Comments on multibrane solutions in cubic superstring field theory

E. Aldo Arroyo

Centro de Ciências Naturais e Humanas, Universidade Federal do ABC
Santo André, 09210-170 São Paulo, SP, Brazil

Abstract

In a previous paper, we have studied multi-brane solutions in the context of cubic superstring field theory. The kinetic term of the action was computed for these multi-brane solutions, and for the evaluation of the energy, the equation of motion contracted with the solutions itself was simply assumed to be satisfied. In this paper, we compute the cubic term of the action and discuss the validity of the previous assumption. Additionally, we evaluate the Ellwood’s gauge invariant observable.
1 Introduction

Schnabl’s work on the first analytic solution in open bosonic string field theory [1] can be considered the first step towards the analytic understanding of string field theory. After the publication of Schnabl’s seminal paper, a remarkable amount of work has been done concerning the analysis of the tachyon vacuum solution and the construction of associated solutions by algebraic techniques [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. For instance, the tachyon vacuum solution was rewritten in terms of basic string fields constructed out of elements in the $\mathcal{KBc}$ subalgebra [15, 16, 17, 18, 19, 20]. Using the elements of this subalgebra, Murata and Schnabl have constructed a family of solutions known as the multi-brane solutions [21]. Depending on the analytic properties of a function which parameterizes the solutions, it has been shown that the evaluation of the energy leads to an answer compatible with solutions that describe multiple coincident D-branes.

Although various calculations associated with the multi-brane solutions, such as the evaluation of gauge-invariant observables, provide expected results, there are subtleties involved in the computations. Since the solutions can have expressions which are either divergent or anomalous, they must be treated with due care. In a recent set of papers [22, 23, 24], the authors have analyzed the existence of possible anomalies in the evaluation of the gauge-invariant observables. The origin of these anomalies are related to the violation of some regularity conditions imposed on the function that parameterizes the solutions [22]. As Murata and Schnabl have pointed out the status of the multi-brane solutions might be analogous to the tachyon vacuum solution without the phantom term.

The construction of analytic solutions in the modified cubic superstring field theory [25] naively follows the prescriptions used in the bosonic case. Since these two theories
have a similar cubic-like interaction term and the string field products are based on
Witten’s associative star product \cite{20}, the bosonic results admit quite straightforward
extensions to the superstring case \cite{27, 28, 30, 31}. For instance, the $KBC$ subalgebra
can be extended to the $KBC\gamma$ subalgebra which includes the superstring ghost field $\gamma$
\cite{32, 33, 34, 35}. Using this subalgebra, we have studied the multi-brane solutions in the
context of the modified cubic superstring field theory \cite{36}.

As in the bosonic case, by evaluating the energy associated to these solutions, we have
shown that the solutions can be interpreted as describing multiple coincident D-branes.
Nevertheless, for the evaluation of the energy, the equation of motion contracted with the
solutions itself was simply assumed to be satisfied. In this paper, we compute the cubic
term of the action and discuss the validity of the previous assumption. Additionally, we
evaluate the Ellwood’s gauge invariant observable \cite{37} for the multi-brane solutions. It
turns out that the energy computed from the action and from the Ellwood’s invariant will
agree provided that the function that parameterizes the multi-brane solutions satisfies
appropriate holomorphicity conditions that are similar to the bosonic case \cite{22}.

The paper is organized as follows. In section 2, we review the multi-brane solutions
in the modified cubic superstring field theory. In section 3, we compute the cubic
term of the action for the multi-brane solutions. In section 4, the Ellwood’s gauge invariant
overlap for the multi-brane solutions will be evaluated. In section 5, a summary and
further directions of exploration are given.

2 Review of the multi-brane solutions in the cubic
superstring field theory

In this section, a short review of the multi-brane solutions in the modified cubic super-
string field theory will be given. In our previous paper \cite{36}, using the prescription
developed in reference \cite{34}, we have derived the multi-brane solutions by performing a
gauge transformation over an identity based solutions. Here instead of employing that
prescription, we will adopt the standard procedure, namely we are going to write the so-
lutions as a pure gauge form. It turns out that solutions given in this way naively satisfy
the string field equation of motion \cite{2}.

Since the algebraic structure of the modified cubic superstring field theory is similar
to the open bosonic string field theory, the bosonic results admit quite straightforward
extensions to the superstring case. For instance, the $KBC$ subalgebra of the bosonic string
field theory can be extended to the $KBC\gamma$ subalgebra which includes the superstring ghost
field $\gamma$ \cite{19, 27, 30, 34}.

Employing the elements of the $KBC\gamma$ subalgebra, we construct a rather generic so-
lution which can be written as a pure gauge form $\Psi = UQU^{-1}$ with the string field $U$
defined by

\[ U = 1 - F B_c F, \quad U^{-1} = 1 + \frac{F}{1 - F^2} B_c F, \]  

(2.1)

where \( F \) is a function of \( K \), and \( B, c \) are the elements of the \( KB_c \gamma \) subalgebra. These basic string fields satisfy the usual algebraic relations

\[
\{ B, c \} = 1, \quad [B, K] = 0, \quad B^2 = c^2 = 0, \\
\partial c = [K, c], \quad \partial \gamma = [K, \gamma], \quad [c, \gamma] = 0, \quad [B, \gamma] = 0,
\]

(2.2)

and have the following BRST variations

\[
Q_K = 0, \quad QB = K, \quad Qc = cKc - \gamma^2, \quad Q\gamma = c\partial\gamma - \frac{1}{2}\gamma\partial c.
\]

(2.3)

Performing some algebraic manipulations with these basic string fields, and using equations (2.1)-(2.3) we can write the following solution

\[
\Psi = F c KB_1 - F_2 c F + F B_2 \gamma^2 F,
\]

(2.4)

which formally satisfies the string field equation of motion \( Q\Psi + \Psi \Psi = 0 \), where \( Q \) is the BRST operator of the open Neveu-Schwarz superstring theory. Since the solution for the superstring case (2.4) is almost similar to the bosonic solution \( \Psi_{bos} = F c \frac{Kc}{1 - F^2} c F \), the second term on the right-hand side of equation (2.4) is commonly known as the superstring correction [27].

