IMPROVEMENT OF THE THEOREM OF HARDY-LITTLEWOOD
ON DENSITY OF ZEROS OF THE FUNCTION $\zeta \left( \frac{1}{2} + it \right)$

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Abstract. In this paper we improve classical Hardy-Littlewood exponent $1/2$
by about 16.6% 62 years after the original result. This result is the first step
to prove the Selberg’s hypothesis (1942). In order to reach our purpose we use
discrete method. This paper is the English version of our paper [4].

1. INTRODUCTION

Let us remind that Hardy and Littlewood have proved (see [1], p. 283) the
following classical estimate

$$(1.1) \quad N_0(T + T^{1/2 + \epsilon} - N_0(T) > A(\epsilon) T^{1/2 + \epsilon}, \quad T \geq T_0(\epsilon)$$

for every fixed small positive $\epsilon$, where $N_0(T)$ denotes the number of zeros of the
function

$$\zeta \left( \frac{1}{2} + it \right), \quad t \in (0, T],$$

and $A(\epsilon)$ is a constant depending on the choice of $\epsilon$.

In this paper we shall prove, by a discrete method, a theorem on number of good
segments (definition of these is placed below). Next, as a Corollary, we obtain the
following estimate

$$(1.2) \quad N_0(T + T^{5/12} \psi \ln^3 T) - N_0(T) > A(\psi) T^{5/12} \psi \ln^3 T,$$

where $\psi = \psi(T)$ means an arbitrarily slowly increasing function unbounded from
above, for example

$$\psi = \ln \ln \ldots \ln T,$$

and $A(\psi)$ is a constant depending upon the choice of $\psi$.

Next, let us remind that A. Selberg (see [5], p. 46, Theorem A) has obtained
the fundamental result by means of the estimate

$$N_0(T + T^{1/2 + \epsilon}) - N_0(T) > A(\epsilon) T^{1/2 + \epsilon} \ln T$$

and he has raised the following hypothesis

$$(1.3) \quad \frac{1}{2} \rightarrow a : a < \frac{1}{2}.$$

Remark. We notice explicitly that

(a) our improvement (1.2)

$$(1.2) \quad \frac{1}{2} \rightarrow \frac{5}{12}$$

of the Hardy-Littlewood exponent in (1.1) represents the part 16.6%,
(b) estimate (1.2) is the first step on a way to prove the Selberg’s hypothesis (1.3).

2. Theorem

Let (see [7], pp. 79, 329)

\[ Z(t) = e^{i\varphi(t)} \zeta\left(\frac{1}{2} + it\right), \]

\[ \varphi(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma\left(\frac{1}{2} + it\right) = \vartheta_1(t) + O\left(\frac{1}{t}\right), \]

\[ \vartheta_1(t) = \frac{\pi}{2} \nu, \nu = 1, 2, \ldots \]

Next, let

\[ \omega = \frac{\pi}{\ln \frac{T}{2\pi}}, \quad U = T^{5/12} \psi \ln^3 T, \quad M_1 = M_1(\delta, T) = \lfloor \delta \ln T \rfloor, \quad \delta > 1. \]

**Definition.** We shall call the segment

\[ [\bar{t}_\nu + k(\nu)\omega, \bar{t}_\nu + (k(\nu) + 1)\omega], \]

where

\[ \bar{t}_\nu \in [T, T + U], \quad 0 \leq k(\nu) \leq M_1, \]

and \( k(\nu) \in \mathbb{N}_0 \), as the **good segment** (comp. [6], [2]) if

\[ Z(\bar{t}_\nu + k(\nu)\omega) \cdot Z(\bar{t}_\nu + (k(\nu) + 1)\omega) < 0. \]

Let \( G(T, U, \delta) \) denote the number of non-intersecting good segments within the segment \([T, T + U]\). The following theorem holds true.

