Critical dynamics and multifractal exponents at the Anderson transition in 3d disordered systems

T. Brandes, B. Huckestein and L. Schweitzer

1Department of Physics, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo 171, Japan
2Institut für Theoretische Physik, Universität zu Köln, Zülpicher Str. 77, 50937 Köln, Germany
3Physikalisch-Technische Bundesanstalt, Bundesallee 100, 38116 Braunschweig, Germany

(March 23, 2022)

Abstract

We investigate the dynamics of electrons in the vicinity of the Anderson transition in \(d=3\) dimensions. Using the exact eigenstates from a numerical diagonalization, a number of quantities related to the critical behavior of the diffusion function are obtained. The relation \(\eta = d - D_2\) between the correlation dimension \(D_2\) of the multifractal eigenstates and the exponent \(\eta\) which enters into correlation functions is verified. Numerically, we have \(\eta \approx 1.3\).

Implications of critical dynamics for experiments are predicted. We investigate the long-time behavior of the motion of a wave packet. Furthermore, electron-electron and electron-phonon scattering rates are calculated. For the latter, we predict a change of the temperature dependence for low \(T\) due to \(\eta\).

The electron-electron scattering rate is found to be linear in \(T\) and depends on the dimensionless conductance at the critical point.

I. INTRODUCTION

The dynamics of non-interacting electrons in the vicinity or directly at the critical point of a metal-insulator transition is still an unsolved problem. In the last years, a number of works \([25,1-3,5,20,6,7]\) revealed strong amplitude fluctuations of the eigenstates near the critical energy \(E_c\). The wave functions turned out to be multifractal objects, described by a set of generalized fractal dimensions \(D_q\). On the other hand, investigation of correlation functions \([8]\) in combination with scaling arguments showed that on certain length and time scales the dynamics near \(E_c\) is governed by anomalous diffusion, described by an exponent \(\eta\). One possible definition \([9]\) for \(\eta\) is the algebraic decay \([10]\) of the static two-particle correlation function \(S(r, E, \omega \to 0)\) near the critical energy \(E \to E_c\)

\[
S(r, E, \omega \to 0) \sim \left(\frac{r}{\xi(E)}\right)^{-\eta}, \quad r \ll \xi(E)
\]

where \(\xi(E)\) is the localization length. In real space, \(S\) is defined as

\[
S(r, E) := \left\langle \sum_{\alpha,\beta} \delta(E^- - E_\alpha)\delta(E^+ - E_\beta)\Psi^\ast_{E^-}(0)\Psi_{E^+}(r)\Psi^\ast_{E^+}(0)\right\rangle,
\]
where \( E^\pm := E \pm \hbar \omega/2 \), \( \alpha, \beta \) label the eigenstates, and the brackets \( \langle \rangle \) denote an impurity average. The correlation function \( S \) is connected to the diffusion function \( D(q, \omega) \) via

\[
S(q, E, \omega) = \frac{\rho(E)}{\pi \hbar} \frac{D(q, \omega)q^2}{\omega^2 + (D(q, \omega)q^2)^2},
\]

where in turn \( D(q, \omega) \) is the generalization of the diffusion constant \( D \) in the metallic case and \( \rho(E) \) is the density of states. Indeed, Eq. (3) is a general expression for the two-particle correlation function compatible with particle conservation [11]. The function \( D(q, \omega) \) appears in the diffusion propagator

\[
P(q, \omega) := \frac{1}{-i\omega + D(q, \omega)q^2},
\]

which is the Fourier transform of the probability distribution \( P(r, t) \) describing the motion of an electron wave packet in a disordered system for \( t > 0 \), initially located at \( r = 0 \) at time \( t = 0 \).

In this work, we determine the correlation dimension \( D_2 \) of the multifractal eigenstates and the exponent \( \eta \) for the Anderson transition in 3 dimensions. The relation

\[
D_2 = d - \eta,
\]

which holds in \( d \) dimensions, is derived from the respective definitions of the exponents and verified numerically. We independently determine \( D_2 \) via the box counting method applied to the spatial amplitude fluctuations of the critical eigenstates on the one side, and \( \eta \) from a correlation function in energy space using a scaling form of \( D(q, \omega) \) on the other side. Furthermore, the relation \( \delta = \tilde{D}_2 = D_2/d \) is established and checked numerically, where \( \delta \) governs the probability of return of a wave packet, \( p(t) \sim t^{-\delta} \), and \( \tilde{D}_2 \) is the correlation dimension of the spectral measure at the critical point.

Two applications which might be relevant for experiments are presented. First, we discuss the long time behavior of a wave packet of eigenstates with energies near \( E_c \) and show that in three dimension this behavior is drastically different from its two-dimensional counterpart [6,7]. Then, it is shown how rates for inelastic electron-electron and electron-phonon scattering are changed due to the critical dynamics as compared with the usual metallic case. We conclude with an overview of the results for various exponents in the two dimensional quantum Hall, the two dimensional symplectic, and the three dimensional orthogonal case.

II. SCALING FORM OF \( D(Q, \omega) \)

The one-parameter scaling hypothesis [12], which underlies a great part of the theoretical analysis of the metal-insulator transition [13], allows to write the diffusion function \( D(q, \omega) \) in \( d \) dimensions near the critical point in the form [14]

\[
D(q, \omega) = \frac{\xi^{2-d}}{\rho} F(\xi/L, \xi q)
\]

(6)
where $\xi$ is the localization length, $\rho = \rho(E)$ the density of states (DOS) near the critical energy $E = E_c$, $F$ is a scaling function, and

$$L_\omega := (\rho h \omega)^{-1/d} = L(h \omega / \Delta)^{-1/d},$$

(7)

is a third length scale, besides $q^{-1}$ and $\xi$, relevant at the critical point ($\Delta = (L^d \rho)^{-1}$ is the mean level spacing).

In addition to $L_\omega$, one can define a second frequency dependent length

$$\tilde{L}_\omega = \left( \frac{D(\omega)}{\omega} \right)^{1/2},$$

(8)

the distance the particle diffuses in the time $1/\omega$. Here $D(\omega) = \lim_{q \to 0} D(q, \omega)$ is explicitly scale-dependent. Apparently, $L_\omega$ and $\tilde{L}_\omega$ scale with different powers of the frequency. However, at the mobility edge this not the case. From the definition eq. (8), the dimensionless conductance for a hypercube of size $\tilde{L}_\omega$ is given by

$$g^* = G \frac{\hbar}{e^2} = \frac{\hbar}{e^2} \sigma(\omega) \tilde{L}_\omega^{-d-2}.$$  

(9)

Here, the frequency-dependent conductivity $\sigma(\omega)$ is related to the diffusion coefficient by the Einstein relation

$$\sigma(\omega) = e^2 \rho D(\omega).$$

(10)

Eliminating $D(\omega)$ from these equations leads to

$$\tilde{L}_\omega = \left( \frac{g^*}{\hbar \rho \omega} \right)^{1/d} = g^{*1/d} L_\omega.$$  

(11)

Thus, at the mobility edge both frequency-dependent length scales differ only by a factor of $g^{*1/d}$.

