Scalar field perturbations of a Lifshitz black hole in conformal gravity in three dimensions

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(Dated: August 4, 2014)

We study the reflection and transmission coefficients and the absorption cross section for scalar fields in the background of a Lifshitz black hole in three-dimensional conformal gravity with $z = 0$, and we show that the absorption cross section vanishes at the low and high frequency limit. Also, we determine the quasinormal modes of scalar perturbations and then we study the stability of these black holes under scalar field perturbations.

PACS numbers:

I. INTRODUCTION

The three-dimensional models of gravity have attracted a remarkable interest in recent years. Apart from the BTZ black hole [1], which is a solution to the Einstein equations with a negative cosmological constant, much attention has been paid to topologically massive gravity (TMG), which generalizes three-dimensional general relativity (GR) by adding a Chern-Simons gravitational term to the Einstein-Hilbert action [2]. In this model, the propagating degree of freedom is a massive graviton. TMG also admits the BTZ black hole as an exact solution. The renewed interest in TMG results from the possibility of constructing a chiral theory of gravity at a special point of the space of parameters, [3].

Another three-dimensional massive gravity theory that has attracted attention in recent years is known as new massive gravity (NMG), where the action is the standard Einstein-Hilbert term plus a specific combination of scalar curvature square term and a Ricci tensor square term [4–8], and at the linearized level it is equivalent to the Fierz-Pauli action for a massive spin-2 field [4]. This model in three dimensions is indeed unitary at tree-level, but the corresponding model in higher dimensions is not due to the appearance of non-unitary massless spin-2 modes [9]. NMG admits warped AdS black holes [10], AdS waves [11, 12], asymptotically Lifshitz black holes [13], gravitational solitons, kinks and wormholes [14]; for further aspects of NMG see [15–20]. Asymptotically AdS and Lifshitz black holes in NMG dressed by a (non)minimally coupled scalar field have been reported recently in [21]. TMG and NMG share common features, however there are different aspects: one of these is the existence of a new type of black hole for a specific combination of parameters in the NMG Lagrangian, known as new type black holes, which are also solutions of conformal gravity in three dimensions [22].

Lifshitz spacetimes have received considerable attention from the point of view of condensed matter, i.e. the search for gravity duals of Lifshitz fixed points due to the AdS/CFT correspondence for condensed matter physics and quantum chromodynamics [23, 24]. There are many invariant scale theories of interest when studying such critical points, such theories exhibit the anisotropic scale invariance $t \rightarrow \lambda^z t$, $x \rightarrow \lambda x$, with $z \neq 1$, where $z$ is the relative scale dimension of time and space, and they are of particular interest in studies of critical exponent theory and phase transitions.

Lifshitz spacetimes are described by the metrics

$$ds^2 = \frac{r^{2z}}{l^{2z}} dt^2 + \frac{l^2}{r^2} dr^2 + \frac{r^2}{l^2} d\vec{x}^2,$$

(1)
where $\vec{x}$ represents a $d - 2$ dimensional spatial vector, with $d$ representing the spacetime dimension while $l$ denotes the length scale in the geometry. For $z = 1$ this spacetime is the usual anti-de Sitter metric in Poincaré coordinates.

The metrics of the Lifshitz black hole asymptotically have the form (1). However, obtaining analytic solutions does not seem to be a trivial task, and therefore constructing finite temperature gravity duals requires the introduction of strange matter content the theoretical motivation of which is not clear. Another way of finding such a Lifshitz black hole solution is by considering carefully-tuned higher-curvature modifications to the Hilbert-Einstein action, as in NMG in 3-dimensions or $R^2$ corrections to GR. This has been done, for instance, in [33, 34]. A 4-dimensional topological black hole with $z = 2$ was found in [28, 29] and a set of analytical Lifshitz black holes in higher dimensions for arbitrary $z$ in [30]. Lifshitz black holes with arbitrary dynamical exponent in Horndeski theory were found in [31] and nonlinearly charged Lifshitz black holes for any exponent $z > 1$ in [32].

