On chiral bosons in 2D and 6D

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Abstract: In the Hamiltonian formulation of chiral $2k$-form electrodynamics, the $2k$-form potential on the $(4k + 1)$-space is defined up to the addition of either (i) a closed $2k$-form or (ii) an exact $2k$-form, depending on the choice of chirality constraint. Case (i) is realized by the Floreanini-Jackiw 2D chiral boson (for $k = 0$) and its Henneaux-Teitelboim generalisation to $k > 0$. For all $k$, but focusing on the 6D case, we present a simple Lorentz-invariant Hamiltonian model that realizes case (ii), and we derive it from Siegel's manifestly Lorentz invariant Lagrangian formulation.

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1 Introduction

The term “chiral boson” is usually taken to mean a scalar field in a two-dimensional (2D) locally-Minkowski spacetime whose excitations travel (at light speed) only to the left (or to the right), but it could equally well be taken to mean (for any non-negative integer \( k \)) a \( 2k \)-form electrodynamics theory in a \((4k + 2)\)-dimensional locally-Minkowski spacetime for which the \((2k + 1)\)-form field-strength is self-dual (or antiself-dual) [1]. For \( k = 0 \), and 2D Minkowski metric

\[
ds^2 = -dt^2 + d\sigma^2,
\]

the self-duality constraint on the one-form \( d\phi \) for scalar field \( \phi(t,\sigma) \) is

\[
\partial_- \phi = 0, \quad \partial_\pm := \frac{1}{\sqrt{2}} (\partial_t \pm \partial_\sigma),
\]

which implies that \( \phi \) is a superposition of left-moving light-speed waves.

However, the equation (1.2) is not the only Lorentz-invariant free-field equation for a 2D chiral boson; another possibility is the Floreanini-Jackiw equation:

\[
\partial_- \phi' = 0, \quad (\dot{\phi}, \phi') := (\partial_t \phi, \partial_\sigma \phi).
\]

This equation is not manifestly Lorentz invariant, because it is the space-derivative of (1.2), but it \textit{is} Lorentz invariant [2]. It is also invariant under the additional restricted gauge transformation

\[
\phi(t,\sigma) \to \phi(t,\sigma) + \alpha(t),
\]

where \( \alpha \) is an arbitrary constant on the one-dimensional space but an arbitrary function of time. In an application to the heterotic string, in which context \( \phi(t,\sigma) \) represents the
displacement of the string worldsheet in one of 16 ‘extra’ space dimensions, the additional restricted gauge invariance implies that the string’s centre of mass (and hence all particle-like oscillation modes) can move only in the nine ‘physical’ space dimensions [3].

This distinction, based on gauge invariance, between the two 2D chiral boson equations is also a feature of the actions that give rise to these equations by variation. The chiral boson equation (1.2) is equivalent to the Euler-Lagrange (EL) equations for the 2D ‘Siegel’ action with Lagrangian density [4]

\[ \mathcal{L}_{\text{Siegel}} = \gamma \partial_- \phi \partial_+ \phi + \frac{1}{2} \lambda^{--} (\partial_- \phi)^2 , \]  

(1.5)

where \( \lambda^{--}(t, \sigma) \) is a Lagrange multiplier field and \( \gamma = 1 \), originally, but here we allow it to be an arbitrary constant. The joint field equations of \( \phi \) and \( \lambda^{--} \) are equivalent to (1.2) for any \( \gamma \), including \( \gamma = 0 \), because \( (\partial_- \phi)^2 = 0 \) implies \( \partial_- \phi = 0 \). Although the action depends on the Lagrange multiplier \( \lambda^{--} \) in addition to the scalar field \( \phi \), it is invariant under the following “Siegel symmetry” gauge transformation with parameter \( \alpha^- \) [4]:

\[ \delta \phi = -\alpha^- \partial_\phi, \quad \delta \lambda^{--} = 2 \gamma \partial_+ \alpha^- + \lambda^{--} \partial_- \alpha^- - \alpha^- \partial_- \lambda^{--} . \]  

(1.6)

This is a “trivial” gauge transformation in the sense that the variation of the dynamical field \( \phi \) is zero on-shell (i.e. on solutions of the dynamical field-equation \( \partial_- \phi = 0 \)) but it allows a gauge to be chosen in which \( \lambda^{--} \) is zero almost everywhere [4].

The Siegel action is not invariant under the (non-trivial) restricted gauge transformation of (1.4), in contrast to the Floreanini-Jackiw action, which has the Lagrangian density [2]

\[ \mathcal{L}_{\text{FJ}} = (\partial_- \varphi) \varphi' , \]  

(1.7)

with the FJ equation (1.3) as its EL equation. Although Lorentz invariance of \( \mathcal{L}_{\text{FJ}} \) is not manifest, it can be made manifest in various ways; e.g. by the introduction of additional fields with additional gauge invariances.2

This distinction between 2D chiral boson theories explained above generalises to chiral 2\( k \)-form electrodynamics in \( (4k + 2) \) dimensions. We shall focus here on the \( k = 1 \) case of 6D chiral 2-form electrodynamics, but we also explain how our results generalise to \( k > 1 \). The 6D analog of the chiral boson equation (1.2) is the manifestly Lorentz-invariant self-duality condition

\[ 0 = \mathcal{F}^+ := \mathcal{F} + *\mathcal{F} , \]  

(1.8)

where \( \mathcal{F} = d\mathcal{A} \) is the 3-form field-strength and \( *\mathcal{F} \) is its Hodge dual. In Minkowski coordinates \((t, \sigma^i; i = 1, \ldots, 5)\), we may write \( \mathcal{A} \) as

\[ \mathcal{A} = A_i dt \wedge d\sigma^i + \frac{1}{2} A_{ij} d\sigma^i \wedge d\sigma^j , \]  

(1.9)

\[ \text{1} \text{A potential legitimate gauge choice is } \lambda^{--} = c\delta (t - t_*) \text{ for some initial time } t_* \text{ and variable constant } c, \text{ which imposes } \partial_- \phi = 0 \text{ as an initial condition preserved by the field equation } \Box \phi = 0. \]

\[ \text{2} \text{A well-known example is the PST method [5, 6]. Another (string-inspired) example, in which Lorentz invariance becomes linearly realized as an ‘internal’ symmetry, can be found in [3].} \]
so \( A_{0i} = \bar{A}_i \) and \( A_{ij} = A_{ij} \), and \( \mathbf{F} \) is invariant under the gauge transformations

\[
A_{ij} \rightarrow A_{ij} + 2\partial_{[i} \alpha_{j]} , \quad \bar{A}_i \rightarrow \bar{A}_i + \dot{\alpha}_i - \partial_i \alpha_0 ,
\]

where the parameters \((\alpha_0, \alpha_i)\) are the components of an arbitrary one-form on the 6D spacetime. We may define the ‘electric’ and ‘magnetic’ components of \( \mathbf{F} \) as follows:

\[
E_{ij} := F_{0ij} \equiv \dot{A}_{ij} - 2\partial_{[i} A_{j]} , \quad B^{ij} := \frac{1}{6} \varepsilon^{ijklm} F_{klm} \equiv (\nabla \times A)^{ij} ,
\]

where we use a notation for which, for any 5-space 2-form \( C \),

\[
(\nabla \times C)^{ij} := \frac{1}{2} \varepsilon^{ijklm} \partial_k C_{lm} .
\]

The self-duality condition (1.8) may now be written as

\[
E = B ,
\]

which has an immediate \( k > 1 \) generalization [7]. This equation is Lorentz invariant because the infinitesimal Lorentz-boost transformations of \((E, B)\) are such that

\[
\delta_\omega (E \pm B) = \pm \omega \times (E \pm B) ,
\]

where \( \omega \) is the constant 5-vector boost parameter, and the 5-space cross product of \( \omega \) with \((E - B)\) is defined, in analogy to the 5-space ‘curl’ of (1.12). For later use, we detail here the 5-space tensor algebra notation used in this paper.

