**α-Amenable Hypergroups**

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**Abstract**

Let $K$ denote a locally compact commutative hypergroup, $L^1(K)$ the hypergroup algebra, and $\alpha$ a real-valued hermitian character of $K$. We show that $K$ is $\alpha$-amenable if and only if $L^1(K)$ is $\alpha$-left amenable. We also consider the $\alpha$-amenability of hypergroup joins and polynomial hypergroups in several variables as well as a single variable.

**Keywords.** $\alpha$-Amenable hypergroups; Koornwinder, associated Legendre, Pollaczek, and disc polynomials

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**Introduction.** Let $K$ denote a locally compact commutative hypergroup, $L^1(K)$ the hypergroup algebra, and $\alpha$ a hermitian character of $K$. It is shown in [7] that $K$ is $\alpha$-amenable if and only if either $K$ satisfies the modified Reiter's condition of $P_1$-type in $\alpha$ or the maximal ideal in $L^1(K)$ generated by $\alpha$ has a bounded approximate identity. For instance, $K$ is always $(1-\alpha)$-amenable, and if $K$ is compact or $L^1(K)$ is amenable, then $K$ is $\alpha$-amenable in every character $\alpha$. It is worth noting, however, that there do exist hypergroups which are not $\alpha(\neq 1)$-amenable; e.g. see [7, 15]. So, the amenability of a hypergroup in a character $\alpha$ cannot in general imply its amenability in other characters even if $\alpha$ is integrable, as illustrated in Section 2. In fact, this kind of amenability of hypergroups depends heavily on the asymptotic behavior of characters as well as Haar measures, as demonstrated in this paper and [2, 3, 7].

The paper is devoted to the character amenability of hypergroups. Sections 1 and 2 contain our main results. First we show that if the character $\alpha$ is real-valued, then $K$ is $\alpha$-amenable if and only if $L^1(K)$ is $\alpha$-left amenable; see Theorem 1.1. We then (Theorem 1.3) consider the $\alpha$-amenability of hypergroup joins. Section 2 is restricted to the polynomial hypergroups. Theorem 2.1 provides a necessary condition for the $\alpha$-amenability of hypergroups; and, subsequently we use this result to examine the $\alpha$-amenability of various polynomial hypergroups. In fact, we show that the majority of common examples of polynomial hypergroups are only 1-amenable, and Example (VI) illustrates just how complicated hypergroups can be.
Parts of this paper are taken from author’s dissertation at Technische Universität München.

**Preliminaries.** Let \((K, p, \sim)\) denote a locally compact commutative hypergroup with Jewett’s axioms [10], where \(p : K \times K \to M^1(K)\), \((x, y) \mapsto p(x, y)\), and \(\sim : K \to K\), \(x \mapsto \bar{x}\), specify the convolution and involution on \(K\) and \(p(x, y) = p(y, x)\) for every \(x, y \in K\). Here \(M^1(K)\) denotes the set of all probability measures on \(K\).

Let us first recall required notions here, which are mainly from [4, 10]. Let \(C_c(K)\), \(C_0(K)\), and \(C^b(K)\) be the spaces of all continuous functions, those which have compact support, vanishing at infinity, and bounded on \(K\), respectively. Both \(C^b(K)\) and \(C_0(K)\) will be topologized by the uniform norm \(\|\cdot\|_\infty\), and by Riesz’s theorem \(C_0(K)^* \cong M(K)\), the space of all complex regular Radon measures on \(K\). The translation of \(f \in C_c(K)\) at the point \(x \in K\), \(T_x f\), is defined by \(T_x f(y) = \int_K f(t) dp(x, y)(t)\), for every \(y \in K\).

Let \(m\) denote the unique Haar measure of \(K\) [16] and \((L^p(K), \|\cdot\|_p)\) \((p \geq 1)\) the usual Banach space. If \(p = 1\), \((L^1(K), \|\cdot\|_1)\) is a Banach \(*\)-algebra where the convolution and involution of \(f, g \in L^1(K)\) are given by \(f \ast g(x) = \int_K f(y) T_y g(x) dm(y)\) \((m\text{-a.e.})\) and \(f^*(x) = \overline{f(x)}\) respectively. If \(K\) is discrete, then \(L^1(K)\) has an identity element; otherwise \(L^1(K)\) has a bounded approximate identity \((b. a. i.)\), i.e. there exists a net \(\{e_i\}_i\) of functions in \(L^1(K)\) with \(\|e_i\|_1 \leq M, \ M > 0\), such that \(\|f \ast e_i - f\|_1 \to 0\) as \(i \to \infty\).

The dual of \(L^1(K)\) can be identified with the usual Banach space \(L^\infty(K)\), and its structure space is homeomorphic to the character space of \(K\), i.e.

\[\mathcal{X}^b(K) := \left\{\alpha \in C^b(K) : \alpha(e) = 1, \ p(x, y)(\alpha) = \alpha(x)\alpha(y), \ \forall x, y \in K\right\}\]

equipped with the compact-open topology. \(\mathcal{X}^b(K)\) is a locally compact Hausdorff space. Let \(\widehat{K}\) denote the set of all hermitian characters \(\alpha\) in \(\mathcal{X}^b(K)\), i.e. \(\alpha(\bar{x}) = \overline{\alpha(x)}\) for every \(x \in K\), with a Plancherel measure \(\pi\). In contrast to the case of groups, \(\widehat{K}\) might not have the dual hypergroup structure and might properly contain \(\mathcal{X} = \text{supp } \pi\).

The Fourier-Stieltjes transform of \(\mu \in M(K)\), \(\widehat{\mu} \in C^b(\widehat{K})\), is given by \(\widehat{\mu}(\alpha) := \int_K \overline{\alpha(x)} d\mu(x)\).

Its restriction to \(L^1(K)\) is called the Fourier transform. We have \(\widehat{f} \in C_0(\widehat{K})\), for \(f \in L^1(K)\), and \(I(\alpha) := \{f \in L^1(K) : \widehat{f}(\alpha) = 0\}\) is the maximal ideal in \(L^1(K)\) generated by \(\alpha\) [5].

\(K\) is called \(\alpha\)-amenable \((\alpha \in \widehat{K})\) if there exists \(m_{\alpha} \in L^\infty(K)^*\) such that (i) \(m_{\alpha}(\alpha) = 1\) and (ii) \(m_{\alpha}(T_x f) = \alpha(x)m_{\alpha}(f)\) for every \(f \in L^\infty(K)\) and \(x \in K\). \(K\) is called amenable if the latter holds for \(\alpha = 1\).

