Lifting Problems in Grothendieck Fibrations

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June 16, 2018

Abstract

Many interesting classes of maps from homotopical algebra can be characterised as those maps with the right lifting property against certain sets of maps (such classes are sometimes referred to as cofibrantly generated). In a more sophisticated notion due to Garner (referred to as algebraically cofibrantly generated) the set of maps is replaced with a diagram over a small category.

We give a yet more general definition where the set or diagram of maps is replaced with a vertical map in a Grothendieck fibration. In addition to an interesting new view of the existing examples above, we get new notions, such as computable lifting problems in presheaf assemblies, and internal lifting problems in a topos.

We show that under reasonable conditions one can define a notion of universal lifting problem and carry out step-one of Garner’s small object argument. We give explicit descriptions of what the general construction looks like in some examples.

Contents

1 Introduction 2

2 Lifting Problems in a Fibration 6
  2.1 Review of Grothendieck Fibrations 6
  2.2 Definition of Lifting Problem 8

3 Universal Lifting Problems 9
  3.1 Review of Hom Objects and Locally Small Fibrations 9
  3.2 Definition and Existence of Universal Lifting Problems 12

4 Step-one of the Small Object Argument 13
  4.1 A Review of Bifibrations 14
  4.2 Compositions of Fibrations and Bifibrations 15
  4.3 Adjunctions, (Co)Pointed Endofunctors and (Co)monads over a Category 17
  4.4 Functorial Factorisations and Law’s’s over a Fibration 19
  4.5 An Abstract Description of Step-one 28
  4.6 Applying the Abstract Description 24
5 Criteria for the Existence of Algebraically Free Rawfs’s and Awfs’s
5.1 The Construction of Adjunctions and Monads . . . . . . . . . . . . . 26
5.2 Algebraic Weak Factorisation Systems . . . . . . . . . . . . . . . . 28
5.3 Algebraically Free (R)awfs’s . . . . . . . . . . . . . . . . . . . . . 28
5.4 Criteria for the Existence of Algebraically Free Awfs’s . . . . . . 29
5.5 Fibred and Strongly Fibred Algebraically Free Awfs’s . . . . . . . 31
6 Fibred Leibniz Construction
6.1 Review of Monoidal Fibrations . . . . . . . . . . . . . . . . . . . . 31
6.2 Definition and Existence of the Fibred Leibniz Construction . . . 32

7 Examples
7.1 Trivial Fibrations . . . . . . . . . . . . . . . . . . . . . . . . . . . 34
7.2 Set Indexed Families . . . . . . . . . . . . . . . . . . . . . . . . . 35
7.3 Category Indexed Families . . . . . . . . . . . . . . . . . . . . . . 35
7.4 Internal Category Indexed Families of Diagrams . . . . . . . . . . . 41
7.4.1 Presheaf Assemblies . . . . . . . . . . . . . . . . . . . . . . 42
7.5 Codomain Fibrations . . . . . . . . . . . . . . . . . . . . . . . . . 44
7.5.1 Individual Morphisms . . . . . . . . . . . . . . . . . . . . . 48
7.5.2 Strongly Fibred Cofibrations and Pushout Product . . . . . . 49
7.5.3 Trivial Fibrations in 01-Substitution Sets . . . . . . . . . . . 52
7.5.4 Trivial Fibrations and Fibrations in CCHM Cubical Sets . . . . 54

8 A Further Generalisation: Lifting Problems for Squares
8.1 Definition . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 55
8.2 Squares over a Codomain Fibration . . . . . . . . . . . . . . . . . . 57

9 Conclusion and Directions for Future Work
9.1 Cofibrantly Generated Awfs’s in IIW-Pretoposes with WISC . . . 58
9.2 Applications to Realizability . . . . . . . . . . . . . . . . . . . . . . 59
9.3 The BCH Cubical Set Model . . . . . . . . . . . . . . . . . . . . . 59
9.4 Relation to Other Generalised Notions of Lifting Problem . . . . . 60

1 Introduction

We first recall some standard notions in homotopical algebra. See e.g. [25, Chapters 11 and 12] for more details.

Given two maps $m: U \to V$ and $f: X \to Y$ in a category $\mathbb{C}$, we say that $m$ has the left lifting property with respect to $f$ and $f$ has the right lifting property with respect to $m$ if for every commutative square, as in the solid lines below (which we refer to as a lifting problem), there is a diagonal filler, which is the dotted line below, making two commutative triangles.

\[
\begin{array}{ccc}
U & \longrightarrow & X \\
\downarrow m & & \downarrow f \\
V & \longrightarrow & Y \\
\end{array}
\]
Definition 1.0.1. A weak factorisation system (wfs) is two classes of maps \( \mathcal{L} \) and \( \mathcal{R} \), which are closed under retracts such that every element of \( \mathcal{L} \) has the left lifting property against every element of \( \mathcal{R} \), and any map factors as an element of \( \mathcal{L} \) followed by an element of \( \mathcal{R} \).

Definition 1.0.2. We say a weak factorisation system \((\mathcal{L}, \mathcal{R})\) is cofibrantly generated if there is some set \( I \) of morphisms in \( \mathbb{C} \) such that \( \mathcal{R} = I^\mathrm{R} \).

A well known result due to Quillen, the small object argument shows that in categories satisfying certain conditions any set \( I \) cofibrantly generates a wfs.

We note however, that the definition of cofibrantly generated still makes sense in the absence of any weak factorisation system. Specifically, we say that a class \( \mathcal{R} \) is cofibrantly generated by a set \( I \) if \( \mathcal{R} = I^\mathrm{R} \). This can still be a useful thing to do as it can give us an easy way to give concise definitions of classes of maps. In this paper we will focus on classes of maps considered in the semantics of homotopy type theory, although the techniques developed may be more widely applicable.

A generalisation of cofibrantly generated was developed by Garner in [8]. The definition was again originally stated for wfs’s, or more precisely a more structured notion called algebraic weak factorisation system (awfs) developed by Grandis and Tholen in [10] (originally referred to as natural weak factorisation system). We again note however, that the definition still makes sense for arbitrary classes of maps.

Definition 1.0.3 (Garner). Let \( \mathcal{C} \) be a category. Let \( \mathcal{A} \) and \( \mathcal{B} \) be small categories and let \( F : \mathcal{A} \to \mathbb{C}^2 \) and \( G : \mathcal{B} \to \mathbb{C}^2 \) be functors. We say \( F \) has the left lifting property against \( G \) and \( G \) has the right lifting property against \( F \) if the following holds. For all objects \( a \) of \( \mathcal{A} \) and \( b \) of \( \mathcal{B} \), and for all lifting problems of \( F(a) \) against \( G(b) \), we have a choice of filler. Furthermore these fillers satisfy a uniformity condition which states that for all morphisms \( \sigma : a \to a' \) in \( \mathcal{A} \), all \( \tau : b \to b' \) in \( \mathcal{B} \) and all commutative cubes with the square \( F(\sigma) \) on the left and \( G(\tau) \) on the right, the resulting “diagonal square” formed by the fillers commutes.

Definition 1.0.4 (Garner). Let \( \mathbb{C} \) be a category. We say a class of maps \( \mathcal{R} \) is algebraically cofibrantly generated if there is a small category \( I \) and a functor \( J : I \to \mathbb{C}^2 \) such that \( f \) belongs to the class if and only if it has the right lifting property against \( J \).

It is natural to also consider the notion of cofibrantly generated category as defined below.

Definition 1.0.5. Let \( \mathbb{C} \) be a category. Let \( \mathcal{I} \) be a small category and \( J : \mathcal{I} \to \mathbb{C}^2 \). Write \( J^\mathrm{R} \) for the category defined as follows. An object of \( J^\mathrm{R} \) is a morphism \( f \) of \( \mathbb{C} \) together with a uniform choice of diagonal fillers of \( J \) against \( f \). A morphism is a commutative square in \( \mathbb{C} \) which is compatible with the fillers.

We say a category \( \mathbb{D} \) and functor \( U : \mathbb{D} \to \mathbb{C}^2 \) is cofibrantly generated if they are isomorphic to the forgetful functor \( J^\mathrm{R} \to \mathbb{C}^2 \) for some \( J : \mathcal{I} \to \mathbb{C}^2 \).

Garner developed an improved version of Quillen’s small object argument, referred to as the algebraic small object argument. As a consequence of this result, under certain assumptions, cofibrantly generated categories are isomorphic.
to categories of algebras over a monad on $C^2$. As part of the proof, known as step-one of the algebraic small object argument, Garner considered a weaker notion of left half of an algebraic weak factorisation system. This is already sufficient to show that cofibrantly generated categories are isomorphic to categories of algebras over a pointed endofunctor on $C^2$.

We will develop a further generalisation of lifting problem, in which the set $I \subseteq C^2$, or diagram $J : I \rightarrow C^2$ is replaced with a vertical map in a Grothendieck fibration. We will also describe some interesting examples that aren’t included under existing definitions of cofibrantly generated. Our main two examples will be category indexed presheaf assemblies and codomain fibrations. The first of these is a variant of Garner’s definition of algebraically cofibrantly generated applied to presheaf categories, but where the choice of fillers for lifting problems must satisfy the additional requirement of being uniformly computable. The second allows us to formalise a notion of lifting problem internal to a topos (or more generally any locally cartesian closed category). This will allow us to better understand certain ideas considered by Van den Berg and Frumin in [30] and by Pitts and Orton in [20]. In both of these cases we note that the underlying category is not necessarily cocomplete, which is a necessary condition required for Garner’s small object argument, even for the relatively simple step-one part.

We will then develop some constructions that can be carried out for the general definition under reasonable assumptions (which do not require the underlying category to be complete or cocomplete). The first is universal lifting problem, in which the collection of all lifting problems between two families of maps corresponds to one single lifting problem in the total category of the fibration, which is uniquely determined by a universal property. We then define an abstract version of step-one of the small object argument that exhibits cofibrantly generated categories as categories of algebras over pointed endofunctors.

We will also show how to generalise the definition of cofibrantly generated algebraic weak factorisation system to our setting. Although we won’t see any new examples of such awfs’s in this paper, we will see some smaller results in this direction. We will show how the existing results by Garner relate to this definition. We will also develop a sufficient criterion for the existence of cofibrantly generated awfs’s in terms of the existence of choices of initial objects for certain categories of algebras. This will then be used in a future paper alongside a new generalisation of dependent $W$-types to construct new examples of awfs’s.

