The observed power spectrum of the cosmic microwave background (CMB) is consistent with inflationary cosmology, which predicts a nearly scale-invariant power spectrum of quantum fluctuations of the inflaton field as they exit the Hubble horizon during inflation. Here we report a very significant correction (of several orders of magnitude) to the predicted amplitude of the power spectrum. This correction does not alter the near scale-invariance of the spectrum, but is crucial for testing predictions of the Hubble parameter during inflation against the observed amplitude of the CMB power spectrum. This novel correction appears because, as we show, the subtractions that renormalize the short-wavelength ultraviolet divergences of the inflaton two-point function have a significant effect on the amplitude of that two-point function at the longer wavelengths characteristic of the Hubble horizon. Earlier conclusions in the literature that certain theories (such as grand unified theories) implied perturbations that were too large by several orders of magnitude will have to be reconsidered in light of the present result.

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I. INTRODUCTION

The primordial perturbations responsible for the anisotropies in the CMB $^1$, $^2$, $^3$ are likely to have originated as quantum fluctuations of a scalar field $\phi$, the inflaton, that was the dominant constituent of the universe during a period of rapid exponential inflation $^4$, $^5$, $^6$, $^7$, $^8$, $^9$, $^{10}$, $^{11}$, $^{12}$ that lasted long enough to allow all parts of our observed universe to be causally connected. The potential-energy density of the inflaton field provided the impetus for the inflationary expansion, while its quantum fluctuations upon leaving the Hubble event horizon during inflation provided the initial conditions for the metric and density perturbations at the beginning of the radiation-dominated stage of the expansion (see, for example, the books $^{13}$, $^{14}$). We consider the following typical slow-roll inflationary scenario, which is consistent with the three-year Wilkinson Microwave Anisotropy Probe (WMAP) observations $^3$. As first shown by the author $^{15}$, $^{16}$, the generation and amplification of quantum fluctuations of a scalar field, such as the inflaton, is inevitable during an expansion of the universe like that of inflation.

The metric during exponential inflation is

$$ds^2 = dt^2 - \exp(2Ht) \left((dx)^2 + (dy)^2 + (dz)^2\right)$$

with $H$ given by

$$H = \sqrt{\frac{8\pi G}{3} V(\phi(0))}.$$  

where $G$ is Newton’s constant and $V(\phi(0))$ is the inflaton potential. The zeroth-order inflaton field $\phi(0)$ is treated classically. It and $H$ change very gradually in slow-roll inflation as $\phi(0)$ rolls slowly down the potential toward a minimum.

The inflaton field to first order is written as

$$\phi(\vec{x},t) = \phi(0)(t) + \delta\phi(\vec{x},t)$$

The first-order perturbation obeys the minimally-coupled linear scalar field equation on the background metric:

$$\partial_t^2 \delta\phi + 3H \partial_t \delta\phi - \exp(-2Ht) \sum_{i=1}^{3} \partial_i^2 \delta\phi + m(\phi(0))^2 \delta\phi = 0$$

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In the present analysis, we are treating the expanding inflationary universe. This vacuum state is a natural one to take for the inflaton perturbation field in the exponentially quantized inflaton perturbation field. By expanding inflationary universe.

The coordinate system \((x, y, z, t)\) that appears in \(ds^2\) above covers a part of deSitter spacetime. The well-known normalized positive-frequency solution of the above differential equation for the field \(\delta \phi\) that is determined to within a constant phase factor by the deSitter isometries of spatial rotations and translations, and by the deSitter extension of time translation, is

\[
f_k^\delta(\vec{x}, t) = (2L^3a(t)^3)^{-1/2}h_k(t)\exp(i\vec{k} \cdot \vec{x})
\]

with \(a(t) \equiv \exp(Ht)\), and

\[
h_k(t) = \sqrt{\frac{\pi}{2H}}H_n^{(1)}(v)
\]

where \(H_n^{(1)}(v)\) is a Hankel function of the first kind, and

\[
n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \quad \text{and} \quad v = kH^{-1}\exp(-Ht).
\]

In the present analysis, we are treating \(H\) and \(m\) as constant to first approximation during inflation. Note that \(v = 2\pi H^{-1}/\lambda(k, t)\) is the ratio of the radius of the Hubble horizon \(H^{-1}\) to the wavelength of the inflaton perturbation \(\lambda(k, t) = 2\pi a(t)/k\). For a given value of \(k\), the convention is to define the time of exit of the corresponding mode from the Hubble horizon as the time when \(v = 1\).

