Period Integrals of Smooth Projective Hypersurfaces and Homotopy Lie Algebras

Jae-Suk Park\(^{1,2}\)*, Jeehoon Park\(^{2}\)**

1 Center for Geometry and Physics, Institute for Basic Science (IBS), 77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk, Korea 790-784
2 Department of Mathematics, Pohang University of Science and Technology (POSTECH), 77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk, Korea 790-784

Abstract. The goal of this paper is to reveal hidden structures on the Griffiths period integrals of differential forms on smooth projective hypersurfaces, studied extensively in [6], in terms of period integrals of Lie algebra representations and strongly homotopy Lie algebras (so called, \(L_\infty\)-homotopy theory).

Let \(X_G\) be a smooth projective hypersurface of dimension \(n - 1\) defined by a homogeneous polynomial \(G(x)\) for \(n \geq 1\). Let \([\gamma] \in H_{n-1}(X_G, \mathbb{Z})\). We consider the Griffiths period integral \(C_\gamma : \mathcal{H}(X_G) \rightarrow \mathbb{C}\), where \(\mathcal{H}(X_G)\) is the \(n\)-th rational de Rham cohomology group of \(\mathbb{P}^n\) regular outside \(X_G\) which is isomorphic to the middle dimensional primitive cohomology of \(X_G\). We construct a differential graded Lie algebra \((\mathcal{A}, \ell)_X\) which is quasi-isomorphic to \((\mathcal{H}(X_G), 0)\) with zero differential, and prove that \(C_\gamma\) can be lifted to a non-trivial \(L_\infty\)-morphism \(\kappa = \kappa_1, \kappa_2, \cdots\), which factorizes through \((\mathcal{A}, \ell)_X\), such that \(\kappa_1 = C_\gamma\) and \(\kappa\) are invariants of the \(L_\infty\)-homotopy types of those morphisms. We use this hidden \(L_\infty\)-homotopy structure to study a new type of extended formal deformation of \(X_G\) and the correlation of its period integrals; in particular, we construct a flat connection on the tangent bundle of a formal deformation space attached to \(X_G\).

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** e-mail: jeehoonpark@postech.ac.kr. Work of Jeehoon Park was partially supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2013023108) and was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2013053914).
1. Introduction

The purpose of this paper is an attempt to establish certain correspondences between the Griffiths period integrals of smooth algebraic varieties and period integrals, defined in section 2, attached
to representations of a finite dimensional Lie algebra on the polynomial algebra. Such correspondences allow us to reveal hidden $L_{\infty}$-homotopy structures in period integrals, leading to their higher generalization. We work out the correspondence in detail when the variety is a smooth projective hypersurface and when the representation is that of Schrödinger representation of the Heisenberg Lie algebra twisted by the Dwork's polynomial associated with the hypersurface. The period integrals of this kind of example have been studied extensively by Griffiths in [6].

Let $n$ be a positive integer. Let $X_G$ be a smooth hypersurface in the complex projective $n$-space $P^n$ defined by a homogeneous polynomial $G(x) = G(x_0, \cdots , x_n)$ of degree $d$ in $\mathbb{C}[x_0, \cdots , x_n]$. Let $H_{n−1}(X_G, \mathbb{Z})$ be the singular homology group of $X_G$ of degree $n−1$ and let $H_{n−1}^{p,q}(X_G, \mathbb{C})$ be the primitive part of the middle dimensional cohomology group $H^{n−1}(X_G, \mathbb{C})$. Then we are interested in the following period integrals

$$C[\gamma] : H_{n−1}^{p,q}(X_G, \mathbb{C}) \rightarrow \mathbb{C}, \quad [\sigma] \mapsto \int_\gamma \sigma, \quad (1.1)$$

where $\gamma$ and $\sigma$ are representatives of the homology class $[\gamma] \in H^{n−1}_{n−1}(X_G, \mathbb{Z})$ and the cohomology class $[\sigma] \in H_{n−1}^{p,q}(X_G, \mathbb{C})$, respectively. We shall often use the shorthand notation $H = H_{n−1}(X_G, \mathbb{C})$ from now on. We also use the notation that

$$\mathcal{F}_j H = H_{n−1,0}^{p,q}(X_G, \mathbb{C}) \oplus H_{n−1,1}^{p,q}(X_G, \mathbb{C}) \oplus \cdots \oplus H_{n−1−j,j}^{p,q}(X_G, \mathbb{C}), \quad 0 \leq j \leq n−1,$$

where $H_{n−1}^{p,q}(X_G, \mathbb{C})$ is the $(p,q)$-th Hodge component of $H$.

Let $H(X_G)$ be the rational de Rham cohomology group defined as the quotient of the group of rational $n$-forms on $P^n$ regular outside $X_G$ by the group of the forms $d\psi$ where $\psi$ is a rational $n−1$ form regular outside $X_G$. For each $k \geq 1$, let $H_k(X_G) \subset H(X_G)$ be the cohomology group defined as the quotient of the group of rational $n$-forms on $P^n$ with a pole of order $\leq k$ along $X_G$ by the group of exact rational $n$-forms on $P^n$ with a pole of order $\leq k$ along $X_G$. Griffiths showed that any rational $n$-forms on $P^n$ with a pole of order $\leq k$ along $X_G$ can be written as a rational differential $n$-form $\frac{F(x)}{G(x)^k}$, where $\Omega_n = \sum (-1)^j x_j dx_0 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_n$ and $F(x)$ is a homogeneous polynomial of degree $kd−(n+1)$. He also showed that $H_n(X_G) = H(X_G)$ and there is a natural injection $H_k(X_G) \subset H_{k+1}(X_G)$ for each $k \geq 1$. Moreover, Griffiths defined the isomorphism (the residue map)

$$Res : H(X_G) \rightarrow H$$

by

$$\frac{1}{2\pi i} \int_{\tau(\gamma)} \frac{\overline{F(x)}}{G(x)^k} \Omega_n = \int_\gamma \overline{Res \frac{F(x)}{G(x)^k}} \Omega_n,$$

where $\tau(\gamma)$ is the tube over $\gamma$, as in (3.4) of [6], such that $Res$ takes the pole order filtration

$$H_1(X_G) \subset H_2(X_G) \subset \cdots \subset H_{n−1−1}(X_G) \subset H_{n−1}(X_G) = H_{n−1+1}(X_G) = \cdots = H(X_G) \quad (1.2)$$

of $H(X_G) (= H_{n+1}(X_G)$ for all $\ell \geq 0$) onto the increasing Hodge filtration $\mathcal{F}_n H$

$$\mathcal{F}_0 H \subset \mathcal{F}_1 H \subset \cdots \subset \mathcal{F}_{n−1} H = H \quad (1.3)$$
of the primitive middle dimensional cohomology $\mathbb{H} = H^m_{\text{prim}}(X_G, \mathbb{C})$. The Griffiths theory provides us with an effective method of studying the period integrals $C_{[\gamma]}$ on the hypersurface $X_G$ as well as an infinitesimal family of hypersurfaces.

In this article, we provide a new homotopy theoretic framework to understand such period integrals. Our first main result is that the period integral $C_{[\gamma]}$ can be enhanced to an $L_\infty$-morphism $\kappa = \kappa_1, \kappa_2, \cdots$, where $\kappa_1 = C_{[\gamma]}$ and $\kappa_m$ is a linear map from the $m$-th symmetric power $S^m \mathbb{H}$ of $\mathbb{H}$ into $\mathbb{C}$ which is a composition of two non-trivial $L_\infty$-morphisms such that $\kappa$ depends only on the $L_\infty$-homotopy types of each factors.\footnote{1}{We recall that an $L_\infty$-morphism $\kappa = \kappa_1, \kappa_2, \cdots$ from $(\mathcal{A}, L_\mathcal{A})_X$ into $(\mathbb{C}, 0)$ – the ground field $\mathbb{C}$ regarded as an $L_\infty$-algebra $(\mathbb{C}, 0)$ with zero $L_\infty$-structure 0, whose $L_\infty$-homotopy type $[\kappa]$ is determined uniquely by the homology class $[\gamma]$ of $\gamma$.}

We use this hidden $L_\infty$-homotopy structure to study certain extended deformations and correlations of period integrals; we develop a new formal deformation theory of $X_G$ which leaves the realm of infinitesimal variations of Hodge structures of $X_G$. This new formal deformation theory has directions that do not satisfy Griffiths transversality.

**Theorem 1.1.** There is a non-trivial differential graded Lie algebra $(\mathcal{A}_X, L_X)_X$ associated to $X_G$, i.e., $(\mathcal{A}_X, L_X)_X$ is an $L_\infty$-algebra with $\ell_3 = \ell_4 = \cdots = 0$, with the following properties:

(a) The cohomology $\mathbb{H} := H^m_{\text{prim}}(X_G, \mathbb{C})$, regarded as an $L_\infty$-algebra $(\mathbb{H}, 0)$ with zero $L_\infty$-structure 0, is quasi-isomorphic to the $L_\infty$-algebra $(\mathcal{A}_X, L_X)_X$.

(b) For each representative $[\gamma] \in H^{m-1}(X_G, \mathbb{Z})$ there is an $L_\infty$-morphism $\phi_{[\gamma]}^\infty = \phi_{[\gamma]}^1, \phi_{[\gamma]}^2, \cdots$ from $(\mathcal{A}_X, L_X)_X$ into $(\mathbb{C}, 0)$ – the ground field $\mathbb{C}$ regarded as an $L_\infty$-algebra $(\mathbb{C}, 0)$ with zero $L_\infty$-structure 0, whose $L_\infty$-homotopy type $[\phi_{[\gamma]}^\infty]$ is determined uniquely by the homology class $[\gamma]$ of $\gamma$.

(c) There is an explicitly constructible $L_\infty$-morphism $\kappa = \kappa_1, \kappa_2, \cdots$ from $(\mathbb{H}, 0)$ into $(\mathbb{C}, 0)$ which is the composition $\kappa := \phi_{[\gamma]}^\infty \cdot \phi_{[\gamma]}^\mathbb{H}$ of an $L_\infty$ quasi-isomorphism $\phi_{[\gamma]}^\mathbb{H} : (\mathbb{H}, 0) \to (\mathcal{A}_X, L_X)_X$ from (a) and the $L_\infty$-morphism $\phi_{[\gamma]}^\infty$ associated to $\gamma$ from (b) such that

(i) $\kappa := \phi_{[\gamma]}^\infty \cdot \phi_{[\gamma]}^\mathbb{H}$ depends only on the $L_\infty$-homotopy types of $\phi_{[\gamma]}^\mathbb{H}$ and $\phi_{[\gamma]}^\infty$ and
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(ii) $\kappa_1 = C_{[\gamma]} = \phi_1^{\mathcal{H}} \circ \varphi_1^{\mathcal{H}}$:

\[
\begin{array}{ccc}
(\mathbb{H}, 0) & \xrightarrow{\varphi_1^{\mathcal{H}} \circ \phi_1^{\mathcal{H}}} & (\mathcal{C}, 0).
\end{array}
\]

Note that $\varphi_1^{\mathcal{H}}$ is a cochain quasi-isomorphism from $(\mathbb{H}, 0)$ to $(\mathcal{S}, \ell_1)_{\mathcal{X}}$, $\phi_1^{\mathcal{H}}$ is a cochain map from $(\mathcal{S}, \ell_1)_{\mathcal{X}}$ to $(\mathcal{C}, 0)$ and both are defined up to cochain homotopies. Within their own homotopy types, a choice of $\varphi_1^{\mathcal{H}}$ corresponds to a choice of representative $\gamma$ of the cohomology class $[\gamma] \in H^*_{prim}(\mathcal{X}, \mathbb{C})$, while a choice of $\phi_1^{\mathcal{H}}$ corresponds to, after dualization, a choice of representative $\gamma$ of the homology class $[\gamma] \in H_{n-1}(\mathcal{X}, \mathbb{Z})$ in the integral $\int \gamma$ in (1.1) such that $\kappa_1([\gamma]) = \phi_1^{\mathcal{H}} \circ \varphi_1^{\mathcal{H}}([\gamma]) = \int \gamma$. From now on we use the notation $\gamma_r := \phi_1^{\mathcal{H}}$.

For each $n \geq 1$, $\kappa_n = \left( \varphi_1^{\mathcal{H}} \bullet \varphi_1^{\mathcal{H}} \right)_n$ in Theorem 1.1 is a $\mathbb{C}$-linear map from the $n$-th symmetric power $S^n \mathbb{H}$ of $\mathbb{H}$ into $\mathcal{C}$. Let $\{e_a\}_{a \in I}$ be a $\mathbb{C}$-basis of $\mathbb{H}$, where $I$ is an index set, and denote the $\mathbb{C}$-dual of $e_a$ by $t^a$. Then we can define the following formal power series

\[
\mathcal{X}_{[\gamma]}(\varphi_1^{\mathcal{H}}) := \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{a_1, \ldots, a_n} t^{a_1} \cdots t^{a_n} \left( \varphi_1^{\mathcal{H}} \bullet \varphi_1^{\mathcal{H}} \right)_n(e_{a_1}, \ldots, e_{a_n}) \right) - 1 \in \mathbb{C}[[\mathcal{X}]],
\]

which depends only on the homology class $[\gamma] \in H_{n-1}(\mathcal{X}, \mathbb{Z})$ of $\gamma$ and the $L_\infty$-homotopy type $[\varphi_1^{\mathcal{H}}]$ of the $L_\infty$-quasi-isomorphism $\varphi_1^{\mathcal{H}} : (\mathbb{H}, 0) \longrightarrow (\mathcal{S}, \ell_1)_{\mathcal{X}}$. This generating power series shall be used to determine the period integrals of an extended formal deformation of the hypersurface $\mathcal{X}_G$.

By a standard basis of $\mathbb{H}$ we mean a choice of basis $e_1, \ldots, e_{\delta_0}, e_{\delta_0+1}, \ldots, e_{\delta_1}, \ldots, e_{\delta_{n-2}}, \ldots, e_{\delta_n}$ for the flag $\mathcal{F}_n \mathbb{H}$ in (1.3) such that $e_1, \ldots, e_{\delta_0}$ gives a basis for the subspace $\mathbb{H}^{n-1, 0} := H_{prim}^{n-1, 0}(\mathcal{X}_G, \mathbb{C})$ and $e_{\delta_{k-1}+1}, \ldots, e_{\delta_k}$, $1 \leq k \leq n-1$, gives a basis for the subspace $\mathbb{H}^{n-1, k} = H_{prim}^{n-1, k}(\mathcal{X}_G, \mathbb{C})$. We denote such a basis by $\{e_a\}_{a \in I}$ where $I = I_0 \cup I_1 \cup \cdots \cup I_{n-1}$ with the notation $\{e_a\}_{a \in I_j} = e_{\delta_{j-1}+1}, \ldots, e_{\delta_j}$ and $\{t^a\}_{a \in I_j} = t^{\delta_{j-1}+1}, \ldots, t^\delta$.

**Theorem 1.2.** For any standard basis $\{e_a\}_{a \in I}$ of $\mathbb{H}$ with $I = I_0 \cup I_1 \cup \cdots \cup I_{n-1}$, there is an $L_\infty$-quasi-isomorphism $f : (\mathbb{H}, 0) \longrightarrow (\mathcal{S}, \ell)_{\mathcal{X}}$ such that

(a) $f = f_1$, i.e. $f_2 = f_3 = \cdots = 0$, 

(b) $f_1$ takes a standard basis $\{e_a\}_{a \in I}$ to a standard basis $\{e_a^{\prime}\}_{a \in I}$, i.e. $e_a^{\prime} = e_a$. 

(c) $f_1$ is a cochain quasi-isomorphism.
(b) for each $0 \leq k \leq n - 1$, the set $\left\{ f_i (e_a^{[k]}) \right\}_{a \in I_k}$ corresponds to a set $\left\{ F_k [a] (x) \right\}_{a \in I_k}$ of homogeneous polynomials of degree $d(k + 1) - (n + 1)$ such that

$$\left\{ \text{Res} \frac{F_k [a] (x)}{G(x)^{k+1}} \Omega_n \right\}_{a \in I_k} = \left\{ e_a^{[k]} \right\}_{a \in I_k}.$$

(c) for each $0 \leq k \leq n - 1$, $f^{[k]} := \int_{\mathbb{H}^{n-k-1}}$ is an $L_{\infty}$-morphism from $(\mathbb{H}^{n-k}, 0)$ into $(\alpha, L)\chi$.

(d) we have the following equation:

$$\int_{\tau(\gamma)} \frac{\Omega_n}{G(x)} \int_{\tau(\gamma)} \frac{\Omega_n}{G(x)} = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{a_1, \ldots, a_n} t^{a_1} \cdots t^{a_n} \left( \phi^{G(x)} \cdot f^{[1]} \right) (e_{a_1}, \ldots, e_{a_n}) \right) - 1,$$

where the right hand side depends only on the $L_{\infty}$-homotopy types of $f^{[1]}$ and $\phi^{G(x)}$, and

$$G_k (x) = G(x) + \sum_{a \in I_k} t^a F_1 [a] (x).$$

(e) we have the following equation:

$$\int_{\tau(\gamma)} \left( \int_0^\infty \chi^L(t) \cdot e^{-y G(x)} dy \right) \Omega_n - \int_{\tau(\gamma)} \frac{\Omega_n}{G(x)}$$

$$= \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{a_1, \ldots, a_n} t^{a_1} \cdots t^{a_n} \left( \phi^{G(x)} \cdot f \right) (e_{a_1}, \ldots, e_{a_n}) \right) - 1,$$

where

$$\chi^L(t) = \exp \left( \sum_{a \in I_k} t^a F_0 [a] (x) + y \sum_{a \in I_1} t^a F_1 [a] (x) + \cdots + y^{n-1} \sum_{a \in I_{n-1}} t^a F_{n-1} [a] (x) \right).$$

Note that $\int_{\tau(\gamma)} \frac{\Omega_n}{G(x)} = 0$ if $d \neq n + 1$, and $\int_{\tau(\gamma)} \frac{\Omega_n}{G(x)} \neq 0$ if $d = n + 1$ which implies that $X_G$ is a Calabi-Yau hypersurface. Note that $\int_{\tau(\gamma)} \frac{\Omega_n}{G(x)}$ in (d) above is a geometrically defined invariant of the formal family of hypersurfaces $X_G$. Recall that $G_k (x) = G(x) + \sum_{a \in I_k} t^a F_1 [a] (x)$ where $F_1 [a] (x)$ are homogenous polynomials of degree $d$ and the image of $\frac{\Omega_n}{G(x)}$ under the residue map represents a holomorphic $(n-1)$-form on $X_G$. On the other hand, the invariants $\int_{\tau(\gamma)} \left( \int_0^\infty \chi^L(t) \cdot e^{-y G(x)} dy \right) \Omega_n$ in (e) contain additional information without a clear geometrical origin. The properties

$$\int_0^\infty \chi^L(t) \cdot e^{-y G(x)} dy \bigg|_{t=0} = \frac{\Omega_n}{G(x)}$$

$$\frac{\partial}{\partial t^a} \int_0^\infty \chi^L(t) \cdot e^{-y G(x)} dy \bigg|_{t=0} = k^a F_k [a] \cdot \Omega_n \frac{\Omega_n}{G(x)^{k+1}}, \quad \forall a \in I_k,$$

(1.6)
for example, imply that Griffiths transversality is violated for $k \geq 2$.

Recall that there is an extended deformation functor $\text{Def}^\mathbb{H}$ attached to any $L_\infty$-algebra that goes from the category $\text{Art}^\mathbb{H}$ of $\mathbb{Z}$-graded commutative local Artinian algebras to the category $\text{Set}$ of sets. The functor $\text{Def}^\mathbb{H}$ assigns to an object $\mathfrak{H}$ of $\text{Art}^\mathbb{H}$ the set of solutions parametrized by the maximal ideal of $\mathfrak{H}$, to the Maurer-Cartan equation (modulo natural gauge equivalence). One can also associate a $\mathbb{Z}$-graded formal moduli space to the $L_\infty$-algebra if the attached functor $\text{Def}^\mathbb{H}$ is pro-representable, for which a sufficient condition is that the $L_\infty$-algebra is quasi-isomorphic to an $L_\infty$-algebra with zero $L_\infty$-structure (See [9]).

Returning to our discussion, the property $(a)$ in Theorem 1.1 implies that the extended deformation functor attached to $(\mathfrak{O}, \ell)_X$ is pro-representable by the completed symmetric algebra $\mathbb{H}^{\hat{\mathfrak{H}}}$ of $\mathbb{H}$. Let $X_{\mathfrak{H}}$ be the associated formal moduli space. Now consider the generating series $\mathcal{Z}_\gamma^\mathfrak{H}((L))$ in (1.5). Note that $L_\infty$-homotopy types of quasi-isomorphisms $\varphi^\mathfrak{H} : (\mathbb{H}, \mathfrak{O}) \longrightarrow (\mathfrak{O}, \ell)_X$ are not unique, though the $L_\infty$-homotopy type of $\varphi^\mathfrak{H}$ is uniquely determined by $[\gamma] \in H_{n-1}(X_{\mathfrak{H}}, \mathbb{Z})$.

**Theorem 1.3.** Let $\{e_a\}_{a \in I}$ be a basis of $\mathbb{H}$ with dual basis $\{t^a\}_{a \in I}$. Then for any $L_\infty$ quasi-isomorphism $\varphi^\mathfrak{H}$ from $(\mathbb{H}, \mathfrak{O})$ to $(\mathfrak{O}, \ell)_X$, we have the following versal solution to the Maurer-Cartan equation of $(\ell, \ell)_X$:

$$
\Gamma((L)) = \sum_{a \in I} t^a \varphi^\mathfrak{H}(e_a) + \sum_{k=2}^\infty \sum_{a_1, \ldots, a_k \in I} t^{a_1} \cdots t^{a_k} \otimes \varphi^\mathfrak{H}_k(e_{a_1}, \ldots, e_{a_k}) \in (C[[L]] \otimes \mathfrak{O})^0
$$

such that $\Gamma((L))$ is gauge equivalent to $\Gamma((\ell))$ if and only if $\varphi^\mathfrak{H}$ is $L_\infty$-homotopic to $\varphi^\mathfrak{H}$, and

$$
\mathcal{Z}_\gamma^\mathfrak{H}((L)) = \varphi^\mathfrak{H} \left( \Gamma((L)) \right) - 1.
$$

Hence the above theorem implies that the generating formal power series $\mathcal{Z}_\gamma^\mathfrak{H}((L))$ gives rise to a function $\mathcal{Z}_\gamma^\mathfrak{H}$ on the moduli space $X_{\mathfrak{H}}$. Note that the dual basis $\{t^a\}_{a \in I}$ is an affine coordinate on $\mathbb{H}$.

**Theorem 1.4.** An $L_\infty$ quasi-isomorphism $\varphi^\mathfrak{H} : (\mathbb{H}, \mathfrak{O}) \longrightarrow (\mathfrak{O}, \ell)_X$ induces a 1-tensor $T(\ell)_{\varphi^\mathfrak{H}} \in C[[\ell]]$ on $X_{\mathfrak{H}}$, which depends only on the $L_\infty$-homotopy type of $\varphi^\mathfrak{H}$, such that the following equations are satisfied:

(a) $\mathcal{Z}_\gamma^\mathfrak{H}((L)) = 1 + \sum_{a \in I} T(\ell)_{\varphi^\mathfrak{H}} C_{\gamma}(e_a)$,

(b) $T(\ell)_{\varphi^\mathfrak{H}} = t^a + \theta(t^2), \quad \forall a \in I$.

From the property $(a)$ in the above, we see that the 1-tensor $T(\ell)_{\varphi^\mathfrak{H}}$ determines the generating series $\mathcal{Z}_\gamma^\mathfrak{H}((L))$ completely if it is combined with the period integral $C_{\gamma} : \mathbb{H} \rightarrow C$. From the
property (b) in the above theorem, \{T^a(\mathcal{T})_x^a\} has the composition inverse \{t^a(\mathcal{T})_x^a\}, which is an affine coordinate system on \mathcal{M}_{X_c}. Hence the \( L_\infty \)-homotopy types of \( L_\infty \)-quasi-isomorphisms give affine coordinate systems on \( \mathcal{M}_{X_c} \). The property (b) in Theorem 1.4 also allows us to define an invertible 2-tensor \( \varphi^a_\beta(\mathcal{T})_x^a \) := \( \partial_a T^\beta(t)^a \) = \( \delta_a^\beta + \mathcal{O}(t) \), where \( \partial_a \) means the partial derivative with respect to \( t^a \), with inverse \( \varphi^{-1}_a(\mathcal{T})_x^a \in \mathbb{C}[\mathcal{T}] \). We can further define a 3-tensor \( [A_\alpha^\beta_\gamma(\mathcal{T})_x^\gamma] \in \mathbb{C}[\mathcal{T}] \) depending only on the \( L_\infty \)-homotopy type of \( \varphi_H \) such that for all \( \alpha, \beta, \sigma \in I \),
\[
A_\alpha^\beta_\gamma(t)_x^\gamma = \sum_{\sigma \in I} \left( \partial_\sigma \varphi^\alpha_\beta(t)_x^\gamma \right) \varphi^{-1}_\sigma(t)_x^\gamma.
\]
(1.7)

It follows as a corollary to Theorem 1.4 that, for all \( \alpha, \beta, \sigma \in I \),
\[
\left( \partial_\alpha \partial_\beta - \sum_{\sigma \in I} A_{\alpha^\beta_\sigma}(t)_x^\sigma \partial_\alpha \right) \mathcal{Z}_{[H]} \left( \left[ \varphi_H \right] \right)(t) = 0,
\]
(1.8)
and, for all \( \alpha, \beta, \sigma, \rho \in I \) the following equations are satisfied:
\[
\partial_\alpha A_{\beta^\rho_\sigma}(t)_x^\sigma - \partial_\beta A_{\alpha^\rho_\sigma}(t)_x^\sigma + \sum_{\nu \in I} A_{\alpha^\nu_\sigma}(t)_x^\nu A_{\beta^\rho_\nu}(t)_x^\nu - \sum_{\nu \in I} A_{\beta^\nu_\sigma}(t)_x^\nu A_{\alpha^\rho_\nu}(t)_x^\nu = 0.
\]
(1.9)

Our approach shall provide an effective algorithm, which can be implemented in a computer algebra system such as SINGULAR, for computing the 3-tensor \( A_{\alpha^\beta_\sigma}(t)_x^\sigma \). Also there is a concrete algorithm to determine the 1-tensor \( [T^a(\mathcal{T})_x^a] \) from the 3-tensor \( \{A_{\alpha^\beta_\sigma}(t)_x^\sigma]\). This means that, for an arbitrary homogeneous polynomial \( f(x) \) of degree \( k d - (n + 1), k \geq 1 \), there is an effective algorithm to compute the period integral \( \int \text{Res} \left( \frac{f(x)}{G_x^f(x) \Omega_n} \right) \) from the finite data \( \{C_{|\gamma}(e_a) \in I \} \). Specializing the generating series \( \mathcal{Z}_{[H]} \left( \left[ \varphi_H \right] \right)(t) \) for any 1-parameter family, by setting \( t^a = 0 \) for all \( a \in I \) except for one parameter \( t^\beta \), the above system of second order partial differential equations (1.8) gives rise to an ordinary differential equation of higher order; this turns out to be the usual Picard-Fuchs equation for the period of 1-parameter family if \( \beta \in I \) in a standard basis of \( H \).

Remark that the properties in (1.9) can be reformulated as an existence of a torsion-free flat connection on the tangent bundle \( T_{\mathcal{M}_{X_c}} \) to the formal moduli space \( \mathcal{M}_{X_c} \) attached to \( X_c \). Recall that a formal Frobenius manifold is a formal manifold whose tangent space has a linear pencil of flat torsion-free connections with an invariant inner product. The following theorem says that \( T_{\mathcal{M}_{X_c}} \) is equipped with a linear pencil of flat torsion-free connections.

**Theorem 1.5.** There is a distinguished \( L_\infty \)-quasi-isomorphism \( \varphi_H^H \), called the quantum descendant, such that the associated 3-tensors \( \{A_{\alpha^\beta_\sigma}(t)_x^\sigma\} \in \mathbb{C}[\mathcal{T}] \) satisfy, for all \( \alpha, \beta, \sigma, \rho \in I \),
\[
A_{\alpha^\beta_\sigma}(t)_x^\sigma = A_{\beta^\rho_\sigma}(t)_x^\rho,
\]
\[
\partial_\alpha A_{\beta^\rho_\sigma}(t)_x^\sigma = \partial_\beta A_{\alpha^\rho_\sigma}(t)_x^\rho,
\]
\[
\sum_{\nu \in I} A_{\alpha^\nu_\sigma}(t)_x^\nu A_{\beta^\rho_\nu}(t)_x^\nu = \sum_{\nu \in I} A_{\beta^\nu_\sigma}(t)_x^\nu A_{\alpha^\rho_\nu}(t)_x^\nu.
\]
(1.10)
We call the affine coordinate system \( \{ t^u (T) \}_{u \in \mathbb{N}} \) defined by the 1-tensor \( \{ T^u (T) \}_{u \in \mathbb{N}} \in \mathbb{C}[\mathbb{L}] \) the quantum coordinate system on \( \mathcal{M}_{X_0} \).

We turn to the general framework behind Theorems. In the introduction, we will briefly indicate how we approach their proofs. We associate to \( X_G \) a representation of a finite dimensional abelian Lie algebra \( g \) of dimension \( n + 2 \) on a polynomial algebra with \( n + 2 \) variables;

\[
\rho_X : g \to \text{End}_k(A), \quad A := k[y, x_0, \ldots, x_n] = k[y, \underline{x}].
\]

This Lie algebra representation comes from the Schrödinger representation of the abelian Lie subalgebra of the Heisenberg Lie algebra of dimension \( 2(n + 2) + 1 \) twisted by the Dwork polynomial \( y \cdot G(\underline{x}) \in k[y, x_0, \ldots, x_n] \) of \( G(\underline{x}) \). Let \( \alpha_{-1}, \alpha_0, \ldots, \alpha_n \) be a \( k \)-basis of \( g_1 \). We also introduce variables \( y_{-1} = y, y_0 = x_0, y_1 = x_1, \ldots, y_n = x_n \) for notational convenience. If we consider the formal operators (twisting \( \rho \) by \( y \cdot G(\underline{x}) \)),

\[
\rho_X(\alpha_i) := \exp(-y \cdot G(\underline{x})) \cdot \frac{\partial}{\partial y_i} \cdot \exp(y \cdot G(\underline{x})), \quad i = -1, 0, \ldots, n,
\]

where \( y \cdot G(\underline{x}) \in A \), then we can see that

\[
\rho_X(\alpha_i) = \frac{\partial}{\partial y_i} + \left[ \frac{\partial}{\partial y_i}, y \cdot G(\underline{x}) \right] + \frac{1}{2} \left[ \left[ \frac{\partial}{\partial y_i}, y \cdot G(\underline{x}) \right], y \cdot G(\underline{x}) \right] + \cdots = \underbrace{\partial y \cdot G(\underline{x})}_{\partial y_i} + \frac{\partial}{\partial y_i}.
\]

With the notion of period integrals of Lie algebra representations, we will show that the period integrals of such representation are the Griffiths’ period integrals of the hypersurface \( X_G \). Then if we use a degree-twisted version of the dual Chevalley-Eilenberg complex \( (\mathcal{A}, K_X) \), which computes the Lie algebra homology associated to \( \rho_X \), then we can realize \( C_\mathbb{C}[y] \) as the homotopy type of a cochain map \( \mathcal{C}_r : (\mathcal{A}, K_X) \to (\mathbb{C}, 0) \) of cochain complexes which are equipped with a super-commutative product.

This leads us to study the the category \( \mathcal{C}_k \) of cochain complexes over a field \( k \) equipped with a super-commutative product. An object of \( \mathcal{C}_k \) is a unital \( \mathbb{Z} \)-graded associative and super-commutative \( k \)-algebra \( \mathcal{A} \) with differential \( K \), denoted \( (\mathcal{A}, K) \). A morphism in \( \mathcal{C}_k \) is a cochain map (note that a morphism is not necessarily a ring homomorphism). We will prove all the main theorems by studying this category systematically.

- The basic principle here is that all Theorems in this article can be derived systematically from a pair \((\mathcal{A}, K) \in \text{Ob}(\mathcal{C}_k)\) and \( \mathcal{C} : (\mathcal{A}, K) \to (k, 0) \in \text{Mor}(\mathcal{C}_k) \).

We sometimes call such a pair a homotopy probability space. This category \( \mathcal{C}_k \) is studied in the context of homotopy probability theory by the first named author, [12]. The failure of derivation of the differential \( K \) of the product - is related to independence in probability theory and the differential \( K \) is related to homotopy theory. But here we will not touch any issues related to probability theory. Instead we will provide a self-contained argument and proofs regarding \( \mathcal{C}_k \).

The category can be seen as a bridge between period integrals of \( X_G \) and \( L_{\infty} \)-homotopy theory. The relationship between the hypersurface, its period integral, and \( \mathcal{C}_k \) is made by the representation
\( \rho_X \) and the cochain level realization \( 'C_\gamma \) and the relationship between \( C_{\Delta} \) and \( L_\infty \)-homotopy theory will be given by the descendant functor. The schematic picture of our theory is as follows:

The category \( C_{\Delta} \)

Descendant functor

Period integrals of Lie algebra representations

Period integrals of hypersurfaces.

Homotopy Lie algebra theory

The following result is the cornerstone to prove Theorem 1.1; we refer to Propositions 2.3 and 4.4, and Theorem 4.1 for its proof.

**Theorem 1.6.** (a) Let \( X_G \) be a projective smooth hypersurface defined by a homogeneous polynomial \( G(x) \) of degree \( d \). There is a Lie algebra representation \( \rho_X \) attached to \( X_G \) and a cochain complex \( (\cal A, \cdot, K) = (\cal A_X, \cdot, K_G) \) associated to \( \rho_X \) such that

\[ \bigoplus_{m \in \mathbb{Z}} H^m(\cal A) \simeq H^0_K(\cal A) \oplus H^{-1}_K(\cal A), \]

- \( \cal A \) is a \( \mathbb{Z} \)-graded vector space with \( \cal A^0 = A \) which has an associative and super-commutative binary product \( \cdot \),
- its cohomology groups are as follows:

\[ \dim_C H^{-1}_K(\cal A) = \begin{cases} 1 & \text{if } d = n + 1, \\ 0 & \text{if } d \neq n + 1. \end{cases} \]

(b) There exists a cochain map \( \cal C_\gamma : (\cal A, K) \to (\mathbb{Z}, 0) \) such that the induced map of \( \cal A_\gamma \) on \( H^0_K(\cal A) \simeq \cal H^n \) is the same as \( C_{[\gamma]} \). In particular, \( \cal C_\gamma \) is a morphism in \( C_{\Delta} \).

The relationship between the category \( C_{\Delta} \) and homotopy Lie algebra theory is given by the descendant functor \( \cal D_{\Delta} \). The descendant functor is a homotopy functor from the category \( C_{\Delta} \) to the category \( \mathcal{L} \) of \( L_\infty \)-algebras (we include Appendix 5.2 explaining notation for the homotopy category of unital \( L_\infty \)-algebras suitable for our purpose), which is defined by using the binary product \( \cdot \) of an object of \( C_{\Delta} \). See Definition 3.1 and Theorem 3.1 for details. This functor can be regarded as an organizing principle (or tool) to understand the correlations among \( \cal C_\gamma(x_1), \cal C_\gamma(x_1 \cdot x_2), \ldots, \cal C_\gamma(x_1 \cdot \ldots \cdot x_m) \), where \( x_1, \ldots, x_m \) are homogeneous elements in \( \cal A_X \) and \( m \geq 1 \). This functor unifies two different failures of compatibility of algebraic structures into one language; we show that measuring how much \( \cdot \) fails to be a derivation of \( K \) induces an \( L_\infty \)-algebra structure on \( \cal A \).
denoted $(\mathcal{A}, \ell^K)$, and measuring how much $\gamma$ fails to be a $k$-algebra homomorphism induces an $L_\infty$-morphism from $(\mathcal{A}, \ell^K)$ to $(\mathcal{A}', \ell'^K)$, denoted $\phi_\gamma^{\mathcal{A}}$. Note that the descendant functor is independent of hypersurfaces and their period integrals and is a general notion which measures incompatibilities of mathematical structures of the category $\mathcal{C}_k$.

In the particular case of the projective smooth hypersurface $X_G$ and $k = \mathbb{C}$, the relevant $L_\infty$-algebra turns out to be a differential graded Lie algebra $(\mathcal{A}_X, \ell^K_X) = (\mathcal{A}_X, K_X, \ell^K_X)$ and the descendant of $\gamma : (\mathcal{A}_X, K_X) \to (k, 0)$ gives an $L_\infty$-morphism $\phi_\gamma^{\mathcal{A}}$ from $(\mathcal{A}_X, K_X, \ell^K_X)$ into $(k, \mathbb{Q})$ (note that the descendant $L_\infty$-algebra of $(k, 0)$ is just $(k, \mathbb{Q})$).

The novel feature here is that we are able to put an associative and super-commutative binary product $\cdot$ on the cochain complex $(\mathcal{A}, K) = (\mathcal{A}_X, K_X)$ which turns out to govern correlations and variations (hidden structures we alluded to) of the period integral $C[\gamma]$. Since $\mathcal{H}(X_G)$ is defined as the cohomology of the de Rham complex with the wedge product, one might think to play a similar game to define hidden correlations. But if one wedges two $n$-forms then the resulting differential form is a $2n$-form which can not be integrated against a fixed cycle $\gamma$.

