Modifications of the Hubble Law
in a Scale-Dependent Cosmology

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Abstract

We study some observational consequences of a recently proposed scale-dependent cosmological model for an inhomogeneous Universe. In this model the Universe is pictured as being inside a highly dense and rapidly expanding shell with the underdense center. For nearby objects (\( z \ll 1 \)), the linear Hubble diagram is shown to remain valid even in this model, which has been demonstrated both analytically and numerically. For large \( z \), we present some numerical results of the redshift–luminosity distance relation and the behavior of the mass density as a function of the redshift. It is shown that the Hubble diagram in this model for a locally open Universe (\( \Omega(t_0, r \sim 0) = 0.1 \)) resembles that of the flat Friedmann cosmology. This implies that study of the Hubble diagram cannot uniquely determine the value of \( q_0 \) or \( \Omega_0 \) in a model-independent way. The model also accounts for the fact that \( \Omega_0 \) is an increasing function of the redshift.

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I. INTRODUCTION

One of the basic assumptions of the standard Friedmann cosmology is the Cosmological Principle which states that the Universe is homogeneous and isotropic \[1\] so that every point of the Universe is equivalent. While isotropy has been reasonably well established from, among others, the observation of the cosmic microwave background radiation by COBE \[2\], homogeneity has been challenged by various observations of the Large Scale Structures (LSS). Recent galaxy redshift surveys such as CfA 1 \[3\], CfA 2 \[4\], SSRS1 \[5\], SSRS2 \[6\] and the pencil beam surveys \[6\] have provided evidence for the LSS such as filaments, sheets, superclusters and voids, up to \(200 - 300h^{-1}\)Mpc. The current interpretation based on the observation of the LSS is that the Cosmological Principle may not be applicable, at least, to the local Universe, although it may be applicable to the Universe on a very large scale with a characteristic distance \(\lambda\). However, one of the most remarkable consequences of the above galaxy surveys is that the scale of the largest structures in each survey is comparable with the extent of the survey itself. Recently, it has been suggested that from pencil beam surveys \[6\] as well as from new deep redshift survey (ESP survey) \[7\], \(\lambda\) should be much larger than the survey limits \(\sim 500 - 600h^{-1}\)Mpc, implying the absence of any tendency towards homogeneity up to the present observational limits. For example, Pietronero \[8\] has attempted to explain these phenomena by suggesting that the LSS shows fractal properties of the Universe. Even though this non–analytical distribution of matter means that the Universe is not homogeneous, the local isotropy preserves the fundamental assumption that every point of the Universe is equivalent.

Unfortunately, however, a rigorous mathematical description of such a Universe is extremely difficult and in practice it is almost impossible. Therefore, it is desirable to simplify the description of this *inhomogeneous* Universe to the extent that its analytical study becomes possible in order to see, at least, qualitative features of the matter distribution and cosmological consequences. History of cosmological models for an inhomogeneous Universe dates back to as early as 1930’s. Lemaitre, Tolman and Dingle \[9\] attempted to describe the
evolution of the fluctuation in the mass distribution. Later in 1947, Bondi [10] elaborated the model and discussed observational consequences. In their model, which we shall call Tolman-Bondi (TB) model, the global Universe is that of the standard Friedmann cosmology, implying homogeneity over the region of order \( \lambda \). Recently, in the framework of the TB model, Moffat and Tatarski [11] studied, in order to describe the local inhomogeneity, cosmology of a local void in the globally Friedmann Universe and its effect on the measurement of the Hubble constant and the redshift–luminosity distance relation. Since the Universe modeled in [11] consists of many expanding voids and we happen to be located at the center of one of them, the shell–crossing singularity occurs, implying that different shells collide and the comoving coordinate become meaningless.

Recently, based on the fact that there is no observational evidence of approaching towards the homogeneity within the survey limit, another cosmological model [12] was proposed, whose global feature is not asymptotic to the Friedmann cosmology. In [12], the observable Universe is modeled as being inside an expanding bubble with the underdense center and matter inside the bubble is isotropically (but inhomogeneously) distributed when viewed by an observer located at the center. They proposed that such a Universe may be described by the following inhomogeneous metric

\[
\text{d}r^2 = \text{d}t^2 - R^2(t, r)[\text{d}r^2 + r^2 \text{d}\Omega^2],
\]

where \( R(t, r) \) is the scale-factor, dependent on \( r \) as well as on \( t \). Therefore, homogeneity in the Cosmological Principle is explicitly violated, whereas isotropy remains intact. Based on the high degree of isotropy of the cosmic microwave background radiation measured by COBE [4], it was assumed that the observer is located at the center of the bubble (or near it), albeit the return of the pre–Copernican notion. Whether this picture is correct or not can only be decided when the results of the model are confronted with the observation. In [12], its cosmological consequences were qualitatively discussed. For example, the Hubble constant, the density parameter and the age of the Universe all became scale–dependent, whereas the analysis in [11] was simplified to avoid the possible position–dependent age.
Moreover, because of the lack of the light propagation solution on which every observation is based, no explicit and testable cosmological results were derived, which can be compared with the observation. Therefore, it is interesting to examine observational consequences of such a model and compare with those of the TB model.

In this paper, we first present a general redshift–luminosity distance relation for a certain class of inhomogeneous cosmological models. Then we apply the result to the model discussed in [12] as an example. The plan of this paper is as follows. In Section II, we present the redshift–luminosity distance relation for the case of one \((t, r)\)–dependent scale factor in the metric. In order to proceed further to derive some specific observable consequences, we have chosen the model proposed in [12] as an example and briefly summarize the model in Section III. In Section IV, we derive modified results of the redshift–luminosity distance relation and show that they reduce to the well-known relations in the standard Friedmann cosmology for small \(z\), i.e., for nearby objects. Section V deals with some numerical results of this model which are applicable for large \(z\). Also discussed in this Section is the observed increase of density parameter with the redshift in the framework of this model. A brief summary and conclusions are given in Section VI.