We will also show how to generalise the Leibniz construction to a fibration, and a generalised notion of lifting problem due to Sattler. Both may have useful application when applying these ideas to the semantics of homotopy type theory (work in progress by the author suggests that the latter can be used to better understand and generalise the implementation of higher inductive types in cubical sets).

We draw attention the following ideas that motivated various aspects of this work that are good to bear in mind when reading.

**The similarities and differences between two generalisations of cubical sets** Cubical sets (from [5]), are inspired by ideas in homotopical algebra, and in particular the simplicial set model [15]. However, over time two distinct approaches have arisen.
In [7], Gambino and Sattler view cubical sets as a special case of general constructions in homotopical algebra, including Garner’s notion of lifting problem and small object argument and the Leibniz construction. We will refer to this as the “algebraic approach.” On the other hand Orton and Pitts view cubical sets as a special case of a construction in the internal logic of a topos. We will refer to this as the “internal logic approach.”

The two approaches are closely connected, and as Gambino and Sattler show, there is a large amount of overlap. However, there are also a number of curious differences. In using Garner’s definition of cofibrantly generated, the algebraic approach makes essential use of the notion of “small category” and thereby to an external notion of set that does not correspond to anything internal to the category we are working with. We see this again with the requirement that the category we work with is cocomplete. In this sense the internal logic approach, which does not require cocompleteness would appear to be more flexible. However, there are also several ways in which the algebraic is more general. For example, much of the general theory in the algebraic approach can be carried out without the category of study being locally cartesian closed: if we need to talk about maps from an object $X$ to an object $Y$, we just use the set of morphisms. This would be impossible in the internal logic approach, where we need local cartesian closedness in order to talk about maps from $X$ to $Y$ internally.

We will see that what the two approaches have in common is that both can be seen as studying part of a locally small bifibration. The differences between the two approaches can be explained in terms of differences between the bifibrations that we are working over. The external notion of set and small category in the algebraic approach arises from the base category of the fibration of “category indexed families.” For the internal logic approach, we are (implicitly) working over a codomain fibration, where the base category is the just the category we are studying. The essential use of infinite colimits in the algebraic approach arise from the opcartesian maps in the fibration we use, which are left Kan extensions. On the other hand the opcartesian maps over a codomain fibration are just given by composition, so we don’t see any kind of infinite colimit over a small category (even internally).

Applications to presheaf assemblies One of the main applications that the author hopes to use in future work is to understanding lifting problems within presheaf assemblies (categories of presheaves constructed internally in categories of assemblies). In particular working with presheaf assemblies should not involve minor but tedious modifications of existing arguments. Instead the same proofs and definitions should apply to both the existing definitions of cubical sets and to new realizability variants. We will in fact see two different approaches that could be used in the future in presheaf assemblies. The first is simply to build on the existing approach of Pitts and Orton1. The other is a more “hands on” approach based on Garner’s definition of algebraic lifting problem. In this approach a functor $J: \mathcal{I} \to \mathcal{C}^2$ (like in definition 1.0.3) is annotated with extra computational information. A filler is then required to be

1Technically their paper does not apply to presheaf assemblies because of the use of subobject classifiers, but the author expects this can be worked around without difficulty. Certainly the main results in this paper do not require subobject classifier.
both uniform in the algebraic sense, while also uniformly computable. This has quite a different character to the internal approach, and might also be useful for some applications.

A key point to make about presheaf assemblies is that they are not cocomplete, even for just countable colimits.

A generalisation of Garner’s algebraic lifting problem that does not mention small categories On seeing the essential role that small categories play in Garner’s definition, one might expect that something analogous, such as internal categories appear in any generalisation. Indeed in early drafts of this paper, that was the approach taken. However, somewhat surprisingly, our general definition will not involve any notion of small category or internal category, and yet still includes Garner’s definition as a special case by choosing the Grothendieck fibration appropriately (using so called category indexed family fibrations).

The distinction between fibred and strongly fibred awfs’s When working over a fibration it’s natural to require that functors preserve cartesian maps. We will see that in this case there are in fact two different senses in which an awfs may preserve cartesian maps, that we refer to as fibred and strongly fibred.

For a general fibration the difference is clear to see. The definition of strongly fibred always involves pullbacks, whereas fibred only mentions cartesian maps from the fibration we are working over.

Over a codomain fibration, it is possible to confuse the two notions since both can roughly be described as “stability under pullback.” However, the distinction is still important to make. We will see that over codomain fibrations, cofibrantly generated lawfs’s are always fibred (and so are cofibrantly generated awfs’s when they exist). In general, cofibrantly generated lawfs’s need not be strongly fibred (and if an awfs fails to be strongly fibred this is already the case for the lawfs at step-one), but we will see a useful special case where they are.

Throughout we pay careful attention to when we know that functors are fibred and when we don’t.

Acknowledgements
I’m grateful to Benno van den Berg for many helpful comments and corrections.

2 Lifting Problems in a Fibration

2.1 Review of Grothendieck Fibrations

We recall the definition of Grothendieck fibration. See e.g. [12, Chapter 1] for a more detailed introduction.

Definition 2.1.1. Let \( p: E \to B \) be a functor. A morphism \( f: A \to B \) in \( E \) is vertical if \( p(A) = p(B) \) and \( p(f) \) is the identity morphism. We will often refer to a vertical map over \( I \in B \) as a family of maps over \( I \).

Definition 2.1.2. Let \( p: E \to B \) be a functor and let \( I \) be an object of \( B \). We define the fibre category over \( I \), \( E_I \) to consist of objects \( X \) of \( E \) such that \( p(X) = I \), and morphisms (vertical) maps \( f \) such that \( p(f) = 1_I \).
Definition 2.1.3. Let \( p : E \to B \) be a functor and \( u : I \to J \) a morphism in \( B \). A morphism \( f : X \to Y \) is cartesian over \( u \) if \( p(f) = u \) and for every \( g : Z \to Y \) in \( E \) with \( p(g) = u \circ w \) for some \( w : p(Z) \to I \) there is a unique \( h : Z \to X \) in \( E \) above \( w \) with \( f \circ h = g \).

We think of cartesian maps as “substitutions.”

Definition 2.1.4. A cloven Grothendieck fibration is a functor \( p : E \to B \) together with a choice of cartesian lifts. That is, for each \( X \in E \) and \( \sigma : I \to p(Y) \) in \( B \) we are given an object \( \sigma^* Y \) in \( E \) and a cartesian morphism \( \overline{\sigma}(X) : u^* Y \to Y \) over \( \sigma \). We call the choice of lifts a cleavage for \( p \). We will write \( \overline{\sigma}(X) \) just as \( \sigma \) if \( X \) is clear from context. We will refer to \( \sigma^* (Y) \) as the reindexing of \( Y \).

We refer to \( B \) as the base and to \( E \) as the total category of the fibration.

Remark 2.1.5. The term cloven refers to the fact that we require a choice of cartesian lifts, not just that there exists at least one cartesian lift in each case. In this paper we will only consider cloven fibrations. In fact the same applies for all categorical structure in this paper. For instance, when we say that a category has certain limits, we really mean that we are given a choice of limit for each diagram of that shape.

Proposition 2.1.6. For each \( \sigma : I \to J \), reindexing along \( \sigma \) defines a functor \( E_J \to E_I \).

An important concept that we will use throughout the paper is that of fibrations of vertical maps, which we define below.

Definition 2.1.7. Let \( p : E \to B \) be a functor. We define the category of vertical maps, \( V(E) \), to be the full subcategory of \( E \) consisting of vertical maps.

Proposition 2.1.8. The functors \( p \circ \text{dom} \) and \( p \circ \text{cod} \) from \( V(E) \) to \( B \) are equal.

Notation 2.1.9. We will sometimes write the functor \( p \circ \text{cod} = p \circ \text{dom} \) as \( p \), when it is clear to do so from context.

Proposition 2.1.10. Let \( p : E \to B \) be a fibration. Then a morphism in \( V(E) \) is cartesian over \( B \) if and only if it is levelwise cartesian over \( p \). The functor \( p \circ \text{cod} : V(E) \to B \) is also a fibration.

Proof. It is easy to check that if a morphism is levelwise cartesian then it is cartesian in \( V(E) \). We can hence give \( p \circ \text{cod} \) the structure of a cloven fibration, by applying the cleavage of \( p \) levelwise. But then any cartesian map in \( V(E) \) must be isomorphic to such a morphism, and so must be levelwise cartesian in \( E \).

Remark 2.1.11. While working with Grothendieck fibrations, we will sometimes refer to the reference by Johnstone [14]. In these cases the results are actually for indexed categories. However, indexed categories and Grothendieck fibrations are equivalent and one may readily switch between them. See [14, Section B1.3] for details.

The main examples of Grothendieck fibrations that we will consider are the following.
Example 2.1.12. Let \( C \) be any category. We define the fibration of set indexed families of \( C \), \( p: \text{Fam}(C) \to \text{Set} \) as follows. Objects of \( \text{Fam}(C) \) are set indexed families of objects of \( C \), \((C_i)_{i \in I} \) (where \( I \) is an indexing set and \( C_i \) are objects of \( C \)).

Vertical maps \( F \) over a set \( I \) consist of set indexed families of morphisms in \( C \).

Cartesian maps are maps in \( \text{Fam}(C) \) that are levelwise isomorphisms, and reindexing is just reindexing in the usual sense.

Example 2.1.13. The set indexed family fibration \( \text{Fam}(C) \to \text{Set} \) extends to a fibration \( \text{Fam}(C) \to \text{Cat} \). In this case we define \( \text{Fam} \) to consist of pairs \( \langle A, X \rangle \) where \( A \) is a small category and \( X \) is a functor \( A \to C \).

A vertical map from \( X: A \to C \) to \( Y: A \to C \) is a natural transformation between the functors, and reindexing is given by composition.

Example 2.1.14. Again let \( C \) be any category. A map is cartesian over the codomain functor \( \text{cod}: C^2 \to C \) if and only if it is a pullback in \( C \). Hence \( \text{cod} \) is a fibration if and only if \( C \) has pullbacks.

2.2 Definition of Lifting Problem
We now give our general definition of lifting problem (which we will actually refer to as family of lifting problems) and fillers, as well as a few equivalent definitions.

In the below, let \( p: E \to B \) be a fibration.