For the sake of completeness, we recall the modified Reiter’s condition of \(P_1\)-type in \(\alpha \in \widehat{K}\) from [7] which is required in Theorem 1.2. By this condition we shall mean for every \(\epsilon > 0\) and every compact subset \(C\) of \(K\) there exists \(g \in L^1(K)\) with \(\|g\|_1 \leq M (M > 0)\) such that \(\widehat{g}(\alpha) = 1\) and \(\|T_x g - \alpha(x)g\|_1 < \epsilon\) for all \(x \in C\). The condition is simply called Reiter’s condition if \(\alpha = 1\) [15].
1 $\alpha$-Left Amenability of $L^1(K)$

Let $X$ be a Banach $L^1(K)$-bimodule and $\alpha \in \hat{K}$. Then, in a canonical way, the dual space $X^*$ is a Banach $L^1(K)$-bimodule. The module $X$ is called a $\alpha$-left $L^1(K)$-module if the left module multiplication is given by $f \cdot x = \tilde{f}(\alpha)x$, for every $f \in L^1(K)$ and $x \in X$. In this case, $X^*$ turns out to be a $\alpha$-right $L^1(K)$-bimodule as well, i.e. $\varphi \cdot f = \tilde{f}(\alpha)\varphi$, for every $f \in L^1(K)$ and $\varphi \in X^*$.

A continuous linear map $D : L^1(K) \to X^*$ is called a derivation if $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$, for every $f, g \in L^1(K)$, and an inner derivation if $D(f) = f \cdot \varphi - \varphi \cdot f$, for some $\varphi \in X^*$. The algebra $L^1(K)$ is called $\alpha$-left amenable if for every $\alpha$-left $L^1(K)$-module $X$, every continuous derivation $D : L^1(K) \to X^*$ is inner; and, if the latter holds for every Banach $L^1(K)$-bimodule $X$, then $L^1(K)$ is called amenable.

As shown in [7], $K$ is $\alpha$-amenable if and only if either $I(\alpha)$ has a b.a.i. or $K$ satisfies the modified Reiter’s condition of $P_1$-type in $\alpha$. In the following theorem we explore the connection between the $\alpha$-amenability of $K$ and $\alpha$-left amenability of $L^1(K)$.

**Theorem 1.1.** Let $K$ be a hypergroup and $\alpha \in \hat{K}$, real-valued. Then $K$ is $\alpha$-amenable if and only if $L^1(K)$ is $\alpha$-left amenable.

**Proof.** Assume $K$ to be $\alpha$-amenable, choose $X$ to be an arbitrary $\alpha$-left $L^1(K)$-module, and suppose that $D : L^1(K) \to X^*$ is a continuous derivation. For fixed $x \in X$ define $\Phi_x \in L^1(K)^*$ by $\Phi_x(f) = D(f)(x)$ for $f \in L^1(K)$. Then for every $f, g \in L^1(K)$

$$\Phi_x(f \cdot g) = D(f \cdot g)(x) = (f \cdot D(g))(x) + (D(f) \cdot g)(x)$$

$$= D(g)(x \cdot f) + \tilde{g}(\alpha)D(f)(x)$$

$$= \Phi_x(f) + \tilde{g}(\alpha)\Phi_x(g).$$

(1)

Moreover, $\Phi_{x+y} = \Phi_x + \Phi_y$, $\Phi_{\lambda \cdot x} = \lambda \Phi_x$, and $\|\Phi_x\| \leq \|D\alpha\| \|x\|$ for $x, y \in X$ and $\lambda \in \mathbb{C}$.

We identify $\Phi \in L^1(K)^*$ with $\eta \in L^\infty(K)$ by the relation

$$\Phi(g) := \int_K g(t)\eta(t)dm(t)$$

for $g \in L^1(K)$. Denote by $\eta_x$ and $\eta_{x \cdot f}$ the elements of $L^\infty(K)$ corresponding to $\Phi_x$ and $\Phi_{x \cdot f}$, respectively. Thus, $\eta_{x+y} = \eta_x + \eta_y$, $\eta_{\lambda \cdot x} = \lambda \eta_x$, and $\|\eta_x\|_\infty \leq \|D\alpha\| \|x\|$ for $x, y \in X$ and $\lambda \in \mathbb{C}$.

Now (1) can be rewritten as

$$\int_K f \cdot g(t)\eta_x(t)dm(t) = \int_K g(t)\eta_{x \cdot f}(t)dm(t) + \tilde{g}(\alpha)\int_K f(t)\eta_x(t)dm(t)$$

(2)
Applying Fubini’s theorem and [4, Thm.1.3.21] yield

\[
\int_K f * g(t) \eta_x(t) dm(t) = \int_K \left( \int_K f(y) T_y g(t) \eta(y) dm(y) \right) \eta_x(t) dm(t)
\]

\[
= \int_K \left( \int_K g(t) T_y \eta_x(t) dm(t) \right) f(y) dm(y)
\]

\[
= \int_K \left( \int_K f(y) T_y \eta_x(t) dm(y) \right) g(t) dm(t)
\]

and

\[
\int_K g(t) \eta_x(t) dm(t) = \int_K \left( \int_K f(y) T_y \eta_x(t) dm(y) \right) g(t) dm(t)
\]

\[
- \int_K g(t) \left( \alpha(t) \int_K f(y) \eta_x(y) dm(y) \right) dm(t).
\]  

(3)

Now, by assumption there exists \( m_\alpha \in L^\infty(K)^* \) such that \( m_\alpha(\alpha) = 1 \) and \( m_\alpha(T_y \eta) = \alpha(y) m_\alpha(\eta) \)
for every \( \eta \in L^\infty(K) \) and \( y \in K \). Let

\[
\varphi(x) := m_\alpha(\eta_x), \quad x \in X.
\]

Then \( \varphi(x + y) = \varphi(x) + \varphi(y) \), \( \varphi(\lambda x) = \lambda \varphi(x) \) and \( |\varphi(x)| \leq \|m_\alpha\| \|D\| \|x\| \). Hence \( \varphi \in X^* \), and for \( f \in L^1(K) \) and \( x \in X \) it follows that

\[
f \cdot \varphi(x) = \varphi(x \cdot f) = m_\alpha(\eta_{x,f}).
\]

By Goldstein’s theorem [6], the functional \( m_\alpha \) is the \( w^* \)-limit of a net of functions \( g \in L^1(K) \), therefore from (3) we obtain that

\[
m_\alpha(\eta_{x,f}) = \int_K f(y) m_\alpha(T_y \eta_x) dm(y) - m_\alpha(\alpha) \Phi_x(f),
\]

and hence

\[
\Phi_x(f) = \hat{\alpha}(\alpha) m_\alpha(\eta_x) - m_\alpha(\eta_{x,f}) \quad \text{for} \ f \in L^1(K), x \in X.
\]

That means \( D(f)(x) = \varphi \cdot f(x) - f \cdot \varphi(x) \), hence \( D \) is an inner derivation, and this gives the \( \alpha \)-left amenability of \( L^1(K) \).

