ARITHMETICALLY COHEN-MACaulAY BUNDLES ON THREEFOLD HYPERSURFACES

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Abstract. We prove that any rank two arithmetically Cohen-Macaulay vector bundle on a general hypersurface of degree at least six in \( \mathbb{P}^4 \) must be split.

1. Introduction

This article is a continuation of the study in [5] of arithmetically Cohen-Macaulay (ACM for short) bundles on hypersurfaces. For a detailed introduction and references, we refer the reader to that paper.

The main result proved here is the following

Main Theorem. Fix degree \( d \geq 6 \). There is a non-empty Zariski open set of hypersurfaces of degree \( d \) in \( \mathbb{P}^4 \), none of which support an indecomposable ACM rank two bundle.

The special case when \( d = 6 \) was proved by Chiantini and Madonna [2]. In [5], we proved the result analogous to the main theorem for hypersurfaces in \( \mathbb{P}^5 \) along with partial results on \( \mathbb{P}^4 \). In the current paper, we shall summarize and use some of these partial results. We will also use the relation between rank two ACM bundles on hypersurfaces and Pfaffians that was observed by Beauville in [1] and which we did not need in [5].

The results from [5] that are important for our proof here are paraphrased below (combining theorem 1.1 (3) and corollary 2.3 of that article).

Theorem 1. Let \( E \) be an indecomposable rank two ACM bundle on a smooth hypersurface \( X \) of degree \( d \) in \( \mathbb{P}^4 \). Then \( H^2(X, \End(E)) \) is a non-zero cyclic module of finite length, with the generator living in degree \(-d\). If \( d \geq 5 \) and \( X \) is general, then \( H^2(X, \End(E)) = 0 \).
2. ACM bundles and Pfaffians

We work over an algebraically closed field of characteristic zero. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d$. Recall that a vector bundle $E$ of rank two on $X$ is called ACM if $H^i(E(k)) = 0$ for all $k$ and $0 < i < n - 1$. By Horrocks’ criterion [4], this is equivalent to saying that $E$ has a resolution,

$$0 \to F_1 \overset{\Phi}{\to} F_0 \overset{\sigma}{\to} E \to 0,$$

where the $F_i$’s are direct sums of line bundles on $\mathbb{P}^n$. We will assume that this resolution is minimal, with $F_0 = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^n}(-a_i)$ where $a_1 \leq a_2 \leq \cdots \leq a_n$. Using [1], we may write $F_1$ as $F_1 \lor \mathcal{O}_{\mathbb{P}^n}(-d)$, where $\mathcal{O}_{\mathbb{P}^n}$ is the first Chern class of $E$, and we may assume that $\Phi$ is a skew-symmetric $n \times n$ matrix with $n$ even. The $(i,j)$-th entry $\phi_{ij}$ of $\Phi$ has degree $d - e - a_i - a_j$. The condition of minimality implies that there are no non-zero scalar entries in $\Phi$ and thus every degree zero entry must be zero.

We quote some facts about Pfaffians and refer the reader to [7] for more details. Let $\Phi = (\phi_{ij})$ be an $n \times n$ even-sized skew symmetric matrix and let $\text{Pf}(\Phi)$ denote its Pfaffian. Then $\text{Pf}(\Phi)^2 = \det \Phi$. Let $\Phi(i,j)$ be the matrix obtained from $\Phi$ by removing the $i$-th and $j$-th rows and columns. Let $\Psi$ be the skew-symmetric matrix of the same size with entries $\psi_{ij} = (-1)^{i+j} \text{Pf}(\Phi(i,j))$ for $0 \leq i < j \leq n$. We shall refer to $\text{Pf}(\Phi(i,j))$ as the $(i,j)$-Pfaffian of $\Phi$. The product $\Phi \Psi = \text{Pf}(\Phi) I_n$ where $I_n$ is the identity matrix.

**Example 1.** Let $n = 4$ above. Then

$$\text{Pf}(\Phi) = \phi_{12} \phi_{34} - \phi_{13} \phi_{24} + \phi_{14} \phi_{23}.$$ 

The following lemma shows the relation between skew-symmetric matrices, ACM rank 2 bundles and the equation defining the hypersurface.