In the framework of the modified cubic superstring field theory, the solution (2.4) has been studied for the specific cases: \( F^2 = e^{-K} \) and \( F^2 = 1/(1 + K) \), where it was shown that the solution characterizes the tachyon vacuum solution [27, 30]. It is interesting to note that, as argued in reference [27], from an analytic perspective the proposed tachyon vacuum solution in the modified cubic superstring field theory appears to be as regular as Schnabl’s original solution for the bosonic string. Nevertheless, from the perspective of the level expansion the situation is unclear, though to be honest, the analysis of the energy for the tachyon vacuum solution using the usual Virasoro \( L_0 \) level expansion has not yet been carried out. Relevant considerations related to the gauge equivalence of the tachyon vacuum solutions were properly analyzed in reference [33].

The evaluation of the energy for a class of analytic solutions of the form (2.4) for a generic function \( F(K) \) was performed in reference [36]. Nevertheless, for the computation of the energy, the equation of motion contracted with the solution itself was simply assumed to be satisfied. To test the validity of this assumption, we need to explicitly show that

\[
\langle \Psi Q \Psi \rangle + \langle \Psi \Psi \Psi \rangle = 0.
\]

(2.5)

In the previous paper [36], only the kinetic term \( \langle \Psi Q \Psi \rangle \) was computed. And therefore, it remains the computation of the cubic term \( \langle \Psi \Psi \Psi \rangle \). This calculation will be performed in the next section.
3 Evaluation of the cubic term of the action

Although the solution (2.4) can be written as a pure gauge form \( \Psi = UQU^{-1} \) such that formally satisfies the string field equation of motion \( Q\Psi + \Psi\Psi = 0 \), it is not a trivial task to test if the equation of motion contracted with the solution itself is satisfied. In general, a priori there is no justification for assuming the validity of \( \langle \Psi Q\Psi \rangle + \langle \Psi\Psi\Psi \rangle = 0 \) without an explicit calculation. Therefore the cubic term of the action must be evaluated.

Before computing the cubic term of the action for the multi-brane solutions, we are going to calculate a correlator that will be very useful for the evaluation of the cubic term. The definition of the considered correlator is as follows

\[
\langle G_1, G_2, G_3 \rangle = \langle \langle BG_1(K)cG_2(K)cG_3(K)\gamma^2 \rangle \rangle, \tag{3.1}
\]

for a general set of functions \( G_i(K) \). The inclusion of notation \( \langle \langle \cdots \rangle \rangle \) refers for a standard correlator with the difference that we have to insert the operator \( Y_{-2} \) at the open string midpoint. The operator \( Y_{-2} \) can be given as the product of two inverse picture changing operators, \( Y_{-2} = Y(i)Y(-i) \), with \( Y(z) = -\partial_\xi e^{-2\phi}c(z) \).

Let us define all functions \( G_i(K) \) as an integral representation of a continuous superposition of wedge states,

\[
G_i(K) = \int_0^\infty dt_i g_i(t_i) e^{-t_i K}. \tag{3.2}
\]

Formally equation (3.2) can be thought as a Laplace transform. The validity of this representation depends on specific holomorphicity conditions imposed on the functions \( G_i(K) \). Detailed discussions regarding to these conditions were studied in reference [8]. However at this point, let us simply assume that the functions \( G_i(K) \) satisfied the preceding conditions.

Replacing the integral representation of the functions \( G_i \)’s (3.2) into (3.1), we obtain the following triple integral

\[
\langle G_1, G_2, G_3 \rangle = \int_0^\infty dt_1 dt_2 dt_3 g_1(t_1)g_2(t_2)g_3(t_3)\langle \langle Be^{-t_1 K}ce^{-t_2 K}ce^{-t_3 K}\gamma^2 \rangle \rangle. \tag{3.3}
\]

The correlator \( \langle \langle Be^{-t_1 K}ce^{-t_2 K}ce^{-t_3 K}\gamma^2 \rangle \rangle \) has been evaluated in references [27, 30, 34]

\[
\langle \langle Be^{-t_1 K}ce^{-t_2 K}ce^{-t_3 K}\gamma^2 \rangle \rangle = \frac{s}{2\pi^2}t_2, \text{ where } s = t_1 + t_2 + t_3. \tag{3.4}
\]

Next we are going to use the s-z trick developed in [21, 22]. Essentially the trick tells us to insert the identity

\[
1 = \int_0^\infty ds \delta \left( s - \sum_{i=1}^3 t_i \right) = \int_0^\infty ds \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} e^{sz} e^{-z \sum_{i=1}^3 t_i}, \tag{3.5}
\]
into the triple integral \((3.3)\). This identity allows us to treat the variable \(s\) as independent of the other integration variables \(t_i\). Employing the correlator \((3.4)\) and inserting the identity \((3.5)\) into \((3.3)\), we get

\[
\frac{1}{2\pi^2} \int_0^\infty dt_1 dt_2 dt_3 g_1(t_1) t_2 g_2(t_2) g_3(t_3) \int_0^\infty ds \int_{i\infty}^{+i\infty} \frac{dz}{2\pi i} se^{sz} e^{-z \sum_{i=1}^3 t_i}. \tag{3.6}
\]

Carrying out the integral over the variables \(t_i\) and rewriting the result in terms of the functions \(G_i(z)\), we obtain

\[
\langle G_1, G_2, G_3 \rangle = -\frac{1}{2\pi^2} \int_0^\infty ds \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} s e^{sz} G'_2(z) G_1(z) G_3(z). \tag{3.7}
\]

Note that this correlator is simpler than the one derived in the bosonic case, where trigonometric functions are involved and produce lengthy results for the corresponding correlator \([21, 22]\). With the aid of the above formula \((3.7)\), we are in position to evaluate the cubic term of the action for the multi-brane solutions.

