Approximate Solution Of Schrodinger Equation For Eckart Potential Combined With Trigonometric Poschl-Teller Non-Central Potential Using Romanovski Polynomials

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Abstract. The approximate analytical solution of Schrodinger equation for Eckart potential combined with trigonometric Poschl-Teller noncentral potential is investigated using Romanovski polynomial. The approximate bound state energy eigenvalues are given in the close form, the corresponding approximate radial eigenfunctions is formulated in term of Romanovski polynomials, and the angular wave function is also expressed in term Romanovski polynomial. The effect of the presence of trigonometric Poschl-Teller potential changes the state of angular wave function level.

1. Introduction

The exact analytical solutions of Schrodinger equations for some physical potentials are very essential since they provide the important information of the quantum system. Recently, considerable efforts have been paid to obtain the exact solution of the non-central potentials. These potentials include ring-shaped oscillator, Modified Kratzer, Coulombic ring-shaped, and ring-shaped non-spherical harmonic oscillator [1-4]. The bound state energy spectra of these potentials have been investigated by various techniques such as factorization methods, [5-7] supersymmetric quantum mechanics, [8-11] hypergeometric type equation,[12-14] Nikiforov-Uvarov method, [15-17] etc. The exact solution of the radial part of the Schrodinger equations for some potentials such as Eckart, Manning Rosen, Hulthen, modified Poschl-Teller potentials, are obtained if the the angular momentum \( l=0 \). However, for \( l \neq 0 \), the radial part of Schrodinger equation can only be solved approximately for different suitable approximation scheme. One of the suitable approximation scheme is conventionally proposed by Greene and Aldrich [18-19].

In this paper we will attempt to solve the Schrodinger equation for Eckart potential combined with Poschl-Teller non-central potential in term of finite Romanovski polynomials. The Romanovski polynomials consist of reducing Schrodinger equation by an appropriate change of the variable to form of generalized hypergeometric equation[11]. The polynomial was discovered by Sir E.J.Routh [20] and by V. I. Romanovski [21]. The notion “finite” refers to the observation that, for any given set of parameters (i.e. in any potential) only a finite of polynomials appear orthogonal [17,23].

Eckart potential and trigonometric Poschl-Teller potentials play the essential roles in interatomic and intermolecular forces [22-25] and can be applied to describe molecular vibrations. This potential is exactly solvable or quasi – exactly solvable and their bound state solutions have been reported,[26]. The Eckart potential combined with trigonometric Poschl-Teller non-central potential is separable potential therefore it
is solved using variable separation method and has potential application in describing the molecular vibration and intermolecular forces for $l \neq 0$.

This paper is organized as follows. In section 1.1, we review the finite Romanovski polynomial briefly. In section 1.2, we find the bound state energy solution in term of new orbital quantum number and radial and also angular wave functions. A brief results and discussion is presented in section four and conclusion in section two.

2. Review of Finite Romanovski Polynomials

The generalized hypergeometric equation is given as [15]

$$\sigma(s) \frac{d^2 y_n}{ds^2} + \tau(s) \frac{dy_n}{ds} + \lambda_n y_n(s) = 0$$

(1)

where $\sigma(s) = as^2 + bs + c$ and $\tau = ds + e$ and $\lambda_n = -\{(n(n-1) + 2n(1-p)\}$

(2)

Equation (1) is described in the textbook by Nikiforov-Uvarov [15] where it is cast into self adjoint form and its weight function, $w(x)$, satisfies Pearson differential equation which is expressed as

$$\frac{d(\sigma(s)w(s))}{ds} = \tau(s)w(s)$$

(3)

The weight function obtained by solving the Pearson differential equation is

$$w(s) = \exp\left(\int (d - 2a)s + (e - b) \frac{ds}{as^2 + bs + c}\right)$$

(4)

The corresponding polynomials to the weight function equation (4) are built up from the Rodrigues representation that is given as

$$y_n = \frac{1}{w(s)} \frac{d^n}{ds^n}\left((as^2 + bs + c)w(s)\right)$$

(5)

For Romanovski polynomial, the values of parameters in equation (4) are:

$$a = 1, b = 0, c = 1, d = 2(1-p) \quad and \quad e = q \quad with \quad p > 0$$

(6)

