Correlation functions of local composite operators from generalized unitarity

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Abstract

We describe the use of generalized unitarity for the construction of correlation functions of local gauge-invariant operators in general quantum field theories and illustrate this method with several calculations in $\mathcal{N} = 4$ super-Yang-Mills theory involving BPS and non-BPS operators. Form factors of gauge-invariant operators and their multi-operator generalization play an important role in our construction. We discuss various symmetries of the momentum space presentation of correlation functions, which is natural in this framework and give examples involving non-BPS and any number of BPS operators. We also discuss the calculation of correlators describing the energy flow in scattering processes as well as the construction of the effective action of a background gravitational field.

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1 Introduction and summary

Correlation functions of gauge-invariant operators are natural observables in both conformal and non-conformal field theories. In the early days of the AdS/CFT correspondence correlation functions of BPS operators played an instrumental role in establishing and testing it. More recently, correlation functions have been shown to exhibit fascinating relations to other quantities: limits in which the gauge invariant operators are null separated are related to expectation values of null polygonal Wilson loops with or without local operators \[ \text{[1, 2, 3, 4]} \] and, in \( \mathcal{N} = 4 \) super-Yang-Mills theory in the planar limit, the same limit is related to scattering amplitudes \[ \text{[5]} \]. Moreover, since any curve may be approximated to arbitrary precision by a null polygon, it is in principle possible (though perhaps difficult in practice) that special limits of correlation functions with arbitrarily many operators can be used to construct the expectation value of Wilson loops of arbitrary shape.

Correlation functions of gauge-invariant operators also have a natural place in non-conformal theories, such as QCD, where \( \text{e.g.} \) suitable correlation functions capture certain inclusive properties of the final state of scattering processes. Among them are the energy correlators, originally introduced in \[ \text{[6]} \]. As discussed in \[ \text{[7]} \], they have a counterpart in conformal field theories, where they describe the final state produced by the time evolution of some localized excitation and may be thought of as particular analytic continuations of regular correlation functions.

Even though in conformal field theories their axioms guarantee that higher-point correlation functions are determined by the two- and three-point functions, an explicit evaluation along this line is not straightforward. It is therefore interesting to devise methods to directly evaluate them – at weak and at strong coupling – both for generic positions of operators or directly in special limits.

The classic approach to the calculation of correlation functions makes use of Feynman diagrams in momentum space, position space or superspace; in \( \mathcal{N} = 4 \) sYM theory three-point functions of scalar operators of dimension \( \Delta \leq 5 \) have been evaluated systematically using such methods in \[ \text{[8]} \]. An efficient reorganization of this approach is the Lagrangian insertion formalism \[ \text{[9, 10, 11]} \]. In this theory, and in general in all theories in which the planar dilatation operator defines an integrable Hamiltonian, the calculation of three-point (as well as that of higher-point) functions benefits from use of integrable model techniques \[ \text{[12, 13, 14, 15, 16]} \]. More recently it was proposed \[ \text{[17]} \] and illustrated for the correlation function of four BPS operators in \( \mathcal{N} = 4 \) sYM theory that light-cone superspace can be efficiently used for this purpose. The correlation functions of chiral stress tensor multiplets in \( \mathcal{N} = 4 \) sYM theory enjoy special properties at the integrand level \[ \text{[18]} \] – a permutation symmetry which becomes manifest when the integrand is constructed in the Lagrangian insertion formalism. This was used to devise an efficient method for the construction of
four-point correlation functions of stress tensor multiplets [19] which led to the evaluation of the five-loop correction to the anomalous dimension of the Konishi multiplet [20]. The result confirms the integrability-based predictions [21, 22].

In conformal field theories that have a string theory dual, certain classes of correlation functions of local operators may be evaluated at strong coupling using semiclassical expansion on the string theory side [23, 24, 25, 26, 27, 28, 29, 30]. It was moreover argued [31, 32] that AdS supergravity scattering amplitudes with suitable boundary conditions follow an on-shell recursion relation of the same type as the gauge theory on-shell recursion relations.

Independently, increasingly efficient perturbative computational techniques – generalized unitarity, on-shell recursion relations, etc. – led to remarkable progress in our understanding of scattering amplitudes of $\mathcal{N} = 4$ sYM theory and to the discovery of new and powerful symmetries – dual superconformal symmetry [33, 34], color/kinematics duality [35] – which severely constrain the scattering matrix. As we will explain in the next section, correlation functions of local gauge invariant operators may be interpreted as special scattering amplitudes of sources in the theory obtained by adding to the action the operators and their sources. One might naturally expect that the modern techniques developed for S-matrix calculations can also be used to calculate correlation functions; as we shall see, this is indeed the case. While correlation functions are naturally functions of the positions of the operators, an approach that mirrors the calculation of scattering amplitudes will yield their Fourier-transformed expressions, i.e. the momentum space correlation functions. Since position space correlation functions are conformally invariant the momentum space correlators should also have this property (albeit non-manifestly) and thus should be annihilated by the Fourier-transform of the conformal group generators.

The relation between correlation functions and scattering amplitudes implies that the conformal symmetry of the former becomes – in the null separation limit – the dual conformal symmetry of the latter [3]. Momentum space expressions for correlation functions may also be used to search for additional symmetries, which are hidden in the position space expressions.

Apart from constructing position-space correlation functions, momentum space correlators can also be used to construct quantum effective actions in background fields. As we shall briefly discuss in § 5, coupling an action with external fields is essentially equivalent to deforming the action by various operators, with the background fields acting as sources. Compared to the deformations relevant for the calculation of correlation functions, the only difference is that, depending on the specifics of background fields, sources may appear non-linearly. Nevertheless, scattering amplitudes of background fields, perhaps expanded for

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3 The relation to amplitudes might suggest a relation between momentum space symmetries of correlation functions and position space symmetries of amplitudes. Such a relation may however be obscured by the null limit, which is not very transparent in momentum space. While such a relation can exist only in the null limit, it is not clear why momentum space correlation functions cannot have additional symmetries.
small momenta/weakly varying external fields, yield the terms relevant for the construction of their off-shell effective action.

Recently, the color/kinematics duality \[35\] has emerged as an important property of color-dressed amplitudes in certain gauge theories with only adjoint fields and antisymmetric structure constant couplings. It states that, given some amplitude, there exists a presentation of its integrand such that, if the color factors of some integrals obey the Jacobi identity, then the numerator factors of those integrals also obey a Jacobi-like identity. There is by now substantial evidence in favor of the duality, both at tree level \[36, 37, 38, 39, 40, 41\] and at loop level \[42\]. A consequence of this duality is the existence of nontrivial relations between the color-ordered partial tree amplitudes of gauge theory \[35\], which have been proven both from field theory \[43\] and string theory \[44\] perspectives. We will explore whether a duality of this type exists for correlation functions. A natural expectation is that, if present, it would be most easily visible in momentum space correlation functions.

The structure of the operators makes it unlikely that color/kinematics duality always holds for all internal edges of the integrals that appear. Indeed, color/kinematics duality acts naturally on integrals associated to graphs constructed from 3-point vertices. Due to the operator insertions however, correlation functions naturally have multi-point vertices. One may attempt to define the duality by resolving the higher-point vertices into sequences of three-point vertices in color space. In general however, these vertices carry both antisymmetric structure constants as well as the symmetric ones, \(d_{abc}\), and their generalizations. It is therefore not clear whether for a generic correlation function Jacobi relations can exist for all internal edges. A notable exception\[4\] which appears consistent with color/kinematics duality is provided by a four-gluon operator with the color structure given by \(d_{abcd}\). If a Jacobi identity is not present and color and kinematic factors are no longer linked, it is still possible that numerator factors of various integrals are nonetheless nontrivially related to each other\[5\]. Regardless of these details, we will argue that color/kinematics duality should hold for all internal edges not directly connected to one of the operators up to contact terms collapsing at least one of these latter edges.

In this paper we discuss the use of generalized unitarity for the construction of momentum space correlation functions of BPS and non-BPS operators. While we will be mainly concerned with describing the details and subtleties of this approach, we also illustrate this method by recovering the known example of the four-point BPS operators and also by constructing infinite classes of new ones: such as the \(n\)-point function of BPS operators and the

\[4\]We thank H. Johansson for providing this example, discussions on this point and for sharing his insight into \[45\].

\[5\]This is reminiscent of the \(\beta\)-deformed \(\mathcal{N} = 4\) sYM theory \[46\], where tree-level kinematic factors are related despite absence of a relation for the corresponding color factors. The importance of such relations remains an open question.
$n$–point function of BPS operators and some number of twist-2 non-BPS operators at the next-to-leading order. In our examples we will focus on the $\mathcal{N} = 4$ sYM theory; as for scattering amplitudes however, the method we describe here may be applied with suitable care to all quantum field theories. Essential ingredients in this construction are the (super-)form-factors of local gauge-invariant operators as well as their generalizations involving several operators. Unitarity and generalized unitarity have already been applied to the construction of higher-loop form factors in \cite{17, 48, 49, 50}.

In § 2 we will describe the similarities between scattering amplitudes and correlation functions and the use of generalized unitarity for the construction of the latter for general gauge-invariant operators, the relevance of generalized form factors and the need for regularization and renormalization. We will also identify the components of correlation functions which may exhibit color/kinematics duality as well as may contain some hidden consequences of dual conformal invariance. In § 3 we collect the known expression of the MHV super-form factor of the chiral stress tensor multiplet and list (while relegating the details to appendices) the generalized (two-chiral stress tensor multiplet) form factor as well as the MHV super-form factor of scalar non-BPS operators and of the general twist-2 non-BPS operators. We will use them in the examples we discuss in § 4.

We construct the leading order and the next-to-leading order correlation function of four BPS operators in momentum space, Fourier-transform it to position space and recover the known results \cite{51, 52}. While the calculations are mainly carried out in components, in § 4.1.3 we illustrate the use of manifestly supersymmetric methods for the construction of correlation functions. We also derive the (connected part of the) next-to-leading order correlation function of $n$ BPS operators and check that its null limit reproduces the $n$-point MHV amplitude, as originally shown in \cite{5}. In § 4.2 we compute (the connected part of) the correlation function of one and two general non-BPS twist-2 operators and an arbitrary number of BPS operators. Similarly to the discussion of correlators of BPS operators, we verify explicitly that their null limits reproduce the $n$-point MHV amplitude. We also discuss the structure of the correlator of $m$ twist-2 and $n$ BPS operators in a split configuration. For the three-point function we discuss the appearance of the anomalous dimension of the non-BPS operator, its renormalization, as well as the vanishing of the three-point function in limits in which the twist-2 operator becomes a conformal descendant of a BPS operator.

In § 5 we include a general discussion of correlation functions involving stress tensors, which may be used to determine the effective action in non-dynamical gravitational background. We also discuss the energy correlators, which capture the energy flow in the time evolution of some composite state. After recalling their definition, we illustrate this non-standard type of correlation functions by evaluating several examples to first nontrivial order:

\footnote{While writing up this paper we received \cite{53} in which the three-point function of one twist-2 non-BPS operator and two BPS operators was computed through a different method.}
the one- and two-point energy correlator in the state created by a BPS scalar operator as well as the one- and two-point energy correlator in the state created by a twist-2 non-BPS operator. Section §6 contains our conclusions and a summary of our results. Several appendices contain detailed derivation of some of the formulae used in the main body of the paper.

2 Generalized unitarity and correlation functions

Generalized unitarity has been developed as a very efficient tool for the calculation of on-shell scattering amplitudes in quantum field theories. From a modern perspective it is interpreted as a specific organization of the Feynman graphs contributing to an amplitude which exposes the vast simplifications occurring when off-shell Green’s functions are amputated and placed on shell: each generalized cut is, on the one hand, a product of tree-level amplitudes and on the other it is the subset of Feynman graphs that contain the cut propagators. From this perspective it appears natural that anything that is expressible in terms of Feynman graphs and is gauge invariant may be constructed through generalized unitarity-type methods.

An example in this direction are form factors; particular form factors have recently been considered at various loop orders in $[47, 48, 54, 55, 50, 49]$ in $\mathcal{N} = 4$ sYM theory as well as in QCD $[56, 57]$.

While this interpretation of generalized unitarity makes clear its applicability, it is not difficult to give form factors the interpretation of scattering amplitudes. In a similar spirit correlation functions, which are the focus of our paper, can be given the same interpretation.

2.1 Correlation functions as scattering amplitudes

Let us start with the path integral expression of an $n$-point correlation function in some quantum field theory defined by the Euclidian action $S_E[\Phi]$,

$$\langle O_1(x_1) \ldots O_n(x_n) \rangle = \int [D\Phi] O_1(x_1) \ldots O_n(x_n) e^{-S_E[\Phi]} ,$$

where $\Phi$ generically denote the fields of the theory and the local operators $O_i$ are gauge-invariant combinations of them. All correlation functions may be packaged into a generating

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\*In the unitarity construction of scattering amplitudes based on the optical theorem the imaginary part of an amplitude has this property manifestly; the real part is constructed through a dispersion integral.

\*It is presumably possible to generalize this statement to gauge-variant quantities at the expense of having non-vanishing contributions from ghosts and unphysical degrees of freedom.

\*From the perspective of the optical theorem, this implies that correlation functions should also be expressible in terms of dispersion integrals.
functional, by introducing local sources for all possible/desired local operators:

\[ Z[J_1, J_2, \ldots] = \int [D\Phi] \ e^{-S_E[\Phi] - \int d^d x \sum_i J_i(x) O_i(x)} . \tag{2} \]

Correlation functions of specific operators are then extracted by differentiating with respect to the relevant sources and setting all sources to zero:

\[ \langle O_1(x_1) \ldots O_n(x_n) \rangle = \frac{\delta^n}{\delta J_1(x_1) \ldots \delta J_n(x_n)} Z[J_1, \ldots] \bigg|_{J_i \to 0} . \tag{3} \]

The sources \( J_i \) may be interpreted as non-dynamical fields. If \( S_E \) is the action of a conformal field theory with a closed string theory dual \[58\], then the sources may be interpreted as the boundary values of the fields describing the closed string states \[59, 60\]. In particular, for BPS operators, they are just the boundary values of the fields of the relevant supergravity theory.

The construction above is formally identical to that of scattering amplitudes; the only difference is that, for scattering amplitudes, \( J \) are sources for the fundamental fields. From this perspective, if we interpret the sources \( J \) as fields, we may also interpret the correlation functions of operators as scattering amplitudes of their associated source-fields \( J \) in the theory with the modified action

\[ S_E^{\text{mod}}[\Phi, J] = S_E[\Phi] + \int d^d x \sum_i J_i(x) O_i(x) . \tag{4} \]

We may therefore view the scattering amplitudes of source-fields \( J_i \) as the correlation functions of the corresponding operators \( O_i \). If the field theory has a string theory dual we may interpret this as the scattering amplitude of closed strings with specified boundary conditions. Similarly, we may interpret the scattering amplitudes of one source-field \( J \) and any number of fundamental fields \( \Phi \) as the form factor of the operator corresponding to that source,

\[ \langle O|\Phi_1 \ldots \Phi_m \rangle , \tag{5} \]

as also mentioned in \[55\]. From the perspective of a string theory dual we may interpret this as the scattering amplitude of a closed string into open strings. Amplitudes with several sources and fundamental fields have a similar interpretation in the context of gauge/string duality; from a field theory perspective we will refer to them as ”generalized form factors” or as ”multi-operator form factors”. \[11\]

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\[10\] Also, scattering amplitudes are constructed in Lorentzian signature with the time-ordered operators. For correlators in Euclidian signature one may formally introduce a radial ordering of operators.

\[11\] It is important to stress that these are not necessarily related to form factors of multi-trace operators.
The gauge invariance of the operators $\mathcal{O}_i$ and of the corresponding source fields $\mathcal{J}_i$ guarantees that this action is BRST invariant. As in the absence of source fields, it is possible to use a sequence of Ward identities \cite{61} to show that scattering amplitudes with external unphysical fields or ghosts do not contribute to unitarity cuts. 12

As described above, while $\mathcal{J}$ are interpreted as fields, they are nevertheless non-propagating and thus they may appear only as external lines of an amplitude and do not have a well-defined on-shell condition. To make an even closer analogy with scattering amplitudes one may formally promote them to propagating fields by assigning them massive quadratic terms with different masses for each source; they may then appear both as internal and external lines. 13

Restricting the integrands of the resulting amplitudes to terms that have no poles as the mass of the sources is varied guarantees that all contributions with internal sources are projected out. Last, the remaining mass dependence is solely associated to the norm of the external momenta and should participate in the Fourier-transform to position space.

### 2.2 On the presentation, structure and symmetries of correlation functions

Correlation functions of operators with definite dimension transform covariantly under position space conformal transformations and are strongly constrained by them. However, as for scattering amplitudes, the generalized unitarity method yields scattering amplitudes for source fields carrying definite momenta and thus gives the momentum space form of correlation functions

$$\langle \tilde{\mathcal{O}}_1(q_1) \ldots \tilde{\mathcal{O}}_n(q_n) \rangle$$

where $\tilde{\mathcal{O}}_i(q_i)$ are the Fourier-transform of the usual position space operators

$$\tilde{\mathcal{O}}(q) = \int \frac{d^d q}{(2\pi)^d} e^{i q \cdot x} \mathcal{O}(x).$$

Position-space conformal invariance will be hidden in momentum space; nevertheless, since momentum space correlation functions should not have holomorphic anomalies\footnote{Holomorphic anomalies are likely to appear when the operator’s momenta are restricted to be null. Since position space correlation functions should be conformally invariant, we expect that the constrained momenta should not play an important role in the Fourier-transform.} \cite{63} (unlike

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\footnote{The required Ward identities do not hold if at least one of the operators – and hence the deformed action \cite{3} – is gauge-variant.}

\footnote{An example in this direction is the interaction term $H \text{Tr}[F^2]$ generated at one-loop level through a top-quark loop in the Standard Model (and it is responsible for the gluon-fusion production of the Higgs boson). The two-loop amplitude $H \rightarrow ggg$ in this theory, i.e. the two-loop form factor of $\text{Tr}[F^2]$ with three external states, was discussed recently in \cite{56, 62, 67}. By formally taking the Higgs mass to infinity one forces the Higgs boson to appear only as an external state. The resulting amplitudes are either generalized (in the sense described below) form factors of $\text{Tr}[F^2]$ or correlation functions of these operators.}

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scattering amplitudes of fundamental fields) one may test whether they are annihilated by the momentum space form of conformal generators – in particular the special conformal generator

$$\sum_{i=1}^{n} K_{i}^\mu$$  \hspace{1cm} (8)

$$K_{i}^\mu = -2(\Delta_{i} - 3) \frac{\partial}{\partial q_{i}^{\mu}} - \left( q_{i}^{\mu} \frac{\partial^2}{\partial q_{i}^{\nu} \partial q_{i}^{\nu}} - 2q_{i}^{\nu} \frac{\partial}{\partial q_{i}^{\nu}} \frac{\partial}{\partial q_{i}^{\mu}} \right) + 2 \frac{\partial}{\partial q_{i}^{\nu}} (\Sigma^{i})_{\nu}^{\mu}$$

where the sum runs over all operators, $\Delta$ is the conformal dimension and $\Sigma^{i}$ represents $SL(2, C)$ on the $i$–th operator. Conformal symmetry should emerge as a manifest symmetry upon inverse Fourier-transform to position space, if the operators are chosen to have definite dimension.

The unitarity method provides an efficient framework for systematically constructing and verifying the expression for any multi-loop amplitude in a massless field theory. This method, along with various refinements, has already been described in some detail elsewhere [64, 65, 66, 67, 68, 69]; see also [70, 71, 72] for recent reviews. Here we will discuss the additional information needed for the construction of correlation functions of local operators. A color-dressed generalized unitarity cut is a sum over products of color-dressed amplitudes $A_{(i)}$,

$$i^{c} \sum_{\text{states}} A_{(1)} A_{(2)} \ldots A_{(m)}$$  \hspace{1cm} (9)

where the $c$ cut lines are placed on shell and we included a factor of $i$ for each cut propagator; each cut line appears twice, leaving one amplitude factor and entering another. While not necessary, the amplitude factors $A_{(i)}$ can be chosen to be at tree level. In our case of cuts of scattering amplitudes of source fields, the factors $A_{(i)}$ are either amplitudes of fundamental fields or (generalized) form factors. The cut construction of an amplitude formally proceeds by matching all cuts onto an ansatz in terms of Feynman integrals.

A systematic strategy, which is designed to keep under control the size of the ansatz, is the maximal cut method in which one begins by first constraining the ansatz to reproduce all the $(3L + (n - 3))$-particle cuts (i.e. the maximal cuts, in which the maximal number of propagators for an $n$-point amplitude are cut) and then systematically proceeds to relax the cut condition on one propagator (next-to-maximal cuts), two propagators (next-to-next-to-maximal cuts) and so on. At each step one may reduce the expression of the cuts to cuts of master integrals by generalizing the methods of refs. [73, 57] beyond two-loop order and to all massive external legs. The advantage of this method is that, for any one cut, only a small part of the ansatz is relevant. For the resulting expression to be correct it must reproduce all generalized cuts. Since some cuts are special cases of others, it suffices to verify that
it reproduces a spanning set of cuts – \textit{i.e.} a set of cuts that guarantee that all the other ones are satisfied. For simple (low-order) cases one may construct amplitudes by analyzing directly the relevant spanning set of cuts.

