centered at the origin.

Lemma 8. (cf. [ACR] Lemma 13.) Let $K$ and $L$ be as in Theorem 2 and let $\zeta \in A_K$. Then $\zeta \in A_L$, and $\Omega$ is everywhere dense in $S^3(\zeta)$. Moreover, for every $w \in \Omega$ we have $\chi_w(\zeta) = \pm \zeta$ and $\omega_K(\zeta) = \omega_L(\zeta)$.

Lemma 9. (cf. [ACR] Lemma 14.) Let $\chi_w$ be the rotation given by (13), and let $w \in \Omega$. Then the rotation $\varphi_w := (\chi_w)^t$ satisfies $\varphi_w(\zeta) = \pm \zeta$ and
\[
\varphi_w(\theta) = \varphi_w(\theta) = h_{\tilde{K}}(\theta) = h_{\tilde{L}}(\theta) \quad \forall \theta \in S^3(w).
\]

Proof of Theorem 1. Consider the closed sets $\Xi = \{w \in S^3(\zeta) : (15) \text{ holds with } \varphi_w(\zeta) = \zeta\}$ and $\Psi = \{w \in S^3(\zeta) : (15) \text{ holds with } \varphi_w(\zeta) = -\zeta\}$. Since the set $\Omega \subset (\Xi \cup \Psi)$ is everywhere dense in $S^3(\zeta)$ by Lemma 8 we have that $\Xi \cup \Psi = S^3(\zeta)$. We have thus reduced matters to Proposition 1 with $f = h_{\tilde{K}}$ and $g = h_{\tilde{L}}$. Therefore, either $h_{\tilde{K}} = h_{\tilde{L}}$ on $S^4$ or $h_{\tilde{K}}(\theta) = h_{\tilde{L}}(-\theta)$ for every $\theta \in S^4$. This means that either $K + a_K = L + a_L$ or $K + a_K = -L - a_L$. \hfill $\square$
Proof of Corollary \[1\]. First, we translate \( K \) and \( L \) by vectors \( a_K, a_L \in \mathbb{R}^n \), obtaining the bodies \( \bar{K} = K + a_K \) and \( \bar{L} = L + a_L \), so that the diameters \( d_K(\zeta) \) and \( d_L(\zeta) \) are centered at the origin. Next, we observe that for any five dimensional subspace \( J \) of \( \mathbb{R}^n \), containing \( \zeta \), the bodies \( \bar{K}|J \) and \( \bar{L}|J \) verify the hypotheses of Theorem \[1\]. Therefore, \( \bar{K}|J = \pm \bar{L}|J \).

Assume that there exist two five dimensional subspaces \( J_1 \) and \( J_2 \), such that \( \bar{K}|J_1 = \bar{L}|J_1 \) and \( \bar{K}|J_2 = -\bar{L}|J_2 \). If \( J_1 \cap J_2 \) has dimension four, then

\[
\bar{L}|(J_1 \cap J_2) = (\bar{L}|J_1)|(J_1 \cap J_2) = (\bar{K}|J_1)|(J_1 \cap J_2) = (\bar{K}|J_2)|(J_1 \cap J_2)
\]

(16)

\[
= (-\bar{L}|J_2)|(J_1 \cap J_2) = -\bar{L}|(J_1 \cap J_2).
\]

Since \(-I \in SO(4)\), equation (16) implies that the projection \( \bar{L}|(J_1 \cap J_2) \) has an \( SO(4) \) symmetry, contradicting the assumptions of the corollary. The same argument shows that if \( J_1 \cap J_2 \) is three dimensional, the projection \( \bar{L}|(J_1 \cap J_2) \) has an \( O(3) \) symmetry (since \(-I \in O(3)\)). Next, assume that \( J_1 \cap J_2 \) is two dimensional, and let \( \{\zeta, v_1, v_2, v_3, v_4\} \) and \( \{\zeta, v_1, v_2, v_3, v_4'\} \) be orthonormal bases for \( J_1 \) and \( J_2 \), respectively. Consider the subspace \( J_0 \) spanned by \( \{\zeta, v_1, v_2, v_3, v_4'\} \). Since \( J_0 \) is five dimensional and contains \( \zeta \), we know that either \( K|J_0 = \bar{L}|J_0 \) or \( \bar{K}|J_0 = -\bar{L}|J_0 \). In the first scenario we have,

\[
\bar{L}|(J_0 \cap J_2) = (\bar{L}|J_0)|(J_0 \cap J_2) = (\bar{K}|J_0)|(J_0 \cap J_2) = (\bar{K}|J_2)|(J_0 \cap J_2)
\]