We have temporarily imposed periodic boundary conditions on the modes \(f_k(\vec{x}, t)\) in a spatial cube of volume \(L^3\), where \(L\) is taken to infinity in the continuum limit. The hermitian quantized field \(\delta \phi\) is expressed as usual in the mode expansion

\[
\delta \phi(\vec{x}, t) = \sum_{\vec{k}} (A_{\vec{k}}f_{\vec{k}}(\vec{x}, t) + \text{h.c.})
\]

where h.c. denotes the hermitian conjugate of the previous term, and \(A_{\vec{k}}\) is an annihilation operator for a particle of the quantized inflaton perturbation field in mode \(f_{\vec{k}}(\vec{x}, t)\). The deSitter or Bunch-Davies vacuum state \(|0\rangle\) is defined by \(A_{\vec{k}}|0\rangle = 0\) for all \(\vec{k}\). We are working in the Heisenberg picture, and expectation values below are taken with respect to this state vector. This vacuum state is a natural one to take for the inflaton perturbation field in the exponentially expanding inflationary universe.

II. ADIABATIC SUBTRACTIONS NECESSARY FOR RENORMALIZATION

The formal two-point function \(\langle 0|\delta \phi(\vec{x}, t)|^2|0\rangle\) obtained from Eq. (8) is

\[
\langle 0|\delta \phi(\vec{x}, t)|^2|0\rangle_{\text{formal}} = \sum_{\vec{k}} |f_{\vec{k}}(\vec{x}, t)|^2 = (4\pi a(t)^3)^{-1}\int_0^{\infty} \left|h_k(t)\right|^2 k^2 dk
\]
where we have taken the continuum limit. From the large-k asymptotic form of the Hankel function, one finds that this integral over \( k \) diverges quadratically at the upper limit of integration.

Well established methods have been developed to deal with such UV divergences that arise in quantum field theory in curved spacetime, as normal ordering of operators is not sufficient in an evolving universe. The method that best suits our present objective of finding the spectrum of perturbations is the method of adiabatic regularization first developed by Parker and Fulling [15, 17, 18, 19, 20]. The method involves a mode by mode subtraction of terms that are related to the behavior of the mode functions in a slowly (i.e., adiabatically) expanding universe. Adiabatic regularization is related to the fact that the particle number is an adiabatic invariant [15, 16] in a slowly expanding universe. Even the rate of the inflationary expansion is slow relative to the high frequencies that give rise to the UV divergence. The adiabatic subtractions are obtained from an adiabatic series for the mode functions. They serve to identify and eliminate the UV divergences that are present in quantities such as the two-point function and expectation values of the energy-momentum tensor. The method is very simple to apply in the Friedmann, Lemaitre, Robertson, Walker (FLRW) background metrics, such as that of Eq. (1).

The adiabatically regularized expression for the two-point function is

\[
\langle 0 | \delta \phi(\vec{x}, t) | 0 \rangle = (4\pi a(t)^3)^{-1} \int_0^\infty \left( |h_k(t)|^2 - \omega_k(t)^{-1} - (W_k(t)^{-1})^{(2)} \right) k^2 dk
\]

where the first adiabatic subtraction term removes the quadratic UV divergence and the second subtraction removes the logarithmic UV divergence that is present in the formal expression of Eq. (10). The full expression in the integrand is well-defined and gives a finite result, even though if one were to break apart the separate integrals, they would each suffer from a UV divergence. Here \( \omega_k(t) = \sqrt{\frac{k^2}{a(t)^2} + m^2} \), and for the present exponentially expanding universe and field equation, the second subtraction term is

\[
(W_k(t)^{-1})^{(2)} = -\frac{5H^2m^4}{8\omega_k(t)^2} + \frac{3H^2m^2}{4\omega_k(t)^2} + \frac{H^2}{\omega_k(t)^3}.
\]

The adiabatic subtractions must be taken at all frequencies, not just large ones. Otherwise fundamental properties would be violated; for example, the covariant four-divergence of the renormalized energy-momentum tensor would not be zero.

