EQUILIBRIUM DISTRIBUTION OF ZEROS OF RANDOM POLYNOMIALS

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Abstract. We consider ensembles of random polynomials of the form $p(z) = \sum_{j=1}^{N} a_j P_j$ where \{a_j\} are independent complex normal random variables and where \{P_j\} are the orthonormal polynomials on the boundary of a bounded simply connected analytic plane domain $\Omega \subset \mathbb{C}$ relative to an analytic weight $\rho(z) |dz|$. In the simplest case where $\Omega$ is the unit disk and $\rho = 1$, so that $P_j(z) = z^j$, it is known that the average distribution of zeros is the uniform measure on $S^1$. We show that for any analytic $(\Omega, \rho)$, the zeros of random polynomials almost surely become equidistributed relative to the equilibrium measure on $\partial \Omega$ as $N \to \infty$. We further show that on the length scale of $1/N$, the correlations have a universal scaling limit independent of $(\Omega, \rho)$.

Introduction

A well-known result due to Hammersley [Ham] (see also Shepp-Vanderbei [SV]) states that the zeros of random complex ‘Kac’ polynomials

$$f(z) = \sum_{j=0}^{N} a_j z^j, \quad z \in \mathbb{C}$$

(1)

tend to concentrate on the unit circle $S^1 = \{|z| = 1\}$ as $N \to \infty$ when the coefficients $a_j$ are independent complex Gaussian random variables of mean 0 and variance 1:

$$E(a_j) = 0, \quad E(a_j \bar{a_k}) = \delta_{jk}, \quad E(a_j a_k) = 0.$$  

(2)

To be precise, (2) defines a Gaussian probability measure $\gamma_N^{S^1}$ on the space $\mathcal{P}_N$ of polynomials of degree $\leq N$ on $\mathbb{C}$, studied long ago by Littlewood-Offord, Erdos-Turan, and in particular by Kac [K1, K2]. We denote expected values relative to $\gamma_N^{S^1}$ by $E_{S^1}^{N}$. We also define the normalized distribution of zeros of $f$ to be the probability measure

$$\tilde{Z}_f^N := \frac{1}{N} \sum_{f(z) = 0} \delta_z.$$  

(3)

Then we may formulate the concentration of zeros on $S^1$ as the weak limit formula:

$$E_{\gamma_N^{S^1}}(\tilde{Z}_f^N) \to \delta_{S^1} \quad \text{as} \quad N \to \infty, \quad (\delta_{S^1}, \varphi) := \frac{1}{2\pi} \int_{S^1} \varphi(e^{i\theta}) \, d\theta.$$  

(4)

The first purpose of this note is to generalize the expected equidistribution result (4) in a suitable sense to any closed analytic curve $\partial \Omega$ in $\mathbb{C}$ which bounds a simply connected plane.
domain Ω. Our method is new even in the case of $S^1$ and gives sharper results than those of [Ham]. We further prove that the equidistribution result is self-averaging in the sense that $Z_{f_N}$ is almost surely asymptotic to the expected distribution. Our second purpose is to show that the correlations between zeros have a universal scaling limit independent of $\Omega, \rho$ on the length scale of $1/N$ and to determine the properties of the limit pair correlation function.

Our starting point is to reinterpret the Gaussian measure (2) on $P_N$ as the one induced by the inner product
\[
\langle f, \bar{g} \rangle_{S^1} = \frac{1}{2\pi} \int_{S^1} f \bar{g} |dz|
\]
on $P_N$. In general, any inner product
\[
\langle f, \bar{g} \rangle_{\mu} = \int_{C} f \bar{g} d\mu
\]
on $P_N$ induces a Gaussian measure on $P_N$ as follows: Let $\{P_j\}$ denote an orthonormal basis of $P_N$ relative to $\langle . , . \rangle_{\mu}$ and write any polynomial in this basis:
\[
f(z) = \sum_{j=0}^{N} a_j P_j.
\]
The Gaussian measure is defined by the condition that the coefficients $a_j$ in this basis are i.i.d. complex Gaussian variables with mean zero and variance one; i.e. it equals $\pi^{-N} e^{-|a|^2} da$ in terms of the coefficients.

The choice of the Gaussian measure $\gamma^N_{S^1}$ in (2) results in the zeros of the random polynomials becoming equidistributed relative to $\delta_{S^1}$. As is well-known, $\delta_{S^1}$ is the equilibrium measure of $S^1$ (or of the closed unit disk). Recall that the equilibrium measure of a compact set $K$ is the unique probability measure $\nu_K$ which minimizes the energy
\[
E(\mu) = -\int_{K} \int_{K} \log |z - w| d\mu(z) d\mu(w).
\]
We claim that the same equidistribution result holds if we replace the unit disk $U = \{ |z| < 1 \}$ by any simply connected, bounded plane domain $\Omega$ with real-analytic boundary. That is, we replace the inner product (3) on $S^1$ with an inner product on $\partial \Omega$ of the form
\[
\langle f, \bar{g} \rangle_{\partial \Omega, \rho} := \int_{\partial \Omega} f(z) \bar{g}(z) \rho(z) |dz| , \quad \rho \in C^\omega(\partial \Omega) .
\]
We denote by $\gamma^N_{\partial \Omega, \rho}$ the Gaussian measure induced by this inner product on $P_N$ as in (8), and we denote the expectation relative to the ensemble $(P_N, \gamma^N_{\partial \Omega, \rho})$ by $E^N_{\partial \Omega, \rho}$.

As in the $S^1$ case, we obtain an asymptotic formula for the expected normalized distribution of zeros $E^N_{\partial \Omega, \rho}(\tilde{Z}_f^N)$;

**Theorem 1.** Suppose that $\Omega$ is a simply-connected bounded $C^\omega$ domain and $\rho$ is a positive $C^\omega$ density on $\partial \Omega$. Then with the above notation,
\[
E^N_{\partial \Omega, \rho}(\tilde{Z}_f^N) = \nu_\Omega + O(1/N) ,
\]
where $\nu_\Omega$ is the equilibrium measure of $\bar{\Omega}$. 
Here we use the expression $O(1/N)$ to mean a distribution $u_N \in \mathcal{D}'(\mathbb{C})$ such that $|\langle u_N, \varphi \rangle| \leq C_\varphi/N$ for all $\varphi \in \mathcal{D}(\mathbb{C})$, where $C_\varphi$ independent of $N$. Note that the limit distribution of zeros concentrates on $\partial \Omega = \text{Supp}(\nu_\Omega)$ and is independent of the density $\rho$ used to define the inner product.

The equilibrium measure may be understood as the measure by which $N$ ‘electric charges’ in $\Omega$ distribute themselves in the limit $N \to \infty$. Thus, at least on average, the zeros of random polynomials in the ensemble $(\mathcal{P}_N, \gamma^N_{\partial \Omega, \rho})$ behave like electric charges. This suggests that the same equidistribution result should hold if we orthonormalize the polynomials within $\Omega$. We restrict to Lebesgue measure in $\Omega$ and define the inner product

$$\langle f, \bar{g} \rangle_\Omega := \int_\Omega f(z)\bar{g}(z) \, dx \, dy \quad (z = x + iy),$$

which induces the Gaussian measure $\gamma^N_{\Omega}$ on $\mathcal{P}_N$.

