Rings with Effects

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Abstract
An e-ring is a pair \((R, E)\) consisting of an associative ring \(R\) with unity 1 together with a subset \(E \subseteq R\) of elements, called effects, with properties suggested by the so-called effect operators on a Hilbert space. We establish the basic properties of e-rings, investigate commutative e-rings called c-rings, relate certain c-rings called b-rings to Boolean algebras, and prove a structure theorem for b-rings.

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1 Introduction

Let \(\mathcal{H}\) be a Hilbert space. In what follows, \(\mathcal{B}(\mathcal{H})\) denotes the *-algebra of all bounded linear operators on \(\mathcal{H}\), and \(\mathcal{G}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})\) is the subgroup of the additive group of \(\mathcal{B}(\mathcal{H})\) consisting of all Hermitian operators on \(\mathcal{H}\). The additive group \(\mathcal{G}(\mathcal{H})\), organized into a partially ordered abelian group as usual, will be called the Hermitian group for \(\mathcal{H}\). The identity operator \(1\) belongs to \(\mathcal{G}(\mathcal{H})\) and satisfies \(0 \leq 1\). We define \(\mathcal{E}(\mathcal{H}) := \{E \in \mathcal{G}(\mathcal{H}) \mid 0 \leq E \leq 1\}\) and (following G. Ludwig \cite{21}) we refer to operators in \(\mathcal{E}(\mathcal{H})\) as effect operators on \(\mathcal{H}\). We also define \(\mathcal{P}(\mathcal{H}) := \{P \in \mathcal{G}(\mathcal{H}) \mid P^2 = P\}\) to be the set of all (orthogonal) projection operators on \(\mathcal{H}\). Then we have

\[0, 1 \in \mathcal{P}(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{H}) \subseteq \mathcal{G}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}).\]

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In mathematical physics, the representation of observables by so-called POVM-measures, i.e., \(E(\mathcal{F})\)-valued measures on \(\sigma\)-fields \([2]\), as well as by the more conventional \(\mathbb{P}(\mathcal{F})\)-valued measures, has now become a commonplace. Consequently, \(E(\mathcal{F}), \mathbb{P}(\mathcal{F})\), and suitable generalizations thereof, have come to be employed as algebraic models for the semantics of both sharp and unsharp quantum logics \([5, 6, 8, 12, 15, 16, 20]\). Thus motivated, we take the pair \((\mathbb{B}(\mathcal{F}), E(\mathcal{F}))\) consisting of the ring \(\mathbb{B}(\mathcal{F})\) and the “effect algebra” \(E(\mathcal{F})\) as a prototype for the following more general notion of an “e-ring.”

1.1 Definition. An e-ring is a pair \((R, E)\) consisting of an associative ring \(R\) with unity 1 and a subset \(E \subseteq R\) of elements called effects such that 0, 1 \(\in E\); \(e \in E \Rightarrow 1 - e \in E\); and the set \(E^+\) consisting of all finite sums \(e_1 + e_2 + \cdots + e_n\) with \(e_1, e_2, ..., e_n \in E\) satisfies the following conditions: For all \(a, b \in E^+\):

(i) \(-a \in E^+ \Rightarrow a = 0\),

(ii) \(1 - a \in E^+ \Rightarrow a \in E\),

(iii) \(ab = ba \Rightarrow ab \in E^+\),

(iv) \(aba \in E^+\),

(v) \(aba = 0 \Rightarrow ab = ba = 0\), and

(vi) \((a - b)^2 \in E^+\).

The notion of an e-ring is mathematically equivalent to the notion of an effect-ordered ring originally introduced in \([9]\), but reformulated to emphasize the role of the “effect algebra” \(E\).

It is easy to see that the prototypic pair \((\mathbb{B}(\mathcal{F}), E(\mathcal{F}))\) is an e-ring as soon as one observes that \(E(\mathcal{F})^+\) is precisely the set of all positive Hermitian operators on \(\mathcal{F}\). (By a slight abuse of language, we call a Hermitian operator \(A\) “positive” if \(A \geq 0\).) Indeed, every effect operator is positive by definition, and a finite sum of positive Hermitian operators is positive. Conversely, if \(A\) is a positive Hermitian operator, \(\|A\|\) is the uniform operator norm of \(A\), and \(n\) is a positive integer with \(n \geq \|A\|\), then \((1/n)A\) is an effect operator, and \(A\) is a sum of \(n\) effect operators, each equal to \((1/n)A\).

In addition to the prototypic example \((\mathbb{B}(\mathcal{F}), E(\mathcal{F}))\), we have the following simple examples of e-rings. (More examples will be given later.) We denote the set of positive integers by \(\mathbb{N} := \{1, 2, 3, ...\}\).

1.2 Example. Let \(R\) be a ring with unity 1 such that \(n \cdot 1 \neq 0\) for all \(n \in \mathbb{N}\), and define \(E := \{0, 1\}\). Then \((R, E)\) is an e-ring.

1.3 Example. If \(R\) is any subfield of the totally ordered field \(\mathbb{R}\) of real numbers and \(E := \{e \in R \mid 0 \leq e \leq 1\}\), then \((R, E)\) is an e-ring.
As Examples 1.2 and 1.3 illustrate, the definition of an e-ring \((R, E)\) does not necessarily provide a strong connection between the algebraic structure of the set \(E\) of effects and the ring structure of \(R\). For instance, if \((R, E)\) is an e-ring and \(\tilde{R}\) is any extension ring of \(R\) such that \(1 \cdot x = x \cdot 1 = x\) for all \(x \in \tilde{R}\), then \((\tilde{R}, E)\) is also an e-ring. The rather weak connection between \(E\) and \(R\) does not concern us here because, motivated by quantum measurement theory and quantum logic, we are mainly interested in the structure of the “effect algebra” \(E\), and the enveloping ring \(R\) is just a convenient environment in which to study this structure.

Section 2 below, which treats the basic properties of e-rings, culminates with a theorem that the set \(P\) of projections in an e-ring acquires (at least) the structure of an orthomodular poset (Theorem 2.15) and a theorem stating that \(P\) indexes a so-called compression base for the directed group generated by the effects (Theorem 2.18). Section 3, which treats commutativity and coexistence in the effect algebra \(E\) of an e-ring, culminates in a structure theorem for a class of e-rings (called b-rings) in which every effect is a projection (Theorem 3.16). In a subsequent paper, we shall study square roots and polar decompositions in e-rings.

2 Basic Properties of e-Rings

We begin by extracting from an e-ring \((R, E)\) an analogue \(G\) of the Hermitian group \(G(\mathfrak{H})\) for the prototype \((\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))\).

