Conservation laws for a class of nonlinear equations with variable coefficients on discrete and noncommutative spaces

M. Klimek
Institute of Mathematics and Computer Science, Technical University of Częstochowa, ul.Dąbrowskiego 73, 42-200 Częstochowa, Poland
klimek@matinf.pcz.czest.pl

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Abstract

The conservation laws for a class of nonlinear equations with variable coefficients on discrete and noncommutative spaces are derived. For discrete models the conserved charges are constructed explicitly. The applications of the general method include equations on quantum plane, supersymmetric equations for chiral and antichiral supermultiplets as well as auxiliary equations of integrable models - principal chiral model and various cases of nonlinear Toda lattice equations.

1 Introduction

Our aim is to present the procedure of derivation of the conservation laws and consequently the conserved charges for a certain class of nonlinear equations with variable coefficients. In the classical field theory the conserved currents and charges follow from the Noether theorem and are connected with the symmetry of the action. In the case of linear equations of motion with constant coefficients the construction of conserved currents by Takahashi and Umezawa method \cite{20} can be applied.
In the previous papers [10, 11, 12, 13] we have extended this procedure to the linear equations on discrete and noncommutative spaces including quantum Minkowski [22, 23, 24] and braided linear spaces [17, 18, 19, 20]. It appears however that we can consider in a similar way a wide class of equations with variable coefficients built within framework of any differential calculus with the Leibnitz rule for partial derivatives deformed via the transformation operator which is multiplicative and invertible. The range of admissible spaces includes the classical space-time with continuous coordinates, the discrete space, the mixed space with discrete and continuous coordinates, superspace including space-time and spinor coordinates, quantum Minkowski and braided linear spaces (with q-Minkowski as a special case).

The equations we shall investigate have the variable coefficients which fulfill the corresponding restriction (3) including the conjugated derivatives. The possibility of derivation the conservation laws for some equations with variable coefficients on noncommutative spaces was indicated earlier [12, 13] but the proposed conditions for coefficients were too strong to construct useful examples. In contrast the new condition (as we show explicitly in the applications) appears to be identical with the nonlinear equations of some of the integrable models for which we derive the conserved currents via the auxiliary linear equations with variable coefficients.

Resuming the proposed method can be applied to nonlinear models in two ways namely we can consider the equation with nonlinear term free from the derivatives (in form of the potential) or alternatively we investigate the auxiliary linear equations with variable coefficients for nonlinear integrable models.

The paper is organized as follows: the section 2 contains the description of the investigated models and the derivation of the conservation laws. In section 3 we extend the procedure to discrete and mixed discrete and continuous models including both types of derivatives initial $\partial$ and its conjugation $\partial^\dagger$. For this class of equations we add the construction of conserved charges following the classical method of integrating the time-component of the conserved current over the subspace excluding the time-coordinate. This result is due to the fact that for the discrete and classical differential calculi the definite integral over mentioned subspace is known to commute with time-derivative (64).

The section 4 includes applications of the derived procedure. We start with a simple example of an equation of the second order with variable coefficients on quantum plane, then we consider the conserved currents and charges for
a pair of chiral and antichiral supermultiplts on D=4 N=1 superspace connected via nonlinear equation of motion. This section is closed with a review of conservation laws and conserved charges for a range of integrable models. We apply our method to the principal chiral model equations and their auxiliary linear equations. In this example only the classical commutative derivatives are involved; nevertheless it illustrates well the connection between the imposed condition (3) and the initial nonlinear (in this case the principal chiral model) equation. The next model is the nonlinear Toda lattice equation for which we use our results for mixed discrete and continuous spaces in order to write explicitly the conserved currents and charges.

2 Nonlinear equations with variable coefficients and their conservation law

In the previous papers we discussed the conservation laws for linear equations with constant coefficients for discrete differential calculus [10, 11] as well as for a wider category of equations acting on noncommutative spaces namely on quantum Minkowski spaces [12, 13] and the braided linear ones [14, 15, 16]. Now we would like to present our results for an extended class of equations with variable coefficients of the form:

\[ \Lambda(\partial)\Phi = 0 \]  
\[ \Lambda(\partial) = \Lambda_0 + \sum_{l=1}^{N} \Lambda_{\mu_1...\mu_l} \partial^{\mu_1}...\partial^{\mu_l} \]  

where coefficients (may be matrices) fulfill the condition:

\[ \partial^\dagger \Lambda_{\mu_1...\mu_l} = 0 \]  

for \( l = 1, ..., N \) and with the conjugated derivative described below (23).

We include into considerations the nonlinear equations provided the only nonlinear term does not depend on derivatives that means:

\[ \Lambda_0 = \Lambda_0(\Phi) \]  

As will be shown in the sequel this class of equations contains the equations of motion for supersymmetric models in superfield formulation as well as auxiliary linear equations yielded by nonlinear integrable models as for example Toda lattice equations or principal chiral model equations (the last being an
Our construction presented below in Propositions 2.1 and 2.2 holds for any differential calculus with the following deformation of the Leibnitz rule:

$$\partial^i(fg) = (\partial^i f)g + (\zeta^i f)\partial^i g$$

(5)

where $\zeta$ is the invertible transformation operator.

Starting with the classical commutative differential calculus (for which $\zeta$ is the identity operator) we can include the discrete differential calculus with derivatives defined by one of the formulas:

$$\partial^i f_k = (\zeta^i - 1)f_k = f_{k+1} - f_k$$

(6)

$$\partial^i f_k = (x_{k+1} - x_k)^{-1}(\zeta^i - 1)f_k = (x_{k+1} - x_k)^{-1}(f_{k+1} - f_k)$$

(7)

where $k$ denotes the position on the lattice in the direction $i$.

The first one is widely used in the discrete models and measures the difference between the values of the function between two points of the lattice in the given direction $i$ while the second one (introduced [10, 11]) is a quotient of the change of value of the function over the distance between the two neighbouring points of the lattice taken in the direction $i$. Both types of the derivatives obey (5) with the transformation operator being the shift operator along the lattice in the direction $i$:

$$\zeta^i f_k = f_{k+1}$$

(8)

where $k$ in the above formula denotes the value of the $i$-th coordinate of the point on the lattice.

It is clear that we can also include mixed models depending both on classical commutative derivatives with respect to continuous coordinates and discrete derivatives (6) or (7) with respect to lattice dimensions.

The Leibnitz rule of the form (5) is also characteristic for noncommutative spaces. We have checked [12, 13] that for quantum Minkowski spaces and for braided linear spaces (including the q-Minkowski space) it is given by the formula (5) with transformation operators determined by their multiplicity property:

$$\zeta^i(fg) = (\zeta^i f)(\zeta^i g)$$

(9)

and the action on monomials of the first order:

$$\zeta^i x^k = R^{jk} x^a - (RZ)^{jk}_i$$

(10)

$$\zeta^i x^k = R^{ji} x^l$$

(11)
The first of the above formulas is valid on the quantum Minkowski and the second on the braided linear space with \( R \) being the matrices fulfilling the QYBE \([24, 17]\). For quantum Minkowski spaces introduced by Podleś and Woronowicz the \( R \) matrix is self-invertible \( R^2 = 1 \) and for braided linear spaces developed by Majid it is bi-invertible:

\[
\begin{align*}
(R^{-1})_{kl}^{ij} R_{ab}^{kl} &= R_{kl}^{ij} (R^{-1})_{ab}^{kl} = \delta_i^j \delta_a^b \\
\tilde{R}_{al}^{ib} R_{jb}^{ak} &= R_{al}^{ib} \tilde{R}_{jb}^{ak} = \delta_j^i \delta_l^k
\end{align*}
\]

(12) \( \quad \) (13)

Let us notice that the important category of models on noncommutative spaces are supersymmetric models in the superspace formulation. In this framework the noncommutative space is divided into the classical commutative Minkowski space coordinates \( x \) (with corresponding indices and metric) and the spinor coordinates \( \theta \) (we shall not specify below the type of spinors - Majorana or Weyl). For all equations within this class the part connected with space-time is described using the classical commutative differential calculus while for spinor coordinates we have anticommuting derivatives. The first part of the components of the transformation operator looks as follows:

\[
\zeta_{\mu} = \delta_{\nu}^\mu \quad \zeta_{\alpha} = \zeta_{\beta} = 0
\]

(14)

where for the supersymmetric models we have denoted the space-time indices as \( \mu, \nu \) while we use \( \alpha \) and \( \beta \) as spinor indices (Weyl or Majorana). The only nontrivial part of the transformation operator for the supersymmetric models are the components which act between the spinor indices namely for the monomials of the first order they are of the form:

\[
\zeta_{\beta}^{\alpha} x_{\mu} = \delta_{\beta}^{\alpha} x_{\mu} \quad \zeta_{\beta}^{\alpha} \theta_{\gamma} = -\delta_{\beta}^{\alpha} \theta_{\gamma}
\]

(15)

We can extend the supersymmetric transformation operator to an arbitrary function by its multiplicity property \([8]\).

For all the considered differential calculi (discrete, supersymmetric, quantum Minkowski and braided) the transformation operators have one common feature namely they are invertible; for discrete models with derivatives \([3]\) or \([7]\) the inverse operators are simply backshift operators in the given direction while for noncommutative models the inverses are given by their multiplicity property:

\[
\zeta_{j_1}^{i_1} (f g) = (\zeta_{j_1}^{i_1} k f)(\zeta_{k_1}^{i_1} g)
\]

(16)
together with their action on monomials of the first order:
\[ \zeta_j^i x^k = R_{ji}^{kl} x^l + Z_{j}^{ki} \]  
(17)
\[ \zeta_j^i x_k = \tilde{R}_{kj}^{li} x_l \]  
(18)
where \( \tilde{R} \) is the second inverse of the \( R \) matrix characteristic for braided linear spaces [17].

The inverse operator for supersymmetric models is given by:
\[ \zeta_\nu^\mu = \delta_\nu^\mu \]  
(19)
with the spinor - spinor part defined by the multiplicity (16) and its action on monomials of the first order:
\[ \zeta_\beta^\alpha x_\mu = \delta_\alpha^\beta x_\mu \]  
\[ \zeta_\beta^\alpha \theta_\gamma = -\delta_\beta^\alpha \theta_\gamma \]  
(20)
All the described above inverse operators fulfill the condition:
\[ \zeta_k^j \zeta_j^i = \delta_i^j \]  
(21)

The properties of the transformation operators \( \zeta \) and \( \zeta^- \) imply the following modification of the Leibnitz rule:
\[ \partial^k [ (\zeta^-_i f )g] = (-\partial^\dagger_i f)g + f(\partial^i g) = f \left( -\partial^\dagger_i + \partial^i \right) g \]  
(22)
where the conjugated derivative \( \partial^\dagger \) is defined as follows:
\[ \partial^\dagger_i := -\partial^k \zeta^-_k \]  
(23)
and we take for the given type of the derivative the connected inverse transformation operator. We see that after modification we deal with the Leibnitz rule where the right-hand side is analogous to the classical differential calculus:
\[ \partial^i (fg) = f \left( -\partial^\dagger_i + \partial^i \right) g \]
for which the conjugated derivative is given by: \( \partial^\dagger_i = -\partial^i \).

