LOCALLY COMPACT, $\omega_1$-COMPACT SPACES

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Abstract. An $\omega_1$-compact space is a space in which every closed discrete subspace is countable. We give various general conditions under which a locally compact, $\omega_1$-compact space is $\sigma$-countably compact, i.e., the union of countably many countably compact spaces. These conditions involve very elementary properties.

Many results shown here are independent of the usual (ZFC) axioms of set theory, and the consistency of some may involve large cardinals. For example, it is independent of the ZFC axioms whether every locally compact, $\omega_1$-compact space of cardinality $\aleph_1$ is $\sigma$-countably compact. Whether $\aleph_1$ can be replaced with $\aleph_2$ is a difficult unsolved problem. Modulo large cardinals, it is also ZFC-independent whether every hereditarily normal, or every monotonically normal, locally compact, $\omega_1$-compact space is $\sigma$-countably compact.

As a result, it is also ZFC-independent whether there is a locally compact, $\omega_1$-compact Dowker space of cardinality $\aleph_1$, or one that does not contain both an uncountable closed discrete subspace and a copy of the ordinal space $\omega_1$.

Set theoretic tools used for the consistency results include the existence of a Souslin tree, the Proper Forcing Axiom (PFA), and models generically referred to as “MM(S)[S]”. Most of the work is done by the P-Ideal Dichotomy (PID) axiom, which holds in the latter two cases, and which requires no large cardinal axioms when directly applied to topological spaces of cardinality $\aleph_1$, as it is in several theorems.

1. Introduction

A space of countable extent, also called an $\omega_1$-compact space, is one in which every closed discrete subspace is countable. Obvious examples of $\omega_1$-compact spaces are countably compact spaces (because in them every closed discrete subspace is finite), and $\sigma$-countably compact spaces, i.e., the union of countably many countably compact spaces. On the other hand, an elementary application of the Baire Category Theorem shows that the space of irrational numbers with the usual topology is not $\sigma$-countably compact, but like every other separable metrizable space, it is $\omega_1$-compact.

The situation is very different when it comes to locally compact spaces. In an earlier version of this paper due to the first author, he asked:

Question 1.1. Is there a ZFC example of a locally compact, $\omega_1$-compact space of cardinality $\aleph_1$ that is not $\sigma$-countably compact? one that is normal?
Here too, local compactness makes a big difference: without it, the space of irrationals is a counterexample under CH, while ZFC is enough to show any cardinality $\aleph_1$ subset of a Bernstein set is a counterexample.

As it is, the second author showed that the answer to Question 1.1 is negative; see Section 2. On the other hand, both the Kunen line and a Souslin tree with the usual topology are consistent locally compact normal examples for Question 1.1. In the case of the Kunen line, its hereditary separability clearly implies $\omega_1$-compactness, and its hereditary realcompactness implies that every countably compact subspace is compact and therefore countable. The case of a Souslin tree will be shown at the end of Section 5. A third consistent example is given in [28]:

**Theorem 1.2.** Assuming ♠, there is a locally compact, locally countable (hence first countable) $\omega_1$-compact space of cardinality $\aleph_1$ which is not $\sigma$-countably compact.

It is not known whether this example is normal. However, in Section 4 we will construct an example with all the stated properties of Theorem 1.2 and which is monotonically normal.

Monotonically normal spaces are, informally speaking, “uniformly normal” [see Definition 3.3 below]. They are hereditarily normal, and this theorem gives another independence result when combined with:

**Theorem 1.3.** In MM(S)/$\mathcal{S}$ models, every hereditarily normal, locally compact, $\omega_1$-compact space is $\sigma$-countably compact.

An even stronger theorem will be shown in Section 3 which includes the statement that the PID is enough to show Theorem 1.3 for monotonically normal spaces. It also puts some limitations on what kinds of Dowker spaces (that is, normal spaces $X$ such that $X \times [0, 1]$ is not normal) are possible if one only assumes the usual (ZFC) axioms of set theory.

In the light of the negative answer to Question 1.1, it is natural to ask:

**Problem 1.** What is the least cardinality of a locally compact, $\omega_1$-compact space which is not $\sigma$-countably compact? one that is normal?

The best ZFC result so far was shown by the first author [27]:

**Example 1.4.** There is a locally compact, normal, $\omega_1$-compact space of cardinality $\mathfrak{b}$ that is not $\sigma$-countably compact.

Previously, the best upper bound for the minimum was $\mathfrak{c}$, using one of E.K. van Douwen’s “honest submetrizable” examples [7]. However, Example 1.4 still leaves a lot unsaid. See Section 5 for a discussion, especially of the following “echo” of Question 1.1

**Problem 2.** Is there a ZFC example of a locally compact, $\omega_1$-compact space of cardinality $\aleph_2$ that is not $\sigma$-countably compact? one that is normal?

The last section gives more information about Example 1.4 and about a related problem and result of Eric van Douwen, under the assumption of $\mathfrak{b} = \mathfrak{c}$.

In between, Section 6 gives some interesting counterpoints to Problem 1 by discussing questions about the greatest cardinality of a locally countable, normal, countably compact
Local countability may seem like a very specialized property, but it actually holds in most of our examples, including Example 1.4 under CH, and it easily implies local compactness in countably compact spaces, and with it, first countability.

The individual sections are only loosely connected with each other, and each can be read with minimal reliance on any of the others.

All through this paper, “space” means “Hausdorff topological space.” All of the spaces described are locally compact, hence Tychonoff; and all are also normal, except for a consistent example at the end.

2. The cardinality \(\aleph_1\) case

The P-Ideal Dichotomy (PID) plays a key role in this section and the following one. It has to do with the following concept.

**Definition 2.1.** A family \(I \subset \mathcal{P}(X)\) is an ideal on a set \(X\), if it is closed under finite unions, contains all singletons, and is closed under taking subsets of its elements. An ideal \(I\) is a P-ideal if for every countable subfamily \(Q\) of \(I\), there exists \(I \in I\) such that \(Q \subset^* I\) for every \(Q \in Q\). Here \(Q \subset^* I\) means that \(Q \setminus I\) is finite.

The PID states that, for every P-ideal \(I\) of subsets of a set \(X\), either

(A) there is an uncountable \(A \subset X\) such that every countable subset of \(A\) is in \(I\), or

(B) \(X\) is the union of countably many sets \(\{B_n : n \in \omega\}\) such that \(B_n \cap I\) is finite for all \(n\) and all \(I \in I\).

The routine proofs of the next lemma and theorem were given in [13]:

**Lemma 2.2.** Let \(X\) be a locally compact Hausdorff space. The countable closed discrete subspaces of \(X\) form a P-ideal if, and only if, the extra point \(\infty\) of the one-point compactification \(X + 1\) of \(X\) is an \(\alpha_1\)-point; that is, whenever \(\{\sigma_n : n \in \omega\}\) is a countable family of sequences converging to \(\infty\), then there is a sequence \(\sigma\) converging to \(\infty\) such that \(\text{ran}(\sigma_n) \subseteq^* \text{ran}(\sigma)\) for all \(n\).

The key is that an ordinary sequence in \(X\) converges to the extra point of \(X + 1\) if, and only if, its range is a closed discrete subspace of \(X\).

**Theorem 2.3.** Assume the PID axiom. Let \(X\) be a locally compact space. Then at least one of the following is true:

(1\(^{-}\)) \(X\) is the union of countably many subspaces \(Y_n\) such that each sequence in \(Y_n\) has a limit point in \(X\).

(2) \(X\) has an uncountable closed discrete subspace

(3\(^{+}\)) The extra point of \(X + 1\) is not an \(\alpha_1\)-point.

The key here is that (A) goes with (2), (B) goes with (1\(^{-}\)), and (3\(^{+}\)) is equivalent to the countable closed discrete subspaces failing to form a P-ideal, by Lemma 2.2.

The following is well known:

**Lemma 2.4.** If \(X\) is a space of character \(< \mathfrak{b}\), then every point of \(X\) is an \(\alpha_1\)-point.
We now have a negative answer to the second part of Question 1.1.

**Theorem 2.5.** Assume the PID and $b > \aleph_1$. Then every locally compact, $\omega_1$-compact, normal space of cardinality $\aleph_1$ is $\sigma$-countably compact.

