DIAGONAL DOUBLE KODAIRA FIBRATIONS WITH MINIMAL SIGNATURE

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ABSTRACT. We study some special systems of generators on finite groups, introduced in previous work by the first author and called diagonal double Kodaira structures, in order to investigate non-abelian, finite quotients of the pure braid group on two strands $P_2(\Sigma_b)$, where $\Sigma_b$ is a closed Riemann surface of genus $b$. In particular, we prove that, if a finite group $G$ admits a diagonal double Kodaira structure, then $|G| \geq 32$, and equality holds if and only if $G$ is extra-special. In the last section, as a geometrical application of our algebraic results, we construct two 3-dimensional families of double Kodaira fibrations having signature 16.

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0. INTRODUCTION

A Kodaira fibration is a smooth, connected holomorphic fibration $f_1 : S \rightarrow B_1$, where $S$ is a compact complex surface and $B_1$ is a compact closed curve, which is not isotrivial (this means that not all fibres are biholomorphic each other). The genus $b_1 := g(B_1)$ is called the base genus of the fibration, and the genus $g := g(F)$, where $F$ is any fibre, is called the fibre genus. A surface $S$ that is the total space of a Kodaira fibration is called a Kodaira fibred surface. For every Kodaira fibration, we have $b_1 \geq 2$ and $g \geq 3$, see [Kas68, Theorem 1.1]. Since the fibration is smooth, the condition on the base genus implies that $S$ contains no rational or elliptic curves; hence $S$ is minimal and, by the sub-additivity of the Kodaira dimension, it is of general type, hence algebraic.

An important topological invariant of a Kodaira fibred surface $S$ is its signature $\sigma(S)$, namely the signature of the intersection form on the middle cohomology group $H^2(S, \mathbb{R})$. Actually, the first examples of Kodaira fibrations (see [Kod67]) were constructed in order to show that $\sigma$ is not multiplicative for fibre bundles. In fact, $\sigma(S) > 0$ for every Kodaira fibration (see the introduction to [LLR20]), whereas $\sigma(B_1) = \sigma(F) = 0$, hence $\sigma(S) \neq \sigma(B_1)\sigma(F)$;

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by [CHS57], this in turn means that the monodromy action of $\pi_1(B)$ on the rational cohomology ring $H^*(S, \mathbb{Q})$ is non-trivial.

Every Kodaira fibred surface $S$ has the structure of a real surface bundle over a smooth real surface, and so $\sigma(S)$ is divisible by 4, see [Mey73]. If, in addition, $S$ has a spin structure, i.e. its canonical class is 2-divisible in Pic$(S)$, then $\sigma(S)$ is a positive multiple of 16 by Rokhlin’s theorem, and examples with $\sigma(S) = 16$ are constructed in [LLR20]. It is not known whether there exists a Kodaira fibred surface with $\sigma(S) \leq 12$.

Kodaira fibred surfaces are a source of fascinating and deep questions at the cross-road between the algebro-geometric properties of a compact, complex surface and the topological properties of the underlying closed, oriented 4-manifold. In fact, they can be studied by using, besides the usual algebro-geometric methods, techniques borrowed from geometric topology such as the Meyer signature formula, the Birman-Hilden relations in the mapping class group and the subtraction of Lefchetz fibrations, see [En98, EKKOS02, St02, L17]. We refer the reader to the survey paper [Cat17] and the references contained therein for further details.

The original example by Kodaira, and its variants described in [At69, Hir69], are obtained by taking ramified covers of products of curves, so they come with a pair of Kodaira fibrations. This leads to the definition of “double” Kodaira fibration, see [Zaal95, LeBrun00, BDS01, BD02, CatRol09, Rol10, LLR20]:

**Definition 0.1.** A double Kodaira surface is a compact, complex surface $S$, endowed with a double Kodaira fibration, namely, a surjective, holomorphic map $f: S \to B_1 \times B_2$ yielding, by composition with the natural projections, two Kodaira fibrations $f_i: S \to B_i$, $i = 1, 2$.

In the sequel, we will describe our approach to the construction of double Kodaira fibrations based on the techniques introduced in [CaPol19, Pol20], and present our results. The main step is to “detopologize” the problem, by transforming it into a purely algebraic one. This will be done in the particular case of diagonal double Kodaira fibrations, namely, Stein factorizations of finite Galois covers

\begin{equation}
\varphi: \pi_1(\Sigma_b \times \Sigma_b - \Delta) \to G,
\end{equation}

up to automorphisms of $G$. Furthermore, the condition that $f$ is branched of order $n$ over $\Delta$ is rephrased by asking that $\varphi(\gamma_\Delta)$ has order $n$ in $G$, where $\gamma_\Delta$ is the homotopy class in $\Sigma_b \times \Sigma_b - \Delta$ of a loop in $\Sigma_b \times \Sigma_b$ that “winds once” around $\Delta$. The requirement $n \geq 2$ means that $\varphi$ does not factor through $\pi_1(\Sigma_b \times \Sigma_b)$; it also implies that $G$ is non-abelian, because $\gamma_\Delta$ is a non-trivial commutator in $\pi_1(\Sigma_b \times \Sigma_b - \Delta)$.

Recall now that the group $\pi_1(\Sigma_b \times \Sigma_b - \Delta)$ is isomorphic to $P_2(\Sigma_b)$, the pure braid group of genus $b$ on two strands; such a group admit a geometric presentation with $4g + 1$ generators

$$\rho_{11}, \tau_{11}, \ldots, \rho_{1b}, \tau_{1b}, A_{12},$$

where $A_{12}$ corresponds to $\gamma_\Delta$, subject to the set of relations written in Section 2, see [GG04, Theorem 7]. Taking the images of these generators via the group epimorphism (2), we get an ordered set

$$\mathcal{G} = (r_{11}, t_{11}, \ldots, r_{1b}, t_{1b}, r_{21}, t_{21}, \ldots, r_{2b}, t_{2b}, z)$$

of $4b + 1$ generators of $G$, such that $o(z) = n$. This will be called a diagonal double Kodaira structure of type $(b, n)$ on $G$, see Definition 2.1. In the light of the previous considerations, we see that the geometric problem of constructing a $G$-cover $f$ as in (1) is now translated into...
the combinatorial-algebraic problem of finding a diagonal double Kodaira structure of type 
(b, n) in G.

It turns out that the G-cover f is a diagonal double Kodaira fibration (namely, the two surjective maps \( f_i : S \rightarrow \Sigma_b \), obtained as composition with the natural projections, have connected fibres) if and only if \( \mathcal{S} \) is strong, an additional condition introduced in Definition

2.8; furthermore, the algebraic signature \( \sigma(\mathcal{S}) \), see Definition 2.7, equals the geometric signature \( \sigma(S) \).

Summing up, classifying diagonal double Kodaira fibrations is equivalent to describing finite groups which admit a diagonal double Kodaira structure. Our first main result in this direction is the following:

**Theorem A** (see Proposition 3.9, 3.11 and Theorem 3.15). Let G be a finite group admitting a diagonal double Kodaira structure. Then \(|G| \geq 32\), with equality if and only if G is extra-special (see Section 1 for the definition). Moreover, the following holds.

1. Both extra-special groups G of order 32 admit 2211840 = 1152·1920 diagonal double Kodaira structures of type \((b, n) = (2, 2)\). Every such a structure \( \mathcal{S} \) is strong and satisfies \( \sigma(\mathcal{S}) = 16 \).
2. If \( G = G(32, 49) = H_3(\mathbb{Z}_2) \), these structures form 1920 orbits under the action of \( \text{Aut}(G) \).
3. If \( G = G(32, 50) = G_5(\mathbb{Z}_2) \), these structures form 1152 orbits under the action of \( \text{Aut}(G) \).

Theorem A should be compared with previous results, obtained by the first author in collaboration with A. Causin, regarding the construction of diagonal double Kodaira structures on some extra-special groups of order at least \(2^7 = 128\), see [CaPol19,Pol20]. It turns out that the examples presented here are really new, in the sense that they cannot be obtained as images of structures on extra-special groups of larger order, see Remark 3.17.

A restatement of Theorem A in terms of surface braid groups is the following, cf. Remark 3.18. First of all, let us say that a quotient map/group epimorphism \( \varphi : \pi_1(\Sigma_b \times \Sigma_b - \Delta) \rightarrow G \) is admissible if \( \varphi(A_{12}) \) has order \( n \geq 2 \), then:

**Theorem A’.** Let G be a finite group admitting an admissible epimorphism \( \varphi : \Pi_2(\Sigma_b) \rightarrow G \). Then \(|G| \geq 32\), with equality if and only if G is extra-special. Moreover, the following holds.

1. For both extra-special groups G of order 32, there are 2211840 = 1152 · 1920 admissible epimorphisms \( \varphi : \Pi_2(\Sigma_2) \rightarrow G \). For all of them, \( \varphi(A_{12}) \) is the generator of \( Z(G) \), so \( n = 2 \).
2. If \( G = G(32, 49) = H_3(\mathbb{Z}_2) \), these epimorphisms form 1920 orbits under the natural action of \( \text{Aut}(G) \).
3. If \( G = G(32, 50) = G_5(\mathbb{Z}_2) \), these epimorphisms form 1152 orbits under the natural action of \( \text{Aut}(G) \).

The geometrical counterpart of Theorems A and A’ can be now expressed in terms of diagonal double Kodaira fibrations as follows:

**Theorem B** (see Theorem 4.7). Let G be a finite group and \( f : S \rightarrow \Sigma_b \times \Sigma_b \) be a Galois cover, with Galois group G, branched on the diagonal \( \Delta \) with branching order \( n \geq 2 \). Then \(|G| \geq 32\), with equality if and only if G is extra-special. Moreover, the following holds.

1. For both extra-special groups of order 32, there exist 2211840 = 1152·1920 distinct G-covers \( f : S \rightarrow \Sigma_2 \times \Sigma_2 \) as above. All of them are diagonal double Kodaira fibrations with \( n = 2 \) and

\[
b_1 = b_2 = 2, \quad g_1 = g_2 = 41, \quad \sigma(S) = 16.
\]

2. If \( G = G(32, 49) = H_3(\mathbb{Z}_2) \), these G-covers form 1920 equivalence classes up to cover isomorphisms.
3. If \( G = G(32, 50) = G_5(\mathbb{Z}_2) \), these G-covers form 1152 equivalence classes up to cover isomorphisms.
As a consequence, we obtain a sharp lower bound for the signature of a diagonal double Kodaira fibration or, equivalently, of a diagonal double Kodaira structure:

**Theorem C** (see Corollary 4.8). Let $f : S \rightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ be a diagonal double Kodaira fibration, associated with a diagonal double Kodaira structure of type $(b, n)$ on a finite group $G$. Then $\sigma(S) \geq 16$, and equality holds precisely when $(b, n) = (2, 2)$ and $G$ is an extra-special group of order 32.

These results yield, as a by-product, new “double solutions” to a problem (stated by G. Mess) from Kirby’s problem list in low-dimensional topology [Kir97, Problem 2.18 A], asking what is the smallest number $b$ for which there exists a real surface bundle over a real surface with base genus $b$ and non-zero signature. We actually have $b = 2$, also for double Kodaira fibrations, as shown in [CaPol19, Proposition 3.19] and [Pol20, Theorem 3.6] by using double Kodaira structures of type $(2, 3)$ on extra-special groups of order 3. Those fibrations had signature 144 and fibre genera 325; we are now able to substantially lower both these values:

**Theorem D** (see Theorem 4.9). Let $S$ be a diagonal double Kodaira surface, associated with a strong diagonal double Kodaira structure of type $(b, n) = (2, 2)$ on an extra-special group $G$ of order 32. Then the real manifold $M$ underlying $S$ is a closed, orientable 4-manifold of signature 16 that can be realized as a real surface bundle over a real surface of genus 2, with fibre genus 41, in two different ways.

In fact, we may ask whether 16 and 41 are the minimum possible values for the signature and the fibre genus of a (non necessarily diagonal) double Kodaira fibration $f : S \rightarrow \Sigma_2 \times \Sigma_2$, cf. Corollary 4.10.

We believe that the results described above are significant for at least two reasons:

(i) although we know that $P_2(\Sigma_b)$ is residually $p$-finite for all prime number $p \geq 2$, see [BarBel09, pp. 1481-1490], so far there has been no systematic work aimed to describe its admissible quotients. The first results in this direction were those of A. Causin and the first author, who showed that both extra-special groups of order $p^{4b+1}$ appear as admissible quotients of $P_2(\Sigma_b)$ for all $b \geq 2$ and all prime numbers $p \geq 5$; moreover, if $p$ divides $b + 1$, then both extra-special groups of order $p^{2b+1}$ appear as admissible quotients, too. As we said before, the smallest admissible quotients detected in [CaPol19] and [Pol20], corresponding to the case $(b, p) = (3, 2)$, have order $2^7 = 128$.

Our Theorem B sheds some new light on this problem, by providing a sharp lower bound for the order of $G$: more precisely, if a finite group $G$ is an admissible quotient of $P_2(\Sigma_b)$ for some $b$, then $|G| \geq 32$, with equality if and only if $G$ is extra-special. Moreover, for both extra-special groups of order 32, Theorem B computes the number of admissible quotient maps $\varphi : P_2(\Sigma_2) \rightarrow G$, and the number of their equivalence classes up to the natural action of $\text{Aut}(G)$;

(ii) constructing (double) Kodaira fibrations with small signature is a rather difficult problem. As far as we know, before the present work the only examples with signature 16 were the ones listed in [LLR20, Table 3, Cases 6.2, 6.6, 6.7 (Type 1), 6.9]. Our examples in Theorem A are new, since both the base genera and the fibre genera are different from the ones in the aforementioned cases. Note that our results also show that every curve of genus 2 (and not only some special curve with extra automorphisms) is the base of a double Kodaira fibration with signature 16. Thus, we obtain two families of dimension 3 of such fibrations that, to the best of our knowledge, provide the first examples of positive-dimensional families of double Kodaira fibrations with small signature.

Finally, this work also contain a Computer Algebra part, concerning the calculation of the group $H_1(S, \mathbb{Z})$, where $S$ is as in Theorem D, by using the software GAP4, see [GAP4]. The result is the following:
Theorem E (see Proposition 4.14). Let \( f: S \longrightarrow \Sigma_2 \times \Sigma_2 \) be the diagonal double Kodaira fibration associated with a diagonal double Kodaira structure of type \((b, n) = (2, 2)\) on an extra-special group \(G\) of order 32. Then
\[
H_1(S, \mathbb{Z}) = \mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4.
\]
In particular, this homology group is independent both on \(G\) and on the chosen structure on it.

Thus \( b_1(S) = 8 \) and, subsequently, the pull-back map \( f^*: H^1(\Sigma_b \times \Sigma_b, \mathbb{Q}) \longrightarrow H^1(S, \mathbb{Q}) \) is an isomorphism. Following [Breg18], we will express this fact by saying that \( f \) is maximal, see Proposition 4.18. For an interpretation of maximality in terms of monodromy, see Corollary 4.17.

Let us now describe how this paper is organized. In Section 1 we introduce some algebraic preliminaries, in particular we discuss the so-called CCT-groups (Definition 1.1), namely, non-abelian groups of order \(32\) and \(7\) groups of order \(32\), see Remark 1.3, and they play a fundamental role in this paper, as we will soon explain. It turns out that there are precisely eight groups \(G\) with \(|G| \leq 32\) that are not CCT-groups, namely \(S_3\) and seven groups of order 32, see Corollary 1.6, Proposition 1.7 and Proposition 1.14.

In Section 2 we define diagonal double Kodaira structures on finite groups and we explain the relation with their counterpart in geometric topology, namely admissible group epimorphisms from pure surface braid groups.

Section 3 is devoted to the study of diagonal double Kodaira structures in groups of order at most 32. One crucial technical result is Proposition 3.4, stating that there are no such structures on CCT-groups. Thus, in order to prove the first part of Theorem A, we only need to exclude the existence of diagonal double Kodaira structures on \(S_3\) and on the five non-abelian, non-CCT groups of order 32; this is done in Proposition 3.9 and Proposition 3.11, respectively. The second part of Theorem A, i.e. the computation of number of structures in each case, is obtained by using some techniques borrowed from [Win72]; more precisely, we exploit the fact that \(V = G/\mathbb{Z}(G)\) is a symplectic vector space of dimension 4 over \(\mathbb{Z}_2\), and that \(\text{Out}(G)\) embeds in \(\text{Sp}(4, \mathbb{Z}_2)\) as the orthogonal group associated with the quadratic form \(q: V \longrightarrow \mathbb{Z}_2\) related to the symplectic form \(\langle \cdot, \cdot \rangle\) by \(q(x, y) = q(x) + q(y) + (x, y)\).

Finally, in Section 4 we establish the relation between our algebraic results and the geometrical framework of diagonal double Kodaira fibrations, and we provide the proofs of Theorems B, C, and D; furthermore, we state Theorem E, discussing some of its consequences.

