RATIONAL PARKING FUNCTIONS AND CATALAN NUMBERS

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Abstract. The “classical” parking functions, counted by the Cayley number \((n + 1)^{n-1}\), carry a natural permutation representation of the symmetric group \(S_n\) in which the number of orbits is the Catalan number \(\frac{1}{n+1} \binom{2n}{n}\). In this paper, we will generalize this setup to “rational” parking functions indexed by a pair \((a, b)\) of coprime positive integers. We show that these parking functions, which are counted by \(b^{a-1}\), carry a permutation representation of \(S_a\) in which the number of orbits is the “rational” Catalan number \(\frac{1}{a+b} \binom{a+b}{a}\). We compute the Frobenius characteristic of the \(S_a\)-module of \((a, b)\)-parking functions. Next we study \(q\)-analogues of the rational Catalan numbers, proposing a combinatorial formula for \(\frac{1}{a+b} \binom{a+b}{a} q^a\) and relating this formula to a new combinatorial model for \(q\)-binomial coefficients. Finally, we discuss \(q, t\)-analogues of rational Catalan numbers and parking functions (generalizing the shuffle conjecture for the classical case) and present several conjectures.

1. Introduction

1.1. Overview of Parking Functions. The goal of this paper is to generalize the theory of “classical” parking functions — counted by \((n + 1)^{n-1}\) — to the theory of “rational” parking functions — counted by \(b^{a-1}\) for coprime positive integers \(a, b \in \mathbb{N}\). The combinatorial foundation for this theory is given by rational Dyck paths, which are lattice paths staying weakly above a line of slope \(a/b\). The classical case corresponds to \((a, b) = (n, n+1)\), which for most purposes is equivalent to considering a line of slope 1.

One major algebraic motivation for studying parking functions comes from representation theory. It was conjectured by Haiman [21] Conjecture 2.1.1 that the \(S_n\)-module of “diagonal coinvariants” has dimension \((n + 1)^{n-1}\) and that the naturally bi-graded character of this module might be encoded by parking functions [21] Conjecture 2.6.3. The Shuffle Conjecture [19] Conjecture 3.1.2 (which is still open) gives such a precise description. We suggest in Definition 24 a way to generalize the combinatorial form of the Shuffle Conjecture to rational parking functions (see also Conjecture 27). However, we do not know what algebraic or geometric objects might underlie the generalized combinatorics. The fact that the combinatorics works out so nicely suggests that there must be an underlying reason. See [14, 15, 23] for related work that arises from a more geometric viewpoint.

The Shuffle Conjecture and related formulas for \(q, t\)-Catalan numbers involve certain statistics on Dyck paths, namely area paired with either dinv or bounce. The area statistic is natural and the dinv and bounce statistics are strange. However, there is a bijection zeta on Dyck paths that sends dinv to area and area to bounce. Thus one could say that there is really just one natural statistic area and one strange map zeta. In this paper we will define an analogous map that we use to define \(q, t\)-analogues of rational parking functions. Experimentally, this map has beautiful combinatorial properties. Conjecture 27 contains several conjectures pertaining to the symmetry and \(t = 1/q\) specializations of these rational \(q, t\)-parking functions. In a forthcoming paper [4] we will show that our map belongs to a whole family of sweep maps that contain a wealth of combinatorial information.

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We will also study $q$-analogues of rational Catalan numbers that are closely related to the $t = 1/q$ specialization mentioned above. These rational $q$-Catalan numbers are given algebraically by $\frac{1}{[a+b]_q^{[a+b]_q}}$ where $\gcd(a, b) = 1$ (see \[1.5\] for the definition of this notation). Although these expressions have long been known to be polynomials in $q$ with nonnegative integer coefficients, it is an open problem to find a combinatorial interpretation for these polynomials. We propose such a combinatorial model in \[4\] along with a related non-standard combinatorial interpretation for general $q$-binomial coefficients.

1.2. Outline of the Paper. The structure of the paper is as follows. We conclude the introduction by reviewing standard notation concerning partitions, symmetric functions, and $q$-binomial coefficients that will be used throughout the paper; for more details, consult \[29, 32, 39\]. \[2\] gives background pertaining to diagonal coinvariants and classical parking functions, deriving several formulas for the ungraded Frobenius character of these modules. \[3\] generalizes this discussion to the case of rational parking functions. The key to deriving the Frobenius characters of these new modules is Proposition \[2\] which enumerates rational parking functions with a specified vertical run structure. \[4\] studies the rational $q$-Catalan numbers, proposing new combinatorial formulas for these polynomials and for general $q$-binomial coefficients based on certain partition statistics. We prove that Conjecture \[8\] which proposes a novel combinatorial interpretation for $q$-binomial coefficients, is equivalent to Conjecture \[9\] which proposes a combinatorial interpretation for rational $q$-Catalan numbers. \[5\] reviews the classical theory of $q, t$-parking functions and $q, t$-Catalan numbers, culminating in the Shuffle Conjecture for the Frobenius series of the doubly-graded module of diagonal coinvariants. \[6\] generalizes this theory to the case of rational $q, t$-Catalan numbers and rational $q, t$-parking functions. Finally, \[7\] includes explicit computations of various polynomials considered in this paper.

1.3. Notation for Partitions. A partition is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ of positive integers. The $i$-th part of $\lambda$ is $\lambda_i$. The area of $\lambda$ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_s$. We say that $\lambda$ is a partition of $n$ (denoted by $\lambda \vdash n$) when $|\lambda| = n$. The length of $\lambda$ is $\ell(\lambda) = s$, the number of parts of $\lambda$. We consider the empty sequence to be the unique partition of $0$; this partition has length zero. It is sometimes convenient to add one or more zero parts to the end of a partition; this does not change the length of the partition. Let $\text{Par}$ be the set of all partitions. For all $j \geq 1$, $m_j(\lambda)$ is the number of parts of $\lambda$ equal to $j$. The integer $z_\lambda$ is defined by

$$z_\lambda = \prod_{j \geq 1} j^{m_j(\lambda)} m_j(\lambda)!.$$ 

For example, $\lambda = (4, 4, 1, 1, 1)$ is a partition with $|\lambda| = 11$, $\ell(\lambda) = 5$, $m_1(\lambda) = 3$, $m_4(\lambda) = 2$, and $z_\lambda = 1^3 4^2 3! 2! = 192$.

1.4. Notation for Symmetric Functions. Let $\text{Sym}$ denote the ring of symmetric functions, which we view as a subring of $\mathbb{C}[x_1, x_2, x_3, \ldots] = \mathbb{C}[\mathbb{N}]$ as in \[32\] Ch. I. We now recall some bases of $\text{Sym}$ used later in the paper. First, the complete homogeneous symmetric functions $h_i(x)$ are defined by the generating function

$$H(t) = \sum_{i \geq 0} h_i(x) t^i = \prod_{j \geq 1} \frac{1}{(1 - x_j t)}.$$

For any partition $\lambda$ of length $l$, let $h_\lambda(x) = h_{\lambda_1}(x) \cdots h_{\lambda_l}(x)$. Then $\{h_\lambda(x) : \lambda \in \text{Par}\}$ is the complete homogeneous basis of $\text{Sym}$. Second, the power sum symmetric functions $p_i(x)$ are defined for all $i \geq 1$ by setting $p_i(x) = x_1^i + x_2^i + \cdots + x_k^i + \cdots$. For any partition $\lambda$ of length $l$, let $p_\lambda(x) = p_{\lambda_1}(x) \cdots p_{\lambda_l}(x)$. Then $\{p_\lambda(x) : \lambda \in \text{Par}\}$ is the power sum basis of $\text{Sym}$. Third, the monomial symmetric function $m_\lambda(x)$ is the sum of all distinct monomials
\(x_1^{\lambda_1}x_2^{\lambda_2} \cdots x_n^{\lambda_n}\) with nonzero exponents \(\lambda_1, \lambda_2, \ldots, \lambda_t\) (in some order). The **monomial basis** of \(\text{Sym}\) is \(\{m_\lambda(x) : \lambda \in \text{Par}\}\). Finally, the **Schur basis** of \(\text{Sym}\) (defined in [32]) is denoted \(\{s_\lambda(x) : \lambda \in \text{Par}\}\).

1.5. **Notation for \(q\)-Binomial Coefficients.** Let \(q\) be a formal variable. For all integers \(n \geq k \geq 0\), define the \(q\)-**integer** \([n]_q = 1 + q + q^2 + \cdots + q^{n-1}\), the \(q\)-**factorial** \([n]_q! = \prod_{j=1}^n [j]_q\), and the \(q\)-**binomial coefficient**

\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.
\]

We usually use the multinomial coefficient notation \(\binom{a+b}{a,b}_q = \frac{[a+b]_q!}{[a]_q! [b]_q!}\) when discussing \(q\)-binomial coefficients.

2. **Classical Parking Functions**

The permutations of the set \([n] = \{1, 2, \ldots, n\}\) are counted by the factorial \(n!\). Two other important objects in combinatorics, trees and parking functions, are counted by the Cayley number \((n+1)^{n-1}\). One can algebraically motivate the progression from \(n!\) to \((n+1)^{n-1}\) as follows.

2.1. **Diagonal Coinvariants.** The symmetric group \(S_n\) acts on the polynomial ring \(\mathbb{C}[x_1, x_2, \ldots, x_n]\) by permuting variables. Isaac Newton knew that the subring of "symmetric polynomials" is generated by the **power sum polynomials**

\[p_k = p_k(x_1, \ldots, x_n) = x_1^k + x_2^k + \cdots + x_n^k \quad (1 \leq k \leq n).\]

Claude Chevalley [8] knew that the quotient ring of coinvariants

\[R_n = \mathbb{C}[x_1, x_2, \ldots, x_n]/\langle p_1, p_2, \ldots, p_n \rangle\]

is isomorphic to a graded version of the regular representation of \(S_n\), and so has dimension \(n!\). Moreover, the Hilbert series of \(R_n\) is the \(q\)-factorial:

\[
\text{Hilb}_{R_n}(q) = \sum_{i \geq 0} \dim(R_n^{(i)}) q^i = \prod_{j=1}^n (1 + q + \cdots + q^{j-1}) = [n]_q!.
\]

Armand Borel [6] knew that \(R_n\) is also the cohomology ring of the complete flag variety. It turns out that the structure of \(R_n\) is closely related to the combinatorial structure of permutations.

More generally, \(S_n\) acts diagonally on the polynomial ring \(\mathbb{C}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]\) by **simultaneously** permuting the \(x\)-variables and the \(y\)-variables. Hermann Weyl [43] knew that the subring of \(S_n\)-invariant polynomials is generated by the **polarized power sums**:

\[p_{k,\ell} = p_{k,\ell}(x_1, \ldots, x_n, y_1, \ldots, y_n) = x_1^ky_1^\ell + x_2^ky_2^\ell + \cdots + x_n^ky_n^\ell \quad (k, \ell \in \mathbb{N}, k + \ell > 0).\]

Mark Haiman [21] Conjecture 2.1.1] conjectured that the quotient ring of **diagonal coinvariants**

\[\text{DR}_n = \mathbb{C}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]/(p_{k,\ell} : k + \ell > 0)\]

has dimension \((n+1)^{n-1}\) as a vector space over \(\mathbb{C}\). Furthermore, this vector space is bi-graded by \(x\)-degree and \(y\)-degree, and he saw hints that the bi-graded Hilbert series

\[
\text{Hilb}_{\text{DR}_n}(q, t) = \sum_{i,j \geq 0} \dim(\text{DR}_n^{(i,j)}) q^i t^j
\]

is closely related to well-known structures in combinatorics. Haiman, along with Adriano Garsia, made several conjectures in this direction [11]. However, because the polarized power sums \(p_{k,\ell}\) are not algebraically independent, it turned out to be quite difficult to prove these conjectures. Some have now been proved, some are still open, and in general the subject remains very active.
2.2. Parking Functions. Arthur Cayley [7] showed that there are \((n + 1)^n - 1\) trees with \(n + 1\) labeled vertices. However, for the purposes of studying diagonal coinvariants, we prefer to discuss parking functions \([25, 37]\), which can be defined as follows. There are \(n\) cars \(C_1, C_2, \ldots, C_n\) that want to park in \(n\) spaces along a one-way street. Each car \(C_i\) has a preferred spot \(a_i \in [n]\) and the cars park in order: \(C_1\) first, \(C_2\) second, etc. When car \(C_i\) arrives it will try to park in spot \(a_i\). If spot \(a_i\) is already taken it will park in the first available spot after \(a_i\). If no such spot exists, the parking process fails. We say that the \(n\)-tuple \((a_1, a_2, \ldots, a_n)\) is a classical parking function if it allows all of the cars to park.

