Norden structures on cotangent bundles

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Abstract. We study prolongation of Norden structures on manifolds to their generalized tangent bundles and to their cotangent bundles. In particular, by using methods of generalized geometry, we prove that the cotangent bundle of a complex Norden manifold, \((M, J, g)\) admits a structure of Norden manifold, \((T^*(M), \tilde{J}, \tilde{g})\). Moreover if \((M, J, g)\) has flat natural canonical connection then \(\tilde{J}\) is integrable, that is \((T^*(M), \tilde{J}, \tilde{g})\) is a complex Norden manifold. Finally we prove that if \((M, J, g)\) is Kähler Norden flat then \((T^*(M), \tilde{J}, \tilde{g})\) is Kähler Norden flat.

Keywords: Complex manifolds, Cotangent bundles, Generalized Geometry, Norden manifolds.

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1 Introduction

Let \(M\) be a smooth manifold, let \(T(M)\) be the tangent bundle and let \(T^*(M)\) be the cotangent bundle of \(M\). \(E = T(M) \oplus T^*(M)\) is called the generalized tangent bundle of \(M\). In [H] Hitchin introduced the concept of generalized complex structure, he considered complex structures of the generalized tangent bundle compatible with the standard metric of neutral signature on \(E\). In previous papers, [N1], [N2], [N3], [N4], [N5], [N6], we defined and studied a class of complex structures on the generalized tangent bundle compatible with the standard symplectic structure of \(E\), that is a class of pseudo calibrated generalized complex structures. In particular in [N6] we described generalized complex structures defined naturally by Norden structures. In this paper we study prolongation of Norden structures on manifolds to their generalized tangent bundles and to their cotangent bundles. Precisely, by using methods of generalized geometry, we prove that the cotangent bundle of a complex Norden manifold, \((M, J, g)\), admits a structure of Norden manifold, \((T^*(M), \tilde{J}, \tilde{g})\). Moreover if \((M, J, g)\) has flat natural canonical connection then \(\tilde{J}\) is integrable, that is \((T^*(M), \tilde{J}, \tilde{g})\) is a complex Norden manifold. Finally we prove that if \((M, J, g)\) is Kähler Norden flat then \((T^*(M), \tilde{J}, \tilde{g})\) is Kähler Norden flat.

The paper is organized as follows. In sections 2 we introduce basics on generalized tangent bundles, generalized complex structures and Norden manifolds.
Original results are contained in section 3, where we construct Norden structures on generalized tangent bundles, and in section 4, where we construct Norden structures on cotangent bundles and we describe curvature properties.

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2 Preliminaries

2.1 Generalized tangent bundle

Let $M$ be a smooth manifold of real dimension $n$ and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of $M$. Smooth sections of $E$ are elements $X + \xi \in C^\infty(E)$ where $X \in C^\infty(T(M))$ is a vector field and $\xi \in C^\infty(T^*(M))$ is a 1-form. $E$ is equipped with a natural symplectic structure, $(\ , \ )$, defined on two elements $X + \xi, Y + \eta \in C^\infty(E)$ by:

$$ (X + \xi, Y + \eta) = -\frac{1}{2}(\xi(Y) - \eta(X)). $$

Moreover $E$ is equipped with a natural indefinite metric, $< , >$, defined by:

$$ < X + \xi, Y + \eta >= -\frac{1}{2}(\xi(Y) + \eta(X)); $$

$< , >$ is non degenerate and of signature $(n, n)$.

A linear connection on $M$, $\nabla$, defines a bracket on $C^\infty(E)$, $[\ , ]_\nabla$, as follows:

$$ [X + \xi, Y + \eta]_\nabla = [X, Y] + \nabla_X \eta - \nabla_Y \xi $$

where $[\ , ]$ means Lie bracket of vector fields.

