TWISTED MOMENTS OF $L$-FUNCTIONS AND SPECTRAL RECIPROCITY

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Abstract. A reciprocity formula is established that expresses the fourth moment of automorphic $L$-functions of level $q$ twisted by the $\ell$-th Hecke eigenvalue as the fourth moment of automorphic $L$-functions of level $\ell$ twisted by the $q$-th Hecke eigenvalue. Direct corollaries include subconvexity bounds for $L$-functions in the level aspect and a short proof of an upper bound for the fifth moment of automorphic $L$-functions.

1. Introduction

1.1. Spectral reciprocity. Let $q, \ell$ be two distinct odd primes. Gauss’ celebrated law of quadratic reciprocity expresses the quadratic Legendre symbol $(\ell/q)$ in terms of $(q/\ell)$. This is a very remarkable statement, because it connects the arithmetic of two unrelated finite fields $\mathbb{F}_q$ and $\mathbb{F}_\ell$. The present paper establishes a reciprocity formula between the spectrum of the Laplacian on two different arithmetic hyperbolic surfaces $\Gamma_0(q)\backslash \mathbb{H}$ and $\Gamma_0(\ell)\backslash \mathbb{H}$ featuring a product of $L$-functions of total degree 8.

Let $F$ be an automorphic form for the group $\text{SL}_3(\mathbb{Z})$ and let generally denote by $f$ a cusp form for a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. Then roughly speaking, we will prove a formula of the shape

$$\sum_{f \text{ of level } q} L(s, f \times F)L(w, f)\lambda_f(\ell) \sim \sum_{f \text{ of level } \ell} L(s', f \times F)L(w', f)\lambda_f(q)$$

where

$$s' = \frac{1}{2}(1 + w - s), \quad w' = \frac{1}{2}(3s + w - 1).$$

In particular, specializing $s = w = 1/2$ and $F = E_0$ the minimal parabolic Eisenstein series with trivial spectral parameters whose Hecke eigenvalues are the ternary divisor function $\tau_3(n)$, we obtain a reciprocity formula for the twisted fourth moment

$$\sum_{f \text{ of level } q} L(1/2, f)^4\lambda_f(\ell) \sim \sum_{f \text{ of level } \ell} L(1/2, f)^4\lambda_f(q).$$

In addition to its aesthetic features linking arithmetic data on different congruence quotients of the upper half plane, this formula is useful for applications, because we can trade the level of the forms for the twisting Hecke eigenvalue. We will demonstrate this is in the next subsection.

We proceed to describe the precise formula. For the rest of the paper let $q, \ell$ be two coprime positive integers, $s, w$ two complex numbers with positive real part, and $F$ a cuspidal or non-cuspidal automorphic form for the group $\text{SL}_3(\mathbb{Z})$ with Fourier coefficients $A(n_1, n_2)$. We need to (a) parametrize the spectrum of level $q$ and identify suitable test functions on the various components, (b) define correction factors at the ramified primes $p \mid q\ell$, and (c) add additional “main terms”.

Irreducible automorphic representations of $Z(\mathbb{A}_Q)\text{GL}_2(\mathbb{Q})/\text{GL}_2(\mathbb{A}_Q)/K_0(q)$ with $K_0(q) = \{(a b \mid c d) \in \text{GL}_2(\mathbb{Z}) \mid c \equiv 0 \text{ (mod } q\mathbb{Z})\}$ can be generated by three types of functions:

- Cuspidal holomorphic newforms $f$ of weight $k = k_f \in 2\mathbb{N}$, level $q' \mid q$ and Hecke eigenvalues $\lambda_f(n) \in \mathbb{R}$; we denote the set of such forms by $B_{\text{hol}}(q)$ and the set of newforms of exact level $q$ by $B_{\text{hol}}^*(q)$;

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• Cuspidal Maass newforms \( f \) of spectral parameter \( t = t_f \in \mathbb{R} \cup [-i\vartheta, i\vartheta] \), level \( q' \mid q \) and Hecke eigenvalues \( \lambda_f(n) \in \mathbb{R} \), where at the current state of knowledge \( \vartheta = 7/64 \) can be taken [KiSa] (though \( \vartheta = 0 \) is expected); we denote the set of such forms by \( \mathcal{B}(q) \) and the set of newforms of exact level \( q \) by \( \mathcal{B}_q^* \).

• Unitary Eisenstein series \( E_{t,\chi} \) for \( \Gamma_0(q) \), where \( t \in \mathbb{R} \) and \( \chi \) is a primitive Dirichlet character of conductor \( c_\chi \) satisfying \( c_\chi^2 \mid q \). Their \( n \)-th Hecke eigenvalue is \( \lambda_{t,\chi}(n) = \sum_{ab=\ell} \chi(ab)(a/b)^{it} \) for \( (n, q) = 1 \).

We write \( \mathcal{T}_0 := (\mathbb{R} \cup [-i\vartheta, i\vartheta]) \times 2\mathbb{N} \) and consider the set of functions \( \mathfrak{h} = (h, h^{\text{hol}}) : \mathcal{T}_0 \to \mathbb{C} \) where

\[
(1.3) \quad h(-t) = h(t) \ll (1 + |t|)^{-15}, \quad t \in \mathbb{R} \cup [-i\vartheta, i\vartheta], \quad h^{\text{hol}}(k) \ll k^{-15}, \quad k \in 2\mathbb{N}.
\]

We call such functions \emph{weakly admissible}. We will define in Section 6 the notion of an \emph{admissible} function \( \mathfrak{h} \) which imposes additional analytic conditions on \( \mathfrak{h} \), see Lemma 8 for explicit examples.

For a cuspidal (holomorph or Maass) newform \( f \) or an Eisenstein series \( E_{t,\chi} \) and \( w \in \mathbb{C} \) define

\[
\Lambda_f(\ell; w) := \sum_{ab=\ell} \frac{\mu(a)}{a^w} \Lambda_f(b), \quad \Lambda_{t,\chi}(\ell; w) := \sum_{ab=\ell} \frac{\mu(a)}{a^w} \lambda_{t,\chi}(b).
\]

We also need local correction factors \( \tilde{L}_q(s, w, f \times F) \) resp. \( \tilde{L}_q(s, w, E_{t,\chi} \times F) \) that are defined in (9.6) and (10.9) and follow the satisfying properties.

\begin{lemma}
For \( f \in \mathcal{B}_q^* \cup \mathcal{B}_q^{\text{hol}} \) we have
\[
(1.5) \quad \tilde{L}_q(s, w, f \times F) = \frac{\phi(q)}{q^2} \prod_{p \mid q} \left( 1 - \frac{A(1, p)}{p^{2s}} + \frac{\lambda_f(p)}{p^{3s}} \right)
\]

(indepedently of \( w \)). For arbitrary \( f \in \mathcal{B}(q) \cup \mathcal{B}_q^{\text{hol}}(q) \), \( t \in \mathbb{R} \), \( \chi \) a primitive Dirichlet character of conductor \( c_\chi \) with \( c_\chi^2 \mid q \), \( \varepsilon > 0 \) and \( Re(s), Re(w) \geq 1/2 \) we have the uniform bound
\[
(1.6) \quad \tilde{L}_q(s, w, f \times F), \quad \tilde{L}_q(s, w, E_{t,\chi} \times F) \ll \varepsilon q^{0 - \frac{1}{2} + \varepsilon}.
\]

Here and henceforth, \( \theta \) (not to be confused with \( \vartheta \)) denotes an admissible exponent towards the Ramanujan conjecture for \( F \). At the current state of knowledge, \( \theta \leq 5/14 \) (cf. [BB]) is known in general, and for \( F = E_0 \) we have \( \theta = 0 \).

For a weakly admissible function \( \mathfrak{h} = (h, h^{\text{hol}}) \), complex numbers \( s, w \) with \( Re(s), Re(w) \geq 1/2 \) and coprime integers \( q, \ell \) we define the twisted spectral mean values

\[
M_{q,\ell}^{\text{Maass,} \pm}(s, w; \mathfrak{h}) := \sum_{f \in \mathcal{B}(q)} \epsilon_f^{(1+\varepsilon)/2} \frac{L(s, f \times F)L(w, f)}{L(1, \text{Ad}^2 f)} \tilde{L}_q(s, w, f \times F) \frac{\Lambda_f(\ell; w)}{\ell^w} h(t_f),
\]

\[
M_{q,\ell}^{\text{hol}}(s, w; \mathfrak{h}) := \sum_{f \in \mathcal{B}_q^{\text{hol}}(q)} \frac{L(s, f \times F)L(w, f)}{L(1, \text{Ad}^2 f)} \tilde{L}_q(s, w, f \times F) \frac{\Lambda_f(\ell; w)}{\ell^w} h^{\text{hol}}(kf_f),
\]

\[
M_{q,\ell}^{\text{Eis}}(s, w; \mathfrak{h}) := \sum_{\chi \in \chi_1^1 \mid q} \int_{\mathbb{R}} \frac{L(s + it, F \times \chi)L(s - it, F \times \bar{\chi})L(w + it, \chi)L(w - it, \bar{\chi})}{L(1 + 2it, \chi^2)L(1 - 2it, \bar{\chi}^2)} \tilde{L}_q(s, w, E_{t,\chi} \times F) \frac{\Lambda_{t,\chi}(f; w)}{\ell^w} h(t) dt, \tag{1.7}
\]

where \( \epsilon_f \in \{ \pm 1 \} \) is the parity of the Maass form \( f \in \mathcal{B}(q) \) and \( Re(s), Re(w) \neq 1 \) in the expression for \( M_{q,\ell}^{\text{Eis}}(s, w; \mathfrak{h}) \). We write
\[
(1.8) \quad M_{q,\ell}^{\pm}(s, w; \mathfrak{h}) = M_{q,\ell}^{\text{Maass,} \pm}(s, w; \mathfrak{h}) + M_{q,\ell}^{\text{Eis}}(s, w; \mathfrak{h}) + \delta_{\pm =} M_{q,\ell}^{\text{hol}}(s, w; \mathfrak{h}).
\]

Absolute convergence follows from (1.3). Weyl’s law, the convexity bound for the respective \( L \)-functions and the lower bounds (11.3). It is not hard to see (cf. Lemma 16) that under suitable circumstances \( M_{q,\ell}^{\text{Eis}}(s, w; \mathfrak{h}) \) for \( Re(s), Re(w) > 1 \) can be continued to an \( \varepsilon \)-neighbourhood of \( Re(s), Re(w) \geq 1/2 \), and this continuation equals \( M_{q,\ell}^{\text{Eis}}(s, w; \mathfrak{h}) \) for \( 1/2 < Re(s), Re(w) < 1 \) plus some polar terms.
Theorem 1. Let $\mathfrak{h}$ be an admissible function, $(q, \ell) = 1$ and let $s, w \in \mathbb{C}$ be such that
\begin{equation}
1/2 \leq \Re s \leq \Re w < 3/4.
\end{equation}
Let $F$ be either cuspidal or the minimal parabolic Eisenstein series\footnote{More general Eisenstein series could be treated in the same way, but we do not have applications in this case.} $E_0$. Then
\begin{equation}
M_{q, \ell}(s, w; \mathfrak{h}) = N_{q, \ell}(s, w; \mathfrak{h}) + \sum_{\pm} M_{\ell, q}^\pm (s', w'; \mathcal{T}_{s', w} \mathfrak{h}) ,
\end{equation}
where $s', w'$ are as in (1.1), the “main term” $N_{q, \ell}(s, w; \mathfrak{h})$ and the integral transform $\mathcal{T}_{s', w} \mathfrak{h}$ are given in (10.3) resp. (10.2). The function $\mathcal{T}_{s', w} \mathfrak{h}$ is weakly admissible, and for $\varepsilon > 0$ we have
\begin{equation}
N_{q, \ell}(s, w; \mathfrak{h}) \ll_{s, w, \mathfrak{h}, \varepsilon} \ell^{0-1+\varepsilon} + q^{0-1+\varepsilon} .
\end{equation}

In the special case $q = \ell = 1$, $F = E_0$, a beautiful formula for the fourth moment was envisaged by Kuznetsov in the late 1980s and completed by Motohashi [Mo]. This formula, however, has different features. Our proof decomposes the fourth moment as $4 = 3 + 1$, whereas Kuznetsov and Motohashi use a decomposition of the form $4 = 2 + 2$. While more symmetric on the surface, their setup leads to convergence problems that are overcome by considering a carefully designed difference of the holomorphic and the Maass spectrum. In particular, everywhere positive test functions $\mathfrak{h}$ as in Lemma 8c) are in general excluded. (As long as $\ell = q = 1$, one can exploit the fact that there are no holomorphic forms of small weight to circumvent this problem, but for more general values of $q$ and $\ell$ this device is not available.) This is probably the reason why this formula seems to not have been used for applications.

Our formula should be seen as a contribution to the rich theory of automorphic forms on $GL(2) \times GL(4)$ in the special situation where the GL(4) function is an isobaric $3 + 1$-sum. A corresponding reciprocity formula for $\mathcal{M}_{q, \ell}^{-\pm}(s, w; \mathfrak{h})$ could also be established, in fact in an analytically simpler fashion (cf. [Mo, end of Section 2]). A spectral reciprocity formula in the archimedean aspect – i.e. the level is kept fixed, but the spectral parameter is varying – for generic automorphic forms $F$ on $GL(4)$ with an additional twist by the parity
\begin{equation}
\sum_{\ell_j \leq T} \epsilon_j L(1/2, f \times \tilde{F}) \sim T \sum_{\ell_j \ll 1} \epsilon_j L(1/2, f \times \tilde{F})
\end{equation}
(where $\tilde{F}$ denotes the dual form) was considered by a different technique in [BLM] along with applications to non-vanishing of $L$-functions. Finally we mention that there is “lower dimensional” version of reciprocity for certain $L$-functions on $GL(1) \times GL(2)$:
\begin{equation}
\sum_{\chi \pmod{q}} |L(1/2, \chi)|^2 \chi(\ell) \sim \sum_{\chi \pmod{t}} |L(1/2, \chi)|^2 \chi(q).
\end{equation}
This was first observed by Conrey, and extended by Young [Yo] and Bettin [Be]. It would be very interesting to investigate if there is a “master formula” that contains all previously mentioned spectral reciprocity formulae as special cases.

1.2. Applications. The most interesting applications of our reciprocity formula concern the twisted fourth moment (1.2), which has a long history in the theory of $L$-functions. The investigation started with seminal work of Duke-Friedlander-Iwaniec [DFI], who used their work on the binary additive divisor problem to obtain\footnote{This is not stated explicitly, but at least for prime level follows very easily from their Corollary 2, for instance.}
\begin{equation}
\sum_f L(s, f)^4 \lambda_f(\ell) \ll_{k,s,\varepsilon} q^{1+\varepsilon} \ell^{-1/2} + q^{11/12+\varepsilon} \ell^{3/4}
\end{equation}
for $\Re s = 1/2$, where the sum is taken over holomorphic newforms of fixed weight $k$ and prime level $q$. Together with the amplification method they deduced the first subconvexity result
\begin{equation}
L(s, f) \ll_{k,s} q^{\frac{1}{2}-\delta}, \quad \delta < 1/192, \quad \Re s = 1/2
\end{equation}
for this family. The then newly developed machinery was taken up by Kowalski-Michel-VanderKam [KMV], who derived a precise asymptotic formula for the left hand side of (1.12) in the case of weight \( k = 2 \) and prime level \( q \) with error term \( O(q^{11/12 + \varepsilon} \ell^{3/4}) \). This allowed them to deduce, via mollification, various non-vanishing results for central \( L \)-values. Quite recently, Balkanova and Frolenkov [BF] improved the error term somewhat. On heuristic grounds, one would expect an error term \( O(q^{1/2 + \varepsilon}) \), and one would expect that the limit of current machinery should be an error term \( O(q^{1/2 + \varepsilon} \ell^{1/2}) \), at least under the Ramanujan conjecture. This is precisely what we prove:

**Theorem 2.** Let \( q \) be prime, \( \varepsilon > 0 \), \( (\ell, q) = 1 \) and \( h = (h, h_{\text{hol}}) \) an admissible function. Then

\[
\sum_{f \in B^*(q)} \frac{L(1/2, f)}{L(1, \mathcal{A}^2 f)} \lambda_f(\ell) h(t_f) + \sum_{f \in B*_{\text{hol}}(q)} \frac{L(1/2, f)}{L(1, \mathcal{A}^2 f)} \lambda_f(\ell) h_{\text{hol}}(k_f) \ll \frac{q}{\ell} \varepsilon \left( q\ell^{-1/2} + q^{1/2} + \varepsilon \ell^{1/2} \right).
\]

A slightly more general result is given in Proposition 17 in Section 11. The bound is optimal for \( \ell \) up to size \( q^{1/2} \) (modulo the Ramanujan conjecture). Since this is the length of an approximate functional equation for \( L(1/2, f) \), we can sum this bound trivially to obtain a bound for the fifth moment.

**Theorem 3.** Let \( q \) be prime, \( \varepsilon > 0 \). Then

\[
\sum_{f \in B^*(q)} L(1/2, f)^5 e^{-t_f^2} \ll q^{1+\vartheta+\varepsilon}.
\]

For holomorphic forms of weight \( k \in \{4, 6, 8, 10, 14\} \) an analogous result was recently obtained by Kikal and Young [KY] in an impressive 80-page tour de force argument. Our version for Maaß forms comes as a special case of a more general framework. Since \( L(1/2, f) \) is non-negative [KaS], Theorem 3 implies immediately the strong subconvexity bound

(1.14) \[
L(1/2, f) \ll q^{(1+\vartheta+\varepsilon)/5}.
\]

This is an example of “self-amplification”, where the fourth moment is amplified by a fifth copy of the \( L \)-function itself. Working with a traditional amplifier, however, is a more robust technique, since one can exploit positivity more strongly. This gives

**Theorem 4.** Let \( q \) be squarefree with \( (6, q) = 1 \), \( \varepsilon > 0 \) and \( f \in B^*_{\text{hol}}(q) \cup B^*(q) \). Then

\[
L(1/2, f) \ll q^{\frac{1}{4} - \frac{1-2\varepsilon}{24} + \varepsilon}
\]

where the implicit constant depends on the archimedean parameter of \( f \), i.e. \( t_f \) or \( k_f \).

Note that this holds both for Maaß forms and for holomorphic forms of arbitrary (fixed) weight, including weight \( k = 2 \). The assumptions on \( q \) could be weakened with more work. While asymptotically weaker than (1.14) for \( \vartheta \to 0 \), with the current value of \( \vartheta = 7/64 \) the exponent 0.217 is numerically a little stronger than 0.221 in (1.14) and appears to be the current record. The generic 1/24-saving, which is 16.6 percent of the convexity bound, is the natural limit in situations where the Iwaniec-Sarnak amplifier based on the relation \( \lambda_f(p)^2 = 1 + \lambda_f(p^2) \) is applied, cf. for instance the corresponding bound [IS] in the sup-norm problem. As we cannot exclude the possibility of many small Hecke eigenvalues, a more efficient amplifier is not available unconditionally, but an optimal amplifier would yield the bound

\[
L(1/2, f) \ll q^{\frac{1}{4} - \frac{1-2\varepsilon}{24} + \varepsilon}
\]

of Burgess quality, which again marks the limit of current technology.

### 1.3. The methods.

The general strategy of the proof is simple to describe. We apply the Petersson and/or Kuznetsov spectral summation formula, followed by the GL(3) Voronoi formula. In this way, the Kloosterman sums become GL(1)-exponentials of the form \( e(n\tilde{a}/c) \), to which we can apply the additive reciprocity formula

(1.15) \[
e\left(\frac{na}{c}\right) = e\left(\frac{n\tilde{a}}{d}\right) e\left(\frac{n}{cd}\right).
\]
Now we reverse all transformations, i.e. we apply the GL(3) Voronoi formula in the other direction and the Kuznetsov formula backwards to arrive at a “dualized” spectral sum. The application of (1.15) makes this procedure non-involutory and yields the desired reciprocity formula.

Experience has shown that this type of simple back-of-an-envelope heuristics can lead to enormous technical challenges. In this paper we have tried to present an approach that exploits structural features (including higher rank tools) as much as possible and circumvents many of the well-known technical problems.

On the analytic side, we avoid stationary phase arguments and intricate asymptotic analysis completely, but let instead analytic continuation do the work for us. We do not start with an approximate functional equation (which also relieves us from any root number considerations), but work always with infinite Dirichlet series in the region of absolute convergence and postpone analytic continuation to the center of the critical strip to the last moment.

On the arithmetic side, we design the set-up carefully to take care of the combinatorial difficulties of higher rank Hecke algebras. This was already a key device in [BLM]. The GL(3) Voronoi formula (or – roughly equivalently – three instances of Poisson summation in the Eisenstein case) is conceptually well-understood, but introduces in practice a whole alphabet of auxiliary variables for additional divisibility and coprimality conditions. The multiple Dirichlet series that we introduce in Section 5 is tailored to the features of the GL(3) Hecke algebra. The combinatorial difficulties do not completely disappear in this way: a shadow remains in the rather complicated local correction factors \( L_q(s, w, f \times F) \) and \( L_q(s, w, E \times \chi \times F) \), but the advantage of our approach is complete symmetry before and after the application of the arithmetic reciprocity formula (1.15). We would like to emphasize the strength of genuine higher rank tools (such as combinatorial and analytic properties of the GL(3) Voronoi formula) even for applications that involve only GL(2) objects (such as Theorems 2–4), and it seems that the present paper is the first instance where higher rank tools are used for the analysis of the twisted fourth moment.

One of the major problems in earlier approaches [KMV, KY] is the presence of very complicated main terms coming from various sources, the most difficult being the various zero frequencies in the multiple applications of the Poisson summation formula. These terms are complicated by nature, but in our approach they arise in a very simple and conceptual way twice as a single residue of a certain multiple Dirichlet series.

It may be instructive to compare the general strategy with other options used in earlier works. As mentioned before, the analysis of [DFI] and [KMV] is based on a decomposition \( 4 = 2 + 2 \) and a treatment of the binary additive divisor problem. The formula of Kuznetsov and Motohashi [Mo] also decomposes \( 4 = 2+2 \) and dualizes both GL(2) components. Unfortunately this leads to an essentially self-dual deadlock situation that is resolved – as mentioned earlier – by subtracting the holomorphic spectrum from the Maaß spectrum in a carefully designed way. Kıral and Young [KY] instead use the decomposition \( 5 = 3 + 1 + 1 \) in the context of the fifth moment. They first dualize one of the GL(1) components, apply arithmetic reciprocity (1.15) and then dualize the GL(3) component. A major technical difficulty is the fact that this leads to Kloosterman sums with multiplicative inverses in the arguments, so that the Kuznetsov formula at general cusps for possibly non-squarefree levels is required. In contrast, our approach never touches the GL(1) components, but dualizes the GL(3) component twice. This more symmetric set-up avoids complications with the Kuznetsov formula and makes a classical amplification procedure possible, which seems to be hard to implement in the work of [KY]. The idea of coupling a \( 4 = 3 + 1 \) structure along with arithmetic reciprocity (1.15) goes back to important work of X. Li [Li1] and was first used for varying levels by the second author in [Kh].

### 1.4. A roadmap.

Sections 2–4 contain essentially known material, in particular the Kuznetsov formula and the Voronoi summation formula tailored in suitable form for later reference. An important technical ingredient in Section 3 is a detailed Fourier expansion of a complete orthonormal basis (including oldforms) in terms of Hecke eigenvalues. Lemma 3 will be used at the very end of the argument to verify that the integral transform \( T_{s', w'} h \) is weakly admissible. To treat holomorphic
functions of small weight as in Theorem 4, we use a combination of the Petersson and the Kuznetsov formula and certain special functions that are also introduced in Section 3.