Plugging the solution \((2.4)\) into the cubic term of the action \(\langle \Psi \Psi \Psi \rangle\) and employing the relations \((2.2)\), after performing some algebraic manipulations, we obtain

\[
\langle \Psi \Psi \Psi \rangle = 3 \langle \frac{K}{G}(1 - G), (1 - G), \frac{K}{G}(1 - G) \rangle, \tag{3.8}
\]

where \(G = 1 - F^2\).

For the correlator given on the right-hand side of equation \((3.8)\) the functions \(G_i\)’s are identified by \(G_1 = \frac{K}{G}(1 - G)\), \(G_2 = (1 - G)\) and \(G_3 = \frac{K}{G}(1 - G)\). Once this identification has been made, the next step is to use the result \((3.7)\). And hence we arrive at the following expression for the cubic term

\[
\langle \Psi \Psi \Psi \rangle = -\frac{1}{2\pi^2} \int_0^\infty ds \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} s e^{sz} \left[ \frac{6G'(z)}{G(z)} - \frac{3G'(z)}{G(z)^2} - 3G'(z) \right]. \tag{3.9}
\]

Since the term inside the brackets does not depend on the variable \(s\), we can evaluate the integral over this variable, which is well defined for values of the variable \(z\) such that \(\text{Re}(z) < 0\). Performing the integral over \(s\), we obtain

\[
\langle \Psi \Psi \Psi \rangle = -\frac{1}{2\pi^2} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \left[ \frac{6G'(z)}{G(z)} - \frac{3G'(z)}{G(z)^2} - 3G'(z) \right]. \tag{3.10}
\]

At this stage, we are going to impose specific conditions on the corresponding functions. The motivation for demanding these conditions, as we are going to see, will be the fact that the energy computed from the action and from the Ellwood’s gauge invariant will agree provided that the function that parameterizes the multi-brane solutions satisfies holomorphicity conditions that are similar to the bosonic case \([22]\).
Let us assume that the function appearing in the expression of the cubic term of the action (3.10), can be written as \( G(z) = 1 + \sum_{n=1}^{\infty} a_n z^{-n} \), namely \( G \) is holomorphic at the point at infinity \( z = \infty \) and has a limit \( G(\infty) = 1 \). Under this condition, it is possible to make the integral along the imaginary axis into a sufficiently large closed contour \( C \) running in the counterclockwise direction by adding a large non-contributing half-circle in the left half plane such that \( \text{Re}(z) < 0 \), and consequently the integral (3.10) can be written as

\[
\langle \Psi \Psi \rangle = -\frac{1}{2\pi^2} \oint_C \frac{dz}{2\pi i} \left[ 6G''(z) - \frac{3G'(z)G(z)}{G(z)^2} - 3G'(z) \right]. \tag{3.11}
\]

Moreover by demanding two additional requirements for the functions \( G \) and \( 1/G \),

- \( G \) and \( 1/G \) are holomorphic in \( \text{Re}(z) \geq 0 \) except at \( z = 0 \).
- \( G \) or \( 1/G \) are meromorphic at \( z = 0 \).

We can stretch the \( C \) contour around infinity, picking up only a possible contribution from the origin,

\[
\langle \Psi \Psi \rangle = -\frac{1}{2\pi^2} \oint_{C_0} \frac{dz}{2\pi i} \left[ 6G''(z) \frac{G(z)}{G(z)} + 3\partial_z \left\{ \frac{1}{G(z)} - G(z) \right\} \right], \tag{3.12}
\]

where \( C_0 \) is a contour encircling the origin in the clockwise direction. As shown explicitly, the second term appearing in the integrand on the right-hand side of (3.12) is a total derivative term with respect to \( z \) such that the contour integral of that term usually vanishes. In fact, since we assume the meromorphicity of \( G(z) \) at the origin, this total derivative term vanishes. Now inverting the direction of the contour \( C_0 \), we finally obtain

\[
\langle \Psi \Psi \rangle = \frac{3}{\pi^2} \oint dz \frac{G'(z)}{2\pi i G(z)}. \tag{3.13}
\]

In order to calculate the contour integral (3.13), we need to follow a closed curve encircling the origin in the counterclockwise direction.

Let us remember that under the same holomorphicity conditions satisfied by the function that parameterizes the multi-brane solutions, the kinetic term of the action was computed in reference [36]

\[
\langle \Psi Q \Psi \rangle = -\frac{3}{\pi^2} \oint dz \frac{G'(z)}{2\pi i G(z)}. \tag{3.14}
\]

Therefore adding equations (3.13) and (3.14), we conclude that the assumption of the validity of the equation of motion contracted with the solution itself was correct provided that the function that parameterizes the multi-brane solutions satisfies the aforementioned holomorphicity requirements.
3.1 Discussing the result for the cubic term

The final result (3.13) for the cubic term relies on the validity of the step from equation (3.9) to (3.10). The integrand in equation (3.9) can have poles at $z = 0$ for a function $G(z)$ satisfying the three holomorphicity conditions previously given. To avoid the singularities at $z = 0$, we have simply shifted the integration over $z$, which is originally along $\text{Re}(z) = 0$, to that along $\text{Re}(z) < 0$. This procedure needs to be justified.