To prove the converse of the theorem we follow the method in [5, p.239]. Assume \( L^1(K) \) to be \( \alpha \)-left amenable and consider \( \alpha \)-left \( L^1(K) \)-module \( L^\infty(K) \) with the module multiplications \( f \cdot \varphi = \hat{f}(\alpha) \varphi \) and \( \varphi \cdot f = \hat{f} \varphi \), for every \( f \in L^1(K) \) and \( \varphi \in L^\infty(K) \). Since \( \alpha \cdot f = \hat{f} \alpha = \hat{\alpha} \alpha \), \( \mathbb{C} \alpha \) is a closed \( L^1(K) \)-submodule of \( L^\infty(K) \). Hence, \( L^\infty(K) = X \oplus \mathbb{C} \alpha \) where \( X \) is
also a closed $L^1(K)$-submodule of $L^\infty(K)$. Choose $\nu \in L^\infty(K)^*$ such that $\nu(\alpha) = 1$, and define $\delta : L^1(K) \to L^\infty(K)^*$, $\delta(f) = f \cdot \nu - \nu \cdot f$, for $f \in L^1(K)$. Then

$$\delta(f)(\alpha) = f \cdot \nu(\alpha) - \nu \cdot f(\alpha)$$

$$= \nu(\alpha \cdot f) - \nu(f \cdot \alpha)$$

$$= \nu(f \ast \alpha) - \hat{f}(\alpha) \nu(\alpha)$$

$$= \hat{f}(\alpha) \nu(\alpha) - \hat{f}(\alpha) \nu(\alpha) = 0, \quad \text{for every } f \in L^1(K).$$

That means $\delta(f) \in (C\alpha)^\perp \subset L^\infty(K)^*$. Let $P : L^\infty(K) \to X$ denote the projection onto $X$ and $P^* : X^* \to L^\infty(K)^*$ the adjoint operator. $P^*$ is an injective $L^1(K)$-bimodule homomorphism; it follows that $(C\alpha)^\perp = (KerP)^\perp = (P^*(X^*))^\perp = P^*(X^*)$. Hence, for each $f \in L^1(K)$ there exists $D(f) \in X^*$ such that $P^*D(f) = \delta(f)$. Since $\delta$ is a continuous derivation on $L^1(K)$, the map $D : L^1(K) \to X^*$ is a continuous derivation as well. By assumption $D$ is inner, that is, there exists $\psi \in X^*$ such that $D(f) = f \cdot \psi - \psi \cdot f$ for all $f \in L^1(K)$. Define $m_\alpha := \nu - P^* \psi$. Then

$$m_\alpha(\alpha) = \nu(\alpha) - P^* \psi(\alpha) = 1 - \psi(P\alpha) = 1$$

and

$$f \cdot (P^* \psi)(\varphi) = P^* \psi(\varphi \cdot f) = \psi(P(\varphi \cdot f)) = f \cdot \psi(P \varphi) = P^*(f \cdot \psi)(\varphi) \quad (\varphi \in X).$$

Similarly $(P^* \psi) \cdot f(\varphi) = P^*(\psi \cdot f)(\varphi)$, thus

$$f \cdot P^* \psi - P^* \psi \cdot f = P^*(f \cdot \psi - \psi \cdot f)$$

$$= P^*Df = \delta(f) = f \cdot \nu - \nu \cdot f.$$

Hence, $f \cdot m_\alpha = f \cdot \nu - f \cdot P^* \nu = \nu \cdot f - P^* \psi \cdot f = m_\alpha \cdot f$. This means

$$m_\alpha(\varphi \ast f) = m_\alpha(\varphi \cdot f) = f \cdot m_\alpha(\varphi) = m_\alpha \cdot f(\varphi) = m_\alpha(f \cdot \varphi) = \hat{f}(\alpha)m_\alpha(\varphi),$$

and hence

$$m_\alpha(T_x \varphi) = m_\alpha(\delta_x \cdot \varphi) = \alpha(x)m_\alpha(\varphi) = \alpha(x)m_\alpha(\varphi),$$

for every $\varphi \in L^\infty(K)$ and $x \in K$, giving the $\alpha$-amenability of $K$.

\[\Box\]

The previous theorem combined with [7] yield Johnson-Reiter’s condition for hypergroups, in the $\alpha$-setting, which reads as follows.

**Theorem 1.2.** Let $K$ be a hypergroup and $\alpha \in \hat{K}$, real-valued. Then the following statements are equivalent:

(i) $K$ is $\alpha$-amenable.
(ii) $L^1(K)$ is $\alpha$-left amenable.

(iii) $I(\alpha)$ has a b.a.i.

(iv) $K$ satisfies the modified $P_1$-condition in $\alpha$.

**Corollary 1.2.1.** If $K$ is $\alpha$-amenable, then every functional $D : L^1(K) \to \mathbb{C}$ such that $D(f \star g) = \hat{f}(\alpha)D(g) + \hat{g}(\alpha)D(f)$, $f, g \in L^1(K)$, is zero (see [2, 5.2]). The converse, however, is in general not true; see Example (II) or [2, 5.5]. The functional $D$ is called a $\alpha$-derivation.

**Remark 1.2.1.** (i) If $\alpha \in L^1(K) \cap L^2(K)$, then

$$m_\alpha(f) := \frac{1}{\|\alpha\|_2^2} \int_K f(x)\overline{\alpha(x)} dm(x), \quad f \in L^\infty(K),$$

is a $\alpha$-mean on $L^\infty(K)$. For example, if $K$ is a hypergroup of compact type [8], the functional $m_\alpha$ is an $\alpha$-mean on $L^\infty(K)$ for every $\alpha \in \hat{K} \setminus \{1\}$; this holds also for $\alpha = 1$ if $K$ is compact. We note that the $\alpha$-means $m_\alpha$, given as above, are unique [3].

