New results on particle filters with adaptive number of particles

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Abstract

In this paper, we present new results on particle filters with adaptive number of particles. First, we analyze a method which is based on generating fictitious observations from an approximated predictive distribution of the observations and where the generated observations are compared to actual observations at each time step. We show how the number of fictitious observations is related to the number of moments assessed between the approximated and the true predictive probability density function. Then, we introduce a new statistic for deciding how to adapt the number of particles in an online manner and without the need of generating fictitious particles. Finally, we provide a theoretical analysis of the convergence of a general class of particle filters with adaptive number of particles.

Index Terms

Particle filtering, sequential Monte Carlo, predictive distributions, convergence analysis, adaptive complexity.

I. INTRODUCTION

In science and engineering, there are many problems that are studied by dynamic probabilistic models, which mathematically describe the evolution of hidden states and their relation with observations that are sequentially acquired. In many of these problems, the objective is to estimate sequentially the posterior distribution of hidden states. A methodology that has gained considerable popularity in the last two and a half decades is particle filtering (also known as sequential Monte Carlo) [1], [2]. This is a Monte Carlo methodology that approximates the distributions of interest by means of random (weighted) samples.

Arguably, a key parameter of particle filters (PFs) is the number of used particles. A larger number of particles improves the approximation of the filter but also increases the computational complexity. However, a priori it is impossible to know the appropriate number of particles to achieve desirable accuracies of estimated parameters and distributions.

A. A brief review of the literature

Until the publication of [3], very few papers had considered the selection/adaptation of the number of particles. In [3], a methodology was introduced to address this problem with the goal of adapting the number of particles in real-time. The method is based on a rigorous mathematical analysis. Other efforts toward the same goal include the use of a Kullback-Leibler divergence-based approximation error [4], where the divergence was defined between the distribution of the PF and a discrete approximation of the true distribution computed on a predefined grid. The idea from [4] was further explored in [5]. A heuristic approach based on the effective sample size was proposed in [6], [7, Chapter 4]. A disadvantage of using the effective sample size is that once a PF loses track of the hidden state, the effective sample size does not provide information for adjusting the number of particles [8]. See other problems with the effective sample size in [9].

A theoretically-based approach for selecting the number of particles was reported in [10], where Feynman-Kac framework was invoked [11]. In [12], an autoregressive model for the variance of the estimators produced by the PF was employed. Both methods operate only in batch modes. In a group of papers on alive PFs, the number of particles is adaptive and based on sampling schemes that ensure a predefined number of particles to have non-zero weights [13], [14], [15]. In [16], a fixed number of particles is adaptively allocated to several candidate models according to their performances. In [17], particle sets of the same size are generated until an estimation criterion for their acceptance is met.

B. A summary of the method proposed in [3]

In [3], we introduced a methodology for assessing the convergence of PFs. The proposed method works online, and is both model- and filter-independent. The method is based on simulating fictitious observations from one-step-ahead predictive distributions approximated by the PF, and comparing them with actual observations that are available at each time step. In the case of one-dimensional observations, a statistic is constructed that simply represents the number of fictitious observations which are smaller than the actual observation. We show that when the filter has converged, the distribution of the statistic is uniform on a discrete support. We propose an algorithm for statistically assessing the uniformity of the statistic, and based on it modifying the number of particles. The method is supported with rigorous theory regarding the particle approximation of the predictive distribution of the observations and the convergence of the distribution of the statistic.

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C. Contributions

In this paper, we describe a generic framework of block-adaptive particle filters based on the philosophy from [3]. The main idea is based on a specific statistic, $A_{K,M,t}$, which is computed at each time step by comparing the received observation and the predictive distribution of the observations approximated by the filter. In this framework, we propose a novel algorithm that assesses the convergence based on the correlation of the statistic $A_{K,M,t}$ [21]. Moreover, we propose a novel algorithm for assessing the quality of the approximation and for adapting the number of particles. Unlike the algorithm from [3], this algorithm is based on a new statistic $B_{M,t}$, whose computation does not require generation of fictitious particles. We analyze the choice of the key parameters of the adaptive framework, $K$, the number of fictitious observations, $M_n$, the number of time steps in each window, and the width of that window $W$. We provide several theoretical results related to these parameters that not only justify the robustness of the method but also point to adequate choices of these parameters. Then, we analyze the convergence of the block-adaptive framework when the number of particles is increased. We show that for a sufficiently large window, the errors of previous windows (with less number of particles) are forgotten and that the convergence rate of the approximation depends on the current number of particles.

D. Organization of the paper

In the next section, we briefly describe particle filtering as a sequential Monte Carlo methodology and we introduce our notation. In Section III, we propose a general online scheme for selecting the number of particles and for adapting the sizes of the observed data blocks for computing statistics. In the following section, we introduce the new statistic for assessing the performance of a PF. We provide convergence results in Section IV. In the last two sections, we present results of numerical experiments and our conclusions, respectively.

II. PARTICLE FILTERING

We consider Markov dynamic systems whose state-space models are described by

$$X_0 \sim p(x_0),$$  \hspace{1cm} (1)

$$X_t \sim p(x_t | x_{t-1}),$$  \hspace{1cm} (2)

$$Y_t \sim p(y_t | x_t),$$  \hspace{1cm} (3)

where

- $t \in \mathbb{N}$ denotes discrete time;
- $X_t$ is a $d_x \times 1$-dimensional (random) hidden process (state) at time $t$, and where $X_t \in \mathcal{X} \subseteq \mathbb{R}^{d_x}$;
- $p(x_0)$ is the a priori pdf of the state;
- $p(x_t | x_{t-1})$ is the conditional density of $X_t$ given $X_{t-1} = x_{t-1}$;
- $Y_t$ is a $d_y \times 1$-dimensional observation vector at time $t$, where $Y_t \in \mathcal{Y} \subseteq \mathbb{R}^{d_y}$ and is assumed conditionally independent of all the other observations given $X_t$;
- $p(y_t | x_t)$ is the conditional pdf of $Y_t$ given $X_t = x_t$. It is often referred to as the likelihood of $x_t$, when it is viewed as a function of $x_t$ given $y_t$.

Based on the model and made assumptions, we want to estimate in a recursive manner the posterior probability distributions $p(x_t | y_{1:t})$, $t = 1, 2, \ldots$. We can write

$$p(x_t | y_{1:t}) \propto p(y_t | x_t) \int p(x_t | x_{t-1})p(x_{t-1} | y_{1:t-1})dx_{t-1},$$

where we see how $p(x_t | y_{1:t})$ is related to its counterpart at the previous time instant $p(x_{t-1} | y_{1:t-1})$.

We represent the filtering and the predictive posterior pdf of the state by

$$\pi_t(dx_t) := p(x_t | y_{1:t})dx_t, \hspace{1cm} \xi_t(dx_t) := p(x_t | y_{1:t-1})dx_t.$$  \hspace{1cm} (4)

The object that plays a central role in our study is the predictive pdf of the observations, $p(y_t | y_{1:t-1})$ with its probability measure given by

$$\mu_t(dy_t) := p(y_t | y_{1:t-1})dy_t.$$  

It is well known that the predictive pdf is instrumental for model inference [19], [20], [21], [22].

