Optimal condition for measurement observable via error-propagation

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Abstract

Propagation of error is a widely used estimation tool in experiments where the estimation precision of the parameter depends on the fluctuation of the physical observable. Thus the observable that is chosen will greatly affect the estimation sensitivity. Here we study the optimal observable for the ultimate sensitivity bounded by the quantum Cramèr–Rao theorem in parameter estimation. By invoking the Schrödinger–Robertson uncertainty relation, we derive the necessary and sufficient condition for the optimal observables saturating the ultimate sensitivity for the single parameter estimate. By applying this condition to Greenberg–Horne–Zeilinger states, we obtain the general expression of the optimal observable for separable measurements to achieve the Heisenberg-limit precision and show that it is closely related to the parity measurement. However, Jose et al (2013 Phys. Rev. A 87 022330) have claimed that the Heisenberg limit may not be obtained via separable measurements. We show this claim is incorrect.

Keywords: propagation of error, optimal measurement, quantum parameter estimation

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1. Introduction

An essential task in quantum parameter estimation is to suppress the fundamental bound on measurement precision imposed by quantum mechanics. Various quantum strategies have been developed to enhance the accuracy of the parameter estimation, which are closely related to some practical applications, such as the Ramsey spectroscopies, atomic clocks, and
Two approaches in common use for high-precision measurements are the parallel protocol with correlated multi-probes [9] and the multi-round protocol with a single probe [10, 11]. Most recently, some novel methods, like environment-assisted metrology [12] and enhanced metrology by quantum error correction [13–16], were raised to achieve high precision in realistic experiments.

Rather than engineering the sensitivity-enhanced strategies, we concentrate on the problem of how to attain the maximal sensitivity in realistic experiments. In general, a noiseless procedure of the quantum single parameter estimation can be abstractly modeled by four steps (see figure 1): (i) preparing the input state $\rho_{\text{in}}$, (ii) parameterizing it under the evolution of the parameter-dependent Hamiltonian, e.g., a unitary evolution $U_{\phi}$ with $\phi$ the parameter to be estimated, (iii) performing measurements of the observable $\hat{\mathcal{O}}$ on the output state $\rho_{\phi}$, and (iv) finally estimating the value of the parameter from the estimator $\phi_{\text{est}}$ as a function of the outcomes of the measurements.

From estimation theory, the estimation precision is statistically measured by the unit-corrected mean-square deviation of the estimator $\phi_{\text{est}}$ from the true value $\phi$ [17, 18]:

$$
(\delta\phi)_{\text{est}}^2 := \left( \frac{\phi_{\text{est}}}{\partial\left\langle q_{\text{est}} \right\rangle_{\text{av}}} - \phi \right)^2_{\text{av}},
$$

where the brackets $\langle \rangle_{\text{av}}$ denote the statistical average and the derivative $\partial\left\langle q_{\text{est}} \right\rangle/\partial\phi$ removes the local difference in the ‘units’ of $q_{\text{est}}$ and $\phi$. Whichever measurement scheme is employed, the ultimate limit to the precision of the unbiased estimate is given by the quantum Cramér–Rao bound (QCRB) from below as:

$$
(\delta\phi)_{\text{est}}^2 \geq \left( vF_{\phi} \right)^{-1},
$$

where $v$ is the repetitions of the experiment and $F_{\phi}$ is the quantum Fisher information (QFI) (see equation (7) for definition), which measures the statistical distinguishability of the parameter in quantum states. This bound is asymptotically achieved for large $v$ under optimal measurements, followed by the maximum likelihood estimator [17–20].

On the other hand, it is well known that error-propagation is a widely acceptable theory in experiments [1, 3, 9, 21–33]. With this theory, to estimate the parameter, $\phi$ is reduced to measuring the average value of a physical observable $\hat{\mathcal{O}}$. After repeating the experiment $v$ times, the real accessible precision on $\phi$ is given by the error-propagation formula as follows [1, 3, 9, 21–24]:

![Figure 1. The schematic representation of a general scheme of (noiseless) quantum parameter estimation is composed of four components: input state $\rho_{\text{in}}$, parametrization process $U_{\phi}$, measurements $\hat{\mathcal{O}}$, and estimator $\phi_{\text{est}}$. Here, we concentrate on the part in shadow to find the optimal $\hat{\mathcal{O}}$ attaining the highest sensitivity to the parameter $\phi$ in $\rho_{\phi}$.](image-url)
\[
\left( \delta \varphi \right)_{\text{ep}}^2 := \frac{1}{\nu} \left\langle \left( \Delta \hat{\varphi} \right)^2 \right\rangle_{\text{av}}^2,
\]

where \( \Delta \hat{\varphi} = \hat{\varphi} - \langle \hat{\varphi} \rangle \) and \( \langle \hat{\varphi} \rangle = \text{Tr} (\rho_{\varphi} \hat{\varphi}) \). Note that the two estimation errors defined in equations (1) and (3) are closely related.

To show the relationship between the two kinds of the estimation errors, \( (\delta \varphi)_{\text{est}}^2 \) and \( (\delta \varphi)_{\text{ep}}^2 \), we introduce \( \Delta \varphi_{\text{est}} := \varphi_{\text{est}} - \langle \varphi_{\text{est}} \rangle_{\text{av}} \). Then, it is easy to show that [17]:

\[
(\delta \varphi)_{\text{est}}^2 = \left( \frac{\left\langle \left( \Delta \varphi_{\text{est}} \right)^2 \right\rangle_{\text{av}}}{\text{av}} \right)^2 + \left( \frac{\varphi_{\text{est}} - \varphi}{\left( \frac{\partial \varphi}{\partial \varphi_{\text{est}} \left\langle \varphi_{\text{est}} \right\rangle_{\text{av}}} \right)_{\text{av}}} \right)^2 \quad (4)
\]

When viewing the arithmetic mean of the measurement outcomes of \( \hat{\varphi} \) over repetitions of the experiment as the estimator in the quantum setting, one has in general \( (\delta \varphi)_{\text{est}}^2 \geq (\delta \varphi)_{\text{ep}}^2 \geq (\nu F_{\varphi})^{-1} \) by noting that \( \langle \left( \Delta \varphi_{\text{est}} \right)^2 \rangle_{\text{av}} = \langle \left( \Delta \hat{\varphi} \right)^2 \rangle_{\text{av}} \) for sufficiently large \( \nu \), according to the central limit theorem [34] and comparison of the two definitions of the errors given by equations (1) and (3). In such a situation, \( (\delta \varphi)_{\text{est}}^2 \) and \( (\delta \varphi)_{\text{ep}}^2 \) have the same QCRB, and the saturation of the former implies that of the latter.

The formula equation (3) indicates that the fluctuation of the observable \( \hat{\varphi} \) propagates to the estimated values of the parameter \( \varphi \). This means that what kinds observable \( \hat{\varphi} \) employed directly affects the estimating precision of the parameter \( \varphi \). The purpose of this paper is to address the following question: With which kind of observable does the estimation error given by equation (3) achieve the QCRB given by equation (2)?

In this paper, we derive the necessary and sufficient (N&S) condition for the optimal observable saturating the QCRB for the single parameter estimation by using the Schrödinger–Robertson uncertainty relation (SRUR). We then apply this condition to Greenberg–Horne–Zeilinger (GHZ) states and find the general form of the optimal observable for separable measurements to achieve the Heisenberg-limit sensitivity (i.e., \( 1/N \)). Moreover, we discuss the relation between the optimal separable observable and parity measurements. However, in a recent work [35] Jose et al made a contradictory conclusion with respect to the above result. They claimed that it is impossible for separable measurements to go beyond the shot-noise limit (i.e., \( 1/\sqrt{N} \)) for any entangled states. To clarify this issue, we revisit the method in [35] and show the causes for this inconsistency.

This paper is structured as follows. In section 2, we first briefly review the single parameter estimation and obtain the N&S condition for the optimal observable. In section 3, we give an application of this condition to obtain the optimal separable observables for GHZ states to saturate the Heisenberg-limit precision. In section 4, we further elucidate the reasons for contradiction between the result given in [35] and ours. We give a conclusion in section 5.