By inserting equation (6) into equation (4) we obtain the weight function as

$$w(s) = \left(1 + s^2\right)^{-p} e^{q\tan^{-1}(s)}$$

(7)

This weight function was first reported by Routh and then by Romanovski. The polynomials associated with equation (7) are Romanovski polynomials and denoted by $R_n^{(p,q)}(s)$. As long the weight function decreases by $x^{-2p}$, integral of the type

$$\int_{-\infty}^{\infty} w^{(p,q)}(s) R_m^{(p,q)}(s) ds$$

(8)

will be convergent only if

$$m' + m < 2p - 1$$

(9)

This means that there are only a finite number of Romanovski polynomials that are orthogonal. The differential equation satisfied by Romanovski Polynomial is

The differential equation satisfied by Romanovski Polynomial obtained by inserting equations (2) and (6) into equation (1) is

$$\left(1 + s^2\right)^2 \frac{d^2 R_n^{(p,q)}(s)}{ds^2} + \left(2s(-p + 1) + q\right) \frac{dR_n^{(p,q)}(s)}{ds} - \left\{n(n-1) + 2n(1-p)\right\} R_n^{(p,q)}(s) = 0$$

(10)

where $y_n = R_n^{(p,q)}(s)$. The Schrodinger equation of the potential of interest will be reduced into the form which is similar to equation (10) by an appropriate transformation of variable, $r = f(s)$, and by introducing a new wave function which is given as

$$\psi(r) = y_n(s) = \left(1 + s^2\right)^{p/2} e^{-\frac{n}{2} w^{(p,q)}(s)} D_n^{(p,q)}(s)$$

(11)

The eigen function in equation (11) is the solution of Schrodinger equation for potential interest where
The Romanovski polynomials obtained from Rodrigues formula with the weight function in equation (5) is expressed as

\[ R_n^{(\beta, \alpha)}(s) = D_n^{(\beta, \alpha)}(s) = \frac{1}{(1 + s^2)^{\frac{n}{2}}} \frac{d^n}{ds^n} \left( (1 + s^2)^{-\frac{n}{2}} e^{\eta \tan^{-1}(s)} \right) \]  

(13)

If the wave function of the nth level in equation (11) is rewritten as [27-28]

\[ \psi_n(r) = \frac{1}{\sqrt{df(s)/ds}} \left( 1 + s^2 \right)^{-\frac{n}{2}} e^{\frac{\eta}{2} \tan^{-1}(s)} R_n^{(\beta, \alpha)}(s) \]  

(14)

then the orthogonality integral of the wave functions expressed in equation (14) gives rise to orthogonality integral of the finite Romanovski polynomials, that is

\[ \int_{-\infty}^{\infty} \psi_n(r)\psi_m(r)dr = \int_{-\infty}^{\infty} \omega[R^2] R_n^{(\beta, \alpha)}(s)R_m^{(\beta, \alpha)}(s)ds \]  

(15)

In this case the values of \( \eta \) and \( q \) are not n-dependence where n is the degree of polynomials. However, if equation (9) and (15) are not fulfilled then the Romanovski polynomials is infinity [27-29].

3. Solution of Schrödinger equation for Eckart potential combined by trigonometric Poschl-Teller non-central potential Using Romanovski Polynomials

The non-central potential which constructed by Eckart potential and trigonometric Poschl-Teller non-central potential is given as

\[ V(r, \theta) = \frac{\hbar^2}{2Md^2} \left( V_0 \frac{e^{-i\alpha \theta}}{1 - e^{-i\alpha \theta}} - V_1 \frac{1 + e^{-i\alpha \theta}}{1 - e^{-i\alpha \theta}} \right) + \frac{\hbar^2}{2Mr^2} \left( \kappa(\kappa - 1) \sin^2 \theta + \eta(\eta - 1) \cos^2 \theta \right) \]  

with \( V_0 \) and \( V_1 \) describe the depth of the potential well and are positives, \( V_1 > V_0, a \) is a positive parameter which to control the width of the potential well, \( 0 < (r/a) < \infty, \kappa > 1, \eta > 1 \).