**Lemma 1.** Let $E$ be a rank 2 ACM bundle on a smooth hypersurface $X \subset \mathbb{P}^4$ of degree $d$ and let $\Phi : F_1 \to F_0$ be the minimal skew-symmetric matrix associated to $E$. Then $X = X_\Phi$, the zero locus of $\text{Pf}(\Phi)$. Conversely, let $\Phi : F_1 \to F_0$ be a minimal skew-symmetric matrix such that the hypersurface $X_\Phi$ defined by $\text{Pf}(\Phi)$ is smooth of degree $d$. Then $E_\Phi$, the cokernel of $\Phi$, is a rank 2 ACM bundle on $X_\Phi$.

**Proof.** Let $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ be the polynomial defining $X$. Since $E$ is supported along $X$, $\det \Phi = f^n$ for some $n$ up to a non-zero constant where $\Phi$ is as in resolution (1). Locally $E$ is a sum of two line bundles and so the matrix $\Phi$ is locally the diagonal matrix $(f, f, 1, \cdots, 1, 1)$. 


Since the determinant of this diagonal matrix is $f^2$, we get $f = \text{Pf}(\Phi)$ (upto a non-zero constant).

To see the converse: let $\Phi$ be any skew-symmetric matrix and $\Psi$ be defined as above. Let $f = \text{Pf}(\Phi)$ be the Pfaffian. Since $\Phi \Psi = fI_n$, this implies that the composite $F_0(-d) \xrightarrow{f} F_0 \rightarrow E_\Phi$ is zero. Thus $E_\Phi$ is annihilated by $f$ and so is supported on the hypersurface $X_\Phi$ defined by $f$. Since $X_\Phi$ is smooth, by the Auslander-Buchsbaum formula, $E_\Phi$ is a vector bundle on $X_\Phi$. Therefore locally $\Phi$ is a diagonal matrix of the form $(f, \cdots, f, 1, \cdots, 1)$ where the number of $f$'s in the diagonal is equal to the rank of $E$. Since $\det(\Phi) = f^2$, we conclude that rank of $E_\Phi$ is 2.

Let $V \subset \text{Hom}(F_1, F_0)$ be the subspace consisting of all minimal skew-symmetric homomorphisms, where $F_i$'s are as above. The following is an easy consequence of the above lemma.

**Lemma 2.** Let $\Phi_0 \in V$ be an element such that $E_{\Phi_0}$ is a rank 2 ACM bundle on a smooth hypersurface $X_{\Phi_0}$. Then there exists a Zariski open neighbourhood $U$ of $\Phi_0$ such that for any $\Phi \in U$, $X_\Phi$ is a smooth hypersurface and $E_\Phi$ is a rank two ACM bundle supported on $X_\Phi$.

### 3. Special cases

The proof of the main theorem will require the study of some special cases, which are listed below.

**Lemma 3.** Consider the following three types of curves in $\mathbb{P}^4$:

- a curve $C$ which is the complete intersection of three general hypersurfaces, two of which are of degree $\leq 2$.
- a curve $D$ which is the locus of vanishing of the principal $4 \times 4$ sub-Pfaffians of a general $5 \times 5$ skew-symmetric matrix $\chi$ of linear forms.
- a curve $C_r$, $r \geq 0$, which is the locus of vanishing of the $2 \times 2$ minors of a general $4 \times 2$ matrix $\Delta$ with one row consisting of forms of degree $1 + r$, and the remaining three rows consisting of linear forms.

The general hypersurface $X$ in $\mathbb{P}^4$ of degree $\geq 6$ cannot contain any curve of the the first two types. The general hypersurface $X$ of degree $d \geq \max\{6, r + 4\}$ cannot contain any curve of the third type.

**Proof.** The curve $C$ is smooth if the hypersurfaces are general. If $\chi$ is general, the curve $D$ is smooth (see [8], page 432 for example). If $\Delta$ is general, the curve $C_r$ is smooth (see op. cit. page 425).
The proof of the lemma is a straightforward dimension count. By counting the dimension of the set of all pairs \((Y, X)\) where \(Y\) is a smooth curve of the described type and \(X\) is a hypersurface of degree \(d\) containing \(Y\), it suffices to show that this dimension is less than the dimension of the set of all hypersurfaces \(X\) of degree \(d\) in \(\mathbb{P}^4\). This can be done by showing that if \(S\) denotes the (irreducible) subset of the Hilbert scheme of curves in \(\mathbb{P}^4\) parameterizing all such smooth curves \(Y\), then the dimension of \(S\) is at most \(h^0(\mathcal{O}_Y(d)) - 1\).