The conjugated derivatives form the conjugated equation which for the class described by conditions (1 - 3) looks as follows:
\[ \Lambda(\partial^\dagger) = \Lambda_0(\Phi) + \sum_{l=1}^{N} \partial^\dagger_\mu_1 \cdots \partial^\dagger_\mu_l (\zeta^-_{\mu_1} \cdots \zeta^-_{\mu_l} \Lambda_{\alpha_1} \cdots \alpha_l) \]  
(24)
The following propositions hold for equation (1 - 3) and are the extension of results derived in [11, 12, 13].
Proposition 2.1 The unique solution of the operator equation:

\[ \sum_{\mu} (\frac{\leftarrow^\dagger}{\partial} \mu + \partial^\mu) \circ \Gamma_\mu(\partial, \leftarrow^\dagger) = \Lambda(\partial) - \Lambda(\leftarrow^\dagger) \]  

(25)

in the class of polynomials of derivatives \(\leftarrow^\dagger\) and \(\partial\) for the equation operator \(\Lambda\) fulfilling (3) is of the form:

\[ \Gamma_\mu(\partial, \leftarrow^\dagger) = (\zeta^\mu_\leftarrow^\dagger - \alpha \Lambda_\alpha) + \]  

(26)

\[ + \sum_{l=1}^{N-1} \sum_{k=0}^l \leftarrow^\dagger \mu_1 \ldots \leftarrow^\dagger \mu_k (\zeta^\mu_\leftarrow^\dagger - \alpha_k \zeta^\mu_k \ldots \zeta^{\alpha_1}_1 \Lambda_{\alpha_1 \ldots \alpha_k \alpha_{k+1} \ldots \mu_l}) \partial^{\mu_{k+1}} \ldots \partial^{\mu_l} \]  

Proof: Let us explain that the "\(\circ\)" operation describes the way the noncommuting derivatives work on monomials of derivatives \(\leftarrow^\dagger\) and \(\partial\), namely:

\[ (\frac{\leftarrow^\dagger}{\partial} \mu + \partial^\mu) \circ [\nu_1, \ldots, \nu_l]a(\vec{x})[\rho_1, \ldots, \rho_k] := \]  

\[ -[\nu_1, \ldots, \nu_l, \mu]a(\vec{x})[\rho_1, \ldots, \rho_k] + [\nu_1, \ldots, \nu_l]\partial^\mu a(\vec{x})[\rho_1, \ldots, \rho_k] \]  

(27)

where the following notation for monomials was used:

\[ [\rho_1, \ldots, \rho_k] := \partial^{\rho_1} \ldots \partial^{\rho_k} \]  

(28)

\[ \leftarrow^\dagger [\nu_1, \ldots, \nu_l] := \leftarrow^\dagger \nu_1 \ldots \leftarrow^\dagger \nu_l \]  

(29)

The further calculations base on the assumption that we consider the general operator of the order \(N - 1\) with respect to the derivatives and on the associativity of the algebra of derivatives. The explicit solution of the condition (23) is enclosed in Appendix A.

We modify the \(\Gamma\) operator due to the deformation of the Leibnitz rule (5, 22) and obtain the operator \(\hat{\Gamma}\) :

\[ \hat{\Gamma}_\mu(\partial, \leftarrow^\dagger) = \zeta^\mu_\leftarrow^\dagger \]  

(30)

\[ + \sum_{l=1}^{N-1} \sum_{k=0}^l \leftarrow^\dagger \mu_1 \ldots \leftarrow^\dagger \mu_k \zeta^\mu_\leftarrow^\dagger \]  

(31)

\[ (\zeta^\mu_\leftarrow^\dagger - \alpha_k \zeta^\mu_k \ldots \zeta^{\alpha_1}_1 \Lambda_{\alpha_1 \ldots \alpha_k \alpha_{k+1} \ldots \mu_l}) \partial^{\mu_{k+1}} \ldots \partial^{\mu_l} \]
which for a pair of arbitrary functions $f$ and $g$ is connected with the $\Gamma$ operator by the equality:

$$\sum_{\mu} \partial^\mu f \hat{\Gamma}_\mu(\partial, \hat{\partial}) g = \sum_{\mu} f \left( - \frac{\partial^\mu}{\partial} + \partial^\mu \right) \circ \Gamma_\mu(\partial, \hat{\partial}) g$$  \hspace{1cm} (31)$$

The immediate consequence of the Proposition 2.1 is therefore the construction of conserved currents yielded by the following statement.

**Proposition 2.2** Let us assume that function $\Phi$ is an arbitrary solution of equation (1, 2) with coefficients fulfilling (3), that means:

$$\Lambda(\partial) \Phi = 0$$  \hspace{1cm} (32)$$

and function $\Phi'$ solves the conjugated equation with the operator of the form (24):

$$\Phi' \Lambda(\hat{\partial}) = 0$$  \hspace{1cm} (33)$$

Then

$$J_\mu = \Phi' \hat{\Gamma}_\mu(\partial, \hat{\partial}) \Phi$$  \hspace{1cm} (34)$$

where the operator $\hat{\Gamma}_\mu$ is defined by (30), is a current which obeys the conservation law:

$$\sum_\mu \partial^\mu J_\mu = 0$$  \hspace{1cm} (35)$$

Proof:

This is a corollary from the (31) property of the $\hat{\Gamma}$ operator, from the operator equation (25) for $\Gamma$ and finally from the fact that functions $\Phi$ and $\Phi'$ fulfill the respective equations (1, 33).

Let us notice that the auxiliary conjugated equation for nonlinear model is now the linear one with respect to the solution $\Phi'$:

$$\Phi' \left( \Lambda_0(\Phi) + \sum_{l=1}^{N} \left[ \frac{\partial^\mu_1}{\partial} \ldots \frac{\partial^\mu_l}{\partial} \zeta_{\mu_1}^{-\alpha_1} \ldots \zeta_{\mu_l}^{-\alpha_l} \Lambda_{\alpha_1\ldots\alpha_l} \right] \right) = 0$$  \hspace{1cm} (36)$$

The solution $\Phi'$ depends on the solution of the initial equation $\Phi$ which defines the potential for the conjugated equation $\Lambda_0(\Phi)$.

Now the interesting question arises whether the presented construction which works on a wide class of spaces can be extended to equations with operators
including both $\partial$ and $\partial^\dagger$ derivatives. As we know from discrete models (see for example [4, 5]) the equations depend explicitly on the forth- and backshifts along the lattice. Also the equations of motion built within the framework of generalized difference derivatives (7) (which was discussed in [9, 11]) include both initial and conjugate derivatives as a consequence of the minimal action principle. The method of derivation of the conserved currents for the discrete and mixed models shall be described in the next section. It is based on the fact that acting on functions of commutative coordinates the operators (6) and (7) yield two symmetric formulas for the Leibnitz rules (both for initial and conjugated derivative).

3 Conserved currents and charges for discrete and mixed models depending on the $\partial$ and $\partial^\dagger$ derivatives

Our aim is now to extend the construction described in Propositions 2.1 and 2.2 to the equations depending on the initial and conjugated derivatives of the form:

$$[\Lambda(\partial) + \tilde{\Lambda}(\partial^\dagger)]\Phi = 0$$  (37)

$$\Lambda(\partial) = \Lambda_0 + \sum_{l=1}^{N} \Lambda_{\mu_1...\mu_l} \partial^{\mu_1...\mu_l}$$  (38)

$$\tilde{\Lambda}(\partial^\dagger) = \tilde{\Lambda}_0 + \sum_{l=1}^{\tilde{N}} \tilde{\Lambda}_{\mu_1...\mu_l} \partial^{\dagger\mu_1...\dagger\mu_l}$$  (39)

Following the previous considerations we restrict the class of equations to variable coefficients fulfilling the extended version of the condition (3):

$$\partial^{\dagger\mu_1} \Lambda_{\mu_1...\mu_l} = 0 \quad \partial^{\mu_1} \tilde{\Lambda}_{\mu_1...\mu_k} = 0$$  (40)

for all $l = 1, ..., N$ and $k = 1, ..., \tilde{N}$.

We include the nonlinear equations provided the nonlinear terms do not depend on the derivatives that means they have a form of potential:

$$\Lambda_0 = \Lambda_0(\Phi) \quad \tilde{\Lambda}_0 = \tilde{\Lambda}_0(\Phi)$$  (41)

As we shall see in the subsequent sections the class of equations fulfilling (40) contains the widely studied in different context linear auxiliary equations connected with nonlinear integrable models such as principal chiral model.
and equations of nonlinear Toda lattice models.

The important factor in our calculations is the fact that for discrete (or mixed - classical and discrete) differential calculus we can write the modified Leibnitz rule symmetricaly in the following form:

\[ \partial^{\alpha} (\zeta^{\beta} f) g = (-\partial^\dagger \beta f) g + f \partial^\beta g = f (-\partial^\dagger + \partial^\beta) g \] (42)

\[ \partial^\dagger \alpha (\zeta^{\beta} f) g = (-\partial^\beta f) g + f \partial^\dagger \beta g = f (-\partial^\dagger + \partial^\beta) g \] (43)

The second of the above equations indicates that the statement analogous to the Proposition 2.1 holds for the operator \( \tilde{\Lambda}(\partial^\dagger) \); namely we shall prove that the operator equation given below has the unique solution by virtue of Proposition 3.1:

\[ \sum_{\mu} (-\partial^\mu + \partial^\dagger \mu) \circ \tilde{\Gamma}_\mu (\partial^\dagger, \partial) = \tilde{\Lambda}(\partial^\dagger) - \tilde{\Lambda}(\partial) \] (44)

where the conjugated operator \( \tilde{\Lambda}(\partial) \) looks as follows:

\[ \tilde{\Lambda}(\partial) = \tilde{\Lambda}_0(\Phi) + \sum_{l=1}^{\tilde{N}} \partial^\mu_1 \ldots \partial^\mu_l \left( \zeta^\alpha_{\mu_1} \ldots \zeta^\alpha_{\mu_l} \Lambda_{\alpha_1 \ldots \alpha_l} \right) \] (45)

**Proposition 3.1** The unique solution of (44) in the class of polynomials of derivatives \( \partial \) and \( \partial^\dagger \) for the equation operator \( \tilde{\Lambda} \) (33) with coefficients fulfilling (40) is of the form:

\[ \tilde{\Gamma}_\mu (\partial^\dagger, \partial) = \] (46)

\[ (\zeta^\alpha_\mu \tilde{\Lambda}_\alpha) + \sum_{l=1}^{\tilde{N}-1} \sum_{k=0}^{l} \partial^\mu_1 \ldots \partial^\mu_k \left( \zeta^\alpha_{\mu_1} \zeta^\alpha_{\mu_k} \ldots \zeta^\alpha_{\mu_l} \Lambda_{\mu_1 \ldots \mu_k \mu_{k+1} \ldots \mu_l} \right) \partial^{\dagger \mu_{k+1}} \ldots \partial^{\dagger \mu_l} \]

Proof:

The formula (46) is the immediate consequence of the proof of Proposition 2.1 enclosed in Appendix A as the following connections between derivatives and transformation operators hold:

\[ \partial^\dagger \alpha f g = (\partial^\dagger \alpha f) g + (\zeta^\alpha_{\beta} \alpha f) \partial^\dagger \beta g \] (47)

\[ (\zeta^{-})^{-1} = \zeta \] (48)

\[ (\partial^\dagger)^\gamma = -(-\partial^\beta \zeta^\alpha_{\beta} \alpha) \zeta^\alpha_\gamma = \partial^\gamma \] (49)
We notice that $\zeta^-$ is the transformation operator for the derivative $\partial^\dagger$, the operator $\zeta$ being its inverse. Therefore all the calculations from the proof of the Proposition 2.1 can be repeated with suitable replacements. In this way we obtain the formula for the $\hat{\Gamma}$ operator (46) which is simply the (26) operator with new derivatives $\partial^\dagger$ and $\partial$ and the operator $\zeta$ acting now as the inverse transformation operator.

