Travelling waves in multivortex configurations

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Abstract

We investigate zero modes of a multivortex background which are due to the energetic degeneracy of the planar static n-vortex solution in Bogomol'nyi limit of the Abelian Higgs model parametrised by a set of 2n real parameters. The zero modes take the form of waves travelling with a speed of light independently along each of the vortices irrespectively to their mutual separations. String description of these zero modes is constructed. We also comment on the remnants of the modes a little outside of the Bogomol'nyi limit.

Introduction

One of the properties of the soliton solutions is that they break Poincaré symmetries of a vacuum. For example the vortex solution in 2 + 1 dimensional Abelian Higgs model [1] breaks translational invariance. When extended to 3 + 1 dimensions it also breaks part of spacial rotations. Whenever the action of the broken symmetry creates out of the given solution a new solution with different free parameters but of the same energy we can expect zero modes associated with this symmetry. In [2, 3] travelling waves on straight linear vortex background were constructed which are due to the broken transversal translations. Also broken rotational symmetries are the origin of a special kind of massless excitations of the vortex background [4].

In the Abelian Higgs model there is such a special choice of the coupling constants - Bogomol'nyi limit [5, 6] at which there are multivortex static planar solutions parametrised by a set of 2n real parameters [7, 8, 9]. These parameters...
can be chosen for example as Cartesian coordinates of the zeros of the Higgs field. It is usually said that these $n$ vortices do not interact. In fact there are no net static forces between them but they do interact by forces which are nonzero for nonzero vortex velocities and are known to lead to nontrivial scattering patterns [8, 9, 10, 11, 12]. The reasoning of Vachaspati [2] and also construction in [3] when applied to the static multivortex solution leads only to overall translational modes in the form of travelling waves - it makes use of only 2 out of $2n$ free parameters. There seems to be a wider possibility to excite independently the whole set of $2n$ parameters. Such an excitation will in general change the intervortex distances. The goal of this paper is to show that it is indeed possible irrespective of the fact that there are nontrivial velocity-dependent interactions between vortices. We also add a comment on what are the remnants of these excitations outside of the Bogomol’nyi limit and construct string description of both scattering of parallel vortices and splitting modes of vortex with winding number 2.

1 General considerations

The Lagrangian of the Abelian Higgs model reads

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi^* D^\mu \phi - \frac{1}{8} \lambda (\phi^* \phi - 1)^2,$$  

(1)

where $D_\mu \phi = \partial_\mu \phi - i A_\mu \phi$. The corresponding field equations are

$$D_\mu D^\mu \phi = -\frac{1}{2} \lambda (\phi^* \phi - 1) \phi,$$  

(2)

$$\partial_\mu F^{\mu\nu} = j^{\nu} \equiv \frac{1}{2} i (\phi^* D^\nu \phi - \text{c.c.}).$$  

(3)

It is known that at the critical value of the coupling constant of $\lambda = 1$ static version of these equations can be reduced to first order Bogomol’nyi equations. They are known to possess static multivortex solutions, which for the topological index $n$ are parametrised by a set of $2n$ real parameters (we take $n$ positive for definiteness)

$$\phi = \phi(x, y; \xi_A), \ A_\beta = A_\beta(x, y; \xi_A), \ A_k = 0,$$  

(4)

where $\xi_A$-s are the parameters. By $\alpha, \beta...$ we denote indices 1, 2 while by $i, k...$ indices 0, 3. The energy of the solution does not depend on the chosen values of parameters $\xi_A$. Any such given configuration possesses $2n$ zero modes [4] due to the variation of $2n$ parameters

$$\delta \phi_A = \frac{\partial}{\partial \xi_A} \phi(x^7; \xi_B), \ \delta A_\beta^{(A)} = \frac{\partial}{\partial \xi_A} A_\beta(x^7; \xi_B).$$  

(5)
They were explicitly calculated for the axially symmetric vortex with winding number \( n \). We would like to show that when we extend the solution (4) to 3+1 dimensions there are 2\( n \) kinds of excitations travelling along the z-axis with the speed of light. We start by very general considerations without specifying explicitly the set of parameters \( \xi_A \) at first and gradually we will become more specific.

Let us assume that the only change in the Higgs field and in x,y components of the gauge potential is due to the \( t, z \) dependence of the parameters

\[
\phi = \phi(x^\alpha; \xi_A(t, z)) \quad , \quad A_\beta(x^\alpha; \xi_A(t, z)).
\]

It is no longer a solution of the field equations (2,3). To make it a solution we will have to introduce non-zero time dependent \( A_k \) components of the gauge potential.

