A Nonstandard Generalization of Envelopes

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ABSTRACT

The generalized envelopes are studied by a given nonstandard definition of envelope of a family of lines defined in a projective homogenous coordinates PHC by: \( u(t)x + v(t)y + w(t)z = 0 \). The new nonstandard concepts of envelope are applied to conic sections. Our goal in this paper is hat for a given conic section curve \( f(x,y)=0 \), we search for the family of lines in which \( f \) is its envelope.

Keywords: infinitesimals, monad, envelope.
Every concept concerning sets or elements defined in the classical mathematics is called **standard** [7].

Any set or formula which does not involve new predicates “standard, infinitesimals, limited, unlimited…etc” is called **internal**, otherwise it is called **external**. [7]

A real number $x$ is called **unlimited** if and only if $|x| > r$ for all positive standard real numbers; otherwise it is called **limited** [4].

A real number $x$ is called **infinitesimal** if and only if $|x| < r$ for all positive standard real numbers $r$ [6], [4].

Two real numbers $x$ and $y$ are said to be **infinitely close** if and only if $x - y$ is infinitesimal and denoted by $x \equiv y$ [8].

If $x$ is a limited number in $\mathbb{R}$, then it is infinitely close to a unique standard real number, this unique number is called the **standard part** of $x$ or **shadow** of $x$ denoted by $st(x)$ or $^0x$ [6], [8].

**Theorem 1.1 : (Extension Principle)** [3]

Let $X$ and $Y$ be two standard sets, $X$ and $Y$ be the subsets constitute of the standard elements of $X$ and $Y$, respectively. If we can associate with every $x \in X$ a unique $y = f(x) \in Y$ then there exists a unique standard $y^* \in Y$ such that $\forall stx \in X, y^* = f(x)$.

Let $\alpha$ and $\beta$ be any two infinitesimal numbers and $r \neq 0$ is a limited real number, then:

1. $\alpha \cdot r$ is an infinitesimal.
2. $\alpha \cdot \beta$ is an infinitesimal.
3. $\alpha + r$ is limited.
4. $\alpha + \beta$ is an infinitesimal (in general the sum of any arbitrary finite number of infinitesimal numbers is infinitesimal) [6].

The **projective plane** over $\mathbb{R}$, denoted by $\mathbb{P}_R^2$ is the set

\[ \mathbb{P}_R^2 = \mathbb{R}^2 \cup \{ \text{one point at } \infty \text{ for each equivalence classes of parallel lines } \}, \]

we denoted it by (PHP) [2].

The **projective homogeneous coordinates** of a point $p(x, y) \in \mathbb{R}^2$ are $[x_{\alpha}, y_{\alpha}, \alpha]$, where $\alpha$ is any nonzero number, we denoted it by (PHC), in this sense the projective homogeneous coordinates of any point is not unique [2].

A curve $\nu$ is called **envelope** of a family of curves $\gamma_{\alpha}$ depending on a parameter $\alpha$, if at each of its points, it is tangent to at least one curve of the family, and if each of its segments is tangent to an infinite set of these curves[2].
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By a parameterized differentiable curve, we mean a differentiable map \( \gamma : \mathbb{I} \rightarrow \mathbb{R}^3 \) of an open interval \( \mathbb{I} = (a, b) \) of the real line \( \mathbb{R} \) in to \( \mathbb{R}^3 \) such that:

\[
\gamma(t) = (x(t), y(t), z(t)) = x(t)e_1 + y(t)e_2 + z(t)e_3,
\]

and \( x, y, \) and \( z \) are differentiable at \( t \); it is also called spherical curve[2].

2. An Envelope of a Family of Lines in a Plane

We consider \( \mathbb{R}^2 \) as a subset of \( \mathbb{P} \mathbb{H} \mathbb{C} \), let \( \{L_t\} \) be a family of lines in \( \mathbb{P} \mathbb{H} \mathbb{C} \) space defined by:

\[
u(t)X + v(t)Y + w(t)Z = 0,
\]

and suppose that the ordered pairs \((u(t), v(t)), (u(t), w(t)), (v(t), w(t))\) ,where \( u, v, w \) are standard functions defined on an interval sub set of \( \mathbb{R} \).

The purpose is to associate a standard curve which is coincident with the envelope to the family \( \{L_t\} \).