**Theorem.** There are \( \delta_0 > 0, \quad A(\psi, \delta_0) > 0, \quad T_0(\psi, \delta_0) > 0 \)

such that

\[ G(T, U, \delta_0) > A(\psi, \delta_0)U, \quad T \geq T_0(\psi, \delta_0). \]

Since the good segment contains a zero point of the odd order of the function

\[ \zeta\left(\frac{1}{2} + it\right) \]

(see (2.1), (2.8)) then the estimate (1.2) follows, where, of course, \( A(\psi) = A(\psi, \delta_0). \)
3. Main lemmas and proof of Theorem

3.1. Let (see [3], (3))

\[
J = \sum_{k=0}^{M} \sum_{l=0}^{M} \sum_{T \leq \bar{t} \nu \leq T + U} Z(\bar{t}_\nu + k \omega)Z(\bar{t}_\nu + l \omega).
\]

We have (see Theorem from [3]) the following

**Lemma α.**

\[
J = AMU \ln^2 T + o(MU \ln^2 T),
\]

where

\[
U = T^{5/12} \psi \ln^3 T, \quad \ln T < M < \sqrt[3]{\psi} \ln T,
\]

and \(A > 0\) is an absolute constant.

Next, we put

\[
N = \sum_{T \leq \bar{t}_\nu \leq T + U} |K|^2,
\]

where

\[
K = \sum_{k=0}^{M} \left\{ e^{-i \theta (\bar{t}_\nu + k \omega)} Z(\bar{t}_\nu + k \omega) - 1 \right\}.
\]

The following lemma holds true

**Lemma β.**

\[
N = O(MU \ln^2 T).
\]

3.2. Now we use our main lemmas to complete the proof of the Theorem. Let

\[
J(\bar{t}_\nu) = \sum_{k=0}^{M} Z(\bar{t}_\nu + k \omega),
\]

\[
L(\bar{t}_\nu) = \sum_{k=0}^{M} |Z(\bar{t}_\nu + k \omega)|,
\]

and

\[
R = R(T, U)
\]

denote the number of \(\bar{t}_\nu^*\) with the property

\[
\bar{t}_\nu^* \in [T, T + U] \Rightarrow |J(\bar{t}_\nu^*)| = L(\bar{t}_\nu^*).
\]

It is clear that the members of the sequence

\[
\{Z(\bar{t}_\nu^* + k \omega)\}_{k=0}^{M}
\]

preserve their signs. Thus

\[
\sum_{\bar{t}_\nu^*} |J| = \sum_{\bar{t}_\nu^*} L.
\]
Next, we have (see (3.2), (3.5))

\[(3.12) \quad \sum_{\bar{t}^*} |J| \leq \sqrt{R} \left( \sum_{\bar{t}^*} J^2 \right)^{1/2} = \sqrt{RJ} < A\sqrt{RMU \ln^2 T}, \]

\[(3.13) \quad L = \sum_{k=0}^{M} |Z(\bar{t}_\nu + k\omega)| \geq \left| \sum_{k=0}^{M} e^{-i\theta(\bar{t}_\nu + k\omega)} Z(\bar{t}_\nu + k\omega) \right| = |K + M + 1| \geq M + 1 - |K|. \]

Now we have (see (3.6))

\[(3.14) \quad \sum_{\bar{t}^*} L \geq (M + 1)R - \sum_{\bar{t}^*} |K| \geq (M + 1)R - \sqrt{R} \sum_{\bar{t}^*} |K|^2 \geq (M + 1)R - \sqrt{RN} > \]

\[(M + 1)R - A\sqrt{RMU \ln^2 T}, \]

and (see (3.11), (3.12), (3.14))

\[(3.15) \quad (M + 1)R < A\sqrt{RMU \ln^2 T}. \]

Consequently, we obtain

\[(3.16) \quad R < A \frac{U \ln^2 T}{M}. \]

Now, we divide the number (comp. (3), (8))

\[(3.17) \quad Q_1 = \frac{1}{\pi} U \ln \frac{T}{2\pi} + O \left( \frac{U^2}{T} \right) \]

of values \(\bar{t}_\nu \in [T, T + U]\)

into

\(\left[ \frac{Q_1}{2M} \right] \)

pairs of abutting parts \(j_1, j_2\), each except the last \(j_2\), of length \(M\) (comp. (4), p. 226). Let now \(\mu\) denote the number of parts \(j_1\) consisting entirely of the points \(\bar{t}_\nu^*\).