The first of the field equations (3) can be rewritten as

\[
D_iD^i\phi = D_\beta D^\beta \phi - \frac{1}{2}(\phi^* \phi - 1) \phi.
\]

The RHS of the above equation is obviously satisfied by the modified fields (6) at any instant of time and at any value of \( z \). The LHS can be rewritten as

\[
\sum_A \frac{\partial \phi}{\partial \xi^A} \frac{\partial^2 \phi}{\partial \xi^A} + \sum_{A,B} \frac{\partial^2 \phi}{\partial \xi^A \partial \xi^B} \frac{\partial \xi^A \partial \xi^B}{\partial x^\beta} = i(\partial_{i}A^i)\phi + 2iA^i \sum_A \frac{\partial \phi}{\partial \xi^A} \frac{\partial \xi^A}{\partial x^\beta}.
\]

Since we are looking for zero modes we impose on \( \xi^A \) following two equations

\[
\partial_{i} \partial^i \xi^A = 0 \quad , \quad \partial_{i} \xi^A \partial^i \xi^B = 0
\]

for any value of \( A, B \). They mean that each \( \xi^A \) is a function of \( (t + z) \) or \( (t - z) \) only. The situation that part of the parameters are left-movers and another part are right-movers is excluded.

\[
\xi^A = \xi^A(t - z) \quad or \quad \xi^A = \xi^A(t + z)
\]

for any \( A=1,..,2n \). Thus we are left only with the RHS of Eq. (8). It can be put identically equal to zero by an Anzatz

\[
A_k = \sum_A F_A(x^\alpha; \xi_B) \partial_k \xi_A.
\]

which implies the gauge condition: \( \partial_\mu A^\mu = 0 \) since we have already \( \partial_\beta A_\beta = 0 \). Only the coordinate dependence of the functions \( F_A \) is specified so far. Otherwise they are arbitrary. With this form of the gauge potential the field strenght components take the form

\[
F^{ik} = 0 , \quad F^{\beta k} = \sum_A (\partial^k \xi_A) \left( \frac{\partial F_A}{\partial x_\beta} - \frac{\partial A_\beta}{\partial \xi^A} \right) , \quad F^{\beta \alpha} = \partial^\beta A^\alpha - \partial^\alpha A^\beta.
\]
With the second of these formulas one can easily check that Eq.(3) is identically fulfilled for \( \nu = \beta \) and it does not impose any extra constraint on \( F_A \). It is not so trivial with the case of \( \nu = i \). Now Eq.(3) can be rewritten as

\[
\sum_A (\partial^i \xi^A)(\partial_\beta \partial_\beta F_A) = \sum_A (\partial^i \xi^A)(\phi^* \phi) [F_A - \frac{\partial \Theta}{\partial \xi^A}] .
\]  

(13)

Since the dependence of particular \( \xi^A \) on \( x^i \) can be chosen arbitrarily this equation can be further reduced to

\[
\left( \frac{\partial}{\partial x^\beta} \frac{\partial}{\partial x^\beta} - \phi^* \phi \right) F_A = -(\phi^* \phi) \frac{\partial \Theta}{\partial \xi^A} ,
\]

(14)

where \( \Theta \) is an actual value of the phase of the Higgs field, \( \phi = |\phi| \exp i\Theta \). It is a static equation for any given value of the parameters \( \xi^A \).

There is one special case when its solution can be given without any effort for any value of the parameters. We can decompose the set of parameters \( \xi^A \) into a set of two overall translational modes \( X, Y \) and the rest of the modes \( \xi^A, A = 1, \ldots, 2n - 2 \), which we will call "splitting modes". Equation (6) now reads

\[
\phi = \phi(x + X, y + Y; \xi^A) , \quad A_\beta = A_\beta(x + X, y + Y; \xi^A) ,
\]

(15)

Equation (14) for the translational modes takes the form

\[
(\partial_\beta \partial_\beta - \phi^* \phi) F(x^\gamma) = -(\phi^* \phi) \frac{\partial \Theta}{\partial x^\gamma} ,
\]

(16)

and its exact solution is

\[
F(x^\gamma) = A_\beta(x^\alpha + X^\alpha; \xi_A) .
\]

(17)

This particular solution is known thanks to Vachaspatis [2], they are overall translational modes. The general form of the gauge potential now reads

\[
A_k = \sum_\beta A_\beta(x^\alpha + X^\alpha; \xi^B) \partial_k X^\beta + \sum_{A=1}^{2n-2} F_A(x^\alpha + X^\alpha; \xi_B) \partial_k \xi_A .
\]

(18)

When we shift \( x^\alpha \) coordinates \( x^\alpha \rightarrow y^\alpha = x^\alpha + X^\alpha \), we can write Eq.(14) for the splitting modes as

\[
\left( \frac{\partial}{\partial y^\beta} \frac{\partial}{\partial y^\beta} - \phi^* \phi \right) F_A(y^\alpha; \xi^B) = -(\phi^* \phi) \frac{\partial \Theta}{\partial \xi^A} ,
\]

