DISCRETENESS OF THE SPECTRUM OF THE
COMPACTIFIED $D = 11$ SUPERMEMBRANE WITH
NON-TRIVIAL WINDING

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Abstract. We analyze the Hamiltonian of the compactified $D = 11$ supermembrane with non-trivial central charge in terms of the matrix model constructed in $[1]$. Our main result provides a rigorous proof that the quantum Hamiltonian of the supersymmetric model has compact resolvent and thus its spectrum consists of a discrete set of eigenvalues with finite multiplicity.

1. Introduction

According to $[2]$, the spectrum of the $SU(N)$ regularized supermembrane on $D = 11$ Minkowski target space is continuous and it consists on the whole interval $[0, \infty)$. Although it was proven for a regularized model, this result led to an interesting interpretation of the supermembrane in terms of a multiparticle theory. It also showed explicitly how the presence of supersymmetry may change completely the spectrum of a bosonic discrete Hamiltonian over a compact world volume. The proof was based on the existence of locally singular configurations, which do not change the energy of the system, and on the presence of supersymmetry.

The situation concerning the spectrum of the compactified supermembrane is, in distinction, very different. Since the closed but not exact modes present in the compactified case do not fit into an $SU(N)$ formulation of the theory $[4]$, the $SU(N)$ regularization of the compactified supermembrane seems not to be possible. In $[4]$, it was suggested that the spectrum of the compactified supermembrane should also be continuous due to the existence of string-like spikes as in the non-compactified case. The presence of those singular configurations, which do not change the energy of the system, is a common property of all p-branes $[5]$. Recently, $[6]$, it was shown that the Hamiltonian formulation of the super M5-brane also contains singular configurations even when they are neither present in the known covariant formulation of the theory $[7]$ nor in $[8]$.

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In [9] and [10], the compactified supermembrane on $M_9 \times S_1 \times S_1$ was formulated as a noncommutative gauge theory. The Hamiltonian was given in terms of a noncommutative super Maxwell theory plus the integral of the curvature of the noncommutative connection on the world volume. In [1], it was explicitly shown that this Hamiltonian allows the presence of string-like spikes in the configuration space of the compactified supermembrane. This is in agreement with the argument in [4]. If however the theory is restricted to a fixed central charge, by describing a sector of the full compactified supermembrane, the Hamiltonian reduces exactly to a noncommutative super Maxwell theory coupled to seven scalar fields representing the transverse directions to the supermembrane. In this case it was shown in [1] that there are no string-like spikes in the configuration space. This result is in agreement with the arguments in [12].

According to [13], the following properties for the bosonic part of this Hamiltonian hold: it is bounded below and it becomes infinite at infinity in every possible direction on the configuration space. In other words, the potential is “basin shaped”. This property ensures that the resolvent of the bosonic Hamiltonian is compact and therefore its spectrum consists of a set of isolated eigenvalues of finite multiplicity whose only accumulation point is plus infinity. Furthermore, it is possible to find upper bounds for the asymptotic distribution of eigenvalues.

Based on two fundamental properties already discussed in [13], the non existence of string-like spikes and the shape of the bosonic potential, in the present paper we provide a rigorous proof of the discreteness of the spectrum of the Hamiltonian of the noncommutative super-Maxwell theory describing the compactified $D = 11$ supermembrane with fixed central charge. To be precise we show that, as in the bosonic case, the resolvent is compact and hence the spectrum consists of a discrete set of eigenvalues of finite multiplicity. For this we provide a criterion (see lemma [1]), that extends to the supersymmetric case, the well known fact that the spectrum of a Hamiltonian is discrete, when the potential is bounded from below and unbounded above in all directions (see [14] and [15]). The latter played a fundamental role for the bosonic case discussed in [13]. We state this criterion in generic form and provide a self-contained proof, so that it might be applied to Hamiltonians whenever the bosonic potential forbids string-like spikes.

To motivate the present analysis, we can comment on the relationship between our fixed central charge model and the free model studied in [2], [3] and [4]. The winding supermembrane in the latter, can be regarded as a “free” model, since it includes in the configuration space all possible wrappings, $n = 0, 1, 2, \ldots$, on the compactified target space. In contrast, the present work concerns only the sector of the full theory which corresponds to a fixed central charge $n \neq 0$. This case is also of physical relevance, since the fixed wrapping is a “topological” condition
with phenomenological applications. The change in the nature of the spectrum is formally analogous to the well known case of the Laplacian acting on a domain of the Euclidean space: when the domain is the whole space, the operator has pure continuous spectrum in contrast to the case of bounded domains, where the operator has always pure discrete spectrum. We will discuss further on the comparison of the two models in section 2.

2. The Hamiltonian of the double compactified supermembrane

Reference [17] is devoted to study the quantization of the compactified supermembrane in an $M_9 \times S^1 \times S^1$ target space, by finding an explicit expression, given in terms of creation and annihilation operators, of the quantum Hamiltonian in the semi-classical regime. In particular, there are no vacuum energy corrections to the mass formula, since the fermionic contribution cancels the bosonic one.

The exact Hamiltonian was computed in [10] (see also [11]), in terms of a symplectic noncommutative geometry. The symplectic structure which gives rise to the geometry, arises from the non-trivial central charge originated in the wrapping of the supermembrane along the compactified directions of the target space. In order to make more transparent the present exposition, we devote this section to review the main ideas which led to the construction of this Hamiltonian.