The paper ends with two appendices. In Appendix A we collect the presentations for the non-abelian groups of order 24 and 32 that we used in our calculations, while Appendix B contains the details about the computational proof of Theorem E.

Notation and conventions. If \(S\) is a complex, non-singular projective surface, then \(c_1(S)\), \(c_2(S)\) denote the first and second Chern class of its tangent bundle \(T_S\), respectively. If \(X\) is a topological space, the fundamental group of \(X\) will be denoted by \(\pi_1(X)\) and its first Betti number by \(b_1(X)\).

Throughout the paper we use the following notation for groups:

- \(\mathbb{Z}_n\): cyclic group of order \(n\).
- \(G = N \rtimes Q\): semi-direct product of \(N\) and \(Q\), namely, split extension of \(Q\) by \(N\), where \(N\) is normal in \(G\).
- \(G = N.Q\): non-split extension of \(Q\) by \(N\).
- \(\text{Aut}(G)\): the automorphism group of \(G\).
- \(D_{p,q,r} = \mathbb{Z}_q \rtimes \mathbb{Z}_p = \langle x, y \mid xp = y^p = 1, xyyx^{-1} = y^r \rangle\): split metacyclic group of order \(pq\). The group \(D_{2, n, -1}\) is the dihedral group of order \(2n\) and will be denoted by \(D_{2n}\).
• If \( n \) is an integer greater or equal to 4, we denote by \( \text{QD}_{2n} \) the quasi-dihedral group of order \( 2^n \), having presentation
\[
\text{QD}_{2n} := \langle x, y \mid x^2 = y^{2^{n-1}} = 1, xyx^{-1} = y^{2^{n-2} - 1} \rangle.
\]
• The generalized quaternion group of order \( 4n \) is denoted by \( \text{Q}_4n \) and is presented as
\[
\text{Q}_4n = \langle x, y, z \mid x^n = y^2 = z^2 = xyz \rangle.
\]
For \( n = 2 \) we obtain the usual quaternion group \( \text{Q}_8 \), for which we adopt the classical presentation
\[
\text{Q}_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle,
\]
denoting by \(-1\) the unique element of order 2.
• \( S_n, A_n \): symmetric, alternating group on \( n \) symbols. We write the composition of permutations from the right to the left; for instance, \( (13)(12) = (123) \).
• \( \text{GL}(n, \mathbb{F}_q), \text{SL}(n, \mathbb{F}_q), \text{Sp}(n, \mathbb{F}_q) \): general linear group, special linear group and symplectic group of \( n \times n \) matrices over a field with \( q \) elements.
• The order of a finite group \( G \) is denoted by \(|G|\). If \( x \in G \), the order of \( x \) is denoted by \( o(x) \) and its centralizer in \( G \) by \( C_G(x) \).
• If \( x, y \in G \), their commutator is defined as \([x, y] = xyx^{-1}y^{-1}\).
• The commutator subgroup of \( G \) is denoted by \([G, G]\), the center of \( G \) by \( Z(G) \).
• If \( S = \{s_1, \ldots, s_n\} \subseteq G \), the subgroup generated by \( S \) is denoted by \( \langle S \rangle = \langle s_1, \ldots, s_n \rangle \).
• \( \text{IdSmallGroup}(G) \) indicates the label of the group \( G \) in the \texttt{GAP4} database of small groups. For instance \( \text{IdSmallGroup}(\text{D}_4) = G(8, 3) \) means that \( \text{D}_4 \) is the third in the list of groups of order 8.
• If \( N \) is a normal subgroup of \( G \) and \( g \in G \), we denote by \( \bar{g} \) the image of \( g \) in the quotient group \( G/N \).

1. **GROUP-THEORETICAL PRELIMINARIES: CCT-GROUPS AND EXTRA-SPECIAL GROUPS**

**Definition 1.1.** A non-abelian, finite group \( G \) is said to be a center commutative-transitive group (or a CCT-group, for short) if commutativity is a transitive relation on the set on non-central elements of \( G \). In other words, if \( x, y, z \in G - Z(G) \) and \([x, y] = [y, z] = 1\), then \([x, z] = 1\).

**Proposition 1.2.** For a finite group \( G \), the following properties are equivalent.

1. \( G \) is a CCT-group.
2. For every pair \( x, y \) of non-central elements in \( G \), the relation \([x, y] = 1\) implies \( C_G(x) = C_G(y) \).
3. For every non-central element \( x \in G \), the centralizer \( C_G(x) \) is abelian.

**Proof.**
1. \( \Rightarrow \) (2) Take two commuting elements \( x, y \in G - Z(G) \) and let \( z \in C_G(x) \). If \( z \) is central then \( z \in C_G(y) \) by definition, otherwise \([x, y] = [x, z] = 1\) implies \([y, z] = 1\) by the assumption that \( G \) is a CCT-group. Therefore we get \( C_G(x) \subseteq C_G(y) \), and exchanging the roles of \( x, y \) we can deduce the reverse inclusion.

2. \( \Rightarrow \) (3) Given any element \( x \in G - Z(G) \), it is sufficient to check that \([y, z] = 1\) for every pair of non-central elements \( y, z \in C_G(x) \). By assumption, \( C_G(y) = C_G(z) \), hence \( y \in C_G(z) \) and we are done.

3. \( \Rightarrow \) (1) Let \( x, y, z \in G - Z(G) \) and suppose \([x, y] = [y, z] = 1\), namely, \( x, z \in C_G(y) \). Since we are assuming that \( C_G(y) \) is abelian, this gives \([x, z] = 1\), hence \( G \) is a CCT-group.

**Remark 1.3.** CCT-groups are of historical importance in the context of classification of finite simple groups, see for instance \( [\text{Suz}61] \), where they are called \text{CA}-groups. Further references on the topic are \( [\text{Schm}70], [\text{Reb}71], [\text{Rocke}73], [\text{Wu}98] \).

**Lemma 1.4.** If \( G \) is a finite group such that \( G/Z(G) \) is cyclic, then \( G \) is abelian.
Proposition 1.5. Let $G$ be a non-abelian, finite group.

(1) If $|G|$ is the product of at most three prime factors (non necessarily distinct), then $G$ is a CCT-group.

(2) If $|G| = p^4$, with $p$ prime, then $G$ is a CCT-group.

(3) If $G$ contains an abelian normal subgroup of prime index, then $G$ is a CCT-group.

Proof. (1) Assume that $|G|$ is the product of at most three prime factors, and take a non-central element $y$. Then the centralizer $C_G(y)$ has non-trivial center, because $1 \neq y \in C_G(y)$, and its order is the product of at most two primes. Therefore the quotient of $C_G(y)$ by its center is cyclic, hence $C_G(y)$ is abelian by Lemma 1.4.

(2) Assume $|G| = p^4$ and suppose by contradiction that there exist three elements $x, y, z \in G - Z(G)$ such that $[x, y] = [y, z] = 1$ but $[x, z] \neq 1$. They generate a non-abelian subgroup $N = \langle x, y, z \rangle$, which is not the whole of $G$ since $y \in Z(N)$ but $y \notin Z(G)$. It follows that $N$ has order $p^3$ and so, by Lemma 1.4, its center is cyclic of order $p$, generated by $y$. The group $G$ is a finite $p$-group, hence a nilpotent group; being a proper subgroup of maximal order in a nilpotent group, $N$ is normal in $G$ (see [Mac12, Corollary 5.2]), so we have a conjugacy homomorphism $G \to \text{Aut}(N)$, that in turn induces a conjugacy homomorphism $G \to \text{Aut}(Z(N)) \simeq \mathbb{Z}_{p-1}$. The image of such a homomorphism must have order dividing both $p^3$ and $p-1$, hence it is trivial. In other words, the conjugacy action of $G$ on $Z(N) = \langle y \rangle$ is trivial, hence $y$ is central in $G$, contradiction.

(3) Let $N$ be an abelian normal subgroup of $G$ such that $G/N$ has prime order $p$. As $G/N$ has no non-trivial proper subgroups, it follows that $N$ is a maximal subgroup of $G$. Let $x$ be any non-central element of $G$, so that $C_G(x)$ is a proper subgroup of $G$; then there are two possibilities.

Case 1: $x \in N$. Then $N \subseteq C_G(x)$ and so, by the maximality of $N$, we get $C_G(x) = N$, which is abelian.

Case 2: $x \notin N$. Then the image of $x$ generates $G/N$, and so every element $y \in G$ can be written in the form $y = ux^r$, where $u \in N$ and $0 \leq r \leq p-1$. In particular, if $y \in C_G(x)$, the condition $[x, y] = 1$ yields $[x, u] = 1$, namely $u \in N \cap C_G(x)$. Since $N$ is abelian, it follows that $C_G(x)$ is abelian, too.

We now want to classify non-abelian, non-CCT groups of order at most 32. First of all, as an immediate consequence of Parts (1) and (2) of Proposition 1.5, we have the following

Corollary 1.6. Let $G$ be a non-abelian, finite group such that $|G| \leq 32$. If $G$ is not a CCT-group, then either $|G| = 24$ or $|G| = 32$.

Let us start by disposing of the case $G = 24$.

Proposition 1.7. Let $G$ be a non-abelian finite group such that $|G| = 24$ and $G$ is not a CCT-group. Then $G = S_4$.

Proof. We start by observing that $S_4$ is not a CCT-group. In fact, $(1234)$ commutes to its square $(13)(24)$, which commutes to $(12)(34)$, but $(1234)$ and $(12)(34)$ do not commute.

What is left is to show that the remaining non-abelian groups of order 24 are all CCT-groups; we will do a case-by-case analysis, referring the reader to the presentations given in Table 1 of Appendix A. Apart from $G = G(24, 3) = \text{SL}(2, \mathbb{F}_3)$, for which we give an ad-hoc proof, we will show that all these groups contain an abelian subgroup $N$ of prime index, so that we can conclude by using Part (3) of Proposition 1.5.

- $G = G(24, 1)$. Take $N = \langle x^2y \rangle \simeq \mathbb{Z}_{12}$. 

- \( G = G(24, 3) \). The action of \( \text{Aut}(G) \) has five orbits, whose representative elements are \( \{1, x, x^2, z, z^2\} \), see [SL(2,3)]. We have \( \langle z^2 \rangle = Z(G) \) and so, since \( C_G(x) \subseteq C_G(x^2) \), it suffices to show that the centralizers of \( x^2 \) and \( z \) are both abelian. In fact, we have

\[
C_G(x^2) = \langle x \rangle \simeq \mathbb{Z}_6, \quad C_G(z) = \langle z \rangle \simeq \mathbb{Z}_4.
\]

- \( G = G(24, 4) \). Take \( N = \langle x \rangle \simeq \mathbb{Z}_{12} \).
- \( G = G(24, 5) \). Take \( N = \langle y \rangle \simeq \mathbb{Z}_{12} \).
- \( G = G(24, 6) \). Take \( N = \langle y \rangle \simeq \mathbb{Z}_{12} \).
- \( G = G(24, 7) \). Take \( N = \langle z, x^2y \rangle \simeq \mathbb{Z}_6 \times \mathbb{Z}_2 \).
- \( G = G(24, 8) \). Take \( N = \langle y, z \rangle \simeq \mathbb{Z}_6 \times \mathbb{Z}_2 \).
- \( G = G(24, 10) \). Take \( N = \langle z, y \rangle \simeq \mathbb{Z}_{12} \).
- \( G = G(24, 11) \). Take \( N = \langle z, i \rangle \simeq \mathbb{Z}_{12} \).
- \( G = G(24, 13) \). Take \( N = \langle z \rangle \times \mathbb{V}_4 \simeq (\mathbb{Z}_2)^3 \), where \( \mathbb{V}_4 = \langle (1 2)(3 4), (1 3)(2 4) \rangle \) is the Klein subgroup.
- \( G = G(24, 14) \). Take \( N = \langle z, w \rangle \times \langle (123) \rangle \simeq \mathbb{Z}_6 \times \mathbb{Z}_2 \).

This completes the proof. \( \square \)

The next step is to classify non-abelian, non-CCT groups \( G \) with \(|G| = 32\); it will turn out that there are precisely seven of them, see Proposition 1.14. Before doing this, let us introduce the following classical definition, see for instance [Gor07, p. 183] and [Is08, p. 123].

**Definition 1.8.** Let \( p \) be a prime number. A finite \( p \)-group \( G \) is called extra-special if its center \( Z(G) \) is cyclic of order \( p \) and the quotient \( V = G/Z(G) \) is a non-trivial, elementary abelian \( p \)-group.

An elementary abelian \( p \)-group is a finite-dimensional vector space over the field \( \mathbb{Z}_p \), hence it is of the form \( V = (\mathbb{Z}_p)^{\dim V} \) and \( G \) fits into a short exact sequence

\[
1 \longrightarrow \mathbb{Z}_p \longrightarrow G \longrightarrow V \longrightarrow 1.
\]

Note that, \( V \) being abelian, we must have \(|G, G| = \mathbb{Z}_p\), namely the commutator subgroup of \( G \) coincides with its center. Furthermore, since the extension (3) is central, it cannot be split, otherwise \( G \) would be isomorphic to the direct product of the two abelian groups \( \mathbb{Z}_p \) and \( V \), which is impossible because \( G \) is non-abelian.

If \( G \) is extra-special, then we can define a map \( \omega : V \times V \longrightarrow \mathbb{Z}_p \) as follows: for every \( v_1, v_2 \in V \), we set \( \omega(v_1, v_2) = [g_1, g_2] \), where \( g_i \) is any lift of \( v_i \) in \( G \). This turns out to be a symplectic form on \( V \), hence \( \dim V \) is even and \( |G| = p^{\dim V + 1} \) is an odd power of \( p \).

For every prime number \( p \), there are precisely two isomorphism classes \( M(p), N(p) \) of non-abelian groups of order \( p^3 \), namely

\[
M(p) = \langle r, t, z | r^p = t^p = 1, z^p = 1, [r, z] = [t, z] = 1, [r, t] = z^{-1} \rangle
\]

\[
N(p) = \langle r, t, z | r^p = t^p = z, z^p = 1, [r, z] = [t, z] = 1, [r, t] = z^{-1} \rangle
\]

and both of them are in fact extra-special, see [Gor07, Theorem 5.1 of Chapter 5].

If \( p \) is odd, then the groups \( M(p) \) and \( N(p) \) are distinguished by their exponent, which equals \( p \) and \( p^2 \), respectively. If \( p = 2 \), the group \( M(p) \) is isomorphic to the dihedral group \( D_8 \), whereas \( N(p) \) is isomorphic to the quaternion group \( Q_8 \).

The classification of extra-special \( p \)-groups is now provided by the result below, see [Gor07, Section 5 of Chapter 5] and [CaPol19, Section 2].

**Proposition 1.9.** If \( b \geq 2 \) is a positive integer and \( p \) is a prime number, there are exactly two isomorphism classes of extra-special \( p \)-groups of order \( p^{2b+1} \), that can be described as follows.
The central product $H_{2b+1}(\mathbb{Z}_p)$ of $b$ copies of $M(p)$, having presentation

$$H_{2b+1}(\mathbb{Z}_p) = \langle r_1, t_1, \ldots, r_b, t_b, z \mid r_i^p = t_i^p = z^p = 1,$$

\begin{align}
[r_j, z] &= [t_j, z] = 1, \\
[r_j, r_k] &= [t_j, t_k] = 1, \\
[r_j, t_k] &= z^{-\delta_{jk}}. \tag{4}\end{align}

If $p$ is odd, this group has exponent $p$ and is isomorphic to the matrix Heisenberg group $H_{2b+1}(\mathbb{Z}_p) \subset \text{GL}(b+1, \mathbb{Z}_p)$ of dimension $2b+1$ over the field $\mathbb{Z}_p$.

The central product $G_{2b+1}(\mathbb{Z}_p)$ of $b-1$ copies of $M(p)$ and one copy of $N(p)$, having presentation

$$G_{2b+1}(\mathbb{Z}_p) = \langle r_1, t_1, \ldots, r_b, t_b, z \mid r_i^p = t_i^p = \ldots = r_{b-1}^p = t_{b-1}^p = z^p = 1,$$

\begin{align}
[r_j, z] &= [t_j, z] = 1, \\
[r_j, r_k] &= [t_j, t_k] = 1, \\
[r_j, t_k] &= z^{-\delta_{jk}}. \tag{5}\end{align}

If $p$ is odd, this group has exponent $p^2$.

**Remark 1.10.** In both cases, from the relations above we deduce

$$[r_j^{-1}, t_k] = z^{\delta_{jk}}, \quad [r_j^{-1}, t_k^{-1}] = z^{-\delta_{jk}}.$$

**Remark 1.11.** For both groups $H_{2b+1}(\mathbb{Z}_p)$ and $G_{2b+1}(\mathbb{Z}_p)$, the center coincides with the derived subgroup and is equal to $\langle z \rangle \simeq \mathbb{Z}_p$.

