On the future of solutions to the massless Einstein-Vlasov system in a Bianchi I cosmology

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Abstract

We show that massless solutions to the Einstein-Vlasov system in a Bianchi I space-time with small anisotropy, i.e. small shear and small trace-free part of the spatial energy momentum tensor, tend to a radiation fluid in an Einstein-de Sitter space-time with the anisotropy $\Sigma_a^b \Sigma^b_a$ and $\tilde{\omega}^i_j \tilde{\omega}^j_i$ decaying as $O(t^{-\frac{1}{2}})$.

1 Introduction

The massless and massive Einstein-Vlasov equations have been objects of intensive study in mathematical general relativity at least since Rendall drew attention to them in the early 1990’s [16] (see the review [1] for material to 2011). More recently, the stability of Minkowski space among solutions of massless [21] and massive [5, 13] Einstein-Vlasov have been shown, and the methods of Friedrich [7] have been extended to establish existence for massless Einstein-Vlasov with data at space-like future-null-infinity $I$ in the presence of positive cosmological constant ($\Lambda > 0$) or data on an asymptotically hyperbolic initial surface with $\Lambda = 0$. [9].

Massless Einstein-Vlasov has better behaviour under conformal rescaling than massive. This was exploited in [9], and in the opposite direction in [2] and [3] to establish local existence with data given at a conformal gauge (or isotropic) cosmological singularity. In [10] long-time existence and asymptotic behaviour for massive Einstein-Vlasov with $\Lambda > 0$ in the spatially homogeneous setting was

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established, and this was extended to the massless case by [22]. The late-time
asymptotics for massive homogeneous solutions of the Einstein-Vlasov system
are well studied close to self-similar and non self-similar solutions [6, 12, 15].
In [17] a set of conditions was given for long-time existence of various matter
models with spatial homogeneity, and massive Vlasov was identified as a case
satisfying these conditions. It is possible to verify that massless Vlasov satisfies
conditions (1), (2) and (7) in [17], so that this model should also allow a proof
of long-time existence even if \( \Lambda = 0 \). That is what we consider in this article,
in the first instance restricting to Bianchi type I. Note that this is not a special
case of [9] as here there is no assumption of asymptotic flatness. However
this assumption means that, with the restriction to a spatially-homogeneous
distribution function, the Vlasov equation is automatically solved by a distribution
function \( f(p_1, p_2, p_3) \) where the \( p_i \) are conserved quantities for the geodesic
equation obtained from the three translational Killing vectors. Assuming that
the anisotropy of both metric and energy-momentum tensor is small we are able
to show that the space-time tends to a radiation fluid in an Einstein-de Sitter
space-time. In other words the space-time isotropises in the metric and the
energy-momentum tensor. These results are made precise in Theorem 1 in Sec-
tion 3. That this happens was already shown for Bianchi I LRS [19] and Bianchi
I reflection symmetric space-times [18, 8]. In [4] decay rates for the reflection
symmetric case were obtained and a formalism developed which will likely to
be useful in the non-homogeneous case. Here we extend the former results to
the case where no additional symmetries other than Bianchi I are imposed and
we also obtain the optimal decay rate at which this happens.

