Strong enhancement of current, efficiency and mass separation in Brownian motors driven by non Gaussian noises

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Abstract
We study a Brownian motor driven by a colored non Gaussian noise source with a $q$-dependent probability distribution, where $q$ is a parameter indicating the departure from Gaussianity. For $q = 1$ the noise is Gaussian (Ornstein–Uhlenbeck), while, for $q > 1$, the probability distribution falls like a power law. In the latter case, we find a marked enhancement of both the current and the efficiency of the Brownian motor in the overdamped regime. We also analyze the case with inertia and show that, again for $q > 1$, a remarkable increase of the ratchet’s mass separation capability is obtained.
The study of noise induced transport by "ratchets" has attracted an increasing number of researchers that have produced a vast literature \[1, 2, 3, 4, 5\]. Among other aspects, the motivation of these studies has been prompted by both their possible biological interest \[1\] as well as their potential technological applications. Since the initial works, besides the built-in ratchet-like bias and correlated fluctuations (see for instance \[2\]), different aspects have been studied like tilting \[3\] and pulsating \[4\] potentials, velocity inversions \[5\], etc. There are some reviews where it is possible to grasp the state of the art \[6, 7\].

So far, almost all studies have used Gaussian noises, with few exceptions that mainly exploited dichotomic processes \[7\]. Here we analyze the effect of a particular class of colored non Gaussian noise on the transport properties of Brownian motors. Such a noise source is based on the so called Tsallis statistics \[8\] with a probability distribution that depends on \(q\), a parameter indicating the departure from the Gaussian \((q = 1)\) behavior. Some of the motivations for studying the effect of non Gaussian noises are, in addition to the intrinsic interest within the realm of noise induced phenomena, different experimental data indicating that for some biologically motivated systems fluctuations can have a non Gaussian character. An example are current measurements through voltage-sensitive ion channels in a cell membrane or experiments on the sensory system of rat skin \[9\]. It is also worth remarking here that recent detailed studies on the source of fluctuations in some biological systems \[10\] clearly indicate that noise sources in general could be non Gaussian and their distribution bounded. Here, again as a consequence of the non Gaussian character of the driven noise, we find a remarkable increase of the current together with an enhancement of the motor efficiency that, additionally, shows an optimum for a given degree of non Gaussianity. Moreover, when the inertia is taken into account we find that, when departing from the Gaussian case, there is a remarkable increment in the mass separation efficiency of these devices.

We start considering the general system

\[
m \frac{d^2x}{dt^2} = -\gamma \frac{dx}{dt} - V'(x) - F + \xi(t) + \eta(t),
\]

where \(m\) is the mass of the particle, \(\gamma\) the friction constant, \(V(x)\) the ratchet potential, \(F\) is a constant "load" force, and \(\xi(t)\) the thermal noise satisfying \(\langle \xi(t)\xi(t') \rangle = 2\gamma K T \delta(t - t')\). Finally, \(\eta(t)\) is the time correlated forcing (with zero mean) that keeps the system out of thermal equilibrium allowing the rectification of the motion. For this type of ratchet model several different kinds of time correlated forcing have been considered in the literature \[3, 4\]. We may distinguish the cases of deterministic (such as the periodic forcing and the "synthetic noise" discussed in the pioneering work of Magnasco \[2\]) and of stochastic forcing. Among the stochastic forcings the more usual noise sources are dichotomic and Ornstein–Uhlenbeck (OU) noises \[4\]. Here, as indicated above, we will study the effects of a different family of stochastic forcing having non Gaussian statistics. The main characteristic introduced by the non Gaussian form of the forcing we consider here is the appearance of arbitrary strong "kicks" with relatively high probability when compared, for example, with the OU Gaussian process. As we shall see, in a general situation (without fine tuning of the parameters), this leads to the above indicated remarkable effects.