Conformal gravity is a four-derivative theory and it is perturbatively renormalizable [33, 34]. Furthermore, it contains ghostlike modes, in the form of massive spin-2 excitations. However, it has been shown that by using the method of Dirac constraints [35] to quantize a prototype second plus fourth order theory, viz. the Pais-Uhlenbeck fourth order oscillator model [36], it becomes apparent that the limit in which the second order piece is switched off is a highly singular one in which the would-be ghost states move off the mass shell [37]. In three spacetime dimensions the equations of motion contain third derivatives of the metric. AdS and Lifshitz black holes with $z = 0$ and $z = 4$ in four-dimensional conformal gravity have been studied in [38]. In this work, we will consider a matter distribution outside the horizon of the Lifshitz black hole in 3-dimensional conformal gravity and dynamical exponent $z = 0$. It is worth mentioning that for $z = 0$ the anisotropic scale invariance, mentioned previously, corresponds to space-like scale invariance with no transformation of time. The matter is parameterized by a scalar field, which we will perturb by assuming that there is no back reaction on the metric. We then obtain the reflection and transmission coefficients, the absorption cross section and the quasinormal frequencies (QNFs) [39–44] for scalar fields, and study their stability under scalar perturbations.

Several studies have contributed to the scattering and absorption properties of waves in the spacetime of black holes. As the geometry of the spacetime surrounding a black hole is non-trivial, the Hawking radiation emitted at the event horizon may be modified by this geometry, so that when an observer located very far away from the black hole measures the spectrum, this will no longer be that of a black body [45]. The factors that modify the spectrum emitted by a black hole are known as greybody factors and can be obtained through classical scattering; their study therefore allows the semiclassical gravity dictionary to be increased, and also offers insight into the quantum nature of black holes and thus of quantum gravity; for an excellent review of this topic see [46]. Also, see for instance [47–49], for decay of Dirac fields in higher dimensional black holes.

On the other hand, the study of the QNFs gives information about the stability of black holes under matter fields that evolve perturbatively in their exterior region, without backreacting on the metric. In general, the oscillation frequencies are complex, where the real part represents the oscillation frequency and the imaginary part describes the rate at which this oscillation is damped, with the stability of the black hole being guaranteed if the imaginary part is negative. On the other hand, the QNFs determine how fast a thermal state in the boundary theory will reach thermal equilibrium according to the AdS/CFT correspondence [50], where the relaxation time of a thermal state of the boundary thermal theory is proportional to the inverse of the imaginary part of the QNFs of the dual gravity background [51]. Furthermore, in [52–55] the authors discuss a connection between Hawking radiation and black hole quasinormal modes, and black hole quasinormal modes can naturally be interpreted in terms of quantum levels. In three-dimensional spacetime, the quasinormal modes (QNM) of BTZ black holes have been studied in [51, 56, 57]; the QNMs for scalar field perturbations in the background of new type black holes in NMG were studied in [58] and for fermionic field perturbations in [59]. Generally, Lifshitz black holes are stable under scalar perturbations, and the QNFs show the absence of a real part [60, 61]. See also [60] were the authors studied the QNMs of a Lifshitz black hole in four-dimensional conformal gravity. On the other hand, the absorption cross section for asymptotically Lifshitz black holes were examined in [62, 63, 64, 65], and particle motion in these geometries in [66, 67]. Fermions on Lifshitz Background have been studied in [62], by using the fermionic Green’s function in 4-dimensional Lifshitz spacetime with $z = 2$. Also, the Dirac QNMs for a 4-dimensional Lifshitz black hole were studied in [68] and for d-dimensional Lifshitz black holes in [69]. The electromagnetic quasinormal modes for a Lifshitz black hole were analyzed in [70].

