Asymptotic Analysis on Spatial Coupling Coding for Two-Way Relay Channels

Satoshi Takabe\(^*\), Yuta Ishimatsu\(^*\), Tadashi Wadayama\(^*\), and Masahito Hayashi\(^{†‡}\)

\(^*\) Department of Computer Science, Faculty of Engineering, Nagoya Institute of Technology

\(^{†}\) Graduate School of Mathematics, Nagoya University

\(^‡\) Centre for Quantum Technologies, National University of Singapore

Email: s_takabe@nitech.ac.jp, 26115016@stn.nitech.ac.jp, wadayama@nitech.ac.jp, masahito@math.nagoya-u.ac.jp

Abstract—Compute-and-forward relaying is effective to increase bandwidth efficiency of wireless two-way relay channels. In a compute-and-forward scheme, a relay tries to decode a linear combination composed of transmitted messages from other terminals or relays. Design for error correcting codes and its decoding algorithms suitable for compute-and-forward relaying schemes are still important issue to be studied. In this paper, we will present an asymptotic performance analysis on LDPC codes over two-way relay channels based on density evolution (DE). Because of the asymmetric nature of the channel, we employ the population dynamics DE combined with DE formulas for asymmetric channels to obtain BP thresholds. In addition, we also evaluate the asymptotic performance of spatially coupled LDPC codes for two-way relay channels. The results indicate that the spatial coupling codes yield improvements in the BP threshold compared with corresponding uncoupled codes for two-way relay channels.

I. INTRODUCTION

A relaying with an appropriate signal processing and decoding are ubiquitous in wireless communications such as satellite communications, mobile wireless communications, and wireless local area networks. Increasing demand for bandwidth efficiency in wireless communications promotes spread of research activities on relaying and forwarding techniques. For example, theoretical limits of efficiencies of relaying techniques such as decode-and-forward \(^1\) and amplify-and-forward \(^2\) have been deeply studied. Recently, Nazar and Gastpar presented a novel concept of compute-and-forward relaying \(^3\). In a compute-and-forward scheme, a relay tries to decode a linear combination composed of transmitted messages from other terminals (or relays) and then the relay forwards a decoded linear combination to another relay or a terminal. That is, the repeater has no intention to decode each message separately. The concept is also termed as wireless network coding or physical layer network coding that has invoked huge research interests \(^4, 5\). Recently, Sula \(^6\) et al. presented a practical decoding scheme for LDPC codes in compute-forward multiple access (CFMA) systems. Ullah et al. \(^7\) derived the random coding error exponent for the uplink phase of a two-way relay channel.

The simplest scenario for a compute-and-forward scheme may be wireless two-way relay channels \(^8\). Two terminals A, B and a relay R are involved in this channel. The terminal A has own message and wishes to send it to the terminal B. Similarly, the terminal B wishes to send own message to A. There is no direct wireless connection between A and B, but a relay R has bi-directional wireless connections to both of A and B. When the relay R can decode a linear combination successfully, it is broadcasted to A and B in the next time slot. The terminals A and B can recover an intended message by subtracting own message from the received message.

In order to obtain a highly reliable estimate of linear combination at the relay, appropriate error correcting codes should be exploited because the received signal is distorted by additive noises. In such a case, the relay R intends to decode a sum of two codewords sent from A and B. One possible candidate of error correcting codes for such a situation is low-density parity-check (LDPC) codes \(^9\). A combination of LDPC codes and belief propagation (BP) decoding has been proved to be very powerful and effective for additive noise channels \(^10\). Sula et al. \(^6\) discussed an appropriate modified BP decoding for the two-way relay channels. They presented a performance analysis on LDPC codes over two-way relay channel based on computer simulations.

The goal of this work is twofold. The first goal is to provide an asymptotic performance analysis for LDPC codes over two-way relay channels based on density evolution (DE). DE \(^11\) is a common theoretical tool to study the asymptotic typical behavior of a BP decoder and it provides BP thresholds of the target channel. Although the BP threshold is below the Shannon limit, a BP threshold indicates a practical achievable rate with low complexity encoding and decoding. One technical challenge for evaluating the BP threshold of two-way relay channels comes from an asymmetric nature of the channel. That is, we cannot rely on the zero codeword assumption commonly used in DE analysis for binary-input memoryless output-symmetric channels \(^11\). In order to overcome this difficulty, we will employ population dynamics DE \(^12\) combined with the DE formula derived by Wang et al. for asymmetric channels \(^13\).