*Proof.* In a locally compact space, character $\leq$ cardinality. Lemmas 2.2 and 2.4 and $\omega_1$-compactness give us alternative (1−) of Theorem 2.3. The closure of each $Y_n$ is easily seen to be pseudocompact (i.e., every continuous real-valued function is bounded). In a normal space, every closed pseudocompact subspace is countably compact, cf. [37, 17J 3]. So the closures of the $Y_n$ witness that $X$ is $\sigma$-countably compact. □

As shown in [13], the hypothesis of normality in Theorem 2.5 can be greatly weakened to “Property wD”. Also, the proof of Theorem 2.5 clearly extends to show that every normal, locally compact, $\omega_1$-compact space of cardinality (or even Lindelöf number) $< b$ is $\sigma$-countably compact. However, this may be a very limited improvement: the PID implies $b \leq \aleph_2$. This is a theorem of Todorčević, whose proof may be found in [21].

The axioms used in Theorem 2.5 follow from the Proper Forcing Axiom ($PFA$) and hold in $PFA(S)[S]$ models. Each of these models is formed from a $PFA(S)$ model by forcing with a coherent Souslin tree $S$ that is part of the definition of what it means to be a $PFA(S)$ model. The rest of the definition states that every proper poset $P$ that does not destroy $S$ when it is forced with, has the following property. For every family of $\leq \aleph_1$ $\uparrow$-dense, $\uparrow$-open sets, there is a $\downarrow$-closed, $\uparrow$-directed subset of $P$ that meets them all. The $PFA$ is similarly defined by omitting all mention of $S$. What remains is very similar to the well-known definition of Martin’s Axiom ($MA$); the only difference is that $MA$ uses “ccc” instead of “proper.”

In this paper, we will use a slight abuse of language with expressions like $PFA(S)[S]$ and $MM(S)[S]$ as though they were axioms. The latter is defined like the former, but with “semi-proper” in place of “proper.”

For our negative answer to the first half of Question 1.1 we needed a strengthening of $b > \aleph_1$ which holds in models of $PFA(S)[S]$ and $MM(S)[S]$.

**Lemma 2.6.** Let $X$ be a $T_3$ space of weight $< \min\{b, s\}$, and let $Y \subset X$. Suppose that no $Z \in [Y]^{\omega}$ is closed discrete in $X$. Then there exists a countably compact $Y' \subset X$ containing $Y$.

*Proof.* Using ideas from [3], for every $Z \in [Y]^{\omega}$ we can find $S_Z \in [Z]^{\omega}$ which converges to some $x_Z \in X$. More precisely, let $Z$ be the family of infinite elements of $\{Z \cap U : U$ is a basic open subset of $X\}$. Since $|Z| < s$, $Z$ is not a splitting subfamily of $[Z]^{\omega}$, so there exists $S_Z \subset Z$ unsplit by $Z$. Then $S_Z$ has at most one limit point in $X$: If $x_1$ and $x_2$ were two of them, pick open disjoint basic neighbourhoods $U_1 \ni x_1$ and $U_2 \ni x_2$. Then both of the sets $U_1 \cap S_Z = (U_1 \cap Z) \cap S_Z$ and $U_2 \cap S_Z = (U_2 \cap Z) \cap S_Z$ are infinite, and hence each of $U_1 \cap Z, U_2 \cap Z \in Z$ splits $S_Z$, a contradiction. Since $S_Z$ has a limit point $x_Z \in X$, it follows that $S_Z$ converges to $x_Z$.

Let $Y' = Y \cup \{x_Z : Z \in [Y]^{\omega}\}$. We claim that $Y'$ is countably compact. Indeed, otherwise there exists a countable $T = \{t_n : n \in \omega\} \subset Y'$ which is closed discrete in $Y'$. Note that

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1The notation is as in [13]. It is the “Israeli” notation, whereby stronger conditions are larger. The topology to which it refers is the one where each $p \in P$ has the one basic neighborhood $\{t : t \geq p\}$. 
$Z = T \cap Y$ is finite, otherwise $x_Z$ would be an accumulation point of $T$ in $Y'$. So we may assume that $T \cap Y = \emptyset$. For every $n$ fix $Z_n \in [Y]^\omega$ such that $t_n = x_{Z_n}$ and set $S_n := S_{Z_n}$. Let $W_n \ni t_n$ be a neighborhood of $t_n$ in $Y'$ such that $W_n \cap W_m = \emptyset$ for all $n \neq m$. [In a $T_3$ space, every countable closed discrete subspace extends to a disjoint open collection.]

For every $y \in Y' \setminus T$ find a basic open neighborhood $U_y$ of $y$ in $Y'$ such that $\overline{U_y} \cap T = \emptyset$. Given an injective enumeration $\langle z^k : k \in \omega \rangle$ of $S_n$ for every $n \in \omega$, let $f_{U_y} : \omega \to \omega$ be such that $\{z^k : k \geq f_{U_y}(n)\} \cap \overline{U_y} = \emptyset$. Since the weight of $Y'$ is $< b$, there are $< b$ of $U_y$’s, and hence there exists $f : \omega \to \omega$ such that $f_{U_y} <^* f$ for all $y \in Y' \setminus T$. Now $Z_\omega = \{z^n_{f(n)} : n \in \omega\} \subset Y$ has no accumulation points in $Y'$, because $Z_\omega \cap W_n = \{z^n_{f(n)}\}$ for all $n \in \omega$, and $Z_\omega \cap U_y$ is finite for each $y \in Y'$. In other words, $x_{Z_\omega}$ does not exist. This contradiction implies that $Y'$ is countably compact.

For locally compact spaces, we can replace the hypothesis in the statement of Lemma 2.6 with $\overline{Y}$ being hereditarily of Lindelöf degree $< p$. That is, if $S \subset \overline{Y}$, and $U$ is a relatively open cover of $S$, then $U$ has a subcover of cardinality $< p$.

Together with regularity this yields that $\overline{Y}$ has character $< p$, and hence for any $Z \in [Y]^\omega$ and limit point $x_Z \in X$ of $Z$ there is a convergent to $x_Z$ sequence $S_Z \in [Z]^\omega$. This is a well-known property of spaces with character $< p$.

Regarding the second part of the proof, the existence of $\mathcal{U}' \subset \mathcal{U} = \{U_y : y \in S\}$ of cardinality $< p$ covering $S = Y' \setminus T$, is enough to make the argument go through.

Now we can finish answering Question 1.1.

**Theorem 2.7.** Assume the PID and $\min\{b, s\} > \aleph_1$. Then every locally compact, $\omega_1$-compact space of weight $\aleph_1$ is $\sigma$-countably compact.

**Proof.** Again Lemmas 2.2 and 2.4 and $\omega_1$-compactness give us alternative (1−) of Theorem 2.3. The rest is clear from Lemma 2.6.

Returning to Theorem 2.5, its proof also gives:

**Theorem 2.8.** Assume the PID. Then every locally compact, $\omega_1$-compact normal space of cardinality $< b$ is countably paracompact.

**Proof.** A normal space $X$ is countably paracompact if, and only if, for each descending sequence of closed sets $\langle F_n \rangle_{n=1}^\infty$ with empty intersection, there is a sequence of open sets $\langle U_n \rangle_{n=1}^\infty$ with empty intersection, with $F_n \subset U_n$ for all $n$. If $X$ is a countable union of countably compact subsets $C_m$, as in Theorem 2.5, then in such a sequence of closed sets $F_n$, we can only have $F_n \cap C_m \neq \emptyset$ for finitely many $n$. [Otherwise, countable compactness of $C_m$ implies $\bigcap_{n=1}^\infty C_m \cap F_n \neq \emptyset$.] In any Tychonoff space, every pseudocompact subspace, and hence every countably compact subspace, has pseudocompact closure, and every normal, pseudocompact space is countably compact; and so the complements of the sets $\overline{C_m}$ form the desired sequence of open sets.

The equivalence in the preceding proof is shown in [37, 21.3] and is due to Dowker, who also showed its equivalence with $X \times [0, 1]$ being normal. In honor of his pioneering work, normal spaces that are not countably paracompact are called “Dowker spaces.” Theorem 2.8.
thus implies the consistency of there being no locally compact, \( \omega_1 \)-compact Dowker spaces of cardinality \( \aleph_1 \). Specialized though this fact is, it is one of the few theorems as to what kinds of Dowker spaces are unattainable in ZFC. Another interesting such result was obtained by Dow and Tall [10]:

**Theorem 2.9.** If \( MM(S)[S] \), then every locally compact, normal, non-paracompact space of Lindelöf number \( \leq \aleph_1 \) includes a perfect preimage of \( \omega_1 \).