**Remark 1.12.** If $p = 2$, we can distinguish the two groups $H_{2b+1}(\mathbb{Z}_p)$ and $G_{2b+1}(\mathbb{Z}_p)$ by counting the number of elements of order 4.

**Remark 1.13.** The groups $H_{2b+1}(\mathbb{Z}_p)$ and $G_{2b+1}(\mathbb{Z}_p)$ are not CCT-groups. In fact, let us take two distinct indices $j, k \in \{1, \ldots, b\}$ and consider the non-central elements $r_j, t_j, t_k$. Then we have $[r_j, t_k] = [t_k, t_j] = 1$, but $[r_j, t_j] = z^{-1}$.

We can now dispose of the case $|G| = 32$.

**Proposition 1.14.** Let $G$ be a non-abelian, finite group such that $|G| = 32$ and $G$ is not a CCT-group. Then $G = G(32, t)$, where $t \in \{6, 7, 8, 43, 44, 49, 50\}$. Here $G(32, 49) = H_5(\mathbb{Z}_2)$ and $G(32, 50) = G_5(\mathbb{Z}_2)$ are the two extra-special groups of order 32, in particular they have nilpotence class 2, whereas the remaining five groups have nilpotence class 3.

**Proof.** We first do a case-by-case analysis showing that, if $t \notin \{6, 7, 8, 43, 44, 49, 50\}$, then $G = G(32, t)$ contains an abelian subgroup $N$ of index 2, so that $G$ is a CCT-group by Part (3) of Proposition 1.1. In every case, we refer the reader to the presentation given in Table 2 of Appendix A.

- $G = G(32, 2)$. Take $N = \langle x, y^2, z \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$.
- $G = G(32, 4)$. Take $N = \langle x, y^2 \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 5)$. Take $N = \langle x, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 9)$. Take $N = \langle x, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 10)$. Take $N = \langle ix, k \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 11)$. Take $N = \langle x, y \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 12)$. Take $N = \langle x^2, y \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 13)$. Take $N = \langle x^2, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 14)$. Take $N = \langle y^2, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 15)$. Take $N = \langle x^2, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 17)$. Take $N = \langle y \rangle \simeq \mathbb{Z}_{16}$. 


\begin{itemize}
  
  \item $G = G(32, 18)$. Take $N = \langle y \rangle \simeq \mathbb{Z}_{16}$. 
  \item $G = G(32, 19)$. Take $N = \langle y \rangle \simeq \mathbb{Z}_{16}$. 
  \item $G = G(32, 20)$. Take $N = \langle x \rangle \simeq \mathbb{Z}_{16}$. 
  \item $G = G(32, 22)$. Take $N = \langle w \rangle \times \langle x, y \rangle \simeq \mathbb{Z}_8 \times (\mathbb{Z}_2)^2$. 
  \item $G = G(32, 23)$. Take $N = \langle z \rangle \times \langle x, y^2 \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$. 
  \item $G = G(32, 24)$. Take $N = \langle x, y \rangle \simeq (\mathbb{Z}_4)^2$. 
  \item $G = G(32, 25)$. Take $N = \langle z \rangle \times \langle y^2 \rangle \simeq (\mathbb{Z}_2)^2$. 
  \item $G = G(32, 26)$. Take $N = \langle z \rangle \times \langle i \rangle \simeq (\mathbb{Z}_4)^2$. 
  \item $G = G(32, 27)$. Take $N = \langle x, y, a, b \rangle \simeq (\mathbb{Z}_4)^4$. 
  \item $G = G(32, 28)$. Take $N = \langle x, y, z \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$. 
  \item $G = G(32, 29)$. Take $N = \langle x, i, z \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$. 
  \item $G = G(32, 30)$. Take $N = \langle x, y, z \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$. 
  \item $G = G(32, 31)$. Take $N = \langle x, y \rangle \simeq (\mathbb{Z}_4)^2$. 
  \item $G = G(32, 32)$. Take $N = \langle y, z \rangle \simeq (\mathbb{Z}_4)^2$. 
  \item $G = G(32, 33)$. Take $N = \langle x, y \rangle \simeq (\mathbb{Z}_4)^2$. 
  \item $G = G(32, 34)$. Take $N = \langle x, y \rangle \simeq (\mathbb{Z}_4)^2$. 
  \item $G = G(32, 35)$. Take $N = \langle x, k \rangle \simeq (\mathbb{Z}_4)^2$. 
  \item $G = G(32, 37)$. Take $N = \langle x, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$. 
  \item $G = G(32, 38)$. Take $N = \langle x, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$. 
  \item $G = G(32, 39)$. Take $N = \langle z \rangle \times \langle y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$. 
  \item $G = G(32, 40)$. Take $N = \langle z \rangle \times \langle y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$. 
  \item $G = G(32, 41)$. Take $N = \langle w \rangle \times \langle x \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$. 
  \item $G = G(32, 42)$. Take $N = \langle x, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$. 
  \item $G = G(32, 46)$. Take $N = \langle z, w \rangle \times \langle y \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$. 
  \item $G = G(32, 47)$. Take $N = \langle z, w \rangle \times \langle i \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$. 
  \item $G = G(32, 48)$. Take $N = \langle x, y, z \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$. 
\end{itemize}

It remains to show that $G = G(32, t)$ is not a CCT-group for $t \in \{6, 7, 8, 43, 44, 49, 50\}$, and to compute the nilpotency class in each case. In the sequel, we will denote by $G = \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \ldots$ the lower central series of $G$. Recall that a group $G$ has nilpotency class $c$ if $\Gamma_c \neq \{1\}$ and $\Gamma_{c+1} = \{1\}$.

For $t = 49$ and $t = 50$ we have the two extra-special cases, that are not CCT-groups by Remark 1.13; their nilpotency class is 2 by Remark 1.11. Let us now deal with the remaining cases. For each of them, we exhibit three non-central elements for which commutativity is not a transitive relation, and we show that $\Gamma_3 \neq \{1\}$ and $\Gamma_4 = \{1\}$; note that this means that $\Gamma_2 = [G, G]$ is not contained in $Z(G)$, whereas $\Gamma_3 = [\Gamma_2, G]$ is contained in $Z(G)$.

\begin{itemize}
  
  \item $G = G(32, 6)$. The center of $G$ is $Z(G) = \langle x \rangle \simeq \mathbb{Z}_2$. We have $[y, w^2] = [w^2, w] = 1$, but $[y, w] = x$. The derived subgroup of $G$ is $\Gamma_2 = [G, G] = \langle x, y \rangle \simeq (\mathbb{Z}_2)^2$, and a short computation gives $\Gamma_3 = Z(G)$, so $c = 3$. 
  \item $G = G(32, 7)$. The center of $G$ is $Z(G) = \langle w \rangle \simeq \mathbb{Z}_2$. We have $[y, z] = [z, u] = 1$, but $[y, w] = w$. The derived subgroup of $G$ is $\Gamma_2 = [G, G] = \langle w, z \rangle \simeq (\mathbb{Z}_2)^2$, and a short computation gives $\Gamma_3 = Z(G)$, so $c = 3$. 
  \item $G = G(32, 8)$. The center of $G$ is $Z(G) = \langle x^4 \rangle \simeq \mathbb{Z}_2$. We have $[x, x^2] = [x^2, y] = 1$, but $[x, y] = z^2$. The derived subgroup of $G$ is $\Gamma_2 = [G, G] = \langle x^4, y \rangle \simeq (\mathbb{Z}_2)^2$, and a short computation gives $\Gamma_3 = Z(G)$, so $c = 3$. 
  \item $G = G(32, 43)$. The center of $G$ is $Z(G) = \langle z^4 \rangle \simeq \mathbb{Z}_2$. We have $[x, x^2] = [x^2, z] = 1$, but $[x, z] = x^4$. The derived subgroup of $G$ is $\Gamma_2 = [G, G] = \langle x^2 \rangle \simeq \mathbb{Z}_4$, and a short computation gives $\Gamma_3 = Z(G)$, so $c = 3$. 
  \item $G = G(32, 44)$. The center of $G$ is $Z(G) = \langle z^2 \rangle \simeq \mathbb{Z}_2$. We have $[x, xk] = [xk, z] = 1$, but $[x, z] = z^2$. The derived subgroup of $G$ is $\Gamma_2 = [G, G] = \langle k \rangle \simeq \mathbb{Z}_4$, and a short computation gives $\Gamma_3 = Z(G)$, so $c = 3$. 
\end{itemize}
This completes the proof. □

2. DIAGONAL DOUBLE KODAIRA STRUCTURES

For more details on the material contained in this section, we refer the reader to [CaPol19] and [Pol20]. Let $G$ be a finite group and let $b, n \geq 2$ be two positive integers.

**Definition 2.1.** A diagonal double Kodaira structure of type $(b, n)$ on $G$ is an ordered set of $4b+1$ generators

$$G = \langle r_{11}, t_{11}, \ldots, r_{1b}, t_{1b}, r_{21}, t_{21}, \ldots, r_{2b}, t_{2b}, z \rangle,$$

with $o(z) = n$, such that the following relations are satisfied. We systematically use the commutator notation in order to indicate relations of conjugacy type, writing for instance $[x, y] = zy^{-1}$ instead of $xyx^{-1} = z$.

- **Surface relations**
  
  $$[r_{1b}^{-1}, t_{1b}^{-1}, t_{1b}^{-1}] t_{1b}^{-1} \cdots [r_{11}^{-1}, t_{11}^{-1}] t_{11}^{-1} (t_{11} t_{12} \cdots t_{1b}) = z$$
  $$[r_{21}^{-1}, t_{21}] t_{21} [r_{22}^{-1}, t_{22}] t_{22} \cdots [r_{2b}^{-1}, t_{2b}] t_{2b} (t_{2b} t_{2b-1} \cdots t_{21}) = z^{-1}$$

- **Conjugacy action of $r_{ij}$**
  
  $$(6) \: [r_{ij}, r_{2k}] = 1 \quad \text{if} \: j < k$$
  $$[r_{ij}, r_{2k}] = 1 \quad \text{if} \: j > k$$
  $$[r_{ij}, t_{2k}] = 1 \quad \text{if} \: j < k$$
  $$[r_{ij}, t_{2k}] = z^{-1} \quad \text{if} \: j > k$$
  $$[r_{ij}, z] = [r_{2k}^{-1}, z]$$

- **Conjugacy action of $t_{ij}$**
  
  $$(7) \: [t_{ij}, r_{2k}] = 1 \quad \text{if} \: j < k$$
  $$[t_{ij}, r_{2k}] = t_{2j}^{-1} z t_{2j} \quad \text{if} \: j > k$$
  $$[t_{ij}, t_{2k}] = 1 \quad \text{if} \: j < k$$
  $$[t_{ij}, t_{2k}] = t_{2j}^{-1} z \quad \text{if} \: j > k$$
  $$[t_{ij}, z] = [t_{2k}^{-1}, z]$$

**Remark 2.2.** From (6) and (7) we can infer the corresponding conjugacy actions of $r_{ij}^{-1}$ and $t_{ij}^{-1}$. We leave the cumbersome but standard computations to the reader.

**Remark 2.3.** Abelian groups admit no diagonal double Kodaira structures. Indeed, the relation $[r_{ij}, t_{2j}] = z^{-1}$ in (6) provides a non-trivial commutator in $G$, because $o(z) = n$.

**Remark 2.4.** If $[G, G] \subseteq Z(G)$, then the relations defining a diagonal double Kodaira structure of type $(b, n)$ assume the following simplified form.
• Relations expressing the centrality of \( z \)

\[
[r_{1j}, z] = [t_{1j}, z] = [r_{2j}, z] = [t_{2j}, z] = 1
\]

• Surface relations

\[
[r_{1b}^{-1}, t_{1b}^{-1}] [r_{1b-1}^{-1}, t_{1b-1}^{-1}] \cdots [r_{11}^{-1}, t_{11}^{-1}] = z
\]

\[
[r_{21}^{-1}, t_{21}] [r_{22}^{-1}, t_{22}] \cdots [r_{2b}^{-1}, t_{2b}] = z^{-1}
\]

• Conjugacy action of \( r_{1j} \)

\[
[r_{1j}, r_{2k}] = 1 \quad \text{for all } j, k
\]

\[
[r_{1j}, t_{2k}] = z^{-\delta_{jk}}
\]

• Conjugacy action of \( t_{1j} \)

\[
[t_{1j}, r_{2k}] = z^{\delta_{jk}}
\]

\[
[t_{1j}, t_{2k}] = 1 \quad \text{for all } j, k
\]

where \( \delta_{jk} \) stands for the Kronecker symbol. Note that, being \( G \) non-abelian by Remark 2.3, the condition \( G, G \subset Z(\Sigma) \) is equivalent to \( G \) having nilpotency class 2, see [Is08, p. 22].

The definition of diagonal double Kodaira structure can be motivated by means of some well-known concepts in geometric topology. Let \( \Sigma_b \) be a closed Riemann surface of genus \( b \) and let \( \mathcal{P} = (p_1, p_2) \) be an ordered set of two distinct points on it. Let \( \Delta \subset \Sigma_b \times \Sigma_b \) be the diagonal. We denote by \( P_2(\Sigma_b) \) the pure braid group of genus \( b \) on two strands, which is isomorphic to the fundamental group \( \pi_1(\Sigma_b \times \Sigma_b - \Delta, \mathcal{P}) \). By Gonçalves-Guaschi’s presentation of surface pure braid groups, see [GG04, Theorem 7], [CaPol19, Theorem 1.7], we see that \( P_2(\Sigma_b) \) can be generated by \( 4g + 1 \) elements

\[
\rho_{11}, \tau_{11}, \ldots, \rho_{1b}, \tau_{1b}, \rho_{21}, \ldots, \rho_{2b}, \tau_{2b}
\]

subject to the following set of relations.

• Surface relations

\[
[r_{21}^{-1}, \tau_{21}^{-1}] \tau_{21}^{-1} [r_{22}^{-1}, \tau_{22}^{-1}] \cdots [r_{2b}^{-1}, \tau_{2b}^{-1}] (\tau_{11} \tau_{12} \cdots \tau_{1b}) = A_{12}^{-1}
\]

\[
[r_{21}^{-1}, \tau_{21}] \tau_{21} [r_{22}^{-1}, \tau_{22}] \cdots [r_{2b}^{-1}, \rho_{2b}] (\tau_{2b}^{-1} \tau_{2b-1} \cdots \tau_{21}^{-1}) = A_{12}^{-1}
\]

• Conjugacy action of \( \rho_{1j} \)

\[
[r_{1j}, \rho_{2k}] = 1 \quad \text{if } j < k
\]

\[
[r_{1j}, \rho_{2j}] = 1
\]

\[
[r_{1j}, \rho_{2k}] = A_{12}^{-1} \rho_{2k} \rho_{2j}^{-1} A_{12} \rho_{2j} \rho_{2k}^{-1}
\]

\[
[r_{1j}, \tau_{2k}] = 1 \quad \text{if } j < k
\]

\[
[r_{1j}, \tau_{2j}] = A_{12}^{-1}
\]

\[
[r_{1j}, \tau_{2k}] = [A_{12}^{-1}, \tau_{2k}]
\]

\[
[r_{1j}, A_{12}] = [\rho_{2j}^{-1}, A_{12}]
\]
• Conjugacy action of $\tau_{ij}$

$[\tau_{1j}, \rho_{2k}] = 1$ if $j < k$

$[\tau_{1j}, \rho_{2j}] = \tau_{2j}^{-1} A_{12} \tau_{2j}$

$[\tau_{1j}, \rho_{2k}] = [\tau_{2j}^{-1}, A_{12}]$ if $j > k$

$[\tau_{1j}, \tau_{2k}] = 1$ if $j < k$

$[\tau_{1j}, \tau_{2j}] = [\tau_{2j}^{-1}, A_{12}]$

$[\tau_{1j}, \tau_{2k}] = \tau_{2j}^{-1} A_{12} \tau_{2j} A_{12}^{-1} \tau_{2k} A_{12} \tau_{2j}^{-1} A_{12}^{-1} \tau_{2k}^{-1}$ if $j > k$

$[\tau_{1j}, A_{12}] = [\tau_{2j}^{-1}, A_{12}]$

Here the elements $\rho_{ij}$ and $\tau_{ij}$ are the braids depicted in Figure 1, whereas $A_{12}$ is the braid depicted in Figure 2.

**Figure 1.** The pure braids $\rho_{1j}$ and $\rho_{2j}$ on $\Sigma_b$. If $\ell \neq i$, the path corresponding to $\rho_{ij}$ and $\tau_{ij}$ based at $p_\ell$ is the constant path.

