On sequences with prescribed metric discrepancy behavior

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Abstract An important result of H. Weyl states that for every sequence \((a_n)_{n \geq 1}\) of distinct positive integers the sequence of fractional parts of \((a_n\alpha)_{n \geq 1}\) is uniformly distributed modulo one for almost all \(\alpha\). However, in general it is a very hard problem to calculate the precise order of convergence of the discrepancy \(D_N\) of \((\{a_n\alpha\})_{n \geq 1}\) for almost all \(\alpha\). By a result of R. C. Baker this discrepancy always satisfies \(ND_N = O(N^{\frac{1}{2}+\varepsilon})\) for almost all \(\alpha\) and all \(\varepsilon > 0\). In the present note for arbitrary \(\gamma \in (0, \frac{1}{2}]\) we construct a sequence \((a_n)_{n \geq 1}\) such that for almost all \(\alpha\) we have \(ND_N = O(N^\gamma)\) and \(ND_N = \Omega(N^{\gamma-\varepsilon})\) for all \(\varepsilon > 0\), thereby proving that any prescribed metric discrepancy behavior within the admissible range can actually be realized.

Keywords Discrepancy theory · Metric number theory

Mathematics Subject Classification 11K38 · 11J83

1 Introduction

Weyl [12] proved that for every sequence \((a_n)_{n \geq 1}\) of distinct positive integers the sequence \((\{a_n\alpha\})_{n \geq 1}\) is uniformly distributed modulo one for almost all reals \(\alpha\). Here, and in the sequel, \(\{\cdot\}\) denotes the fractional part function. The speed of convergence

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towards the uniform distribution is measured in terms of the discrepancy, which—for an arbitrary sequence \((x_n)_{n \geq 1}\) of points in \([0,1)\)—is defined by

\[
D_N = D_N(x_1, \ldots, x_N) = \sup_{0 \leq a < b \leq 1} \left| \frac{A_N([a,b))}{N} - (b-a) \right|
\]

where \(A_N([a,b)) := \# \{1 \leq n \leq N \mid x_n \in [a,b)\}\). For a given sequence \((a_n)_{n \geq 1}\) it is usually a very hard and challenging problem to give sharp estimates for the discrepancy \(D_N\) of \((\{a_n\alpha\})_{n \geq 1}\) valid for almost all \(\alpha\). For general background on uniform distribution theory and discrepancy theory see for example the monographs [6,9].

A famous result of Baker [3] states that for any sequence \((a_n)_{n \geq 1}\) of distinct positive integers for the discrepancy \(D_N\) of \((\{a_n\alpha\})_{n \geq 1}\) we have

\[
ND_N = O(N^{1/2} (\log N)^{3/2+\varepsilon}) \quad \text{as} \quad N \to \infty
\]

for almost all \(\alpha\) and for all \(\varepsilon > 0\).

Note that (1) is a general upper bound which holds for all sequences \((a_n)_{n \geq 1}\); however, for some specific sequences the precise typical order of decay of the discrepancy of \((\{a_n\alpha\})_{n \geq 1}\) can differ significantly from the upper bound in (1). The fact that (1) is essentially optimal (apart from logarithmic factors) as a general result covering all possible sequences can for example be seen by considering so-called lacunary sequences \((a_n)_{n \geq 1}\), i.e., sequences for which \(\frac{a_{n+1}}{a_n} \geq 1+\delta\) for a fixed \(\delta > 0\) and all \(n\) large enough. In this case for \(D_N\) we have

\[
\frac{1}{4\sqrt{2}} \leq \limsup_{N \to \infty} \frac{ND_N}{\sqrt{2N} \log \log N} \leq c_\delta
\]

for almost all \(\alpha\) (see [10]), which shows that the exponent 1/2 of \(N\) on the right-hand side of (1) cannot be reduced for this type of sequence. For more information concerning possible improvements of the logarithmic factor in (1), see [5].

Quite recently in [2] it was shown that also for a large class of sequences with polynomial growth behavior Baker’s result is essentially best possible. For example, the following result was shown there: let \(f \in \mathbb{Z}[x]\) be a polynomial of degree larger or equal to 2. Then for the discrepancy \(D_N\) of \((\{f(n)\alpha\})_{n \geq 1}\) for almost all \(\alpha\) and for all \(\varepsilon > 0\) we have

\[
ND_N = \Omega(N^{1/2-\varepsilon}).
\]

