Numerical validation of novel scaling laws for air entrainment in water

Daniele Catucci, Riccardo Briganti and Valentin Heller

Environmental Fluid Mechanics and Geoprocesses Research Group, Department of Civil Engineering, University of Nottingham, Nottingham NG7 2RD, UK

The Froude scaling laws have been used to model a wide range of water flows at reduced size for almost a century. In such Froude scale models, significant scale effects for air–water flows (e.g. hydraulic jumps or wave breaking) are typically observed. This study introduces novel scaling laws, excluding scale effects in the modelling of air–water flows. This is achieved by deriving the conditions under which the governing equations are self-similar. The one-parameter Lie group of point-scaling transformations is applied to the Reynolds-averaged Navier–Stokes equations, including surface tension effects. The scaling relationships between variables are derived for the flow variables, fluid properties and initial and boundary conditions. Numerical simulations are conducted to validate the novel scaling laws for (i) a dam break flow interacting with an obstacle and (ii) a vertical plunging water jet. Results for flow variables, void fraction and turbulent kinetic energy are shown to be self-similar at different scales, i.e. they collapse in dimensionless form. Moreover, these results are compared with those obtained using the traditional Froude scaling laws, showing significant scale effects. The novel scaling laws are a more universal and flexible alternative with a genuine potential to improve laboratory modelling of air–water flows.

1. Introduction

Physical modelling at reduced size is one of the oldest and most important design tools in hydraulic engineering. For processes of engineering interest involving free surface flows, the Froude scaling laws have been used since they were introduced by Moritz Weber in 1930 [1]. They ensure that the ratio between
the inertial and gravity force, namely the Froude number \( F_r \), is the same in the model and in nature, i.e. in the prototype. Other force ratios, such as the Reynolds number \( R_e \) (inertial force to viscous force) and the Weber number \( W_e \) (inertial force to surface tension force), are represented incorrectly if the fluid properties are the same in the prototype and its model [2–4]. Froude scaling laws are particularly useful for laminar flows (\( R_e \to 0 \)) and also for fully turbulent flows (\( R_e \to \infty \)) to investigate Reynolds number invariant fluid parameters [5,6].

However, the Froude scaling laws are also used for flows involving air entrainment, e.g. air bubble entrainment into water flows, hereafter referred to as air–water flows. For these, density, viscosity and surface tension between water and air play a central role such that \( F_r, R_e \) and \( W_e \) are all important [7,8]. Indeed, most studies suggest that the Froude scaling laws without scaled fluid properties underestimate air entrainment because the effects of viscosity and surface tension are over-represented in the model [9–11].

Air–water flows are observed in many hydraulic phenomena such as spillway flows, hydraulic jumps, wave breaking and plunging jets, which are still modelled with Froude scaling laws, despite their limitations [12–14]. Moreover, air entrainment occurs at the free surface of oceans, rivers and streams as an important mechanism for the transport of oxygen and carbon dioxide, which is critical for the survival of these ecosystems [15,16]. Despite many studies, turbulent air–water flows remain not well understood such that costly case-specific, large-scale investigations are commonly required to avoid scale effects [2].

An analytical approach to derive novel scaling laws can be based upon self-similarity of the governing equations. A self-similar object is identical to a part of itself. As such, the scaling of an object that follows suitable laws results in a self-similar scaled copy of the object itself [17–20] and a self-similar process behaves the same way at different scales, such that scale effects are avoided [21]. For example, a scaled model and the prototype of a hydraulic jump are self-similar if dimensionless results are identical. This implies that the dimensionless velocity field and the void fraction are variables that are invariant when self-similarity is achieved.

Self-similar conditions for phenomena with negligible surface tension effects have previously been derived by applying the one-parameter Lie group of point scaling transformations [22] (hereafter referred to as Lie group transformations). Lie group transformations were originally used to reduce the number of independent variables of an initial boundary value problem by transforming it in a new space where the solution of the problem is the same as the original [20,23,24]. This approach has been applied by [25] to derive the conditions under which various hydrological processes are self-similar through the change in size. Consequently, the Lie group transformations can be used to find the scaling laws of the variables that can guarantee self-similarity of a phenomenon. The advantage of this approach is that it gives a complete picture of the requirements that must be satisfied for self-similarity, in contrast to a classical dimensional analysis, based on Buckingham \( \Pi \) theorem, which is only applied to the dynamics in the interior of the domain and not to the boundary conditions [26,27].

Self-similar conditions of a depth-averaged two-dimensional hydrodynamic equations system and the three-dimensional (3D) Reynolds-averaged Navier–Stokes (RANS) equations were derived in [28,29]. They showed self-similar conditions of the variables in the RANS equations with \( k-\epsilon \) closure for phenomena that are dominated by gravity and viscous effects. The scaling laws found by [29] were applied numerically to a lid-driven cavity flow, showing self-similar behaviour. In both [28] and [29], computational fluid dynamics (CFD) was used to demonstrate that the proposed scaling laws involve no scale effects. Indeed, CFD can be used to investigate scale effects numerically and the scale and properties of fluids are more easily controlled than in laboratory experiments [30–33].

To the knowledge of the authors, there are no studies addressing the analytical conditions for which the governing equations involving viscous and surface tension effects are simultaneously self-similar when scaled in size. In the present study, we derive novel scaling laws by applying the Lie group transformations to the governing equations of air–water flows, including surface tension effects. No other assumptions are made in the application of the Lie group
transformations. This article shows, by simulating a range of scales, that the derived self-similar conditions for air–water flows and their boundary conditions can be used to achieve self-similarity.

The derived scaling laws are applied numerically to two air–water flow processes, namely (i) a dam break flow interacting with an obstacle, generating large deformations of the free surface [34], and (ii) a vertical circular plunging water jet impinging on quiescent water characterized by significant air entrainment, based on the experimental results of [13]. The 3D RANS equations govern phenomena in which both viscosity and surface tension play a central role.

Air–water flows are here simulated by using interFoam, a numerical solver for two-phase incompressible fluid flows based on the volume of fluid (VOF) method implemented in the OpenFOAM v.1706 CFD package [35]. In these simulations, all the boundary and initial conditions, including the properties of the fluid, are transformed using the novel scaling laws at different geometrical scales with scale factors $\lambda = l_p/l_m$, where $l_p$ is a characteristic length in the prototype (subscript $p$) and $l_m$ is the corresponding one in the model (subscript $m$). The processes are scaled using values of $\lambda$ for which a correct representation of surface tension and viscous effects is essential to avoid significant scale effects. The two processes are also simulated with the commonly applied Froude scaling laws using ordinary water and air in the model, as is common in laboratory experiments (herein called traditional Froude scaling), and the Froude scaling laws in which the properties of the fluids are strictly scaled (herein called precise Froude scaling). It is demonstrated that the novel scaling laws involve no scale effects, in contrast to traditional Froude scaling, and they are also more universal and flexible than precise Froude scaling.

This article is organized as follows: in §2, the Lie group transformations are applied to the governing equations and the novel scaling laws are derived. The numerical model is presented in §3. Subsequently, the two CFD case studies are illustrated in §4, including the set-up, the application of the novel scaling laws and the results. The findings of this research are discussed in §5 and the conclusions and recommendations for future work are given in §6. Finally, appendix A includes the details of the derivation of the novel scaling laws and the self-similar conditions owing to the initial and boundary conditions.

2. Analytical derivation of the novel scaling laws

(a) Governing equations

Air–water flows are here described by the RANS equations for incompressible fluids

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \nu \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right) - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + g_i + \frac{1}{\rho f_\sigma} \right), \quad i, j = 1, 2, 3, \tag{2.2}$$

where $i$ is the free index, $j$ is the dummy index, following Einstein’s notation, $t$ is time, $x_i$ and $x_j$ are the spatial coordinates, $U_i$ and $U_j$ are the Reynolds-averaged flow velocity components, $u_i$ and $u_j$ are the fluctuating velocity components, $\overline{u_iu_j}$ is the Reynolds stress term, $p$ is the Reynolds-averaged pressure, $\nu$ is the kinematic viscosity, $\rho$ is the density of the fluid, $g_i = (g_1, g_2, g_3)$ is the gravitational acceleration vector and $f_\sigma$ is the surface tension force per unit volume, defined as

$$f_\sigma = \sigma \kappa \frac{\partial \gamma}{\partial x_i}. \tag{2.3}$$

In equation (2.3), $\sigma$ is the surface tension constant, $\kappa$ is the curvature of the free surface and $\gamma$ is the phase fraction. This is a dimensionless variable with values between 0 and 1 that is used to identify any air–water interface (see §3). The $k$-$\epsilon$ model is here applied for the Reynolds stresses.
in equation (2.2) (see [36] for more details), for which
\[
- \overline{u_i u_j} = \nu_l \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3k\delta_{ij}},
\]
(2.4)
where \(k\) is the turbulent kinetic energy, \(\delta_{ij}\) is the Kronecker delta and \(\nu_l\) is the eddy viscosity,
\[
\nu_l = \frac{C_\mu k^2}{\epsilon}.
\]
(2.5)
k and its rate of dissipation \(\epsilon\) are calculated from
\[
\frac{\partial k}{\partial t} + U_i \frac{\partial k}{\partial x_i} = P_k - \epsilon + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_l}{C_{\alpha_l}} \right) \frac{\partial k}{\partial x_j} \right]
\]
(2.6)
and
\[
\frac{\partial \epsilon}{\partial t} + U_i \frac{\partial \epsilon}{\partial x_i} = C_{\epsilon_1} \frac{\epsilon}{k} P_k - C_{\epsilon_2} \frac{\epsilon^2}{k} + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_l}{C_{\alpha_l}} \right) \frac{\partial \epsilon}{\partial x_j} \right].
\]
(2.7)
P\(_{K}\) = \(u_i(\partial U_i/\partial x_j)(\partial U_j/\partial x_i + \partial U_i/\partial x_i)\) and \(C_{\epsilon_1} = 1.44, C_{\epsilon_2} = 1.92, C_\mu = 0.09, C_{\alpha_l} = 1.0\) and \(C_{\alpha_l} = 1.3\) are the standard model coefficients used in the \(k-\epsilon\) turbulence model [37]. The approach used here for turbulence modelling, combined with the VOF method, has been recognized to overestimate \(k\) [38], although this does not affect the derivation of the scaling laws and the self-similarity of the representation of the process.