(17)

\[
= (-\bar{L}|J_2)|(J_0 \cap J_2) = -\bar{L}|(J_0 \cap J_2).
\]

Given that the dimension of \( J_0 \cap J_2 \) is 3, equation (17) shows that \( \bar{L}|(J_0 \cap J_2) \) has an \( O(3) \) symmetry, which is a contradiction. For the second scenario, we apply the same argument to \( J_0 \) and \( J_1 \), showing that \( \bar{L}|J_0 \cap J_1 \) has an \( SO(4) \) symmetry (since the dimension of \( J_1 \cap J_0 \) is 4). The case where \( J_1 \cap J_2 \) is one dimensional can be dealt with in a similar way. We conclude that either \( \bar{K}|J = \bar{L}|J \) for every five dimensional subspace \( J \) containing \( \zeta \), or \( \bar{K}|J = -\bar{L}|J \) for all such \( J \). By Theorem 3.1.1 from [Ga] page 99, it follows that \( \bar{K} = \bar{L} \) or \( \bar{K} = -\bar{L} \). Thus, \( K = L + a_L - a_K \) or \( K = -L - a_L - a_K \). \[ \square \]

5. Proofs of Theorem \[2\] and Corollary \[2\]

We are now considering star-shaped bodies with respect to the origin. Let \( \zeta \in S^4 \) be the direction given in Theorem \[2\]. The hypotheses imply that for every \( w \in S^3(\zeta) \) there exists \( \chi_w \in SO(4, S^4(w)) \) and \( a_w \in w^\perp \) such that

\[
\chi_w(K \cap w^\perp) = (L \cap w^\perp) + a_w.
\]

(18)

Let \( l(\zeta) \) denote the one dimensional subspace containing \( \zeta \). As in Section \[3\] we let \( \mathcal{A}_K \subset S^4 \) be the set of directions that are parallel to the diameters of \( K \) (similarly for \( L \)). Note that it is possible for star-shaped bodies to contain several parallel diameters. We consider the set \( \Omega \), defined as in (14), and the set \( \Omega' \), defined by

\[
\Omega' = \{w \in \Omega : K \cap w^\perp \text{ and } L \cap w^\perp \text{ have only one diameter}\).
\]

(19)
Then it follows from the hypothesis of Theorem 2 that if \( w \in \Omega^r \), then \( d_K(\zeta) \) must be the unique diameter of \( K \cap w^\perp \). We will use the notation \( v_K(\zeta) = \rho_K(\zeta) + \rho_K(-\zeta) \) for the length of the diameter \( d_K(\zeta) \). As in the previous Section, it can be shown that for most directions \( w \), \( K \cap w^\perp \), \( L \cap w^\perp \) contain exactly one diameter parallel to \( \zeta \) and passing through the origin.

Lemma 10. (cf. \([ACR]\) Lemma 15.) Let \( K \) and \( L \) be as in Theorem 3. Then \( L \) has a diameter \( d_L(\zeta) \) passing through the origin, and \( \Omega^r \) is everywhere dense in \( S^3(\zeta) \). Moreover, for every \( w \in \Omega^r \) we have \( \chi_w(\zeta) = \pm \zeta \) and \( v_K(\zeta) = v_L(\zeta) \).

We now wish to argue as in the proof of Theorem 1 and translate the body \( L \) so that its diameter \( d_L(\zeta) \), given by Lemma 10 coincides with \( d_K(\zeta) \). However, the translate of a star-shaped body with respect to the origin may no longer be a star-shaped body with respect to the origin. The next Lemma, which is similar to Lemma 16 in \([ACR]\) shows that, under our hypotheses, the translated body is still star-shaped (this Lemma is not necessary if \( K \) and \( L \) are convex). We include the proof here, since the rotation \( \chi_w \) is now in \( SO(4, S^3(w)) \), and the argument is slightly different.

Lemma 11. There exists a vector \( a \in \mathbb{R}^5 \), parallel to \( \zeta \), such that the body \( \tilde{L} = L + a \) is star-shaped with respect to the origin, and \( d_K(\zeta) = d_{\tilde{L}}(\zeta) \).

