Renormalization of Non-Semisimple Gauge Models
with the Background Field Method

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Abstract

We study the renormalization of non-semisimple gauge models quantized in the ‘t Hooft-background gauge to all orders. We analyze the normalization conditions for masses and couplings compatible with the Slavnov-Taylor and Ward-Takahashi Identities and with the IR constraints. We take into account both the problem of renormalization of CKM matrix elements and the problem of CP violation and we show that the Background Field Method (BFM) provides proper normalization conditions for fermion, scalar and gauge field mixings. We discuss the hard and the soft anomalies of the Slavnov-Taylor Identities and the conditions under which they are absent.

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1 Introduction

The Standard Model (SM)\cite{1} is the widely accepted Quantum Field Theory (QFT) describing the physics of elementary particles up to the TeV scale. As any interacting, local QFT in four space-time dimensions it has a singular high energy limit leading to UV divergences in perturbative calculations. The removal of these divergences through the process of renormalization can be achieved in various ways and hence it is important to keep control of possible inconsistencies and ambiguities accompanying such procedures. The most satisfactory way to do so, is to put up a system of “axioms” of renormalization theory, with the aim to characterize the result through physical requirements. This approach to the problem, motivated by development of axiomatic QFT, led to a rather satisfactory framework\cite{2} culminating in the work of Epstein and Glaser\cite{3}. However, this unfortunately does not exhaust the problem of renormalization for the SM, since it involves the introduction of various “unphysical” fields required for the explicitly local formulation of gauge theories, whose decoupling in physical processes has to be guaranteed for the renormalized theory. Also this problem found a satisfactory solution in the form of certain symmetry relations\cite{4}, the Slavnov-Taylor Identities (STI) expressing the BRST invariance of the theory. In order to construct a renormalized perturbative series satisfying these identities the most natural strategy is to find an explicit BRST invariant regularization and subtraction procedure. Despite many attempts in the past no really satisfactory such method is known to the author. Hence it offers itself to make recourse again to the “axiomatic” approach and try to construct a BRST invariant renormalized perturbation series making use of the full freedom allowed by the axioms of renormalization theory. This strategy, known as “algebraic” renormalization (since the result is characterized by the symmetry relations pertaining to some Lie algebra), advocated mainly by the “BRS school”\cite{5,6,7,8} will be adopted in the present work. Although this strategy may seem unduly complicated for practical purposes, it is the only one surpassing unresolved consistency problems of all the hitherto proposed “invariant” regularizations. In fact, combined with efficient strategies to determine the required non-invariant counter terms, it may turn out that “algebraic” renormalization is a liable method for the SM or even its generalizations like its supersymmetric extension.

As a first case, the “algebraic renormalization”\cite{8} has been applied to the renormalization of the Abelian Higgs-Kibble model. This models describes some aspects of the SM, namely the spontaneous symmetry breaking, and it was the first example of gauge theory handled with the BRST formalism\cite{9}.

The essential results of those papers were to establish a formalism independent of the regularization and based on the locality and Lorentz invariance to proof the renormalizability of the gauge theories and the unitarity of the S-matrix. Later, the formalism was generalized to non-abelian gauge group in\cite{10,11,12,13,14,15,16}. However, up to now, although the main problems of the renormalization of the SM\cite{11,12,13,14,15,16} are solved, some peculiar feature of physical interesting models are not yet taken into account, in particular we want to recall

1. The IR problems of off-shell Green’s functions. They have been recently examined in\cite{16}.
2. The CP violation and the renormalization of the mixing. Although these are well known from phenomenological point of view, a complete (algebraic renormalization) analysis is still missing.
3. Unstable particles.

In this paper we use the word minimal in order to distinguish the field content of the SM from that of its extensions like the Two Higgs Doublets Model (2HDM)\cite{18}, the Minimal Supersymmet-
ric Standard Model (MSSM) \cite{17}, the Grand Unified Theory (GUT) $SU(5)$ models \cite{19} and their supersymmetric versions.

However, in order to easily generalize our results for the SM, we shall use a general framework as long as possible and we shall specify the equations when the features of a particular model are considered. We, therefore, examine a non-semisimple gauge model with spontaneous symmetry breaking coupled to scalar fields and to chiral fermion fields. This comprises field mixings such as the fermion mixing, scalar mixing and the system as the photon and the $Z^0$ vector boson; furthermore we do not require the $P$ and $C$ discrete symmetries, but only the hermiticity, locality and covariance which imply the $CPT$ invariance. In this general framework the mass eigenstates do not coincide with the gauge eigenstates and the mass matrices are only semi-definite positive.

The invariance under the local gauge transformations essentially guarantees the consistency between a covariant formalism for relativistic quantum fields and the physical degrees of freedom of vector particles. However for quantization purposes, it is necessary to choose a gauge which breaks this invariance and introduces new unphysical ghost fields. The gauge fixing procedure is conveniently realized \textit{a la} BRST \cite{4} requiring the invariance of the theory under the BRST transformations \cite{5}. At the quantum level this invariance ensures the decoupling of unphysical states for the physical Fock space, guaranteeing the unitarity of the scattering operator and the gauge independence of the physical observables (see \cite{4, 5}).

Regarding the choice of gauge fixing the background-'t Hooft type \cite{16, 20, 21, 22, 23} is found to be particularly convenient for the SM. In that case, in fact, the $S$ – matrix computed with the BFM is completely defined in terms of (background) gauge invariant Green’s functions \cite{24}. Moreover the background gauge invariance, implemented at the quantum level by means of Ward-Takahashi Identities (WTI), provides very useful constraints on the possible counterterms (CP violating counterterms, renormalization of the mixings \cite{25, 26} and wave function renormalizations (w.f.r.)).

As is well known the quantization of gauge models can be consistently performed if the tree level BRST symmetry can be implemented at the quantum level and it is not anomalous. In a general case two kinds of anomaly candidates could be present: the hard anomalies characterized by local operators with the highest dimension and the soft anomalies \cite{6, 27} with operators of lower dimension. In the case of the Adler-Bardeen-Jackiw anomaly \cite{28} –a candidate for hard anomalies– the non-renormalization theorem \cite{1} and the fermion content of the SM ensures the vanishing of its coefficients. The other hard candidates are ruled out by introducing new constraints as will be discussed later and, in absence of hard anomalies, the Callan-Symanzik equation guarantees the vanishing of soft ones \cite{6, 16, 27}. However some of the candidates for the soft anomalies can also be IR dangerous (IR anomalies), \textit{i.e.} the counterterms which can remove them can produce IR divergence. This means that their coefficient must vanish and we will check this explicitly in the Sec. 4.2. It turns out that as a consequence of normalization conditions and of the invariances of the model the IR anomalies can be excluded in the case of the SM and for extended models.

Besides IR problems the normalization conditions play a prominent rôle in defining the quantum theory and require a separate discussion. Our analysis is divided into three steps.

First, we solve the symmetry constraints in order to single out the free parameters of the models without taking into account the IR constraints. This allows us to define the space of free parameters independently of the basis chosen for the fields (\textit{e.g.} it appears convenient to work in the basis of gauge eigenstates because, in that case, the result is already available in the literature \cite{1, 10}).

Second, we establish a set of normalization conditions which is compatible with the symmetry constraints and which ensures the particle content of the model (definition of the physical states). Much emphasis is posed on the IR normalization conditions which are relevant in order to ensure
the absence of IR anomalies. Concerning the normalization conditions, we study the problem of the renormalization of the mixing angles for fermions (CKM) and we propose a scheme to fix the mixing angles among scalars in the case of extended models (2HDM, MSSM etc.). This is based on the study of the cohomology classes with background fields and, subsequently, by imposing the background gauge invariance through the WTI. The latter implements severe constraints on the renormalization of mixings. Third, we check the absence of IR anomalies.

Regarding the normalization conditions and the choice of counterterms we have to take into account the CP violation. It arises only from the complex Yukawa couplings [29], but new UV divergent CP-odd Green’s functions –as for instance the mixed two-point function for the Higgs $H$ and the would-be-Goldstone boson $G_0$ [30]– and the CP-odd tadpoles [31] appear at higher orders. These new divergences require new counterterms and new normalization conditions. In the paper we analyze the CP-odd counterterms and CP-odd anomalies. The number of CP-odd counterterms and their normalization conditions are considerable reduced by using the WTI for the background gauge invariance. On the other side the WTI are not enough to ensure the absence of CP-odd anomalies. The problem is solved by a further functional identity, the Abelian Antighost Equation (AAE) [32], which guarantees the cancellation of the anomaly coefficients order-by-order.

Concerning the system of scalar components of the neutral vector fields $Z^0, \gamma$ and the scalar fields $G_0, H$, the CP-violation makes the usual definition of the Higgs mass, the only physical parameter of the system, meaningless. If the CP symmetry were conserved, (the real part of) the zero of the two-point function for the Higgs would correspond to its physical mass and the pole structure of the unphysical fields would be fixed by means of the STI. However, due to the CP violation, which generates a mixing between the Higgs and the Goldstone field $G_0$, the mass of the Higgs can not be identified with the zero of the Higgs two-point functions. This must be replaced with non-trivial zero of the eigenvalues of the two-point function matrix for the system $Z - \gamma - H - G_0$.

Concerning the BFM, we have to recall two relevant results. The first is the algebraic proof of the stability under radiative corrections of the splitting of the gauge field into a quantum part and into a classical background [22]. The proof also includes the absence of anomalies beyond the conventional Adler-Bardeen-Jackiw anomaly [28], excluded by a suitable content of fermions of the model. The second fact is the extension of this result to scalar fields in order to generalize the ’t Hooft gauge fixing to a background-'t Hooft gauge fixing. This problem was also covered in [16] where the background scalar fields are introduced without imposing the (local) background gauge invariance. The present work follows the lines of [23, 32] where to each bosonic field corresponds a background partner and the background gauge invariance is implemented.

Besides the Slavnov-Taylor Identities [1, 8], which implement the BRST symmetry at the quantum level, the Nakanishi-Lautrup equations [33], the equations of motion of the Faddeev-Popov ghost fields, the Abelian Antighost Equation [32] and the Ward-Takahashi Identities for the background gauge invariance are very useful to renormalize the SM. However, the complexity of the problem requires a method to simplify the system of equations by reducing the problem to a restricted functional space (the cohomology $H^k(S|d)$ [11], in the space of background gauge invariant functionals). We propose the following hierarchy: i) we first implement the linear identities, ii) we use the WTI for the background gauge invariance, and finally iii) we study the STI. As a result we are able to enumerate the free parameters which must be tuned to renormalize the theory on a finite number.

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2Related to this point we have to recall that the algebraic proof of the renormalization of non-semisimple gauge model, given in [11], does not rely on any discrete symmetry (up to CPT). The authors, by introducing some supplementary constraints in order to avoid the CP-odd mixing with the abelian ghost field and the rest of the field content and to cancel the anomalies beyond the ABJ anomaly, are able to prove the renormalizability of the theory only by means of BRST symmetry.
of experimental data.

By imposing the WTI, the number of free parameters is reduced and, consequently, a particular care has to be devoted to check if the number of parameters is enough to ensure the correct IR behavior of two- and three-point functions. In particular it can not be easily achieved in the renormalization of two-point functions for the ghost fields. In fact, in that sector, the matrix of two-point functions is non symmetric and the number of the free parameters to fix their IR behavior turns out to be larger than the number of parameters needed for the system ($\gamma, Z^0$).

Furthermore it is interesting to clarify the relations among the two-point functions with only external background fields and the two-point functions with quantum fields. The main point is to compare their pole structures and to show that the normalization conditions fix consistently both of them.

A possible approach is to avoid finite renormalizations (except those needed to exclude IR divergences) and to rely only on a subtraction scheme such as BPHZL or MS in dimensional regularization $\gamma, Z^0$. Although these schemes seem to be very convenient from a theoretical and computational point of view, some difficulties arise when the comparison with the physical data has to be performed. In fact the MS or BPHZL parameters have to be expressed in terms of the physical input parameters and this implies that some algebraic equations are to be solved $\gamma, Z^0$. A different way of proceeding is to impose normalization conditions such that the correct parameterization is fixed from the beginning. An available scheme of this type is the on-shell scheme applied to the SM in $[11]$ and rigorously studied in $[16]$.

The advantages of the on-shell scheme are due to the fact that the “renormalized” fields coincide with the asymptotic fields whose normalization conditions are fixed by the LSZ conditions $[37]$. However this implies a large number of normalization conditions which appear unnecessary. In addition, the success of the on-shell scheme is mostly due to the construction of physical amplitudes at the one loop order without the computation of LSZ conditions for external particles. One price to pay working with the on-shell scheme for the SM is the deformation of the STI and of the WTI $[16]$ needed to avoid the IR anomalies. One possible alternative consists in requiring a minimal set of normalization conditions for the physical parameters and the remaining parameters are fixed by a consistent subtraction scheme, like for MS.

Our choice will be to keep the WTI and the STI in their tree level form. This reduces the space of free parameters and therefore we can only establish a partially on-shell scheme where a small set of normalization conditions are necessary for the physical computations. We divide the complete set into four different classes of normalizations:

1. At first in order to have spontaneous symmetry breaking one has to renormalize the tadpoles. This is essential in order to implement the Higgs mechanism at higher orders. Furthermore due to the CP violation of the SM some CP-odd tadpoles could emerge from radiative corrections and these have to be fixed to zero in order to distinguish the physical Higgs boson from the unphysical scalars.

2. The second essential class of normalization conditions are the masses of particles. There are three subclasses of masses which have to be fixed:

   (a) The masses of physical asymptotic particles. Only the electron, the neutrinos and the photon are asymptotic fields. For them there is no width and their only physical parameter is their mass.

   (b) The masses of unstable particles. For unstable particles the problem of correct definition of masses was already taken into account in $[38, 39]$. Instead of a real pole, a complex
pole has to be considered. Clearly only the real part of the pole has to be compared with the measured physical masses.

(c) The masses of unphysical particles. Their masses are obviously un-observable, but their definition is very important to guarantee the correct pole structure of two-point functions ensuring the unitarity of the model.

3. IR constraints.

As we will discuss in the next paragraphs, the IR constraints are essential to guarantee the correct pole structure of the massless fields. In fact by ensuring the vanishing of the two-point functions for the massless fields and their mixed two-point functions with massive fields at zero momentum we can avoid any IR divergences caused by regularization and subtraction. In particular we have to discuss the following three classes of IR normalization conditions:

(a) Transverse component of two-point functions for vector fields.
(b) Two-point functions for the longitudinal component of vector fields and scalar fields. Since in the presence of CP violation (as for the SM) there is no discrete quantum number which allows us to disentangle the complete scalar sector into unphysical (would-be-Goldstone) bosons and physical ones (Higgs or its partners in the extended versions), we have to discuss those fields in the same class. We have furthermore to clarify how to implement correctly these conditions in order to avoid completely the IR problems to all orders.
(c) Ghost fields. For the ghost fields which are the unphysical partners of massless gauge fields attention has to be paid in order to get rid of the non-symmetric two-point functions.

In the following we will discuss the partially on-shell scheme comparing our results with those given in the paper [16].

4. Couplings.

The most delicate set of normalization conditions for the non-semi-simple gauge models are the normalization of the couplings. Generically the problem of fixing the gauge couplings and the Yukawa couplings is a hard problem. In fact the best situation which can happen is a one-to-one correspondence among parameters and measurable physical quantities. For couplings it is in general quite difficult to find out a correspondence among Green’s functions (computed at a specific point and at fixed perturbation order) with a single measurable quantity [40]. Generically it turns out that the most commonly used combinations of Green’s functions are gauge parameter dependent forbidding their comparison with physical quantities. Alternative approaches may be applied to by-pass this problem [40], however at the moment there is no definitive unique procedure to fix the gauge couplings at some physical value (for instance the Fermi’s constant is established by means of the $\mu$-decay amplitude, which is satisfactory from practical point of view, but it depends on the specific process considered).

For the couplings we have to single out three different classes of normalization conditions

(a) Gauge Couplings.

In this class, as is well known, due to the spontaneous symmetry breaking mechanism some of the gauge couplings can be computed in terms of some physical masses and they can be fixed by measurable quantities (for instance the Weinberg’s angle $\theta_W$). However not all gauge couplings can be obtained in this way and some vertices have to be taken into account (like for the electric charge $e$ and the QCD strong coupling $g_s$). This corresponds
to consider three- or four-point functions which generically depend on gauge parameters in a complicated way and the best defined objects, also in the context of pure Yang-Mills theory, are the S-matrix elements [41].

(b) Yukawa couplings.

For the Yukawa couplings we have to take into consideration two different types of couplings. Those couplings which can be expressed as fermion masses and those which are expressed in terms of fermion mixing matrices (CKM). In general some Yukawa couplings may not even generate any contributions to fermion mass matrices, and they have to be considered as genuine Yukawa couplings. A proposal is fixing them by computing S-matrix elements involving the scalar and fermion coupling.

(c) Higgs potential.

For the Higgs potential a similar considerations as for the above classes has to be done. Some of the couplings of the Higgs potential are fixed by means of the scalar masses (e.g. for the Higgs mass in the minimal SM) and by means of vanishing of the tadpoles. Furthermore some couplings have to be defined in terms of S-matrix elements and this requires the inspection of scalar self-couplings. Notice that in extended models like 2HDM and the MSSM the Higgs potential is parameterized by the mixing angles among physical Higgs particles. Therefore they correspond to physical measurable quantities like the CKM. We propose a scheme to fix them by using the WTI for the background gauge invariance.

The paper is organized as follows: in Sec. 2, we briefly review the notations, we introduce the gauge fixing and we discuss the relevant symmetries (BRST, background gauge invariance and the Abelian Antighost Equation). In Sec. 3 we discuss the free parameters by solving the complete set of identities. Then we systematically analyze the normalization conditions sector by sector. The hard and soft anomalies are studied in Sec. 4. In particular the cancellation of IR anomalies is explained. Finally, in the App. A we provide the complete set of identities and their algebra; in App. B and in App. C we discuss the renormalization of Nakanishi-Lautrup, the equations of the ghost field equations (Faddeev-Popov equation and the Abelian Antighost Equation), respectively.

2 General Settings

2.1 Gauge Group, Representations and Fields

The field content is specified by the quantized gauge vectors \( W^a_\mu \), their background partners \( \hat{W}^a_\mu \), the scalars \( \phi_i \), the background scalars \( \hat{\phi}_i \), the fermions \( \{ \psi^L_I, \psi^R_I \} \), the Faddeev-Popov ghosts \( c^a, \bar{C}^a \), the Nakanishi-Lautrup multipliers \( b^a \), the BRST sources \( \gamma^a_\mu, \gamma^a_i, \eta^{L/R}_I, \eta^{L/R}_i \), \( \zeta^a \) for quantized fields and the external fields \( \Omega^a_\mu, \Omega_i \). The component of the fields \( W^a_\mu, \gamma^a_\mu, \gamma^a_i, \eta^{L/R}_I, \eta^{L/R}_i, \zeta^a \) for quantized fields and the external fields \( \Omega^a_\mu, \Omega_i \). The component of the fields \( W^a_\mu, \gamma^a_\mu, \gamma^a_i, \eta^{L/R}_I, \eta^{L/R}_i, \zeta^a \) for quantized fields and the external fields \( \Omega^a_\mu, \Omega_i \). The component of the fields \( W^a_\mu, \gamma^a_\mu, \gamma^a_i, \eta^{L/R}_I, \eta^{L/R}_i, \zeta^a \) for quantized fields and the external fields \( \Omega^a_\mu, \Omega_i \). The component of the fields \( W^a_\mu, \gamma^a_\mu, \gamma^a_i, \eta^{L/R}_I, \eta^{L/R}_i, \zeta^a \) for quantized fields and the external fields \( \Omega^a_\mu, \Omega_i \).

\[ \text{3The index I for the fermion fields is a multi-index for isospin, flavour, color (for quarks).} \]
In order to describe the general model it is also useful to introduce the charge matrices involved in the coupling of the gauge fields $W^a_{\mu}$. First of all we specify the symmetric, positive definite charge matrix $e_{ab}$ on adjoint representation of the algebra $G$. Clearly $e_{ab}$ has no elements connecting the semi-simple factors $G_S$ to the abelian ones $G_A$. Furthermore the restriction of $e_{ab}$ to each simple component is proportional to the Killing form, and, in a basis where the latter is diagonal, we have $e_{ab} = e_S \delta_{ab}$. $e_S$ is identified with the charges of the simple factors. The charge matrix $e_{ab}$ of the abelian factors $G_A$ must be symmetric and positive definite.

The gauge group generators $t^a, T^a_{R/L}$ in the scalar and fermion representations obey

\[(t^a)^T = -t^a, \quad (T^a)^\dagger_{R/L} = -T^a_{R/L},\]
\[[t^a, t^b] = f^{abc} t^c, \quad [T^a_{R/L}, T^b_{R/L}] = f^{abc} T^c_{R/L}, \quad [T^a_{R}, T^b_{L}] = 0\]

where $t^a$ are real and $f^{abc}$ are the structure constants of $G$, with $f^{abc} = 0$. The couplings of the gauge fields $W^a_{\mu}$ (for each simple factor $G_S$) and the couplings with scalar and fermion fields can now be expressed in terms of the tensors

\[(e_f)^{aSbScS} = e_S f^{abc} T^c_{R/L}, \quad (e_t)^a = e_{ab} t^b, \quad (e_T)^a_{R/L} = e_{ab} T^b_{R/L}.\]

The tree level action $\Gamma_0$ is a local Lorentz invariant hermitian functional

$$\Gamma_0 = \int d^4x L(W^a_{\mu}, \ldots, \Omega^a_{\mu})(x)$$

where $L(W^a_{\mu}, \ldots, \Omega^a_{\mu})(x)$ is a local function of fields, of sources and their derivatives.