The second goal of this paper is to provide DE analysis for spatially coupled LDPC (SC-LDPC) codes over two-way relay channels. It is known that appropriately designed spatially coupled codes yield improvements in BP thresholds compared with those of uncoupled regular LDPC codes with comparable parameters \(^14, 15\). In many cases, we can observe threshold
saturation \cite{16}, i.e., a phenomenon that the BP threshold converges to the MAP threshold. The same is true for the spatially coupled coding for two-way erasure multiple access channels for a joint compute-and-forward scheme \cite{17}. As far as the authors know, typical behavior of BP decoding of spatially coupled LDPC codes over the two-way relay channels except for erasure ones is unknown. We consider that it is worth studying not only from practical interests but also from theoretical interests to provide an example of the DE analysis for general asymmetric channels. In this work, we will extend the population dynamics DE to protograph codes \cite{18} and perform numerical evaluations. Recent work by Hayashi et al. \cite{19} shows that efficient codes for two-way relay channels are useful to establish secure communication by Hayashi et al. \cite{19} shows that efficient codes for two-way relay channels are useful to establish secure communication.

In this paper, we focus on decoding methods for recovering $x_A \oplus x_B$ from the received word $y$. In this paper, we focus on decoding methods for recovering $x_A \oplus x_B$.

C. IID assumption-based belief propagation

Assume that two stochastic processes

$$\{X_A^{(1)}, X_A^{(2)}, \ldots, X_A^{(n)}, \ldots\}, \{X_B^{(1)}, X_B^{(2)}, \ldots, X_B^{(n)}, \ldots\}$$

are IID and that $X_A^{(t)}$ and $X_B^{(t)}$ are independent. For simplicity, we here assume that $Pr[X_A^{(t)} = 1] = Pr[X_B^{(t)} = 1] = 1/2$ holds for any $t$. From these assumptions, we have the probability of events:

$$Pr[\mu(X_A^{(t)}) + \mu(X_B^{(t)}) = 0] = \frac{1}{2}, \quad (3)$$

$$Pr[\mu(X_A^{(t)}) + \mu(X_B^{(t)}) = -2] = \frac{1}{4}, \quad (4)$$

$$Pr[\mu(X_A^{(t)}) + \mu(X_B^{(t)}) = 2] = \frac{1}{4}. \quad (5)$$

Let $Z^{(t)} = X_A^{(t)} \oplus X_B^{(t)}$. From the IID assumption, $Z^{(t)}$ is also a memoryless stochastic process. We now consider a virtual channel whose input and output symbols are $Z^{(t)}$ and $Y^{(t)}$, respectively. It is evident that the prior probability of $Z^{(t)}$ is given by $Pr(Z^{(t)} = 0) = Pr(Z^{(t)} = 1) = 1/2$. Under the IID assumptions, the conditional PDF representing the channel statistics of the virtual channel is given by

$$Pr[Y^{(t)} = y|Z^{(t)} = 1] = F(y; 0, \sigma^2),$$

$$Pr[Y^{(t)} = y|Z^{(t)} = 0] = \frac{1}{2} F(y; -2, \sigma^2) + \frac{1}{2} F(y; 2, \sigma^2), \quad (6)$$

where $F(y; m, \sigma^2)$ is the Gaussian distribution with mean $m$ and variance $\sigma^2$ defined by

$$F(y; m, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y-m)^2}{2\sigma^2}\right).$$

From this conditional PDF, symbol log likelihood ratio (LLR) can be easily derived:

$$\lambda^{(t)}(y) = \ln \frac{Pr[Y^{(t)} = y|Z^{(t)} = 0]}{Pr[Y^{(t)} = y|Z^{(t)} = 1]} = \ln \frac{\cosh \frac{2y}{\sigma^2}}{\frac{2}{\sigma^2}} - \frac{2}{\sigma^2}. \quad (7)$$

If the IID assumption is valid, we can make the best estimation on $Z^{(t)}$ only from $\lambda^{(t)}$. Note that this LLR expression is a special case of the LLR expression derived by Sula et al. \cite{6}.