For example, \((a_1, a_2, a_3, a_4, a_5) = (2, 4, 1, 2, 1)\) is a parking function. The order of the parked cars is shown here:

\[
\begin{array}{cccccc}
C_3 & C_1 & C_4 & C_2 & C_5 \\
1 & 2 & 3 & 4 & 5
\end{array}
\]

One may check that \((a_1, a_2, \ldots, a_n) \in [n]^n\) is a parking function if and only if its increasing rearrangement \(b_1 \leq b_2 \leq \cdots \leq b_n\) satisfies \(b_i \leq i\) for all \(i \in [n]\). This leads to a few observations.

First, the set of parking functions \((a_1, a_2, \ldots, a_n)\) is closed under permuting subscripts. Let \(PF_n\) denote both the set of parking functions and the corresponding permutation representation of \(S_n\) [38, Def. 1.3.2]. Haiman conjectured that, after a sign twist, the ungraded \(S_n\)-module \(DR_n\) of diagonal coinvariants is isomorphic to \(PF_n\). We describe the graded version of this conjecture in §2.6 below.

Second, the increasing parking functions \((a_1 \leq a_2 \leq \cdots \leq a_n)\) are in bijection with the set of Dyck paths. We define a classical Dyck path of order \(n\) as a sequence in \(\{N, E\}^n\) with the property that every initial subsequence has at least as many \(N\)'s as \(E\)'s. If we read from left to right, interpreting \(N\) as “go north” and \(E\) as “go east”, we can think of this as a lattice path in \(\mathbb{Z}^2\) from \((0,0)\) to \((n,n)\) staying weakly above the diagonal line \(y = x\). Now, given an increasing parking function \(P = (a_1 \leq a_2 \leq \cdots \leq a_n)\), let \(r_i\) be the number of times that \(i\) occurs in \(P\). Then we associate \(P\) with the Dyck path

\[
\begin{aligned}
&N \cdots N \quad E \quad N \cdots N \quad E \quad \cdots \quad E \quad N \cdots N \quad E.
\end{aligned}
\]

For example, the increasing parking function \((1 \leq 1 \leq 2 \leq 2 \leq 4) \in PF_5\) corresponds to the Dyck path \(NNENNEE\):

The number of Dyck paths of order \(n\) is the Catalan number

\[
\text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}.
\]

Third, generalizing the previous observation, we associate (possibly non-increasing) parking functions with labeled Dyck paths. Given a parking function \(P = (a_1, a_2, \ldots, a_n)\) we first draw the Dyck path corresponding to the increasing rearrangement. The \(i\)-th vertical run of the path has length \(r_i\) (possibly zero), which corresponds to the number of occurrences of \(i\) in \(P\). If we have \(r_i = k\) with \(i = a_{j_1} = a_{j_2} = \cdots = a_{j_k}\) then we will label the \(i\)-th vertical run by the set of indices...
\{j_1, j_2, \ldots, j_k\}. We do this by filling the boxes to the right of the \(i\)-th vertical run with the labels \(j_1 < j_2 < \cdots < j_k\), increasing vertically. For example, the parking function \((2, 4, 1, 2, 1) \in \text{PF}_5\) corresponds to the following labeled Dyck path:

This is the standard way to encode parking functions in the literature on diagonal coinvariants. We will use this encoding to compute the structure of the \(S_n\)-module \(\text{PF}_n\).

### 2.3. Frobenius Characteristic of \(\text{PF}_n\)

We have observed that \(S_n\) acts on parking functions \((a_1, \ldots, a_n)\) by permuting subscripts. Translating this to an action on labeled Dyck paths, \(S_n\) acts on a labeled path by permuting labels and then reordering labels in each column so that labels still increase reading up each column. From this description, we see that the orbits of this action are in bijection with Dyck paths, hence are counted by the Catalan number \(\text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}\). Suppose \(\lambda \vdash n\) and a given Dyck path has \(m_i = m_i(\lambda)\) vertical runs of length \(i\) (for \(1 \leq i \leq n\)). Then the orbit corresponding to this Dyck path has stabilizer isomorphic to the Young subgroup \(S^\lambda \leq S_n\), where

\[
S^\lambda \cong S_1^{m_1} \times S_2^{m_2} \times \cdots \times S_n^{m_n}.
\]

The number of Dyck paths with this vertical run structure is known to equal

\[
\frac{n!}{m_0!m_1!m_2!\cdots m_n!},
\]

where we define \(m_0\) so that \(\sum_{i=0}^n m_i = n + 1\); more specifically, \(m_0 = n + 1 - \ell(\lambda)\). (Equation 4 follows, for instance, from a more general result proved in Proposition 2 below.)

Next, recall that the Frobenius characteristic map is an isomorphism from the \(\mathbb{C}\)-algebra of symmetric group representations to the \(\mathbb{C}\)-algebra \(\text{Sym}\). This map sends a class function \(\chi : S_n \to \mathbb{C}\) to the symmetric function \(\sum_{\mu \vdash n} \chi(\mu) \frac{p_\mu(x)}{z_\mu}\), and the map sends the irreducible character \(\chi^\lambda\) to the Schur function \(s_\lambda(x)\) [32, §4.7]. Let \(\text{Frob}_{\text{PF}_n}\) denote the image of the character of \(\text{PF}_n\) under the Frobenius characteristic map.

Each orbit of \(\text{PF}_n\) with stabilizer isomorphic to \(S^\lambda\) contributes a term \(h_\lambda(x)\) to \(\text{Frob}_{\text{PF}_n}\) [32, p. 113–114]. The number of such orbits is given by (4), and hence

\[
\text{Frob}_{\text{PF}_n} = \sum_{\lambda \vdash n} \frac{1}{n+1} \binom{n+1}{n-\ell(\lambda)+1, m_1(\lambda), \ldots, m_n(\lambda)} h_\lambda(x)
\]

where we use the multinomial theorem, the generating function (1), and the fact that \(h_0(x) = 1\), we obtain the identity

\[
\text{Frob}_{\text{PF}_n} = \frac{1}{n+1} \left[ H(t) \right]^{n+1} \bigg|_{t^n}.
\]
24. Character of $\text{PF}_n$. To extract information from (7), we recall the Cauchy product

\begin{equation}
\Pi(x, y) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{(1 - x_i y_j)}.
\end{equation}

The following well-known lemma shows how the Cauchy product can be used to detect dual bases of $\text{Sym}$.

**Lemma 1.** Define the Hall inner product $\langle \cdot, \cdot \rangle$ on $\text{Sym}$ by requiring that $\{h_\lambda : \lambda \in \text{Par}\}$ and $\{m_\lambda : \lambda \in \text{Par}\}$ be dual bases. For any two bases $\{v_\lambda : \lambda \in \text{Par}\}$ and $\{w_\lambda : \lambda \in \text{Par}\}$ of $\text{Sym}$ such that all $v_\lambda$ and $w_\lambda$ are homogeneous of degree $|\lambda|$, we have $\Pi(x, y) = \sum_{\lambda \in \text{Par}} v_\lambda(x)w_\lambda(y)$ if and only if these two bases are dual with respect to the Hall inner product. In particular,

\begin{equation}
\Pi(x, y) = \sum_{\lambda \in \text{Par}} h_\lambda(x)m_\lambda(y) = \sum_{\lambda \in \text{Par}} p_\lambda(x)p_\lambda(y) = \sum_{\lambda \in \text{Par}} s_\lambda(x)s_\lambda(y).
\end{equation}

It follows from (7) and (8) that by setting $n + 1$ of the $y$ variables equal to $t$ and the rest equal to zero, $\Pi(x, y)$ specializes to $[H(t)]^{n+1}$. Furthermore, one sees directly that $p_\lambda(y)$ specializes to $(n + 1)\ell(\lambda)\ell(\lambda)$. These facts, combined with (7) and (9), yield

\begin{equation}
\text{Frob}_{\text{PF}_n} = \frac{1}{n + 1}[H(t)]^{n+1} \bigg|_{t^n} = \frac{1}{n + 1} \sum_{\lambda \in \text{Par}} (n + 1)^\ell(\lambda)\ell(\lambda)\frac{p_\lambda(x)}{z_\lambda} \bigg|_{t^n}.
\end{equation}

Hence

\begin{equation}
\text{Frob}_{\text{PF}_n} = \sum_{\lambda \in \text{Par}} (n + 1)^\ell(\lambda)\ell(\lambda)\frac{p_\lambda(x)}{z_\lambda}.
\end{equation}

Applying the inverse of the Frobenius map, this formula tells us the character $\chi$ of the $S_n$-module $\text{PF}_n$. On one hand, for $w \in S_n$, $\chi(w)$ is the number of parking functions in $\text{PF}_n$ fixed by the action of $w$. On the other hand, if $w$ has cycle type $\lambda$ (so that $w$ is a product of $\ell(\lambda)$ disjoint cycles), the preceding formula shows that $\chi(w) = (n + 1)^\ell(\lambda)\ell(\lambda)$. In particular, taking $w$ to be the identity permutation in $S_n$, which has cycle type $\lambda = (1^n)$ and fixes all parking functions in $\text{PF}_n$, we see that $|\text{PF}_n| = (n + 1)^{n-1}$. (There are easier ways to obtain this result, but we wanted to illustrate the power of the generating function method.)

25. Schur Expansion of $\text{PF}_n$. The derivation of the Frobenius series of $\text{PF}_n$ can be redone using the Schur symmetric functions instead of the power sum basis. As above, we specialize $y$ to consist of zeros along with $n + 1$ copies of $t$. Since $s_\lambda(y)$ is homogeneous of degree $|\lambda|$, this specialization changes $s_\lambda(y)$ into $\ell(\lambda)s_\lambda(1^{n+1})$, where $s_\lambda(1^{n+1})$ indicates that $n + 1$ variables have been set to 1 and the rest to zero. By (9),

\begin{equation}
\text{Frob}_{\text{PF}_n} = \sum_{\lambda \in \text{Par}} \frac{s_\lambda(1^{n+1})}{n + 1}s_\lambda(x).
\end{equation}

This gives the decomposition of the parking function module into irreducible constituents. In the special case of hook shapes $\lambda = (k, 1^{n-k})$ we can compute the coefficients explicitly (see [19], p. 364):

\begin{equation}
\frac{s_{(k, 1^{n-k})}(1^{n+1})}{n + 1} = \frac{1}{n + 1}\frac{(n - 1)(n + k)}{\binom{n}{k}}.
\end{equation}

These integers are called the Schröder numbers, and they also describe the $f$-vector of the associahedron (see Pak-Postnikov [39] formula (1–6)). Setting $k = n$, we verify that the multiplicity
of the trivial representation in $\text{PF}_n$ is the classical Catalan number:

$$\frac{s(n)(1^{n+1})}{n+1} = \text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}.$$ 

These computations were originally done by Haiman [21], Pak-Postnikov [36], and Stanley [40].

2.6. The Graded Version of $\text{PF}_n$. As mentioned earlier, it is known that the parking function module $\text{PF}_n$ and the diagonal coinvariant ring $\text{DR}_n$ are isomorphic as (ungraded) $S_n$-modules. However, $\text{DR}_n$ also comes with a symmetric bi-grading by $x$-degree and $y$-degree, and it is natural to ask whether we can explain this bi-grading in terms of $\text{PF}_n$. This is the content of the famous Shuffle Conjecture, described in §5 below. This section studies the simpler case where we grade $\text{DR}_n$ by the $x$-degree only.

There is a straightforward combinatorial construction that turns the module $\text{PF}_n$ into a singly-graded vector space. The grading is defined on basis vectors by the statistic $\text{area}$:

$\text{PF}_n \rightarrow \mathbb{N}$, which is the number of boxes fully contained between the labeled Dyck path and the diagonal $y = x$. For example, the following figure shows that $\text{area}(2, 4, 1, 2, 1) = 5$:

Since permuting the labels of a labeled Dyck path leaves the area unchanged, we have given $\text{PF}_n$ the structure of a graded $S_n$-module. Haiman conjectured and eventually proved [11, 21, 22] that this module is isomorphic to $\epsilon \otimes \text{DR}_n$ graded by $x$-degree only, where $\epsilon$ denotes the sign character.

3. Rational Parking Functions

In this section we will generalize the $S_n$-module of classical parking functions $\text{PF}_n$, labeled by a single positive integer $n \in \mathbb{N}$ to an $S_a$-module of rational parking functions, denoted $\text{PF}_{a,b}$, labeled by two coprime positive integers $a$ and $b$. The classical parking functions will correspond to the case $(a, b) = (n, n+1)$. To define $\text{PF}_{a,b}$ we must first discuss rational Dyck paths.