II. REDSHIFT IN A GENERAL METRIC WITH ONE SCALE FACTOR

With the exception of astronomical neutrinos and possible future gravitational waves, most of the cosmological measurements are based on the electromagnetic waves, which travel along the null geodesic, i.e., \(d\tau = 0\). Considering only the radial propagation \((d\Omega = 0)\), we have, from Eq.(1),

\[

\frac{dr}{dt} = -\frac{1}{R(t, r)},
\]

where the minus sign is chosen since \(r\) decreases as \(t\) increases. With a given \(R(t, r)\), Eq.(2) appears as a simple first-order differential equation, yielding \(r\) as a function of \(t\). However, complexity of solving this differential equation becomes immediately apparent because we
will be dealing with the case in which $R(t,r)$ is not factorized into a separable form of $a(t)f(r)$. The boundary condition to be imposed is as follows. Since we measure a signal at $r = 0$, the boundary condition is $r(t = t_{\text{received}}) = 0$. More specifically, we treat the solution of Eq.(2) as $r = r(t,t_0)$ which is a function of $t$ with the boundary condition $r(t = t_0, t_0) = 0$.

In order to define the redshift, we consider two successive wave crests, both of which leave $r$ and reach us ($r = 0$) at different times. Suppose that two wave crests were emitted at time $t$ and $t + \Delta t$ and received by us at time $t_0$ and $t_0 + \Delta t_0$, respectively. Then, from Eq.(2) and the definition of $r(t,t_0)$, we have

$$r = \int_t^{t_0} \frac{dt'}{R(t',r(t',t_0))} = \int_{t + \Delta t}^{t_0 + \Delta t_0} \frac{dt'}{R(t',r(t',t_0 + \Delta t_0))} .$$

It is to be noted that for each wave crest a proper boundary condition has to be applied, which is explicitly expressed in the form of $r(t, t_{\text{received}})$. Since $\Delta t$ and $\Delta t_0$ are extremely small compared with the cosmological time scale, it is sufficient to consider only up to the first order in $\Delta t$ or $\Delta t_0$. Then, we have, from Eq.(3),

$$\frac{\Delta t}{R(t,r(t,t_0))} = \frac{\Delta t_0}{R(t_0, r(t_0,t_0))} \left[ 1 + R(t_0, 0) \int_t^{t_0} \frac{dt'}{R(t', r(t', t_0))} \frac{1}{\frac{\partial}{\partial t_0} \left( \frac{1}{R(t', r(t', t_0))} \right)} \right] ,$$

yielding the defining relation of the redshift, $z$, as

$$1 + z \equiv \frac{\Delta t_0}{\Delta t} = \frac{R(t_0, 0)}{R(t,r(t,t_0))} \left[ 1 + R(t_0, 0) \int_t^{t_0} \frac{dt'}{R(t', r(t', t_0))} \frac{1}{\frac{\partial}{\partial t_0} \left( \frac{1}{R(t', r(t', t_0))} \right)} \right]^{-1} ,$$

where the boundary condition, $r(t_0, t_0) = 0$ has been used. Here and hereafter $\frac{\partial}{\partial t_0}$ explicitly means $\frac{\partial t}{\partial t_0}$. In the case of the Robertson–Walker metric where $R(t,r) \equiv S(t)$, the scale factor of the standard Friedmann cosmology, it is easy to see that Eq.(5) reduces to the well-known relation, $(1 + z) = S(t_0)/S(t)$.

In the real observation, the most important definition of distance is the luminosity distance. As is well known, if a source at comoving distance $r$ emits light at time $t$ and a detector at $r = 0$ receives the light at time $t_0$, the luminosity distance, $d_L$, of a source in the standard Friedmann cosmology is
\[ d_L = rS(t_0)(1 + z) = rS(t)(1 + z)^2, \]  

where the second equality is due to the relation \( S(t_0) = S(t)(1 + z) \) in the standard Friedmann cosmology.

The luminosity distance in a spherically symmetric but inhomogeneous Universe was first examined by Bondi \[10\] in 1947. In order to avoid the non-zero pressure, however, two different scale factors were introduced in the metric, as was originally done by Lemaitre, Tolman and Dingle \[9\],

\[ \text{d} \tau^2 = \text{d} t^2 - X^2(t, r) \text{d} r^2 - Y^2(t, r) \text{d} \Omega^2. \]  

(7)

One of the special features of the TB model is that the pressure is always zero, as originally designed so as to be applicable in the matter dominated era only. The situation considered in \[10\] is such that a standard source is at the center and an observer at \((t, r, \theta, \varphi)\). As was shown in \[10\], the ratio of the absolute luminosity to the apparent luminosity is simply given by \( Y^2(t, r)(1 + z)^2 \). Two comments are in order here. First, we note that the light source is at the center, implying that the light propagates out spherically with constant surface energy density at any given time, which is, in general, not the case in an inhomogeneous Universe. Secondly, \((t, r, \theta, \varphi)\) is a coordinate of the observer, not of the source. That is, in the standard cosmology notation, \( Y(t, r) \) physically corresponds to \( rS(t_0) \).

Since we consider the situation in which the position of the observer is located at the center, the light from its source off the center does not even propagate outward in a spherically symmetrical manner. Following the picture of the Universe in \[12\] where the Universe is inside a bubble with the underdense center (where the observer is located) and with the highly dense shell, the light would feel attraction toward the shell, implying that the path of the light propagation is, in general, not a straight line. Moreover, its energy is not uniformly distributed over the non-spherical shell at any given time. Nevertheless, since the position of the observer is fixed at the center (i.e., \( r = 0 \)) and he/she receives the light that propagates on a straight line, we shall use the following definition of luminosity distance

\[ d_L \equiv r(t, t_0)R(t_0, r(t = t_0, t_0))(1 + z), \]  

(8)
where the \((1 + z)\) factor comes from the correction factor, \((1 + z)^2\), that appears in the relationship between the absolute luminosity and the apparent luminosity (hence only one factor out of \((1 + z)^2\) in the distance). One factor of \((1 + z)\) in the luminosity relation is due to the decrease of energy because of the redshift, the other factor coming from the increase of the time interval from \(\Delta t\) to \(\Delta t_0\), which is also just \((1 + z)\) by definition. Of course, the above definition has to be justified by performing the coordinate transformation from the observer to the source in the inhomogeneous Universe. This, however, is beyond the scope of this paper and thus, based on its plausibility, we shall assume its validity in this paper. We caution the reader that the luminosity distance should not be simply written as \(r(t, t_0)R(t, r(t, t_0))(1 + z)^2\), for \(R(t, r(t, t_0))\) is not simply given by \(R(t_0, 0)\) times a factor, \((1 + z)\), as can be seen in Eq.(5).