It is therefore clear that an essential ingredient in the construction of the (momentum space form of) correlation functions are the form factors and generalized form factors of the corresponding operators. While it is obvious that there exists a very close relation between off-shell Green’s functions of fundamental fields and correlation functions of local operators, the generalized unitarity makes it clear that a similar though slightly weaker relation exists between scattering amplitudes of fundamental fields and correlation functions. Let us consider the generalized cut shown in fig. 1: the external lines are attached to blobs representing color-dressed form factors while Feynman diagrammatics guarantees that the blob with no external lines is some multi-loop scattering amplitude of fundamental fields. This structure is independent of the number of external legs of each form factor, with the highest-loop amplitude appearing together with the form factors with the smallest number of external legs. Other cuts are necessary in order to identify the terms in which propagators exposed in fig. 1 are collapsed. It is moreover necessary to consider cuts involving generalized form factors. The terms captured only by these cuts are those in which there is no (sequence of) propagator(s) between two operators; upon Fourier-transform to position space, such terms lead to contact terms which we know are relevant only in the OPE limit; it is possible that at least some contact terms may be inferred from the requirement that the position space Euclidean correlator satisfies all relevant Ward identities (see \textit{e.g.} [74] for a discussion of 3-point functions and the determination of contact terms and [75] for the contact terms required by conformal invariance).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Generalized unitarity guarantees the presence of multi-loop amplitudes in the correlation function calculation. Additional contributions involves at least one form factor with more fields than the corresponding operator.}
\end{figure}
This structure of correlation functions points to the fact that, in theories whose color-dressed scattering amplitudes of fundamental fields exhibit larger symmetries, such as $\mathcal{N} = 4$ sYM theory, momentum space correlation functions should be at least partly constrained by these symmetries. Among the remarkable properties of $\mathcal{N} = 4$ sYM theory are the dual conformal invariance and color/kinematics duality of its scattering amplitudes. They may have some consequences on correlation functions as well. Indeed, the cut in fig. 1 in which the form factors have the smallest number of external lines implies that there should exist a presentation of correlation functions such that the numerator factors of the contributing Feynman integrals obey Jacobi identities for all internal lines not attached to an operator, up to contact terms that collapse one of these lines. Depending on the structure of the operators, in the complete correlation function the internal legs directly attached to any one of them may not necessarily obey a Jacobi identity and a case by case study appears necessary. For example, it was shown \cite{76} that scattering amplitudes with an insertion of $\text{Tr}[F^3]$ operator – i.e. the zero-momentum form factors of this operator – exhibit color/kinematics duality; the arguments above imply that we should expect that the correlation functions of several such operators should also exhibit color/kinematics duality in the zero-momentum limit. The fate of color/kinematics duality for the correlation function of $\text{Tr}[F^3]$ operators with generic momenta is an interesting question, albeit one which we will not explore here. In general, color Jacobi identities exist only in the special case in which the color factors of operators contain at most one factor of the symmetric structure constants or their higher-rank generalizations and thus the correlation functions of only such operators may be expected to obey color/kinematics duality for all internal lines; the possible appearance of contact terms in Jacobi transformations on internal lines requires nevertheless a case by case study to ascertain whether color/kinematics duality is in fact realized. This structure is relatively similar to the properties of amplitudes in the $\beta$-deformed $\mathcal{N} = 4$ sYM theory \cite{46}.

Similarly, if excising the operators leads to a planar amplitude\cite{15}, then the terms in the correlation function which contribute to this cut may exhibit some trace of the dual conformal invariance of that amplitude. In the complete correlator the non-planar terms (if present) as well as the presence of the operator insertions will break the usual dual conformal invariance.

Regularization and renormalization are important issues which need to be carefully considered. Since all external lines of momentum space correlation functions (or, equivalently, of source field amplitudes) are effectively massive, no infrared divergences can appear and thus infrared regularization is not necessary. Ultraviolet divergences have two possible origins: divergences due to the structure of the undeformed Lagrangian and divergences due

\footnote{It is important to note that, since the source fields $J$ are color-singlets, planar integrals are only a subset of the leading terms of the larger $N_c$ expansion. Indeed, if the non-planarity of a integral arises solely because external legs (or sources) are attached to internal lines, then that diagram yields, in fact, leading-color contributions. This has already appeared in the calculation of higher-loop form factors \cite{48,50}.}
to the presence of the deformation by the gauge-invariant operators. The former may be eliminated by standard renormalization of the Lagrangian and lead to $\beta$-functions for the various coupling constants. In $\mathcal{N} = 4$ sYM theory they are absent. The later are eliminated by renormalizing the deformations and are solely related to the anomalous dimensions of operators. In particular, if one used renormalized operators, with the renormalization factors given in dimensional regularization by the usual expression in terms of the anomalous dimension of the operator

$$Z_\gamma = \exp\left(\frac{1}{\epsilon} \int_0^1 \frac{dt}{2t} \gamma(t g^2)\right),$$

($(g$ is the gauge coupling constant), the correlation functions would be finite in the ultraviolet as well. In general the anomalous dimension $\gamma$ requires a separate calculation. In $\mathcal{N} = 4$ sYM in the planar limit they are provided by integrability.

One therefore has several options for accounting for the required renormalization: (1) one uses from the outset dimensional regularization and renormalized operators or (2) one carries out the calculation in four dimensions, regularizes dimensionally (if necessary) the integrals appearing in the final result and searches for potentially missing $\mu$-type integral$^{16}$ such that multiplication by the appropriate $Z$-factors $^{16}$ renders it finite as $\epsilon \to 0$ or (3) one uses another regulator, such as a higher-derivative regulator, for both the calculation of the amplitude and the calculation of the renormalization factors.

While the strategy outlined here applies to generic quantum field theories, in the following we will restrict ourselves to $\mathcal{N} = 4$ sYM theory. In this case the sources break maximal supersymmetry to the subalgebra that leaves invariant the deformation ($^4$). BPS operators preserve some amount of supersymmetry, their anomalous dimension vanishes identically and their $Z$-factors equal unity. In their case no regularization is necessary. For the calculation of correlation functions that include non-BPS operators the deformed action ($^4$) is formally nonsupersymmetric$^{17}$. Nevertheless, since supersymmetry breaking is confined to the operator insertions, we expect that most of the features of maximally supersymmetric calculations will continue to exist here as well. Since to leading order (tree-level in position space) all correlation functions are rational functions and at the next-to-leading order only one-loop bubble integrals are divergent, a systematic regularization procedure starts being necessary only at the next-to-next-to-leading.

Quite generally, operators with definite anomalous dimension are (complicated) linear combinations of simpler single-term operators with coefficients given $^{17}$, in the planar limit

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16 We recall here that $\mu$-type integrals are integrals whose integrand vanishes identically when evaluated in four dimensions.

17 The same is true, in fact, for the action deformed by several BPS operators each of which preserves a different non-overlapping or partly overlapping subsets of supercharges.
and for operators with at least one large charge, in terms of the solution to the Bethe equations \[78\]. The form factors of the former operators are linear combinations of the form factors of the latter operators with the same coefficients\[5\]. To compute correlation functions of such operators we evaluate the correlation functions of the generic terms in each of them and then sum them with the appropriate integrability-determined coefficients. This strategy applies both to 3-point as well as to higher-point functions.

As noted in \[54\], scattering amplitudes may be interpreted as the (generalized) form factors of the zero-momentum Lagrangian. Similarly, form factors with more external legs than fields in the operator may be interpreted (e.g. through the MHV vertex expansion) as generalized form factors of the operator and a suitable number of zero-momentum on-shell Lagrangians. In particular, scattering amplitudes with more than four external fields can be interpreted as generalized form factors of several on-shell Lagrangians. It then follows that the generalized unitarity-based construction of the integrand of the next \(k\)-to-leading order momentum space correlation functions of some operators is the same as the construction of the leading order correlation function of those operators and \(k\) on-shell Lagrangians\[19\]. We therefore see a formal parallel between the generalized unitarity calculation and the Lagrangian insertion method of \[9, 10, 11\].

The various monomials that enter the expression of operators with definite anomalous dimension may not always have the same number of fields. A simple example is the stress tensor, whose expression contains a bilinear in the field strength which has two, three and four fields. This is, in fact, the generic structure of terms on the higher levels of supersymmetry multiplets with the increase in the number of fields being due to non-linear terms in supersymmetry transformations. Clearly, these terms contribute to form factors with different number of external legs; in the standard organization of the Lagrangian, in which the coupling constant appears as an overall factor, these terms contribute to different loop orders.\[20\] It has been suggested in \[79, 80, 81\] their contribution cancels against contact terms arising from terms proportional to the equations of motion from Lagrangian insertions and, at least for the purpose of constructing the null limit of correlation functions, they may be set them aside. Their close relation to contact terms (which in a momentum space framework should appear as Feynman integrals with cancelled propagators) suggests that a shortcut to determining the contribution of nonlinear terms in supersymmetry transformations to correlation functions may be imposing the Ward identities of the various symmetries on (the Euclidian) correlation functions found in the absence of these terms \[74, 75, 11\].

\[18\]Since the \(L\)-loop mixing coefficients are proportional to \(\lambda^{L-1}\), the \(L\)-loop form factor requires use of the \((L+1)\)-loop eigenvectors of the dilatation operator.

\[19\]At each order beyond the leading order one needs to add one additional internal line and hence one additional operator with at least three fields.

\[20\]From a position space perspective one might be tempted to interpret all of them as contributing to the same order – e.g. at tree level – as they do not involve any integration.
3 Some single-operator and multi-operator form factors

We have argued in the previous section that tree-level form factors and generalized form factors are essential for the construction of correlation functions. They may be constructed in several ways, mirroring the various methods for the construction of tree-level scattering amplitudes. As in that case it is useful to assemble them into super-form factors; they are labeled by the coordinates of two superspaces: the usual on-shell $\mathcal{N} = 4$ superspace, with Grassmann coordinates $\eta^A$ and an off-shell superspace for the multiplet of operators, with Grassmann coordinates $\gamma$. A convenient one is the harmonic superspace \[82\]; the on-shell superspace fields\[21\] may also be written in harmonic superspace form \[80, 81\]. Whenever necessary, we will follow the notation there as well as in \[54, 83\] and collect some details in Appendix A.

A possible approach to the construction of (super-)form factors is to use the MHV vertex rules. This strategy was first applied to the construction of form factors of a particular gluon operator in \[84\] and more recently in \[85\]. This method requires independent information on MHV form factors; they are defined to be those that have the minimal number of fermionic coordinates $\eta$. This definition mirrors that of MHV amplitudes.

BCFW-like recursion relations may also be used if a suitable shift can be found. In this approach only the tree-level form factor with the minimal number of fundamental fields is necessary. As in the case of scattering amplitudes of fundamental fields, different BCFW shifts yield different presentations of the same form factors.

In this section we collect several examples of form factors and generalized form factors with arbitrary number of external states which will be useful in the examples of correlation functions we will discuss in the next section.

3.1 The tree-level MHV form factors and generalized form factors of the chiral stress tensor multiplet

The tree-level MHV super-form factor $F_{T}^{\text{MHV}}$ of the chiral part of the stress tensor multiplet $\mathcal{T}$ was found in ref. \[54, 83\] through a combination of symmetry constraints and explicit calculations:

$$F_{T}^{\text{MHV}} = \prod_{i=1}^{n} \frac{1}{(q, q + 1)} \delta^{(4)}(q - \sum_i \lambda_i \lambda_{i+1}) \delta^{(4)}(\gamma_+ - \sum_i \lambda_i \eta_{+i}) \delta^{(4)}(\sum_i \lambda_i \eta_{-i})$$ \hspace{1cm} (11)

---

\[21\] Specific on-shell external fields may be extracted in the usual way, by specifying their position in the on-shell $\mathcal{N} = 4$ multiplet

$$\Phi(\eta) = g_+ + \eta_A \psi^A + \frac{1}{2!} \eta_A \eta_B \phi^{AB} + \frac{1}{3!} \eta_A \eta_B \eta_C \epsilon^{ABCD} \psi^D + \frac{1}{4!} \eta_A \eta_B \eta_C \eta_D \epsilon^{ABCD} g_+$$.
The two $\delta$–functions may be combined into a single one by multiplying their arguments with the suitable harmonic variables:

$$F_{\text{MHV}}^{\text{MHV}T} = \prod_{i=1}^{n} \frac{1}{(i, i + 1)} \delta^{(4)}(q - \sum_{i} \lambda_i \bar{\lambda}_i) \delta^{(8)}(\sum_{i=1}^{n} \eta_{Ai} \lambda_i^a - 1^{+}_{Aa} \gamma_{1+}^{\alpha \alpha}) .$$  \hfill (12)

The form factor of the chiral primary operator is extracted as the coefficient of $(\gamma_+)^4$ while that of the on-shell Lagrangian as the coefficient of $(\gamma_+)^0$.

The simplicity of the MHV form factor resembles that of MHV amplitudes. As discussed in [54], the conjugate super-form factor may be obtained by conjugation and Grassmann Fourier transform of the super-form factor of the chiral primary operator:

$$F_{\text{MHV}}^{\text{maxMHV}} = \delta^{4}(q - \sum_{i} \lambda_i \bar{\lambda}_i) \delta^{(4)}(\gamma_+ - \sum_{i} \lambda_i \eta^+ \gamma_{+i}) \prod_{i=1}^{n} \frac{1}{[i, i + 1]} \int \prod_{i} d^4 \eta_i e^{i n_{Ai} \bar{\eta}_i a} \delta^{(4)}(\sum_{i} \lambda_i \bar{\eta}_i A_{+i}) .$$ \hfill (13)

An alternative expression, in terms of a single auxiliary Grassmann integral, is

$$F_{\text{MHV}}^{\text{maxMHV}} = \delta^{4}(q - \sum_{i} \lambda_i \bar{\lambda}_i) \delta^{(4)}(\gamma_+ - \sum_{i} \lambda_i \eta^+ \gamma_{+i}) \prod_{i=1}^{n} \frac{1}{[i, i + 1]} \int d^4 \omega_{\alpha} \delta^{(2)}(\omega_{\alpha} \lambda_i^\alpha - \eta_{-i}) .$$ \hfill (14)

One might wonder whether there exist generalized MHV form factors – i.e. MHV form factors with several operator insertions. The supersymmetry Ward identities discussed in [54] may be easily extended to such cases: for the chiral stress tensor multiplet, a possible solution is obtained by simply replacing $\gamma_+$ by the sum of the $\gamma_+$ coordinates of all the operators, $\sum_i \gamma_{+i}$:

$$F_{\text{MHV}}^{\text{MHV}}(\gamma_+^{\alpha} \gamma_+^{\beta}) = F \delta^{(4)}(q - \sum_{i} \lambda_i \bar{\lambda}_i) \delta^{(4)}(\sum_{j=1}^{k} \gamma_{+i}^{\alpha} - \sum_{i} \lambda_i \eta^+ \gamma_{+i}) \delta^{(4)}(\sum_{i} \lambda_i \eta^{-i}) ,$$ \hfill (15)

with some coefficient $F$. This expression however contains at most four $\gamma_+$ coordinates and therefore will not contain e.g. the generalized form factor of two chiral primary operators (CPO-s) (which would require eight $\gamma_+$ coordinates).

We may alternatively consider a generalization of eq. (12) in which we replace $1^{+}_{Aa} \gamma_{1+}^{\alpha \alpha}$ by $\sum_{k=1}^{n} k^{+}_{+} A_{a} \gamma_{k+}^{\alpha \alpha}$. However, extracting various components – such as the two-CPO component – out of this expression one quickly finds that they cannot be generated by Feynman diagrams; for example, the two-CPO form factor will not have any scalar external lines.

---

\textsuperscript{22}The supersymmetry Ward identities discussed in [54] suggest that $F = \prod_{m=1}^{n} (m, m + 1)^{-1}$.  
\textsuperscript{23}It contains however the generalized form factor of the CPO and any number of chiral Lagrangians.
Last, one could potentially imagine an MHV form factor with different harmonic variables for each operator that would not allow for a representation in terms of a single eight-dimensional Grassmann $\delta-$function as it would only satisfy the supersymmetric Ward identities up to equations of motion. However, simple counting suggests that the generalized form factor of two CPO-s cannot appear in an MHV generalized form factor because the relevant form factor should have at least twelve Grassmann $\delta-$functions (eight to saturate the eight $\gamma$ integrals isolating the two chiral primaries and four isolating two external scalar fields).

3.2 Generalized tree-level NMHV form factor of two BPS operators

Generalized form factors, as well as the $N^k$MHV form factors of $(k+1)$ BPS operators, are an important ingredient in the construction of higher-loop corrections to correlation functions of CPO-s; in particular, they are necessary to identify potentially degenerate contributions in which propagators between operators are canceled. The evaluation of this form factor turns out to be most efficient through the MHV vertex expansion $[86, 84]$. We should sum over all graphs with two vertices in which either vertex is a super-form factor, fig. 2(a), or one vertex is a generalized two-operator MHV form factor and the other one is a regular MHV amplitude, fig. 2(b):

$$F^{NMHV} = F^{NMHV}_{(a)} + F^{NMHV}_{(b)}.$$  \hspace{1cm} (16)

Since, as we argued in the previous section, an MHV generalized form factor for the lowest component of the chiral stress tensor multiplet does not exist, the diagrams of the second type have vanishing contributions

$$F^{NMHV}_{(b)} = 0.$$ \hspace{1cm} (17)

For higher components of the chiral stress tensor multiplet this contribution may be nonvanishing; we will however not be interested in it here.

Relegating the details to Appendix 13 we list here the final result for $F^{NMHV}_{(a)}$:

$$F^{NMHV}_{(a)} = \delta^8(1^{a_1 a_2} \gamma^{a_3}_{1+} + 2^{a_1 a_2} \gamma^{a_3}_{2+} - \sum_{m=1}^{n} \eta m \lambda m) \delta^4(q_1 + q_2 - p_1 \cdots - p_n) \times \frac{1}{3! \prod_{m=1}^{n} (m m + 1)} \left[ \sum_{w=1}^{2} \sum_{i=1}^{n} \sum_{j=i+1}^{i-2} A_{ij}^{\tilde{w}} + \sum_{w=1}^{2} \sum_{i=1}^{n} D_{i}^{\tilde{w}} \right],$$ \hspace{1cm} (18)
Figure 2: The two generic terms in the MHV vertex expansion of the generalized super-form factor with two insertions of the chiral stress tensor. Grey blobs represent regular single-operator MHV super-form factors, the hashed blob represents an MHV generalized super-form factor and the white blob represents a regular MHV scattering super-amplitude. The momenta carried by the two operators are $q_1$ and $q_2$.

where $A_{ij}^\omega$ and $B_i^\omega$ are defined as:

$$A_{ij}^\omega = \varepsilon^{TABC}(\eta_{Ti}(Q_{ij}^\omega)^2 - w_{Ta}^+(\gamma_{\bar{w}_+}^a Q_{ij}^\omega|i) + \sum_{r=i}^j \eta_{Tr}(r|Q_{ij}^\omega|i)(w_{Ab}^+(\gamma_{\bar{w}_+}^b ij) - \sum_{r=i}^j \eta_{Ar}(ri))$$

$$\times \left(w_{Bc}^+(\gamma_{\bar{w}_+}^c i) - \sum_{r=i}^j \eta_{Br}(ri))\right)(w_{Cd}^+(\gamma_{\bar{w}_+}^d i) - \sum_{r=i}^j \eta_{Cr}(ri)) \frac{1}{(Q_{ij}^\omega)^2(Q_{i+1j}^\omega)^2} \langle jj + 1 \rangle \langle ji \rangle \langle ij + 1 \rangle$$

$$\text{and}$$

$$B_i^\omega = -\frac{3w_{Ab}^+ w_{Be}^+ \varepsilon^{be'} \varepsilon^{TABC}}{2(Q_{ii}^\omega)^2} \left( - \eta_{Ti}\eta_{Ci}\epsilon_{be} \langle \gamma_{\bar{w}_+}^c i \rangle^2 - \frac{1}{4} \epsilon^{a'd'} w_{Ta}^+ w_{Cd}^+ \langle \gamma_{\bar{w}_+}^a i \rangle \langle \gamma_{\bar{w}_+}^d i \rangle^2 \right)$$

It is not difficult to check that the simplest such form factor has two external scalars and it has no momentum dependence, as one might expect based on Feynman diagram calculation.

### 3.3 Tree-level MHV form factor of a non-BPS scalar operator

As discussed in §2, one of the building blocks of correlation functions of non-BPS operators are their tree-level form factors and generalized form factors. Non-BPS operators with definite scaling dimension are typically linear combinations of simpler single-term operators with the same quantum numbers. Their relative coefficients are functions of the coupling constant and, at least for long operators, may be determined using the integrability of the dilatation operator of $\mathcal{N} = 4$ sYM theory [77, 78]. The form factors of non-BPS operators may be therefore interpreted — order by order in perturbation theory — as the sum of form factors of these single-term operators with the same coupling-dependent coefficients. We thus need to focus on them.
The operator mixing determining the eigenvectors of the dilatation operator is, in general, rather complicated with the only constraints arising from charge conservation and the fact that such mixing can occur only between operators with the same classical dimension. There exist, however, "closed sectors" \[87\] in which operator mixing involves a very restricted class of operators. An example is the so-called $SU(2)$-sector which contains operators of any dimension constructed out of two complex scalar fields which are not conjugate of each other.