**Theorem 2.** Suppose that $\Omega$ is a simply-connected bounded $C^\omega$ domain and let $E^N_{\Omega}$ denote the expectation with respect to the Gaussian measure $\gamma^N_{\Omega}$ on $\mathcal{P}_N$. Then

$$E^N_{\Omega}(\tilde{Z}^f_N, \varphi) = \nu_\Omega + O(1/N).$$

As in [ShZ1], the expectation results of Theorems 1–2 can be improved to give an almost sure limit distribution of zeros of random sequences of polynomials of increasing degree:

**Theorem 3.** Suppose that $\Omega$ is a simply-connected bounded $C^\omega$ domain. Let $\gamma_N$ be the Gaussian measure on $\mathcal{P}_N$ induced by the inner product (7) or (8), and let $\mu := \prod_{n=1}^\infty \gamma_N$ be the product probability measure on $S := \prod_n \mathcal{P}_N$. Then $\tilde{Z}^f_N \to \nu_\Omega$ in the measure sense, for $\mu$-almost all sequences $\{f_N\} \in S$; i.e.,

$$\frac{1}{N} \sum_{\{z \in \mathbb{C} : f_N(z) = 0\}} \varphi(z) \to \int_{\partial \Omega} \varphi \, d\nu_\Omega \quad \forall \varphi \in \mathcal{C}(\hat{\mathbb{C}}), \quad \mu\text{-almost surely.}$$

The principal ingredients in the proofs of Theorems 1 and 2 are classical results of Szegö [Sz1] and of Carleman [Ca] on the asymptotics of orthogonal polynomials normalized on the boundary and on the interior of a domain, respectively. We will recall these results in §1. The proofs of the two theorems are then essentially the same.

Theorems 1–3 say that in the limit the zeros tend to be uniformly distributed along $\partial \Omega$ (with respect to the equilibrium measure). However, as in our earlier work with P. Bleher [BSZ1, BSZ2], the zeros do not behave as if they were thrown down at random; indeed, the zeros are correlated. To quantify this correlation, we consider as in [BSZ1, BSZ2] the $\ell$-point correlation functions

$$K^N_{\partial \Omega, \rho}(z_1, \ldots, z_\ell) := \frac{E^N_{\partial \Omega, \rho}(\tilde{Z}^N_f \times \cdots \times \tilde{Z}^N_f)}{E^N_{\partial \Omega, \rho}(\tilde{Z}^N_f) \times \cdots \times E^N_{\partial \Omega, \rho}(\tilde{Z}^N_f)}, \quad (z_1, \ldots, z_\ell) \in \mathbb{C}_\ell, \quad (9)$$

where

$$\mathbb{C}_\ell = \{(z_1, \ldots, z_\ell) \in \mathbb{C}^\ell : z_j \neq z_k \text{ for } j \neq k\}.$$

(We punch out the big diagonal in $\mathbb{C}_\ell$ since the numerator in (9) has a singular part there.) The correlation functions $K^N_{\Omega}(z, w)$ are similarly defined.
In particular, the pair correlation function \( K^{2N}_{\partial \Omega, \rho}(z, w) \) (or \( K^N_{\Omega}(z, w) \)) can be interpreted as the conditional probability of finding a zero at \( w \) given that there is a zero at \( z \), normalized by dividing by the unconditional probability of finding a zero at \( w \). For example, if \( K^{2N}_{\partial \Omega, \rho}(z, w) = 1 \), then the existence of a zero at \( z \) has no influence on the probability of finding a zero at \( w \). This is always the case in the limit as \( N \to \infty \) if \( z, w \) are fixed. However, we have nontrivial correlations if the distance between points is \( O(1/N) \). To describe this phenomenon, let \( z_0 \in \partial \Omega \) be fixed, and choose the complex coordinate \( \zeta = \Phi(z) \), where \( \Phi \) is the Riemann mapping function mapping the exterior of \( \Omega \) to the exterior of the unit disk, mapping \( z_0 \) to 1, and taking \( \infty \) to itself. (This exterior Riemann mapping function is a basic ingredient in the Szegö and Carleman asymptotics used in this paper.) We then have universal scaling limit zero correlation functions:

**Theorem 4.** There exist universal functions \( K^{\ell \infty} : \mathbb{C} \to \mathbb{R}^+ \) independent of \( \Omega, z_0, \rho \) such that

\[
\hat{K}^{\ell N}_{\partial \Omega, \rho} \left( \frac{\zeta_1}{N}, \ldots, 1 + \frac{\zeta_\ell}{N} \right) \to K^{\ell \infty}(\zeta_1, \ldots, \zeta_\ell)
\]

as \( N \to \infty \), where \( \hat{K}^{\ell N}_{\partial \Omega, \rho} = K^{\ell N}_{\partial \Omega, \rho} \circ \Phi^{-1} \) is the correlation function written in terms of the complex coordinate \( \zeta \) described above. Similarly, \( \hat{K}^{\ell N}_{\Omega} \left( \frac{\zeta_1}{N}, \ldots, 1 + \frac{\zeta_\ell}{N} \right) \to K^{\ell \infty}(\zeta_1, \ldots, \zeta_\ell) \).

The universal scaling limit pair correlation function \( K^{2\infty} \) is given by formula (33).

In our work with P. Bleher [BSZ1, BSZ2], we showed that the zero correlation functions can be given by general (universal) formulas involving only the Szegö kernel and its first and second derivatives. In §3, we study the partial Szegö kernel

\[
S_N(z, w) = \sum_{k=0}^{N} P_k(z) \overline{P_k(w)}
\]

which gives the orthogonal projection onto the span of the first \( N \) orthogonal polynomials associated to the inner product (7). We show (Proposition 11) that the scaling asymptotics of \( S_N \) have the form

\[
\frac{1}{N} \hat{S}_N \left( \frac{\zeta_1}{N}, 1 + \frac{\zeta_2}{N} \right) \to C_{\Omega, \rho, z_0} G(\zeta_1 + \zeta_2)
\]

where \( \hat{S}_N = S_N \circ \Phi^{-1} \) and

\[
G(z) = \frac{e^z - 1}{z}.
\]

It is natural to separately consider the ‘tangential’ scaling asymptotics along the boundary and the ‘normal’ scaling asymptotics orthogonal to the boundary. In the tangential case, we set \( \zeta_1 = i\theta, \zeta_2 = 0 \) and obtain the tangential scaling limit kernel \( G(e^{i\theta}) = e^{i\theta/2}(\sin \theta/2)/\theta/2) \).

This modified sine-kernel is reminiscent of the scaling asymptotics of the projection kernels \( K_N \) for the orthogonal polynomials which occur in the theory of random matrices (see [De], Theorem 8.16). However, our actual result is somewhat different and the methods have little in common.