Let us start by considering the $D = 11$ supermembrane Hamiltonian in the light cone gauge as in [2]. In this model, the potential is given by

$$V(X) = \{X^m, X^n\}^2, \quad m, n = 1, \ldots, 9$$

where $\{X^m, X^n\} \equiv (\epsilon^{ab}/\sqrt{W})\partial_a X^m \partial_b X^n$. Here, the scalar density $\sqrt{W}$ appears in the formulation as consequence of the light cone gauge fixing procedure.

Let $\Sigma$ be the spatial part of the world volume. We always assume that $\Sigma$ is a compact Riemann surface of genus $g$. If one of the target space spatial coordinates is compactified on $S^1$, the natural winding condition is given by

$$\oint_{c_j} dX = 2\pi m_j$$

(1)

where $m_j$ are integers and $c_j$ is a basis of homology on $\Sigma$. Analogously, when the target space has two compactified directions on $S^1 \times S^1$, we may consider $X_r, r = 1, 2$, as angular coordinates on each $S^1$. In order to have a well defined map over $S^1$, we must impose as before the condition

$$\oint_{c_j} dX_r = 2\pi m_{jr}, \quad r = 1, 2.$$  

(2)
Assume that the image of $\Sigma$ under $(X_1, X_2)$ describes a torus. Then we should impose an additional constraint. If $\Sigma$ itself is a generic torus, where $w_1$ and $w_2$ denote the normalized basis of homology on $\Sigma$, we have

$$dX_r = m_1r w_1 + m_2r w_2 + da_r \quad r = 1, 2$$

where $m_{jr}$, $j = 1, 2$, are the same integers introduced in (2) and $da_r$ are exact one-forms. The requirement that the image of $\Sigma$ is a torus, may be interpreted as the independence of the one-forms $m_1r w_1 + m_2r w_2$ for $r = 1, 2$, i.e.

$$\det(m_{jr}) \neq 0.$$ 

This condition is equivalent to requiring

$$Z = \int_\Sigma (dX_r \wedge dX_s) \epsilon^{rs} = 2\pi n \neq 0.$$ 

(3)

The factor $2\pi$ corresponds to normalization of the area of $\Sigma$. We remark that, since $Z$ becomes

$$\int_\Sigma g_1^{-1}dg_1 \wedge g_2^{-1}dg_2$$

where $G = (g_1, g_2) \in U(1) \times U(1)$, the integer $n$ is the winding number of the group $U(1) \times U(1)$ over $\Sigma$.

Notice that the integral in condition (3), also corresponds to a realization of the central charge of the supersymmetric algebra of the supermembrane. Hence the condition $n \neq 0$ is equivalent to having non-trivial central charge. If the $X_r$ fulfill condition (2), then (3) holds automatically. However non-triviality of (2), does not necessarily imply non-triviality of (3). We show below, that the configuration space of the compactified supermembrane with fixed central charge is completely characterized by the integer $n$, only the determinant of $(m_{jr})$ is relevant.

In order to describe the winding of the supermembrane in terms of maps from $\Sigma$ onto $S^1 \times S^1$ satisfying (3), together with maps $(X^m)^T_{m=1} : \Sigma \rightarrow \mathbb{R}^7$, we interpret (3) in terms of geometrical objects. Let

$$F := dX_r \wedge dX_s \epsilon^{rs}$$

(4)

be a closed two-form on $\Sigma$, such that $F$ satisfies (3). Since $dF = 0$ and $\int_\Sigma F = 2\pi n$, there always exists a $U(1)$ principal bundle over $\Sigma$ and a one-form connection on it, such that $F$ is the curvature two-form. The integer $n$ characterizes the bundle. If $n \neq 0$, the bundle is non-trivial and the connection one-form must have non-trivial transitions. In this bundle, there are particular connection one-forms satisfying

$$^{*}\tilde{F} \equiv \frac{\epsilon^{ab}}{\sqrt{W}} \tilde{F}_{ab} = n$$

(5)
at any point of $\Sigma$, where $a, b = 1, 2$ are indices associated to local coordinates on $\Sigma$. These are the so called Dirac monopoles over Riemann surfaces. It turns out that these monopoles together with the constraint $X^m = 0$, $m = 1, \ldots, 7$, are configurations where the Hamiltonian of the supermembrane have local minima, [9]. Moreover, there is only one local minimum for each $n$.

Condition (5) implies that $\widehat{F}_{ab}$ is non-degenerate. Any non-degenerate closed two-form, can always be decomposed as

$$\widehat{F}_{ab} = \partial_a \widehat{X}_r \partial_b \widehat{X}_s \epsilon^{rs},$$

in the sense that there exists a Darboux atlas for $\Sigma$ such that the above holds on each open set. The $\widehat{X}_r$, $r = 1, 2$ are harmonic maps over $\Sigma$ with metric

$$g_{ab} = \partial_a \widehat{X}_r \partial_b \widehat{X}_s \delta^{rs}$$

the pull-back of the Euclidean metric over $S^1 \times S^1$. Notice that this is the metric arising from the supermembrane action. The $d\widehat{X}_r$ are harmonic one-forms over $\Sigma$. If $\Sigma$ is any given torus, we may then consider these one forms as a normalized basis of homology over $\Sigma$. Clearly the maps $\widehat{X}_r$ are not uniquely determined. In fact, we are allowed to change the basis by an element of $SL(2, \mathbb{Z})$, the modular group, and $\widehat{F}$ remains invariant. This change of basis corresponds to a conformal diffeomorphism over $\Sigma$. From (5) we obtain

$$n \sqrt{W} = \epsilon^{ab} \widehat{F}_{ab} = \epsilon^{ab} \partial_a \widehat{X}_r \partial_b \widehat{X}_s \epsilon^{rs},$$

which is invariant under this conformal diffeomorphism. Consequently it is an area preserving diffeomorphism and hence it corresponds to a gauge symmetry of the supermembrane action.

