LOCALLY CONFORMALLY SYMPLECTIC CONVEXITY

F. BELGUN, O. GOERTSCHES, AND D. PETRECCA

Abstract. We investigate special lcs and twisted Hamiltonian torus actions on strict lcs manifolds and characterize them geometrically in terms of the minimal presentation. We prove a convexity theorem for the corresponding twisted moment map, establishing thus an analog of the symplectic convexity theorem of Atiyah and Guillemin-Sternberg. We also prove similar results for the symplectic moment map (defined on the minimal presentation) whose image is then a convex cone. In the special case of a compact toric Vaisman manifold, we obtain a structure theorem.

1. Introduction

In the 1980’s, Atiyah and Guillemin–Sternberg independently proved the celebrated convexity theorem in symplectic geometry, see [2, 15]. It states that if a compact torus \( T \) acts in a Hamiltonian fashion on a compact symplectic manifold \((M, \omega)\) with moment map \( \mu: M \to \mathfrak{t}^* \), then the image \( \Delta = \mu(M) \subset \mathfrak{t}^* \) is a convex polytope given by the convex hull of the points in \( M \) fixed by \( T \), and the fiber \( \mu^{-1}(\alpha) \) of every \( \alpha \in \Delta \) is connected.

The approach followed in Atiyah’s proof is based on the fact that a generic component of the moment map is a Morse-Bott function on \( M \). An alternative technique used for the proof of the above theorem makes use of the local-global principle, that relates the openness and convexity of a vector-valued map on a topological space to its local openness and convexity, under relatively mild global topological assumptions (properness), see [17] or [6]. This principle is the link between the local form of a moment map and its global convexity.

The symplectic convexity theorem has variants in a multitude of other geometries. They range from odd-dimensional analogs of symplectic geometry, as contact [22] or cosymplectic geometry [3] to generalizations of symplectic structures, as \( b^m \)-symplectic [14] or generalized H-twisted complex geometry [28].

We focus on compact locally conformally symplectic (lcs) manifolds. They are compact manifolds together with a non-degenerate 2-form \( \omega \) that, around every point, is conformal to a symplectic form. Their study was initiated in [33]. Often one investigates the important special cases of locally conformally Kähler manifolds (lcK) and Vaisman manifolds, where the 2-form \( \omega \) is the fundamental form of a Hermitian metric, which in the Vaisman case also admits a non-zero parallel vector field.

The kind of transformations that we consider are conformal with respect to this structure and satisfy a twisted Hamiltonian condition, making possible to consider a twisted moment map. We refer to Section 2 for an introduction to the content needed for this paper (see also [9] for further details).

The first named author was supported by a grant of Ministry of Research and Innovation, CNCS - UEFISCDI, project number PN-III-P4-ID-PCE-2016-0065, within PNCDI III.
The study of transformations on lcs manifolds was considered by Haller and Rybicki [16], where they study lcs reduction, and recently specialized to lcK manifolds by Istrati [18] and to Vaisman manifolds by Pilca [31].

In this paper we prove a convexity theorem for twisted Hamiltonian actions on locally conformally symplectic manifolds, see Theorem 5.1 and Proposition 5.8:

\textbf{Theorem 1.1.} Consider a twisted Hamiltonian action of Lee type of a compact torus $T$ on a compact lcs manifold $(M, \omega, \theta)$, with twisted moment map $\mu : M \to \mathfrak{t}^*$. Then $\mu : M \to \mu(M)$ is an open map with connected level sets, whose image is a convex polytope.

While the condition of being twisted Hamiltonian is invariant under conformal changes, the convexity of the moment image is not, see Example 7.2. This is reflected in our theorem by the fact that the Lee type condition, i.e. the condition that the anti-Lee (or $s$-Lee) vector field is a fundamental vector field of the action, is not conformally invariant either. We consider, however, also the symplectic moment map (see below) which is a conformal invariant.

The paper is organized as follows: In Section 2 we review the necessary notions of lcs geometry, including the minimal presentation $\hat{M}$ of an lcs manifold $M$, and the Vaisman-Sasaki correspondence, and also give a topological characterization of the rank one case (Proposition 2.8, generalized to Proposition 2.9).

In Section 3 we introduce the lcs, special lcs and twisted Hamiltonian vector fields. The difference between the latter two kinds is measured by the Morse-Novikov cohomology, which is, however, difficult to compute. We provide criteria characterizing special lcs vector fields which are twisted Hamiltonian (Proposition 3.9, Proposition 3.13 and their corollaries).

In Section 4 we discuss Lie group actions on lcs manifolds and show three essential results for the sequel: in Proposition 4.2 we show that if the Lee vector field preserves infinitesimally some metric, then it generates a special lcs action of a torus, in Lemma 4.6 we show that a special lcs action of $G$ on $M$ lifts to a symplectic action of $G$ on the minimal covering $\hat{M}$, and, in Propositions 4.10 and 4.12, we show that a twisted Hamiltonian action of Lee type of a torus has an equivariant twisted moment map.

In Section 5, we prove our main results (Theorem 1.1) and, moreover, discuss another object defined by a twisted Hamiltonian action: the symplectic moment map, corresponding to the symplectic (in fact, Hamiltonian) action of $G$ on the minimal covering $\hat{M}$. The advantage of $\mu$ is that it does not depend on any choice of an lcs form $\omega$, but only on its conformal class (thus on the lcs structure).

Investigating the convexity of $\mu$ directly is, however, more difficult, since this map is usually not proper (see Lemma 5.14) (recall that properness is an essential ingredient for applying the local-global principle).

However, we can prove that properness, thus also the convexity of the symplectic moment map holds for torus actions on lcs manifolds of rank one (Theorem 5.15), under some extra assumption.

For higher rank, we were able to prove, also under some extra assumptions, that the image of the symplectic moment map is a cone over the image of the twisted moment map (Theorem 5.15), and is, therefore, convex.
In Section 6 we specialize to the case of a Vaisman manifold and relate the moment image to that of the associated Sasakian manifold and to the one of the minimal Kähler covering (Proposition 6.1).

In Section 6.1, we provide a structure theorem (Theorem 6.11) for toric Vaisman manifolds. Such a manifold has been already proved in [25] to be a mapping torus of a toric Sasakian manifold. Using toric geometry on Kähler cones, we are able to prove, moreover, that the transformation defining the mapping torus actually lies in the acting torus. This leads to the consequence that every toric Vaisman manifold is diffeomorphic, as a smooth manifold, to the product of a toric Sasakian manifold with a circle.

Finally, we provide in Section 7 examples of twisted Hamiltonian actions on lcs, lcK and Vaisman manifolds and compute their moment images.

Acknowledgments. The authors would like to thank Liviu Ornea, Alexandra Otiman and Mihaela Pilca for the fruitful discussions and their useful remarks.

2. Preliminaries on lcs, lcK and Vaisman manifolds

Let \((M,\omega)\) be an almost symplectic manifold of real dimension greater than 2, where \(\omega\) is a non-degenerate 2-form. Often \(\omega := g(J\cdot,\cdot)\) will be the fundamental form of an (almost) Hermitian metric \(g\) on \((M,J)\), where \(J: TM \to TM\) is an (almost) complex structure on \(M\). We will usually consider the complex case, when \(J\) is integrable.

If every point of \(M\) admits a neighborhood \(U\) and a smooth function \(f_U: U \to \mathbb{R}\) such that the two-form \(e^{-f_U}\omega|_U\) is closed, we call \((M,\omega)\) a locally conformally symplectic manifold (lcs). If \(\omega\) is the fundamental (or Kähler) form of a Hermitian manifold \((M,g,J)\), then we call it a locally conformally Kähler manifold (lcK).

From the definition, it follows that the local 1-forms \(d f_U\) glue together to a global 1-form \(\theta\), called the Lee form, satisfying on \(M\)

\[
(1) \quad d \omega = \theta \wedge \omega.
\]

Thus, the 2-form \(\omega\) is closed with respect to the so-called twisted differential \(d_\theta := d - \theta \wedge \cdot\). Because \(\theta\) is closed, we have \(d_\theta^2 = 0\), thus the twisted differential defines a cohomology (called the Morse-Novikov cohomology) \(H^*(M,d_\theta)\).

In general, the local functions \(f_U\) do not glue together to a global function \(f\) on \(M\). When this happens, or equivalently if \(\theta\) is exact, the structure is called globally conformally symplectic (gcs), resp. globally conformally Kähler (gcK), as this means that \(e^{-f}\omega\) is a symplectic form, resp. \(e^{-f}g\) is a Kähler metric on \(M\). If a connected manifold is gcs, resp. gcK, then all symplectic, resp. Kähler structures in the given conformal class are homothetic to each other.

We call an lcs structure strict if it is not gcs, and an lcK structure strict if it is not gcK. Note that the both the lcs and the lcK condition are invariant under conformal changes (multiplication by a positive function) of \(\omega\), resp. \(g\). In this paper, unless explicitly mentioned, all the considered lcs, resp lcK structures will be strict.

According to the classical result of P. Gauduchon [11], one can always find a metric (called Gauduchon metric) in a conformal lcK class, such that the corresponding Lee form is harmonic (w.r.t. that metric itself). Such Gauduchon metrics are homothetic to each other, hence a connected group that preserves the conformal
and the complex structure of an lcK manifold $$(M, J, \omega)$$ acts by isometries of any Gauduchon metric (see also [25, Proposition 3.8]).

If, for a strict lcK manifold $$M$$, the form $$\theta$$ is parallel with respect to the metric $$g$$, then $$M$$ is called a Vaisman manifold. For simplicity, we will make the convention that the Lee form of a Vaisman manifold has unit length.

The anti-Lee vector field of an lcs manifold $$(M, \omega, \theta)$$ is the unique vector field $$V$$ such that
\[ \omega(V, \cdot) = -\theta. \]
In the lcK setting it is $$V = J\theta^\sharp$$, where $$\theta^\sharp$$, the Lee vector field, is dual to the Lee form $$\theta$$ with respect to the metric $$g$$. As there is no Lee vector field in the lcs setting, we propose to call $$V$$ the s-Lee vector field (the prefix “s” coming from symplectic) in the most general setting.

2.1. Presentations of lcs and lcK manifolds. We characterize now lcs, resp. lcK structures by symplectic, resp. Kähler structures on a covering.

**Definition 2.1.** A presentation $$(M_0, \omega_0, \Gamma, \rho, \lambda)$$ of an lcs structure on $$M$$ consists of a symplectic manifold $$(M_0, \omega_0)$$, a discrete group $$\Gamma$$ acting freely and properly on $$M_0$$ such that $$M = M_0/\Gamma$$ and
\[ \gamma^*\omega_0 = e^{-\rho(\gamma)}\omega_0 \]
for all $$\gamma \in \Gamma$$, where $$\rho : \Gamma \to \mathbb{R}$$ is a group homomorphism, and $$\lambda : M_0 \to \mathbb{R}$$ is a $$\rho$$-equivariant smooth function, i.e.
\[ \lambda \circ \gamma - \lambda = \rho(\gamma) \]
for all $$\gamma \in \Gamma$$.

One says that (4) means that $$\lambda$$ is $$\rho$$-equivariant, and (3) that $$\Gamma$$ acts on $$M_0$$ by symplectic homotheties.

The correspondence between the lcs structure $$(M, \omega)$$ and the above presentation is the following: first, $$M = M_0/\Gamma$$, and $$p : M_0 \to M$$ is a Galois covering. Then,
\[ p^*\omega = e^{\lambda}\omega_0, \quad p^*\theta = d\lambda. \]
More precisely, given a presentation as above, the right hand sides of the above equalities are $$\Gamma$$-invariant forms on $$M_0$$ and define thus $$\omega$$ and $$\theta$$ on $$M = M_0/\Gamma$$.

Conversely, given an lcs structure on $$M$$, and a Galois covering $$p : M_0 \to M$$ with group $$\Gamma$$ such that $$p^*\theta$$ is exact, we define $$\lambda$$ as a primitive of $$p^*\theta$$. Then the group homomorphism $$\rho$$ is defined by the relation (4) (note that the right hand side has to be a constant depending on $$\gamma$$). Setting
\[ \omega_0 := e^{-\lambda}p^*\omega, \]
we clearly get a symplectic form $$\omega_0$$ and the equation (3).

If, moreover, $$M$$ has an lcK structure with complex structure $$J$$, then we consider on $$M_0$$ the lifted complex structure and we obtain that $$(\omega_0, J)$$ is a Kähler structure and $$\Gamma$$ is a discrete group of holomorphic homotheties. Conversely, if a presentation $$(M_0, \omega_0, \Gamma, \rho, \lambda)$$ as above admits a $$\Gamma$$-invariant complex structure $$J$$ on $$M_0$$, then the quotient space $$(M, J)$$ inherits an lcK metric $$g := \omega(\cdot, J\cdot)$$. Note that for the Kähler metric $$g_0 := \omega_0(\cdot, J\cdot)$$, $$\Gamma$$ acts, in particular, by (metric) homotheties.
Lemma 2.7. If $(M,\omega,\theta)$ is an lcs manifold, let $\pi: \tilde{M} \to M$ be the universal covering $M$ with the pull-back structure. Let $\lambda$ be a primitive of $\pi^*\theta$. The fundamental group of $M$ leaves $\pi^*\theta = d\lambda$ invariant, so there is a homomorphism $\tilde{\rho}: \pi_1(M) \to \mathbb{R}$ that maps $\gamma \in \pi_1(M)$ to the number $c_\gamma$ such that $\gamma \circ \lambda = \lambda + c_\gamma$.

Example 2.4. Let $M$ be a connected manifold. Consider
$$\Gamma := \pi_1(M)/[\pi_1(M), \pi_1(M)] \simeq H_1(M, \mathbb{Z})$$
to be quotient of of the fundamental group by its commutator $[\pi_1(M), \pi_1(M)]$, the result being the integer homology group. Then $\Gamma$ is the (Abelian) group of a Galois covering $\tilde{\rho}: \tilde{M} \to M$, called the homological covering $\overline{M} := \tilde{M}/[\pi_1(M), \pi_1(M)]$. As above, if $(M,\omega,\theta)$ is lcs, then we get a presentation on the homological covering $\overline{M}$ because all the objects that we define on the universal covering are $[\pi_1(M), \pi_1(M)]$-invariant. Note that, like the universal covering, the homological covering has the property that the pull-back through $\tilde{\rho}$ of any closed form on $M$ is exact (on $\overline{M}$).