We will consider the dynamics of \(\eta(t)\) as described by the Langevin equation \[11, 12\]

\[
\frac{d\eta}{dt} = -\frac{1}{\tau} \frac{d}{d\eta} V_q(\eta) + \frac{\eta}{\tau},
\]

with \(\langle \zeta(t)\zeta(t') \rangle = 2D \delta(t - t')\) and

\[
V_q(\eta) = \frac{D}{\tau(q-1)} \ln\left[1 + \frac{\tau}{\tau(q-1)} \eta^2 \right].
\]
Previous studies of such processes in connection with stochastic resonance problems [11, 12] and dynamical trapping [13] have shown that the non Gaussianity of the noise leads to interesting effects. For $q = 1$, the process $\eta$ is the (Gaussian) Ornstein–Uhlenbeck one (with correlation time equal to $\tau$), while for $q \neq 1$ it is a non Gaussian process. For $q < 1$ the probability distribution has a cut–off at $\omega = \sqrt{(1-q)\tau/(2D)}$, and is given by:

$$P_{q<1}(\eta) = \left\{ \begin{array}{ll} \frac{1}{Z_q}[1 - \frac{(2\omega^2)^{1-q}}{4}] & \text{if } |\eta| < \omega, \\ 0 & \text{otherwise} \end{array} \right. \quad \text{(4)}$$

where $Z_q$ is the normalization constant. For $1 < q < 3$, the probability distribution (for $-\infty < \eta < \infty$) is

$$P_{q>1}(\eta) = \frac{1}{Z_q} \left[ 1 + \frac{\tau}{2D}(q-1)\eta^2 \right]^{1-q}. \quad \text{(5)}$$

While keeping $D$ constant, the width or dispersion of the distribution increases with $q$. This means that, the higher the $q$, the stronger the “kicks” that the particle will receive. The second moment of the distribution (or intensity of the non Gaussian noise, that we call $D_{ng}$) diverges for $q \geq 5/3 \approx 1.66$ while, for $q \leq 5/3$, is given by

$$D_{ng} \equiv \langle \eta^2 \rangle = \frac{2D}{\tau(5-3q)}. \quad \text{(6)}$$

Here, and in what follows, we shall keep $q < 5/3$.

For the ratchet potential we will first consider the same form as in [2] (with period $2\pi$)

$$V(x) = -\int dx (\exp[\alpha \cos(x)]/J_0(\alpha) - 1), \quad \text{(7)}$$

with $\alpha = 16$. The integrand is the ratchet force (-$V'(x)$) appearing in Eq. (1).

Firstly, we will analyze the overdamped regime setting $m = 0$ and $\gamma = 1$. We are interested on analyzing the dependence of the mean current $J = \langle \frac{dx}{dt} \rangle$ and the efficiency $\varepsilon$ on the different parameters. In particular, their dependence on $q$, the parameter indicating the degree of non Gaussianity of the noise distribution. The efficiency of the ratchet system is defined as the ratio of the work (per unit time) done by the particle “against” the load force $F$, into the mean power injected to the system through the external forcing $\eta$ [14].

$$\varepsilon = \frac{1}{T_f} \int_{x=x(0)}^{x=x(T_f)} F dx(t) \frac{\langle dt \rangle}{\gamma T_f} \int_{x=x(0)}^{x=x(T_f)} \eta(t) dx(t) \quad \text{(8)}$$

For the numerator we get $F\langle \frac{dx}{dt} \rangle = FJ$, while, for the denominator

$$\frac{1}{T_f} \int_{0}^{T_f} \eta(t) \frac{dx}{dt} dt = \frac{1}{\gamma T_f} \int_{0}^{T_f} \eta(t)^2 dt = \frac{2D}{\gamma T_f \tau(5-3q)}. \quad \text{(9)}$$

Interesting and complete discussions on the thermodynamics and energetics of ratchet systems can be found in [15].

