Limit Theorems for Sums of $p$-Adic Random Variables

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Abstract

We study $p$-adic counterparts of stable distributions, that is limit distributions for sequences of normalized sums of independent identically distributed $p$-adic-valued random variables. In contrast to the classical case, non-degenerate limit distributions can be obtained only under certain assumptions on the asymptotic behaviour of the number of summands in the approximating sums. This asymptotics determines the “exponent of stability”.

1 Introduction

The studies in infinitesimal systems of probability measures on locally compact groups (see [He] and references therein) are concentrated mainly on the problem of convergence to the Gaussian distribution. Not much is known (see a review in [Kh]) about analogues of stable distributions and the systems converging to them. Of course this is connected with the fact that standard normalization procedures (see e.g. [F]) are not possible for general groups.

This paper is devoted to an important example of a group for which an analogue of the classical theory can be constructed though the results are quite different from the ones for the group $\mathbb{R}$. Namely, we shall consider the additive group of the field $\mathbb{Q}_p$ of $p$-adic numbers.

Note that there is no Gaussian measure on $\mathbb{Q}_p$ (in the sense of Parthasarathy) since $\mathbb{Q}_p$ is totally disconnected. On the other hand, the distributions $G_{a,\alpha}$ on $\mathbb{Q}_p$ having the functions $g_{a,\alpha}(t) = \exp(-a|t|_p^\alpha)$, $a > 0$, $\alpha > 0$ ($|\cdot|_p$ is the $p$-adic absolute value; see Sect. 2) as their Fourier transforms, were used recently by several authors ([B], [I], [Ha], [K1], [K2], [Va], [VVZ]) as $p$-adic counterparts of
stable distributions, being the basis of the $p$-adic stochastic analysis initiated in the above papers and related to $p$-adic models of mathematical physics; for other approaches to $p$-adic stochastic processes see [AK], [E1], [E2].

It is natural to try to obtain these and more general distributions “of stable type” as limits of certain normalized sums of independent identically distributed random variables with values in $Q_p$. Even the above model example shows differences between the $p$-adic and real cases. Since the $p$-adic absolute value can equal only an integer power of $p$, the classical definitions of a stable distribution (as well as the generalizations proposed in [T]) do not make sense for $Q_p$.

Let $X_1, \ldots, X_n, \ldots$ be a sequence of independent identically distributed $Q_p$-valued random variables, and $B_1, \ldots, B_n, \ldots$ a sequence of $p$-adic numbers. We consider the normalized sums

$$S_n = B_n^{-1} (X_1 + \cdots + X_{k(n)}) \quad n = 1, 2, \ldots$$

where \{\(k(n)\)\} is an increasing sequence of natural numbers. Let $F_n$ be a distribution of $S_n$; suppose that $F_n \to G$ in the weak sense. Our main aim is to describe some distributions $G$ (or their characteristic functions $g(t)$) which may appear this way. Thus we confine ourselves to the “strictly stable” case. A significance of centering is not clear for $p$-adic random variables (for which, by the way, an expectation is not defined).

If $|B_n|_p \leq \text{const}$ then the distribution $G$ is either degenerate (equal to the delta measure) or equal to a cutoff of the Haar measure of the additive group of $Q_p$. An interesting case is the one when $|B_n|_p \to \infty$, so that $G$ is infinitely divisible [PRV]. The answer depends on the behaviour of the sequence

$$\rho_n = \frac{k(n)}{k(n+1)} \quad n = 1, 2, \ldots$$

Passing to subsequences we may assume that $\rho_n \to \beta$, $0 \leq \beta \leq 1$.

If $\beta = 1$ (as in the ’classical” case $k(n) = n$) then, in sharp contrast to the case of real-valued random variables, $G$ is degenerate. Another extreme case is $\beta = 0$ when either $G$ is degenerate or $g$ has a compact support.

The $p$-adic counterparts of stable distributions emerge when $0 < \beta < 1$. These include $G_{a,\alpha}$ (for which $\beta = p^{-\alpha}$). We find a class of distributions (defined by a functional equation for their Lévy measures) which correspond to weak limits of sequences (1). Its subclass consisting of symmetric distributions coincides with the set of distributions corresponding to weak limits
of sequences (1) for which the random variables appearing in (1) have sym-
metric distributions. We have not found such a complete description for the
non-symmetric case. The difficulty here may be related to the fact of non-
uniqueness of the Lévy-Khinchin representation of an infinitely divisible dis-
tribution on \( Q_p \) (as on any Abelian group possessing compact subgroups; see
[PRV]).