2.1 Theorem. Let \((R, E)\) be an e-ring and let \(E^+\) be the set of all sums of finite sequences \(f\) elements of \(E\). Then

\[
G := E^+ - E^+ = \{a - b \mid a, b \in E^+\}
\]

is a subgroup of the additive group of the ring \(R\), and \(G\) is a directed partially ordered abelian group with positive cone \(E^+\) = \(\{g \in G \mid 0 \leq g\}\). Moreover, \(E \subseteq G\) and \(E\) generates the group \(G\).

Proof. Since \(E^+\) is closed under addition, it is clear that \(G = E^+ - E^+\) is a subgroup of the additive group of the ring \(R\). Also, \(0 \in E \subseteq E^+ \subseteq G\), and by Definition 1.1(i), if both \(a\) and \(-a\) belong to \(E^+\), then \(a = 0\). Therefore, \(G\) is a partially ordered abelian group with positive cone \(E^+\), the partial order being given by \(g \leq h \iff h - g \in E^+\) for \(g, h \in G\) [14 p. 3]. By the
definition of $G$, every element $g \in G$ can be written in the form $g = a - b$ with $a, b \in E^+$, i.e., $G$ is directed \cite{14} p. 4. Thus, $E^+$ generates the group $G$, and since $E$ generates $E^+$ as an additive semigroup, it follows that $E$ generates $G$ as a group.

If $(R, E)$ is an e-ring then, as an additive abelian group, $R$ is partially ordered (but not necessarily directed) with $E^+$ as its positive cone; however, unless $R$ is commutative, it is not necessarily a partially ordered ring (as usually understood) because $E^+$ need not be closed under multiplication.

2.2 Definition. Let $(R, E)$ be an e-ring and let $E^+$ be the set of all sums of finite sequences of elements of $E$. Then:

(i) The partially ordered additive abelian group $G := E^+ - E^+$ (Theorem 2.1) is called the directed group of $(R, E)$.

(ii) Idempotent elements $p = p^2 \in G$ are called projections.

For the prototype e-ring $(\mathbb{B}(\mathcal{H}), \mathbb{E}(\mathcal{H}))$, the directed group is the Hermitian group $G(\mathcal{H})$, and $P(\mathcal{H})$ is the set of projections. In Example 1.2 the directed group $G = \{n \cdot 1 \mid n \in \mathbb{Z}\}$ of $(R, E)$ is isomorphic to the totally ordered additive group of the ring $\mathbb{Z}$ of integers. In Example 1.3 $G$ is the additive subgroup of the field $R$ with the total order inherited from $\mathbb{R}$. In both Examples 1.2 and 1.3 the only projections are 0 and 1.

The e-ring $(\mathbb{B}(\mathcal{H}), \mathbb{E}(\mathcal{H}))$ is a special case (when $A$ is a type-I von Neumann factor) of the e-ring $(A, E)$ in the following example.

2.3 Example. Let $A$ be a unital $C^*$-algebra and let

$$E := \{aa^* \mid a \in A \text{ and } \exists b \in A, aa^* + bb^* = 1\}.$$ 

Then $(A, E)$ is an e-ring, $E^+ = \{aa^* \mid a \in A\}$, and the directed group $G$ for $(A, E)$ is the additive group of self-adjoint elements in $A$.

In the sequel, we assume that $(R, E)$ is an e-ring, 1 is the unity element in $R$, $E^+$ is the set of all sums of finite sequences of elements of $E$, $G = E^+ - E^+$ is the directed group of $(R, E)$, $G$ is partially ordered with positive cone $E^+$, and

$$P = \{p \in G \mid p = p^2\}$$

is the set of all projections in $G$. It is understood that $P$ and $E$ are partially ordered by the restrictions of the partial order $\leq$ on $G$. 

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2.4 Lemma. Let \( g, h \in G \) and let \( p \in P \). Then:

(i) \( g^2 \in E^+ \).
(ii) \( gh + hg \in G \).
(iii) \( g \in E^+ \Rightarrow ghg \in G \).
(iv) \( php \in G \).
(v) \( h \in E^+ \Rightarrow php \in E^+ \).

Proof. (i) By hypothesis, \( G \) is directed, hence \( g \) is a difference of two elements in \( E^+ \), so \( g^2 \in E^+ \) by Definition 1.1 (vi).
(ii) By (i), \( gh + hg = (g + h)^2 - g^2 - h^2 \in G \).
(iii) As \( G \) is directed, there exist \( a, b \in E^+ \) such that \( h = a - b \). By Definition 1.1 (iv), \( gag, gbg \in E^+ \), whence \( ghg = gag - gbg \in G \).
(iv) As \( p = p^2 \in G \), (i) implies that \( p \in E^+ \); hence (iv) follows from (iii).
(v) As in (iv), \( p \in E^+ \), so (v) follows from Definition 1.1 (iv).

2.5 Lemma. (i) \( E = \{ e \in G \mid 0 \leq e \leq 1 \} \).
(ii) \( 0, 1 \in P \).
(iii) \( p \in P \Rightarrow 1 - p \in P \).

Proof. (i) By Definition 1.1, we have \( 0, 1 \in E \). Evidently, if \( e \in E \), then \( e, 1 - e \in E \subseteq E^+ = \{ g \in G \mid 0 \leq g \} \), whence \( e \in G \) with \( 0 \leq e \leq 1 \). Conversely, suppose \( e \in G \) with \( 0 \leq e \leq 1 \). Then \( e, 1 - e \in E^+ \), and it follows from Definition 1.1 (ii) that \( e \in E \). Thus, \( E = \{ e \in G \mid 0 \leq e \leq 1 \} \).
(ii) \( 0 = 0^2 \in G \) and \( 1 = 1^2 \in G \).
(iii) If \( p \in G \) with \( p = p^2 \), then \( 1 - p \in G \) with \( (1 - p)^2 = 1 - 2p + p^2 = 1 - p \).
(iv) Suppose \( p \in P \). By Lemma 2.4 (i), \( p = p^2 \in E^+ \), i.e., \( 0 \leq p \). Thus, by (iii), \( 0 \leq 1 - p \), whence \( 0 \leq p \leq 1 \), so \( p \in E \) by (i).

In view of Lemma 2.5 (i), we shall refer to \( E \) as the unit interval in \( G \). We have

\[
0, 1 \in P \subseteq E \subseteq E^+ \subseteq G \subseteq R.
\]

Equipped with the partially defined binary operation \( \oplus \) obtained by restricting + on \( G \) to \( E \), the unit interval \( E \) forms a so-called effect algebra. The effect algebras arising from e-rings in this way are rather special in that they admit a (perhaps only partial) multiplicative structure (cf. [7]).

2.6 Lemma. Let \( d, e, f \in E \) with \( ef = fe \). Then:

(i) \( 0 \leq ef \leq e, f \leq 1 \).
(ii) \( 0 \leq efe \leq e^2 \leq e \leq 1 \).
(iii) \( 0 \leq e, f \leq e + f - ef \leq 1 \).
(iv) \( 0 \leq e - e^2 \leq e, 1 - e \leq 1 \).

