SMOOTH SUBSOLUTIONS OF THE DISCOUNTED HAMILTON-JACOBI EQUATIONS

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Abstract. For the discounted Hamilton-Jacobi equation
\[ \lambda u + H(x, du) = 0, \quad x \in M, \]
we construct \( C^{1,1} \) subsolutions which are indeed solutions on the projected Aubry set. The smoothness of such subsolutions can be improved under additional hyperbolicity assumptions. As applications, we can use such subsolutions to identify the maximal global attractor of the associated conformally symplectic flow, and to control the convergent speed of the Lax-Oleinik semigroups.

1. Introduction

Let \( M \) be a \( C^\infty \) connected, closed Riemannian manifold. A function \( H \in C^k(T^*M, \mathbb{R}), k \geq 2 \) is called a Tonelli Hamiltonian if for all \( x \in M \),

(H1) (Positive Definite) \( H_{pp} \) is positive definite everywhere on \( T^*_xM \);

(H2) (Superlinear) \( \lim_{|p|_x \to +\infty} H(x, p)/|p|_x = +\infty \), where \( |\cdot|_x \) is the norm on \( T^*_xM \) induced by the Riemannian metric.

For a fixed constant \( \lambda > 0 \), we consider the ODE system on \( T^*M \) associated with \( H \), which can be expressed in coordinates as

\[
\begin{cases}
\dot{x} = H_p(x, p), \\
\dot{p} = -H_x(x, p) - \lambda p.
\end{cases}
\]

1991 Mathematics Subject Classification. 35B40, 37J55 49L25, 70H20.
Key words and phrases. discounted equation, Hamilton-Jacobi equation, global attractor, viscosity solution, subsolution, Lax-Oleinik semigroup, Aubry-Mather theory.

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Physically, this system describes the mechanical motion of masses with friction proportional to the velocity. System (1) can also be found in other subjects, e.g. astronomy \[8\], transportation \[24\] and economics \[3\]. It is remarkable that the earliest research of system (1) can trace back to Duffing’s work on explosion engines \[11\]. The nonlinear oscillation he concerned inspires the qualitative theory of dynamical systems in the following decades \[20\].

Due to (H1)-(H2), the local phase flow \(\Phi^t_{H,\lambda}\) of (1) is forward complete, namely, it is well defined for all \(t \in \mathbb{R}^+\). Besides, the direct computation shows that \(\Phi^t_{H,\lambda}\) transports the standard symplectic form \(\Omega = dp \wedge dx\) into a multiple of itself:

\[
(\Phi^t_{H,\lambda})^* \Omega = e^{\lambda t} \Omega, \quad t \in \mathbb{R}^+.
\]

(2)

That is why system (1) is also called conformally symplectic \[24\] or dissipative \[16\] in some literatures. The attracting invariant sets of twist map satisfying (2) \((t \in \mathbb{Z})\) have been revealed by Le Calvez \[16\] and Casdagli \[7\]. Besides, the existence of KAM tori for system (1) was investigated in \[5\].

1.1. Viscosity subsolutions of discounted H-J equations. Following the ideas of Aubry-Mather theory \[19\] and weak KAM theory \[13\], the authors of \[10, 18\] initiate the investigation of variational methods associated to (1). They try to seek a viscosity solution of the discounted Hamilton-Jacobi equation

\[
\lambda u + H(x, du) = 0, \quad x \in M
\]

which is related with the variational minimal curves starting from each \(x \in M\). It is well-known to specialists in PDE \[2, 10\], such a viscosity solution is unique but usually not \(C^1\). To understand such a solution, we introduce the following variational principle: let’s define the Legendre transformation by

\[
L_H : T^* M \to TM; (x, p) \mapsto (x, H_p(x, p))
\]

which is a diffeomorphism due to (H1)-(H2). Accordingly, the Tonelli Lagrangian \(L \in C^k(TM, \mathbb{R})\)

\[
L(x, v) := \max_{p \in T^*_x M} \{ \langle p, v \rangle - H(x, p) \},
\]

(3)

is well defined and the maximum is attained at \(\tilde{p} \in T^*_x M\) such that \(\tilde{p} = L_v(x, v)\).

**Definition 1.1.** A function \(u \in C(M, \mathbb{R})\) is called \(\lambda\)-dominated by \(L\) and denoted by \(u \prec_{\lambda} L\), if for any absolutely continuous \(\gamma : [a, b] \to M\), there holds

\[
e^{\lambda b} u(\gamma(b)) - e^{\lambda a} u(\gamma(a)) \leq \int_a^b e^{\lambda t} L(\gamma(t), \dot{\gamma}(t)) \, dt.
\]

(4)

We can denote by \(\mathcal{S}^-\) the set of all \(\lambda\)-dominated functions of \(L\).

**Remark 1.2.** Recall that any \(\lambda\)-dominated function \(u\) has to be Lipschitz (Proposition 6.3 of \[10\]). Therefore, we can prove

\[
\lambda u(x) + H(x, du(x)) \leq 0, \quad a.e. \ x \in M,
\]

which implies \(u\) is an almost everywhere subsolution of (1) (see Lemma \[2, 14\]). On the other side, the equivalence between almost everywhere subsolution and viscosity subsolution was proved in bunch of references e.g. \[1, 2, 10, 12, 22\]. So we get the equivalence among the three:

\[
a.e. \ subsolution \iff \text{viscosity subsolution} \iff \lambda\text{-dominated function}
\]
Definition 1.3. \[ \gamma \in C^{ac}(\mathbb{R}, M) \] is called globally calibrated by \( u \in S^- \), if for any \( a < b \in \mathbb{R} \),
\[
e^{\lambda b} u(\gamma(b)) - e^{\lambda a} u(\gamma(a)) = \int_a^b e^{\lambda t} L(\gamma(t), \dot{\gamma}(t)) \, dt.
\]

The Aubry set \( \tilde{A} \) is an \( \Phi_{L,\lambda} \)-invariant set defined by
\[
\tilde{A} = \bigcup_{u \in S^-} \bigcup_{\gamma} \{(\gamma, \dot{\gamma}) | \gamma \text{ is globally calibrated by } u\} \subset TM
\]
and the projected Aubry set can be defined by \( A = \pi \tilde{A} \subset M \), where \( \pi : TM \to M \) is the canonical projection.

Remark 1.4. Here the definition of the Aubry set is equivalent to the definition in [18], see Appendix B for the proof. Therefore, \( \pi^{-1} : A \to \tilde{A} \subset TM \) is a Lipschitz graph, as described in Theorem (ii) of [18].

The relation between \( u^- \) and \( S^- \) can be revealed by the following conclusion:

Theorem 1.5 (proved in Appendix A). The viscosity solution of (D) is the pointwise supreme of all smooth, i.e., \( C^\infty \), viscosity subsolutions, namely we have
\[
u^- (x) = \sup_{u \in S^-} u(x) = \sup_{u \in C^\infty \cap S^-} u(x).
\]

1.2. Constrained subsolutions & Main results. Notice that for any \( w \in C^1(M, \mathbb{R}) \), there always exists a constant \( c > 0 \) large enough, such that \( w - c \) is a subsolution of (D). Therefore, we are easy to get tons of subsolutions, which couldn’t tell us any information about \( \tilde{A} \) yet. So we need a further selection in \( S^- \).

Let us denote \( \mathcal{M}_\lambda \) by the set of all \( \Phi_{L,\lambda} \)-invariant measures (w.r.t the (E-L) flow, see Sec. 2). It is non-empty, since there exists at least an invariant probability measure \( \mu \) supported on \( \tilde{A} \), due to the \( \Phi_{L,\lambda} \)-invariance of \( \tilde{A} \) and the Krylov-Bogolyubov’s theorem (see Remark 10 of [18]). Then we can give the following definition.

Definition 1.6. \( u \in S^- \) is called a constrained subsolution of (D), if
\[
\inf_{\mu \in \mathcal{M}_\lambda} \int \mathcal{L} - \lambda u \, d\mu = 0.
\]
We denote by \( S^-_c \) the set of constrained subsolutions.

The first conclusion shows the fine properties of the constrained subsolutions:

Theorem A.

1. \( u^- \in S^-_c \), which implies \( S^-_c \neq \emptyset \).
2. For each \( u \in S^-_c \), there exists an \( \Phi_{L,\lambda} \)-invariant subset \( \tilde{A}(u) \subset \tilde{A} \), such that \( u = u^- \) on \( \pi \tilde{A}(u) \).

