TOPOLOGICAL PROPERTIES FROM EINSTEIN’S EQUATIONS?

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Abstract

In this work we propose a new procedure for to extract global information of a space-time. We considered a space-time immersed in a higher dimensional space and we formulate the equations of Einstein through of the Frobenius conditions to immersion. Through of an algorithm and the implementation into algebraic computing system we calculate normal vectors from the immersion to find out the second fundamental form. We make a application for space-time with spherical symmetry and static. We solve the equations of Einstein to the vacuum and we obtain space-times with different topologies.

1 Introduction

We are living a fertile period of attempts to solve great problems not solved of the physics. We can mention some of them, for instance in the cosmology area: What is the true cause of the accelerating expansion we seem to it observes now, and is it likely to it continues indefinitely into the future? What lack in the theory of the general relativity to explain this accelerated expansion in a consistent way? If the data astrophysicists are correct we needed to do a revision in the physics foundations, as for instance in the theory of the general relativity.

The Einstein’s equations are local field equations describing local structure of the space-time. The central idea in the formulation of the field equations is in the question of how mass-energy generates curvature. One of

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requirement essential this formulation is conservation of the mass-energy and it results the need of we define a geometric quantity that satisfies this condition. This quantity is the Einstein tensor that is composed as a combination of the Ricci’s curvature and scalar of curvature, which are contractions of the tensor of Riemann. This last one is calculated with the metric. The solution of the equation of Einstein is a metric one that represents the gravitational field. Of this formulation we don’t have information which is the shape of the space-time, solution of the field equations. From Einstein equations we have no global or topological information. The Einsteins equations being partial differential equations, they describe only local properties of space-time. The reason is in the fact that when we calculated the tensor of Riemann with the metric we cannot distinguish the shape of the space-time. A trivial example is in the geometry of surfaces, where we don’t get to differentiate the shape between a plan and a cylinder from the Gaussian curvature. However, when we observed these surfaces immersed in $\mathbb{R}^3$, we can distinguish the shape of the surfaces through the extrinsic curvature, when we used the equation of Gauss. This is a subtle idea that results in an 'expansion of the equations of Einstein', as we will see along this note.

The idea of extra dimensions at the space-time is very old, we can mention the work of Kaluza, a theory that requests high energies. Today we are reviving this idea in other approaches, as in the theory of strings, brane-world or even in other independent models of these where we considered a space-time immersed in a space of larger dimension, confinement of the gauge fields and the fact of only gravitons to escape for the extra dimensions, besides requesting energy in a 'low scale'.[1][5]

In this communication we propose a procedure where the space-times are immersed in higher dimensional spaces, natural hypothesis requested for us to acquire topological information of the space-time comings of the solution of the equations of Einstein. As example, we will take two isometric immersions of a space-time with spherical symmetry and static in a space pseudo-euclidean, in each case, we calculate the second fundamental form and through the equation of Gauss we calculate the tensor of Einstein. Soon after we solve the equations of Einstein to the vacuum we obtain space-times with different topologies.
Now we make the usual construction of the a metric with spherical symmetry and we solve the equations of field of the gravitation to vacuum with static source and spherical symmetry. The solution is assumed to be spherically symmetric, static and into vacuum. Simplifying the general metric with spherical symmetry, for the usual method to find out the metric,

\[ ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \] (1)

Consider two isometric immersions of a space-time with metric above from the isometric condition,[6]

\[ g_{ij} = Y_{i\mu}Y_{j\nu}\eta_{\mu\nu}, \] (2)

into a pseudo Euclidean manifold of six dimensions, with different signatures:

- The first immersion:

\[ ds^2 = dY_1^2 + dY_2^2 - dY_3^2 - dY_4^2 - dY_5^2 - dY_6^2. \]

- The second immersion:

\[ ds^2 = dY'_1^2 - dY'_2^2 - dY'_3^2 - dY'_4^2 - dY'_5^2 - dY'_6^2. \]

Respectively given by

\[
\begin{align*}
Y_1 &= r\sin \theta \sin \phi \\
Y_2 &= r\sin \theta \cos \phi \\
Y_3 &= r\cos \theta \\
Y_4 &= \alpha(r) \\
Y_5 &= A(r)^{1/2} \cos t \\
Y_6 &= A(r)^{1/2} \sin t \\
\end{align*}
\]

and

\[
\begin{align*}
Y'_1 &= r\sin \theta \sin \phi \\
Y'_2 &= r\sin \theta \cos \phi \\
Y'_3 &= r\cos \theta \\
Y'_4 &= \beta(r) \\
Y'_5 &= A(r)^{1/2} \cos \theta t \\
Y'_6 &= A(r)^{1/2} \sin \theta t \\
\end{align*}
\]

where \( d\alpha/dr = F(A(r), B(r)) \) and \( d\beta/dr = F(A(r), B(r)) \)