Alan Dow has improved “perfect preimage” to “copy” [private communication]. Combining this with Theorem 2.8 we have:

**Corollary 2.10.** If \( MM(S)[S] \), then every locally compact Dowker space of cardinality \( \leq \aleph_1 \) includes both a copy of \( \omega_1 \) and an uncountable closed discrete subspace.

This corollary may be vacuous; in fact, the following problem is open:

**Problem 3.** Is there a model of PID + \( c = \aleph_2 \) in which there is a locally compact Dowker space of cardinality \( \aleph_1 \)?

The consistency of PID was shown using forcing from a ground model with a supercompact cardinal. There are versions for spaces of weight \( \aleph_1 \), hence all locally compact spaces of cardinality \( \aleph_1 \), which require only the consistency of ZFC. One restricted version of the PID axiom is designated \((*)\) in [1], and is adequate for Theorem 2.8. But it is still an open problem whether the main results of our next section are ZFC-equiconsistent.

### 3. When hereditary normality implies \( \sigma \)-countable compactness

For the main theorem of this section, we recall the following concepts:

**Definition 3.1.** Given a subset \( D \) of a set \( X \), an **expansion** of \( D \) is a family \( \{U_d : d \in D\} \) of subsets of \( X \) such that \( U_d \cap D = \{d\} \) for all \( d \in D \). A space \( X \) is [strongly] **collectionwise Hausdorff** (abbreviated [s]cwH) if every closed discrete subspace has an expansion to a disjoint [resp. discrete] collection of open sets.

The properties of \( \omega_1 \)-[s]cwH only require taking care of those \( D \) that are of cardinality \( \leq \omega_1 \).

A well-known, almost trivial fact is that every normal, cwH space is scwH: if \( D \) and \( \{U_d : d \in D\} \) are as in 3.1, let \( V \) be an open set containing \( D \) whose closure is in \( \bigcup \{U_d : d \in D\} \); then \( \{U_d \cap V : d \in D\} \) is a discrete open expansion of \( D \).

As is well known [11, 2.1.7], a space is hereditarily normal if (and only if) every open subspace is normal. A similar statement holds for the cwH property:

**Theorem 3.2.** The following are equivalent for a space \( X \).

1. \( X \) is hereditarily cwH.
2. Every open subspace of \( X \) is cwH.
3. Every discrete subspace of \( X \) expands to a disjoint collection of open sets.
Proof. To show (2) implies (3), let \( D \) be a discrete subspace of \( X \). Then \( D' = D \setminus D \) is a closed subset of \( X \), and \( D \) is a closed discrete subspace of the open subspace \( Y = X \setminus (D \setminus D) \) of \( X \). Clearly, any expansion of \( D \) to disjoint open sets in \( Y \) is also a disjoint open expansion in \( X \).

To show (3) implies (1), let \( S \) be a subspace of \( X \), and let \( D \) be a relatively closed, discrete subspace of \( S \). Then \( D \) is also a discrete subspace of \( X \), and the trace on \( S \) of a disjoint open expansion of \( D \) in \( X \) is a disjoint relatively open expansion of \( D \) in \( S \). \( \square \)

The analogous theorems for scwH, \( \omega_1 \)-scwH and \( \omega_1 \)-cwH also hold, and the proofs are similar. We will be using the one for \( \omega_1 \)-cwH in the main theorem of this section.

The following class of normal spaces plays a big role in this section and in the following one.

Definition 3.3. A space \( X \) is monotonically normal if there is an operator \( G(\mathcal{A}) \) assigning to each ordered pair \( (F_0, F_1) \) of disjoint closed subsets an open set \( G(F_0, F_1) \) such that
\[ a) \quad F_0 \subset G(F_0, F_1) \]
\[ b) \quad \text{If } F_0 \subset F'_0 \text{ and } F'_1 \subset F_1 \text{ then } G(F_0, F_1) \subset G(F'_0, F'_1) \]
\[ c) \quad G(F_0, F_1) \cap G(F_1, F_0) = \emptyset \]

Monotone normality is a hereditary property; that is, every subspace inherits the property. This is not so apparent from this definition, but it follows almost immediately from the following criterion, due to Borges [4].

Theorem 3.4. A space is monotonically normal if, and only if, there is an assignment of an open neighborhood \( h(x, U) =: U_x \) containing \( x \) to each pair \( (x, U) \) such that \( U \) is an open neighborhood of \( x \), and such that, if \( U_x \cap V_y \neq \emptyset \), then either \( x \in V \) or \( y \in U \).

Every monotonically normal space is (hereditarily) countably paracompact [32, Theorem 2.3] and (hereditarily) scwH: the Borges criterion easily give cwH, and normality does the rest.

The main theorem of this section, Theorem 3.7, also involves the following concepts.

Definition 3.5. An \( \omega \)-bounded space is one in which every countable subset has compact closure. A space is \( \sigma\omega \)-bounded if it is the union of countably many \( \omega \)-bounded subspaces.

Clearly, every \( \omega \)-bounded space is countably compact. Theorem 3.7 below makes use of the following axiom, which is shown in [13] to follow from PID and whose numbering is aligned with that of Theorem 2.3:

Axiom 3.6. Every locally compact space has either:
(1) A countable collection of \( \omega \)-bounded subspaces whose union is the whole space or
(2) An uncountable closed discrete subspace or
(3) A countable subset with non-Lindelöf closure.

Of course, (1) and (2) are mutually exclusive, but each is compatible with (3).

Part (iii) of our main theorem is the promised strengthening of Theorem 1.3.
Theorem 3.7. Let $X$ be a locally compact, $\omega_1$-compact space. If either
(i) $X$ is monotonically normal and the P-Ideal Dichotomy (PID) axiom holds, or
(ii) $X$ is hereditarily $\omega_1$-scwH, and either PFA or PFA$(S)[S]$ hold, or
(iii) $X$ is hereditarily normal, and MM$(S)[S]$ holds,
THEN $X$ is $\sigma$-$\omega$-bounded, and is either Lindelöf (and hence $\sigma$-compact) or contains a copy of $\omega_1$.

Proof. The $\sigma$-$\omega$-bounded property in case (i) follows from the fact that, in a monotonically normal space, every countable subset has Lindelöf closure \[31\], and from Axiom 3.6.

The PID, and hence Axiom \[3.6\] holds in any model of PFA or PFA$(S)[S]$, so $\sigma$-$\omega$-boundedness in case (ii) follows from the following two facts. First, every point (and hence every countable subset) of a locally compact space has an open Lindelöf neighborhood (see \[25\] Lemma 1.7). Second, in these models, every open Lindelöf subset of a hereditarily $\omega_1$-scwH space has Lindelöf closure (see \[25\] and \[29\], respectively). So again, (3) in Axiom \[3.6\] fails outright, and $\sigma$-$\omega$-boundedness for (ii) follows from (1) of Axiom \[3.6\].

As for (iii), Dow and Tall \[10\] have shown that MM$(S)[S]$ implies that every normal, locally compact space is $\omega_1$-cwH. This implies that every hereditarily normal, locally compact space is hereditarily $\omega_1$-scwH by the following reasoning. First, every open subspace of a locally compact space is locally compact. Therefore, if an open subspace is normal, then by the Dow-Tall theorem, it is $\omega_1$-cwH. Then, by a trivial variation on Theorem 3.2, the space is hereditarily $\omega_1$-cwH. Finally, by the comment following Definition \[3.1\] it is hereditarily $\omega_1$-scwH. Now we can continue as for (ii).

That $X$ is either Lindelöf or contains a copy of $\omega_1$ in case (i), is immediate from the following fact, whose proof is deferred to the end of this section:

Every locally compact, monotonically normal space is either paracompact or contains a closed copy of a regular uncountable cardinal.

The Lindelöf alternative for case (i) uses the fact \[11\] 5.2.17] that every locally compact, paracompact space is the union of a disjoint family of (closed and) open Lindelöf subspaces. Now $\omega_1$-compactness makes the family countable, and so $X$ is Lindelöf.

The same either/or alternative for (ii) and (iii) uses the ZFC theorem that any $\omega$-bounded, locally compact space is either Lindelöf or contains a perfect preimage of $\omega_1$, along with the reduction of character in Lemma 1.2 of \[25\], which uses MA$(\omega_1)$, which in turn is implied by the PFA. Moreover, the proof of this lemma carries over to PFA$(S)[S]$. This proof shows that in a locally compact, hereditarily $\omega_1$-scwH space, every open Lindelöf subset has Lindelöf closure and hereditarily Lindelöf (hence first countable) boundary.

