Multidimensional Gravity on the Principal Bundles

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Abstract

The multidimensional gravity on the total space of principal bundle is considered. In this theory the gauge fields arise as nondiagonal components of multidimensional metric. The spherically symmetric and cosmology solutions for gravity on SU(2) principal bundle are obtained. The static spherically symmetric solution is wormhole-like solution located between two null surfaces, in contrast to 4D Einstein-Yang-Mills theory where corresponding solution (black hole) located outside of event horizon. Cosmology solution (at least locally) has the bouncing off effect for spatial dimensions. In spirit of Einstein these solutions are vacuum solutions without matter.

Key words: multidimensional gravity, principal bundle, wormhole-like and cosmology solutions.
I. INTRODUCTION

As well known the Yang-Mills gauge field is the geometrical object: a connection on the principal bundle. The base of this bundle is spacetime and the fibres are structural group. If we include gravity then it acts only on the base of this bundle and this is the Einstein-Yang-Mills theory. The simplest extension of such 4D gravity is assumption that the gravity acts on total space of the principal bundle. The relation between two these theories give the following theorem [1]-[2]:

Let $G$ group be the fibre of principal bundle. Then there is the one-to-one correspondence between $G$-invariant metrics on the total space $\mathcal{X}$ and the triples $(g_{\mu\nu}, A^a_\mu, h_{\gamma ab})$. Where $g_{\mu\nu}$ is Einstein’s pseudo-Riemannian metric on the base; $A^a_\mu$ is gauge field of the $G$ group (the nondiagonal components of multidimensional metric); $h_{\gamma ab}$ is symmetric metric on the fibre.

The such multidimensional (MD) gravity differs from standard MD gravity by following manner:

1. The extremal dimensions (fibres of bundle) are not equivalent to spacetime dimensions, as they make up group.

2. Any physical fields on the total space of bundle (including MD metric) can depend only on base (spacetime) coordinates.

3. In this vacuum theory the gauge fields appears by natural way as a nondiagonal components of MD metric.

In standard MD gravity the gauge field is added as an external matter field. See, for
example [3]-[4], where the spherically symmetric and cosmological solutions are obtained in MD gravity coupling with generalized Maxwell field. In [5] an inhomogeneous multidimensional cosmological model with a higher dimensional space-time manifold \( M = M_0 \times \prod_{i=1}^{n} M_i \) \((n \geq 1)\) are investigated under dimensional reduction to tensor-multi-scalar theories. \((M_0\) is Einstein’s spacetime, \(M_i\) are internal spaces).

**II. GRAVITY EQUATIONS**

We note that the metric on the fibre can have only following view:

\[
ds_{\text{fiber}}^2 = h(x^\mu)\sigma^a \sigma_a,
\]

where conformal factor \( h(x^\mu) \) depends only on spacetime coordinates \( x^\mu; \mu = 0, 1, 2, 3; \sigma_a = \gamma_{ab} \sigma^b; \gamma_{ab} \) is Euclidean metric; \( a = 4, 5, \ldots, N \) index on fibre (internal space). This follows from the fact that the fibre is a symmetrical space (gauge group). \( \sigma^a \) are one-form satisfies Maurer - Cartan structure equations:

\[
d\sigma^a = f^a_{bc} \sigma^b \wedge \sigma^c,
\]

where \( f^a_{bc} \) is a structural constant of gauge group. Thus, MD metric on the total space can be written in the following view:

\[
ds^2 = ds_{\text{fiber}}^2 + 2G_{A\mu} dx^A dx^\mu,
\]

where \( A = 0, 1, \ldots, N \) is multidimensional index on the total space.

Hence we have only following independent degrees of freedom: conformal factor \( h(x^\mu) \) and MD metric \( G_{A\mu} \). Varying with respect to these variables leads to the following gravity equations:
\[ R_{A\mu} - \frac{1}{2} G_{A\mu} R = 0, \quad (4) \]

\[ R^\alpha_\alpha = R^1_1 + R^5_5 + R^6_6 = 0. \quad (5) \]

These equations are Einstein’s MD equations for gravity on the principal bundle in vacuum. Below we consider two cases: spherically symmetric and cosmology solutions.