Proof. Assume the hypotheses.

(i) By Definition 1.1 (iii), \( 0 \leq ef \). Likewise, \( 0 \leq e(1 - f) = e - ef \), so \( ef \leq e \), and by symmetry, \( ef \leq f \).
(ii) By Definition [1.1 (iv)], \(0 \leq ede\) and \(0 \leq e(1-d)e = e^2 - ede\). Also, by (i) with \(f = e\), we have \(e^2 \leq e \leq 1\).

(iii) By (i), \(0 \leq f - ef\), so \(0 \leq e + f - ef\), and by symmetry, \(f \leq e + f - ef\). Also, by Definition [1.1 (iii)], \(0 \leq (1-e)(1-f) = 1 - e - f + ef\), whence \(e + f - ef \leq 1\).

(iv) By (ii), \(0 \leq e - e^2\). Also, by Lemma [2.5 (i)], \(0 \leq e^2\), whence \(e - e^2 \leq e\). Finally, by (iii) with \(f = e\), we have \(2e - e^2 \leq 1\), so \(e - e^2 \leq 1 - e\). \(\square\)

2.7 Lemma. Let \(g, h, k \in E^+, p \in P\), and \(n \in \mathbb{N}\). Then: (i) \(gh = 0 \Rightarrow hg = 0\). (ii) If \(gk = kg\) and \(hk = kh\), then \(g \leq h \Rightarrow gk \leq hk\). (iii) If \(gh = hg\), then \(g \leq h \Rightarrow g^2 \leq h^2\). (iv) \(g \leq np \Rightarrow g = gp = pg\). (v) \(g^n = 0 \Rightarrow g = 0\).

Proof. (i) \(gh = 0 \Rightarrow ghg = 0 \Rightarrow hg = 0\) by Definition [1.1 (v)].

(ii) Assume the hypotheses. Then \(h - g \in E^+\) and \(hk - gk = (h-g)k = k(h-g) \in E^+\) by Definition [1.1 (iii)].

(iii) Assume the hypotheses. By (ii), \(g^2 \leq gh\) and \(gh \leq h^2\), so \(g^2 \leq h^2\).

(iv) Assume the hypotheses. Then \(g, np - g \in E^+\), whence \((1-p)g(1-p), (1-p)(np-g)(1-p) = -(1-p)g(1-p) \in E^+\) by Lemma [2.2 (v)]. Therefore, \((1-p)g(1-p) = 0\) by Definition [1.1 (i)], and it follows from Definition [1.1 (v)] that \((1-p)g = g(1-p) = 0\), i.e., \(g = pg = gp\).

(v) We may assume that \(n\) is the smallest positive integer such that \(g^n = 0\). If \(n\) is even and \(k = n/2\), we have \(g^k \cdot 1 \cdot g^k = 0\), so \(g^k = g^k \cdot 1 = 0\) by Definition [1.1 (v)], contradicting our assumption. Therefore \(n\) is odd. If \(n = 1\), we are done, so we may assume that \(n = 2k + 1\) where \(k \in \mathbb{N}\). Then \(g^kg^k = 0\), so \(g^{k+1} = g^kg = 0\) by Definition [1.1 (v)], again contradicting our assumption. \(\square\)

According to part (i) of the following lemma, 1 is a so-called order unit in \(G\) [14] p. 4.

2.8 Lemma. (i) If \(g \in G\), there exists \(n \in \mathbb{N}\) such that \(g \leq n \cdot 1\). (ii) If \(a_1, a_2, \ldots, a_n \in E^+\) and \(a_1 + a_2 + \cdots + a_n = 0\), then \(a_1 = a_2 = \cdots = a_n = 0\).

Proof. (i) Write \(g = a - b\) with \(0 \leq a, b\). Then \(0 \leq b = a - g\), whence \(g \leq a\). As \(a \in E^+\), there exist \(e_1, e_2, \ldots, e_n \in E\) with \(a = e_1 + e_2 + \cdots + e_n\). By Lemma [2.5 (i)], \(e_i \leq 1\) for \(i = 1, 2, \ldots, n\), and it follows that \(g \leq a \leq n \cdot 1\).

(ii) Assume the hypotheses. It will be sufficient to prove that \(a_1 = 0\). But, \(-a_1 = a_2 + \cdots + a_n \in E^+\), so \(a_1 = 0\) by Definition [1.1 (i)]. \(\square\)
2.9 Theorem. Let $e \in E$ and $p \in P$. Then the following conditions are mutually equivalent: (i) $e \leq p$, (ii) $e = ep = pe$, (iii) $e = pep$, (iv) $e = ep$, (v) $e = pe$.

Proof. (i) $\Rightarrow$ (ii). Assume that $e \leq p$ and let $d := p - e$. Then $e, d \in E^+$, $e + d = p$, and

$$(1 - p)e(1 - p) + (1 - p)d(1 - p) = (1 - p)p(1 - p) = 0.$$  

By Lemma 2.5 (iii), $1 - p \in P$, whence by Lemma 2.4 (v),

$$(1 - p)e(1 - p), (1 - p)d(1 - p) \in E^+,$$

and it follows from Lemma 2.8 (ii) that $(1 - p)e(1 - p) = (1 - p)d(1 - p) = 0$. Therefore, by Definition 1.1 (v), $(1 - p)e = e(1 - p) = 0$, i.e., $e = pe = ep$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Follows from $p = p^2$.

(iv) $\Leftrightarrow$ (v). By Lemma 2.7 (i), $e = pe \Rightarrow (1 - p)e = 0 \Rightarrow e(1 - p) = 0 \Rightarrow e = ep$, and the converse implication follows by symmetry.

(v) $\Rightarrow$ (i). Assume (v). Since (iv) $\Leftrightarrow$ (v), we have $pe = ep = e$, so $(1 - e)p = p(1 - e) = p - e$, whence $p - e \in E^+$ by Definition 1.1 (iii), and therefore $e \leq p$. \hfill \Box

2.10 Corollary. Let $e \in E$ and $p \in P$. Then the following conditions are mutually equivalent: (i) $p \leq e$, (ii) $p = ep = pe$, (iii) $p + pep = pe + ep$, (iv) $p = ep$, (v) $p = pe$.

Proof. Replace $e$ by $1 - e$ and $p$ by $1 - p$ in Theorem 2.9 noting that $p \leq e \Leftrightarrow 1 - e \leq 1 - p$. \hfill \Box

2.11 Theorem. Let $p, q \in P$. Then the following conditions are mutually equivalent: (i) $p + q \in E$, (ii) $p + q \leq 1$, (iii) $pq = 0$, (iv) $pq = qp = 0$, (v) $p + q \in P$.

Proof. (i) $\Rightarrow$ (ii) Follows from Lemma 2.6 (i).