Then by (3.16) we have

\[(3.18) \quad \mu M < A \frac{U \ln^2 T}{M} \Rightarrow \mu < AU \left( \frac{\ln T}{M} \right)^2. \]

If (see (2.5))

\(M = M_1 = [\delta \ln T], \)

then we obtain by (2.5), (3.17) and (3.18)

\[(3.19) \quad \left[ \frac{Q_1}{2M} \right] - \mu > A_1 \frac{U \ln T}{M} - A_2 \frac{U^2}{MT} - \mu > \frac{1}{\delta_0} \left( A_3 - A_4 \right) U - A_5 > \]

\(> A(\psi, \delta_0)U. \)
for sufficiently big $\delta$, (say). This inequality gives the estimate from below for
the number of parts $j_{1}$ such that every of these contains at least one point $\bar{t}_{\nu}$ for
which (see (3.7) – (3.9))

\begin{equation}
|J(\bar{t}_{\nu})| \neq L(\bar{t}_{\nu}).
\end{equation}

Of course, for such a point we have that the members of the sequence

\begin{equation}
\{Z(\bar{t}_{\nu} + k\omega)\}_{k=0}^{M}
\end{equation}

change sign, i.e. there is

\[ k(\nu) \in [0, M(\delta_{0})] \]

such that (2.8) holds true. Since there is at least (see (3.19))

\[ A(\psi, \delta_{0}) \]

of parts $j_{1}$ of this kind, then (2.9) follows.

Proof of Lemma $\beta$ is in what follows.

4. Decomposition of the sum $N$

Let (comp. (3.3))

\begin{equation}
K_{1} = \text{Re} \left\{ e^{-i\theta(\bar{t}_{\nu} + k\omega)} Z(\bar{t}_{\nu} + k\omega) - 1 \right\} \cdot \left\{ e^{i\theta(\bar{t}_{\nu} + l\omega)} Z(\bar{t}_{\nu} + l\omega) - 1 \right\} =
\end{equation}

\begin{equation}
= \text{Re} \left\{ (e^{-i\theta_{k}} Z_{k} - 1)(e^{i\theta_{l}} Z_{l} - 1) \right\} =
\end{equation}

\[ = Z_{k} Z_{l} \cos(\theta_{k} - \theta_{l}) - Z_{k} \cos \theta_{k} - Z_{l} \cos \theta_{l} + 1. \]

Putting

\begin{equation}
Z_{k} = 2 \cos \theta_{k} + \bar{Z}_{k}, \quad \bar{t}_{k} = \bar{t}_{\nu} + k\omega,
\end{equation}

where (see [3], (117))

\begin{equation}
\bar{Z}_{k} = 2 \sum_{2 \leq n < P_{0}} \frac{1}{\sqrt{n}} \cos(\theta_{k} - \bar{t}_{k} \ln n) + O(T^{-1/4}), \quad P_{0} = \sqrt{\frac{T}{2\pi}},
\end{equation}

we obtain

\begin{equation}
K_{1} = \bar{Z}_{k} \bar{Z}_{l} \cos(\theta_{k} - \theta_{l}) +
\end{equation}

\[ + 2 \bar{Z}_{k} \cos \theta_{l} \cos(\theta_{k} - \theta_{l}) + 2 \bar{Z}_{l} \cos \theta_{k} \cos(\theta_{k} - \theta_{l}) -
\]

\[ - \bar{Z}_{k} \cos \theta_{k} - \bar{Z}_{l} \cos \theta_{l} +
\]

\[ + 4 \cos \theta_{k} \cos \theta_{l} \cos(\theta_{k} - \theta_{l}) - 2 \cos^{2} \theta_{k} - 2 \cos^{2} \theta_{l} + 1. \]