Finally, we describe the domains of attraction for the above distributions
giving conditions for the weak convergence of the sequence (1).

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2 Preliminaries

In this section we give some basic information from \( p \)-adic analysis. See [VVZ]
for further details.

Let \( p \) be a prime number. The field of \( p \)-adic numbers is the completion
\( Q_p \) of the field of rational numbers, with respect to the absolute value \( |x|_p \)
defined by setting \( |0|_p = 0 \) and \( |x|_p = p^{-\nu} \) if \( x = p^{-\nu} m/n \) where \( \nu, m, n \in \mathbb{Z} \) and
\( m, n \) are prime to \( p \).

The absolute value \( |x|_p \), \( x \in Q_p \), has the following properties: \( |x|_p = 0 \) if
and only if \( x = 0 \); \( |xy|_p = |x|_p |y|_p \); \( |x + y|_p \leq \max(|x|_p, |y|_p) \). If \( |x|_p = p^N \)
then \( x \) admits the canonical representation

\[
x = p^{-N} \left( x_0 + x_1 p + x_2 p^2 + \cdots \right)
\]

where \( x_0, x_1, x_2, \ldots \in \{0, 1, \ldots, p - 1\} \), \( x_0 \neq 0 \). The series is convergent with
respect to the topology defined by the metric \( |x - y|_p \).

\( Q_p \) is a complete, separable, totally disconnected, locally compact metric
space. We shall denote by \( dx \) the Haar measure on the additive group of \( Q_p \)
normalized in such a way that

\[
\int_{|x|_p \leq 1} dx = 1.
\]

If \( a \in Q_p \), \( a \neq 0 \), then \( d(ax) = |a|_p dx \). The measure of a ball \( \{x \in Q_p : |x|_p \leq p^N\} \) equals \( p^N \). Note that a ball, as well as a sphere \( \{x \in Q_p : |x|_p = p^N\} \),
are open and simultaneously closed (compact) sets.
The canonical additive character of the field $Q_p$ is defined by the formula

$$\chi(x) = \exp(2\pi i \{x\}_p)$$

where $\{x\}_p$ is the fractional part of $x \in Q_p$; if $x$ has the representation (2) then

$$\{x\}_p = p^{-N} \left(x_0 + x_1p + \cdots + x_{N-1}p^{N-1}\right)$$

if $N > 0$, and $\{x\}_p = 0$ if $N \leq 0$.

The character $\chi$ is an example of a locally constant function on $Q_p$: $\chi(x+x') = \chi(x)$ for any $x \in Q_p$, if $|x'|_p \leq 1$. In general a function $f : Q_p \to \mathbb{C}$ is called locally constant if there exists such $n \in \mathbb{Z}$ that $f(x+x') = f(x)$ for any $x \in Q_p$, if $|x'|_p \leq p^n$.

The Fourier transform of a complex-valued function $\varphi \in L_1(Q_p)$ is defined by

$$\hat{\varphi}(\xi) = \int_{Q_p} \chi(\xi x)\varphi(x) \, dx , \quad \xi \in Q_p .$$

(3)

The inverse transform is

$$\varphi(x) = \int_{Q_p} \chi(-\xi x)\hat{\varphi}(\xi) \, d\xi , \quad x \in Q_p ,$$

(4)

if $\hat{\varphi} \in L_1(Q_p)$. In particular, the relations (3), (4) are valid for $\varphi \in D(Q_p)$ where $D(Q_p)$ is the space of locally constant functions with compact supports. In this case $\varphi \in D(Q_p)$ implies $\hat{\varphi} \in D(Q_p)$. Note that $D(Q_p)$ contains, in particular, indicator functions of all open compact subsets of $Q_p$.

Let $\mu$ be a probability measure on the Borel $\sigma$-algebra of $Q_p$. Its characteristic function is defined as usual:

$$\hat{\mu}(t) = \int_{Q_p} \chi(tx)\mu(dx) .$$

If $\mu$ is symmetric, that is $\mu(-M) = \mu(M)$ for any Borel set $M \subset Q_p$, then $\hat{\mu}$ is a real-valued function. If $\mu = \delta_\xi$ is a delta measure concentrated at $\xi \in Q_p$ then $\hat{\mu}(t) = \chi(\xi t)$.