For us, the main example of a presentation, besides the universal and the homological covering, is the minimal covering $\tilde{M} := \tilde{M}/\Gamma_0$, where $\Gamma_0 := \ker \tilde{\rho}$ is the (normal) subgroup of $\pi_1(M)$ that consists of the elements that act on $(\tilde{M}, \omega_0)$ by holomorphic isometries (in the lcs case only symplectomorphisms). The corresponding group $\tilde{\Gamma} := \pi_1(M)/\Gamma_0$ is then a free Abelian $\mathbb{Z}^k$, because the induced group homomorphism $\tilde{\rho}: \tilde{\Gamma} \to \mathbb{R}$ is injective. The number $k$ is called the lcK rank (or the lcs rank) of $(M,\omega)$ and it is a number between 1 and $b_1(M)$, see also [13 Prop. 2.12].

Remark 2.5. The lcs (lcK) rank of $(M,\omega)$ is the rank of the free abelian subgroup $\rho(\Gamma) \subset \mathbb{R}$, which is the same for all presentations $(M_0,\omega_0,\Gamma,\rho,\lambda)$.

Remark 2.6. The minimal covering, and thus the lcK or lcs rank defined above depend only on the closed, non-exact 1-form $\theta$ (more precisely on its cohomology class in $H^1(M, \mathbb{R})$): $M_0$ is the quotient of $\tilde{M}$ by the subgroup of $\pi_1(M)$ that preserves a primitive $\lambda$ of $\theta$ (hence any).

The case where the lcs (lcK) rank is 1 is special from a topological viewpoint.

Lemma 2.7. If $(M,\omega)$ has lcs rank 1, then there is a map $\overline{\lambda}: M \to S^1$ such that, if we denote by $q$ the projection from $\mathbb{R}$ to $\mathbb{R}/\rho(\Gamma) \simeq S^1$, and by $\tilde{\rho}: \tilde{M} \to M$ the projection of the minimal covering, we have
$$q \circ \lambda = \overline{\lambda} \circ \tilde{\rho}.$$  

Proof. From Remark 2.5, $\Gamma \simeq \mathbb{Z}$. The lemma follows now from the $\rho$-equivariance of $\lambda$. \qed

As a consequence, we prove the following proposition.

Proposition 2.8. Let $(M,\omega)$ be an lcs manifold. The primitive $\lambda: M_0 \to \mathbb{R}$ of the pull-back of the Lee form $\theta$ to a presentation $M_0$ is a proper map if and only if the lcs rank is 1, $M$ is compact and $M_0$ is (a finite covering of) the minimal presentation of $M$.  

Remark 2.2. Note that, if $M$ is a strict lcs or lcK manifold, then the group homomorphism $\rho$ is non-trivial, thus $\Gamma$ must be infinite.
Proof. Assume that $\lambda : M_0 \to \mathbb{R}$ is proper. We first show that the lcs rank of $M$ is 1, i.e., by Remark 2.5, that $\rho(\Gamma)$ is a cyclic subgroup of $\mathbb{R}$. By (4) this is true if $\lambda(\Gamma \cdot x) \subset \mathbb{R}$ is closed for some arbitrary fixed $x \in M_0$. So suppose there is a sequence $\gamma_n \cdot x$, with $\gamma_n \in \Gamma$, such that $\lambda(\gamma_n \cdot x)$ is a Cauchy sequence. Let $A$ be the closure of $\{\lambda(\gamma_n \cdot x) \mid n \in \mathbb{N}\}$ in $\mathbb{R}$. Then $A$ is compact, and thus, using the properness of $\lambda$, the preimage $\lambda^{-1}(A) \subset M_0$ is compact as well.

The properness of the free action of the discrete group $\Gamma$ on $M_0$ implies that the sequence $(\gamma_n)_{n \in \mathbb{N}}$ contains only finitely many elements of $\Gamma$. Thus the set $\{\gamma_n \cdot x \mid n \in \mathbb{N}\}$ is finite and so is $A$. This implies that $\lambda(\Gamma \cdot x) \subset \mathbb{R}$ is closed, and hence the lcs rank of $M$ is 1.

Taking the above sequence $(\gamma_n)_{n \in \mathbb{N}}$ in the normal subgroup $\Gamma_0 := \ker \rho \subset \Gamma$ we obtain using (3) and the properness of $\lambda$ that $\Gamma_0$ is finite, and thus $M_0$ is a finite covering of the minimal covering $\tilde{M} \simeq M_0/\Gamma_0$.

To prove compactness of $M$, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $M$ and let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $M_0$ such that $p(y_n) = x_n$ for all $n \in \mathbb{N}$. Let $a > 0$ be a generator of $\rho(\Gamma) \subset \mathbb{R}$. By shifting the points $y_n \in M_0$ by elements of $\Gamma$, we can suppose that $\lambda(y_n)$ is contained in the interval $[0, a]$ for all $n \in \mathbb{N}$. By the properness of $\lambda$, $(y_n)$ is a sequence in the compact set $\lambda^{-1}([0, a])$. Therefore $(y_n)_{n \in \mathbb{N}}$ and thus also $(x_n)_{n \in \mathbb{N}}$ admit converging subsequences.

Let us now show the converse direction. We use the notation from Lemma 2.7. For a compact interval $I$ of length less than $a$, the positive generator of the cyclic group $\rho(\Gamma) \subset \mathbb{R}$, $\lambda^{-1}(I)$ is homeomorphically projected by $\tilde{\rho}$ onto $\lambda^{-1}(q(I))$. This latter set is closed in $M$, thus compact. As any compact set in $\mathbb{R}$ can be covered by finitely many sets of type $I$, we conclude that $\lambda : M_0 \to \mathbb{R}$ is proper.

We formulate here a very general topological fact that generalizes both Lemma 2.7 and Proposition 2.8.

Proposition 2.9. Let $\overline{\rho} : \overline{M} \to M$ be the homological covering of a compact connected manifold $M$.

Let $\alpha \in C^\infty(\Lambda^1 \overline{M} \otimes H_1(M, \mathbb{R}))$ be a closed form with values in $H_1(M, \mathbb{R}) \simeq H^1(M, \mathbb{R})^*$, representing the tautological class $1\in H^1(M, \mathbb{R}) \otimes H_1(M, \mathbb{R})$. Then the pull-back of $\alpha$ to $\overline{M}$ is exact and let $\overline{\alpha} : \overline{M} \to H_1(M, \mathbb{R})$ be a primitive of $\overline{\rho} \alpha$.

Then there exists a smooth map $F : M \to T$, where $T := H_1(M, \mathbb{R})/\rho(H_1(M, \mathbb{Z}))$ is a torus (here $\rho : H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{R})$ is the canonical map; denote by $\pi : H_1(M, \mathbb{R}) \to T$ the projection), such that $\pi \circ \overline{F} = F \circ \overline{\rho}$. In particular $\overline{F}$ is proper.

Proof. The claims are straightforward, once we prove that the objects $\overline{\rho}$, $\rho$, $F$, etc. are well-defined. The simplest way is to remark that, if we define the primitive $\overline{F} : \overline{M} \to H_1(M, \mathbb{R})$ of the pull-back to the universal cover $\overline{M}$ of the chosen “tautological” 1-form $\alpha \in C^\infty(\Lambda^1 \overline{M} \otimes H_1(M, \mathbb{R}))$, then $\overline{F} \circ \gamma$ is another primitive of the same form as $\gamma^*\overline{\rho} \alpha = \overline{\rho} \alpha$. Recall that such a primitive $\overline{F}$ can be defined by choosing $x_0 \in \overline{M}$ and setting

$$\overline{F}(y) := \int_{x_0}^{y} \overline{\rho} \alpha,$$

where the path from $x_0$ to $y$ in $\overline{M}$ can be chosen arbitrarily, as all such paths are homotopic.
Thus \( \tilde{F} \circ \gamma - \tilde{F} = \rho(\gamma) \), for some constant in \( \rho(\gamma) \in H_1(M, \mathbb{R}) \), and we see, using the definition of \( \tilde{F} \), that \( \rho(\gamma) \) is the integral of \( \alpha \) along a loop in \( M \) representing \( \gamma \in \pi_1(M) \), and thus \( \rho : \pi_1(M) \to H_1(M, \mathbb{R}) \) is a group homomorphism.

To see that \( \rho \) induces the canonical homomorphism \( H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{R}) \simeq H^1(M, \mathbb{R})^* \), note that, for any cohomology class \( [\beta] \in H^1(M, \mathbb{R}) \) (de Rham cohomology) and homology class \( [\gamma] \in H_1(M, \mathbb{Z}) \), we have (using angular brackets for the dual pairing):

\[
\langle \rho(\gamma), [\beta] \rangle = \int_\gamma \langle \alpha, [\beta] \rangle = \int_\gamma [\beta] = \langle [\gamma], [\beta] \rangle.
\]

Also the map \( F : M \to T \) is induced by \( \tilde{F} \) and the action of \( \Gamma = H_1(M, \mathbb{Z}) \) on \( \overline{M} \), resp. \( H_1(M, \mathbb{R}) \).

**Remark 2.10.** Lemma \( \text{2.7} \) and Proposition \( \text{2.8} \) follow from the above result because the minimal cover is a quotient of \( \overline{M} \). The circle appearing in Lemma \( \text{2.7} \) is then a subgroup of the torus \( T \) above.

### 2.2. The Vaisman-Sasaki correspondence.

We recall (see, for example, [4, Theorem 3] and [13, Prop. 4.2]) that an lcK manifold is Vaisman if and only if it admits a presentation of the form \((C(S), \Gamma)\) where \( S \) is a Sasakian manifold and \( \Gamma \) is a discrete subgroup of holomorphic homotheties of the Kähler metric cone over \( S \), \( C(S) := \mathbb{R} \times S \) (see the description of the metric below) acting freely and properly, such that the action commutes with the flow of \( \partial_t \) (here \( t \) is the standard coordinate on \( \mathbb{R} \), extended to a smooth function on \( C(S) = \mathbb{R} \times S \) and the function \( \lambda \) of the presentation is just \( t \)). In this case, all presentations are of this form.

Recall that a Sasakian metric is an odd-dimensional analog of a Kähler metric, more precisely it is a contact metric manifold \((S, g_S)\), with Reeb field \( R \) (the metric dual of the contact form) which is Killing and has unit length, and its covariant derivative defines a CR structure \( J \) on \( R^2 \). We refer to [11] for details. Here we will mainly use the fact that the metric cone \( C(S) := (\mathbb{R} \times S, e^t(dt^2 + g_S)) \) is Kähler (one extends \( J \) by \( J\partial_t := R \) to an (integrable) almost complex structure on \( C(S) \)).

We infer that, for a Vaisman metric, its pull-back to any presentation is the product metric on \( C(S) = \mathbb{R} \times S \), the (conformally related) Kähler metric is \( J \) inserted into the conic metric, and the pull-back of the Lee form is \( dt \).

This Sasakian manifold \( S \) covers then every leaf \( W \) of the Sasakian foliation of \( M \) (obtained by integrating the distribution \( \ker \theta \)), the covering group of this covering being the normal subgroup \( \ker \rho \subset \Gamma \). Therefore, if \( C(S) \) is the minimal covering of \((M, \omega, \theta)\), then \( S \cong W \). Such an \( S \) is then called the associated Sasakian manifold to \( M \).

Conversely, if \( W \) is a Sasakian manifold, the product manifold \( M = W \times S^1 \) of \( W \) with a circle (or, more generally, the mapping torus of a Sasakian automorphism of \( S \), see Theorem [6.11]) admits a natural Vaisman structure (of lcK rank 1).

In [25], the authors prove that if the lcK rank of a compact Vaisman manifold \( M \) is one, then \( M \) is a mapping torus over its associated Sasakian \( W \) and that, hence, \( W \) is compact. This result, as well as the converse of this statement can be retrieved as a direct consequence of Proposition [2.8] because the level sets of \( \lambda \) are all diffeomorphic to \( S \).

**Corollary 2.11.** A Vaisman manifold \( M \) has a compact associated Sasakian manifold if and only if the lcK rank of \( M \) is one.
3. Twisted Hamiltonian vector fields on lcs manifolds

In this section we describe the infinitesimal automorphisms of an lcs manifold, and in the next one the group actions preserving the lcs structure. We will focus on the general lcs case; in the lcK setting, we require that lcK vector fields and transformations are lcs and holomorphic. We will briefly comment on the lcK case in Subsection 4.1.

The lcs transformations of an lcs manifold $\mathcal{M},\omega$ are the diffeomorphisms $\varphi: \mathcal{M} \to \mathcal{M}$ such that $\varphi^*\omega = F\omega$, for $F$ a function on $\mathcal{M}$.

Proposition 3.1. Let $X$ be a vector field on an lcs manifold $(\mathcal{M},\omega)$ whose flow consists of lcs transformations. Let $(\mathcal{M}_0,\omega_0)$ be a presentation of $(\mathcal{M},\omega)$ and lift $X$ to a vector field on $\mathcal{M}_0$, denoted again by $X$. Then
\[ L_X\omega_0 = a(X)\omega_0, \]
for some $a(X) \in \mathbb{R}$.

Proof. $d\omega_0 = 0$ implies that $L_X\omega = a(X)\omega_0$ has to be closed as well, hence $a(X)$ is constant. \qed

This leads to the following definitions.

Definition 3.2 (following [25]). An infinitesimal lcs automorphism of $(\mathcal{M},\omega)$ is called special if its lift to a (hence any) presentation is infinitesimal symplectomorphic.

One can read off whether $X$ is special lcs without going to any presentation, as follows.

Proposition 3.3. A vector field $X$ on an lcs manifold $(\mathcal{M},\omega)$ is special lcs if and only if
\[ d\theta(\omega(X,\cdot)) = L_X\omega - \theta(X)\omega = 0. \]

Proof. The first equality in (7) is an identity, following directly from the definition of $d\theta$ and the Cartan formula for $L_X\omega$.

Consider a presentation $(\mathcal{M}_0,\omega_0)$ of the lcs manifold $\mathcal{M}$, and by $X$ again the lift of $X$ to $\mathcal{M}_0$. Being special for $X$ is equivalent to $L_X\omega_0 = 0$, where $\omega_0 = e^{-\lambda}p^*\omega$ (recall that $\lambda : \mathcal{M}_0 \to \mathbb{R}$ is a primitive of (the pull-back to $\mathcal{M}_0$ of) the Lee form $\theta$). Then the equality
\[ L_X\omega_0 = e^{-\lambda}(-\theta(X)\omega + L_X\omega) \]
implies that $X$ is special if and only if the second term in (7) vanishes. \qed

Definition 3.4. A vector field $X$ on an lcs manifold $(\mathcal{M},J,\theta,\omega)$ is twisted Hamiltonian if there exists a smooth function $h_X$ on $\mathcal{M}$ such that $\iota_X\omega = d\theta h_X$. This function is called the twisted Hamiltonian of the vector field $X$.