Suppose that \( t \) ranges over the interval \( E \subset \mathbb{R} \) so that for every \( t \in E \), there exists \( \alpha > 0 \) such that \( \forall s \in [t - \alpha, t + \alpha] \), \( L_t \neq L_s \)

This is equivalent to \( L_t \neq L_{t+\varepsilon} \ \forall t \in E \), where \( \varepsilon \) is an infinitesimal real number.

Also, at each standard \( t \), we can associate two lines \( L_t \) and \( L_{t+\varepsilon} \) such that \( L_t \neq L_{t+\varepsilon} \), where \( L_t \) and \( L_{t+\varepsilon} \) are taken in \( \mathbb{P} \mathbb{H} \mathbb{C} \).

Let \( \gamma(t) \) be the envelope curve of the family \( \{L_t\} \). By using the principle of extension we have:

There exists a unique standard application \( \alpha : E \rightarrow \mathbb{P} \mathbb{R}^2 \) such that \( \gamma(t) \equiv \alpha(t) \ \forall^d t \in E \).

Now, let the families \( \{L_t\} \) and \( \{L_{t+\varepsilon}\} \) be given as follows:

\[
\begin{align*}
L_t & : u(t)X + v(t)Y + w(t)Z = 0 \\
L_{t+\varepsilon} & : u(t+\varepsilon)X + v(t+\varepsilon)Y + w(t+\varepsilon)Z = 0.
\end{align*}
\]

Then the intersection point of \( \{L_t\} \) and \( \{L_{t+\varepsilon}\} \) in \( \mathbb{P} \mathbb{H} \mathbb{C} \) is given by:

\[
\begin{align*}
X & (t+\varepsilon) = v(t+\varepsilon)w(t) - v(t)w(t+\varepsilon) \\
Y & (t+\varepsilon) = w(t+\varepsilon)u(t) - w(t)u(t+\varepsilon) \\
Z & (t) = u(t+\varepsilon)v(t) - u(t)v(t+\varepsilon)
\end{align*}
\]

Suppose that the functions \( u, v, \) and \( w \) are differentiable functions each of order at least \( n \), then by expanding each of \( u(t+\varepsilon), v(t+\varepsilon), \) and \( w(t+\varepsilon) \) using Taylor development, we get

\[
\begin{align*}
X & (t+\varepsilon) = v(t+\varepsilon)w(t) - v(t)w(t+\varepsilon)
\end{align*}
\]
\[(v'(t)w(t) - w'(t)v(t)) + \cdots + (v^{(n)}(t)w(t) - w^{(n)}(t)v(t)) \frac{\varepsilon^n}{n!} + \delta_1 \varepsilon^n \]

\[Y \varepsilon (t) = w(t + \varepsilon)u(t) - w(t)u(t + \varepsilon) \]

\[= (w'(t)u(t) - u'(t)w(t)) + \cdots + (w^{(n)}(t)u(t) - u^{(n)}(t)w(t)) \frac{\varepsilon^n}{n!} + \delta_2 \varepsilon^n \quad \cdots (2.2)\]

\[Z \varepsilon (t) = u(t + \varepsilon)v(t) - u(t)v(t + \varepsilon) \]

\[= (u'(t)v(t) - v'(t)u(t)) + \cdots + (u^{(n)}(t)v(t) - v^{(n)}(t)u(t)) \frac{\varepsilon^n}{n!} + \delta_3 \varepsilon^n , \]

where \( \delta_1, \delta_2, \delta_3 \) are infinitesimals. In general, put

\[
\begin{align*}
p_n(t) &= v^{(n)}(t)w(t) - w^{(n)}(t)v(t) \\
r_n(t) &= w^{(n)}(t)u(t) - u^{(n)}(t)w(t) \\
q_n(t) &= u^{(n)}(t)v(t) - v^{(n)}(t)u(t)
\end{align*}
\]

\[\left\{ \begin{array}{l}
\vdots
\end{array} \right. \quad \cdots (2.3)\]

The following cases are related to the last assumption

**Case 1.** If \( q_1(t) \neq 0 \) and \( p_1(t) \) and \( r_1(t) \) are not both zero, then the PHC points of envelope curve \( \gamma(t), (p_1(t), r_1(t), q_1(t)), \) are independent on \( \varepsilon \), and the triple \((p_1(t), r_1(t), q_1(t))\) represents the classical definition of an envelope curve.