**Lemma 1** If $|\hat{\mu}(t_0)| = 1$ for some $t_0 \in Q_p$, $t_0 \neq 0$, then $\hat{\mu}$ is a locally constant function. If $|\hat{\mu}(t)|$ takes only two values, 0 and 1, then there exists $\xi \in Q_p$, $N \in \mathbb{Z}$ such that

$$\hat{\mu}(t) = \chi(t\xi)\Omega_N(t)$$

(5)
where
\[ \Omega_N(t) = \begin{cases} 1, & \text{if } |t|_p \leq p^N \\ 0, & \text{if } |t|_p > p^N \end{cases} \]

In this case \( \mu(dx) = p^N \Omega_N(x - \xi)dx \). If \( |\hat{\mu}(t)| \equiv 1 \) then \( \hat{\mu}(t) = \chi(t\xi) \), \( \mu = \delta_\xi \), \( \xi \in Q_p \).

**Proof.** Let \( t_0 \in Q_p \) be such that \( t_0 \neq 0 \), \( |\hat{\mu}(t_0)| = 1 \). Denote by \( R_p \) the set of rational numbers of the form \( p^{-n}(a_0 + a_1p + \cdots + a_{n-1}p^{n-1}) \), \( n \geq 1 \); \( a_0, \ldots, a_{n-1} \in \{0, 1, \ldots, p-1\} \), \( a_0 \neq 0 \).

Suppose that \( \hat{\mu}(t_0) = e^{2\pi ir} \), \( 0 \leq r < 1 \). Then by the definition of \( \chi \)
\[
eq r = \int_{Q_p} \exp(2\pi i\{t_0x\}_p)\mu(dx)
\]
whence
\[
\int_{Q_p} (1 - \exp(2\pi i\{t_0x\}_p - r)))\mu(dx) = 0.
\]
In particular,
\[
\int_{Q_p} (1 - \cos 2\pi(\{t_0x\}_p - r))\mu(dx) = 0.
\]
This means that either \( r \in R_p \) or \( r = 0 \).

In both cases there exists \( \xi \in Q_p \) such that \( r = \{t_0\xi\}_p \), that is \( \hat{\mu}(t_0) = \chi(t_0\xi) \). As above, we obtain that
\[
\int_{Q_p} (1 - \cos 2\pi(\{t_0(x - \xi)\}_p))\mu(dx) = 0,
\]
so that \( \mu \) is concentrated on the set of those \( x \) for which \( \{t_0(x - \xi)\}_p = 0 \), that is on the set \( \{x \in Q_p : |x - \xi|_p \leq |t_0|_p^{-1}\} \).

Now
\[
\hat{\mu}(t) = \int_{|x - \xi|_p \leq |t_0|_p^{-1}} \chi(tx)\mu(dx) = \chi(t\xi) \int_{|x - \xi|_p \leq |t_0|_p^{-1}} \chi(t(x - \xi))\mu(dx)
\]
so that \( \hat{\mu} \) is locally constant and \( \hat{\mu}(t) = \chi(t\xi) \) if \( |t|_p \leq |t_0|_p \).

Suppose that \( |\hat{\mu}(t)| = 0 \) or \( 1 \). If the set \( \{t \in Q_p : |\hat{\mu}(t)| = 1\} \) is unbounded then it coincides with \( Q_p \), and in this case \( \hat{\mu}(t) = \chi(t\xi) \), \( \mu = \delta_\xi \). Otherwise we come to (5). The expression for \( \mu \) follows from well-known integration formulas (see [VVZ]). \( \Box \)
If $\mu$ is an infinitely divisible distribution then it follows from the general result of [PRV] that

$$
\hat{\mu}(t) = \chi(t\xi)\Omega_N(t) \exp \int_{Q_p\setminus\{0\}} (\chi(tx) - 1)\Phi(dx), \quad t \in Q_p,
$$

where $\xi \in Q_p$, $N \in \mathbb{Z} \cup \{\infty\}$ ($\Omega_{\infty}(t) \equiv 1$), $\Phi$ is a Borel measure on $Q_p \setminus \{0\}$ which is finite on the complement of any neighbourhood of zero. Formula (6) differs from a similar formula for $\mathbb{R}$ in two respects - possible presence of the factor $\Omega_N(t)$ (thus $\hat{\mu}$ may vanish on an open set), and non-uniqueness of the Lévy measure $\Phi$. However, $\Phi$ can be uniquely determined if the integral under the exp is given.

