PARthENoPE : Public Algorithm Evaluating the Nucleosynthesis of Primordial Elements

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We describe a program for computing the abundances of light elements produced during Big Bang Nucleosynthesis which is publicly available at [http://parthenope.na.infn.it/]. Starting from nuclear statistical equilibrium conditions the program solves the set of coupled ordinary differential equations, follows the departure from chemical equilibrium of nuclear species, and determines their asymptotic abundances as function of several input cosmological parameters as the baryon density, the number of effective neutrino, the value of cosmological constant and the neutrino chemical potential. The program requires commercial NAG library routines.

Program summary

Title of program: PARthENoPE
Program URL: http://parthenope.na.infn.it/
Program obtainable from: parthenope@na.infn.it
Computers: PC-compatible running Fortran on Unix, MS Windows or Linux
Operating systems under which the program has been tested: Windows 2000, Windows XP, Linux
Programming language used: Fortran 77
External routines/libraries used: NAG libraries
No. of lines in distributed program, including input card and test data: 4969
No. of bytes in distributed program, including input card and test data: 192 Kb
Distribution format: tar.gz
Nature of physical problem: Computation of yields of light elements synthesized in the primordial universe.
Method of solution: BDF method for the integration of the ODE’s, implemented in a NAG routine
Typical running time: 90 sec with default parameters on a Dual Xeon Processor 2.4GHz with 2 GB RAM

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“I’l alta fantasia qui mancò possa;
ma già volgeva il mio disio e ’l velle,
sì come rota ch’igualmente è mossa,
l’amor che move il sole e l’ altre stelle.”

Dante Alighieri, “Commedia” - Paradiso, Canto XXXIII, 142-145

I. INTRODUCTION

Big Bang Nucleosynthesis (BBN) is one of the fundamental pillars of the Cosmological Standard Model. In the very early Universe, when the temperature of the primordial plasma decreased from a few MeV down to \( \sim 10 \) keV, light nuclides as \(^2\text{H}\), \(^3\text{He}\), \(^4\text{He}\) and, to a smaller extent, \(^7\text{Li}\) were produced via a network of nuclear processes. The relative abundances of these nuclear “ashes” with respect to hydrogen can be determined via several observational techniques and in different astrophysical environments. In the standard cosmological scenario and in the framework of the electroweak Standard Model, the dynamics of this phase is controlled by only one free parameter, the baryon to photon number density, which can thus be fixed by fitting experimental observations. This parameter can be also independently measured with very high precision by Cosmic Microwave Background anisotropies \(^1\) and the agreement with BBN result is quite remarkable. For reviews see e.g. \(^2\) in \(^3\) or \(^4\).

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More in general, the quality and quantity of new cosmological and astrophysical data available in the last decade has led to an overall consistent picture of the evolution of the Universe, usually referred to as the “concordance” model. This is based on standard physics plus a few phenomenological parameters, for which an underlying theory is however still missing. At present, one thus faces the intriguing possibility that one might test models which go beyond our present understanding of fundamental interactions, in a way which is complementary to traditional earth-based laboratory and accelerator approaches. This is illustrated e.g. by the search for new light degrees of freedom which might contribute to the total energy density in the Universe in addition to photons and neutrinos. To pursue this programme, it is crucial to achieve a high level of accuracy in theoretical predictions for cosmological observables, at least at the level of experimental uncertainties. In the case of BBN, many steps have been done in this direction by a careful analysis of several key aspects of the physics involved in the phenomenon. The accuracy of the weak reactions which enter the neutron/proton chemical equilibrium has been pushed well below the percent level \[6, 7, 8, 9\]. Similarly, the neutrino decoupling has been carefully studied by several authors by explicitly solving the corresponding kinetic equations \[10, 11, 12, 13, 14, 15\]. These two issues are mainly affecting the prediction of \(^4\text{He}\) mass fraction, which presently has a very small uncertainty, of the order of 0.1 %, due to the experimental uncertainty on neutron lifetime. Finally, much study has been devoted to the analysis of several nuclear reaction rates entering the BBN network, as well as the corresponding uncertainties. This task involves a careful study of the available data or predictions on each reaction, including an update in light of new relevant experimental measurements, the choice of a reasonable protocol to combine them in order to obtain a best estimate and an error and, finally, the calculation of the corresponding thermal averaged rates. This issue has been extensively discussed in \[9\], whose results have been used in the program described in the present paper, and \[16, 17\]. An important benchmark in this development has been represented by the compilation of the NACRE Collaboration database \[18\].

In view of all these recent developments, we believe that the scientific community interested in BBN, in itself or as a tool to constrain new physics beyond the Standard Model, might find useful a new public BBN code which updates the pioneering achievements of \[19, 20, 21\]. For this reason we have publicly released a code we have developed and continuously updated over almost a decade, which we named \texttt{PArthENoPE} and can be obtained at the URL \url{http://parthenope.na.infn.it/}. The aim of the present paper is to give a general description of the program and how to use it. After briefly summarizing in Section II the theoretical framework of BBN and all major improvements implemented in \texttt{PArthENoPE}, we discuss in Section III a few extensions of the minimal standard BBN scenario which are also included in the code. In Section IV the main structures of \texttt{PArthENoPE} are outlined, while a comparison with the public code of \[20\] is discussed in Section V. Finally, in Section VI we report our conclusions. Hereafter we use natural units where the reduced Planck constant, the speed of light and the Boltzmann constant are fixed to 1, i.e. \(\hbar = c = k_B = 1\).

\section{II. THE THEORY OF BIG BANG NUCLEOSYNTHESIS}

\subsection{A. The set of equations}

We consider \(N_{\text{nuc}}\) species of nuclides, whose number densities, \(n_i\), are normalized with respect to the total number density of baryons \(n_B\),

\[ X_i = \frac{n_i}{n_B} \quad i = n, p, ^2\text{H}, \ldots \quad . \tag{1} \]

The list of all nuclides which are typically included in BBN analyses and considered in \texttt{PArthENoPE} is reported in Table I. In the (photon) temperature range of interest for BBN, \(10\text{ MeV} > T > 0.01\text{ MeV}\), electrons and positrons are kept in thermonuclear equilibrium with photons by fast electromagnetic interactions and distributed according to a Fermi-Dirac distribution function \(f_{\pm}^z\), with chemical potential \(\mu_{\pm}\), parameterized in the following by the function \(\phi_{\pm} \equiv \mu_{\pm}/T\). The pressure and energy density of the electromagnetic plasma \(\left( e^\pm \text{ and } \gamma \right)\) is calculated in \texttt{PArthENoPE} by including the effect of finite temperature QED corrections \[13\]. Furthermore, electromagnetic and nuclear scatterings keep the non-relativistic baryons in kinetic equilibrium, and their energy density \(\rho_B\) and pressure \(p_B\) are given by

\[ \rho_B = \left[ M_u + \sum_i \left( \Delta M_i + \frac{3}{2} T \right) X_i \right] n_B \quad , \tag{2} \]

\[ \rho_B = \left[ M_u + \sum_i \left( \Delta M_i + \frac{3}{2} T \right) X_i \right] n_B \quad . \tag{2} \]

\footnote{It is a pleasure to acknowledge the public code of \[20\] as the starting point for many scholars interested in BBN, including the authors of the present paper.}
with $\Delta M_i$ and $M_n$ the i-th nuclide mass excess and the atomic mass unit, respectively.