(b) One-parameter Lie group transformations

The Lie group is defined as
\[
\phi = \beta^{a_0}\phi^*.
\]
(2.8)
Equation (2.8) transforms the variable \(\phi\) in the original space into the variable \(\phi^*\) in the transformed (*) space, \(\beta\) is the scaling parameter and \(a_0\) is the scaling exponent of the variable \(\phi\). The scaling ratio of the variable \(\phi\) is \(r_\phi = \phi/\phi^* = \beta^{a_0}\) [29,39].

All the variables of equations (2.1)–(2.7) in the original domain are written in the transformed domain as
\[
x_1 = \beta^{a_x} x^*_1, \quad x_2 = \beta^{a_x} x^*_2, \quad x_3 = \beta^{a_x} x^*_3, \quad t = \beta^{a_t} t^*, \quad U_1 = \beta^{a_t} U^*_1, \quad U_2 = \beta^{a_t} U^*_2, \quad U_3 = \beta^{a_t} U^*_3, \quad p = \beta^{a_p} p^*,
\]
\[
\gamma_i = \beta^{a_\rho} \gamma^*_i, \quad \rho = \beta^{a_\rho} \rho^*, \quad \nu = \beta^{a_v} \nu^*, \quad \sigma = \beta^{a_\sigma} \sigma^*, \quad \kappa = \beta^{a_\kappa} \kappa^*,
\]
\[
\gamma_1 = \beta^{a_\gamma_1} \gamma^*_1, \quad \gamma_2 = \beta^{a_\gamma_2} \gamma^*_2, \quad \gamma_3 = \beta^{a_\gamma_3} \gamma^*_3,
\]
\[
k = \beta^{a_k} k^*, \quad \epsilon = \beta^{a_\epsilon} \epsilon^*, \quad \nu_l = \beta^{a_{\nu_l}} \nu^*_l, \quad P_k = \beta^{a_{p_k}} P^*_k.
\]
(2.9)
Self-similar conditions are obtained when the governing equations in the original domain, subjected to the Lie group transformations, remain invariant. The Lie group transformations for equation (2.1) yield the following equation in the transformed domain:
\[
\frac{\partial \beta^{a_{U_1}} U^*_1}{\partial \beta^{a_{x_1}} x^*_1} + \frac{\partial \beta^{a_{U_2}} U^*_2}{\partial \beta^{a_{x_2}} x^*_2} + \frac{\partial \beta^{a_{U_3}} U^*_3}{\partial \beta^{a_{x_3}} x^*_3} = 0,
\]
(2.10)
which, with \(\beta\) being a constant parameter, is rearranged as
\[
\beta^{a_{U_1}} U^*_1 + \beta^{a_{U_2}} U^*_2 + \beta^{a_{U_3}} U^*_3 = 0.
\]
(2.11)
Self-similarity is achieved if equation (2.11) can be obtained from equation (2.1) by means of a simple scaling process. Therefore, all terms of equation (2.11) must be transformed by using the same scaling ratios
\[
\beta^{a_{U_1}} - a_{\alpha_1} = \beta^{a_{U_2}} - a_{\alpha_2} = \beta^{a_{U_3}} - a_{\alpha_3} \Rightarrow a_{\alpha_1} = a_{\alpha_2} = a_{\alpha_3} = a_{\alpha_1} - a_{\alpha_2} = a_{\alpha_1} - a_{\alpha_3}.
\]
(2.12)
To attain self-similarity of air–water flows, the exponents for length, velocity and fluctuating velocity components have to be identical for the \(i\)th axis. This is shown in appendix A, where
the detailed derivation of equations (2.2)–(2.7) is presented. Hereafter, $\alpha_x$, $\alpha_U$ and $\alpha_u$ are used to indicate the scaling exponents of length, Reynolds-averaged velocity and fluctuating velocity components on the $i$th axis. Similarly, $\alpha_\rho$, $\alpha_\nu$ and $\alpha_\sigma$ are derived by applying the Lie group transformations to equation (2.3). Furthermore, based on equations (2.5)–(2.7), the scaling conditions for the turbulent parameters are derived. In addition, the detailed derivations of the self-similar conditions for the initial and boundary conditions are also shown in appendix A. The scaling conditions derived above are summarized in the second column of table 1. They are consistent with those reported in tables 1 and 2 in [29] with addition of the surface tension and the curvature of the free surface. All the exponents are written in terms of three independent scaling exponents, namely $\alpha_x$, $\alpha_t$ and $\alpha_\rho$, meaning that they are user defined (their choice is flexible). In fact, the solution of air–water flow equations can be mapped to solutions in other transformed domains with different $\lambda = \beta^{\alpha_x}$ by selecting the scaling parameter $\beta$ and changing the $\alpha$ of three independent variables.

It is possible to assign the value of one or two of the three $\alpha$ while still preserving self-similarity. For example, in table 1, it is shown that choosing $\alpha_g = 0$ implies that $\alpha_{\nu} = \alpha_x - 2\alpha_t$. Therefore, the unscaled $g$ requires that $\alpha_t = 0.5\alpha_x$. In this configuration, the remaining scaling exponents are written in terms of $\alpha_x$, $\alpha_\rho$ and $\alpha_g = 0$ (fourth and fifth columns of table 1). Hence, keeping $g$ invariant in a scaled model requires the time and flow velocities to be scaled and the properties of the fluids to be changed to obtain a self-similar behaviour. A further restriction can be imposed on the density of the fluids, namely $\alpha_\rho = 0$. This restriction leads to the well-known precise Froude scaling laws [3], as a particular case of the novel scaling laws, where $g$ is constant and $\nu$ and $\rho$ are scaled by keeping Re and We invariant.

3. Numerical model

Air–water flows are simulated by using the two-phase flow solver *interFoam*, based on the VOF method, implemented in the OpenFOAM v1706 CFD package [35]. A single system of RANS equations is solved with the pressure and velocity fields shared among both phases. The interface between water and air is identified by a value of the phase fraction $\gamma$ between $\gamma = 1$ (water) and $\gamma = 0$ (air). The fluid properties used in the equations are mapped in all domains as a weighted average using $\gamma$ as weight, e.g. for $\rho$ and $\nu$

$$\rho = \gamma \rho_w + (1 - \gamma) \rho_a \quad (3.1)$$

and

$$\nu = \gamma \nu_w + (1 - \gamma) \nu_a \quad (3.2)$$

where subscripts $w$ and $a$ refer to the water and air phase, respectively. $\sigma$ appears in equation (2.3) to model the surface tension force per unit volume, as stated in the continuum surface force method proposed by [40]. The curvature of the interface between two fluids $\kappa$ is defined as

$$\kappa = -\frac{\partial}{\partial x_i} \left( \frac{\partial \gamma / \partial x_i}{|\partial \gamma / \partial x_i|} \right). \quad (3.3)$$

$\gamma$ is transported as a scalar by the flow field and the interface location (e.g. the free surface) is updated by solving the volume fraction equation

$$\frac{\partial \gamma}{\partial t} + \frac{\partial (\gamma U_i)}{\partial x_i} = 0. \quad (3.4)$$

The interface reconstruction technique used by *interFoam* is MULES [41]. The free surface can also be captured by using alternative techniques, such as the isoAdvector method [42]. However, the governing equations remain the same and the self-similarity of the representation of the process under the novel scaling laws is not affected by the interface reconstruction technique.
Table 1. Novel and precise Froude scaling laws for the variables of the RANS equations and the $k$-$\varepsilon$ turbulence model obtained by applying the Lie group transformations.