The R-symmetry properties of the $\mathcal{N} = 4$ SYM fields implies that, at tree-level, it is more efficient to evaluate the form factor of scalar operators of arbitrary charges rather than restricting ourselves to specific charges. We carry out this calculation in Appendix \[B\] using a BCFW recursion relation. Defining

$$H_{aAbB} = \eta_{Aa}\eta_{Bb} - \eta_{Ba}\eta_{Ab} + \delta_{ab}\eta_{Ba}\eta_{Ab},$$

(21)

$$\left(\Sigma_{a_{1}b_{1}}\right)^{\alpha\gamma} = \lambda_{a_{1}}^{\alpha}\lambda_{b_{1}}^{\gamma} \varepsilon_{\beta\gamma},$$

(22)

the MHV super-form factor of the operator $O_{A_{1}B_{1}...A_{k}B_{k}} = \text{Tr}(\phi_{A_{1}B_{1}}...\phi_{A_{k}B_{k}})$ with $n$ external fields is given by:

$$\langle \tilde{O}_{A_{1}B_{1}...A_{k}B_{k}}(q)|1...n\rangle = \frac{\delta^{4}(q - \sum_{i=1}^{n}p_{i})}{\prod_{m=1}^{n}\langle m, m+1 \rangle} \sum_{\{a_{1}, b_{1}...a_{k}, b_{k}\}} \left(\prod_{i=1}^{k} H_{a_{i}A_{i}b_{i}B_{i}}\right) \text{Sp}\left(\prod_{j=1}^{k} \Sigma_{a_{j}b_{j}}\right),$$

(23)

where the first sum runs over all over the sets $\{a_{1}, b_{1}...a_{k}, b_{k}\}$ with $a_{1} \leq b_{1} < a_{2} \leq b_{2}...b_{k-1} < a_{k} \leq b_{k}$ and its cyclic permutations and $\text{Sp}$ stands for the trace over the $SL(2,C)$ indices of the product of $\Sigma$ matrices. By restricting the pairs $(A_{i}, B_{i})$ to only two values, e.g. $\{(1,2), (1,3)\}$ or $\{(1,2), (1,4)\}$, one finds the $SU(2)$ sector operators and their form factors.

The tree-level form factors \[23\] can be used to construct correlation functions\[24\]; one first constructs the correlation functions of the single-term operators $O_{A_{1}B_{1}...A_{k}B_{k}}$, then one takes appropriate linear combinations with integrability-determined coefficients and finally one renormalizes the result by multiplying with the corresponding factors $Z_{\gamma_{i}}$, cf. eq. \[10\].

### 3.4 Tree-level MHV form factor of a general twist-2 operator

In the next section we will construct examples of $n$-point functions with one twist-2 spin-$S$ operator. These operators are linear combinations of $O_{2, S, x}^{AB, CD} = \text{Tr}(D_{+}^{x}\phi^{AB}D_{-}^{S-x}\phi^{CD})$

$$O_{2, S}^{AB, CD} = \sum_{n=0}^{S} c_{S,n}(\lambda)O_{2, S, n}^{AB, CD},$$

(24)

\[24\] They could, of course, also be used to construct higher-loop form factors for these operators, determine their mixing and anomalous dimensions, etc.
where the coefficients $c_n(\lambda)$ are determined by requiring that these operators have definite anomalous dimensions. At one loop and if $\phi^{AB} = \phi^{CD} = Z$, they are given\textsuperscript{25} in terms of the Gegenbauer polynomials\textsuperscript{25}:

$$O_{2,S} = (n \cdot D_1 + n \cdot D_2)^S C_S^{1/2} \left. \left( \frac{n \cdot D_1 - n \cdot D_2}{n \cdot D_2 + n \cdot D_1} \right) \text{Tr}[Z(\xi_2)Z(\xi_1)] \right|_{\xi_1 = \xi_2},$$

(25)

where $n \cdot D$ are covariant derivatives in the adjoint representation in the light-like direction specified by the vector $n$; one possible choice is $n \cdot D = D_+$. The one-loop coefficients in (24) are $c_{S,n,0} = (-1)^n \left( \frac{S}{2} \right)$; the two-loop coefficients may be found in\textsuperscript{39}.

We will describe in Appendix B the evaluation of the form factors of the operators $O_{2,S}^{AB,CD}$ through a BCFW recursion relation. It is convenient to express the result in terms of the $\eta$-dependent combinations $H$ in eq. (21):

$$\langle \tilde{O}_{2,S,x}^{AB,CD} | 1 \ldots n \rangle = \frac{\delta^4(q - \sum_{i=1}^n p_i)}{\prod_{m=1}^n (m, m+1)} \sum_{\{a,b,c,d\}} H_{aABB}H_{cCDd} \sum_{k=b}^{c-1} \sum_{l=d}^{a-1} \left( \sum_{r=l+1}^{k} p_r \right)^x \left( \sum_{s=k+1}^{l} p_s \right)^{S-x} \times \left( \frac{\langle b | \sigma^- p_k | c \rangle}{2p_k} + \frac{\langle b | \sigma^- p_{k+1} | c \rangle}{2p_{k+1}} - \langle bc \rangle \right) \left( \frac{\langle d | \sigma^- p_l | a \rangle}{2p_l} + \frac{\langle d | \sigma^- p_{l+1} | a \rangle}{2p_{l+1}} - \langle da \rangle \right),$$

(26)

where $q$ is the momentum of $\tilde{O}$, the sum runs over all sets $\{a, b, c, d\}$ where $a \leq b < c \leq d$ or $d < a \leq b < c$ etc. For small number of external particles this expression may be easily verified using Feynman graphs. As in the case of the scalar non-BPS operators, taking the appropriate linear coupling constant-dependent combinations of these expressions yields the form factors of twist-2 spin-$S$ operators with definite anomalous dimensions.

4 Examples of correlation function construction

Using the form factors constructed in the previous section we shall construct examples of correlation functions to leading order (LO) and next-to-leading order (NLO). Some of these correlators have been known for some time and we reproduce their expressions. We will also discuss various limits and properties of our results.

4.1 Correlators of BPS operators

Correlation functions of four chiral stress tensor multiplets have been evaluated to high loop order\textsuperscript{19, 20} using a hidden symmetry of their position space integrand\textsuperscript{18}. We will discuss
here, to a much lower loop order, a unitarity-based approach to the same correlator. As we will see, this construction generalizes quite easily to correlators of any number of operators.

The simple structure of the CPO-s implies that the construction of their 4-point function through generalized unitarity is almost identical to that in terms of Feynman diagrams. From (11) it follows that the (MHV or anti-MHV) form factor with two external lines of the chiral primary operator has no momentum dependence, which is the same as the vertex containing the source of the operator and two scalars. For more operators we need to use the generalized form factors; nevertheless, for scalar operators, they are quite similar to the results of Feynman diagram calculations.

R-charge conservation implies that the correlation function of four CPO-s (two-index symmetric traceless dimension-2 operators) is determined by six functions of the positions of the operators [51]:

\[
\langle \text{Tr}[\phi^{I_1}\phi^{J_1}]\text{Tr}[\phi^{I_2}\phi^{J_2}]\text{Tr}[\phi^{I_3}\phi^{J_3}]\text{Tr}[\phi^{I_4}\phi^{J_4}] \rangle = a_1 \delta_{12}^2 \delta_{34}^2 + a_2 \delta_{13}^2 \delta_{24}^2 + a_3 \delta_{14}^2 \delta_{23}^2 + b_1 \delta_{13} \delta_{14} \delta_{23} \delta_{24} + b_2 \delta_{12} \delta_{14} \delta_{32} \delta_{34} + b_3 \delta_{12} \delta_{13} \delta_{42} \delta_{43} .
\]

with

\[
\delta_{ij}^2 = \delta_{\{I_i\}}^I \delta_{\{J_i\}}^J, \quad \delta_{13} \delta_{14} \delta_{23} \delta_{24} = \delta_{\{I_{14}\}} \delta_{\{J_{14}\}} \delta_{\{I_{23}\}} \delta_{\{J_{23}\}} ,
\]

where \(I_i\) and \(J_i\) are \(SO(6)\) indices. As mentioned before, since the sources are singlets under the \(\mathcal{N} = 4\) gauge group, each term in the momentum space correlator may be extracted from un-ordered source-field scattering amplitudes.

**4.1.1 Leading order**

As it is well-known, at this order the contributions to the (Fourier-transform of the) coefficient functions \(a_i\) come only from disconnected graphs. In our language, the generalized (quadruple) cut contributing to e.g. \(a_1\) is shown in fig. 3(a). The \(b_i\) coefficients receive contributions only from connected graphs. To determine them one may consider two-particle
cuts with the 2-operators form factors discussed in § 3.2. A close inspection of these form factors with two external fields implies however that there is always a propagator between the two operators; this in turn implies that the leading order correlator is determined by its quadruple cuts, such as the one shown in fig. 3(b) and its non-cyclic permutations. Depending on the specific choice of R-charge carried by the four operators only some of these cuts may exist.

Choosing as representatives of the four CPO-s the operators \( \mathcal{O}_1 = \text{Tr}[\phi^{12}\phi^{12}] \), \( \mathcal{O}_2 = \text{Tr}[\phi^{34}\phi^{34}] \), \( \mathcal{O}_3 = \text{Tr}[\phi^{13}\phi^{13}] \) and \( \mathcal{O}_4 = \text{Tr}[\phi^{24}\phi^{24}] \) only the disconnected quadruple cut is non-zero. This correlation function determines the coefficient \( a_1 \). The relevant form factors which determine the quadruple cut are extracted from (11) (by appropriately choosing the harmonic variables) and are constant; it is therefore trivial to find that the momentum space representation of this correlator and thus of \( a_3 \) is

\[
\tilde{a}_3 = \langle \tilde{\mathcal{O}}_1(q_1)\tilde{\mathcal{O}}_2(q_2)\tilde{\mathcal{O}}_3(q_3)\tilde{\mathcal{O}}_4(q_4) \rangle
\]

\[
= \delta(q_1 + q_2)\delta(q_3 + q_4) \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2(p + q_1)^2} \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2(l + q_3)^2} .
\]

(29)

(30)

It is easy to Fourier-transform this expression to position space before the \( p \) and \( l \) integrals are carried out, with the expected result. By permuting the labels of the various operators it is not difficult to find the coefficients \( a_1 \) and \( a_2 \).

Choosing as representatives of the four CPO-s the operators \( \mathcal{O}_1 = \text{Tr}[\phi^{12}\phi^{23}] \), \( \mathcal{O}_2 = \text{Tr}[\phi^{14}\phi^{24}] \), \( \mathcal{O}_3 = \text{Tr}[\phi^{13}\phi^{13}] \) and \( \mathcal{O}_4 = \text{Tr}[\phi^{24}\phi^{34}] \) the only non-zero quadruple cut is the one shown in fig. 3(b). From the R-charge assignment it is easy to see that this correlator determines the coefficient \( b_2 \) in eq. (27). The quadruple cut may be easily evaluated to be consistent with

\[
\tilde{b}_2 = \langle \tilde{\mathcal{O}}_1(q_1)\tilde{\mathcal{O}}_2(q_2)\tilde{\mathcal{O}}_3(q_3)\tilde{\mathcal{O}}_4(q_4) \rangle = \delta^4 \sum_{i=1}^4 q_i \mathcal{I}_{4m}[1](q_1, q_2, q_3, q_4) .
\]

(31)

where \( \mathcal{I}_{4m}[1](q_1, q_2, q_3, q_4) \) is the standard 4-mass scalar box integral. Previous arguments, based on the structure of the generalized form factor with two CPO insertions, imply that this should be the complete result. One may nevertheless check the relevant 2-particle cuts are correctly reproduced. Thus, this is the complete momentum-space form \( \tilde{b}_2 \) of \( b_2 \). As in the case of the disconnected contribution, it is not difficult to Fourier-transform this expression\(^{27}\) and recover the standard position-space form of \( b_2 \):

\[
b_2 = \frac{1}{x_{12}^4 x_{23}^4 x_{34}^4 x_{41}^4} .
\]

(32)

\(^{26}\)Some UV regularization is assumed.

\(^{27}\)To this end one treats independently the momentum of each propagator and introduces a \( \delta \)-function for each three-point vertex. Using an integral representation of these four \( \delta \)-functions makes all integrals trivial.
This expression is annihilated by the special conformal generators; consequently, their momentum space expression should annihilate the 4-mass box integral.

4.1.2 Next-to-leading order

As in the case of regular scattering amplitudes, we may either proceed systematically through the maximal cut method or, in simple cases, bypass some steps analyze directly a spanning set of color-dressed cuts. In the case at hand – the correlation function of four CPO-s – we may use the fact that we expect that it should be UV-finite. Thus, all contributing Feynman integrals should have at least five propagators. Ignoring disconnected and non-1PI contributions, this implies that the contributions to the momentum space form of the connected part of the coefficient functions should be determined by the cuts in fig. 4 as well as the permutations of their external leg labels. As explained in §2, each blob is either a color-dressed form factor or a color-dressed amplitude; depending on the specifics of operators, color-dressed form-factors of single-trace operators may be – up to the color factor – the same as color-ordered form factors. An example of such an operator is the CPO.

Figure 4: Color-dressed cuts for connected part of the next-to-leading order 4-point correlator. There are 2 cuts of type (a), 4 cuts of type (b), 4 cuts of type (c) and 2 cuts of type (d). Other members of each set are related by relabeling. Each blob with an external line is a (generalized) color-dressed form factor; blobs with no external lines are color-dressed on-shell scattering amplitudes.

The color algebra is nontrivial only for the cuts in figs. 4(a) and 4(b). Writing the 4-point amplitude as

\[ A(l_1, l_2, l_3, l_4) = M_1 f_{a_1a_2d} f_{a_3a_4d} + M_2 f_{a_1a_3} f_{a_4a_2d} + M_3 f_{a_1a_4} f_{a_2a_3d} \]  

28 Depending on the complexity of the correlator one may also use integral reduction strategy of generalized to all-massive external legs, which reduces integrals to a basis at the level of maximal cuts.

29 A divergence would signal the presence of an anomalous dimension for at least one of the operators, which contradicts the fact that the operators are chiral primaries.

30 The cut in fig. 4(a) also captures the non-1PI correction to the correlation function. The additional cut that is needed to determine the disconnected part may be interpreted as determining the next-to-leading order correction to the two-point function of two of the four operators.
and using the fact that the 2-point form factors are just Kronecker delta-functions in color space (with indices in the adjoint representation – e.g. the color factors of the two form factors at the left of the cut in fig. 4(a) are just δ_{a_1a_4}) it follows that the cuts in figs. 4(a) and 4(b) are proportional to

\[(M_2 - M_1)f^{a_1a_4}f^{a_4a_2d}\delta_{a_1a_4}\delta_{a_3a_2}.\]  

(34)

Further using the relation between \(M_i\) and color-ordered amplitudes it follows that the cuts in figs. 4(a) and 4(b) are

\[
C_{4(a)} = \tilde{F}(q_1, l_1, l_2)\tilde{F}(q_2, l_6, l_2)\tilde{F}(q_3, l_3, l_6)\tilde{F}(q_4, l_5, l_4)A(l_1, l_3, l_4, l_2) \tag{35}
\]

\[
C_{4(b)} = \tilde{F}(q_1, l_1, l_4)\tilde{F}(q_2, l_5, l_2)\tilde{F}(q_3, l_6, l_5)\tilde{F}(q_4, l_3, l_6)A(l_1, l_3, l_4, l_2) \tag{36}
\]

where \(A(l_1, l_3, l_4, l_2)\) is the color-ordered amplitude\(^{31}\) with the relevant external legs\(^\text{32}\). This expression may also be justified using the photon decoupling identity.

As in the leading order case, choosing as representatives of the four CPO-s the operators \(\mathcal{O}_1 = \text{Tr}[\phi^{12}\phi^{13}], \mathcal{O}_4 = \text{Tr}[\phi^{24}\phi^{34}], \mathcal{O}_2 = \text{Tr}[\phi^{13}\phi^{15}]\) and \(\mathcal{O}_3 = \text{Tr}[\phi^{24}\phi^{25}]\) determines the contribution to the coefficient \(a_3\). It is not difficult to see that the cuts in figs. 4(b), (c) and (d) vanish identically because \(SO(6)\) invariance forbids propagators between form factors.

The cut in fig. 4(a) is

\[
C_{4(a)} = 1 \tag{37}
\]

which implies that \(\tilde{a}_3\) is given by the bow tie integral (see fig. 5(a)):

\[
\tilde{a}_3 = \int \frac{d^4p d^4l}{(2\pi)^8} \frac{1}{p^2(p + q_1)^2(p + q_1 + q_4)^2 l^2(l + q_2)^2(l + q_2 + q_3)^2} = \text{BTie}(4,1|2,3) ; \tag{38}
\]

This may be Fourier-transformed to position space with the result

\[
a_3 = \frac{1}{x_1^2 x_2^2 x_3^2} \int \frac{d^4x_0}{x_{01}^2 x_{02} x_{03}^2 x_{04}^2} . \tag{39}
\]

Permuting the labels of the operators we may similarly obtain \(a_1\) and \(a_2\).

To determine the NLO correction to \(b_2\) we choose as representatives of the four CPO-s the operators to be \(\mathcal{O}_1 = \text{Tr}[\phi^{12}\phi^{23}], \mathcal{O}_2 = \text{Tr}[\phi^{14}\phi^{24}]\mathcal{O}_3 = \text{Tr}[\phi^{13}\phi^{15}]\) and \(\mathcal{O}_4 = \text{Tr}[\phi^{24}\phi^{34}]\).

All cuts are nontrivial:

\[
C_{4(a)} = \frac{\langle l_1 l_4 \rangle \langle l_2 l_3 \rangle}{\langle l_1 l_3 \rangle \langle l_4 l_2 \rangle} = \frac{(q_1 + q_4)^2}{(l_1 + l_3)^2} , \quad C_{4(b)} = \frac{\langle l_1 l_4 \rangle \langle l_2 l_3 \rangle}{\langle l_1 l_3 \rangle \langle l_4 l_2 \rangle} = \frac{q_1^2}{(l_1 + l_3)^2} \tag{40}
\]

\[
C_{4(c)} = \frac{1}{2l_1 \cdot l_2} \left[ \frac{\langle l_1 l_3 \rangle \langle l_1 l_4 \rangle}{\langle l_2 l_3 \rangle \langle l_2 l_4 \rangle} + \frac{\langle l_2 l_3 \rangle \langle l_2 l_4 \rangle}{\langle l_1 l_3 \rangle \langle l_1 l_4 \rangle} + 2 \right] , \quad C_{4(d)} = \frac{\langle l_1 l_3 \rangle \langle l_4 l_5 \rangle}{\langle l_1 l_2 \rangle \langle l_2 l_3 \rangle \langle l_2 l_5 \rangle \langle l_4 l_2 \rangle} . \tag{41}
\]

\(^{31}\)This becomes the corresponding super-amplitude if one uses super-form factors.

\(^{32}\)The coefficients \(M_i\) in the four-point amplitude \((33)\) may be chosen to obey color/kinematics duality \((35)\) on the unique off-shell leg of this amplitude. Thus, the corresponding cuts in figs. 4(a) and 4(b) obey it as well.
Using the methods developed for the construction of scattering amplitudes of fundamental fields (e.g. matching onto an ansatz, or simply expressing the cuts in terms of momenta and inspecting the result) it is not difficult to find (the integral representation of) a function that has these cuts:

\[ \tilde{b}_2 = \delta^{(4)} \left( \sum_{i=1}^{4} q_i \right) \left[ (q_1 + q_2)^2 DB(1, 2|3, 4) + (q_1 + q_4)^2 DB(4, 1|2, 3) + q_3^2 TriP(1|2, 3, 4) + q_4^2 TriP(2|3, 4, 1) + q_3^2 TriP(3|4, 1, 2) + q_4^2 TriP(4|1, 2, 3) \right. \\
- TriB(1|2, 3, 4) - TriB(2|1, 3, 4) - TriB(2|3, 4, 1) - TriB(3|2, 4, 1) - TriB(3|4, 1, 2) - TriB(4|3, 1, 2) - TriB(4|1, 2, 3) - TriB(1|4, 2, 3) \]

Here DB(a, b|c, d) is the double-box integral with legs a and b on one one-loop box sub-integral, TriP(a|b, c, d) is the triangle-pentagon integral with external leg a on the triangle point and TriB(a|b|c, d) is the triangle-box integral with external leg a at the 4-point vertex and external leg b on the triangle point; these integrals are shown in figs. (b), (c) and (d), respectively.

The expression for \( \tilde{b}_2 \) in eq. (42) can be Fourier-transformed to position space; we immediately recover the expression for the integral representation of the coefficient \( b_2 \) found in [51]. The easiest way to carry out the Fourier-transform is to (1) treat the momentum of each propagator as independent and introduce the appropriate \( \delta \)-functions enforcing momentum conservation at each vertex (2) use an integral parametrization of the \( \delta \)-functions to carry out the momentum integrals and (3) at this stage the Fourier-transform becomes trivial and identifies the integration parameters for the \( \delta \)-functions of vertices with external lines with the position of the operators. The resulting integrals (over the position of the vertices with no external lines) may be further reduced using a strategy outlined in the Appendix A of [90] and we find the expected result [52]; we collect some of the details in Appendix C.