Lemma 2.1. (N1) For all $X, Y \in C^\infty(T(M))$, for all $\xi, \eta \in C^\infty(T^*(M))$ and for all $f \in C^\infty(M)$ we have:

1. $[X + \xi, Y + \eta]_\nabla = -[Y + \eta, X + \xi]_\nabla$
2. $[f(X + \xi), Y + \eta]_\nabla = f[X + \xi, Y + \eta]_\nabla - Y(f)(X + \xi)$
3. Jacobi’s identity holds for $[\ , ]_\nabla$ if and only if $\nabla$ has zero curvature.

2.2 Pseudo calibrated generalized complex structures

Definition 2.2. A generalized complex structure on $M$ is an endomorphism $\hat{J} : E \to E$ such that $\hat{J}^2 = -I$.

Definition 2.3. A generalized complex structure $\hat{J}$ is called pseudo calibrated if it is $(\ , \ )$-invariant and if the bilinear symmetric form defined by $(\ , \hat{J})$ on $T(M)$ is non degenerate. Moreover $\hat{J}$ is called calibrated if it is pseudo calibrated and $(\ , \hat{J})$ is positive definite.
From the definition we obtain that a generalized pseudo calibrated complex structure has the following block matrix form:

\[
\hat{J} = \begin{pmatrix} J & -(I + J^2)g^{-1} \\ g & -J^* \end{pmatrix}
\]

where \( g : T(M) \to T^*(M) \) is identified to the bemolle musical isomorphism of a pseudo Riemannian metric \( g \) on \( M \), \( J : T(M) \to T(M) \) is a \( g \)-symmetric operator and \( J^* : T^*(M) \to T^*(M) \) is the dual operator of \( J \) defined by \( J^*(\xi)(X) = \xi(J(X)) \).

\( \hat{J} \) is calibrated if and only if \( g \) is a Riemannian metric, namely:

\[
(g(X))(Y) = g(X, Y) = 2(X, \hat{J}Y).
\]

2.3 Integrability

**Lemma 2.4.** ([N2]) Let \( \hat{J} : E \to E \) be a generalized complex structure on \( M \) and let

\[
N^\nabla(\hat{J}) : C^\infty(E) \times C^\infty(E) \to C^\infty(E)
\]

defined for all \( \sigma, \tau \in C^\infty(E) \) by:

\[
N^\nabla(\hat{J})(\sigma, \tau) = [\hat{J}\sigma, \hat{J}\tau]_\nabla - \hat{J}[\hat{J}\sigma, \tau]_\nabla - \hat{J}[\sigma, \hat{J}\tau]_\nabla - [\sigma, \tau]_\nabla.
\]

\( N^\nabla(\hat{J}) \) is a skew symmetric tensor called the Nijenhuis tensor of \( \hat{J} \) with respect to \( \nabla \).

Let \( E^C = (T(M) \oplus T^*(M)) \otimes \mathbb{C} \) be the complexified generalized tangent bundle. The splitting in \( \pm i \) eigenspaces of \( \hat{J} \) is denoted by: \( E^C = E^{1,0}_\hat{J} \oplus E^{0,1}_\hat{J} \), with \( E^{0,1}_\hat{J} = \overline{E^{1,0}_\hat{J}} \). Let \( P_+ : E^C \to E^{1,0}_\hat{J} \) and \( P_- : E^C \to E^{0,1}_\hat{J} \) be the projection operators: \( P_\pm = \frac{1}{2}(I \mp i\hat{J}) \). The following holds.

**Lemma 2.5.** ([N2]) For all \( \sigma, \tau \in C^\infty(E^C) \) we have:

\[
P_\pm[P_\pm(\sigma), P_\pm(\tau)]_\nabla = -\frac{1}{4} P_\pm(N^\nabla(\hat{J})(\sigma, \tau)).
\]

**Corollary 2.6.** For any linear connection \( \nabla \) on \( M \) \( E^{1,0}_\hat{J} \) and \( E^{0,1}_\hat{J} \) are \([,]_\nabla\)-involutive if and only if \( N^\nabla(\hat{J}) = 0 \).