On the other hand there is the classical example of the Kronecker sequence, i.e., \(a_n = n\), which shows that the actual metric discrepancy behavior of \((\{a_n\alpha\})_{n \geq 1}\) can differ vastly from the general upper bound in (1). Namely, for the discrepancy of the sequence \((\{n\alpha\})_{n \geq 1}\) for almost all \(\alpha\) and for all \(\varepsilon > 0\) we have

\[
ND_N = O(\log N (\log \log N)^{1+\varepsilon}),
\]

for almost all \(\alpha\) in the sense of (2).
which follows from classical results of Khintchine in the metric theory of continued fractions (for even more precise results, see [11]). The estimate (2) of course also holds for \( a_n = f(n) \) with \( f \in \mathbb{Z}[x] \) of degree 1. In [2] further examples for \( (a_n)_{n \geq 1} \) were given, where \( (a_n)_{n \geq 1} \) has polynomial growth behavior of arbitrary degree, such that for the discrepancy of \( (\{ a_n \alpha \})_{n \geq 1} \) we have

\[
ND_N = \mathcal{O}((\log N)^{2+\varepsilon})
\]

for almost all \( \alpha \) and for all \( \varepsilon > 0 \); see there for more details.

These results may seduce to the hypothesis that for all choices of \( (a_n)_{n \geq 1} \) for the discrepancy of \( (\{ a_n \alpha \})_{n \geq 1} \) for almost all \( \alpha \) we either have

\[
ND_N = \mathcal{O}(N^{\varepsilon})
\]  

or

\[
ND_N = \Omega(N^{\frac{1}{2}-\varepsilon}).
\]

This hypothesis, however, is wrong as was shown in [1]: let \( (a_n)_{n \geq 1} \) be the sequence of those positive integers with an even sum of digits in base 2, sorted in increasing order; that is \( (a_n)_{n \geq 1} = (3, 5, 6, 9, 10, \ldots) \). Then for the discrepancy of \( (\{ a_n \alpha \})_{n \geq 1} \) for almost all \( \alpha \) we have

\[
ND_N = \mathcal{O}(N^{\kappa+\varepsilon})
\]

and

\[
ND_N = \Omega(N^{\kappa-\varepsilon})
\]

for all \( \varepsilon > 0 \), where \( \kappa \) is a constant with \( \kappa \approx 0.404 \). Interestingly, the precise value of \( \kappa \) is unknown; see [8] for the background.

The aim of the present paper is to show that the example above is not a singular counter-example, but that indeed “everything” between (3) and (4) is possible. More precisely, we will show the following theorem.

**Theorem 1** Let \( 0 < \gamma \leq \frac{1}{2} \). Then there exists a strictly increasing sequence \( (a_n)_{n \geq 1} \) of positive integers such that for the discrepancy of the sequence \( (\{ a_n \alpha \})_{n \geq 1} \) for almost all \( \alpha \) we have

\[
ND_N = \mathcal{O}(N^\gamma)
\]

and

\[
ND_N = \Omega(N^{\gamma-\varepsilon})
\]

for all \( \varepsilon > 0 \).
2 Proof of the theorem

For the proof we need an auxiliary result which easily follows from classical work of Behnke [4].

**Lemma 1** Let \((e_k)_{k \geq 1}\) be a strictly increasing sequence of positive integers. Let \(\varepsilon > 0\). Then for almost all \(\alpha\) there is a constant \(K(\alpha, \varepsilon) > 0\) such that for all \(r \in \mathbb{N}\) there exist \(M_r \leq e_r\) such that for the discrepancy of the sequence \((\{n^2 \alpha\})_{n \geq 1}\) we have

\[
M_r D_M \geq K(\alpha, \varepsilon) \sqrt{\frac{e_r}{(\log e_r)^{1+\varepsilon}}}.
\]

**Proof** For \(\alpha \in \mathbb{R}\) let \(a_k(\alpha)\) denote the \(k\)th continued fraction coefficient in the continued fraction expansion of \(\alpha\). Then it is well-known that for almost all \(\alpha\) we have \(a_k(\alpha) = \mathcal{O}(k^{1+\varepsilon})\) for all \(\varepsilon > 0\). Let \(\varepsilon > 0\) be given and let \(\alpha\) and \(c(\alpha, \varepsilon)\) be such that

\[
a_k(\alpha) \leq c(\alpha, \varepsilon) k^{1+\varepsilon}
\]

for all \(k \geq 1\).

Let \(q_l\) the \(l\)th best approximation denominator of \(\alpha\). Then

\[
q_{l+1} \leq (c(\alpha, \varepsilon) l^{1+\varepsilon} + 1)q_l.
\]

Since \(q_l \geq 2^l\) in any case, we have \(l \leq \frac{2 \log q_l}{\log 2}\), and we obtain

\[
q_{l+1} \leq c_1(\alpha, \varepsilon) q_l (\log q_l)^{1+\varepsilon},
\]

for an appropriate constant \(c_1(\alpha, \varepsilon)\). In [4] it was shown in Satz XVII that for every real \(\alpha\) we have

\[
\left| \sum_{n=1}^{N} e^{2\pi i n^2 \alpha} \right| = \Omega(N^{\frac{1}{2}}).
\]