| variables                      | scaling conditions in terms of $\alpha_x$, $\alpha_t$ and $\alpha_p$ (novel scaling laws) | scaling conditions in terms of $\alpha_x$, $\alpha_t$ and $\alpha_p = 0$ | scaling conditions in terms of $\alpha_x$, $\alpha_t$ and $\alpha_p = 0$ (precise Froude scaling laws) |
|-------------------------------|---------------------------------|---------------------------------|---------------------------------|
|                               | exponents                      | scaling ratios                  | exponents                      | scaling ratios                  |
| length (m)                    | $\alpha_x$                     | $\beta^{\alpha_x}$             | $\alpha_x$                     | $\beta^{\alpha_x}$             |
| time (s)                      | $\alpha_t$                     | $\beta^{\alpha_t}$             | $\alpha_t = 0.5\alpha_x$       | $\beta^{0.5\alpha_x}$          |
| density (kg m$^{-3}$)         | $\alpha_p$                     | $\beta^{\alpha_p}$             | $\alpha_p = 0$                 | $\beta^0 = 1$                  |
| velocity (m s$^{-1}$)         | $\alpha_u = \alpha_x - \alpha_t$ | $\beta^{\alpha_u-\alpha_t}$  | $\alpha_u = 0.5\alpha_x$       | $\beta^{0.5\alpha_x}$          |
| pressure (Pa)                 | $\alpha_p = 2\alpha_x - 2\alpha_t + \alpha_p$ | $\beta^{2\alpha_x-2\alpha_t+\alpha_p}$ | $\alpha_p = \alpha_x + \alpha_p$ | $\beta^{\alpha_x+\alpha_p}$    |
| gravitational acceleration (m s$^{-2}$) | $\alpha_g = \alpha_x - 2\alpha_t$ | $\beta^{\alpha_g-2\alpha_t}$ | $\alpha_g = 0$                 | $\beta^0 = 1$                  |
| viscosity (m$^2$ s$^{-1}$)    | $\alpha_v = 2\alpha_x - \alpha_t$ | $\beta^{2\alpha_x-\alpha_t}$  | $\alpha_v = 1.5\alpha_x$       | $\beta^{1.5\alpha_x}$          |
| surface tension (N m$^{-1}$)  | $\alpha_\sigma = 3\alpha_x - 2\alpha_t + \alpha_p$ | $\beta^{3\alpha_x-2\alpha_t+\alpha_p}$ | $\alpha_\sigma = 2\alpha_x + \alpha_p$ | $\beta^{2\alpha_x+\alpha_p}$    |
| curvature of the free surface (1 m$^{-1}$) | $\alpha_k = 2\alpha_x - 2\alpha_t$ | $\beta^{2\alpha_x-2\alpha_t}$ | $\alpha_k = 2\alpha_x$         | $\beta^{2\alpha_x}$            |
| eddy viscosity (m$^3$ s$^{-1}$) | $\alpha_v = 2\alpha_x - \alpha_t$ | $\beta^{2\alpha_x-\alpha_t}$  | $\alpha_v = 1.5\alpha_x$       | $\beta^{1.5\alpha_x}$          |
| Reynolds stresses (m$^2$ s$^{-2}$) | $\alpha_{(u,u)} = 2\alpha_x - 2\alpha_t$ | $\beta^{2\alpha_x-2\alpha_t}$ | $\alpha_{(u,u)} = \alpha_x$    | $\beta^{\alpha_x}$             |
| turbulent kinetic energy (m$^3$ s$^{-2}$) | $\alpha_k = 2\alpha_x - 2\alpha_t$ | $\beta^{2\alpha_x-2\alpha_t}$ | $\alpha_k = \alpha_x$         | $\beta^{\alpha_x}$             |
| dissipation (m$^3$ s$^{-3}$)  | $\alpha_\varepsilon = 2\alpha_x - 3\alpha_t$ | $\beta^{2\alpha_x-3\alpha_t}$ | $\alpha_\varepsilon = 0.5\alpha_x$ | $\beta^{0.5\alpha_x}$          |
| production of turbulence due to horizontal velocity gradients (m$^2$ s$^{-3}$) | $\alpha_{p_1} = 2\alpha_x - 3\alpha_t$ | $\beta^{2\alpha_x-3\alpha_t}$ | $\alpha_{p_1} = 0.5\alpha_x$     | $\beta^{0.5\alpha_x}$          |

Where $\alpha_x$, $\alpha_t$, and $\alpha_p$ are the exponents of the scaling conditions, $\beta$ is the scaling ratio, and $\lambda$ is the precise Froude scaling ratio.
Figure 1. Initial set-up of the dam break flow prototype and a scaled numerical domain to schematically illustrate the novel scaling laws. The flow parameters at a specified time and space can be transformed to the corresponding time and space in the self-similar domain. (Online version in colour.)

4. Numerical results

The self-similar conditions of the novel scaling laws are validated with the simulation of two physical processes: (i) a dam break flow interacting with an obstacle and (ii) a vertical plunging water jet. The simulations for both processes involve the prototype and a number of scaled models up to large geometrical scale factors of $\lambda = 16$.

(a) Dam break flow

Dam break flows have been widely investigated numerically and the specific case addressed herein is chosen because it is a well-known test to validate the modelling of large deformations of free surfaces [43,44]. The solver used in the present study has been validated with this particular test case by [34]. In this study, $\gamma = 0.1$ is selected to identify the air–water interface in the VOF method. $\gamma = 0.1$ is obtained by considering the value between 0 and 1 providing the best fit with the experimental void fraction distribution in §4b.

(i) Numerical set-up

The initial condition at $t = 0$ consists of a quiescent water column of volume $1.228 \times 0.550 \times 1.000 \text{ m}^3$, located at the left side of a $3.220 \times 1.000 \times 1.000 \text{ m}^3$ tank (figure 1). A prismatic fixed obstacle with a volume of $0.160 \times 0.160 \times 0.403 \text{ m}^3$ is located at $x_1 = 2.395 \text{ m}$. The water column is released instantaneously at $t = 0$. Subsequently, the flow impacts the obstacle and creates a complex two-phase flow. The top wall of the domain is modelled as an open, fully transmissive boundary at atmospheric pressure and all the remaining walls as no-slip boundary conditions. The water density is $\rho_w = 1000 \text{ kg m}^{-3}$, its kinematic viscosity is $\nu_w = 1 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$ and the surface tension constant is $\sigma = 0.07 \text{ N m}^{-1}$.

A 180 (length) × 60 (width) × 80 (height) Cartesian computational grid was used, apart from the obstacle. Note that, because of the orientation of the reference frame, for this case $g_i = (0, 0, -g)$ in equation (2.2). The time step $\Delta t$ was set equal to 0.001 s at the start of the simulation and it was...
Two self-similar domains, namely D8 and D16, are created with geometrical scale factors of \( \lambda = \beta^{\alpha_\lambda} = 8 \) and 16, respectively. To achieve this, it is assumed that \( \alpha_\lambda = 1 \), such that \( \beta = 8 \) (D8) and 16 (D16), respectively. All variables and parameters are transformed by the scaling exponents in the fourth and fifth columns of table 2 (with scaling conditions in terms of \( \alpha_x, \alpha_t, \alpha_\rho \) and \( \alpha_\sigma = 0 \)). Their specific values for the prototype and the scaled models, obtained by applying the conditions in table 2, are presented in table 3. The prototype is also scaled by using precise Froude scaling (D8PFr and D16PFr) and traditional Froude scaling (D8TFr and D16TFr) using the same \( \lambda \) as in the self-similar domains.
(iii) Results

For the purpose of this work, it is interesting to analyse the time when gravity, inertial, viscous and surface tension effects are all relevant. This happens when the dam break flow impacts the obstacle and creates an elongated water tongue. Figure 2 shows this process with snapshots of the prototype and the scaled domains at $x'_{2} = x_{2}/h_{w} = 0$ (figure 1) and dimensionless time $t' = t\sqrt{g/h_{w}} = 2.7$. The contours in figure 2 represent the dimensionless velocity magnitude $U' = U/\sqrt{gh_{w}}$, where $U = \sqrt{U_{1}^{2} + U_{2}^{2} + U_{3}^{2}}$. The prototype shows a large free surface deformation after impacting the obstacle (figure 2a). The self-similar domains and the domains scaled with precise Froude scaling all simulate the water tongue of the prototype correctly. Moreover, the dimensionless velocity magnitudes in the prototype and in the self-similar domains are the same, despite the increasing $\lambda$ (figure 2b–e). On the other hand, traditional Froude scaling does not model the free surface correctly owing to Re and We scale effects, i.e. the water tongue becomes less prolonged with increasing $\lambda$ (figure 2f,g).

The differences between the prototype and the scaled domains are quantified using the root mean square error along the plane $x'_{2} = 0$ for $U'$ (RMSE$_{U'}$)

$$\text{RMSE}_{U'} = \sqrt{\frac{\sum_{b=1}^{n} (U'_{b,p} - U'_{b,m})^2}{n}},$$

where $U'_{b,p}$ are the cell values of $U'$ in the prototype, $U'_{b,m}$ are the scaled domains and $n = 14283$ is the number of cells in the cross-section $x'_{2} = 0$. As shown in table 4, the RMSE$_{U'}$ values for D8 and D16 confirm a nearly perfect self-similarity with respect to the prototype. $k$ is used to assess turbulence because it shows significant scale effects if $\nu$ is not scaled. Air entrainment is assessed by using $\gamma$, which is expected to deviate from the prototype if the surface tension is over-represented in the scaled domain. Figure 3 shows the dimensionless turbulent kinetic energy $k' = k/(g h_{w})$ at point RW (figure 1) versus $t'$ and the variation of $\gamma$ is shown in figure 4. After $t' = 2.7$, the water tongue collapses and creates a complex flow characterized by strong turbulence and air entrainment. The flow reaches the downstream wall, where it is reflected at $t' = 3.25$. At a later stage, the dam break wave is re-reflected at the upstream wall and it reaches point RW again at $t' = 23.6$. 