The general relation between the conductivity $\sigma$ and the conductance $G$ for a sample in the form of a hypercube with sides of length $L$, $G(L) = \sigma(L)L^{d-2}$, and the vanishing of the $\beta$-function $\partial G/\partial L = 0$ at $E_c$ [12] requires the conductivity and therefore the diffusion function $D \sim \sigma$ to scale like

$$D \sim L^{2-d}$$

(12)

at the critical point.

A direct consequence of this scaling is a certain behavior of $D(q, \omega)$ in different regions of the ($q, \omega$)-plane, as discussed by Chalker [15]. In the limit of small $q \to 0$, namely $q L_\omega \ll 1$, $L_\omega$ sets the shortest length scale (region (A) in Fig. 1), and

$$D(q \to 0, \omega) \sim L_\omega^{2-d} \sim \omega^{(d-2)/d}, \quad \omega \to 0.$$  

(13)

This behavior has been predicted by Wegner in 1976 [10]. Recently, $D(\omega) \sim \omega^{1/3}$ has been confirmed numerically for $d = 3$ [16].
For larger $q$ (region (B)), the length scale determining the scaling of $D(q, \omega)$ is the inverse wave vector itself, if the localization length is not shorter than $1/q$. This leads to

$$D(q, \omega) \sim q^{d-2}. \quad \text{(14)}$$

It has turned out in the last years that there are corrections to the above scaling form due to strong fluctuations of the wave function amplitudes near the critical point. These fluctuations are observed numerically on small length scales corresponding to large $q$-vectors. In the $(q, \omega)$-plane, they give rise to another region (C) where $D(q, \omega)$, Eq. (14), acquires an additional factor $q^{-\eta}$, the scale of which is given by either the localization length $\xi$ or the length $L_\omega$, namely

$$D(q, \omega) \sim q^{d-2} \left[ q \cdot \min(\xi, L_\omega) \right]^{-\eta}. \quad \text{(15)}$$

For our numerical analysis, we use the following scaling form of the diffusion function:

$$D(q, \omega) = \frac{q^{d-2}}{h\rho} f(x), \quad x := [q \cdot \min(\xi, L_\omega)]^d, \quad \text{(16)}$$

which satisfies the scaling relation ($\xi(E) > L_\omega$)

$$D(q, \omega) = b^{2-d} D(bq, \omega b^d) \quad \text{(17)}$$

which is required for a transformation of length scales $r \mapsto r/b$. Note that in contrast to the conductance at $E_c$, the diffusion function itself is scale dependent.

From these arguments it is not evident where the transition between regions (B) and (C) occurs. It is conceivable that instead of the scaling behavior in region (B) a smooth crossover from region (A) to (C) takes place. From the present numerical results we cannot address this question.

The regions (A), (B), and (C) correspond to three different regimes of the function $f(x)$: First, in order to have $D(q \to 0, \omega) \sim L_\omega^{2-d}$ (regime A), the function $f(x)$ must follow $f(x) \sim x^{(2-d)/d}$ in the limit $x \to 0$. Second, for large $x$, one has $f \sim x^{-\eta/d}$ in order to fulfill the behavior Eq. (17), regime C. The scaling form in the intermediate region (B), $D(q, \omega) \sim q^{d-2}$, requires that $f(x) = \text{const}$ there. We thus can write the correlation function $S(q, E, \omega)$ as

$$S(q, E, \omega) = \frac{\rho(E)}{\pi\hbar\omega} \frac{x f(x)}{1 + [x f(x)]^2}, \quad x = [q \cdot \min(\xi, L_\omega)]^d, \quad \text{(18)}$$

where

$$f(x) = \begin{cases} 
  c_\alpha \cdot x^{(2-d)/d} : & x \to 0 \ (A) \\
  c_\beta & : \text{in between} \ (B) \\
  c_\gamma \cdot x^{-\eta/d} : & x \to \infty \ (C)
\end{cases} \quad \text{(19)}$$

with constants $c_\alpha$, $c_\beta$, and $c_\gamma$. In the brackets () we indicate the corresponding area in Fig. 1. The boundaries between the three regions and the constants have to be extracted from numerical calculations. For the latter, we work close to $E_c$ so that $\xi > L_\omega$ and the scaling variable $x = [q \cdot L_\omega]^d$. The case $\xi < L_\omega$ gives rise to the two remaining areas in Fig. 1 with
$D \sim \xi^{2-d}$ (left to A) and $D \sim q^{d-2}(q\xi)^{-\eta}$ (left to C), which are not relevant in our analysis. In the case of two dimensions, $d = 2$, regions A and B coincide and $f$ is constant there. The numerical analysis (see below) shows that the large-$x$ regime of $f(x)$, governed by the power law involving the exponent $\eta$, is indeed relevant over several decades of $x$.

We notice that upon Fourier transforming $S(q, \omega)$ into real space, large values of $x$ correspond to small $r \ll \min(\xi, L_\omega)$, where the strong fluctuations of the eigenstate amplitudes become important. In this region (corresponding to $x = [\min(\xi, L_\omega)/r]^{d} \gg 1$), one finds

$$S(r, \omega) \sim \frac{1}{\omega r^{d}} x^{-1+\eta/d} \sim \frac{1}{\omega r^{d}} [\min(\xi, L_\omega)/r]^{-d+\eta} \quad (20)$$

which for $\omega \to 0$ ($\xi < L_\omega$) gives $S(r, \omega) \sim 1/(\omega^{d} \cdot (r/\xi)^{-\eta}$, Eq. (1). For $\xi > L_\omega$ one recovers the form $S(r, \omega) \sim r^{-\eta} \omega^{-\eta/d}$ given by Wegner [18].

III. A WAVE PACKET AT THE CRITICAL POINT

In order to illustrate the implications of critical dynamics, we use the relation between the diffusion function $D(q, \omega)$ and the diffusion propagator $P(q, \omega)$, Eq. (4). The latter is the Fourier transform of the probability distribution $P(r, t)$ describing the motion of an electron wave packet constructed from eigenstates with energies close to a fixed energy $E$. In the metallic region, $P(r, t)$ is the solution of the diffusion equation in $d$ dimensions

$$\frac{\partial P(r, t)}{\partial t} - D \nabla^2 P(r, t) = 0, \quad t > 0 \quad (21)$$

with the diffusion constant $D$,

$$P(r, t) = \frac{e^{-r^2/(4Dt)}}{(4\piDt)^{d/2}}, \quad t > 0. \quad (22)$$

Due to causality, $P(q, \omega)$ has to be analytical in the upper half plane as a function of complex $\omega$: The integral

$$P(q, t) = \frac{1}{2\pi} \int d\omega \frac{e^{-i\omega t}}{-i\omega + D(q, \omega)q^2} \quad (23)$$