Although $\widehat{X}_r$, $r = 1, 2$, are not unique as homotopic maps from $\Sigma$ onto $S^1 \times S^1$, they are all equivalent on the configuration space of the compactified supermembrane. Then one realizes that the problem of handling the closed, but not exact, one-forms in the quantum analysis of the compactified $M_9 \times S^1 \times S^1$ supermembrane has been solved. In fact the maps $X_r$ decompose as the sum of $\widehat{X}_r$ plus a homotopically trivial map to be quantized. The $\widehat{X}_r$ will then be conveniently incorporated to the general description of the action. In so doing, we will end up with a formulation of the theory as a symplectic non-commutative Yang-Mills action.

The zwei-vein

$$e^a_r \equiv \frac{\widehat{\Pi}^a_r}{\sqrt{W}} \equiv \epsilon^{ab} \partial_b \widehat{X}_r,$$

allows us to write down all the geometrical objects in the spatial world volume in terms of the corresponding objects in the tangent space. We can express the curvature $F$ of any connection over the mentioned
bundle (characterized by \( n \)) in terms of \( \hat{F} \), as
\[
F = \hat{F} + f,
\]
where \( f = da \) satisfies \( \int_{\Sigma} f = 0 \). The one-form \( a \) is a one-form connection on a trivial \( U(1) \) bundle, it has no transitions over \( \Sigma \) and \( f \) is an exact two-form.

Let
\[
D_r := \frac{\hat{\Pi}_r}{\sqrt{W}} \partial_a.
\]
Let \( A_r \) be such that
\[
X_r = \hat{X}_r + A_r.
\]
Then, by computing \( dX_1 \wedge dX_2 \), the decomposition (6) for (4) yields
\[
*F = *\hat{F} + *F,
\]
where
\[
*F = \epsilon^{rs} F_{rs},
\]
\[
F_{rs} = D_r A_s - D_s A_r + \{ A_r, A_s \}.
\]
It turns out, \( [9] \), that under the area preserving diffeomorphism, the residual gauge symmetry of the supermembrane Hamiltonian in the light cone gauge, \( A_r \), transforms as
\[
\delta A_r = D_r \xi
\]
where the covariant derivative
\[
D_r \cdot = D_r \cdot + \{ A_r, \cdot \}.
\]
Then, the term \( F_{rs} \) is interpreted as the curvature of a symplectic non-commutative connection. The results of \( [11] \) describe the relationship between this connection and the ones arising from a non-commutative product on the Weyl algebra bundle.

The condition \( \int_{\Sigma} f = 0 \) yields
\[
\int_{\Sigma} F = 0.
\]
This allows to write the Hamiltonian of the supermembrane, only in terms of \( A_r \) and \( X^m \). Identity \( [7] \) arises by imposing fixed central charge, or, analogously, by considering a fixed \( U(1) \) principal bundle on \( \Sigma \). Hence, according to \( [9] \),
\[
H = \int_{\Sigma} \left( \frac{1}{2} \sqrt{W} \right) \left[ (P_m)^2 + (\Pi_r)^2 + \frac{1}{2} W \{ X^m, X^n \}^2 + W (D_r X^m)^2 + \right. \\
+ \frac{1}{2} W (F_{rs})^2 \right] + \int_{\Sigma} \left[ (1/8) \sqrt{W} n^2 - \Lambda (D_r \Pi_r + \{ X^m, P_m \}) \right] + \\
- \frac{1}{4} \int_{\Sigma} \sqrt{W} n^* F, \quad n \neq 0
\]
(8)
together with its fermionic contribution

\[ \int_{\Sigma} \sqrt{\mathcal{W}} \left[ -\bar{\psi} \Gamma_{\Gamma_r} \mathcal{D}_r \psi + \bar{\psi} \Gamma_{\Gamma_m} \{ X^m, \psi \} + \Lambda \{ \bar{\psi} \Gamma_{\Gamma}, \psi \} \right]. \]  \hspace{1cm} (9)

Here \( P_m \) and \( \Pi_r \) denote the momenta conjugate to \( X^m \) and \( \mathcal{A}_r \) respectively. By \( \Psi \) we denote the Majorana spinors of the \( D = 11 \) formulation which may be decomposed in terms of a complex 8-component spinor of \( SO(7) \times U(1) \).