The three dimensional time-independent Schrödinger equation for Eckart potential combined with trigonometric Poschl-Teller non-central potential is

\[ -\frac{\hbar^2}{2M} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \psi(r, \theta, \phi) + \frac{\hbar^2}{2Md^2} \left( V_0 \frac{e^{-i\alpha \theta}}{1 - e^{-i\alpha \theta}} - V_1 \frac{1 + e^{-i\alpha \theta}}{1 - e^{-i\alpha \theta}} \right) \psi(r, \theta, \phi) + \frac{\hbar^2}{2Mr^2} \left( \kappa(\kappa - 1) \sin^2 \theta + \eta(\eta - 1) \cos^2 \theta \right) \psi(r, \theta, \phi) = E \psi(r, \theta, \phi) \]  

(17)

Equation (17) is solved using variable separation method so we get radial, polar and azimuthal parts of Schrödinger equation as following:

\[ \frac{1}{R} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{P \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\Phi} \frac{\partial^2}{\partial \phi^2} = \lambda \]

(19)

\[ \frac{1}{R} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{r^2}{a^2} \left( V_0 \frac{e^{-i\alpha \theta}}{1 - e^{-i\alpha \theta}} - V_1 \frac{1 + e^{-i\alpha \theta}}{1 - e^{-i\alpha \theta}} \right) + \frac{2Mr^2}{\hbar^2} E = \lambda = l(l+1) \]  

(18)

The azimuthal part of wave function obtained from equation (19) is given, as usual, as

\[ \Phi = A_n e^{im\phi} \]  

(21)

The radial and polar parts of the Schrödinger equations are solved using Romanovski Polynomials.
3.1 Solution of radial part of Schrödinger equation.

By substitution \( \frac{2M}{\hbar^2} E = -\epsilon^2 \) and \( R = \frac{X(r)}{r} \) in equation (14) we get

\[
\frac{\partial^2}{\partial r^2} X(r) - \frac{1}{a^2} \left( V_0 \frac{e^{\epsilon r/2a}}{(1 - e^{\epsilon r/2a})^2} - V_1 \frac{1 + e^{\epsilon r/2a}}{1 - e^{\epsilon r/2a}} \right) X(r) - \epsilon^2 X(r) - \frac{l(l+1)}{2a^2} \chi(r) = 0 \tag{22}
\]

For \( \frac{r}{a} \ll 1 \) the approximation of the centrifugal term [18–19] in equation (22) is given as

\[
\frac{1}{r^2} \approx \frac{1}{4a^2} \left( d_0 + \frac{1}{\sinh^2(r/2a)} \right) \quad \text{with} \quad d_0 = 1/12.
\]

In term of hyperbolic functions, equation (22) is re written as

\[
\frac{d^2 X(r)}{dr^2} - \frac{l(l+1)}{4a} \left( e^{\frac{r}{2a}} + \frac{1}{\sinh^2(r/2a)} \right) X(r) - \frac{1}{a} \left( V_0 \frac{1}{4 \sinh^2(r/2a)} - V_1 \coth(r/2a) \right) X(r) - \epsilon^2 X(r) = 0 \tag{23}
\]

and by making an appropriate change of variable, \( r = f(x) = 2a \coth^{-1}(ix) \) in equation (23), we get

\[
(1 + x^2)^2 \frac{d^2 X}{dx^2} + 2x \frac{dX}{dx} - \left\{ V_0 + l(l+1) + \frac{4V_1 ix}{1 + x^2} - \frac{l(l+1)d_0 + 4\epsilon^2 a^2}{1 + x^2} \right\} X = 0 \tag{24}
\]

To solve equation (24) in terms of Romanovski polynomial, equation (7) suggests the substitution in equation (24) as [29]

\[
\chi(f(x)) = g_n(x) = \left(1 + x^2\right)^{\frac{\beta}{2}} e^{\frac{-\pi - i \pi}{2}} D_n(\beta x)
\]

where \( 1 \leq ix \leq \infty \)

By inserting equation (25) into equation (24) we obtain

\[
\left\{ +x^2 \frac{\partial D}{\partial x} + \frac{1}{2} x(\beta + 1) - \alpha \frac{\partial}{\partial x} \right\} \left\{ - \frac{\alpha^2}{4} + \beta^2 - (l(l+1)d_0 + 4\epsilon^2 a^2) \right\} D = 0 \tag{26}
\]