III. SCALE–DEPENDENT COSMOLOGY

In order to obtain some specific results of cosmological consequences of the proposed inhomogeneous metric, Eq.(1), we shall consider, as an example, the model of [12]. In this Section, we shall briefly summarize the model. First, given the metric in Eq.(1), in order to accommodate the \(r\)-dependence on the Ricci tensors, the Einstein equation was also generalized as

\[
R^\mu{}\nu - \frac{1}{2} g^\mu{}\nu R = -8\pi [GT]^\mu{}\nu (t, r),
\]  

where the \((t, r)\) dependence of the combination, \([GT]^\mu{}\nu\], was explicitly noted. When the non-vanishing elements of Ricci tensor calculated from Eq.(1) are substituted into the generalized Einstein equation in Eq.(9), we obtain the following non-vanishing components.

\[
\frac{3 \dddot{R}(t, r)}{R(t, r)} - 2 \frac{\ddot{R}(t, r)}{R(t, r)} + \frac{\dot{R}(t, r)^2}{R(t, r)^3} - 4 \frac{\dot{R}(t, r)}{R(t, r) R^3(t, r)} = 8\pi G \rho
\]

\[
2 \frac{\dddot{R}(t, r)}{R(t, r)} + \frac{\ddot{R}(t, r)}{R(t, r)} - 2 \frac{\dot{R}(t, r)^2}{R(t, r)^3} - \frac{2 \dot{R}(t, r)}{R(t, r) R^3(t, r)} = -8\pi G \rho_r
\]

\[
2 \frac{\dddot{R}(t, r)}{R(t, r)} + \frac{\ddot{R}(t, r)}{R(t, r)} - 2 \frac{\dot{R}(t, r)^2}{R(t, r)^3} - \frac{2 \dot{R}(t, r)}{R(t, r) R^3(t, r)} = -8\pi G \rho_0
\]
where dots and primes denote, respectively, derivatives with respect to $t$ and $r$. Another non-vanishing Ricci tensor $R_{01}$ yields

$$R_{01} = 2 \left( \frac{\dot{R}(t,r)}{R(t,r)} - \frac{\dot{R}(t,r)R'(t,r)}{R^2(t,r)} \right) = -8\pi [GT_{01}] . \tag{14}$$

As was discussed in [12], in order to maintain an inhomogeneous matter distribution, it is essential to keep pressures and $T_{01}$ to be finite so that $R(t,r)$ is kept from being factored out as $a(t)f(r)$, in which case the Robertson–Walker metric is recovered. This feature distinguishes this model from the TB model, in which pressure was set to be zero to begin with. Moreover, to avoid the shear force, it was assumed in [12] that $p_r = p_\theta (= p_\phi)$. Then a constraint on $R(t,r)$ is uniquely determined from Eqs.(11) and (12) as

$$R(t,r) = \frac{a(t)}{1 - B(t)r^2} , \tag{15}$$

where $a(t)$ and $B(t)$ are positive, arbitrary functions of $t$ alone. The negative sign on the right hand side of Eq.(15) is chosen to avoid a locally closed Universe (see below Eq.(16)). Inserting Eq.(15) into Eq.(10) gives

$$\left[ \frac{\dot{R}(t,r)}{R(t,r)} \right]^2 = \frac{8\pi}{3} [G\rho](t,r) + \frac{4B(t)}{a^2(t)} . \tag{16}$$

The term, $4B(t)/a^2(t)$, was interpreted as a time-varying vacuum energy density in [12]. It should be noted here that the appearance of this term, admittedly very surprising, is a consequence of the metric in Eq.(1). Let us briefly discuss physical implications of Eq.(16). In our neighborhood (i.e., $r \ll 1$), the left-hand side of Eq.(16) is reduced to $\dot{(a/a)}^2$, implying that $a(t)$ represents more or less the scale factor for our local Universe. Moreover, since the standard Friedmann cosmology has been successful in describing our local neighborhood, any modifications of the standard cosmology in this model must be small in the local Universe.

The next question is whether our local neighborhood is flat or open. If it is flat, $B(t)$ should be treated, as can be seen on the right–hand side of Eq.(16), as being small in
accord with the assumption of small modifications on the local neighborhood. If the local neighborhood is open, however, $B(t)$ itself is not small. Upon writing $B(t)$ as $[1 + b(t)]/4$, Eq.(16) becomes

$$\left[ \frac{\dot{R}(t, r)}{R(t, r)} \right]^2 = \frac{8\pi}{3} [G\rho](t, r) + \frac{1}{a^2(t)} + \frac{b(t)}{a^2(t)} ,$$

implying that in the local neighborhood (i.e., for $r \ll 1$, or equivalently for $(\dot{R}/R)^2 \simeq (\dot{a}/a)^2$), small is $b(t)$, but not $B(t)$.

It is interesting to mention here that the metric given by Eq.(1) with the Einstein equation dictates the behavior of the energy density, pressure and momentum density of the Universe. We first note that the constraint on $R(t, r)$, Eq.(15), severely restricts the behavior of $G\rho$, $Gp$ and $GT_{01}$ as functions of $t$ and $r$. Substituting Eq.(15) into Eqs.(10), (11) and (14), we obtain the following explicit expressions:

$$8\pi G\rho = \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{a^2} - \frac{b}{a^2} + 2 \left( \frac{\dot{a}}{a} \right) \left( \frac{\dot{b}r^2/4}{1 - 1 + b/4} \right) + \left( \frac{\dot{b}r^2/4}{1 - 1 + b/4} \right)^2 \tag{18}$$

$$8\pi Gp = \frac{1}{a^2} - 2 \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 + \frac{b}{a^2} - \left( 6 \frac{\dot{a}}{a} + 2 \frac{\ddot{b}}{b} \right) \left( \frac{\dot{b}r^2/4}{1 - 1 + b/4} \right) - 5 \left( \frac{\dot{b}r^2/4}{1 - 1 + b/4} \right)^2 \tag{19}$$

$$8\pi GT_{01} = \frac{\dot{b}r}{a^2} . \tag{20}$$

The following comments are in order here. From Eqs.(18) and (19), it is easy to see that as $r$ approaches $\sqrt{\frac{4}{1+b}}$, we have $p/\rho = -\frac{5}{3}$, implying a bizarre equation of state. In the standard inflationary scenario in which a constant vacuum energy is responsible for generating an inflationary period, one has $p/\rho = -1$. Therefore, in the model under consideration, an unusual scalar sector with time-dependent vacuum energy, which is as yet to be understood, may be responsible for the inflation which is much more rapid than the usual inflation. By using the arbitrariness of $b(t)$, however, this singularity with the unusual $p/\rho$ ratio can be pushed far away from the particle horizon, so that the local value of $p/\rho$ at the present matter dominated era can be made to be positively small, as generally expected. For this reason, in spite of this bizarre behavior, we shall proceed to use this model as an example for our following discussions.
IV. PERTURBATIVE APPROACH