The set of differential equations ruling primordial nucleosynthesis is the following (see for example [7, 8, 19]):

$$\frac{\dot{n}_B}{n_B} = -3H, \quad \frac{\dot{\rho}}{\rho} = -3H (\rho + p),$$

$$\dot{X}_i = \sum_{j, k, l} N_i \left( \Gamma_{kl \rightarrow ij} \frac{X_i^{N_i} X_j^{N_j}}{N_i! N_j!} - \Gamma_{ij \rightarrow kl} \frac{X_i^{N_i} X_j^{N_j}}{N_i! N_j!} \right) \equiv \Gamma_i,$$

$$n_B \sum_j Z_j X_j = n_{e^-} - n_{e^+} \equiv L \left( m_e/T, \phi_e \right) \equiv T^3 \tilde{L} \left( m_e/T, \phi_e \right),$$

where $\rho$ and $p$ denote the total energy density and pressure, respectively,

$$\rho = \rho_\gamma + \rho_e + \rho_{\nu} + \rho_B, \quad p = \rho_\gamma + \rho_e + \rho_{\nu} + p_B,$$

while $i, j, k, l$ denote nuclear species, $N_i$ the number of nuclides of type $i$ entering a given reaction (and analogously $N_j, N_k, N_l$), and the $\Gamma$’s denote symbolically the reaction rates. For example, in the case of decay of the species $i$, $N_i = 1$, $N_j = 0$ and $\sum \Gamma_{i \rightarrow kl}$ is the inverse lifetime of the nucleus $i$; for binary collisions, $N_i = N_j = N_k = N_l = 1$ and $\Gamma_{ij \rightarrow kl} = \langle \sigma_{ij \rightarrow kl} v \rangle$, i.e. it represents the thermal average of the cross section for the reaction $i + j \rightarrow k + l$ times the relative velocity of $i$ and $j$. In Eq. (8), $Z_i$ is the charge number of the i-th nuclide, and the function $\tilde{L}(\xi, \omega)$ is defined as

$$\tilde{L}(\xi, \omega) \equiv \frac{1}{\pi^2} \int_{\xi}^{\infty} d\zeta \, \zeta \sqrt{\xi^2 - \zeta^2} \left( \frac{1}{e^{\xi - \omega} + 1} - \frac{1}{e^{\xi + \omega} + 1} \right).$$

Equation (4) is the definition of the Hubble parameter $H$, $a$ denoting the scale factor of the Friedmann-Robertson-Walker-Lemaître metric, with $G_N$ the gravitational constant, whereas Eqs (5) and (6) state the total baryon number and entropy conservation per comoving volume, respectively. The set of $N_{nu}$ Boltzmann equations (7) describes the density evolution of each nuclide species, with $\Gamma_{kl \rightarrow ij}$ the rate per incoming particles averaged over kinetic equilibrium distribution functions. Finally, Eq. (8) states the Universe charge neutrality in terms of the electron chemical potential, with $L(m_e/T, \phi_e)$ the charge density in the lepton sector in unit of the electron charge.

The neutrino energy density and pressure are defined in terms of their distributions in momentum space as

$$\rho_\nu = 3 p_\nu = 2 \int \frac{d^3 p}{(2\pi)^3} |\vec{p}| \left[ f_{\nu_e} + 2 f_{\nu_\mu} \right],$$

Indeed, in the default scenario we assume a vanishing neutrino chemical potential, so that $f_{\nu_e} = f_{\bar{\nu}_e}$ and $f_{\nu_\mu} \equiv f_{\nu_\mu} \equiv f_{\bar{\nu}_\mu} = f_{\bar{\nu}_\mu}$. The nuclide evolution can be followed in PArthENoPE also for finite neutrino chemical potential, see Section [III] below.
As well known, neutrinos decouple from the electromagnetic plasma at temperatures of a few MeV. Soon after, when the onset of $e^+ - e^-$ annihilations takes place, $e^\pm$ are still partially coupled to neutrinos. The neutrino distributions are thus slightly distorted, especially in their high energy tail (and the $e-$flavor more than the other two, since the former also interacts via charged current). To get BBN predictions accurate at the sub-percent level it is necessary to follow in details this residual out of equilibrium neutrino heating by solving the kinetic equations for neutrino distributions. Remarkably, baryons provide a negligible contribution to the dynamics of the Universe at the BBN epoch as the baryon to photon number density is very small, $\eta < 10^{-9}$, and therefore Boltzmann equations for neutrino species can be solved together with equations (4) and (6) only, ignoring the dynamics of nuclear species. This allows one to solve the evolution of the neutrino species first, and then to substitute the resulting neutrino distribution into the remaining equations. We do not consider neutrino oscillations, whose effect has been studied and shown to be sub-leading in [14]. The reader can find further details on the neutrino decoupling stage in [9, 14, 15].

**B. Numerical solution of the BBN set of equations**

The BBN set of equations (4)-(8) can be recast in a form more convenient for a numerical solution, which follows the evolution of the $N_{\text{nuc}} + 1$ unknown quantities ($\phi_e$, $X_i$) as functions of the dimensionless variable $z = m_e/T$. In this framework, Eq. (8) provides $n_B$ as a function of $\phi_e$. In particular, the set of differential equations implemented in PArTHENoPE is the following:

$$\frac{d\phi_e}{dz} = \frac{1}{z} \hat{L} \kappa_1 + \left( \hat{\rho}_{e\gamma B} + \hat{\rho}_{e\gamma B} + \frac{N(z)}{3} \right) \kappa_2,$$

$$\frac{dX_i}{dz} = X_i \frac{dt}{dz} = \frac{\widehat{\Gamma}_i}{3zH} \left[ \hat{L} \frac{\partial \phi_e}{\partial \phi_e} - \frac{\partial L}{\partial \phi_e} \left( \hat{\rho}_{e\gamma B} + \hat{\rho}_{e\gamma B} + \frac{N(z)}{3} \right) \right],$$

where

$$\kappa_1 = 4 \left( \hat{\rho}_e + \hat{\rho}_\gamma \right) + \frac{3}{2} \left( \hat{\rho}_B + \frac{\partial \hat{\rho}_B}{\partial z} - \hat{\rho}_z \right) + \frac{1}{L} \left( 3 \hat{L} - z \frac{\partial \hat{L}}{\partial z} \right) \hat{\rho}_B - \frac{z^2 \hat{L}}{\sum_j Z_j X_j} \sum_i \left( \Delta \hat{M}_i + \frac{3}{2} \hat{\rho}_B \right) \widehat{\Gamma}_i,$$

$$\kappa_2 = z \frac{\partial \hat{L}}{\partial z} - 3 \hat{L} - z \hat{L} \sum_j \frac{Z_j \widehat{\Gamma}_i}{X_j}.$$

See Appendix A for notations and the explicit derivation of this set of equations. Equations (13) and (14) are solved by imposing the following initial conditions at $z_{in} = m_e/(10\,\text{MeV})$:

$$\phi_e(z_{in}) = \phi_e^0,$$

$$X_1(z_{in}) = X_n(z_{in}) = (\exp \{ \hat{q} z_{in} \} + 1)^{-1},$$

$$X_2(z_{in}) = X_p(z_{in}) = (\exp \{- \hat{q} z_{in} \} + 1)^{-1},$$

$$X_i(z_{in}) = \left( \frac{q_i}{2} \right) \left( \zeta(3) \sqrt{\frac{8}{\pi}} \right) A_i \left( \frac{m_e}{M_{N z_{in}}} \right)^{\frac{1}{2}} (A_i - 1) \eta_i^{A_i - 1} X_p (z_{in}) X_{z_i} (z_{in}) \times X_{A_i - z_i} (z_{in}) \exp \{ \hat{B}_i z_{in} \}, \quad i = 2\,\text{H, 3H,...}.$$  