3.1. Rational Dyck Paths. Given positive integers $a, b \in \mathbb{N}$, let $\mathcal{R}(N^aE^b)$ denote the subset of $\{N,E\}^{a+b}$ consisting of words containing $a$ copies of $N$ and $b$ copies of $E$. We can think of such a word as a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(b,a)$ by reading from left to right, interpreting $N$ as “go north” and $E$ as “go east.” An element of $\mathcal{R}(N^aE^b)$ is called an $(a,b)$-Dyck path if it stays weakly above the diagonal line $y = \frac{a}{b}x$. Let $\mathcal{D}(N^aE^b)$ denote the set of $(a,b)$-Dyck paths. An example of a $(5,8)$-Dyck path is shown here:
Recall that the \((n,n)\)-Dyck paths are called “classical” and they are counted by the Catalan number \(\text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}\). Every \((n, n+1)\)-Dyck path must end with an east step, and by removing this east step we obtain a canonical bijection

\[
\mathcal{D}(N^n E^{n+1}) \rightarrow \mathcal{D}(N^n E^n).
\]

In the case that \(a, b\) are coprime, the enumerative theory of Dyck paths is quite nice. In particular, it was known as early as 1954 (see Bizley [5]) that when \(a, b\) are coprime we have

\[
|\mathcal{D}(N^n E^b)| = \frac{1}{a+b} \binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}.
\]

We call this number the **rational Catalan number**, denoted \(\text{Cat}_{a,b}\). When \(a, b\) are coprime, we call \(\mathcal{D}(N^n E^b)\) the set of **rational Dyck paths** in \(\mathcal{R}(N^n E^b)\). Observe that \(\text{Cat}_{n,n+1} = \text{Cat}_n\).

The main result of this section is a formula enumerating rational Dyck paths with a specified vertical run structure. In this setting, a “vertical run of length \(i \geq 0\)” is a subword \(N^i E\) such that this subword is either preceded by \(E\) or is the beginning of the entire word. The classical (i.e., non-rational) version of this formula goes back at least to Kreweras’ work in 1972 [26] Theorem 4 in the context of noncrossing partitions.

**Proposition 2.** Let \(a\) and \(b\) be coprime positive integers. Let \(m_0, m_1, m_2, \ldots, m_a\) be nonnegative integers such that \(\sum_{i \geq 0} im_i = a\) and \(\sum_{i \geq 0} m_i = b\). Then the number of \((a,b)\)-Dyck paths in \(\mathcal{D}(N^n E^b)\) with \(m_i\) vertical runs of length \(i\) is given by

\[
\frac{1}{b} \binom{\sum_{i \geq 0} m_i}{m_0, m_1, \ldots, m_a} = \frac{(b-1)!}{m_0!m_1! \cdots m_a!}.
\]

**Proof.** *Step 1.* Let \(Y\) be the set of paths \(\pi \in \mathcal{R}(N^n E^b)\) such that \(\pi\) has \(m_i\) vertical runs of length \(i\) for all \(i \geq 0\), and \(\pi\) ends in an east step. We show

\[
|Y| = \binom{\sum_{i \geq 0} m_i}{m_0, m_1, \ldots, m_a}.
\]

Let \(X\) denote the set \(\mathcal{R}(v_0^{m_0} v_1^{m_1} \cdots v_a^{m_a})\) of words containing \(m_i\) copies of \(v_i\) for each \(i\). Define \(f : X \rightarrow \{N, E\}^*\) by replacing each letter \(v_i\) in a word by \(N^i E\). Since \(\sum_i im_i = a\) and \(\sum m_i = b\), we see that \(f\) maps \(X\) into \(\mathcal{R}(N^n E^b)\) and that \(f(w)\) ends in an east step for all \(w \in X\). It is now clear that \(f\) is a bijection of \(X\) onto \(Y\). Since \(|X| = (m_0, m_1, \ldots, m_a)\), Step 1 is complete.

*Step 2.* Let \(\pi \in Y\). We associate a **level** \(l_i\) to the \(i\)-th lattice point on a path \(\pi\) as follows. Set \(l_0 = 0\). For each \(i > 0\), set

\[
l_i = \begin{cases} 
    l_{i-1} + b, & \text{if the } i\text{-th step of } \pi \text{ is a north step}; \\
    l_{i-1} - a, & \text{if the } i\text{-th step of } \pi \text{ is an east step}.
\end{cases}
\]

Note that the level of a lattice point \((x,y)\) is \(by - ax\). We show that the levels \(l_1, l_2, \ldots, l_{a+b}\) for a given \(\pi \in Y\) are all **distinct**. For suppose \(l_i = l_j\) for some \(i < j\); we prove that \(i\) must be 0 and \(j\) must be \(a + b\). Let \((x_i, y_i)\) and \((x_j, y_j)\) be the lattice points reached by the \(i\)-th and \(j\)-th steps of \(\pi\). We know \(0 \leq x_i \leq x_j \leq 0 \leq y_i \leq y_j \leq a\); also, \(y_i = y_j - ax_i \) and \(l_j = y_j - ax_j\). Since \(l_i = l_j\), \(b(y_j - y_i) = a(x_j - x_i)\). But \(a\) and \(b\) are coprime, so their least common multiple is \(ab\). This forces \(a\) to divide \(y_j - y_i\) and \(b\) to divide \(x_j - x_i\). So we must have \(x_i = y_i = 0\), \(x_j = b\), and \(y_j = a\), giving \(i = 0\) and \(j = a + b\) as claimed.

*Step 3.* Define an equivalence relation \(\sim\) on \(Y\) by letting \(\pi_1 \sim \pi_2\) iff there exist \(w_1, w_2 \in X\) such that \(f(w_1) = \pi_1\), \(f(w_2) = \pi_2\), and \(w_2\) is a cyclic shift of \(w_1\). We show that every equivalence class has size \(b\) and contains exactly one \((a,b)\)-Dyck path. Fix \(\pi \in Y\) and write \(\pi = f(w)\) for some \(w \in X\). By cyclically shifting \(w\) by 0, 1, 2, \ldots, \(b - 1\), we obtain a list of \(b\) paths \(\pi = \pi_0, \pi_1, \pi_2, \ldots, \pi_{b-1}\) (not yet known to be distinct), which are the paths in \(Y\) equivalent to \(\pi\). Suppose the list of steps in
\pi has east steps at positions \(i_1 < i_2 < \cdots < i_b = a + b\). Define \(i_0 = 0\). Cyclically shifting \(w\) by \(k\) steps, where \(0 \leq k < b\), has the effect of cyclically shifting \(\pi\) by \(i_k\) steps. This cyclic shift will replace the sequence of levels \(l_0, l_1, \ldots\) for \(\pi\) by the new sequence
\[
(10) \quad l_{i_k} - l_{i_k}, l_{i_k+1} - l_{i_k}, \ldots, l_{a+b} - l_{i_k}, l_1 - l_{i_k}, l_2 - l_{i_k}, \ldots, l_{i_k-1} - l_{i_k}.
\]
If \(m_0\) is the minimum level in \(\pi\), it follows that \(m_0 - l_{i_k}\) is the minimum level in \(\pi_k\). Since \(l_{i_1}, \ldots, l_{i_b}\) are distinct by Step 2, it follows that the \(b\) paths \(\pi_0, \ldots, \pi_{b-1}\) all have distinct minimum levels, hence these \(b\) paths must be pairwise distinct. Furthermore, the minimum level \(m_0\) in \(\pi\) must occur at the end of an east step, so \(m_0 = l_{i_j}\) for some \(j\). For every \(k\), the minimum level in \((10)\) is \(l_{i_j} - l_{i_k}\), which is nonnegative iff \(l_{i_j} \geq l_{i_k}\). But since \(l_{i_j}\) is the minimum level in \(\pi\) and all levels are distinct, the minimum level in \(\pi_k\) is nonnegative iff \(k = j\). This means that exactly one of the paths in the equivalence class of \(\pi\) is an \((a, b)\)-Dyck path.

**Step 4.** Step 3 shows that \(Y\) decomposes into a disjoint union of \(b\)-element subsets, each of which meets \(D(N^aE^b)\) in exactly one point. So the cardinality \(|D(N^aE^b)|\) can be found by dividing the multinomial coefficient in Step 1 by \(b\).

### 3.2. Rational Parking Functions

An \((a, b)\)-parking function is an \((a, b)\)-Dyck path together with a labeling of the north steps by the set \(\{1, 2, \ldots, a\}\) such that labels increase in each column going north. For example, here is a \((5, 8)\)-parking function.

![Diagram of a 5,8-parking function]

The symmetric group \(S_a\) acts on \((a, b)\)-parking functions by permuting labels and then reordering the labels within columns if necessary. Let \(PF_{a,b}\) denote the set, and also the \(S_a\)-module, of \((a, b)\)-parking functions. Consistent with our terminology for \((a, b)\)-Dyck paths, we call \(PF_{a,b}\) the set of rational parking functions in the case where \(a\) and \(b\) are coprime. We make this assumption of coprimality for the rest of this section. In this case, we can compute the Frobenius characteristic of \(PF_{a,b}\) by the same method used in §2.3 to compute the Frobenius characteristic of \(PF_n\) (which is essentially \(PF_{n,n+1}\)).

**Theorem 3.** The Frobenius characteristic of the \(S_a\)-module of parking functions \(PF_{a,b}\) has the following expansions in terms of complete homogeneous, power sum, and Schur symmetric functions, respectively:
\[
\text{Frob}(PF_{a,b}) = \sum_{\lambda \vdash b} \frac{1}{b} \left( b - \ell(\lambda), m_1(\lambda), m_2(\lambda), \ldots, m_a(\lambda) \right) h_\lambda(x) = \sum_{\lambda \vdash a} b^{\ell(\lambda)-1} \frac{p_\lambda(x)}{z_\lambda} = \sum_{\lambda \vdash a} \frac{s_\lambda(1^b)}{b} s_\lambda(x).
\]

**Proof.** The \(h\)-expansion follows from Proposition \(2\) in the same way that \((5)\) followed from \((4)\). (Note that the hypothesis that \(a\) and \(b\) are coprime is necessary here.) By the multinomial theorem, \(\text{Frob}_{PF_{a,b}}\) is the coefficient of \(t^a\) in the generating function \(\frac{1}{b} [H(t)]^b\). The power sum and Schur expansions then follow from \((8)\) and \((9)\), by the same arguments used in §2.4 and §2.5. \(\square\)
Recall that the $p$-expansion tells us the character of the permutation action of $S_a$ on $PF_{a,b}$. We can express this as follows.

**Corollary 4.** Let $w \in S_a$ be a permutation with $k$ cycles. Then the number of elements of $PF_{a,b}$ fixed by $w$ equals $b^{k-1}$. In particular, taking $w$ to be the identity permutation in $S_a$ (which has $a$ cycles), $|PF_{a,b}| = b^{a-1}$.

Using [39, page 364], we obtain the following formula for the coefficient of $s_\lambda$ when $\lambda$ is a hook shape.

**Corollary 5.** For $0 \leq k \leq a - 1$, the multiplicity of the hook Schur function $s_{(k+1,1^{a-k-1})}(x)$ in $\text{Frob}_{PF_{a,b}}$ is

$$\frac{s_{(k+1,1^{a-k-1})}(1^b)}{b} = \frac{1}{b} \binom{a-1}{k} \binom{b+k}{a}.$$ 

We call these integers the rational Schröder numbers, denoted $\text{Sch}_{a,b,k}$. We note that these numbers also occur as the $f$-vector of the recently discovered rational associahedron [3].

We already knew a special case of this result. Namely, when $k = a - 1$, we find that the multiplicity of the trivial character in $PF_{a,b}$ is the rational Catalan number

$$\text{Cat}_{a,b} = \frac{1}{b} \binom{b+a-1}{a,b-1} = \frac{(a+b-1)!}{a!b!}.$$ 

But since $PF_{a,b}$ is a permutation module, the multiplicity of the trivial character is the number of orbits. Since the orbits are represented by $(a,b)$-Dyck paths, we recover Bizley’s result regarding the number of $(a,b)$-Dyck paths [5]. This result can also be proved by an argument similar to the one in Proposition 2 [see 29, §12.1].

Observe from Corollary 5 that the sign character of $S_a$ occurs in $PF_{a,b}$ if and only if $b \geq a$. More precisely, we see that the smallest value of $k$ for which $s_{(k+1,1^{a-k-1})}(x)$ occurs in $\text{Frob}_{PF_{a,b}}$ is $k = \max\{0, a-b\}$.

### 3.3. Graded Version of $PF_{a,b}$

It is straightforward to generalize the area statistic on classical Dyck paths to rational Dyck paths. For coprime $a, b \in \mathbb{N}$ and $D \in \mathcal{D}(N^aE^b)$, $\text{area}(D)$ is the number of boxes fully contained between $D$ and the diagonal line $y = \frac{a}{b}x$. For example, the rational Dyck path displayed at the beginning of §3.2 has area 5. The action of $S_a$ on $PF_{a,b}$ preserves the area statistic, so this statistic turns $PF_{a,b}$ into a singly-graded $S_a$-module, as in the classical case.