**Lemma 2** If

$$
\varphi(t) = \int_{Q_p\setminus\{0\}} (\chi(tx) - 1)\Phi(dx), \quad t \in Q_p,
$$

then for any open compact subset $M \subset Q_p \setminus \{0\}$

$$
\Phi(M) = \int_{Q_p} \varphi(y)m(y)\,dy
$$

where $m$ is an inverse Fourier transform of the indicator function $\omega_M$ of the set $M$.

**Proof.** We have

$$
\omega_M(x) = \int_{Q_p} \chi(xy)m(y)\,dy, \quad x \in Q_p,
$$

whence $m \in D(Q_p)$ and

$$
\int_{Q_p} m(y)\,dy = \omega_M(0) = 0.
$$

Using (7) and (9) we obtain that

$$
\int_{Q_p\setminus\{0\}} \omega_M(x)\Phi(dx) = \int_{Q_p\setminus\{0\}} \Phi(dx) \int_{Q_p} (\chi(xy) - 1)m(y)\,dy = \int_{Q_p} \varphi(y)m(y)\,dy
$$

which is equivalent to (8). Our use of the Fubini theorem was based on the fact that $\supp m \subset \{y \in Q_p : |y|_p \leq p\}$ for some $l \in \mathbb{Z}$, and $\chi(xy) - 1 = 0$ for $|y|_p \leq p^l$, $|x|_p \leq p^{-l}$ while the measure $\Phi$ is finite on the set $\{x \in Q_p : |x|_p > p^{-l}\}$. □
3 Limits of Normalized Sums

Let us consider the normalized sums (1) with $|B_n|_p \to \infty$ and $\rho_n \to \beta$, $0 \leq \beta \leq 1$. Let $\gamma_n = \frac{B_n}{B_{n+1}}$. Since $|B_n|_p \to \infty$, there exists a subsequence $\{\gamma_n\}$ for which $|\gamma_n|_p \leq p^{-1}$. We may assume (passing if necessary to a subsequence once more) that $\gamma_n \to \gamma_0$ in $Q_p$, $|\gamma_0|_p \leq p^{-1}$.

Suppose that the distributions $F_n$ of the normalized sums $S_n$ converge weakly, $F_n \to G$, and $g(t)$ is a characteristic function of $G$. Let $f(t)$ be the characteristic function of each of the (independent, identically distributed) random variables $X_n$. Then

$$
(f\left(\frac{t}{B_n}\right))^{k(n)} \to g(t) , \ n \to \infty ,
$$

uniformly on compact subsets of $Q_p$. The left-hand side of (10) will be denoted $f_n(t)$.

**Proposition 1**

(i) If $\beta \neq 0$ then uniformly on compact subsets of $Q_p$

$$
|f_n(\gamma nt)| \to |g(t)|^\beta .
$$

If $\beta = 0$, this relation holds for those $t$ where $g(t) \neq 0$.

(ii) The identity

$$
|g(\gamma_0 t)| = |g(t)|^\beta
$$

is valid for any $t \in Q_p$, if $\beta \neq 0$, and for any $t$ with $g(t) \neq 0$ if $\beta = 0$.

**Proof.** Let us consider a random variable $S'_n = B_{n+1}^{-1}(X_1 + \cdots + X_{k(n)})$. Its characteristic function equals

$$
(f\left(\frac{t}{B_{n+1}}\right))^{k(n)} = f_n(\gamma nt) .
$$

On the other hand,

$$
|f\left(\frac{t}{B_{n+1}}\right)|^{k(n)} = |f_{n+1}(t)|^{\rho_n} ,
$$

so that (10) and (13) imply (11).

Given $\varepsilon > 0$, we find, for any fixed $t \in Q_p$, such $n_0$ that

$$
|f_n(\gamma nt) - g(\gamma nt)| < \varepsilon \ \text{if} \ n \geq n_0
$$

(since the sequence $\{\gamma nt\}_{n \geq 0}$ is pre-compact). Thus $f_n(\gamma nt) \to g(\gamma_0 t)$ by continuity of $g$, and (12) follows from (11). $\square$
Corollary 1 If $\beta \neq 0$ then $g(t) \neq 0$ for any $t \in Q_p$.