Figure 2. Snapshots of the dam break flow at the cross-section $x'_{2} = 0$ and dimensionless time $t' = 2.7$ of the (a) prototype and scaled with (b,c) the novel scaling laws, (d,e) precise Froude scaling and (f,g) traditional Froude scaling.
Table 3. Parameters for the dam break flow in the prototype and the scaled domains.

| variable                  | prototype D1 | domain D8 | domain D16 | domain D8Fr | domain D16Fr | domain D8Fr | domain D16Fr |
|---------------------------|--------------|-----------|------------|-------------|--------------|-------------|--------------|
| tank length (m)           | 3.22         | 0.4025    | 0.20125    | 0.4025      | 0.20125      | 0.4025      | 0.20125      |
| water column height (m)   | 0.55         | 0.06875   | 0.034375   | 0.06875     | 0.034375     | 0.06875     | 0.034375     |
| computational time (s)    | 6            | 2.12      | 1.5        | 2.12        | 1.5          | 2.12        | 1.5          |
| gravitational acceleration (m s⁻²) | 9.81        | 9.81      | 9.81       | 9.81        | 9.81         | 9.81        | 9.81         |
| water density (kg m⁻³)    | 1000         | 125       | 62.5       | 1000        | 1000         | 1000        | 1000         |
| water viscosity (m² s⁻¹)  | 10⁻⁶         | 4.42 × 10⁻⁸ | 1.56 × 10⁻⁸ | 4.42 × 10⁻⁸ | 1.56 × 10⁻⁸ | 10⁻⁶        | 10⁻⁶         |
| air density (kg m⁻³)      | 1            | 0.125     | 0.0625     | 1           | 1            | 1           | 1            |
| air viscosity (m² s⁻¹)    | 1.48 × 10⁻⁵  | 6.54 × 10⁻⁷ | 2.31 × 10⁻⁷ | 6.54 × 10⁻⁷ | 2.31 × 10⁻⁷ | 1.48 × 10⁻⁵ | 1.48 × 10⁻⁵ |
| surface tension (N m⁻¹)   | 0.07         | 1.37 × 10⁻⁴ | 1.70 × 10⁻³ | 1.09 × 10⁻³ | 2.73 × 10⁻⁴ | 0.07        | 0.07         |
Figure 3. $k'$ time histories in the dam break flow at point RW for (a) domains $D_1$, $D_8$, $D_{16}$, $D_{8\text{PFr}}$ and $D_{16\text{PFr}}$ and (b) $D_1$, $D_{8\text{TFr}}$ and $D_{16\text{TFr}}$. (Online version in colour.)

Table 4. RMSE$_{U'}$ for the dam break flow for the domains $D_1$ and $D_8$, $D_{16}$, $D_{8\text{PFr}}$, $D_{16\text{PFr}}$, $D_{8\text{TFr}}$ and $D_{16\text{TFr}}$, for the snapshots in figure 2.

| RMSE$_{U'}$ |
|-------------------|
| $D_1$–$D_8$ | $D_1$–$D_{16}$ | $D_1$–$D_{8\text{PFr}}$ | $D_1$–$D_{16\text{PFr}}$ | $D_1$–$D_{8\text{TFr}}$ | $D_1$–$D_{16\text{TFr}}$ |
| $2.98 \times 10^{-4}$ | $2.10 \times 10^{-4}$ | $3.00 \times 10^{-4}$ | $2.21 \times 10^{-4}$ | 0.067 | 0.119 |

The perfect collapse of the data for $D_1$, $D_8$ and $D_{16}$ affirms the self-similar behaviour of $k'$ for the novel scaling laws. The self-similar behaviour is also confirmed for $D_{8\text{PFr}}$ and $D_{16\text{PFr}}$. On the other hand, $k'$ shows scale effects using traditional Froude scaling; the first $k'$ peak is either
under- or overestimated (D8_{TFr} and D16_{TFr}, respectively), while the magnitude of the second peak decreases with increasing $\lambda$.

As demonstrated in figure 4, where $\gamma$ is shown as a proxy for surface tension, air entrainment is correctly scaled in the self-similar domains as it controls the air–water interface and the free surface curvature. While the results in the domains D1, D8, D16, D8_{PFr} and D16_{PFr} essentially collapse, the domains scaled with traditional Froude scaling show significant differences in the region where air entrainment is most important. $\gamma$ starts to increase close to $t' = 4$, meaning that the wave reaches RW consistently at the same time in all domains except for D8_{TFr} and D16_{TFr} (figure 4). Subsequently, $\gamma$ increases to reach 1 less rapidly than in the prototype when using traditional Froude scaling. These differences become more visible at a later stage of the simulation when the dam break wave is re-reflected at $t' = 23.6$, showing significant scale effects.

(b) Plunging water jet

In this section, the same scaling laws as in the previous test case are applied to the plunging water jet presented in [13]. This involves free-surface instabilities, air entrainment and turbulence.

(i) Numerical set-up

The set-up is based on the experiments of [13], consisting of a jet from a circular orifice impinging on a prismatic column of water. However, in this study, the symmetry of the problem with respect to two orthogonal vertical planes is used to simulate only a quarter of the domain, in order to reduce the computational cost. Figure 5 shows the numerical domain and the variables used in the prototype. A plunging water jet is ejected from a nozzle having a radius $r_{in} = 0.0125$ m. Here, the subscript $in$ indicates the quantities at the nozzle, i.e. at the inlet of the numerical domain, while the subscript $im$ indicates values of variables at the still water level, i.e. $x_1 = 0$. The receiving pool is 0.15 m wide and 1.80 m deep and at the start of the simulation the distance between the water surface and the nozzle is $l_1 = 0.10$ m. The velocity of the jet at $x_1 = 0$ is $U_{im} = 4.10$ m s$^{-1}$. Here, a Cartesian coordinate system with $x_1$ pointing downwards is used; therefore, $g_i = (g, 0, 0)$.

The inlet boundary condition, namely the nozzle, is at the top boundary. The velocity at the inlet $U_{in}$ and both $k_{in}$ and $\epsilon_{in}$ are prescribed, while the outlet is located at the bottom boundary, having the same flow rate magnitude as the inlet.

$U_{in}$ is calculated starting from the jet impact velocity using Bernoulli’s theorem $U_{in} = \sqrt{U_{im}^2 - 2gl_1} = 3.85$ m s$^{-1}$. At the outlet (subscript out) $U_{out} = U_{in}$ and $r_{out} = r_{in}$. $k_{in}$ and $\epsilon_{in}$ are calculated as

$$k_{in} = \frac{3}{2} (U_{in} l) \epsilon = 0.000471 \text{ m}^2 \text{s}^{-2}$$

and

$$\epsilon_{in} = C_\mu \frac{k_{in}^{3/2}}{l_t} = 0.00105 \text{ m}^2 \text{s}^{-3},$$

where $l = 0.46\%$ is the turbulence intensity following [13], and $l_t$ is the turbulence mixing length approximated with $l_t = 0.07 r_{in}$. The part of the top boundary of the domain not occupied by the inlet was modelled as a fully transmissive open boundary at atmospheric pressure. Since only a quarter of the domain is simulated, a symmetry boundary condition is used at the symmetry boundary walls and no-slip conditions are applied at the remaining walls, including the bottom wall outside the outlet cells (figure 5).

A structured orthogonal mesh is used with a finer resolution for the area in which the water jet impacts the free surface down to a depth of 0.6 m. The smallest observed bubble size was 1 mm and the minimum cell size was 0.625 mm to increase the interface sharpness around the bubbles [13,46]. This mesh resolution is not fine enough to resolve the smallest bubbles present in the flow. However, the main focus of this work is to show the relative differences in the results of the application of different scaling laws for air–water flows, rather than to perfectly resolve the dynamics of individual bubbles.
Figure 5. Schematic illustration of the computational domain and mesh of the plunging water jet. (Online version in colour.)

The simulation time was 300 s, the same duration as that used by [13] to compute the
distribution of the void fraction from the laboratory measurements, and the time step varied
with respect to the CFL condition. \( C_{\text{max}} \) was set equal to 0.3. The simulations were run on
the University of Nottingham HPC cluster Augusta. The number of cells in the computational
domain was \( 1.89 \times 10^6 \) and the corresponding cores and memory were 10 and 36 GB, respectively.
It required 168 h to simulate 300 s actual time (also for the corresponding times at reduced scales).