Due to the intrinsic properties of the Lax-Oleinik semigroup, we can find smooth constrained subsolutions:

Theorem B. There exists a \( u \in C^{1,1}(M, \mathbb{R}) \cap S^-_c \) which is a solution on \( A \).

The smoothness of constrained subsolutions can be further improved, if additional hyperbolicity of \( \tilde{A} \) is supplied:
Theorem C. Assume \( \tilde{A} \) consists of finitely many hyperbolic equilibria or periodic orbits, then there exists a sequence \( \{u_i \in C^0(M, \mathbb{R})\}_{i \in \mathbb{N}} \subseteq \mathcal{S}_c^- \) converging to \( u^- \) as \( i \to +\infty \) w.r.t. the \( C^0 \)-norm, such that each \( u_i \) equals \( u^- \) on \( A \) and satisfies \( \lambda u_i(x) + H(x, du_i(x)) < 0 \) for \( x \notin A \).

1.3. Applications of constrained subsolutions. As the first application of constrained subsolutions, we show how to locate the maximal global attractor of (11) by using elements in \( \mathcal{S}_c^- \cap C^{1,1} \). We will see that the smoothness plays a crucial role.

Definition 1.7. \([18 \text{ page } 5]\) A compact \( \Phi_{H,\lambda}^t \)-invariant set \( \Omega \subset T^*M \) is called a global attractor of \( \Phi_{H,\lambda}^t \), if for any point \( (x, p) \in T^*M \) and any open neighborhood \( \mathcal{U} \) of \( \Omega \), there exists \( T(x, p, \mathcal{U}) > 0 \) such that for all \( t \geq T \), \( \Phi_{H,\lambda}^t(x, p) \in \mathcal{U} \). Moreover, if \( \Omega \) is not contained in any larger global attractor, then it is called a maximal global attractor.

Theorem D. For any initial point \( (x, p) \in T^*M \), the flow \( \Phi_{H,\lambda}^t(x, p) \) tends to a maximal global attractor \( \mathcal{K} \) as \( t \to +\infty \). Moreover, \( \mathcal{K} \) can be identified as the forward intersectional set of the following region:

\[
\Sigma_c^- := \bigcap_{u \in \mathcal{S}_c^- \cap C^{1,1}(M, R)} \{(x, p) \in T^*M | \lambda u(x) + H(x, p) \leq 0\},
\]

i.e.

\[
\mathcal{K} = \bigcap_{t \geq 0} \Phi_{H,\lambda}^t(\Sigma_c^-).
\]

Another application of the constrained subsolutions, we show how \( \mathcal{S}_c^- \) can be used to control the convergent speed of the Lax-Oleinik semigroup, with the hyperbolic assumptions:

Theorem E. Assume \( \tilde{A} \) consists of a unique hyperbolic equilibrium \( (x_0, 0) \in TM \) with \( \mu > 0 \) being the minimal positive eigenvalue, then there exists \( K > 0 \) which guarantees

\[
\|T_0^- 0(x) - u^-(x) + e^{-\lambda t} \alpha\| \leq K \exp\left(-\left(\mu + \lambda\right)t\right), \quad \forall t \geq 0.
\]

where \( T_0^- \) is the Lax-Oleinik semigroup operator (see (11) for the definition) and

\[
\alpha = \int_{-\infty}^0 e^{\lambda t} L(x_0, 0) dt = u^-(x_0)
\]

is a definite constant.

Corollary F. Assume \( \tilde{A} \) consists of a unique hyperbolic periodic orbit, then there exists a constant \( K > 0 \) and a constant \( \mu > 0 \) being the minimal Lyapunov exponent of the hyperbolic periodic orbit, such that

\[
\liminf_{t \to +\infty} \frac{\|T_0^- 0(x) - u^-(x) + e^{-\lambda t} \alpha\|}{\exp\left(-\left(\mu + \lambda\right)t\right)} \leq K
\]

with \( \alpha = u^-(x_0) \) for a fixed \( x_0 \in A \).

Remark 1.8. For any \( \psi \in C^0(M, \mathbb{R}) \), the convergent rate

\[
\|T_t^- \psi(x) - u^-(x)\| \sim O(e^{-\lambda t}), \quad t \geq 0
\]

has been easily proved in a bunch of references, e.g. \([10, 18]\). However, as \( \lambda \to 0^+ \), this inequality becomes ineffective in constraining the convergent speed of the Lax-Oleinik semigroup.

On the other side, for the case \( \lambda = 0 \) \([15, 21, 23]\) has shown the exponential convergence of the Lax-Oleinik semigroup, with the assumption that Aubry set consists of finitely many hyperbolic
equilibria. So we have chance to generalize the idea in \cite{15, 21} to $0 < \lambda \ll 1$, then prove Theorem E and Corollary F.

Notice that due to Theorem D of \cite{17}, for generic $H(x, p)$ we can guarantee the hyperbolic equilibrium (resp. periodic orbit with fixed homology class) of (1) with $\lambda = 0$ is unique, so the uniqueness of equilibrium (or periodic orbit) is not artificial. Nonetheless, we can still generalize Theorem E (resp. Corollary F) to several hyperbolic equilibriums (resp. periodic orbits), by replacing the constant $\alpha$ to a piecewise function

$$\alpha(x) = \int_{-\infty}^{0} e^{\lambda t} L(z_x, 0) dt, \quad x \in M.$$ 

where $z_x$ is an arbitrary point in the $\alpha$–limit set of the backward calibrated curve $\gamma_x^-$ ending with $x$. This $\alpha(x)$ is posteriorily decided by the equilibriums (or periodic orbits), which dominate the asymptotic behavior of backward calibrated curves.

1.4. Organization of the article. The paper is organized as follows: In Sec\[2\], we give a brief review of weak KAM theory for equation (1), then in Sec. \[3\] we give the proof of Theorem A, B and C. In Sec \[4\] we discuss the global attractor and prove Theorem D. In Sec \[5\] we discuss the convergent speed of the Lax-Oleinik semigroup and prove Theorem E and Corollary F. For the consistency of the proof, some technical conclusions are moved to the Appendix.

Acknowledgements: The first author is supported by National Natural Science Foundation of China (Grant No.11571166, No.11901293) and by Startup Foundation of Nanjing University of Science and Technology (No. AE89991/114). The second author is supported by the National Natural Science Foundation of China (Grant No. 11901560), and the Special Foundation of the Academy of Science (No. E0290103).

2. Weak KAM theory of discounted systems

In this section, we shall discuss some details about the weak KAM theory for the discounted Hamilton-Jacobi equation (1) and its relationship with viscosity solutions.

Lemma 2.1. $u \prec_{\lambda} L$ if and only if $u$ is a viscosity subsolution of (1).

Proof. Assume that $u : M \rightarrow \mathbb{R}$ is a viscosity subsolution of (1), then it’s Lipschitzian (Proposition 2.4 of \cite{10}), which is therefore differentiable almost everywhere. Suppose $x \in M$ is a differentiable point of $u$, for any $C^1$ curve $\gamma : [a, b] \rightarrow M$ with $\gamma(a) = x$, we can take the directional derivative by

$$\left. \frac{d}{dt} (e^{\lambda t} u(\gamma(t))) \right|_{t \rightarrow a^+} = e^{\lambda a} \langle du(x), \dot{\gamma}(a) \rangle + \lambda e^{\lambda a} u(x)$$

$$\leq e^{\lambda a} [L(x, \dot{\gamma}(a)) + H(x, du(x)) + \lambda u(x)]$$

$$\leq e^{\lambda a} L(x, \dot{\gamma}(a))$$

which implies $u \prec_{\lambda} L$ by using \cite{13} Proposition 4.2.3].
Conversely, $u \prec \lambda L$ implies that $u \in Lip(M, \mathbb{R})$, so for any differentiable point $x \in M$ of $u$ and $C^1$ curve $\gamma : [a, b] \rightarrow M$ with $\gamma(a) = x$,

$$\lim_{b \rightarrow a^+} \frac{1}{b-a} [e^{\lambda b}u(\gamma(b)) - e^{\lambda a}u(\gamma(a))] \leq \lim_{b \rightarrow a^+} \frac{1}{b-a} \int_a^b e^{\lambda s L(\gamma(s), \dot{\gamma}(s))} ds$$

which leads to

$$e^{\lambda a}(du(x), \dot{\gamma}(a)) + \lambda e^{\lambda a} u(x) \leq e^{\lambda a} L(x, \dot{\gamma}(a)).$$