\section{The massless Einstein-Vlasov system}

In this section we introduce the massless Einstein-Vlasov system. Consider a
four-dimensional oriented and time oriented Lorentzian manifold \((\mathcal{M}, g)\) and a
distribution function \( f \), then the massless Einstein-Vlasov system is written as
\[
G_{\alpha\beta} = T_{\alpha\beta},
\]
\[
\mathcal{L}f = 0,
\]
where \( \mathcal{L} \) is the Liouville operator, \( G_{\alpha\beta} \) is the Einstein tensor and \( T_{\alpha\beta} \) is the energy-momentum tensor defined by
\[
T_{\alpha\beta} = \int_{\mathcal{H}(t)} f p_\alpha p_\beta \omega_p.
\]
Here, the integration is over the future pointing light-cone \( \mathcal{H} \) at a given space-
time point which is defined by
\[
p_\alpha p_\beta g^{\alpha\beta} = 0, \quad p^0 > 0
\]
with the apex removed, and \( \omega_p \) is the Lorentz invariant measure on the light-
cone of \( p \). The basic equations we shall use can be found in Sections 7.3–7.4 and
Chapter 25 of [20]. We also refer to this book for an introduction to the Einstein-
Vlasov system. Let \( \Sigma \) be a spacelike hypersurface in \( \mathcal{M} \) with \( n \) its future directed
unit normal. We define the second fundamental form as \( k(X, Y) = \langle \nabla_X n, Y \rangle \)
for vectors $X$ and $Y$ tangent to $\Sigma$, where $\nabla$ is the Levi-Civita connection of $^4g$. Here and throughout the paper we assume that Greek letters run from 0 to 3, while Latin letters from 1 to 3, and also follow the sign conventions of [20].

The Hamiltonian and momentum constraint equations are, as usual:

$$\mathcal{R} = k_{ij}k^{ij} + k^2 = 2\rho,$$
$$\nabla_i k_{ji} - \nabla_j k = -J_i,$$

where $g$ is the induced metric on $\Sigma$, $k = k_{ab}g^{ab}$ the trace of the second fundamental form, $\mathcal{R}$ and $\nabla$ the scalar curvature and the Levi-Civita connection of $g$ respectively, and matter terms are given by $\rho = T_{\alpha\beta}n^{\alpha}n^{\beta}$ and $J_i X^i = -T_{\alpha\beta}n^{\alpha}X^3$ for $X$ tangent to $\Sigma$.

2.1 The massless Einstein-Vlasov system with Bianchi I symmetry

The metric of a Bianchi spacetime in a left-invariant frame is written as

$$^4g = -dt \otimes dt + g_{ij}(t)\xi^i \otimes \xi^j$$
on $\mathcal{M} = I \times G$ with $e_0$ future oriented. We will need equations (25.17)–(25.18) of [20] (without scalar field) with the notation $T_{ab} = S_{ab}$ and since the 3-metric is flat for Bianchi I with $R_{ab} = R = 0$:

$$g_{ab} = 2k_{ab},$$
$$\dot{k}_{ab} = 2k^i_k a_{ki} - k_{ab} + S_{ab},$$

where the dot means the derivative with respect to time $t$. Note that in the massless case $g^{ab}S_{ab} = \rho$. Since $k_{ab} = k_{ab}(t)$ the constraint equations are as follows:

$$2\rho = -k_{ij}k^{ij} + k^2,$$
$$J_i = 0.$$

The time-derivative of the mixed version of the second fundamental form is given by

$$\dot{k}^a_b = -k^a_b + S^a_b,$$

and by taking the trace of (3) we have

$$\dot{k} = -k^2 + \rho.$$

It is convenient to express the second fundamental form as

$$k_{ab} = \sigma_{ab} + Hg_{ab},$$

where $\sigma_{ab}$ is the trace free part and $H = \frac{1}{3}k$ is the Hubble parameter. Then (4) becomes

$$\dot{H} = -3H^2 + \frac{1}{3}\rho.$$
and (1) becomes, introducing $\Omega$:

$$\Omega := \frac{\rho}{3H^2} = 1 - \frac{1}{6} F,$$

where $F = \frac{\sigma_{ab}^{\alpha\beta}}{H^2}$. In terms of the trace free part [3], transforms into

$$\dot{\delta}^a_b = -3H\sigma^a_b + S^a_b - (3H^2 + \dot{H})\delta^a_b = -3H\sigma^a_b + \pi^a_b,$$

where $\pi^a_b$ is the trace free part of $S^a_b$. By (1) one can eliminate the energy density such that (4) reads:

$$\dot{k} = -\frac{1}{2} k^2 - \frac{1}{2} k_{ij} k^{ij}.$$

Using the trace free part of $k_{ab}$, the Hubble variable and (5) we obtain

$$\frac{d}{dt} \left( \frac{1}{H} \right) = -\frac{\dot{H}}{H^2} = 2 + \frac{1}{6} F = 3 - \Omega.$$