The paper is organized as follows. In Sec. I we give a brief review of an asymptotically Lifshitz black hole in three-dimensional conformal gravity. In Sec. III we calculate the reflection and transmission coefficients, the absorption cross section and the quasinormal modes of scalar perturbations for the three-dimensional Lifshitz black hole with $z = 0$. We conclude with Final Remarks in Sec. IV.
II. LIFSHITZ BLACK HOLE IN THREE-DIMENSIONAL CONFORMAL GRAVITY WITH \( z = 0 \)

In this work we will consider a matter distribution described by a scalar field outside the event horizon of an asymptotically Lifshitz black hole in three-dimensional conformal gravity with \( z = 0 \). In three dimensions, the field equations of conformal gravity are given by the vanishing of the Cotton tensor:

\[
C^\alpha_\beta = \epsilon^{\alpha\sigma\alpha} \nabla_\rho \left( R_{\sigma\beta} - \frac{1}{4} g_{\sigma\beta} R \right) = 0 ,
\]

where \( R \) is the Ricci scalar. The Cotton tensor is a traceless tensor that vanishes if and only if the metric is locally conformally flat. Solutions to this theory have been studied in [22, 76, 77]. The Lifshitz metric (1) solves the field equations for \( z = 0 \), and also the following Lifshitz black hole metric with \( z = 0 \)

\[
ds^2 = - f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 ,
\]

\[
f(r) = 1 - \frac{M}{r^2} .
\]

(3)

The requirement \( r_+ = \sqrt{M} > 0 \) implies that \( M > 0 \). The Kretschmann scalar is given by

\[
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 4 + \frac{20 M^2}{r^4} ,
\]

therefore there is a curvature singularity at \( r = 0 \) for \( M \neq 0 \). Other asymptotically Lifshitz black hole solutions in four-dimensional and six-dimensional conformal gravity are found in [38] and [78], respectively.

In the next section, we will determine the reflection coefficient, the transmission coefficient and the absorption cross section. Then, we will compute the QNFs, which coincide with the poles of the transmission coefficient and we will study the linear stability of these black holes under scalar field perturbations.

III. REFLECTION COEFFICIENT, TRANSMISSION COEFFICIENT, ABSORPTION CROSS SECTION AND QUASINORMAL MODES OF \( z = 0 \) LIFSHITZ BLACK HOLE

The scalar perturbations in the background of an asymptotically Lifshitz black hole in three-dimensional conformal gravity with dynamical exponent \( z = 0 \) are given by the Klein-Gordon equation of the scalar field solution with suitable boundary conditions at the event horizon and at the asymptotic infinity. The Klein-Gordon equation in curved spacetime is given by

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \right) \phi = m^2 \phi ,
\]

where \( m \) is the mass of the scalar field \( \phi \), which is minimally coupled to curvature. By means of the following ansatz

\[
\phi = e^{-i \omega t} e^{i \kappa \theta} R(r) ,
\]

(6)

the Klein-Gordon equation reduces to the following differential equation for the radial function \( R(r) \)

\[
r^2 \left( r^2 - M \right) \frac{d^2 R(r)}{dr^2} + 2r^3 \frac{dR(r)}{dr} + \left( \frac{\omega^2 r^4}{r^2 - \kappa^2} - m^2 r^2 \right) R(r) = 0 .
\]

(7)

Now, by considering \( R(r) = r^{-1/2} G(r) \) and by introducing the tortoise coordinate \( x \), given by \( dx = \frac{dr}{rf(r)} \), the latter equation can be rewritten as a one-dimensional Schrödinger equation

\[
\left[ \partial_x^2 + \omega^2 - V_{eff}(r) \right] G(x) = 0 ,
\]

where the effective potential is given by

\[
V_{eff}(r) = f(r) - \frac{3}{4} f(r)^2 + \frac{\kappa^2}{r^2} f(r) + m^2 f(r) .
\]

(9)
In Fig. 1 we plot the effective potential for $M = 1$, $m = 1$ and different values of the angular momentum $\kappa = 0, 1, 2$. Note that, when $r \to \infty$ the effective potential goes to $1/4 + m^2$.