Definition 2.2.1. Let \( m: U \to V \) and \( f: X \to Y \) be vertical maps over objects \( I \) and \( J \) of \( B \) respectively. A family of lifting problems from \( m \) to \( f \) consists of an object \( K \) of \( B \), morphisms \( \sigma: K \to I \), \( \tau: K \to J \), together with morphisms making the top and bottom of a commutative square in \( E_K \) as below.

\[
\begin{array}{ccc}
\sigma^*(U) & \rightarrow & \tau^*(X) \\
\sigma^*(m) \downarrow & & \tau^*(f) \\
\sigma^*(V) & \rightarrow & \tau^*(Y)
\end{array}
\] (2.1)

We will refer to such commutative squares as families of lifting problems from \( m \) to \( f \) over \( K \).

Definition 2.2.2. A solution, or choice of diagonal fillers for a family of lifting problems is a map \( j: \sigma^*(V) \to \tau^*(X) \) in \( E_K \), making (2.1) into two commutative triangles:

\[
\begin{array}{ccc}
\sigma^*(U) & \rightarrow & \tau^*(X) \\
\sigma^*(m) \downarrow & \nearrow & \tau^*(f) \\
\sigma^*(V) & \rightarrow & \tau^*(Y)
\end{array}
\]

(Note that by considering the image of the diagram in \( B \) we see \( j \) is necessarily a vertical map.)

Example 2.2.3. Over a trivial fibration \( C \to 1 \), a family of lifting problems is just a lifting problem in \( C \) and a solution is a diagonal filler.
We will often use one of the reformulations of lifting property below. Each is easy to check using the cartesianness of $\tau(f)$ and the characterisation of cartesian maps in $V(\mathbb{E})$ as levelwise cartesian maps in $\mathbb{E}$.

**Proposition 2.2.4.** Families of lifting problems from $m$ to $f$ correspond precisely to an object $K$ of $\mathcal{B}$, morphisms $\sigma: K \to I$, $\tau: K \to J$, together with a vertical map $\sigma^*(m) \to \tau^*(f)$ in $V(\mathbb{E})_K$.

**Proposition 2.2.5.** Families of lifting problems from $m$ to $f$ correspond precisely to an object $K$ of $\mathcal{B}$, morphisms $\sigma: K \to I$, $\tau: K \to J$, together with a vertical map $\sigma^*(U) \to \tau^*(V)$ in $V(\mathbb{E})_K$.

Moreover, solutions of (2.1) correspond precisely to maps $j: \sigma^*(V) \to X$ making (2.2) into two commutative triangles.

(Note that by considering the image of the diagram in $\mathcal{B}$ we see $j$ necessarily lies over $\tau$.)

**Proposition 2.2.6.** Families of lifting problems from $m$ to $f$ correspond precisely to an object $K$ of $\mathcal{B}$, morphisms $\sigma: K \to I$, $\tau: K \to J$, together with a morphism $\sigma^*(m) \to f$ in $V(\mathbb{E})$ lying over $\tau$.

**Proposition 2.2.7.** Families of lifting problems from $m$ to $f$ correspond precisely to isomorphism classes of spans $m \leftarrow \cdot \to f$ in $V(\mathbb{E})$ where the map $\cdot \to m$ is cartesian.

### 3 Universal Lifting Problems

#### 3.1 Review of Hom Objects and Locally Small Fibrations

We recall the definitions of hom object and locally small fibration. We essentially follow Jacobs’ presentation in [12, Section 9.5], although we will change the terminology a little for convenience. We will also make some basic observations that will be used later.

**Definition 3.1.1.** For any objects $X$ and $Y$ in $\mathbb{E}$ with $p(X) = p(Y) = I$, we say the hom object from $X$ to $Y$ exists if the functor from $(\mathcal{B}/I)^{op}$ to $\text{Set}$ sending $u: J \to I$ to $\mathbb{E}(u^*(X), u^*(Y))$ is representable. We write the representing object as $h_0: \text{Hom}_I(X, Y) \to I$, and $h_1$ for the canonical map $h_0^*(X) \to h_0^*(Y)$ corresponding to the identity on $h_0$. We will refer to $(\text{Hom}_I(X, Y), h_0, h_1)$ as the hom object from $X$ to $Y$.

Note that for each $X$ and $Y$, the hom object from $X$ to $Y$ is unique up to isomorphism when it exists. This justifies referring to it as “the” hom object.

**Proposition 3.1.2.** To specify a hom object from $X$ to $Y$ over $I$ is to specify a span from $X$ to $Y$, $X \leftarrow Z \to Y$ where the left map $Z \to X$ is cartesian, which
is universal, in the sense that for every span $X \leftarrow Z' \to Y$ where $Z' \to X$ is cartesian, there is a unique map $t: Z' \to Z$ in the commutative diagram below.

![Diagram](image.png)

**Notation 3.1.3.** We will sometimes write the object $Z$ appearing in proposition 3.1.2 as $\text{Hom}_I(X, Y)$ when it is clear to do so from context.

**Definition 3.1.4.** A fibration $p: E \to B$ is locally small if we have a choice of hom object $(\text{Hom}_I(X, Y), h_0, h_1)$ for all $I \in B$ and all $X, Y \in E_I$.

**Proposition 3.1.5.** Let $p: E \to B$ be a fibration, and suppose that $B$ has all finite limits. The $p$ is locally small if and only if the following holds. For all $I, J \in B$, all $X \in E_I$ and for all $Y \in E_J$ there is a choice of object $\text{Hom}(X, Y) \in B$ together with maps $\sigma: \text{Hom}(X, Y) \to I$, $\tau: \text{Hom}(X, Y) \to J$ and $h: \sigma^*(X) \to Y$ over $\tau$, which are universal in the following sense. For any $K$, together with maps $\sigma': K \to X$, $\tau': K \to Y$ and $h': \sigma'^*(X) \to Y$ over $\tau'$, there is a unique map $t: K \to \text{Hom}(X, Y)$ such that $t$ and the canonical map $\sigma^*(X) \to \sigma'^*(X)$ over $t$ make the following diagrams commute.

![Diagram](image.png)

**Proof.** Given $X$ over $I$ and $Y$ over $J$, we define $\text{Hom}(X, Y)$ as follows. Write $\pi_0$ and $\pi_1$ for the projections from $I \times J$. We take $\text{Hom}(X, Y)$ to be $\text{Hom}_{I \times J}(X, Y)$.

Conversely, given $X$ and $Y$ both in $E_I$, note that we have a map $\text{Hom}(X, Y) \to I \times I$ given by $\sigma$ and $\tau$ above. We define $\text{Hom}_I(X, Y)$ to be the pullback of $\text{Hom}(X, Y)$ along the diagonal map $I \to I \times I$.

The remaining details are left as an exercise for the reader (see [12] Exercise 9.5.2, and also see [13, Lemma A.2], although the formulation of hom object used there is a little different to one we use).

**Notation 3.1.6.** We will follow the convention that whenever we write in the subscript object from $B$, as in $\text{Hom}_I(X, Y)$, we are using definition 3.1.1. Whenever we drop the subscript as in $\text{Hom}(X, Y)$ we are following the alternative definition from proposition 3.1.5.

**Notation 3.1.7.** We will sometimes write the $\sigma^*(X)$ from proposition 3.1.5 as $\text{Hom}(X, Y)$ when it is clear to do so from context.
Remark 3.1.8. Just as for lifting problems, we can give an alternative formulation of $\text{Hom}(X, Y)$, where instead of specifying a map $h: \sigma^*(X) \to Y$ over $\tau$, we can specify a vertical map $h': \sigma^*(X) \to \tau^*(Y)$.

We will use the following lemmas later.

Lemma 3.1.9. Suppose that $p: E \to B$ is locally small and $B$ has pullbacks. Then $V(E) \to B$ is also locally small.

Proof. Let $m, f$ be objects of $V(E)$. Then we can view them as vertical maps $m: U \to V$ and $f: X \to Y$. We take $\text{Hom}_I(m, f)$ to be the pullback below.

\[
\text{Hom}_I(m, f) \longrightarrow \text{Hom}_I(U, X) \\
\downarrow \\
\text{Hom}_I(V, Y) \longrightarrow \text{Hom}_I(U, Y)
\]

It is straightforward to verify that this does give a hom object. \qed

Lemma 3.1.10. Let $X$, $Y$ and $Y'$ be objects of $E$ over $I$, $J$ and $J'$ respectively. Let $f: Y' \to Y$ be cartesian. Then the following square is a pullback, where the map $\text{Hom}(X, Y') \to \text{Hom}(X, Y)$ is the canonical map corresponding to composition with $f$.

\[
\text{Hom}(X, Y') \xrightarrow{\tau'} J' \\
\downarrow \\
\text{Hom}(X, Y) \xrightarrow{\tau} J
\]

Proof. Suppose we are given a diagram in $B$ of the following form.

\[
\text{Hom}(X, Y') \xrightarrow{\tau'} J' \\
\downarrow \\
\text{Hom}(X, Y) \xrightarrow{\tau} J
\]

This gives us the following diagram in $E$.

\[
\text{Hom}(X, Y') \xrightarrow{\tau'} J' \\
\downarrow \\
\text{Hom}(X, Y) \xrightarrow{\tau} J
\]
Since \( f \) is cartesian, there is a unique map \( \beta^*\sigma^*(X) \to Y' \) over \( \beta \) allowing us to extend the diagram to the one below.

But now this gives us a map \( K \to \text{Hom}(X,Y') \) extending (3.1), which is unique by a similar argument, and so the square is a pullback, as required.

### 3.2 Definition and Existence of Universal Lifting Problems

Universal lifting problem is an important concept that allows us to view the class of all lifting problems as a single uniquely determined ordinary lifting problem in the total category of the fibration.

**Definition 3.2.1.** Let \( m \) and \( f \) be vertical maps over \( I \) and \( J \), as before. A coherent choice of solutions consists of a choice of solution for each family of lifting problems, satisfying the following condition. Suppose that we are given the diagram below

and suppose that we are given a family of lifting problems over \( K' \).

Then we also have a family of lifting problems over \( K' \) given by applying \( k^* \) to the family of lifting problems over \( K \). The choice of solutions must satisfy that the choice of solution to the lifting problems over \( K' \) is given by applying \( k^* \) to the choice of solution to the lifting problems over \( K \).

**Definition 3.2.2.** Let \( \mathcal{B} \) have finite limits, let \( p \) be a locally small fibration and let \( m \) and \( f \) be vertical morphisms over \( I \) and \( J \) respectively. We define a family of lifting problems denoted the universal family of lifting problems, or just universal lifting problem. Take \( K \) to be \( \text{Hom}(m,f) \) (the hom object in \( V(\mathcal{E}) \), as defined in proposition 3.1.5) and take commutative square to be the morphism \( h_1: \sigma^*(m) \to \tau^*(f) \) in \( V(\mathcal{E}) \) given together with \( \text{Hom}(m,f) \).