(ii) $\Rightarrow$ (iii). If $p + q \leq 1$, then $p \leq 1 - q$, and it follows from Theorem 2.9 that $p = p(1 - q) = p - pq$, whence $pq = 0$.

(iii) $\Rightarrow$ (iv). Follows from Lemma 2.7 (i).

(iv) $\Rightarrow$ (v). If $pq = qp = 0$, then $(p + q)^2 = p^2 + pq + qp + q^2 = p + q$.

(v) $\Rightarrow$ (i). Follows from Lemma 2.5 (iv). \hfill \Box
2.12 Theorem. If \( p, q \in P \), then the following conditions are mutually equivalent: (i) \( pq \in P \), (ii) \( pq \in E \), (iii) \( pq = qp \), (iv) \( pq = pqp \). Moreover, if any—hence all—of these conditions hold, then \( pq = p \land q \) is the infimum (greatest lower bound) of \( p \) and \( q \) both in \( P \) and in \( E \).

Proof. (i) \(\Rightarrow\) (ii). Assume that \( e := pq \in E \). Since \( e = pe = eq \), Theorem 2.9 implies that \( e \leq p, q \) and \( e = ep = eq \). Thus, \( e^2 = epq = e \), so \( e \in P \). As \( e \leq p, q \), it follows that \( p - e, q - e \in E^+ \). Furthermore, \( (p - e)(q - e) = pq - pe - eq + e^2 = 0 \), whence Lemma 2.7 (i) implies that \( 0 = (q-e)(p-e) = qp - e \), i.e., \( qp = e = pq \).

(ii) \(\Rightarrow\) (iii). Assume that \( p = q, p \) and \( p = q, q \). Then \( pq \in P \) with \( pq \leq p, q \). Also, if \( e \in E \) with \( e \leq p, q \), then \( e = ep = eq \), whence \( e(pq) = e \), i.e., \( e \leq pq \).

(iii) \(\Rightarrow\) (iv). If \( pq = qp \), then \( pq = p(pq) = pqp \).

(iv) \(\Rightarrow\) (i). If \( pq = pqp \), then \( (pq)^2 = pqpq = (pq)q = pq \), so \( pq \in P \).

Suppose that any, hence all of the conditions (i)–(iv) hold. Then \( pq \in P \) with \( pq \leq p, q \). Also, if \( e \in E \) with \( e \leq p, q \), then \( e = ep = eq \), whence \( e(pq) = e \), i.e., \( e \leq pq \).

2.13 Corollary. If \( p, q \in P \) and \( pq = qp \), then \( p + qpq \) is the supremum (least upper bound) of \( p \) and \( q \) both in \( P \) and in \( E \).

Proof. The mapping \( r \mapsto (1-r) \) is order inverting and of period 2 on \( P \). Also \( pq = qp \iff (1-p)(1-q) = (1-q)(1-p) \), and \( 1-(1-p)(1-q) = p + q - pq \).

2.14 Corollary. Let \( p, q \in P \). Then \( q - p \in E \iff p \leq q \iff q - p \in P \). Moreover, if \( p \leq q \), then \( q - p = q \land (1 - p) \) is the infimum of \( q \) and \( 1 - p \) both in \( P \) and in \( E \).

Proof. If \( q - p \in E \), then \( 0 \leq q - p \), whence \( p \leq q \). Suppose that \( p \leq q \). Then \( p = q - p \) by Theorem 2.9, so \( q(1 - p) = (1 - p)q = q - p \) and it follows from Theorem 2.12 that \( q - p \in P \) and that \( q - p \) is the infimum both in \( E \) and \( P \) of \( q \) and \( 1 - p \).

2.15 Theorem. With \( p \mapsto 1 - p \) as orthocomplementation, \( P \) is an orthomodular poset (OMP) and, for \( p, q \in P \) with \( p \leq 1 - q \), the supremum in \( P \) of \( p \) and \( q \) is \( p \lor q = p + q \).

Proof. We have \( 0 \leq p \leq 1 \) for all \( p \in P \), and \( p \mapsto 1 - p \) is an order-reversing mapping of period 2 on \( P \). Let \( p, q \in P \). If \( p \leq 1 - q \), then \( p + q \leq 1 \), whence \( p + q \in E \), so \( p + q \in P \) with \( pq = qp = 0 \) by Theorem 2.11 and it follows from Corollary 2.13 that \( p + q = p + q - pq \) is the supremum of \( p \) and \( q \) in \( P \). Now suppose that \( p \leq q \). By Corollary 2.14 \( q - p \in P \), whence \( q = p + (q - p) \) is the orthomodular identity.
2.16 Lemma. Suppose that \(d, e, f, d+e+f \in E\) with \(d+e, d+f \in P\). Then \(d, e, f \in P\).

Proof. Assume the hypotheses and let \(p := d+e \in P\) and \(q := d+f \in P\). Then \(p+f = d+e+f \leq 1\), so \(f \leq 1-p \in P\); hence by Theorem 2.9 \(f = (1-p)f = pf\), and therefore \(pf = 0\). Also, \(d \leq d+e = p\), so \(pd = d\) by Theorem 2.9. Consequently, \(pq = p(d+f) = d \in E\), and it follows from Theorem 2.12 that \(d = pq = pq \in P\). As \(d \leq p\) and \(d, p \in P\), it follows from Corollary 2.14 that \(e = p - d \in P\), and likewise \(f = q - d \in P\). \(\square\)

The following theorem provides useful conditions—not directly involving multiplication—for determining whether an effect is a projection.

2.17 Theorem. If \(e \in E\), then the following conditions are mutually equivalent: (i) If \(a, b, a+b \in E\), then \(a, b \leq e \Rightarrow a+b \leq e\). (ii) If \(d \in E\) with \(d \leq e, 1-e\), then \(d = 0\). (iii) \(e \in P\).

Proof. (i) \(\Rightarrow\) (ii). Assume (i) and the hypotheses of (ii). Then \(d+e \leq 1\) with \(d, e \leq e\), whence \(d+e \leq e\), and it follows that \(d = 0\).

(ii) \(\Rightarrow\) (iii). Assume (ii). By Lemma 2.6 (iv), \(0 \leq e - e^2 \leq e, 1-e\), so \(e - e^2 = 0\), i.e., \(e = e^2\).

(iii) \(\Rightarrow\) (i). Suppose \(e \in P\) and assume the hypotheses of (i). By Corollary 2.10 \(a = ae\) and \(b = be\), whence \((a+b)e = a+b\), and it follows that \(a+b \leq e\). \(\square\)

If \(p \in P\) and \(g \in G\), then by Lemma 2.4 (iv), \(pgp \in G\); hence we can define the mapping \(J_p: G \rightarrow G\) by \(J_p(g) = pgp\) for all \(g \in G\). Thus, owing to Lemmas 2.4, 2.5, 2.16 and Theorem 2.11 we have the following theorem (see [10, 11]).