**III. WORMHOLE-LIKE SOLUTIONS**

**A. U(1) case**

We remind the solution for 5D Kaluza - Klein’s theory derived in [6]. The metric is:

\[ ds^2 = e^{2\nu(r)} dt^2 - e^{2\psi(r)} (d\chi - \omega(r) dt)^2 - dr^2 - a^2(r)(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (6) \]

where \( \chi \) is the 5th supplementary coordinate; \( r, \theta, \varphi \) are 3D polar coordinates; \( t \) is the time. The solution of 5D Einstein’s equations is:

\[ a^2 = r_0^2 + r^2, \quad (7) \]

\[ e^{-2\psi} = e^{2\nu} = \frac{2r_0 r^2}{\frac{q}{r_0^2 - r^2}}, \quad (8) \]

\[ \omega = \frac{4r_0^2 r}{q \frac{r_0^2}{r^2} - r^2}. \quad (9) \]

This solution is the wormhole-like object located between two null surfaces \( r = \pm r_0 \).

We note that this solution is nonsingular in \( |r| \leq r_0 \). Really, determinant of this MD metrics is equal to:

\[ \det(G_{AB}) = \sin^2 \theta(r_0^2 + r^2)^2 \quad (10) \]

this is indirect confirmation that multidimensional metrics doesn’t have singularity on null surface \( r = \pm r_0 \). Also we can say that from Einstein’s equations follows that \( R^A_0 = 0 \) and
hence $R^B_A R^A_B < \infty \ (A, B = 0, 1, 2, 3, 4)$. At last it can shown that invariant $R^A_{BCD} R^B_{ACD} < \infty$.

Such solution we can name as cutting off wormhole (WH) in contrast to standard WH joining two asymptotically flat regions.

**B. SU(2) monopole-like case**

We can introduce the Euler’s angles $\alpha, \beta, \gamma$ on fibre ($SU(2)$ group). Then one-forms $\sigma^a$ can be written as a follows:

\[
\sigma^1 = \frac{1}{2}(\sin \alpha d\beta - \sin \beta \cos \alpha d\gamma),
\]

\[
\sigma^2 = -\frac{1}{2}(\cos \alpha d\beta + \sin \beta \sin \alpha d\gamma),
\]

\[
\sigma^3 = \frac{1}{2}(d\alpha + \cos \beta d\gamma),
\]

where $0 \leq \beta \leq \pi, 0 \leq \gamma \leq 2\pi, 0 \leq \alpha \leq 4\pi$. We see a solution of the form:

\[
ds^2 = e^{2\nu(r)} dt^2 - r_0^2 e^{2\psi(r)} \sum_{a=1}^{3} (\sigma^a - A^a_{\mu}(r) dx^\mu)^2 - dr^2 - a^2(r) \left(d\theta^2 + \sin^2 \theta d\varphi^2\right).
\]

We choose the “potentials” $A^a_{\mu}$ in following monopole-like form:

\[
A^a_{\varphi} = \frac{1}{2} (f(r) + 1) \{\sin \varphi \cos \theta; \sin \varphi \sin \theta; -\cos \theta\},
\]

\[
A^a_\theta = \frac{1}{2} (f(r) + 1) \sin \theta \{\cos \varphi \cos \theta; \sin \varphi \cos \theta; -\sin \theta\},
\]

\[
A^a_t = v(r) \{\sin \theta \cos \varphi; \sin \theta \sin \varphi; \cos \theta\},
\]