Now, given a perfect surjective map $\varphi : W \to \omega_1$, let $Y$ be the union of the boundaries of the subsets $\varphi^+[0, \alpha)$ where $\alpha$ is a limit ordinal. Letting $\Lambda$ be the set of all limit countable ordinals, note that $\psi : Y \to \Lambda$ such that $\psi^+\left(\{\alpha\}\right) = \partial(\varphi^+[0, \alpha)) \subset \varphi^+\left(\{\alpha\}\right)$ (here the boundary is computed of course in $W$) is a perfect map with first-countable fibers, the compactness of $\varphi^+[0, \alpha]$ guaranteeing that $\partial(\varphi^+[0, \alpha)) \neq \emptyset$ and thus the surjectivity of $\psi$. Since perfect preimages of locally compact first-countable spaces under maps with first-countable fibers are also first-countable, we can apply to $Y$ the theorem \[9\] that PFA$(S)[S]$
implies that every first countable perfect preimage of $\omega_1 \cong \Lambda$ contains a copy of $\omega_1$. That the PFA also implies this is a well-known theorem of Balogh, shown in [8].

Remark 3.8. In the conclusion of Theorem 3.7 one cannot expect the copy of $\omega_1$ to be closed. The ordinal $\omega_2$ is monotonically normal (as is any linearly orderable space) and $\omega$-bounded, but is not Lindelöf, and every copy of $\omega_1$ inside it is bounded, hence not closed.

The following theorem can be derived from Theorem 3.7 (ii) in the same way that Theorem 2.8 is derived from Theorem 2.5.

Theorem 3.9. Let $X$ be a locally compact, $\omega_1$-compact, normal, hereditarily $\omega_1$-scwH space. If either PFA or PFA($S$)$[S]$ holds, then $X$ is countably paracompact.

To put it positively, every ZFC example of a locally compact, hereditarily $\omega_1$-scwH Dowker space must contain an uncountable closed discrete subspace. However, to be absolutely certain of this, we need a negative answer to the second part of the following question:

Problem 4. Do we need the large cardinal strength of (at least) PID to obtain any or all of the conclusions of Theorem 3.7 or Theorem 3.9?

Where Theorem 3.9 is concerned, the other applications of PFA, etc. are taken care of by axioms $MA(\omega_1)$ and $MA(S)[S]$ [25], [27], both of which are equiconsistent with ZFC. However, we may need something more for the conclusion about the copy of $\omega_1$, or for the theorem [10] that $MM(S)[S]$ implies that every locally compact normal space is $\omega_1$-cwH, or for the following corollary of this fact and of Theorem 3.9:

Corollary 3.10. If $MM(S)[S]$, then every locally compact, hereditarily normal Dowker space must contain an uncountable closed discrete subspace. □

We now complete the proof of Theorem 3.7. First, we recall a powerful theorem of Balogh and Rudin [2].

Theorem 3.11. Let $X$ be a monotonically normal space. The following are equivalent.

1. $X$ is paracompact.
2. $X$ does not have a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.

Lemma 3.12. Let $\delta$ be an ordinal of uncountable cofinality, and let $E$ be a locally compact stationary subset of $\delta$. There is a tail (final segment) $T$ of $E$ which is a closed (hence club) subset of $\delta$.

Proof. Each $x \in E$ has a compact open neighborhood $H_x$ of the form $[\beta_x, x] \cap E$ whose least element is isolated in $E$. Then the pressing down lemma gives us a single $\beta$ which works for a cofinal subset $S$ of $E$, and so $\bigcup \{[\beta, x] \cap E : x \in S\}$ is a tail of $E$ and a club in $\delta$. □

Lemma 3.12 gives the strengthening of Theorem 3.11 for locally compact spaces that was used in the proof of Theorem 3.7.

Corollary 3.13. Let $X$ be a monotonically normal, locally compact space. The following are equivalent.

1. $X$ is paracompact.
2. $X$ does not contain a closed copy of a regular uncountable cardinal.
Proof. We show that \((ii)\) is equivalent to (2) of Theorem 3.11. Every ordinal is a stationary subset of itself; hence if (2) is true then so is \((ii)\). Conversely, every closed subset of \(X\) is locally compact. Therefore, if \(E\) is a closed copy in \(X\) of a stationary subset of the regular uncountable cardinal \(\delta\), then the \(T\) obtained in Lemma 3.12 is itself a copy of \(\delta\). \(\square\)

4. MONOTONICALLY NORMAL EXAMPLES UNDER ♣ FOR QUESTION 1.1

This section features a remarkably simple construction of consistent examples of locally compact, \(\omega_1\)-compact, monotonically normal spaces of cardinality \(\aleph_1\) that are not \(\sigma\)-countably compact.

Our construction gives the ordinal \(\omega_1\) a locally compact topology \(\tau\) that is finer than the usual (order) topology, in which the \(\tau\)-relative topology on the subspace \(\Lambda\) of limit ordinals is its usual order topology. As for the successor ordinals \(< \lambda \in \Lambda\), they will include closed discrete subspaces with supremum \(\lambda\), and sequences converging to \(\lambda\). The axiom ♣ is used to ensure that these two behaviors result in both \(\omega_1\)-compactness and failure of \(\sigma\)-countable compactness.

The following concept is part of the most widely used statement of ♣.

Definition 4.1. Given a countable limit ordinal \(\alpha\), a ladder at \(\alpha\) is an infinite subset \(L_\alpha\) of \(\alpha\) such that \(|L_\alpha \cap \beta| < \omega\) for all \(\beta \in \alpha\). A ladder system on \(\omega_1\) is a family

\[\mathcal{L} = \{L_\alpha : \alpha \in \Lambda\}\]

where each \(L_\alpha\) is a ladder at \(\alpha\).

Axiom 4.2. Axiom ♣ states that there is a ladder system \(\mathcal{L}\) on \(\omega_1\) such that, for any uncountable subset \(S\) of \(\omega_1\), there is \(L_\alpha \in \mathcal{L}\) such that \(L_\alpha \subset S\).

We now define a general family \(\mathcal{S}\) of spaces similar to the two families in [20].

Notation 4.3. Let \(\mathcal{S}\) designate the set of topologies \(\tau\) on \(\omega_1\) in which, to each point \(\alpha\) there are associated \(B(\alpha) \subset [0, \alpha]\) and \(B(\alpha, \xi) = B(\alpha) \cap (\xi, \alpha]\) for each \(\xi < \alpha\), such that:

1. \(\{B(\alpha, \xi) : \xi < \alpha\}\) is a base for the neighborhoods of \(\alpha\) [we allow \(\xi = -1\) in case \(\alpha = 0\)].
2. If \(\alpha \in \Lambda\) and \(\beta > \alpha\), then there exists \(\xi < \alpha\) such that \(B(\alpha, \xi) = B(\beta) \cap (\xi, \alpha]\). [In particular, \(\alpha \in B(\beta)\).]
3. There is a ladder system \(\mathcal{L} = \{L_\alpha : \alpha \in \Lambda\}\), such that if \(M_\alpha = \{\xi + 1 : \xi \in L_\alpha\}\), then \(\alpha = \sup[M_\alpha \cap B(\alpha)] = \sup[M_\alpha \setminus B(\alpha)]\).

One role of \(M_\alpha\) is to simplify the construction of spaces in \(\mathcal{S}\) by not having to deal with splitting \(L_\alpha\) between limit and isolated ordinals. It has the novel effect of translating the ladders in ♣ to \(\omega_1 \setminus \Lambda\), but preserving the action of ♣ to give \((\omega_1, \tau)\) the desired properties.

We do not need ♣ for the following lemma:

Lemma 4.4. Every space in \(\mathcal{S}\) is locally compact, and the \(\tau\)-relative topology on \(\Lambda\) is the usual order topology.
Proof. The second conclusion is immediate from (2). Obviously, \( B(0) = \{0\} \) and \( B(\xi + 1, \xi) \) are singletons for all successor ordinals \( \xi + 1 \). If \( \beta \in \Lambda \), then, since \( B(\beta) \) is countable, it is enough to show that \( B(\beta) \) is countably compact to show that it is compact.

Let \( A \) be an infinite subset of \( B(\beta) \). Then \( A \) contains a strictly ascending sequence \( \sigma \) of ordinals. Let \( sup(ran(\sigma)) = \alpha \). If \( \alpha = \beta \) then \( \sigma \to \alpha \) by (1), while if \( \alpha < \beta \), then (1) and (2) have the same effect, implying that \( \alpha \) is a limit point of \( A \) in \( B(\beta) \). \( \square \)

Next, we show how to construct spaces in \( \mathcal{G} \) by induction.