(ii) Application of the novel scaling laws

The two self-similar domains P8 and P16 were simulated with geometrical scale factors of \( \lambda = 8 \)
and 16, respectively. Similarly to the dam break case, the scaling exponent for length is \( \alpha_x = 1 \)
so that \( \beta = 8 \) (P8) and 16 (P16). The scaling ratios and parameters obtained by applying
the conditions in the second column of table 1 are shown in table 5. The domains \( \text{P8}_{\text{PR}} \) and \( \text{P16}_{\text{PR}} \)
refer to precise Froude scaling and \( \text{P8}_{\text{TF}} \) and \( \text{P16}_{\text{TF}} \) to traditional Froude scaling (table 5).

(iii) Results

Figure 6 shows the time-averaged \( \gamma \) along the section A–A’ for domains P1, P8, \( \text{P8}_{\text{PR}} \) and \( \text{P8}_{\text{TF}} \).
The prototype shows a distribution of the time-averaged void fraction that is consistent with
Table 5. Scaling parameters and exponents used to scale the plunging jet prototype values to the corresponding values in the domains P8 and P16.

| variable                              | prototype | domain P8 | domain P16 | domain P8_{Fr} | domain P16_{Fr} | domain P8_{Fr} | domain P16_{Fr} |
|---------------------------------------|-----------|-----------|------------|----------------|-----------------|----------------|-----------------|
| inlet radius (m)                      | 0.0125    | 0.0015625 | 7.81 × 10^{-4} | 0.0015625 | 7.81 × 10^{-4} | 0.0015625 | 7.81 × 10^{-4} |
| computational time (s)                | 300       | 106       | 75         | 106            | 75              | 106            | 75              |
| impact velocity (m s\(^{-1}\))       | 4.10      | 1.45      | 1.025      | 1.45           | 1.025           | 1.45           | 1.025           |
| gravitational acceleration (m s\(^{-2}\)) | 9.81    | 9.81      | 9.81       | 9.81           | 9.81            | 9.81           | 9.81            |
| water density (kg m\(^{-3}\))        | 1000      | 125       | 62.5       | 1000           | 1000            | 1000           | 1000            |
| water viscosity (m\(^2\) s\(^{-1}\)) | 10^{-6}   | 4.42 × 10^{-8} | 1.56 × 10^{-8} | 4.42 × 10^{-8} | 1.56 × 10^{-8} | 10^{-6}        | 10^{-6}         |
| air density (kg m\(^{-3}\))          | 1         | 0.125     | 0.0625     | 1              | 1               | 1              | 1               |
| air viscosity (m\(^2\) s\(^{-1}\))   | 1.48 × 10^{-5} | 6.54 × 10^{-7} | 2.31 × 10^{-7} | 6.54 × 10^{-7} | 2.31 × 10^{-7} | 1.48 × 10^{-5} | 1.48 × 10^{-5} |
| surface tension (N m\(^{-1}\))       | 0.07      | 1.37 × 10^{-4} | 1.70 × 10^{-5} | 1.09 × 10^{-3} | 2.73 × 10^{-4} | 0.07           | 0.07            |
| inlet turbulent kinetic energy (m\(^2\) s\(^{-2}\)) | 4.71 × 10^{-4} | 5.89 × 10^{-5} | 2.94 × 10^{-5} | 5.89 × 10^{-5} | 2.94 × 10^{-5} | 5.89 × 10^{-5} | 2.94 × 10^{-5} |
| inlet energy dissipation rate \(\epsilon_{in}\) (m\(^2\) s\(^{-3}\)) | 1.05 × 10^{-3} | 3.71 × 10^{-4} | 2.63 × 10^{-4} | 3.71 × 10^{-4} | 2.63 × 10^{-4} | 3.71 × 10^{-4} | 2.63 × 10^{-4} |
the description of high Re plunging jets provided by [47]. In particular, the flow shows the characteristic conical shape of the air-entrainment layer and the dispersion of bubbles due to the buoyancy effects outside the cone. The consequence of air entrainment in the flow is a rise of the free surface with respect to the initial conditions (figure 6a–c). Domains P8 and P8PFr have an identical shape to the air-entrainment layer, showing that the free surface reaches the same level, while P8TFr shows clear differences.

The following results are all shown along section A–A’ at ($x_l - l_1$)/$r_{im} = 1.60$. The distribution of the void fraction is compared with the experimental results of [13] in figure 7. The computed distribution and that measured in [13] are shown to have a close agreement. The novel scaling laws and precise Froude scaling reproduce the distribution of the void fraction of the prototype correctly, in terms of both the shape and magnitude. On the other hand, the traditional Froude scaling fails to describe the void fraction distribution.

Figure 8 shows the time-averaged dimensionless velocity magnitude $\overline{U'}$, where for this case $U' = U/U_{im}$. In the prototype, the maximum value of $\overline{U'}$ is at the jet centreline and $\overline{U'}$ follows qualitatively the same velocity distribution as found in [48]. While the results of the domains P1, P8, P16, P8PFr and P16PFr are identical, $\overline{U'}$ for the domains P8TFr and P16TFr are lower than in the prototype.

Figure 9 shows the time-averaged dimensionless turbulent kinetic energy $\overline{k'}$, where $k' = k/(gr_{im})$. In the prototype and self-similar domains the maximum value is $\overline{k'} = 10$ at $s/r_{im} = 1.0$, beyond which $\overline{k'}$ decreases to less than 4.0 at $s/r_{im} = 2.0$. On the other hand, the behaviour in the domains based on traditional Froude scaling is different. Indeed, $\overline{k'}$ in P8TFr does not show a clear
5. Discussion

Self-similarity has been achieved for the governing equations of air–water flows, including surface tension expanding the scaling conditions reported in [28,29]. An advantage of this approach is that the scaling conditions are directly derived from the governing equations. This
leads to more universal scaling laws than the Froude scaling laws \[49\]. Furthermore, the choice of the scaling exponents \(\alpha_x\), \(\alpha_t\) and \(\alpha_\rho\) in the second column of table 1 are user defined (flexible). This implies that novel scaling laws can also be written in terms of a set of other variables to find different configurations. For example, it is shown that precise Froude scaling is obtained as a special case of the novel scaling laws. The CFD simulations conducted herein demonstrated that both the novel scaling laws and precise Froude scaling result in self-similar air–water flows, which would also be the case for another set of variables.

In the dam break flow, a significant deformation of the free surface is shown in the prototype after the flow impacts the obstacle, with a characteristic water tongue projected downstream of the obstacle. This behaviour is captured in all the domains scaled with the novel scaling laws; figure 3 show that \(k'\) is the same by using the novel scaling laws and \(k\) is thus self-similar. The phase fraction is also self-similar. This is a strong indication that surface tension effects are self-similar as well (figure 4) and it is also true for the domains \(D8_{PFr}\) and \(D16_{PFr}\), since precise Froude scaling is a special case of the novel scaling laws. On the other hand, the commonly applied traditional Froude scaling, relying on the same fluids as in the prototype, fails to reproduce the behaviour of the prototype. Indeed, figure 2 shows that the water tongue is not well predicted. After \(t' = 2.7\), it collapses and the flow is reflected at the downstream wall. Scale effects are observed in \(k'\) and \(\gamma\) at point RW. Furthermore, the flow reaches point RW later than in the prototype with increasing \(\lambda\). Scale effects are also observed after the flow is re-reflected, particularly at the second peak of \(k'\).

For the plunging jet, air entrainment plays a central role. Figures 8a and 9a demonstrate that the novel scaling laws result in self-similarity for \(\overline{U'}\) and \(\overline{k'}\), i.e. these results collapse for \(P1, P8, P16, P8_{PFr}\), and \(P16_{PFr}\), while this is not the case for \(P8_{TFr}\) and \(P16_{TFr}\). The self-similarity of the distribution of the void fraction depends on density, viscosity and surface tension effects. The prototype simulation captures the mechanism of air entrainment by a plunging jet (figure 5), including the formation of an air cavity between the impinging jet and the surrounding fluid, which collapses and reforms intermittently, entraining air bubbles that are transported by the flow. At this stage, air bubbles are advected in a turbulent shear flow and they are broken into smaller bubbles, creating a conical air-entrainment layer. Subsequently, buoyancy determines the re-surfacing of bubbles in the portion of the flow outside the air layer \[7,8,12\]. This complex mechanism causes the air-entrainment layer in figure 6, where the novel scaling laws guarantee self-similarity. This is also true for the void fraction in figure 7, which is a consequence of
the mechanism described above. On the other hand, figure 7b demonstrates that traditional Froude scaling fails to reproduce the void fraction distribution. By using ordinary water, the surface tension and viscosity are over-represented; therefore, the distribution of the void fraction gradually decreases with increasing λ. As expected, for increasing λ the flow regime changes, transitioning from high \( \text{Re} = 50840 \) in the prototype to \( \text{Re} = 800 \) in P16TFr, calculated by using \( U_{\text{im}}, r_m \) and \( \nu_w \). The use of \( k-\epsilon \), this case, introduces also model, in addition to scale effects [2,3,47], which explain the results in figures 8b and 9b.