Performing the change of variable $z = 1 - M r^2$, the radial equation can be written as

$$z(1 - z) \frac{\partial^2 R(z)}{\partial z^2} + \left(1 - \frac{3}{2} z\right) \frac{\partial R(z)}{\partial z} + \frac{1}{4} \left[\frac{\omega^2}{z(1 - z)} - \frac{\kappa^2}{M} - \frac{m^2}{1 - z}\right] R(z) = 0 .$$

Using the decomposition $R(z) = z^\alpha (1 - z)^\beta F(z)$, with

$$\alpha_{\pm} = \pm \frac{i\omega}{2} ,$$

$$\beta_{\pm} = \frac{1}{4} \left(1 \pm \sqrt{1 + 4(m^2 - \omega^2)}\right) ,$$

allows us to write as a hypergeometric equation for $F(z)$

$$z(1 - z) \frac{\partial^2 F(z)}{\partial z^2} + [c - (1 + a + b)z] \frac{\partial F(z)}{\partial z} - abF(z) = 0 ,$$

where the coefficients are given by

$$a = \frac{1}{4} + \alpha + \frac{1}{4} \sqrt{1 - \frac{4\kappa^2}{M}} ,$$

$$b = \frac{1}{4} + \alpha + \beta = \frac{1}{4} \sqrt{1 - \frac{4\kappa^2}{M}} ,$$

$$c = 1 + 2\alpha .$$

The general solution of the hypergeometric equation is

$$F(z) = C_1 F_1(a, b, c; z) + C_2 z^{-\alpha} F_1(a - c + 1, b - c + 1, 2 - c; z) ,$$

and it has three regular singular points at $z = 0$, $z = 1$, and $z = \infty$. $2F_1(a, b, c; z)$ is a hypergeometric function and $C_1$ and $C_2$ are integration constants. Thus, the solution for the radial function $R(z)$ is

$$R(z) = C_1 z^\alpha (1 - z)^\beta F_1(a, b, c; z) + C_2 z^{-\alpha} (1 - z)^\beta F_1(a - c + 1, b - c + 1, 2 - c; z) .$$

So, in the vicinity of the horizon, $z = 0$ and using the property $F(a, b, c, 0) = 1$, the function $R(z)$ behaves as

$$R(z) = C_1 e^{\alpha \ln z} + C_2 e^{-\alpha \ln z} ,$$

FIG. 1: The effective potential as a function of $r$, for $M = 1$, $m = 1$ and $\kappa = 0, 1, 2$. 

In Fig. 1 we plot the effective potential for $M = 1$, $m = 1$ and different values of the angular momentum $\kappa = 0, 1, 2$. Note that, when $r \to \infty$ the effective potential goes to $1/4 + m^2$. 

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Using the decomposition $R(z) = z^\alpha (1 - z)^\beta F(z)$, with

$$\alpha_{\pm} = \pm \frac{i\omega}{2} ,$$

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$$a = \frac{1}{4} + \alpha + \frac{1}{4} \sqrt{1 - \frac{4\kappa^2}{M}} ,$$

$$b = \frac{1}{4} + \alpha + \beta = \frac{1}{4} \sqrt{1 - \frac{4\kappa^2}{M}} ,$$

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The general solution of the hypergeometric equation is

$$F(z) = C_1 F_1(a, b, c; z) + C_2 z^{-\alpha} F_1(a - c + 1, b - c + 1, 2 - c; z) ,$$

and it has three regular singular points at $z = 0$, $z = 1$, and $z = \infty$. $2F_1(a, b, c; z)$ is a hypergeometric function and $C_1$ and $C_2$ are integration constants. Thus, the solution for the radial function $R(z)$ is

$$R(z) = C_1 z^\alpha (1 - z)^\beta F_1(a, b, c; z) + C_2 z^{-\alpha} (1 - z)^\beta F_1(a - c + 1, b - c + 1, 2 - c; z) .$$