Once we get an $L_\infty$-algebra $(\mathcal{A}, \ell^K) = (\mathcal{A}_X, \ell^K_X)$, we can study a formal deformation functor attached to it. This deformation turns out include the classical geometric deformation and have new directions which violate Griffiths transversality. A careful analysis of $(\mathcal{A}, \ell^K)$ and the descendant $L_\infty$-morphism $\phi^{\mathcal{A}}_\gamma$ will enable us to prove Theorem 1.2; see Subsection 4.6 for its proof. The other theorems 1.3, 1.4, and 1.5 can be derived by applying the general theory on the category $\mathcal{C}_k$ to the particular example $\gamma$ provided by Theorem 1.6.

As an application of our new approach to understanding $C[\gamma]$, we were able to prove the existence of cochain level realization of the Hodge filtration (see Proposition 4.8), which is not available in the de Rham complex $(\Omega(X_G), -, d)$. This could provide a new optic to understand the (infinitesimal) variation of polarized Hodge structures at the level of cochains and we will pursue this issue in another paper.

Now we explain the contents of each section of the paper. The paper consists of 3 main sections and the appendix. In the first main section, Section 2, we explain the general theory of period integrals associated to a Lie algebra representation. In Subsection 2.1, we define the notion of period integrals of a Lie algebra representation $\rho$ (see Definition 2.1). Then, in Subsection 2.2, we explain how to construct a cochain complex associated to $\rho$ which is dual to the Chevalley-Eilenberg complex, and a way of associating a morphism into $(k, 0)$ in the category $\mathcal{C}_k$ of cochain complexes with super-commutative product to its period integral. Then we illustrate by an example why the dual Chevalley-Eilenberg complex is crucial and more suitable to understand the period integral of $\rho$ than the cohomology Chevalley-Eilenberg complex attached to $\rho$ in Subsection 2.3.

The second main section, Section 3 is about the general theory of the category $\mathcal{C}_k$. The key concepts are the descendant functor, generating power series, and flat connections. In Subsection 3.1, we explain the basic philosophy of the descendant functor. In Subsection 3.2, we provide a way to understand the category $\mathcal{C}_k$ in terms of $L_\infty$-homotopy theory; we construct the homotopy descendant functor from the category $\mathcal{C}_k$ to the category $\mathcal{L}$ of $L_\infty$-algebras. Then we show that a descen-
dant $L_{\infty}$-algebra is smooth-formal in Subsection 3.3. In Subsection 3.4, we attach a deformation problem to the descendant $L_{\infty}$-algebra of $(\mathcal{A}, \cdot, K)$ and explain what we deform. In Subsection 3.5, we define a notion of the generating power series, which organizes various correlations and deformations of period integrals into one power series in the deformation parameters, and show that they are $L_{\infty}$-homotopy invariants. Then we verify the generating power series attached to a versal formal deformation satisfies a system of partial differential equations (Theorem 3.3) with respect to derivatives of deformation parameters and show the coefficients appearing in the differential equations are homotopy invariants, in Subsection 3.6. In Subsection 3.7, we provide a way to compute the generating power series explicitly, together with the proof of Theorem 1.4. Finally, in Subsection 3.8, this system of partial differential equation is interpreted as the existence of a $\mathbb{Z}$-graded flat connection on a formal deformation space attached to $(\mathcal{A}, \cdot, K)$. Section 2 can be regarded as a general way to provide examples of objects along with morphisms to the initial object (the period integrals of Lie algebra representations) in the category $\mathcal{C}_k$.

In the third main section, Section 4, we apply all the general machinery of the previous sections to reveal hidden structures on the Griffiths period integrals of smooth projective hypersurfaces. Section 4 can be viewed as a source of explicit examples of non-trivial period integrals of certain Lie algebra representations and consequently gives non-trivial examples of objects and morphisms into the initial object in $\mathcal{C}_k$. In Subsection 4.1, we apply the general theory to the toy model to illustrate our homotopical viewpoint of understanding the Griffiths period integral. In Subsection 4.2, we explain how to attach a Lie algebra representation to a projective smooth hypersurface. In Subsection 4.3, we briefly recall Griffiths’ theory of period integrals of smooth projective hypersurfaces and construct a non-trivial period integral of its associated Lie algebra representation. Then, in Subsection 4.4, we explicitly describe the cochain complex $(\mathcal{A}, \cdot, K) = (\mathcal{A}_X, \cdot, K_X)$ with super-commutative product attached to the hypersurface $X_G$ and compute its $K$-cohomology $H_K(\mathcal{A})$ of $\mathcal{A}$ to describe its precise relation to the cohomology $H^0(X_G) \cong H$; this is Theorem 1.5. In Subsection 4.5, we prove Theorem 1.1. We verify Theorems 1.2 and 1.3 in Subsection 4.6. In Subsection 4.7, we explain how to compute variations of the Griffiths period integrals. The Subsection 4.8 is devoted to to the proof of Theorem 1.5, i.e. an explicit construction of a special quantum flat connection on the tangent bundle of a formal deformation space attached to $X_G$. Finally, in Subsection 4.9, we show that there is a cochain level realization of the Hodge filtration by putting a certain (pole order) filtration on $(\mathcal{A}, \cdot, K)$.

The main idea of this paper originated from the first named author’s work on the algebraic formalism of quantum field theory, [11]. Thus, in the appendix, Section 5, we decided to add an explanation of the quantum origin of the Lie algebra representation attached to a given hypersurface $X_G$ (Subsection 5.1). Finally, we include Subsection 5.2 on $L_{\infty}$-algebras, $L_{\infty}$-morphisms, and $L_{\infty}$-homotopies in order to explain the notation and conventions used throughout the paper.

Before finishing the introduction, we mention two things. Firstly, our theory can be generalized to complete intersections from hypersurfaces and conjecturally to any algebraic varieties. It would be a good project to carry out the details and the relevant references which play a similar role as [6] would be [1], [2], [4], and [15]. Secondly, a more important comment is that the category $\mathcal{C}_k$ seems more fundamental than the category of $L_{\infty}$-algebras; in our point of view, the $L_{\infty}$-homotopy data
appearing here are just shadow information inherited from objects and morphisms in the category $\mathcal{C}_{/kbb}$. It may seem artificial to study $\mathcal{C}_{/kbb}$ at a first glance; we study the category $\mathcal{C}_{/kbb}$ whose objects are $(\mathfrak{g}/Acal, 1_{\mathfrak{g}/Acal}, \cdot, K)$, where we do not require compatibility between the super-commutative product $\cdot$ and the differential $K$, and whose morphisms are not structure preserving maps in the sense that they preserve only additive and differential structures (i.e. they are cochain maps) not super-commutative ring structure (note a difference between the category $\mathcal{C}_{/kbb}$ and the category of CDGAs, i.e. commutative differential graded algebras, where all the structures are compatible). But this category $\mathcal{C}_{/kbb}$ is worth investigating and studying; the Griffiths period integral of the hypersurface $X_G$ can be interpreted (very neatly) as a morphism from $(\mathfrak{g}/Acal_{X}, \cdot, K_{X})$ to the initial object $(kbb, \cdot, 0)$ in the category $\mathcal{C}_{/kbb}$ and the shadow $L_\infty$-homotopy information obtained by applying the descendant functor measures a failure of compatibilities among structures and reveals hidden structures on the period integral $C_{[\gamma]}$ such as its correlations and (a new type of) formal deformation with $L_\infty$-homotopy invariance and differential equations.

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2. Lie algebra representations and period integrals

2.1. Period integrals of Lie algebra representations

Let $kbb$ be a field of characteristic 0 and $g$ be a finite dimensional Lie algebra over $kbb$. Let $\rho : g \to \text{End}_k(A)$ be a $kbb$-linear representation of $g$. We assume that $A$ is a commutative associative $kbb$-algebra (with unity) throughout the paper.

**Definition 2.1.** We call a $kbb$-linear map $C : A \to kbb$ a period integral\(^2\) attached to $\rho$ if $C(x) = 0$ for every $x$ in the image of $\rho(g)$ for every $g \in g$.

Note that such a map $C$ becomes zero if $A$ is an irreducible $g$-module. For a given Lie algebra representation $\rho$, it would be an interesting question to find non-trivial period integrals. Here we present a simple non-trivial example.

**Example 2.1.** Let $g$ be a one-dimensional Lie algebra $kbb = \mathbb{R}$ generated by $e$. Let $\rho$ be a Lie algebra representation on $A = kbb[x]$ given by $\rho(e) = \frac{\partial}{\partial x} - x \in \text{End}_k(A)$. Then we consider the Gaussian probability measure $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ and define a $kbb$-linear map

\[
C : kbb[x] \to kbb,
\]

\[
f(x) \mapsto C(f(x)) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\frac{x^2}{2}} dx.
\]

\(^2\) We use this terminology in a different sense than arithmetic geometers (a comparison of rational structures of relevant cohomology groups); we simply choose this terminology since the period integrals of smooth hypersurfaces can be understood as an example.
This is an example of a period integral of $\rho$, since
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \frac{\partial f(x)}{\partial x} - x f(x) \right) e^{-x^2} dx = 0.
\]

The above Gaussian period integral is a special example of a more general kind. Let $A = k[q] = k[q^1, q^2, \cdots, q^m]$ be a polynomial ring with $m$ variables for $m \geq 1$. Let $S = S(q) \in A$. Let $J_S \subset A$ be the Jacobian ideal of $S(q)$, i.e. the ideal generated by $\frac{\partial S(q)}{\partial q^1}, \frac{\partial S(q)}{\partial q^2}, \cdots, \frac{\partial S(q)}{\partial q^m}$. Let $g = g_S$ be the finite dimensional abelian Lie algebra over $k$ of dimension $m$, generated by $a_1, a_2, \cdots, a_m$. We define the following Lie algebra representation $\rho_S: g \to \text{End}_k(A)$:

\[
\rho_S(a_i) = \exp \left( -S(q) \right) \cdot \frac{\partial}{\partial q} \cdot \exp \left( S(q) \right) = \frac{\partial}{\partial q} + \frac{\partial S(q)}{\partial q}, \quad i = 1, 2, \cdots, m. \tag{2.2}
\]

We remark that this representation $\rho_S$ is obtained by twisting the Schrödinger representation of (a certain abelian Lie subalgebra) of the Heisenberg Lie algebra by the polynomial $S(q)$; see Subsection 5.2 for details. This motivates us to call $\rho_S$ as the quantum Jacobian Lie algebra representation associated to $S(q) \in A$. It turns out that there are many interesting non-trivial examples of period integrals of $\rho_S$.

**Example 2.2.** We give an example for which $m = 1$, which generalizes the previous Gaussian example. Let $S(x) \in \mathbb{R}[x] = A$ be a polynomial such that
\[
\lim_{x \to -\infty} f(x) e^{S(x)} = \lim_{x \to \infty} f(x) e^{S(x)} = 0
\]
for every $f(x) \in \mathbb{R}[x]$. Let $g$ be a one-dimensional Lie algebra generated by $e$. Then the $\mathbb{R}$-linear map
\[
C : \mathbb{R}[x] \to \mathbb{R},
\]
\[
f(x) \mapsto C(f(x)) := \int_{-\infty}^{\infty} f(x) e^{S(x)} dx,
\tag{2.3}
\]
is an example of a period integral of $\rho_S$, since
\[
C \left( \rho_{G(x)}(e)(f(x)) \right) = \int_{-\infty}^{\infty} \left( \frac{\partial f(x)}{\partial x} + \frac{\partial S(x)}{\partial x} f(x) \right) e^{S(x)} dx = 0, \quad \forall f(x) \in \mathbb{R}[x].
\]

Such a period integral $C$ attached to $\rho = \rho_S$ gives rise to a map $C : A/N_\rho \to k$, where $N_\rho := \sum_{a \in g} \text{im} \rho(a)$. Note that, in general, $C$ fails to be an algebra homomorphism and $N_\rho$ fails to be an ideal of $A$. This failure will play a pivotal role in studying the period integral $C$ via $L_\infty$-homotopy theory.

Our main example, which we will focus on in section 4, is when the Lie algebra representation $\rho_S$ is constructed out of $S = y \cdot G(x)$, the so called Dwork polynomial of $G(x)$, where $G(x)$ is the defining
equation of a smooth projective hypersurface. Then the rational period integral of a smooth projective hypersurface $X_G$, which was extensively studied by Griffiths and Dwork, can be interpreted as the period integral of $\rho_S$ (see Subsection 4.3).

We remark that studying a non-trivial period integral of a Lie algebra representation of a non-abelian Lie algebra would be a very interesting question, though we limit all the examples to the abelian case in this article.

2.2. Cochain map attached to a period integral

We now explain our strategy to study period integrals, assuming such a nonzero period integral is given. The main idea is to enhance the $\mathbf{k}$-linear map $C : A \rightarrow \mathbf{k}$ to the cochain complex level (we do this in this Subsection) and develop an infinity homotopy theory (see section 3) by analyzing the failure of $C$ being an algebra homomorphism systematically. In this paper, the relevant homotopy theory will be the $L_\infty$-homotopy theory (this fact is related with the assumption that $A$ is a commutative $\mathbf{k}$-algebra).

Let $C : A \rightarrow \mathbf{k}$ be a nontrivial period integral attached to $\rho : g \rightarrow \text{End}_k(A)$. We will construct a cochain complex $(\mathcal{A}, \cdot, K) = (\mathcal{A}_\rho, \cdot, K_\rho)$ with super-commutative product $\cdot$ whose degree 0 part is $A$ and a cochain map $\mathcal{C} : (\mathcal{A}, K) \rightarrow (\mathbf{k}, 0)$, where we view $(\mathbf{k}, 0)$ as a cochain complex which has only degree 0 part and zero differential.

Let $\alpha_1, \cdots, \alpha_n$ be a $\mathbf{k}$-basis of $g$ where $n$ is the dimension of $g$. We consider $g$ as a $\mathbb{Z}$-graded $\mathbf{k}$-vector space with only degree 0 part. Then $g[1]$ is a $\mathbb{Z}$-graded $\mathbf{k}$-vector space with only degree -1 part. Let $\eta_1, \cdots, \eta_n$ be the $\mathbf{k}$-basis of $g[1]$ corresponding to $\alpha_1, \cdots, \alpha_n$ so that they have degree -1. We consider the following $\mathbb{Z}$-graded supersymmetric algebra

$$S(g[1]) = T(g[1])/J$$

where $T(g[1])$ is the tensor algebra of $g[1]$ and $J$ is the ideal of $T(g[1])$ generated by the elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$ with $x, y \in T(g[1])$. Note that $|x|$ here means the degree of $x$. We can also view $A$ as a $\mathbb{Z}$-graded $\mathbf{k}$-algebra concentrated on degree zero part. Then we define the $\mathbb{Z}$-graded $\mathbf{k}$-algebra $\mathcal{A}_\rho$, as the supersymmetric tensor product of $A$ and $S(g[1])$.

**Proposition 2.1.** The $\mathbf{k}$-algebra $\mathcal{A} = \mathcal{A}_\rho$ is a $\mathbb{Z}$-graded super commutative algebra and we have a decomposition of $\mathcal{A}$ into

$$\mathcal{A}^m = \mathcal{A}^{-n} \oplus \mathcal{A}^{-(n-1)} \oplus \cdots \oplus \mathcal{A}^{-1} \oplus \mathcal{A}^0$$

where $\mathcal{A}^m$ is the $\mathbf{k}$-subspace of $\mathcal{A}$ consisting of degree $m$ elements, with $\mathcal{A}^0 = A$.

**Proof.** The fact that $\mathcal{A}$ is super commutative $(xy = (-1)^{|x||y|} yx$ for every $x, y \in \mathcal{A}$) follows from the construction. It is clear that $\eta^2_i = 0$ for $i = 1, \cdots, n$, since $2\eta^2_i = 0$ (recall that $\eta_1, \cdots, \eta_n$ is a $\mathbf{k}$-basis of $g[1]$ whose degree is -1) and the characteristic of $\mathbf{k}$ is not 2. Therefore the smallest degree which the elements of $\mathcal{A}$ can have is $-n$ (for example, $\eta_1 \cdots \eta_2 \cdots \eta_n$ has degree $-n$). After these observations, the rest is obvious to check. □
Now we construct a differential $K = K_{\rho}$ coming from the Lie algebra representation $\rho$:

$$K_{\rho} : \mathfrak{g}^m \to \mathfrak{g}^{m+1},$$

where $m \in \mathbb{Z}$. Define

$$K_{\rho} = \sum_{i=1}^{n} \rho_{a_i} \otimes \frac{\partial}{\partial \eta_i} - 1 \otimes \sum_{i,j,k=1}^{n} \frac{1}{2} f_{ij}^k \eta_k \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \eta_j} : \mathfrak{g}^m \to \mathfrak{g}^{m+1}, \quad (2.4)$$

where $\{ f_{ij}^k \} \in \mathbb{k}$ are the structure constants of the Lie algebra $\mathfrak{g}$ defined by the relation $[\alpha_i, \alpha_j] = \sum_{k=1}^{n} f_{ij}^k \alpha_k$ and $\rho_{a_i} = \rho(a_i)$. Note that $\rho_{a_i}$ only acts on the degree zero part $A = \mathfrak{g}^0$ via the representation $\rho$ and the partial derivative operator $\frac{\partial}{\partial \eta_i}$ increases degree 1, since $\eta_i$ has degree -1. We shall often omit the tensor product symbol in the expression of $K_{\rho}$. Next, we show that $K_{\rho}$ is actually a differential of $\mathfrak{g}^\rho$.

**Proposition 2.2.** We have that $K_{\rho}^2 = K_{\rho} 1_{\mathfrak{g}^\rho} = 0$.

**Proof.** This follows from that $\rho : \mathfrak{g} \to \text{End}_\mathbb{k}(A)$ is a Lie algebra representation. \(\square\)

**Example** We extend our period map $C : A \to \mathbb{k}$ to $\mathcal{C} : \mathfrak{g}^\rho \to \mathbb{k}$ by setting $\mathcal{C}(x) = C(x)$ if $x \in \mathfrak{g}^0 = A$ and $\mathcal{C}(x) = 0$ otherwise.

**Proposition 2.3.** We can extend any period map $C$ attached to $\rho$ to a cochain map $\mathcal{C}$ from $(\mathfrak{g}^\rho, K_{\rho})$ to $(\mathbb{k}, 0)$, i.e. $\mathcal{C} \circ K_{\rho} = 0$.

**Proof.** We only have to check $\mathcal{C}(K_{\rho}(x)) = 0$ when $K_{\rho}(x) \in A$. Let us write the general element $x$ in $\mathfrak{g}^{\rho - 1}$ as

$$x = \sum_{i=1}^{n} F_i \cdot \eta_i, \text{ where } F_i \in \mathfrak{g}^0_{\rho}.$$  

Then $K_{\rho}(x) = \sum_{i=1}^{n} \rho(a_i)(F_i) \in A$. By Definition of the period integral attached to $\rho$, we immediately see that $\mathcal{C}(K_{\rho}(x)) = 0$ for any $x \in \mathfrak{g}^{\rho - 1}$. \(\square\)

**Example 2.3.** We illustrate the above construction for Example 2.2. In this case, the cochain complex $\mathfrak{g}^\rho$ with super-commutative product, associated to $\rho = \rho_{G(x)}$, is given by

$$\mathfrak{g}^\rho = \mathbb{k}[x][\eta], \quad \eta^2 = 0, \quad \eta x = x \eta,$$

where $\eta$ is an element of degree -1 (the so called ghost component). Then $\mathfrak{g}^m_{\rho} = 0$ unless $m = 0, -1$. The differential $K_{\rho}$ is given by

$$K_{\rho} = \rho(e) \frac{\partial}{\partial \eta} = \left( \frac{\partial}{\partial x} + \frac{\partial G(x)}{\partial x} \right) \frac{\partial}{\partial \eta}.$$
The period integral $C$ in (2.3) can be enhanced to a cochain map $\mathcal{C} : (\mathcal{A}, r, K_\rho) \to (k, 0)$ by Proposition 2.3. Then $\mathcal{C}$ induces the map $\overline{\mathcal{C}} : H_k(\mathcal{A}, \rho) \to \mathbb{R}$, where $H_k(\mathcal{A}, \rho) = \bigoplus_{i \leq 0} H_k^i(\mathcal{A}, \rho)$ with $i$-th degree cohomology $H_k^i(\mathcal{A}, \rho)$ of $(\mathcal{A}, K_\rho)$. Then it turns out that $H_k(\mathcal{A}, \rho) = H^0(\mathcal{A}, \rho)$ and is a finite dimensional $\mathbb{R}$-vector space of dimension equal to the degree of $G(x)$. In section 3, we will provide a general machinery to understand important properties (correlations and deformations) of such an enhanced period integral $\mathcal{C}$ by using the $L_\infty$-homotopy theory. We refer to Subsection 4.1 for a detailed analysis of Example 2.3 via the $L_\infty$-homotopy theory.

When we study the period integral of a Lie algebra representation $\rho$ of $\mathfrak{g}$, we are particularly interested in the case where $\rho(a)$ is a differential operator of order $n$ for $a \in \mathfrak{g}$. Recall Definition of Grothendieck:

**Definition 2.2.** Let $\mathcal{A}$ be a $\mathbb{Z}$-graded $k$-vector space. Let $\pi \in \text{End}_k(\mathcal{A})$. We call $\pi$ a differential operator of order $n$, if $n$ is the smallest positive integer such that $\ell_n \neq 0$ and $\ell_{n+1} = 0$, where

$$\ell_n(x_1, x_2, \cdots, x_n) = [\cdots [[\pi, L_{x_1}], L_{x_2}], \cdots], L_{x_n}]_{(1, \mathcal{A})},$$

for $x_1, \cdots, x_n \in \mathcal{A}$. Here $L_x : \mathcal{A} \to \mathcal{A}$ is left multiplication by $x$ and the commutator $[L, L'] := L \cdot L' - (-1)^{|L||L'|} L' \cdot L \in \text{End}_k(\mathcal{A})$ and $1_{\mathcal{A}}$ is the identity element of $\mathcal{A}$.

If the Lie algebra $\mathfrak{g}$ is non-abelian, then $K_\rho$ for its arbitrary Lie algebra representation $\rho$ has order at least 2, because of the term $\frac{\delta}{\epsilon_{\mathfrak{g}}} \cdot \frac{d}{\epsilon_{\mathfrak{g}}} \frac{d}{\epsilon_{\mathfrak{g}}}$ in (2.4). In general, $K_\rho$ for any Lie algebra representation $\rho$ has order at least the order of $\rho(a_i) + 1$ for any $a_i \in \mathfrak{g}$, because of the term $\rho(a_i)_{\epsilon_{a_i}}$ in (2.4).

### 2.3. The origin of the cochain complex associated to $\rho$

In the previous Subsection 2.2, we constructed a cochain complex $(\mathcal{A}, K_\rho)$ associated to $\rho$ and explained how to enhance the period integral $C$ of $\rho$ to a cochain map $\mathcal{C}$. This can be seen as a degree-twisted cochain complex of the (homology version of) the Chevalley-Eilenberg complex in [3]; see Proposition 2.4. For a given Lie algebra representation $\rho$, there are two kinds of standard complexes, the cochain complex for the Lie algebra cohomology $H^k(\mathfrak{g}, A)$ and the chain complex for the Lie algebra homology $H_k(\mathfrak{g}, A)$. It will be crucial to use the Lie algebra homology complex instead of the cohomology complex for our analysis of period integrals, for which we will explain the reason.

We briefly recall the Chevalley-Eilenberg complex for cohomology. We define a $\mathbb{Z}$-graded vector space

$$C(\mathfrak{g}; \rho) = \bigoplus_{p \geq 0} C^p(\mathfrak{g}; \rho), \text{ where } C^p(\mathfrak{g}; \rho) := A \otimes_k A^p \mathfrak{g}^*.$$

If $\beta \in \mathfrak{g}^* = \text{Hom}_k(\mathfrak{g}, k)$, define $\epsilon(\beta) : A^p \mathfrak{g}^* \to A^{p+1} \mathfrak{g}^*$, for any integer $p \geq 0$, by wedging with $\beta$:

$$\epsilon(\beta) \omega = \beta \wedge \omega.$$
Similarly, if $X \in \mathfrak{g}$, then define $\iota(X) : \Lambda^p \mathfrak{g}^* \to \Lambda^{p-1} \mathfrak{g}^*$, for any integer $p \geq 1$, by contracting with $X$:

$$\iota(X)\beta = \beta(X), \quad \text{for } \beta \in \mathfrak{g}^*$$

and extending it as an odd derivation

$$\iota(X)(\alpha \wedge \beta) = \iota(X)\alpha \wedge \beta + (-1)^{(|\alpha|)}\alpha \wedge \iota(X)\beta$$

to the exterior algebra $\Lambda^* \mathfrak{g}^*$ of $\mathfrak{g}^*$. Here $\alpha \in \Lambda^p \mathfrak{g}^*$ if and only if $|\alpha| = p$. Notice that $\epsilon(\alpha)(X) + \iota(X)\epsilon(\alpha) = \alpha(X)\id$. If we let $\{\alpha_i\}$ and $\{\beta^j\}$ be canonically dual bases for $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively, then it is well-known that the Chevalley-Eilenberg differential on $C(\mathfrak{g}; \rho)$ can be written as

$$d_{\rho} = d = \sum_i \rho(\alpha_i) \otimes \epsilon(\beta^i) - \text{id} \otimes \frac{1}{2} \sum_{i,j} \tilde{f}^i_{ij} \cdot \iota(\alpha_k) \epsilon(\beta^i) \epsilon(\beta^j) : C^p(\mathfrak{g}; \rho) \to C^{p+1}(\mathfrak{g}; \rho). \quad (2.5)$$

This construction gives the cochain complex $(C(\mathfrak{g}; \rho), d_{\rho})$, called the Chevalley-Eilenberg complex attached to $\rho$, which computes the Lie algebra cohomology of $\rho$. If you compare $d_{\rho}$ with $K_\rho$, then the wedging operator $\epsilon(\beta^i)$ corresponds to $\frac{2}{\beta^i}$ and the contracting operator $\iota(\alpha_k)$ corresponds to multiplication by $\eta_k$. Note that the Chevalley-Eilenberg complex is obtained by adding degree 1 elements $\{\beta^i\}$ to $A$ and the cochain complex $(A_\rho, K_\rho)$ is obtained by adding degree -1 elements $\{\eta_i\}$ to $A$. Thus $C(\mathfrak{g}; \rho)$ has no negative degree components and $A_\rho$ has no positive degree components. This duality between $K_\rho$ and $d_{\rho}$ leads us to prove that $(A_\rho, K_\rho)$ is, in fact, a degree-twisted version of the Lie algebra homology Chevalley-Eilenberg complex. In proving such a result, we also briefly recall the (dual) Chevalley-Eilenberg complex which computes the Lie algebra homology; we consider a $\mathbb{Z}$-graded vector space

$$E(\mathfrak{g}; \rho) = \bigoplus_{p \geq 0} E_p(\mathfrak{g}; \rho), \quad \text{where } E_p(\mathfrak{g}; \rho) := A \otimes_k \Lambda^p \mathfrak{g},$$

and equip it with the differential $\delta_{\rho}$ defined by

$$\delta_{\rho} (a \otimes (x_1 \wedge \cdots \wedge x_n)) = \sum_{i=1}^n (-1)^{i+1} \rho(x_i)(a) \otimes (x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n)$$

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} a \otimes ([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n). \quad (2.6)$$

Then we twist\footnote{The degree twisting is needed for consistency with degree convention of the $L_\infty$-morphisms which we will consider in Section 3; we want our $L_\infty$-morphisms have degree 1 instead of -1.} the degree of the chain complex $E(\mathfrak{g}; \rho)$ in order to make a cochain complex $\tilde{E}(\mathfrak{g}; \rho) = \bigoplus_{p \leq 0} \tilde{E}_p(\mathfrak{g}; \rho)$;

$$\tilde{E}_p(\mathfrak{g}; \rho) := E_{-p}(\mathfrak{g}; \rho), \quad p \leq 0.$$  

Then $\tilde{E}(\mathfrak{g}; \rho)$ has only negative degrees up to the $k$-dimension of $\mathfrak{g}$ and becomes a cochain complex.

**Proposition 2.4.** The cochain complex $(A_\rho, K_\rho)$ is isomorphic to the cochain complex $(\tilde{E}(\mathfrak{g}; \rho), \delta_{\rho})$. 

\[\text{Proposition 2.4.} \quad \text{The cochain complex } (A_\rho, K_\rho) \text{ is isomorphic to the cochain complex } (\tilde{E}(\mathfrak{g}; \rho), \delta_{\rho}).\]
Proof. If we denote a \( k \)-basis of \( g \) by \( \alpha_1, \cdots, \alpha_n \), the \( k \)-linear map sending \( \alpha_i \) to \( \eta_i \), for each \( i = 1, \cdots, n \), clearly defines an \( A \)-module isomorphism (a \( k \)-vector space isomorphism, in particular). The commutativity with differentials follows from a direct comparison between (2.4) and (2.6). □

Since we assume that \( A \) has a commutative associative product in addition to the module structure, this induces a natural super-commutative product on \( E(g; \rho) \), i.e. the tensor product algebra of \( A \) and the alternating algebra \( A^*g \). Then it is clear that the module isomorphism in Proposition 2.4 also respects the algebra structure. The binary product of \( A \) has important information about correlation of period integrals. We emphasize the key difference between these two constructions (homology and cohomology) with respect to the binary product structure; \( d_\rho \) is a (graded) derivation of the tensor and wedge product of \( C(g; \rho) \) but \( K_\rho \) is not a derivation of the super-commutative product of \( \mathcal{A}_\rho \).

Because of the fact that the Chevalley-Eilenberg complex \((E(g; \rho), d_\rho)\) becomes a differential graded algebra with respect to the wedge product, it has been studied more often by algebraic topologists and algebraic geometers rather than \((\mathcal{A}_\rho, \bar{\rho})\) (or equivalently \((\mathcal{A}_\rho, K_\rho)\)).\(^4\) But, in our theory, the failure of \( K_\rho \) being a derivation will play a key role in deriving an \( L_\infty \)-algebra from \((\mathcal{A}_\rho, \cdot, K_\rho)\) by the descendant functor (in Section 3) and studying the corresponding formal deformation theory.

We add a simple justification why \((\mathcal{A}_\rho, \cdot, K_\rho)\) is more suitable than \((C(g; \rho), \wedge, d_\rho)\) for understanding the period integral \( C \) of \( \rho \). Observe that we can similarly enhance \( C : A \to \mathbb{k} \) to a cochain map \( \mathcal{C} : (C(g; \rho), d_\rho) \to (\mathbb{k}, 0) \) by setting \( \mathcal{C}(x) = C(x) \) if \( x \in C^0(g; \rho) = A \) and \( \mathcal{C}(x) = 0 \) otherwise; note that \( \mathcal{C} \) is not a derivation of \( \mathcal{C} \). But this cochain map \( \mathcal{C} \) loses the key information of the period integral \( C \); we illustrate this by using the toy example 2.1. The induced map of \( \mathcal{C} \) on the 0-th cohomology \( H^0(g, A) = H^0_{d_\rho}(C(g; \rho)) \) contains all the information of \( C \). In Example 2.1, we see that \( H^0(\mathcal{g}, A) := \ker(d_\rho) \cap C^0(g; \rho) = 0 \), since the differential equation

\[
\frac{\partial f(x)}{\partial x} - x f(x) = 0
\]

does not have a non-zero solution in \( A = \mathbb{k}[x] \). This means that the map \( \overline{\mathcal{C}} : H^*(g, A) \to \mathbb{R} \) induced from the cochain map \( \mathcal{C} \) is zero; so it is not a good cochain level realization of \( C \) in (2.1). On the other hand, the 0-th cohomology \( H^0_{K_\rho}(\mathcal{A}_\rho) \) is isomorphic to the \( \mathbb{k} \)-vector space \( \mathbb{k}[x]/\mathcal{N}_\rho \), where

\[
\mathcal{N}_\rho = \{ \frac{\partial f(x)}{\partial x} - x f(x) : f(x) \in \mathbb{k}[x] \}.
\]

The induced map \( \overline{\mathcal{C}} \) of \( \mathcal{C} \) on the cohomology \( \mathbb{k}[x]/M_\rho \) has substantial information about \( C \). This justifies our use of the (twisted) homological version \((\mathcal{A}_\rho, K_\rho)\) of the Chevalley-Eilenberg complex rather than the cohomological version.

\(^4\) For example, the Chevalley-Eilenberg complex \( C(g; \rho)_{\mathcal{A}(\rho)} \) of the quantum Jacobian Lie algebra representation \( \rho \cdot \mathcal{A}(\rho) \) in (2.2) where \( q_1 = y, q_2 = x_n, q_3 = x_{n-1}, \cdots, q_m = x_1 \) and \( G(x) \) is a defining polynomial of a smooth projective hypersurface \( X_0 \) of dimension \( n - 1 \), turns out to be the algebraic Dwork complex studied in several articles, including [1], [2], and [8].
3. The descendant functor and homotopy invariants

This section is about the general theory of the descendant functor from the category $\mathcal{C}/k$ of commutative homotopy probability algebras over $k$ to the category $\mathcal{L}/k$ of $L_\infty$-algebras over $k$. This theory will provide the general strategy for analyzing the period integrals of a Lie algebra representation and its associated cochain complex and cochain map via $L_\infty$-homotopy theory.

3.1. Why descendant functors?

Here we explain how the notion of a descendant functor arises in the study of period integrals of Lie algebra representations. This provides the general framework behind the main theorems of this paper. Let $k$ be a field of characteristic 0 and $g$ be a finite dimensional Lie algebra over $k$. Let $\rho : g \rightarrow \text{End}_k(A)$ be a $k$-linear representation of $g$, where $A$ is a commutative associative $k$-algebra, such that $\rho(g)$ acts on $A$ as a linear differential operator for all $g \in g$. We called a $k$-linear map $C : A \rightarrow k$ a period integral attached to $\rho : g \rightarrow \text{End}_k(A)$, $x \in \mathcal{N}_\rho := \sum_{g \in g} \text{im}(\rho(g))$. Hence such a period integral $C$ induces a map $P_C : A/\mathcal{N}_\rho \rightarrow k$. Two closely related properties of our period integrals are that, in general,

(i) $C : A \rightarrow k$ fails to be an algebra homomorphism,

(ii) $\mathcal{N}_\rho$ fails to be an ideal of $A$.

The descendant functor $\mathcal{D}_{\text{es}}$ is designed to provide a general framework to understand such period integrals and their higher structures by exploiting those failures and successive failures systematically. The domain category of $\mathcal{D}_{\text{es}}$ is the category $\mathcal{C}/k$ of commutative homotopy probability algebras (CHPA) over $k$ and the target category is the category $\mathcal{L}/k$ of $L_\infty$-algebras over $k$.

The category $\mathcal{C}/k$ is defined such that objects are triples $(A, \cdot, K)$, where the pair $(A, \cdot)$ is a $\mathbb{Z}$-graded super-commutative associative $k$-algebra while the pair $(A, K)$ is a cochain complex over $k$, and morphisms are cochain maps. We note that two of the salient properties of the category $\mathcal{C}/k$ are that

(i) morphisms are not required to be algebra homomorphisms,

(ii) the differential and multiplication in an object have no compatibility condition.

Then the functor $\mathcal{D}_{\text{es}} : \mathcal{C}/k \longrightarrow \mathcal{L}/k$ takes

(i) a morphism $f$ in $\mathcal{C}_k$ to an $L_\infty$-morphism $\phi^f = \phi^f_1, \phi^f_2, \phi^f_3, \cdots$, where $\phi^f_1 = f$ and $\phi^f_2, \phi^f_3, \cdots$ measure the failure and higher failures of $f$ being an algebra homomorphism,

(ii) an object $(\mathcal{A}, \cdot, K)$ in $\mathcal{C}_k$ to an $L_\infty$-algebra $(\mathcal{A}, \ell^K = \ell^K_1, \ell^K_2, \ell^K_3, \cdots)$, where $\ell^K_1 = K$ and $\ell^K_2, \ell^K_3, \cdots$ measure the failure and higher failures of $K$ being a derivation of the multiplication in $\mathcal{A}$.

Morphisms in both categories $\mathcal{C}_k$ and $\mathcal{L}_k$ come with a natural notion of homotopy and the descendant functor induces a well defined functor from the homotopy category $h\mathcal{C}_k$ to the homotopy category $h\mathcal{L}_k$. 
The general construction, developed in Subsection 2.2, associates to a representation \( \rho : \mathfrak{g} \to \text{End}_k(A) \) an object \((\mathcal{A}, \cdot, K)\rho\) of the category \( \mathcal{C}_k \) such that the subalgebra \( \mathcal{A}_0^\rho \) is isomorphic to \( A \) and the 0-th cohomology \( H_0^\rho(\mathcal{A}_\rho) \) of the cochain complex \((\mathcal{A}, K)\rho\) is isomorphic to the quotient \( A/\mathcal{N}_\rho \). Then, a period integral \( C : A \to k \) attached to \( \rho \) gives a morphism \( \mathcal{C} \) up to homotopy (see Proposition 2.3) from the object \((\mathcal{A}, \cdot, K)\rho\) onto the initial object \((k, \cdot, 0)\) of the category \( \mathcal{C}_k \), i.e. \( \mathcal{C} \) is a \( k \)-linear map of degree 0 from \( \mathcal{A} \) onto \( k \) such that \( \mathcal{C} \circ K = 0 \). We then realize the period map \( \mathcal{P}_C^0 : A/\mathcal{N}_\rho \to k \) by the following commutative diagram

\[
\begin{array}{ccc}
H_K(\mathcal{A}_\rho) & \xrightarrow{\mathcal{P}_C(0)} & k \\
\downarrow{f} & & \downarrow{\mathcal{C}} \\
\mathcal{A}_\rho & & 
\end{array}
\]

where \( f \) is a cochain quasi-isomorphism from \( H_K(\mathcal{A}_\rho) \), considered as a cochain complex with zero differential, into the cochain complex \((\mathcal{A}, K)\rho\) which induces the identity map on cohomology. Note that \( \mathcal{P}_C(0) := f \circ \mathcal{C} : H_K(\mathcal{A}_\rho) \to k \) depends only on the cochain homotopy types of \( f \) and \( \mathcal{C} \).