It is convenient to introduce a dimensionless time variable $\tau$ as follows:

$$\frac{dt}{d\tau} = H^{-1},$$

and denote derivation with respect to that variable by a prime. Sometimes it is also useful to use the variable $q$:

$$q = -1 - \frac{\dot{H}}{H^2} = 1 + \frac{1}{6} F,$$

where we have used (7) in the last equation. The evolution equation for $\Sigma^a_b = \frac{\sigma^a_b}{H}$ is

$$(\Sigma^a_b)' = -\left(3 + \frac{\dot{H}}{H^2}\right) \Sigma^a_b + \frac{S^a_b - \frac{1}{3} \delta^a_b \rho}{H^2}.$$

Using (7) and (5) we have

$$(\Sigma^a_b)' = -\Omega (\Sigma^a_b - 3w^a_b + \delta^a_b),$$

with $w^a_b = \frac{S^a_b}{\rho}$. Since $\Sigma^a_b$ is trace free it is convenient to work with $\Sigma_+$ and $\Sigma_-$ defined by

$$\Sigma_+ = \frac{1}{2H} \left( \sigma^2_2 + \sigma^3_3 \right), \Sigma_- = \frac{1}{2\sqrt{3}H} \left( \sigma^2_2 - \sigma^3_3 \right).$$

The evolution equations for $\Sigma_-$ and $\Sigma_+$ can be found in [12]. We use the constraint equation to substitute $\Sigma_-$ in $S_\pm$ and define $w_\pm$ analogously to $\Sigma_\pm$ by

$$w_+ = \frac{\pi^2_2 + \pi^3_3}{2\rho}, w_- = \frac{\pi^2_2 - \pi^3_3}{2\sqrt{3}\rho}.$$

Our equation are thus

$$\Sigma_+ = (q - 2)\Sigma_+ + 3w_+ \Omega,$$

$$\Sigma_- = (q - 2)\Sigma_- + 3w_- \Omega.$$
2.2 The Vlasov equation with Bianchi symmetry

Since we use a left-invariant frame, and assume spatial homogeneity, \( f \) will not depend on \( x^a \). Moreover for Bianchi I the components \( p_a \) of momenta in the invariant frame are constants of motion for the geodesic equation, so that the distribution function can be taken to be \( f(p_1, p_2, p_3) \) with no explicit time dependence. We will assume that \( f \) has compact support in momentum for simplicity. Since \( g^{00} = g^{00} = -1 \) and \( g^{0a} = 0 \), we have \( p^0 = -p_0 = \sqrt{p_a p_b g^{ab}} \), \( \rho = T_{00} \), and \( J_a = -T_{0a} = 0 \). The frame components of the energy-momentum tensor are thus

\[
\rho = (\det g)^{-\frac{1}{2}} \int f p^0 dp, \quad S_{ij} = (\det g)^{-\frac{1}{2}} \int f p_i p_j p^0 dp,
\]

where the distribution function is understood as \( f = f(p) \) with \( p = (p_1, p_2, p_3) \).

The Vlasov equation is simply

\[
\frac{\partial f}{\partial t} = 0. \tag{12}
\]

Consider the derivative of the following quantity:

\[
\omega^{ij} = \frac{S_{ij}}{\rho} = \frac{\int f p_i p_j g^{ab} (p^0)^{-1} dp}{\int f p^0 dp} . \tag{13}
\]

The derivative of \( \omega^{ij} \) with respect to \( \tau \) is

\[
(\omega^{ij})' = -2 w^a \Sigma^a_{ij} + w^d \Sigma^d_{ij} + \Sigma^e_{ij} \xi^{ed}, \tag{14}
\]

where

\[
\xi^{jd} = \int f p_i p_j p^0 g^{ab} (p^0)^{-3} dp, \tag{15}
\]

The derivative of \( \xi^{jd} \) is

\[
(\xi^{jd})' = -2 \Sigma^a_{ij} \xi^{jd}_{ij} - 2 \Sigma^d_{ij} \xi^{jd} + 3 \Sigma^e_{ij} \int f p_i p_j p^0 g^{ab} (p^0)^{-3} dp + \xi^{jd}_{ij} w_i w_j. \tag{16}
\]

As one can see from the last equation, our system is not closed. The derivative of \( \xi^{jd} \) involves higher momentum terms.

