We solve a problem posed by A. Bonilla and K.-G. Grosse-Erdmann \[7\] by constructing an entire function $f$ that is frequently hypercyclic with respect to the differentiation operator, and satisfies $M_f(r) \leq ce^r r^{-1/4}$, where $c > 0$ be chosen arbitrarily small. The obtained growth rate is sharp. We also obtain optimal results for the growth when measured in terms of average $L^p$-norms. Among other things, the proof applies Rudin-Shapiro polynomials and heat kernel estimates.

1. INTRODUCTION

A linear operator $T$ on a separable topological vector space $E$ is hypercyclic if there exists $x \in E$ such that the set of iterates $\{T^n x : n \geq 1\}$ is dense in $E$. In this situation $x$ is sometimes called an universal element. We refer to \[11\] for basic facts on hypercyclicity.

Recently there has been interest on a related, more stringent notion. The operator $T$ (and likewise the element $x \in E$) is called frequently hypercyclic if $T^n x$ visits any given neighbourhood with a relatively constant rate. More precisely, given any open set $U \subset E$ one asks that the set

$$A = \{ n \geq 1 : T^n x \in U \}$$

has positive density, i.e.

$$\liminf_{n \to \infty} \frac{1}{n} \#(A \cap \{1, \ldots, n\}) > 0.$$ 

This notion was introduced by Bayart and Grivaux \[3\], \[4\] and has been studied in many papers devoted to operators in Hilbert, Banach, or general topological vector spaces. We refer to \[10\] and \[1\] and especially for the references therein for more information on the known results.

Classical examples of hypercyclic operators are the translation and differentiation operators in the space $E$ of entire functions on the complex plane $\mathbb{C}$, equipped with the standard compact-open topology. We shall consider only the differentiation operator
$D : \mathcal{E} \to \mathcal{E}$, where $Df(z) := f'(z)$ (see [1] for results on the translation operator). Let us recall that hypercyclicity of $D$ is a classical result due to MacLane [17].

In this note we study the following problem: how slowly can a $D$-frequently hypercyclic entire function grow? This question was raised by Bonilla and Grosse-Erdmann [7], and at the same time they gave concrete estimates for the minimal growth of such a function (it was proven in [8] that indeed $D$ is frequently hypercyclic). In [6] the same authors generalized a well-known result of Godefroy and Shapiro [9] by showing that every operator on $\mathcal{E}$ which commutes with $D$, and is not a multiple of the identity, is frequently hypercyclic. A couple of years later Blasco, Bonilla and Grosse-Erdmann [11] improved on the earlier results and showed that a $D$-frequently hypercyclic entire function $f$ satisfies

$$\liminf_{r \to \infty} \frac{M_f(r)}{r^{1/4}} > 0,$$

where $M_f(r) := \sup_{\theta} |f(re^{i\theta})|$. Moreover, given any function $\phi : \mathbb{R} \to [1, \infty)$ with $\lim_{r \to \infty} \phi(r) = \infty$ they proved the existence of a $D$-frequently hypercyclic entire function $f$ with $M_f(r) \leq e^r \phi(r)$ for $r \geq 1$.

The paper [1] also considered growth in terms of the average $L^p$-norms. Thus, for $p \in [1, \infty)$ and $r > 0$ let

$$M_{f,p}(r) := \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p},$$

and augment this with $M_{f,\infty}(r) = M_f(r)$. In [11], the authors also showed that, given $a > 0$, an inequality

$$M_{f,p}(r) \leq C_a e^{\gamma r - a}$$

could only be valid if $a \leq \max(1/2p, 1/4)$. Moreover, if $C$ is replaced by a factor $\phi(r) \uparrow \infty$, examples with the growth rate $a = \min(1/2p, 1/4)$ were constructed.

Quite recently Bonet and Bonilla [5] constructed almost optimal examples in the range $p \in [1, 2)$ by showing that $a = 1/2p$ can be achieved, again requiring the factor $\phi(r) \uparrow \infty$. Their result is sharp in the special case $p = 1$.

Our main result determines the optimal growth rate of entire $D$-frequently hypercyclic functions. It turns out that the sharp result actually corresponds to the lowest possible rates consistent with [11] for all $p$. Both [11] and [5] employ fairly sophisticated tools from the general theory of frequent hypercyclicity. In contrast, our construction relies only on basic complex analysis, elementary heat kernel estimates and on two classes of classical polynomials whose properties reflect the role that $p$ plays in these results. Thus for large $p$, our construction patches together Rudin-Shapiro polynomials $p_m$, having coefficients $\pm 1$, but whose supnorm is minimal; for $1 \leq p \leq 2$ an analogous role is played by the de la Vallée-Poussin polynomials $p^*_{m_k}$.

We present our main result in three parts, with the most interesting case $p = \infty$ meriting its own statement.

**Theorem 1.1.** (i) For any $c > 0$ there is an entire frequently hypercyclic function $f$ such that

$$M_f(r) \leq c \frac{e^r}{r^{1/4}} \quad \text{for all } r > 0.$$
This estimate is optimal: every such function satisfies $\limsup_{r \to \infty} r^{1/4} e^{-r} M_f(r) > 0$.