**Figure 2.** The pure braid $A_{12}$ on $\Sigma_b$

**Remark 2.5.** Under the identification of $P_2(\Sigma_b)$ with $\pi_1(\Sigma_b \times \Sigma_b - \Delta, \mathcal{P})$, the generator $A_{12} \in P(\Sigma_b)$ represents the homotopy class $\gamma_\Delta \in \pi_1(\Sigma_b \times \Sigma_b - \Delta, \mathcal{P})$ of a loop in $\Sigma_b \times \Sigma_b$ that “winds once” around the diagonal $\Delta$.

We can now state the following

**Proposition 2.6.** A finite group $G$ admits a diagonal double Kodaira structure of type $(b, n)$ if and only if there is a surjective group homomorphism

(8) $\varphi : P_2(\Sigma_b) \to G$

such that $\varphi(A_{12})$ has order $n$.

**Proof.** If such a $\varphi : P_2(\Sigma_b) \to G$ exists, we can obtain a diagonal double Kodaira structure on $G$ by setting

(9) $r_{ij} = \varphi(\rho_{ij}), \quad t_{ij} = \varphi(\tau_{ij}), \quad z = \varphi(A_{12})$.

Conversely, if $G$ admits a diagonal double Kodaira structure, then (9) defines a group homomorphism $\varphi : P_2(\Sigma_b) \to G$ with the desired properties. $\square$
The braid group $\mathbb{P}_2(\Sigma_b)$ is the middle term of two split short exact sequences

\begin{equation}
1 \longrightarrow \pi_1(\Sigma_b - \{p_i\}, p_j) \longrightarrow \mathbb{P}_2(\Sigma_b) \longrightarrow \pi_1(\Sigma_b, p_i) \longrightarrow 1,
\end{equation}

where $\{i, j\} = \{1, 2\}$, induced by the two natural projections of pointed topological spaces

\begin{equation}
(\Sigma_b \times \Sigma_b - \Delta, \mathcal{P}) \longrightarrow (\Sigma_b, p_i),
\end{equation}

see [GG04, Theorem 1]. Since we have

\begin{align*}
\pi_1(\Sigma_b - \{p_2\}, p_1) &= \langle \rho_{11}, \tau_{11}, \ldots, \rho_{1b}, \tau_{1b}, A_{12} \rangle \\
\pi_1(\Sigma_b - \{p_1\}, p_2) &= \langle \rho_{21}, \tau_{21}, \ldots, \rho_{2b}, \tau_{2b}, A_{12} \rangle,
\end{align*}

it follows that the two subgroups

\begin{align*}
K_1 &= \langle r_{11}, t_{11}, \ldots, r_{1b}, t_{1b}, z \rangle \\
K_2 &= \langle r_{21}, t_{21}, \ldots, r_{2b}, t_{2b}, z \rangle
\end{align*}

are both normal in $G$, and that there are two short exact sequences

\begin{equation}
\begin{aligned}
1 &\longrightarrow K_1 \longrightarrow G \longrightarrow Q_2 \longrightarrow 1 \\
1 &\longrightarrow K_2 \longrightarrow G \longrightarrow Q_1 \longrightarrow 1,
\end{aligned}
\end{equation}

such the elements $r_{21}, t_{21}, \ldots, r_{2b}, t_{2b}$ yield a complete system of coset representatives for $Q_2$, whereas the elements $r_{11}, t_{11}, \ldots, r_{1b}, t_{1b}$ yield a complete system of coset representatives for $Q_1$.

Let us now give a couple of definitions, whose geometrical meaning will become clear in Section 4, see in particular Proposition 4.3 and Remark 4.4.

**Definition 2.7.** Let $\mathcal{S}$ be a diagonal double Kodaira structure of type $(b, n)$ on a finite group $G$. Its signature is defined as

\[ \sigma(\mathcal{S}) = \frac{1}{3} |G| (2b - 2) \left( 1 - \frac{1}{n^2} \right). \]

**Definition 2.8.** A diagonal double Kodaira structure on $G$ is called strong if $K_1 = K_2 = G$.

For later use, let us write down the special case consisting of a diagonal double Kodaira structure of type $(2, n)$. It is an ordered set of nine generators of $G$

\[ (r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, t_{22}, z), \]
with $o(z) = n$, subject to the following relations.

(S1) $[r_{12}^{-1}, t_{12}^{-1}]_1 t_{12}^{-1} [r_{11}^{-1}, t_{11}^{-1}]_1 t_{11}^{-1} (t_{11} t_{12}) = z$

(S2) $[r_{21}^{-1}, t_{21}]_1 t_{21} [r_{22}^{-1}, t_{22}]_1 t_{22} (t_{22}^{-1} t_{21}^{-1}) = z^{-1}$

(R1) $[r_{11}, r_{22}] = 1$

(R2) $[r_{11}, r_{21}] = 1$

(R3) $[r_{11}, t_{22}] = 1$

(R4) $[r_{11}, t_{21}] = z^{-1}$

(R5) $[r_{11}, z] = [r_{21}^{-1}, z]$

(R6) $[r_{12}, r_{22}] = 1$

(R7) $[r_{12}, r_{21}] = z^{-1} r_{21}^{-1} r_{22}^{-1} z r_{22}^{-1}$

(R8) $[r_{12}, t_{22}] = z^{-1}$

(R9) $[r_{12}, t_{21}] = [z^{-1}, t_{21}]$

(R10) $[r_{12}, z] = [r_{22}^{-1}, z]$

(T1) $[t_{11}, r_{22}] = 1$

(T2) $[t_{11}, r_{21}] = t_{21}^{-1} t_{21}^{-1}$

(T3) $[t_{11}, t_{22}] = 1$

(T4) $[t_{11}, t_{21}] = [t_{21}^{-1}, z]$

(T5) $[t_{11}, z] = [t_{21}^{-1}, z]$

(T6) $[t_{12}, r_{22}] = t_{22}^{-1} z t_{22}$

(T7) $[t_{12}, r_{21}] = [t_{22}^{-1}, z]$

(T8) $[t_{12}, t_{22}] = [t_{22}^{-1}, z]$

(T9) $[t_{12}, t_{21}] = t_{22}^{-1} z t_{22} z^{-1} t_{21} z t_{22} z^{-1} t_{22} t_{21}^{-1}$

(T10) $[t_{12}, z] = [t_{22}^{-1}, z]$

\[(13)\]

Remark 2.9. When $[G, G] \subseteq Z(G)$, we have

$[r_{11}, z] = [t_{11}, z] = [r_{12}, z] = [t_{12}, z] = 1$

$[r_{21}, z] = [t_{21}, z] = [r_{22}, z] = [t_{22}, z] = 1$

and the previous relations become

(S1') $[r_{12}^{-1}, t_{12}^{-1}]_1 [r_{11}^{-1}, t_{11}^{-1}]_1 = z$

(S2') $[r_{21}^{-1}, t_{21}]_1 [r_{22}^{-1}, t_{22}]_1 = z^{-1}$

(R1') $[r_{11}, r_{22}] = 1$

(R2') $[r_{11}, r_{21}] = 1$

(R3') $[r_{11}, t_{22}] = 1$

(R4') $[r_{11}, t_{21}] = z^{-1}$

(R5') $[r_{11}, z] = [r_{21}^{-1}, z]$

(R6') $[r_{12}, r_{22}] = 1$

(R7') $[r_{12}, r_{21}] = z^{-1} r_{21}^{-1} r_{22}^{-1} z r_{22}^{-1}$

(R8') $[r_{12}, t_{22}] = z^{-1}$

(R9') $[r_{12}, t_{21}] = [z^{-1}, t_{21}]$

(R10') $[r_{12}, z] = [r_{22}^{-1}, z]$

\[(14)\]

3. Structures on Groups of Order at Most 32

3.1. Prestructures.

Definition 3.1. Let $G$ be a finite group. A prestructure on $G$ is an ordered set of nine elements

$$(r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, t_{22}, z),$$

with $o(z) = n \geq 2$, subject to the relations (R1), ..., (R10), (T1), ..., (T10) in (13).
In other words, the nine elements must satisfy all the relations defining a diagonal double Kodaira structure of type \((2, n)\), except the surface relations. In particular, no abelian group admits prestructures. Note that we are not requiring that the elements of the prestructure generate \(G\).

**Proposition 3.2.** If a finite group \(G\) admits a diagonal double Kodaira structure of type \((b, n)\), then it admits a prestructure with \(o(z) = n\).

**Proof.** Consider the ordered set of nine elements \((r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, t_{22}, z)\) in Definition (2.1) and the relations satisfied by them, with the exception of the surface relations. □

**Remark 3.3.** Let \(G\) be a finite group that admits a prestructure. Then \(z\) and all its conjugates are non-trivial elements of \(G\) and so, from relations (R4), (R8), (T2), (T6), it follows that \(r_{11}, r_{12}, r_{21}, t_{12}, t_{21}, t_{22}\) are non-central elements of \(G\).

**Proposition 3.4.** If \(G\) is a CCT-group, then \(G\) admits no prestructures and, subsequently, no diagonal double Kodaira structures.

**Proof.** The second statement is a direct consequence of the first one (Proposition 3.2), hence it suffices then to check that \(G\) admits no prestructures. Otherwise, keeping in mind Remark 3.3, we see that (R6) and (T1) imply \([r_{12}, t_{11}] = 1\). From this and (T3) we get \([r_{12}, t_{22}] = 1\), that contradicts (R8).

Given a finite group \(G\), we define the socle of \(G\), denoted by \(\text{soc}(G)\), as the intersection of all non-trivial, normal subgroups of \(G\). For instance, \(G\) is simple if and only if \(\text{soc}(G) = G\).

**Definition 3.5.** A finite group \(G\) is called monolithic if \(\text{soc}(G) \neq \{1\}\). Equivalently, \(G\) is monolithic if it contains precisely one minimal non-trivial, normal subgroup.

**Example 3.6.** If \(G\) is an extra-special \(p\)-group, then \(G\) is monolithic and \(\text{soc}(G) = Z(G)\). Indeed, since \(Z(G) \cong \mathbb{Z}_p\) is normal in \(G\), by definition of socle we always have \(\text{soc}(G) \subseteq Z(G)\). On the other hand, every non-trivial, normal subgroup of an extra-special group contains the center (see [Rob96, Exercise 9 p. 146]), hence \(Z(G) \subseteq \text{soc}(G)\).

**Proposition 3.7.** The following holds.

1. Assume that \(G\) admits a prestructure, whereas no proper quotient of \(G\) does. Then \(G\) is monolithic and \(z \in \text{soc}(G)\).

2. Assume that \(G\) admits a prestructure, whereas no proper subgroup of \(G\) does. Then the elements of the prestructure generate \(G\).

**Proof.** (1) Let \(\mathcal{S} = (r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, t_{22}, z)\) be a prestructure in \(G\). Assume that there is a non-trivial normal subgroup \(N\) of \(G\) such that \(z \notin N\). Then \(\bar{z} \in G/N\) is non-trivial, and so \(\bar{S} = (\bar{r}_{11}, \bar{t}_{11}, \bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}, \bar{t}_{21}, \bar{r}_{22}, \bar{t}_{22}, \bar{z})\) is a prestructure in the quotient group \(G/N\), contradiction. Therefore we must have \(z \in \text{soc}(G)\), in particular, \(G\) is monolithic.

(2) Clear, because every prestructure \(\mathcal{S}\) in \(G\) is also a prestructure in the subgroup \((\mathcal{S})\). □

**Corollary 3.8.** Given a prestructure on an extra-special \(p\)-group \(G\), the element \(z\) is a generator of \(Z(G) \cong \mathbb{Z}_p\).

**Proof.** If \(G\) is extra-special, every proper quotient of \(G\) is abelian, hence it admits no prestructures. The result now follows from Example 3.6 and Proposition 3.7 (1).

Note that, by Corollary 3.8, in the case of extra-special \(p\)-groups the choice of calling \(z\) the element in the prestructure is coherent with presentations (4) and (5). The case of diagonal double Kodaira structures on extra-special groups of order 32 will be studied in Subsection 3.4.
3.2. The case $|G| < 32$.

**Proposition 3.9.** If $|G| < 32$, then $G$ admits no diagonal double Kodaira structures.

**Proof.** By Corollary 1.6, Proposition 1.7 and Proposition 3.4, it remains only to check that the symmetric group $S_4$ admits no prestructures. We start by observing that
\[
\text{soc}(S_4) = V_4 = \langle (12)(34), (13)(24) \rangle
\]
and so, by part (1) of Proposition 3.7, if $\mathcal{G}$ is a prestructure on $S_4$ then $z \in V_4$. Let $x, y \in S_4$ be such that $[x, y] = z$. Examining the tables of subgroups of $S_4$ given in [S4], by straightforward computations and keeping in mind that the cycle type determines the conjugacy class, we deduce that either $x, y \in C_{S_4}(z) \simeq D_8$ or $x, y \in A_4$. Every pair in $A_4$ includes at least a 3-cycle and so, if $[x, y] = z$ and both $x$ and $y$ have even order, then $x$ and $y$ centralize $z$.

If $x \in S_4$ is a 3-cycle, then $C_{S_4}(x) \simeq Z_3$. So, from relations (R1), (R2), (R3), (R6), it follows that, if one of the elements $r_{11}, r_{12}, r_{21}, r_{22}, t_{22}$ is a 3-cycle, then all these elements generate the same cyclic subgroup. This contradicts (R8), hence $r_{11}, r_{12}, r_{21}, r_{22}, t_{22}$ all have even order.

Let us look now at relation (R8). Since $r_{12}, t_{22}$ have even order, from the previous remark we infer $r_{12}, t_{22} \in C_{S_4}(z)$. Let us consider $r_{11}$. If $r_{11}$ belongs to $A_4$, being an element of even order it must be conjugate to $z$, and so it commutes with $z$; otherwise, by (R4), both $r_{11}$ and $t_{21}$ commute with $z$. Summing up, in any case we have $r_{11} \in C_{S_4}(z)$.

Relation (R5) can be rewritten as $r_{11}r_{21} \in C_{S_4}(z)$, hence $r_{21} \in C_{S_4}(z)$. Analogously, relation (R10) can be rewritten as $r_{12}r_{22} \in C_{S_4}(z)$, hence $r_{22} \in C_{S_4}(z)$.

Using relation (R9), we get $r_{12}z \in C_{S_4}(t_{21})$. Since $r_{12}$ and $z$ commute and their orders are powers of 2, it follows that $o(r_{12}z)$ is also a power of 2. Therefore $t_{21}$ cannot be a 3-cycle, otherwise $C_{S_4}(t_{21}) \simeq Z_3$ and so $r_{12}z = 1$ that, in turn, would imply $[r_{12}, t_{22}] = 1$, contradicting (R8). It follows that $t_{22}$ has even order and so, since $r_{11}$ has even order as well, by (R4) we infer $t_{21} \in C_{S_4}(z)$.

Now we can rewrite (T2) as $[t_{11}, r_{21}] = z$. If $t_{11}$ were a 3-cycle, from (T1) we would get $r_{22} \in C_{S_4}(t_{11}) \simeq Z_3$, a contradiction since $r_{22}$ has even order. Thus $t_{11}$ has even order and so it belongs to $C_{S_4}(z)$, because $r_{21}$ has even order, too. Analogously, by using (T6) and (T7), we infer $t_{12} \in C_{S_4}(z)$.

Summarizing, if $\mathcal{G}$ were a prestructure on $S_4$ we should have
\[
\langle \mathcal{G} \rangle = C_{S_4}(z) \simeq D_8,
\]
contradicting part (2) of Proposition 3.7. \qed

3.3. The case $|G| = 32$ and $G$ non-extra-special. We start by proving the following partial strengthening of Proposition 3.4.

**Proposition 3.10.** Let $G$ be a non-abelian finite group, and let $H$ be the subgroup of $G$ generated by those elements whose centralizer is non-abelian. If $H$ is abelian and $[H : Z(G)] \leq 4$, then $G$ admits no prestructures with $z \in Z(G)$.

**Proof.** First of all, remark that $Z(G)$ is a (normal) subgroup of $H$ because $G$ is non-abelian. Assume now, by contradiction, that the elements $(r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, t_{22}, z)$ form a prestructure on $G$, with $z \in Z(G)$. Then these elements satisfy relations (R1′), (R2′), (T1′), ..., (T9′) in (14). As $H$ is abelian, (R4′) implies that at least one between $r_{11}, t_{21}$ does not belong to $H$.