In the previous equations $\hat{q} = (M_N - M_p)/m_e$, and the quantities $A_i$ and $\hat{B}_i$ denote the atomic number and the binding energy of the $i$-th nuclide normalized to electron mass, respectively. Also note that Eq. (20) is only applied if the resulting abundance is greater than the numerical zero assumed (variable YMIN, whose default setting is $10^{-30}$). Finally, $\eta_i$ is the initial value of the baryon to photon number density ratio at $T = 10\,\text{MeV}$ (for a discussion of how it is related to the final value after $e^+ - e^-$ annihilation stage see e.g. Section 4.2.2 in [3]), and $\phi_e^0$ the solution of the implicit equation

$$\hat{L}(z_{in}, \phi_e^0) = \frac{2 \zeta(3)}{\pi^2} \eta_i \sum_i Z_i X_i (z_{in})$$  \hspace{1cm} (21)
In Tables III and IV are reported the nuclear processes considered in PArthEnoPE. The enumeration shown in the first column of the tables correspond to the order in which they appear in the program. See [9] for the relevant formalism concerning the thermally averaged nuclear rates and an analysis of the main experimental reaction rates. Reactions included in Table II are used when running PArthEnoPE in its simpler version (small network), while those of Tables III and IV are added in the intermediate and complete network running options, which also follows the evolution of the nuclides heavier than $^7\text{Be}$ and $^{12}\text{N}$, respectively. Using the small network gives values of the lighter nuclides like $^2\text{H}$, $^3\text{He}$, $^4\text{He}$ and $^7\text{Li}$ which differ from the results obtained with the complete network for less than 0.02%, for default values of the input cosmological parameters. With respect to the database used in [9], there are a few minor upgrades implemented here, namely the three reactions (98, 99, 100) have been inserted following the analysis of the extended network reported in [22]. Also, we have added the recent data reported in [23] to the regressions for the rates (28, 29), with results in good agreement with those adopted in [9].

### III. NON-STANDARD PHYSICS

In the standard scenario the only free parameter entering the BBN dynamics is the value of the baryon to photon number density $\eta$, or equivalently the baryon energy density parameter $\Omega_Bh^2$, see e.g. [9] for the relation between these parameters. If one goes beyond the standard framework, the BBN predictions may be altered by non-standard physics entering e.g. the neutrino [11, 12, 24, 27] or gravity sector [26, 27], or more generically by the presence in the plasma of other degrees of freedom besides the Standard Model ones [28, 29, 30, 31, 32, 33, 34]. For an earlier review, see [37]. Typically, constraints on non-minimal and/or exotic scenarios require model-dependent modifications of the equations ruling BBN. In PArthEnoPE we implement a few of them described below which are general enough to be commonly used/referred to in the specialized literature.
The allowed range for this parameter in units of MeV

\[ \rho \leq \left( \frac{\rho}{\text{MeV}^4} \right) \leq 1. \]

The reactions used in the intermediate network in addition to those of Table II.

**TABLE III**

A. Energy density of the vacuum, \( \rho_\Lambda \)

As in the original Kawano code [29], we allow for a non-zero cosmological constant term at the BBN epoch. We parameterize it by means of \( \rho_\Lambda \) entering the equations only via

\[ 3H \rightarrow 3H = \sqrt{24 \pi G_N \left( \frac{m_e}{2} \right)^4 \frac{\dot{\rho}}{\rho_\Lambda}}. \]

The allowed range for this parameter in units of \( \text{MeV}^4 \) is \( 0 \leq (\rho_\Lambda/\text{MeV}^4) \leq 1. \)

**TABLE IV**

No. Reaction Type No. Reaction Type
74 \(^{12}\text{C} + n \leftrightarrow \gamma + ^{13}\text{C}\) (n,\(\gamma\))  88 \(^{12}\text{C} + ^4\text{He} \leftrightarrow \gamma + ^{16}\text{O}\) (\(\alpha,\gamma\))
75 \(^{13}\text{C} + n \leftrightarrow \gamma + ^{14}\text{C}\) (n,\(\gamma\))  89 \(^{10}\text{B} + ^4\text{He} \leftrightarrow p + ^{13}\text{C}\) (\(\alpha,p\))
76 \(^{14}\text{N} + n \leftrightarrow \gamma + ^{15}\text{N}\) (n,\(\gamma\))  90 \(^{11}\text{B} + ^4\text{He} \leftrightarrow p + ^{14}\text{C}\) (\(\alpha,p\))
77 \(^{13}\text{N} + n \leftrightarrow p + ^{13}\text{C}\) (n,p)  91 \(^{11}\text{C} + ^4\text{He} \leftrightarrow p + ^{14}\text{N}\) (\(\alpha,p\))
78 \(^{14}\text{N} + n \leftrightarrow p + ^{14}\text{C}\) (n,p)  92 \(^{12}\text{N} + ^4\text{He} \leftrightarrow p + ^{15}\text{O}\) (\(\alpha,p\))
79 \(^{15}\text{O} + n \leftrightarrow ^{14}\text{N}\) (n,p)  93 \(^{13}\text{N} + ^4\text{He} \leftrightarrow p + ^{16}\text{O}\) (\(\alpha,p\))
80 \(^{16}\text{O} + n \leftrightarrow ^4\text{He} + ^{12}\text{C}\) (n,\(\alpha\))  94 \(^{10}\text{B} + ^4\text{He} \leftrightarrow n + ^{13}\text{N}\) (\(\alpha,n\))
81 \(^{12}\text{C} + p \leftrightarrow \gamma + ^{13}\text{N}\) (p,\(\gamma\))  95 \(^{11}\text{B} + ^4\text{He} \leftrightarrow n + ^{14}\text{N}\) (\(\alpha,n\))
82 \(^{13}\text{C} + p \leftrightarrow \gamma + ^{14}\text{N}\) (p,\(\gamma\))  96 \(^{12}\text{B} + ^4\text{He} \leftrightarrow n + ^{15}\text{N}\) (\(\alpha,n\))
83 \(^{14}\text{C} + p \leftrightarrow \gamma + ^{15}\text{N}\) (p,\(\gamma\))  97 \(^{13}\text{C} + ^4\text{He} \leftrightarrow n + ^{16}\text{O}\) (\(\alpha,n\))
84 \(^{13}\text{N} + p \leftrightarrow \gamma + ^{14}\text{O}\) (p,\(\gamma\))  98 \(^{11}\text{B} + ^2\text{H} \leftrightarrow p + ^{12}\text{B}\) \(^2\text{H}\) Strip.
85 \(^{14}\text{N} + p \leftrightarrow \gamma + ^{15}\text{O}\) (p,\(\gamma\))  99 \(^{12}\text{C} + ^2\text{H} \leftrightarrow p + ^{13}\text{C}\) \(^2\text{H}\) Strip.
86 \(^{15}\text{N} + p \leftrightarrow \gamma + ^{16}\text{O}\) (p,\(\gamma\)) 100 \(^{13}\text{C} + ^2\text{H} \leftrightarrow p + ^{14}\text{C}\) \(^2\text{H}\) Strip.
B. Extra degrees of freedom, $\Delta N_{\text{eff}}$

We parameterize the radiation density in non-electromagnetically interacting particles at the BBN epoch by an additional radiation energy density $\rho_X$ entering $H$. This is related to the “number of extra effective neutrino species” customarily used in the literature $\Delta N_{\text{eff}}$ by the equation

$$\rho_X = \frac{7}{8} \frac{\pi^2}{30} \Delta N_{\text{eff}} T_X^4,$$

(23)

where, from the entropy conservation, $T_X = T = m_e/z$ at temperatures higher than the effective neutrino decoupling temperature, chosen as $T_d = 2.3$ MeV, or else

$$T_X = \frac{T}{\rho_{\gamma,e,B}(T_d) + \rho_{\gamma,e,B}(T_d)} \left[ \frac{\rho_{\gamma,e,B}(T)}{\rho_{\gamma,e,B}(T_d) + \rho_{\gamma,e,B}(T_d)} \right]^{1/3}, \quad T < T_d.$$  

(24)

The user may input a value of $\Delta N_{\text{eff}}$ in the range $-3.0 \leq \Delta N_{\text{eff}} \leq 15.0$.