- Cross product. For any 5-vector \( w \) and 2-forms \((C, C')\),

\[
(w \times C)^{ij} := \frac{1}{2} \varepsilon^{ijklm} w_k C_{lm} , \quad (C \times C')^i := \frac{1}{4} \varepsilon^{ijklm} C_{jk} C'_{lm} .
\]

Notice that \( C \times C' = C' \times C \).

- Dot product and norm.

\[
C \cdot C' := \frac{1}{2} C^{ij} C'_{ij} , \quad |C|^2 := C \cdot C .
\]

This is a special case of the standard inner-product on forms of arbitrary degree.

- Triple scalar product. From \((w, C, C')\) we may construct a scalar in two potentially different ways, but the following identity implies their equivalence:

\[
(w \times C) \cdot C' \equiv w \cdot (C \times C') .
\]

The parentheses in the expressions on either side are optional because neither \( w \times (C \cdot C') \) nor \( w \cdot C \) is defined.
A manifestly Lorentz invariant action that yields the self-duality equation $E = B$ was proposed by Siegel [4]; we shall consider it in detail later. All we need to know for the moment is that the equation $E = B$ is invariant under the gauge transformations (1.10).

By taking the 5-space ‘curl’ of this equation we arrive (since $\nabla \times E \equiv \dot{B}$) at the Henneaux-Teitelboim (HT) equation [7]

$$\dot{B} - \nabla \times B = 0.$$  \hspace{1cm} (1.18)

This is an equation for the 2-form $A$ alone, invariant under the gauge transformation

$$A_{ij} \rightarrow A_{ij} + \beta_{ij}, \quad \partial_i[\beta_{jk}] = 0,$$  \hspace{1cm} (1.19)

where $\beta_{ij}$ are the components of an arbitrary closed 5-space 2-form $\beta$, which is also an arbitrary function of time. In the special case that $\beta = d\alpha$ (where $d$ is the 5-space exterior derivative) we recover the gauge transformation of $A$ in (1.10), and in this sense the gauge invariance of (1.18) is ‘enlarged’ relative to that of (1.13). This is also a feature of the HT action for which (1.18) is the EL equation; its Lagrangian density is [7]

$$L_{HT} = \dot{A} \cdot B - |B|^2.$$  \hspace{1cm} (1.20)

There are various ways to verify that $L_{HT}$ is Lorentz invariant, despite appearances [7]. Here we simply observe that it is the free-field case of the general phase-space action with Hamiltonian density satisfying the conditions required for Lorentz invariance [8].

The HT self-duality equation is the 6D analog of the 2D FJ chiral-boson equation. Alternatively, one may view the FJ equation as the $k = 0$ case of the generalization of the 6D HT equation to arbitrary $k$. As confirmation of this claim, we observe that the parameter of the FJ gauge transformation (1.4) is a closed 1-space 0-form (i.e., a constant on space but still an arbitrary function of time). The non-existence of any exact 1-space 0-form accords with the fact that the ‘standard’ 2D chiral boson equation ($\partial_- \phi = 0$) is not invariant under this transformation.

The main purpose of this paper is to explore this distinction between two ‘types’ of chiral $2k$-form electrodynamics in the context of their Hamiltonian formulations.\footnote{Specifically, phase-space formulations in which the $2k$-form potential on the $(4k + 1)$-space is one of the canonical variables; it is possible to deduce an alternative canonical formalism directly from the gauge-invariant field equations but this leads ultimately to the FJ/HT formulation [9].} For $k = 0$ we will be following in the footsteps of other authors who have found the time-reparametrization Hamiltonian formulation of both the FJ and Siegel formulations of the 2D chiral boson, and noticed that the Hamiltonian chirality constraint is linear in the first case and quadratic in the second case. However, the significance of this fact has not previously been properly appreciated, in our opinion; we discuss this in the following section.

We then move on to 6D chiral 2-form electrodynamics i.e. $k = 1$, limiting the discussion to the free field case. As we review here, the HT version can be seen as a modification of the non-chiral theory to include a linear phase-space chirality constraint. A new result of this paper is the Hamiltonian formulation of Siegel’s 6D chiral 2-form theory [4]; we show that the phase-space action is gauge-equivalent to one with a simple quadratic chirality constraint, which leads to a considerable simplification of the “trivial” but non-linear
“Siegel symmetry” gauge invariances. However, our main conclusion is that the different implementation of the chirality constraint in the Hamiltonian formulations of the 6D Siegel and HT chiral 2-form theories leads (as for $k = 0$) to different classical theories, at least for periodic boundary conditions.

2 2D chiral bosons and Hamiltonian constraints

Consider the following first-order 2D Lagrangian density with independent auxiliary field $\pi(t, x)$:

$$\tilde{L}^{(2)} = \pi(\dot{\phi} - \phi') - \frac{\mu}{2}(\pi - \gamma\phi')^2.$$  \hspace{1cm} (2.1)

The subscript $(2)$ is to remind us that the chirality constraint imposed by the Lagrange multiplier $\mu$ is quadratic in the fields $(\phi', \pi)$. Elimination of $\pi$ by its algebraic field equation takes us back to the 2D Siegel Lagrangian density of (1.5), with

$$\lambda^{--} = \mu^{-1} - \gamma.$$  \hspace{1cm} (2.2)

Although this appears to show that equivalence of (2.1) to (1.5) requires $\mu \neq 0$, the EL equations for (2.1) are jointly equivalent to

$$\partial_- \phi = 0, \quad \pi = \gamma\phi',$$  \hspace{1cm} (2.3)

with $\mu$ undetermined, and hence equivalent in dynamical content to (1.2). We may write (2.1) in the form

$$\tilde{L}^{(2)} = \pi \dot{\phi} - H - \mu \Phi,$$  \hspace{1cm} (2.4)

where

$$H = \pi \phi', \quad \Phi = \frac{1}{2}(\pi - \gamma\phi')^2.$$  \hspace{1cm} (2.5)

This shows that $\tilde{L}^{(2)}$ is a phase-space Lagrangian density for phase-space fields $(\pi, \phi)$ with Hamilton density $\mathcal{H}$ and phase-space chirality constraint $\Phi \approx 0$, imposed by a Lagrange multiplier $\mu$.

We have used Dirac’s “weak equality” notation ($\approx$) because it serves to remind us that constraints may differ in their PB relations with functions on phase-space even when they are equivalent as equations; i.e. when they have the same solution space. As we shall see below, this point is nicely illustrated by consideration of the alternative linear constraint

$$\chi \approx 0, \quad \chi := \pi - \gamma\phi'.$$  \hspace{1cm} (2.6)

The equation $\Phi = 0$ is equivalent to the equation $\chi = 0$ but replacing $\Phi$ by $\chi$ in (2.4) yields the alternative phase-space Lagrangian density

$$\tilde{L}^{(1)} = \pi(\dot{\phi} - \phi') - \mu(\pi - \gamma\phi').$$  \hspace{1cm} (2.7)

Notice that $(\pi, \mu)$ now form a pair of auxiliary fields because their joint field equations are

$$\pi = \gamma\phi', \quad \mu = \partial_- \phi,$$  \hspace{1cm} (2.8)
which allows their consistent elimination from the action. The result is
\[ \tilde{L}(1) \rightarrow \gamma (\partial_\varphi \varphi)', \] (2.9)
which is the FJ Lagrangian density (1.7) for \( \gamma = 1 \).

