SPECTRAL RADIUS MINUS AVERAGE DEGREE: A BETTER Bound

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Abstract. Collatz and Sinogowitz had proposed to measure the departure of a graph $G$ from regularity by the difference of the (adjacency) spectral radius and the average degree: $\epsilon(G) = \rho(G) - \frac{2m}{n}$. We give here new lower bounds on this quantity, which improve upon the currently known ones.

1. Introduction

1.1. Motivation. Let $G$ be a graph that has $n$ vertices and $m$ edges. The average degree is $\bar{d} = \frac{2m}{n}$. Suppose now that $G$ has adjacency matrix $A$ and let us denote its spectral radius (i.e. the largest modulus of an eigenvalue) by $\rho$. A classic 1957 result of Collatz and Sinogowitz is:

Theorem 1. [7] Let $G$ be a graph with average degree $\bar{d}$ and spectral radius $\rho$. Then

$$\rho \geq \bar{d}$$

and equality holds if and only if $G$ is regular.

Theorem 1 has served as the departure point for several interesting inquiries. As one particularly impressive recent example we may mention the independent discovery by Babai and Guiduli [4] and by Nikiforov [17] of a spectral counterpart to the classic Kővari-Sós-Turán [13] bound for the Zarankiewicz problem.

Another point of view inspired by Theorem 1 is to consider the difference

$$\epsilon(G) = \rho - \bar{d}$$

as a measure for the irregularity of the graph $G$. This irregularity measure has been studied by various authors [3, 5, 6, 9, 15, 16].

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1.2. A brief digression about irregularity measures. The simplest irregularity measure is that provided by the difference of the maximum and minimum degree (denoted, by $\Delta$ and $\delta$, respectively):

$$\Delta - \delta.$$ 

Though very simply defined and thus perhaps considered by some as too crude to be of use, this measure is actually quite useful in some contexts (cf. [20] for an example).

Let us now introduce yet another irregularity measure, the variance of degrees:

$$\text{var}(G) = \frac{1}{n} \sum_{u \in V(G)} \left( d_u - \frac{2m}{n} \right)^2.$$ 

Bell [5] compares $\epsilon(G)$ and $\text{var}(G)$ for various classes of graphs.

We wish to remark that the following relationship between $\Delta - \delta$ and $\text{var}(G)$ is easily established by applying inequalities due to Popoviciu (cf. [18, (1.4)]) and Nagy (cf. [18, (1.5)]):

$$\frac{(\Delta - \delta)^2}{2n} \leq \text{var}(G) \leq \frac{(\Delta - \delta)^2}{4n}.$$ 

The upper bound in [1] has also been observed in [8, p. 62].

For more alternative notions of graph irregularity we refer the interested reader to [1, 2, 8].

1.3. Main result. Our purpose in this paper is to improve the extant lower bounds for $\epsilon(G)$, using rather elementary methods. The best bound to be found in the literature is due to Nikiforov [15]:

**Theorem 2.** [15] For every graph $G$,

$$\epsilon(G) \geq \frac{\text{var}(G)}{\sqrt{8m}}.$$ 

For example, as can be easily ascertained using [1], it implies the following bound obtained by Cioabă and Gregory in [6]:

**Corollary 1.** [6, Corollary 3] For every graph $G$,

$$\epsilon(G) \geq \frac{(\Delta - \delta)^2}{4n\Delta}.$$ 

We shall prove, using elementary methods, the following new bound:

**Theorem 3.** For every graph $G$,

$$\epsilon(G) \geq \frac{\text{var}(G)\sqrt{n}}{\sqrt{8m\Delta}}.$$ 

As $n > \Delta$, the new bound of [3] is always strictly better than [2].
2. Subregular graphs

There is one very special case which merits separate treatment.

Definition 1. [16] Let $G$ be a graph with $\Delta - \delta = 1$. If there is either exactly one vertex of degree $\Delta$ or exactly one vertex of degree $\Delta - 1$, then $G$ is called subregular.

Clearly, subregular graphs are very close to being regular. We will find it convenient to distinguish between their two varieties thus:

Definition 2. Let $G$ be a subregular graph.

- If there is exactly one vertex of degree $\Delta$, $G$ is high subregular.
- If there is exactly one vertex of degree $\Delta - 1$, $G$ is low subregular.

For subregular graphs the bounds discussed so far yield estimates which are far too pessimistic. However, there is another bound due to Cioabă and Gregory [6] which performs better in this case.