The main idea of particle filtering is to estimate sequentially the probability measures $\{\pi_t\}_{t \geq 1}$ from the observations $\{y_t\}_{t \geq 1}$. The basic method for accomplishing this is known as bootstrap filter (BF) [23]. The BF implements three steps, 1) drawing particles $x_t^{(m)}$, $m = 1, 2, \ldots, M$ with $M$ being the number of particles, from the conditional pdf, $p(x_t | x_{t-1})$, where the conditioning state has values at previously drawn particles, 2) computation of the normalized importance weights of the

\footnote{We presented this algorithm in the conference paper [18].}
TABLE I: General algorithm for adapting the number of particles (on a BPF)

1) [Initialization]
   a) Initialize the particles and the weights of the filter as
   \[ x_0^{(m)} \sim p(x_0) \], \quad m = 1, \ldots, M_0, \\
   \[ w_0^{(m)} = 1/M_0, \quad m = 1, \ldots, M_0, \]
   and set \( n = 1 \).
2) [For \( t = 1 : T \)]
   a) Bootstrap particle filter:
      - Resample \( M_n \) samples of \( x_{t-1}^{(m)} \) with weights \( u_{t-1}^{(m)} \) to obtain \( x_t^{(m)} \).
      - Propagate \( x_t^{(m)} \) \( \sim p(x_t|x_{t-1}^{(m)}) \), \( m = 1, \ldots, M_n \).
      - Compute the non-normalized weights \( \tilde{w}_t^{(m)} = p(y_t|x_t^{(m)}) \), \( m = 1, \ldots, M_n \).
      - Normalize the weights \( \tilde{w}_t^{(m)} \) to obtain \( w_t^{(m)} \), \( m = 1, \ldots, M_n \).
   b) Fictitious observations:
      - Draw \( \hat{y}_t^{(k)} \sim p^M(\cdot|y_{t-1}^{(k)}), \quad k = 1, \ldots, K \).
      - Compute \( a_{K,M,t} = A_{K,M,t} \) i.e., the position of \( y_t \) within the set of ordered fictitious observations \( \{\hat{y}_t^{(k)}\}_{k=1}^K \).
   c) If \( t = nW \), (assessmnet of convergence):
      - With some specific algorithm from Section III analyze the sequence \( S_t = \{ a_{K,M,t}, a_{K,M,t-1}, \ldots, a_{K,M,t-W+1} \} \).
      - Increase/decrease/keep fixed the number of particles \( M_t \).
      - Set \( n = n + 1 \).
   d) If \( t < Wn \), set \( t = t + 1 \) and go to 2. Otherwise, end.

Once the particles and their weights are computed, we can obtain approximations of various probability measures. The filter measure \( \pi_t \) can be approximated as
\[ \tilde{\pi}_t^M = \sum_{m=1}^M w_t^{(m)} \delta_{x_t^{(m)}} \]
where \( \delta_x \) represents the Dirac delta measure at \( x \in \mathcal{X} \). Further, given \( Y_{1:t-1} = y_{1:t-1} \), the predictive pdf’s of \( X_t \), \( \tilde{p}_t(x_t) := p(x_t|y_{1:t-1}) \) and \( Y_t, p_t(y_t) := p(y_t|y_{1:t-1}) \), are approximated by
\[ \tilde{p}_t^M(x_t) := \sum_{m=1}^M w_t^{(m)} p(x_t|x_{t-1}^{(m)}), \quad x_t \in \mathcal{X}, \]
\[ p_t^M(y_t) := \frac{1}{M} \sum_{m=1}^M p(y_t|x_{t}^{(m)}), \quad y_t \in \mathcal{Y}, \quad \text{and} \]
\[ \tilde{p}_t^M(y_t) := \sum_{m=1}^M w_t^{(m)} p(y_t|x_{t}^{(m)}), \quad y_t \in \mathcal{Y}. \]

III. General online selection of the number of particles: Block adaptation

The generic block-adapting method for selecting the number of particles is summarized in Table I. We implement it on the standard BPF, but the methodology is the same for any other PF. In Step 1(a), the filter is initialized with \( M_0 \) particles. The filter works at each time step in a standard manner, as described in Step 2(a). However, the first modification w.r.t. the BPF comes in Step 2(b), where \( K \) fictitious observations \( \{\hat{y}_t^{(k)}\}_{k=1}^K \) are simulated from the predictive distribution of the observations \( p^M(\cdot|y_{t-1}) \), (see Section IV-A). These fictitious observations are used to compute the statistic of \( A_{K,M,t} = a_{K,M,t} \), where \( a_{K,M,t} \) is the position of \( y_t \) within the set of ordered fictitious observations \( \{\hat{y}_t^{(k)}\}_{k=1}^K \).

The algorithm works with windows of size \( W \), where at the end of the window (Step 2(c)), the sequence \( S_t = \{ a_{K,M,t-W+1}, a_{K,M,t-W+2}, \ldots, a_{K,M,t-1}, a_{K,M,t} \} \) is processed for assessment of convergence of the filter. The number of particles is adapted (increased, decreased, or kept constant) based on the assessment.

Under the assumption of no approximation error in the observation predictive pdf, i.e., \( p_t^M(y_t) = p_t(y_t) = p(y_t|y_{1:t-1}) \), the fictitious observations \( \{\hat{y}_t^{(k)}\}_{k=1}^K \) are drawn from the same pdf as the actual observation \( y_t \). In that case, the statistic does not depend on the filter approximation, i.e., \( A_{K,M,t} \equiv A_{K,M} \). Two preliminary theoretical results about the statistic \( A_{K,M,t} \) are proved in [3].
Proposition 1. If \( y_t, \tilde{y}_t^{(1)}, \ldots, \tilde{y}_t^{(K)} \) are i.i.d. samples from a common continuous (but otherwise arbitrary) probability distribution, then the pmf of the r.v. \( A_{K,t} \) is
\[
\mathbb{Q}_K(n) = \frac{1}{K+1}, \quad n = 0, \ldots, K.
\] (9)

Proposition 2. If the r.v.s \( y_t, \tilde{y}_t^{(1)}, \ldots, \tilde{y}_t^{(K)} \) are i.i.d. with common pdf \( p_t(y) \), then the r.v.s in the sequence \( \{A_{K,t}\}_{t \geq 1} \) are independent.

Since in practice, \( p_t^M(y_t) \) is just an approximation of the predictive observation pdf \( p_t(y_t) \), it is also shown that the pmf \( \mathbb{Q}_{K,M,t} \) of \( A_{K,M,t} \) converges to a uniform pmf when \( K \to \infty \).

In the following, we describe two different assessment methods, one that tests the uniformity and the other the correlation of \( A_{K,M,t} \), respectively.

A. Algorithm 1: Uniformity of the statistic \( A_{K,M,t} \)

The algorithm assesses the uniformity of the empirical distribution of the statistic \( A_{K,M,t} \) within the block, exploiting Proposition 1. Under the null hypothesis (perfect convergence of the filter), the r.v. \( A_{K,M,t} \) is uniform on \( \{0, 1, \ldots, K\} \), i.e., the pmf of the r.v. \( A_{K,M,t} \) is the pmf of Eq. (9). Therefore, a statistical test is performed for checking whether \( S_t \) is a sequence of samples from the uniform distribution.

B. Algorithm 2: Correlation of the statistic \( A_{K,M,t} \)

If the filter has converged, the samples of \( S_t \) at the end of the block are i.i.d., and distributed uniformly on \( \{0, 1, \ldots, K\} \). Since independence implies absence of correlation, we can test if the samples of \( S_t \) are correlated [18].