**Definition 2.7.** Let \( \hat{J} : E \to E \) be a generalized complex structure on \( M \), \( \hat{J} \) is called \( \nabla \)-integrable if \( N^\nabla(\hat{J}) = 0 \).
Let \((M, g)\) be a pseudo Riemannian manifold, let \(\nabla\) be a linear connection on \(M\), the torsion of \(\nabla\), \(T^{\nabla}\), and the exterior differential associated to \(\nabla\) acting on \(g\), \((d^{\nabla} g)\), are defined on \(X, Y \in C^\infty(T(M))\) respectively by:

\[
T^{\nabla}(X, Y) = \nabla X Y - \nabla Y X - [X, Y]
\]

\[
(d^{\nabla} g)(X, Y) = (\nabla X g)(Y) - (\nabla Y g)(X) + g(T^{\nabla}(X, Y)).
\]

Let \(J\) be a \(g\)-symmetric operator on \(T(M)\) and let \(N(J)\) be the Nijenhuis tensor of \(J\), defined on \(X, Y \in C^\infty(T(M))\) by:

\[
N(J)(X, Y) = [JX, JY] - H[JX, Y] - J[X, JY] + J^2[X, Y].
\]

Let us suppose \(J^2 = -I\). Let \(\hat{J}\) be the pseudo calibrated generalized complex structure defined by \(g\) and \(J\):

\[
\hat{J} = \begin{pmatrix} J & 0 \\ g & -J^* \end{pmatrix}.
\]

\(\nabla\) - integrability of \(\hat{J}\) is described as follows.

**Theorem 2.8.** ([N5], [N6]) The pseudo calibrated generalized complex structure \(\hat{J} = \begin{pmatrix} J & 0 \\ g & -J^* \end{pmatrix}\) is \(\nabla\) -integrable if and only if for all \(X, Y \in C^\infty(T(M))\) the following conditions hold:

\[
\begin{cases}
N(J) = 0 \\
\nabla J = 0 \\
(d^{\nabla} g)(JX, Y) + (d^{\nabla} g)(X, JY) = 0.
\end{cases}
\]

### 2.4 Norden manifolds

Norden manifolds were introduced by Norden in [N] and then studied also as almost complex manifolds with B-metric and anti Kählerian manifolds, [BFV], [GM]. They have applications in mathematics and in theoretical physics.

**Definition 2.9.** Let \((M, J)\) be an almost complex manifold and let \(g\) be a pseudo Riemannian metric on \(M\) such that \(J\) is a \(g\)-symmetric operator, \(g\) is called Norden metric and \((M, J, g)\) is called Norden manifold.

**Definition 2.10.** Let \((M, J, g)\) be a Norden manifold, if \(J\) is integrable then \((M, J, g)\) is called complex Norden manifold.

The following result is well known.
Theorem 2.11. (GM) Let $(M, J, g)$ be a complex Norden manifold, there exists a unique linear connection $D$ with torsion $T$ on $M$ such that:

\[
\begin{align*}
(D_X g)(Y, Z) &= 0, \\
T(JX, Y) + T(X, JY) &= 0, \\
g(T(X, Y), Z) + g(T(Y, Z), X) + g(T(Z, X), Y) &= 0.
\end{align*}
\]

for all $X, Y, Z \in C^\infty(T(M))$. $D$ is called the natural canonical connection.

Remark 2.12. The natural canonical connection $D$ is defined by:

\[
D_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J)Y
\]

where $\nabla$ is the Levi Civita connection of $g$. In particular $DJ = 0$.

Then, from Theorem 2.8 and Theorem 2.11, we get the following.

Proposition 2.13. (NA) Let $(M, J, g)$ be a complex Norden manifold and let $D$ be the natural canonical connection on $M$, the generalized complex structure on $M$ defined by $J$ and $g$:

\[
\hat{J} = \begin{pmatrix} J & 0 \\ g & -J^* \end{pmatrix}
\]

is $D$ -integrable.