By taking $\dot{\gamma}(a) = H_p(x, du(x))$, we get

$$L(x, \dot{\gamma}(a)) + H(x, du(x)) = \langle du(x), \dot{\gamma}(a) \rangle$$

which implies

$$\lambda u(x) + H(x, du(x)) \leq 0,$$

then $u : M \rightarrow \mathbb{R}$ is an almost everywhere subsolution, then has to be a viscosity subsolution. □

Let’s define the action function by

$$h^1_t(x, y) := \inf_{\gamma \in C^{ac}((0,t], M)} \int_0^t e^{\lambda s L(\gamma(s), \dot{\gamma}(s))} ds, \quad t \geq 0$$

of which the infimum $\gamma_{\min} : [a, b] \rightarrow M$ is always available and is $C^k$–smooth (by Weierstrass Theorem of [15]). Moreover, $\gamma_{\min}$ has to be a solution of the Euler-Lagrange equation

$$(E-L) \quad \frac{d}{dt} L_v(\gamma, \dot{\gamma}) + \lambda L_v(\gamma, \dot{\gamma}) = L_x(\gamma, \dot{\gamma}).$$

For any point $(x, v) \in TM$, we denote by $\Phi^t_{L,\lambda}(x, v)$ the Euler-Lagrange flow, which satisfies $\Phi^t_{L,\lambda} \circ L_H = L_H \circ \Phi^t_{H,\lambda}$ in the valid time domain of $\Phi^t_{H,\lambda}$.

The backward Lax-Oleinik semigroup operator $T^-_t : C^0(M, \mathbb{R}) \rightarrow C^0(M, \mathbb{R})$ can be expressed by

$$(11) \quad T^-_t \psi(x) := e^{-\lambda t} \min_{y \in M} \{ \psi(y) + h^1_t(y, x) \}$$

which works as a viscosity solution of the following evolutionary equation:

$$(12) \quad \begin{cases} \partial_t u(x, t) + H(x, \partial_x u) + \lambda u = 0, \\ u(\cdot, 0) = \psi(\cdot). \end{cases}$$

As $t \rightarrow +\infty$, $T^-_t \psi(x)$ converges to a unique limit function

$$(13) \quad u^-(x) := \lim_{t \rightarrow +\infty} T^-_t \psi(x) = \inf_{\gamma \in C^{ac}(-\infty,0], M)} \int_{-\infty}^0 e^{\lambda r L(\gamma, \dot{\gamma})} dr.$$ 

which is exactly viscous solution of (12) if we take $\alpha = 0$. Similarly, we can define the forward Lax-Oleinik semigroup operator $T^+_t : C^0(M, \mathbb{R}) \rightarrow C^0(M, \mathbb{R})$ by

$$(14) \quad T^+_t \psi(x) := \max_{y \in M} \{ e^{\lambda t} \psi(y) - h^1_t(x, y) \}$$

for later use.
Definition 2.2. A continuous function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is called semiconcave with linear modulus if there exists $C > 0$ such that

$$f(x+h) + f(x-h) - 2f(x) \leq C|h|^2$$

for all $x \in U, h \in \mathbb{R}^n$. Here $C$ is called a semiconcavity constant of $f$. Similarly we can define the semiconvex functions with linear modulus if we change $\leq$ to $\geq$ in (15).

Definition 2.3. Assume $u \in C(M, \mathbb{R})$, for any $x \in M$, the closed convex set

$$D^-u(x) = \left\{ p \in T^*_x M : \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}$$

(resp. $D^+ u(x)$

is called the sub-differential (resp. super-differential) set of $u$ at $x$.

Lemma 2.4. [6] Theorem 3.1.5] $f : U \subset \mathbb{R}^d \to \mathbb{R}$ is semiconcave (w.r.t. semiconvex), then $D^+f(x)$ (w.r.t. $D^-f(x)$) is a nonempty compact convex set for any $x \in U$.

Proposition 2.5.

1. $T_{t+s}^- = T_t^- \circ T_s^-;
2. \text{if } u \leq v, T_t^+ u \leq T_t^+ v;
3. u \prec \lambda L \text{ if and only if } u \leq T_t^- u, \text{ then } T_t^- u \prec \lambda L.
4. u \prec \lambda L \text{ if and only if } T_t^+ u \leq u, \text{ then } T_t^+ u \prec \lambda L.
5. For any $\psi \in C^0(M, \mathbb{R})$, $T_{t}^- \psi$ is semiconcave for $t > 0$. Similarly, $T_{t}^+ \psi$ is semiconvex for $t > 0$.

Proof. The idea is borrowed from [4], with necessary adoptions.

(1) For any $t, s > 0$, we have

$$T_{t+s}^- \psi(x) = e^{-\lambda(t+s)} \min_{y \in M} \left\{ \psi(y) + h_{\lambda}^{t+s}(y, x) \right\}$$

$$= e^{-\lambda t} \min_{y \in M} \{ \psi(y) + \min_{z \in M} \{ h_{\lambda}^t(y, z) + e^{\lambda s} h_{\lambda}^s(z, x) \} \}$$

$$= e^{-\lambda t} \min_{y \in M, z \in M} \{ e^{-\lambda s} \psi(y) + e^{-\lambda s} h_{\lambda}^s(y, z) + h_{\lambda}^s(z, x) \}$$

$$= e^{-\lambda t} \min_{z \in M} \{ e^{-\lambda s} \min_{y \in M} \{ \psi(y) + h_{\lambda}^s(y, z) \} + h_{\lambda}^s(z, x) \}$$

$$= T_{t}^- \psi(x) + h_{\lambda}^s(z, x)$$

$$= T_{t}^- \circ T_{s}^- \psi(x)$$

It is similar for $T_{s+t}^- = T_{s}^- \circ T_{t}^-$. (2) It is an immediate consequence of the definition of $T_{t}^- \text{ and } T_{t}^+$, i.e.

$$T_{t}^- u = e^{-\lambda t} \min_{y \in M} \{ u(y) + h_{\lambda}^t(y, x) \} \leq e^{-\lambda t} \min_{y \in M} \{ v(y) + h_{\lambda}^t(y, x) \} = T_{t}^- u.$$

(3) On one hand, if $u \leq T_{t}^- u$, according to the definition of $T_{t}^-$, we have that

$$u(x) \leq T_{t}^- u(x) = \inf_{y \in M} \left\{ e^{-\lambda t} u(y) + e^{-\lambda t} h_{\lambda}^t(y, x) \right\}$$

which means that $e^{\lambda t} u(x) - u(y) \leq h_{\lambda}^t(y, x)$ for any $x, y \in M$. Therefore, $u \prec \lambda L$. 

On the other hand, if \( u \prec \lambda L \), we have that \( u(x) \leq e^{-\lambda t}u(y) + e^{-\lambda t}h_{\lambda}^{h}(y, x) \) for any \( x, y \in M \) which implies that \( u \leq T_t u \) by taking the infimum of \( y \). In summary, for every \( t' > 0 \), one obtains

\[
T_{t'} u - T_t u \leq T_{t'} T_{t'} u - T_t u = T_{t'} u
\]

which implies that \( T_{t'} u \prec \lambda L \).

(4) Similar as above, \( T_{t'}^{+} u \leq u \) if and only if \( u \prec \lambda L \). Hence, for every \( t' > 0 \),

\[
T_{t'}^{+} u \geq T_{t'}^{+} \left[ T_{t'}^{+} u \right] = T_{t'}^{+} u = T_{t'}^{+} u
\]

which implies that \( T_{t'}^{+} u \prec \lambda L \).

(5) As is shown in [13, Proposition 6.2.1] or [6], \( h_{\lambda}^{h}(x, y) \) is semiconcave w.r.t. \( x \) (resp. \( y \)), since \( M \) is compact. For any fixed \( t > 0 \) and \( y \in M \), \( h_{\lambda}^{h}(y, \cdot) \) and \( h_{\lambda}^{h}(\cdot, y) \) are both semiconcave, then \( \psi(y) + h_{\lambda}^{h}(y, \cdot) \) is semiconcave and \( e^{\lambda t}(\psi(y) - h_{\lambda}^{h}(\cdot, y)) \) is semiconcave. Due to [6] Proposition 2.1.5, \( \min_{y \in M} (\psi(y) + h_{\lambda}^{h}(y, x)) \) preserves the semiconcavity, so \( T_{t} \psi(x) \) is also semiconcave. Similar proof implies \( T_{t} \psi(x) \) is semiconcave.