4.1.3 Brief comments on supersymmetric methods

While the calculation in the previous subsection was carried out in components, it is not difficult to streamline it by making use of the super-form factor of the chiral stress tensor
multiplet. In such an approach it is not necessary to choose representatives of the CPOs; rather, products of the harmonic variables appearing in each form factor play the role of the various $\delta$–functions appearing in eq. (27). In the following we will continue constructing component correlation functions; in this subsection however we will illustrate the supersymmetric methods and recover the LO and NLO correction to the four-point correlation function of chiral stress tensor multiplets.

The quadruple cut in fig. 3(b) is given by

$$
\langle \mathcal{T}(q_1)\mathcal{T}(q_2)\mathcal{T}(q_3)\mathcal{T}(q_4)\rangle \mid_{4\text{-cut}} = \delta^4(\sum q_i) \prod_{i=1}^{4} \frac{1}{\langle l_{i+1} l_i \rangle} \int d^4\eta_1 d^4\eta_2 d^4\eta_3 d^4\eta_4 \tag{43}
$$

$$
\begin{align*}
\delta^{(4)}(\gamma_{1+} - l_1\eta[1]_{+;1} + l_2\eta[1]_{+;2})
\delta^{(4)}(l_1\eta[1]_{-;1} - l_2\eta[1]_{-;2})
\delta^{(4)}(l_2\eta[2]_{+;2} + l_3\eta[2]_{+;3})
\delta^{(4)}(l_2\eta[2]_{-;2} - l_3\eta[2]_{-;3})
\delta^{(4)}(\gamma_{3+} - l_3\eta[3]_{+;3} + l_4\eta[3]_{+;4})
\delta^{(4)}(l_3\eta[3]_{-;3} - l_4\eta[3]_{-;4})
\delta^{(4)}(\gamma_{4+} - l_4\eta[4]_{+;4} + l_1\eta[4]_{+;1})
\delta^{(4)}(l_4\eta[4]_{-;4} - l_1\eta[4]_{-;1})
\end{align*}
$$

Carrying out the $\eta$ integrals (either directly or though the methods described in [91]) we find the cut of a 4-mass box integral multiplied by the product of harmonic variables (12)(23)(34)(41) and $\delta^{(4)}(\gamma_{1+})\delta^{(4)}(\gamma_{2+})\delta^{(4)}(\gamma_{3+})\delta^{(4)}(\gamma_{4+})$, implying that only the CPO components have non-vanishing four-point function [3].

Through similar manipulations and using the higher-point form factors [11] it is not difficult to compute the cuts in fig. 4 up to a factor of $\delta^{(4)}(\gamma_{1+})\delta^{(4)}(\gamma_{2+})\delta^{(4)}(\gamma_{3+})\delta^{(4)}(\gamma_{4+})$, cuts (a) and (d) are:

$$
C_{[3,a]} = (14)(23) + (12)(23)(34)(41) \frac{(q_1 + q_4)^2}{(l_1 + l_2)^2} + (13)(32)(24)(41) \frac{(q_1 + q_4)^2}{(l_1 + l_2)^2} \tag{44}
$$

$$
C_{[3,d]} = (12)(23)(34)(41) \left[ -\frac{(q_1 + q_4)^2}{(l_1 + l_2)^2} \frac{(l_2 - l_3)^2}{(l_2 - l_3)^2} + \frac{l_4^2}{(l_2 + l_3)^2} - \frac{(q_1 + q_4)^2}{(l_1 + l_2)^2} \frac{(l_2 - l_3)^2}{(l_2 - l_3)^2} \right] \tag{45}
$$

We recognize the various momentum-dependent factors as the component cuts found previously. Together with the cuts obtained by permuting the labels of the external lines these expressions can be used to recover the results of the §4.1.2.

4.1.4 Higher-point correlation function of BPS operators

The calculation of the correlation function of four scalar BPS operators may be easily extended to the correlator of any number of operators. The main difference is that, unlike

33We use the standard convention $p \rightarrow -p \rightarrow |p| \rightarrow -|p|, |p| \rightarrow |p|$. The Grassmann coordinates $\eta[i]$ and $\eta[j]$ have different harmonic coordinates since they correspond to different multiplets – $\eta[x]^{\alpha}_{\nu,i} = \eta_{\nu,i} \, \eta_{x,\alpha}^{\nu}$ – where $x$ labels the operator to which $\eta$ belongs to.
the four-point correlator, the R-symmetry constraints are more difficult to write in compact form. Similarly to eq. (27), one is to identify the \(SO(6)\) singlets in the product of \(n\ 20'\)-dimensional representations of \(SO(6)\). By picking suitable combinations of operators and evaluating their correlation functions we may then extract the coefficient of each individual \(SO(6)\) singlet.

\[
\begin{align*}
&I_{0i}^{[k]} & I_{1i}^{[k]} & I_{2i}^{[k]} \\
&i & i & i & i \\
&i & i+1 & i & i+1 \\
&i-1 & i-1 & i-1 & i-1 \\
&i+k & i+k & i+k & i+k \\
&i+k+1 & i+k+1 & i+k+1 & i+k+1 \\
\end{align*}
\]

Figure 6: General two-loop integrals: \(I_{0i}^{[k]}, I_{1i}^{[k]}, I_{2i}^{[k]}\). The integral \(I_{1i}^{[k]}\) includes the numerator factor \(s_i^{[k]} = (\sum_{j=0}^{k} q_{i+j})^2\).

The cuts that need to be evaluated are similar to the cuts in fig. 4 except that each form factor is replaced with a generalized form factor with arbitrary number of operators subject to the constraint that the sum of the number of operators is fixed to \(n\). It is not difficult to see that the building blocks are identified by eqs. (37), (40), (41) and they lead to the integrals shown in fig. 6.

Based on the calculation of the 4-point correlation function one can see that integrals of the type \(I_{0i}^{[k]}\) enter only in the partly-connected (in R-symmetry space) components of the correlator and integrals of the type \(I_{1i}^{[k]}\) and \(I_{2i}^{[k]}\) appear in the connected components. Analyzing the cuts we find that the analog of the \(b_i\) coefficients in eq. (27), i.e. the coefficient function for the maximally connected \(SO(6)\) singlet, defined by the tensor structure

\[
\prod_{i=1}^{n} \delta_{i,i+1} ,
\]

is given by

\[
\tilde{b}_{2,\text{max}} = \sum_{i=1}^{n} \sum_{k=0}^{n-2} I_{1i}^{[k]} - \sum_{i=1}^{n} \sum_{k=1}^{n-2} I_{2i}^{[k]}
\]

where the integral \(I_{1i}^{[k]}\) has a numerator factor \(s_i^{[k]} = (q_i^{[k]})^2\) with \(q_i^{[k]} = \sum_{j=0}^{k} q_{i+j}\) and the first index on \(b\) denotes the fact that the index contraction is analogous to that of \(b_2\).

While kinematic restrictions forbid a parity-odd part (i.e. containing Levi-Civita tensors) in the four-point correlation function, one may wonder whether such terms should be present
in eq. (47). An analysis of cuts similar to those in figs. 4(a), 4(b) and 4(c) shows that the integral topologies detected by these cuts do not exhibit a parity-odd component. Cuts involving two three-point form factors have two components, in which either one of the form factors is MHV while the other is $\overline{\text{MHV}}$. While each component has a nontrivial parity-odd part, it cancels in the sum. Thus, to this order, no parity-odd part should be present in eq. (47).

While it is not difficult (and perhaps useful for specific applications) to Fourier transform this expression to position space using the details in Appendix C and check its conformal invariance, the result is rather complicated. It is more instructive to carry out the Fourier transform under the assumption that we also take the null limit of the resulting expression and make contact with the $n$-point 1-loop MHV amplitude thus reproducing the results of [5]. Indeed, it is not difficult to see that the integrals $I_{0_i}^{[k]}$ and $I_{2_i}^{[k]}$ are not proportional to $\prod_{i=1}^{n} x_{i,i+1}^{-2}$ and thus are not sufficiently singular to contribute to this limit. The Fourier transform of integrals of the type $I_{1_i}^{[k]}$ is (cf. eqs. (C.16) and (C.14))

$$\int (\prod_j d^4 q_j e^{ix_j \cdot q_j}) I_{1_i}^{[k]} = \frac{x_{i+k,i-1}^2 - x_{i+k,i-1,i+k+1}^2}{\prod_{j=1}^{n} x_{j,j+1}^2} + O(x_{l,l+1}^2) \int dx_0 x_{i,0}^2 x_{i+k,0}^2 x_{i+k+1,0}^2 x_{i-1,0}^2$$

for $k \neq 1$ and $k \neq n - 1$ and subleading for these two cases. Assembling $b_{1,\text{max}}$, the Fourier-transform of $\tilde{b}_{1,\text{max}}$ and after a trivial change of summation indices we recover, as expected, eq (4.11) of [5]. It is important to note that all contact term integrals in momentum space – i.e. integrals with cancelled propagators – do not contribute in the null separation limit of the Fourier-transform of the momentum space correlator. This might have been expected since at this order any contact term lies along a path between two null-separated operators and such terms have been argued [3] to be subleading.

### 4.2 Correlators of BPS and non-BPS operators

The methods described in previous sections can equally well be applied to correlation functions involving non-BPS operators, such as a general twist-2 operator. As discussed in §2.2 since these operators are nontrivial sums of monomials with coupling constant-dependent coefficients, the strategy for the evaluation of their correlation function is to evaluate the correlator of generic monomials and subsequently sum these components with the appropriate coefficients. We will illustrate here this construction with the calculation of the three-point function of one non-BPS twist-2 operator and two BPS scalar operators. We will then generalize this calculation to an arbitrary number of scalar BPS operators as well as to two twist-2 operators. Compared to the correlation function of BPS operators, the additional
form factors which are needed are special cases of the general form factor found in § 3.4.

4.2.1 3-point correlators with one twist-2 operator

For operators with general R-symmetry indices,

\[ O_1 = \text{Tr}(D_x^I \phi^I D_x^{S-x} \phi^I) \quad O_2 = \text{Tr}(\phi^{I_2} \phi^{I_2}) \quad O_3 = \text{Tr}(\phi^{I_3} \phi^{I_3}) \] (48)

the R-index structure of the correlator is

\[ \langle O_1 O_2 O_3 \rangle = c_1 (0) \delta_{\{J_2\}} \delta_{\{J_1\}} + c_2 (0) \delta_{\{J_3\}} \delta_{\{J_1\}} . \] (49)

Depending on the choice of operators, both, one or none of the structures above are allowed. Whenever both are allowed the coefficient functions \( c_1 \) and \( c_2 \) are related by the transformation \( x \leftrightarrow S - x \). Here we will choose a particular non-BPS representative in which the two scalar fields are the same, \( I_1 = J_1 \); in this case the factorization occurs on a term by term basis and its momentum/position dependence is given by \( (c_1^x + c_2^x) \).

![Figure 7: Triple cut determining the leading order three-point function in eq. (49). The black dot denotes the non-BPS operator while the gray dots denote BPS operators.](image)

It is convenient to define

\[ S(a, b, x) = (a^-)^{S-x} (b^-)^x + (b^-)^{S-x} (a^-)^x , \] (50)

where the upper index "−" denotes a projection onto a null direction (e.g. \( a^- = a_0 + a_3 \), etc). The leading order correlation function is determined by the triple cut in fig. 7 to be

\[ \tilde{c}_1 (0) + \tilde{c}_2 (0) = \int \prod_{i=1}^3 \frac{d^4 l_i}{(2\pi)^4} \delta^4 (q_3 - l_2 - l_3) \delta^4 (q_2 + l_3 - l_1) \delta^4 (q_1 + l_1 + l_2) \frac{S(l_1, l_2, x)}{l_1^2 l_2^2 l_3^2} \] (51)

The Fourier transform to position space yields the expected result which may then be used to assemble the correlation function of the full twist-2 spin-\( S \) operator with two BPS operators by summing them with the coefficients in eq. (25).
Figure 8: Cut topologies which determine the next-to-leading order correction to the three-point function in eq. (49). As in fig. 7 the black dot denotes the non-BPS operator while the gray dots denote BPS operators.

To construct the next-to-leading order correction to this correlation function we may proceed with the maximal cut method with the expectation that we will need to consider maximal and next-to-maximal cuts. At this order however the calculations are sufficiently simple to allow us to proceed directly to next-to-maximal and next-to-next-to-maximal cuts; up to the interchange of the two BPS operators (or, alternatively, up to the transformation $q_1 \leftrightarrow q_2$) they are shown in fig. 8. These cuts are not difficult to evaluate using the form factors described in § 3. They are:

\[ C_{\text{fig. 8} (a)} = S(l_1, l_2, x) \frac{\langle l_1 l_2 \rangle \langle l_3 l_5 \rangle}{\langle l_2 l_3 \rangle \langle l_5 l_1 \rangle} = -S(l_1, l_2, x) \frac{q_1^2}{(l_2 + l_3)^2} \quad (52) \]

\[ C_{\text{fig. 8} (b)} = S(l_1, l_2, x) \frac{\langle l_3 l_1 \rangle \langle l_5 l_4 \rangle}{\langle l_1 l_5 \rangle \langle l_4 l_3 \rangle} = -S(l_1, l_2, x) \frac{q_3^2}{(l_1 + l_5)^2}. \quad (53) \]

Due to the presence of two three-field form factors, the last two cuts receive contributions from the two configurations in which either one of them is MHV and the other is $\overline{\text{MHV}}$. We find:

\[ C_{\text{fig. 8} (c)} = \left( \frac{l_3}{l_2} - \frac{1}{(l_2 + l_3)^2} \right) \left( S(l_1 + l_2 + l_3, x) - S(l_1 + l_2, l_3, x) \right) \]

\[ + \frac{l_4}{l_2} \frac{1}{(l_2 + l_3)^2} \left( S(l_1 + l_2, l_3, x) - S(l_1, l_2 + l_3, x) \right) + S(l_1 + l_2, l_3, x) \]

\[ \times \left( \frac{q_1^2}{(l_2 + l_3)^2(l_1 + l_2)^2} - \frac{1}{(l_1 + l_2)^2} - \frac{1}{(l_2 + l_3)^2} + \frac{q_3^2}{(l_2 + l_4)^2(l_1 + l_2)^2} \right) \]

\[ + S(l_1, l_2 + l_3, x) \left( \frac{q_2^2}{(l_2 + l_3)^2(l_2 + l_4)^2} - \frac{1}{(l_2 + l_4)^2} + \frac{1}{(l_2 + l_3)^2} \right) \quad (54) \]
Thus, these integrals require regularization; we may simply promote to a BPS operator we consider acquires a nontrivial anomalous dimension (and also mixes with D-dimensions the result of the four-dimensional calculation. The test of the validity of this procedure is the emergence of the correct anomalous dimension contribution to the correlator.

As usual, each diagram stands for the product of scalar propagators corresponding to the graph, the product of the momentum conserving delta function at each vertex and an integral over all internal momenta with the numerator factor explicitly shown.

While the spin of the operators can be arbitrarily large and thus the numerator factors can have arbitrarily high powers of loop momenta (cf. eq. (50)), Lorentz invariance guarantees that of all the integrals present in eq. (56) only those with an explicit one-loop bubble sub-integral are divergent in the UV; this divergence is related to the fact that the non-BPS operator we consider acquires a nontrivial anomalous dimension (and also mixes with other operators). Thus, these integrals require regularization; we may simply promote to D-dimensions the result of the four-dimensional calculation. The test of the validity of this procedure is the emergence of the correct anomalous dimension contribution to the correlator.

It is not difficult to evaluate the one-loop bubble subintegral in dimensional regularization; from the $l_1^-$ and $(q_1 + l_1)^-$ dependence of the result one can reconstruct the mixing matrix.
of the monomials $O_1 = O_{2,S,x}^{AB,AB}$ in (48). Diagonalizing it one finds the correct one-loop anomalous dimension – proportional to the harmonic number $h(S)$ – and the corresponding eigenvectors, as well as the anomalous dimensions and eigenvectors of the descendants of twist-2 operators of lower spin. After taking the appropriate linear combinations (25) of $\tilde{c}_1^{(1)} + \tilde{c}_2^{(1)}$, to isolate an operator with definite anomalous dimension, multiplication of the full (LO + NLO) correlator by the appropriate renormalization factor (10) will render it finite as the regulator is removed. To complete the calculation of the three-point function it is necessary to also include the tree-level correlation function (51) summed against the two-loop mixing coefficients (89).

It is relatively easy to check that the expression (56) vanishes in the appropriate limits. In particular, as $S, x \to 0$ the non-BPS operator becomes BPS and the three-point function should vanish identically (92). Indeed, in this limit the coefficients of the last two integrals vanish identically and thus the correlator is finite; after using momentum conservation the remainder is proportional to

$$q_1^{\underline{1}} q_2^{\underline{1}} q_3^{\underline{1}} q_4^{\underline{1}} q_5^{\underline{1}} = 0; \quad (57)$$

We may similarly construct the correlation function of the spin-$S$ descendants of the BPS operator by noticing that it is given by the sum

$$\sum_x C_S^x O_{2,S,x}^{AB,AB}, \quad (58)$$

where $C_S^x$ are the binomial coefficients; summing $\tilde{c}_1^{(1)} + \tilde{c}_2^{(1)}$ over integer values of $x$ between 0 and $S$ the divergent integrals cancel out and the remainder is again proportional to (57).

### 4.2.2 Correlators of one non-BPS twist-2 operator and $(n - 1)$ BPS

The calculation of the correlator of one twist-2 and two BPS operators can be generalized to correlators with any number of BPS operators. As in the case of correlators of only BPS operators, the cut calculation is almost identical, but in this case we need to evaluate the two additional cuts shown in fig. 9. These cut are very similar to the cuts in figs. 8(c) and 8(d), respectively, in which the intermediate state is restricted to only gluons and scalars. Similarly to the discussion in §4.1.4, the R-index structure is more complicated than that

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34Thus, at this order the regulator dependence is completely accounted for by promoting the integration measure to be $D$-dimensional.

35Upon Fourier-transform to position space this multiplication also restores the combination $\gamma_1 \ln \frac{\mu (x_{12})}{|x_{12}|}$ which is required by conformal invariance.
in eq. (49) and it is given by all the $SO(6)$ singlets that appear in the (non-symmetrized) tensor product of the representations of the BPS and non-BPS operators. We will focus on the completely connected part of the correlator and, as in the case of the three-point function, we will choose a non-BPS twist-2 operator with two identical scalar fields.

![Figure 9: Additional cuts necessary to determine the correlator of one twist-2 scalar operator (with momentum $q_1$) and an arbitrary number of dimension-2 scalar BPS operators. Momentum labels are slightly changed compared to those in fig. 8.](image)

When non-vanishing, the leading-order connected component of the momentum space correlator of one spin-$S$ twist-2 scalar operator and $(n-1)$ scalar BPS operators of dimension 2 is

$$
\tilde{C}^{(0)} = \int \prod d^4 l_i \delta(q_i + l_i - l_{i+1}) \frac{S(l_1, l_2, x)}{\prod l_i^2} \tag{59}
$$

where the external leg $q_1$ corresponds to the non-BPS operator. Denoting by $l_1$ and $l_2$ the momenta of the internal lines immediately adjacent to the external line $q_1$ – corresponding to the source-field of the non-BPS operator – in the integrals in which this line is not shown explicitly in fig. 10, the NLO correction to this correlation function is:

$$
\tilde{C}^{(1)} = S(l_1, l_2, x) \left( \sum_{i=1}^{n} \sum_{k=1}^{n-2} I_1^{[k]} - \sum_{i=2}^{n} \sum_{k=1}^{n-2} I_2^{[k]} \right) \\
+ \left( \frac{p_1^-}{l_2} S(t_1 + t_2, t_3, x) + \frac{p_2^-}{l_2} S(t_1 + t_2 + t_3, x) \right) \sum_{k=1}^{n-2} I_{21}^{[k]} \\
+ \frac{p_2^- + p_3^-}{p_2^-} \left( S(p_1 + p_2 + p_3, x) - S(p_1 + p_2, p_3, x) \right) I_{3;1} \\
+ \frac{p_1^- + p_2^-}{p_2^-} \left( S(p_1 + p_2 + p_3, x) - S(p_1 + p_2 + p_3, x) \right) I_{4;1} \tag{60}
$$

The integrals appearing in this expression and their internal momenta are defined in fig. 10. A potentially non-vanishing odd part present in various contributions to the cuts in fig. 9

32
Figure 10: Integrals $I_{1i}^{[k]}$, $I_{2i}^{[k]}$, $I_{3i}$, $I_{4i}$ entering the $n$-point correlator of one non-BPS scalar twist-2 operator and any number of dimension-2 BPS operators. The first two integrals are the same as $I_{1i}^{[k]}$ and $I_{2i}^{[k]}$ in fig. 6.

cancels in the complete expression of the cut. This is consistent with the absence of a parity-odd component of all the other cuts.