We modify the operator $\hat{\Gamma}$ similarly as in (30) and obtain the operator $\hat{\hat{\Gamma}}$:

$$\hat{\hat{\Gamma}}(\partial^\dagger, \partial) = \zeta_j (\zeta^\alpha \Lambda_\alpha) +$$

$$+ \sum_{l=1}^{N-1} \sum_{k=0}^l \partial^\dagger \partial^\mu \zeta_j \partial^\mu_k \ldots \partial^\mu_k \ldots \partial^\mu_l \zeta^\alpha \Lambda^\alpha_{\alpha \ldots \alpha_k \ldots \alpha_l}$$

which in turn fulfills the equality:

$$\sum_\mu \partial^\dagger \mu \left( f R_{\mu} (\partial^\dagger, \partial) g \right) = \sum_\mu f \left( (- \partial^\mu + \partial^\dagger \mu) \circ \hat{\hat{\Gamma}}(\partial^\dagger, \partial) \right) g$$

for a pair of arbitrary functions $f$ and $g$.

We can use the constructed operators $\hat{\Gamma}$ and $\hat{\hat{\Gamma}}$ to derive the conserved currents by the following proposition:

**Proposition 3.2** Let us assume that the function $\Phi$ is an arbitrary solution of equation (37) with coefficients fulfilling (40) which means:

$$[\Lambda(\partial) + \tilde{\Lambda}(\partial^\dagger)] \Phi = 0$$

and the function $\Phi'$ solves the conjugated equation with operators (24, 45):

$$\Phi'[\Lambda(\partial^\dagger) + \tilde{\Lambda}(\partial)] = 0$$

Then:

$$J_\mu = \Phi' \hat{\Gamma}_{\mu} \Phi \quad \tilde{J}_\mu = \Phi' \hat{\hat{\Gamma}}_{\mu} \Phi$$

where the operators $\hat{\Gamma}$ and $\hat{\hat{\Gamma}}$ are given by (30, 50) is a current that obeys the conservation law:

$$\sum_\mu \partial^\mu J_\mu + \sum_\mu \partial^\dagger \mu \tilde{J}_\mu = 0$$
Proof:
The conservation law (55) is implied by the properties (31, 51) of the applied operators $\hat{\Gamma}$ and $\hat{\tilde{\Gamma}}$:

$$\partial^\mu \Phi' \hat{\Gamma}_\mu \Phi + \partial^\dagger \mu \Phi' \hat{\tilde{\Gamma}}_\mu \Phi = (56)$$

$$\Phi' \left( \sum_\mu ( - \hat{\tilde{\Gamma}}_\mu \Phi \partial^\dagger + \partial^\mu) \circ \hat{\tilde{\Gamma}}_\mu (\partial, \hat{\tilde{\Gamma}}_\mu \Phi \partial^\dagger) \right) \Phi =$$

$$\Phi' \left( \hat{\tilde{\Lambda}}(\partial^\dagger) - \hat{\tilde{\Lambda}}(\partial) + \Lambda(\partial) - \Lambda(\partial^\dagger) \right) \Phi = 0$$

Corollary 3.3 The components of the conserved current for (37) fulfilling the condition (40) can be written also in the form:

$$J'_\mu = \Phi' \hat{\Gamma}_\mu \Phi - \zeta^{-\nu}_\mu \nu \left( \Phi' \hat{\tilde{\Gamma}}_\nu \Phi \right) = (57)$$

They obey the conservation law including only the $\partial$ derivatives:

$$\sum_\mu \partial^\mu J'_\mu = 0 = (58)$$

provided $\Phi$ is the solution of (37) and the function $\Phi'$ solves the conjugate equation with operators (24, 45).

Let us notice that the presented construction holds when we work within the framework of classical commutative differential calculus (for which the conjugation means $\partial^\dagger = -\partial$ and the transformation operator is the identity) as well as for mixed models where part of the coordinates is continuous and the other part is discrete with suitable difference derivatives in the form (6) or (7).

In the sequel we shall use the construction (30, 50, 54) in order to derive the conserved currents and in consequence the conserved charges for mixed discrete nonlinear Toda model and for double discrete Toda model as well. As we know from the classical field theory we can use the conserved currents in construction of the conserved charges. This is also the case for our discrete and mixed discrete models with derivatives defined by (3) or (7). We shall integrate the time component of the conserved current over subspace excluding the time-coordinate. We use in derivation the conserved currents
given by Proposition 3.2 or accordingly by the Corollary 3.3 if the conjugated derivative with respect to time-coordinate appears in the model. Integrating we must remember to take the corresponding integral namely for continuous coordinates we understand \( \int dx_i \) as the Lebesque integral while for the discrete derivative \( \zeta \) the definite integral is given by:

\[
\int dx_i := \sum_{k=-\infty}^{+\infty} (\zeta_i)^k
\]  

(59)

If the equation includes the discrete derivative of the type \( \zeta \) we use the definite integral in the form:

\[
\int dx_i := \sum_{k=-\infty}^{+\infty} (x_{k+1}^i - x_k^i)(\zeta_i)^k
\]  

(60)

In the above formulas for discrete integrals the \( \zeta_i \) is the shift operator in the direction \( i \) along the lattice.

Let us denote the integral over subspace with excluded time-coordinate as \( \int_{sub} \). Then the conserved charges for discrete and mixed discrete models with variable coefficients can be derived using the following proposition:

**Proposition 3.4** Let us assume that in the model described by the equation \( (37) \) the conjugated discrete derivative with respect to the time-coordinate does not appear. Then the charge

\[
Q = \int_{sub} J_t
\]  

(61)

where \( J_t \) is the time component of the conserved current described in the Proposition 3.2 is conserved:

\[
\partial^t Q = 0
\]  

(62)

Proof:

The conservation of the charge \( (61) \) is implied by the conservation law from Proposition 3.2 namely:

\[
\partial^t Q = \int_{sub} \partial^t J_t = \int_{sub} - \sum_{k\neq t} (\partial^k J_k + \partial^t k \tilde{J}_k) =
\]

= boundary terms = 0
provided the currents vanish at the infinity in the corresponding space-like dimensions. We have used here also the property of the discrete and mixed discrete differential calculus:

\[ \partial_t^d \int_{\text{sub}} = \int_{\text{sub}} \partial_t^d \]  

(64)

Let us point out that the equality (64) does not apply in general to non-commutative spaces (the superspace being an exception) and this fact is the major difficulty in construction of the conserved charges for an arbitrary model on noncommutative space [14, 16].

If in the equation (37) both initial and conjugate time-derivatives appear we derive the conserved charge using the Corollary 3.3 namely the following statement is then valid (with proof being the copy of the above calculations for Proposition 3.4):

**Corollary 3.5** Let us assume that in the model described by the equation (37) both discrete derivatives \( \partial_t^d \) and \( \partial_t^{\dagger} \) appear. Then the charge

\[ Q = \int_{\text{sub}} J_t' \]  

(65)

where \( J_t' \) is the time component of the conserved current described in the Corollary 3.3 is conserved:

\[ \partial_t^d Q = 0 \]  

(66)

4 Applications

4.1 The second order equation on quantum plane with variable coefficients

Let us start with the simple example of the second order equation on quantum plane with coefficients depending on coordinates \( x \) and \( y \). The commutation relations for derivatives and coordinates look as follows [17]:

\[ yx = qxy \quad \text{\( \partial^y \partial^x = q^{-1} \partial^x \partial^y \)} \]  

(67)
The Leibnitz rule (5) in differential calculus is defined by the following $R$-matrix:

$$ R = \begin{bmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2 - 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{bmatrix} \quad R' = q^{-2}R $$

Form the above matrix we can deduce the explicit action of the transformation operator $\zeta$ on the monomials of the first order (11). It yields the following formulas for the inverse transformation $\zeta^{-1}$:

$$ \zeta^{-1} x = q^{-2}x \quad \zeta^{-1} y = q^{-1}y \quad \zeta^{-1} \cdot x = (q^{-2} - 1)y \quad \zeta^{-1} \cdot y = 0 $$

The conjugated derivatives are defined using the inverse transformation operator:

$$ \partial^\dagger x = -\partial_x \zeta_x^{-1} x - \partial^\mu \zeta_{x \mu}^{-1} \quad \partial^\dagger y = -\partial_y \zeta_y^{-1} y - \partial^\mu \zeta_{y \mu}^{-1} $$

Now let us discuss an arbitrary equation of the second order with the coefficients depending on $x$ and $y$ and check which of them fulfill the condition (3):

$$ \Lambda(\partial^\dagger x, \partial^\dagger y) = \Lambda_{xx} \partial^\dagger \partial^\dagger x + \Lambda_{xy} \partial^\dagger \partial^\mu + \Lambda_{yx} \partial^\mu \partial^\dagger y + \Lambda_{yy} \partial^\mu \partial^\mu $$

The condition (3) for our simple case reads as follows:

$$ \partial^\dagger x \Lambda_{xx} + \partial^\dagger y \Lambda_{yx} = 0 \quad \partial^\dagger y \Lambda_{yy} + \partial^\dagger x \Lambda_{xy} = 0 $$

Let us assume $\Lambda_{xy} = \Lambda_{yx} = 0$. Then we can show that $\Lambda_{xx} = \phi(y)$ and $\Lambda_{yy} = \psi(x)$ obey the conditions (3), namely:

$$ \partial^\dagger x \Lambda_{xx} = (-\partial^\dagger \zeta_x^{-1} x - \partial^\mu \zeta_{x \mu}^{-1}) \phi(y) = -\partial^\mu \phi(q^{-1}y) = 0 $$

$$ \partial^\dagger y \Lambda_{yy} = (-\partial^\dagger \zeta_y^{-1} y - \partial^\mu \zeta_{y \mu}^{-1}) \psi(x) = -\partial^\mu \psi(q^{-1}x) = 0 $$

Now we can apply our construction to obtain the $\Gamma$ operator:

$$ \Gamma_x = \frac{\epsilon^{-1} x}{\partial} \phi(q^{-2}y) + \phi(q^{-1}y) \partial^\dagger x \quad \Gamma_y = \frac{\epsilon^{-1} y}{\partial} \psi(q^{-2}x) + \psi(q^{-1}x) \partial^\mu $$

It is easy to check that this operator fulfills the operator equation (22):
\[ = - \left( \frac{\partial}{\partial x} \phi(q^{-2}y) + \frac{\partial}{\partial y} \psi(q^{-2}x) \right) + \phi(y) \partial^x \partial^x + \psi(x) \partial^y \partial^y \] (77)

where the operator \( \Lambda(\partial) \) gives the conjugated equation:

\[ \Lambda(\partial) = \frac{\partial}{\partial x} \phi(q^{-2}y) + \frac{\partial}{\partial y} \psi(q^{-2}x) \] (78)

The modified \( \hat{\Gamma} \) operator is given by the formula:

\[
\begin{align*}
\hat{\Gamma}_x &= \frac{\partial}{\partial x} \zeta \phi(q^{-2}y) + \zeta \phi(q^{-1}y) \partial^x \\
\hat{\Gamma}_y &= \frac{\partial}{\partial y} \zeta \phi(q^{-2}y) + \zeta \phi(q^{-1}y) \partial^x + \frac{\partial}{\partial y} \zeta \psi(q^{-2}x) + \zeta \psi(q^{-1}x) \partial^y
\end{align*}
\] (80)

Having the explicit form of the \( \Gamma \) and \( \hat{\Gamma} \) operators we can construct the conserved currents:

\[
\begin{align*}
J_x &= \Phi' \hat{\Gamma}_x \Phi \\
J_y &= \Phi' \hat{\Gamma}_y \Phi
\end{align*}
\] (81)

where the functions \( \Phi' \) and \( \Phi \) solve the respective equations:

\[
\Phi' \left( \frac{\partial}{\partial x} \phi(q^{-2}y) + \frac{\partial}{\partial y} \psi(q^{-2}x) \right) = 0 \] (82)

\[
(\phi(y) \partial^x \partial^x + \psi(x) \partial^y \partial^y) \Phi = 0 \] (83)

According to the Proposition 2.2 the above current obeys the conservation law:

\[ \partial^x J_x + \partial^y J_y = 0 \] (84)

provided functions \( \Phi' \) and \( \Phi \) are the solutions of the corresponding equations (82, 83).