**Examples 4.5.** Let \( \tau \) be a topology defined using \( L \), and using the base \( B(\alpha) \) and \( B(\alpha, \xi) \) defined by recursion as follows:

Let \( B(0) = \{0\} \) and, if \( \alpha = \xi + 1 \), let \( B(\alpha) = B(\xi) \cup \{\alpha\} \). Given \( L_\alpha \in L \), let \( M_\alpha = \{ \xi + 1 : \xi \in L_\alpha \} \) be listed in strictly ascending order as \( \{ \alpha_n : n \in \omega \} \). If \( \alpha = \nu + \omega \) where \( \nu \) either is 0 or a limit ordinal, let \( B(\alpha) = B(\nu) \cup (\nu, \alpha) \setminus \{ \alpha_{2n} : n \in \omega \} \).

If \( \alpha \in \Lambda \) is not of the form \( \nu + \omega \), let \( S(\alpha, 0) = B(\alpha_0) \) and, for \( n > 0 \), let \( S(\alpha, n) = B(\alpha_n, \alpha_{n-1}) \) and let

\[
B(\alpha) = \bigcup_{n=0}^{\infty} (S(\alpha, n) \cup \{\alpha_{2n+1} : n \in \omega \}) \setminus \{\alpha_{2n} : n \in \omega \}.
\]

**Claim** \((\omega_1, \tau) \in \mathcal{G} \).

\( \vdash \) Assuming (2) in Notation 4.3 we first show (1). If \( \alpha \in B(\beta, \eta) \cap B(\gamma, \nu) \) and \( \alpha > \nu \), \( \alpha \notin \Lambda \) then obviously \( B(\alpha, \alpha - 1) = \{\alpha\} \) works, i.e., \( \alpha \in B(\alpha, \alpha - 1) \subset B(\beta, \eta) \cap B(\gamma, \nu) \). Otherwise, we can find a basic open neighborhood of \( \alpha \) inside the intersection by using (2). Just choose \( \xi \) greater than \( \eta \) and \( \nu \) and large enough so that \( B(\alpha, \xi) = B(\beta) \cap (\xi, \alpha] \) and also \( B(\alpha, \xi) = B(\gamma) \cap (\xi, \alpha] \).

Next we show (2) by induction. Suppose it is false, and that \( \beta \) is the first ordinal for which there is a limit ordinal \( \alpha < \beta \) where it fails. Clearly \( \beta \in \Lambda \). Only successor ordinals are in \( [0, \beta] \setminus B(\beta) \), so \( \alpha \in B(\beta) \). Let \( \alpha \in S(\beta, n) = B(\gamma, \nu) \). By minimality of \( \beta \), there exists \( \mu \geq \nu \) such that \( B(\alpha, \mu) = B(\gamma, \nu) \cap (\mu, \alpha] \). However, \( S(\beta, n) \) only contains finitely many members of \( L_\beta \); once \( \xi \) is above all the members of \( L_\beta \) that are above \( \alpha \), we must have \( B(\beta) \cap (\xi, \alpha] = B(\gamma, \nu) \cap (\xi, \alpha] \), and if \( \xi \geq \mu \) then this equals \( B(\alpha, \xi) \), a contradiction.

Finally, (3) is true by construction since \( \alpha \) is the supremum of the points of \( M_\alpha \) that have odd subscripts and also of the ones that have even subscripts. \( \vdash \)

Now we come to the main theorem of this section.

**Theorem 4.6.** Suppose that \( \tau \in \mathcal{G} \) and \( L \) is a ladder system such that item (3) of Notation 4.3 holds for \( \tau \) and \( L \). If in addition \( L \) witnesses \( \clubsuit \), then \((\omega_1, \tau) \) is monotonically normal and \( \omega_1 \)-compact but not \( \sigma \)-countably compact.

**Proof.** Recall the Borges criterion for monotone normality, Theorem 3.3:

There is an assignment of an open neighborhood \( h(x, U) =: U_x \) containing \( x \) to each pair \((x, U)\) such that \( U \) is an open neighborhood of \( x \), and such that, if \( U_x \cap V_y \neq \emptyset \), then either \( x \in V \) or \( y \in U \).
The choice of \( h(z, U) = \{ z \} \) for all isolated points \( z \) works for the case where either \( x \) or \( y \) is not a limit ordinal. So it is enough to take care of the case where \( x \) and \( y \) are both limit ordinals and \( x < y \). Given \( \alpha \in U \), let \( U_\alpha = B(\alpha, \xi) \) for some \( \xi \) such that \( B(\alpha, \xi) \subset U \). It follows from (2) that \( x \in U_y \) whenever \( U_x \cap U_y \neq \emptyset \).

It also follows from (2) that the \( \tau \)-relative topology on \( \Lambda \) is its usual order topology. So, to show \( \omega_1 \)-compactness it is enough to show that every uncountable set \( S \) of successor ordinals has an ascending sequence \( \langle s_n : n \in \omega \rangle \) that \( \tau \)-converges to its supremum.

Finally, to show that \( (\omega_1, \tau) \) is not \( \sigma \)-countably compact, let \( \omega_1 = \bigcup_{n=1}^{\infty} A_n \). It is clearly enough to show that any \( A_n \) that contains uncountably many successor ordinals is not countably compact, since there is at least one such \( A_n \). Let one of these be \( S \), and let \( R \) and \( M_\alpha \) be as before. Then the even-numbered members of \( M_\alpha \) are an infinite \( \tau \)-closed discrete subset of \( \omega_1 \). \( \square \)

The way that the Borges criterion was shown for the spaces in Theorem 4.6 makes it clear that they witness the following property.

**Definition 4.7.** A space \( X \) is **utterly ultranormal** if there is a system of basic neighborhoods \( B_x \) for each \( x \in X \) satisfying

\[(\dagger) \quad \text{If } B_x \in B_x \text{ and } B_y \in B_y \text{ and } B_x \cap B_y \neq \emptyset, \text{ then either } x \in B_y \text{ or } y \in B_x.\]

For the more general concept of **utterly normal** see [5]. Every utterly normal space is monotonically normal, but the converse is a long-standing unsolved problem.

5. **The minimum cardinality theme**

We return now to Problems 1 and 2, repeated here for convenience:

**Problem 1.** What is the minimum cardinality of a locally compact, \( \omega_1 \)-compact space that is not \( \sigma \)-countably compact? one that is normal?

**Problem 2.** Is there a ZFC example of a locally compact, \( \omega_1 \)-compact space of cardinality \( \aleph_2 \) that is not \( \sigma \)-countably compact? one that is normal?

Of course, if \( b = \aleph_1 \), then Example 1.4 works for Problem 1, and thus the topological direct sum thereof with a compact space of cardinality \( \aleph_2 \) works for Problem 2. Similarly, if \( b = \aleph_2 \), then the answer to Problem 2 is Yes, and hence the answer to Problem 1 is \( \leq \aleph_2 \).

Theorem 2.7 gave us extra set-theoretic conditions under which the answer to Problem 1 is \( \geq \aleph_2 \). Let us note that the proof of Theorem 2.7 for spaces of size \( \aleph_1 \) only used the weakening of PID to \( P \)-ideals on sets of size also \( \aleph_1 \), designated \( PID_{\aleph_1} \). Since \( PID_{\aleph_2} \) implies \( b \leq \aleph_2 \) by a result of Todorčević, it is impossible to make the same technique as in the proof of Theorem 2.7 work to give us a model where the answer to Problem 1 would be \( \aleph_3 \), as we would need the inconsistent assumption \( PID_{\aleph_2} + b > \aleph_2 \). For the sake of completeness we prove here the above-mentioned inconsistency. First, some notation and a definition.