Note also that \( \mathcal{P}_C(0) \) is a zero map on \( H_i^K(\mathcal{A}_\rho) \) unless \( i = 0 \) so that it can be identified with \( \mathcal{P}_C^0 \) via isomorphism between \( H_0^K(\mathcal{A}_\rho) \) and \( A/\mathcal{N}_\rho \).

Our general framework together with various propositions will allow us to enhance the above commutative diagram as follows:

\[
\begin{array}{ccc}
S(H_K(A_\rho)) & \xrightarrow{\mathcal{K}} & k \\
\downarrow{\phi^\mathcal{C}} & & \downarrow{\mathcal{C}} \\
S(\mathcal{A}_\rho) & & 
\end{array}
\]

where the dotted arrows are the following \( L_\infty \)-morphisms

1. the \( L_\infty \)-morphism \( \phi^\mathcal{C} = \phi_1^\mathcal{C}, \phi_2^\mathcal{C}, \cdots \) is the descendent \( \mathcal{D}_\mathcal{C}(\mathcal{C}) \) of \( \mathcal{C} \) such that \( \phi_1^\mathcal{C} = \mathcal{C} \),
2. \( \varphi = \varphi_1, \varphi_2, \cdots \) is an \( L_\infty \)-quasi-isomorphism from \( H_K(\mathcal{A}_\rho) \) considered as a zero \( L_\infty \)-algebra into the \( L_\infty \)-algebra \( \mathcal{D}_\mathcal{C}(\mathcal{A}_\rho) \) such that \( \varphi_1 = f \).
3. the \( L_\infty \)-morphism \( \kappa = \kappa_1, \kappa_2, \cdots \) is the composition \( \phi^\mathcal{C} \circ \varphi \) in \( \mathcal{C}_k \) such that \( \kappa_1 = \phi_1^\mathcal{C} \circ \varphi_1 = \mathcal{P}_C(0) : H_K(\mathcal{A}_\rho) \to k \).

Note that the \( L_\infty \)-morphism \( \phi^\mathcal{C} = \mathcal{D}_\mathcal{C}(\mathcal{C}) \) is determined by \( \mathcal{C} \), while there are many different choices of \( L_\infty \)-quasi-isomorphism \( \varphi \). One of our main theorem show that \( L_\infty \)-morphism \( \kappa := \phi^\mathcal{C} \circ \varphi \) depends only on the cochain homotopy type of \( \mathcal{C} \) and the \( L_\infty \)-homotopy type of \( \varphi \).
3.2. Explicit description of the homotopy descendant functor

Here we shall give details about the homotopy descendant functor from the homotopy category of \( \mathcal{C}_k \) to the homotopy category of \( \mathcal{L}_k \) of \( L_\infty \)-algebras over \( k \) by using the binary product as a crucial ingredient.

When the representation space of \( \rho \) has an associative and commutative binary product, \((\mathcal{A}_\rho, K_\rho)\) attached to a Lie algebra representation \( \rho \) also has a \( k \)-algebra structure. This implies that \((\mathcal{A}_\rho, K_\rho)\) and a period integral \( \phi : (\mathcal{A}_\rho, K_\rho) \to (k, 0) \) is an object and a morphism of \( \mathcal{C}_k \) respectively.

An \( L_\infty \)-algebra structure on \( \mathcal{A} \) is a sequence of \( k \)-linear maps \( \ell = \ell_1, \ell_2, \cdots \) such that \( \ell_n : S^n(\mathcal{A}) \to \mathcal{A} \) satisfy certain relations (see Definition 5.3) and an \( L_\infty \)-morphism from an \( L_\infty \)-algebra \((\mathcal{A}', \ell')\) to another \( L_\infty \)-algebra \((\mathcal{A}, \ell)\) is a sequence of \( k \)-linear maps \( \phi = \phi_1, \phi_2, \cdots \) such that \( \phi_n : S^n(\mathcal{A}) \to \mathcal{A}' \) satisfy certain relations (see Definition 5.4). We use a variant of the standard \( L_\infty \)-algebra such that every \( k \)-linear map \( \ell_n \) for \( n = 1, 2, \cdots \), is of degree 1. See Subsection 5.2 for our presentation of \( L_\infty \)-algebras, \( L_\infty \)-morphisms, and \( L_\infty \)-homotopies (suitable for the purpose of describing correlations), which is based on partitions of \( \{1, 2, \cdots, n\} \).

As we already indicated, our algebraic analysis is based on two facts; the differential \( K \) is not a \( k \)-derivation of the (super-commutative) product of \( \mathcal{A} \) and the cochain map (period integral) \( f : (\mathcal{A} \cdot, K) \to (k, 0) \) is not a \( k \)-algebra map in general. Therefore it is natural to propose the following definition for \( \ell^K_1 : \mathcal{A} \to \mathcal{A} \) and \( \ell^K_2 : S^2(\mathcal{A}) \to \mathcal{A} \);

\[
\ell^K_1(x) = Kx, \\
\ell^K_2(x, y) = K(x \cdot y) - Kx \cdot y - (-1)^{|x|} x \cdot Ky.
\] (3.1)

so that \( \ell^K \) measures the failure of \( K \) being a derivation of the product. In the case of morphisms, we propose the following definition for \( \phi^K_1 : \mathcal{A} \to \mathcal{A}' \), \( \phi^K_2 : S^2(\mathcal{A}) \to \mathcal{A}' \) in the same vein:

\[
\phi^K_1(x) = f(x), \\
\phi^K_2(x, y) = f(x \cdot y) - f(x) \cdot f(y).
\] (3.2)

so that \( \phi^K \) measures the failure of \( f \) being an algebra map. But then there would be many choices for how to measure higher failures\(^5\), i.e. how to define \( \ell^K_3, \ell^K_4, \cdots \) and \( \phi^K_3, \phi^K_4, \cdots \) in a systematic way. We provide one particular way so that resulting \( \ell^K \) becomes an \( L_\infty \)-algebra and \( \phi^K \) becomes an \( L_\infty \)-morphism in a functorial way; see Theorem 3.1. We refer to the appendix for notation regarding the partition \( P(n) \) appearing in the following definition.

---

\(^5\) A homotopy associative algebra (so called, \( A_\infty \)-algebra) can be also used as a target category; we can construct a \( A_\infty \)-descendant functor from \( \mathcal{C}_k \) to the category of \( A_\infty \)-algebras over \( k \), which even works if we drop the super-commutativity of the multiplication in an object of the category \( \mathcal{C}_k \). The descendant functor is a more general notion; we recently found that the pseudo character (a generalization of the trace of a group representation) is also a sort of descendant which measures different higher failures. But we limit our study only to \( L_\infty \)-descendant functor in this article.
**Definition 3.1.** For a given object $(\mathcal{A}, \cdot, K)$ in $\mathcal{E}_k$, we define $\text{Des}(\mathcal{A}, \cdot, K) = (\mathcal{A}, \ell^K)$, where $\ell^K = \ell^K_1, \ell^K_2, \cdots$ is the family of linear maps $\ell^K : S^n(\mathcal{A}) \to \mathcal{A}$, inductively defined by the formula

$$K(x_1 \cdots x_n) = \sum_{i \in \mathbb{P}(n)} e(\pi, i) \cdot x_{B_1} \cdots x_{B_{n-1}} \cdot \ell^K(x_{B_i}) \cdot x_{B_{i+1}} \cdots x_{B_n},$$

(3.3)

for any homogeneous elements $x_1, x_2, \cdots, x_n \in \mathcal{A}$. Here we use the following notation:

- $x_B = x_j \otimes \cdots \otimes x_{j_r}$ if $B = \{j_1, \cdots, j_r\}$.
- $\ell(x_B) = \ell_r(x_j, \cdots, x_{j_r})$ if $B = \{j_1, \cdots, j_r\}$.
- $e(\pi, i) = e(\pi)(-1)^{|x_{B_i}| + |1 + x_{B_{n-1}}|}$.

For a given morphism $f : (\mathcal{A}, \cdot, K) \to (\mathcal{A}', \cdot, K')$ in $\mathcal{E}_k$, we define a morphism $\text{Des}(f)$ as the family $\phi^f = \phi^f_1, \phi^f_2, \cdots$ constructed inductively as

$$f(x_1 \cdots x_n) = \sum_{\pi \in \mathbb{P}(n)} e(\pi)\phi^f(x_{B_1}) \cdots \phi^f(x_{B_n}), \quad n \geq 1,$$

(3.4)

where $\phi^f(x_B) = \phi^f(x_j, \cdots, x_{j_r})$ if $B = \{j_1, \cdots, j_r\}$, $1 \leq j_1, \cdots, j_r \leq n$, for any homogeneous elements $x_1, \cdots, x_n \in \mathcal{A}$. Here $\phi^f : S^n(\mathcal{A}) \to \mathcal{A}'$ is a $\mathbb{k}$-linear map defined on the super-commutative symmetric product of $\mathcal{A}$.

**Remark 3.1.** We will call the above $\text{Des}(\mathcal{A}, \cdot, K)$ and $\text{Des}(f)$ a descendant $L_\infty$-algebra and a descendant $L_\infty$-morphism respectively. Note that not every $L_\infty$-algebra over $\mathbb{k}$ is a descendant $L_\infty$-algebra over $\mathbb{k}$; for example, an induced minimal $L_\infty$-algebra structure on the cohomology $H_K(\mathcal{A})$ of a descendant $L_\infty$-algebra $(\mathcal{A}, \ell^K)$ is always trivial; see Proposition 3.1.

According to Definition, we have (3.1). For $n \geq 2$, the following holds:

$$\ell^K_n(x_1, \cdots, x_{n-1}, x_n) = \ell^K_{n-1}(x_1, \cdots, x_{n-2}, x_{n-1} \cdot x_n)$$

$$-\ell^K_{n-1}(x_1, \cdots, x_{n-1}) \cdot x_n = (-1)^{|x_{n-1}| |1 + x_1| + \cdots + |x_{n-2}|} x_{n-1} \cdot \ell^K_{n-1}(x_1, \cdots, x_{n-2}, x_n).$$

If $\mathcal{A}$ has a unit $1_\mathcal{A}$ and $K(1_\mathcal{A}) = 0$, then one can also easily check that

$$\ell^K_n(x_1, x_2, \cdots, x_n) = [\cdots [[K, L_{x_1}], L_{x_2}], \cdots], L_{x_n}] (1_\mathcal{A})$$

(3.5)

for any homogeneous elements $x_1, x_2, \cdots, x_n \in \mathcal{A}$. Here $L_{x} : \mathcal{A} \to \mathcal{A}$ is left multiplication by $x$ and $[L, L'] := L \cdot L' - (-1)^{|L||L'|} L' \cdot L \in \text{End}_{\mathcal{A}}(\mathcal{A})$, where $|L|$ means the degree of $L$.

Unravelling Definition also shows (3.2). For $n \geq 2$, we have

$$\phi^f_n(x_1, \cdots, x_n) = \phi^f_{n-1}(x_1, \cdots, x_{n-2}, x_{n-1} \cdot x_n) - \sum_{\pi \in \mathbb{P}(n) \setminus \{\pi = 2\}} \phi^f(x_{B_1}) \cdot \phi^f(x_{B_2}) .$$
Let $\text{Art}_k$ denote the category of unital $\mathbb{Z}$-graded commutative Artinian local $k$-algebras with residue field $k$. Let $\gamma \in (m_a \otimes \mathcal{A})^0$, where $a \in \text{Ob}(\text{Art}_k)$ and let $m_a$ denote the unique maximal ideal of $a$; the tensor product $m_a \otimes \mathcal{A}$ also has a natural induced $\mathbb{Z}$-grading and $(m_a \otimes \mathcal{A})^n$ denotes the $k$-subspace of homogeneous elements of degree $n$.

**Lemma 3.1.** For every $\gamma \in (m_a \otimes \mathcal{A})^0$ and for any homogeneous element $\lambda \in a \otimes \mathcal{A}$ whenever $a \in \text{Ob}(\text{Art}_k)$, we have identities in $a \otimes \mathcal{A}$;

$$K(e^\gamma - 1) = e^\gamma \cdot L_k^1(\gamma),$$

where $L_k^1(\gamma) = \sum_{n \geq 1} \ell_n^k(\gamma, \cdots, \gamma)$, and

$$K(\lambda \cdot e^\gamma) = L_k^1(\lambda) \cdot e^\gamma + (-1)^{|\lambda|} \cdot K(e^\gamma - 1),$$

where $L_k^1(\lambda) := K\lambda + \ell_n^k(\gamma, \lambda) + \sum_{n=3}^{\infty} \frac{1}{(n-1)!} \ell_n^k(\gamma, \cdots, \gamma, \lambda)$.

**Proof.** Note that the above infinite sum is actually a finite sum, since $m_a$ is a nilpotent $k$-algebra. Also note that we use the following notation (see Definition 5.1);

$$\ell_n^k(a_1 \otimes v_1, \cdots, a_n \otimes v_n) = (-1)^{|a_1|+|a_2|(|v_1|+|v_2|)+\cdots+|a_n|(|v_1|+|v_2|+\cdots+|v_n|)} a_1 \cdots a_n \otimes \ell_n^k(v_1, \cdots, v_n)$$

for $a_i \otimes v_i \in (m_a \otimes \mathcal{A})^0$ with $i = 1, 2, \cdots, n$. Then the first formula follows from a simple combinatorial computation by plugging in $\gamma = \sum_{i=1}^n a_i \otimes v_i$; you may regard (3.6) as an alternative definition of the $L_\infty$-descendant algebra $(\mathcal{A}, \ell^k)$.

For the second equality, let $a = k[\epsilon]/(\epsilon^2)$ be the ring of dual numbers, which is an object of $\text{Art}_k$. Denote the power series $e^X - 1$ by $P(X)$ and let $P'(X) = e^X$ be the formal derivative of $P(X)$ with respect to $X$. Then it is easy to check that $P(\gamma + \epsilon \cdot \lambda) = P(\gamma) + (\epsilon \cdot \lambda) \cdot P'(\gamma)$ for $\epsilon \cdot \lambda \in (a \otimes \mathcal{A})^0$. This says that

$$P'(\gamma + \epsilon \cdot \lambda) \cdot L_k^1(\gamma + \epsilon \cdot \lambda) = K(P(\gamma + \epsilon \cdot \lambda)) \quad \text{by (3.6)}$$

$$= K(P(\gamma)) + K((\epsilon \cdot \lambda) \cdot P'(\gamma))$$

$$= P'(\gamma) \cdot L_k^1(\gamma) + K((\epsilon \cdot \lambda) \cdot P'(\gamma)) \quad \text{by (3.6)}$$

On the other hand, we have that

$$P'(\gamma + \epsilon \cdot \lambda) \cdot L_k^1(\gamma + \epsilon \cdot \lambda) = P'(\gamma) \cdot L_k^1(\gamma) + (\epsilon \otimes \lambda) \cdot P'(\gamma) \cdot L_k^1(\gamma) + P'(\gamma) \cdot L_k^1(\epsilon \cdot \lambda)$$

where $L_k^1(\epsilon \cdot \lambda) = K(\epsilon \cdot \lambda) + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \ell_n^k(\gamma, \cdots, \gamma, (\epsilon \cdot \lambda))$. By comparison, we get that

$$K((\epsilon \cdot \lambda) \cdot P'(\gamma)) = (\epsilon \cdot \lambda) \cdot P'(\gamma) \cdot L_k^1(\gamma) + P'(\gamma) \cdot L_k^1(\epsilon \cdot \lambda) = L_k^1(\epsilon \cdot \lambda) \cdot P'(\gamma) + (\epsilon \cdot \lambda) \cdot K(P(\gamma)).$$

Then this implies the second identity according to our sign convention. □

**Lemma 3.2.** The descendants $\ell^k$ define an $L_\infty$-algebra structure over $k$ on $\mathcal{A}$.

Proof. For every $\gamma \in (m_\mathfrak{g} \otimes \mathfrak{g})^0$, we consider $L^K(\gamma) = \sum_{n \geq 1} \ell^K_n(\gamma, \cdots, \gamma)$. Then (3.6) says that

$$K(e^\gamma - 1) = e^\gamma \cdot L^K(\gamma) = L^K(\gamma) \cdot e^\gamma.$$  

Applying $K$ to the above, we obtain that

$$0 = K(L^K(\gamma) \cdot e^\gamma) = L^K(\gamma) \cdot e^\gamma - L^K(\gamma) \cdot K(e^\gamma - 1) = L^K(\gamma) \cdot e^\gamma,$$

where we have used $K^2 = 0$ for the 1st equality. The 2nd equality follows from Lemma 3.1 and the 3rd equality results from the fact that $L^K(\gamma) \cdot K(e^\gamma - 1) = L^K(\gamma)^2 \cdot e^\gamma$ vanishes, since $L^K(\gamma)^2 = 0$ by the super-commutativity of the product (note that $L^K(\gamma)$ has degree 1). Hence the following expression

$$\chi(\gamma) := L^K(\gamma) = K(L^K(\gamma)) + \ell^K_1(\gamma, L^K(\gamma)) + \sum_{n=3}^{\infty} \frac{1}{(n-1)!} \ell^K_n(\gamma, \cdots, L^K(\gamma))$$

vanishes for every $\gamma \in (m_\mathfrak{g} \otimes \mathfrak{g})^0$ whenever $a \in \text{Ob}(\text{Ar}_k^\mathfrak{g})$. We then consider a scaling $\gamma \rightarrow \lambda \cdot \gamma$, $\lambda \in k^*$, and the corresponding decomposition $\chi(\gamma) = \chi_1(\gamma) + \chi_2(\gamma) + \chi_3(\gamma) + \cdots$ such that $\chi_n(\lambda \cdot \gamma) = \lambda^n \chi_n(\gamma)$, i.e., we have for $n \geq 1$

$$\chi_n(\gamma) = \sum_{k=1}^n \frac{1}{(n-k)!k!} \ell^K_{n-k+1}(\ell^K_1(\gamma, \cdots, \gamma), \gamma, \cdots, \gamma).$$

It follows that $\chi_n(\gamma) = 0$, for all $n \geq 1$ and for all $\gamma \in (m_\mathfrak{g} \otimes \mathfrak{g})^0$ and every $a \in \text{Ob}(\text{Ar}_k^\mathfrak{g})$. Hence $(\mathfrak{g}, L^K, 1, \mathfrak{g})$ is an $L_\infty$-algebra over $k$ by Definition 5.1. \(\square\)

Lemma 3.3. For every $\gamma \in (m_\mathfrak{g} \otimes \mathfrak{g})^0$ and for any homogeneous element $\lambda \in \mathfrak{g} \otimes \mathfrak{g}$, we have identities in $\mathfrak{g} \otimes \mathfrak{g}$;

$$f(e^\gamma - 1) = e^{\Phi^f(\gamma)} - 1,$$  \hspace{1cm} (3.8)

where $\Phi^f(\gamma) = \sum_{n \geq 1} \frac{1}{n!} \Phi^f_n(\gamma, \cdots, \gamma)$, and

$$f(\lambda \cdot e^\gamma) = \Phi^f(\lambda) \cdot e^{\Phi^f(\gamma)},$$  \hspace{1cm} (3.9)

where $\Phi^f(\lambda) := \Phi^f_1(\lambda) + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \Phi^f_{n-1}(\gamma, \cdots, \gamma, \lambda)$.

Proof. Recall the sign convention from Definition 5.2;

$$\Phi_n(a_1 \otimes v_1, \cdots, a_n \otimes v_n) = (-1)^{|a_2||v_1|+\cdots+|a_n||v_1|+\cdots+|a_{n-1}||v_1|} a_1 \cdots a_n \otimes \Phi_n(v_1, \cdots, v_n).$$

for $a_i \otimes v_i \in (m_\mathfrak{g} \otimes \mathfrak{g})^0$ with $i = 1, 2, \cdots, n$, whenever $a \in \text{Ob}(\text{Ar}_k^\mathfrak{g})$. The first equality is a simple combinatorial reformulation of Definition 3.1 with the above sign convention. We leave this as an
exercise (plug in $\gamma = \sum_{i=1}^{n} a_i \otimes v_i$); you can regard (3.8) as an alternative definition of a descendant $L_\infty$-morphism $\phi_f$.

Let $a = k[\epsilon]/(\epsilon^2) \in \text{Ob}(\text{Art}_k^a)$ be the ring of dual numbers. Denote the power series $e^X - 1$ by $P(X)$ and let $P'(X) = e^X$ be the formal derivative of $P(X)$ with respect to $X$. Note that $P(\gamma + \epsilon \cdot \lambda) = P(\gamma) + (\epsilon \cdot \lambda) \cdot P'(\gamma)$ for $\epsilon \cdot \lambda \in (a \otimes \mathcal{A})^0$. Inside $a \otimes \mathcal{A}$ we have that

\[
 f(P(\gamma)) + f((\epsilon \cdot \lambda) \cdot P'(\gamma)) = f\left(P(\gamma + \epsilon \cdot \lambda)\right) = P\left(\phi_f(\gamma + \epsilon \cdot \lambda)\right) = P\left(\phi_f(\gamma) + \phi_f'(\epsilon \cdot \lambda)\right) = P(\phi_f(\gamma)) + P'(\phi_f(\gamma)) \cdot \phi_f(\epsilon \cdot \lambda).
\]

Then this finishes the proof of (3.9) by applying our sign convention. \qed

**Lemma 3.4.** The descendants $\phi_f$ define an $L_\infty$-morphism from $(\mathcal{A}, L^K)$ into $(\mathcal{A}', L^K')$.

**Proof.** Applying $K'$ to the relation $f(e^\gamma - 1) = e^{\phi_f(\gamma)} - 1$ in (3.8), we obtain that

\[
 f\left(K(e^\gamma - 1)\right) = K'(e^{\phi_f(\gamma)} - 1),
\]

where we have used $K'f = f K$. Then (3.6) implies that

\[
 f\left(L^K(\gamma) \cdot e^\gamma\right) = L^K'\left(\phi_f(\gamma)\right) \cdot e^{\phi_f(\gamma)}.
\]

Then (3.10) combined with (3.9) says that

\[
 \phi_f(L^K(\gamma)) \cdot e^{\phi_f(\gamma)} = L^K'\left(\phi_f(\gamma)\right) \cdot e^{\phi_f(\gamma)}.
\]

Therefore the following expression

\[
 \zeta(\gamma) := \left(\phi_f\left(L^K(\gamma)\right) + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \phi_{f,n}^{(n)}(\gamma, \ldots, \gamma, L^K(\gamma))\right) - L^K'\left(\sum_{n=1}^{\infty} \frac{1}{n!} \phi_{f,n}^{(n)}(\gamma, \ldots, \gamma)\right)
\]

vanishes for every $\gamma \in (m_a \otimes \mathcal{A})^0$ whenever $a \in \text{Ob}(\text{Art}_k^a)$. Then we consider a scaling $\gamma \rightarrow \lambda \cdot \gamma$, $\lambda \in k^*$, and the corresponding decomposition $\zeta(\gamma) = \zeta_1(\gamma) + \zeta_2(\gamma) + \zeta_3(\gamma) + \cdots$ such that $\zeta_n(\lambda \cdot \gamma) = \lambda^n \zeta_n(\gamma)$, where

\[
 \zeta_n(\gamma) = \sum_{j_1 + j_2 = n} \frac{1}{j_1! j_2!} \phi_{f,j_1+1}^f (\ell^K_{j_1}(\gamma, \ldots, \gamma), \ldots, \gamma) 
 - \sum_{j_1 + \cdots + j_r = n} \frac{1}{r! j_1! \cdots j_r!} L_{j_1}^{j_2} \left(\phi_{f,j_1}^f(\gamma, \ldots, \gamma), \ldots, \phi_{f,j_r}^f(\gamma, \ldots, \gamma)\right).
\]

It follows that $\zeta_n(\gamma) = 0$, for all $n \geq 1$ and all $\gamma \in (m_a \otimes \mathcal{A})^0$ whenever $a \in \text{Ob}(\text{Art}_k^a)$. Thus, by using Definition 5.2, it follows that the sequence $\phi_f$ defines an $L_\infty$-morphism between $L_\infty$-algebras. \qed
**Lemma 3.5.** Let $f : (\mathcal{A}', K) \to (\mathcal{A}', K')$ and $f' : (\mathcal{A}', K') \to (\mathcal{A}'', K'')$ be two morphisms in $\mathcal{C}_k$. Then we have
\[
\phi_{f' \circ f} = \phi_{f'} \bullet \phi_f,
\]
where $\bullet$ is the composition of $L_{\infty}$-morphisms (see Definition 5.5).

**Proof.** Consider $\gamma \in (m_a \otimes \mathcal{A})^0$ whenever $a \in \text{Ob}(\text{Art}_k^+ \mathcal{C})$. Then (3.8) implies that
\[
e^{\Phi_\gamma}(e^{\Phi_\gamma} - 1) = (f' \circ f)(e^{\gamma} - 1) = e^{\Phi_{\gamma} \circ \Phi_\gamma} - 1.
\]
Therefore, if we set $X(\gamma) = \Phi_{\gamma} \circ \Phi_{\gamma}(\gamma)$ and $Y(\gamma) = \Phi_{\gamma} \circ \Phi_{\gamma}(\gamma)$, then we have $X(\gamma) = Y(\gamma)$ for every $\gamma \in (m_a \otimes \mathcal{A})^0$. Consider the decomposition $X(\gamma) = X_1(\gamma) + X_2(\gamma) + \cdots$ and $Y(\gamma) = Y_1(\gamma) + Y_2(\gamma) + \cdots$, where $X_n(\lambda \cdot \gamma) = \lambda^n \cdot X_n(\gamma)$ and $Y_n(\lambda \cdot \gamma) = \lambda^n \cdot Y_n(\gamma)$, $\lambda \in \mathbb{k}$. Then $X_n(\gamma) = Y_n(\gamma)$ for all $n \geq 1$. The equality (3.8) implies that
\[
X_n(\gamma) = \sum_{j_1 + \cdots + j_n = n} \frac{1}{j_1! \cdots j_n!} \phi_{f' \circ f}(\gamma_{j_1}, \cdots, \gamma_n) = \frac{1}{n!} \left( \phi_{f' \circ f} \right)_n(\gamma_{j_1}, \cdots, \gamma_n).
\]

We hence conclude that $\phi_{f' \circ f}(\gamma, \cdots, \gamma) = \left( \phi_{f' \circ f} \right)_n(\gamma, \cdots, \gamma)$ for all $n \geq 1$. It follows that $\phi_{f' \circ f} = \phi_{f'} \bullet \phi_f$. \(\Box\)

Now we turn our attention to cochain homotopies in $\mathcal{C}_k$ and $L_{\infty}$-homotopies in $\mathcal{C}_k$.

**Definition 3.2.** Two morphisms $f : (\mathcal{A}', K) \to (\mathcal{A}', K')$ and $\bar{f} : (\mathcal{A}', K) \to (\mathcal{A}', K')$ in $\mathcal{C}_k$ are called homotopic, denoted by $f \sim \bar{f}$, if there is a polynomial family $F(\tau) : (\mathcal{A}', K) \to (\mathcal{A}', K')$ in $\tau$ of morphisms with $F(0) = f$ and $F(1) = \bar{f}$ satisfying
\[
\frac{d}{d\tau} F(\tau) = K' \sigma(\tau) + \sigma(\tau) K
\]
(3.11)

for some polynomial family $\sigma(\tau)$ in $\tau$ belonging to $\text{Hom}(\mathcal{A}', \mathcal{A}')(\mathcal{A})$. In this case, we call $\sigma(\tau)$ as a homotopy between $f$ and $\bar{f}$.

**Definition 3.3.** For given two morphisms $f : (\mathcal{A}', K) \to (\mathcal{A}', K')$ and $\bar{f} : (\mathcal{A}', K) \to (\mathcal{A}', K')$ in $\mathcal{C}_k$, and a homotopy $\sigma = \sigma(\tau)$ and a polynomial family $F = F(\tau)$ in (3.11), we define the descendant homotopy $\lambda_n = \lambda_n^{F, \sigma} \in \text{Hom}(S_n \mathcal{A}, \mathcal{A}')(\mathcal{A})$ for each $n \geq 1$, by the following formula
\[
\sigma(\gamma - 1) = e^{\Phi_\gamma} \cdot A^{F, \sigma}(\lambda),
\]
(3.12)

where $A^{F, \sigma}(\lambda) := \sum_{n=1}^{\infty} \frac{1}{n!} \lambda_n(\gamma, \cdots, \gamma)$.

Compare the defining formula (3.12) with the defining formula (3.6) for the descendant $L_{\infty}$-algebra and the defining formula (3.8) for the descendant $L_{\infty}$-morphism.
Lemma 3.6. Let \( f, \tilde{f} : (\mathcal{A}, K, \cdot) \to (\mathcal{A}', K', \cdot) \) be morphisms which are homotopic in \( \mathcal{C}_k \) in the sense of Definition 3.2. For every \( \gamma \in (m_0 \otimes \mathcal{A})^0 \) and any homogeneous element \( \mu \in a \otimes \mathcal{A} \), we have the following identity in \( a \otimes \mathcal{A} \):

\[
\sigma (e^r \cdot \mu) = e^{\phi^F(r)} \cdot \Lambda^E_{F_0}(\mu) + (-1)^{\mu|e^{\phi^F(r)} \cdot \mu} \cdot \Lambda^E_{F_0}(\gamma),
\]

(3.13)

where \( \Lambda^E_{F_0}(\mu) = \lambda_1(\mu) + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \lambda_n (\gamma, \ldots, \gamma, \mu) \).

Proof. Again, let \( a = k[\epsilon]/(\epsilon^2) \in \text{Ob} \text{Art}_{\mathcal{A}}^{\mathbb{C}} \) be the ring of dual numbers. Denote the power series 
\( e^X - 1 \) by \( P(X) \) and let \( P'(X) = e^X \) be the formal derivative of \( P(X) \) with respect to \( X \). For \( \epsilon \cdot \mu \in (a \otimes \mathcal{A})^0 \), we have the following identities in \( a \otimes \mathcal{A} \):

\[
\sigma (P(\gamma)) + \sigma (P'(\gamma) \cdot \epsilon \mu) = \sigma (P(\gamma + \epsilon \cdot \mu)) = P'(\Phi_F(\gamma + \epsilon \cdot \mu)) \cdot \Lambda^E_{F_0}(\gamma + \epsilon \cdot \mu)
\]

\[
= \left( P'(\Phi_F(\gamma)) + P'(\Phi_F(\epsilon \cdot \mu)) \right) \cdot \left( \Lambda^E_{F_0}(\gamma) + \Lambda^E_{F_0}(\epsilon \cdot \mu) \right)
\]

\[
= \sigma (P(\gamma)) + P'(\Phi_F(\epsilon \cdot \mu)) \cdot \Lambda^E_{F_0}(\gamma) + P'(\Phi_F(\gamma)) \cdot \Lambda^E_{F_0}(\epsilon \cdot \mu).
\]

Then (3.13) follows from this computation combined with our sign convention. \( \square \)

Lemma 3.7. The descendants of morphisms which are homotopic in \( \mathcal{C}_k \) are \( L_\infty \)-homotopic in the sense of definition 5.9.

Proof. By Definition 3.2, there is a polynomial family \( F(\tau) \) in \( \tau \) of morphisms with \( F(0) = f \) and \( F(1) = \tilde{f} \) satisfying

\[
\frac{\partial}{\partial \tau} F(\tau) = K' \sigma(\tau) + \sigma(\tau) K
\]

(3.14)

for some polynomial family \( \sigma(\tau) \) in \( \tau \) belonging to \( \text{Hom}(\mathcal{A}, \mathcal{A}')^{-1} \). Recall that

\[
\phi_F^\tau(\gamma) = \sum_{n \geq 1} \frac{1}{n!} \phi_n^F(\gamma, \ldots, \gamma) \quad \text{and} \quad F(\epsilon^r - 1) = e^{\phi^F(\gamma)} - 1,
\]

where \( \phi_0^F = \phi_0^F(\gamma) = \phi_1^F, \phi_2^F, \ldots \) is the descendant of \( F = F(\tau) \). Then the identity (3.14) implies that

\[
\frac{\partial}{\partial \tau} e^{\phi^F(\gamma)} = K' \sigma(\tau) e^{\epsilon^r - 1} + \sigma(\tau) K(\epsilon^r - 1) = K' \left( P'(\Phi_F(\gamma)) \cdot \Lambda^E_{F_0}(\gamma) \right) + \sigma(\gamma) K(\epsilon^r \cdot L^K(\gamma))
\]

(3.15)

where \( F = F(\tau) \) and \( \sigma = \sigma(\tau) \). The formulas (3.7) and (3.13) say that the right-hand side of (3.15) is the same as (with cancellation of middle terms)

\[
L^K_{\phi^F(\gamma)} \left( \Lambda^E_{F_0}(\gamma) \right) \cdot P'(\Phi_F(\gamma)) + P'(\Phi_F(\gamma)) \cdot \Lambda^E_{F_0}(L^K(\gamma)).
\]

The left-hand side of (3.15) is the same as \( P'(\Phi_F(\gamma)) \cdot \frac{\partial}{\partial \tau} \Phi_F(\gamma) \). Therefore we eventually get

\[
\frac{\partial}{\partial \tau} \phi^F(\gamma) = L^K_{\phi^F(\gamma)} \left( \Lambda^E_{F_0}(\gamma) \right) + \Lambda^E_{F_0}(L^K(\gamma)).
\]

(3.16)
Decomposing the equality (3.16) by $\mathbb{k}^*$-action $\gamma \to a\gamma, a \in \mathbb{k}^*$, we have the following form of the flow equation appearing in Definition 5.9 of $L_\infty$-homotopy of $L_\infty$-algebras; for every $\gamma \in (\mathfrak{m}_0 \otimes \mathcal{A})^0$ whenever $a \in \text{Ob}(\mathcal{C}_\mathbb{k})$

\[
\frac{\partial}{\partial \tau} \phi^F_n(\tau)(\gamma, \cdots, \gamma) = \sum_{k=1}^{n} \sum_{j_1+j_2=n-k} \frac{1}{r! j_1! 1} \ell_{j_1+1}^K(\phi^F_{j_1}(\gamma, \cdots, \gamma), \cdots, \phi^F_{j_2}(\gamma, \cdots, \gamma), \lambda_k(\gamma, \cdots, \gamma)) \\
+ \sum_{j_1+j_2=n} \frac{1}{j_1 j_2} \lambda_{j_1+1} \ell_{j_1+1}(\gamma, \cdots, \gamma, \ell_{j_2}^K(\gamma, \cdots, \gamma)),
\]

where $\phi^F_n(0) = \phi^F_n$ and $\phi^F_n(1) = \phi^F_n$, for all $n \geq 1$. \square

Remark 3.2. The explicit description of $L_\infty$-homotopies in Definition 5.9, which is hard to come up with in the literature, is motivated by our formalism of the descendant functor; we have defined the $L_\infty$-homotopy (motivated by the derivation of the equality (3.16) from the natural notion of cochain homotopy $\sigma$) so that $\text{Des}$ becomes a homotopy functor.

Let $h\mathcal{C}_k$ be the homotopy category of $\mathcal{C}_k$, i.e. objects in $h\mathcal{C}_k$ are the same as those in $\mathcal{C}_k$ and morphisms in $h\mathcal{C}_k$ are homotopy classes of morphisms in $\mathcal{C}_k$ in the sense of Definition 3.2. We also define the category $\mathcal{C}_k$ (respectively, $h\mathcal{C}_k$) to be the (respectively, homotopy) category of $L_\infty$-algebras over $k$ in the same way by using Definition 5.9 for $L_\infty$-homotopy. If we combine all of the above lemmas, we have the following result.

Theorem 3.1. The assignment $\text{Des}$ is a homotopy functor from the homotopy category $h\mathcal{C}_k$ to the homotopy category $h\mathcal{C}_k$, which we call the descendant functor.

The functor $\text{Des}$ is faithful but not fully faithful; two non-isomorphic objects in $\mathcal{C}_k$ can give isomorphic $L_\infty$-algebras under $\text{Des}$, the trivial $L_\infty$-algebra $(V, \varnothing)$ can not be a descendant $L_\infty$-algebra (unless $V$ has a commutative and associative binary product), and an arbitrary $L_\infty$-morphism between descendant $L_\infty$-algebras need not be a descendant $L_\infty$-morphism.

3.3. Cohomology of a descendant $L_\infty$-algebra

Here we prove that a descendant $L_\infty$-algebra is smooth-formal in the sense of Definition 5.8. By Theorem 5.1, on the cohomology of an $L_\infty$-algebra $(V, \mathcal{L})$ there is a minimal $L_\infty$-algebra structure $(H, \mathcal{L}^H)$ together with an $L_\infty$-morphism $\varphi$ from $H$ to $V$. If an $L_\infty$-algebra $(V, \mathcal{L})$ is a descendant $L_\infty$-algebra, i.e. $(V, \mathcal{L}) = (\mathcal{A}, \mathcal{L}^K) = \text{Des}(\mathcal{A}, \cdot, K)$ for some object $(\mathcal{A}, \cdot, K)$ in $\mathcal{C}_k$, then this minimal $L_\infty$-algebra structure on the cohomology $H_K$ of $(\mathcal{A}, \cdot, K)$ is trivial.