With this expression in hand we may test whether the non-BPS nature of one operator affects the relation between the null limit of the correlator and the MHV $n$-point amplitude; general arguments in a specific (non-standard) regularization scheme [3] suggest that the relation is unaffected. The similarity of the contact term integral $I_{21}^{[k]}$ with the integral with the same name entering the correlation function of BPS operators (47) and the fact that the Fourier-transform of the latter is subleading in the null implies that the integral $I_{21}^{[k]}$ here will also yield only subleading terms; as in that case, it is not difficult to check this explicitly. Inspecting (60) and comparing it with (47) we notice that, up to a factor $S(l_1, l_2, x) = (l_1^{-1})^{S-x}(l_2^{-1})^x + (l_2^{-1})^x(l_1^{-1})^{S-x}$ the integrals are identical. This factor is present in the leading order correlator (59) and this cancels out in normalization. Of the other integrals in (60), $I_3$ and $I_4$ could also give contributions of the same order. If however the null limit is taken in the presence of a finite dimensional regulator, these integrals are subleading by $x_{12}^2$ and $x_{i11}^2$, respectively, and thus drop out as well [3, 3]. We therefore explicitly confirm that the null limit of the correlator of one non-BPS and $(n - 1)$ BPS twist-2 operators is the same as that of the correlator of $n$ BPS twist-2 operators and the same as the MHV gluon amplitude. We also note that, as in the case of correlators of BPS operators, momentum space contact term integrals do not contribute to this limit.

4.2.3 The correlation function of two twist-2 and any number of BPS primaries

Generalizing slightly the calculation it is not difficult to construct the correlator of two non-BPS twist-2 operators and any number of BPS operators. For general choices of BPS scalar operators and twist-2 operators one would have several possible choices of R-index

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36 If the null limit is taken after the correlator is expanded in small $\epsilon$ at finite $x^2_{i,i+1}$, the contribution of these integrals should factorize into a scheme-dependent coefficient function [3].
structures; the coefficient of each one of them may be obtained either by choosing suitable representatives or by using supersymmetric methods. We will focus here on the structures that have a nonzero position-space tree-level term if the twist-2 operators are constructed from $\text{Tr}[D_x^n \phi^{AB} D_x \phi^{AB}]$ and its conjugate.\footnote{\text{i.e. the scalars in the two BPS operators are conjugates of each other.}} There are two classes of R-index structures that can appear: those in which the two twist-2 operators are adjacent (\text{i.e.} their R-indices are contracted) and those in which the two twist-2 operators are not (\text{i.e.} their R-indices are not contracted but rather are contracted with the R-indices of BPS operators).

![Diagram](https://via.placeholder.com/150)

Figure 11: Additional cuts needed to find the correlation function if two non-BPS operators and any number of BPS operators at NLO. Cut (a) is absent if the scalar fields in the two twist-2 operators are not conjugates of each other.

Apart from the cuts already used to construct the correlators in the previous section, we need the information provided by the cuts in fig. 11(a) in order to find the correlators in the first class. Without loss of generality we will assume that the two non-BPS operators are at locations 1 and 2; whenever there are only two internal lines attached to the non-BPS operators, we denote their momenta by $l_1$, $l_2$ and $l_3$, respectively. The result for the NLO contribution to this correlator is:

$$\tilde{C}_1^{(1)} = S(l_1, l_2, x)S(l_2, l_3, y) \left( \sum_{i=1}^{n} \sum_{k=1}^{n-2} I_{1i}^{[k]} - \sum_{i=3}^{n} \sum_{k=1}^{n-2} I_{2i}^{[k]} \right)$$
\[
+ \sum_{k=2}^{n-2} \left( \frac{p_1^-}{t_2^2} S(t_1 + t_2, t_3, x) + \frac{p_2^-}{t_2^2} S(t_1, t_2 + t_3, x) \right) S(l_2, l_3, y) I_2^{[k]}
\]
\[
+ \sum_{k=1}^{n-2} \left( \frac{p_1^-}{t_2^2} S(t_1 + t_2, t_3, y) + \frac{p_2^-}{t_2^2} S(t_1, t_2 + t_3, y) \right) S(l_1, l_2, x) I_2^{[k]}
\]
\[
+ \frac{p_1^- + p_2^-}{p_2} \left( S(p_1, p_2 + p_3, y) - S(p_1 + p_2, p_3, y) \right) S(p_3, p_3 - q_1, x) I_{3,2}
\]
\[
+ \frac{p_2^- + p_3^-}{p_2} \left( S(p_1 + p_2, p_3, y) - S(p_1, p_2 + p_3, y) \right) S(p_2 + p_3, p_2 + p_3 - q_1, x) I_{4,2}
\]
\[
+ \frac{p_1^- + p_2^-}{p_2} \left( S(p_1, p_2 + p_3, x) - S(p_1 + p_2, p_3, x) \right) S(p_1 + p_2, p_1 + p_2 - q_2, y) I_{3,1}
\]
\[
+ \frac{p_2^- + p_3^-}{p_2} \left( S(p_1 + p_2, p_3, x) - S(p_1, p_2 + p_3, x) \right) S(p_1, p_1 - q_2, y) I_{4,1}
\]
\[
(61)
\]

Note that the position of the bubble integral in the line connecting the two non-BPS operators is ambiguous; we have chosen it such that the coefficients of \( I_{4,2} \) and \( I_{3,1} \) are the ones that appear in the anomalous dimension of the twist-2 operators. Such an organization exists for any operator.

It is not difficult to extend this result to an arbitrary number of BPS and non-BPS twist-2 operators in a split configuration (i.e. with all adjacent BPS operators). The only difference compared to eq. (61) is the proliferation of integrals \( I_3 \) and \( I_4 \): each twist-2 non-BPS operator contributes two such integrals in a pattern easily identifiable by comparing the past two terms in eq. (60) and the last four lines of eq. (61).

For the correlators in the second class, in which the non-BPS operators are not adjacent, we need to also evaluate the cut fig. 11(b), in which both such operators enter via a three-point form factor. It turns out however that this cut does not reveal any new terms. Let us assume that the two non-BPS operators are at positions 1 and \( j \) and that, whenever there are only two internal lines attached to each of them, their momenta are \( l_1, l_2 \) and \( l_j, l_{j+1} \), respectively. The NLO contribution to the correlator correlator is given by:

\[
\tilde{C}^{(1)}_2 = S(l_1, l_2, x) S(l_j, l_{j+1}, y) \left( \sum_{i=1}^{n} \sum_{k=1}^{n-2} I_{1_i}^{[k]} - \sum_{i=2, i \neq j}^{n} \sum_{k=1}^{n-2} I_{2_i}^{[k]} \right)
\]
In both cases the potential parity-odd components cancel (up to total derivatives) in the complete expression of cuts. As in the case of correlators with a single non-BPS operator, it is not difficult to Fourier-transform eqs. (61) and (62) in the limit in which the operator insertions are null-separated. As in the case of a single non-BPS operator, the null limit is to be taken at finite dimensional regulator.

5 On effective actions and energy flow correlators

As discussed in the introduction, an interesting application of the techniques we described is the construction of effective actions in background fields — in particular in non-dynamical gravitational background — as well as the construction of correlation functions describing the energy and charge flow in scattering processes of gauge-singlet states [6, 7]. Since the relevant correlators are somewhat different in the two cases we will discuss them separately.

5.1 Effective actions

The linearized coupling of a field theory with a background gravitational field is universal:

$$S = S_0 + \int d^d x \, h^{\mu \nu} T_{\mu \nu} + \ldots$$

(63)

where $S_0$ is the flat space action, $h^{\mu \nu}$ is the departure of the background metric from that of the flat Minkowski space, $T_{\mu \nu}$ is the stress tensor and the ellipsis stand for terms non-linear in $h$. The effective action for the gravitational field is obtained by integrating out all the fields except for $h^{\mu \nu}$. We may interpret this as the evaluation of the scattering amplitudes of the background gravitons $h^{\mu \nu}$. As discussed in § 2 up to terms in which two gravitons emerge
from the same vertex, this is the same as the evaluation of the stress tensor correlation functions; that is, up to contact terms that may perhaps be constructed on the basis of general coordinate invariance and other symmetries, the effective action for the background gravitational field is nothing but the collection of the stress tensor correlation functions. To construct them in our approach we need to understand how to extract the stress tensor form factors. We will describe this in the next section. We will however leave for the future the complete evaluation of effective actions, including the determination of contact terms either by requiring that all Ward identities are satisfied or by finding and using the multi-stress tensor form factors.

5.2 The MHV stress tensor form factor

Thus, both for the evaluation of effective actions as well as for the evaluation of energy and charge correlators it is necessary to evaluate correlation functions involving the stress tensor perhaps together with other operators, perhaps with additional constraints in the structure of the contributing diagrams. They may – in principle – be constructed by a careful application of supersymmetry transformations on correlation functions in which the stress tensors are replaced by dimension-2 BPS operators and separation of the contributions of the descendants of lower levels in the supersymmetry multiplet. Extraction of the stress tensor component is, however, not obvious since it is necessary to separate from the coefficient of $\theta^2 \bar{\theta}^2$ all the contribution of the supersymmetry descendants of the operators appearing on lower levels in the multiplet.

To evaluate such correlators in our approach it is necessary to know the form factors and generalized form factors of the stress tensor; their MHV components are part of the MHV super-form factor of the complete stress tensor multiplet constructed in \[54\], which is a natural generalization of the super-form factor of the chiral stress tensor multiplet:

$$G = \prod_{i=1}^{n} \frac{1}{\langle i, i+1 \rangle} \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i) \delta^{(4)}(\gamma_+ - \sum_i \lambda_i \eta_{+,i}) \delta^{(4)}(\gamma_- - \sum_i \lambda_i \eta_{-,i}) \quad (64)$$

$$= \prod_{i=1}^{n} \frac{1}{\langle i, i+1 \rangle} \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\gamma - \sum_i \lambda_i \eta_i) . \quad (65)$$

This super-form factor and its generalizations may be used to construct super-correlation functions from which the desired components can then be extracted. Alternatively, one construct directly the component correlation function by first extracting the relevant form factors.

A naive extraction of the component form factor from \(65\) as the coefficient of $\gamma_+ \sigma^\mu \gamma_- \gamma_+ \sigma^\nu \gamma_-$ leads to expressions which violate the expected conformal Ward identities related to the tracelessness and conservation of the stress tensor. This is a reflection of the difficulty mentioned
above, that apart from the stress tensor, the coefficient of $\theta^2 \bar{\theta}^2$ also contains descendants of the operators at lower levels in the supersymmetry multiplet. It turns out that the stress tensor form factor (i.e. the coefficient of $\theta^2 \bar{\theta}^2$ from which the descendant contribution is separated) may be extracted as

$$\langle 0\lvert T_{\mu\nu}\lvert 1,\ldots,n \rangle = \frac{1}{4!} \int d^8\gamma \epsilon^{ABCD} \gamma_A \bar{\gamma}^B \gamma_C \bar{\gamma}^D \mathcal{G}.$$ \hspace{1cm} (66)

Conservation and tracelessness follow from the fact that $\epsilon^{ABCD} \gamma_A = 0$, which is a consequence of the Grassmann nature of $\gamma$ and of the antisymmetry of the Lorentz contraction. We have checked that this expression agrees with a Feynman diagram and BCFW-based evaluation of this form factor.

As discussed in section 2 to construct correlation functions though generalized unitarity we also need generalized form factors with more operator insertions. In the case of the stress tensor they are particularly important to consider because the derivatives present in the stress tensor lead to contact terms which are missed if only regular form factors are used; it is possible that such contact terms – which are important for the construction of effective actions – can be determined by requiring that correlators exhibit conformal invariance.

### 5.3 Energy and charge flow correlators

Energy and charge correlation functions [6, 93, 94, 95] are particular examples of event shapes, which are important observables in QCD. For some some initial state $X$ and observable $e(N)$ that depends on the final state, they are defined as weighted cross sections:

$$\frac{1}{\sigma_{tot}} \frac{d\sigma_{tot}}{de} = \sum_N d\sigma_{X \rightarrow N} \delta(e - \hat{e}(N))$$ \hspace{1cm} (67)

and describe the distribution of $e(N)$ in the final state of some scattering process while being insensitive to the particular composition of this state. Popular choices for $\hat{e}$ is the energy or (some) charge flowing in some direction, e.g.

$$\mathcal{E}(\vec{n})|k_1,\ldots,k_N\rangle = \sum_{i=1}^N \delta(\cos \theta - \cos \theta_i) \delta(\phi - \phi_i) k_i^0 |k_1,\ldots,k_N\rangle$$ \hspace{1cm} (68)

and similarly for some charge $Q$. The angles ($\theta_i, \phi_i$) define the unit vector $\vec{n}_i = \vec{k}_i/||\vec{k}_i||$ with respect to some reference direction ($\theta, \phi$). The operator $\mathcal{E}$ admits presentations [96, 7] in terms of the stress tensor:

$$\hat{e} = \int d^3\vec{n} \delta(\vec{n}^2 - 1) w(\vec{n}) \mathcal{E}(\vec{n})$$

$$\mathcal{E}(\vec{n}) = \lim_{R \rightarrow \infty} R^2 \int_{-\infty}^{+\infty} dt \ n^0 T_i^0(t, R\vec{n}^*)$$ \hspace{1cm} (69)
Using the expression of the differential cross section in terms of squared S-matrix elements one casts eq. (67) into an expression whose building blocks are special correlation functions of the operators $E$. In the definition above $E$ is understood to contain a projection onto states at $t = +\infty$.

In a conformal field theory the initial state $X$ is created by the action on the in vacuum state of some local gauge invariant operator:

$$|X\rangle = \mathcal{O}|0\rangle_{\text{in}}$$  \hspace{1cm} (70)

Then, the analog of the total weighted cross section is

$$\frac{1}{\sigma_{\text{tot}}} \frac{d\sigma_{\text{tot}}}{d\theta} \sim \frac{\langle 0|\mathcal{O}^\dagger \delta(e - \int d^3\vec{n} \ u(\vec{n})\mathcal{E}(\vec{n}))\mathcal{O}|0\rangle_{\text{in}}}{\langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle_{\text{in}}}.$$  \hspace{1cm} (71)

This expression may be refined by assigning a different $\delta-$function factor to each S-matrix element. Using an integral representation of the $\delta-$function it is easy to see that the expectation value (71) is defined by the more general correlators (94):

$$G(\vec{n}_1, \ldots \vec{n}_n, \mathcal{O}) = \langle \mathcal{E}(\vec{n}_1) \ldots \mathcal{E}(\vec{n}_n) \rangle \equiv \frac{\langle 0|\mathcal{O}^\dagger \mathcal{E}(\vec{n}_1) \ldots \mathcal{E}(\vec{n}_n)\mathcal{O}|0\rangle_{\text{in}}}{\langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle_{\text{in}}}.$$  \hspace{1cm} (72)

The fact that the vacuum states are both in-states makes this correlator different than the standard ones which were interpreted as scattering amplitudes of source fields. Rather, as mentioned in §5.3, both the numerator and denominator have the interpretation of cut diagrams; we emphasize that these are regular unitarity cuts that have an imaginary part interpretation: in this case it is the imaginary part of the two-point function $\langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle$. The fact that $\mathcal{E}$ contains a projection onto out-states implies that, in these unitarity cuts, the operators $\mathcal{E}$ are to be inserted only on the cut lines; this is a reflection of the definition (67) which assigns the energy weight factors only on the final state $|N\rangle$. For the same reason, as well as because $\mathcal{E}$ acts in the final state on widely separated particles, all operators $\mathcal{E}$ commute with each other and thus no ordering needs to be specified. Singularities may nevertheless arise if some $\vec{n}_i = \vec{n}_j$ and the corresponding operators act on the same particle.

We will avoid such situations by keeping $\vec{n}_i \neq \vec{n}_j$ for all $i \neq j$.

5.4 On the evaluation of energy correlators at weak and strong coupling

In contrast to correlation functions of stress tensors, energy correlators provide a class of observables which are insensitive to contact terms. Indeed, as discussed in detail in §5.3

$^{38}$One may similarly define charge correlators, in which (some of) the energy detectors $\mathcal{E}$ are replaced by the R-charge detectors

$$Q(\theta) = \lim_{r \to \infty} r^2 \int_{-\infty}^{+\infty} dt \ n^i J_i(t, r\vec{n}) \ .$$

$^{39}$This is the same as the cut diagram interpretation of inclusive cross sections.
the stress tensors featuring in eq. (72) are connected only to cut lines and therefore, all potential contact terms are set to zero. Moreover, since the energy measurement is made at infinity it affects only one particle at a time and thus only the two-point form factor of the stress tensor is necessary.

As described earlier in this section, the energy correlators in the state created by the operator $O$ may be interpreted as the unitarity cut of the two-point function of $O$ and $O^\dagger$ dressed with energy- and orientation-dependent factors. The discussion in §2 implies that this construction may be given an interpretation in terms of the form factors of $O$. Indeed, inserting complete sets of out-states we have

$$\langle 0|O^\dagger O|0\rangle_{in} = \sum_X \langle 0|O^\dagger|N\rangle\langle N|O|0\rangle_{in} ,$$

$$\langle 0|O^\dagger \mathcal{E}(\vec{n}_1) \ldots \mathcal{E}(\vec{n}_n) O|0\rangle_{in} = \sum_{N,N'} \langle 0|O^\dagger|N'\rangle\langle N'|\mathcal{E}(\vec{n}_1) \ldots \mathcal{E}(\vec{n}_n)|N\rangle\langle N|O|0\rangle_{in} ,$$

where the sums run over all physical Fock space states $|X\rangle$ of some fixed momentum; we recognize each factor as the form factor of $O$ and its conjugate.

To evaluate the right-hand side of (74) one may use the original definition (68) of $E$, superficially perhaps at the expense of not using supersymmetric methods to evaluate the sum over intermediate states. We may alternatively use the stress tensor super-form factor and carefully take the limit $R \to \infty$ limit at the end while enforcing the relevant on-shell conditions. As we shall see shortly, the limit $R \to \infty$ requires that the momentum of the stress tensor operator vanishes; in this case it turns out that the form factor is just the traceless part of $p_\mu p_\nu$, where $p$ is the momentum of one of the external fields and thus factorizes in the sum over intermediate states. One may show along the lines of [97] that, for $x = (t, R\vec{n})$,

$$\lim_{R\to\infty} R^2 \int dt \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \delta^{(+)}(p_1^0)\delta^{(+)}(p_2^0) (p_1^0 n_i p_2^i) e^{ip_1 \cdot x} e^{-ip_2 \cdot x} F(p_1, p_2, \ldots)$$

$$= \int \frac{d^3p}{(2\pi)^3 2\omega_p} \omega_p \delta^2(\vec{n} - \frac{\vec{p}}{|p|}) F(p, p, \ldots) .$$

The argument of the integral on the left-hand side of the equation above is the typical term in the momentum space correlation function corresponding to (72) after the state sum was performed and before the limit $R \to \infty$ is taken. $F$ represents the contribution of the two form factors and of the other stress tensors except for one; the contribution of the last stress tensor is just $(p_1^0 n_i p_2^i)$. Evaluating the angular integrals in the saddle-point approximation (justified by the large value of $R$) leads to the orientation of the space-like parts of $p_1$ and $p_2$ be the same as the external vector $\vec{n}$. The integral in the time direction sets to zero the energy carried by stress tensor operator thus setting $p_1^0 = p_2^0$ and thus also sets equal the
nons of the space-like components of the momenta $p_1$ and $p_2$ and thus also $|X\rangle = |X'\rangle$. This identity therefore allows us to use supersymmetric methods to evaluate (74).

In [7] it was suggested that the energy correlation function may be extracted by suitable analytic continuation of the time-ordered correlation function. It is in principle possible to use the form factor interpretation described above to evaluate them at strong coupling; the construction the form factors [98] in terms of a $Y$—system would be an essential ingredient. A difficulty arises however in the treatment and cancellation of infrared divergences: while the energy correlators are infrared-safe observables, form factors exhibit infrared divergences. In a fixed order calculation they cancel against real emission contributions – lower-loop higher-points contributions to form factors. It is not obvious how the cancellation proceeds at strong coupling; we leave this important problem for the future.

5.5 Examples of energy correlators

We will discuss examples of energy correlators for two-field operators. The simplest example involves a single operator $\mathcal{E}$. At leading order the calculation is trivial; we need the explicit expression of the two-point form factors of the desired operator and we need to carry out the phase space integral. For definiteness let us consider the scalar BPS operator, also discussed in [7]. In this case the form factors are unity; assuming that the state created by the operator has 4-momentum $(q, \vec{0})$, using the notation $\hat{p} = \vec{p}/|\vec{p}|$ and using the symmetry under the interchange of the two cut lines as well as the on-shell conditions we find

$$in\langle 0|\mathcal{O}^\dagger \mathcal{E}(\vec{n})\mathcal{O}|0\rangle_{in}^{(0)} = 2 \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \omega_1 \delta(2)(\vec{n} - \hat{p}_1) \delta(\omega_1 + \omega_2 - q)$$

$$= \frac{2}{(2\pi)^6} \int \frac{\omega_1^2 d\omega_1}{4\omega_1} \delta(2 \omega_1 - q) = \frac{2}{(2\pi)^6} \frac{q}{16}$$

(76)

$$in\langle 0|\mathcal{O}^\dagger \mathcal{O}|0\rangle_{in}^{(0)} = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \delta(\omega_1 + \omega_2 - q)$$

$$= \frac{4\pi}{(2\pi)^6} \int \frac{\omega_1^2 d\omega_1}{4\omega_1^2} \delta(2 \omega_1 - q) = \frac{4\pi}{(2\pi)^6} \frac{1}{8}$$

(77)

Taking the ratio (72) we find [7]

$$\langle \mathcal{E}(\vec{n})\rangle_{BPS}^{(0)} = \frac{q}{4\pi}. \quad (78)$$

The spherical symmetry is, of course, due to the state created by $\mathcal{O}_{BPS} = \text{Tr}[\phi^{12}]$ being spherically symmetric [7].

