SERIES EXPANSIONS FOR ANY REAL POWERS OF (HYPERBOLIC) SINE FUNCTIONS IN TERMS OF WEIGHTED STIRLING NUMBERS OF THE SECOND KIND

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ABSTRACT. In the paper, by virtue of the Faà di Bruno formula and in terms of weighted Stirling numbers of the second kind, the author derives the series expansion of any integer power of the sine function, finds a closed-form formula of a specific partial Bell polynomial, and establishes series expansions of any real powers of the sine and hyperbolic sine functions.

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1. Motivations

In mathematics, ones usually most possibly consider basic elementary functions
\[ e^x, \ln(1 + x), \sin x, \csc x, \cos x, \sec x, \tan x, \cot x, \]
\[ \arcsin x, \arccos x, \arctan x, \sinh x, \csch x, \cosh x, \sech x, \]
\[ \tanh x, \coth x, \arcsinh x, \arccosh x, \arctanh x \]
and their series expansions at \( x = 0 \). Their series expansions at \( x = 0 \) can be found in mathematical handbooks such as [1, 9, 17]. What are series expansions at \( x = 0 \) of integer or real powers of these functions?

It is combinatorial knowledge [7, 8] that coefficients of the series expansion of the power function \((e^x - 1)^k\) for \( k \in \mathbb{N} = \{1, 2, \ldots\} \) are the Stirling numbers of the second kind, while coefficients of the series expansion of the power function \(|\ln(1 + x)|^k\) for \( k \in \mathbb{N} \) are the Stirling numbers of the first kind. In other words, the power functions \((e^x - 1)^k\) and \(|\ln(1 + x)|^k\) for \( k \in \mathbb{N} \) are generating functions of the Stirling numbers of the first and second kinds.

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In the paper [6], among other things, Carlitz introduced the notion of weighted Stirling numbers of the second kind $R(n, k, r)$. Carlitz also proved in [6] that the numbers $R(n, k, r)$ can be generated by

$$\left(\frac{e^z - 1}{z}\right)^k e^{\lambda z} = \sum_{n=k}^{\infty} R(n, k, \lambda) \frac{z^n}{n!}$$

and can be explicitly expressed by

$$R(n, k, r) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (r+j)^n$$

for $r \in \mathbb{R}$ and $n \geq k \geq 0$. Specially, when $\lambda = 0$, the quantities $R(n, k, 0)$ become the Stirling numbers of the second kind $S(n, k)$.

In the handbook [9], series expansions of the functions arcsin$^m x$, arcsinh$^m x$, sin$^2 x$, cos$^2 x$, sin$^3 x$, and cos$^3 x$ are collected.

In the papers [4, 10, 11, 15, 20, 22] and plenty of references collected therein, the series expansions of the functions

$$\arcsin^m x, \quad \operatorname{arcsinh}^m x, \quad \arctan^m x, \quad \operatorname{arctanh}^m x$$

for $m \in \mathbb{N}$ have been established, applied, reviewed, and surveyed.

In the papers [5, 18], explicit series expansions of the functions tan$^2 x$, tan$^3 x$, cot$^2 x$, cot$^3 x$, sin$^m x$, cos$^m x$ for $m \in \mathbb{N}$ were written down.

In the papers [2, 3, 13, 14, 16, 23, 24], series expansions of the functions $I_{\mu}(x)I_{\nu}(x)$ and $[I_{\nu}(z)]^2$ were explicitly written out, while the series expansion of the power function $[I_{\nu}(z)]^r$ for $\nu \in \mathbb{C} \setminus \{-1, -2, \ldots\}$ and $r, z \in \mathbb{C}$ was recursively formulated, where $I_{\nu}(z)$ denotes modified Bessel functions of the first kind.

In the paper [19], series expansions at $x = 0$ of the functions $(\arccos x)^r$ and $(\arcsin x)^r$ were established for $r \in \mathbb{R}$. In [20], a series expansion at $x = 1$ of the function $[(\arccos x)^r]^r$ was invented for $r \in \mathbb{R}$.

In this paper, by virtue of the Faà di Bruno formula (1) and in terms of weighted Stirling numbers of the second kind $R(n, k, r)$, we will discover series expansions at $x = 0$ of the power functions $(\sin x)^r$ and $(\sinh x)^r$ for $x \in (-\pi, \pi)$ and fixed $r \in \mathbb{R}$?