Equation (26) reduces to differential equation that satisfied by Romanovski polynomials if the numerator of \( \frac{1}{1 + x^2} \) is set to be zero, that are

\[
- \frac{\alpha^2}{4} + \beta^2 - (l(l+1)d_0 + 4\epsilon^2 a^2) = 0 \quad \text{and} \quad \beta \alpha + 4V_1 i = 0 \tag{27}
\]

and then equation (26) becomes

\[
\left\{ +x^2 \frac{\partial D}{\partial x} + \frac{1}{2} x(\beta + 1) - \alpha \frac{\partial}{\partial x} \right\} \left\{ V_0 + l(l+1) - \beta^2 - \beta \right\} D = 0 \tag{28}
\]

By comparing the parameters between equations (10) and (28) we obtain the following relation:

\[
V_0 + l(l+1) - \beta^2 - \beta = n(n-1) + 2n(1-p) \quad \Rightarrow \quad 2(\beta + 1) = 2(-p + 1) \quad \text{and} \quad \alpha = -q \tag{29}
\]

To have more physical meaning the value of \( \beta \) obtained from equations (29) is

\[
\beta = \beta_n = -\sqrt{V_0 + \left( l + \frac{1}{2} \right)^2 - n - \frac{1}{2}} \tag{30}
\]

and from equations (27) and (30) we obtain

\[
\alpha^2 = -2(\beta + 1)d_0 + 4\epsilon^2 a^2 + 2\sqrt{(l(l+1)d_0 + 4\epsilon^2 a^2)^2 - 16V_1^2} \tag{31}
\]
From equations (31) and (32) we obtain the energy spectrum of the system given as

\[ E_i = \frac{-\hbar^2}{2M} \left( \frac{\left( V_0 + \frac{l+\frac{1}{2}}{2} \right)^2 + n + \frac{1}{2}}{4a^2} \right) + \frac{V_i}{a} \left( \frac{\left( V_0 + \frac{l+\frac{1}{2}}{2} \right)^2 + n + \frac{1}{2}}{4a^2} \right)^2 \]

The energy spectra in equation (33) is in agreement with the result obtained using NU method.

The energy spectra of Eckart potential obtained using NU method for Eckart plus Hulthen potential is given as

\[ E_i = \frac{-\hbar^2}{2M} \left( \frac{\left( V_0 + \frac{l+\frac{1}{2}}{2} \right)^2 + n + \frac{1}{2}}{4a^2} \right) + \frac{V_i}{a} \left( \frac{\left( V_0 + \frac{l+\frac{1}{2}}{2} \right)^2 + n + \frac{1}{2}}{4a^2} \right)^2 \]

The energy obtained using NU method given in equation (34) is a little different with the energy calculated using Romanovski polynomials in equation (33). This difference is caused by the different approximation for the centrifugal term which is approximated by \( \frac{1}{r^2} = \frac{1}{4a^2} \left( \frac{1}{\sinh^2(r/2a)} \right) \) for NU method.

To determine the wave function, equations (30) and (32) are inserted into equations (7) and (11) so that we obtain the weight function \( w(x) \) and the Romanovski polynomials \( R_n^{(\beta, \alpha)}(x) \) as

\[ w^{(\beta, \alpha)} = \left( 1 + x^2 \right) \left( \frac{\left( n \right)^{\frac{1}{2}}}{\left( n \right)^{\frac{1}{2}}} \right) \exp \left( -\frac{4iV_1}{\sqrt{V_0 + \frac{l+\frac{1}{2}}{2} + n + \frac{1}{2}}} \tan^2(x) \right) \]

and \( D_n^{(\beta, \alpha)}(x) = R_n^{(\beta, \alpha)}(x) = \frac{1}{\left( 1 + x^2 \right)^2} e^{\alpha \tan^{-1}(x)} \frac{d^n}{dx^n} \left( 1 + x^2 \right)^{\frac{n}{2}} e^{\alpha \tan^{-1}(x)} \}

where \( \beta_n \) and \( \alpha_n \) are expressed in equations (30) and (32). As a result, the wave function of the nth level is given by

\[ \chi(f(x)) = g_n(x) = \left( 1 + x^2 \right)^{\frac{\beta_n}{2}} e^{\frac{-\alpha_n \tan^{-1}(x)}{2}} R_n^{(\beta, \alpha)}(x) \]

By using trigonometric-hyperbolic functions relation we have

\[ e^{-\frac{\alpha_n \tan^{-1}(x)}{2}} = e^{-\frac{\alpha_n \tan^{-1}(x)}{2}} \left( e^{i\coth(r/2a)} \right)^{\frac{i\alpha_n}{4}} \]