A version of this analysis was used by Faddeev and Jackiw to argue for equivalence of the 2D Siegel and Floreanini-Jackiw chiral boson theories [10]. Starting with \( \tilde{L}(2) \), for \( \gamma = 1 \), they argued that the equivalence of the equation \( \Phi = 0 \) to \( \chi = 0 \) justifies the substitution \( \pi \rightarrow \phi' \) which, as we have seen, leads to \( L_{FJ} \). The problem with this argument is that the equation of one variable (\( \mu \)) is being used to solve for another one (\( \pi \)), which is fine in the field equations but back-substitution into the action is not guaranteed to yield an equivalent action, and is therefore not generally legitimate. If the substitution were legitimate in this case it would be so for any \( \gamma \) including \( \gamma = 0 \), but for \( \gamma = 0 \) it leads to a zero action. The resolution of this difficulty is that the substitution is not legitimate, and for this reason we are not required to choose the same value for \( \gamma \) in \( L(1) \) and \( \tilde{L}(2) \). Indeed, \( \gamma = 0 \) is not allowed in \( \tilde{L}(1) \), whereas it is allowed in \( \tilde{L}(2) \).

In principle, the substitution \( \pi \rightarrow \gamma \phi' \) might also be illegitimate in \( \tilde{L}(1) \) but in this context it yields the same result as the (legitimate) simultaneous elimination of the auxiliary pair \((\pi, \mu)\). This exception to the general rule may be understood from a path-integral perspective: given a path integral over the fields \((\phi, \pi; \mu)\) with a measure determined by \( \tilde{L}(1) \) we can first do the functional integral over \( \mu \) to get the delta-functional \( \delta[\chi] \) in the measure; the functional integral over \( \pi \) is then trivial and the result is (provided \( \gamma \neq 0 \)) a path-integral over \( \phi \) with a measure determined by the FJ action. An argument along these lines was used by Bernstein and Sonnenschein but for \( \tilde{L}(2) \) [11]. In essence, they assumed that \( \Phi \) can be replaced by \( \chi \) for reasons similar to those of Faddeev and Jackiw; they then showed that the resulting path integral is equivalent to one determined by the FJ action. As we have seen, this conclusion is correct given the premise, but the premise is false because \( \delta[\Phi] \) is not generally equivalent to \( \delta[\chi] \), just as \( \delta(x^2) \) in the measure of an integral over real number \( x \) is not generally equivalent to \( \delta(x) \).

We conclude this discussion by summarising how the distinction between the 2D Siegel and Floreanini-Jackiw chiral boson theories arises in the Hamiltonian formulation from the different effects of quadratic and linear chirality constraints that are equivalent as equations.

- \( \Phi \approx 0 \). The PB relation of \( \Phi(\sigma) \) with any phase-space function, including \( \Phi(\sigma') \), is zero on the constraint surface. The constraints are therefore “first-class” and they generate gauge transformations which, however, are “trivial” because they are zero on the constraint surface. The Hamiltonian field equations are equivalent to the standard chiral boson equation \( \partial_+ \phi = 0 \) for any value of the constant \( \gamma \).

- \( \chi \approx 0 \). For \( \gamma \neq 0 \) the Hamiltonian field equations are equivalent to the FJ chiral boson equation. The infinite set of Fourier modes of \( \chi \) is a basis for mixed first and second class constraints. The non-zero modes come in pairs and they are a basis for a set of second-class constraints, but the lone zero mode is first class and it generates the restricted gauge invariance of the FJ theory [3].
Here we must stress that the above discussion concerns the classical field theory. In the quantum theory, the quadratic Siegel constraints are no longer first-class because of a quantum anomaly, so modifications are required [12], and complications associated with the second-class constraints of the FJ formulation have led to other modifications (e.g. [13, 14]). However, none of these modifications negate the fact that the physical phase spaces of the Siegel and FJ formulations differ for periodic boundary conditions because of the additional restricted gauge invariance of the latter, and we may expect this difference to be important to the quantum theory. This expectation is confirmed by a Lagrangian-based derivation of the partition function for a chiral boson on a 2-torus [15].

3 6D chiral 2-form electrodynamics

For Minkowski coordinates $x^\mu = (t, \sigma^i)$, the manifestly Lorentz invariant free-field Lagrangian density for the non-chiral 6D 2-form electrodynamics is

$$\mathcal{L} = -\frac{1}{12} F^{\mu\nu\rho} F_{\mu\nu\rho} \quad \left( F_{\mu\nu\rho} = 3 \partial\llbracket \mu A_{\nu\rho} \rrbracket \right),$$

(3.1)

where a Minkowski metric of ‘mostly plus’ signature is used to raise Lorentz-vector indices. Using the definitions (1.11) of electric and magnetic components of $F$, we find the alternative expression

$$\mathcal{L} = \frac{1}{2} (|E|^2 - |B|^2).$$

(3.2)

Only the invariance under 5-space rotations is now obvious but we still have a linearly-realized invariance under the Lorentz boost transformations of (1.14), so the Lorentz invariance is still “manifest” in this sense.

The EL equations obtained from (3.1) or (3.2) propagate a total of 6 modes in the $(0,3) \oplus (3,0)$ representation of the 6D lightlike “little group” $SO(4)$. In contrast, the chiral 2-form equation $E = B$ (which, we recall, is the self-duality condition on the spacetime 3-form $\mathcal{F}$) propagates only three modes. The attempt to construct an action for this equation by means of a Lagrange multiplier that imposes $E = B$ fails because the resulting action then includes 3 additional modes propagated by the Lagrange multiplier; this is because $E = B$ is a dynamical equation, not a constraint. This difficulty is most simply solved, at the cost of a loss of manifest Lorentz invariance,\(^4\) by passing to the Hamiltonian formulation because a chirality condition may then be imposed as a (non-dynamical) phase-space constraint. However, the Hamiltonian route to an off-shell chiral 2-form electrodynamics may be implemented in essentially two distinct ways: we may choose the chirality constraint to be linear in the natural phase-space variables, or we may choose it to be quadratic. These two options do not preclude the possibility of more complicated non-linear chirality constraints but they are the simplest choices that lead to the two possible 6D free-field chiral 2-form equations, distinguished by their distinct set of gauge invariances, in close analogy to our discussion of the 2D chiral boson.

\(^4\)It can also be solved while maintaining manifest Lorentz invariance, as for 2D; however, as pointed out in [8], there is always some non-manifest symmetry or gauge invariance.
We shall begin with a review of the Hamiltonian formulation of non-chiral 2-form electrodynamics and how the HT chiral 2-form theory is obtained from it by imposing a linear chirality constraint. We then investigate the consequences of replacing this linear constraint by a simple quadratic one that preserves a non-linearly realized Lorentz invariance. We then derive this simple ‘quadratic’ Hamiltonian formulation from Siegel’s manifestly Lorentz invariant Lagrangian formulation. This result exposes the close analogy of the 6D Siegel and HT formulations of 6D chiral 2-form electrodynamics to the 2D Siegel and FJ formulations of the 2D chiral boson, in addition to greatly simplifying the “Siegel symmetry” gauge invariances of the Siegel formulation.

3.1 Hamiltonian formulation: non-chiral case

The Hamiltonian density for the non-chiral theory is the Legendre transform of the Lagrangian density of (3.2) with respect to $E$:

$$\mathcal{H}(D, B) = \sup_E \{D \cdot E - \mathcal{L}(E, B)\} = \frac{1}{2} \left( |D|^2 + |B|^2 \right), \quad (3.3)$$

where the antisymmetric tensor field $D^{ij}$ is the momentum variable canonically conjugate to $A_{ij}$. The Hamiltonian field equations are the Euler-Lagrange equations for the phase-space Lagrangian density

$$\tilde{\mathcal{L}} = E \cdot D - \mathcal{H}(D, B) \quad \text{(non-chiral)}.$$  

In this free-field case, the field equation for $D$ is $D = E$, and we then recover (3.2) by (consistent) substitution. Notice that

$$E \cdot D = \dot{A} \cdot D - \partial_i(\dot{A}_j D^{ij}), \quad G_i = \partial_j D^{ij}, \quad (3.5)$$

which shows that $\dot{A}$ is a 5-vector Lagrange multiplier for a (first class) 5-vector phase-space constraint; the 5-vector constraint function $G$ generates the gauge transformation of the 5-space 2-form potential $A$. To verify this, it is useful to consider a functional basis parametrised by smooth 1-forms $\alpha(\sigma)$; i.e.

$$G[\alpha] = \int d\sigma \alpha_i G^i. \quad (3.6)$$

Assuming that the 1-forms $\alpha$ have compact support (which allows us to ignore surface terms when integrating by parts) we may use the canonical PB relations

$$\{A_{ij}(\sigma), D^{kl}(\tau)\}_{PB} = 2\delta^{k}_{[i} \delta^{l]_{j}} \delta(\sigma - \tau), \quad (3.7)$$

to show that $G[\alpha]$ has a zero PB with the Hamiltonian (because it is gauge-invariant) and that $\{G[\alpha], G[\alpha']\}_{PB} = 0$. The set of functionals $\{G\}$ is therefore trivially first-class, and the infinitesimal gauge transformation generated by $G[\alpha]$ is

$$\delta_\alpha A_{ij} = \{A_{ij}, G[\alpha]\}_{PB} = 2\partial_i[\alpha_j], \quad (3.8)$$

- 8 -
which is the usual gauge transformation in which the 5-space 2-form potential $A$ is shifted by an exact 2-form $d\alpha$.