In particular, a twisted Hamiltonian vector field is lcs (even special lcs), see below.

Remark 3.5. As an example, in the lcs setting, the s-Lee field $V$ is twisted Hamiltonian for the (constant) function $h_V := 1$, see [2].

In the definition above, we refer to $h_X$ as to the twisted Hamiltonian of $X$. The uniqueness of a twisted Hamiltonian for a given lcs vector field is due to the following standard result.
Lemma 3.6. [25] Lemma 2.1] Let \((M, \omega, \theta)\) be a strict lcs manifold. Then the twisted differential \(d_\theta : C^\infty(M) \to \Omega^1(M)\) is injective.

Clearly, if \(X\) is twisted Hamiltonian, then \(\omega(X, \cdot)\) is \(d_\theta\)-exact, which means it is \(d_\theta\)-closed, thus \(X\) is special lcs by Proposition 3.3. Conversely, for a special lcs vector field to be twisted Hamiltonian, the twisted cohomology class of \(\omega(X, \cdot)\) (for the twisted differential \(d_\theta\); this is also called the Morse-Novikov cohomology \(H^1(M, d_\theta)\)) must vanish.

Therefore, if \(H^1(M, d_\theta) = 0\), then all special lcs vector fields on \(M\) are twisted Hamiltonian. However, the computation of the Morse-Novikov cohomology is not straightforward [19], so this cohomological criterion has limited applicability.

On the other hand, we know that special lcs vector fields exist on an Inoue surface (which is lcK as well, and the vector field is special lcK, i.e., holomorphic as well), that are not twisted Hamiltonian [29, Example 5.17].

Remark 3.7. As shown in [31, Rmk. 3.5] for the lcK setting, it is easy to see that if \(X\) is twisted Hamiltonian on \((M, \omega)\), then it remains so for all conformally equivalent 2-forms \(\omega' = e^f \omega\), and the corresponding twisted Hamiltonian functions change as \(h'_X = h_X e^f\).

Another approach to twisted Hamiltonians of special lcs vector fields on \((M, \omega)\) is to consider the Hamiltonians of their lifts to the universal covering, and, possibly, to other presentations.

Proposition 3.8. Let \(X\) be a special lcs vector field on \((M, \omega)\). The field \(X\) is twisted Hamiltonian if and only if, for some (and then, for any) presentation \((M_0, \omega_0, \Gamma, \rho)\), there exists a Hamiltonian \(F\) for its lift to \(M_0\) such that

\[
F \circ \gamma = e^{-\rho(\gamma)} F
\]

for all \(\gamma \in \Gamma\).

Proof. Let \(F\) be an \(e^{-\rho}\)-equivariant Hamiltonian for the lift of \(X\), as described by [8]. If we set \(h_X := e^\lambda F\), for \(\lambda : M_0 \to \mathbb{R}\), \(d\lambda = \theta\) and \(\lambda \circ \gamma = \lambda + \rho(\gamma)\), for all \(\gamma \in \Gamma\), then \(h_X\) is \(\Gamma\)-invariant and hence a function on \(M\) satisfying \(d_\theta h_X = e^\lambda dF = \omega(X, \cdot)\).

Conversely, if \(X\) has \(h_X\) as associated twisted Hamiltonian, then \(F := e^{-\lambda} h_X\) is a Hamiltonian for \(X\) on any presentation:

\[
dF = e^{-\lambda}(-\theta + dh_X) = e^{-\lambda} d_\theta h_X = e^{-\lambda} \omega(X, \cdot) = \omega_0(X, \cdot).
\]

Moreover, \(F \circ \gamma = e^{-\rho(\gamma)} F\), as required. \(\square\)

Of course, we know for sure that, for any special lcs vector field \(X\) on \(M\), its lift to the universal covering of \(M\) admits a Hamiltonian. This Hamiltonian function turns out to satisfy a more complicated \(\pi_1(M)\)-equivariance relation (see Proposition 3.9 below) than [8], which characterizes the twisted Hamiltonian case.

In the following proposition, we denote by \(\text{Aff}_+(\mathbb{R})\) the group of orientation-preserving affine transformations of \(\mathbb{R}\), i.e., the transformations of the form \(t \mapsto at + b\), for \(a > 0\) and \(b \in \mathbb{R}\). The group \(\mathbb{R}_+^\ast\) of positive homotheties is the abelianization of \(\text{Aff}_+(\mathbb{R})\), i.e. the quotient of \(\text{Aff}_+(\mathbb{R})\) by its commutator, which is \(\mathbb{R}\) itself, seen as the group of translations in \(\mathbb{R}\).
Proposition 3.9. Let \((M, \omega)\) be a strict lcs manifold and \((\tilde{M}, \omega_0, \rho)\) the corresponding universal presentation. For every special lcs vector field \(X\) on \(M\), there is a group homomorphism

\[
\varphi_X : \pi_1(M) \to \text{Aff}_+(\mathbb{R}),
\]

that extends \(e^{-\rho} : \pi_1(M) \to \mathbb{R}^*_+\).

In particular, \(X\) is twisted Hamiltonian if and only if the image of \(\varphi_X\) is Abelian (thus if and only if, up to an affine reparametrization of \(\mathbb{R}\), \(\varphi_X\) coincides with its abelianization \(e^{-\rho}\)).

Proof. As \(X\) is special, there exists \(F : \tilde{M} \to \mathbb{R}\) such that \(dF = \omega_0(X, \cdot)\) (unique up to an additive constant). Thus, for every \(\gamma \in \pi_1(M)\),

\[
d(F \circ \gamma - e^{-\rho(\gamma)}F) = 0,
\]
as the lift of \(X\) is \(\gamma\)-invariant and \(\gamma^*\omega_0 = e^{-\rho(\gamma)}\omega_0\). If we denote by \(b_{\gamma} := F \circ \gamma - e^{-\rho(\gamma)}F\) and by

\[
\varphi_X(\gamma)(t) := e^{-\rho(\gamma)}t + b_{\gamma},\; t \in \mathbb{R},
\]
we obtain thus a map \(\varphi : \pi_1(M) \to \text{Aff}_+(\mathbb{R})\) satisfying

\[
F \circ \gamma = e^{-\rho(\gamma)}F + b_{\gamma} = \varphi_X(\gamma) \circ F.
\]

To prove that \(\varphi_X\) is a group homomorphism, we compute

\[
b_{\gamma'} = F \circ \gamma \circ \gamma' - e^{-\rho(\gamma'\gamma')}F
\]

\[
= (F \circ \gamma - e^{-\rho(\gamma)}F) \circ \gamma' + e^{-\rho(\gamma)}(F \circ \gamma' - e^{-\rho(\gamma')}F)
\]

\[
= b_{\gamma} + e^{-\rho(\gamma)}b_{\gamma'},
\]

which is precisely the formula for the composition of affine maps.

Finally, \(X\) is twisted Hamiltonian if and only if there exists a constant \(b \in \mathbb{R}\) such that \(F - b\) is \(e^{-\rho}\)-equivariant, i.e., \(\varphi_X = T_b \circ e^{-\rho} \circ T_b^{-1}\), where \(T_b : \mathbb{R} \to \mathbb{R}\) denotes the translation with \(b\).

Note that, since \(\rho\) is nontrivial, this is the only possibility for the image of \(\varphi_X\) to be Abelian.

This proposition helps us to establish various criteria for a special lcs vector field to be twisted Hamiltonian:

Corollary 3.10. A vector field on \((M, \omega)\) is twisted Hamiltonian if and only if its lift to the minimal covering, or to any presentation with Abelian group \(\Gamma\), is Hamiltonian.

Proof. If \(X\) has \(h_X : M \to \mathbb{R}\) as twisted Hamiltonian, then its lift to any presentation \((M_0, \omega_0, \Gamma)\) admits \(h_X e^{-\lambda}\) as a Hamiltonian (as usual, \(d\lambda = \theta\)).

Conversely, suppose \(X\) is Hamiltonian on \((M_0, \omega_0, \Gamma)\), with \(\Gamma\) Abelian. Then, following the proof of Proposition 3.9, we obtain a group homomorphism from \(\Gamma\) to \(\text{Aff}_+(\mathbb{R})\). As its image is Abelian, it follows that \(X\) has an \(e^{-\rho}\)-equivariant Hamiltonian, thus \(X\) is twisted Hamiltonian by Proposition 3.9.

We give some examples of properties of the fundamental group of \(M\) that imply that every special lcs vector field is twisted Hamiltonian:

Corollary 3.11. If the fundamental group of \(M\) is semisimple or nilpotent, then every special lcs vector field on \((M, \omega)\) is twisted Hamiltonian.
Proof. With the above hypotheses, the image of $\varphi_X$ in $\text{Aff}_+(\mathbb{R})$ is also semisimple or nilpotent. This, in turn, we can show it implies that this image $H \subseteq \text{Aff}_+(\mathbb{R})$ is Abelian: let $\varphi_i(t) := a_i t + b_i$, $i = 1, 2$ be two elements in $\text{Aff}_+(\mathbb{R})$. We compute their commutator:

$$
\varphi_1 \circ \varphi_2 \circ \varphi_1^{-1} \circ \varphi_2^{-1}(t) = a_1(a_2a_1^{-1}(a_2^{-1}(t - b_2) - b_1) + b_2) + b_1 = t + b_1(1 - a_2) - b_2(1 - a_1).
$$

This implies that

- The commutator of any subgroup $H$ of $\text{Aff}_+(\mathbb{R})$ is included in the subgroup of translations and is thus Abelian;
- If $\varphi_1$, with $a_1 = 1$ and $b_1 \neq 0$, is in the center of $H$, then $H$ consists only of translations;
- If $\varphi_1$, with $a_1 \neq 1$, is in the center of $H$, then all elements of $H$ must fix the unique fixed point of $\varphi_1$, in particular $H$ contains no nontrivial translations and thus its commutator is trivial.

If $H$ is nilpotent, then its center is nontrivial, thus $H$ is Abelian. If $H$ is a direct product of simple groups, then its commutator is the product of its nonabelian factors (if any), but this has to be contained in the subgroup of translations, it (and therefore $H$ itself) is thus Abelian. \hfill \Box

Another application of Proposition 3.9 is when the commutator of $\pi_1(M)$ is “small enough”:

**Corollary 3.12.** Suppose the torsion-free part of the abelianization of the commutator of $\pi_1(M)$ is trivial or cyclic. Then every special lcs vector field on $(M, \omega)$ is twisted Hamiltonian.

**Proof.** If we denote by $H$ the image of $\varphi_X$, for $X$ a special lcs vector field, then the restriction of $\varphi_X$ to the commutator of $\pi_1(M)$ is a group homomorphism into $\mathbb{R}$, thus it induces a map $\varphi$ from the abelianization of the commutator of $\pi_1(M)$ to $\mathbb{R}$. Its image is the commutator $H_1$ of $H$, and it is a subgroup of $\mathbb{R}$ generated by at most 1 element.

If the abelianization of the commutator of $\pi_1(M)$ is pure torsion, $H_1$ is trivial thus $H$ is Abelian.

If we suppose that $H_1$ is cyclic, generated by $b \in \mathbb{R}$, then we use that $H_1$ must be invariant by conjugation with non-translation elements of $H$. But conjugation of the translation $t \mapsto t + b$ by $t \mapsto at + b$, with $a < 1$ (such elements exist in $H$), is the translation by $ab$, which does not belong to the group generated by $b$. Thus, the case, $H_1$ cyclic does not occur, hence $H$ is Abelian. \hfill \Box

There is another instance where a special lcs vector field is twisted Hamiltonian that will turn out useful in the Vaisman case:

**Proposition 3.13.** Let $(M, \omega)$ be lcs and let $\omega = d\theta$, where $\theta$ is the Lee form of $\omega$. Then every vector field $X$ on $M$ that satisfies $\theta(X) = 0$ and whose flow preserves $\beta$ admits $-\beta(X)$ as the twisted Hamiltonian.

**Proof.** We compute

$$
d\beta(-\beta(X)) = -d(\beta(X)) + \beta(X)\theta = d\beta(X, \cdot) - \theta \wedge \beta(X, \cdot) = \omega(X, \cdot),
$$

where we have used the Cartan formula for $\mathcal{L}_X \beta = 0$ and that $\theta(X) = 0$. $X$ is thus twisted Hamiltonian. \hfill \Box
Corollary 3.14. A special lK (i.e., special lcs and holomorphic) vector field on a Vaisman manifold is twisted Hamiltonian.

Proof. It is well-known (see, for example [9]), that the Vaisman 2-form $\omega$ equals $d\theta(\theta \circ J)$. If $X$ is special lcs, then it is conformal Killing, thus by [25, Proposition 3.8] its flow preserves the Gauduchon metric, which is the Vaisman metric itself. Thus $\mathcal{L}_X\omega = 0$, which implies $\mathcal{L}_X\theta = 0$ and $\mathcal{L}_X(\theta \circ J) = 0$, because $X$ is holomorphic. As $\theta(X)$ is constant and $X$ is special lK (lcs), then $\theta(X) \equiv 0$ and we conclude by Proposition 3.13. \[\square\]

4. Group actions on lcs manifolds

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ acting on an lcs manifold $(M, \omega)$. We will denote by $\underline{X} \in \mathfrak{X}(M)$ the fundamental vector field associated to $X \in \mathfrak{g}$.

Definition 4.1. The action of $G$ is lcs on $(M, \omega)$ if and only if $\mathfrak{g}^*\omega$ is proportional with $\omega$, for $g \in G$. The action is special lcs, resp. (weakly) twisted Hamiltonian if every $X \in \mathfrak{g}$ induces a special lcs, resp. twisted Hamiltonian field $\underline{X}$ on $M$. The action is of Lee type if the s-Lee field is induced by an element of the Lie algebra of $G$.