By inserting equation (38) into equation (37) we get the radial wave function given as

\[ R_n^{(\beta, \alpha)}(r) = \frac{e^{\frac{l}{2}}}{\left( 1 - \coth(r/2a) \right)^{\frac{n}{2}} \left( 1 + \coth(r/2a) \right)^{\frac{n}{2}}} \frac{d^n}{dx^n} \left( 1 + x^2 \right)^{\frac{n}{2}} e^{\alpha \tan^{-1}(x)} \}

The radial wave function for Eckart potential with centrifugal term obtained from the solution of Eckart plus Hulthen potential with centrifugal term using NU method is given as

\[ R_n^{(\beta, \alpha)}(r) = \frac{e^{\frac{l}{2}}}{\left( 1 - \coth(r/2a) \right)^{\frac{n}{2}} \left( 1 + \coth(r/2a) \right)^{\frac{n}{2}}} \frac{d^n}{dx^n} \left( 1 - \coth(r/2a) \right)^{\frac{n}{2}} \left( 1 + \coth(r/2a) \right)^{\frac{n}{2}} \]
\[
\chi_n(r) = \frac{1}{r^{1/2}} \mathcal{Y} \left( \frac{1 + \cosh(r/2a)}{2} \right)^{1/2} \mathcal{Y} \left( \frac{1 - \cosh(r/2a)}{2} \right)^{1/2} \chi_n \left( \frac{1 - \cosh(r/2a)}{2} \right)
\]

(41)

\[
\chi_n(r) = \frac{C \sum_{l=0}^{\infty} \frac{(l+1)^{1/2}}{l!} \frac{d}{\rho} \left( \frac{1 + \cosh(r/2a)}{2} \right)^{1/2} \mathcal{Y} \left( \frac{1 - \cosh(r/2a)}{2} \right)^{1/2} \chi_n \left( \frac{1 - \cosh(r/2a)}{2} \right)}{1 - \cosh(r/2a)}
\]

(42)

with

\[
p = \left( \sqrt{\frac{V_0(l+1)^2}{2} + n + \frac{1}{2}} \right)
\]

(43)

Equations (40), (41), (42), and (43) show that the radial wave functions for Eckart potential with centrifugal term obtained using Romanovski polynomial and NU method are the same.

Since the \( \beta_n \) and \( \alpha_n \) parameters, expressed in equations (30) and (32), are n-dependence then the orthogonality of the wave functions may not produce to the orthogonality integral of the polynomials[29]. The orthogonality integral of the wave functions is given as

\[
\int_0^\infty \chi_n(r) \chi_{n'}(r) dr = \delta_{nn'}
\]

(44)

By carrying out the differentiations of equation (36), we find the lowest four Romanovski polynomials given as

\[
R_0^{(-\beta_0,-\alpha_0)}(x) = 0
\]

(45a)

\[
R_1^{(-\beta_0,-\alpha_0)}(x) = (\beta_1 + 1) x - \alpha_1
\]

(45b)

\[
R_2^{(-\beta_0,-\alpha_0)}(x) = 2(\beta_2 + 2)(\beta_2 + 3)x^2 - 2\alpha_2 (2\beta_2 + 3)x + \alpha_2^2 + 2\beta_2 + 4
\]

(45c)

\[
R_3^{(-\beta_0,-\alpha_0)}(x) = 4x(\beta_3 + 3)(2\beta_3 + 5)(\beta_3 + 2) - 6\alpha_3 x^2 (2\beta_3 + 5)(\beta_3 + 2) + 2\alpha_3 x \beta_3 + 3\alpha_3 \beta_3 + 28\beta_3 + 6\alpha_3^2 + 34) - 2\alpha_3 (2\beta_3 + 5) - \alpha(\alpha_3^2 + 2\beta_3 + 6)
\]

(45d)

where \( \beta_n \) and \( \alpha_n \) are expressed in equations (30) and (32). The lowest four degrees of radial wave functions for arbitrary values of \( l \) are calculated by using equations (39) and (45).