The linear relationship between the redshift and luminosity distance with a constant coefficient (the Hubble constant) for nearby objects (for \(z \ll 1\)) has been well established by various observations. In this Section, we investigate whether or not the redshift–luminosity distance relation for small \(z\) in an inhomogeneous cosmological model discussed in the previous Section still remains the same as in the standard cosmology.

A. Locally flat Universe

We start with the light propagation equation

\[
\frac{dr}{dt} = -\frac{1}{a(t)} + \frac{B(t)}{a(t)} r^2, \tag{21}
\]

which is determined by two arbitrary functions of \(t, a(t)\) and \(B(t)\). But unfortunately Eq.(21) cannot be solved analytically because it is non-linear in \(r\). For a locally flat Universe with small perturbations to the standard cosmology, \(B(t)\) should be treated as being very small. Therefore, we have, from Eq.(21),

\[
r(t, t_0) = \int_t^{t_0} \frac{dt'}{a(t')} - \int_t^{t_0} \frac{dt'}{a(t')} \left[ \int_{t'}^{t_0} \frac{dt''}{a(t'')} \right]^2 + \mathcal{O}(B^2), \tag{22}
\]

yielding

\[
\frac{\partial}{\partial t_0} \left[ \frac{1}{R(t, r(t, r_0))} \right] = -\frac{2B(t)}{a(t)a(t_0)} \int_t^{t_0} \frac{dt'}{a(t')} + \mathcal{O}(B^2), \tag{23}
\]

where we have used Eq.(15). Now, the redshift is given by

\[
1 + z \equiv \frac{\Delta t_0}{\Delta t} \tag{24}
\]

\[
= \frac{a(t_0)}{a(t)} \left[ 1 - B(t) \left( \int_t^{t_0} \frac{dt'}{a(t')} \right)^2 + 2 \int_t^{t_0} \frac{dt'}{a(t')} \int_{t'}^{t_0} \frac{dt''}{a(t'')} + \mathcal{O}(B^2) \right].
\]

In a special case where \(B(t)\) is a constant, which corresponds to the standard Friedmann cosmology, the redshift simply reduces to the standard relation.
\[ 1 + z = \frac{a(t_0)}{a(t)} , \] 

where we have used the relation

\[ 2 \int_t^{t_0} \frac{dt'}{a(t')} \int_t^{t_0} \frac{dt''}{a(t'')} = \left[ \int_t^{t_0} \frac{dt'}{a(t')} \right]^2 . \]

That is, the well-known redshift expression in the standard cosmology is reproduced, as expected. Therefore, this result strongly suggests that \( a(t) \) plays a role of, more or less, the scale factor of the standard Friedmann cosmology. In the case of a locally flat Universe, therefore, the behavior of \( a(t) \) cannot be much different from that of the standard Friedmann cosmology with \( k = 0 \), which is proportional to \( t^{\frac{2}{3}} \) in the matter dominated era. For mathematical simplicity and illustrative purposes, we assume that \( a(t) \) behaves the same as that in the standard cosmology and \( B(t) \) can be expressed by a simple power law. That is, we assume that \( a(t) \) and \( B(t) \) are, in the matter dominated era, of the form

\[ a(t) = \alpha t_0 \left( \frac{t}{t_0} \right)^{\frac{2}{3}} , \quad B(t) = \beta \left( \frac{t}{t_0} \right)^n \quad (n \geq 0) , \] 

where \( \alpha \) and \( \beta \) are dimensionless parameters to be determined and \( n \) is set to be non-negative because of the observed increase of the matter density as a function of \( r \). [12] Substituting Eq.(27) into Eq.(22) yields

\[ r(t, t_0) = \frac{3}{\alpha} \left[ 1 - \left( \frac{t}{t_0} \right)^{\frac{1}{3}} \right] - \frac{9}{\alpha^2} \left[ \frac{2}{9(n + \frac{1}{3})(n + \frac{2}{3})(n + 1)} \right] \]

\[ - \frac{1}{(n + \frac{1}{3}) (n + \frac{1}{3})} \left( \frac{t}{t_0} \right)^{n+\frac{1}{3}} + \frac{2}{(n + \frac{1}{3}) (n + \frac{1}{3})} \left( \frac{t}{t_0} \right)^{n+\frac{1}{3}} - \frac{1}{(n + 1) (n + \frac{1}{3}) (n + \frac{1}{3})} \left( \frac{t}{t_0} \right)^{n+1} \] 

+ \mathcal{O}(\beta^2) .

In practice, what is measured is the redshift. Therefore, we must express \( (t/t_0) \) in terms of the red shift. By substituting Eqs.(27) into Eq.(24), we have

\[ 1 + z = \left( \frac{t}{t_0} \right)^{-\frac{2}{3}} \left[ 1 - \frac{9}{\alpha^2} \left( \frac{t}{t_0} \right)^{n} \left( 1 - \left( \frac{t}{t_0} \right)^{\frac{1}{3}} \right)^2 \right] \]

\[ + \beta \frac{2}{\alpha^2 (n + \frac{1}{3})(n + \frac{2}{3})} \left[ 1 + 3(n + \frac{1}{3}) \left( \frac{t}{t_0} \right)^{n+\frac{1}{3}} - 3(n + \frac{2}{3}) \left( \frac{t}{t_0} \right)^{n+\frac{1}{3}} \right] + \mathcal{O}(\beta^2) , \]

yielding
\[
\left( \frac{t}{t_0} \right) \equiv T(0) + \beta T(1) + O(\beta^2) ,
\]
where, for notational simplicity, we define \( T(0)(z) \) and \( T(1)(z) \) as

\[
T(0) = (1 + z)^{-\frac{2}{3}} \quad (31)
\]

\[
T(1) = \frac{3}{2\alpha^2} \left[ \frac{2}{(n + \frac{1}{3})(n + \frac{2}{3})}(1 + z)^{-\frac{3}{2}} + \frac{18n}{(n + \frac{1}{3})}(1 + z)^{-\frac{3}{2}(n + \frac{4}{3})} 
- \frac{9n}{(n + \frac{2}{3})}(1 + z)^{-\frac{3}{2}(n + \frac{5}{3})} - 9(1 + z)^{-\frac{3}{2}(n + 1)} \right] .
\]