We note that in estimating the autocorrelation, longer windows (larger value of \( W \)) allow for more accurate estimation. However, in that case we lose on the responsiveness in the adaptation.

C. Sharing of moments of the approximating and true predictive distributions

A key parameter in the assessment framework is the number of fictitious observations, \( K \). In the following, we show that if the pmf of \( A_{K,M,t} \) is uniform, then the first \( K \) moments of \( p_t^M(y) \) and \( p_t(y) \) defined by \( ((F_t^M)^n, p_t^M) \) and \( ((F_t)^n, p_t) \), respectively, for \( n = 0, 1, 2, \ldots, K \) are identical, and where \( F_t^M \) denotes the cdf of \( p_t^M \).

Lemma 1. For any pdf \( p_t^M \) and its associated cdf \( F_t^M \),
\[
((F_t^M)^n, p_t^M) = \frac{1}{n+1}, \quad \forall n \in \{0, \ldots, K\}.
\] (10)

Proof:
\[
((F_t^M)^n, p_t^M) = \int_{-\infty}^{\infty} (F_t^M(y))^n p_t^M(y) dy
\] (11)
\[
= \int_{-\infty}^{\infty} (F_t^M(y))^n dF_t^M(y)
\] (12)
\[
= \int_{0}^{1} x^n dx
\] (13)
\[
= \frac{1}{n+1},
\] (14)
where we applied the change of variable \( x = F_t^M(y) \).

Lemma 2. Assume that the pmf \( \mathbb{Q}_{K,M,t} \) of \( A_{K,M,t} \) is uniform. Then the following \( K \) equalities hold:
\[
((F_t^M)^n, p_t) = \frac{1}{n+1}, \quad \forall n \in \{0, 1, \ldots, K\}.
\] (15)

Proof: The proof can be found in Appendix A

Theorem 1. Assume that the pmf \( \mathbb{Q}_{K,M,t} \) of \( A_{K,M,t} \) is uniform. Then \( p_t^M \) has at least the following \( K \) moments equal to the respective moments of the true pdf \( p_t \).
\[
((F_t^M)^n, p_t^M) = ((F_t^M)^n, p_t) = \frac{1}{n+1}, \quad \forall n \in \{0, 1, \ldots, K\}.
\] (16)

Proof: See Lemma 1 and Lemma 2
Remark 1. Theorem 2 shows that the first K noncentral moments of $F^M_t$ with respect to $p^M_t$ and $p_t$ are identical.

In the next section we propose a new method for assessing the convergence of the filter. It is based on a new statistic, $B_{M,t}$, which follows the same distribution as $A_{K,M,t}$ when $K \to \infty$. Further, we show that the complexity of computing $B_{M,t}$ is $O(M)$.

IV. A NEW METHOD FOR ASSESSMENT OF CONVERGENCE

We introduce a new method for assessing convergence of a PF, which does not simulate fictitious observations. It is based on a new statistic that we define below. Let us assume first that there is no approximation error. Then we can write $p^M_t(y_t) = p_t(y_t) = p(y_t|y_{1:t-1})$. We define the r.v. $B_t := F_t(y_t)$, where $F_t(y_t)$ is the cdf of $y_t$. Next, we prove two propositions about $B_t$.

Proposition 3. If $y_t$ is distributed according to some continuous probability distribution, then the pdf of the r.v. $B_t$ is $U(0,1)$.
Proof: This is known as the probability integral transform, and its simple proof is widely known, see, e.g. [23, p. 139].

Proposition 4. The r.v.'s in the sequence $\{B_t\}_{t\geq 1}$ are independent.

The sequence of r.v.'s $\{B_t\}_{t\geq 1}$ are constructed to be independent, similarly as the sequence $\{A_{K,t}\}_{t\geq 1}$, as is shown in Appendix B of [3]. Note that in practice, the predictive pdf $p_t(y_t)$ is approximated by the particle filter by $p^M_t(y_t)$ with associated cdf $F^M_t$, i.e., the cdf of $y_t$ is not exactly $F_t$. However, if the assumptions of Theorem 1 in [3] are met, the theorem guarantees the a.s. convergence of the approximate measure $\mu^M_t(dy_t) = p^M_t(y_t)dy_t$, and it enables us to describe the error between $F^M_t(y_t)$ and $F_t(y_t)$. Note that the error goes to zero when $M \to \infty$. To be specific, we introduce the r.v. $B_{M,t} := F^M_t(y_t)$ with cdf $F_{B_{M,t}}$. First, we make the same assumptions as in [3]. Section III, (Σ), (Ω), and (Σ). Two of these assumptions are related to the likelihood of the state. Namely, the likelihood has to be bounded, (Σ), and differentiable with respect to $y$ with bounded derivatives up to order $d_y$, (Ω). The assumption (Σ) requires that the random probability measure $\mu_t$ satisfies a certain inequality. We have the following result for $F_{B_{M,t}}$.

Theorem 2. Let $y_t$ be a sample from $p_t(y_t)$, and let the observations $y_{1:t-1}$ be fixed. Further, let the Assumptions (Σ), (Ω) and (Σ) hold. Then, there exists a sequence of non-negative r.v.'s $\{\varepsilon^M_t\}_{t\in\mathbb{N}}$ such that $\lim_{M \to \infty} \varepsilon^M_t = 0$ a.s. and

$$B_t(y_t) - \varepsilon^M_t \leq B_{M,t}(y_t) \leq B_t(y_t) + \varepsilon^M_t. \quad (17)$$

In particular, $\lim_{M \to \infty} B_{M,t}(y_t) = B_t(y_t)$ a.s. Moreover, the pdf of the r.v. $B_{M,t}(Y_t)$ when $M \to \infty$ converges to $U(0,1)$.

Proof: The result is a consequence of Theorem 1 in [3], where it is proved that

$$|\langle h, \mu^M_t \rangle - \langle h, \mu_t \rangle| \leq \frac{W_t}{M^{2\varepsilon/\beta}} \eta^\varepsilon,$$

where $h = I_{(-\infty,y_t]}$ is the indicator function defined as

$$I_A(y) = \begin{cases} 1, & \text{if } y \in A \\ 0, & \text{otherwise} \end{cases}. \quad (18)$$

Note that, $\|I_A\| = 1 < \infty$ independently of the set $A$ and, therefore, Theorem 1 in [3] can be applied and $\lim_{M \to \infty} B_{M,t}(y_t) = B_t(y_t)$ a.s., which implies the convergence of their cdfs as $\lim_{M \to \infty} F_{B_{M,t}}(b) = F_{B_t}(b)$. Moreover, due to Proposition 3 $\lim_{M \to \infty} F_{B_{M,t}}(b) = b$, with $b \in (0, 1)$, i.e., its pdf converges to $U(0,1)$.

Remark 2. The computation of the statistic $B_t$ requires evaluation of the approximated predictive cdf $F^M_t$ at $y_t$. This will require the evaluation of $M$ kernels (cdfs), one for each particle, which may become a heavy computational load if the number of particles is high.

