A few results concerning the Schur stability of the Hadamard powers and the Hadamard products of complex polynomials

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Abstract

For a complex polynomial

\[ f(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \]

and for a rational number \( p \), we consider the Schur stability problem of the \( p \)-th Hadamard power of \( f \)

\[ f^{[p]}(s) = s^n + a_{n-1}^ps^{n-1} + \ldots + a_1^ps + a_0^p. \]

We show that there exist two numbers \( p^* \geq 0 \geq p_* \) such that \( f^{[p]} \) is Schur stable for every \( p > p^* \) and is not Schur stable for \( p < p_* \) (or vice versa, depending on \( f \)). Also, we give simple sufficient conditions for the Schur stability of the Hadamard product of two complex polynomials. Numerical examples complete and illustrate the results.

1 Introduction

Over two decades ago, in 1996, Garloff and Wagner [1] provided an interesting property of the Hurwitz stable polynomials. They proved that the Hadamard product (i.e. element-wise multiplication) of two real Hurwitz stable polynomials is again Hurwitz stable. An immediate consequence of the Garloff-Wagner result is that the stability of \( f \) implies that of \( f^{[p]} \), the \( p \)-th Hadamard power of \( f \), for every positive integer \( p \). Gregor and Tišer [4] claimed that even more is true, that is, that the \( p \)-th Hadamard power of a Hurwitz stable polynomial is Hurwitz stable for every real power \( p > 1 \). Unfortunately, as Bialas and

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Białas-Cież proved in their recent work [5], they were wrong, i.e. for a stable polynomial \( f \), the polynomial \( f[p] \) does not need to be Hurwitz stable for \( p > 1 \).

Motivated by the work of Białas and Białas-Cież, we will focus the attention on the Schur stability problem of the Hadamard powers of complex polynomials. It is known that the result of Garloff and Wagner does not extend neither to the complex case nor to the class of the Schur stable polynomials (see Bose and Gregor [2] or again Garloff and Wagner [1]). The main aim of this work is to show that for a very wide class of complex polynomials including, among others, unstable elements, it is possible to find two numbers \( p^* \geq 0 \geq p_* \) depending on \( f \) and such that the \( p \)-th Hadamard power of \( f \) is Schur stable for every \( p > p^* \) and is not Schur stable for every \( p < p_* \) (or vice versa). Some attention is also paid to possibility of construction of families of the Schur stable polynomials with complex coefficients that are closed under the Hadamard multiplication. The obtained results complete and generalize those given by Garloff and Wagner [1], Gregor and Tišer [4] and Białas and Białas-Cież [5].

2 Preliminary results

2.1 Basic notations

We use standard notation: \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \) stand for the set of rational numbers, real numbers and complex numbers, respectively; \( \pi_n(\mathbb{C}) \) stands for the family of \( n \)-th degree monic polynomials with complex coefficients; \( |·| \) denotes the moduli of a complex number and \( i \) stands for the imaginary unit.

2.2 Stable polynomials

A polynomial is said to be Schur stable (shortly stable) if all its zeros lie in the open unit disc. From among many sufficient conditions for the stability of a polynomial we recall the one following from the bound for the moduli of zeros of polynomials given by Fujiwara [6]: a polynomial \( f \in \pi_n(\mathbb{C}) \) of the form \( f(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \) is stable if it satisfies the stability condition, i.e. there exist \( \{\lambda_k\} \), a sequence of positive numbers whose sum does not exceed 1, such that the following condition holds

\[
|a_k| < \lambda_k, \quad \text{for } k \in N_f
\]

where \( N_f = \{k \in \{0, \ldots, n-1\} : a_k \neq 0\} \) (the proof of sufficiency of (1) for the stability of \( f \) can be easily derived from Fujiwara’s work [6], but for the sake of completeness of the article we present it in Appendix A). It seems to be interesting and important to note that the stability condition (1) is sharp in the sense that for every \( \varepsilon \geq 0 \) and for every sequence of positive numbers \( \lambda_0, \ldots, \lambda_{n-1} \) summing up to \( 1 + \varepsilon \), the polynomial \( s \to s^n - \sum_{k=0}^{n-1} \lambda_k s^k \) is unstable.
2.3 The Hadamard product and the Hadamard powers of polynomials