It is worth mentioning here that the “effective Markovian approximation” introduced in [12] is not adequate for the present case as it only properly works in the low $\tau$ limit. Here, in the overdamped regime we are able to give an approximate analytical solution for the problem, which is expected to be valid in the large correlation time regime ($\frac{1}{T_f} << 1$): we perform the adiabatic approximation of solving the Fokker-Planck equation associated to Eq. (1) assuming a constant value of $\eta$ [16]. This leads us to obtain a value for the current $J(\eta)$ which depends
on $\eta$ and, in order to get the final result, we should perform an average of $J(\eta)$ over $\eta$ using the distribution $P_q(\eta)$ with the desired value of $q$.

In Fig. 1, we show typical analytical results for the current and the efficiency as functions of $q$ together with results coming from numerical simulations (for the complete system given by Eqs. (1) and (2)). Calculations have been done in a region of parameters similar to the one studied in 1 but considering (apart from the difference provided by the non Gaussian noise) a non–zero load force that leads to a non–vanishing efficiency. As can be seen, although there is not a quantitative agreement between theory and simulations, the adiabatic approximation predicts qualitatively very well the behavior of $J$ (and $\varepsilon$) as $q$ is varied. As shown in the figure, the current grows monotonously with $q$ (at least for $q < 5/3$) while there is an optimal value of $q$ ($> 1$) which gives the maximum efficiency. This fact is interpreted as follows: when $q$ is increased, the width of the $P_q(\eta)$ distribution grows and high values of the non Gaussian noise become more frequent, this leads to an improvement of the current. Although the mean value of $J$ increases monotonously with $q$, the grow of the width of $P_q(\eta)$ leads to an enhancement of the fluctuations around this mean value. This is the origin of the efficiency’s decay that occurs for high values of $q$: in this region, in spite of having a large (positive) mean value of the current, for a given realization of the process, the transport of the particle towards the desired direction is far from being assured.

In Fig. 2 we show results from simulations for $J$ and $\varepsilon$ as functions of $q$ for different values of $D$, the intensity of the white noise in Eq. (2). The results correspond to $K T = 0$, hence, the only noise present in the system is the non Gaussian one. On the curve corresponding to the results for $J$ (Fig. 2.a.), we indicate with error bars the dispersion of the results. The huge growth of the dispersion occurring for $q > 1.3$, as well as the decay of the efficiency for the same values of $q$, is apparent.

It is worth recalling here that $q = 1$ corresponds to the Gaussian OU noise (analyzed, for example in 3 and 17). Hence, our results show that the transport mechanism becomes more efficient when the stochastic forcing has a non Gaussian distribution with $q > 1$.

In the previous calculations, we have analyzed the values of $J$ and $\varepsilon$ as functions of $q$ for fixed values of $D$. However, there are situations where we should consider that the non Gaussian noise source is the “primary” source, for instance see 10 for biologically motivated problems. Besides this, there could also be situations of clear technological interest. For these reasons we wondered which would be the dependence of $J$ and $\varepsilon$ on $q$ when we fix the intensity of the non Gaussian noise $D_{ng}$ defined in Eq.(8). In such case, the energy per unit time supplied to the ratchet is independent of $q$. Our results (not shown) indicate that, at variance to what occurs in calculations for fixed $D$, for constant $D_{ng}$, the optimum value of $q$ which maximizes the efficiency also maximizes the mean current $J$, which decays for larger values of $q$. This result is easy to understand by observing that the width of the $P_q(\eta)$ distribution decreases with $q$ when $D_{ng}$ is kept constant.

Now we turn to study the $m \neq 0$ case, that is, the situations in which the inertia effects are relevant. In Fig. 3 we show the dependence of the current $J$ on the mass $m$ for different values of $q$. The results are from simulations for zero temperature and without load force. It can be seen that, as $m$ is increased from 0, the inertial effects initially contribute to increase the current, until an optimal value of $m$ is reached. As it is expected, for high values of $m$, the motion of the particle becomes difficult and, for $m \to \infty$, the current vanishes.