A. Relation between $A_{K,t}$ and $B_t$

Note that $A_{K,t}$ represents the number of fictitious observations that are smaller than $y_t$, while $B_t$ represents the probability mass of $p^M_t$ in the interval $(\infty, y_t)$. The connection between the two statistics is strong. Intuitively, when the number of simulated fictitious observations $K$ goes to infinity, the rate of these observations that are smaller than $y_t$ should tend to the probability of a fictitious observation being smaller than $y_t$. More precisely,

Theorem 3. If we grow the number of fictitious observations, $K$, to infinity, the statistic $A_{K,M,t}$ divided by $K$ becomes the statistic $B_{M,t}$, i.e.,

$$\lim_{K \to \infty} \frac{A_{K,M,t}}{K} = B_{M,t}. \quad (19)$$
TABLE II: General BPF with block-adaptive number of particles, $M_n$.

1) [Initialization]
   a) Draw independent samples $x_t^{(m)}$ from the prior $\pi_0(dx_0)$ and assign uniform weights $w_0^{(m)} = 1/M_0$, $m = 1, \ldots, M_0$.
   b) Set $k = 0$ (block counter) and choose $W_0 > 0$ (size of the first block).

2) [For $t = 1, 2, \ldots$]
   a) Bootstrap particle filter:
      - Propagate the particles $x_t^{(m)} \sim \kappa_t(dx_t|x_{t-1}^{(m)})$, $m = 1, \ldots, M_n$.
      - Compute normalized weights $\tilde{w}_t^{(m)} \propto g_t(k_t^{(m)})$, $m = 1, \ldots, M_n$.
   b) Fictitious observations:
      - Draw $\bar{y}_t^{(k)} \sim p_t^M(y_t|k_t)$, $k = 1, \ldots, K$.
      - Compute $A_{K,M_n,t} = a_{K,M_n,t}$, i.e., the position of the actual observation $y_t$ within the set of ordered fictitious observations $\{\bar{y}_t^{(k)}\}_{k=1}^K$.
   c) Assessment of convergence: If $t = \sum_{j=0}^n W_j - 1$ (end of the $n$-th block) then:
      - Analyze with some specific algorithm from Section III S$a\{a_{K,M_n,t}; a_{K,M_n,t-1}, \ldots, a_{K,M_n,t-W_n+1}\}$.
      - Set $n = n + 1$.
      - Select the number of particles $M_n > 0$.
      - Select the block size $W_n > 0$.
      - Resample $M_n$ particles with replacement, from the weighted set $\{\tilde{x}_t^{(m)}, \tilde{w}_t^{(m)}\}_{m=1}^{M_n-1}$, to obtain $\{\hat{x}_t^{(m)}\}_{m=1}^{M_n}$.
      - Else:
        - Resample $M_n$ particles with replacement, from the weighted set $\{\tilde{x}_t^{(m)}, \tilde{w}_t^{(m)}\}_{m=1}^{M_n}$, to obtain $\{\hat{x}_t^{(m)}\}_{m=1}^{M_n}$.

**Proof**: Recall that $B_{M,t} = F_t(y_t) \equiv (h, \mu_t^M)$, where $h = I_{(-\infty,y_t]}$ is the indicator function defined in Eq. (18). It is possible to estimate this integral by drawing $K$ samples $\bar{y}_t^{(k)}$ from $\mu_t^M$ and building the raw Monte Carlo estimator $1/K \sum_{k=1}^K I_{(-\infty,y_t]}(\bar{y}_t^{(k)}) \equiv \frac{A_{K,M,t}}{K}$. Note that this estimator is unbiased, i.e., according to the law of large numbers,

$$\lim_{K \to \infty} \frac{A_{K,M,t}}{K} \equiv \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^K I_{(-\infty,y_t]}(\bar{y}_t^{(k)}) = B_{M,t}. \quad (20)$$

**Remark 3.** Note that the complexity of $A_{K,t}$ when $K \to \infty$ grows to infinity as well, while the computation of $B_t$ depends on the number of particles $M$.

V. CONVERGENCE OF THE BOOTSTRAP PARTICLE FILTER WITH ADAPTIVE NUMBER OF PARTICLES

In this section we obtain explicit error rates for a general class of PFs that update the number of particles, $M_n$, at the end of blocks of observations of length $W_n$. Let

$$\kappa_t(dx_t|x_{t-1}) := p(x_t|x_{t-1}) dx_{t-1} \quad (21)$$

denote the Markov kernel that governs the dynamics of the state sequence $\{x_t\}_{t \geq 0}$ and write

$$g_t^e(x_t) := p(y_t|x_t) \quad (22)$$

to indicate the conditional pdf of the observations. While the notation in (22) implies that the kernel $\kappa_t$ has a density w.r.t. the Lebesgue measure, the results to be presented in this section actually hold for a broader class of kernels.

Table III outlines the general algorithm we analyze here. This is a BPF with $M_n$ particles in the $n$th block of observations whose length is $W_n$. The theoretical results we introduce are valid for variable block sizes, $W_n$, as well as for fixed block sizes, $W_n = W$ for every $n$ (i.e., Table III is a particular case of the general algorithm). Our analysis also holds independently of the update rule for $M_n$, as long as only positive values are permitted. Specifically, we assume that there is a positive lower bound $M$ such that $M_n \geq M$ for every $n \geq 0$. In practice, we usually have a finite upper bound $M \geq M_n$ as well (but this plays no role in the analysis).

Intuitively, we aim at proving that the error bounds for the approximate filter $\pi_t^{M_n} = \frac{1}{M_n} \sum_{m=1}^{M_n} \delta_{x_t^{(m)}}$ change as we update the number of particles $M_n$. For a real r.v. $Z$ with probability measure $\alpha$, the $L_p$ norm of $Z$, with $p \geq 1$, is

$$\|Z\|_p := \left( \int |z|^p \alpha(dz) \right)^{1/p}. \quad (23)$$

Since the approximate measures $\pi_t^{M_n}$ produced by the algorithm in Table III are random, the approximation error $(f, \pi_t^{M_n}) - (f, \pi_t)$, where $f$ is some integrable $X \to \mathbb{R}$ function, is a real r.v. and we can assess its $L_p$ norm. In the sequel, we show that by the end of the $n$th block of observations, the upper bound of the $L_p$ error $\|(f, \pi_t^{M_n}) - (f, \pi_t)\|_p$, ...
where \( t_n = \sum_{j=0}^{n} W_j - 1 \) and \( f \) is a bounded real function, depends on \( M_n \), but \textit{not} on the past values \( M_{n-1}, M_{n-2}, \ldots \) or the lower bound \( \underline{M} \). This is true under certain regularity assumptions that we introduce below.

We start with constructing the prediction-update (PU) operators \( \Psi_t \) that produce the sequence of filtering probability measures given a prior measure \( \pi_0 \), the sequence of kernels \( \kappa_t \) and the likelihoods \( g_t^i \).

**Definition 1.** Let \( \mathcal{P}(\mathcal{X}) \) be the set of probability measures on the space \((\mathcal{B}(\mathcal{X}), \mathcal{X})\) and let the sequence of operators \( \Psi_t : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X}) \) satisfy the relationships

\[
(f, \Psi_t(\alpha)) = \frac{(fg_t^i, \kappa_t\alpha)}{(g_t, \kappa_t\alpha)}, \quad t = 1, 2, \ldots
\]

for any \( \alpha \in \mathcal{P}(\mathcal{X}) \).