(ii) Observe that $\hat{K}$ might contain some positive characters $\alpha \neq 1$ in which case $K$ is $\alpha$-amenable; see Example (VI).

Our next topic is about the $\alpha$-amenability of hypergroup joins, and Theorem 1.3 generalizes [15, 3.12] to the $\alpha$-setting. For the sake of convenience, we first recall the definition of hypergroup joins and some known facts about their dual spaces. Let $(H, \ast)$ be a compact hypergroup with a normalized Haar measure $m_H$, $(J, \cdot)$ a discrete hypergroup with a Haar measure $m_J$, and suppose that $H \cap J = \{e\}$, where $e$ is the identity of both hypergroups. The hypergroup joins $(H \vee J, \circ)$ is the set $H \cup J$ with the unique topology for which $H$ and $J$ are closed subspace of, where the convolution $\circ$ is defined as follows:

1. $\varepsilon_x \circ \varepsilon_y$ agrees with that on $H$ if $x, y \in H$,

2. $\varepsilon_x \circ \varepsilon_y = \varepsilon_x \cdot \varepsilon_y$ if $x, y \in J, x \neq \tilde{y}$,

3. $\varepsilon_x \circ \varepsilon_y = \varepsilon_y \circ \varepsilon_x$ if $x \in H, y \in J \setminus \{e\}$, and

4. if $y \in J$ and $y \neq e$,

$$\varepsilon_y \circ \varepsilon_y = c_e m_H + \sum_{w \in J \setminus \{e\}} c_w \varepsilon_w$$

where $\varepsilon_y \cdot \varepsilon_y = \sum_{w \in J} c_w \varepsilon_w$, $c_w \geq 0$, only finitely many $c_w$ are nonzero, and $\sum_{w \in J} c_w \varepsilon_w = 1$. 
Theorem 1.3. Let $K$ be as above, $|J| \geq 2$, and $\alpha \in \hat{J}$. Then $J$ is $\alpha$-amenable if and only if $K$ is $\alpha$-amenable. Moreover, if $H$ and $J$ are strong hypergroups, then $\hat{H}$ is $\beta$-amenable if and only if $\hat{K}$ is $\beta$-amenable ($\beta \in \hat{H}$).

Proof. Let $x \in J^* := J \setminus \{e\}$. By [15, 3.15] for $f \in L^\infty(K)$, we have

$$T_x f = T_x(f|_{J^*}) + T_x(1_H) \int_H f(t)dm_H(t). \quad (4)$$

Now, take $\alpha \in \hat{J}$ and assume $J$ to be $\alpha$-amenable. Then there exists $m_\alpha : \ell^\infty(J) \to \mathbb{C}$ such that $m_\alpha(\alpha) = 1$ and $m_\alpha(T_x f) = \alpha(x)m_\alpha(f)$, for all $f \in \ell^\infty(J)$ and $x \in J$. The character $\alpha$ can be extended to $K$ by letting $\gamma(x) := 1$ for all $x \in H$. Define

$$M_\gamma : L^\infty(K) \to \mathbb{C}, \quad M_\gamma(f) := m_\alpha(f|_{J^*}), \quad f \in L^\infty(K).$$

We have $M_\gamma(\gamma) = m_\alpha(\gamma|_{J^*}) = m_\alpha(\alpha) = 1$, and (4) implies that

$$M_\gamma(T_x f) = M_\gamma(T_x(f|_{J^*})) + M_\gamma(T_x(1_H)) \int_H f(t)dm_H(t)$$

$$= m_\alpha(T_x f|_{J^*}) = \alpha(x)m_\alpha(f|_{J^*}) = \gamma(x)M_\gamma(f), \quad \text{for all } f \in L^\infty(K), x \in J^*. \quad (5)$$

Since $\gamma|_H = 1$ and $(T_x f)|_{J^*} = f|_{J^*}$ for $x \in H$, the equality (5) is valid for all $x \in K$. Therefore, $K$ is $\gamma$-amenable.

To prove the converse, let $\gamma \in \hat{K}$ and assume $K$ to be $\gamma$-amenable. If $\gamma|_H = 1$, then by $\hat{K} = \hat{H} \cup \hat{J}$ we have $\gamma \in \hat{J}$. Define

$$m_\gamma : \ell^\infty(J) \to \mathbb{C}, \quad m_\gamma(f) := M_\gamma(f), \quad f \in L^\infty(K),$$

where $M_\gamma$ is a $\gamma$-mean on $L^\infty(K)$. Obviously $m_\gamma(\gamma) = 1$, and

$$m_\gamma(T_x f) = M_\gamma(T_x f) = M_\gamma(T_x(f|_{J^*})) = \gamma(x)M_\gamma(f) = \gamma(x)m_\gamma(f),$$

for all $f \in \ell^\infty(J)$ and $x \in J^*$. If $\gamma(x) \neq 1$ for some $x \in H$, then $\gamma$ is a nontrivial character of $H$ and $\hat{K} = \hat{H} \cup \hat{J}$ implies that $\gamma|_{J^*} = 1$. Since $J$ is commutative, $J$ must be amenable [15], and the amenability of $H$ in $\gamma$ follows from Remark 1.2.1.

The proof of the second part can be obtained from the first part and the fact that $\hat{K} \cong \hat{H} \vee \hat{J}$ if $H$ and $J$ are strong hypergroups.


\section{\(\alpha\)-Amenability of polynomial hypergroups}

In this section we restrict our discussion to the polynomial hypergroups. First we consider polynomial hypergroups in several variables which have been already studied by several authors (e.g. see \cite{12, 24}). The translation operators of these hypergroups seem to be complicated, and the study of their character amenability via the modified Reiter’s condition, in contrast to the one variable case \cite{7}, may require sophisticated calculations. In Theorem 2.1, however, we provide a necessary condition to the \(\alpha\)-amenability of these hypergroups. Hence we point out that the majority of common examples of polynomial hypergroups do not satisfy this condition.