has to be performed in the upper half plane for $t < 0$ whence $P(q, t < 0) = 0$. On the other hand, for $t > 0$ we can write

$$P(q, t) = \frac{1}{2\pi} \int d\omega e^{-i\omega t} \left\{ \frac{1}{-i\omega + D(q, \omega)q^2} + \frac{1}{[-i\omega + D(q, \omega)q^2]^*} \right\}, \quad t > 0 \quad (24)$$

since for $t > 0$ the integral is performed in the lower half plane where the second term gives zero contribution. For real $\omega$, $D(q, \omega)$ is real and one can write

$$P(q, t > 0) = \frac{1}{\pi} \int d\omega e^{-i\omega t} \frac{D(q, \omega)q^2}{\omega^2 + (D(q, \omega)q^2)^2} = \frac{\hbar}{\rho} \int d\omega e^{-i\omega t} S(q, E, \omega). \quad (25)$$
Recalling that \( S(q, E, \omega) \) is an even function in \( \omega \) and introducing the dimensionless variable 
\[ x = (qL_\omega)^d \equiv q^d/\hbar \rho, \]
we obtain
\[
P(q, t > 0) = 2 \int_0^\infty d\omega \frac{\hbar}{\rho} \cos(\omega t) S(q, E, \omega) = \frac{2}{\pi} \int_0^\infty \frac{dx}{x} \cos \left( \frac{q^d t}{\hbar \rho x} \right) \frac{x f(x)}{1 + [xf(x)]^2}. \tag{26}
\]
Normalization of the wave packet requires 
\[
P(q = 0, t > 0) = 1, \quad \text{i.e.}
\]
\[
\frac{2}{\pi} \int_0^\infty \frac{dx}{x} \frac{xf(x)}{1 + [xf(x)]^2} = 1 \tag{27}
\]
which is an additional condition to be fulfilled by the function \( f(x) \) Eq. (19). Since the exact form of \( f(x) \) is not known, we cannot determine \( P(q, t) \) exactly. However, for large \( q^d t \) the probability \( P(q, t) \) in Eq. (26) is determined by large \( x \) values due to the rapidly oscillating cosine-term. For large \( x \), on the other hand, the form of \( f(x) \) is known, and we can write
\[
P(q, t) \sim \frac{2}{\pi} \int_{x_-}^{x_+} \frac{dx}{x} \cos \left( \frac{q^d t}{\hbar \rho x} \right) \frac{1}{c_x x^{1-\eta/d}} \sim (q^d t)^{-1+\eta/d} = (q^d t)^{-D_2/d}, \quad q^d t/\hbar \rho \gg 1. \tag{28}
\]
Here, we used the relation \( D_2 = d - \eta \) (see the following section). Furthermore, the lower (high frequency) cutoff \( x_- \) is determined by the inverse of a microscopic time scale \( \tau \), below which the motion of the electron is ballistic and not described by the diffusion pole \( P(q, \omega) \).

In the same line of argument, one obtains the probability of return of a wave packet to the origin in the long time limit. This quantity is defined by the \( r = 0 \) value of \( P(r, t) \) and can therefore simply be recovered from \( P(q, t) \) by Fourier transformation. Again, the upper limit of the resulting \( q \)-integral is determined by a microscopic cutoff \( \sim 1/l \). One has
\[
p(t) = \frac{1}{(2\pi)^d} \int dq^d P(q, t) \sim \int_0^{1/l} dq q^{d-1} (q^d t)^{-D_2/d} \sim t^{-D_2/d}. \tag{29}
\]
This dependence of \( p(t) \) is not valid for very short time scales corresponding to a (quasi)-ballistic motion of the wave packet, and times so long that the system boundaries become important.

We can compare the critical behavior of \( p(t) \) in \( d = 2 \) and \( d = 3 \) to the metallic case, where 
\[
p(t) = (4\pi Dt)^{-d/2}:
\]
\[
p(t) \sim \begin{cases} 
t^{-1} & : \text{2d metallic} 
\frac{t^{-(1-\eta/2)}}{t^{-3/2}} & : \text{2d critical} 
\frac{t^{-(1-\eta/3)}}{t^{-3/2}} & : \text{3d metallic} 
\end{cases}
\tag{30}
\]
In particular, this demonstrates the different role of the exponent \( \eta \) in two and three dimensions: While in two dimensions, \( \eta \) essentially can be regarded as a more or less small correction to the ordinary metallic diffusion pole, the situation in three dimensions is drastically different. There, due to the scaling form of \( D(q, \omega) \), the dynamical behavior of electrons near \( E_c \) is qualitatively very different from the metallic case.
IV. RELATION BETWEEN THE CORRELATION DIMENSION $D_2$ AND THE EXPONENT $\eta$

The relation Eq. (3) has been derived previously [19,20]; we shortly outline our proof since it comprises the definitions concerning the multifractal dimension $D_2$ entering the analysis of the wave function at $E_c$. One first defines the probability

$$P_i(\lambda) := \int_{\Omega_i(\lambda)} d^d x |\Psi_E(x)|^2$$

(31)

To find the electron of fixed energy $E$ in a finite region (box $i$) of real space denoted by $\Omega_i(\lambda) = l^d$ as a subspace of the total volume $\Omega = L^d$, where $l = \lambda L, 0 < \lambda \leq 1$. The scaling with $\lambda$ is used to define so-called fractal dimensions $D_q$. One considers the averaged $q$th power of $P_i(\lambda)$ :

$$P(\lambda, q) := \frac{1}{N(\lambda)} \sum_{i=1}^{N(\lambda)} [P_i(\lambda)]^q \sim \lambda^{D_0 + (q-1)D_q}.$$  

(32)

Here, $N(\lambda) = \lambda^{-d}$ is the number of small cubes into which the original total cube of volume $L^d$ was split. Requiring that at $E_c$, the summation in Eq. (32) over the different boxes labeled by $i$ is equivalent to a disorder average in one (arbitrarily) fixed box (say $i = 1$),

$$P(\lambda, 2) = \langle P_i(\lambda)^2 \rangle,$$

(33)

one has (omitting the index $i = 1$ now)

$$P(\lambda, 2) = \int_{\Omega(\lambda)} d^d x \int_{\Omega(\lambda)} d^d x' \langle |\Psi_E(x)|^2 |\Psi_E(x')|^2 \rangle.$$  

(34)

Because of the disorder average, the product of the squares should depend on the difference $x - x'$ only. One can then introduce coordinates $r = x - x'$ and $R = (x + x')/2$. The Jacobian of this transformation is 1, but because the integration region is finite, the integration limits are also changed which makes the evaluation tedious. However, for large enough $\Omega(\lambda)$ the integral over $R$ simply gives the volume $\Omega(\lambda)$ itself, and we have

$$P(\lambda, 2) = \Omega(\lambda) \cdot \int_{\Omega(\lambda)} d^d r \langle |\Psi_E(0)|^2 |\Psi_E(r)|^2 \rangle.$$  

(35)

