New results for algebraic tensor reduction of Feynman integrals

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We report on some recent developments in algebraic tensor reduction of one-loop Feynman integrals. For 5-point functions, an efficient tensor reduction was worked out recently and is now available as numerical C++ package, PJFry, covering tensor ranks until five. It is free of inverse 5-point Gram determinants and inverse small 4-point Gram determinants are treated by expansions in higher-dimensional 3-point functions. By exploiting sums over signed minors, weighted with scalar products of chords (or, equivalently, external momenta), extremely efficient expressions for tensor integrals contracted with external momenta were derived. The evaluation of 7-point functions is discussed. In the present approach one needs for the reductions a \((d+2)\)-dimensional scalar 5-point function in addition to the usual scalar basis of 1- to 4-point functions in the generic dimension \(d = 4 - 2\varepsilon\). When exploiting the four-dimensionality of the kinematics, this basis is sufficient. We indicate how the \((d+2)\)-dimensional 5-point function can be evaluated.

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1. Introduction

In recent years, we worked out a tensor reduction formalism for one-loop Feynman integrals with more than four legs. Tensor integrals are

$$I_{\mu_1 \cdots \mu_R}^n = \int \frac{d^d k}{i \pi^{d/2}} \frac{\prod_{r=1}^{R} k^{\mu_r}}{\prod_{j=1}^{n} c_j},$$

with denominators $c_j = (k - q_j)^2 - m_j^2 + i \varepsilon$ and chords $q_j$.

An efficient and stable numerical implementation is of high relevance for the description of multi-leg final states at the LHC. The state of the art has been regularly summarized at the “Les Houches Workshops on Physics at TeV Colliders”, see e.g. [1] or the forthcoming proceedings of the 2012 edition. For a broader context, see also [2–4] and references therein.

The basics of our algebraic approach were formulated in [5–7]. In a series of papers, we performed the explicit reduction of 5- and 6-point tensors [8–15]. Arbitrary internal masses and external virtualities are allowed. Numerical singularities arising from inverse small 5-point Gram determinants are avoided and those from inverse small 4-point Gram determinants are safely evaluated by series of 3-point functions in higher dimensions with improvements by Padé approximants [14]. Based on that, the numerical C++ package PJFry was made available open-source [14, 16, 17]. An instructive example of typical output for a 6-point function is figure 1; for details on the kinematics see [17].

Since then, the interesting case of contractions of tensor integrals with external momenta was studied. After such contractions, the resulting scalar quantities are compact linear combinations of the basic scalar integrals with factorizing, simple combinations of signed minors and scalar products of external momenta [19, 20].
These developments were summarized at the conference, but because they were described already in earlier write-ups, we restrict ourselves here to the above quotations. Quite recently, we also studied the question of how to extend the tensor reductions to more than \( n = 6 \) external legs. We again relied on the earlier treatment given in [7]. The reductions may be performed consistently in terms of 1- to 4-point scalar functions, as was done for \( n \leq 6 \), but one has in this framework as an additional element of the basis the 5-point function \( I_5^{d+} \) in \( d = 6 - 2\varepsilon \) dimensions.

This is described in section 2. The additional integral \( I_5^{[d+]} \) is finite, but it is not contained in packages with scalar integrals like LoopTools/FF [21, 22], QCDDLoop/FF [23, 22], or OneLOop [24]. In section 3, we indicate how to evaluate this integral directly.

After the conference we studied the approach due to [25, 26]. It allows to avoid the use of \( I_5^{[d+]} \). This, together with the effects of contractions with external momenta, is studied for 7- and 8-point tensor integrals in [27], but the details for higher tensor ranks have to be worked out yet.