A similar observation for the result in the bosonic case [21, 22] has been made in Hata and Kojita’s paper [24]. To treat the points at $z = 0$, we use the property that the eigenvalue distribution of $K$ is restricted to real and non-negative [23, 24], and so we can replace $K \rightarrow K + \epsilon$, with $\epsilon$ being a positive infinitesimal. Now if we compute the cubic term with $K$ replaced by $K + \epsilon$ and take the limit $\epsilon \rightarrow 0$, we obtain

$$\langle \Psi \Psi \Psi \rangle = -\frac{1}{2\pi^2} \int_0^\infty ds \int_{i\infty}^{+i\infty} \frac{dz}{2\pi i} s e^{sz} \left[ \frac{6G'(z)}{G(z)} - \frac{3G'(z)}{G(z)^2} - 3G'(z) \right],$$

(3.15)

where the integration over $z$ is along a line parallel to the pure-imaginary axis with $\text{Re}(z) > 0$. Since $\epsilon > 0$, it is easy to see why in this case the integration must be along $\text{Re}(z) > 0$.

In order to simplify the notation, let us define the function $J(z)$ as

$$J(z) = -\frac{z^2}{2\pi^2} \left[ \frac{6G'(z)}{G(z)} - \frac{3G'(z)}{G(z)^2} - 3G'(z) \right].$$

(3.16)

Employing this definition (3.16) into equation (3.15), we write the cubic term as follows

$$\langle \Psi \Psi \Psi \rangle = \int_0^\infty ds \int_{C_>\infty} \frac{dz}{2\pi i} s e^{sz} J(z),$$

(3.17)

where the notation $C_>$ represents the curve corresponding to the line parallel to the pure-imaginary axis with $\text{Re}(z) > 0$. Let us also denote $C_<$ as the curve corresponding to the line parallel to the pure-imaginary axis with $\text{Re}(z) < 0$. Note that the integration over $z$ along the curve $C_<$ corresponds to the one used in passing of the step from equation (3.9) to (3.10).

By inverting the direction of the curve $C_<$ and joining its end points with the end points of curve $C_>$, we construct a large closed curve running in the counterclockwise direction. Since the integrand $se^{sz}J(z)$ can have poles at $z = 0$, the integration over $z$ along this large closed curve is equivalent to the integration along a closed curve encircling the origin in the counterclockwise direction,

$$\int_0^\infty ds \int_{C_>\infty} \frac{dz}{2\pi i} s e^{sz} J(z) - \int_0^\infty ds \int_{C_<\infty} \frac{dz}{2\pi i} s e^{sz} J(z) = \int_0^\infty ds \oint \frac{dz}{2\pi i} s e^{sz} J(z),$$

(3.18)

where we have the minus sign because by construction the left hand side of the large closed curve goes in the opposite direction of $C_<$. 

8
Employing equations (3.17) and (3.18), we see that the cubic term of the action is given by

\[ \langle \Psi \Psi \Psi \rangle = \int_0^\infty ds \oint \frac{dz}{2\pi i} s e^{sz} J(z) + \int_0^\infty ds \oint \frac{dz}{2\pi i} s e^{sz} J(z). \] (3.19)

The first term on the right hand side of equation (3.19) precisely corresponds to the term on the right hand side of equation (3.9), with the integration over \( z \) along \( \text{Re}(z) < 0 \). The desired result (3.10) is obtained provided that the second term on the right hand side of equation (3.19) vanishes. Thus, we need to prove that \( \mathcal{I} = 0 \), where \( \mathcal{I} \) is defined as the following integral

\[ \mathcal{I} = \int_0^\infty ds \oint \frac{dz}{2\pi i} s e^{sz} J(z). \] (3.20)

The result given by equation (3.19) is quite similar to the one obtained in the bosonic context [24]. Actually, using the \( K_\epsilon \)-regularization and the function

\[ G(K) = \left( \frac{K}{1 + K} \right)^n, \] (3.21)

the evaluation of the cubic term leads to the result

\[ \frac{\pi^2}{3} \langle \Psi \Psi \Psi \rangle_{\text{bosonic}} = n + \mathcal{A}_n, \] (3.22)

with

\[ \mathcal{A}_n = \frac{\pi^2}{3} n(1 - n^2) \text{Re}_\epsilon \mathcal{F}_1(2 - n, 4; 2\pi i), \] (3.23)

where \( \mathcal{F}_1(a, b; z) \) is the confluent hypergeometric function. Note that the expected result, \( \frac{\pi^2}{3} \langle \Psi \Psi \Psi \rangle_{\text{bosonic}} = n \), is obtained only for values of \( n \) such that \( n = 0, \pm 1 \). Let us see what happens for the superstring case.

Performing the replacement \( K \rightarrow K + \epsilon \), we obtain the following expression for the integral \( \mathcal{I} \rightarrow \mathcal{I}_\epsilon \)

\[ \mathcal{I}_\epsilon = \int_0^\infty ds \oint \frac{dz}{2\pi i} s e^{(z-\epsilon)s} J(z), \] (3.24)

where the part \( e^{-\epsilon s} \) comes from \( K \rightarrow K + \epsilon \). To evaluate this integral (3.24), we are going to use the function (3.21) which satisfies the aforementioned three holomorphicity conditions. This is the same function that has been used in the analysis of multibrane solutions in the bosonic case [21, 22, 23, 24].