Considering the definition of the two-particle correlation function Eq. (2) for $\omega \to 0$, one obtains

$$P(\lambda, 2) = (\Omega \rho(E))^2 \Omega(\lambda) \cdot \int_{\Omega(\lambda)} d^d r S(r, E, \omega \to 0).$$  

(36)

where $\Omega$ is the total volume $\Omega = \Omega(\lambda = 1)$ of the system. Using Eq. (1), one has

$$\int_{\Omega(\lambda)} d^d r r^{-\eta} \sim \int_0^{\lambda L} d r r^{d-1} r^{-\eta} \sim (\lambda L)^{d-\eta}.$$  

(37)

Because $\Omega(\lambda) = (\lambda l)^d$, we find
By comparison with Eq. (32) one reads off \( d + (d - \eta) = D_0 + D_2 \). Noticing that \( D_0 = d \) is the dimension of the total support of the wave function, we have the announced relation Eq. (3) between the exponent \( \eta \) and the fractal dimension

\[
D_2 = d - \eta.
\]

This fundamental relation relates properties of correlation functions to the spatial structure of the wave functions at the critical point. Furthermore, if \( \eta \) is larger than 0 at a critical point of a metal-insulator transition [21], by Eq. (3) it follows \( D_2 < d \) and the wave functions must be multifractal [25].

We obtain another exponent relation by exploiting the long-time behavior \( p(t) \) of the return probability of a wave packet, constructed from eigenstates near \( E_c \). First, the exponent \( \delta \) introduced via \( p(t) \sim t^{-\delta} \) was generally proven by Ketzmerick et al. [22] to be equal to the generalized dimension \( \tilde{D}_2 \) of the spectral measure. Just as \( D_2 \) describes spatial correlations, \( \tilde{D}_2 \) describes the correlations of the local density of states of the system as a function of energy. In [22] it was shown that \( \delta = \tilde{D}_2 \). On the other hand, we have seen that \( \delta = D_2/d \), Eq. (29), and therefore

\[
\delta = \frac{D_2}{d} = \tilde{D}_2.
\]

V. NUMERICAL ANALYSIS

A. Model for the numerical investigation

The dynamics of non interacting electrons in the presence of disorder is studied within the framework of the Anderson model described by the Hamiltonian

\[
\mathcal{H} = \sum_r \epsilon_r |r\rangle\langle r| + \sum_{\langle r,r' \rangle} V_{r,r'} |r\rangle\langle r'|,
\]

where the diagonal disorder potentials \( \epsilon_r \) are independent random numbers with a constant probability distribution in the range \(-W/2 \leq \epsilon_r \leq W/2\) and the non diagonal transfer matrix elements between nearest neighbors, \( V_{r,r'} \), are taken to be the unit of energy. The vectors \( r \) denote the sites of a simple cubic lattice with lattice constant \( a \) and periodic boundary conditions are applied in all directions.

Eigenvalues and eigenvectors have been obtained for systems of size up to \((L/a = 40)^3\) sites by direct diagonalization using a Lanczos algorithm. The critical regime in the middle of the disorder broadened tight binding band, where both the localization length and the correlation length diverge, is known to correspond to a critical disorder of \( W_c/V \simeq 16.4 \) [26,28] which separates localized \((W > W_c)\) from metallic behavior \((W < W_c)\).
B. The function $Z(\omega)$ and the exponent $\mu = \eta/d$

For the numerical analysis, we defined the function

$$Z(E, E') := \int d^d x \left| \Psi_E(x) \right|^2 \left| \Psi_{E'}(x) \right|^2$$

for eigenstates with energy $E$ and $E'$, where $d$ denotes the spatial dimension. In the following, we will be interested in the case where the localization length is the largest length scale in the system so that $\xi(E) > L_\omega$. We show that one can directly extract the quantity $\eta$ from $Z(E, E')$ which is easier to obtain numerically than, e.g., a direct determination from the diffusion function $D(q, \omega)$ itself. Indeed, the two-particle correlation function in real space for $r = 0$

$$S(r = 0, E, \omega) := \left\langle \sum_{\alpha, \beta} \delta(E^- - E_\alpha) \delta(E^+ - E_\beta) \Psi_E(0) \Psi_{E^-}(0) \Psi_{E^+}(0) \Psi_{E^+}(0) \right\rangle$$

(43)

can be related to $Z(E, E')$ assuming that the disorder average in Eq. (43) is equivalent to a spatial average for one fixed impurity configuration. In this case,

$$S(r = 0, E, \omega) = \Omega^{-1} \left( \Omega \rho(E^-) \right) \left( \Omega \rho(E^+) \right) \int d^d x \left| \Psi_{E^-}(x) \right|^2 \left| \Psi_{E^+}(x) \right|^2,$$

(44)

with $\hbar \omega = E^+ - E^-$ and $E = (E^+ + E^-)/2$. The equivalence of these averages is underlying the subsequent analysis of the function $Z(E, E')$. The relation

$$Z(E^+, E^-) = \frac{1}{(2\pi)^d \Omega \rho(E^+) \rho(E^-)} \int d^d q S(q, E, E^+ - E^-),$$

(45)

obtained by Fourier transformation, is used in the following. Keeping $E$ fixed at the critical energy, the quantity $Z$ depends on $\omega = (E^+ - E^-)/\hbar$ only. Since the numerical calculation is performed on a finite lattice, the $q$-integrals have to be cut off at $q = 2\pi/a$, where $a$ is the lattice constant. A lower cutoff is given by the system size $L$ itself. In terms of the scaling variable $x$, the lower cutoff is $x_-(\omega) := ((L/2\pi)^d \rho \hbar \omega)^{-1} = (2\pi)^d \Delta/\hbar \omega$. The upper cutoff is $x_+(\omega) := ((a/2\pi)^d \rho \hbar \omega)^{-1} = x_-(\omega)(L/a)^d$. In a finite system, energy transfers $\hbar \omega$ from one state with energy $E$ to another with energy $E + \hbar \omega$ satisfy $\hbar \omega \gtrsim \Delta$. In terms of the variable $x$, this means $x_-(\omega) \lesssim (2\pi)^d$. For the function $Z(\omega)$, $\hbar \omega = E^+ - E^-$ in Eq. (45), and we obtain

$$Z(\omega) = \frac{s_d}{(2\pi)^d \Omega \rho^2} \int_{2\pi/a}^{2\pi/L} dq q^d S(q, E, \omega) = \frac{s_d}{d(2\pi)^d \Omega} \int_{x_-(\omega)}^{x_+(\omega)} dx \frac{x f(x)}{1 + |x f(x)|^2}. $$

(46)

Here, $s_{d=2} = 2\pi$ and $s_{d=3} = 4\pi$. For small $\omega$ and large $L$ the integral is dominated by the large $x$ behavior of $f(x)$ (region (C)).

$$Z(\omega) \approx \frac{s_d}{d(2\pi)^d \Omega} \int_{x_-(\omega)}^{x_+(\omega)} dx \left( x c_\gamma x^{-\eta/d} \right)^{-1} \approx \frac{s_d}{\pi(2\pi)^d \Omega c_\gamma \eta} \left( \frac{2\pi L}{a} \right)^\eta \left( \frac{\Delta}{\hbar \omega} \right)^{\eta/d}. $$

(47)

For small $\omega$, there is a range where $Z(\omega) \sim \omega^{-\mu}$ with $\mu = \eta/d$. A logarithmic plot of $Z(\omega)$ thus yields the exponent $\mu = \eta/d$, furthermore from Eq. (47) one can determine the coefficient $c_\gamma$. 