Theorem 4. [6]

\[
\epsilon(G) \geq \frac{1}{n(\Delta + 2)}.
\]

We will prove:

Theorem 5. Let $G$ be a connected subregular graph on $n \geq 7$ vertices and with maximum degree $\Delta$. Then:

- If $G$ is high subregular, then:

\[
\epsilon(G) \geq \frac{n^2 - 2n + 3}{n^3 \Delta}.
\]

- If $G$ is low subregular, then:

\[
\epsilon(G) \geq \frac{2n^2 - 4n - 3}{2n^3(\Delta - 1 + \frac{1}{\Delta})}.
\]

Example 1. Consider the high subregular graph $G$ depicted in Figure 1. We have the following lower bounds for $\epsilon(G)$:

| $\epsilon(G)$ | (2)   | (3)   | (4)   | (5)   |
|---------------|-------|-------|-------|-------|
|               | 0.0461| 0.0137| 0.0209| 0.0286| 0.0364|

3. Proof of Theorem 3

We begin by collecting a number of lemmata.
Lemma 1 (Hofmeister [10]).

\[ \rho \geq \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2}. \]

Lemma 2.

\[ \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2} \geq \frac{2m}{n}. \]

Proof. Cauchy-Shwarz. \qed

Lemma 3. [15, p. 352]

\[ \frac{1}{n} \sum_{i=1}^{n} d_i^2 - \left( \frac{2m}{n} \right)^2 = \text{var}(G). \]

Let \( A(G) \) and \( D(G) \) be the adjacency matrix and the diagonal matrix of vertex degrees, respectively, of \( G \). Then \( Q(G) = A(G) + D(G) \) is called the \textit{signless Laplacian matrix}. The following claim is stated by Liu and Liu [14] only for connected graphs but in fact their proof does not use the connectedness assumption.

Lemma 4. [14, Theorem 2.1] Let \( G \) be a graph. If \( \rho \) is the spectral radius of \( Q(G) \), then

\[ \sum_{i=1}^{n} d_i^2 \leq m \rho. \]
Lemma 5. [14, Lemma 2.4] Let $G$ be a graph. If $\rho$ is the spectral radius of $Q(G)$, then
\[ \rho \leq 2\Delta. \]

We can now easily deduce:

Lemma 6. Let $G$ be a graph. Then
\[ \sum_{i=1}^{n} d_i^2 \leq 2m\Delta. \]

Proof of Theorem 3 First of all, in light of Lemma 1 and 2 we have:
\[ \epsilon(G) = \rho(G) - \frac{2m}{n} \geq \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2 - \frac{2m}{n}} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2 - \left(\frac{2m}{n}\right)^2} \]
\[ \geq \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2 + \frac{2m}{n}}. \]

Now apply Lemma 3 and then Lemma 2 once more:
\[ \epsilon(G) \geq \frac{1}{n} \sum_{i=1}^{n} d_i^2 - \left(\frac{2m}{n}\right)^2 \leq \frac{\text{var}(G)}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2 + \frac{2m}{n}}} \]
\[ \geq \frac{\text{var}(G)}{2\sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2}}. \]

Finally, use Lemma 6
\[ \epsilon(G) \geq \frac{\text{var}(G)}{2\sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2}} \geq \frac{\text{var}(G)}{2\sqrt{\frac{2m\Delta}{n}}} = \frac{\text{var}(G)\sqrt{n}}{\sqrt{8m\Delta}}. \]

\[ \square \]

4. Proof of Theorem 5

Our approach will be similar to that taken in the proof of Theorem 3 but instead of Hofmeister’s bound for $\rho$ we shall need a more powerful one, due to Yu, Lu, and Tian [19]. To state it, we define the 2-degree $t_i$ of the vertex $v_i$ as the sum of the degrees of the vertices adjacent to $v_i$. That is:
\[ t_i = \sum_{j \sim i} d_j. \]

Lemma 7. [19] Let $G$ be a connected graph. Then,
\[ \rho \geq \sqrt{\frac{\sum_{i=1}^{n} t_i^2}{\sum_{i=1}^{n} d_i^2}} \geq d. \]

Lemma 8. Let $G$ be a high subregular graph on $n$ vertices and with maximum degree $\Delta$. Then $\Delta \leq n - 2$. 