Let us assume $r_{11} \notin H$. Thus $C_G(r_{11})$ is abelian, and so (R2′) and (R3′) yield $[r_{21}, t_{22}] = 1$. From this, using (T2′) and (T3′), we infer that $C_G(t_{21})$ is non-abelian. Similar considerations show that $C_G(r_{21})$ and $C_G(r_{22})$ are non-abelian, and so we have $r_{21}, r_{22}, t_{22} \in H$. Using (T2′), (T6′), (R8′), together with the fact that $H$ is abelian, we deduce $t_{11}, t_{12}, r_{12} \notin H$. In particular, $C_G(r_{12})$ is abelian, so (R7′) and (R9′) yield $[r_{21}, r_{22}] = 1$; therefore (T2′) and (T4′) imply that $C_G(t_{21})$ is non-abelian, and so $t_{21} \in H$. Summing up, we have proved that the four elements
$r_{21}, t_{21}, r_{22}, t_{22}$ belong to $H$; since they are all non-central, we infer that they yield four non-trivial elements in the quotient group $H/Z(G)$. On the other hand, we have $|H : Z(G)| \leq 4$, and so $H/Z(G)$ contains at most three non-trivial elements; it follows that (at least) two among the elements $r_{21}, t_{21}, r_{22}, t_{22}$ have the same image in $H/Z(G)$. This means that these two elements are of the form $g, gz$, with $z \in Z(G)$, and so they have the same centralizer. But this is impossible: in fact, relations (14) show that each element in the set $\{r_{21}, t_{21}, r_{22}, t_{22}\}$ fails to commute with exactly one element in the set $\{r_{11}, t_{11}, r_{12}, t_{12}\}$, and no two elements in $\{r_{21}, t_{21}, r_{22}, t_{22}\}$ fail to commute with the same element in $\{r_{11}, t_{11}, r_{12}, t_{12}\}$.

The remaining case, namely $t_{21} \notin H$, can be dealt with in an analogous way. Indeed, in this situation we obtain $\{r_{11}, t_{11}, r_{12}, t_{12}\} \subseteq H$, that leads to a contradiction as before. \hfill \Box

We can now rule out the non-extra-special groups of order 32.

**Proposition 3.11.** Let $G$ be a finite group of order 32 which is not extra-special. Then $G$ admits no diagonal double Kodaira structures.

**Proof.** If $G$ is a CCT-group, then the result follows from Proposition 3.4. Thus, by Proposition 1.14, we must only consider the cases $G = G(32, t)$, where $t \in \{6, 7, 8, 43, 44\}$. Standard computations using the presentations in Table 2 of Appendix A show that all these groups are monolithic, and that for all of them $soc(G) = Z(G) \cong \mathbb{Z}_2$, cf. the proof of Proposition 1.14. Since no proper quotients of $G$ admit diagonal double Kodaira structures (Proposition 3.9), it follows from Proposition 3.7 that every diagonal double Kodaira structure on $G$ is such that $z$ is the generator of $Z(G)$. Let $H$ be the subgroup of $G$ generated by those elements whose centralizer is non-abelian; then, by Proposition 3.10, we are done, provided that in every case $H$ is abelian and $|H : Z(G)| \leq 4$. Let us now show that this is indeed true, leaving the straightforward computations to the reader.

- $G = G(32, 6)$. In this case $soc(G) = Z(G) = \langle x \rangle$ and $H = \langle x, y, w^2 \rangle$. Then $H \simeq (\mathbb{Z}_2)^3$ and $|H : Z(G)| = 4$.
- $G = G(32, 7)$. In this case $soc(G) = Z(G) = \langle w \rangle$ and $H = \langle z, u, w \rangle$. Then $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$ and $|H : Z(G)| = 4$.
- $G = G(32, 8)$. In this case $soc(G) = Z(G) = \langle x^4 \rangle$ and $H = \langle x^2, y, z^2 \rangle$. Then $H \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $|H : Z(G)| = 4$.
- $G = G(32, 43)$. In this case $soc(G) = Z(G) = \langle x^4 \rangle$ and $H = \langle x^2, z \rangle$. Then $H \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $|H : Z(G)| = 4$.
- $G = G(32, 44)$. In this case $soc(G) = Z(G) = \langle z^2 \rangle$ and $H = \langle x, k \rangle$. Then $H \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $|H : Z(G)| = 4$.

This completes the proof. \hfill \Box

### 3.4. The case $|G| = 32$ and $G$ extra-special

We are now ready to address the case where $|G| = 32$ and $G$ is extra-special. Let us first recall some results on extra-special $p$-groups, referring the reader to [Win72] for more details.

Let $G$ be an extra-special $p$-group of order $p^{2b+1}$ and $x, y \in G$. Setting $(\bar{x}, \bar{y}) = \bar{a}$ where $[x, y] = z^r$, the quotient group $V = G/Z(G) \simeq (\mathbb{Z}_p)^{2b}$ becomes a non-degenerate symplectic vector space over $\mathbb{Z}_p$. Looking at (4) and (5), we see that in both cases $G = H_{2b+1}(\mathbb{Z}_p)$ and $G = G_{2b+1}(\mathbb{Z}_p)$ we have

$$
(\bar{r}_j, \bar{r}_k) = 0, \quad (\bar{t}_j, \bar{t}_k) = 0, \quad (\bar{r}_j, \bar{t}_k) = -\delta_{jk}
$$

for all $j, k \in \{1, \ldots, b\}$, so that

$$
(15) \quad r_1, \bar{t}_1, \ldots, r_b, \bar{t}_b
$$

is an ordered symplectic basis for $V \simeq (\mathbb{Z}_p)^{2b}$. If $p = 2$, we can also set $q(\bar{x}) = \bar{c}$, where $x^2 = z^c$ and $c \in \{0, 1\}$; this is a quadratic form on $V$. If $x \in G/Z(G)$ is expressed in coordinates,
with respect to the symplectic basis (15), by the vector \((\xi_1, \psi_1, \ldots, \xi_b, \psi_b) \in (\mathbb{Z}_2)^{2b}\), then a straightforward computation yields

\[
q(\bar{x}) = \begin{cases} 
\xi_1 \psi_1 + \cdots + \xi_b \psi_b, & \text{if } G = H_{2b+1}(\mathbb{Z}_2) \\
\xi_1 \psi_1 + \cdots + \xi_b \psi_b + \xi_b^2 + \psi_b^2 & \text{if } G = G_{2b+1}(\mathbb{Z}_2).
\end{cases}
\]

These are the two possible normal forms for a non-degenerate quadratic form over \(\mathbb{Z}_2\). Moreover, in both cases the symplectic and the quadratic form are related by

\[q(\bar{x}\bar{y}) = q(\bar{x}) + q(\bar{y}) + (\bar{x}, \bar{y}) \text{ for all } \bar{x}, \bar{y} \in V.\]

If \(\phi \in \text{Aut}(G)\), then \(\phi\) induces a linear map \(\tilde{\phi} \in \text{End}(V)\); moreover, if \(p = 2\), then \(\phi\) acts trivially on \(Z(G) = [G, G] \simeq \mathbb{Z}_2\), and this in turn implies that \(\phi\) preserves the symplectic form on \(V\). In other words, if we identify \(V\) with \((\mathbb{Z}_2)^{2b}\) via the symplectic basis (15), we have \(\tilde{\phi} \in \text{Sp}(2b, \mathbb{Z}_2)\).

We are now in a position to describe the structure of \(\text{Aut}(G)\), see [Win72, Theorem 1].

**Proposition 3.12.** Let \(G\) be an extra-special group of order \(2^{2b+1}\). Then the kernel of the group homomorphism \(\text{Aut}(G) \rightarrow \text{Sp}(2b, \mathbb{Z}_2)\) given by \(\phi \mapsto \tilde{\phi}\) is the subgroup \(\text{Inn}(G)\) of inner automorphisms of \(G\). Therefore \(\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)\) embeds in \(\text{Sp}(2b, \mathbb{Z}_2)\). More precisely, \(\text{Out}(G)\) coincides with the orthogonal group \(O_\epsilon(2b, \mathbb{Z}_2)\), of order

\[
|O_\epsilon(2b, \mathbb{Z}_2)| = 2^{(b-1)(b+1)}(2^b - \epsilon) \prod_{i=1}^{b-1} (2^{2i} - 1),
\]

associated with the quadratic form (16). Here \(\epsilon = 1\) if \(G = H_{2b+1}(\mathbb{Z}_2)\) and \(\epsilon = -1\) if \(G = G_{2b+1}(\mathbb{Z}_2)\).

**Corollary 3.13.** Let \(G\) be an extra-special group of order \(2^{2b+1}\). We have

\[
|\text{Aut}(G)| = 2^{(b+1)(b+1)}(2^b - \epsilon) \prod_{i=1}^{b-1} (2^{2i} - 1).
\]

**Proof.** By Proposition 3.12 we get \(|\text{Aut}(G)| = |\text{Inn}(G)| \cdot |O_\epsilon(2b, \mathbb{Z}_2)|\). Since \(\text{Inn}(G) \simeq G/Z(G)\) has order \(2^{2b}\), the claim follows from (17).

In particular, plugging \(b = 2\) in (18), we can compute the orders of automorphism groups of extra-special groups of order 32, namely

\[
|\text{Aut}(H_5(\mathbb{Z}_2))| = 1152, \quad |\text{Aut}(G_5(\mathbb{Z}_2))| = 1920.
\]

Assume now that \(\mathcal{G} = (r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, z)\) is a diagonal double Kodaira structure of type \((2, n)\) on an extra-special group \(G\) of order 32; by Corollary 3.8, the element \(z\) is the generator of \(Z(G) \simeq \mathbb{Z}_2\), hence \(n = 2\). Then

\[
\mathcal{G} = (\bar{r}_{11}, \bar{t}_{11}, \bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}, \bar{t}_{21}, \bar{r}_{22}, \bar{t}_{22})
\]

is an ordered set of generators for the symplectic \(\mathbb{Z}_2\)-vector space \(V = G/Z(G) \simeq (\mathbb{Z}_2)^4\), and (14) yields the relations

\[
(\bar{r}_{12}, \bar{t}_{12}) + (\bar{r}_{11}, \bar{t}_{11}) = 1,
\]

\[
(\bar{r}_{21}, \bar{t}_{21}) + (\bar{r}_{22}, \bar{t}_{22}) = 1,
\]

\[
(\bar{r}_{1j}, \bar{t}_{2k}) = \delta_{jk}, \quad (\bar{r}_{1j}, \bar{r}_{2k}) = 0
\]

Conversely, given any set of generators \(\mathcal{G}\) of \(V\) as in (20), whose elements satisfy (21), a diagonal double Kodaira structure of type \((b, n) = (2, 2)\) on \(G\) inducing \(\mathcal{G}\) is necessarily of the form

\[
\mathcal{G} = (r_{11}z^{a_{11}}, t_{11}z^{b_{11}}, r_{12}z^{a_{12}}, t_{12}z^{b_{12}}, r_{21}z^{a_{21}}, t_{21}z^{b_{21}}, r_{22}z^{a_{22}}, t_{22}z^{b_{22}}, z),
\]

where \(a_{ij}, b_{ij} \in \{0, 1\}\). This proves the following...
Lemma 3.14. The total number of diagonal double Kodaira structures of type $(b, n) = (2, 2)$ on an extra-special group $G$ of order 32 is obtained multiplying by $2^b$ the number of ordered sets of generators $\mathfrak{S}$ of $V$ as in (20), whose elements satisfy (21). In particular, such a number does not depend on $G$.

We are now ready to state the main result of this section.

Theorem 3.15. A finite group $G$ of order 32 admits a diagonal double Kodaira structure if and only if $G$ is extra-special. In this case, the following holds.

1. Both extra-special groups of order 32 admit $2211840 = 1152 \cdot 1920$ distinct diagonal double Kodaira structures of type $(b, n) = (2, 2)$. Every such a structure $\mathfrak{S}$ is strong and satisfies $\sigma(\mathfrak{S}) = 16$.
2. If $G = G(32, 49) = H_5(\mathbb{Z}_2)$, these structures form 1920 orbits under the action of $\text{Aut}(G)$.
3. If $G = G(32, 50) = G_5(\mathbb{Z}_2)$, these structures form 1152 orbits under the action of $\text{Aut}(G)$.

Proof. We already know that non-extra-special groups of order 32 admit no diagonal double Kodaira structures (Proposition 3.11) and so, in the sequel, we can assume that $G$ is extra-special.

Looking at the first two relations in (21), we see that we must consider four cases:

(a) $(\bar{r}_{12}, \bar{t}_{12}) = 0$, $(\bar{r}_{11}, \bar{t}_{11}) = 1$, $(\bar{r}_{21}, \bar{t}_{21}) = 0$, $(\bar{r}_{22}, \bar{t}_{22}) = 1$;
(b) $(\bar{r}_{12}, \bar{t}_{12}) = 1$, $(\bar{r}_{11}, \bar{t}_{11}) = 0$, $(\bar{r}_{21}, \bar{t}_{21}) = 1$, $(\bar{r}_{22}, \bar{t}_{22}) = 0$;
(c) $(\bar{r}_{12}, \bar{t}_{12}) = 0$, $(\bar{r}_{11}, \bar{t}_{11}) = 1$, $(\bar{r}_{21}, \bar{t}_{21}) = 1$, $(\bar{r}_{22}, \bar{t}_{22}) = 0$;
(d) $(\bar{r}_{12}, \bar{t}_{12}) = 1$, $(\bar{r}_{11}, \bar{t}_{11}) = 0$, $(\bar{r}_{21}, \bar{t}_{21}) = 0$, $(\bar{r}_{22}, \bar{t}_{22}) = 1$.

Case (a). In this case the vectors $\bar{r}_{11}, \bar{t}_{11}, \bar{r}_{22}, \bar{t}_{22}$ are a symplectic basis of $V$, whereas the subspace $W = (\bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}, \bar{t}_{21})$ is isotropic, namely the symplectic form is identically zero on it. Since $V$ is a symplectic vector space of dimension 4, the Witt index of $V$, i.e. the dimension of a maximal isotropic subspace of $V$, is $\frac{1}{2} \dim(V) = 2$, see [Ar62, Théorèmes 3.10, 3.11]. On the other hand, we have $(\bar{r}_{12}, \bar{t}_{22}) = 1$ and $(\bar{r}_{12}, \bar{t}_{22}) = 0$, hence $\bar{r}_{12}, \bar{t}_{12}$ are linearly independent and so they must generate a maximal isotropic subspace; it follows that $W = (\bar{r}_{12}, \bar{t}_{12})$. Let us set now

$$(\bar{r}_{11}, \bar{r}_{12}) = a, \quad (\bar{r}_{11}, \bar{t}_{12}) = b, \quad (\bar{r}_{12}, \bar{t}_{11}) = c, \quad (\bar{t}_{11}, \bar{t}_{12}) = d,$$

$$(\bar{r}_{21}, \bar{r}_{22}) = e, \quad (\bar{r}_{21}, \bar{t}_{22}) = f, \quad (\bar{r}_{22}, \bar{t}_{21}) = g, \quad (\bar{t}_{21}, \bar{t}_{22}) = h,$$

where $a, b, c, d, e, f, g, h \in \mathbb{Z}_2$, and let us express the remaining vectors of $\mathfrak{S}$ in terms of the symplectic basis. Standard computations yield

$$\bar{r}_{12} = a\bar{r}_{11} + a\bar{t}_{11} + \bar{r}_{22}, \quad \bar{t}_{12} = d\bar{r}_{11} + b\bar{t}_{11} + \bar{t}_{22},$$
$$\bar{r}_{21} = \bar{r}_{11} + f\bar{r}_{22} + c\bar{t}_{22}, \quad \bar{t}_{21} = \bar{r}_{11} + h\bar{r}_{22} + g\bar{t}_{22}.$$

Now recall that $W$ is isotropic; then, using the expressions in (22) and imposing the relations

$$(\bar{r}_{12}, \bar{t}_{12}) = 0, \quad (\bar{r}_{12}, \bar{t}_{21}) = 0, \quad (\bar{r}_{21}, \bar{t}_{21}) = 0,$$
$$(\bar{r}_{21}, \bar{t}_{12}) = 0, \quad (\bar{r}_{21}, \bar{t}_{21}) = 0, \quad (\bar{r}_{21}, \bar{t}_{21}) = 0,$$

we get

$$ad + bc = 1, \quad a + e = 0, \quad c + g = 0,$$
$$b + f = 0, \quad d + h = 0, \quad eh + fg = 1.$$

Summing up, the elements $\bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}, \bar{t}_{21}$ can be determined from the symplectic basis via the relations

$$\bar{r}_{12} = c\bar{r}_{11} + a\bar{t}_{11} + \bar{r}_{22}, \quad \bar{t}_{12} = d\bar{r}_{11} + b\bar{t}_{11} + \bar{t}_{22},$$
$$\bar{r}_{21} = \bar{r}_{11} + f\bar{r}_{22} + a\bar{t}_{22}, \quad \bar{t}_{21} = \bar{r}_{11} + d\bar{r}_{22} + c\bar{t}_{22},$$

where $a, b, c, d \in \mathbb{Z}_2$ and $ad + bc = 1$. Conversely, given any symplectic basis $\bar{r}_{11}, \bar{t}_{11}, \bar{r}_{22}, \bar{t}_{22}$ of $V$ and elements $\bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}, \bar{t}_{21}$ as in (23), with $ad + bc = 1$, we get a set of generators $\mathfrak{S}$ of
$V$ having the form (20), and whose elements satisfy (21). Thus, the total number of such $\tilde{G}$ in this case is given by

$$|\text{Sp}(4, \mathbb{Z}_2)| \cdot |\text{GL}(2, \mathbb{Z}_2)| = 720 \cdot 6 = 4320$$

and so, by Lemma 3.14, the corresponding number of diagonal double Kodaira structures is $2^8 \cdot 4320 = 1105920$. All these structures are strong; in fact, we have

$$K_1 = \langle r_{11}, t_{11}, r_{12}, t_{12} \rangle = \langle r_{11}, t_{11}, r_{12}^{t_{11}}, r_{12}^{t_{11}} \rangle = \langle r_{11}, t_{11}, r_{22}, t_{22} \rangle = G$$

$$K_2 = \langle r_{21}, t_{21}, r_{22}, t_{22} \rangle = \langle r_{11}^{t_{22}}, r_{22}^{t_{22}}, t_{11}, t_{22} \rangle = \langle r_{11}, t_{11}, r_{22}, t_{22} \rangle = G,$$

the last equality following in both cases because $\langle r_{11}, t_{11}, r_{22}, t_{22} \rangle = V$ and $[r_{11}, t_{11}] = z$.