C. Chemical potential of the neutrinos, $\xi$

The usual argument in favor of a cosmic lepton asymmetry is that sphaleron effects before electroweak symmetry breaking equilibrate the lepton and baryon asymmetries to within a factor of order unity, thus producing the observed baryon density. In principle, however, the electron-neutrino degeneracy parameter $\xi = \mu_{\nu_e}/T_{\nu_e}$ as well as the degeneracy parameters of the other neutrino flavors are not determined within the Standard Model, and should be constrained observationally. Recently, it has been realized that the measured neutrino oscillation parameters imply that neutrinos reach approximate chemical equilibrium before the BBN epoch. Thus, all neutrino chemical potentials can be taken to be equal, i.e. they are all characterized by the same degeneracy parameter $\xi$ that applies to $\nu_e$ [36, 37, 38]. In light of these results it is meaningful to assume a single and shared value $\xi$ as the only free input parameter. Also, to achieve an approximate agreement between the observed and predicted light element abundances a possible lepton asymmetry must be small, $|\xi| \ll 1$. For such small $\xi$ values the most important impact on BBN is a shift of the beta equilibrium between protons and neutrons. A subleading effect is a modification of the radiation density,

$$\Delta N_{\text{eff}}(\xi) = 3 \left[ \frac{30}{7} \left( \frac{\xi}{\pi} \right)^2 + \frac{15}{7} \left( \frac{\xi}{\pi} \right)^4 \right].$$

(25)

Moreover, the neutrino decoupling temperature is higher than in the standard case [39, 40], so that in principle one could get a non-standard $T_{\nu}(T)$ evolution, but such effects are completely negligible for our case. A non-zero $\xi$ slightly modifies the partial neutrino reheating following the $e^+e^-$ annihilation [41], again a completely negligible effect for the range of $\xi$ of our interest. In the code, we allow the user to select among 21 possible values of $\xi$, between -1.0 and +1.0, spaced by 0.1. The changes in the weak reactions are then automatically implemented, as in [8]. The associated change in $\Delta N_{\text{eff}}$ of Eq. (25) is also accounted for. Note that to derive results on a finer grid a perturbative approach as the one in [31] would be required. This possibility is left for an implementation in a future upgraded version of the code.

IV. THE STRUCTURE OF PARTHENOPH

The code is divided in two files, main.f and parthenope.f, the former one containing the main program and the latter the remaining subroutines. While all the physics is implemented in parthenope.f, the file main.f is an interface which can be possibly adapted to the user needs. The user can choose between two running modes: an interactive one, with parameter selections given on the screen, and a card mode requiring an input card, an example of which is provided as the file input (see also Table VI). The program links to the NAG libraries [42] for some algebraic operations and the evaluation of special functions.

The logical structure of the code is depicted in Figure 1. In the following we detail the aim of each block.
FIG. 1: The logical structure of PArthENoPE.

A. MAIN

MAIN contains the interface which allows the user to choose the physical and network input parameters and to customize the output.

Physical parameters presently are: baryon density, number of additional neutrino species, neutron lifetime, neutrino chemical potential, energy density of the vacuum at the BBN epoch.

Network parameters include: the choice among a small (9 nuclides and 40 reactions), an intermediate (18 nuclides and 73 reactions), and a complete network (26 nuclides and 100 reactions). Moreover, the user can change the rates of each reaction included in the chosen network, selecting a ‘LOW’ or a ‘HIGH’ value, based on the experimental or theoretical uncertainties, or a customized multiplicative ‘FACTOR’.

Finally, the output options include: the choice of the nuclides whose evolution has to be followed versus $z$, the name of the output files (a first one with the final results and a second with the evolution of the selected nuclides), and the possibility to follow the status of the evolution on the screen.

| KEYWORD | DESCRIPTION | DEFAULT | RANGE/OPTIONS |
|---------|-------------|---------|---------------|
| OMEGABH | Baryon density $\Omega_B h^2$ | 0.0223 | 0.01 $\div$ 0.03 |
| DNNU    | Number of additional neutrino species | 0. | -3. $\div$ 15. |
| TAU     | Neutron lifetime | 885.7 s | 880. $\div$ 890. |
| IXIE    | Integer fixing the electron neutrino chemical potential | 11 | 1 $\div$ 21 |
| RHOLMBD | Energy density of the vacuum, $\rho_\Lambda$, in MeV$^4$ | 0. | 0. $\div$ 1. |
| NETWORK | Number of nuclides in the network | 9 | 9,18,26 |
| FOLLOW  | Option for following the evolution on the screen | F | T,F |
| OVERWRITE | Option for overwriting the output files | F | T,F |
| FILES   | Name of the output files | parthenope.out | 20 bit string |
|         |                          | nuclides.out | 20 bit string |
| OUTPUT  | Evolution of nuclides | first 9 nuclides | see text |
| RATES   | Details on changed rates | No change | see text |
| EXIT    | Closing keyword in the input card | | |

TABLE V: The list of the possible keywords in the input card, their default values and corresponding ranges/options.
RATES 3 (12 1 0.) (28 3 .4) (29 2 0) options for changing the nuclear rates
RATES 2 (3 2 0.) (5.3 6) options for changing the nuclear rates
TAU 885.7 experimental value of neutron lifetime
DNNU .0 number of extra neutrinos
IXIE 11 integer giving the value of $\nu_e$ chemical potential
RHOLMBD .0 value of cosmological constant energy density at the BBN epoch
OVERWRITE T option for overwriting the output files
FOLLOW T option for following the evolution on the screen
OMEGABH .0223 value of $\Omega_B h^2$
NETWORK 9 number of nuclides in the network
FILES parthenope1.out nuclides1.out names of the two output files
OUTPUT T 3 2 3 4 options for customizing the output
EXIT terminates input

| TABLE VI: An example of input card |
|------------------------------------|

All this information can be provided either interactively, following the on-screen instructions, or by an input card, with the format of the example card included in the distribution. In particular, each line in this card must start with an allowed key and the last line key has to be ‘EXIT’. In order to be recognized, each key must start at the first bit of the line. The allowed keywords are listed in Table V together with the default values adopted by the code whenever the corresponding key is not explicitly set.

An example of input card is shown in Table VI. While the keys OMEGABH, DNNU, TAU, IXIE, RHOLMBD, NETWORK, FOLLOW, and OVERWRITE have only one argument, the keyword FILES has two arguments, that is the two names of the output files, each one at most 20 bits long. Some more details deserve the two keywords OUTPUT and RATES. OUTPUT can have at most $N_{nucl} + 2$ arguments ($N_{nucl}$ being the number of nuclides of the chosen network), which are: 1) a bit equal to ‘T’ or ‘F’, if the user wants or not to store in the output the evolution of a given set of nuclide abundances, 2) the total number of such nuclides, and 3) the identity of these nuclides (given as the corresponding number in Table I). In the example of Table VI with the sequence T 3 2 3 4 the user has chosen to store in the output the three nuclides $p$, $^2H$ and $^3H$. The keyword RATES allows to change the default values of the nuclear rates used in the chosen network. The input card can have more than one line with this keyword, as in the example of Table VI. Each line contains: 1) an integer $k$, giving the number of reactions whose change is specified on that line; 2) the kind of change for the $k$ reactions with the syntax $(m i f)$, indicating that the reaction number $m$ (see Tables II, III and IV) has to be changed according to the type of change $i$, with the factor $f$. Obviously, $m$ can assume the values 1,...,M (M being the number of reactions of the chosen network), while $i=1,2,3$ corresponds to the ‘LOW’, ‘HIGH’, or ‘FACTOR’ type of change, respectively. Whenever a statistically sound analysis is possible, as it is the case for most of the main reactions, the ‘LOW’/‘HIGH’ rates represent 1 $\sigma$ lower/upper limits to the rate, as compiled in I. For most of the subleading reactions, they represent estimated ranges of variability, obtained from the literature. The main (or unique) reference from which the rate has been taken is reported as a comment in the code next to the related reaction line. Finally, if $i=3$ the real number $f$ is the value of the multiplicative factor applied to the chosen reaction rate (not considered if the options $i=1,2$ are selected). For example, the first line of the input card of Table VI specifies that 3 reaction rates should be changed in running ParthEnoPE as follows

$$p + n \leftrightarrow \gamma + ^2H \quad \text{low rate}$$
$$^2H + ^2H \leftrightarrow n + ^3He \quad \text{rate multiplied by the factor 0.4}$$
$$^2H + ^2H \leftrightarrow p + ^3H \quad \text{high rate}$$

Notice that it is possible to add comments after the parameters in the input card and the order of the lines with different keywords is not important.