This argument was carried out in [6] where \(Y\) is any complete intersection curve in \(\mathbb{P}^4\). The case where \(Y\) equals the first type of curve \(C\) in the list above is Case 2 of [6]. Hence we will only consider the types of curves \(D\) and \(C_r\) here.

If \(Y\) is of type \(D\) in the list, the sheaf \(\mathcal{I}_D\) has the following free resolution ([8], page 427):

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-5) \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 5} \xrightarrow{X} \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 5} \to \mathcal{I}_D \to 0.
\]

From this one computes the dimension of the set of such cubic scrolls to be 18, since the 30 dimensional space of all \(3 \times 2\) linear matrices is acted on by automorphisms of \(\mathcal{O}_{\mathbb{P}^n}(-3)^2\) and \(\mathcal{O}_{\mathbb{P}^n}(-2)^3\), with scalars giving the stabilizer of the action. Furthermore, by dualizing the resolution, we get a resolution for \(\omega_S\):

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-5) \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 3} \xrightarrow{\theta^\vee} \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 3} \to \mathcal{I}_S \to 0.
\]

A section of \(\omega_S(r + 3)\) gives a lift \(\mathcal{O}_{\mathbb{P}^n}(-r - 3) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 2}\), and we obtain a \(4 \times 2\) matrix \(\begin{pmatrix} \theta & \alpha \end{pmatrix}\) of the required type. Hence \(C_r\) is a curve in the linear series \(|K_S + (r + 3)H|\), where \(H\) is the hyperplane section on \(S\). Intersection theory on \(S\) gives \(K_S.K_S = 8\), \(K_S.H = -5\) and \(H.H = 3\). Using this, we may compute the dimension of the linear system of \(C_r\) on \(S\), and we get the dimension of the set \(S\) of all such \(C_r\) in \(\mathbb{P}^4\) to be 21 if \(r = 0\) and \((3/2)r^2 + (13/2)r + 24\) otherwise.
The ideal sheaf of $C_r$ has a free resolution given by the Eagon-Northcott complex \[3\]
\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-r - 4)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(-r - 3)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^n}(-r - 2)^{\oplus 3} \to \mathcal{I}_{C_r} \to 0.
\]

Let $d \geq \max\{6, r + 4\}$ be chosen as in the statement of the lemma. Then $d = r + s + 4$ where $s \geq 0$ ($s \geq 2$ when $r = 0$; $s \geq 1$ when $r = 1$).

Using the above resolution, a calculation gives
\[
h^0(\mathcal{O}_{C_r}(d)) = 2 \frac{3}{2} r^2 + 29 \frac{3}{2} r + 3rs + 4s + 17.
\]
The required inequality $\dim S < h^0(\mathcal{O}_{C_r}(d))$ is now evident. \qed

4. PROOF OF MAIN THEOREM

In this section, $E$ will be an indecomposable ACM bundle of rank two and first Chern class $e$ on a smooth hypersurface $X$ of degree $d$ in $\mathbb{P}^4$. The minimal resolution \([\mathbb{I}]\) gives $\sigma : F_0 \to E \to 0$, and we may describe $\sigma$ as $[s_1, s_2, \ldots, s_n]$ where $s_1, s_2, \ldots, s_n$ is a set of minimal generators of the graded module $H^0_*(E)$ of global sections of $E$, with degrees $a_1 \leq a_2 \leq \cdots \leq a_n$.

**Lemma 4.** If $E$ is an indecomposable rank 2 ACM bundle with first Chern class $e$ on a general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 6$, then there is a relation in degree $3 - e$ among the minimal generators of $S^2E$.

**Proof.** Consider the short exact sequence
\[
0 \to \mathcal{O}_X \to \mathcal{E}nd(E) \to (S^2E)(-e) \to 0.
\]

$S^2E(-e)$ has the same intermediate cohomology as $\mathcal{E}nd(E)$ since the sequence splits in characteristic zero.

Choose a minimal resolution of $S^2E$:
\[
0 \to B \to C \to S^2E \to 0,
\]
where $C$ is a direct sum of line bundles on $X$ and $B$ is a bundle on $X$ with $H^1(X, B) = 0$.