(ii) More generally, given $c > 0$ and $p \in (1, \infty]$ there is an entire $D$-frequently hypercyclic function $f$ with

$$M_{f,p}(r) \leq c e^{-r \alpha(p)}$$

for all $r > 0$, where $\alpha(p) = 1/4$ for $p \in [2, \infty]$ and $\alpha(p) = 1/(2p)$ for $p \in (1, 2]$. This estimate is optimal: every such function satisfies $\limsup_{r \to \infty} r^{\alpha(p)} e^{-r} M_{f,p}(r) > 0$.

(iii) Given $\phi(r) \uparrow \infty$, there is an entire $D$-frequently hypercyclic function $f$ with

$$M_{f,1}(r) \leq \phi(r) e^{-r}$$

for all $r > 0$.

This estimate is optimal: every such function satisfies $\limsup_{r \to \infty} r^{1/2} e^{-r} M_{f,1}(r) = \infty$.

The sharp conclusion (i) provides the converse of (1), yielding a considerable strengthening of the best previously known growth (2). In turn, (ii) sharpens the main result of [5] by removing the unnecessary increasing factor $\phi(r) \uparrow \infty$. Finally, assertion (iii) is already due to Bonet and Bonilla [5, Corollary 2.4].

2. THE CONSTRUCTION OF THE $D$-FREQUENTLY-HYPERCYCLIC FUNCTION $f$

For any given polynomial $q$ with Taylor series

$$q(z) = \sum_{j=0}^{d} \frac{q_j z^j}{j!}, \quad d = \deg(q),$$

set

$$\tilde{q}(z) := \sum_{j=0}^{d} q_j z^j \quad \text{and} \quad \|\tilde{q}\|_{\ell_1} := \sum_{j=0}^{d} |q_j|.$$

For our purposes it will be useful to divide the Taylor series $f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$ of a given entire function $f$, into blocks of specified size. It turns out that the correct partition is the decomposition

$$f(z) = \sum_{n=0}^{\infty} P_n f,$$

where

$$P_n f(z) = \sum_{k=n^2}^{(n+1)^2 - 1} \frac{a_k}{k!} z^k \quad (n \geq 0).$$

In fact, the precise size of the blocks $P_k$ is one of the key ingredients of our argument. The motivation for the choice (3) will be discussed later on in Remark 4.4.
2.1. Long polynomials with controlled norm. Explicit polynomials having a fixed proposition of coefficients 1 and with small $L^p$-norm (up to order of magnitude) will be needed for construction of the blocks $P_k f$. It is here where the cases $1 \leq p \leq 2$ and $p \geq 2$ bifurcate. The first part of the next lemma records the beautiful result of Rudin and Shapiro [18] which produces polynomials $\{p_m\}$ of each degree $m$ having coefficients $\pm 1$ and with optimal growth of sup norm (see also [15, §6, Chapter 5], [14] and [2] for further results on unimodular polynomials). That this growth is indeed optimal follows immediately by the Parseval formula. For $p \in [1, 2]$, the $\{p_m\}$ are replaced by the de la Vallée-Poussin polynomials $p_m^*$. Below $p'$ stands for the exponent conjugate to $p$.

Lemma 2.1. (i) For each $m \geq 1$ there is a trigonometric polynomial $p_m$

$$p_m = \sum_{k=0}^{m-1} b_{m,k} e^{ik\theta},$$

where $b_{m,k} = \pm 1$ for all $0 \leq k \leq m - 1$, with at least half of the coefficients of $f$ being $+1$, and with

$$\|p_m\|_p \leq 5\sqrt{m} \quad \text{for} \quad p \in [2, \infty].$$

(ii) Corresponding to each $m \geq 1$ is a polynomial

$$p_m^* = \sum_{k=0}^{m-1} b_{m,k}^* e^{ik\theta},$$

where $|b_{m,k}| \leq 1$ for all $0 \leq k \leq m - 1$, and with at least $\lfloor m/4 \rfloor$ coefficients being $+1$, and with

$$\|p_m^*\|_p \leq 3m^{1/p'} \quad \text{for} \quad p \in [1, 2].$$

Proof. In case $p = \infty$ assertion (i) just records the main result of [18]. Since $\|p_m\|_2 = \sqrt{m}$, the claim for exponents $p \in (2, \infty)$ is immediate. In turn, for assertion (ii) we may assume that $m \geq 4$. Set $k = \lfloor m/4 \rfloor$ and choose let $p_m^*(e^{i\theta}) = e^{2k\theta}(2F_{2k}(e^{i\theta}) - F_k(e^{i\theta}))$, where $F_k$ is the $k$:th Fejer kernel. In other words, $p_m^*$ corresponds to a shifted de la Vallée Poussin kernel [15, p.16]. Then $\|p_m^*\|_1 \leq 3$ and the statement for $p \in (1, 2]$ is obtained by interpolating (i.e. applying Hölder’s inequality) with the obvious estimate $\|p_m^*\|_2 \leq \sqrt{m}$. \qed