**Lemma 3.2.3.** For any family of lifting problems over \( K \), there is a unique map \( t: K \to \text{Hom}(m,f) \) such that the family of lifting problems is given by applying \( t^* \) to the universal family.

**Proof.** This follows directly from the characterisation of locally small in proposition 3.1.5. \( \square \)
Proposition 3.2.4. Suppose that a fibration $p: E \to B$ is locally small and $B$ has all finite limits. Then the universal family of lifting problems from $m$ to $f$ exists for any vertical maps $m$ and $f$.

Proof. By lemma 3.1.9.

Finally, we give a simple, yet powerful result that allows us to characterise the coherent existence of a filler for every lifting problem as a single filler for the universal lifting problem.

Proposition 3.2.5. Let $B$ have finite limits, let $p$ be a locally small fibration and let $m$ and $f$ be vertical morphisms. Then the following are equivalent.

1. Every family of lifting problems from $m$ to $f$ has a solution.

2. The universal family of lifting problems from $m$ to $f$ has a solution.

3. There is a coherent choice of solutions to all families of lifting problems from $m$ to $f$.

Proof. Follows easily from the definition of universal family of lifting problems.

Definition 3.2.6. If $m$ and $f$ are families of maps over $I$ and $J$ respectively and one of the equivalent conditions in proposition 3.2.5 holds, then we say $m$ has the fibred left lifting property against $f$ and $f$ has the fibred right lifting property against $m$.

4 Step-one of the Small Object Argument

The overall aim of this section is theorem 4.6.3 where we will produce a more general version of step-one of Garner’s small object argument as appears in [8] (see also [25, Chapter 12]). This will produce for each vertical map $m$ a left half of an awfs ($L_1, R_1$), whose right maps are families of maps with the fibred right lifting property against $m$ (and moreover $R_1$-algebra structures correspond precisely to solutions to the universal lifting problem from $m$ to $f$). For this we will need the Grothendieck fibration to have the additional structure of a bifibration. As in the previous section, we also assume that the fibration is locally small.

A motivation for doing this is that the main role that awfs’s play in the cubical set model of type theory is not in constructing Kan fibrations, but in giving an elegant definition of maps with the structure of a fibration in terms of algebras over a monad. However, this can already be done with an awfs, except that fibrations are algebras over a pointed endofunctor rather than a monad. Using this idea it is possible to give a simple way to combine a split comprehension category with an awfs to obtain a new split comprehension category, and the cubical set model of type theory can be viewed as an instance of this construction.

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2This will appear as part of a future paper by the author.
4.1 A Review of Bifibrations

We review some basic definitions and observations on bifibrations. See e.g. [12, Section 9.1] for more details.

**Definition 4.1.1.** Let \( p : E \rightarrow B \) be a functor. A morphism \( X \rightarrow Y \) in \( E \) is opcartesian if the corresponding map \( Y \rightarrow X \) is cartesian in \( p^{\text{op}} : E^{\text{op}} \rightarrow B^{\text{op}} \). We say \( p \) is an opfibration if \( p^{\text{op}} : E^{\text{op}} \rightarrow B^{\text{op}} \) is a fibration. We say \( p \) is a bifibration if it is both a fibration and an opfibration.

**Proposition 4.1.2.** Let \( p : E \rightarrow B \) be a cloven fibration. Then opcartesian maps over \( u : I \rightarrow J \) correspond precisely to left adjoints to \( u^* : E_J \rightarrow E_I \) (which we will write as \( \bigvee u \dashv u^* \)).

**Proof.** See [12, Lemma 9.1.2] for full details. Here we just note that given \( \bigvee u \dashv u^* \) with unit \( \eta \) and \( X \in p^{-1}(I) \), the map \( \bar{u}(\bigvee u X) \circ \eta X : X \rightarrow \bigvee u X \) is opcartesian.

**Proposition 4.1.3.** Let \( p : E \rightarrow B \) be a bifibration. Suppose that we are given the following diagram in \( E \).

\[
\begin{array}{ccc}
X & \xrightarrow{f} & U \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & V
\end{array}
\]

(4.1)

If the top map is opcartesian and we are given a map \( t : p(U) \rightarrow p(V) \) making a commutative square in \( B \) then there is a unique map \( U \rightarrow V \) over \( t \) making a commutative square.

**Proof.** This is easy to show from the definition of opcartesian.

In particular, let \( p \) be cloven, and suppose we are given the following square in \( B \).

\[
\begin{array}{ccc}
A & \xrightarrow{e} & B \\
\downarrow s & & \downarrow t \\
C & \xrightarrow{u} & D
\end{array}
\]

(4.2)

Then if we are given \( f : X \rightarrow Y \) over \( s \), this gives us a canonical map \( \coprod u X \rightarrow \coprod u Y \). It is easy to check that in fact this is functorial using the uniqueness in proposition 4.1.3.

**Definition 4.1.4.** We say that \( p \) satisfies the Beck-Chevalley condition if the following holds.

Suppose we are given a square in \( E \) of the following form.

\[
\begin{array}{ccc}
X & \xrightarrow{h} & V \\
\downarrow f & & \downarrow g \\
Y & \xrightarrow{k} & W
\end{array}
\]

If the image of the square in \( B \) is a pullback, \( g \) is cartesian and \( k \) is opcartesian then \( h \) is opcartesian if and only if \( f \) is cartesian.
Proposition 4.1.5. Suppose that \( p \) satisfies the Beck-Chevalley condition, that \( \mathbb{P} \) is a pullback square and that \( f \) is cartesian. Then the canonical map \( \coprod_p X \to \coprod_u Y \) is cartesian.

Proof. We consider the following commutative square in \( \mathbb{E} \), where the bottom map is the canonical opcartesian map and the top is the unique map over \( r \) making the square commute.

\[
\begin{array}{ccc}
X & \xrightarrow{t^*(\coprod_u Y)} & \coprod_u Y \\
\downarrow^{f} & & \downarrow^{t(\coprod_u Y)} \\
Y & \to & \coprod_u Y
\end{array}
\]

Then by assumption \( f \) is cartesian, so applying Beck-Chevalley, the top map is opcartesian. We deduce that there is a vertical isomorphism \( \coprod_p X \cong t^*(\coprod_u Y) \). But this implies that the canonical map \( \coprod_p X \to \coprod_u Y \) is cartesian. \( \square \)

4.2 Compositions of Fibrations and Bifibrations

In this section we will prove some useful lemmas about cartesian maps over compositions of functors. We phrase the statements of the lemmas so that they can be easily dualised to give corresponding results for opcartesian maps. We will then apply these results to the composition of fibrations \( V(\mathbb{E}) \xrightarrow{\text{cod}} \mathbb{E} \xrightarrow{p} \mathbb{B} \) and to the composition of opfibrations \( V(\mathbb{E}) \xrightarrow{\text{dom}} \mathbb{E} \xrightarrow{p} \mathbb{B} \). We will in particular see that opcartesian maps over \( \text{dom} \) are characterised as a square with both horizontal maps opcartesian followed by a pushout. This will be key to relating our general definition of step-one to Garner’s definition.

Throughout this section, we assume that we have functors \( p, q \) and \( r \) as below.

\[
\begin{array}{ccc}
\mathbb{F} & \xrightarrow{q} & \mathbb{E} \\
\downarrow^{r} & & \downarrow^{p} \\
\mathbb{B} & & \\
\end{array}
\]

Lemma 4.2.1. Let \( r \) and \( p \) be bifibrations, let \( q \) preserve opcartesian maps, let \( I \in \mathbb{B} \) and let \( f : X \to Y \) in \( \mathbb{F}_I \) be cartesian over \( q_I : \mathbb{F}_I \to \mathbb{E}_I \). Then \( f \) is also cartesian as a map in \( \mathbb{F} \) over \( q \).

Proof. A straightforward diagram chase. \( \square \)

Lemma 4.2.2. Suppose that \( r \) and \( p \) are bifibrations, and \( q \) preserves cartesian maps. Then every map \( f \) in \( \mathbb{F} \) cartesian over \( r \) is also cartesian over \( q \).

Proof. We need to show \( f \) is cartesian over \( q \). Suppose we have the following three diagrams in \( \mathbb{F}, \mathbb{E} \) and \( \mathbb{B} \) respectively.

\[
\begin{array}{ccc}
Z & \xrightarrow{q} & q(Z) \\
X & \xrightarrow{f} & Y \\
\end{array} \quad \begin{array}{ccc}
q(Z) & \xrightarrow{q(\gamma)} & q(Y) \\
q(X) & \xrightarrow{q(f)} & q(Y) \\
\end{array} \quad \begin{array}{ccc}
r(Z) & \xrightarrow{r(\gamma)} & r(Y) \\
r(X) & \xrightarrow{r(f)} & r(Y) \\
\end{array}
\]
Then there is a unique map \( t \) over \( p(h) \) making the left hand diagram commute. However, using the fact that \( q(f) \) is cartesian in \( E \) over \( r(f) \), we have that \( q(t) = h \). Hence \( t \) lies over \( h \), and is clearly the unique such map. Hence \( f \) is cartesian over \( q(f) \).

**Lemma 4.2.3.** Suppose that \( r \) and \( p \) are bifibrations, and \( q \) preserves both cartesian and opcartesian maps, and is a (cloven) isofibration. Suppose further that for all \( I \in \mathcal{B} \), \( q_I \) is a (cloven) fibration (and the cleavage is uniform in \( I \)).

Then \( q \) is a fibration, and moreover every cartesian map over \( q \) can be factored as a map vertical over \( r \) (and necessarily cartesian over \( q \)) followed by a map cartesian over both \( r \) and \( q \).

**Proof.** Let \( f : X \rightarrow q(Y) \) in \( E \). Since \( p \) is a fibration, we may factor \( f \) as a map \( f_1 \) vertical over \( p \), followed by \( f_2 \) cartesian over \( p \). Then, since \( r \) is a fibration we have a cartesian map \( g_2 \) over \( p(f) \). Since \( q(g_2) \) is cartesian and \( q \) an isofibration, we may assume without loss of generality that \( q(g_2) = f_2 \). We also have that \( g_2 \) is cartesian over \( q \) by lemma 4.2.2.

Next, since \( q_{p(X)} \) is a fibration, we have a vertical map \( g_1 \) which is cartesian over \( f_1 \) in \( q_{p(X)} \), and hence also cartesian as a map over \( q \) by lemma 4.2.1, with \( \text{cod}(f_1) = \text{dom}(f_2) \).

Now note that \( f_2 \circ f_1 \) is a composition of cartesian morphisms and so cartesian over \( q \).