Due to [13], the following properties of \( u^{-} \) can be easily proved:

**Proposition 2.6.** [18] Proposition 5.7

- \( u^{-} \) is Lipschitz on \( M \), with the Lipschitz constant depending only on \( L \).
- \( u^{-} \prec \lambda L \).
- For every \( x \in M \), there is a backward calibrated curve \( \gamma_{t}^{-} : (-\infty, 0] \rightarrow M \) which achieves the minimum of \( (13) \).
- For any \( t < 0 \),

\[
u^{-}(x) = e^{\lambda t}u^{-}(\gamma_{t}^{-}(t)) + \int_{t}^{0} e^{\lambda s}L(\gamma_{s}^{-}(s), \gamma_{s}^{-}(s))ds,
\]

and there is a uniform upper bound \( K \) depending only on \( L \) such that \( |\gamma_{t}^{-}| \leq K \).
- For every \( t < 0 \), \( u^{-} \) is differentiable at \( \gamma_{t}^{-}(t) \) and

\[
\lambda u^{-}(\gamma_{t}^{-}(t)) + H(\gamma_{t}^{-}(t), du^{-}(\gamma_{t}^{-}(t))) = 0.
\]

Any continuous function simultaneously satisfies bullets 2 and 3 are called a backward weak KAM solution of \( (13) \).

3. Proof of Theorem A, B and C

3.1. Proof of Theorem A. (1) For any \( \nu \in \mathfrak{M}_{\lambda} \) and any \( u \in \mathcal{S}^{-} \),

\[
\int_{T_{M}} \lambda u d\nu \leq \int_{T_{M}} \lambda u^{-} d\nu = \lambda \int_{T_{M}} \int_{-\infty}^{0} e^{\lambda s}L(\Phi_{L, \lambda}^{s}(x, \nu))dsd\nu
\]

\[
= \lambda \int_{-\infty}^{0} e^{\lambda s} \left( \int_{T_{M}} L(\Phi_{L, \lambda}^{s}(x, \nu), \nu)ds \right) d\nu
\]

\[
= \lambda \int_{-\infty}^{0} e^{\lambda s} \left( \int_{T_{M}} L(x, \nu)d\nu \right) ds
\]

\[
= \lambda \int_{-\infty}^{0} e^{\lambda s} ds \cdot \int_{T_{M}} L(x, \nu)d\nu = \int_{T_{M}} L(x, \nu)d\nu,
\]

(16)
which implies
\[
\int_{\mathcal{TM}} L(x, v) - \lambda u \, d\nu \geq 0, \quad \forall \nu \in \mathfrak{M}_\lambda.
\]
Moreover, for any ergodic measure \(\mu \in \mathfrak{M}_\lambda\) supported in \(\widetilde{\mathcal{A}}\), for every \((x, v) \in \text{supp } \mu\),
\[
\int_{\mathcal{TM}} \lambda u^- \, d\mu = \lambda \int_{\mathcal{TM}} \int_{-\infty}^0 e^{\lambda s} L(\Phi_{L, \lambda}^s(x, v)) \, ds \, d\mu
\]
\[= \lambda \int_{-\infty}^0 e^{\lambda s} \int_{\mathcal{TM}} L(\Phi_{L, \lambda}^s(x, v)) \, d\mu \, ds
\]
\[= \lambda \int_{-\infty}^0 e^{\lambda s} \int_{\mathcal{TM}} L(x, v) \, d\mu \, ds
\]
\[= \lambda \int_{-\infty}^0 e^{\lambda s} ds \cdot \int_{\mathcal{TM}} L(x, v) \, d\mu = \int_{\mathcal{TM}} L(x, v) \, d\mu.
\]
So \(u^- \in \mathcal{S}_c^-\).

(2). For \(u \in \mathcal{S}_c^\ast\), if there is a \(\mu \in \mathfrak{M}_\lambda\) such that each step in (10) becomes an equality, then for \(\mu-\text{a.e. } (x, v) \in \mathcal{TM}\), we have \(u = u^-\). Therefore, for a.e. \((x, v) \in \text{supp}(\mu)\),
\[
u(t) = \inf_{\gamma(0) = x} \int_{-\infty}^0 e^{\lambda s} L(\gamma(s), \dot{\gamma}(s)) ds
\]
\[= \int_{-\infty}^0 e^{\lambda s} L(\Phi_{L, \lambda}^s(x, v)) ds
\]

thus \(\pi \Phi_{L, \lambda}^t(x, v)\) is a \(u^-\)-calibrated curve for \(t \in (-\infty, 0]\). Since \(\mu\) is invariant, \(\pi \Phi_{L, \lambda}^t(x, v)\) is globally calibrated, i.e. \(\Phi_{L, \lambda}^t(x, v) \in \mathcal{A}\). So \(\mathcal{A}(u) := \text{supp}(\mu) \subset \mathcal{A}\). □

3.2. Proof of Theorem B. Due to Proposition [2.5] as long as \(t > 0\) and \(s > 0\) sufficiently small, \(\mathcal{T}_s \mathcal{T}_t^+ u^-(x)\) is a subsolution of (13). Another thing is that \(u^- (x) = \mathcal{T}_t^+ u^-(x) = \mathcal{T}_s \mathcal{T}_t^+ u^-(x)\) for any \(x \in \mathcal{A}\). This is because \(u^- \in \mathcal{S}_c^\ast\), by Lemma [2.1], we have that \(u^- \preceq L\) which implies that \(\mathcal{T}_t^\pm u^- \geq u^-\) and \(\mathcal{T}_t^\pm u^- \leq u^-\) due to Proposition [2.3]. For \(x \in \mathcal{A}\), by the Definition [1.3], there exist a curve \(\gamma_x^- : [0, t] \to M\) such that \(e^\lambda u^- (\gamma_x^-(t)) - u^- (\gamma_x^-(0)) = \int_0^t e^{\lambda s} L(\gamma_x^-(s), \dot{\gamma}_x^-(s)) ds\) with \(u^- (\gamma_x^-(0)) = x\). Hence,
\[
u(t) = \mathcal{T}_t^+ u^-(x) = \sup_{\gamma \in C^\omega([0, t], M)} e^{\lambda t} u^- (\gamma(t)) - \int_0^t e^{\lambda s} L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau.
\]
\[\leq e^{\lambda t} u^- (\gamma_x^-(0)) + \int_0^t e^{\lambda s} L(\gamma_x^-(s), \dot{\gamma}_x^-(s)) d\tau
\]
\[= u^- (\gamma_x^-(0)) = u^- (x).
\]
which implies that \(u^- (x) = \mathcal{T}_t^+ u^- (x)\) for any \(x \in \mathcal{A}\). Similarly we can prove \(u^- (x) = \mathcal{T}_t^- u^- (x)\). So \(\mathcal{T}_s \mathcal{T}_t^\pm u^-\) is indeed a solution on \(\mathcal{A}\). Recall that \(\mathcal{T}_t^\pm u^- (x)\) is always semiconvex, and for sufficiently small \(s > 0\), it is proven in following Lemma [3.1] that \(\mathcal{T}_s^\pm \psi\) keeps the semiconvexity for any semiconvex function \(\psi(x)\), then \(\mathcal{T}_s \mathcal{T}_t^\pm u^- (x)\) is both semiconcave and semiconvex, thus has to be \(C^{1,1}\). So we finish the proof. □

Lemma 3.1. Assume \(H \in C^k(T^* M, \mathbb{R})\) is a Tonelli Hamiltonian, for each semiconvex function \(\psi : M \to \mathbb{R}\) with a linear modulus, there is \(t_0 > 0\) such that \(\mathcal{T}_s^{-t} \psi\) is semiconvex for \(t \in [0, t_0]\).
Proof. We follow the proof of [4] Lemma 4 to prove that, there exists \( t_0 > 0 \) such that, for \( t \in [0, t_0] \), \( T^t_\psi \) is supreme of a family of \( C^2 \) functions with a uniform \( C^2 \)-bound. Then semiconvexity of \( T^-_t \psi \) is a direct corollary of that.

Since \( \psi \) is semiconvex with a linear modulus, by [4] Proposition 10 or [6] Theorem 3.4.2, there exists a bounded subset \( \Psi \subset C^2(M, \mathbb{R}) \) such that

1. \( \psi = \max_{\varphi \in \Psi} \varphi \),
2. for each \( x \in M \) and \( p \in D^-\psi(x) \), there exists a function \( \varphi \in \Psi \) satisfying \((\varphi(x), d\varphi(x)) = (\psi(x), p)\).