Inserting Eqs.(30) and (31) into the right-hand side of Eq.(28) gives

\[
r(t, t_0) = r(0) + \beta r(1) + O(\beta^2) ,
\]

where \( r(0) \) and \( r(1) \) are defined as

\[
r(0) \equiv \frac{3}{\alpha}[1 - T(0)^{\frac{4}{3}}] \quad (33)
\]

\[
r(1) \equiv -\frac{1}{\alpha^3} \left[ T(0)^{-\frac{4}{3}} T(1) \alpha^2 + \frac{2}{(n + \frac{1}{3})(n + \frac{2}{3})(n + 1)} 
- \frac{9}{(n + \frac{2}{3})} T(0)^{n + \frac{4}{3}} + \frac{18}{(n + \frac{2}{3})} T(0)^{n + \frac{5}{3}} - \frac{9}{(n + 1)} T(0)^{n + 1} \right] .
\]

It is easy to see that the \( r(0) \) term reproduces the result in the standard Friedmann cosmology whereas \( r(1) \) represents a correction term. It is interesting to note that for small \( z \), \( r(1) \) is zero up to the second order in \( z \), as can easily be seen by substituting Eq.(31) into Eq.(33). That is, there is no modification of small \( r(t) \) (i.e., for \( z \ll 1 \)) due to small \( B \), up to the second order. Since the luminosity distance is \( r(t, t_0)R(t_0, 0)(1 + z) \), the redshift–luminosity distance relation for small \( z \) remains intact, at least, up to the first order in \( B(t) \). This is a consequence of the plausible assumption that \( a(t) \) has very similar behavior of the scale factor of the standard Friedmann cosmology in the matter dominated era.

### B. Locally Open Universe.

Before we proceed, we make a brief comment on the status of the density parameter \( \Omega \).

One of the most challenging tasks in the observational astrophysics is the measurement of
the mass density of the Universe, which is supposed to be a constant at any given time in the standard Friedmann cosmology. Various observations, however, indicate that the mass density indeed appears to increase as we probe farther out \[13\], \[14\]. From direct observations, the fraction of critical density associated with luminous galaxies is $\Omega_{LUM} \leq 0.01$. When extending the observation to distances beyond the luminous part of galaxies, we found that there exist galactic halos which have a mass corresponding to $\Omega_{HALO} \simeq 0.1$. On a larger scale such as the Virgo cluster, modeling the local distortion of the Hubble flow around the cluster yields $\Omega_{CLUSTER} = 0.1$ to 0.2. Recently, using the redshift measurements for the catalogue of galaxies by the Infrared Astronomy Satellite (IRAS) \[15\], it became apparent that galaxies out to about 100 Mpc flow towards the Great Attractor with high peculiar velocity. It was concluded that the observed dynamics on this scale requires $\Omega_{IRAS} \sim 1 \pm 0.6$. Based on the above observations, we have a picture of the Universe in which our local neighborhood is underdense and the mass density increases with scale.

Therefore, we present, in this Subsection, the redshift–luminosity distance relation for the locally open Universe. As was discussed before, $a(t)$ is more or less the scale-factor of the locally open Universe in the standard Friedmann cosmology. Therefore, it is more transparent to rewrite $B(t)$ as $[1 + b(t)]/4$. Hereafter, a small perturbation to the locally open Universe is represented by $b(t)$ rather than by $B(t)$. It is convenient to introduce a new coordinate $\Phi$ as defined by $r(t, t_0) = 2 \tanh \Phi(t, t_0)$. The light propagation equation is then reduced to

$$\frac{d\Phi}{dt} = -\frac{1}{2a(t)} + b(t) \frac{1}{2a(t)} \sinh^2 \Phi.$$  \hspace{1cm} (34)

We consider two successive wave crests that leave a comoving coordinate $\Phi$ at time $t$ and $t + \Delta t$ and arrive at $\Phi = 0$ at times $t_0$ and $t_0 + \Delta t_0$, respectively, which yields the following equality:

$$\Phi = \int_t^{t_0} dt' \left[ \frac{1}{2a(t')} - \frac{b(t')}{2a(t')} \sinh^2 \Phi(t', t_0) \right]$$

$$= \int_{t + \Delta t}^{t + \Delta t_0} dt' \left[ \frac{1}{2a(t')} - \frac{b(t')}{2a(t')} \sinh^2 \Phi(t', t_0 + \Delta t_0) \right],$$  \hspace{1cm} (35)
from which the redshift relation is given by

\[
1 + z \equiv \Delta t_0 = \frac{a(t_0)}{a(t)} \left[ \frac{1 - b(t) \sinh^2 \Phi(t, t_0)}{1 - a(t_0) \int_t^{t_0} dt' \frac{b(t')}{a(t')} \frac{\partial \Phi(t', t_0)}{a(t_0)} \sinh 2\Phi(t', t_0)} \right].
\] (36)

For an open Friedmann Universe where \( b(t) = 0 \), \((1 + z) \) simply becomes \( a(t_0)/a(t) \), which is the standard result. Since the light propagation equation is non-linear, we will again use the perturbation method by treating \( b(t) \) as being small. Then, the redshift in this picture becomes

\[
1 + z \simeq \frac{a(t_0)}{a(t)} \left[ 1 - b(t) \sinh^2 \int_t^{t_0} \frac{dt'}{2a(t')} + \int_t^{t_0} \frac{dt'}{2a(t')} b(t') \sinh \int_t^{t_0} \frac{dt''}{a(t'')} + O(b^2) \right],
\] (37)

where we have used the relation, \( \partial \Phi(t, t_0)/\partial t_0 = 1/2a(t_0) \). Again to proceed further we need specific functional forms of \( a(t) \) and \( b(t) \). Since \( a(t) \) cannot be too different from the scale factor in the Friedmann cosmology with \( k = -1 \), we assume that, in the matter-dominated era, \( a(t) \) satisfies the following differential equation as in the standard Friedmann cosmology with \( k = -1 \):

\[
2 \frac{\ddot{a}(t)}{a(t)} + \left( \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{1}{a^2(t)} \simeq 0.
\] (38)

Here, only one of the two initial conditions can be fixed as \( a(t = 0) = 0 \). The solution of Eq.(38) may be parameterized by an angle, \( \Psi \), as

\[
a(\Psi) = \alpha t_0 \cosh \Psi - 1
\] (39)

\[
t(\Psi) = \alpha t_0 \sinh \Psi - \Psi,
\]

where \( t_0 \) is the age of the Universe and \( \alpha \) is a dimensionless parameter to be determined.