When $a$ and $b$ are coprime, the maximum value of $\text{area}(D)$ over all $(a,b)$-Dyck paths is $(a - 1)(b - 1)/2$. Indeed, the diagonal of the $a \times b$ rectangle intersects a ribbon of $a + b - 1$ boxes, as shown here:

![Rational Dyck Path](image)

Note that this ribbon divides the rest of the rectangle into two equal pieces of size

$$\frac{ab - (a + b - 1)}{2} = \frac{(a - 1)(b - 1)}{2}.$$
4. Rational q-Catalan Numbers

This section studies a q-analogue of the rational Catalan number $\text{Cat}_{a,b}$ obtained by replacing $\frac{1}{a+b} (a+b)$ by $\frac{1}{[a+b]} q^{[a+b]}$. We conjecture combinatorial interpretations for these polynomials, as well as a new combinatorial formula for all q-binomial coefficients, based on certain partition statistics. The main theorem of this section shows that the conjecture for q-binomial coefficients implies the conjecture for rational q-Catalan numbers.

4.1. Background on q-Binomial Coefficients. The q-binomial coefficients defined by $[\frac{a}{b}]$ are in fact polynomials in $\mathbb{N}[q]$. One can prove this algebraically by checking that the q-binomial coefficients satisfy the recursions

$$ [a+b]_{a,b} = q^a [a+b-1]_{a,b-1} + [a+b-1]_{a-1,b} = [a+b-1]_{a,b-1} + q^b [a+b-1]_{a-1,b} \quad (a,b > 0) \quad (11) $$

and initial conditions $[\frac{a}{a,0}] = [\frac{b}{0,b}] = 1$.

Recall the following well-known combinatorial interpretation of q-binomial coefficients. Let $R(a,b)$ denote the set of partitions whose diagrams fit in the box with corners $(0,0)$, $(b,0)$, $(b,a)$, and $(0,a)$. More precisely, $\mu \in R(a,b)$ iff $\mu = (\mu_1, \mu_2, \ldots, \mu_a)$ where the parts $\mu_j$ are integers satisfying $b \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_a \geq 0$. Note that $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_a$ is the number of boxes in the diagram of $\mu$. One may verify that

$$ [a+b]_{a,b} = \sum_{\mu \in R(a,b)} q^{w(\mu)} \quad (12) $$

by showing that the right side of $(12)$ satisfies the recursions in $(11)$ (see, e.g., [29, §6.7]). We refer to $(12)$ as the standard combinatorial interpretation for the q-binomial coefficients, to distinguish it from the non-standard interpretation presented below.

4.2. Rational q-Catalan Numbers. Suppose $a$ and $b$ are positive integers with gcd$(a,b) = 1$. The rational q-Catalan number $\text{Cat}_{a,b}(q)$ for the slope $a/b$ is

$$ \text{Cat}_{a,b}(q) = \frac{1}{[a+b]q} [a+b]_{a,b} = \frac{[a+b-1]!}{[a]![b]!} q^{[a+b-1]} [a,b-1]_{a,b-1} = \frac{1}{[a]!} \frac{[a+b-1]}{[a-1,b]} \quad (13) $$

It is known that $\text{Cat}_{a,b}(q)$ is a polynomial with coefficients in $\mathbb{N}$. Haiman gives an algebraic proof of this fact in [21, Prop. 2.5.2] and also gives an algebraic interpretation for these polynomials as the Hilbert series of a suitable quotient ring of a polynomial ring [21, Prop. 2.5.3 and 2.5.4]. This polynomial also appears to be connected to certain modules arising in the theory of rational Cherednik algebras. Our purpose here is to propose a combinatorial interpretation for the rational q-Catalan numbers. More precisely, given $a,b > 0$ with gcd$(a,b) = 1$, our problem is to find a set $X(a,b)$ of combinatorial objects and a statistic wt : $X(a,b) \rightarrow \mathbb{N}$ such that

$$ \text{Cat}_{a,b}(q) = \sum_{w \in X(a,b)} q^{\text{wt}(w)}. $$

For certain special choices of $a$ and $b$, this problem has been solved. For instance, when $a = n$ and $b = n + 1$, we may take $X(a,b)$ to be the set $DW_n$ of Dyck words consisting of $n$ zeroes and $n$ ones, such that every prefix of the word has at least as many zeroes as ones. (Dyck words are obtained from classical Dyck paths by replacing each N by 0 and each E by 1.) Given a word $w = w_1w_2 \cdots w_{2n} \in DW_n$, define the major index of $w$, denoted maj$(w)$, to be the sum of all
i < 2n with \( w_i > w_{i+1}. \) MacMahon \cite{macMahon} proved that
\[
\sum_{w \in DW_n} q^{\text{maj}(w)} = \frac{1}{[n+1]_q} \left[ \frac{2n}{n,n} \right]_q = \text{Cat}_{n,n+1}(q).
\]

Another known case is \( a = n \) and \( b = mn + 1, \) where \( m,n \) are fixed positive integers. In this case, we may take \( X(a,b) \) to be the set of lattice paths from \((0,0)\) to \((mn,n)\) that never go below the line \( x = my. \) The statistic needed here is obtained by taking the \( t = 1/q \) specialization of the higher-order \( q,t \)-Catalan numbers \cite{loehr2009}. Using the area and bounce statistics defined in \cite{comtet1974}, it is shown in \cite{comtet1974} §3.3 that
\[
\sum_{w \in X(n,mn+1)} q^{|mn(n-1)/2 + \text{area}(w) - \text{bounce}(w)|} = \frac{1}{[mn+1]_q} \left[ \frac{mn+n}{mn,n} \right]_q = \text{Cat}_{n,mn+1}(q).
\]

One can replace the power of \( q \) here by \( mn(n-1)/2 + h(w) - \text{area}(w), \) where \( h \) (also called \( \text{dinv}_m \)) is defined in \cite{comtet1974}.

### 4.3. Conjectured Combinatorial Formula.

We can specialize the rational \( q,t \)-Catalan numbers described in \cite{comtet1974} §7 to give a conjectured combinatorial interpretation for \( \text{Cat}_{a,b}(q) \) for any \( a,b \) with \( \gcd(a,b) = 1. \) We also introduce a related, non-standard combinatorial interpretation for the \( q \)-binomial coefficient \( \left[ \frac{a+b}{a,b} \right]_q \) that does not require the hypothesis \( \gcd(b,a) = 1. \)

To state these conjectures, fix \( a,b \in \mathbb{N}. \) Let \( D(a,b) \) be the set of integer partitions whose diagrams fit in the triangle with vertices \((0,0),\) \((0,a),\) and \((b,a).\) More precisely, \( \mu \in D(b,a) \) iff \( \mu = (\mu_1, \mu_2, \ldots, \mu_a) \) where \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_a \geq 0 \) and \( \mu_j \leq b(a-j)/a \) for all \( j. \) For example, the figure below shows that \( \mu = (6, 3, 2, 0, 0) \in D(5,8). \)

![Triangular Diagram](image)

For each cell \( c \) in the diagram of \( \mu, \) let \( \text{arm}(c) \) be the number of cells in the same row to the right of \( c, \) and let \( \text{leg}(c) \) be the number of cells in the same column below \( c. \) Let \( h_{b,a}^+(\mu) \) be the number of cells \( c \) in the diagram of \( \mu \) such that \( -a < a \cdot \text{arm}(c) - b \cdot \text{leg}(c) \leq b. \) Let \( h_{b,a}^-(\mu) \) be the number of cells \( c \) in the diagram of \( \mu \) such that \( -a \leq a \cdot \text{arm}(c) - b \cdot \text{leg}(c) < b. \) As above, \( \lfloor \mu \rfloor \) is the total number of cells in the diagram of \( \mu. \) For the example partition shown above, we compute \( \lfloor \mu \rfloor = 11 \) and \( h_{8,5}^+(\mu) = h_{8,5}^-\mu) = 9. \) (In \cite{comtet1974}, \( h_{b,a}^+ \) and \( h_{b,a}^- \) are denoted \( h_{b/a}^+ \) and \( h_{b/a}^- \), respectively.)

**Conjecture 6.** For all \( a,b \in \mathbb{N} \) with \( \gcd(a,b) = 1, \)
\[
\text{Cat}_{a,b}(q) = \sum_{\mu \in D(a,b)} q^{\lfloor \mu \rfloor + h_{b,a}^+(\mu)}.
\]

**Remark 7.** We can use either \( h_{b,a}^+ \) or \( h_{b,a}^- \) in this conjecture since the condition \( \gcd(a,b) = 1 \) ensures that the “critical” values \( -a \) and \( b \) for \( a \cdot \text{arm}(c) - b \cdot \text{leg}(c) \) cannot actually occur. The table below illustrates how each partition in \( \text{D}(3,5) \) contributes to \( \text{Cat}_{3,5}(q,t) = 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8. \)

Next we describe our non-standard combinatorial interpretation for \( q \)-binomial coefficients. Given \( \mu \in R(a,b), \) the **frontier** of \( \mu \) is the lattice path from \((0,0)\) to \((b,a)\) obtained by taking north and east steps along the boundary of the diagram of \( \mu. \) For fixed \( a,b \in \mathbb{N}, \) assign the **level** \( by - ax \) to the lattice point \((x,y) \in \mathbb{N}^2. \) (These are the same levels for lattice points used in the proof...
Let \( h_{b,a}(\mu) \) be the minimum level appearing on the frontier of \( \mu \). Note that \( ml_{b,a}(\mu) \leq 0 \), with equality iff \( \mu \in D(a,b) \).

**Conjecture 8.** For all \( a, b \in \mathbb{N} \),

\[
[a + b]_{q} = \sum_{\mu \in R(a,b)} q^{\mu + ml_{b,a}(\mu) + h_{b,a}^{+}(\mu)} = \sum_{\mu \in R(a,b)} q^{\mu + ml_{b,a}(\mu) + h_{b,a}^{-}(\mu)}.
\]

**Example 9.** We compute the middle expression in Conjecture 8 for \( a = 2, b = 3 \). The corresponding polynomial in \( q \) is

\[
\frac{[2 + 3]}{[2,3]} = \frac{[5]}{[2]} = 1 + q(1 - q) + q^{2} = 1 + q + 2q^{2} + 2q^{3} + 2q^{4} + q^{5} + q^{6}.
\]

We have confirmed Conjectures 6 and 8 for \( a, b \leq 12 \) using a Sage worksheet. The rest of this section is devoted to a bijective proof of the following theorem.

**Theorem 10.** Conjecture 6 holds for given coprime \( a, b \in \mathbb{N} \) if and only if Conjecture 8 holds for this choice of \( a, b \).

### 4.4. Alternate Description of \( h^{+} \) and \( h^{-} \)

We begin with a lemma showing how to compute \( h_{b,a}^{+}(\mu) \) and \( h_{b,a}^{-}(\mu) \) from the levels on the frontier of \( \mu \).

**Lemma 11.** Given \( a, b \in \mathbb{N} \) and \( \mu \in R(a,b) \), let the frontier of \( \mu \) be \( w_{1}w_{2} \cdots w_{a+b} \in \{N,E\}^{a+b} \), and let the levels on the frontier of \( \mu \) be \( l_{0} = 0, l_{1}, l_{2}, \ldots, l_{a+b} = 0 \).

(a) \( h_{b,a}^{+}(\mu) \) is the number of pairs \( i < j \) with \( w_{i} = E, w_{j} = N \) and \( 1 \leq l_{i} - l_{j-1} \leq a + b \).
(b) $h_{b,a}(\mu)$ is the number of pairs $i < j$ with $w_i = E$, $w_j = N$, and $1 \leq l_j - l_i \leq a + b$.

Proof. (a) First note that there is a bijection between the set of cells $c$ in the diagram of $\mu$ and the set of pairs $i < j$ with $w_i = E$ and $w_j = N$. This bijection maps $c$ to the unique pair $(i, j)$ such that $w_i$ is the east step on the frontier due south of $c$, and $w_j$ is the north step on the frontier due east of $c$. We can use the coordinates of the starting vertices of these steps to determine whether $c$ contributes to $h_{b,a}(\mu)$: Suppose $w_i$ goes from $(u, v)$ to $(u + 1, v)$. By definition of $\text{arm}(c)$ and $\text{leg}(c)$, we see that $w_j$ must go from $(u + \text{arm}(c) + 1, v + \text{leg}(c))$ to $(u + \text{arm}(c) + 1, v + \text{leg}(c) + 1)$. Using these coordinates, we have $l_{i-1} = bv - au$ and $l_{j-1} = b(v + \text{leg}(c)) - a(u + \text{arm}(c) + 1)$, so $l_{i-1} - l_{j-1} = a + a \cdot \text{arm}(c) - b \cdot \text{leg}(c)$. This cell $c$ contributes to $h_{b,a}(\mu)$ iff $-a < a \cdot \text{arm}(c) - b \cdot \text{leg}(c) \leq b$ iff $0 < l_{i-1} - l_{j-1} \leq a + b$.