Hence, (see (3.4), (3.5), (3.11), (3.3))

\begin{equation}
N = \sum_{T \leq \bar{t}_{\nu} \leq T + U} \sum_{k=0}^{M} \sum_{l=0}^{M} K_{1} = \sum_{\bar{t}_{\nu}} \sum_{k} \sum_{l} \bar{Z}_{k} \bar{Z}_{l} \cos(\theta_{k} - \theta_{l}) +
\end{equation}

\[ + 4 \sum_{\bar{t}_{\nu}} \sum_{k} \sum_{l} \bar{Z}_{k} \cos \theta_{l} \cos(\theta_{k} - \theta_{l}) - 2 \sum_{\bar{t}_{\nu}} \sum_{k} \sum_{l} \bar{Z}_{k} \cos \theta_{k} +
\]

\[ + \sum_{\bar{t}_{\nu}} \sum_{k} \sum_{l} \{ 4 \cos \theta_{k} \cos \theta_{l} \cos(\theta_{k} - \theta_{l}) - 4 \cos^{2} \theta_{k} + 1 \} =
\]

\[ = w_{1} + w_{2} + w_{3} + w_{4}. \]
5. Estimate of \( w_4 \)

Next, we obtain, (see (2.2) – (2.4) and [3], (118), (119)) that

\[
4 \cos \vartheta_k \cos \vartheta_l \cos (\vartheta_k - \vartheta_l) = \\
= 2 \cos^2 (\vartheta_k - \vartheta_l) + 2 \cos (\vartheta_k + \vartheta_l) \cos (\vartheta_k - \vartheta_l) = \\
= 1 + \cos \{2(\vartheta_k - \vartheta_l)\} + \cos (2\vartheta_k) + \cos (2\vartheta_l) = \]

\[
= 1 + \cos \{2(\vartheta_{1,k} - \vartheta_{1,l})\} + \cos (2\vartheta_{1,k}) + \cos (2\vartheta_{1,l}) + \mathcal{O} \left( \frac{1}{T} \right) = \\
= 1 + \cos \{2(k - l)\omega \ln P_0\} + \cos (\pi \nu + 2k\omega \ln P_0) + \\
+ \cos (\pi \nu + 2\nu \omega \ln P_0) + \mathcal{O} \left( \frac{MU}{T \ln T} \right) + \mathcal{O} \left( \frac{1}{T} \right) = \\
= 1 + (-1)^{k+l} + (1)^{\nu+k} + (1)^{\nu+l} + \mathcal{O} \left( \frac{MU}{T \ln T} \right),
\]

\[
\text{(5.1)}
\]

\[
-4 \cos^2 \vartheta_k = -2 - 2 \cos (2\vartheta_k) = \\
= -2 - 2 \cos (\pi \nu + 2k\omega \ln P_0) + \mathcal{O} \left( \frac{MU}{T \ln T} \right) = \\
= -2 - 2(-1)^{\nu+k} + \mathcal{O} \left( \frac{MU}{T \ln T} \right),
\]

\[
\text{(5.2)}
\]

since (see (2.5), (4.3))

\[
2 \omega \ln P_0 = \pi.
\]

Finally, we have (see [3], (17), (45), (5.1), (5.2))

\[
w_4 = \sum_{t, \nu} \sum_{k} \sum_{l} (-1)^{k+l} + \mathcal{O} \left( M^2 U \ln T \frac{MU}{T \ln T} \right) = \\
= \mathcal{O} (U \ln T) + \mathcal{O} \left( \frac{M^3 U^2}{T} \right).
\]

\[
\text{(5.4)}
\]

6. Estimate of \( w_1 \)

Further, we have (see (2.2), (2.3), (4.3) and [3], (119), (120))

\[
Z_k Z_l = 2 \sum_{m} \sum_{n} \frac{1}{\sqrt{mn}} \cos (\vartheta_k + \vartheta_l - \bar{t}_k \ln n - \bar{t}_l \ln m) + \\
+ 2 \sum_{m} \sum_{n} \frac{1}{\sqrt{mn}} \cos (\vartheta_k - \vartheta_l - \bar{t}_k \ln n + \bar{t}_l \ln m) + \mathcal{O} (T^{-1/12} \ln T) = \\
= 2 \sum_{m} \sum_{n} \frac{(-1)^{\nu}}{\sqrt{mn}} \cos \left( \bar{t}_\nu \ln (mn) - (k + l)\omega \ln P_0 + k\omega \ln n + l\omega \ln m \right) + \\
+ 2 \sum_{m} \sum_{n} \frac{1}{\sqrt{mn}} \cos \left( \bar{t}_\nu \ln \frac{n}{m} + (l - k)\omega \ln P_0 + k\omega \ln n - l\omega \ln m \right) + \\
+ \mathcal{O} \left( \frac{MU}{\sqrt{T \ln T}} \right) + \mathcal{O} (T^{-1/12} \ln T).
\]

\[
\text{(6.1)}
\]