In the following, \( \Psi_0 \) is defined as \( t(\Psi_0) = t_0 \), which would correspond to \((1 - q_0)/q_0 \) in the Friedmann cosmology with \( k = -1 \), where \( q_0 \) is the deceleration parameter. For illustrative purposes, we consider, in this Section, a simple case where the arbitrary function \( b(t) \) behaves as

\[
b(\Psi) = \beta \Psi,
\] (40)
where $\beta$ is a dimensionless parameter to be treated as a perturbation. Then, from Eqs. (35) and (37), we have

$$\frac{a(t)}{a(t_0)} = \frac{1}{1 + z} \left[ 1 + \frac{\beta}{2} \{ \sinh(\Psi_0 - \Psi) - (\Psi_0 - \Psi) \} \right] + \mathcal{O}(\beta^2) \quad (41)$$

and

$$\Phi(\Psi, \Psi_0) = \frac{1}{2}(\Psi_0 - \Psi) \quad (42)$$

$$- \frac{\beta}{4} \left[ \cosh(\Psi_0 - \Psi) + \Psi \sinh(\Psi_0 - \Psi) - \frac{\Psi_0^2 - \Psi^2}{2} - 1 \right] + \mathcal{O}(\beta^2) .$$

Now, we are ready to obtain the redshift–luminosity relation based on $d_L = rR(t_0, 0)(1 + z)$. Recalling $r(t, t_0) = 2 \tanh \Phi(t, t_0)$, $(\Psi_0 - \Psi)$ should be expressed in terms of the redshift.

From the parameterized solution Eq. (39), $\Psi$ is

$$\Psi = \cosh^{-1} \left[ 1 + (\cosh \Psi_0 - 1) \frac{a(t)}{a(t_0)} \right] . \quad (43)$$

Substituting Eq. (41) into Eq. (43), we find

$$\frac{\Psi_0 - \Psi}{\Psi_0} = \Psi_0 - \cosh^{-1} \left[ 1 + \frac{\cosh \Psi_0 - 1}{1 + z} \right] \quad (44)$$

$$- \frac{\beta}{4} \sqrt{\cosh \Psi_0 - 1} \left[ \sinh(\Psi_0 - \Psi) - (\Psi_0 - \Psi) \right] + \mathcal{O}(\beta^2) .$$

For simplicity, we define the zeroth order term of $\Psi$, $\Psi^{(0)}$, as

$$\Psi^{(0)} = \cosh^{-1} \left[ 1 + \frac{\cosh \Psi_0 - 1}{1 + z} \right] . \quad (45)$$

Then, from Eqs. (42), (44) and (45), we have

$$\Phi(\Psi, \Psi_0) = \frac{1}{2}(\Psi_0 - \Psi^{(0)}) \quad (46)$$

$$+ \frac{\beta}{4} \sqrt{\cosh \Psi_0 - 1} \left[ (\Psi_0 - \Psi^{(0)}) - \sinh(\Psi_0 - \Psi^{(0)}) \right]$$

$$- \frac{\beta}{4} \left[ \cosh(\Psi_0 - \Psi^{(0)}) + \Psi^{(0)} \sinh(\Psi_0 - \Psi^{(0)}) - \frac{\Psi_0^2 - (\Psi^{(0)})^2}{2} - 1 \right]$$

$$+ \mathcal{O}(\beta^2)$$

$$\equiv \Phi^{(0)}(z) + \frac{\beta}{4} \Phi^{(1)} + \mathcal{O}(\beta^2) .$$
In our neighborhood \((z \ll 1)\), \(\Phi^{(0)}(z)\) becomes
\[
\Phi^{(0)} = \frac{1}{2} \sqrt{\frac{\cosh \Psi_0 - 1}{\cosh \Psi_0 + 1}} \left[ z - \frac{\cosh \Psi_0 + 2}{2(\cosh \Psi_0 + 1)} z^2 \right] + O(z^3). \tag{47}
\]
Using Eqs.(44), (45) and (46), it can easily be seen that \(\Phi^{(1)}\) is zero up to the second order of \(z\). Finally, \(r(t, t_0)\) is expressed in terms of the redshift by
\[
r(t, t_0) = 2 \tanh \Phi(t, t_0) = 2 \tanh \Phi^{(0)}(z) + \frac{\beta}{2} \Phi^{(1)}(z)(1 - \tanh^2 \Phi^{(0)}(z)) + O(\beta^2)
= \sqrt{\frac{\cosh \Psi_0 - 1}{\cosh \Psi_0 + 1}} \left[ z - \frac{\cosh \Psi_0 + 2}{2(\cosh \Psi_0 + 1)} z^2 \right] + O(z^3, \beta^2). \tag{48}
\]
If \(\cosh \Psi_0 = (1 - q_0)/q_0\) as in the Friedmann cosmology, the standard result, \(r = \sqrt{1 - 2q_0|z - (1 + q_0)z^2/2| + O(z^3)}\), is recovered, implying that the luminosity distance, \(rR(t_0)(1 + z)\), has the same dependence on \(z\) for the small redshift as in the standard model.

To summarize, the standard results of the redshift–luminosity distance relation in the Friedmann cosmology for \(z \ll 1\) are preserved in this model, regardless of whether the local Universe is assumed to be open or flat.

