CONTACT SURGERY GRAPHS

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ABSTRACT. We define a graph encoding the structure of contact surgery on contact 3-manifolds and analyze its basic properties and some of its interesting subgraphs.

1. Introduction

In an unpublished work, William Thurston defined a graph consisting of a vertex $v_M$ for every diffeomorphism type of a closed, orientable 3-manifold $M$. Two vertices $v_M$ and $v_{M'}$ are connected by an edge if there exist a Dehn surgery between $M$ and $M'$. In [HW15], this graph is called the big Dehn surgery graph and studied in various ways.

In this paper we define a directed graph encoding the structure of contact $(\pm 1)$-surgeries on contact 3-manifolds. The contact surgery graph $\Gamma$ is the graph consisting of a vertex $v_{(M,\xi)}$ for every contactomorphism type of a contact 3-manifold $(M,\xi)$. Whenever there exists a Legendrian knot $L$ in $(M_1,\xi_1)$ such that contact $(-1)$-surgery along $L$ yields $(M_2,\xi_2)$ we introduce a directed edge pointing from $v_{(M_1,\xi_1)}$ to $v_{(M_1,\xi_1)}$.

The contact surgery graph is closely tied to the properties of Stein or Weinstein cobordisms and fillings of contact manifolds since contact $(-1)$-surgery along a Legendrian knot in a contact 3-manifold $(M,\xi)$, also known as Legendrian surgery, corresponds to the attachment of a Weinstein 2-handle to the symplectization of $(M,\xi)$. The inverse operation of a contact $(-1)$-surgery is called contact $(+1)$-surgery.

1.1. Properties of the contact surgery graph. In Section 2 we study the basic properties of $\Gamma$. First, we observe that the contact surgery graph $\Gamma$ is connected by the work of Ding–Geiges [DG04] who showed that any contact 3-manifold can be constructed from the standard tight contact structure $\xi_{st}$ on $S^3$ by a sequence of $(\pm 1)$-contact surgeries. On the other hand, we know from [E90, G98] that a contact manifold $(M,\xi)$ is Stein fillable if and only if there exists a directed path from $v_{(\#_nS^1\times S^2,\xi_{st})}$ to $v_{(M,\xi)}$ and thus $\Gamma$ is not strongly connected since there exists non-Stein fillable contact manifolds. (Recall that an oriented graph is strongly connected if for any pair of vertices $(v_1, v_2)$ there exist a path from $v_1$ to $v_2$ following the orientations of the edges.) Etnyre–Honda [EH02] showed that there exists a directed path from any vertex corresponding to a given overtwisted contact manifold to any other vertex, i.e. $\Gamma$ is strongly connected from any vertex corresponding to an overtwisted contact manifold. The question if there exists a vertex such that $\Gamma$ is

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strongly connected to that vertex is equivalent to the open question if there exists a maximal element with respect to the Stein cobordism relation \[\text{We14}\].

We show that \(\Gamma\) stays connected after removing an arbitrary finite collection of vertices and edges. Recall that a graph is called \(k\)-\(connected\) if it is still connected after removing \(k\) arbitrary vertices and \(k\)-edge-\(connected\) if it remains connected after removing \(k\) arbitrary edges.

**Theorem 1.1.** The contact surgery graph \(\Gamma\) is \(k\)-\(connected\) and \(k\)-edge-\(connected\) for any integer \(k \geq 0\).

We equip \(\Gamma\) with its graph metric \(d\). Then the distance from \(v(S^3, \xi_{st})\) to another vertex \(v(M, \xi)\) equals the contact \((\pm 1)\)-surgery number \(cs_{\pm 1}(M, \xi)\), i.e. the minimal number of components of a contact \((\pm 1)\)-surgery link \(L\) in \((S^3, \xi_{st})\) describing \((M, \xi)\) \[EKO22\].

**Proposition 1.2.** The contact surgery graph \(\Gamma\) has infinite diameter, infinite in-degree and infinite outdegree.

Next, we study the existence of Euler- and Hamiltonian walks and paths. We first recall the necessary definitions. A \emph{track} \(t\) in \(\Gamma\) is an infinite sequence whose terms are alternately vertices and edges of \(\Gamma\) starting at a vertex and such that any edge in \(t\) joins the vertices preceding and following the edge in \(t\). A \emph{Hamiltonian walk} of \(\Gamma\) is a track running through any vertex of \(\Gamma\) at least once. A \emph{Hamiltonian path} (Eulerian path) of \(\Gamma\) is a track running through any vertex (edge) of \(\Gamma\) exactly once. If a track is following the direction of the oriented edges it is called \emph{ditrack} and then the definitions of Hamiltonian diwalks, and Hamiltonian- and Eulerian dipaths are obvious.