**Case (b).** In this situation, the elements $\{r_{12}, \bar{t}_{12}, \bar{r}_{21}, \bar{t}_{21}\}$ form a symplectic basis for $V$, whereas $W = \langle r_{11}, t_{11}, \bar{r}_{22}, \bar{t}_{22} \rangle$ is an isotropic subspace. The same calculations as in case (a) show that there are again 1105920 diagonal double Kodaira structures.

**Case (c).** This case do not occur. In fact, in this situation the subspace $W = \langle \bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21} \rangle$ is isotropic. Take a linear combination of its generators giving the zero vector, namely

$$a\bar{r}_{12} + b\bar{t}_{12} + c\bar{r}_{21} = 0.$$

Pairing with $\bar{t}_{21}, \bar{t}_{22}, \bar{r}_{22}$, we get $c = a = b = 0$. Thus, $\bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}$ are linearly independent, and $W$ is an isotropic subspace of dimension 3 inside the 4-dimensional symplectic space $V$, contradiction.

**Case (d).** This case is obtained from (c) by exchanging the indices 1 and 2, so it does not occur, either.

Summarizing, we have found 1105920 diagonal double Kodaira structures in cases (a) and (b), and no structure at all in cases (c) and (d). So the total number of diagonal double Kodaira structures on $G$ is 2211840, and this concludes the proof of part (1).

Now observe that, since every diagonal double Kodaira structure $G$ generates $G$, the only automorphism $\phi$ of $G$ fixing $G$ elementwise is the identity. This means that $\text{Aut}(G)$ acts freely on the set of diagonal double Kodaira structures, hence the number of orbits is obtained dividing 2211840 by $|\text{Aut}(G)|$. Parts (2) and (3) now follow from (19), and we are done. □

**Example 3.16.** Let us give an explicit example of diagonal double Kodaira structure on an extra-special group $G$ of order 32, by using the construction described in the proof of part (1) of Theorem 3.15. Referring to the presentations for $H_5(\mathbb{Z}_2)$ and $G_5(\mathbb{Z}_2)$ given in Proposition 1.9, we start by choosing in both cases the following elements, whose images give a symplectic basis for $V$:

$$r_{11} = r_1, \quad t_{11} = t_1, \quad r_{22} = r_2, \quad t_{22} = t_2.$$  

Choosing $a = d = 1$ and $b = c = 0$ in (23), we find the remaining elements, obtaining the diagonal double Kodaira structure

$$r_{11} = r_1, \quad t_{11} = t_1, \quad r_{12} = r_2 t_1, \quad t_{12} = r_1 t_2$$  

$$r_{21} = r_1 t_2, \quad t_{21} = r_2 t_1, \quad r_{22} = r_2, \quad t_{22} = t_2.$$

**Remark 3.17.** Theorem 3.15 should be compared with previous results of [CaPol19] and [Pol20], regarding the construction of diagonal double Kodaira structures on some extra-special groups of order at least $2^7 = 128$. The examples on extra-special groups of order 32 presented here are really new, in the sense that they cannot be obtained by taking the image of structures on extra-special groups of bigger order: in fact, an extra-special group admits no non-abelian proper quotients, cf. Example 3.6.
Remark 3.18. Although it is known that the pure braid group $P_2(\Sigma_b)$ is residually $p$-finite for all prime number $p \geq 2$, see [BarBel09, pp. 1481-1490], it is a non-trivial problem to understand its non-abelian, finite quotients. The extra-special examples of order at least 128 cited in Remark 3.17 were the outcome of the first (as far as we know) systematic investigation of this matter. Our approach in the present work sheds some new light on this problem, providing a sharp lower bound for the order of a non-abelian quotient $G$ of $P_2(\Sigma_b)$, under the assumption that the quotient map $\varphi: P_2(\Sigma_b) \rightarrow G$ does not factor through $\pi_1(\Sigma_b \times \Sigma_b, \mathcal{P})$; indeed, by [CaPol19, Remark 1.7 (iv)], the factorization occurs if and only if $\varphi(A_{12})$ is trivial. More precisely, we showed that, for every such a quotient, the inequality $|G| \geq 32$ holds, with equality if and only if $G$ is extra-special: in fact, both extra-special groups of order 32 do appear as quotients of $P_2(\Sigma_b)$. Moreover, for these groups, Theorem 3.15 also computes the number of distinct group epimorphisms $\varphi: P_2(\Sigma_2) \rightarrow G$ such that $\varphi(A_{12}) = z$, and the number of their equivalence classes up to the natural action of $\text{Aut}(G)$.

4. Geometrical Application: Diagonal Double Kodaira Fibrations

Recall that a Kodaira fibration is a smooth, connected holomorphic fibration $f_1: S \rightarrow B_1$, where $S$ is a compact complex surface and $B_1$ is a compact complex curve, which is not isotrivial. The genus $b_1 := g(B_1)$ is called the base genus of the fibration, whereas the genus $g := g(F)$, where $F$ is any fibre, is called the fibre genus.

**Definition 4.1.** A double Kodaira surface is a compact complex surface $S$, endowed with a double Kodaira fibration, namely a surjective, holomorphic map $f: S \rightarrow B_1 \times B_2$ yielding, by composition with the natural projections, two Kodaira fibrations $f_i: S \rightarrow B_i$, $i = 1, 2$.

The aim of this section is to show how the existence of diagonal double Kodaira structures is equivalent to the existence of some special double Kodaira fibrations, that we call diagonal double Kodaira fibrations. We closely follow the treatment given in [Pol20, Section 3].

With a slight abuse of notation, in the sequel we will use the symbol $\Sigma$ to indicate both a smooth complex curve of genus $b$ and its underlying real surface. By Grauert-Remmert's extension theorem and Serre's GAGA, the group epimorphism $\varphi: P_2(\Sigma_b) \rightarrow G$ described in Proposition 2.6 yields the existence of a smooth, complex, projective surface $S$ endowed with a Galois cover

$$f: S \rightarrow \Sigma_b \times \Sigma_b,$$

with Galois group $G$ and branched precisely over $\Delta$ with branching order $n$, see [CaPol19, Proposition 3.4]. Composing the left homomorphism in (10) with $\varphi: P_2(\Sigma_b) \rightarrow G$, we get two homomorphisms

$$\varphi_1: \pi_1(\Sigma_b - \{p_2\}, p_1) \rightarrow G, \quad \varphi_2: \pi_1(\Sigma_b - \{p_1\}, p_2) \rightarrow G,$$

whose respective images coincide with the subgroups $K_1$ and $K_2$ defined in (12). By construction, these are the homomorphisms induced by the restrictions $f_i: \Gamma_i \rightarrow \Sigma_b$ of the Galois cover $f: S \rightarrow \Sigma_b \times \Sigma_b$ to the fibres of the two natural projections $\pi_i: \Sigma_b \times \Sigma_b \rightarrow \Sigma_b$. Since $\Delta$ intersects transversally at a single point all the fibres of the natural projections, it follows that both such restrictions are branched at precisely one point, and the number of connected components of the smooth curve $\Gamma_i \subset S$ equals the index $m_i := [G: K_i]$ of $K_i$ in $G$.

So, taking the Stein factorizations of the compositions $\pi_i \circ f: S \rightarrow \Sigma_b$ as in the diagram below

\begin{equation}
\begin{array}{c}
S \xrightarrow{\pi_i \circ f} \Sigma_b \\
\downarrow f_i \quad \downarrow \theta_i \\
\Sigma_b \end{array}
\end{equation}

(24)
we obtain two distinct Kodaira fibrations $f_i: S \to \Sigma_{b_i}$, hence a double Kodaira fibration by considering the product morphism
\[ f = f_1 \times f_2: S \to \Sigma_{b_1} \times \Sigma_{b_2}. \]

**Definition 4.2.** We call $f: S \to \Sigma_{b_1} \times \Sigma_{b_2}$ the *diagonal double Kodaira fibration* associated with the diagonal double Kodaira structure $\mathcal{E}$ on the finite group $G$. Conversely, we will say that a double Kodaira fibration $f: S \to \Sigma_{b_1} \times \Sigma_{b_2}$ is of diagonal type $(b, n)$ if there exists a finite group $G$ and a diagonal double Kodaira structure $\mathcal{E}$ of type $(b, n)$ on it such that $f$ is associated with $\mathcal{E}$.

Since the morphism $\theta_i: \Sigma_{b_i} \to \Sigma_b$ is étale of degree $m_i$, by using the Hurwitz formula we obtain
\[ b_1 - 1 = m_1(b - 1), \quad b_2 - 1 = m_2(b - 1). \]
Moreover, the fibre genera $g_1, g_2$ of the Kodaira fibrations $f_1: S \to \Sigma_{b_1}, f_2: S \to \Sigma_{b_2}$ are computed by the formulæ
\[ 2g_1 - 2 = \frac{|G|}{m_1}(2b - 2 + n), \quad 2g_2 - 2 = \frac{|G|}{m_2}(2b - 2 + n), \]
where $n := 1 - 1/n$. Finally, the surface $S$ fits into a diagram
\[
\begin{array}{c}
S \\
\downarrow f \\
\Sigma_{b_1} \times \Sigma_{b_2} \\
\end{array}
\quad (\theta_1 \times \theta_2)^{-1}(\Delta) = \Sigma_{b_1} \times \Sigma_{b_2},
\]
so that the diagonal double Kodaira fibration $f: S \to \Sigma_{b_1} \times \Sigma_{b_2}$ is a finite cover of degree $\frac{|G|}{m_1m_2}$, branched precisely over the curve
\[
(\theta_1 \times \theta_2)^{-1}(\Delta) = \Sigma_{b_1} \times \Sigma_{b_2}.
\]
Such a curve is always smooth, being the preimage of a smooth divisor via an étale morphism. However, it is reducible in general, see [CaPol19, Proposition 3.11]. The invariants of $S$ can be now computed as follows, see [CaPol19, Proposition 3.8].

**Proposition 4.3.** Let $f: S \to \Sigma_{b_1} \times \Sigma_{b_2}$ be a diagonal double Kodaira fibration, associated with a diagonal double Kodaira structure $\mathcal{E}$ of type $(b, n)$ on a finite group $G$. Then we have
\[ c_1^2(S) = |G|(2b - 2)(4b - 4 + 4n - n^2) \]
\[ c_2(S) = |G|(2b - 2)(2b - 2 + n) \]
where $n = 1 - 1/n$. As a consequence, the slope and the signature of $S$ can be expressed as
\[ \nu(S) = \frac{c_1^2(S)}{c_2(S)} = 2 + \frac{2n - n^2}{2b - 2 + n} \]
\[ \sigma(S) = \frac{1}{3} \left( c_1^2(S) - 2c_2(S) \right) = \frac{1}{3} |G|(2b - 2) \left( 1 - \frac{1}{n^2} \right) = \sigma(\mathcal{E}) \]

**Remark 4.4.** By definition, the diagonal double Kodaira structure $\mathcal{E}$ is strong if and only if $m_1 = m_2 = 1$, that in turn implies $b_1 = b_2 = b$, i.e., $f = f$. In other words, $\mathcal{E}$ is strong if and only if no Stein factorization as in (24) is needed or, equivalently, if and only if the Galois cover $f: S \to \Sigma_b \times \Sigma_b$ induced by (8) is already a double Kodaira fibration, branched on the diagonal $\Delta \subset \Sigma_b \times \Sigma_b$. 
Remark 4.5. Every Kodaira fibred surface $S$ satisfies $\sigma(S) > 0$, see the introduction to [LLR20]; moreover, since $S$ is a differentiable 4-manifold that is a real surface bundle, its signature is divisible by 4, see [Mey73]. In addition, if $S$ is associated with a diagonal double Kodaira structure of type $(b, n)$, with $n$ odd, then $K_S$ is 2-divisible in Pic($S$) and so $\sigma(S)$ is a positive multiple of 16 by Rohklin’s theorem, see [CaPol19, Remark 3.9].

Remark 4.6. Not all double Kodaira fibration are of diagonal type. In fact, if $S$ is of diagonal type then its slope satisfies $\nu(S) = 2 + s$, where $s$ is rational and $0 < s < 6 - 4\sqrt{2}$, see [Pol20, Proposition 3.12 and Remark 3.13].

We are now ready to give a geometric interpretation of Proposition 3.9, Proposition 3.11 and Theorem 3.15 in terms of double Kodaira fibrations.

Theorem 4.7. Let $G$ be a finite group and

$$f : S \longrightarrow \Sigma_b \times \Sigma_b$$

be a Galois cover with Galois group $G$, branched over the diagonal $\Delta$ with branching order $n \geq 2$. Then the following hold.

1. We have $|G| \geq 32$, with equality precisely when $G$ is extra-special.
2. If $G = G(32, 49) = H_5(Z_2)$ and $b = 2$, there are 1920 $G$-covers of type (28), up to cover isomorphisms.
3. If $G = G(32, 50) = G_5(Z_2)$ and $b = 2$, there are 1152 $G$-covers of type (28), up to cover isomorphisms.

Finally, in both cases (2) and (3), we have $n = 2$ and each cover $f$ is a double Kodaira fibration with

$b_1 = b_2 = 2, \quad g_1 = g_2 = 41, \quad \sigma(S) = 16$.

Proof. By the result of Section 4, a cover as in (28), branched over $\Delta$ with order $n$, exists if and only if $G$ admits a double Kodaira structure of type $(b, n)$, and the number of such covers, up to cover isomorphisms, equals the number of structures up the natural action of Aut($G$). Then, (1), (2) and (3) can be deduced from the corresponding statements in Theorem 3.15. The same theorem tells us that all double Kodaira structures on an extra-special group of order 32 are strong, hence the cover $f$ is already a double Kodaira fibration and no Stein factorization is needed (Remark 4.4). The fibre genera, the slope and the signature of $S$ can be now computed by using (25) and (27).

As a consequence, we obtain a sharp lower bound for the signature of a diagonal double Kodaira fibration or, equivalently, of a diagonal double Kodaira structure.

Corollary 4.8. Let $f : S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ be a diagonal double Kodaira fibration, associated with a diagonal double Kodaira structure of type $(b, n)$ on a finite group $G$. Then $\sigma(S) \geq 16$, and equality holds precisely when $(b, n) = (2, 2)$ and $G$ is an extra-special group of order 32.

Proof. Theorem 3.15 implies $|G| \geq 32$. Since $b \geq 2$ and $n \geq 2$, from (27) we get

$$\sigma(S) = \frac{1}{3} |G| (2b - 2) \left(1 - \frac{1}{n^2}\right) \geq \frac{1}{3} \cdot 32 \cdot (2 \cdot 2 - 2) \left(1 - \frac{1}{2^2}\right) = 16,$$

and equality holds if and only if we are in the situation described in Theorem 4.7, namely, $b = n = 2$ and $G$ an extra-special group of order 32.

These results provide, in particular, new “double solutions” to a problem, posed by G. Mess, from Kirby’s problem list in low-dimensional topology [Kir97, Problem 2.18 A], asking what is the smallest number $b$ for which there exists a real surface bundle over a real surface with base genus $b$ and non-zero signature. We actually have $b = 2$, also for double Kodaira fibrations, as shown in [CaPol19, Proposition 3.19] and [Pol20, Theorem 3.6] by using double Kodaira structures of type $(2, 3)$ on extra-special groups of order $3^5$. Those
fibrations had signature 144 and fibre genera 325; by using our new examples, we are now able to substantially lower both these values.

**Theorem 4.9.** Let $S$ be the diagonal double Kodaira surface associated with a strong diagonal double Kodaira structure of type $(b, n) = (2, 2)$ on an extra-special group $G$ of order 32. Then the real manifold $M$ underlying $S$ is a closed, orientable 4-manifold of signature 16 that can be realized as a real surface bundle over a real surface of genus 2, with fibre genus 41, in two different ways.