3 Main result

We will consider that we are close to the case that \( g_{ab} \) and \( f \) are isotropic. In that case \( \xi^{jd} \) takes some specific values which we will denote by \( \tilde{\xi}^{jd} \). Let us consider the trace free part \( \tilde{w}^i_j = w^i_j - \frac{1}{3} \delta^i_j \) and define \( \tilde{\xi}^{jd} = \xi^{jd} - \tilde{\xi}^{jd} \). Then \( \text{(14)} \) turns into

\[
(\tilde{w}^i_j)' = -\frac{2}{3} \Sigma^i_j + 2 \tilde{w}^i_j \Sigma_j^i + \tilde{w}^i_j \Sigma^a_j (\tilde{w}^i_j + \frac{1}{3} \delta^i_j) + \Sigma^e_j (\tilde{\xi}^{jd} + \tilde{\xi}^{jd} \delta^i_j) . \tag{17}
\]
For the diagonal terms we have

\[ w^\prime_+ = -\frac{2}{3} \Sigma^\prime_+ - \tilde{w}_2^2 \Sigma^\prime_a - \tilde{w}_3^3 \Sigma^\prime_a + \tilde{w}_4^4 \Sigma^\prime_a \left( \Sigma^\prime_+ \frac{1}{3} \right) + \frac{1}{2} \Sigma^\prime_d \left( \xi^2_{2c} + \xi^2_{3c} + \xi^3_{4d} \right), \]

\[ w^\prime_- = -\frac{2}{3} \Sigma^- - \frac{1}{\sqrt{3}} \left( \tilde{w}_2^2 \Sigma^-_a - \tilde{w}_3^3 \Sigma^-_a \right) + \tilde{w}_4^4 \Sigma^-_a \Sigma^- + \frac{1}{2} \Sigma^\prime_d \left( \xi^2_{2c} + \xi^2_{3c} - \xi^3_{4d} \Sigma^\prime_2 \right). \]

In the isotropic case \( g_{ij} = R^2 \delta_{ij} \). The factor \( R^2 \) will disappear in that case in \( \xi_{ic}^{jd} \) due to cancellation, which means the expression is basically an Euclidean one. Denoting by a hat the isotropic case we have

\[ \hat{\xi}_{ic}^{jd} = \delta^{ja} \delta^{ib} \int \hat{f} p_a p_b p_0 |p|^{-3} dp \int \hat{f} |p| dp. \]

with \(|p| = \sqrt{p_a p_b \delta^{ab}}\). For that case the different integrals which can appear are

\[ I_0 = \int \hat{f} |p| dp, \quad I_1 = \int \hat{f} p_2 |p|^{-3} dp = \frac{1}{5} I_0, \quad I_3 = \int \hat{f} p_1^2 p_2^2 |p|^{-3} dp = \frac{I_0}{15} \]

\[ I_2 = \int \hat{f} p_1^2 p_2^2 |p|^{-3} dp = 0, \quad I_4 = \int \hat{f} p_1^2 p_2 p_3 |p|^{-3} dp = 0. \]

Suspending the Einstein summation convention for the next two equations, the only non-vanishing expressions are

\[ \hat{\xi}_{aa}^{ab} = \frac{1}{5}, \quad a = 1, 2, 3 \]

\[ \hat{\xi}_{ab}^{ab} = \hat{\xi}_{ab}^{ba} = \xi_{aa}^{bb} = \frac{1}{15}, \quad a \neq b, a, b = 1, 2, 3. \]