### 4.2 Nonlinear equation of motion for chiral and antichiral supermultiplets

The supersymmetric models in the superfield formulation yield interesting examples of the equations of motion with coefficients depending on variables \([2, 25]\). Let us recall that in such a framework the fields depend on superspace...
coordinates including the space-time and spinor variables while in construction of the action the covariant derivatives are used which explicitly depend on spinor coordinates.

We shall study in this section the D=4 N=1 chiral and antichiral superfields obeying the nonlinear equation of motion resulting from the supersymmetric action:

\[ I = \int d^4x d^2\theta d^2\bar{\theta} \Phi \Phi + \int d^4x d^2\bar{\theta} \left( \frac{m}{2} \Phi^2 + \frac{g}{3} \Phi^3 \right) + \int d^4x d^2\bar{\theta} \left( \frac{m}{2} \bar{\Phi}^2 + \frac{g}{3} \bar{\Phi}^3 \right) \]

(85)

The integration over spinor variables \( \theta \) and \( \bar{\theta} \) can be expressed in terms of the covariant derivatives:

\[
\int d^2\theta = -\frac{1}{4} D_2^2 \\
\int d^2\bar{\theta} = -\frac{1}{4} \bar{D}_2^2 \\
\int d^2\theta d^2\bar{\theta} = \frac{1}{16} D_2^2 \bar{D}_2^2 = \frac{1}{16} D_2^2 D_2^2 = \frac{1}{16} \bar{D}_2 D_2 \bar{D}_2 D_2 = \frac{1}{16} \bar{D}_2 D_2 \bar{D}_2 D_2
\]

(86)

(87)

The covariant derivatives used in supersymmetric models are built from the basic derivatives with respect to the space-time and spinor coordinates:

\[
D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\beta} \bar{\theta}^\beta \partial_\mu \\
D_\alpha = \epsilon^{\alpha\beta} D_\beta \\
\bar{D}_\dot{\alpha} = -\partial_{\dot{\alpha}} - i\theta^\beta \sigma^\mu_{\dot{\alpha}\dot{\beta}} \partial_\mu \\
\bar{D}_\dot{\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}_\beta
\]

(88)

(89)

Due to the fact that chiral \( \Phi \) and antichiral \( \bar{\Phi} \) superfields fulfill the following condition:

\[ \bar{D}_\dot{\alpha} \Phi = 0 \quad D_\alpha \bar{\Phi} = 0 \]

(90)

we obtain from the action (85) the equations of motion in the form:

\[
\begin{pmatrix}
\frac{1}{4} D_2^2 & m + g\Phi \\
m + g\bar{\Phi} & \frac{1}{4} \bar{D}_2^2
\end{pmatrix}
\begin{pmatrix}
\Phi \\
\bar{\Phi}
\end{pmatrix} = 0
\]

(91)

We have written the equations for the superfields \( \Phi \) and \( \bar{\Phi} \) in the matrix form. In this example we deal with the coefficients of the equation depending explicitly on \( \theta \) and \( \bar{\theta} \) variables due to the form of the covariant derivatives (88, 89):

\[
D_2^2 = D_\alpha D_\alpha = -\partial^\alpha \partial_\alpha + 2i\bar{\theta}^\beta \sigma^\mu_{\alpha\beta} \partial_\mu \partial^\alpha - \bar{\theta}^2 \Box \\
\bar{D}_2^2 = \bar{D}_{\dot{\alpha}} \bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} \partial_{\dot{\alpha}} - 2i\theta^\beta \sigma^\mu_{\dot{\alpha}\dot{\beta}} \partial_\mu \partial^\beta + \theta^2 \Box
\]

(92)

(93)
The second important feature of the chiral-antichiral superfield equation of motion is its nonlinearity for \( g \neq 0 \). However the nonlinear term does not depend on the derivatives so we can apply our method of derivation the conservation law provided the condition (3) for the kinetic part of the operator is fulfilled. Let us extract the \( \zeta \) and \( \zeta^- \) operators from the Leibnitz rule for basic derivatives:

\[
\partial^\mu (fg) = (\partial^\mu f)g + f\partial^\mu g \tag{94}
\]

\[
\partial^\alpha (fg) = (\partial^\alpha f)g + (\zeta^\beta f)\partial^\beta g \tag{95}
\]

\[
\bar{\partial}^\dot{\alpha} (fg) = (\bar{\partial}^\dot{\alpha} f)g + (\zeta^\dot{\beta} f)\bar{\partial}^\dot{\beta} g \tag{96}
\]

It is clear that \( \zeta^\mu_\nu = \delta^\mu_\nu \) and the following components of the transformation operator vanish:

\[
\zeta^\mu_\alpha = \zeta^\alpha_\mu = 0 \quad \zeta^\dot{\mu}_\alpha = \zeta^\dot{\alpha}_\mu = 0 \quad \zeta^\alpha_\dot{\alpha} = \zeta^\dot{\alpha}_\alpha = 0 \tag{97}
\]

The remaining components of the \( \zeta \) operator are defined by its action on the monomials of the first order in superspace variables:

\[
\zeta^\alpha_\beta x_\nu = \delta^\alpha_\beta x_\nu \quad \zeta^\alpha_\beta \theta_\gamma = -\delta^\alpha_\beta \theta_\gamma \quad \zeta^\alpha_\beta \bar{\theta}_\gamma = -\delta^\alpha_\beta \bar{\theta}_\gamma \tag{98}
\]

The analogous formulas hold also for dotted indices. Hence we get for the inverse transformation operator:

\[
\zeta^-_\beta x_\nu = \delta^-_\beta x_\nu \quad \zeta^-_\beta \dot{\alpha} \theta_\gamma = -\delta^-_\beta \dot{\theta}_\gamma \quad \zeta^-_\beta \dot{\alpha} \bar{\dot{\theta}}_\gamma = -\delta^-_\beta \dot{\dot{\theta}}_\gamma \tag{99}
\]

with the corresponding expressions with dotted indices of the same form.

Now having the explicitly derived inverse transformation operator we arrive at the conjugated derivatives given by the formulas:

\[
\bar{\partial}^\dagger \mu = -\partial^\mu \quad \bar{\partial}^\dagger \alpha = -\partial^\beta \zeta^-_\beta \alpha \quad \bar{\partial}^\dagger \dot{\alpha} = -\bar{\partial}^\dot{\beta} \zeta^-_\beta \dot{\alpha} \tag{100}
\]

Using the conjugated derivatives we obtain the condition (3) written for our equation for the chiral and antichiral superfields:

\[
\bar{\partial}^\dagger \alpha \Lambda_\alpha \mu + \bar{\partial}^\dagger \dot{\alpha} \Lambda_\dot{\alpha} \mu + \partial^\dagger \nu \Lambda_\nu \mu = 0 \tag{101}
\]

\[
\bar{\partial}^\dagger \alpha \Lambda_\alpha \beta + \partial^\dagger \mu \Lambda_\mu \beta = 0 \tag{102}
\]

\[
\bar{\partial}^\dagger \dot{\alpha} \Lambda_\dot{\alpha} \dot{\beta} + \partial^\dagger \mu \Lambda_\mu \dot{\beta} = 0 \tag{103}
\]
One can easily check that the above conditions are fulfilled using the explicit form of the coefficients from equation (91):

\[
\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha} = 0 \quad (104)
\]

\[
\Lambda_{\alpha\beta} = -\epsilon_{\alpha\beta} \left( \begin{array}{ccc} \frac{i}{4} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad (105)
\]

\[
\Lambda_{\dot{\alpha}\dot{\beta}} = -\epsilon_{\dot{\alpha}\dot{\beta}} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \frac{i}{4} & 0 \end{array} \right) \quad (106)
\]

\[
\Lambda_{\mu\alpha} = \Lambda_{\alpha\mu} = \left( \frac{i}{4} \bar{\theta}^\beta \sigma^\nu_{\alpha\beta} 0 \\ 0 \right) g_{\nu\mu} \quad (107)
\]

\[
\Lambda_{\mu\dot{\alpha}} = \Lambda_{\dot{\alpha}\mu} = \left( 0 0 0 \\ 0 -\frac{i}{4} \bar{\theta}^2 \sigma^\nu_{\dot{\beta}\dot{\alpha}} \right) g_{\nu\mu} \quad (108)
\]

\[
\Lambda_{\mu\nu} = \left( -\frac{i}{4} \bar{\theta}^2 0 \\ 0 -\frac{i}{4} \bar{\theta}^2 \right) g_{\mu\nu} \quad (109)
\]

The next step is therefore the application of the general formula (26) for the \( \Gamma \) operator to our model:

\[
\Gamma_\mu = \left( \frac{\partial}{\partial \nu} \Lambda_{\nu\mu} + \Lambda_{\mu\nu} \partial^\nu \right) \quad (110)
\]

\[
\Gamma_\alpha = -\left( \frac{\partial}{\partial \gamma} \Lambda_{\gamma\alpha} + \Lambda_{\alpha\gamma} \partial^\gamma - \Lambda_{\alpha\mu} \partial^\mu \right) \quad (111)
\]

\[
\Gamma_{\dot{\alpha}} = -\left( \frac{\partial}{\partial \bar{\gamma}} \Lambda_{\bar{\gamma}\dot{\alpha}} + \Lambda_{\dot{\alpha}\bar{\gamma}} \partial^\bar{\gamma} - \Lambda_{\dot{\alpha}\mu} \partial^\mu \right) \quad (112)
\]