We may introduce an asymmetry by considering the state created by an e.g. twist-2 operator. Since in this case the operator is a sum of terms, to reconstruct both the operator and its conjugate, we need to consider the mode general building block $in\langle 0|\mathcal{O}_{2,s,y}^\dagger \mathcal{E}(\vec{n})\mathcal{O}_{2,s,x}|0\rangle_{in}$
with the unspecified R-symmetry indices chosen such that the total R-charge vanishes. It evaluates to:

\[
\text{in}(0)\mathcal{O}_{1, S, y}^1 \mathcal{E}(\vec{n}) \mathcal{O}_{2, S, x}^1 |0\rangle_{\text{in}}^0 = 2 \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \omega_1 \delta^{(2)}(\vec{n} - \hat{p}_1)(p_{1+}^x p_{2+}^x)(p_{1+}^y p_{2+}^y) \delta^3(\vec{p}_1 + \vec{p}_2) \delta(\omega_1 + \omega_2 - q)
\]

\[
= \frac{2}{(2\pi)^6} \frac{q}{16} \left(\frac{q}{2}\right)^{2S} \left(1 + n_3\right)^{2S} \left(1 - n_3\right)^{x+y} \quad (79)
\]

\[
\text{in}(0)\mathcal{O}_{1, S, y}^1 \mathcal{O}_{2, S, x}^1 |0\rangle_{\text{in}}^0 = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \omega_1 \delta^{(2)}(\vec{n} - \hat{p}_1)(p_{1+}^x p_{2+}^y)(p_{1+}^y p_{2+}^x) \delta^3(\vec{p}_1 + \vec{p}_2) \delta(\omega_1 + \omega_2 - q)
\]

\[
= \frac{1}{(2\pi)^6} \frac{1}{8} \left(\frac{q}{2}\right)^{2S} \int d^2\Omega (1 + \cos \theta)^{2S} \left(\frac{1 - \cos \theta}{1 + \cos \theta}\right)^{x+y} \quad (80)
\]

where we used that \(p_{i,+} \equiv p_{i,0} + p_{i,3}\). Summing over \(x \) and \(y \) the ratio of the two expressions above with coefficients given by (79) and using the orthonormality of Gegenbauer polynomials to evaluate the angular integral we find that the leading order of the energy one-point function in the state created by a non-BPS twist-2 operator is

\[
\langle \mathcal{E}(\vec{n}) \rangle_{2, S}^0 = \frac{q}{2\pi} \left(S + \frac{1}{2}\right) C_3^{1/2}(n_3)^2 .
\]

For twist-2 operators expressed in terms of an arbitrary null vector \(\ell = (\ell_0, \ell)\) the energy one-point function is obtained from the above by the replacement \(n_3 \mapsto \ell \cdot \vec{n}\). By appropriately changing the coefficients we may also construct the same one-point function in the states created by conformal descendants of twist-2 lower-spin operators.

The correlator of two (or several) energy flow operators is an observable with nontrivial parameter dependence even in the case of a spherically-symmetric state. Let us briefly discuss the correlator \(\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\rangle_{\text{BPS}}\). We find:

\[
\text{in}(0)\mathcal{O}_{\text{BPS}} \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) \mathcal{O}_{\text{BPS}} |0\rangle_{\text{in}}^0 = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \omega_1 \delta^{(2)}(\vec{n}_1 - \hat{p}_1)\omega_2 \delta^{(2)}(\vec{n}_2 - \hat{p}_2) \delta^3(\vec{p}_1 + \vec{p}_2) \delta(\omega_1 + \omega_2 - q)
\]

\[
+ \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \omega_1 \delta^{(2)}(\vec{n}_1 - \hat{p}_1)\omega_1 \delta^{(2)}(\vec{n}_2 - \hat{p}_2) \delta^3(\vec{p}_1 + \vec{p}_2) \delta(\omega_1 + \omega_2 - q)
\]

\[
+ \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \omega_2 \delta^{(2)}(\vec{n}_2 - \hat{p}_2)\omega_2 \delta^{(2)}(\vec{n}_1 - \hat{p}_2) \delta^3(\vec{p}_1 + \vec{p}_2) \delta(\omega_1 + \omega_2 - q)
\]

\[
+ \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \omega_1 \delta^{(2)}(\vec{n}_1 - \hat{p}_1)\omega_1 \delta^{(2)}(\vec{n}_1 - \hat{p}_1) \delta^3(\vec{p}_1 + \vec{p}_2) \delta(\omega_1 + \omega_2 - q)
\]

\[
= \frac{2}{(2\pi)^b} \frac{q^2}{32} \left(\delta^{(2)}(\vec{n}_1 + \vec{n}_2) + \delta^{(2)}(\vec{n}_1 - \vec{n}_2)\right) .
\]

\[42\]
together with \((77)\) it implies that

\[
\langle E(\vec{n}_1)E(\vec{n}_2)\rangle^{(0)}_{\text{BPS}} = \frac{q^2}{8\pi} \left( \delta^{(2)}(\vec{n}_1 + \vec{n}_2) + \delta^{(2)}(\vec{n}_1 - \vec{n}_2) \right).
\] (83)

Thus, this correlator vanishes identically as long as \(\vec{n}_1 \neq \pm \vec{n}_2\). The same \(\delta\)-function dependence on the fixed vectors \(\vec{n}_i\) seems to be generic for the leading order two-\(E\) correlator in the state created by any two-field operator. At higher orders this appears to change; below we will explore the next order – which is also the first nontrivial order – assuming that \(\vec{n}_1 \neq \pm \vec{n}_2\).

![Figure 12](image-url)

Figure 12: The four components of the cut contributing to the energy correlators at the NLO in the state created by a BPS two-field operator. The operators \(E(\vec{n})\) are placed on the cut propagators.

The components of the cut contributing to energy correlators at the first nontrivial order (we will refer to it as the LO) in the state created by a BPS two-field operator are shown in fig. 12. They may be evaluated directly using the one-loop two-point form factor and the tree-level three-point form factor of dimension-2 BPS operators or they may be obtained by constructing the cut of the momentum-space NLO two-point function. Using the discussion in secs. 2, 3 and 4 it is easy to find that, up to a factor of the square of the momentum carried by the operator, the two-point function is given by the two-loop scalar bubble integral.\(^{40}\) From the integral form of the two-point function it is easy to find the four components of the cut such that momentum labels are aligned.

A calculation similar to the one leading to eq. \((82)\) shows that the contribution of the first two components is proportional to \(\delta^{(2)}(\vec{n}_1 \pm \vec{n}_2)\), which vanishes due to our assumption that \(\vec{n}_1 \neq \pm \vec{n}_2\). The last two components are the same up to relabeling of momenta; we will analyze only the first one and double it. The contribution of both energy operators are inserted on the same cut line is proportional to \(\delta^{(2)}(\vec{n}_1 - \vec{n}_2)\) and thus may also be ignored.

\(^{40}\)This is consistent with the non-renormalization of the two-point function of BPS operators: the two-loop bubble integral is proportional to \(q^{-2}\) and therefore the momentum-space two-point function is a constant. Fourier-transforming to position space yields only contact terms, proportional to \(\delta^4(x_1 - x_2)\). It is interesting to note that in this case the energy correlator is related to such contact terms. This is special to states created by BPS operators.
The remaining terms are:

\[
\text{in} \langle 0 | \mathcal{O}_{\text{BPS}}^\dagger \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) \mathcal{O}_{\text{BPS}} | 0 \rangle_{\text{in}}^{(1)} = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \delta(\omega_1 + \omega_2 + \omega_3 - q)
\]

\[
\times \frac{1}{2(\omega_1 \omega_3 - \vec{p}_1 \cdot \vec{p}_2)} \frac{1}{2(\omega_2 \omega_3 - \vec{p}_2 \cdot \vec{p}_3)} \times \left( \omega_1 \omega_2 \delta(\vec{n}_1 - \vec{p}_1) \delta(\vec{n}_2 - \vec{p}_2) + \delta(\vec{n}_2 - \vec{p}_1) \delta(\vec{n}_1 - \vec{p}_2) \right)
\]

\[
+ \omega_1 \omega_3 \delta(\vec{n}_1 - \vec{p}_1) \delta(\vec{n}_2 - \vec{p}_3) + \delta(\vec{n}_2 - \vec{p}_1) \delta(\vec{n}_3 - \vec{p}_2)
\]

\[
+ \omega_2 \omega_3 \delta(\vec{n}_1 - \vec{p}_2) \delta(\vec{n}_2 - \vec{p}_3) + \delta(\vec{n}_2 - \vec{p}_2) \delta(\vec{n}_3 - \vec{p}_1)
\)

The six integrals are essentially the same; we will discuss only the ones on the first line of the parenthesis. The integral over the angular parts of \(\vec{p}_{1,2}\) sets \(\vec{p}_{1,2} = \omega_{1,2} \vec{n}_{a,b}\) with \(a \neq b \in \{1, 2\}\). Then the \(\vec{p}_3\) integral sets \(\vec{p}_3 = - (\omega_1 \vec{n}_a + \omega_2 \vec{n}_b)\) and \(\omega_3 = |\vec{p}_3|\). Last, we can do the integral over \(|\vec{p}_2| = \omega_2\); the \(\delta\)-function fixes \(\omega_2\) and introduces a Jacobian factor \(J = \omega_3/(q - \omega_1(1 - \vec{n}_1 \cdot \vec{n}_2))\). Most factors cancel and we are left with (for \(0 < \vec{n}_1 \cdot \vec{n}_2 < 1\))

\[
I_{1,2} = \frac{2}{32(2\pi)^9} \frac{1}{1 - \vec{n}_1 \cdot \vec{n}_2} \int_0^{q/2} \frac{\omega_1 d\omega_1}{(q - \omega_1(1 - \vec{n}_1 \cdot \vec{n}_2))^2}
\]

\[
= \frac{q^2}{16(2\pi)^9} \frac{(1 - \vec{n}_1 \cdot \vec{n}_2) \ln(2/(1 + \vec{n}_1 \cdot \vec{n}_2))}{(1 - \vec{n}_1 \cdot \vec{n}_2)^3(1 + \vec{n}_1 \cdot \vec{n}_2)}
\]

Similarly, the integrals on the second and third lines of the parenthesis are

\[
I_{1,3} = \frac{q^2}{16(2\pi)^9} \frac{-2(1 - \vec{n}_1 \cdot \vec{n}_2) + (3 + \vec{n}_1 \cdot \vec{n}_2) \ln(2/(1 + \vec{n}_1 \cdot \vec{n}_2))}{(1 - \vec{n}_1 \cdot \vec{n}_2)^3(1 + \vec{n}_1 \cdot \vec{n}_2)}
\]

\[
I_{2,3} = I_{1,2}
\]

Combining them, accounting for the component in fig. 12(d) and normalizing\(^{41}\) by dividing by \(^{17}\) leads to\(^{42}\)

\[
\langle \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) \rangle_{\text{BPS}}^{(1)} = \frac{q^2}{2(2\pi)^4} \frac{\ln(2/(1 + \vec{n}_1 \cdot \vec{n}_2))}{(1 - \vec{n}_1 \cdot \vec{n}_2)^2(1 + \vec{n}_1 \cdot \vec{n}_2)}.
\]

It should not be difficult to proceed to higher orders or to evaluate the energy correlators in states created by other operators. For example, at the next order one may start from the presentation of the two-point function in terms of the three-loop ladder and crossed-ladder propagator integrals, construct their three- and four-particle cuts and dress them with

\(^{41}\)Since the LO correlator \(\text{in} \langle 0 | \mathcal{O}_{\text{BPS}}^\dagger \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) \mathcal{O}_{\text{BPS}} | 0 \rangle_{\text{in}}^{(0)}\) the cut of the NLO 2-point function contributes only to NLO energy correlator.

\(^{42}\)This logarithmic dependence is the same as in QCD while the coefficient of the logarithm is different. We thank Lance Dixon for emphasizing this point to us.
Figure 13: Plot of the two-energy correlator in the state created by a twist-2 spin-$S$ operator for $S = 0$ (BPS), $S = 4$ and $S = 8$ and for $S = 16$, $S = 32$ and $S = 64$ as a function of $\vec{n}_1 \cdot \vec{n}_2$, for $\vec{\ell} \cdot \vec{n}_1 = 0 = \vec{\ell} \cdot \vec{n}_2$. The energy scale is set to $q = 1$ and may be restored to arbitrary values by conformal invariance.

energy factors. In evaluating the remaining integrals one sound observe the cancellation of IR divergences between the contribution of these two cuts.

We may also easily evaluate the two-energy correlator to first nontrivial order in the state created by a non-BPS twist-2 spin-$S$ operator. The relevant dressed cut is:

\[
\begin{align*}
\left\langle 0\left| \mathcal{O}_{2,3}^{\gamma}(\vec{n}_1) \mathcal{E}(\vec{n}_2) \mathcal{O}_{2,3}^{\gamma}(\vec{n}_2) \right| 0 \right\rangle_{\text{in}} & = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} \frac{\delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)\delta(\omega_1 + \omega_2 + \omega_3 - q)}{16(\omega_1\omega_2 - \vec{p}_1 \cdot \vec{p}_2)(\omega_2\omega_3 - \vec{p}_2 \cdot \vec{p}_3)} S(p_1 + p_2, p_3, x) S(p_1, p_2 + p_3, y) \\
& + \frac{p_{1+} + p_{2+}}{8p_{2+}(\omega_1\omega_2 - \vec{p}_1 \cdot \vec{p}_2)} \left( S(p_1 + p_2, p_3, x)(S(p_1, p_2 + p_3, y) - S(p_1 + p_2, p_3, y)) \\
& + S(p_1 + p_2, p_3, y)(S(p_1, p_2 + p_3, x) - S(p_1 + p_2, p_3, x)) \right) \\
& \times \left( \omega_1\omega_2(\delta^2(\vec{n}_1 - \hat{\vec{p}}_1)\delta^2(\vec{n}_2 - \hat{\vec{p}}_2) + \delta^2(\vec{n}_1 - \hat{\vec{p}}_2)\delta^2(\vec{n}_2 - \hat{\vec{p}}_1)) \\
& + \omega_1\omega_3(\delta^2(\vec{n}_1 - \hat{\vec{p}}_1)\delta^2(\vec{n}_2 - \hat{\vec{p}}_3) + \delta^2(\vec{n}_1 - \hat{\vec{p}}_3)\delta^2(\vec{n}_2 - \hat{\vec{p}}_1)) \\
& + \omega_2\omega_3(\delta^2(\vec{n}_1 - \hat{\vec{p}}_2)\delta^2(\vec{n}_2 - \hat{\vec{p}}_3) + \delta^2(\vec{n}_1 - \hat{\vec{p}}_3)\delta^2(\vec{n}_2 - \hat{\vec{p}}_2)) \right).
\end{align*}
\]

The evaluation of integrals proceeds following the same steps as for $\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) \rangle_{\text{BPS}}$. The resulting expressions, which are now functions of four variables – $S$, $\vec{n}_1 \cdot \vec{n}_2$, $\vec{\ell} \cdot \vec{n}_1$ and $\vec{\ell} \cdot \vec{n}_2$ where $\vec{\ell}$ defines the null direction in the twist-2 spin-$S$ operators – are quite lengthy. Multiplication by $8/(2\pi)^2(2S + 1)/q^{2S}$ yields the desired two-energy correlator, $\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) \rangle_{2,S}$.

Its expression simplifies somewhat by choosing the vector $\vec{\ell}$ defining the light-like direction
to be orthogonal to the two unit vectors \( \vec{n}_{1,2} \), i.e. for \( \vec{\ell} \cdot \vec{n}_1 = 0 = \vec{\ell} \cdot \vec{n}_2 \). In fig. 13 we show the graph of \( \langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) \rangle_{2,S} \) for \( S = 0, 4, 8 \) and for \( S = 16, 32, 64 \) for this choice of \( \vec{\ell} \), as a function of \( \vec{n}_1 \cdot \vec{n}_2 \). Similarly to the two-energy correlation in the state created by a BPS operator (recovered here as the \( S = 0 \) case), it has a simple pole at \( \vec{n}_1 \cdot \vec{n}_2 = -1 \) and a pole of order \( (2S + 4) \) (\( S \neq 0 \) and even) at \( \vec{n}_1 \cdot \vec{n}_2 = +1 \). The numerator exhibits the same logarithmic dependence as in the case of a state created by BPS operators, i.e. it depends on \( \ln(2/(1 + \vec{n}_1 \cdot \vec{n}_2)) \). Unlike that case however, the coefficient of the logarithm is a polynomial of degree \( (2S + 2) \) in \( \vec{n}_1 \cdot \vec{n}_2 \); the numerator also contains a polynomial of degree \( (2S + 3) \) in \( \vec{n}_1 \cdot \vec{n}_2 \) which factorizes as \( (1 - \vec{n}_1 \cdot \vec{n}_2)P_{2S+2}(\vec{n}_1 \cdot \vec{n}_2) \) and thus lowers the strength of the pole at \( \vec{n}_1 \cdot \vec{n}_2 = 1 \). These polynomial factors are a consequence of the momentum factors present in the operators. Indeed, setting \( S = 0 \) we recover the expression in eq. (88). By comparing figs. 13(a) and 13(b) we also note that the details of the operator with definite anomalous dimension are important for the two-energy correlator in a large spin state and are not important at low spin. Indeed, it is easy to see that the oscillatory behavior of Gegenbauer polynomials is not reflected by \( \langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) \rangle_{2,S} \) at low spin, fig. 13(a), while it is obvious at larger spins, fig. 13(b).

The same general features persist also if \( \vec{\ell} \cdot \vec{n}_1 \neq 0 \neq \vec{\ell} \cdot \vec{n}_2 \). These scalar products do not affect the pole structure of the logarithmic terms of the two-energy correlator; they appear only in the various polynomial factors due to the momentum dependence of the operator creating the state.

6 Outlook

Correlation functions of local gauge-invariant operators are an important class of observables in both conformal and non-conformal field theories. In this paper we extended the generalized unitarity method to the calculation of momentum space correlation functions; apart from the tree-level amplitudes of fundamental fields, correlation functions of gauge-invariant operators \( \mathcal{O}_i \) are determined by their on-shell form factors and generalized (multi-operator) form factors. This approach makes use of the methods developed for efficient construction of on-shell scattering amplitudes and exposes properties of some of the contributions to the integrand of momentum space correlation functions. In particular, we have argued for the presence of color/kinematics duality for all internal propagators which do not begin or end at an operator insertion up to contact terms that collapse at least one line connected to an operator. Using this approach we have recovered the known expression \([51, 52]\) for the four-point function of scalar BPS operators of dimension 2, we have found the \( n \)-point functions of such operators at the next-to-leading order and discussed the limit in which the operator insertion points are light-like separated. As expected \([5]\), we found that in this limit the correlator reduces to the one-loop \( n \)-point MHV amplitudes of \( \mathcal{N} = 4 \) theory. The methods
described in this paper are not restricted to correlators of BPS operators. To illustrate this we have computed the next-to-leading order correction to correlation functions of one and two non-BPS twist-2 spin-$S$ operators and any number of scalar BPS operators of dimension 2 and described the features of the correlator of $m$ twist-2 and $n$ BPS operators in a split configuration. We have moreover discussed the $\mathcal{N}=4$ realization of the energy correlators and evaluated the one-point and two-point correlators to leading nontrivial order in states created by the action of the chiral primary and twist-2 non-BPS operators. It is not difficult to extend these calculations to higher orders. It would be very interesting to understand the relation between the standard approach we followed and the analytic continuation prescription suggested in [7]. It would also be very interesting to understand how to carry out the same calculation at strong coupling, perhaps by making use of the available techniques [98] for the evaluation of form factors.

The use of generalized unitarity for the construction of correlation functions follows closely the construction of scattering amplitudes and thus it produces momentum space correlation functions in an algorithmic way. It would be interesting to formulate generalized unitarity directly in position space; unlike its momentum space version, such a formulation would likely manifestly preserve conformal symmetry. It is possible that it will bring further simplifications to the Lagrangian insertion method.

As suggested and illustrated in [5, 80, 81] position space $n$-point super-correlation functions of the chiral stress tensor multiplet are, in the null-separation limit, proportional to $n$-point superamplitudes. We have reproduced this result to leading nontrivial order using the momentum space correlation functions constructed here; we have also explicitly shown that the presence of non-BPS operators does not spoil this conclusion.

The calculation of general correlation functions at large 't Hooft coupling remains an interesting open problem. The use of form factors, for which Y-system-based methods have been developed [98], together with some variant of unitarity is a possible approach, as discussed in §5.4. Worldsheet form factors, recently discussed in [99], should provide an alternative approach to the same quantities. To this end it is necessary to find the worldsheet form factors of the vertex operators dual to gauge theory local operators and use them to construct correlation functions of vertex operators using worldsheet generalized unitarity.