### 2. N&S condition for optimal observable in single parameter estimation

We start by a brief review of quantum single parameter estimation via the general estimator. Consider a parametric family of density matrices \( \rho_{\varphi} \) containing an unknown parameter \( \varphi \) to be estimated. Suppose that the general quantum measurement performed on \( \rho_{\varphi} \) is characterized by a positive-operator-valued measure (POVM) \( M := \{ M_x \} \), with \( x \) the results of measurement. The value of the parameter is inferred via an estimator \( \varphi_{\text{est}} \), which maps the measurement outcomes to the estimated value. After repeating the experiment \( \nu \) times, the
standard estimation error $(\delta \phi)^2_{est}$ in equation (1) is bounded from below as:

$$
(\delta \phi)^2_{est} \geq \left(\nu F_{\phi}\right)^{-1},
$$

where

$$
F_{\phi} := \sum_{x} p_{\phi}(x) \left[ \partial_{\phi} \ln p_{\phi}(x) \right]^2
$$

is the (classical) Fisher information of the measurement-induced probability distribution $p_{\phi}(x) = \text{Tr}(\rho_{\phi} \hat{M}_x)$. The maximization over all POVMs gives rise to the so-called QFI, which is defined by:

$$
\hat{F}_{\phi} := \text{Tr}(\rho_{\phi} \hat{L}_{\phi}^2).
$$

Hence, a tighter bound of equation (5) is given by equation (2). Here $\hat{L}_{\phi}$ is the symmetric logarithmic derivative (SLD) operator, which is a Hermitian operator determined by:

$$
\partial_{\phi} \rho = \frac{1}{2} \left[ \rho, \hat{L}_{\phi} \right],
$$

with $[\cdot, \cdot]$ denoting the anti-commutator (see [17]). It is remarkable that $\hat{L}_{\phi}$ may not be uniquely determined by equation (8) when $\rho_{\phi}$ is not of full rank [36].

However, in general the value of the parameter $\phi$ may not be directly measured. The most general method of estimating the value of $\phi$ in practice involves measurements corresponding to a physical observable $\hat{O}$ which is generally $\phi$-independent. In such cases, the estimation error is given by the error-propagation formula equation (3), in which the fluctuations on the observable $\hat{O}$ propagate to the uncertainty in the estimation of $\phi$. In the following, we follow Hotta and Ozawa [24] to derive the achievable lower bound of the estimation error $(\delta \phi)^2_{est}$ by using the SRUR.

Let us first recall the SRUR [37, 38], which states that the uncertainty of two non-commuting observables $\hat{X}$, $\hat{Y}$ must obey the following inequality:

$$
\left\langle \left(\Delta \hat{X}\right)^2 \right\rangle \left\langle \left(\Delta \hat{Y}\right)^2 \right\rangle \geq \frac{1}{4} \left\langle \left[\left[\hat{X}, \hat{Y}\right]\right]\right\rangle^2 + \frac{1}{4} \left\langle \left[\Delta \hat{X}, \Delta \hat{Y}\right]\right\rangle^2,
$$

where $[\cdot, \cdot]$ denotes the commutator. The SRUR follows from the Schwarz inequality for the Hilbert–Schmidt inner product and naturally reduces to the Heisenberg uncertainty relation under the condition $\left\langle [\Delta \hat{X}, \Delta \hat{Y}]_{\phi}\right\rangle = 0$. By substituting $\hat{X} (\hat{Y})$ with $\hat{O} (\hat{L}_{\phi})$ and utilizing

$$
\hat{F}_{\phi} = \left\langle \hat{L}_{\phi}^2 \right\rangle = \left\langle \left(\Delta \hat{L}_{\phi}\right)^2 \right\rangle,
$$

as a result of $\langle \hat{L}_{\phi} \rangle = 2 \partial_{\phi} \text{Tr}(\rho_{\phi}) = 0$, equation (9) becomes:

$$
\left\langle \left(\Delta \hat{O}\right)^2 \right\rangle \hat{F}_{\phi} \geq \frac{1}{4} \left\langle \left[\hat{O}, \hat{L}_{\phi}\right]\right\rangle^2 + \frac{1}{4} \left\langle \left[\hat{O}, \hat{L}_{\phi}\right]\right\rangle^2.
$$
Moreover, since the observable operator $\hat{O}$ is independent of $\phi$, we have:

$$\langle [\hat{O}, \hat{L}_\phi] \rangle_\rho = \text{Tr} \left( [\hat{O}, \hat{L}_\phi] \rho \right) = \text{Tr} \left( \hat{O} \left[ L_\phi, \rho \right] \right) = 2 \partial_\phi \langle \hat{O} \rangle,$$  