As an aside on conventions, we remark that $2\delta^k_{[i} \delta^l_{j]}$ is the identity matrix acting on the 10-dimensional space of antisymmetric $5 \times 5$ matrices since

$$\frac{1}{2} \left[ 2\delta^k_{[i} \delta^l_{j]} \right] \left[ 2\delta^p_{[r} \delta^q_{s]} \right] = 2\delta^p_{r} \delta^q_{s},$$

where the factor of $\frac{1}{2}$ is needed to avoid overcounting; this is the same factor of $\frac{1}{2}$ that appears in (1.16).

### 3.2 Linear chirality constraint: the HT case

A chiral version of the free-field 2-form electrodynamics may be found by using a Lagrange multiplier field to impose the linear chirality constraint [7]

$$\chi_{ij} := (D - B)_{ij} \approx 0. \quad (3.10)$$

This yields the following phase-space Lagrangian density:

$$\tilde{L}_{(1)} = \dot{A} \cdot D - \frac{1}{2} \left( |D|^2 + |B|^2 \right) - \mu \cdot \chi, \quad (3.11)$$

where the subscript “(1)” serves to remind us that the chirality constraint imposed by the 5-space 2-form Lagrange multiplier $\mu$ is linear in the phase-space fields $(E,D)$. We have omitted the $A \cdot G$ term because it can be removed (ignoring total derivatives) by a redefinition of the Lagrange multiplier $\mu$; the ‘Gauss law’ constraint $G \approx 0$ is thus subsumed into the chirality constraint.

The 2-form fields $(D,\mu)$ form an auxiliary pair that may be eliminated by their joint algebraic field equations. This yields the Henneaux-Teitelboim chiral 2-form Lagrangian density of (1.20). Direct substitution $D \rightarrow B$ yields the same result, as in the 2D case for a linear chirality constraint, and this can again be explained by path-integral considerations. The Lagrangian density $\tilde{L}_{(1)}$ is therefore equivalent to $\tilde{L}_{HT}$ of (1.20), but has the advantage that its gauge invariances are now associated to first-class constraints, as we now explain.

As for the non-chiral theory, it is convenient to choose a functional basis for the constraints, which are now parametrised by 5-space 2-forms $\beta_{ij}$:

$$\chi[\beta] = \int d^5 \sigma \beta \cdot \chi. \quad (3.12)$$

Using the canonical Poisson bracket relations

$$\left\{ A_{ij}(\sigma), D^{kl}(\sigma') \right\}_{PB} = 2\delta^k_{[i} \delta^l_{j]} \delta(\sigma - \sigma'), \quad (3.13)$$

which implies\(^5\)

$$\left\{ B^{ij}(\sigma), D^{kl}(\sigma') \right\}_{PB} = \varepsilon^{ijklm} \partial_m \delta(\sigma - \sigma'), \quad (3.14)$$

\(^5\)The substitution $D \rightarrow B$ in this PB relation does not give the correct result for the PB relations involving only components of $B$; these are all zero in the current context, but the corresponding Dirac brackets are (necessarily) the same as the Poisson brackets deduced directly from (1.20).
we find that \[ \{ \chi[\beta], \chi[\beta'] \} \right|_{PB} = 2 \int \beta \wedge d\beta' , \] (3.15)

which shows that we have a set of mixed first-class and second-class constraints. The first-class subset is parametrized by closed 2-forms \( \beta \), and these generate the infinitesimal gauge transformations of \( A \):

\[
\delta_\beta A_{ij} = \{ A_{ij}, \chi[\beta] \} \right|_{PB} = \beta_{ij} , \quad \partial_{[k} \beta_{ij]} = 0 .
\] (3.16)

For \( \beta = d\alpha \) we recover the gauge transformation (3.8) but now we have a larger set of gauge transformations because the closed 2-forms \( \beta \) include the harmonic 5-space 2-forms, as has been previously emphasized in [17], and in [18] in the context of a PST-covariantization of the HT action.

Whether there are any harmonic 2-forms depends on the boundary conditions; these have been left unspecified but there certainly are harmonic 2-forms if we impose periodic boundary conditions in at least two independent directions. For example, we could take the 5-space to be a flat 5-torus \( T^5 \) or \( \mathbb{R} \times T^4 \), in which case there is a consistent dimensional-reduction/truncation to the FJ chiral boson on \( S^1 \) or \( \mathbb{R} \), respectively [8].

3.3 Interlude: a quadratic chirality constraint

Possibly the simplest action with the self-duality condition \( E = B \) as its EL equations is the action with Lagrangian density

\[
L = \frac{\gamma}{2} \left( |E|^2 - |B|^2 \right) + \frac{\lambda}{2} |E - B|^2 ,
\] (3.17)

where \( \gamma \) is an arbitrary constant (which we could set to zero) and \( \lambda \) is a (rotation scalar) Lagrange multiplier. The field equation found from variation of \( A \) is implied by the equation found from variation of \( \lambda \), which is \( E = B \) (because \( |E - B| = 0 \) iff \( E - B = 0 \)). The action is also Lorentz invariant, but this is not manifest because it is realized non-linearly on the Lagrange multiplier \( \lambda \). To verify this, we observe that the Lorentz-boost transformations of (1.14) yield

\[
\delta_\omega |E - B|^2 = -2\omega \cdot (E - B) \times (E - B) ,
\] (3.18)

from which it follows that the Lagrange multiplier term of (3.17) is Lorentz invariant if the infinitesimal Lorentz-boost transformation of \( \lambda \) is taken to be

\[
\delta_\omega \lambda = 2\lambda \omega \cdot N \times N ,
\] (3.19)

where

\[
N = (E - B)/|E - B| .
\] (3.20)

Although \( N \) is ambiguous when \( E = B \), the product \( \delta_\omega \lambda |E - B|^2 \) is unambiguously zero when \( E = B \). It also follows from (1.14) that

\[
\delta_\omega N = -\omega \times N - (\omega \cdot N \times N)N ,
\] (3.21)
and this may be used to verify that the commutator of two Lorentz boosts acting on \( \lambda \) with 5-vector parameters \( \omega \) and \( \omega' \) is zero, as it should be since \( \lambda \) is a rotation scalar.

Our motivation for analysing the model defined by (3.17) is that it is a useful preliminary to an analysis of Siegel’s manifestly Lorentz invariant Lagrangian formulation because it is possible to bring the Siegel Lagrangian density to the form (3.17) by a partial fixing of the “Siegel” gauge invariances; this will be demonstrated in the following subsection. The infinitesimal transformations of the residual Siegel gauge invariance are

\[
\delta \xi A = -\xi (E - B), \quad \delta \xi \lambda = \gamma (\dot{\xi} + \nabla \cdot N \times N) + \lambda \dot{\xi} - \lambda \nabla \cdot N \times N,
\]

where we use the notation

\[
\lambda \, d \xi \equiv \lambda d\xi - \xi d\lambda.
\]

Notice that the variation of \( A \) is zero “on-shell”, making this a “trivial” gauge invariance. In addition both \( \lambda \) and the parameter \( \xi \) are arbitrary functions of both time and space, so we may choose a gauge in which \( \lambda \) is zero almost everywhere. For our purposes, gauge invariances with these two properties are “Siegel” gauge invariances.