2.18 Theorem. The family \((J_p)_{p \in P}\) is a compression base for \(G\).

The partially ordered abelian group \(G\) is said to be archimedean iff, whenever \(g, h \in G\) and \(ng \leq h\) for all \(n \in \mathbb{N}\), it follows that \(g \leq 0\) [14, p. 20]. An order-preserving group endomorphism \(J: G \rightarrow G\) is called a retraction iff \(J(1) \in E\) and, for all \(e \in E\), \(e \leq J(1) \Rightarrow J(e) = e\). If \(p \in P\), it is clear that \(J_p\) is a retraction on \(G\). Conversely, as a consequence of [9, Corollary 4.6], we have the following theorem.

2.19 Theorem. If \(G\) is archimedean, then every retraction \(J\) on \(G\) has the form \(J = J_p\) with \(p = J(1) \in P\).
3 Commuting Elements of $G$

We maintain our standing hypothesis that $(R, E)$ is an e-ring, $G$ is its directed group, and $P$ is the OMP of projections in $G$.

3.1 Definition. Let $g, h \in G$. We write $gCh$ iff $gh = hg$ and we define the commutant of $g$ in $G$ by $C(g) := \{h \in G \mid gCh\}$. More generally, if $X \subseteq G$, then $C(X) := \bigcap_{x \in X} C(x)$ is called the commutant of $X$.

In contrast with more-or-less standard usage, e.g., in operator theory, we use the notion of the commutant only in relation to elements of $G$, and not to general elements of the enveloping ring $R$.

If $L$ is any OMP, then two elements $p, q \in L$ are said to be Mackey compatible iff there exist pairwise orthogonal elements $p_1, q_1, d \in L$ with $p = d \lor p_1$, and $q = d \lor q_1$ [8]. By Theorem 2.15, projections $p, q$ in the OMP $P$ are Mackey compatible iff there exist projections $p_1, q_1, d \in P$ with $d + p_1 + q_1 \leq 1$, $p = d + p_1$, and $q = d + q_1$. The next lemma provides a useful condition—not directly involving multiplication—for determining whether two projections commute.

3.2 Lemma. If $p, q \in P$, then $pCq$ iff $p$ and $q$ are Mackey compatible in $P$.

Proof. Suppose that $pCq$. By Theorem 2.12, $pq \in P$ with $pq \leq p, q$; by Corollary 2.13, $p + q - pq \in P$; and by Corollary 2.14, $p_1 := p - pq \in P$ and $q_1 := q - pq \in P$. Thus, with $d := pq$, we have $d + p_1 + q_1 = p + q - pq \in P$ with $p = d + p_1$, and $q = d + q_1$.

Conversely, suppose there exist $p_1, q_1, d \in P$ such that $d + p_1 + q_1 \in P$, $p = d + p_1$, and $q = d + q_1$. Then $d + p_1 \leq d + p_1 + q_1 \leq 1$, whence $dCp_1$ by Theorem 2.11. Likewise, $dCq_1$, and $p_1Cq_1$, whence $pCq$. \hfill \Box

In the following definition, the condition in Lemma 3.2 is generalized to effects $e, f \in E$.

3.3 Definition. Effects $e, f \in E$ are said to be coexistent iff there exist effects $d, e_1, f_1 \in E$ such that $d + e_1 + f_1 \in E$, $e = d + e_1$, and $f = d + f_1$.

The terminology “coexistent” is borrowed from the quantum theory of measurement [2]. (Some authors also refer to coexistent effects as being “Mackey compatible,” but, since coexistent effects need not commute, we prefer not to follow this practice.)
3.4 Lemma. Let \( e, f \in E \). Then: (i) If \( eCf \), then \( e \) and \( f \) are coexistent. (ii) If \( e + f \leq 1 \), then \( e \) and \( f \) are coexistent.

Proof. (i) Let \( d := ef, e_1 := e - ef, \) and \( f_1 := f - ef \). By Lemma 2.6 (i), \( d, e_1, f_1 \in E \). Also, \( d + e_1 + f_1 = e + f - ef \in E \) by Lemma 2.6 (iii).

(ii) If \( e + f \in E \), then \( 0 + e + f \in E \) with \( e = 0 + e \) and \( f = 0 + f \).

In general, the converse of Lemma 3.4 (i) is false. For instance, in the prototype \( E(H) \), choose two effect operators \( A \) and \( B \) that do not commute. Then \( \frac{1}{2}A \) and \( \frac{1}{2}B \) are non-commuting effect operators; yet, since \( \frac{1}{2}A + \frac{1}{2}B \leq 1 \), they are coexistent. However, we do have the following result.

3.5 Theorem. Let \( p, q \in P \). Then, regarded as effects in \( E \), the projections \( p \) and \( q \) are coexistent iff \( pCq \).

Proof. Combine Lemma 2.16 and Lemma 3.2.

In a Boolean algebra (i.e., a bounded complemented distributive lattice), every element has a unique complement; hence if an OMP is a Boolean algebra, the Boolean complementation mapping coincides with the orthocomplementation mapping. It is well known that an OMP is a Boolean algebra iff its elements are pairwise Mackey compatible \([8]\); hence we have the following.

3.6 Corollary. The OMP \( P \) is a Boolean algebra iff \( P \subseteq C(P) \). Moreover, if \( P \) is a Boolean algebra, then \( p \mapsto 1 - p \) is the Boolean complementation mapping on \( P \).

In what follows, we shall be considering the condition \( G \subseteq C(G) \) and the weaker condition \( G \subseteq C(P) \). For instance, if the enveloping ring \( R \) is commutative, then \( G \subseteq C(G) \). Since \( E \) generates the group \( G \), it follows that \( G \subseteq C(G) \Leftrightarrow E \subseteq C(E) \) and that \( G \subseteq C(P) \Leftrightarrow E \subseteq C(P) \). Also, if \( G \subseteq C(P) \), then \( P \subseteq C(P) \), whence \( P \) is a Boolean algebra by Corollary 3.6. If the unital C*-algebra \( A \) in Example 2.3 satisfies virtually any version of the spectral theorem (e.g., if \( A \) is a von Neumann algebra, or even an AW*-algebra), then \( P \subseteq C(P) \) will imply that \( G \subseteq C(G) \).

Suppose that \( G \subseteq C(G) \), let \( g, h \in G \), and choose \( a, b, c, d \in E^+ \) such that \( g = a - b \) and \( h = c - d \). By Definition 1.1 (iii), \( ac, ad, bc, bd \in E^+ \subseteq G \), whence \( gh = ac - ad - bc + bd \in G \), and it follows that \( G \) is not only an additive abelian group, but a commutative subring of \( R \). Clearly, with \( G \) thus organized into a ring, \((G, E)\) is an e-ring, \( G \) is a partially ordered
commutative ring with unity 1, the partially ordered additive group \( G \) is the directed group of \( (G, E) \), and \( (G, E) \) is a \( c \)-ring as per the following definition.