Let us go back to the argument on the case where terminals A and B employ a binary linear code $C$. Due to linearity of the code $C$, it is clear that $(Z^{(1)}, \ldots, Z^{(n)})$ also belongs to $C$. From this fact, IID-assumption based maximum likelihood (ML) decoding can be defined as

$$(z_1, \ldots, z_n) = \arg \max_{z \in C} \prod_{t=1}^{n} L(y_t|z_t), \quad (8)$$

where the likelihood functions are defined by

$$L[y_1] = F(y; 0, \sigma^2),$$

$$L[y_0] = \frac{1}{2} F(y; -2, \sigma^2) + \frac{1}{2} F(y; 2, \sigma^2). \quad (9)$$
This ML rule is sub-optimal because the likelihood is based on the IID assumption. Regardless of its sub-optimality, the IID assumption makes the structure of a decoder simple, and it also makes it easier to exploit known channel coding techniques developed for memoryless channels.

Belief propagation (BP) decoding for LDPC codes can be regarded as an approximation of ML decoding as a message passing form. It would be natural to develop a BP decoding algorithm for the binary compute-and-forward channel based on the IID assumption-based ML rule \([6]\). It is not hard to see that the IID assumption-based BP coincides with the conventional log-domain BP algorithm \([11]\) with symbol LLR expression \([7]\). This type of BP decoder has already discussed in \([6,7]\). A significant advantage of the IID assumption-based BP is that it can be easily implemented based on a practical BP decoder for the additive white Gaussian noise (AWGN) channel just by replacing an LLR computation unit.

### III. Density Evolution for Binary Two-Way Relay Channels with IID Assumption

We employ DE to study BP thresholds of binary two-way relay channels with the IID assumption. In this section, we first introduce the population dynamics DE and estimate the BP threshold for uncoupled regular LDPC codes. The BP threshold for SC-LDPC is then evaluated.

#### A. Density Evolution for Asymmetric Channels

For simplicity, we here focus on \((d_c, d_r)\)-regular LDPC codes, where \(d_c\) and \(d_r\) represent the variable and check node degrees, respectively. Extension to irregular codes is straightforward. It is noteworthy that we need to handle signal dependent noises \([6]\) for two-way relay channels with the IID assumption. This means that we cannot rely on the zero code assumption in a DE analysis. In the following, we follow the Wang’s DE formulation \([13]\) to overcome this difficulty.

The conditional PDF \(P^{(l)}(m|z)\) (resp. \(Q^{(l)}(\hat{m}|z)\)) denote the PDF of a message \(m\) from a variable node to a check node (resp. \(\hat{m}\) from a check node to a variable node) with transmitted word \(z\) at the \(l\)-th step. The distribution of LLR of the virtual channel is denoted by \(P^{(0)}(z)\). Note that those PDFs depend on a transmitted word because of the asymmetric nature of the channel. For symmetric channels, in contrast, the zero code assumption omits the dependence. Let \(\Gamma(P_A) \triangleq P_A \circ \gamma^{-1}\) be a density transformation for a random variable \(A\) with distribution \(P_A\) \([13]\) where \(\gamma : \mathbb{R} \rightarrow \{0,1\} \times [0,\infty), \gamma(m) = (1_m \llcorner 0, \ln \coth \lceil \bar{m}/2 \rceil)\) with an indicator function \(1_{\{.\}}\).

The DE equations for binary asymmetric channels \([13]\) are given by

\[
P^{(l)}(m|z) = P^{(0)}(z) \otimes \left( Q^{(l-1)}(\hat{m}|z) \otimes (d_r-1) \right), \tag{10}
\]

\[
Q^{(l)}(\hat{m}|z) = \Gamma^{-1} \left( \left[ \Gamma \left( P^{(l)}(m|0) + P^{(l)}(m|1) \right) \right] \otimes (d_c-1) \right) + (-1)^{\bar{m}} \left[ \Gamma \left( P^{(l)}(m|0) - P^{(l)}(m|1) \right) \right] \otimes (d_c-1), \tag{11}
\]

where \(\otimes\) denotes the convolution operator on PDFs. Although these convolutions of PDFs can be efficiently evaluated with fast Fourier transformation, numerical evaluation requires huge computational costs. We use an alternative approach, population dynamics \([12]\), to reduce computational complexity because the DE analysis for SC-LDPC codes deals with a number of DE equations simultaneously.