Proof. Suppose that $g(t_0) = 0$ for some $t_0 \in Q_p$. By (12) we find that

$$|g(t_0)| = |g(\gamma_0 t_0)|^{1/\beta} = |g(\gamma_0^2 t_0)|^{1/\beta^2} = \ldots = |g(\gamma_0^n t_0)|^{1/\beta^n}.$$ 

Since $\gamma_0^\beta \to 0$ and $g(0) = 1$, we obtain that $g(\gamma_0^n t_0) \neq 0$ for a certain $n$, so that we come to a contradiction. $\square$

4 Distributions of Stable Type

In this section we consider certain distributions on $Q_p$, seen as counterparts of classical stable distributions.

Theorem 1 (i) Let $\Phi$ be a Borel measure on $Q_p \setminus \{0\}$ which is finite outside any neighbourhood of zero and satisfies the relation

$$\Phi(M) = \beta \Phi(\gamma_0 M),$$

with $0 < \beta < 1$, $\gamma_0 \in Q_p$, $0 \neq |\gamma_0|_p \leq p^{-1}$, for any compact open subset $M \subset Q_p \setminus \{0\}$. Then a function $g(t)$ of the form

$$g(t) = \exp \int_{Q_p \setminus \{0\}} (\chi(ty) - 1) \Phi(dy)$$

is a characteristic function of a distribution which is a weak limit of some sequence (1) with $\rho_n \to \beta$, $\gamma_n \to \gamma_0$.

(ii) If distributions $F_n$ for a sequence (1) with independent symmetric identically distributed random variables $X_n$, $|B_n|_p \to \infty$, $\gamma_n \to \gamma_0$, $0 \neq |\gamma_0|_p \leq p^{-1}$, $\rho_n \to \beta$, $0 < \beta < 1$, converge weakly to a distribution $G$, then its characteristic function is of the form (15) where the Lévy measure $\Phi$ is symmetric and satisfies (14).

Proof. (i) By [PRV], the function (15) is a characteristic function corresponding to a random variable $X$. Let $X_1, X_2, \ldots$ be independent copies of $X$. Set $B_n = \gamma_0^{-n}$, $k(n) = [\beta^{-n}]$ where $[\cdot]$ means the integer part. Then

$$f_n(t) = (g((\gamma_0^n t))^{[\beta^{-n}]},$$

$$= \exp \beta^n [\beta^{-n}] \int_{Q_p \setminus \{0\}} (\chi(ty) - 1) \Phi(dy) \to g(t), \quad n \to \infty,$$
uniformly on compact subsets, since \([\beta^{-n}] = \beta^{-n} + O(1), \ n \to \infty\).

(ii) Let us proceed from the relation (10). Since \(f\) is real-valued, continuous, and \(f(0) = 1\), we see that \(f_n(t) > 0\) for each fixed \(t\), if \(n\) is large enough. Hence, \(g(t) \geq 0\), and by Corollary 1, \(g(t) > 0\) for all \(t \in Q_p\). The sequence \(\{\log f_n(t)\}\) is bounded, uniformly with respect to \(t\) from any compact subset of \(Q_p\). Thus

\[
k(n) \left( (f_n(t))^{1/k(n)} - 1 \right) = k(n) \left( \exp \left( \frac{1}{k(n)} \log f_n(t) \right) - 1 \right)
= \log f_n(t) + O\left( \frac{1}{k(n)} \right), \ n \to \infty,
\]

so that

\[
k(n) \left( (f_n(t))^{1/k(n)} - 1 \right) \longrightarrow \log g(t), \ n \to \infty,
\]

uniformly on compact subsets of \(Q_p\).

Introducing the measures

\[
\Phi_n(dy) = k(n) F(d(B_n y))
\]

where \(F\) is the distribution of each random variable \(X_1, \ldots, X_n, \ldots\), we may rewrite (16) as

\[
\int_{Q_p \setminus \{0\}} (\chi(ty) - 1) \Phi_n(dy) \longrightarrow \log g(t).
\]

Note that for each \(t \neq 0\) the integral in the left-hand side of (18) is actually taken over the set of those \(y\) for which \(|y|_p > |t|^{-1}_p\).