The tangential scaling limit for pair correlations between zeros,

\[
\kappa^T(\alpha) := K^{2\infty}(0, i\alpha) = \lim_{N \to \infty} \hat{K}^{2N}_{\partial \Omega, \rho}(1, 1 + \frac{i\alpha}{N}) = \lim_{N \to \infty} \hat{K}^{2N}_{\partial \Omega, \rho}(1, e^{i\alpha/N})
\]
measures the probability (density) of finding a pair of zeros in small disks around two points on \( \partial \Omega \) in terms of the \( \frac{1}{N} \)-scaled angular distance \( \alpha \) between them. As is illustrated by the graph of \( \kappa^T \) (drawn using Maple\textsuperscript{TM}) in Figure 1, in tangential directions the zeros ‘repel’ as \( \alpha \to 0 \) and are oscillatory with \( 1/\alpha^2 \) decay over long scaled distances, very much as for correlations between eigenvalues of Gaussian random Hermitian matrices.

\[ \kappa^T(\alpha) \]

In the normal directions, we obtain the scaling limit:

\[
\kappa^\perp(\tau) := K^{2\infty}(0, \tau) = \lim_{N \to \infty} \hat{K}^{2N}_{\partial \Omega, \rho} \left( 1, 1 + \frac{\tau}{N} \right).
\]

The graph of \( \kappa^\perp \) in Figure 2 is reminiscent of the Bleher-Di correlation function [BD] for real Kac polynomials. Although the normal correlation is not oscillatory, we again have zero repulsion and \( 1/\tau^2 \) decay.

\[ \kappa^\perp(\tau) \]

**Figure 1.** The tangential pair correlation function \( \kappa^T(\alpha) \)

**Figure 2.** The normal pair correlation function \( \kappa^\perp(\tau) \)
We end the introduction with some remarks on and comparison to prior results in this area. Real Kac polynomials, i.e. polynomials as in (1) but with real Gaussian coefficients of mean zero and variance one, were shown by Kac \[ K1, K2 \] to have on average around \( \log N \) real zeros. The same was proved for more general Kac polynomials in \[ IM \]. As mentioned above, Hammersley \[ Ham \] obtained the concentration of zeros of complex Kac polynomials on the unit circle in a somewhat less precise sense than the one we state here. Later, Shepp and Vanderbei \[ SV \] showed that the complex zeros of random real Kac polynomials also tend to concentrate on the unit circle, explaining in a qualitative way why so few zeros are real. Other classical studies of random real polynomials are given in \[ EO, LO, ET, EK \]. The probability of a real Kac polynomial having no real zeros is estimated in \[ DPSZ \]. Other results on hole probabilities for complex zeros are given in \[ So \].

Recently, a number of results were obtained on the distribution of and correlations between zeros of Gaussian random polynomials in one variable (and more general holomorphic sections) in which the inner product comes from a hermitian metric \( h \) on a positive line bundle \( L \to M \) over a Riemann surface \( M \) (see \[ Han, BR, ShZ1, BSZ1 \]). For simplicity, let us mention only the case of SU(2) polynomials, in which \( M = \mathbb{C}P^1, L = O(N), \) and where \( h = h_{FS}^N \) is the \( N^{th} \) power of the Fubini-Study metric. It is obvious from the SU(2) symmetry of the expected distribution of zeros in this case that it must equal the standard (Fubini-Study) area form, in sharp contrast to the very singular equidistribution result we find in Theorems 1–3. Moreover, although the pair correlation function for SU(2) polynomials given by \[ Han \] (see \[ BSZ1 \] for a generalization) exhibits the same repulsive behavior near 0, its scaling limit occurs on the length scale of \( 1/\sqrt{N} \) rather than \( 1/N \). Thus, the correlations are quite incomparable in the two types of ensembles.

To explain the strong differences in the results, it should be observed that the ensemble \((P_N, \gamma_N)\) in the SU(2) case comes from an inner product \([,]_N\) on \( P_N \) which changes with each \( N \), according to which \( \|z^k\|^2_N = \binom{N}{k} \). These inner products \([,]_N\) derive from a single inner product on CR (Cauchy-Riemann) functions on the boundary of the unit ball in \( \mathbb{C}^2 \) rather than on polynomials in one complex variable. This suggests that polynomials in the SU(2) ensemble should be thought of as homogeneous polynomials \( \sum_{j=1}^{N} a_j z_0^{N-j} z_1^j \) in two variables and that their equidistribution law and correlations reflect the extra dimensionality. Because of the concentration of zeros on curves, the ensembles studied here share a one-dimensionality with the ensembles of GUE and CUE random matrices and with the observed correlations between zeros of the Riemann zeta function.

We note that in random matrix theory, orthogonal polynomials on \( \mathbb{R} \) with respect to weights \( \rho_N = e^{-NV(x)}dx \) depending on the degree \( N \) are quite important (see \[ De \]). This suggests that ensembles \((P_N, \gamma_N)\) of random polynomials on curves \( \partial \Omega \) where the inner product is given by a weight of the form \( \rho_N = \rho(z)^N |dz| \) might be interesting. Such ensembles have features in common with both the SU(2) ensemble (and its generalization to other Riemann surfaces) and with the ensembles of this paper. How are the zeros distributed and correlated? Do the zeros concentrate on \( \partial \Omega \) or are they diffuse? Our methods do not seem to adapt in a simple way to such ensembles, and perhaps they belong to different universality classes than the ones studied here.

Another interesting direction would be to generalize our results to higher dimensions. Clearly the methods we use, based on the Riemann mapping theorem, have no simple generalization.
1. Asymptotics of orthogonal polynomials

The proofs of Theorems 1-2 are based on asymptotic properties of orthogonal polynomials associated to \( \Omega \), and to their relations with the interior and exterior Riemann mapping functions. These imply asymptotic properties of partial Szegö and Bergman kernels for \( \Omega \). We recall the results we need in this section. Classical references are [Sz1, Ca] while more recent references (with more precise results) are in [SL, Su].

1.1. Szegö kernel and orthogonal polynomials. Let \( \Omega \subset \mathbb{C} \) be a smooth bounded domain. The Szegö kernel of \( \Omega \) with respect to a measure \( \rho|dz| \) on \( \partial \Omega \) is the orthogonal projection

\[
S : L^2(\partial \Omega, \rho|dz|) \to H^2(\partial \Omega, \rho|dz|)
\]

onto the Hardy space of boundary values of holomorphic functions in \( \Omega \) which belong to \( L^2(\partial \Omega, |dz|) \). The Schwartz kernel of \( S \) is denoted \( S(z,w) \). According to [Bel], Theorem 24.3, \( S(z, w) \) admits an analytic continuation (holomorphic in \( z \) and anti-holomorphic in \( w \)) to \( \overline{\Omega} \times \overline{\Omega} \setminus \Delta \), where \( \Delta \) is the diagonal in \( \partial \Omega \times \partial \Omega \). We will refer to this as the regularity theorem.

Let

\[
\left\{ P_j(z) = a_{j0} + a_{j1}z + \cdots + a_{jj}z^j \right\}
\]

be the orthonormal basis of orthogonal polynomials for \( L^2(\partial \Omega, \rho|dz|) \) obtained by applying Gram-Schmidt to \( \{1, z, z^2, \ldots, z^j, \ldots\} \). Since \( S \) is an orthogonal projection, we may express it in terms of any orthonormal basis. Hence, we have

\[
S(z, w) = \sum_{k=0}^{\infty} P_k(z)\overline{P_k(w)}, \quad (z, w) \in \overline{\Omega} \times \overline{\Omega}
\]  

(13)

By the regularity theorem, one has that \( S(z, z) < \infty \) for \( z \in \Omega \), and thus \( P_N \to 0 \) on \( \Omega \). Hence,

\[
S_N(z, z) \to S(z, z), \quad \text{uniformly on compact subsets of } \Omega,
\]

where \( S_N(z, w) = \sum_{k=0}^{N} P_k(z)\overline{P_k(w)} \) is the partial Szegö kernel given in (10).