The above Hamiltonian describes a non-commutative Maxwell connection coupled to the transverse scalar fields to the supermembrane world volume. The first class constraint generating the area preserving diffeomorphisms realizes as the noncommutative Gauss constraint. The presence of the integral of the noncommutative curvature is highly relevant, since it explains why the nature of the spectrum in our model differs from the model studied in \([2, 3] \) and \([4] \). Indeed, we can use the very same formulation \( (8) \) of the Hamiltonian, which is also valid for the free winding supermembrane. We emphasize that this is the exact expression for the Hamiltonian and not an approximation. In the free winding case, \( \mathcal{A}_r \) is a multi-valued connection over \( \Sigma \), unlike the fixed central charge case, where it is a single-valued object. In this sense, our model corresponds to a restriction in the space of all possible configurations of the free winding case. In the fixed central charge model, the term \( \sqrt{\mathcal{W}} n^* \mathcal{F} \) is a total derivative, hence its integral cancels out. In this case the condition of zero hamiltonian density in an open region implies both, zero curvature and hence trivial \( \mathcal{A}_r \), and constant \( X^m \). This ensures the absence of singular configurations. These results were obtained in \([1] \) in explicit manner. On the other hand, when \( \mathcal{A}_r \) is multi-valued, the latter term in expression \( (8) \) can be non-zero and it is not difficult to check that non-trivial singular configurations can arise in general. This is in agreement with the results of \([4] \).

Reference \([1] \) is devoted to finding an \( SU(N) \) regularized formulation of the Hamiltonian for a fixed non-trivial central charge. In the case of free winding, the closed, but not exact, modes seem not to fit in an \( SU(N) \) model, cf. \([4] \). Unfortunately, the present work does not contribute to this situation. However, as we mentioned earlier, in the case of fixed central charge, the close one forms are given in unique manner in terms of a given basis of homology. The configuration space is then described in terms of exact one-forms. The regularization procedure leads then to a model formulated in terms of \( SU(N) \)-valued geometrical objects. The resulting model for a toroidal supermembrane is given
by

\[
H = \text{Tr} \left( \frac{1}{2N^3} (P_m^0 T_0 P_m^0 T_0 + \Pi_r^0 T_0 \Pi_r^{-0} T_0 + (P_m)^2 + (\Pi_r)^2) + \\
+ \frac{n^2}{16\pi^2 N^3} [X^m, X^n]^2 + \frac{n^2}{8\pi^2 N^3} \left( \frac{i}{N} [T_{V_r}, X^m] [T_{-V_r} - [A_r, X^m]] \right)^2 + \\
+ \frac{n^2}{16\pi^2 N^3} \left( [A_r, A_s] + \frac{i}{N} [T_{V_r}, A_r] T_{-V_s} - [T_{V_r}, A_s] T_{-V_r} \right)^2 + \frac{1}{8} n^2 + \\
+ \frac{n}{4\pi^2 N^3} \Lambda \left( [X^m, P_m] - \frac{i}{N} [T_{V_r}, \Pi_r] T_{-V_s} + [A_r, \Pi_r] \right) + \\
+ \frac{in}{4\pi^2 N^3} (\Psi \Gamma_\rho \Gamma_m [X^m, \Psi] - \overline{\Psi} \Gamma_\rho [A_r, \Psi] + \Lambda [\overline{\Psi} \Gamma_\rho, \Psi] + \\
- \frac{i}{N} \overline{\Psi} \Gamma_\rho [T_{V_r}, \Psi] T_{-V_s}) \right)
\]

subject to

\[
A_1 = A_1^{(a_1, 0)} T_{(a_1, 0)}, \\
A_2 = A_2^{(a_1, a_2)} T_{(a_1, a_2)} \quad \text{with} \quad a_2 \neq 0. \quad (11)
\]

Here \( A = (a_1, a_2), \) where the indices \( a_1, a_2 = 0, \ldots, N - 1 \) exclude the pair \( (0, 0), \) \( V_1 = (0, 1), \) \( V_2 = (1, 0) \) and \( T_0 \equiv T_{(0, 0)} = N \mathbb{I}. \) We agree in the following convention

\[
X^m = X^{mA} T_A, \quad P_m = P_m^A T_A, \\
A_r = A_r^A T_A, \quad \Pi_r = \Pi_r^A T_A,
\]

where \( T_A \) are the generators of the \( SU(N) \) algebra:

\[
[T_A, T_B] = f_{ABC}^A T_C.
\]

The condition (11) correspond to a truncation of a gauge fixing of (8) and (9), when the geometrical objects are given in a complete orthonormal basis of \( L^2(\Sigma). \) Gauge fixing conditions of the same kind were first used in [2].

We showed in [3] (see also [13]) that the regularized Hamiltonian (10) has an associated mass operator with no string-like spikes. The gauge fixing conditions together with the constraint

\[
([X^m, P_m]) = -\frac{i}{N} [T_{V_r}, \Pi_r] T_{-V_s} + [A_r, \Pi_r] + [\overline{\Psi} \Gamma_\rho, \Psi] = 0
\]

allow, [13], a canonical reduction of the Hamiltonian. Therefore the conjugate pairs

\[
A_1^{a,b}, \Pi_1^{a,b}, \quad b \neq 0 \\
A_2^{a,0}, \Pi_2^{a,0},
\]

do not appear in (10). After this reduction the terms \( |\Pi_1^{a,b}|^2, b \neq 0 \) and \( |\Pi_2^{a,0}|^2 \) become nontrivial, however, since they are positive, we
can bound the mass operator by an operator $\mu$ without such terms. Notice that the imposition of the gauge fixing conditions (11) together with the elimination of the associated conjugate momenta, is valid only in the interior of an open cone $K$. Our model considers then wavefunctions with support on the interior of $K$. In order to show that the spectrum of $\mu$ is discrete, we will consider $\mu$ as an operator acting on the whole configuration space. The discreteness of the spectrum for the latter implies the same property for the restriction to any hypercone. We notice that the assumption that the quantum problem is formulated only on an open cone $K$ of all configuration space, with Dirichlet boundary conditions, is also implicit in $[2]$.

