SPECIAL HERMITIAN METRICS AND LIE GROUPS

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Abstract. A Hermitian metric on a complex manifold is called strong Kähler with torsion (SKT) if its fundamental 2-form $\omega$ is $\partial\bar{\partial}$-closed. We review some properties of strong KT metrics also in relation with symplectic forms taming complex structures. Starting from a $2n$-dimensional SKT Lie algebra $\mathfrak{g}$ and using a Hermitian flat connection on $\mathfrak{g}$ we construct a $4n$-dimensional SKT Lie algebra. We apply this method to some 4-dimensional SKT Lie algebras. Moreover, we classify symplectic forms taming complex structures on 4-dimensional Lie algebras.

1. Introduction

A $J$-Hermitian metric $g$ on a complex manifold $(M, J)$ is called SKT (strong Kähler with torsion) or pluriclosed if the fundamental 2-form $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ satisfies

$$\partial\bar{\partial}\omega = 0$$

(see for instance [18]). For complex surfaces a Hermitian metric satisfying the SKT condition is standard in the terminology of Gauduchon [20] and on a compact manifold a standard metric can be found in the conformal class of any given Hermitian metric. However, the theory is completely different in higher dimensions. The study of SKT metrics is strictly related to the study of the geometry of the Bismut connection. Indeed, any Hermitian manifold $(M, J, g)$ admits a unique connection $\nabla^B$ preserving $g$ and $J$ and such that the tensor

$$c(X, Y, Z) = g(X, T^B(Y, Z))$$

is totally skew-symmetric, where by $T^B$ we denote the torsion of $\nabla^B$ (see [19]). This connection was introduced by Bismut in [5] to prove a local index formula for the Dolbeault operator for non-Kähler manifolds. The torsion 3-form $c$ is related to the fundamental form $\omega$ of $g$ by

$$c(X, Y, Z) = -d\omega(JX, JY, JZ)$$

and it is well known that $\partial\bar{\partial}\omega = 0$ is equivalent to $dc = 0$.

SKT metrics have a central role in type II string theory, in 2-dimensional supersymmetric $\sigma$-models (see [18, 34]) and they have also relations with generalized Kähler geometry (see for instance [18, 22, 25, 1, 6, 15]). Indeed, by [22, 1] it follows that a generalized Kähler structure on a $2n$-dimensional manifold $M$ is equivalent to a pair of SKT structures $(J_+, g)$ and $(J_-, g)$ such that $d_+^-\omega_+ = -d^-_+\omega_-$, where $\omega_{\pm}(\cdot, \cdot) = g(J_{\pm}\cdot, \cdot)$ are the fundamental 2-forms associated to the Hermitian structures $(J_{\pm}, g)$ and $d_+^- = i(\partial_{\pm} - \partial_{\pm})$. The closed 3-form $d_+^-\omega_+$

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is called the *torsion* of the generalized Kähler structure and the structure is said *untwisted* or *twisted* according to the fact that the cohomology class \([d^c_+ \omega_+] \in H^3(M, \mathbb{R})\) vanishes or not. In particular, any Kähler metric \((J, g)\) determines a generalized Kähler structure by setting \(J_+ = J\) and \(J_- = \pm J\).

Recently, SKT metrics have been studied by many authors. For instance, new simply-connected compact SKT examples have been constructed by Swann in [35] via the twist construction and SKT structures on \(S^1\)-bundles over almost contact manifolds have been studied in [12]. Moreover, in [16] it has been shown that the blow-up of an SKT manifold at a point or along a compact submanifold admits an SKT metric.

For real Lie groups admitting left-invariant SKT metrics there are some classification results in dimension 4, 6 and 8. More precisely, 6-dimensional (resp. 8-dimensional) SKT nilpotent Lie groups have been classified in [14] (resp. in [11] and for a particular class in [31]) and a classification of SKT solvable Lie groups of dimension 4 has been obtained in [30].

General results are known also for nilmanifolds, i.e. compact quotients of simply connected nilpotent Lie groups \(G\) by discrete subgroups \(\Gamma\). Indeed, in [11] it has been proved that if \((M = G/\Gamma, J)\) is a nilmanifold (not a torus) endowed with an invariant complex structure \(J\) and if there exists a \(J\)-Hermitian SKT metric \(g\) on \(M\), then \(G\) must be 2-step nilpotent and \(M\) is a total space of a principal holomorphic torus bundle over a torus.

No general restrictions for the existence of SKT and generalized Kähler structures are known in the case of solvmanifolds, i.e. compact quotients of solvable Lie groups by discrete subgroups. A structure theorem by [23] states that a solvmanifold carries a Kähler structure if and only it is covered by a complex torus which has a structure of a complex torus bundle over a complex torus.

As far as we know, the only known solvmanifolds carrying a generalized Kähler structure are the Inoue surface of type \(S^0\) defined in [27] and \(T^{2k}\)-bundles over the Inoue surface of type \(S^0\) constructed in [15].