Theorem 4.7 also implies the following partial answer to [CaPol19, Question 3.20].

**Corollary 4.10.** Let $g_{\text{min}}$ and $\sigma_{\text{min}}$ be the minimal possible fibre genus and signature for a double Kodaira fibration $f : S \rightarrow \Sigma_2 \times \Sigma_2$. Then we have

$$g_{\text{min}} \leq 41, \quad \sigma_{\text{min}} \leq 16.$$

In fact, it is an interesting question whether 16 and 41 are the minimum possible values for the signature and the fibre genus of a (non necessarily diagonal) double Kodaira fibration $f : S \rightarrow \Sigma_2 \times \Sigma_2$, but we will not address this problem here.

**Remark 4.11.** Constructing (double) Kodaira fibrations with small signature is a rather difficult problem. As far as we know, before our work the only examples with signature 16 were the ones listed in [LLR20, Table 3, Cases 6.2, 6.6, 6.7 (Type 1), 6.9]. The examples provided by Theorem 4.7 are new, since both the base genera and the fibre genera are different. Note that our results also show that every curve of genus 2 (and not only some special curve with extra automorphisms) is the base of a double Kodaira fibration with signature 16. Thus, we obtain two families of dimension 3 of such fibrations that, to the best of our knowledge, provide the first examples of a positive-dimensional families of double Kodaira fibrations with small signature.

**Remark 4.12.** Let $f : S \rightarrow \Sigma_b \times \Sigma_b$ be a double Kodaira fibration, associated with a strong diagonal double Kodaira structure of type $(b, n)$ on a finite group $G$. Then the branch locus of $f$ is $\Delta \subset \Sigma_b \times \Sigma_b$, namely the graph of the identity map $\Sigma_b \rightarrow \Sigma_b$, and so, adopting the terminology in [CatRol09], we say that $f$ is very simple. Let us denote by $\mathfrak{M}_S$ the connected component of the Gieseker moduli space of surfaces of general type containing the class of $S$, and by $\mathcal{M}_b$ the moduli space of smooth curves of genus $b$. Thus, by applying [Rol10, Thm. 1.7], we infer that every surface in $\mathfrak{M}_S$ is still a very simple double Kodaira fibration and that there is a natural map of schemes

$$\mathcal{M}_b \rightarrow \mathfrak{M}_S,$$

which is an isomorphism on geometric points. Roughly speaking, since the branch locus $\Delta \subset \Sigma_b \times \Sigma_b$ is rigid, all the deformations of $S$ are realized by deformations of $\Sigma_b \times \Sigma_b$ preserving the diagonal, hence by deformations of $\Sigma_b$, cf. [CaPol19, Proposition 3.22]. In particular, this shows that $\mathfrak{M}_S$ is a connected and irreducible component of the Gieseker moduli space.

### 4.1. The computation of $H_1(S, \mathbb{Z})$

We end this section by computing the first homology group $H_1(S, \mathbb{Z})$, where $f : S \rightarrow \Sigma_2 \times \Sigma_2$ is the diagonal double Kodaira fibration associated with a diagonal double Kodaira structure of type $(b, n) = (2, 2)$ on an extra-special group of order 32. To this purpose, we will make use of the following result, which is a consequence of [Fox57, Theorem p. 254].

**Proposition 4.13.** Let $G$ be a finite group and $\varphi : P_2(\Sigma_b) \rightarrow G$ be a surjective group homomorphism, such that $\varphi(A_{12})$ has order $n \geq 2$. If $f : S \rightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ is the diagonal double Kodaira fibration associated with $\varphi$, then $\pi_1(S)$ sits into a short exact sequence

$$1 \rightarrow \pi_1(S) \rightarrow P_2(\Sigma_2)^{\text{orb}} \xrightarrow{\varphi^{\text{orb}}} G \rightarrow 1,$$

where $P_2(\Sigma_b)^{\text{orb}}$ is the quotient of $P_2(\Sigma_b)$ by the normal closure of the cyclic subgroup $\langle A_{12}^n \rangle$, and $\varphi^{\text{orb}} : P_2(\Sigma_b)^{\text{orb}} \rightarrow G$ is the group epimorphism naturally induced by $\varphi$. 

Proposition 4.13 allows one, at least in principle, to compute \( \pi_1(S) \), and so its abelianization \( H_1(S, \mathbb{Z}) \). However, doing all the calculations by hand seems quite difficult, so we resorted to the Computer Algebra System GAP4, see [GAP4]. The reader can find the idea behind the calculation and the corresponding script in Appendix B, while here we just report the result.

**Proposition 4.14.** Let \( f : S \to \Sigma_2 \times \Sigma_2 \) be the diagonal double Kodaira fibration associated with a diagonal double Kodaira structure of type \((b, n) = (2, 2)\) on an extra-special group \( G \) of order 32. Then

\[
H_1(S, \mathbb{Z}) = \mathbb{Z}^8 \oplus (\mathbb{Z}/2)^4.
\]

In particular, this homology group is independent both on \( G \) and on the chosen structure on it.

**Corollary 4.15.** Let \( f : S \to \Sigma_2 \times \Sigma_2 \) be the diagonal double Kodaira fibration associated with a diagonal double Kodaira structure of type \((b, n) = (2, 2)\) on an extra-special group of order 32. Then

\[
c_1^2(S) = 368, \quad c_2(S) = 160, \quad p_0(S) = 47, \quad q(S) = 4.
\]

**Proof.** The integers \( c_1^2(S), c_2(S) \) can be computed by using (26). Since \( b_1(S) = 8 \) (Proposition 4.14), it follows \( q(S) = 4 \). On the other hand, Noether formula gives

\[
1 - q(S) + p_g(S) = \chi(\mathcal{O}_S) = \frac{c_1^2(S) + c_2(S)}{12} = 44,
\]

hence \( p_g(S) = 47 \).

Finally, we want to relate Proposition 4.14 to some facts about monodromy. Recall that, by [CatRol09, Proposition 2.5], given a Kodaira fibration \( S \to \Sigma_b \), with base genus \( b \) and fibre genus \( g \), there is a short exact sequence of fundamental groups

\[
1 \to \pi_1(\Sigma_g) \to \pi_1(S) \to \pi_1(\Sigma_b) \to 1,
\]

which induces, by conjugation, a monodromy representation \( \pi_1(\Sigma_b) \to \text{Out}(\pi_1(\Sigma_g)) \). Taking the abelianization of the right side, and dualizing over \( \mathbb{Q} \), we obtain a monodromy representation

\[
\rho_{\pi_1(\Sigma_b)} : \pi_1(\Sigma_b) \to \text{Aut}(H^1(\Sigma_g, \mathbb{Q})),
\]

whose invariant subspace will be denoted by \( H^1(\Sigma_g, \mathbb{Q})^{\pi_1(\Sigma_b)} \).

Now, let \( f : S \to \Sigma_b \times \Sigma_b \) be a diagonal double Kodaira fibration, associated with a group epimorphism \( \varphi : P_2(\Sigma_b) \to G \) such that \( \varphi(A_{12}) \) has order \( n \geq 2 \). Set \( \{i, j\} = \{1, 2\} \) and let \( f_i : S \to \Sigma_b \) be the Kodaira fibration obtained composing \( f \) with the natural projection \( \pi_i : \Sigma_b \times \Sigma_b \to \Sigma_b \) onto the \( i \)th factor. Assume moreover that the induced group homomorphism \( \varphi_j : \pi_1(\Sigma_b - \{p_i\}) \to G \) is surjective. Then \( \pi_1(\Sigma_\alpha) \) fits into a short exact sequence

\[
1 \to \pi_1(\Sigma_g) \to \pi_1(\Sigma_b - \{p_i\})^{\text{orb}} \xrightarrow{\varphi_j^{\text{orb}}} G \to 1,
\]

where

\[
\pi_1(\Sigma_b - \{p_i\})^{\text{orb}} = \langle \rho_{j1}, \tau_{j1}, \ldots, \rho_{jb}, \tau_{jb}, A_{12} | A_{12} \prod_{t=1}^b [\rho_{jt}, \tau_{jt}] = 1, A_{12} = 1 \rangle,
\]

see (11), and \( \varphi_j^{\text{orb}} : \pi_1(\Sigma_b - \{p_i\})^{\text{orb}} \to G \) is the group epimorphism naturally induced by \( \varphi_j \). Correspondingly, we have a monodromy representation

\[
\rho_G : G \to \text{Aut}(H^1(\Sigma_g, \mathbb{Q})),
\]

whose invariant subspace will be denoted by \( H^1(\Sigma_g, \mathbb{Q})^G \).

**Proposition 4.16.** Let \( f : S \to \Sigma_b \times \Sigma_b \) be a diagonal double Kodaira fibration as above. Then the following holds.
(1) \( \dim H^1(S, \mathbb{Q}) = \dim H^1(\Sigma_g, \mathbb{Q})^{\pi_1(\Sigma_b)} + 2b \)
(2) \( H^1(\Sigma_g, \mathbb{Q})^G \subseteq H^1(\Sigma_g, \mathbb{Q})^{\pi_1(\Sigma_b)} \)

Proof. (1) If \( \iota : \Sigma_g \rightarrow S \) is the inclusion, the Hochschild-Serre spectral sequence in cohomology associated with (29) provides a short exact sequence of \( \mathbb{Q} \)-vector spaces

\[
0 \rightarrow H^1(\Sigma_b, \mathbb{Q}) \xrightarrow{f^*} H^1(S, \mathbb{Q}) \xrightarrow{\iota^*} H^1(\Sigma_g, \mathbb{Q})^{\pi_1(\Sigma_b)} \rightarrow 0,
\]

see for instance [Breg18, p. 5] or [Vid19, p. 740], so the claim follows.

(2) The \( G \)-cover \( h : \Sigma_g \rightarrow \Sigma_b \) define an injective pull-back map \( h^* : H^1(\Sigma_b, \mathbb{Q}) \rightarrow H^1(\Sigma_g, \mathbb{Q}) \), whose image is precisely \( H^1(\Sigma_g, \mathbb{Q})^G \). So it suffices to check that \( h^* \) is invariant under the monodromy map \( \rho_{\pi_1(\Sigma_b)} \) and, to this purpose, we consider the factorization of \( h \) given as follows:

\[
\Sigma_g \xrightarrow{\iota} S \xrightarrow{f} \Sigma_b \times \Sigma_b \xrightarrow{\pi_i} \Sigma_b.
\]

By (30), the image of \( \iota^* : H^1(S, \mathbb{Q}) \rightarrow H^1(\Sigma_g, \mathbb{Q}) \) coincides with the invariant subspace \( H^1(\Sigma_g, \mathbb{Q})^{\pi_1(\Sigma_b)} \); thus, given any automorphism \( \xi : H^1(\Sigma_g, \mathbb{Q}) \rightarrow H^1(\Sigma_g, \mathbb{Q}) \) in the image of \( \rho_{\pi_1(\Sigma_b)} \), we get \( \xi \circ \iota^* = \iota^* \). Using (31), this in turn implies

\[
\xi \circ h^* = \xi \circ (\iota^* \circ f^* \circ \pi_i^*) = (\xi \circ \iota^*) \circ f^* \circ \pi_i^* = \iota^* \circ f^* \circ \pi_i^* = h^*,
\]

so \( h^* \) is \( \rho_{\pi_1(\Sigma_b)} \)-invariant and we are done. \( \square \)

Corollary 4.17. Let \( f : S \rightarrow \Sigma_b \times \Sigma_b \) be a diagonal double Kodaira fibration as above. Then the following are equivalent.

(1) The pull-back map \( f^* : H^1(\Sigma_b \times \Sigma_b, \mathbb{Q}) \rightarrow H^1(S, \mathbb{Q}) \) is an isomorphism.
(2) \( H^1(\Sigma_g, \mathbb{Q})^G = H^1(\Sigma_g, \mathbb{Q})^{\pi_1(\Sigma_b)} \).

Proof. It is sufficient to show that both conditions are equivalent to \( b_1(S) = 4b \). For (1), this follows from the injectivity of the pull-back in cohomology associated with a finite \( G \)-cover. On the other hand, by Proposition 4.16, equality (2) holds if and only if

\[
\dim H^1(\Sigma_b, \mathbb{Q}) = \dim H^1(S, \mathbb{Q}) - 2b,
\]

namely, if and only if \( \dim H^1(S, \mathbb{Q}) = 4b \), as claimed. \( \square \)

Borrowing the terminology from [Breg18, Definition 2.8], we say that a diagonal double Kodaira fibration \( f : S \rightarrow \Sigma_b \times \Sigma_b \) is maximal if it satisfies one of the equivalent conditions in Corollary 4.17. Since \( b_1(\Sigma_2 \times \Sigma_2) = 8 \), Proposition 4.14 implies the following

Proposition 4.18. Let \( f : S \rightarrow \Sigma_2 \times \Sigma_2 \) be a diagonal double Kodaira fibration, associated with a diagonal double Kodaira structure of type \((b, n) = (2, 2)\) on an extra-special group \( G \) of order 32. Then \( f \) is maximal.

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https://mathoverflow.net/questions/357453
https://mathoverflow.net/questions/366044
https://mathoverflow.net/questions/366771
https://mathoverflow.net/questions/368628
https://mathoverflow.net/questions/371181
### Appendix A. Non abelian groups of order 24 and 32

Table 1. Nonabelian groups of order 24.