We first show that $B^\vee(e + d - 5)$ is not regular. For this, consider the dual sequence $0 \to (S^2E)^\vee \to C^\vee \to B^\vee \to 0$.

By Serre duality and Theorem \([\mathbb{I}]\)
\[
H^1(X, (S^2E)^\vee(d + e - 5)) = 0.
\]
Therefore
\[
H^0(X, C^\vee(d + e - 5)) \to H^0(X, B^\vee(d + e - 5))
\]
is onto. If $B^\vee(d + e - 5)$ were regular, the same would be true for
\[ H^0(X, C^\vee(d + e - 5 + k)) \to H^0(X, B^\vee(d + e - 5 + k)) \quad \forall k \geq 0. \]
However, this is false for $k = d$ since by Serre duality and Theorem \[1\],
\[ H^1(X, (S^2E)^\vee(2d + e - 5)) \neq 0. \] Thus $B^\vee(e + d - 5)$ is not regular. Now
\[ H^1(X, B^\vee(e + d - 6)) \cong H^2(X, B(1 - e)) \cong H^1(X, S^2E(1 - e)) \]
\[ \cong H^1(X, E_{nd}(E)(1)). \]
By Serre duality,
\[ H^1(X, E_{nd}(E)(1)) \cong H^2(X, E_{nd}(E)(d - 6)) \]
which by Theorem \[1\] equals zero for $d \geq 6$ (this is the main place where we use the hypothesis that $d \geq 6$). Furthermore, $H^2(X, B^\vee(e + d - 7)) = 0$ since $H^1(X, B) = 0$. Since $B^\vee(e + d - 5)$ is not regular, we must have $H^3(X, B^\vee(e + d - 8)) \neq 0$.

In conclusion, $H^0(X, B(3 - e)) \neq 0$. In other words, there is a relation in degree $3 - e$ among the minimal generators of $S^2E$. \[\square\]

**Lemma 5.** Let $E$ be as above. Then $1 \leq a_1 + a_2 + e \leq a_1 + a_3 + e \leq 2$.

**Proof.** The resolution \([1]\) for $E$ gives an exact sequence of vector bundles on $X$: $0 \to G \to F_0 \twoheadrightarrow E \to 0$, where $F_0 = F_0 \otimes \mathcal{O}_X$ and $G$ is the kernel. This yields a long exact sequence,
\[ 0 \to \wedge^2 G \to F_0 \otimes G \to S^2F_0 \to S^2E \to 0. \]
From the arguments after Lemma 2.1 of \[5\] (using formula (5)), it follows that $H^2(\wedge^3 G) = 0$. Hence the map $S^2F_0 \to S^2E$ is surjective on global sections. The image of this map picks out the sections $s_1, s_2$ of degree $a_1 + a_2$ in $S^2E$. Observe that the lowest degree minimal sections $s_1, s_2$ of $E$ induce an inclusion of sheaves $\mathcal{O}_X(-a_1) \oplus \mathcal{O}_X(-a_2) \xhookrightarrow{s_1, s_2} E$ whose cokernel is supported on a surface in the linear system $|\mathcal{O}_X(a_1 + a_2 + e)|$ on $X$ (a nonempty surface when $E$ is indecomposable). Hence $1 \leq a_1 + a_2 + e$. There is an induced inclusion
\[ S^2[\mathcal{O}_X(-a_1) \oplus \mathcal{O}_X(-a_2)] \hookrightarrow S^2E. \]
Therefore the three sections of $S^2E$ given by $s_1^2, s_1s_2, s_2^2$ cannot have any relations amongst them. Since these are also three sections of $S^2E$ of the lowest degrees, they can be taken as part of a minimal system of generators for $S^2E$. It follows that the relation in degree $3 - e$ among the minimal generators of $S^2E$ obtained in the previous lemma must include minimal generators other than $s_1^2, s_1s_2, s_2^2$. Since the other minimal generators have degree at least $a_1 + a_3$, and since
we are considering a relation amongst minimal generators, we get the inequality \( a_1 + a_3 \leq 2 - e \).

**Lemma 6.** For any choice of \( 1 \leq i < j \leq n \), the \((i, j)-\text{Pfaffian of } \Phi\) is non-zero. Consequently, its degree (which is \( a_i + a_j + e \)) is at least \((n - 2)/2\).