**3.7 Definition.** A \( c \)-ring is an \( e \)-ring \( (G, E) \) such that \( G \) is a commutative ring and \( G = E^+ - E^+ \).

If \( G \subseteq C(G) \) and \( R \neq G \), we can disregard the enveloping ring \( R \) and drop down to the \( c \)-ring \( (G, E) \). Evidently, the passage from the \( e \)-ring \( (R, E) \) to the \( c \)-ring \( (G, E) \) affects neither the structure of the effect algebra \( E \) nor of the Boolean algebra \( P \).

As a consequence of the Gelfand representation theorem \cite[Theorem 4.4.3]{19}, the following example of a \( c \)-ring may be regarded as the commutative version of Example \cite{23}.

**3.8 Example.** Let \( X \) be a compact Hausdorff space, define \( C(X, \mathbb{R}) \) to be the ring of all continuous real-valued functions \( f: X \rightarrow \mathbb{R} \) with pointwise operations, and let

\[
E(X, \mathbb{R}) := \{e \in C(X, \mathbb{R}) \mid 0 \leq e(x) \leq 1, \forall x \in X\}.
\]

Then \( (C(X, \mathbb{R}), E(X, \mathbb{R})) \) is a \( c \)-ring, the partial order on \( C(X, \mathbb{R}) \) is the pointwise partial order, \( C(X, \mathbb{R}) \) is archimedean, and the Boolean algebra

\[
P(X, \mathbb{R}) := \{p \in C(X, \mathbb{R}) \mid p(x) \in \{0, 1\}, \forall x \in X\}
\]

of projections in \( (C(X, \mathbb{R}), E(X, \mathbb{R})) \) consists of the characteristic set functions \( \chi_K \) of compact open subsets \( K \) of \( X \).

In the following example of a \( c \)-ring, the effects are “fuzzy subsets” of \( X \) in the sense of L. Zadeh \cite{20}.

**3.9 Example.** Let \( \mathcal{F} \) be a \( \sigma \)-field of subsets of a nonempty set \( X \), define \( \mathcal{B}(X, \mathcal{F}, \mathbb{R}) \) to be the ring under pointwise operations of all bounded real-valued \( \mathcal{F} \)-measurable functions \( f: X \rightarrow \mathbb{R} \), and let

\[
\mathcal{E}(X, \mathcal{F}, \mathbb{R}) := \{e \in \mathcal{B}(X, \mathcal{F}, \mathbb{R}) \mid 0 \leq e(x) \leq 1, \forall x \in X\}.
\]

Then \( (\mathcal{B}(X, \mathcal{F}, \mathbb{R}), \mathcal{E}(X, \mathcal{F}, \mathbb{R})) \) is a \( c \)-ring, the partial order on \( \mathcal{B}(X, \mathcal{F}, \mathbb{R}) \) is the pointwise partial order, \( \mathcal{B}(X, \mathcal{F}, \mathbb{R}) \) is archimedean, and the Boolean algebra

\[
\mathcal{P}(X, \mathcal{F}, \mathbb{R}) := \{p \in \mathcal{B}(X, \mathcal{F}, \mathbb{R}) \mid p(x) \in \{0, 1\}, \forall x \in X\}
\]
of projections in $\mathcal{E}(X, \mathcal{F}, \mathbb{R})$ consists of the characteristic set functions $\chi_M$ of sets $M \in \mathcal{F}$.

Recall that a partially ordered abelian group $G$ is said to be lattice ordered, or for short, is an $\ell$-group, iff every pair of elements $g, h \in G$ has an infimum $g \wedge_G h$ and a supremum $g \vee_G h$ in the partially ordered set $G$. The additive partially ordered abelian groups $C(X, \mathbb{R})$ and $B(X, \mathcal{F}, \mathbb{R})$ in Examples 3.8 and 3.9 are $\ell$-groups with pointwise minimum and maximum as the infimum and supremum, respectively.

If $G$ has the property that, for every $a, b, c, d \in G$ with $a, b \leq c, d$ (i.e., $a \leq c, a \leq d, b \leq c,$ and $b \leq d$), there exists $t \in G$ with $a, b \leq t \leq c, d$, then $G$ has the Riesz interpolation property, or for short, $G$ is an interpolation group [14, Chapter 2]. If $G$ is an $\ell$-group, then it is an interpolation group. (Just take $t$ to be any element between $a \vee_G b$ and $c \wedge_G d$.) Thus, the directed groups $C(X, \mathbb{R})$ and $B(X, \mathcal{F})$ are interpolation groups.

The so-called MV-algebras, which play an important role in the analysis of many-valued logics [3, 4] and in the classification of AF $C^*$-algebras [22], can be characterized as the effect algebras that are realized as unit intervals in abelian $\ell$-groups with order units. Thus, the unit intervals $E(X, \mathbb{R})$ and $\mathcal{E}(X, \mathcal{F}, \mathbb{R})$ in Examples 3.8 and 3.9 are MV-algebras. Not every MV-algebra can be realized as the unit interval in a c-ring, but the author does not know a perspicuous characterization of those that can.

In the theory of operator algebras, there are well-known connections between commutativity and lattice structure. For instance, by a theorem of S. Sherman [25], a unital $C^*$-algebra $A$ (Example 2.3) is commutative iff the directed group $G$ of self-adjoint elements in $A$ is an $\ell$-group. On the other hand, by a result of R. Kadison [18], if $A$ is a von Neumann algebra, then $A$ is a factor iff the directed group $G$ is an antilattice (i.e., only pairs of comparable elements can have an infimum or a supremum in $G$). Under suitable hypotheses (borrowed from the theory of operator algebras) similar results can be obtained for groups with compression bases [13]; hence for e-rings. If $P$ is a Boolean algebra, then it is a lattice, so Corollary 3.6 already furnishes a hint of the commutativity-lattice connection for e-rings; further evidence is provided by Theorems 3.10, 3.11, and 3.12 below.

3.10 Theorem. Suppose that $G$ is an $\ell$-group (or more generally, an interpolation group [14]). Then, (i) $G \subseteq C(P)$ and (ii) $P$ is a Boolean algebra.

Proof. Assume that $G$ is an interpolation group.
(i) Let \( 0 \leq g \in G \) and \( p \in P \). As \( G \) is directed, it will be sufficient to prove that \( gCp \). By Lemma 2.8 (i), there exists \( n \in \mathbb{N} \) such that \( 0 \leq g \leq n \cdot 1 \), whence \( 0 \leq g \leq np + n(1 - p) \). As \( G \) is an interpolation group, there exist \( x, y \in G \) with \( 0 \leq x \leq np, 0 \leq y \leq n(1 - p) \), and \( g = x + y \) (see [14, Proposition 2.1 (b)]). Thus, by Lemma 2.7 (iv), \( x = xp = px \) and \( y = y(1 - p) = (1 - p)y \), and it follows that \( yp = py = 0 \) and \( gp = pg = x \).