(19)

which is formally the same as the original equation (13). Thus we can see that the splitting modes can be investigated on the background of the translational modes. The only effect of the Vachaspatis modes is the time dependent shift of the coordinates in Eq.(19). Another effect is a correlation, which states that both translational modes and splitting modes have to be exclusively right-movers or left-movers. For the analysis of Eq.(19) itself we can forget about translational modes and think of them as if they were put equal to zero.
2 Solutions to the effective equation

Analysis of equation (14) or (19) is quite nontrivial due to a very limited knowledge about the background field. Nevertheless it can be completed in certain limit cases.

In general the n-vortex configuration in 2 + 1 dimensions can be described by

$$\phi = (z^n - \sum_{k=0}^{n-1} \lambda_k z^k) W(z, z^*) \ ,$$  \hspace{1cm} (20)

where \( z = x + iy = re^{i\theta} \) and \( W(z, z^*) \) is a positively definite real function. The \( \lambda \)-polynomial itself determines both the phase of the Higgs field and the positions of its zeros. There is one to one correspondence between the field configuration and the set of parameters \( \lambda_k \) \[9\]. The phase of the Higgs field is given by:

$$\Theta(z; \lambda_k) = \text{Arg}(z^n - \sum_{k=0}^{n-1} \lambda_k z^k) \ , \ \lambda_k = \lambda_1^k + i\lambda_2^k \ .$$  \hspace{1cm} (21)

First we will analyse the limit case of all \( \lambda_k^n = 0 \).

2.1 Case of n coincident vortices

In the limit of vanishing \( \lambda \)-s, which corresponds to \( n \) vortices sitting on top of each other, the desired derivatives of \( \Theta \) with respect to the parameters are

$$\frac{\partial\Theta}{\partial\lambda_1^k} = \frac{\sin(m\theta)}{r^m} \ , \ \frac{\partial\Theta}{\partial\lambda_2^k} = -\frac{\cos(m\theta)}{r^m} \ , \ m = n - k \ .$$  \hspace{1cm} (22)

When the background field configuration is \( \phi = f(r) \exp in\theta \), eq.(14) can be written in the form

$$\left(\nabla^2 - f^2(r)\right) F_{1k}(r, \theta) = -\frac{f^2(r)}{r^m} \sin(m\theta) \ ,$$

$$\left(\nabla^2 - f^2(r)\right) F_{2k}(r, \theta) = \frac{f^2(r)}{r^m} \cos(m\theta) \ .$$  \hspace{1cm} (23)

With a decomposition

$$F_{1k}(r, \theta) = F_k(r) \sin(m\theta) \ , \ F_{2k}(r, \theta) = -F_k(r) \cos(m\theta) \ ,$$  \hspace{1cm} (24)

the above equations will be reduced to a single radial equation

$$F_k'' + \frac{1}{r} F_k' - \frac{m^2}{r^m} F_k - f^2 F_k = -\frac{f^2}{r^m} \ , \ m = n - k \ .$$  \hspace{1cm} (25)
With a help of an asymptotics \( f(r) \sim f_0 r^n \) one can analyse this equation near the origin \( r \sim 0 \):

\[
F_k(r) = \frac{a}{r^m} + br^m + cr^{2m} - \frac{f_0^2}{4(k+1)(n+1)}r^{2k+2} + \ldots . \quad (26)
\]

The point is that there is only one singular term \( O(\frac{1}{r^m}) \). We will show that the coefficient \( a \) of this term can be tuned to zero.

We can regard equation (25) as an inhomogenous linear differential equation. For large \( r \) where \( f^2 \sim 1 \), the homogenous part of the equation approaches the modified Bessel equation. We choose the asymptotically vanishing solution as

\[
F_{k}^{\text{hom}}(r) \sim d e^{-r \sqrt{r}}, \quad r \to \infty . \quad (27)
\]

Since for small \( r \) \( f^2 \to 0 \) and \( \frac{m^2}{r^2} \) diverges, this asymptotically vanishing solution is transferred into a linear combination

\[
F_{k}^{\text{hom}}(r) \sim d(\frac{e}{r^m} + \ldots) + d(gr^m + \ldots), \quad r \to 0 . \quad (28)
\]

\( d \) is an overall constant which will be chosen later. We expect \( e \) to be nonzero. An indirect proof of this fact can be extracted from considerations in [13]. We need this singular term to remove an eventual singularity of the total solution.