By the definition of \( T^-_t \) and (1), we have

\[
T^-_t \psi \geq \sup_{\varphi \in \Psi} T^-_t \varphi.
\]

On the other hand, for the family \( \Psi \), there exists \( t_0 > 0 \) such that, for each \( t \in [0, t_0] \), the image \( T^-_t (\Psi) \) is also a bounded subset of \( C^2(M, \mathbb{R}) \) and for all \( \varphi \in \Psi \) and \( x \in M \),

\[
T^-_t \varphi(\gamma(t)) = e^{-\lambda t} \varphi(x) + \int_0^t e^{\lambda(\tau-t)} L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau
\]

where \( \gamma(t) = \pi \circ \Phi_{H,\lambda}^t (x, d\varphi(x)) \).

Let \( (\gamma(t), p(t)) : [0, t] \to T^*M \) be a trajectory of (11) which is optimal for \( h^1_\lambda(y, x) \), i.e., \( \gamma(0) = y, \gamma(t) = x \) and

\[
h^1_\lambda(y, x) = \int_0^t e^{\lambda \tau} L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau.
\]

It is not difficult to see that \( p(0) \) is a super-differential of the function \( z \mapsto h^1_\lambda(z, x) \) at \( y \). Since the function \( z \mapsto e^{-\lambda t}[\psi(z) + h^1_\lambda(z, x)] \) is minimal at \( y \), then \( p(0) \in D^-\psi(y) \).

We consider a function \( \varphi \in \Psi \) such that \((\varphi(y), d\varphi(y)) = (\psi(y), p(0))\), then we have \((\gamma(t), p(t)) = \Phi_{H,\lambda}^t (y, d\varphi(y)) \) and

\[
T^-_t \varphi(x) = e^{-\lambda t} \varphi(y) + \int_0^t e^{\lambda(\tau-t)} L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau = e^{-\lambda t}[\psi(y) + h^1_\lambda(y, x)] = T^-_t \psi(x).
\]

Thus for each \( x \in M \), there exists a function \( \varphi \in \Psi \) such that \( T^-_t \varphi(x) = T^-_t \psi(x) \), therefore

\[
T^-_t \psi \leq \sup_{\varphi \in \Psi} T^-_t \varphi.
\]

So we complete the proof. \(\square\)

Remark 3.2. Actually, obtaining \( C^{1,1} \)–functions via the forward Lax-Oleinik semigroup and backward semigroup is a typical application of the Lasry-Lions regularization method. For the discounted Hamilton-Jacobi equation, the readers can refer to [9] for more applications of this method.

3.3. Proof of Theorem C. To prove Theorem C, the following Lemma is needed:

Lemma 3.3 (\( C^k \) graph). Assume \( \bar{A} \) consists of finitely many hyperbolic equilibrium or periodic orbits, then there exists a neighborhood \( V \supset A \), such that for all \( x \in V \), \((x, du^- (x)) \) lies exactly on the local unstable manifold \( W^u_{loc}(\bar{A}) \) (which is actually \( C^{k-1} \)–graphic).
Proof. We claim that:

For any neighborhood $V$ of $\mathcal{A}$, there always exists an open neighborhood $U \subset V$ containing $\mathcal{A}$, such that for any $x \in U$, the associated backward calibrated curve $\gamma_x^-(\lambda) : (-\infty, 0] \to M$ would lie in $V$ for all $t \in (-\infty, 0]$.

Otherwise, there exists a $V_\ast$ neighborhood of $\mathcal{A}$ and a sequence $\{x_n \in V_\ast\}_{n \in \mathbb{N}}$ converging to some point $z \in \mathcal{A}$, such that the associated backward calibrated curve $\gamma_n^-$ ending with $x_n$ can go outside $V_\ast$ for all $n \in \mathbb{N}$. Namely, we can find a sequence $\{T_n \geq 0\}_{n \in \mathbb{N}}$ such that $\gamma_n^-(T_n) \in \partial V_\ast$. Due to item 2 of Proposition 2.6 any accumulating curve $\gamma_\infty$ of the sequence $\{\gamma_n^-.\}_{n \in \mathbb{N}}$ is also a calibrated curve in 2 cases:

Case 1: the accumulating value $T_\infty$ of associated $\gamma_\infty^-$ is finite, which implies $\gamma_\infty^- : [-T_\infty, 0] \to M$ connecting $z$ and $\partial V_\ast$. Since $z \in \mathcal{A}$ and $\mathcal{A}$ is $\Phi_{1,\lambda}^+$ invariant, then $\gamma_\infty^- : \mathbb{R} \to M$ is contained in $\mathcal{A}$ as well. That’s a contradiction.

Case 2: the accumulating value $T_\infty$ of associated $\gamma_\infty^-$ is infinite, then $\eta_\infty^- (t) := \gamma_n^- (t - T_n) : (-\infty, T_n] \to M$ accumulates to a $\eta_\infty^- : \mathbb{R} \to M$ which is globally calibrated by $u^-$. Due to the definition of $\mathcal{A}$, $\eta_\infty^-$ has to be contained in $\mathcal{A}$. That’s a contradiction.

After all, the claim holds. Since $W^n_{\text{loc}}(\mathcal{A})$ has to be $C^{k-1}$-graphic in a suitable neighborhood of $\mathcal{A}$ (Proposition B of [14]), then our claim actually indicates there exists a suitable $V \supset \mathcal{A}$, such that for all $x \in V$, the backward calibrated curve $\gamma_x^- : (-\infty, 0] \to M$ is unique and $(x, du^-(x)) \in W^n_{\text{loc}}(\mathcal{A})$. \qed

Due to Lemma 3.3, $u^-$ has to be a $C^k$ graph in a small neighborhood $U$ of $\mathcal{A}(H)$ and $(x, du^-(x)) \in W^n(\mathcal{A}(H))$ for all $x \in U$. Notice that there exists a nonnegative, $C^\infty$-smooth function $V : M \to \mathbb{R}$ which is zero on $\mathcal{A}(H)$ and keeps positive outside $\mathcal{A}(H)$. Moreover, $\|V\|_{C^k}$ can be taken sufficiently small, so for the new Hamiltonian

$$\tilde{H}(x, p) := H(x, p) + V(x),$$

the hyperbolicity of $\mathcal{A}(\tilde{H})$ persists and $\mathcal{A}(\tilde{H}) = \mathcal{A}(H)$ due to the upper semicontinuity of the Aubry set (see Lemma 3.3). So if we denote by $\tilde{u}^-$ the viscosity solution of $\tilde{H}$, $\tilde{u}^-$ is also $C^k$ on $U$. We can easily see that $\tilde{u}^-$ is a strict subsolution of $H$ in $U \setminus \mathcal{A}(H)$. Outside $U$ we can convolute $\tilde{u}^-$ with a $C^\infty$ function, and keeps $\tilde{u}^-$ invariant on $U$. Without loss of generality, let us denote by $\hat{u}^-$ the modified function, then for any $x \notin U$ being a differentiable point of $\tilde{u}^-$, we have

$$\lambda \hat{u}^- (x) + H(x, d\hat{u}^-(x)) \leq \lambda \tilde{u}^- (x) + \tilde{H}(x, d\tilde{u}^- (x)) - V(x)$$

$$\leq \lambda \tilde{u}^- (x) + \tilde{H}(x, d\tilde{u}^- (x)) - V(x) + \lambda |\tilde{u}^- (x) - \tilde{u}^- (x)|$$

$$+ \max_{\theta \in [0, 1]} |H_p \left( x, \theta d\tilde{u}^- (x) + (1 - \theta) d\tilde{u}^- (x) \right) | \cdot |d\tilde{u}^- (x) - d\tilde{u}^- (x)|$$

$$\leq - V(x) + C \cdot |\hat{u}^- (x) - \tilde{u}^- (x)| + |d\tilde{u}^- (x) - d\tilde{u}^- (x)|$$

$$\leq - V(x)/2 < 0,$$

since $|\tilde{u}^- (x) - \tilde{u}^- (x)|$ and $|d\tilde{u}^- (x) - d\tilde{u}^- (x)|$ can be made sufficiently small. Recall that $\tilde{u}^-. |_{\partial U} = \tilde{u}^- |_{\partial U}$, so $\tilde{u}^-$ is a $C^k$ smooth constrained subsolution of [12] which is a solution on $\mathcal{A}(H)$ and strict subsolution outside. \qed
4. Global attractors and the proof of Theorem D

Another usage of $S_c^-$ is to identify the maximal global attractor. Due to Theorem B, $S_c^- \cap C^1(M, \mathbb{R})$ is nonempty. Moreover, by sending $s, t \to 0_+$ in the Lax-Oleinik operators $T_s^-$ and $T_t^+$ as in the proof of Theorem B, we can find a sequence of $C^1$ constrained subsolutions converging to $u^-$ w.r.t. the $C^0$-norm.

