Twin Paired Domination number of a graph

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Abstract. A new domination parameter "Twin paired domination number" is introduced in this paper. The set \( S \subseteq V \) is called as a twin paired dominating set, if \( S \) is a paired dominating set and \(< S >\) has a perfect matching. The minimum cardinality over all \( S \) is called as twin paired domination number and it is denoted by \( TPD(G) \). In this paper, we investigate this parameter for some standard, special type graphs and for some power graphs.

Key words: Paired Domination number, Complementary independent twin paired domination number.

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1. Introduction
T.W. Haynes, et.al., [5, 8] Introduced a concept of paired domination number. A paired-dominating set is a set \( S \subseteq V \) must be a dominating set and the subgraph \(< S >\) induced by \( S \) contains a perfect matching. Paulraj Joseph et.al., initiated the concept of complementary perfect domination number and many results are discussed \([3, 5–7]\) . A set \( S \subseteq V \) is called a Complementary perfect dominating set, if \( S \) is a dominating set of \( G \) and the induced subgraph \(< V - S >\) has a perfect matching.

The minimum cardinality taken over \( S \) (Mentioned in above definitions) is called as paired- domination number and complementary perfect domination number respectively. As a motivation of above definitions, G. Mahadevan et.al., in \([9]\) introduced a new concept called complementary independent twin paired domination number. The set \( S \subseteq V \) is said to be Complementary independent twin paired dominating set, if \( S \) is a paired dominating set and \(< V - S >\) is a set of independent edges. The minimum cardinality taken over all the Complementary independent twin paired dominating set is called as Complementary independent twin paired domination number and it is denoted by \( CITD(G) \). The square graph \( G^2 \) of \( G \) has \( v(G^2) = v(G) \), where \( u \) and \( v \) are adjacent in \( G^2 \) whenever \( d_g(u, v) \leq 2 \). The cube graph \( G^3 \) of \( G \) has \( v(G^3) = v(G) \), where \( u \) and \( v \) are adjacent in \( G^3 \) whenever \( d_g(u, v) \leq 3 \). The total graph \( T(G) \) of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \) and two vertices are adjacent whenever they are either adjacent or incident in \( G \). The central graph \( C(G) \) is obtained by subdividing each edge exactly once and joining all the non adjacent vertices of \( G \). The middle graph of \( M_d(G) \) is a graph whose vertex
set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other an edge incident with it. The total graph $T(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in $G$. The Shadow graph $D_2(G)$ of a connected graph $G$ is obtained by taking two copies of $G$, say $G'$ and $G''$ join each vertex $u'$ in $G'$ to the neighbours of corresponding vertex $u''$ in $G''$.

**Preliminary result:**

**Theorem 1** [9] For even path, $CITD(P_n) = 2 \left\lfloor \frac{n}{4} \right\rfloor + 2$, where $n \geq 6$

**Theorem 2** [9] For even cycle, $CITD(C_n) = \begin{cases} 2 \left\lfloor \frac{n}{4} \right\rfloor + 2 & \text{if } n \equiv 2 \pmod{4} \\ 2 \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 0 \pmod{4} \end{cases}$

2. TPD-number for some standard type of graphs

**Definition 1** The set $S \subseteq V$ is called as be twin paired dominating set, if $S$ is a paired dominating set and $<S^c>$ has a perfect matching. The minimum cardinality over all $S$ is called as twin paired domination number and it is denoted by $TPD(G)$.

**Example**

![Illustration](image)

**Illustration** In the above figure 2.1, TPD-set is $S = \{v_2, v_3\}$ $TPD(G) = 2$

**Observation 1** $TPD(G) \leq CITD(G)$ and bounds are sharp
Example

In figure 2.2, $S_1 = \{v_1, v_2, v_7, v_8\}$ is the twin paired dominating set and $|S_1| = 4$. $S_2 = \{v_1, v_2, v_5, v_6, v_7, v_8\}$ is the complementary independent twin paired dominating set and $|S_2| = 6$. Thus $TPD(G) \leq CITD(G)$, and the sharpness is obtained for path and cycle.