2. Tensor reduction for 7-point functions with recurrence relations

Let us shortly repeat the reasoning for 6-point functions, and then switch to \( n > 6 \). In [12] we have set up the tensor reduction for the 5-point functions of rank \( R \), expressing any \((5,R)\) pentagon plus \((4,R-1)\) boxes:

\[
I_5^{\mu_1\ldots\mu_{R-1}\mu} = I_5^{\mu_1\ldots\mu_{R-1}} Q_0^\mu - \sum_{s=1}^{5} I_4^{\mu_1\ldots\mu_{R-1},s} Q_s^\mu.
\]  

(2.1)

In [14] we have also given the corresponding homogeneous formula for 6-point functions

\[
I_6^{\mu_1\ldots\mu_{R-1}\mu} = -\sum_{s=1}^{6} I_5^{\mu_1\ldots\mu_{R-1},s} Q_s^{0,\mu},
\]

(2.2)

with \( p_6 = 0 \) and

\[
Q_s^{0,\mu} = \sum_{i=1}^{5} q_i^{\mu} \binom{6}{d} \binom{0}{0}. \qquad (2.3)
\]

Now we follow [7]. For a reduction of 5-point functions two recursion relations are in principle sufficient:

\[
v_j i_j I_n^{(d+2)} = \frac{1}{()}_n \left[ -\binom{j}{0}_n + \sum_{k=1}^{n} \binom{j}{k}_n k^- \right] I_n^d,
\]

(2.4)

\[
(d - \sum_{i=1}^{n} v_i + 1) I_n^{(d+2)} = \frac{1}{()}_n \left[ \binom{0}{0}_n - \sum_{k=1}^{n} \binom{0}{k}_n k^- \right] I_n^d,
\]

(2.5)

where \( i^\pm, j^\pm, k^\pm \) act by shifting the indices \( v_i, v_j, v_k \) by \( \pm 1 \). They do not work out for 6-point functions since \( ()_n = 0 \) for \( n \geq 6 \). A further recursion relation, which does not decrease dimension, reads

\[
v_j i_j I_n^{(d)} = \frac{1}{()}_n \sum_{k=1}^{n} \binom{0j}{0k}_n. \qquad (2.6)
\]

1See also [28].
If reduction of dimension is acceptable, then the above double sum is often reduced by means of
\[
\sum_{j=1}^{n} v_{ij} I_{n,j}^{(d+2)} = -I_{n}^{(d)}.
\] (2.7)

Further one uses (4.4) of [10] together with (4.5),
\[
\sum_{i=1}^{s} q^{\mu}_{j} \left( \begin{array}{c} 0 \\ i \end{array} \right)_{6} = 0.
\] (2.8)

Collecting the above, 6-point functions may be reduced.

For 7-point functions, things are a bit more involved because we have \( (\_\_\_\_\_\_) \gamma = 0 \) and \( (\_\_\_\_\_\_\_\_) \gamma = 0 \). Nevertheless one can apply recursion relations based on (2.6) and (2.7), see [7, 10]:
\[
\left( \begin{array}{c} 0 \\ 0 \end{array} \right)_{n} I_{n,ij}^{(d+1)} = [n+1-(d+2x)] \left( \begin{array}{c} 0 \\ i \end{array} \right)_{n} I_{n,j}^{(d+1)} + \sum_{r=1}^{n} \left( \begin{array}{c} 0j \\ 0r \end{array} \right)_{n} I_{n-1,i}^{(d+1)},
\] (2.9)
\[
\left( \begin{array}{c} 0 \\ 0 \end{array} \right)_{n} v_{ij} I_{n,j}^{(d+1)} = [n+2-(d+2x)] \left( \begin{array}{c} 0 \\ j \end{array} \right)_{n} I_{n,j}^{(d+1)} + [i]^{j} + \sum_{r=1}^{n} \left( \begin{array}{c} 0j \\ 0r \end{array} \right)_{n} I_{n-1,j}^{(d+1)},
\] (2.10)
\[
\left( \begin{array}{c} 0 \\ 0 \end{array} \right)_{n} v_{ijk} I_{n,jkl}^{(d+1)} = [n+3-(d+2x)] \left( \begin{array}{c} 0 \\ i \end{array} \right)_{n} I_{n,jj}^{(d+1)} + [ij]^{k}_{\text{red}} + \sum_{r=1}^{n} \left( \begin{array}{c} 0k \\ 0r \end{array} \right)_{n} I_{n-1,ij}^{(d+1)},
\] (2.11)
\[
\left( \begin{array}{c} 0 \\ 0 \end{array} \right)_{n} v_{ijk} I_{n,jkl}^{(d+1)} = [n+4-(d+2x)] \left( \begin{array}{c} 0 \\ l \end{array} \right)_{n} I_{n,jkl}^{(d+1)} + [ijk]_{\text{red}} + \sum_{r=1}^{n} \left( \begin{array}{c} 0l \\ 0r \end{array} \right)_{n} I_{n-1,ijkl}^{(d+1)},
\] (2.12)