Indeed, if we follow the proof of this theorem we find that even the following was shown: for every \(\alpha\) and for every best approximation denominator \(q_l\) of \(\alpha\) there exists an \(Y_l < \sqrt{q_l}\) such that \(\left| \sum_{n=1}^{Y_l} e^{2\pi i n^2 \alpha} \right| \geq c_{abs} \sqrt{q_l}\). Here \(c_{abs}\) is a positive absolute constant (not depending on \(\alpha\)).
Let now \( r \in \mathbb{N} \) be given and let \( l \) be such that \( q_l \leq e_r < q_{l+1} \), and let \( M_r := Y_l \) from above. Then by (6) and (7) we obtain, for an appropriate constant \( c_2(\alpha, \varepsilon) \),

\[
\left| \sum_{n=1}^{M_r} e^{2\pi i n^2 \alpha} \right| \geq c_{\text{abs}} \sqrt{q_l} \\
\geq c_2(\alpha, \varepsilon) \sqrt{q_l} \\
\geq c_2(\alpha, \varepsilon) \sqrt{e_l}.
\]

By the fact that (see Chapter 2, Corollary 5.1 of [9])

\[
M_r D_{M_r} \geq \frac{1}{4} \left| \sum_{n=1}^{M_r} e^{2\pi i n^2 \alpha} \right|,
\]

which is a special case of Koksma’s inequality, the result follows. \( \square \)

Now we are ready to prove the main theorem.

**Proof of Theorem 1** Let \((m_j)_{j \geq 1}\) and \((e_j)_{j \geq 1}\) be two strictly increasing sequences of positive integers, which will be determined later. We will consider the following strictly increasing sequence of positive integers, which will be our sequence \((a_n)_{n \geq 1}:\)

\[
1, 2, 3, \ldots, \underbrace{m_1},
\]

\[
A_1 + 1^2, A_1 + 2^2, A_1 + 3^2, A_1 + 4^2, \ldots, A_1 + \underbrace{e_1^2}, : = B_1
\]

\[
B_1 + 1, B_1 + 2, B_1 + 3, \ldots, \underbrace{B_1 + m_2},
\]

\[
A_2 + 1^2, A_2 + 2^2, A_2 + 3^2, A_2 + 4^2, \ldots, A_2 + \underbrace{e_2^2}, : = B_2
\]

\[
B_2 + 1, B_2 + 2, B_2 + 3, \ldots, \underbrace{B_2 + m_3},
\]

\[
A_3 + 1^2, A_3 + 2^2, A_3 + 3^2, A_3 + 4^2, \ldots, A_3 + \underbrace{e_3^2}, : = B_3
\]

\[
: : : 
\]
Furthermore, let

\[ F_s := \sum_{i=1}^{s} m_i + \sum_{i=1}^{s-1} e_i \quad \text{and} \quad E_s := \sum_{i=1}^{s} m_i + \sum_{i=1}^{s} e_i. \]

The sequence \((a_n)_{n \geq 1}\) is constructed in such a way that it contains sections where it grows like \((n)_{n \geq 1}\) as well as sections where it grows like \((n^2)_{n \geq 1}\). By this construction we exploit both the strong upper bounds for the discrepancy of \((\lfloor n\alpha \rfloor)_{n \geq 1}\) and the strong lower bounds for the discrepancy of \((\lfloor n^2\alpha \rfloor)_{n \geq 1}\), in an appropriately balanced way, in order to obtain the desired discrepancy behavior of the sequence \((a_n\alpha)_{n \geq 1}\).

In our argument we will repeatedly make use of the fact that

\[ D_N(x_1, \ldots, x_N) = D_N([x_1 + \beta], \ldots, [x_N + \beta]) \]

for arbitrary \(x_1, \ldots, x_N \in [0, 1]\) and \(\beta \in \mathbb{R}\), which allows us to transfer the discrepancy bounds for \((\lfloor n\alpha \rfloor)_{n \geq 1}\) and \((\lfloor n^2\alpha \rfloor)_{n \geq 1}\) directly to the shifted sequences \((\lfloor (M+n)\alpha \rfloor)_{n \geq 1}\) and \((\lfloor (M+n^2)\alpha \rfloor)_{n \geq 1}\) for some integer \(M\).

Let \(\alpha\) be such that it satisfies (5) with \(\varepsilon = \frac{1}{2}\). Then it is also well-known (see for example [9]) that for the discrepancy \(D_N\) of the sequence \((\lfloor n\alpha \rfloor)_{n \geq 1}\) we have

\[ ND_N \leq c_1(\alpha) (\log N)^{\frac{3}{2}} \]

for all \(N \geq 2\).