(ii) Follows from (i) and Corollary 3.6.

The c-ring in Example 3.9 satisfies the conditions in the following theorem. Of course, the c-ring in Example 3.8 satisfies condition (i), and it satisfies condition (ii) if \( X \) is basically disconnected (i.e., the closure of every open \( F_\sigma \) subset of \( X \) remains open).

3.11 Theorem. Suppose that (i) \( G \subseteq C(P) \), and (ii) for every \( g \in G \), there exists \( p \in P \) with \( (1 - p)g \leq 0 \leq pg \). Then \( G \) is an \( \ell \)-group.

Proof. The proof is adapted from [14, Proposition 8.9]. Let \( g, h \in G \). It will be sufficient to prove that the supremum \( g \vee_G h \) exists in \( G \). By hypothesis, there exists \( p \in P \) such that \( (1 - p)(g - h) \leq 0 \leq p(g - h) \). Put

\[ s := pg + (1 - p)h = p(g - h) + h = (1 - p)(h - g) + g. \]

Evidently, \( h, g \leq s \). Furthermore, if \( k \in G \) with \( g, h \leq k \), then by Lemma 2.7 (i), \( pg \leq pk \) and \( (1 - p)h \leq (1 - p)k \), whence \( s \leq pk + (1 - p)k = k \). Therefore, \( s = g \vee_G h \). \( \square \)

A subset \( A \subseteq P \) is said to be orthogonal iff, for all \( a, b \in A, a \neq b \Rightarrow ab = 0 \). If \( A \) is an orthogonal subset of \( P \), then by Corollary 2.13 and induction on \( n \), the sum \( p := a_1 + a_2 + \cdots + a_n \) of finitely many distinct elements \( a_1, a_2, \ldots, a_n \in A \) belongs to \( P \) and coincides with the supremum \( p := a_1 \lor a_2 \lor \cdots \lor a_n \) both in \( P \) and in \( E \).

3.12 Theorem. The following conditions are mutually equivalent:

(i) \( P \) is a Boolean algebra and \( P \) generates the group \( G \).

(ii) \( G \subseteq C(G) \) and, for each \( g \in G \), there is a finite orthogonal set \( A \subseteq P \) such that \( g \) is a linear combination with integer coefficients of the elements of \( A \).

(iii) \( G \) is an \( \ell \)-group and \( E = P \).
(iv) $G$ is an interpolation group and $1$ is a minimal order unit in $G$.

(v) $E$ is a Boolean algebra with $e \mapsto 1 - e$ as the Boolean complementation mapping.

(vi) $E = P$.

Proof. (i) $\Rightarrow$ (ii). Assume (i). By Corollary 3.6, $P \subseteq C(P)$ and, since $P$ generates $G$, it follows that $G \subseteq C(G)$. Let $g \in G$. Then there are projections $p_i \in P$, $1 \leq i \leq n$, and integer coefficients $c_i$ such that $g = \sum_{i=1}^{n} c_i p_i$. Let $B$ be the sub-Boolean algebra of $P$ generated by $p_i$, $1 \leq i \leq n$. Since $B$ is a finitely generated Boolean algebra, it is finite. Let $A$ be the set of atoms (minimal nonzero elements) in $B$. Then, if $a, b \in A$ with $a \neq b$, we have $ab = ba = a \land b = 0$ (Theorem 2.12), so $A$ is a finite orthogonal subset of $P$. Also, each element in $B$, and in particular each $p_i$, can be written as a sum of certain of the projections in $A$. Thus, by gathering terms, we can write $g = \sum_{i=1}^{n} c_i p_i = \sum_{a \in A} k_a a$ with integer coefficients $k_a$ for all $a \in A$.

(ii) $\Rightarrow$ (iii). Assume (ii). Then $G \subseteq C(G) \subseteq C(P)$. Let $g \in G$, and let $A$ be a finite orthogonal subset of $P$ such that $g = \sum_{a \in A} k_a a$ for integer coefficients $k_a$. Define $A_+ := \{a \in A \mid k_a > 0\}$, $A_- := \{a \in A \mid k_a < 0\}$, and $p := \sum_{a \in A_+} a$. Then $p \in P$, $a \in A_+ \Rightarrow pa = k_a a$, and $a \in A_- \Rightarrow pa = 0$. Thus, $pg = \sum_{a \in A_+} k_a a \geq 0$ and $(1 - p)g = g - pg = \sum_{a \in A_-} k_a a \leq 0$; hence $G$ is an $\ell$-group by Theorem 3.11. Now suppose $g \in E$. Then, if $a \in A_-$, we have $0 \leq ga = ag$, whence $0 \leq ga = k_a a \leq 0$, so $k_a a = 0$. Consequently, $g = \sum_{a \in A_+} k_a a$. Also, if $a \in A_+$, we have $a \leq k_a a = ga \leq a$ (Lemma 2.10 (i)), so $k_a a = a$. Consequently, $g = \sum_{a \in A_+} k_a a = \sum_{a \in A_+} a = p \in P$. Therefore, $E = P$.

(iii) $\Rightarrow$ (iv). Assume (iii). Then $G$ is an interpolation group. Suppose $p$ is an order unit in $G$ and $p \leq 1$. Then $0 \leq p$ and there exists $n \in \mathbb{N}$ such that $0 \leq 1 - p \leq np$. As $0 \leq p \leq 1$, we have $p \in E = P$; hence by Lemma 2.7 (iv), $1 - p = (1 - p)p = 0$, i.e., $p = 1$. Thus 1 is a minimal order unit in $G$.

(iv) $\Rightarrow$ (v). Assume (iv). By Theorem 3.10, $G \subseteq C(P)$ and $P$ is a Boolean algebra with $p \mapsto 1 - p$ as the Boolean complementation. It will be enough to show that $E = P$. Thus, let $e \in E$ and suppose that $d \in E$ with $d \leq e, 1 - e$. Then $1 - e \leq 1 - d$ and $e \leq 1 - d$. Adding the last two inequalities, we find that $1 \leq 2(1 - d)$. Thus, if $g \in G$, there exists $n \in \mathbb{N}$ such that $g \leq n \cdot 1 \leq 2n(1 - d)$, and it follows that $1 - d$ is an order unit in
1 − d ≤ 1; hence, by hypothesis, 1 − d = 1, i.e., d = 0. Consequently, e ∈ P by Theorem 2.17 and we conclude that E = P.

