Long-term stability studies of nonlinear Hamiltonian systems require symplectic integration algorithms which are both fast and accurate. In this paper, we study a symplectic integration method wherein the symplectic map representing the Hamiltonian system is refactorized using polynomial symplectic maps. This method is analyzed in detail for the three degree of freedom case. We obtain explicit formulas for the action of the constituent polynomial maps on phase space variables.

Keywords: Symplectic integration; polynomial maps; Lie perturbation theory

1. Introduction

Numerical integration algorithms are essential to study the long term single particle stability of nonlinear, nonintegrable Hamiltonian systems. However, standard numerical integration algorithms can not be used since they are not symplectic \(^1\). This violation of the symplectic condition can lead to spurious chaotic or dissipative behavior. Numerical integration algorithms which satisfy the symplectic condition are called symplectic integration algorithms \(^1\). Several symplectic integration algorithms have been proposed in the literature \(^2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21\). Some of these directly use the Hamiltonian whereas others use the symplectic map \(^22,23\) representing the nonlinear Hamiltonian system. For complicated systems like the Large Hadron Collider which has thousands of elements, using individual Hamiltonians for each element can drastically slow down the integration process. One the other hand, the map based approach is very fast in such cases \(^24,25\).

One class of the map-based methods uses jolt factorization \(^6,11,17,19\). But there are still unanswered questions on how to best choose the underlying group and elements in the group \(^26\). Further, some of these methods \(^11,17,19\) can be quite difficult to generalize to higher dimensions. Another class of methods uses solvable maps \(^12,21\) or monomial maps \(^18\). Even though they are fairly straightforward to generalize to higher dimensions, they tend to introduce spurious poles and branch
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points not present in the original map\textsuperscript{26}.

We investigate a new symplectic integration method where the symplectic map is
refactorized using “polynomial maps” (maps whose action on phase space variables
gives rise to polynomials). This does not introduce spurious poles and branch
points. Moreover, it is easy to generalize to higher dimensions. Further, since it is
map-based, it is also very fast. In this paper, we describe in detail the theoretical
underpinnings of the polynomial map factorization of symplectic maps. We also
apply it to Hamiltonian systems.

2. Preliminaries

We restrict ourselves to three degrees of freedom nonlinear Hamiltonian system.
The effect of a Hamiltonian system on a particle can be formally expressed as the
action of a symplectic map $\mathcal{M}$ that takes the particle from its initial state $z^{in}$ to
its final state $z^{fin}$:

$$z^{fin} = \mathcal{M} z^{in}. \quad (1)$$

Here $z$ represents the collection of six phase space variables:

$$z = (q_1, q_2, q_3, p_1, p_2, p_3). \quad (2)$$

Using the Dragt-Finn factorization theorem\textsuperscript{27,22}, the symplectic map $\mathcal{M}$ can be
factorized as shown below:

$$\begin{align*}
\mathcal{M} &= \hat{M} e^{f_3} e^{f_4} \ldots e^{f_n} \ldots .
\end{align*} \quad (3)$$

Here $f_n(z)$ denotes a homogeneous polynomial (in $z$) of degree $n$ uniquely deter-
mined by the factorization theorem. The Lie transformation $e^{f(z)}$ is given by

$$e^{f(z)} = \sum_{n=0}^{\infty} \frac{f(z)^n}{n!}, \quad (4)$$

where

$$\begin{align*}
:f(z):g(z) &= [f(z), g(z)].
\end{align*} \quad (5)$$

Here $[f(z), g(z)]$ denotes the usual Poisson bracket of the functions $f(z)$ and $g(z)$. Further $\hat{M}$ gives the linear part of the map and hence has an equivalent representa-
tion in terms of the Jacobian matrix $\hat{M}$ of the map $\mathcal{M}$\textsuperscript{22}:

$$\hat{M} z_i = M_{ij} z_j = (M z)_i. \quad (6)$$

The infinite product of Lie transformations $\exp(\hat{f}_n)$ ($n = 3, 4, \ldots$) in Eq. (3)
represents the nonlinear part of $\mathcal{M}$.

As an application, let us consider a charged particle particle storage ring which
typically comprises thousands of elements (drifts, quadrupoles, sextupoles etc.) Us-
ing the above procedure, one can represent each element in the storage ring by a
symplectic map. By concatenating\textsuperscript{22} these maps together using group-theoretical
methods, we obtain the so-called ‘one-turn’ map representing the entire storage ring. The one-turn map gives the final state \( z^{(1)} \) of a particle after one turn around the ring as a function of its initial state \( z^{(0)} \):

\[
  z^{(1)} = M z^{(0)}. \tag{7}
\]

To obtain the state of a particle after \( n \) turns, one has to merely iterate the above mapping \( N \) times i.e.

\[
  z^{(n)} = M^n z^{(0)}. \tag{8}
\]

Since \( M \) is explicitly symplectic, this gives a symplectic integration algorithm. Further, since the entire ring can be represented by a single (or at most a few) symplectic map(s), numerical integration of particle trajectories using symplectic maps is very fast.

To obtain a practical symplectic integration algorithm, we follow the perturbative approach and truncate \( M \) after a finite number of Lie transformations:

\[
  M \approx \hat{M} e^{\mathcal{F}_3} e^{\mathcal{F}_4} \ldots e^{\mathcal{F}_P}. \tag{9}
\]

The symplectic map is said to be truncated at order \( P \). This map is still symplectic. However, each exponential \( e^{\mathcal{F}_n} \) in \( M \) still contains an infinite number of terms in its Taylor series expansion. We get around the above problem by refactorizing \( M \) in terms of simpler symplectic maps which can be evaluated exactly without truncation. We use ‘polynomial maps’ which give rise to polynomials when acting on the phase space variables. This avoids the problem of spurious poles and branch points present in generating function methods, solvable map and monomial refactorizations.

### 3. Symplectic Polynomial Maps

In this section we study symplectic polynomial maps in some detail. We start by describing the difference between monomial maps and polynomial maps with respect to presence of poles and branch points. This difference can be illustrated using the following examples. Consider the monomial symplectic map \( \exp(\cdot q_1^2 p_1 \cdot) \). Its action on \( q_1, p_1 \) in a two dimensional phase space is given as follows:

\[
  q_1' = \exp(\cdot q_1^2 p_1 \cdot)q_1 = \frac{q_1}{1 + q_1}; \quad p_1' = \exp(\cdot q_1^2 p_1 \cdot)p_1 = p_1(1 + q_1)^2. \tag{10}
\]

This map has a pole at \( q_1 = -1 \).

On the other hand, consider the symplectic map \( \exp(\cdot a_1 q_1^3 + a_2 p_1 \cdot) \) where \( a_1, a_2 \) are real constants. We determine its action on phase space variables as follows. Note that the symplectic map is of the form \( \exp(\cdot h(z) \cdot) \) where \( h(z) \) is a function which depends only on the phase space variables \( z \) and is independent of time \( t \). If we take \( h(z) \) to be the Hamiltonian function, then solving the Hamilton’s equations...
of motion for this Hamiltonian from time \( t = t^i \) to time \( t = t^f \) is equivalent to the following symplectic map action:

\[
z(t = t^f) = \exp\left[-(t^f - t^i) : h(z) \right] z(t = t^i).
\] (11)

Equivalently, obtaining the action of the symplectic map \( \exp\left[-(t^f - t^i) : h(z) \right] \) on the phase space variables is the same as solving the Hamilton’s equations of motion with \( h(z) \) as the Hamiltonian from time \( t^i \) to \( t^f \). Setting \( t^i = 0 \) and \( t^f = -1 \) we have the following equivalence: Obtaining the action of the symplectic map \( \exp( : h(z) : ) \) on phase space variables is equivalent to solving the Hamilton’s equations of motion using \( h(z) \) as the Hamiltonian from time \( t = 0 \) to time \( t = -1 \). In this case, \( z(0) \) will correspond to the initial values of the phase space variables and \( z(-1) \) to the final values obtained after the action of the map \( \exp( : h(z) : ) \).