This provides a cartesian lift witnessing that \( q \) is a fibration. However, note also that every cartesian map is isomorphic to one of this form, and the property we are considering is preserved by isomorphism, so every cartesian map factors as described.

**Lemma 4.2.4.** Suppose that \( r \) and \( p \) are bifibrations, and \( q \) preserves both cartesian and opcartesian maps, and is a (cloven) isofibration. Suppose further that for all \( I \in \mathcal{B} \), \( q_I \) is a (cloven) opfibration. Then \( q \) is an opfibration, and moreover every opcartesian map over \( q \) can be factored as a map opcartesian over both \( r \) and \( q \) followed by a map vertical over \( r \) (and necessarily opcartesian over \( q \)).

**Proof.** Dual to lemma 4.2.3.

**Lemma 4.2.5.** Suppose we are given the diagram of functors below.

\[
\begin{array}{ccc}
F_1 & \xrightarrow{q_1} & F_2 \\
\downarrow r_1 & & \downarrow q_2 \\
\mathbb{B} & \xrightarrow{E} & \mathbb{E} \\
\uparrow r_2 & & \uparrow r_2 \\
\end{array}
\]

Suppose further that \( p, r_1 \) and \( r_2 \) are (cloven) bifibrations, that \( q_1 \) and \( q_2 \) preserve cartesian and opcartesian maps over \( \mathbb{B} \) and are (cloven) isofibrations. Suppose further that for all \( I \in \mathcal{B} \), \( q_{1,1} \) and \( q_{2,1} \) are (cloven) fibrations. Then \( \chi \) preserves all cartesian maps over \( \mathbb{E} \) if and only if it preserves cartesian maps over \( \mathbb{B} \) and for each \( I \in \mathcal{B} \), \( \chi_I \) preserves cartesian maps over \( \mathbb{E}_I \).
Proof. Follows easily from the characterisation of cartesian maps in lemma 4.2.3.

Lemma 4.2.6. Suppose that $p: E \to B$ is a bifibration and that for each $I \in B$, $E$ has pullbacks. Then cod: $V(E) \to E$ is a fibration and for any map $s$ in $V(E)$, the following are equivalent.

1. $s$ is a pullback, regarded as a square in $E$.
2. $s$ is cartesian over cod.
3. $s$ factors as a map vertical over $p \circ$ cod and cartesian over cod followed by a map cartesian over both cod and $p \circ$ cod.
4. $s$ (regarded as a square in $E$) factors as pullback with all maps vertical over $B$, followed by a (pullback) square where both horizontal maps are cartesian in $E$ over $p$.

Proof. The usual proof that cartesian maps over cod are exactly pullback squares easily generalises to give us $1 \iff 2$. Together with the observation that cartesian maps in $V(E)$ are exactly the levelwise cartesian maps, this also gives us $3 \iff 4$. (And it is easy to check that levelwise cartesian maps in $V(E)$ are pullbacks in $E$.)

Finally we get $2 \iff 3$ by lemma 4.2.3. (It is easy to check that cod satisfies the necessary conditions to apply the lemma.)

Lemma 4.2.7. Suppose that $p: E \to B$ is a bifibration and that for each $I \in B$, $E$ has pushouts. Then dom: $V(E) \to E$ is an opfibration and for any map $s$ in $V(E)$, the following are equivalent.

1. $s$ is a pushout, regarded as a square in $E$.
2. $s$ is opcartesian over dom.
3. $s$ factors as a map opcartesian over both dom and $p \circ$ dom followed by a map vertical over $p \circ$ dom and opcartesian over dom.
4. $s$ (regarded as a square in $E$) factors as a (pushout) square where both horizontal maps are opcartesian in $E$, followed by a pushout with all maps vertical over $B$.

Proof. Dual to lemma 4.2.6.

4.3 Adjunctions, (Co)Pointed Endofunctors and (Co)monads over a Category

We give definitions of adjunctions, (co)pointed endofunctors and (co)monads over a fibration. These are fairly standard, although we drop the requirement that they preserve cartesian maps, which is often assumed in other places (see e.g. [12, Chapter 1]).

Definition 4.3.1. Let $p: E \to B$ and $q: F \to B$ be fibrations. An adjunction over $B$ consists of an adjunction $F \dashv G$ from $E$ to $F$ such that $q \circ F = p$, $p \circ G = q$ and the unit and counit maps are vertical.
Remark 4.3.2. We do not require that $F$ and $G$ in an adjunction over $B$ preserve cartesian maps. However, one can deduce from the definition that $G$ always does (see [12, Exercise 1.8.5]). If $F$ also preserves cartesian maps, we say the adjunction is a fibred adjunction.

We will use later the following internal version of the usual isomorphism between maps $FX \to Y$ and maps $X \to GY$.

Lemma 4.3.3. Suppose that $p: E \to B$ and $q: F \to B$ are fibrations and $F \dashv G$ is a fibred adjunction from $E$ to $F$ over $B$. Suppose that $X$ is an object of $E$ over $I$ and $Y$ is an object of $F$ over $J$. Suppose that the hom object from $FX$ to $Y$ exists, and is of the form below.

\[
\begin{array}{ccc}
I & \overset{\sigma}{\to} & J \\
\downarrow & & \downarrow \\
\Hom(FX,Y) & \overset{\pi(X)}{\to} & F(\sigma(X)) \\
\end{array}
\]

Then the hom object from $X$ to $GY$ (exists and) is of the form below.

\[
\begin{array}{ccc}
I & \overset{\sigma}{\to} & J \\
\downarrow & & \downarrow \\
\Hom(FX,Y) & \overset{\pi(X)}{\to} & GY \\
\end{array}
\]

In particular $\Hom(FX,Y) \cong \Hom(X,GY)$.

Proof. Straightforward.

Lemma 4.3.4. Suppose that $p: E \to B$ and $q: F \to B$ are locally small fibrations, $B$ has finite limits and $F \dashv G$ is a fibred adjunction from $E$ to $F$ over $B$.

Suppose that $m$ is a vertical map in $E$ over $I$ and $f$ is a vertical map in $F$ over $J$. Then solutions to the universal lifting property from $Fm$ to $f$ correspond precisely to solutions of the universal lifting property from $m$ to $Gf$.

Proof. By applying lemma 4.3.3 to the lift of the adjunction to a fibred adjunction from $V(E)$ to $V(F)$ and the usual properties of an adjunction.

Definition 4.3.5. Let $p: E \to B$ be a fibration. An endofunctor over $B$ is a functor $T: E \to E$ such that $p \circ T = p$. A pointed endofunctor over $B$ is a an endofunctor over $B$ together with a natural transformation $\eta: 1 \Rightarrow T$ which is pointwise vertical. We define monad over $B$ to be a pointed endofunctor with pointwise vertical multiplication $\mu: T^2 \Rightarrow T$ satisfying the usual axioms for a monad. We dually define copointed endofunctor over $B$ and comonad over $B$.

Remark 4.3.6. We again don’t require an endofunctor necessarily to preserve cartesian maps. If it does we say it is fibred.

Definition 4.3.7. Let $T$ be an endofunctor over $B$. An algebra over $T$ is an object $X$ of $E$ together with a vertical map $\alpha: TX \to X$. 

18
If $T, \eta$ is a pointed endofunctor, we define an algebra over $T, \eta$ to be an algebra over the endofunctor which additionally satisfies the unit law:

$$X \xrightarrow{\eta_X} TX \xrightarrow{\alpha} X$$

If $T, \eta, \mu$ is a monad over $B$, we define an algebra over $T, \eta, \mu$ to be an algebra over the pointed endofunctor which additionally satisfies the multiplication law:

$$T^2X \xrightarrow{T\alpha} TX \xrightarrow{\mu_X} TX \xrightarrow{\alpha} X$$

**Remark 4.3.8.** For algebras over a pointed endofunctor and over a monad we can drop the requirement that $\alpha$ is vertical, since it follows automatically from the unit law together with the axiom that the unit is vertical.

### 4.4 Functorial Factorisations and Lawfs’s over a Fibration

In this section, we give generalisations of the usual definitions of functorial factorisation, lawfs and awfs, as appear for instance in [8].

We write $E^3_{\text{vert}}$ for the full subcategory of $E^3$ where both maps in an object are vertical.

**Definition 4.4.1.** Let $p: E \to B$. A functorial factorisation over $B$ is a functor $V(E) \to E^3_{\text{vert}}$ over $B$ that is a section of the composition functor $E^3_{\text{vert}} \to V(E)$. If $f: X \to Y$ is a vertical map in $E$ we will usually write the factorisation of $f$ as $X \xrightarrow{Lf} Kf \xrightarrow{Rf} Y$.

**Lemma 4.4.2.** Let $p: E \to B$ be a fibration. Assume further that $p$ has all finite limits, and so cod: $V(E) \to E$ is also a fibration. The following categories are isomorphic.

1. Functorial factorisations $(L, R)$ over $p$.
2. Pointed endofunctors $(R, \lambda)$ over $\text{cod}: V(E) \to E$.
3. Copointed endofunctors $(L, \rho)$ over $\text{dom}: V(E) \to E$.

**Proof.** We can take $\lambda_f$ to be $Lf$ and vice versa to show 1 and 2 are isomorphic.

Similarly $\rho_f$ and $Rf$ can be swapped to show 1 and 3 are isomorphic.

**Definition 4.4.3.** A left half of an algebraic weak factorisation system over $B$ (lawfs) is a comonad over $\text{dom}: V(E) \to E$.

We dually define right half of an algebraic weak factorisation system over $B$ (rawfs) as a monad over $\text{cod}: V(E) \to E$. 

19
Proposition 4.4.4. Lawfs’s correspond precisely to a functorial factorisations over $\mathbb{B}$ with a vertical natural transformation $\Sigma: L \Rightarrow L^2$ making $L$ into a comonad over dom: $V(E) \to E$.

Rawfs’s correspond precisely to a functorial factorisations over $\mathbb{B}$ with a vertical natural transformation $\Pi: R^2 \Rightarrow R$ making $R$ into a monad over cod: $V(E) \to E$.

Proposition 4.4.5. Let $(L, R)$ be a functorial factorisation over $\mathbb{B}$. The following are equivalent.

1. The underlying functor between fibrations $V(E) \to E$ is fibred over $\mathbb{B}$.
2. The endofunctor $L$ is fibred over $p \circ \text{dom}$.
3. The endofunctor $R$ is fibred over $p \circ \text{cod}$.

Definition 4.4.6. We say a functorial factorisation is fibred if one of the equivalent conditions in proposition 4.4.5 holds.