The adverb weakly refers to the possible lack of an equivariant twisted moment map (see below). We will however omit it in the sequel, and refer simply to twisted Hamiltonian actions. In [31], the author requires for the definition of twisted Hamiltonian an additional condition with respect to the weakly twisted Hamiltonian setting. This condition is that the moment map, seen as a map from $\mathfrak{g}$ to $C^\infty(M)$, is a Lie algebra homomorphism for the twisted Poisson structure

\begin{equation}
\{f, g\}_\theta := \omega^{-1}(d\theta f, d\theta g).
\end{equation}

If fact, it is easy to see that this condition is automatically satisfied on strict lcs manifolds, which we consider here, hence we omit it in the definition. For a complete study of equivalent formulations of (9), and also for their automatic occurrence on a strict lcs manifold, see [16, Section 3].

Proposition 4.2. Suppose that the closure of the flow of the s-Lee field $V$ on a compact lcs manifold $(M, \omega)$ is a compact torus $T$ (equivalently, the flow of $V$ consists of isometries of some Riemannian metric on $M$). Then the action of $T$ is special lcs.

Proof. The flow of a vector field $V$ on a compact manifold $M$ has a compact closure $T$ in the diffeomorphism group if and only if $V$ is Killing with respect to some Riemannian metric. Clearly, as the closure of an abelian group, $T$ is Abelian, thus a torus. We will compare $T$, the closure of the flow of $V$ on $M$, with $\hat{T}$, the closure of the flow of the lift $\hat{V}$ of $V$ on the minimal covering $\hat{M}$ (in the isometry group of the lift of some $T$-invariant Riemannian metric).

Two possibilities occur: either $\hat{T}$ is a torus as well (i.e., the closure of the flow of $\hat{V}$ is compact as well), or $\hat{T}$ is a closed, non-compact subgroup of the isometry group of $\hat{M}$ isomorphic to $\mathbb{R}$. We will first show that the second case does not occur.

It is known that a closed subgroup of the isometry group of a Riemannian manifold acts properly, see [30, Proposition 5.2.4], and hence all isotropy groups of the
$\hat{T}$-action on $\hat{M}$ are compact. To show compactness of $\hat{T}$ it is therefore sufficient to show compactness of the $\hat{T}$-orbits (in fact, of a single $\hat{T}$-orbit).

As $\omega$ is invariant under the flow of the $s$-Lee field $V$, it is invariant under its closure $T$ as well. We choose a $T$-invariant associated almost Hermitian metric $g:=\omega(\cdot,J\cdot)$ (for $J$ the corresponding almost complex structure), and consider the (vector-valued) 1-form $\alpha$ in Proposition 2.9 to be harmonic with respect to $g$. Its components are thus harmonic forms and therefore $\alpha(V)$ is constant, because $V$ is Killing (see, e.g. [25, Lemma 5.1]). If we show that $\alpha(V) = 0$, then the orbits of $V$ (the lift of $V$) on the homological covering $\hat{M}$ are contained in the fibers of the proper map $\hat{\pi}: \hat{M} \to H_1(M,\mathbb{R})$, see Proposition 2.9 and are thus relatively compact. Of course, this means that the orbits of $\hat{V}$ on $\hat{M}$, which is a quotient of $\hat{M}$, are relatively compact.

So we need to prove that, for every harmonic form $\beta$, the constant $\beta(V)$ is zero. Suppose $\beta(V) > 0$, then $\int_M \beta(V) > 0$, and this is the integral of the scalar product of $\beta$ with $V^\flat$, the metric dual of $V$, which is $-\theta \circ J$, where $J$ is the almost complex structure compatible with $\omega$ and the metric.

Now, on an almost Hermitian manifold of dimension $2n$, the Hodge star operator $*: \Lambda^k M \to \Lambda^{2n-k} M$ has the following expression in degree 1: for all $X \in TM$ we have

$$\tag{10} *X^b = -\frac{1}{(n-1)!}\omega^{n-1} \wedge \omega(X, \cdot).$$

For $X = V$, we obtain:

$$*V^b = -\frac{1}{(n-1)!}\omega^{n-1} \wedge \theta = \frac{1}{(n-1)!}(n-1)! d(\omega^{n-1}),$$

which is thus exact, i.e., $V^b$ is co-exact. But the $L^2$-scalar product of a co-exact form with a harmonic form vanishes, thus $\alpha(V) = 0$ for all harmonic 1-forms $\alpha$ on $M$, thus the orbits of $V$ on $\hat{M}$ are included in the fibers of $\hat{\pi}$, which are compact.

This shows that the closure of the flow of $\hat{V}$ on $\hat{M}$ is compact, as claimed.

From (7), the action of $T$ is special lcs if and only if $L_X \omega = \theta(X) \omega$ for all $X \in \mathfrak{t}$. But $L_X \omega = 0$ implies $L_X \theta = 0$, thus $\theta(X) = d\lambda(X)$ is constant for all $X \in \mathfrak{t}$, where $\lambda: \hat{M} \to \mathbb{R}$ is a primitive of $\theta$.

As $\hat{V}$ is compact, for every $X \in \mathfrak{t}$ the constant $\theta(X) = d\lambda(X)$ vanishes, as $\lambda$ is bounded on the orbits of $\hat{T}$ (which were seen to be compact above). Thus the $T$-action is special lcs.

**Corollary 4.3.** A compact lcs manifold $M$ admits a special lcs torus action of Lee type if and only if the $s$-Lee vector field $V$ is Killing with respect to some Riemannian metric.

**Corollary 4.4.** The torus obtained as the closure of the flow of the $s$-Lee vector field $V$ on a compact Vaisman manifold $M$ acts on $M$ in a holomorphic, twisted Hamiltonian fashion.

**Proof.** As $V$ is a Killing vector field, the torus $T$ from Proposition 4.2 consists of isometries that preserve the Vaisman 2-form $\omega$. All fundamental vector fields $\mathcal{X}$, for $X \in \mathfrak{t}$ are thus holomorphic and (from Proposition 4.2) special lcs. By Corollary 3.14 they are twisted Hamiltonian. □
Remark 4.5. Note that the projection from $\hat{M}$ to $M$ induces a group homomorphism $\psi : \hat{T} \to T$, whose image contains the flow of $V$, thus it is dense in $T$. We have shown that $\hat{T}$ is compact, thus $\psi(\hat{T})$ is compact and dense, hence $\psi : \hat{T} \to T$ is surjective.

On the other hand, all elements in $\ker \psi$ (which are of finite order) must be induced by the deck transformation group $\tilde{\Gamma} = \pi_1(X)/\Gamma$, which is free abelian. This means that $\psi : \hat{T} \to T$ is a Lie group isomorphism.

We show now that it is a feature of special lcs actions of $G$ to lift to actions of $G$ on the minimal covering (thus $G_0 = G$ in Remark 4.8).

Lemma 4.6. Consider a special lcs action of a connected Lie group $G$ on the lcs manifold $(M, \omega)$. Then this action lifts to a symplectic action of $G$ on $(\hat{M}, \omega_0)$.

Proof. By lifting the (special lcs) fundamental fields of $G$ to the minimal covering $\hat{M}$, we obtain a symplectic action of $\tilde{G}$, the universal covering of $G$ on $\hat{M}$. Denote by $p : \tilde{G} \to G$ the projection. If $p(\check{g}) = 1 \in G$ for some $\check{g} \in \tilde{G}$, then $\check{g}$ maps the fibers of $\pi$ to themselves.

But the covering $\pi : \hat{M} \to M$ is Galois, with group $\Gamma$, thus for all $x \in \hat{M}$ there exists a unique $\gamma_x \in \Gamma$ such that $\check{g}.x = \gamma_x(x)$. By continuity, $\gamma_x$ is actually independent of $x$ and thus equal to a single element $\gamma \in \Gamma$.

All nontrivial elements of $\Gamma$ are strict symplectic homotheties, but $x \mapsto \check{g}.x$ is a symplectomorphism, thus $\gamma = 1$ and $\check{g}$ acts trivially on $\hat{M}$. Thus the action of $\tilde{G}$ factors to an action of $G$ on $\hat{M}$ as claimed. \hfill \Box

The above result has been proven previously for the case of a compact torus in [18] and, in the lcK setting, for compact $G$, in [25].

We now recall the map that bundles together all the twisted Hamiltonian functions, in analogy with symplectic geometry:

Definition 4.7. Let $G$ be a Lie group that acts on a lcs manifold and such that every $X \in \mathfrak{g}$ is twisted Hamiltonian. A map $\mu : M \to \mathfrak{g}^*$ such that $d\theta_0 \mu^X = i X \omega$ is a twisted moment map.

In the symplectic setting, it is known that the moment map of a Hamiltonian $G$-action on a symplectic manifold is unique up to constants in $[\mathfrak{g}, \mathfrak{g}]$. In the setting of twisted Hamiltonian actions on strict lcs manifolds, the uniqueness of the twisted moment map follows automatically from Lemma 4.6.

Remark 4.8. If a connected Lie group $G$ acts by special lcs transformations on $(M, \omega)$, then its fundamental fields $\mathfrak{X}$, for $X \in \mathfrak{g}$, lift (we denote their lifts to $M_0$ equally by $\mathfrak{X}$) to induce on any presentation $(M_0, \omega_0)$ a symplectic action of a covering $G_0$ of the group $G$. Proposition 3.5 implies that the action of $G$ is twisted Hamiltonian, if and only if the action of $G_0$ is Hamiltonian and admits a (necessarily unique) $e^{-\rho}$-equivariant (w.r.t. the action of $\Gamma$ on $M_0$) moment map $\mu_0 : M_0 \to \mathfrak{g}$. This moment map (that we call the symplectic moment map of a twisted Hamiltonian action) is related, cf. [23], to the twisted moment map $\mu$ of the action of $G$ on $M$:

$$\mu_0 = e^{-\lambda} \mu \circ p,$$

where $p : M_0 \to M$ denotes the covering map. We see thus that, while the twisted moment map depends on the choice of $\omega$ in its conformal class (and of the Lee
form \( \theta \), the symplectic moment map depends only on the lcs structure (i.e., the conformal class of \( \omega \) and, therefore, on the cohomology class of the Lee form \( \theta \)).

As in symplectic geometry, we consider the problem of \( G \)-equivariance of the moment map. First, we show the \( G_0 \)-equivariance of the symplectic moment map.

**Proposition 4.9.** Let a connected Lie group \( G \) act in twisted Hamiltonian fashion on \((M, \omega)\). For every presentation \((M_0, \omega_0)\), if we denote by \( \mu_0 : M_0 \to g \) the moment map of the symplectic action of \( G_0 \) on \( M_0 \), then \( \mu_0 \) is \( G_0 \)-equivariant w.r.t. the action on \( M_0 \) and the coadjoint action on \( g \).

**Proof.** As \( G \) and its covering \( G_0 \) are connected, the equivariance of \( \mu_0 \) holds if and only if
\[
(X_\mu_0)_Y = (X_\mu_0)_Y = \mu_0([X,Y])
\]
for all \( X, Y \in g \). To prove this, we show first that the left hand side is a Hamiltonian for the fundamental field defined by \([X, Y] \) on \( M_0 \), in other words, that the difference to the right hand side is a constant:
\[
d(X_\mu_0)_Y = d(\omega_0(Y, X)) = L_X(\omega_0(Y, \cdot)) = \omega_0([X, Y], \cdot) = d(\mu_0)_Y,
\]
because \( \omega_0 \) is closed and \( X_\mu_0 \)-invariant. Moreover, \( X_\mu_0 \) is again \( e^{-\rho} \)-equivariant, thus it coincides with \( \mu_0([X,Y]) \). Thus \( X_\mu_0 = ad_X^*(\mu_0) \).

Let us now discuss the \( G \)-equivariance of the twisted moment map.

**Proposition 4.10.** Let \((M, \omega, \theta)\) be a strict lcs manifold and \( G \) be a connected Lie group acting in twisted Hamiltonian fashion on \( M \). Then the following conditions are equivalent:

1. The twisted moment map \( \mu : M \to g^* \) is \( G \)-equivariant, i.e.,
\[
\mu(g \cdot x) = \text{Ad}_{g^{-1}}^*(\mu(x))
\]

for all \( g \in G \).
2. \( \omega \) is \( G \)-invariant.
3. \( \theta(X) = 0 \) for all \( X \in g \).

**Proof.** The equivalence of the first two conditions follows from the equivariance of \( \mu_0 \) (Proposition 4.9) and equation (11), see also [16, Proposition 2] for another proof.

The equivalence of the second and third condition is clear because by Proposition 3.3 we have \( \mathcal{L}_X \omega = \theta(X) \omega \) for the fundamental vector fields \( X \) of the action.

If the twisted moment map is \( G \)-equivariant, then we call the \( G \)-action strongly twisted Hamiltonian.

**Remark 4.11.** In [16], the authors use a different, weaker definition of \( G \)-equivariance:
\[
\mu(g \cdot x) = a_g \text{Ad}_{g^{-1}}^*(\mu(x))
\]
for all \( g \in G \), where \( a_g \in (0, \infty) \) also occurs as a factor in the conformal \( G \)-invariance of \( \omega : g^* \omega = a_g^{-1} \omega \). We would rather call this notion conformal \( G \)-equivariance, and, in fact, it is shown already in [16] to be equivalent to (9) and thus to hold automatically on every strict lcs manifold, [16, Proposition 2].
For a given twisted Hamiltonian action of a compact group $G$ on $(M, \omega)$, one can integrate over $G$ (for a bi-invariant volume form on $G$) the function $g \mapsto g^* \omega$ and get a conformally equivalent 2-form $\omega'$ which is $G$-invariant. (Recall that $g^* \omega$ is conformal to $\omega$ by definition of an lcs transformation.) 

**Proposition 4.12.** Let $T$ be a torus acting on a strict lcs manifold $(M, \omega)$ by special lcs transformations and suppose the action is of Lee type. Then $\omega$ is $T$-invariant.

**Proof.** By Proposition 4.11 it suffices to show that $\theta(X) = 0$ for all $X \in \mathfrak{t}$. We have $[\overline{X}, V] = 0$ for all $X \in \mathfrak{t}$, where $V$ is the s-Lee vector field. We write also

$$d(\theta(\overline{X})) = \mathcal{L}_{\overline{X}} \theta = -\mathcal{L}_{\overline{X}} (\omega(V, \cdot)) = -(\mathcal{L}_{\overline{X}} \omega)(V, \cdot) = \theta(\overline{X}) \theta.$$ 

This implies $\theta(\overline{X}) = 0$ by the following lemma (see also the argument from the proof of [25, Lemma 2.1]) \hfill $\square$

**Lemma 4.13.** Let $N$ be a connected manifold, $\eta$ a closed, non-exact form on $N$, and $f : N \to \mathbb{R}$ a function satisfying $df = f \eta$ on $N$. Then $f \equiv 0$.