2. Series expansions and partial Bell polynomials

For discovering series expansions at the point $x = 0$ of the power functions $(\sin x)^r$ and $(\sinh x)^r$ on $(-\pi, \pi)$ for fixed $r \in \mathbb{R}$, we need the following lemmas.

Lemma 1 ([7, Definition 11.2 and Theorem 11.4] and [8, pp. 134 and 139, Theorems A and C]). For $n \in \mathbb{N}$, the Faà di Bruno formula can be described in terms of partial Bell polynomials $B_{n,k}$ by

$$\frac{d^n}{dx^n} f \circ h(x) = \sum_{k=1}^{n} f^{(k)}(h(x)) B_{n,k}(h'(x), h''(x), \ldots, h^{(n-k+1)}(x)),$$  (2)
Proof. Replacing $\ell$ by $2\ell$ in (2), the point $x$ and $\ell$ by $2\ell$ and by $2\ell$ and simplifying result in the series expansions

$$
\sin^{2\ell} x = \frac{(-1)^\ell}{2^{\ell}} \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell}{q} \cos[(2q-\ell)x] \cos \left(\frac{\ell}{2} \pi \right) + \sin[(2q-\ell)x] \sin \left(\frac{\ell}{2} \pi \right)
$$

(5)

Replacing $\ell$ by $2\ell - 1$ and by $2\ell$ and simplifying result in the series expansions

$$
\sin^{2\ell-1} x = \frac{(-1)^\ell}{2^{\ell}} \sum_{j=0}^{\ell} (-1)^j \left(2\ell - 1\right) \left(2q - 2\ell + 1\right)^{2j-1} \frac{x^{2j-1}}{(2j-1)!}
$$

(5)

and

$$
\sin^{2\ell} x = \frac{(-1)^\ell}{2^{\ell}} \sum_{j=0}^{\ell} (-1)^j 2^{2j} \left(2\ell\right) \left(2q - \ell\right)^{2j} \frac{x^{2j}}{(2j)!}.
$$

(6)
The series expansions (5) and (6) can be reformulated as
\[
\left(\frac{\sin x}{x}\right)^{2\ell-1} = \frac{1}{2^{2\ell-1}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2\ell+2j-1)!} \left[ \sum_{q=0}^{2\ell-1} (-1)^q \binom{2\ell-1}{q} (2q-2\ell+1)^{2\ell+2j-1} \right] x^{2j}
\]
and
\[
\left(\frac{\sin x}{x}\right)^{2\ell} = \frac{1}{2^{2\ell}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2\ell+2j)!} \left[ \sum_{q=0}^{2\ell} (-1)^q \binom{2\ell}{q} (2q-2\ell)^{2\ell+2j} \right] x^{2j},
\]
for \( \ell \in \mathbb{N} \) and \( x \in \mathbb{R} \). These two series expansions can be unified and rearranged as (3). Lemma 2 is thus proved. □

**Lemma 3.** For \( n \geq k \geq 1 \), partial Bell polynomials \( B_{n,k} \) satisfy
\[
B_{2m-1,k}\left(0, \frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{(-1)^m}{2m-k+1} \cos \frac{k\pi}{2}\right) = 0
\]
and
\[
B_{2m,k}\left(0, \frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{(-1)^m}{2m-k+2} \sin \frac{k\pi}{2}\right)
= (-1)^{m+k} \frac{2m}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} R\left(2m+j, j, -\frac{j}{2}\right)
\]
for \( m \in \mathbb{N} \), where \( R(2m+j, j, -\frac{j}{2}) \) is given by (1).