In the case of the unit disk \( U \) (with the weight \( \rho \equiv 1 \)), one has:

\[
S_U^U(z, w) = \frac{1}{2\pi(1 - zw)}.
\]

We observe that \( S(z, w) \) admits a meromorphic continuation to the exterior as well as the interior of \( \Omega \). However, the behavior is quite different with regard to the partial Szegö kernels. Indeed, The orthogonal polynomials are \( P_k(z) = z^k \); hence

\[
|P_N(z)|^2 = |z|^{2N}, \quad S_N^U(z, z) = \sum_{k=0}^{N-1} |z|^{2k} = \frac{1 - |z|^{2N}}{1 - |z|^2}.
\]  

(14)

Clearly, \( S_N(z, z) \to \infty \) at an exponential rate in the exterior of \( \Omega \).
1.2. Partial Szegő kernels and the exterior Riemann mapping function. We will need the behavior of $S_N(z, z)$ in the exterior of $\Omega$. The relevant result was proved by Szegő [Sz1, Sz2].

Let $\Omega \subset \mathbb{C}$ denote a simply connected bounded plane domain with $C^\infty$ boundary $\partial \Omega$. The exterior domain $\hat{\Omega} \setminus \Omega$ is also a simply connected domain in $\hat{\mathbb{C}}$ and we denote by
\[
\Phi : \hat{\mathbb{C}} \setminus \Omega \to \hat{\mathbb{C}} \setminus U, \quad \Phi(z) = cz + c_0 + c_1 z^{-1} + \cdots
\]
the (unique) exterior Riemann mapping function with $\Phi(\infty) = \infty$, $\Phi'(\infty) \in \mathbb{R}^+$. We recall that the equilibrium measure $\nu_\Omega$ of $\Omega$ is given by
\[
\nu_\Omega = \Phi^* \delta_{S^1} ; \quad \text{i.e.,} \quad \int \varphi \, d\nu_\Omega = \frac{1}{2\pi} \int_0^{2\pi} \varphi \circ \Phi^{-1}(e^{i\theta}) \, d\theta .
\]

Remark: If we fix a point $x_0 \in \Omega$, we may define as well the interior Riemann mapping function $f : \Omega \to U$ with $f(x_0) = 0$, $f'(x_0) \in \mathbb{R}^+$. The functions $f|_{\partial \Omega}$ and $\Phi|_{\partial \Omega}$ are generally different.

Let us assume that $\partial \Omega \in C^\omega$. One can define a (unique) nonvanishing holomorphic function $\Delta_e$ on $\mathbb{C} \setminus \Omega$ satisfying:
\[
|\Delta_e|_{\partial \Omega}^2 = \rho .
\]
(See [Sz2, §§10.2,16.1].) In the following, $L = \text{Length}(\partial \Omega)$, and
\[
T_\varepsilon(\partial \Omega) = \{ x \in \mathbb{C} : d(x, \partial \Omega) < \varepsilon \} .
\]

Theorem 5. [Sz2, Theorems 16.4–16.5] Let $\{P_n\}$ denote the orthonormal polynomials relative to $(\partial \Omega, \rho |dz|)$, where $\Omega$, $\rho$ are as in Theorem 3. Normalize $P_n$ so that its leading coefficient is positive. Then there exist $\varepsilon > 0$ and $0 < \delta < 1$ such that for $z \in (\mathbb{C} \setminus \Omega) \cup T_\varepsilon(\partial \Omega)$, we have
\[
P_n(z) = \left( \frac{L}{2\pi} \right)^{1/2} \Delta_e(z)^{-1} \Phi'(z)^{1/2} \Phi(z)^n + O(\delta^n) .
\]

The proof is based on the properties of Faber polynomials $F_n$ associated to $G_n(z) := \left( \frac{L}{2\pi} \right)^{1/2} \Delta_e(z)^{-1} \Phi'(z)^{1/2} \Phi(z)^n$; i.e., $F_n(z)$ is the polynomial part of the Laurent expansion of $G_n(z)$ about $z = \infty$. Suppose that the Riemann mapping function $\Phi^{-1}$ extends analytically to $\{|z| > r\}$ where $r < 1$. The main estimates are:
\[
P_n = F_n + O(n^{1/2}r^{n/2}) = G_n + O(n^{1/2}r^{n/2}) .
\]
This shows that the Faber polynomials (with weights) are asymptotic to the orthogonal polynomials (with weights).

1.3. Bergman kernel and orthogonal polynomials. In the proof of Theorem 2 on the expected distribution of zeros, where the inner product is chosen to be $\langle f, g \rangle_\Omega = \int_\Omega f \bar{g} \, dx \, dy$, the role of the Szegő kernel is played by the Bergman kernel, i.e. the orthogonal projection from $L^2(\Omega)$ onto the subspace $H^2(\Omega)$ spanned by the $L^2$ holomorphic functions.

As before, we assume $\Omega \subset \mathbb{C}$ is a simply connected, bounded plane domain with $C^\omega$ boundary. We denote by
\[
\{P_j(z) = a_{j0} + a_{j1}z + \cdots + a_{jj} z^j \}
\]
the orthonormal basis of orthogonal polynomials for $L^2(\Omega, dx dy)$ with positive leading coefficient. The Bergman kernel may be expressed in terms of the orthogonal polynomials by:

$$B(z, w) = \sum_{k=0}^{\infty} P_k(z)\overline{P_k(w)}, \quad (z, w) \in \Omega \times \Omega.$$  

(17)

We let $B_N(z, w) = \sum_{k=0}^{N} P_k(z)\overline{P_k(w)}$; as in the case of the Szegő kernel, we have $B_n(z, w) \rightarrow B(z, w)$, $B(z, z) > 0$.

Orthogonal polynomials over a domain were studied by Carleman after Szegő's work on polynomials orthogonal on $\partial \Omega$. A key element in our proof is the following asymptotic formula for $P_N(z)$:

**Theorem 6.** (Carleman [Ca]; cf. [SL], Theorems 1–2, p. 290) Let $\{P_n\}$ denote the orthonormal polynomials relative to $\langle \Omega, dx dy \rangle$, where $\Omega$ is a simply-connected bounded $C^\omega$ domain. Normalize $P_n$ so that its leading coefficient is positive. Then there exist $\varepsilon > 0$ and $0 < \delta < 1$ such that:

$$P_n(z) = \begin{cases} 
\left(\frac{N+1}{\pi}\right)^{1/2} \Phi'(z)\Phi(z)^n + O(\delta^n), & z \in T_\varepsilon(\partial \Omega) \cap \Omega \\
\left(\frac{N+1}{\pi}\right)^{1/2} \Phi'(z)\Phi(z)^n[1 + O(\delta^n)], & z \in \mathbb{C} \setminus \Omega 
\end{cases}$$

2. **Proof of the asymptotic formulas**

We now prove Theorems 1–2. The proofs are basically the same in the boundary and interior cases, granted the theorems of Szegő and Carleman. We therefore only give details in the boundary case.

We first relate zero distributions to Szegő kernels, as in our previous papers [ShZ1, ShZ2].