Although we adopt the BPHZL \[8, 34\] as a renormalization scheme in order to get rid of divergences, our results can be extended to other renormalization schemes \[35\]. We will use the usual symbol $\Gamma$ to denote the effective action which generates the one particle irreducible Green’s functions \[8, 37\].

According to the BPHZL the fields are characterized by their UV and IR degrees so that the power counting renormalizability of the theory and the IR finiteness of the off-shell Green’s functions requires

$$d_{UV} \Gamma_0 \leq 4 \quad \text{and} \quad d_{IR} \Gamma_0 \geq 4.$$  

We also assume the charge neutrality for $\Gamma_0$ with respect to QED charge, Faddeev-Popov charge, lepton numbers and baryon number. The tree level action $\Gamma_0$, in this paper, is the bookkeeping of all (invariant and non-invariant) counterterms of the model and we refer to the classical terms of the action $\Gamma_0$ as the Lagrangian.

As is well known \[4, 5, 7\] the renormalization of gauge theories requires external sources coupled to non-linear BRST transformations. The BRST-sources or, equivalently, the anti-fields enter on the same footing as the quantum fields and they also require their own IR and UV degree. If $\gamma_{\phi}$ is the BRST-source for the field $\phi$, Eqs. \(2.3\) imply

$$d_{UV} \gamma_{\phi} \leq 4 - d_{UV} \phi \quad \text{and} \quad d_{IR} \gamma_{\phi} \geq 4 - d_{IR} \phi$$

with the condition $d_{UV} \leq d_{IR}$.

At tree level the gauge eigenstates are defined by the background gauge symmetry, but they have no definite IR power counting. This is clearly due to non-diagonal non-negative definite

\[4\]In the SM this applies to the mixing between the photon field $A$ and the $Z$ bosons. Moreover also the ghost system $c^A - c^E$ has no definite IR power counting in the gauge eigenstates basis. For the quark fields it is reasonable to work in the gauge eigenstates, but this is not very practical.
mass matrices. A meaningful subtraction procedure with the correct IR power counting can only be achieved by expressing the fields in terms of combinations corresponding to mass eigenstates. Then we also introduce the tree level mass eigenstates \( A^\mu_a, Z^\mu_a, G^\mu, H^\mu, \tilde{f}_I, f_I, c^\alpha_a, c^a_2 \) for massless and massive gauge fields for would-be Goldstone bosons and Higgs fields, for fermions and for massless/massive ghost fields respectively. \( \Xi^\alpha_a, \tilde{A}^\mu_a, \tilde{Z}^\mu_a, \tilde{G}^\mu, \tilde{H}^\mu, \Theta^\alpha_a, \tilde{c}^a_1, \tilde{b}^a_1 \) are the corresponding BRST-sources, background fields, the BRST variations of the background fields, the antighost fields and their BRST variations in the mass eigenstates representation. The definition of the mass-eigenstates will be extended at the quantum level in the next sections.

### 2.2 Symmetries and Background Gauge Fixing

To implement the BRST symmetry \([4, 5, 9, 11, 16]\) at the quantum level, the non-linear BRST transformations are coupled to the BRST-sources (summation over repeated indices is understood)

\[
\mathcal{L}^{ST} = \gamma^{\mu}_{as} \left( \partial_\mu c_{as} - (ef)^{ab} b_{bs} W^{bs}_{\mu} c_{cs} \right) + \gamma^i \left( c_a (et)^a_{ij} (\phi_j + v_j) \right) + \zeta^{as} \left( \frac{1}{2} (ef)^{as} b_{bs} c_{cs} \right) + \eta^R \{ (et)^a_{R,ij} \psi_j^R + \psi_I^c (et)^a_{R,ij} \eta_j^R + \text{h.c.} \}
\]

The invariance of the vertex functional \( \Gamma \) is expressed by the Slavnov-Taylor Identities \((L/R \) is suppressed, but the summation over the two types of fermion is understood)

\[
\mathcal{S}(\Gamma) = \int d^4 x \left[ \frac{\partial \Gamma}{\partial \gamma^{\mu}_{as}} \frac{\partial \Gamma}{\partial W^{as}_{\mu}} + \partial_\mu c^a + \frac{\partial \Gamma}{\partial C^a_{bs}} \frac{\partial \Gamma}{\partial e^{as}} + \frac{\partial \Gamma}{\partial \phi} + \frac{\partial \Gamma}{\partial \psi^+} + \frac{\partial \Gamma}{\partial \psi^-} \right] = 0.
\]

The general symmetric and invertible matrix \( G^{ab} \) was introduced in the papers by Becchi et al. \([4, 5]\). It can be used to get rid of the possible IR problems in the on-shell renormalization scheme, proposed by Aoki et al. in \([11]\), for the SM as pointed out by E.Kraus in \([16]\). However we can introduce the “rotated” anti-ghost fields \( \bar{c}^a = (G^{-1})^{db} C_b \) in order to simplify our derivations and we will use this matrix discussing the normalization conditions for the ghost fields in Sec. 3 and in App. C.

The nilpotency of the BRST transformations is a crucial ingredient in the analysis of the quantum theory; however, in the case of non-semisimple gauge models, this is not sufficient to guarantee that the gauge group of the renormalized theory and, in particular, its scalar and fermion representations coincide with the tree level ones \([3]\). This is equivalently expressed by saying that the couplings among the matter fields and abelian gauge bosons with scalars and fermions are not protected against the mixing with other conserved abelian currents \([19]\). For that reason, we impose the Abelian Antighost Equation (AAE)

\[
\mathcal{G}_{a_A}(\Gamma) = \frac{\partial \Gamma}{\partial c^a_{A}} - \dot{\gamma}_j (et)^a_{jk} (\phi_j + v_k) = \partial^\phi \bar{c}^a + \gamma_j (et)^a_{jk} (\phi_j + v_k) + \eta_j^R (et)^a_{R,ij} \psi_j^R + \text{h.c.} \equiv \Delta_{cA}^{\mathcal{G}}
\]

as supplementary constraints on \( \Gamma \) (see \([3, 22]\) for further details). In App. A (Eq. \((4.113)\) it will be shown that the commutation relation between the STI \((2.5)\) and the AAE coincides with the WTI

\[
\text{In the paper we will follow the convention given by} \quad [11] \quad \text{for the SM:} \quad Z^\mu = c_w W^\mu_3 - s_w W^\mu_\mu.
\]
for the background gauge invariance with respect to abelian factors. Those WTI can be used to fix
the abelian couplings in the same way as the Eq. (2.7) as in [10].

The linear terms in Eq. (2.5), proportional to the external fields \( \Omega^a \), \( \Omega_i \), which implement
the non-trivial BRST transformation of the background fields \( W_{\mu}^a = \phi_i \), are essential for proving
the renormalizability [16, 22]. In fact the introduction of background fields \( W_{\mu}^a \), \( \phi_i \) triggers new
couplings with dimension less or equal to four among background and quantum fields: the latter are
absent in other formalisms and they require a proper renormalization. The extension of the BRST
symmetry to the background fields is a necessary condition to ensure the renormalizability of the
model, but it is not sufficient to guarantee the equivalence between the conventional approach (that
is without background fields) and the BFM. The equivalence can be only achieved requiring the
background gauge invariance of the simple factors \( G^a \)

\[
W_{as}(\Gamma) = -\nabla^a_{\mu} \frac{\delta \Gamma}{\delta W^b_{\mu}} - \nabla^a_{b} \frac{\delta \Gamma}{\delta W^b_{\mu}} +
\]

\[
+ (et)^{a}_{ij} \left[ (\phi + v)_j \frac{\delta \Gamma}{\delta \phi_i} + (\phi + v)_j \frac{\delta \Gamma}{\delta \phi_i} + \Omega_j \frac{\delta \Gamma}{\delta \Omega_i} + \gamma_j \frac{\delta \Gamma}{\delta \gamma_i} \right] +
\]

\[
+ (et)^{a}_{R,ij} \left[ \eta^R_i \delta \Gamma + \gamma^R_i \frac{\delta \Gamma}{\delta \gamma_i} + \frac{\delta \Gamma}{\delta \eta^R_i} \eta^R_i + \frac{\delta \Gamma}{\delta \eta^R_i} \eta^R_i \right] + (L \rightarrow R) +
\]

\[
+ (ef)^{a}_{bc} \left[ \Omega^b_{\mu} \frac{\delta \Gamma}{\delta \Omega^\mu_c} + \gamma^b \frac{\delta \Gamma}{\delta \gamma_c} + \gamma^b \frac{\delta \Gamma}{\delta \gamma_c} + \gamma^b \frac{\delta \Gamma}{\delta \gamma_c} \right] = 0
\]

where \( \nabla^a_{\mu} \), \( \nabla^a_{b} \) are the covariant derivatives with respect to the gauge fields \( W_{\mu}^a \) and to the
backgrounds \( \hat{W}_{\mu}^a \). The invariance under background gauge transformations of the action implies
also the invariance under rigid transformation of the group \( G^a \).

The quantization of the classical gauge invariant action \( \Gamma_0^{inv}[W, \phi, \psi] \) requires a gauge fixing
which breaks the local gauge symmetry. In particular for the BFM the choice[6]

\[
L^{gf} = b^b \left[ \delta^{ab} \nabla^b_{\mu} (W - \hat{W})^\mu_{bs} + \delta^{ba} \partial_{\mu} (W - \hat{W})^\mu_{ab} + \rho^{bc} \frac{(\phi + v)_i (et)^{c}_{ij} \frac{\delta \Gamma}{\delta \gamma_j} + \frac{\Lambda^{bc}}{2} b_c} \right]
\]

is invariant under background gauge transformations (2.7) if the ’t Hooft parameters \( \rho^{ab} \) and the
gauge parameters \( \Lambda^{ab} \) satisfy

\[
\Lambda^{ab} = \xi_s \delta^{as} \delta^{bs} + \xi_{a} \delta^{ab} \partial_{\mu} \delta^{as} \delta^{bs} + \rho^{ab} \frac{(\phi + v)_i (et)^{c}_{ij} \frac{\delta \Gamma}{\delta \gamma_j} + \frac{\Lambda^{bc}}{2} b_c} \]

Here a single gauge parameter \( \xi_s \) and a single ’t Hooft parameter \( \rho_s \) is allowed for each simple factor
\( G^a \). The symmetric and positive definite matrices \( \xi_{a} \), \( \rho^{ab} \) fix the gauge for the abelian factors
\( G^a \).

\[\text{For the computations of radiative corrections [23], it is convenient to introduce the background gauge fields for the abelian gauge bosons } A^a_{\mu}. \text{ However those background fields are unessential from a theoretical point of view, since their equations of motion are}
\]

\[
\frac{\delta \Gamma}{\delta W_{\mu}^a} = \partial_\mu b^a\]

and they are free after removing the \( b_A \) fields. This equation has to be compared with the analogous equations for the non-abelian background gauge fields [22] and for the scalar fields. They are non-trivial and require the sources \( \Omega^a_{\mu} \), \( \Omega_i \).

\[\text{The ’t Hooft parameters, in the tree level approximation, are proportional to ghost masses. If some masses vanish, as, for instance, for the ghost } c_A \text{ for the photon, the ’t Hooft parameters } \rho^{ab} \text{ are constrained to ensure this feature to all orders.}\]
We assume that $\Gamma_0^{\text{inv}}[W, \phi, \psi]$ does not explicitly depend on the background fields $\hat{W}_\mu^{as}, \phi_i$. The latter enter only in the gauge fixing terms (2.8). This, together with the WTI (2.7), ensures the equivalence between the physical observables computed in the conventional approach (namely with the 't Hooft gauge fixing) and those computed by the BFM (see for example [24] for further details). Finally, the variation of the gauge fixing with respect to the BRST provides the following Faddeev-Popov terms

$$\mathcal{L}^{\Phi \Pi} = -\bar{c}_a \left[ \delta^{as} \hat{\nabla}_\mu \hat{\nabla}_\nu c_s + \delta^{aa} \delta^{cc} \nabla \nabla \nabla \nabla + \rho^{ab} (\hat{\phi} + \phi) (et)_{ij}^b (et)_{jk}^c (\phi + v)_{ik} + \nabla \nabla \nabla \nabla \nabla + \rho^{ab} \Omega_i (et)_{ij}^b (\phi + v)_{jk} \right]$$

(2.10)

Non-semisimple gauge models, quantized in the 't Hooft-background gauge, are completely determined by the complete set of identities, derived (see App. A) by computing the commutators among STI (2.5), the AAE (2.6) and the WTI (2.7) for simple factors, and by normalization conditions which will be discussed in the next section.

3 Normalization Conditions and Parameterization

Renormalization Scheme

In the next sections we study the free parameters and the normalization conditions according to the following scheme.

1. We derive the solution to the functional equations (2.5), (2.7), (2.6), (A.119), (A.111) in the space of local integrated functionals. We assume that the identities are not spoiled by anomalies and this hypothesis will be verified in the Sec. 4. Furthermore we assume that these identities maintain their tree level form [4, 5, 6]. This implies severe constraints on the renormalizations of the field mixings and on the renormalization of the background fields. As a result we are able to classify the free parameters of the model and to derive the classical action $\Gamma_{Cl}$ including all counterterms.

2. In order to separate the normalization conditions for the unphysical states like ghost fields from the physical ones (masses and couplings of physical particles), it is convenient to start considering the ghost sector. There, we discuss the normalization conditions which prevent IR problems and provide a reasonable definition of the Z-ghost mass. As a consequence we find that the free parameters compatible with the complete set of functional identities are not sufficient to fix the proper normalization conditions. Nevertheless we show that ghost equations (A.118)-(2.6) must be slightly deformed in order to be consistent with the normalization conditions. Therefore the equations of motion for the $b$ fields (NL Eqs. (A.111)) are accordingly modified to higher orders. These deformations do not influence either the STI and the WTI nor the physical amplitudes.

3. By using the renormalized ghost equations (A.118) and the NL Eqs. (A.111) we can simplify the STI for the longitudinal (scalar) gauge bosons and the mixing with scalar fields introducing the reduced functional $\tilde{\Gamma}$. Within this sector we discuss the renormalization of tadpoles, the IR problems (related to the IR anomalies studied in Sec. 4.2) and we discuss the renormalization of the would-be Goldstone masses. As a consequence of the STI we are also able to identify the masses of the Higgs fields and to provide a gauge parameter independent definition for
them. In the sector of scalar fields we have to consider the possible mixing and the CP-odd counterterms and renormalizations.

4. Having fixed the unphysical degrees of freedom and the Higgs masses, we consider the gauge bosons. By means of STI for two-point Green’s functions, we discuss the renormalization of massless and massive gauge bosons and their mixings. We show that by using a minimal set of free parameters compatible with the WTI we are able to avoid completely the IR problems. Furthermore, independently of the mixings and in the case of CP violation we are also able to fix the proper, gauge parameter independent mass renormalizations.

5. For fermion fields we have to select the normalization conditions which must be fulfilled in order to compute gauge invariant amplitudes: the renormalization of the mass and of the CKM mixing matrices. We provide a gauge independent definition of the mass renormalization which is also appropriate in the presence of mixings. Moreover, we define the mass eigenstates and in terms of them we consider the WTI. Finally by using the WTI expressed in the mass eigenstates, we are able to fix the CKM renormalizations.

6. Finally we show that the renormalization Green’s functions with external background fields is related to the renormalization of Green’s functions with external quantum fields. In particular we show that the zeros of two-point functions with external background fields coincide with zeros of two-point functions with external quantum partners. We also show that the constraints coming form the WTI are not sufficient to fix the IR problems for quantum Green’s functions and proper normalization conditions must be used (see Item (4) above).

7. At the end we discuss the gauge coupling renormalization.

3.1 General Solution of the Symmetry Constraints at Tree Level

The unknown quantity of the problem is a Lorentz-invariant CPT-even local functional

\[ \Gamma_{Cl} = \int d^4x L_{Cl} \]

with zero Faddeev-Popov charge. Furthermore the power counting renormalizability requires

\[ d_{UV} \Gamma_{Cl} \leq 4 \]

and

\[ d_{IR} \Gamma_{Cl} \geq 4 \]

and, since we perform the present calculation in the symmetric variables, we derive the most general solution which respect to the UV power counting. In the next sections we show that also the IR constraint can be satisfied by a suitable choice of normalization conditions.

Due to the complexity of the problem it is convenient to establish a precise hierarchy among functional identities, solving the linear constraints at the first step, then solving the WTI and, finally, the STI. This hierarchy shows how the abelian anti-ghost equation (2.6) fixes the abelian couplings among the quantum fields \( \phi^i, \psi^R_i, \psi^L_i \) and their BRST sources \( \gamma_i, \eta^R_i, \eta^L_i \) and the renormalization of the abelian ghost fields \( c^a_A \).

The solution can be expressed in terms of the following separated terms

\[ \Gamma_{Cl} = \Gamma_{Cl}^Q + \Gamma_{Cl}^{S.T.} + \Gamma_{Cl}^\Omega + \Gamma_{Cl}^{g.f.} \]  
(3.11)

\footnote{In the case of the SM we cannot assume the P and the CP invariance because they are broken by the presence of explicit chiral vertices and by the CP-violating phase (and the non-degeneracy of the quark masses) of the CKM matrix.}

\footnote{This functional must not be confused with \( \Gamma_0 \). This new quantity contains all the counterterms needed to impose the normalization conditions \( 8, 9 \).}
where $\Gamma_{CI}^Q$ depends only on quantum fields, $\Gamma_{CI}^{S,T}$ contains the BRST-sources couplings and the w.f.r. of the quantum fields, $\Gamma_{CI}^S$ contains the couplings among the BRST-sources and the BRST variation of the background fields $\Omega^{gs}_\mu$, $\Omega_i$ and $\Gamma_{CI}^{g,f}$ is

$$
\Gamma_{CI}^{g,f} = \int d^4x \left( b_c \left[ \delta^{ca} \tilde{\Omega}^{ab}_{\mu} b_s (W - \hat{W})^{\mu}_{bs} + \delta^{ca} \tilde{\Omega}^{ab}_{\mu} (W - \hat{W})^{\mu}_{aA} \right] + \rho^{cb} (\dot{\phi} + v)^i (\tilde{e})^{b}_{ij} (\dot{\phi} + v)^j + \frac{A^{ca}}{2} b_c b_a - \tilde{c}_a \tilde{c}_a \right) 
$$

contains the gauge fixing terms. Furthermore by the Eqs. (A.118) and Eq. (2.6) the reduced functional $\hat{\Gamma} = \Gamma - \Gamma_{CI}^{g,f}$ depends on the new variables

$$
\hat{\gamma}_{i\mu} = \gamma_{i\mu} + \tilde{\Omega}^{ab}_{\mu} e^{bs}, \quad \hat{\gamma}_i = \gamma_i + \tilde{c}_a \rho_{ab}(\tilde{e})^{b}_{ij} (\dot{\phi} + v)^j, \quad \hat{\Omega}_i = \Omega_i + (\tilde{e})^{aA}_{ij} (\dot{\phi} + v)^j e^{aA}.
$$

As a consequence we derive the renormalization of the gauge fields and the splitting among the quantum parts from the background ones

$$
\hat{W}^{gs}_{\mu} = Z^{W,gs}_{\mu} \left( W^{bs}_{\mu} + X_S \hat{W}^{bs}_{\mu} \right), \quad \hat{\phi}^i = \phi^i + X_0 (\hat{R}_i^j + \hat{R}_i^k - \delta_{ij})^k
$$

where $X_{gsb}, X_{ij}$ are arbitrary non singular and, respectively, real and complex matrices. These are constrained by the WTI which imply (assuming the irreducibility of the scalar field representation)

$$
X_{gsb} = X_S \delta^{gsb}, \quad X_{ij} = X_0 \delta_{ij} + \sum_{aA} X_{aA} t_{ij}^{aA}
$$

where $X_S$ are single free real parameters for each simple factor $G_S$ of the group and $X_0, X_{aA}$ for the scalar representation. For a complex representation (as for the Higgs minimal sector of the SM) $X_0, X_{aA}$ are complex. For this reason there is no counterterms for the mixing between the would-be-Goldstone boson and the Higgs field in the case of the minimal SM where only a restricted Higgs potential is allowed. In the case of reducible representations for the scalar fields (such as in the 2HDM and in the Higgs sector of the MSSM) we have a set of free complex constants $\{X_{aA} \}$ for each irreducible representation $\alpha$. It is convenient to rewrite the $X_{ij}$ in terms of a product of a multiplicative factor times a rotation $X_{ij} = X_0 R_{ij}$. The rotation is generated by the abelian generators $t_{ij}^{aA}$ in the scalar representation. Finally by imposing the STI and the WTI we immediately get

$$
\hat{W}^{gs}_{\mu} = Z^{W,gs}_{\mu} \left( W^{bs}_{\mu} + X_S \hat{W}^{bs}_{\mu} \right), \quad \hat{\phi}^i = \phi^i + X_0 (\hat{R}_i^j + \hat{R}_i^k - \delta_{ij})^k
$$

where $Z^{W,gsb}, Z^{gsb,ij}$ are arbitrary non singular matrices and the $X$ are related to them by the equations

$$
Z^{W}_{gsb} = Z_S^{W} \delta^{gsb}, \quad Z^{gsb,ij} = Z^{gsb,ij}_{0} \delta_{ij} + \sum_{aA} Z^{gsb,ij}_{aA} t_{ij}^{aA} \equiv Z^{gsb,ij}_{0} R_{ij}
$$

As a consequence we derive the renormalization of the gauge fields and the splitting among the quantum parts from the background ones

$$
\hat{W}^{gs}_{\mu} = Z^{W,gs}_{\mu} \left( W^{bs}_{\mu} - \hat{W}^{bs}_{\mu} \right) + \hat{W}^{gs}_{\mu}, \quad \hat{\phi}^i = \phi^i + \hat{\phi}^i.
$$
This implies that the background field part of the gauge fields and of scalar fields \( \hat{W}_\mu^{as} \), \( \bar{\phi}^j \) do not get independent radiative corrections and only the quantum part of these fields is renormalized by a wave function \( Z_S, Z_0 R_{ij} \). By rescaling the gauge fields \( \hat{W}_\mu^{as} \rightarrow e_s W_\mu^{as} \) and their background partners \( \hat{Z}_S^{as} \rightarrow e_s \hat{Z}_S^{as} \) and introducing the wave function renormalization for the background gauge fields \( \hat{Z}_S^{W} \delta^{as} \), the usual relation \( \hat{Z}_S^{as} \) between the charge renormalization and the two-point functions with external background gauge fields is easily recovered.