So, in the vicinity of the horizon, $z = 0$ and using the property $F(a, b, c, 0) = 1$, the function $R(z)$ behaves as

$$R(z) = C_1 e^{\alpha \ln z} + C_2 e^{-\alpha \ln z} ,$$
and the scalar field $\phi$, for $\alpha = \alpha_-$, can be written as follows

$$\phi \sim C_1 e^{-i\omega(t+\ln z)} + C_2 e^{-i\omega(t-\ln z)},$$

where the first term represents an ingoing wave and the second one an outgoing wave in the black hole. So, by imposing that only ingoing waves exist at the horizon, this fixes $C_2 = 0$. The radial solution then becomes

$$R(z) = C_1 e^{\alpha \ln z} (1-z)^{\beta} _2F_1(a, b, c; z) = C_1 e^{-i\omega \ln z} (1-z)^{\beta} _2F_1(a, b, c; z).$$

(21)

The reflection and transmission coefficients depend on the behavior of the radial function, both at the horizon and at the asymptotic infinity, and they are defined by

$$R = \left| \frac{F_{\text{out, asymp}}}{F_{\text{in, asymp}}} \right|; \quad T = \left| \frac{F_{\text{in, asymp}}}{F_{\text{in, asymp}}} \right|,$$

(22)

where $F$ is the flux, and is given by

$$F = \frac{1}{2i} \sqrt{-g} g^{rr} \left( R(r)^{+} \frac{dR(r)}{dr} - R(r) \frac{dR(r)^{+}}{dr} \right),$$

(23)

where $\sqrt{-g} = 1$. The behavior at the horizon is given by (19), and using (23), we get the flux at the horizon

$$F_{\text{in, hor}} = -\omega \sqrt{M} |C_1|^2.$$

(24)

Now, in order to obtain the asymptotic behavior of $R(z)$, we take the limit $z \rightarrow 1$ in (10). Thus, we obtain the following solution

$$R(z) = B_1 (1-z)^{\beta_+} + B_2 (1-z)^{\beta_-}.$$

(25)

Thus, the flux (23) at the asymptotic region is given by

$$F_{\text{asymp}} = -\sqrt{M} \sqrt{\omega^2 - m^2 - \frac{1}{4}} \left( |A_1|^2 - |A_2|^2 \right),$$

(26)

where $\omega^2 \geq m^2 + \frac{1}{4}$. On the other hand, by replacing Kummer’s formula [79], in (21),

$$ _2F_1(a, b, c; z) = \frac{\Gamma (c) \Gamma (c-a-b)}{\Gamma (c-a) \Gamma (c-b)} _2F_1(a, b, a+b-c; 1-z) +$$

$$(1-z)^{c-a-b} \frac{\Gamma (c) \Gamma (a+b-c)}{\Gamma (a) \Gamma (b)} _2F_1(c-a, c-b, c-a-b+1; 1-z),$$

(27)

and by using Eq. (23) we obtain the flux

$$F_{\text{asymp}} = -\sqrt{M} \sqrt{\omega^2 - m^2 - \frac{1}{4}} \left( |A_1|^2 - |A_2|^2 \right),$$

(28)

where,

$$A_1 = C_1 \frac{\Gamma (c) \Gamma (a+b-c)}{\Gamma (a) \Gamma (b)} = B_1,$$

(29)

$$A_2 = C_1 \frac{\Gamma (c) \Gamma (c-a-b)}{\Gamma (c-a) \Gamma (c-b)} = B_2.$$

Therefore, the reflection and transmission coefficients are given by

$$R = \frac{|A_1|^2}{|A_2|^2},$$

(30)

$$T = \frac{\omega |C_1|^2}{\sqrt{\omega^2 - m^2 - \frac{1}{4}|A_2|^2}}.$$
and the absorption cross section $\sigma_{\text{abs}}$, becomes

$$\sigma_{\text{abs}} = \frac{T}{\omega} = \frac{|C_1|^2}{\sqrt{\omega^2 - m^2 - \frac{1}{4}|A_2|^2}}. \quad (32)$$