For two polynomials $f, g \in \pi_n(\mathbb{C})$,
\begin{align*}
f(s) &= s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \\
g(s) &= s^n + b_{n-1}s^{n-1} + \ldots + b_1s + b_0
\end{align*}
we define their Hadamard product $f \circ g$ as an $n$-th degree polynomial of the form
\[(f \circ g)(s) = s^n + a_{n-1}b_{n-1}s^{n-1} + \ldots + a_1b_1s + a_0b_0.\]

In turn, for $p \in \mathbb{Q}$ the polynomial $f[p]$ given by
\[f[p](s) = s^n + a_{n-1}p s^{n-1} + \ldots + a_1p s + a_0p,
\]
is called the $p$-th Hadamard power of $f$ (we put, by definition, that $0^p = 0$ for $p \in \mathbb{Q}$). If $p$ is an integer then $f[p]$ is a polynomial. However, if $p$ is a non-integer rational number, say $p = k/m$ with $k$ and $m$ relatively prime integers, then $p$-th power of the complex number $a_i$ is not a number but is a set of $m$ complex numbers whose $m$-th power gives $a_{i,k}^m$. In other words, for $a_j = |a_j|(\cos \alpha_j + i \sin \alpha_j)$ we have $a_j^p = \{a_{j,0}^p, \ldots, a_{j,m-1}^p\}$, where
\[a_{j,l}^p = |a_j|^p (\cos (p \alpha_j + 2\pi l/m) + i \sin (p \alpha_j + 2\pi l/m)),
\]
for $l = 0, \ldots, m - 1$. In that case, the $p$-th Hadamard power of a polynomial should be understood as a set of $m^n$ polynomials
\[s \to s^n + a_{n-1,1}^p s^{n-1} + \ldots + a_1^p s + a_0^p\]
whose coefficients are calculated as in (3).

3 Main results

3.1 The Schur stability of the Hadamard powers of a polynomial

Let, for $f$ as in (2) and for $N_f$ as on page 2,
\[\Lambda_f = \left\{ \{\lambda_k\}_{k \in N_f} : \lambda_k \in (0,1], \sum_{k \in N_f} \lambda_k \leq 1 \right\}.
\]

We are now ready to formulate the main result of this work.

Theorem 1 For $f \in \pi_n(\mathbb{C})$ as in (2) the following hold:

(a) if $N_f$ is non-empty and $|a_k| < 1$ for $k \in N_f$, then $f[p]$ is Schur stable for every $p > p^*_\max$ where
\[p^*_\max = \inf_{\lambda_k \in \Lambda_f} \max_{k \in N_f} \frac{\ln \lambda_k}{\ln |a_k|};\]
(b) if \( N_f \) is non-empty and \( |a_k| > 1 \) for \( k \in N_f \), then \( f^{[p]} \) is Schur stable for every \( p < p^*_\text{min} \leq 0 \) where

\[
p^*_\text{min} = \sup_{(\lambda_k) \in \Lambda_f} \min_{k \in N_f} \frac{\ln \lambda_k}{\ln |a_k|};
\]

(c) if \( N_f \) is empty, then \( f^{[p]} \) is stable for every \( p \in \mathbb{Q} \).

**Proof.** If \( N_f \) is empty then the result is obvious. Suppose thus that \( N_f \) is non-empty. We can restrict our considerations to the real polynomial \( s \to s^n + \sum_{k=0}^{n-1} |a_k| s^k \) and its \( p \)-th power \( s \to s^n + \sum_{k=0}^{n-1} r_k s^k \) being a polynomial with nonnegative coefficients. Indeed, if the real polynomial \( s \to s^n + \sum_{k=0}^{n-1} r_k s^k \) satisfies the stability condition, then every complex polynomial whose \( k \)-th coefficient has the moduli equal to \( r_k \) (for \( k = 0, \ldots, n-1 \)) satisfies it too. In other words, the polynomial \( s \to s^n + \sum_{k=0}^{n-1} |a_k|^p s^k \) satisfies the stability condition if and only if each polynomial of the form \( (4) \), and thus \( f^{[p]} \), does.