Proof of Theorem D: Due to Theorem 1.5 for any $u \in S_c^- \cap C^1(M, \mathbb{R})$, we have $u \leq u^-$. Therefore,
\[ \{(x, p) \in T^*M | \lambda u(x) + H(x, p) \leq 0\} \supset \{(x, p) \in T^*M | \lambda u^-(x) + H(x, p) \leq 0\}, \]
which accordingly indicates
\[ \Sigma_c^- = \{(x, p) \in T^*M | \lambda u^-(x) + H(x, p) \leq 0\}. \]
On the other side, let us denote
\[ F_u(x, p) := \lambda u(x) + H(x, p), \]
then we can prove that
\[
\frac{d}{dt} \bigg|_{t=0} F_u(\Phi_{H, \lambda}(x, p)) = \lambda(du(x), \dot{x}) + H_x(x, p)\dot{x} + H_p(x, p)p \\
= \lambda(du(x), \dot{x}) + H_x(x, p)H_p(x, p) + H_p(x, p)(-H_x(x, p) - \lambda p) \\
= \lambda(du(x), \dot{x}) - \lambda(\dot{x}, p) \\
\leq \lambda(H(x, du(x)) + L(x, \dot{x})) - \lambda(\dot{x}, p) \\
= \lambda[H(x, du(x)) + (\dot{x}, p) - H(x, p)] - \lambda(\dot{x}, p) \\
= -\lambda[H(x, p) - H(x, du(x))] \\
\leq -\lambda[\lambda u(x) + H(x, p)] \\
= -\lambda F_u(x, p)
\]
where the second equality is according to equation (1) and the first inequality is due to Fenchel transform. It implies that every trajectory of (1) tends to $\Sigma_c^-$ as in the proof of Theorem B, we can find a sequence of $C^1$ constrained subsolutions converging to $u^-$ w.r.t. the $C^0$-norm.

Remark 4.1. A similar conclusion as Theorem D was firstly proved by Maro and Sorrentino in [15], where they used a complicated real analysis method to handle with the low regularity of $u^-$. For each $u$ contained in $S^-_c \cap C^1(M, \mathbb{R})$, we can take the derivative of $u$ directly and avoid this difficulty.
5. Exponential convergence of the Lax-Oleinik semigroup

Proof of Theorem E: Recall that
\[ T_t^{-} 0(x) = \inf_{\gamma: [-t,0] \to M} \int_{-t}^{0} e^{\lambda s} L(\gamma, \dot{\gamma}) ds, \]
then we have
\[ u^-(x) - T_t^{-} 0(x) \geq \int_{-\infty}^{-t} e^{\lambda s} L(\gamma^x, \dot{\gamma}_x) ds \]
where \( \gamma^x : (-\infty, 0] \to M \) is the backward calibrated curve by \( u^- \) and ending with \( x \). On the other side, suppose \( \hat{\gamma} : [0,t] \to M \) is the infimum achieving \( T_t^{-} 0(x) \), then
\[ u^-(x) - T_t^{-} 0(x) \leq \int_{-\infty}^{-t} e^{\lambda s} L(\eta, \dot{\eta}) ds + \int_{-t}^{0} e^{\lambda s} L(\hat{\gamma}, \hat{\dot{\gamma}}) ds - T_t^{-} 0(x) \]
\[ = \int_{-\infty}^{-t} e^{\lambda s} L(\eta, \dot{\eta}) ds \]
where \( \eta : (-\infty, -t] \to M \) is the backward calibrated curve by \( u^- \) and ending with \( \hat{\gamma}(-t) \).
Recall that \( \mathcal{A} \) consists of a unique hyperbolic equilibrium, without loss of generality, we assume \( \mathcal{A} = \{ z = (x_0,0) \} \). Combining (18) and (19) we get
\[ |u^-(x) - T_t^{-} 0(x) - e^{-\lambda t} \alpha| \leq \max \left\{ \left| \int_{-\infty}^{-t} e^{\lambda s} L(\eta, \dot{\eta}) ds - e^{-\lambda t} \alpha \right|, \left| \int_{-\infty}^{-t} e^{\lambda s} L(\gamma^x, \dot{\gamma}_x) ds - e^{-\lambda t} \alpha \right| \right\} \]
with
\[ \alpha := \int_{-\infty}^{0} e^{\lambda t} L(x_0,0) dt = u^-(x_0). \]
On the other side, we can claim the following conclusion:

Claim: For any neighborhood \( V \supset \mathcal{A} \), there exists a uniform time \( T_V > 0 \), such that for any \( x \in M \), the associated backward calibrated curve \( \gamma^x : (-\infty, 0] \to M \) will not stay outside \( V \) for a time longer than \( T_V \).

Otherwise, there must be a neighborhood \( V_* \supset \mathcal{A} \) and a sequence \( \{ x_n \} \subset M \), such that the associated backward calibrated curve \( \gamma_n \) would stay outside of \( V_* \) for a time \( T_n \), with \( T_n \to +\infty \) as \( n \to +\infty \). With almost the same analysis as in Lemma 3.3, we can show that any accumulating curve of \( \{ \gamma_n \} \) would lie outside \( V_* \) for infinitely long time, which implies \( V_* \cap \mathcal{A} \neq \emptyset \). This contradiction lead to the claim.

Now we choose a suitably small neighborhood \( \tilde{U} \supset \mathcal{A} \), such that both the Hartman Theorem is available in \( \tilde{U} \) and \( W^u(\mathcal{A}) \cap \tilde{U} \) is \( C^{k-1} \)-graphic. Due to Lemma 3.3 there exists a constant \( K_1 > 0 \), such that for any \( x \in U := \pi \tilde{U} \), the associated backward calibrated curve \( \gamma^x : (-\infty, 0] \to M \) can be estimated by the following:
\[ \| \gamma^x(-t) - z \| \leq K_1 \exp(-\mu t), \quad \forall \ t \geq 0 \]
with \( \mu > 0 \) being the largest negative Lyapunov exponent of the hyperbolic equilibrium. Due to our claim and Lemma 5.3 there exists a constant \( K_2 \geq K_1 \), such that for any \( x \in M \), the associated backward calibrated curve \( \gamma^x : (-\infty, 0] \to M \) satisfies
\[ \| \gamma^x(-t) - z \| \leq K_2 \exp(-\mu t), \quad \forall \ t \geq 0. \]
Due to Theorem C, we can find a sequence of $C^k$ subsolutions $\{u_n \in S_n^-\}_{n \in \mathbb{N}}$ approaching to $u^-$ w.r.t. $C^0$-norm. Then for any $x \in M$, and the associated backward calibrated curve (with a time shift) $\eta_x^\ast : (-\infty, -t) \to M$ ending with it, we have
\[
\left| \int_{-\infty}^{-t} e^{\lambda s} L(\eta_x^\ast, \eta_x^\ast) ds - e^{-\lambda t} \alpha \right| = \lim_{n \to +\infty} \left| \int_{-\infty}^{-t} \frac{d}{ds} \left( e^{\lambda s} u_n(\eta_x^\ast(s)) - e^{\lambda s} u_n(x_0) \right) ds \right|
= \lim_{n \to +\infty} \left| e^{-\lambda t} (u_n(\eta_x^\ast(-t)) - u_n(x_0)) \right|
= \lim_{n \to +\infty} e^{-\lambda t} \left| u_n(\eta_x^\ast(-t)) - u_n(x_0) \right|
\leq \lim_{n \to +\infty} e^{-\lambda t} \left| \|u_n\| \cdot \|\eta_x^\ast(-t) - x_0\| \right|
\leq \lim_{n \to +\infty} e^{-\lambda t} \left| \|u_n\| \cdot K_2 \exp(-\mu t) \right|
\leq C \cdot K_2 \cdot \exp(-\mu + \lambda t)
\]
due to the uniform semiconcavity of $\{u_n\}_{n \in \mathbb{N}}$. We can apply this inequality to both (18) and (19), then prove this Theorem. \hfill \square

**Proof of Corollary F:** We can totally borrow previous analysis, with only the following adaptation: since now $\tilde{A}$ is a periodic orbit $\{(\gamma_{p}(t), \dot{\gamma}_{p}(t))| t \in [0, T_p]\}$ which may no longer be an equilibrium, so we can just assume $z \in \tilde{A}$ being one point such that $u^-(z) = 0$. Therefore, we can only guarantee the existence of a constant $K_3 > 0$, such that for any $x \in M$, the associated backward calibrated curve $\gamma_x^\ast : (-\infty, 0) \to M$ satisfies
\[
\liminf_{t \to +\infty} \frac{\|\gamma_x^\ast(-t) - z\|}{\exp(-\mu t)} \leq K_3, \quad \forall \, t \geq 0.
\]
That’s different from (20), but other parts of the analysis follows. \hfill \square

**Appendix A. Proof of Theorem 1.5**

**Lemma A.1.** [10] Lemma 2.2] Assume $G \in C(T^* M, \mathbb{R})$ is fiberwise convex in $p$ and $v$ is a Lipschitz subsolution of equation
\[
G(x, d_x v) = 0, \quad x \in M,
\]
then for any $\varepsilon > 0$, there exists $v_\varepsilon \in C^\infty(M, \mathbb{R})$ such that $\|v - v_\varepsilon\|_{C^0} \leq \varepsilon$ and $H(x, d_x v) \leq \varepsilon$ for all $x \in M$.