Through of an algorithm and a implementation into algebraic computing system we calculate normal vectors from the immersion to find out the second fundamental form,[6][7]

\[ b_{ijA} = Y_{ij}^\mu N^\nu A_{\mu\nu}. \]
We substitute those components in the equation of Gauss,

\[ R_{nijk} = g^{AB}(b_{ikA}b_{jnB} - b_{ijA}b_{knB}), \]

for to find the tensor of Riemann and soon after the tensor of Einstein \( G_{ij} \).

We use the field equations \( G_{ij} = 0 \) to find \( A(r) \) and \( B(r) \). We have been finding two relative metrics to the immersions mentioned above, \( g \) and \( g' \) metrics respectively:

\[ ds^2 = (1 - 2m/r)dt^2 - (1 - 2m/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (4) \]

\[ ds^2 = (32m^3/r)\exp(-r/2m)(dv^2 - du^2) - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (5) \]

where \( u = u(Y_6') \) and \( v = v(Y_5') \).

3 Topological Properties from Einstein’s Equations

The Schwarzschild’s solution representing the empty space-time with spherical symmetry outside of a body with spherical mass. Using spherical coordinates \((t, r, \theta, \phi)\) this solution is given by [8]

\[ ds^2 = (1 - 2m/r)dt^2 - (1 - 2m/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (6) \]

where \( M = c^2mG^{-1} \), \( c \) is the speed of light and \( G \) is the gravitational constant, and it coincides with (4). Notice that in these coordinates regions \( r = 0 \) and \( r = 2m \) are singular and need to be removed. When we remove the surface \( r = 2m \), the manifold becomes separated in two disconnected components, one for \( 2m < r < \infty \) and the other for \( 0 < r < 2m \). Since we are dealing with the existence of the metric associated to a physical space, we require a connected space. Therefore, we define the following regions:

a) The exterior Schwarzschild space-time \((V_4, g)\):
\[ V_4 = P^2_I \times S^2; \quad P^2_I = \{(t, r) \in \mathbb{R}^2 \mid r > 2m\} \]
b) The Schwarzschild black hole \((B_4, g)\):
\[ B_4 = P^2_{II} \times S^2; \quad P^2_{II} = \{(t, r) \in \mathbb{R}^2 \mid 0 < r < 2m\} \]

In both cases, \( S^2 \) is the sphere of radius \( r \) and the metric \( g \) is given from (6).

Consider two isometric immersions of space-time \((E, g) = ([P^2 \cup P^2_{II}] \times S^2, g)\), into a pseudo Euclidean manifold of six dimensions, with different signatures:
- The Kasner immersion: \[ds^2 = dY_1^2 + dY_2^2 - dY_3^2 - dY_4^2 - dY_5^2 - dY_6^2.\]

- The Fronsdal immersion: \[ds^2 = dY'_1^2 - dY'_2^2 - dY'_3^2 - dY'_4^2 - dY'_5^2 - dY'_6^2,\]

Respectively given by

\[
\begin{align*}
Y_1 &= r \sin \theta \sin \phi \\
Y_2 &= r \sin \theta \cos \phi \\
Y_3 &= r \cos \theta \\
Y_4 &= \alpha(r) \\
Y_5 &= (1 - 2m/r)^{1/2} \cos t \\
Y_6 &= (1 - 2m/r)^{1/2} \sin t
\end{align*}
\]

where \((d\alpha/dr)^2 = \frac{2m^3 + m^2}{r^3 - 2mr^2}\).

\[
\begin{align*}
Y'_1 &= r \sin \theta \sin \phi \\
Y'_2 &= r \sin \theta \cos \phi \\
Y'_3 &= r \cos \theta \\
Y'_4 &= \beta(r) \\
Y'_5 &= (1 - 2m/r)^{1/2} \cosh t \\
Y'_6 &= (1 - 2m/r)^{1/2} \sinh t
\end{align*}
\]

where \((d\beta/dr)^2 = \frac{2m^3 - m^2}{r^4 - 2mr^2}\).