B. Parthenope

This subroutine drives the resolution of the BBN set of equations. It starts calling the initialization routine INIT, then the NAG solver, finally the output printing routine OUTEND. The NAG resolution parameters, controlling for example the resolution method and the numerical accuracy, have been chosen to optimize the performances of the
NAG solver. Any change of these parameters should be implemented only after a careful reading of the NAG manual.

Further relevant parameters are $z_{\text{in}}$ and $z_{\text{end}}$, setting respectively the initial and final value of the independent variable $z$. Their present values correspond to the two temperatures of $T_i = 10\text{ MeV}$ and $T_f = 1/130\text{ MeV}$. Note that a few settings depend on these values, which then should be varied with caution.

C. INIT

Besides initializing the nuclear parameters, this subroutine calculates the initial values for all nuclide abundances and the electron chemical potential, the latter requiring the inversion of the implicit equation with a NAG routine.

D. FCN, THERMO, RATE, EQSLIN

The subroutine FCN is required by the NAG solver to calculate the right hand side of the differential BBN equations. In order to do this, the thermodynamical quantities which appear in the equations are evaluated with a call to the subroutine THERMO. The second step is the calculation of the reaction rates with the subroutine RATE. Then the linearization of the set of equations is performed, with the construction of a corresponding $N_{\text{nuc}} \times N_{\text{nuc}}$ matrix ($N_{\text{nuc}}$ being the number of nuclides). In this way, the unknown functions appear in a linear equation system, solved by Gaussian elimination in the subroutine EQSLIN (this method is very similar to the one applied in the Kawano code).

E. OUTEVOL, OUTEND

The subroutine OUTEVOL is called during the evolution, for printing the intermediate values of the chosen nuclide abundances in one of the output files. Moreover, if requested by the user, this subroutine allows to follow the resolution evolution, printing some physical quantities on the screen. Finally, OUTEND prints the final values of the nuclide abundances and electron chemical potential in the other output file along with some technical information on the differential evolution resolution. The final yield of the $i$-th nuclide is expressed as the ratio $X_i/X_p$, i.e. number density normalized to hydrogen. The only exceptions are Hydrogen expressed as $X_p$ and $^4\text{He}$, which is conventionally reported in terms of the (approximate) mass fraction $Y_p = 4X_{^4\text{He}}$.

V. MAIN DIFFERENCES WITH RESPECT TO THE WAGONER-KAWANO CODE

As we have previously emphasized, the public Kawano code was the starting point for the development of PArthEnoPE. So, they have a similar structure, like the subdivision in several subroutines which contain, for example, the interface with the user, the calculation of nuclear rates, of the thermodynamical quantities, the differential equation solver, and the production of the output. Here we summarize the main physical and numerical differences with respect to the original code:

- The interface menu and the options are now different. In particular, the user can more easily implement changes in the nuclear reaction network. A card-mode input is available. The output is easily customized.

- Several numerical routines have been replaced with more efficient algorithms, most of which using NAG routines.

- Improved calculations for the $n-p$ reactions are implemented via new fits, not as effective corrections added a posteriori. They also include effects of finite nucleon mass and non-thermal neutrino spectral distortions. The same holds for the case with neutrino asymmetry.

- An improved and more accurate calculation of thermodynamical variables is implemented, both in the electromagnetic sector and the neutrino one. In particular, for the latter case we do not simply impose entropy conservation, but take into account entropy transfer during the $e^+ - e^-$ annihilation phase.

- We implemented an updated nuclear reaction network, obtained using new data and reduction techniques. Also, new reactions and new types of reactions (different number of nuclides in the initial and final states) are now included. The new code has been restructured so that it is easier to implement new reactions and that only
FIG. 2: The relative difference in percent of nuclei abundances between PArthEnOPE and the Kawano code, $\Delta X_i/\bar{X}_i \equiv 2(X_i^P - X_i^K)/(X_i^P + X_i^K)$, versus baryon density $\Omega_B h^2$. The solid (dashed) curves are for positive (negative) values.

fundamental nuclear data are needed as input. All derived quantities are calculated in the code. This is aimed at simplifying future updates.

• We improved the numerical resolution of the coupled BBN equations, by using a multistep method, belonging to the class of Backward Differentiation Formulas (instead of the traditional Runge-Kutta solver of the Kawano code which is a single step-method) implemented by a NAG routine. The default values of the accuracy parameters have been chosen to guarantee a good compromise between the accuracy goal and a reasonable running time. In particular the relative accuracy reached on $^4$He mass fraction is of the order of $10^{-4}$, thus keeping the numerical error below the level of theoretical uncertainties.

In Figure 2 we show the relative variation (in percent) of nuclear abundances computed with PArthEnOPE and using the original Kawano code vs. baryon density, for the fiducial values of all the other parameters. To give an idea of the differences introduced by accounting for the major updates to the physics implemented in the code, in Table VII we report, for $\Omega_B h^2 = 0.022$ (i.e. consistent with the value singled out by the WMAP Collaboration [1]), the relative variation with respect to the Kawano code results of the theoretical prediction of abundances in three cases: 1) improved treatment of n-p weak rates, as presently in PArthEnOPE, but original nuclear network as in Kawano code (first column); 2) improved treatment of n-p weak rates as presently in PArthEnOPE and updated nuclear network as in Ref. 9 (second column); 3) PArthEnOPE code, with improved calculation of thermodynamical variables, both in the electromagnetic sector and the neutrino one, and complete improvement of the nuclear network (third column).

In summary, the PArthEnOPE prediction for $^4$He differs with respect to the original Kawano code by 0.7%, mainly due to the improved treatment of radiative corrections, finite temperature and finite nucleon mass corrections in neutron-proton weak rates. Actually, this value is larger than the theoretical accuracy of PArthEnOPE on $Y_p$, which is of the order of 0.2%, see also 9. On the other hand, our results are, for example, in very good agreement with 43, where all mentioned corrections to weak rates are included as discussed in details in 35. Rescaling the value of neutron lifetime to the values $\tau_n = 886.7$ s adopted in 43 and using their fit for $Y_p$ the agreement is at the 0.1% level. Concerning the other nuclides, the variation of the theoretical predictions with respect to the Kawano code reaches larger values, of the order of 3% for $^6$Li, at the 1% level for $^7$Li and $^7$Be, while it is very small for $^2$H, as in this case
| Nuclide | weak rates (%) | nucl. rates (%) | Parthenope (%) |
|---------|----------------|----------------|----------------|
| $^3$H   | 1.5            | 3.2            | 0.2            |
| $^3$H   | 3.3            | 2.3            | -0.7           |
| $^4$He  | 0.1            | -3.4           | -1.1           |
| $^4$He  | 0.4            | 0.4            | 0.7            |
| $^6$Li  | -2.7           | 12.2           | 2.8            |
| $^7$Li  | -1.8           | 2.9            | 1.3            |
| $^7$Be  | -2.0           | 2.5            | 1.6            |
| $^9$Be  | -0.1           | 200.0          | 200.0          |
| $^{10}$B| 1.3            | -150.0         | -153.8         |
| $^{11}$C| 200.0          | 146.0          | 153.8          |
| $^{12}$C| 0.4            | 0.5            | -43.6          |