The need for novel scaling laws for scaling fluid properties requires the modification or replacement of ordinary water in laboratory experiments, e.g. for values of λ comparable to the highest used here, i.e. \( \lambda = 16, \) where \( \rho_w = 62.5 \text{ kg m}^{-3}, \nu_w = 1.56 \times 10^{-8} \text{ m}^2 \text{ s}^{-1} \) and \( \sigma = 1.70 \times 10^{-5} \text{ N m}^{-1} \) (tables 3 and 5). There are options to alter the relevant fluid properties: the surface tension can be modified by adding ethanol to water [11] and the viscosity can also be reduced, e.g. Rouse et al. [50] modelled a hydraulic jump with air. A more recent approach to change the water properties is based on nanofluids, i.e. nanoparticles are added to water [51,52]. A key advantage of the novel scaling laws is that fluids of different density from water, e.g. ethanol, now also qualify as potential candidates for laboratory experiments.

6. Conclusion

The Froude scaling laws have been applied to model water flows at reduced size for almost a century. A significant disadvantage of Froude scaling is the potential for scale effects. This article shows how such scale effects in air–water flows are avoided with novel scaling laws based upon self-similarity of the governing equations. Lie group transformations are applied to the Reynolds-averaged Navier–Stokes equations where surface tension effects are included as a source term. This allows the modelling of hydrodynamic phenomena at small scale without viscous and surface tension scale effects. These novel scaling laws are more universal and flexible than the precise Froude scaling laws because different scaling configurations can be obtained, e.g. by also scaling the density of the fluid. In this study, the gravitational acceleration is kept constant and the scaling exponents of the variables are expressed as a function of the scaling exponents of the length \( \alpha_x \), time \( \alpha_t \) and gravitational acceleration \( \alpha_g = 0 \).

The derived novel scaling laws were validated with the simulations of two air–water flow phenomena: (i) a dam break flow interacting with an obstacle and (ii) a plunging water jet. The numerical simulations demonstrated that the processes are correctly scaled, and showed perfect agreement at different scales for air entrainment and kinematic properties. The results of the precise Froude scaling, where the properties of the fluids are strictly scaled, demonstrate that a particular configuration of the novel scaling laws is also able to result in self-similarity. By contrast, the simulations based on traditional Froude scaling using ordinary water and air, as is common in laboratory studies, show significant scale effects, as expected.

While this study provides a thorough numerical validation of the proposed scaling laws, future work aims to identify suitable fluids satisfying the novel scaling laws, which would enable the scaling of air–water flows without scale effects for the first time in a laboratory environment.

Data accessibility. All the OpenFOAM set-ups used for this article are available via Dryad: https://datadryad.org/stash/share/ny73jIRnKRNgqSmX2J3V4fSywvEIGxUeZSXiyn1k.

Competing interests. We declare we have no competing interests.

Funding. The work was carried out as part of Daniele Catucci’s PhD study, funded by the University of Nottingham Pro-Vice Chancellor Research Excellence Scholarship. The simulations were conducted on the University of Nottingham HPC clusters Augusta.

Acknowledgements. The authors thank Dr David Hargreaves for helpful suggestions.

Appendix A. Derivation of the novel scaling laws

The remaining scaling conditions in table 1, in addition to the ones presented in §2b, are derived here. The Lie group transformations for equation (2.2) yield the following equations in the
transformed domain:

\[
\beta^{a_{u_i}-\alpha} \frac{\partial U_i^*}{\partial t^*} + \beta^{a_{u_j}+\alpha_{u_i}-\alpha_{j}} U_i \frac{\partial U_j^*}{\partial x_j^*} = \beta^{a_{u_i}+\alpha_{i}-2\alpha_{x_i}} \frac{\partial}{\partial x_i^*} \left( v^* \frac{\partial U_i^*}{\partial x_i^*} \right) - \beta^{a_{x_j}-2\alpha_{x_j}} \frac{\partial U_i^*}{\partial x_i^*} + \beta^{a_{\rho_i}} \frac{\partial p^*}{\partial \rho^*}.
\] (A 1)

Self-similarity is guaranteed if the scaling ratios of all terms in equation (A 1) are the same, implying that the exponents of all terms must be the same

\[
\alpha_{U_i} = \alpha_i = \alpha_{U_j} + \alpha_{U_i} - \alpha_{j} = \alpha_{U_j} + \alpha_{U_1} - \alpha_{3} = 2\alpha_{x_1} + \alpha_{x_1} - \alpha_{x_3}
\]

\[
\alpha_{U_2} = \alpha_i = \alpha_{U_1} + \alpha_{U_2} - \alpha_{x_1} = \alpha_{U_1} + \alpha_{U_2} - 2\alpha_{x_2} = \alpha_{U_1} + \alpha_{U_2} - 2\alpha_{x_3}
\]

\[
\alpha_{U_3} = \alpha_i = \alpha_{U_1} + \alpha_{U_3} - \alpha_{x_1} = \alpha_{U_1} + \alpha_{U_3} - 2\alpha_{x_2} = \alpha_{U_1} + \alpha_{U_3} - 2\alpha_{x_3}
\]

\[
\alpha_{\rho} = \alpha_{\rho} - \alpha_{x_1}
\]

\[
\alpha_{f} = \alpha_{f} - \alpha_{\rho}
\]

\[
\alpha_{f_j} = \alpha_{f} - \alpha_{\rho}
\]

and

\[
\alpha_{U_4} = \alpha_i = \alpha_{U_1} + \alpha_{U_4} - \alpha_{x_1} = \alpha_{U_1} + \alpha_{U_4} - 2\alpha_{x_2} = \alpha_{U_1} + \alpha_{U_4} - 2\alpha_{x_3}
\]

\[
\alpha_{U_5} = \alpha_i = \alpha_{U_1} + \alpha_{U_5} - \alpha_{x_1} = \alpha_{U_1} + \alpha_{U_5} - 2\alpha_{x_2} = \alpha_{U_1} + \alpha_{U_5} - 2\alpha_{x_3}
\]

\[
\alpha_{\rho} = \alpha_{\rho} - \alpha_{x_3}
\]

\[
\alpha_{f} = \alpha_{f} - \alpha_{x_3}
\]

\[
\alpha_{f_j} = \alpha_{f} - \alpha_{x_3}
\]

The Lie group transformations for equation (2.3) result in

\[
\beta^{\alpha_{f \sigma}} f_{\sigma} = \beta^{a_{\sigma} + \alpha_{\sigma} - \alpha_{x_1}} \sigma^{*} \frac{\partial \gamma^{*}}{\partial x_1^*}.
\] (A 5)

The dimension \( \kappa \) is the inverse of a length such that \( \alpha_{\kappa} = -\alpha_{x_1} \). Furthermore, \( \alpha_{\gamma} = 0 \) because \( \gamma \) is dimensionless. Hence, equation (A 5) reduces to

\[
\alpha_{f \sigma} = \alpha_{\gamma} - 2\alpha_{x_1}.
\] (A 6)

From equations (A 2)–(A 4), the scaling exponents of the length dimensions along the \( i \)th axis are obtained as

\[
\alpha_{U_1} - \alpha_i = \alpha_{U_1} + \alpha_{U_1} - 2\alpha_{x_1} \Rightarrow \alpha_{x_1} = \frac{\alpha_i + \alpha_{x_1}}{2},
\] (A 7)

\[
\alpha_{U_2} - \alpha_i = \alpha_{U_2} + \alpha_{U_2} - 2\alpha_{x_2} \Rightarrow \alpha_{x_2} = \frac{\alpha_i + \alpha_{x_2}}{2},
\] (A 8)

and

\[
\alpha_{U_3} - \alpha_i = \alpha_{U_3} + \alpha_{U_3} - 2\alpha_{x_3} \Rightarrow \alpha_{x_3} = \frac{\alpha_i + \alpha_{x_3}}{2}.
\] (A 9)
In other words, the scaling exponents of the length scale must be identical for \( i = 1, 2, 3 \) because the fluids are considered isotropic; therefore,

\[
\alpha_{x_1} = \alpha_{x_2} = \alpha_{x_3} = \alpha_x. \tag{A 10}
\]

Similarly, \( \alpha_{U_1}, \alpha_{U_2} \) and \( \alpha_{U_3} \) are obtained from equations (A 2)–(A 4) as follows:

\[
\alpha_{U_1} - \alpha_t = \alpha_{U_1} + \alpha_{U_1} - \alpha_s \Rightarrow \alpha_{U_1} = \alpha_x - \alpha_t, \tag{A 11}
\]

\[
\alpha_{U_2} - \alpha_t = \alpha_{U_2} + \alpha_{U_2} - \alpha_s \Rightarrow \alpha_{U_2} = \alpha_x - \alpha_t \tag{A 12}
\]

and

\[
\alpha_{U_3} - \alpha_t = \alpha_{U_3} + \alpha_{U_3} - \alpha_s \Rightarrow \alpha_{U_3} = \alpha_x - \alpha_t. \tag{A 13}
\]

Hence, \( \alpha_{U_1}, \alpha_{U_2} \) and \( \alpha_{U_3} \) are also equal;

\[
\alpha_{U_1} = \alpha_{U_2} = \alpha_{U_3} = \alpha_U = \alpha_x - \alpha_t. \tag{A 14}
\]