Denote the left-hand side of (18) by \(\varphi_n(t)\). If \(M\) is a compact open subset of \(Q_p \setminus \{0\}\), and \(m\) is an inverse Fourier transform of the indicator \(\omega_M\), then by Lemma 2

\[
\Phi_n(M) = \int_{Q_p} \varphi_n(t) m(t) \, dt,
\]

and by virtue of (18)

\[
\int_{Q_p \setminus \{0\}} \omega_M(x) \Phi_n(dx) \longrightarrow \int_{Q_p} m(t) \log g(t) \, dt, \ n \to \infty.
\]

It follows from (20) that the sequence

\[
\int_{Q_p \setminus \{0\}} \omega(x) \Phi_n(dx)
\]
converges for any locally constant function $\omega$ with a compact support not containing the origin. Every continuous function with a compact support on $\mathbb{Q}_p \setminus \{0\}$ can be approximated uniformly by such functions (see [VVZ]).

By (19), the sequence of measures $\{\Phi_n\}$ is bounded on compact subsets of $\mathbb{Q}_p \setminus \{0\}$. This means that $\{\Phi_n\}$ is a Cauchy sequence with respect to the vague topology [He], which is sequentially complete. Thus $\Phi_n$ is vaguely convergent to a symmetric Radon measure $\Phi$ on $\mathbb{Q}_p \setminus \{0\}$.

Now, in order to prove the representation (15), it is sufficient to show that $\Phi_n \to \Phi$ in the weak sense on each set $M_{i,\infty} = \{x \in \mathbb{Q}_p : |x|_p > p^i\}$, $i \in \mathbb{Z}$. By Theorem 1.1.9 of [He], that will be proved if we show that

$$\Phi_n(M_{i,\infty}) \longrightarrow \Phi(M_{i,\infty}) < \infty, \quad n \to \infty. \quad (21)$$

Simultaneously (21) would imply the required finiteness of $\Phi$ outside any neighbourhood of the origin.

Consider the set

$$M_{l,i} = \{x \in \mathbb{Q}_p : p^{i+1} \leq |x|_p \leq p^l\}, \quad l > i.$$

Let us compute $\Phi_n(M_{i,l})$ using (19) where $m(t)$ corresponds to the set $M_{l,i}$. This set is a set-theoretic difference of two balls. The Fourier transform of the indicator function of a ball is computed in [VVZ]. Thus $m(t) = m_l(t) - m_i(t)$ where

$$m_j(t) = \begin{cases} 1, & \text{if } |t|_p \leq p^{-j} \\ 0, & \text{if } |t|_p > p^{-j} \end{cases}$$

and we obtain that

$$\Phi_n(M_{i,l}) = - \int_{p^{-l+1} \leq |t|_p \leq p^{-i}} \varphi_n(t) \, dt \quad (22)$$

whence

$$\Phi_n(M_{i,\infty}) = - \int_{|t|_p \leq p^{-i}} \varphi_n(t) \, dt \quad (23)$$

It follows from (18) and (22) that

$$\Phi(M_{i,l}) = - \int_{p^{-l+1} \leq |t|_p \leq p^{-i}} \log g(t) \, dt .$$

This yields

$$\Phi(M_{i,\infty}) = - \int_{|t|_p \leq p^{-i}} \log g(t) \, dt .$$
Comparing with (23) we come to (21).

It remains to prove the relation (14). As we have seen, \( \varphi_n(t) \sim \log f_n(t) \), \( n \to \infty \), uniformly on compact subsets of \( Q_p \). Thus by Proposition 1,

\[
\varphi_n(\gamma_n t) \longrightarrow \beta \log g(t) , \quad n \to \infty, \tag{24}
\]

uniformly on compact subsets.