A quaternionic analogous of Kähler manifolds is given by *hyper-Kähler with torsion* (shortly *HKT*) manifolds, that are hyper-Hermitian manifolds \((M^{4n}, J_1, J_2, J_3, h)\) admitting a hyper-Hermitian connection with totally skew-symmetric torsion, i.e. for which the three Bismut connections associated to the three Hermitian structures \((J_r, h)\), \(r = 1, 2, 3\) coincide. This geometry was introduced by Howe and Papadopoulos [20] and later studied for instance in [21, 13, 3, 4, 35]. In [4] it was shown that the tangent Lie algebra of an HKT Lie algebra may admit an HKT structure, constructing in this way a family of new compact strong HKT manifolds.

In this paper we adapt the previous construction to the SKT and generalized Kähler case. Starting from a \(2n\)-dimensional SKT (generalized Kähler) Lie algebra \(\mathfrak{g}\) and using a suitable connection on \(\mathfrak{g}\) we construct a \(4n\)-dimensional SKT (generalized Kähler) Lie algebra. We apply the previous procedure to some of the 4-dimensional SKT Lie algebras, obtaining in this way new SKT examples in dimension 8 and recovering the generalized Kähler example found in [15].

The existence of an SKT metric \(\omega\) on a complex manifold \((M, J)\) such that \(\partial \omega = \overline{\partial} \beta\) for a \(\partial\)-closed \((2,0)\)-form \(\beta\) is equivalent to the existence of a symplectic form taming \(J\) ([11]).
We recall that an almost complex structure $J$ on a compact $2n$-dimensional symplectic manifold $(M, \Omega)$ is said to be tamed by $\Omega$ if

$$\Omega(X, JX) > 0$$

for any non-zero vector field $X$ on $M$. When $J$ is a complex structure (i.e. $J$ is integrable) and $\Omega$ tames $J$, the pair $(\Omega, J)$ has been called a Hermitian-symplectic structure in [32]. Although any symplectic structure always admits tamed almost complex structures, it is still an open problem to find an example of a compact complex manifold admitting a taming symplectic structure but no Kähler structures. From [32, 28] there exist no compact examples in dimension 4. Moreover, the study of taming symplectic structures in dimension 4 is related to a more general conjecture of Donaldson (see for instance [8, 36, 28]).

In [11] some negative results for the existence of taming symplectic structures on compact quotients of Lie groups by discrete subgroups were obtained. It was shown that if $M$ is a nilmanifold (not a torus) endowed with an invariant complex structure $J$, then $(M, J)$ does not admit any symplectic form taming $J$.

The taming symplectic structures are related to static solutions of a new metric flow on complex manifolds (see [32]). Indeed, Streets and Tian constructed an elliptic flow using the Ricci tensor associated to the Bismut connection instead of the Levi-Civita connection, and it turns out that this flow preserves the SKT condition and that the existence of some particular type of static SKT metrics implies the existence of a taming symplectic structure on the complex manifold ([33]). Static SKT metrics on Lie groups have been also recently studied in [10].

In the last section of the paper we prove that a 4-dimensional Lie algebra $\mathfrak{g}$ endowed with a complex structure $J$ admits a taming symplectic structure if and only if $(\mathfrak{g}, J)$ admits a Kähler metric. Moreover, under this condition, every SKT metric induces a symplectic form taming $J$.

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2. Preliminaries

Let $M$ be a $2n$-dimensional manifold. We recall that an almost complex structure $J$ on $M$ is integrable if and only if the Nijenhuis tensor

$$N(X, Y) = J([X, Y] - [JX, JY]) - ([JX, Y] + [X, JY])$$

vanishes for all vector fields $X, Y$. In this case $J$ is called a complex structure on $M$.

A Riemannian metric $g$ on a complex manifold $(M, J)$ is said to be Hermitian if it is compatible with $J$, i.e. if $g(JX, JY) = g(X, Y)$ for any $X, Y$. In [19], Gauduchon proved that if $(M, J, g)$ is an Hermitian manifold, then there is a 1-parameter family of canonical Hermitian connections on $M$ characterized by the properties of their torsion tensor. In particular, the Bismut connection is the unique connection $\nabla^B$ such that

$$\nabla^B J = \nabla^B g = 0$$
and its torsion tensor

\[ c(X, Y, Z) = g(X, T^B(Y, Z)) \]

is totally skew-symmetric, where \( T^B \) is the torsion of \( \nabla^B \). The geometry associated to the Bismut connection is called KT geometry and when \( c = 0 \) it coincides with the usual Kähler geometry.

**Definition 2.1.** Let \( (M, J, g) \) be a Hermitian manifold. If the torsion 3-form \( c \) of the Bismut connection is \( d \)-closed or equivalently if \( \partial \dbar \omega = 0 \), then the Hermitian metric \( g \) on a complex manifold \( (M, J) \) is called strong Kähler with torsion (shortly SKT).

An interesting case is when \( g \) is compatible with two complex structures \( J_+ \) and \( J_- \). We recall the following

**Definition 2.2.** A Riemannian manifold \( (M, g) \) is called generalized Kähler if it has a pair of SKT structures \( (J_+, g) \) and \( (J-, g) \) for which \( c_- = -c_+ \), where \( c_\pm \) denotes the torsion 3-form of the Bismut connection associated to the SKT structure \( (J_\pm, g) \).