On the other hand we can look for a special solution of the whole inhomogenous equation, which for large \( r \) can be solved by

\[
F_{k}^{\text{inhom}}(r) \sim \frac{1}{r^m}, \quad r \to \infty . \quad (29)
\]

For small \( r \) it is in general transferred into a singularity

\[
F_{k}^{\text{inhom}}(r) \sim \frac{s}{r^m} + \ldots, \quad r \to 0 . \quad (30)
\]

If the coefficient \( s \) happens to be equal to 0, we will also choose \( d = 0 \) and there will be no singularity in Eq. (24). If \( s \neq 0 \), we will have to choose \( de = -s \) and once again singularity will be removed. Eq. (26) shows that there is no other singular term to be removed. Thus potentials \( A_i \)

\[
A_i = \sum_{k=0}^{n-1} F_k(r)[(\partial_i \lambda_k^1) \sin(n-k)\theta - (\partial_i \lambda_k^2) \cos(n-k)\theta], \quad (31)
\]

are regular for small \( r \). The asymptotics for large \( r \) is dominated by a contribution from the inhomogenous part: \( F_k \sim \frac{1}{r^m} \) without any undetermined coefficients

\[
A_i \sim \sum_{m=1}^{n} [\partial_i \lambda_k^1 \frac{\sin m\theta}{r^m} - \partial_i \lambda_k^2 \frac{\cos m\theta}{r^m}], \quad m = n-k , \quad (32)
\]

6
This asymptotic form is an exact result. We expect it to be also valid for finite values of \( \lambda \). An asymptotic form of the derivatives of the phase of the Higgs field for any finite values of \( \lambda \) is still given by Eq. (22). For large \( r \) also general \( \phi^* \phi \sim 1 \) and the "homogenous" solutions must be taken to be asymptotically exponentially vanishing. Thus Eq. (32) is a general asymptotic result.

The existence of solutions to equation (14) in the limit of all \( \lambda^\alpha_k = 0 \) strongly suggests that we can expect them to exist also for small but finite \( \lambda \).

## 2.2 Small fluctuations around \( n \)-vortex background

With a help of the Bogomol’nyi equations at the critical value of the coupling constant

\[
(D_1 + iD_2)\phi = 0 , \quad F_{12} = \frac{1}{2} (1 - \phi^* \phi)
\]

one can obtain a general perturbation of the Higgs field [8, 12] around the background of \( n \) coincident vortices \( f(r)e^{in\theta} \):

\[
\phi + \delta \phi = f(r)e^{in\theta}[1 + \sum_{s=0}^{n-1} \lambda_s H_{n-s}(r)e^{-i(n-s)\theta}] + O(\lambda^2) , \tag{34}
\]

where functions \( H_p(r) \) satisfy

\[
H_p'' + \frac{1}{r} H_p' - \frac{p^2}{r^2} H_p - f^2 H_p = 0 , \tag{35}
\]

which happens to be identical with the homogenous part of Eq. (25). We choose the normalisation of the solution (27,28) in such a way that

\[
H_p(r) \sim -\frac{1}{rp} , \quad r \to 0 , \tag{36}
\]

With this normalisation \( \lambda_s \) in Eq. (34) can be identified with those in Eq. (20,21), because for small \( r \) the Higgs field is well approximated by

\[
\phi + \delta \phi \approx (z^n - \sum_{s=0}^{n-1} \lambda_s z^s) \tag{37}
\]

With Eq. (34) the approximate phase of the Higgs field reads

\[
\Theta = n\theta + \frac{1}{2i} \ln[1 + \sum_{s=0}^{n-1} \lambda_s H_{n-s}(r)e^{-i(n-s)\theta}] , \tag{38}
\]

Eq. (14) for any particular \( \lambda^\alpha_k \) can be always transformed into equation for \( \lambda^\alpha_k \) by an appropriate rotation of the reference frame, so without loss of generality

7
we can restrict to this case. The derivative of the phase of the Higgs field with respect to $\lambda^k_1$ is

$$\frac{\partial \Theta}{\partial \lambda^k_1} = \frac{H_{n-k}(r) \Im \left[ e^{i(k-n)\theta} + \sum_{l=0}^{n-1} \lambda^k_1 H_{n-l}(r) e^{i(k-l)\theta} \right]}{1 + \sum_{s=0}^{n-s} \lambda^s H_{n-s}(r) e^{-i(n-s)\theta} \right]} . \quad (39)$$