Returning to our symplectic map, we obtain its action by first solving the Hamilton’s equations of motion from time \( t = 0 \) to \( t = -1 \) using the argument of the Lie transformation, \( h = a_1q_1^2 + a_2p_1 \), as the Hamiltonian. The Hamilton’s equations of motion are given by:

\[
\frac{dq_1}{dt} = \frac{\partial h}{\partial p_1},
\]

\[
\frac{dp_1}{dt} = -\frac{\partial h}{\partial q_1}.
\] (12)

Solving these simple equations, we obtain:

\[
q_1(t) = q_1(0) + a_2t;
\]

\[
p_1(t) = p_1(0) - a_1a_2^2t^3 - 3a_1a_2q_1(0)t^2 - 3a_1q_1(0)^2t.
\] (13)

where \( q_1(0) \) and \( p_1(0) \) denote the values of \( q_1 \) and \( p_1 \) at time \( t = 0 \). To obtain the action of the map \( \exp( : a_1q_1^2 + a_2p_1 : ) \) on phase space variables, we set \( t = -1 \) in the above equations and denote \( q_1(-1) \), \( p_1(-1) \) by \( q_{1i}^{\text{fin}} \), \( p_{1i}^{\text{fin}} \) and \( q_1(0) \), \( p_1(0) \) by \( q_{1i}^{\text{in}} \), \( p_{1i}^{\text{in}} \) respectively. Thus we get

\[
q_{1i}^{\text{fin}} = q_{1i}^{\text{in}} - a_2, \quad p_{1i}^{\text{fin}} = p_{1i}^{\text{in}} + a_1a_2^2 - 3a_1a_2q_{1i}^{\text{in}} + 3a_1(q_{1i}^{\text{in}})^2.
\] (14)

Using Eq. (4), we can easily verify that the above result is indeed correct. We note that the final values of the phase space variables are polynomial functions of the initial variables and therefore involve no poles or branch points. This is an example of a polynomial map.

We now determine the classes of symplectic maps which are also polynomial maps. We obtain the following simple principles which are equally applicable in higher dimensions.

1. All polynomials of the form \( h(z) \) where both a phase space variable and its canonically conjugate variable do not occur simultaneously give rise to symplectic polynomial maps via \( \exp( : h(z) : ) \). We will call such \( h(z) \)'s as polynomials of the first type.
2. If a canonically conjugate pair $q_i, p_i$ is present in the polynomial $h(z)$ and it appears either in the form $[a(\hat{z})q_i + g(p_i, \hat{z})]m$ or $[a(\hat{z})p_i + g(q_i, \hat{z})]m$ (where $m = 1, 2, \ldots, \hat{z} = \{q_j, p_k\}$ with $j \neq k \neq i$ and $a$, $g$ are polynomials in the indicated variables), then this polynomial $h(z)$ again gives rise to a symplectic polynomial map via $\exp(: h(z) : )$. If a product/sum of such factors appears in $h(z)$, each term in the product/sum is a function of different canonically conjugate pairs. We will call $h(z)$'s of the form described above as polynomials of the second type.

We can prove the above results as follows. Let $\hat{z}$ denote a collection of phase space variables $\{q_j, p_k\}$ with $j \neq k$. Thus polynomials of the first type are of the form $h(\hat{z})$. The polynomial map is then given by $\exp(: h(\hat{z}) : )$. As described earlier, its action on the phase space variables is given by solving the Hamilton’s equations of motion from time $t = 0$ to $t = -1$ using $h(\hat{z})$ as the Hamiltonian. From classical mechanics $2^9$ each of the variables in the collection $\hat{z}$ is a cyclic variable and is therefore conserved by the Hamiltonian. Thus $\hat{z}(t) = \hat{z}(0)$. Consequently

$$\hat{z}^{fin} = \exp(: h(\hat{z}) : )\hat{z}^{in} = \hat{z}^{in}. \tag{15}$$

Next we consider the action of this map on the variable $\hat{z}_j$ which is canonically conjugate to $\hat{z}_j$. Solving Hamilton’s equations of motion with $h(\hat{z})$ as the Hamiltonian from time $t = 0$ to $t = -1$ we get

$$\hat{z}_j(-1) = \hat{z}_j(0) - (-1)^r \frac{dh}{d\hat{z}_j}(\hat{z}_j(0)), \tag{16}$$

where $r$ is zero if $\hat{z}_i$ is a coordinate variable and 1 otherwise. As before, the action of $\exp(: h(\hat{z}) : )$ on $\hat{z}_j$ is obtained by setting $\hat{z}_j(0)$ as $\hat{z}_j^{in}$ and $\hat{z}_j(-1)$ as $\hat{z}_j^{fin}$:

$$\hat{z}_j^{fin} = \hat{z}_j^{in} - (-1)^r \frac{dh}{d\hat{z}_j}(\hat{z}_j^{in}). \tag{17}$$

Since $h(\hat{z})$ is a polynomial in $\hat{z}$, the right hand side of the above equation is a polynomial in $\hat{z}$ and $\hat{z}$. From Eqs. (15) and (17) we conclude that $\exp(: h(\hat{z}) : )$ is a polynomial map. Further, since all Lie transformations are symplectic maps $2^2$, $\exp(: h(\hat{z}) : )$ is a symplectic polynomial map.

Next we consider polynomials $h(z)$ of the second type described in item 2) above. Let $h(z) = [a(\hat{z})q_i + g(p_i, \hat{z})]m$ where $q_i, p_i$ is a canonically conjugate pair. Further $m$ is a positive integer, $\hat{z} = \{q_j, p_k\}$ with $j \neq k \neq i$ and $a$, $g$ are polynomials in the indicated variables. We will show that $\exp(: h(z) : )$ is a polynomial map. The proof for the case $h'(z) = [a(\hat{z})p_i + g(q_i, \hat{z})]m$ is similar. A concrete example of the type of $h(z)$ that we are considering is given by $h(z) = (a_1p_2 + a_2p_3 + a_3p_2^2 + a_4p_3p_4 + a_5p_3^2 + \alpha_0q_3p_2)^m$ where $\alpha$'s are real constants.

As before we first solve the Hamilton’s equations of motion with $h(z)$ as the Hamiltonian. Since each of the variables in the collection denoted by $\hat{z}$ is cyclic, we have $\hat{z}(t) = \hat{z}(0)$ and

$$\hat{z}^{fin} = \exp(: h(\hat{z}) : )\hat{z}^{in} = \hat{z}^{in}. \tag{18}$$
Solving the Hamilton’s equation of motion for $p_i$ we get
\[
\frac{dp_i}{dt} = -\frac{\partial h(z)}{\partial q_i} = -m[a(\bar{z}(0))q_i(0) + g(p_i(0), \bar{z}(0))]^{m-1}a(\bar{z}(0)),
\]
where we have used the fact that $[a(\bar{z})q_i + g(p_i, \bar{z})]$ is a conserved quantity under this Hamiltonian flow i.e. $[a(\bar{z})q_i + g(p_i, \bar{z})] = [a(\bar{z}(0))q_i(0) + g(p_i(0), \bar{z}(0))]$. Solving this simple equation we obtain
\[
p_i(t) = p_i(0) - m[a(\bar{z}(0))q_i(0) + g(p_i(0), \bar{z}(0))]^{m-1}a(\bar{z}(0))t.
\]
Note that $p_i(t)$ is a polynomial in both $z$ and $t$. Setting $t = -1$ and denoting $z(0), z(-1)$ by $z^{in}, z^{fin}$ respectively, we see that $p_i^{fin} = \exp(\cdot h(z) :)p_i^{fin}$ is a polynomial in $z^{in}$;
\[
p_i^{fin} = p_i^{in} + m[a(\bar{z}(0))q_i(0) + g(p_i(0), \bar{z}(0))]^{m-1}a(\bar{z}^{in}).
\]
The equation of motion for $q_i$ gives us
\[
\frac{dq_i}{dt} = \frac{\partial h(z)}{\partial p_i} = -m[a(\bar{z}(0))q_i(0) + g(p_i(0), \bar{z}(0))]^{m-1}\frac{dg(p_i, \bar{z})}{dp_i}.
\]
Since $p_i(t)$ is a polynomial in $z$ and $t$, the right hand side of the above equation is also a polynomial in $z$ and $t$. Therefore, the above differential equation can be easily integrated to give $q_i(t)$ which is guaranteed to be a polynomial in $z$ and $t$. Setting $t = -1$ and denoting $z(0), z(-1)$ by $z^{in}, z^{fin}$ respectively, we obtain the result that $q_i^{fin} = \exp(\cdot h(z) :)q_i^{fin}$ is a polynomial in $z^{in}$. Finally, the equation of motion for $\bar{z}_j$, the variable canonically conjugate to $\bar{z}_j$, is given by
\[
\frac{d\bar{z}_j}{dt} = (-1)^r \frac{\partial h(z)}{\partial \bar{z}_j} = (-1)^rm[a(\bar{z}(0))q_i(0) + g(p_i(0), \bar{z}(0))]^{m-1} \frac{d[a(\bar{z})q_i + g(p_i, \bar{z})]}{d\bar{z}_j}(\bar{z} = \bar{z}(0)),
\]
where $r$ is zero if $\bar{z}_j$ is a coordinate variable and 1 otherwise. The right hand side is a polynomial function of $q_i, p_i$ both of which in turn are polynomials in $t$. Hence the equation can be integrated giving $\bar{z}_j(t)$ as a polynomial in $z$ and $t$. Setting $t = -1$ and denoting $z(0), z(-1)$ by $z^{in}, z^{fin}$ respectively, we find that $\bar{z}_j^{fin} = \exp(\cdot h(z) :)\bar{z}_j^{fin}$ is a polynomial in $z^{in}$. Thus we have proved that $\exp(\cdot h(z) :)$ is a (symplectic) polynomial map.