3.2 The solution of angular Schrodinger equation for Eckart potential combined with non central Poschl-Teller potential.

To solve the polar Schrodinger equation expressed in equation (19), we set the angular wave function as

\[
P = \frac{Q(\theta)}{\sin \theta} \frac{\partial P}{\partial \theta} = \frac{dQ}{\sin \theta} \frac{1}{\cos \theta} \frac{1}{\sqrt{(\sin \theta)}}
\]

(46)

where \( Q(\theta) \) is the new angular wave function. By inserting equation (46) into equation (19) we obtain

\[
\frac{d^2 Q}{d\theta^2} - \left( \frac{\kappa(\kappa - 1) + m^2}{\sin^2 \theta} - \frac{\eta(\eta - 1)}{\cos^2 \theta} \right) Q = (l(l+1) + (1/4))Q = 0
\]

(47)

Equation (47) to be solved using Romanovski polynomials, therefore we have to substitute the variable \( \theta \) and introduce a new wave function such that equation (47) reduces to general hypergeometric type equation expressed in equation (1) or into second order differential equation of Romanovski polynomials expressed in equation (10). By making a change of variable in equation (47), \( \cos 2\theta = is \) then equation (47) becomes

\[
\left( 1 + s^2 \right) \frac{d^2 Q}{d\xi^2} + s \frac{dQ}{d\xi} + \left[ \frac{\kappa(\kappa - 1) + m^2}{2(1 + s^2)} - \frac{\eta(\eta - 1)}{2(1 + s^2)} \right] Q = 0
\]

(48)

Equation (48) will be reduced into differential equation of Romanovski polynomial by setting
\[ Q(\theta) = G_n(s) = (1 + s^2)^{\frac{\theta}{2}} e^{-s \tan^{-1}(s)} D_{s}^{\theta} \]  

(49)

By inserting equation (49) into equation (48) we get

\[ (1 + s^2) \frac{\Delta D}{\Delta s} + (2 \beta + 1 - \alpha) \frac{\partial D}{\partial s} - \alpha \left( \frac{l}{2s} + \frac{\beta^2}{4} - \beta^2 \right) = 0 \]  

(50a)

Equation (50a) reduces to the differential equation satisfied by Romanovski polynomials given as

\[ (1 + s^2) \frac{\Delta D}{\Delta s} + s(2\beta + 1 - \alpha) \frac{\partial D}{\partial s} - \left( \frac{l}{2s} + \frac{\beta^2}{4} - \beta^2 \right) = 0 \]  

(50b)

when the numerator whose its denominator \(2(1 + s^2)\) is set to be zero, that are

\[ -\left( \kappa(\kappa - 1) + m^2 - \frac{1}{4} \eta(\eta - 1) \right) - \frac{\alpha^2}{2} + 2\beta^2 - 2\beta = 0 \]  

(51)

\[ -\left( \kappa(\kappa - 1) + m^2 - \frac{1}{4} \eta(\eta - 1) \right) l + 2\beta\alpha - \alpha = 0 \]  

(52)

By comparing equations (50b) and (10) we obtain

\[ (2\beta + 1) = 2(-p + 1) ; \alpha = -q \]  

and

\[ \left( \frac{l}{2s} + \frac{\beta^2}{4} - \beta^2 \right) = n(n - 1) + 2n(1 - p) \]  

(54)

By using equations (51) and (52) and setting

\[ \left( \kappa - \frac{1}{2} \right)^2 + m^2 - \frac{1}{4} = \kappa(\kappa - 1) + m^2 = F, (\eta - \frac{1}{2})^2 = G \]  

(55)

we obtain \( \alpha = \pm i(\sqrt{F} \pm \sqrt{G}) \) and

\[ \beta - \frac{1}{2} = \frac{(F - G)}{\pm 2i(\sqrt{F} \pm \sqrt{G})} \]  

(56)

To have physical meaning, the proper choice of the values of \( \alpha \) and \( \beta \) in equation (56) are

\[ \alpha = \pm \left( \sqrt{\kappa(\kappa - 1) + m^2} + \left( \eta - \frac{1}{2} \right) \right) \]  

and \( \beta = \pm \frac{\sqrt{\kappa(\kappa - 1) + m^2} + (\eta + \frac{1}{2})}{2} \)  

(57)

From equations (54) and (57) we have

\[ l = \left( \left( \kappa - \frac{1}{2} \right)^2 + m^2 - \frac{1}{4} + \eta + 2n \right) \]  

(58)