Eq. (14) when we make an expansion $F_{\alpha k} \rightarrow F_{\alpha k} + \delta F_{\alpha k} + O(\lambda^2)$, $(F_{\alpha k}$ corresponds to $\lambda^\alpha_0 )$, takes the form

$$(\nabla^2 - f^2) F_{\alpha k}(r, \theta) = -f^2 \frac{\partial \Theta}{\partial \lambda^\alpha_1} , \quad (40)$$

$$(\nabla^2 - f^2) \delta F_{\alpha k}(r, \theta) = (\delta \phi^* \phi) F_{\alpha k} - \delta (\phi^* \phi \frac{\partial \Theta}{\partial \lambda^\alpha_1}) , \quad (41)$$

where the first equation comes from the terms $O(\lambda^0)$, the second equation from those linear in $\lambda$-s. We have omitted higher order equations in this hierarchy. The first equation has been already analysed in section 2.1

$$F_{1k}(r, \theta) = F_k(r) \sin(n - k)\theta , \quad F_{2k}(r, \theta) = -F_k(r) \cos(n - k)\theta . \quad (42)$$

Upon substitution of this solution, Eq. (41) once again becomes linear inhomogeneous equation. In this way any equation in the hierarchy, which begins with (40,41), can be step by step cast in a form of linear inhomogeneous equation of d’Alembert type and when solved provide a complete input for the next equation in order of the powers of $\lambda$. This property in principle provides us with a systematic calculational scheme. Nevertheless for practical calculations for any given set of parameters we think Eq. (14) to be much easier manageable by numerical methods. We will use the hierarchy only to analyse solutions for small $\lambda$-s.

When we take into account that

$$| \phi + \delta \phi |^2 = f^2(r) \left| 1 + \sum_{s=0}^{n-1} \lambda^s H_{n-s}(r) e^{-i(n-s)\theta} \right|^2 \quad (43)$$

Eq. (41) in our case can be explicitly written as

$$(\nabla^2 - f^2) \delta F_{1k}(r, \theta) = 2f^2 F_k \sin(n - k)\theta \sum_{s=0}^{n-1} H_{n-s}(r) [\lambda^1_1 \cos(n - s)\theta + \lambda^2_1 \sin(n - s)\theta]$$

$$+ f^2 H_{n-k} \sum_{s=0}^{n-1} H_{n-s}[\lambda^1_1 \cos(k - l)\theta - \lambda^2_1 \sin(k - l)\theta]$$

$$\equiv \sum_{l=0}^{2n} [W_l(r) \cos l\theta + U_l(r) \sin l\theta] \quad (44)$$

where we have rewritten the RHS in a form of its most general Fourier transform in $\theta$. $W_l(r)$ and $U_l(r)$ are regular functions quickly vanishing at infinity. Some of them are identically equal to zero. With a decomposition

$$\delta F_{1k}(r, \theta) = \sum_{l=0}^{2n} [g_l(r) \cos l\theta + h_l(r) \sin l\theta] , \quad (45)$$

8
we arrive at a set of linear inhomogenous differential equations

\[ g''_l + \frac{1}{r} g'_l - \frac{l^2}{r^2} g_l - f^2 g_l = W_l , \]  

\[ h''_l + \frac{1}{r} h'_l - \frac{l^2}{r^2} h_l - f^2 h_l = U_l . \]

The regularity of their solutions at \( r = 0 \) can be established by a similar analysis as in section 2.1.

A natural question arises whether we can expect regularity of solutions for any finite values of \( \lambda \)-s. In principle we could repeat our method of analysis to any finite order of the hierarchy beginning with (40,41). An indication that the result of such an analysis can be expected to be positive is that we can give an approximate solution to Eq.(14) for large separations of vortices.

### 2.3 Well separated vortices

Let us concern a situation in which there is certain number of vortices labelled by \( s \), with winding numbers \( n_s \) and positions \( \vec{R}_s \). This planar configuration is to be regarded as a background input to Eq.(14) so in reality it can quite as well fit to a situation when at certain value of \( z \) at certain moment of time the cross-section through the vortex spaghetti is a planar configuration of well separated vortices.

Vortices are well separated when distances between them

\[ | \vec{R}_s - \vec{R}_{s'} | \gg 1 \]  

for any values of \( s \) and \( s' \). In our dimensionless Lagrangian the mass scales are of order 1, so our condition means that solutions around separate \( \vec{R}_s \) approach asymptotics very quickly as compared to the intervortex distances. In this case the background fields can be approximated by

\[ \phi(x, y; \vec{R}_s) \approx \prod_s \phi^{(s)}(x - X_s, y - Y_s) , \]

\[ A_\beta(x, y; \vec{R}_s) \approx \sum_s A^{(s)}_\beta(x - X_s, y - Y_s) , \]

where \( \phi^{(s)} \) and \( A^{(s)}_\beta \) are fields of separate vortices with winding numbers \( n_s \). This is a good approximation up to terms vanishing exponentially with intervortex distances. With the same degree of accuracy an approximate solution for \( A_i \) can be given

\[ A_i \approx - \sum_{s, \beta} (\partial_i \vec{R}_s) A^{(s)}_\beta (x - X_s, y - Y_s) . \]