Proof. Suppose that $\Delta = n - 1$. Then we have that all vertices but one are of degree $n - 1$. But this means that the remaining vertex has $n - 1$ neibours as well. This is a contradiction. \hfill $\Box$

Lemma 9. \cite{12} Let $G$ be a connected graph on $n$ vertices and $m$ edges, with maximum degree $\Delta$ and minimum degree $\delta$. Then,

$$\rho \leq \frac{\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - \delta n)}}{2}.$$ 

Corollary 2. Let $G$ be a connected low subregular graph with maximum degree $\Delta$. Then

$$\rho \leq \Delta - 1 + \frac{1}{\Delta}.$$ 

Proof. By Lemma\cite{9} we have

$$\rho \leq \frac{\Delta - 2 + \sqrt{\Delta^2 + 4}}{2}.$$ 

Our conclusion follows by observing that $\Delta^2 + 4 \leq (\Delta + \frac{2}{\Delta})^2$. \hfill $\Box$

Proof of Theorem \cite{5}

Case: $G$ is high subregular

Let $v_1$ be the single vertex of degree $\Delta - 1$ and let $v_2, \ldots, v_\Delta$ be its neighbours. Then we have:

$$t_1 = \Delta(\Delta - 1),$$

$$t_i = \Delta^2 - 1, \quad 2 \leq i \leq \Delta,$$

$$t_i = \Delta^2, \quad \Delta + 1 \leq i \leq n.$$ 

Applying Lemma\cite{7} we get:

$$\rho \geq \sqrt{\frac{\Delta^2(\Delta - 1)^2 + (\Delta - 1)(\Delta^2 - 1)^2 + (n - \Delta)\Delta^4}{(\Delta - 1)^2 + (n - 1)\Delta^2}} =$$

$$= \sqrt{\frac{n\Delta^4 - 4\Delta^3 + 3\Delta^2 + \Delta - 1}{n\Delta^2 - 2\Delta + 1}}.$$ 

The average degree in this case is:

$$d = \Delta - \frac{1}{n}.$$ 

Consider now the following quantity:

$$L(n, \Delta) = \frac{n\Delta^4 - 4\Delta^3 + 3\Delta^2 + \Delta - 1}{n\Delta^2 - 2\Delta + 1} - \left(\Delta - \frac{1}{n}\right)^2.$$
Algebraic manipulation yields:

\[ L(n, \Delta) = \frac{(2\Delta^2 + \Delta - 1)n^2 + (2\Delta - 5\Delta^2)n + 2\Delta - 1}{n^2(n\Delta^2 - 2\Delta + 1)}. \]

This expression is hardly manageable, but it simplifies dramatically upon observing that \( L(n, \Delta) \) is a non-increasing function of \( \Delta \) (this is verified by taking the partial derivative with respect to \( \Delta \), we omit the simple but tedious details). Therefore, using Lemma 8 we have:

\[ L(n, \Delta) \geq L(n, n-2) = \frac{2n^4 - 12n^3 + 27n^2 - 22n - 5}{n^5 - 4n^4 + 2n^3 + 5n^2} \geq \frac{1}{n^2}(2n-4+\frac{6}{n}). \]

Now we can complete the argument, using the well-known fact that \( \Delta \geq \rho \):

\[ \epsilon(G) = \rho - \bar{d} = \frac{\rho^2 - \bar{d}^2}{\rho + \bar{d}} \geq \frac{L(n, \Delta)}{\rho + \bar{d}} \geq \frac{1}{n^2}(2n-4+\frac{6}{n}) = \frac{n^2 - 2n + 3}{n^3\Delta}. \]

**Case: G is low subregular**

As before, let \( v_1 \) be the single vertex of degree \( \Delta \). We have:

\[
\begin{align*}
t_1 &= \Delta(\Delta - 1), \\
t_i &= \Delta^2 - 2\Delta + 2, & 2 \leq i \leq \Delta + 1, \\
t_i &= (\Delta - 1)^2, & \Delta + 2 \leq i \leq n.
\end{align*}
\]

Thus

\[ \rho \geq \sqrt{\frac{\Delta^2(\Delta - 1)^2 + \Delta(\Delta^2 - 2\Delta + 2)^2 + (n - \Delta - 1)(\Delta - 1)^4}{\Delta^2 + (\Delta - 1)^2(n - 1)}} = \]

\[ = \sqrt{\frac{n\Delta^4 - (4n - 4)\Delta^3 + (6n - 9)\Delta^2 - (4n - 7)\Delta + n - 1}{n\Delta^2 - (2n - 2)\Delta + n - 1}}. \]

Keeping in mind that \( \bar{d} = \Delta - 1 + \frac{1}{n} \) we define \( L(n, \Delta) \) to be:

\[
L(n, \Delta) = \frac{n\Delta^4 - (4n - 4)\Delta^3 + (6n - 9)\Delta^2 - (4n - 7)\Delta + n - 1}{n\Delta^2 - (2n - 2)\Delta + n - 1} - \left(\Delta - 1 + \frac{1}{n}\right)^2.
\]