**Theorem 1.3.** The contact surgery graph \(\Gamma\) admits Eulerian and Hamiltonian paths and also Hamiltonian walks. On the other hand, there exists no Hamiltonian diwalk, no Hamiltonian dipath and no Eulerian dipath in \(\Gamma\).

1.2. **Contact geometric subgraphs of \(\Gamma\).** In Section 3 we study interesting contact geometric subgraphs of \(\Gamma\). We define the subgraphs \(\Gamma_{OT}\), \(\Gamma_{\text{tight}}\), \(\Gamma_{\text{Stein}}\), \(\Gamma_{\text{strong}}\), \(\Gamma_{\text{weak}}\) and \(\Gamma_{c \neq 0}\) consisting of vertices (and the corresponding edges connecting two such vertices) of \(\Gamma\) representing contact manifolds which are overtwisted, tight, Stein fillable, strongly fillable, weakly fillable or with a non-vanishing contact class \(c\) in Heegaard Floer homology. We analyze some of the basic properties of these subgraphs and in particular prove the following results.

**Theorem 1.4.** \(\Gamma_{OT}\) is strongly connected. Each of \(\Gamma_{\text{tight}}, \Gamma_{\text{Stein}}, \Gamma_{\text{strong}}, \Gamma_{\text{weak}}\) and \(\Gamma_{c \neq 0}\) is connected, but not strongly connected.

**Theorem 1.5.** There exists no Hamiltonian diwalk in each of \(\Gamma_{\text{tight}}, \Gamma_{\text{Stein}}, \Gamma_{\text{strong}}, \Gamma_{\text{weak}}\) and \(\Gamma_{c \neq 0}\) and thus also no Hamiltonian- or Eulerian dipath. On the contrary, \(\Gamma_{OT}\) admits an Eulerian dipath and thus also a Hamiltonian diwalk.

1.3. **Topological subgraphs of \(\Gamma\).** In Section 4 we concentrate on topological subgraphs of the contact surgery graph. Let \(\Gamma_M\) denote the subgraph that consists of vertices corresponding to contact manifolds where the underlying topological manifolds are all diffeomorphic to a fixed manifold \(M\). Similarly we denote by \(\Gamma_{(M, s)}\) the subgraph of \(\Gamma_M\) consisting of all contact structures on a fixed manifold \(M\) lying in the same spin\(^c\) structure \(s\).
Theorem 1.6. The connected components of $\Gamma_M$ are given by $\Gamma_{(M,\xi)}$, and thus the connected components of $\Gamma_M$ are in bijection with $H_1(M)$.

The link $\text{lk}(M, \xi)$ of a contact 3-manifold $(M, \xi)$ is defined as

$$\text{lk}(M, \xi) := \text{lk}(v(M, \xi)) := \{v(N, n) | d(v(M, \xi), v(N, n)) = 1\}.$$ 

In [HW15] it is shown that the link of $S^3$ in the topological surgery graph is connected and of bounded diameter. The main question still remains open: Is the link of any topological 3-manifold connected? It turns out that we can answer this question for contact 3-manifolds.

Theorem 1.7. The link $\text{lk}(M, \xi)$ of any contact 3-manifold $(M, \xi)$ is connected and of diameter less than 4.

Conventions. Throughout, this paper we work in the smooth category. We assume all 3-manifolds to be connected, closed, oriented, and smooth; all contact structures are positive and coorientable. For background on contact surgery and symplectic and Stein cobordisms we refer to [GS99, DG04, DGS04, Ge08, We14, DK16, Ke17, CEK21, EKO22]. Legendrian links in $(S^3, \xi_{st})$ are always presented in their front projection. We choose the normalization of the $d_3$-invariant as in [CEK21, EKO22] which differs from the normalizations in [Go98, DGS04, DK16] by $1/2$. Using our normalization we see that contact structures on homology spheres have integral $d_3$-invariants (in particular $d_3(S^3, \xi_{st}) = 0$) and that the $d_3$-invariant is additive under connected sums.

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2. Properties of the Contact Surgery Graph

We start by discussing the basic properties of the contact surgery graph $\Gamma$.

Proof of Proposition 1.2. $\Gamma$ has infinite diameter since a single surgery can change the rank of the first homology at most by one.