We introduce the redefined sources \( \tilde{\gamma}_\mu^{as} = Z_S^{W,-1} \gamma^{as} \), \( \tilde{z}_i = Z_\phi^{-1} \delta_i \), the BRST-sources part \( \Gamma_{ST}^{CL} \) is given by

\[
\Gamma_{ST}^{CL} = \int d^4x \left[ \sum_s \tilde{\gamma}_\mu^{as} \nu^{as}_{\mu \nu} \left( Z_{bsd \nu}^{as} c_{ds}^{bs} \right) + \gamma_i Z_{as \nu}^{as} c_{\nu i}^{as} \left( \bar{\phi} + v \right)_i \right. \\
+ \sum_s \zeta_s Z_{as \nu}^{as} c_{\nu i}^{as} \left( Z_{bsd \nu}^{as} c_{ds}^{bs} \right) \right] + \gamma_i Z_{as \nu}^{as} c_{\nu i}^{as} \left( \phi + v \right)_i
\]

where

\[
\nu^{as}_{\mu \nu} = (e_f)_{as \nu \mu} \left( Z_{bsd \nu}^{as} (W - \hat{W})^{bs}_{\mu} + \hat{W}^{as}_{\mu} \right) c_{ds}^{bs}
\]

with an arbitrary constant \( Z_S^C \) for each simple factor.

For the fermions we introduce \( Z_{1J}^{\psi R}, Z_{1J}^{\psi L}, Z_{1J}^{\bar{\psi} R}, Z_{1J}^{\bar{\psi} L} \) and applying the WTI to the fermionic terms we have

\[
Z_{1J}^{\psi R} = Z_{1J}^{\bar{\psi} R} \delta_{1J} + \sum_{aA} Z_{1J}^{\bar{\psi} R, I \alpha} T^{aA}_{\alpha I} \] , \[ Z_{1J}^{\psi L} = Z_{1J}^{\bar{\psi} L} \delta_{1J} + \sum_{aA} Z_{1J}^{\bar{\psi} L, I \alpha} T^{aA}_{\alpha I} \] (3.20)

for each independent multiplet of fermions.

In order to compare these free parameters with the conventional formalism of the SM, we translate the compact notation \( \psi_I, \bar{\psi}_I \) for fermions used here into physical fields

\[
\psi_I^L = \{ Q_{\alpha, a, i}, L_{\alpha, i}^L \} , \quad \psi_I^R = \{ u_{\alpha, a}, d_{\alpha, a}, e_{\alpha}^R \} \] (3.21)

where \( \alpha \) is the flavour index, \( a \) is the colour index and \( i \) is the \( SU(2) \) isospin. In this formalism, we obtain

\[
Z_{1J}^{\psi L} = Z_{\alpha, I, \beta}^{\psi, L, \gamma} \delta_{ij} \delta_{a b} + \sum_{aA} Z_{\alpha, I, \beta, \gamma}^{L, aA} T^{aA}_{\alpha I} \delta_{ij} \delta_{a b} , \] (3.22)

for \( Z_{1J}^{\psi L} \) and analogous equations hold for \( Z_{1J}^{\bar{\psi} R}, Z_{1J}^{\psi, R} \). In the above equation the sum runs over the five abelian conserved currents (see App. A). Notice that without the BFM, and therefore, without imposing the WTI, we can add a further set of parameters \( Z_{\alpha, I, \beta}^{L, aA} \) where \( \sigma^3 \) is the third component of the \( SU(2) \) factor of the gauge group.
Finally the terms $\Gamma^Q_{C_1}$ involving the quantum fields only are (compare with \[3\] and \[10\]):

$$
\Gamma^Q_{C_1} = \int d^4x \left[ -\frac{1}{4} \sum_S F^{aS}_{\mu\nu} F^{aS}_{\mu \nu} - \frac{1}{4} F^{aA}_{\mu\nu} F^{aA}_{\mu \nu} + \nabla^\mu (\bar{\phi} + v)^i \nabla^\nu (\phi + v) + \mu_{ij}(\bar{\phi} + v)^i (\phi + v)^j + \lambda_{ijkl} (\bar{\phi} + v)^i (\phi + v)^j (\bar{\phi} + v)^k (\phi + v)^l + \frac{\tilde{\omega}^R_{iJJ}}{\gamma^\mu} \nabla^R_{\mu} \bar{\psi}_J + \frac{\tilde{\omega}^L_{iJJ}}{\gamma^\mu} \nabla^L_{\mu} \bar{\psi}_J + \frac{\tilde{\omega}^{R^*}_{iJJ}}{\gamma^\mu} \psi^R_J + \frac{\tilde{\omega}^{L^*}_{iJJ}}{\gamma^\mu} \psi^L_J \right] (\phi + v) + \text{h.c.} 
$$

(3.23)

where $F^{aS}_{\mu\nu}$ is the field strength tensor for the non-abelian gauge fields $\tilde{W}^{aS}_\mu$ and $F^{aA}_{\mu\nu}$ for the abelian ones $W^{aA}_\mu$. $\bar{\psi}_J$ are the rescaled fermions $\bar{\psi}_J = Z^L_{JK} \psi^L_J$. The covariant derivatives are given by

$$
\nabla^\mu_{ij} \phi^j = \partial^\mu \phi_i + \left[ (et)^{aS}_{ij} \left( Z^W_s (W - \tilde{W})^a_s + \tilde{W}_s^a \right) + (et)^{aA}_{ij} W^{aA}_\mu \right] \phi^j 
$$

and analogously for the left-handed fermions.

The remaining free parameters (in the gauge sector) are the gauge couplings $e_S$ for each simple factor and the gauge couplings $e_{a_A b_A}$ for the abelian factors. Notice that for several abelian gauge fields the kinetic terms of the action could mix the gauge fields. For the SM, for a single $U(1)$ factor, this is obviously excluded. A complete discussion on the free abelian gauge fields and abelian factors is given in paper [11].

The free invariant parameters $\mu_{ij}, \lambda_{ijkl}$ are the quadratic and the quartic couplings for the scalar fields, respectively. They satisfy the following algebraic relations [9]

$$
(\text{et})^a_{ik} \mu_{kj} + \mu_{ik} (\text{et})^a_{kj} = 0 
$$

(3.24)

$$
(\text{et})^a_{im} \lambda_{ijkl} + (\text{et})^a_{jm} \lambda_{imkl} + (\text{et})^a_{km} \lambda_{ijml} + (\text{et})^a_{lm} \lambda_{ijkm} = 0.
$$

(3.25)

The number of these free parameters depends on the number of scalar multiplets. For the minimal SM the number of free parameters are just one quadratic term ($\mu$) and one quartic coupling ($\lambda$), however for the non-minimal SM such as the 2HDM the number of invariant scalar terms increases.

Concerning the CP-violating counterterms we find the following result: without the background gauge invariance the CP-violation permits the existence of non-vanishing CP-odd Green’s functions –for instance the mixing between the would-be-Goldstone boson and the Higgs– and the STI allows for CP-violating counterterms which fix these new divergences. By using the BFM in the case of CP-violation induced only by fermion couplings, these CP-odd Green’s functions are automatically fixed by the background gauge invariance. For extended models, where the Higgs potential violates the CP symmetry, the background gauge invariance is not sufficient to fix all these CP-odd couplings and new symmetries must be invoked.

In order to respect the symmetry constraints the Yukawa coupling $Y^{IJ}_i$ have to satisfy the algebraic constraints [9]

$$
(\text{et})^a_{iK} Y^{KJ}_i + Y^{IK}_i (\text{et})^L_{KJ} + (\text{et})^a_{ij} Y^{IK}_J = 0
$$

(3.26)

and this restricts further the number of free parameters.

Notice that the covariant kinetic terms for fermions and scalars have no free parameter, beyond the w.f.r., in contrast to the kinetic terms for the gauge fields or mass terms for the scalar fields. This is due to the linear dependence among these terms and the invariant counterterms of the form

$$
S_0 \int d^4x \gamma_1 \bar{\phi} + v \phi, \quad S_0 \int d^4x (\bar{\eta}_I \psi_J + \psi_I \bar{\eta}_J)
$$

(3.27)
The analysis of this linear dependence it is essential to separate the unphysical parameters such as the w.f.r. and the physical CKM matrix elements.

The Yukawa matrices for the SM in the basis \( [3,2] \) are given by (the indices of the representation of \( SU(2) \) for scalar fields and fermions are the same because they live in the same representation)

\[
Y_{iJK} = \begin{pmatrix}
Y_{Qu,i}^{(a,a,j)} & Y_{Qd,i}^{(a,a,j)} & Y_{Qe,i}^{(a,a,j)} \\
Y_{Lu,i}^{(a,j)} & Y_{Ld,i}^{(a,j)} & Y_{Qc,i}^{(a,j)}
\end{pmatrix}
\]

and by the covariance with respect to \( SU(3) \) gauge symmetry we have \( Y_{Qu,i}^{(a,a,j)} = Y_{Lu,i}^{(a,j)} = Y_{Ld,i}^{(a,j)} = Y_{Qc,i}^{(a,j)} \) and \( Y_{Qu,i}^{(a,a,j)} = Y_{Qd,i}^{(a,a,j)} = Y_{Ld,i}^{(a,j)} = Y_{Qc,i}^{(a,j)} \)\( \delta_{ab} \). The covariance with respect to the \( SU(2) \) is more delicate and the instabilities of the fermion representation of the abelian factor \( U(1) \) have to be taken into account.

### 3.2 Ghost Sector

This section deals with the normalization conditions for the ghost fields and it is divided into two parts. In the first part we discuss the normalization conditions on two-point functions with a special emphasis on the renormalization of the mixings \( \bar{c}^A - c^Z, \bar{c}^2 - c^A \). In the second part we analyze the Faddeev-Popov equations \((A.118)\) and their deformations. In particular we show that the number of free parameters \( (Z_2^A, \bar{Z}_{A,bA}^c) \), constrained by Faddeev-Popov equations \((A.118)\) and by the AAE \((2.6)\) is not enough to fix the two-point functions in order to avoid the IR divergences. We discuss the minimal extension of the set of parameters, namely the introduction of the w.f.r. for the antighost fields \( \bar{Z}_S^c, \bar{Z}_{A,bA}^c \) (related to the matrix \( G_{ab} \) discussed in Sec. \( 2.2 \)), which respects the STI and the WTI and the corresponding normalization conditions.

The ghost masses are independent parameters of the theory and their renormalization can be achieved by adjusting the independent ‘t Hooft parameters \( \rho^S, \rho^{A,bA} \). For practical calculations, in order to avoid the double poles, it is advantageous to set the ghost masses \( m^{Gh} \) equal to the masses of the would-be-Goldstone bosons (restricted ‘t Hooft gauge fixing). That corresponds, at tree level, to setting all the ‘t Hooft parameters \( \rho^S, \rho^{A,bA} \) equal to the gauge parameters \( \xi^{aS}, \xi^{A,bA} \). However, at higher orders, this degeneracy cannot be maintained since, to exclude IR divergences, the ‘t Hooft parameters must be used to enforce

\[
\Gamma_{e^A e^A}(p) \bigg|_{p^2 = 0} = 0 \quad \Gamma_{e^A e^Z}(p) \bigg|_{p^2 = 0} = 0 \quad \Gamma_{e^Z e^A}(p) \bigg|_{p^2 = 0} = 0 \quad (3.29)
\]

as normalization conditions. Notice that the ghosts \( (e^A_1, e^Z_2) \) are the massless ghosts and \( e^{A''}_2 \) are the massive ones) and the antighost fields \( (\bar{e}^A_1, \bar{e}^Z_2) \) have independent degrees of freedom and the two-point function matrix is non-symmetric. As a consequence, unlike the gauge bosons, we must nullify both the mixed two-point functions. To this end, we firstly check that we can impose such normalization conditions by tuning the free parameters \( \rho_{A,bA}, \rho_S, Z_2^A \) and the w.f.r. for antighost fields \( \bar{Z}_{A,bA}^c, Z_2^A \).

We consider the two-point functions matrix

\[
\begin{pmatrix}
\Gamma_{e^A e^A}(p) & \Gamma_{e^A e^Z}(p) \\
\Gamma_{e^Z e^A}(p) & \Gamma_{e^Z e^Z}(p)
\end{pmatrix}
\]

\( (3.30) \)
Furthermore, we consider tree level parameters appearing in the action $\Gamma^0$, namely $\bar{Z}_c^c, Z_{aAbA}^c$, $Z_c^c, Z_{aAbA}^c$ and setting $m_{aSbS}^G \equiv \rho_S e_S^2 v^i_{as} v^j_{bs} k_v$, and analogously for $m_{aGbA}^G, m_{aGbS}^G, m_{aGbA}^G$, the first equation of (3.29) is rewritten in the form

\[
\begin{aligned}
\bar{R}_{aSb'} \left[ Z_c^c m_{aSbS}^G Z_c^c + \bar{R}_{a} \left( Z_{aAbA}^c m_{cAbA}^G Z_{aAbA}^c + \Sigma_{c,a}^{(n)}(0) \right) \right] = 0,
\end{aligned}
\]

where the ghost fields corresponding to the gauge eigenstates are rescaled. Equivalent equations are derived for $\Gamma_{a'c',c',a}^c(0)$ substituting the matrices $\bar{R}_{aSb'}$, $\bar{R}_{a}^{a'}$ with $\bar{R}_{aSb'}$, $\bar{R}_{a}^{a''}$ (where the index $a''$ runs over the set of massive ghost fields) and $\bar{R}_{bSb'}, \bar{R}_{b}^{a} \bar{R}_{bSb'}, \bar{R}_{b}^{b'}$. In Eq. (3.31), $\Sigma_{c,a}^{(n)}(0)$ is the $n$-loop contribution to the two-point function $\Gamma_{a'c',a}^c(0)$ computed in the BPHZL subtraction scheme.\footnote{In the present section we always use the notation $\Gamma^{(1)} \equiv \Gamma_{a}^{(1)}$ to indicate the tree level action, $\Sigma^{(n)}$ to denote the $n$-loop correction which satisfy the normalization conditions up to $n - 1$ order. The field $\phi(p)$ indicates the Fourier transform of the field with incoming momentum $p$ and the momentum conservation is understood.}

The determinant of the system which corresponds to the normalization conditions (3.29) vanishes. This is due to the presence of massless particles and it is ensured by the STI. Therefore some equations are redundant and, for instance, $\Gamma_{a'c',a}^c(0)$ can be discarded. Nevertheless the system is still solvable in terms of w.f.r. $\bar{Z}_c^c, Z_{aAbA}^c, Z_c^c$ and in terms of the 't Hooft parameters. On the other side $Z_{aAbA}^c$ is fixed by the WTI and the AAE to the charge renormalization $\epsilon_{aAbA}^c$. As an example we consider the system $c^A - c^Z$ of the SM, and we impose that $\bar{R}$ and $\bar{R}$ coincide with the tree level rotations, namely the Weinberg angle $\theta_W$. The latter follows from our request of preserving the STI and the WTI in their tree level form in the non-minimal sector. In fact, we use the common identifications\footnote{The symbols $s_w, c_w$ denote the sine and cosine of the Weinberg’s angle.}

\[
\begin{aligned}
\bar{R}_3 A = \bar{R}_3 A = s_w, & \quad \bar{R}_3 Z = \bar{R}_3 Z = c_w, \\
\bar{R}_0 A = \bar{R}_0 A = c_w, & \quad \bar{R}_0 Z = \bar{R}_0 Z = -s_w,
\end{aligned}
\]

for the rotations and

\[
\begin{aligned}
\bar{m}_{33}^{33} &= \bar{Z}_c^c Z_c^c Z_c^c = Z_c^c Z_c^c \rho_2 g_2 v^2, & \quad \bar{m}_{30}^{30} &= \bar{Z}_c^c Z_c^c \rho_2 g_2 v^2, \\
\bar{m}_{30}^{03} &= \bar{Z}_c^c Z_c^c \rho_2 g_2 v^2, & \quad \bar{m}_{10}^{00} &= \bar{Z}_c^c \rho_1 g_1 v^2
\end{aligned}
\]

for the renormalized mass parameters. There we put $\bar{Z}_c^c \equiv \bar{Z}_{SU(2)}^c, \bar{Z}_c^c \equiv \bar{Z}_{U(1)}^c$ for the antighost and $Z_c^c \equiv Z_{SU(2)}^c, Z_c^c \equiv Z_{U(1)}^c$ for the ghosts; we have also used the parameters $\rho_2 \equiv \rho_{SU(2)}^c, \rho_1 \equiv \rho_{U(1)}$ and the couplings $g_2 \equiv g_{SU(2)}, g_1 \equiv g_{U(1)}$. Therefore in the neutral ghost sector of the SM, we have five real parameters. Indeed, $Z_1$ is fixed by the renormalization of the abelian gauge fields and, by means of the AAE (2.4), coincides with the renormalization of the $U(1)$ gauge coupling.
Then introducing this definition into the equation (3.31) we finally obtain the system

\[ \begin{align*}
\tilde{m}_{Gh}^{03} &= -\gamma_w^2 \Sigma_{\bar{c}acZ}(0) + c_W^2 \Sigma_{\bar{c}ZcA}(0) - s_W c_W (\Sigma_{\bar{c}ZcZ}(0) - \Sigma_{\bar{c}Aac}(0)) \\
\tilde{m}_{\bar{c}h}^{30} &= s_W^2 \Sigma_{\bar{c}ZcA}(0) - c_W^2 \Sigma_{\bar{c}Aac}(0) - s_W c_W (\Sigma_{\bar{c}ZcZ}(0) - \Sigma_{\bar{c}Aac}(0))
\end{align*} \]  

(3.34)

which can be solved in terms of two parameters. One of the remaining parameters is used to impose

\[ \lim_{p^2 \to 0} -i \frac{p_\mu}{p^2} \Gamma_{\gamma^A_{\mu}cA}(p) = s_W \]  

(3.35)

where \( \gamma^\mu_{\bar{c}} \) is the BRST source for the third component of \( SU(2) \) triplet of gauge bosons, which is essential to guarantee the absence of IR anomalies (see Sec. 4.2). The remaining parameters are needed to have a consistent renormalization of the Eqs. (A.118) as we will be shown below, and to fix the mass of the \( Z \)-ghost. Therefore there is no free parameter to fix also the mass of the \( W \)-ghost as also discussed in \([16]\). To fix those masses independently the WTI must be deformed. On the other hand, the degeneracy among the ghost masses and masses of the system of Goldstones and scalar components of gauge bosons is ensured by the STI.

In conclusion, this analysis shows that for the SM the free parameters, constrained by the equation of motion for the ghost fields (Faddeev-Popov equations and AAE), are not enough to avoid the IR problems and to fix the \( Z \)-ghost mass. If a new abelian factor is added and the new gauge boson has to be massless, the WTI and the STI are necessarily modified\(^{12}\).