**Proof:**

Using (2.2), we get:

\[X \varepsilon (t) = (v'(t)w(t) - w'(t)v(t)) \varepsilon + \cdots + (v^{(n)}(t)w(t) - w^{(n)}(t)v(t)) \frac{\varepsilon^n}{n!} + \delta_1 \varepsilon^n \]

\[= \varepsilon (v'(t)w(t) - w'(t)v(t)) + \cdots + (v^{(n)}(t)w(t) - w^{(n)}(t)v(t)) \frac{\varepsilon^{n-1}}{n!} + \delta_1 \varepsilon^{n-1} \]

Taking the shadow of the of the last equation we obtain

\[oX \varepsilon (t) = \varepsilon (v'(t)w(t) - w'(t)v(t)) \]

In the same way, we have

\[oY \varepsilon (t) = \varepsilon (w'(t)u(t) - u'(t)w(t)) \]

and

\[oZ \varepsilon (t) = \varepsilon (u'(t)v(t) - v'(t)u(t)) \]

Therefore,

\[\begin{align*}
(X \varepsilon (t), Y \varepsilon (t), Z \varepsilon (t)) &> (0X \varepsilon (t), oY \varepsilon (t), oZ \varepsilon (t)) \\
&= (\varepsilon (v'(t)w(t) - w'(t)v(t)), \varepsilon (w'(t)u(t) - u'(t)w(t)), \varepsilon (u'(t)v(t) - v'(t)u(t)))
\end{align*}\]

Now, using the properties of the PHC, we deduce that any point of the form \((\lambda a, \lambda b, \lambda c)\) is equivalent with the point \((a, b, c)\) for any parameter \(\lambda\).
Therefore, the PHC of $\gamma(t)$ is $(X(\varepsilon(t)), Y(\varepsilon(t)), Z(\varepsilon(t)))$, and it is equal to
\[
(v'(t)w(t)-w'(t)v(t), w'(t)u(t) - u'(t)w(t), u'(t)v(t) - v'(t)u(t))
\]
\[
= (p_1(t), r_1(t), q_1(t))
\]
\[
\text{... (2.4)}
\]
That is, $(p_1(t), r_1(t), q_1(t))$ represents the classical definition of an envelope curve which does not depend on $\varepsilon$.
Moreover, the Cartesian coordinates of points of $\gamma$ are given by
\[
(x(t), y(t)) = \left[ \begin{array}{c}
X(\varepsilon(t)) \\
Y(\varepsilon(t)) \\
Z(\varepsilon(t))
\end{array} \right]
\]
\[
= \left[ \begin{array}{c}
v(t)u(t) - u(t)v(t) \\
u(t)v(t) - v(t)u(t) \\
u(t)v(t) - v(t)u(t)
\end{array} \right]
\]
\[
\text{... (2.5)}
\]
This is the classical form of the envelope curve of the family of straight lines.

**Case 2.** If $q_1(t) = 0$, $p_1(t)$, and $r_1(t)$ are not both zeroes, then the PHC points of the envelope curve $\gamma(t)$ are infinitely large, and the corresponding tangents $\{L_i\}$ are asymptotes of $\gamma(t)$.

**Case 3.** If $p_1(t) = r_1(t) = q_1(t) = 0$ and $p_2(t) \neq 0, r_2(t) \neq 0 = q_2(t) \neq 0$ then, the PHC points of the envelope curve $\gamma(t)$ are $(p_2(t), r_2(t), q_2(t))$, and $(p_1(t), r_1(t), q_1(t))$ is an inflection point of the envelope curve $\gamma(t)$.

**Case 4.** If $p_1(t) = r_1(t) = q_1(t) = 0$ for $1 \leq k < n$ (n standard) and $p_n(t), r_n(t), q_n(t)$ are not all zeroes, then the PHC points of $\gamma(t)$ are of the form $(p_n(t), r_n(t), q_n(t))$ which does not depend on $\varepsilon$. Thus, we get the generalized nonclassical form of the envelope curve $\gamma(t)$ as follows:
\[
(x(t), y(t)) = \left[ \begin{array}{c}
X(\varepsilon(t)) \\
Y(\varepsilon(t)) \\
Z(\varepsilon(t))
\end{array} \right]
\]
\[
= \left[ \begin{array}{c}
v(n)(t)u(t) - u(n)(t)v(t) \\
u(n)(t)v(t) - v(n)(t)u(t) \\
u(n)(t)v(t) - v(n)(t)u(t)
\end{array} \right]
\]
**Case 5.** If $p_k(t) = r_k(t) = q_k(t) = 0$ for any value of $k$, then we can not say anything about the generalization of the envelope curve. In the following sections, by $p(\hat{t}), r(\hat{t})$ and $q(\hat{t})$ we mean $p_1(t), r_1(t)$ and $q_1(t)$ respectively.
3. Applications to Conic Sections

We restrict our study on a family of straight lines only, other studies on envelops and singularity of envelops, for example can be found in [1]. Our goal is that for a given conic section curve $f(x,y)=0$, we search for a family of lines in which $f$ is its envelope.