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Next, (see (6.1), comp. [3], (121))

\[
\bar{Z}_k \bar{Z}_l \cos(\vartheta_k - \vartheta_l) = \bar{Z}_k \bar{Z}_l \cos\{(k - l)\omega \ln P_0\} + \mathcal{O}\left(\frac{MU \ln T}{T^{2/3}}\right) = \\
= \sum_{m} \sum_{n} \frac{(-1)^{\nu}}{\sqrt{mn}} \cos\left(\bar{t}_\nu \ln \frac{n}{m} + k\omega \ln n - \omega \ln m\right) + \\
+ \sum_{m} \sum_{n} \frac{(-1)^{\nu}}{\sqrt{mn}} \cos\left(\bar{t}_\nu \ln \frac{n}{m} - 2k\omega \ln P_0 + k\omega \ln n + \omega \ln m\right) + \\
(6.2) + \sum_{m} \sum_{n} \frac{1}{\sqrt{mn}} \cos\left(\bar{t}_\nu \ln \frac{n}{m} + k\omega \ln n - \omega \ln m\right) + \\
+ \sum_{m} \sum_{n} \frac{1}{\sqrt{mn}} \cos\left(\bar{t}_\nu \ln \frac{n}{m} - 2(k - l)\omega \ln P_0 + k\omega \ln n - \omega \ln m\right) + \\
+ \mathcal{O}\left(\frac{MU}{\sqrt{T \ln T}}\right) + \mathcal{O}\left(T^{-1/2} \ln T\right) + \mathcal{O}\left(\frac{MU}{T^{2/3}}\right) = \\
= s_5 + s_6 + s_7 + s_8 + \mathcal{O}\left(\frac{MU}{\sqrt{T \ln T}}\right) + \mathcal{O}(T^{-1/12} \ln T).
\]

First of all

(6.3) \[
\sum_{m} \sum_{n} \bar{\sum}_{k} \sum_{l} (s_5 + s_6 + s_7^{(m\neq n)} + s_8^{(m\neq n)}) = \mathcal{O}(M^3 T^{5/12} \ln^3 T)
\]

by lemmas of type B and C from [3]. Next, (comp. [3], (94), (104))

(6.4) \[
s_7 = \sum_{k} \sum_{l} s_7^{(m\neq n)} = \sum_{2 \leq n < P_0} \frac{1}{n} G(M + 1, \omega \ln n).
\]

Since (see (2.5))

\[
0 < \frac{1}{2} \omega \ln n = \frac{\pi}{4} \ln n < \frac{\pi}{4},
\]

then

(6.5) \[
\sin\left(\frac{1}{2} \omega \ln n\right) > A \omega \ln n,
\]

and

(6.6) \[
s_7 = \mathcal{O}\left(\frac{1}{\omega^2} \sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}\right) = \mathcal{O}(\ln^3 T).
\]

Consequently, (see (3.17))

(6.7) \[
\bar{\sum}_{k} \bar{\sum}_{l} s_7 = \mathcal{O}(U \ln T \ln^2 T) = \mathcal{O}(U \ln^3 T).
\]

Next we obtain by similar way that

(6.8) \[
s_1 = \sum_{k} \sum_{l} s_8^{(m=n)} = \sum_{2 \leq n < P_0} \frac{1}{n} G\left(M + 1, \omega \ln \frac{P_0^2}{n}\right).
\]

Since

\[
\frac{1}{2} \omega \ln \frac{P_0^2}{n} = \frac{\pi}{2 \ln P_0^2} \frac{P_0^2}{n},
\]