Our approach suggests that, at least in certain situations, recursive methods could be used to construct the higher-loop integrand of momentum space correlation functions. To this end a necessary ingredient is the existence of a BCFW shift of operator momenta for which the tree-level form factors and generalized form factors are well-defined at infinity. Most operators explicitly discussed in this paper have this property, as can be seen from secs. 3.2, 3.3, 3.4 and Appendix B.3. Clearly, if such a recursion relation exists it will determine simultaneously scattering amplitudes, form factors of operators, as well as their correlation functions. While we believe that a recursion relation exists – if only because,
for correlation functions of BPS operators at generic positions, recursion relations have been argued to exist at strong coupling \[31, 32\] – the nonplanar nature of correlation functions appears to be a (hopefully temporary) obstruction to formulating such a recursion relation in weak coupling perturbation theory due to the non-uniqueness of the parametrization of non-planar integrands.

As we illustrated briefly in §4.1.3, supersymmetric methods initially developed for the construction of form factors \[54, 83\] apply naturally to the calculation of correlation functions and are quite efficient if all operators are annihilated by the same supersymmetry generators. In this case the presence of Grassmann \(\delta\)–functions in form factors and scattering amplitudes simplifies the evaluation of sums over internal states and reduces them to simply an exercise in solving linear algebraic equations \[91, 100\]. While for non-BPS operators such convenient presentations of form factors do not exist, it is possible to construct a coherent state representation for them once a component is found by some means. Adopting such a representation \[101\] for both the form factors of BPS operators amplitudes seems likely to simplify the evaluation of state sums for correlation functions involving non-BPS operators.

Acknowledgments

We are grateful to Z. Bern, J.C. Collins, J. Maldacena, G. Travaglini and A. Tseytlin for useful discussions and to A. Belitsky, J.J. Carrasco, L. Dixon, H. Johansson, G. Korchemsky, E. Sokatchev and A. Tseytlin for comments on a preliminary version of this paper. This work is supported by the US Department of Energy under contract de-sc0008745.

Appendix A: Harmonic superspace conventions

The harmonic superspace \[82\] provides a bridge between a non-chiral \(\mathcal{N} = 4\) off-shell superspace and the chiral on-shell momentum superspace used to specify external states of scattering amplitudes and form factors. By introducing harmonic variables \((u^+_a, u^-_{a'})\) valued in the coset

\[
\frac{SU(4)}{SU(2) \times SU(2)' \times U(1)}
\]

an \(SU(4)\) index is split into indices transforming under a local \(SU(2) \times SU(2)' \times U(1)\) subgroup:

\[
\theta^A \mapsto \theta^+ = \theta^A u^+_a \quad \theta^- = \theta^A u^-_{a'}.
\]

\[43\]This holds both in \(\mathcal{N} = 4\) sYM theory as well as in theories with fewer supercharges.

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By gauge-fixing the local symmetries one obtains a parametrization of the coset in which the $SU(4)$ symmetry is broken. Keeping the harmonic variables generic allows one to carry out the projections \((A.2)\) while maintaining an unbroken $SU(4)$ symmetry.

As an element of $SU(4)$ the $4 \times 4$ matrix \((u^+_A a, u^-_{A'})\) is unitary and thus its $4 \times 2$ blocks have the following properties:

\[
\bar{u}^+_A a u^+_{A'} = \delta^a_{a'}, \quad \bar{u}^-_{A'} u^-_{A'} = \delta^a_{a'}, \quad \bar{u}^+_A a u^-_{A'} = \bar{u}^+_A a u^-_{A'} = 0, \tag{A.3}
\]

where the bar stands for complex conjugation. Similarly,

\[
u^+_A a u^+_B a + u^-_{A'} a u^-_{B'} = \delta^a_{B}. \tag{A.4}
\]

Unitarity also requires that the matrix have unit determinant:

\[
\frac{1}{4} \epsilon^{ABCD} u^+_A a \epsilon^{ab} u^+_B b u^-_{C} c \epsilon^{d} d u^-_{D} d' = 1, \tag{A.5}
\]

where we use the convention:

\[
\epsilon^{12} = \epsilon^{1'2'} = 1 = -\epsilon_{12} = -\epsilon_{1'2'}. \tag{A.6}
\]

Eq. \((A.5)\) leads to the relation:

\[
\frac{1}{2} \epsilon^{ABCD} u^+_A a \epsilon^{ab} u^+_B b = -\bar{u}^+_C c u^+_D d. \tag{A.7}
\]

There exists one set of harmonic variables for each operator; rather than attaching another decoration to $u$, it is convenient to use the operator label (or the insertion point label) instead:

\[
u^+_A a \mapsto k^+_A a. \tag{A.8}
\]

The on-shell superspace coordinates $\eta$ can be projected similarly to \((A.2)\):

\[
\eta_A \mapsto \eta^+_a = \bar{u}^+_A a \eta_A \quad \eta^-_{a'} = \bar{u}^-_{A'} a' \eta_A, \tag{A.9}
\]

and similarly project the $N = 4$ fields such that the on-shell superfield is invariant; this amounts to inserting a suitable number of identity matrices in the form \((A.4)\) in the superfield in footnote \(21\) Fourier-transforming the coordinates $\eta^a$ leads to the projective (non-chiral) superspace of ref. \cite{102} which may be obtained by dimensional reduction from a six-dimensional $(1,1)$ superspace. The superfield is then given by \cite{102}:

\[
\Phi = \phi^{21} + \eta^a \bar{\psi}_{a'} + \bar{\eta}_a \psi^{a+} + \bar{\eta}_a \eta^a \phi_{a'}^{a+} + \eta^2 g^+ + \bar{\eta}^2 g^- + \frac{1}{2} \eta^2 \bar{\eta}^2 \phi^{43} \tag{A.10}
\]

where $\eta \equiv \eta^+$, $\bar{\eta}$ is the Fourier conjugate of $\eta^-$, $\eta^2 = \frac{1}{2} \eta^a \bar{\eta}_a$ and $\bar{\eta}^2 = \frac{1}{2} \eta_{a'} \bar{\eta}^a'$. 

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Appendix B:

In this Appendix we summarize the calculation of the various form factors and generalized form factors used in our examples of calculation of correlation functions of local operators through the generalized unitarity method. The following shorthand will be convenient:

\[ P_{ij} = -(p_i + p_{i+1} + \cdots + p_j), \quad Q_{ij}^k = q_k + P_{ij}. \] (B.1)

Furthermore, we will use \( \varepsilon \) for spinor indices and \( \epsilon \) for the \( SU(2) \) indices of the harmonic variables and in addition to the definitions from Appendix A we introduce the following conventions:

\[ \lambda_i^a \varepsilon_{\alpha\beta} \lambda_j^\beta = \langle ij \rangle \quad \epsilon_{12}^1 = +1. \] (B.2)

B.1 Tree-level (NMHV) generalized form factor of the stress two tensor multiplets

As may be easily seen using Feynman diagrams, the simplest component of the generalized form factor with two insertions of the chiral stress tensor multiplet involves two scalar operators connected by a propagator. We may also interpret this as the result of an MHV vertex expansion; this particular generalized form factor arises from two MHV vertices, as shown in fig. 2(a); we will therefore refer to it as NMHV.

In computing the NMHV form factor, one \( SU(4) \) index will be treated differently than the others. We will denote it by \( T \), while we will sum over the indices \( A, B, \) and \( C \). By breaking manifest \( SU(4) \) symmetry we can get a relatively compact expression while still having the possibility of making a convenient choice of \( T \). For the derivation of the non-contact part of the color-dressed components of this form factor we will use the MHV vertex expansion, in which one sums up the contribution of MHV diagrams with the topologies shown in fig. 2(a). It is not difficult to see that

\[ \mathcal{F}_{(a)}^{NMHV} = \prod_{m=1}^n \frac{1}{\langle m, m + 1 \rangle^8} \delta(8)(1_a^+ \gamma_1^a + 2_a^+ \gamma_2^a - \sum_{m=1}^n \eta_m \lambda_m) \delta(4)(q_1 + q_2 - p_1 \cdots - p_n) \]

\[ \sum_{i=1}^{n} \sum_{j=1}^{i-2} \delta^4(1_a^+ \gamma_1^a Q_{ij}^1) - \sum_{r=i}^j \eta_r \langle r Q_{ij}^1 \rangle \frac{1}{(Q_{ij}^1)^2} \frac{\langle j+1 \rangle \langle i-1 \rangle}{\langle j Q_{ij}^1 \rangle \langle Q_{ij}^1 \rangle} \frac{\langle i-1 \rangle}{\langle i Q_{ij}^1 \rangle}, \] (B.3)

\[ \langle j Q_{ij}^1 \rangle = \langle Q_{ij}^1 \rangle \langle j Q_{ij}^1 \rangle + 1 \langle i-1 \rangle, \] (B.4)

where we have already combined the Grassmann \( \delta \)–functions of the two MHV form factors to expose the overall super-momentum conservation constraint. The factor \( \delta^{(4)} \) may also be
written as the four-fold antisymmetric product of its argument. Using two such factors we partial fraction the ratio of spinor products as

\[ F_{(a)}^{NMHV} = \frac{1}{3! \prod_{m=1}^{n} \langle m, m+1 \rangle} \delta^8(1^+_a \gamma^a_{1+} + 2^+_a \gamma^a_{2+} - \sum_{m=1}^{n} \eta_m \lambda_m) \delta^4(q_1 + q_2 - p_1 \cdots - p_n) \]

\[ \times \epsilon^{TABC} \sum_{i=1}^{n} \sum_{j=1}^{i-2} (1^+_T \langle \gamma^a_{1+} Q^i_{ij} \rangle - \sum_{r=i}^{j} \eta_{Tr} \langle r Q^i_{ij} \rangle) (1^+_A \langle \gamma^b_{1+} Q^i_{ij} \rangle - \sum_{r=i}^{j} \eta_{Ar} \langle r Q^i_{ij} \rangle) \]

\[ \times (1^+_B \gamma^c_{1+} - \sum_{r=i}^{j} \eta_{Br} \lambda^c_{r}) (1^+_C \gamma^d_{1+} - \sum_{r=i}^{j} \eta_{Cr} \lambda^d_{r}) \frac{\varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta}}{(Q^i_{ij})^4} \]

\[ \times \left[ \frac{\lambda^\delta_{\beta} \lambda^\delta_{\gamma}}{\langle jQ^i_{ij} \rangle (i - 1 Q^i_{ij})} + \frac{\lambda^\delta_{\beta} \lambda^\delta_{\gamma}}{\langle Q^i_{ij} \rangle (j Q^i_{ij})} + \frac{\lambda^\delta_{\beta} \lambda^\delta_{\gamma}}{\langle Q^i_{ij} \rangle (i - 1 Q^i_{ij})} + \frac{\lambda^\delta_{\beta} \lambda^\delta_{\gamma}}{\langle Q^i_{ij} \rangle (j Q^i_{ij})} \right]. \]

(B.5)

To simplify somewhat this expression we introduce a factor of \( \langle ij \rangle / \langle ij \rangle \) or similar except in the cases where this is factor takes the value 0/0 (perhaps due to restricted kinematic configuration). Using repeatedly Schouten’s identity as well as the relation

\[ \gamma^{1\alpha}_{\bar{a}+} \gamma^{1\gamma}_{\bar{a}+} \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} = \frac{1}{2} \langle \gamma^{1}_{\bar{a}+} \rangle \varepsilon_{\beta\delta}. \]

(B.6)
we find, after a lengthy calculation, that

\[ \mathcal{F}^{\text{NMHV}}_{(a)} = \frac{1}{3! \prod_{m=1}^{n} (m, m + 1)} \delta^8 (1^+_a \gamma^a_1 + 2^+_a \gamma^a_2 - \sum_{m=1}^{n} \eta_m \lambda_m) \delta^4 (q_1 + q_2 - p_1 \cdots - p_n) \]

\[ \epsilon^{TABC} \sum_{i=1}^{n} i - 2 \sum_{j=1}^{j=i} \left( \frac{1}{1} + \gamma^a_1 + \sum_{r=1}^{j} \eta_{Tr} \langle r Q^1_{ij} \rangle \right) \left[ \sum_{j=1}^{j=i} \left( 1^+_a \gamma^a_1 + \sum_{r=1}^{j} \eta_{Ar} \lambda^a_r \right) \langle j + 1 \rangle \right] \]

\[ + \frac{\lambda^a_r \lambda^a_j \langle i \rangle}{Q^1_{ij}} \left( j - 1 \right) \left( j - 1 \right) + \sum_{r=1}^{j} \left( 1^+_a \gamma^a_1 + \sum_{r=1}^{j} \eta_{Cr} \lambda^a_r \right) \langle i \rangle \]

\[ \left[ \sum_{j=1}^{j=i} \left( 1^+_a \gamma^a_1 + \sum_{r=1}^{j} \eta_{Cr} \lambda^a_r \right) \langle i \rangle \right] \]

\[ + \eta_{rl} \langle r \rangle \langle l \rangle \left( \sum_{i=1}^{n} \frac{3 e^{u \gamma^u} e^{v \gamma^v} e^{w \gamma^w}}{2} \langle i \rangle \langle j \rangle \right) \left( \sum_{j=1}^{j=i} \left( 1^+_a \gamma^a_1 + \sum_{r=1}^{j} \eta_{Cr} \lambda^a_r \right) \langle i \rangle \right) \]

The relation \( \langle i Q^1_{ij} \rangle = \langle i Q^1_{i+1} \rangle \) implies that we can shift the summation index and further simplify the expression of the form factor. In the end we find that the color-ordered \( \overline{\text{MHV}} \) form factor is given by

\[ \mathcal{F}^{\text{NMHV}}_{(a)} = \frac{1}{3! \prod_{m=1}^{n} (m, m + 1)} \delta^8 (1^+_a \gamma^a_1 + 2^+_a \gamma^a_2 - \sum_{m=1}^{n} \eta_m \lambda_m) \delta^4 (q_1 + q_2 - p_1 \cdots - p_n) \]

\[ \sum_{a=1}^{n} \sum_{j=i}^{j=i} \sum_{l=1}^{l=i} A^a_{ij} + \sum_{a=1}^{n} \sum_{j=1}^{j=i} B^a_{ij} \]
where we have defined the expressions:

\[
A_{ij} = \epsilon^{TABC} \left( \eta_{Ti}(Q_{ij}^w)^2 - w_{Ta}^+(\gamma_i^a | Q_{ij}^w | i) + \sum_{r=1}^{j} \eta_{Tr}(r|Q_{ij}^w|i)(w_{Ab}^+(\gamma_i^b | i) - \sum_{r=1}^{j} \eta_{Ar}(r_i)) \right) \\
\times (w_{Bc}^+(\gamma_i^c | i) - \sum_{r=1}^{j} \eta_{Br}(r_i))(w_{Cd}^+(\gamma_i^d | i) - \sum_{r=1}^{j} \eta_{Cr}(r_i)) \frac{1}{(Q_{ij}^w)^2} \langle jj + 1 \rangle \langle jj + 1 \rangle.
\]

Here the harmonic variable \( w \) takes two values corresponding the the harmonic variables of the two chiral stress-tensor multiplets.

This analysis is sufficient for the construction of the generalized form factor of two CPOs. For operators at higher levels in the chiral stress tensor multiplet it is possible that MHV generalized form factors exist. In this case it is necessary to supplement the terms above by the contribution of MHV diagrams in fig. 2(b). We leave the construction of the relevant MHV form factor and of \( \mathcal{F}_{(b)}^{N=MHV} \) to the interested reader.

### B.2 MHV form factor of a scalar non-BPS operator

The \( SU(2) \) sector of \( \mathcal{N} = 4 \) sYM theory contains operators constructed out of two complex scalar fields whose two-point function vanishes; it is perhaps the simplest sector which is closed under renormalization group flow. Operators in this sector are labeled by the number of fields of each type; operators with definite anomalous dimensions are linear combinations of single-trace monomials with fields ordered in different ways. In this Appendix we consider a generic single-trace monomial, \( \mathcal{O}_{A_1B_1...A_kB_k} = \text{Tr}(\phi_{A_1B_1} \cdots \phi_{A_kB_k}) \); and construct its MHV super-form factor; we will find the result quoted in eq. (23). By restricting the pairs \((A_i, B_i)\) to take only two values one finds the building blocks of operators in the \( SU(2) \) sector.

The MHV form factor can be computed easily when there are no gluons by just computing a single Feynman diagram; the minimum number of external legs is \( k \). To add gluons, we will rely on a BCFW recursion relation. To preserve the MHV character of the form factor the added gluons must all have positive helicity. We will choose to shift the momenta of two scalar fields; because of this only three-point scalar-gluon amplitudes can appear in the recursion relation.

---

44While in its "standard" presentation all operators are constructed out of two holomorphic scalar fields, this sector may be embedded in the \( SO(6) \) R-symmetry in many different ways.

45Here MHV stands for \( 2k \) Grassmann parameters – two for each scalar field.
Similarly to amplitudes, form factors carry color structure. It may be possible to generalize the construction of \cite{103} to color-dressed form factors (the generalization at least partly bypass the use of multi-peripheral color decomposition \cite{104} since the presence of the operator makes the form factor potentially depend on other color tensors apart from the antisymmetric structure constants). Here we will find the color-ordered form factors from which the complete color-dressed one may be constructed.

Let us focus on the form factor with external gluons and scalars and start by adding a gluon between two adjacent scalar legs, \( i \) and \( j \), (in the sense that there is no other scalar between them); the rest of the external legs will be other scalars. We choose to shift the momenta of the scalars,:

\[
|\hat{i}\rangle = |i\rangle + z|j\rangle,
\]

\[
|\hat{j}\rangle = |j\rangle - z|i\rangle .
\]  

This ensures that there is no boundary term at infinity. Indeed, since the gluon has to be attached to either the leg \( i \) or the leg \( j \), there will be one propagator dependent on \( z \) and the form factor goes to 0 for \( z \to \infty \). The two BCFW contributions, \( T_i \) and \( T_j \), are:

\[
T_i = \frac{[\hat{i} + 1][i + 1\hat{P}_{ii+1}]}{[\hat{i}\hat{P}_{ii+1}]} \frac{1}{s_{ii+1}} = \frac{1}{s_{ii+1}} [i + 1\langle ij\rangle = \frac{1}{s_{ii+1}} \frac{\langle ij\rangle}{\langle ii+1\rangle \langle i+1j\rangle} .
\]  

\[
T_j = \frac{[\hat{P}_{i+1j}i + 1][i + 1\hat{j}]}{[\hat{P}_{i+1j}]} \frac{1}{s_{i+1j}} = 0 .
\]

The vanishing of the second contribution is a consequence of the fact that \( |\hat{j}\rangle = |j\rangle \) and the existence of the necessary three-point amplitude requires that \( \langle i+1, j \rangle = 0 \) i.e. the external legs cannot be in a generic kinematic configuration.
Proceeding to add one more gluon suggests that the expected expression for the color-ordered form factor with external scalars and gluons is

$$F = \frac{\langle ai \rangle \langle ij \rangle}{\prod_{k=a}^{i-1} \langle kk + 1 \rangle} C$$  \hspace{1cm} (B.13)$$

where $i$ and $j$ are the adjacent shifted legs, $a$ labels the first scalar line before $i$ (see figs. 14, 15) and $C$ denotes the contribution to the form factor that depends on the external lines $j, j+1, \ldots a$. To prove this conjecture we will show that the BCFW recursion relation preserves its form.

Let us consider an arbitrary configuration of external lines such that there are at least two gluons between $i$ and $j$ and at least two gluons between $a$ and $i$ (other configurations must be considered separately). The possible contributions shown in figs. 14(a) and 14(b), in which there exists a tree-level amplitude with at least two gluons, vanish identically because the (desired) MHV nature of the form factor required all gluons to have positive helicities and thus requires that the amplitude factor vanishes identically (being related to an amplitude with a single negative helicity gluon). The BCFW contribution shown in fig. 14(c) vanished because it requires that the momentum of the new gluon be related to the momentum of the leg $j$. The only non-zero diagrams are the ones shown in fig. 15. Notice also that the computation does not depend on what exactly is on the right side of the graph between $j$ and $a$; we will denote this part by $C$. The two contributions are:

$$T_{\text{fig. 15}}^{(a)} = \frac{[\hat{i}i + 1][i + 1 \hat{P}_{i+1}]}{[\hat{i}P_{i+1}]} \frac{1}{s_{i+1}} \frac{\langle ai \rangle \langle ij \rangle}{\prod_{k=a}^{i-1} \langle kk + 1 \rangle} \langle i - 1 \hat{P}_{i+1} \rangle \langle -\hat{P}_{i+1} \rangle \langle -\hat{P}_{i+1} \hat{j} \rangle \langle i + 1 \rangle \langle i - 1 \hat{P}_{i+1} \hat{i} \rangle \langle i - 1 \rangle \langle i + 2 \rangle$$

$$= \frac{C \langle ij \rangle}{\prod_{k=a}^{i-1} \langle kk + 1 \rangle} \langle i - 1 \hat{P}_{i+1} \hat{i} \rangle \langle i - 1 \hat{P}_{i+1} \rangle \langle i - 1 \rangle \langle i + 1 \rangle \langle i + 2 \rangle$$  \hspace{1cm} (B.14)$$

$$T_{\text{fig. 15}}^{(b)} = \frac{[\hat{i}i - 1][i - 1 \hat{i}]}{[\hat{i}i]} \frac{1}{s_{i-1}} \frac{\langle ai \rangle \langle ij \rangle}{\prod_{k=a}^{i-2} \langle kk + 1 \rangle} \langle i - 2 \hat{P}_{i-1} \rangle \langle -\hat{P}_{i-1} \hat{i} \rangle \langle -\hat{P}_{i-1} \hat{j} \rangle \langle i - 1 \hat{P}_{i-1} \hat{i} \rangle \langle i - 1 \rangle$$

$$= \frac{C \langle ij \rangle}{\prod_{k=a}^{i-2} \langle kk + 1 \rangle} \langle i - 1 \hat{P}_{i-1} \hat{i} \rangle \langle i - 1 \hat{P}_{i-1} \rangle \langle i - 1 \rangle \langle i + 1 \rangle \langle i + 2 \rangle$$  \hspace{1cm} (B.15)$$

In these expressions we used (B.13) for the necessary form factors.