Let \( \{\lambda_k\} \) be an arbitrary element of \( \Lambda_f \). The stability condition applied to the polynomial \( f^{[p]} \) gives

\[
p \ln |a_k| < \ln \lambda_k
\]

for \( k \in N_f \). If \( |a_k| < 1 \) for \( k = 0, \ldots, n-1 \), then \( (7) \) leads to

\[
p > \max_{k \in N_f} \frac{\ln \lambda_k}{\ln |a_k|}
\]

and in case \( |a_k| > 1 \) for \( k \in N_f \), it leads to

\[
p < \min_{k \in N_f} \frac{\ln \lambda_k}{\ln |a_k|}.
\]

Since it is sufficient for the stability of \( f^{[p]} \) that inequality \( (8) \) or inequality \( (9) \) holds for at least one sequence \( \{\lambda_k\} \), we can repeat the same for every \( \{\lambda_k\} \in \Lambda_f \) and take in \( (8) \) infimum over all \( \{\lambda_k\} \) and supremum over all \( \{\lambda_k\} \) in \( (9) \). This yields to (a) and (b).

**Remark 1** As it is known (see Example 5.3 in Saydy et al. [7]), in the entire family of real polynomials having all roots in the closed unit disc, the so-called guardian map

\[
\Phi : f \to f(1)f(-1) \det D_f,
\]

where \( D_f \) is some real matrix of order \( n-1 \) formed from the coefficients of \( f \), vanishes if and only if \( f \) is unstable (has a root on the unit circle). Thus, when for a real polynomial \( f \) with nonnegative coefficients there exists, as in Theorem [7] a number \( p^* \) for which \( f^{[p]} \) is stable for \( p > p^* \) (or for \( p < p^* \)) then the minimal (maximal) value of such \( p^* \) can be calculated as the maximal (minimal) real zero of the function

\[
\Phi_f : p \to f^{[p]}(1)f^{[p]}(-1) \det D_{f^{[p]}}.
\]
In case of a complex polynomial \( f \) and its integer Hadamard powers, such \( p^* \), if any, can be calculated as the maximal (minimal) real zero of the function

\[
\Phi_f : p \rightarrow f_{\text{Re}}(1) f_{\text{Re}}(-1) \det D_f^{[p]} ,
\]

where \( f_{\text{Re}} = \bar{f} \cdot f \) and \( \bar{f} \) is a polynomial whose coefficients are complex conjugates of these of \( f \).

The next theorem shows that the assumptions of Theorem 1 are relevant.

**Theorem 2** For \( f \in \pi_n(\mathbb{C}) \) as in (2) the following hold:

(a) if the set \( N_f \) is non-empty and \( k^* = \min \{ k : k \in N_f \} \), then \( f^{[p]} \) is not Schur stable for every \( p \leq 0 \) if \( |a_{k^*}| \leq 1 \) and for every \( p \geq 0 \) if \( |a_{k^*}| \geq 1 \);

(b) if the set \( N_{f,1+} = \{ k \in N_f : |a_k| > 1 \} \) is non-empty, then \( f^{[p]} \) is not Schur stable for every \( p \geq \beta_{\text{max}}^* \geq 0 \), where

\[
\beta_{\text{max}}^* = \min_{k \in N_{f,1+}} \frac{\ln \left( \binom{n}{k} \right)}{\ln |a_k|} .
\]

(c) if the set \( N_{f,1-} = \{ k \in N_f : |a_k| < 1 \} \) is non-empty, then \( f^{[p]} \) is not Schur stable for every \( p \leq \beta_{\text{min}}^* \leq 0 \) where

\[
\beta_{\text{min}}^* = \max_{k \in N_{f,1-}} \frac{\ln \left( \binom{n}{k} \right)}{\ln |a_k|} .
\]

**Proof.** Since \( a_{k^*} \) is, with accuracy to the sign, a product of all nonzero roots of the polynomial \( f \), condition (a) is obvious. To prove (b) and (c), recall that a necessary condition for the stability of \( f \) is that \( |a_k| < \binom{n}{k} \), for \( k = 0, \ldots, n-1 \). In other words, if for some \( k \in N_f \),

\[
p \ln |a_k| \geq \ln \binom{n}{k} ,
\]

then \( f^{[p]} \) is not stable. For \( k \in N_{f,1+} \) (10) follows from \( p \geq \beta_{\text{max}}^* \) proving (b) and for \( k \in N_{f,1-} \) from \( p \leq \beta_{\text{min}}^* \) proving (c).