Generalized Kähler structures were introduced in [18] and studied by M. Gualtieri in his PhD thesis [22] in the more general context of generalized complex geometry, which contains complex and symplectic geometry as extremal special cases and shares important properties with them.

When \( M \) is 4-dimensional, by [1] there are two classes of generalized Kähler structures, according to whether the complex structures \( J_+ \) and \( J_- \) induce the same or different orientations on \( M \). In [1] compact 4-dimensional generalized Kähler manifolds \( (M^4, J_\pm, g) \) for which \( J_+ \) and \( J_- \) commute have been classified.

In this paper we consider Lie algebras endowed with SKT structures which induce left-invariant SKT structures on the corresponding simply connected Lie groups.

Let \( g \) be a Lie algebra with an (integrable) complex structure \( J \) and an inner product \( g \) compatible with \( J \). If the associated Kähler form \( \omega(X, Y) = g(JX, Y) \) satisfies \( d\omega = 0 \), where

\[ d\omega(X, Y, Z) = -\omega([X, Y], Z) - \omega(Y, Z), X) - \omega([Z, X], Y), \]

for any \( X, Y, Z \in g \), the Hermitian Lie algebra \( (g, J, g) \) is Kähler. Equivalently, \( (g, J, g) \) is Kähler if and only if \( \nabla^g J = 0 \), where \( \nabla^g \) is the Levi-Civita connection of \( g \). If \( \partial \dbar \omega = 0 \), the Hermitian Lie algebra \( (g, J, g) \) is SKT. We recall that for a Lie group \( G \) with a left-invariant Hermitian structure \( (J, g) \) the Bismut connection \( \nabla^B \) on \( G \) is given by the following equation

\[ g(\nabla^B_X Y, Z) = \frac{1}{2}\{g([X, Y] - [JX, JY], Z) - g([Y, Z] + [JY, JZ], X) \]

\[ - g([X, Z] - [JX, JZ], Y) \}

for any \( X, Y, Z \in g \) (see [9]).

If \( G \) is nilpotent and admits a left-invariant SKT structure \( (J, g) \), then by [11] \( G \) has to be 2-step nilpotent and \( J \) has to preserve the center of \( G \). Moreover, nilpotent Lie groups cannot admit any left-invariant generalized Kähler structure unless they are abelian [7].

In the solvable case there is a classification of SKT Lie groups of dimension 4 [30], but there are no general results in higher dimensions. Examples of SKT and generalized Kähler solvable Lie groups admitting compact quotients have been shown in [15, 17].
3. SKT structures on tangent Lie algebras

Let \( \mathfrak{g} \) be a \( 2n \)-dimensional Lie algebra endowed with a complex structure \( J \) and an inner product \( g \) compatible with \( J \). Assume that \( D \) is a flat connection on \( \mathfrak{g} \) preserving the Hermitian structure, i.e. such that \( DG = 0 \) and \( DJ = 0 \).

Consider the tangent Lie algebra \( T_D \mathfrak{g} := \mathfrak{g} \ltimes_D \mathbb{R}^{2n} \) endowed with the Lie bracket

\[
[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], D_X Y_2 - D_Y X_2)
\]

and with the complex structure

\[
\mathcal{J}(X_1, X_2) = (JX_1, JX_2).
\]

Since \( D \) is flat, the Lie bracket (2) on \( T_D \mathfrak{g} \) satisfies the Jacobi identity. The integrability of the complex structure \( \mathcal{J} \) on \( T_D \mathfrak{g} \) follows from the fact that \( J \) is integrable and parallel with respect to \( D \) (see [2, Proposition 3.3]).

Let \( \hat{g} \) be the inner product on \( T_D \mathfrak{g} \) induced by \( g \) such that \((\mathfrak{g}, 0)\) and \((0, \mathfrak{g})\) are orthogonal. Then \( \hat{g} \) is compatible with \( \mathcal{J} \), that is, \((T_D \mathfrak{g}, \mathcal{J}, \hat{g})\) is a Hermitian Lie algebra.

In a similar way as for the HKT case (see [4, Proposition 4.1]) we can prove the following

**Proposition 3.1.** Let \((\mathfrak{g}, J, g)\) be a Hermitian Lie algebra and \( D \) a flat connection such that \( DG = 0 \) and \( DJ = 0 \). Then the Hermitian structure \((\mathcal{J}, \hat{g})\) on \( T_D \mathfrak{g} \) is SKT if and only if \((J, g)\) is SKT on \( \mathfrak{g} \).

**Proof.** The proof is already contained in [4]. Indeed, by a direct computation we have that the Bismut connection \( \nabla^B \) of the new Hermitian structure \((\mathcal{J}, \hat{g})\) on \( T_D \mathfrak{g} \) is related to the Bismut connection \( \nabla^B \) of the Hermitian structure \((J, g)\) on \( \mathfrak{g} \) by

\[
\hat{g}(\nabla^B_{(X_1, X_2)}(Y_1, Y_2), (Z_1, Z_2)) = g(\nabla^B_{X_1} Y_1, Z_1) + g(D_X Y_2, Z_2),
\]

for any \( X_i, Y_i, Z_i \in \mathfrak{g}, i = 1, 2, 3 \). Therefore, denoting by \( \tilde{c} \) and \( c \) the torsion 3-forms of the Bismut connections on \( T_D \mathfrak{g} \) and \( \mathfrak{g} \), respectively, we obtain

\[
\tilde{c}((X_1, X_2), (Y_1, Y_2), (Z_1, Z_2)) = c(X_1, Y_1, Z_1),
\]

and

\[
d\tilde{c}((X_1, X_2), (Y_1, Y_2), (Z_1, Z_2), (W_1, W_2)) = dc(X_1, Y_1, Z_1, W_1).
\]