9
1. Numerical data

We used the numerical data to obtain the exponent $\mu = \eta/d$ from the function $Z(\omega)$. The result for a system of size $\Omega = (40a)^3$ is shown in Fig. 2. One clearly observes a power law with an exponent $\mu = 0.5 \pm 0.1$. This yields $\eta = \mu \cdot d \approx 1.5$. In the energy range $\omega_c \approx 300\Delta > |E - E'| \gtrsim \Delta$, we can fit the function $Z$ by $Z(\omega) = Z_0|E - E'|^{-0.5}$ with $Z_0 = 2.5 \cdot 10^{-5} \cdot \Delta/V$ is about $2.7 \cdot 10^{-4}$ in the present system. For even larger values of $\omega$ or correspondingly smaller $x$, one should enter regions (A) and (B) which are beyond the applicability of eq. (47).

On the other hand, for small $\omega < \sim \Delta$, $Z(\omega)$ saturates at the value of the inverse participation ratio $P(2)$ of $\Psi_E(x)$, $\xi_0 = \xi_0|E - E'|^{-\nu} \gg L$. We used our previously obtained eigenvalues and eigenfunctions of a system of size $125 \times 125$ and a magnetic field which corresponds to $1/3$ flux quanta per plaquette (see [6] for details) to calculate the function $Z(\omega)$ shown in Fig. 3. Again, a power law relation can be observed with a saturation at $\omega \approx \Delta = 3 \cdot 10^{-4}V$. The exponent $\mu = \eta/2 = 0.26 \pm 0.05$ is somewhat larger than the value obtained from the exponent $\delta$ of the temporal decay of a wavepacket ($\eta/2 = 1 - \delta = 0.19$) built from the same critical eigenvectors [6].

2. Comparison to the quantum Hall case ($d = 2$)

In the quantum Hall case there exists no complete Anderson transition, but for finite systems a critical behavior and multifractal eigenstates [2–4] can be observed in an energy range about the center of the disorder broadened Landau band, $E_0$, where the localization length exceeds the system size, $\xi = \xi_0|E - E_0|^{-\nu} \gg L$. We used our previously obtained eigenvalues and eigenfunctions of a system of size $125 \times 125$ and a magnetic field which corresponds to $1/3$ flux quanta per plaquette (see [6] for details) to calculate the function $Z(\omega)$ shown in Fig. 3. Again, a power law relation can be observed with a saturation at $\omega \approx \Delta = 3 \cdot 10^{-4}V$. The exponent $\mu = \eta/2 = 0.26 \pm 0.05$ is somewhat larger than the value obtained from the exponent $\delta$ of the temporal decay of a wavepacket ($\eta/2 = 1 - \delta = 0.19$) built from the same critical eigenvectors [6].

C. Exponents $D_2$, $\tilde{D}_2$, $\delta$.

1. The Wave packet

The correlation function $C(t)$ related to the probability of return $p(t)$ of a wave packet, constructed from eigenstates with energies $E \approx E_c$ of a 3d Anderson model, is shown in Fig. 4, where $C(t)$ is plotted as function of $t\Delta/\hbar$. A power law $p(t) \sim t^{-\delta}$ can be clearly identified, we obtain $\delta = 0.6 \pm 0.05$ in the region where $t > 2 \cdot 10^{-3}\hbar/\Delta$. The condition $q^4t/\hbar\rho$ in Eq. (28) can be checked if we use $q = 2\pi/L$ as the smallest $q$ value in the system ($L/a = 35$). Together with the density of states, which is $\rho = 0.058/(V a^3)$, we obtain from Eq. (28) the condition $t \gg 4 \cdot 10^{-3}\hbar/\Delta$ which is in accordance with the observed behavior in Fig. 4.

2. Multifractality exponents

We calculated the correlation dimension of the multifractal wavefunction $D_2$, Eq (12), and the correlation dimension of the spectral measure, $\tilde{D}_2$, defined by
\[ \gamma(q, \varepsilon) = \lim_{\varepsilon \to 0} 1/L^2 \sum_{\mathbf{r}} \sum_{\mathbf{i}} \left( \sum_{\mathbf{E} \in \Omega_i(\varepsilon)} |a_{\mathbf{E}}|^2 \right)^q \sim \varepsilon^{(q-1)D_q}, \]  

(48)

using a box-counting method. The result of the scaling behavior of the latter is shown in Fig. [3]. The exponent \( \delta \) from the probability of return of a wave packet (Fig. [4] and references therein) related to the generalized fractal dimensions \( D_q \) which completely describe all the moments of the spatial amplitude fluctuations of the critical wave functions have also been calculated. Our results are compiled in Table 1. The relations concerning the exponent \( \eta \) which have to be fulfilled according to Eq. (29), Eq. (5), and the definition \( \mu = \eta/d \), are 

\[ \eta = d(1 - \delta) = 1.2 \pm 0.15, \quad \eta = d - D_2 = 1.3 \pm 0.2, \quad \text{and} \quad \eta = 1.5 \pm 0.3, \]

respectively. Thus, one can say that within the numerical uncertainty the different methods of determining \( \eta \) yield the same value \( \eta \approx 1.3 \). We note that our value for \( D_2 \) is very close to a result of previous numerical work by Soukoulis and Economou using a different method [1] who obtained \( D_2 = 1.7 \pm 0.3 \).

**VI. INELASTIC SCATTERING AT THE ANDERSON-TRANSITION**

In this section we investigate the implications of critical dynamics at the Anderson transition on inelastic scattering rates. Inelastic scattering rates were calculated previously [24] for a quantum Hall system where the critical energy coincides with the Landau band center. It could be shown that the exponent \( \eta \) describing the eigenfunction correlation showed up in the temperature dependence of the scattering rate and the energy loss rate of electrons with acoustical phonons. On the other hand, the temperature dependence of the electron-electron scattering rate was similar to that in a two-dimensional disordered metal without a magnetic field. It is therefore of some interest to study these quantities for the three-dimensional system, too.

**A. Electron-phonon scattering rate**

The electron-phonon (e-p) scattering rate is calculated in the standard way [31] to second order in \( V_q \), the e-p coupling matrix element. It is assumed that the impurity potential is not changed by the lattice motion. In a metal this assumption is not fulfilled a priori: The impurities are embedded in the lattice and therefore move in phase with the other lattice atoms [32]. Here, we address to a situation where the random potential is generated by external sources like donor atoms far away from the electron gas.