Consequently, \( u_1, u_2 \) and \( u_3 \) have the same exponents in all directions as well because they are transformed by using the velocity ratio

\[
\alpha_{u_1} = \alpha_{u_2} = \alpha_{u_3} = \alpha_u. \tag{A 15}
\]

The results in equations (A 10)–(A 15) are important because the selection of unique scaling exponents for length and velocity scales in the \( i \)th axis is necessary to achieve self-similarity of air–water flows. \( \alpha_g, \alpha_p, \) and \( \alpha_v \) are obtained from equations (A 2)–(A 4) and can be written in terms of \( \alpha_x, \alpha_t \) and \( \alpha_p \) as

\[
\alpha_{U_1} - \alpha_t = \alpha_g = \alpha_x - 2\alpha_t, \tag{A 16}
\]

\[
\alpha_{U_1} - \alpha_t = \alpha_p - \alpha_p - \alpha_x \Rightarrow \alpha_p = 2\alpha_x - 2\alpha_t + \alpha_p \tag{A 17}
\]

and

\[
\alpha_{U_1} - \alpha_t = \alpha_{U_1} + \alpha_v - 2\alpha_x \Rightarrow \alpha_v = 2\alpha_x - \alpha_t. \tag{A 18}
\]

By using equations (A 2) and (A 6)

\[
\alpha_{U_1} - \alpha_t = \alpha_{f_p} - \alpha_p \Rightarrow \alpha_x - 2\alpha_t = \alpha_{f_p} - 2\alpha_x - \alpha_p, \tag{A 19}
\]

from which

\[
\alpha_{f_p} = 3\alpha_x - 2\alpha_t + \alpha_p. \tag{A 20}
\]

Similarly, equations (2.4)–(2.7) are transformed by keeping \( C_{\epsilon_1}, C_{\epsilon_2}, C_{\mu}, C_{\sigma_k} \) and \( C_{\sigma_v} \) as dimensionless coefficients

\[
-\beta^{\alpha_{\epsilon_1}, \alpha_{\epsilon_2}} U_{\epsilon_1} U_{\epsilon_2} = \beta^{\alpha_{\epsilon_1} + \alpha_{\epsilon_2} - \alpha_1} \frac{\partial U_{\epsilon_1}}{\partial x_1} + \beta^{\alpha_{\epsilon_1} + \alpha_{\epsilon_2} - \alpha_3} \frac{\partial U_{\epsilon_1}}{\partial x_3} - \frac{2}{3} \beta^{\alpha_{\epsilon_1}, \alpha_{\epsilon_2}} \delta_{ij}, \tag{A 21}
\]

\[
\beta^{\alpha_{\epsilon_1}, \alpha_{\epsilon_2}} \epsilon_i = \beta^{2\alpha_{\epsilon_1} - \alpha_k} C_{\mu} \frac{k^{*2}}{\epsilon_1}, \tag{A 22}
\]

\[
\beta^{\alpha_{\epsilon_1}, \alpha_{\epsilon_2}} \frac{\partial k^*}{\partial t^*} + \beta^{\alpha_{\epsilon_1} + \alpha_{\epsilon_2} - \alpha_1} \frac{\partial k^*}{\partial x_1} = \beta^{\alpha_{\epsilon_1} + \alpha_{\epsilon_2} - \alpha_3} \frac{\partial k^*}{\partial x_3} = \beta^{\alpha_{\epsilon_1} - \alpha_1} \frac{\partial k^*}{\partial x_1} + \beta^{\alpha_{\epsilon_1} - \alpha_3} \frac{\partial k^*}{\partial x_3} \tag{A 23}
\]

and

\[
\beta^{\alpha_{\epsilon_1}, \alpha_{\epsilon_2}} \frac{\partial \epsilon^*}{\partial t^*} + \beta^{\alpha_{\epsilon_1} + \alpha_{\epsilon_2} - \alpha_1} \frac{\partial \epsilon^*}{\partial x_1} = \beta^{\alpha_{\epsilon_1} + \alpha_{\epsilon_2} - \alpha_1} \frac{\partial \epsilon^*}{\partial x_1} + \beta^{\alpha_{\epsilon_1} - \alpha_1} C_{\epsilon_1} \frac{\partial \epsilon^*}{\partial x_1} = \beta^{\alpha_{\epsilon_1} - \alpha_1} C_{\epsilon_1} \frac{\partial \epsilon^*}{\partial x_1} \tag{A 24}
\]
For equations (A 21)–(A 24) to be self-similar, the following conditions must hold:

\[
\begin{align*}
\alpha_{uu} &= \alpha_{v_i} + \alpha_{U} - \alpha_x \\
&= \alpha_{v_i} + \alpha_{U} - \alpha_x \\
&= \alpha_k, \\
\alpha_{v_i} &= 2\alpha_k - \alpha_{\epsilon}, \quad (A 25) \\
\alpha_k - \alpha_t &= \alpha_{U} + \alpha_k - \alpha_x \\
&= \alpha_{p_k} \\
&= \alpha_{\epsilon} \\
&= \alpha_{v_i} + \alpha_k - 2\alpha_x \\
&= \alpha_{v_i} + \alpha_k - 2\alpha_x \\
\end{align*}
\]

and

\[
\begin{align*}
\alpha_{\epsilon} - \alpha_t &= \alpha_{U} + \alpha_{\epsilon} - \alpha_x \\
&= \alpha_{\epsilon} + \alpha_{p_k} - \alpha_k \\
&= 2\alpha_{\epsilon} - \alpha_k \\
&= \alpha_{v_i} + \alpha_{\epsilon} - 2\alpha_x \\
&= \alpha_{v_i} + \alpha_{\epsilon} - 2\alpha_x. \\
\end{align*}
\]

\(v_t\) has the same dimension as \(v_i\); therefore, equation (A 17) yields

\[
\alpha_{v_i} = 2\alpha_x - \alpha_t. \quad (A 29)
\]

\(\alpha_{uu}\) is the same in all directions and is calculated from equations (A 25)–(A 29) as

\[
\alpha_{uu} = \alpha_{v_i} + \alpha_{U} - \alpha_x \Rightarrow \alpha_{uu} = 2\alpha_x - 2\alpha_t. \quad (A 30)
\]

From equation (A 25), \(\alpha_k\) is obtained (\(\alpha_k = 2\alpha_x - 2\alpha_t\)). Finally, \(\alpha_{\epsilon}\) and \(\alpha_{p_k}\) are obtained from equations (A 26) and (A 27) as

\[
\begin{align*}
\alpha_{\epsilon} &= 2\alpha_k - \alpha_{v_i} \Rightarrow \alpha_{\epsilon} = 2\alpha_x - 3\alpha_t \\
\alpha_{p_k} &= \alpha_k - \alpha_t \Rightarrow \alpha_{p_k} = 2\alpha_x - 3\alpha_t. \\
\end{align*}
\]

The Lie group transformations are also applied to the initial and boundary conditions. The initial velocity \(U(x_i, t = 0) = U_{i_0}(x_i)\) and pressure fields \(p(x_i, t = 0) = p_0(x_i)\) are transformed as

\[
\begin{align*}
U_{i_0}^*(x_i^*) &= \beta^{-\alpha_{U}} U_{i_0}(x_i) = \beta^{-\alpha_{U}} U_{i_0}(\beta^{\alpha_x} x_i^*) \\
p_0^*(x_i^*) &= \beta^{-\alpha_{P}} p_0(x_i) = \beta^{-\alpha_{P}} p_0(\beta^{\alpha_x} x_i^*). \\
\end{align*}
\]

Another boundary condition is the zero gradient \((\partial/\partial x_i)\phi = 0\) for a flow variable \(\phi\). This gradient condition is transformed as \(\beta^{\alpha_{\phi} - \alpha_{x}} (\partial/\partial x_i)\phi^* = 0\). Since \(\beta \neq 0\), this does not pose any limitation in the scaling conditions \(((\partial/\partial x_i)\phi^*) = 0\).