2.2. Explicit formula for $f$. Let $0 < c < 1$ be given, let $p \in (1, \infty]$. Denote by $\mathcal{P}$ the (countable) set of all polynomials with rational coefficients, and consider pairs $(q, \ell)$ with $q \in \mathcal{P}$ and $\ell \in \mathbb{N}$ with $\ell \geq \|q\|_{\ell_1}$, exhibited in a single sequence $(q_k, \ell_k) \ k \geq 1$. Let us record the important fact

$$\|\tilde{q}_k\|_{\ell_1} \leq \ell_k \quad \text{for every} \quad k \geq 1.\tag{4}$$

Next partition the even integers in $\mathbb{N}$ into countably many infinite disjoint arithmetic sequences by setting $2\mathbb{N} = \bigcup_{k \geq 1} \mathcal{A}_k$ where

$$\mathcal{A}_k = \{2^k(2j - 1) : j \in \mathbb{N}\}.$$

Next, for any $k \geq 1$ denote by $\alpha_k$ the integer

$$\alpha_k := 1 + \left\lceil \max \left((10^{10}\ell_k/c)^{\max(2,p')}, 2d_k + 8\ell_k\right) \right\rceil,$$

with $d_k$ the degree of $q_k$.\hfill\qed
We have noted that
\[ f = \sum_{j=1}^{\infty} a_j z^j := \sum_{n=1}^{\infty} P_n f \]
is uniquely determined by the blocks \( P_n f \). First, consider the case \( p \geq 2 \). Set first \( a_0 = 0 \), or, in other words \( P_0 f = 0 \). Fix \( n \geq 1 \). If \( n \) is odd, take \( P_n f = 0 \). When \( n \) is even and \( n \in A \) we use the Rudin-Shapiro polynomials to write
\[ \overline{P_n f} := \begin{cases} 0 & \text{if } n < 10\alpha_k, \\ z^{n^2} p_{[n/\alpha_k]}(z^{\alpha_k})\tilde{q}_k(z) & \text{otherwise.} \end{cases} \]  
For \( 1 < p < 2 \), one just modifies the second line in (6) by setting for even \( n \), \( n \in A_k, n \geq 10\alpha_k \)
\[ \overline{P_n f} = z^{n^2} p^*_{[n/\alpha_k]}(z^{\alpha_k})\tilde{q}_k(z). \] 
We shall treat the case \( p = 1 \) in Remark 3.4 below. In what follows we shall show that the function \( f \) we just defined satisfies the assertions (i) and (ii) of Theorem 1.1.

3. The Proof

The proof of Theorem 1.1 will be based on two auxiliary results, Propositions 3.3 and 3.4 below. We begin by recording two simple auxiliary observations.

**Lemma 3.1.** Let \( a \in [0, 1] \). Then for any \( m \geq 1 \),
\[ \sum_{n=1}^{\infty} e^{n^2} n^{-2a} \left( \frac{m^2}{n^2} \right)^{n^2} \leq 10 e^{m^2} m^{-2a}. \]

**Proof.** Write the elementary inequality \( 1 + \log x \leq x \) \( (0 < x \leq \infty) \) as \( 1 + \log(x^2) \leq x^2 - (1 - x)^2 \). By making the substitution \( x = m/n \) and multiplying both sides by \( n^2 \) we obtain
\[ n^2 + n^2 \log(m^2/n^2) \leq m^2 - (n - m)^2. \] 
Thus \( e^{n^2}(m^2/n^2)^{n^2} \leq e^{m^2} e^{-(n-m)^2} \), and hence the sum has the upper bound
\[ e^{m^2} m^{-2a} \left( \sum_{n=1}^{m} \left( m/n \right)^{2a} e^{-(n-m)^2} + \sum_{n=m+1}^{\infty} e^{-(n-m)^2} \right) \]
\[ \leq e^{m^2} m^{-2a} \left( 2 \sum_{j=0}^{\infty} (j+1)^2 e^{-j^2} \right) \leq 10 e^{m^2} m^{-2a}, \]
where we have used the simple inequality \( (m/n)^{2a} \leq ((m-n)+1)^{2a} \leq ((m-n)+1)^2 \) when \( n \leq m \).

**Lemma 3.2.** Assume that \( 0 < x_0 < x_1 \) and the function \( u : [x_0, x_1] \to \mathbb{R} \) is of the form
\[ u(x) = a \log(x) + b - (cx + d) \]
with \( a > 0 \) and \( u(x_0) = u(x_1) = 0 \). Then \( u(x) \leq a(x_1/x_0 - 1)^2/8 \) for all \( x \in [x_0, x_1] \).

**Proof.** Simply observe that \( u(x) - (x - x_0)(x_1 - x)ax_0^{-2}/2 \leq 0 \) as the left hand side is convex and vanishes at the endpoints.
The following proposition contains an important underlying principle for bounding the growth of a given entire function $g$: correct size for each $P_n g$ on both $\{|z| = n^2\}$ and on $\{|z| = (n+1)^2\}$ is enough to guarantee the desired growth for $g$.