The operator \( \hat{\Gamma} \) differs from the \( \Gamma \) by insertion of the \( \zeta^\alpha \) operator in the middle of the monomials of derivatives:

\[
\hat{\Gamma}_\mu = \Gamma_\mu \quad (113)
\]

\[
\hat{\Gamma}_\beta = -\left( \frac{\partial}{\partial \zeta_{\beta}} \Lambda_{\mu\alpha} + \frac{\partial}{\partial \zeta_{\alpha}} \Lambda_{\gamma\alpha} + \zeta_{\alpha} \Lambda_{\alpha\gamma} \partial^\gamma - \zeta_{\beta} \Lambda_{\alpha\mu} \partial^\mu \right) \quad (114)
\]

\[
\hat{\Gamma}_{\dot{\beta}} = -\left( \frac{\partial}{\partial \zeta_{\dot{\beta}}} \Lambda_{\mu\dot{\alpha}} + \frac{\partial}{\partial \zeta_{\dot{\alpha}}} \Lambda_{\gamma\dot{\alpha}} + \zeta_{\dot{\alpha}} \Lambda_{\dot{\alpha}\gamma} \partial^\gamma - \zeta_{\dot{\beta}} \Lambda_{\dot{\alpha}\mu} \partial^\mu \right) \quad (115)
\]

We apply the obtained operator \( \hat{\Gamma} \) to derive the currents:

\[
J_\mu = (\Phi', \Phi') \hat{\Gamma}_\mu \left( \begin{array}{c} \Phi \\ \bar{\Phi} \end{array} \right) \quad (116)
\]

\[
J_\alpha = (\Phi', \Phi') \hat{\Gamma}_\alpha \left( \begin{array}{c} \Phi \\ \bar{\Phi} \end{array} \right) \quad (117)
\]

\[
J_{\dot{\alpha}} = (\bar{\Phi}', \Phi') \hat{\Gamma}_{\dot{\alpha}} \left( \begin{array}{c} \Phi \\ \bar{\Phi} \end{array} \right) \quad (118)
\]
which obey the conservation law:

\[ \partial^\mu J_\mu + \partial^\alpha J_\alpha + \bar{\partial}^\dot{\alpha} J_{\dot{\alpha}} = 0 \]  

(119)

provided the fields used in construction fulfill the corresponding equation of motion (91) and the pair of superfields \( \Phi' \) and \( \Phi' \) its conjugated version:

\[
(\Phi', \Phi') \begin{pmatrix}
\frac{1}{4} \tilde{D}^2 & m + g\Phi \\
m + g\Phi & \frac{1}{4} \tilde{D}^2
\end{pmatrix} = 0
\]

(120)

Let us notice that this equation is linear with respect to superfields \( \Phi' \) and \( \Phi' \) and its solution depends on the potential given by the chiral and antichiral superfields \( \Phi \) and \( \bar{\Phi} \).

Due to the properties of covariant derivatives built form the selfconjugated operators we conclude that the conjugated set of equations for superfields \( (\Phi', \Phi') \) has at least following solutions:

- for the case \( g \neq 0 \) we can take \( \bar{\Phi}' = \Phi \) and \( \Phi' = \bar{\Phi} \) where \( \Phi \) and \( \bar{\Phi} \) are solutions for (91).

- when \( g = 0 \) the superfield \( \bar{\Phi}' \) is an arbitrary solution of chiral and \( \Phi' \) of antichiral part of the set of equations (111).

The above solutions give in turn the explicit form of the conserved currents (116-118) where we replace the multiplet \( (\Phi', \Phi') \) with \( (\Phi, \bar{\Phi}) \) for \( g \neq 0 \) and with \( (\delta \Phi, \delta \bar{\Phi}) \) for \( g = 0 \). Let us notice that for linear model \( g = 0 \) we obtain the full set of conserved currents connected with symmetry operators of chiral-antichiral supermultiplet equation:

\[
J_\mu = (\delta \Phi, \delta \bar{\Phi}) \tilde{\Gamma}_\mu \begin{pmatrix} \Phi \\ \bar{\Phi} \end{pmatrix}
\]

(121)

\[
J_\alpha = (\delta \Phi, \delta \bar{\Phi}) \tilde{\Gamma}_\alpha \begin{pmatrix} \Phi \\ \bar{\Phi} \end{pmatrix}
\]

(122)

\[
J_{\dot{\alpha}} = (\delta \Phi, \delta \bar{\Phi}) \tilde{\Gamma}_{\dot{\alpha}} \begin{pmatrix} \Phi \\ \bar{\Phi} \end{pmatrix}
\]

(123)

where the symmetries \( \delta \) include supersymmetric transformations:

\[
Q_\alpha = -i\partial_\alpha - \sigma^l_{\alpha \beta} \bar{\theta}^\beta \partial_l \\
\bar{Q}^\dot{\alpha} = -i\bar{\partial}^\dot{\alpha} + \sigma^l_{\alpha \dot{\beta}} \bar{\theta}^\dot{\beta} \partial_l
\]

(124)
and the operators from Poincaré algebra: momenta, angular momentum and boosts for four-dimensional Minkowski space. We can use the obtained conserved current (116 - 118) to construct the integral of motion by integrating the time-component of the current (116) over the subspace:

\[ Q = \int d^3x d^2\theta d^2\bar{\theta} J_0 \]  

(125)

as the following equality holds:

\[ \partial^0 Q = \int d^3x d^2\theta d^2\bar{\theta} \partial^0 J_0 = \]

(126)

\[ \int d^3x d^2\theta d^2\bar{\theta} \left( -\partial^k J_k - \partial^\alpha J_\alpha - \partial^{\dot{\alpha}} J_{\dot{\alpha}} \right) = \text{boundary terms} = 0 \]

Let us point out that the developed method allows immediate construction of conserved charges. The obtained charges are supermultiplets built from component charges which are also conserved separately. The application of our method to the equations obeying the condition (3) is an alternative to the procedure used in supersymmetric models (see for example [1] for chiral superfields and [7] for supersymmetric principal chiral model) where the conservation laws include covariant spinor derivatives \( D \) and \( \bar{D} \) instead of basic derivatives \( \partial^\mu \), \( \partial^\alpha \) and \( \partial^{\dot{\alpha}} \) which we have used. The consequence of the fact that the time-derivative does not appear explicitly in the conservation law is an additional procedure required to derive the component conservation laws and conserved charges.

4.3 Nonlinear integrable models and their consistency condition

4.3.1 Principal chiral model equation

We shall discuss the application of the construction given by Propositions 2.1 and 2.2 to the classical problem of nonlocal conserved currents and charges for the principal chiral model. The model is built using the commutative differential calculus, nevertheless its linearized version yields an interesting example of the linear equation with coefficients fulfilling the condition (3). Dimakis and Müller-Hoissen [5, 6] treated this case within their framework of
the gauged bi-covariant noncommutative differential calculus and obtained
the linearized version of the equations in the form:

\[ \partial_t \chi = \lambda (\partial^x + g^{-1}(\partial^x g)) \chi \] (127)
\[ \partial^x \chi = \lambda (\partial^t + g^{-1}(\partial^t g)) \chi \] (128)

where \( \chi \) is an \( N \times N \) matrix of smooth functions depending on \( t \) and \( x \) while
the field \( g \) denotes the principal chiral field and the initial nonlinear equation
of the principal chiral model looks as follows:

\[ \partial^x (g^{-1}(\partial^x g)) = \partial^t (g^{-1}(\partial^t g)) \] (129)

In the paper [6] the integrability condition for the set (127,128) was used and
it appeared that it is first order in derivatives therefore yields the conservation
law with currents linear with respect to the field \( \chi \) and depending on the \( \lambda \)
parameter.
We propose in addition to consider the consistency condition derived from
the set (127,128) which has the following form:

\[ \left( \partial^x \partial^x - \partial^t \partial^t + g^{-1}(\partial^x g)\partial^x - g^{-1}(\partial^t g)\partial^t \right) \chi = 0 \] (130)

As we see the operator of this equation is free of the parameter \( \lambda \) and the con-
dition (3) becomes identical with the principal chiral model equation (129):

\[ \partial^x \Lambda_x + \partial^t \Lambda_t = \partial^x g^{-1}(\partial^x g) - \partial^t g^{-1}(\partial^t g) = 0 \] (131)

the remaining coefficients \( \Lambda \) of the operator \( \Lambda(\partial) \) being constant.
The conjugated equation for (130) together with connected conjugated set
for (127,128) look as follows:

\[ \chi' \left[ \frac{\partial^x}{\partial x} - \frac{\partial^t}{\partial t} + \frac{\partial^t}{\partial x} g^{-1}(\partial^x g) + \frac{\partial^t}{\partial t} g^{-1}(\partial^t g) \right] = 0 \] (132)
\[ -\partial^t \chi' = \lambda \left[ -\partial^x \chi' + \chi' g^{-1}(\partial^x g) \right] \] (133)
\[ -\partial^x \chi' = \lambda \left[ -\partial^t \chi' + \chi' g^{-1}(\partial^t g) \right] \] (134)

We work in this example with classical commutative basic derivatives so the
transformation operator is simply the identity operator and the operators \( \Gamma \)
and \( \hat{\Gamma} \) coincide:

\[ \hat{\Gamma}_x = \Gamma_x = g^{-1}(\partial^x g) - \frac{\partial^x}{\partial x} + \partial^x \] (135)
\[ \hat{\Gamma}_t = \Gamma_t = -g^{-1}(\partial^t g) + \frac{\partial^t}{\partial t} - \partial^t \] (136)
Our consistency condition allows the construction of the conserved current according to the Proposition 2.2. The conserved current is given by the derived operator $\Gamma$ and the solution $\chi$ of the equation (130) while $\chi'$ denotes the solution of the conjugated equation (132-134):

$$J_x = \chi' \Gamma_x \delta \chi$$
$$J_t = \chi' \Gamma_t \delta \chi$$

(137)

where we have denoted as $\delta$ the symmetry operator of the equation (130).

The operator $\delta$ can be derived using the extended vector fields method described in [21]. We have checked that the following operators transform solution $\chi$ of the consistency condition (130) into solution $\delta \chi$:

$$\delta^0 = C \frac{\partial}{\partial \chi} \quad \delta^1 = \chi \frac{\partial}{\partial \chi} \quad \delta^2 = g^{-1} \frac{\partial}{\partial \chi} \quad \delta^3 = (\partial^x)^{-1}(g^{-1}\partial^y g) \frac{\partial}{\partial \chi}$$

(138)

where $C$ is a constant matrix $N \times N$.

The above symmetry operators of the equation (131) imply the following currents:

$$J^0_\mu = \chi' \Gamma_\mu C \quad J^1_\mu = \chi' \Gamma_\mu \chi \quad J^2_\mu = \chi' \Gamma_\mu g^{-1} \quad J^3_\mu = \chi' \Gamma_\mu (\partial^x)^{-1}(g^{-1}\partial^y g)$$

(139)

which are conserved:

$$\partial^x J^i_x + \partial^t J^i_t = 0$$

(140)

thereby yielding the conserved charges after the integration of the time-components of the currents (139) with respect to the space variable $x$:

$$Q^0 = \int dx \ \chi' \Gamma_t C$$
$$Q^1 = \int dx \ \chi' \Gamma_t \chi$$
$$Q^2 = \int dx \ \chi' \Gamma_t g^{-1}$$
$$Q^3 = \int dx \ \chi' \Gamma_t (\partial^x)^{-1}(g^{-1}\partial^y g)$$

(141-144)

where the operator $\Gamma_t$ is given by formula (136).