**Proof.** From
\[
\frac{\sin x}{x} = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} x^{2j} \quad x \in \mathbb{R},
\]
it follows that
\[
\left. \left(\frac{\sin x}{x}\right)^{(2j)} \right|_{x=0} = \frac{(-1)^j}{2j+1} \quad \text{and} \quad \left. \left(\frac{\sin x}{x}\right)^{(2j-1)} \right|_{x=0} = 0 \quad \text{for} \ j \in \mathbb{N}.
\]
On [8, p. 133], the identity
\[
\frac{1}{m!} \left( \sum_{\ell=1}^{\infty} \frac{x^\ell}{\ell!} \right)^m = \sum_{n=m}^{\infty} B_{n,m}(x_1, x_2, \ldots, x_{n-m+1}) \frac{t^n}{n!}
\]
is given for \( m \geq 0 \). The formula (8) implies that
\[
B_{n+k,k}(x_1, x_2, \ldots, x_{n+1}) = \binom{n+k}{k} \lim_{t \to 0} \frac{d^n}{dt^n} \left[ \sum_{\ell=0}^{\infty} \frac{x^{\ell+1}}{(\ell+1)!} t^\ell \right]^k
\]
for \( n \geq k \geq 0 \). Substituting \( x_{2j} = \frac{(-1)^j}{2j+1} \) and \( x_{2j-1} = 0 \), that is, \( x_j = \frac{1}{j+1} \cos \left( \frac{j\pi}{2} \right) \), for \( j \in \mathbb{N} \) into (9) results in
\[
B_{n+k,k}\left(0, \frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{n+2} \cos \left( \frac{n+1}{2} \frac{\pi}{2} \right)\right)
= \binom{n+k}{k} \lim_{t \to 0} \frac{d^n}{dt^n} \left[ \sum_{\ell=0}^{\infty} \frac{1}{(\ell+2)!} \cos \left( \frac{\ell+1}{2} \frac{\pi}{2} \right) t^\ell \right]^k
\]
functions

proof of Lemma 3 is complete. □

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Theorem 1.

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the point,

\[
\sin x
\]

for \(n\geq k\geq 1\), where we used the series expansion (3) in Lemma 2. The
proof of Lemma 3 is complete. □

3. Series expansions of real powers

We now start out to establish series expansions at the point \(x = 0\) of the power
functions \((\sin x)^r\) and \((\sinh x)^r\) for \(r \in \mathbb{R}\).

Theorem 1. For \(r \in \mathbb{R}\), assume that the value of the power function \((\sin x)^r\) at
the point \(x = 0\) is 1. Then

(1) for \(r \geq 0\), the series expansion

\[
(\sin x)^r = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k=1}^{2m} \sum_{j=1}^{k} \frac{(-r)_k}{k!} \frac{1}{2^j (j + 2m)!} \left(\frac{1}{2m + j, -i^j}{(2m + j)}\right) R(2m + j, j, -i^j) \left(\frac{1}{2m!}\right)
\]

(10)
Proof. The series expansion (11) follows from replacing \( \sin x \) then substituting

\[
(r)_k = \prod_{\ell=0}^{k-1} (r+\ell) = \begin{cases} r(r+1)\cdots(r+k-1), & k \geq 1 \\ 1, & k = 0 \end{cases}
\]

and \( R(2m+j,j,-\frac{j}{2}) \) is given by (1);

(2) for \( r < 0 \), the series expansion (10) is convergent in \( x \in (-\pi, \pi) \).

Proof. By virtue of the Faá di Bruno formula (2) in Lemma 1, we obtain

\[
\frac{d^j}{dx^j} \left( \frac{\sin x}{x} \right)^r = \sum_{k=1}^{j} \frac{d^k u^r}{du^k} B_{j,k} \left( \left( \frac{\sin x}{x} \right)^{r-k}, \left( \frac{\sin x}{x} \right)^{r-k} \right) 
\]

\[
= \sum_{k=1}^{j} \langle r \rangle_k B_{j,k} \left( \left( \frac{\sin x}{x} \right)^{r-k}, \left( \frac{\sin x}{x} \right)^{r-k} \right) \sin \left( \frac{j-k}{2} \pi \right), \quad x \to 0
\]

\[
= \begin{cases} 0, & j = 2m - 1 \\ \sum_{k=1}^{2m} \langle r \rangle_k B_{2m,k} \left( 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{j-k+2} \right), & j = 2m \end{cases}
\]

for \( m \in \mathbb{N} \), where \( u = u(x) = \frac{\sin x}{x} \) and we used the derivatives in (7). Therefore, with the help of Lemma 3, we arrive at

\[
\left( \frac{\sin x}{x} \right)^r = 1 + \sum_{j=1}^{\infty} \left[ \lim_{x \to 0} \frac{d^j}{dx^j} \left( \frac{\sin x}{x} \right)^r \right] \frac{x^j}{j!}
\]

\[
= 1 + \sum_{m=1}^{\infty} \left[ \sum_{k=1}^{2m} \langle r \rangle_k B_{2m,k} \left( 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{j-k+2} \right) \right] \frac{x^{2m}}{(2m)!}.
\]

The proof of Theorem 1 is thus complete. \( \square \)

**Corollary 1.** For \( r \in \mathbb{R} \), assume the value of the function \( \left( \frac{\sin x}{x} \right)^r \) at \( x = 0 \) is 1.