To show that the outdegree of $\Gamma$ is infinite, let $v(M, \xi)$ be a vertex of $\Gamma$. For an even integer $n$, choose a Legendrian knot $L$ in a Darboux-ball in $(M, \xi)$ with Thurston–Bennequin invariant $tb = 1$ and rotation number $rot = n$. Denote the contact manifold obtained by contact ($-1$)-surgery along $L$ by $L(-1)$. A calculation as for example in [DGS04, DK16] shows that the homology of $L(-1)$ is

$$H_1(L(-1)) = H_1(M) \oplus \mathbb{Z}_{\mu_L},$$

where the $\mathbb{Z}$-summand is generated by a meridian $\mu_L$ of $L$ and that the Poincaré dual of the Euler class $e(L(-1))$ is given by

$$e(L(-1)) = e(M, \xi) + n\mu_L,$$

where $e(M, \xi)$ denotes the Poincaré dual of the Euler class of $(M, \xi)$. Thus, we get infinitely many different contact manifolds by contact $(-1)$-surgery from $M$. The same construction with contact $(+1)$-surgery along a Legendrian knot with Thurston–Bennequin invariant $tb = -1$ provides the fact that the indegree is also infinite.
Proof of Theorem 1.1. Let \( E \) be a set consisting of \( k \)-different vertices in \( \Gamma \) and let \((M_0, \xi_0)\) and \((M_n, \xi_n)\) be two contact 3-manifolds representing vertices in \( \Gamma \setminus E \). We consider a path

\[(M_0, \xi_0) \to (M_1, \xi_1) \to (M_2, \xi_2) \to \cdots \to (M_n, \xi_n)\]

in \( \Gamma \) between \((M_0, \xi_0)\) and \((M_n, \xi_n)\). We denote by \( L_0 \) a Legendrian surgery link in \((S^3, \xi_0)\) representing \((M_0, \xi_0)\). Let \( K_{i+1} \) be a Legendrian knot in \((M_i, \xi_i)\) such that contact \((\pm 1)\)-surgery along \( K_i \) is contactomorphic to \((M_{i+1}, \xi_{i+1})\). By Lemma 4.7.1 in [Ke17] we can represent \( K_1 \) as a Legendrian knot in the complement of \( L_0 \) and thus \( L_1 := L \cup K_1 \) is a surgery link in \((S^3, \xi_0)\) for \((M_1, \xi_1)\). By induction we get surgery links \( L_i \) in \((S^3, \xi_0)\) representing \((M_i, \xi_i)\) describing the above path in \( \Gamma \). We will construct a path from \((M_0, \xi_0)\) to \((M_n, \xi_n)\) in \( \Gamma \setminus E \).

Let \( p \geq 2 \) be an integer such that no contact structure on \( M_i \# L(p, 1) \) is in \( E \) for all \( i = 0, \ldots, n \). (Such a \( p \) exists because \( E \) is finite.) We define a new sequence of Legendrian surgery links as follows. We set \( L'_0 = L_0 \) and \( L'_{i+1} = L_i \sqcup U_p(-1) \) where \( \sqcup \) denotes the disjoint union of knots and \( U_p(-1) \) denotes the contact \((-1)\)-surgery along a Legendrian unknot with \( \text{tb} = 1 - p \). It follows that \( L'_{i+1} \) represents a contact structure on \( M_i \# L(p, 1) \). Finally we set \( L'_{n+2} \) to be the disjoint union of \( L_n \) with \( U_p(-1) \) together with a \((+1)\)-framed meridian \( \mu_p \) of \( U_p \). Since \( U_p(-1) \sqcup \mu_p(+1) \) yields \((S^3, \xi_n)\) by [AV13], we see that \( L'_{n+2} \) represents \((M_n, \xi_n)\). Thus, a path in \( \Gamma \setminus E \) between \((M_0, \xi_0)\) and \((M_n, \xi_n)\) is constructed. The same argument shows that \( \Gamma \) is also \( k \)-edge connected.

\( \square \)

In fact, the above proof directly implies the following corollary.

Corollary 2.1. Let \( E \) be a finite set of vertices in \( \Gamma \) and let \((M_1, \xi_1)\) and \((M_2, \xi_2)\) be two vertices in \( \Gamma \setminus E \). Then their distances in the corresponding graphs are related as

\[ d_{\Gamma \setminus E}(M_1, M_2) \leq d_\Gamma(M_1, M_2) + 2. \]

With the above results we study the existence of Eulerian and Hamiltonian paths and walks.

Proof of Theorem 1.3. Since \( \Gamma \) is connected, \( k \)-connected and \( k \)-edge-connected, for any natural number \( k \), and of infinite degree the main results from [EGV36, EGV38, Na71] immediately imply that \( \Gamma \) contains Eulerian and Hamiltonian paths.