**Proposition 3.3.** Let $p \in (1, \infty]$ and $a \in [0, 1]$. Assume that there is a constant $b > 0$ such that for each $n \geq 1$ the blocks of a given entire function $g$ with $g(0) = 0$ satisfies

$$M_{P_n g,p}(n^2) \leq b e^{n^2} n^{-2a}. \quad (8)$$

and

$$M_{P_n g,p}((n + 1)^2) \leq b e^{(n+1)^2} (n + 1)^{-2a}. \quad (9)$$

Then $g$ itself satisfies

$$M_{g,p}(r) \leq 10^9 b e^r r^{-2a} \quad (r > 0).$$

**Proof.** We start by recalling [13, p.76] that for any entire function $g$, the $p$-means $\log(M_{g,p}(r))$, $p > 0$ are increasing, convex functions of $\log r$ (alternatively, in our range $p \in [1, \infty]$ this is just Hadamard’s three circles theorem combined with a simple duality argument). Especially, one may apply the maximum principle for the $p$-means. When $h = P_n g$, we have that $z^{-n^2} h(z)$ is holomorphic in $\{|z| \leq n^2\}$ and that $z^{-(n+1)^2} h(z)$ is holomorphic in $\mathbb{C} \setminus \{|z| < (n+1)^2\}$. Hence (8) and (9) imply by the maximum principle that

$$M_{P_n g,p}(m^2) \leq b e^{n^2} n^{-2a} \left(\frac{m^2}{n^2}\right)^n \quad \text{for } 1 \leq m \leq n.$$  

and

$$M_{P_n g,p}(m^2) \leq b e^{(n+1)^2} (n + 1)^{-2a} \left(\frac{m^2}{(n + 1)^2}\right)^{n+1} \quad \text{for } 1 \leq n < m.$$  

By summing over $n$, observing that $P_0 g = 0$, and and invoking Lemma 3.1 we obtain the desired estimate when $r = m^2$, with $m$ a positive integer

$$M_{g,p}(m^2) \leq \sum_{n=m}^{\infty} b e^{n^2} n^{-2a} \left(\frac{m^2}{n^2}\right)^n + \sum_{n=1}^{m-1} b e^{(n+1)^2} (n + 1)^{-2a} \left(\frac{m^2}{(n + 1)^2}\right)^n \leq 11 b e^{m^2} m^{-2a}. $$

The log-convexity of the $p$-means together with Lemma 3.2 allow this estimate to be interpolated inside each interval $(m^2, (m + 1)^2)$. Observe first that the effect of the term $m^{-2a}$ can be ignored since $\log(r^{-2a})$ is a linear function of $\log r$. Thus assume that $a = 0$ and $m \geq 1$. Denote $r_0 = m^2$ and $r_1 = (m + 1)^2$. We obtain for $r \in (r_0, r_1)$ that

$$M_{g,p}(r) \leq \log 11 + \frac{\log r_1 - \log r}{\log r_1 - \log r_0} + \frac{\log r - \log r_0}{\log r_1 - \log r_0} r_1 : = C + \frac{r_1 - r_0}{\log(r_1/r_0)} \log r.$$  

If one replaces the right hand side by a function that is linear in $r$ with the same values at the endpoints $r_0, r_1$, Lemma 3.2 yields that the error so induced is less than

$$\left(\frac{r_1 - r_0}{\log(r_1/r_0)}\right) (r_1/r_0 - 1)^2/8 \leq \frac{3(r_1 - r_0)^2}{8r_0} \leq (3/2)^3,$$

where we have also applied the inequality $\log(r_1/r_0) \geq (r_1/r_0 - 1)/3$. The stated result follows by observing that $11 e^{(3/2)^3} \leq 10^3$. 

Finally, the claim for values \( r \in (0, 1] \) follows immediately from the maximum principle as we already know that \( M_{g,p}(1) \leq 11b \) and since \( 11e \leq 1000 \). \( \square \)

The relation between the degree-2n polynomial determining \( P_n g \) and the growth of \( P_n g \) is characterized in the following proposition, where the main trick is the surprising appearance of the heat kernel.

**Proposition 3.4.** Assume that \( n \geq 1 \), \( p \in [1, \infty] \) and \( g \) is an arbitrary entire function \( g(z) = \sum_{k=0}^{\infty} \frac{b_k}{k!} z^k \). Then, if \( B := \| \sum_{k=0}^{2n} \frac{b_{n+k} e^{ik\theta}}{k!} \|_{L^p(T)} \),

one has

\[
M_{P_n g, p}(n^2) \leq 20 Be^{n^2}n^{-1}
\]

and

\[
M_{P_n g, p}(n + 1)^2 \leq 20 Be^{(n+1)^2}(n + 1)^{-1}.
\]

**Proof.** We consider first (10), and may assume that \( n \geq 2 \). Observe that one may write

\[
|P_n g(n^2 e^{i\theta})| = \left( \frac{n^2}{n^2} \right)^{\frac{n^2}{n^2}} \left| \sum_{k=0}^{2n} \lambda_{n,k} b_{n+k} e^{ik\theta} \right|,
\]

where

\[
\lambda_{n,k} := \left( \frac{n^2}{n^2} \right)^{\frac{n^2}{n^2}} \left( \frac{n^2}{n^2 + 1} \right)^{\frac{n^2}{n^2 + 2}} \cdots \left( \frac{n^2}{n^2 + k} \right).
\]

Stirling’s formula yields the rate of growth:

\[
\frac{(n^2)^{n^2}}{(n^2)!} \leq \frac{e^{n^2}}{\sqrt{2\pi n}}.
\]

Hence, we need to verify that the Fourier multiplier operator

\[
\sum_{k=0}^{2n-1} c_k e^{ik\theta} \mapsto \sum_{k=0}^{2n-1} \lambda_{n,k} c_k e^{ik\theta},
\]

acting on the \( L^p \)-space of trigonometric polynomials of degree 2n, has norm bounded independent of \( n \) and \( p \).