TABLE VII: The relative variation in percent of the theoretical predictions between PArthENoPE and the Kawano code, $\Delta X_i/X_i \equiv 2(X_P^i - X_K^i)/(X_P^i + X_K^i)$, is shown for three cases (see text). The results are shown for $\Omega_B h^2 = 0.022$ and standard values for all other physical parameters.

the effects of improvements on weak rates treatment and updated nuclear rate network have different sign with respect to the one in plasma and neutrino treatment and almost cancel out accidentally. We notice that the introduction of new processes (reactions 98, 99, 100 of Table IV, see [22], and the update of those already considered in the Kawano code result in a large difference in the BBN theoretical prediction for metallicity, although probably insufficient to change the chemistry of primordial clouds. Finally, the reader may want to consider similar comparisons performed in the literature between updated versions of BBN codes used, e.g. in [44], [45] and the already quoted [43], and the results of the Kawano code presented in [21].

VI. CONCLUSIONS

In this paper we have described the general structure and features of PArthENoPE, a new numerical code which computes the theoretical abundances of nuclei produced during BBN, as function of several input cosmological parameters. This code has been recently made public and can be obtained at the URL http://parthenope.na.infn.it/. The code evaluates the abundances of 26 nuclide in the standard BBN scenario, as well as in extended models allowing for extra relativistic particles or neutrino chemical potential. We checked for a few fiducial cases that the results for the $^4$He abundance agree very closely with those of Ref. [43], where the appropriate corrections [35] are applied to the original Wagoner/Kawano code [13][20].

In view of the improved data coming from astrophysical observations, accurate tools providing theoretical predictions on cosmological observables are required to check the overall consistency of the picture of the evolution of the Universe, as well as for investigating and constraining new physics beyond the present framework of fundamental interactions.

Much effort has been put in the recent years by several groups in order to increase the level of accuracy of theoretical prediction on nuclide abundances, in particular by improving the estimate of the neutron to proton weak conversion rates and the nuclear network rates. The results of these studies, along with a to date analysis of experimental results on relevant nuclear reactions have been implemented in PArthENoPE, which hopefully will turn useful as an accurate tool for BBN-related studies.

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APPENDIX A: DERIVATION OF THE PARTHENOPE SET OF EQUATIONS

We define \( z \equiv m_e/T, \ x = m_e a, \ \tilde{z} = x/z = aT = T/T_\nu, \ \hat{n}_B = m_e^{-3} n_B \) and introduce the following quantities:

\[
\mathcal{N}(z) = \frac{1}{z} \left( x \frac{d}{dx} \hat{\rho}_\nu \right) \bigg|_{x=x(z)} , \quad \hat{\rho}_\nu = a^4 \rho_\nu = \left( \frac{x}{m_e} \right)^4 \rho_\nu , \quad (A1)
\]

\[
\rho = \rho_{e\gamma B} + \rho_\nu , \quad p = p_{e\gamma B} + p_\nu , \quad (A2)
\]

\[
\dot{\rho} = T^{-4} \rho = \left( \frac{z}{m_e} \right)^4 \rho , \quad \dot{p} = T^{-4} p = \left( \frac{z}{m_e} \right)^4 p . \quad (A3)
\]

Starting from Eq.s (5) and (6),

\[
\text{For the neutrino energy density one gets}
\]

\[
\dot{\rho}_\nu = \frac{d\rho_\nu}{dx} = m_e \frac{d\rho_\nu}{dx} = H x \frac{d\rho_\nu}{dx} = \left( \frac{m_e}{x} \right)^4 H x \frac{d\rho_\nu}{dx} - 4 \dot{\rho}_\nu , \quad (A4)
\]

\[
\dot{\rho}_\nu = 4 \dot{p}_\nu - 3 H (\rho_{e\gamma B} + \rho_\nu) - 4 H (\rho_\nu + 4 H \rho_\nu) . \quad (A5)
\]

and separating the neutrino contribution one gets

\[
\dot{\rho}_{e\gamma B} = -3 H \rho_{e\gamma B} + \rho_\nu - 3 H (\rho_{e\gamma B} + \rho_\nu) - 4 H \rho_\nu , \quad (A6)
\]

where in (A7) we have used \( \rho_\nu = 3 p_\nu \). From Eq. (A6) one gets the time derivative

\[
\dot{\rho} = \frac{d\rho}{dz} = m_e \frac{d\rho}{dz} = H \frac{d\rho}{dz} = \left( \frac{m_e}{x} \right)^4 H \frac{d\rho}{dz} - 4 \rho \quad \text{and thus}
\]

\[
\dot{\rho}_{e\gamma B} = -3 H (\dot{\rho}_{e\gamma B} + \dot{p}_{e\gamma B}) + 4 \dot{\rho}_{e\gamma B} - 3 H (\rho_{e\gamma B} + \rho_\nu) - 4 H (\rho_\nu + 4 H \rho_\nu) . \quad (A7)
\]

For the neutrino energy density one gets

\[
\dot{\rho}_\nu = \frac{d\rho_\nu}{dz} = m_e \frac{d\rho_\nu}{dz} = H \frac{d\rho_\nu}{dz} = \left( \frac{m_e}{x} \right)^4 H \frac{d\rho_\nu}{dz} - 4 \rho_\nu + 4 H \rho_\nu , \quad (A8)
\]

Hence substituting (A10) in (A9) we obtain

\[
\dot{\rho}_{e\gamma B} = 4 \frac{z}{z} \dot{\rho}_{e\gamma B} - 3 H (\dot{\rho}_{e\gamma B} + \dot{p}_{e\gamma B}) - H \mathcal{N}(z) . \quad (A11)
\]

On the other hand, the total time derivative of \( \dot{\rho}_{e\gamma B} \) can be expressed via the partial derivatives with respect to \( z, \dot{\phi}_e \) and \( X_i \),

\[
\dot{\rho}_{e\gamma B} = \frac{\partial \rho_{e\gamma B}}{\partial z} \dot{z} + \frac{\partial \rho_{e\gamma B}}{\partial \dot{\phi}_e} \dot{\phi}_e + \sum_i \frac{\partial \rho_{e\gamma B}}{\partial X_i} \dot{X}_i = \left( \frac{\partial \rho_{e\gamma B}}{\partial z} + \frac{\partial \rho_{e\gamma B}}{\partial \dot{\phi}_e} \frac{d\phi_e}{dz} + \sum_i \frac{\partial \rho_{e\gamma B}}{\partial X_i} \frac{dX_i}{dz} \right) \dot{z} . \quad (A12)
\]

Thus, equating the r.h.s of (A11) and (A12) after some rearrangement reads

\[
\left( \frac{\partial \rho_{e\gamma B}}{\partial z} - 4 \frac{z}{z} \rho_{e\gamma B} + \frac{\partial \rho_{e\gamma B}}{\partial \dot{\phi}_e} \frac{d\phi_e}{dz} + \sum_i \frac{\partial \rho_{e\gamma B}}{\partial X_i} \frac{dX_i}{dz} \right) \dot{z} = -3 z H (\dot{\rho}_{e\gamma B} + \dot{p}_{e\gamma B}) - H \mathcal{N}(z) . \quad (A13)
\]