To conclude, we consider the case where a sum/product of factors of the form $[a(\bar{z})q_i + g(p_i, \bar{z})]^m$ or $[a(\bar{z})p_i + g(q_i, \bar{z})]^m$ appear in $h(z)$. If a phase space variable appears in one factor of the sum/product, by assumption, neither this variable nor its canonically conjugate variable appears in the remaining factors of the sum/product. Therefore each term in the sum/product is independently conserved by the Hamiltonian flow generated by $h(z)$ and acts only on the phase space variables appearing
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in that term. Consequently each term can be considered separately and since each
term is of the form \([a(\bar{z})q_i + g(p_i, \bar{z})]^m\) or \([a(\bar{z})p_i + g(q_i, \bar{z})]^m\), the argument given
above can be immediately applied proving that \(\exp(\cdot h(z)\cdot)\) is a symplectic poly-
nomial map even in this case. This completes the proof of the claims made in items
1) and 2). We conjecture that all symplectic polynomial maps have one of the two
forms enumerated above.

4. Symplectic Integration using Polynomial Maps

In this section, we return to the problem of symplectic integration. We restrict
ourselves to symplectic maps in a six dimensional phase space trunca-
ted at order 4. The results obtained below can be generalized to both higher orders and higher
dimensions using symbolic manipulation programs. The Dragt-Finn factorization
of the symplectic map is given by:

\[
\mathcal{M} = \hat{M} e^{\text{f}_3}; e^{\text{f}_4};
\]

where

\[
f_3 = a_{28}q_1^3 + a_{29}q_1^2 p_1 + \cdots + a_{83}p_3^3,
\]

\[
f_4 = a_{84}q_1^4 + a_{85}q_1^3 p_1 + \cdots + a_{209}p_4^4.
\]

Here the coefficients \(a_{28}, \ldots, a_{209}\) can be explicitly computed given a Hamiltonian
system\(^2\(^\)\) and are therefore known to us. The numbering of these monomial coef-
ficients follows the standard Giorgilli scheme\(^3\(^\)\). The above map captures the leading
order nonlinearities of the system. Since the action of the linear part \(\hat{M}\) on phase
space variables is well known [cf. Eq. (6)] and is already a polynomial action, we
only refactorize the nonlinear part of the map using \(N\) polynomial maps\(^3\(^1\)\). This
is done as follows:

\[
\mathcal{M} \approx \mathcal{P} = \hat{M} e^{h_1}; e^{h_2}; \cdots e^{h_N};
\]

where \(e^{h_i}\)'s are symplectic polynomial maps and the numeral appearing in the
subscript indexes the polynomial maps. The polynomial maps are determined by
requiring that \(\mathcal{P}\) agree with \(\mathcal{M}\) up to order 4. That is, when the \(N\) polynomial
maps are combined, the resulting symplectic map should have all the monomials
present in \(f_3\) and \(f_4\) with the correct coefficients up to order 4.

The basic idea in obtaining the required refactorization is to group the monomial
terms present in \(f_3\) and \(f_4\) [cf. Eq. (25)] such that the Lie transformation corre-
sponding to each grouping gives a polynomial map. It is obviously easy to handle
monomials where both members of the canonically conjugate pair are not present
simultaneously. They can be grouped into different polynomials (for example, mon-
omials involving only the coordinate variables \(q_i\)'s in one group and those involving
only the momentum variables \(p_i\)'s in another group etc.) so that each one of these
is a polynomial of the first type. Since a product of two Lie transformations \((e^{f_3};
\text{and } e^{f_4})\) is being refactorized as a product of many simpler polynomial symplec-
tic maps, the coefficients multiplying the monomials in each individual polynomial
map will be in general different from the coefficient multiplying the corresponding monomial in Eq. (25). The relation between these coefficients is easily obtained using the CBH theorem. Monomials \( s(z) \) where both members of the canonically conjugate pair are present simultaneously (like for example, \( s(z) = q_i^3 p_1 \)) are more difficult to handle since the corresponding Lie transformations \( e^{i s(z)} \) typically give rise to poles and branch points which we wish to avoid. But by using a product of two polynomial maps of type 2 with carefully chosen coefficients, these can also be generated.

Using the above procedure, it turns out that we require 23 polynomial maps for refactorization:

\[
\mathcal{M} \approx \mathcal{P} = M e^{h_1}; e^{h_2}; \ldots; e^{h_{23}},
\]

The \( h_i \)'s are given as follows:

\[
\begin{align*}
\mathbf{h}_1 &= q_3^3 b_{28} + q_2^2 q_2 b_{30} + q_2^2 q_4 b_{31} + q_1 q_2^3 b_{41} + q_1 q_2^2 b_{46} + q_2^2 b_{64} + q_2^2 q_3 b_{66} + q_2 q_3^2 b_{71} + q_3^3 b_{80} + q_1^4 b_{84} + q_1^3 q_2 b_{86} + q_3^3 q_3 b_{88} + q_2^2 h_{55} + q_2^2 q_2 q_3 h_{67} + q_1 q_2^3 h_{102} + q_1 q_2^2 h_{120} + q_1 q_2^3 h_{122} + q_1^2 q_3^2 h_{127} + q_1 q_3^3 h_{136} + q_2^3 h_{175} + q_3^3 q_3 h_{177} + q_2^2 q_3^3 h_{182} + q_2 q_3^3 h_{191} + q_4^3 h_{205}, \\
\mathbf{h}_2 &= [(b_{29} + b_{34}) + q_2 (b_{91} + b_{106}) + p_2 (b_{92} + b_{107}) + q_3 (b_{93} + b_{108}) + p_3 (b_{94} + b_{109})](p_1 + q_1)^3, \\
\mathbf{h}_3 &= [(-b_{29} + b_{34}) + q_2 (-b_{91} + b_{106}) + p_2 (-b_{92} + b_{107}) + q_3 (-b_{93} + b_{108}) + p_3 (-b_{94} + b_{109})](p_1 + q_1)^3, \\
\mathbf{h}_4 &= [(b_{65} + b_{68}) + q_1 (b_{121} + b_{124}) + p_1 (b_{156} + b_{159}) + q_3 (b_{180} + b_{186}) + p_3 (b_{181} + b_{187})](p_2 + q_2)^3, \\
\mathbf{h}_5 &= [(-b_{65} + b_{68}) + q_1 (-b_{121} + b_{124}) + p_1 (-b_{156} + b_{159}) + q_3 (-b_{180} + b_{186}) + p_3 (-b_{181} + b_{187})](p_2 + q_2)^3, \\
\mathbf{h}_6 &= [(b_{81} + b_{82}) + q_1 (b_{137} + b_{138}) + p_1 (b_{172} + b_{173}) + q_2 (b_{192} + b_{193}) + p_2 (b_{202} + b_{203})](p_3 + q_3)^3, \\
\mathbf{h}_7 &= [(-b_{81} + b_{82}) + q_1 (-b_{137} + b_{138}) + p_1 (-b_{172} + b_{173}) + q_2 (-b_{192} + b_{193}) + p_2 (-b_{202} + b_{203})](p_3 + q_3)^3, \\
\mathbf{h}_8 &= (p_1 + q_1)^2 (q_2 b_{35} + q_3 b_{37} + q_2^2 b_{110} + q_2 q_3 b_{112} + p_3 q_2 b_{113} + q_3^2 b_{117}), \\
\mathbf{h}_9 &= (p_1 + q_1)^2 (p_2 b_{36} + q_3 b_{38} + q_2^2 b_{114} + p_2 q_3 b_{115} + p_2 q_3 b_{116} + q_3^2 b_{119}), \\
\mathbf{h}_{10} &= (p_2 + q_2)^2 (q_1 b_{40} + q_3 b_{49} + q_1^2 b_{96} + q_1 q_3 b_{125} + p_3 q_1 b_{126} + q_3^2 b_{188}), \\
\mathbf{h}_{11} &= (p_2 + q_2)^2 (p_1 b_{55} + q_3 b_{70} + p_1^2 b_{146} + p_1 q_3 b_{160} + p_1 q_3 b_{161} + q_3^2 b_{190}), \\
\mathbf{h}_{12} &= (p_3 + q_3)^2 (q_1 b_{47} + q_2 b_{72} + q_1^2 b_{103} + q_1 q_2 b_{128} + q_3 b_{134} + q_2^2 b_{183}), \\
\mathbf{h}_{13} &= (p_3 + q_3)^2 (p_1 b_{62} + q_2 b_{78} + p_1^2 b_{153} + p_1 q_2 b_{163} + p_3 q_1 b_{163} + q_2^2 b_{199}), \\
\mathbf{h}_{14} &= p_2 q_3^2 b_{31} + p_3 q_2^2 b_{33} + p_2 q_1^2 b_{43} + p_2 q_1 b_{45} + p_3 q_1 b_{48} + p_2 q_3^2 b_{87} +
\end{align*}
\]
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\[ h_{15} = p_3 q_1^3 b_{89} + p_2 q_1^2 b_{90} + p_2 p_3 q_1^2 b_{101} + p_3 q_1^2 b_{104} + p_2^2 q_1 b_{130} + p_2^2 p_3 q_1 b_{132} + p_2 p_3^2 q_1 b_{135} + p_3^2 q_1 b_{139}, \]

\[ h_{16} = p_1^2 q_2 b_{50} + p_1^2 q_3 b_{52} + p_1 q_2^2 b_{54} + p_1 q_3 b_{56} + p_1 q_2 b_{60} + p_1^2 q_2 b_{141} + p_1^2 q_3 b_{143} + p_1^2 q_2^2 b_{145} + p_1^2 q_2 q_3 b_{147} + p_1^2 q_2^3 b_{152} + p_1 q_2^3 b_{155} + p_1^2 q_2 q_3 b_{157} + p_1 q_2^3 b_{162} + p_1 q_2^3 b_{171}, \]

\[ h_{17} = p_2 q_3 q_4 b_{44} + p_2^2 q_3 b_{75} + p_2 q_2^2 b_{77} + p_2 q_3^2 b_{100} + p_2 q_2 q_3 b_{131} + p_2 q_1^2 q_3 b_{133} + p_2 q_1 q_3^2 b_{196} + p_2^2 q_2 q_3 b_{198} + p_2 q_3^2 b_{201}, \]

\[ h_{18} = p_1 p_2 q_3 b_{59} + p_1^2 p_2 q_3 b_{150} + p_1 p_2 q_3 b_{166} + p_1 p_2 q_3 b_{168}, \]

\[ h_{19} = p_3 q_1 q_2 b_{42} + p_3 q_1^2 q_2 b_{98} + p_3^2 q_1 b_{123} + p_3^2 q_2 b_{129}, \]

\[ h_{20} = p_3^3 b_{49} + p_1^3 p_2 b_{51} + p_1^2 p_3 b_{53} + p_1^2 p_2 b_{58} + p_1 p_2 p_3 b_{60} + p_1 p_2 b_{63} + p_2 p_3 + p_2^2 p_3 b_{76} + p_2 p_3^2 b_{79} + p_3^2 b_{93} + p_1^2 b_{140} + p_1 p_2 b_{142} + p_1^2 p_3 b_{144} + p_2^2 p_2 b_{149} + p_2 p_3 b_{151} + p_2 p_3^2 b_{154} + p_1^2 b_{165} + p_1 p_2 p_3 b_{167} + p_1 p_2^2 b_{170} + p_1 p_3^2 b_{174} + p_2^4 b_{195} + p_3^3 p_3 b_{197} + p_2^2 p_3^2 b_{200} + p_2 p_3^3 b_{204} + p_3^3 b_{209}, \]

\[ h_{21} = (p_1 + q_1 + q_1^2 b_{105})^3 + (p_2 + q_2 + q_2^2 b_{185})^3 + (p_3 + q_3 + q_3^2 b_{208})^3, \]

\[ h_{22} = (-p_1 - q_1 + q_1^2 b_{85})^3 + (-p_2 - q_2 + q_2^2 b_{176})^3 + (-p_3 - q_3 + q_3^2 b_{206})^3, \]

\[ h_{23} = (p_1 + q_1)^4 b_{90} + (p_1 + q_1)^2 (p_2 + q_2)^2 b_{111} + (p_1 + q_1)^2 (p_3 + q_3)^2 b_{118} + (p_2 + q_2)^4 b_{179} + (p_2 + q_2)^2 (p_3 + q_3)^2 b_{189} + (p_3 + q_3)^4 b_{207}. \]

Here \( b_i \)'s are at present unknown coefficients. As mentioned above, by forcing the refactorized form \( P \) to equal the original map \( M \) up to order 4 and using the CBH theorem\cite{28}, we can easily compute these unknown coefficients in terms of the known \( a_i \)'s. These expressions are available from the author as part of a FORTRAN program implementing the above algorithm.

The explicit actions of the polynomial maps on phase space variables can be obtained and they are given below. This completely determines the refactorized map \( P \). Each \( \exp(h_i) \) is a polynomial map which can be evaluated exactly and is explicitly symplectic. Thus by using \( P \) instead of \( M \) in Eq. (8), we obtain an explicitly symplectic integration algorithm. Further, it is fast to evaluate and does not introduce spurious poles and branch points. The above factorization is not unique. However, the principles outlined earlier impose restrictions on the possible forms and this cases considerably the task of refactorization. Moreover, we require the coefficients \( b_i \) to be polynomials in the known coefficients \( a_i \). Otherwise this can lead to divergences when \( a_i \)'s take on certain special values. Finally, we minimize the number of polynomial maps in the refactorized form. Our studies show that
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different polynomial map refactorizations obeying the above restrictions do not lead to any significant differences in their behavior.