where
\[
[i]^{j} = \left( \begin{array}{c} 0j \\ 0i \end{array} \right)_{n} I_{n}^{(d+1)}
\] (2.13)
\[
[ij]^{k} = \left( \begin{array}{c} 0k \\ 0i \end{array} \right)_{n} I_{n,j}^{(d+1)} + \left( \begin{array}{c} 0k \\ 0j \end{array} \right)_{n} I_{n,i}^{(d+1)},
\] (2.14)

and the \([ij]_{\text{red}}^{k}\) is \([i]^{j}k\), but without repetition of equal indices \(i, j\).

In [7] an idea of how to proceed for 7-point functions was formulated - details, however, were not given. For the 7-point vector one obtains from [5]
\[
I_{i}^{\mu} = -\sum_{j=1}^{7} q^{\mu}_{j} I_{\gamma,j}^{(d)}.
\] (2.15)

The (2.11) with \( (\_\_\_\_\_\_) \gamma = 0 \) and \( (\_\_\_\_\_\_\_\_) \gamma = 0 \) yields for \( i = j = k \) and \( x = 2 \)
\[
\left( \begin{array}{c} 0i \\ 0i \end{array} \right)_{n} I_{n,j}^{(d+1)} + \sum_{r=1}^{n} \left( \begin{array}{c} 0i \\ 0r \end{array} \right)_{n} I_{n-1,ii}^{(d+1)} = 0.
\] (2.16)

Since for the 6-point function \( (\_\_\_\_\_\_\_\_) \gamma = 0 \), we have from eq. (55) of [7]
\[
I_{6,ii}^{(d+1)} = \sum_{s=1}^{7} \left( \begin{array}{c} 0r \\ 0r \end{array} \right)_{n} I_{6,ii}^{(d+1,rs)} + \left( \begin{array}{c} 0r \\ 0r \end{array} \right)_{n} I_{6,ii}^{(d+1,r)}, \quad R = \text{any value of } 0, \ldots, 7.
\] (2.17)
Applying in standard manner the recursion for the 5-point function

\[ V_{ij}I_{5,ij}^{d+[\nu]} = -\binom{0}{j}_5 I_{5,i}^{d+(\nu-1)} + \sum_{s=1}^{5} \binom{\nu}{s} \binom{0}{s}_5 I_{4, sj}^{d+(\nu-1)} + \binom{\nu}{5} \binom{0}{5}_5 I_{5}^{d+(\nu-1)}, \]

we replace \( I_{5,ij}^{d+} \) by integrals of the type \( I_{5,i} \) and \( I_{s,ij}^{4} \), the dimension of which must not be reduced. Therefore we have to apply recursion (2.6) in the form

\[
\binom{0}{n} I_{n,j} = -[d-(n+1)] \binom{0}{j}_n I_{n} - \sum_{i,k,i\neq k} \binom{0}{j}_0 \binom{0}{k}_n I_{n-1,i,k}, \quad n = 5, 4, 3, 2,
\]

i.e. starting at \( n = 2 \) and increasing \( n \) step by step, we obtain the desired integrals \( I_{5,i} \) and \( I_{s,ij}^{4} \). For \( n = 2 \) we have

\[
\binom{0}{2} I_{2,j} = -[d-3] \binom{0}{j}_2 I_{2} + \binom{0}{j}_{02} I_{1,1} - \binom{0}{j}_{01} I_{1,2},
\]

with

\[
I_{1,1}^{2} = \frac{d-2}{2m_{1}^{2}} I_{1}(m_{1}^{2}), \quad I_{1,2}^{2} = \frac{d-2}{2m_{2}^{2}} I_{1}(m_{2}^{2}).
\]

The problematic case is the integral \( I_{6,ij}^{d+[\nu]} \) for which we can write similarly to (2.17)

\[
I_{6,ij}^{d+[\nu]} = \sum_{s=1}^{7} \binom{R_{sj}}{s} I_{5, sj}^{d+[\nu]} + \binom{R_{sj}}{6} I_{6,ij}^{d+[\nu]}.
\]

The \( I_{6,ij}^{d+[\nu]} \) cannot so easily be eliminated as in the case of the 6-point function, where the vanishing of (2.8) was used. Inserting (2.23) into (2.17), there occurs \( \binom{R_{ij}}{6}^{2} \), i.e. quadratic, so that the right hand side of (2.8) does not vanish.