Separate \( X_s, Y_s \) vary independently due to general results of section 1. All of them have to be exclusively left-movers or right-movers.
Putting together all the results of section 2 we think justified expectation that Eq. (14) possesses solution for any value of the parameters $\xi^A$. Let us give three examples of particularly symmetric excitations travelling with a speed of light along vortex with winding number two $\phi = (z^2 - \lambda)W(z, z^*)$.

a) Let us take $\lambda = R^2 \sin(k_i x^i)$, $k_i k^i = 0$. Corresponding positions of the zeros of the Higgs field are

$$z_{1,2} = \pm R \sqrt{\sin(k_i x^i)} \quad z_{1,2} = \pm i R \sqrt{-\sin(k_i x^i)},$$

for the positive and negative $\sin(k_i x^i)$ respectively. This solution describes travelling splitting wave. At the points where the actual amplitude is equal to zero (coincident vortices) the orientation of the wave is twisted by the right angle.

b) Another example is $\lambda = R^2 e^{2i k_i x^i}$ which corresponds to positions of vortices

$$z_1 = R e^{i (k_i x^i)} \quad z_2 = R e^{i (k_i x^i + \pi)},$$

This excitation has a form of double helix moving up or down $z$-axis with a speed of light.

c) Our final example is $\lambda = R^2 e^{2i \Theta(k_i x^i)}$, where $\Theta(k_i x^i) = \arctan(k_i x^i)$ interpolates between $-\pi/2$ for negative argument and $\pi/2$ for positive $k_i x^i$. Vortices at $R \exp i \Theta$ and $R \exp i (\Theta + \pi)$ are parallel lines far from $k_i x^i \approx 0$ and around this point they make a twist by an angle of $\pi$. The twist moves with a speed of light.

In all three cases we have taken into account only splitting modes. There is a wider possibility to take these splitting modes on a translational background.

3 String description and adiabatic approximation

Vortices in 2 + 1 dimensions can be regarded as point-like particles while those in 3 + 1 dimensions were invented to be field-theoretical realisation of strings. Scattering of vortices on plane was successfully described with a help of Manton’s approximation which amounts to reduction of the full field-theoretical dynamics in n-vortex sector to mechanics of n point particles on curved surface. For example dynamics of pair of vortices can be described in their center of mass frame by an effective nonrelativistic Lagrangian

$$L_{eff} = h_{\alpha \beta}(\xi) \frac{d\xi^\alpha}{dt} \frac{d\xi^\beta}{dt},$$

where positions of vortices on the plane are $(\xi^1, \xi^2)$ and $(-\xi^1, -\xi^2)$. $h_{\alpha \beta}(\xi)$ is a metric tensor, symmetric with respect to $\xi \to -\xi$, which incorporates effects of vortex mutual interactions. For large separations it is asymptotically
flat $h_{\alpha\beta} \approx \delta_{\alpha\beta}$ while for small separations it tends to $R^2(\dot{R}^2 + R^2\ddot{R}^2)$ in polar coordinates and is responsible for the right-angle scattering in the head-on collision. This approximation gives results in accordance with direct numerical simulations up to velocity 0.4 [1]. The effective Lagrangian (53) was derived in adiabatic approximation so an infinite series of terms of higher order in velocity was systematically neglected. We would like now turn attention to the 3 + 1 dimensional case and unify this nonrelativistic theory with our essentially relativistic results presented in previous sections.

Translational waves on an isolated vortex coincide with analogous excitations of the Nambu-Goto string [2]. Let us imagine two strings parametrised for example by

$$
\xi_{\alpha}^{(1)} = [\tau, \xi^1(\tau, \sigma), \xi^2(\tau, \sigma), \sigma], \quad \xi_{\alpha}^{(2)} = [\tau, -\xi^1(\tau, \sigma), -\xi^2(\tau, \sigma), \sigma].
$$

A point on $z$-axis is a “center-of-mass” for each cross section by a plane of constant $z$ through this configuration of two strings. The action for the Nambu-Goto string is constructed out of a metrics induced on its world-sheet, namely $\gamma_{\alpha\beta} = \eta_{\mu\nu} \partial_\mu \xi^\alpha \partial_\nu \xi^\beta$. In this way we would obtain two Nambu-Goto strings with nontrivial mutual interactions. How to geometrise these interactions? Let us take a cross section through our system at any fixed $z$, thus we obtain 2-particle mechanical system in $C-M$ frame. We propose to replace Minkowskian metrics in definition of $\gamma_{\alpha\beta}$ by an effective metrics which incorporates mutual interactions

$$
G_{\mu\nu} = \left[ \eta_{\alpha\beta}(x^\gamma) \right].
$$

The effective induced metrics now read

$$
g_{ik}^{(1)} = \eta_{mn} \frac{\partial \xi^m}{\partial \tau^i} \frac{\partial \xi^n}{\partial \tau^k} - h_{\alpha\beta}(\xi) \frac{\partial \xi^\alpha}{\partial \tau^i} \frac{\partial \xi^\beta}{\partial \tau^k} = g_{ik}^{(2)} = g_{ik},
$$

The effective two string action becomes

$$
S_{eff}^{2+1} = -2\pi \int d\tau d\sigma \sqrt{-g} = -2\pi \int d\tau d\sigma \sqrt{-g},
$$

where $g = \text{det}(g_{ik})$ and prefactors $\pi$ are string tensions equal to linear energy density of a vortex in Bogomol’nyi limit.