(b) Keep the notation from (a). We find that $l_i = bv - a(u + 1)$ and $l_j = b(v + \text{leg}(c) + 1) - a(u + \text{arm}(c) + 1)$, so $l_j - l_i = b \cdot \text{leg}(c) - a \cdot \text{arm}(c) + b$. The cell $c$ contributes to $h_{b,a}(\mu)$ iff $-a \leq a \cdot \text{arm}(c) - b \cdot \text{leg}(c) < b$ iff $-a - b \leq a \cdot \text{arm}(c) - b \cdot \text{leg}(c) - b < 0$ iff $0 < l_j - l_i \leq a + b$. □

Henceforth, we discuss only the statistic $h_{b,a}^+$; the arguments needed for $h_{b,a}^-$ are analogous.

4.5. Cyclic Shift Analysis. Define a cyclic-shift map $C$ on words by letting $C(w_1w_2 \cdots w_{a+b}) = w_2 \cdots w_{a+b}w_1$. We get an associated bijection $C$ on $R(a, b)$ as follows. Given $\mu \in R(a, b)$, let $w$ be the frontier of $\mu$; then let $C(\mu)$ be the unique partition in $R(a, b)$ with frontier $C(w)$. The next lemma reveals how the cyclic-shift map affects $h_{b,a}^+$.

Lemma 12. Given $a, b \in \mathbb{N}$ and $\mu \in R(a, b)$, let $w$ be the frontier of $\mu$ be $w_1w_2 \cdots w_{a+b} \in \{N, E\}^{a+b}$, with corresponding vertex levels $l_0, \ldots, l_{a+b}$. Let $\Delta h^+ = h_{b,a}^+(C(\mu)) - h_{b,a}^+(\mu)$.

(a) If $w_1 = N$, then

$$\Delta h^+ = |\{i > 0 : w_i = E \text{ and } 1 \leq l_{i-1} \leq a + b\}| = |\{k > 0 : 1 \leq l_{k-1} \leq b\}|.$$

(b) If $w_1 = E$, then

$$\Delta h^+ = |\{j > 0 : w_j = N \text{ and } 1 \leq -l_{j-1} \leq a + b\}| = |\{k > 0 : 1 \leq -l_{k-1} \leq a\}|.$$

Proof. (a) Assume $w_1 = N$. The frontier of $C(\mu)$ is $w_1'w_2' \cdots w_{a+b}'$, where $w_i' = w_{i+1}$ for $1 \leq i < a + b$. The levels of the vertices on the frontier of $C(\mu)$ are $l_0', l_1', \ldots, l_{a+b}'$, where $l_{a+b}' = 0$ and $l_i' = l_{i+1} - b$ for $0 \leq i < a + b$. Let us use Lemma 11 to compute $h_{b,a}^+(C(\mu))$. Note that $(i, j)$ satisfies $0 < i < j < a + b$, $w_i' = E$, $w_j' = N$, and $1 \leq l_{i+1}' - l_{j+1}' \leq a + b$ iff $(i + 1, j + 1)$ satisfies $1 < i + 1 < j + 1 \leq a + b$, $w_{i+1} = E$, $w_{j+1} = N$, and $1 \leq l_i - l_j \leq a + b$. So the contributions to $h_{b,a}^+$ from these pairs will cancel in the computation of $\Delta h^+$. We still need to consider pairs $(i, j)$ contributing to $h_{b,a}^+(C(\mu))$ where $j = a + b$, and pairs $(i, j)$ contributing to $h_{b,a}^+(\mu)$ where $i = 1$. But, since $w_1 = N$, no pair of the second type causes a contribution to $h_{b,a}^+(\mu)$. Since $w_{a+b}' = N$ and $l_{a+b}' = -b$, a pair $(i, a + b)$ contributes to $h_{b,a}^+(C(\mu))$ iff $w_i' = E$ and $1 \leq l_{i+1}' - b \leq a + b$ iff $w_{i+1} = E$ and $1 \leq l_i \leq a + b$. Replacing $i$ by $i - 1$ gives the first formula in (a).

To prove the second formula in (a), let $A$ be the set of lattice points on the frontier of $\mu$ with levels in $[1, a + b]$ and followed by an east step, and let $B$ be the set of lattice points on the frontier of $\mu$ with levels in $[1, b]$. It suffices to define a bijection $f : A \to B$. Given $(x, y) \in A$ with level $i \in [1, a + b]$, the next point on the frontier of $\mu$ is $(x + 1, y)$, which has level $i - a \in [1 - a, b]$. If this level is positive, then set $f(x, y) = (x + 1, y) \in B$. Otherwise, keep stepping forward along the frontier (wrapping back at the end from $(b, a)$ to $(0, 0)$ if needed) until a positive level is reached, and let $f(x, y)$ be the location of this level. Since north steps increase the level value in increments of $b$, the positive level reached must lie in $[1, b]$. Also, such a level must exist, as $(x, y)$ has a positive level. The inverse bijection is similar: given $(u, v) \in B$, scan backwards along the frontier until one
hits a point with level in \([1, a+b]\) that is reached by following an east step backwards. One sees that \(f(x, y)\) is always the first point after \((x, y) \in A\) with positive level, whereas \(g(u, v)\) is always the first point before \((u, v) \in B\) with positive level. It follows that \(f \circ g\) and \(g \circ f\) are identity maps.

(b) Keep the notation from (a). In this case, since \(w_1 = E\), the new levels \(l'_0, \ldots, l'_{a+b}\) satisfy \(l'_i = l_{i+1} + a\) for \(0 \leq i < a+b\). As in (a), a pair \((i, j)\) with \(w_i = E\) and \(w_j = N\) and \(0 < i < j < a+b\) contributes to \(h_{b,a}^+(C(\mu))\) iff the pair \((i+1, j+1)\) contributes to \(h_{b,a}^+(\mu)\). Since \(w'_{a+b} = w_1 = E\), no pairs \((i, j)\) with \(j = a+b\) contribute to \(h_{b,a}^+(C(\mu))\). On the other hand, a pair \((1, j)\) contributes to \(h_{b,a}^+(\mu)\) iff \(w_j = N\) and \(1 \leq l_0 - l_{j-1} \leq a+b\). So \(-\Delta h^+\) is the number of \(j > 0\) with \(w_j = N\) and \(1 \leq -l_{j-1} \leq a+b\).

To prove the second formula in (b), let \(A\) be the set of lattice points on the frontier of \(\mu\) with levels in \([-b,a]\) and followed by a north step, and let \(B\) be the set of lattice points on the frontier of \(\mu\) with levels in \([-a,1]\). Define a bijection \(f : A \to B\) by letting \(f(x, y)\) be the first point following \((x, y) \in A\) along the frontier that has a negative level. The inverse bijection \(g : B \to A\) sends \((u, v) \in B\) to the first point preceding \((u, v)\) along the frontier that has a negative level. Both maps “wrap around” at \((b,a)\) and \((0,0)\) when needed. One checks that \(f\) maps \(A\) into \(B\), \(g\) maps \(B\) into \(A\), and the two maps are inverses. \(\square\)

4.6. **Cyclic Shift Orbits when \(\gcd(a,b) = 1\).** We now impose the hypothesis \(\gcd(a,b) = 1\). In this case, the argument in Step 2 of the proof of Proposition\(^2\) shows that the levels \(l_0, l_1, \ldots, l_{a+b-1}\) on any lattice path from \((0,0)\) to \((b,a)\) are all distinct. Because each shifted path has a different minimum level, it follows that the \(a+b\) possible cyclic shifts of this lattice path are all distinct. Moreover, exactly one of the \(a+b\) cyclic shifts has minimum level zero, and all other cyclic shifts have a negative minimum level (see the proof of Proposition\(^2\) and \[29, \S 12.1\]).

We can rephrase these results in terms of partitions. Given \(\mu \in \mathbb{R}(a,b)\), there exists a unique partition \(\mu^0 \in D(a,b)\) that can be obtained by cyclically shifting the frontier of \(\mu\). The \(a+b\) partitions \(\mu^0, C(\mu^0), \ldots, C^{a+b-1}(\mu^0)\) are all distinct. By induction on the number of cyclic shifts needed to pass from \(\mu^0\) to \(\mu\), one may check that \(|\mu^0| = |\mu| + \text{ml}_{b,a}(\mu)\). The following lemma is the final ingredient needed for the proof of Theorem\(^10\).

**Lemma 13.** Assume \(\gcd(a,b) = 1\). Given \(\mu^0 \in D(a,b)\), let \(\mu^0, \mu^1, \ldots, \mu^{a+b-1}\) be the \(a+b\) distinct partitions obtained by cyclically shifting the frontier of \(\mu^0\). Then

\[
\sum_{i=0}^{a+b-1} q^{h_{b,a}^+(\mu^i) - h_{b,a}^+(\mu^0)} = q^{\text{ml}_{b,a}(\mu)}(a+b)\, q.
\]

More precisely, suppose the \(a+b\) (distinct) levels on the frontier of \(\mu^0\) are \(0 = i_0 < i_1 < i_2 < \cdots < i_{a+b-1}\). For \(0 \leq k < a+b\), if \(\mu^k\) is the cyclic shift of \(\mu^0\) with minimum level \(-i_k\), then \(h_{b,a}^+(\mu^k) - h_{b,a}^+(\mu^0) = k\).

**Proof.** Fix a partition \(\mu\) in the orbit of \(\mu^0\) with minimum level \(-i_k\). Let \(\nu = C(\mu)\), and consider two cases.

**Case 1:** The frontier of \(\mu\) begins with a north step. Then \(\nu\) has minimum level \(-i_k - b\). In the diagram for \(\mu\), the lattice point \((0,0)\) has level zero. This lattice point was moved to the origin by cyclically shifting some lattice point \((x, y)\) on the frontier of \(\mu^0\). Since \(\text{ml}_{b,a}(\mu) = -i_k\) and \(h_{b,a}^+(\mu^0) = 0\), it follows that \((x,y)\) has level \(i_k\). Similarly, the point \((0,0)\) in the diagram of \(\nu\) arose by cyclically shifting some point \((u,v)\) on the frontier of \(\mu^0\) with level \(i_k + b\), where \(i_k + b = i_j\) for some \(j > k\). Now, keeping in mind that all levels are distinct, \(j - k\) is precisely the number of levels on the frontier of \(\mu^0\) whose values are between \(i_k\) and \(i_k + b\), excluding \(i_k\) and including \(i_k + b\). Cyclically shifting \(\mu^0\) to \(\mu\), this means that \(j - k\) is the number of levels on the frontier of \(\mu\) with values in the interval \([1, b]\). By Lemma\(^{12}\)(a), we have \(h_{b,a}^+(\nu) - h_{b,a}^+(\mu) = j - k\).
Case 2: The frontier of $\mu$ begins with an east step. Then $\nu$ has minimum level $-i_k + a$. As in Case 1, the origin in the diagram of $\mu$ is the cyclic shift of a point $(x, y)$ on the frontier of $\mu^0$ at level $i_k$, whereas the origin in the diagram of $\nu$ is the cyclic shift of a point $(u, v)$ on the frontier of $\mu^0$ at level $i_k - a$. We know $i_k - a = i_j$ for some $j < k$. Here, $|j - k|$ is the number of levels in $\mu^0$ with values between $i_j = i_k - a$ (inclusive) and $i_k$ (exclusive). Cycling $\mu^0$ to $\mu$, we see that $|j - k|$ is the number of levels in $\mu$ with values in the interval $[-a, -1]$. By Lemma $[12](b)$, $h_{b,a}^+(\nu) - h_{b,a}^+(\mu) = -|j - k| = j - k$.

Repeatedly using the two cases, the claimed formula $h_{b,a}^+(\mu) - h_{b,a}^+(\mu^0) = k$ now follows by induction on the number of cyclic shift steps required to go from $\mu^0$ to $\mu$. $\square$

We know $R(a, b)$ is the disjoint union of the orbits of the cyclic-shift map $C$, where each orbit has a unique representative $\mu^0 \in D(a, b)$. Combining this fact with the last lemma, we see that

$$
\sum_{\mu \in R(a, b)} q^{\mu + ml_{b,a}(\mu) + h_{b,a}^+(\mu)} = (a + b)_q \sum_{\mu^0 \in D(a, b)} q^{\mu^0 + h_{b,a}^+(\mu^0)}.
$$

This equation shows the equivalence of Conjecture 6 and Conjecture 8 for each fixed $a, b$ with $\gcd(a, b) = 1$. So the proof of Theorem 10 is complete.

Below we illustrate Lemma 13 for $\mu^0 = (2, 1, 0) \in D(3, 5)$. Cells contributing to $h_{b,a}^+$ are labeled with the corresponding value of $i_{l-1} - l_{j-1}$ as explained in Lemma 11. The vertex labels contributing to $\Delta h^+$ are italicized and written in blue (red) for paths starting with an east (north) step.