At this point we are able to study the Faddeev-Popov equations and the renormalization of the Green’s functions which contain the BRST sources. In particular, by the Faddeev-Popov equations we have the following relations among two-point functions

\[ \Gamma_{\psi_{\mu}e_{\mu}A}(p) = -\delta^{\mu}_{\alpha} p + \rho^{\alpha A}_{\mu} v_i (et)_{ij} \Gamma_{\gamma_{\mu}e_{\mu}A}(p^2) \]

\[ \Gamma_{\psi_{\mu}e_{\mu}S}(p) = \rho^{\alpha A}_{\mu} v_i (et)_{ij} \Gamma_{\gamma_{\mu}e_{\mu}S}(p^2) \]

\[ \Gamma_{\psi_{\mu}e_{\mu}A}(p) = -i p_\mu \Gamma_{\gamma_{\mu}e_{\mu}A}(p^2) + \rho_S v_i (et)_{ij} \Gamma_{\gamma_{\mu}e_{\mu}A}(p^2) \]

\[ \Gamma_{\psi_{\mu}e_{\mu}S}(p) = -i p_\mu \Gamma_{\gamma_{\mu}e_{\mu}S}(p^2) + \rho_S v_i (et)_{ij} \Gamma_{\gamma_{\mu}e_{\mu}S}(p^2) \]  

(3.36)

The UV dimensions of BRST-sources \( \gamma_{\mu}^{\psi_{\mu}e_{\mu}A} \) and the Lorentz invariance imply the superficial divergence of two-point functions \( \Gamma_{\gamma_{\mu}e_{\mu}A}, \Gamma_{\gamma_{\mu}e_{\mu}S} \) \( \Gamma_{\gamma_{\mu}e_{\mu}A}, \Gamma_{\gamma_{\mu}e_{\mu}S} \). They are renormalized by the w.f.r.s \( \bar{Z}^c_{aA}, \bar{Z}^c_{aS}, Z^c_S \) and the mass renormalization of ghost fields. Furthermore, in order to clarify the relation among the kinetic terms of ghost two-point functions and the \( \Gamma_{\gamma_{\mu}e_{\mu}S} \), we differentiate the forth equation in the system (3.36) with respect to momentum \( p \) and we consider the limit \( p \to \infty \):

\[ \lim_{p \to \infty} \partial_{p^2} \Gamma_{\psi_{\mu}e_{\mu}A}(p^2) = \lim_{p \to \infty} \left[ i \partial_{p^2} \left( p_\mu \Gamma_{\gamma_{\mu}e_{\mu}A}(p^2) \right) + \rho_S v_i (et)_{ij} \partial_{p^2} \Gamma_{\gamma_{\mu}e_{\mu}S}(p^2) \right] = \lim_{p \to \infty} \left( 2i \partial_{p^2} + p_\mu \partial_{p^2} \right) \Gamma_{\gamma_{\mu}e_{\mu}A}(p^2) \]  

(3.37)

The Weinberg’s theorem \([12]\) is used to discard the convergent two-point functions. On the other hand, because of the AAE Eq. (2.6), the Green’s functions \( \Gamma_{\gamma_{\mu}e_{\mu}A}(p) \) are superficially finite to all orders and no normalization is required. This can be easily shown by differentiating Eq. (2.6) with respect to the sources \( \gamma_{\mu}^{\psi_{\mu}e_{\mu}}(p) \)

\[ \partial_{p_\mu} \Gamma_{\gamma_{\mu}e_{\mu}A}(p) + v_i (et)_{ij} \partial_{p_\mu} \Gamma_{\gamma_{\mu}e_{\mu}A}(p^2) = 0 \]  

(3.38)

\(^{12}\)In that case new w.f.r. \( Z^c \) for ghost fields are needed to impose the normalization conditions.
and observing that $\partial_{\mu} \Gamma_{\gamma_a}^{\alpha_a \rho_1}(p^2)$ is superficially convergent by power counting. We conclude that also $\Gamma_{\gamma_a}^{\alpha_a \rho_1}(p)$ are superficially convergent to all orders. As a consequence, the w.f.r. of abelian ghost fields $Z_{\alpha a b A}$ is not independent of the w.f.r. of the abelian gauge fields.

Finally to check the consistency among the Eq. (3.36) and the normalization conditions (3.29), we insert the expressions (3.36), computed at zero momentum, in (3.31). However, since the system has sensible solutions only for specific models, we concentrate on the $c^A, c^2$ system of the SM. From identities (3.36), by using the identifications (3.32-3.33) we obtain the following system

$$
\begin{align*}
\Gamma_{c^A, c^A}(p) &= -p^2 (c_W^2 + s_W \Gamma_{c^A c^A}) + (s_W Z_2^2 g_2 p_2 + c_W Z_1^1 g_1 \rho_1) \Gamma_{c^A c^A} \\
\Gamma_{c^A, c^2}(p) &= -p^2 (c_W s_W - s_W \Gamma_{c^A c^2}) + (s_W Z_2^2 g_2 p_2 + c_W Z_1^1 g_1 \rho_1) \Gamma_{c^A c^2} \\
\Gamma_{c^2, c^4}(p) &= -p^2 (s_W c_W - c_W \Gamma_{c^2 c^4}) - (c_W Z_1^1 g_1 \rho_1 - s_W Z_2^2 g_2 p_2) \Gamma_{c^2 c^4}
\end{align*}
$$

(3.39)

where $\gamma_0$ is the BRST source coupled to the BRST variation of the neutral Goldstone boson $G_0$. The vanishing of the terms on the l.h.s. at zero momentum implies $\Gamma_{c^A c^A}(0) = 0$ and $(c_W Z_2^2 g_2 p_2 + s_W Z_1^1 g_1 \rho_1) = 0$. The latter imposes a relation among the ’t Hooft parameters and the w.f.r. which reduces the number of free parameters. Furthermore those relations imply that also the first Eq. (3.39) is automatically satisfied showing that $\Gamma_{c^A c^A}(0) = 0$. This is a consequence of the STI (which can also be checked from (3.36)) and the normalization conditions (3.29) are compatible with the Faddeev-Popov equations. The resulting condition $\Gamma_{\gamma_0 c^A}(0) = 0$ turns out to be useful to exclude the IR anomalies.

Clearly the Faddeev-Popov Eqs. (A.118) are deformed by the mass renormalization of the mass of $c^2$ and by the conditions (3.29). This means that the parameters $\rho_1, \rho_2$ in Eqs. (3.39) do not coincide with to tree level parameters. Moreover, the constraints coming from the STI and the WTI are not affected by these deformations.

### 3.3 Scalar Sector

The unitarity of the S-matrix relies on the quartet mechanism [3, 33]; that is, the scalar degrees of freedom of gauge bosons and their Goldstone partners must be degenerate with the system of ghosts and anti-ghosts. Therefore, it must be implemented at higher orders, and this requires a proper renormalization of tadpoles, of the Nakamichi-Lautrup Eqs. (A.111) and, finally, of the STI.

First, we recall that a generic subtraction scheme (for instance the MS scheme for the Dimensional Regularization) shifts the tadpole contribution to Green’s functions. Consequently, by adjusting the vacuum expectation value $v_i$ and the free parameters $\mu_{ij}, \lambda_{ijkl}$, defined in (3.24), the tadpole normalization conditions

$$
\delta \Gamma^{(n)} \bigg|_{\phi=0} = \left( \mu_{ij} v_i + \frac{1}{6} \lambda_{ijkl} v_j^i v_k^l \right)^{(n)} + \Sigma_{\phi_i}^{(n)}(0) = 0
$$

(3.40)

must be imposed to all orders. On the r.h.s., $\Sigma_{\phi_i}^{(n)}(0)$ denotes n-loop contribution to tadpoles.

Notice that the normalization condition (3.40) involves both the tadpoles of physical Higgs fields ($H_i'$ where $i' = 1 \ldots N_{Higgs}$) and the tadpoles of unphysical Goldstone fields ($G_{i''}$ with $i'' = 1 \ldots N_{Goldstone}$) [31]. In fact, although the latter are absent at tree level, due to the CP invariance of the scalar terms in the action (3.23), they are shifted by CP-odd radiative corrections involving the CKM couplings. Furthermore notice that for extended models there is a flat direction of the Higgs
potential ensured by the following STI
\[ \Gamma_c^{\alpha\gamma}(0) \Gamma\phi_i(0) = 0 \quad \text{(3.41)} \]
where \( c_2^{\alpha\gamma} \) are the neutral massive ghost fields. For the SM, neglecting CP violation, the above equation implies that \( \Gamma_{G_0}(0) = 0 \) where \( G_0 \) is the neutral would-be-Goldstone boson. For the SM, considering the CP violation, we have that \( \Gamma_{H}(0) = 0 \) implies \( \Gamma_{G_0}(0) = 0 \). And finally, for extended model, by a suitable rotation among the scalar fields, one tadpole is automatically nullified.

For two-point functions we have to consider the complete matrix
\[ \mathcal{M}_{\alpha\beta}^{L}(p) = \begin{pmatrix} \Gamma_{W^\alpha A W^b A}(p) & \Gamma_{W^\alpha A W^b S}(p) & \Gamma_{W^a A \phi_j}(p) \\ \Gamma_{W^a S W^b A}(p) & \Gamma_{W^a S W^b S}(p) & \Gamma_{W^a S \phi_j}(p) \\ \Gamma_{\phi_i W^b A}(p) & \Gamma_{\phi_i W^b S}(p) & \Gamma_{\phi_i \phi_j}(p) \end{pmatrix} \quad \text{(3.42)} \]
where \( \Gamma_{W^\alpha W^b}(p) \) are the longitudinal part of two-point functions for the gauge bosons,
\[ \Gamma_{W^\alpha W^b}(p) = \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Gamma_{W^\alpha W^b}(p) + \frac{p_\mu p_\nu}{p^2} \Gamma_{W^\alpha W^b}(p); \]
in the same way
\[ \Gamma_{W^a \phi_i}(p) = -i p_\mu \Gamma_{W^a \phi_i}(p) \]
are the mixed two-point functions for gauge bosons and scalars and \( \Gamma_{\phi_i \phi_j}(p) \) are the two-point functions matrix for scalars.

In Sec. 2 the gauge fixing was given in terms of the \( b_a \) fields. However, for practical computation these fields are commonly integrated out and the resulting tree level contribution to mixed gauge -scalar Green’s functions and for scalar two-point functions of \( \text{(3.42)} \) are
\[ \Gamma_{W^a S \phi_j}(p) = \left( 1 - \frac{\rho_S}{\xi_S} \right) v_k t^a_{k_i} v_i, \quad \Gamma_{W^a A \phi_j}(p) = \left( 1 - \frac{1}{\xi_{a,A b A}} \rho_{b A C A} \right) v_k t^A_{k_i} \]
\[ \Gamma_{\phi_i \phi_j}(0) = \mu_{ij} + \frac{1}{2} \lambda_{ijkl} v_k v_l + \frac{p^2}{\xi_S} v_k t^a_{k_i} t^a_{j l} v_i + \rho_{a A b A} \frac{1}{\xi_{b A C A}} \rho_{c A d A} v_k t^a_{k_i} t^a_{j l} v_l \]
The masses of the Goldstone \( m^i_G \) correspond to the expression in the brackets of \( \Gamma_{\phi_i \phi_j}(0) \). Notice that, since the relations among the Goldstone masses and gauge parameters are invertible, the Goldstone masses can be read as gauge parameters. The restricted ‘t Hooft gauge fixing corresponds to setting the masses of the Goldstones equal to the masses of the massive ghosts. Clearly this is not true for the gauge parameters of massless gauge fields.

Another very convenient choice is the ‘t Hooft-Feynman gauge fixing \( (R_\xi \, \text{gauge}) \) which corresponds to identify the masses of Goldstone scalars with the masses of ghosts and with the masses of the gauge bosons. This feature can not be achieved in the BFM because the background gauge invariance implies the degeneracy among the gauge parameters: only a single gauge parameter \( \xi_S, \rho_S \) can be introduced for each simple factor.

In conclusion the Nakanishi-Lautrup Eqs. (3.111) cannot be maintained in their tree level form and the parameters which appear in those equations are fixed on the masses of the Goldstone \((3.43)\) and by the normalization conditions (3.29) for the ghost fields.\footnote{Due to the IR power counting and the zero momentum configuration this STI cannot be spoiled by breaking terms due to the regularization scheme.}
Besides the normalization of the Goldstone masses, we have to verify that the equations
\[
\lim_{p \to 0} \frac{-ip\mu}{p^2} \Gamma_{A_{\mu}^\nu \phi_i}(p) = \Gamma_{A_{\mu}^\nu \phi_i}(0) = 0,
\]
where \(a'\) runs over the set of massless gauge fields and \(i\) over the set of scalar fields, are satisfied. This conditions are valid at tree level, ensuring that the massless gauge fields remain massless after fixing the gauge and spontaneous symmetry breaking. However they must also be maintained at higher orders to guarantee the absence of IR anomalies for the STI.

In particular for the minimal SM the equation
\[
\lim_{p \to 0} \frac{-ip\mu}{p^2} \Gamma_{A_{\mu}^\nu G_\phi}(p) \equiv \Gamma_{AG_\phi}(0) = 0
\]
expresses that the spontaneous symmetry breaking mechanism and the gauge fixing does not generate a mass term for the photon field. However at the quantum level this conditions must be ensured by the choice of the gauge fixing, by the STI and eventually by using the WTI of the BFM. In particular we are able to prove it as a non-trivial consequence of WTI, of STI and of the normalization conditions for transverse component of the two-point functions for the gauge bosons. In the case of CP violation, this is also related to the mixing between the Goldstone field and the Higgs as it will be shown below.

On the other hand, the equation
\[
\lim_{p \to 0} \frac{-ip\mu}{p^2} \Gamma_{A_{\mu}^\nu H}(p) \equiv \Gamma_{AH}(0) = 0,
\]
where \(H\) is the Higgs of the minimal SM, is not relevant for IR anomaly cancellations, but we are able to prove it as a consequence of normalization conditions, WTI and STI. Notice that, at tree level there is no bilinear coupling between the Higgs field and the longitudinal degree of freedom of the photon because of the CP invariance of the Lagrangian of the scalar and gauge sector. However, at the quantum level, the CP violating radiative corrections might generate that Green’s function.

Finally, we can study the definition of the Higgs masses. These are the only physical parameters entering in the present sector. In absence of CP-violation the two sectors, namely the Higgs fields and the unphysical scalar sector can be analyzed separately. However, the CP-violation introduces CP-odd radiative corrections which mix the scalar fields. For instance for the minimal SM we must consider the complete matrix
\[
\left( \begin{array}{cc}
\Gamma_{G_\phi G_\phi}(p) & \Gamma_{G_\phi H}(p) \\
\Gamma_{HG_\phi}(p) & \Gamma_{HH}(p)
\end{array} \right)
\]
where the mixings \(\Gamma_{G_\phi H}\) are different from zero only starting from three loops.

The aim of this paragraph is to show that, despite several mixings among the unphysical fields and the Higgses, it is always possible to define the Higgs masses. To this purpose we use the STI\(^{\text{14}}\)
\[
\hat{\Gamma}^L_{V^a_{\mu} A_{\nu}^b} + \hat{\Gamma}^L_{V^a_{\mu} A_{\nu}^b} + \hat{\Gamma}^L_{V^c_{\mu} V^b_{\nu} A} + \hat{\Gamma}^L_{V^c_{\mu} V^b_{\nu} A} = 0
\]
\[
\hat{\Gamma}^L_{V^a_{\mu} A_{\nu}^b} + \hat{\Gamma}^L_{V^a_{\mu} A_{\nu}^b} + \hat{\Gamma}^L_{V^c_{\mu} V^b_{\nu} A} + \hat{\Gamma}^L_{V^c_{\mu} V^b_{\nu} A} = 0
\]
\[
-\mu^2 \left( \hat{\Gamma}_{V^a_{\mu} A_{\nu}^b} \phi_j + \hat{\Gamma}_{V^a_{\mu} A_{\nu}^b} \phi_j \right) + \hat{\Gamma}_{V^a_{\mu} A_{\nu}^b} \phi_j = 0
\]
\[
\hat{\Gamma}^L_{V^a_{\mu} A_{\nu}^b} + \hat{\Gamma}^L_{V^a_{\mu} A_{\nu}^b} + \hat{\Gamma}^L_{V^c_{\mu} V^b_{\nu} A} + \hat{\Gamma}^L_{V^c_{\mu} V^b_{\nu} A} = 0
\]
\[
-\mu^2 \left( \hat{\Gamma}_{V^a_{\mu} A_{\nu}^b} \phi_j + \hat{\Gamma}_{V^a_{\mu} A_{\nu}^b} \phi_j \right) + \hat{\Gamma}_{V^a_{\mu} A_{\nu}^b} \phi_j = 0
\]
\(^{14}\)The IR anomalies of this equation will be discussed in the Sec. \(\text{[1]}\)
where the reduced functional $\hat{\Gamma}$ defined by $\hat{\Gamma} \equiv \Gamma - \Gamma^{g.f.}$ replace $\Gamma$; to define $\Gamma^{g.f.}$ at the quantum level the Nakanishi-Lautrup (A.111) and the Faddeev-Popov equations (A.118) are used. Here $\Gamma^{g.f.}$ is defined by the renormalized NL Eqs. (A.111) and by the renormalized ghost equations (A.118)-(2.6).

By means of these identities it is easy to prove that

$$\text{Rank} \left( M^2(p) \right) = \text{Rank} \left( \hat{\Gamma}_{\phi_i \phi_j}(p) \right).$$

This means that non-trivial physical information is only contained in the scalar sub-matrix $\hat{\Gamma}_{\phi_i \phi_j}(p)$.

Furthermore by using the STI (3.51)-(3.53) computed at zero momentum (static STI) it is easy to show that

$$\text{Rank} \left( \hat{\Gamma}_{\phi_i \phi_j}(0) \right) = \text{Rank} \left( \hat{\Gamma}_{H_i' H_j'}(0) \right)$$

(3.54)

where $i', j'$ run over the only set of physical scalar fields.

For the minimal SM, where only the Higgs $H$ and the neutral would-be-Goldstone $G_0$ are involved, the Eq. (3.54) coincides with

$$\hat{\Gamma}_{G_0 G_0}(0) \hat{\Gamma}_{H H}(0) - \hat{\Gamma}_{G_0 H}^2(0) = 0$$

(3.55)

and, therefore, $\hat{\Gamma}_{G_0 G_0}(0) = 0$ if $\hat{\Gamma}_{G_0 H}(0) = 0$. This last condition is needed to cancel the IR anomalies of the STI and it is a consequence of the WTI. This will be shown in Sec. 4.2. Notice that this particular normalization condition states that the would-be-Goldstone bosons and the Higgs are decoupled on the mass-shell of the unphysical state. As a consequence this confirms that the Higgs field is the physical field of the present sector. In the case of extended models similar normalization conditions must be established in order to prevent the theory from IR anomalies.

Finally we have only to fix the normalization conditions for the non-trivial eigenvalues $\lambda^{(n)}_{ij}(p), i' = 1, \ldots, N_{Higgs}$. The complex zeros of the eigenvalues $\lambda^{(n)}_{ij}(p)$, namely $\lambda^{(n)}_{ij}(p^*_{ij}) = 0$, can be identified with the masses $M_{H_{i'}}$ and the width $\Gamma_{H_{i'}}$ of the physical Higgses. Clearly these normalization conditions fix partially the free parameters $\mu_{ij}, \lambda_{ijkl}$.

As in the case of fermions (see next sections) two or more scalar doublets (e.g. in the case of the 2HDM [18] and of the MSSM [17]) can generate physical mixing angles whose renormalization must be discussed. The normalization conditions for the masses – by comparing the zeros of the eigenvalues with the physical masses – is independent of the renormalization of the mixing angles guaranteeing that the pole mass definitions can be always achieved.

For the renormalization of the mixing angle we use again the WTI for the background gauge invariance. There, indeed, the mixing angles appear as constant parameters and, on the basis of the mass eigenstates they can be identified with the renormalized mixing angles.

To be more precise we consider the WTI, neglecting gauge, fermion and ghost terms,

$$W_a(\hat{\Gamma}) = (\text{gauge} - \text{fermion} - \text{ghost terms}) + (et)_{ij}^a \left[ (\phi + v)_j \frac{\delta \hat{\Gamma}}{\delta \phi^i} + (\phi + v)_j \frac{\delta \hat{\Gamma}}{\delta \phi^j} \right] = 0$$

(3.56)

and we define the mass eigenvectors by requiring

$$\hat{\Gamma}_{\phi_i \phi_j}(p) u_j^{(i)} \bigg|_{p^2 = (p^*)^2} = 0$$

(3.57)

15For the SM $i' = j' = H$, i.e. the Higgs field, for the 2HDM [18], $i', j' = H, h, A, H^{\pm}$, where $H$ is the Higgs, $h$ is the second neutral scalar field, $A$ is a pseudo-scalar (CP-odd) field and $H^{\pm}$ are the charged physical scalar fields
where \( \hat{\Gamma}_{\phi_i \phi_j}(p) \) are the two-point functions for scalar fields including all the counter terms computed in the Sec. 3.1. In the case of the Goldstone fields \( p_i^* = 0 \). As a consequence, the eigenvectors \( u_j^{(i)} \) are functions of the free parameters of the Higgs potential \( \mu_{ij} \) and \( \lambda_{ijkl} \) and of the allowed w.f.r. \( Z_0^\phi, Z_{\phi_A}^\phi \) of Eq. (3.10).

Finally we can rewrite the WTI (3.56) in the mass eigenstates \( \phi'_i = \sum_j u_j^{(i)} \phi_i \) and correspondently for the background fields \( \phi_i = \sum_j u_j^{(i)} \phi_i \)

\[
W_a(\Gamma) = (\text{gauge} - \text{fermion} - \text{ghost terms}) + T_{ij}^a \left[ (\phi' + v')_j \frac{\delta \hat{\Gamma}}{\delta \phi'^{,i}} + (\phi'^* + v'^*)_j \frac{\delta \hat{\Gamma}}{\delta \phi'^{,i}} \right] = 0 \tag{3.58}
\]

where \( T_{ij}^a = u_k^{(i)} (et)^{p_k}_{kl}(u^{-1})^l_j \) coincide, at tree level with the physical angles among the scalar fields, and at the quantum level they are functions of the free parameters \( \mu_{ij}, \lambda_{ijkl}, Z_0^\phi, Z_{\phi_A}^\phi \). In order to impose normalization conditions only on the angle the matrix \( u_j^{(i)} \) should be unitary: \( u_j^{(i)} u_j^{(k)} = \delta_{ij} \).

This can be achieved by adjusting the free parameters \( \mu_{ij}, \lambda_{ijkl}, Z_0^\phi, Z_{\phi_A}^\phi \).

The mixing angle can now be fixed by requiring that at the quantum level \( T_{ij}^a \) coincide with the renormalized mixing angles. Only by representing the WTI on the basis of the mass eigenstates the quantities \( T_{ij}^a \) can be identified with renormalized mixing angles avoiding the problem of w.f.r. for scalar fields.