According to the results of $[13]$, the bosonic part of the operator $\mu$ has compact resolvent. Moreover, the bosonic potential is what we can formally call “basin shaped”. We will prove in the forthcoming sections that the operator $\mu$ including the fermionic sector, has compact resolvent and consequently the mass operator of the Hamiltonian (10) has also a compact resolvent.

3. The Fermionic Potential

We consider the metric

$$
\eta_{ab} = \begin{pmatrix}
0 & -1 & & \\
-1 & 0 & & \\
& & 1 & \\
& & & . . . \\
& & & & 1
\end{pmatrix}
$$

in the light cone. In the $2 + 2 + 7$ decomposition of the space-time indices, $\Gamma^\mu = (\Gamma^+, \Gamma^-, \Gamma^1, \Gamma^2, \Gamma^i)$ where

$$(\Gamma^+)^2 = (\Gamma^-)^2 = 0 \quad \text{and} \quad \{\Gamma^+, \Gamma^-\} = 2\mathbb{I}.$$ 

An appropriate representation is found by considering

$$\Gamma^\pm = \sigma_\pm \otimes \mathbb{I} \otimes \mathbb{I}_{8 \times 8},$$

where

$$\sigma_\pm = \sqrt{\frac{1}{2}}(\sigma_1 \pm i\sigma_2),$$

$$\Gamma^r = \sigma_3 \otimes i\sigma_r \otimes \mathbb{I}_{8 \times 8}, \quad r = 1, 2,$$

$$\Gamma^i = \sigma_3 \otimes \sigma_3 \otimes \gamma^i_{8 \times 8} \quad , i = 3, \ldots 9$$

and

$$(\gamma^i)^T = -\gamma^i,$$

in order to ensure that $C\Gamma^i$ are symmetric. The charge conjugation is antisymmetric and it is given by

$$C = \sigma_2 \otimes \sigma_1 \otimes \mathbb{I}_{8 \times 8}.$$
The Majorana condition and the light cone gauge fixing condition associated to the kappa symmetry

$$\Psi = \Psi^T C, \quad \text{and} \quad \Gamma^+ \Psi = 0,$$

respectively, allow to rewrite $\Psi$ in terms of a complex spinor of $SO(7)$

$$\Psi = \begin{pmatrix} i\chi^* \\ \chi \\ 0 \\ 0 \end{pmatrix}$$

where $\chi_\alpha$ and $\chi^+_{\beta}$ are anticommuting variables, and $\alpha, \beta = 1, \ldots, 8$ are $SO(7)$ spin indices. The canonical quantization rules for $\chi$, considering the second class Dirac constraint on the spinor fields, turns out to be

$$\{\chi_\alpha, \chi^*_{\beta}\} = \delta_{\alpha\beta}$$

$$\{\chi_\alpha, \chi_{\beta}\} = \{\chi^*_\alpha, \chi^*_{\beta}\} = 0. \quad (12)$$

In terms of the matrix model, the canonical quantization reads

$$\{\chi^{A\dagger}_\alpha, \chi^{\dagger}_{\beta B}\} = \delta_{\alpha\beta} \delta^{A}_B$$

$$\{\chi^{A\dagger}_\alpha, \chi^{B}_{\beta}\} = \{\chi^{\dagger}_\alpha A, \chi^{\dagger}_{\beta B}\} = 0.$$

This allows us to compute the fermionic potential in terms of the $\chi$ fields. We have

$$\langle \overline{\Psi} \Gamma^- \Gamma_m X^m \partial \Psi \rangle = \langle -\sqrt{2}(-i\chi^{\dagger} \gamma_m \{X^m, \chi\} + i\chi^{\dagger} \gamma_m \{X^m, \chi^*\}) \rangle$$

and in the matrix model

$$\frac{-n}{4\pi N^3} f^{AB}_{BC} [\chi^{\dagger}_A \gamma_m X^{mB} \chi^C + \chi^{TC} \gamma_m X^{mB} \chi^*_A].$$

The other terms are constructed in a similar manner. Spinors will be represented by $2^{8D} \times 2^{8D}$ matrices, with $D$ equals to the dimension of the symmetry group, in this case $SU(N)$. The explicit construction of such matrices can be performed inductively. To avoid dispersing from our main task, we show this procedure in detail in section 6.

4. “Basin shaped” potentials

The criterion we present below extends the well-known fact that if a potential is bounded from below and $V(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$, then the Hamiltonian $-d^2/dx^2 + V$ as a closed operator acting on $L^2(\mathbb{R}^n)$ has discrete spectrum (see for instance [15, theorem XIII.16]). For completeness of our exposition, we provide a self-contained proof of this result.