To summarize, our system is

\[ (\Sigma^\prime_a) = -\Omega \left( \Sigma^\prime_a - 3 \xi^a_b \right), \quad (\tilde{w}_i^i)^{\prime} = -\frac{2}{3} \tilde{w}_i^i - 2 \tilde{w}_i^a \Sigma^\prime_a + \tilde{w}_i^b \Sigma^\prime_a \left( \tilde{w}_j^j + \frac{1}{3} \delta_j^j \right) + \Sigma^\prime_d \left( \xi_{ic}^{jd} + \xi_{ic}^{jd} \right). \]

For the diagonal terms we have

\[ \Sigma^\prime_+ = \Omega \left( 3 w^\prime_+ - \Sigma^\prime_+ \right), \]

\[ \Sigma^\prime_- = \Omega \left( 3 w^\prime_- - \Sigma^\prime_- \right). \]

**Lemma 1.** Consider the massless Einstein-Vlasov system with Bianchi I symmetry. Suppose \( \Sigma^\prime_0(\tau_0) = \epsilon_\Sigma, \quad \tilde{w}^i_0(\tau_0) = \epsilon_w, \quad \text{and} \quad \hat{\xi}_{ic}^{jd}(\tau_0) = \epsilon_\xi \) are small initially. Then the following estimates hold:

\[ \Sigma^\prime_a = O\left( \epsilon e^{-\frac{1}{2} \tau} \right), \]

\[ \tilde{w}^j_i = O\left( \epsilon e^{-\frac{1}{2} \tau} \right), \]

where \( \epsilon \) only depends on \( \epsilon_\Sigma \) and \( \epsilon_w \).

**Proof.** Let us assume for an interval \([\tau_0, \tau_1]\) that

\[ \Sigma^\prime_0 = e^{-\frac{1}{2} \tau}, \]

\[ \tilde{w}^j_i = e^{-\frac{1}{2} \tau}. \]
From (15) we have the following bound using the fact the higher order momenta terms are bounded by $\rho$ since $p_a p^b \leq C(p^0)^2$:

$$\int f p_i p_j p^b p^c p^d (p^0)^{-5} dp \leq C \int f p^i p^j p^d (p^0)^{-3} dp \int f p^i dp = C \xi^{jd}_{ic}. \quad (26)$$

Similarly $\xi^{jd}_{ic}$ is also bounded since

$$\xi^{jd}_{ic} \leq C \int f p_i p_j p^b (p^0)^{-1} dp \int f p^i dp \leq C. \quad (27)$$

Using assumption (24) for $\Sigma_{ab}$ and (26)-(27) that

$$\left( \xi^{jd}_{ic} \right)' \leq C |\Sigma_{ab}| \leq \epsilon e^{-\frac{\tau}{4}},$$

for all indices. The same bound holds for $\tilde{\xi}^{jd}_{ic}$. Integrating we obtain that

$$\tilde{\xi}^{jd}_{ic}(\tau) \leq \tilde{\xi}^{jd}_{ic}(\tau_0) + C \epsilon. \quad (28)$$

This means that this quantity does not necessarily get smaller, but we can bound it by a small quantity.

If we focus on the first order terms in (18) and (19) we have that

$$\left( \Sigma_{ab} \right)' = -\Sigma_{ab} + 3 \tilde{w}_a^b + O_{\Sigma}, \quad (29)$$

$$\left( \tilde{w}_i^j \right)' = -\frac{2}{3} \Sigma^j_i + \Sigma^c_i \tilde{\xi}^{jd}_{ic} + O_w, \quad (30)$$

with

$$O_{\Sigma} = \frac{1}{6} \Sigma_{ab} \Sigma_{ac} \left( \Sigma_{bc} - 3 \tilde{w}_a^b \right),$$

$$O_w = -2 \tilde{w}_a^b \Sigma^j_i \left( \tilde{w}_i^j + \frac{1}{3} \delta^j_i \right) + \Sigma^c_i \tilde{\xi}^{jd}_{ic}.$$ 