Now following [6] we can expand the fields $\chi$ and $\chi'$ in terms of powers of the parameter $\lambda$ and obtain the infinite set of conserved charges connected with the nonlinear equation (129) of the principal chiral model:

$$\chi = \sum_{m=0}^{\infty} \lambda^m \chi^{(m)} \quad \chi' = \sum_{m=0}^{\infty} \lambda^m \chi'^{(m)}$$

(145)
Using equations (127, 128) and (133, 134) we arrive at the following set of equations for coefficients \( \chi^{(m)} \) and \( \chi'^{(m)} \) (we assume after [6] that \( \chi^{(0)}, \chi^{(1)}, \chi'^{(0)} \) and \( \chi'^{(1)} \) are the unital \( N \times N \) matrices):

\[
\partial_t \chi^{(m)} = [\partial_t + g^{-1}(\partial_t g)]\chi^{(m-1)} \quad m \geq 2 \tag{146}
\]

\[
\partial_x \chi^{(m)} = [\partial_t + g^{-1}(\partial_t g)]\chi^{(m-1)} \quad m \geq 2 \tag{147}
\]

\[
-\partial_t \chi'^{(m)} = -\partial_t \chi'^{(m-1)} + \chi'^{(m-1)}g^{-1}(\partial^x g) \quad m \geq 2 \tag{148}
\]

\[
-\partial_x \chi'^{(m)} = -\partial_x \chi'^{(m-1)} + \chi'^{(m-1)}g^{-1}(\partial^x g) \quad m \geq 2 \tag{149}
\]

which give after solution:

\[
\chi^{(2)} = (\partial^x)^{-1}g^{-1}(\partial^x g) = (\partial^t)^{-1}g^{-1}(\partial^t g) = -\chi'^{(2)} \tag{150}
\]

\[
\chi^{(m)} = [(\partial^x)^{-1}(\partial^t + g^{-1}(\partial^x g))]^{m-2} (\partial^x)^{-1}g^{-1}(\partial^x g) \tag{151}
\]

\[
\chi'^{(m)} = (-1)^{m-2}\chi'^{(2)} \left[ \left(-\frac{\partial^t}{\partial^x} + g^{-1}(\partial^x g) \right) (\partial^x)^{-1} \right]^{m-2} \tag{152}
\]

Inserting the solution (151,152) into the conserved currents (139) we obtain after expansion with respect to the parameter \( \lambda \) the infinite tower of conserved charges for the field \( g \) of the principal chiral model:

\[
Q^0 \ (m) = \int dx \ \chi'^{(m)} \Gamma_t C \tag{153}
\]

\[
Q^1 \ (m) = \int dx \ \sum_{k=0}^{m} \chi'^{(k)} \Gamma_t \chi'^{(m-k)} \tag{154}
\]

\[
Q^2 \ (m) = \int dx \ \chi'^{(m)} \Gamma_t g^{-1} \tag{155}
\]

\[
Q^3 \ (m) = \int dx \ \chi'^{(m)} \Gamma_t (\partial^x)^{-1}g^{-1}(\partial^x g) \tag{156}
\]

4.3.2 Generalization of the principal chiral model equation

We can easily extend the results of the previous section to the case of the principal chiral model with the fields depending smoothly on variables \( t, x \) and \( y \). Let us recall the linearized version for \( N \times N \) matrices \( \chi \) with entries depending on the mentioned variables [6]:

\[
\partial^t \chi = \lambda(\partial^x + g^{-1}(\partial^x g))\chi \tag{157}
\]

\[
\partial^y \chi = \lambda(\partial^t + g^{-1}(\partial^t g))\chi \tag{158}
\]

with the conjugated set looking as follows:

\[
\partial^t \chi' = \lambda[\partial^x \chi' - \chi'g^{-1}(\partial^x g)] \tag{159}
\]

\[
\partial^y \chi' = \lambda[\partial^t \chi' - \chi'g^{-1}(\partial^t g)] \tag{160}
\]
The integrability condition of the set (157, 158) yields the currents discussed in [6]. Following the previous example we propose to investigate the consistency condition for (157, 158) which becomes:

\[
\left[ \partial^x \partial^y - \partial^t \partial^t + g^{-1}(\partial^x g) \partial^y - g^{-1}(\partial^t g) \partial^t \right] \chi = 0 \quad (161)
\]

with its conjugation in the sense (24) of the form:

\[
\left[ \partial^x \partial^y - \partial^t \partial^t \right] \chi' + \chi' \left[ - \frac{\partial}{\partial y} g^{-1}(\partial^x g) + \frac{\partial}{\partial t} g^{-1}(\partial^t g) \right] = 0 \quad (162)
\]

The principal chiral model equation coincides with the condition (3) for variable coefficients:

\[
\partial^t \Lambda_y + \partial^t \Lambda_t = \partial^y g^{-1}(\partial^x g) - \partial^t g^{-1}(\partial^t g) = 0 \quad (163)
\]

We can write down the components of the \( \Gamma \) operator taking into account that due for the fact of the basic derivatives being commutative, the transformation operator is the identity so the operators \( \Gamma \) and \( \hat{\Gamma} \) coincide:

\[
\hat{\Gamma}_x = \Gamma_x = \frac{1}{2} \left( - \frac{\partial}{\partial y} + \partial^y \right) \quad (164)
\]

\[
\hat{\Gamma}_y = \Gamma_y = g^{-1}(\partial^x g) + \frac{1}{2} \left( - \frac{\partial}{\partial x} + \partial^x \right) \quad (165)
\]

\[
\hat{\Gamma}_t = \Gamma_t = - g^{-1}(\partial^t g) + \frac{\partial}{\partial t} \quad (166)
\]

The current built using the above operators and the solutions \( \chi \) and \( \chi' \) of the respective equations (157 - 160) looks as follows:

\[
J^t_x = \chi' \Gamma_x \delta \chi \quad J^t_y = \chi' \Gamma_y \delta \chi \quad J^t_t = \chi' \Gamma_t \delta \chi \quad (167)
\]

with \( \delta \) being the symmetry operator from the set:

\[
\delta^0 = C \frac{\partial}{\partial \chi} \quad \delta^1 = \chi \frac{\partial}{\partial \chi} \quad \delta^2 = g^{-1} \frac{\partial}{\partial \chi} \quad \delta^3 = (\partial^y)^{-1} (g^{-1} \partial^t g) \frac{\partial}{\partial \chi} \quad (168)
\]

where \( C \) is a constant \( N \times N \) matrix.

According to the Proposition 2.2 the currents (167) obey the conservation law:

\[
\partial^t J^t_x + \partial^x J^t_y + \partial^y J^t_t = 0 \quad (169)
\]
Thus the currents (167) produce the infinite tower of conserved charges similarly to the previous case. After expansion of the fields $\chi$ and $\chi'$ in terms of powers of the parameter $\lambda$ (145) we obtain:

$$Q^0 (m) = \int dxdy \chi^{(m)} \Gamma_t C$$

(170)

$$Q^1 (m) = \int dxdy \sum_{k=0}^{m} \chi^{(k)} \Gamma_t \chi^{(m-k)}$$

(171)

$$Q^2 (m) = \int dxdy \chi^{(m)} \Gamma_t g^{-1}$$

(172)

$$Q^3 (m) = \int dxdy \chi^{(m)} \Gamma_t (\partial^y)^{-1} g^{-1} (\partial^t g)$$

(173)

where the explicit expressions for the component fields given by the set of equations (157 - 160) look as follows (with $\chi^{(0)}$, $\chi^{(1)}$, $\chi'^{(0)}$ and $\chi'^{(1)}$ being the unital $N \times N$ matrices as before):

$$\chi^{(2)} = -\chi'^{(2)} = (\partial^y)^{-1} g^{-1} (\partial^t g)$$

(174)

$$\chi^{(m)} = [((\partial^y)^{-1} (\partial^t + g^{-1} (\partial^t g))]^{m-2} (\partial^y)^{-1} g^{-1} (\partial^t g) \quad m \geq 2$$

(175)

$$\chi'^{(m)} = (-1)^{m-2} \chi'^{(2)} \left[ \left( -\partial_t + g^{-1} (\partial^t g) \right) (\partial^y)^{-1} \right]^{m-2} \quad m \geq 2$$

(176)

### 4.3.3 Nonlinear Toda lattice equation

The interesting case of application of the Proposition 3.2 is the linear set of equations which originates from nonlinear Toda lattice equation. The set of auxiliary equations derived in the paper [6] from gauged bicovariant differential calculus reads as follows:

$$\dot{\chi}_k = \lambda (e^{q_{k-1} - q_k} \chi_{k-1} - \chi_k) \zeta$$

(177)

$$\chi_{k+1} - \chi_k = -\lambda (\dot{\chi}_k + \dot{q}_k \chi_k) \zeta$$

(178)

where we have used our notation for the transformation operator which for the mixed model like the one considered in this section is simply shifting along the lattice $\zeta f_k = f_{k-1}$ with respect to the space-like dimension. Again we shall discuss the consistency condition which for the set of equations (177,178) becomes an equation of second order including classical derivative with respect to the time-variable and the discrete one with respect to the space-lattice variable:

$$\ddot{\chi}_k + \dot{q}_k \dot{\chi}_k - e^{q_{k-1} - q_k} (\chi_{k-1} - \chi_k) - (\chi_{k+1} - \chi_k) = 0$$

(179)
Let us notice that this equation is free of the parameter $\lambda$ and the operator $\zeta$.
We can rewrite this equation using the notation:

\[ \chi_{k-1} - \chi_k = (\zeta - 1)\chi_k = \partial \chi_k \quad \chi_{k+1} - \chi_k = (\zeta^\dagger - 1)\chi_k = \partial^\dagger \chi_k \] (180)

\[ q_{k-1} - q_k = (\zeta - 1)q_k = \partial q_k \quad \dot{f} = \partial^t f \] (181)

and it has the following form:

\[ \left[ \partial^t \partial^t + (\partial^t q)\partial^t - e^{(\partial^t q)\partial^t} \right] \chi = 0 \] (182)

In this form we see clearly that the equation involves both $\partial$ and $\partial^\dagger$ derivatives so we should follow the construction for discrete models given in Proposition 3.1 and 3.2.

Let us observe that the discussed equation fulfills the restriction (3) and it coincides in this case with the nonlinear Toda lattice equation:

\[ \partial^\dagger t \Lambda_t + \partial^t \Lambda_x = -\dot{q}_k - e^{q_k-q_{k+1}} + e^{q_{k-1}-q_k} = 0 \] (183)

We construct the $\Gamma$ operator according to (26, 30, 46, 50) taking into account that the transformation operator acts as follows:

\[ \zeta_t f_k(t) = f_k(t) \quad \zeta f_k(t) = f_{k-1}(t) \quad \zeta^- f_k(t) = f_{k+1}(t) \] (184)

\[ \hat{\Gamma}_t = \Gamma_t = -\partial^t + \partial^t + (\partial^t q) \] (185)

\[ \hat{\Gamma}_x = -\zeta^\dagger e^{-(\partial^t q)} \] (186)

\[ \hat{\Gamma}_x = \zeta \] (187)

The components of the $\hat{\Gamma}$ operator yield the corresponding components of the current:

\[ J^t_t = \delta \chi \hat{\Gamma}_t \chi \quad J^t_x = \delta \chi \hat{\Gamma}_x \chi \quad J^\dagger_x = \delta \chi \hat{\Gamma}_x \chi \] (188)

which obey the conservation law including the conjugated derivative:

\[ \partial^t J^t_t + \partial^t J^t_x + \partial^\dagger J^\dagger_x = 0 \] (189)

provided the field $\chi$ fulfills the consistency condition (182), the field $\chi'$ its conjugation:

\[ \chi' \left[ \dot{\chi}' - \partial^t \left( \partial^t q \right) - \partial^\dagger e^{-(\partial^t q)} \right] = 0 \] (190)
and the operator $\delta$ is now the symmetry operator of the conjugated equation (190).