2.1. **Zero distributions.** For convenience, we let

$$Z_f^N = \sum_{f(z)=0} \delta_z$$

denote the zero distribution of a polynomial $f$ so that recalling (3), we have $\tilde{Z}_f = \frac{1}{N} Z_f$. (Strictly speaking, we should count the zeros with multiplicities, but polynomials in $\mathcal{P}_N$ almost surely have only simple zeros.) The expected zero distribution has a simple expression:

**Proposition 7.** We have

$$E_{\partial \Omega, \rho}^N(Z_f) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log S_N(z, z).$$

This proposition is a special case of Proposition 4.1 in [ShZ2] (which is a variation of a result in our earlier work [ShZ1]). For completeness, we give the proof here: We first note that since

$$Z_f = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |f|^2,$$
we have

\[ E_{\partial \Omega, \rho}^N(Z_f) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} E_{\partial \Omega, \rho}^N(\log |f|^2) \]

To calculate the expectation, we write \( f \) in terms of the orthonormal basis \( \{ P_j \} \) of \( \mathcal{P}_N \):

\[ f(z) = \sum_{j=0}^N a_j P_j(z) = \langle a, p(z) \rangle, \]

where \( a = (a_0, \ldots, a_N) \), \( P = (P_0, \ldots, P_N) \). Then,

\[ E_{\partial \Omega, \rho}^N(Z_f) = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \int_{\mathbb{C}^{N+1}} \log |\langle a, P(z) \rangle| \frac{1}{\pi^{N+1}} e^{-\|a\|^2} \, da. \]

We write

\[ P(z) = \|P(z)\|u(z), \quad \|P(z)\|^2 = \sum_{j=0}^N |P_j(z)|^2 = S_N(z, z), \quad \|u(z)\| = 1. \]

Then,

\[ \log |\langle a, P(z) \rangle| = \log \|P(z)\| + \log |\langle a, u(z) \rangle|. \]

We observe that

\[ \int_{\mathbb{C}^{N+1}} \log |\langle a, u(z) \rangle| e^{-\|a\|^2} \, da = \text{constant} \]

since for each \( z \) we may apply a unitary coordinate change so that \( u(z) = (1, 0, \ldots, 0) \). Hence the derivative equals zero, and we have

\[ E_{\partial \Omega, \rho}^N(Z_f) = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log \|P(z)\| = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log S_N(z, z). \]

By exactly the same argument, we also have:

**Proposition 8.** We have

\[ E_{\Omega}^N(Z_f) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log B_N(z, z). \]

2.2. **The circular ensemble.** We start with the fundamental case of the circle; in the next section we will reduce the other cases to the circular case. Before proving our precise result for the circle (Proposition 9), we first give a simple ad hoc argument that the expected limit distribution of zeros in this case is the measure \( \delta_{S^1} \) given in (4): By (14),

\[ \frac{1}{N} \partial \bar{\partial} \log S_N(z, z) \sim \frac{1}{N} \partial \bar{\partial} \log(1 - |z|^{2N}). \]

Clearly, in any annulus \( |z| \leq r < 1, (1 - |z|^{2N}) \to 1 \) rapidly with its derivatives, and the limit equals zero. In any annulus \( |z| \geq r > 1 \) we may write \( (1 - |z|^{2N}) = |z|^{2N}(1 - |z|^{-2N} - 1) \) and separate the factors after taking log. The second again tends to zero rapidly, while the first factor, \( \log |z|^{2N} \), is killed by \( \partial \bar{\partial} \) (note that \( z \neq 0 \) in this part). It follows that the limit measure must be supported on \( S^1 \). Since it is \( \text{SO}(2) \)-invariant (radial), and since it is a probability measure, it must be \( \frac{1}{2\pi} d\theta \), which we henceforth denote by \( \nu \).

We have the following explicit formula and asymptotics for the circular case:
Proposition 9. Let $\nu = \frac{d\theta}{2\pi}$ denote Haar measure on $S^1$. Then

$$E^N_\nu(Z_f) = \left[ \frac{1}{(|z|^2 - 1)^2} - \frac{(N+1)^2|z|^{2N}}{(|z|^{2N+2} - 1)^2} \right] \sqrt{-1} \frac{1}{2\pi} \, dz \wedge d\bar{z},$$

Furthermore, $E^N_\nu(Z_f) = N\nu + O(1)$; i.e., for all test forms $\varphi \in \mathcal{D}(\mathbb{C})$, we have

$$E^N_\nu \left( \sum_{\{z: f(z) = 0\}} \varphi(z) \right) = \frac{N}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta}) \, d\theta + O(1).$$

In particular, $E^N_\nu(Z^\nu_f) \to \nu$ in $\mathcal{D}'(\mathbb{C})$.

The formula for $E^N_\nu(Z_f)$ in Theorem 9 agrees with the one given by Hammersley (see also [EK, SV]).

Proof. By Proposition 4,

$$E^N_\nu(Z_f) = \sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} \log \frac{1 - |z|^{2N+2}}{1 - |z|^2}. \tag{18}$$

We write $\zeta = \rho + i\theta = 2 \log z$, and we let

$$h_N(\rho) = S^U_N(e^{i^\nu\rho}, e^\nu\rho) = \sum_{n=0}^{N} e^{n\rho} = \frac{1 - e^{(N+1)\rho}}{1 - e^\rho}. \tag{19}$$

Then

$$E^N_\nu(Z_f) = \frac{i}{2\pi} (\log h_N)^\nu(\rho) \, d\zeta \, d\bar{\zeta}. \tag{20}$$

We let

$$g_N := (\log h_N)^\nu = \frac{e^{-\rho}}{(1 - e^{-\rho})^2} - \frac{(N+1)^2}{1 - e^{-(N+1)\rho}}. \tag{21}$$

The desired formula for $E^N_\nu(Z_f)$ follows by substituting $d\zeta = \frac{2}{z}dz$ into (20)–(21).

We easily see that $g_N > 0$ (which also follows from the strict subharmonicity of $\log S^U_N(z, z) = \log(\sum_{n=0}^{N} |z|^{2n})$) and that $g_N(-\rho) = g_N(\rho)$. Now suppose that $\psi \in \mathcal{C}_\infty(\mathbb{R}) \cap \mathcal{L}_\infty(\mathbb{R})$ such that $\psi(0) = 0$. We claim that

$$\int g_N \psi = O(1). \tag{22}$$

To verify (22), we break up the integral into 3 pieces: $\rho \leq -1$, $-1 \leq \rho \leq 1$, and $\rho \geq 1$. First, we have

$$\int_{-\infty}^{-1} g_N = \int_1^{+\infty} g_N = \frac{1}{e-1} - \frac{N+1}{e^{N+1} - 1} \leq \frac{1}{e-1},$$

and thus

$$\left| \int_{-\infty}^{-1} g_N \psi \right| + \left| \int_1^{+\infty} g_N \psi \right| = O(1).$$

To estimate the integral over $[-1, 1]$, we write $\psi(\rho) = a\rho + O(\rho^2)$ and we note that

$$g_N(\rho) \leq \frac{1}{(1 - e^{-\rho})^2} \leq \frac{4}{\rho^2} \quad \text{for} \quad 0 \leq \rho \leq 1.$$
Since \( g_N \) is even, we then have
\[
\left| \int_{-1}^{1} g_N \psi \right| \leq C \int_{0}^{1} \rho^2 g_N(\rho) \, d\rho \leq 4C, 
\]
which completes the proof of (22).