The imaginary part of the self energy gives the inelastic lifetime of an electron of energy \( E \)

\[ \tau_{ep}^{-1}(E) = \frac{2\pi}{\hbar \Omega} \sum_{\mathbf{q},E'} |V_q|^2 |M_{E,E'}(\mathbf{q})|^2 \times \]

\[ \times \left\{ n(\omega_q) + f(E') \delta(E - E' + \hbar \omega_q) - (\omega_q \rightarrow -\omega_q) \right\} \]

(49)

Here, \( n(\omega_q) \) denotes the Bose function for phonon frequency \( \omega_q \) and \( f \) the Fermi function. The terms in the curly brackets correspond to phonon emission (+\( \hbar \omega_q \)) and -absorption
potential $\Xi$, and mass density $\rho$

$$\sum_{E'} |M_{E,E'}(q)|^2 \delta(E - E' + \hbar\omega_q) = \Omega \rho(E + \hbar\omega_q)|M_{E,E+\hbar\omega_q}(q)|^2,$$

(50)

where $\rho$ is the three-dimensional density of states and $\Omega$ is the total volume of the system. We express the matrix elements by the two-particle correlation function

$$\Omega \rho(E + \hbar\omega)|M_{E,E+\hbar\omega}(q)|^2 = \frac{1}{\rho(E)} S(q; E + \frac{\hbar\omega}{2}, \omega)$$

(51)

We restrict the discussion to energies $E = E_F$, where $E_F$ is the Fermi energy. Using $n(\omega_q) + f(E + \hbar\omega_q) = 1/\sinh(\beta\hbar\omega_q)$, we get

$$\tau_{ep}^{-1} = \frac{4}{2\pi \hbar} \int_0^{\omega_D} d\omega \frac{\alpha^2 F(\omega) \ S(\omega_s, E + \frac{\hbar\omega}{2}, \omega)}{\sinh \beta \hbar\omega \ \rho(E)}$$

(52)

where we defined the function $\alpha^2 F(\omega) = \alpha^2 \omega^n$ for acoustical phonons with coupling constant $\alpha^2$. In the case of non-piezoelectric materials, only the deformation potential coupling is relevant. Then, $n = 3$ and $\alpha^2 = (\Xi^2 \hbar)/(2\rho_M c_S^3)$ with speed of sound $c_S$, deformation potential $\Xi$, and mass density $\rho_M$. The temperature dependence at the critical energy $E = E_c$ is determined by the form of the correlation function, defined via Eq. (19) with $d = 3$. The phonon dispersion $\omega = c_s q$ leads to $x = \omega^2/(c_s^2 \rho h)$. The dimensionless variable $x$ ($d = 3$) can be written as $x = (\omega/\omega_{cr})^2$ by introducing the crossover frequency

$$\omega_{cr} := \left(\frac{\hbar c_s^3}{\rho}\right)^{1/2}.$$  

(53)

The exponent for the $T$-dependence of $\tau_{ep}^{-1}$ can be extracted in two limits: For frequencies $\omega \gg \omega_{cr}$ one has $x \gg 1$ and the correlation function $S \sim (xf(x))^{-1}/\omega \sim \omega^{-3+2n/3}$. In the opposite case $\omega \ll \omega_{cr}$ and $x \ll 1$, $S \sim (xf(x))/\omega \sim \omega^{4/3-1}$. This leads to different behaviors of the scattering rate $\tau_{ep}^{-1}$ in the limit $k_B T \ll \hbar\omega_{cr}$ and $k_B T \gg \hbar\omega_{cr}$, respectively. We can evaluate Eq. (52), introducing $\beta \hbar\omega$ as new variable. First, for temperatures $k_B T \gg \hbar\omega_D \gg \hbar\omega_{cr}$ larger than the Debye energy, the dependence is linear in $T$. This is (as in an ordinary metal) the trivial high-temperature case. The more interesting limit $k_B T \ll \hbar\omega_D$ gives, together with $\alpha^2 F(\omega) \sim \omega^n$ and the knowledge of the correlation function $S$, the limiting forms

$$\tau_{ep}^{-1} \sim \begin{cases} \frac{T^{n+4/3}}{T^{n-2+2n/3}} : k_B T \ll \hbar\omega_{cr} \\ f & k_B T \gg \hbar\omega_{cr}. \end{cases}$$

(54)

These two regimes correspond to the case (A) and (C) in Eq. (19). The full temperature dependence of $\tau_{ep}^{-1}$ can be obtained from a numerical evaluation of Eq. (52).

The result $\tau_{ep}^{-1} \sim T^{3+4/3}$ for deformation potential scattering ($n = 3$) in the low temperature limit is a direct consequence of the ‘Wegner-scaling’ of the diffusion function $D(q, \omega) \sim \omega^{1/3}$ in regime (A). It should be compared to the dependence $\tau_{ep}^{-1} \sim T^4$ for the corresponding rate in the disordered metallic case [32]. We also note that as in the quantum Hall case [23], the exponent $\eta$ appears only in an intermediate and not in the low-temperature regime.
B. Electron-electron scattering

Electron-electron (e-e) scattering rates at the Anderson transition have been calculated first by Belitz and Wysokinski [33], who used the so-called 'exact eigenstate formalism' and an RPA-like approximation for the Coulomb interaction. The advantage of this method is that it is non-perturbative in the disorder and, at least in principle, applicable to a large number of systems. By this one can generalize results for e-e scattering rates which are already known in the ballistic or the diffusive, weakly localized case. However, one has to keep in mind that this approach is perturbative in the interaction between the electrons.

The general expression for the electron scattering rate $\tau_{ee}^{-1}$ at finite temperature $k_B T = 1/\beta > 0$, given by the on-shell electronic self-energy at Fermi energy $E = E_F$ [33,33]:

$$\tau_{ee}^{-1} = 2 \int_0^\infty d\omega \frac{\phi(\omega)}{\sinh \beta \hbar \omega}$$

$$\phi(\omega) = -\frac{1}{\pi \hbar \rho \Omega} \sum_q \text{Im} V(q, \omega) \text{Im} \Phi(q, \omega)$$