Let \(\{P_n\}_{n\in\mathcal{H}}\) be a set of orthogonal polynomials on \(\mathbb{C}^d\) with respect to a measure \(\pi \in M^1(\mathbb{C}^d)\) such that \(P_n(u) = 1\) for some \(u \in \mathbb{C}^d\), where \(\mathcal{H} := \mathbb{N}_0^m\) with the discrete topology, \(m, d \in \mathbb{N}\), and \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\). Assume \(\mathcal{P}_n\) denotes the set of all polynomials \(P' \in \mathbb{C}[z_1, z_2, \ldots, z_d]\) with degree less or equal than \(n\) and \(\mathcal{H}_n := \{n \in \mathcal{H} : P_n \in \mathcal{P}_n\}\). Suppose that for every \(n \in \mathbb{N}\) the set \(\{P_n : n \in \mathcal{H}_n\}\) is a basis of \(\mathcal{P}_n\), and for every \(n, m \in \mathcal{H}\) the product \(P_n \cdot P_m\) admits the unique non-negative linearization formula, i.e.

\[ P_n \cdot P_m := \sum_{t \in \mathcal{H}} g(n, m, t)P_t \]  

(6)

where \(g(n, m, t) \geq 0\). Assume further that there exists a homeomorphism \(n \rightarrow \hat{n}\) on \(\mathcal{H}\) such that \(P_n = P_{\hat{n}}\) for every \(n \in \mathcal{H}\). In this case \(\mathcal{H}\) with the convolution of two point measures defined by \(\varepsilon_n * \varepsilon_m(\varepsilon_t) := p(n, m)(\varepsilon_t) := g(n, m, t)\) is a hypergroup which is called a polynomial hypergroup in \(d\) variables. The hypergroup \(\mathcal{H}\) is obviously commutative and the identity element \(e\) is the constant polynomial \(P_0 \equiv 1\). The character space \(\widehat{\mathcal{H}}\) can be identified with the set \(\{x \in \mathbb{C}^d : |\alpha_x(n)| \leq 1, \alpha_x(\hat{n}) = \overline{\alpha_x(n)} \forall n \in \mathcal{H}\}\), where \(\alpha_x(n) := P_n(x)\) for \(x \in \mathbb{C}^d\) and \(n \in \mathcal{H}\). For more on polynomial hypergroups in several variables we refer the reader to e.g. \cite{4, 12, 24}.

\textbf{Theorem 2.1.} Let \(\{P_n(x)\}_{n\in\mathcal{H}}\) define a polynomial hypergroup in \(d\) variables on \(\mathcal{H} := \mathbb{N}_0^m\) and \(\alpha_x \in \widehat{\mathcal{H}}\) with \(\pi(\{\alpha_x\}) = 0\). If \(\alpha_x \in C_0(\mathcal{H})\), then \(\mathcal{H}\) is not \(\alpha_x\)-amenable.

\textbf{Proof.} Assume to the contrary that \(\mathcal{H}\) is \(\alpha_x\)-amenable and \(m_{\alpha_x}\) is a \(\alpha_x\)-mean on \(\ell^\infty(\mathcal{H})\). Due to

\[ T_n \varepsilon_0(m) = \sum_{t \in \mathcal{H}} \varepsilon_0(t)p(n, m)(t) = p(n, m)(0)\varepsilon_0(m) = \frac{1}{h(n)}\varepsilon_\hat{n}(m), \]

we have \(T_n \varepsilon_0 = \frac{1}{h(n)}\varepsilon_\hat{n}\) for every \(n \in \mathcal{H}\). Therefore,

\[ m_{\alpha_x}(\varepsilon_\hat{n}) = h(n)m_{\alpha_x}(T_n \varepsilon_0) = h(n)\alpha_x(n)m_{\alpha_x}(\varepsilon_0). \]  

(7)

8
Let $M > 0$ be a bound for $m_{\alpha_x}$ and $\xi_n = \frac{P_n(x)}{|P_n(x)|}$ for $P_n(x) \neq 0$. Then by the linearity of $m_{\alpha_x}$ and (7) we have

$$M \geq |m_{\alpha_x} \left( \sum_{n \in M} \xi_n \epsilon_n \right)| = \left| \sum_{n \in M} \xi_n m_{\alpha_x} (\epsilon_n) \right| = \left| \sum_{n \in M} |P_n(x)| h(n) m_{\alpha_x} (\epsilon_0) \right| \geq \sum_{n \in M} |P_n(x)|^2 h(n) |m_{\alpha_x} (\epsilon_0)|,$$

where $M$ is an arbitrary finite subset of $\mathcal{K}$. If $m_{\alpha_x} (\epsilon_0) \neq 0$, then the previous inequalities show that $\alpha_x \in \ell^1(\mathcal{K}) \cap \ell^2(\mathcal{K})$, hence $\pi(\alpha_x) > 0$ (see [4, Proposition 2.5.1]) which is a contradiction. If we now define $\{\alpha^m_x\}_{m \in \mathcal{H}}$ by

$$\alpha^m_x(n) := \begin{cases} 0 & n_i < m_i \ (1 \leq i \leq d), \\ \alpha_x(n) & \text{other}, \end{cases}$$

then $\alpha_x(n) = (P_n(x))_{n \in \mathcal{H}}$ can be written as follows

$$\alpha_x = \sum_{0 \leq t \leq m_i} \epsilon_t P_t(x) + \alpha^m_x.$$ 

Hence,

$$m_{\alpha_x}(\alpha_x) = \sum_{0 \leq t \leq m_i} m_{\alpha_x}(\epsilon_t) P_t(x) + m_{\alpha_x}(\alpha^m_x)$$

which implies that

$$|m_{\alpha_x}(\alpha_x)| = |m_{\alpha_x}(\alpha^m_x)| \leq M\|\alpha^m_x\|.$$ 

The latter shows that if $\alpha_x \in C_0(\mathcal{H})$, then $\alpha^m_x \in C_0(\mathcal{H})$ for all $m \in \mathcal{H}$, hence $m_{\alpha_x}(\alpha_x) = 0$ which is a contradiction.

\[\square\]

**Remark 2.1.1.**

1. Observe that in the preceding theorem neither of the assumptions $\pi(\{\alpha_x\}) = 0$ nor $\alpha_x \in C_0(\mathcal{H})$ can be omitted. For example, a hypergroup of compact type is $\alpha$-amenable in every character $\alpha$ while 1 is the only character in $\mathcal{K}$ with the vanishing Plancherel measure \[8, 15\]; see also Example (VI).