Proof. Consider the sets

\[
R_1 = \{ w \in \Omega^r : \chi_w(d_K(\zeta)) = d_K(\zeta) \}, \\
R_2 = \{ w \in \Omega^r : \chi_w(d_K(\zeta)) \neq d_K(\zeta) \},
\]

where \( \chi_w \) is the rotation in \( SO(4, S^3(w)) \) as in \([18]\). By Lemma 1, since \( \chi_w(\zeta) = \pm \zeta \), we have that either \( \chi_w = \pm I \), or that \( \chi_w \) has an invariant two dimensional subspace \( \Pi^\perp \), containing \( \zeta \), such that \( \chi_w|\Pi^\perp = \pm I \). If \( \chi_w = I \) or \( \chi_w|\Pi^\perp = I \), or if \( d_K(\zeta) \) is centered at the origin, then \( w \in R_1 \). The only case in which \( w \in R_2 \) is if \( d_K(\zeta) \) is not centered at the origin, and either \( \chi_w = -I \) or \( \chi_w|\Pi^\perp = -I \).

Assume, at first, that \( \Omega^r = R_1 \). Since the diameter \( d_K(\zeta) \) is fixed by \( \chi_w \), and \( d_L(\zeta) \) contains the origin, it follows that the vector \( a_w \) in \([18]\) is independent of \( w \in \Omega^r \) and \( a_w = a_1 = (\rho_K(\zeta) - \rho_L(\zeta)) \zeta \). The translated section \( (L \cap w^\perp) + a_1 \) coincides with \( \chi_w(K \cap w^\perp) \), and therefore \( (L \cap w^\perp) + a_1 \) is star-shaped with respect to the origin for every \( w \in \Omega^r \). Since \( \Omega^r \) is dense in \( S^3(\zeta) \), we conclude that the translated body \( \tilde{L} = L + a \), with \( a = a_1 \), is also star-shaped with respect to the origin.

Secondly, assume that \( \Omega^r = R_2 \). Then, \( a_w \) is independent of \( w \in \Omega^r \) and \( a_w = a_2 = (\rho_K(-\zeta) - \rho_L(\zeta)) \zeta \). We conclude that \( \tilde{L} = L + a \), with \( a = a_2 \), is star-shaped with respect to the origin.

Finally, we show that the case where \( R_1 \) and \( R_2 \) are both nonempty does not occur under the assumptions of Theorem 2. Since \( R_1 \cup R_2 = \Omega^r \), we
have $S^3(\zeta) = R_1 \cup R_2 \subseteq R_1 \cup R_2 \subseteq S^3(\zeta)$. Hence, there exists $w_0 \in R_1 \cap R_2$, i.e., there is a rotation $\chi_{w_0}$ such that $\chi_{w_0}(d_K(\zeta)) = d_K(\zeta)$ and

\begin{equation}
\chi_{w_0}(K \cap w_0^\perp) = L \cap w_0^\perp + a_1,
\end{equation}

and a rotation $\tilde{\chi}_{w_0}$ such that $\tilde{\chi}_{w_0}(d_K(\zeta)) \neq d_K(\zeta)$ and

\begin{equation}
\tilde{\chi}_{w_0}(K \cap w_0^\perp) = L \cap w_0^\perp + a_2.
\end{equation}

In particular, since $\tilde{\chi}_{w_0}$ does not fix $d_K(\zeta)$, this diameter cannot be centered at the origin, and it follows that the other rotation $\chi_{w_0}$ must be the identity, at least on a two dimensional subspace containing $\zeta$. By (20) and (21) we have

\begin{equation}
K \cap w_0^\perp = \chi_{w_0}^{-1}\left(\tilde{\chi}_{w_0}(K \cap w_0^\perp)\right) + b,
\end{equation}

where $b \in \mathbb{R}^5$. Observe that the rotation $\chi_{w_0}^{-1} \circ \tilde{\chi}_{w_0}$ is not the identity, since $\chi_{w_0}^{-1} \circ \tilde{\chi}_{w_0}(\zeta) = -\zeta$. Therefore, $K \cap w_0^\perp$ has a rotational symmetry. This contradicts the hypothesis of Theorem 2. The Lemma is proven.

In order to finish the argument, we need one further Lemma.