Equation (58) shows that the values of \( l \) depend on the potential parameters, \( \kappa \) and \( \eta \), and the degree of the Romanovski polynomial, \( n=n \). The weight function obtained from equations (7) and (57) is given as
The Romanovski polynomials are obtained by using equations (13) and (61) as

\[ R_n^{(-\beta, -\alpha)}(s) = \frac{1}{(1 + s^2) \left( \sqrt{s^2 + G} \right)^{\frac{1}{2}}} e^{-i \left( \sqrt{s^2 + G} \right) \tan^{-1} s} \]  

(60)

and the angular wave functions obtained from equations (11) and (60) is given as

\[ Q(\theta) = g_n(s) = (1 + s^2)^{\frac{(\sqrt{s^2 + G}) + 1}{4}} e^{-i \frac{1}{2} \left( \sqrt{s^2 + G} \right) \tan^{-1} s} R_n^{(-\beta, -\alpha)}(s) \]  

(61)

The angular eigenfunction obtained by using equations (46) and (61) is given as

\[ P_{\nu=\eta, p}^{(n)}(\theta) = \frac{Q_{\nu=\eta, p}^{(n)}(s)}{\sin \theta} = \frac{1}{\sin \theta} \sqrt{(1 + s^2)^{\frac{(\sqrt{s^2 + G}) + 1}{2}}} e^{-i \frac{1}{2} \left( \sqrt{s^2 + G} \right) \tan^{-1} s} R_n^{(-\beta, -\alpha)}(s) \]  

(62a)

or

\[ P_{\nu=\eta, p}^{(n)}(\theta) = \frac{1}{\sin \theta} \left( 1 - \cos 2\theta \right)^{\frac{\sqrt{s^2 + G}}{2}} \left( 1 + \cos 2\theta \right)^{\frac{\sqrt{s^2 + G}}{2}} R_n^{(-\beta, -\alpha)}(-i \cos 2\theta) \]  

(62b)

The orthogonality integral of the angular wave function obtained from equation (62) is given as

\[ \int_0^\pi P_{\nu=\eta, p}^{(n)}(\theta) \sin \theta d\theta = \int_0^\pi \left( 1 + s^2 \right)^{\frac{(\sqrt{s^2 + G}) + 1}{2}} e^{-i \frac{1}{2} \left( \sqrt{s^2 + G} \right) \tan^{-1} s} R_n^{(-\beta, -\alpha)}(s) R_n^{(-\beta, -\alpha)}(-s) ds \]  

(63)

The orthogonality of Romanovski polynomials is produced from the orthogonality of wave function but equation (63) is not convergent [29-30] since

\[ n + n' < 2p - 1 \text{ or } n < \frac{1}{2} \left( \frac{k^2}{2} + m^2 \right) - \frac{1}{4} + \eta + \frac{1}{2} \]  

(64)

Construction of Romanovski polynomials. The first four Romanovski polynomials obtained from equation(60) are given as:

\[ R_n^{(-\beta, -\alpha)}(s) = 1 \]  

(65)

\[ R_n^{(-\beta, -\alpha)}(s) = (\sqrt{s^2 + G} + 2) s - i(\sqrt{s^2 + G}) \]  

(66)

\[ R_n^{(-\beta, -\alpha)}(s) = \left[ (\sqrt{s^2 + G} + 3) (\sqrt{s^2 + G} + 4) - 3i^2 (\sqrt{s^2 + G} + 5) \right] \]  

(67)

\[ R_n^{(-\beta, -\alpha)}(s) = \left[ 3 \sqrt{s^2 + G} + (\sqrt{s^2 + G} + 6) \left( \sqrt{s^2 + G} + 4 \right) \right] \]  

(68)

If there is no the presence of trigonometric Poschl-Teller potential, where \( \kappa = 0 \) and \( \eta = 0 \) then the angular wave function reduces to associated Legendre polynomials and the orbital quantum number equation (36) becomes \( l = m + 2n \). However the associated Legendre polynomial obtained from this non-central potential are only those polynomials whose values of \( l \) and \( m \) are differed by even numbers since \( l = m + 2n \). The effect of the presence of Poschl-Teller non-central potential to spherical harmonics is illustrated using the three dimensional representation and the polar diagram of the absolute value of un-normalized angular wave functions obtained from equations (55), (62), and (68). The 3D representations and polar diagram of \( |Y_n| \) visualized using Mat Lab 7 are shown in Figure 1 for \( n = 3, \kappa = 0, \eta = 0, l = 6, m = 0 \), Figure 2 for...
By comparing Figures 2 and 3 with Figure 1, it is concluded that there is a state change in angular wave function. Therefore it may be concluded that the number of the degeneracy of the system changes. The un-normalized angular wave functions illustrated in Figures 1-3 is in agreement with the result calculated using NU method [31]. By putting the new value of the orbital quantum number expressed in equation (68), the energy of Eckart potential combined with trigonometric Poschl-Teller non-central potential is rewritten as