Then, for \( x, r \in \mathbb{R} \), we have

\[
\left( \frac{\sin x}{x} \right)^r = 1 + \sum_{m=1}^{\infty} \left[ \sum_{k=1}^{2m} \frac{(-r)_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} R(2m+j,j,-\frac{j}{2}) \right] \frac{(2x)^{2m}}{(2m)!}.
\]

where \( R(2m+j,j,-\frac{j}{2}) \) is given by (1).

Proof. The series expansion (11) follows from replacing \( \sin x \) by \( \frac{\sin(x+i)}{i} \) in (10) and then substituting \( x+i \) for \( x \). \( \square \)
4. Remarks

Finally we list several remarks about related things.

Remark 1. From Lemma 2 and its proof, we can derive the following identities

\[
R\left(2j - 1, 2\ell - 1, -\frac{2\ell - 1}{2}\right) = \begin{cases} 
0, & 1 \leq j \leq \ell - 1 \\
1, & j = \ell 
\end{cases}
\]

and

\[
R(2j, 2\ell, -\ell) = \begin{cases} 
0, & 1 \leq j \leq \ell - 1 \\
1, & j = \ell 
\end{cases}
\]

for \( \ell \in \mathbb{N} \).

Remark 2. The formula (4) can also be found at the websites https://math.stackexchange.com/a/4331451/945479 and https://math.stackexchange.com/a/4332549/945479 respectively.

The formulation of the series expansion in Lemma 2 is better and simpler than corresponding ones in [5, Section 3, pp. 798–799].

Remark 3. As long as the function \( f(u) \) is infinitely differentiable at \( u = 1 \), Lemma 3 can be used to compute series expansions at \( x = 0 \) of the functions \( f\left(\sin \frac{x}{\ell}\right) \). For example, making use of the Faà di Bruno formula (2) in Lemma 1, the derivatives in (7), and Lemma 3, we acquire

\[
\exp\left(\frac{\sin x}{x}\right) = \sum_{k=0}^{\infty} \lim_{x \to 0} \frac{d^k}{dx^k} \exp\left(\frac{\sin x}{x}\right) \frac{x^k}{k!}
\]

\[
= e + \sum_{k=1}^{\infty} \sum_{j=1}^{k} \frac{B_{k,j}}{k!} \left[ \left. \frac{\sin x}{x} \right|_{x=0} \cdot \left. \left(\frac{\sin x}{x}\right)'' \right|_{x=0} \cdot \ldots \cdot \left. \left(\frac{\sin x}{x}\right)^{(k-j+1)} \right|_{x=0} \right] \frac{x^k}{k!}
\]

\[
= e + \sum_{k=1}^{\infty} \sum_{j=1}^{k} \frac{B_{k,j}}{k!} \left( 0, \frac{1}{3}, \ldots, \frac{1}{k-j+2} \cos \left( \frac{k-j+1}{2} \pi \right) \right) \frac{x^k}{k!}
\]

\[
= e + \sum_{k=1}^{\infty} \sum_{j=1}^{2k} \frac{B_{2k,j}}{2k!} \left( 0, -\frac{1}{3}, \frac{1}{2k-j+2} \cos \left( \frac{2k-j+1}{2} \pi \right) \right) \frac{x^{2k}}{(2k)!}
\]

\[
= e + \sum_{k=1}^{\infty} \frac{(-1)^{k+j}}{j!} \sum_{\ell=1}^{k} \frac{(-1)^\ell}{\ell!} \frac{R(2k + \ell, \ell, -\frac{4}{\ell})}{(2k+\ell)!} \frac{x^{2k}}{(2k)!}
\]