### 3.4 Gauge Boson Sector

For the transverse components of the two-point functions for the gauge bosons we have to impose the following normalization conditions

\[
\Gamma^{(n),T}_{A^{a'}A^{a'}}(p) \bigg|_{p^2 = 0} = 0 \quad \Gamma^{(n),T}_{A^{a'}Z^{b'}}(p) \bigg|_{p^2 = 0} = 0. \tag{3.59}
\]

in order to avoid the IR divergences. Then, recalling that the BFM leaves only \( Z_s^W \) as free parameters for each gauge multiplet, it seems almost impossible that these normalization conditions can be achieved. However, as we will show below, for the SM (a non-semisimple gauge model with a single abelian factor) the Eqs. (3.59) can be solved.

We define the new two-point functions by rotating, at fixed \( p \), the matrices of two-point functions matrix of the mass eigenstates

\[
\begin{pmatrix}
\Gamma^T_{W^{a'}A^{a'}}(p) \\
\Gamma^T_{W^{a'}W^{a'}A}(p) \\
\Gamma^T_{W^{a'}W^{a'}S}(p) \\
\Gamma^T_{W^{a'}W^{a'}A}(p)
\end{pmatrix} = \mathcal{R}^{-1}(p) \begin{pmatrix}
\Gamma^T_{A^{a'}A^{a'}}(p) \\
\Gamma^T_{A^{a'}Z^{b'}}(p) \\
\Gamma^T_{A^{a'}Z^{b'}}(p) \\
\Gamma^T_{A^{a'}Z^{b'}}(p)
\end{pmatrix} \mathcal{R}(p)
\]

where the matrix \( \mathcal{R}(p) \) of two-point functions has the following block structure

\[
\mathcal{R}(p) = \begin{pmatrix}
\mathcal{R}_{a'b}(p) & \mathcal{R}_{a'b'}(p) \\
\mathcal{R}_{a' b}(p) & \mathcal{R}_{a' b'}(p)
\end{pmatrix} \tag{3.60}
\]

The indices \( a', b' \) run over \( 1, \ldots, N_a \) (where \( N_a \) is the number of massless gauge field), \( a'', b'' \) run over \( 1, \ldots, N_Z \) (where \( N_Z \) is the number of massive gauge field; \( a_A, b_A, a_S, b_S \) run over \( 1, \ldots, N_A \) and \( 1, \ldots, N_S \) where \( N_A, N_S \) are the number of the abelian factors and of simple factors, respectively.

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The presence of massless particles is detected by computing the eigenvalues of the two-point function matrix (3.60) and the STI ensure that the number of those particles, namely \( N_\gamma \), is left invariant under renormalization. This means that

\[
\text{Rank} \left( \begin{array}{cc}
\Gamma^T_{W^a A^b d_a}(0) & \Gamma^T_{W^c S^d A^b}(0) \\
\Gamma^T_{W^a A^b d_a}(0) & \Gamma^T_{W^c S^d A^b}(0)
\end{array} \right) = \text{Rank} \left( \Gamma^T_{Z_a^\nu Z_b^\nu}(0) \right)
\]

which implies that if \( \Gamma^T_{Z_a^\nu A^b(0)} = 0, \forall a'' = 1, \ldots, N_Z \cup \forall a'' = 1, \ldots, N_\gamma \), then \( \Gamma^T_{Z_a^\nu A^b(0)} = 0, \forall a', b' = 1, \ldots, N_\gamma \). Therefore the normalization conditions (3.59) are not independent and they require fewer free parameters to be fixed. In particular, recalling that we have one free w.f.r. for each simple factor \( Z_S^W \) and, by using the invertibility of the matrix \( R \equiv R(0) \) computed at zero momentum, the normalization conditions \( \Gamma^T_{Z_a^\nu A^b(0)} = 0, \forall a'' = 1, \ldots, N_Z \cup \forall b' = 1, \ldots, N_\gamma \) corresponds to

\[
\Sigma_{A^a Z^\nu}(0) - M^2_{Z^\nu} \sum_{S=1}^{N_S} \delta Z^W_{S} \sum_{c_S} R_{a' c_S} \mathcal{R}^{-1}_{c_S b'} = 0
\]

where \( \Sigma_{Z_a^\nu A^b(0)} \) is the renormalized amplitude up to \( n-1 \) order and \( \mathcal{R}^{-1}_{c_S b'} \equiv (\mathcal{R}^{-1})_{c_S b'} \). This equation is solved in terms for the unknown quantities \( \delta Z^W_{S} \) if \( N_S \geq N_\gamma \times N_Z \). The non-singularity of \( \sum_{c_S} R_{a' c_S} \mathcal{R}^{-1}_{c_S b'} \) is insured by the the fact that \( \text{Rank} \left( \Gamma^T_{Z_a^\nu A^b(0)} \right) = N_Z \) and by the invertibility of \( R \) (existence of non-trivial tree-level mixing). As an example for the SM, where \( N_\gamma = N_Z = N_{SU(2)} = 1, Z^W_{SU(2)} = Z^W_3 \) and \( a' = A, b'' = Z, c_S = 3 \) we get

\[
\delta Z^W_3 = \frac{1}{M^2 Z^W_3 R_{A3}} \Sigma_{AZ}(0)
\]

where \( R_{A3} = s_w, R^{-1}_{AZ} = c_w \).

Besides the conditions (3.59) we have also to impose some mass normalization conditions. Therefore, we consider the sub-matrix \( \Gamma^T_{Z_a^\nu Z_b^\nu} \) involving only the massive gauge fields. This remaining matrix is real\footnote{We use the definition \( Z^W_S = 1 + \sum_{n \geq 1} h^{(n)} \delta Z^W_{S}^{(n)} \).}, symmetric and non-diagonal, and it can be diagonalized by means of an orthogonal transformation restricted only to the subspace of massive fields. As a consequence, the only relevant information is contained in its eigenvalues \( \lambda_{Z_a^\nu}(p) \) and, in particular, in their zeros \( p_{Z_a^\nu}^* \). Notice that, for the transverse components of the two-point for gauge bosons, the existence of \( N_Z \) (number of massive gauge bosons) non-trivial eigenvalues \( \lambda_{Z_a^\nu}^{(n)}(p) \) is ensured by the STI. Then we compare the real part (in the case of unstable particles) of zeros \( p_{Z_a^\nu}^* \) with the experimental mass \( M_{Z_a^\nu} \).

\[
\text{Re}(p_{Z_a^\nu}^*) = M_{Z_a^\nu}.
\]

This definition is independent of the gauge parameters \( \xi, \rho \) [26, 30] and of the mixings. At the tree level the mixing of the gauge bosons in the SM concerns only the Weinberg angle, but, at higher orders, the radiative corrections generate new mixings. Those mixings carry no physical information. Finally we can define the mass eigenstates which appear in the LSZ formalism. They are obtained\footnote{The two-point functions are real only below the particle creation thresholds.}
by requiring
\[
\begin{pmatrix}
0 & \Gamma^T_{Z^i a^i Z^{i'}(0)} \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
u_{a^i} \\
u_{a^{i'}}
\end{pmatrix}
\begin{pmatrix}
\Gamma^T_{Z^i a^i Z^{i'}(0)} \\
\Gamma^T_{Z^i a^i Z^{i'}(0)}
\end{pmatrix}
\begin{pmatrix}
u_{a^i} \\
u_{a^{i'}}
\end{pmatrix} = 0,
\]
\[
\begin{pmatrix}
\Gamma^T_{A^i a^i A^{i'}}(p_{Z^i a^i}^*) \\
\Gamma^T_{A^i a^i A^{i'}}(p_{Z^i a^i}^*)
\end{pmatrix}
\begin{pmatrix}
\Gamma^T_{Z^i a^i Z^{i'}(0)} \\
\Gamma^T_{Z^i a^i Z^{i'}(0)}
\end{pmatrix}
\begin{pmatrix}
\Gamma^T_{Z^i a^i Z^{i'}(0)} \\
\Gamma^T_{Z^i a^i Z^{i'}(0)}
\end{pmatrix}
\begin{pmatrix}
u_{a^i} \\
u_{a^{i'}}
\end{pmatrix} = 0.
\]
\[(3.63)\]

As a consequence the metric is not diagonal
\[
\begin{pmatrix}
(u_{A^i a^i}, u_{A^{i'}, a^{i'}}) & (u_{A^i a^i}, u_{Z^{i'}, a^{i'}}) \\
(u_{Z^{i'}, a^{i'}}, u_{A^i a^i}) & (u_{Z^{i'}, a^{i'}}, u_{Z^{i'}, a^{i'}})
\end{pmatrix}
= \begin{pmatrix}
\delta^{a^i, a^{i'}} & K^{a^i, a^{i'}} \\
H^{a^i, a^{i'}} & K^{a^i, a^{i'}}
\end{pmatrix}
\]
\[(3.64)\]
where \((..,.)\) is the scalar product, and no free parameter can be adjusted in order to achieve its diagonalization.

### 3.5 Fermion Sector

In the fermion sector we have several free parameters (the Yukawa couplings \(Y_{ij}^f\), and the w.f.r. \(Z_{ij}^{R/L} , \tilde{Z}_{ij}^{R/L}\) for the fermions) which can be tuned in order to satisfy the normalization conditions. The free parameters are selected by the BRST symmetry and, in particular, by the invariance under the background gauge transformations. As a consequence the normalization conditions must respect those symmetries to all orders. Moreover in the present sector there are two kinds of physical normalization conditions which must be implemented in order to compare the model with the experimental data: the fermion masses and the mixing parameters.

At the tree level those parameters can be easily expressed in terms of the Yukawa couplings. In fact we can diagonalize the mass matrices (after the spontaneous symmetry breaking) \(M_{ij} = Y_{ij}^f v_i\) by bi-unitary transformations \(U_{ij}^{R/L}\) acting on the space of fermions and the resulting eigenvalues \(\lambda_i^Y v_i\) must be compared with the fermion masses \(M_i\). As is well known the diagonalization of the Yukawa matrices modifies the interaction terms (\(\Gamma^{\psi W \psi}\)) among the fermion and the charged gauge fields
\[
\Gamma^{\psi W \psi} = \int dx \left( \bar{\psi}_{R} L_{U}^{L,R} T_{IJ}^{L,R} U_{IJ}^{L,R} W^{a}_{\psi R} + \bar{\psi}_{R} L_{U}^{L,R} T_{IJ}^{L,R} U_{IJ}^{L,R} W^{a}_{\psi R} + \text{h.c.} \right)
\]
and the resulting matrices \(U_{IJ}^{L,R} T_{IJ}^{L,R}\) are interpreted as the CKM angles. In the case of the minimal SM, the matrices \(U_{IJ}^{L,R}\) split into up and down parts and they commute with the third generator of the \(SU(2)\) and with \(U(1)\). Therefore we recover the usual CKM matrix. Furthermore, since the right fermions couple only to the \(U(1)\) gauge group (and to \(SU(3)\)), the \(U_{IJ}^{R}\) commute and, therefore, no CKM is present for the right-handed part. Notice that the only relevant physical CKM angles are obtained by a suitable rephasing of fermions.

At the quantum level we have to translate those requirements in terms of normalization conditions. First we have to define the generations intrinsically. At the tree level this can be easily done by considering the masses of the fermions and organizing the fermions according to the hierarchy of the masses. At the quantum level we consider the two-point functions \(\Gamma^{\psi_I \psi_J}(p)\) and their decomposition into Lorentz-invariant terms:
\[
\Gamma^{\psi_I \psi_J}(p) = \Sigma_{IJ}^{D}(p) P_L + p P_L \Sigma_{IJ}^{L}(p) + P_R \Sigma_{IJ}^{D}(p) + p P_R \Sigma_{IJ}^{R}(p)
\]
\[(3.66)\]
where \(\Sigma_{IJ}^{D}(p), \Sigma_{IJ}^{L}(p), \Sigma_{IJ}^{R}(p)\) are Lorentz invariant matrices. In order to avoid heavy notation we suppress the labels \(\psi_I\) and only the index \(I\) of the flavor (and of the color, in the case of quarks) is
used. Then, we can form the following new Lorentz invariant hermitian (below particle production thresholds) matrix \[ 43 \]

\[
K_{IJ}(p) \equiv p^2 \Sigma^L_{IJ}(p) - \Sigma^D_{IJ}(p) \Sigma^R_{IJ}(p)
\]  

(3.67)

and its eigenvalues \( \lambda_I(p) \) correspond to fermion two-point functions with flavour mixing. Moreover the zeros of eigenvalues \( \lambda_I(p_f^2) = 0 \), or equivalently the zeros of the determinant

\[
\text{Det} [K_{IK}(p)]|_{p^2=(p_f^2)} = 0,
\]  

(3.68)

are gauge independent and their real and imaginary parts are compared with the masses of fermions \( M_I \) and their width \( \Gamma_{\psi_I} \). This fixes a possible choice of normalization conditions for quark and lepton masses. Although the QCD corrections affect the pole structure, this definition is used in the explicit computations of electroweak radiative corrections (switching off the strong interactions). As a consequence we are able to organize the fermions into generations intrinsically by fixing the zeros of the eigenvalues of the matrix \( K_{IJ}(p) \) and this choice fixes some of the free parameters \( Y^i_{J} \).

At this point we are able to define the mass eigenstates for the fermion fields in the case of mixing. We define mass eigenstates by

\[
\tilde{u}^{(I)}_K \Gamma_{KJ}(p)|_{p^2=p_f^2} = 0, \quad \Gamma_{KJ}(p)u^{(I)}_J|_{p^2=p_f^2} = 0
\]  

(3.69)

where \( \tilde{u}^{(I)} \), \( u^{(I)} \) are independent eigenvectors due to the CP violation. Those vectors depend of the free parameters \( Y^i_{J}, Z^L_{IJ}, \bar{Z}^L_{IJ} \) (cf. Eqs. \( 3.20 \)-\( 3.20 \)) and the metric \( (\bar{u}^{(I)}, u^{(J)}) \) is not diagonal at the quantum level. As a consequence we can define the mass eigenfield by projecting the fermion fields \( \psi_I, \bar{\psi}_I \) onto the vectors \( \tilde{u}^{(I)}, u^{(I)} \): \( f'_I = \tilde{u}^{(I)}_K \psi_K \) and \( \bar{f}'_I = \bar{\psi}_J u^{(J)}_J \). Notice that we can introduce the matrices \( U_{IJ} = u^{(I)}_K \) and \( \bar{U}_{IJ} = \tilde{u}^{(I)}_K \) which are complicate functions of the free parameters \( Y^i_{J}, Z^L_{IJ}, \bar{Z}^L_{IJ} \) (except those which are already fixed by the mass renormalization) and we require that the matrices are unitary in order to single out only the relevant mixing angles. It is easy to show that the system is invertible at tree level and, by the theorem of the implicit function for formal power series, this is true also at the quantum level.

By following Aoki \textit{et al.} we would be forced to impose normalization conditions on the mass eigenstates \( \tilde{u}^{(I)}, u^{(I)} \), but, due to the constraints of BFM, this can not be achieved without deforming the WTI. Therefore we choose to fix the remaining parameters, namely the CKM matrix, by using the WTI themselves. In fact, by recalling that the CKM angles appear as couplings among charged gauge fields and fermions and that the couplings among background and quantum gauge fields with fermions coincide in the BFM, we are able to fix the CKM by imposing the WTI in the mass eigenstate representations \( f'_I, \bar{f}'_I \)

\[
W_a(\Gamma) = (\text{scalar} + \text{gauge} + \text{ghost terms}) + \\
\quad + (eT)^a_{\mu J} \left[ \tilde{f}'_Im^L_{\mu a}(U^L)^{\ast}_{IJ} \frac{\delta \Gamma}{\delta f'^L_{IJ}} + \frac{\Gamma}{\delta f'^L_{IJ}} \right] (L \rightarrow R)
\]  

(3.70)

\[
= (\text{scalar} + \text{gauge} + \text{ghost terms}) + \\
\quad + \left[ \tilde{f}'_Im^L_{\mu a}(V^L)^{\ast}_{IJ} \frac{\delta \Gamma}{\delta f'^L_{IJ}} + \frac{\Gamma}{\delta f'^L_{IJ}} \right] (L \rightarrow R) = 0
\]

\[ 18 \]The definition of masses has been taken into account in \[ 48 \] and in the case of fermion in \[ 44 \], where the problem of gauge parameter independence is considered. From a more rigorous point of view we refer to papers \[ 44 \]. An extension to flavour mixings is discussed in \[ 28 \].
where only the fermion terms are displayed and \( V_{I,J}^{L,a} = U_{K,I}^*(eT)^a_{L,KM}U_{M,J} \) (and analogously for the right part and for \( \tilde{V}_{I,J}^{L,a} \)) are the renormalized CKM matrices. The above equation fixes the free parameters \( Y_{I,J}^a \) up to unphysical rescaling and up to unphysical w.f.r.

Analogously to the scalar mixing the quantities \( V_{I,J}^{L,a} = U_{K,I}^*(eT)^a_{L,KM}U_{M,J} \) and \( \tilde{V}_{I,J}^{L,a} \) can be identified with the renormalized mixing angles avoiding the problem of the w.f.r. and the mixing among physical and unphysical quantities.

### 3.6 Background Field Renormalization

In this section we discuss the normalization conditions for Green’s functions with external background fields. As is clear from [20] some Green’s functions with external background gauge and scalar fields are divergent, then they require their proper normalization conditions. However two orthogonal approaches can be assumed: the first one is to fix these normalization conditions independently of those of quantum fields, the second one corresponds to choose the normalization conditions which respect the form of WTI (2.7) and (A.119). The first alternative has been adopted in [16] and it requires some “unphysical” normalization conditions for the background fields; namely the free parameters \( X_S, X_{ij} \) are fixed independently of w.f.r. \( Z_W, Z^\phi \) of quantum fields. Here, we choose the second alternative in order to minimize the number of normalization conditions and the amount of work to renormalize the model.

Furthermore, since the BFM is a powerful tool to compute S-matrix elements from Green’s functions with external bosonic background fields [24], it is crucial to show that the normalization conditions imposed on those Green’s functions directly correspond to physical normalization conditions on the poles of quantum Green’s functions. Therefore, in the next paragraphs we explicitly prove the degeneracy among the zeros of two-point functions with external background fields and the corresponding two point functions with quantum fields.

The parameters \( X_S, X_{ij} \) are respectively related to the two-point functions with external BRST-sources \( \Gamma_{ij}^{g_S c_S, b_S} (p), \Gamma_{ij}^{g_S, g_S} (p) \) which are superficially divergent; on the contrary, the other two-point functions \( \Gamma_{ij}^{g_S, b_S} (p), \Gamma_{ij}^{g_S, g_S} (p) \) are superficially convergent by power counting and Lorentz covariance.

To compare these Green’s functions we firstly use the WTI to show that the renormalization of \( \Gamma_{ij}^{g_S c_S, b_S} (p), \Gamma_{ij}^{g_S, g_S} (p) \) depends on the renormalization of vertex functions with external quantum fields. In fact, by taking the derivative of (2.7) with respect to \( \bar{\gamma}_{ij}^{b_S} (p) \) and \( e^{c_S} (q) \), we get\(^{19}\)

\[
\frac{\delta^2 (W_{aS\Gamma})}{\delta \bar{\gamma}_{ij}^{b_S} (p) \delta e^{c_S} (q)} \bigg|_{\phi=0} = i(p + q)_{\mu} \left( \Gamma_{ij}^{c_S, g_S b_S} (p, q) + \Gamma_{ij}^{g_S, a_S b_S} (p, q) \right) + \left( \phi_{ij}^{g_S} \right) c_S (p, q) + f^{c_S a_S d_S} \Gamma_{ij}^{b_S, c_S} (p, q) + f^{b_S c_S d_S} \Gamma_{ij}^{g_S, c_S} (q) = 0. \tag{3.71}
\]

Exploring the region of very high momenta and using the Weinberg’s theorem [42] from the expression

\[
\lim_{\rho \rightarrow \infty} \frac{\partial \rho^\mu}{\partial \rho^\mu} \frac{\delta^2 (W_{aS\Gamma})}{\delta \bar{\gamma}_{ij}^{b_S} (p) \delta e^{c_S} (-\rho p)} \bigg|_{\phi=0} = \Gamma_{ij}^{g_S, a_S b_S} \rho^S + \Gamma_{ij}^{g_S, g_S b_S} \rho^S + f^{a_S b_S d_S} \Gamma_{ij}^{b_S, c_S} \rho^S \rho^\mu \Gamma_{ij}^{b_S, c_S} \rho^\mu, \tag{3.72}
\]

\(^{19}\)In the present section we re-absorb the couplings \( e_S, e_{aS b_S} \) from WTI by rescaling the gauge fields and we impose that the v.e.v. \( \nu_i \) in the WTI coincides with its tree level value.
we can isolate the asymptotic behaviour of the Green’s like $\Gamma_{\gamma_\mu}^{\infty} W_{\gamma_\nu}^{aS} c_{S}$ to be compared with the renormalized Green’s functions with external quantum fields.