### 4.7. Remarks on Conjecture 8

A natural approach to proving Conjecture 8 is to show that the combinatorial formulas in (14) satisfy the recursions (11). One difficulty here is determining how to divide $R(a, b)$ into two disjoint sets that are $q$-counted by the two terms in these recursions. The approach used for the standard interpretation of $q$-binomial coefficients (removing the first step or the last step of the frontier) does not work. Another major obstacle is the fact that when the dimension of the rectangle is reduced from $(b, a)$ to $(b, a - 1)$ or $(b - 1, a)$, the statistic changes from $h_{b,a}^+$ to $h_{b,a-1}^+$ or $h_{b-1,a}^+$, so that the power of $q$ for a particular partition changes unpredictably.

### 5. The Shuffle Conjecture

To prepare for our discussion of rational $q, t$-parking functions in the next section, this section reviews some combinatorial conjectures for the Hilbert series and Frobenius series of $DR_n$ involving classical $q, t$-parking functions. These conjectures first appeared in [10][20].
5.1. **The dinv Statistic.** Let \( P \in \mathbf{PF}_n \) be a labeled Dyck path. We have already defined \( \text{area}(P) \) in §2.6. We now define a second statistic \( \text{dinv}(P) \) (the **diagonal inversion count** of \( P \)) as follows.

Use \( P \) to define vectors \((g_1, \ldots, g_n)\) and \((p_1, \ldots, p_n)\) where \( g_i \) is the number of area cells in the \( i \)-th row from the bottom, and \( p_i \) is the label in the \( i \)-th row from the bottom. For example, the labeled Dyck path shown at the end of §2.2 has \((g_1, \ldots, g_5) = (0, 1, 1, 2, 1)\) and \((p_1, \ldots, p_5) = (3, 5, 1, 4, 2)\).

Let \( \text{dinv}(P) \) be the number of pairs \( i < j \) such that either \( g_i = g_j \) and \( p_i < p_j \), or \( g_i = g_j + 1 \) and \( p_i > p_j \). Our example object has \( \text{dinv}(P) = 2 \), corresponding to the pairs \((i, j) = (3, 5)\) and \((i, j) = (4, 5)\).

Haglund, Haiman, and Loehr conjectured the following formula for the Hilbert series of the doubly-graded diagonal coinvariant module.

**Conjecture 14** ([20]). For all \( n > 0 \),

\[
\text{Hilb}_{DR_n}(q, t) = \sum_{P \in \mathbf{PF}_n} q^{\text{area}(P)} t^{\text{dinv}(P)}.
\]

5.2. **The zeta Map.** The unusual statistic \( \text{dinv} \) is actually a disguised version of the natural statistic \( \text{area} \). To explain this, we recall the definition of a bijection \( \zeta \) between two different encodings of parking functions, which are two different ways to label Dyck paths. We are already familiar with the first labeling scheme (filling the vertical runs with increasing labels). Call this the **coset notation**.

A parking function under the second labeling scheme will also consist of a Dyck path \( D \) paired with a permutation. In this case, the entries of the permutation will be written along the diagonal \( y = x \). However, just as the labels in the coset notation must increase along each vertical run, there are restrictions on the permutation here as well. To describe these restrictions, we label each square above \( y = x \) by the ordered pair \((i, j)\) where \( i \) is the permutation entry in the same column and \( j \) is the permutation entry in the same row. The pair consisting of the Dyck path and the permutation is a parking function if and only if each square lying both immediately above an east step of \( D \) and immediately to the left of a north step of \( D \) has a label \((i, j)\) satisfying \( i < j \). (We call such squares **left-turns** of \( D \).) The figure below shows an order-5 parking function using this new labeling system. Call this the **root notation** for parking functions.

Note that the pairs labeling the left-turns are 35 and 14, which are indeed increasing. We remark that parking functions in root notation are in natural bijection with regions of the Shi hyperplane arrangement (see Armstrong [1]).

To define the map \( \zeta \), begin with a parking function \( P \in \mathbf{PF}_n \) in coset notation. Define the **diagonal reading word** of \( P \), denoted \( \text{drw}(P) \), by reading labels along diagonals of slope 1, working from higher diagonals to lower diagonals, and scanning each diagonal from northeast to southwest. For example, the \( P \) displayed in §2.6 (in coset notation) has \( \text{drw}(P) = 42153 \). Let \( l_1l_2\cdots l_n \) be the reverse of \( \text{drw}(P) \). Define a set \( \text{valleys}(P) \) consisting of the ordered pairs \((j, k)\) such that the label \( l_j \) occurs in the box just below \( l_k \) in coset notation. Finally, let \( \zeta(P) \) be the parking function in root notation with

- word \( l_1l_2\cdots l_n \) on the diagonal and
- Dyck path having left turns at the squares containing the labels \( \text{valleys}(P) \).
One may check that the elements of valleys\((P)\) are “non-nesting” so that such a path exists. The following figure gives an example of this map:

![Diagram]

Remark 15. We regard the coset notation and the root notation as analogous to the two combinatorial ways to view an affine Weyl group — via minimal coset representatives for the finite Weyl group, and via the semi-direct product of the finite Weyl group and the root lattice. For more details, see [1].

5.3. The area' Statistic. Next, we define the area' statistic. If \(P \in \text{PF}_n\) is a parking function in coset notation with corresponding parking function \(\text{zeta}(P)\) in root notation, we let \(\text{area}'(\text{zeta}(P))\) be the number of boxes fully between the path and the diagonal in \(\text{zeta}(P)\) that are labeled by increasing pairs. For our running example \(P = (2, 4, 1, 2, 1)\), we have \(\text{area}'(\text{zeta}(P)) = 2\), coming from the boxes labeled 12 and 24. It can be shown that \(\text{dinv}(P) = \text{area}'(\text{zeta}(P))\) for all \(P \in \text{PF}_n\) (cf. [20], which proves this using a different description of the map \(\text{zeta}\)). Accordingly, we can restate the Hilbert series conjecture as follows.

Conjecture 16 ([20]). For all \(n > 0\),

\[
\text{Hilb}_{\text{DR}_n}(q, t) = \sum_{P \in \text{PF}_n} q^{\text{area}(P)} t^{\text{area}'(\text{zeta}(P))}.
\]

5.4. The Shuffle Conjecture. We need two more ingredients to state the Shuffle Conjecture [19] giving a combinatorial formula for the Frobenius series of the doubly-graded module \(\text{DR}_n\). First, we say that a formal power series in \(\mathbb{C}[[x]]\) is quasisymmetric iff it is invariant under shifts of the variables that preserve the order of indices. Let \(\text{QSym} \subseteq \mathbb{C}[[x]]\) denote the subalgebra of quasisymmetric functions. Gessel’s fundamental basis of \(\text{QSym}\) is indexed by pairs \((n, S)\) where \(S \subseteq [n-1] = \{1, 2, \ldots, n-1\}\). This basis is defined by

\[
F_{n,S}(x) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1}x_{i_2}\cdots x_{i_n}.
\]

Second, given a parking function \(P \in \text{PF}_n\) with diagonal reading word \(\text{drw}(P)\), define the inverse descent set \(\text{IDes}(P) \subseteq [n-1]\) by letting \(j \in \text{IDes}(P)\) if and only if \(j + 1\) occurs to the left of \(j\) in \(\text{drw}(P)\), for all \(j \in [n-1]\). Our running example has \(\text{drw}(P) = 42153\) and \(\text{IDes}(P) = \{1, 3\}\).

Conjecture 17 (Shuffle Conjecture [19]). The bi-graded Frobenius characteristic of the ring of diagonal coinvariants \(\text{DR}_n\) (considered as an \(S_n\)-module bi-graded by \(x\)-degree and \(y\)-degree) satisfies

\[
\text{Frob}_{\text{DR}_n}(x; q, t) = \sum_{P \in \text{PF}_n} q^{\text{area}(P)} t^{\text{area}'(\text{zeta}(P))} F_{n,\text{IDes}(P)}(x) = \sum_{P \in \text{PF}_n} q^{\text{area}(P)} t^{\text{dinv}(P)} F_{n,\text{IDes}(P)}(x).
\]

Recall that increasing parking functions (in coset notation) correspond bijectively with unlabeled Dyck paths. One can check that when \(D\) is increasing, \(\text{dinv}(D)\) is the number of \(i < j\) with
\(g_i - g_j \in \{0, 1\}\), and \(\text{area}(\text{zeta}(D)) = \text{area}(\text{zeta}(D))\). As a special case of the Shuffle Conjecture, the coefficient of the sign representation in \(\text{Frob}_{DR, n}\) is given by the \(q, t\)-Catalan number

\[
\text{Cat}_n(q, t) = \sum_{D \in \mathcal{D}(N^n E^n)} q^{\text{area}(D)} t^{\text{area}(\text{zeta}(D))} = \sum_{D \in \mathcal{D}(N^n E^n)} q^{\text{area}(D)} t^{\text{dinv}(D)}.
\]

This special case of the conjecture has been proved by Garsia and Haglund \[10\].

The full Shuffle Conjecture is quite remarkable since it contains many famous combinatorial polynomials at various specializations of \(q, t, \text{and } x\). It has also been exceptionally difficult to prove. One issue is that the combinatorial formula is experimentally seen to be symmetric in \(q\) and \(t\) (as it must be to satisfy the conjecture); however, this symmetry is not visible in the combinatorics. In fact, it is an open problem to prove combinatorially that the expression on the right side of (15) is symmetric in \(q\) and \(t\).

## 6. Rational \(q, t\)-Catalan Numbers

Throughout this section, fix coprime integers \(a, b \in \mathbb{N}\). We now generalize the combinatorics of \(q, t\)-parking functions and \(q, t\)-Catalan numbers from the classical parking functions \(\text{PF}_{n, n+1}\) to the rational parking functions \(\text{PF}_{a, b}\). In doing so, we will define a rational version of the shuffle conjecture \((15)\). Our new formula should be the Frobenius series for some natural representation of \(S_a\) with a symmetric bi-grading, generalizing the ring of diagonal coinvariants. A conjecturally suitable representation of \(S_a\) has been constructed using the theory of rational Cherednik algebras \[35\], but at present the construction is very difficult. It is an open problem to construct this representation more directly.

### 6.1. The Sweep Map

Recall from \[16\] that the classical \(q, t\)-Catalan numbers of Garsia and Haiman \[11\] can be defined in terms of classical Dyck paths \(\mathcal{D}(N^n E^n)\) using one natural statistic (called \(\text{area}\)) and one strange map (called \(\text{zeta}\)). We already generalized the \(\text{area}\) statistic to rational Dyck paths in \[3, 3\]. Next we define a map, called the sweep map, that will play the role of \(\text{zeta}\) in the rational case. This map is studied in detail in \[4\]; see also \[2\].

**Definition 18** (The Sweep Map). Consider a lattice path \(w = w_1 w_2 \cdots w_{a+b} \in \mathcal{D}(N^a E^b)\). As in Step 2 of the proof of Proposition \[2\] define levels \(l_0, l_1, \ldots, l_{a+b}\) of lattice points on the path by setting \(l_0 = 0\) and, for \(1 \leq i \leq a+b\), setting \(l_i = l_{i-1} + b\) if \(w_{i-1} = N\) and \(l_i = l_{i-1} - a\) if \(w_{i-1} = E\). For each step \(w_i\) in the path, let \(l_{i-1}\) be the wand label associated to this step. Define \(\text{sweep}(w)\) to be the path obtained by sorting the steps \(w_i\) into increasing order according to their wand labels, and then erasing all the labels.

For example, the figure below computes \(\text{sweep}(w)\) for the \((7, 10)\)-Dyck path

\[
w = NENENNENNEEEEEEE \in \mathcal{D}(N^7 E^{10}).
\]

Next to each step in \(\text{sweep}(w)\), we have written the wand label of the corresponding step in \(w\), although these labels are not retained as part of the final output.
Since sweep acts by reordering path steps, it is immediate from the definition that if $D \in D(N^aE^b)$, then $\text{sweep}(D) \in R(N^aE^b)$. It is true but non-obvious that $\text{sweep}(D)$ is actually an $(a,b)$-Dyck path; see [4] for a proof.

6.2. Rational $q,t$-Catalan Numbers. We use the area statistic and the sweep map to define the rational $q,t$-Catalan number

$$\text{Cat}_{a,b}(q,t) = \sum_{D \in D(N^aE^b)} q^{\text{area}(D)}t^{\text{area}(\text{sweep}(D))}.$$  

For example, we have

$$\text{Cat}_{5,8}(q,t) = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 & 1 & 1 \\
\ldots & \ldots & \ldots & 2 & 2 & 2 & 1 & 1 & \ldots \\
\ldots & \ldots & 1 & 3 & 3 & 2 & 1 & 1 & \ldots \\
\ldots & 2 & 4 & 3 & 2 & 1 & 1 & \ldots & \ldots \\
\ldots & 2 & 4 & 3 & 2 & 1 & 1 & \ldots & \ldots \\
\ldots & 1 & 4 & 3 & 2 & 1 & 1 & \ldots & \ldots \\
\ldots & 3 & 3 & 2 & 1 & 1 & \ldots & \ldots & \ldots \\
\ldots & 2 & 3 & 2 & 1 & 1 & \ldots & \ldots & \ldots \\
\ldots & 2 & 2 & 1 & 1 & \ldots & \ldots & \ldots & \ldots \\
1 & 2 & 1 & 1 & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix},$$

where the entry in the $i$-th row from the top and the $j$-th column from the left (starting from 0) gives the coefficient of $q^i t^j$, and dots denote zeroes. Observe that this matrix is symmetric, a fact that does not follow obviously from (17). We state this symmetry as a conjecture.