$$
E_n = -\frac{\hbar^2}{2M} \left\{ \frac{\sum_{l} l \left(V_n + (\sqrt{(\kappa + 1)^2 + \eta + 2l} + \frac{1}{2} + n_r + \frac{1}{2} \right)^2}{4a^2} - \frac{l(l+1)d_0}{r} \right\}
$$

(69)

where \(l\) is a new orbital quantum number and its values are non-negative integer, while \(n_r\) is radial quantum number and is nonnegative integer. From equation (65) we can calculate the energy for special case, for Eckart potential, we set \(\kappa = \eta = m = n_r = 0\), therefore the energy spectrum of Eckart potential is

$$
E_n = -\frac{\hbar^2}{2M} \left\{ \frac{\sum_{l} l \left(V_n + (\frac{1}{2} + \frac{1}{2})^2 + n_r + \frac{1}{2} \right)^2}{4a^2} - \frac{l(l+1)d_0}{r} \right\}
$$

(70)

The total un-normalized wave function of the system obtained from equations (39) and (65b) is given as

$$
\psi(r, \theta, \phi) = \left( 1 - \coth(r/2a) \right)^{\frac{\beta_r + \beta_0}{4}} \left( 1 + \coth(r/2a) \right)^{\frac{\beta_r - \beta_0}{4}} R_{\frac{n_r}{2}}(\rho - \sqrt{\frac{1}{2}}, -\rho - \sqrt{\frac{1}{2}}) \left( -i \coth(r/2a) \right) \\
\times \frac{1}{\sqrt{\sin \theta}} (1 - \cos 2\theta)^{\frac{1}{4}} (1 + \cos 2\theta)^{\frac{1}{4}} R_{\frac{n}{2}}(\rho - \sqrt{\frac{1}{2}}, -\rho - \sqrt{\frac{1}{2}}) \left( -i \cos 2\theta \right) e^{im\phi}
$$

(71)
4. Results and Discussion

The energy spectra for Eckart plus trigonometric Poschl-Teller non-central potential reduce to energy spectra of Eckart potential plus centrifugal term by setting the Poschl-Teller potential’s parameter, $\kappa = \eta = m = 0$ in equation (69). The ground state wave function of Eckart potential is produced from radial wave function for $n_r = 0$ and $l = 0$. The radial wave function for Eckart potential with centrifugal term obtained using Romanovski polynomials is in agreement with the result obtained using NU method. The effect of Poschl-Teller non-central potential is dominantly coming from the $\sec^2 \theta$ term which changes the state of the system, as shown in Figure 3, while the effect of $\cos \theta \sec \theta$ is similar with the effect of quantum magnetic number, $m$, as shown in equation (58). The energy eigenvalue decreases by the increase of the values of $\kappa$, $m$, and $\eta$. The angular wave function of Eckart potential plus trigonometric Poschl-Teller non-central potential obtained using Romanovski polynomials is in agreement with the result calculated using NU method. The associated Legendre polynomials is produced from angular wave function by setting $\kappa = \eta = 0$. This potential can be applied in an optical media, particularly as a model for graded refractive index [32].

5. Conclusion

The three dimensional Schrodinger equation for Eckart plus trigonometric Poschl-Teller non-central potential is solved approximately using Romanovski polynomials with suitable coordinate transformation. The energy spectrum is obtained in the closed form, the radial wave function is solved approximately and given in term of Romanovski polynomials. The angular wave function is expressed in term of Romanovski polynomials. The presence of Poschl-Teller non-central potential causes the change of the angular state wave function and the decrease of the energy eigenvalue.

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Acknowledgement
This work is partially supported by DIPA BLU Post Graduate Program Sebelas Maret University grant No. 2340/UN27.10/PG/2012