Since the \( z \)-integration is a contour integral performed around a closed curve encircling the origin in the counterclockwise direction, to compute the integral over this variable \( z \), we need to write the Laurent series of the integrand around \( z = 0 \)

\[ s e^{(z-\epsilon)s} J(z) = \frac{I_n(s, \epsilon)}{z} + \sum_{p \neq -1} I_{p,n}(s, \epsilon) z^p, \] (3.25)
and pick up the coefficient $I_n(s, \epsilon)$ in front of the term $1/z$. Then by performing the $s$-integration, we obtain the value of the integral (3.24), namely $I_\epsilon = \int_0^\infty ds I_n(s, \epsilon)$.

With the aid of equations (3.16), (3.21) and (3.25), we are in position to explicitly evaluate the coefficient $I_n(s, \epsilon)$ for various values of $n$. It turns out that the coefficient $I_n(s, \epsilon)$ vanishes identically for values of $n = 0, \pm 1$. Let us see what happens if $|n| \geq 2$. For instance, with the values of $n = \pm 2, \pm 3, \pm 4$, we obtain the following expressions for the coefficients

\[
I_{\pm 2}(s, \epsilon) = \pm \frac{3se^{-\epsilon s}}{\pi^2},
\]
\[
I_{\pm 3}(s, \epsilon) = \pm \frac{9s(s + 2)e^{-\epsilon s}}{2\pi^2},
\]
\[
I_{\pm 4}(s, \epsilon) = \pm \frac{3s(s^2 + 6s + 6)e^{-\epsilon s}}{\pi^2}.
\]

Since $I_\epsilon = \int_0^\infty ds I_n(s, \epsilon)$, the value of the integral $I_\epsilon \to I$ in the limit $\epsilon \to 0$ is non-vanishing and divergent except for the cases where $n = 0, \pm 1$. In fact, with $n = \pm 2$ we obtain $I_\epsilon = \mp (3/\pi^2) \int_0^\infty ds se^{-\epsilon s} \propto 1/\epsilon^2$. This result, together with equation (3.19) implies that the validity of the step from equation (3.9) to (3.10) only follows when $n = 0, \pm 1$.

4 Evaluation of the Ellwood’s gauge invariant

In this section, the Ellwood’s gauge invariant overlap for the multi-brane solutions will be evaluated. A similar computation was done in reference [31] for the half-brane solution. The Ellwood’s gauge invariant overlap is given by

\[
W(\Psi, \mathcal{V}) = \text{Tr}(\Psi),
\]

where the notation $\text{Tr}(\cdots)$ is defined in the same way as the correlator (3.1) except the picture changing operator $Y_{-2}$ is replaced by an on shell closed string vertex operator $\mathcal{V}(i)$ inserted at the midpoint, $\text{Tr}(\Psi) = \langle \mathcal{V}(i) \Psi \rangle$. We assume the same $\mathcal{V}$ used in reference [31], this field is an NS-NS closed string vertex operator of the form

\[
\mathcal{V}(z) = c\bar{c}e^{-\phi}e^{-\bar{\phi}}\mathcal{O}^m,
\]

where $\mathcal{O}^m$ is a weight $(\frac{1}{2}, \frac{1}{2})$ superconformal matter primary field. As argued by Ellwood [37], the gauge invariant overlap represents the shift in the closed string tadpole of the solution relative to the perturbative vacuum.

Replacing the multi-brane solution (2.4) into the definition of the gauge invariant overlap (4.1), the term $\text{Tr}(FB\gamma^2F)$ does not contribute since we need three $c$’s fields
to saturate the corresponding correlator, and the insertion $V$ already has two $c$’s fields. Therefore, we obtain

$$W(\Psi, V) = \text{Tr}(F c \frac{K B}{1 - F^2} c F).$$

(4.3)

As for the evaluation of the cubic term of the action, let us write the functions $F$ and $K/G$ as an integral representation of a continuous superposition of wedge states,

$$F = \int_0^\infty dt f(t) e^{-tK},$$

(4.4)

$$K/G = \int_0^\infty dt g(t) e^{-tK},$$

(4.5)

where $G = 1 - F^2$. The validity of this assumption depends on the holomorphicity conditions satisfied by the functions. Replacing equations (4.4) and (4.5) into (4.3), we obtain

$$W(\Psi, V) = \int_0^\infty dt_1 dt_2 dt_3 f(t_1) g(t_2) f(t_3) \text{Tr}(e^{-t_1K} ce^{-t_2K} B ce^{-t_3K}).$$

(4.6)

The correlator $\text{Tr}(e^{-t_1K} ce^{-t_2K} B ce^{-t_3K})$ has been evaluated in reference [31] by using the usual scaling argument

$$\text{Tr}(e^{-t_1K} ce^{-t_2K} B ce^{-t_3K}) = (t_1 + t_3) \text{Tr}(c \Omega),$$

(4.7)

where $\Omega = e^{-K}$ and $\text{Tr}(c \Omega) = \langle V(i\infty)c(0) \rangle_{C_1}$ is the expected result of the closed string tadpole on the disk.