**Conjecture 19.** The rational $q,t$-Catalan number is jointly symmetric in $q$ and $t$:

$$\text{Cat}_{a,b}(q,t) = \text{Cat}_{a,b}(t,q).$$

The weaker symmetry property $\text{Cat}_{a,b}(q,1) = \text{Cat}_{a,b}(1,q)$ would follow from the definition if we knew that sweep were a bijection on $D(N^aE^b)$. This is known for certain choices of $a, b$ (see [4]), but not for general coprime $a, b$.

**Example 20.** The following table shows the contributions of each path $D \in D(N^3E^5)$ to $\text{Cat}_{3,5}(q,t)$. Horizontal divisions demarcate the orbits of $D(N^3E^5)$ under sweep. We conclude that $\text{Cat}_{3,5}(q,t) = q^4 + q^3 t + q^2 t^2 + q^2 t + qt^2 + qt^3 + t^4$. 


tions (labeled rational Dyck paths). The goal is to define a representation of $S$ the sweep
Remark 22. One can use [31, Thm. 16] and [34] along with the relationships among varations on §
and third authors [31, 6.3. A Rational Frobenius Series. Although these definitions use different notation and constructions, they lead to the same $q,t$-polynomials. We give the details of this equivalence in [4].

| $D$ | sweep($D$) | area($D$) | area(sweep($D$)) |
|-----|-------------|-----------|------------------|
| NENENEEE NENENEEE | 1 | 3 |
| NENENEEE NENENEEE | 3 | 1 |
| NENENEEE NENENEEE | 0 | 4 |
| NENENEEE NENENEEE | 4 | 0 |
| NENENEEE NENENEEE | 2 | 1 |
| NENENEEE NENENEEE | 1 | 2 |
| NENENEEE NENENEEE | 2 | 2 |

Upon substituting $t = 1/q$ and rescaling by the maximum value of area (see §3.3), we conjecture that the rational $q,t$-Catalan number specializes to the rational $q$-Catalan number studied in [4].

**Conjecture 21.**

$$q^{(a-1)(b-1)/2} \text{Cat}_{a,b}(q, 1/q) = \frac{1}{[a+b]_q} \left[ \frac{a+b}{a,b} \right].$$

**Remark 22.** One can use [31, Thm. 16] and [34] along with the relationships among variations on the sweep map found in [1] to prove the formula $h_{a,b}^1(\mu) = (a-1)(b-1)/2 - \text{area(sweep(\mu))}$. This equality implies that Conjectures 6 and 21 are equivalent. Conjectures 6 and 21 have been checked computationally [42] for $a, b \leq 12$.

**Remark 23.** Rational $q,t$-Catalan numbers have been defined three times before: by the second and third authors [31, §7, Def. 21]; by Gorsky and Mazin [12, 13]; and by Armstrong, Hanusa, and Jones [2]. Although these definitions use different notation and constructions, they lead to the same $q,t$-polynomials. We give the details of this equivalence in [4].

6.3. **A Rational Frobenius Series.** This section develops $q,t$-analogues of rational parking functions (labeled rational Dyck paths). The goal is to define a representation of $S_n$ with a symmetric bi-grading such that the bi-graded Hilbert series is a $q,t$-analogue of $b^{a-1}$. We give a combinatorial formula for the Frobenius characteristic, rather than constructing the representation itself.

To each $(a, b)$-parking function $P \in \text{PF}_{a,b}$, we assign three pieces of data: a $q$-weight, a $t$-weight and a subset of $[a-1]$ (used to index a fundamental quasisymmetric function). The $q$-weight is the usual area statistic on the underlying Dyck path of $P$. The $t$-weight is a “$\text{dinv}$” statistic obtained as follows.

Since $\gcd(a, b) = 1$, there exist integers $x$ and $y$ such that $xa + yb = 1$. Furthermore, any integer solution $x'$ and $y'$ to $x'a + y'b = 1$ satisfies $x' = x + kb$ and $y' = y - ka$ for some integer $k$. It follows that there is a unique such solution with $-b < x \leq 0$ and $0 \leq y < a$. We then construct a $(|xa|, yb)$-parking function $P' \in \text{PF}_{|xa|, yb}$ by replacing each north step with $|x|$ copies of itself and each east step with $y$ copies of itself. The repeated north steps retain their original labels from $\{1, 2, \ldots, a\}$ (so that the north-step labels now come from the multiset containing $|x|$ copies of each number from 1 to $a$). Since the last step of $P'$ is an east step and $|xa| = -xa = yb - 1$, the first $|xa| + yb$ steps of $P'$ naturally encode a parking function $P'' \in \text{PF}_{|xa|}$.

For an $(a, b)$-parking function $P$ with underlying $(a, b)$-Dyck path $D$, define $d(P) = \text{area(sweep}(D))$. As we run over all parking functions with a given underlying $(a, b)$-Dyck path $D$, there is a maximum value $m(P)$ of $\text{dinv}(P'')$. We set $\text{dinv}(P) = \text{dinv}(P'') + d(P) - m(P)$. Note that the adjustment ensures that the maximum value of $\text{dinv}(P)$ is $d(P)$ for fixed underlying $(a, b)$-Dyck path $D$.

Finally, we compute the subset of $[a-1]$ assigned to $P$ as follows. Recall from the proof of Proposition 2 that each lattice point $(x, y)$ receives a level by $-ax$. Then the north (resp. east) steps in $P$ inherit the level of their bottom (resp. left) endpoints. The **diagonal reading word** of $P$, $\text{drw}(P)$, is then defined to be the word obtained by reading the labels of the north steps in
increasing order according to level. The inverse descent set \( \mathrm{IDes}(P) \) is computed from \( \mathrm{drw}(P) \) exactly as in \( \S 5.4 \).

**Definition 24.** Define

\[
\mathrm{PF}_{a,b}(q,t) = \sum_{P \in \mathrm{PF}_{a,b}} q^{\text{area}(P)} t^{\text{dinv}(P)} F_{n,\mathrm{IDes}(P)}(x).
\]

**Remark 25.** One can give an equivalent definition of the rational parking functions in terms of the LLT polynomials of Lascoux, Leclerc, and Thibon \[27, 28\] by assigning a suitably scaled LLT polynomial to each rational Dyck path. While the LLT definition somewhat obscures the combinatorics, it has the advantage of showing immediately that the \( \mathrm{PF}_{a,b}(q,t) \) are Schur-positive symmetric functions.

**Remark 26.** The diagonal reading word of a parking function \( P \in \mathrm{PF}_{n,n+1} \) is the reverse of the diagonal reading word obtained when \( P \) is viewed as an element of \( \mathrm{PF}_n \). Our convention for \( \mathrm{PF}_{a,b}(q,t) \) is consistent with the existing literature. Our convention for \( \mathrm{PF}_{a,b}(q,t) \) is motivated by the combinatorics.

Let \( \omega \) denote the endomorphism on the ring of symmetric functions determined by \( p_k \mapsto (-1)^{k-1} p_k \). It is a standard fact that \( \omega(s_\lambda) = s_{\lambda'} \) where \( \lambda' \) denotes the transpose of the partition \( \lambda \) (see, e.g., [17, Theorem 1.20.2]). Choosing the opposite convention for computing the diagonal reading word for rational parking functions (i.e., reading the labels of the north steps in decreasing order) leads to \( \omega(\mathrm{PF}_{a,b}(q,t)) \).

Presumably \( \mathrm{PF}_{a,b}(q,t) \) is the Frobenius series of some naturally occurring bi-graded version of \( \epsilon \otimes \mathrm{PF}_{a,b} \). We do not know any natural construction of this bi-graded \( S_n \)-module analogous to the diagonal coinvariant ring, but see Oblomkov and Yun [35]. For now, we make the following conjectures.

**Conjecture 27.** Let \( \mathrm{Sch}_{a,b;k}(q,t) \) denote the coefficient of the hook Schur function \( s_{(k+1,1^{a-k-1})} \) in \( \mathrm{PF}_{a,b}(q,t) \). We conjecture that the Schur-positive symmetric function \( \mathrm{PF}_{a,b}(q,t) \) satisfies the following properties.

1. \( \mathrm{PF}_{a,b}(q,t) = \mathrm{PF}_{a,b}(t,q) \).
2. \( (q^{a-1} t^{b-1}) \frac{\mathrm{PF}_{a,b}(q,1/t)}{\mathrm{PF}_{a,b}(q,1/q)} = [b]_q^{-a} \).
3. \( q^{(2ak-k^2+b-a^2-b+1)/2} \frac{\mathrm{Sch}_{a,b;k}(q,1/q)}{\mathrm{PF}_{a,b}(q,t)} = \frac{1}{[b]_q [a]_q} \binom{a-1}{k} \binom{b+k}{a} \).

**Example 28.** The following table illustrates the three parking functions that contribute to \( \mathrm{PF}_{2,3}(q,t) \). Column headings refer to the paragraphs leading up to Definition 24. Note that north-step labels in the second column are listed from bottom to top. Also note that \( |x| = y = 1 \) (see \( \S 6.3 \)). Since \( F_{2,0} = s_2 \) and \( F_{2,1} = s_{1,1} \), we conclude that \( \mathrm{PF}_{2,3}(q,t) = (q + t)s_2 + s_{1,1} \).

| \( P \) | labels | area | \( \text{dinv}(P') \) | \( d(P) \) | \( m(P) \) | \( \text{dinv}(P) \) | \( \text{IDes} \) | Contribution |
|-------|--------|------|-----------------|--------|--------|-----------------|--------|------------|
| NNEEE | 12     | 1    | 0               | 0      | 0      | 0               | 0      | \( qF_{2,0}(x) \) |
| NENEE | 12     | 1    | 1               | 0      | 0      | 0               | 0      | \( tF_{2,0}(x) \) |
| NENEE | 21     | 0    | 1               | 1      | 0      | 0               | 1      | \( F_{2,1}(x) \) |

\( F_{2,0} = s_2 \) and \( F_{2,1} = s_{1,1} \), we conclude that \( \mathrm{PF}_{2,3}(q,t) = (q + t)s_2 + s_{1,1} \).

**Example 29.** In this example we compute the contribution of a given parking function to \( \mathrm{PF}_{5,8}(q,t) \). Let \( D = \text{NNENENNEEEE} \) and let \( P \) be the parking function with underlying path \( D \) and with north-step labels 4, 5, 1, 3, 2 when read from bottom to top. We see that \( \text{area}(P) = 9 \). Also, \( \text{sweep}(D) = \text{NENENENNEEEE} \), so \( d(P) = \text{area}(	ext{sweep}(D)) = 3 \). The values of \(|x|\) and \( y \) used in the construction of \( P'' \) are 3 and 2, respectively. So \( P'' \) has underlying Dyck path...
$D'' = N^6E^2N^6E^4N^3E^9$. A direct computation shows that $\text{dinv}(P'') = 5$. One can also check that $\text{dinv}$ achieves a maximum value of 6 on $D''$ with a label ordering such as 1, 2, 3, 5, 4. Hence $\text{dinv}(P) = 5 + 3 - 6 = 2$. Finally, since $\text{drw}(P) = 45123$, we see that $\text{IDes}(P) = \{3\}$. In summary, the ultimate contribution of $P$ to $\text{PF}_{5,8}(q,t)$ is $q^9t^2F_{5,\{3\}}(x)$.