Let us test this construction on two examples. First one is a case of slow motion of parallel strings: $\xi^\alpha = \xi^\alpha(\tau)$. Now the action (57) reduces to

$$
S_{eff}^{2+1} = -2\pi \int d\tau \sqrt{g_{\tau\tau}}, \quad g_{\tau\tau} = (\frac{d\xi^0}{d\tau})^2 - h_{\alpha\beta}(\xi) \frac{d\xi^\alpha}{d\tau} \frac{d\xi^\beta}{d\tau},
$$

which is parametrisation invariant. We can introduce a 1-dim Vielbein on the worldline $g_{\tau\tau} = e_0^0 \eta_{00} e_0^0 \equiv e^2$ as an additional dynamical variable

$$
S_{eff}^{2+1} = -2\pi \int d\tau [e - \mu(e^2 - g_{\tau\tau})],
$$

11
where $\mu$ is a Lagrange multiplier and $g_{\tau \tau}$ is defined in equation (58). Variation with respect to $e$ leads to an equation $\mu = \frac{1}{2e}$. When we fix the gauge $e = 1$ and drop constant terms from the action it will take quadratic form

$$S_{\text{eff}}^{2+1} = -\pi \int d\tau \left[ \left( \frac{d\xi^0}{d\tau} \right)^2 \right]$$

with a constraint

$$\left( \frac{d\xi^0}{d\tau} \right)^2 = 1 + h_{\alpha \beta}(\xi) \frac{d\xi^\alpha}{d\tau} \frac{d\xi^\beta}{d\tau}.$$  

(61)

$\xi^0$ component is fully determined by the constraint and otherwise decouples, while the dynamics of $\xi^\alpha$-s is governed by a Lagrangian

$$L_{\text{eff}} \approx h_{\alpha \beta}(\xi) \frac{d\xi^\alpha}{d\tau} \frac{d\xi^\beta}{d\tau},$$

(62)

which is identical with (53) except for parametrisation by the proper time $\tau$ instead of $t$. From the constraint one can find relation between $\tau$ and $t$. From (12) we will obtain the same geodesic curve as from (53). The only difference negligible for small velocities is a different parametrisation. It does not change deflection angles in scattering of vortices but from the point of view of the action (58) there is a different correspondence between velocity used in adiabatic approximation (12), which is velocity with respect to $\tau$ and that in direct numerical simulation which is velocity with respect to laboratory time $t$. An important difference arises when we go a little outside of the Bogomol’nyi limit like in [14]. In this case due to potential forces between vortices they no longer follow geodesics but now their scattering angle depends both on impact parameter and on initial velocity. In the Bogomol’nyi limit up to velocity 0.4 there is a good agreement between analytic results and those from numerical simulation [8], while outside of the critical coupling there is a systematic discrepancy [14] vanishing for small velocities and smoothly rising as they go from 0 to 0.4. This systematic error can be explained from our point of view. For velocities greater then 0.4 scattering is no longer elastic so point particle approximation in both cases does not work any longer.

Thus we can see that this construction fits to 2 + 1 dimensional results but what about travelling waves? Variation of the action (57) leads to equation of motion

$$\frac{1}{\sqrt{-g}} \frac{d}{d\tau} \left( \sqrt{-g} g^{mn} \frac{\partial g_{mn}}{\partial \xi^i} \right) = g^{mn} \frac{\partial g_{mn}}{\partial \xi^i}.$$  

(63)

With a parametrisation (54) the effective metric reads

$$g_{mn} = \eta_{mn} - h_{\alpha \beta}(\xi) \frac{\partial \xi^\alpha}{\partial \tau^m} \frac{\partial \xi^\beta}{\partial \tau^n}.$$  

(64)

When we impose a condition $\xi^\alpha = \xi^\alpha(\tau + \sigma)$ or $\xi^\alpha = \xi^\alpha(\tau - \sigma)$ exclusively, then one can easily check that $g = -1$ and thus simplified equation (63) is identically
fulfilled. This reasoning is not sensitive to the precise form of the metric $h_{\alpha\beta}$, it is enough to assume for the whole construction its parity and that it is a symmetric invertible matrix. Thus we can see that also in this construction vortices can be independently translationally excited and they do not feel interactions essential for the scattering of parallel strings.