### 3.2 The Schur stability of the Hadamard product of polynomials

Now, we will focus the attention on the stability of the Hadamard product \( f \circ g \) of two complex polynomials \( f, g \in \pi_n(\mathbb{C}) \). As mentioned in the introductory section, the Hadamard product of two stable (real or complex) polynomials does not have to be stable. In case of real polynomials, Gregor and Bose [2] noted that when multiplying, in the Hadamard sense, the Hadamard product \( f \circ g \) of two Schur stable polynomials \( f \) and \( g \) by the polynomial \( h(x) = \sum_{k=0}^{n-1} \binom{n}{k} x^k \),
then the product $f \circ g \circ h$, called sometimes the Szegö product of $f$ and $g$, becomes Schur stable.

The following theorem gives simple sufficient conditions for the Schur stability of both the Hadamard and the Szegö product of two complex polynomials.

**Theorem 3** Let $f, g \in \pi_n(\mathbb{C})$ be two polynomials of the form (2).

(a) If $f$ satisfies the stability condition and $|b_k| \leq 1$ for $k \in N_g \cap N_f$, then both the Szegö product and the Hadamard product of $f$ and $g$ satisfy the stability condition (and thus are stable).

(b) If $f$ satisfies the stability condition and $|b_k| \leq {n \choose k}$ for $k \in N_g \cap N_f$, then the Szegö product of $f$ and $g$ satisfies the stability condition (and thus is stable). In particular, if $f$ satisfies the stability condition and $g$ is stable, then the Szegö product of $f$ and $g$ satisfies the stability condition (and thus is stable).

(c) If $f$ and $g$ satisfy the following condition

$$\max \{|a_k|, |b_k|\} < \sqrt{\lambda_k}, \text{ for every } k \in N_f \cap N_g,$$

where $\{\lambda_k\}$ is a sequence of positive numbers whose sum does not exceed 1, then both the Szegö product and the Hadamard product of $f$ and $g$ satisfy the stability condition (and thus are stable).

Instead of the proof, which is a simple consequence of the stability condition, we make some remarks.

Firstly, note that the assumptions on $g$ in Theorem 3(a) and Theorem 3(b) do not imply its stability. It means that for the Schur stability of the Hadamard product or the Szegö product of two complex polynomials $f$ and $g$, it suffices to require slightly more than the stability of $f$ and slightly less than the stability of $g$. As we know, the stability of $f$ and $g$ does not suffice.

Note also, that the assumption on $f$ and $g$ in Theorem 3(c) does not guarantee their stability. Theorem 3(c) can be thus viewed as a sufficient condition for the stability of the Hadamard product and the Szegö product of two (unstable) polynomials.

We close this part with the following conclusion (its simple proof based on the stability condition is omitted).

**Conclusion 4** For every non-zero polynomial $f \in \pi_n(\mathbb{C})$ there exists a stable polynomial $g \in \pi_n(\mathbb{C})$ such that both the Szegö product and the Hadamard product of $f$ and $g$ are stable.

### 3.3 Does it work for polynomials of fractional orders?

At the end, let us note that all the above results can also be applied to fractional-order polynomials.
Recall that a fractional-order polynomial is a function of the form
\[ f : s \rightarrow s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0, \]  
where \( a_0, \ldots, a_{n-1} \) are known coefficients and \( \sigma_n > \sigma_{n-1} > \ldots > \sigma_1 > 0 \) are known powers being real numbers. The polynomials of non-integer order play an important role in the stability analysis of linear time-invariant fractional-order systems (e.g. Matignon [3]) and have recently attracted lots of attention in the control theory literature.