Proposition 3.1. Let $(\mathcal{A}, \cdot, K)$ be an object of $\mathcal{C}_k$ and let $(\mathcal{A}, \mathcal{L}^K)$ be its descendant $L_\infty$-algebra. Then the minimal $L_\infty$-algebra structure on the cohomology $H_K$ of $(\mathcal{A}, \mathcal{L}^K)$ is trivial, i.e. $\ell^H_2 = \ell^H_3 = \cdots = 0$, and there is an $L_\infty$-quasi-isomorphism $\varphi^H$ from $(H_K, \mathcal{L})$ into $(\mathcal{A}, \mathcal{L}^K)$. 
Proof. Let $H = H_k$ for simplicity. Let $f : H \to \mathcal{S}$ be a cochain quasi-isomorphism between cochain complexes $(H, 0)$ and $(\mathcal{S}, K)$, which induces the identity map on $H$. Note that $f$ is defined up to cochain homotopy. Let $h : \mathcal{S} \to H$ be a homotopy inverse to $f$ such that $h \circ f = I_H$ and $f \circ h = L_{\mathcal{S}} + K \beta + \beta K$ for some $k$-linear map $\beta$ from $\mathcal{S}$ into $\mathcal{S}$ of degree $-1$. Let $1_H = h(1_{\mathcal{S}})$. We need to establish that (i) the minimal $L_{\infty}$-structure $v_k = v_2, v_3, \ldots$ on $H$ is trivial (we use the notation $v_k = \ell^H K$, i.e., $v_2 = v_3 = \cdots = 0$, (ii) there is a family $\varphi^H = \varphi^H_1, \varphi^H_2, \ldots$, where $\varphi^H_1 = f$ and $\varphi^H_k$ is a $k$-linear map from $S^k(H)$ into $\mathcal{S}$ of degree $0$ for all $k \geq 2$ such that, for all homogeneous elements $a_1, \ldots, a_n$ of $H$ and for all $n \geq 1$

$$\sum_{\pi \in P(n)} e(\pi)\ell^K_{[m]}(\varphi^H(a_{B_1}), \ldots, \varphi^H(a_{B_{\pi}})) = 0.$$  

We shall use mathematical induction. Set $\varphi^H_1 = f$. Then we have

$$K \varphi^H_1 = 0.$$  

Define a $k$-linear map $L_2 : S^2(H) \to \mathcal{S}$ of degree $1$ such that, for all homogeneous elements $a_1, a_2 \in H$,

$$L_2(a_1, a_2) := \ell^K_2(\varphi^H_1(a_1), \varphi^H_1(a_2)).$$

Then $\text{Im} L_2 \subset \text{Ker} K \cap \mathcal{S}$ by Definition of $\ell^K_2$ and thus we can define $v_2 : S^2(H) \to H$ by $v_2(a_1, a_2) := h \circ L_2(a_1, a_2)$. On the other hand, it follows that $h \circ L_2(a_1, a_2) = 0$ for every homogeneous elements $a_1, a_2 \in H$, because we have

$$L_2(a_1, a_2) = K(\varphi^H_1(a_1), \varphi^H_1(a_2)) - K \varphi^H_1(a_1) \cdot \varphi^H_1(a_2) - (-1)^{|a_1|} \varphi^H_1(a_1) \cdot K \varphi^H_1(a_2)$$

$$= K(\varphi^H_1(a_1), \varphi^H_1(a_2)).$$

Hence $v_2 = 0$. From $h \circ L_2(a_1, a_2) = 0$, we have $0 = f \circ h \circ L_2(a_1, a_2) = L_2(a_1, a_2) + K \varphi^H_1(a_1, a_2)$ where $\varphi^H_2(a_1, a_2) := \beta \circ L_2(a_1, a_2)$ is a $k$-linear map from $S^2(H)$ to $\mathcal{S}$ of degree $0$. Hence we have, for all homogeneous $a_1, a_2 \in H$,

$$K \varphi^H_2(a_1, a_2) + \ell^K_2(\varphi^H_1(a_1), \varphi^H_1(a_2)) = 0.$$  

Fix $n \geq 2$ and assume that $v_2 = \cdots = v_n = 0$ and there is a family $\varphi^{[n]} = \varphi^H_1, \varphi^H_2, \ldots, \varphi^H_n$, where $\varphi^H_k$ is a linear map $S^k(H) \to \mathcal{S}$ of degree $0$ for $1 \leq k \leq n$, $\varphi^H_1 = f$ and such that, for all $1 \leq k \leq n$,

$$\sum_{\pi \in P(k)} e(\pi)\ell^K_{[m]}(\varphi^{[n]}(a_{B_1}), \ldots, \varphi^{[n]}(a_{B_{\pi}})) = 0. \quad (3.17)$$

Define the linear map $L_{n+1} : S^{n+1}(H) \to \mathcal{S}$ of degree $1$ by

$$L_{n+1}(a_1, \ldots, a_{n+1}) := \sum_{\pi \in P(n+1)} e(\pi)\ell^K_{[m]}(\varphi^{[n]}(a_{B_1}), \ldots, \varphi^{[n]}(a_{B_{\pi}})).$$

Then $\text{Im} L_{n+1} \subset \text{Ker} K \cap \mathcal{S}$. Define the linear map $v_{n+1} : S^{n+1}(H) \to H$ of degree $1$ by

$$v_{n+1}(a_1, \ldots, a_{n+1}) := h \circ L_{n+1}(a_1, \ldots, a_{n+1}).$$
Now (3.17) implies that
\[
L_{n+1}(a_1, \cdots, a_{n+1}) = K \sum_{\pi \in P(n+1)} e(\pi)\phi^{[n]}(a_{R_1}) \cdots \phi^{[n]}(a_{B_{m}})
\]
and \(L_{n+1}(a_1, \cdots, a_n, 1_H) = 0\). Hence we get
\[
h \circ L_{n+1}(a_1, \cdots, a_{n+1}) = 0,
\]
and \(v_{n+1} = 0\). It follows that, by the assumption, \(v_2 = \cdots = v_n = v_{n+1} = 0\). By applying the map \(f : H \to \mathcal{A}\) to (3.18), we obtain
\[
K \circ \beta \circ L_{n+1}(a_1, \cdots, a_{n+1}) + L_{n+1}(a_1, \cdots, a_{n+1}) = 0
\]
Set \(\varphi^H_{n+1} = \beta \circ L_{n+1} : S^{n+1}(H) \to \mathcal{A}\), which is a \(k\)-linear map of degree 0. Hence
\[
L_{n+1}(a_1, \cdots, a_{n+1}) + K\varphi^H_{n+1}(a_1, \cdots, a_{n+1}) = 0.
\]
If we set \(\phi^{[n+1]} = \phi^{[n]}, \varphi^H = \varphi^H_1, \cdots, \varphi^H_n, \varphi^H_{n+1}\), then the above identity can be rewritten as follows:
\[
\sum_{\pi \in P(n+1)} e(\pi)\epsilon^K(\phi^{[n+1]}(a_{R_1}), \cdots, \phi^{[n+1]}(a_{B_{m}})) = 0.
\]
This finishes the proof. \(\Box\)

### 3.4. Deformation functor attached to a descendant \(L_{\infty}\)-algebra

In general, one can always consider a deformation functor associated to an \(L_{\infty}\)-algebra. Here we consider a deformation problem attached to the descendant \(L_{\infty}\)-algebra \((\mathcal{A}, \ell^K)\) of an object \((\mathcal{A}, \ell)\) in \(\mathcal{C}_k\). Recall that \(\text{Art}^G_k\) is the category of \(\mathbb{Z}\)-graded commutative artinian local \(k\)-algebras with residue field \(k\). Let \(a \in \text{Ob}(\text{Art}^G_k)\) and \(m_a\) denote the maximal ideal of \(a\).

**Definition 3.4.** Let \(\Gamma, \tilde{\Gamma} \in (m_a \otimes \mathcal{A})^0\) such that
\[
K(e^{\Gamma} - 1) = 0 = K(e^{\tilde{\Gamma}} - 1).
\]
Then we say that \(\Gamma\) is homotopy equivalent (or gauge equivalent) to \(\tilde{\Gamma}\), denoted by \(\Gamma \sim \tilde{\Gamma}\), if there is a one-variable polynomial solution \(\Gamma(\tau) \in (m_a \otimes \mathcal{A})^0[\tau]\) with \(\Gamma(0) = \Gamma\) and \(\Gamma(1) = \tilde{\Gamma}\) to the following flow equation
\[
\frac{\partial}{\partial \tau} e^{\Gamma(\tau)} = K \left(\lambda(\tau) \cdot e^{\Gamma(\tau)}\right)
\]
for some one-variable polynomial \(\lambda(\tau) \in (m_a \otimes \mathcal{A})^{-1}[\tau]\).
Note that the above infinite sum is actually a finite sum, since \( m_a \) is a nilpotent \( k \)-algebra. We used the product structure on \( \mathcal{A} \) to define the homotopy equivalence and

\[
K(e^\ell - 1) = 0 \quad \text{is equivalent to} \quad \sum_{n=1}^{\infty} \frac{1}{n!} \ell^K_n(\Gamma, \ldots, \Gamma) = 0.
\]

Define a covariant functor \( \mathcal{D} \mathcal{E} \mathcal{F}(\mathcal{A}_k, \ell^K) \) from \( \mathbb{A} \mathbb{R} \) to \( \mathbb{S} \mathbb{E} \mathbb{T} \), where \( \mathbb{S} \mathbb{E} \mathbb{T} \) is the category of sets, by

\[
a \mapsto \mathcal{D} \mathcal{E} \mathcal{F}(\mathcal{A}_k, \ell^K)(a) = \{ I_a \in (m_a \otimes \mathcal{A})^0 : \sum_{n=1}^{\infty} \frac{1}{n!} \ell^K_n(\Gamma_a, \ldots, \Gamma_a) = 0 \} / \sim
\]

(3.19)

where \( \sim \) is the equivalence relation given in Definition 3.4. One can send a morphism in \( \mathbb{A} \mathbb{R} \) to a morphism in \( \mathbb{S} \mathbb{E} \mathbb{T} \) in an obvious way. Let \( \mathcal{D} \mathcal{E} \mathcal{F}(\mathcal{A}_k, \ell^K) \) be the extension of \( \mathcal{D} \mathcal{E} \mathcal{F}(\mathcal{A}_k, \ell^K) \) to the category of \( \mathbb{Z} \)-graded complete commutative noetherian local \( k \)-algebras with residue field \( k \); see [7], p 83. We want to study this deformation functor. Any \( L_\infty \)-structure on \( H_K = H_K(\mathcal{A}) \), induced from the descendant \( (\mathcal{A}, \ell^K) \) via Theorem 5.1, is trivial by Proposition 3.1, i.e. \( (\mathcal{A}_k, \ell^K) \) is smooth-formal (see Definition 5.8). Then this implies that \( \mathcal{D} \mathcal{E} \mathcal{F}(\mathcal{A}_k, \ell^K) \) is pro-representable by a complete noetherian local \( k \)-algebra \( \tilde{\mathcal{H}} := \lim \left\{ \bigoplus_{k=0}^{n} S^k(H^K_k) \right\} \) with \( H^K_k = \text{Hom}(H_K, k) \), if \( H_K \) is finite dimensional over \( k \). The theorem 5.1 also implies that there is an \( L_\infty \)-quasi-isomorphism \( \varphi^H = \varphi_1^H, \varphi_2^H, \ldots \) from \( (H_K, \mathcal{O}) \to (\mathcal{A}_k, \ell^K) \), i.e. \( \varphi_m^H \) is a \( k \)-linear map from \( S^m(H_K) \) into \( \mathcal{A} \) of degree 0 for all \( m \geq 1 \) such that

\[
\sum_{\pi \in P(n)} \epsilon(\pi) \ell^K_\pi(\varphi^H(a_{B_1}), \ldots, \varphi^H(a_{B_n})) = 0
\]

for every set of homogeneous elements \( a_1, \ldots, a_n \) of \( H_K \) and for all \( n \geq 1 \). In fact, the \( L_\infty \)-homotopy types of \( L_\infty \)-morphisms from \( (H_K, \mathcal{O}) \) into \( (\mathcal{A}_k, \ell^K) \) are classified by \( \mathcal{D} \mathcal{E} \mathcal{F}(\mathcal{A}_k, \ell^K)(\tilde{\mathcal{H}}) \). To be more precise, we have the following results:

**Proposition 3.2.** Assume that \( H_K = H(\mathcal{A}) \) is finite dimensional over \( k \).

(a) The deformation functor \( \mathcal{D} \mathcal{E} \mathcal{F}(\mathcal{A}_k, \ell^K) \) is pro-representable by \( \tilde{\mathcal{H}} := \lim \left\{ \bigoplus_{k=0}^{n} S^k(H^K_k) \right\} \) with \( H^K_k = \text{Hom}(H_K, k) \), i.e. there is an isomorphic natural transformation from \( \text{Hom}(\tilde{\mathcal{H}}, \cdot) \) to \( \mathcal{D} \mathcal{E} \mathcal{F}(\mathcal{A}_k, \ell^K)(\cdot) \).

(b) There is a bijection between \( \mathcal{D} \mathcal{E} \mathcal{F}(\mathcal{A}_k, \ell^K)(\tilde{\mathcal{H}}) \) and the set

\[
\{ \varphi : \varphi = \varphi_1, \varphi_2, \ldots \text{ is an } L_\infty \text{-morphism from } (H_K, \mathcal{O}) \text{ to } (\mathcal{A}_k, \ell^K) \} / \sim
\]

where \( \sim \) means the \( L_\infty \)-homotopy equivalence relation given in Definition 5.9.

**Proof.** (a) This is a special case of Theorem 5.5 of [10], because \( (\mathcal{A}, \ell^K) \) is smooth-formal by Proposition 3.1.

(b) Let \( \{ e_\alpha : \alpha \in I \} \) be a homogeneous \( k \)-basis of the \( \mathbb{Z} \)-graded \( k \)-vector space \( H_K = H_K(\mathcal{A}) \), where \( I \) is an index set. Let \( t^\alpha \) be the \( k \)-dual of \( e_\alpha \), where \( e_\alpha \) varies in \( \{ e_\alpha : \alpha \in I \} \). Then \( \{ t^\alpha : \alpha \in I \} \) is
the dual $\mathbb{k}$-basis of $\{e_\alpha : \alpha \in I\}$ and the degree $|t^\alpha| = -|e_\alpha|$ where $e_\alpha \in H^{|e_\alpha|}_K$. We consider the power series ring $k[[t^\alpha]] = k[[t^\alpha : \alpha \in I]]$. Let $I$ be the unique maximal ideal of $k[[t^\alpha]]$, so that

$$a_N := k[[t^\alpha]]/I^{N+1} \in \text{Ob}(\mathcal{Art}_k)$$

(3.20)

for arbitrary $N \geq 0$. Then $\mathcal{SH}$ is isomorphic to $\varinjlim_N a_N$. For a given $L_\infty$-homotopy type of $\varphi$, we define

$$\xi = \xi(\varphi) = \left[ \sum_{n=1}^\infty \frac{1}{n!} \sum_{a_1, \ldots, a_n} t^{a_n} \cdots t^{a_1} \otimes \varphi_n(e_{a_1}, \ldots, e_{a_n}) \right] \in \mathcal{D}_{\ell}(\mathcal{SH}),$$

where $[\cdot]$ means the homotopy equivalence class. One can check that this is a well-defined map, i.e. it sends $L_\infty$-homotopy types to homotopy equivalence classes. Let $\tilde{\varphi}$ be $L_\infty$-homotopic to $\varphi$ and $\tilde{\xi} := \xi(\tilde{\varphi})$. By Definition 5.9, there exists a polynomial solution $\Phi(\tau) = \Phi_1(\tau), \Phi_2(\tau), \ldots$ of the flow equation with respect to a polynomial family $\hat{\lambda}(\tau) = \hat{\lambda}_1(\tau), \hat{\lambda}_2(\tau), \ldots$ such that $\Phi(0) = \varphi$ and $\Phi(1) = \tilde{\varphi}$. If we set

$$\Xi(\tau) = \sum_{n \geq 1} \frac{1}{n!} \sum_{a_1, \ldots, a_n} t^{a_n} \cdots t^{a_1} \otimes \Phi_n(\tau)(e_{a_1}, \ldots, e_{a_n}),$$

$$A(\tau) = \sum_{n \geq 1} \frac{1}{n!} \sum_{a_1, \ldots, a_n} t^{a_n} \cdots t^{a_1} \otimes \hat{\lambda}_n(\tau)(e_{a_1}, \ldots, e_{a_n}),$$

then we have $\Xi(0) = \xi$ and $\Xi(1) = \tilde{\xi}$. Moreover, the flow equation implies (by a direct computation) that

$$\frac{\partial}{\partial \tau} e^{\Xi(\tau)} = K(\tau) \cdot e^{\Xi(\tau)}.$$ 

Hence $\Xi(0) = \xi$ and $\Xi(1) = \tilde{\xi}$ are homotopic to each other in the sense of Definition 3.4. Conversely, for a given equivalence class $[\xi]$ of a solution of the Maurer-Cartan equation

$$\xi = \left[ \sum_{n=1}^\infty \frac{1}{n!} \sum_{a_1, \ldots, a_n} t^{a_n} \cdots t^{a_1} \otimes v_{a_1, \ldots, a_n} \right] \in \mathcal{D}_{\ell}(\mathcal{SH}),$$

define $\varphi[\xi](e_{a_1}, \ldots, e_{a_n}) := v_{a_1, \ldots, a_n}$ for any $n \geq 1$ and extend it $\mathbb{k}$-linearly. Then $\varphi[\xi]$ is an $L_\infty$-morphism and is also a well-defined map by a similar argument. 

This proposition, combined with Proposition 3.1, says that there exists an $L_\infty$-quasi-isomorphism $\varphi^H$ such that the pair $(\mathcal{SH}, \Gamma)$ where

$$\Gamma = \sum_{n=1}^\infty \frac{1}{n!} \sum_{a_1, \ldots, a_n} t^{a_n} \cdots t^{a_1} \otimes \varphi^H_n(e_{a_1}, \ldots, e_{a_n}) \in \mathcal{D}_{\ell}(\mathcal{SH}),$$

is a universal family in the sense of Definition 14.3, [7]. If $\Gamma \in (m_{\mathcal{SH}} \otimes \mathcal{SH})^0$ corresponds to $\varphi$, we will use the notation $\Gamma = \Gamma[\varphi]$. 
Remark 3.3. (a) This proposition can be generalized to any $L_\infty$-algebra $(V,\mathcal{L})$, if we replace $\text{Art}^\mathbb{Z}_k$ by the category of unital Artinian local $\mathbb{k}$-CDGA (commutative differential graded algebra) with residue field $\mathbb{k}$; see Theorem 5.5, [10].

(b) Let $\mathcal{C}_k\mathcal{A}$ be the full subcategory of $\mathcal{C}_k$ whose objects consist of unital $\mathbb{Z}$-graded commutative artinian local $\mathbb{k}$-algebra with residue field $\mathbb{k}$ and a differential. Then we can construct a (generalized) deformation functor $\mathcal{P}\text{Def}_{(\mathcal{A}, \mathcal{K})}$ from $\mathcal{C}_k\mathcal{A}$ to $\textbf{Set}$, by associating, for any object $(\mathcal{A}, \mathcal{K})$ of $\mathcal{C}_k$,

$$a \to \mathcal{P}\text{Def}_{(\mathcal{A}, \mathcal{K})}(a) = \{I_\circ \in (m_\circ \otimes \mathcal{A})^0 : K e^{I_\circ} = 0 \}/ \sim$$

where $\sim$ is the equivalence relation given in Definition 3.4. The key point here is that $\mathcal{C}_k$ forms a monoidal category and so a morphism $f : (\mathcal{A}, K) \to (\mathcal{A}', K')$, in $\mathcal{C}_k\mathcal{A}$ (recall that these are not ring homomorphisms) sends the solution $I_\circ$ of $K e^{I_\circ} = 0$ to the solution $\Phi f (I_\circ)$ of $K e^{\Phi f (I_\circ)} = 0$. We will study this functor more carefully in the sequel paper.

Now we examine what we are deforming by the functor $\mathcal{P}\text{Def}_{(\mathcal{A}, \mathcal{K})}$. A solution of the functor $\mathcal{P}\text{Def}_{(\mathcal{A}, \mathcal{K})}$ gives a formal deformation of $(\mathcal{A}, \mathcal{K}) \in \text{Ob}(\mathcal{C}_k)$ inside the category $\mathcal{C}_k$; see Lemma 3.8 below. We use new notation $K_f$ for $L_f^k$ in (3.7)

$$K_f(\lambda) := L_f^k(\lambda) := \left( K\lambda + t_2^k (\Gamma, \lambda) + \sum_{n=3}^{\infty} \frac{1}{(n-1)!} t_n^k (\Gamma, \cdots, \Gamma, \lambda) \right),$$

(3.21)

for any homogeneous element $\lambda \in a \otimes \mathcal{A}$.

Lemma 3.8. Let $a \in \text{Ob}(\mathcal{C}_k)$ and $\Gamma \in (m_\circ \otimes \mathcal{A})^0$ be a solution of the Maurer-Cartan equation in (3.19). Then $K_f$ is a $\mathbb{k}$-linear map on $a \otimes \mathcal{A}$ of degree 1 and satisfies

$$K_f^2 = 0.$$

In other words, $(a \otimes \mathcal{A}, K_f)$ is also an object of the category $\mathcal{C}_k$.

Proof. If we plug in $\lambda = K_f(\nu)$ for any homogeneous element $\nu \in a \otimes \mathcal{A}$ in (3.7), then

$$K(K_f(\nu) \cdot e^{I_\circ}) = K_f(K_f(\nu)) \cdot e^{I_\circ} = K_f^2(\nu) \cdot e^{I_\circ}$$

since $K e^{I_\circ} = 0$. But the left hand side of the above equality is

$$K(K(\nu \cdot e^{I_\circ}) - (-1)^{|\nu|} \nu \cdot K e^{I_\circ}) = K^2(\nu \cdot e^{I_\circ}) = 0$$

by using (3.7) again. Therefore $K_f^2(\nu) \cdot e^{I_\circ} = 0$, which implies that $K_f^2 = 0$. It is obvious that $K_f$ has degree 1 by its construction. □

By the above lemma, we can consider the cohomology $H_{K_f}(a \otimes \mathcal{A})$ of the cochain complex $(a \otimes \mathcal{A}, K_f)$. Moreover, we can formally deform a morphism $\'\mathcal{A} : (\mathcal{A}, K) \to (\mathbb{k}, 0)$ by using a solution of $\mathcal{P}\text{Def}_{(\mathcal{A}, \mathcal{K})}$. 


Lemma 3.9. Let \( \mathcal{C} : (\mathfrak{g}, K) \to (k, 0) \) be a cochain map. Let \( \Gamma \in (m_\mathfrak{g} \otimes \mathfrak{g})^0 \) be the solution of the Maurer-Cartan equation in (3.19). If we define
\[
\mathcal{C}_\Gamma(x) = \mathcal{C}(x \cdot e^\Gamma), \quad x \in \mathfrak{a} \otimes \mathfrak{g},
\]
then \( \mathcal{C}_\Gamma : (\mathfrak{a} \otimes \mathfrak{g}, K_\Gamma) \to (\mathfrak{a} \otimes k, 0) \) is also a cochain map, where \( K_\Gamma \) is given in (3.21).

Proof. We have to check that \( \mathcal{C}_\Gamma \circ K_\Gamma = 0 \). This follows from (3.7);
\[
(\mathcal{C}_\Gamma \circ K_\Gamma)(x) = \mathcal{C}(K_\Gamma(x) \cdot e^\Gamma) = \mathcal{C}(K(x \cdot e^\Gamma)) = 0,
\]
for any homogeneous element \( x \in \mathfrak{a} \otimes \mathfrak{g} \).

3.5. Invariants of homotopy types of \( L_\infty \)-morphisms

Throughout Subsection we assume that the cohomology space \( H_K = H_K(\mathfrak{g}) \) is a finite dimensional \( k \)-vector space; let \( \{ e_\alpha : \alpha \in I \} \) be a homogeneous \( k \)-basis of \( H_K \) where \( I \subseteq \mathbb{N} \) is a finite index set. Let \( \{ t^\alpha : \alpha \in I \} \) be the dual \( k \)-basis of \( \{ e_\alpha : \alpha \in I \} \). Recall that the degree of \( t^\alpha = -|e_\alpha| \) where \( e_\alpha \in H_K^{\deg e_\alpha} \).

Proposition 3.3. Let \( \mathcal{C} : (\mathfrak{g}, \cdot, K) \to (k, \cdot, 0) \) be a morphism in the category \( \mathfrak{C}_k \). Let \( C : H_K \to k \) be the induced map on cohomology. Then \( C : H_K \to k \) can be enhanced to an \( L_\infty \)-morphism \( \phi^H \circ \phi^H \) for some \( L_\infty \)-quasi-isomorphism \( \phi^H \), i.e. in the following diagram
\[
\begin{align*}
(H_K, 0) & \xrightarrow{\phi^H} (\mathfrak{g}, \ell^K) \xrightarrow{\phi^H} (k, 0), \\
\end{align*}
\]
we have \( C = \mathcal{C} \circ \phi^H_1 \).

Proof. Theorem 3.1 gives us the descendant \( L_\infty \)-morphism \( \phi^H \) of the cochain map \( \mathcal{C} \). Proposition 3.1 supplies us with an \( L_\infty \)-quasi-morphism \( \phi^H \) (which is not a descendant \( L_\infty \)-morphism), which corresponds to a solution \( \Gamma^H \) of the Maurer-Cartan equation of the descendant \( L_\infty \)-algebra \( (\mathfrak{g}, \ell^K) \) (see Proposition 3.2). Let \( \phi^H_\mathcal{C} \circ \phi^H \) be the \( L_\infty \)-morphism from \( (H_K, 0) \) to \( (k, 0) \) which is defined to be the composition of \( L_\infty \) morphisms \( \phi^H \) and \( \phi^H_\mathcal{C} \). Then the desired result \( C = \mathcal{C} \circ \phi^H_1 \) follows, since the first piece \( \phi^H_1 \) of \( \phi^H \) is a cochain quasi-isomorphism. \( \square \)

Definition 3.5. Let \( \mathcal{C} : (\mathfrak{g}, \cdot, K) \to (k, \cdot, 0) \) be a morphism in the category \( \mathfrak{C}_k \).

(a) A solution \( \Gamma \in (m_\mathfrak{g} \otimes \mathfrak{g})^0 \) of the Maurer-Cartan equation \( \sum_{n=1}^\infty \frac{1}{n!} \ell_n(\Gamma, \cdots, \Gamma) = 0 \), i.e. \( K(e^\Gamma) = 0 \), is called a versal solution, if the corresponding \( L_\infty \)-morphism \( \phi[\Gamma] \) is an \( L_\infty \)-quasi-isomorphism.
(b) Let $\Gamma \in (m_{\hat{\Gamma}} \otimes \mathcal{C})^0$ be a solution of $K(e^\Gamma - 1) = 0$ corresponding to an $L_\infty$-morphism $\varphi$ (by Proposition 3.2);

$$\Gamma = \Gamma(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\alpha_1, \ldots, \alpha_n} t^{\alpha_n} \cdots t^{\alpha_1} \otimes \varphi_n(e_{\alpha_1}, \ldots, e_{\alpha_n}) \in (m_{\hat{\Gamma}} \otimes \mathcal{C})^0. \tag{3.23}$$

We define the generating power series attached to $\mathcal{C}$ and $\Gamma$, as the following power series in $t = \{ t^\alpha : \alpha \in I \}$ of $m_{\hat{\Gamma}}$;

$$\mathcal{C}(e^\Gamma - 1) = \mathcal{C}(e^{\Gamma(t)} - 1) \in \mathcal{S}^H.$$ 

**Lemma 3.10.** Let $\mathcal{C} : (\mathcal{C}, \Lambda) \rightarrow (k, \Omega)$ be a morphism in the category $\mathcal{C}_k$. If we define $\Omega_{\alpha_1 \cdots \alpha_n} \in \mathcal{k}$ by the following equality

$$\mathcal{C}(e^{\Gamma(t)} - 1) = e^{\mathcal{C}(\varphi_{\Gamma}(t))} - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\alpha_1, \ldots, \alpha_n} t^{\alpha_n} \cdots t^{\alpha_1} \otimes \Omega_{\alpha_1 \cdots \alpha_n} \in \mathcal{S}^H,$$ 

then we have

$$\mathcal{C}(\varphi_{\Gamma}(t)) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\alpha_1, \ldots, \alpha_n} t^{\alpha_n} \cdots t^{\alpha_1} \otimes (\varphi_{\Gamma})_{n}(e_{\alpha_1}, \ldots, e_{\alpha_n}) \in m_{\hat{\Gamma}}^0, \tag{3.25}$$

where $e_B = e_{j_1} \otimes \cdots \otimes e_{j_n}$ for $B = \{ j_1, \ldots, j_n \}$.

**Proof.** We leave this combinatorial lemma as an exercise; it follows from Definition 5.5 of the composition of $L_\infty$-morphisms. \qed

**Definition 3.6.** For a given $L_\infty$-morphism $\mathcal{Z} = \kappa_1, \kappa_2, \cdots$, from an $L_\infty$-algebra $(V, \mathcal{Z})$ to $(k, \Omega)$, we define the following power series in $t = \{ t^\alpha : \alpha \in I \}$ of $m_{\hat{\Gamma}}$;

$$\mathcal{Z}_{[\Omega]}(t) := \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\alpha_1, \ldots, \alpha_n} t^{\alpha_n} \cdots t^{\alpha_1} \otimes \kappa_n(e_{\alpha_1}, \ldots, e_{\alpha_n}) \right) - 1 \in \mathcal{S}^V,$$

where $\{ e_\alpha : \alpha \in I \}$ is a basis of a $k$-vector space $V$ and $\{ t^\alpha : \alpha \in I \}$ is its dual $k$-basis. Here we use the notation $\mathcal{S}^V := \lim_{\to n} \oplus_{k=0}^{n} S^k(V^*)$ with $V^* = \text{Hom}(V, k)$.

If we let $V = H_k$ and $\mathcal{S}^V = \mathcal{S}^H$, then Lemma 3.10 implies that

$$\mathcal{Z}_{[\varphi^\star \Omega]}(t) = \mathcal{C}(e^{\Gamma(t)} - 1). \tag{3.26}$$
The main theorem here is that the generating power series \( \mathcal{G}(e^{\Gamma} - 1) \) is an invariant of the homotopy types of \( \mathcal{G} \) and \( \Gamma(\mathcal{L}) \). Accordingly, we show that \( Z_{[\mathcal{G}, \mathcal{L}]}(\mathcal{L}) \) is an invariant of the \( L_\infty \)-homotopy types of the two \( L_\infty \)-morphisms. Let \( \Gamma, \tilde{\Gamma} \in (m_\mathbb{A} \otimes V)^0 \) such that

\[
\sum_{n=1}^{\infty} \frac{1}{n!} \ell_n(\Gamma, \ldots, \Gamma) = 0 = \sum_{n=1}^{\infty} \frac{1}{n!} \ell_n(\tilde{\Gamma}, \ldots, \tilde{\Gamma}).
\]

**Theorem 3.2.** Let \( \Gamma \) be homotopy equivalent to \( \tilde{\Gamma} \) (see Definition 3.4). Let \( \mathcal{G} \) be a cochain map which is cochain homotopic to \( \mathcal{G} \). Then we have

\[
\mathcal{G}(e^{\Gamma} - 1) = \mathcal{G}(e^{\tilde{\Gamma}} - 1). \tag{3.27}
\]

Similarly, if \( \mathcal{G} \) (corresponding to \( \Gamma \)) is \( L_\infty \)-homotopic to some \( \tilde{\mathcal{G}} \) (corresponding to \( \tilde{\Gamma} \)) and \( \mathcal{G}^{\mathcal{L}} \) is \( L_\infty \)-homotopic to some \( \tilde{\mathcal{L}} \)-morphism \( \tilde{\mathcal{L}} \), then we have

\[
Z_{[\mathcal{G}, \mathcal{L}]}(\mathcal{L}) = Z_{[\tilde{\mathcal{G}}, \tilde{\mathcal{L}}]}(\tilde{\mathcal{L}}). \tag{3.28}
\]

**Proof.** We first prove (3.28). If \( \mathcal{G} \) (corresponding to \( \Gamma \)) is \( L_\infty \)-homotopic to some \( \tilde{\mathcal{G}} \) (corresponding to \( \tilde{\Gamma} \)) and \( \mathcal{G}^{\mathcal{L}} \) is \( L_\infty \)-homotopic to some \( \tilde{\mathcal{L}} \)-morphism \( \tilde{\mathcal{L}} \), then Lemma 5.1 implies that \( \mathcal{G}^{\mathcal{L}} \cdot \mathcal{G} \) is \( L_\infty \)-homotopic to \( \tilde{\mathcal{L}} \cdot \tilde{\mathcal{G}} \). Because both \( \mathcal{G}^{\mathcal{L}} \cdot \mathcal{G} \) and \( \tilde{\mathcal{L}} \cdot \tilde{\mathcal{G}} \) are defined from \( (H_K, \mathcal{0}) \) into \( (k, \mathcal{0}) \), i.e. both \( H_K \) and \( k \) have zero \( L_\infty \)-algebra structures, they should be the same by Definition 5.9;

\[
\mathcal{G}^{\mathcal{L}} \cdot \mathcal{G} = \tilde{\mathcal{L}} \cdot \tilde{\mathcal{G}}.
\]

Therefore we have the equality (3.28);

\[
Z_{[\mathcal{G}, \mathcal{L}]}(\mathcal{L}) = Z_{[\tilde{\mathcal{G}}, \tilde{\mathcal{L}}]}(\tilde{\mathcal{L}}).
\]

We now prove (3.27) by using the equalities (3.26) and (3.28). Since Proposition 3.2, (b) gives a correspondence between the homotopy types of \( \mathcal{G} = \Gamma_\mathcal{G} \) and the \( L_\infty \)-homotopy types of \( \mathcal{G} \) and Lemma 3.6 says that \( \mathcal{G}^{\mathcal{L}} \) is \( L_\infty \)-homotopic to \( \tilde{\mathcal{L}} \cdot \mathcal{G} \), we see the invariance \( \mathcal{G}(e^{\Gamma} - 1) = \mathcal{G}(e^{\tilde{\Gamma}} - 1) \) as follows by using (3.26) and (3.28);

\[
\mathcal{G}(e^{\Gamma} - 1) = Z_{[\mathcal{G}, \mathcal{L}]}(\mathcal{L}) = Z_{[\tilde{\mathcal{L}} \cdot \mathcal{G}, \mathcal{L}]}(\mathcal{L}) = \mathcal{G}(e^{\tilde{\Gamma}} - 1) = \mathcal{G}(e^{\tilde{\Gamma} - 1}).
\]

We also provide an alternative proof of (3.27) in order to illustrate a key idea behind \( L_\infty \)-homotopy invariance in a better way. If \( \mathcal{G} : (\mathcal{G}, K) \rightarrow (k, \mathcal{0}) \) is cochain homotopic to \( \tilde{\mathcal{G}} \), then \( \mathcal{G} = \mathcal{G} + \mathcal{Y} \circ K \) where \( \mathcal{Y} \) is a cochain homotopy. Since \( \mathcal{G} \circ K = 0 \), we have

\[
\mathcal{G}(e^{\Gamma} - 1) = \mathcal{G}(e^{\tilde{\Gamma}} - 1).
\]
If $\Gamma$ is homotopy equivalent to $\tilde{\Gamma}$, Definition 3.4 says that there is a polynomial solution $\Gamma(\tau) \in (m_{\mathcal{SH}} \otimes \mathcal{A})^0[\tau]$ with $\Gamma(0) = \Gamma$ and $\Gamma(1) = \tilde{\Gamma}$ to the following flow equation

$$\frac{\partial}{\partial \tau} e^{\Gamma(\tau)} = K \left( \lambda(\tau) \cdot e^{\Gamma(\tau)} \right)$$

for some 1-variable polynomial $\lambda(\tau) \in (m_{\mathcal{SH}} \otimes \mathcal{A})^0[\tau]$. This implies that $e^{\tilde{\Gamma}} - e^\Gamma = K \left( \int_0^1 \lambda(\tau) \cdot e^{\Gamma(\tau)} d\tau \right)$, which in turn implies that $\mathcal{C}(e^{\tilde{\Gamma}}) = \mathcal{C}(e^\Gamma)$. $\square$

Let $\mathcal{Z}$ be the quotient $k$-vector space of all (the degree 0) cochain maps from $(\mathcal{A}, K)$ to $(k, 0)$ modulo the subspace of all the maps of the form $\mathcal{J} \circ K$ where $\mathcal{J} : \mathcal{A} \to k$ varies over $k$-linear maps of degree -1. In other words, $\mathcal{Z}$ is the space of cochain homotopy classes of maps from $(\mathcal{A}, K)$ to $(k, 0)$.

**Proposition 3.4.** Let $(\mathcal{A}, K) = (\mathcal{A}_\rho, K_\rho)$ be associated to a Lie algebra representation $\rho$. The $k$-vector space $\mathcal{Z}$ is isomorphic to the $k$-dual of $H^0_k(\mathcal{A})$.

**Proof.** Since every representative $\mathcal{C}$ of an element of $\mathcal{Z}$ has degree 0 and $(k, 0)$ is concentrated in degree 0, all the homogeneous elements except for degree 0 elements in $\mathcal{C}$ vanish under the map $\mathcal{C}$. In fact, there is no $k$-linear map $\mathcal{J} : \mathcal{A} \to k$ of degree -1, since $\mathcal{A}$ does not have positive degree, i.e. $\mathcal{A}^1 = \mathcal{A}^2 = \cdots = 0$. Then it is clear that $\mathcal{Z}$ is isomorphic to the $k$-dual of $(\mathcal{A}/(\mathcal{A}^1)) =: H^0_k(\mathcal{A})$. $\square$

### 3.6. Differential equations attached to variations of period integrals

The main goal of this Subsection is to prove that the generating power series $\mathcal{C}(e^{\Gamma(\mathcal{U})_l^H} - 1) = \mathcal{Z}(\mathcal{U}^* \otimes m_{\mathcal{SH}}(\mathcal{U})$ attached to $\mathcal{C}$ and a Maurer-Cartan solution $\Gamma(\mathcal{U})_l^H$ corresponding to an $L_\infty$-quasi-isomorphism $\varphi^H$ satisfies a system of second order partial differential equations.