The Hamiltonian we shall consider is as follows. Let

$$H = -\Delta + V$$
acting as a closed operator on $L^2(\mathbb{R}^N) \otimes \mathbb{C}^n$, where $\Delta \equiv \Delta_x \otimes \mathbb{I}_{n \times n}$ and $V(x) = V(x)^* \in \mathbb{C}^{n \times n}$, $x \in \mathbb{R}^N$. In addition, assume that $V$ is measurable and satisfies $V(x) \geq c$ in the sense that

$$V(x)w \cdot w \geq c, \quad x \in \mathbb{R}^N, \: w \in \mathbb{C}^n,$$

where $c \in \mathbb{R}$ is constant. This ensures that the operator $H$ can be defined by means of a symmetric quadratic form and it is bounded from below. Once we have defined rigorously operator $H$, the validity of the following criterion is ensured.

**Lemma 1.** Let $v_k(x)$ be the eigenvalues of $V(x)$. If all $v_k(x) \to +\infty$ as $|x| \to \infty$, then the spectrum of $H$ is discrete.

**Proof.** Without loss of generality we assume that $v_k(x) \geq 0$. Since $H$ is bounded from below, one can apply the Raleigh-Ritz principle to find the eigenvalues below the essential spectrum. Let

$$\lambda_m(T) := \inf \left( \sup_{\Phi \in L} \frac{\langle T\Phi, \Phi \rangle}{\|\Phi\|^2} \right)$$

where the infimum is taken over all $m$-dimensional subspaces $L \subseteq L^2(\mathbb{R}^N) \otimes \mathbb{C}^n$. Then the bottom of the essential spectrum of $T$ is $\lim_{m \to \infty} \lambda_m(T)$. Notice that if this limit is $+\infty$, then the spectrum of $T$ is discrete. Above

$$\langle \Phi, \Psi \rangle = \int_{\mathbb{R}^n} \Phi(x) \cdot \Psi(x) dx.$$

and $\| \cdot \|^2 = \langle \cdot, \cdot \rangle$.

The hypothesis of the lemma is equivalent to the following condition: for all $c > 0$, there exists a ball $S$ (of possibly very large radius) such that

$$V(x)w \cdot w \geq c|w|^2, \quad \text{all } x \not\in S.$$

Let

$$W(x) := \begin{cases} -c & x \in S \\ 0 & x \not\in S \end{cases}.$$

Then for all $\Phi$ smooth and with compact support,

$$V(x)\Phi(x) \cdot \Phi(x) \geq c\Phi(x) \cdot \Phi(x) + W(x)\Phi(x) \cdot \Phi(x)$$

so that

$$\langle V\Phi, \Phi \rangle \geq \langle (c + W)\Phi, \Phi \rangle.$$

Thus

$$\lambda_m(H) \geq c + \lambda_m(-\Delta + W)$$

for all $l = 1, 2, \ldots$.

Since $W(x)$ is a bounded potential with compact support, by Weyl’s theorem, the essential spectrum of $-\Delta + W$ is $[0, \infty)$. Therefore by Raleigh-Ritz criterion, there exists $\tilde{M} > 0$ such that

$$\lambda_m(-\Delta + W) \geq -1, \quad m \geq \tilde{M}.$$
Thus, $\lambda_m(H) \geq c - 1$ for all $m \geq M$. Since we can take $c$ very large, necessarily $\lambda_m \to +\infty$ as $m$ increases and the proof of the lemma is complete.

5. Discreteness of the spectrum

By using lemma 1, we show in this section that the resolvent of the operator $\mu$ is compact and hence it has discrete spectrum.

To this end, decompose $\mu$ as

$$\mu = -\Delta + V_B I + V_F$$

where $V_B$ and $V_F$ denote the bosonic and fermionic potentials respectively. Then $V_F$ is the sum of a linear homogeneous part $M(X, A)$ corresponding to

$$\frac{in}{4\pi N^3} (\overline{\Psi}_{\Gamma_m} [X^m, \Psi] - \overline{\Psi}_{\Gamma_r} [A_r, \Psi])$$

and a constant matrix $C$ corresponding to

$$\frac{-n}{4\pi N^4} (\overline{\Psi}_{\Gamma_r} [T_V, \Psi] T_{-V_r}).$$

Put $T := -\Delta + V_B + M(X, A)$. Since $T$ is Hermitian (a self-adjoint operator in its domain), $(T - i\rho)$ is invertible for all $\rho > 0$ and we can choose $\rho$ large enough such that the resolvent $(T - i\rho)^{-1}$ satisfies

$$\| (T - i\rho)^{-1} \| \leq \| C \|^{-1}/2.$$

Here $\| \cdot \|$ is the supremum norm for operators acting on Hilbert spaces. Hence

$$\mu - i\rho = T + C - i\rho = (T - i\rho)(I + (T - i\rho)^{-1} C).$$

Because of $\| (T - i\rho)^{-1} C \| \leq 1/2$, the latter term is invertible and so

$$(\mu - i\rho)^{-1} = (I + (T - i\rho)^{-1} C)^{-1} (T - i\rho)^{-1}.$$

Since the first term at the right hand side is bounded, the resolvent of $\mu$ is compact if and only if the resolvent of $T$ is compact.

