A new proof of the Nonrationality of Cubic Threefolds

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Abstract

A new proof of the non-rationality of a generic cubic threefold is given as follows: If a generic cubic threefold were rational then the associated intermediate Jacobian would be a product of Jacobians of curves. We degenerate a generic cubic threefold to the Segre Cubic Threefold and so there is a degeneration of intermediate Jacobians as well. Associated to the degenerating family of Pryms is a unimodular system of vectors. Rationality of the generic cubic threefold would imply that the unimodular system would be cographic dicing. However, we show that the unimodular system obtained is a well known symmetric non-cographic dicing called $E_5$.

0 Introduction and History

The aim of this paper is to use methods of degenerations of Prym varieties of $ABH$ to prove the following theorem:

**Theorem 1** A generic cubic threefold is not birational to $\mathbb{P}^3$.

There have been several proofs of this theorem. The first proof was by Clemens and Griffiths in $CG$. However, even though their proof does use degenerations, it does not use them directly in the proof of Theorem $B$ So we cannot compare their proof to the others mentioned here. The advantage which their proof has over the ones listed here is that it applies to any cubic threefold.

Collino gives a degeneration proof in $C$. He looks at a family $X/S$ of cubic threefolds where $S$ is a smooth but not necessarily complete curve. The family has the property that for each $s \in S$ and $s \neq s_0$ the threefold $X_s$ is smooth, and $X_{s_0}$ has exactly one ordinary double point. To this family he associates a family of generalized Prym varieties $A/S$ such that when $s \neq s_0$, $A_s = J(X_s)$, the intermediate Jacobian of $X_s$ and $A_{s_0}$ is an extension of a Jacobian $B$ of a curve $C$ by a torus $\tau$. On the family $A/S$ he constructs a relative cartier divisor $D$ such that when $s \neq s_0$, $D_s$ induces the same polarization as the theta divisor of $J(X_s)$ and $D_{s_0} = 2\Theta$ in $NS(B)$, the Neron-Severi group. Here $\Theta$ is a divisor which induces a principal polarization on $B$. The irrationality is proved by showing that if all $X_s (s \neq s_0)$ were rational then the pair $(A_{s_0}, D_{s_0})$ should be a polarized generalized Jacobian of a curve with ordinary double points. The curve $C$ would have to be both hyperelliptic and of genus 4 embedded in $\mathbb{P}^3$ which is a contradiction. This proof works over all algebraically closed fields where the characteristic is not 2.

Bardelli also does a degeneration proof in $B$. In his case he starts out with a family $X/\mathbb{P}^1$ of cubic threefolds. For $t \neq 0$ the fiber $X_t$ is a smooth cubic threefold and $X_0 = \bigoplus_{j=1}^3 X_0^j$ where each $X_0^j$ is isomorphic to $\mathbb{P}^3$. The generalized intermediate jacobian of $X_0$ is an extension of $\bigoplus_{j=1}^3 J(X_0^j)$ by a torus $\tau$. Then for $t \neq 0$ $H^3(X_t)$ is polarized by a cup product which is denoted $\Theta(t)$. The family $\{\Theta(t)\}$ specializes to the natural polarization on $\bigoplus_{j=1}^3 J(X_0^j)$. The Hodge structure corresponding to $\tau$ is polarized by a bilinear symmetric form $\psi_{X_0}$. It is shown that $\psi_{X_0}$ is defined in terms of $\Theta_t$ and the local monodromy of the family of threefolds. By assuming $X_t$ is rational for $t \neq 0$ he shows that $\psi_{X_0}$ is not the natural polarization on the space of transverse 1-cycles of any semistable curve. Because of the use of Hodge theory, this proof only works over $\mathbb{C}$.

The proof in this paper is different in that it is maximal in the following sense: The Prym variety arising from double covers of stable curves is an extension of an abelian part by a torus. For the proofs above there is an extension of a Jacobian by a torus $\tau$. In Bardelli’s proof $\tau$ is $(\mathbb{C}^*)^2$ ($B$ 6.2.1) and in Collino it is...
\((k^*)^1\) where \(k\) is an algebraically closed field. For the proof presented here there is no abelian part, so the
torus is \((k^*)^5\) which is the maximum it can be.