These differential equations, which are obtained by analyzing the binary product structure on $\mathcal{A}$ and the differential $K$, are governed by the underlying infinity homotopy structure on $(\mathcal{A}, L^K)$, namely the descendant $L_\infty$-algebra $(\mathcal{A}, L^K)$; we will show that the differential equations themselves are *invariants of the $L_\infty$-homotopy type* of the solution of the Maurer-Cartan equation. See Theorem 3.3 for details. Moreover, these differential equations will lead to a $Z$-graded flat connection on the tangent bundle of the formal deformation space of $\mathcal{D}(\mathcal{A}, L^K)$; see Theorem 3.4.

By Proposition 3.2, a (homotopy type of) solution $\Gamma = \Gamma_\varphi \in (m_{\mathcal{SH}} \otimes \mathcal{A})^0$ of the Maurer-Cartan equation

$$\sum_{n=1}^{\infty} \frac{1}{n!} L^K_n(\Gamma', \ldots, \Gamma) = 0, \quad (3.29)$$
gives us an \((L_\infty\)-homotopy type of) \(L_\infty\)-morphism \(\varphi\) from \((H_K, \emptyset)\) to \((\mathcal{A}, \mathcal{E})\). We will make differential equations with respect to parameter \(\{t^a\}\) in the complete local \(k\)-algebra \(\widehat{SH}\). Here recall that \(\widehat{SH} := \lim_{n \to \infty} \bigoplus_{k=0}^n S^k(H_K^\infty)\) which is isomorphic to \(k[[t]]\).

**Theorem 3.3.** Let \(\psi : (\mathcal{A}, \cdot, K) \to (k, \cdot, \emptyset)\) be a morphism in the category \(\mathcal{C}_k\). Let \(\Gamma = \Gamma(t) : \in (m_{\widehat{SH}} \otimes \mathcal{A})^0\) be a solution of the Maurer-Cartan equation (3.29) corresponding to an \(L_\infty\)-quasi-isomorphism \(\varphi^K\) from \((H_K, \emptyset)\) to \((\mathcal{A}, \mathcal{E})\). Assume that \(H_K(\mathcal{A})\) is a finite dimensional \(k\)-vector space and \(H_K(\widehat{SH} \otimes \mathcal{A})\) is a free \(\widehat{SH}\)-module satisfying

\[
\dim_k H_K(\mathcal{A}) = \text{rk}_{\widehat{SH}} H_K(\widehat{SH} \otimes \mathcal{A}).
\]

(3.30)

Then there exists an element \(A_{\alpha\beta} \tilde{\Gamma}(t) = A_{\alpha\beta} \tilde{\Gamma}(t)|t = \tilde{t}\) depending on \(\tilde{t}\), where \(\tilde{t} = \{t^a : \alpha \in I\}\), such that

\[
\left(\partial_a \tilde{\Gamma} - \sum_{\gamma} A_{\alpha\beta} \tilde{\Gamma}(t) \partial_{\gamma}\right) \psi(e^{t^|I - 1} = 0, \text{ for } \alpha, \beta \in I,
\]

(3.31)

where \(\partial_a\) means the partial derivative with respect to \(t^a\) where \(\alpha \in I\). Moreover, if \(\tilde{\Gamma}\) and \(\Gamma\) are homotopy equivalent, then \(A_{\alpha\beta} \tilde{\Gamma}(t) = A_{\alpha\beta} \tilde{\Gamma}(t)|t\).

**Proof.** Note that \(\Gamma = \Gamma(t)\) depends on \(\tilde{t}\). Then the condition \(K(e^{t^|I - 1} = 0\) implies that

\[K(\partial_a e^{t^|I}) = K(\partial_a \Gamma(t) \cdot e^{t^|I}) = 0,
\]

which says, by the equality (3.7), that

\[K_{\Gamma(t)}(\partial_a \Gamma(t)) = 0.
\]

Therefore, combined with the condition (3.30), we may assume that \(\{[\partial_a \Gamma(t)] : \alpha \in I\}\) is an \(\widehat{SH}\)-basis of \(H_K(\widehat{SH} \otimes \mathcal{A})\), since \(\varphi^K\) is a cochain quasi-isomorphism. Here \(\{\cdot\}\) means the cohomology class. The condition \(K(e^{t^|I}) = 0\) also implies that

\[K(\partial_a \partial_{\gamma} e^{t^|I}) = K\left((\partial_a \partial_{\beta} \Gamma(t) + \partial_a \Gamma(t) \partial_{\beta} \Gamma(t)) \cdot e^{t^|I}\right) = 0,
\]

where \(\partial_{\alpha} = \frac{\partial^2}{\partial_{t_a} \partial_{t_\beta}}\), and the equality (3.7) says that

\[K_{\Gamma(t)}(\partial_a \partial_{\beta} \Gamma(t) + \partial_a \Gamma(t) \partial_{\beta} \Gamma(t)) \cdot e^{t^|I} = 0.
\]

Thus we should be able to write down \([\partial_a \Gamma(t) + \partial_a \Gamma(t) \partial_{\beta} \Gamma(t)]\) as an \(\widehat{SH}\)-linear combination of \([\partial_a \Gamma(t)]\)'s, i.e. there exists a unique 3-tensor \(A_{\alpha\beta} \Gamma(t) \in \widehat{SH}\) such that

\[\partial_a \Gamma(t) + \partial_a \Gamma(t) \partial_{\beta} \Gamma(t) = \sum_{\gamma} A_{\alpha\beta} \tilde{\Gamma}(t) \partial_{\gamma} \Gamma(t) + K_{\Gamma(t)}(A_{\alpha\beta} \Gamma(t))
\]

(3.32)
for some $\Lambda_{\alpha\beta}(t)\in \widehat{SH} \otimes \mathcal{A}$. Then this is equivalent to

$$\partial_\alpha \partial_\beta \gamma(t) e^{\Gamma(t)} - \sum_\gamma A_{\alpha\beta\gamma}(t) \partial_\gamma e^{\Gamma(t)} = K(\Lambda_{\alpha\beta}(t) \cdot e^{\Gamma(t)}).$$

We finish the proof by applying $\mathcal{C}_{\cal H}$ to the above equality and using the fact that $\mathcal{C}_{\cal H}$ is a cochain map, i.e. $\mathcal{C}_{\cal H} \circ K = 0$.

If $\tilde{\Gamma}(t)$ and $\Gamma(t)$ are homotopy equivalent, then, according to Definition 3.4, we have

$$e^{\tilde{\Gamma}(t)} - e^{\Gamma(t)} = K \left( \int_0^1 \lambda(\tau) e^{\Gamma(t)(\tau)} d\tau \right),$$

where $\Gamma(t)(1) = \tilde{\Gamma}(t)$ and $\Gamma(t)(0) = \Gamma(t)$. By using (3.33) we can derive the following;

$$\partial_\alpha \partial_\beta (e^{\tilde{\Gamma}(t)} - e^{\Gamma(t)}) + \left( \sum_\gamma A_{\alpha\beta\gamma}(t) \partial_\gamma e^{\Gamma(t)} - \sum_\gamma A_{\alpha\beta\gamma}(t) \partial_\gamma e^{\tilde{\Gamma}(t)} \right)$$

$$= K(\Lambda_{\alpha\beta}(t) \cdot e^{\Gamma(t)} - \Lambda_{\alpha\beta}(t) \cdot e^{\tilde{\Gamma}(t)}).$$

Therefore (3.34) implies that

$$\sum_\gamma A_{\alpha\beta\gamma}(t) \partial_\gamma e^{\Gamma(t)} - \sum_\gamma A_{\alpha\beta\gamma}(t) \partial_\gamma e^{\tilde{\Gamma}(t)} \in \text{Im } K.$$

After we add $\sum_\gamma A_{\alpha\beta\gamma}(t) \partial_\gamma e^{\Gamma(t)}$ and subtract to the above, we can apply (3.34) to prove that

$$D := \left( \sum_\gamma (A_{\alpha\beta\gamma}(t) \Lambda_{\alpha\beta\gamma}(t) \partial_\gamma \Gamma(t)) \cdot e^{\Gamma(t)} \right) \in \text{Im } K.$$

Note that $D$ has the form $K(\xi \cdot e^{\Gamma(t)})$ for some $\xi$. Then (3.7) implies that

$$\sum_\gamma (A_{\alpha\beta\gamma}(t) \Lambda_{\alpha\beta\gamma}(t) \partial_\gamma \Gamma(t)) \partial_\gamma \Gamma(t) \in \text{Im } K_{\Gamma(t)}.$$

Since $\{[\partial_i \Gamma(t)] : i \in I\}$ is an $\widehat{SH}$-basis of $H_{K_{\cal H}}(\widehat{SH} \otimes \mathcal{A})$, the desired result $A_{\alpha\beta\gamma}(t) = A_{\alpha\beta\gamma}(t)$ follows if we take the $K_{\Gamma(t)}$-cohomology of the above. $\square$

The method used in the proof can be made into an effective algorithm (see Subsection 4.1 for a toy model case and Subsection 4.7 for a smooth projective hypersurface case) to compute $\mathcal{C}_{\cal H}(e^{\Gamma(t)})$ and it leads to the Picard-Fuchs type differential equation for a family of hypersurfaces, if we interpret the period integrals of hypersurfaces as period integrals of quantum Jacobian Lie algebra representations attached to hypersurfaces; see Subsection 4.6.
3.7. Explicit computation of the generating power series

The goal of this Subsection is to reduce the problem of computing the generating power series \( \mathcal{G}(e^I - 1) \) attached to \( \mathcal{G} \) and \( I = \sum \mathcal{G}^{\alpha} \) to the problem of computing the 3-tensor \( A_{\alpha\beta\gamma}(\mathcal{L})_\ell \) for \( \alpha, \beta, \gamma \in I \) that appeared in (3.31), together with the proof of Theorem 1.4. Let \( \mathcal{G} : \langle \mathcal{A} \rangle, K \rightarrow \langle k \rangle, 0 \) be a morphism in the category \( \mathcal{C}_K \). Let \( I = \sum \mathcal{G}^{\alpha} \in (m_0 \otimes \mathcal{A})^0 \) be a solution of the Maurer-Cartan equation (3.29) corresponding to an \( L_\infty \)-quasi-isomorphism \( \varphi^H \) from \( (H, 0) \) to \( (\mathcal{A}, \mathcal{L}^K) \). Assume that \( H_{K}(\mathcal{A}) \) is a finite dimensional \( k \)-vector space and \( H_{K}(\widehat{SH} \otimes \mathcal{A}) \) is a free \( \widehat{SH} \)-module satisfying (3.30). Let us write \( I \) as

\[
I = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{a_1, \ldots, a_n} t^{a_n} \cdots t^{a_1} \otimes \varphi^H_n(e_{a_1}, \ldots, e_{a_n}) \in (m_{\widehat{SH}} \otimes \mathcal{A})^0.
\]  

(3.35)

**Lemma 3.11.** There exist \( T^\gamma(\mathcal{L}) \in \widehat{SH} \cong k[[t^a]] \) and \( \Lambda \in \widehat{SH} \otimes \mathcal{A} \) such that

\[
e^I = 1 + \sum_{\gamma} T^\gamma(\mathcal{L}) \cdot \varphi^H_\gamma(e_{\gamma}) + K(\Lambda) \in (\widehat{SH} \otimes \mathcal{A})^0.
\]

(3.36)

**Proof.** Recall that \( \{ e_{a_\alpha} : \alpha \in I \} \) is a \( \mathbb{Z} \)-graded homogeneous basis of \( H_{K} \). Since \( \varphi^H_\gamma \) is a cochain quasi-isomorphism, \( \varphi^H_\gamma(e_{\gamma}) \) forms a complete set of representatives of \( H_{K} \). Therefore the result follows, because \( K(e^I) = 0 \) and \( K \) is \( \widehat{SH} \)-linear. \( \square \)

**Remark 3.4.** Since \( \mathcal{G} \circ K = 0 \), we can express the generating power series as

\[
\mathcal{G}(e^I)^2 \mu - 1 = \sum \gamma T^\gamma(\mathcal{L}) \cdot \mathcal{G}(\varphi^H_\gamma(e_{\gamma})).
\]

(3.37)

So the above lemma gives us Theorem 1.4, combined with Theorems 1.1 and 1.3. Moreover, its explicit expression (1.7) in terms of \( A_{\alpha\beta\gamma}(\mathcal{L})_\ell \) in (3.31) can be derived by a direct computation.

Note that \( \mathcal{G}(\varphi^H_\gamma(e_{\gamma})) \) does not depend on \( \mathcal{L} \) and \( T^\gamma(\mathcal{L}) \) does not depend on \( \mathcal{G} \). Thus we only need to know the values \( \mathcal{G}(\varphi^H_\gamma(e_{\gamma})) \) at the basis \( \{ e_{\alpha} : \alpha \in I \} \) of the cohomology group \( H_{K} \) and the formal power series \( T^\gamma(\mathcal{L}) = T^\gamma(\mathcal{L})_\ell \), which depend only on the homotopy type of \( \varphi^H \), in order to compute \( \mathcal{G}(e^I - 1) \). We will explain how to compute \( T^\gamma(\mathcal{L}) \); we reduce its computation to the computation of \( A_{\alpha\beta\gamma}(\mathcal{L})_\ell \). Let us write \( T^\gamma(\mathcal{L}) \) as a power series in \( \mathcal{L} \) as follows;

\[
T^\gamma(\mathcal{L}) = t^\gamma + \sum_{n=2}^{\infty} \frac{1}{n!} t^{a_n} \cdots t^{a_1} \cdot M_{a_1, a_2, \ldots, a_n}, \quad \text{for } \gamma \in I,
\]

for a uniquely determined \( M_{a_1, a_2, \ldots, a_n} \in k \). We define a \( k \)-multilinear operation \( M_n : S^n(H_{K}) \rightarrow H_{K} \) as follows

\[
M_n(e_{a_1}, \ldots, e_{a_n}) = \sum_{\gamma} M_{a_1, a_2, \ldots, a_n} \gamma e_{\gamma}, \quad n \geq 2.
\]
We also write that the 3-tensor $A_{\alpha\beta\gamma}(L)_r$ in (3.31) as follows;

$$A_{\alpha\beta\gamma}(L)_r = m_{\alpha\beta\gamma} + \sum_{n=1}^{\infty} \frac{1}{n!} \sum t^{a_1} \cdots t^{a_n} m_{a_1 \cdots a_n \alpha\beta\gamma}$$

for a uniquely determined $m_{a_1 \cdots a_n \alpha\beta\gamma} \in \mathbb{k}$. We define a $\mathbb{k}$-multilinear operation $m_n : T^n(H_K) \to H_K$ as follows

$$m_n(e_{a_1}, \ldots, e_{a_n}) = \sum_{\gamma} m_{a_1 \cdots a_n \alpha\beta\gamma} e_\gamma, \quad n \geq 2.$$

**Proposition 3.5.** Let $M_1$ be the identity map on $H_K$. Then we have the following relationship between $m_n$ and $M_n$;

$$M_n(v_1, \ldots, v_n) = \sum_{\pi \in P(n)} e(\pi)M_{|\pi|}(v_{B_1}, \ldots, v_{B_{|\pi|}}, m(v_{B_{|\pi|}}))$$

for any homogeneous element $v_1, \ldots, v_n \in H_K$ and $n \geq 2$. The following is also true;

- $M_n$ is a $\mathbb{k}$-linear map from $S^n(H_K)$ into $H_K$ of degree zero for all $n \geq 1$
- $M_{n+1}(v_1, \ldots, v_n, 1_H) = M_n(v_1, \ldots, v_n)$ for all $n \geq 1$, where $1_H$ is a distinguished element corresponding to $1,_{\alpha}$.

**Proof.** We leave this as an exercise. □

This explicit combinatorial formula says that the data of $m_n$, $n \geq 2$ completely determines $M_n$, $n \geq 2$, and vice versa; knowing $A_{\alpha\beta\gamma}(L)_r$ for $\alpha, \beta, \gamma \in I$, is equivalent to knowing $T^n(L)$ for $\gamma \in I$. By Proposition 3.5, it is enough to give an algorithm to compute the 3-tensor $A_{\alpha\beta\gamma}(L)_r$ in order to compute the generating power series $\mathcal{G}(e^L - 1)$, in addition to the values $\mathcal{G}(e^L_1(e_\alpha))$, $\alpha \in I$. We will provide an effective algorithm for computing $A_{\alpha\beta\gamma}(L)_r$, when a versal Maurer-Cartan solution $\Gamma$ is associated to $(\alpha_X, \cdot, K_X)$, which is attached to a projective smooth hypersurface $X_G$; see Subsection 4.7.

### 3.8. A flat connection on the tangent bundle of a formal deformation space

Here we will show that the system of differential equation (3.31) can be reformulated to give a flat connection on the tangent bundle of a formal deformation space. The proposition 3.2, (a) says that the deformation functor $\mathcal{D}e_{(\alpha_X, \cdot, L_X)}$ is pro-representable by $\hat{SH}$. Thus we can consider a formal deformation space $\mathcal{M}$ attached to the universal deformation ring $\hat{SH}$ of $\mathcal{D}e_{(\alpha_X, \cdot, L_X)}$, so that $\Omega^0(\mathcal{M}) = \hat{SH}$. Let $T\mathcal{M}$ be the tangent bundle of $\mathcal{M}$ and $T^*\mathcal{M}$ be the cotangent bundle of $\mathcal{M}$. Let $\Gamma(\mathcal{M}, T\mathcal{M})$ be the $\mathbb{k}$-space of global sections of $T\mathcal{M}$ and $\Omega^p(\mathcal{M})$ be the $\mathbb{k}$-space of differential $p$-forms on $\mathcal{M}$. We list some important properties of the 3-tensor $A_{\alpha\beta\gamma}(L)_r$ in (3.31), which we use to prove the existence of a flat connection on $T\mathcal{M}$. 
**Lemma 3.12.** The 3-tensor $A_{\alpha\beta}^\gamma := A_{\alpha\beta}^\gamma(t)_{I'}$ in Theorem 3.3 satisfies the following properties.

\[
A_{\alpha\beta}^\gamma - (-1)^{\varepsilon_\alpha\varepsilon_\beta} A_{\alpha\beta}^\gamma = 0,
\]

\[
\partial_\alpha A_{\beta\gamma}^\sigma - (-1)^{\varepsilon_\alpha} \partial_\beta A_{\alpha\gamma}^\sigma + \sum_\rho \left( A_{\beta\gamma}^\rho A_{\rho\sigma}^\alpha - (-1)^{\varepsilon_\alpha} A_{\alpha\gamma}^\rho A_{\beta\rho}^\sigma \right) = 0,
\]

for all $\alpha, \beta, \gamma, \sigma \in I$, where $\partial_\alpha$ means the partial derivative with respect to $t^\alpha$.

**Proof.** The first one follows from the super-commutativity of the binary product of $\mathcal{S}$. The second one can be proved by taking the derivatives of (3.31) with respect to $t$ and using the associativity of the binary product of $\mathcal{S}$. We leave this as an exercise. \(\square\)

Note that $T(M)$ is a trivial bundle and let us write $T(M) = M \times V$, where $V$ is isomorphic to $H^*_K$. The most general connection is of the form $d + A$, where $A$ is an element of $\Omega^1(M) \otimes \text{End}_K(V)$. Here $\otimes$ denotes the completed tensor product. We define a 1-form valued matrix $A_I$ by

\[
(A_I)_\beta^\gamma := -\sum_\alpha dt^\alpha \cdot A_{\alpha\beta}^\gamma (t)_{I'}, \quad \beta, \gamma \in I,
\]

where $A_{\alpha\beta}^\gamma (t)_{I'}$ is given in (3.31). Then $A_I \in \Omega^1(M) \otimes \text{End}_K(V)$.

**Theorem 3.4.** Let $\Gamma = \Gamma \in \mathcal{D}_{\mathcal{H}, L}^{\infty} \mathcal{L}^K$ which corresponds to an $L^\infty$-quasi-isomorphism $\varphi^H$. Then the $\mathbb{K}$-linear operator $D_I := d + A_I$ defined in (3.38)

\[
D_I := d + A_I : \Gamma(M, T(M)) \rightarrow \Omega^1(M) \otimes \mathbb{K} \Gamma(M, T(M)),
\]

is a $\mathbb{Z}$-graded flat connection on $T(M)$.

**Proof.** This obviously defines a $\mathbb{Z}$-graded connection. We compute the curvature of the connection $D_I$ as follows;

\[
D_I^2 = dA_I + A_I^2.
\]

Then a simple computation confirms that $dA_I + A_I^2 = 0$ is equivalent to the second equality of Lemma 3.12 by using the first equality of Lemma 3.12. Thus $dA_I + A_I^2 = 0$ and $D_I$ is flat. \(\square\)

**Proposition 3.6.** Let $\varphi^H$ be an $L^\infty$-quasi-isomorphism from $(H_K, \mathcal{L})$ to $(\mathcal{S}, \mathcal{L}^K)$. The generating power series $\mathcal{C}(e^t - 1)$ attached to $\mathcal{C}$ and $\Gamma = \Gamma_{\mathbb{Z}^n}$ satisfies the equation

\[
d(\partial_\gamma \mathcal{C}(e^t - 1)) = -A_I(\partial_\gamma \mathcal{C}(e^t - 1)), \quad \text{i.e. } D_I(\partial_\gamma \mathcal{C}(e^t - 1)) = 0,
\]

where we view $\partial_\gamma \mathcal{C}(e^t - 1)$ as a column vector indexed by $\gamma \in I$.

**Proof.** If we multiply (3.31) by $t^\alpha$ and sum over $\alpha \in I$, then we get

\[
\left( \sum_\alpha dt^\alpha \partial_\alpha \delta_\beta^\gamma - \sum_\gamma \sum_\alpha dt^\alpha \cdot A_{\alpha\beta}^\gamma (t) \right) (\partial_\gamma \mathcal{C}(e^t - 1)) = 0, \quad \text{for } \beta \in I,
\]

(3.40)

where $\delta_\beta^\gamma$ is the Kronecker delta symbol. Then the desired equality follows from Definition of $A_I$ in (3.38), by noting that $d = \sum_\alpha dt^\alpha \partial_\alpha$. \(\square\)
4. Period integrals of smooth projective hypersurfaces

In [6], P. Griffiths extensively studied the period integrals of smooth projective hypersurfaces. We use his theory to give a non-trivial example of a period integral of a certain Lie algebra representation and reveal its hidden structures namely its correlations and variations, by applying the general theory we developed so far. We start with a toy example in order to illustrate our applications of the general theory more transparently.

4.1. Toy model

We go back to the example 2.2. We have a (quantum Jacobian) Lie algebra representation $\rho_G$ attached to a polynomial $G(x) \in \mathbb{R}[x]$ of degree $d + 1$. The associated cochain complex $(\mathcal{A}^*_\rho_G, K_{\rho_G})$ is given by $\mathcal{A}^*_\rho_G = \mathcal{A}^{d-1}_G \oplus \mathcal{A}^0_G$ and $K_{\rho_G} = \left( \frac{d}{dx} + G(x) \right).$ The map $C(f) = \int_{-\infty}^\infty f(x)e^{G(x)} \, dx$ can be enhanced to $\mathcal{C}_G : (\mathcal{A}^*_\rho_G, K_{\rho_G}) \to (\mathcal{K}, 0)$ by Proposition 2.3.

**Proposition 4.1.** The cohomology group $H^{d-1}_\mathcal{K}(\mathcal{A}^*_G)$ vanishes and $H^0_\mathcal{K}(\mathcal{A}^*_G)$ is a finite dimensional $\mathcal{K}$-vector space. Its dimension is the degree of the polynomial $G(x)$.

**Proof.** Even though the proof is straightforward, we decide to provide details in order to give a clue about the more complicated multi-variable version, Lemma 4.7. Note that $K_G$ consists of the quantum part $\Delta = \frac{d}{dx}$ and the classical part $Q = G(x)\frac{d}{dx}$. The image of $\mathcal{A}^{d-1}$ under the classical part $Q$ defines the Jacobian ideal of $\mathbb{R}[x]$. We first solve an ideal membership problem (just the Euclidean algorithm in the one variable case); for a given any $f(x) \in \mathbb{R}[x]$, there is an effective algorithm to find unique polynomials $q^{0}(x), r^{0}(x) \in \mathbb{R}[x]$ such that

$$f(x) = G'(x)q^{0}(x) + r^{0}(x), \quad \deg(r^{0}(x)) \leq d - 1.$$ 

(4.1)

Then we use the quantum part $\Delta$ to rewrite (4.1);

$$f(x) = -\frac{\partial q^{0}(x)}{\partial x} + (G'(x) + \frac{\partial}{\partial x})q^{0}(x) + r^{0}(x) = -\frac{\partial q^{0}(x)}{\partial x} + r^{0}(x) + K_{\rho_G}(q^{0}(x) \cdot \eta).$$

Then we again use the Euclidean algorithm to find unique $q^{1}(x), r^{1}(x) \in \mathbb{R}[x]$ such that

$$-\frac{\partial q^{0}(x)}{\partial x} = G'(x)q^{1}(x) + r^{1}(x), \quad \deg(r^{1}(x)) \leq d - 1.$$

This implies that

$$f(x) = G'(x)q^{1}(x) + r^{1}(x) + r^{0}(x) + K_{\rho_G}(q^{0}(x) \cdot \eta)$$

$$= -\frac{\partial q^{1}(x)}{\partial x} + r^{1}(x) + r^{0}(x) + K_{\rho_G}(q^{1}(x) \cdot \eta + q^{0}(x) \cdot \eta).$$
We can continue this process, which stops in finite steps, which shows that the dimension of $H^d_{\cal K_G}(\cal A_G)$ is not bigger than $d$. But $1,x,\ldots,x^{d-1}$ can not be in the image of $K_G$ because of degree. Thus the result follows. The vanishing of $H^{-1}$ is trivially derived, since any solution $u(x)$ to the differential equation $\frac{du(x)}{dx} + G'(x) \cdot u(x) = 0$ can not be a polynomial. □

This type of interaction between quantum and classical components of $K_G$ will be the key technique to compute effectively (a fancy way of doing integration by parts) the generating power series $\mathcal{C}(e^f - 1)$ of period integrals of the quantum Jacobian Lie algebra representation. Therefore $(\mathcal{A}_G, K_G)$ is quasi-isomorphic to the finite dimensional space $(H^0_{\cal K_G}(\mathcal{A}_G), 0)$ with zero differential; this is a perfect example to apply all the results developed in section 3. We record a general result in order to compute descendant $L_\infty$-algebras.

**Proposition 4.2.** Let $S \in \mathbb{k}[q^1, \ldots, q^m]$ be a multi-variable polynomial. The descendant $L_\infty$-algebra of cochain complex $(\mathcal{A}, K)_{\rho_S} = (\mathcal{A}_{\rho_S}, K_{\rho_S})$ associated to the quantum Jacobian Lie algebra representation $\rho_S$ in (2.2) is a differential graded Lie algebra over $\mathbb{k}$, i.e. $\ell^1_{\mathcal{K}_{\rho_S}} = \ell^2_{\mathcal{K}_{\rho_S}} = \cdots = 0$. Moreover, $(\mathcal{A}_{\rho_S}, \ell^2_{\mathcal{K}_{\rho_S}})$ is a Poisson algebra over $\mathbb{k}$.

**Proof.** The equality (3.5) implies the result, since $K_{\rho_S}$ is a differential operator on $\mathcal{A}_{\rho_S}$ of order 2. For the second claim, we have to show that $\ell^2_{\mathcal{K}_{\rho_S}}$ is a derivation of the product;

$$\ell^2_{\mathcal{K}_{\rho_S}}(a \cdot b, c) = (-1)^{|a|} a \cdot \ell^2_{\mathcal{K}_{\rho_S}}(b, c) + (-1)^{|b|+|c|} \cdot \ell^2_{\mathcal{K}_{\rho_S}}(a, c) \cdot b,$$

for any homogeneous elements $a, b, c \in \mathcal{A}_{\rho_S}$. This follows from a direct computation, which uses again the fact that $K_{\rho_S}$ is a differential operator of order 2. □

**Remark 4.1.** Such a quadruplet $(\mathcal{A}_{\rho_S}, \cdot, K, \ell^2_{\mathcal{K}_{\rho_S}})$ is called a BV algebra (Batalin-Vilkovisky algebra). Of course, in general, the cochain complex $(\mathcal{A}, \cdot, K)_{\rho}$ associated to $\rho$ is neither a differential graded Lie algebra nor a Poisson algebra. So the quantum Jacobian Lie algebra representations are very special ones, while they are general enough to study period integrals of algebraic varieties.

Thus, by Proposition 4.2, the descendant $L_\infty$-algebra $(\mathbb{R}[x][\eta], \ell_2)$, where

$$\ell_2(u, v) := \ell^2_{\mathcal{K}_G}(u, v) = K_G(u \cdot v) - K_G(u) \cdot v - (-1)^{|u|} u \cdot K_G(v),$$

is a differential graded Lie algebra (no higher homotopy structure) which is quasi-isomorphic to $(H, 0)$ where $H = H_{\cal K_G}(\mathcal{A}_G)$. But note that the descendant $L_\infty$-morphism $\varphi^\mathcal{C}_G$ from $(\mathbb{R}[x][\eta], \ell_2)$ to $(\mathbb{R}, 0)$ does have a non-trivial higher homotopy structure; $\varphi^\mathcal{C}_G, \varphi^\mathcal{C}_G^2, \cdots$ do not vanish. This higher structure governs the correlation of the period integral $\int_{-\infty}^{\infty} u(x) e^{G(x)} dx$. Let $\{e_0, \ldots, e_{d-1}\}$ be an $\mathbb{R}$-basis of $H$. We define an $L_\infty$-morphism $f = f_1, f_2, \cdots$ by

$$f_1(e_i) = x^i, \text{ for } i = 0, 1, \ldots, d - 1, \quad f_2 = f_3 = \cdots = 0,$$
which is clearly an $L_\infty$-quasi-isomorphism. The version of Theorem 1.1 for the toy model (which follows from Proposition 3.3 and Theorem 3.2) can be summarized as the following commutative diagram:

\[
\begin{array}{c}
(H, 0) \\
\downarrow f \\
(R[x][\eta], \ell_2) \quad \phi \circ f \\
\downarrow \phi \\
(R, 0)
\end{array}
\]

(4.3)

Then $\Gamma(f) := \sum_{n=1}^{\infty} \frac{1}{n!} \sum t_a \cdots t_a f_n(e_a, \cdots, e_a) = \sum_{i=0}^{d-1} t_i \cdot x^i \in \mathbb{R}[x][\ell], \text{ where } \{t_a\} = \{t_0, \cdots, t_{d-1}\}$ is an $\mathbb{R}$-dual basis to $\{e_0, \cdots, e_{d-1}\}$. Hence the generating power series for $\mathcal{C}$ and $f$ is

\[ \mathcal{Z}_{\phi \circ f}(f) = \mathcal{C}(e^{\Gamma(f)} - 1) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} \phi^n \left( \sum_{i=0}^{d-1} t_i \cdot x^i \right)^n \right) - 1, \]

which is an $L_\infty$-homotopy invariant (Theorem 3.2) so that it gives rise to a function $f \mapsto \mathcal{Z}_{\phi \circ f}(f)$ on the formal versal deformation space attached to $(R[x][\eta], \ell_2)$. The lemma 3.11 says that there exists a power series $T^{(i)}(f) = T^{(i)}(f) \in \mathbb{R}[[x]]$ for each $i = 0, 1, \cdots, d - 1$ such that

\[ \mathcal{Z}_{\phi \circ f}(f) = \sum_{i=0}^{d-1} T^{(i)}(f) \cdot \left( \int_{-\infty}^{\infty} x^i e^{G(x)} dx \right). \]

Note that the 1-tensor $T^{(i)}(f)$ has all the information of the integrals $\int_{-\infty}^{\infty} x^m e^{G(x)} dx$ for $m \geq d$, and is completely determined by the cochain complex $(R[x][\eta], (\partial_x + G(x)) \frac{\phi}{\partial \eta})$ with super-commutative multiplication. The Euclidean algorithm enhanced with quantum component in Proposition 4.2, which we generalize to the Griffiths period integral (Lemma 4.7), can be used to compute $T^{(i)}(f)$ effectively and consequently compute all the moments $\int_{-\infty}^{\infty} x^m e^{G(x)} dx, \forall m \geq d$ from the finite data

\[ \int_{-\infty}^{\infty} e^{G(x)} dx, \int_{-\infty}^{\infty} x e^{G(x)} dx, \cdots, \int_{-\infty}^{\infty} x^{d-1} e^{G(x)} dx. \]

Our general theory says that this determination mechanism is governed by $L_\infty$-homotopy theory, more precisely, the interplay between the 1-tensor $T^1(f)$ and the 3-tensor $A_{a, \beta}(f)_r$ (Proposition 3.5).
4.2. The Lie algebra representation $\rho_X$ associated to a smooth projective hypersurface $X_G$

Let $\mathbb{k}$ be the complex field $\mathbb{C}$ from now on. Let $n$ be a positive integer. We use $x = [x_0, x_1, \ldots, x_n]$ as a projective coordinate of the projective $n$-space $\mathbb{P}^n$. Let $G(x) \in \mathbb{k}[x]$ be the defining homogeneous polynomial equation of degree $d \geq 1$ for a smooth projective hypersurface, denoted $X = X_G$, of $\mathbb{P}^n$. Let $\mathfrak{g}$ be an abelian Lie algebra of dimension $n + 2$. Let $\alpha_{-1}, \alpha_0, \ldots, \alpha_n$ be a $\mathbb{k}$-basis of $\mathfrak{g}$. Let

$$A := \mathbb{k}[y, x_0, \ldots, x_n] = \mathbb{k}[y, x]$$

be a commutative polynomial $\mathbb{k}$-algebra generated by $y, x_0, \ldots, x_n$. We also introduce variables $y_{-1} = y, y_0 = x_0, y_1 = x_1, \ldots, y_n = x_n$ for notational convenience. For a given $G(x)$ of degree $d$, we associate a Lie algebra representation $\rho_X = \rho_{X_G}$ of $A$ of $\mathfrak{g}$ as follows:

$$\rho_i := \rho_X(\alpha_i) : = \frac{\partial}{\partial y_i} + \frac{\partial S(y, x)}{\partial y_i}, \text{ for } i = -1, 0, \ldots, n, \tag{4.4}$$

where $S(y, x) = y \cdot G(x)$. In other words, this is the quantum Jacobian Lie algebra representation $\rho_{S(y, x)}$ associated to $S(y, x) = y \cdot G(x)$ in (2.2). Of course, we extend this $\mathbb{k}$-linearly to get a map $\rho_X : \mathfrak{g} \rightarrow \text{End}_A(A)$. This is clearly a Lie algebra representation of $\mathfrak{g}$. We will show that Griffiths’ period integrals of the projective hypersurface $X_G$ are, in fact, period integrals of $\rho_X$ in the sense of Definition 2.1.

4.3. Period integrals attached to $\rho_X$

We briefly review Griffiths’ theory and find a nonzero period integral attached to $\rho_X$.

**Proposition 4.3.** Let $\mathbb{k}$ be a non negative integer. Every rational differential $n$-form $\omega$ on $\mathbb{P}^n$ with a pole order of $\leq k$ along $X_G$ (regular outside $X_G$ with a pole order of $\leq k$) can be written as

$$\omega = \frac{F(x)}{G(x)^k} \Omega_n,$$

where $\Omega_n = \sum_{i=0}^n (-1)^i x_i (d x_0 \wedge \cdots \wedge \hat{d} x_i \wedge \cdots d x_n)$ and $F$ is a homogeneous polynomial such that $\text{deg } F + n + 1 = k \text{deg } G = k d$.

Griffiths defined a surjective $\mathbb{k}$-linear map, called a tubular neighborhood map $\tau$, (3.4) in [6]

$$\tau : H_{n-1}(X_G, \mathbb{Z}) \rightarrow H_n(\mathbb{P}^n - X_G, \mathbb{Z}),$$

where $H_i$’s are singular homology groups of the topological spaces $X_G(\mathbb{C})$ and $\mathbb{P}^n(\mathbb{C}) - X_G(\mathbb{C})$. It is known that this map is an isomorphism, if $n$ is even. He also studied the following $\mathbb{k}$-linear map, called the residue map

$$\text{Res} : \mathfrak{h}(X_G) \rightarrow H^{n-1}(X_G, \mathbb{C}),$$

$$\omega \mapsto (\gamma \mapsto \frac{1}{2\pi i} \int_{\gamma(\tau)} \omega)$$
where $\mathcal{H}(X_G)$ is the rational de Rham cohomology group defined as the quotient of the group of rational $n$-forms on $\mathbb{P}^n$ regular outside $X_G$ by the group of the forms $d\psi$ where $\psi$ is a rational $n-1$ form regular outside $X_G$. It turns out that there is an increasing filtration of $\mathcal{H}(X_G)$ (see (6.1) of [6]):

$$
\mathcal{H}_1(X_G) \subset \cdots \subset \mathcal{H}_{n-1}(X_G) \subset \mathcal{H}_n(X_G) = \mathcal{H}_{n+1}(X_G) = \cdots \simeq \mathcal{H}(X_G),
$$

where $\mathcal{H}_k(X_G)$ is the cohomology group defined as the quotient of the group of rational $n$-forms on $\mathbb{P}^n$ with a pole of order $\leq k$ along $X_G$ by the group of exact rational $n$-forms on $\mathbb{P}^n$ with a pole of order $\leq k$ along $X_G$. The isomorphism $\mathcal{H}_k(n) \simeq \mathcal{H}_k$ follows from Theorem 4.2, [6] and the fact that the natural map $\mathcal{H}^{(k)}_n \rightarrow \mathcal{H}^{(k+1)}_n$ is injective follows from Theorem 4.3, [6]. Moreover, Griffiths proved that $\text{Res}$ sends this filtration given by pole orders to the Hodge filtration of $H^{n-1}(X_G, \mathbb{C})$. For each $k \geq 1$, if we define $\mathcal{F}_k \subset H^{n-1}(X_G, \mathbb{C})$ to be the $k$-vector space consisting of all $(n-1,0),(n-2,1),\cdots,(n-1-k,k)$-forms. Then

$$
\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-2} \subset \mathcal{F}_{n-1} = H^{n-1}(X_G, \mathbb{C}).
$$

Theorem 8.3, [6], says that $\mathcal{H}_{k}(X_G)$ goes into the primitive part of $\mathcal{F}_{k-1}$ under $\text{Res}$ and $\text{Res}$ is a $k$-vector space isomorphism from $\mathcal{H}_k$ to the primitive part $H^{n-1}_{\text{prim}}(X_G, \mathbb{C})$ of $H^{n-1}(X_G, \mathbb{C})$. For Definition of the primitive part, see page 488, [6].