that is,

\[
\exp\left(\frac{\sin x}{x} - 1\right) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2k}{3} \sum_{j=1}^{2k} \frac{(-1)^j}{j!} \sum_{\ell=1}^{j} \frac{(-1)^\ell}{\ell!} \frac{R(2k + \ell, \ell, -\frac{4}{\ell})}{(2k+\ell)!} \frac{x^{2k}}{(2k)!}
\]

\[
= 1 - \frac{x^2}{6} + \frac{x^4}{45} - \frac{1073x^6}{54360} + \frac{1189x^8}{5443200} - \frac{1633x^{10}}{89812800} + \cdots,
\]

where \( R(2k + \ell, \ell, -\frac{4}{\ell}) \) is given by (1).
Remark 4. The first identity in Lemma 3 is a special case of the following general conclusion in [11, Theorem 1.1].

For \( k, n \geq 0, m \in \mathbb{N}, \) and \( x_m \in \mathbb{C}, \) we have
\[
B_{2n+1,k} \left( 0, x_2, 0, x_4, \ldots, \frac{1 + (-1)^k}{2} x_{2n-k+2} \right) = 0.
\]

Remark 5. From the proof of Lemma 3, we derive the identity
\[
\sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{R(2\ell + j, j, \frac{j}{2})}{(2\ell+j)!} = 0
\]
for \( k \geq 2 \) and \( 1 \leq \ell \leq k - 1. \)

Remark 6. We guess that
\[
\sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{R(2m + j - 1, j, \frac{j}{2})}{(2m+j-1)!} = 0, \quad k, m \in \mathbb{N} \quad (12)
\]
and
\[
\sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{R(2m + j, j, \frac{j}{2})}{(2m+j)!} = 0, \quad k > m \geq 1. \quad (13)
\]
If these two guesses were proved to be true, then we can reformulate Lemma 3 as that, for \( n \geq k \geq 1, \) partial Bell polynomials \( B_{n,k} \) satisfy
\[
B_{n,k} \left( 0, \frac{1}{3}, 0, \frac{1}{5}, \ldots, \frac{1}{n-k+2} \cos \left( \frac{n-k+1}{2} \pi \right) \right) = (-1)^k \frac{2^n}{k!} \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j \binom{k}{j} \frac{R\left( n+j, j, \frac{j}{2} \right)}{\binom{n+j}{j}},
\]
where \( R(n+j, j, \frac{j}{2}) \) is given by (1) and \( \lfloor r \rfloor \) for \( r \in \mathbb{R} \) denotes the floor function whose value is equal to the largest integer less than or equal to \( r. \)

These guesses (12) and (13) have been announced at https://mathoverflow.net/questions/420121/ for confirming or denying.

Remark 7. The series expansion (3) in Lemma 2 or the series expansion (10) in Theorem 1 can be used to answer two questions at https://math.stackexchange.com/q/2267836 and https://math.stackexchange.com/q/3673133.

Remark 8. Let \( r > 0 \) and \( k \geq 0. \) Then, by virtue of the Faá di Bruno formula (2) in Lemma 1 and employing the formula
\[
B_{n,k}(x, 1, 0, \ldots, 0) = \frac{1}{2^{n-k} k!} \binom{k}{n-k} x^{2k-n}
\]
collected in [21, Section 1.4], we obtain
\[
\left[ \frac{1}{(1 + x^2)^r} \right]^{(k)} = \sum_{j=0}^{k} \frac{d^j}{du^j} \left( \frac{1}{u^r} \right) B_{k,j}(x, 2, 0, \ldots, 0) = \sum_{j=0}^{k} \frac{(-r)_j}{u^{r+j}} 2^j B_{k,j}(x, 1, 0, \ldots, 0)
\]
\[
\sum_{j=0}^{k} \frac{(-r)_j}{(1 + x^2)^{r+j}} 2^j \frac{k!}{j!} \left( \frac{j}{k-j} \right) x^{2j-k}
\]
\[
= \frac{k!}{2^k x^k (1 + x^2)^{r}} \sum_{j=0}^{k} \frac{(-r)_j}{j!} 2^j \left( \frac{j}{k-j} \right) x^{2j} (1 + x^2)^{-j},
\]
where \( u = u(x) = 1 + x^2 \). See also texts at the site https://math.stackexchange.com/a/4418636.

5. Declarations

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