Since contact \((\pm 1)\)-surgery preserves tightness by [Na66], a Hamiltonian diwalk has to start at an overtwisted contact manifold and has to run first through all overtwisted contact manifolds before reaching to a tight contact manifold. But since there exists infinitely many overtwisted contact manifolds this is not possible.

\( \square \)

Using the main result of [Na66] we conclude the following.

Corollary 2.2. \( \Gamma \) is biased, i.e. there exists a subset \( X \) of the vertices of \( \Gamma \) such that there are infinitely many edges oriented from \( X \) to its complement \( X^c \), but only finitely many edges oriented from \( X^c \) to \( X \).

It would be interesting to find such a set \( X \) explicitly.

Proof of Corollary 2.2. The main result of [Na66] says that an oriented graph admits an Eulerian dipath based at a vertex \( v \) if and only if the graph is countable, connected, 1-coherent, \( v \)-solenoidal and unbiased. We refer to [Na66] for the definitions. Since the contact surgery graph \( \Gamma \) is connected and countable but admits
no Eulerian dipath it is enough to check that $\Gamma$ is 1-coherent and $v$-solenoidal. 1-coherency is a condition of the underlying unoriented graph and since $\Gamma$ admits an undirected Euler path, $\Gamma$ is 1-coherent. Finally, it follows that $\Gamma$ is $v$-solenoidal for any vertex since the indegree and outdegree are infinite for any vertex.

3. Contact geometric subgraphs of $\Gamma$

Here we study the subgraphs $\Gamma_{\text{OT}}$, $\Gamma_{\text{tight}}$, $\Gamma_{\text{Stein}}$, $\Gamma_{\text{strong}}$, $\Gamma_{\text{weak}}$ and $\Gamma_{c \neq 0}$.

Proof of Theorem 1.4. We first show that $\Gamma_{\text{OT}}$ is strongly connected. Let $(M_1, \xi_1)$ and $(M_2, \xi_2)$ be two overtwisted contact manifolds. By [EH02], there exist a directed path $p$ from $(M_1, \xi_1)$ to $(M_2, \xi_2)$ in $\Gamma$. We argue that any vertex in $p$ corresponds to an overtwisted contact manifold. Let us assume the contrary. Since $(M_2, \xi_2)$ is overtwisted, there exists an overtwisted contact manifold $(M_{\text{OT}}, \xi_{\text{OT}})$ that can be obtained by contact $(-1)$-surgery from a tight contact manifold $(M_{\text{tight}}, \xi_{\text{tight}})$ contradicting Wand’s result which says that contact $(-1)$-surgery preserves tightness [Wa15].

Next, we consider $\Gamma_*$ with $* = \text{tight}$, Stein, strong or weak and $c \neq 0$. To show that $\Gamma_*$ is connected we show that there exist an undirected path in $\Gamma_*$ from $(S^3, \xi_{\text{st}})$ to any other contact manifold $(M, \xi)$ with property $*$. We consider the contact $(+1)$-surgery along the Legendrian unknot with Thurston–Bennequin invariant $tb = -2$ and rotation number $rot = 1$ in $(S^3, \xi_{\text{st}})$. It is well-known that the resulting contact manifold is the overtwisted contact structure $\xi_1$ on $S^3$ with normalized $d_3$-invariant equal to 1 [DGS04]. Then by [EH02], there exists a directed path in $\Gamma$ of contact $(-1)$-surgeries from $(S^3, \xi_1)$ to $(M, \xi)$. However, this path runs through at least one overtwisted contact manifold and hence it is not in $\Gamma_*$. In total we get a surgery link $L$ in $(S^3, \xi_{\text{st}})$ describing $(M, \xi)$ with only a single contact surgery coefficient $(+1)$. Now we change the order of the surgeries and first perform all contact $(-1)$-surgeries and at the end we perform the single contact $(+1)$-surgery. Since contact $(-1)$-surgery is known to preserve any of the properties $*$ by [Wa15], [EH00], [We91], [OS05], we get a path in $\Gamma_*$ from $(S^3, \xi_{\text{st}})$ to $(M, \xi)$.

The contact surgery subgraphs $\Gamma_{\text{tight}}$, $\Gamma_{\text{strong}}$, $\Gamma_{\text{weak}}$ and $\Gamma_{c \neq 0}$ are not strongly connected since there exists in each of this graphs a contact manifold $(M, \xi)$ which is not Stein fillable [El96], [Ch08] and then there cannot be a directed path from $(S^3, \xi_{\text{st}})$ to $(M, \xi)$.