We give here some details of the proof that the transformations (3.22) leave invariant the Lagrangian density of (3.17) if we ignore total derivative terms in its variation. This will also serve to illustrate the utility of the 5-vector algebra notation summarized in the Introduction:

- The variation \( \delta \xi A = -\xi (E - B) \) of (3.22) implies that
  \[
  \delta \xi E = -\partial_t [\xi (E - B)], \quad \delta \xi B = -\nabla \times [\xi (E - B)],
  \]
  and hence
  \[
  \delta \xi \left\{ \frac{1}{2} (|E|^2 - |B|^2) \right\} = -E \cdot \partial_t [\xi (E - B)] + B \cdot \nabla \times [\xi (E - B)]
  \]
  \[
  = \xi \left[ \dot{E} - \nabla \times B \right] \cdot (E - B) + \text{total derivative}. \quad (3.25)
  \]

Omitting the total derivative we then have

\[
\delta \xi \left\{ \frac{1}{2} (|E|^2 - |B|^2) \right\} = \xi (E - B) \cdot \left[ \partial_t (E - B) + (\dot{B} - \nabla \times B) \right]
\]

\[
= \xi (E - B) \cdot \left( \partial_t + \nabla \times \right) (E - B)
\]

\[
= \frac{\xi}{2} \left\{ \partial_t |E - B|^2 + \nabla \cdot [(E - B) \times (E - B)] \right\}. \quad (3.26)
\]

Integrating by parts and again omitting total derivatives, we arrive at

\[
\delta \xi \left\{ \frac{1}{2} (|E|^2 - |B|^2) \right\} = - \left( \hat{\xi} + \nabla \xi \cdot N \times N \right) \left[ \frac{1}{2} |E - B|^2 \right]. \quad (3.27)
\]

Next we compute

\[
\delta \xi \left\{ \frac{1}{2} |E - B|^2 \right\} = -(E - B) \cdot (\partial_t - \nabla \times) [\xi (E - B)]
\]

\[
= - \left( \hat{\xi} - \nabla \xi \cdot N \times N \right) |E - B|^2
\]

\[
- \frac{\xi}{2} \left\{ \partial_t |E - B|^2 - \nabla \cdot [(E - B) \times (E - B)] \right\}, \quad (3.28)
\]

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and hence, omitting total derivatives,
\[ \delta \xi \left\{ \frac{\lambda}{2} |E - B|^2 \right\} = \left[ \delta \xi \lambda - 2 \lambda (\dot{\xi} - \nabla \xi \cdot N \times N) \right] \left[ \frac{1}{2} |E - B|^2 \right] + \left[ \partial_t (\lambda \xi) - \nabla (\lambda \xi) \cdot N \times N \right] \left[ \frac{1}{2} |E - B|^2 \right] \] (3.29)

Putting these results together we find that the variation of \( L \) is a total derivative provided that \( \delta \xi \lambda \) is given by the expression in (3.22).

We now turn to the Hamiltonian formulation. Consider the following first order Lagrangian density
\[ \tilde{L}^{(2)} = D \cdot (E - B) - \frac{\mu}{2} |D - \gamma B|^2, \] (3.30)
where \( D \) is an independent auxiliary 5-space 2-form field, and
\[ \mu = \frac{1}{\lambda + \gamma}. \] (3.31)
The subscript “(2)” serves to remind us that the (rotation scalar) Lagrange multiplier \( \mu \) imposes a quadratic chirality constraint. Elimination of \( D \), by means of its field equation
\[ D - \gamma B = \mu^{-1} (E - B), \] (3.32)
yields the second-order Lagrangian density of (3.17), so we now have an equivalent first-order version of it. Moreover, Lorentz invariance is preserved in the passage from the second order \( L^{(2)} \) to the first-order \( \tilde{L}^{(2)} \); the infinitesimal Lorentz-boost transformation of \( D \) is
\[ \delta_\omega D = (1 - \gamma \mu) \omega \times (D - \gamma B) + \gamma \omega \times B, \] (3.33)
and the Lorentz-boost transformation of \( \mu \) is
\[ \delta_\omega \mu = -2 \mu (1 - \gamma \mu) \omega \cdot \tilde{N} \times \tilde{N}, \quad \tilde{N} = (D - \gamma B)/|D - \gamma B|. \] (3.34)
On elimination of \( D \) this transformation of \( \mu \) implies that of \( \lambda \) in (3.18) since the \( D \) field equation implies \( \tilde{N} = N \). In contrast, the variation (3.33) of \( D \) as an independent field does not agree with its variation as the function of \( (E, B, \mu) \) implied by (3.32) unless we also use the dynamical equation \( E = B \), but this fact is perfectly compatible with the off-shell Lorentz-boost invariance of the Lagrangian density \( \tilde{L}^{(2)} \) obtained from \( \tilde{L}^{(1)} \) by substitution for \( D \); it is just an indication that Lorentz-boost invariance of the field equations of \( \tilde{L}^{(1)} \) is not achieved by a separate invariance of its algebraic and dynamical equations.

We may rewrite (3.30) as
\[ \tilde{L}^{(2)} = E \cdot D - \mathcal{H} - \mu \Phi(D, B), \] (3.35)
where
\[ \mathcal{H} = D \cdot B, \quad \Phi := \frac{1}{2} |D - \gamma B|^2. \] (3.36)
We recognise this as a phase-space Lagrangian density with Hamiltonian density $\mathcal{H}$ and phase-space constraint $\Phi \approx 0$. The surface in phase-space defined by this constraint is exactly the same as it was for the linear constraint; i.e. $D = \gamma B$, and hence\footnote{Recall that $\approx$ indicates “weak equality” in Dirac’s sense.}

$$\tilde{\mathcal{H}} \approx \left( \frac{\gamma}{\gamma + 1} \right) \left( |D|^2 + |B|^2 \right). \quad (3.37)$$

For $\gamma = 1$ this is the Hamiltonian density of $\tilde{\mathcal{L}}_{(1)}$.

We should not forget that $E \cdot D$ includes an $\hat{A} \cdot \vec{G}$ term, which can not now be removed by a redefinition of the Lagrange multiplier for the chirality constraint. We therefore have two (sets of) constraints, one with a basis of functionals $G[\alpha]$, for 5-space 1-form parameters $\alpha = \alpha_i d\sigma^i$, and another with a basis of functionals

$$\Phi[\beta] = \int d^5 \sigma \beta(\sigma) \Phi(\sigma), \quad (3.38)$$

where the parameter $\beta$ is an inverse-scalar-density; as before we assume that the parameters are smooth and have finite support but are otherwise arbitrary. As in the non-chiral theory, $G[\alpha]$ generates the standard gauge transformation $A \to A + d\alpha$ of the 5-space 2-form potential $A$. As $\Phi[\beta]$ is invariant under this gauge transformation, its Poisson bracket with $G[\alpha]$ is zero.