In the same way, by considering the derivative of WTI (2.5) with respect to $\tilde{\Omega}$, we obtain

$$\lim_{\rho \to \infty} \frac{\delta^2 (W_{aS} \Gamma)}{\delta \tilde{\Omega}^1(p) \delta \tilde{\Omega}^1(-p) \phi_{\nu}} = v_{ij} \frac{\delta^2 (W_{aS} \Gamma)}{\delta \tilde{\Omega}^1(p) \delta \tilde{\Omega}^1(-p) \phi_{\nu}} + v_{ij} \Gamma_{\gamma_\mu}^{\infty} W_{\gamma_\nu}^{aS} c_{S} + \Gamma_{\gamma_\mu}^{\infty} \Gamma_{\gamma_\nu}^{\infty} + f^{aS bS dS} \Gamma_{\gamma_\nu}^{\infty} + f^{aS bS dS} \Gamma_{\gamma_\nu}^{\infty}$$

(3.73)

which relate the $\Gamma_{\gamma_\mu}^{\infty}$ to Green’s functions with quantum fields. Equations (3.72) and (3.73) are the all-order version of relations between $X_{\gamma}, X_{\nu}$, $T_{ij}$ and $Z_{\gamma}, Z_{\nu}$, and they reduce to them at the tree level.

Furthermore, the Green’s functions $\Gamma_{\gamma_\mu}^{bS} c_{S}, \Gamma_{\gamma_\nu}^{cS}, \Gamma_{\gamma_\gamma}^{aS} bS, \Gamma_{\phi_\gamma}^{bS} c_{S}$ contain the radiative corrections to the structure constants $f^{aS bS cS}$ of the gauge group $G$ and to the generators $t_{ij}^{\gamma}$. Therefore, their correct renormalization ensures that the model has the same gauge group and the same matter representation to all orders $\Gamma_{\gamma_\mu}^{bS} c_{S}, \Gamma_{\gamma_\nu}^{cS}$.

Moreover, in the case of non-invariant regularization schemes, general non-invariant counterterms

$$\Gamma_{BRST}^{C.T.} = \int d^4 x \left( x_{aS bS} \delta_{\mu}^{cS} \Gamma_{\gamma_\mu}^{aS} c_{S} + x_{aS bS cS} \delta_{\mu}^{cS} \Gamma_{\gamma_\mu}^{aS} c_{S} + x_{aS bS cS} \delta_{\mu}^{aS} c_{S} + x_{aS bS cS} \delta_{\mu}^{aS} c_{S} \phi \right)$$

(3.74)

must be added to the action in order to restore the STI and the WTI. The coefficients $x_{aS bS}, x_{aS bS cS}$, $x_{aS bS cS}, x_{aS bS}, x_{aS bS}$ are computed in terms of the breakings to the STI. This necessarily implies the analysis of STI for the three-points functions involving, at least, two ghost fields and it requires a great amount of work to compute these coefficients at higher orders. However, by using the WTI (3.72), (3.73), $x_{aS bS}, x_{aS bS cS}, x_{aS bS}, x_{aS bS}$ can be computed from the breakings of WTI, and only some STI for two-point functions are needed [13]. Furthermore, since up to this moment, only a subset of two-loop electroweak radiative corrections have been computed [40], two-loop and one-loop WTI and one-loop STI are sufficient to evaluate all possible counterterms, including those of Eq. (3.74).

To complete our program we have to show the relation among the Green’s functions $\Gamma_{\phi_\gamma}^{bS} c_{S}, \Gamma_{\gamma_\mu}^{aS} bS, \Gamma_{\gamma_\mu}^{cS}$, and the two-point functions $\Gamma_{\gamma_\mu}^{aS} bS, \Gamma_{\gamma_\mu}^{cS}$. For this purpose we consider the STI

$$\Gamma_{\gamma_\mu}^{aS} bS, \Gamma_{\gamma_\mu}^{cS}, \Gamma_{\gamma_\mu}^{aS} bS, \Gamma_{\gamma_\mu}^{cS}$$

(3.75)

obtained by differentiating (2.3) with respect to $\tilde{\Omega}_{\mu}^{aS} (-p - q), \tilde{\Omega}_{\mu}^{bS} (p), \tilde{\Omega}_{\mu}^{cS} (q)$ and with respect to $\tilde{\Omega}_{\gamma}^{aS} (-p - q), \tilde{\Omega}_{\gamma}^{bS} (p), \tilde{\Omega}_{\gamma}^{cS} (q)$. Since $\Gamma_{\phi_\gamma}^{bS} c_{S}$ and $\Gamma_{\phi_\gamma}^{cS} c_{S}$ are superficially convergent by power counting, we immediately deduce the relations

$$\Gamma_{\gamma_\mu}^{aS} bS, \Gamma_{\gamma_\mu}^{cS}, \Gamma_{\gamma_\mu}^{aS} bS, \Gamma_{\gamma_\mu}^{cS}$$

(3.76)

These equations fix completely the local parts of Green’s with background fields in terms of the Green’s functions with quantum fields. Furthermore we have to recall that the renormalization of
for the process. The computation of background two-point functions is enough to derive the complete S-matrix element.

Additionally, quantum two-point functions are also consistently fixed. Needless to say, in several applications the complex zeros in the case of unstable particles) of background two-point functions, the zeros of radiative corrections with the BFM [21]: by fixing the mass renormalization on the zeros (or on the complex zeros in the case of unstable particles) of background two-point functions, the zeros of quantum two-point functions are also consistently fixed. Needless to say, in several applications the computation of background two-point functions is enough to derive the complete S-matrix element for the process.

In the last part of the present section, we discuss the relation among the zeros of the background two-point functions, $\hat{\Gamma}_{W}^{a\bar{a}}(p), \hat{W}_{a\bar{a}}^{b\bar{b}}(-p)$

$$\Gamma_{W_i}^{a\bar{a}}W_{a\bar{a}}^{b\bar{b}} = \Gamma_{W_i}^{a\bar{a}}(p)\Gamma_{W_i}^{b\bar{b}}(p) + \Gamma_{W_i}^{a\bar{a}}(p)\Gamma_{W_i}^{b\bar{b}}(p)$$

where we have suppressed the argument since they are all two-point functions with same momentum. The two-point functions $\Omega_{W_i}^{a\bar{a}}\gamma_{2}\gamma_{S}$ have been already discussed in the previous paragraphs and only the quantum abelian gauge fields $W_{a\bar{a}}^{b\bar{b}}$ enter into the discussion, since their background are completely decoupled. In the same way, we get

$$\Gamma_{W_i}^{a\bar{a}}W_{a\bar{a}}^{b\bar{b}}(p) = \Gamma_{W_i}^{a\bar{a}}(p)\Gamma_{W_i}^{b\bar{b}}(p) + \Gamma_{W_i}^{a\bar{a}}(p)\Gamma_{W_i}^{b\bar{b}}(p)$$

by differentiating with respect to $\tilde{\Omega}_{\mu_{i}}^{a\bar{a}}(p), \tilde{W}_{\nu_{i}}^{b\bar{b}}(-p)$ and $\tilde{\Omega}_{\mu_{i}}^{a\bar{a}}(p), \tilde{W}_{\nu_{i}}^{b\bar{b}}(-p)$. Therefore, by decomposing the two-point functions into a transverse and a longitudinal part, the Eqs. (3.78) can be put in the form

$$U_{ab}(p) = \left(\begin{array}{cc}
\Gamma_{W_{i}^{a}W_{i}^{b}}^{T}(p) & \Gamma_{W_{i}^{a}W_{i}^{b}}^{T}(p) \\
\Gamma_{W_{i}^{a}W_{i}^{b}}^{T}(p) & \Gamma_{W_{i}^{a}W_{i}^{b}}^{T}(p)
\end{array}\right)$$

where the matrix $U_{ab}(p)$ is

$$U_{ab}(p) \equiv \left(\begin{array}{cc}
\delta_{a,b} & 0 \\
0 & \Omega_{\mu_{i}}^{a\bar{a}}\gamma_{S}(p)
\end{array}\right).$$

This matrix $U_{ab}(p)$ is a non-singular matrix for $p \neq 0$ and $\Omega_{\mu_{i}}^{a\bar{a}}\gamma_{S}(p) \neq 0$ starting at one loop. As a consequence, the zeros of the eigenvalues of the background two-point functions and the
corresponding quantum partners coincide\textsuperscript{20}. This is a crucial property for the self-energies to ensure that the S-matrix defined with the BFM is equivalent to the conventional one \[24\].

Moreover we can immediately see that the diagonalization of the two-point functions matrix for the background fields does not imply the diagonalization of the two-point functions for the quantum fields. This is only achieved by a proper choice of normalization for the matrix $U_{ab}(p)$. In particular the IR behavior might be different. Therefore, we have to study the two-point functions at zero momentum. By using the Eq. (3.60) and by defining, in the same way, for the background fields, a matrix $\mathcal{R}(p)$ the relation among the background two-point functions and the quantum two-point functions in the mass eigenstates is given by

$$
\begin{pmatrix}
\Gamma^T_{Aa',Aa'}(p) & \Gamma^T_{Za',Aa'}(p) \\
\Gamma^T_{Aa'Za'}(p) & \Gamma^T_{Za'Za'}(p)
\end{pmatrix} = (\mathcal{R}^{-1} \mathcal{U} \mathcal{R})(p) \begin{pmatrix}
\Gamma^T_{Aa',Aa'}(p) & \Gamma^T_{Za',Aa'}(p) \\
\Gamma^T_{Aa'Za'}(p) & \Gamma^T_{Za'Za'}(p)
\end{pmatrix}
$$

and we set $\mathcal{U}(p) \equiv (\mathcal{R}^{-1} \mathcal{U} \mathcal{R})(p)$. Since the local WTI for the background gauge transformations ensure that

$$
\Gamma^T_{Wa'Wb'}(p) = 0 \quad a', b' = 1 \ldots N \gamma
$$

we immediately get

$$
\Gamma^T_{Wa'Wb'}(p) \bigg|_{p^2=0} = \sum_{a'',b''}(0)\mathcal{U}_{a'b''}(0)\mathcal{U}_{a''b''}(0), \quad a', b' = 1 \ldots N \gamma
$$

from where $\mathcal{U}_{a'a''}(0) = 0$ for $a' = 1 \ldots N \gamma$, $a'' = 1 \ldots N \gamma$. The matrices $\mathcal{U}_{a'a''}$ depend on the already discussed Green’s functions $\Gamma^T_{\gamma_S \gamma_S}(p)$ computed at zero momentum. These Green’s functions are superficially divergent (only the transverse parts, while the longitudinal part is finite) and are fixed by the normalization conditions \[3.84\]. However we saw that the Green’s functions $\Gamma^T_{\Omega_S \gamma_S}$ are related to the w.f.r. of the multiplets of quantum fields for each simple single sector $G_S$ of the group. Hence the IR requirements given by \[3.84\] for the quantum fields fix their w.f.r.s $Z_S$.

The BFM restricts the number of free parameters to a single w.f.r. for each simple factor and this bounds the number of massless particles which can be coupled to the model without modifying the STI and the local WTI. In fact increasing the number of massless particles the number of normalization conditions increases and the bound $N \gamma N \gamma \leq N_S$ (where $N_S$ is the number of simple factors) can not be respected any longer.

### 3.7 Renormalization of Couplings

For what concerns the coupling renormalization it is very difficult to discuss general properties, however we can divide the problem into two different categories: the couplings which are related to

\textsuperscript{20} Notice that, here, we used the representation $W, W$ to denote the quantum field and the background field. However the S-matrix elements obtained in the BFM approach are constructed with quantum fields $Q^W = W - W$ and with the background fields $W$. Therefore the two-point functions with external background described in the literature \[24 \[22 \[23\] are related to ours by the simple relation $\Gamma^W_{Wa'}(p) = \Gamma_{WaN}(p) + 2\Gamma_{WN}(p) + \Gamma_{WN}(p)$ where $\Gamma^W_{ab}$ is defined in \[20\]. By Eq. (3.84), it is easy to show that $\Gamma^W_{ab}$ has the same zeros of $\Gamma_{WW}(p)$.
the physical masses (in the case of spontaneous symmetry breaking) and those which are true gauge couplings. For the first set, by using the gauge invariant definition of masses we are able to provide a proper definition, but for the others a construction of an invariant charge (see for example [37] in chapter 13) is required [40].

The couplings $\lambda_{ijkl}, \mu_{ij}$ of the scalar sector are related to the renormalization of the tadpoles and the masses of Higgs fields (see Sec. 3.3). By identifying the masses of the Higgs fields as the real part of the zeros of non-trivial eigenvalues of the two-point function matrix some of the couplings $\lambda_{ijkl}, \mu_{ij}$ are fixed in a gauge independent way.

In the sector of the gauge bosons of the SM, $M_Z, M_W$ define the v.e.v. $v$ and the weak angle $c_W$. The procedure of comparing the real part of the zeros of eigenvalues for the two-point function matrices with the physical masses is gauge parameter independent and therefore the weak angle $c_W$, defined by the conditions $c_W^2 = \frac{M_W^2}{M_Z^2}$ [36] is clearly independent of the gauge parameters. As discussed in the previous section, the zeros of two-point functions with external background fields and with quantum ones coincide, therefore, in order to fix the mass renormalization only the background two-point functions are required. This has been already observed in reference [21] where the one loop analysis has been performed.

However due the precision of the experimental measurement of the $\mu$-decay, the Fermi’s constant $G_F$ is used to calculate $M_W$. The definition of $G_F$ is given in terms of the $\mu$-decay amplitude which is clearly gauge independent [36].

The QED charge $\alpha_{QED}$ is fixed in terms of the photon two-point function computed at zero momentum transfer. It is easy to see that this definition with the normalization conditions given in Sec. 3.4 is gauge independent to all orders. The definition of the QCD coupling constant $\alpha_S$ requires a detailed analysis. We refer to the literature [20, 40, 41] on the subject.

Finally, concerning the charge renormalization, we have to stress again that the renormalization of the two-point functions for background gauge fields and for background scalar fields can be immediately related to the charge renormalization or to the v.e.v. renormalization, respectively. This provides a very simple way to compute the beta functions of the Renormalization Group equation and the Callan-Symanzik equation. This advantage has been extensively used in the QCD [20], in the SM [21] and for extended models.

4 Anomalies

The issue of algebraic renormalization of identities has been widely analyzed in the literature [6, 9, 10, 16, 22, 47], therefore, here, we only recall the main results about the hard anomalies for non-semisimple gauge models with background fields and we discuss the problem of IR anomalies of STI in the minimal SM. Besides the hard anomalies and the IR anomalies, some soft anomalies can be present in the SM. However they are rule out by the Callan-Symanzik equation as shown in [6, 9, 16].

4.1 Hard Anomalies

Due to their relevance in the renormalization procedure, we describe briefly the hard anomalies of STI and of WTI for a non-semisimple gauge model.

As proved in [6, 10] the structure of the general solution $A$ of the consistency conditions $S_0(A) = 0$
is given by

\[
A = \int d^4x A^{ABJ}(x) + \int d^4x A_1(x) + \int d^4x A_2(x) + S_0 \int d^4x \mathcal{B}(x)
\]  

(4.85)

\[
A^{ABJ}(x) = \sum_i r_i \epsilon^{i\mu\nu\rho\sigma} \left[ D_i^{abc} e^a \partial_\mu W^b_\nu \partial_\rho W^c_\sigma + \frac{F_{abcd}^{i}}{12} (\partial_\mu e^a) W^b_\nu W^c_\rho W^d_\sigma \right]
\]

(4.86)

\[
A_1(x) = \epsilon^{aA} R_{aA}(x)
\]

(4.87)

\[
A_2(x) = w_{aA,bA}^\alpha \left( j_\alpha^A W^a_\mu c^b A + \frac{1}{2} P_{\alpha\beta}^{\mu\nu} c^A c^b A + \frac{1}{2} P_{\alpha\gamma}^{\mu\nu} e^A c^b A + \frac{1}{2} P_{\alpha\eta}^{\mu\nu} e^A e^b A + h.c. \right)
\]

(4.88)

where \( A^{ABJ}(x) \) is the well known Adler-Bardeen-Jackiw anomaly \[23\] and the \( r_i \) are its coefficients; the tensors \( F_{abcd} \) are defined by

\[
F_{abcd} = D_t^{abx} (\epsilon f)^{xcd} + D_t^{adx} (\epsilon f)^{xbc} + D_t^{acx} (\epsilon f)^{xdb}
\]

with \( D^{abc} \) invariant symmetric tensors of rank three on the algebra of the gauge group. \( \mathcal{B}(x) \) in Eq. (4.85) is a generic polynomial with dimension \( \leq 4 \) and Faddeev-Popov charge zero. \( R_{aA}(x) \) are a set of BRST invariant polynomials. The latter are absent if the discrete CP invariance is not violated. In the expression \( A_2 \) of Eq. (4.88) the coefficients \( w_{aA,bA}^\alpha \) are constant and antisymmetric in the abelian indices \( a, b \) for each value of \( \alpha \) and \( P_{\alpha\mu,i}^{\mu\nu}, \ldots, P_{\alpha}^{I} \) are defined in App. A.

As observed by Barnich et al. \[10\] the anomaly terms (4.88) are trivial if and only if the conserved currents \( j_\mu^A \) are trivial, that is, are equal to an identically conserved total divergence when the equation of motion are satisfied; however as already observed, in the SM, the conserved lepton and baryon numbers provide four non-trivial examples of \( j_\mu^A \) \[49\]. About the actual presence of anomalies as (4.87) and (4.88) at higher orders there is no evidence from one- and two-loop calculations and the only example where (4.87) occurs in the literature is given in \[50\]. In fact by choosing an appropriate gauge fixing, one can avoid these anomalies as in ’t Hooft gauge fixing as proved in \[3\] or as in ’t Hooft-Background gauge fixing as proved in paper \[23\]. Generically for an arbitrary choice of the gauge fixing the abelian ghosts are coupled to scalars in order to protect the model against IR divergences of massless would-be Goldstone fields, as a consequence anomalies as \( A_1, A_2 \) might appear.

To summarize we have the following anomaly cancellation mechanisms

- The coefficients \( r_i \) of \( A^{ABJ} \) depend on the charges of fermion fields and on the gauge group structures. Therefore for a proper choice of fermion content of the theory, the coefficients \( r_i \) are zero. In addition, by the non-renormalization theorem (see \[1\] and the references therein), if \( r_i \) are zero at one loop, they will be absent to all orders.

- The candidates \( A_1 \) are excluded by the AAE \[24\]. In fact it is easy to show that, solving the constraints hierarchically, the AAE rules out the candidates which depend on the abelian ghost fields \( c_{aA} \) \[22\].

- The candidates \( A_2 \) are absent because of the AAE. This can also easily proved by using the WTI (A.119) for the abelian factors which is a consequence of the STI and the AAE.

- As proved in \[22\] the hard anomalies of the WTI are related to those of the STI. Therefore they are cancelled by the same mechanisms.

\[21\]It provides the non-invariant counterterms to cancel the spurious anomalies coming from a non-symmetric renormalization scheme. Those counterterms which are IR dangerous will be discussed in the next section.
4.2 IR Anomalies

In this last section we present the analysis of the IR anomalies and we derive the conditions under which their coefficients vanish. We also prove that, under these conditions, the IR anomalies which depend on background fields are absent to all orders. Here, we consider only the IR anomalies to STI; the IR anomalies to WTI are related to the STI anomalies by consistency conditions. The IR anomalies to the Nakanishi-Lautrup equations (4.111), to the Faddeev-Popov equations (4.118) and to the Abelian Antighost equation (2.6) will be discussed in App. B and in App. C, respectively.

As a consequence of the Quantum Action Principle (QAP) [4, 7, 51], a non-invariant renormalization scheme breaks the STI

\[ S(\Gamma) = \Delta^3_{S,4} \cdot \Gamma \]  

(4.89)

by local (integrated) terms \( \Delta^3_{S,4} = \int d^4x \Delta^3_{S,4}(x) \) whose invariance properties \( (S_0 \Delta^3_{S,4} = 0) \), UV and IR power counting degrees\(^{22} \) \( (d_{UV} \Delta^3_{S,4} = 4, d_{IR} \Delta^3_{S,4} = 3) \) and Faddeev-Popov charge are fixed by the algebraic properties of the functional operator \( S_0 \) (cf. App. (5)). If the theory is renormalizable, it must be possible to re-absorb the breaking terms \( \Delta^3_{S,4} \) by suitable non-invariant counterterms.

However some of the anomaly candidates which can be written as a variation of a local counterterms

\[ \Delta^3_{S,4} = S_0 \int d^4x \left( x_{\alpha' \beta'} A_{\alpha'}^i Z^{\beta'} + x_{\alpha' i}^2 \right) + \hat{\Delta}^3_{S,4}, \]  

(4.90)

whose IR degree is smaller than four, cannot be removed by counterterms. Indeed, these counterterms would generate IR divergences at the next order. As a consequence, the STI is anomaly-free only if the coefficients \( x_{\alpha' \beta'}, x_{\alpha' i}^2 \) of the breaking terms vanish to all orders. The breaking terms \( \hat{\Delta}^3_{S,4} \) contain other candidates which are removed by counterterms. Notice that also some terms containing one or two background fields (e.g. \( \int d^4x A_{\mu}^i Z^{\mu} \)) might appear as IR anomalies to STI; however, due to the fact that the background fields do not propagate, the corresponding counterterms do not produce IR divergences. Finally, we do not assume the CP symmetry as an invariance of the model and the basis of monomials for \( \Delta^3_{S,4} \) contains both CP-odd and CP-even terms.

To derive the minimal set of conditions under which the IR dangerous part of \( \Delta^3_{S,4} \) vanishes we have, first, to differentiate the STI with respect to those fields entering in the monomials \( S_0 \int d^4x A_{\mu}^i Z^{\mu} \) and \( S_0 \int d^4x c_{\alpha'}^i \), and, then, to choose a convenient configuration for momenta to compute the coefficients \( x_{\alpha' \beta'}, x_{\alpha' i}^2 \). It turns out that the most convenient point is at zero momentum and the requirement that \( x_{\alpha' \beta'} = x_{\alpha' i}^2 = 0 \) implies sensible normalization conditions for two-point Green’s functions at zero momentum. In particular, among these conditions there are independent equations which must be solved in terms of the free parameters of the model. The existence of a solution to those equations has been already covered in Sec. (5).