Finally we show that $\Gamma_{\text{Stein}}$ is not strongly connected. We show that there exists no directed path of contact $(-1)$-surgeries from $(S^3, \xi_{\text{st}})$ to $(S^3 \times S^2, \xi_{\text{st}})$. Let us assume the contrary. Then we get a simply-connected Stein cobordism from $(S^3, \xi_{\text{st}})$ to $(S^3 \times S^2, \xi_{\text{st}})$. We glue this Stein cobordism to the standard 4-ball filling of $(S^3, \xi_{\text{st}})$ to get a simply connected Stein filling $(W, \omega_{\text{st}})$ of $(S^3 \times S^2, \xi_{\text{st}})$. However, by [El90] any Stein filling of $(S^3 \times S^2, \xi_{\text{st}})$ is diffeomorphic to $S^3 \times D^1$ which is not simply-connected.

Remark 3.1. We remark that we have shown that $\Gamma_{\text{OT}}$ is a strong connected component of $\Gamma$ and we wonder what the other strong connected components of $\Gamma$ are. Using Wendl’s theorem on symplectic fillings of planar contact manifolds [We10] Plamenevskaya [PT12] deduced that any planar Stein fillable contact manifold $(M, \xi)$ cannot be obtained from itself by a sequence of contact $(-1)$-surgeries. It follows that any planar Stein fillable contact manifold is its own strong connected component.
Question 3.2. Is $\Gamma_{\text{OT}}$ the only non-trivial strong connected component of $\Gamma$?

Remark 3.3. We know that each of $\Gamma_{\text{OT}}$, $\Gamma_{\text{tight}}$, $\Gamma_{\text{Stein}}$, $\Gamma_{\text{strong}}$, $\Gamma_{\text{weak}}$ and $\Gamma_{c \neq 0}$ has infinite diameter and infinite in- and outdegree. That $\Gamma_{\text{tight}}$ and $\Gamma_{c \neq 0}$ have infinite indegree follows from the work of Lisca–Stipsicz [LS04]. As part of their main theorem they describe infinitely many different Legendrian knots such that contact (+1)-surgery on them yield tight contact manifolds with non-vanishing contact class. That $\Gamma_{\text{strong}}$, $\Gamma_{\text{weak}}$, and $\Gamma_{\text{Stein}}$ have infinite indegree follows similarly from [CET17]. The other statements follow from the arguments in the proof of Proposition 1.2.

For the proof of Theorem 1.5 we need the following lemma.

Lemma 3.4. Let $(M, \xi^M)$ be an overtwisted manifold and $(M, \xi^M_{\text{stab}})$ be its stabilization, i.e. $(M, \xi^M_{\text{stab}}) = (M, \xi^M) \# (S^3, \xi_1)$. Let $(N, \xi^N)$ be an overtwisted contact manifold which can be obtained from $(M, \xi^M_{\text{stab}})$ by a single contact $(-1)$-surgery. Then we can obtain $(M, \xi^M)$ by a contact $(-1)$-surgery from $(N, \xi^N)$.

Proof of Lemma 1.4. Let $L$ be a Legendrian knot in $(M, \xi^M_{\text{stab}})$ such that $L(-1) = (N, \xi^N)$. Let $L^*$ be the dual surgery knot of $L$ in $(N, \xi^N)$. By the cancellation lemma, $L^*(+1)$ is again contactomorphic to $(M, \xi^M_{\text{stab}})$. Now we choose a loose Legendrian realization $L'$ of $L^*$ such that if we stabilize $L'$ once positive and once negative we get a Legendrian knot which is formally isotopic to $L^*$. (This is possible since we can destabilize any loose Legendrian knot.)

We claim that $L'(-1)$ yields $(M, \xi^M)$. Since $L'$ is topologically isotopic to $L^*$ and its contact framing and the contact framing of $L^*$ differ by 2, the contact $(-1)$-surgery along $L'$ yields topologically the same manifold as the contact $(+1)$-surgery along $L^*$ which is in fact $M$. Note that $L'(-1)$ is overtwisted since $L'$ is a loose knot. Then a straightforward computation in a local model as in [DGS05] shows that the homotopical invariants of the contact structures agree. Thus by Eliashberg’s classification of contact structures [Eli89] it follows that the contact structures are contactomorphic.