An interesting effect appears for $q = 1.3$: for a well defined interval of the value of the mass (ranging approximately from $m = 100$ to $m = 5000$), a negative current is observed. This can be explained as a consequence of the high value of the mass, that makes the inertial effects much more important than those of the ratchet potential. Due to the high value of $m$, most of the time the particle will be almost at rest around the minimum of the ratchet potential. However, a very high value of $\eta$ may occur (that may last for an interval of time of the order
of \( \tau \) causing the particle to move a period of the potential to the right or to the left. The obvious question is, what is the value of the force \( \eta \) needed in order to make the particle jump a period of the potential to the left (right), during a time \( \tau \)? Assuming no friction (which is of no importance for high values of \( m \)), a constant (average) value of the ratchet force \( V' \), and performing a classical calculation we get:

\[
\eta = |V'| + \frac{2md}{\tau^2}, \tag{10}
\]

where \( d \) is the distance from one minimum of \( V(x) \) to the next left (right) maximum of \( V(x) \). In our system, for a jump to the left, we have \( |V'| \sim 4.5 \) and \( d \sim 1.1 \), while, for a jump to the right, we have \( |V'| \sim .96 \) and \( d \sim 5.2 \). Considering \( \tau = 100/(2\pi) \), for \( m = 800 \) (the value for which in Fig. 3 the minimum of the current is obtained for \( q = 1.3 \)) we find that the required value of \( \eta \) for a jump to the left (right) is \( |\eta_l| \sim 6.9 \) (\( |\eta_r| \sim 32.4 \)). Hence, for this value of the mass, the jumps to the left are more probable than the jumps to the right, and a negative current is to be expected. In contrast, for a value of the mass \( m = 100 \) (approximately where the current change from being positive to being negative) we find \( |\eta_l| \sim 5.4 \) and \( |\eta_r| \sim 5.1 \), i.e. both jumps are almost equiprobable. Note that, for \( m = 800 \), the large value of \( m \) makes the second term in Eq. (10) more important than the contribution of \( |V'| \) (the inertial effect dominates over the ratchet force), hence, the difference in the value of \( \eta \) necessary for a jump to the left or to the right comes, essentially, from the difference on the distances \( (d) \) that the heavy particle should be displaced to each side.

Until this point, the analysis makes no mention of the value of \( q \) of the noise distribution. The fact that the \( (J < 0) \)–effect appears for \( q = 1.3 \) and not for \( q = 1 \) is understood as a consequence of the fact that, for \( q = 1 \), the occurrence of a value of noise \( |\eta_L| \) is highly improbable in the regime where the inertia dominates over the effect of the ratchet force. Hence, the particle remains essentially motionless.

Now we analyze in more detail the problem of mass separation, an aspect that has been studied in some works \[13, 14\]. In view of the results discussed above, it is reasonable to expect that non Gaussian noises may improve the capability of mass separation in ratchets in more general situations. Reference \[13\] was one of the primary works discussing mass separation by ratchets. There, the authors analyzed a ratchet system like the one described by Eq. (1), considering OU noise as external forcing (in our case it corresponds to \( q = 1 \)). They studied (both numerically and analytically) the dynamics for different values of the correlation time of the forcing \( \tau \), finding that there is a region of parameters where mass separation occurs. This means that the direction of the current is found to be mass–dependent: the “heavy” species moves in the negative sense while the “light” one, do so in the positive sense.