Part [1] of the conjecture implies that the coefficient of each Schur function in $\text{PF}_{a,b}(q,t)$ is symmetric in $q$ and $t$. For example, here is the coefficient of $s_{(3,2)}$ in $\text{PF}_{5,8}(q,t)$, where the entry in the $i$-th row and $j$-column (starting from 0) gives the coefficient of $q^i t^j$ in $\text{PF}_{5,8}(q,t)$:

\[
\begin{bmatrix}
. . . . . . . . & 1 & 1 & 1 & 1 & . . . . . . . . \\
. . . . . . . . & 2 & 4 & 4 & 3 & 2 & 1 & . . . . . . . . \\
. . . . . . . . & 1 & 5 & 8 & 6 & 4 & 2 & 1 & . . . . . . . . \\
. . . . . . . . & 2 & 7 & 10 & 7 & 4 & 2 & 1 & . . . . . . . . \\
. . . . . . . . & 1 & 7 & 11 & 7 & 4 & 2 & 1 & . . . . . . . . \\
. . . . . . . . & 5 & 10 & 7 & 4 & 2 & 1 & . . . . . . . . \\
. . . . . . . . & 2 & 8 & 7 & 4 & 2 & 1 & . . . . . . . . \\
. . . . . . . . & 4 & 6 & 4 & 2 & 1 & . . . . . . . . \\
1 & 4 & 4 & 2 & 1 & . . . . . . . . \\
1 & 3 & 2 & 1 & . . . . . . . . \\
1 & 2 & 1 & . . . . . . . . \\
1 & 1 & . . . . . . . . \\
1 & . . . . . . . . . . \\
\end{bmatrix}
\]

Conjecture [27] has been checked for coprime $a, b \leq 9$. Note that the Schur expansion can be obtained combinatorially from the fundamental quasisymmetric function expansion given in the definition of rational parking functions by the methods in [9]. We end by posing the following question.

**Question 30.** We know $\text{PF}_{a,b}(q,t)$ is the Frobenius characteristic of some doubly-graded $S_a$-module of dimension $b^{a-1}$, whereas $\text{PF}_{b,a}(q,t)$ is the Frobenius characteristic of some doubly-graded $S_b$-module of dimension $a^{b-1}$. What is the algebraic and combinatorial relationship between $\text{PF}_{a,b}(q,t)$ and $\text{PF}_{b,a}(q,t)$?

7. **Appendix: Computations**

The following expansions were computed using Sage [31]. The Sage worksheet we used to do this can be found on the third author’s web page [42].

7.1. **Rational Catalan numbers.**

\[
\text{Cat}_{2,3}(q,t) = \text{Cat}_{3,2}(q,t) = \begin{bmatrix}
1 \\
1 \\
\end{bmatrix},
\]

\[
\text{Cat}_{3,5}(q,t) = \text{Cat}_{5,3}(q,t) = \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
\end{bmatrix}
\]

\[
\text{Cat}_{3,5}(q,t) = \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
\end{bmatrix}
\]

\[
\text{Cat}_{3,5}(q,t) = \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
\end{bmatrix}
\]
\[ \text{Cat}_{3,7}(q,t) = \text{Cat}_{7,3}(q,t) = \begin{bmatrix} \ldots & \ldots & 1 \\ \ldots & 1 & 1 \\ \ldots & 1 & 1 \\ \ldots & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \]

\[ \text{Cat}_{4,7}(q,t) = \text{Cat}_{7,4}(q,t) = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & 1 & 2 & 1 \\ \ldots & \ldots & \ldots & 1 & 2 & 1 \\ \ldots & \ldots & \ldots & 1 & 2 & 1 \\ \ldots & \ldots & \ldots & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \]

\[ \text{Cat}_{5,8}(q,t) = \text{Cat}_{8,5}(q,t) = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 & 2 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 1 & 2 & 1 & 1 \\ \ldots & \ldots & \ldots & \ldots & 1 & 2 & 1 & 1 & 1 \\ \ldots & \ldots & \ldots & 1 & 2 & 1 & 1 & 1 & 1 \\ \ldots & \ldots & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ \ldots & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \]

### 7.2. Rational Parking Functions.

\[ \text{PF}_{2,3}(q,t) = \begin{bmatrix} \ldots & \ldots & 1 \\ \ldots & 1 & 1 \\ \ldots & 1 & 1 \\ \ldots & 1 & 1 \end{bmatrix} s_{(2)} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} s_{(1,1)}, \]

\[ \text{PF}_{2,5}(q,t) = \begin{bmatrix} \ldots & \ldots & 1 \\ \ldots & 1 & 1 \\ \ldots & 1 & 1 \\ \ldots & 1 & 1 \end{bmatrix} s_{(2)} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} s_{(1,1)}, \]

\[ \text{PF}_{3,5}(q,t) = \begin{bmatrix} \ldots & \ldots & 1 \\ \ldots & 1 & 1 \\ \ldots & 1 & 1 \end{bmatrix} s_{(3)} + \begin{bmatrix} \ldots & \ldots & 1 \\ \ldots & 1 & 1 \\ \ldots & 1 & 1 \end{bmatrix} s_{(2,1)} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} s_{(1,1,1)}, \]
\[ \text{PF}_{5,3}(q,t) = \begin{bmatrix} \ldots & 1 \\ \ldots & 1 & 1 \\ \ldots & 1 \\ 1 & \ldots \\ 1 & \ldots & \ldots \end{bmatrix} s_{(5)} + \begin{bmatrix} \ldots & 1 \\ \ldots & 2 & 1 \\ \ldots & 1 \\ 1 & \ldots \\ 1 & \ldots & \ldots \end{bmatrix} s_{(4,1)} + \begin{bmatrix} 1 & 1 \\ 1 & \ldots \\ 1 & \ldots & \ldots \end{bmatrix} s_{(3,2)} + \begin{bmatrix} 1 \\ 1 \\ \ldots \end{bmatrix} s_{(3,1,1)} + [1] s_{(2,2,1)}, \]

\[ \text{PF}_{4,7}(q,t) = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 1 & 1 \\ \ldots & \ldots & \ldots & \ldots & 1 & 2 & 1 \\ \ldots & \ldots & \ldots & 1 & 2 & 1 & 1 \\ \ldots & \ldots & 1 & 1 & 1 & \ldots & \ldots \\ \ldots & 1 & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} s_{(4)} + \begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 2 & 3 \\ \ldots & \ldots & \ldots & \ldots & 1 & 3 & 4 \\ \ldots & \ldots & \ldots & 1 & 3 & 4 & 2 \\ \ldots & \ldots & 2 & 4 & 2 & 1 & \ldots \\ \ldots & 3 & 2 & 1 & \ldots & \ldots & \ldots \\ 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} s_{(3,1)} + \begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} s_{(3,1)}, \]

\[ \begin{bmatrix} \ldots & 1 \\ \ldots & 1 & 2 & 1 \\ \ldots & 1 & 3 & 1 & 1 \\ \ldots & 1 & 3 & 1 & 1 \\ \ldots & 2 & 1 & \ldots \\ \ldots & 1 & 1 & \ldots \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ 1 & \ldots & \ldots \\ 1 & \ldots & \ldots \end{bmatrix} s_{(2,2)} + \begin{bmatrix} \ldots & 1 \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ 1 & \ldots & \ldots \\ 1 & \ldots & \ldots \end{bmatrix} s_{(2,1,1)} + \begin{bmatrix} \ldots & 1 \\ \ldots & 1 \\ \ldots & 1 \\ \ldots & 1 \\ \ldots & 1 \\ \ldots & 1 \\ \ldots & 1 \\ \ldots & 1 \\ \ldots & 1 \\ \ldots & 1 \\ 1 & \ldots \end{bmatrix} s_{(1,1,1,1)}, \]
\[ \text{PF}_{7,4}(q, t) = s(7) + s(6, 1) + s(5, 2) + s(5, 1, 1) + s(4, 2, 1) + s(4, 1, 1) + s(3, 2, 1) + s(3, 2, 1, 1) + s(2, 2, 1) \]

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\begin{pmatrix}
\ldots & 1 & 2 & 1 & 1 \\
\ldots & 1 & 3 & 4 & 2 & 1 \\
\ldots & 1 & 4 & 5 & 2 & 1 \\
\ldots & 3 & 5 & 2 & 1 \\
\ldots & 4 & 2 & 1 \\
\ldots & 2 & 2 & 1 \\
\ldots & 1 & 1 \\
\ldots & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\ldots & 1 & 2 & 1 & 1 \\
\ldots & 1 & 3 & 2 & 1 \\
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\ldots & 3 & 2 & 1 \\
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\end{pmatrix}
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\begin{pmatrix}
\ldots & 1 & 2 & 2 & 1 \\
\ldots & 2 & 4 & 3 & 1 \\
\ldots & 1 & 4 & 3 & 1 \\
\ldots & 2 & 3 & 1 \\
\ldots & 2 & 1 \\
\ldots & 1 \\
\ldots & 1
\end{pmatrix}
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\]

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\ldots & 1 \\
\ldots & 1 \\
\ldots & 1 \\
\ldots & 1
\end{pmatrix}
\]
\[
\text{PF}_{5,8}(q, t) = \begin{bmatrix}
\ldots & \ldots & \ldots & 1 1 1 1 \\
\ldots & \ldots & \ldots & 2 2 2 1 1 \\
\ldots & \ldots & \ldots & 1 3 3 2 1 1 \\
\ldots & \ldots & \ldots & 2 4 3 2 1 1 \\
\ldots & \ldots & \ldots & 2 2 1 1 \\
1 1 1 1 \\
1 1 1 \\
1 \\
1 \\
\end{bmatrix}
\]

\[s(5) + \begin{bmatrix}
\ldots & \ldots & \ldots & 1 3 4 3 2 1 \\
\ldots & \ldots & \ldots & 1 4 7 6 4 2 1 \\
\ldots & \ldots & \ldots & 2 7 9 7 4 2 1 \\
\ldots & \ldots & \ldots & 2 8 10 7 4 2 1 \\
\ldots & \ldots & \ldots & 1 7 10 7 4 2 1 \\
\ldots & \ldots & \ldots & 4 9 7 4 2 1 \\
1 7 7 4 2 1 \\
1 2 1 \\
1 \\
1 \\
\end{bmatrix}
\]

\[s(4,1) + \begin{bmatrix}
\ldots & \ldots & \ldots & 1 1 2 1 1 \\
\ldots & \ldots & \ldots & 2 4 5 4 2 1 \\
\ldots & \ldots & \ldots & 1 4 8 7 5 2 1 \\
\ldots & \ldots & \ldots & 1 5 10 8 5 2 1 \\
\ldots & \ldots & \ldots & 4 7 5 2 1 \\
\ldots & \ldots & \ldots & 1 5 2 1 \\
1 4 2 1 \\
2 2 1 \\
1 1 \\
1 \\
\end{bmatrix}
\]

\[s(3,2) + \begin{bmatrix}
\ldots & \ldots & \ldots & 1 1 1 1 1 \\
\ldots & \ldots & \ldots & 2 4 4 3 2 1 \\
\ldots & \ldots & \ldots & 1 5 8 6 4 2 1 \\
\ldots & \ldots & \ldots & 2 7 10 7 4 2 1 \\
\ldots & \ldots & \ldots & 1 7 11 7 4 2 1 \\
\ldots & \ldots & \ldots & 5 10 7 4 2 1 \\
\ldots & \ldots & \ldots & 2 8 7 4 2 1 \\
\ldots & \ldots & \ldots & 4 6 4 2 1 \\
\ldots & \ldots & \ldots & 1 4 4 2 1 \\
\ldots & \ldots & \ldots & 1 3 2 1 \\
\ldots & \ldots & \ldots & 1 2 1 \\
\ldots & \ldots & \ldots & 1 \\
\ldots & \ldots & \ldots & 1 \\
\ldots & \ldots & \ldots & 1 \\
\ldots & \ldots & \ldots & 1 \\
\ldots & \ldots & \ldots & 1 \\
\ldots & \ldots & \ldots & 1 \\
\end{bmatrix}
\]

\[s(3,1,1) + \begin{bmatrix}
\ldots & \ldots & \ldots & 1 1 1 1 1 \\
\ldots & \ldots & \ldots & 1 3 4 3 2 1 \\
\ldots & \ldots & \ldots & 2 6 6 4 2 1 \\
\ldots & \ldots & \ldots & 2 7 7 4 2 1 \\
\ldots & \ldots & \ldots & 1 6 7 4 2 1 \\
\ldots & \ldots & \ldots & 3 6 4 2 1 \\
\ldots & \ldots & \ldots & 1 4 4 2 1 \\
\ldots & \ldots & \ldots & 1 3 2 1 \\
\ldots & \ldots & \ldots & 1 2 1 \\
\ldots & \ldots & \ldots & 1 \\
\end{bmatrix}
\]

\[s(2,2,1) + \begin{bmatrix}
\ldots & \ldots & \ldots & 1 1 1 1 \\
\ldots & \ldots & \ldots & 1 3 3 2 1 \\
\ldots & \ldots & \ldots & 2 4 4 2 1 \\
\ldots & \ldots & \ldots & 1 4 4 2 1 \\
\ldots & \ldots & \ldots & 3 4 2 1 \\
\ldots & \ldots & \ldots & 1 3 2 1 \\
\ldots & \ldots & \ldots & 1 2 1 \\
\ldots & \ldots & \ldots & 1 \\
\end{bmatrix}
\]

\[s(2,1,1,1) + \begin{bmatrix}
\ldots & \ldots & \ldots & 1 \\
\ldots & \ldots & \ldots & 1 \\
\ldots & \ldots & \ldots & 1 \\
\end{bmatrix}
\]

\[s(1,1,1,1,1)
\]

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