Source: groupprops.subwiki.org/wiki/Groups_of_order_24

| IdSmallGroup(G) | G | Presentation |
|-----------------|---|-------------|
| G(24, 1) | D_{8,3,-1} | \langle x, y \mid x^8 = y^3 = 1, xyx^{-1} = y^{-1} \rangle |
| G(24, 3) | SL(2, \mathbb{F}_3) | \langle x, y, z \mid x^3 = y^3 = z^2 = xyz \rangle |
| G(24, 4) | Q_{24} | \langle x, y, z \mid x^2 = y^2 = z^2 = 1 \rangle |
| G(24, 5) | D_{12,5} | \langle x, y \mid x^2 = y^{12} = 1, xyx^{-1} = y^{-1} \rangle |
| G(24, 6) | D_2 \times D_{12,5} | \langle x, y \mid x^2 = y^{12} = 1, xyx^{-1} = y^{-1} \rangle |
| G(24, 7) | \mathbb{Z}_2 \times D_{1,5,-1} | \langle z \mid z^2 = 1 \rangle \times \langle x, y \mid x^4 = y^3 = 1, xyx^{-1} = y^{-1} \rangle |
| G(24, 8) | ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3) \times \mathbb{Z}_2 | \langle x, y, z, w \mid x^2 = y^3 = z^2 = w^3 = 1, \ [y, z] = [y, w] = [z, w] = 1, \ xyx^{-1} = y, xzx^{-1} = zy, xwx^{-1} = w^{-1} \rangle |
| G(24, 10) | \mathbb{Z}_3 \times D_3 | \langle z \mid z^3 = 1 \rangle \times \langle x, y \mid x^2 = y^3 = 1, xyx^{-1} = y^{-1} \rangle |
| G(24, 11) | \mathbb{Z}_3 \times Q_8 | \langle z \mid z^3 = 1 \rangle \times \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle |
| G(24, 12) | S_4 | \langle x, y \mid x = (12), y = (1234) \rangle |
| G(24, 13) | \mathbb{Z}_2 \times A_4 | \langle z \mid z^2 = 1 \rangle \times \langle x, y \mid x = (12)(34), y = (123) \rangle |
| G(24, 14) | (\mathbb{Z}_2)^2 \times S_3 | \langle z, w \mid z^2 = w^2 = 1 \rangle \times \langle x, y \mid x = (12), y = (123) \rangle |
| IdSmallGroup(G) | \(G\) | Presentation |
|----------------|----------------|----------------|
| G(32, 2)     | \((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4\) | \(\langle x, y, z \mid x^4 = y^2 = z^2 = 1,\) \(x, y = z, [x, z] = [y, z] = 1\rangle\) |
| G(32, 4)     | D_{4,8.5}       | \(\langle x, y \mid x^4 = y^8 = z^2 = w^2 = 1,\) \([x, y] = 1, [x, z] = 1, [y, z] = 1\rangle\) |
| G(32, 5)     | \((\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2\) | \(\langle x, y, z \mid x^2 = y^2 = z^2 = 1,\) \([x, y] = 1,\) \(z x x z^{-1} = x y y z^{-1} = y z^{-1}\) |
| G(32, 6)     | \((\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_4\) | \(\langle x, y, z, w \mid x^2 = y^2 = z^2 = w^2 = 1,\) \([x, y] = 1, [x, z] = 1, (y u^{-1})^2 = 1,\) \(y u^{-1} = y^{-1}\) |
| G(32, 7)     | \((\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2\) | \(\langle x, y, z, u, w \mid u^2 = w^{-1}, x^2 = u, (y u^{-1})^2 = 1,\) \(y u^{-1} = y^{-1}\) |
| G(32, 8)     | \((\mathbb{Z}_2)^2 \cdot (\mathbb{Z}_4 \rtimes \mathbb{Z}_2)\) | \(\langle x, y, z \mid x^4 = y^2 = z^2 = w^2 = 1,\) \(x x x x^{-1} = j, x j x x^{-1} = k, x k x k^{-1} = k^{-1}\) |
| G(32, 9)     | \((\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2\) | \(\langle x, y, z \mid x^2 = y^2 = z^2 = 1,\) \([x, y] = 1,\) \(z x x z^{-1} = x z^3 y, y z y z^{-1} = y^{-1}\) |
| G(32, 10)    | Q_{8} \rtimes \mathbb{Z}_4 | \(\langle i, j, k, x \mid i^2 = j^2 = k^2 = 1, i x x^{-1} = j, x j x x^{-1} = i, x k x k^{-1} = k^{-1}\) |
| G(32, 11)    | \((\mathbb{Z}_4)^2 \rtimes \mathbb{Z}_2\) | \(\langle x, y, z \mid x^4 = y^2 = z^2 = 1,\) \(z x x z^{-1} = y, y z y z^{-1} = x\) |
| G(32, 12)    | D_{8,4.3}       | \(\langle x, y \mid x^4 = y^2 = 1, x y x^{-1} = y^{-1}\) |
| G(32, 13)    | D_{8,4.3}       | \(\langle x, y \mid x^4 = y^2 = 1, x y x^{-1} = y^{-1}\) |
| G(32, 14)    | D_{8,4.3} -1    | \(\langle x, y \mid x^4 = y^2 = 1, x y x^{-1} = y^{-1}\) |
| G(32, 15)    | \(\mathbb{Z}_4 \cdot D_8\) | \(\langle x, y, z, u, w \mid u^2 = w^{-1}, z^2 = u^2 = w^{-1},\) \(x x x x^{-1} = j, x j x x^{-1} = k, x k x k^{-1} = k^{-1}\) |
| G(32, 17)    | D_{2,16.9}      | \(\langle x, y \mid x^2 = y^{10} = 1, x y x^{-1} = y^{-1}\) |
| G(32, 18)    | D_{32}          | \(\langle x, y \mid x^4 = y^2 = 1, x y x^{-1} = y^{-1}\) |
| G(32, 19)    | QD_{32}         | \(\langle x, y \mid x^4 = y^{10} = 1, x y x^{-1} = y^{-1}\) |
| G(32, 20)    | Q_{32}          | \(\langle x, y, z \mid x^2 = y^2 = z^2 = 1\) |
| G(32, 22)    | \(\mathbb{Z}_2 \times ((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)\) | \(\langle w \mid w^2 = 1\rangle \times \langle x, y, z \mid x^4 = y^2 = z^2 = 1,\) \([x, y] = 1,\) \(x x x x^{-1} = y, y z y z^{-1} = y\) |
| G(32, 23)    | \(\mathbb{Z}_2 \times D_{4,4.3}\) | \(\langle z \mid z^2 = 1\rangle \times \langle x, y \mid x^4 = y^2 = 1, x y x^{-1} = y^{-1}\) |
| G(32, 24)    | \((\mathbb{Z}_4)^2 \rtimes \mathbb{Z}_2\) | \(\langle x, y \mid x^4 = y^2 = z^2 = 1,\) \([x, y] = 1,\) \(z x x z^{-1} = x, y z y z^{-1} = x^{-1}\) |
| G(32, 25)    | \(\mathbb{Z}_4 \times D_8\) | \(\langle z \mid z^2 = 1\rangle \times \langle x, y \mid x^2 = y^2 = 1, x y x^{-1} = y^{-1}\) |
| G(32, 26)    | \(\mathbb{Z}_4 \times Q_8\) | \(\langle z \mid z^2 = 1\rangle \times \langle i, j, k \mid i^2 = j^2 = k^2 = i j k\) |
| G(32, 27)    | \((\mathbb{Z}_2)^3 \rtimes (\mathbb{Z}_2)^2\) | \(\langle x, y, z, a, b \mid x^2 = y^2 = z^2 = a^2 = b^2 = 1,\) \([x, y] = [y, z] = [x, z] = [a, b] = 1,\) \(a x a^{-1} = x, a y a^{-1} = y, a z a^{-1} = x z,\) \(b x b^{-1} = x, b y b^{-1} = y, b z b^{-1} = y z\) |
| G(32, 28)    | \((\mathbb{Z}_4 \times (\mathbb{Z}_2)^2) \rtimes \mathbb{Z}_2\) | \(\langle x, y, z, w \mid x^4 = y^2 = z^2 = w^2 = 1,\) \([x, y] = [x, z] = [y, z] = 1,\) \(w x w^{-1} = x^{-1}, w y w^{-1} = z, w z w^{-1} = y\) |
| G(32, 29)    | \((\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2\) | \(\langle x, i, j, k, z \mid x^2 = z^2 = 1, i^2 = j^2 = k^2 = i j k,\) \([x, i] = [x, j] = [x, k] = 1,\) \(z x x z^{-1} = x, z y y z^{-1} = z, i j z = x y z^{-1}\) |
| G(32, 30)    | \((\mathbb{Z}_4 \times (\mathbb{Z}_2)^2) \rtimes \mathbb{Z}_2\) | \(\langle x, y, z, w \mid x^2 = y^2 = z^2 = w^2 = 1,\) \([x, y] = [y, z] = [z, w] = 1,\) \(w x w^{-1} = x y, w y w^{-1} = y, w z w^{-1} = x^2 z\) |
| \(\text{IdSmallGroup}(G)\) | \(G\) | Presentation |
|----------------|----------------|-------------|
| \((\mathbb{Z}_4)^2 \times \mathbb{Z}_2\) | \(\langle x, y, z \mid x^4 = y^4 = [x, y] = 1, z^2 = 1,\) \(\langle xyz^{-1} = xy^2, yz^{-1} = x^2y^2\rangle\) | \((32, 31)\) |
| \((\mathbb{Z}_2)^4 \cdot (\mathbb{Z}_2)^4\) | \(\langle x, y, z, u, w \mid u^2 = w^2 = 1,\) \(u = z^2, u = x^2, w = y^2,\) \(xy^{-1} = x^{-1}, [y, z] = 1, xxyz = 1\rangle\) | \((32, 32)\) |
| \((\mathbb{Z}_4)^2 \times \mathbb{Z}_2\) | \(\langle x, y, z \mid x^4 = y^4 = [x, y] = 1, z^2 = 1,\) \(\langle xxz^{-1} = xy^2, yz^{-1} = x^2y^2\rangle\) | \((32, 33)\) |
| \((\mathbb{Z}_4)^2 \times \mathbb{Z}_2\) | \(\langle x, y, z \mid x^4 = y^4 = [x, y] = 1, z^2 = 1,\) \(\langle xxz^{-1} = x^{-1}, yz^{-1} = y^{-1}\rangle\) | \((32, 34)\) |
| \(\mathbb{Z}_4 \times Q_8\) | \(\langle x, i, j, k \mid x^4 = 1, i^2 = j^2 = k^2 = ijk,\) \(\langle xix^{-1} = x^{-1}, jxj^{-1} = x^{-1}, kxk^{-1} = x\rangle\) | \((32, 35)\) |
| \((\mathbb{Z}_8 \times \mathbb{Z}_2) \times \mathbb{Z}_2\) | \(\langle x, y, z \mid x^8 = y^2 = z^2 = 1, [x, y] = 1,\) \(\langle xxz^{-1} = x^5, yz^{-1} = y\rangle\) | \((32, 36)\) |
| \((\mathbb{Z}_8 \times \mathbb{Z}_2) \times \mathbb{Z}_2\) | \(\langle x, y, z \mid x^8 = y^2 = z^2 = 1, [x, y] = 1,\) \(\langle xxz^{-1} = x, yz^{-1} = x^{-1}\rangle\) | \((32, 37)\) |
| \(\mathbb{Z}_2 \times D_{16}\) | \(\langle z \mid z^2 = 1 \rangle \times \langle x, y \mid x^4 = y^8 = 1, xyx^{-1} = y^{-1}\rangle\) | \((32, 38)\) |
| \(\mathbb{Z}_2 \times QD_{16}\) | \(\langle z \mid z^2 = 1 \rangle \times \langle x, y \mid x^4 = y^8 = 1, xyx^{-1} = y^{-1}\rangle\) | \((32, 39)\) |
| \(\mathbb{Z}_2 \times Q_{16}\) | \(\langle w \mid w^4 = 1 \rangle \times \langle x, y, z \mid x^4 = y^2 = z^2 = xyz\rangle\) | \((32, 40)\) |
| \((\mathbb{Z}_8 \times \mathbb{Z}_2) \times \mathbb{Z}_2\) | \(\langle x, y, z \mid x^8 = y^2 = z^2 = 1, [x, y] = 1,\) \(\langle xxz^{-1} = x^3, yz^{-1} = y\rangle\) | \((32, 41)\) |
| \(\mathbb{Z}_8 \times (\mathbb{Z}_2)^4\) | \(\langle x, y, z \mid x^8 = 1, y^2 = z^2 = [y, z] = 1,\) \(\langle xxy^{-1} = x^{-1}, zzx^{-1} = x^3\rangle\) | \((32, 42)\) |
| \((\mathbb{Z}_2 \times Q_8) \times \mathbb{Z}_2\) | \(\langle x, i, j, k, z \mid x^2 = z^2 = 1, i^2 = j^2 = k^2 = ijk,\) \(\langle [x, i] = [x, j] = [x, k] = 1,\) \(\langle xxz^{-1} = x^2, yz^{-1} = j, zjz^{-1} = i\rangle\) | \((32, 43)\) |
| \((\mathbb{Z}_2)^2 \times D_8\) | \(\langle z, w \mid z^2 = w^2 = [z, w] = 1\rangle\) \(\times (\langle x, y \mid x^2 = y^1 = 1, xyx^{-1} = y^{-1}\rangle\) | \((32, 44)\) |
| \((\mathbb{Z}_2)^2 \times Q_8\) | \(\langle z, w \mid z^2 = w^2 = [z, w] = 1\rangle\) \(\times (\langle i, j, k \mid i^2 = j^2 = k^2 = ijk\rangle\) | \((32, 45)\) |
| \((\mathbb{Z}_4 \times (\mathbb{Z}_2)^2) \times \mathbb{Z}_2\) | \(\langle x, y, z \mid x^4 = y^2 = z^2 = 1,\) \([x, y] = [x, z] = [y, z] = 1,\) \(\langle wxw^{-1} = x, wyw^{-1} = y, wzw^{-1} = x^2z\rangle\) | \((32, 46)\) |
| \(H_5(\mathbb{Z}_2)\) | \(\langle r_1, r_2, t_2, z \mid r_1 = t_2^2 = z^2 = 1,\) \(\langle r_1, z, t_2 \rangle = 1,\) \(\langle r_1, r_2 \rangle = 1,\) \(\langle r_1, t_2 \rangle = 1,\) \(\langle r_1, t_2 \rangle = 1,\) \(\langle r_1, t_2 \rangle = 1\rangle,\) \(\text{see (4)}\) | \((32, 47)\) |
| \(G_5(\mathbb{Z}_2)\) | \(\langle r_1, r_2, t_2, z \mid r_1 = t_2^2 = z^2 = 1,\) \(\langle r_1, z, t_2 \rangle = 1,\) \(\langle r_1, r_2 \rangle = 1,\) \(\langle r_1, t_2 \rangle = 1,\) \(\langle r_1, t_2 \rangle = 1\rangle,\) \(\text{see (5)}\) | \((32, 48)\) |

Table 2. Nonabelian groups of order 32.
Source: groupprops.subwiki.org/wiki/Groups_of_order_32
Appendix B. The Computation of $H_1(S, \mathbb{Z})$

Let $G$ be an extra-special group of order 32, and $S$ the diagonal double Kodaira surface associated with the diagonal double Kodaira structure of type $(b, n) = (2, 2)$ on $G$ given in Example 3.16. We want to show how to compute the first homology group $H_1(S, \mathbb{Z})$ by using the computer algebra software GAP4. In the sequel we will assume $G = G(32, 49)$; the script for $G = G(32, 50)$ only requires minimal modifications.

We start by constructing the group $G$:

```gap
ColorPrompt(true);
# redefine commutators
comm := function ( x, y ) return x*y*x^-1*y^-1; end;

# presentation of G(32, 49)
f := FreeGroup("r1", "t1", "r2", "t2", "z");;
AssignGeneratorVariables(f);
G49 := f / [ r1^2, t1^2, r2^2, t2^2, z^2,
            comm( r1, z ), comm( t1, z ), comm( r2, z ), comm( t2, z ),
            comm( r1, r2 ), comm( t1, t2 ), comm( r1, t2 ), comm( r2, t1 ),
            comm( r1, t1 ) * z, comm( r2, t2 ) * z ];;
AssignGeneratorVariables(G49);
```

Then we construct the group $P_2(\Sigma_2)^{\text{orb}}$:

```gap
F := FreeGroup("r11", "t11", "r12", "t12", "r21", "t21", "r22", "t22", "A12");;
AssignGeneratorVariables(F);
S1 := comm( r12^-1, t12^-1 ) * t12^-1 * comm( r11^-1, t11^-1 ) * t12 * A12^-1;
S2 := comm( r21^-1, t21 ) * t21 * comm( r22^-1, t22 ) * t21^-1 * A12;
R1 := comm( r11, r22 );;
R2 := comm( r11, r21 );;
R3 := comm( r11, t22 );;
R4 := comm( r11, t21 ) * A12;
R5 := comm( r11, A12 ) * comm( A12, r21^-1 );;
R6 := comm( r12, r22 );;
R7 := A12^-1 * r21 * r22^-1 * A12 * r22 * r21^-1 * comm( r21, r12 );;
R8 := comm( r12, t22 ) * A12;
R9 := comm( r12, t21 ) * comm( t21, A12^-1 );;
R10 := comm( r12, A12 ) * comm( A12, r22^-1 );;
T1 := comm( t11, r22 );;
T2 := comm( t11, r21 ) * t21^-1 * A12^-1 * t21;
T3 := comm( t22, t11 );;
T4 := comm( t11, t21 ) * comm( A12, t21^-1 );;
T5 := comm( t11, A12 ) * comm( A12, t21^-1 );;
T6 := comm( t12, r22 ) * t22^-1 * A12^-1 * t22;
T7 := comm( t12, r21 ) * comm( A12, t22^-1 );;
T8 := comm( t12, t22 ) * comm( A12, t22^-1 );;
T9 := t22^-1 * A12 * t22 * A12^-1 * t21 * A12 * t22^-1 * A12^-1 * t22 * t21
     * comm( t21, t12 );;
T10 := comm( t12, A12 ) * comm( A12, t22^-1 );;
P2_orb := F / [ S1, S2, R1, R2, R3, R4, R5, R6, R7, R8, R9, R10,
             T1, T2, T3, T4, T5, T6, T7, T8, T9, T10, A12 ];;
AssignGeneratorVariables( P2_orb );;
```

Next, we define the group epimorphism $\varphi^{\text{orb}} : P_2(\Sigma_2)^{\text{orb}} \to G$, and we compute its kernel, which is isomorphic to $\pi_1(S)$:

```gap
# computation of the map $\psi_{\text{orb}}$

$$\psi_{\text{orb}}_{49} := \text{GroupHomomorphismByImages}( P2_{\text{orb}}, G49, \{r11, t11, r12, t12, r21, t21, r22, t22, A12\}, \{r1, t1, r2*t1, r1*t2, r1*t2, r2*t1, r2, t2, z\} );$$

# computation of the fundamental group of $S$

$$K49 := \text{Kernel}( \psi_{\text{orb}}_{49} );;$$

$$I49 := \text{Image}( \text{IsomorphismFpGroup}( K49 ) );;$$

$$\pi_1S_{49} := \text{Range}( \text{IsomorphismSimplifiedFpGroup}( I49 ) );;$$

$$\text{Size}( \text{FreeGeneratorsOfFpGroup}( \pi_1S_{49} ) );$$

$$\text{Size}( \text{RelatorsOfFpGroup}( \pi_1S_{49} ) );$$

The output tells us that $\pi_1(S)$ has 18 generators and 365 relations. Finally, we compute its abelianization, via the command

# computation of the first homology group of $S$

$$H_1S_{49} := \text{AbelianInvariants}( \pi_1S_{49} );$$

The output is $[0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2]$, and this means that we have an isomorphism

$$H_1(S, \mathbb{Z}) = \mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4.$$}

In the same way, we checked that the result does not depend on the chosen diagonal double Kodaira structure. More details about the calculation will appear in a subsequent paper.

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