Now we construct a $k$-linear map $C_\gamma : A \rightarrow k$ for each $\gamma \in H_{n-1}(X_G, \mathbb{Z})$, which is a period integral attached to $\rho_X$. For $y^{k-1}F(x)$ where $F(x)$ is a homogeneous polynomial of degree $dk-(n+1)$ and $k \geq 1$, define

$$
C_\gamma(y^{k-1}F(x)) = -\int_{\tau(\gamma)} \left( \int_0^\infty y^{k-1}F(x) \cdot e^{yG(x)} \, dy \right) \Omega_n
$$

$$
= (-1)^{k-1} (k-1)! \int_{\tau(\gamma)} \frac{F(x)}{G(x)^k} \Omega_n.
$$

In the second equality, we used the Laplace transform:

$$
\int_0^\infty y^{k-1}e^{-yT} \, dy = \frac{(k-1)!}{T^k}.
$$

Note that $\frac{F(x)}{G(x)^k} \Omega_n$ is a representative of an element of $\mathcal{H}_k(x_G)$ and $\int_{\tau(\gamma)} \frac{F(x)}{G(x)^k} \Omega_n$ is well-defined. If $x \in A$ is not of the form as above, we simply define $C_\gamma(x) = 0$. Because the elements $y^{k-1}F(x)$, where $k \geq 1$ and $F(x)$ varies over any homogeneous polynomials in $k[x]$, constitute a $k$-basis of $A$, the above procedure, extended $k$-linearly, gives a map $C_\gamma : A \rightarrow k$. Then we claim that this is a period integral attached to the Lie algebra representation $\rho_X$.

**Proposition 4.4.** Let $\gamma \in H_{n-1}(X_G, \mathbb{Z})$. The $k$-linear map $C_\gamma$ is a period integral attached to $\rho_X$, i.e. $C_\gamma((\rho_x)(f)) = 0$ for every $f \in A$. 

Proof. Recall that

\[ \rho_{-1} = G(x) + \frac{\partial}{\partial y}, \]
\[ \rho_i = y \frac{\partial}{\partial x_i} G(x) + \frac{\partial}{\partial x_i}, \quad i = 0, \cdots, n. \]

It is enough to check the statement when \( \rho_i(x) \) is a \( k \)-linear combination of the forms \( y^{k-1}F(x) \), where \( k \geq 1 \) and \( F(x) \) varies over any homogeneous polynomials in \( k[x] \) such that \( \deg F + n + 1 = kd \), since \( C_\gamma \) is already zero for other forms of polynomials. Let \( f(x) \in k[x] \) be a homogeneous polynomial. Then for each \( k \geq 1 \) we have

\[ \rho_i(y^{k-1}f(x)) = \frac{\partial G(x)}{\partial x_i} \cdot y^{k} f(x) + y^{k-1} \frac{\partial f(x)}{\partial x_i}. \]

We compute

\[
C_\gamma \left( \frac{\partial G(x)}{\partial x_i} \cdot y^{k} f(x) + y^{k-1} \frac{\partial f(x)}{\partial x_i} \right) \\
= (-1)^k k! \int_{\tau(\gamma)} \frac{\partial G(x)}{\partial x_i} \cdot f(x) \Omega_n + (-1)^{k-1} (k-1)! \int_{\tau(\gamma)} \frac{\partial f(x)}{\partial x_i} G(x)^k \Omega_n \\
= (-1)^{k-1} k! \int_{\tau(\gamma)} \frac{k \cdot f(x) \cdot \partial G(x)}{\partial x_i} - G(x) \cdot \frac{\partial f(x)}{\partial x_i} \Omega_n \\
= \int_{\tau(\gamma)} d \left( \frac{(-1)^k (k-1)!}{G(x)^k} \sum_{0 \leq i < j \leq n} (x_i A_j(x) - x_j A_i(x)) \cdots d x_i \cdots d x_j \cdots \right),
\]

where \( A_i(x) = f(x) \) and \( A_j(x) = 0 \) for \( j \neq i \). The last equality is a simple differential calculation, which can be found in (4.4) and (4.5) of [6]. Therefore this expression has the form \( \int_{\tau(\gamma)} d \omega \) and this is zero, which is what we want. Now we look at

\[ \rho_{-1}(y^{k-1}f(x)) = G(x) y^{k-1} f(x) + (k-1) y^{k-2} f(x). \]

Then we can similarly check (this is much easier and follows from the factor \( (-1)^k (k-1)! \) and sign) that

\[ C_\gamma \left( G(x) y^{k-1} f(x) + (k-1) y^{k-2} f(x) \right) = 0, \]

for all \( k \geq 1 \) and any homogeneous polynomial in \( k[x] \). 
\( \square \)
4.4. The cochain complex with super-commutative product attached to a smooth hypersurface

We now have two data: the Lie algebra representation $\rho_X$ attached to the smooth projective hypersurface $X_G$ and a period integral $C_\gamma : A \to k$ attached to $\rho_X$ for $\gamma \in H_{n-1}(X_G, \mathbb{Z})$. In Subsection 2.1, we constructed a cochain complex $(\mathcal{O}^\rho_X, K^\rho_X)$ with the super-commutative product $\cdot$ whose degree 0 part is $A$ (in fact, an object in the category $\mathcal{C}_k$) and a cochain map $(\mathcal{O}^\rho_X, K^\rho_X) \to (k, 0)$. We drop the $\rho_X$ from the notation for simplicity, if there is no confusion. We would like to compute the cohomology group of the abelian Lie algebra $X$ from the notation for simplicity, if there is no confusion. We would like to compute the cochain complex attached to $\rho_X$ and $\pi_X$ should be the same, since we use the same Lie algebra $\mathfrak{g}$. The differentials $K := K^\rho_X$ and $Q := K^\rho_X$ have many differences. The differentials $K$ and $Q$ are given as follows:

$$K := K^\rho_X = \sum_{i=1}^n \left( \frac{\partial S(y, x)}{\partial y_i} + \frac{\partial}{\partial y_i} \frac{\partial}{\partial \eta_i} \right) : \mathcal{O} \to \mathcal{O},$$

$$Q := K^\rho_X = \sum_{i=1}^n \frac{\partial S(y, x)}{\partial y_i} \frac{\partial}{\partial \eta_i} : \mathcal{O} \to \mathcal{O}.$$

Since $\frac{\partial S(y, x)}{\partial y_i}$ is a differential operator of order 1, the differential $Q$ is a derivation of the product of $\mathcal{O}$. But $K$ is not a derivation of the product, as we have already pointed out: the differential operator $\frac{\partial}{\partial y_i} \frac{\partial}{\partial \eta_i}$ has order 2. Another difference is that the period integral $\mathcal{O}_\gamma : (\mathcal{O}, K) \to (k, 0)$ attached to $\rho_X$ is not a period integral attached to $\pi_X$. We also introduce the $k$-linear map

$$\Delta := K - Q = \sum_{i=1}^n \frac{\partial}{\partial y_i} \frac{\partial}{\partial \eta_i} : \mathcal{O} \to \mathcal{O}.$$

Note that $\Delta$ is a also a differential of degree 1, i.e. $\Delta^2 = 0$. Therefore we have

$$\Delta Q + Q\Delta = 0.$$

**Proposition 4.5.** Let $H^n_\mathcal{C}(\mathcal{O})$ be the $n$-th cohomology group of $(\mathcal{O}, Q)$. Then we have

$$\bigoplus_{m \in \mathbb{Z}} H^n_\mathcal{C}(\mathcal{O}) = H^0_\mathcal{C}(\mathcal{O}) \oplus H^1_\mathcal{C}(\mathcal{O}).$$

Moreover, we get

$$H^1_\mathcal{C}(\mathcal{O}) \simeq k[y, x], \quad H^0_\mathcal{C}(\mathcal{O}) \simeq k[y, x]/J_S,$$

where $J_S$ is the Jacobian ideal defined as the ideal of $A = k[y, x]$ generated by $G(x), y \frac{\partial G}{\partial x_0}, \ldots, y \frac{\partial G}{\partial x_n}$. 
Proof. It is clear that we have

\[ H_Q^0(\mathcal{A}) \approx \mathbb{k}[y, \bar{x}]/I_S. \]

In order to prove \( H_Q^{-1}(\mathcal{A}) \approx \mathbb{k}[y, \bar{x}] \), we recall the result of Dwork, Lemma 3.1 of [5];

**Lemma 4.1.** If \( g_0, \ldots, g_n \) are non-constant homogeneous forms in \( \mathbb{k}[x_0, \ldots, x_n] \) with no common zero in \( \mathbb{P}^n \) over \( k \) and if \( \{P_i\}_{i \in I} \) is a set of polynomials indexed by a subset \( I \) of \( \{0, \ldots, n\} \) such that \( \sum_{i \in I} P_i g_i = 0 \), then there exists a skew symmetric set \( S_{i,j} \) in \( \mathbb{k}[x_0, \ldots, x_n] \) indexed by \( I \) such that \( P_i = \sum_{j \in I} S_{i,j} g_j \) for each \( i \in I \). Furthermore if \( \{P_i\} \) consists of homogeneous elements such that \( \deg(P_i g_i) = m \) is independent of \( i \), then each \( S_{i,j} \) may be chosen to be homogeneous of degree \( m - \deg(g_i g_j) \).

We first compute the cohomology group \( H_Q^{-2}(\mathcal{A}) \) using this lemma. Note that every element of \( \mathcal{A}^{-2} \) can be written as

\[ A = \sum_{-1 \leq i < j \leq n} A_{i,j}(y) \cdot \eta_i \cdot \eta_j, \quad \text{with} \quad A_{i,j} = -A_{j,i}. \]

Since we have

\[ Q(A) = \sum_{\ell=-1}^{n} \left( \sum_{a=-1}^{n} \frac{\partial S(y)}{\partial y_a} 2A_{a,\ell}(y) \right) \cdot \eta_{\ell}, \]

the element \( A \in \text{Ker}(Q) \cap \mathcal{A}^{-2} \) if and only if

\[ \sum_{a=-1}^{n} \frac{\partial S(y)}{\partial y_a} A_{a,\ell}(y) = 0, \quad \text{for every} \quad \ell = -1, 0, \ldots, n. \]

Since \( S(y) = y \cdot G(x) \), the above condition is equivalent to (recall \( y = y_{-1} = y_0 = x_0, \ldots, y_n = x_n \))

\[ G(x) \cdot A_{-1,\ell}(y) + \sum_{a=-1}^{n} \frac{\partial G(x)}{\partial x_a} A_{a,\ell}(y) = 0, \quad \text{for every} \quad \ell = -1, 0, \ldots, n. \]

Thus finding the condition on \( A_{i,j}(y) \) for \( A \in \text{Ker}(Q) \cap \mathcal{A}^{-2} \) is equivalent to solving the following differential equation for \( B_{a,\ell}(x) \) with \( B_{a,\ell}(x) = -B_{a,\ell}(x) \) and \( B_{a,\ell}(x) | A_{a,\ell}(y) = A_{a,\ell}(y, x); \)

\[ G(x) \cdot B_{-1,\ell}(x) + \sum_{a \in I} \frac{\partial G(x)}{\partial x_a} B_{a,\ell}(x) = 0, \quad \text{for every} \quad \ell = -1, 0, \ldots, n, \]

where \( I \) varies over a subset of \( \{0, \ldots, n\} \). If \( B_{-1,\ell}(x) \) is not zero for some \( \ell \), the only solutions of this equation should have the form \( B_{-1,\ell} = -d \cdot \epsilon, B_{0,\ell} = x_0 \cdot \epsilon, \ldots, B_{n,\ell} = x_n \cdot \epsilon \), where \( \epsilon \) is any constant polynomial of \( \mathbb{k}[y, \bar{x}] \) depending on \( \ell \). But these can not be solutions, since they do not satisfy the
skew symmetric property. The remaining case is that $B_{-1,\ell}(x)$ is zero for all $\ell$ (i.e. $A_{-1,\ell}(x) = 0$ for all $\ell$); then we have
\[
\sum_{a,I} \frac{\partial G(x)}{\partial x_a} B_{a,\ell}(x) = 0, \quad \text{for every } \ell = -1, 0, \cdots, n.
\]
Now we can apply Lemma 4.1 by letting $g_a = \frac{\partial G(x)}{\partial x_a}$ for $a \in I$, since $X_G$ is smooth. Therefore for each $\ell$ there exist a skew symmetric matrix $S_{a,\beta}(x)$ of homogeneous forms such that
\[
B_{a,\ell}(x) = \sum_{\beta \in I} S_{a,\beta}(x) \cdot \frac{\partial G(x)}{\partial x_\beta} \quad \text{for every } \ell = -1, 0, \cdots, n.
\]
If we consider
\[
\Theta := \sum_{-1 \leq k \leq n, \ i,j \in I, i < j < k} S_{j,j}^{(k)} \cdot \eta_i \cdot \eta_j \cdot \eta_k \in \mathfrak{a}_-^3,
\]
then
\[
Q(\Theta) = 3 \sum_{i \in I, i < j} \left( \sum_{\beta \in I} \frac{\partial G(x)}{\partial x_\beta} \cdot S_{i,j}^{(j)} \right) \eta_i \cdot \eta_j + 3 \sum_{i,j \in I, i < j} B_{i,j}(y) \cdot \eta_i \cdot \eta_j.
\]
This computation shows that $A = \sum_{-1 \leq j \leq n} A_{i,j}(y) \cdot \eta_i \cdot \eta_j \in \text{Ker } Q \cap \mathfrak{a}_-^{-2}$ with $A_{-1,j}(y) = 0$ for all $j$ is a $Q$-exact form. Because we also saw that if $A_{-1,j}(y) \neq 0$ for some $j$ then $A$ can not $Q$-closed, we conclude $H_Q^{-1}(\mathfrak{a}) = 0$. Exactly the same computation shows that $H_Q^m(\mathfrak{a}) = 0$ for $m \leq -3$. Note that by Definition of the cochain complex $(\mathfrak{a}, Q)$, we know $H_Q^m(\mathfrak{a}) = 0$ for $m < -n - 3$ or $m > 0$. Therefore we have proved that
\[
\bigoplus_{m \in \mathbb{Z}} H_Q^m(\mathfrak{a}) = H_Q^{-1}(\mathfrak{a}) \oplus H_Q^0(\mathfrak{a}).
\]
Now let us compute $H_Q^{-1}(\mathfrak{a})$. Let $\lambda = \sum_{\ell=1}^n A_\ell(y) \cdot \eta_\ell$ be any element of $\mathfrak{a}_-^{-1} \cap \text{Ker } Q$. Then
\[
Q(\lambda) = \sum_{a=1}^n \frac{\partial S(y)}{\partial y_a} \cdot A_a(y) = G(x)A_{-1}(y) + \sum_{a=0}^n \frac{\partial G(x)}{\partial x_a} A_a(y) = 0.
\]
Solving for $A_a(y)$ in this differential equation divides into two cases according to whether $A_{-1}(y)$ is zero or not. Firstly, we look at the case $A_{-1}(y) \neq 0$. The only solutions of this equation have the form
\[
A_{-1}(y) = -d \cdot y \cdot u(y), \quad A_i = x_i \cdot u(y) \quad \text{for } \ i = 0, 1, \cdots, n,
\]
where $u(y)$ is any polynomial in $k[y, x]$. Secondly, we look at the case $A_{-1}(y) = 0$. In this case, we look at the coefficients (which belongs to $k[x]$) of all the powers of $y$ of $Q(\lambda)$; then Lemma 4.1 says that there exists a skew-symmetric matrix $S_{a,\beta}(x)$ of homogeneous forms such that
\[
\sum_{\beta \in I} \frac{\partial S(x)}{\partial x_\beta} \cdot S_{\beta,a}(x) = A_a(x),
\]
where $I$ is a finite subset of $\{0, \cdots, n\}$. Note that if
$$\theta = \sum_{i,j \in I, i < j} S_{i,j}(\Sigma) \cdot \eta_i \cdot \eta_j \in \mathcal{A}^{-2},$$
then
$$Q(\theta) = 2 \sum_{i \in I} \left( \sum_{a \in I} \frac{\partial G(x)}{\partial x_a} S_{a,i}(\Sigma) \right) \cdot \eta_i = 2 \sum_{i \in I} A_i(\Sigma) \cdot \eta_i.$$

This computation shows us two things: the element $R_u := -d \cdot y \cdot u(y) \cdot \eta_{-1} + \sum_{a=0}^n x_i \cdot u(y) \cdot \eta_a \in \text{Ker} \, Q \cap \mathcal{A}^{-1}$
can not be $Q$-exact and any element which is not of the form $R_u$ for $u(y) \in \mathbb{k}[y, \Sigma]$ is $Q$-exact. Hence we conclude $H_{-1}^{Q}(\mathcal{A}) \cong \mathbb{k}[y, \Sigma]$. \qed

**Proposition 4.6.** The cohomology group $H_{-1}^{Q}(\mathcal{A})$ is both an $\mathcal{A}^0$-module and a $H_{0}^{Q}(\mathcal{A})$-module.

**Proof.** We consider $R \in \mathcal{A}^{-1}$ such that $QR = 0$. For any $f \in \mathcal{A}^0 = A$, we have $Q(f \cdot R) = 0$, since $A \subseteq \text{Ker} \, Q$. Let $S = Q\sigma$ where $\sigma \in \mathcal{A}^{-2}$. Then we have
$$S \cdot f = Q(\sigma \cdot f), \quad \text{for } f \in \mathcal{A}^0.$$

Therefore $H_{-1}^{Q}(\mathcal{A})$ is an $A$-module. Note that $H_{0}^{Q}(\mathcal{A})$ has a $\mathbb{C}$-algebra structure inherited from $\mathcal{A}$, since $Q$ is a derivation of the product of $\mathcal{A}$. Then one can similarly check that $H_{-1}^{Q}(\mathcal{A})$ is also a $H_{0}^{Q}(\mathcal{A})$-module. \qed

But notice that $H_{-1}^{Q}(\mathcal{A})$ is not an $\mathcal{A}^0$-module under the product of $\mathcal{A}$. We consider $R \in \mathcal{A}^{-1}$ such that $QR = 0$. For any $f \in \mathcal{A}^0$, the equation (3.1) says that
$$K(R \cdot f) = \ell^K_2(R, f) + KR \cdot f. \quad (4.6)$$

Because $\ell^K_2$ is not zero, $H_{-1}^{Q}(\mathcal{A})$ does not necessarily have an $A$-module structure. In fact, this plays an important role in understanding the complex $(\mathcal{A}, K)$. Recall that $H_{-1}^{Q}(\mathcal{A}) \cong \mathbb{k}[y, \Sigma]$ by Proposition 4.5. Let $R \in \mathcal{A}^{-1}$ be a representative of a cohomology class of $H_{-1}^{Q}(\mathcal{A})$. We may choose $R$ as
$$R := -d \cdot y \eta_{-1} + \sum_{i=0}^n x_i \eta_i \in \mathcal{A}^{-1}. \quad (4.7)$$

The reason for this is
$$QR = \left( \sum_{i=0}^n x_i \frac{\partial}{\partial x_i} - d \cdot y \frac{\partial}{\partial y} \right) S(y, \Sigma) = 0,$$
which follows from the fact that \( G(x) \) is a homogeneous polynomial of degree \( d \), and \( R \) can not be \( Q \)-exact for degree reasons. Then the straightforward computation says that

\[
KR = n + 1 - d.
\]

Now we prepare for computing \( H_K(\mathcal{A}_K) \). The idea is to find a simpler cochain complex (by charge decomposition) which is quasi-isomorphic to \( (\mathcal{A}, K) \). We define a \( \mathbb{k} \)-linear map \( \partial_R : \mathcal{A} \rightarrow \mathcal{A} \) by

\[
\partial_R(x) = \ell_2^K(R, x) + KR \cdot x = \ell_2^K(R, x) + (n + 1 - d) \cdot x, \quad x \in \mathcal{A}.
\]

(4.8)

It is clear that \( \partial_R \) preserves degree, since \( R \in \mathcal{A}^{-1} \) and \( \ell_2^K \) is a degree one map; for any \( m \in \mathbb{Z} \), we have \( \partial_R : \mathcal{A}^m \rightarrow \mathcal{A}^m \). Proposition 4.2 implies that \( \ell_2^K \) is a derivation of the product;

\[
\ell_2^K(a \cdot b, c) = (-1)^{|a|} a \cdot \ell_2^K(b, c) + (-1)^{|b|} c \cdot \ell_2^K(a, c) \cdot b,
\]

for any homogeneous elements \( a, b, c \in \mathcal{A} \). Using this one can compute that

\[
\ell_2^K(R, F) = -d \cdot \frac{\partial F}{\partial y} \cdot y + (-1)^n \frac{\partial F}{\partial \eta_{-1}} \cdot \eta_{-1} + \sum_{i=0}^n \frac{\partial F}{\partial x_i} \cdot x_i + (-1)^m \frac{\partial F}{\partial \eta_i} \cdot \eta_i,
\]

for any \( F \in \mathcal{A}^m \) (we sometimes say that \( F \) has ghost number \( m \)). Using this computation, we can diagonalize the \( \mathbb{k} \)-linear map \( \partial_R \) (i.e. \( \partial_R \) is a semi-simple operator on the infinite dimensional \( \mathbb{k} \)-vector space \( \mathcal{A}^m \) for each \( m \)); the eigenvectors for \( \partial_R \) are given by monomials of \( \mathcal{A} \). For example, \( \eta_{-1} y^2 x_1^2 \) and \( \eta_3 \eta_4 y^3 x_2^3 \) are monomials of degree 5 and 8 respectively in \( \mathcal{A} \). If \( F = F(\eta, y) \in \mathcal{A}^m \) is a monomial, then there exists \( c_F \in \mathbb{Z} \) such that

\[
\partial_R(F) = c_F \cdot F.
\]

We call \( c_F \) the charge of \( F \) and let \( \mathcal{A}^m[c] \) be the \( c \)-th \( \mathbb{k} \)-linear charge space of \( \mathcal{A}^m \). We have the following charge decomposition of \( \mathcal{A}^m \) (the decomposition of \( \mathcal{A}^m \) into \( \partial_R \)-eigen spaces)

\[
\mathcal{A}^m = \bigoplus_{c \in \mathbb{Z}} \mathcal{A}^m[c] = \ker(\partial_R)^m \oplus \im(\partial_R)^m, \quad \ker(\partial_R)^m = \mathcal{A}^m[0], \im(\partial_R)^m = \bigoplus_{c \neq 0} \mathcal{A}^m[c].
\]

We will reduce the computation of the cohomology to the charge zero part computation, Lemma 4.3.

**Lemma 4.2.** The map \( \partial_R \) preserves the degree of \( \mathcal{A} \). Moreover, we have

\[
\partial_R \circ K = K \circ \partial_R \quad \partial_R(\mathcal{A}^m) \cap \ker K \subseteq K(\mathcal{A}^{m-1}) \subseteq \mathcal{A}^m
\]

for each \( m \in \mathbb{Z} \).
Injectivity follows from the decomposition $\delta y$. Let us denote the complex $(\mathcal{A}, R)$ of $(\mathcal{A}, K)$ to relate the degree 0 cohomology group of $(\mathcal{A}, K)$ to the middle dimensional primitive cohomology of the smooth projective hypersurface $X_G$. The main theorem of this Subsection is the following:

**Lemma 4.3.** The pair $(\text{Ker} \delta R, K)$ is a cochain complex and the natural inclusion map from $(\text{Ker} \delta R, K)$ to $(\mathcal{A}, K)$ is a quasi-isomorphism.

**Proof.** The relation $\delta R \circ K = K \circ \delta R$, Lemma 4.2, says that if $x \in \text{Ker} (\delta R)$ then $Kx \in \text{Ker} (\delta R)$. Thus $K$ is a linear map from $\text{Ker} \delta R$ to $\text{Ker} \delta R$. Since $K^2 = 0$, we see that $(\text{Ker} \delta R, K)$ is a cochain complex. If we index $K$ by $K_m : \mathcal{A}^m \to \mathcal{A}^{m+1}$ for each $m \in \mathbb{Z}$, then the inclusion map from $(\text{Ker} \delta R, K)$ to $(\mathcal{A}, K)$ induces a linear map

$$H^m_\delta (\text{Ker} (\delta R)) = \frac{\text{Ker} (K_m) \cap \text{Ker} (\delta R)}{K_{m-1}(\mathcal{A}^{m-1}) \cap \text{Ker} (\delta R)} \to \frac{\text{Ker} (K_m)}{K_{m-1}(\mathcal{A}^{m-1})} =: H^m_\delta (\mathcal{A}).$$

Injectivity follows from the decomposition $\mathcal{A} = \text{Ker} (\delta R) \oplus \text{im} (\delta R)$ and surjectivity follows from $\delta R (\mathcal{A}^m) \cap \text{Ker} K \subseteq K_{m-1}(\mathcal{A}^{m-1})$ in Lemma 4.2. Therefore we conclude that it is a quasi-isomorphism.

**Remark 4.2.** Note that the existence of this quasi-isomorphism follows from the homogeneity of $G(\mathfrak{g})$; the existence of a non-trivial cohomology class $R$ of $H^{-1}_Q (\mathcal{A})$ relies on the fact that $G(\mathfrak{g})$ is homogeneous. We could have developed a theory with any polynomial $S(y, \mathfrak{z})$ instead of the special form $y \cdot G(\mathfrak{z})$.

Let us denote the complex $(\text{Ker} \delta R, K)$ by $(B, K)$. Then

$$B = \text{Ker} \delta R \simeq \bigoplus_{m \in \mathbb{Z}} B^m = B^{-n-2} \oplus \cdots \oplus B^0$$

where $B^m$ is the degree $m$ part of $B$. We use this complex $(B, K)$ to relate the degree 0 cohomology group of $(\mathcal{A}, K)$ to the middle dimensional primitive cohomology of the smooth projective hypersurface $X_G$. The main theorem of this Subsection is the following:
**Theorem 4.1.** Let $H^n_K(\mathcal{A})$ be the $n$-th cohomology group of the cochain complex $(\mathcal{A}, K := K_p\mathcal{C})$. Then we have

$$\bigoplus_{m \in \mathbb{Z}} H^n_K(\mathcal{A}) \simeq H^n_K(\mathcal{A}) \oplus H^{-1}_K(\mathcal{A}).$$

Moreover, $H^n_K(\mathcal{A})$ is isomorphic to $H_{prim}^{n-1}(X_G, \mathbb{C})$ as a $\mathbb{C}$-vector space, where $H_{prim}^{n-1}(X_G, \mathbb{C})$ means the primitive part of $H^{n-1}(X_G, \mathbb{C})$, and

$$\dim_{\mathbb{C}} H^{-1}_K(\mathcal{A}) = \begin{cases} 1 & \text{if } d = n + 1, \\
0 & \text{if } d \neq n + 1. \end{cases}$$

**Proof.** It is clear $H^n_K = 0$ if $m \neq -(n + 2), -(n + 1), \cdots, 1, 0$. First we compute $H^{-1}_K(\mathcal{A})$. The idea of its computation is essentially the same as that for $H_{Q}^{-1}$; The main tool is again Lemma 4.1. Since the cochain complex $(\ker \delta_R, K)$ is quasi-isomorphic to $(\mathcal{A}, K)$, we use $(\mathcal{B}, K) = (\ker \delta_R, K)$ to compute the cohomology. By noting that

$$\ell^K_2(R, x) = -(n + 1 - d) \cdot x \quad \text{if and only if} \quad x \in \ker(\delta_R),$$

we can easily check that any element $\Lambda \in \ker(\delta_R) \cap \mathcal{A}^{-1} = \mathcal{B}^{-1}$ is a $\mathbb{C}$-linear combination of homogeneous polynomials of the form $y^{k-1} F_{-1}(x)$ and $y^{k-1} F_a(x)$, $a = 0, \cdots, n$, where the degree of $F_{-1}(x)$ is $d(k-1) - (n + 1)$ and the degree of $F_a(x)$ is $d k_a - n$ for every $a = 0, \cdots, n$. A simple computation shows that $\Lambda = \sum_{a=-1}^{n} y^{k_a} F_a(x) \cdot \eta_a \in \mathcal{B}^{-1} \cap \ker(K)$ is equivalent to

$$K(\Lambda) = K\left( y^{k-1} F_{-1}(x) \cdot \eta_{-1} + \sum_{a=0}^{n} y^{k_a} F_a(x) \cdot \eta_a \right)$$

$$= (y \cdot G(x) + k_{-1} - 1) F_{-1}(x) y^{k_{-1} - 2} + \sum_{a=0}^{n} \left( \frac{\partial G(x)}{\partial x_a} F_a(x) y + \frac{\partial F_a(x)}{\partial x_a} \right) y^{k_a} - 1$$

$$= 0.$$ 

In order to solve this differential equation, we have to let the coefficients of every power of $y$ equal 0. By comparing every possible configuration of all the powers of $y$, one can reduce this equation to the following two cases:

$$(k_{-1} - 1) F_{-1}(x) + \sum_{a=0}^{n} \frac{\partial F_a(x)}{\partial x_a} = 0,$$

$$G(x) F_{-1}(x) + \sum_{a=0}^{n} \frac{\partial G(x)}{\partial x_a} F_a(x) = 0,$$  \hspace{1cm} (4.10)

or

$$F_{-1}(x) = 0, \quad \sum_{a \in I} \frac{\partial F_a(x)}{\partial x_a} = 0, \quad \sum_{a \in I} \frac{\partial G(x)}{\partial x_a} F_a(x) = 0,$$  \hspace{1cm} (4.11)
where $I$ is a subset of $\{0, \cdots, n\}$. The solution of the second equation in (4.10) should have the following form as we already saw

$$F_{-i}(x) = -d \cdot u(y), \quad F_a(x) = x_a \cdot u(y) \quad \text{for } a = 0, \cdots, n,$$

(4.12)

where $u(y)$ is any polynomial in $k[y] = k[y, x]$. But the first equation of (4.10) implies that if $d = n+1$ then $u(y)$ should be a constant function, and if $d \neq n+1$ then (4.12) can not be a solution of our differential equation. We analyze (4.11); here Lemma 4.1 says that there exists a skew-symmetric matrix $S_{a, \beta}(x) \in k[x]$ of homogeneous polynomials such that

$$F_a(x) = \sum_{\beta \in I} S_{a, \beta}(x) \frac{\partial G(x)}{\partial x_{\beta}}.$$

If we use $S_{a, \beta}(x)$, then a similar computation to Proposition 4.5 shows that an element $A$ in $B^{-1}$ satisfying (4.11) is $K$-exact. Also the same kind of computation as above shows that $H^m_{K}(\mathcal{A}) = 0$ for $m \leq -2$.

Now we begin the argument for $H^0_{K}(\mathcal{A})$. A simple computation shows that $B^0$ is spanned (as a $k$-vector space) by homogeneous polynomials of the form $y^{k-1}F(x)$, where the degree of $F(x)$ is $kd - (n+1)$ with $k \geq 1$. Then we define a $\mathbb{C}$-linear map $J$ by

$$J : B^0 \rightarrow H^{n-1}(X_G, \mathbb{C})$$

$$y^{k-1}F(x) \rightarrow \left\{ \gamma \rightarrow (-1)^{k-1} \int_{\tau(\gamma)} \left( \int_0^\infty y^{k-1} e^{-y G(x)} dy \right) F(x) \Omega_n = (-1)^{k-1}(k-1)! \int_{\tau(\gamma)} \frac{F(x)}{G(x)^k} \Omega_n \right\},$$

and extending it $k$-linearly. Proposition 4.4 says that $K(B^{-1})$ goes to zero under the map $J$ and so $J$ induces a $\mathbb{K}$-linear map $H^0_K(\mathcal{B}) \rightarrow H^{n-1}(X_G, \mathbb{C})$. Now recall that $\mathcal{H}(X_G)$ was defined to be the rational De Rham cohomology group defined as the quotient of the group of exact rational $n$-forms on $\mathbb{P}^n$ regular outside $X_G$ by the group of exact rational $n$-forms on $\mathbb{P}^n$ regular outside $X_G$. Theorem 8.3, [6], tells us that the residue map $Res$ induces an isomorphism between $\mathcal{H}(X_G)$ and $H^{n-1}_{\text{prim}}(X_G, \mathbb{C})$. Thus for our proof of Theorem, it is enough to show that the following map (extended $k$-linearly)

$$J' : B^0 \rightarrow \Omega(V)^n$$

$$y^{k-1}F(x) \rightarrow (-1)^{k-1} \int_0^\infty y^{k-1} e^{-y G(x)} dy \cdot F(x) \Omega_n = (-1)^{k-1}(k-1)! \frac{F(x)}{G(x)^k} \Omega_n,$$

(4.13)

where $\Omega(V)^n$ is the group of rational $n$-forms on $\mathbb{P}^n$ regular outside $X_G$, induces an isomorphism $J' : H^0_K(\mathcal{B}) \rightarrow \mathcal{H}(X_G)$, i.e. $J$ factors through the isomorphism $J'$. This follows from Corollary 2.11, (4.4), and (4.5) in [6] with a computation below. An arbitrary homogeneous element of $B^{-1}$ can be written as $A = \sum_{i=0}^n A_i(y, \underline{x}) \eta_i + B(y, \underline{x}) \eta_{n-1}$, where $A_i(y, \underline{x}) = y^{k} \cdot M_i(\underline{x})$ and $B(y, \underline{x}) = y^{l} \cdot N(\underline{x})$ are homogeneous polynomials of $A = \mathbb{C}[y, \underline{x}]$. Then we have

$$K A = \sum_{i=0}^n A_i(y, \underline{x}) \frac{\partial G}{\partial x_i} y + G(\underline{x}) B(y, \underline{x}) + \sum_{i=0}^n \frac{\partial A_i(y, \underline{x})}{\partial x_i} + \frac{\partial B(y, \underline{x})}{\partial y}$$

$$= \sum_{i=0}^n y^{k+1} M_i(\underline{x}) \frac{\partial G}{\partial x_i} + \sum_{i=0}^n y^k \frac{\partial M_i(\underline{x})}{\partial x_i} + G(\underline{x}) y^l N(\underline{x}) + l y^{l-1} N(\underline{x}).$$
If we apply $J'$ to $KA$, then a simple computation shows that
\[ J'(KA) = k! (-1)^{k-1} \frac{(k+1) \sum_{i=0}^n M_i(x) \frac{\partial G(x)}{\partial x_i} - G(x) \sum_{i=0}^n \frac{\partial M_i(x)}{\partial x_i}}{G(x)^{k+2}} \cdot \Omega_n. \]

Note that $J'(G(x)^n N(x) + 1) = 0$. The relation (4.5), [6] says that $J'(KA)$ is an exact rational differential form. Thus $J'$ induces a $k$-linear map $J' : H^0_K(\mathcal{B}) \to \mathcal{H}(X_G)$. This is easily checked to be an isomorphism; subjectivity follows from Proposition 4.3 and any exact differential form in $\Omega(V)^n$ comes from the image of $K$ under the map $J'$ by a similar computation, which shows injectivity. □

4.5. The $L_\infty$-homotopy structure on period integrals of smooth projective hypersurfaces

Here we prove the main theorem, Theorem 1.1, which describes how to understand the period integrals of smooth projective hypersurfaces in terms of homotopy theory, by applying the general theory developed in sections 2 and 3. Fix a homology cycle $\gamma \in H_{n-1}(X_G, \mathbb{Z})$. Then we have a $k$-linear map, the Griffiths period integral;
\[ C[\gamma] : \mathcal{H}(X_G) \to k \]
\[ \left[ \frac{F(x)}{G(x)^k \Omega_n} \right] \to \frac{1}{2\pi i} \int_{\gamma(x)} \frac{F(x)}{G(x)^k \Omega_n}, \]
where $[\cdot]$ means the cohomology class, $k$ is a positive integer, and $\deg F + n + 1 = k \deg G = kd$ (see Proposition 4.3). We associated the Lie algebra representation $\rho_x$ to $X_G$ and constructed a cochain complex $(\mathcal{A}^\bullet, K \rho_x)$ with product, an object of the category $\mathfrak{c}_K$. For simplicity, let us use the shorthand notation $\mathcal{H}_n = \mathcal{H}(X_G) \simeq H^{n-1}_{\text{prim}}(X_G, \mathbb{C})$. We again use the notation $(\mathcal{A}, K) = (\mathcal{A}^\bullet, K \rho_x)$. Then Proposition 2.3 says that we can enhance $C[\gamma] : A \to k$ to a cochain map $\gamma' : (\mathcal{A}, K) \to (k, 0)$ such that $\gamma'_x(x) = C[\gamma](x)$ for $x \in \mathcal{A}$ and $\gamma'_x(x) = 0$ otherwise. If we apply the descendant functor (Theorem 3.1) to $(\mathcal{A}, K)$, then Proposition 4.2 implies that $(\mathcal{A}, K, l^1_k)$ is a differential graded Lie algebra (DGLA) over $k$. The $k$-module $\mathcal{A}^{-1}$ with the bracket $l^1_k(\cdot, \cdot)$ becomes a Lie algebra over $k$.