We apply lemma 1 in order to show compactness for the resolvent of $T$. If we denote by $R$ the normal vectors in the configuration space so that $X = R\phi$ and $A = R\psi$, according to [13] §3, $R = 0$ is a double zero of $V_B(R\phi, R\psi)$ and

$$V_B(R\phi, R\psi) \geq kR^2$$

for some constant $k > 0$. The eigenvalues of the matrix

$$V_B + M(X, A)$$

are the $\lambda \in \mathbb{R}$ such that

$$\det [\lambda - V_B - M(X, A)] = 0.$$
By virtue of the homogeneity of $M$, $\lambda$ must satisfy
\[
\det \left[ \frac{\lambda - V_B}{R} \mathbb{I} - M(\phi, \psi) \right] = 0, \quad R > 0.
\]

Therefore if $\lambda$ are the eigenvalues of $M(\phi, \psi)$, then
\[
\lambda = V_B(R\phi, R\psi) + R\lambda.
\]
Consequently, $\lambda \to +\infty$ whenever $R \to \infty$. Notice that $V$ is continuous, hence it is automatically bounded from below. This ensures that the resolvent of $T$ is compact as a consequence of lemma \[1\].

6. Matrix representation of spinors

In this final section we show how to construct a basis $\chi_\alpha$, $\alpha = 0, \ldots, n-1$ of size $2^n \times 2^n$ satisfying the anti-commutative relations
\[
\{\chi_\alpha, \chi_\beta^s\} = \delta_{\alpha\beta}, \\
\{\chi_\alpha, \chi_\beta\} = 0.
\]

This ensures \[12\].

The entries of the matrices $\chi_\alpha$ are either 0 or $\pm 1$. Adopting a common notation in combinatorics, the list
\[
[(m_1, n_1)_{\pm}; \ldots; (m_j, n_j)_{\pm}]
\]
denotes a matrix full of zeros except that it has either $+1$ or $-1$ at the entries $(m_1, n_1), \ldots, (m_j, n_j)$. For $l = 0, 1, \ldots$ and $\alpha > 0$, let
\[
B_0(l) = (2l + 1, 2l + 2)_+ \\
B_\alpha(l) = [(2^{\alpha+1}l+1, 2^{\alpha+1}l+1+2^\alpha)_s; \ldots; (2^{\alpha+1}l+(2m+1), 2^{\alpha+1}l+(2m+1)+2^\alpha)_s; \\
\ldots ; (2^{\alpha+1}l+(2^\alpha-1), 2^{\alpha+1}l+(2^\alpha-1)+2^\alpha)_s; (2^{\alpha+1}l+2, 2^{\alpha+1}l+2+2^\alpha)_s; \\
\ldots ; (2^{\alpha+1}l+(2m+2), 2^{\alpha+1}l+(2m+2)+2^\alpha)_s; \ldots ; (2^{\alpha+1}l+2^\alpha, 2^{\alpha+1}l+2^\alpha+2^\alpha)_s]
\]
be blocks of size $2^{\alpha+1} \times 2^{\alpha+1}$, where the sign $s$ of the non-zero entry $(m, n)_s$ of $B_\alpha(0)$ is determined according to the rule
\[
s = (-1)^{\#(q)_2}(-1)^{m+1}, \quad m = 2q + 1 \text{ or } m = 2q + 2 \quad (14)
\]
and the sign distribution of $B_\alpha(l)$ for $l \geq 1$ is the same as that of $B_\alpha(0)$. The integer $\#(q)_2$ is the number of ones in the binary representation of $q$. Then we construct the desired basis as the block diagonal matrices
\[
\chi_\alpha = [B_\alpha(0); \ldots; B_\alpha(2^n/2^{\alpha+1} - 1)], \quad \alpha = 0, \ldots, n-1
\]
of size $2^n \times 2^n$. In order to illustrate this procedure take $n = 4$, then the basis is
\[
\chi_0 = [(1, 2)_+; (3, 4)_+; (5, 6)_+; (7, 8)_+; (9, 10)_+; (11, 12)_+; (13, 14)_+; (15, 16)_+] \\
\chi_1 = [((1, 3)_+; (2, 4)_-); [(5, 7)_+; (6, 8)_-]; [(9, 11)_+; (10, 12)_-]; [(13, 15)_+; (14, 16)_-]] \\
\chi_2 = [[[1, 5)_+; (3, 7)_-; (2, 6)_-; (4, 8)_+]; [(9, 13)_+; (11, 15)_-; (10, 14)_-; (12, 16)_+]] \\
\chi_3 = [(1, 9)_+; (3, 11)_-; (5, 13)_-; (7, 15)_+; (2, 10)_-; (4, 12)_+; (6, 14)_+; (8, 16)_-].
We now show that the $\chi_0, \ldots, \chi_{n-1}$ satisfy (13). In order to simplify notation, we represent the product of two matrices whose rows and columns have only one non-zero entry whose value is $\pm 1$, as a signed permutation of the group of integer numbers. To be more precise, it is easy to see that the only non-zero entries of this product are the pairs $(m, o)_s$ with sign $s = rt$ where the entries $(m, n)_t$ are non-zero with sign $t$ on the first matrix, and the entries $(n, o)_r$ are non-zero with sign $r$ on the second matrix. This will be described pictorially as $m \xrightarrow{t} n \xrightarrow{r} o$.