This shows that the strong condition is preserved. \( \square \)

**Remark 3.2.** To construct the tangent Lie algebra we consider the flat connection \( D \) as a representation of \( \mathfrak{g} \) on \( \mathbb{R}^{2n} \). If we choose the adjoint representation \( \text{ad} \) of \( \mathfrak{g} \) on \( \mathfrak{g} \) then the semidirect product \( \mathfrak{g} \ltimes_{\text{ad}} \mathbb{R}^{2n} \) is the Lie algebra of the Lie group \( TG \), the tangent bundle over \( G \) [2]. In this case the conditions \( DJ = DG = 0 \) are satisfied if and only if \((\mathfrak{g}, J)\) is a complex Lie algebra and the inner product \( g \) is bi-invariant. Therefore this construction allows us to lift an invariant SKT structure \((\hat{g}, \hat{J})\) from a Lie group \( G \) to its tangent bundle \( TG \) if and only if \((G, \hat{J})\) is a complex Lie group and \( \hat{g} \) is a bi-invariant metric.

Since a generalized Kähler structure on a \( 2n \)-dimensional Lie algebra \( \mathfrak{g} \) is equivalent to a pair of SKT structures \((J+, g)\) and \((J-, g)\), such that \( c_+ = -c_- \), as a consequence of the previous proposition we can prove the following
Corollary 3.3. Let \((\mathfrak{g}, J_\pm, g)\) be a generalized Kähler Lie algebra and \(D\) a flat connection such that \(Dg = 0\) and \(DJ_\pm = 0\). Then, \((\tilde{J}_\pm, \tilde{g})\) on \(T_D \mathfrak{g}\) is generalized Kähler.

Proof. We know that \(\tilde{J}_-\) and \(\tilde{J}_+\) are integrable and that the metric \(\tilde{g}\) is still compatible with both complex structures. Moreover, if \((\mathfrak{g}, J_\pm, g)\) is a generalized Kähler Lie algebra, then \(e_+ = -e_-\) and \(dc_+ = dc_- = 0\). Therefore, equations (5) and (6) yield \(\tilde{e}_+ = -\tilde{e}_-\) and \(d\tilde{c}_+ = d\tilde{c}_- = 0\), i.e. \((T_D \mathfrak{g}, \tilde{J}_\pm, \tilde{g})\) is a generalized Kähler Lie algebra. \(\square\)

For Hermitian Lie algebras \(\mathfrak{g}\) such that the commutator \([\mathfrak{g}, \mathfrak{g}]\) does not coincide with \(\mathfrak{g}\), we can show that it is always possible to find a Hermitian flat connection \(D\).

Proposition 3.4. Let \((\mathfrak{g}, J, g)\) be a 2n-dimensional SKT Lie algebra such that \(\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}\). Then \(\mathfrak{g}\) admits a flat connection \(D\) such that \(DJ = Dg = 0\).

Proof. Let \(X \in \mathfrak{g} \setminus \mathfrak{g}^1\), and choose a basis \(\{e_i\}\) of \(\mathfrak{g}\) such that \(e_{2n} = X\). We define
\[
\begin{align*}
D_{e_i} Y &= 0 & i &= 1, \ldots, 2n - 1 \\
D_{e_{2n}} Y &= JY.
\end{align*}
\]
It is easy to verify that \(D\) is flat. Moreover, it satisfies the conditions
\[
g(D_X Y, Z) = -g(Y, D_X Z) \quad J(D_X Y) = D_X (JY)
\]
since \(g\) is Hermitian. \(\square\)

Remark 3.5. Note that the strict inclusion \([\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}\) holds for every solvable Lie algebra \(\mathfrak{g}\). Therefore, by applying the previous proposition, we have that every SKT solvable Lie algebra admits an SKT tangent Lie algebra.

Next we will apply the previous construction to 4-dimensional SKT Lie algebras. We recall that in the solvable case a 4-dimensional SKT Lie group is unimodular if and only if it admits a compact quotient by a discrete subgroup (see [30]).

By [24] a complex (non-Kähler) surface diffeomorphic to a 4-dimensional compact homogeneous manifold \(X = \Theta \setminus L\), where \(\Theta\) is a uniform discrete subgroup of \(L\), and which does not admit any Kähler structure is one of the following:

a) Hopf surface;
b) Inoue surface of type \(S^0\);
c) Inoue surface of type \(S^\pm\);
d) primary Kodaira surface;
e) secondary Kodaira surface;
f) properly elliptic surface with first odd Betti number.