**Proof.** Let $\beta : \tilde{N} \to \mathbb{R}$ such that $d\beta = p^* \eta$, for $p : \tilde{N} \to N$ the universal covering on $N$. Then we have

$$d(e^{-\beta} f \circ p) = e^{-\beta} (d\beta f \circ p + df \circ p) = 0$$

on $\tilde{N}$, thus $e^{-\beta} f \circ p = k$ is constant on $\tilde{N}$. This implies $f \circ p = ke^\beta$, but $\beta$ and $e^\beta$ are not $\pi_1(N)$-invariant (otherwise $\eta$ would be exact), so the only possibility is $k = 0$. \hfill $\square$

**Corollary 4.14.** A twisted Hamiltonian action of Lee type of a torus $T$ has a $T$-equivariant twisted moment map.

As we are interested in actions of compact groups, mainly in torus actions, we may consider actions that are are extended by continuity from actions of some dense subgroups (like a torus action can be the closure of the flow of one vector field). We show the following:

**Proposition 4.15.** Let $H \subset G$ be a dense subgroup in a compact Lie group $G$ and assume that $H$ acts by special lcs transformations on a compact lcs manifold $(M, \omega)$ of lcs rank 1. Then $G$ acts by special lcs transformations of $(M, \omega)$.

**Proof.** It is clear that the action can be extended to $G$ and that this action is lcs (here we need that $M$ is compact). Possibly after integrating $g^* \omega$ over $G$ as described above, we can suppose that $\omega$ is $G$-invariant, thus $\mathcal{L}_{\overline{X}} \theta = 0$, i.e., $\theta(X)$ is constant for any $X \in \mathfrak{g}$. Moreover, if $X \in \mathfrak{h}$, then $\theta(X) = 0$ from Proposition 4.2.

But this is equivalent, cf. Lemma 2.7, to $d\overline{X}(\overline{X}) = 0$, where we recall that $\overline{X} : M \to S^1$ is induced by a primitive $\lambda : \tilde{M} \to \mathbb{R}$ of $\theta$, in the case of lcs rank 1. This means that the function $\overline{X}$ is constant along the $H$-orbits; by continuity it is constant along the $G$-orbits. Hence $\theta(X) = 0$ for all $X \in \mathfrak{g}$ which shows that the $G$-action is special lcs by Proposition 3.3. \hfill $\square$

**Remark 4.16.** We could have tried to apply the strategy from Proposition 4.2 to show that the action of $G$ lifts to $\tilde{M}$, and possibly drop the restriction on the lcs rank.
For this, one would first lift the action of $H$ to $\hat{M}$ (this is possible, see Lemma 4.6), then one would need to show that its orbits are included in the fibers of the proper map $\overline{F} : \overline{M} \to H_1(M, \mathbb{R})$, thus are relatively compact. This was achieved in Proposition 4.2 because $\ast V^\flat$ is exact (for $V$ the $s$-Lee vector field), but if we consider in (10) an arbitrary vector field $X$, for $X \in g$, then $\ast X^\flat$ is not necessarily exact, therefore the map $\overline{F}$ might not necessarily be constant along the $H$-orbits and the method fails.

Remark 4.17. The property of being twisted Hamiltonian does not necessarily extend from a dense (disconnected) subgroup $H$ to $G$: if, for example $H \simeq \mathbb{Z}$ is dense in a circle $S^1$ acting on an Inoue surface by special lcK transformations (see [29, Example 5.17]), then the action of the discrete group $H$ is trivially twisted Hamiltonian, while the action of $S^1$ is not.

However, if we consider the special case where $H$ is a group of lcK transformations on a compact Vaisman manifold of rank 1, then $G$ is also special lcK and thus, by Corollary 3.14 twisted Hamiltonian.

4.1. Remarks concerning the lcK setting. An lcK transformation is conformal and biholomorphic, or, equivalently, lcs and biholomorphic. In particular, all the above types of vector fields (lcs, special lcs and twisted Hamiltonian) can be encountered in lcK geometry, but they are lcK if and only if they are, additionally, holomorphic (as usual, we say that a real vector field $X$ is holomorphic if $L_X J = 0$).

Moreover, a group $G$ acts on $(M, \omega, J)$ by lcK transformations (possibly special lcK or twisted Hamiltonian lcK) if, additionally to the conditions considered above in the lcs setting, $G$ acts by biholomorphisms of $(M, J)$. This is a serious restriction for actions of Lee type, because the Lee field needs to be holomorphic. This holds for Vaisman structures, but there are very few non-Vaisman lcK examples with holomorphic Lee field, see [27].

On the other hand, the lcK action of a group $G$ on a compact lcK manifold $(M, \omega, J)$ can always be compactified: $G$, acting by conformal biholomorphisms on $(M, J, g)$, acts by isometries with respect to the Gauduchon metric, thus the closure $\overline{G}$ of $G$ in the (compact) group of isometries of this metric is compact. The Lee type condition (although rare, see [27]) clearly passes to $\overline{G}$, but it is not automatic that the action of $\overline{G}$ is still special or twisted Hamiltonian (if the action of $G$ was of this kind), see, however, Corollary 3.14 and Remark 4.17 above.

5. Convexity of the twisted moment map

The goal of this section is to prove an lcs analog of the classical symplectic convexity theorem of Atiyah [2] and Guillemin-Sternberg [15]:

**Theorem 5.1.** Consider a twisted Hamiltonian action of Lee type of a compact torus $T$ on a compact lcs manifold $(M, \omega, \theta)$, with twisted moment map $\mu : M \to \mathbb{R}^r$. Then $\mu : M \to \mu(M)$ is an open map with connected level sets, and convex image.

In order to prove this theorem we will use the local-global-principle [17], see also [3] [6]. In a form suitable for our purposes it reads

**Theorem 5.2** (Local-global principle). Let $X$ be a connected Hausdorff space and $\mu : X \to \mathbb{R}^k$ a proper map. If every $x \in X$ admits a neighborhood $U$ such that $\mu|_U : U \to \mu(U)$ is open and convex, then $\mu : X \to \mu(X)$ is open and convex.
For completeness, we recall the notion of convexity for maps which is used here.

**Definition 5.3.** A continuous path \( c : [0, 1] \to \mathbb{R}^k \) is *monotone straight* if it is a weakly monotone parameterization of a (possibly degenerate) closed interval, that is if \( c(t_2) \) belongs to the segment from \( c(t_1) \) to \( c(t_3) \) whenever \( 0 \leq t_1 \leq t_2 \leq t_3 \leq 1 \).

A continuous map \( f : X \to \mathbb{R}^k \) is *convex* if for all \( x, y \in X \) there exists a continuous path \( c : [0, 1] \to X \) such that \( c(0) = x, c(1) = y \) and \( f \circ c \) is monotone straight.

Note that this is different from the notion of convexity of real valued functions. We will also call a map \( \mu \) satisfying the assumptions of Theorem 5.2 locally open and locally convex.

This purely topological principle was used to reduce the statement of the classical symplectic convexity to the following well-known local statement, which one obtains by establishing a local normal form for the moment map.

**Lemma 5.4.** [see e.g. Prop. 29 of [6]] A moment map of a Hamiltonian action of a torus \( T \) on a not necessarily compact symplectic manifold \( M \) is locally open and locally convex.

In the situation of Theorem 5.1 we will achieve convexity of the twisted moment map through the following steps:

1. Lift the \( T \)-action to a Hamiltonian action on the minimal covering using Proposition 4.6. Its moment map \( \hat{\mu} \) is locally open and locally convex by Lemma 5.4.
2. Show that \( \mu \) is locally open and locally convex using the projection \( \pi : \hat{M} \to M \) and the relation (11) between the twisted vs. symplectic moment map.
3. Conclude that \( \mu \) is open and convex using the local-global principle in Theorem 5.2.

Observe that in order to apply the local-global principle, it is essential that the map in question is proper. This is automatic for \( \mu \), as we assume the manifold \( M \) to be compact, but not for \( \hat{\mu} \) (see below).

To finish the proof of Theorem 5.1 it remains to show the second step above. For this, we now use the assumption on the action that it is of Lee type: let \( Y \in t \) be the element that induces the \( s \)-Lee vector field. By Remark 3.5, \( \mu : M \to t^\ast \) takes values in the affine subspace \( \{ \alpha \in t^\ast \mid \alpha(Y) = 1 \} \). Recall that by (11), the moment maps \( \mu \) and \( \hat{\mu} \) are related by

\[
\hat{\mu} = e^{-\lambda} \mu \circ \pi,
\]

where \( \pi : \hat{M} \to M \) is the minimal covering. This shows that

\[
\mu \circ \pi = \Psi \circ \hat{\mu},
\]

where the rescaling map \( \Psi : \{ \alpha \mid \alpha(Y) > 0 \} =: \mathcal{H} \to t^\ast \) with image \( \{ \alpha \in t^\ast \mid \alpha(Y) = 1 \} \) is given by

\[
\Psi(\alpha) = \frac{\alpha}{\alpha(Y)}.
\]

We have the following fact.

**Lemma 5.5.** The map \( \Psi \) is open and maps weakly monotone parameterizations of segments in \( \mathcal{H} \) to weakly monotone parameterizations of segments in \( t^\ast \).
Proof. Openness follows by taking a basis \( \{ e_i \} \) of \( t \) and its dual \( \{ e^i \} \) such that \( Y = e_1 \). In these coordinates we have

\[
\Psi(y_1, \ldots, y_k) = \left( 1, \frac{y_2}{y_1}, \ldots, \frac{y_k}{y_1} \right)
\]

and we can see that \( \Psi \) is open.

Clearly, \( \Psi \) is a projective transformation (it is the central projection from the origin onto an affine hyperplane), and as such it maps lines into lines. By continuity, it maps interiors of segments into interiors of segments, thus it is monotone straight. \( \gamma : [0, 1] \to \mathcal{H} \) be a monotone parameterization of a segment. This means that for all \( 0 \leq t_1 \leq t_2 \leq t_3 \leq 1 \), the point \( \gamma(t_2) \) belongs to the segment \( [\gamma(t_1), \gamma(t_3)] \). □

Using the local openness and convexity of \( \hat{\mu} \), we can deduce now the same properties for \( \mu \).

**Lemma 5.6.** The map \( \mu : M \to t^* \) is locally open and locally convex.

Proof. Let \( p \in M \) and choose a point \( \hat{p} \in \hat{M} \) in the fiber over \( p \). By Lemma 5.5 there exists a neighborhood \( \hat{U} \subset \hat{M} \) of \( \hat{p} \) such that the restriction of \( \hat{\mu} \) to \( \hat{U} \) is locally open and locally convex.

Let \( U := \pi(\hat{U}) \), let \( x, y \in U \) and choose \( \hat{x}, \hat{y} \in \hat{U} \) in their fibers. The convexity of \( \hat{\mu} | \hat{U} \) means that there exists a continuous path \( \hat{\gamma} : [0, 1] \to \hat{U} \) such that \( \hat{\gamma}(0) = \hat{x} \), \( \hat{\gamma}(1) = \hat{y} \) and \( \hat{\mu} \circ \hat{\gamma} \) is a monotone parameterization. Let \( \gamma = \hat{\gamma} \circ \pi \) be the projection of the path \( \hat{\gamma} \), connecting \( x \) to \( y \).

From the relation (14), we have that \( \mu \circ \gamma = \Psi \circ \hat{\mu} \circ \hat{\gamma} \) and by Lemmas 5.5 and 5.4 we conclude that \( \mu \circ \gamma \) is a monotone parameterization of a segment. The local openness of \( \mu \) also follows from the one of \( \hat{\mu} \). □

As explained above, this concludes the proof of Theorem 5.1.

 Conversely, the Lee type property can be read off in terms of the image of the twisted moment map.

**Proposition 5.7.** Let \( M \) be an lcs manifold and \( T \) a torus action on it (effectively) in a twisted Hamiltonian fashion and having \( \mu : M \to t^* \) as the twisted moment map. Suppose \( \mu(M) \subset t^* \) is convex and has no interior points. Then the action of \( T \) on \( M \) is of Lee type.

Proof. If \( \mu(M) \) is convex, then there exists an affine subspace \( W \subset t^* \) such that \( \mu(M) \subset W \), and \( W \) is minimal with this property. By assumption, \( W \neq t^* \).

If \( W \subset t^* \) has codimension at least 2, or if \( 0 \in W \), there exists \( X \in t \), \( X \neq 0 \), such that \( \mu(M) \) lies in the annihilator of \( X \), or, equivalently, \( \mu_X \equiv 0 \). This is, however, impossible since the action is effective. Therefore, \( W \) is an affine hyperplane, thus there exists a unique element \( V \in t \) such that \( W = \{ \alpha \in t^* \mid \alpha(V) = 1 \} \). But then \( \mu_V \equiv 1 \), which means that \( V \) is the \( s \)-Lee vector field. □

Note that convexity of \( \mu \) alone does not characterize the Lee type property: as an example, consider \( t_0 \) a complement in \( t \) to \( \mathbb{R} V \), where \( T \) is a torus acting in twisted Hamiltonian fashion of Lee type on \( M \), with twisted moment map \( \mu \). For a countable choice of \( t_0 \), it is the Lie algebra of a subtorus \( T_0 \subset T \). The twisted moment map \( \mu_0 \) of the action of \( T_0 \) is obtained from \( \mu \) after projecting onto \( t_0 = t^*/\text{Ann}(t_0) \). It is thus convex as well. On the other hand, the corresponding projection induces an isomorphism from \( \mathcal{H} \) to \( t_0^* \), thus the image of \( \mu_0 \) contains a
nonempty open set. Depending on the choice of \( T_0 \), 0 may be in the image of \( \mu_0 \) or not.

Finally, we wish to show that the moment image is a convex polytope:

**Proposition 5.8.** In the situation of Theorem 5.1, the moment image \( \mu(M) \) is a convex polytope. It is the convex hull of \( \mu(C) \), where

\[
C = \{ p \in M \mid \overline{X}_{p} \in \mathbb{R} : V_{p} \text{ for all } X \in t \}
\]

is the set of points whose \( T \)-orbits equal their \( V \)-orbits, where \( V \) is the \( s \)-Lee vector field. (Thus the \( T \)-orbits in \( C \) are at most one-dimensional).