$\{\chi_\alpha, \chi_\alpha\} = 0$ and $\{\chi_\alpha, \chi_\alpha^*\} = I$. Notice that each $\chi_\alpha = (m_1, n_1)_s; \ldots; (m_j, n_j)_s$, where each index $m_p$ or $n_p$ is different from all other indices. This ensures the first identity. The second identity is consequence of the fact that the only non-zero entries of the product $\chi_\alpha \chi_\alpha^*$ are represented by $m_p \xrightarrow{s} n_p \xrightarrow{s} m_p$ and those of $\chi_\alpha^* \chi_\alpha$ by $n_p \xrightarrow{s} m_p \xrightarrow{s} n_p$.

In order to show that $\{\chi_\alpha, \chi_\beta\} = 0$ when $\alpha \neq \beta$, it is enough to check this property only when $0 \leq \alpha < n - 1$ and $\beta = n - 1$, whereas the other cases follow from an inductive argument. Furthermore since our basis consists of block diagonal matrices, we only have to verify how the first block of $\chi_\alpha, B_\alpha(0)$, multiplies with the suitable elements of $\chi_\beta$. Notice that the only possibly non-zero entries of $\{\chi_\alpha, \chi_\beta\}$ are $(m, n)$ consequence of the sum

\[ m \xrightarrow{s} o \xrightarrow{t} n + m \xrightarrow{n} p \xrightarrow{v} n. \]  

(15)

We claim that this entry will also be zero, due to the fact that the product of signs $stuv$ is always negative. Let us show this claim.

$\{\chi_0, \chi_{n-1}\} = 0$. Put

\[ m = 1, \ o = 2, \ n = 2 + 2^{n-1}, \ p = 1 + 2^{n-1} \]

in (15). The signs $s$ and $v$ are positive because of all the non-zero entries of $\chi_0$ are equal to one. According to the rule (14), $t = (-1)^{\#(0)2}(-1)^2$ and $u = (-1)^{\#(0)2}(-1)^3$. This ensures that the product of signs is always negative in this case.

$\{\chi_\alpha, \chi_{n-1}\} = 0, \ 0 < \alpha < n - 1$. If $m$ is odd, the indices in (15) are

\[ m = 2q+1, \ o = 2q+1+2^a, \ n = 2q+1+2^a+2^{n-1}, \ p = 2q+1+2^{n-1} \]

for $0 \leq q \leq 2^{a-1} - 1$. Then

\[ s = u = (-1)^{\#(q)2} = v, \quad t = (-1)^{\#(q+2^{n-1})2}. \]

For the third inequality notice that by construction the sign rule of $B_\alpha(l)$ for $l > 0$ copies the one of $B_\alpha(0)$. Since the binary representation
of \( q \) has at most \( 2^{\alpha-2} \) digits, then \( \#(q + 2^{\alpha-1})_2 = \#(q)_2 + 1 \) and thus the product of signs is always negative. If \( m \) is even, for reasons similar to the odd case, also \( s = u = v \) and \( t = -s \).

This completes the proof of our claim.

\[ \{ \chi_\alpha, \chi_\beta^* \} = 0, \alpha \neq \beta. \] By using arguments involving the diagram (15), this can be shown in a similar manner as the previous identity. One should take into account that due to the transposition in the second term, the signs of \( s \) and \( v \) should be determined not from the first but from the second entry of the suitable pair.

**Conclusions**

We have shown that the quantum Hamiltonian of the compactified supermembrane on \( M_9 \times S_1 \times S_1 \) with non-trivial central charge, that is with irreducible winding, has compact resolvent and consequently its spectrum consists of a discrete set of eigenvalues with finite multiplicity.

We have considered a regularized \( SU(N) \) model of the compactified supermembrane with non-trivial central charge. The condition of having a non-trivial winding, determines a sector of the full compactified supermembrane. In the explicit formulas we use a toroidal supermembrane, however, the result still holds for other non-trivial topologies (the spherical supermembrane has been considered recently in \([18]\) in terms of an \( SU(2) \) model). Our approach is valid for the analysis of the compactified supermembrane with target space \( M_9 \times S_1 \times S_1 \). In this case the class of maps defining the configuration space of the supermembrane are determined by the central charge which becomes proportional to \( n \), the winding number of the supermembrane. The existence of a non-trivial central charge leads to a re-formulation of the problem in terms of a symplectic non-commutative super Yang-Mills theory. In particular our regularized Hamiltonian is a consequence of this construction.

The lemma we have established in section 5, seems to be the appropriate strategy to investigate any compactified supersymmetric models where no string-like configurations are present. The assumptions on the bosonic potential are very mild, we only require the potential to be measurable, bounded from below and unbounded above in every direction (if the potential is continuous the unbounded assumption ensures the boundedness from below). For instance, for a quantum mechanical potential of the form

\[ V = V_B(x)\mathbb{I} + V_F(x) \in \mathbb{C}^{2^n \times 2^n}, \]

where \( x \in \mathbb{R}^L \), \( V_B(x) \) is continuous with the asymptotic behaviour

\[ V_B \geq c\|x\|^{2p}, \quad c > 0, \]
and the fermionic matrix potential satisfies
\[ V_F \leq V_F\|x\|=1\|x\|^q, \]
for all \( \|x\| > R_0 \) with \( 2p > q \), the Hamiltonian of the quantum system has spectrum consisting exclusively of isolated eigenvalues of finite multiplicity.

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