All the previous complex (non-Kähler) surfaces admit an invariant SKT structure (i.e., induced by a SKT structure on the Lie algebra of \(L\)) and by [11] [15] an Inoue surface of type \(S^0\) admits an invariant generalized Kähler structure. A \(\mathbb{T}^2\)-bundle over the Inoue surface of type \(S^0\) was considered in [15] in order to construct a 6-dimensional compact solvmanifold with a non-trivial generalized Kähler structure. A similar construction can be done for any of the non-Kähler complex homogeneous surfaces, using the description of \(L\) and \(\Theta\) in [24]. Indeed, in [10] it was proved that on any non-Kähler compact homogeneous complex surface
X = Θ\L there exists a non-trivial compact T²-bundle M carrying a locally conformally balanced SKT metric.

In the sequel we will use tangent Lie algebras associated to 4-dimensional solvmanifolds to produce examples of SKT metrics and generalized Kähler structures in higher dimensions.

**Example 3.6.** Consider the 4-dimensional solvable Lie algebra \( \mathfrak{g}_1 \) with structure equations

\[
\begin{align*}
de_1 &= e^2 \wedge e^4 \\
de_2 &= -e^1 \wedge e^4 \\
de_3 &= e^1 \wedge e^2 \\
de_4 &= 0.
\end{align*}
\]

To simplify the notations, we will denote \( \mathfrak{g}_1 \) by \( (e^{24}, -e^{14}, e^{12}, 0) \). On \( \mathfrak{g}_1 \) we define the integrable complex structure \( J e^{2i-1} = e^{2i} \) for \( i = 1, 2 \) and the SKT inner product \( g = \sum_{j=1}^4 e^j \otimes e^j \). By [24], \( (\mathfrak{g}_1, J) \) is the Lie algebra corresponding to the secondary Kodaira surface.

Since \( e_4 \notin [\mathfrak{g}_1, \mathfrak{g}_1] \), we can consider the flat connection \( D \) defined in the proof of Proposition 3.4. The tangent Lie algebra \( T_D \mathfrak{g}_1 \) has structure equations

\[
(f^{24}, -f^{14}, f^{12}, 0, -f^{46}, f^{45}, -f^{48}, f^{47}).
\]

Combining Propositions 3.3 and 3.4 the induced Hermitian structure \( (\tilde{J}, \tilde{g}) \) is SKT. Moreover, it is easy to verify that the Lie algebra \( \tilde{\mathfrak{g}}_1 = T_D \mathfrak{g}_1 \) is 3-step solvable and unimodular. The simply connected Lie group \( \tilde{G}_1 \) with Lie algebra \( \tilde{\mathfrak{g}}_1 \) is isomorphic to the semidirect product \( \mathbb{R} \ltimes_\mu (H_3 \times \mathbb{C}^2) \) where \( H_3 \) is the real 3-dimensional Heisenberg Lie group and \( \mu \) is the automorphism

\[
\mu(t) : (x + iy, u, z_1, z_2) \to (e^{it}x, e^{it}y, u, e^{it}z_1, e^{it}z_2)
\]

by identifying the matrix

\[
\begin{pmatrix}
1 & x & u \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]

in \( H_3 \) with \( (x + iy, u) \in \mathbb{C} \ltimes \mathbb{R} \). Arguing as in [17] it is possible to show that \( \tilde{G}_1 \) admits a uniform discrete subgroup.

More in general, a flat Hermitian connection \( \hat{D} \) is on \( (\mathfrak{g}_1, J, g) \) then it is given with respect to the basis \( \{e_i\} \) by

\[
(7) \quad \hat{D}_{e_i} = 0, \ i = 1, 2, 3, \quad \hat{D}_{e_4} = \begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
-a_{12} & 0 & -a_{14} & a_{13} \\
-a_{13} & a_{14} & 0 & a_{34} \\
-a_{14} & -a_{13} & -a_{34} & 0
\end{pmatrix}, \ a_{ij} \in \mathbb{R}
\]

The connection \( D \) considered before coincides with \( \hat{D} \) when \( a_{1,2} = a_{3,4} = 1, \ a_{1,3} = a_{1,4} = 0 \). Note that different choices of the coefficients can lead to non-isomorphic Lie algebras. Indeed, when \( a_{1,2} = a_{1,3} = a_{1,4} = 0 \) we obtain that \( T_D \mathfrak{g}_1 \cong \mathbb{R}^2 \times \mathfrak{h} \) for a 6-dimensional Lie algebra \( \mathfrak{h} \), so \( T_D \mathfrak{g}_1 \not\cong \mathfrak{g}_1 \).
Example 3.7. We start by considering the 4-dimensional solvable Lie algebra

\[ \mathfrak{g}_2 = (ae^{14} + be^{24}, -be^{14} + ae^{24}, 2ae^{34}, 0), \quad a, b \in \mathbb{R} - \{0\}, \]

endowed with the two integrable complex structures \( J_\pm \) defined by

\[ J_\pm e^1 = e^2, \quad J_\pm e^3 = \pm e^4 \]