**Proof.** We first note that by the Krein-Milman theorem \( \mu(M) \) is the convex hull of its extremal points, i.e. those points \( \alpha \in \mu(M) \) such that whenever \( \alpha = (1-t)\beta + t\gamma \) for some \( 0 < t < 1 \) and \( \beta, \gamma \in \mu(M) \), then \( \beta = \gamma = \alpha \).

On the other hand, any point in \( \mu(M) \) contains a neighborhood in \( \mu(M) \) which looks like the product of a \( k \)-plane with a cone – this follows from the normal form for symplectic moment maps and the fact that by Lemma 5.5 the map \( \Psi \) sends subsets of this form to subsets of this form. This implies that for any extremal point \( \alpha = \mu(p) \) of \( \mu(M) \) we have \( d\mu p = 0 \). Then one has, by the definition of the twisted moment map,

\[
\overline{X}_{p} = -\mu^X(p)\theta_{p} = -\mu^X(p)\iota_{V}\omega_{p}
\]

for all \( X \in t; V \) denotes the \( s \)-Lee vector field. This means that \( \overline{X}_{p} = -\mu^X(p)V \), hence the \( T \)-orbit through \( p \) coincides with the \( V \)-orbit. Thus \( \mu(M) \) is the convex hull of \( \mu(C) \).

It remains to show that \( \mu(M) \) has only finitely many extremal points. For this we note that \( C \) is the union of the (possibly empty) \( T \)-fixed point set \( M^T \) with the fixed point sets \( M^H \), where \( H \) runs through the (finitely many) codimension one isotropy subgroups whose Lie algebra \( \mathfrak{h} \) does not contain \( V \). Each of these fixed point sets is a finite union of submanifolds of \( M \).

We first claim that \( \mu \) sends every component \( N \) of the \( T \)-fixed point set to a single point. Indeed, for any \( p \in N \) we have \( \theta_{p} = 0 \) because the \( s \)-Lee vector field vanishes along \( N \), and hence \( d\mu p = 0 \).

For any codimension one subgroup with \( V \notin \mathfrak{h} \) we observe that the minimal covering \( \pi : \tilde{M} \to M \) restricts, by equivariance of \( \pi \), to a covering \( \pi : \tilde{M}^H \to M^H \). We fix an arbitrary component \( N \) of \( M^H \), as well as a component \( \tilde{N} \) of \( \tilde{M}^H \) that is sent by \( \pi \) to \( N \). Consider the symplectic moment map \( \tilde{\mu} : \tilde{M} \to t^{\star} \) of the lifted \( T \)-action on the minimal covering \( \tilde{M} \). We know that the image of \( d\tilde{\mu}_{q} : T_{q}\tilde{M} \to t^{\star} \), for all \( q \in \tilde{N} \), is contained in the annihilator of \( \mathfrak{h} \), which is a one-dimensional subspace of \( t^{\star} \). In particular, \( \tilde{\mu}(\tilde{N}) \) is a connected, closed subspace of a straight line in \( t^{\star} \).

Using (14)

\[
\mu \circ \pi = \Psi \circ \tilde{\mu}
\]

and that \( \Psi \) sends segments to segments by Lemma 5.5 we conclude that \( \mu(N) \) is a connected, closed subspace of a straight line in \( t^{\star} \) as well. It follows that \( \mu(N) \) can contain at most two extremal points of \( \mu(M) \). As there are in total only finitely many components \( N \) of relevant fixed point submanifolds \( M^H \), we conclude that \( \mu(M) \) has only finitely many extremal points, hence is a convex polytope.
In the proof we observed that the components of $MT$ (if there exist any) are sent by $\mu$ to single points. The same holds for those components $N$ of $M^H$ that are strict lcs submanifolds.

**Proposition 5.9.** Let $T$ be a torus acting in twisted Hamiltonian fashion of Lee type on the lcs manifold $M$, and let $H \subset T$ be a closed subgroup of $T$, such that $\mathfrak{h} \oplus \mathbb{R}V = t$, and let $N$ be a connected component of $M^H := \{x \in M \mid H.x = x\}$ such that $\theta|_N$ is not exact (we say that $N$ is strict lcs). Then $\mu|_N$ is constant.

Note that, from Lemma 5.11 below, we see that $\omega|_N$ is nondegenerate, which means that $(N, \omega|_N)$ is lcs with Lee form $\theta|_N$. It is thus equivalent to say that $N$ is strict lcs and that $\theta|_N$ is not exact.

**Proof.** Let us show that $d\mu_p|_{T_pN} : T_pN \to t^*$ is the zero map for all $p \in N$.

For the $s$-Lee vector field $V$ we have $d\mu_p^V = 0$ because $\mu^V \equiv 1$ by Remark 3.3. For $X \in \mathfrak{h}$ we first claim that $\mu^X(p) = 0$. Indeed, on $N$ we have the equality $d\mu^X = \mu^X \theta$. If $N$ is strict lcs, then the restriction (pullback) of $\theta$ on $N$ is not exact, thus Lemma [4.13] shows that $\mu^X = 0$ on $N$ for all $X \in \mathfrak{h}$.

Therefore, if $p \in N$, then $d\mu_p(X) = 0$, for all $X \in t$, in particular $\mu|_N$ is constant.

**Corollary 5.10.** With the notations above, $N$ is strict lcs if $\theta$ does not vanish on $N$ (here, we mean not just $\theta|_{T_pN} \neq 0$, but $\theta_p \neq 0$ for all $p \in N$).

**Proof.** Suppose $N$ is gcs, thus $\theta|_N = d\beta$, for $\beta : N \to \mathbb{R}$. As $N$ is compact, $\beta$ has at least an extremal point $q$ such that $\theta_q = 0$ on $T_qN$.

On the other hand, we have the following Lemma.

**Lemma 5.11.** Suppose $\omega$ is $T$-invariant, $H \subset T$ is a subgroup and let $q \in M^H$. Then $\theta_q$ is zero on a complement of $T_qM^H$ in $T_qM$.

**Proof.** Consider the decomposition of $T_pM$ into irreducible submodules of the isotropy representation of the identity component of $H$: $T_pM = T_pM^H \oplus \bigoplus_{i=1}^r W_i$, where $\mathfrak{h}$ acts nontrivially on the two-dimensional submodules $W_i$. Note also that $T_pN = \{v \in T_pM \mid [\overline{X}, v]|_p = 0 \text{ for all } X \in \mathfrak{h}\}$.

Note that, because $\overline{X}$ vanishes at $p$, the bracket with $\overline{X}$ at $p$ induces a well-defined linear endomorphism of $T_pM$. Then, any $w \in W_i$ can be written as $w = [X, u]|_p$ for some $X \in \mathfrak{h}$ and $u \in W_i$, and because $L_X\omega = \theta(\overline{X})\omega = 0$ along $N$, we compute $\theta(w) = \omega(V, w) = \omega(V, [\overline{X}, u]) = -\omega(\overline{X}, V, u) = 0$, where we have used the $T$-invariance of $\omega$.

Thus $\theta_q = 0$ on $T_qM$, which is a contradiction, as the $s$-Lee vector field by assumption does not vanish at $q$.

Therefore, we have a clear description of the extremal points of $\mu(M)$, in case the Lee form is non-vanishing.

**Corollary 5.12.** Let $M, T$ as above. If the Lee form $\theta$ does has no zeros, then the extremal points of $\mu(M)$ are $\mu(N)$, for $N$ a connected component of $M^H$, for $H$ a codimension one subgroup of $T$ (that does not contain the flow of the $s$-Lee field $V$).

At the moment, we have no knowledge of an lcs manifold, admitting a twisted Hamiltonian torus action of Lee type, whose Lee form has zeros.
5.1. The image of the symplectic moment map. Let a compact torus $T$ act in twisted Hamiltonian fashion on a compact lcs manifold $(M,\omega)$. Recall that $\hat{\mu}:\tilde{M}\to\mathfrak{t}^*$ is the symplectic moment map of $T$ (acting symplectically on $\tilde{M}$, cf. Proposition 4.9), and that this moment map does not depend on the choice of $\omega$ in its conformal class, moreover it is $T$-equivariant (Proposition 4.9), which means it is constant on $T$-orbits.

This symplectic moment map is thus an object of great interest, and it would be interesting to prove a convexity result for it. As we show below in Example 7.3 its image is not always convex, so additional conditions need to be considered for such a result:

Theorem 5.13. Let $T$ be a compact torus acting in twisted Hamiltonian fashion on a compact, connected lcs manifold $(M,\omega)$ of rank 1, such that the symplectic moment map $\hat{\mu}:\tilde{M}\to\mathfrak{t}^*$ has no zero (in particular if the action is of Lee type for some conformally equivalent $\omega'=e^J\omega$).

Then $\hat{\mu}$ is convex, its image $\hat{\mu}(\tilde{M})\subset\mathfrak{t}^*\setminus0$ is closed, and a convex cone.

If there exists $\omega'=e^J\omega$ such that the action of $T$ on $(M,\omega')$ is of Lee type, then $\hat{\mu}(\tilde{M})$ is the cone over the convex polytope $\mu(M)$, where $\mu$ is the twisted moment map of the $T$-action on $(M,\omega')$.

Proof. As noticed above, the difficulty of applying the local-global principle (Theorem 5.2) to $\hat{\mu}$ is the fact that this map is usually not proper. We show, however, the following.

Lemma 5.14. The symplectic moment map $\hat{\mu}:\tilde{M}\to\mathbb{R}$ of a twisted Hamiltonian action of a compact Lie group $G$ on a compact, connected lcs manifold $(M,\omega)$ is proper if and only if the lcs rank of $M$ is 1 and $\hat{\mu}$ (or, equivalently, $\mu$) has no zero.

Proof. Suppose the lcs rank is 1, and $\mu(M)\subset\mathfrak{g}^*\setminus0$. Let $(x_n)_{n\in\mathbb{N}}$ be sequence in $\tilde{M}$ such that $(y_n)_{n\in\mathbb{N}}$, for $y_n:=\hat{\mu}(x_n)$ is convergent in $\mathfrak{g}^*$. If we denote by $a_n:=e^{-\lambda(x_n)}$ and $z_n:=\mu(\pi(x_n))$, for $\pi:\tilde{M}\to M$, then (11) implies that $y_n=a_nz_n$.

Without loss of generality we can assume that $(\pi(x_n))_{n\in\mathbb{N}}$ is convergent, thus $z_n\to z_0\in\mathfrak{g}^*\setminus0$. This implies that $a_n$ converges as well. Now, using the properness of the map $\lambda$ (Proposition 2.3), we conclude that the sequence $x_n$ admits a converging subsequence, which proves the properness of $\hat{\mu}$.

Conversely, if $\hat{\mu}$ is proper, then we choose the sequence $(x_n)_{n\in\mathbb{N}}$ above such that $\lambda(x_n)$ converges. Thus $(a_n)_{n\in\mathbb{N}}$ converges to $a_0>0$. Moreover, since $M$ is compact, we assume $\pi(x_n)$, thus also $(z_n)$, are convergent sequences. It follows that $y_n$ is convergent thus the properness of $\hat{\mu}$ implies that $(x_n)_{n\in\mathbb{N}}$ admits a converging subsequence, thus $\lambda$ is proper. By Proposition 2.3 the lcs rank of $M$ is 1.

Note that if $\hat{\mu}$ (thus $\mu$) has a zero, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\mu(x_n)\neq0$ and $\mu(x_n)\to0$ (the twisted moment map $\mu$ cannot be constant zero, otherwise all fundamental fields vanish everywhere - contradiction). By possibly replacing $x_n$ by $\gamma_n(x_n)$, with $\gamma_n\in\Gamma$ such that

$$\lim_{n\to\infty}\lambda(x_n)+\rho(\gamma_n)=+\infty,$$

we may assume that $\lambda(x_n)\to+\infty$, which means that $x_n$ cannot have any convergent subsequence. This, combined with $\hat{\mu}(x_n)\to0$, contradicts the properness of $\hat{\mu}$. □
In the lcs rank 1 case we can apply the local-global principle (Theorem 5.2) to the proper map \( \hat{\mu} \) (which is locally open and locally convex by Lemma 5.4), and we conclude that it is convex.

From the convexity of \( \hat{\mu} \) we obtain that its image is convex, i.e. it contains the rays \( \mathbb{R}_+^* \hat{\mu}(x) \), for \( x \in \hat{M} \): indeed, if \( \gamma \) is a generator of \( \Gamma \), \( \hat{\mu}(\gamma^n(x)) = e^{-n\rho(\gamma)} \hat{\mu}(x) \in \hat{\mu}(\hat{M}) \). The image of \( \hat{\mu} \) is thus a cone over a compact set, therefore it is closed in \( \mathfrak{t}^* \). If there additionally exists a 2-form \( \omega \) for which the \( T \)-action is of Lee type, this cone is the cone over a compact set, therefore it is closed in \( \mathfrak{t}^* \). Finally, if the \( T \)-action is of Lee type for \( \omega' = e^F \omega \), the conclusion follows from Theorem 5.15. □

For higher lcs rank, we know that \( \hat{\mu}(\hat{M}) \subset \mathbb{R}_+^* \mu(M) \) is included in the cone over the compact set \( \mu(M) \), and it is dense in this set: as \( \rho(\Gamma) \simeq \mathbb{Z}^k \), with \( k > 1 \), this subgroup is dense in \( \mathbb{R} \), thus

\[
\{ \hat{\mu}(\gamma(x)) \mid \gamma \in \Gamma \}
\]

is dense in the ray \( \mathbb{R}_+^* \mu(x) \). In general, we do not know whether the image of \( \hat{\mu} \) is a cone. We have, however, the following result.

**Theorem 5.15.** Let \( T \) act in twisted Hamiltonian fashion on the compact, connected lcs manifold \( M \) of rank \( k \geq 2 \), such that the Lee form is nowhere-vanishing. Suppose the symplectic moment map (or equivalently, the twisted moment map) has no zero. Then the image of the symplectic moment map \( \hat{\mu} : \hat{M} \to \mathfrak{t}^* \) is a cone without the apex. If there additionally exists a 2-form \( \omega \) for which the \( T \)-action is of Lee type, this cone is the cone over \( \mu(M) \) and is thus convex.

**Proof.** Note that in all regular points of the \( T \)-action on \( \hat{M} \) (denote by \( M_0 \) the open set of these points), the moment map is locally open to its image in \( \mathfrak{t}^* \), so by the density argument above, this implies that \( \hat{\mu}(M_0) \) is a cone.