Definition 2.14. Let $(M, J, g)$ be a Norden manifold and let $\nabla$ be the Levi Civita connection of $g$, $(M, J, g)$ is called Kähler Norden manifold if $\nabla J = 0$.

Remark 2.15. For a Kähler Norden manifold $(M, J, g)$ the structure $J$ is integrable and the natural canonical connection is the Levi Civita connection of $g$.

3 Norden structures on generalized tangent bundles

3.1 Norden metric on $T(M) \oplus T^*(M)$

Let $(M, J, g)$ be a Norden manifold and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of $M$. Let $X + \xi$, $Y + \eta$ be smooth sections of $E$ and let $\flat : T^*(M) \to T(M)$ be the inverse of the bemolle musical isomorphism of $g$, $\flat : T^*(M) \to T^*(M)$, $\flat(X)(Y) = g(X, Y)$, for all $X, Y \in C^\infty(T(M))$.

We define:

\[
g(X + \xi, Y + \eta) = g(X, Y) + \frac{1}{2} g(JX, \flat\eta) + \frac{1}{2} g(\flat\xi, JY) + g(\flat\xi, \flat\eta).
\]

We have the following.
Proposition 3.1. \( \hat{g} \) is symmetric and the pseudo calibrated generalized complex structure on \( M \) defined by \( g \) and \( J, \hat{J} = \begin{pmatrix} J & 0 \\ g & -J^* \end{pmatrix} \), is a \( \hat{g} \)-symmetric operator.

Proof. The symmetry of \( \hat{g} \) follows immediately from the definition. In order to verify that \( \hat{J} \) is a \( \hat{g} \)-symmetric operator let us compute:

\[
\hat{g}(\hat{J}(X + \xi), Y + \eta) = g(JX + g(X) - J^*(\xi), Y + \eta)
\]

\[
= g(JX, Y) + \frac{1}{2}g(-X, J\eta) + \frac{1}{2}g(X - J^*(\xi), JY) + g(X - J^*(\xi), J\eta)
\]

\[
= g(JX, Y) + \frac{1}{2}g(X, J\eta) + \frac{1}{2}g(X, JY) - \frac{1}{2}g(J^*(\xi), JY) - g(J^*(\xi), J\eta).
\]

On the other hand we have:

\[
\hat{g}(X + \xi, \hat{J}(Y + \eta)) = g(X + \xi, JY + g(Y) - J^*(\eta))
\]

\[
= g(X, JY) + \frac{1}{2}g(X, JY - J^*(\eta)) + \frac{1}{2}g(J\xi, -Y) + g(J\xi, JY - J^*(\eta))
\]

\[
= g(X, JY) + \frac{1}{2}g(X, JY) - \frac{1}{2}g(JX, J^*(\eta)) + \frac{1}{2}g(J\xi, Y) - g(J^*(\xi), J\eta).
\]

From the property \( J^* = J^* \) and from the fact that \((M, J, g)\) is a Norden manifold we get the statement. \( \square \)

Definition 3.2. Let \((M, J, g)\) be a Norden manifold, \((\hat{J}, \hat{g})\) defined as before is called \textit{Norden structure on} \( T(M) \oplus T^*(M) \) defined by \((J, g)\).

3.2 Natural canonical connection on \( T(M) \oplus T^*(M) \)

Let \((M, J, g)\) be a complex Norden manifold and let \( D \) be the natural canonical connection on \( M \), we define

\[
(3.2) \quad \hat{D} : C^\infty(T(M)) \times C^\infty(T(M) \oplus T^*(M)) \rightarrow C^\infty(T(M) \oplus T^*(M))
\]

by:

\[
(3.3) \quad \hat{D}_X(Y + \eta) = D_XY + D_X\eta
\]

for all \( X \in C^\infty(T(M)) \), \( Y + \eta \in C^\infty(T(M) \oplus T^*(M)) \).