2. Theorem 2.1 is known for $m = d = 1$ in [7].

We continue the section by examining the $\alpha$-amenability of various polynomial hypergroups. Let us first start with polynomial hypergroups in two variables which have been extensively studied by T. H. Koornwinder in [12].
(1) **Koornwinder Class V hypergroups:** In this case \( \mathcal{H} := \{(n,k) \in \mathbb{N}^2_0 : n \geq k \} \) and the characters are given by

\[
P_n(x,y) := p_{(n,k)}^{\alpha,\beta,\gamma,\eta}(x,y) := P_n^{(\alpha,\beta)}(x)P_{n-k}^{(\gamma,\eta)}(y), \quad n = (n,k),
\]

where \( P_n^{(\alpha,\beta)} \) denote the Jacobi polynomials, \((\alpha,\beta), (\gamma,\eta) \in V, P_n^{(\alpha,\beta,\gamma,\eta)}(1,1) = 1, \) and

\[
V := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha \geq \beta > -1, (\alpha + \beta + 1)(\alpha + \beta + 4)(\alpha + \beta + 6)
\geq (\alpha - \beta)^2 \cdot (\alpha^2 - 2\alpha\beta + \beta^2 - 5\alpha - 5\beta - 30) \}.
\]

The support of the Plancherel measure \( d\pi(x,y) = (1-x)^{\alpha}(1+x)^{\beta}(1-y)^{\gamma}(1+y)^{\eta} \) \( dx \, dy \)

is \( D := \{(x,y)| -1 \leq x \leq 1, -1 \leq y \leq 1 \} \).

Since \( |p_n^{(\alpha,\beta)}(y)| = c(n^{\alpha - \frac{1}{2}}) \) as \( n \to \infty \) [9], we have

\[
|p_n^{(\alpha,\beta,\gamma,\eta)}(x,y)| = |p_n^{(\gamma,\eta)}(y)| \rightarrow 0 \quad (n \to \infty)
\]

when \( (x,y) \in [-1,1] \times (-1,1) \) and \( \alpha, \eta > -\frac{1}{2} \). So, from Theorem 2.1 it follows that \( \mathcal{H} \) is not \( \alpha_{(x,y)} \)-amenable.

For \( (x,y) \in \{(1,1),(1,-1)(-1,1)\} \), if \( \alpha > \beta \) and \( \gamma > \eta \), \( \alpha = \beta \) and \( \gamma > \eta \), or \( \alpha > \beta \) and \( \gamma = \eta \) since

\[
P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{n+\beta}{n} / \binom{n+\alpha}{n},
\]

we have \( |p_{2n}^{(\alpha,\beta,\gamma,\eta)}(x,y)| \rightarrow 0 \) as \( n \to \infty \), hence \( \mathcal{H} \) is not \( \alpha_{(x,y)} \)-amenable.

The hypergroup \( \mathcal{H} \) is, in fact, the product of two Jacobi polynomial hypergroups with parameters \((\alpha,\beta)\) and \((\gamma,\eta)\) on \( \mathbb{N}^2_0 \) [24]. Theorem 2.1 combined with [23] implies that \( \ell^1(\mathbb{N}_0) \) is amenable if and only if \( \alpha = \beta = \gamma = \eta = -\frac{1}{2} \). Thus, since \( \ell^1(\mathcal{H}) \cong \ell^1(\mathbb{N}_0) \otimes_p \ell^1(\mathbb{N}_0) \), the algebra \( \ell^1(\mathcal{H}) \) is amenable and its maximal ideals have b.a.i.; see [5, 11]. Consequently, Theorem 1.2 results in the \( \alpha_{(x,y)} \)-amenability of \( \mathcal{H} \) for \( (x,y) \in D \) and \( x,y = \pm 1 \).

**Remark 2.1.2.**

(i) Let \( (x,y_0) \in [-1,1] \times [-1,1] \) be as above fixed. For \( \gamma > \frac{1}{2} \) one can show that the usual derivation of the Fourier transform gives a rise to a nonzero bounded \( \alpha_{(x,y_0)} \)-derivation on \( \ell^1(\mathcal{H}) \). So, it follows from Remark 2.1.1 that \( \mathcal{H} \) is not \( \alpha_{(x,y_0)} \)-amenable and \( \{\alpha_{(x,y_0)}\} \) is not a spectral set.
(ii) Similar to the previous case, one can show that hypergroups of Koornwinder class III, VI, and some related hypergroups in two variables which are mentioned in [4, 3.1.16-20] are not \( \alpha_k \)-amenable if \( \alpha_k \neq 1 \).

(II) **Disc Polynomial Hypergroups:** For \( \alpha' \geq 0 \) the disc polynomials

\[
P_{m,n}^{\alpha'}(z, \bar{z}) = \begin{cases} 
P_n^{(\alpha', m-n)}(2z\bar{z} - 1)z^{m-n}, & \text{for } m \geq n, \\
P_m^{(\alpha', n-m)}(2z\bar{z} - 1)z^{n-m}, & \text{for } n \geq m, 
\end{cases}
\]

induce a hypergroup structure on \( \mathcal{H} := \mathbb{N}_0^2 \). The support of the Plancherel measure with the density \( (z_1, z_2) \rightarrow c\alpha'(1 - |z_1|^2)\alpha' \) is \( \mathcal{D} := \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = \bar{z}_1, |z_1| < 1\} \). From Theorem 2.1 and

\[
P_{n,n}^{\alpha'}(z, \bar{z}) = P_n^{(\alpha', 0)}(2z\bar{z} - 1) = P_n^{(\alpha', 0)}(2|z|^2 - 1) = \mathcal{O}(n^{-\alpha'-1/2}) \quad (z \in \mathcal{D}),
\]

as \( n \to \infty \), we infer that \( \mathcal{H} \) is \( \alpha_c \)-amenable if and only if \( \alpha_c = 1 \). Observe that \( \mathcal{H} := \{(n, n) : n \in \mathbb{N}_0\} \) is a supernormal subhypergroup of \( \mathcal{H} \) which is isomorphic to the Jacobi hypergroup with the character set \( \{P_{n,n}^\alpha(x)\}_{n \in \mathbb{N}_0} \). In this case we see also that \( \mathcal{H} \) is \( \alpha_c \)-amenable if and only if \( \alpha_c = 1 \) despite the fact that for every \( x \in (-1, 1) \) the singleton \( \{\alpha_s\} \) is a spectral for \( \mathcal{H} \) if \( \alpha' < \frac{1}{2} \); see [23]. In other words, if \( \alpha' < \frac{1}{2} \) then every bounded \( \alpha_s \)-derivation on \( \ell^1(\mathcal{H}) \) is zero, however \( \mathcal{H} \) is only 1-amenable.