(12)

where the second equality is obtained by employing the cyclic property of the trace operation, and the third equality is due to the SLD equation (8). Provided that $\langle \hat{O} \rangle$ is nonzero, combining equations (3), (11), and (12) yields:

$$\left( \delta \phi \right)^2_{\text{ep}} \geq \frac{1}{\nu F_\phi} \left( 1 + \frac{\left| \langle [\hat{O}, \hat{L}_\phi] \rangle \right|^2}{\left| \langle \hat{O}, \hat{L}_\phi \rangle \rangle_\rho \right|} \right)$$

$$= \frac{1}{\nu F_\phi} \left( 1 + \frac{\text{Im} \langle \hat{O} \hat{L}_\phi \rangle}{\text{Re} \langle \hat{O} \hat{L}_\phi \rangle} \right)^2$$

$$\geq \left( \frac{1}{\nu F_\phi} \right)^{-1}.$$  

(13)

$$\geq \left( \frac{1}{\nu F_\phi} \right)^{-1}.$$  

(14)

The bound in equation (13) describes the achievable sensitivity of $\phi$ when employing an observable $\hat{O}$. The bound in equation (15) gives the highest precision for $\phi$ for the optimal observable $\hat{O}_{\text{opt}}$, which coincides with the QCRB in equation (2). It is shown that the estimation error $\left( \delta \phi \right)^2_{\text{ep}}$ achieves the QCRB only when the two equalities in equations (13) and (15) hold simultaneously.

Below, we consider the attainability of the above bounds and give the N&S condition for optimal observables. From the N&S condition for equality in the SRUR, the equality in equation (13) holds if and only if

$$\Delta \hat{O} \sqrt{\rho_\phi} = \alpha \hat{L}_\phi \sqrt{\rho_\phi}$$

(16)

is satisfied with a nonzero complex number $\alpha$. Note that we restrict here $\alpha \neq 0$ at the request of $\langle [\hat{O}, \hat{L}_\phi] \rangle \neq 0$ in the denominator of equation (13). Furthermore, the equality in equation (15) holds if and only if

$$\text{Im} \langle \hat{O} \hat{L}_\phi \rangle = 0.$$  

(17)

This condition can be combined into the condition (16) by restricting $\alpha$ to be a nonzero real number, i.e.,

$$\Delta \hat{O} \sqrt{\rho_\phi} = \alpha \hat{L}_\phi \sqrt{\rho_\phi} \quad \text{with} \quad \alpha \in \mathbb{R} \setminus \{0\}.$$  

(18)

This is the N & S condition of the optimal observable for density matrix $\rho_\phi$. It implies that the estimation error achieves the QCRB given by the QFI for $\rho_\phi$ only when the observable that we choose satisfies equation (18). This is the main result of the paper. For pure states $\rho_\phi = |\psi_\phi \rangle \langle \psi_\phi |$, the condition (18) is equivalent to:
\[ \Delta \hat{\Theta} \left| \varphi \right> = a \hat{L}_\varphi \left| \varphi \right> \quad \text{with } \alpha \in \mathbb{R} \setminus \{0\}. \quad (19) \]

If we assume that the parameter \( \varphi \) here is imprinted via a unitary operation \([9]\), i.e.,
\[ \rho_\varphi = \exp\left(-i\hat{G}\varphi\right)\rho_{in} \exp\left(i\hat{G}\varphi\right) \]
with \( \hat{G} \) the generator, associating with the equality \( \partial_\varphi \rho_\varphi = -i[\hat{G}, \rho_\varphi] \), then condition (19) further reduces to:
\[ \Delta \hat{\Theta} \left| \varphi \right> = -2i\alpha \hat{G} \left| \varphi \right> \quad \text{with } \alpha \in \mathbb{R} \setminus \{0\}. \quad (20) \]

This condition was alternatively obtained in \([31]\). It is important to note that their proof is only valid in the case of unitary parametrization for pure states and cannot be generalized to obtain the condition (18).