Proof of Theorem 1.5. First we remark that by following the proof of Theorem 1.1 we conclude that each of the subgraphs $\Gamma_{\text{OT}}$, $\Gamma_{\text{tight}}$, $\Gamma_{\text{Stein}}$, $\Gamma_{\text{strong}}$, $\Gamma_{\text{weak}}$ and $\Gamma_{c \neq 0}$ is $k$-connected and $k$-edge-connected for any natural number $k$. Thus, each of these subgraphs admits Eulerian and Hamiltonian paths and also Hamiltonian walks. For the directed paths we conclude as in the proof of Theorem 1.3 that $\Gamma_*$ does not admit a Hamiltonian diwalk, a Hamiltonian dipath and a Eulerian dipath for $* = \text{tight}$, Stein, strong, weak or $c \neq 0$.

On the other hand, we show that an Eulerian dipath exists in $\Gamma_{\text{OT}}$. As in the proof of Corollary 2.2 it follows that $\Gamma_{\text{OT}}$ is 1-coherent and $v$-solenoidal. By the main result of [Na66], it is therefore enough to show that $\Gamma_{\text{OT}}$ is unbiased. Assume there exists a subset $X$ of the vertices of $\Gamma_{\text{OT}}$ such that there are infinitely many edges oriented from $X$ to its complement $X^c$, but only finitely many edges oriented from $X^c$ to $X$. As a first step, we show that there exists for any $i \in \mathbb{N}$ a Legendrian knot $K_i$ in an overtwisted contact manifolds $(M_i, \xi_i)$ in $X$, such that all $M_i$ are pairwise non-diffeomorphic and the contact $(-1)$-surgery $K_i(-1)$ along $K_i$ lies in $X^c$. (This step will not use the overtwistedness.)

By assumption we know that there are infinitely many edges pointing out of $X$. For any $i \in \mathbb{N}$, we choose some Legendrian knot $K'_i$ in an overtwisted contact
manifold $(M, \xi)$ in $X$ such that $K'_i(-1)$ lies in $X^c$. We show that we can obtain infinitely many of the $K'_i(-1)$ by a contact $(-1)$-surgery from infinitely many different manifolds in $X$. For that we first observe, that we can get an overtwisted contact structure $\xi_{OT}$ on $M \# L(i+1, 1)$ by a single contact $(+1)$-surgery along a Legendrian unknot $U_i$ in a Darboux ball in $M \setminus K_i$. Since only finitely many edges point into $X$ we conclude that an infinite subset of the $(M \# L(i+1, 1), \xi_{OT})$ are elements of $X$ again. Performing a contact $(-1)$-surgery along $K_i$ in $(M \# L(i+1, 1), \xi_{OT})$ yields $K_i(-1) \# L(i+1, 1)$ with an overtwisted contact structure. And canceling the contact $(+1)$-surgery along $U_i$ by a contact $(-1)$-surgery along a push-off of $U_i$ yields $K_i(-1)$. Thus we can conclude that there exists an infinite family of Legendrian knots $K_i$ in different overtwisted contact manifolds $(M_i, \xi_i)$ in $X$ such that $K_i(-1)$ are elements of $X^c$.

In the next step, which only works for overtwisted contact manifolds, we show that we can get back from the $K_i(-1)$ to $X$ by contact $(-1)$-surgeries. We write $(M_i, \xi_i)$ as $(M, \xi'_i) \# (S^3, \xi_0)$. By Lemma 3.4 there exists a contact $(-1)$-surgery along a Legendrian knot in $K_i(-1)$ yielding $(M_i, \xi'_i)$ and from $(M_i, \xi'_i)$ we can get back to $(M_i, \xi_i)$ by another contact $(-1)$-surgery. Thus we have constructed an infinite family of edges pointing from $X^c$ into $X$ contradicting the assumption and finishing the proof of Theorem 1.5.

Next, we study the difference of the distance functions.

**Theorem 3.5.** Given two overtwisted contact manifolds $(M_1, \xi_1)$ and $(M_2, \xi_2)$.

1. The distance between $(M_1, \xi_1)$ and $(M_2, \xi_2)$ in $\Gamma_{OT}$ is at most 2 larger than their distance in $\Gamma$.