In the rest of the section we deal with the polynomial hypergroups in one variable, i.e. the system \( \{P_n\}_{n \in \mathcal{H}} \) consists of polynomials of one variable and the index set \( \mathcal{H} \) is \( \mathbb{N}_0 \). The linearization formula in (6) can be expressed in the three term recursion formula

\[
P_1(x)P_n(x) = a_nP_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x),
\]

for \( n \in \mathbb{N} \) and \( P_0(x) = 1 \), and we take \( P_n(1) = 1 \), \( P_1(x) = \frac{1}{a_0}(x - b_0) \) with \( a_n > 0 \), \( b_n \in \mathbb{R} \), and \( c_{n+1} > 0 \) for all \( n \in \mathbb{N}_0 \). The existence of the orthogonality measure is due to Favard’s theorem [9] and applying it to the relation (9) results in \( a_n + b_n + c_n = 1 \) and \( a_0 + b_0 = 1 \). The identity map defines an involution to these hypergroups and their Haar weights are given by \( h(0) = 1 \) and \( h(n) = \left( \int_{\mathbb{R}} P_n^2(x)d\pi(x) \right)^{-1/2} (n \geq 1) \) [4, Theorem.1.3.26]. We consider the \( \alpha \)-amenability of following polynomial hypergroups.

(III) **Associated Legendre hypergroups:** For \( v \in \mathbb{R}_0 \), let \( \gamma_n := \frac{v+1}{2^v(v+\frac{1}{2})n} \left( 1 + \sum_{k=1}^n \frac{v}{k+v} \right) \), \( a_n := \frac{\gamma_{n+1}}{\gamma_n} \), \( b_n := 0 \), and \( c_n := 1 - a_n \) if \( n \geq 1 \) and \( \gamma_0 = 1 \). The polynomial \( P_n \) associated to the
sequences \((a_n)_{n \geq 1}, (b_n)_{n \geq 1}, (c_n)_{n \geq 1}\) in the recursion formula \((9)\) is the \(n\)-th associated Legendre polynomial with parameter \(v\). The Haar weights of the induced hypergroup on \(\mathbb{N}_0\) are given by \(h(0) = 1\) and \(h(n) = \frac{(2n + 2\eta + 2\mu + 1)(2\eta + 1)n}{(2\eta + 2\nu + 1)n!} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{(2\mu)^k}{(2\eta + 1)_k}\right)^2\), \(n \geq 1\), and the support of the Plancherel measure can be identified with \([-1, 1]\); see \([4]\). If \(x \in (-1, 1)\), \(\pi(\{\alpha_x\}) = 0\) and \(\alpha_x \in C_0(\mathbb{N}_0)\), so it follows from Theorem 2.1 that \(\mathbb{N}_0\) is \(\alpha_x\)-amenable if and only if \(\alpha_x = 1\).

(IV) **Pollaczek polynomials hypergroup**: The Pollaczek polynomials \(\{P_{n}^{(\eta, \mu)}\}_{n \in \mathbb{N}_0}\) depending on the parameters \(\eta \geq 0, \mu > 0\) or \(-\frac{1}{2} < \eta < 0\) and \(0 \leq \mu < \eta + \frac{1}{2}\) induce a hypergroup structure on \(\mathbb{N}_0\) \([13]\). The Haar weights are given by \(h(0) = 1\) and

\[
h(n) = \frac{(2n + 2\eta + 2\mu + 1)(2\eta + 1)n}{(2\eta + 2\nu + 1)n!} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{(2\mu)^k}{(2\eta + 1)_k}\right)^2,
\]

and the Plancherel measure with the support \(\mathcal{S} \cong [-1, 1]\) is given by \(d\pi(x) = A(x)dx\) where \(A(\cos t) = (\sin t)^{2\eta}|\Gamma(\eta + \frac{1}{2} + i\mu \cot(t))|^2 \exp((2t - \pi)\mu \cot(t)), \ 0 \leq t \leq \pi\). Given \(x \in (-1, 1)\), since \(\pi(\{\alpha_x\}) = 0\) and \(\alpha_x \in C_0(\mathbb{N}_0)\), by Theorem 2.1 we see that \(\mathbb{N}_0\) is \(\alpha_x\)-amenable if and only if \(\alpha_x = 1\).

(V) **Generalized Soradi hypergroups**: These are polynomial hypergroups of type \([V]\) on \(\mathbb{N}_0\) \([4]\) with the characters

\[
P_n(\cos \theta) = \frac{\sin((n + 1)\theta) - k \sin n\theta}{(nk + n + 1)\sin \theta} \quad (n \geq 1),
\]

and the density of the Plancherel measure on the dual space \(\widehat{\mathbb{N}_0} \cong [-1, 1]\) is given by \(\rho(x) = \frac{2(1-x^2)^{1/2}}{\pi(1+k^2-2\overline{k}x)} \ (k > 1)\). For \(x \in [-1, 1]\), since \(\pi(\{\alpha_x\}) = 0\) and \(\alpha_x \in C_0(\mathbb{N}_0)\), Theorem 2.1 implies that \(\mathbb{N}_0\) is \(\alpha_x\)-amenable if and only if \(\alpha_x = 1\).

(VI) **Hypergroups associated with infinite distance-transitive graphs**: They are polynomial hypergroups on \(\mathbb{N}_0\) depending on \(a, b \in \mathbb{R}\) with \(a, b \geq 2\); and, one can associate them with infinite distance-transitive graphs if \(a, b\) are integers. These hypergroups have been thoroughly studied by M. Voit \([20]\). For \(b > a \geq 2\) (see below) they provide a rare and interesting case of \(\alpha\)-amenability of hypergroups. Their Haar weights and characters are given by

\[
h^{(a,b)}(0) := 1, \ h^{(a,b)}(n) = a(a-1)^{n-1}(b-1)^n \quad (n \geq 1),
\]

and

\[
P_n^{(a,b)}(x) = \frac{a-1}{a((a-1)(b-1))^{n/2}} \left(U_n(x) + \frac{b-2}{((a-1)(b-1))^{1/2}}U_{n-1}(x) - \frac{1}{a-1}U_{n-2}(x)\right),
\]