We say it is strongly fibred if it is fibred as a pointed endofunctor over cod.

Lemma 4.4.7. A functorial factorisation $(L, R)$ is strongly fibred if and only if it is both fibred and for each $I \in \mathbb{B}$, the restriction $R_I: V(E)_I \to V(E)_I$ preserves cartesian maps over $E_I$. (Note that $V(E)_I$ is just $E^2_I$, and the fibration $V(E)_I \to E_I$ is just codomain, so this says $R_I$ preserves pullbacks.)

Proof. By lemma 4.2.5.

Definition 4.4.8. We say an lawfs or rawfs is (strongly) fibred if the underlying functorial factorisation is (strongly) fibred.

We will see that it is often easy to show functorial factorisations are fibred under mild assumptions. Strongly fibred functorial factorisations are much rarer. We will however see later some useful lemmas for producing a few interesting examples of strongly fibred awfs’s.

4.5 An Abstract Description of Step-one

Ultimately we want to construct an lawfs over a fibration, which is a comonad over the opfibration dom: $V(E) \to E$. We will first give a more general construction of a comonad where we replace dom: $V(E) \to E$ with an arbitrary opfibration $q$ in the diagram below.

$$
\begin{array}{c}
F \\
\downarrow q \\
E
\end{array}
\quad
\begin{array}{c}
\downarrow p \\
\mathbb{B}
\end{array}
$$

(4.3)

This will allow us to give a clean proof that our claimed comonad really is a comonad, including the construction of comultiplication and the proof that it satisfies the “square” comonad law, which is somewhat cumbersome to do directly. This description will also be useful when we show that the generating family of left maps has a coalgebra structure and (under suitable conditions) that the resulting lawfs is fibred. However, we will see that the abstract description
can easily be reduced to give an explicit description of the action of the comonad on objects in the special case we are interested in. The reader may prefer to skip ahead to section 4.6 to see how the abstract description is used before looking at this section in detail.

Assume that we are given functors as in diagram \((4.3)\). Suppose further that \(r\) is a locally small fibration and \(q\) is an opfibration. We will use this to construct a comonad \(L_1\) over \(q\). The overall idea is to first construct an adjunction \(F \dashv G\), and then take \(L_1\) to be \(F G\).

**Definition 4.5.1.** Let \(X \in F\). We define the category, \(\text{Bicart}(X)\) of bicartesian spans from \(X\) as follows. An object consists of \(Y, Z \in F\) together with \(h: Z \to X\) cartesian over \(r\) and \(k: Z \to Y\) opcartesian over \(q\). A morphism from \(X \xleftarrow{h} Z \xrightarrow{k} Y\) to \(X \xleftarrow{h'} Z' \xrightarrow{k'} Y'\) consists of diagrams of the following form.

\[
\begin{array}{ccc}
X & \xleftarrow{h} & Z \\
\downarrow{k} & & \downarrow{k'} \\
Y & \xrightarrow{m} & \text{Hom}(X, Y)
\end{array}
\]

We define \(F: \text{Bicart}(X) \to F\) to be the functor sending \(X \xleftarrow{h} Z \xrightarrow{k} Y\) to \(Y\). We will define a right adjoint \(G\) to \(F\), giving us a comonad \(FG\) on \(F\).

Given \(Y \in F\), we construct \(G(Y)\) as follows. Using the local smallness of \(r\) we have a span as below, where \(l\) is cartesian.

\[
\begin{array}{ccc}
X & \xleftarrow{i} & \text{Hom}(X, Y) \\
\downarrow{l} & & \downarrow{m} \\
Y & \xrightarrow{n} & \text{Hom}(X, Y')
\end{array}
\]

We then get an opcartesian map \(\text{Hom}(X, Y) \to \prod_{q(m)} \text{Hom}(X, Y)\) given by the opfibration structure on \(q\). We then take \(G(Y)\) to be \(X \xleftarrow{h} \text{Hom}(X, Y) \xrightarrow{k} \prod_{q(m)} \text{Hom}(X, Y)\). Given \(n: Y \to Y'\), we define \(G(n)\) by first applying the universal property of \(\text{Hom}(X, Y')\), and then applying the universal property of the opcartesian map \(\text{Hom}(X, Y') \to \prod_{q(m)} \text{Hom}(X, Y')\).

We now check that \(F \dashv G\). Suppose that we are given an object of \(\text{Bicart}(X)\) of the form \(X \xleftarrow{h} Z \xrightarrow{k} Y\) an object \(Y'\) of \(F\), and a map \(n: Y \to Y'\). This gives
us the following diagram.

We then apply the universal property of \( \text{Hom}(X, Y') \) followed by the opcartesianess of \( k \), to extend the diagram as below (and the right hand triangle commutes, again using the opcartesianess of \( k \)).

But we now have a morphism from \( X \xrightarrow{h} Z \xrightarrow{k} Y \) to \( G(Y') \). This operation is invertible by composition with the canonical vertical map \( \prod_{q(m)} \text{Hom}(X, Y') \to Y' \), and one can check that it is natural. Hence this does give us an adjunction \( F \dashv G \).

**Definition 4.5.2.** We assume that we are given functors in the diagram below.

\[
\begin{array}{c}
\mathbb{F} \xrightarrow{q} \mathbb{E} \\
\downarrow r \quad \downarrow p \\
\mathbb{B}
\end{array}
\]

Suppose further that \( r \) is a locally small fibration and \( q \) is an opfibration, and we are given \( X \in \mathbb{F} \). We will use this to construct a comonad over \( q \).

Let \( F \dashv G \) be the adjunction above. We write \( L_1 \) for the resulting comonad \( FG \). Note that in fact \( F \) and \( G \) can be viewed as functors over \( \mathbb{E} \) and the unit and counit are both vertical, giving us an adjunction over \( \mathbb{E} \). Hence the counit and comultiplication are vertical over \( \mathbb{E} \) and so \( L_1 \) is a comonad over \( \mathbb{E} \).

We also get an explicit description of the counit of the comonad. Namely, it is the map corresponding to the identity on \( GY \) under the adjunction, which
unfolding the description above, is the vertical map in the factorisation of the map \( \text{Hom}(X, Y) \to Y \) as opcartesian map followed by vertical map.

We now show that \( X \) admits an \( FG \)-coalgebra structure.

**Lemma 4.5.3.** Let \( X \) be the object appearing in the definition of \( L_1 \). Then \( X \) admits an \( L_1 \)-coalgebra structure.

**Proof.** Define \( \bar{X} \) to be the span \( X \xleftarrow{1_X} X \xrightarrow{1_X} X \). The identity map is both cartesian and opcartesian, so this gives an object of \( \text{Bicart}(X) \). We clearly have \( X = F(\bar{X}) \).

Recall that for any adjunction \( F \dashv G : \mathcal{C} \to \mathcal{D} \) and any object \( Z \) of \( \mathcal{C} \), \( F(Z) \) admits an \( FG \)-coalgebra structure, given by \( F\eta_Z \), where \( \eta \) is the unit of the adjunction.

Applying this to \( \bar{X} \) gives us an \( FG \)-coalgebra structure on \( F(\bar{X}) = X \).

Finally, we show a general result about when the comonad is fibred.

**Definition 4.5.4.** We say \( q \)-opcartesian maps are \( r \)-fibred if the following holds. Suppose we are given a diagram of the following form in \( F \).

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{k} \\
W & \xrightarrow{g} & Z
\end{array}
\]

Suppose further that \( k \) and \( h \) are cartesian over \( r \), that \( g \) is both vertical over \( r \) and opcartesian over \( p \), and that \( f \) is vertical over \( r \). Then the condition states that for every such square \( f \) is also opcartesian over \( q \).

**Lemma 4.5.5.** Suppose that \( r \) is a bifibration satisfying the Beck-Chevalley condition, that \( q \)-opcartesian maps are \( r \)-fibred and that \( f : Y' \to Y \) is cartesian over \( r \). Then \( L_1 f : L_1 Y' \to L_1 Y \) is also cartesian over \( r \).

**Proof.** We consider the following square.

\[
\begin{array}{ccc}
\sigma^*(X) & \xrightarrow{\prod_{q(m')} \text{Hom}(X, Y')} & \\
\downarrow{\sigma^*(X)} & & \downarrow{\prod_{q(m)} \text{Hom}(X, Y)} \\
\sigma^*(X) & \xrightarrow{\prod_{q(m')} \text{Hom}(X, Y')} & \\
\downarrow{\sigma^*(X)} & & \downarrow{\prod_{q(m)} \text{Hom}(X, Y)}
\end{array}
\] (4.4)

The horizontal maps are opcartesian over \( q \) so by lemma [4.2.4] we can factorise each of them as maps opcartesian over \( r \) followed by maps opcartesian over \( q \) and vertical over \( r \). Using the opcartesianess of the maps on the left, we also get a canonical map \( t \) splitting the square into two, as below.

\[
\begin{array}{ccc}
\sigma^*(X) & \xrightarrow{\prod_{q(m')} \text{Hom}(X, Y')} & \\
\downarrow{\sigma^*(X)} & & \downarrow{\prod_{q(m)} \text{Hom}(X, Y)} \\
\sigma^*(X) & \xrightarrow{\prod_{q(m')} \text{Hom}(X, Y')} & \\
\downarrow{\sigma^*(X)} & & \downarrow{\prod_{q(m)} \text{Hom}(X, Y)}
\end{array}
\]
Since the horizontal maps on the right are both vertical over \( r \), the left hand square lies over the same square in \( \mathbb{B} \) as (4.4) did, which is the one below.

\[
\begin{array}{ccc}
\text{Hom}(X, Y') & \to & r(Y') \\
\downarrow & & \downarrow \\
\text{Hom}(X, Y) & \to & r(Y)
\end{array}
\]

However, by lemma 3.1.10 and the cartesianness of \( f \) this is a pullback square. Using this together with the Beck-Chevalley condition and proposition 4.1.5 we see that the \( t \) must be cartesian.

Since we now know \( t \) is cartesian, we can use the assumption that \( q \)-opcartesian maps are \( r \)-fibred together with the same argument that we used in the proof of proposition 4.1.5 to show that the right hand map in (4.4) is cartesian over \( r \), as required.

\[ \square \]

### 4.6 Applying the Abstract Description

Suppose we are given a vertical map \( m : U \to V \) over \( I \in \mathbb{B} \).

We will use this to define a lawfs over \( p \) denoted \textit{step-one of the small object argument}, \((L_1, R_1)\).