V. NUMERICAL RESULTS

As was demonstrated in the previous section, the most fundamental cosmological relation predicted by this model, i.e., the redshift–luminosity relation, is practically the same as in the standard cosmology for small \(z\). This is due to our plausible assumption that \(a(t)\) behaves more or less the same as the scale factor in the standard cosmology. We now proceed to explore the relationship for large \(z\). Unfortunately, it is impossible to do so analytically, due partly to our inability to analytically solve the light propagation equation. In this Section, therefore, we resort to numerical method and present some numerical answers in the form of figures depicting possible modifications of the redshift–luminosity distance relation for large \(z\). Since our local Universe appears to be open as discussed before, we only consider the second case of the previous section. Thus, \(a(t)\) is parameterized by an angle \(\Psi\) as given in
Eq.(39). Since $b(t)$ is completely arbitrary except for being small, we consider the following three representative cases: (A) $b(\Psi) = \beta \Psi$, (B) $b(\Psi) = \beta \Psi^4$ and (C) $b(\Psi) = \beta [\cosh \Psi - 1]^2$, where $\beta$ is a dimensionless parameter to be determined. Note that the case (C) corresponds to the picture in which there is a constant vacuum energy density in the Universe, as can be seen from Eq.(17). Recalling that the Hubble expansion rate of the local Universe at the present epoch, $\bar{H}_0 \equiv \dot{R}/R|_{t=t_0, r \sim 0}$, is $[\dot{a}(t)/a(t)]|_{t=t_0}$, the density parameter in our neighborhood is, from Eq.(18),

$$\bar{\Omega}_0 \equiv \frac{8\pi G \rho(t = t_0, r \simeq 0)}{3\bar{H}_0^2} = 1 - \dot{a}_0^2 - b(t_0)\dot{a}_0^2 ,$$

(49)

where the bar denotes the local value and the subscript zero represents the present value. With the definitions of $\bar{\Omega}_{0,a} \equiv 1 - \dot{a}_0^2$ and $\bar{\Omega}_{0,b} \equiv b_0\dot{a}_0^2$, the density parameter and the pressure of the local Universe can be expressed by

$$\bar{\Omega}_0 = \bar{\Omega}_{0,a} - \bar{\Omega}_{0,b} ,$$

(50)

$$8\pi G p_0 = \bar{\Omega}_{0,b}\bar{H}_0^2 ,$$

(51)

where Eq.(19) is used.

From the observation of small peculiar velocities of nearby galaxies, we assume that the pressure of the local Universe is relatively small compared with the energy density. That is, $\bar{\Omega}_{0,b}$ is very small. In the numerical calculations to be presented in this Section, we assume $\bar{\Omega}_0 = 0.1$ and, for the sake of definiteness, choose values $\bar{\Omega}_{0,a} = 0.1001$ and $\bar{\Omega}_{0,b} = 0.0001$, specifying the numerical values of $\alpha$ and $\beta$. The light propagation equation can then be numerically integrated with the boundary condition $r(t = t_0, t_0) = 0$. To this end, we divide the numerical solution of $r(t, t_0)$ into $N$ intervals and fit each interval using

$$r_i(t, t_0) = \delta_i [t_0^{\gamma_i} - t_i^{\gamma_i}] \quad \text{for} \quad \frac{i-1}{N} < \frac{t}{t_0} < \frac{i}{N} \quad (i = 1, 2, 3...),$$

(52)

which then yields the numerical values of $\delta_i$ and $\gamma_i$ for each interval. We have used $N = 200$ in our numerical calculations. Using the definition of the comoving distance, i.e., $r_i(t, t_0) = r_i(t + \Delta t, t_0 + \Delta t_0)$, we have the redshift, for each interval, as
Then, the luminosity distance, \( d_L = rR(t_0)(1+z) \), for each interval, can easily be calculated from Eqs.(52) and (53). In Fig.1, the results of the three cases discussed above are presented by solid lines, while the standard results with \( \Omega = 0.1 \) and \( \Omega = 1.0 \) by the dashed lines and dotted lines, respectively. First, it is to be noted that on small scale, the linear Hubble diagrams are preserved for all of the three cases, as was shown by perturbative calculations. In the cases of (A) and (C), the redshift–luminosity distance relations in this model are almost indistinguishable and furthermore correspond to the standard results with the density parameter in the range \( \Omega \sim 0.8 - 0.9 \), even though the local mass density in the model is set to be \( \Omega_0 = 0.1 \). In the case (B), however, it appears to be the standard result with \( \Omega > 1 \) although the difference is not very significant. Nevertheless, this qualitative feature was totally unexpected. In this model, therefore, a precise measurement of the redshift–luminosity distance relation alone cannot provide information on \( q_0 \), which is related to \( \Omega_0 \) as \( 2q_0 = \Omega_0 \), contrary to the case of the standard cosmology.

Now it would be appropriate to discuss the meaning of the observed increase of \( \Omega_0 \) as we look farther out. Every light signal we are receiving right now contains information about the past in time. That is, what we measure are, for example, the redshift and \( G\rho(t) \), not \( G\rho(t_0) \). Thus, we deduce physical quantities at the present time \( t_0 \), using the standard cosmological evolution equations. Recalling that \( G\rho(t) \) in the matter dominated era in the standard Friedmann cosmology is proportional to \( 1/S^3(t) \), and \( S(t) \) is just \( S(t_0)/(1+z) \), \( G\rho(t_0) \) is obtained from \( G\rho(t) \) by multiplying \( 1/(1+z)^3 \). Dividing \( G\rho(t_0) \) with the constant critical density in the standard cosmology, \( G\rho_c^{(s)} \), we can deduce the density parameter at present time, which we shall call \( \Omega_0^{\text{obs}} \), as

\[
\Omega_0^{\text{obs}} \equiv \frac{G\rho^{\text{obs}}(t_0)}{G\rho_c^{(s)}} = \frac{G\rho(t, r(t))}{G\rho_c^{(s)}(1+z)^3}.
\]  

(54)

Using Eqs.(52) and (53) for each interval in Eq.(18), we have calculated \( \Omega_0^{\text{obs}} \) versus the redshift. The results are shown in Fig.2. As before, we have taken the local value \( \Omega_0^{\text{obs}} = 0.1 \).
In all three cases, the calculated values of $\Omega_0^{\text{obs}}$ are increasing functions of $z$. (A naive value in the standard cosmology is supposed to be a constant.) We can see that the cases of (A) and (C) are practically identical, whereas the case (B) shows a faster increase of $\Omega_0^{\text{obs}}$. As can be seen in Fig.2, however, even in the case (B) the increase is too slow to explain the IRAS data [13] where $\Omega_0^{\text{obs}}$ is compatible to unity (but with large errors) at the distance of about several 100 Mpc ($z \sim 0.0166$). To fit the increase of $\Omega_0^{\text{obs}}$ up to unity at $\sim 100$ Mpc requires drastic (perhaps unrealistic) changes in the form of and parameters in $a(t)$ and $b(t)$, which then would modify our predictions on the Hubble law.