**Theorem 3** For even path, $TPD(P_n) = 2\left\lfloor \frac{n}{4} \right\rfloor + 2$, where $n \geq 6$

**Proof.** The proof follows from Theorem 1

**Theorem 4** For even cycle, $TPD(C_n) = \begin{cases} 2\left\lfloor \frac{n}{4} \right\rfloor + 2 & \text{if } n \equiv 2 \pmod{4} \\ 2\left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 0 \pmod{4} \end{cases}$

**Proof.** The proof follows from Theorem 2

**Observation 2**

(i) For complete graph, $TPD(K_n) = 2$, where $n$ is even.

(ii) For Bipartite graph, $TPD(K_{n,m}) = 2$ where $n+m$ is even.

(iii) For wheel graph, $TPD(W_n) = 2$ where $n$ is odd.

(iv) For Book graph, $TPD(B_n) = 2$.

3. TPD-number for square and cube graphs of path and cycle.

**Theorem 5** For even number of vertices, $TPD(P_n^2) = TPD(C_n^2) = \begin{cases} \frac{n}{2} & n \equiv 0 \pmod{4} \\ \frac{n-2}{2} & n \equiv 2 \pmod{4} \end{cases}$

**Proof.** Let $S_1 = \{v_i : i \equiv 1, 2 \pmod{4}\}, S_2 = \{v - i : i \equiv 0, 3 \pmod{4}\}$.

**Case 1** $n \equiv 0 \pmod{4}$, For both the square graphs of path and cycle $S = S_1$ is the twin paired dominating set. Hence $|S| = \frac{n}{2}$.

**Case 2** $n \equiv 2 \pmod{4}$, For both the square graphs of path and cycle $S = S_2$ or $S_1$ is the twin paired dominating set. Hence $|S| = \frac{n-2}{2}$.

Clearly in above cases $TPD(G) \leq |S|$.

If $T \subseteq S$ is the twin paired dominating set, then $T$ fails to satisfy the definition. Therefore $S$ is the minimum Complementary independent twin paired dominating set. Implies $|S| \leq TPD(G)$. Hence $|S| = TPD(G)$. 

Figure 2.2
Example

![Figure 3.1](image1)

Illustration Here filled vertices are TPD-number of a graph. In figure 3.1 \( n = 8 \equiv 0 \text{(mod4)} \), hence \( TPD(C_n^2) = \frac{n}{2} \), which implies \( TPD(C_8^2) = \frac{8}{2} = 4 \).

In figure 3.2 \( n = 6 \equiv 2 \text{(mod4)} \), hence \( TPD(P_n^3) = \frac{n-2}{2} \), which implies \( TPD(P_6^3) = \frac{6-2}{2} = 2 \).

**Theorem 6** For even number of vertices, \( TPD(P_n^3) = \left\{ \begin{array}{ll} \frac{n}{2} & n \equiv 0 \text{(mod4)} \\ \frac{n-2}{2} & n \equiv 2 \text{(mod4)} \end{array} \right. \)

**Proof.** Follows same as theorem 5

**Theorem 7** For even number of vertices, \( TPD(C_n^3) = \left\{ \begin{array}{ll} 2 \lfloor \frac{n-1}{4} \rfloor & n \equiv 0 \text{(mod4)} \\ \frac{n-2}{2} & n \equiv 2 \text{(mod4)} \end{array} \right. \)

**Proof.** Let \( S_1 = \{v_i : i \equiv 1, 2 \text{(mod8)}\} \), \( S_2 = \{v - i : i \equiv 0, 3 \text{(mod4)}\} \).

**Case 1** \( n \equiv 0 \text{(mod4)} \), For both the square graphs of path and cycle \( S = S_1 \) is the twin paired dominating set. Hence \( |S| = 2 \lfloor \frac{n-1}{4} \rfloor \).

**Case 2** \( n \equiv 2 \text{(mod4)} \), For both the square graphs of path and cycle \( S = S_2 \) or \( S_1 \) is the twin paired dominating set. Hence \( |S| = \frac{n-2}{2} \).