We now derive the explicit actions of the polynomial maps on phase space variables. First consider \( \exp(\cdot h_1 \cdot) \). We obtain its action on the phase space variables by following the procedure outlined in the paragraph before Eq. (12). We notice that \( h_1 = h_1(q_1, q_2, q_3) \) depends only on the coordinate variables which are therefore cyclic variables. Hence we immediately obtain: \( q_i(t) = q_i(0) \) \( (i = 1, 2, 3) \). Solving the Hamilton’s equations of motion for \( p_i \) with \( h_1 \) as the Hamiltonian from \( t = 0 \) to \( t = -1 \) we get:

\[
p_i(-1) = p_i(0) + \frac{\partial h_1}{\partial q_i}(q_1(0), q_2(0), q_3(0)), \quad i = 1, 2, 3. \tag{28}
\]

Denoting \( z_i(0) \) and \( z_i(-1) \) by \( z_i^{in} \) and \( z_i^{fin} \) respectively, we finally obtain the action of \( \exp(\cdot h_1 \cdot) \):

\[
q_i^{fin} = q_i^{in}, \quad p_i^{fin} = p_i^{in} + \frac{\partial h_1}{\partial q_i}(q_1^{in}, q_2^{in}, q_3^{in}), \quad i = 1, 2, 3. \tag{29}
\]

Next consider the action of \( \exp(\cdot h_2 \cdot) \). The 2 factors in the product are independently conserved under the Hamiltonian flow. Consequently, we have

\[
A_2 = [(b_2 + b_{34}) + q_2(b_{91} + b_{106}) + p_2(b_{92} + b_{107}) + q_3(b_{93} + b_{108}) + p_3(b_{94} + b_{109})]
\]

\[
= [(b_2 + b_{34}) + q_2(0)(b_{91} + b_{106}) + p_2(0)(b_{92} + b_{107}) + q_3(0)(b_{93} + b_{108}) + p_3(0)(b_{94} + b_{109})], \tag{30}
\]

\[
B_2 = (q_1 + p_1) = (q_1(0) + p_1(0)).
\]

Solving Hamilton’s equation of motion with \( h_2 \) as the Hamiltonian from \( t = 0 \) to \( t = -1 \) we get

\[
q_1(-1) = q_1(0) - 3A_2B_2^2; \quad p_1(-1) = p_1(0) + 3A_2B_2^2;
\]

\[
q_2(-1) = q_2(0) - B_2^3(b_{92} + b_{107}); \quad p_2(-1) = p_2(0) + B_2^3(b_{91} + b_{106}); \tag{31}
\]

\[
q_3(-1) = q_3(0) - B_2^3(b_{94} + b_{109}); \quad p_3(-1) = p_3(0) + B_2^3(b_{93} + b_{108}).
\]

Denoting \( z_i(0) \) and \( z_i(-1) \) by \( z_i^{in} \) and \( z_i^{fin} \) respectively, we obtain the action of \( \exp(\cdot h_2 \cdot) \):

\[
q_1^{fin} = q_1^{in} - 3A_2B_2^2; \quad p_1^{fin} = p_1^{in} + 3A_2B_2^2;
\]

\[
q_2^{fin} = q_2^{in} - B_2^3(b_{92} + b_{107}); \quad p_2^{fin} = p_2^{in} + B_2^3(b_{91} + b_{106}); \tag{32}
\]

\[
q_3^{fin} = q_3^{in} - B_2^3(b_{94} + b_{109}); \quad p_3^{fin} = p_3^{in} + B_2^3(b_{93} + b_{108}).
\]

where \( A_2, B_2 \) are now functions of \( z^{in} \). The actions of \( \exp(\cdot h_i \cdot) \), \( i = 3, 4, \ldots, 7 \) on the phase space variables are obtained in a similar fashion and these actions are listed in the Appendix.
We now consider the action of \( \exp(\cdot; h_8 :) \). From the Hamilton’s equations of motion, we have the following conserved quantities:

\[
A_8 = (q_2 b_{35} + q_3 b_{37} + q_2^2 b_{110} + q_2 q_3 b_{112} + p_3 q_2 b_{113} + q_3^2 b_{117}) \\
= (q_2(0)b_{35} + q_3(0)b_{37} + q_2^2(0)b_{110} + q_2(0)q_3(0)b_{112} + \\
p_3(0)q_2(0)b_{113} + q_3^2(0)b_{117}),
\]  

(33)  

\[
B_8 = (q_1 + p_1) = (q_1(0) + p_1(0)).
\]

Solving the equations of motion for \( q_i \)’s with \( h_8 \) as the Hamiltonian from \( t = 0 \) to \( t \) we get:

\[
q_1(t) = q_1(0) + 2A_8 B_8 t; \quad q_2(t) = q_2(0); \\
q_3(t) = q_3(0) + B_8^2 b_{113} q_2(0) t.
\]  

(34)

For the momentum variables we get the following differential equations:

\[
\frac{dp_1(t)}{dt} = -2A_8 B_8, \\
\frac{dp_2(t)}{dt} = -B_8^2 [b_{35} + 2b_{110} q_2(t) + b_{112} q_3(t) + b_{113} p_3(t)], \\
\frac{dp_3(t)}{dt} = -B_8^2 [b_{37} + b_{112} q_2(t) + 2b_{113} q_3(t)].
\]  

(35)

The first equation can be trivially solved to obtain \( p_1(t) \). After substituting for \( q_2(t) \) and \( q_3(t) \) which are known, we can next solve the last equation for \( p_3(t) \). Substituting this in the second equation, we finally get \( p_2(t) \). Setting \( t = -1 \) and denoting \( z_i(0), z_i(-1) \) by \( z_i^{\text{in}}, z_i^{\text{fin}} \) respectively, we obtain the action of \( \exp(\cdot; h_8 :) \):

\[
q_1^{\text{fin}} = q_1^{\text{in}} - 2A_8 B_8; \quad p_1^{\text{fin}} = p_1^{\text{in}} + 2A_8 B_8; \\
q_2^{\text{fin}} = q_2^{\text{in}}; \quad p_2^{\text{fin}} = p_2^{\text{in}} + B_8^2 [b_{35} + 2b_{110} q_2^{\text{in}} + b_{112} q_3^{\text{in}} + b_{113} p_3^{\text{in}} + \\
B_8^2 b_{113} (b_{37}/2 + b_{113} q_2^{\text{in}} - B_8^2 b_{113} q_2^{\text{in}}/3)]; \\
q_3^{\text{fin}} = q_3^{\text{in}} - B_8^2 b_{113} q_2^{\text{in}}; \\
p_3^{\text{fin}} = p_3^{\text{in}} + B_8^2 [b_{37} + b_{112} q_2^{\text{in}} + 2b_{113} q_3^{\text{in}} - B_8^2 b_{113} b_{117} q_2^{\text{in}}],
\]  

(36)

where \( A_8, B_8 \) are now functions of \( z^{\text{in}} \). The actions of \( \exp(\cdot; h_i :)\), \( i = 9, 10, \ldots, 13 \) on the phase space variables are obtained in a similar fashion and these actions are listed in the Appendix.

Next consider the action of \( \exp(\cdot; h_{14} :) \). We notice that \( h_{14} = h_{14}(q_1, p_2, p_3) \) is independent of \( p_1, q_2 \) and \( q_3 \). Hence \( q_1, p_2, p_3 \) are cyclic variables and are conserved under the action of the Hamiltonian. Solving the Hamilton’s equations of motion for \( p_1, q_2 \) and \( q_3 \) with \( h_{14} \) as the Hamiltonian from \( t = 0 \) to \( t = -1 \) we get:

\[
p_1(-1) = p_1(0) + \frac{\partial h_{14}}{\partial q_1}(q_1(0), p_2(0), p_3(0)),
\]
respectively, we obtain the action of \( \exp(\cdot) \): \( h \)

The actions of \( \exp(\cdot) \), \( i = 15, 16, \ldots, 20 \) on the phase space variables are obtained in a similar fashion and these actions are listed in the Appendix.

We now consider the action of \( \exp(\cdot) : h_{21} \). From the Hamilton’s equations of motion, we have the following conserved quantities:

\[
\begin{align*}
A_{21} &= (p_1 + q_1 + p_1^2 b_{105}) = (p_1(0) + q_1(0) + p_1^2(0) b_{105}), \\
B_{21} &= (p_2 + q_2 + p_2^2 b_{185}) = (p_2(0) + q_2(0) + p_2^2(0) b_{185}) \\
C_{21} &= (p_3 + q_3 + p_3^2 b_{208}) = (p_3(0) + q_3(0) + p_3^2(0) b_{208}).
\end{align*}
\]

Solving the equations of motion for \( p_i \)’s with \( h_{21} \) as the Hamiltonian from \( t = 0 \) to \( t \) we get:

\[
\begin{align*}
p_1(t) &= p_1(0) - 3A_{21}^2 t; & p_2(t) &= p_2(0) - 3B_{21}^2 t; \\
p_3(t) &= p_3(0) - 3C_{21}^2 t.
\end{align*}
\]

The equations of motion for \( q_i \)’s are given as:

\[
\begin{align*}
\frac{dq_1(t)}{dt} &= 3A_{21}^2 (1 + 2b_{105}p_1(t)), \\
\frac{dq_2(t)}{dt} &= 3B_{21}^2 (1 + 2b_{185}p_2(t)), \\
\frac{dq_3(t)}{dt} &= 3C_{21}^2 (1 + 2b_{208}p_3(t)).
\end{align*}
\]

Substituting the expressions for \( p_1(t), p_2(t), p_3(t) \) obtained earlier, the above equations can be easily solved. Setting \( t = -1 \) and denoting \( z_i(0), z_i(-1) \) by \( z_i^{\text{in}}, z_i^{\text{fin}} \) respectively, we obtain the action of \( \exp(\cdot) : h_{21} \):
where $A_{21}, B_{21}, C_{21}$ are now functions of $z^{in}$. The action of \( \exp(h_{22}) \) on the phase space variables is obtained in a similar fashion and is listed in the Appendix.