Section 5 features a multiple Dirichlet series in three variables that contains both GL(3) Fourier coefficients and Kloosterman sum. It is carefully designed as the combinatorial hinge between the Kuznetsov formula and the Voronoi formula, so that both formulas can be applied without extra technical complications. If \( F = E_0 \) is an Eisenstein series, we take some time to compute the Laurent expansion of the triple pole in one of the variables that will contribute to the main term \( \mathcal{N}_{q,t}(s,w;b) \) in Theorem 1. These are related to the “fake main terms” in [KY]. After a somewhat involved computation it turns out that the Laurent coefficients (as a function of the two other variables) have a beautiful formulation as a quotient of Riemann zeta-functions and their derivatives.

Section 6 gives a precise definition of admissible functions and establishes some technical properties of these functions. Section 7 is also of analytic nature and studies in detail a certain integral transform that will later become the central part of the transform \( \mathcal{F}^\pm_{s',w'} \) in (10.2). The key point here is to obtain analytic continuation and suitable decay properties which is achieved by careful contour shifts.

Section 8 features a prototype of the reciprocity formula based on the triad Voronoi-reciprocity-Voronoi. At this point we still work with points \( s, w \) whose real part is sufficiently large. The use of Kuznetsov formula at the beginning and at the end to obtain the full 5-step procedure outlined in the previous subsection, as well as the analytic continuation to \( \Re w \geq \Re s \geq 1/2 \), is postponed to Section 10 where the proof of Theorem 1 is completed. The necessary local computations to deal with the ramified primes \( p \mid \ell q \) are provided in Section 9. It is then a simple task to derive Theorems 2–4 in the final two sections.

Finally we mention that while the abstract procedure Kuznetsov-Voronoi-reciprocity-Voronoi-Kuznetsov is completely symmetric, there are certain differences on a technical level before and after the reciprocity formula. In the latter case, some local factors at primes \( p \mid \ell q \) and the archimedean place converge absolutely only slightly to the right of \( \Re w \geq \Re s \geq 1/2 \). To get the desired analytic continuation, some extra maneuvers are necessary, as can be seen in the proof of Lemma 7 and in particular in Lemma 10 that will be applied both in Sections 8 and 10.

1.5. Notation and conventions. Throughout, the letter \( \varepsilon \) denotes an arbitrarily (and sufficiently) small positive real number, not necessarily the same at each occurrence. All implied constants may depend on \( \varepsilon \) (where applicable), but this is suppressed from the notation. We will frequently encounter multiple sums and integrals of holomorphic functions in one or more variables. All expressions of this type will be absolutely convergent by which we mean in addition locally uniformly convergent in the auxiliary variables, so that they represent again holomorphic functions. Also implied constants depending on complex variables are always understood to depend locally uniformly on them. By an \( \varepsilon \)-neighbourhood of a strip \( c_1 \leq \Re s \leq c_2 \) we mean the open set \( c_1 - \varepsilon < \Re s < c_2 + \varepsilon \), and similarly for multidimensional tubes. Occasionally we will encounter meromorphic functions in multidimensional tubes which will be holomorphic outside a finite set of polar divisors given by affine hyperplanes. By \( v_p \) we denote the usual \( p \)-adic valuation. We write \( a \mid b^\infty \) to mean that \( a \) divides some power of \( b \). We write \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

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2. Basic analysis

2.1. Mellin transform. Throughout this paper we denote by

\[
\hat{W}(s) = \int_0^\infty W(x)x^s \frac{dx}{x}
\]

the Mellin transform of a function \( W : [0, \infty) \to \mathbb{C} \) of class \( C^J \) for some \( J \in \mathbb{N}_0 \) satisfying \( x^j W^{(j)}(x) \ll_{j,a,b} \min(x^{-a}, x^{-b}) \) for some \( -\infty < a < b < \infty \) and all \( 0 \leq j \leq J \). The function \( \hat{W} \) is then (initially) defined in \( a < \Re s < b \) as an absolutely convergent integral and satisfies
\( \hat{W}(s) \ll (1 + |s|)^{-J} \) in this region (in some cases it may be continued meromorphically to a larger region).

The inverse Mellin transform of a function \( \mathcal{W} \) that is holomorphic in a strip containing \( a \leq \Re s \leq b \) and bounded by \( \mathcal{W}(s) \ll (1 + |s|)^{-r} \) for some \( r > n \in \mathbb{N} \), is given by

\[
(2.2) \quad \hat{W}(x) = \int_{(c)} \mathcal{W}(s)x^{-s}\frac{ds}{2\pi i}
\]

where here and in the following (c) denotes the line \( \Re s = c \) with \( a \leq c \leq b \). We have

\[
(2.3) \quad x^j\hat{W}^{(j)}(x) \ll \min(x^{-a},x^{-b}),
\]

for \( j = 0, 1, \ldots, n - 1 \).

### 2.2. The Gamma function

We recall the reflection, recursion and duplication formula

\[
\Gamma(s)\Gamma(1-s) = \pi\sin(\pi s)^{-1}, \quad \Gamma(s+1) = s\Gamma(s), \quad \Gamma(s+1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s).
\]

For fixed \( \sigma \in \mathbb{R} \), real \( |t| \geq 3 \), and any \( M > 0 \) we recall Stirling’s formula

\[
(2.4) \quad \Gamma(s) = e^{\frac{\pi}{2}|t|^{\sigma/2}}\exp\left(it\log\frac{|t|}{e}\right)g_{\sigma,M}(t) + O_{\sigma,M}(|t|^{-M}),
\]

where

\[
(2.5) \quad g_{\sigma,M}(t) = \sqrt{2\pi}\exp\left(\frac{\pi}{4}(2\sigma - 1)i\text{sgn}(t)\right) + O_{\sigma,M}(|t|^{-1})
\]

and also

\[
(2.6) \quad |t|^j g_{\sigma,M}^{(j)}(t) \ll j,\sigma,M
\]

for all fixed \( j \in \mathbb{N}_0 \). We define \( \Gamma_R(s) = \pi^{-s/2}\Gamma(s/2) \) and for \( j \in \{0, 1\} \) we write

\[
G_j(s) = \frac{\Gamma_R(s+j)}{\Gamma_R(1-s+j)} = 2(2\pi)^{-s}\Gamma(s)\begin{cases} \cos(\pi s/2), & j = 0, \\ \sin(\pi s/2), & j = 1, \end{cases}
\]

where the last identity follows from the reflection and duplication formula of the gamma function. We also need the linear combination

\[
(2.7) \quad G^\pm(s) = \frac{1}{2}G_0(s) \pm \frac{i}{2}G_1(s) = \Gamma(s)(2\pi)^{-s}\exp(\pm i\pi s/2),
\]

which by (2.4) and (2.6) satisfies for \( |t| \geq 3 \) the bound

\[
(2.8) \quad G^\pm(s) \ll (1 + |s|)^{\Re s - 1/2}e^{-\pi\max(0,|\Im s|)/2}
\]

and the asymptotic formula

\[
(2.9) \quad G^\pm(s + it) = |t|^{\sigma/2}\exp\left(it\log\frac{|t|}{2\pi e}\right)\tilde{g}_{\sigma,M}(t) + O_{\sigma,M}(|t|^{-M})
\]

with \( \tilde{g}_{\sigma,M} \) satisfying (2.6). Note that \( G^\pm(s + it) \) is exponentially decaying for \( \pm t \to \infty \) by (2.8).

From the theory of hypergeometric integrals we quote the following formula: if \( a, b, c, d \in \mathbb{C} \) satisfy \( \Re(c + d) - 1 > \Re(a + b) > 0 \) then

\[
(2.10) \quad \int_{(c)} \frac{\Gamma(a+s)\Gamma(b-s)}{\Gamma(c+s)\Gamma(d-s)}\frac{ds}{2\pi i} = \frac{\Gamma(a+b)\Gamma(c+d-1-a-b)}{\Gamma(c-a)\Gamma(d-b)\Gamma(d+c-1)}
\]

where the integral on the left is absolutely convergent and the path of integration separates the poles, i.e. \( \Re b > \sigma > -\Re a \), for instance \( \sigma = \frac{1}{2}(b - a) \). This follows from [Sl, (4.5.1.2)] together with [GR, 9.122.1], or by computing the Mellin transform of the product of two Bessel J-functions using [GR, 6.574.2].
2.3. Integration by parts. We quote a useful integration by parts lemma [BKY, Lemma 8.1], which follows immediately from [BKY, (8.6)].

Lemma 2. Let \( Y \geq 1, X, Q, U, R > 0, B \in \mathbb{N} \), and suppose that \( w \) is a smooth function with support on some interval \([\alpha, \beta]\), satisfying
\[
w^{(j)}(t) \ll XU^{-j}
\]
for \( 0 \leq j \leq B \). Suppose \( H \) is a smooth function on \([\alpha, \beta]\) such that
\[
|H'(t)| \gg R, \quad H^{(j)}(t) \ll YQ^{-j} \text{ for } j = 2, 3, \ldots, B.
\]
Then
\[
I = \int_{\mathbb{R}} w(t)e^{iH(t)}dt \ll_B (\beta - \alpha)X \left[ (QR/\sqrt{Y})^{-B/2} + (RU)^{-B/2} \right].
\]

3. The Kuznetsov formula

Let \( N \in \mathbb{N} \) and write
\[
N\nu(N) := [\Gamma_0(1) : \Gamma_0(N)], \quad \text{i.e. } \nu(N) = \prod_{p|N} \left(1 + \frac{1}{p}\right).
\]
We equip \( \Gamma_0(N) \backslash \mathbb{H} \) with the inner product
\[
(f, g) := \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z)\overline{g(z)}\frac{dz}{y^2}.
\]
The Kuznetsov/Petersson formulae require a sum over an orthonormal basis of automorphic forms of a given level \( N \). While vectors in different representation spaces of \( Z(\mathbb{A}_Q)\text{GL}_2(\mathbb{Q})/\text{GL}_2(\mathbb{A}_Q)/K_0(N) \) are always orthogonal, such a space may contain more than one \( L^2 \)-normalized vector of given \( K_\infty \)-type. Classically this corresponds to the existence of oldforms. In the following we give explicit formulae for the Fourier coefficients of a complete orthonormal basis.

3.1. Fourier expansion of Eisenstein series. The Fourier expansion of Eisenstein series is computed explicitly in [KL, Section 5] (see [BH, Section 2.7] for similar calculations over number fields). The unitary Eisenstein spectrum of \( \Gamma_0(N) \) can be parametrized by a continuous parameter \( s = 1/2 + it \) together with pairs \((\chi, M)\), where \( \chi \) is a primitive Dirichlet character of conductor \( c_\chi \) and \( M \in \mathbb{N} \) satisfies \( c_\chi^2 | M \mid N \). For our purposes this adelic parametrization is more convenient than the classical parametrization by cusps. For fixed \( t \) and \( \chi \) the Eisenstein series for various \( M \) belong to the same representation space. In the notation of [KL] we have \( M = \prod p^{\nu_p} \). We define as in [KL, (5.22)]
\[
n^2(M) := \frac{1}{M} \prod_{p|N} \frac{p}{p+1} \prod_{p|(M,N/M)} \frac{p-1}{p+1} = \frac{1}{M} \tilde{n}^2(M),
\]
say. Next, as in [KL, Section 5.5] we define
\[
N_1 = \prod_{p|N/M} p^{\nu_p(N)}.
\]
The normalized Eisenstein series \( E_{X, M, N}(z, s) \) of level \( N \) corresponding to \((\chi, M)\) has the Fourier expansion
\[
E_{X, M, N}(z, 1/2 + it) = \rho^{(0)}_{X, M, N}(t, y) + \frac{2\pi^{1/2+it}y^{1/2}}{\Gamma(1/2 + it)} \sum_{n \neq 0} \rho_{X, M, N}(n, t)K_{it}(2\pi|n|y)e(nz),
\]
where for \( n \neq 0 \) we have
\[
\rho_{X, M, N}(n, t) = \frac{C(\chi, M)}{(N\nu(N))^{1/2}|M(L(N))|^{(1 + 2it, \chi^2)}} \cdot \frac{|n|^{it}}{M^{1 + 2it}} \sum_{c|n} \chi(c) \sum_{(c, N_1) = 1} \frac{\chi(d)e \left(\frac{dn/c}{M}\right)}{d(d, M) = 1}. 
\]
for a constant $|C(\chi, M)| = 1$. As usual, the superscript $L^{(N)}$ denotes that the Euler factors at primes dividing $N$ are omitted. This follows from [KL, (5.32), (5.34)] after re-normalizing by $n(M)$, cf. [KL, (5.22)], and taking into account that the adelic inner product in [KL] differs from (3.2) by a factor (3.1). We write

$$M = c_\chi M_1 M_2,$$

where $(M_2, c_\chi) = 1$, $M_1 | c_\chi^\infty$, so that $c_\chi | M_1$ and $(M_1, M_2) = 1$. Then by the Chinese Remainder Theorem the $d$-sum equals

$$r_{M_2}(n/c) \sum_{d |_{(c_\chi M_1)}} \chi(d)e \left( \frac{dM_2(n/c)}{c_\chi M_1} \right) = \delta_{c_\chi M_1 | n} r_{M_2}(n/c) \chi(M_2) \bar{\chi}\left( \frac{n}{c_\chi M_1} \right) M_1 \sum_{d |_{(c_\chi)}} \chi(d)e \left( \frac{d}{c_\chi} \right),$$

where the Gauß sum on right hand side has absolute value $c_\chi^{1/2}$ and

$$r_M(n) = \sum_{d \text{ (mod } M \text{)}} e \left( \frac{dn}{M} \right) = \sum_{d | (n, M)} d \mu \left( \frac{M}{d} \right)$$

is the Ramanujan sum. Recalling (3.3), we conclude

$$\rho_{x,M,N}(n,t) = \frac{C(\chi, M, t) |n|^it}{(N\nu(N))^{1/2} \nu(N) L(N)(1 + 2it, \chi^2)} \left( \frac{M_1}{M_2} \right)^{1/2} \sum_{c_\chi M_1 | n \mod (c, N_1) = 1} \chi(c) c^{2it} r_{M_2}(n/c) \bar{\chi}\left( \frac{n}{c_\chi M_1} \right)$$

where $|C(\chi, M, t)| = 1$. We note that the condition $(c, N_1) = 1$ is equivalent to $(c, N/M) = 1$. Finally we insert the expression on the right of (3.5) for the Ramanujan sum getting the final formula

$$\rho_{x,M,N}(n,t) = \frac{C(\chi, M, t) |n|^it}{(N\nu(N))^{1/2} \nu(N) L(N)(1 + 2it, \chi^2)} \left( \frac{M_1}{M_2} \right)^{1/2} \sum_{\delta | M_2} \delta \mu(M_2/\delta) \bar{\chi}(\delta) \sum_{(c_\chi, M) = 1} \chi(c) c^{2it} \bar{\chi}(f).$$

with the notation (3.1), (3.3), (3.4). In particular, using the third bound of (11.3) below, we find for $t \in \mathbb{R}$ that

$$\rho_{x,M,N}(n,t) \ll (1 + |t|)N|n| \epsilon \left( \frac{M_1 M_2}{N^{1/2}} \right) \leq ((1 + |t|)N|n|)^\epsilon.$$

### 3.2. Fourier expansion of cusp forms.

The cuspidal spectrum is parametrized by pairs $(f, M)$ of $\Gamma_0(N)$-normalized newforms $f$ of level $N_0 | N$ and integers $M | N/N_0$. This comes from orthonormalizing the set $\{ z \mapsto f(Mz) : M | N/N_0 \}$ (which belongs to the same representation space) by Gram-Schmidt. If $f$ is a Maass form, we write the Fourier expansion of the pair $(f, M)$ as

$$(2 \cosh(\pi t)y)^{1/2} \sum_{n \neq 0} \rho_{f,M,N}(n) K_d(2\pi |n|y)e(nx).$$

A standard Rankin-Selberg computation shows (verbatim as in [Bl, (2.8) - (2.9)]) that for $M = 1$, i.e. for newforms, one has

$$|\rho_{f,1,N}(1)|^2 = \frac{1}{L(1, Ad^2 f) N \nu(N) \prod_{p | N_0} \left( 1 - \frac{1}{p^2} \right)}$$

and $\rho_{f,1,N}(n) = \rho_{f,1,N}(1) \lambda_f(n)$ for $n \in \mathbb{N}$. For general $M$, the Fourier coefficients of the orthonormal basis were computed in [BM, Lemma 9]. Define

$$r_f(c) = \sum_{b \mid c} \frac{\mu(b) \lambda_f(b)^2}{b} \left( \sum_{d | (c, N_0)} \frac{1}{d} \right)^{-2}, \quad \alpha(c) = \sum_{b \mid c} \frac{\mu(b)}{b^2}, \quad \beta(c) = \sum_{b \mid c} \frac{\mu^2(b)}{b^2},$$

3The conditions $(d, N_0) = 1$ in the definition of $r_f$ and $(b, N_0) = 1$ in the definitions of $\alpha$ and $\beta$ were erroneously missing there, see www.uni-math.gwdg.de/blomer/corrections.pdf.
define \( \mu_f(c) \) as the Dirichlet series coefficients of \( L(s, f)^{-1} \) and let

\[
\xi_f' (M, d) = \frac{\mu(M/d) \lambda_f(M/d)}{r_f(M)^{1/2} \beta(M/d)}, \quad \xi_f (M, d) = \frac{\mu_f(M/d)}{r_f(M)^{1/2} \alpha(M)^{1/2}}.
\]

Write \( M = M_1 M_2 \) where \( M_1 \) is squarefree, \( M_2 \) is squarefull and \( (M_1, M_2) = 1 \). Then

\[
(3.9) \quad \xi_f(M, d) := \xi_f'(M_1, (M_1, d)) \xi''_f(M_2, (M_2, d)) \ll M^\vartheta(M/d)^\vartheta
\]

and

\[
(3.10) \quad \rho_{f, M, N}(n) = \frac{1}{L(1, \text{Ad}^2 f)^{1/2} (Nv(N))^{1/2}} \prod_{p \nmid N_0} \left( 1 - \frac{1}{p^2} \right)^{1/2} \sum_{d \mid M} \xi_f(M, d) \frac{d}{M^{1/2}} \lambda_f(n/d)
\]

for \( n \in \mathbb{N} \) with the convention \( \lambda_f(n) = 0 \) for \( n \not\in \mathbb{Z} \). For \( -n \in \mathbb{N} \) we define \( \rho_{f, M, N}(n) = \epsilon_f \rho_{f, M, N}(-n) \) where \( \epsilon_f \in \{ \pm 1 \} \) is the parity of \( f \). Using a standard Rankin-Selberg bound as well as the standard lower bound (11.3) below, we obtain

\[
(3.11) \quad \sum_{n \leq x} |\rho_{f, M, N}(n)|^2 \ll (N x (1 + |t_f|))^\vartheta x.
\]

If \( f \) is a holomorphic newform of weight \( k \) and level \( N_0 \mid N \), we write the Fourier expansion of the pair \((f, M)\) as

\[
\left( \frac{2\pi^2}{\Gamma(k)} \right)^{1/2} \sum_{n \geq 0} \rho_{f, M, N}(n)(4\pi n)^{(k-1)/2} e(nz).
\]

If \( M = 1 \), then (3.8) remains true (cf. e.g. [BKY, (2.1)]), and so does (3.10) for \( n \in \mathbb{N} \) and arbitrary \( M \mid N/N_0 \) as well as (3.11) with \( k_f \) in place of \((1 + |t_f|)\). For negative \( n \) we define \( \rho_{f, M, N}(n) = 0 \).

### 3.3. Versions of the Kuznetsov formula.

For \( x > 0 \) we define the integral kernels

\[
\mathcal{J}^+(x, t) := \frac{\pi i}{\sinh(\pi t)} (J_{2it}(4\pi x) - J_{-2it}(4\pi x)),
\]

\[
\mathcal{J}^-(x, t) := \frac{-\pi i}{\sinh(\pi t)} (I_{2it}(4\pi x) - I_{-2it}(4\pi x)) = 4 \cosh(\pi t) K_{2it}(4\pi x),
\]

\[
\mathcal{J}^{\text{hol}}(x, k) := 2\pi i^k J_{k-1}(4\pi x) = \mathcal{J}^+(x, (k-1)/(2i)), \quad k \in 2\mathbb{N}.
\]

For future reference we record the Mellin transforms of these kernels:

\[
(3.12) \quad \mathcal{J}^+(., t)(u) = \frac{\pi i (2\pi)^{-u}}{2 \sinh(\pi t)} \left( \frac{\Gamma(u/2 + it)}{\Gamma(1 - u/2 + it)} - \frac{\Gamma(u/2 - it)}{\Gamma(1 - u/2 - it)} \right)
\]

\[
= (2\pi)^{-u} \Gamma(u/2 + it) \Gamma(u/2 - it) \cos(\pi u/2),
\]

\[
\mathcal{J}^-(., t)(u) = (2\pi)^{-u} \Gamma(u/2 + it) \Gamma(u/2 - it) \cos(\pi t),
\]

\[
\mathcal{J}^{\text{hol}}(., k)(u) = i^k (2\pi)^{-u} \pi \Gamma((u + k - 1)/2) \Gamma((1 + k - u)/2)^{-1}.
\]

These formulae follow from [GR, 17.43.16 & 18] together with the reflection formula for the gamma function.