Let \( M \) be a compact open subset of \( Q_p \setminus \{0\} \). Then

\[
\Phi(\gamma_0^{-1}M) = \lim_{n \to \infty} \Phi_n(\gamma_n^{-1}M). \tag{25}
\]

Indeed, let \( \omega_n \) be an indicator of the set \( \gamma_n^{-1}M \), \( n = 0, 1, 2, \ldots \). Writing the action of a measure as a functional we get

\[
\langle \Phi_n, \omega_n \rangle - \langle \Phi, \omega_0 \rangle = \langle \Phi_n - \Phi, \omega_n \rangle + \langle \Phi, \omega_n - \omega_0 \rangle. \tag{26}
\]

For large \( n \) \( \gamma_n^{-1}M \subset M_{l', l''} \) where \( l', l'' \) are certain fixed numbers. As above, this means that the supports of the inverse Fourier transforms \( \tilde{\omega}_n \) of all \( \omega_n \) lie in a certain compact set \( N \), so that

\[
|\langle \Phi_n - \Phi, \omega_n \rangle| \leq \int_N |\varphi_n(t) - \log g(t)| \, dt \longrightarrow 0
\]
due to the uniform convergence. The second summand on the right in (26) tends to zero due to the dominated convergence theorem, so (25) has been proved.

Next, by (19)

\[
\Phi_n(\gamma_n^{-1}M) = \int_{Q_p} \varphi_n(t) \tilde{\omega}_n(t) \, dt ,
\]

\[
\tilde{\omega}_n(t) = \int_{\gamma_n^{-1}M} \chi(-ty) \, dy = |\gamma_n|^{-1} m(\gamma_n^{-1}t) ,
\]

whence

\[
\Phi_n(\gamma_n^{-1}M) = \int_{Q_p} \varphi_n(\gamma_n t) m(t) \, dt .
\]

Now it follows from (24), (25) and (20) that

\[
\Phi(\gamma_0^{-1}M) = \beta \int_{Q_p} m(t) \log g(t) \, dt = \beta \Phi(M)
\]

which is equivalent to (14). \( \square \)
Example. If $|\gamma_0|_p = p^{-1}$ then the relation (14) means that $\Phi$ is determined by its restriction to the group of units $U_p = \{x \in Q_p : |x|_p = 1\}$.

For a particular example, let the above restriction be proportional to the restriction of the Haar measure:

$$\Phi(M_0) = a \frac{p^\alpha - 1}{1 - p^{-\alpha - 1}} \int_{M_0} dx, \quad \alpha > 0, \ a > 0,$$

for any open and closed subset $M_0 \subset U_p$. Suppose that $\beta = p^{-\alpha}$, $k(n) = [p^{\alpha n}]$, $\gamma_0 = p$.

If $M$ is a compact open subset of $Q_p \setminus \{0\}$ then it may be written as a finite union

$$M = \bigcup_{N} M \cap S_N, \quad S_N = \{x \in Q_p : |x|_p = p^N\}.$$

In accordance with (14),

$$\Phi(M) = \sum_{N} \Phi(M \cap S_N) = \sum_{N} p^{-\alpha N} \Phi(p^N M \cap S_0)$$

$$= \sum_{N} p^{-\alpha N} \int_{p^N M \cap S_0} dx = \sum_{N} p^{-(\alpha + 1) N} \int_{M \cap S_N} dy,$$

so that

$$\Phi(M) = a \frac{p^\alpha - 1}{1 - p^{-\alpha - 1}} \int_M |x|_p^{-\alpha - 1} dx.$$

Now the identity

$$|t|_p^\alpha = \frac{1 - p^\alpha}{1 - p^{-\alpha - 1}} \int_{Q_p} |s|_p^{-\alpha - 1} (\chi(st) - 1) ds$$

(see [VVZ]) shows that in this case the limit characteristic function $g(t)$ coincides with the function $g_{a,\alpha}(t)$ mentioned in the introduction.

Let us consider conditions for the weak convergence of the sequence (1) with $|B_n|_p \to \infty$.

**Theorem 2** In order that the sequence (1) be weakly convergent, it is sufficient that the measures (17) converge weakly on each set $M_i, \infty$, $i \in Z$, to a measure $\Phi$ on $Q_p \setminus \{0\}$, finite outside any neighbourhood of zero. If the random variables $X_1, X_2, \ldots$ are symmetric and $\beta \neq 0$, this condition is also necessary.
Proof. The necessity was proved in the course of proving Theorem 1. To prove the sufficiency, write \( f_n(t) \) in the form

\[
f_n(t) = \left( 1 + \frac{1}{k(n)} \int_{Q_p \setminus \{0\}} (\chi(ty) - 1)\Phi_n(dy) \right)^{k(n)}.
\]