For that end we first compute (recall that \( k \leq 2n \))

\[
- \log(\lambda_{n,k}) = \sum_{j=1}^{k} \log \left( \frac{n^2 + j}{n^2} \right) = \sum_{j=1}^{k} \frac{j}{n^2} + \sum_{j=1}^{k} \left| \log \left( \frac{n^2 + j}{n^2} \right) - \frac{j}{n^2} \right|
\]

\[
= \frac{k^2}{2n^2} + \varepsilon'_{n,k},
\]

and since \( |\log(1 + x) - x| \leq x^2/2 \) for \( x > 0 \), we have that

\[
|\varepsilon'_{n,k}| \leq \frac{k}{2n^2} + \sum_{j=1}^{k} \left| \log \left( \frac{n^2 + j}{n^2} \right) - \frac{j}{n^2} \right| \leq \frac{k}{2n^2} + \frac{1}{2n^4} \sum_{j=1}^{k} j^2 \leq \frac{3k}{n^2} \leq \frac{6}{n}
\]
for $0 \leq k \leq 2n$. Since $|e^x - 1| \leq 2|x|$ for $|x| \leq 1$ estimates (14) and (15) yields that
\[
\lambda_{n,k} = e^{-k^2/2n^2} + \varepsilon_n'' \quad \text{with} \quad |\varepsilon_n''| \leq 12/n.
\]
The multiplier corresponding to the sequence $(\varepsilon_{n,k}'')$, has norm less than $(2n + 1) \cdot 12/n \leq 30$, since our polynomials have degree $2n$. We next consider the main term, i.e. the Fourier multiplier
\[
\sum_{k=0}^{\infty} c_k e^{ik\theta} \mapsto \sum_{k=0}^{\infty} e^{-k^2/2n^2} c_k e^{ik\theta},
\]
(observe that now we allow polynomials of arbitrary degree). This map is the convolution operator $f \mapsto g \ast f$, where $g$ is the positive function
\[
g(\theta) = \sum_{\ell \in \mathbb{Z}} \sqrt{2\pi n} e^{-n^2(x-2\pi\ell)^2}/2.
\]
Especially, $\int_{\mathbb{T}} g(\theta) \, d\theta = 1$. The norm of such a convolution operator is 1 on all the spaces $L^p(\mathbb{T})$, and this finishes the proof of the Lemma, in view of the inequality $(30 + 1)/\sqrt{2\pi} \leq 20$.

Finally, the verification of (11) uses the identity
\[
|P_n g((n+1)^2 e^{i\theta})| = \frac{(n+1)^2}{(n+1)^2} \sum_{k=1}^{2n+1} \lambda'_{n,k} b_{(n+1)^2-k} e^{-ik\theta},
\]
with
\[
\lambda'_{n,k} := \left(\frac{(n+1)^2}{(n+1)^2}\right) \cdots \left(\frac{(n+1)^2 - k + 1}{(n+1)^2}\right).
\]
The previous argument applies with minor modifications, and we obtain
\[
\| \sum_{k=1}^{2n+1} \lambda'_{n,k} b_{(n+1)^2-k} e^{-ik\theta} \|_p = \| \sum_{k=1}^{2n+1} \lambda'_{n,k} b_{(n+1)^2-k} e^{ik\theta} \|_p \leq 20 \| \sum_{k=1}^{2n+1} b_{(n+1)^2-k} e^{ik\theta} \|_p
\]
\[
= 20 \| \sum_{k=1}^{2n+1} b_{(n+1)^2-k} e^{-ik\theta} \|_p = 20 \| \sum_{k=0}^{2n} b_{n^2+k} e^{i\theta} \|_p.
\]

3.1. **Proof of Theorem 1.1.** We are finally ready to show that our function $f$ constructed in Section 2.2 is has the desired properties. In what follows we show in the case $p \in (1, \infty]$ that $f$ has the desired growth and is D-frequently hypercyclic. As mentioned in the introduction, the case $p = 1$ and the optimality of Theorem 1.1 for any $p$ are already known after [1] and [5], but for the reader’s convenience we present their result using our techniques in Remarks 4.1 and 4.2.

**Bounding the growth of $f$.** Let first $2 \leq p \leq \infty$. Actually, since $\| \cdot \|_p$ increases with $p$ it is then enough to consider only the case $p = \infty$. Assume first that $n \in \mathcal{A}_k$ is positive and even. By construction, by Proposition 3.3 and by Lemma 2.1 we first obtain that
\[
M_{p,n}(f^2) \leq 20 e^{n^2} n^{-1} \|p|_{n/\alpha_k} (\cdot) \tilde{g}_k(\cdot)\|_\infty \leq 20 \cdot 5 e^{n^2} n^{-1} \sqrt{n/\alpha_k} \|\tilde{g}_k\|_\ell^1
\]
\[
\leq e^{n^2} n^{-1/2} 100 \ell_k \alpha_k^{-1/2} \leq c 10^{-3} e^{n^2} n^{-1/2}.
\]
where one applied (3) and the first condition in definition (5) of the sequence $(\alpha_k)_{k \geq 1}$. Trivially the same bound applies for odd $n$, or for $n = 0$, since then $P_n f = 0$. In a similar manner, Proposition 3.4 yields that

$$M_{P_n f}((n + 1)^2) \leq c 10^{-3} e^{(n+1)^2(n + 1)^{-1/2}}.$$

At this stage Proposition 3.3 applies, and we deduce that $f$ satisfies the desired growth, i.e. $M_f(r) \leq c e^{r^{-1/4}}$ for all $r > 0$.