Now let \( \varphi \in \mathcal{D}(\mathbb{R}) \). An easy computation gives \( \int_{\mathbb{R}} g_N \, d\rho = N \). (This also follows from the fact that the expected number of zeros of a degree-\( N \) polynomial is \( N \).) Thus by (22),
\[
\int_{\mathbb{R}} g_N \varphi = \varphi(0) \int_{\mathbb{R}} g_N + \int_{\mathbb{R}} g_N [\varphi - \varphi(0)] = N\varphi(0) + O(1) .
\]

**Remark:** Recalling (19), we have
\[
g_N^{-1}(0) = h_N^{-1}(0)h''_{N-1}(0) - h'_{N-1}(0)^2 = \frac{N \sum_{k=0}^{N-1} k^2 - (\sum_{k=0}^{N-1} k)^2}{N^2} = \frac{N^2 - 1}{12} .
\]

One can easily check that \( g_N'(\rho) < 0 \) for \( \rho > 0 \) and since \( g_N \) is even, we have \( g_N(0) = \sup g_N \).

In fact, a computation using Maple\textsuperscript{TM} gives:
\[
g_N^{-1} = \frac{N^2 - 1}{12} - \frac{N^4 - 1}{240} \rho^2 + \frac{N^6 - 1}{6048} \rho^4 - \frac{N^8 - 1}{172800} \rho^6 + \frac{N^{10} - 1}{532240} \rho^8 - \frac{691(N^{12} - 1)}{118879488000} \rho^{10} \ldots
\]

### 2.3. The general case.

We now prove Theorem \[4\]. We must show that
\[
\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log S_N(z, z) = N\nu_\Omega + O(1) . \tag{23}
\]

We shall break up the proof of (23) into two regions: \( \Omega \) and a neighborhood \( W_\varepsilon \) of \( \hat{C} \setminus \Omega \). The estimate (23) is obvious on \( \Omega \). Indeed, by Theorem 16.3 in [S2], \( S_N(z, z) \rightarrow S(z, z) \) uniformly in compact subsets of \( \Omega \). Furthermore, \( S(z, z) > 0 \) on \( \Omega \) (see e.g., [B, Th. 12.3]), and hence \( \log S_N(z, z) \rightarrow \log S(z, z) \). Thus
\[
\partial \bar{\partial} \log S(z, z) \rightarrow \partial \bar{\partial} \log S(z, z) + o(1) = O(1) \text{ in } \mathcal{D}'(\Omega) .
\]
(In fact, \( S_N(z, w) \rightarrow S(z, w) \) on \( \Omega \times \Omega \), and hence by the Cauchy Integral formula, we have normal convergence of all derivatives; it then follows that \( \partial \bar{\partial} \log S_N(z, z) \rightarrow \partial \bar{\partial} \log S(z, z) \) uniformly on compact subsets of \( \Omega \).)

Next we verify (23) on the domain
\[
W_\varepsilon := \left( \hat{C} \setminus \Omega \right) \cup T_\varepsilon(\partial \Omega) ,
\]
where \( \varepsilon \) is chosen sufficiently small so that Theorem \[4\] holds and \( |\Phi| \geq \delta' > \delta \) on \( W_\varepsilon \).

Let
\[
A_N(z) = S_N(z, z) / \Phi_N S_N^U(z, z) . \tag{24}
\]

We claim that there is a positive constant \( C \) such that
\[
0 < 1/C \leq A_N(z) \leq C < +\infty \quad \text{for } z \in W_\varepsilon . \tag{25}
\]
To verify (25), we let $n$ as above. Then by Theorem 5, there exists $z$ it then follows from Szegő’s Theorem 5 that $\sup_{z \in W_\varepsilon} \frac{1}{\psi}$

and we note that $\psi$ and $\frac{1}{\psi}$ are bounded on $W_\varepsilon$. Recalling that

$$\Phi_N^* S_N^U(z, z) = \sum_{n=0}^N |\Phi(z)|^{2n},$$

it then follows from Szegő’s Theorem 5 that $\sup_{z \in W_\varepsilon} A_N(z) \leq C < +\infty$.

The lower bound for $A_N(z)$ follows from [Sz2, §16.5]. Alternately, let $G_n(z) := \psi(z)\Phi(z)^n$ as above. Then by Theorem 5, there exists $n_0 \geq 2$ such that $|P_n(z)|^2 \geq \frac{1}{2}|G_n(z)|^2$ for $z \in W_\varepsilon$, $n \geq n_0$. If $\delta' < |\Phi(z)| \leq 1$ we have $|p_0(z)|^2 = c \geq c'(|G_0(z)|^2 + \cdots + |G_{n_0}(z)|^2)$ and hence

$$S_N(z, z) \geq \min\{1/2, c\} \sum_{n=0}^N |G_n(z)|^2 \geq c'\Phi_N^* S_N^U(z, z).$$

On the other hand, if $|\Phi(z)| \geq 1$ then the required estimate follows from

$$S_N(z, z) \geq |P_{n_0}(z)|^2 + \cdots + |P_N(z)|^2 \geq \frac{1}{2} \left[ |G_{n_0}(z)|^2 + \cdots + |G_N(z)|^2 \right] \geq \frac{1}{4} \left[ |G_0(z)|^2 + \cdots + |G_N(z)|^2 \right] = \frac{1}{4} |\psi(z)|^2 \Phi_N^* S_N^U(z, z),$$

for $N \geq 2n_0$.

It follows from (25) that

$$(\partial \bar{\partial} \log A_N, \varphi) = (\log A_N, \partial \bar{\partial} \varphi) = O(1), \quad \text{for} \quad \varphi \in \mathcal{D}(W_\varepsilon).$$

Recalling (16), we then have

$$\frac{i}{2\pi} \partial \bar{\partial} \log S_N(z, z) = \Phi^* \left( \frac{i}{2\pi} \partial \bar{\partial} \log S_N^U(z, z) \right) - \frac{i}{2\pi} \partial \bar{\partial} \log A_N(z)$$

$$= \Phi^*(N\nu + O(1)) + O(1) = N\nu_\Omega + O(1).$$

\[\square\]

2.4. **The interior case.** The proof of Theorem 2 is exactly the same as in the boundary case, using Carleman’s theorem in place of Szegő’s, and the Bergman kernel and Proposition 8 in place of the Szegő kernel and Proposition 7. We omit the details.

We also note that the conclusion of Theorem 2 holds for inner products on $\Omega$ with certain analytic weights; see [SL].

2.5. **Proof of Theorem 3.** The proof of almost sure convergence to the average is exactly the same as the proof of the similar statement for sections of positive line bundles in [ShZ1]. We summarize the proof here: Let

$$\omega_N = \begin{cases} 
\mathbb{E}_{\partial_\Omega, \rho}^N(\hat{Z}_f^N) = \frac{1}{2\pi N} \partial \bar{\partial} \log S_N(z, z) & \text{for the boundary case}, \\
\mathbb{E}_{\Omega}^N(\hat{Z}_f^N) = \frac{1}{2\pi N} \partial \bar{\partial} \log B_N(z, z) & \text{for the interior case}.
\end{cases}$$
We then have the following variance estimate:

**Lemma 10.** Let \( \varphi \) be any smooth test form. Then

\[
E \left( \left( \tilde{Z}^N_{f, \omega} - \omega_N, \varphi \right)^2 \right) = O(N^{-2}).
\]

Lemma 10 is given as Lemma 3.3 in [ShZ1] and has exactly the same proof. Continuing as in [ShZ1], by Theorem 1 or Theorem 2, it suffices to show that

\[
(\tilde{Z}^N_{f, \omega}, \varphi) \to 0 \quad \text{almost surely .}
\]

Consider the random variables \( X_N \) on \( S \) given by:

\[
X_N(\{f_N\}) = (\tilde{Z}^N_{f, \omega}, \varphi)^2 \geq 0.
\]

By Lemma 10, \( \int_S X_N \, d\mu = O(N^{-2}) \), and therefore

\[
\int_S \sum_{N=1}^{\infty} X_N \, d\mu = \sum_{N=1}^{\infty} \int_S X_N \, d\mu < +\infty.
\]

Hence, \( X_N \to 0 \) almost surely. The conclusion follows by considering a countable \( C^0 \)-dense family of test forms.