Let \( h \) be an even function \( h(t) \ll (1 + |t|)^{-2-\delta} \) for \( t \in \mathbb{R} \cup [-i\theta, i\theta] \), and let \( h^{\text{hol}} \colon 2\mathbb{N} \to \mathbb{C} \) be a function satisfying \( h^{\text{hol}}(k) \ll k^{-2-\delta} \) for some \( \delta > 0 \). Then for \( q \in \mathbb{N} \), \( n, m \in \mathbb{Z} \setminus \{0\} \) we define

\[
A_q^{\text{Maass}}(n, m; h) := \sum_{q_0 \mid Mq} \sum_{f \in B^+(q_0)} \rho_{f, M, q}(n) \rho_{f, M, q}(m) h(t_f),
\]

\[
A_q^{\text{Eis}}(n, m; h) := \sum_{c^2 \mid Mq} \int_{\mathbb{R}} \rho_{\chi, M, q}(n, t) \rho_{\chi, M, q}(m, t) h(t) \frac{dt}{2\pi},
\]

\[
A_q^{\text{hol}}(n, m; h^{\text{hol}}) := \sum_{q_0 \mid Mq} \sum_{f \in B^{\text{hol}}_q(q_0)} \rho_{f, M, q}(n) \rho_{f, M, q}(m) h^{\text{hol}}(k_f).
\]
(Note that by definition $A_{q}^\text{hol}(n, m; h^\text{hol}) = 0$ if $n$ or $m$ are negative.) If in addition $h$ is holomorphic in an $\varepsilon$-neighbourhood of $|t| \leq 1/2$ and still satisfies the bound $h(t) \ll (1 + |t|)^{-2-\delta}$ in this region, then for $n, m \in \mathbb{N}$ the Bruggeman-Kuznetsov formula states (e.g. [IK, Theorem 16.3], cf. [IK, (16.19)] for the normalization there)

\begin{equation}
A_{q}^\text{Mas}^\text{hol}(n, m; h) + A_{q}^\text{Eis}(n, m; h) = \delta_{n, m} \int_{-\infty}^{\infty} h(t) \frac{t \tanh(\pi t) dt}{2\pi^2} + \sum_{q \mid c} S(n, m, c) K_{\mathcal{H}}(\sqrt{\frac{nm}{c}}),
\end{equation}

where

\begin{equation}
K_{\mathcal{H}}(x) := \int_{-\infty}^{\infty} J^+(x, t) h(t) t \tanh(\pi t) \frac{dt}{2\pi^2}
\end{equation}

and $S(m, n, c) \ll (m, n, c)^{1/2} \tau(c)c^{1/2}$ is the Kloosterman sum. Absolute convergence of the $c$-sum in (3.13) follows from shifting the contour in (3.14) to $\Im t = \pm 3/8$, say. The formula (3.13) is complemented by the Petersson formula [IK, Proposition 14.5]

\begin{equation}
A_{q}^\text{hol}(n, m; \delta = k_0) = \frac{k_0}{2\pi^2} \left( \delta_{n, m} + 2\pi i \delta - k_0 \sum_{q \mid c} S(n, m, c) J_{k_0-1}(4\pi \sqrt{nm}/c) \right)
\end{equation}

for $k_0 \in 2\mathbb{N}$, $n, m \in \mathbb{N}$. Sometimes it is useful to apply the Petersson formula (3.15) and the Kuznetsov formula (3.13) simultaneously. With this in mind, for a pair of functions $\mathfrak{h} = (h, h^\text{hol})$ satisfying the above conditions, we define

\begin{equation}
\mathcal{K}^\ast \mathfrak{h}(x) := \mathcal{K} h(x) + \sum_{k \in 2\mathbb{N}} i^{-k} k \frac{1}{\pi} h^\text{hol}(k) J_{k-1}(4\pi x).
\end{equation}

Then

\begin{equation}
A_{q}(n, m; \mathfrak{h}) := A_{q}^\text{Mas}(n, m; h) + A_{q}^\text{Eis}(n, m; h) + A_{q}^\text{hol}(n, m; h^\text{hol})
\end{equation}

\begin{equation}
= \delta_{n, m} A \mathfrak{h} + \sum_{q \mid c} S(n, m, c) \mathcal{K}^\ast \mathfrak{h} \left( \frac{\sqrt{nm}}{c} \right),
\end{equation}

for $n, m \in \mathbb{N}$ with

\begin{equation}
A \mathfrak{h} := \int_{-\infty}^{\infty} h(t) t \tanh(\pi t) \frac{dt}{2\pi^2} + \sum_{k \in 2\mathbb{N}} k \frac{1}{2\pi^2} h^\text{hol}(k).
\end{equation}

Conversely, if $H \in C^3((0, \infty))$ satisfies\footnote{There are various assumptions in the literature, e.g. [IK, (16.38)], [Ku, Theorem 2], [Mo, (2.4.6)]. We follow the latter, although the precise exponents make no difference for our argument.} $x^j H^{(j)}(x) \ll \min(x, x^{-3/2})$ for $0 \leq j \leq 3$ and $n, m, q \in \mathbb{N}$, then we have [IK, Theorem 16.5]

\begin{equation}
\sum_{q \mid c} S(\pm n, m, c) H \left( \frac{\sqrt{nm}}{c} \right) = A_{q}(\pm n, m; \mathcal{L} \pm H)
\end{equation}

where

\begin{equation}
\mathcal{L} \pm H = (\mathcal{L} \pm H, \mathcal{L} \text{hol} H), \quad \mathcal{L}^\ast \mathfrak{h} = \int_{0}^{\infty} J^+(x, \cdot) h(x) \frac{dx}{x}
\end{equation}

for $\cdot \in \{+, -, \text{hol}\}$. The formulas (3.17) and (3.19) are inverses to each other, and so are the corresponding integral transforms: for $H \in C^2((0, \infty))$ with $H^{(j)}(x) \ll \min(x^{1/2}, x^{-5/2})$ for $j = 0, 1, 2$ we have the Sears-Titchmarsh inversion formula (cf. [ST, (4.9)]) or [Ku, (A.4)])

\begin{equation}
\mathcal{K}^\ast \mathcal{L} \pm H = \mathcal{K}^\ast (\mathcal{L} \pm H, \mathcal{L} \text{hol} H) = H.
\end{equation}
To treat holomorphic cusp forms of small weight, we use the following special functions, borrowed from [BHM, Section 2]. For integers $3 < b < a$ with $a \equiv b \pmod{2}$ let

$$H(x) = H_{a,b}(x) = i^{b-a} J_a(4\pi x)(4\pi x)^{-b}.$$  

By [BHM, (2.21)] we have

$$\mathcal{L}^+ H(t) = \frac{b!}{2^b} \prod_{j=0}^{b} \left( t^2 + \left( \frac{a+b}{2} - j \right)^2 \right)^{-1}, \quad \mathcal{L}^{\text{hol}} H(k) = \frac{b!}{2^b} \prod_{j=0}^{b} \left( \left( \frac{1-k}{2} \right)^2 + \left( \frac{a+b}{2} - j \right)^2 \right)^{-1}.$$  

Obviously we have

$$\mathcal{L}^+ H(t) > 0, \quad \mathcal{L}^+ H(t) \asymp (1 + |t|)^{-2b-2}, \quad t \in \mathbb{R} \cup [-i\vartheta, i\vartheta]$$

and

$$\mathcal{L}^{\text{hol}} H(k) > 0 \text{ for } 2 \leq k \leq a - b, \quad |\mathcal{L}^{\text{hol}} H(k)| \asymp k^{-2b-2}.$$  

We choose a constant $c(a, b)$ such that

$$h_{\text{pos}}^{\text{hol}}(k) := \mathcal{L}^{\text{hol}} H_{a,b}(k) + \delta_{k > a-b} c(a, b) k^{-2b-1} > 0$$  

for all $k \in 2\mathbb{N}$, and we put $h_{\text{pos}} := \mathcal{L}^+ H_{a,b}$. Then by (3.22) and (3.23), the pair $h_{\text{pos}} = (h_{\text{pos}}^{\text{hol}}, h_{\text{pos}})$ (which depends on $a, b$, but this is suppressed from the notation) is strictly positive on the entire spectrum $T_0$, and we have by (3.20), (3.21) and (3.23) the identity

$$\mathcal{C} h_{\text{pos}}(x) = i^{b-a} J_a(4\pi x)(4\pi x)^{-b} + \sum_{a-b < k \in 2\mathbb{N}} \frac{c(a, b)}{k^{2b+1}} J_{k-1}(4\pi x).$$

For future reference we state the following lemma, for which we recall the notation (2.2).

**Lemma 3.** Let $0 \leq \vartheta \leq 7/64$ and $A \geq 5$ an integer. Let $\Phi$ be a function that is holomorphic in $-2\vartheta - \varepsilon < \Re u < A$ for some $\varepsilon > 0$ and satisfies $\Phi(u) \ll (1 + |u|)^{-A}$ in this region.

a) We have

$$\mathcal{L}^\pm \check{\Phi}(t) \ll_A (1 + |t|)^{-A}, \quad t \in \mathbb{R} \cup [-i\vartheta, i\vartheta], \quad \mathcal{L}^{\text{hol}} \check{\Phi}(k) \ll_A k^{-A}, \quad k \in 2\mathbb{N}.$$  

b) If $0 \leq \tau < 1$ and $\Phi$ is in addition meromorphic in $\Re u \geq -2\tau - \varepsilon$ with finitely many poles at $u_1, \ldots, u_n \in \mathbb{C}$, then $\mathcal{L}^\pm \check{\Phi}(t)$ has meromorphic continuation to $|\Im t| < \tau$ with poles at most at $t = \pm iu_j/2, j = 1, \ldots, n$.

**Proof.** a) Using the definitions (3.19) and (2.2) and exchanging integrals, we have for $t \in \mathbb{R} \cup [-i\vartheta, i\vartheta]$ that

$$\mathcal{L}^\pm \check{\Phi}(t) = \int_{(2\vartheta + \varepsilon)/2} \mathcal{J}^\pm(\cdot, t)(u) \Phi(-u) \frac{du}{2\pi i}$$

as an absolutely convergent integral (by Stirling’s formula), where $\mathcal{J}^\pm(\cdot, t)(u)$ is given in (3.12). This contour is within the region of holomorphicity of $\Phi(-u)$, but to the right of the poles of $\mathcal{J}^\pm(\cdot, t)$ for $t \in \mathbb{R} \cup [-i\vartheta, i\vartheta]$. To deduce the required bound, we may assume without loss of generality that $|t| \geq 1$. We shift the contour to the left to $\Re u = -A + 1/2$. On the way we pick up poles at $u = -2n \pm 2it, n \in \{0, 1, \ldots, \left\lfloor \frac{A}{2} \right\rfloor \}$ with residues of the shape

$$\pm 2(2\pi)^{2n+2it} \cosh(\pi t) \Gamma(-n \mp 2it) \Phi(2n \pm 2it) \ll (1 + |t|)^{-A-1/2}$$

with various sign combinations. We estimate the remaining integral by Stirling’s formula (2.4) as

$$\ll \int_{(-A+1/2)} \left( (1 + |3u + 2t|)(1 + |3u - 2t|) \right)^{-(2A+1)/4} (1 + |u|)^{-A} |du| \ll (1 + |t|)^{-A-1/2},$$

which completes the proof for $\diamond \in \{+, -\}$. 

The holomorphic case is similar (and a little easier). Here we have nothing to show for \( k \leq A \), and for \( k \geq A \), the contour shift to the left \( \Re u = -A + 1 \in \mathbb{N} \) does not produce any poles. The remaining integral can be estimated in the same way using the stronger bound
\[
\mathcal{J} \text{vol}(k)(-A + 1 + iw) \ll \frac{\Gamma((-A + k + iw)/2)}{\Gamma((k + A - iw)/2)} \ll (k + |w|)^{-A}
\]
for \( A \in \mathbb{N} \) by the recursion formula for the gamma function.

b) We fix some \( t_0 \in \mathbb{C} \) with \( |\Im t_0| < \tau \). By assumption we can bend the contour in (3.25) to be to the right of \( 2t_0 \) (within the region of meromorphicity of \( \Phi \)), but the unbounded part still runs at real part equal to \( 2\theta + \varepsilon/2 \). We may pick up poles with residue \( \mathcal{J}(.,\imath)(-u_j) \), which are meromorphic functions with poles at most at \( t = \pm iu_j/2 \), while the remaining integral is holomorphic in a neighbourhood of \( t_0 \).

4. The GL(3) Voronoi summation formula

Let \( F \) be a cuspidal automorphic form for the group \( \text{SL}_3(\mathbb{Z}) \). As in the introduction, let \( \theta \leq 5/14 \) be an admissible bound for the Ramanujan-Petersson conjecture for \( F \). We denote its archimedean Langlands parameters by \( \mu = (\mu_1, \mu_2, \mu_3) \) with
\[
(4.1) \quad \mu_1 + \mu_2 + \mu_3 = 0, \quad |\Re \mu_j| \leq \theta,
\]
and we denote its Fourier coefficients by \( A(n, m) \) as in [Go]. They satisfy \( A(n, m) = \overline{A(m, n)} \),
\[
(4.2) \quad A(n, m) = \sum_{d|(n,m)} \mu(d) A(n/d, 1) A(1, m/d)
\]
as follows from [Go, Theorem 6.4.11] and Möbius inversion), and the Rankin-Selberg bound
\[
(4.3) \quad \sum_{nm^2 \leq x} |A(n, m)|^2 \ll x.
\]
Individually, we only know
\[
(4.4) \quad A(n, m) \ll (nm)^{\theta + \varepsilon}.
\]
We will always regard \( F \) as fixed, and all implied constants may depend on \( F \), in particular on \( \mu \), but we suppress this from the notation.

The Voronoi summation formula [MS, Theorem 1.8] states the following: for \( c, d, m \in \mathbb{Z} \) with \( (c, d) = 1 \), \( c, m > 0 \) and a smooth compactly supported weight function \( w : \mathbb{R}_{>0} \to \mathbb{C} \) we have
\[
(4.5) \quad \sum_n A(m, n) e\left(\frac{nd}{c}\right) w(n) = c \sum_{\pm} \sum_{n_1 \mid c \text{cm } n_2 > 0} A(n_2, n_1) S(md, \pm n_2, mc/n_1) W(\frac{n_2^2}{c^2 m}) J(\frac{n_2^2}{c^2 m}),
\]
where
\[
W(\pm)(x) = \int_{(1)} x^{-s} \mathcal{G}_\mu^\pm(s + 1) \tilde{w}(-s) \frac{ds}{2\pi i}
\]
with
\[
\mathcal{G}_\mu^\pm(s) = \frac{1}{2} \prod_{j=1}^3 G_0(s + \mu_j) \pm \frac{1}{2i} \prod_{j=1}^3 G_1(s + \mu_j)
\]
\[
(4.6) \quad = 4(2\pi)^{-3s} \prod_{j=1}^3 \Gamma(s + \mu_j) \left( \prod_{j=1}^3 \cos \left( \frac{\pi(s + \mu_j)}{2} \right) \pm \frac{1}{i} \prod_{j=1}^3 \sin \left( \frac{\pi(s + \mu_j)}{2} \right) \right).
\]
(To compare this with the formula in [MS], write \( w = w_{\text{even}} + w_{\text{odd}} \) with \( w_{\text{even}}(x) = \frac{1}{2}(w(x) + w(-x)) \), \( w_{\text{odd}}(x) = \frac{1}{2}(w(x) - w(-x)) \), and observe that \( M_0 w_{\text{even}} = M_1 w_{\text{odd}} = \tilde{w} \).) The following lemma summarizes the analytic properties of \( \mathcal{G}_\mu^\pm \).

\(^5\)to ensure absolute convergence since a priori we have no growth condition for \( |\Im t| > 2\theta + \varepsilon \) available.
Lemma 4. The functions $G_n^{\pm}$ are meromorphic on $C$ with poles at $s = -n - \mu_k$, $n \in \mathbb{N}_0$, $k \in \{1, 2, 3\}$. In particular, they are holomorphic in $\Re s > \theta$. Away from poles they satisfy the bound

\[ G_n^{\pm}(s) \ll_{\Re s} (1 + |s|)^{3\Re s - \frac{3}{2}} e^{-\pi \max(0, \pm \Re s/2)}. \]

In particular, $G_n^{\pm}(s)$ is exponentially decaying for $\pm \Re s \to \infty$. For $|t| \geq 3$ sufficiently large we have the asymptotic formula

\[ G_n^{\pm}(\sigma + it) = |t|^3|\sigma - \frac{3}{2}| \exp\left(3it \log \frac{|t|}{2\pi e}\right) w_{\sigma, M, n}(t) + O_{\sigma, M}(|t|^{-M}) \]

with

\[ |t|^j w_{\sigma, M, n}(t) \ll_{j, \sigma, M, 1} \]

for all $j, M \in \mathbb{N}_0$. Finally, for $n \in \mathbb{N}_0$ and $\epsilon_2 \in \{\pm 1\}$ we have

\[ \sum_{\epsilon_1 \in \{\pm 1\}} e^{\epsilon_1 \pi n/2} G_{\pm, \mu}^{\epsilon_1}(1 - v + n) G_n^{\epsilon_1, \epsilon_2}(v) = \frac{1 + \epsilon_2}{2} (2\pi)^n \prod_{k=1}^{3} \prod_{j=1}^{n} (v - j + \mu_k). \]

Proof. Equations (4.7) and (4.8) follow directly from from (4.1) and Stirling’s formula (2.4). For the proof of (4.9) we use the recursion and reflection formula of the gamma function to see that $G_{\mu}^{\epsilon_1, \epsilon_2}(1 - v + n) G_n^{\epsilon_1, \epsilon_2}(v)$ equals

\[ 16(2\pi)^{-3+n} \left[ \prod_{k=1}^{3} \prod_{j=1}^{n} \frac{(-v + j - \mu_k)}{\sin(\pi(v + \mu_k))} \right] \left[ \prod_{k=1}^{3} \cos\left(\frac{\pi(v + \mu_k)}{2}\right) + \epsilon_1 \epsilon_2 \prod_{k=1}^{3} \sin\left(\frac{\pi(v + \mu_k)}{2}\right) \right] \]

\[ \left[ \prod_{k=1}^{3} \cos\left(\frac{(1 - v - n + \mu_k)}{2}\right) - \epsilon_1 \epsilon_2 \prod_{k=1}^{3} \sin\left(\frac{(1 - v - n + \mu_k)}{2}\right) \right]. \]

Hence $e^{\epsilon_1 \pi n/2} G_{\mu}^{\epsilon_1, \epsilon_2}(1 - v + n) G_n^{\epsilon_1, \epsilon_2}(v)$ equals

\[ 2(2\pi)^n \left[ \prod_{k=1}^{3} \prod_{j=1}^{n} \frac{(-v + j - \mu_k)}{\sin(\pi(v + \mu_k))} \right] \left[ \prod_{k=1}^{3} \cos\left(\frac{\pi(v + \mu_k)}{2}\right) + \epsilon_1 \epsilon_2 \prod_{k=1}^{3} \sin\left(\frac{\pi(v + \mu_k)}{2}\right) \right] \]

\[ (-1)^n \left[ \epsilon_1 \epsilon_2 \prod_{k=1}^{3} \cos\left(\frac{\pi(v + \mu_k)}{2}\right) + \prod_{k=1}^{3} \sin\left(\frac{\pi(v + \mu_k)}{2}\right) \right]. \]

Summing over $\epsilon_1 \in \{\pm 1\}$, we can drop all terms depending linearly on $\epsilon_1$, and (4.9) follows easily from the addition theorem of the sin-function.

We rewrite the summation formula in terms of Dirichlet series as follows. For positive integers $c, d, m$ with $(c, d) = 1$ let

\[ \Phi(c, \pm d, m; v) := \sum_{n>0} A(m, n) e\left(\pm \frac{nd}{c}\right) n^{-v}. \]

By (4.2) – (4.4) this is absolutely convergent in $\Re v > 1$ and satisfies the uniform bound

\[ \Phi(c, \pm d, m; v) \ll \alpha(m) := m^\varepsilon \max_{d|m}|A(d, 1)|, \quad \Re v \geq 1 + \varepsilon. \]

Moreover, let

\[ \Xi(c, \pm d, m; v) := c \sum_{n_1 | cm} \sum_{n_2 > 0} \frac{A(n_2, n_1)}{n_2 n_1} S(\pm md, n_2, mc/n_1) \left(\frac{n_2 n_1^2}{c e^m}\right)^{-v}. \]

This is absolutely convergent in $\Re v > 0$ and satisfies the uniform bound

\[ \Xi(c, \pm d, m; v) \ll (mc)^{1/2 + \Re v + \varepsilon}, \quad \Re v \geq \varepsilon. \]

\[ \text{Recall that all implied constants may depend on } \mu. \]
using Weil’s bound for Kloosterman sums and again (4.2) – (4.4).

By converting additive characters into multiplicative characters we see that the functions \( \Phi(c, \pm d, m; \cdot) \) and \( \Xi(c, \pm d, m; \cdot) \) are, up to finitely many Euler factors at primes dividing \( m \) that are holomorphic in \( \Re v > \theta \) resp. \( \Re v \geq \theta - 1 \), linear combinations of \( L \)-functions corresponding to \( F \times \chi \) for Dirichlet characters \( \chi \). This shows that \( \Phi(c, \pm d, m; \cdot) \) is of finite order and analytic in \( \Re v \geq 1/2 \) and \( \Xi(c, \pm d, m; \cdot) \) is of finite order and analytic in \( \Re v \geq -1/2 \). The Voronoi formula is equivalent to the vector-valued functional equation

\[
\begin{pmatrix} \Phi(c, d, m; v) \\ \Phi(c, -d, m; v) \end{pmatrix} = \begin{pmatrix} G^+_\mu(1 - v) & G^-_\mu(1 - v) \\ G^-_\mu(1 - v) & G^+_\mu(1 - v) \end{pmatrix} \begin{pmatrix} \Xi(c, d, m; -v) \\ \Xi(c, -d, m; -v) \end{pmatrix}.
\]

Inverting the “scattering matrix” (using (4.9) with \( n = 0 \)), we obtain

\[
\begin{pmatrix} \Xi(c, d, m; v) \\ \Xi(c, -d, m; v) \end{pmatrix} = \begin{pmatrix} G^\pm_\mu(-v) & G^\mp_\mu(-v) \\ G^\mp_\mu(-v) & G^\pm_\mu(-v) \end{pmatrix} \begin{pmatrix} \Phi(c, d, m; -v) \\ \Phi(c, -d, m; -v) \end{pmatrix}.
\]

In particular, both \( \Xi(c, \pm d, m; \cdot) \) and \( \Phi(c, \pm d, m; \cdot) \) are of finite order and entire. By the Phragmén-Lindelöf principle, (4.11), (4.13) and (4.7) we obtain

\[
\begin{align*}
\Phi(c, d, m; v) &< R_v \alpha(m) (mc^2(1 + |3v|)^3)^{\max(\frac{1}{2}, \frac{1}{4} + (1 - \Re v, 0), 0) + \varepsilon}, \\
\Xi(c, d, m; v) &< R_v \alpha(m) (mc^3)^{\max(\frac{1}{2}, \frac{1}{4} + (1 - \Re v, 0), 0) + \varepsilon} \big(1 + |3v|^3\big)^{\max(0, -\Re v - \frac{1}{2}, -\Re v)} + \varepsilon.
\end{align*}
\]

The Voronoi formula is a consequence of the functional equation of the twisted \( L \)-functions \( L(s, F \times \chi) \) and relations in the unramified \( GL(3) \) Hecke algebra. Therefore the functional equations (4.14) and (4.15) must continue to hold if \( F \) is an Eisenstein series, except that in this case the functions \( \Xi(c, \pm d, m; \cdot) \) and \( \Phi(c, \pm d, m; \cdot) \) are not necessarily entire any more. We are only interested in the case when \( F = E_0 \) is the minimal parabolic Eisenstein series with trivial parameters \( \mu = (0, 0, 0) \). In this case, \( A(n, 1) = A(1, n) = \tau_3(n) \) is the ternary divisor function, and in general \( A(n, m) \) is given by (4.2). As above, we see that \( \Phi(c, \pm d, m; \cdot) \) can only have a (triple) pole at \( v = 1 \) in \( \Re v \geq 1/2 \) and \( \Xi(c, \pm d, m; \cdot) \) can only have a (triple) pole at \( v = 0 \) in \( \Re v \geq -1/2 \). (A classical Voronoi formula for \( E_0 \) analogous to (4.5) with extra polar terms has been worked out in [Li2].)

5. A multiple Dirichlet series

Fix two coprime numbers \( \ell, q \in \mathbb{N} \) and three complex numbers \( s, w, u \in \mathbb{C} \). If

\[
\Re(w + u/2) > 1, \quad \Re(3s + u/2) > 2, \quad \Re(3s - u/2) > 4, \quad \Re u < -1/2,
\]

we define

\[
\mathcal{D}^\pm_{q, \ell}(s, u, w) := \sum_{(c, d) = 1} \sum_{\ell | mc} \Xi(c, \pm d, m; 1 + s + u/2)
\]

and

\[
\tilde{\mathcal{D}}^\pm_{q, \ell}(s, u, w) := \sum_{(c, d) = 1} \sum_{\ell | mc} \Phi(c, \pm d, m; -1 - s - u/2)
\]

with \( \Phi \) as in (4.10) and \( \Xi \) as in (4.12), satisfying the functional equations

\[
\begin{align*}
\mathcal{D}^\pm_{q, \ell}(s, u, w) &= G^\pm_{\mu}(1 - s - u/2)\tilde{\mathcal{D}}^\pm_{q, \ell}(s, u, w) + G^\pm_{\mu}(1 - s - u/2)\mathcal{D}^\pm_{q, \ell}(s, u, w), \\
\tilde{\mathcal{D}}^\pm_{q, \ell}(s, u, w) &= G^\pm_{\mu}(s + u/2)\mathcal{D}^\pm_{q, \ell}(s, u, w) + G^\pm_{\mu}(s + u/2)\tilde{\mathcal{D}}^\pm_{q, \ell}(s, u, w)
\end{align*}
\]

by (4.15) and (4.14). Recall that the numerator of (5.2) and (5.3) is holomorphic in the region (5.1) if \( F \) is cuspidal.