As an example, the two conditions \( \Gamma_{c_{\alpha'}^i}(0) = 0 \) and \( \Gamma_{A_{\mu}^i}(0) = 0 \) are required to cancel the IR dangerous parts of \( \Delta^3_{S,4} \), but they cannot be solved by adjusting free parameters. It will be shown that from the explicit form of the breaking terms (4.90), by using the nilpotency of the Slavnov-Taylor operator \( S_0 \) and the WTI they are not independent and they are implemented simultaneously. It is important to stress that this particular cancellation of anomalies works only for the SM which contains one neutral massless field.

Let us consider the STI in terms of the mass eigenstates [11, 16]

\[
S(\Gamma) = \int d^4x \left( c_w \partial_\mu c_A - s_w \partial_\mu c_Z \right) \left( c_w \frac{\delta \Gamma}{\delta A_\mu} - s_w \frac{\delta \Gamma}{\delta Z_\mu} \right) + s_w \frac{\delta \Gamma}{\delta \gamma^i} \left( s_w \frac{\delta \Gamma}{\delta A_\mu} + c_w \frac{\delta \Gamma}{\delta Z_\mu} \right)
\]

\(^{22}\)We underline that we use the UV and IR power counting degrees defined within the BPHZL scheme [8], but our considerations remain valid for any renormalization scheme.
where the ellipsis collect those terms which are not involved in the present analysis. Here the symbol \( \Gamma \) stands for the reduced functional given in Eq. (3.12) and the sources \( \gamma_3^A, \gamma_0^A \) — for the third component of the triplet gauge field \( W_3^A \) and for the Goldstone — coincide with the modified ones defined in (3.13). This choice is very convenient, although not strictly necessary, since it automatically takes into account the Faddeev-Popov Eqs. (A.118) and the abelian antighost Eq. (2.6).

The IR dangerous breaking terms are given by the following local monomials

\[
\Delta^3_{S,A} = S_0 \int d^4x \left( x_1 A_\mu Z^\mu + x_2 \gamma_0 c_A + x_3 \gamma_H c_A \right) = \int d^4x \left( x_1 (p) - p \partial_\mu c_A + x_2 (0) \partial \gamma_0 c_A + x_3 \partial \gamma_H s_w \partial \delta_0 \right) \]

where terms with higher powers of fields are not shown.

In order to compute the coefficients \( x_i, i = 1, 2, 3 \) we differentiate both sides of the Eq. (4.91) with respect to \( \tilde{c}_A(p) \) and \( \tilde{Z}_A(-p) \) and with respect to the ordinary derivative \( \partial_\mu \). Then, we evaluate the entire expression at zero momentum

\[
x_1 + v x_2 = i \partial_\mu \frac{\delta^2}{\delta \tilde{c}_A(p) \delta \tilde{Z}_A(-p)} S(\Gamma) \bigg|_{\phi = 0, p = 0} = \]

\[
\begin{align*}
&= \left( c^2_w + s_w \Gamma_{\gamma_c c_A}(0) \right) \Gamma_{\gamma_A}(0) + \left( -s_w + \Gamma_{\gamma_c c_A}(0) \right) c_w \Gamma_{\gamma_A}(0) \\
&- \Gamma_{\gamma_5 c_A}(0) \Gamma_{G_0}(0) - \Gamma_{\gamma_5 c_A}(0) \Gamma_{H_0}(0)
\end{align*}
\]

The conventional decomposition in terms of longitudinal and transverse parts of the gauge two-point functions and the definition for the scalar-gauge mixed two-point functions (Sec. 3) below the Eq. (3.42) are used.

Notice that the two-point functions \( \Gamma_{\gamma_0 c_A}, \Gamma_{\gamma_H c_A} \) must vanish at zero momentum transfer because of the IR degrees of the fields and the BRST sources. The first one, namely \( \Gamma_{\gamma_0 c_A}(0) = 0 \) is ensured by the ghost equations (see Sec. 3), but the second one is a consequence of the STI and the AAE as we will show below. Furthermore to ensure the correct IR behavior of the \( Z - A \) mixing, as discussed in Sec. 3, \( \Gamma_{\gamma_5}^L(0) = 0 \) and, finally, the vanishing of \( x_1 + v x_2 \) is achieved only if \( \Gamma_{\gamma_5 c_A}(0) = s_w \).

The other anomaly coefficients are determined by the following STI

\[
-p^2 x_2 = \frac{\delta^2}{\delta \tilde{c}_A(p) \delta \tilde{G}_0(-p)} S(\Gamma) \bigg|_{\phi = 0} = \]

\[
\begin{align*}
&= -p^2 \left[ \left( c^2_w + s_w \Gamma_{\gamma_5 c_A}(p) \right) \Gamma_{\gamma_5 G_0}(p) + \left( -s_w + \Gamma_{\gamma_5 c_A}(p) \right) c_w \Gamma_{G_0}(p) \right] \\
&- \Gamma_{\gamma_5 c_A}(p) \Gamma_{G_0}(p) - \Gamma_{\gamma_5 c_A}(p) \Gamma_{H_0}(p)
\end{align*}
\]

\[
(M^2_H - p^2) x_3 = \frac{\delta^2}{\delta \tilde{c}_A(p) \delta \tilde{H}(-p)} S(\Gamma) \bigg|_{\phi = 0} = \]

\[
\begin{align*}
&= -p^2 \left[ \left( c^2_w + s_w \Gamma_{\gamma_5 c_A}(p) \right) \Gamma_{\gamma_5 A_0}(p) + \left( -s_w + \Gamma_{\gamma_5 c_A}(p) \right) c_w \Gamma_{A_0}(p) \right] \\
&- \Gamma_{\gamma_5 c_A}(p) \Gamma_{A_0}(p) - \Gamma_{\gamma_5 c_A}(p) \Gamma_{H_0}(p)
\end{align*}
\]
where $M_H$ is the Higgs mass. From the second one computed at zero momentum, we immediately get $x_3 = 0$ as a consequence of $\Gamma_{\gamma c \Gamma}(0) = \Gamma_{\gamma H c \Gamma}(0) = 0$. However the Eqs. (4.93) are not sufficient for our purposes, in fact, in order to verify that also $x_2 = 0$, it is necessary to differentiate the Eq. (4.93) with respect to $p^2$ and to study the system of scalar fields. This has been already analyzed in Sec. 3, however the corresponding identities are derived in the gauge eigenstates. In particular here we show that the WTI and the STI implies that vanishing of the mixed Green’s function $\tilde{\Gamma}_{G_H}(p)$ at zero momentum.

Here we need the following two equations

$$\frac{\delta^2}{\delta c(p)\delta G_0(-p)} S(\Gamma) \bigg|_{\phi=0, p=0} = -\Gamma_{\gamma_0 c \Gamma}(0)\Gamma_{G_0 G_0}(0) - \Gamma_{\gamma H c \Gamma}(0)\Gamma_{H G_0}(0) = 0$$

(4.97)

$$\frac{\delta^2}{\delta c(p)\delta H(-p)} S(\Gamma) \bigg|_{\phi=0, p=0} = -\Gamma_{\gamma_0 c \Gamma}(0)\Gamma_{G_0 H}(0) - \Gamma_{\gamma H c \Gamma}(0)\Gamma_{H H}(0) = 0$$

which can be spoiled by breaking terms reabsorbable without any IR obstruction. As a consequence of these equations, the determinant of the matrix of two-point functions for the scalars (3.47) vanishes (see Eq. 3.54). Finally, by the normalization condition $\Gamma_{G_0 H}(0) = 0$, the two-point function for the neutral Goldstone $\Gamma_{G_0 G_0}(0) = 0$ vanishes at zero momentum. Going back to Eqs. (4.95), we can compute the derivative with respect to $p^2$ and, setting $p = 0$, we deduce that $\Gamma_{AG_0}(0) = 0$ is a necessary and sufficient conditions to guarantee that the coefficients $x_2$ is zero. To complete the proof we have to derive $\Gamma_{G_0 H}(0) = \Gamma_{AG_0}(0) = 0$.

The condition $\Gamma_{G_0 H}(0) = 0$ is satisfied as a consequence of the invariance under the background gauge transformations and the relations among the two-point function with external background fields and the two-point functions with external quantum fields. In particular by the STI

$$\frac{\delta^2}{\delta G_0(p)\delta \Omega_0(-p)} S(\Gamma) \bigg|_{\phi=0, p=0} = -\Gamma_{\gamma_0 \Omega_0}(0)\Gamma_{G_0 H}(0) - \Gamma_{\gamma H \Omega_0}(0)\Gamma_{H H}(0) + \Gamma_{\tilde{G}_0 H}(0) = 0,$$

(4.98)

where $\Omega_0$ is the BRST variation of the background Goldstone $\tilde{G}_0$, and the WTI

$$\left(\Gamma_{G_0 H}(0) + \Gamma_{G_0 H}(0)\right) = 0,$$

(4.99)

obtained by differentiating the functional identities (2.7) with respect to $\tilde{H}(0)$, we immediately get

$$(1 + \Gamma_{\gamma_0 \Omega_0}(0)) \Gamma_{G_0 H}(p) = \Gamma_{\gamma H \Omega_0}(0)\Gamma_{H H}(0).$$

(4.100)

This implies that, if the CP-violation is induced only by means of the CKM matrix for fermions, $\Gamma_{G_0 H}(0) = 0$. In fact this equation relates the CP-odd Green’s function $\Gamma_{G_0 H}(0)$ with external quantum fields which are directly coupled to the fermions to the CP-odd Green’s function $\Gamma_{\gamma H \Omega_0}(0)$ whith external classical fields which do not couple directly to fermions. Therefore we can conclude that $\Gamma_{G_0 H}(0) = \Gamma_{\gamma H \Omega_0}(0) = 0$. Notice that in the case of CP-violation Higgs potential (see for example [18]) this conclusion cannot be achieved any longer.

\[\text{Notice that both the eqs. (4.98)-(4.99) can be spoiled by breaking terms, however they can be removed by counterterms without introducing any IR divergences.}\]
To prove that $\Gamma_{AG_0}(0) = 0$ we consider the following WTI

$$- i p_\mu \left( \Gamma^{L}_{\hat{A}_\gamma} Z_\nu(p) + \Gamma^{L}_{\hat{A}_\beta} Z_\nu(p) \right) = 0$$

(4.101)

which expresses the transversality of the mixed two-point functions. Notice that this identity can be broken by the renormalization procedure by terms of the form $\Delta_{A_\beta} Z_\nu(p) = y_1 p_\mu + y_2 p_\mu^2 p_\mu$ which must be removed by the following counterterms

$$\int d^4 x \left( y_1 \hat{A}_\mu Z_\nu + y_2 \partial^\mu \hat{A}_\mu \partial_\nu Z_\nu \right).$$

(4.102)

Only the first terms has IR degree equal to 3. However, since this term is linear in the quantum field $Z$, it does not contribute to the irreducible Green’s functions and it never produces IR divergences.

From Eq. (4.101), by using the decomposition for two-point functions into longitudinal and transverse part, we arrive to

$$\left( \Gamma^{L}_{\hat{A}_\gamma} + \Gamma^{L}_{\hat{A}_\beta} \right)(p) = 0$$

(4.103)

and the normalization condition $\Gamma^{L}_{\hat{A}_\gamma}(0) = 0$ implies that $\Gamma^{L}_{\hat{A}_\beta}(0) = 0$. Finally, by analyticity of Green’s functions, we obtain that $\Gamma^{T}_{\hat{A}_\beta}(0) = 0$.

Furthermore we know from Sec. 3.4 that the two-point functions with external background fields are in general related, by means of the STI, to the two-point functions with quantum fields. and in particular we have

$$\Gamma^{T}_{\hat{A}_\beta}(p) \propto \Gamma^{T}_{\hat{W}_3}(p) = \Gamma^{T}_{\Omega_{3\gamma_3}}(s_w \Gamma^{T}_{\hat{A}_\gamma}(p) + c_w \Gamma^{T}_{\hat{Z}_Z}(p))$$

(4.104)

where $\Gamma^{T}_{\Omega_{3\gamma_3}}(p)$ is the transverse part of $\Gamma^{T}_{\Omega_{3\gamma_3}}(p)$. This equation implies that $\Gamma^{T}_{\Omega_{3\gamma_3}}(0) \Gamma^{T}_{\hat{Z}_Z}(0) = 0$ to all orders, and, since $\Gamma^{T}_{\hat{Z}_Z}(0) \neq 0$ already at tree level, we deduce that $\Gamma^{T}_{\Omega_{3\gamma_3}}(0)$ must vanish. Notice that this is a consequence of the normalization conditions for the transverse part of the two-point functions for the gauge fields and, it follows from imposing the background gauge invariance as in its tree level form. Removing this essential constraint we cannot deduce this result and the STI must be deformed as discussed in [1]. Notice that the STI (4.104) can be spoiled by breaking terms, however as proved in [22] these breaking terms are compensated by suitable counterterms dependent on the background fields. This means that, as in the case of (4.101), the corresponding counterterms do not produce any IR divergence.

Analogously to the mixed two-point function we have also the following STI for mixed Goldstone-photon two-point function

$$\Gamma_{\hat{A}G_0}(p) \propto \Gamma_{\hat{W}_3G_0}(p) = \Gamma^{L}_{\Omega_{3\gamma_3}}(p) \Gamma_{\hat{W}_3G_0}(p) + \Gamma^{L}_{\Omega_{3\gamma_0}}(p) \Gamma_{\hat{G}_0G_0}(p) + \Gamma^{L}_{\Omega_{3\gamma_H}}(p) \Gamma_{\hat{H}G_0}(p).$$

(4.105)

Notice that if the CP invariance were a symmetry of the model the last terms would be absent. By recalling that $\Gamma_{\hat{H}G_0}(0) = \Gamma_{\hat{G}_0G_0}(0) = 0$, that $\Gamma^{T}_{\hat{A}_\gamma}(0) = \Gamma^{L}_{\Omega_{3\gamma_3}}(0) = 0$ by analyticity and $\Gamma^{(0)}_{\hat{W}_3G_0}(p) \neq 0$, we deduce that $\Gamma_{\hat{A}G_0}(0) = 0$.

To establish that $\Gamma_{\hat{A}G_0}(0) = 0$ we can use again the WTI for the background gauge invariance

$$\left( \Gamma^{L}_{\hat{A}G_0}(p) + \Gamma^{L}_{\hat{A}G_0}(p) \right) = 0$$

(4.106)

24 Notice that for convenience we use the notation $\hat{A}_\mu$ to denote the background field for the photon. However in our approach without an explicit background field for the $U(1)$ gauge boson we have that $\hat{A}_\mu A_\nu = -s_w \hat{W}_3 A_\nu$, where $\hat{W}_3$ is the background field for the $SU(2)$ factor. Using the background field for the photon the following equations stay unchanged.
which immediately implies our claim. Notice that also in this case the WTI can be broken by local terms \( \Delta \lambda A G_0(p) = y_3 p^2 \) (a constant breaking terms is excluded by IR power counting) which is removed by using counterterms of the form

\[
\int d^4 x \left( y_3 \hat{\lambda} \mu \delta^{\mu} G_0 \right)
\]  

(4.107)

which is not IR dangerous and has no effect in irreducible Green’s functions.

Finally we have to show that \( \Gamma_{\gamma H e A}^{(n)}(0) = 0, \forall n \). This can be achieved by considering the AAE and the STI. In particular, by taking the functional derivative of AAE (2.6) with respect to \( \tilde{\gamma}^{H}(p) \) at zero momentum, we have

\[
c_w \Gamma_{\gamma H e A}(0) - s_w \Gamma_{\gamma H e Z}(0) - \nu \Gamma_{\gamma \Omega_0}(0) = 0.
\]  

(4.108)

By considering Eq. (4.98) and \( \Gamma_{G_0 e H}(0) = 0, \Gamma_{\gamma e A}(0) = 0 \), from the second equation of (4.97) we get \( \Gamma_{\gamma H e Z}(0) = 0, \forall n, \) and finally from eq. (4.108) and \( \Gamma_{\gamma H e A}(0) = 0 \) we obtain \( \Gamma_{\gamma H e A}(0) = 0 \) which is our claim.

Finally, by this minimal set of normalization conditions we are able to exclude any IR anomalies and, by considering the new identity

\[
- i \partial_{\mu} \frac{\delta^2}{\delta \hat{c}_A(p) \partial_{\mu} (-p)} S(\Gamma) \bigg|_{\phi=0, p=0} =
\]

\[
= \left[ -s_w \left( c_w + \Gamma_{\gamma e Z}(0) \right) \Gamma_{AA}(0) + \left( s_w^2 - c_w \Gamma_{\gamma e Z}(0) \right) \Gamma_{ZA}(0) \right]
\]

\[
- \Gamma_{\gamma e Z}(0) \Gamma_{G_0 e A}(0) - \Gamma_{\gamma H e Z}(0) \Gamma_{HA}(0) = 0
\]

we obtain the well-known result [11]

\[
\left( \Gamma_{AA}(0) \Gamma_{ZA}(0) - (\Gamma_{ZA}(0))^2 \right) = 0.
\]  

(4.110)

for the \( Z - A \) mixing. Notice that this can be only achieved in restricted set of models. Therefore, by imposing \( \Gamma_{AA}(0) = 0 \), we immediately get \( \Gamma_{AA}(0) = 0 \) from (1.110) and \( \Gamma_{AH}(0) = 0 \) from (1.109).

We conclude the section reminding the reader that this particular cancellation of the IR anomaly can be only achieved in the minimal SM, because there is only one abelian factor.

The analysis of the IR anomalies cancellation in the case of extended models as the 2HDM and in the MSSM requires a more involved proof because any kind of mixing among the scalar fields are admitted by symmetries and CP violation. In the same way the corresponding BRST sources \( \gamma_i \) can mixed among themselves and we have to prove that \( \Gamma_{c A \gamma_i}(0) = 0 \) for all \( i \). The analysis of these model will be presented in a forthcoming paper.

5 Conclusions

We have shown that the BFM allows for a consistent renormalization of non-semisimple gauge theories and the Standard Model. We have discussed the relation between the background gauge invariance, expressed in terms of Ward-Takadashi Identities on the Green’s function and the normalization conditions. In particular, we have analyzed the IR problems and we have defined a “minimal” set of normalization conditions (partially on-shell scheme) compatible with the background gauge invariance which avoids the (off-shell) IR singularities.
Furthermore we have studied the anomalies of STI and WTI with special attention to their IR anomalies. We have shown that the IR anomalies for the SM can be excluded by a suitable choice of free parameters. This implies that, in the case of extended models, the STI and the WTI must be deformed.

In addition, we have considered the problem of mixings in all sectors of non-semisimple gauge models and we discuss the renormalization conditions for the CKM matrix elements and the mixing angles. We are able to formulate the renormalization independently of the CP-invariance, showing that new counterterms are fixed by the WTI for the background gauge invariance and by a proper renormalization of tadpoles.

In conclusion, we have analyzed the renormalization of the background fields and we have proved that the normalizations of their two-point functions are related to the renormalizations of quantum fields. This ensures that there is no independent renormalization of the background fields and the S-matrix elements built by background external Green’s functions, as prescribed by the Background Field Method, have the same poles as the S-matrix constructed in the conventional way.

Acknowledgments

The author is deeply indebted to D. Maison for discussions, suggestions and for a careful reading of the manuscript. I also want to thank C. Becchi and R. Ferrari for useful remarks and comments. Moreover, I would like to acknowledge P. Gambino for useful clarifications on the electroweak radiative calculations and, finally, E. Kraus and K. Sibold concerning the algebraic renormalization.

Appendix A: Functional Operators and their Algebra

In the present section we will present the functional operators and their algebra in the gauge eigenfield formalism. The expressions of those operators in the mass eigenfield formalism is provided in [11, 16].

Starting from the STI (2.5), from the WTI (2.7) for simple factors, from the choice of the gauge fixing and from the AAE discussed in Sec. 2.2, and by requiring the closure of the algebra of the functional operators we derive the complete set of identities.

The choice of the gauge fixing (2.8) is implemented at the quantum level as equations of motion for the $b_a$ fields

$$\frac{\delta \Gamma}{\delta b_a} = \left[ \frac{\delta^{ab} \hat{\nabla}_\mu b^c S}{\delta \gamma^a S} (W - \hat{W})^\mu c S + \frac{\delta^{ab} \partial_\mu (W - \hat{W})}{\delta b_a}^\mu + \frac{\rho^{ab} \hat{\phi} + v}{\delta b_a} + \frac{\lambda^{ac}}{\delta c_a S} \right] \equiv \Delta_{\text{Cl}} b_a. \tag{A.111}$$

For deriving the consistency conditions among the STI and other functional identities we introduce the linearized Slavnov-Taylor operator:

$$S_0 = \int d^4x \left[ \frac{\delta \Gamma_0^a}{\delta S^a} \frac{\delta}{\delta W^{aS}_\mu} + \frac{\delta \Gamma_0}{\delta \gamma^a S} \frac{\delta}{\delta W^{aA}_\mu} + \frac{\delta \Gamma_0}{\delta c^a S} \frac{\delta}{\delta W^{aA}_\mu} + \frac{\delta \Gamma_0}{\delta \phi^a S} \frac{\delta}{\delta \gamma^a S} + \frac{\delta \Gamma_0}{\delta \phi^a S} \frac{\delta}{\delta \gamma^a S} + \frac{\delta \Gamma_0}{\delta \phi^a S} \frac{\delta}{\delta \gamma^a S} + \frac{\delta \Gamma_0}{\delta \phi^a S} \frac{\delta}{\delta \gamma^a S} + \frac{\delta \Gamma_0}{\delta \phi^a S} \frac{\delta}{\delta \gamma^a S} \right]$$

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For deriving the consistency conditions among the STI and other functional identities we introduce the linearized Slavnov-Taylor operator:
where $\Gamma_0$ is the tree level action (2.2). Notice that because of invariance of $\Gamma_0$ under BRST transformations the operator $S_0$ is nilpotent: $S_0^2 = 0$.