**Lemma 12.** (cf. ACR, Lemma 17.) For every $w \in \Omega^r$ there exists $\varphi_w = \chi_{w}^{-1} \in SO(4, S^3(w))$, $\varphi_w(\zeta) = \pm \zeta$, such that

\begin{equation}
\rho_K \circ \varphi_w(\theta) = \rho_L(\theta) \quad \forall \theta \in S^3(w).
\end{equation}

**Proof of Theorem 2.** Consider the sets

\begin{equation}
\Xi^r = \{w \in S^3(\zeta) : \quad (22) \text{ holds with } \varphi_w(\zeta) = \zeta\}
\end{equation}

and

\begin{equation}
\Psi^r = \{w \in S^3(\zeta) : \quad (22) \text{ holds with } \varphi_w(\zeta) = -\zeta\}.
\end{equation}

By definition, $\Omega^r \subset (\Xi^r \cup \Psi^r)$. Therefore, Lemma 10 implies that $\Xi^r \cup \Psi^r = S^3(\zeta)$. Now we can apply Proposition 1 (with $f = \rho_K$, $g = \rho_L$, and $\Xi = \Xi^r$, $\Psi = \Psi^r$) obtaining that either $\rho_K = \rho_L$ on $S^4$, or $\rho_K(\theta) = \rho_L^\perp(-\theta)$ for all $\theta \in S^4$. In the first case, $K = \tilde{L}$, and in the second, $K = -\tilde{L}$. Thus, either $K = L + a$, or $K = -L - a$. This finishes the proof of Theorem 2.

**Proof of Corollary 2.** The proof is similar to the one of Corollary 1. One has only to consider the sections $K \cap J$, $\tilde{L} \cap J$, instead of the projections $K|J$, $\tilde{L}|J$, and Theorem 7.1.1 from Ga, page 270, instead of Theorem 3.1.1 from Ga, page 99.

**References**

[A] A. D. Alexandrov, *On the theory of mixed volumes of convex bodies II. New inequalities between mixed volumes and their applications* [in Russian], Mat. Sbornik, 2 (1937), 1205-1238.

[ACR] M. A. Alfonseca, M. Cordier, D. Ryabogin, *On bodies with directly congruent projections and sections*, Israel J. Math. 215 (2016), no. 2, 765-799.

[Ga] R. J. Gardner, *Geometric tomography*, second edition. Encyclopedia of Mathematics and its Applications, 58, Cambridge University Press, Cambridge, 2006.
[Go] V. P. Golubyatnikov, *Uniqueness questions in reconstruction of multidimensional objects from tomography type projection data. Inverse and Ill-posed problems series*. De Gruyter, Utrecht-Boston-Köln-Tokyo, 2000.

[Gr] H. Groemer, *Geometric Applications of Fourier Series and Spherical Harmonics*, Encyclopedia of Mathematics and its Applications, 61. Cambridge University Press, Cambridge, 1996.

[Ha] H. Hadwiger, *Seitenrisse konvexer Körper und Homothetie*, Elem. Math. 18 (1963), 97-98.

[My] S. Myroshnychenko, *On a functional equation related to a pair of hedgehogs with congruent projections or sections*, J. Math. Anal. Appl. 445 (2017), no. 2, 1492–1504.

[MyR] S. Myroshnychenko and D. Ryabogin, *On polytopes with congruent projections and sections*, Adv. Math. 325 (2018), 482–504.

[Pa] R. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) 73 (1961), 295-323.

[R] D. Ryabogin, *On the continual Rubik's cube*, Adv. Math. 231 (2012), 3429-3444.

[R1] D. Ryabogin, *A Lemma of Nakajima and Süss on convex bodies*, Amer. Math. Monthly 122 (2015), no. 9, 890-892.

[R2] D. Ryabogin, *On symmetries of projections and sections of convex bodies*, Springer Contributed Volume “Discrete Geometry and Symmetry” dedicated to Karoly Bezdek and Egon Schulte on the occasion of their 60-th birthdays, to appear.

[Sa] H. Samelson, *A Theorem on differentiable manifolds*, Portugaliae Math. 10 (1951), 129-133.

[Sch1] R. Schneider, *Convex bodies with congruent sections*, Bull. London Math. Soc. 312 (1980), 52-54.

[Si] W. Sierpiński, *Un théorème sur les continus*, Tohoku Math. J. 13 (1918), 300–303.

[St] J. Stillwell, *Naive Lie Theory*, Undergraduate Texts in Mathematics, Springer, 2008.

[Zh] N. Zhang, *On bodies with congruent sections or projections*, https://arxiv.org/abs/1711.10445.

Department of Mathematics, North Dakota State University, Fargo, ND 58108, USA
*E-mail address*: maria.alfonseca@ndsu.edu

Department of Mathematics, Chatham University, Pittsburgh, PA 15232, USA
*E-mail address*: m.doyle@chatham.edu

Department of Mathematics, Kent State University, Kent, OH 44242, USA
*E-mail address*: ryabogin@math.kent.edu