### III. THE SPECTRUM OF INFLATON PERTURBATIONS

The two-point function is related to the spectrum of inflaton perturbations \( \Delta^2_\phi(k, t) \), also denoted as \( P_\phi(k, t) \) by

\[
\langle 0 | \delta \phi(\vec{x}, t) | 0 \rangle = \int_0^\infty \Delta^2_\phi(k, t) k^{-1} dk
\]

Comparing with Eq. (10), one finds

\[
\Delta^2_\phi(k, t) = (4\pi a(t)^3)^{-1} k^3 \left( |h_k(t)|^2 - \omega_k(t)^{-1} - (W_k(t)^{-1})^{(2)} \right)
\]

This is equivalent to the definition of the spectrum as [1]:

\[
\Delta^2_\phi(k, t) = \frac{k^3}{(2\pi^2)} \langle 0 | \delta \phi_k(t) | 0 \rangle
\]

where we define the renormalized momentum-space expectation value as

\[
\langle 0 | \delta \phi_k(t) | 0 \rangle = (2\pi^2)(4\pi a(t)^3)^{-1} \left( |h_k(t)|^2 - \omega_k(t)^{-1} - (W_k(t)^{-1})^{(2)} \right).
\]

As we have seen, consistency between the physically relevant renormalized two-point functions in position space and momentum space demands that the physically relevant perturbations in momentum space are given by Eq. (13).

Having obtained the momentum-space two-point function of Eq. (13) directly from the renormalized position-space two-point function, one may ask if there is an alternative argument for the physical relevance of the subtractions that deals with the individual momentum-space modes in the expansion of the quantized field? Such an argument can be found in the author’s Ph.D. thesis [12, pp. 140–163], in which quantum measurement theory was used to redefine the creation and annihilation operators that are measurable for each mode in the expansion of the field analogous...
to Eq. [8]. The modes in the field expansion, when expressed in terms of these physically relevant creation and annihilation operators, would lead to the result for the momentum-space two-point function given in Eq. [15], with the vacuum state being the one annihilated by the physical annihilation operators. In the present case, the connection with quantum measurement theory can be thought of in terms of the classical curvature perturbations that give the initial conditions for the radiation-dominated stage of our universe. The classical curvature perturbations that arise from the quantum inflaton perturbations can be regarded as making a measurement on them, with the readout of this measuring instrument being the observed power spectrum of the CMB.

Simply defining the spectrum $\Delta^2_\phi(k, t)$ by using the Fourier components in the field expansion of Eq. [8] would give for $(0|\delta\phi_<(t)|2)$ the result on the right-hand-side of Eq. [15], but without any of the adiabatic subtractions. Then the position space two-point function $(0|\delta\phi(\vec{x}, t)|2)$ obtained from Eq. [12] would be infinite. One may try to tame this infinity by the ad hoc prescription of demanding that $\delta\phi_k(t)$ approach 0 rapidly at sufficiently large $k$. In this procedure, one obtains for the modes of interest:

$$\Delta^2_\phi(k, t)_{\text{Hankel}} \equiv (4\pi a(t)^3)^{-1} k^3 |h_k(t)|^2$$

However, this ad hoc procedure does not correctly take into account the fact that the physically relevant spectrum of Eq. [13] is significantly different from the ad hoc spectrum obtained by this cut-off procedure.

Another often used procedure [13] is to combine the ad hoc cut-off idea with a method of putting boundary conditions on the mode functions such that the end result is to find the same result as Eq. [16], but with $|h_k(t)|^2 = \omega_k(t)^{-1}$. This gives, for the modes of interest, the spectrum

$$\Delta^2_\phi(k, t)_{\omega} \equiv (4\pi a(t)^3)^{-1} k^3 \omega_k(t)^{-1}.$$  

Without the cut-off, this would also give an infinite position-space two-point function.

At horizon exit ($v = 1$) it turns out that for small values of $m^2$, the last two spectra are related by $\Delta^2_\phi(k, t)_{\text{Hankel}} \approx 2\Delta^2_\phi(k, t)_{\omega}$. It is our contention that neither of these last two spectra are correct because they do not take account of the subtraction terms that are necessary for renormalization in curved spacetime.

Next we compare the actual physically relevant spectrum with the two ad hoc spectra. We will find that the near scale-independence of the spectral components at the times of exit from the Hubble horizon is not altered, but the magnitudes are quite different. Thus, the agreement with CMB observations is not altered. But for comparison of the observations with theories that predict the magnitude of the Hubble parameter $H$ during inflation, the difference in amplitude between the actual spectrum and the two ad hoc spectra is quite important.