Clearly in above cases \( TPD(G) \leq |S| \).

If \( T \subseteq S \) is the twin paired dominating set, then \( T \) fails to satisfy the definition. Therefore \( S \) is the minimum Complementary independent twin paired dominating set. Implies \( |S| \leq TPD(G) \). Hence \( |S| = TPD(G) \).

Example

![Figure 3.3](image2)

Illustration Here filled vertices are TPD-number of a graph. In figure 3.3 \( n = 8 \equiv 0 \text{(mod4)} \), hence \( TPD(C_n^3) = 2 \lfloor \frac{n-1}{4} \rfloor \), which implies \( TPD(C_8^3) = 2 \lfloor \frac{8-1}{4} \rfloor = 2 \).

In figure 3.4 \( n = 6 \equiv 2 \text{(mod4)} \), hence \( TPD(C_n^3) = \frac{n-2}{2} \), which implies \( TPD(C_6^3) = \frac{6-2}{2} = 2 \).
4. TPD- number for middle, Central and total graph.

**Theorem 8** For a Middle graph, \( TPD(Md(C_n)) = \begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd} \end{cases} \)

**Proof.** Let \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \). Let \( V(Md(C_n)) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\} \). Let \( S_1 = \{v_i, v'_i: i \text{ is odd} \} \) and \( S_2 = \{v_n, v'_n\} \). If \( n \) is even \( S = S_1 \) is the twin paired dominating set whose cardinality \( |S| = n \). If \( n \) is odd \( S = S_1 \cup S_2 \) is the twin paired dominating set whose cardinality \( |S| = n + 1 \). Clearly in above cases \( TPD(G) \leq |S| \). If \( T \subseteq S \) is the twin paired dominating set, then \( T \) fails to satisfy the definition. Therefore \( S \) is the minimum Complementary independent twin paired dominating set. Implies \( |S| \leq TPD(G) \). Hence \( |S| = TPD(G) \).

**Observation 3** We cannot find TPD- number for middle graph of path as the total number of vertices of \( Md(P_n) \) is odd always.

**Example**

![Figure 4.1](image)

**Illustration** Here filled vertices are TPD-number of a graph. In figure 4.1, \( n = 4 \) is even hence \( TPD(Md(C_n)) = n = 4 \).

**Theorem 9** For a Central graph, \( TPD(C(C_n)) = \begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd} \end{cases} \)

**Proof.** We take the same TPD- set as of theorem 8, Hence the TPD- number follows from theorem 8.

**Observation 4** We cannot find TPD- number for Central graph of path as the total number of vertices of \( C(P_n) \) is odd always.
Example

Illustration Here filled vertices are TPD-number of a graph. In figure 4.2, n=4 is even hence $TPD(C(C_n)) = n = 4$.
In figure 4.3, n=5 is odd hence $TPD(C(C_n)) = n + 1 = 6$.

Theorem 10 For a Total graph, $TP(T(C_n)) = \begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd} \end{cases}$

Proof. Let $\{v_1,v_2,\ldots,v_n\}$ be the vertex set of cycle $C_n$. Let $\{v_1,v_2,\ldots,v_n,v'_1,v'_2,\ldots,v'_n\}$ be the vertex set of $TPD(T(C_n))$. Let $S_1 = \{v_i,v'_i : i \text{ is odd}\}$. If n is even $S = S_1$ is the twin paired dominating set whose cardinality $|S| = n$. If n is odd $S = S_1$ is the twin paired dominating set whose cardinality $|S| = n - 1$. Clearly in above cases $TPD(G) \leq |S|$. If $T \subseteq S$ is the twin paired dominating set, then T fails to satisfy the definition. Therefore S is the minimum Complementary independent twin paired dominating set. Implies $|S| \leq TPD(G)$. Hence $|S| = TPD(G)$.

Observation 5 We cannot find TPD-number for Total graph of path as the total number of vertices of $T(P_n)$ is odd always.