If \( F = E_0 \), the only polar divisor of \( \mathcal{D}^\pm_{q, \ell}(s, u, w) \) in \( \Re(-1 + s + u/2) \geq -1/2 \) can occur at \( -1 + s + u/2 = 0 \), and the only polar divisor of \( \tilde{\mathcal{D}}^\pm_{q, \ell}(s, u, w) \) in \( \Re(1 - s - u/2) \geq 1/2 \) can be at
We will see in a moment in Lemmas 6 and 7 that the Laurent expansions at these poles are independent of the ± sign. Since \( G^{\pm}_i(s) + G^{-}_j(s) \) has a triple zero at \( s = 1 \), the functional equations (5.4) imply that no other polar divisors of \( D_{q,\ell}^\pm(s, u, w) \) and \( \tilde{D}_{q,\ell}^\pm(s, u, w) \) can exist in the domain (5.1) of definition.

By (4.16) the triple sums in (5.2) and (5.3) are absolutely convergent: the first condition in (5.1) ensures absolute convergence of the \( d \)-sum, the other three conditions ensure absolute convergence of the \( c \)-sum; the \( m \)-sum requires \( \Re(s + w) > 1 \), \( \Re(w - \frac{1}{2}u) > 1/2 \) and \( \Re(\frac{1}{2}s + w - \frac{1}{4}u) > 1 \) for absolute convergence, which also follows from (5.1). Again by (4.16) we have the bounds

\[
\begin{align*}
D_{q,\ell}^\pm(s, u, w) &\ll_{s, w} (1 + |u|)^3 \max(0, \frac{1}{2} - \Re(s + \frac{1}{2}u), \frac{1}{2} - \Re(\frac{1}{2}s + \frac{1}{4}u)) + \varepsilon, \\
\tilde{D}_{q,\ell}^\pm(s, u, w) &\ll_{s, w} (1 + |u|)^3 \max(0, -\frac{1}{2} + \Re(s + \frac{1}{2}u), \Re(\frac{1}{2}s + \frac{1}{4}u)) + \varepsilon
\end{align*}
\]

for \( s, u, w \) satisfying (5.1) (away from the pole if \( F = E_0 \)). If in addition \( \Re(s + u/2) > 1 \), we can insert the definition (4.12) into (5.2). This gives the following alternative representation.

**Lemma 5.** Let \( (q, \ell) = 1 \) and let \( s, u, w \in \mathbb{C} \) satisfy

\[
\Re(w + u/2) > 1, \quad \Re(s + u/2) > 1, \quad \Re u < -1/2.
\]

Then

\[
D_{q,\ell}^\pm(s, u, w) = \sum_{\ell | r} \sum_{q | mc} \sum_{n_1} \frac{A(n_2, n_1) S(\pm r, n_2, c)}{n_2^{s + u/2} n_1^{1 - u - w + u/2}}.
\]

**Proof.** The conditions (5.6) imply (5.1). By absolute convergence we can re-arrange the sums to see that \( D_{q,\ell}^\pm(s, u, w) \) equals

\[
\begin{align*}
\sum_d \sum_{\ell | md} \frac{1}{(md)^{w + u/2}} &\sum_{n_1} \sum_{q | mc} \frac{1}{(mc)^{1 - u}} \sum_{n_2} \frac{A(n_2, n_1)}{n_2^{s + u/2} n_1^{1 - u - w + u/2}} S(\pm md, n_2, mc/n_1) \\
= \sum_d \sum_{\ell | md} \frac{1}{(md)^{w + u/2}} &\sum_{fgd = m} \sum_{n_1} \frac{1}{(mc)^{1 - u}} \sum_{n_2} \frac{A(n_2, \nu_1 f)}{n_2^{s + u/2} (\nu_1 f)^{2s + u - 1}} S(\pm md, n_2, \frac{mc}{\nu_1}) \\
= \sum_d \sum_{\ell | fgd} \frac{1}{(fgd)^{w + u/2}} &\sum_{q | fgd \nu_1} \frac{1}{(vc)^{1 - u}} \sum_{n_2} \frac{A(n_2, \nu_1 f)}{n_2^{s + u/2} (\nu_1 f)^{2s + u - 1}}.
\end{align*}
\]

There is a bijection between integer quintuples

\((d, f, g, \nu_1, \gamma)\) satisfying \( \ell \mid fgd, q \mid fgd \nu_1 \gamma, (\nu_1, g) = 1, (\nu_1 \gamma, d) = 1 \)

and integer triples

\((n_1, r, c)\) satisfying \( q \mid n_1 c, \ell \mid r \)

given by

\((n_1, r, c) = (\nu_1 f, fgd, g\gamma)\)

with inverse map

\((d, f, g, \nu_1, \gamma) = \left( \frac{r}{n_1 c}, \frac{1}{n_1 r}, \frac{n_1 c}{n_1 r}, \frac{n_1}{n_1 r} \right) \cdot \left( \frac{c(n_1 r)}{n_1 c}, \frac{n_1}{n_1 r} \right) \cdot \left( \frac{c(n_1 r)}{n_1 c}, \frac{n_1}{n_1 r} \right).
\]

This shows the desired formula (5.7).
Next we compute the Laurent expansion of $D_{q,t}^\pm(s,u,w)$ at $u = 2 - 2s$ if $F = E_0$. We write

$$D_{q,t}^\pm(s,u,w) = \sum_{j=1}^3 \frac{R_{q,t,j}(s,w)}{(u - (2 - 2s))^j} + O(1).$$

We will see in the proof of the next lemma that $R_{q,t,j}(s,w)$ is independent of the $\pm$ sign.

**Lemma 6.** Let $F = E_0$. Then the Laurent coefficients $R_{q,t,j}(s,w)$ are meromorphic in $\mathbb{C} \times \mathbb{C}$. In an $\varepsilon$-neighbourhood of the region $\Re w \geq \Re s \geq 1/2$, the function

$$(s - \frac{1}{2})(w-s)(s+w-1))^4 R_{q,t,j}(s,w)$$

is holomorphic and bounded by $O_{s,w}((q\ell)^{-1+\varepsilon})$.

**Proof.** In the region (5.6) we have (recall (4.2) and $A(n,1) = A(1,n) = \tau_3(n)$)

$$D_{q,t}^\pm(s,u,w) = \sum_{\ell | r} \sum_{q | n_1 c} \sum_{d_1, d_2, c} \frac{\tau_3(n_1) \mu(d) S(\pm r, db_1 b_2, c)}{n_1^2 d^{2s+u/2}(b_1 b_2)} \rho_j \left( \frac{b_1}{c}, \frac{b_2}{c} \right)$$

where

$$\zeta(s, \alpha) = \sum_{(n+\alpha)>0} \frac{1}{(n+\alpha)^s} = \frac{1}{s-1} - \psi(\alpha) - \gamma(\alpha)(s-1) + \ldots$$

is the Hurwitz zeta function for some functions $\psi, \gamma$. We note that

$$\sum_{b \mod m} \zeta(s, b/m) = m^s \zeta(s)$$

$$= \frac{m}{s-1} + m \log m + \gamma + \left( \frac{1}{2} m \log^2 m + \gamma m \log m - m \gamma_1 \right) (s-1) + \ldots$$

for Euler's constant $\gamma = 0.577\ldots$ and another constant $\gamma_1 \in \mathbb{R}$.

Inserting (5.11) into (5.10), we compute the Laurent coefficients as

$$R_{q,t,j}(s,w) = \sum_{\ell | r} \sum_{q | n_1 c} \sum_{d_1, d_2, c} \frac{\tau_3(n_1) \mu(d) S(\pm r, db_1 b_2, c)}{n_1^2 d^{2s+u/2}(b_1 b_2)} \rho_j \left( \frac{b_1}{c}, \frac{b_2}{c} \right)$$

where

$$\rho_1(n_1, n_2, m) = 8, \quad \rho_2(n_1, n_2, m) = -12 \psi(n_1) - 4 \log m,$$

$$\rho_3(n_1, n_2, m) = 6 \psi(n_1) \psi(n_2) - 3 \gamma(n_1) + (\log m)^2 + 6 \psi(n_1) \log m.$$

We open the Kloosterman sum and evaluate the $b_3$-sum getting

$$R_{q,t,j}(s,w) = \sum_{\ell | r} \sum_{q | n_1 c} \sum_{d_1, d_2, c} \frac{\tau_3(n_1) \mu(d) S(\pm r, 0, 0)}{n_1^2 d^{2s+u/2+1}c^{1+2s}w+1-s} \rho_j \left( \frac{b_1}{c}, \frac{b_2}{c} \right)$$

At this point it is clear that the $\pm$-sign plays no role. We order this by $\beta_i = (b_i, c)$ getting

$$R_{q,t,j}(s,w) = \sum_{q | n_1 c} \sum_{d_1, d_2, c} \frac{\tau_3(n_1) \mu(d) \mu(c/a)}{n_1^2 a^{w-s}d^{2s+1}c^{1+2s}w+1-s} R_j \left( \frac{c}{\beta_1}, \frac{c}{\beta_2}, d \right)$$
where
\[ R_1(n_1, n_2, m) = 8\phi(n_1)\phi(n_2), \]
\[ R_2(n_1, n_2, m) = -12\Psi^*(n_1, \phi(n_2)) - 4\phi(n_1)\phi(n_2) \log m, \]
\[ R_3(n_1, n_2, m) = 6\Psi^*(n_1)\Psi^*(n_2) - 3G^*(n_1)\phi(n_2) + \phi(n_1)\phi(n_2)(\log m)^2 + 6\Psi^*(n_1)\phi(n_2) \log m \]
with
\[ \Psi^*(n) = \sum_{b \mod n \atop (b, n) = 1} \psi(b/n), \quad G^*(n) = \sum_{b \mod n \atop (b, n) = 1} \gamma(b/n). \]

Let
\[ (5.15) \]
\[ \phi_j(n) = \sum_{ab=n} \mu(a)b(\log b)^j \]
denote the Dirichlet series coefficient of \((-1)^j\zeta^{(j)}(s-1)/\zeta(s)\). In particular, \(\phi_0 = \phi\) is the Euler phi-function. Then
\[ \Psi(m) := \sum_{b \mod m} \psi(b/m) = -m(\gamma + \log m) \]
by \((5.12)\), so that by Möbius inversion
\[ \Psi^*(n) = \sum_{ab=n} \mu(a)\Psi(b) = -\gamma\phi_0(n) - \phi_1(n). \]

Similarly, we have
\[ G(m) := \sum_{b \mod m} \gamma(b/m) = -\frac{1}{2} m \log^2 m - \gamma m \log m + m\gamma_1, \]
so that
\[ G^*(n) = \sum_{ab=n} \mu(a)\Psi(b) = \gamma_1\phi_0(n) - \gamma\phi_1(n) - \frac{1}{2} \phi_2(n). \]

Altogether,
\[ R_j(n_1, n_2, m) = \sum_{\nu+\mu+\kappa \leq 3-j} C_{\nu, \mu, \kappa, j}\phi_\nu(n_1)\phi_\mu(n_2)(\log m)^\kappa \]
for certain constants \(C_{\nu, \mu, \kappa, j}\). Substituting back into \((5.14)\) we obtain a linear combination of
\[ \sum_{q|n_1} \sum_{c} \sum_{d} \sum_{a|c} \sum_{\ell|ra} \sum_{\beta_1, \beta_2|c} \sum_{\ell|\beta_1, \beta_2} \sum_{r} \frac{\tau_3(n_1)\mu(d)\mu(c/a)}{n_1^2d|w-c|d_2+1_{2s+1,2r+w+1-s}} \phi_\nu(c/\beta_1)\phi_\mu(c/\beta_2)(\log dcar)^\kappa \]
with \(\nu + \mu + \kappa \leq 3 - j\). We make several changes of variables. We write \(\beta_i\gamma_i = c\), switch to the co-divisor \(\gamma_i\) and open the \(\phi_\nu\)-functions as in \((5.15)\), writing \(\gamma_i = a_ib_i\). Next we replace \(c\) with \(ac\) and recast the previous display as
\[ \sum_{q|n_1} \sum_{d} \sum_{a|c} \sum_{\ell|ra} \sum_{\beta_1, \beta_2|c} \sum_{\ell|\beta_1, \beta_2} \sum_{r} \frac{\tau_3(n_1)\mu(d)\mu(c)a_1\mu(a_2)b_1b_2}{n_1^{2s}(cd)^{2s+1}d_2^{1s+w+1-s}} (\log b_1)^\nu(\log b_2)^\mu(\log da^2cr)^\kappa. \]

We introduce the following generalized function
\[ Z_{q, \ell}(x_1, x_2, x_3) := \sum_{q|n_1} \sum_{d} \sum_{a|c} \sum_{\ell|ra} \sum_{\beta_1, \beta_2|c} \sum_{\ell|\beta_1, \beta_2} \sum_{r} \frac{\tau_3(n_1)\mu(d)\mu(c)a_1\mu(a_2)b_1b_2}{n_1^{2s}(cd)^{2s+1}d_2^{1s+w+1-s}} b_1^{x_1}b_2^{x_2}(da^2cr)^x_3 \]
so that \(R_{q, \ell}(s, w)\) is a linear combination of
\[ \frac{\partial^{\nu}}{\partial x_1^{\nu}} \frac{\partial^{\mu}}{\partial x_2^{\mu}} \frac{\partial^{\kappa}}{\partial x_3^{\kappa}} Z_{q, \ell}(x_1, x_2, x_3)|_{x_1=x_2=x_3=0} \]
with \( \nu + \mu + \kappa = 3 - j \leq 2 \). The function \( Z_{q, \ell}(x_1, x_2, x_3) \) can be written as an Euler product, and for a generic prime \( p \nmid \ell q \) the p-Euler factor equals

\[
(1-p^{-2s})^{-3}(1-p^{-(1+w-s-x_3)})^{-1} \sum_{\nu_1, \alpha, \gamma = 0}^{1} \sum_{\alpha_1, \alpha_2, \gamma, \delta = 0}^{1} \sum_{\alpha_1 + \beta_1 + \alpha_2 + \beta_2 \leq \alpha + \gamma} \frac{(-1)^{\bar{\nu} + \gamma + \alpha_1 + \alpha_2} \zeta_r(1+x_1) + \zeta_r(1+x_2)}{p^{(\gamma + \delta)(2s+1-x_3) + \alpha(1+s+w-2x_3)}}.
\]

This can be expressed in closed form using geometric series, for instance by distinguishing the 5 disjoint cases (i) \( \alpha = \gamma = 0 \), (ii) \( \alpha + \gamma \geq 1 \), (iii) \( \alpha = \beta_1 = \beta_2 = 0 \), (iv) \( \alpha + \gamma \geq 1 \), \( \alpha_1 = \beta_1 \geq 1 \), \( \alpha_2 + \beta_2 \geq 0 \) and (v) \( \alpha + \gamma \geq 1 \), \( \alpha_1 = \beta_1 \geq 1 \), \( \alpha_2 + \beta_2 \geq 1 \). After a lengthy, but completely straightforward computation we obtain the beautiful expression

\[
(1-p^{-2s})^{-3}(1-p^{-(1+w-s-x_3)})^{-1}(1-p^{-(s+w-x_2-2s)})^{-1}(1-p^{-(s+w-x_2-2x_3)})^{-1}.
\]

The Euler factor at primes \( p \mid q \ell \) can again be computed by a more complicated, but finite computation with geometric series, and it is clear that it is a rational function in \( p^{-s} \), \( p^{-w} \) and \( p^{-x_j} \), \( j = 1, 2, 3 \), in particular meromorphic. Since the product of (5.16) over all primes is a quotient of zeta-functions, it follows that \( \mathcal{R}_{q, \ell}(s, w) \) is a meromorphic function.

In an \( \varepsilon \)-neighbourhood of the region \( \Re w > \Re s \geq 1/2 \) and \( x_1 = x_2 = x_3 = 0 \), the Euler factors at primes \( p \mid \ell q \) converge absolutely. In particular, they are holomorphic, and we see after taking derivatives and putting \( x_1 = x_2 = x_3 = 0 \) that (5.9) is holomorphic. To get the desired bound, we estimate the Euler factors at \( p \mid q \) trivially as

\[
4 \sum_{\nu_1, \alpha, \rho = 0}^{1} \sum_{\nu_1 + \alpha + \rho \geq \varepsilon(\nu_1 + 1)/2} \frac{(\nu_1 + 2)\nu_1 + 1/2}{\nu_1 + 2(\alpha + \rho)} \ll (p^\nu(q))^{-1+O(\varepsilon)}.
\]

The same argument with the condition \( \alpha + \rho \geq \varepsilon_0(\ell q) \) instead of \( \nu_1 + \alpha + \gamma \geq \varepsilon_0(\ell q) \) applies for Euler factors at \( p \mid \ell \). This completes the proof of the lemma.

We also need to study the Laurent expansion of the companion function

\[
\tilde{D}_{q, \ell}^+(s, u, w) = \sum_{j=1}^{3} \frac{\tilde{R}_{q, \ell}^+(s, w)}{(u + 2s)^j} + O(1)
\]

at \( s + u/2 = 0 \) in the case when \( F = E_0 \) (again the Laurent coefficients are independent of the \( \pm \) sign).

**Lemma 7.** Let \( F = E_0 \). Then the Laurent coefficients \( \tilde{R}_{q, \ell}^+(s, w) \) are meromorphic in \( \mathbb{C} \times \mathbb{C} \). In an \( \varepsilon \)-neighbourhood of the region \( \Re w > \Re s \geq 1/2 \), the function

\[
((s - 1/2)(w - s)(s + w - 1))^3 \tilde{R}_{q, \ell}^+(s, w)
\]

is holomorphic and bounded by \( O_{s, w} ((\ell q)^{-1}q^{-1}) \).

**Proof.** The proof is similar, so we can be brief. We have

\[
\Phi(c, \pm d, m, n) = \sum_n A(m, n) e \left( \pm \frac{nd}{c} \right) n^{-w} = \sum_{r_1 \neq r_2 = m} \frac{\mu(r_1)A(r_2, 1)}{r_1^2} \sum_n \frac{A(1, n) c(mr_2d/c)}{n^{w}}
\]

\[
= \sum_{r_1 \neq r_2 = m} \frac{\mu(r_1)r_2(r_2)}{r_1^2} \frac{1}{c^{3w}} \sum_{a_1, a_2, a_3 \equiv (0, \pm c)} e \left( \pm \frac{a_1a_2a_3r_1d}{c} \right) \prod_j \zeta \left( \frac{a_j}{c} \right),
\]

so that

\[
\tilde{D}_{q, \ell}^+(s, u, w) = \sum_{r_1 \neq r_2 = m} \frac{r_2(r_2)\mu(r_1)}{r_1^2} \frac{1}{c^{3w}} \sum_{a_1, a_2, a_3 \equiv (0, \pm c)} e \left( \pm \frac{a_1a_2a_3r_1d}{c} \right) \prod_j \zeta \left( 1 - \frac{s - u/2}{c} \frac{a_j}{c} \right).
\]
We conclude
\[
\tilde{R}_{q,t}(s,w) = \sum_{(c,d)=1 \atop \ell \mid rd} \frac{\tau_3(r_2) \mu(r_1)}{r_1^{1+w+s} r_2^{1+2w+2s} c^{1+2s} d^{w-s}} \sum_{a_1,a_2,a_3 \mod c} e \left( \pm \frac{a_1a_2a_3r_1d}{c} \right) \tilde{\rho}_j \left( \frac{a_1}{c}, \frac{a_2}{c}, \frac{d}{rc^2} \right)
\]
with \( \tilde{\rho}_j(n_1, n_2, m) = (-1)^j \rho_j(n_1, n_2, m) \) as in (5.13). We evaluate the \( a_3 \)-sum getting
\[
\tilde{R}_{q,t}(s,w) = \sum_{(c,d)=1 \atop \ell \mid rd} \frac{\tau_3(r_2) \mu(r_1)}{r_1^{1+w+s} r_2^{1+2w+2s} c^{1+2s} d^{w-s}} \sum_{a_1,a_2 \mod c} \tilde{\rho}_j \left( \frac{a_1}{c}, \frac{a_2}{c}, \frac{d}{rc^2} \right).
\]
Again we order by \( \alpha_i = (a_i, c) \) getting
\[
\tilde{R}_{q,t}(s,w) = \sum_{(c,d)=1 \atop \ell \mid rd} \frac{\tau_3(r_2) \mu(r_1)}{r_1^{1+w+s} r_2^{1+2w+2s} c^{1+2s} d^{w-s}} \sum_{\alpha_1, \alpha_2 \mod c} \tilde{R}_j \left( \frac{a_1}{c}, \frac{a_2}{c}, \frac{d}{rc^2} \right)
\]
with
\[
\tilde{R}_j(n_1, n_2, m) = (-1)^j R_j(n_1, n_2, m) = \sum_{\nu+\mu+\kappa \leq 3-j} \tilde{C}_{\nu,\mu,\kappa,j} \phi_\nu(n_1) \phi_\mu(n_2) \phi_\kappa(m)^\kappa
\]
for certain constants \( \tilde{C}_{\nu,\mu,\kappa,j} \). Removing the coprimality condition \( (c,d) = 1 \) by Möbius inversion and changing variables similarly as in the preceding proof we are left with terms of the form
\[
\sum_{q \mid \ell, r \mid d} \frac{\tau_3(r_2) \mu(r_1) \mu(b)}{r_1^{1+w+s} r_2^{1+2w+2s} c^{1+2s} d^{w-s}} \sum_{\alpha_1, \alpha_2 \mod \ell} \phi_\nu(\alpha_1) \phi_\mu(\alpha_2) (\log d(br_1)^{-1} c^{-2})^\kappa
\]
As before we define the generalized function
\[
(5.19) \quad \tilde{Z}_{q,t}(x_1, x_2, x_3) = \sum_{q \mid \ell, r \mid d} \frac{\tau_3(r_2) \mu(r_1) \mu(b) \mu(\alpha_2) h_1 b_2^2}{r_1^{1+w+s} r_2^{1+2w+2s} c^{1+2s} d^{w-s}} b_1 b_2 x_1 x_2 (d(br_1)^{-1} c^{-2}) x_3.
\]
We compute the generic Euler factor at primes \( p \nmid \ell q \) to be
\[
\frac{(1 - p^{-s-w})^{-3} (1 - p^{-(w-s-x_3)})^{-1} (1 - p^{-(2s-2x_1+2x_3)})^{-1} (1 - p^{-(2s-2x_2+2x_3)})^{-1}}{(1 - p^{-s+w+x_3-x_1})^{-1} (1 - p^{-s+w+x_3-x_2})^{-1} (1 - p^{-2s+2x_1+2x_3})^{-1}}.
\]
(In fact, this can be deduced from the computation in the previous lemma by a change of variables.) As before, we conclude that \( \tilde{R}_{q,t}(s,w) \) is meromorphic.

The estimation of (5.18) requires slightly more care, because the region \( \Re w \geq \Re s \geq 1/2, x_1 = x_2 = x_3 = 0 \) just fails to be inside the region of absolute convergence of the individual Euler factors. However, the only problem is caused by the \( d \)-sum in (5.19).

Let first \( p \mid q \). Summing over \( d \) (restricted to powers of \( p \)) first, the same computation as before shows that the \( p \)-Euler factor equals \( (1 - p^{-s-x_3})^{-1} \) times a holomorphic expression that is bounded by \( \ll (p^s q^r(n))^{-1+\varepsilon} \) in an \( \varepsilon \)-neighbourhood of \( \Re w \geq \Re s \geq 1/2, x_1 = x_2 = x_3 = 0 \).

Suppose now that \( p \nmid \ell \). We split the sum into \( v_\ell(\ell) + 1 \) terms with \( p^\lambda \mid d \) for \( 0 \leq \lambda \leq v_\ell(\ell) \). In each of these we sum over \( d \) first, and then estimate the rest as before. This gives an expression of the form \( p^{\lambda(s-w-x_3)} (1 - p^{-(w-s-x_3)})^{-1} \) times an absolutely convergent sum that is uniformly bounded in an \( \varepsilon \)-neighbourhood of \( \Re w \geq \Re s \geq 1/2, x_1 = x_2 = x_3 = 0 \).