Given a relation \( R \) on \( \omega \) and \( x, y \in \omega^\omega \), we denote the set \( \{ n \in \omega : x(n) Ry(n) \} \) by \( [x Ry] \).
Definition 5.1. Let $\kappa, \lambda$ be regular cardinals. A $(\kappa, \lambda)$-pregap $\langle \{f_\alpha\}_{\alpha < \kappa}, \{g_\beta\}_{\beta < \lambda}\rangle$ is a pair of transfinite sequences $\{f_\alpha : \alpha < \kappa\}$ and $\{g_\beta : \beta < \lambda\}$ of nondecreasing sequences $f_\alpha, g_\beta$ of natural numbers such that $f_{\alpha_1} \leq^* f_{\alpha_2} \leq^* g_{\beta_2} \leq^* g_{\beta_1}$ for all $\alpha_1 \leq \alpha_2 < \kappa$ and $\beta_1 \leq \beta_2 < \lambda$. As usual, $f \leq^* g$ means that the set $\{f > g\}$ is finite. A $(\kappa, \lambda)$-pregap is called a $(\kappa, \lambda)$-gap, if there is no $h \in \omega^\omega$ such that $f_\alpha \leq^* h \leq^* g_\beta$ for all $\alpha, \beta$.

Lemma 5.2 is a slight improvement of [21] Lemma 1.12, which in its turn is modelled on the second paragraph of [36].

Lemma 5.2. Suppose that there exists a $(\kappa, \lambda)$-gap such that $\kappa, \lambda$ are regular uncountable.

1. If $\kappa > \aleph_1$ then $\text{PID}_\lambda$ fails.
2. If $\lambda > \aleph_1$ then $\text{PID}_\kappa$ fails.

Proof. The proofs of the two parts of the lemma are essentially the same, but we present both of them for the sake of completeness.

1. Assume that $\kappa > \aleph_1$ but $\text{PID}_\lambda$ holds. Fix a $(\kappa, \lambda)$-gap $\langle \{f_\alpha\}_{\alpha < \kappa}, \{g_\beta\}_{\beta < \lambda}\rangle$ and set

$$I = \{ A \in [\lambda]^{\aleph_0} : \exists \alpha \in \kappa \forall n \in \omega (\{ \beta \in A : [f_\alpha > g_\beta] \subseteq n \} < \aleph_0)\}.$$ 

Note that if $\alpha$ witnesses $A \in I$, then any $\alpha' > \alpha$ has also this property because $f_\alpha \leq^* f_{\alpha'}$. We claim that $I$ is a $P$-ideal. Indeed, let $\{A_i : i \in \omega\}$ be a sequence of elements of $I$, $\alpha_i$ be a witness for $A_i \in I$, and $\alpha = \sup\{\alpha_i : i \in \omega\}$. Then $\alpha$ witnesses $A_i \in I$ for all $i \in \omega$. Let $B_i = \{ \beta \in A_i : [f_\alpha > g_\beta] \subseteq i \}$. Then $B_i$ is a finite subset of $A_i$ by the definition of $I$. Set $A = \bigcup_{i \in \omega} A_i \setminus B_i$ and fix $n \in \omega$. If $\beta \in A_i$ is such that $[f_\alpha > g_\beta] \subseteq n$ and $i \geq n$, then $\beta \in B_i$. Therefore

$$\{ \beta \in A : [f_\alpha > g_\beta] = n \} \subseteq \{ \beta \in \bigcup_{i < n} A_i : [f_\alpha > g_\beta] \subseteq n \},$$

and the latter set is finite by the definition of $I$ and our choice of $\alpha$.

Applying $\text{PID}_\lambda$ to $I$ we conclude that one of the following alternatives is true:

1.a. There exists $S \in [\lambda]^{\aleph_1}$ such that $[S]^{\aleph_0} \subseteq I$. Passing to an uncountable subset of $S$ if necessary, we can assume that $S = \{ \beta_\xi : \xi < \omega_1 \}$ and $\beta_\xi < \beta_\eta$ for any $\xi < \eta < \omega_1$. For every $\xi$ we denote by $S_\xi$ the set $\{ \beta_\xi : \xi < \xi_0 \}$.

By the definition of $I$ for every $\xi$ there exists $\alpha_\xi \in \kappa$ witnessing for $S_\xi \subseteq I$. Let $\alpha = \sup\{\alpha_\xi : \xi < \omega_1\}$. There exists $n \in \omega$ such that the set $C = \{ \xi < \omega_1 : [f_\alpha > g_{\beta_\xi}] \subseteq n \}$ is uncountable. Let $\xi_0$ be the $\omega$-th element of $C$. Then $\alpha \geq \alpha_{\xi_0}$ and for all $\xi \in C \cap \xi_0$ we have $[f_\alpha > g_{\beta_\xi}] \subseteq n$. On the other hand, $S_{\xi_0} \subseteq I$, and hence there should be only finitely many such $\xi \in \xi_0$, a contradiction.

1.b. $\lambda = \bigcup_{m \in \omega} S_m$ such that $S_m$ is orthogonal to $I$ for all $m \in \omega$. This means that for every $m \in \omega$ and $\alpha \in \kappa$ there exists $n_{m, \alpha} \in \omega$ such that $[f_\alpha > g_\beta] \subseteq n_{m, \alpha}$ for all $\beta \in S_m$. (If there is no such $n = n_{m, \alpha}$, construct a sequence $\langle \beta_n : n \in \omega \rangle \in S_m^{\aleph_0}$ such that $[f_\alpha > g_{\beta_n}] \not\subseteq n$ and note that $\alpha$ witnesses $\{ \beta_n : n \in \omega \} \subseteq I$, which is impossible because $S_m$ is orthogonal to $I$. Since $\lambda$ is regular uncountable, there exists $m \in \omega$ such that $S_m$ is cofinal in $\lambda$. Let $n \in \omega$ be such that the set $J = \{ \alpha \in \kappa : n_{m, \alpha} = n \}$ is cofinal in $\kappa$. For every $k$ let $h(k) = \max\{f_{\alpha}(k) : \alpha \in J\}$. Note that $h(k) \in \omega$ for $k \geq n_{m, k}$ because $f_{\alpha}(k) \leq g_\beta(k)$ for all $\alpha \in J$, $\beta \in S_m$, and $k \geq n_{m, \alpha}$. For $k < n_{m, \alpha}$ the fact that $h(k) \in \omega$ follows from $f_\alpha$'s being non-decreasing. From the above it follows that $[g_\beta < h] \subseteq n$ for all $\beta \in S_m$, and hence $h$ contradicts the fact that $\langle \{f_\alpha\}_{\alpha \in J}, \{g_\beta\}_{\beta \in S_m}\rangle$ is a gap.
2. Assume that \( \lambda > \aleph_1 \) but PID\(_\kappa\) holds. Fix a \((\kappa, \lambda)\)-gap \( \langle \{f_\alpha\}_{\alpha<\kappa}, \{g_\beta\}_{\beta<\lambda}\rangle \) and set
\[
\mathcal{I} = \{A \in [\kappa]^{\aleph_1} : \exists \beta \in \lambda \forall n \in \omega (|\{\alpha \in A : [f_\alpha > g_\beta] \subset n\}| < \aleph_0)\}.
\]
Note that if \( \beta \) witnesses \( A \in \mathcal{I} \), then any \( \beta' > \beta \) has also this property because \( f_{\beta'} \leq^* f_\beta \).

We claim that \( \mathcal{I} \) is a \( P \)-ideal. Indeed, let \( \{A_i : i \in \omega\} \) be a sequence of elements of \( \mathcal{I} \), \( \beta_i \) be a witness for \( A_i \in \mathcal{I} \), and \( \beta = \sup\{\beta_i : i \in \omega\} \). Then \( \beta \) witnesses \( A_i \in \mathcal{I} \) for all \( i \in \omega \). Let \( B_i = \{\alpha \in A_i : [f_\alpha > g_\beta] \subset i\} \). Then \( B_i \) is a finite subset of \( A_i \) by the definition of \( \mathcal{I} \). Set \( A = \bigcup_{i \in \omega} A_i \setminus B_i \) and fix \( n \in \omega \). If \( \alpha \in A_i \) is such that \( [f_\alpha > g_\beta] \subset n \) and \( i \geq n \), then \( \alpha \in B_i \).

Therefore
\[
\{\alpha \in A : [f_\alpha > g_\beta] \subset n\} \subset \{\alpha \in \bigcup_{i<n} A_i : [f_\alpha > g_\beta] \subset n\},
\]
and the latter set is finite by the definition of \( \mathcal{I} \) and our choice of \( \beta \).

Applying PID\(_\kappa\) to \( \mathcal{I} \) we conclude that one of the following alternatives is true:

2a. There exists \( S \in [\kappa]^{\aleph_1} \) such that \( [S]^{\aleph_0} \subset \mathcal{I} \). Passing to an uncountable subset of \( S \) if necessary, we can assume that \( S = \{\alpha_\xi : \xi < \omega_1\} \) and \( \alpha_\xi < \alpha_\eta \) for any \( \xi < \eta < \omega_1 \). For every \( \xi \) we denote by \( S_\xi \) the set \( \{\alpha_\zeta : \zeta < \xi\} \).