(v) ⇒ (vi). Assume (v) and let e ∈ E. If d ∈ E with d ≤ e, 1 − e, then since 1 − e is the Boolean complement of e in the Boolean algebra E, it follows that d = 0; hence e ∈ P by Theorem 2.17. Consequently, E = P.

(vi) ⇒ (i). Assume (vi) and let p, q ∈ P = E. Since E generates the group G, it will be sufficient by Corollary 3.6 to prove that pCq. Let d := pqp. By Lemma 2.6 (ii), d ∈ E = P, and it is clear that p = dp = pd, whence d ≤ p. Thus, dqd = dpqpd = d3 = d, and it follows that d(1 − q)d = 0. Therefore, d(1 − q) = 0, i.e., d = dq, so d ≤ q. By Corollary 2.14 p1 := p − d ∈ P and q1 := q − d ∈ P. Also, pq1p = pqp − pdp = d − d = 0, so pq1 = 0, p = d + p1, q = d + q1, and by Theorem 2.11 d + p1 + q1 = p + q1 ∈ P. Consequently, p and q are Mackey compatible, so pCq by Lemma 3.2.

If the e-ring (R, E) satisfies any, hence all, of the conditions (i)–(v) in Theorem 3.12, then G ⊆ C(G) by condition (ii), and we can drop down to the c-ring (G, E), which of course will continue to satisfy conditions (i)–(v).

3.13 Definition. A b-ring is a c-ring (G, E) satisfying any, hence all, of the conditions (i)–(v) in Theorem 3.12.

The b-ring in the following example is a modification of Example 3.9 in which the totally ordered field R is replaced by the totally ordered ring Z of integers and the σ-field F is replaced by any field of sets.

3.14 Example. Let F be a field of subsets of a nonempty set X, define B(X, F, Z) to be the ring under pointwise operations of all bounded functions f : X → Z such that f−1(z) ∈ F for all z ∈ Z, and let

\[ \mathcal{E}(X, F, Z) := \{ e ∈ B(X, F, Z) \mid e(x) ∈ \{0, 1\}, ∀ x ∈ X \}. \]

Then (B(X, F, Z), \mathcal{E}(X, F, Z)) is a b-ring, the partial order on B(X, F, Z) is the pointwise partial order, and B(X, F, Z) is archimedean. The effects in \mathcal{E}(X, F, Z), which coincide with the projections for the b-ring (B(X, F, Z), \mathcal{E}(X, F, Z)), are the characteristic set functions χ_M of sets M ∈ F.

Under set-inclusion, a field F of subsets of a nonempty set X is a Boolean algebra, and in Example 3.14 the Boolean algebra \mathcal{E}(X, F, Z) is isomorphic to F. By the Stone representation theorem, every Boolean algebra B is isomorphic to the field F of compact open subsets of a compact Hausdorff
totally-disconnected space $X$; hence, every Boolean algebra can be realized as the Boolean algebra of projections in a b-ring.

The functions $f : X \to \mathbb{Z}$ in Example 3.14 can be regarded as “signed multisets” by thinking of $f(x)$ as the “signed multiplicity of $x$ in $f$.” In [17], T. Hailperin suggests that, in contemporary algebraic terms, the true realization of Boole’s original ideas is not what is now called a Boolean algebra, but rather it is an algebra of signed multisets forming a commutative ring with unity and with no nonzero additive or multiplicative nilpotents. Our b-ring $(B(X, \mathcal{F}, \mathbb{Z}), E(X, \mathcal{F}, \mathbb{Z}))$ is precisely such an algebra, and the “b” in “b-ring” is meant to suggest this Boolean connection.

**3.15 Theorem.** Let $(G, E)$ and $(H, F)$ be b-rings and let $\phi : E \to F$ be a Boolean homomorphism from the Boolean algebra $E$ to the Boolean algebra $F$. Then $\phi$ admits a unique extension to a group homomorphism $\Phi : G \to H$ of the additive group $G$ into the additive group $H$. Moreover, $\Phi : G \to H$ is an order-preserving ring homomorphism with $\Phi(1) = 1$.

*Proof.* The Boolean homomorphism $\phi : E \to F$ preserves 0, 1, finite infima, and finite suprema. For $p, q \in E = P$, we have $p \land q = pq$; hence $\phi(pq) = \phi(p) \land \phi(q) = \phi(p) \phi(q)$, i.e., $\phi$ preserves products of projections. Also, if $p + q \in E$, then $p \lor q = p + q$; hence $\phi(p + q) = \phi(p \lor q) = \phi(p) \lor \phi(q) = \phi(p) + \phi(q)$, i.e., $\phi : E \to H$ preserves existing sums in $E$. Since $G$ is an interpolation group, a theorem of S. Pulmannová [23] implies that $\phi$ admits a unique extension to a group homomorphism $\Phi : G \to H$. As $\phi(E) \subseteq F$, it follows that $\Phi(E^+) \subseteq F^+$, whence $\Phi$ is order preserving. Every element in $G$ is a finite linear combination of projections with integer coefficients, and since $\Phi$ preserves products of projections, it follows that $\Phi$ preserves products. Obviously, $\Phi(1) = \phi(1) = 1$. \[\square\]

The following is the fundamental structure theorem for b-rings.

**3.16 Theorem.** Let $(G, E)$ be a b-ring, let $X$ be the Stone space of the Boolean algebra $E$, and let $\mathcal{F}$ be the field of compact open subsets of $X$. Then there is an order and ring isomorphism $\Phi : G \to \mathcal{B}(X, \mathcal{F}, \mathbb{Z})$ such that the restriction $\phi$ of $\Phi$ to $E$ is a Boolean isomorphism of $E$ onto $\mathcal{E}(X, \mathcal{F}, \mathbb{Z})$.

*Proof.* The projections in $\mathcal{E}(X, \mathcal{F}, \mathbb{Z})$ are characteristic set functions $\chi_K$ of compact open subsets $K$ of $X$; hence by Stone’s representation theorem, there is a Boolean isomorphism $\phi : E \to \mathcal{E}(X, \mathcal{F}, \mathbb{Z})$. By Theorem 3.15, $\phi$ can be extended to an order-preserving ring homomorphism $\Phi : G \to \mathcal{B}(X, \mathcal{F}, \mathbb{Z})$.
and $\phi^{-1}: \mathcal{E}(X, \mathcal{F}, \mathbb{Z}) \to E$ can be extended to an order-preserving ring homomorphism $\Psi: \mathcal{B}(X, \mathcal{F}, \mathbb{Z}) \to G$. The ring endomorphism $\Psi \circ \Phi: G \to G$ is the identity on $E$, and $E$ generates $G$; hence $\Psi \circ \Phi: G \to G$ is the identity on $G$. Likewise $\Phi \circ \Psi$: is the identity on $\mathcal{B}(X, \mathcal{F}, \mathbb{Z})$, so $\Phi$ is an order-preserving ring isomorphism with $\Psi = \Phi^{-1}$.

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