It is not hard to see that Definition 1 implies that \( \pi_t = \Psi_t(\pi_{t-1}) \). In order to represent the evolution of the sequence of filtering measures over several time steps, we introduce the composition of operators

\[
\Psi_{t|t-r}(\alpha) := (\Psi_t \circ \Psi_{t-1} \circ \cdots \circ \Psi_{t-r+1})(\alpha).
\]

It is apparent that \( \pi_t = \Psi_{t|t-r}(\pi_{t-r}) \). The composition operator \( \Psi_{t|t-r} \) is most useful to represent the filters obtained after \( r \) consecutive steps when we start from different probability measures at time \( t-r \), i.e., to compare \( \Psi_{t|t-r}(\alpha) \) and \( \Psi_{t|t-r}(\beta) \) for \( \alpha, \beta \in \mathcal{P}(\mathcal{X}) \).

For our analysis, we assume that the sequences of probability measures generated by \( \Psi_t, t \geq 1 \), are \textit{stable}. The intuitive meaning is that such sequences “forget” their initial condition over time. This is formalized below.

**Assumption 1.** For every \( f \in \mathcal{B}(\mathcal{X}) \) and every \( r > 0 \) there exists \( \varepsilon(f, r) \) such that

\[
\sup_{\alpha, \beta \in \mathcal{P}(\mathcal{X})} \left| (f, \Psi_{t|t-r}(\alpha)) - (f, \Psi_{t|t-r}(\beta)) \right| \leq \varepsilon(f, r),
\]

\[
\lim_{r \to \infty} \varepsilon(f, r) = 0.
\]

The strongest assumption in our analysis is that the sequence of likelihoods is uniformly bounded away from zero (as well as upper bounded), as specified below.

**Assumption 2.** There exists a positive constant \( G < \infty \) such that

\[
0 < \frac{1}{G} < g_t^i(x) < G < \infty
\]

for every \( t \geq 1 \) and every \( x \in \mathcal{X} \).

Note that Assumption 2 depends not only on the form of the pdf \( g_t^i(x) = p(y_i|x_i) \) but also on the specific sequence of observations \( y_1, y_2, \ldots \). While Assumption 2 may appear restrictive, it is typical in the analysis of uniform convergence of PFs and is expected to hold naturally when the state space \( \mathcal{X} \) is compact.

The main result in this section is stated next.

**Theorem 4.** Let \( t_n = \sum_{j=0}^{n} W_j - 2 \) and let \( \pi_{t_n}^{M_n} \) be the particle approximation of the filtering measure \( \pi_{t_n} \) produced by the block-adaptive BPF in Table 2. If assumptions 1 and 2 hold, then for any \( 0 < \epsilon < \frac{1}{2} \) there exists \( W_n = \mathcal{O}(\epsilon \log(M_n)) \) such that

\[
\left\| (f, \pi_{t_n}^{M_n}) - (f, \pi_{t_n}) \right\| < \frac{c\|f\|_{\infty}}{M_n^2} \epsilon + \bar{\varepsilon}(f, M_n),
\]

where

\[
\lim_{M_n \to \infty} \bar{\varepsilon}(f, M_n) = 0.
\]

for every \( f \in \mathcal{B}(\mathcal{X}) \), every \( n \geq 1 \) and a constant \( c < \infty \) independent of \( \epsilon, t_n \) and \( M_n \).

See Appendix for a proof.

We make a few remarks about Theorem 4:

- The theorem states that \( W_n \) can be chosen in such a way that the “inherited error” due to a low number of particles \( M_{n-1} \) in the \((n-1)\)th block can be forgotten when a larger number \( M_n > M_{n-1} \) is selected in the \(n\)th block (due to the stability property of the PU operator \( \Psi_t \)). In particular, if \( M_n \to \infty \), then \( W_n \to \infty \) and \( \left\| (f, \pi_{t_n}^{M_n}) - (f, \pi_{t_n}) \right\| \to 0 \).
- The “big oh” notation \( W_n = \mathcal{O}(\epsilon \log(M_n)) \) indicates that, for some constant \( C < \infty \),

\[
W_n \leq C \epsilon \log(M_n).
\]
In practice, the variability of the window lengths \(W_n\) can be a drawback (or just an unwanted complication). The inequality (31) can obviously be satisfied as well for a constant \(W_n = W < \infty\). If we choose a constant window length, \(W\), then the second error term in (29) has the form
\[
\varepsilon(f, M_n) = \varepsilon(f, C \epsilon \log(M_n)) \leq \varepsilon(f, W_n) = \varepsilon(f, W).
\] (32)

Hence, we simply need to choose \(W\) sufficiently large to ensure that \(\varepsilon(f, W)\) is small enough (the error will not vanish as \(M_n \to \infty\), however).

- If the sequence of probability measures generated by the PU operators \(\Psi_t\) is exponentially stable \([11]\) and the \(W_n\)'s are suitably chosen, then the inequality (29) reduces to
\[
\left\| \left( f, \pi_{t_n}^{M_n} \right) - \left( f, \pi_{t_n} \right) \right\|_p \leq \frac{c' \|f\|_\infty}{\sqrt{M_n}}
\] (33)
for some constant \(c' < \infty\).

VI. NUMERICAL EXPERIMENTS

In the first experiment, we show the relation between the correlation coefficient of \(A_{K,M,t}\) and the MSE of estimator built by the particle approximation in a non-linear state-space model. Then, we complement the results of III-C, showing numerically some properties of \(A_{K,M,t}\) for different values of \(K\) and \(M\), and their connection to the statistic \(B_{M,t}\). Third, we show numerically the convergence of the block-adaptive BPF.

![Stochastic growth model: MSE, p-value of the Pearson's \(\chi^2\), and Pearson's correlation coefficient \(r\).](image.png)
A. Assessing convergence from the correlation of $A_{K,M,t}$.

Consider the stochastic growth model \cite{19].

\[ x_t = \frac{x_{t-1}}{2} + \frac{25x_{t-1}}{1 + x_{t-1}^2} + 8 \cos(\phi t) + u_t, \]  
\[ y_t = \frac{x_t^2}{20} + v_t, \]

where $\phi = 0.4$, and $u_t$ and $v_t$ are independent Gaussian r.v.’s with zero mean, and variance $\sigma_u^2$ and $\sigma_v^2$, respectively. At this point, we define two models:

- Model 1: $\sigma_u = 1$ and $\sigma_v = 0.5$,
- Model 2: $\sigma_u = 2$ and $\sigma_v = 0.1$.

In this example, we ran the BPF for $T = 5,000$ time steps, always with a fixed number of particles. We tested different values of $M \in \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384\}$. In order to evaluate the behavior of $A_{K,M,t}$, we set $K = 7$ fictitious observations. Figure 1(a) shows the mean squared error (MSE) in the estimate of the posterior mean for each value of $M$, which obviously decreases with $M$. Figure 1(b) displays the p-value of the Pearson’s $\chi^2$ test for assessing the uniformity of $A_{K,M,t}$ (in the domain $A_{K,M,t} \in \{0, ..., K + 1\}$) in windows of $W = 20$ (Algorithm 1 of Section III-A see more details in \cite{3}). Clearly, increasing the number of particles also increases the p-value, i.e., the statistic distribution becomes closer to the uniform distribution. Figure 1(c) is related to Algorithm 2 of Section III-B. We show the sample Pearson’s correlation coefficient $r$, using the whole sequence of statistics $\{a_{K,M,t}\}_{t=1}^{T}$, computed with a lag $\tau = 1$. All results are averaged over 200 independent runs.

We observe that when we increase $M$, the correlation between consecutive statistics decreases. It is interesting to note that the curve of the correlation coefficient $r$ has a very similar shape to the MSE curve. While $r$ can easily be computed, the MSE is always unknown, which shows the utility of Algorithm 2.