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then
\[ \frac{\pi}{4} < \frac{1}{2} \omega \ln \frac{P_0^2}{n} < \frac{\pi}{2}, \]
and
\[ \sin \left( \frac{1}{2} \omega \ln \frac{P_0^2}{n} \right) > A > 0. \]
Hence
\[ s_{s1} = O \left( \sum_{2 \leq n < P_0} \frac{1}{n} \right) = O(\ln T), \]
and consequently,
\[ \sum_{t_{\nu}} s_{s1} = O(U \ln^2 T). \]
Finally, we obtain (see (4.5), (6.2), (6.3), (6.7), (6.11)) in the case (3.3) that
\[ w_1 = O(M^3 T^{5/12} \ln T) + O(U \ln^3 T) + \]
\[ + O \left( \frac{M^3 U^2}{\sqrt{T}} \right) + O(M^2 U T^{-1/12} \ln T) = O(M U \ln^2 T). \]

7. Estimates of \( w_2, w_3 \)

First of all, we have (see (4.2), (4.3), comp. (4.5))
\[ 4 \tilde{Z}_k \cos \vartheta_k \cos (\vartheta_k - \vartheta_l) = \]
\[ = \sum_n \frac{1}{\sqrt{n}} \cos (t_{\nu} \ln n) + \sum_n \frac{1}{\sqrt{n}} \cos (2 \vartheta_k - \tilde{t}_{\nu} \ln n) + \]
\[ + \sum_n \frac{1}{\sqrt{n}} \cos (2 \vartheta_l - \tilde{t}_{\nu} \ln n) + \sum_n \frac{1}{\sqrt{n}} \cos (2 \vartheta_k - 2 \vartheta_l - \tilde{t}_k \ln n) + \]
\[ + O(T^{-1/4}) = \]
\[ = w_{21} + w_{22} + w_{23} + w_{24} + O(T^{-1/4}), \quad \tilde{t}_k = \tilde{t}_{\nu} + k \omega. \]
Next, we have (see (2.2), (2.3), (3.17), (5.3) and [3], (118))
\[ w_{221} = \sum_{T \leq \tilde{t}_\nu \leq T + U} \]
\[ = \sum_n \frac{1}{\sqrt{n}} \sum_{t_{\nu}} \cos (\pi \nu + 2k \omega \ln P_0 - \tilde{t}_k \ln n) + \]
\[ + O \left\{ U \ln T \cdot T^{1/4} \left( \frac{1}{T} + \frac{M U}{T \ln T} \right) \right\} = \]
\[ = (-1)^k \sum_n \frac{1}{\sqrt{n}} \sum_{t_{\nu}} \cos (\pi \nu - \tilde{t}_{\nu} \ln n - k \omega \ln n) + O(M^2 U T^{-1/4}) = \]
\[ = (-1)^k \sum_n \frac{1}{\sqrt{n}} \sum_{t_{\nu}} \cos (\pi \nu + \tilde{t}_{\nu} \ln n + k \omega \ln n) + O(M^2 U T^{-1/4}). \]
The following function corresponds to the \( \tilde{t}_{\nu} \)-sum (comp. [6], pp. 99, 100)
\[ \chi(\nu) = \frac{1}{2\pi} (\pi \nu + \tilde{t}_{\nu} + k \omega \ln n), \quad \tilde{t}_{\nu} \in [T, T + U]. \]
Consequently, (see (2.3), (2.4))
\[
\chi'(\nu) = \frac{1}{2} + \frac{1}{2\pi} \ln \frac{\sqrt{2\pi n}}{2\pi} = \frac{1}{2} + \frac{1}{4\ln P_0} + O\left(\frac{U}{T \ln T}\right),
\]
\[
\chi''(\nu) < 0.
\]
Hence
\[
\chi'(\nu) \in \left(\frac{1}{2}, \frac{3}{4} + \epsilon\right), \epsilon \in (0, 1/4),
\]
and (see (7.2), comp. [6], p. 100)
\[
w_{221} = O\left(\sum \frac{1}{\sqrt{n}}\right) + O(M^2T^{-3/4}) = O(T^{1/4}).
\]
Similar estimates can be obtained also for
\[
w_{211}, w_{231}, w_{241}.
\]
Consequently, (see (4.5), (7.1))
\[
(7.3) \quad w_2 = O(M^2T^{1/4}),
\]
and similarly we obtain the estimate
\[
(7.4) \quad w_3 = O(M^2T^{1/4}).
\]
Finally, we obtain by (2.5), (4.5), (5.4), (6.12), (7.3) and (7.4) that
\[
N = O(MU \ln^2 T),
\]
and the estimate (3.6) holds true.

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