By the definition of \( \mathcal{I} \) for every \( \xi \) there exists \( \beta_\xi \in \lambda \) witnessing for \( S_\xi \in \mathcal{I} \). Let \( \beta = \sup\{\beta_\xi : \xi < \omega_1\} \) and note that it witnesses \( S_\xi \in \mathcal{I} \) for all \( \xi < \omega_1 \). On the other hand, there exists \( n \in \omega \) such that the set \( C = \{\xi < \omega_1 : [f_{\alpha_\xi} > g_\beta] \subset n\} \) is uncountable. Let \( \xi_0 \) be the \( \omega \)-th element of \( C \). Then \( \beta \) fails to witness \( S_{\xi_0} \in \mathcal{I} \), a contradiction.

2b. \( \kappa = \bigcup_{m \in \omega} S_m \) such that \( S_m \) is orthogonal to \( \mathcal{I} \) for all \( m \in \omega \). This means that for every \( m \in \omega \) and \( \beta \in \lambda \) there exists \( n_{m,\beta} \in \omega \) such that \( [f_\alpha > g_\beta] \subset n_{m,\beta} \) for all \( \alpha \in S_m \). (If there is no such \( n = n_{m,\beta} \), construct a sequence \( \langle \alpha_n : n \in \omega\rangle \in S_m^{\omega} \) such that \( [f_{\alpha_n} > g_\beta] \not\subset n \) and note that \( \beta \) witnesses \( \{\alpha_n : n \in \omega\} \in \mathcal{I} \), which is impossible because \( S_m \) is orthogonal to \( \mathcal{I} \).) Since \( \kappa \) is regular uncountable, there exists \( m \in \omega \) such that \( S_m \) is cofinal in \( \kappa \). Let \( n \in \omega \) be such that the set \( J = \{\beta \in \lambda : n_{m,\beta} = n\} \) is cofinal in \( \lambda \). For every \( k \) let \( h(k) = \min\{g_\beta(k) : \beta \in J\} \). From the above it follows that \( [f_\alpha > h] \subset n \) for all \( \alpha \in S_m \), and hence \( h \) contradicts the fact that \( \langle \{f_\alpha\}_{\alpha \in S_m}, \{g_\beta\}_{\beta \in J}\rangle \) is a gap.

This proof also shows that the following weakening of PID\(_\lambda\) fails in (1):

**Axiom 5.3.** PID\(_w^\lambda\) states that for each \( P \)-ideal \( \mathcal{I} \) on a set \( X \) of cardinality \( |X| \geq \lambda \), either there is an uncountable set \( A \subset X \) with \( [A]^\omega \subset \mathcal{I} \) or a set \( B \) of cardinality \( \lambda \) such that \( B \cap I \) is finite for all \( I \in \mathcal{I} \).

Similarly, PID\(_w^\kappa\) fails in (2).

The proof of the next lemma can be found in [16] page 578.

**Lemma 5.4.** If \( b > \aleph_2 \) then there is an \((\aleph_2, \lambda)\)-gap for some uncountable regular \( \lambda \).

As a corollary we get the following strengthening of a result usually attributed to Todorčević for PID\(_{\aleph_2}\).

**Theorem 5.5.** (PID\(_{\aleph_2}^w \land\) PID\(_{\aleph_1}^w\)) implies \( b \leq \aleph_2 \).

**Proof.** Suppose that \( b > \aleph_2 \) and nevertheless PID\(_{\aleph_2}^w\) holds. By Lemma 5.4 there is an \((\aleph_2, \lambda)\)-gap for some uncountable regular \( \lambda \). If \( \lambda = \aleph_1 \), then PID\(_{\aleph_1}^w\) fails by Lemma 5.2(1). If \( \lambda > \aleph_1 \), then PID\(_{\aleph_2}^w\) fails by Lemma 5.2(2). \( \square \)
Clearly, new ideas for the solution of Problem 2 are needed!

At the opposite extreme from Problem 2, we have:

**Problem 5.** Can $b$ be “arbitrarily large” and still be the minimum cardinality of a locally compact, $\omega_1$-compact space that is not $\sigma$-countably compact? of one that is also normal?

The question of what happens under Martin’s Axiom (MA) is especially interesting since it implies $b = c$. It also implies that ♣ fails and that there are no Souslin trees. Now a Souslin tree with the interval topology is of cardinality $\aleph_1$, and is locally compact, locally countable, $\omega_1$-compact and hereditarily collectionwise normal [23, 4.18] (hence hereditarily normal and hereditarily strongly cwH) but is not $\sigma$-countably compact.

To show $\omega_1$-compactness, use the fact that every closed discrete subspace in a tree with the interval topology is a countable union of antichains, and the fact that a Souslin tree is, by definition, an uncountable tree in which every chain and antichain is countable. To show that a Souslin tree is not the union of countably many countably compact subspaces, use these facts together with the fact that at least one of these subspaces would have to be uncountable, and the observation that the Erdős - Radó theorem implies that every uncountable tree must either contain an uncountable chain, or an infinite antichain, which would be a closed discrete subspace, contradicting countable compactness. [The interval topology on a subtree is not always the relative topology, but the relative topology is finer, so new infinite closed discrete subspaces could arise.]

It is worth noting here that adding a single Cohen real adds a Souslin tree but leaves $b$ as the same aleph that it is in the ground model. So $b$ can be arbitrarily large while the minimum cardinality for Problem 1 is $\aleph_1$.

### 6. A Maximum Cardinality Theme

Since most of the examples mentioned are locally countable, it is natural to inquire what happens if local countability is added to local compactness and $\omega_1$-compactness. The following very simple problem is a dramatic counterpoint to Question 1.1.

**Problem 6.** Is there a ZFC example of a normal, locally countable, countably compact space of cardinality greater than $\aleph_1$?

Although this problem does not mention local compactness, that is easily implied by normality (indeed regularity), local countability, and countable compactness.

In a paper in preparation, [30], the first author shows that there does exist a consistent example of a locally countable, normal, $\omega$-bounded (hence countably compact) space of cardinality $\aleph_2$, under the axiom $\square_{\aleph_1}$. This axiom is consistent if ZFC is consistent, and the equiconsistency strength of its negation is that of a Mahlo cardinal [16, Exercise 27.2]. But the following question, a counterpoint to Problem 2, is completely open — no consistency results are known either way:

**Problem 7.** Is there a normal, locally countable, countably compact space of cardinality greater than $\aleph_2$?
Of course, one could also ask whether there is an upper bound on the cardinalities of normal, locally countable, countably compact spaces. If “countably compact” is strengthened to “\(\omega\)-bounded” then it is consistent, modulo large cardinals, that there is an upper bound. In fact, it has long been known that every regular, locally countable, \(\omega\)-bounded space is of cardinality \(< \aleph_{\omega}\) if the Chang Conjecture variant \((\aleph_{\omega+1}, \aleph_{\omega}) \rightarrow (\aleph_1, \aleph_0)\) holds \[19\]. It has recently been shown by Monroe Eskew and Yair Hayut \[12\] that the consistency of a huge cardinal implies that this variant is consistent. Previously, the best large cardinal axiom known was the existence of a 2-huge cardinal.

On the other hand, it has long been known that there are examples of regular, locally countable, \(\omega\)-bounded spaces of cardinality \(\aleph_n\) for all finite \(n\) just from ZFC, as was shown in \[18\]. However, the construction in \[18\] is so general that it could even produce non-normal closed subspaces of cardinality \(\aleph_1\), and it is difficult to see how these can be avoided in ZFC without many additional details, if at all.

We conjecture that it is consistent, modulo large cardinals, that every normal, locally countable, countably compact space is of cardinality \(< \aleph_{\omega}\), leaving only the consistency of examples of cardinality \(\aleph_n\) unknown for \(3 \leq n < \omega\). This is because of the Chang Conjecture variant result \[19\] for the \(\omega\)-bounded case and the following lemma:

**Lemma 6.1.** The PFA implies every normal, first countable, countably compact space is \(\omega\)-bounded.

**Proof.** This follows from the following three statements, the first two being consequences of the PFA:

1. every first countable, countably compact space is either compact or contains a copy of \(\omega_1\) \[8, 6.5, 6.6\],
2. \(p = c = \aleph_2\), and:
3. \[15, 3.9\] \(p = c\) is equivalent to the following statement.  