We have:
**Proposition 3.3.** The following condition hold:

\[
\begin{align*}
\hat{D}_X \hat{J} &= 0 \\
\hat{D}_X \hat{g} &= 0.
\end{align*}
\]

for all \( X \in C^\infty(T(M)) \). Moreover if \( D \) is flat then \( \hat{D} \) is flat. \( \hat{D} \) is called the natural canonical connection of \( T(M) \oplus T^*(M) \).

**Proof.** From the definition of \( \hat{J} \) and from the properties of \( D \) we get:

\[
\begin{align*}
\hat{D}_X \hat{J} (Y + \eta) &= \hat{D}_X (JY + g(X) - J^* \eta) = D_X JY + D_X (g(Y) - J^* \eta) \\
&= JD_X Y + g(D_X Y) - J^* (D_X \eta) = \hat{J}(D_X Y) + \hat{J}(D_X \eta) = \hat{J} (\hat{D}_X (Y + \eta))
\end{align*}
\]

for all \( X \in C^\infty(M), Y + \eta \in C^\infty(T(M) \oplus T^*(M)) \).

Moreover from the definition of \( \hat{g} \) we get:

\[
\begin{align*}
X \hat{g}(Y + \eta, Z + \zeta) - \hat{g}(\hat{D}_X(Y + \eta), Z + \zeta) - \hat{g}(Y + \eta, \hat{D}_X(Z + \zeta))
&= X \{ g(Y, Z) + \frac{1}{2} g(JY, \zeta) + \frac{1}{2} g(\eta, JZ) + g(\eta, \zeta) \} + \\
&\quad - \{ g(D_X Y, Z) + \frac{1}{2} g(\hat{J}(D_X Y), \zeta) + \frac{1}{2} g(\eta, JZ) + g(\eta, \zeta) \} + \\
&\quad - \{ g(Y, D_X Z) + \frac{1}{2} g(JY, \zeta(D_X \zeta) + \frac{1}{2} g(\eta, J(D_X Z)) + g(\eta, \zeta(D_X \zeta)) \}
&= 0
\end{align*}
\]

for all \( X, Y, Z \in C^\infty(T(M)), Y + \eta, Z + \zeta \in C^\infty(T(M) \oplus T^*(M)) \).

Finally, if \( R^D \) denotes the curvature tensor of \( D \):

\[
R^D(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}
\]

and \( R^{\hat{D}} \) the curvature tensor of \( \hat{D} \):

\[
R^{\hat{D}}(X, Y)(Z + \zeta) = \hat{D}_X \hat{D}_Y (Z + \zeta) - \hat{D}_Y \hat{D}_X (Z + \zeta) - \hat{D}_{[X,Y]} (Z + \zeta),
\]

we have:

\[
R^{\hat{D}}(X, Y)(Z + \zeta) = R^D(X, Y)Z + R^D(X, Y)(\zeta)
\]

for all \( X, Y, Z \in C^\infty(T(M)), \zeta \in C^\infty(T^*(M)) \).

Then the proof is complete. \( \square \)
4 Norden structures on cotangent bundles

4.1 Complex Norden structure on $T^*(M)$

Let $(M, J, g)$ be a Norden manifold and let $\nabla$ be a linear connection on $M$. $\nabla$ defines the decomposition in horizontal and vertical subbundles of $T(T^*(M))$:

\begin{equation}
T(T^*(M)) = T^H(T^*(M)) \oplus T^V(T^*(M)).
\end{equation}

Let $\{x^1, ..., x^n\}$ be local coordinates on $M$, let $\{\tilde{x}^1, ..., \tilde{x}^n, y_1, ..., y_n\}$ be the corresponding local coordinates on $T^*(M)$ and let $\left\{\frac{\partial}{\partial \tilde{x}^1}, ..., \frac{\partial}{\partial \tilde{x}^n}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\right\}$ be local frames on $T(T^*(M))$. We have:

\begin{align*}
(4.1) & \quad \left(\frac{\partial}{\partial \tilde{x}^i}\right)^H = \frac{\partial}{\partial \tilde{x}^i} + y_k \Gamma^k_{il} \frac{\partial}{\partial y_l}, \\
(4.2) & \quad \left(\frac{\partial}{\partial \tilde{x}^i}\right)^V = -y_k \Gamma^k_{il} \frac{\partial}{\partial y_l}, \\
(4.3) & \quad \left(\frac{\partial}{\partial y_i}\right)^H = 0, \\
(4.4) & \quad \left(\frac{\partial}{\partial y_i}\right)^V = \frac{\partial}{\partial y_i},
\end{align*}

where $i, k, l$ run from 1 to $n$, $\Gamma^k_{il}$ are Christoffel’s symbols of $\nabla$ and we used Einstein’s convention on repeated indices.

Let us denote $X_i = \frac{\partial}{\partial \tilde{x}^i}$, we have:

\begin{align*}
(4.5) & \quad [X^H_i, X^H_j] = y_k R^k_{ijl} \frac{\partial}{\partial y_l}, \\
(4.6) & \quad [X^H_i, \frac{\partial}{\partial y_j}] = -\Gamma^j_{il} \frac{\partial}{\partial y_i},
\end{align*}

where $R$ is the curvature tensor of $\nabla$:

\begin{equation}
R(X_i, X_j)X_l = R^k_{ijl} X_k = (\nabla_{X_i} \nabla_{X_j} - \nabla_{X_j} \nabla_{X_i} - \nabla_{[X_i, X_j]}) X_l.
\end{equation}

We recall the following.
**Proposition 4.1.** ([N2]) Let $M$ be a smooth manifold and let $\nabla$ be a linear connection on $M$, there is a bundle morphism:

\[
\Phi^\nabla : T(M) \oplus T^*(M) \to T(T^*(M))
\]

which is an isomorphism on the fibres and such that

1. $(\Phi^\nabla)^*(\Omega) = -2( , )$ if and only if $\nabla$ has zero torsion,
2. $(\Phi^\nabla)([ , ]) = [\Phi^\nabla, \Phi^\nabla]$ if and only if $\nabla$ has zero curvature,

where $\Omega$ is the canonical symplectic form on $T^*(M)$ defined by the Liouville 1-form.

In local coordinates we have the following expressions:

\[
\Phi^\nabla \left( \frac{\partial}{\partial x^i} \right) = X_i^H
\]

\[
\Phi^\nabla (dx^j) = \frac{\partial}{\partial y^j}.
\]

Let $(\tilde{J}, \tilde{g})$ be the Norden structure on $T(M) \oplus T^*(M)$ defined in previous section, the isomorphism $\Phi^\nabla$ allows us to define an almost complex structure $\tilde{J}$ and a neutral metric $\tilde{g}$ on $T^*(M)$ as in the following.

We define $\tilde{J} : T(T^*(M)) \to T(T^*(M))$ by

\[
\tilde{J} = (\Phi^\nabla) \circ \tilde{J} \circ (\Phi^\nabla)^{-1}
\]

and the pseudo Riemannian metric $\tilde{g}$ on $T^*(M)$ by

\[
\tilde{g} = ((\Phi^\nabla)^{-1})^*(\tilde{g}).
\]