**Proof.** On \( X \), \( E \) has an infinite resolution

\[
\cdots \to \mathcal{F}'_0(e - 2d) \to \mathcal{F}_0(-d) \to \mathcal{F}'_0(e - d) \to \mathcal{F}_0 \to E \to 0.
\]

We also have

\[
\begin{align*}
\mathcal{F}'_0(e - 2d) & \xrightarrow{\varphi} \mathcal{F}_0(-d) \xrightarrow{\delta} E(-d) \\
& \downarrow \cong \\
\quad E'(e - d) & \xrightarrow{\sigma'} \mathcal{F}'_0(e - d) \xrightarrow{\varphi'} \mathcal{F}_0.
\end{align*}
\]

Let \( \Theta = \sigma' \alpha \sigma \). Since \( \sigma = (s_1, \ldots, s_n) \), we may express the \((i, j)-\)th entry of \( \Theta \) as \( \theta_{ij} = s_i \wedge s_j \) (suppressing the canonical isomorphism \( \alpha \)).

\( \Phi' = -\Phi \) and \( \alpha \sigma : \mathcal{F}_0(-d) \to E'(e - d) \) is surjective on global sections. Hence we have a commuting diagram

\[
\begin{align*}
\mathcal{F}'_0(e - 2d) & \xrightarrow{\varphi} \mathcal{F}_0(-d) \xrightarrow{\delta} \mathcal{F}'_0(e - d) \xrightarrow{\varphi'} \mathcal{F}_0 \to E \to 0 \\
\downarrow \cong & \\
\mathcal{F}'_0(e - 2d) & \xrightarrow{\varphi} \mathcal{F}_0(-d) \xrightarrow{\delta} \mathcal{F}'_0(e - d) \xrightarrow{\varphi'} \mathcal{F}_0 \to E \to 0
\end{align*}
\]

It is easy to see that \( B \) is an isomorphism. As a result, every column of \( B \) has a non-zero scalar entry.

Now suppose that \( \overline{\psi}_{ij} = 0 \) for some \( i, j \) so that \( \sum_k s_i \wedge s_k b_{kj} = 0 \). Let \( Y_i \) be the curve given by the vanishing of the minimal section \( s_i \) with the exact sequence

\[
0 \to \mathcal{O}_X(-a_i) \xrightarrow{s_i} E \xrightarrow{\lambda'_i} I_{Y_i/X}(a_i + e) \to 0.
\]

Hence \( s_i \wedge s_i = 0 \) and \( s_i \wedge s_k \) for \( k \neq i \) give minimal generators for \( I_{Y_i/X} \).

It follows that no \( b_{kj} \) can be a non-zero scalar for \( k \neq i \). Hence \( b_{ij} \) has to be a non-zero scalar and the only one in the \( j \)-th column. However, \( \overline{\psi}_{jj} = 0 \). So by the same argument, \( b_{jj} \) is the only non-zero scalar. To avoid contradiction, \( \overline{\psi}_{ij} \) and hence \( \psi_{ij} \neq 0 \) for \( i \neq j \).

We now complete the proof of the Main Theorem. As in the previous lemmas, assume that \( X \) is general of degree \( d \geq 6 \), with \( E \) an indecomposable rank two ACM bundle on \( X \). We will show that the inequalities of Lemma 5 lead us to the special cases of Lemma 3, giving a contradiction.
Let $\mu = a_1 + a_2 + e$. By Lemma 5 $1 \leq \mu \leq 2$.

**Case $\mu = 1$.** In this case, in order for the $(1, 2)$-Pfaffian of $\Phi$ to be linear, by Lemma 6 $n$ must equal 4. In the $4 \times 4$ matrix $\Phi$, the $(1, 2)$-Pfaffian is the entry $\phi_{34}$ which we are claiming is linear. Likewise the $(1, 3)$-Pfaffian is the entry $\phi_{24}$ which by Lemma 5 has degree $a_1 + a_3 + e \leq 2$. By Lemma 2 we may assume that $\phi_{14}, \phi_{24}, \phi_{34}$ define a smooth complete intersection curve and $X$ contains this curve by example 1.

By Lemma 3 $X$ cannot be general.