Replacing the correlator (4.7) into (4.6), we obtain the following expression for the gauge invariant overlap

$$W(\Psi, V) = \int_0^\infty dt_1 dt_2 dt_3 f(t_1) g(t_2) f(t_3)(t_1 + t_3) \text{Tr}(c \Omega).$$

(4.8)

To evaluate the right-hand side of equation (4.8), we use again the $s$-$z$ trick. Inserting the identity (3.5) into the triple integral (4.8), we obtain

$$\int_0^\infty dt_1 dt_2 dt_3 f(t_1) g(t_2) f(t_3) \int_0^\infty ds \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} e^{sz} e^{-z \sum_{i=1}^3 t_i} \text{Tr}(c \Omega).$$

(4.9)

Evaluating the integral over the variables $t_i$ and rewriting the result in terms of the functions $F(z)$ and $z/G(z)$, we get

$$W(\Psi, V) = -2 \int_0^\infty ds \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} e^{sz} F(z) F'(z) G(z) \text{Tr}(c \Omega)$$

$$= \int_0^\infty ds \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} e^{sz} G'(z) G(z) \text{Tr}(c \Omega),$$

(4.10)
where $G(z) = 1 - F^2(z)$. Evaluating the integral over the variable $s$, which is well defined for $\text{Re}(z) < 0$, we obtain

$$W(\Psi, \mathcal{V}) = -\int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \frac{G'(z)}{G(z)} \text{Tr}(c\Omega). \quad (4.11)$$

Employing the same holomorphicity conditions used in the evaluation of the cubic term of the action, we can take the integral along the imaginary axis into a sufficiently large closed contour $C$ running in the counterclockwise direction by adding a large non-contributing half-circle in the left half plane $\text{Re}(z) < 0$. So that the Ellwood’s gauge invariant overlap for the multi-brane solution (2.4) can be written as the following contour integral

$$W(\Psi, \mathcal{V}) = -\oint_{C} \frac{dz}{2\pi i} \frac{G'(z)}{G(z)} \text{Tr}(c\Omega). \quad (4.12)$$

Furthermore we can stretch the $C$ contour around infinity, picking up only a possible contribution from the origin,

$$W(\Psi, \mathcal{V}) = -\oint_{C_0} \frac{dz}{2\pi i} \frac{G'(z)}{G(z)} \text{Tr}(c\Omega), \quad (4.13)$$

where $C_0$ is a contour encircling the origin in the clockwise direction. Now inverting the direction of the contour $C_0$, we finally obtain

$$W(\Psi, \mathcal{V}) = \oint_{C_0} \frac{dz}{2\pi i} \frac{G'(z)}{G(z)} \text{Tr}(c\Omega). \quad (4.14)$$

As in the case of the expression for the cubic term (3.13), to compute the contour integral (4.14), we need to follow a closed curve encircling the origin in the counterclockwise direction.

Note that the final result for the Ellwood’s gauge invariant (4.14) depends on the holomorphicity conditions imposed on the function that parameterizes the multi-brane solutions. As in the bosonic case, it should be nice to analyze if the violation of some of these holomorphicity conditions leads to the appearance of anomalies associated to the evaluation of the gauge-invariant observable [22, 23, 24].

4.1 Discussing the result for the Ellwood’s gauge invariant

The final result for the Ellwood’s gauge invariant (4.14) relies on the validity of the step from equation (4.10) to (4.11). The integrand in equation (4.10) can have poles at $z = 0$ for a function $G(z)$ satisfying the three holomorphicity conditions previously given. To avoid the singularities at $z = 0$, we have simply shifted the integration over $z$, which is
originally along Re(\(z\)) = 0, to that along Re(\(z\)) < 0. As in the case of the cubic term, we need to justify this procedure.

Employing the same arguments developed for the case of the cubic term, we show that the Ellwood’s gauge invariant can be written as

\[
W(\Psi, \mathcal{V}) = \int_{0}^{\infty} ds \oint_{C_{<}} \frac{dz}{2\pi i} e^{sz} z \frac{G'(z)}{G(z)} \text{Tr}(c\Omega) + \int_{0}^{\infty} ds \oint_{C_{<}} \frac{dz}{2\pi i} e^{sz} z \frac{G'(z)}{G(z)} \text{Tr}(c\Omega). \tag{4.15}
\]

The first term on the right hand side of equation (4.15) precisely corresponds to the term on the right hand side of equation (4.10), with the integration over \(z\) along Re(\(z\)) < 0. The desire result (4.11) is obtained provided that the second term on the right hand side of equation (4.15) vanishes. Therefore, we need to prove that \(K = 0\), where \(K\) is defined by

\[
K = \int_{0}^{\infty} ds \oint_{C_{<}} \frac{dz}{2\pi i} e^{sz} z \frac{G'(z)}{G(z)} \text{Tr}(c\Omega). \tag{4.16}
\]

As for the cubic term, by performing the replacement \(K \rightarrow K + \epsilon\), we obtain the following expression for the integral \(K \rightarrow K_{\epsilon}\)

\[
K_{\epsilon} = \int_{0}^{\infty} ds \oint_{C_{<}} \frac{dz}{2\pi i} e^{s(z-\epsilon)} z \frac{G'(z)}{G(z)} \text{Tr}(c\Omega). \tag{4.17}
\]

where the part \(e^{-\epsilon s}\) comes from \(K \rightarrow K + \epsilon\). To evaluate this integral (4.17), let us use the function \(G(z)\) defined by equation (3.21).

Since the \(z\)-integration is a contour integral performed around a closed curve encircling the origin in the counterclockwise direction, to compute the integral over this variable \(z\), we need to write the Laurent series of the integrand around \(z = 0\)

\[
e^{s(z-\epsilon)} z \frac{G'(z)}{G(z)} \text{Tr}(c\Omega) = \left[ \frac{K_{n}(s, \epsilon)}{z} + \sum_{p \neq -1} K_{p,n}(s, \epsilon) z^{p} \right] \text{Tr}(c\Omega), \tag{4.18}
\]

and pick up the coefficient \(K_{n}(s, \epsilon)\) in front of the term \(1/z\). Then by performing the \(s\)-integration, we obtain the value of the integral (4.17), namely \(K_{\epsilon} = \int_{0}^{\infty} ds K_{n}(s, \epsilon) \text{Tr}(c\Omega)\).