If $d \neq n + 1$, then Theorem 4.1 implies that $(\mathcal{A}, K, l^1_k)$ is a non-trivial differential graded Lie algebra which is quasi-isomorphic to $(\mathcal{H}, 0)$. So this proves the part $(a)$ of Theorem 1.1, when $d \neq n + 1$.

If $d = n + 1$, then $H^{-1}_K(\mathcal{A}) \simeq k \cdot R$, where $R$ is given in (4.7), according to Theorem 4.1. Thus $(\mathcal{A}, K, l^1_k)$ is not quasi-isomorphic to $(\mathcal{H}, 0)$. We kill the non-trivial cohomology class $[R]$ by a non-derivation version of the Koszul-Tate resolution, in order to prove part $(a)$. Define
\[ \mathcal{J} := k[\theta] \otimes_k \mathcal{B}, \]
where $\theta$ is a formal element of degree $-2$ introduced to kill $[R]$. Recall that $\mathcal{B} = \text{Ker}(\partial_R)$, where $\partial_R$ is given in (4.8). Note that $\mathcal{J}$ does not inherit a product structure from $k[\theta] \otimes \mathcal{A}$, since $\mathcal{B}$...
is not closed under multiplication in $\mathcal{A}$. But it has an $\mathcal{A}^{0}$-module structure inherited from the multiplication of $\mathcal{A}$. Then we define the $\mathbb{k}[\theta]$-linear map

$$
\tilde{K} = K + R \frac{\partial}{\partial \theta} : \mathcal{A} \to \mathcal{A}
$$

(4.14)

so that $\tilde{K}(\theta) = K(\theta) + R = \theta \cdot K(1) + R = R$. Note that $\mathcal{B}$ has a distinguished element 1 which is the unit of $\mathcal{A}$ and $K(1) = 0$ by definition. We have

$$\mathcal{A} = \bigoplus_{n \leq 0} \mathcal{A}^n = \mathcal{A}^{0} \oplus \mathcal{A}^{-1} \oplus \mathcal{A}^{-2} \oplus \cdots,$$

where $\mathcal{A}$ has arbitrarily large negative degree (non-zero) elements because of $\theta$.

**Lemma 4.4.** The pair $(\mathcal{A}, \tilde{K})$ is a cochain complex over $\mathbb{k}$.

**Proof.** We abbreviate $\theta^{n} \otimes b \in \mathcal{A} = \mathbb{k}[\theta] \otimes \mathcal{B}$ by $\theta^{n} \cdot b$. Let us compute

$$
\tilde{K}^{2}(\theta^{n} \cdot b) = \tilde{K} \left( (K + R \frac{\partial}{\partial \theta})(\theta^{n} \cdot b) \right) = \tilde{K} \left( \theta^{n} K(b) + R n \theta^{n-1} b \right)
$$

$$
= (K + R \frac{\partial}{\partial \theta}) \left( \theta^{n} K(b) + n \theta^{n-1} R b \right) = n \theta^{n-1} R \cdot K(b) + n \theta^{n-1} K(R b) = n \theta^{n-1} \delta_R(b),
$$

where we used $R \cdot R = 0$ and $K^2 = 0$. Since $b \in \operatorname{Ker}(\delta_R)$, the result follows. $\Box$

We now define a $\mathbb{k}[\theta]$-linear map $\tilde{l}_2 : S^2(\mathcal{A}) \to \mathcal{A} \otimes \mathbb{k}[\theta]$ by

$$
\tilde{l}_2(\theta^n \cdot b, \theta^m \cdot b') = \theta^{n+m} \cdot l_2^K(b, b'), \quad b, b' \in \mathcal{B}, \; n, m \geq 0.
$$

(4.15)

**Lemma 4.5.** The image of $\tilde{l}_2$ lands in $\mathcal{A}$. Moreover, $\tilde{l}_2 : S^2(\mathcal{A}) \to \mathcal{A}$ defines a graded Lie algebra structure on $\mathcal{A}$.

**Proof.** The first claim follows from (4.2) by a direct computation. The second one follows since the descendant $l_2^K$ defines a graded Lie algebra structure on $\mathcal{B} = \operatorname{Ker}(\delta_R)$. $\Box$

**Proposition 4.7.** The triple $(\mathcal{A}, \tilde{K}, \tilde{l}_2)$, associated to the hypersurface $X_G$, is a differential graded Lie algebra quasi-isomorphic to $(\mathbb{H}, 0)$.

**Proof.** It remains to check that $\tilde{K}$ is a graded derivation of $\tilde{l}_2$;

$$
\tilde{K} \left( \tilde{l}_2(\theta^n \cdot b, \theta^m \cdot b') \right) + \tilde{l}_2(K(\theta^n \cdot b), \theta^m \cdot b') + (-1)^{|b|}(\theta^n \cdot b, \tilde{K}(\theta^m \cdot b')) = 0,
$$

(4.16)

where $b, b' \in \mathcal{B} = \operatorname{Ker}(\delta_R), \; n, m \geq 0$. This follows from two facts: (i) that $K$ is a graded derivation of $l_2^K$ and (ii) that $(\mathcal{A}, *, \tilde{l}_2^K)$ is the poisson structure (4.2). Thus $(\mathcal{A}, \tilde{K}, \tilde{l}_2)$ is a differential graded Lie algebra with using Lemmas 4.4 and 4.5.
In order to show that \((\mathcal{A}, \bar{K})\) is quasi-isomorphic to \((\mathbb{H}, 0)\), we need the following claim;

\[
\theta^n \cdot b \in \mathcal{A}^k, \ Kb = Rb = 0, n \geq 0 \quad \text{implies} \quad \theta^n \cdot b \in \text{im}(\bar{K}),
\]

(4.17) for any homogeneous element \(b \in \mathcal{B}\) and \(k \leq -1\). We prove this by induction. If \(n = 0\), then the claim follows, since \((\mathcal{A}, K)\) does not have non-zero cohomology when the degree \(\leq -1\), except for a \(k\)-multiple of \(R\) which becomes the image of \((a \cdot k\)-multiple of) \(\theta\) under \(\bar{K}\). Now assume that the claim is true for \(n \geq 1\). Then the degree of \(b\) is less than or equal to -1. There are two cases; (i) \(b\) is a \(k\)-multiple of \(R\), and (ii) \(b\) is not. The first case follows from the following computation

\[
\bar{K} \left( \frac{1}{n + 2} \cdot \theta^{n+2} \right) = \theta^{n+1} \cdot R.
\]

In the second case, \(Kb = 0\) says that we can find \(a \in \mathcal{B}\) such that \(Ka = b\), since \((\mathcal{A}, K)\) has only 0-th cohomology except for a \(k\)-multiple of \(R\). Then we use

\[
\bar{K}(\theta^{n+1}a) = \theta^{n+1}Ka + (n + 1)\theta^n Ra = \theta^{n+1}b + (n + 1)\theta^n Ra,
\]

and the induction hypothesis that \(\theta^n Ra\) is \(\bar{K}\)-exact (because \(K(Ra) = -R(Ka) = -Rb = 0\). Then \(\theta^{n+1}b\) is also \(\bar{K}\)-exact), which proves the claim for \(n + 1\).

If \(\bar{K}(\theta^n \cdot b) = \theta^n K(b) + n\theta^{n-1}Rb = 0\), then \(K(b) = 0\) and \(Rb = 0\). Then by claim (4.17), \(\theta^n \cdot b\) is \(\bar{K}\)-exact. One can argue in a similar way that if \(\bar{K}(\sum_{k=1}^m \theta^k \cdot b_k) = 0\) for \(m \geq 1\) and \(b_k \in \mathcal{B}\), then \(\sum_{k=1}^m \theta^k \cdot b_k\) is also \(\bar{K}\)-exact by using (4.17). This means that \((\mathcal{A}, \bar{K})\) does not generate new non-trivial cohomology elements. Also note that \(H_{\bar{K}}^{-1}(\mathcal{B}) = 0\) and \(H_{\bar{K}}^0(\mathcal{B}) \simeq \mathbb{H}\). This finishes the proof.

\[\Box\]

This motivates us to define \((\mathcal{A}, \bar{L})_X\) as be the differential graded Lie algebra

\[
(\mathcal{A}, \bar{L})_X := \begin{cases} 
(\mathcal{A}, K, \ell^X_k) & \text{if } d \neq n + 1, \\
(\mathcal{A}, \bar{K}, \bar{\ell}_2) & \text{if } d = n + 1.
\end{cases}
\]

(4.18)

Then Theorem 4.1 and Proposition 4.7 imply the following theorem, part (a) of Theorem 1.1;

**Theorem 4.2.** The triple \((\mathcal{A}, \bar{L})_X\) is a non-trivial differential graded Lie algebra associated to \(X_G\), which is quasi-isomorphic to \((\mathbb{H}, 0)\), where \(\mathbb{H} = H_{\text{prim}}^{-1}(X_G, \mathbb{C})\).

For the proof of the statements \((b)\) and \((c)\) in Theorem 1.1, we analyze the cochain map \(\mathcal{C}_r : (\mathcal{A}, K) \to (\mathbb{K}, 0)\) using the super-commutative product \(\cdot\), by applying the descendant functor \(\mathcal{D}\) in Theorem 3.1 and applying all the general results (for example, Theorems 3.3 and 3.2) which we have proved for objects and morphisms in the category \(\mathcal{C}_k\). The key point here is that we are able to construct a cochain complex \((\mathcal{A}, K)\) which has a super-commutative associative algebra structure and realize the \(k\)-linear map \(\mathcal{C}_{[r]} : \mathcal{H}(X_G) \to \mathbb{C}\) as the homotopy type of the cochain map \(\mathcal{C}_r : (\mathcal{A}, K) \to (\mathbb{K}, 0)\) (see Proposition 3.4) so that we can compare \(\mathcal{C}(u), \mathcal{C}(v)\), and \(\mathcal{C}(u \cdot v)\) and study their higher correlations. The strategy we have developed so far suggests that we have to
understand $C_{[\gamma]}$ using the composition of $L_\infty$-morphisms; Proposition 2.3, Proposition 3.3, Theorem 3.2, Proposition 4.4, and Theorem 4.1 imply the following theorem, the parts (b) and (c) of the main theorem 1.1;

**Theorem 4.3.** For each $\gamma \in H_{n-1}(X_G, \mathbb{C})$, the induced map of $\mathcal{C} = \mathcal{C}_\gamma : (\mathcal{A}, \mathcal{E})_X \to (\mathbb{k}, 0)$ on the cohomology $\mathbb{H}$ is the same as the Griffiths period integral $C_{[\gamma]}$. Moreover, $C_{[\gamma]}$ can be enhanced to the composition of $L_\infty$-morphisms through the differential graded Lie algebra $(\mathcal{A}, \mathcal{E})_X$; we have the following diagram of $L_\infty$-morphisms of $L_\infty$-algebras;

\[
\begin{array}{ccc}
\mathbb{H} \oplus (\mathbb{k}, 0) & \xrightarrow{\mathcal{C}_{[\gamma]}} & (\mathbb{k}, 0) \\
\mathcal{E}_H & \xrightarrow{\phi_{[\gamma]} \cdot \psi_H} & \phi_{[\gamma]} \cdot \psi_H \\
(\mathcal{A}, \mathcal{E})_X & \xrightarrow{\phi_{[\gamma]} \cdot \psi_H} & (\mathbb{k}, 0)
\end{array}
\]

where $C_{[\gamma]}$ is the same as $(\phi_{[\gamma]} \cdot \psi_H)_1 = \mathcal{C}_\gamma \cdot \psi_H$. The $L_\infty$-morphism $\mathcal{K} := \phi_{[\gamma]} \cdot \psi_H$ depends only on the $L_\infty$-homotopy types of $\psi_H$ and $\phi_{[\gamma]}$.

The composition $\phi_{[\gamma]} \cdot \psi_H$ is not a descendant $L_\infty$-morphism; we do not have a super-commutative associative binary product on $\mathbb{H}$ (the differential $K$ is not a derivation of the product of $\mathcal{A}$ so that it does not induce a product on $\mathbb{H}$). Note that $\psi_H$ is an $L_\infty$-quasi-isomorphism: such quasi-isomorphisms are classified by the versal solutions $\Gamma \in (m_{[\mathbb{H}]} \otimes \mathcal{A})^0 = m_{[\mathbb{H}]} \otimes \mathcal{A}^0$ of the Maurer-Cartan equation (see Proposition 3.2)

\[K(\Gamma) + \frac{1}{2} \ell_k^2(\Gamma, \Gamma) = 0.\]

So we found hidden structure of the period integral $C_{[\gamma]} : \mathbb{H} \simeq \mathcal{H}(X_G) \to \mathbb{k}$ for a fixed $\gamma \in H_{n-1}(X_G, \mathbb{Z})$; there is an $L_\infty$-quasi-morphism from the $L_\infty$-algebra $(\mathbb{H}, 0)$ to the $L_\infty$-algebra $(\mathcal{A}, \mathcal{E})_X$, and a sequence of $\mathbb{k}$-linear maps $(\phi_{[\gamma]} \cdot \psi_H)_m : S^m(\mathbb{H}) \to \mathbb{k}$, which reveals hidden correlations of $C_{[\gamma]}$, such that $C_{[\gamma]} = (\phi_{[\gamma]} \cdot \psi_H)_1$. Then the theory of $L_\infty$-algebras suggests that we can study a (new type of) formal variation of the Griffiths period integral. We discuss this issue in the next Subsections.

### 4.6. Variations of period integrals and a generalization of Picard-Fuchs equations

In this Subsection we prove Theorem 1.2 and Theorem 1.3. Theorem 1.3 follows easily from Proposition 3.2, Definition 3.6, and Lemma 3.10. Hence we concentrate on Theorem 1.2. Recall that by a
standard basis of $\mathbb{H}$ we mean a choice of basis $e_1, \cdots, e_{\delta_0}, e_{\delta_0+1}, \cdots, e_{\delta_1}, \cdots, e_{\delta_{n-2}+1}, \cdots, e_{\delta_{n-1}}$ for the flag $\mathcal{F}_n\mathbb{H}$ in (1.3) such that $e_1, \cdots, e_{\delta_0}$ gives a basis for the subspace $\mathbb{H}^{-1-0.0} := H_{\operatorname{prim}}^{n-1,0}(X_G, \mathbb{C})$ and $e_{\delta_{k-1}+1}, \cdots, e_{\delta_k}$, $1 \leq k \leq n - 1$, gives a basis for the subspace $\mathbb{H}^{-n-1-k,k} := H_{\operatorname{prim}}^{n-1-k,k}(X_G, \mathbb{C})$. We denote such a basis by $\{e_a\}_{a \in I}$ where $I = I_0 \cup I_1 \cup \cdots \cup I_{n-1}$ with the notation $\{e_a\}_{a \in I_j} = e_{\delta_{j-1}+1}, \cdots, e_{\delta_j}$ and $\{t^a\}_{a \in I_j} = t^{\delta_j-1+1}, \cdots, t^{\delta_j}$. For part (a), (b) and (c) of Theorem 1.2, we define $k$-linear maps $\phi : (\mathbb{H}, \mathcal{O}) \longrightarrow (\mathcal{O}, \mathcal{L})$;

$$f = f_1, \quad f_1(e_a) := y^k \cdot F_{\{1\}}(x) \in \mathcal{O}^0, \quad f_2 = f_3 = \cdots = 0, \quad (4.20)$$

where $F_{\{1\}}(x)$ can be chosen to be any homogeneous polynomial of degree $d(k + 1) - (n + 1)$ such that $\{F_{\{1\}}(x) : a \in I, \ 0 \leq k \leq n - 1\}$ is a set of representatives of all the cohomology classes of $\mathcal{H}(X_G) \cong \mathbb{H}$. Then $f$ is clearly a $L_\infty$-quasi-isomorphism satisfying (a), (b) and (c) by the Griffiths theorem on $\mathcal{H}(X_G)$ and our general theory.

Let $G_{\{1\}}(x) = G(x) + \sum_{a \in I_1} t^a F_{\{1\}}(x)$. Part (d) follows from the following computation:

$$\int_{\tau(\gamma)} \frac{\Omega_n}{G_{\{1\}}(x)} - \int_{\tau(\gamma)} \frac{\Omega_n}{G(x)} = \int_{\tau(\gamma)} \int_0^\infty e^{-y G_{\{1\}}(x)} dy \Omega_n - \int_{\tau(\gamma)} \int_0^\infty e^{-y G(x)} dy \Omega_n$$

$$= \mathcal{C}_\gamma(e^{\sum_{a \in I} t^a F_{\{1\}}(x)} - 1)$$

$$= \exp \left( \phi_\mathcal{C}_\gamma \left( \sum_{a \in I} t^a F_{\{1\}}(x) \right) \right) - 1$$

$$= \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{a_1, a_2, a_n} t^{a_n} \cdots t^{a_1} \left( \phi_\mathcal{C}_\gamma \bullet f_{\{1\}} \right)_n (e_{a_1}, \cdots, e_{a_n}) \right) - 1,$$

where we used Definition of $\mathcal{C}_\gamma$ and the equalities (3.24) and (3.25). Let $\chi^L(t) = \exp \left( \sum_{a \in I_0} t^a F_{\{0\}}(x) + y \sum_{a \in I_1} t^a F_{\{1\}}(x) + \cdots + y^{n - 1} \sum_{a \in I_{n-1}} t^a F_{\{n-1\}}(x) \right)$. Part (e) follows from the similar following computation:

$$\int_{\tau(\gamma)} \left( \int_0^\infty \chi^L(t) \cdot e^{-y G(x)} dy \right) \Omega_n - \int_{\tau(\gamma)} \frac{\Omega_n}{G(x)}$$

$$= \exp \left( \phi_\mathcal{C}_\gamma \left( \sum_{a \in I_0} t^a F_{\{0\}}(x) + y \sum_{a \in I_1} t^a F_{\{1\}}(x) + \cdots + y^{n - 1} \sum_{a \in I_{n-1}} t^a F_{\{n-1\}}(x) \right) \right) - 1$$

$$= \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{a_1, \cdots, a_n} t^{a_n} \cdots t^{a_1} \left( \phi_\mathcal{C}_\gamma \bullet f_{\{1\}} \right)_n (e_{a_1}, \cdots, e_{a_n}) \right) - 1.$$
The invariants \( \int_\gamma \left( \int_0^\infty \chi^L(t) \cdot e^{y \cdot G(x)} dy \right) \Omega_n \) in \( (e) \) contain additional information, not contained in \( (d) \), without a clear geometrical origin. The properties, which can be easily checked,

\[
\int_0^\infty \chi^L(t) \cdot e^{-y \cdot G(x)} dy = \frac{\Omega_n}{G(x)},
\]

\[
\frac{\partial}{\partial t_k} \int_0^\infty \chi^L(t) \cdot e^{-y \cdot G(x)} dy \bigg|_{t=0} = k^2 \frac{F_{ij}^k \cdot \Omega_n}{G(x)^{k+1}}, \quad \forall a \in I_k,
\]

for example, imply that Griffiths transversality is violated for \( k \geq 2 \).

Note that the deformation in part \( (d) \) is a geometric deformation of the complex structure of \( X_G \), a family of hypersurfaces. Theorem 1.2 shows that the period integral of deformations is the same as the generating power series \( \mathcal{C}(e^{f[1]}) - 1 \), which is an \( L_\infty \)-homotopy invariant, for a particular \( L_\infty \)-morphism \( f[1] \) from \( (\mathbb{H}^{n-2}, 0) \) to \( (\mathcal{O}_I, \mathcal{L}_X) \). Thus this supports our point of view that the generating power series is a natural generalization of the geometric invariant, the Griffiths period integral of a family of smooth hypersurfaces, in addition to its \( L_\infty \)-homotopy invariance. Note that the geometric deformation will not be an \( L_\infty \)-quasi-isomorphism on the full cohomology \( (\mathbb{H}, 0) \) and so this does not satisfy the system of second order partial differential equation (3.31). This suggests the necessity of enlarging the category (the category \( \mathcal{C}_k \) or \( \mathcal{L}_k \) rather than the category of algebraic varieties) in order to capture a new variation and correlation information of the Griffiths’ period integral \( C_{\mathcal{C}_k} \) of \( X_G \). Though such a geometric deformation does not satisfy (3.31), a similar algorithm in Lemma 4.7 still gives a differential equation, which leads to the Picard-Fuchs type equation resulting from the Gauss-Manin connection of a family of hypersurfaces.

### 4.7. Explicit computation of variations of Griffiths period integrals

Let \( (\mathcal{O}_I, K) \in \mathcal{O}(\mathcal{C}_k) \) be the cochain complex attached to \( X_G \). Let \( H_K = H_k(\mathcal{O}_I) \) be its cohomology group. Let \( \mathcal{C}^H \) be an \( L_\infty \)-quasi-isomorphism from \( (H_K, 0) \) to \( (\mathcal{O}_I, \mathcal{L}_X) \) by Theorem 4.2. Then we see that the system of second order partial differential equations (3.31) holds for a uniquely determined 3-tensor \( A_{\alpha \beta}^\gamma(t) \in k[[t^n]] \cong \mathcal{S}^H \) for \( \Gamma = \Gamma(t) \gamma, \) where \( \mathcal{S}^H := \lim_{n \to \infty} \bigoplus_{k=0}^n \mathcal{S}^k(H_K^*) \) with \( H_K^* = \text{Hom}(H_K, k) \); the assumption in Theorem 3.3 can be checked for \( (\mathcal{O}_I, K) \). In Subsection 3.3, we saw that the explicit computation problem of the generating power series \( \mathcal{C}(e^{\ell - 1}) \) reduces to the problem of computing \( A_{\alpha \beta}^\gamma(t) \), in addition to the data \( \mathcal{C}(\varphi_{\gamma}^H(\ell))) \), \( \alpha \in I \). Here we provide an algorithm to compute \( A_{\alpha \beta}^\gamma(t) \), generalizing the method (a systematic way of doing integration by parts in the one-variable case) in Proposition 4.1. We recall that \( \mathcal{O}_I = \mathcal{O}_{-n-2} \oplus \cdots \oplus \mathcal{O}_{-1} \oplus \mathcal{O}_0 \), \( K = \Delta + Q \), and

\[
K_{I'} = \Delta + Q_I,
\]

\[
\Delta = \sum_{i=-n}^n \frac{\partial}{\partial y_i} \frac{\partial}{\partial \eta_i},
\]

\[
Q_I = \sum_{i=-n}^n \frac{\partial(y \cdot G(x))}{\partial y_i} \frac{\partial}{\partial \eta_i} + \ell^k_{2}(\Gamma, \cdot) = Q + \ell^k_{2}(\Gamma, \cdot).
\]
If $A = \sum_{i=1}^{n} \lambda_i \eta_i \in \mathcal{A}^{-1}$, where $\lambda_i \in \mathcal{A}^0$, $i = -1, 0, \cdots, n$, then

$$Q_I(A) = \sum_{i=1}^{n} \lambda_i \frac{\partial(yG(x))}{\partial y_i} + \ell^2_2(\Gamma, A), \quad \Delta(A) = \sum_{i=1}^{n} \frac{\partial \lambda_i}{\partial y_i}.$$

Note that

$$K^2_I = \Delta^2 = 0, \quad K(\mathcal{A}) = Q(\mathcal{A}) = \Delta(\mathcal{A}) = 0,$$

where $1_{\mathcal{A}}$ is the identity element in $\mathcal{A}^0 = k[y, x]$.\

**Lemma 4.6.** For any $L_\infty$-quasi-isomorphism $\varphi^H$ from $(H_K(\mathcal{A}), 0)$ to $(\mathcal{A}_\mathcal{X}, K\mathcal{X})$, we have that

$$\Delta(\Gamma(t)_{\varphi^H}) = 0. \quad (4.21)$$

**Proof.** Recall that

$$\Gamma = \Gamma(t)_{\varphi^H} = \sum_{n=1}^{\infty} \frac{1}{n! a_1 \cdots a_n} t^{a_n} \cdots t^{a_1} \otimes \varphi^H_n(e_{a_1}, \cdots, e_{a_n}) \in (m_{\mathcal{H}} \otimes \mathcal{A})^0.$$

By Theorem 4.1, $H_K = H_{K^{-1}} \oplus H_{K^0}$. Thus the image of $\varphi^H$ is in $\mathcal{A}^{-1} \oplus \mathcal{A}^0$, since $\Gamma_{\varphi^H}$ has degree zero. If $\varphi^H$ has its image in $\mathcal{A}^0$, then (4.21) is trivially true. If $\varphi^H$ has its image in $\mathcal{A}^{-1}$ (this occurs only if $d = n + 1$), then $\varphi^H_n(e_{a_1}, \cdots, e_{a_n})$ is a $k$-linear combination of the following form (since $H_{K^{-1}}$ is a one-dimensional space spanned by $R$)

$$R + K(A), \quad R = -(n + 1) \cdot y \eta_{n-1} + \sum_{i=0}^{n} x_i \eta_i \in \mathcal{A}^{-1},$$

where $A = \sum_{1 \leq i < j \leq n} A_{i,j}(y) \cdot \eta_i \cdot \eta_j \in \mathcal{A}^{-2}$, with $A_{i,j} = -A_{j,i}$. Then a direct computation shows that $\Delta(R + K(A)) = 0$. This finishes the proof. \hfill $\Box$

Lemma 4.6 implies that

$$Q(\Gamma) + \frac{1}{2} \ell^2_2(\Gamma, \Gamma) = 0,$$

because $K(\Gamma) + \frac{1}{2} \ell^2_2(\Gamma, \Gamma) = Q(\Gamma) + \Delta(\Gamma) + \frac{1}{2} \ell^2_2(\Gamma, \Gamma) = 0$, which is equivalent to $K(e^\Gamma - 1) = 0$. Then this says that

$$\Delta Q_I + Q_I \Delta = Q_I^2 = 0.$$ 

So we have a cochain complex $(\mathcal{S} \mathcal{H} \otimes \mathcal{A}, Q_I)$ with super-commutative product. Note that $Q_I$ is a derivation of the binary product of $\mathcal{S} \mathcal{H} \otimes \mathcal{A}$. We tacitly think $Q_I$ as the classical component of the differential $K_I = Q_I + \Delta$; we view $\Delta$ as the quantum component of $K_I$ on the other hand. The key point in the algorithm of computing $A_{ab}^\varphi_{I}(\Gamma)$ is that we can use the ideal membership problem which is susceptible to Gröbner basis methods to compute the answer to a problem that involves only $Q_I$ and relate it the corresponding problem on $K_I$, by utilizing the quantum component $\Delta$. 
Lemma 4.7. Let $\Gamma = \Gamma^{(\Gamma)}_{\cal H}$ be a versal Maurer-Cartan solution. Then there is an algorithm to compute a finite sequence $A^{(M)\gamma}_{\alpha\beta}(\mathcal{L}) \in (\mathcal{S}H \otimes \mathcal{A})^0$, where $m = 0, 1, \ldots, M$ for some positive integer $M$, such that

$$A_{\alpha\beta}^{\gamma}(\mathcal{L}) = A^{(0)\gamma}_{\alpha\beta}(\mathcal{L}) - A^{(1)\gamma}_{\alpha\beta}(\mathcal{L}) - \cdots - A^{(M)\gamma}_{\alpha\beta}(\mathcal{L}) \in \widehat{\mathcal{S}H},$$

where $A_{\alpha\beta}^{\gamma}(\mathcal{L})$ is in (3.31).

Proof. We use the notation $\Gamma = \partial_a \mathcal{L}(\mathcal{L})$ and $\Gamma_a = \partial_a \mathcal{L}(\mathcal{L})$. One can check that $H_{Q_t} = H_{Q_t}(\mathcal{S}H \otimes \mathcal{A})$ is a finitely generated $\mathcal{S}H \otimes \mathcal{A}^0$-module, whose generators are given by $\{\Gamma_\gamma : \gamma \in \mathcal{I}\}$, since $Q_t$ is a derivation of the binary product. Then, for $\Gamma_a \cdot \Gamma_\beta \in (\mathcal{S}H \otimes \mathcal{A})^0$, we can find $A^{(0)\gamma}_{\alpha\beta}(\mathcal{L}) \in \mathcal{S}H \otimes \mathcal{A}^0$ and $A^{(0)\gamma}_{\alpha\beta}(\mathcal{L}) \in (\mathcal{S}H \otimes \mathcal{A})^0$ (this is an ideal membership problem) such that

$$\Gamma_a \cdot \Gamma_\beta = \sum_{\gamma} A^{(0)\gamma}_{\alpha\beta}(\mathcal{L}) \Gamma_\gamma + Q_t(A^{(0)\gamma}_{\alpha\beta}(\mathcal{L})).$$

Then the relation (4.22) can be rewritten as

$$\Gamma_a \cdot \Gamma_\beta = \sum_{\gamma} A^{(1)\gamma}_{\alpha\beta}(\mathcal{L}) \Gamma_\gamma + K_t(A^{(1)\gamma}_{\alpha\beta}(\mathcal{L})) - \Delta(A^{(0)\gamma}_{\alpha\beta}(\mathcal{L})).$$

Let $R^{(1)} = \Delta(A^{(0)\gamma}_{\alpha\beta}(\mathcal{L}))$ and we can find $A^{(1)\gamma}_{\alpha\beta}(\mathcal{L}) \in \mathcal{S}H \otimes \mathcal{A}^0$ and $A^{(1)\gamma}_{\alpha\beta}(\mathcal{L}) \in (\mathcal{S}H \otimes \mathcal{A})^0$ (again by an ideal membership problem) such that

$$R^{(1)} = \sum_{\gamma} A^{(1)\gamma}_{\alpha\beta}(\mathcal{L}) \Gamma_\gamma + Q_t(A^{(1)\gamma}_{\alpha\beta}(\mathcal{L})).$$

Set $R^{(2)} = \Delta(A^{(1)\gamma}_{\alpha\beta}(\mathcal{L}))$ and we can find $A^{(2)\gamma}_{\alpha\beta}(\mathcal{L}) \in \mathcal{S}H \otimes \mathcal{A}^0$ and $A^{(2)\gamma}_{\alpha\beta}(\mathcal{L}) \in (\mathcal{S}H \otimes \mathcal{A})^0$ similarly such that

$$R^{(2)} = \sum_{\gamma} A^{(2)\gamma}_{\alpha\beta}(\mathcal{L}) \Gamma_\gamma + Q_t(A^{(2)\gamma}_{\alpha\beta}(\mathcal{L})).$$

We can continue this way and observe that this process stops after a finite number of steps. Then Theorem 3.3 guarantees that we can choose $A^{(M)\gamma}_{\alpha\beta}(\mathcal{L})$ such that $\Delta(A^{(M)\gamma}_{\alpha\beta}(\mathcal{L})) = \Gamma_a$ so that we have

$$\Gamma_a \cdot \Gamma_\beta = \sum_{\gamma} A^{(M)\gamma}_{\alpha\beta}(\mathcal{L}) \Gamma_\gamma - \Delta(A^{(M)\gamma}_{\alpha\beta}(\mathcal{L})) + K_t(L_{\alpha\beta}(\mathcal{L})),$$

where

$$A^{(M)\gamma}_{\alpha\beta}(\mathcal{L}) = A^{(M)\gamma}_{\alpha\beta}(\mathcal{L}) \Gamma_\gamma = A^{(0)\gamma}_{\alpha\beta}(\mathcal{L}) \Gamma_\gamma - A^{(1)\gamma}_{\alpha\beta}(\mathcal{L}) \Gamma_\gamma - \cdots - A^{(M)\gamma}_{\alpha\beta}(\mathcal{L}) \Gamma_\gamma \in \widehat{\mathcal{S}H},$$

$$L_{\alpha\beta}(\mathcal{L}) = A^{(0)\gamma}_{\alpha\beta}(\mathcal{L}) - A^{(1)\gamma}_{\alpha\beta}(\mathcal{L}) - \cdots - A^{(M)\gamma}_{\alpha\beta}(\mathcal{L}) \in (\mathcal{S}H \otimes \mathcal{A})^0$$

for some positive integer $M$. Note that this equality is same as (3.32), which finishes the proof. $\square$

This enables us to compute a new formal deformation of the Griffiths period integral, i.e. the generating power series $\varphi_{\gamma}(e^\Gamma_{\cal H} - 1)$ attached to $C_{\gamma}$ and a formal deformation data $\varphi_{\gamma}^H$. 
4.8. A special flat connection on the formal deformation space of $X_G$

Here we will prove Theorem 1.5; we can deform (in an extended sense) a projective smooth hypersurface $X_G$ in a particular way so that the corresponding flat connection $A_f$, in Theorem 3.4, has a special property, a so called quantum property.

**Theorem 4.4.** Let $(\mathcal{A}, \cdot, K)$ be the object of $\mathcal{C}_k$ associated to the projective smooth hypersurface $X_G$. Then we can choose a special $L_\infty$-quasi-isomorphism $\hat{\phi}$ such that the flat connection $D_{I_{2}^\Lambda} := d + A_{I_{2}^\Lambda}$ satisfies:

$$dA_{I_{2}^\Lambda} = 0, \quad A_{I_{2}^\Lambda}^2 = 0.$$  

(4.23)

**Proof.** We use the argument and the notation used in the proof of Lemma 4.7. Let $f : H_K \to \mathcal{A}$ be a cochain quasi-isomorphism between cochain complexes $(H_K, 0)$ and $(\mathcal{A}, K)$, which induces the identity map on $H = H_K$. Let $h : \mathcal{A} \to H$ be a homotopy inverse of $f$ such that $h \circ f = I_H$ and $f \circ h = I_\mathcal{A} + K\beta + \beta K$ for some $k$-linear map $\beta$ from $\mathcal{A}$ to $\mathcal{A}$ of degree $-1$. First, we define $\varphi^H_1(e_a) = f(e_a)$ for $a \in I$, which is a cochain quasi-isomorphism. Recall that, for $\Gamma_a \cdot \Gamma_{\beta} \in (\hat{S} \hat{H} \otimes \mathcal{A})$, we can find $A_{a\beta}(\underline{t})_{|\Gamma} \in \hat{S} \hat{H} \otimes \mathcal{A}$ and $A_{a\beta}(\underline{t}) \in (\hat{S} \hat{H} \otimes \mathcal{A})^{-1}$ (this is an ideal membership problem) such that

$$\Gamma_a \cdot \Gamma_{\beta} = \sum_{\gamma} A_{a\beta}(\underline{t})_{|\Gamma} \cdot \Gamma_\gamma + Q_\gamma(A_{a\beta}(\underline{t})).$$  

(4.24)

We write $A_{a\beta}(\underline{t})$ as follows:

$$A_{a\beta}(\underline{t}) = \lambda_{2}^{(0)}(e_a, e_\beta) + \sum_{n=3}^{\infty} \frac{1}{(n-2)!} t^{\rho_{n-2}} \cdots t^{\rho_1} \cdot \lambda_{n}^{(0)}(e_a, e_\beta, e_{\rho_1}, \ldots, e_{\rho_{n-2}}),$$

where $\lambda_n^{(0)} : S^n(H) \to \mathcal{A}$. Then we define $\varphi^H_2(e_a, e_\beta)$ such that $\partial_a \partial_\beta \Gamma_{\underline{t}}|_{\Gamma} = \varphi^H_2(e_a, e_\beta) = \Delta(A_{a\beta}(\underline{t})) = \lambda_{n}^{(0)}(e_a, e_\beta) (\text{we choose this definition because of (3.32)})$ and define $\varphi^H_{n} = \Delta(\lambda_n^{(0)})$ for $n \geq 3$. Such a choice of $\varphi^H$ satisfies $K(e^{\Gamma_{\underline{t}} - 1}) = 0$ (so that it is an $L_\infty$-quasi-isomorphism) and

$$\partial_a \partial_\beta \Gamma_{\underline{t}} = \Delta(A_{a\beta}(\underline{t})),$$

because the equality (3.35) implies that

$$\partial_a \partial_\beta \Gamma_{\underline{t}} = \varphi^H_2(e_a, e_\beta) + \sum_{n=3}^{\infty} \frac{1}{(n-2)!} t^{\rho_{n-2}} \cdots t^{\rho_1} \cdot \varphi^H_{n}(e_a, e_\beta, e_{\rho_1}, \ldots, e_{\rho_{n-2}}).$$

Therefore we can conclude that

$$\Gamma_a + \Gamma_a \cdot \Gamma_{\beta} = \sum_{\gamma} A_{a\beta}^{(0)}(\underline{t})_{|\Gamma} \cdot \Gamma_\gamma + Q_\gamma(A_{a\beta}(\underline{t})), \quad A_{a\beta}^{(0)}(\underline{t})_{|\Gamma} = A_{a\beta}^{(0)}(\underline{t}),$$

in Theorem 4.8. A special flat connection on the formal deformation space of $X_G$. 

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The associativity of the product \((\Gamma_\alpha \cdot \Gamma_\beta) \cdot \Gamma_\gamma = \Gamma_\alpha \cdot (\Gamma_\beta \cdot \Gamma_\gamma)\), combined with the equality (4.24), implies that

\[ A_2 X^{\alpha} \hat{\phi} H = 0. \]

Since we already know \(d A_2 X^{\alpha} \hat{\phi} H + A_2 X^{\alpha} \hat{\phi} H = 0 \) from Theorem 3.4, we also have \(d A_2 X^{\alpha} \hat{\phi} H = 0\). □

This proves Theorem 1.5 in the case \(d \neq n + 1\), by checking that (4.23) is equivalent to (1.10). When \(d = n + 1\), the argument can be modified by using the differential graded Lie algebra \((\mathcal{G}, \hat{K}, \hat{\ell}_2)\).