Let us introduce tetrads $e^A_{\dot{A}}$:

\[
ds^2 = \eta_{\dot{A}\dot{B}} \Sigma^{\dot{A}} \Sigma^{\dot{B}},
\]

\[
\Sigma^{\dot{A}} = e^A_{\dot{A}} dx^A,
\]
where $\bar{A}, \bar{B} = 0, 1, \ldots, 6$ are tetrads indexes; $\eta_{AB}$ is 7D Minkowski metric. The input equations are written below in the following form:

$$R_{\bar{A} \mu} = 0, \quad (20)$$

$$R^a_a = 0. \quad (21)$$

7D gravity equations become:

$$\nu'' + \nu'^2 + 3\nu' \psi' + 2\frac{a' \nu'}{a} - \frac{r_0^2}{2} e^{2(\psi - \nu)} \nu'^2 - \frac{r_0^2}{a^2} \psi'^2 e^{2(\psi - \nu)} = 0, \quad (22)$$

$$\nu'' + \nu'^2 + 3\psi'' + 3\psi'^2 + 2\frac{a''}{a} - \frac{r_0^2}{2} e^{2(\psi - \nu)} \nu'^2 + \frac{r_0^2}{2a^2} \nu'^2 e^{2(\psi - \nu)} + \frac{r_0^2}{8a^4} (f^2 - 1)^2 = 0, \quad (23)$$

$$\frac{a''}{a} + \frac{a'}{a} (\nu' + 3\psi') + \frac{a'^2}{a^2} - \frac{1}{a^2} + \frac{r_0^2}{8a^2} e^{2\psi} f'^2 - \frac{r_0^2}{2a^2} \nu'^2 e^{2(\psi - \nu)} + \frac{r_0^2}{8a^4} (f^2 - 1)^2 = 0, \quad (24)$$

$$\psi'' + 3\psi'^2 + 2\frac{a' \psi'}{a} + \psi' \nu' + \frac{r_0^2}{6} e^{2(\psi - \nu)} \nu'^2 - \frac{2}{r_0^2} e^{-2\psi} - \frac{r_0^2}{12a^2} e^{2\psi} + \frac{r_0^2}{3} \frac{v'^2 f'^2}{a^2} e^{2(\psi - \nu)} - \frac{r_0^2}{24a^4} (f^2 - 1)^2 = 0, \quad (25)$$

$$f'' + f'(\nu' + 5\psi') + 2v^2 f e^{-2\nu} = \frac{f}{2a^2} (f^2 - 1), \quad (26)$$

$$v'' - v' (\nu' - 5\psi') - 2\frac{a'}{a} = 2\frac{v}{a} f^2, \quad (27)$$

here the Eq’s(26) and (27) are “Yang - Mills” equations for nondiagonal components of the MD metric. For simplicity we consider $f = 0$ case. This means that we have “color electrical” field $A^a_i$ only (i=1,2,3). In this case it is easy to integrate Eq.(27):

$$v' = \frac{q}{r_0 a^2} e^{\nu - 5\psi}, \quad (28)$$

where $q$ is the constant of the integration (“color electrical” charge). Let us examine the most interesting case when the linear dimensions of fibres $r_0$ are vastly smaller than the space dimension $a_0$ and “charge” $q$ is sufficiently large:

$$\left( \frac{q}{a_0} \right)^{1/2} \gg \left( \frac{a_0}{r_0} \right)^2 \gg 1, \quad (29)$$

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where \( a_0 = a(r = 0) \) is the throat of the WH.

On this approximation we deduce the equations system:

\[
\begin{align*}
\nu'' + \nu'^2 + 3 \nu' \psi' + 2 \frac{a' \nu'}{a} - \frac{q^2}{2a^4} e^{-8\psi} &= 0, \\
\nu'' + \nu'^2 + 3 \psi'' + 3 \psi'^2 + 2 \frac{a''}{a} - \frac{q^2}{2a^4} e^{-8\psi} &= 0, \\
\psi'' + 3 \psi'^2 + 2 \frac{a' \psi'}{a} + \psi' \nu' + \frac{q^2}{6a^4} e^{-8\psi} &= 0, \\
\frac{a''}{a} + \frac{a'}{a} (\nu' + 3 \psi') + \frac{a'^2}{a^2} - \frac{1}{a^2} &= 0.
\end{align*}
\]