**VI. SUMMARY AND CONCLUSIONS**

We have studied how cosmological observables are modified in an isotropic but inhomogeneous Universe compared with those of the standard model. In particular, the luminosity distance and the density parameter as functions of the redshift have been examined in the generalized Robertson–Walker spacetime with only one $(t, r)$-dependent scale factor and they were compared with the standard results.

When $R(t, r)$ is not factorized into the form of $a(t)f(r)$, the simple redshift–scale factor relation such as $(1 + z) = a(t_0)/a(t)$ remains no longer valid. First by solving light propagation equation, Eq.(2), for radially propagating light with the boundary condition $r(t = t_{\text{received}}) = 0$ and then considering two wave crests emitted at time $t$ and $t + \Delta t$ which are received at $t_0$ and $t_0 + \Delta t_0$, respectively, we have obtained the general redshift–scale factor relation given by Eq.(5). The result is valid in an inhomogeneous Universe and is shown to be reduced to the simple form $(1 + z) = a(t_0)/a(t)$ in the case of the homogeneous spacetime, i.e., the standard Robertson–Walker spacetime. Our general relations agree with the results obtained in [10] and [11].

We have applied the general redshift–scale factor relation to the cosmological model in [12] where the Universe is pictured as being inside a highly dense and rapidly expanding shell with the underdense center. First, for the nearby objects ($z \ll 1$), the luminosity distances
as functions of the redshift are obtained analytically, using a perturbative method for two cases where the underdense center is either flat or open according to the definition of the standard Friedmann cosmology. One of the most interesting features in [12] is that the scale factor $R(t, r) = a(t)/(1 - B(t)r^2)$ is specified by two arbitrary functions, $a(t)$ and $B(t)$ (or $b(t)$), and $a(t)$ is very similar to the scale factor of the standard Friedmann cosmology and $B(t)$ (or $b(t)$) is the perturbation to the locally flat (or open) Universe. Under the assumption that $a(t)$ behaves the same as that in the standard cosmology, it is shown analytically that the standard redshift–luminosity distance relations in the Friedman cosmology for $z \ll 1$ remains intact for both cases. Specifically, since the corrections of order $O(\beta)$ of these expressions can be expanded as a power series of $\sum_{i=3}^{\infty} c_i z^i$ with some coefficients $c_i$ (that is, zero up to the second order of $z$), it has been shown that for nearby objects, in spite of its different metric, the cosmological model of [12] is not much different from the standard cosmology as far as the Hubble law is concerned. It is also interesting to note that in spite of the special location of the observer, i.e., the return of the pre-Copernican notion in the model [12], the results are almost the same as those of the standard model.

As for large $z$, the redshift–luminosity distance relations given in Eqs. (32) and (48) are different from those of the standard cosmology and moreover it is expected that the corrections would be larger when $O(\beta^2)$ terms are included. In this case, as we mentioned repeatedly, we cannot use the perturbative method and thus we have solved them numerically and obtained the results as shown in Fig.1. and Fig.2.

Figure 1 shows the redshift–luminosity distance relation in the cosmological model of [12]. Comparing them with the standard cosmology with $k = 0$ (dotted curve) and $k = -1$ (dashed curve), we can easily see that for small $z$, the redshift–luminosity distance relation of model [12] denoted by the solid lines is almost the same as the standard one, as was also shown in the explicit perturbative calculation. But for $1 < z < 3$, the Hubble law of the model is very similar to that of the standard cosmology with $k = 0$, not with that with $k = -1$, in spite of the fact that the mass density of the local Universe is set to be $\Omega_0 = 0.1$. It is to be noted that although there is no substantial deviation from the standard model in
our redshift–luminosity distance relation in Fig.1, the deviation in the case of the TB model is more prominent and is different from ours, both qualitatively and quantitatively as was shown in [11]. Figure 2 shows that the calculated density parameter at the present time, $\Omega_0^{obs}$, is an increasing function of the redshift.

Although, admittedly, our perturbative calculations may not be rigorous in the sense that the functions $a(t)$ and $B(t)$ that appear in the scale factor $R(t, r)$ could not be determined by the equation of state, and furthermore our numerical results can only explain the IRAS data qualitatively, we feel that the qualitative nature of our results are robust because we have considered the general case with $B(t) = \beta(t/t_0)^n \ (n > 0)$, $b(t) = \beta\Psi(t)$, $b(t) = \beta\Psi^4(t)$ and $b(t) = \beta[\cosh\Psi - 1]^2$ with a variety of numerical values of the parameters involved, all of which have led to similar results.

In summary, although the scale-dependent cosmology for the inhomogeneous Universe as modeled in [12] implies the explicit running of $H_0$, $\Omega_0$ and $t_0$ as functions of $r$ because of the non-Robertson-Walker metric, as far as the observables such as redshift–luminosity distance relations are concerned, the results are hardly different from those of the standard model in our neighborhood, i.e. for small $z$. Even for large $z$, the difference between the model considered and the standard model with $k = 0$ still remains small but the model can be tested when the data from galaxy redshift survey at long distance become available and are compared with the predictions of the model on the matter distribution and on the age of the Universe. [12]

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Figure Captions

Fig. 1 The luminosity distance, $d_L$, as a function of the redshift $z$ for the three cases (A), (B) and (C) as discussed in the text. Solid lines denote the results in our model, whereas dashed and dotted lines indicate the standard results with $\Omega = 0.1$ and $\Omega = 1.0$, respectively.

Fig. 2 The calculated density distribution, $\Omega^{obs}_i(z)$ as a function of the redshift $z$. The numerical results for the cases (A) and (C) are practically indistinguishable.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9508336v1
$d_L$ in units of $c/H_0$
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9508336v1