Suppose we are given a vertical map \( f : X \to Y \) over \( J \).

First we view \( m \) and \( f \) both as elements of \( V(E) \) over \( I \) and \( J \) respectively.

We then apply the abstract version of step-one to the following diagram.

\[
\begin{array}{ccc}
V(E) & \xrightarrow{\text{dom}} & E \\
\downarrow & & \downarrow \text{p} \\
\mathbb{B} & \xrightarrow{\text{p cod = p dom}} & \mathbb{B}
\end{array}
\]

(4.5)

Unfolding the abstract definition of \( L_1 \) in this case, we see that for a fixed \( m \) over \( I \), we define the comonad \( L_1 \) over \( \text{dom} \) as follows. Given a vertical \( f \) we construct \( L_1f \) by first taking the hom object from \( m \) to \( f \), which is just the universal lifting problem from \( m \) to \( f \) and consists of the object \( \text{Hom}(m, f) \), maps \( \sigma : \text{Hom}(m, f) \to I \) and \( \tau : \text{Hom}(m, f) \to J \), and a map \( h_1 : \sigma^*(m) \to f \) in \( V(E) \) over \( \tau \) (which recall we can also view as a square in \( E \) where the horizontal maps lie over \( \tau \)). We then factor \( h_1 \) as an opcartesian map followed by a vertical map, and the vertical map is the value of \( L_1f \) together with the counit at \( f \).

Since \( L_1 \) is a comonad over \( \text{dom} \) it is an lawfs.

**Lemma 4.6.1.** Let \((L_1, R_1)\) be the lawfs obtained by applying the abstract version of step-one to (4.5). Then \( R_1 \)-algebra structures on \( f \) correspond precisely to solutions of the universal lifting problem of \( m \) against \( f \).

**Proof.** As stated above, in the first part of the abstract version of step-one we take the universal lifting problem and in the second part we factor the map \( \text{Hom}(m, f) \to f \) as a map which is opcartesian over \( \text{dom} \), followed by a map which is vertical over \( \text{dom} \). By lemma 4.2.7 this gives us the following diagram in \( E \) where the rectangle is the universal lifting problem, the right hand square
is the factorisation of \( f \) given by step-one, and the left hand square is a pushout in the category \( E \).

\[
\begin{array}{ccc}
\sigma^*(U) & \longrightarrow & X \\
\sigma^*(m) & \downarrow & \downarrow L_1 f \\
\sigma^*(V) & \longrightarrow & K_1 f \\
\end{array}
\]

It is now easy to see from the universal property of the pushout that diagonal fillers of the whole rectangle correspond to diagonal fillers of the right hand square. However, diagonal fillers of the right hand square are necessarily vertical over \( p \) and are precisely \( R_1 \)-algebra structures on \( f \).

**Lemma 4.6.2.** Let \( m \) be the family of generating left maps for \( (L_1, R_1) \). \( m \) has the structure of an \( L_1 \)-coalgebra.

**Proof.** By lemma 4.5.3.

We summarise the above as the following theorem.

**Theorem 4.6.3** (Step-one of the small object argument). Let \( p: E \rightarrow B \) be a locally small bifibration. Let \( m \) be a vertical map over \( p \). Then there is an \( \text{lawfs} \) \( (L_1, R_1) \) such that \( R_1 \)-algebra structures on a vertical map \( f \) correspond precisely to solutions of the universal lifting problem from \( m \) to \( f \), and \( m \) can be given the structure of an \( L_1 \)-coalgebra.

**Proof.** An \( \text{lawfs} \) is a comonad over \( p \circ \text{dom} \). We construct this as the comonad obtained by applying the abstract version of step-one (definition 4.5.2) to the fibrations in diagram (4.5). The rest of the theorem is then lemmas 4.6.1 and 4.6.2.

**Theorem 4.6.4.** Suppose that \( p: E \rightarrow B \) is a locally small bifibration that satisfies the Beck-Chevalley condition and has fibred pushouts. Then step-one of the small object argument, \( (L_1, R_1) \) is a fibred \( \text{lawfs} \).

**Proof.** We aim to apply lemma 4.5.5.

Since cartesian maps and opcartesian maps in \( V(E) \) are just maps that are levelwise cartesian and opcartesian respectively, the Beck-Chevalley condition for \( p \) implies the same for \( \text{dom} \circ p \). Next, note that the condition that dom-opcartesian maps are \( p \)-fibred is exactly the assumption that pushouts are fibred.

Hence by lemma 4.5.5 we can deduce that \( L_1 \) is fibred over \( B \) and so is a fibred \( \text{lawfs} \).

### 5 Criteria for the Existence of Algebraically Free Rawfs’s and Awfs’s

In this section we show that the existence of algebraically free awfs’s is equivalent to the existence of initial algebras for certain pointed endofunctors. The motivation for doing this is that initial algebras allow us to formalise in category theory the notion of inductively generated object. We can then focus on arguments that produce objects that are intuitively the “least” objects satisfying some definition, then formalise this idea by showing they are initial algebras,
then deduce that we get awfs’s by applying the results of this section. This will be used by the author in a future paper alongside a generalisation of dependent $W$-types, in which dependent polynomial endofunctors are generalised to a certain class of pointed endofunctors that will include as a special case the pointed endofunctors appearing in this section in codomain fibrations.

### 5.1 The Construction of Adjunctions and Monads

The following lemma is based on an observation by Van den Berg and Garner in [31, Proposition 3.3.6].

**Lemma 5.1.1.** Suppose that $T: E \to E$ is a (not necessarily fibred) pointed endofunctor over the fibration $p: E \to B$. Write $U: T-\text{Alg} \to E$ for the forgetful functor. Then for every cartesian map $f: X \to U(Y)$ in $E$ there is a unique cartesian map $g$ in $T-\text{Alg}$ such that $U(g) = f$ and $\text{cod} g = Z$.

Consequently, $p \circ U$ is a fibration, and $U$ is a fibred functor, regardless of whether or not $T$ is fibred.

**Proof.** Let $f: X \to Y$ be cartesian in $E$ and let $\alpha: TY \to Y$ be a $T$-algebra on $Y$. Then we have the following diagram.

$$
\begin{array}{ccc}
T(X) & \xrightarrow{T(f)} & TY \\
\downarrow & & \downarrow \alpha \\
X & \xrightarrow{f} & Y
\end{array}
$$

Using the fact that $f$ is cartesian, there is a unique vertical map $T(X) \to X$ making a commutative square. It is straightforward to check that this satisfies the unit law, and so is a $T$-algebra structure on $X$ and that the resulting morphism of $T$-algebras is cartesian in $T-\text{Alg}$.

**Lemma 5.1.2.** Suppose we are given a fibred functor $G: F \to E$ over $B$. Suppose further that we are given a (not necessarily fibred) choice of initial object of $(X \downarrow G_I)$ for each $I \in B$ and each $X \in E_I$ (where $G_I$ is the restriction $F_I \to E_I$). Then we have a (not necessarily fibred) adjunction $F \dashv G$.

Furthermore, if the initial object at $X$ is $(FX, X \xrightarrow{\eta_X} GFX)$, then these form the action on objects of the left adjoint and the unit of the adjunction respectively.

**Proof.** This is a straightforward variation on the standard result for adjunctions.

**Remark 5.1.3.** It is necessary in lemma 5.1.2 to assume that $G$ is fibred, since this is the case for any adjunction by remark [4.3.2]. If we do have a fibred choice of initial object then the same construction clearly gives us a fibred left adjoint (see [21, Proposition 2.2.2]).

**Definition 5.1.4.** Let $T$ be a pointed endofunctor over a fibration $p: E \to B$. A monad $M$ over $p$ is algebraically free on $T$ if there is a morphism of pointed endofunctors $\xi: T \to M$ such that $\xi$ is levelwise vertical and the canonical map $\xi: M-\text{Alg} \to T-\text{Alg}$ is an isomorphism.
Proposition 5.1.5. Let $T$ be a pointed endofunctor over a fibration $p: E \to B$. If the forgetful functor $U: T\text{-Alg} \to E$ has a left adjoint $F$ over $p$, then the monad $UF$ over $p$ is algebraically free on $T$.

Proof. Except for the requirement that we are working over $B$, the result is exactly the one proved by Kelly in [16, Theorem 22.3].

However, it is easy to check that if the adjunction $F \dashv U$ is an adjunction over $p$ then $UF$ is a monad over $p$, and that the map constructed by Kelly is levelwise vertical.

Lemma 5.1.6. Suppose that $T, \eta$ is a (not necessarily fibred) pointed endofunctor over $p: E \to B$. Suppose further that we are given a choice of initial object of $(X \downarrow U_I)$ for each $I \in B$ and $X \in E_I$ (where $U_I$ is the forgetful functor, restricted to the fibre of $I$). Then we can construct a monad algebraically free on $T$.

Proof. The basic idea is the follow [6, Section 6.1] with a few modifications.

First, note that by lemma 5.1.1, $U$ is a fibred functor. We can therefore apply lemma 5.1.2 to construct a left adjoint $F$ to $U$ over $B$. Note that this is in particular an ordinary adjunction $F \dashv U$. Furthermore, note that forgetful functors $U$ from algebras over a pointed endofunctor always create coequalizers for $U$-split pairs. Hence we can apply (the ordinary version of) Beck’s monadicity theorem to show that the canonical functor $T\text{-Alg} \to UF\text{-Alg}$ is an isomorphism, and so the monad $UF$ is algebraically free on $T$.

Lemma 5.1.7. Suppose that $T, \eta$ is a (not necessarily fibred) pointed endofunctor over $p: E \to B$. Then algebraically free monads on $T$ are unique up to isomorphism.

(In particular every algebraically free monad on $T$ is isomorphic to the one constructed in lemma 5.1.6)

Proof. Let $(M, \mu)$ be a monad and $\xi: T \to M$ a morphism of pointed endofunctors witnessing that $M$ is algebraically free on $T$. For each $I$, the forgetful functor $M\text{-Alg}_I \to E_I$ has a left adjoint, which sends $X$ to $MX$ together with the $M$-algebra structure $\mu_X: M^2X \to MX$. Composing with the isomorphism $T\text{-Alg} \cong M\text{-Alg}$ gives a left adjoint to the forgetful functor $T\text{-Alg}_I \to E_I$. The result now follows from uniqueness of adjoints.

Lemma 5.1.8. Suppose that $T, \eta$ is a fibred pointed endofunctor and $p: E \to B$ has all dependent products (which do not need to satisfy Beck-Chevalley). Then the algebraically free monad on $T$ is fibred.