Now let us make a bold extension of this construction to the case of slightly curved vortex with winding number 2. Let us define a "center of mass frame" in this case by

$$x^\mu = R^\mu(\tau, \sigma) + \rho^\alpha n^\mu_{(\alpha)}(\tau, \sigma),$$

(65)

where $n^2_{(\alpha)} = -1$, $\eta_{\mu\nu}n^\mu_{(\alpha)}R_{i}^\mu_{(\alpha)}$ and we assume the reference frame to be torsion-free $\eta_{\mu\nu}n^\mu_{(\alpha)}n^\nu_{(\beta)} = 0$. The strings in coordinates $\{\tau, \sigma, \rho^\alpha\}$ are parametrised as

$$\xi^a_{(1)}(\tau, \sigma) = [\xi^0, \xi^1, \xi^2, \xi^3], \quad \xi^a_{(2)}(\tau, \sigma) = [\xi^0, -\xi^1, -\xi^2, \xi^3],$$

(66)

what defines the frame used. When $\xi^\alpha = 0$ both strings coincide with the worldsheet $R^\mu$ or $\rho = 0$. For $\xi^\alpha \neq 0$ they are uniformly splitting from $R^\mu$. The metric in new coordinates

$$G_{ab} = \begin{bmatrix} \gamma_{ik} & 0 \\ 0 & -\delta_{\alpha\beta} \end{bmatrix} + O(\rho) \rightarrow G_{ab}^{\text{eff}} = \begin{bmatrix} \gamma_{ik} & 0 \\ 0 & -h_{\alpha\beta}(\xi) \end{bmatrix} + O(\xi)$$

(67)

is once again replaced by an effective metric incorporating effects of interactions of separate strings. Terms $O(\xi)$ are small for small external curvature of the worldsheet $R^\mu$ and sufficiently small $\xi^\alpha$. $\gamma_{ik}$ is a metric induced on the line $R^\mu(\tau, \sigma)$. Effective metric tensors are

$$g^{(1)}_{ik} = G_{ab}^{\text{eff}}(\xi) \frac{\partial \xi^a}{\partial \tau^i} \frac{\partial \xi^a}{\partial \tau^k} = g^{(2)}_{ik} = g_{ik}$$

(68)

and they give rise to the effective action

$$S_{\text{eff}} = -2\pi \int d\tau d\sigma \sqrt{-g},$$

(69)

which contains as dynamical variables $R^\mu(\tau, \sigma)$ and $\xi^a(\tau, \sigma)$ a part of which can be removed by gauge fixing. This action can be thought of as a candidate to qualitatively describe excitations of slightly curved vortex with winding number 2 due to its internal multivortex structure.

4 Conclusion

We have investigated zero modes travelling with a speed of light which are due to the well known degeneracy of the general planar n-vortex solution in the Bogomol’nyi limit due to the existence of 2n free parameters. The 2n zero modes are independent except of a restriction that they all have to be exclusively
left-movers or right-movers. When we identify 2n parameters with positions of vortices we can say that the general excitation looks like n vortices which are independently translationally excited. This is a nontrivial result because there is no restriction as to the distance between separate vortices.

What happens when we go outside of the Bogomol’nyi limit? For well separated vortices we expect this picture to be preserved. For distances comparable with mass scales of the theory vortices will attract or repel depending on whether the strength of the nonlinearity of the Higgs field is smaller or greater than in the Bogomol’nyi limit \[1\], \[14\]. When they repel they tend to be well separated but when there are attractive forces between them they will collapse to one n-vortex configuration. Small excitations of such a configuration at critical coupling were described in sections 2.1 and 2.2. The dispersion relation in the Bogomol’nyi limit for the splitting modes is \(k_i k^i = 0\). From a recent work by P.A.Shah \[14\] on extension of the Manton’s approximation a little outside of the critical coupling one can extract that since for small \(\lambda\)-s the nonrelativistic potential is of the harmonic oscillator type the dispersion relation for small \(\lambda\)-s will be \(k_i k^i = m^2\). Thus we can expect the splitting modes to be preserved but this time with a massive dispersion relation. It would be interesting to incorporate them into the newly invented by T.Dobrowolski membrane description of a vortex with a higher winding number \[15\]. Our examples of particularly symmetric splitting modes in the end of section 2 will be modified in this case. Examples a) and b) will be preserved but with a massive \(k_i k^i = m^2\) since they are sinus waves when expressed in \(\lambda\)-s. This time they will move up or down the \(z\)-axis with a superluminal phase velocity. The twist of c) no longer exists as a coherent state.

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