Finally, we consider the action of \( \exp(h_{23}) \). From the Hamilton’s equations of motion, we have the following conserved quantities:

\[
A_{23} = (p_1 + q_1) = (p_1(0) + q_1(0)), \\
B_{23} = (p_2 + q_2) = (p_2(0) + q_2(0)), \\
C_{23} = (p_3 + q_3) = (p_3(0) + q_3(0)).
\]

From the equations of motion with $h_{23}$ as the Hamiltonian we get:

\[
\begin{align*}
\frac{dq_1(t)}{dt} &= -\frac{dp_1(t)}{dt} = 4b_{90}A_{23}^3 + 2b_{111}A_{23}B_{23}^2 + 2b_{118}A_{23}C_{23}^2, \\
\frac{dq_2(t)}{dt} &= -\frac{dp_2(t)}{dt} = 4b_{179}B_{23}^3 + 2b_{111}A_{23}B_{23} + 2b_{189}B_{23}C_{23}, \\
\frac{dq_3(t)}{dt} &= -\frac{dp_3(t)}{dt} = 4b_{207}C_{23}^3 + 2b_{118}A_{23}C_{23} + 2b_{189}B_{23}^2C_{23}.
\end{align*}
\]

Solving these equations from $t = 0$ to $t = -1$ and denoting $z_i(0), z_i(-1)$ by $z_i^{in}, z_i^{fin}$ respectively, we obtain the action of \( \exp(h_{23}) \):

\[
\begin{align*}
q_1^{fin} &= q_1^{in} - [4b_{90}A_{23}^3 + 2b_{111}A_{23}B_{23}^2 + 2b_{118}A_{23}C_{23}^2]; \\
p_1^{fin} &= p_1^{in} + [4b_{90}A_{23}^3 + 2b_{111}A_{23}B_{23}^2 + 2b_{118}A_{23}C_{23}^2]; \\
q_2^{fin} &= q_2^{in} - [4b_{179}B_{23}^3 + 2b_{111}A_{23}B_{23} + 2b_{189}B_{23}C_{23}^2]; \\
p_2^{fin} &= p_2^{in} + [4b_{179}B_{23}^3 + 2b_{111}A_{23}B_{23} + 2b_{189}B_{23}C_{23}^2]; \\
q_3^{fin} &= q_3^{in} - [4b_{207}C_{23}^3 + 2b_{118}A_{23}C_{23} + 2b_{189}B_{23}^2C_{23}]; \\
p_3^{fin} &= p_3^{in} + [4b_{207}C_{23}^3 + 2b_{118}A_{23}C_{23} + 2b_{189}B_{23}^2C_{23}],
\end{align*}
\]

where $A_{23}, B_{23}, C_{23}$ are now functions of $z^{in}$.

Substituting in Eq. (27) the explicit formulas for the actions of the polynomial maps listed above and in the Appendix, we can evaluate the action of $\mathcal{P}$ without violating the symplectic condition. Using this explicitly symplectic map in Eq. (8), we have the desired symplectic integration algorithm.

5. Applications

We have applied the method to a large particle storage ring for storing charged particles. This storage ring consists of 5109 individual elements (where these elements could be drifts, bending magnets, quadrupoles or sextupoles). If one tries to numerically integrate the trajectory of a charged particle through this ring using a conventional integration algorithm, one has to go through the ring element by element where each element is described by its own Hamiltonian. This is cumbersome and slow and further, does not respect the Hamiltonian nature of the system. On the other hand, a map based approach where one represents the entire storage ring
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in terms of a single map is much faster \(^{24,25}\). When this is combined with our polynomial map refactorization, one obtains a symplectic integration algorithm which is both fast and accurate and is ideally suited for such complex real life systems. The \(q_1 - p_1\) phase plot for one million turns around the ring using our polynomial map method is given in Figure 1. In this case, \(q_1\) and \(p_1\) represent the deviations from the closed orbit coordinate and momentum respectively. From theoretical considerations, we expect the so-called betatron oscillations in these variables. This manifests itself as ellipses in the phase space plot of \(q_1\) and \(p_1\) variables. In Figure 1, we observe the expected betatron oscillations. We also see the thickening of the ellipses caused by nonlinearities present in the sextupoles.

\[\text{Fig. 1. This figure shows the } q_1 - p_1 \text{ phase space plot for one million turns around a storage ring using the polynomial map method (only every 1000th point is plotted).}\]

6. Conclusions
To conclude, we described in detail a new symplectic integration algorithm based on polynomial map refactorization. We enumerated the types of symplectic maps which give rise to polynomial actions on phase space variables. For a six dimensional phase space, we obtained the refactorization of a given symplectic map in terms of 23 polynomial maps. The explicit actions of these polynomial maps were derived. This polynomial map method can be used to study long term stability of complicated
nonlinear Hamiltonian systems.

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Appendix A

The actions of polynomial maps \( \exp(h_i) \) which were not listed in the main text are given in this Appendix.

The action of \( \exp(h_3) \) is given as follows:

\[
\begin{align*}
q_1^{fin} &= q_1^{in} + 3A_3B_3^2; \\
p_1^{fin} &= p_1^{in} + 3A_3B_3^2; \\
q_2^{fin} &= q_2^{in} - B_3^1(-b_{92} + b_{106}); \\
p_2^{fin} &= p_2^{in} + B_3^1(-b_{91} + b_{106}); \\
q_3^{fin} &= q_3^{in} - B_3^3(-b_{94} + b_{109}); \\
p_3^{fin} &= p_3^{in} + B_3^3(-b_{93} + b_{108}),
\end{align*}
\]

where

\[
\begin{align*}
A_3 &= \left( -b_{29} + b_{34} \right) + q_2^{in}(-b_{91} + b_{106}) + p_2^{in}(-b_{92} + b_{107}) + q_3^{in}(-b_{93} + b_{108}) + p_3^{in}(-b_{94} + b_{109}), \\
B_3 &= (q_1^{in} - p_1^{in}).
\end{align*}
\]

The action of \( \exp(h_4) \) is given as follows:

\[
\begin{align*}
q_1^{fin} &= q_1^{in} - B_4^3(b_{156} + b_{159}); \\
p_1^{fin} &= p_1^{in} + B_4^3(b_{121} + b_{124}); \\
q_2^{fin} &= q_2^{in} - 3A_4B_4^2; \\
p_2^{fin} &= p_2^{in} + 3A_4B_4^2; \\
q_3^{fin} &= q_3^{in} - B_4^3(b_{181} + b_{187}); \\
p_3^{fin} &= p_3^{in} + B_4^3(b_{180} + b_{186}),
\end{align*}
\]

where

\[
\begin{align*}
A_4 &= \left( b_{65} + b_{68} \right) + q_1^{in}(b_{121} + b_{124}) + p_1^{in}(b_{156} + b_{159}) + q_3^{in}(b_{180} + b_{186}) + p_3^{in}(b_{181} + b_{187}), \\
B_4 &= (q_2^{in} + p_2^{in}).
\end{align*}
\]