**Lemma 3.1**

Consider a standard family of lines $\{L_t\}$ defined by

$$u(t)x + v(t)y + w(t)z = 0 \quad \ldots \ (3.1.1)$$

where $u$, $v$, and $w$ are standard real polynomials of at most second degree, then the equation of $\{L_t\}$ can be written as follows:

$$A(x,y)t^2 + B(x,y)t + C(x,y) = 0, \quad \ldots \ (3.1.2)$$

in which $A$, $B$, and $C$ are linear equations of the variables $X$ and $Y$ and $Z$ belonging to $PHP$. And conversely every equation of the form (3.1.2) represents a family of lines of the form (3.1.1)

**Proof:**

Obvious

**Theorem 3.2**

If $\{L_t\}$ is a family of lines defined by:

$$u(t)x + v(t)y + w(t)z = 0,$$

where $u$, $v$, and $w$ are standard real polynomials of at most second degree, then the envelope of $\{L_t\}$ is a cone of the form:

$$B^2(x,y) - 4A(x,y)C(x,y) = 0, \quad \ldots \ (3.2.1)$$

in which $A$, $B$, and $C$ are linear equations of the variables $x$ and $y$ belonging to $R[x,y]$.

Moreover Equation (3.2.1) represents a general form of a second degree equation of two variables $x$ and $y$ and conversely.

**Proof:**

Consider the families of lines $\{L_t\}$ and $\{L_{t+\varepsilon}\}$

By using Lemma 3.1, we get that:

$L_t : A(x,y)t^2 + B(x,y)t + C(x,y) = 0$

$L_{t+\varepsilon} : A(x,y)(t+\varepsilon)^2 + B(x,y)(t+\varepsilon) + C(x,y) = 0$

Then solving $L_t$ and $L_{t+\varepsilon}$ as an instantaneous system to omit $t^2$ we get:

$$2\varepsilon A(x,y)t + A(x,y)\varepsilon^2 + B(x,y)\varepsilon = 0.$$
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Therefore,
\[2A(x,y)t + A(x,y) + B(x,y) = 0 \quad \ldots (3.2.2)\]

Taking the shadow of (3.2.2), we get \( t = \frac{-B(x,y)}{2A(x,y)} \) and then putting it in \( L_t \) we obtain the required result.

For the second part, since \( A, B, \) and \( C \) are linear equations of the variables \( x \) and \( y \) belonging to \( \mathbb{R} [x,y] \), so Putting

\[
A(x,y) = a_1x + a_2y + a_3 \\
B(x,y) = b_1x + b_2y + b_3 \\
C(x,y) = c_1x + c_2y + c_3 .
\]
in Equation (3.2.1), we get the following equation

\[B^2(x,y) - 4A(x,y)C(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0,\]

where

\[
a = (b_1^2 - 4a_1c_1) \\
b = 2(b_1b_2 - 2(a_1c_2 + a_2c_1)) \\
c = (b_2^2 - 4a_2c_2) \\
d = 2(b_1b_3 - 2(a_1c_3 + a_3c_1)) \\
e = 2(b_2b_3 - 2(a_2c_3 + a_3c_2)) \\
f = (b_3^2 - 4a_3c_3)
\]

This is a general form of second degree equation in two variables \( x \) and \( y \).

Conversely, assuming that we have a second degree equation of two variables \( x \) and \( y \) such as:

\[ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad \ldots (3.2.3)\]

By a suitable changing of coordinate's axis; if \( b \neq 0 \) then a rotation of axis through the angle \( \alpha \) determined by the equation \( \cot 2\alpha = \frac{A-C}{B} \) will transform Equation (3.2.3) to the following equation

\[a^*x^2 + b^*y^2 + e^*x + d^*y + e^* = 0 \quad \ldots (3.2.4)\]