and the inner product \( g = \sum_{j=1}^4 e^j \otimes e^j \). By [24], \((\mathfrak{g}_2, J_+ )\) corresponds to the Inoue surface of type \( S^0 \). Defining \( \omega_\pm (\cdot, \cdot) = g(J_\pm \cdot, \cdot) \) we obtain \( d_+^c \omega_+ = -d_+^c \omega_- = 2ae^{123} \) and \( dd_+^c \omega_+ = dd_+^c \omega_- = 0 \), so \((J_\pm , g)\) is a generalized Kähler structure on \( \mathfrak{g}_2 \). We note that \( e_4 \notin [\mathfrak{g}_2, \mathfrak{g}_2] \), so applying Proposition 3.4 the connection \( \hat{D} \) defined by

\[
\begin{cases}
    D_{e_i} = 0 & i = 1, 2, 3 \\
    D_{e_4} = J_+
\end{cases}
\]

is flat and satisfies \( D J_+ = D g = 0 \). Thus, the induced Hermitian structure \((\tilde{J}_+, \tilde{g})\) on the Lie algebra \( T_D \mathfrak{g}_2 \) with structure equations

\[ (af^{14} + bf^{24}, -bf^{14} + af^{24}, -2ae^{34}, 0, -f^{46}, f^{45}, -f^{48}, f^{47}) \]

is SKT. Moreover, since \( J_+ \) and \( J_- \) commute \( D J_- = 0 \), hence by Corollary 3.3 \((\tilde{J}_-, \tilde{g})\) is a generalized Kähler structure on \( T_D \mathfrak{g}_2 \). Again, \( T_D \mathfrak{g}_2 \) is 2-step solvable and unimodular. This generalized Kähler Lie algebra was already introduced in [15] and it was shown that the corresponding simply connected Lie group admits a compact quotient by a discrete subgroup.

More in general, a flat Hermitian connection \( \hat{D} \) is expressed by (7) with respect to the basis \( \{e_i\} \). Moreover, for every choice of the coefficients \( a_{1,2}, a_{1,3}, a_{1,4}, a_{3,4} \) we find that \( \hat{D} \) and \( J_- \) commute, i.e. \( \hat{D} J_- = 0 \). So \((T_D \mathfrak{g}_2, \tilde{J}_-, \tilde{g})\) is a GK Lie algebra for every \( \hat{D} \).

Example 3.8. Consider the 4-dimensional nilpotent Lie algebra

\[ \mathfrak{g}_3 = (0, 0, e^{12}, 0) \]

endowed with the integrable complex structure \( J \) defined by \( J e^{2i-1} = e^{2i} \) for \( i = 1, 2 \) and the SKT inner product \( g = \sum_{j=1}^4 e^j \otimes e^j \) corresponding to the primary Kodaira surface. This is a 2-step nilpotent Lie algebra. Indeed \( [\mathfrak{g}_3, \mathfrak{g}_3] = \text{span} \langle e_3 \rangle \), so, in order to generate the action induced by a Hermitian flat connection \( \hat{D} \) on \( \mathbb{R}^4 \), we need not only \( e_4 \) (as in the previous example) but also \( e_1 \) and \( e_2 \). In fact, every connection in the form

\[
D_{e_3} = 0, \quad D_{e_i} = \begin{pmatrix}
0 & a_{i,1} & a_{i,2} & a_{i,3} \\
-a_{i,1} & 0 & -a_{i,3} & a_{i,2} \\
-a_{i,2} & a_{i,3} & 0 & a_{i,4} \\
-a_{i,3} & -a_{i,2} & -a_{i,4} & 0
\end{pmatrix} \quad i = 1, 2, 4
\]

with \( a_{i,j} \in \mathbb{R} \) satisfying the conditions

\[
a_{2,2}(a_{1,1} - a_{1,4}) = a_{1,2}(a_{2,1} - a_{2,4}), \quad a_{1,3} a_{2,2} = a_{1,2} a_{2,3}, \quad a_{2,2}(a_{3,1} - a_{3,4}) = a_{3,2}(a_{2,1} - a_{2,4})
\]

is flat and \( D g = D J = 0 \). The tangent Lie algebra \( T_D \mathfrak{g}_3 \) is 2-step solvable and unimodular as in the previous cases, but in general it is not nilpotent.
4. Taming symplectic forms on 4-dimensional Lie groups

We recall that on a complex manifold \((M, J)\) a taming symplectic form is a symplectic form \(\Omega\) on \(M\) such that \(\Omega(X, JX) > 0\) for every non-zero vector field \(X\) of \(M\). This is equivalent to the existence of an SKT metric \(\omega\) such that \(\partial\omega = \overline{\partial}\beta\) for a \(\partial\)-closed \((2,0)\)-form \(\beta\) \([11]\). If \(M\) is compact and \((M, J)\) admits a Kähler metric, the converse is also true:

Proposition 4.1. Let \((M, J)\) be a compact complex manifold that admits a Kähler metric. Then every SKT metric induces a taming symplectic form.