Equations \((10)\) and \((11)\) have equivalent forms called the replica-symmetric cavity equations \([12]\), which read

\[
P^{(l)}(m|z) = \int dy L_{y|z} \prod_{s=1}^{d_r-1} d\hat{m}^{(s)} Q^{(l-1)}(\hat{m}^{(s)}|z) \times \delta \left( m - \lambda(y) - \sum_{s=1}^{d_r-1} \hat{m}^{(s)} \right), \tag{12}
\]

\[
Q^{(l)}(\hat{m}|z) = \frac{1}{2^{d_c-2}} \sum_{(z|s) \in S} \prod_{s=1}^{d_r-1} dm^{(s)} P^{(l)}(m^{(s)}|z^{(s)}) \times \delta \left( \hat{m} - 2 \tanh^{-1} \left[ \prod_{s=1}^{d_r-1} \tanh \left( \frac{m^{(s)}}{2} \right) \right] \right), \tag{13}
\]

where \(\lambda(y)\) denotes the LLR defined as the r.h.s. of \((7)\) and \(S \triangleq \{ (z^{(s)}) \in \{0,1\}^{d_c}; \bigoplus_{s=1}^{d_r} z^{(s)} = 0, z^{(d_c)} = z \}\).

In Algorithm 1, we describe a procedure of the population dynamics DE. In population dynamics, the PDFs \(P^l(\cdot|z)\) and \(Q^l(\cdot|z)\) are approximated to histograms (populations) of \(N\) samples denoted by, e.g., \(\{\nu_i^k\} (i \in [N] \triangleq \{1, \ldots, N\})\). The parameter \(N\) is called the population size and the DE equations are exactly solved in the large-\(N\) limit. Each sample is recursively updated by an update rule written in a delta function \(\delta(\cdot)\) in \((12)\) or \((13)\). After each iteration finishes, we can estimate bit error rate (BER) at the step. Although the recursion should continue until every population converges, it stops at the maximum iteration step \(T\) in practice.

We evaluate a BP threshold defined as a threshold of \(\sigma\) in the virtual channel \([6]\) below which LDPC codes are typically decodable by a BP decoder. As a MAP threshold, we use the symmetric information rate \(\sigma_{sym}(R)\) defined as a solution of \(C_{sym}(\sigma_{sym}(R)) = R\) for code rate \(R\), where

\[
C_{sym}(\sigma) = - \int_{-\infty}^{\infty} P(y) \log_2 P(y) dy + \frac{1}{2} \int_{-\infty}^{\infty} L_{y|0} \log_2 L_{y|0} dy + \frac{1}{4} \log_2 (2\pi\sigma^2 e), \tag{14}
\]

denotes the symmetric information rate of the two-way relay channel under the IID assumption and \(P(y) = (1/2)L_y|0 + (1/2)L_y|1\) is the PDF of a received symbol under the assumption.

The BP thresholds of various regular LDPC ensembles versus the code rate are shown in Fig. \([1]\). We search BP thresholds by evaluating BER using the population dynamics DE with \(N = 10^5\) and \(T = 2000\). It is confirmed that the estimation is accurate up to the third decimal place. The BP thresholds have a gap to symmetric information rate as predicted in \([6]\).
Algorithm 1 Population Dynamics DE

Input: Population size $N$, Maximum iteration $T$
Output: Populations $\{\nu^1_s\}$, $\{\nu^1_t\}$, and $\{\nu^1_i\}$ ($i \in [N]$)
1: Initialization: $\nu^0_s = \nu^0_t = \nu^0_i = 0$
2: for $l = 1$ to $T$ do
3:   for $z = 0$ to $1$ do  
4:      for $i = 1$ to $N$ do
5:         Draw $z(1), \ldots, z(d_r - 1)$ uniformly in $\{0, 1\}$
6:         to satisfy $z \oplus (\bigoplus_{s=1}^{d_r-1} z(s)) = 0$.
7:         Draw $i(1), \ldots, i(d_r - 1)$ uniformly in $[N]$.
8:         $\nu^t_i \leftarrow 2 \tanh^{-1} \left[ \prod_{s=1}^{d_r-1} \tanh \left( \nu^{z(s)}_i / 2 \right) \right]$
9:      end for
10:  end for
11: end for