\( \square \)

### 3. Pair correlations of zeros

In our previous work with P. Bleher [BSZ1, BSZ2], we derived the scaling limit pair correlation functions for zeros of SU\((m + 1)\)-polynomials and showed that these correlations are ‘universal.’ In this section, we apply the general formulas from [BSZ1, BSZ2] to prove Theorem 4 and to describe our universal pair correlation function \( K^2_{\infty} \).

#### 3.1. The scaled Szegő kernel

In [BSZ1, §2.3], we gave a general formula for the \( \ell \)-point correlation of zeros in terms of the Bergman-Szegő kernel and its first and second derivatives. In order to apply this formula to find the scaling limit correlations, we need the following universal scaling limit Szegő and Bergman kernels for plane domains:

**Proposition 11.** Let \( \Omega \) be a simply-connected bounded \( C^\omega \) domain and let \( \rho \) be a \( C^\omega \) density.

i) Let \( S_N \) be the orthogonal projection for the inner product \( \langle , \rangle_{\partial \Omega, \rho} \). Then

\[
\frac{1}{N} \hat{S}_N \left( 1 + \frac{\zeta_1}{N}, 1 + \frac{\zeta_2}{N} \right) \to |\psi(z_0)|^2 G(\zeta_1 + \bar{\zeta}_2), \quad G(z) = \frac{e^z - 1}{z},
\]

where \( \hat{S}_N = S_N \circ \Phi^{-1} \) is the projection kernel written in terms of the complex coordinate \( \zeta = \Phi(z) \), \( \psi \) is given by (27) and \( z_0 = \Phi^{-1}(1) \).

ii) Let \( B_N \) be the orthogonal projection for the inner product \( \langle , \rangle_{\Omega} \). Then

\[
\frac{1}{N^2} \hat{B}_N \left( 1 + \frac{\zeta_1}{N}, 1 + \frac{\zeta_2}{N} \right) \to \frac{1}{\pi} |\Phi'(z_0)|^2 G(\zeta_1 + \bar{\zeta}_2),
\]

where \( \hat{B}_N = B_N \circ \Phi^{-1} \) is the projection kernel written in terms of the complex coordinate \( \zeta = \Phi(z) \) and \( z_0 = \Phi^{-1}(1) \).

The above limits are uniform when \( |\zeta_1| + |\zeta_2| \) is bounded.
Proof. We begin as before with the case of the disk $U$ with density $\rho = 1$. Then

$$\frac{1}{N} S_N^U(1 + \zeta_1/N, 1 + \zeta_2/N) = \frac{1}{N} \sum_{k=0}^{N} (1 + \zeta_1/N)^k (1 + \tilde{\zeta}_2/N)^k$$

$$= \left[(1 + \zeta_1/N)(1 + \tilde{\zeta}_2/N)\right]^{\frac{N+1}{2}} - 1$$

$$\rightarrow \frac{e^{\zeta_1 \bar{\zeta}_2} - 1}{\zeta_1 + \tilde{\zeta}_2} = G(\zeta_1 + \tilde{\zeta}_2) . \quad (28)$$

We now consider the inner product $\langle \cdot, \cdot \rangle_{\partial \Omega, \rho}$. By Szegö’s Theorem \[ we can choose $\varepsilon, \lambda > 0$ such that

$$P_k(z) = \psi(z)\left[\Phi(z)^k + q_k(z)\right], \quad |q_k(z)| \leq C_1 e^{-\lambda k} \text{ for } 1 - \varepsilon \leq |\Phi(z)| \leq 2,$$

and hence

$$\frac{\widehat{S}_N(1 + \frac{\zeta_1}{N}, 1 + \frac{\zeta_2}{N})}{N \widehat{\psi}(1 + \frac{\zeta_1}{N})\widehat{\psi}(1 + \frac{\zeta_2}{N})} = \frac{1}{N} \hat{s}_N^U(1 + \frac{\zeta_1}{N}, 1 + \frac{\zeta_2}{N}) + \frac{1}{N} \sum_{k=0}^{N} (1 + \frac{\zeta_1}{N})^k \hat{q}_k(1 + \frac{\zeta_2}{N})$$

$$+ \frac{1}{N} \sum_{k=0}^{N} (1 + \frac{\zeta_2}{N})^k \hat{q}_k(1 + \frac{\zeta_1}{N}) + \frac{1}{N} \sum_{k=0}^{N} \hat{q}_k(1 + \frac{\zeta_1}{N})\hat{q}_k(1 + \frac{\zeta_2}{N}).$$

By (28), the first term on the right side approaches $G(\zeta_1 + \tilde{\zeta}_2)$. Thus it remains to show that the other terms tend to zero. Suppose that $|\zeta_1| \leq A < +\infty$. Then,

$$\frac{1}{N} \sum_{k=0}^{N} (1 + \frac{\zeta_1}{N})^k q_k(1 + \frac{\zeta_2}{N}) \leq \frac{C_1}{N} \sum_{k=0}^{N} e^{(1 + \frac{N}{N-\lambda})k} \leq \frac{C_2}{N} + \frac{C_1}{N} \sum_{k=k_0}^{N} e^{-1/2\lambda k} = O \left(\frac{1}{N}\right).$$

Exactly the same estimate holds for the next term, and the last term is clearly also $O(\frac{1}{N})$.

The same argument holds for the partial Bergman kernel, using Carleman’s Theorem instead of Szegö’s. \[ 3.2. \textbf{The scaled zero density.} \quad \text{We define the expected zero density function } D_{\partial \Omega, \rho}^N \text{ by}

$$E_{\partial \Omega, \rho}(Z_f^N) = D_{\partial \Omega, \rho}^N(z) \left(\frac{i}{2} dz \wedge d\bar{z}\right).$$

We recall that by Proposition \[ we have

$$D_{U, 1}^N(z) \rightarrow \frac{1}{\pi(|z|^2 - 1)^2} \quad (|z| \neq 1), \quad \text{ and } \quad \frac{1}{N} D_{U, 1}^N \rightarrow \nu .$$

We now give a third limit for $D_{U, 1}^N$, the scaling limit, which we find to be universal:

**Proposition 12.** Let $\Omega, \rho, \Phi$ be as above and let $\hat{D}^N := (D^N \circ \Phi^{-1})(|\Phi^{-1}|^2)$ be the expected zero density for the inner product $\langle \cdot, \cdot \rangle_{\partial \Omega, \rho}$ or for $\langle \cdot, \cdot \rangle_{\Omega}$ with respect to the coordinate $\zeta = \Phi(z)$. Then

$$\frac{1}{N^2} \hat{D}^N \left(1 + \frac{\tau}{N}\right) \rightarrow D^\infty(\tau) \quad \text{ as } N \rightarrow \infty ,$$
where

\[ D^\infty(\tau) = \frac{1}{\pi} (\log G)'(2\tau) = \frac{e^{4\tau} - (2 + 4\tau^2) e^{2\tau} + 1}{4\pi (e^{2\tau} - 1)^2 \tau^2} = \frac{1}{\pi \tau^2} - \frac{e^{2\tau}}{4\pi (e^{2\tau} - 1)^2}. \]

In particular, the scaling limit density \( D^\infty(\tau) \) has \( 1/\tau^2 \) decay as \( \tau \to \pm \infty \).