It can be seen that both algorithms can identify a malfunctioning of the filter when the number of particles is insufficient. We note that Algorithm 2 works better for Model 1 than for Model 2 because the autocorrelation of the statistics is more sensitive in detecting the malfunctioning for low $M$. However, Algorithm 1 works better for Model 2 because the p-value of the uniformity test is always smaller than in Model 1, i.e., it is more discriminative. Therefore, there is no clear superiority of one algorithm over the other.

B. The three-dimensional Lorenz system

Table III shows results of the Lorenz example described in \cite{3} Section V-A] with fixed number of particles $M \in \{8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384\}$. We show the MSE in the approximation of the posterior mean, averaged over 200 runs. Again $r$ is the sample Pearson’s correlation coefficient, using the whole sequence of statistics $\{a_{K,M,t}\}_{t=1}^{T}$, with a lag $\tau = 1$, and p-val is the p-value of the Pearson’s $\chi^2$ test for assessing the uniformity of the same set. Similar conclusions than in previous example can be obtained.

C. Behavior of $A_{K,M,t}$ and its relation with $B_{M,t}$

In Fig. 2 we show the histograms of $A_{K,M,t}$ and $B_{M,t}$ for the stochastic growth model described in \cite{35}, \cite{35}. We set $K \in \{3, 5, 7, 10, 20, 50, 100, 1000, 5000\}$. The BPF is with fixed $M = 2^{14}$. When $K$ grows, the pmf seems to converge to the pdf of $B_{M,t}$. In Table IV] we show the averaged absolute error (distance) between the realizations of r.v.s $A_{K,M,t}/K$ and $B_{M,t}$ for the stochastic growth model with fixed $M = 2^{14}$. The results are averaged over $T = 100$ time steps in 100 independent runs. It is clear that when $K$ grows, the deviation between both r.v.s, which take values in $(0, 1)$, decreases. Thus, for $K = 5000$, the difference is on average 0.433%.

| $M$ | $\Delta^2$ | $\Delta t$ | $\Delta x$ | $\Delta u$ | $\Delta v$ |
|-----|-------------|-------------|-------------|-------------|-------------|
| 8   | 105.63      | 75.56       | 40.19       | 15.69       | 5.90        |
| 16  | 0.6927      | 0.4939      | 0.2595      | 0.1132      | 0.0463      |
| 32  | 0.0393      | 0.1276      | 0.2923      | 0.4279      | 0.4823      |
| 64  | 0.0156      | 0.5016      | 0.5117      | 0.5106      | 0.4998      |
| 128 | 0.0195      | 0.0195      | 0.0195      | 0.0195      | 0.0195      |
| 256 | 0.0195      | 0.0195      | 0.0195      | 0.0195      | 0.0195      |
| 512 | 0.0195      | 0.0195      | 0.0195      | 0.0195      | 0.0195      |
| 1024| 0.0195      | 0.0195      | 0.0195      | 0.0195      | 0.0195      |
| 2048| 0.0195      | 0.0195      | 0.0195      | 0.0195      | 0.0195      |
| 4096| 0.0195      | 0.0195      | 0.0195      | 0.0195      | 0.0195      |
| 8192| 0.0195      | 0.0195      | 0.0195      | 0.0195      | 0.0195      |
| 16384| 0.0195 | 0.0195 | 0.0195 | 0.0195 | 0.0195 |

TABLE III: Lorenz Model: $\Delta = 10^{-3}$, $T_{obs} = 200\Delta$, $\sigma^2 = 0.5$. Algorithm details: $W = 20$, $K = 7$. MSE in the approximation of the posterior mean, the averaged $\hat{R}(1)$, and the averaged p-value of the Pearson’s chi-square test on the uniformity on $S_t$.

TABLE IV: (Ex. of Section VI-D) Averaged absolute error (distance) between the realizations of the r.v.s $A_{K,M,t}$ and $B_{M,t}$ for the stochastic growth model $M = 2^{14}$. The results are averaged over $T = 100$ time steps in 100 independent runs.
D. Forgetting property in the block-adaptive bootstrap particle filter

In this section we simulate the behavior in the approximation errors when the block-adaptive BPF increases the number of particles. To that end, we run two specific state-space models where in the first half of time steps, the number of particles is set to \( M_1 \) while in the second half, the number of particles is \( M_2 > M_1 \). We then compute the MSE of predicted observations (in the last quarter of the time steps), and we compare it with the case of BPF with \( M_2 \) particles used from the beginning.

Table [V] shows the MSE of a BPF run on the linear Gaussian model described by

\[
x_t = ax_{t-1} + u_t, \quad y_t = x_t + v_t,
\]

with \( T = 1000, \sigma_u = \sqrt{0.5}, \sigma_v = 1, \) and \( a = 0.9 \). We simulate one example with \( M_1 = 100 \) and \( M_2 = 1000 \) (left part of the table), and another with \( M_1 = 1000 \) and \( M_2 = 10000 \) (right part of the table). In both cases, we are able to show that the BPF achieves in the last quarter of time steps (from \( t = 750 \) to \( t = T = 1000 \)) the same MSE as if the biggest number of particles was set at the very beginning.

Table [VI] presents analogous results for the stochastic growth model described in the first experiment, with \( T = 1000, \sigma_x = 1, \sigma_y = 0.1, \) and \( \phi = 0.4 \). Now we simulate the BPF with the following pairs of number of particles \((M_1, M_2) \in \{(50, 1000), (200, 4000), (1000, 20000)\}\). We make similar conclusions.

\[
\begin{array}{c|cccccccc}
M_1 & 100 & 1000 & 1000 & 1000 & 1000 & 10000 & 100000 & 1000000 \\
M_2 & 1000 & 1000 & 1000 & 1000 & 1000 & 1000 & 1000 & 1000 \\
MSE (last \( T/4 \)) & 8.9010^{-6} & 9.0210^{-4} & 8.9910^{-4} & 9.0210^{-4} & 8.9310^{-4} & 8.6910^{-4} & \\
\end{array}
\]

\text{TABLE V: Linear Gaussian Model:} \( T = 1000, \sigma_x = \sqrt{0.5}, \sigma_y = 1, \) \( a = 0.9 \). \( M_1 \) particles for \( t \in \{1, \ldots, \frac{T}{2}\} \) and \( M_2 \) particles for \( t \in \{\frac{T}{2} + 1, \ldots, T\} \).

\[
\begin{array}{c|cccccccc}
M_1 & 50 & 1000 & 500 & 2000 & 20000 & 1000 & 1000 & 1000 \\
M_2 & 1000 & 1000 & 1000 & 10000 & 10000 & 1000 & 1000 & 1000 \\
MSE (last \( T/4 \)) & 1.493 & 1.386 & 1.374 & 1.348 & 1.345 \\
\end{array}
\]

\text{TABLE VI: Stochastic Growth Model:} \( T = 1000, \sigma_x = 1, \sigma_y = 0.1, \phi = 0.4 \). \( M_1 \) particles for \( t \in \{1, \ldots, \frac{T}{2}\} \) and \( M_2 \) particles for \( t \in \{\frac{T}{2} + 1, \ldots, T\} \).
VII. Conclusions

In this paper, we provided new results of PFs whose number of particles vary with time as needed. First, we proposed a method for assessing convergence of a statistic of a r.v. based on the correlation of the statistic, where the r.v. represents the position of an observation within a set of ordered fictitious observations. Then we proposed a novel algorithm based on a new statistics for assessing the quality of the filtering and for adapting the number of particles. Further, we provided several theoretical results about the parameters of the proposed methods. Finally, we showed that the convergence rate of the approximation depends on the current number of particles.