If at least one power in (11) is non-integer, then the fractional-order polynomial \( f \) is a multivalued function. Supposing that \( \sigma_k = \alpha k \) for some positive number \( \alpha \) (\( f \) is then said to be of a commensurate order) and substituting \( s^\alpha = w \) in (11), we obtain an integer-order polynomial \( F_f \) associated with \( f \)
\[ F_f (w) = w^n + a_{n-1}w^{n-1} + \ldots + a_1w + a_0. \]

As \( \alpha \) is a rational number, every root of \( F_f \) gives a finite set of roots of \( f \) (as in (3)). Moreover, according to \( s^\alpha = w \), \( f \) is Schur stable if and only if \( F_f \) is. This shows that Theorems 1–3 and Conclusion 4 can be applied to both integer-order and fractional-order polynomials.

4 Numerical experiments

In closing, we shall give two numerical examples completing and illustrating the results developed in this work.

**Example 1** Consider two real polynomials \( f \) and \( g \)
\[ f (s) = s^5 + 0.9s^2 + 0.2s + 0.7, \]
\[ g (s) = s^5 + 2.5s^2 + 2s + 3, \]
both having zeros outside the unit disc and thus unstable. In order to illustrate Theorem 4 we need to approximate value \( \lambda_k \) for \( f \) and value \( \lambda_k \) for \( g \). The approximations were obtained by generating sequences \( \{ \lambda_k \} \) of the form \( \{ ml/n^2, m(n-l)/n^2, (n-m)/n \} \) for \( n = 10^3 \) and \( m, l = 1, \ldots, n-1 \), and performing necessary computations. The approximation of (4) for \( f \) is \( p_{\text{max}}^* \approx 3.40372 \), whereas the minimal value of \( p^* \) such that \( f[p] \) is stable for every \( p > p^* \) (see Remark 1) is equal to \( p^* \approx 3.35457 \). The approximation of (6) for \( g \) is \( p_{\text{min}}^* \approx -1.24121 \), whereas the minimal value of \( p^* \) such that \( g[p] \) is stable for every \( p < p^* \) is equal to \( p^* \approx -1.01579 \).

**Example 2** Consider two complex polynomials \( f \) and \( g \)
\[ f (s) = s^4 + (0.2 - 0.4i)s^3 + 0.7s - 0.9i, \]
\[ g (s) = s^4 - 1.5s^3 + (2 - i)s^2 + 1 - 0.5i. \]
Proceeding as in Example 1 we get $p^*_{\text{max}} \approx 3.69323$ for $f$ and $p^*_{\text{min}} \approx -3.40696$ for $g$. To confirm the results we have plotted in Fig. 1 the zeros of $f^{[p]}$ and $g^{[q]}$ for integer values of $p \in \{1, \ldots, 10^2\}$ and $q \in \{-10^2, \ldots, -1\}$. The $p$-th Hadamard power of $f$ occurs unstable for $1 \leq p \leq 3$ and becomes stable for $p \geq 4$, as expected. Similarly, the $q$-th Hadamard power of $g$ occurs unstable for $-1 \leq q \leq -3$ and becomes stable for $q \leq -4$.

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Appendix A

To prove that \[ (1) \] is a sufficient condition for the Schur stability of the complex polynomial $f$ of the form

$$ f(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 $$

note that

$$ |f(s)| \geq |s|^n - \sum_{k \in N_f} |a_k| |s|^k, $$

where $N_f = \{k \in \{0, \ldots, n-1\} : a_k \neq 0\}$. It means that if for every $k \in N_f$

$$ \lambda_k |s|^n > |a_k| |s|^k, $$
where \( \{ \lambda_k \} \) is a sequence of positive numbers whose sum does not exceed 1, then \( f(s) \neq 0 \). In other words, if \( s \) is a zero of \( f \) then

\[
|s| \leq \max_{k \in N_f} \left( \frac{|a_k|}{\lambda_k} \right)^{1/n - k}.
\]

If \( |a_k| < \lambda_k \) for \( k \in N_f \), then every zero of \( f \) has moduli less than 1 and thus \( f \) is Schur stable. This completes the proof.

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