**Proof.** By Proposition 7,

\[ \hat{D}^N(r) = \frac{1}{\pi} \frac{d^2}{dr^2} \log \hat{S}^N(r, r). \]

The conclusion follows by substituting \( r = 1 + \frac{\tau}{N} \) and applying Proposition 11.

The graph of the scaling limit density function is given in Figure 3 below.

**Figure 3.** The scaled zero density \( D^\infty(\tau) \)

### 3.3. The scaling limit zero correlation functions.

Theorem 4 is an immediate consequence of Propositions 11–12 and Theorem 2.4 in [BSZ1] (with \( k = m = 1, n = \ell \)), which gives a universal formula for the \( \ell \)-point zero correlations in terms of the projection kernel and its first and second derivatives. (See also [BSZ2, Theorem 1.1].)

For the reader’s convenience, we derive the formula for the pair correlation case \( \ell = 2 \):

Using the coordinate \( \zeta = \Phi(z) \), we define the \( 2 \times 2 \) matrices

\[ A^N(\zeta, \eta) = \left( \begin{array}{cc} \hat{S}^N(\zeta, \zeta) & \hat{S}^N(\zeta, \eta) \\ \hat{S}^N(\eta, \zeta) & \hat{S}^N(\eta, \eta) \end{array} \right), \]

\[ B^N(\zeta, \eta) = \frac{\partial}{\partial \bar{\eta}} A^N(\zeta, \eta), \quad C^N(\zeta, \eta) = \frac{\partial^2}{\partial \zeta \partial \bar{\eta}} A^N(\zeta, \eta). \]

As in [BSZ1, BSZ2] we let

\[ \Lambda^N = C^N - (B^N)^*(A^N)^{-1}B^N. \]

Writing

\[ E^N_{\partial \Omega, \rho}(Z^N_f \times Z^N_f) = \lambda_N(\zeta, \eta) d\text{Vol}_{\mathbb{C}^2}, \]
we have by [BSZ1, (91)] or [BSZ2, (23)],
\[
\lambda_N = \frac{\Lambda_1^N \Lambda_2^N + \Lambda_1^N \Lambda_2^N}{\pi^2 \det A^N}.
\] (29)

We consider the case of the inner product \( \langle , \rangle_{\partial \Omega, \rho} \). (The interior case is similar.) By Proposition 11,
\[
\frac{c}{N} A^N (1 + \zeta_1/N, 1 + \zeta_2/N) \rightarrow \begin{pmatrix} G(\zeta_1 + \bar{\zeta}_1) & G(\zeta_1 + \bar{\zeta}_2) \\ G(\zeta_2 + \bar{\zeta}_1) & G(\zeta_2 + \bar{\zeta}_2) \end{pmatrix} \overset{\text{def}}{=} A^\infty(\zeta_1, \zeta_2),
\] (30)
where \( c = |\psi(z_0)|^{-2} \). Differentiating (30), we obtain
\[
\frac{c}{N} B^N (1 + \zeta_1/N, 1 + \zeta_2/N) \rightarrow \begin{pmatrix} G'(\zeta_j + \bar{\zeta}_k) \end{pmatrix}_{j,k=1,2} \overset{\text{def}}{=} B^\infty(\zeta_1, \zeta_2)
\]
and
\[
\frac{c}{N^3} C^N (1 + \zeta_1/N, 1 + \zeta_2/N) \rightarrow \begin{pmatrix} G''(\zeta_j + \bar{\zeta}_k) \end{pmatrix}_{j,k=1,2} \overset{\text{def}}{=} C^\infty(\zeta_1, \zeta_2).
\] (31)

We write
\[
\Lambda^\infty(\zeta_1, \zeta_2) = C^\infty(\cdot)(B^\infty)^{-1} B^\infty = \lim_{N \to \infty} \frac{c}{N^3} A^N (1 + \zeta_1/N, 1 + \zeta_2/N).
\] (32)

Hence by (29),
\[
\frac{1}{N^4} \lambda_N (1 + \zeta_1/N, 1 + \zeta_2/N) \rightarrow \frac{\Lambda_1^\infty \Lambda_2^\infty + \Lambda_1^\infty \Lambda_2^\infty}{\pi^2 \det A^\infty}.
\]

Therefore by Proposition 12,
\[
\tilde{K}^{2N}_{\partial \Omega, \rho} \left( 1 + \frac{\zeta_1}{N}, 1 + \frac{\zeta_2}{N} \right) \rightarrow K^{2\infty}(\zeta_1, \zeta_2) = \frac{(\Lambda_1^\infty \Lambda_2^\infty + \Lambda_1^\infty \Lambda_2^\infty)(\zeta_1, \zeta_2)}{D^{\infty}(\text{Re } \zeta_1) D^{\infty}(\text{Re } \zeta_2) \det A^\infty(\zeta_1, \zeta_2)}.
\] (33)

The explicit expansion of (33) in terms of \( \zeta_1, \zeta_2 \) is quite complicated. Using Maple\textsuperscript{TM}, we can compute the expansions of the tangential and normal correlations for short distances:
\[
\kappa^T(\alpha) = K^{2\infty}(0, i\alpha) = \frac{1}{150} \alpha^2 + \frac{11}{4200} \alpha^4 + \frac{23}{529200} \alpha^6 + \frac{107}{582120000} \alpha^8 - \frac{6659}{953512560000} \alpha^{10} \cdots,
\]
\[
\kappa^\perp(\tau) = K^{2\infty}(0, \tau) = \frac{1}{150} \tau^2 + \frac{1}{1200} \tau^4 - \frac{101}{5292000} \tau^6 - \frac{10289}{5821200000} \tau^8 + \frac{7249481}{4767562800000} \tau^{10} \cdots.
\]
Thus zeros on \( \partial \Omega \) repel as in the case of SU(2)-polynomials considered in [Han, BSZ]. The behavior over long distances is illustrated in Figures 1–2 in the introduction. Note that \( \tilde{K}^{2N} \) is invariant (only) under the \( S^1 \) action, and thus
\[
K^{2\infty}(i\alpha_1, i\alpha_2) = K^{2\infty}(0, i(\alpha_2 - \alpha_1))
\]
(However, \( K^{2\infty}(\tau_1, \tau_2) \neq K^{2\infty}(0, \tau_2 - \tau_1) \).
)

Remark: The function
\[
\det A^\infty(0, i\alpha) = 1 - \left( \frac{\sin \alpha/2}{\alpha/2} \right)^2,
\]
which appears in the denominator of our formula for the tangential limit correlation, happens to be the limit pair correlation function for the eigenvalues of random Hermitian matrices (see [D3, (5.74)]). However, we are unaware of a good interpretation for \( \det A^\infty \) in our setting.
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