\(\textbf{H:}\) If \(D\) is a countable dense subset of a \(T_1\) space \(K\), and there exists an open cover \(\mathcal{U}\) of \(K\) such that \(\{U \cap D : U \in \mathcal{U}\}\) has cardinality \(< c\), and admits no finite subcover of \(D\), then \(D\) has an infinite closed discrete subset.

Let \(X\) be a normal, first countable, countably compact space with a countable dense subset \(D_0\). If \(X\) is not compact, then by (1), identify \(\omega_1\) with a subspace of \(X\). By first countability, \(\omega_1\) is closed in \(X\). For each \(\alpha \in \omega_1\), let \(V_\alpha\) be an open neighborhood of \([0, \alpha]\) in \(X\) whose closure misses \(\omega_1 \setminus [0, \alpha]\). This does not require normality, only the elementary fact that in a Hausdorff space, a compact set and a closed set disjoint from it can be put into disjoint open sets.

Let \(V = \{V_\alpha : \alpha \in \omega_1\}\) and let \(V = \bigcup V\). Assuming normality, let \(G\) be an open neighborhood of \(\omega_1\) whose closure \(K\) is a subset of \(V\) and let \(D = D_0 \cap G\). Then \(D\), \(K\), and \(\mathcal{U} = \{V_\alpha \cap K : \alpha \in \omega_1\}\) are as in statement \(\textbf{H}\), so in \(K\) there is an infinite closed discrete subset of \(D\). But \(K\) is a closed subset of \(X\), contradicting countable compactness of \(X\). \(\square\)

**Remark 6.2.** In the light of Lemma 6.1, the questions in this section have natural generalizations for first countable, locally compact spaces. The key is that every point in such a space either has a a countable neighborhood, or a compact neighborhood of cardinality \(c\).
For instance, our conjecture can be rephrased:

**Conjecture 6.3.** It is consistent, modulo large cardinals, that every normal, locally compact, first countable, countably compact space is of Lindelöf degree $< \aleph_\omega$ and hence of cardinality $< \max\{\aleph_\omega, c^+\}$.

If “normal” is replaced by “locally hereditarily Lindelöf,” the answer is Yes: the Chang Conjecture variant does it [19].

### 7. Locally compact, quasi-perfect preimages

Example [14] was the first nontrivial example for the following problem by van Douwen [6].

**Problem 8.** Is ZFC enough to imply that each first countable regular space of cardinality at most $c$ is a quasi-perfect image of some locally compact space?

**Definition 7.1.** A continuous map $f : X \to Y$ is **quasi-perfect** if it is surjective, and closed, and each fiber $f^{-1}\{x\}$ is countably compact.

The following theorem was the key to Example [14]

**Theorem 7.2.** [27] Let $E$ be a stationary, co-stationary subset of $\omega_1$. There is a locally compact, normal, quasi-perfect preimage of $E$, of cardinality $b$. If $b = \aleph_1$, then this preimage can also be locally countable, hence first countable.

The stated properties of Example [14] follow from this theorem and from the following simple facts.

**Theorem 7.3.** Let $f : Y \to X$ be a continuous surjective map.

(i) If $X$ is not $\sigma$-countably compact, neither is $Y$.

(ii) If $X$ is $\omega_1$-compact, and $f$ is closed, and each fiber $f^{-1}\{x\}$ is $\omega_1$-compact, then $Y$ is $\omega_1$-compact.

**Proof.** Statement (i) easily follows by contrapositive and the elementary fact that the continuous image of a countably compact space is countably compact.

To show (ii), let $A$ be an uncountable subset of $Y$. Then either $A$ meets some fiber in an uncountable subset, in which case it is not closed discrete in $Y$, or $f^{-1}[A]$ is uncountable and so it is not closed discrete. Let $B$ be a subset of $f^{-1}[A]$ that is not closed, and let $p \in \overline{B} \setminus B$.

Let $A_0 = \{x_b : b \in B\}$ be a subset of $A$ such that $f(x_b) = b$ for all $b \in B$. Because $f$ is closed, and $B$ is not closed, neither is $A_0$ and, in fact, it has an accumulation point in $f^{-1}\{p\}$. \[\square\]

If $E$ is co-stationary, then all countably compact subsets of $E$, being closed, are countable. So $E$ cannot be $\sigma$-countably compact unless it is countable. And if $E$ is stationary, it has limit points in the closure of every uncountable subset of $\omega_1$, and so it is $\omega_1$-compact. It therefore follows from Theorem [7.3] that any quasi-perfect preimage of $E$ is $\omega_1$-compact, but not $\sigma$-countably compact.
Problem 8 was motivated by a theorem in 13.4 of [6], which stated that the preimages it asks for do exist if $b = c$. The preimages van Douwen constructed were locally countable. But it is also consistent that some of them are not normal:

**Theorem 7.4.** Let $X$ be a locally compact, locally countable, quasi-perfect preimage of the space $\mathbb{P}$ of irrationals with the relative topology. If the PFA holds, then $X$ is not normal.

*Proof.* Let $p \in \mathbb{P}$ and let $\pi : X \to \mathbb{P}$ be a surjective quasi-perfect map. Let $\{\sigma_\alpha : \alpha < \omega_1\}$ be a family of injective sequences in $X$ with disjoint $\pi$-images that converge to $p$. Let $A_\alpha$ be the set of all limit points of $\sigma_\alpha$. Clearly $A_\alpha \subset \pi^{-1}\{p\}$, so $A_\alpha \cup \text{ran}(\sigma_\alpha)$ is a separable, closed, countably compact subspace of $X$.

**Case 1.** $A_\alpha$ is uncountable for some $\alpha$. Then $\text{ran}(\sigma_\alpha)$ is a countable subspace of $A_\alpha \cup \text{ran}(\sigma_\alpha)$ that does not have compact closure, but now Lemma 6.1 shows that $A_\alpha \cup \text{ran}(\sigma_\alpha)$, and hence $X$, is not normal.

**Case 2.** $A_\alpha$ is countable for all $\alpha$. By induction, build sequences $\tau_\alpha(\alpha < \omega_1)$ in $X$ whose $\pi$-images converge to $p$, and such that $\text{ran}(\sigma_\beta) \subset^* \text{ran}(\tau_\alpha)$ for all $\beta < \alpha$. If some $\tau_\alpha$ has uncountably many limit points, argue as in Case 1. Otherwise, the set $C_\alpha$ of limit points of each $\tau_\alpha$ is compact and countable, so it is contained in a countable, compact, open subset $V_\alpha$ of $X$. Let $V = \bigcup_{\alpha < \omega_1} V_\alpha$. Since the $C_\alpha$ form an increasing chain, their union $C$ is $\omega$-bounded, hence countably compact, hence closed in $X$ by first countability of $X$. However, inasmuch as the $\sigma_\alpha$ have disjoint ranges, $\pi[N \setminus C]$ is uncountable for any neighborhood $N$ of $C$. And so, the $\pi$-image of every closed neighborhood of $C$ is an uncountable, closed subset of $\mathbb{P}$; hence it must be of cardinality $\mathfrak{c}$. But $|V| = \aleph_1$, so $V$ cannot contain a closed neighborhood of $C$, contradicting normality. \[\square\]

The italicized consequence of $p > \aleph_1$ in the proof of Lemma 6.1 is actually equivalent to it. The familiar Franklin-Rajagopalan space $\gamma^\mathbb{N}$ has a countable dense set of isolated points, and is normal, locally compact, and locally countable, hence first countable. It can be made countably compact if, and only if, $p = \aleph_1$. In [22] there is an extended treatment of when $\gamma^\mathbb{N}$ can be hereditarily normal. Theorem 3.7 excludes $MM(S)[S]$ models, but we do not know whether these models can be substituted for the PFA in Theorem 7.4. In these models, $p = \aleph_1$ while $b = c = \aleph_2$.

In the absence of $b = c$, we still have very little idea of which first countable spaces have locally compact quasi-perfect preimages. It is even unknown whether there is a model of $b < c$ in which the space $\mathbb{P}$ of irrationals has such a preimage, or whether there is a model in which it does not have one. It is also unknown whether there is a locally compact, locally countable, quasi-perfect preimage of $[0, 1]$ if $b < c$. Such a space would solve the following problem:

**Problem 9.** Is there a ZFC example of a scattered, countably compact, regular space that can be mapped continuously onto $[0, 1]$?

See [26] for discussion of this problem, including an explanation why the answer is affirmative if “regular” is omitted.
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