Appendix A

Proof of Lemma 2

First, we express the pmf \( Q_{K,M,t} \) of \( A_{K,M,t} \) as a function of the predictive pdf of the observations. We recall that \( Q_K(n) \) is the probability that exactly \( n \) fictitious observations are smaller than the actual \( y_t^{(1)} \). Hence, \( \forall n \in \{0,...,K\} \),

\[
Q_{Ak}(n) = \int_{-\infty}^{\infty} P(A_K = n|y^{(1)} = z) p_t(z) dz \\
= \int_{-\infty}^{\infty} \left( \frac{K}{n} \right)^n F_t^M(z)^n (1 - F_t^M(z))^{K-n} p_t(z) dz \\
= \left( \frac{K}{n} \right) \int_{-\infty}^{\infty} F_t^M(z)^n \left( \sum_{i=0}^{K-n} \binom{K-n}{i} (-1)^i F_t^M(z)^i \right) \times p_t(z) dz \\
= \left( \frac{K}{n} \right) \sum_{i=0}^{K-n} \int_{-\infty}^{\infty} \left( \binom{K-n}{i} (-1)^i F_t^M(z)^{n+i} p_t(z) dz, \right. \tag{38}
\]

where \( F_t^M(z) \) is the probability of a single fictitious observation of being smaller than \( z \). We want to prove \( 15 \). The case for \( n = K \) is obvious by rewriting \( 38 \) as

\[
Q_K(K) = \int_{-\infty}^{\infty} F_t^M(z)^K p_t(z) dz \equiv ((F_t^M)^K, p_t) = \frac{1}{K+1} \tag{39}
\]

Assume that for a specific \( n \in \{1,...,K\} \), \( 15 \) holds for all \( i \in \{n,...,K\} \). Then the goal is proving by induction that it also holds for \( n - 1 \). Let us write the pmf of \( 38 \) at \( n - 1 \) as

\[
Q_K(n - 1) \\
= \left( \frac{K}{n-1} \right) \sum_{i=0}^{K-n+1} \int_{-\infty}^{\infty} \left( \binom{K-n+1}{i} (-1)^i \times F_t^M(z)^{n+i} p(z) dz, \right. \tag{40}
\]

where in the first step, all the integrals have been replaced (assumption for \( n \) except that corresponding to \( n - 1 \), in the second step we split the series between the terms \( i > 0 \) and the term \( i = 1 \), and in the third step we have substituted the series using the identity of \( 27 \), Eq. 1.41]. Now we use \( 1 \) and rewrite \( 40 \) as

\[
\frac{1}{K+1} = \left( \frac{K}{n-1} \right) \frac{1}{n} \left( \frac{1}{K-n+1} - 1 \right) \\
+ \left( \frac{K}{n-1} \right) \int_{-\infty}^{\infty} F_t^M(z)^{n-1} p(z) dz. \tag{41}
\]
Hence,
\[
\int_{-\infty}^{\infty} F_t^M(z)^{n-1} p(z) dz = \frac{1}{K+1} \left( \frac{1}{K+1} \right) \left( \frac{1}{K+1} \right) \left( \frac{1}{K+1} \right) = \frac{1}{n} \cdot \Box
\]  

\text{(42)}

\section*{Appendix B}
\textbf{Proof of Theorem}\[28\]

\textbf{A. Notation and preliminary results}

It is often convenient to re-write integrals of the form \( (f, \Psi_t)_k(\alpha)) \), where \( f \in B(\mathcal{X}) \) and \( \alpha \in \mathcal{P}(\mathcal{X}) \), in a manner which is easier for certain manipulations. With this aim, we introduce the following notation:

\textbf{Definition 2. For any} \( f \in B(\mathcal{X}) \), \textbf{we define the map} \( \Gamma_{t|t-k} : B(\mathcal{X}) \rightarrow B(\mathcal{X}) \), \( k \geq 0 \), \textbf{recursively as}

\[
\begin{align*}
\Gamma_{t|t}(f) & \triangleq f, \\
\Gamma_{t|t-k}(f) & \triangleq (g_{t-k}^{\nu} \Gamma_{t-k+1}(f), \kappa_{t-k+1}), \quad 1 \leq k \leq t.
\end{align*}
\]

It is not difficult to show that the transformation \( \Gamma_{t|t-k} \) can be used to provide an alternative expression for the composition of maps \( \Psi_t^\beta \circ \cdots \circ \Psi_{t-k+1}^\beta \). In particular,

\[
(f, \Psi_{t|t-k}(\alpha)) = \frac{\Gamma_{t|t-k}(f, \alpha)}{\Gamma_{t|t-k}(1, \alpha)},
\]  

\text{(43)}

for any \( f \in B(\mathcal{X}) \), any \( \alpha \in \mathcal{P}(\mathcal{X}) \) and \( 1(x) = x \) for every \( x \in \mathcal{X} \). Moreover, assuming \( \sup_{t \geq 1} ||g_t^\nu||_{\infty} < G < \infty \), it readily follows from the recursive definition of \( \Gamma_{t|t-k} \) that

\[
\sup_{t \geq k} ||\Gamma_{t|t-k}(f)||_{\infty} \leq G^k ||f||_{\infty} \quad \text{and} \quad \inf_{t \geq k} \Gamma_{t|t-k}(1) \geq G^{-k}.
\]  

\text{(44)}

Let \( \alpha \) and \( \beta \) be probability measures in \( \mathcal{P}(\mathcal{X}) \), any \( f \in B(\mathcal{X}) \) and any \( h \in B(\mathcal{X}) \) such that \( h(x) > 0 \) for every \( x \in \mathcal{X} \). It is straightforward to show that

\[
\frac{(fh, \alpha)}{(h, \alpha)} - \frac{(fh, \beta)}{(h, \beta)} = \frac{(fh, \alpha)}{(h, \alpha)} - \frac{(fh, \beta)}{(h, \beta)} + \frac{(fh, \beta)}{(h, \alpha)} - \frac{(fh, \beta)}{(h, \alpha)} + \frac{(fh, \beta)}{(h, \alpha)} - \frac{(fh, \beta)}{(h, \alpha)} - \frac{(fh, \beta)}{(h, \alpha)} - \frac{(fh, \beta)}{(h, \alpha)},
\]

hence

\[
\left| \frac{(fh, \alpha)}{(h, \alpha)} - \frac{(fh, \beta)}{(h, \beta)} \right| \leq \frac{|(fh, \alpha) - (fh, \beta)|}{(h, \alpha)} + \frac{||f||_{\infty}}{(h, \alpha)} \left| (h, \beta) - (h, \alpha) \right|.
\]  

\text{(45)}

\textbf{B. Proof}

The proof is based on the classical argument in \[28\]. At the last time step of the \( n \)th block of observations, namely \( t_n = \sum_{j=0}^{n} W_j - 1 \), the difference \( (f, \pi_{t_n}^M) - (f, \pi_{t_n}) \) can be expressed as the “telescopic” sum

\[
(f, \pi_{t_n}^M) - (f, \pi_{t_n}) = \sum_{k=0}^{W_n-2} (f, \Psi_{t_n|t_n-k} \left( \pi_{t_n-k}^M \right))
\]