The action of \( \exp(h_5) \) is given as follows:

\[
\begin{align*}
q_1^{fin} &= q_1^{in} - B_5^3(-b_{156} + b_{159}); \\
p_1^{fin} &= p_1^{in} + B_5^3(-b_{121} + b_{124}); \\
q_2^{fin} &= q_2^{in} + 3A_5B_5^2; \\
p_2^{fin} &= p_2^{in} + 3A_5B_5^2; \\
q_3^{fin} &= q_3^{in} - B_5^3(-b_{181} + b_{187}); \\
p_3^{fin} &= p_3^{in} + B_5^3(-b_{180} + b_{186}),
\end{align*}
\]

where

\[
\begin{align*}
A_5 &= \left[ -b_{65} + b_{68} \right] + q_1^{in}(-b_{121} + b_{124}) + p_1^{in}(-b_{156} + b_{159}) + q_3^{in}(-b_{180} + b_{186}) + p_3^{in}(-b_{181} + b_{187}), \\
B_5 &= (q_2^{in} - p_2^{in}).
\end{align*}
\]
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The action of \( \exp(h_6) \) is given as follows:
\[
q_{1}^{\text{fin}} = q_{1}^{\text{in}} - B_{3}^{3} (b_{172} + b_{173}) ; \quad p_{1}^{\text{fin}} = p_{1}^{\text{in}} + B_{3}^{3} (b_{137} + b_{138}) ; \\
q_{2}^{\text{fin}} = q_{2}^{\text{in}} - B_{3}^{3} (b_{202} + b_{203}) ; \quad p_{2}^{\text{fin}} = p_{2}^{\text{in}} + B_{3}^{3} (b_{192} + b_{193}) ; \\
q_{3}^{\text{fin}} = q_{3}^{\text{in}} - 3A_{6} B_{6}^{3} ; \quad p_{3}^{\text{fin}} = p_{3}^{\text{in}} + 3A_{6} B_{6}^{3} ,
\]
where
\[
A_{6} = \left[ (b_{81} + b_{82}) + q_{1}^{\text{in}} (b_{137} + b_{138}) + p_{1}^{\text{in}} (b_{172} + b_{173}) + q_{2}^{\text{in}} (b_{192} + b_{193}) + p_{2}^{\text{in}} (b_{202} + b_{203}) \right] , \\
B_{6} = (q_{3}^{\text{in}} + p_{3}^{\text{in}}).
\]
The action of \( \exp(h_7) \) is given as follows:
\[
q_{1}^{\text{fin}} = q_{1}^{\text{in}} - B_{3}^{3} (-b_{172} + b_{173}) ; \quad p_{1}^{\text{fin}} = p_{1}^{\text{in}} + B_{3}^{3} (-b_{137} + b_{138}) ; \\
q_{2}^{\text{fin}} = q_{2}^{\text{in}} - B_{3}^{3} (-b_{202} + b_{203}) ; \quad p_{2}^{\text{fin}} = p_{2}^{\text{in}} + B_{3}^{3} (-b_{192} + b_{193}) ; \\
q_{3}^{\text{fin}} = q_{3}^{\text{in}} + 3A_{7} B_{7}^{3} ; \quad p_{3}^{\text{fin}} = p_{3}^{\text{in}} + 3A_{7} B_{7}^{3} ,
\]
where
\[
A_{7} = \left[ (-b_{81} + b_{82}) + q_{1}^{\text{in}} (-b_{137} + b_{138}) + p_{1}^{\text{in}} (-b_{172} + b_{173}) + q_{2}^{\text{in}} (-b_{192} + b_{193}) + p_{2}^{\text{in}} (-b_{202} + b_{203}) \right] , \\
B_{7} = (q_{3}^{\text{in}} + p_{3}^{\text{in}}).
\]
The action of \( \exp(h_9) \) is given as follows:
\[
q_{1}^{\text{fin}} = q_{1}^{\text{in}} - 2A_{9} B_{9} ; \quad p_{1}^{\text{fin}} = p_{1}^{\text{in}} + 2A_{9} B_{9} ; \\
q_{2}^{\text{fin}} = q_{2}^{\text{in}} + B_{3}^{3} \left[ -b_{36} - 2b_{114} p_{2}^{\text{in}} - b_{115} q_{3}^{\text{in}} - b_{116} p_{3}^{\text{in}} + B_{9}^{2} b_{115} (b_{38}/2 + b_{119} p_{3}^{\text{in}} + B_{3}^{2} c_{119} p_{2}^{\text{in}} / 3) \right] ; \quad p_{2}^{\text{fin}} = p_{2}^{\text{in}} \\
q_{3}^{\text{fin}} = q_{3}^{\text{in}} - B_{3}^{3} \left[ b_{38} + b_{116} p_{2}^{\text{in}} + 2b_{119} p_{3}^{\text{in}} + B_{3}^{2} b_{115} b_{119} p_{2}^{\text{in}} / 3 \right] ; \\
p_{3}^{\text{fin}} = p_{3}^{\text{in}} + B_{3}^{2} b_{115} p_{2}^{\text{in}} ,
\]
where
\[
A_{9} = \left( p_{2}^{\text{in}} b_{36} + p_{3}^{\text{in}} b_{38} + (p_{2}^{\text{in}})^{2} b_{114} + p_{2}^{\text{in}} q_{3}^{\text{in}} b_{115} + p_{2}^{\text{in}} p_{3}^{\text{in}} b_{116} + (p_{3}^{\text{in}})^{2} b_{119} \right) , \\
B_{9} = (q_{1}^{\text{in}} + p_{1}^{\text{in}}).
\]
The action of \( \exp(h_{10}) \) is given as follows:
\[
q_{1}^{\text{fin}} = q_{1}^{\text{in}} ; \quad p_{1}^{\text{fin}} = p_{1}^{\text{in}} + B_{10}^{2} \left[ b_{40} + 2b_{96} q_{1}^{\text{in}} + b_{125} q_{3}^{\text{in}} + b_{126} p_{3}^{\text{in}} + B_{10}^{2} b_{126} (b_{69}/2 + b_{188} q_{1}^{\text{in}} - B_{10}^{2} b_{126} b_{188} q_{1}^{\text{in}} / 3) \right] ; \\
q_{2}^{\text{fin}} = q_{2}^{\text{in}} - 2A_{10} B_{10} ; \quad p_{2}^{\text{fin}} = p_{2}^{\text{in}} + 2A_{10} B_{10} ; \\
q_{3}^{\text{fin}} = q_{3}^{\text{in}} - B_{10}^{2} b_{126} q_{1}^{\text{in}} ; \\
p_{3}^{\text{fin}} = p_{3}^{\text{in}} + B_{10}^{2} \left[ b_{69} + b_{125} q_{1}^{\text{in}} + 2b_{188} q_{3}^{\text{in}} - B_{10}^{2} b_{126} b_{188} q_{1}^{\text{in}} \right] ,
\]
where

\[
A_{10} = (q_1^{in} b_{10} + q_3^{in} b_{69} + (q_1^{in})^2 b_{96} + q_1^{in} q_3^{in} b_{125} + p_3^{in} q_1^{in} b_{126} \\
+ (q_3^{in})^2 b_{188}) \\
B_{10} = (q_2^{in} + p_2^{in}).
\]  

(14)

The action of \(\exp(h_{11})\) is given as follows:

\[
q_1^{fin} = q_1^{in} + B_{11}^2 \left[-b_{55} - 2b_{146}q_1^{in} - b_{160}q_3^{in} - b_{161}p_3^{in} + B_{11}^2 b_{160}b_{190}p_1^{in}/3\right]; \quad p_1^{fin} = p_1^{in};
\]

\[
q_2^{fin} = q_2^{in} - 2A_{11}B_{11}; \quad p_2^{fin} = p_2^{in} + 2A_{11}B_{11};
\]

\[
q_3^{fin} = q_3^{in} - B_{11}^2 \left[b_{70} + b_{161}q_1^{in} + 2b_{190}p_3^{in} + B_{11}^2 b_{160}b_{190}p_1^{in}\right];
\]

\[
p_3^{fin} = p_3^{in} + B_{11}^2 b_{160}p_1^{in},
\]

where

\[
A_{11} = (p_1^{in} b_{55} + p_3^{in} b_{70} + (p_1^{in})^2 b_{146} + q_1^{in} q_3^{in} b_{160} + p_1^{in} p_3^{in} b_{161} + (p_3^{in})^2 b_{190});
\]

\[
B_{11} = (q_2^{in} + p_2^{in}).
\]