We proceed taking integrals with respect to the discrete spatial variable (we have denoted it as $x$) and obtain the charges:

$$Q^\delta = \int dx \ J_t = \int dx \ \delta \hat{\chi} t \chi$$  \hspace{1cm} (191)

which are also conserved:

$$\partial^t Q = 0$$  \hspace{1cm} (192)

In the above formula we understand the discrete definite integral $\int dx$ as given by (58).

Analogously to the examples for principal chiral model we can check that the following operators transform solution $\chi'$ of the equation (190) into the solution $\delta \chi'$:

$$\delta^0 = c \frac{\partial}{\partial \chi'}$$  
$$\delta^1 = e^q \frac{\partial}{\partial \chi'}$$  
$$\delta^2 = [-\partial^{-1}(\partial'q) + t] \frac{\partial}{\partial \chi'}$$  \hspace{1cm} (193)

where the indefinite discrete integral $\partial^{-1}$ is given by:

$$\partial^{-1} = \frac{1}{\zeta - 1} = -\sum_{k=0}^{\infty} \zeta^k$$  \hspace{1cm} (194)

and $c$ is a constant.

The derived symmetries of the equation (182) lead to the following expressions for charges:

$$Q^0 = \int dx \ c \Gamma_t \chi$$  \hspace{1cm} (195)

$$Q^1 = \int dx \ e^q \Gamma_t \chi$$  \hspace{1cm} (196)

$$Q^2 = \int dx \ [-\partial^{-1}(\partial'q) + t] \Gamma_t \chi$$  \hspace{1cm} (197)

After expanding the solutions $\chi$ with respect to powers of the operators $\lambda \zeta$ and $\chi'$ in terms of $\lambda \zeta^{-}$ we shall be able to rewrite the expressions (191) as infinite set of conserved charges:

$$Q^{0(m)} = \int dx \ c \Gamma_t \chi^{(m)}$$  \hspace{1cm} (198)

$$Q^{1(m)} = \int dx \ e^q \Gamma_t \chi^{(m)}$$  \hspace{1cm} (199)

$$Q^{2(m)} = \int dx \ [-\partial^{-1}(\partial'q) + t] \Gamma_t \chi^{(m)}$$  \hspace{1cm} (200)

where the components $\chi^{(m)}$ read as follows:

$$\chi^{(m)} = (-1)^{m-1} \left[ (\partial^t)^{-1}(\partial^t + (\partial^t q)) \right]^{m-2} (\partial^t)^{-1}(\partial'q) \quad m \geq 2$$  \hspace{1cm} (201)
4.3.4 Nonlinear Toda lattice equation - generalization

Our next example is the generalization of nonlinear Toda model sketched in [6]. The functions depend now smoothly on one time variable $t$, one spacelike variable $y$ and one discrete space-like coordinate $x$. The bicovariant differential calculus is in this case defined as follows [6]:

$$\delta f = [S^{-1}, f] \tau - f' \xi$$
$$df = \dot{f} \tau + [S, f] \xi$$  \hspace{1cm} (202)

$$A = X \tau + (Y - 1) S \xi$$  \hspace{1cm} (203)

where $X$ and $Y$ are matrices with entries being smooth functions of $t$ and $y$ and discrete in the coordinate $x$. The condition $\delta A = 0$ leads to the following generalizd Toda model nonlinear equations:

$$X'_k = Y_k - Y_{k-1} \quad \dot{Y}_k = Y_k X_{k+1} - X_k Y_k$$  \hspace{1cm} (204)

The linear version for auxiliary field $\chi$ can be deduced following [6] from the set of equation given by:

$$\delta \chi = \lambda (d + A) \chi$$  \hspace{1cm} (205)

and looks as follows:

$$(S^{-1} - 1) \chi = \lambda [\dot{\chi} + X \chi] S$$  \hspace{1cm} (206)

$$-\chi' = \lambda [(S - 1) \chi + (Y - 1) (S \chi)] S$$  \hspace{1cm} (207)

We can rewrite the above set of equations using our notations:

$$S = \zeta \quad s^{-1} = \zeta^-$$
$$\zeta \chi_k(t, y) = \chi_{k+1}(t, y) \quad \zeta^- \chi_k(t, y) = \chi_{k-1}(t, y)$$  \hspace{1cm} (208)

$$(\zeta - 1) \chi = \partial \chi \quad (\zeta^- - 1) \chi = \partial^\dagger \chi$$  \hspace{1cm} (209)

$$\chi' = \partial^y \chi \quad \dot{\chi} = \partial^\dagger \chi$$  \hspace{1cm} (210)

and they are given by the formulas:

$$\partial^\dagger \chi = \lambda [\partial^\dagger + X] \chi \zeta$$  \hspace{1cm} (212)

$$-\partial^y \chi = \lambda [\partial \chi + (Y - 1) (\zeta \chi)] \zeta$$  \hspace{1cm} (213)

The consistency equation for the above set of equations reads as:

$$[\partial^y \partial^\dagger + X \partial^y - Y \partial - \partial^\dagger] \chi = 0$$  \hspace{1cm} (214)
Let us check the condition (3) for the variable coefficients of the operator of the equation:

$$\partial^t y \Lambda_y + \partial^t \Lambda_x = -\partial^y X - Y + \zeta Y = 0$$  \hspace{1cm} (215)

As before our equation fulfills the restriction (3) due to the fact that X and Y obey the generalization of the nonlinear Toda model (204). The condition (40) also holds as the coefficient for the derivative $\partial^t$ is constant.

Similarly to the previous examples we can apply the Propositions 3.1 and 3.2 for the discrete models depending on initial and conjugate discrete derivatives. In this way we obtain the $\hat{\Gamma}$ operator in the form:

$$\hat{\Gamma}_t = \Gamma_t = \frac{1}{2}(-\partial^t + \partial^y) \quad \hat{\Gamma}_y = \Gamma_y = \frac{1}{2}(-\partial^t + \partial^y) + X$$  \hspace{1cm} (216)

$$\hat{\Gamma}_x = -\zeta (\zeta Y) \quad \hat{\Gamma}_x = -\zeta$$  \hspace{1cm} (217)

We use the above components of the $\hat{\Gamma}$ and $\hat{\tilde{\Gamma}}$ operators to construct the respective components of the currents:

$$J^\delta_t = \delta \chi \hat{\Gamma}_t \chi \quad J^\delta_y = \delta \chi \hat{\Gamma}_y \chi$$  \hspace{1cm} (218)

$$J^\delta_x = \delta \chi \hat{\Gamma}_x \chi \quad J^\delta_x = \delta \chi \hat{\tilde{\Gamma}}_x \chi$$  \hspace{1cm} (219)

where we have denoted $\delta$ as the symmetry operator of the conjugated equation (220) and $\chi$ is the solution of (212 - 214) while $\chi'$ solves its conjugation:

$$[-\partial^y \partial^t + \partial] \chi' + (\partial^y \chi') X + (\partial^\dagger \chi')(\zeta Y) = 0$$  \hspace{1cm} (220)

As the currents fulfill the conservation law:

$$\partial^t J^\delta_t + \partial^y J^\delta_y + \partial_x J^\delta_x = 0$$  \hspace{1cm} (221)

they yield the conserved charges (with the integral $\int dx$ in the sense of (59)):

$$Q^\delta = \int dydx \quad J^\delta_t = \int dydx \quad \delta \chi \hat{\Gamma}_t \chi$$  \hspace{1cm} (222)

Following the procedure applied earlier we expand the solutions $\chi$ and $\chi'$ in terms of powers of the operator $\lambda \lambda$ and $\lambda \lambda$ respectively:

$$\chi = \sum_{m=0}^{\infty} (\lambda \zeta)^m \chi^{(m)} \quad \chi' = \sum_{m=0}^{\infty} (\lambda \zeta)^m \chi'^{(m)}$$  \hspace{1cm} (223)
After assuming the first components in the form of the unital $N \times N$ matrix:

\[
\chi^{(0)} = \chi'^{(0)} = \chi^{(1)} = \chi'^{(1)} = 1_{N \times N}\]  

we obtain the following equations for the subsequent components:

\[
\partial^{\dagger} \chi^{(m)} = [\partial^t + X] \chi^{(m-1)} \quad m \geq 2 \]  
\[
-\partial^y \chi^{(m)} = [\partial + (Y-1)\zeta] \chi^{(m-1)} \quad m \geq 2
\]

Inserting the expansion (223) into the expressions for currents and charges we arrive at the infinite towers of the conserved currents and charges:

\[
J^\delta_{\mu}^{(m)} = \delta \chi^{(m)} \hat{\Gamma}_{\mu} \]  
\[
Q^\delta^{(m)} = \int dx dy \delta \chi^{(m)} t
\]

with the following explicit expressions for the components of the fields $\chi$:

\[
\chi^{(2)} = \chi'^{(2)} = (\partial^{\dagger})^{-1} X \]  
\[
\chi^{(m)} = [(\partial^{\dagger})^{-1} (\partial^t + X)]^{m-2} (\partial^{\dagger})^{-1} X \quad m \geq 2
\]

and the initial solutions for the conjugated equation (220) related to the symmetry operators in the form:

\[
\delta^0 \chi' = C \quad \delta^1 \chi' = \partial^{-1} X - y
\]

with $C$ being a constant $N \times N$ matrix.