**Case $\mu = 2$.** In this case $a_2 = a_3$. By Lemma 6 $n$ must be 4 or 6. The case $n = 4$ is ruled out again by the arguments of the above paragraph since $\Phi$ has two entries of degree 2 in its last column. We will therefore assume that $n = 6$. The matrix

$$
\Phi = \begin{pmatrix}
0 & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & \phi_{16} \\
0 & 0 & \phi_{23} & \phi_{24} & \phi_{25} & \phi_{26} \\
* & * & 0 & \phi_{34} & \phi_{35} & \phi_{36} \\
* & * & * & 0 & \phi_{45} & \phi_{46} \\
* & * & * & * & 0 & \phi_{56} \\
* & * & * & * & * & 0
\end{pmatrix}
$$

is skew-symmetric and by our choice of ordering of the $a_i$’s, the degrees of the upper triangular entries are non-increasing as we move to the right or down.

As remarked before, the degree of $\phi_{ij}$ is $d - e - a_i - a_j$. The $(1, 2)$-Pfaffian (which is a non-zero quadric when $\mu = 2$) is given by the expression (see example 1)

$$
(2) \quad \text{Pf}(\Phi(1, 2)) = \phi_{34}\phi_{56} - \phi_{35}\phi_{46} + \phi_{36}\phi_{45}.
$$

We shall consider the following two sub-cases, one where $\phi_{56}$ has positive degree (and hence can be chosen non-zero by Lemma 2) and the other where it has non-positive degree (and hence is forced to be zero):

- $d - e - a_5 - a_6 > 0$.

Since $\phi_{34} \cdot \phi_{56}$ is one term in the $(1, 2)$-Pfaffian of $\Phi$, and since degree $\phi_{34}$ is at least degree $\phi_{56}$, they are both forced to be linear. Therefore $\phi_{34}, \phi_{35}, \phi_{36}$ have the same degree (=1) and so $a_4 = a_5 = a_6$. Likewise, $a_3 = a_4 = a_5$. Therefore $a_2 = a_3 = a_4 = a_5 = a_6$. Hence $\Phi$ has a principal $5 \times 5$ submatrix $\chi$ (obtained by deleting the first row and column in $\Phi$) which is a skew symmetric matrix of linear terms, while its first row and first column have entries of degree $1 + r, r \geq 0$. 
By Lemma 2 we may assume that the ideal of the $4 \times 4$ Pfaffians of $\chi$ defines a smooth curve $C$. $X$ is then a degree $d = 3 + r$ hypersurface containing $C$. By Lemma 3, $X$ cannot be general when $d \geq 6$.

In this case, the entry $\phi_{56} = 0$. Suppose $\phi_{46}$ is also zero. Then both $\phi_{36}$ and $\phi_{45}$ must be linear and non-zero since the (1,2)-Pfaffian of $\Phi$ (see equation 2) is a non-zero quadric. Since $a_2 = a_3$, $\phi_{26}$ is also linear. Thus using Lemma 2, $X$ contains the complete intersection curve given by the vanishing of $\phi_{16}$ and the two linear forms $\phi_{36}, \phi_{26}$. By Lemma 3, $X$ cannot be general.

So we may assume that $\phi_{46} \neq 0$. Since $\phi_{35}$ is also non-zero, both must be linear. Hence $a_3 + a_5 = a_4 + a_6$, and so $a_3 = a_4$ and $a_5 = a_6$.

After twisting $E$ by a line bundle, we may assume that $a_2 = a_3 = a_4 = 0 \leq a_5 = a_6 = b$. The linearity of the entry $\phi_{46}$ gives $d - e - b = 1$. The condition $d - e - a_5 - a_6 \leq 0$ yields $1 \leq b$. Taking first Chern classes in resolution 1 gives $e = 2 - a_1$.

Let $r = -a_1$, $s = b - 1$. Then $r, s \geq 0$, and $d = r + s + 4$. If we inspect the matrix $\Phi$, the non-zero rows in columns 5 and 6 give a $4 \times 2$ matrix $\Delta$ with top row of degree $1 + r$ and the other entries all linear. By Lemma 2, we may assume that the $2 \times 2$ minors of this $4 \times 2$ matrix define a smooth curve $C_r$ as described in Lemma 3. Since $X$ contains this curve, $X$ cannot be general when $d \geq 6$.

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