**Lemma A.2.** [10] Theorem 2.5] Assume $H \in C(T^* M, \mathbb{R})$ such that $H(x, p) \to \infty$ as $|p|_x \to \infty$ uniformly in $x \in M$. If $v, u$ are, respectively, a sub and supersolution of (12), then $v \leq u$ in $M$.

**Proof of Theorem 1.5** Since $u^-$ is a supersolution of (12), based on Lemma A.2 we obtain that
\[
u^-(x) \geq \sup_{u \in S^-} u(x) \geq \sup_{u \in C^\infty(M, \mathbb{R}) \cap S^-} u(x).
\]
Since $u^-$ is also a subsolution, we have that $u^-(x) \leq \sup_{u \in S^-} u(x)$, then

$$u^-(x) = \sup_{u \in S^-} u(x).$$

To prove $u^-(x) = \sup_{u \in \mathcal{C}_1^\infty(M, \mathbb{R}) \cap S^-} u(x)$, it suffices, for any $\delta > 0$, to construct a subsolution $U_{\delta} \in \mathcal{C}_1^\infty(M, \mathbb{R})$ of (13) such that

$$u^-(x) - \delta \leq U_{\delta}(x) \leq u^-(x) \quad \text{on} \quad M.$$

Set $\varepsilon = \frac{\lambda}{1 + 2\lambda} \cdot \delta$ and $v(x) = u^-(x) - \frac{\varepsilon}{1 + 2\lambda} \cdot \delta,$

$$G(x, p) = \lambda \cdot u^-(x) + H(x, p),$$

then $v$ is a subsolution of $G(x, d_x u) = 0$. Due to Lemma A.1, we obtain $v_{\varepsilon} \in \mathcal{C}_1^\infty(M, \mathbb{R})$ and set $U_{\delta} = v_{\varepsilon}$, thus

$$u^-(x) - \delta = v(x) - \varepsilon \leq U_{\delta}(x) \leq v(x) + \varepsilon = u^-(x) - \frac{1}{1 + 2\lambda} \cdot \delta < u^-(x),$$

and

$$\lambda U_{\delta}(x) + H(x, dU_{\delta}) = \lambda[U_{\delta}(x) - u^-(x)] + G(x, dU_{\delta})$$

$$= \lambda[v_{\varepsilon}(x) - u^-(x)] + G(x, d\varepsilon) \leq -\lambda \cdot \frac{1}{1 + 2\lambda} \cdot \delta + \varepsilon = 0.$$

Since $\delta > 0$ is arbitrarily chosen, we proved the conclusion. \qed

**Appendix B. More about the Aubry set**

In [18], the Aubry set is defined by

$$\bar{\mathcal{A}} = \bigcup_{\gamma} \{(\gamma, \dot{\gamma})| \gamma \text{ is globally calibrated by } u^-\} \subset TM.$$

To show this definition is equivalent to (5), it suffices to prove any globally calibrated curve $\gamma$ by $u^-$ minimizes $h_{\lambda}^{b-a}(\gamma(a), \gamma(b))$ for all $a < b \in \mathbb{R}$. For this purpose, we can assume that $\text{diam}(M) = 1$, then make use of the $\mathcal{C}^0$-convergence of the Lax-Oleinik semigroup.

**Definition B.1.** $\gamma \in \mathcal{C}_1^{ac}(\mathbb{R}, M)$ is called a global minimizer, if for any $a < b \in \mathbb{R},$

$$e^{\lambda x} h_{\lambda}^{b-a}(\gamma(a), \gamma(b)) = \int_a^b e^{\lambda t} L(\gamma(t), \dot{\gamma}(t))dt$$

where $L : TM \to \mathbb{R}$ is the Tonelli Lagrangian associated with $H(x, p)$, see (5). Due to the Weierstrass Theorem in [19], any global minimizer $\gamma : \mathbb{R} \to M$ has to satisfy (E-L), and has to be as smooth as $H(x, p)$.

Recall that

$$\mathcal{T}_{\gamma}^{-0}(x) = \inf_{\xi \in \mathcal{C}_1^{ac}([0, t], M)} \int_{t}^0 e^{\lambda s} L(\xi(s), \dot{\xi}(s))ds$$

and a function $F_{\gamma}(t) : [0, +\infty) \to \mathbb{R}$ by

$$F_{\gamma}(t) := \int_{-t}^0 e^{\lambda s} L(\gamma(s), \dot{\gamma}(s))ds - \mathcal{T}_{\gamma}^{-0}(\gamma(0)).$$
associated to an arbitrary global minimizer $\gamma : \mathbb{R} \to M$. Apparently we have $\lim_{t \to \infty} \mathcal{T}_t^{-0}(x) = u^-(x)$, and $F_\gamma(t)$ is nonnegative. If we set
\[ C_0 := \sup \{ |L(x, v)| : |v|_x \leq 1 \} < \infty, \]
then we get the following:

**Lemma B.2.** For any global minimizer $\gamma : \mathbb{R} \to M$,
\[ \lim_{t \to \infty} F_\gamma(t) = 0. \]

*Proof.* For any $\epsilon > 0$, since $\lim_{t \to \infty} \mathcal{T}_t^{-0}(x) = u^-(x)$, there is $T_1 > 0$ such that $\forall t_0 \geq t_1 \geq T_1$,
\[ |\mathcal{T}_{t_0}^{-0}(x) - \mathcal{T}_{t_1}^{-0}(x)| \leq \frac{\epsilon}{2}. \]
Let’s set
\[ T = T(\epsilon) := \max \{ T_1, \frac{1}{\lambda} \ln \left( \frac{2C_0}{\lambda \epsilon} \right) \}. \]
By compactness of $M$, there is a $C^k$ curve $\eta_1 : s \in [-T, 0] \to M$ attains the infimum in the definition of $h(x, T)$, i.e., $\eta_1(s)|_{s=0} = \gamma(s)|_{s=0} = x$ and
\[ \mathcal{T}_t^{-0}(x) = \int_{-T}^0 e^{\lambda s} L(\eta_1(s), \dot{\eta}_1(s))ds. \]
For $t \geq T + 1$, there is a geodesic $\eta_2 : [-t, -T] \to M$ satisfying
\[ \eta_2(-t) = \gamma(-t), \quad \eta_2(-T) = \eta_1(-T), \]
and $\|\eta_2(-t)\|_1 \leq 1$ since $\text{diam}(M) \leq 1$. We consider the concatenated curve $\eta : [-t, 0] \to M$ by $\eta = \eta_2 \ast \eta_1$, then $\eta : [-t, 0] \to M$ with $\eta(-t) = \gamma(-t)$, $\eta(0) = \gamma(0)$. Since $\gamma$ is a global minimizer, we obtain that
\[
\int_{-t}^{0} e^{\lambda s} L(\gamma(s), \dot{\gamma}(s))ds
\leq \int_{-t}^{0} e^{\lambda s} L(\eta(s), \dot{\eta}(s))ds
= \int_{-T}^{0} e^{\lambda s} L(\eta_2(s), \dot{\eta}_2(s))ds + \int_{-T}^{0} e^{\lambda s} L(\eta_1(s), \dot{\eta}_1(s))ds
= \int_{-T}^{0} e^{\lambda s} L(\eta_2(s), \dot{\eta}_2(s))ds + \mathcal{T}_t^{-0}(0(0))
\leq \frac{C_0}{\lambda} e^{-\lambda T} + \mathcal{T}_T^{-0}(0(0)) \leq \frac{\epsilon}{2} + \mathcal{T}_t^{-0}(0(0)) \leq \epsilon + \mathcal{T}_t^{-0}(0(0)),
\]
where the third and last inequality follow from (21) and (22). By the definition and nonnegative property of $F_\gamma$, the above inequality shows that
\[ 0 \leq F_\gamma(t) \leq \epsilon, \quad \text{for } t \geq T + 1. \]
That completes the proof. \qed

As a direct corollary of above lemma, we obtain

**Corollary B.3.** Let $\gamma : \mathbb{R} \to M$ be any global minimizer, then
\[ u^-(x) = u^-(\gamma(0)) = \int_{-\infty}^{0} e^{\lambda s} L(\gamma(s), \dot{\gamma}(s))ds. \]
That implies every global minimizer is globally calibrated by \( u^- \), so we get the equivalence of two definitions of the Aubry set.