Let us evaluate BP thresholds for SC-LDPC codes with finite $L$. In a protograph, each variable and check nodes respectively have a PDF $P(l)(m|z)$ and $Q(l)(\hat{m}|z)$ of messages as in the last subsection. Those PDFs are propagated as messages on a protograph. From a symmetric structure in each bundle, $P_{s \rightarrow y}^{L}(m|z)$ denotes the PDF of message $m$ as a message from a variable node in the $i$-th bundle to a check node $a$ at the $l$-th step. Similarly, let us denote the PDF of message $\hat{m}$ as a message from a check node $a$ to a variable node in the $i$-th bundle by $Q_{y \rightarrow a}^{l}(\hat{m}|z)$. DE equations of two-way relay channels and $(d_t, d_r, L)$-LDPC codes then read

$$P_{i \rightarrow a}^{(l)}(m|z) = \int dy L[y|z] \prod_{b \in N(i)} dm_b \frac{Q_{b \rightarrow i}^{(l-1)}(\hat{m}_b|z)}{\delta \left( m - \lambda(y) - \sum_{b \in N(i)} \hat{m}_b \right)}$$

$$Q_{a \rightarrow y}^{(l)}(\hat{m}|z) = \frac{1}{2^{d_s - 2}} \sum_{\{i_j(s)\} \in S'} \prod_{a=1}^{k} \prod_{j \in N(a)} \int d_{m_j(s)}^{(s)} P_{j \rightarrow a}^{(l)}(m_j(s)|z(s))$$

$$\times \prod_{j \in N(a)} \left[ \prod_{a=1}^{k} \frac{d_{m_j(s)}^{(s)} P_{j \rightarrow a}^{(l)}(m_j(s)|z(s))}{\delta \left( \hat{m} - 2 \tanh^{-1} \left[ \prod_{(j,s) \neq (i,k)} \tanh \left( \frac{m(s)}{2} \right) \right] \right)} \right]$$,

where $N(z)$ is a set of neighboring nodes in a protograph and $S' \triangleq \{ \{i_j(s)\} \}_{j \in N(a)} \in \{0, 1\}^{d_s}; \bigoplus_{j,s} \hat{m}_j(s) = 0, \hat{m}_j(k) = z \}$. A protograph of uncoupled LDPC codes recovers (12) and (13).

Population dynamics is implemented as an extension of Algorithm 1. In this case, we prepare $4d_l$ populations with size $N$ to approximate PDFs $P_{i \rightarrow a}^{(l)}(m|z)$ and $Q_{a \rightarrow y}^{(l)}(\hat{m}|z)$. Fig. 3 shows dynamics of BER of each variable node in $(3, 6, 25)$-LDPC codes when $N = 10^4$ and $\sigma = 0.78$. It is apparent that they decrease from each side of the chain, as observed in the symmetric channel case [20]. BERs vanish after the 169th step indicating that the code is decodable.

Fig. 4 shows the BP threshold of $(3, 6, L)$-LDPC codes and symmetric information rate corresponding to the de-
BER

10

10

0.75

0.95

1.00

10

−

−

10

−

30

15

5

SC-LDPC codes for two-way relay channels are studied. Coding achieves facts suggest that the spatial coupling coding successfully for spatially coupled codes needs a large number of iterations is not sufficient in general. It is known that a BP decoder our evaluation underestimates BP thresholds because

\[ T(3) \]

the same is true for \( T(3) \) while the BP threshold

\[ \text{Symmetric information rate} \]

\[ \text{Spatially coupled LDPC} \]

\[ \text{Symmetric information rate} \]

\[ \text{uncoupled LDPC} \]

sign rate. In population dynamics, we use \( N = 10^5 \) and \( T = 2000 \). The results indicate that the BP threshold is monotonously decreasing as \( L \) increases. The limiting value is estimated as 0.785 by extrapolation, which lies between the BP threshold 0.742 of the uncoupled (3,6)-LDPC codes and the correspondent symmetric information rate 0.805. The same is true for (3,9, \( L \))-LDPC codes: the spatially coupling coding achieves 0.647 (\( L \rightarrow \infty \)) while the BP threshold and the symmetric information rate of uncoupled codes are respectively given by 0.624 and 0.666. It is noteworthy that our evaluation underestimates BP thresholds because \( T = 2000 \) is not sufficient in general. It is known that a BP decoder for spatially coupled codes needs a large number of iterations before convergence especially around the threshold. These facts suggest that the spatial coupling coding successfully improves BP thresholds although whether it achieves the MAP threshold or not is still left to open.

IV. Summary

In this paper, asymptotic behavior of LDPC codes and SC-LDPC codes for two-way relay channels are studied.

Combining the population dynamics DE with DE formulas for asymmetric channels, BP thresholds of regular LDPC codes are evaluated. In addition, we provide the DE equations of \((d_1, d_2, L)\)-LDPC codes and performed the population dynamics DE. The results show that the spatial coupling coding successfully improves the BP thresholds of two-way relay channels.

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