**Proposition 4.2.** $(T^*(M), \tilde{J}, \tilde{g})$ is a Norden manifold.

**Proof.** For all $X, Y \in C^\infty(T(T^*(M)))$ we have:

\[
\tilde{g}(\tilde{J}(X), Y) = \tilde{g}((\Phi^\nabla)^{-1}((\Phi^\nabla) \circ \tilde{J} \circ (\Phi^\nabla)^{-1})(X), (\Phi^\nabla)^{-1}(Y))
\]

\[
= \tilde{g}(\tilde{J}((\Phi^\nabla)^{-1}(X)), (\Phi^\nabla)^{-1}(Y)) = \tilde{g}((\Phi^\nabla)^{-1}(X), \tilde{J}((\Phi^\nabla)^{-1}(Y)))
\]

\[
= \tilde{g}((\Phi^\nabla)^{-1}(X), (\Phi^\nabla)^{-1}((\Phi^\nabla) \circ \tilde{J} \circ (\Phi^\nabla)^{-1})(Y))) = \tilde{g}(X, \tilde{J}(Y)). \square
\]

Direct computations give the following local expressions for $\tilde{J}$ and $\tilde{g}$:

\[
\tilde{J} \left( X_i^H \right) = J_i^k X_k^H + g_{ik} \frac{\partial}{\partial y_k}
\]

\[
\tilde{J} \left( \frac{\partial}{\partial y_j} \right) = -J_k^j \frac{\partial}{\partial y_k}.
\]
Moreover, if we denote by $\tilde{N}$ the Nijenhuis tensor of $\tilde{J}$, the following hold:

\begin{align}
\tilde{N} \left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) &= \Phi \nabla \left( N \nabla (J) (dx^i, dx^j) \right) \\
\tilde{N} \left( X^H_i, \frac{\partial}{\partial y_j} \right) &= -((\nabla_J X, J) X_k - J (\nabla_J X) X_k) \frac{\partial}{\partial y_k} \\
\tilde{N} \left( X^H_i, X^H_j \right) &= \Phi \nabla \left( N \nabla (\tilde{J}) (X_i, X_j) \right) + \\
&+ y_l \left( J^r_i J^s_j R^l_{khr} + J^r_i J^s_j R^l_{kjr} - J^r_i J^s_j R^l_{kjr} - R^l_{ijkl} \right) \frac{\partial}{\partial y_l}.
\end{align}

Thus we get the following.

**Proposition 4.3.** Let $(M, J, g)$ be a complex Norden manifold with flat natural canonical connection then $(T^*(M), \tilde{J}, \tilde{g})$ is a complex Norden manifold.

### 4.2 Kähler Norden structure on $T^*(M)$

A direct computation gives the following.

**Lemma 4.4.** Let $(M, J, g)$ be a Norden manifold and let $(\tilde{J}, \tilde{g})$ be the Norden structure defined on $T^*(M)$ as in Proposition 4.2.. Let $\nabla$ and $\tilde{\nabla}$ be the Levi Civita connection of $g$ and $\tilde{g}$ respectively, then:

\begin{align}
\tilde{\nabla}_{X^H_i} X^H_j &= \{ \Gamma^r_{ij} - \frac{1}{10} (\nabla J) \frac{\partial}{\partial x^r} \} + \\
&+ (\nabla J) \frac{\partial}{\partial x^r} + J^r_p \frac{\partial}{\partial x^p} + \frac{1}{5} g^{rs} y_k \left( R^i_{js} J^s_l - 2 R^k_{ils} J^s_l - 2 R^k_{jls} J^s_l \right) \frac{\partial}{\partial y_k} + \\
&+ \frac{1}{5} \left( g_{rs} \left( (\nabla J) \frac{\partial}{\partial x^l} + (\nabla J) \frac{\partial}{\partial x^l} + (\nabla J) \frac{\partial}{\partial x^l} + (\nabla J) \frac{\partial}{\partial x^l} \right) + y_k \left( 3 R^i_{js} J^s_l + J^i_j J^r_s R^l_{str} + J^i_j J^r_s R^l_{str} \right) \frac{\partial}{\partial y_s}.
\end{align}
\[
\n\hat{\nabla}_{j} X_{i}^{H} = \frac{1}{5} \{ g^{r k} (\nabla \partial J) \frac{\partial}{\partial x^{r}} - (\nabla \partial J) \partial \frac{\partial}{\partial x^{r}} \}^{j} + \\
-2 g^{r k} g^{l j} R_{i i}^{l} \} X_{k}^{H} + \frac{1}{10} \{ -J_{r}^{i} (\nabla \partial J) \frac{\partial}{\partial x^{r}} + \\
- (\nabla \partial J) \frac{\partial}{\partial x^{r}} \}^{i} + \\
+ 2 J_{s}^{i} g^{j k} R_{r s t}^{k} \} \frac{\partial}{\partial y^{s}}
\]