Here, in order to compare results, we analyze the same system studied in \[13, 14\] but considering the non Gaussian forcing described by Eq. (1). Hence, we study the system in Eq. (1) with \( V(x) = -[\sin(2\pi x) + .25\sin(4\pi x)]/(2\pi) \) as the ratchet potential. We focus on the region of parameters where, in \[13\] (for \( q = 1 \)), separation of masses was found. We fix \( \gamma = 2, KT = .1, \tau = .75, \) and \( D = .1875 \) and consider the values of the masses \( m = m_1 = 0.5 \) and \( m = m_2 = 1.5 \) as in \[13\]. Our main result here is that the separation of masses is enhanced when a non–Gaussian noise with \( q > 1 \) is considered. In Fig. 4.a. we show \( J \) as function of \( q \).
for $m_1 = 0.5$ and $m_2 = 1.5$. It can be seen that there is an optimum value of $q$ that maximizes the difference of currents. In that figure, this value, which is close to $q = 1.25$, is indicated with a vertical double arrow. Another double arrow indicates the separation of masses occurring for $q = 1$ (Gaussian OU forcing). The calculations are for a load force $F = 0.25$. We have observed that, when the value of the load force is varied, the difference between the curves remain approximately constant but both are shifted together to positive or negative values (depending on the sign of the variation of the loading). By controlling this parameter it is possible to achieve, for example, the situation shown in Fig. 4.b., where, for the value of $q$ at which the difference of currents is maximal, the heavy “species” remains static on average (has $J = 0$), while the light one has $J > 0$. Also it is possible to get the situation shown in Fig. 4.c, at which the two species moves in the opposite direction with equal absolute velocity.

Summarizing, we have systematically studied the effect of a colored non Gaussian noise source on the transport properties of a Brownian motor. What we have found is that a departure from Gaussian behavior, given by a value of $q$ larger than 1, induces a remarkable increase of the current together with an enhancement of the motor efficiency. The latter shows, in addition, an optimum value for a given degree of non Gaussianity. When inertia is taken into account we also find a considerable increment in the mass separation capability.

It is worth mentioning that in the studies of the influence of non Gaussian noises in stochastic resonance and other related phenomena, the system’s response enhancement occurs for $q < 1$, as discussed in \[1, 2, 3\]. The reason can be traced back to the dependence of the mean-first-passage-time with $q$, and the interplay between transition rates and modulation frequency. In contrast, in the present work, and due to the “favorable” influence of high values of the noise for increasing the transport effect, we observe that the relevant effect occurs in the opposite case, that is for $q > 1$.

These studies could be of interest for possible relation with biologically motivated problems \[1, 3, 10, 21\] as well as for potential technological applications, for instance in ”nanomechanics” \[3, 7\]. More specific studies of aspects in one or other area will be the subject of further work.

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[16] In doing this, we use a piecewise linear approximation for the ratchet potential which is

\[ V(x) = 2.5 - (5/1.1)x \text{ for } x < 1.1 \text{ and } V(x) = -2.5 + [5/(2\pi - 1.1)](x - 1.1) \text{ for } 1.1 < x < 2\pi. \]

We solve the Fokker–Planck equation for 0 ≤ x ≤ 2π with periodic boundary condition asking for continuity of the distribution and the current at x = 1.1.
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Figure 1: Current (a) and efficiency (b) as functions of $q$. The solid line corresponds to the analytical results in the adiabatic approximation while the line with squares shows results from simulations. All calculations are for $m = 0, \gamma = 1, KT = 0.5, F = 0.1, D = 1$ and $\tau = 100/(2\pi)$.

Figure 2: Current (a) and efficiency (b) as functions of $q$. Results from simulations at $KT = 0$ for $D = 1$ (circles), $D = 10$ (squares), and $D = 20$ (triangles). All calculations are for $m = 0, \gamma = 1, F = 0.1$ and $\tau = 100/(2\pi)$.

Figure 3: Current as function of the mass for different values of $q$: circles $q = 1$ (Gaussian noise case), squares $q = 1.3$.

Figure 4: Separation of masses: results from simulations for the current as a function of $q$ for particles of masses $m = 0.5$ (hollow circles) and $m = 1.5$ (solid squares). Calculations for three different values of the load force: $F = .025$ (a), $F = 0.02$ (b) and $F = 0.03$ (c).
Bouzat & Wio. Fig. 1.
Bouzat & Wio. Fig. 4.