The outline of this paper is as follows: In section 1 we gather the results we need from [A] and [ABH] and [V],
which we will use later in the paper. Section 2 presents the information we need to know about unimodular
systems and \(E_5\) in particular. Section 3 gives the relationship between cubic threefolds and Prym varieties.
Finally, in section 4, we compute the unimodular system for our degeneration, put everything together and
prove Theorem 1.

Acknowledgements: I would like to thank Prof. Valery Alexeev for his help and attention to my work
and Prof. Berndt Sturmfels for his help with Macaulay2. I would also like to thank Prof. Roy Smith and
Prof. Robert Varley for lots of help and suggestions in the course of preparing this paper.

1 Degeneration results

In this section we recall the results we need in order for our degeneration to work. The results are proved in
the papers [A], [ABH] and [N]. All results work over algebraically closed fields with characteristic not 2.

1.1 Jacobians

Suppose we have a 1-parameter family of smooth curves degenerating to a stable curve with dual graph
\(\Gamma\). Then by [A] (or [ABH]) we get induced data. For us the object of interest is the cell decomposition
obtained \((A) 5.5.\) This cell decomposition is obtained by intersecting the subspace
\(H_1(\Gamma, \mathbb{R}) \subset C_1(\Gamma, \mathbb{R})\). This decomposition, called a cographic dicing, does not depend on
the 1-parameter family and can be obtained from a unimodular system \((ABH), 2.3 (J6))\).

1.2 Pryms

Suppose we have a 1-parameter family of curves \((\tilde{C}, \iota)\)/\(S\) such that the generic fiber is a smooth curve
with a base-point-free involution and the degenerate curve \((\tilde{C}_0, \iota)\) is a stable curve. Then degenerate data
can be obtained for this family \((ABH), Section 2.4 PP0–PP6)\). In particular the data we need is the cell
decomposition of PP6 which we now describe.

For the rest of this section we will the drop the subscript 0 from \(\tilde{C}_0\) and \(\tilde{\Gamma}_0\). Let the dual graph of \(\tilde{C}\) be \(\tilde{\Gamma}\).
To get this cell decomposition we do the following: We define a map
\[
\pi^- : H_1(\tilde{\Gamma}, \mathbb{Z}) \longrightarrow H_1(\tilde{\Gamma}, \frac{1}{2}\mathbb{Z})
\]
\[
h \mapsto \frac{1}{2}(h - \iota(h))
\]

Let \(X^- := \pi^-(H_1(\tilde{\Gamma}, \mathbb{Z})\). The space \(X^- \otimes \mathbb{R}\) is contained in \(C_1(\tilde{\Gamma}, \mathbb{R})\). Each edge \(e_j\) of \(\tilde{\Gamma}\) defines a coordinate function \(z_j\)
in \(C_1(\tilde{\Gamma}, \mathbb{R})\). Let \(m_j = 1\) if \(z_j : X^- \rightarrow \mathbb{Z}\) is surjective and \(m_j = 2\) if \(z_j : X^- \rightarrow \frac{1}{2}\mathbb{Z}\) is surjective.
The functions \(m_j z_j\) define a cell decomposition of \(X^- \otimes \mathbb{R}\).

The cell decomposition defined on \(X^- \otimes \mathbb{R}\) is independent of the 1-parameter family if the vertices of the
cell decomposition are precisely the points of \(X^-\), i.e. the linear functions define a dicing of \(X^\). This
is condition (*) in [ABH]. An equivalent, easier to verify, condition is given in [N] as Theorem 0.1. This
condition can be stated as follows:

Lemma 2 ([N] Theorem 0.1) The cell decomposition depends only on the degenerate fiber \((\tilde{C}, \iota)\) if the
following is true: There do not exist two connected subgraphs \(\Gamma_0, \Gamma_1\) of the dual graph of \(\tilde{C}\) such that
\(\iota(\Gamma_i) = \Gamma_i, (i = 0, 1)\) and there are at least four edges connecting \(\Gamma_0\) and \(\Gamma_1\).
2 Unimodular systems

Definition 3 ([DG]) A system of \( m \geq n \) vectors \( R \) generating \( \mathbb{R}^n \) is called a unimodular system (U-system) if when we write any \( n \) vectors of \( R \) as columns in terms of a basis \( B \subset R \) we obtain a matrix which is totally unimodular, i.e the maximal minors are either 0, 1 or \(-1\).