Example

Illustration Here filled vertices are TPD-number of a graph. In figure 4.4, n=4 is even hence $TPD(T(C_n)) = n = 4$.
In figure 4.5, n=5 is odd hence $TPD(T(C_n)) = n - 1 = 4$. 

\[ v_1 \quad v'_1 \quad v_2 \quad v'_2 \quad v_3 \quad v'_3 \]

Figure 4.2

\[ v_5 \quad v'_5 \quad v_1 \quad v'_1 \quad v_2 \quad v'_2 \quad v_3 \quad v'_3 \]

Figure 4.3

\[ v_1 \quad v'_1 \quad v_2 \quad v'_2 \quad v_3 \quad v'_3 \]

Figure 4.4

\[ v_5 \quad v'_5 \quad v_1 \quad v'_1 \quad v_2 \quad v'_2 \quad v_3 \quad v'_3 \]

Figure 4.5
5. TPD- number of shadow graph and prism graph.

**Theorem 11** For the Shadow graph of path \( P_n \), \( TPD(D_2(P_n)) = \begin{cases} n & \text{if } n \text{ is even} \\ n + 3 & \text{if } n \text{ is odd} \end{cases} \)

**proof** Let \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \). Let \( \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\} \) be the vertex set of \( D_2(P_n) \). Let \( S_1 = \{v_i : i \equiv 2, 3 \text{(mod4)}\} \), \( S_2 = \{v'_i : i \equiv 0, 1 \text{(mod4)}\} \) and \( S_3 = \{v_{n-2}, v_{n-1}, v_n, v'_{n-2}, v'_{n-1}, v'_n\} \). If \( n \) is even \( S = S_1 \cup S_2 \) is twin paired dominating set whose cardinality \( |S| = n \). If \( n \) is odd \( S = S_1 \cup S_2 \cup S_3 \) is twin paired dominating set whose cardinality \( |S| = n + 3 \). Clearly in above cases \( TPD(G) \leq |S| \). If \( T \subseteq S \) is the twin paired dominating set, then \( T \) fails to satisfy the definition. Therefore \( S \) is the minimum Complementary independent twin paired dominating set. Implies \( |S| \leq TPD(G) \). Hence \( |S| = TPD(G) \).

**Example**

**Illustration** Here filled vertices are TPD-number of a graph. In figure 5.6, \( n = 4 \) is even hence \( TPD(D_2(P_n)) = n = 4 \).

In figure 5.7, \( n = 5 \) is odd hence \( TPD(D_2(P_n)) = n + 3 = 8 \).

**Theorem 12** For the Prism graph \( TPD(Pr_n) = \begin{cases} n & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd} \end{cases} \)

**proof** Let \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \). Let \( \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\} \) be the vertex set of \( Pr_n \). Let \( S_1 = \{v_i : i \equiv 1, 3 \text{(mod4)}\} \), \( S_2 = \{v'_i : i \equiv 1, 3 \text{(mod4)}\} \). If \( n \) is even \( S = S_1 \cup S_2 \) is twin paired dominating set whose cardinality \( |S| = n \). If \( n \) is odd \( S = S_1 \cup S_2 \) is twin paired dominating set whose cardinality \( |S| = n - 1 \). Clearly in above cases \( TPD(G) \leq |S| \). If \( T \subseteq S \) is the twin paired dominating set, then \( T \) fails to satisfy the definition. Therefore \( S \) is the minimum Complementary independent twin paired dominating set. Implies \( |S| \leq TPD(G) \). Hence \( |S| = TPD(G) \).

**Example**
Illustration Here filled vertices are TPD-number of a graph. In figure 5.8, n=3 is odd hence $TPD(P_r n) = n - 1 = 2$.

In figure 5.9, n=4 is even hence $TPD(P_r n) = n = 4$.

6. Conclusion
Throughout this paper, we introduced a new theory called Twin Paired Dominion Number and found its number for power graphs, central, Middle, Total graphs and shadow graphs of paths and cycles. we have also obtained this number for some product graphs and compare its result with other domination parameters that will be stated in subsequent papers.

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