3. Numerical evaluation of higher-dimensional scalar integrals

Since in the described approach the higher dimensional integral \( I_{6,ij}^{d+[\nu]} \) cannot be eliminated, it is needed to investigate the possibility for its numerical evaluation – if one wants to continue with this approach. First of all one would reduce the 6-point function to 5-point functions by

\[
I_{6,ij}^{d+[\nu]} = \sum_{s=1}^{7} \binom{R_{sj}}{s} I_{5, sj}^{d+[\nu]},
\]
such that the problem is shifted to the numerical evaluation of the $I_5^{[d+\varepsilon]}$. This, however, is a well known pathological case, since reducing the 6-dimensional 5-point function to 4-dimensional 5- and 4-point functions, one has

$$I_5^{[d+\varepsilon]} = \left[ \frac{C_5}{d} I_5 - \sum_{s=1}^{5} \frac{C_s}{d} I_4^s \right] \frac{1}{d-4},$$

(3.2)
i.e. for $d = 4$ one meets a division by zero, $\frac{0}{0}$. In [29], however, an interesting approach has been proposed, which can also be applied to handle this case. In fact, the idea is to go to even higher dimensions, which provides good numerical stability.

One-loop $n$-point integrals in arbitrary dimension $d$ can be expressed in standard manner in terms of Feynman parameters as

$$I_n^{(d)} = \Gamma(n - \frac{d}{2}) \int_0^1 dx_1 \ldots \int_0^1 dx_{n-1} J_n x_n^{d/2-n},$$

(3.3)

where

$$J_n = x_{n-2} x_{n-3} \cdots x_1^{n-2},$$

(3.4)

and $h_n$ is a polynomial in the integration variables as well, containing also masses and momenta squared. The idea is to transform these integrals into integrals of higher dimension $D = d - 2\varepsilon + 2n - 2$. For small $\varepsilon$ the expansion of $I_n^{(d + 2n - 2)}$ in $\varepsilon$ then reads [29]

$$I_n^{(d + 2n - 2)} = \Gamma(1 + \varepsilon) \left[ -\frac{s_n}{\varepsilon} - s_n - R_n + O(\varepsilon) \right],$$

(3.5)

where $s_n$ can be written as

$$s_n = \frac{1}{(n+1)!} \sum_{i,j=1}^{n} Y_{ij},$$

(3.6)

with $Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2$ and it is

$$R_n = \int_0^1 dx_1 \ldots \int_0^1 dx_{n-1} J_n h_n \ln(h_n).$$

(3.7)

Obviously in such an integral no infinities occur anymore in the integrand and numerical integration is straightforward. It might even be possible and useful to evaluate it analytically. In our case of a 5-point function we have $n = 5$, i.e. this formula applies for the integral $I_5^{[d+\varepsilon]}$. Applying recursion relations, we can express (3.2) as

$$I_5^{[d+\varepsilon]} = (d-2) d(d+2) \left[ \frac{1}{5} I_5^{[d+\varepsilon]} + (d-2) d \frac{1}{S} \sum_{s=1}^{5} \left( \begin{array}{c} s \cr 0 \end{array} \right) I_4^{[d+\varepsilon]_s} \right]$$

$$+ (d-2) \frac{1}{S} \sum_{s=1}^{5} \left( \begin{array}{c} s \cr 0 \end{array} \right) I_4^{[d+\varepsilon]_s} + \frac{1}{S} \sum_{s=1}^{5} \left( \begin{array}{c} s \cr 0 \end{array} \right) I_4^{[d+\varepsilon]_s}.$$

(3.8)

Here we have now the desired representation of $I_5^{[d+\varepsilon]}$ in terms of higher-dimensional 4-point functions, for which standard recursions can be used to reduce them to integrals in generic dimension $d$. In this way 7-point functions could finally be dealt with.

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2 Other approaches to the numerical treatment of higher-dimensional scalar functions are e.g. [30, 31].
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