This yields the desired bound for (5.18) and completes the proof.

6. Admissible functions

We call a function \( H : \mathbb{R}_{>0} \to \mathbb{C} \) admissible of type \((A, B)\) for some \( A, B > 5 \) if it is of one of the following two types:
• [first type] we have \( H(x) = \mathcal{X} h(x) \) (with \( \mathcal{X} \) as in (3.14)) where \( h \) is even and holomorphic in \( |\Im t| < A \), such that \( h(t) \ll (1 + |t|)^{-B-2} \) and \( h \) has zeros at \( \pm(n + 1/2)i \), \( n \in \mathbb{N}_0 \), \( n + 1/2 < A \);

• [second type] we have

\[
H(x) = \alpha_0 J_0(4\pi x)(4\pi x)^{-b} + \sum_{a - b < k \in 2\mathbb{N}} \alpha_k J_{k-1}(4\pi x)
\]

for a constants \( \alpha_0 \in \mathbb{C}, a, b \in \mathbb{N} \) with \( a - b \geq A \), and \( \alpha(k) \ll k^{-B-2} \).

If \( \{H_j\} = \{\mathcal{X} h_j\} \) is a family of admissible functions of type \((A, B)\) (of the first type), we call it uniformly admissible of type \((A, B)\) if the implied constant in the required bound \( h_j(t) \ll (1 + |t|)^{-B-2} \) can be chosen independently of \( j \).

We call a weakly admissible (i.e. satisfying (1.3)) pair \( \mathfrak{h} = (h, h^\text{hol}) \) admissible if \( h(t) \) is holomorphic in an \( \varepsilon \)-neighbourhood of \( |\Im t| \leq 1/2 \) and \( \mathcal{X}^* \mathfrak{h} \) (with \( \mathcal{X}^* \) as in (3.16)) is admissible of type \( 7 \) (500, 500), i.e. of one of the above two types. Such a pair satisfies in particular the assumptions of (3.17). We call a family of pairs of the special shape \( \{h_j = (h_j, 0)\} \) uniformly admissible if the family \( \{h_j\} \) is uniformly admissible of type \((500, 500)\).

From now on, we agree on the convention that all implied constants may depend on admissible weight functions \( \mathfrak{h} \) where applicable; however, if \( \{h_j = (h_j, 0)\} \) is a uniformly admissible family, implied constants can always be chosen independently of \( j \) (this applies also to (1.11), for instance).

We will consider a uniformly admissible family only once in this paper, in the proof of Theorem 3.

**Lemma 8.** A pair \( \mathfrak{h} = (h, h^\text{hol}) \) is admissible if

a) \( \mathfrak{h} = (0, \delta_{k=k_0}) \) for some \( k_0 > 500 \);

b) \( \mathfrak{h} = (h, 0) \), where \( h \) is even and holomorphic in \( |\Im t| < 500 \), such that \( h(t) \ll (1 + |t|)^{-502} \) and \( h \) has zeros at \( \pm(n + 1/2)i \), \( n \in \mathbb{N}_0 \), \( n + 1/2 < 500 \);

c) \( \mathfrak{h} = (h_{\text{pos}}, h_{\text{pos}}^\text{hol}) \) as in (3.24) with \( a = 1000 \), \( b = 400 \). This function is strictly positive on \( T_0 = (\mathbb{R} \cup [-i\theta, i\theta]) \times 2\mathbb{N} \).

This is clear from the definition, cf. also (3.22). A typical function satisfying the hypotheses of Lemma 8b) is

\[
h(t) = e^{-t^2} \prod_{n \leq 500} (t^2 + (n + \frac{1}{2})^2).
\]

We need two technical properties of admissible functions that are presented in the following two lemmas.

**Lemma 9.** Let \( H \) be an admissible function of type \((A, B)\).

a) The Mellin transform \( \hat{H} \) is holomorphic in \( \Re u > -A \) and in this region bounded by

\[
\hat{H}(u) \ll \Re u (1 + |u|)^{\Re u - 1}.
\]

b) For any \( M > 0 \), we have the asymptotic formula

\[
\hat{H}(\sigma + it) = |t|^{\sigma - 1} \exp \left( it \log \frac{|t|}{4\pi e} \right) j_\sigma(t) + O_s(|t|^{\sigma - 1 - B})
\]

for \( \sigma > -A \), \( |t| \geq 30 \) and a smooth function \( j_\sigma \) satisfying for all \( \nu \leq B \) the bound

\[
|t|^{\nu} j_\sigma^{(\nu)}(t) \ll 1.
\]

As per our convention, if \( \{H_j\} = \{\mathcal{X} h_j\} \) is uniformly admissible of type \((A, B)\), then all implied constants in the previous lemma can be chosen independently of \( j \).

**Proof.** We treat the two types of admissible functions separately.

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\( ^7 \)We made no effort to optimize the requirements on \( \mathfrak{h} \); most likely the number 500 could be reduced to 50, say, with no additional effort.
1) Suppose that $H = \mathcal{X} h$ is of the first type. For $\Re u > 0$ we have by (3.12) and (3.14) that
\begin{equation}
\hat{H}(u) = \frac{i}{4\pi} \int_0^\infty (2\pi)^{-u} \left( \frac{\Gamma(u/2 + i\tau)}{\Gamma(1 - u/2 + i\tau)} - \frac{\Gamma(u/2 - i\tau)}{\Gamma(1 - u/2 - i\tau)} \right) \frac{h(\tau)\tau}{\cosh(\pi\tau)} d\tau
\end{equation}
(6.5)
\[ = \frac{i}{2\pi} \int_{-\infty}^{\infty} (2\pi)^{-u} \frac{\Gamma(u/2 + i\tau)}{\Gamma(1 - u/2 + i\tau)} \frac{h(\tau)\tau}{\cosh(\pi\tau)} d\tau,
\]
as an absolutely convergent integral. We shift the contour to the line $\Re \tau = -A + 1/8$. This does not cross poles of $1/cosh(\pi t)$ because of the zeros of $h$. In this way see that the integral is holomorphic in $\Re u > -2A + 1/4$ and bounded by $O((1 + |u|)^{\Re u - 1})$, with a uniform implied constant if $\{\mathcal{X} h_j\}$ is a uniform family. This confirms a).

Moreover, for $u = \sigma + it$ with $\sigma > -A$, $|t| \geq 30$ and $r = (\sigma - 1)/2$ (say) we have
\[ \hat{H}(\sigma + it)|t|^{-\sigma - 1} \exp \left( -it \log \frac{|t|}{4\pi e} \right)
\]
\[ = \frac{i}{2\pi} \int_{-\infty}^{\infty} (2\pi)^{-\sigma}|t|^{-\sigma - 1} \exp \left( -it \log \frac{|t|}{2e} \right) \frac{\Gamma(\sigma/2 - r + i(\frac{1}{2}t + \tau))}{\Gamma(1 - \frac{1}{2}\sigma - r + i(\frac{1}{2}t))} \frac{h(\tau)\tau}{\cosh(\pi(\tau + ir))} d\tau,
\]
having shifted the contour in (6.5) to imaginary part $r$ (as we need to obtain the analytic continuation to $\Re u = \sigma$). We want to show that except for an error of $O(|t|^{-B})$, this function satisfies (6.4) for $\nu \leq B$ (and fixed $r$ and $\sigma$). By trivial estimates using $h(t) \ll (1 + |t|)^{-B - 2}$, the portion $|\tau| > \frac{1}{2}|t|$ contributes $O(|t|^{-B})$. For the remaining portion we can insert Stirling’s formula (2.4) for the gamma quotient. The error term in (2.4) can be bounded by any negative power of $|t|$, and the main term gives us a phase $\exp(i\phi_{\tau}(t))$ with
\[ \phi_{\tau}(t) = -t \log \frac{|t|}{2e} + \frac{1}{2} t \log \frac{t^2 - (2\tau)^2}{(2e)^2} + \tau \log \frac{|t + 2\tau|}{t - 2\tau}
\]
satisfying
\[ \phi_{\tau}(t) = \frac{1}{2} \log \left| \frac{t^2 - (2\tau)^2}{t^2} \right| \ll \frac{|\tau|}{|t|}, \quad \phi_{\tau}^{(j)}(t) \ll \frac{|\tau|}{|t|^j} \quad (j \geq 2)
\]
uniformly in $|\tau| \leq \frac{1}{2}|t|$. Now we differentiate under the integral sign, and since
\[ |\tau|^\nu \frac{d^\nu}{dt^\nu} \left( |t|^{-\sigma - 1} \exp \left( -it \log \frac{|t|}{2e} \right) \frac{\Gamma(\sigma/2 - r + i(\frac{1}{2}t + \tau))}{\Gamma(1 - \frac{1}{2}\sigma - r + i(\frac{1}{2}t))} \frac{h(\tau)\tau}{\cosh(\pi(\tau + ir))} \right) \ll 1 + |\tau|,
\]
and $\int_{\mathbb{R}} |h(\tau + ir)|(1 + |\tau|)^2 d\tau \ll 1$, we obtain (6.4) as desired. This confirms b).

2) If $H$ is of the form (6.1), then
\begin{equation}
\hat{H}(u) = a_0 \frac{\Gamma(\frac{1}{2}(a + u - b))}{2(2\pi)^{a-b} \Gamma(\frac{1}{2}(2 + a - u + b))} + \sum_{a-b<k \in \mathbb{N}} \alpha_k \frac{\Gamma(\frac{1}{2}(k - 1 + u))}{2(2\pi)^k \Gamma(\frac{1}{2}(1 + k - u))},
\end{equation}
and a) is clear. By Stirling’s formula, the first term on the right hand side satisfies (6.3) and (6.4). To treat the second term, we write for $k \in \mathbb{N}$ and $u = \sigma + it$ the last fraction in the form
\[ \frac{\Gamma(\frac{1}{2}(k - 1 + u))}{\Gamma(\frac{1}{2}(1 + k - u))} = \frac{\Gamma(\frac{1}{2}(1 + \sigma + it))}{\Gamma(\frac{1}{2}(3 - \sigma - it))} \prod_{n=1}^{k-1} \frac{k - 1 + \sigma - 2n + it}{k - 1 - \sigma - 2n - it}.
\]
We use Stirling’s formula for the gamma fraction on the right hand side. In order to verify (6.3) and (6.4), it remains to show that
\begin{equation}
\sum_{k \in \mathbb{N}} \left| \alpha_k t^\nu \frac{d^\nu}{dt^\nu} \prod_{n=1}^{k-1} \left( \frac{k - 1 + \sigma - 2n + it}{k - 1 - \sigma - 2n - it} \right) \right| \ll 1
\end{equation}
for $\nu \leq B$. We have
\[ \frac{d^\nu}{dt^\nu} \left( \frac{k - 1 + \sigma - 2n + it}{k - 1 - \sigma - 2n - it} \right) = 2i^\nu \nu! \frac{k - 1 - 2n}{(k - 1 - 2n - \sigma - it)^{\nu + 1}},
\]
and so by Leibniz’ rule the left hand side of (6.7) is bounded by $\sum_k |\alpha_k| k^r \ll 1$ for $\nu \leq B$.

**Lemma 10.** Let $c, d \in \mathbb{C}$, $x \in \mathbb{R} \setminus \mathbb{N}_0$. Let $H$ be an admissible function of type $(A, B)$. Suppose that $2x + \Re c > -A$, $\Re d > \theta + \max(x, (x + 1)/2)$. Then

$$G^\pm(c, d) := \int_{(x)} G^\pm(\xi) \hat{H}(c + 2\xi) G_{-\mu}^\pm(d - \xi) \frac{d\xi}{2\pi i}$$

with $G^\pm$ as in (2.7) and $G_{-\mu}^\pm$ as in (4.6) is holomorphic for $\Re(c + 3d) < 2$ and has a meromorphic continuation to $\Re(c + 3d) < 3$ with a simple polar divisor at most at $c + 3d = 2$.

**Proof.** We recall that the spectral parameter $\mu = (\mu_1, \mu_2, \mu_3)$ of the automorphic form $F$ satisfies $|\Re \mu_j| \leq \theta \leq 5/14$. Throughout the proof we always assume that $c$ and $d$ satisfy

$$2x + \Re c > -A, \quad \Re d > \theta + \max(x, (x + 1)/2).$$

The first condition ensures that we are in the domain of holomorphicity of $\hat{H}$, the condition $\Re d > \theta + x$ ensures that $G_{-\mu}^\pm(d - \xi)$ is holomorphic, and the condition $\Re d > \theta + (x + 1)/2$ will be needed later.

Holomorphy in $\Re(c + 3d) < 2$ follows immediately from (2.8), (4.7) and Lemma 9a, but for $\Re(c + 3d) \geq 2$, the integral fails to converge absolutely. Fix any $\beta > 0$. Clearly it suffices to continue the restricted integral with the compact region $|\Im \xi| \leq D := 2\beta + 2\max_j |\Im \mu_j| + 2$ removed, meromorphically to the region $|\Im d| \leq \beta$ and $\Re(c + 3d) < 3$.

To this end, we will approximate the integrand for $|\Im \xi| \geq D$ by a simpler expression. The error in this approximation will decay more quickly by an additional power of $1/|\xi|$, which buys us absolute convergence in $\Re(c + 3d) < 3$. We will then complete the simplified integral by re-inserting the portion $|\Im \xi| \leq D$ and evaluate it explicitly by (2.10). In the following we treat only the case $G^+(c, d)$ and drop the superscript, the other case is analogous.

Using the definitions (2.7) and (4.6), we have

$$G^+(\xi) G_{-\mu}^-(d - \xi) = \Gamma(\xi)(2\pi)^{2\xi} \left( \prod_{j=1}^3 \Gamma(d - \xi - \mu_j) \right) \sum_{\nu=-3}^4 \gamma_\nu(d) e^{i\xi^\nu \xi}.$$

for certain holomorphic functions $\gamma_\nu(d)$ with $\gamma_4(d) = (2\pi)^{-3d}$ (that in general depend also on the spectral parameter $\mu$ which we suppressed from the notation). The terms $-3 \leq \nu \leq 3$ lead to exponentially decaying integrals and can easily be continued homomorphically to $\Re(c + 3d)$ arbitrarily large. For notational simplicity let us write $d_j := d - \mu_j$. We have

$$\prod_{j=1}^3 \Gamma(d_j - \xi) = \Gamma(1 - \xi) \Gamma(d_1 + d_2 - 1 - \xi) \frac{\Gamma(d_1 - \xi) \Gamma(d_2 - \xi)}{\Gamma(1 - \xi) \Gamma(d_1 + d_2 - 1 - \xi) \sin(\pi(d_3 - \xi)) \Gamma(1 - d_3 + \xi)} \frac{\pi}{\Gamma(1 - \xi) \Gamma(d_1 + d_2 - 1 - \xi) \sin(\pi(d_3 - \xi)) \Gamma(1 - d_3 + \xi)},$$

by the reflection formula, and by another application of the reflection formula we see that the term corresponding to $\nu = 4$ equals

$$(2\pi)^{2\xi} \Gamma(d_1 + d_2 - 1 - \xi) \frac{\pi^2 (2\pi)^{-3d} e^{2\pi i \xi}}{\Gamma(1 - d_3 + \xi) \sin(\pi \xi) \sin(\pi(d_3 - \xi)) \Gamma(1 - \xi) \Gamma(d_1 + d_2 - 1 - \xi)} \Gamma(d_1 - \xi) \Gamma(d_2 - \xi).$$

The basic observation is now that the last gamma fraction is asymptotically constant as $|\Im \xi| \to \infty$. More precisely, by (2.4) – (2.5), the difference

$$\frac{\Gamma(d_1 - \xi) \Gamma(d_2 - \xi)}{\Gamma(1 - \xi) \Gamma(d_1 + d_2 - 1 - \xi)} - 1,$$

is holomorphic in $|\Im d| < \beta$, $|\Im \xi| > D$ and bounded by $O_d, \Re \xi(|\xi|^{-1})$. Consequently, the difference

$$\frac{\pi^2 (2\pi)^{-3d} e^{2\pi i \xi}}{\sin(\pi \xi) \sin(\pi(d_3 - \xi)) \Gamma(1 - \xi) \Gamma(d_1 + d_2 - 1 - \xi)} \Gamma(d_1 - \xi) \Gamma(d_2 - \xi) = \begin{cases} 0, & \Im \xi > D, \\ 4\pi^2 (2\pi)^{-3d} e^{i\pi d}, & \Im \xi < -D, \end{cases}$$
is holomorphic in $|3d| < \beta$ (because we stay away from zeros for of the sin-function) and bounded by $O_{d,\Re\xi}(\xi^{-1})$. Since
\[
(2\pi)^2 \frac{\Gamma(d_1 + d_2 - 1 - \xi)}{\Gamma(1 - d_3 + \xi)} \hat{H}(c + 2\xi) |\xi|^{-1} \ll_{\Re, d, \Re c} |\xi|^{\Re(c + 3d) - 4}
\]
by (6.2), this portion can be continued to $\Re(c + 3d) < 3$ as an absolutely (and locally uniformly in $d$) convergent integral over $|3\xi| > D$. Hence we are left with
\[
4\pi^2 (2\pi)^{-3d} e^{i\pi d} \int_{|3\xi| > D} \hat{H}(c + 2\xi) (2\pi)^{2\xi} \frac{\Gamma(d_1 + d_2 - 1 - \xi)}{\Gamma(1 - d_3 + \xi)} (1 - \text{sgn}(3\xi)) \frac{d\xi}{2\pi i}
\]
and of course the constant in front of the integral plays no role. In order to remove the unwanted factor $1 - \text{sgn}(3\xi)$, we use that
\[
-i \cdot \text{sgn}(3\xi) - \frac{\Gamma(1 - d_3 + \xi) \Gamma(d_1 + d_2 - 1/2 - \xi)}{\Gamma(3/2 - d_3 + \xi) \Gamma(d_1 + d_2 - 1 - \xi)}
\]
is holomorphic in $|3d - \beta| < 1$ for $|3\xi| > D$ and bounded by $O_{d,\Re\xi}(\xi^{-1})$, which follows again from (2.4) – (2.5). Hence our job is reduced to continuing
\[
\int_{|3\xi| > D} \hat{H}(c + 2\xi) (2\pi)^{2\xi} \frac{\Gamma(d_1 + d_2 - 1 + \kappa - \xi)}{\Gamma(1 + \kappa - d_3 + \xi)} \frac{d\xi}{2\pi i}
\]
for $\kappa \in \{0, 1/2\}$, and we may now re-insert the compact interval $|3\xi| > D$. At this point the condition $\Re d > \theta + (x + 1)/2$ comes handy, since the completed contour is still to the right of all poles. Thus we let
\[
\mathcal{J}_\kappa(c, d) := \int_{(x)} \hat{H}(c + 2\xi) (2\pi)^{2\xi} \frac{\Gamma(d_1 + d_2 - 1 + \kappa - \xi)}{\Gamma(1 + \kappa - d_3 + \xi)} \frac{d\xi}{2\pi i}
\]
and treat the two types of admissible functions separately.

If $H = \mathcal{K} h$ is of the first type, then by (6.5) we have
\[
\mathcal{J}_\kappa(c, d) = (2\pi)^{-c-1} \int_{\mathbb{R} \xi = x} \int_{|3\tau| = 1/2} \frac{\Gamma(d_1 + d_2 - 1 + \kappa - \xi)}{\Gamma(1 + \kappa - d_3 + \xi)} \frac{d\xi}{2\pi i} h(\tau) \frac{d\tau}{\cosh(\pi \tau)}
\]
where the exchange of integration is easily justified by absolute convergence and we are allowed to shift the $\tau$-contour since $H$ is admissible of type $(A, B)$. The conditions (6.8) ensure that the arguments of the gamma factors in the numerator are to the right of all poles. By (2.10) and (4.1), we can evaluate the $\xi$-integral getting
\[
\mathcal{J}_\kappa(c, d) = (2\pi)^{-c-1} \int_{\mathbb{R} \xi = x} \int_{|3\tau| = 1/2} \frac{\Gamma(d_1 + d_2 - 1 + \kappa + \xi)}{\Gamma(2 - c - 3d)} \frac{d\tau}{\cosh(\pi \tau)} h(\tau)
\]
This expression is meromorphic in $\Re(c + 3d) < 2$ (intersected with (6.8)) with its only polar divisor at $c + 3d = 2$. (Note that (6.8) implies that the argument of the first gamma factor in the numerator has positive real part.)

If $H$ is of the second type, we argue similarly: by (6.6), $\mathcal{J}_\kappa(c, d)$ equals
\[
\frac{1}{2} \int_{(x)} \left[ a_0 (2\pi)^{b-c-1} \Gamma(\xi + 1/2 (a - b + c)) + \sum_{a-b-c\in2\mathbb{N}} a_k (2\pi)^{-c-1} \Gamma(\xi + 1/2 (k + 1 - c)) \frac{\Gamma(d_1 + d_2 - 1 + \kappa - \xi)}{\Gamma(1 + \kappa - d_3 + \xi)} \frac{d\xi}{2\pi i} \right]
\]
The first term in parentheses yields an integral that is absolutely convergent in $\Re(c + 3d) < 2 + b$. For the second term we exchange sum and integration and evaluate the $\xi$-integral explicitly by (2.10), which provides the meromorphic continuation to $\Re(c + 3d) < 3$ with a polar divisor only at $c + 3d = 2$. 

7. AN INTEGRAL TRANSFORM

Fix $s, w \in \mathbb{C}$ and suppose that $H$ is admissible of type $(A, B)$. With later applications in mind, we define the following contour consisting of four line segments

\begin{equation}
C = (\left\{ -\frac{3}{5} - i\infty, -\frac{3}{5} - i \right\} \cup \left\{ -\frac{3}{5} - i + \frac{1}{10} \right\} \cup \left\{ -\frac{3}{5} + i \right\} \cup \left\{ -\frac{3}{5} + i + i\infty \right\}).
\end{equation}

Then for all $u$ satisfying

\begin{equation}
- A < \Re\left(3s - w + u + \frac{1}{2} i\right), \quad \theta < \Re(s + \frac{1}{2} u) < \frac{1}{2}, \quad \Re(u + w) > 0
\end{equation}

the integral transforms

\begin{equation}
(\gamma_{s, w}^\pm \hat{H})(u) := \int_C \hat{H}(3s - w - 1 + u + 2i) \left[ G^+(\xi)G^-_{-\mu}(1 - s - \frac{1}{2} u - \xi)G^+_{\mu}(s + \frac{1}{2} u) \right]
+ G^-(\xi)G^+_{-\mu}(1 - s - \frac{1}{2} u - \xi)G^-_{\mu}(s + \frac{1}{2} u) \frac{d\xi}{2\pi i}
\end{equation}

with $G^\pm$ as in (2.7) and $G^\pm_{\mu}$ as in (4.6) define an absolutely convergent integral. Indeed, the first condition in (7.2) implies that the argument of $\hat{H}$ is in the region of holomorphicity as provided in Lemma 9a, while the third condition ensures, by Lemma 4 and (6.2), absolute convergence. It is important to note that this condition is independent of $\xi$ and in particular not affected by possible contour shifts. Finally, the second condition in (7.2) ensures that as $u$ varies no poles are crossed.

We would like to continue $(\gamma_{s, w}^\pm \hat{H})(u)$ holomorphically to a larger region of $(u, s, w)$ and also obtain good bounds in terms of $u$. All implied constants in the following may depend on $s, w, A, B, \mu$.