The diagonal terms for $\Sigma_{ab}$ can be expressed by $\Sigma_{\pm}$, so that

$$\Sigma'_+ = -\Sigma_+ + 3 w_+ + O_{\Sigma}, \quad (31)$$

$$\Sigma'_- = -\Sigma_- + 3 w_- + O_{\Sigma}. \quad (32)$$

Since most of the quantities $\tilde{\xi}^{jd}_{ic}$ vanish we have simple expressions. The expressions can be separated into diagonal and non diagonal terms. For $w_{\pm}$ we obtain

$$w'_+ = -\frac{8}{15} \Sigma_+ + O_w, \quad (33)$$

$$w'_- = -\frac{8}{15} \Sigma_- + O_w. \quad (34)$$

This means we have the same system for $\Sigma_{\pm}, w_{\pm}$, namely

$$\begin{pmatrix} \Sigma_{\pm} \\ w_{\pm} \end{pmatrix}' = \begin{pmatrix} -1 \frac{3}{5} \\ \frac{8}{15} \end{pmatrix} \begin{pmatrix} \Sigma_{\pm} \\ w_{\pm} \end{pmatrix} + \begin{pmatrix} O_{\Sigma} \\ O_w \end{pmatrix}, \quad (35)$$
The non-diagonal terms satisfy for any $i \neq j$

$$
\begin{pmatrix}
\Sigma_j^i \\
\bar{\Sigma}_j^i \\
\bar{w}_j^i \\
\bar{w}_j^i
\end{pmatrix}' =
\begin{pmatrix}
-1 & 0 & 3 & 0 \\
0 & -1 & 0 & 3 \\
-9 & 1 & 0 & 0 \\
-15 & 15 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\Sigma_j^i \\
\bar{\Sigma}_j^i \\
\bar{w}_j^i \\
\bar{w}_j^i
\end{pmatrix} +
\begin{pmatrix}
O_{\Sigma} \\
O_{\Sigma} \\
O_{w} \\
O_{w}
\end{pmatrix}.
$$

(36)

As one can see from this, except for the error term, the diagonal and the non-diagonal part are independent.

Consider the diagonal case first. The eigenvalues of the linearised system are:

$$
\lambda = \frac{1}{10}(-5 \pm 3\sqrt{15}).
$$

Making the canonical transformation to

$$
\mathbf{Y} = \begin{pmatrix}
\bar{\Sigma}_\pm \\
\bar{w}_\pm 
\end{pmatrix} = P^{-1} \begin{pmatrix}
\Sigma_\pm \\
\bar{w}_\pm 
\end{pmatrix},
$$

with

$$
P^{-1} = \begin{pmatrix}
-\frac{16}{15} & \frac{1}{4} \sqrt{15} \\
0 & \frac{1}{4} \sqrt{15}
\end{pmatrix}
$$

(38)

we obtain

$$
\mathbf{Y}' = \frac{1}{10} \begin{pmatrix}
-5 & -3\sqrt{15} \\
3\sqrt{15} & -5
\end{pmatrix} \mathbf{Y} + \begin{pmatrix}
O_{\Sigma} + O_{w} \\
O_{w}
\end{pmatrix}.
$$

(39)

Defining

$$
R = \|\mathbf{Y}\|^2,
$$

we obtain

$$
R' = -R + \|\mathbf{Y}\| (O_w + O_{\Sigma}).
$$

Introducing spherical coordinates for $\bar{\Sigma}_\pm$ and $\bar{w}_\pm$ we obtain the desired estimate for the diagonal components up to an epsilon, since the error term $\|\mathbf{Y}\| (O_w + O_{\Sigma})$ is of third order and we can choose the initial value for $\xi_{id}^{ij}(\tau_0)$ smaller than $\epsilon_{\Sigma}$. Choosing $\epsilon_{\Sigma}$ and $\epsilon_w$ small enough we have improved the bootstrap assumptions (24)-(25) for the diagonal components. For the non-diagonal components we obtain the eigenvalues

$$
\lambda = \frac{1}{2}(-1 \pm i\sqrt{7}), \frac{1}{10}(-5 \pm 3i\sqrt{15}).
$$

Proceeding as for the diagonal components we obtain the desired estimates for the non-diagonal ones up to an $\epsilon$ as well. Now we can do another loop as for instance in Section 5 of [13] to get rid of the epsilon and thus to obtain the desired estimates.