For $1 < p \leq 2$ we use a similar analysis, based on replacing the polynomial $p_m$ with $p_m^*$ in (3), again with $m = \lfloor n/\alpha_k \rfloor$. Since now $p' > 2$, the above computation takes the form

$$M_{P_n f, p}(n^2) \leq 20 e^{n^2} n^{-1} \| p_{n/\alpha_k}^* (z_k^2) \|_p \| \tilde{q}_k \|_{\ell_1} = 20 e^{n^2} n^{-1} \| p_{n/\alpha_k}^* \|_p \| \tilde{q}_k \|_{\ell_1} \leq 20 e^{n^2} n^{-1+1/p'} \ell_k \alpha_k^{-1/p'} \leq c 10^{-3} e^{n^2} n^{-1/p}.$$

Together with an analogous estimate for $M_{P_n f, p}((n + 1)^2)$ we obtain the growth $M_{f, p}(r) \leq c e^{r^{-1/2p}}$ again with the aid of Proposition 3.3.

$f$ is $D$-frequently hypercyclic. This part of the argument $f$ is independent of $p \in (1, \infty]$. We start by observing the simple coefficient bound for the Taylor coefficients of $f$:

$$|a_j| \leq j \quad \text{for all } j \geq 1.$$

This is an immediate consequence of the bound (3) together with definitions (5) and (6) (resp. (7)) since the absolute value of the coefficients of the polynomials $p_m$ (resp. $p_m^*$) do not exceed 1, and if $j \in A_k$ with $a_j \neq 0$, then

$$|a_j| \leq \| \tilde{q}_k \|_{\ell_1} \leq \ell_k < \alpha_k < j.$$

For any fixed even integer $n \in A_k$ with $n \geq \alpha_k$, denote by $B_n$ the set of indices $s$ such that the coefficient of $z^s$ in the polynomial $z^n p_{2n/\alpha_j}(z_k^n)$ is 1. With the $\{(q_k, \ell_k)\}$ used to define $f$ (so that $\ell_k \uparrow \infty$), we claim that for each $k \geq 1$,

$$\sup_{|z| = \ell_k} |q_k(z) - \left( \frac{d}{dz} \right)^s f(z) | \leq \frac{1}{\ell_k} \quad \text{for any } s \in B_n \quad \text{and for any } n \in A_k.$$

This clearly suffices as $P_n$ dense in the space of entire functions, and obviously for any fixed $k \geq 1$ the set

$$\{ s : s \in B_n, n \in A_k, n \geq \alpha_k \}$$

has positive density since for large $n$ we have arranged that

$$\#(B_n) \geq (\alpha_k)^{-1} n \geq (3\alpha_k)^{-1} \#(\{ n^2, n^2 + 1, \ldots (n + 1)^2 - 1 \}),$$

and since $A_k$ contains an arithmetic progression of $n$.

Towards (17), fix an even $n \geq 1$ and $s \in B_n$, where we suppose that $n \in A_k$ with $n \geq \alpha_k$. Our construction of $P_n f$ shows that the coefficients $a_s, a_{s+1}, \ldots a_{s+\deg(q_k)}$ coincide precisely with those of $q_k$, and by the choice of $\alpha_k$ at least the next $8\ell_k$
coefficients among the $a_j$ are zero. Since $(n+1)$ is odd, we have that $a_j = 0$ for $(n+1)^2 \leq j \leq (n+2)^2 - 1$, and so

$$
\left( \frac{d}{dz} \right)^s f(z) - q_k(z) = \sum_{j=s+8\ell_k}^{(n+1)^2-1} \frac{a_j z^{j-s}}{(j-s)!} + \sum_{j=(n+2)^2}^{\infty} \frac{a_j z^{j-s}}{(j-s)!} =: S_1(z) + S_2(z).
$$

By definition, for $n^2 \leq j \leq (n+1)^2 - 1$ one has $|a_j| \leq \|q_k\|_\ell \leq \ell_k$, so the first sum is bounded by

$$
\sup_{|z|=\ell_k} |S_1(z)| \leq \|q_k\|_\ell \sum_{m=8\ell_k}^{\infty} \frac{\ell_m}{m!} \leq \frac{2^{8\ell_k+1}}{(8\ell_k)!} \frac{\ell_{8\ell_k}}{2\ell_k}.
$$

In the above computation one observed that the ratio of any two consecutive terms in the last written series is less than $1/2$, and the very last step was due to the estimate

$$
\frac{x^{8x}}{(8x)!} \leq \frac{1}{4x^3} \quad \text{for} \quad x \geq 2.
$$

In turn, the inequality (18) is an easy consequence of Stirling’s formula.