**Lemma 11.** Let $s, w \in \mathbb{C}$, $A, B > 2$, suppose that $H$ is admissible of type $(A, B)$ and suppose that the set $u$ satisfying (7.2) is non-empty. Then $(\gamma_{s, w}^\pm \hat{H})(u)$ can be extended holomorphically to all $(u, s, w) \in \mathbb{C}^3$ satisfying

\begin{equation}
\theta < \Re(s + \frac{1}{2} u) < \frac{1}{2} A + \frac{1}{2} \Re(s - w), \quad \Re(u + w) > 0
\end{equation}

and satisfies\(^8\)

\begin{equation}
(\gamma_{s, w}^\pm \hat{H})(u) \ll (1 + |u|)^{3(|\Re s|+|\Re w|+|\Re u|)-\min(1, A)}
\end{equation}

in this region. Moreover, $(\gamma_{s, w}^\pm \hat{H})(u)$ can be extended meromorphically to

\begin{equation}
-1 + \theta < \Re(s + \frac{1}{2} u) < \frac{1}{2} A + \frac{1}{2} \Re(s - w), \quad \Re(u + w) > -1
\end{equation}

with poles at most at $u/2 + w = 0$ and $u/2 + s + \mu_j = 0$, $j = 1, 2, 3$.

**Proof.** Before we start with the proof, we use (4.9) to compute explicitly the residue of the integrand in (7.3) at $\xi = -n$, $n \in \mathbb{N}$, as

\begin{equation}
\begin{cases}
0, \\
\hat{H}(3s - w - 1 + u - 2n)(2\pi)^{-2n}(n!)^{-1} \prod_{j=1}^n \prod_{k=1}^3 (s + \frac{1}{2} u + \mu_k - j)^3, & \pm = +,
\end{cases}
\end{equation}

The proofs of (7.5) and (7.6) need a slightly different treatment depending on the sign, and we start with $(\gamma_{s, w}^- \hat{H})(u)$. Let initially $(u, s, w)$ be in the region (7.2). We straighten the $\xi$-contour and shift it to the far left to $\Re \xi = -A_1 \not\in \mathbb{N}$. Since $\Re(s + u/2) > 0$, this does not leave the domain of holomorphicity of $\hat{H}$ provided

\begin{equation}
\Re(s - w - 1 - 2A_1) > -A.
\end{equation}

We pick up possible poles at $\xi = -n$, $n \in \mathbb{N}$, $n_0 < A_1$, whose residues are by (7.7) holomorphic functions in $u, s, w$ in the region (7.8). The remaining integral is holomorphic in the region

\begin{equation}
\theta < \Re(s + \frac{1}{2} u) < 1 - \theta + A_1, \quad \Re(u + w) > 0.
\end{equation}

\(^8\)Again, if $\{H_j\}$ is uniformly admissible of type $(A, B)$, then the implied constants can be chosen independently of $j$.  

This gives a holomorphic continuation of \((Y_{s,w}^{-})\bar{H}(u)\) in \((u,s,w)\) to the intersection of the regions (7.8) and (7.9). Choosing \(A_1 = \frac{1}{2} (A + R(s - v) - 1) - 1/100\), we obtain a region containing (7.4).

The same argument combined with Lemma 10 shows that \((Y_{s,w}^{-})\bar{H}(u)\) has a meromorphic continuation to

\[-1 + \theta < R(s + \frac{1}{2}u) < 1 - \theta + \frac{1}{2}A_1, \quad R(\frac{1}{2}u + w) > -1\]

with poles at most at \(u/2 + w = 0\) and \(u/2 + s + \mu_j = 0\), \(j = 1, 2, 3\).

Now we fix \(s\) and \(w\) and proceed to estimate \((Y_{s,w}^{-})\bar{H}(u)\) for \(u\) satisfying (7.4) with the aim of establishing (7.5). We write \(v = \frac{1}{2} 3u\) and may assume that \(v > 0\) is sufficiently large (in terms of \(\mu\)), for if \(|v|\) is bounded there is nothing to prove, and the case of negative \(v\) is similar. We may then focus our attention to the second term in (7.3), since by (4.7) the first term, which contains a factor \(G_{\mu}^+(s + \frac{1}{2}u)\), is exponentially decaying as \(v \to +\infty\).

We shift the \(\xi\)-contour back to the far right, to \(\Re \xi = A_2 > 0\), say, making sure that this line does not cross poles. This cancels all residues from (7.7) that we picked up earlier, but introduces potentially new residues at \(1 - s - \frac{1}{2}u - \xi = -\mu_j - n, n \in \mathbb{N}_0, j \in \{1, 2, 3\}\). We do not compute them explicitly, but observe that a priori the residues must be meromorphic in \((u,s,w)\); since we know already that \((Y_{s,w}^{-})\bar{H}(u)\) is holomorphic, their joint contribution must be holomorphic, too. Moreover, for \(1 - s - \frac{1}{2}u - \xi = O(1)\) but off the poles, the integrand in (7.3) is

\[\ll v^{O(1)} e^{-\pi v/2}\]

by (2.8) and (4.7); by Cauchy’s theorem this bound also holds for the residues at \(1 - s - \frac{1}{2}u - \xi = -\mu_j - n\), which is in agreement with (7.5). This time we used the exponential decay of \(G^{-}(\xi)\) in the second term of (7.3) under our current assumption \(v > 0\).

It remains to estimate the remaining integral on the line \(\Re \xi = A_2\). We split the integral smoothly into two pieces as follows. Let \(w : \mathbb{R} \to [0, 1]\) be a smooth function that is constantly 1 on \([-1, 1]\) and vanishes outside \([-2, 2]\). Then it suffices to estimate

\[I_1^- := \int_{(A_2)} \hat{H}(3s - w - 1 + u + 2\xi) G^{-}(\xi) G_{\mu}^+(1 - s - \frac{1}{2}u - \xi) G_{\mu}^-(s + \frac{1}{2}u) w \left(\frac{3\xi}{V}\right) \frac{d\xi}{2\pi i},\]

and

\[I_2^- := \int_{(A_2)} \hat{H}(3s - w - 1 + u + 2\xi) G^{-}(\xi) G_{\mu}^+(1 - s - \frac{1}{2}u - \xi) G_{\mu}^-(s + \frac{1}{2}u) \left(1 - w \left(\frac{3\xi}{V}\right)\right) \frac{d\xi}{2\pi i},\]

where \(1 \leq V \leq v\) will be chosen in a moment. We estimate \(I_1^-\) trivially by (6.2), (2.8) and (4.7) getting

\[I_1^- \ll v^{R(3s - w - 2 + u - A_2)} V^{A_2 + \frac{1}{2}}.\]

Since we are free to choose \(A_2\) as large as we wish, this is admissible for (7.5) provided \(V \leq v^{1-\delta}\) for some fixed \(\delta > 0\), which we assume from now on. For the estimation of \(I_2^-\), we first observe that we can restrict to the branch \(3\xi > 0\), as the branch \(3\xi < 0\) can be estimated trivially by (7.10)

\[\ll v^{O(1)} e^{-\frac{1}{2} \pi V},\]

which is admissible for (7.5) provided \(V > v^\delta\) for some fixed \(\delta > 0\). The treatment of the remaining case \(3\xi > 0\), which we assume from now on, is the only point where properties (6.3) and (6.4) of \(\hat{H}\) are required. We insert the asymptotic formula (6.3) along with the asymptotic formulae (2.9) and (4.8). The error terms corresponding to (2.9) and (4.8) save arbitrarily many powers of \(V > v^\delta\) which is admissible for (7.5). The error term corresponding to (6.3) contributes at most

\[\int_{x > V} (x + v)^{-R(s + \frac{1}{2}u) - A_2 - \frac{1}{2} - B} e^{-2\frac{3R(s - \frac{1}{2}u)}{4} - \frac{1}{4} - B} dV \ll v^{R(3s - 2s - w) - \frac{1}{4} - B}.\]

For the main terms, we obtain an integral of the shape

\[G_{\mu}^+(s + \frac{1}{2}u) \int_0^\infty (x + v)^{-\frac{1}{4} - A_2 - R(u - \frac{1}{2}n, u) A_2 - \frac{1}{2} \omega(x) e^{i\theta(x)} \left(1 - w \left(\frac{3\xi}{V}\right)\right) dx\]

where \(\omega(x) = -\frac{3\xi}{V}\).
with $x^j \omega^{(j)}(x) \ll j$ for all $j \in \mathbb{N}_0$ and

\begin{equation}
\phi(x) = x \log \frac{|x|}{2\pi e} - (x + v) \log \frac{|x + v|}{2\pi e}.
\end{equation}

We compute

\begin{equation}
\phi'(x) = \log \left| \frac{x}{x + v} \right|, \quad \phi^{(j)}(x) \ll \frac{|v|(|v| + |x|)^{j-2}}{|x|^j v + |x|^{j-1}}, \quad j \geq 2.
\end{equation}

We attach another smooth partition of unity and decompose the integral smoothly into dyadic pieces supported on $Z < x < 4Z$ with $Z \geq V$. For each piece we can apply Lemma 2 with

\begin{equation}
\beta - \alpha \gg U = Q = Z, \quad Y = \frac{v Z}{v + Z}, \quad R = \frac{v}{v + Z}, \quad X = (v + Z)^{-\frac{1}{2} - \frac{1}{2}(1 - 2\Im - \frac{1}{2}\Re)} Z^{1 - \frac{1}{2}}.
\end{equation}

(note that for $Z \gg x \gg v$ we have $\log x / (x + v) \approx v / (v + Z)$) and bound it by

\begin{equation}
\ll v^{3(\Re - \frac{1}{2}\Re - \frac{1}{2})} (v + Z)^{-\frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} \Re} Z^{2 - \frac{1}{2}} \left( \frac{\sqrt{Z} v}{v + Z} \right)^{-B/2}.
\end{equation}

We estimate $Z v / (v + Z) \gg \min(v, Z) \gg V$ and bound the previous display by

\begin{equation}
\ll v^{3(\Re - \frac{1}{2}\Re - \frac{1}{2})} V^{-\frac{1}{2} - B} Z^{-\Re + \frac{1}{2}u}.
\end{equation}

Summing this over $Z = 2^\nu$, we get a convergent sum by the last condition in (7.4), and choosing $V > v^{3/4}$, we obtain a total bound of

\begin{equation}
v^{3(\Re - \frac{1}{2}\Re - \frac{1}{2})} - \frac{1}{2} B.
\end{equation}

We proceed to treat $(\mathcal{H}_+ \mathcal{H})_0(u)$, which similar, but slightly simpler. Let initially $(u, s, w)$ be in the region (7.2). We straighten the $\zeta$-contour and shift it to the far left to $\Re z = -A_1 \not \in \mathbb{N}$ satisfying (7.8). By (7.7) we do not pick up any poles. As before this gives analytic continuation to the region (7.4) and meromorphic continuation to the region (7.6) with poles at most at $u/2 + w = 0$ and $u/2 + s + \mu_j = 0$, $j = 1, 2, 3$. In the present case, we do not shift back to the right, but estimate the integral on the line $\Re \zeta = A_1$. We continue to write $v = \frac{1}{2} 3u > 0$ and assume that $v$ is sufficiently large. We focus now on the first term in (7.3), since the second is exponentially decreasing in $v$. Again we split the $\zeta$-integral into two pieces and consider

\begin{equation}I_1^+ := \int_{(-A_1)} \mathcal{H}(3s - u - 1 + u + 2\xi)G^+(\xi)G^-_\mu(1 - s - \frac{1}{2} u - \xi)G^-_\mu(s + \frac{1}{2} u) \left( \frac{v + \Im \xi}{V} \right) \frac{d\xi}{2\pi i}
\end{equation}

and

\begin{equation}I_2^+ := \int_{(-A_1)} \mathcal{H}(3s - u - 1 + u + 2\xi)G^+(\xi)G^-_\mu(1 - s - \frac{1}{2} u - \xi)G^-_\mu(s + \frac{1}{2} u) \left( 1 - w \frac{v + \Im \xi}{V} \right) \frac{d\xi}{2\pi i},
\end{equation}

that is, we distinguish between $\Im (u + 2\xi)$ small or large. As before we assume $v^{\delta} < V < v^{1-\delta}$. We estimate $I_1^+$ trivially getting

\begin{equation}I_1^+ \ll v^{3(\Re + \frac{1}{2} u - \frac{1}{2})} v^{-A_1 - \frac{1}{2} V^{-\frac{1}{2} - \frac{1}{2} \Re - \frac{1}{2} u} + A_1}.
\end{equation}

For the estimation of $I_2^+$ we can restrict ourselves to the branch $v + \Im \xi < -V$, for the other branch satisfies (7.10). Again we now use the asymptotic formula (6.3) which contributes an error term of at most

\begin{equation}\int_{-\infty}^{-V-v} \left| v + x \right|^{\Re(\frac{1}{2} u + v) - \frac{1}{2} + A_1 - B} \left| x \right|^{-A_1 - \frac{1}{4} v^{3(\Re + \frac{1}{2} u) - \frac{1}{2}}} d x
\end{equation}

\begin{equation}\ll v^{3(\Re - \frac{1}{2} u - v) - \frac{1}{2} - B} + v^{3(\Re + \frac{1}{2} u) - \frac{1}{2} - A_1 - \frac{1}{4} V^{-\Re(\frac{1}{2} u + v) + \frac{1}{2} A_1 - B}.
\end{equation}

The main term produces an integral of the shape

\begin{equation}G^-_\mu(s + \frac{1}{2} u) \int_{-\infty}^{0} (x + v)^{-\frac{1}{2} + A_1 - \Re(\frac{1}{2} u + v) u - A_1 - \frac{1}{4} \omega(x) e^{i\phi(x)} \left( 1 - w \frac{v + x}{V} \right) d x
\end{equation}
with the same phase function (7.12) as before and \(|x + v|^j \omega_j(x) \ll j 1. We restrict the integral to smooth dyadic ranges \(|x + v| \gg Z \geq V\), so \(|x| \gg Z + v\). By the same application of Lemma 2 each such dyadic region can be bounded by

\[
Z^{\frac{1}{2} + \frac{1}{2}A_1 - \Re(v + \frac{1}{2}u)}(Z + v)^{-A_1 - \frac{1}{2}v\Re(s + \frac{1}{2}u) - \frac{3}{2} \left(\frac{Zv}{v + Z}\right)^{B/2}} \ll v^{3\Re(s + \frac{1}{2}u) - \frac{1}{2} Z^{-\Re(v + \frac{1}{2}u) V - \frac{1}{4} B}},
\]

and summing this over \(Z = 2^n\), we obtain a total bound of

\[
(7.16) \quad v^{3\Re(s + \frac{1}{2}u) - \frac{5}{4} V - \frac{1}{4} B}.
\]

We now choose \(V = v^{1/2}\), \(A_1 = \frac{1}{4}(A + \Re(s - w) - 2)\) in agreement with (7.8). Then the bounds (7.14), (7.15), (7.16) become \(O(v^E)\) with

\[
E = \max \left(\frac{4}{7}\Re(11s + 5u - w - 3) - \frac{1}{4} A, \Re(3s - u - w) - \frac{1}{2} B, \frac{1}{4}\Re(11s + 5u - w - 5) - \frac{1}{4} A - \frac{5}{2} B, 3\Re(s + \frac{1}{2}u) - \frac{3}{2} - \frac{1}{2} B\right).
\]

Combining this with the bounds (7.11) and (7.13), we complete the proof of (7.5).

8. A PRELIMINARY RECIPROCITY FORMULA

As outlined in the introduction, Theorem 1 is a consequence of the five-step procedure Kuznetsov-Voronoi-reciprocity-Voronoi-Kuznetsov. In the following proposition we consider the middle triplet Voronoi-reciprocity-Voronoi.

For \(s, w \in \mathbb{C}\) with \(\Re s, \Re w > 3/2\), \(q, \ell \in \mathbb{N}\) coprime and a function \(H\) satisfying \(H(x) \ll x^{2/3}\), we define the absolutely convergent expression

\[
\mathcal{E}^\pm_{q, \ell}(s, w; H) := \sum_{\ell | r} \sum_{q | n_1} \sum_{n_2} \frac{A(n_2, n_1)S(\pm r, n_2, c)}{n_2^2 n_1^2 c^{w+1}} H \left(\frac{\sqrt{n_2}}{c}\right).
\]

**Proposition 12.** Let \(H\) be an admissible function of type (500,500). Let \(3/2 < \Re s < 2, 4 < \Re w < 5\) and suppose that \(q, \ell \in \mathbb{N}\) are coprime. Then

\[
(8.1) \quad \mathcal{E}^+_q(s, w; H) = \mathcal{N}^{(1)}_{q, \ell}(s, w; H) - \mathcal{N}^{(2)}_{q, \ell}(s, w; H) + \sum_{\pm} \mathcal{E}^\pm_\ell(s', w'; \mathcal{Y}_{s', w; \ell}^\pm H)
\]

where \(s', w'\) are as in (1.1), the right hand side employs the notation (2.1), (2.2) and (7.3), and the “main terms” \(\mathcal{N}^{(1)}_{q, \ell}(s, w; H)\) and \(\mathcal{N}^{(2)}_{q, \ell}(s, w; H)\) are given in (8.6) and (8.9); they vanish if \(F\) is cuspidal, and if \(F = \mathcal{E}_0\), they have meromorphic continuation to an \(\varepsilon\)-neighbourhood of \(\Re w \geq 3\) \(1/2\) and satisfy the bounds

\[
(8.2) \quad \left((s - \frac{1}{2})(w - s)(s + w - 1)\right)^4 \mathcal{N}^{(1)}_{q, \ell}(s, w; H) \ll_{s, w} (q\ell)^{-1 + \varepsilon},
\]

\[
\left((s - \frac{1}{2})(w - s)(s + w - 1)\right)^6 \mathcal{N}^{(2)}_{q, \ell}(s, w; H) \ll_{s, w} (\ell q)^{\varepsilon \ell - 1}
\]

Moreover, the function \(\mathcal{Y}_{s', w; \ell}^\pm H\) is holomorphic in

\[
(8.3) \quad 0 < \Re s < 2, \quad 0 < \Re w < 5, \quad \max(2\Re s', -2\Re w) < \Re u < 15
\]

and in this region bounded by

\[
(8.4) \quad \mathcal{Y}_{s', w; \ell}^\pm H(u) \ll_{s, w} (1 + |u|)^{-15}.
\]

In addition, it is meromorphic in

\[
(8.5) \quad 0 < \Re s < 2, \quad 0 < \Re w < 5, \quad \max(-2 - 2\Re s', -2 - 2\Re w) < \Re u < 15
\]

with poles at most at \(u \in \{-2w', -2s' - 2\mu_1, -2s' - 2\mu_2, -2s' - 2\mu_3\}\).
Proof. We first note that if $H$ is admissible of type $(A,B)$, then by Lemma 9a) and (2.3) we have $H(x) \ll x^{2/3}$, so that $\mathcal{E}_{q,t}^+(s,w;H)$ makes sense. By Mellin inversion and (5.7) we have

$$\mathcal{E}_{q,t}^+(s,w;H) = \int_{(-1)} \hat{H}(u) D_{q,t}^+(s,u,w) \frac{du}{2\pi i}.$$ 

By (5.5) and (6.2) the $u$-integral is absolutely convergent provided that

$$\Re u < 0, \quad \Re(3s-u/2) > 3, \quad \Re(3s+u/2) > 3/2,$$

which is automatically satisfied, provided (5.1) holds. We shift the $u$-contour to $\Re(s+u/2) = -2/3$ (for the moment any negative number would suffice, but later we need $\Re(s+u/2) < -1/2$). This is still in agreement with (5.1). On the way we may pick up a pole at $u = 2 - 2s$, which contributes

$$(8.6) \quad N_{q,t}^{(1)}(s,w;H) := \text{res}_{u=2-2s} \hat{H}(u) D_{q,t}^+(s,u,w) = \left\{ \begin{array}{ll} \sum_{j=1}^{3} \mathcal{R}_{q,t} j(s,w) \hat{H}^{(j-1)}(2-2s), & F = \text{E}_0, \\ 0, & F \text{ cuspidal} \end{array} \right.$$ 

with the notation as in (5.8). Lemma 6 provides analytic continuation of this term as well as the first bound in (8.2).

Having shifted the contour to $\Re(s+u/2) = -2/3$, we can insert the first functional equation in (5.4) and apply the definition (4.10) since we are in the region of absolute convergence. In this way we conclude

$$\mathcal{E}_{q,t}^+(s,w;H) = N_{q,t}^{(1)}(s,w;H) + \int_{(-\frac{4}{3} - 2\Re s)} \hat{H}(u) \sum_{\pm} G_{-\mu}(1-s-u/2) \sum_{(c,d)=1}^{\infty} \sum_{q|m,c} A(m,n)e(\pm \frac{n}{cd}) \frac{du}{2\pi i}.$$ 

At this point we insert artificially a factor

$$1 = e \left( \mp \frac{n}{cd} \right) e \left( \pm \frac{n}{cd} \right) = e \left( \mp \frac{n}{cd} \right) \int_{C} G^\pm(\xi) \left( \frac{n}{cd} \right)^{-\xi} d\xi,$$

where $G^\pm$ and $C$ were defined in (2.7) resp. (7.1). The contour is designed so that the integral is absolutely convergent, but the contour is to the right of the pole at $\xi = 0$. By the reciprocity formula (1.15) the integral in (8.7) equals

$$\int_{(-\frac{4}{3} - 2\Re s)} \int_{C} \hat{H}(u) \sum_{\pm} G^\pm(\xi) G_{-\mu}(1-s-u/2) \sum_{(c,d)=1}^{\infty} \sum_{q|m,c} \Phi(d, \mp c, m; 1-s-u/2 + \xi) \frac{du}{2\pi i} = \int_{C} G^\pm(\xi) G_{-\mu}(1-s-u/2) \frac{du}{2\pi i},$$

and the entire expression is still absolutely convergent. Here we used that $\Re(1-s-u/2 + \xi) \geq 16/15 > 1$ on the entire $\xi$-contour, so that we are in the region of absolute convergence of $\Phi$. It is now convenient to interchange the $u$- and $\xi$-integration and to replace the straight $u$-contour with a polygonal contour $C(\xi)$ (depending on $\xi$) such that $\Re(1-s-u/2 + \xi) = 16/15$. The resulting expression is still absolutely convergent. We now introduce the new variables

$$(8.8) \quad s' = \frac{1}{2}(1-s+w), \quad w' = \frac{1}{2}(3s+w-1), \quad u' = 3s+w-1+u-2\xi.$$ 

This has the following effect: the exponents $(3s+\frac{1}{2}u-1-\xi, s+w, \frac{1}{2}u-\xi)$ of $(c,m,d)$ become $(w'+\frac{1}{2}u', s'+w', 3s'+\frac{1}{2}u'-1)$, and the contour $C(\xi)$ given by $\Re(1-s-u/2 + \xi) = 16/15$ becomes $s'+\frac{1}{2}u' = -\frac{1}{10}$ (independently of $\xi$). Recalling the definition (5.3), we can recast the integral in (8.7) as

$$\int_{C} \int_{(-\frac{4}{3} - 2\Re s')} \hat{H}(3s'-w'-1+u'+2\xi) \sum_{\pm} G^\pm(\xi) G_{-\mu}(1-s'-\frac{1}{2}w'-\xi) \frac{du'}{2\pi i} \frac{d\xi}{2\pi i}.$$
Now we shift the \( u' \)-integral the right to \( \Re(s' + u'/2) = 1/2 \). Again we may pick up a pole with residue
\[
\mathcal{N}_{\ell,q}^{(2)}(s, w; H) := \int_{\nu' w = -2
u} \text{Res} \, \hat{H}(3s' - w' - 1 + u' + 2\xi) \sum_{\pm} G^\pm(\xi) \hat{\tilde{g}}^\pm_{\mu}(1 - s' - \frac{1}{2}u' - \xi) \mathcal{D}^\pm_{\ell,q}(s', u', w') \frac{d\xi}{2\pi i}
\]
(which counts negative because of the right shift). If \( F \) is cuspidal, this vanishes, and if \( F = E_0 \), then by (5.17) it equals
\[
(8.9) \sum_{j=1}^{3} \mathcal{R}_{\ell,q;j}(s', u') \sum_{\nu_1 + \nu_2 = j - 1} \left(-\frac{1}{2}\right)^{\nu_2} \pm \int_{(1/10)} \frac{d\xi}{2\pi i} G^\pm(\xi) \hat{\tilde{g}}^\pm(\nu_1)(-2s + 2\xi) \hat{H}((1 - s) = 1 - \xi) \frac{d\xi}{2\pi i},
\]
where we straightened the contour \( C \). Note that the conditions \( \Re w \geq \Re s \geq 1/2 \) and \( \Re w' \geq \Re s' \geq 1/2 \) are equivalent and \((s' - 1/2)(w' - s')(s' + w' - 1) = (s - 1/2)(w - s)(w + s - 1)\), so Lemma 7 and Lemma 10 with \( x = 1/10, \, \ell = -2s, \, d = 1 \) provide meromorphic continuation of this term as well as the second bound in (8.2). (Note that \( \ell \) and \( q \) are interchanged relative to Lemma 7 and the \( \xi \)-integral contributes at most a triple pole at \( s = 1/2 \).)