This system has the following solution:

\[
\begin{align*}
\nu &= -3\psi, \\
a^2 &= a_0^2 + r^2, \\
e^{-\frac{3}{2}\nu} &= \frac{q}{2a_0} \cos \left( \sqrt{\frac{8}{3}} \arctan \frac{r}{a_0} \right), \\
v &= \sqrt{6} \frac{a_0}{r_0q} \tan \left( \sqrt{\frac{8}{3}} \arctan \frac{r}{a_0} \right).
\end{align*}
\]

Let us define value \( r \) in which metric has null surfaces. From condition:

\[
G_{tt}(r_g) = e^{2\nu(r_g)} - r_0^2 e^{2\psi(r)} \sum_{a=1}^{3} (A^a_t(r_g))^2 = 0
\]

it follows that:

\[
\frac{r_g}{a_0} = \tan \left( \sqrt{\frac{3}{8}} \arcsin \sqrt{\frac{2}{3}} \right) \approx 0.662.
\]

It is easy to verify that \( \exp(2\nu) = 0 \) (\( \exp(2\psi = \infty) \) by \( r/a_0 = \tan(\pi \sqrt{3/32}) \approx 1.434 \). This value lies beyond the null surfaces. This means that the small terms in (22), (25) will stay also small even near the null surfaces.
IV. SU(2) COSMOLOGY SOLUTION

Analogous to spherically symmetric metric (14) we will search the MD cosmology metric on the total space as follows:

\[ ds^2 = dt^2 - b^2(t) \sum_{a=1}^{3} \left( \sigma^a - A^a_{\mu}(r)dx^\mu \right)^2 - a^2(t) \left( \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \right), \]

\[ \sigma_x = \frac{1}{2}(\sin \psi d\theta - \sin \theta \cos \psi d\varphi), \]

\[ \sigma_y = \frac{1}{2}(-\cos \psi d\theta - \sin \theta \sin \psi d\varphi), \]

\[ \sigma_z = \frac{1}{2}(d\psi + \cos \theta d\varphi). \]

Here \( \psi, \theta, \varphi \) are Euler’s angles on \( S^3 \) (3D sphere is spacelike section of Universe). We write down the non-diagonal components of MD metric in following instanton-like form:

\[ A^a_x = \frac{1}{4} \{- \sin \theta \cos \varphi; - \sin \theta \sin \varphi; \cos \theta\} (f(t) - 1), \]

\[ A^a_\theta = \frac{1}{4} \{- \sin \varphi; - \cos \varphi; 0\} (f(t) - 1), \]

\[ A^a_\varphi = \frac{1}{4} \{0; 0; 1\} (f(t) - 1). \]

After substitution to initial gravity equations (20)-(21) we have:

\[ \frac{\ddot{a}}{a} = \frac{2}{a^2} + \frac{4}{b^2} + 2\frac{\dot{a}^2}{a^2} + 4\frac{\dot{b}^2}{b^2} + \frac{9}{ab} + \frac{3}{8} f^2 \frac{b^2}{a^2}, \]

\[ \frac{\ddot{b}}{b} = -\frac{2}{a^2} - \frac{4}{b^2} - 2\frac{\dot{a}^2}{a^2} - 4\frac{\dot{b}^2}{b^2} - \frac{9}{ab} + \frac{1}{4} f^2 \frac{b^2}{a^2}, \]

\[ \ddot{f} = -\dot{f} \left( \frac{\dot{b}}{b} + \frac{\dot{a}}{a} \right) + \frac{2}{a^2} f \left( 1 - f^2 \right), \]

\[ \frac{1}{a^2} + \frac{1}{b^2} + \frac{\dot{a}^2}{a^2} + \frac{\dot{b}^2}{b^2} + 3\frac{\dot{ab}}{ab} - \frac{1}{16} f^2 \frac{b^2}{a^2} - \frac{1}{16 a^4} (f^2 - 1)^2 = 0. \]