Suppose \( R \) is a set of vectors spanning \( \mathbb{R}^n \). Then these vectors define a family \( H(R) \) of parallel hyperplanes \( H(r, z) = \{ x \in \mathbb{R}^n : x \cdot r = z \}, z \in \mathbb{Z}, r \in R \). If \( B \subset R \) is a basis for \( \mathbb{R}^n \) then the intersection points of hyperplanes of \( H(R) \) is a lattice and \( H(R) \) is then called a lattice dicing. The set of intersection points of hyperplanes of \( H(R) \) is a lattice if and only if \( R \) is a unimodular system.

There are unimodular systems which are not cographic. In our case, we are interested in a very nice and exceptional one called \( E_5 \). It is represented by the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

The fact that it is not cographic is proved as Cor 13.2.5 in [O].

3 Cubic Threefolds and Plane Quintics

In this section we show the connection between cubic threefolds and plane quintics for nonsingular threefolds and the Segre threefold. Most of the information here is put together from the following sources: in [DG] Sections 4.8 and 5.17, [H] Section 3.1 and 3.2, [M1], [M2], [CG] and [SR].

3.1 Smooth Cubic Threefolds

Let \( X \) be a smooth cubic threefold in \( \mathbb{P}^4 \). The lines in \( X \) form a surface \( F \) called a Fano surface ([M1], 1.1; [CG], Theorem 7.8). Pick a generic enough line \( \ell \) in \( F \subset X \) (satisfying conditions in [M1], Prop 1.25). The space of planes through \( \ell \) is parametrized by a projective space \( Y = \mathbb{P}^2 \). Let \( C_\ell \subset Y \) be the planes \( L \) such that \( L \cap X \) consists of three lines. Then \( C_\ell \) is a plane quintic curve. Also, let \( \hat{C}_\ell = \{ \ell' \in F | \ell \cap \ell' \neq \emptyset \} \). Then \( \hat{C}_\ell \) is a curve in \( F \) and there is a natural map \( q : \hat{C}_\ell \to C_\ell \). We get an involution \( \iota \) on \( \hat{C}_\ell \) as follows: for a point \( L \in \hat{C}_\ell \) \( q^{-1}(L) = \{ \ell', \ell'' \} \). So for \( \ell' \in \hat{C}_\ell \) we have \( \iota(\ell') = \ell'' \). From this double cover we obtain a Prym variety \( P(\hat{C}_\ell, \iota) \). However the Prym variety is independent of \( \ell \), so we can write \( P(X) \). This construction is in [M2] Section 2, and [CG], Appendix C.

3.2 The Segre Threefold

Now we construct a singular plane quintic and a double cover.

Definition 4 The Segre cubic threefold \( S \) is given by the following equation in \( \mathbb{P}^5 \):

\[
\left\{ (x_0, \ldots, x_5) : \sum_{j=0}^{5} x_j^3 = \sum_{j=0}^{5} x_j = 0 \right\}.
\]
This equation only works with fields of characteristic not 3. In all characteristics not 2 the following works:

\[
\begin{cases}
(x_0, \ldots, x_4) : \\
\sum_{i, j, k} 2x_i x_j x_k + \sum_{i \neq j} x_i^2 x_j = 0, \quad i, j, k \in \{0, \ldots, 4\}
\end{cases}
\]

The Segre threefold is the unique (up to isomorphism) threefold with ten nodes. The equation above shows that the cubic threefold is invariant under the action of the symmetric group $S_6$ on the coordinates. One node is given by the coordinates $(1, 1, 1, -1, -1)$ and the other nine are obtained by the action of $S_6$ on the coordinates.

We can obtain $\mathcal{S}$ as follows: Let $p_1, \ldots, p_5$ be points in general position in $\mathbb{P}^3$ and for each pair of points $\ell_{ij}$ is the line joining $p_i$ and $p_j$. Blow up $\mathbb{P}^3$ at the five points and then blow down the proper transforms of the ten lines $\ell_{ij}$. The image is $\mathcal{S}$ in $\mathbb{P}^4$.

There are 15 planes contained in $\mathcal{S}$. Ten of them come from proper transforms of the planes in $\mathbb{P}^3$ containing three points $p_i$, $p_j$ and $p_k$. These are labeled $\Pi_{ijk}$ ($i < j < k$). The other five are images of the exceptional divisors coming from the blowup. These are labeled $\Pi_i$ ($1 \leq i \leq 5$). Each of the 15 planes contains 4 nodes.