Having shifted the contour to \( \Re(s' + u'/2) = 1/2 \), we apply the second functional equation in (5.4) and recall the definition (7.3) to obtain
\[
\mathcal{E}_{q,\ell}^\pm(s, w; H) = \mathcal{N}_{\ell,q}^{(1)}(s, w; H) - \mathcal{N}_{\ell,q}^{(2)}(s, w; H) + \sum_{\pm} \int_{(1/2 \Re s')} (\mathcal{V}^\pm_{s,w} \tilde{H})(u') \mathcal{D}^\pm_{\ell,q}(s', u', w') \frac{du'}{2\pi i}.
\]

Lemma 11 gives us analytic continuation and decay conditions for \( \mathcal{V}^\pm_{s,w} \tilde{H} \), so that we can continue to shift the contour to the right into the region of absolute convergence of \( \mathcal{D}^\pm_{\ell,q}(s', u', w') \) to, say, \( \Re(s' + u'/2) = 3/2 \). We write \( \mathcal{D}^\pm_{\ell,q}(s', u', w') \) in terms of its Dirichlet series representation (5.7), and by Mellin inversion we obtain the formula (8.1).

Now we observe that for \( 0 < \Re s < 2, \, 0 < \Re w < 5 \) we have \(-1/2 < \Re s', \, \Re w' < 5 \). By (7.4) – (7.5) with \( A, B \geq 500, \, |\Re s|, \, |\Re w| \leq 5 \), we obtain the holomorphicity of \( \mathcal{V}^\pm_{s,w} \tilde{H} \) in the region (8.3) and the bound (8.4). In particular, if \( 3/2 < \Re s < 2 \) and \( 4 < \Re w < 5 \), then \( \Re s' \geq 3/2 \) and \( \Re w' \geq 7/2 \), so that (8.3) and (8.4) in combination with (2.3) imply that \((\mathcal{V}^\pm_{s,w} \tilde{H})(x) \ll x^{2/3}, \) so that the rightmost term in (8.1) makes sense. Meromorphic continuation of \( \mathcal{V}^\pm_{s,w} \tilde{H}(u) \) to (8.5) and the location of poles follows from the statement containing (7.6). This completes the proof.

We end this section by relating \( \mathcal{E}^\pm(s, w; H) \) to spectral sums. By (3.17) we have for \( \Re s, \Re w > 3/2, \) \( q, \ell \) coprime and \( h \) admissible that
\[
(8.10) \sum_{d_1d_2 = q} \sum_{d} \sum_{n_1} A(n_2, n_1) \mathcal{A}_{d_2}(r, n_2, h) = \sum_{\ell} \sum_{n_2} A(n_2, n_1) \mathcal{A}_{d_2}(r, n_2, h) = \mathcal{E}^\pm_{q,\ell}(s, w, \mathcal{F}^\pm h).
\]

Conversely, if \( h \) satisfies \( x^2H^{(j)}(x) \ll \min(x, x^{-3/2}) \) for \( 0 \leq j \leq 3 \) and if in addition \( \mathcal{H} \) is holomorphic in \(-2\Re \varepsilon < \Re u < \Re u < 5 \) and satisfies \( \mathcal{H}(u) \ll (1 + |u|)^{-\delta} \), say, then by (3.18) for \( \Re s', \Re w' > 3/2 \) we have
\[
(8.11) \mathcal{E}^\pm_{\ell,q}(s', u'; H) = \sum_{d_1d_2 = q} \sum_{d} \sum_{n_2} A(n_2, n_1) \mathcal{A}_{d_2}(r, n_2, h) = \mathcal{E}^\pm_{q,\ell}(s, w, \mathcal{F}^\pm h),
\]
where by (3.7), (3.11), (4.3), Lemma 3a and Weyl’s law, the various sums in (8.10) and (8.11) are absolutely convergent. The next section is devoted to relating the left hand side of (8.10) and the right hand side of (8.11) to \( \mathcal{L} \)-functions.

9. Local factors

9.1. Local computations. For a prime \( p \) let \( \alpha_{f,\nu}(p) (\nu = 1, 2), \, \alpha_{F,j}(p) (j = 1, 2, 3) \) denote the Satake parameters of \( f \) and \( F \) at \( p \) satisfying
\[
\alpha_{f,1}(q)\alpha_{f,2}(q) = \alpha_{F,1}(q)\alpha_{F,2}(q)\alpha_{F,3}(q) = 1.
\]
We have
\[ \lambda_f(p^\nu) = \frac{\alpha_{f,1}(p)\nu+1 - \alpha_{f,2}(p)^{\nu+1}}{\alpha_{f,1}(p) - \alpha_{f,2}(p)} \]
and
\[ (9.2) \quad A(p^\nu, p^\mu) = \det \left( \begin{pmatrix} \alpha_{f,1}(p)\nu+1 & \alpha_{f,2}(p)^{\nu+1} \\ \alpha_{f,1}(p) & \alpha_{f,2}(p)^{\nu+1} \end{pmatrix} \right) V_F(p)^{-1} \]
where
\[ V_F(p) = \det \left( \begin{pmatrix} \alpha_{f,1}(p)^2 & \alpha_{f,2}(p)^2 \\ \alpha_{f,1}(p) & \alpha_{f,2}(p)^2 \end{pmatrix} \right); \]
in particular
\[ \lambda_f(p) = \alpha_{f,1}(p) + \alpha_{f,2}(p), \quad A(p, 1) = \alpha_{f,1}(p) + \alpha_{f,2}(p) + \alpha_{f,3}(p), \]
\[ A(1, p) = \alpha_{f,1}(p)\alpha_{f,2}(p) + \alpha_{f,1}(p)\alpha_{f,3}(p) + \alpha_{f,2}(p)\alpha_{f,3}(p). \]
Note that (9.2) remains formally true for \( \nu \) or \( \mu = -1 \) if we define \( A(p^\nu, p^\mu) = 0 \) in this case. Let
\[ L_p(p^{-s}, f \times X) = \sum_{n \in \mathbb{N}_0} \frac{A(n, m)\lambda_f(n)}{(nm^2)^s} = \prod_{j=1}^{3} \prod_{\nu=1}^{3} \left( 1 - \frac{\alpha_{f,j}(p)\alpha_{f,\nu}(p)}{p^s} \right)^{-1} \]
denote the local Rankin-Selberg factor at \( p \). We start with the following combinatorial lemma.

**Lemma 13.** For \( p \) prime and \( \Re s > \theta + \vartheta \), the following identities hold:
\[ \sum_{\nu \in \mathbb{N}_0} \frac{A(p^\nu, 1)\lambda_f(p^\nu)}{p^{qs}} = \left( 1 - \frac{A(1,p)}{p^{3s}} + \frac{\lambda_f(p)}{p^{3s}} \right) L_p(p^{-s}, f \times X), \]
\[ \sum_{\nu \in \mathbb{N}_0} \frac{A(p^{\nu+1}, 1)\lambda_f(p^{\nu})}{p^{qs}} = \left( A(p, 1) - \frac{A(1,p)\lambda_f(p)}{p^s} + \frac{\lambda_f(p^2)}{p^{2s}} \right) L_p(p^{-s}, f \times X), \]
\[ \sum_{\nu_1, \nu_2 \in \mathbb{N}_0} \frac{A(p^{\nu_2}, p^{\nu_1+1})\lambda_f(p^{\nu_2})}{p^{s(\nu_2+2\nu_1)}} = \left( A(1,p) - \frac{\lambda_f(p)}{p^s} \right) L_p(p^{-s}, f \times X). \]

**Proof.** A simple computation based on (9.2) and geometric series shows the power series identity
\[ \sum_{\nu \in \mathbb{N}_0} A(p^\nu, 1)\lambda_f(p^\nu)X^\nu = \left[ 1 - (\alpha_{f,1}(p)\alpha_{f,2}(p) + \alpha_{f,1}(p)\alpha_{f,3}(p) + \alpha_{f,2}(p)\alpha_{f,3}(p))\alpha_{f,1}(p)\alpha_{f,2}(p)X^2 \right. \]
\[ \left. + \alpha_{f,1}(p)\alpha_{f,2}(p)\alpha_{f,3}(p)(\alpha_{f,1}(p) + \alpha_{f,2}(p))\alpha_{f,1}(p)\alpha_{f,2}(p)X^3 \right] L_p(X, f \times X) \]
which by (9.1) and (9.3) with \( X = p^{-s} \) gives the first formula. The other two are proved in the same way.

**Remark:** Note that \( \theta + \vartheta \leq 5/14 + 7/64 < 1/2 \). This is useful numerical coincidence, but \( f \times X \) is known to correspond to an automorphic representation on \( \text{GL}(6) \) [KiSh], therefore such a relation follows at every place from general bounds towards the Ramanujan conjecture on \( \text{GL}(n) \) [LRS].

We need a similar technical lemma that we apply later for the contribution of Eisenstein series.

**Lemma 14.** Let \( M, d, g_1, g_2, q \in \mathbb{N} \) with \( g_1 \mid g_2 \) and \( M \mid q \). Then the series
\[ \sum_{c \in \mathbb{C}_1, \nu \in \mathbb{N}, \nu+1} \frac{A(cfM, nd)}{c} f^n \nu^{\nu+v} \prod_{p \mid q} \prod_{j=1}^{3} \left( 1 - \frac{\alpha_{f,j}(p)}{p^u} \right) \left( 1 - \frac{\alpha_{f,j}(p)}{p^v} \right), \]
initially defined for \( \Re u, \Re v > \theta \) as an absolutely convergent series, has a holomorphic extension to an \( \varepsilon \)-neighbourhood of \( \Re u, \Re v \geq 0, \Re(u + v) \geq 1/2 \) and is bounded by \( O(q^\varepsilon(dM)^\theta) \) in this region.
Proof. The sum factorizes into a product of $p \mid q$, and it suffices to consider each factor separately. Let $m = v_p(M)$, $k = v_p(d)$, and put $X = p^{-u}$, $Y = p^{-v}$. The local $p$-factor equals

$$E(p) := \sum_{\beta,\gamma,\delta \in \mathbb{N}_0 \atop (p^\beta \cdot g_1) = (p^\alpha, g_2) = 1} A(p^{\beta+\gamma+m},p^{\delta+k})X^{\beta}Y^{\gamma}(XY)^{\delta} \prod_{j=1}^{3}(1 - \alpha_{F,j}(p)X)(1 - \alpha_{F,j}(p)Y).$$

By (4.2) we have

$$A(p^{\beta+\gamma+m},p^{\delta+k}) = A(p^{\beta+\gamma+m},1)A(p^{\delta+k},1) - A(p^{\beta+\gamma+m-1},1)A(p^{\delta+k-1},1)$$

(with the above convention $A(p^{-1}, 1) = 0$). We treat the first summand on the right hand side, the second one is similar. Depending on whether (i) $p \mid g_1$, (ii) $p \nmid g_2$, but $p \nmid g_1$ or (iii) $p \nmid g_2$, we need to compute one, two or three geometric series. In cases (i) and (ii) we obtain

$$E(p) = \sum_{i=0}^{2} \sum_{j=0}^{3} \frac{P_{i,j}(\alpha_{F,1}(p), \alpha_{F,2}(p), \alpha_{F,3}(p))}{V_F(p)} X^i Y^j$$

where $P_{i,j}$ is a homogeneous polynomial of degree $3 + k + i + j$. Since $E(p)$ is an entire function in $\alpha_{F,j}$ for $X, Y$ sufficiently small, each $P_{i,j}$ must be divisible $V_F(p)$, and we obtain

$$A(p^{k}, 1) \sum_{i=0}^{2} \sum_{j=0}^{3} \tilde{P}_{i,j}(\alpha_{F,1}, \alpha_{F,2}, \alpha_{F,3})X^i Y^j$$

with a homogeneous polynomial $\tilde{P}_{i,j}$ of degree $k + i + j$. In case (iii) a similar argument shows

$$E(p) = \sum_{i=0}^{2} \sum_{j=0}^{3} \frac{\tilde{Q}_{i,j}(\alpha_{F,1}(p), \alpha_{F,2}(p), \alpha_{F,3}(p))\tilde{R}_{i,j}(\alpha_{F,1}(p), \alpha_{F,2}(p), \alpha_{F,3}(p))}{(1 - \alpha_{F,1}(p)XY)(1 - \alpha_{F,2}(p)XY)(1 - \alpha_{F,3}(p)XY)} X^i Y^j$$

for homogeneous polynomials $\tilde{Q}_{i,j}$, $\tilde{R}_{i,j}$ of degrees $k + i + j$ and $m + i + j$ respectively.

Since $\max_{\alpha_{F,j}(p)} \leq p^\theta$, we can continue each $E(p)$ to an $\varepsilon$-neighbourhood of $\Re u, \Re v \geq 0$, $\Re(u + v) \geq 1/2$ and bound it by $O(p^{\theta(3+m)+\varepsilon})$ in this region. This completes the proof.

Finally, using the formula for Bump’s double Dirichlet series [Go, Proposition 6.6.3], we have

$$\sum_{\ell|n_2} \sum_{\ell|n_1} A(n_2, n_1) \frac{L(s + w, F)L(2s, \tilde{F})}{\zeta(3s + w)} \sum_{n_1, n_2|\ell^\infty} A(\ell n_2, n_1) \frac{A(n_2, n_1)}{n_2^{s+w} n_1^{2s}}$$

which provides analytic continuation of the left hand side, initially defined in $\Re s, \Re w > 1/2$ to the region $\Re(3s + w) > 1$, $\Re s, \Re w > \theta$ (with polar divisors at most at $s = 1/2$, $s + w = 1$ if $F = E_0$),

and it also provides the bound

$$O_s,w(\ell^{-\Re(s+w)+\theta+\varepsilon})$$

in this region, away from polar divisors.

9.2. The cuspidal case. We start by considering

$$\sum_{d_1, d_2 = q} \sum_{\ell|r} \sum_{(n_1, q) = d_1} \sum_{n_2} A(n_2, n_1) \frac{A_{Maab}(\pm r, n_2, \ell)}{n_2^{s+w} r^w} d_2^{1+|1|/2} h(t_f) S_{q,\ell}(s, w; f)$$

with

$$S_{q,\ell}(s, w; f) := \sum_{d_1 d_2 = q} \sum_{M|d_2/d_0} \sum_{\ell|r} \sum_{(n_1, q) = d_1} \sum_{n_2} A(n_2, n_1) \rho_{f, M, d_2}(r) \frac{\rho_{f, M, d_2}(n_2)}{n_2^{s+w} r^w}$$

The sum factorizes into a product of $p \mid q$, and it suffices to consider each factor separately. Let $m = v_p(M)$, $k = v_p(d)$, and put $X = p^{-u}$, $Y = p^{-v}$. The local $p$-factor equals
for \( f \in \mathcal{B}^*(d_0) \). We insert (3.10) and obtain
\[
L(1, \text{Ad}^2 f) \prod_{p|d_0} (1 - p^{-2}) S_{q, \ell}(s, w; f)
\]
\[
= \sum_{d_1, d_2 = \ell r \ (n_1, q) = d_1} \sum_{d_0 \mid d_1 \delta_2 | M} \frac{\xi_f(M, \delta_1) \xi_f(M, \delta_2) \delta_1 \delta_2 A(n_2, n_1) \lambda_f(\ell) \lambda_f(n_2/\delta_2)}{d_0 \nu_d(d_2)} M \sum_{d_2 / d_2} \frac{\xi_f(M, \delta_1) \xi_f(M, \delta_2) \delta_1^{-1} \delta_2^{-1} d_1^{-2s} \sum_{r \ (n_1, d_1) = 1} A(n_2 \delta_2, n_1 d_1) \lambda_f(\ell) \lambda_f(n_2)}{n_2^2 n_1^2 r^w}.
\]

We open \( \lambda_f(\ell r) \) using the Hecke relations and recognize the \( r \)-sum as \( \Lambda_f(\ell; w)L(w, f) \) with the notation as in (1.4). We also recognize the \( n_1, n_2 \)-sum as \( L(s, f \times F) \) up to Euler factors at primes dividing \( q \). Hence
\[
S_{q, \ell}(s, w; f) = \frac{L(w, f) L(s, f \times F)}{\ell w L(1, \text{Ad}^2 f)} \Lambda_f(\ell; w) L_q(s, w, f \times F)
\]
with
\[
\tilde{L}_q(s, w, f \times F) = \frac{1}{q} \prod_{p|d_0} \prod_{j=1}^{\nu} \left( 1 - \frac{\alpha_f(p) \alpha_f(\nu)(p)}{p^s} \right) \sum_{d_1, d_2 = \ell r \ (n_1, q) = d_1} \sum_{d_0 \mid d_1 \delta_2 | M} \frac{\xi_f(M, \delta_1) \xi_f(M, \delta_2) \delta_1^{-1} \delta_2^{-1} d_1^{-2s}}{\nu_d(d_2)} \sum_{d_2 / d_2} \frac{\xi_f(M, \delta_1) \xi_f(M, \delta_2) \delta_1^{-1} \delta_2^{-1} d_1^{-2s}}{\nu_d(d_2)} \sum_{(n_1, d_1) = 1} A(n_2 \delta_2, n_1 d_1) \lambda_f(n_2)}{n_2^2 n_1^2 r^w} \prod_{p|d_0} (1 - p^{-2}).
\]

In particular, using the notation (1.7) we have
\[
\tilde{L}_q(s, w, f \times F) \ll q^{s-1} \sum_{d_1, d_2 = \ell r \ (n_1, q) = d_1} \sum_{d_0 \mid d_1 \delta_2 | M} \sum_{(\delta_1, d_2) = 1} \frac{1}{d_1 \delta_2} (d_1 \delta_2)^{-1} \left( \frac{M^2}{d_1 \delta_2} \right)^\theta \frac{1}{M} (d_1 \delta_2)^\theta \ll q^{t-1+\varepsilon}
\]
for \( \Re s, \Re w \geq 1/2 \). This proves (1.6) when \( f \) is Maass. For \( f \in \mathcal{B}^*(q) \) we have \( d_0 = q \), hence \( d_2 = q \), \( d_1 = M = \delta_1 = \delta_2 = 1 \), and the \( n_1, n_2 \)-sum over powers of primes dividing \( q \) in (9.6) can be computed explicitly using the first formula in Lemma 13. In this way one obtains (1.5) for Maass newforms \( f \) of level \( q \). With slightly more computational effort one shows
\[
\tilde{L}_q(s, f \times F) = \frac{1}{q} \prod_{p|d_0} \left( \frac{p-1}{p} \left( 1 - A(1, p) + \lambda_f(p) \right) / p^{3/2} \right) \prod_{p|d_0} \left( 1 + A(1, p) - A(1, p) / p - 1 \right)
\]
if \( q \) is squarefree and \( f \) is a newform of level \( d_0 | q \).

The same computation holds verbatim for \( f \in \mathcal{B}^*_\text{cusp}(d_0) \), which completes the proof Lemma 1 in the cuspidal case.

### 9.3. The Eisenstein case
Similarly as in the previous subsection we consider
\[
\sum_{d_1, d_2 = \ell r \ (n_1, q) = d_1} \sum_{n_2} \frac{A(n_2, n_1)}{n_2^2 n_1^2 r^w} A_{d_2}^{\text{Eis}}(\pm r, n_2, h) = \sum_{d_0 | d_2 \chi \text{ primitive}} \int_{\mathbb{R}} S_{q, \ell}(s, w; (t, \chi)) h(t) \frac{dt}{2\pi}
\]
with
\[
S_{q, \ell}(s, w; (t, \chi)) := \sum_{d_1, d_2 = \ell r \ (n_1, q) = d_1} \sum_{(n_2, d_2 \chi \text{ (mod } d_0))} \sum_{(\delta_1, d_2) = 1} \sum_{n_2} \frac{A(n_2, n_1) \rho_{\chi, d_0 M_2}(r, t) \rho_{\chi, d_0 M_1 M_2}(n_2, t)}{n_2^2 n_1^2 r^w}.
\]
for $t \in \mathbb{R}$ and $\chi$ a primitive Dirichlet character modulo $d_0$. We define

$$L(\chi, t, d_2) := \prod_{p \mid d_2} \left( 1 - \frac{\chi^2(p)}{p^{1+2it}} \left( 1 - \frac{\overline{\chi}(p)}{p^{1-2it}} \right) \right)^{-2},$$

insert (3.6) and recast $|L(1 + 2it, \chi)|^2 \mathcal{S}_d \ell(s, w; (t, \chi))$ as

$$\sum_{d_1 d_2 = q} L(\chi, t, d_2) \sum_{d_0 \mid d_1 \mid d_0^{\infty}} \sum_{d_0 d_1 \mid d_2} \sum_{\delta_1, \delta_2 \mid d_2} M_1 \delta_1 \delta_2 \mu(M_2/\delta_1) \mu(M_2/\delta_2) \bar{\chi}(\delta_1) \chi(\delta_2)$$

$$\prod_{n \mid q} \frac{1}{\mu(n)} \frac{\chi(n)}{n^{1/2}}.$$
with the notation (9.9). In particular, we can write
\begin{equation}
\sum_{d_1d_2=q} \sum_{(n_1,q)=d_1} \sum_{n_2} \frac{A(n_2,n_1)}{n_2^2n_1^{a+r}} A_{d_2}^{\text{triv}}(\pm r, n_2, \mathfrak{h}) = M_{q,\ell}^{\text{triv}}(s, w; \mathfrak{h}).
\end{equation}

For \(R_s, \Re w \geq 1/2, t \in \mathbb{R}\) we estimate trivially (using (4.4))
\begin{equation}
\hat{L}_q(s, w; E_{t,\chi} \times F) \ll q^{s-1} \sum_{d_1d_2=q} \sum_{|d_1|M_1|d_2|d_0M_2} \sum_{d_2} \sum_{\delta_1,\delta_2|M_2} \frac{(\delta_1\delta_2)^{1/2}}{M_2}(M_1\delta d_1)^\theta \ll q^{\theta-1+\epsilon}
\end{equation}
confirming (1.6) in the case of Eisenstein series. For an application in the next section we will also need analytic continuation of \(\hat{L}_q(s, w; E_{t,\text{triv}} \times F)\) to certain complex values of \(t\) for the trivial character \(\chi = \text{triv}\) modulo 1, i.e. the constant function with value 1.

**Lemma 15.** The functions \(\hat{L}_q(s, w; E_{\pm(1-s)/i,\text{triv}} \times F), \hat{L}_q(s, w; E_{\pm(1-w)/i,\text{triv}} \times F)\), initially defined in \(R_s, \Re w > 1\) as absolutely convergent series, have meromorphic continuation to an \(\varepsilon\)-neighbourhood of \(R_s, \Re w \geq 1/2\) with polar divisors at most at \(s = 1/2, w = 1/2\) and satisfy the bounds
\begin{equation}
\begin{cases}
(s-1/2)\hat{L}_q(s, w; E_{\pm(1-s)/i,\text{triv}} \times F) \\ (w-1/2)\hat{L}_q(s, w; E_{\pm(1-w)/i,\text{triv}} \times F)
\end{cases} \ll_{s,w} q^{\theta+\varepsilon-1}
\end{equation}
for \(1/2 - \varepsilon \leq R_s, \Re w < 1\).