By the above mentioned general result of Baker, that is by (1), we know that for almost all \(\alpha\) for the discrepancy \(D_N\) of the sequence \((\lfloor n^2\alpha \rfloor)_{n \geq 1}\) we have

\[ ND_N \leq c_3(\alpha, \varepsilon) N^{\frac{1}{2}} (\log N)^{\frac{3}{2} + \varepsilon} \]

for all \(\varepsilon > 0\) and for all \(N \geq 2\), for an appropriate constant \(c_3(\alpha, \varepsilon)\). Actually an even slightly sharper estimate was given for the special case of the sequence \((\lfloor n^2\alpha \rfloor)_{n \geq 1}\) by Fiedler et al. [7], who proved that

\[ ND_N \leq c_4(\alpha, \varepsilon) N^{\frac{1}{4}} (\log N)^{\frac{1}{4} + \varepsilon} \]

for almost all \(\alpha\) and for all \(\varepsilon > 0\) and all \(N \geq 2\).

Assume that \(\alpha\) satisfies (10) with \(\varepsilon = \frac{1}{8}\). Then

\[ ND_N \leq c_2(\alpha) N^{\frac{1}{8}} (\log N)^{\frac{3}{8}} \]

for all \(N \geq 2\). Now for such \(\alpha\) and for arbitrary \(N\) we consider the discrepancy \(D_N\) of the sequence \((\lfloor a_n\alpha \rfloor)_{n \geq 1}\).

**Case 1** Let \(N = F_l\) for some \(l\). Then \(ND_N \leq E_{l-1} D_{E_{l-1}} + (N - E_{l-1}) D_{E_{l-1}, F_l}\), where \(D_{x,y}\) denotes the discrepancy of the point set \((\lfloor a_n\alpha \rfloor)_{n=x+1, x+2, \ldots, y}\). Hence by (8), (9) and by the trivial estimate \(D_{B_{l-1}} \leq 1\) we have
\[ ND_N \leq E_{l-1} + c_1(\alpha) (\log m_l)^{\frac{3}{2}} \]
\[
\leq 2 (\log m_l)^2
\leq 2 (\log N)^2
\]

for all \( l \) large enough, provided that [condition (i)] \( m_l \) is chosen such that \((\log m_l)^2 \geq E_{l-1}\).

**Case 2** Let \( F_l < N \leq E_l \) for some \( l \). Then by Case 1 and by (8) and (11) we have for \( l \) large enough that
\[
ND_N \leq F_l D_{F_l} + (N - F_l) D_{F_l,N}
\leq 2 (\log F_l)^2 + c_2(\alpha) (N - F_l)^{\frac{1}{2}} (\log (N - F_l))^\frac{3}{8}.
\]

Note that \( 0 < N - F_l < e_l \).

We choose [condition (ii)]
\[
e_l := \left\lceil \frac{F_l^{2\gamma}}{\log (F_l^{2\gamma})} \right\rceil.
\]

Note that conditions (i) and (ii) do not depend on \( \alpha \). Now assume that \( l \) is so large that \( 2 (\log F_l)^2 < \frac{F_l^{\gamma}}{2} \). Then
\[
\frac{F_l^{\gamma}}{2} \leq 2 (\log F_l)^2 + (e_l \log e_l)^{\frac{1}{2}} \leq 2F_l^{\gamma}
\]
and (note that \( \gamma \leq \frac{1}{2} \))
\[
F_l < N \leq E_l = F_l + e_l \leq 2F_l.
\]

Hence
\[
ND_N \leq \max (1, c_2(\alpha)) 2F_l^{\gamma}
\]
\[
\leq \max (1, c_2(\alpha)) 2N^{\gamma}.
\]

**Case 3** Let \( E_l < N < F_{l+1} \) for some \( l \). Then by Case 2 and by (8) and (9) we have
\[
ND_N \leq E_l D_{E_l} + (N - E_l) D_{E_l,N}
\leq 2 \max (1, c_2(\alpha)) E_l^{\gamma} + c_1(\alpha) (\log (N - E_l))^2
\leq 3 \max (1, c_2(\alpha)) N^{\gamma}
\]
for \( N \) large enough.

It remains to show that for every \( \varepsilon > 0 \) we have \( ND_N \geq N^{\gamma - \varepsilon} \) for infinitely many \( N \). Let \( l \) be given and let \( M_l \leq e_l \) with the properties given in Lemma 1. Let \( N := F_l + M_l \). Then by Lemma 1, Case 1, (8), (12) and (13) for \( l \) large enough we have
\[ ND_N \geq M_l D_{F_l,N} - F_l D_{F_l} \]
\[ \geq K(\alpha, \varepsilon) \sqrt{\frac{e_l}{(\log e_l)^{1+\varepsilon}}} - 2 (\log m_l)^2 \]
\[ \geq \frac{F_l^\gamma}{(\log F_l)^3} \]
\[ \geq N^{\gamma-\varepsilon}. \]

This proves the theorem. \(\square\)

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