(55)

can be derived by starting from the exact eigenstates of the unperturbed system [14]. Here, $\Phi(q, \omega)$ is the density relaxation function of which the imaginary part is related to that of the density response function $\chi(q, \omega)$ by $\text{Im} \Phi(q, \omega) = (1/\omega) \text{Im} \chi(q, \omega) = \pi S(q; E, \omega)$ where the energy $E$ appears in the density of states $\rho = \rho(E)$. Notice that $\text{Im} \Phi(q, \omega) = \text{Im} \Phi(q, -\omega)$ and $\text{Im} V(q, \omega) = -\text{Im} V(q, -\omega)$ because $\text{Re} \chi(q, \omega)$ is an even and $\text{Im} \chi(q, \omega)$ is an odd function of $\omega$. One therefore can restrict the integration to positive values of $\omega$. The above formula can be viewed as the rate for the scattering of one electron at the fluctuations of the electromagnetic field caused by the motion of all the other electrons [35], the function $\phi$ being an effective density of states of these fluctuations. It corresponds in the case of electron-phonon scattering to the Eliashberg-function $\alpha^2 F$ which there essentially gives the phonon density of states. In the Coulomb potential, screening is included via the RPA approximation

$$V(q, \omega) = \frac{V_0(q)}{1 + \chi(q, \omega) V_0(q)}$$

(56)

with the bare Coulomb potential $V_0(q) = 4\pi e^2/q^2$. Notice that only the inclusion of dynamical screening leads to an non vanishing imaginary part of $V(q, \omega)$ and thus to a nonzero e-e scattering rate. The bare Coulomb potential would render the electron self-energy merely real and $\tau_{ee}^{-1}$ would be zero. The influence of arbitrary disorder on the screening here is incorporated in the correlator $\chi$ which contains all the information about the unperturbed system. In [33], the $q = 0$ limit of the diffusion function $D(q = 0, \omega)$ was used for the evaluation of Eq. (55) at the critical point. Although the integration requires both the $q$- and the $\omega$-dependence of the diffusion function, it turned out that this approximation affects only the prefactor of the scattering rate and not its temperature dependence.

In the following, we will use the full form of the diffusion function as it enters into the density-density correlation function, which at the critical point can be written as

$$\chi(q, \omega) = i\rho \frac{xf(x)}{1 + ixf(x)}, \quad x = \frac{q^3}{\hbar \omega \rho}$$

(57)
The function $\Phi(\omega)$ can then be evaluated; its zero-frequency limit is given by

$$\Phi(\omega = 0) \equiv \Phi_0 = \frac{1}{6\pi^3} \int_0^\infty dx \frac{1}{1 + (xf(x))^2}. \quad (58)$$

As in the two-dimensional diffusive and Quantum Hall case [36,24], the dimensionless quantity $\Phi_0$ can be expressed by an integral over a function of the variable $x = q^d/(\hbar \omega \rho)$ where $d$ is the dimension. As in the above cases, the scattering rate according to Eq. (55) formally diverges because the function $\Phi(\omega)$ describing the DOS of the electromagnetic fluctuations from the electron 'bath' does not vanish at zero frequency. However, for finite temperatures such that the scattering rate $\tau_e^{-1}$ is larger than the frequency $\omega$, the description of the critical dynamics in terms of $\chi(q^3/\omega)$ can no longer be valid and the integral should be cut off at $\tau_e^{-1}$ [33]. This leads to a self-consistent equation (for low $T$, $\Phi(\omega)$ can be replaced by $\Phi_0$)

$$\tau_e^{-1} = 2\Phi_0 \int_{\tau_e^{-1}}^\infty d\omega \frac{1}{\sinh \beta \hbar \omega}, \quad (59)$$

of which the solution can be written as

$$\tau_e^{-1} = \frac{k_B T}{\hbar} \gamma, \quad \gamma = -2\Phi_0 \ln \tanh(\gamma/2). \quad (60)$$

This is in fact the result [33] of Belitz and Wysokinski, differing only in the prefactor given by

$$\gamma = -\frac{\sqrt{3}}{4c} \ln \tanh(\gamma/2) \quad \text{Ref. [33]}, \quad (61)$$

where $c = 13.4$ [4] or 1 [33], respectively. In view of our analysis, Eq. (19) and Eq. (58), we find that the prefactor depends on the microscopic details of the system, namely the constants $c_\alpha, c_\gamma$, and the dimensionless conductance at the critical point $g^*$. In two-dimensional quantum Hall systems the critical conductance was found to be universal [17]. However, we know of no argument that shows the critical conductance to be universal in three dimensions. Thus we find that the universality of the rate $\tau_e^{-1}$ claimed in [33] is related to the universality or lack thereof of the critical conductance $g^*$ [37].

**VII. CONCLUSION**

We have discussed some aspects of multifractal fluctuations near the Anderson transition in three-dimensional electron systems. The behavior of the dynamical diffusion function $D(q, \omega)$ was reviewed. For large wavevectors $q$ and small frequencies $\omega$ multifractal fluctuations lead to the occurrence of an anomalous diffusion exponent $\eta$ in the diffusion function. We have found $\eta \approx 1.3$ in a numerical analysis of eigenfunctions obtained by numerical diagonalization, and verified that $\eta$ is related to the correlation dimension $D_2$ of the multifractal fluctuations by $\eta = d - D_2$.

The multifractal eigenfunction fluctuations influence the temperature dependence of the electron-phonon and electron-electron scattering rates. We found that the low-temperature exponent of the former is modified by the finite value of $\eta$, while the latter is linear in $T$ and depends on the dimensionless conductance.
VIII. ACKNOWLEDGMENTS

T.B. would like to acknowledge support by the EU STF9 fellowship and stimulating discussions with A. Kawabata and Y. Hirayama. B.H. would like to acknowledge the support through the Sonderforschungsbereich 341 of the Deutsche Forschungsgemeinschaft and fruitful discussions with M. Janßen.
REFERENCES