Completing the square for each uncompleted square related to the variables \( x \) and \( y \) in Equation (3.2.4) and simplifying the result, we get:
Now put

\[
A(x,y) = \begin{bmatrix}
1 & y + \frac{d^*}{2b} \\
2 & c^* b + d^* a - 4 a^* b^* e
\end{bmatrix}
\]

\[
B(x,y) = \begin{bmatrix}
x + \frac{c^*}{2a} \\
\frac{c^* b + d^* a - 4 a^* b^* e}{4 a^* b^*}
\end{bmatrix}
\]

\[
C(x,y) = \begin{bmatrix}
1 & y + \frac{d^*}{2b} \\
2 & c^* b + d^* a - 4 a^* b^* e
\end{bmatrix}
\]

If \(a^*, b^* = 0\) or \(\frac{c^* b + d^* a - 4 a^* b^* e}{4 a^* b^*} < 0\) then we obtain undefined or imaginary values which are unacceptable cases in real homogenous projective plane.

Thus we assume that \(a^*, b^* \neq 0\) and \(\frac{c^* b + d^* a - 4 a^* b^* e}{4 a^* b^*} \geq 0\).

Hence we get the required result.

**Remark 3.3**

If the last conditions of the previous theorem are not valid or \(a^* = b^*\) we deduce that Equation (3.2.4) represents a standard conic section which can be transferred to the form (3.2.1) as it is shown in the following table:
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| Conic Section | General Form | Standard Form | A(x,y) | B(x,y) | C(x,y) |
|---------------|--------------|---------------|--------|--------|--------|
| Circle        | $x^2 + y^2 + ax + by + c = 0$ | $r^2 = x^2 + y^2$ | $y$ | $x$ | $0$ |
| Parabola      | $y^2 + ax + by + c = 0$ | $y = 4ax$ | $a$ | $x$ | $y$ |
| Ellipse       | $ax^2 + by^2 + cx + dy + e = 0$ | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ | $\frac{y}{b}$ | $\frac{1}{2}x/a$ | $y/2b$ |
| Hyperbolic    | $ax^2 + by^2 + cx + dy + e = 0$ | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ | $1$ | $\frac{x}{2a}$ | $y/2b$ |

### Example 3.4

The circle $x^2 + y^2 = 1$ is an envelope curve of the family of lines $(1-t^2)x + (2t)y + (t^2 + 1) = 0$, such as shown in the Figure 3.1. By applying Theorem 3.2 to the equation of the given circle we get

$$y^2 - 4(1/2-x/2)(1/2+x/2) = 0$$

Therefore $$(1/2-x/2)y^2 + yt + 1/2 + x/2 = 0,$$

which is an equation of a family of lines.

---

\(a = -6.3\) (family of 99, step 0.12)

\(y = \frac{a}{2} = \frac{-6.3}{2} = 3.15\)

\(x^2 + y^2 = 1\)
Figure 3.1

Note that we can show that the given circle equation $x^2 + y^2 = 1$ is an envelope equation of the founded family classically or by nonstandard tools. In the following, we give a nonstandard method for such purpose.

\[
\begin{align*}
\mathcal{L}_t : (1-t^2)x + (2t)y + (t^2 + 1) &= 0 \\
\mathcal{L}_{t+\varepsilon} : (1 - (t+\varepsilon)^2)x + 2(t+\varepsilon)y + (t+\varepsilon)^2 + 1 &= 0
\end{align*}
\] …(3.4.1)

Solving equations $\mathcal{L}_t$ and $\mathcal{L}_{t+\varepsilon}$ instantaneously, we get:

\[
2\varepsilon tx + \varepsilon^2 x - 2\varepsilon y - 2\varepsilon t - \varepsilon^2 = 0.
\]

Therefore

\[
2tx + \varepsilon^2 x - 2y - 2t + \varepsilon = 0.
\]

Taking the shadow, we get

\[
2tx - 2y - 2t = 0 \quad \text{... (3.4.2)}
\]

Now, remove the variable $t$ form Equations (3.4.1) and (3.4.2), we get the required result.

**Example 3.5**

Consider the curve $x + y^2 - 1 = 0$

By applying Theorem 3.2 to the given equation, we get \( y^2 - 4(1/4)(1-x) = 0 \).

Now use Lemma 3.1 we get

\( 1/4t^2 + yt + 1 - x = 0 \)

which is a family of lines whose envelope is the given equation, such as shown in the Figure 3.2

Figure 3.2
Remark: The graphs in Figs 3.1 and 3.2 are plotted with specific softwares: Omnigraph V3.1b-2005, Function Grapher V2.8-2006.

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