After simplification we get:

\[
L(n, \Delta) = \frac{(2\Delta^2 - 3\Delta + 2)n^2 - (5\Delta^2 - 8\Delta + 3)n - 2\Delta + 1}{(\Delta^2 - 2\Delta + 1)n^3 + (2\Delta - 1)n^2}.
\]
This function is also non-increasing with respect to $\Delta$ and thus we have:

$$L(n, \Delta) \geq L(n, n-1) = \frac{2n^3 - 10n^2 + 15n - 3}{n^2(n^2 - 3n + 3)} \geq \frac{1}{n^2}(2n - 4 - \frac{3}{n}).$$

To complete the argument we resort to Corollary 2:

$$\epsilon(G) = \rho - \delta = \frac{\rho^2 - \delta^2}{\rho + \delta} \geq \frac{L(n, \Delta)}{\rho + \delta} \geq \frac{1}{n^2}(2n - 4 - \frac{3}{n}) \geq \frac{2n^2 - 4n - 3}{2n^3(\Delta - 1 + \frac{1}{\Delta})}.$$

\[ □ \]

5. ADDENDUM

Hong [11] raises the following problem (Problem 3 in his list):

**Question 1.** Let $G$ be the graph with the smallest value of $\epsilon(G)$ among non-regular graphs with $n$ vertices and $m$ edges. Is it true that $\Delta(G) - \delta(G) = 1$?

We remark that Bell [5] has solved the problem of determining the graph with $n$ vertices and $m$ edges that has maximal $\epsilon(G)$.

**References**

[1] Y. Alavi, G. Chartrand, F. R. K. Chung, P. Erdős, R. L. Graham, and O. R. Oellermann. Highly irregular graphs. *J. Graph Theory*, 11(2):235–249, 1987.
[2] M. O. Albertson. The irregularity of a graph. *Ars Comb.*, 46:219–225, 1997.
[3] M. Aouchiche, F. K. Bell, D. Cvetković, P. Hansen, P. Rowlinson, S. K. Simić, and D. Stevanović. Variable neighborhood search for extremal graphs. XVI. Some conjectures related to the largest eigenvalue of a graph. *European J. Oper. Res.*, 191:661–676, 2008.
[4] L. Babai and B. Guiduli. Spectral extrema for graphs: the Zarankiewicz problem. *Electron. J. Comb.*, 16(1):R123, 2009.
[5] F. K. Bell. Eigenvalues and degree deviation in graphs. *Linear Algebra Appl.*, 161:45–54, 1992.
[6] S. M. Cioabă and D. A. Gregory. Large matchings from eigenvalues. *Linear Algebra Appl.*, 422(1):308–317, 2007.
[7] L. Collatz and U. Sinogowitz. Spekter endlicher Grafen. *Abh. Math. Sem. Univ. Hamburg*, 21:63–77, 1957.
[8] C. Elphick and P. Wocjan. New measures of graph irregularity. *Electron. J. Graph Theory Appl.*, 2(1):52–65, 2014.
[9] F. Goldberg. New results on eigenvalues and degree deviation. [http://arxiv.org/abs/1403.2629](http://arxiv.org/abs/1403.2629), 2014.
[10] M. Hofmeister. Spectral radius and degree sequence. *Math. Nachr.*, 139:37–44, 1988.
[11] Y. Hong. Bounds of eigenvalues of graphs. *Discrete Math.*, 123(1–3):65–74, 1993.
Y. Hong, J.-L. Shu, and K. Fang. A sharp upper bound of the spectral radius of graphs. *J. Combin. Theory Ser. B*, 81(2):177–183, 2001.

T. Kővári, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloq. Math.*, 3:50–57, 1954.

M. Liu and B. Liu. New sharp upper bounds for the first Zagreb index. *MATCH Commun. Math. Comput. Chem.*, 62(3):689–698, 2009.

V. Nikiforov. Eigenvalues and degree deviation in graphs. *Linear Algebra Appl.*, 414(1):347–360, 2006.

V. Nikiforov. Bounds on graph eigenvalues II. *Linear Algebra Appl.*, 427(2–3):183–189, 2007.

V. Nikiforov. A contribution to the Zarankiewicz problem. *Linear Algebra Appl.*, 432(6):1405–1411, 2010.

R. Sharma, M. Gupta, and G. Kapoor. Some better bounds on the variance with applications. *J. Math. Inequal.*, 4(3):355–363, 2010.

A. Yu, M. Lu, and F. Tian. On the spectral radius of graphs. *Linear Algebra Appl.*, 387:41–49, 2004.

R. Yuster. Maximum matching in regular and almost regular graphs. *Algorithmica*, 66(1):87–92, 2013.

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