### 4.9. A cochain level realization of the Hodge filtration on \(H^{n-1}_{\text{prim}}(X_G)\)

In this Subsection, we define a certain filtration on \((\mathcal{G}, K) = (\mathcal{G}_X, K_X)\) which is compatible with the Hodge filtration of \(H^{n-1}_{\text{prim}}(X_G, \mathbb{C}) \cong H^n_K(\mathcal{G})\). Let us start by defining a new grading \(\deg'\) on \(\mathcal{G}\) by

\[ \deg'(y) = 1, \quad \deg'(x_i) = 0, \quad \deg'(\eta_{-1}) = 0, \quad \deg'(\eta_i) = 1, \text{ for } i = 0, 1, \ldots, n. \]

For each \(m \in \mathbb{Z}\), let \(\mathcal{G}_m\) be the \(\deg'\)-th \(\mathbb{k}\)-linear space of \(\mathcal{G}\) under the above grading \(\deg'\), i.e. we define a \(\mathbb{k}\)-subspace of \(\mathcal{G}\) by

\[ \mathcal{G}_m := \{ f \in \mathcal{G} : f \text{ is } \mathbb{k}\text{-linearly spanned by } \eta_{-1} \prod_{k=0}^{n} \eta_{l_k} \cdot g(x, \eta_{-1}) \text{ for } g(x, \eta_{-1}) \in \mathbb{k}[x][\eta_{-1}] \text{ and } m = n + \sum_{k=0}^{n} l_k, \text{ with } n, l_k \geq 0 \}. \]

for \(m \geq 0\) and \(\mathcal{G}_m = 0\) for \(m \leq -1\). Then we have

\[ \mathcal{G} \cong \bigoplus_{m \geq 0} \mathcal{G}_m = \bigoplus_{m \in \mathbb{Z}} \mathcal{G}_m. \]

We define an increasing filtration on \(\mathcal{G}\) by \(\mathcal{F}_q(\mathcal{G}) = \bigoplus_{m \leq q} \mathcal{G}_m\) for each \(q \in \mathbb{Z}\). Let \(\Delta := K - Q\) so that \(K = Q + \Delta\). Then a simple computation confirms that

\[ Q : \mathcal{G}_m \to \mathcal{G}_m, \quad \Delta : \mathcal{G}_m \to \mathcal{G}_{m-1}, \quad m \in \mathbb{Z}, \]

and we have

\[ K : \mathcal{G}_m \to \mathcal{G}_{m} \oplus \mathcal{G}_{m-1}, \quad m \in \mathbb{Z}. \quad (4.25) \]

Therefore we have \(K : \mathcal{F}_q(\mathcal{G}) \to \mathcal{F}_q(\mathcal{G})\) and so \((\mathcal{F}_q(\mathcal{G}), K)\) is a cochain subcomplex of \((\mathcal{G}, K)\) for any \(q \in \mathbb{Z}\). In other words, \((\mathcal{G}, K)\) is a filtered cochain complex with the filtration \((\mathcal{F}_q(\mathcal{G}), K)\), \(q \in \mathbb{Z}\), which is separated and exhaustive;

\[ 0 = \mathcal{F}_{-1}(\mathcal{G}) \subset \mathcal{F}_0(\mathcal{G}) \subset \mathcal{F}_1(\mathcal{G}) \subset \cdots, \quad \text{and } \mathcal{G} \cong \bigcup_{q \in \mathbb{Z}} \mathcal{F}_q(\mathcal{G}). \]
Recall the increasing Hodge filtration on $H^{n-1} = H^{n-1}(X_G, \mathbb{C})$;

$$\mathcal{F}_q H^{n-1} = \bigoplus_{i+j=n-1} H^{i,j} = H^{n-1,0} \oplus H^{n-2,1} \oplus \cdots \oplus H^{n-q,q-1} \oplus H^{n-1,q},$$

where $H^{i,j}$ is the cohomology of $(i,j)$-forms on the hypersurface $X_G$. Theorem 4.1 says that the map $J : \mathcal{B}^0 = \mathcal{A}^0 \cap \text{Ker} \delta_R \rightarrow H^{n-1}(X_G, \mathbb{C})$ induces an isomorphism $J : H^0_{\mathcal{A}}(\mathcal{A}) \rightarrow H^{n-1}_{\text{prim}}(X_G, \mathbb{C})$. The following proposition says that $J$ preserves the above filtrations.

**Proposition 4.8.** The filtration $\mathcal{F}_q(\mathcal{A})$ on $\mathcal{A}$ induces an increasing filtration $\mathcal{F}_q H^n_{\mathcal{A}}$ on $H^n_{\mathcal{A}}(\mathcal{A})$ for every $n \in \mathbb{Z}$ and this induced filtration on $H^n_{\mathcal{A}}(\mathcal{A})$ is compatible with the increasing Hodge filtration on $H^{n-1}_{\text{prim}}(X_G, \mathbb{C})$ under the isomorphism $J$, i.e. $J(\mathcal{F}_q H^n_{\mathcal{A}}) \subseteq \mathcal{F}_q H^{n-1}_{\text{prim}} = \mathcal{F}_q H^{n-1} \cap H^{n-1}_{\text{prim}}(X_G, \mathbb{C})$ for every integer $q$ with $0 \leq q \leq n - 1$.

**Proof.** Since we already proved that $(\mathcal{F}_q(\mathcal{A}), K)$ is a cochain complex, we can define

$$\mathcal{F}_q H^n_{\mathcal{A}} : = \text{the } n\text{-th cohomology of } (\mathcal{F}_q(\mathcal{A}), K).$$

Then it is clear that this defines an increasing filtration on $H^n_{\mathcal{A}}(\mathcal{A})$. If we recall the $\mathbb{C}$-linear map $J'$ in (4.13), then

$$J'(\mathcal{F}_q(\mathcal{A}) \cap \mathcal{B}^0) \subseteq \Omega(V)^n_{(q+1)},$$

where $\Omega(V)^n_{(q)}$ is the group of rational $n$-forms on $\mathbb{P}^n$ with pole order $\leq q$ along $X_G$. Note that the Griffiths residue map satisfies

$$\text{Res}(\mathcal{H}_{q+1}(X_G)) \subseteq \mathcal{F}_q \cap H^{n-1}_{\text{prim}}(X_G, \mathbb{C})$$

where we recall that $\mathcal{F}_q = \mathcal{F}_q H^{n-1}$ is the increasing Hodge filtration in (4.5). Therefore $J = \text{Res} \circ J'$ satisfies $J(\mathcal{F}_q H^n_{\mathcal{A}}) \subseteq \mathcal{F}_q H^{n-1}_{\text{prim}}$ for every for every integer $q$ with $0 \leq q \leq n - 1$. \hfill $\square$

**Remark 4.3.** The key point of the above proposition is that we can realize the Hodge filtration on $H^{n-1}_{\text{prim}}(X_G, \mathbb{C})$ in the cochain level; we constructed a filtered cochain complex $(\mathcal{A}, K)$ with a filtration $(\mathcal{F}_q(\mathcal{A}), K)$ whose 0-th cohomology $H^0_{\mathcal{A}}(\mathcal{A}) \simeq H^{n-1}_{\text{prim}}(X_G, \mathbb{C})$ has the induced increasing Hodge filtration $\mathcal{F}_q H^{n-1}$.

5. **Appendix**

5.1. **The quantum origin of the Lie algebra representation $\rho_X$**

In this section, we explain how we arrive at the key definition in (4.4), the definition of the Lie algebra representation $\rho_X$ attached to a hypersurface $X_G$. An interesting thing is that this has a quantum origin. It is well known that representation theory of the Heisenberg group plays a crucial role...
in quantum field theory and quantum mechanics. We will focus on the Lie algebra representation of the Heisenberg Lie algebra in order to explain our motivation for Definition in (4.4). For each integer \( m \geq 1 \), we consider the universal enveloping algebra of the Heisenberg Lie algebra \( \mathcal{H}_m \) over \( \mathbb{k} \) as follows;

\[
U(\mathcal{H}_m) = \mathbb{k}\langle q^1, \cdots, q^m; p_1, \cdots, p_m; z \rangle/I
\]

where \( I \) is an ideal of the free \( \mathbb{k} \)-algebra \( \mathbb{k}\langle q^1, \cdots, q^m; p_1, \cdots, p_m; z \rangle \), generated by \([q^i, p_j] - z\delta^i_j\), \([q^i, z] \), \([p_i, z] \), \([q^i, q^j] \), and \([p_i, p_j] \) for all \( i, j = 1, \cdots, m \). Here, \([x, y] = x \cdot y - y \cdot x \) and \( \delta^i_j \) is the usual Dirac delta symbol. Then \((\mathcal{H}_m, [\cdot, \cdot])\) is a nilpotent Lie algebra whose \( \mathbb{k} \)-dimension is \( 2m + 1 \). Let \( I_P \) be the left ideal of \( U(\mathcal{H}_m) \) generated by \( p_1, \cdots, p_m \). Then the Schrödinger representation of \( U(\mathcal{H}_m) \) is given by

\[
\rho_{\text{Sch}} : U(\mathcal{H}_m) \to \text{End}_{\mathbb{k}}(U(\mathcal{H}_m)/I_P), \quad f \mapsto (g + I_P \to f \cdot g + I_P).
\]

This celebrated representation has attained much attention from both physicists and mathematicians, since it can be used to derive both the Heisenberg matrix formulation and the Schrödinger wave differential equation formulation of quantum mechanics through Dirac’s transformation theory and also plays a crucial role in the study of theta functions and modular forms via the oscillator (also called Weil) representation coming from the Schrödinger representation. We have the Schrödinger Lie algebra representation \( \rho_{\text{Sch}} \) (with the same notation) of \( \mathcal{H}_m \) on the same \( \mathbb{k} \)-vector space \( U(\mathcal{H}_m)/I_P \) from (5.1). Let \( P \) be the abelian \( \mathbb{k} \)-sub Lie algebra of \( \mathcal{H}_m \) spanned by \( p_1, \cdots, p_m \). If we restrict \( \rho_{\text{Sch}} \) to \( P \), we get

\[
\rho_{\text{Sch}}|_P : P \to \text{End}_{\mathbb{k}}(U(\mathcal{H}_m)/I_P),
\]

which is a Lie algebra representation of \( P \). This Lie algebra representation of \( P \) corresponding to \( \rho_{\text{Sch}} \) is our starting point to arrive at Definition in (4.4). Note that the representation space \( U(\mathcal{H}_m)/I_P \) does not have a \( \mathbb{k} \)-algebra structure, since \( I_P \) is not a two-sided ideal. This is a very important point in a mathematical algebraic formulation of quantum field theory (see the first author’s algebraic formalism of quantum field theory [11] for related issues). But we decided to simplify the quantum picture by introducing the Weyl algebra.\(^6\) The \( m \)-th Weyl algebra (which is introduced to study the Heisenberg uncertainty principle in quantum mechanics), denoted \( A_m \), is the ring of differential operators with polynomial coefficients in \( m \) variables. It is generated by \( q^1, \cdots, q^m \) and \( \frac{\partial}{\partial q^i}, \cdots, \frac{\partial}{\partial q^m} \). Then we have a surjective \( \mathbb{k} \)-algebra homomorphism

\[
r : U(\mathcal{H}_m) \to A_m,
\]

defined by sending \( z \) to 1, \( q^i \) to \( q^i \), and \( p_i \) to \( \frac{\partial}{\partial q^i} \) for \( i = 1, \cdots, m \) and extending the map in an obvious way. Note that \( z, q^1, \cdots, q^m; p_1, \cdots, p_m \) are \( \mathbb{k} \)-algebra generators of \( U(\mathcal{H}_m) \). Therefore the \( m \)-th Weyl algebra is a quotient algebra of \( U(\mathcal{H}_m) \). We further project the representation \( \rho_{\text{Sch}} \) to \( A_m \) to get a Lie algebra representation of \( A_m \) on \( A_m/r(I_P) \), denoted \( \rho_{\text{Wey}} \),

\[
\rho_{\text{Wey}} : A_m \to \text{End}_{\mathbb{k}}(A_m/r(I_P)) \approx \text{End}_{\mathbb{k}}(\mathbb{k}[q^1, \cdots, q^m]).
\]

\(^6\) This has the benefit of dramatically reducing the length of our paper, although it hides certain important quantum features of the theory. In later papers, we will pursue how this phenomenon (\( I_P \) is only a left \( U(\mathcal{H}_m) \)-ideal, not both-sided) is related to understanding period integrals, which seems very important in order to connect our theory to the theory of modular forms.
The benefit of working with the Weyl algebra is that $A_m/r(I_P)$ is isomorphic to $k[q^1, \cdots, q^m]$ as a $k$-vector space and so $A_m/r(I_P)$ has the structure of a commutative associative $k$-algebra which is inherited from $k[q^1, \cdots, q^m]$. Recall that we assumed the representation space of the Lie algebra representation was a commutative associative $k$-algebra in Definition 2.1. Note that $\rho_{\text{Weyl}}$ restricted to $r(I_P)$ is isomorphic to the representation on $k[q^1, \cdots, q^m]$ obtained by applying the differential operators $\partial/\partial q_i, \cdots, \partial/\partial q_m$. Now let $m = n + 2$ and put $y = q^1, x_0 = q^2, \cdots, x_n = q^{n+2}$ in order to connect to $\rho\rho_X$, where $G(X)$ is the defining equation of the smooth projective hypersurface $X_G$. Then when $X_G = P^n$, the representation $\rho\rho_X$ is isomorphic to $\rho_{\text{Weyl}}$. Recall that $A = k[y, X] = k[y_1, y_0, \cdots, y_n]$. Dirac's transformation theory in quantum mechanics suggests considering the following deformation of $\rho_{\text{Weyl}} : A_m \to \text{End}_k(A_m/r(I_P))$: for $F \in A$, consider the formal operators

$$\rho[i] := \exp(-F(y)) \cdot \frac{\partial}{\partial y_i} \cdot \exp(F(y)), \quad i = -1, 0, \cdots, n.$$ 

These operators $\rho[i], i = -1, 0, \cdots, n,$ will act on $\exp(-F(y)) \cdot A/r(I_P) \simeq A_m/r(I_P) \simeq A$ (as $k$-vector spaces). Formally, we can write and check

$$\rho[i] = \frac{\partial}{\partial y_i} + \left[ \frac{\partial}{\partial y_i}, F \right] + \frac{1}{2} \left[ \left[ \frac{\partial}{\partial y_i}, F \right], F \right] + \cdots \quad (5.4)$$

where $F = F(y)$. For any $F \in A = k[y]$, define $\rho[F] : r(I_P) \to \text{End}_k(A_m/r(I_P))$ via the rule

$$r(p_i) \mapsto (Q + r(I_P) \to \rho[i] \cdot Q + r(I_P)), \quad i = -1, 0, \cdots, n.$$ 

Proposition 5.1. We have a canonical isomorphism between two Lie algebra representations $\rho[F]$ above and $\rho\rho_X$ defined in (4.4), if $F = y \cdot G(X) \in A$.

Proof. The abelian Lie algebra $\mathfrak{g}$ in Definition of $\rho\rho_X$ is isomorphic to $r(I_P)$ and it is clear from (5.4) that the natural $k$-vector space isomorphism $A_m/r(I_P) \simeq A$ realizes the Lie algebra representation isomorphism between $\rho[F]$ and $\rho\rho_X$. \qed

5.2. The homotopy category of $L_\infty$-algebras

An $L_\infty$-algebra is a generalization of an $\mathbb{Z}$-graded Lie-algebra such that the graded Jacobi identity is only satisfied up to homotopy. An $L_\infty$-algebra is also known as a strongly homotopy Lie algebra in [13], or Sugawara Lie algebra. It has also appeared, albeit the dual version, in [14]. It is also the Lie version of an $A_\infty$-algebra (strongly homotopy associative algebra), which is the original

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Footnote 7: This assumption is why we encounter $L_\infty$-homotopy theory, when we analyze period integrals. If it is not commutative, then we would need a different type of homotopy theory.
example of homotopy algebra due to Stasheff. In this paper we encounter a variant of $L_\infty$-algebra such that its Lie bracket has degree one. In other words, a structure of $L_\infty$-algebra on $V$ in our paper is equivalent to that of the usual $L_\infty$-algebra on $V[1]$, where $V[1]$ means that $V[1]^m = V^{m+1}$ for $m \in \mathbb{Z}$. We should also note that the usual presentation of $L_\infty$-algebras and $L_\infty$-morphisms via generators and relations relies on unshuffles, which can be checked to be equivalent to our presentation based on partitions.

Let $\text{Art}^Z_k$ denote the category of $\mathbb{Z}$-graded artinian local $k$-algebras with residue field $k$ and $\widehat{\text{Art}^Z_k}$ be the category of complete noetherian local $k$-algebras. Let $R \in \text{Ob}(\widehat{\text{Art}^Z_k})$ concentrated in degree zero. For $A \in \text{Ob}(\text{Art}^Z_k)$, $m_A$ denotes the maximal ideal of $A$ which is a nilpotent $\mathbb{Z}$-graded super-commutative and associative $k$-algebra without unit. Let $V = \bigoplus_{i \in \mathbb{Z}} V^i$ be a $\mathbb{Z}$-graded vector space over a field $k$ of characteristic 0. If $x \in V^i$, we say that $x$ is a homogeneous element of degree $i$; let $|x|$ be the degree of a homogeneous element of $V$. For each $n \geq 1$ let $S(V) = \bigoplus_{n=0}^\infty S^n(V)$ be the free $\mathbb{Z}$-graded super-commutative and associative algebra over $k$ generated by $V$, which is the quotient algebra of the free tensor algebra $T(V) = \bigoplus_{n=0}^\infty T^n(V)$ by the ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$. Here $T^0(V) = k$ and $T^n(V) = V^\otimes n$ for $n \geq 1$.

**Definition 5.1** ($L_\infty$-algebra). The triple $V_\ell = (V, \ell, 1_V)$ is a unital $L_\infty$-algebra over $k$ if $1_V \in V^0$ and $\ell = \ell_1, \ell_2, \cdots$ be a family such that

- $\ell_n \in \text{Hom}(S^n V, V)^1$ for all $n \geq 1$.
- $\ell_n(v_1, \cdots, v_{n-1}, 1_V) = 0$, $v_1, \cdots, v_{n-1} \in V$, for all $n \geq 1$.
- for any $A \in \text{Ob}(\text{Art}^Z_k)$ and for all $n \geq 1$

$$\sum_{k=1}^n \frac{1}{(n-k)!k!} \ell_{n-k+1}(\ell_k(\gamma, \cdots, \gamma), \gamma, \cdots, \gamma) = 0,$$

whenever $\gamma \in (m_A \otimes V)^0$, where

$$\ell_n(a_1 \otimes v_1, \cdots, a_n \otimes v_n) = (-1)^{|a_1| + |a_2| + \cdots + |a_n| + |v_1| + \cdots + |v_n|} a_1 \cdots a_n \otimes \ell_n(v_1, \cdots, v_n).$$

Every $L_\infty$-algebra in this paper is assumed to satisfy $\ell_n = 0$ for all $n > N$ for some natural number $N$.

**Definition 5.2** ($L_\infty$-morphism). A morphism of unital $L_\infty$-algebras from $V_\ell$ into $V'_\ell$ is a family $\phi = \phi_1, \phi_2, \cdots$ such that

- $\phi_n \in \text{Hom}(S^n V, V')^0$ for all $n \geq 1$.
- $\phi_1(1_V) = 1_{V'}$ and $\phi_n(v_1, \cdots, v_{n-1}, 1_V) = 0$, $v_1, \cdots, v_{n-1} \in V$, for all $n \geq 2$. 

regarding the

5.4 imply, after simple
and

5.2 5.3

It is convenient to introduce an equivalent presentation of $L$-partitions

combinatorics, the claimed relations in Definitions 3.1 and 3.2.

Let us set up some notation related to partitions. A partition $\pi = \{1, 2, \ldots, n\}$ is a decomposition of $[n]$ into a pairwise disjoint non-empty subsets $B_i$, called blocks. Blocks are ordered by the minimum element of each block and each block is ordered by the ordering induced from the ordering of natural numbers. The notation $|\pi|$ means the number of blocks in a partition $\pi$ and $|B|$ means the size of the block $B$. If $k$ and $k'$ belong to the same block in $\pi$, then we use the notation $k \sim_\pi k'$. Otherwise, we use $k \not\sim_\pi k'$. Let $P(n)$ be the set of all partitions of $[n]$. For a permutation $\sigma$ of $[n]$, we define a map $\hat{\sigma} : T^n(V) \rightarrow T^n(V)$ by

$$\hat{\sigma}(x_1 \otimes \cdots \otimes x_n) = \epsilon(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

for homogeneous elements $x_1, \ldots, x_n \in V$, where $\epsilon(\sigma) = \epsilon(\sigma, \{x_1, \ldots, x_n\}) = \pm 1$ is the Koszul sign determined by decomposing $\hat{\sigma}$ as composition of adjacent transpositions $\tau$, where $\hat{\tau} : x_1 \otimes x_j \rightarrow (-1)^{\varepsilon(\tau)x_i=x_j} x_j \otimes x_i = \epsilon(\tau, \{x_i, x_j\}) x_j \otimes x_i$, i.e. the Koszul sign is defined as the product of the signs $\epsilon(\tau, \{x_i, x_j\})$ of such adjacent transpositions. Note that $\epsilon(\sigma)$ depends on the degrees of $x_1, \ldots, x_n$ but we omit such dependence in the notation for simplicity. The Koszul sign $\epsilon(\tau)$ for a partition $\tau = B_1 \cup B_2 \cup \cdots \cup B_m \in P(n)$ is defined by the Koszul sign $\epsilon(\sigma)$ of the permutation $\sigma$ determined by

$$x_{B_1} \otimes \cdots \otimes x_{B_m} = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},$$

where $x_B = x_{j_1} \otimes \cdots \otimes x_{j_r}$ if $B = \{j_1, \ldots, j_r\}$. Let $|x_B| = |x_{j_1}| + \cdots + |x_{j_r}|$. Then it can be checked that the following definitions 5.3 and 5.4 are equivalent to the above ones; for an arbitrary finite set $\{v_1, \ldots, v_N\}$ of homogeneous elements in $V$, consider $k[[t_1, \ldots, t_N]]$, where $|t_j| = |v_j|$, $1 \leq j \leq N$. Let $J$ be the maximal ideal of $k[[t_1, \ldots, t_N]]$, so that $A := k[[t_1, \ldots, t_n]]/J^{N+1} \in Art_k$. Let $\gamma$ be $\sum_{j=1}^N t_j v_j$, which is an element of $(m_A \otimes V)^0$. Definitions 5.1 and 5.2 imply, after simple combinatorics, the claimed relations in Definitions 5.3 and 5.4 for all $n$ satisfying $1 \leq n \leq N$. It is clear that the converse is also true.
Definition 5.3 \((L_\infty\text{-algebra})\). An \(L_\infty\text{-algebra}\) is a \(\mathbb{Z}\)-graded vector space \(V = \bigoplus_{n \in \mathbb{Z}} V^m\) over a field \(k\) with a family \(\ell = \ell_1, \ell_2, \cdots\), where \(\ell_k : S^k(V) \to V\) is a linear map of degree 1 on the super-commutative \(k\)-th symmetric product \(S^k(V)\) for each \(k \geq 1\) such that
\[
\sum_{\pi \in P(n)} e(\pi, i) \ell_{|\pi|}(x_{B_1}, \cdots, x_{B_{|\pi|}}, \ell(x_{B_i}), x_{B_{i+1}}, \cdots, x_{B_{|\pi|}}) = 0. \tag{5.5}
\]
Here we use the following notation:
\[
\ell(x_B) = \ell_r(x_{j_1}, \cdots, x_{j_r}) \text{ if } B = \{j_1, \cdots, j_r\}
\]
\[
e(\pi, i) = e(\pi)(-1)^{|\pi|_{x_B}|+\cdots+|x_{B_{|\pi|}}|}.
\]
An \(L_\infty\text{-algebra}\) with unity (or a unital \(L_\infty\text{-algebra}\)) is a triple \((V, \ell, \epsilon_1)\), where \((V, \ell)\) is an \(L_\infty\text{-algebra}\) and \(\epsilon_1\) is a distinguished element in \(V^0\) such that \(\ell_n(x_1, \cdots, x_{n-1}, 1) = 0\) for all \(n \geq 1\) and every \(x_1, \cdots, x_{n-1} \in V\).

Definition 5.4 \((L_\infty\text{-morphism})\). A morphism from an \(L_\infty\text{-algebra}\) \((V, \ell)\) to an \(L_\infty\text{-algebra}\) \((V', \ell')\) over \(k\) is a family \(\phi = \phi_1, \phi_2, \cdots\), where \(\phi_k : S^k(V) \to V'\) is a \(k\)-linear map of degree 0 for each \(k \geq 1\), such that
\[
\sum_{\pi \in P(n)} e(\pi) \ell_{|\pi|}'(\phi(x_{B_1}), \cdots, \phi(x_{B_{|\pi|}})) = \sum_{\pi \in P(n)} e(\pi, i) \phi_{|\pi|}(x_{B_1}, \cdots, x_{B_{n-1}}, \ell(x_{B_i}), x_{B_{i+1}}, \cdots, x_{B_{|\pi|}}).
\]
A morphism of \(L_\infty\text{-algebras}\) with unity from \((V, \ell, 1_V)\) to \((V', \ell', 1_{V'})\) over \(k\) is a morphism \(\phi\) of \(L_\infty\text{-algebras}\) such that \(\phi_1(1_V) = 1_{V'}\) and \(\phi_n(x_1, \cdots, x_{n-1}, 1_V) = 0\) for all \(n \geq 2\) and every \(x_1, \cdots, x_{n-1} \in V\).

Remark 5.1. If one uses an interval partition of \([n]\) instead of \(P(n)\), we can prove that this formalism gives an equivalent definition to the usual definition of \(A_\infty\text{-algebras}\) and \(A_\infty\text{-morphisms}\).

If we let \(\pi = B_1 \cup \cdots \cup B_{n-1} \cup B_1 \cup B_{i+1} \cup \cdots \cup B_{|\pi|} \in P(n)\), then the condition \(|B_1| = n - |\pi| + 1\) in the summation implies that the set \(B_1, \cdots, B_{n-1}, B_{i+1}, \cdots, B_{|\pi|}\) are singletons. Let \(\ell_1 = K\). For \(n = 1\), the relation (5.5) says that \(K^2 = 0\). For \(n = 2\), the relation (5.5) says that
\[
K\ell_2(x_1, x_2) + \ell_2(Kx_1, x_2) + (-1)^{|x_1|}\ell_2(x_1, Kx_2) = 0.
\]
For \(n = 3\), we have
\[
\ell_2(\ell_2(x_1, x_2), x_3) + (-1)^{|x_1|}\ell_2(x_1, \ell_2(x_2, x_3)) + (-1)^{|x_1|+|x_2|}\ell_2(x_2, \ell_2(x_1, x_3))
= -K\ell_3(x_1, x_2, x_3) - \ell_3(Kx_1, x_2, x_3)
- (-1)^{|x_1|}\ell_3(x_1, Kx_2, x_3) - (-1)^{|x_1|+|x_2|}\ell_3(x_1, x_2, Kx_3).
\]
Because the vanishing of the left hand side is the graded Jacobi identity for \(\ell_2\), we see that \(\ell_2\) fails to satisfy the graded Jacobi identity. Thus the failure of \(\ell_2\) being a graded Lie algebra is measured by the homotopy \(\ell_3\).
Corollary 5.1. A \(\mathbb{Z}\)-graded vector space over \(k\) is an \(L_\infty\)-algebra with \(\ell = 0\), which we refer to as a trivial \(L_\infty\)-algebra. A cochain complex \((V, K)\) is an \(L_\infty\)-algebra with \(\ell = 1\), where \(\ell_1 = K\). A differential graded Lie algebra (DGLA) \((V, K, [\cdot, \cdot])\) is an \(L_\infty\)-algebra with \(\ell = 1, \ell_2\), where \(\ell_1 = K\), and \(\ell_2(\cdot, \cdot) = [\cdot, \cdot]\).

Let \(\ell_1 = K\) and \(\ell' = K'\). For \(n = 1\), the relation in Definition 5.4 says that

\[ \phi_1 K = K' \phi_1. \]

For \(n = 2\), we have

\[ \phi_1(\ell_2(x_1, x_2)) - \ell'(\phi_1(x_1), \phi_1(x_2)) = K' \phi_2(x_1, x_2) - \phi_2(Kx_1, x_2) - (-1)^{|x_1|} \phi_2(x_1, Kx_2). \]

Hence \(\phi_1\) is a cochain map from \((V, K)\) to \((V', K')\), which fails to be an algebra map. The failure of being an algebra map is measured by the homotopy \(\phi_2\). For \(n = 3\), the above says

\[ \phi_1(\ell_3(x_1, x_2, x_3)) - \ell'_3(\phi_1(x_1), \phi_1(x_2), \phi_1(x_3)) + \phi_2(\ell_2(x_1, x_2, x_3) + (-1)^{|x_1|} \phi_2(x_1, \ell_2(x_2, x_3)) + (-1)^{|x_2| + |x_3|} \phi_2(x_2, \ell_2(x_1, x_3)) - \ell'_2(\phi_2(x_1, x_2), \phi_2(x_1, x_3)) - (-1)^{|x_1| |x_2|} \phi_2(\ell_2(x_1, x_2), \phi_2(x_1, x_3)) \]

\[ = K' \phi_3(x_1, x_2, x_3) - \phi_3(Kx_1, x_2, x_3) - (-1)^{|x_1|} \phi_3(x_1, Kx_2, x_3) - (-1)^{|x_2|} \phi_3(x_1, x_2, Kx_3). \]

Now we define composition of morphisms.

Definition 5.5. The composition of \(L_\infty\)-morphisms \(\phi : V \to V'\) and \(\phi' : V' \to V''\) is defined by

\[ (\phi \circ \phi')(x_1, \ldots, x_n) = \sum_{\pi \in P(n)} \epsilon(\pi) \phi'_{|\pi|}(\phi(x_{|\pi|}), \ldots, \phi(x_{|\pi|})), \]

for every homogeneous elements \(x_1, \ldots, x_n \in V\) and all \(n \geq 1\).

Then it can be checked that unital \(L_\infty\)-algebras over \(k\) and \(L_\infty\)-morphisms form a category.

Definition 5.6. The cohomology \(H\) of the \(L_\infty\)-algebra \((V, K)\) is the cohomology of the underlying complex \((V, K)\). An \(L_\infty\)-morphism \(\phi\) is a quasi-isomorphism if \(\phi_1\) induces an isomorphism on cohomology.

Definition 5.7. An \(L_\infty\)-algebra \((V, K)\) is called minimal if \(\ell_1 = 0\).

We recall the following well-known theorem, sometimes called the transfer lemma.

Theorem 5.1. There is the structure of a minimal \(L_\infty\)-algebra \((H, \ell_H^H = \ell_2^H, \ell_3^H, \ldots)\) on the cohomology \(H\) of an \(L_\infty\)-algebra \((V, K)\) over \(k\) together with an \(L_\infty\)-quasi-isomorphism \(\phi\) from \((H, \ell_H^H)\) to \((V, \ell_K^K)\). Both the minimal \(L_\infty\)-algebra structure and the \(L_\infty\)-quasi-isomorphism are not unique, while \(\ell_H^H\) is defined uniquely.
**Definition 5.8.** An $L_\infty$-algebra $(V,\ell)$ is smooth-formal if a minimal $L_\infty$-algebra structure on $H$ is trivial, i.e. $\ell^H = 0$.

**Definition 5.9.** Two $L_\infty$-morphisms $\phi$ and $\tilde{\phi}$ of unital $L_\infty$-algebras from $(V,\ell,1_V)$ into $(V',\ell',1_{V'})$ are $L_\infty$-homotopic, denoted by $\phi \sim_{\infty} \tilde{\phi}$, if there is a polynomial family $\lambda(\tau) = \lambda_1(\tau), \lambda_2(\tau), \cdots$ in $\tau$, where

- $\lambda_n(\tau) \in \text{Hom}(S^nV,V)^{-1}$ for all $n \geq 1$.
- $\lambda_n(\tau)(v_1,\cdots,v_{n-1},1_V) = 0$, $v_1,\cdots,v_{n-1} \in V$, for all $n \geq 1$,

and a polynomial family $\Phi(\tau) = \Phi_1(\tau),\Phi_2(\tau),\cdots$, where

- $\Phi_n(\tau) \in \text{Hom}(S^nV,V')^0$ for all $n \geq 1$,
- $\Phi_n(1_V) = 1_{V'}$ and $\Phi_n(\tau)(v_1,\cdots,v_{n-1},1_V) = 0$, $v_1,\cdots,v_{n-1} \in V$, for all $n \geq 1$,

and $\Phi(0) = \phi$ and $\Phi(1) = \tilde{\phi}$, such that $\Phi$ satisfies the following flow equation

$$\frac{\partial}{\partial \tau} \Phi_n(\tau)(\gamma,\cdots,\gamma) = \sum_{k=1}^n \sum_{j_1+\cdots+j_r=n-k} \frac{1}{r! j_1! \cdots j_r!} \left( \Phi_{j_1}(\tau)(\gamma,\cdots,\gamma),\cdots,\Phi_{j_r}(\tau)(\gamma,\cdots,\gamma), \lambda_k(\tau)(\gamma,\cdots,\gamma) \right)$$

$$+ \sum_{j_1+j_2=n} \frac{1}{j_1! j_2!} \lambda_{j_1+j_2}(\tau)(\gamma,\cdots,\gamma,j_{j_1}(\gamma,\cdots,\gamma))$$

for all $n \geq 1$ and $\gamma \in (m_A \otimes V)^0$ whenever $A \in \text{Ob}(\text{Art}^\infty_{\ell})$.

It can be checked that $\sim_{\infty}$ is an equivalence relation (see 4.5.2 of [9] for another form of the above definition).

**Lemma 5.1.** Consider $L_\infty$-morphisms $\phi : V_L \to V_L'$ and $\phi' : V_L' \to V_L''$. Let $\tilde{\phi} \sim_{\infty} \phi$ and $\tilde{\phi}' \sim_{\infty} \phi'$. Then $\tilde{\phi}' \circ \tilde{\phi} \sim_{\infty} \tilde{\phi'} \circ \phi$.

**Proof.** Let $\Phi(\tau)$, where $\Phi(0) = \phi$ and $\Phi(1) = \tilde{\phi}$, be a polynomial solution to the flow equation with a polynomial family $\lambda(\tau)$. Let $\Phi'(\tau)$, where $\Phi'(0) = \phi'$ and $\Phi'(1) = \tilde{\phi}'$, be a polynomial solution to the flow equation with polynomial family $\lambda'(\tau)$. Then $\Phi'(\tau) := \Phi'(\tau) \cdot \Phi(\tau)$ is a polynomial family in $\text{Hom}(V,V'')^0$ such that $\Phi'(0) = \phi' \cdot \phi$ and $\Phi'(1) = \tilde{\phi}' \cdot \tilde{\phi}$. It can be checked that, for all $n \geq 1$ and
\[ \gamma \in (m_A \otimes V)^I \text{ whenever } A \in \text{Ob}(\mathcal{A})^I, \]

\[ \frac{\partial}{\partial \tau} \gamma''(\gamma, \cdots, \gamma) = \frac{1}{1! \cdots j_1! j_2!} \sum_{j_1 + \cdots + j_n = n-k} \lambda''_{j_1, j_2, \cdots} \left( \Phi_{j_1}(\gamma, \cdots, \gamma), \cdots, \Phi_{j_n}(\gamma, \cdots, \gamma) \right) \]

where \( \lambda''_{j_1, j_2, \cdots} \) is the polynomial family in \( \text{Hom}(V, V'')^{-1} \) given by

\[ \lambda''_{j_1, j_2, \cdots} = \frac{1}{1! \cdots j_1! j_2!} \sum_{j_1 + \cdots + j_n = n-k} \lambda''_{j_1, j_2, \cdots} \left( \Phi_{j_1}(\gamma, \cdots, \gamma), \cdots, \Phi_{j_n}(\gamma, \cdots, \gamma) \right) \]

This proves Lemma. \( \square \)

The above lemma implies that the homotopy category of unital \( L_\infty \)-algebras is well-defined, where the morphisms in that category consists of \( L_\infty \)-homotopy classes of \( L_\infty \)-morphisms.

Let \( (V, [\cdot, \cdot]) \) and \( (V', [\cdot, \cdot']) \) be an ordered pair of DGLAs over \( k \). Then a morphism \( f : V \to V' \) as DGLAs is an \( L_\infty \)-morphism \( \phi = \phi_1 \) such that \( \phi_1 = f \). If \( f : V \to V' \) is a DGLA morphism, then \( \tilde{f} = f + K s + s K \) is a cochain map homotopic to \( f \) by the cochain homotopy \( s \). In this case, there is an \( L_\infty \)-morphism \( \phi = \phi_1, \phi_2, \cdots \) which is homotopic to \( \phi = \phi_1 = f \) by an \( L_\infty \)-homotopy \( \lambda = \lambda_1, \lambda_2, \cdots \) such that \( \lambda_1 = s \) and \( \phi_1 = \tilde{f} \). Let \( \tilde{f} : V \to V' \) be a cochain map and \([\tilde{f}]\) be the cochain homotopy class of \( \tilde{f} \). Then there is a representative \( f \) of \([\tilde{f}]\) which is a DGLA morphism if and only if \( \tilde{f} \) can be extended to an \( L_\infty \)-morphism \( \phi \), i.e. \( \phi_1 = f \) which is \( L_\infty \)-homotopic to a cochain map.

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