For every fixed \( t \) we have

\[
\int_{Q_p \setminus \{0\}} (\chi(ty) - 1)\Phi_n(dy) \longrightarrow \int_{Q_p \setminus \{0\}} (\chi(ty) - 1)\Phi(dy)
\]

since \( \chi(ty) - 1 = 0 \) when \( |y|_p \leq |t|_p^{-1} \). Recalling that \( \left( 1 + \frac{z}{k(n)} \right)^{k(n)} \rightarrow e^z \) uniformly on compact sets, we find that

\[
f_n(t) \longrightarrow \exp \left( \int_{Q_p \setminus \{0\}} (\chi(ty) - 1)\Phi(dy) \right), \quad n \rightarrow \infty.
\]

Since the function in the right-hand side of (27) is continuous, Theorem 3.3.1 of \([G]\) implies weak convergence of \( S_n \). \( \square \)

Remark. Theorems 1 and 2 can be extended easily to the case of general non-Archimedean local fields.

5 Degenerate Cases

Let us consider the “extreme” cases \( \beta = 0 \) and \( \beta = 1 \) in the weakly convergent scheme (1).

Proposition 2 If \( \beta = 1 \) then \( G \) is degenerate. If \( \beta = 0 \) then either \( G \) is degenerate or its characteristic function \( g \) has a compact support. If \( \beta = 0 \) and \( |\gamma_0|_p = p^{-1} \) then \( g(t) \) coincides with the right-hand side of (5).

Proof. Let \( \beta = 1 \). As before, we may assume that \( \gamma_n \rightarrow \gamma_0, |\gamma_0|_p \leq p^{-1} \).

By Proposition 1, we have

\[
|g(\gamma_0 t)| = |g(t)|
\]

for any \( t \in Q_p \), so that

\[
|g(t)| = |g(\gamma_0^n t)|, \quad n = 1, 2, \ldots.
\]
Since \( \gamma_0^n \to 0 \) for \( n \to \infty \), \( g \) is continuous and \( g(0) = 1 \), this implies the identity \( |g(t)| \equiv 1 \). Hence, by Lemma 1 \( G \) is degenerate.

If \( \beta = 0 \) then the same reasoning shows that \( |g(\gamma_0 t)| = 1 \) as soon as \( g(t) \neq 0 \). Thus if \( g(t) \neq 0 \) for all \( t \) then \( G \) is degenerate. Otherwise \( g \) has a compact support in accordance with (6). The last assertion of the proposition follows from Lemma 1. \( \square \)

Finally, let us consider the case of a weakly convergent sequence (1) with \( |B_n|_p \nrightarrow \infty \). Then there exists a subsequence \( B_{n_l} \to b \in Q_p \).

**Proposition 3** If \( b \neq 0 \) then \( g \) coincides with the right-hand side of (5). If \( b = 0 \) then \( G \) is degenerate.

**Proof.** Let \( b \neq 0 \). It follows from (10) that the inequality \( |f(b^{-1}t)| < 1 \) implies the equality \( g(t) = 0 \). If \( |f(b^{-1}t)| = 1 \) for some \( t \) then Lemma 1 shows that \( f \) is locally constant. Thus \( f(s) = 1 \) when \( s \) belongs to a certain neighbourhood of the point \( b^{-1}t \); in particular, \( |f(B_{n_l}^{-1}t)| = 1 \) for large \( l \) whence \( |g(t)| = 1 \). It remains to use Lemma 1.

Let \( b = 0 \). Take, for an arbitrary \( s \in Q_p \), a compact set \( C \) containing the subsequence \( \{B_{n_l}s\} \). Given \( \varepsilon > 0 \), we find such a natural number \( l_0 \) that for \( l \geq l_0 \)

\[
| f \left( \frac{t}{B_{n_l}} \right)^{n_l} - |g(t)| < \varepsilon , \quad t \in C.
\]

In particular, for \( t = B_{n_l}s \) we obtain that

\[
| |f(s)|^{n_l} - |g(B_{n_l}s)| | < \varepsilon , \quad l \geq l_0 .
\]

Since \( g(B_{n_l}s) \to 1 \), we see that \( |f(s)|^{n_l} \to 1 \) for \( l \to \infty \), whence \( |f(s)| \equiv 1 \). By (10) \( G \) is degenerate. \( \square \)

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