Adding the two contributions $T_{\text{fig. 15}}^{(a)}$ and $T_{\text{fig. 15}}^{(b)}$ leads to

$$F = \frac{\langle ai \rangle \langle ij \rangle}{\prod_{k=a}^{j-1} \langle kk + 1 \rangle} C$$  \hspace{1cm} (B.16)$$

which is consistent with the expectation (B.13). It is not difficult to repeat the analysis above for the special cases when the number of gluons between legs $a$ and $i$ and/or between $i$ and $j$ is 0 or 1 and confirm the expression (B.16) for these cases as well.
The computations with fermion legs are very similar. Making use of the fact that (B.16) is not sensitive to the R-symmetry index structure of the operator $O_{A_1 B_1, \ldots, A_k B_k} = \text{Tr}(\phi_{A_1 B_1} \cdots \phi_{A_k B_k})$, the form factors with $n$ external scalars, fermions and gluons may be combined with (B.16) into the MHV super-form factor

$$F_{O_{A_1 B_1, \ldots, A_k B_k}} = \frac{\delta^4(q - \sum_{i=1}^n p_i)}{\prod_{m=1}^n \langle m, m+1 \rangle} \sum_{\{a_1, b_1, \ldots, a_k, b_k\}} \left( \prod_{i=1}^k H_{a_i A_i b_i B_i} \right) \text{Sp} \left( \prod_{j=1}^k \Sigma_{a_j b_j} \right), \quad (B.17)$$

where

$$H_{a A b B} \equiv \eta_{Aa} \eta_{Bb} - \eta_{Ba} \eta_{Ab} + \delta_{ab} \eta_{Ba} \eta_{Ab}, \quad (B.18)$$

$$(\Sigma_{a_1 b_1})^{\alpha \gamma} \equiv \lambda_{a_1}^\alpha \lambda_{b_1}^\beta \varepsilon_{\beta \gamma} \quad (B.19)$$

and the first sum runs over all over the sets $\{a_1, b_1, \ldots, a_k, b_k\}$ with particle labels valued between 1 and $k$ and ordered as $a_1 \leq b_1 < a_2 \leq b_2, \ldots, b_{k-1} < a_k \leq b_k$ or $b_k < a_1 \leq b_1 < a_2 \leq b_2, \ldots, b_{k-1} < a_k$ etc, as stated in §3.3.

### B.3 Tree-level MHV form factor of twist-2 operators

Another set of operators that is closed under renormalization group flow forms the so-called $SL(2)$ sector; these operators are linear combinations of monomials of the type

$$O_{2, S, x}^{AB, CD} = \text{Tr}(D_x^+ \phi^{AB} D_{-x}^{S-x} \phi^{CD}) \quad (B.20)$$
Figure 16: An example of a form factor with two external fermions and one external scalar

![Diagram of form factor with two external fermions and one external scalar]

Figure 17: The two classes of BCFW terms contributing to the form factor of twist-2 operators.

![Diagram illustrating two classes of BCFW terms]

as stated in eq. (B.20). In this appendix we show that the tree-level form factor of these monomials are given by eq. (26)

\[
\mathcal{F}_{O_{AB,CD}} = \frac{\delta^4(q - \sum^n_{i=1} p_i)}{\prod_{m=1}^{n} \langle m, m + 1 \rangle} \sum_{\{a,b,c,d\}} H_{aAbB} H_{cCdD} \left( \sum^{c-1}_{k=b} \sum^{a-1}_{l=d} \left( \sum^{k}_{r=l+1} p_r^{-} \right)^x \left( \sum^{l}_{s=k+1} p_s^{-} \right)^{S-x} \right) \]

(B.21)

\[
\frac{\langle b|\sigma^{-}p_k|c \rangle}{2p_k^{-}} + \frac{\langle b|p_{k+1}\sigma^{-}|c \rangle}{2p_{k+1}^{-}} - \langle bc \rangle \left( \frac{\langle d|\sigma^{-}p_l|a \rangle}{2p_l^{-}} + \frac{\langle d|p_{l+1}\sigma^{-}|a \rangle}{2p_{l+1}^{-}} - \langle da \rangle \right);
\]

the first sum runs over all sets \{a, b, c, d\} where \(a \leq b < c \leq d\) or \(d < a \leq b < c\) etc. Fig. 16 summarizes the notation we use.

We will focus on the case when all external lines are either scalars of gluons. The form factor with two external scalars and no gluons can be easily constructed by inspection: it is just the product of the “−” components of the momenta of the two scalars to the appropriate powers.

To prove eqs. (26), (B.21) we will use a variant of a BCFW shift. In choosing the shift we would like to avoid a boundary term arising from \(z \to \infty\) region; this can be accomplished by the following shift that involves the momentum of a gluon and of the operator itself:

\[
|i\rangle = |i\rangle + z|\phi\rangle,
\]

(B.22)

\[
\hat{q}^\mu = q^\mu + \frac{1}{2} z [i|\sigma^\mu|\phi\rangle,
\]

(B.23)
where \( |\phi\rangle \) is chosen to satisfy \( |i|\sigma^-|\phi\rangle = 0 \); such a choice is always possible. With this shift the BCFW recursion relation for the MHV form factor of \( O_{2,S,x}^{AB,CD} \) has only the two contributions shown in fig. 17:

\[
T_{fig. 17(a)} = \frac{[a][i - \hat{P}_{ia}]}{[a - \hat{P}_{ia}]} \left( \hat{P}_{ia} \right)^x (p^{-}_a) (p^{-}_b)^{S-x} = \frac{\langle a\phi \rangle}{\langle ai \rangle \langle i\phi \rangle} (p^{-}_a + p^{-}_i) (p^{-}_b)^{S-x}
\]

\[
T_{fig. 17(b)} = \frac{[-\hat{P}_{ib}][ib]}{-\hat{P}_{ib}^2} (p^{-}_b)^x (\hat{P}_{ib})^{S-x} = \frac{\langle \phi b \rangle}{\langle ib \rangle \langle \phi i \rangle} (p^{-}_a)^x (p^{-}_i + p^{-}_b)^{S-x}
\]

(B.25)

in deriving these expressions we used that

\[
\frac{\langle a\phi \rangle}{\langle ai \rangle \langle i\phi \rangle} = \frac{\langle i|\sigma^-|ai \rangle}{2p^{-}_i \langle i\phi \rangle} = \frac{\langle a|\sigma^-|i \rangle}{2p^{-}_i}
\]

(B.26)

which follows from the Schouten identity and the choice of \( |\phi\rangle \) such that \( |i|\sigma^-|\phi\rangle = 0 \). Adding the \( T_{fig. 17(a)} \) and \( T_{fig. 17(b)} \) we find a result that is indeed consistent with the general expression shown in eqs. (26) and (B.21):

\[
\langle \tilde{O}_{2,S,x}^{AB,CD} |\phi_{AB} A^+ \phi_{CD} \rangle = \frac{\langle ba \rangle}{\langle ai \rangle \langle ib \rangle \langle ba \rangle} \left( (p^{-}_a)^x (p^{-}_i + p^{-}_b)^{S-x} \frac{\langle a|\sigma^-|b \rangle}{2p^{-}_i} + (p^{-}_a + p^{-}_i)^x (p^{-}_b)^{S-x} \frac{\langle a|\sigma^-|i \rangle}{2p^{-}_i} \right)
\]

(B.27)

This expression was used in the evaluation of some of the cuts in \( \text{§4.2} \). It is also possible to check that this agrees with the Feynman diagram evaluation of \( \langle \tilde{O}_{2,S,x}^{AB,CD} |\phi_{AB} A^+ \phi_{CD} \rangle \). It is also easy to see that in the limit \( S, x \to 0 \) eq. (B.27) reduces, as it should, to the three-field form factor of the BPS scalar operator.

Let us proceed to the general case, in which we add one gluon to a form factor which already has some arbitrary number of gluons present; the two non-vanishing contributions...
to the color-ordered form factor in the ordering in which all gluons are adjacent are shown in fig. 18:

\[ T_1^{\text{fig. 18a}} = \frac{[ai][i \rightarrow P_{ia}]}{[a-P_{ia}]} \frac{1}{P_{ia}^2 \prod_{m=a}^{b} \langle m, m + 1 \rangle \langle P_{ia} \rangle} \left[ \sum_{k=i+1}^{b-1} \left( \frac{\hat{P}_{ia} - \sum_{r=i+1}^{k} p_r - 1}{2p_k} \right) + \left( \frac{\hat{P}_{ia} \sigma^{-} p_k \sigma^{-} |b|}{2p_{k+1}} - \langle \hat{P}_{ia} \rangle \right) + \left( \frac{\hat{P}_{ia} \sigma^{-} p_{k+1} \sigma^{-} |b|}{2p_{k+1}} \right) \right] \]

\[ = \frac{1}{\prod_{m=a}^{b-1} \langle m, m + 1 \rangle} \left( \frac{\langle a | \sigma^{-} p_b | b \rangle}{2p_b} + \frac{\langle a | p_{b+1} \sigma^{-} | b \rangle}{2p_{b+1}} - \langle ab \rangle \right) \]

\[ T_2^{\text{fig. 18b}} = \frac{[i i + 1]^4}{[a P_{i i + 1}]} \frac{1}{[1 - i + 1 P_{i i + 1}]} \frac{1}{P_{i i + 1}^2 \prod_{m=a+2}^{b-1} \langle m, m + 1 \rangle \langle a P_{i i + 1} \rangle \langle P_{i i + 1} i + 2 \rangle} \]

\[ \left( \frac{p_b}{2} \right)^x \left( \hat{P}_{i i + 1} - \sum_{s=i+2}^{b} p_m \right)^{S-x} \left( \frac{\langle a | \sigma^{-} P_{i i + 1} \sigma^{-} | b \rangle}{2P_{i i + 1}} \right) \]

Adding together the two contributions we find exactly the scalar-gluon component of eqs. (26) and (B.21). We described explicitly here the construction of the form factor with adjacent gluons; a very similar calculation yields the form factor with some gluons above and some gluons below the scalar line. Among the MHV form factors there also some with two external fermions; For the appropriate color ordering it combines with the scalar-gluon form factor and yield eqs. (26) and (B.21).
Appendix C: On the Fourier-transform of the NLO four-point function of CPO-s

Let us begin by Fourier-transforming the double-box integral to position space.

\[
\text{DB}(1, 2|3, 4) = \int \prod_{i=1}^{7} \frac{d^4 l_i}{(2\pi)^4 l_i^2} (2\pi)^{24} \delta^4(q_1 + l_1 + l_2) \delta^4(q_2 + l_3 - l_2) \delta^4(q_4 + l_4 + l_5) \\
\delta^4(q_3 + l_6 - l_5) \delta^4(l_7 - l_3 - l_6) \delta^4(-l_1 - l_3 - l_7) \tag{C.1}
\]

\[
= \int \prod_{j=1}^{6} d^4 x_j \int \prod_{i=1}^{7} \frac{d^4 l_i}{(2\pi)^4 l_i^2} e^{ix_1 \cdot (q_1 + l_1 + l_2)} e^{ix_2 \cdot (q_2 + l_3 - l_2)} \\
e^{ix_4 \cdot (q_4 + l_4 + l_5)} e^{ix_5 \cdot (q_3 + l_6 - l_5)} e^{ix_6 \cdot (-l_1 - l_3 - l_7)} \tag{C.2}
\]

\[
= \int \prod_{j=1}^{6} d^4 x_j \ e^{\sum_{i=1}^{4} x_i \cdot (I_{16} I_{12} I_{25} I_{35} I_{34} I_{46} I_{56})} \tag{C.3}
\]

where \( I_{ij} \) are standard position space propagators\(^{46}\).

\[
I_{ij} = \frac{1}{(2\pi)^2 (x_i - x_j)^2}. \tag{C.4}
\]

Fourier-transforming \[(C.3)\] to position space simply freezes the integrals over \( x_1, x_2, x_3 \) and \( x_4 \) to the location of the operator insertions. Momentum space numerator factors become differential operators acting on the integrand.

Repeating the derivation above for the other integrals in eq. (42) we find that, up to derivative and numerator factors, the integrals that can appear in their Fourier transform are

\[
Y_{123} = \int d^4 w I_{1w} I_{2w} I_{3w} \tag{C.5}
\]
\[
X_{1234} = \int d^4 w I_{1w} I_{2w} I_{3w} I_{4w} \tag{C.6}
\]
\[
H_{12;34} = \int d^4 u d^4 v I_{1u} I_{2u} I_{3v} I_{4v} \tag{C.7}
\]

The correlation functions depend only on \((\partial_1 + \partial_2)^2 H_{41;23}\) or, perhaps more symmetrically,

\[
((\partial_1 + \partial_2)^2 + (\partial_3 + \partial_4)^2) H_{41;23}; \tag{C.8}
\]

following the ideas of [90], in this appendix we spell out some details of the reduction of the relevant derivatives of the \( H \) integral.

\(^{46}\)In Minkowski space the propagator has an additional factor of \( i \).
This integral does not require any regularization as it is finite both in the UV and in the IR. Furthermore, the terms in which two derivatives act on the same coordinate reduce immediately to integrals of the type \( Y_{ij} \) upon use of

\[
\partial^2_i I_{ij} = \delta^4(x_i - x_j) ;
\]

We therefore need to reduce only

\[
I(x_1, x_2, x_3, x_4) = \int \prod_{i=1}^{4} \frac{d^4 p_i}{(2\pi)^4} e^{i \Sigma_{i=1}^4 p_i \cdot x_i} \frac{p_1 \cdot p_2 + p_3 \cdot p_4}{p_1^2 p_2^2 p_3^2 p_4^2 (p_1 + p_4)^2} (2\pi)^4 \delta^4 \left( \sum_{i=1}^{4} p_i \right)
\]

(C.9)

Following [90], we multiply and divide by \((x_1 - x_4)^2(x_2 - x_3)^2\) and rewrite the numerator in terms of derivatives on the integrand:

\[
I = \frac{(x_1 - x_4)^2(x_2 - x_3)^2}{(x_1 - x_4)^2(x_2 - x_3)^2} I,
\]

\[
= \frac{1}{x_1^2 x_2^2 x_3^2 x_4^2} \int \prod_{i=1}^{4} \frac{d^4 p_i}{(2\pi)^4} \left( (\partial_{p_1} - \partial_{p_4})^2 e^{ip_1 \cdot x_1 + ip_4 \cdot x_4} \right) \left( (\partial_{p_2} - \partial_{p_3})^2 e^{ip_2 \cdot x_2 + ip_3 \cdot x_3} \right)
\]

\[
\times \frac{p_1 \cdot p_2 + p_3 \cdot p_4}{p_1^2 p_2^2 p_3^2 p_4^2 (p_1 + p_4)^2} (2\pi)^4 \delta^4 (p_1 + p_2 + p_3 + p_4).
\]

(C.10)

We then integrate by parts \((\partial_{p_1} - \partial_{p_4})^2\) setting aside the factor \((p_2^2 p_3^2 (p_1 + p_4)^2)^{-1}\) on which this combination of derivatives act trivially and find that the integrand becomes

\[
(\partial_{p_1} - \partial_{p_4})^2 \frac{p_1 \cdot p_2 + p_3 \cdot p_4}{p_1^2 p_4^2} = - \frac{4(p_1 + p_4)^2(p_2 \cdot p_2 + p_3 \cdot p_4)}{p_1^4 p_4^4} - \frac{2(p_2 + p_3)^2}{p_1^4 p_4^4} \left( \frac{1}{p_1^4} + \frac{1}{p_4^4} \right)
\]

\[
- \frac{2}{p_1^4 p_4^4} \left( \frac{p_2^4}{p_1^4} - \frac{p_3^4}{p_4^4} + \frac{p_3^4}{p_1^4} - \frac{p_2^4}{p_4^4} \right).
\]

(C.11)

The main difference between the first two and the last terms above is that in the former the denominator factor \((p_1 + p_4)^2\) cancels out. We will therefore treat these terms separately.

Plugging the last term in the original integral (C.12), extracting the remaining derivatives as \(x_{23}^2\) and adding to the result its image under the combined transformations \(x_1 \leftrightarrow x_2\) and \(x_3 \leftrightarrow x_4\) it is not difficult to see that this term leads to \(Y_{i,j,k}\)-type integrals multiplied by ratios of distances \(x_{ij}^2\):

\[
\frac{1}{x_1^2 x_2^2 x_3^2 x_4^2} \left( (x_{34}^2 - x_{13}^2) Y_{134} - (x_{24}^2 - x_{12}^2) Y_{124} + (x_{34}^2 - x_{24}^2) Y_{234} - (x_{13}^2 - x_{12}^2) Y_{134} \right).
\]

(C.14)

A useful relation is

\[
x_{12}^2 Y_{123} = 4 \int d^4 y \int \prod_{i=1}^{3} \frac{d^4 p_i}{(2\pi)^4} e^{ip_1 \cdot (x_1 - y) + ip_2 \cdot (x_2 - y) + ip_3 \cdot (x_3 - y)} \frac{1}{p_1^2 p_2^2} \frac{1}{p_1^2 + p_2^2} \left( \frac{1}{p_1^2} + \frac{1}{p_2^2} \right)
\]

(C.15)

\[
= 4 \int \frac{d^4 p_1 d^4 p_2}{(2\pi)^8} \frac{e^{ip_1 \cdot x_1 + ip_2 \cdot x_2}}{p_1^2 p_2^2} - 4 \int \frac{d^4 p_1}{(2\pi)^4} \frac{e^{ip_1 \cdot x_1}}{p_1^2} - 4 \int \frac{d^4 p_2}{(2\pi)^4} \frac{e^{ip_2 \cdot x_2}}{p_2^2}.
\]

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which may be proven by starting with a Fourier transform representation for $Y_{123}$.

Returning to the first two terms in (C.13) we notice that upon restoring the factor of $(p_2^2 p_3^2 (p_1 + p_4)^2)^{-1}$ the mixed-momentum propagator cancels out. Acting with the remaining derivatives, $(\partial_{p_2} - \partial_{p_3})^2$, we find

$$
(\partial_{p_2} - \partial_{p_3})^2 \left( \frac{-4(p_1 \cdot p_2 + p_3 \cdot p_4)}{p_1^2 p_2^2 p_3^2 p_4^2} \right) - \frac{2}{p_1^2 p_2^2 p_3^2 p_4^2} \left( \frac{1}{p_4^2} + \frac{1}{p_1^2} \right) = \frac{-32}{p_1^4 p_2^4 p_3^4 p_4^4} (p_1 \cdot p_2 p_3 \cdot p_4 + p_1 \cdot p_4 p_2 \cdot p_3 - p_1 \cdot p_3 p_2 \cdot p_4).
$$

Integrating this expression with the measure factor following from eq. (C.12) one can check that the result may be reorganized as

$$
\frac{x_{13}^2 x_{24}^2 - x_{14}^2 x_{23}^2 - x_{13}^2 x_{23}^2}{2 x_{14}^2 x_{23}^2} \int [d^4 p] e^{i p_1 \cdot x_1 + i p_2 \cdot x_2 + i p_3 \cdot x_3 + i p_4 \cdot x_4} \frac{1}{p_1^2 p_2^2 p_3^2 p_4^2} = \frac{1}{x_{13} x_{24} x_{14} x_{23}} \int [d^4 p] e^{i p_1 \cdot x_1 + i p_2 \cdot x_2 + i p_3 \cdot x_3 + i p_4 \cdot x_4} \frac{1}{p_1^2 p_2^2 p_3^2 p_4^2}.
$$

Combining eqs. (C.16) and (C.14) it is easy to see that

$$
-(\partial_1 \cdot \partial_2 + \partial_3 \cdot \partial_4) H_{41,23} = \frac{X_{1234}}{I_{14} I_{23}} - \frac{X_{1234}}{I_{34} I_{12}} - \frac{X_{1234}}{I_{23} I_{14}} + G_{3,41} - G_{2,41} + G_{4,32} - G_{1,32}.
$$

where

$$
G_{3,41} = \frac{Y_{341}}{I_{31}} - \frac{Y_{341}}{I_{34}}.
$$

Using similar techniques, or as a limit of (C.17) by noticing that in the limit $x_1 \rightarrow x_2$ the right-hand side of eq. (C.16) becomes identically zero, it is possible to show that

$$
\partial_1^2 H_{41;13} = \frac{I_{13} I_{14}}{I_{34}} Y_{134}.
$$

which was originally proven in [51] by explicit integration.
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