12
respectively, where $U_n(\cos t) = \frac{\sin((a+1)t)}{\sin t}$ are the Tchebychev polynomials of the second kind and $u_{-1} = u_{-2} := 0$. The dual space $\mathbb{N}_0^{(a,b)}$ can be identified with $[-s_1, s_1]$, where $s_1 := \frac{ab-a-b+1}{2\sqrt{(a-1)(b-1)}}$. The normalized orthogonality measure $\pi \in M^1(\mathbb{R})$ is

$$d\pi(x) = A(x)dx||_{-1,1} \quad \text{for } a \geq b \geq 2,$$

and

$$d\pi(x) = A(x)dx||_{-1,1} + \frac{b-a}{b} ds_0 \quad \text{for } b > a \geq 2$$

with $A(x) := \frac{a}{2\pi} \frac{(1-x^2)^{1/2}}{(s_1-x)(x-s_0)}$, $s_0 = \frac{2-a-b}{2\sqrt{(a-1)(b-1)}}$. Note that

$$P_n^{(a,b)}(s_1) = 1 \quad \text{and} \quad P_n^{(a,b)}(s_0) = (1-b)^{-n} \quad \text{for } n \geq 0.$$

**Proposition 2.1.** Let $\mathbb{N}_0^{(a,b)}$ denote the above hypergroup. Then

(i) for $a \geq b \geq 2$, $\mathbb{N}_0^{(a,b)}$ is $\alpha_x$-amenable if and only if $x = s_1$.

(ii) for $b > a \geq 2$, $\mathbb{N}_0^{(a,b)}$ is $\alpha_x$-amenable if and only if $x = s_1$ or $x = s_0$.

**Proof.** (i) If $x \in (-s_1, s_1)$, then $\pi(\{\alpha_x\}) = 0$ and $\alpha_x \in C_0(\mathbb{N}_0^{(a,b)})$. So, applying Theorem 2.1 yields that $\mathbb{N}_0^{(a,b)}$ is $\alpha_x$-amenable if and only if $x = s_0$, as $\alpha_{s_0} = 1$.

(ii) As in part (i), we can show that if $x \neq s_0$, then $\mathbb{N}_0^{(a,b)}$ is $\alpha_x$-amenable if and only if $x = s_1$. In the case of $x = s_0$, obviously $\alpha_{s_0} \in l^1(\mathbb{N}_0^{(a,b)})$ (see also [20, Remark 1.1]) which implies, by Remark 1.2.1 (ii), that $\mathbb{N}_0^{(a,b)}$ is $\alpha_{s_0}$-amenable.

**Remark 2.1.3.** Notice that in the previous example $\hat{K}$ contains two positive characters $\alpha_{s_0}$ and $\alpha_{s_1}$ with diverse behaviours. Indeed, Part (i) shows that $\mathbb{N}_0^{(a,b)}$ is $\alpha_{s_1}$-amenable but not $\alpha_{s_0}$-amenable if $a \geq b \geq 2$, whereas Part (ii) shows that $\mathbb{N}_0^{(a,b)}$ is $\alpha_{s_1}$ and $\alpha_{s_0}$-amenable for $b > a \geq 2$. In latter case, the functional $m_{\alpha_{s_0}}$, given in Remark 1.2.1 (ii), is a unique $\alpha_{s_0}$-mean on $l^\infty(\mathbb{N}_0^{(a,b)})$ while the cardinality of $\alpha_{s_1}$-means on $l^\infty(\mathbb{N}_0^{(a,b)})$ is infinity; see [3, 15].

**References**

[1] A. Azimifard, $\alpha$-Amenability of Banach algebras on commutative hypergroups. Dissertation, Technische Universität München, 2006.

[2] A. Azimifard, On the $\alpha$-Amenability of Hypergroups. Montash. Math. (to appear 2008)
[3] A. Azimifard, Hypergroups with the unique $\alpha$-means. C. R. Math. Acad. Sci. Soc. R. Can. (to appear 2008)

[4] W. R. Bloom and H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups. De Gruyter, 1994.

[5] F. F. Bonsall and J. Duncan, Complete Normed Algebras. Springer-Verlag, New York-Heidelberg, 1973.

[6] N. Dunford and J. T. Schwartz, Linear Operators I. Wiley & Sons, 1988.

[7] F. Filbir, R. Lasser and R. Szwarc, Reiter’s condition $P_1$ and approximate identities for hypergroups. Monat. Math. 143 (2004) 189–203.

[8] F. Filbir, R. Lasser and R. Szwarc, Hypergroups of compact type. J. Comp. and App. Math. 178 (1) (2005) 205–214.

[9] M. Ismail, Classical and quantum orthogonal polynomials in one variables. Cambridge University Press, 2005.

[10] R. I. Jewett, Spaces with an abstract convolution of measures. Adv. in Math. 18 (1975) 1–101.

[11] B. E. Johnson, Cohomology in Banach algebras. Mem. Amer. Math. Soc. 127 (1972).

[12] T. H. Koornwinder, Two-variable analogues of the classical orthogonal polynomials. In: Theory And Application Of Special Functions, (R. Askey, ed.), pp. 435 – 495, Academic Press, New York, 1975.

[13] R. Lasser, Orthogonal polynomials and hypergroups II-the symmetric case. Trans. Amer. Math. Soc. 341 (1994) 749–770.

[14] P. G. Nevai, Orthogonal Polynomials. Mem. Amer. Math. Soc. 213, 1979.

[15] M. Skantharajah, Amenable hypergroups. Illinois J. Math. 36 (1) (1992) 15–46.

[16] R. Spector, Aperçu de la théorie des hypergroupes. In Anal. harmon. Groupes de Lie, LNM 497 (1975) 643–673.

[17] M. Voit, Positive characters on commutative hypergroups and some applications. Math. Z. 198 (1988) 405–421.

[18] M. Voit, On the dual space of a commutative hypergroup. Arch. Math. 56 (4) (1991) 380–385.

[19] M. Voit, Factorization of probability measures on symmetric hypergroups. J. Aust. Math. Soc. (Series A) 50 (1991) 417 – 467.

[20] M. Voit, A product formula for orthogonal polynomials associated with infinite distance-transitive graphs. J. App. Theory. 120 (2003) 337–354.
[21] M. Vogel, Spectral synthesis on algebras of orthogonal polynomial series. Math. Z. 194 (1) (1987) 99–116.

[22] R. C. Vrem, Hypergroups joins and their dual objects. Pacific J. Math. 111 (1984) 483–495.

[23] S. Wolfenstetter, Jacobi-Polynome und Bessel-Funktionen unter dem Gesichtspunkt der harmonischen Analyse. Dissertation, Technische Universität München, 1984.

[24] H. Zeuner, Polynomial hypergroups in several variables. Arch. Math. 58 (1992) 425–434.