\text{where}

\[
(f, \pi_{t_n}^M) = f, \Psi_{t_n|t_n-k} \left( \pi_{t_n-k}^M \right) + \left( f, \Psi_{t_n|t_n-W_n+1} \left( \pi_{t_n-W_n+1}^M \right) \right) - \left( f, \Psi_{t_n|t_n-W_n+1} \left( \pi_{t_n-W_n+1} \right) \right).
\]  

\text{(46)}
Note that \( \pi_{t_n - W_n + 1}^{M_n} \) is the first approximate measure constructed with \( M_n \) particles (while at time \( t_n - W_n \) we have the particle approximation \( \pi_{t_n - W_n}^{M_n} = \pi_{t_n - W_n - 1}^{M_n} \)). The last term in (46) can be upper bounded using \( A.1 \) namely
\[
\left| \left( f, \Psi_{t_n|t_n - W_n + 1}^{M_n} \left( \pi_{t_n - W_n + 1} \right) \right) - \left( f, \Psi_{t_n|t_n - W_n - 1}^{M_n} \left( \pi_{t_n - W_n + 1} \right) \right) \right| \\
\leq \varepsilon(f, W_n),
\]
(47)
and combining (47) and (46) yields
\[
\left| \left( f, \pi_{t_n - W_n + 1}^{M_n} \right) - \left( f, \pi_{t_n - W_n - 1}^{M_n} \right) \right| \leq \sum_{k=0}^{W_n - 2} \left| \left( f, \Psi_{t_n|t_n - k}^{M_n} \left( \pi_{t_n - k} \right) \right) - \left( f, \Psi_{t_n|t_n - k - 1}^{M_n} \left( \pi_{t_n - k - 1} \right) \right) \right| + \varepsilon(f, W_n).
\]
(48)
The error \( \varepsilon(f, W_n) \) is handled easily by resorting to assumption \( A.1 \). Indeed, any choice of \( W_n \) such that \( \lim_{M_n \to \infty} W_n = \infty \) ensures that \( \lim_{M_n \to \infty} \varepsilon(f, W_n) = 0 \) for any \( f \in B(\mathcal{X}) \), and hence the task is to find suitable upper bounds for the \( W_n \) terms in the summation of (48).

We rewrite the \( k \)th error term in (48) using the map \( \Gamma \) in Definition 2 which yields
\[
\left| \left( f, \Psi_{t_n|t_n - k}^{M_n} \right) - \left( f, \Psi_{t_n|t_n - k - 1}^{M_n} \right) \right| = \frac{\left| \left( (\Gamma_{t_n|t_n - k}(f), \pi_{t_n - k}^{M_n}) - (\Gamma_{t_n|t_n - k}(f), \pi_{t_n - k - 1}^{M_n}) \right) \right|}{\left( (\Gamma_{t_n|t_n - k}(1), \pi_{t_n - k}^{M_n}) \right)}.
\]
(49)
From (49) and the inequality (45) we readily obtain
\[
\left| \left( f, \pi_{t_n - k}^{M_n} \right) - \left( f, \pi_{t_n - k - 1}^{M_n} \right) \right| \leq \frac{\left| \left( (\Gamma_{t_n|t_n - k}(f), \pi_{t_n - k}^{M_n}) - (\Gamma_{t_n|t_n - k}(f), \pi_{t_n - k - 1}^{M_n}) \right) \right|}{\left( (\Gamma_{t_n|t_n - k}(1), \pi_{t_n - k}^{M_n}) \right)}.
\]
(50)
and, via (44) and Minkowski’s inequality, we arrive at
\[
\left| \left( f, \pi_{t_n - k}^{M_n} \right) - \left( f, \pi_{t_n - k - 1}^{M_n} \right) \right| \leq \left| \left( \Gamma_{t_n|t_n - k}(f), \pi_{t_n - k}^{M_n} \right) - \left( \Gamma_{t_n|t_n - k}(f), \pi_{t_n - k - 1}^{M_n} \right) \right| + \left| \left( (\Gamma_{t_n|t_n - k}(1), \pi_{t_n - k}^{M_n}) \right) \right|.
\]
(51)
However, since \( \pi_{t_n - k}^{M_n} \) is a Monte Carlo estimate of \( \pi_{t_n - k}^{M_n} \), hence for any \( h \in B(\mathcal{X}) \) it is straightforward to prove that
\[
\left| \left( h, \pi_{t_n - k}^{M_n} \right) - \left( h, \pi_{t_n - k - 1}^{M_n} \right) \right| \leq \frac{\left| \left( \Gamma_{t_n|t_n - k}(f), \pi_{t_n - k}^{M_n} \right) - \left( \Gamma_{t_n|t_n - k}(f), \pi_{t_n - k - 1}^{M_n} \right) \right|}{\left( (\Gamma_{t_n|t_n - k}(1), \pi_{t_n - k}^{M_n}) \right)}.
\]
(52)
where \( C < \infty \) is a constant independent of \( t_n \) and \( M_n \). If we plug (52) into (51) (with \( h = \Gamma_{t_n|t_n - k}(f) \) for the first term on the rhs and \( h = \Gamma_{t_n|t_n - k}(1) \) for the second one) and recall that \( \left| \Gamma_{t-n}(f) \right| \leq G^k \| f \| \) from (44), then we obtain
\[
\left| \left( f, \pi_{t_n - k}^{M_n} \right) - \left( f, \pi_{t_n - k - 1}^{M_n} \right) \right| \leq 2CG^k \| f \| \frac{M_n}{M_n^2}.
\]
(53)
which is the upper bound for the \( k \)th term in the sum of (48).

At this point, we simply substitute the latter inequality backwards. Indeed, noticing that there are \( W_n \) terms indexed by \( k = 0, \ldots, W_n - 2 \) in the sum of (48), we can replace every term by the bound in (53) and, by way of Minkowski’s inequality, obtain the error bound
\[
\left| \left( f, \pi_{t_n}^{M_n} \right) - \left( f, \pi_{t_n} \right) \right| \leq \frac{2CW_nG^2W_n}{M_n^2} \| f \| + \varepsilon(f, W_n).
\]
(54)
To conclude the proof, choose any \( \varepsilon \in (0, \frac{1}{2}) \) and let the \( n \)th block length \( W_n \) be a function of the number of particles \( M_n \).
In particular, let us choose
\[
W_n = \frac{\varepsilon \log M_n}{1 + 2 \log G},
\]
(55)
which indeed implies $W_n = O(\epsilon \log M_n)$. From (55) we readily have the inequality
\[ \epsilon \log M_n = W_n + 2W_n \log G \geq \log W_n + 2W_n \log G, \]
and, as a consequence, $M_n' \geq W_n G^{2W_n}$, and
\[ \frac{1}{M_n^{\frac{1}{2} - \epsilon}} \geq \frac{W_n G^{2W_n}}{M_n^{\frac{1}{2}}}. \] (56)

Finally, if we combine the inequalities (56) and (54) we obtain
\[ \| (f, \pi_{t_n}) - (f, \pi_{t_n}) \|_p \leq 2C \epsilon + \epsilon (f, M_n), \]
where
\[ \epsilon (f, M_n) := \epsilon \left( f, \frac{\epsilon \log M_n}{1 + 2 \log G} \right). \]

and, therefore, $\lim_{M_n \to \infty} \epsilon (f, M_n) = 0$. □

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