Proof. Since \((M, J)\) is compact and admits a Kähler metric, the \(\overline{\partial}\)-lemma holds. Let \(\omega\) be the fundamental 2-form of an SKT metric, i.e. \(\partial\overline{\partial}\omega = 0\). Applying the \(\overline{\partial}\overline{\partial}\)-lemma to \(\partial\omega\) we obtain \(\partial\omega = \overline{\partial}\gamma\) for some \((1,0)\)-form \(\gamma\) on \(M\). Then \(\partial\omega = \overline{\partial}(\partial\gamma)\) with \(\partial(\partial\gamma) = 0\), so \(\omega\) induces a taming symplectic form. \(\square\)

If \((M, J)\) is compact and 4-dimensional, then it admits a taming symplectic form if and only if it admits a Kähler metric \([28, \text{Theorem 1.5}]\), so every SKT metric on a compact 4-dimensional Kähler manifold induces a symplectic form that tames the complex structure. One can wonder if it also holds for non-compact manifolds. We verify that is still true in the case of invariant Hermitian structures on 4-dimensional simply connected Lie groups.

Proposition 4.2. Let \((\mathfrak{g}, J)\) be a 4-dimensional Lie algebra endowed with a complex structure. Then:

1. \(\mathfrak{g}\) admits a taming symplectic form if and only if it admits a Kähler metric;
2. if \(\mathfrak{g}\) admits a Kähler metric, every SKT metric induces a taming symplectic form.

Proof. It is well known that a non-solvable Lie algebra of dimension 4 is unimodular, so by \([29]\) it does not admit any symplectic structure. Moreover, an SKT solvable Lie algebra of dimension 4 admits a compact quotient if and only if it is unimodular. Therefore, to study taming symplectic forms on non-compact Lie algebras it is sufficient to consider the non-unimodular case. Using the classification of SKT structures on 4-dimensional Lie algebras in \([30]\), for every non-unimodular Lie algebra \(\mathfrak{g}\) that admits an SKT structure \((J, \omega)\) we provide a \(\partial\)-closed \((2,0)\)-form \(\beta\) such that \(\partial\omega = \overline{\partial}\beta\) and a \(J\)-compatible Kähler metric \(\omega_k\).

Before starting, we note that since \(\mathfrak{g}\) is 4-dimensional, the space of \((2,0)\)-forms is generated by \(a e^{12}\), where \(\{\alpha^1, \alpha^2\}\) is a basis for \((1,0)\)-forms. Thus \(\beta\) is \(\partial\)-closed and is in the form \(a \alpha^{12}\), where \(a \in \mathbb{C}\).

Following the notations of \([30]\), we study the non-unimodular SKT 4-dimensional Lie algebras:

- \(\mathbb{R} \times \mathfrak{su}_3 = (0, e^{21}, 0, 0)\). Every SKT metric has the form \(\omega = e^{12} + e^{34}\) with \(Je^1 = e^2\), \(Je^3 = e^4\) and structure equations
  \[
  (0, 0, 0, u^2 e^{12} + \sqrt{u_1 w_1} (e^{14} - e^{23}) + w_1 e^{34})
  \]
  where the coefficients are real and \(w_1 > 0\), \(u_1 \geq 0\). We find that \(\partial\omega = \overline{\partial}(a \alpha^{12})\) with \(a = i \frac{\sqrt{u_1 w_1}}{2w_1}\).
  Moreover, the \((1,1)\)-form
  \[
  \omega_k = ne^{12} + me^{34} + m \frac{\sqrt{u_1 w_1}}{w_1} (e^{14} - e^{23})
  \]
with the conditions \( n > m \frac{34}{\omega_1}, m > 0 \) is closed and positive, so it is a Kähler metric with respect to \( J \).

- \( \mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}} = (0, e^{21}, 0, e^{43}) \). Every SKT metric has the form \( \omega = e^{12} + e^{34} + t(e^{13} + e^{24}) \) with \( J e^1 = e^2, J e^3 = e^4 \) and structure equations
  \[
  (0, 0, x_1 e^{12} + x_3(e^{14} - e^{23}) + y_2 e^{34}, u_1 e^{12} + u_3(e^{14} - e^{23}) + v_2 e^{34}),
  \]
  where \( de^2 \) and \( de^4 \) are linearly independent and the real coefficients satisfy
  \[
  y_2 x_1 - y_2 u_3 + v_2 x_3 - x_3^2 = 0 \quad u_1 v_2 - u_1 x_3 + u_3 x_1 - u_3^2 = 0
  \]
  \[
  u_3 x_3 - y_2 u_1 = 0 \quad (u_1 - x_3)(v_2 + x_3) - (u_3 + x_1)(u_3 - y_2) = 0.
  \]
  We find that \( \partial \omega = \overline{\partial}(a \alpha^{12}) \) with \( a = -\frac{i}{2} + i \frac{y_2}{2(x_1 + y_3)} \). More in general, we have that every \( d \)-closed 3-form is exact because \( b_3^{\text{inv}} = \dim H^3(\mathfrak{g}) = 0 \).
  Moreover, the \((1,1)\)-form
  \[
  \omega_k = n e^{12} + m e^{34} + p(e^{14} - e^{23})
  \]
  with the conditions \( n x_3 - m u_1 + p(u_3 - x_1) = 0, nm > p^2 \) and \( m > 0 \) is closed and positive, so it is a Kähler metric with respect to \( J \). Note that the conditions above admit a solution for every choice of the coefficients \( x_1, x_3, u_1, u_3 \). Indeed, fixing \( p \), the first condition can be written as \( n x_3 = m u_1 - p(u_3 - x_1) \), so we can choose \( n \) and \( m \) as large as we need in order to satisfy \( nm > p^2 \).