2. The minimal lengths of directed paths from $(M_1, \xi_1)$ to $(M_2, \xi_2)$ agree in $\Gamma$ and $\Gamma_{OT}$.

**Proof.** (1) Let $p$ be a minimal (undirected) path between $(M_1, \xi_1)$ and $(M_2, \xi_2)$ in $\Gamma$. If every vertex in $p$ corresponds to an overtwisted contact manifold the distances in $\Gamma$ and $\Gamma_{OT}$ agree. In general, however, the path $p$ might run through tight contact manifolds. To prevent this we choose an ordered surgery link $L$ in $(M_1, \xi_1)$ such that contact $(\pm 1)$-surgery in that given order along $L$ corresponds to the path $p$. We denote by $\xi_q$ the overtwisted contact structure on $S^3$ with vanishing normalized $d_3$-invariant. A 2-component surgery diagram of $(S^3, \xi_0)$ is shown in Figure 1 (center). We add this surgery diagram in a Darboux ball in the exterior of $L$ to the surgery link, where we first perform the surgeries along the two new Legendrian knots and afterwards the surgeries corresponding to $p$. Then any contact manifold in the path $p$ is replaced by a connected sum with $(S^3, \xi_0)$. By Eliashberg’s classification of overtwisted contact structures [Eli89], it follows that we have constructed a new path $p'$ from $(M_1, \xi_1)$ to $(M_2, \xi_2)$ in $\Gamma_{OT}$ of length 2 larger than the length of $p$.

(2) By Wand’s theorem [Wal15], any directed path in $\Gamma$ between two overtwisted contact manifolds cannot run through a tight contact manifold, see proof of Theorem 1.4.

4. Topological subgraphs of $\Gamma$

Before we prove Theorem 1.6 we first discuss the corresponding result for $S^3$.

**Lemma 4.1.** $\Gamma_{S^3}$ is connected.
Figure 1. Contact surgery diagrams of contact structures on $S^3$.
Left: $(S^3, \xi_{-1})$, center: $(S^3, \xi_0)$, right: $(S^3, \xi_1)$ \cite{DGS04, EKO22}.

Proof. Recall that by the work of Eliashberg $\xi_{st}$ is the unique tight contact structure on $S^3$ \cite{El92} and that the overtwisted contact structures are in one-to-one correspondence to the homotopy classes of tangential 2-plane fields \cite{El89}, which are on homology spheres in bijection with the integers via their normalized $d_3$-invariants \cite{Go98}. We denote the unique overtwisted contact structure on $S^3$ with normalized $d_3$-invariant equal to $n$ by $\xi_n$.

Contact surgery diagrams for all contact structures on $S^3$ where explicitly described in \cite{DGS04}. A contact surgery diagram of $\xi_1$ is given by the contact (+1)-surgery along the Legendrian unknot with Thurston–Bennequin invariant $t = -2$ and rotation number $r = 1$ shown on the right of Figure 1. The contact (±1)-surgery diagram along the 2-component link shown on the left of Figure 1 represents $\xi_{-1}$. The disjoint union of two surgery diagrams describes a connected sum of the underlying contact manifolds. Since the $d_3$-invariant behaves additively under the connected sum, we get contact surgery diagrams of all contact structures on $S^3$ by taking appropriate disjoint unions of the contact surgery diagrams of $\xi_{1}$ and $\xi_{-1}$.

It follows that we can get $(S^3, \xi_1)$ by a single contact (+1)-surgery from $(S^3, \xi_{st})$ and that there exists a contact (+1)-surgery on $(S^3, \xi_k)$ yielding $(S^3, \xi_{k+1})$ and thus conversely a contact (−1)-surgery from $(S^3, \xi_k)$ to $(S^3, \xi_{k-1})$. □

Proof of Theorem 1.6. First we discuss the case that $M$ is a homology sphere. Then we can get any overtwisted contact structure on $M$ from a fixed contact structure on $M$ by connected summing with overtwisted contact structures of $S^3$ \cite{DGS04}. It follows that any two overtwisted contact structures on $M$ can be connected in $\Gamma_M$. Now let $\xi_{tight}$ be some tight contact structure on $M$. Then there there exists a single contact (+1)-surgery along a Legendrian unknot in $(M, \xi_{tight})$ yielding $(M, \xi_{tight}) \# (S^3, \xi_1)$ which is overtwisted and thus it follows that $\Gamma_M$ is connected.

If the underlying manifold is not a homology sphere it gets slightly more complicated since the classification of tangential 2-plane fields is more involved. As a further invariant we have the $spin^c$ structure of a contact structure. However, it is known that for a given contact structure with $spin^c$ structure $s$ we can get any other overtwisted contact structure with the same $spin^c$ structure $s$ by connected summing with the overtwisted contact structures on $S^3$ \cite{DGS04}. Thus, we can apply the same argument as in the homology sphere case to deduce that $\Gamma_{(M, s)}$ is connected.