**Lemma B.4** (Upper semicontinuity). As a set-valued function, 
\[
(0, 1] \times C^r(TM, \mathbb{R}) \ni (\lambda, L) \mapsto \tilde{A}(L) \subset TM
\]
is upper semicontinuous w.r.t. the Hausdorff distance in \( TM \).

**Proof.** It suffices to prove that for any \((\lambda_n, L_n)\) accumulating to \((\lambda_*, L_*)\), the upper limit for the sequence \(\tilde{A}(\lambda_n, L_n)\) is contained in \(\tilde{A}(\lambda_*, L_*)\). In other words, we can individually prove that as \(\lambda_n \to \lambda_*\) (resp. \(L_n \to L_*\)), the accumulating curve of \(\{\gamma_n \in \tilde{A}(\lambda_n)\}\) (resp. \(\{\gamma_n \in \tilde{A}(L_n)\}\)) would be contained in \(\tilde{A}(\lambda_*)\) (resp. \(\tilde{A}(L_*)\)).

Due to item 2 of Proposition 2.6 for any \(\lambda_n \in (0, 1]\), \(\tilde{A}(\lambda_n)\) is uniformly compact in the phase space. Therefore, for any sequence \(\{\gamma_n\}\) each of which is globally calibrated, the accumulating curve \(\gamma_*\) satisfies the Euler-Lagrange equation as well. On the other side, the associated \(u^-_*\) converges to \(u^-_*\) w.r.t. the \(C^0\)-norm, with \(u^-_*\) being the weak KAM solution for \(\lambda_*\)-Hamiltonian equation. For any \(t < s \in \mathbb{R}\), we have
\[
e^{\lambda_*}u^-_n(\gamma_n(s)) - e^{\lambda_*}u^-_n(\gamma_n(t)) = \int_t^s e^{\lambda_*}L(\gamma_n, \dot{\gamma}_n) d\tau
\]
then the limit
\[
e^{\lambda_*}u^-_*(\gamma_*(s)) - e^{\lambda_*}u^-_*(\gamma_*(t)) = \int_t^s e^{\lambda_*}L(\gamma_*, \dot{\gamma}_*) d\tau
\]
implies \(\gamma_*\) is globally calibrated.

For \(L_n \to L_*\), the proof is similar; We just need to assume that \(\lim_{n \to +\infty} \|L_n - L_*\|_{C^2} = 0\) on the whole phase space \(TM\), then repeat previous process. \(\square\)

**References**

[1] Martino Bardi and Italo Capuzzo-Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Systems and Control: Foundations & Applications. Birkhäuser Boston Inc., 1997. doi:10.1007/978-0-8176-4755-1

[2] Guy Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*, volume 17 of Mathématiques & Applications. Springer-Verlag, Paris, 1994.

[3] Alain Bensoussan. Perturbation methods in optimal control. *Wiley/Gauthier-Villars Ser. Modern Appl. Math.*, John Wiley & Sons Ltd. Chichester, translated from the French by C. Tomson., 1988.

[4] Patrick Bernard. Existence of \(C^{1,1}\) critical sub-solutions of the Hamilton-Jacobi equation on compact manifolds. *Ann. Sci. École Norm. Sup. (4)*, 40(3):445–452, 2007. URL: https://doi.org/10.1016/j.ansens.2007.01.004

[5] Renato Calleja, Alessandra Celletti, and Rafael De la Llave. A kam theory for conformally symplectic systems: Efficient algorithms and their validation. *Journal of Differential Equations*, 5(255):978–1049, 2013. doi:10.1016/j.jde.2013.05.001

[6] Piermarco Cannarsa and Carlo Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, volume 58 of Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston Inc., Boston, MA, 2004.

[7] Martin Casdagli. Periodic orbits for dissipative twist maps. *Ergodic Theory and Dynamical Systems*, 7:165–173, 1987. doi:10.1017/S0143385700003916

[8] Alessandra Celletti and Luigi Chierchia. Quasi-periodic attractors in celestial mechanics. *Archive for Rational Mechanics and Analysis*, 191:311–345, 2009. doi:10.1007/s00205-008-0141-5
[9] Cui Chen, Wei Cheng, and Qi Zhang. Lasry–Lions approximations for discounted Hamilton-Jacobi equations. *J. Differential Equations*, 265(2):719–732, 2018. URL: https://doi.org/10.1016/j.jde.2018.03.010

[10] Andrea Davini, Albert Fathi, Renato Iturriaga, and Maxime Zavidovique. Convergence of the solutions of the discounted Hamilton-Jacobi equation: convergence of the discounted solutions. *Invent. Math.*, 206(1):29–55, 2016. URL: https://doi.org/10.1007/s00222-016-0648-6

[11] G. Duffing. *Erzwungene Schwingungen bei Veränderlicher Eigenfrequenz und ihre Technische Bedeutung*, volume 134 of *Sammlung Vieweg*, Vieweg & Sohn, Braunschweig, 1918.

[12] Albert Fathi. Weak kam from a pde point of view: Viscosity solutions of the hamilton—jacobi equation and aubry set. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 142:1193–1236, 2012. doi:10.1017/S030821050000064

[13] Albert Fathi. *Weak KAM theorem in Lagrangian dynamics*. Cambridge University Press, version 10, 2020.

[14] Renato Iturriaga Gonzalo Contreras. Convex hamiltonians without conjugate points. *Ergodic Theory and Dynamical Systems*, 19:901–952, 1999.

[15] Renato Iturriaga and Hector Sanchez Morgado. Hyperbolicity and exponential convergence of the lax–oleinik semigroup. *Journal of Differential Equations - J DIFFERENTIAL EQUATIONS*, 246:1744–1753, 2009. doi:10.1016/j.jde.2008.12.012

[16] Patrice Le Calvez. Propriétés des attracteurs de birkhoff. *Ergodic Theory and Dynamical Systems - ERGOD THEOR DYN SYST*, 8, 1988. doi:10.1017/S0143385700004442

[17] Ricardo Mané. Generic properties and problems of minimizing measures of lagrangian systems. *Nonlinearity*, 9:273–310, 1996.

[18] Stefano Marmo and Alfonso Sorrentino. Aubry-mather theory for conformally symplectic systems. *Communications in Mathematical Physics*, 354:775–808, 2017. doi:10.1007/s00220-017-2900-3

[19] John N. Mather. Action minimizing invariant measures for positive definite Lagrangian systems. *Math. Z.*, 207(2):169–207, 1991. URL: https://doi.org/10.1007/BF02571383

[20] F. Moon and P. Holmes. A magnetoelastic strange attractor. *Journal of Sound and Vibration - J SOUND VIB*, 65:275–296, 1979. doi:10.1016/0022-460X(79)90520-0

[21] Hector Sanchez Morgado. Hyperbolicity and exponential long-time convergence for space-time periodic hamilton-jacobi equations. *Proceedings of the American Mathematical Society*, 143, 2012. doi:10.1090/0002-9939-2014-12290-8

[22] Antonio Siconolfi. *Hamilton–Jacobi Equations and Weak KAM Theory*. Volume 1–3. Springer, New York, 2012. doi:10.1007/978-0-387-30440-3(\_\_288)

[23] Kaizhi Wang and Jun Yan. The rate of convergence of new lax-oleinik type operators for time-periodic positive definite lagrangian systems. *Nonlinearity*, 25(7), 2012. doi:10.1088/0951-7715/25/7/2039

[24] Maciej Wojtkowski and Carlangelo Liverani. Conformally symplectic dynamics and symmetry of the lyapunov spectrum. *Communications in Mathematical Physics*, 194(1):47–70, 1997. doi:10.1007/s002200500347