Here, we discuss the relations between the saturation of the QCRB with respect to \( (\delta \varphi)^2_{est} \) and with respect to \( (\delta \varphi)^2_{ep} \). Following Braunstein and Caves \([17]\), the saturation of the QCRB with respect to the error \( (\delta \varphi)^2_{ep} \) can be separated as the saturation of a classical Cramér–Rao bound (CCRB) equation (5) and finding an optimal measurement attaining the QFI. The CCRB can always be asymptotically achieved by the maximum likelihood estimator, so whether the QCRB can be asymptotically saturated is determined by whether the measurement attains the QFI. The N&S condition for the optimal measurement attaining the QFI reads \([17]\):
\[ \sqrt{M_x} \sqrt{\rho_p} = u_x \sqrt{M_x \hat{L}_\varphi} \sqrt{\rho_p}, \quad (21) \]
where \( \{\hat{M}_x\} \) denotes the POVM of the measurement and \( u_x \) are real numbers. In the following, we show that the N&S condition (18) for the saturation of the QCRB with respect to \( (\delta \varphi)^2_{ep} \) identifies an optimal measurement attaining QFI. Let \( \hat{O}_{opt} \) be the optimal observable satisfying equation (18) and \( P_x \) the eigenprojectors of \( \hat{O}_{opt} \) with the eigenvalues \( x \). Left multiplying \( P_x \) on both sides of equation (18), it is easy to see that \( \{P_x\} \) is the optimal measurement attaining the QFI. That is to say, the projective measurement \( \{P_x\} \), followed by the maximum likelihood estimator of the measurement outcomes, saturates the QCRB with respect to the standard estimation error \( (\delta \varphi)^2_{est} \).

### 3. Optimal separable observable for GHZ states

Below, we apply the N&S condition to show the general optimal observable for GHZ states. Let us specifically consider an experimentally realizable Ramsey interferometry to estimate the transition frequency \( \omega \) of the two-level atoms loaded in the ion trap \([1, 2]\). The Hamiltonian of the system with \( N \) atoms is
\[ H = (\omega/2) \sum_{i=1}^{N} \hat{\sigma}_z^i \]
where \( \hat{\sigma}_z^i \) is the Pauli matrix acting on the \( i \)th particle. In this setup, the measurements are limited to be performed separately on each atom. The observable operator may be described as a tensor product of Hermitian matrices \( \hat{O} = \hat{O}_q^\otimes N \), with \( \hat{O}_q = a_0 \mathbb{I} + a \cdot \hat{\sigma} \) dependent on four real coefficients \( \{a_0, a_1, a_2, a_3\} \), where \( \mathbb{I} \) is the identity matrix of dimension 2.

Considering the input state as the GHZ state, which provides the Heisenberg-limit scaling-sensitivity of frequency estimation in the absence of noise \([1, 9, 39]\). Under the time evolution \( \hat{U} = \exp\left(-iHt\right) \), the output state can be represented as:
\[ \left| \psi_{\text{GHZ}}(\varphi) \right> = \frac{1}{\sqrt{2}} \left( \left| 0 \right>^{\otimes N} + e^{i\varphi} \left| 1 \right>^{\otimes N} \right) \], \quad (22)
up to an irrelevant global phase with \( \varphi = \omega t \). Here, we adopt the standard notation where \(|0\rangle\) and \(|1\rangle\) are the eigenvectors of \( \sigma_z \) corresponding to eigenvalues \(+1\) and \(-1\), respectively. To determine the optimal separable observable \( \hat{\mathcal{O}} \), we need to find the solutions of the coefficients \( \{a_0, a_1, a_2, a_3\} \) to satisfy equation (19). With \( \hat{L}_\varphi = 2\hat{\delta}_\varphi (|\psi\rangle\langle\psi|) \) for pure states, the SLD operator for the state of equation (22) is given by:

\[
\hat{L}_\varphi = -iNe^{-i\varphi} |0\rangle\langle 1|^{\otimes N} + iNe^{i\varphi} |1\rangle\langle 0|^{\otimes N}.
\]