**Proof.** The possible polar divisors at \(s = 1/2\) or \(w = 1/2\) come from \(L(\text{triv}, t, d_2)\) at \(t = \pm(1-s)/i\) or \(\pm(1-w)/i\) defined in (9.9). An application of Lemma 14 shows that the rest can be continued holomorphically to an \(\varepsilon\)-neighbourhood of \(R_s, \Re w \geq 1/2\). For \(it \in \{\pm(1-s), \pm(1-w)\}\) and \(\delta_1, \delta_2 | M_2\) we have \(|\delta_1^{1-s-it}\delta_2^{1-w+it}| \leq \max(M_2^{1/2-2\min(\Re s, \Re w)}, M_2^{(\Re w-\Re s)})\), so that (9.12) follows from (9.10) and
\begin{equation}
\frac{1}{q^{1-\varepsilon}} \sum_{d_1d_2=q} \sum_{d_0M_1d_0M_2} \sum_{\delta_1,\delta_2|M_2} \frac{M_2^{2-2\min(\Re s, \Re w)} + M_2^{(\Re w-\Re s)}}{M_2}(M_1\delta d_1)^\theta \ll q^{\theta+\varepsilon-1}
\end{equation}
for \(1/2 - \varepsilon \leq R_s, \Re w < 1\). This completes the proof.

We end this section by combining (9.7) (and the corresponding formula for the holomorphic case) and (9.11) with (1.8) to obtain
\begin{equation}
\sum_{d_1d_2=q} \sum_{(n_1,q)=d_1} \sum_{n_2} \frac{A(n_2,n_1)}{n_2^2n_1^{a+r}} A_{d_2}^{\pm}(\pm r, n_2, \mathfrak{h}) = M_{q,\ell}^{\pm}(s, w; \mathfrak{h}).
\end{equation}

10. **Proof of Theorem 1**

We have now prepared the scene for a quick proof of Theorem 1. Let initially be \(3/2 < R_s < 2, 4 < \Re w < 5\) and let \(\mathfrak{h} = (h, h^{\text{hol}})\) be admissible. Then by definition, \(\mathcal{X}^{*}\mathfrak{h}\) is admissible of type \((500, 500)\), and moreover \(3/2 < R_s' < 2, 7/2 < \Re w' < 5\). Combining (8.1), (8.10), (8.11), (9.13), we obtain
\begin{equation}
M_{q,\ell}^{+}(s, w; \mathfrak{h}) = \sum_{\ell|n_2} \sum_{n_1} \frac{A(n_2,n_1)}{n_2^2n_1^{a+r}} A_{d_2}^\mathfrak{h} - \sum_{j \in \{1, 2\}} (-1)^j \mathcal{N}_{q,\ell}^{(j)}(s, w; \mathcal{X}^{*}\mathfrak{h}) + \sum_{\pm} \mathcal{M}_{q,\ell}^{\pm}(s', w', \mathcal{G}_{s', w}^\pm; \mathcal{X}^{*}\mathfrak{h})
\end{equation}
where
\begin{equation}
\mathcal{G}_{s', w}^\pm := \mathcal{L}^\pm(\mathcal{Y}_{s', w}^\pm; \mathcal{X}^{*}\mathfrak{h})
\end{equation}
using the notation (3.16), (7.3), (3.19) and (2.1), (2.2). Note, however, that we cannot simply insert the various formulas into each other to compute the transform on the right of (10.2), because there is a process of analytic continuation in Lemma 11. By (8.3), (8.4) and (2.3) we conclude that (8.11) is indeed applicable under the current assumption \(3/2 < R_s' < 2, 7/2 < \Re w' < 5\).
Write \( \mathcal{D}_{s', w}^\pm \mathfrak{h} = (h_{s', w'}, \pm h_{s', w'}^{hol}) \). Combining (8.3) – (8.4) with Lemma 3a, we conclude that \( \mathcal{D}_{s', w}^\pm \mathfrak{h} \) is weakly admissible as in (1.3) provided that
\[
\Re s' > \theta + \vartheta, \quad \Re w > \vartheta, \quad 0 < \Re s < 2, \quad 0 < \Re w < 5
\]
(clearly \( h_{s', w'}, \pm \)(t) is even in t). Since \( \theta + \vartheta < 1/2 \), this includes in particular the region \( 1/2 \leq \Re s \leq \Re w < 3/4 \). Moreover, combining (8.5) with Lemma 3b, we see that \( h_{s', w', \pm}(t) \) is meromorphic in an \( \varepsilon \)-neighbourhood of \( |\Im t| < 1/2 \) with poles at most at \( \pm it \in \{w', s' + \mu_1, s' + \mu_2, s' + \mu_3\} \). We will need this observation in a moment after having proved the next lemma.

It remains to continue all terms in (10.1) to a domain containing (1.9). The analytic continuation of the cuspidal contribution of the terms \( \mathcal{M}_{q, \ell}^+(s, w; \mathfrak{h}) \) and \( \mathcal{M}_{q, \ell}^\pm(s', w', \mathcal{D}_{s', w}^\pm \mathfrak{h}) \) is clear. For the Eisenstein contribution, we appeal to the following lemma.

**Lemma 16.** Let \( \mathfrak{h} = (h, h^{hol}) \) be weakly admissible, and suppose that \( h \) has a meromorphic continuation to an \( \varepsilon \)-neighbourhood of \( |\Im t| < 1/2 \) with at most finitely many poles. Then the term \( \mathcal{M}_{q, \ell}^{Eis}(s, w; \mathfrak{h}) \), initially defined in \( \Re s, \Re w > 1 \) continues meromorphically to an \( \varepsilon \)-neighbourhood of \( \Re w, \Re s \geq 1/2 \) with at most finitely many polar divisors. If \( 1/2 \leq \Re s, \Re w < 1 \), its analytic continuation is given by \( \mathcal{M}_{q, \ell}^{Eis}(s, w; \mathfrak{h}) + R_{q, \ell}(s, w; \mathfrak{h}) \) where \( R_{q, \ell}(s, w; \mathfrak{h}) \) is defined as
\[
\sum_{it = \pm(1-\varepsilon)} \operatorname{res}_{it = \pm(1-u)} \frac{L(s + it, F)\zeta(w + it)}{\zeta(1 + 2it)\zeta(1 - 2it)} \hat{L}_q(s, w, E_{t, \text{triv}} \times F) \frac{\Lambda_{E, \text{triv}}(F; w)}{\ell^w} h(t).
\]

**Proof.** For \( t \in \mathbb{R} \) choose \( 0 < \sigma(t) < 1/4 \) in a continuous way such that \( L(1 - 2\sigma + 2it, \chi) \neq 0 \) for \( 0 < \sigma < \sigma(t) \) and all primitive Dirichlet characters \( \chi \) of conductor \( c \), such that \( c^2 \mid q \) and in addition \( h(t - i\sigma) \) is pole-free for \( 0 < \sigma < \sigma(t) \). Let initially \( 1 < \Re w, \Re s < 1 + \sigma(3w) \). In the defining integral of \( \mathcal{M}_{q, \ell}^{Eis}(s, w; \mathfrak{h}) \) we shift the \( t \)-contour down to \( \Im t = -\sigma(\Re t) \). We pick up a pole at \( w - it = 1 \) if \( \chi = \text{triv} \) and a (triple) pole at \( s - it = 1 \) if in addition \( F = E_0 \). The remaining integral is holomorphic in \( 1 - \sigma(\Re t) < \Re w, \Re s < 1 + \sigma(\Re t) \). Now choosing \( s, w \) with \( 1 - \sigma(\Re t) < \Re w, \Re s < 1 \), we shift the \( t \)-contour back to \( \Im t = 0 \) picking up a pole at \( w + it = 1 \) if \( \chi = \text{triv} \) and a (triple) pole at \( s + it = 1 \) if in addition \( F = E_0 \). This proves the desired formula for \( 1 - \sigma(\Re t) < \Re w, \Re s < 1 \), but then it follows for \( s, w \) in an \( \varepsilon \)-neighbourhood of \( \Re w, \Re s \geq 1/2 \) by analytic continuation, using Lemma 15 for the term \( \hat{L}_q(s, w, E_{t, \text{triv}} \times F) \). This completes the proof.

Note that this lemma is applicable both for the admissible function \( \mathfrak{h} \) and for \( \mathcal{D}_{s', w}^\pm \mathfrak{h} \), since the latter satisfies the assumption of meromorphic continuation. We recall that condition (1.9) implies \( 1/2 \leq \Re s' \leq \Re w' < 1 \). We conclude that the analytic continuation of \( \mathcal{M}_{q, \ell}^{Eis}(s, w; \mathfrak{h}) \) and \( \mathcal{M}_{q, \ell}^{Eis}(s', w', \mathcal{D}_{s', w}^\pm \mathfrak{h}) \) infers two extra main terms, and we obtain the reciprocity formula (1.10) with
\[
(10.3) \quad \sum_{\ell \mid n_2} \sum_{n_1} \frac{A(n_2, n_1)}{n_2^{s+w} n_1^{2s}} \mathcal{N}^\mathfrak{h} = \sum_{j \in \{0, 1\}} (-1)^j \mathcal{N}_{q, \ell}^{(j)}(s, w; \mathcal{D}_{s', w}^\pm \mathfrak{h}) - R_{q, \ell}(s, w; \mathfrak{h}) + R_{q, \ell}(s', w'; \mathcal{D}_{s', w}^\pm \mathfrak{h}).
\]

A (meromorphic) continuation of these terms follows from (9.4) and Proposition 12 and Lemma 16, and the bounds (9.12), (9.5), (8.2) confirm (1.11), away from polar divisors, in an \( \varepsilon \)-neighbourhood of (1.9). While the individual main terms may have polar lines, their joint contribution must be holomorphic, because the rest of terms in the reciprocity formula are holomorphic. Hence the various (possible) polar divisors must cancel, so that by standard complex analysis (e.g. Cauchy’s integral formula in the \( s \) and \( w \) variable) the estimates (1.11) are valid in the entire region (1.9). This completes the proof.
11. Proof of Theorem 2

Before we start with the proof, we recall that a standard application of the large sieve (cf. e.g. [IK, Theorem 7.35]) shows that

\[ \sum_{f \in {B^*(q)}} L(1/2, f)^4 |h(t_f)| + \sum_{f \in \mathcal{B}_{hol}^*(q)} L(1/2, f)^4 |\overline{h}(k_f)| \ll q^{1+\varepsilon} \]

for any \( q \in \mathbb{N} \) (not necessarily prime), whenever \( \mathfrak{h} = (h, \overline{h}) \) satisfies (1.3). Moreover, by another standard application of the large sieve ([IK, Theorem 7.34]) we have

\[ \sum_{c_x \le q^{1/2}} \int_{-\infty}^{\infty} |L(1/2 + it, \chi)|^8 |h(t)| dt \ll q^{1+\varepsilon}. \]

We will also frequently use the standard bounds

\[ (q(1 + |t_f|))^{-\varepsilon} \ll L(1, \text{Ad}^2 f) \ll (q(1 + |t_f|))^\varepsilon, \quad f \in B^*(q), \]

\[ (q k_f)^{-\varepsilon} \ll L(1, \text{Ad}^2 f) \ll (q k_f)^\varepsilon, \quad f \in B_{hol}^*(q), \]

\[ (q(1 + |t|))^{-\varepsilon} \ll |L(1 + 2it, \chi)|, \quad t \in \mathbb{R}, c_x^2 \mid q. \]

Finally we recall that the central values \( L(1/2, f) \) for \( f \in B^*(q) \cup B_{hol}^*(q) \) are non-negative [KZ, KaS], for arbitrary \( q \in \mathbb{N} \).

Now let \( F = E_0 \) be the minimal parabolic Eisenstein series with trivial spectral parameters. By Theorem 1 with \( \theta = 0 \) we have for \( (q, \ell) = 1, \mathfrak{h} \) admissible that

\[ q \sum_{a = \ell} \mathcal{M}_{q,a}^+(1/2, 1/2, \mathfrak{h}) \left( \frac{a}{b} \right)^{1/2} \ll (q \ell)^\varepsilon \sum_{a = \ell} \left( \frac{a}{b} \right)^{1/2} \left( \frac{1}{a} + \frac{1}{q} + \sum_{\pm} |\mathcal{M}_{q,a}^+(1/2, 1/2; \mathcal{T}_{1/2,1/2}^\pm)\right). \]

For \( \Lambda_f(a; w) \) as in (1.4) we have

\[ \sum_{a = \ell} \Lambda_f(a; 1/2) \left( \frac{a}{b} \right)^{1/2} = \lambda_f(\ell). \]

Now let \( q \) be squarefree. Then we can use (11.2) with \( q = 1 \) together with (1.6) and the last bound in (11.3) to estimate the Eisenstein contribution in (1.7). In this way we see that

\[ q \sum_{a = \ell} \mathcal{M}_{q,a}^+(1/2, 1/2, \mathfrak{h}) \left( \frac{a}{b} \right)^{1/2} = q \sum_{f \in B(q)} \frac{L(1/2, f)^4}{L(1, \text{Ad}^2 f)} \tilde{L}_q(1/2, 1/2, f \times G_0) \lambda_f(\ell) h(t_f) \]

\[ + q \sum_{f \in B_{hol}(q)} \frac{L(1/2, f)^4}{L(1, \text{Ad}^2 f)} \tilde{L}_q(1/2, 1/2, f \times G_0) \lambda_f(\ell) h_{hol}(k_f) + O(\varepsilon). \]

On the other hand, again by (1.7), (1.8), (1.4) with \( q \) in place of \( \ell \) and (1.6) with \( \theta = 0 \) and \( a \) in place of \( q \) we have

\[ \mathcal{M}_{a,q}^+(1/2, 1/2; \mathcal{T}_{1/2,1/2}^\pm) \ll a^{-1} q^{1/2 + \varepsilon} \left( \sum_{q \in B(a)} \frac{|L(1/2, f)|^4}{L(1, \text{Ad}^2 f)} (1 + |t_f|)^{-15} \right. \]

\[ + \sum_{f \in B_{hol}(a)} \frac{|L(1/2, f)|^4}{L(1, \text{Ad}^2 f)} k_f^{15} + \sum_{\chi \mod a, q \ll q^{1/2}} \int_{\mathbb{T}} \frac{|L(1/2 + it, \chi)|^8}{|L(1 + 2it, \chi)|^2} (1 + |t|)^{-15} \frac{dt}{2\pi} \right). \]

By (11.1) – (11.3) and the non-negativity of \( L(1/2, f) \) we conclude

\[ \mathcal{M}_{a,q}^+(1/2, 1/2; \mathcal{T}_{1/2,1/2}^\pm) \ll q^{1/2 + \varepsilon} a^{-1}, \]

so that the right hand side of (11.4) is \( \ll (q \ell)^\varepsilon (q \ell^{-1/2} + q^{1/2 + \theta} \ell^{1/2}) \). We have shown
Proposition 17. For $q$ squarefree, $\varepsilon > 0$, $(\ell, q) = 1$ and $\mathfrak{h}$ admissible, we have

\[
q \sum_{f \in \mathcal{B}(q)} \frac{L(1/2, f)^4}{L(1, \Ad^2 f)} L_q(1/2, 1/2, f \times E_0) \lambda_f(\ell) h(t_f) \ll (q \ell)^{\varepsilon} \left( \frac{q}{\ell^{1/2}} + q^{1/2+\varepsilon} \ell^{1/2} \right).
\]

(11.5)

If $q$ is prime, by Lemma 1 we have

\[
\tilde{L}_q(1/2, 1/2, f \times E_0) = 1/q + O(1/q^2), \quad f \in \mathcal{B}^*(q) \cup \mathcal{B}_{\text{hol}}^*(q),
\]

(11.6)

By (11.7) the contribution of the level 1 forms in (11.5) is $O((q\ell)^{\varepsilon} \ell^\theta)$. For $f \in \mathcal{B}^*(q) \cup \mathcal{B}_{\text{hol}}^*(q)$ we use (11.6) and estimate the contribution of the error term by $O((q\ell)^{\varepsilon} \ell^\theta)$, again using (11.1) – (11.3). This completes the proof of Theorem 2.

12. Proofs of Theorems 3 and 4

12.1. Proof of Theorem 3. Let $f \in \mathcal{B}^*(q)$. By an approximate functional equation\(^9\), cf. [IK, Theorem 5.3], we have

\[
L(1/2, f) = (1 + \omega_f) \sum_{\ell} \lambda_f(\ell) \frac{\ell^{1/2}}{q^{1/2}} W_f \left( \frac{\ell}{q^{1/2}} \right)
\]

(12.1)

where $\omega_f \in \{\pm 1\}$ is the root number and we can choose

\[
W_f(x) = \frac{1}{2\pi i} \int (\varepsilon) \pi^{-s} L_{\infty}(1/2 + s, f) \frac{G_f(s)}{G_f(0)} e^{-t_f s} ds, \quad \varepsilon > 0,
\]

with $L_{\infty}(s, f) = \pi^{-s} \prod_{j=0}^{1000} \prod_{\varepsilon_1, \varepsilon_2 \in \{\pm 1\}} \left( \frac{1}{2} + \varepsilon_1 s + i\varepsilon_2 t_f + j \right)$.

The choice for the particular weight function $G_f$ will become clear in a moment. Since $L(1/2, f) \geq 0$ with equality if $\omega_f = -1$, we can get rid of the root number in (12.1) and obtain by (11.3) that

\[
\sum_{f \in \mathcal{B}^*(q)} L(1/2, f)^5 e^{-t_f^2} \ll \sum_{f \in \mathcal{B}^*(q)} L(1/2, f)^4 e^{-t_f^2} \left( \frac{q(1 + |t_f|)}{L(1, \Ad^2 f)} \right) \cdot 2 \frac{\lambda_f(\ell)}{x^{1/2}} W_f \left( \frac{\ell}{q^{1/2}} \right)
\]

= \frac{2^5}{2\pi i} \int (1) \pi^{-s} q^{s/2} \sum_{\ell} \ell^{-1/2-s} \sum_{f \in \mathcal{B}^*(q)} L(1/2, f)^4 \frac{L_{\infty}(1/2 + s, f)}{L_{\infty}(1/2, f)} \lambda_f(\ell) \frac{G_f(s)}{G_f(0)} \epsilon^{-t_f^2} (1 + |t_f|)^\varepsilon ds.
\]

Shifting the contour to the far right, we can truncate the $\ell$-sum at $q^{1/2+\varepsilon}$ at the cost of an error $O(q^{-10})$. Note that now automatically $(q, \ell) = 1$. Having done this, we shift the contour back to $\Re s = \varepsilon$, and by the exponential decay of

\[
\frac{L_{\infty}(1/2 + s, f)}{G_f(1/2, f)} e^{-t_f^2}
\]

along vertical lines (uniformly in $t_f$) we can truncate the contour at $|s| \leq (\log q)^2$ with the same error. Hence

\[
\sum_{f \in \mathcal{B}^*(q)} L(1/2, f)^5 e^{-t_f^2} \ll q^\varepsilon \max_{|\ell| \leq (\log q)^2} \left| \sum_{\ell \in q^{1/2+\varepsilon}} \frac{1}{\ell^{1/2}} \left| \sum_{f \in \mathcal{B}^*(q)} \frac{L(1/2, f)^4}{L(1, \Ad^2 f)} \lambda_f(\ell) h_\varepsilon(t_f) \right| + q^{-10}
\]

\(^9\)This is the only point where an approximate functional equation is used explicitly although it is implicit in (11.1).
where
\[ h_\tau(t_f) = \frac{L_{\infty}(1/2 + \varepsilon + i\tau, f) \cdot G_f(\varepsilon + i\tau)}{L_{\infty}(1/2, f) \cdot G_f(0)} e^{-\varepsilon t_f (1 + |t_f|)^{\varepsilon}}. \]

By construction, the family \( \{ h_\tau = (h_\tau, 0) : |\tau| \leq (\log q)^2 \} \) is uniformly admissible. Theorem 2 now yields the desired bound.

12.2. Proof of Theorem 4. Let \( f_0 \in \mathcal{B}^t(q) \cup \mathcal{B}_{\text{hol}}(q) \). Let \( q^{1/100} \leq L < q \) be a parameter to be determined later in terms of \( q \). In the following all implied constants may depend on the archimedean parameter of \( f_0 \), and \( p \) denotes a prime number. For \( f \in \mathcal{B}(q) \cup \mathcal{B}_{\text{hol}}(q) \) we choose the following amplifier

\[ A_f := \left| \sum_{p \leq L, p \not| q} \lambda_f(p)x(p) \right|^2 + \left| \sum_{p \leq L, p \not| q} \lambda_f(p^2)x(p^2) \right|^2, \quad x(n) = \text{sgn}(\lambda_f(n)). \]

Then
\[ A_{f_0} = \left( \sum_{p \leq L, p \not| q} |\lambda_{f_0}(p)| \right)^2 + \left( \sum_{p \leq L, p \not| q} |\lambda_{f_0}(p^2)| \right)^2 \geq \frac{1}{2} \left( \sum_{p \leq L, p \not| q} |\lambda_{f_0}(p)| + |\lambda_{f_0}(p^2)| \right)^2 \gg \frac{L^2}{\log L} \]

by the prime number theorem and the Hecke relation \( \lambda_{f_0}(p)^2 = 1 + \lambda_{f_0}(p) \). On the other hand,
\[ A_f = \sum_{p \leq L, p \not| q} (x(p)^2 + x(p^2)^2) + \sum_{p_1, p_2 \leq L, p_1 \not| p_2} \lambda_{f_1}(p_1) x(p_1^2) x(p_2^2) + \sum_{p_1, p_2 \leq L, p_1 \not| p_2} x(p_1)^2 x(p_2)^2 \lambda_f(p_1 p_2). \]

Now suppose that \( q \) is squarefree and \( (6, q) = 1 \). Then by (9.8) with \( A(1, p) = A(p, 1) = \tau_3(p) = 3 \) and \( |\lambda_f(p)| \leq p^{\varepsilon} + p^{-\varepsilon} \) we see that
\[ L_q(1/2, 1/2, f \times E_0) > 0, \quad f \in \mathcal{B}(q) \cup \mathcal{B}_{\text{hol}}(q). \]

We employ now the admissible function \( h_{\text{pos}} = (h_{\text{pos}}, h_{\text{pos}}^{\text{hol}}) \) with \( a = 1000, b = 400 \), the non-negativity of central values and (11.3) to conclude
\[ A_{f_0} L(1/2, f_0)^4 \ll q^\varepsilon \sum_{f \in \mathcal{B}^t(q)} \frac{L(1/2, f)^4}{L(1, Ad^2 f)} L_q(1/2, 1/2, f \times E_0) A_f h_{\text{pos}}(t_f) \]
\[ + q^\varepsilon \sum_{f \in \mathcal{B}_{\text{hol}}(q)} \frac{L(1/2, f)^4}{L(1, Ad^2 f)} L_q(1/2, 1/2, f \times E_0) A_f h_{\text{pos}}^{\text{hol}}(k_f). \]

We insert the lower bound (12.2) on the left hand side and the exact formula for \( A_f \) on the right hand side. By Proposition 17 with \( \ell = 1, p_1 p_2, p_1^2 p_2^2 \), we obtain
\[ L^2 L(1/2, f)^4 \ll q^\varepsilon (q L + q^{1/2 + \varepsilon} L^4). \]

Choosing \( L = q^{(1-2\varepsilon)/6} \), we complete the proof.

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