(16)

The action of \(\exp(h_{12})\) is given as follows:

\[
q_1^{fin} = q_1^{in}; \quad p_1^{fin} = p_1^{in} + B_{12}^2 \left[b_{47} + 2b_{103}q_1^{in} + b_{128}q_2^{in} + b_{134}p_2^{in} + \right.\]

\[
B_{12} b_{134}(b_{72}/2 + b_{183}q_2^{in} - B_{12}^2 b_{134}b_{183}q_1^{in}/3\right]);
\]

\[
q_2^{fin} = q_2^{in} - B_{12}^2 b_{134}q_1^{in};
\]

\[
p_2^{fin} = p_2^{in} + B_{12}^2 \left[b_{72} + b_{128}q_1^{in} + 2b_{183}q_2^{in} - B_{12}^2 b_{134}b_{183}q_1^{in}\right];
\]

\[
q_3^{fin} = q_3^{in} - 2A_{12}B_{12}; \quad p_3^{fin} = p_3^{in} + 2A_{12}B_{12},
\]

where

\[
A_{12} = (q_1^{in} b_{47} + q_2^{in} b_{72} + (q_1^{in})^2 b_{103} + q_1^{in} q_2^{in} b_{128} + p_2^{in} q_1^{in} b_{134} + \right.\]

\[
(q_2^{in})^2 b_{183});
\]

\[
B_{12} = (q_3^{in} + p_3^{in}).
\]

(18)

The action of \(\exp(h_{13})\) is given as follows:

\[
q_1^{fin} = q_1^{in} + B_{13}^2 \left[-b_{62} - 2b_{153}p_1^{in} - b_{163}q_2^{in} - b_{169}p_2^{in} + \right.\]

\[
B_{13} b_{163}(b_{78}/2 + b_{199}p_2^{in} + B_{13}^2 b_{163}b_{199}p_1^{in}/3\right]);
\]

\[
q_2^{fin} = q_2^{in} - B_{13}^2 \left[b_{78} + b_{169}p_1^{in} + 2b_{199}p_2^{in} + B_{13}^2 b_{163}b_{199}p_1^{in}\right];
\]

\[
p_2^{fin} = p_2^{in} + B_{13}^2 b_{163}p_1^{in};
\]

\[
q_3^{fin} = q_3^{in} - 2A_{13}B_{13}; \quad p_3^{fin} = p_3^{in} + 2A_{13}B_{13},
\]

(19)
where

\[
A_{13} = (p_1^{\text{in}} b_{62} + p_2^{\text{in}} b_{78} + (p_1^{\text{in}})^2 b_{153} + p_1^{\text{in}} q_2^{\text{in}} b_{163} + p_1^{\text{in}} p_2^{\text{in}} b_{169} + (p_2^{\text{in}})^2 b_{199}),
\]

\[
B_{13} = (q_3^{\text{in}} + p_3^{\text{in}}).
\]

The action of \(\exp(: h_{15} :)\) is given as follows:

\[
q_1^{\text{fin}} = q_1^{\text{in}} - \frac{\partial h_{15}}{\partial p_1}(q_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}); \quad p_1^{\text{fin}} = p_1^{\text{in}},
\]

\[
q_2^{\text{fin}} = q_2^{\text{in}}; \quad p_2^{\text{fin}} = p_2^{\text{in}} + \frac{\partial h_{15}}{\partial q_2}(p_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}),
\]

\[
q_3^{\text{fin}} = q_3^{\text{in}}; \quad p_3^{\text{fin}} = p_3^{\text{in}} + \frac{\partial h_{15}}{\partial q_3}(q_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}).
\]

The action of \(\exp(: h_{16} :)\) is given as follows:

\[
q_1^{\text{fin}} = q_1^{\text{in}}; \quad p_1^{\text{fin}} = p_1^{\text{in}} + \frac{\partial h_{16}}{\partial q_1}(q_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}),
\]

\[
q_2^{\text{fin}} = q_2^{\text{in}}; \quad p_2^{\text{fin}} = p_2^{\text{in}} + \frac{\partial h_{16}}{\partial q_2}(q_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}), \quad p_3^{\text{fin}} = p_3^{\text{in}},
\]

The action of \(\exp(: h_{17} :)\) is given as follows:

\[
q_1^{\text{fin}} = q_1^{\text{in}}; \quad p_1^{\text{fin}} = p_1^{\text{in}}; \quad p_2^{\text{fin}} = p_2^{\text{in}} + \frac{\partial h_{17}}{\partial q_2}(q_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}),
\]

\[
q_3^{\text{fin}} = q_3^{\text{in}}; \quad p_3^{\text{fin}} = p_3^{\text{in}} + \frac{\partial h_{17}}{\partial q_3}(q_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}).
\]

The action of \(\exp(: h_{18} :)\) is given as follows:

\[
q_1^{\text{fin}} = q_1^{\text{in}}; \quad p_1^{\text{fin}} = p_1^{\text{in}} + \frac{\partial h_{18}}{\partial q_1}(q_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}),
\]

\[
q_2^{\text{fin}} = q_2^{\text{in}}; \quad p_2^{\text{fin}} = p_2^{\text{in}} + \frac{\partial h_{18}}{\partial q_2}(q_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}), \quad p_3^{\text{fin}} = p_3^{\text{in}},
\]

The action of \(\exp(: h_{19} :)\) is given as follows:

\[
q_1^{\text{fin}} = q_1^{\text{in}}; \quad p_1^{\text{fin}} = p_1^{\text{in}} + \frac{\partial h_{19}}{\partial q_1}(q_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}),
\]

\[
q_2^{\text{fin}} = q_2^{\text{in}}; \quad p_2^{\text{fin}} = p_2^{\text{in}} + \frac{\partial h_{19}}{\partial q_2}(q_1^{\text{in}}, q_2^{\text{in}}, q_3^{\text{in}}), \quad p_3^{\text{fin}} = p_3^{\text{in}}.
\]
The action of $\exp(h_{20})$ is given as follows:

\[
q_1^{fin} = q_1^{in} - \frac{\partial h_{20}}{\partial p_1}(p_1^{in}, p_2^{in}, p_3^{in}); \quad p_1^{fin} = p_1^{in},
\]
\[
q_2^{fin} = q_2^{in} - \frac{\partial h_{20}}{\partial p_2}(p_1^{in}, p_2^{in}, p_3^{in}); \quad p_2^{fin} = p_2^{in},
\]
\[
q_3^{fin} = q_3^{in} - \frac{\partial h_{20}}{\partial p_3}(p_1^{in}, p_2^{in}, p_3^{in}); \quad p_3^{fin} = p_3^{in}.
\]  \hspace{1cm} (26)

The action of $\exp(h_{22})$ is given as follows:

\[
q_1^{fin} = q_1^{in} + 3A_{22}^{2}; \quad p_1^{fin} = p_1^{in} - 3A_{22}^{2}(1 - 2b_{85}q_1^{in} - 3A_{22}^{2}b_{85});
\]
\[
q_2^{fin} = q_2^{in} + 3B_{22}^{2}; \quad p_2^{fin} = p_2^{in} - 3B_{22}^{2}(1 - 2b_{176}q_2^{in} - 3B_{22}^{2}b_{176});
\]
\[
q_3^{fin} = q_3^{in} + 3C_{22}^{2}; \quad p_3^{fin} = p_3^{in} - 3C_{22}^{2}(1 - 2b_{206}q_3^{in} - 3C_{22}^{2}b_{206}).
\]  \hspace{1cm} (27)

where

\[
A_{22} = (-p_1^{in} - q_1^{in} + (q_1^{in})^2 b_{85}),
\]
\[
B_{22} = (-p_2^{in} - q_2^{in} + (q_2^{in})^2 b_{176})
\]
\[
C_{22} = (-p_3^{in} - q_3^{in} + (q_3^{in})^2 b_{206}).
\]  \hspace{1cm} (28)

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