$\mathcal{S}$ contains 6 two dimensional families of lines, $R_i$ ($1 \leq i \leq 5$) which are proper transforms of lines through $p_i$ and $R_0$ which is the family of twisted cubics through the five points. Each line in $R_i$ goes through the five planes $\Pi_{ijk}$ ($i < j < k$). Each line in $R_0$ goes through $\Pi_i$ ([SR] VIII 2.32).

We choose a line $\ell$ in $R_1$ which does not go through a node and goes through exactly 5 planes. We project from $\ell$ onto $\mathbb{P}^2$. The images of each of the five planes is in the degeneracy locus for $\mathcal{S}$. Each line meets each of the other four lines at a node. Each node is where the preimage is a plane containing a node on $\mathcal{S}$. The five lines form a pentagon in $\mathbb{P}^2$ which we call $C_0$. The dual graph of $C_0$ is the complete graph on five vertices $K_5$.

As in the smooth case, we can construct a double cover for the curve $C$ above. This double cover will lie in $G(2, 5)$. The preimage of each point in $C$ consists of two lines, excluding $\ell$, which are points in the double cover. The whole cover can be described as follows: Take ten copies of $\mathbb{P}^1$, $L_i^\epsilon$ where $1 \leq i \leq 5$ and $\epsilon$ is either 0 or 1. Each line $L_j^\epsilon$ has four points marked on it $p_{i,j}^\epsilon$ where $1 \leq j \leq 5$ and $j \neq i$. In the following the notation $(p_{i,j}^0 \sim p_{j,i}^1)$ means that the points $p_{i,j}^0$ and $p_{j,i}^1$ are identified. The double cover $\tilde{C}$ is

$$
\tilde{C} = \left( \coprod_{i, \epsilon} L_i^\epsilon \right) / (p_{i,j}^0 \sim p_{j,i}^1)
$$

The dual graph $\Gamma_0 = \Gamma(\tilde{C}_0)$ is shown below. If the vertices of $\Gamma(\tilde{C}_0)$ are labelled $v_j$, the the vertices $a_j$ and $b_j$ both map to $v_j$. The map of edges is given by the map of vertices.
The edges are named as follows: \(e_1 = (b_1, a_2), e_2 = (a_4, b_2), e_3 = (a_5, b_3), e_4 = (a_3, b_4), e_5 = (a_5, b_1), e_6 = (a_4, b_3), e_7 = (b_3, a_1), e_8 = (b_2, a_5), e_9 = (a_1, b_4), e_{10} = (b_2, a_1)\). The rest of the edges are named as follows: if edge \(e_i\) is \((a_j, b_k)\) respectively the edge \(e_i'\) is \((b_j, a_k)\) respectively.

The tree used to form the basis of \(H_1(\Gamma, \mathbb{Z})\) is given by the edges \(e_6 = (a_4, b_3), e_7 = (b_3, a_1), e_8 = (b_2, a_5), e_9 = (a_1, b_4), e_{10} = (b_2, a_1)\) and \(e'_{10}(a_3, b_1), e'_5 = (b_1, a_4), e'_4 = (a_2, b_3)\).

### 4 Proof of the Main Theorem

We now prove the irrationality of cubic threefolds by making use of the following lemma which relates cubic threefolds and Prym varieties.

**Lemma 5 (\[M2\] Thm 3.11)** Let \(\text{char}(k) \neq 2\). Let \(X\) be a nonsingular cubic threefold in \(\mathbb{P}^4\), defined over \(k\). If there exists a birational transformation between \(X\) and \(\mathbb{P}^3\) then the canonically polarized prym variety \((P(X), \Xi)\) associated with \(X\) is isomorphic, as a polarized abelian variety, to a product of canonically polarized Jacobian varieties of curves.

Given a smooth plane quintic curve with a double cover we degenerate it to the stable curve \(C_0\) above. The generalized Prym \(P(\tilde{C}, \iota)\) can be easily shown to be isomorphic to \((k^*)^5\).