It remains to show that there is no edge connecting two different $spin^c$ structures on $M$. For that we use Gompf’s $\Gamma$-invariant which classifies $spin^c$ structures \cite{Go98}. Let $s$ be a $spin^c$ structure on $M$ and $\xi$ be a contact structure inducing $s$. Let $L_0$ in $(M, \xi)$ be a Legendrian knot such that contact (+1)- or contact (−1)-surgery along
L₀ yields another contact structure ξ' on M. We want to show that ξ' induces the same spinc structure s. For that we choose a spin structure t, describe (M, ξ) by a contact surgery diagram along a Legendrian link L = L₁ ∪ ... ∪ Lₙ in (S³, ξₛ), and present L₀ as a knot in the exterior of L. To show that ξ and ξ' induce the same spinc structures it is enough to compute that Gompf’s Γ invariants of ξ and ξ' with respect to t agree. We present t via a characteristic sublink (Lⱼ)ⱼ∈J of L₀. Since the homologies of M and L₀(±1) agree we deduce that µ₀ is nullhomologous in L₀(±1) and that J is also a characteristic sublink of the surgery diagram L₀ ∪ L of L₀(±1). Then we can use the formula for computing Gompf’s Γ-invariant from [EKO22] to compute

\[ \Gamma(ξ′, t) - \Gamma(ξ, t) = \frac{1}{2} \left( \sum_{i=0}^{n} r_i \mu_i + \sum_{j \in J} (Q′ \mu)_j \right) - \frac{1}{2} \left( \sum_{i=1}^{n} r_i \mu_i + \sum_{j \in J} (Q \mu)_j \right) = 0, \]

where rᵢ denotes the rotation number and μᵢ the meridian of Lᵢ, Q' and Q denotes the linking matrices of L₀ ∪ L and L, and all non-canceling terms are multiples of μ₀ = 0.

We now turn to the proof of Theorem 1.7. For the proof, we need the following lemma.

**Lemma 4.2.** Let L be a Legendrian knot in some contact 3-manifold (M, ξ) and U be a Legendrian meridian of L with tb = -1. Let L± denote the positive/negative stabilization of L (similarly for U). Then contact (-1)-surgery along L followed by contact (+1)-surgery along U± is contactomorphic to the contact (+1)-surgery along L± (see Figure 2), i.e.

\[ L(-1) \cup U_± (+1) = L_± (+1). \]

![Figure 2](image)

**Figure 2.** From a contact (-1)-surgery to a contact (+1)-surgery.

**Proof.** We relate the two contact surgery diagrams in Figure 2 by first performing two handle slides [DG09, Av13, CEK21] followed by a lantern destabilization [LS11], cf. [EKO22]. We first slide the red knot L over U± as in Figure 3(i). Then, we slide U± over the red one as in Figure 3(ii), where the handle slides are indicated via the arrows. After isotoping Figure 3(iii) to get Figure 3(iv), we apply the lantern destabilization to get Figure 3(v).

**Proof of Theorem 1.7.** Let (N, ξₙ) be a contact manifold in the link of (M, ξ). We construct a path of length one or two in lk(M, ξ) from (N, ξₙ) to (M, ξ) # (S³, ξ₁). (We recall that (M, ξ) # (S³, ξ₁) can be obtained from (M, ξ) by a single contact (+1)-surgery along a Legendrian unknot with tb = -2 in a standard Darboux ball in (M, ξ),)
First, we consider the case that we can obtain \((N, \xi_N)\) by a contact \((+1)\)-surgery along a Legendrian knot \(K\) in \((M, \xi)\). In [Av13] it is shown that performing a contact \((+1)\)-surgery \(K\) followed by a contact \((+1)\)-surgery along a Legendrian meridian \(U\) of \(K\) with \(tb = -1\) corresponds to a negative stabilization of \((M, \xi)\) and thus yields \((M, \xi) \# (S^3, \xi_1)\).

The case that \((N, \xi_N)\) arises as contact \((-1)\)-surgery from \((M, \xi)\) can be reduced to the first case by applying Lemma 4.2 once. □

The proof of Theorem 1.7 directly implies the following corollary.

**Corollary 4.3.** (1) If \((N, \xi)\) is in the link of \((M, \xi)\). Then

\[
\text{d}_{\text{lk}(M, \xi)}((N, \xi), (M, \xi) \# (S^3, \xi_1)) \leq 2.
\]

(2) If \((N, \xi)\) can be obtained from \((M, \xi)\) by a single contact \((+1)\)-surgery, then \((N, \xi)\) can be obtained from \((M, \xi) \# (S^3, \xi_1)\) by a single contact \((-1)\)-surgery.

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