In order to estimate the second sum $S_2(z)$ we write $j = s + m$ with $m \geq (n+2)^2 - (n+1)^2 = 2n+3$ and observe that (17) yields that $|a_j| = |a_{s+m}| \leq (n+1)^2 + m \leq m(m-1)$. Hence we obtain

$$
\sup_{|z|=\ell_k} |S_2(z)| \leq \left| \sum_{j=(n+2)^2}^{\infty} \frac{a_j \ell^{j-s}_k}{(j-s)!} \right| \leq \sum_{m=2n+3}^{\infty} \frac{(n+1)^2 + m}{m!} \frac{\ell_m}{\ell_k} \leq \frac{\ell^2_k}{\ell_k} \sum_{m=2n+1}^{\infty} \frac{\ell_m}{m!} \leq \frac{1}{2\ell_k}.
$$

Above one applied the knowledge $n \geq \alpha_k \geq 8\ell_k$ from (5) and the last sum was estimated as before by a geometric series and (18).

Put together, the estimates we obtained for $S_1$ and $S_2$ yield (17) and the proof of Theorem 1.1 is completed.

\[ \Box \]

4. REMARKS

**Remark 4.1.** It is instructive to analyze what happens with the above argument in the special case $p = 1$. A key point in the proof is the fact that $\hat{p}_n f$ contains the product $p_{(n/\alpha_k)}(z_k^n)\bar{q}_k(z)$ (resp. $p_{(n/\alpha_k)}(z_k^n)\bar{q}_k(z)$ if $p \in (1,2)$). Hence by choosing $\alpha_k$ large enough we may decrease the $L^1$-norm of the first factor to compensate for the possibly increasing size of $\bar{q}_k$. However, this does not work for $p = 1$ since obviously the $L^1$-norm of any polynomial must exceed the sup-norm of its coefficients!

However, a small change in the argument produces the optimal result also in the case $p = 1$ (and hence provides an alternative proof of part (iii) of Theorem 1.1). Assume that we are given an increasing function $\phi(r) : (0, \infty) \to [1, \infty)$ with $\lim_{r \to \infty} \phi(r) = \infty$. One constructs $f$ as before in case $p \in (1,2)$, the only change is that initially in the definition of the sequence $(\alpha_k)$ one replaces condition (5) by the single demand
\[ \alpha_k \geq 2d_k + 8\ell_k. \] By writing \( f \) as
\[ f = \sum_{k=1}^{\infty} f_k \quad \text{with} \quad f_k = \sum_{n \in A_k} P_n f, \]
the argument of Section 3 applies as such to each piece \( f_k \): one deduces for each \( n \geq 1 \) the bound \( M_{f,k} \leq 6(2^k)^{-1} \|q_k\|_1 \), whence Proposition 3.3 yields the growth \( M_{f,k,1}(r) \leq e^{r^2/2} \cdot 10^{6} \|q_k\|_1 \). Note that the \( j \)-th Taylor coefficients of \( P_k \) can be nonzero only if \( j \geq \alpha_k \). Hence, by further increasing the size of \( \alpha_k \), and by recalling (16) we may in addition force \( f_k \) as small as we want in any compact region. In particular, we may demand
\[ M_{f,k,1}(r) \leq 2^{-k} \phi(r)e^{r^{1/2}}. \]
Summing up we obtain the growth \( M_{f,1}(r) \leq \phi(r)e^{r^{1/2}}. \)

**Remark 4.2.** For the reader’s convenience we sketch the proof of optimality in Theorem 1.1, although this is already contained in [1]. Assume that \( f \) is entire and \( D \)-frequently hypercyclic. In particular, \( f \) frequently approximates the constant function 2 up to precision 1 in \( B(0,2) \), which implies that the density of the set \( H \) is positive, where \( H := \{ k \geq 1 : |a_k| \geq 1 \} \). A fortiori, there is a constant \( c_1 > 0 \) such that for infinitely many \( n \geq 1 \)
\[ \#(H \cap \{ n^2, n^2 + 1, \ldots, n^2 + 2n \}) \geq c_1 n. \]
Next, write down an analogue of the identity (12)
\[ |f(n^2 e^{i\theta})| = \frac{(n^2)^n}{(n^2)!} \sum_{k=-n^2}^{n^2} \lambda_{n,k} a_{n^2+k} e^{ik\theta}, \]
where one has extended in a natural way definition (13) of \( \lambda_{n,k} \) to all values \( k \geq -n^2 \). Observe that the estimates (14) and (15) show that \( \lambda_{n,k} \geq c_2 \) for \( k = 0, \ldots, 2n \), where \( c_2 \) is independent of \( n \). Hence, for \( p > 1 \) the optimality follows simply by considering (20) for arbitrarily large \( n \) that satisfy (19), applying Stirling’s formula, and the standard estimate \( \| \sum_{k=-\infty}^{\infty} b_k e^{i\theta k} \|_p \geq \| (b_k)_{k=-\infty}^{\infty} \|_{\max(p',2)} \), obtained from the Hausdorff-Young inequality see [10]. In case \( p = 1 \) one approximates similarly any given constant \( A \) (instead of just \( A = 1 \)) and obtains that \( \limsup_{r \to \infty} M_{f,1}(r)^{1/2}e^{-r} \geq cA \). Since \( A > 1 \) is arbitrary, the necessity of the increasing factor \( \phi(r) \) in case (iii) of Theorem 1.1 follows.