We find that equation (19) is always satisfied for \( a_0 = a_3 = 0 \) and arbitrary real numbers \( a_1, a_2 \), which do not vanish simultaneously. Therefore, the general expression of the optimal separable observable is given by:

\[
\hat{\mathcal{O}}_{opt} = (a_1\hat{\sigma}_x + a_2\hat{\sigma}_y)^{\otimes N},
\]

which is independent of the parameter \( \varphi \), i.e., globally optimal in the whole range of the parameter. It is easy to check that such observables saturate the Heisenberg-limit sensitivity. Actually, according to the error-propagation formula equation (3), we have:

\[
\delta \varphi_{GHZ} = \frac{1}{\sqrt{\delta}} \left| \frac{\{\hat{\mathcal{O}}_{opt}^2\} - \{\hat{\mathcal{O}}_{opt}\}^2}{\{\hat{\mathcal{O}}_{opt}\}} \right| = \frac{1}{\sqrt{\delta} N}.
\]

as a result of:

\[
\{\hat{\mathcal{O}}_{opt}\} = \text{Re} \left[ e^{-iN\varphi} (a_1 + ia_2)^N \right].
\]

When setting \( a_1 = 1, a_0 = a_2 = a_3 = 0 \), the optimal observable in equation (24) reduces to \( \hat{\sigma}_x^{\otimes N} \), as given in [9]. Note that here, measuring the observable \( \hat{\sigma}_x^{\otimes N} \) fails to attain the Heisenberg limit for the cases of \( \varphi = k\pi/N \) \( (k \in \mathbb{Z}) \), in which equation (25) becomes singular. Besides, we note that measuring the spin observable \( \hat{\sigma}^{\otimes N} \) also fails in these cases when \( N \) is even, and it is useful except for the cases of \( \varphi = (2k + 1)\pi/2N \) \( (k \in \mathbb{Z}) \), when \( N \) is odd.

We next show that the optimal observable in the form of equation (24) is closely related to the parity measurement proposed originally by Bollinger et al [3]. As is well known, in the standard Ramsey interferometry there are generally two Ramsey pulses applying before and after the free evolution (with an accumulated phase \( \varphi \)), and measurements often take place after the second pulse [1, 3]. Here the action of the pulse is modeled by a \( \pi/2 \)-rotation operation about the \( y \) axis, i.e., \( R_y \left[ \frac{\pi}{2} \right] = \exp \left[ -i \hat{\phi} \frac{\pi}{2} \right] \), and the measurement observable is denoted as the operator \( \hat{\mathcal{O}}_f \). With equation (24), one has:

\[
\hat{\mathcal{O}}_f = R_y \left[ \frac{\pi}{2} \right] \hat{\mathcal{O}} \hat{R}_y \left[ \frac{\pi}{2} \right] = (a_1\hat{\sigma}_z + a_2\hat{\sigma}_y)^{\otimes N}.
\]

When setting \( a_1 = 1, a_2 = 0 \), equation (28) reduces to:

\[
\hat{\mathcal{O}}_f = \hat{\sigma}_z^{\otimes N} \equiv (-1)^{j-j'},
\]

with \( j = N/2 \), which is the so-called parity measurement [3]. It is shown that only a parity measurement is necessary for the optimal estimate of the phase parameter \( \varphi \) for GHZ states,
and it is more experimentally feasible than the detection strategy, as discussed in [9], which applies local operations and classical communication.

4. Further discussions

However, in a recent work [35], it was pointed out that the separable measurement (the restricted readout procedure) might not be able to go beyond the shot-noise limit, even for arbitrary entangled states. It seems that this conclusion is inconsistent with ours in the above discussion. In what follows, we clarify this issue by revisiting the method in [35] and showing the causes for this inconsistency.

For simplicity, let us consider the two-qubit parametric GHZ state:

\[ |\psi_{\text{GHZ}}^{(2)}(\varphi)\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + e^{2i\varphi} |11\rangle \right). \]  

(30)

Following [35], we restrict the separable measurement to be the projective measurements \( |1\rangle \langle 1|, |0\rangle \langle 0| \) for each qubit with:

\[ | \pm \rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle). \]  

(31)

According to the condition of equation (21), the question of whether the above restricted measurement presented by equation (31) is the optimal measurement saturating the QCRB can be tested by asking whether or not the operators of the form:

\[ \hat{K} = \lambda_{\pm+} |\pm\rangle \langle \pm| + \lambda_{\pm-} |\pm\rangle \langle \pm| \]

\[ + \lambda_{\pm+} |\pm\rangle \langle \pm| + \lambda_{\pm-} |\pm\rangle \langle \pm| \]

(32)

can be the SLD operator for the state of equation (30). By demonstrating that for the state in equation (30) with \( \varphi = 0 \), there is no solution of the SLD equation (8) for the coefficients \( \lambda_{\pm+}, \lambda_{\pm-} \) in equation (32), the authors in Ref. [35] claimed that the projective measurement about \( |++\rangle, |+\rangle, |-\rangle \) is not the optimal measurement for the state of equation (30).