Using the results on degeneration above we can compute the unimodular system for the Delaunay decomposition of the degeneration. By constructing the dual graph of the double cover and using the algorithm in \[ABH\], outlined in Section 1.2 we prove the following theorem:

**Theorem 6** The unimodular system for cell decomposition associated to a family of cubic threefolds degenerating to the Segre Threefold is \(E_5\).

Before we do the proof we need to verify that if a family of cubic threefolds degenerate to \(\mathcal{S}\) the the family of double covers satisfies Lemma 3.

Let \(\Gamma\) be as above. Suppose we have the two subgraphs \(\Gamma_1\) and \(\Gamma_2\) of \(\Gamma\). Then the first possible case is that \(\Gamma_1\) has 2 vertices and \(\Gamma_2\) has 8 vertices. This is not possible because then \(\Gamma_1\) would not be connected. Suppose the vertices in \(\Gamma_1\) correspond to the lines \(L_0^i\) and \(L_1^i\) (using the notation from above). The line \(L_0^i\) and \(L_1^i\) do not meet, so on the dual graph their corresponding vertices do not have an edge between them. The second possible case is if \(\Gamma_1\) and \(\Gamma_2\) have 4 and 6 vertices respectively. Suppose without loss of generality the lines \(L_1^5, L_1^3, L_2^5\) and \(L_2^3\) are in \(\Gamma_1\). This would imply \(\Gamma_1\) is not connected because the connected subgraph with \(L_1^5\) and \(L_1^3\) is not connected with the subgraph with \(L_2^5\) and \(L_2^3\). So the Delaunay decomposition is independent of the 1-parameter family, therefore we can obtain a unimodular system.
Proof
[of Theorem 1]

We now compute \( X^- \subset C_1(\Gamma, \mathbb{Z}) \).

The following is a basis for \( H_1(\Gamma, \mathbb{Z}) \).

\[
\begin{align*}
    h_1 &= e_6' - e_7 - e_9 + e_6 + e_7 + e_9 \\
    h_2 &= e_1' + e_7' - e_9 + e_6 + e_7 + e_{10} \\
    h_3 &= e_1 - e_{10}' - e_9 + e_6 \\
    h_4 &= e_2 + e_6 + e_7 + e_{10} \\
    h_5 &= e_2' - e_{10}' - e_9 + e_6 + e_7 - e_9
\end{align*}
\]

The basis for \( X^- \) is as follows

\[
\begin{align*}
    \ell_1 &= \frac{1}{2}(h_2 - \iota(h_2)) = \frac{1}{2}(h_3 - \iota(h_3)) \\
    \ell_2 &= \frac{1}{2}(h_4 - \iota(h_4)) = \frac{1}{2}(h_5 - \iota(h_5)) \\
    \ell_3 &= \frac{1}{2}(h_9 - \iota(h_9)) = \frac{1}{2}(h_9' - \iota(h_9')) \\
    \ell_4 &= \frac{1}{2}(h_8 - \iota(h_8)) = \frac{1}{2}(h_8' - \iota(h_8')) \\
    \ell_5 &= \frac{1}{2}(h_6 - \iota(h_6)) = \frac{1}{2}(h_7 - \iota(h_7))
\end{align*}
\]

and \( X^- = \langle \ell_1, ..., \ell_5 \rangle \). The unimodular system is obtained by seeing how the edges \( e_j \) restrict to \( X^- \).

The unimodular matrix for the dicing of \( X^- \otimes \mathbb{R} \) is \((a_{ij})\) where \( a_{ij} \) is defined to be 1 if \( \ell_i \) contains \( e_j \). This matrix is \( E_5 \). \( \square \)

We are now in a position to prove Theorem 1.
Let \( \ell_0 \) be a line in \( S \) (chosen as in Section 3.3). Choose a smooth cubic threefold \( X \) such that it also contains \( \ell_0 \). We have a pencil of cubic threefolds which contain \( \ell_0 \):

\[
X_{a,b} = aX + bS \quad (a : b) \in \mathbb{P}^1.
\]

By restricting to some open subset \( S \) of \( \mathbb{P}^1 \) we get a family \( X/S \) of cubic threefolds. For each \( s \) there is a Prym \( P(X_s) \). If the threefolds in our family were rational then by Section 3.3 and Lemma 2 the family of Pryms we obtain should give a cographic unimodular system. But by Theorem 1 we get \( E_5 \) which we know is not cographic. So our original supposition that the cubic threefolds were rational is false. \( \square \)

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