We still need the PB relations of the functionals $\Phi[\beta]$. Using (3.14) we find that

$$\{ \Phi[\beta'], \Phi[\beta] \}_PB = 2\gamma \Phi \left[ (\beta \nabla \beta' \cdot \tilde{N} \times \tilde{N} \right], \quad (3.39)$$

which shows that the functionals $\Phi[\beta]$ span a first-class set of constraints, and therefore generate gauge transformations of the canonical variables. To verify this claim we compute

$$\delta_\beta D \equiv \{ D, \Phi[\beta] \}_PB = -\gamma \nabla \times [\beta(D - \gamma B)]$$

$$\delta_\beta A \equiv \{ A, \Phi[\beta] \}_PB = \beta(D - \gamma B), \quad (3.40)$$

from which we deduce

$$\delta_\beta E = \partial_t [\beta(D - \gamma B)] , \quad \delta_\beta B = \nabla \times [\beta(D - \gamma B)] . \quad (3.41)$$

With these variations in hand, we find that $\delta_\beta \tilde{\mathcal{L}}_{(2)} = 0$ (omitting total derivatives) provided that\footnote{Again, this variation is not defined when $D = \gamma B$ but the product $(\delta_\beta \mu)\Phi$ is unambiguous.}

$$\delta_\beta \mu = \dot{\beta} - \nabla \beta \cdot \tilde{N} \times \tilde{N}. \quad (3.42)$$

However, the $\beta$-gauge transformations of the canonical variables are weakly zero and hence do not reduce the dimension of the physical phase space; in other words, $\Phi[\beta]$ generates “trivial” gauge invariances that have no effect on the field equations. This trivial gauge invariance is just the Hamiltonian version of the invariance of the Lagrangian $\mathcal{L}$ under the gauge transformations (3.22) (the parameters are related by $\beta = -\mu \xi$). The only non-trivial gauge invariance is the one generated by $G[\alpha]$, as in the non-chiral theory.
3.4 Siegel’s chiral 2-form electrodynamics

The stress-energy tensor of the non-chiral 2-form electrodynamics theory may be written as the sum
\[ T_{\mu\nu} = T^+_{\mu\nu} + T^-_{\mu\nu}, \quad T^\pm_{\mu\nu} := \frac{1}{8} F^\pm_{\mu\lambda\nu} F^\pm_{\nu\lambda\rho}. \] (3.43)
We get a chiral theory by setting to zero one of the two terms in the sum. Implementing this by a Lagrange multiplier, we arrive at the manifestly Lorentz invariant Lagrangian density
\[ \mathcal{L} = -\frac{\gamma}{12} F^{\mu\rho} F_{\mu\rho} + \lambda^{\mu\nu} T^+_{\mu\nu}, \] (3.44)
where \( \gamma \) is a constant and \( \lambda^{\mu\nu} \) a symmetric tensor Lagrange multiplier, which we may assume to be traceless (i.e. \( \eta_{\mu\nu} \lambda^{\mu\nu} = 0 \), where \( \eta \) is the Minkowski metric). For \( \gamma = 1 \) this is the Lagrangian density proposed by Siegel [4] (after a rescaling of the Lagrange multiplier) but the field equations are equivalent to the free-field chiral 2-form equation \( F^+ = 0 \) for any value of \( \gamma \), including \( \gamma = 0 \). As Siegel showed, there are gauge invariances that are “trivial” but which act non-trivially on the traceless-tensor Lagrange multiplier field, such that it may almost be gauged away; this is the 6D analog of the 2D “Siegel symmetry” with transformations (1.6).

To pass to the Hamiltonian formulation, we first make a time/space split in order to rewrite (3.44) in terms of the electric and magnetic components of \( F \). The corresponding components of \( F^+ \) are
\[ F^{\pm}_{0ij} = (M_-)_{ij}, \quad F^{\pm}_{ijk} = -\frac{1}{2} \epsilon_{ijklm} (M_+)_{lm}, \] (3.45)
where we use the notation
\[ M_{\pm} = E \pm B. \] (3.46)
We now find that the Lagrangian density of (3.44) is
\[ \mathcal{L} = \frac{\gamma}{2} M_+ \cdot M_- + \frac{1}{2} \left\{ \lambda^{00} |M_-|^2 - v \cdot M_- \times M_- + \lambda^{ij} \left[ M^2_+ \right]_{ij} \right\}, \] (3.47)
where we use the notation
\[ u^i = \lambda^{0i}, \] (3.48)
and \( M^2_\pm \) is the matrix square of \( M_\pm \) (so that \( \text{tr} M^2_\pm = -2 |M_\pm|^2 \)). As \( \lambda^{00} = \lambda^k_k \), we have
\[ \lambda^{ij} = \tilde{\lambda}^{ij} + \frac{1}{5} \delta^{ij} \lambda^{00}, \quad \tilde{\lambda}^k_k \equiv 0, \] (3.49)
and we may rewrite the Lagrangian density as
\[ \mathcal{L} = \frac{\gamma}{2} M_+ \cdot M_- + \frac{1}{2} \left\{ \frac{3}{5} \lambda^{00} |M_-|^2 - v \cdot M_- \times M_- + \tilde{\lambda}^{ij} \left[ M^2_+ \right]_{ij} \right\}. \] (3.50)
In this form we see clearly that the original Lagrange multiplier, which transforms as an irreducible \( 20 \) of the 6D Lorentz group, decomposes into the \( 1 \oplus 5 \oplus 14 \) of the \( \text{SO}(5) \) rotation group. The field equations are still equivalent to the standard 6D self-duality relation, which now takes the form \( M_- = 0 \).
In the current notation, the infinitesimal Lorentz-boost transformations of (1.14) are
\[ \delta \omega_{M \pm} = \pm \omega \times M_{\pm}. \quad (3.51) \]

The infinitesimal Lorentz-boost transformations of the Lagrange multiplier fields are
\[ \delta \omega_{\lambda^{00}} = -2 \omega \cdot \upsilon, \quad \delta \upsilon^i = -2 \omega^{(i} \lambda^{j)}, \quad \delta \omega_{\lambda^{ij}} = -2 \omega^{(i} \upsilon^{j)}. \quad (3.52) \]

Notice that these are linear transformations; in this respect Lorentz invariance is still “manifest”. To verify Lorentz boost invariance, it is convenient to use the identities
\[ M_{[i} = \delta \upsilon_{M_{+}} \equiv \frac{1}{6} \epsilon^{ijklm} (M_{\pm} \times M_{\pm})_m, \]
\[ (\omega \times M_{\pm}) \times M_{\pm} \equiv (M_{\pm}^2 + |M_{\pm}|^2 I_5) \omega, \quad (3.53) \]
to verify that
\[ \delta \omega_{(M_{\pm}^2)_{ij}} = \pm \left[ \omega_i (M_{-} \times M_{-})_{j} - (\omega \cdot M_{-} \times M_{-}) \delta_{ij} \right], \]
\[ \delta \omega (M_{\pm} \times M_{\pm}) = \pm 2 (M_{\pm}^2 + |M_{\pm}|^2 I_5) \omega. \quad (3.54) \]
The first of these equations implies that\(^8\)
\[ \delta \omega |M_{\pm}|^2 = \pm 2 \omega \cdot M_{-} \times M_{-}, \quad (3.55) \]
which also follows immediately from (3.51).

Observe now that the Lagrangian density of (3.50) can be further rewritten as
\[ \mathcal{L} = \frac{\gamma}{2} M_{-} M_{-} + \frac{\lambda}{2} |M_{-}|^2, \quad (3.56) \]
where
\[ \lambda = \frac{3}{5} \lambda^{00} - \upsilon \cdot N \times N + \tilde{\lambda}^{ij} (N^2)_{ij}, \quad (3.57) \]
with \( N = M_{-}/|M_{-}| \), as in (3.34), and \( N^2 \) is its matrix square. This is formally the same as the Lagrangian density (3.17) in the previous subsection, but here \( \lambda \) is not an independent variable, so its Lorentz boost transformation can be calculated from (3.52) and (3.21), using the following formulae analogous to those of (3.54):
\[ \delta \omega (N^2)_{ij} = (\omega \cdot N \times N) (2 N^2 + I_5)_{ij} - \omega_i (N \times N)_j, \]
\[ \delta \omega (N \times N) = 2 (\omega \cdot N \times N) (N \times N) - 2 (N^2 + I_5) \omega. \quad (3.58) \]
The result is
\[ \delta \omega \lambda = 2 \lambda (\omega \cdot N \times N), \quad (3.59) \]
which is exactly that of (3.19), as it had to be because we already know that this is required by Lorentz invariance.