(4.20)

\[
\hat{\nabla}_{X^{H}} \frac{\partial}{\partial y^{j}} = \hat{\nabla}_{j} X^{H} = \Gamma_{j}^{i} \frac{\partial}{\partial y^{s}}
\]

(4.21)

\[
\hat{\nabla} \frac{\partial}{\partial y^{j}} = 0
\]

(4.22)

In particular the following hold.

**Proposition 4.5.** Let \((M, J, g)\) be a flat Kähler Norden manifold then \((T^{*}(M), \hat{J}, \hat{g})\) is a flat Kähler Norden manifold.

**Proof.** From Lemma 4.4, under the assumption \((M, J, g)\) is Kähler Norden flat, we get the following expression of \(\hat{\nabla}\):

\[
\hat{\nabla}_{X^{H}} X^{H} = \Gamma_{r}^{i} X^{H}
\]

(4.23)

\[
\hat{\nabla} \frac{\partial}{\partial y^{j}} X^{H} = 0
\]

(4.24)

\[
\hat{\nabla}_{X^{H}} \frac{\partial}{\partial y^{j}} = -\Gamma_{s}^{i} \frac{\partial}{\partial y^{s}}
\]

(4.25)

\[
\hat{\nabla} \frac{\partial}{\partial y^{j}} = 0.
\]

(4.26)
Then we get:

\begin{align}
(\tilde{\nabla}_{X^H_i} \tilde{J})X^H_j &= (\nabla_{\partial_j} \tilde{J})\frac{\partial}{\partial x^i}Y^r X^H_r = 0 \\
(4.27)
\end{align}

\begin{align}
(\tilde{\nabla}_{\partial_j} \tilde{J})X^H_i &= 0 \\
(4.28)
\end{align}

\begin{align}
(\tilde{\nabla}_{X^H_i} \tilde{J})\frac{\partial}{\partial y^j} &= -((\nabla_{\partial_i} \tilde{J})\frac{\partial}{\partial x^j})\frac{\partial}{\partial y_j} = 0 \\
(4.29)
\end{align}

\begin{align}
(\tilde{\nabla}_{\partial_i} \tilde{J})\frac{\partial}{\partial y^j} &= 0. \\
(4.30)
\end{align}

Hence \( \tilde{\nabla}\tilde{J} = 0 \).

Moreover let \( \tilde{R} \) be the Riemann curvature tensor of \( \tilde{g} \), we have:

\begin{align}
\tilde{R}(X^H_i, X^H_j)X^H_k &= R^l_{ijk}X^H_l = 0 \\
(4.31)
\end{align}

\begin{align}
\tilde{R}(X^H_i, X^H_j)\frac{\partial}{\partial y_k} &= R^k_{jir}\frac{\partial}{\partial y_r} = 0 \\
(4.32)
\end{align}

\begin{align}
\tilde{R}(X^H_i, \frac{\partial}{\partial y_k})X^H_j &= 0 \\
(4.33)
\end{align}

\begin{align}
\tilde{R}(X^H_i, \frac{\partial}{\partial y_j})\frac{\partial}{\partial y_k} &= 0 \\
(4.34)
\end{align}

\begin{align}
\tilde{R}(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})\frac{\partial}{\partial y_k} &= 0. \\
(4.35)
\end{align}

Thus \( \tilde{R} = 0 \) and the proof is complete. \( \Box \)
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