References

1. Hager WH, Castro-Orgaz O. 2017 William Froude and the Froude number. J. Hydraul. Eng. **143**, 02516005. (doi:10.1061/(ASCE)HY.1943-7900.0001213)
2. Heller V. 2011 Scale effects in physical hydraulic engineering models. J. Hydraul. Res. **49**, 293–306. (doi:10.1080/00221686.2011.578914)
3. Hughes SA. 1993 Physical models and laboratory techniques in coastal engineering, vol. 7. Singapore: World Scientific.
4. Ali SZ, Dey S. 2017 Origin of the scaling laws of sediment transport. Proc. R. Soc. A 473, 20160785. (doi:10.1098/rsapa.2016.0785)
5. Frisch U. 1995 Turbulence: the legacy of AN Kolmogorov. Cambridge, UK: Cambridge University Press.
6. Heller V. 2017 Self-similarity and Reynolds number invariance in Froude modelling. J. Hydraul. Res. 55, 293–309. (doi:10.1080/00221686.2016.1250832)
7. Biň AK. 1993 Gas entrainment by plunging liquid jets. Chem. Eng. Sci. 48, 3585–3630. (doi:10.1016/0009-2509(93)81019-R)
8. Kiger KT, Duncan JH. 2012 Air-entrainment mechanisms in plunging jets and breaking waves. Annu. Rev. Fluid Mech. 44, 563–596. (doi:10.1146/annurev-fluid-122109-160724)
9. Felder S, Chanson H. 2017 Scale effects in microscopic air-water flow properties in high-velocity free-surface flows. Exp. Therm Fluid Sci. 83, 19–36. (doi:10.1016/j.expthermflusci.2016.12.009)
10. Heller V, Hager WH, Minor H-E. 2008 Scale effects in subaerial landslide generated impulse waves. Exp. Fluids 44, 691–703. (doi:10.1007/s00348-007-0427-7)
11. Stagonas D, Warbrick D, Muller G, Magagna D. 2011 Surface tension effects on energy dissipation by small scale, experimental breaking waves. Coastal Eng. 58, 826–836. (doi:10.1016/j.coastaleng.2011.05.009)
12. Blenkinsopp C, Chaplin J. 2007 Void fraction measurements in breaking waves. Proc. R. Soc. A 463, 3151–3170. (doi:10.1098/rspa.2007.1901)
13. Chanson H, Aoki S, Hoque A. 2004 Physical modelling and similitude of air bubble entrainment at vertical circular plunging jets. Chem. Eng. Sci. 59, 747–758. (doi:10.1016/j.ces.2003.11.016)
14. Wang H, Chanson H. 2015 Air entrainment and turbulent fluctuations in hydraulic jumps. Urban Water J. 12, 502–518. (doi:10.1080/1573062X.2013.847464)
15. Leighton TG, Coles DG, Srokosz M, White PR, Woolf DK. 2018 Asymmetric transfer of CO₂ across a broken sea surface. Sci. Rep. 8, 1–9. (doi:10.1038/s41598-018-25818-6)
16. Mustaffa NIH, Ribas-Ribas M, Banko-Kubis HM, Wurl O. 2020 Global reduction of in situ CO₂ transfer velocity by natural surfactants in the sea-surface microlayer. Proc. R. Soc. A 476, 20190763. (doi:10.1098/rspa.2019.0763)
17. Barenblatt GI. 1996 Scaling, self-similarity, and intermediate asymptotics: dimensional analysis and intermediate asymptotics. Cambridge, UK: Cambridge University Press.
18. Barenblatt GI. 2003 Scaling, vol. 34. Cambridge, UK: Cambridge University Press.
19. Henriksen RN. 2015 Scale invariance: self-similarity of the physical world. Weinheim, Germany: John Wiley & Sons.
20. Zohuri B. 2015 Dimensional analysis and self-similarity methods for engineers and scientists. Cham, Switzerland: Springer.
21. Polyanin AD, Manzhirov AV. 2008 Handbook of integral equations. London, UK: Chapman and Hall.
22. Lie S. 1880 Theorie der Transformationsgruppen I. Mathematische Annalen 16, 441–528. (doi:10.1007/BF01446218)
23. Bluman GW, Cole JD. 1974 Similarity methods for differential equations, vol. 13. New York, NY: Springer.
24. Bluman G, Anco S. 2002 Symmetry and integration methods for differential equations, vol. 154. New York, NY: Springer.
25. Haltas I, Kavvas M. 2011 Scale invariance and self-similarity in hydrologic processes in space and time. J. Hydrol. Eng. 16, 51–63. (doi:10.1061/(ASCE)HE.1943-5584.0000289)
26. Polisinelli J, Kavvas ML. 2016 A comparison of the modern Lie scaling method to classical scaling techniques. Hydrol. Earth Syst. Sci. 20, 2669–2678. (doi:10.5194/hess-20-2669-2016)
27. Ercan A, Kavvas ML, Haltas I. 2014 Scaling and self-similarity in one-dimensional unsteady open channel flow. Hydrol. Processes 28, 2721–2737. (doi:10.1002/hyp.9822)
28. Ercan A, Kavvas ML. 2015 Scaling and self-similarity in two-dimensional hydrodynamics. Chaos 25, 075404. (doi:10.1063/1.4913852)
29. Ercan A, Kavvas ML. 2017 Scaling relations and self-similarity of 3-dimensional Reynolds-averaged Navier-Stokes equations. Sci. Rep. 7, 6416. (doi:10.1038/s41598-017-06669-z)
30. Huang W, Yang Q, Xiao H. 2009 CFD modeling of scale effects on turbulence flow and scour around bridge piers. Comput. Fluids 38, 1050–1058. (doi:10.1016/j.compfluid.2008.01.029)
31. Oliveira FS, Contente J. 2013 Scale effects in numerical modelling of beach profile erosion. J. Coastal Res. 65, 1815–1820. (doi:10.2121/jcoastres.65.2.1815-1820.2013)
32. Torres C, Borman D, Sleigh A, Neeve D. 2018 Investigating scale effects of a hydraulic physical model with 3D CFD. In Smart Dams and Reservoirs—Proc. of the 20th Biennial Conf. of the British Dam Society, Swansea, UK, 13–15 September 2018, pp. 89–101. London, UK: ICE Publishing.

33. Carr K, Erkan A, Kavvas M. 2015 Scaling and self-similarity of one-dimensional unsteady suspended sediment transport with emphasis on unscaled sediment material properties. J. Hydraul. Eng. 141, 04015003. (doi:10.1061/(ASCE)HY.1943-7900.0000994)

34. Zhainakov AZ, Kurbanaliev A. 2013 Verification of the open package OpenFOAM on dam break problems. Thermophys. Aeromech. 20, 451–461. (doi:10.1134/S0869864313040082)

35. Greenshields CJ. 2019 The OpenFOAM foundation user guide 7.0. London, UK: The OpenFOAM Foundation Ltd.

36. Pope SB. 2000 Turbulent flows. Cambridge, UK: Cambridge University Press.

37. Launder B, Spalding D. 1974 The numerical computation of turbulent flows. Comput. Methods Appl. Mech. Eng. 3, 269–289. (doi:10.1016/0045-7825(74)90029-2)

38. Fan W, Anglart H. 2020 varRhoTurbVOF: a new set of volume of fluid solvers for turbulent isothermal multiphase flows in OpenFOAM. Comput. Phys. Commun. 247, 106876. (doi:10.1016/j.cpc.2019.106876)

39. Erkan A, Kavvas ML. 2015 Self-similarity in incompressible Navier-Stokes equations. Chaos 25, 123126. (doi:10.1063/1.4938762)

40. Brackbill JU, Kothe DB, Zemach C. 1992 A continuum method for modeling surface tension. J. Comput. Phys. 100, 335–354. (doi:10.1016/0021-9991(92)90240-Y)

41. Deshpande SS, Anumolu L, Trujillo MF. 2012 Evaluating the performance of the two-phase flow solver interFoam. Comput. Sci. Discov. 5, 014016. (doi:10.1088/1749-4699/5/1/014016)

42. Roenby J, Bredmose H, Jasak H. 2016 A computational method for sharp interface advection. R. Soc. Open Sci. 3, 160405. (doi:10.1098/rsos.160405)

43. Issakhov A, Zhandaulet Y, Nogaeva A. 2018 Numerical simulation of dam break flow for various forms of the obstacle by VOF method. Int. J. Multiphase Flow 109, 191–206. (doi:10.1016/j.ijmultiphaseflow.2018.08.003)

44. Kleefsman K, Fekken G, Veldman A, Iwanowski B, Buchner B. 2005 A volume-of-fluid based simulation method for wave impact problems. J. Comput. Phys. 206, 363–393. (doi:10.1016/j.jcp.2004.12.007)

45. Courant R, Friedrichs K, Lewy H. 1967 On the partial difference equations of mathematical physics. IBM J. Res. Dev. 11, 215–234. (doi:10.1147/rd.112.0215)

46. Boualouache A, Zidouni F, Mataoui A. 2018 Numerical visualization of plunging water jet using volume of fluid model. J. Appl. Fluid Mech. 11, 95–105. (doi:10.29252/jafm.11.01.27861)

47. Hassan SH, Guo T, Vlachos PP. 2019 Flow field evolution and entrainment in a free surface plunging jet. Phys. Rev. Fluids 4, 104603. (doi:10.1103/PhysRevFluids.4.104603)

48. McKeogh E, Ervine D. 1981 Air entrainment rate and diffusion pattern of plunging liquid jets. Chem. Eng. Sci. 36, 1161–1172. (doi:10.1016/0009-2509(81)85064-6)

49. Kline SJ. 1965 Similitude and approximation theory. London, UK: McGraw-Hill.

50. Rouse H, Siao TT, Nagaratnam S. 1958 Turbulence characteristics of the hydraulic jump. J. Hydraul. Div. 84, 1–30. (doi:10.1061/JYCEAJ.0000161)

51. Lu G, Duan YY, Wang XD. 2014 Surface tension, viscosity, and rheology of water-based nanofluids: a microscopic interpretation on the molecular level. J. Nanopart. Res. 16, 2564. (doi:10.1007/s11051-014-2564-2)

52. Xu M, Liu H, Zhao H, Li W. 2013 How to decrease the viscosity of suspension with the second fluid and nanoparticles? Sci. Rep. 3, 3137. (doi:10.1038/srep03137)