Based on the previous lemma we can obtain estimates of other variables.
**Lemma 2.** Consider initial data which correspond to a massless solution of the Einstein-Vlasov system with Bianchi I symmetry which expands initially, i.e. \( H(t_0) > 0 \) and satisfies the conditions of Lemma 1, then

\[
H = \frac{1}{2} t^{-1} \left( 1 + O(\epsilon^2 t^{-\frac{1}{3}}) \right),
\]

\[
g_{ab} = t(G_{ab} + O(\epsilon t^{-\frac{1}{3}})),
\]

\[
g^{ab} = t^{-1} (G^{ab} + O(\epsilon t^{-\frac{1}{3}})),
\]

where \( G_{ab} \) and \( G^{ab} \) are constant matrices and \( \epsilon \) a small quantity.

**Proof.** From (7) we obtain

\[
\frac{1}{H} \leq 3(t - t_0) + \frac{1}{H(t_0)}.
\]

Using (8) and setting \( t_0 = 1/(2H(t_0)) \) we obtain

\[
\frac{dt}{d\tau} \leq 3t.
\]

Integrating and doing some computations we obtain

\[
e^{-\tau} \leq Ct^{-\frac{1}{3}}.
\]

From (7) we also obtain

\[
H = \frac{1}{2} t^{-1} \frac{1}{1 + \frac{1}{2} t^{-1} I},
\]

with

\[
I = \frac{1}{6} \int_{t_0}^{t} \Sigma_{ab} \Sigma_{ab}^{\ast} (s) ds.
\]

Using now (45) we obtain

\[
I \leq C \epsilon^2 t^{\frac{2}{3}}.
\]

As a result we obtain the estimate for \( H \)

\[
H = \frac{1}{2} t^{-1} \left( 1 + O \left( \epsilon^2 t^{-\frac{1}{3}} \right) \right).
\]

Using this in (8) again we obtain

\[
e^{-\tau} \leq Ct^{-\frac{1}{3}}.
\]

Doing another loop of the computations we obtain the desired estimate of \( H \).

To obtain the estimate of \( g_{ab} \) we do the same computations as in [11] concerning the estimate of the metric. Note that

\[
|H - \frac{1}{2} t^{-1}| \leq C \epsilon^2 t^{-\frac{1}{3}}
\]

(50)

\[
(\sigma_{ab} \sigma^{ab})^{\frac{1}{2}} \leq C \epsilon t^{-\frac{2}{3}},
\]

(51)

and the estimate for \( g_{ab} \) and \( g^{ab} \) follow by the same computations as in [11]. \( \square \)
Let us summarize our results in the following theorem:

**Theorem 1.** Consider initial data which correspond to a massless solution of the Einstein-Vlasov system with Bianchi I symmetry which expands initially, i.e. $H(t_0) > 0$. Suppose $\Sigma_j^i(\tau_0) = \epsilon \Sigma$, $\tilde{w}_j^i(\tau_0) = \epsilon w$, and $\tilde{\xi}_j^d(\tau_0) = \epsilon \xi$ are small initially. Then the following estimates hold:

$$\Sigma_j^i = O(\epsilon t^{-\frac{7}{4}}),$$  \hspace{1cm} (52)
$$\tilde{w}_j^i = O(\epsilon t^{-\frac{7}{4}}),$$  \hspace{1cm} (53)
$$H = \frac{1}{2}t^{-1}(1 + O(\epsilon^2 t^{-\frac{1}{2}}),$$  \hspace{1cm} (54)
$$g_{ab} = t(G_{ab} + O(\epsilon t^{-\frac{1}{4}})),$$  \hspace{1cm} (55)
$$\tilde{g}^{ab} = t^{-1}(G^{ab} + O(\epsilon t^{-\frac{1}{4}})),$$  \hspace{1cm} (56)

where $\epsilon$ only depends on the initial data.

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