Now, we will carry out a numerical analysis of the reflection coefficient $R$, transmission coefficient $T$, and absorption cross section $\sigma_{\text{abs}}$ of $z = 0$ Lifshitz black holes, for scalar fields. So, we plot the reflection and transmission coefficients and the absorption cross section in Fig. (2), with $M = 1$, $m = 1$ and $\kappa = 0, 1, 2$. Essentially, we found that the reflection coefficient is one at the low frequency limit, that is $\omega \approx \sqrt{m^2 + \frac{1}{4}}$, and at the high frequency limit this coefficient is null. The behavior of the transmission coefficient is the opposite, with $R + T = 1$. Also, the absorption cross section is null at the low and high-frequency limits, but there is a range of frequencies for which the absorption cross section is not null, and it also has a maximum value.

The QNFs are defined as the poles of the transmission coefficient, which is equivalent to impose that only outgoing waves exist at the asymptotic infinity. These poles are given by $A_2 = 0$, and this occurs when $c - a + n = 0$ or $c - b + n = 0$, with $n = 0, 1, 2 \ldots$. Therefore, the quasinormal frequencies are given by

$$\omega = -i \frac{-2m^2(1 + 2n) + 1 + 10n + 24n^2 + 16n^3 + 2\frac{m^2}{\kappa} + 4n\frac{m^2}{\kappa} \pm \sqrt{1 - 4\frac{m^2}{\kappa} \left(1 + m^2 + 4n + 4n^2 + \frac{m^2}{\kappa}\right)}}{3 + 16n + 16n^2 + 4\frac{m^2}{\kappa}}. \quad (33)$$
FIG. 4: The absorption cross section $\sigma_{abs}$ as a function of $\omega$; for $M = 1$, $m = 1$ and $\kappa = 0, 2, 4$.

We observe that for $1 - 4\kappa^2 M > 0$ the QNFs are purely imaginary and for $1 - 4\kappa^2 M < 0$ they acquire a real part. Also, we observe that for some values of the scalar field mass $m$ the quasinormal frequencies can have a positive imaginary part, therefore we conclude that this black hole is not stable under scalar field perturbations.

IV. FINAL REMARKS

In this work we studied scalar perturbations for an asymptotically Lifshitz black hole in three-dimensional conformal gravity with dynamical exponent $z = 0$, where in this case the anisotropic scale invariance corresponds to a space-like scale invariance with no transformation of time, and we calculated the reflection and transmission coefficients, the absorption cross section and the quasinormal modes. The results obtained show that the absorption cross section vanishes at the low frequency limit as well as at the high frequency limit. This means that a wave emitted from the horizon, with low or high frequency, does not reach spatial infinity and therefore is totally reflected, because the fraction of particles penetrating the potential barrier vanishes. We have also shown in Fig. (2) that there is a range of frequencies where the absorption cross section is not null. On the other hand, the reflection coefficient is one at the low frequency limit and null at high frequencies; the behavior of the transmission coefficient is the opposite, where $R + T = 1$. Furthermore, we have shown that the absorption cross section decreases for higher values of angular momentum, and decreases when the mass $m$ of the scalar field increases; however, for high frequencies the difference is negligible, see Figs. (3, 4).

Furthermore, we calculated analytically the QNFs of scalar perturbations, which coincide with the poles of the transmission coefficient, and we found two sets of quasinormal frequencies. However, some of these can have a positive imaginary part, depending of the value of $m$, and therefore the black hole is not stable under scalar field perturbations.

Acknowledgments

This work was funded by the Comisión Nacional de Investigación Científica y Tecnológica through FONDECYT Grant 11121148 (YV, MC) and also partially funded by Dirección de investigación, Universidad de La Frontera (MC).

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