Due to the presence of several multiplets of scalars and fermions, the classical action turns out to be invariant under some global symmetries (here labeled by $\alpha$) [19]. The following WTI for the global symmetries implement those symmetries at the quantum level

$$W_\alpha \Gamma = \int d^4x \left[ P_{a,\alpha,\mu}^a \frac{\delta \Gamma}{\delta W_{a,\alpha,\mu}} + P_{\alpha}^i \frac{\delta \Gamma}{\delta \phi^i} + \bar{P}_\alpha^I \frac{\delta \Gamma}{\delta \bar{\psi}^I} + \frac{\delta \Gamma}{\delta \psi^I} P_\alpha^I \right] = 0 \quad (A.113)$$

where $P_{a,\alpha,\mu}^a, P_{\alpha}^i, \bar{P}_\alpha^I, P_\alpha^I$ are local polynomials for each $\alpha$ and $P_{a,\alpha,\mu}^a, \ldots, P_\alpha^I$ are the generators of the global symmetry in the field representation.

In the SM there are four different global currents which correspond to conserved quantum numbers (at the level of perturbation theory), namely the individual lepton numbers $L_e, L_\mu, L_\tau$ and the baryon number $B$. The corresponding currents are given by

$$j_B^\mu = \sum_\beta \left( \bar{Q}_{L,\beta} \gamma_\mu Q_{L,\beta} + \bar{u}_{R,\beta} \gamma_\mu u_{R,\beta} + \bar{d}_{R,\beta} \gamma_\mu d_{R,\beta} \right) \quad (A.114)$$

$$j_L^{\alpha \mu} = \left( T_{L,\alpha} \gamma_\mu L_{L,\alpha} + \bar{e}_{R,\alpha} \gamma_\mu e_{R,\alpha} \right),$$

where $\alpha = e, \mu, \nu$ and $\beta$ counts the generations, which can mix with the current of the hypercharge

$$j_Y^\mu = \sum_\alpha \left( \frac{1}{6} \bar{Q}_{L,\alpha} \gamma_\mu Q_{L,\alpha} + \frac{2}{3} \bar{u}_{R,\alpha} \gamma_\mu u_{R,\alpha} - \frac{1}{3} \bar{d}_{R,\alpha} \gamma_\mu d_{R,\alpha} - \frac{1}{2} L_{L,\alpha} \gamma_\mu L_{L,\alpha} - \bar{e}_{R,\alpha} \gamma_\mu e_{R,\alpha} \right) \quad (A.115)$$

as discussed in [3, 4, 16, 22]. The currents (A.114) provide the four corresponding WTI

$$W_B \Gamma = \int d^4x \partial^\mu j_B^\mu, \quad W_L^{\alpha} \Gamma = \int d^4x \partial^\mu j_L^{\alpha} \quad (A.116)$$

which commute with the other functional equations and among themselves.

Among the functional identities we have the following commutation relations (here expressed for a generic functional $\mathcal{F}$)

$$S_\mathcal{F} S(\mathcal{F}) = 0, \quad S_\mathcal{F} W_{a,s}(\mathcal{F}) - W_{a,s} S(\mathcal{F}) = 0$$

$$\frac{\delta}{\delta b_0} \left( G_{a,a} (\mathcal{F}) - \Delta_{c,A}^{\text{cl}} \right) - G_{a,a} \left( \frac{\delta}{\delta b_0} (\Gamma) - \Delta_{b}^{\text{cl}} \right) = 0$$

$$[W_{a,s}, W_{b,s}] (\mathcal{F}) = (ef)^{a b s c} W_{c,s} (\mathcal{F})$$

$$\left[ W_{a,}, \left( \frac{\delta}{\delta b_\alpha} - \Delta_{b_\alpha}^{\text{cl}} \right) \right] (\mathcal{F}) = (ef)^{a b c} \left( \frac{\delta}{\delta b_\alpha} - \Delta_{b_\alpha}^{\text{cl}} \right) (\mathcal{F}) \quad (A.117)$$

which respectively express the nilpotency of the Slavnov-Taylor operator, the invariance of the Slavnov-Taylor operator under the background gauge transformations of the simple factors, the compatibility between the AAE and the gauge fixing and, finally, the covariance of the Eq. (A.111) with respect to the background gauge transformations.

However the system of the above equations does not close under the (anti-)commutation relations and in particular the Faddeev - Popov equations of motion ($\Phi \Pi$ equations) are derived by requiring
the involution of the system:

\[
\mathcal{G}_a(\Gamma) \equiv \frac{\delta}{\delta b_a} \mathcal{S}(\Gamma) - \mathcal{S}_\Gamma \left( \frac{\delta}{\delta b_a} (\Gamma) - \Delta_{\tilde{c}_a}^{CL} \right) = \\
= \frac{\delta}{\delta c_a} + \delta^{a\alpha} \nabla^\alpha b_a \frac{\delta \Gamma}{\delta b_a} + \rho^{ab}(\phi + v)_{ij} \frac{\delta \Gamma}{\delta \gamma^j} = \\
= -\delta^{a\alpha} \partial^2 c_a - \delta^{a\alpha} \nabla^\alpha b_a \nabla^\beta \phi_{ij} - \rho^{ab}\Omega_i(e^b)(\phi + v)_{ij} \equiv \Delta_{\tilde{c}_a}^{CL}
\]

Although these equations are not independent of the NL Eqs. \((A.111)\) and of STI, they provide a direct method for eliminating the ghost fields from the vertex functional.

The anti-commutation relation between the STI \((2.5)\) and the AAE \((2.6)\) is more interesting, since this implies the following WTI

\[
W_{a,\mu}(\Gamma) \equiv \mathcal{G}_{a,\mu} \mathcal{S}(\Gamma) - \mathcal{S}_\Gamma \left( \mathcal{G}_{a,\mu} (\Gamma) - \Delta_{\tilde{c}_a,\mu}^{CL} \right) = -\partial_\mu \left( \frac{\delta \Gamma}{\delta W^{\alpha}_{a,\mu}} + \frac{\delta \Gamma}{\delta W^{a}_{a,\mu}} \right) + \\
+ (e^b)^{aA}_{ij} \left[ (\Phi + v)_{ij} \frac{\delta \Gamma}{\delta \Phi} + (\Phi + v)_{ij} \frac{\delta \Gamma}{\delta \Phi} + \Omega_j \frac{\delta \Gamma}{\delta \Omega^\mu} + \gamma_j \frac{\delta \Gamma}{\delta \gamma^j} \right] + \\
+ (e^b)^{aA}_{ij} \left[ \eta^R_{ij} \frac{\delta \Gamma}{\delta \eta^R_{ij}} + \psi^R_{ij} \frac{\delta \Gamma}{\delta \psi^R_{ij}} + \eta^R_{ij} \frac{\delta \Gamma}{\delta \eta^R_{ij}} + \psi^R_{ij} \frac{\delta \Gamma}{\delta \psi^R_{ij}} \right] + (R \to L)
\]

which describe the background gauge invariance for the abelian factors \(\mathcal{G}_A\). By eliminating the abelian background gauge fields we get

\[
W_{a,\mu}(\Gamma) = \partial^2 b_{a,\mu}
\]

and, since the l.h.s. is linear in quantum fields \(b_{a,\mu}\) it does not require any new external source to fix its renormalization.

At this point it is easy to check that the functional operators generates an algebra over the space of local and integrated functionals, in particular we have to supply also the following remaining relations

\[
\mathcal{S}_\mathcal{F} \left( \mathcal{G}_a(\mathcal{F}) - \Delta_{\tilde{c}_a}^{CL} \right) + \mathcal{G}_a \mathcal{S}(\mathcal{F}) = 0, \quad \frac{\delta}{\delta b_a} \left( \mathcal{G}_a(\mathcal{F}) - \Delta_{\tilde{c}_a}^{CL} \right) - \mathcal{G}_a \left( \frac{\delta}{\delta b_a} (\Gamma) - \Delta_{\tilde{c}_a}^{CL} \right) = 0,
\]

\[
\mathcal{S}_\mathcal{F} W_{a,\mu}(\mathcal{F}) - W_{a,\mu} \mathcal{S}(\mathcal{F}) = 0, \quad [W_{a,\mu}, W_{b,\nu}](\mathcal{F}) = 0,
\]

\[
[W_{a,\mu}, (\mathcal{G}_b - \Delta_{\tilde{c}_b}^{CL})](\mathcal{F}) = (e^f)^{abc} \left( \mathcal{G}_c - \Delta_{\tilde{c}_c}^{CL} \right)(\mathcal{F}),
\]

\[
[W_{a,\mu}, \left( \frac{\delta}{\delta b_a} - \Delta_{\tilde{c}_b}^{CL} \right)](\mathcal{F}) = (e^f)^{abc} \left( \frac{\delta}{\delta b_c} - \Delta_{\tilde{c}_c}^{CL} \right)(\mathcal{F}),
\]

\[
[W_{a,\mu}, (\mathcal{G}_a - \Delta_{\tilde{c}_a}^{CL})](\mathcal{F}) = 0, \quad \left\{ (\mathcal{G}_a - \Delta_{\tilde{c}_a}^{CL}), (\mathcal{G}_b - \Delta_{\tilde{c}_b}^{CL}) \right\}(\mathcal{F}) = 0,
\]

\[
\left\{ (\mathcal{G}_a - \Delta_{\tilde{c}_a}^{CL}), (\mathcal{G}_b - \Delta_{\tilde{c}_b}^{CL}) \right\}(\mathcal{F}) = 0.
\]

This ensures the integrability of the system (in the sense of the Fröbenius’ theorem).

**Appendix B: Renormalization of NL Equations**

In this section we will discuss the renormalization of the NL Eqs. \((A.111)\) by means of the algebraic technique with special care to the problems of IR divergences. In fact, although the renormalization of the NL equations does not present any algebraic obstruction, we have to be sure that the
introduction of non-invariant counter term does not introduce some IR anomalies. In particular we
do not analyze the complete algebraic renormalization of the NL Eqs. (A.11), but we look only for
the lower dimension terms. The renormalization of these equations is also studied in book [7] and
in recent paper [11].

By using the relation among the $b_a$ and the corresponding anti-ghost fields $\bar{c}^a$, we introduce
the “massless” $b_A^\phi$ and the “massive” $b_Z^{\bar{c}}$ Nakanishi-Lautrup which are the BRST variation of the
anti-ghost fields: $s\bar{c}^a_A = b_A^\phi$, $s\bar{c}^{\bar{c}}_Z = b_Z^\phi$ with the same IR and UV degree.

Then the quantum extension of the NL equations to the all orders gives

$$\frac{\delta \Gamma}{\delta b_A^\phi} = \Delta_{b_A}^{\phi,a} + \left[ Q_{b_A}^a \cdot \Gamma \right]_2^2, \quad \frac{\delta \Gamma}{\delta \bar{b}_Z^{\bar{c}}} = \Delta_{b_Z}^{\phi,a} + \left[ Q_{b_Z}^{\bar{c}} \cdot \Gamma \right]_2$$

(B.1)

where the classical terms $\Delta_{b_A}^{\phi,a}, \Delta_{b_Z}^{\phi,a}$ are specified in Eqs. (A.1). The

Recursively assuming that the lower order breaking terms of the Eqs. (B.1) up to order $h^{n-1}$ are
compensated by means of counterterms we deduce

$$\left[ Q_{b_A}^{a} \cdot \Gamma \right]_2^2 = h^n Q_{b_A}^{a} + O(h^{n+1} Q_{b_A}), \quad \left[ Q_{b_z}^{\bar{c}} \cdot \Gamma \right]_2^1 = h^n Q_{b_z}^{\bar{c}} + O(h^{n+1} Q_{b_z})$$

(B.2)

where $Q_{b_A}^{a}$, $Q_{b_z}^{\bar{c}}$ are local polynomials in terms of fields and their derivatives with zero Faddeev-
Popov charge. By UV and IR power counting, by covariance and by Faddeev-Popov charges, the possible candidates for $Q_{b_A}^{a}$, $Q_{b_z}^{\bar{c}}$ are

$$Q_{b_A}^{a} = Y_{a,1} \partial_{\mu} W_{\mu} + Y_{Z,a} \phi_{\phi} + Y_{Z,2} \phi_{\phi} + Y_{Z,3} \phi_{\phi} + Y_{Z,4} \phi_{\phi} + Y_{Z,5} \phi_{\phi}$$

$$Q_{b_z}^{\bar{c}} = Y_{a,1} \partial_{\mu} W_{\mu} + Y_{A,2} \phi_{\phi} + Y_{A,3} \phi_{\phi} + Y_{A,4} \phi_{\phi} + Y_{A,5} \phi_{\phi}$$

(B.3)

where the coefficients $Y_{A,a}, Y_{Z,a}$ are constant. The only dangerous terms are the $Y_{a,2} \phi_{\phi}$ massless (at least one of them). Nevertheless the non-invariant counter terms $b_A^\phi Y_{A,3} \phi_{\phi}, b_A^\phi Y_{A,3} \phi_{\phi}$ are IR convergent because of the IR degree of the $b_A^\phi$ fields.

Moreover no other obstruction occurs to the compensation of the breaking terms to the NL
equations by means of local counter terms. In the next appendix we will see that for the ghost
equations (2.4) and (A.11), this possibility occurs and a suitable rotation among massless and
massive fields has to be taken into account in order to avoid the IR divergences.

We have to remind that the presence of the non-linear breaking terms $Y_{Z,4} \phi_{\phi}, Y_{Z,9} \phi_{\phi}, Y_{A,4} \phi_{\phi}, Y_{A,9} \phi_{\phi}$ introduce new Feynman rules that have to be considered at the higher orders.

Appendix C: Renormalization of $\Phi\Pi\Phi$ and AAE

In this appendix we briefly discuss the renormalization of the functional equations for the ghost and
anti-ghost fields (2.6) and (A.11). Although in the main text we underlined the implication of the
AAE, here both the Faddeev-Popov equation and the AAE turn out to be necessary to compute
ghost dependent counterterms. The present discussion follows essentially the lines of the proof of
absence of anomalies for the Faddeev-Popov equation and AAE in [8] and in the paper by T.Clark [12].
By using the QAP the Faddeev-Popov equation and the AAE at higher order are given by

\[ \bar{\mathcal{G}}^{ab} \Gamma = \Delta^{CL,c}_{a} + \left[ \bar{Q}^{a} \cdot \Gamma \right]_{2}^{1}, \quad \mathcal{G}^{aa} \Gamma = \Delta^{CL,c}_{a} + [\bar{Q}^{aa} \cdot \Gamma]_{4}^{3} \]  

(C.1)

where \( \Delta^{CL,c}_{a} \) is the left hand side of Eq. (A.118) and \( \Delta^{CL,c}_{a} \) is given in Eq. (2.4). Recursively, by assuming that lower order breaking terms \( \bar{Q}^{a}, Q^{aa} \) are compensated by means of suitable counterterms up to order \( h^{n-1} \) the r.h.s. of Eq. (C.1) become

\[ [\bar{Q}^{a} \cdot \Gamma]_{2}^{1} = h^{n} \bar{Q}^{a} + O(h^{n+1} \bar{Q}) \]
\[ [Q^{aa} \cdot \Gamma]_{4}^{3} = h^{n} Q^{aa} + O(h^{n+1} Q) \]  

(C.2)

where \( \bar{Q}^{a}, Q^{aa} \) are local polynomials in terms of fields and their derivatives with Faddeev-Popov charge +1 and -1 respectively. By UV and IR power counting, by the Lorentz covariance and by Faddeev-Popov charges conservation, the possible candidates for \( \bar{Q}^{a}, Q^{aa} \) are

\[ \bar{Q}^{a} = \bar{X}_{1}^{ab} c_{b} + \bar{X}_{2}^{a[bc]} c_{b} c_{c} \bar{d} + \bar{X}_{3}^{a \gamma} \Omega_{3} + \bar{X}_{\mu, A}^{a b} Q_{b}^{\mu} \]  

(C.3)

\[ Q^{aa} = \bar{X}_{1}^{a b} \bar{c}_{b} + \bar{X}_{2}^{a[bc]} \bar{c}_{b} \bar{c}_{c} \bar{d} + \bar{X}_{3}^{a \gamma \gamma} \gamma_{3} + \bar{X}_{\mu, A}^{a b} \gamma_{b}^{\mu} + \bar{X}_{\mu, 5}^{a f} \eta_{I} + h.c. \]  

(C.4)

where \( \bar{X}_{1}^{ab}, \bar{X}_{2}^{a[bc]}, \bar{X}_{1}^{a b}, \bar{X}_{3}^{a \gamma}, \bar{X}_{\mu, A}^{a b}, \bar{X}_{\mu, 5}^{a f} \) are polynomials of quantum fields while the constant coefficients \( \bar{X}_{1}^{abc} \) are totally antisymmetric tensors of the Lie algebra of \( \mathcal{G} \). This follows from the consistency conditions derived by the commutation relations (A.121):

\[ \mathcal{G}_{(x)}^{a} \mathcal{G}_{(y)}^{b} + \mathcal{G}_{(y)}^{b} \mathcal{G}_{(x)}^{a} = 0, \quad \mathcal{G}_{(x)}^{a} \mathcal{G}_{(y)}^{b} + \mathcal{G}_{(y)}^{b} \mathcal{G}_{(x)}^{a} = 0 \]  

(C.5)

which imply relations among the coefficients of \( \bar{Q}^{a}, Q^{aa} \).

As is well known (see for further details the papers [1] and [2]) the breaking terms (C.3)–(C.4) can be removed by means of counterterms and no anomaly appears for equations (C.1). But, although from an algebraic point of view there are local counterterms which cancel the apparent breaking terms, we have to be sure that those counterterms do not introduce any IR divergences. To this purpose we have to check the structure of the lower dimensional terms in the explicit decomposition (C.3)–(C.4), that is \( \bar{X}_{1}^{ab}, \bar{X}_{1}^{a b} \) and verify if the corresponding counter terms could give IR problems. It is easy to see that the only dangerous candidates are

\[ \mathcal{L}^{c.t.}_{\rho IR \leq 3}(x) = \bar{c}^{\prime \prime}_{A} K^{1}_{a, b'} b'_{A} + \bar{c}^{\prime \prime}_{A} K^{2}_{a, b'} b'_{A} + \bar{c}^{\prime \prime}_{A} K^{3}_{a, b'} (\phi, \hat{\phi}) c'_{A} \]  

(C.6)

with IR degree \( \rho_{IR} \leq 3 \). Only \( K^{3}_{a, b'} (\phi, \hat{\phi}) \) could depend on the scalars if they are massless, otherwise the coefficients are constant and represent mass terms for the massless ghost fields \( c'_{A}, \bar{c}^{\prime \prime}_{A} \). However the last term of the \( \mathcal{L}^{c.t.}_{\rho IR \leq 3} \) is not necessary; in fact IR power counting implies that

\[ \bar{Q}_{\rho IR \leq 1}^{a} = \bar{X}_{1, con}^{ab} b'_{Z}, \quad Q^{aa}_{\rho IR \leq 3} = X_{1, con}^{a b''} b''_{Z} \]  

(C.7)

where \( \bar{X}_{1, con}^{ab}, X_{1, con}^{a b''} \) are the field independent constant parts of the polynomials \( \bar{X}_{1}^{ab}, X_{1}^{a b''} \) and the index \( b'' \) runs only over the indices of massive ghost fields \( b''_{Z} \). Furthermore the consistency conditions (C.3) impose the constraint \( \bar{X}_{1, con}^{a b''} + X_{1, con}^{a b''} = 0 \), reducing the only free coefficients to \( \bar{X}_{1, con}^{a b''} \). In the case \( \bar{X}_{1, con}^{a b''} \not= 0 \) the corresponding counterterms (C.4) cannot be introduced in the tree level action. However another solution can be found. We can use the matrix \( \mathcal{G}^{ab} \) introduced above in order to remove the anomaly terms (C.7), or equivalently, to fix the normalization conditions

\[ \Gamma_{A}^{a_{a'} b''_{Z}} (p^{2} = 0) = 0, \quad \Gamma_{A}^{a_{a'} b''_{Z}} (p^{2} = 0) = 0, \quad \Gamma_{A}^{a_{a'} b''_{Z}} (p^{2} = 0) = 0 \]  

(C.8)
ensuring the correct normalization properties of massless ghost fields. As is well known the IR problems arise when radiative corrections mix massive and massless fields. Therefore the anomalies in the functional equations for the ghost fields can be removed by rotating the anti-ghost fields. Finally we would like to stress that the coefficients \( X_{1,co} \) computed within BPHZL scheme, as a consequence, are zero and the normalization conditions (C.8) are automatically satisfied. On the other side the choice of other normalization conditions for physical fields (as for Standard Model with on-shell normalization conditions) might spoil the (C.8) and spurious anomalies as (C.3)-(C.4) might appear.

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