Consider now \( p \) a point with nontrivial isotropy \( H \). Then \( M^H \) is a finite union of strict lcs submanifolds \( N \) (as in Corollary 5.10 but with arbitrary dimension of \( H \)). Then \( \mu^X \) is zero along \( N \), for all \( X \in \mathfrak{h} \), with the same argument as in Proposition 5.4. This implies that \( \hat{\mu}^X \) is zero along \( \hat{M}^H \). Now, for the symplectic moment map, the image of \( d\hat{\mu}_p \) is the annihilator of the isotropy subalgebra \( t_p = \mathfrak{h} \). This means: we have shown that \( \hat{\mu}(p) \) itself is in the annihilator of \( \mathfrak{h} \), and thus \( \hat{\mu}(p) \) is in the image of \( d\hat{\mu}_p \).

Therefore, if \( \hat{\mu}(p) \neq 0 \), then the vector \( \hat{\mu}(p) \) lies in the image of \( d\hat{\mu}_p \), and thus a small interval around \( \hat{\mu}(p) \), in radial direction, is in the image of \( \hat{\mu} \). Combined with the density property above, this shows that the full radial line \( \mathbb{R}_+^* \hat{\mu}(p) \) is in the image of \( \hat{\mu} \).

Clearly, if the! the action of \( T \) is of Lee type with respect to \( \omega \), then \( \hat{\mu}(\hat{M}) \), which is a cone itself, and is included in the cone over \( \mu(M) \) as a dense subset in it, must coincide with \( \mathbb{R}_+^* \cdot \mu(M) \).

**Remark 5.16.** It is always possible to obtain a twisted Hamiltonian torus action such that the twisted moment map has zeroes: take an action where \( \hat{\mu}(M) \ni 0 \) and consider a subtorus \( T' \subset T \) such that the annihilator of \( t' \) in \( \mathfrak{t}^* \) lies in \( \hat{\mu}(M) \). Then
the twisted moment map of the action of $T'$ is $\mu' = q \circ \mu$ (where $q : t^* \to t^*$ is the dual of the inclusion $t' \hookrightarrow t$) and it has zeroes.

6. THE TWISTED MOMENT MAP OF VAISMAN MANIFOLDS

In this section we consider an effective, holomorphic and twisted Hamiltonian action of a compact connected Lie group $G$ on a compact Vaisman manifold $(M, g, J, \theta)$, with twisted moment map $\mu : M \to t^*$.

In the Vaisman setting, such an action is automatically isometric and every field induced by $g$ lies in the kernel of the Lee form, because a Vaisman metric is Gauduchon (cf. comments at the end of Section 2, see also [31, Lemma 4.2]).

We assume without loss of generality that the Lee vector field is of unit length. Let $S$ be the associated Sasakian manifold, as a codimension one submanifold of the minimal covering $\hat{M}$ of $M$. In Section 2.1 it is explained that $\hat{M}$ is the cone $C(S)$.

The explicit expression of the twisted moment map on $M$ is, for $X \in g$, given by Proposition 3.13 (see also [31, Lemma 4.4] with a different sign convention)

$$\mu^X = -\theta(JX).$$

Let $\lambda$ be the coordinate on the $\mathbb{R}$-factor on the minimal covering $\hat{M} = \mathbb{R} \times S$. On the other hand, on $\hat{M}$ the lift of $\theta$ is exactly $d\lambda$ and the Kähler form is exact and given by $d(e^{-2h}\eta)$, where $\eta = -d^c h$ (see e.g. [32]). Note that $\eta$ is precisely the contact form of the associated Sasakian manifold $S$. As shown in Lemma 4.6, see also [25], the $G$-action lifts to $\hat{M}$, and the lifted action respects the Sasakian leaves. The contact moment map for the induced action on $S$ is

$$X \mapsto \eta(X) = -d^c t(X) = -\theta(JX).$$

As the moment map $\mu$ is $\theta^2$-invariant, the following follows.

**Proposition 6.1.** The twisted Hamiltonian action of a compact connected Lie group $G$ on a complete Vaisman manifold by Vaisman automorphisms and the induced action on its associated Sasakian manifold share the same moment image in $g^*$.  

**Remark 6.2.** By Proposition 2.11 in the case of Vaisman manifolds of lcK rank one, the associated Sasakian manifold is compact. Thus, in this case the Convexity Theorem 5.1 follows trivially from known convexity results for compact Sasakian manifolds and Proposition 6.1.

For the remaining of this section, we focus on the relation between the twisted moment image of $\mu : M \to g^*$ and the Kähler moment image of $\hat{\mu} : \hat{M} \to g^*$. Consider the vector field $X$ given by the gradient of $h$ with respect to the gcK metric $\pi^* g$. This is nothing but the metric dual with respect to $\pi^* g$ of the lift of the Lee form, and it satisfies $\pi_* X = \theta^2$. Let $\varphi : \mathbb{R} \times \hat{M} \to \hat{M}$ be the flow of $X$.

**Lemma 6.3.** We have $\hat{\mu}^Z \circ \varphi_* = e^{-h}\hat{\mu}^Z$ for all $Z \in g$.

**Proof.** We have that $\mu$ is $\theta^2$-invariant and, by Equation (11), that $\hat{\mu} = e^{-h} \mu \circ \pi$. Then we compute

$$X(\mu^Z) = X(e^{-h} \mu^Z \circ \pi) = X(e^{-h}) \mu^Z \circ \pi = -X(h) \mu^Z = -\hat{\mu}^Z,$$

using that the Lee vector field has constant length 1. $\square$

The following result is, in fact, a particular case of Theorem 5.15.
Proposition 6.4. We have $\hat{\mu}(\hat{M}) = \mathbb{R}^0 \cdot \mu(M)$.

Proof. Consider the associated Sasakian manifold $S = h^{-1}(0)$. Because $\hat{\mu} = e^{-h} \mu \circ \pi$ and $\mu$ is $\theta^\sharp$-invariant, we have $\hat{\mu}(S) = \mu(\pi(S)) = \mu(M)$. Then Lemma 6.3 implies the claim. □

A Vaisman manifold is strongly regular if both $\theta^\sharp$ and $J\theta^\sharp$ induce a free circle action. In this case we can consider the projection space $\pi : M \to \hat{M} = M/\langle \theta^\sharp, J\theta^\sharp \rangle$ to the orbit space of the associated $T^2$-action, which is a Kähler manifold. As before, we consider the action of a compact connected Lie group $G$ on $M$ by Vaisman automorphisms which is twisted Hamiltonian, with moment map $\mu : M \to \mathfrak{g}^*$. We have then the following result, essentially contained in [31].

Proposition 6.5. Let $(M, g, J, \theta)$ be a strongly regular Vaisman manifold with an effective twisted Hamiltonian $G$-action by Vaisman automorphisms. Then the $G$-action descends to the Kähler manifold $\hat{M}$. It is Hamiltonian, with a moment map $\hat{\mu} : \hat{M} \to \mathfrak{g}^*$ defined by $\mu = \hat{\mu} \circ \pi$. In particular, the moment images satisfy $\hat{\mu}(\hat{M}) = \mu(M)$.

Proof. As the action is by Vaisman automorphisms, the vector fields $\theta^\sharp$ and $J\theta^\sharp$ are central in $\text{aut}(M, J, g)$. So the $G$-action descends to a Kähler action on $\hat{M}$.

The moment map $\mu : M \to \mathfrak{g}^*$ descends to a map $\hat{\mu} : \hat{M} \to \mathfrak{g}^*$. Indeed, by the definition of twisted moment maps and recalling Equation (15) and that $\mathfrak{g} \subseteq \ker \theta$, we can verify that it is invariant under the action of $\theta^\sharp$ and $J\theta^\sharp$.

The proof that $\hat{\mu}$ is a moment map for the $G$-action on $\hat{M}$ is done along the same line of the first part of the proof of Theorem 5.1 of [31]. □

Remark 6.6. If the action is additionally of Lee type, then the subgroup spanned by (the element that induces) $J\theta^\sharp$ is a subgroup $H \subset G$ isomorphic to $S^1$, which is central in $G$ and acts trivially on $\hat{M}$. We define $\hat{G} = G/H$, and consider the projection $p : G \to \hat{G}$.

For $Z \in \mathfrak{h}$ we have $d_\theta \mu^Z = i_{\pi^*Z} \omega = 0$, because $H$ acts trivially on $\hat{M}$. As $d_\theta$, by Lemma 3.6, is injective on functions, this implies that $\mu^Z = 0$. It follows that we can define a map $\hat{\mu} : M \to \hat{g}^*$ such that $p^* \circ \hat{\mu} = \hat{\mu}$; it is a moment map for the $\hat{G}$-action on $\hat{M}$. Then the moment images are related by $p^*(\hat{\mu}(\hat{M})) = \hat{\mu}(\hat{M}) = \mu(M)$.

6.1. Toric Vaisman manifolds. In this subsection, we focus on toric Vaisman manifolds.

Definition 6.7. A connected, compact lcs, resp. lcK manifold of real dimension $2n$ is toric if it admits an effective (for lcK also holomorphic) and twisted Hamiltonian action of the $n$-dimensional compact torus.

Remark 6.8. In [18] it is shown that toric lcK manifolds admit a toric Vaisman structure, and an example of a toric lcs manifold is given which does not admit a toric lcK structure. We consider this example below in Example 7.3.

Let $M$ be a toric Vaisman manifold, acted on by a compact torus $T$. In [25] Prop. 5.4], it is proven that $M$ is the mapping torus of its associated toric Sasakian
manifold $S$ with respect to a $T$-equivariant Sasakian automorphism $\psi$ such that the map $f(t, p) = (t + \alpha, \psi(p))$ generates the $\mathbb{Z}$-action on the Kähler cone $\mathbb{R} \times S$. The purpose of this section is to show that the Sasakian automorphism is necessarily given by an element of $T$.

Let $S$ be a $(2n + 1)$-dimensional toric Sasakian manifold acted by a $(n + 1)$-dimensional torus $T$. We denote a moment map of the induced action on the Kähler cone by $\mu: C(S) \to \mathbb{R}^{n+1}$. Let $C$ be the image of the moment map of the Kähler cone $C(S)$. Then the open set

$$C(S)^0 = \mu^{-1}(C^0) = \{(t, p) \in C(S) \mid T_{(t,p)} = 1\},$$

consists of the points in which the $T$-action is free.

Following Abreu [1], $C(S)^0$ is described by action-angle coordinates $x = \mu \in C^0$ and $y \in T$. On $C(S)^0$, the $T$-action is given by torus translations on the second component.

Moreover, the moment map has the form $\mu^X(t, p) = e^{-2t} \eta_p(\mathcal{X}_p)$, where $\eta$ is the contact form of $S$ and by the bar we mean the induced field on $S$.

**Lemma 6.9.** For any $\alpha \in \mathbb{R}$, the transformation $f(t, p) = (t + \alpha, \psi(p))$ is a $T$-equivariant holomorphic homothety of the toric Kähler cone $C(S)$ if and only if it is induced by an element of $T$.

**Proof.** In the $(x, y)$-coordinates, the Kähler form on $C(S)^0$ has the form

$$\omega_{(x,y)} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

and the complex structure is

$$J_{(x,y)} = \begin{pmatrix} 0 & -S(x)^{-1} \\ S(x) & 0 \end{pmatrix}$$

where $S(x)$ is the Hessian of a smooth function $s: \mathbb{C}^0 \to \mathbb{R}$, degree $-1$, i.e. $S(e^{2t} x) = e^{-2t} S(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{C}^0$.

Being $f(t, p) = (t + \alpha, \psi(p))$, that is a translation on the $\mathbb{R}$-factor and a $T$-equivariant contactomorphism on $S$, it is clear that it satisfies $\mu \circ f = e^{-2\alpha} \mu$, so in action-angle coordinates it has the form

$$f(x, y) = (e^{-2\alpha} x, f_2(x, y)).$$

Imposing $T$-equivariance, we have the condition $f_2(x, y + y_0) = f_2(x, y) + y_0$ and we obtain, by differentiation, that

$$\frac{\partial f_2}{\partial y} (x, y) = I.$$

Note that here we have identified $f_2: \mathbb{C}^0 \times T \to T$ with its lift $f_2: \mathbb{C}^0 \times \mathbb{R}^n \to \mathbb{R}^n$ and that the plus sign denotes addition in $\mathbb{R}^n$.

---

1The Riemannian geometric part of this statement follows from the classical result in [21 Chap. VI, Sec. 3, Lemma 2], that was extended to the classification of homotheties of a Riemannian cone in [13 Thm. 5.1]. The fact that it is a contact transformation is equivalent to the cone homothety being holomorphic and follows from the well-known relations between contact structures and their symplectizations, see e.g. [11 Chap. 6].
Denote $A(x, y) = \frac{\partial x}{\partial t}(x, y)$. Swapping partial derivatives in [19] we obtain that $A$ does not depend on $y$. Hence the differential of $f$ has the form

$$
df_{(x, y)} = \begin{pmatrix} e^{-2\alpha} I & 0 \\ A(x) & 1 \end{pmatrix}. $$

The condition that $f$ is holomorphic, i.e. $df_{(x, y)} J (x, y) = J f_{(x, y)} df_{(x, y)}$ translates into

$$
\begin{pmatrix} e^{-2\alpha} I & 0 \\ A(x) & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -S(x)^{-1} \\ S(x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -S(e^{-2\alpha}x)^{-1} \\ S(e^{-2\alpha}x) & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{-2\alpha} I & 0 \\ A(x) & 1 \end{pmatrix}
$$

that implies $A = 0$. Note that $df$ is a homothety of $\omega$ with factor $e^{-2\alpha}$.

Hence, on $C(S)^0$, the map $f$ has the form $f(x, y) = (e^{-2\alpha}x, y + y_0)$ for some $y_0 \in T$. So, $f$ coincides on $C(S)^0 \cap S$ with the transformation induced by the action of the torus element $y_0$. □

We can thus infer the following.

**Lemma 6.10.** Any equivariant Sasakian automorphism of a toric Sasakian manifold is equivariantly isotopic to the identity.

**Proof.** Given such an automorphism $\psi$, extend it to a homothety $f(t, p) = (t + \alpha, \psi(p))$ on the Kähler cone.