With the aid of equations (3.21) and (4.18), we are in position to explicitly evaluate the coefficient \(K_{n}(s, \epsilon)\) for various values of \(n\). It turns out that the coefficient \(K_{n}(s, \epsilon)\) vanishes identically for any integer value of \(n\) (while for the case of the cubic term, only the coefficients with \(n = 0, \pm 1\) vanish identically). Since \(K_{\epsilon} \rightarrow K\) in the limit \(\epsilon \rightarrow 0\), we conclude that \(K = 0\). This result together with equation (4.15) justify the validity of the step from equation (4.10) to (4.11).

5 Summary and conclusions

Given the following list of holomorphicity conditions imposed on the function that parameterizes the multi-brane solutions
i) $G$ and $1/G$ are holomorphic in $\text{Re}(z) \geq 0$ except at $z = 0$.

ii) $G$ or $1/G$ are meromorphic at $z = 0$.

iii) $G$ is holomorphic at the point at infinity $z = \infty$ and has a limit $G(\infty) = 1$.

We have evaluated the cubic term of action for the multi-brane solutions. The result is given in terms of a contour integral

$$\langle \Psi \Psi \Psi \rangle = \frac{3}{\pi^2} \oint \frac{dz}{2\pi i} \frac{G'(z)}{G(z)}, \quad (5.1)$$

Now by employing the result coming from the evaluation of kinetic term of the action, which has been performed in reference [36]

$$\langle \Psi Q \Psi \rangle = -\frac{3}{\pi^2} \oint \frac{dz}{2\pi i} \frac{G'(z)}{G(z)}, \quad (5.2)$$

we can write the following expression for the energy

$$E = \frac{1}{2} \langle \Psi Q \Psi \rangle + \frac{1}{3} \langle \Psi \Psi \Psi \rangle = -\frac{1}{2\pi^2} \oint \frac{dz}{2\pi i} \frac{G'(z)}{G(z)}. \quad (5.3)$$

Using the same holomorphicity conditions i)-iii), we have also computed the Ellwood’s gauge invariant overlap for the multi-brane solutions and we have found the result

$$W(\Psi, \mathcal{V}) = \oint \frac{dz}{2\pi i} \frac{G'(z)}{G(z)} \text{Tr}(c\Omega). \quad (5.4)$$

Let us remember that to compute the above contour integrals, we need to follow a closed curve encircling the origin in the counterclockwise direction.

Comparing equations (5.3) and (5.4), we conclude that the energy computed from the action and from the Ellwood’s invariant will agree provided that the function that parameterizes the multi-brane solutions satisfies the holomorphicity conditions i)-iii). This conclusion turns out to be true as long as the values of the integer $n$ appearing in the definition of the function $G(z) = [z/(1 + z)]^n$ are restricted to the values $n = 0, \pm 1$. This result is similar to the bosonic case [24].

Prior the proposed multi-brane solutions, in the framework of the modified cubic superstring field theory, solutions of the form

$$\Psi = F e^{KB \over 1 - F^2} e^{\gamma F} + F B \gamma^2 F, \quad (5.5)$$

have been considered for the specific cases: $F^2 = e^{-K}$ and $F^2 = 1/(1 + K)$, where it was shown that the solutions characterize the tachyon vacuum solution [27, 30]. It is
interesting to note that, as argued in reference [27], from an analytic perspective the suggested tachyon vacuum solution appears to be as regular as Schnabl’s original solution in the open bosonic string field theory [1]. Nevertheless, from the perspective of the level expansion the situation is unclear, though to be honest, the analysis of the energy for the tachyon vacuum solution using the usual Virasoro $L_0$ level expansion has not yet been carried out. In this respect, the situation for the multi-brane solutions is similar and therefore it should be a good research project to analyze the solutions using the Virasoro $L_0$ level expansion.

Finally, we would like to comment about Berkovits non-polynomial open superstring field theory [39], since this theory is based on Witten’s associative star product, its mathematical setup shares the same algebraic structure of both string field theories, the open bosonic string field theory and the modified cubic superstring field theory, and hence the strategy and prescriptions studied in this work should be directly extended to that theory. Recently, the construction of the tachyon vacuum solution in Berkovits superstring field theory based on elements in the $KBc\gamma\gamma^{-1}$ subalgebra has been proposed by T. Erler [40].

Acknowledgements

I would like to thank Ted Erler, Masaki Murata, Martin Schnabl and Michael Kroyter for useful discussions. This work is supported by CNPq grant 303073/2012-8.

References

[1] M. Schnabl, Analytic solution for tachyon condensation in open string field theory, Adv. Theor. Math. Phys. 10, 433 (2006), hep-th/0511286.
[2] Y. Okawa, Comments on Schnabl’s analytic solution for tachyon condensation in Witten’s open string field theory, JHEP 0604, 055 (2006), hep-th/0603159.
[3] E. Fuchs and M. Kroyter, On the validity of the solution of string field theory, JHEP 0605, 006 (2006), hep-th/0603195.
[4] L. Rastelli and B. Zwiebach, Solving Open String Field Theory with Special Projectors, JHEP 0801, 020 (2008), hep-th/0606131.
[5] Y. Okawa, L. Rastelli and B. Zwiebach, Analytic Solutions for Tachyon Condensation with General Projectors, hep-th/0611110.
[6] T. Erler and C. Maccaferri, The Phantom Term in Open String Field Theory, JHEP 1206, 084 (2012), [arXiv:1201.5122].
[7] I. Ellwood, Singular gauge transformations in string field theory, JHEP 0905, 037 (2009), [arXiv:0903.0390].
[8] M. Schnabl, *Algebraic solutions in Open String Field Theory - a lightning review*, [arXiv:1004.4858].