**Remark 4.3.** By employing the identity (20) exactly as in the previous remark, and by recalling that each set \( A_k \) contains an arithmetic sequence of indices (it enough to consider just any single \( A_k \) with \( q_k \neq 0 \)), one checks that for \( p \in (1, \infty] \) our function \( f \) verifies the lower bound
\[ M_{f,p}(r) \geq \tilde{c} e^{r^{-\max(1/2p,1/4)}} \quad \text{for} \quad r > 1, \]
where \( \tilde{c} \) is a positive constant. Hence one has \( M_{f,p}(r) \simeq e^{r^{-\max(1/2p,1/4)}} \) for all \( r > 1 \). However, in general a frequently hypercyclic function \( f \) needs not satisfy (21), since one could easily modify our construction to impose infinitely many large (dyadic) gaps to the Taylor series of \( f \).
Remark 4.4. A main ingredient in our construction is the selection of the right size for the blocks $P_n f$. This is actually rather delicate issue: if the blocks are suitably sparse, one obtains that the blocks $P_n f$ are independent in the sense of Proposition 3.3. This can be most easily understood by comparing the growth of a single term inside the block with the exponential growth. Indeed, the ratio $r^n e^{-r}$ takes its maximal value at $|z| = n$ and starts to decay rapidly as soon as $|z - n| >> \sqrt{n}$ (or, equivalently, $\lambda_{n,k}$ in (13) decreases quickly for $k >> n$). This suggests that the block structure should be at least as sparse as in (3), which choice also enables one to make full advantage of the use of Lemma 2.1. On the other hand, if the blocks were more sparse, the application of Hadamard’s three circles theorem in Proposition 3.3 would be defective. All said, for our proof the choice in (3) is essentially unique.

Remark 4.5. The question of the minimal growth of $D$-hypercyclic entire functions is considerably easier since then there is no need to try to build in cancellation. The sharp result [19], [12] says that any $D$-frequently hypercyclic entire function satisfies $\limsup_{r \to \infty} M_f(r)e^{-r\sqrt{r}} = \infty$ and this result is optimal.

References

[1] O. Blasco, A. Bonilla and K.-G. Grosse-Erdmann: On the growth of frequently hypercyclic functions, Proc. Edinb. Math Soc 53 (2010), 39–59.
[2] E. Bombieri and J. Bourgain: On Kahane’s ultraflat polynomials, J. Eur. Math. Soc. 11 (2009), 627–703.
[3] F. Bayart and S. Grivaux: Hypercyclicite: le role du spectre ponctuel unimodulaire, C. R. Math. Acad. Sci. Paris 338 (2004), 703–708.
[4] F. Bayart and S. Grivaux: Frequently hypercyclic operators, Trans. Amer. Math. Soc. 358 (2006), 5083–5117.
[5] J. Bonet and A. Bonilla: Chaos of the differentiation operator on weighted Banach spaces of entire functions, to appear in Complex Anal. Oper. Theory.
[6] A. Bonilla and K.-G. Grosse-Erdmann: On a theorem of Godefroy and Shapiro, Integral Equations Operator Theory 56 (2006), 151-162.
[7] A. Bonilla and K.-G. Grosse-Erdmann: A problem concerning the possible rates of growth of frequently hypercyclic entire functions. In: Topics in complex analysis and operator theory, 155–158, Univ. Malaga, Malaga 2007.
[8] A. Bonilla and K.-G. Grosse-Erdmann: Frequently hypercyclic operators and vectors, Ergodic Theory Dyman. Systems 27 (2007), 383–404.
[9] G. Godefroy and J. H. Shapiro: Operators with dense, invariant, cyclic vector manifolds, J. Funct. Anal. 98 (1991), 229–269.
[10] Sophie Grivaux: A new class of frequently hypercyclic operators, Indiana Univ. Math. J. (to appear).
[11] K.-G. Grosse-Erdmann: Universal families and hypercyclic operators, Bull. Amer. Math. Soc. (N.S.) 36 (1999), 345-381.
[12] K.-G. Grosse-Erdmann: On the universal functions of G. R. MacLane, Complex Variables Theory Appl. 15 (1990), 193–196.
[13] W. K. Hayman and P. B. Kennedy: Subharmonic Functions, vol. 1. Academic Press, 1976.
[14] J.-P. Kahane: Sur les polynômes à coefficients unimodulaires (French). Bull. London Math. Soc. 12 (1980), 321-342
[15] J.-P. Kahane: Some random series of functions. Second edition. Cambridge University Press 1985.
[16] Y. Katznelson: An Introduction to Harmonic Analysis. Dover, 1976.
[17] G. R. MacLane: Sequences of derivatives and normal families, J. Analyse Math. 2 (1952/53), 72–87.
[18] W. Rudin: *Some theorems on Fourier coefficients*, Proc. Amr. Math. Soc. 10 (1959), 855–859.
[19] S. A. Shkarin: *On the growth of D-universal functions*, Moscow Univ. Math.Bull. 48 (1993), no. 6, 49–51.

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