Starting from (A5) and proceeding in the same way leads to

\[
\left( \frac{\partial \hat{n}_B}{\partial z} + \frac{\partial \hat{n}_B}{\partial \dot{\phi}_e} \frac{d\phi_e}{dz} + \sum_i \frac{\partial \hat{n}_B}{\partial X_i} \frac{dX_i}{dz} \right) \dot{z} = -3 z H \dot{\hat{n}}_B . \quad (A14)
\]
Obtaining \( \dot{z} \) from (A14) and substituting into (A13) we obtain

\[
- 3 H \hat{n}_B \frac{\partial \hat{n}_B}{\partial z} - \frac{\dot{3}}{2} \hat{\rho}_{e\gamma} B + \frac{\partial \hat{n}_B}{\partial \phi_e} \frac{d\phi_e}{dz} + \sum_i \frac{\partial \hat{n}_B}{\partial X_i} \frac{dX_i}{dz} = - 3 z H (\hat{\rho}_{e\gamma} B + \hat{\rho}_{e\gamma} B) - H \mathcal{N}(z) .
\]  

(A15)

By using Eq. (S)

\[
\dot{n}_B = \frac{\hat{L}(z, \phi_e)}{z^3 \sum_i Z_i X_i},
\]  

(A16)

one can express \( \dot{n}_B \) and its derivatives as function of \( \hat{L}(z, \phi_e) \) which is defined in (11). By solving Eq. (A15) with respect to \( d\phi_e/dz \) one gets

\[
\frac{d\phi_e}{dz} = \frac{1}{\hat{\rho}_{e\gamma} B} \frac{\hat{L} \kappa_1 + \left( \hat{\rho}_{e\gamma} B + \hat{\rho}_{e\gamma} B + \frac{\mathcal{N}(z)}{3} \right) \kappa_2}{\hat{L} \frac{\partial \hat{\rho}_{e\gamma} B}{\partial \phi_e} + \hat{\rho}_{e\gamma} B + \frac{\mathcal{N}(z) + \mathcal{N}(z)}{3}},
\]  

(A17)

where

\[
\kappa_1 = 4 (\hat{\rho}_e + \hat{\gamma}) + \frac{3}{2} \hat{\rho}_B - \frac{3 \hat{\rho}_{e\gamma} B - \frac{1}{L} \left( 3 \hat{L} - \frac{3 \hat{L}}{z} \right) \hat{\rho}_B - \sum_i Z_i X_i \left( \Delta \hat{M}_i + \frac{3}{2z} \right) \hat{\Gamma}_i,
\]  

(A18)

\[
\kappa_2 = \frac{z \frac{\partial \hat{L}}{\partial z} - \frac{3 \hat{L} - \frac{3 \hat{L}}{z} \sum_i Z_i X_i}{\frac{\partial \hat{L}}{\partial z} \sum_j Z_j X_j}}.
\]  

(A19)

According to our notations, \( \Delta \hat{M}_i \) and \( \hat{M}_i \) stand for the i-th nuclide mass excess and the atomic mass unit, respectively, normalized to \( m_e \), whereas \( H \equiv m_e \bar{H} \) and \( \Gamma_i \equiv m_e \bar{\Gamma}_i \). By substituting in (A14) the expression obtained for \( d\phi_e/dz \) in (A17) we get

\[
\dot{z} = -3 H \frac{\partial \hat{n}_B}{\partial \phi_e} \left( \frac{\hat{\rho}_{e\gamma} B + \hat{\rho}_{e\gamma} B + \frac{\mathcal{N}(z)}{3}}{3 \hat{L} \frac{\partial \hat{\rho}_{e\gamma} B}{\partial \phi_e} + \hat{\rho}_{e\gamma} B + \frac{\mathcal{N}(z) + \mathcal{N}(z)}{3}} \right),
\]  

(A20)

namely

\[
\frac{dt}{dz} = -3 H \Delta \hat{M}_i \frac{\partial \hat{n}_B}{\partial \phi_e} \left( \hat{\rho}_{e\gamma} B + \hat{\rho}_{e\gamma} B + \frac{\mathcal{N}(z)}{3} \right).
\]  

(A21)

The equations for the abundances (7) then become

\[
\frac{dX_i}{dz} = \frac{\hat{G}_i}{dz} \frac{dt}{dz} = -3 H \frac{\partial \hat{n}_B}{\partial \phi_e} \left( \hat{\rho}_{e\gamma} B + \hat{\rho}_{e\gamma} B + \frac{\mathcal{N}(z)}{3} \right) \]  

(A22)

The solution of neutrino dynamics performed in [13, 14, 15] allows to compute the quantity

\[
\mathcal{N}(z) = \left\{ \begin{array}{ll} 
\exp \left( \sum_{t=1}^{13} m_t z^t \right) & \text{if } z < 4 \\
0 & \text{if } z \geq 4 
\end{array} \right.
\]  

(A23)

Notice that \( \mathcal{N}(z) \) would vanish for purely thermal neutrinos, and it is strictly related to the small entropy transfer to neutrinos during the \( e^+ - e^- \) annihilation stage. In the code, \( \mathcal{N}(z) \) is calculated by using the following fit, which is accurate to better than 0.1\% in the relevant range:

\[
\mathcal{N}(z) = \left\{ \begin{array}{ll} 
\exp \left( \sum_{t=1}^{13} m_t z^t \right) & \text{if } z < 4 \\
0 & \text{if } z \geq 4 
\end{array} \right.
\]  

(A24)
with

\[ n_0 = -10.21703221236002 \quad n_1 = 61.24438067531452 \]
\[ n_2 = -340.3323864212157 \quad n_3 = 1057.2707914654834 \]
\[ n_4 = -2045.577491331372 \quad n_5 = 2605.9087171012848 \]
\[ n_6 = -2266.152815470196 \quad n_7 = 1374.2623075963388 \]
\[ n_8 = -586.0618273295763 \quad n_9 = 174.87532902234145 \]
\[ n_{10} = -35.715878215468045 \quad n_{11} = 4.7538967685808755 \]
\[ n_{12} = -0.3713438862054167 \quad n_{13} = 0.012908416591272199 \]