\subsection*{4.3.5 Double discrete nonlinear Toda lattice equation}

Let us end the review of the integrable models with fully discrete Toda lattice equation. The set of auxiliary linear equations given in \[5\] has the following form:

\[
\chi_k(n) - \chi_k(n-1) = -\frac{\gamma}{c} [g_k^{-1}(n)g_{k+1}(n)\chi_{k+1}(n) - \chi_k(n)] \]  
\[
\chi_k(n) - \chi_{k-1}(n) = -\gamma c [g_k^{-1}(n)g_k(n+1)\chi_k(n+1) - \chi_k(n)]
\]

We rewrite the above equations using our notation:

\[
f_k(n) - f_{k-1}(n) = -\partial^x f_k(n) \quad \zeta_x f_k(n) = f_{k+1}(n)\]  
\[
f_k(n) - f_k(n-1) = -\partial^y f_k(n) \quad \zeta_t f_k(n) = f_k(n+1)\]
Then the set of equations (232,233) looks as follows:
\[
-\partial_t \chi = -\gamma c [g^{-1}(\zeta_x g)(\zeta_x \chi) - \chi] \\
-\partial^x \chi = -\gamma c [g^{-1}(\zeta_t g)(\zeta_t \chi) - \chi]
\] (236)

The consistency equation for the above set reads as:
\[
-g^{-1}(\zeta_x^{-} g) \partial^x \chi - \partial^x \chi + c^2 g^{-1}(\zeta_t^{-} g) \partial^t \chi + c^2 \partial^t \chi = 0
\] (238)

The condition (40) for our example is fulfilled:
\[
\partial_t \Lambda_x + \partial^t \Lambda_t = 0 \\
\partial^x \tilde{\Lambda}_x + \partial^x \tilde{\Lambda}_t = -\partial^x g^{-1}(\zeta_x^{-} g) + c^2 \partial^x g^{-1}(\zeta_t^{-} g) = 0
\] (239)

provided the double discrete nonlinear Toda lattice equation is valid for the field \( g \):
\[
\partial^x g^{-1}(\zeta_x^{-} g) = c^2 \partial^x g^{-1}(\zeta_t^{-} g)
\] (241)

We construct the conserved currents following Propositions 3.1 and 3.2. As both initial and conjugated derivatives appear in the equation we use the \( \Gamma \) and \( \tilde{\Gamma} \) operators:
\[
\Gamma_x = -1 \quad \tilde{\Gamma}_x = -(\zeta_x g^{-1}) g \\
\Gamma_t = c^2 \quad \tilde{\Gamma}_t = c^2 (\zeta_t g^{-1}) g
\] (242)

and obtain the \( \hat{\Gamma} \) and \( \hat{\tilde{\Gamma}} \) operators with the following components:
\[
\hat{\Gamma}_x = -\zeta_x^{-} \quad \hat{\tilde{\Gamma}}_x = -\zeta_x^{-} (\zeta_x g^{-1}) g \\
\hat{\Gamma}_t = c^2 \zeta_t^{-} \quad \hat{\tilde{\Gamma}}_t = c^2 \zeta_t^{-} (\zeta_t g^{-1}) g
\] (245)

The above operators give in turn the currents:
\[
J^\delta_\mu = \delta \chi \hat{\Gamma}_\mu \chi \quad \bar{J}^\delta_\mu = \delta \chi \hat{\tilde{\Gamma}}_\mu \chi
\] (246)

which are conserved according to the Proposition 3.2:
\[
\partial_t J_t^\delta + \partial^x J_x^\delta + \partial^t \bar{J}_t^\delta + \partial^x \bar{J}_x^\delta = 0
\] (247)

provided the function \( \chi \) solves the equation (238), \( \chi' \) the conjugated equation:
\[
\chi' \left[ \partial^x \left( \zeta_x g^{-1} \right) g + c^2 \partial^t \left( \zeta_t g^{-1} \right) g - c^2 \partial^t \right] = 0
\] (248)
and $\delta$ is the symmetry operator for the conjugated equation. Following the Corollary 3.3 we can reformulate the currents (246) in order to obtain the conservation law including only the $\partial$ derivatives:

\begin{align*}
J_x^\delta &= J_x^\delta - \zeta_x^-(\tilde{J}_x^\delta) \\
J_t^\delta &= J_t^\delta - \zeta_t^-(\tilde{J}_t^\delta)
\end{align*}
(249)

The current $J'$ obeys the conservation law:

$$\partial^x J^\delta_t + \partial^x J^\delta_x = 0$$
(250)

Now we apply the (57) version of the conserved current so as to obtain the conserved charges:

$$Q^\delta = \int dx J^\delta_t = \int dx \left( \delta \chi T_t \chi - \zeta_t^-(\delta \chi T_t \chi) \right)$$
(251)

where the integral $\int dx$ is defined as in (59).

\section{Final remarks}

We have discussed the method of derivation of the conservation laws for a class of equations with variable coefficients. It can be applied to models built using supersymmetric, discrete, mixed discrete and noncommutative (quantum Minkowski or braided) differential calculus.

The conserved charges were constructed explicitly for considered supersymmetric and discrete equations. The problem of derivation of such charges for noncommutative models is open. The integration over noncommutative spaces namely braided linear space (including q-Minkowski) is well developed \[3, 4, 8, 14\] but subintegrals and commutation rules for subintegrals and derivatives need further study. In particular the generalized version of the property (64)

$$\partial^x \int_{sub} = \int_{sub} \partial^x$$

must be derived. As we have shown for models on quantum planes \[14\] it is crucial in the construction of the conserved charges.
6 Appendix A

Proof of the Proposition 2.1:
Let us recall the notation for monomials of derivatives:
\[ [\rho_1...\rho_k] := \partial^{\rho_1}...\partial^{\rho_k} \]
\[ [\nu_1...\nu_m] := \partial^{\langle \nu_{1} \rangle}...\partial^{\langle \nu_{m} \rangle} \] (252)

Due to the modification of the Leibnitz rule we are to consider the solution of the operator equation for the \( \Gamma \) operator in the form of the polynomial of order \( N - 1 \):
\[ \Gamma_{\mu}(\partial, \langle \partial \rangle) = a^0_{\mu} + \sum_{l=1}^{N-1} \sum_{k=0}^{l} [\mu_1, ..., \mu_k, \mu] a^k_{\mu_1...\mu_l} [\mu_{k+1}, ..., \mu_l] \] (253)

where the coefficients \( a^k \) depend on the coordinates \( \vec{x} \).
The condition (25) from the section 2 applied to the above polynomial yields the equations for coefficients \( a^k_{\mu_1...\mu_l} \):
\[ \sum_{\mu} (- \langle \partial \mu \rangle + \partial^{\mu}) \circ \Gamma_{\mu}(\partial, \langle \partial \rangle) = \]
\[ - \sum_{l=1}^{N-1} \sum_{k=0}^{l} \sum_{\mu} [\mu_1, ..., \mu_k, \mu] a^k_{\mu_1...\mu_l} [\mu_{k+1}, ..., \mu_l] \]
\[ + \sum_{l=1}^{N-1} \sum_{k=0}^{l} [\mu_1, ..., \mu_k] \sum_{\mu} (\zeta^\mu_{\nu} a^k_{\mu_1...\mu_l}) [\nu, \mu_{k+1}, ..., \mu_l] \]
\[ + \sum_{l=1}^{N-1} \sum_{k=0}^{l} [\mu_1, ..., \mu_k] \sum_{\mu} (\partial^{\mu} a^k_{\mu_1...\mu_l}) [\mu_{k+1}, ..., \mu_l] \]
\[ - \sum_{\mu} [\mu] a^0_{\mu} + \sum_{\mu} (\zeta^\mu_{\nu} a^0_{\mu}) [\nu] + \sum_{\mu} (\partial^{\mu} a^0_{\mu}) = \]
\[ = \Lambda(\partial) - \Lambda(\langle \partial \rangle) \]

The procedure is analogous to the one used in the proof of the Proposition 4.1 from [13] which describes the derivation of conservation law for the equation with constant coefficients or fulfilling the strong condition (see (23) and...
(24) from \[13\]). Now we have decided to change the form of the conjugated equation. This results in the weaker restrictions for coefficients of the equation (1) and changes the set of equations for functions $a^k_{\mu_0\ldots\mu_l}$ defining the operator $\Gamma$ for the following one:

$$\partial^\mu a^0_\mu = 0$$  \hspace{0.5cm} (255)

$$a^0_\mu = \zeta^-_\mu \Lambda_\alpha$$  \hspace{0.5cm} (256)

$$\partial^\alpha a^0_\alpha + \zeta^-_\mu a^0_\mu = \Lambda_\mu$$  \hspace{0.5cm} (257)

$$\zeta^a_\mu a^0_\alpha\mu_1\ldots\mu_l + \partial^\alpha a^0_\alpha\mu_1\ldots\mu_l = \Lambda_{\mu_1\ldots\mu_l}$$  \hspace{0.5cm} (258)

$$-a^k_{\mu_1\ldots\mu_l} + \zeta^a_\mu a^{k+1}_{\mu_1\ldots\mu_l} + \partial^\alpha a^{k+1}_{\alpha\mu_1\ldots\mu_l} = 0$$  \hspace{0.5cm} (259)

$$a^l_{\mu_1\ldots\mu_l} + \partial^\alpha a^l_{\alpha\mu_1\ldots\mu_l} = \zeta^-_\mu \zeta^-_\mu \ldots \zeta^-_\mu \Lambda_{\alpha_1\ldots\alpha_l}$$  \hspace{0.5cm} (260)

with $l = 1, \ldots, N - 1$, $k = 0, \ldots, l - 1$.

We see that the equations (255, 256, 260) yielded by the coefficients of the conjugated equation differ from the case studied earlier in [12, 13]. We begin to solve this set of equations by deriving the coefficients $a^0_{\mu_1\ldots\mu_l}$ from equations (256, 257, 258). Namely for $l = N - 1$ we have:

$$\zeta^a_\mu a^0_{\alpha_1\ldots\mu_{N-1}} = \Lambda_{\mu_1\ldots\mu_{N-1}}$$  \hspace{0.5cm} (261)

Applying the inverse operator $\zeta^-$ we obtain:

$$a^0_{\mu_1\ldots\mu_{N-1}} = \zeta^-_\mu \Lambda_{\alpha_1\ldots\mu_{N-1}}$$  \hspace{0.5cm} (262)

We insert this solution into (258) for $l = N - 2$ and solve the next equation:

$$\zeta^a_\mu a^0_{\alpha_1\ldots\mu_{N-2}} - \partial^\alpha \Lambda_{\alpha_1\ldots\mu_{N-2}} = \Lambda_{\mu_1\ldots\mu_{N-2}}$$  \hspace{0.5cm} (263)

By assumption (3) after using $\zeta^-$ operator and (21) we derive $a^0_{\mu_1\ldots\mu_{N-2}}$ as:

$$a^0_{\mu_1\ldots\mu_{N-2}} = \zeta^-_\mu \Lambda_{\alpha_1\ldots\mu_{N-2}}$$  \hspace{0.5cm} (264)

Passing to the next equation from the subset (258) and solving them in the similar way we obtain the unique solution for coefficients $a^0$:

$$a^0_{\mu_1\ldots\mu_l} = \zeta^-_\mu \Lambda_{\alpha_2\ldots\mu_l}$$  \hspace{0.5cm} $l = 1, \ldots, N$  \hspace{0.5cm} (265)

This solution for initial coefficients allows us to evaluate the remaining ones using (259, 260), namely we obtain the $a^1$ coefficients after writing the subset
The same method applied to subsets of $\{259, 260\}$ for $k = 1, \ldots, N - 2$ produces the unique solution of the set of equations for coefficients in the form:

$$a^k_{\mu_1 \ldots \mu_l} = \zeta_{-}^{\alpha} \zeta_{\mu_k}^{\alpha_k} \ldots \zeta_{\mu_1}^{\alpha_1} \Lambda_{\alpha_1 \alpha_2 \ldots \mu_l} = 1, \ldots, N - 1$$

The derivation of the explicit formulas for unique solution of the coefficients of the operator $\Gamma_{\mu}$ concludes the proof of Proposition 2.1.

Let us notice once more that in the derivation of coefficients for the $\Gamma_{\mu}$ operator the crucial factors were the properties of coefficients of the equation $(1 - 3)$ which enabled us to solve the equation $(25)$ in the explicit form.

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