However, as we showed in section 3, \( \sigma_0 \otimes \sigma_0 \) is an optimal observable saturating the QCRB with respect to \( (\delta \varphi)^2 \) for the states (30). Although the estimation error considered in [35] is \( (\delta \varphi)^2_{\text{est}} \), a contradiction still arises, as the projective measurement of \( \sigma_0 \otimes \sigma_0 \) attains the QFI of states (30) (see the end in section 2), so that \( \{ |++\rangle, |+\rangle, |-\rangle \} \) is the optimal measurement regarding the estimation error \( (\delta \varphi)^2_{\text{est}} \). Below, we shall show that actually for any other point except for \( \varphi = k\pi/2 \), \( k \in \mathbb{Z} \) in the range of the parameter, the SLD operator exists in the form of equation (32).

First, note that the SLD operator for the non-full-rank density matrices is not uniquely determined, but \( \hat{L}_{\psi} \) (or \( \hat{L}_{\psi} \otimes \sigma_0 \) for pure state) is uniquely determined. Second, from equation (23), we see that:

\[ \hat{L}_{\psi} = -2ie^{-2i\varphi} |00\rangle \langle 11| + 2ie^{2i\varphi} |11\rangle \langle 00| \]

(33)

is an SLD operator for the state of equation (30). Third, since \( \hat{L}_{\psi} \otimes \sigma_0 \) is uniquely determined, then \( \hat{K} \) is the SLD operator for \( |\psi_{\varphi}\rangle \) if and only if

\[ \hat{L}_{\varphi} |\psi_{\varphi}\rangle = \hat{K} |\psi_{\varphi}\rangle \]  

(34)
is satisfied. Thus, substituting equations (30)–(33) into equation (34), we obtain the solutions for the coefficients as:

$$
\lambda_{++} = \lambda_{--} = -2 \tan \varphi, \quad \lambda_{+-} = \lambda_{-+} = 2 \cot \varphi.
$$

The above solutions are singular for $\varphi = k\pi/2$, $(k \in \mathbb{Z})$, which coincide with the results discussed below in equation (27). Note that here the $\varphi = 0$, $(k = 0)$ case is just considered in Ref. [35]. For a general value of the parameter except those singular points, the restricted separable measurement considered here indeed saturates the Heisenberg-limit-scaling sensitivity for the parametric state of equation (30). Moreover, it is easy to check that the same results of equation (35) can be obtained when restricting the separable measurement to be the projective measurements $\{ |+\rangle, |-\rangle \}$ for each qubit with:

$$
| \pm \rangle_y = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle),
$$

the eigenvectors of $\sigma_y$. This coincides with the result shown below in equation (27), that measuring the observable $\sigma_y^{\otimes N}$ fails to attain the Heisenberg limit for the $\varphi = k\pi/N$, $(k \in \mathbb{Z})$ cases when $N$ is even.

5. Conclusion

We have addressed the optimization problem of measurements for achieving the ultimate sensitivity determined by the QCRB. From the propagation of error, we derive the N&S condition of the optimal observables for a single parameter estimate by using the SRUR. As an application of this condition, we examine the optimal observables for GHZ states to achieve the ultimate sensitivity at the Heisenberg limit. We consider an experimentally feasible case that the observable operators are restricted to separably acting on the subsystem. We then find the general expression of the optimal separable observable by applying the N&S condition and show that it is exactly equivalent to the parity measurement when applying a $\pi/2$ pulse operation. However, Jose et al in [35] gave a contradictory conclusion with respect to ours, that separable measurements are impossible to beat the shot-noise limit even for entangled states. We show that for the GHZ state case, their conclusion is established only for some particular values of the parameter. Our results may be helpful for further investigation of quantum metrology.

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