(A25)
### APPENDIX B: COMMON VARIABLES USED IN PARTHENOPE

| VARIABLE       | DESCRIPTION                                                                 | COMMON  |
|----------------|-----------------------------------------------------------------------------|---------|
| AA(NNUC)       | Nuclide atomic numbers, $A_i$                                               | ANUM    |
| FACTOR(NREC)   | Multiplicative factor for the rate of reaction $i$-th                      | CHRATE  |
| HCHRAT(NREC)   | Type of changes adopted for reaction $i$-th                                |         |
| NCHRATE        | Number of reactions to be changed                                           |         |
| WCHRAT(NREC)   | Reactions to be changed                                                    |         |
| ALF            | Fine structure constant, $\alpha$                                          | CONSTANTS |
| COEF(4)        | Unity conversion factors                                                   |         |
| GN             | Newton constant, $G_N$, in MeV$^{-2}$                                      |         |
| ME             | Electron mass, $m_e$                                                       |         |
| MU             | Atomic mass unit, $M_u$                                                    |         |
| PI             | $\pi$                                                                      |         |
| IFCN           | Counter                                                                    | COUNTS  |
| IFCN1          | Counter                                                                    |         |
| ISAVE1         | Counter                                                                    |         |
| ISAVE2         | Counter                                                                    |         |
| ISTEP          | Counter                                                                    |         |
| DZ             | Stepsize of the independent variable in the resolution of the nucleosynthesis equations | DELTAZ |
| DZ0            | Initial value for DZ                                                       |         |
| DM(0:NNUC)     | Mass excesses in MeV, $\Delta M_i$                                        | DMASS   |
| DMH(NNUC)      | Adimensional mass excesses, $\Delta M_i/m_e$                              | DMASSH  |
| PHI            | Adimensional electron chemical potential, $\phi_e$                        | ECHPOT  |
| AG(NNUC,4)     | Nuclear partition function coefficients                                     | GPART   |
| YY0(NNUC+1)    | Initial values of electron chemical potential, $\phi_e$, and nuclide abundances, $X_i$ | INABUN  |
| CMODE          | Flag for the choice of the running mode                                    | INPCARD |
| FOLLOW         | Option for following the evolution on the screen (card mode)              |         |
| OVERW          | Option for overwriting the output files (card mode)                        |         |
| INC            | Maximum value of the flag for the convergence of the matrix inversion     | INVFLAGS|
| MBAD           | Error flag for the matrix inversion                                        |         |
| LH0            | Initial value for the adimensional electron/positron asymmetry, $L$        | INVPHI  |
| Z0             | Initial value for the evolution variable $z = m_e/T (=ZIN)$                |         |
| AMAT(NNUC,NNUC)| Matrix involved in the linearization of the relation between $X_i(z+dz)$ and $X_i(z)$ | LINCOEF |
| BVEC(NNUC)     | Vector involved in the linearization of the relation between $X_i(z-dz)$ and $X_i(z)$ (contains $X_i$ in reverse order) |         |
| YX(NNUC)       | $X_i(z)$ in reverse order                                                  |         |
| YMIN           | Numerical zero of nuclide abundances                                       | MINABUN |
| DNNU           | Number of extra effective neutrinos, $\Delta N_{eff}$                     | MODPAR  |
| DNNUXI         | Contribution to the number of extra effective neutrinos from a non zero neutrino chemical potential, $\Delta N_{eff}$ of Eq. (25) |         |
| ETAF           | Final value of the baryon to photon density ratio, $\eta_f$               |         |
| IXIE           | A positive integer fixing the electron neutrino chemical potential         |         |
| RHOLMBD        | Energy density corresponding to a cosmological constant, $\rho_{\Lambda}$  |         |
| TAU            | Value of neutron lifetime in seconds, $\tau_n$                            |         |
| XIE            | Electron neutrino chemical potential, $\xi$ ($=XIE0$)                     |         |
| XIE0(NXIE)     | Electron neutrino chemical potential, $\xi$                               | NCHPOT  |
| INUC           | Number of nuclides in the selected network                                 | NETWRK  |
| IREC           | Number of reactions among nuclides in the selected network                 |         |
| IXT(30)        | Code of the nuclides whose evolution has to be followed (ixt(30)=control integer) |         |
| IXE            | Number of additional nuclides whose evolution has to be followed           |         |
| VARIABLE         | DESCRIPTION                                                      | COMMON   |
|------------------|------------------------------------------------------------------|----------|
| BYY(NNUC+1)      | Text strings for the output                                      | NSYMB    |
| MN               | Neutron mass, $M_n$                                              | NUCMASS  |
| MP               | Proton mass, $M_p$                                               |          |
| NAMEFILE1        | Name of the output file for the final values of the nuclide abundances | OUTFILES |
| NAMEFILE2        | Name of the output file for the evolution of the nuclides whose evolution has to be followed |          |
| NBH              | Adimensional baryon number density, $n_B/m_e^3$                 | OUTVAR   |
| THETAH           | Adimensional Hubble function times 3, $3\dot{H}$               |          |
| TXH              | Neutrino to photon temperature ratio, $T_X/T$                   |          |
| DZP              | Previous iteration value of the step-size of the independent variable in the resolution of the nucleosynthesis equations | PREVVAL  |
| SUMMYP           | Value of the linear combination $\sum_i \left( \Delta M_i + \frac{2}{3} \right) \hat{\Gamma}_i$ |          |
| SUMZYP           | Value of the linear combination $\sum_i Z_i \hat{\Gamma}_i$    |          |
| ZP               | Previous iteration value of the evolution variable $z = m_e/T$ |          |
| CFLAG            | Flag for the input variable type in the card reading (card mode) | READINP  |
| IKEY             | Progressive argument key number in the card reading (card mode) |          |
| ISTART           | Starting point of the line in the card reading (card mode)      |          |
| DNCHRAT          | Number of reactions to be added to the changed ones             |          |
| LINE             | Line input from the card file (card mode)                       |          |
| IFORM(NREC)      | Reaction type (1-12)                                            | RECPAR   |
| NG(NREC)         | Number of incoming nuclides of type TG                          |          |
| NH(NREC)         | Number of incoming nuclides of type TH                          |          |
| NI(NREC)         | Number of incoming nuclides of type TI                          |          |
| NJ(NREC)         | Number of incoming nuclides of type TJ                          |          |
| NK(NREC)         | Number of incoming nuclides of type TK                          |          |
| NL(NREC)         | Number of incoming nuclides of type TL                          |          |
| Q9(NREC)         | Energy released in reaction (in unit of $10^9$ K)               |          |
| REV(NREC)        | Reverse reaction coefficient                                     |          |
| TG(NREC)         | Incoming nuclide type                                           |          |
| TH(NREC)         | Outgoing nuclide type                                           |          |
| TI(NREC)         | Incoming nuclide type                                           |          |
| TJ(NREC)         | Incoming nuclide type                                           |          |
| TK(NREC)         | Outgoing nuclide type                                           |          |
| TL(NREC)         | Outgoing nuclide type                                           |          |
| RATEPAR(NREC,13) | Reaction parameter values ($=\text{IFORM}+\text{TI}+...+\text{NI}+...$) | RECPAR0  |
| RSTRING(NREC)    | Reaction text strings                                           | RSTRINGS |
| LH               | Function $\hat{L}$                                              | THERMQ   |
| LHPHI            | Derivative of $\hat{L}$ with respect to $\phi_e$                |          |
| LHZ              | Derivative of $\hat{L}$ with respect to $z$                     |          |
| NAUX             | Neutrino auxiliary function, $N(z)$                            |          |
| PBH              | Adimensional baryon pressure, $\hat{\rho}_B$                   |          |
| PEH              | Adimensional electron pressure, $\hat{\rho}_e$                  |          |
| PGH              | Adimensional gamma pressure, $\hat{\rho}_\gamma$               |          |
| RHOBH            | Adimensional baryon energy density, $\hat{\rho}_B$             |          |
| ROEH             | Adimensional electron energy density, $\hat{\rho}_e$            |          |
| ROEHPHI          | Derivative of $\hat{\rho}_e$ with respect to $\phi_e$          |          |
| ROEHZ            | Derivative of $\hat{\rho}_e$ with respect to $z$               |          |
| ROGH             | Adimensional gamma energy density, $\hat{\rho}_\gamma$         |          |
| ROGHZ            | Derivative of $\hat{\rho}_\gamma$ with respect to $z$          |          |
| RHOH             | Adimensional total energy density, $\hat{\rho}$                 |          |
| GNUC(NNUC)       | Nuclide spin degrees of freedom                                 | SPINDF   |
| VARIABLE     | DESCRIPTION                                                                 | COMMON |
|--------------|------------------------------------------------------------------------------|--------|
| A(13)        | Forward weak reaction best-fit parameters (non degenerate case)              |        |
| B(10)        | Reverse weak reaction best-fit parameters (non degenerate case)             |        |
| DA(12,NXIE)  | Forward weak reaction best-fit parameters (degenerate case)                 |        |
| DB(12,NXIE)  | Reverse weak reaction best-fit parameters (degenerate case)                 |        |
| QNP          | Forward weak reaction best-fit exponent parameter                           |        |
| QNP1         | Forward weak reaction best-fit exponent parameter                           |        |
| QPN          | Reverse weak reaction best-fit exponent parameter                           |        |
| QPN1         | Reverse weak reaction best-fit exponent parameter                           |        |
| ZZ(NNUC)     | Nuclide atomic charges, $Z_i$                                               | ZNUM   |

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