By Lemma 6.9, $\psi$ is defined by the action of an element $\sigma \psi \in T$. Then the map $\psi_t(x) = \sigma(t)$ is a $T$-equivariant isotopy between $\psi$ and the identity, where $\sigma(t)$ is any smooth curve joining $\sigma$ and $1$ in $T$. □

**Theorem 6.11.** Let $M$ be a compact toric Vaisman manifold. Then the following hold.

1. $M$ is isomorphic to the mapping torus of a toric Sasakian manifold $S$ by an element $g \in T$: it is $M = S_{\alpha, g} := \mathbb{R} \times (t, p) \sim (t + \alpha, gp)$ with the natural Vaisman structure.
2. $M$ is diffeomorphic, as a smooth manifold, to the direct product of a toric Sasakian manifold and a circle.
3. Two toric Vaisman manifolds $S_{\alpha, g}$ and $W_{\beta, h}$ are $T$-equivariantly isomorphic as Vaisman manifolds if and only if $S = W$, $\alpha = \beta$ and $g = h$.

**Proof.** The first statement is [23 Prop. 5.4], together with Lemma 6.9. The second one is Lemma 6.10. For the last one, observe first that an isomorphism of Vaisman manifolds induces an isomorphism of the associated Sasakian manifolds. The number $\alpha$ is just the length of the flow of the Lee vector field, so in particular an invariant of the Vaisman structure. Now, considering two Vaisman manifolds $S_{\alpha, g}$ and $S_{\alpha, h}$, any equivariant isomorphism $f$ lifts to an equivariant holomorphic homothety of the associated toric Kähler cone $C(S)$. Then, by Lemma 6.9 there exists a $\sigma \in T$ such that $F$ is induced by $(t, p) \mapsto (t + \gamma, \sigma(p))$, for some $\gamma \in \mathbb{R}$. The Kähler cone transformations defined by $g$ and $h$ are conjugated under $F$, so in $T$, the elements $g$ and $h$ are conjugated under $\sigma$. But, $T$ being Abelian, it follows that $g = h$. □

**Remark 6.12.** It is well-known, see e.g. [10] or [26], that the choice of a strongly convex polyhedral cone $C$ together with a point $\xi$ in its interior $C^0$ defines a toric Sasakian manifold and that there is no uniqueness in this construction. All toric Sasakian manifolds with given Reeb field and moment cone are parameterized by...
Let $\mathcal{C}_0^* \times \mathcal{H}(1)$, where $\mathcal{C}_0^*$ is the interior of the dual cone $\mathcal{C}^*$ and $\mathcal{H}(1)$ is the space of smooth homogeneous degree one functions on $\mathcal{C}$, such that their Hessian is strictly convex. Then the theorem shows that the collection of toric Vaisman manifolds, up to $T$-equivariant Vaisman isomorphism, with moment image $\mathcal{C} \cap \chi_\xi$ is in a one-to-one correspondence with $\mathcal{C}_0^* \times \mathcal{H}(1) \times T$.

**Remark 6.13.** Part (2) of Theorem 6.11 can give information about the topology of a compact toric Vaisman manifold.

Combining Theorem 6.11 with results about the topology of toric contact manifolds (see e.g. [8, 23]) we can easily obtain information about the topology of a toric Vaisman manifold $M$. In particular, we reobtain the fact that $b_1(M) = 1$, already proven in [25], and one can compute the first and second homotopy group of $M$ in terms of the combinatorial data of the moment cone. Moreover, in [24], it is proved that the fundamental group of a toric contact manifold of Reeb type is finite cyclic of order, say, $k$ and it is explained how to read $k$ from the moment data.

### 7. Examples

In this section we provide examples of twisted Hamiltonian actions on various lcs and lcK structures. We begin with the most standard example of a Vaisman manifold.

**Example 7.1 (The weighted Hopf manifold).** Consider the action on $\mathbb{C}^n \setminus \{0\}$ given by

$$\gamma(z) = (a_1z_1, \ldots, a_nz_n),$$

for some complex numbers $a_j$ such that $0 < |a_j| < 1$ and let $\Gamma$ be the subgroup of holomorphic isometries of $\mathbb{C}^n \setminus \{0\}$ generated by $\gamma$.

In [20, Thm. B], it is proved that $M = (\mathbb{C}^n \setminus \{0\})/\Gamma$ carries a Vaisman structure, whose associated Sasakian manifold is the weighted Sasakian sphere $S_w^{2n-1}$ with weights $w_j = \log |a_j|$, Reeb field

$$\xi_w = \sum_{j=1}^{n} w_j \left( x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right)$$

and contact form

$$\eta_w = \frac{1}{\sum_j w_j |z_j|^2} \eta_0,$$

where $\eta_0$ is the standard contact form on $S^{2n-1}$.

On both $M$ and $S_w^{2n-1}$, the standard torus $T^n$ is taken with its standard action. The image of the twisted moment map of $M$ is, by Theorem 6.4, the same as the contact moment map, that is given by the intersection of the first octant $\{x \in \mathbb{R}^n \mid x_j > 0\}$ with the characteristic hyperplane $\{x \mid \langle x, w \rangle = 1\}$.

The construction of [20] generalizes the one of [12] for $n = 2$. Moreover, in [24], it is proven that this weighted Hopf manifold, for $n = 2$, is the only toric Vaisman surface.

**Example 7.2.** Changing conformally the form $\omega$ leads to a corresponding change of the twisted moment map, see Remark 3.7. We give here an example where the process leads to a twisted moment map with non-convex image (the Lee type condition is lost as well): We start with the standard Vaisman structure $(g, J)$ on a diagonal Hopf surface $M$, i.e., we let $n = 2$ and $a_1 = a_2$ in Example 7.1. Let
Let $d\theta$ be the twisted moment map of the standard $T^2$-action which is of Lee type, given by
\[
\mu(z) = \frac{1}{\|z\|^2}(|z_1|^2, |z_2|^2).
\]

Let $f: M \to \mathbb{R}$ be a smooth function of the variable $t = |z_1|^2/\|z\|^2$. The new lcK structure $(e^{-f}g, J)$ is then still lcK and acted upon by the same two-dimensional torus, with twisted moment map $\tilde{\mu} := e^{-f}\mu: M \to \mathbb{R}^2$. The action on the new structure is not of Lee type.

The image of $\tilde{\mu}$ is
\[
e^{-f}(\mu(M) = \{e^{-f(t)}(t, 1-t) \mid t \geq 0\}
\]
and we note that it is a convex subset of the plane if and only if it is a piece of a straight line, which happens exactly when $f$ is constant.

**Example 7.3.** The following example was given in [18 Ex. 6.4], as a toric lcs manifold that does not admit a toric lcK metric. Let the standard 2-torus $T = \mathbb{T}^2$ act on the compact manifold $M$ and we note that it is a convex subset of the plane if and only if it is a piece of a straight line.

The following example was given in \[27, Section 3\], the authors start with a Vaisman structure with Lee field $V$ and deform it, in a non-conformal way, to a non-Vaisman lcK structure with the same Lee field. They use a non-constant function $f: M \to \mathbb{R}$ with $f > -1$ and gradient collinear with $V$.

$\mu: M \to \mathbb{R}^2$ be the twisted moment map of the standard $T^2$-action which is of Lee type, given by
\[
\mu(z) = \frac{1}{\|z\|^2}(|z_1|^2, |z_2|^2).
\]

They use a non-constant function $f: M \to \mathbb{R}$ to deform it, in a non-conformal way, to a non-Vaisman lcK structure with the same Lee field. They use a non-constant function $f: M \to \mathbb{R}$ with $f > -1$ and gradient collinear with $V$. 
In detail, they take the new lcK form \( \omega = \omega + f \theta \wedge J \theta \), with Lee form \( \theta = (1 + f) \theta \).
We claim that if \((M, J, \omega, \theta)\) admits a holomorphic twisted Hamiltonian action of Lee type of a Lie group \( G \), then the same action is still twisted Hamiltonian with respect to the new lcK structure, with the same moment map.

Let \( X \in \mathfrak{g} \). The twisted moment map of the \( G \)-action, with respect to the Vaisman structure, is (see (15)) given by \( \mu^X = i_X \theta = -\theta(J X) \), and we note that \( X \in \ker \theta \), by [31, Lemma 4.2]. We compute
\[
\begin{align*}
\iota_X \omega &= \iota_X \omega + f \iota_X (\theta \wedge J \theta) \\
&= \iota_X \omega - f \theta \wedge \iota_X J \theta \\
&= d_\theta \mu^X - f \mu^X \theta \\
&= d \mu^X - \mu^X \theta - f \mu^X \theta \\
&= d \mu^X - (1 + f) \mu^X \theta \\
&= d_\theta \mu^X.
\end{align*}
\]

The authors give an explicit example of such a function \( f \) on the complex \( n \)-dimensional Hopf manifold. In this way they obtain a toric non-Vaisman lcK structure on the Hopf manifold. Its moment map is the same as in the Vaisman case, see Example 7.1.

References

[1] M. Abreu, Kähler-Sasaki geometry of toric symplectic cones in action-angle coordinates, Port. Math. 67 (2010), no. 2, 121–153. MR 2662864
[2] M. F. Atiyah, Convexity and commuting Hamiltonians, Bull. London Math. Soc. 14 (1982), no. 1, 1–15. MR 642416
[3] G. Bazzoni and O. Goertsches, Toric actions and convexity in cosymplectic geometry, arXiv:1712.08141.
[4] F. A. Belgun, On the metric structure of non-Kähler complex surfaces, Math. Ann. 317 (2000), no. 1, 1–40. MR 1760667
[5] P. Birtea, J.-P. Ortega, and T. S. Ratiu, Openness and convexity for momentum maps, Trans. Amer. Math. Soc. 361 (2009), no. 2, 603–630. MR 2452817
[6] C. Bjorndahl and Y. Karshon, Revisiting Tietze-Nakajima: local and global convexity for maps, Canad. J. Math. 62 (2010), no. 5, 975–993. MR 2730351
[7] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, second ed., Progress in Mathematics, vol. 203, Birkhäuser Boston, Inc., Boston, MA, 2010. MR 2682326
[8] C. Boyer and K. Galicki, Sasakian Geometry, Oxford Science Publications, 2007.
[9] S. Dragomir and L. Ornea, Locally Conformal Kähler Geometry, Progress in Mathematics, Birkhäuser Boston, 2012.
[10] A. Futaki, H. Ono, and G. Wang, Transverse Kähler geometry of Sasaki manifolds and toric Sasaki–Einstein manifolds, J. Diff. Geom. 83 (2009), 585–635.
[11] P. Gauduchon, Calabi’s extremal Kähler metrics: An elementary introduction, To appear.
[12] P. Gauduchon and L. Ornea, Locally conformally Kähler metrics on Hopf surfaces, Ann. Inst. Fourier (Grenoble) 48 (1998), no. 4, 1107–1127. MR 1656010 (2000g:53088)
[13] R. Gini, L. Ornea, M. Parton, and P. Piccinni, Reduction of Vaisman structures in complex and quaternionic geometry, J. Geom. Phys. 56 (2006), no. 12, 2501–2522. MR 2252875
[14] V. Guillemin, E. Miranda, and J. Weitsman, Convexity of the moment map image for torus actions on \( b^n \)-symplectic manifolds, arXiv:1801.01097.
[15] V. Guillemin and S. Sternberg, Symplectic techniques in physics, Cambridge University Press, 1984.
[16] S. Haller and T. Rybicki, Reduction for locally conformal symplectic manifolds, J. Geom. Phys. 37 (2001), no. 3, 202–271. MR 1807280
[17] J. Hilgert, K. Neeb, and W. Plank, Symplectic convexity theorems, Sem. Sophus Lie 3 (1993), no. 2, 123–135. MR 1270171
[18] N. Istrati, A characterisation of toric LCK manifolds, arXiv:1612.03832, 2016.
[19] N. Istrati and A. Otiman, De Rham and twisted cohomology of Oeljeklaus-Toma manifolds, arXiv:1711.07847, 2017.
[20] Y. Kamishima and L. Ornea, Geometric flow on compact locally conformally Kähler manifolds, Tohoku Math. J. (2) 57 (2005), no. 2, 201–221. MR 2137466
[21] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Interscience Publisher, 1963.
[22] E. Lerman, A convexity theorem for torus actions on contact manifolds, Illinois J. Math. 46 (2002), no. 1, 171–184. MR 1936083
[23] Homotopy groups of K-contact toric manifolds, Trans. Amer. Math. Soc. 356 (2004), no. 10, 4075–4083. MR 2058839
[24] H. Li, The fundamental group of toric contact manifolds, arXiv:1703.07277v2.
[25] F. Madani, A. Moroianu, and M. Pilca, On toric locally conformally Kähler manifolds, Ann. Global Anal. Geom. 51 (2017), no. 4, 401–417. MR 3648998
[26] D. Martelli, J. Sparks, and S.-T. Yau, The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds, Comm. Math. Phys. 268 (2006), no. 1, 39–65. MR 2249795
[27] A. Moroianu, S. Moroianu, and L. Ornea, Locally conformally Kähler manifolds with holomorphic Lee field, arXiv:1712.05821, 2017.
[28] Y. Nitta, Convexity properties of generalized moment maps, J. Math. Soc. Japan 61 (2009), no. 4, 1171–1204. MR 2588508
[29] A. Otiman, Locally conformally symplectic bundles, arXiv:1510.02770, to appear in J. Sympl. Geom., 2015.
[30] R.S. Palais and C. Terng, Critical Point Theory and Submanifold Geometry, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 1988.
[31] M. Pilca, Toric Vaisman manifolds, J. Geom. Phys. 107 (2016), 149–161.
[32] J. Sparks, Sasaki-Einstein manifolds, Surv. Differ. Geom. 16 (2011), 265–324.
[33] I. Vaisman, Locally conformal symplectic manifolds, Internat. J. Math. Math. Sci. 8 (1985), no. 3, 521–536. MR 809073

(F. BELGUN) INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY - 21 CALEA GRIVITEI STREET, 010702 BUCHAREST, ROMANIA.
E-mail address: florin.belgun at math.uni-hamburg.de

(O. GOERTSCHES) FACHBEREICH MATHEMATIK UND INFORMATIK DER PHILIPPS-UNIVERSITÄT MARBURG - HANS-MEERWEIN-STRASSE 6, MARBURG, GERMANY.
E-mail address: goertsch at mathematik.uni-marburg.de

(D. PETRECCA) INSTITUT FÜR DIFFERENTIALGEOMETRIE, LEIBNIZ UNIVERSITÄT HANNOVER - WELFENGARTEN 1, HANNOVER, GERMANY.
E-mail address: petrecca at math.uni-hannover.de