- \( \mathfrak{r}_{4, \lambda, 0} = (0, \lambda e^{21}, e^{41}, -e^{31}) \), with \( \lambda > 0 \). Every SKT metric has the form \( \omega = e^{12} + e^{34} \) with \( J e^1 = e^2, J e^3 = e^4 \) and structure equations
  \[
  (0, x_1 e^{12}, y_1 e^{12} + y_3 e^{14}, -y_3 e^{13})
  \]
  where the coefficients are real, \( y_3 \neq 0 \), \( x_1 > 0 \) and \( y_1 > 0 \). Under these conditions \( \lambda = \frac{y_1}{y_3} \). We find that \( \partial \omega = \overline{\partial}(a \alpha^{12}) \) with \( a = -\frac{y_1}{2(x_1 + y_3)}(x_1 + iy_3) \).
  Moreover, the \((1,1)\)-form
  \[
  \omega_k = n e^{12} + m e^{34} + \frac{y_1 y_3}{x_1^2 + y_3^2}(e^{14} - e^{23}) - m \frac{y_1 x_1^2}{x_1^2 + y_3^2}(e^{13} + e^{24})
  \]
  with the conditions \( n > m \frac{y_1}{x_1^2 + y_3^2}(y_3 - x_1), n, m > 0 \) is closed and positive, so it is a Kähler metric with respect to \( J \).

- \( \mathfrak{d}_{4, 2} = (0, 2 e^{21}, -e^{31}, e^{41} + e^{32}) \). Every SKT metric has the form \( \omega = e^{12} + e^{34} \) with \( J e^1 = e^2, J e^3 = e^4 \) and structure equations
  \[
  (0, x_1 e^{12}, y_1 e^{12} - \frac{1}{2} x_1 e^{13}, u_1 e^{12} + \frac{1}{2} x_1 e^{14} - x_1 e^{23})
  \]
  where the coefficients are real and \( x_1 > 0 \). We find that \( \partial \omega = \overline{\partial}(a \alpha^{12}) \) with \( a = \frac{1}{3 x_1^2}(-y_1 + i u_1) \).
  Moreover, the \((1,1)\)-form
  \[
  \omega_k = n e^{12} + m e^{34} + \frac{2 u_1}{3 x_1}(e^{14} - e^{23})
  \]
with the conditions \( n > m \frac{4q^2}{9z_1^2}, \) \( m > 0 \) is closed and positive, so it is a Kähler metric with respect to \( J \).

- \( \mathfrak{d}_{4,\lambda} = (0, \lambda e^{21} + e^{31}, -e^{21} + \lambda e^{31}, 2\lambda e^{41} + e^{32}) \), with \( \lambda > 0 \). Every SKT metric has the form \( \omega = e^{12} + e^{34} + t(e^{13} + e^{24}) \) with \( Je^1 = e^2, Je^3 = e^4 \) and structure equations
  \[
  \begin{cases}
  de^1 = 0 \\
  de^2 = -k(1 + q^2)e^{12} - kqr(e^{14} - e^{23}) - kr^2 e^{34} \\
  de^3 = \frac{3q}{r} e^{12} - \frac{k}{2} e^{13} + z_3 e^{14} \\
  de^4 = \frac{q}{r}(kq^2 + \frac{k}{2}) e^{12} - z_3 e^{13} + (kq^2 - \frac{k}{2}) e^{14} - kq^2 e^{23} + kqr e^{34},
  \end{cases}
  \]
with \( q, r, k \in \mathbb{R} \) such that \( q^2 + r^2 = 1, \ r > 0 \) and \( k, z_3 \neq 0 \). Under these conditions \( \lambda = \frac{k}{2z_3} \). We find that \( \partial \omega = \overline{\partial}(a \alpha^{12}) \) with \( a = -\frac{t}{2} - i\frac{q}{2r} \). More in general, we have that every \( d \)-closed 3-form is exact because \( b_3^{\text{inv}} = \dim H^3(g) = 0 \).

Moreover, the \((1,1)\)-form
  \[
  \omega_k = m \left( 1 + \frac{q^2}{r^2} \right) e^{12} + me^{34} + m \frac{q}{r} (e^{14} - e^{23})
  \]
with the condition \( m > 0 \) is closed and positive, so it is a Kähler metric with respect to \( J \).

- \( \mathfrak{d}_{4,4} = (0, \frac{1}{2} e^{21}, \frac{1}{2} e^{31}, e^{41} + e^{32}) \). Every SKT metric has the form \( \omega = e^{12} + e^{34} + t(e^{13} + e^{24}) \) with \( Je^1 = e^2, Je^3 = e^4 \) with the structure equations consider in the latter case with the additional condition \( z_3 = 0 \). Like before, we find that \( \partial \omega = \overline{\partial}(a \alpha^{12}) \) with \( a = -\frac{t}{2} - i\frac{q}{2r} \) and that the \((1,1)\)-form
  \[
  \omega_k = m \left( 1 + \frac{q^2}{r^2} \right) e^{12} + me^{34} + m \frac{q}{r} (e^{14} - e^{23})
  \]
with the condition \( m > 0 \) is a Kähler metric with respect to \( J \).

\( \square \)

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