\(^8\)Notice that \( \delta \omega (|M_{\pm}|^4 - |M_{\pm} \times M_{\pm}|^2) = 0 \), so these particular quartic scalar functions of \( (E, B) \) are Lorentz invariants.
So far, we have shown that the Lagrangian density (3.44) of Siegel’s 6D chiral 2-form theory can be written in the form (3.17), but it still depends on all 20 independent Lagrange-multiplier components, which (we recall) decompose into the $\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{14}$ of $\text{SO}(5)$. However, the relation (3.57) that determines the $\text{SO}(5)$ scalar $\lambda$ in terms of these $\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{14}$ Lagrange-multiplier components is unchanged if we make arbitrary changes to the $\mathbf{5} \oplus \mathbf{14}$ provided we also make an appropriate change to the $\mathbf{1}$, i.e. to $\lambda^{00}$. Specifically, $\lambda$ is invariant if

$$\frac{3}{5} \delta_{S} \lambda^{00} = (\delta_{S} v) \cdot N \times N - \left( \delta_{S} \tilde{\lambda}^{ij} \right) (N^2)_{ij}, \quad (3.60)$$

where the subscript $S$ indicates that variations of the Lagrange multipliers satisfying this condition are “Siegel symmetry” gauge transformations in the sense that they are “trivial” gauge invariances of the Lagrangian density (3.56) that allow us to set to zero components of the original Siegel Lagrange multiplier, in this case the $\mathbf{5} \oplus \mathbf{14}$ components. In this gauge we have

$$\lambda = \frac{3}{5} \lambda^{00}, \quad (3.61)$$

and this residual Lagrange multiplier is subject to the gauge transformation (3.22), which we may now interpret as the residual “Siegel symmetry” gauge transformation that allows us to set $\lambda = 0$ almost everywhere. We have not made any attempt to compare these “Siegel symmetry” gauge transformations with those given (to first order in an expansion in powers of Lagrange multiplier components) in [4]; we presume that they are equivalent since their effects are equivalent.

The gauge choice leading to (3.61) breaks Lorentz invariance, and this is reflected in the fact that the Lorentz-boost transformation of $\lambda^{00}$ given in (3.52) does not agree with the Lorentz transformation of ($5/3$ times) $\lambda$ given in (3.59). However, this does not mean that the gauge-fixed Lagrangian density is not Lorentz invariant. Instead, it means that a Lorentz transformation must be accompanied by a compensating Siegel gauge transformation, with $\omega$-dependent parameters; let us use $\delta_{S(\omega)}$ to denote these compensating Siegel transformations, which must be such that

$$\delta_{\omega} v^{i} + \delta_{S(\omega)} v^{i} = 0, \quad \delta_{\omega} \tilde{\lambda}^{ij} + \delta_{S(\omega)} \tilde{\lambda}^{ij} = 0. \quad (3.62)$$

As $\lambda^{00}$ also transforms under the compensating Siegel transformation, its Lorentz transformation is modified to

$$\delta_{\omega} \left[ \lambda^{00} \right] = \delta_{\omega} \lambda^{00} + \delta_{S(\omega)} \lambda^{k}_{k}$$
$$= \delta_{\omega} \lambda^{00} + \frac{5}{3} \left[ (\delta_{S(\omega)} v) \cdot N \times N - \left( \delta_{S(\omega)} \right) \tilde{\lambda}^{ij} (N^2)_{ij} \right]$$
$$= \delta_{\omega} \lambda^{00} - \frac{5}{3} \left[ \delta_{\omega} v \cdot N \times N - \delta_{\omega} \tilde{\lambda}^{ij} (N^2)_{ij} \right]$$
$$= \frac{5}{3} \delta_{\omega} \lambda, \quad \text{(3.63)}$$

and hence the relation (3.61) is preserved by the combination of a Lorentz transformation and the compensating Siegel transformation needed to maintain the gauge choice.
To summarise, the Lorentz invariance realised linearly on the $1 \oplus 5 \oplus 14$ Lagrange-multiplier components prior to setting to zero the $5 \oplus 14$ components (as a partial Siegel gauge choice) becomes a Lorentz invariance that is realized non-linearly on the surviving singlet Lagrange-multiplier $\lambda^{00}$, and hence on $\lambda$. This partial gauge fixing of Siegel invariances reduces the Lagrangian density (3.50), which is just a rewriting of 6D Siegel Lagrangian density (3.44), to the much simpler Lagrangian density of (3.17) with the single Lagrange multiplier $\lambda$ and a residual Siegel gauge invariance with transformations given by (3.22).

4 Higher dimensions

For $2k$-form electrodynamics in a $(4k + 2)$-dimensional locally Minkowski spacetime we have a $(2k + 1)$-form field-strength $\mathcal{F} = d\mathcal{A}$ for a $2k$-form potential $\mathcal{A}$, which decomposes into $2k$-form potential $\mathcal{A}$ and a $(2k - 1)$-form potential $\mathcal{A}$ on the $(2k + 1)$-dimensional flat space, and we can define analogs of electric and magnetic fields by direct analogy with the $k = 1$ case of (1.11):

$$
E_{i_1 \ldots i_{2k}} := F_{0i_1 \ldots i_{2k-1}} \equiv \dot{\mathcal{A}}_{i_1 \ldots i_{2k}} - 2k \partial_{i_1} \mathcal{A}_{i_2 \ldots i_{2k}} ,
$$

$$
B^{i_1 \ldots i_{2k}} := \frac{1}{(2k + 1)!} \varepsilon^{i_1 \ldots i_{2k}j_1 \ldots j_{2k+1}} F_{j_1 \ldots j_{2k+1}} \equiv (\nabla \times \mathcal{A})_{i_1 \ldots i_{2k}} .
$$

Equivalently,

$$
E = \dot{\mathcal{A}} - (2k)d\mathcal{A} , \quad B = *d\mathcal{A} ,
$$

where $d$ is the exterior derivative on the $(2k + 1)$-space and $*$ is the Hodge dual with respect to its (flat) $(2k + 1)$-metric. The 5-vector algebra used in this paper also generalises: for any 1-form $w$ and 2-forms $(C, C')$,

$$
(w \times C) := * (w \wedge C) , \quad C \times C' := *(C \wedge C') ,
$$

and

$$
C \cdot C' := \frac{1}{(2k)!} C_{i_1 \ldots i_{2k}} C'_{i_1 \ldots i_{2k}} .
$$

The unique scalar constructible from $(w, C, C')$ is $*(w \wedge C \wedge C')$, and this can be rewritten in either of the two forms given in (1.17).

With these definitions/conventions, the HT action for chiral $2k$-form electrodynamics has a Lagrangian density that is formally identical to $L_{HT}$ of (1.20), and its field equation is formally identical to (1.18). In other words, the HT formulation applies for any $k \geq 1$ (in addition to reducing to the FJ formulation of the 2D chiral boson for $k = 0$) [7]. We therefore focus on the Siegel formulation for $k > 1$.