[9] T. Erler and M. Schnabl, *A Simple Analytic Solution for Tachyon Condensation*, JHEP 0910, 066 (2009), [arXiv:0906.0979].

[10] E. Aldo Arroyo, *The Tachyon Potential in the Sliver Frame*, JHEP 0910, 056 (2009), [arXiv:0907.4939].

[11] E. A. Arroyo, *Cubic interaction term for Schnabl’s solution using Pade approximants*, J. Phys. A 42, 375402 (2009), [arXiv:0905.2014].

[12] E. A. Arroyo, *Conservation laws and tachyon potentials in the sliver frame*, JHEP 1106, 033 (2011), [arXiv:1103.4830].

[13] L. Bonora and S. Giaccari, *Generalized states in SFT*, [arXiv:1304.2159].

[14] T. Masuda, T. Noumi and D. Takahashi, *Constraints on a class of classical solutions in open string field theory*, JHEP 1210, 113 (2012), [arXiv:1207.6220].

[15] T. Erler, *Split string formalism and the closed string vacuum*, JHEP 0705, 083 (2007), hep-th/0611200.

[16] T. Erler, *Split string formalism and the closed string vacuum. II*, JHEP 0705, 084 (2007), hep-th/0612050.

[17] S. Zeze, *Application of KBc subalgebra in string field theory*, Prog. Theor. Phys. Suppl. 188, 56 (2011).

[18] S. Zeze, *Tachyon potential in KBc subalgebra*, Prog. Theor. Phys. 124, 567 (2010), [arXiv:1004.4351].

[19] E. A. Arroyo, *Comments on regularization of identity based solutions in string field theory*, JHEP 1011, 135 (2010), [arXiv:1009.0198].

[20] E. Aldo Arroyo, *Level truncation analysis of regularized identity based solutions*, JHEP 1111, 079 (2011), [arXiv:1109.5354].

[21] M. Murata and M. Schnabl, *On Multibrane Solutions in Open String Field Theory*, Prog. Theor. Phys. Suppl. 188, 50 (2011), [arXiv:1103.1382].

[22] M. Murata and M. Schnabl, *Multibrane Solutions in Open String Field Theory*, JHEP 1207, 063 (2012), [arXiv:1112.0591].

[23] H. Hata and T. Kojita, *Winding Number in String Field Theory*, JHEP 1201, 088 (2012), [arXiv:1111.2389].

[24] H. Hata and T. Kojita, *Singularities in K-space and Multi-brane Solutions in Cubic String Field Theory*, JHEP 1302, 065 (2013), [arXiv:1209.4406].

[25] I. Y. Arefeva, P. B. Medvedev and A. P. Zubarev, *New Representation For String Field Solves The Consistency Problem For Open Superstring Field Theory*, Nucl. Phys. B 341, 464 (1990).

[26] E. Witten, *Noncommutative Geometry And String Field Theory*, Nucl. Phys. B 268, 253 (1986).
T. Erler, *Tachyon Vacuum in Cubic Superstring Field Theory*, JHEP **0801**, 013 (2008), [arXiv:0707.4591].

M. Kroyter, *Comments on superstring field theory and its vacuum solution*, JHEP **0908**, 048 (2009), [arXiv:0905.3501].

I. Y. Aref’eva, R. V. Gorbachev and P. B. Medvedev, *Tachyon Solution in Cubic Neveu-Schwarz String Field Theory*, Theor. Math. Phys. **158**, 320 (2009), [arXiv:0804.2017].

R. V. Gorbachev, *New solution of the superstring equation of motion*, Theor. Math. Phys. **162**, 90 (2010), [Teor. Mat. Fiz. **162**, 106 (2010)].

T. Erler, *Exotic Universal Solutions in Cubic Superstring Field Theory*, JHEP **1104**, 107 (2011), [arXiv:1009.1865].

I. Y. Aref’eva, R. V. Gorbachev and P. B. Medvedev, *Pure Gauge Configurations and Solutions to Fermionic Superstring Field Theories Equations of Motion*, J. Phys. A **42**, 304001 (2009), [arXiv:0903.1273].

I. Y. Arefeva and R. V. Gorbachev, *On Gauge Equivalence of Tachyon Solutions in Cubic Neveu-Schwarz String Field Theory*, Theor. Math. Phys. **165**, 1512 (2010), [arXiv:1004.5064].

E. A. Arroyo, *Generating Erler-Schnabl-type Solution for Tachyon Vacuum in Cubic Superstring Field Theory*, J. Phys. A **43**, 445403 (2010), [arXiv:1004.3030].

S. Inatomi, I. Kishimoto and T. Takahashi, *Homotopy Operators and Identity-Based Solutions in Cubic Superstring Field Theory*, JHEP **1110**, 114 (2011), [arXiv:1109.2406].

E. Aldo Arroyo, *Multibrane solutions in cubic superstring field theory*, JHEP **1206**, 157 (2012), [arXiv:1204.0213].

I. Ellwood, *The Closed string tadpole in open string field theory*, JHEP **0808**, 063 (2008), [arXiv:0804.1131].

T. Masuda, *Comments on new multiple-brane solutions based on Hata-Kojita duality in open string field theory*, [arXiv:1211.2649].

N. Berkovits, *SuperPoincare invariant superstring field theory*, Nucl. Phys. B **450**, 90 (1995), [Erratum-ibid. B **459**, 439 (1996)], hep-th/9503099.

T. Erler, *Analytic Solution for Tachyon Condensation in Berkovits’ Open Superstring Field Theory*, [arXiv:1308.4400].