[1] C. M. Soukoulis, E. N. Economou: Phys. Rev. Lett. 52 (1984) 565
[2] W. Pook, M. Janßen: Z. Phys. B 82 (1991) 295
[3] B. Huckestein, L. Schweitzer: in High Magnetic Fields in Semiconductor Physics III, Würzburg 1990, Vol. 101 of Springer Series in Solid–State Sciences, edited by G. Landwehr (Springer, Berlin, 1992), p. 84.
[4] B. Huckestein, B. Kramer, L. Schweitzer: Surf. Sci. 263 (1992) 125
[5] M. Schreiber, H. Grussbach: Phys. Rev. Lett. 67 (1991) 607
[6] B. Huckestein, L. Schweitzer: Phys. Rev. Lett. 72 (1994) 713
[7] L. Schweitzer: J. Phys.: Condens. Matter 7 (1995) L281
[8] J. Chalker, G. Daniell: Phys. Rev. Lett. 61 (1988) 593
[9] One has to be careful when comparing results for $\eta$ by other authors using the definition $d−2+\eta$ instead of $\eta$ in Eq. (1). In fact, in 1964 Fisher [M. A. Fisher, Journ. Math. Phys. 5, 944, (1964)] introduced an exponent $\eta$ (which we here denote by $\tilde{\eta}$ in order to avoid confusions) for the behavior of the pair correlation function $G(r)$ describing the density fluctuations of a fluid at the critical point of a second order phase transition. There, the correlations decay algebraically (instead of exponentially $\sim \exp(-r/\xi)$) because of $r \ll \xi \rightarrow \infty$: $G(r) \sim 1/r^{d−2+\tilde{\eta}}$, where $\tilde{\eta} > 0$ describes deviations from the Ornstein-Zernicke prediction.
[10] F. J. Wegner: Z. Phys. B 25 (1976) 327
[11] D. Forster: Hydrodynamic Fluctuations, Broken Symmetry and Correlation Functions, Frontiers in Physics, 47 (Benjamin, London, 1975).
[12] E. Abrahams, P. W. Anderson, D. C. Liciardello, V. Ramakrishnan: Phys. Rev. Lett. 42 (1979) 673
[13] For a review, see, e.g. : B. Kramer and A. MacKinnon: Rep. Prog. Phys. 56 (1994) 1469
[14] E. Abrahams, P. A. Lee: Phys. Rev. B 33 (1986) 683
[15] J. Chalker: Physica A 167 (1990) 253
[16] P. Lambrianides, H. B. Shore: Phys. Rev. B 50 (1994) 7268
[17] Y. Huo, R. E. Hetzel, R. N. Bhatt: Phys. Rev. Lett. 70 (1993) 481
[18] The case $\xi > L_\omega$ with $S(r, \omega) \sim r^{-\eta_\omega-\eta/d}$ is identical to Eq. (5.15) in [10], where the exponent $\kappa'$ corresponds to $\eta$.
[19] T. Brandes, L. Schweitzer, B. Kramer: in Quantum Dynamics of Submicron Structures, Vol. 291 of NATO ASI Series E, edited by H. Cerdeira, B. Kramer and G. Schön (Kluwer Academic Publishers, Dordrecht, 1994), p. 115.
[20] M. Janssen: Int. Journ. Mod. Phys. B 8 (1994) 943
[21] In [10], $0 \leq \eta \leq d$ is shown by means of Schwarz’s inequality. Note that the exponent $\kappa'$ in [10] is $\eta$ in our notation, furthermore $\kappa' = -x_{2s}$ in [25], cp. Eq. (3.21) and (1.6) there.
[22] R. Ketzmerick, G. Petschel, T. Geisel: Phys. Rev. Lett. 69 (1992) 695
[23] M. Kaveh, N. F. Mott: Phil. Mag. B 55 (1986) 9
[24] T. Brandes, L. Schweitzer, B. Kramer: Phys. Rev. Lett. 72 (1994) 3582; T. Brandes: Phys. Rev. B 52 (1995) 8391
[25] F. Wegner: Z. Phys. B 36 (1980) 209
[26] E. Hofstetter, M. Schreiber: Europhys. Lett. 21 (1993) 933

16
[27] A. MacKinnon: J. Phys. Condens. Matter 6 (1994) 2511
[28] I. Kh. Zharekeshev, B. Kramer: Phys. Rev. 51 (1995) 17356
[29] J. Bauer, T.-M. Chang, J.L. Skinner: Phys. Rev. B 42 (1990) 8121
[30] A. B. Harris: J. Phys. C 7 (1974) 1671
[31] G. D. Mahan: Many–Particle Physics (Plenum Press, New York, 1990).
[32] S. Chakravarty, A. Schmid: Phys. Rep. 140 (1986) 193
[33] D. Belitz, K. Wysokinski: Phys. Rev. B 36 (1987) 9333
[34] E. Abrahams, P. Anderson, P. Lee, T. Ramakrishnan: Phys. Rev. B 24 (1981) 6783
[35] W. Eiler: Journal of Low Temperature Physics 56 (1984) 481
[36] S. C. Y. Siak: J. Phys. C 3 (1991) 4213
[37] For a quantitative estimate of $\gamma$, one has to know the full form of $f(x)$ which, in principle, can be extracted from the numerical calculation. However, we find no direct relation of the constants $c_\alpha$ and $c_\gamma$ to an experimentally relevant quantity like the dimensionless conductance $g^*$.
[38] B. Huckestein: Europhys. Lett. 20 (1992) 451
[39] U. Fastenrath: Physica A 189 (1992) 27
### TABLE I

The calculated exponents related to the critical states at the metal-insulator transitions in two and three dimensional systems. The results for the critical exponent of the localisation length $\nu$ are taken from [38] (QHE), [39] (2d symplectic), and [26,27] (3d orthogonal). The remaining values for 2d systems not calculated in the present work were taken from [3] (QHE) and [4] (2d symplectic).

| System     | $\nu$  | $\delta$ | $\alpha_0$ | $D_2$  | $\tilde{D}_2$ | $\mu = \eta/d$ |
|------------|--------|----------|------------|--------|---------------|----------------|
| 2d QHE     | 2.35   | 0.81 ± 02| 2.29 ± 02 | 1.62 ± 02| 0.80 ± 05    | 0.26 ± 05      |
| 2d Sympl.  | 2.75   | 0.83 ± 03| 2.19 ± 03 | 1.66 ± 05| 0.83 ± 03    | 0.175 ± 03     |
| 3d Ortho.  | 1.45   | 0.6 ± 05 | $\sim 4.0$| 1.7 ± 0.2| 0.55 ± 05    | 0.5 ± 0.1      |
FIG. 1. Five different regimes (separated by solid lines) for the diffusion function $D(q, \omega)$ after Chalker [15]. Relevant for our numerical analysis are the three regimes A,B,C ($\xi \to \infty$) with the behavior of $D(q, \omega) \sim \omega^{(d-2)/d}$, $\sim q^{d-2}$, and in particular $q^{d-2}(qL_\omega)^{-\eta}$ with the exponent $\eta = D_2 - d$ related to the multifractality of the eigenstates. According to the value of the scaling variable $x = (qL_\omega)^d$, the function $f(x)$, Eq. [19], follows a different power-law in $x$. The dotted lines indicate this crossover between A, B and C. Since $q^\xi = x^{1/d}(L_\omega^{-1}\xi)$, (a) corresponds to small, (b) to intermediate and (c) to large $x$. The latter case is relevant for the numerical determination of $\eta$. 
FIG. 2. Energy correlation function $Z(\omega)$ of eigenstates taken from the critical regime ($W_c/V = 16.3$) of a 3d Anderson model of system size $(L/a = 40)^3$. We used 371 eigenstates with energies $E$ from the interval $-0.05 \leq E/V \leq 0.05$ to extract the exponent $\mu = \eta/3 = 0.5$ from the power law decay.

FIG. 3. Energy correlation function $Z(\omega)$ for a QHE-system of size $(L/a = 125)^2$. 330 eigenstates taken from an energy interval $\Delta E = 0.1V$ around the center of the lowest Landau band were used to determine the exponent $\mu = \eta/2 = 0.26$. 

FIG. 4. Temporal autocorrelation function $C(t)$ for a 3d critical Anderson model ($W_c/V = 16.5$) of size $(L/a = 35)^3$ versus time $t$ in units of the Heisenberg time $t_H = \hbar/\Delta$. To construct the wave packet at the initial site a total of 1194 eigenstates from the energy interval $[-0.5, 0.0]$ have been used.

FIG. 5. The correlation dimension $\tilde{D}_2 = 0.55 \pm 0.5$ of the spectral measure as obtained from the scaling relation of $\gamma_2$ (see Eq. 48) versus the length of the energy interval $\varepsilon$ for a critical system of size $(L/a = 40)^3$. 