### IV. COMPARISON OF SPECTRAL AMPLITUDES

As noted, exit from the Hubble horizon is defined in the literature as occurring for each inflaton perturbation mode at the time when $v = 1$, where $v$ is defined in Eq. [7]. Expressing each of the spectra given by Eqs. [13], [16], and [17] in terms of $v$, we find that the spectrum we are proposing is

$$\Delta^2_\phi(k, t) = \frac{H^2v^3}{32\pi^2} \left(4\pi |H_n^{(1)}(v)|^2 - \frac{8m_H^6 + 3m_H^4(3 + 8v^2) + 2m_H^2v^2(11 + 12v^2) + 8(v^4 + v^6)}{(m_H^2 + v^2)^{7/2}} \right)$$

where $n = \sqrt{(9/4) - m_H^2}$ and $m_H = m/H$. The other two spectra are, respectively,

$$\Delta^2_\phi(k, t)_{\text{Hankel}} = \frac{H^2v^3}{8\pi} |H_n^{(1)}(v)|^2$$

and

$$\Delta^2_\phi(k, t)_{\omega} = \frac{H^2v^3}{4\pi^2 \sqrt{v^2 + m_H^2}}$$

When $v = 1$ and $m \ll H$, one has for the spectrum of Eq. (20), $\Delta^2_\phi(k, t)_{\omega} = \frac{H^2}{2\pi^2}$, which is the result used in the literature [1, 2, 3, 13, 14]. We are proposing that instead one should use the result obtained from Eq. (18).

All three of these spectra depend on the mode $k$ only through the variable $v$. Therefore, the spectra obtained from the modes as they exit the Hubble horizon will be scale-invariant in each case and will give rise to scale-invariant initial curvature perturbations at the beginning of the radiation-dominated stage of the expansion. (Because $H$ depends on
m, and for the values of $v$ out to be exactly zero instead of infinity. To our knowledge, this is a new result. The existence of a similar IR divergence for the Bunch-Davies de Sitter vacuum state, that the formal two-point function of the massless, minimally-coupled scalar field is finite. The spectra of Eqs. (19) and (20) increase with $v$, and in the absence of an explicit cut-off, lead to an infinite position-space two-point function.

A notable feature of the proposed physical two-point function spectrum is that the amplitude of the perturbations is significantly smaller than the value used in the literature. Earlier conclusions in the literature (for example, [11, 12]) that the amplitude predicted by certain theories (such as grand unified theories) was too large by several orders of magnitude will have to be reconsidered in light of the present correction to the spectrum. In TABLE I, the values of each of the three spectra at horizon exit ($v = 1$) are shown for four small values of $m^2/H^2$, ranging from 0.1 to 0.0001. We see that the proposed physical spectrum has an amplitude at horizon exit that decreases with decreasing $m$ and, for the values of $m$ shown, is from one- to four-powers of 10 smaller than the unrenormalized spectra.

It is not difficult to prove from Eq. (18) that when $m_H^2 = 0$, the subtraction terms give the exact result that $\Delta_2^2(k, t) = 0$ for all $v$. This is a remarkable result, in view of the fact that without the adiabatic subtractions the massless minimally-coupled field has a formal two-point function in position space that is infinite in the de Sitter (Bunch-Davies) vacuum state because of an IR divergence (as well as the UV divergence). The adiabatic subtractions are determined by the need to cancel the UV divergences. There is no freedom to change them at the IR end of the spectrum. The cancellation of the IR divergence by the same subtractions that tame the UV divergence appears to be fortuitous, but may have a deeper significance, perhaps pointing to a duality between IR and UV behavior.

Although, in slow-roll inflation $m$ is not 0, the tensor perturbations (gravitons) of each polarization that are
created by the inflationary expansion of the universe are governed in the Lifschitz gauge by an equation like that of a minimally-coupled 0-mass inflaton. A gauge invariant analysis would be necessary to determine if the above result for the massless scalar field may be relevant to the spectrum of tensor perturbations resulting from inflation.

V. CONCLUSIONS

We have shown that established methods of quantum field theory in curved spacetime give a major correction to the predicted amplitude of the perturbation spectrum that results from inflation. This correction will be crucial in comparing theories that predict the magnitude of the Hubble parameter $H$ during inflation to the beautiful observations of the cosmic microwave background and the large scale structure of the universe.