The manifestly Lorentz-invariant Siegel Lagrangian density for $k > 1$ is a straightforward generalization of (3.44):

$$
\mathcal{L} = - \frac{\gamma}{2(2k + 1)!} \mathcal{F}^{\mu_1 \ldots \mu_{2k+1}} \mathcal{F}_{\mu_1 \ldots \mu_{2k+1}} + \frac{1}{4(2k)!} \lambda^{\mu \nu} \mathcal{F}^+_{\mu \rho_1 \ldots \rho_{2k}} \mathcal{F}^+_{\nu \rho_1 \ldots \rho_{2k}} ,
$$

(4.5)
where $\lambda^{\mu\nu}$ is again traceless in the Minkowski spacetime metric. The $k > 1$ generalization of (3.45) is
\begin{align}
F^+_{i_1 \ldots i_{2k}} &= (E - B)_{i_1 \ldots i_{2k}}, \\
F^+_{i_1 \ldots i_{2k+1}} &= -\frac{1}{(2k)!} \varepsilon_{i_1 \ldots i_{2k+1}j_1 \ldots j_{2k}} (E - B)_{j_1 \ldots j_{2k}},
\end{align}
(4.6)
With respect the SO($4k+1$) rotation group, the Siegel Lagrange multiplier decomposes into the following sum of irreps:
\begin{equation}
1 \oplus (4k + 1) \oplus 2k(4k + 3),
\end{equation}
(4.7)
which is the $1 \oplus 5 \oplus 14$ decomposition for $k = 1$. Using the same notation as we used for the $k = 1$ case, we again find that the Siegel Lagrangian density can be rewritten in the form (3.56), except that we now have
\begin{equation}
\lambda = \begin{pmatrix} 4k - 1 \\ 4k + 1 \end{pmatrix} \lambda^{00} - v \cdot N_\times N_- + \tilde{\lambda}^{ij} \left[N^2\right]_{ij},
\end{equation}
(4.8)
where $N_- = (E - B)/|E - B|$, as for $k = 1$, but
\begin{equation}
\left[N^2\right]_{ij} = -\frac{1}{(2k-1)!} (E - B)_{i_1 \ldots i_{2k-1}} (E - B)_{j_1 \ldots j_{2k-1}}.
\end{equation}
(4.9)
For $k = 1$ this is the matrix square of $(E - B)$, but the matrix interpretation of $(E - B)$ applies only for $k = 1$. As for the $k = 1$ case, the Siegel gauge invariances may be partially fixed by setting to zero all Lagrange multipliers except the singlet $\lambda^{00}$ which is proportional to $\lambda$, and we then have the higher-dimensional version of the simple ‘quadratic’ model of subsection 3.3.

Notice that for $k = 0$ the formula of (4.7) would give two Lagrange multipliers rather than one. In light-cone coordinates for the 2D Minkowski spacetime these are the $\lambda^{--}$ and $\lambda^{++}$ components of $\lambda^{\mu\nu}$, but only the $\lambda^{--}$ component appears in the action (because now $N^2 \equiv 1$).

5 Summary and discussion

The original aim of this paper was to revisit Siegel’s proposal for manifest Lorentz invariant actions for chiral $2k$-form electrodynamics within the context of a Hamiltonian formulation in order to facilitate comparison with the Henneaux-Teitelboim (HT) formulation, which is first-order and hence essentially ‘already Hamiltonian’. One obvious question of interest is whether the two formulations are equivalent, which would allow us to view Siegel’s proposal as a means of making manifest the Lorentz invariance of the HT formulation. In the $k = 0$ case, for which the HT formulation reduces to the Floreanini-Jackiw (FJ) formulation of the 2D chiral boson theory, there appeared to be a consensus that the Siegel and FJ formulations are equivalent.

However, the claims of equivalence for the Siegel and FJ formulations of the 2D boson are at odds with the fact that the FJ chiral boson has an additional gauge invariance since this implies, for periodic boundary conditions, that the physical phase spaces are essentially...
different. This difference has rarely played a role in discussions of the quantum theory of chiral bosons (e.g. [19] which contains a useful review of the literature in the immediate aftermath of the work of Floreanini and Jackiw). An exception is the much later work of Chen et al. in which the additional gauge invariance of the FJ action is shown to be crucial to a ‘Lagrangian’ derivation of the partition function for a chiral boson on a 2-torus [15].

Other differences between the 2D Siegel and FJ chiral boson theories are certainly well appreciated. The Hamiltonian constraints differ since those of the 2D Siegel theory are first class classically but not quantum mechanically, whereas those of the FJ theory are almost all second class. The “almost” qualification is often omitted but there must be, and there is, one first class constraint to generate the additional restricted gauge invariance of the FJ theory [3]. It may be that some consistent quantization of the 2D Siegel chiral boson will force its quantum equivalence with the FJ chiral boson, but any such “quantization” is likely to include a procedure that effectively first converts it into the FJ chiral boson because “the existence of an additional gauge symmetry is a salient feature of all Lagrangian formulations for chiral bosons” [15]. In any case, it is our contention that they are distinct as classical theories.

Most of this paper has been concerned with an examination of these issues for the 6D chiral 2-form electrodynamics, for which there are again two distinct Lagrangian formulations. One is Siegel’s generalization to 6D of his Lagrangian formulation of the 2D chiral boson, and the other is a 6D generalization by Henneaux and Teitelboim of the FJ chiral boson. These authors have also shown that the 6D case is the $k = 1$ example of a chiral 2$k$-form electrodynamics in a spacetime of $(4k + 2)$ dimensions, and that the 2D chiral boson is the $k = 0$ case.

We have pointed out here that the essential physical difference between the Siegel and HT formulations is that the HT formulation has additional gauge invariances, first noticed in [17], and that this difference is associated with different implementations of the chirality constraint in the Hamiltonian formulation. The linearity of the HT formulation means that it is essentially ‘already’ Hamiltonian; its time-reparametrization invariant formulation includes a linear chirality constraint function whose modes are parametrized by time-dependent 2-forms on the 5-dimensional space. The subset with closed 2-forms is first-class and it generates a gauge invariance parametrized by closed 2-forms.

Here we have found a Hamiltonian formulation of Siegel’s Lagrangian formulation of chiral 6D electrodynamics in which a simple quadratic chirality constraint is imposed by a rotation-scalar Lagrange multiplier $\lambda$ that is a non-linear function of the phase-space fields and the Lagrangian Lagrange multipliers. The “trivial” Siegel-symmetry gauge invariances of the Lagrangian formulation can now be used to set to zero all but one component, proportional to $\lambda$ in this gauge, on which Lorentz invariance is now non-linearly realized. The remaining Siegel-symmetry gauge invariance is generated by the chirality constraint function, while the 2-form gauge transformation is generated as for the non-chiral theory, with a parameter that is an exact 2-form.

We have also shown how this difference generalises to chiral 2$k$-form electrodynamics. In the Hamiltonian formulation, the “additional” gauge invariances of the HT action are parametrized by 2$k$ forms on the $(4k + 1)$-dimensional space that are closed but not exact.
These are elements of the de Rham cohomology group $H^{2k}(X)$, where $X$ is the $(4k + 1)$-dimensional space. This result applies equally for $k = 0$, for which $X$ is either $\mathbb{R}$ or $S^1$, for which $H^0(X)$ has dimension 0 and 1 respectively; the one additional gauge invariance for $X = S^1$ is the additional gauge invariance of the FJ chiral boson theory with periodic boundary conditions.

If $H^{2k}(X)$ is trivial, there is no physical difference between the field equation $E = B$ and the HT field equation $\dot{B} = \nabla \times B$, as pointed out in [17]. This is consistent with the fact, established in [20], that the unique unitary irreducible representation of the 6D Poincaré group specified by zero mass and spin in the $(3, 0)$ irrep of the SO(4) “little group” corresponds to the free-field theory with field equation $E = B$ because the 6D Lorentz subgroup is broken by identifications whenever $H^2(X)$ is non-trivial (and an analogous observation applies for any $k > 1$). Such identifications preserve the local metric structure of spacetime, which is therefore locally (but not globally) Minkowski; it is the “local” qualification that has allowed us to consider periodic boundary conditions (and hence closed strings for 2D and toroidal compactifications for higher dimensions).

We have restricted our analysis in this paper to free-field chiral $2k$-form electrodynamics in a $(4k + 2)$-dimensional locally-Minkowski spacetime. These restrictions have been imposed for reasons of simplicity of presentation; we expect a generic classical inequivalence of the Siegel and HT formulations to survive the introduction of self-interactions (e.g., in the context of M5-brane dynamics for $k = 1$) or gravitational interactions leading to spacetimes that are not locally Minkowski.

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