These equations have the following interesting properties. In 4D case Friedman-Walker-Robertson (FRW) solutions is not the solutions with bouncing off (they don’t have even
though local minimum). Let us consider the time moment in which the all functions $a(t), b(t), f(t)$ have the local extreme and analyze this extremum. In this point we have the following expression for $\ddot{a}_0, \ddot{b}_0$ and $\ddot{f}_0$:

\[
\frac{\ddot{a}_0}{a_0} = \frac{2}{a_0^2} + \frac{4}{b_0^2},
\]

\[
\frac{\ddot{b}_0}{b_0} = -\frac{2}{a_0^2} - \frac{4}{b_0^2},
\]

\[
\ddot{f}_0 = \frac{2}{a_0^2} f_0 \left(1 - f_0^2\right),
\]

here sign 0 indicate that the value of corresponding function at $t = 0$ is given. From these equations we see that 3D Universe (time section of 4D base of principal bundle) has, at least locally, a bouncing off effect in contrast with 4D case. This leads from the fact that the effective 4D stress-energy tensor derived from metric on extremal dimension evidently violate the strong energy conditions. The total space of principal bundle behaves as MD Kasner’s Universe with expanding space dimensions and contracting extremal dimensions (at least locally). But unlike to standard MD gravity the space coordinates can be only expand and respectively the extremal dimensions (fibre of principal bundle) only shrink.

The general solution of Eq’s (47-50) has a singularity. Let us investigate the behaviour of functions $a(t), b(t), f(t)$ near this singularity. We will search solution in this region in the following form:

\[
a(t) \approx a_\infty (t - t_0)^\alpha,
\]

\[
b(t) \approx b_\infty (t - t_0)^\beta,
\]

\[
f(t) \approx f_\infty + f_1 (t - t_0)^\gamma,
\]

$a_\infty, b_\infty, f_\infty, f_1$ are some constants. The simple calculations give us the following results:
\[
\alpha = \frac{1 \pm \sqrt{5}}{6},
\]
(57)

\[
\beta = \frac{1 \pm \sqrt{5}}{6},
\]
(58)

\[
\gamma = \frac{5 \pm \sqrt{5}}{3},
\]
(59)

The initial equations are very difficult for analytical investigations and hence we solve these equations only numerically with following bouncing off initial conditions:

\[
a(0) = a_0, \quad \dot{a}(0) = 0,
\]

\[
b(0) = b_0, \quad \dot{b}(0) = 0,
\]

\[
f(0) = f_0, \quad \dot{f}(0) = 0,
\]
(60)

Without loss of generality we can take \(a_0 = 1\). The condition for \(b_0\) follows from initial conditions Eq. (50):

\[
\left(\frac{b_0}{a_0}\right)^2 = 8 \left[1 + \sqrt{1 + \frac{(1-f_0^2)^2}{4}}\right] \left(1 - f_0^2\right)^2
\]
(61)

Thus, this system has only own independent parameter \(f_0\). The typically solution of these equations is presented on Fig.1-3. From these Fig’s we see that our solution has a local bouncing off effect by \(t = 0\) and singularity by some \(t_0\).

**V. CONCLUSION**

Finally, we can to say following: 4D Einstein-Yang-Mills theory and corresponding MD gravity on the principal bundle conform to each another in some sense. But the dynamic of these theories is sufficiently another. In static spherically symmetric case the 4D Einstein-Yang-Mills theory has solution outside of event horizon (black hole filled by
Yang-Mills gauge field Ref’s (7) whereas analogously solution in MD gravity there is under null surfaces. Further, 4D FRW solution doesn’t have the bouncing off but MD gravity on the principal bundle has (at least locally) bouncing off effect in general solution. Most likely this take place from the fact that the MD gravity on principal bundle can violate the energy condition.

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