In order to determine the metric \(g'\) (extension of \(g\)), define the new coordinates \(u\) and \(v\) by:

- For \(r > 2m\),
  \[v = \frac{1}{4m} (\frac{r}{2m})^{1/2} \exp(\frac{r}{4m}) Y'_5 \quad \text{and} \quad u = \frac{1}{4m} (\frac{r}{2m})^{1/2} \exp(\frac{r}{4m}) Y'_6.\]

- For \(0 < r < 2m\),
  \[v = \frac{1}{4m} (-\frac{r}{2m})^{1/2} \exp(\frac{r}{4m}) Y'_5 \quad \text{and} \quad u = \frac{1}{4m} (-\frac{r}{2m})^{1/2} \exp(\frac{r}{4m}) Y'_6.\]

where

\[u^2 - v^2 = (\frac{r}{2m} - 1) \exp(\frac{r}{2m}) \iff Y'_5^2 - Y'_6^2 = 16m^2 (1 - \frac{2m}{r}).\]

Now \(r = r(Y'_5, Y'_6)\) is implicitly defined by equation (10), while \(t = t(Y'_5, Y'_6)\) is implicitly defined by

\[Y'_5/Y'_6 = \tgh(\frac{t}{4m}).\]

Now we can write the \(g'\) metric,
ds^2 = (32m^3/r)exp(-r/2m)(dv^2 - du^2) - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (12)

We notice that \((E', g')\) is the extension of \((E, g)\), it calculated through of the coordinates of immersion and it coincides with (5). The space \((E', g')\) don’t has a singularity at \(r = 2m\). We know that \((E, g)\) is disconnected because it is composed by two connected components. When we calculated the extension \((E', g')\) through the Fronsdal immersion we see that it is connected.

We use the isometric immersion formalism to establish the extension of \((E, g) = ([P_2^I \cup P_2^II] \times S^2, g)\), denoted by \((E', g') = (Q^2 \times S^2, g')\), where \(Q^2\) is the Kruskal plane.\textsuperscript{12} The topology of a gravitational field outside of a body with spherical symmetry is given by \(\mathbb{R}^2 \times S^2\). We have that the topology of \((E, g)\) is given by \(\mathbb{R}^2 - \{(t, r) \in \mathbb{R}^2 \mid r = 2m\}\) \(\times S^2\), is different from the topology \(\mathbb{R}^2 \times S^2\) of \((E', g')\).\textsuperscript{13}

4 Conclusion

We start of one space-time immersed in a space pseudo-euclidean. We compose the tensor of Riemann through of the Frobenius conditions to immersion. In particular we solved the field equations for the vacuum and we assume a space-time with spherical symmetry and static. We found the immersion coordinates and later we calculate the second fundamental form through an algorithm using algebraic computation. Soon after we calculate the tensor of Riemann and Einstein through the equation of Gauss. We found the solutions of the equations of Einstein for the vacuum: the metric of Schwarzschild in the usual coordinates and Kruskal metric in this approach. Finally we found the topological characteristics associated the those solutions. We conclude that is possible to extract global information of the solutions of the equations of Einstein with this new procedure. This procedure results a source of global information to the gravitation field, not supplied by the solutions of the Einstein equations of the general relativity. We suggest that those additional information to gravitational field brought from of the geometric entities of the immersion can be associated to physical amounts that help, for instance, to solve current problems in cosmology.
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References

[1] T. Kaluza, *Sitzunger. Preuss. Akad. Wiss. Berlin, Phys. Math. K* 1, 33 (1921).
[2] Nima Arkani-Hamed et al., *Phys. Lett. B* 429, 263 (1998).
[3] L. Randall and R. Sundrum, *Phys. Rev. Lett.* 83, 3370 (1999).
[4] K. Maeda et al., *Phys. Rev. D* 62, 24012 (2000).
[5] M. D. Maia and E. M. Monte, *Phys. Lett. A* 297, 9 (2002).
[6] S. Kobayaschi and K. Nomizu, *Foundations of Differential Geometry*, (John Wiley and Sons, Inc. New York, 1963).
[7] M. P. Carmo, *Riemannian Geometry*, (Birkhuser, Boston, 1993).
[8] S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time*, (Cambridge Univ. Press, Cambridge, 1973).
[9] B. O’Neill, *Semi-Riemannian Geometry*, (Academic Press, New York, 1970).
[10] E. Kasner, *Am. J. Math.* 43, 130 (1921).
[11] C. Fronsdal, *Phys.Rev.*, 116, 778 (1959).
[12] M. Kruskal, *Phys.Rev.*, 119, 1743 (1960).
[13] M. D. Maia and E. M. Monte, *Mat. Contemp.*, 13, 229 (1997).