Proof. Let $\sigma: I \to J$ in $B$. Then we have seen that the reindexing functor $\sigma^*: E_J \to E_I$ lifts to a functor $\sigma^*: (T\text{-Alg})_J \to (T\text{-Alg})_I$. However, when $T$ is fibred the dependent product $\Pi_\sigma: E_I \to E_J$ also lifts to a functor $(T\text{-Alg})_I \to (T\text{-Alg})_J$, and this is right adjoint to $\sigma^*$ (see [14, Lemma B1.4.15(i)] for the result when $T$ has the structure of a monad and note the same proof holds for pointed endofunctors).

Then we can lift this to get an adjunction between $(Y \downarrow U_I)$ and $(\sigma^*(Y) \downarrow U_I)$ for $Y \in E_J$. Since $\sigma^*: (Y \downarrow U_J) \to (\sigma^*(Y) \downarrow U_I)$ is a left adjoint it preserves colimits and in particular initial objects. Hence any choice of initial objects is fibred, and so the $F$ constructed using such initial objects is also fibred.
5.2 Algebraic Weak Factorisation Systems

We now the definition of algebraic weak factorisation over a fibration. This is a generalisation of the well known definition due to Grandis and Tholen \[10\] (originally referred to as natural weak factorisation system).

**Definition 5.2.1.** An algebraic weak factorisation system over \( p \) (awfs) is a functorial factorisation over \( p \) together with a both a vertical natural transformation \( \Sigma: L \Rightarrow L^2 \) making \( L \) into a comonad over \( \text{dom} \): \( V(\mathbb{E}) \rightarrow \mathbb{E} \) and a vertical transformation \( \Pi: R^2 \rightarrow R \) making \( R \) into a monad over \( \mathbb{B} \), and the canonical natural transformation \( LR \Rightarrow RL \) is a distributive law (see e.g. \[24, Definition 4.12\] for an elaboration of this part of the definition).

**Definition 5.2.2.** An awfs is (strongly) fibred if the underlying functorial factorisation is (strongly) fibred.

5.3 Algebraically Free (R)awfs’s

**Definition 5.3.1.** A morphism of functorial factorisations over \( B \) is a vertical natural transformation \( \xi \) between the functors \( V(\mathbb{E}) \rightarrow \mathbb{E} \). A morphism of lawfs’s is a morphism of functorial factorisations that also respects the comultiplication. A morphism of awfs’s is a morphism of lawfs’s that also respects the multiplications.

We define morphisms of rawfs’s dually to those of lawfs’s. We write the corresponding categories as \( \text{FF}(p), \text{LAWFS}(p), \text{AWFS}(p) \) and \( \text{RAWFS}(p) \) respectively.

**Proposition 5.3.2.** Any morphism of lawfs’s \( \xi: (L, R) \Rightarrow (L', R') \) induces a functor \( R'-\text{Alg} \rightarrow R-\text{Alg} \) over \( B \).

**Definition 5.3.3.** Let \( (L_0, R_0) \) be a functorial factorisation over a fibration \( p \). An rawfs \( (L, R) \) over \( p \) is algebraically free on \( (L_0, R_0) \) if the corresponding monad over \( p \) is algebraically free on the corresponding pointed endofunctor.

**Definition 5.3.4.** Let \( (L, R, \Sigma) \) be an lawfs over \( B \). An algebraically free awfs on \( (L, R, \Sigma) \) is an awfs \( (L', R', \Sigma', \Pi) \) over \( B \) together with a morphism of lawfs’s \( \xi: (L, R, \Sigma) \Rightarrow (L', R', \Sigma') \) such that the composition of the functor \( R'-\text{Alg} \rightarrow R-\text{Alg} \) with the forgetful functor \( (R', \lambda, \Pi)-\text{Alg} \rightarrow (R', \lambda)-\text{Alg} \) is an isomorphism.

**Definition 5.3.5.** Let \( m \) be a vertical map over \( p \). We say that an awfs \( (L, R) \) is cofibrantly generated by \( m \) if it is algebraically free on step-one.

**Theorem 5.3.6.** Suppose that we are given a functorial factorisation \( (L_0, R_0) \). Write \( U: R_0-\text{Alg} \rightarrow V(\mathbb{E}) \) for the forgetful functor. Suppose we are given an initial object of \( f \downarrow U_Y \) for every object \( Y \) of \( \mathbb{E} \), and every vertical map \( f: X \rightarrow Y \). Then there is an rawfs \( (L, R) \) algebraically free on \( (L_0, R_0) \).

**Proof.** This is a special case of lemma \[5.1.6\].

**Theorem 5.3.7.** Suppose that \( p: \mathbb{E} \rightarrow \mathbb{B} \) is a fibration with dependent products (right adjoints to reindexing functors) and \( (L_0, R_0) \) is a fibred functorial factorisation. If the algebraically free rawfs on \( (L_0, R_0) \) exists (say \( (L, R) \)), then it is also fibred.
Proof. First note that the fibration \( p \circ \text{cod}: V(\mathcal{E}) \to \mathcal{B} \) also has dependent products, which are just defined levelwise. Recall that an rawfs \((L, R)\) is algebraically free on a functorial factorisation \((L_0, R_0)\) if \(R\) is algebraically free on \(R_0\) over \(E\). However, this implies that \(R\) is also algebraically free on \(R_0\) over \(B\). (In fact the algebras over \(B\) are the same as the corresponding algebras over \(E\) by remark 4.3.8.)

Therefore we can apply lemma 5.1.8 to show that \(R\) is fibred.

Theorem 5.3.8. Suppose that \(p: \mathcal{E} \to \mathcal{B}\) is a fibration with dependent products (right adjoints to reindexing functors), that each fibre category \(\mathcal{E}_I\) is locally cartesian closed and \((L_0, R_0)\) is a strongly fibred functorial factorisation. If the algebraically free rawfs on \((L_0, R_0)\) exists (say \((L, R)\)), then it is also strongly fibred.

Proof. We can construct dependent products for \(\text{cod}: V(\mathcal{E}) \to \mathcal{E}\) using the dependent products for \(p\) and the dependent products in the fibre categories as follows. We need to show that for every map \(u: X \to Y\) in \(\mathcal{E}\), the reindexing map \(u^*\) has a right adjoint. However, we know by the characterisation of cartesian maps in lemma 4.2.6 that this is the same as reindexing along \(p(u)\) over \(p \circ \text{cod}\), followed by pulling back along the vertical component of \(u\). The former has a right adjoint by the same argument as in theorem 5.3.7 and the latter has a right adjoint by the existence of dependent products. Hence each \(u^*\) has a right adjoint by the composition of adjoints.

Therefore we can apply lemma 5.1.8 to show \(R\) is fibred over \(\text{cod}\), and so \((L, R)\) is strongly fibred.

5.4 Criteria for the Existence of Algebraically Free Awfs’s

We now use the observation by Garner that awfs’s can be characterised in terms of double categories. We will follow the description by Bourke and Garner in \([4, \text{Section 2 and 3}]\) and by Riehl in \([24, \text{Section 2.5 and Section 6.2}]\).

The result is essentially that for a given rawfs \(R\), the additional structure making it an awfs corresponds precisely to a natural composition operation, assigning an \(R\)-algebra structure to \(g \circ f\) given \(R\)-algebra structures on \(f\) and \(g\). We express this composition operation as an extension of the category \(R\)-Alg to a double category.

In this paper we adapt the results as follows. First of all, we are of course working over a fibration rather than a category. Secondly, the descriptions in \([24]\) and \([4]\) consider awfs’s over different base categories, which requires a more sophisticated notion of morphism (so called lax morphisms of awfs’s). On the other hand, we fix a single fibration throughout, since that is all we need here. Finally, we make a minor observation that does not appear in those papers. If instead of an rawfs we are given a functorial factorisation, then one direction of the correspondence still holds. Namely, given an awfs we can produce a composition operation on the \(R\)-algebra structures, where \(R\) is now just a pointed endofunctor rather than a monad.

Note that \(V(\mathcal{E})\) has the structure of a double category by taking vertical maps to be vertical maps in \(\mathcal{E}\), taking squares to be squares in \(\mathcal{E}\) (with left and right maps vertical), and taking composition to be composition in \(\mathcal{E}\) in each direction. (Note that we chose to orientate the double category so that vertical
Lemma 5.4.1. If \( L \) is an awfs, and \( R \) is the corresponding monad, then \( R \)-algebras can be composed naturally. Formally, there is a double category \( T \) whose vertical maps are \( R \)-algebras, whose squares are morphisms of \( R \)-algebras such that the forgetful functor \( R \text{-Alg} \to V(\mathbb{E}) \) is a double functor (i.e. it preserves the vertical composition).

Proof. The usual proof generalises to fibrations without modification. See e.g. [24, Definition 2.21 and lemma 2.22].

Lemma 5.4.2. If \((L, R)\) is an lawfs, then \( R \)-algebras can be composed naturally. Formally, there is a double category \( T \) whose vertical maps are \( R \)-algebras (where these are now just pointed endofunctor algebras), whose squares are morphisms of \( R \)-algebras such that the forgetful functor \( R \text{-Alg} \to V(\mathbb{E}) \) is a double functor (i.e. it preserves the vertical composition).

Proof. Observe that the explicit description in [24, Definition 2.21] makes no use of the multiplication on \( R \). A lengthy but straightforward diagram chase verifies that this gives a functorial composition operation for \( R \)-algebras using only the lawfs axioms. The generalisation to fibrations is again straightforward.

Lemma 5.4.3. If \( R \) is an rawfs, and \( R \)-algebras can be composed naturally, then the corresponding copointed endofunctor \( L \) can be given the structure of a comonad (and so \((L, R)\) the structure of an awfs).

Proof. Again the usual proof generalises easily to fibrations. See e.g. [24, Theorem 2.24] or [4, Proposition 4].

Lemma 5.4.4. Suppose that we are given a lawfs \((L_1, R_1)\) and an awfs \((L, R)\) together with a morphism of pointed endofunctors \( \xi: (L_1, R_1) \to (L, R) \). Suppose that the corresponding map \( \bar{\xi}: R \text{-Alg} \to R_1 \text{-Alg} \) preserves vertical composition (i.e. it is a double functor). Then \( \xi \) respects comultiplication, and so is a morphism of lawfs’s.

Proof. In the second half of [24, Theorem 6.9], Riehl gives an explicit proof of this when \((L_1, R_1)\) is also an awfs. However, the multiplication on \((L_1, R_1)\) is never needed and in fact the same proof applies when \((L_1, R_1)\) is just an lawfs (and once again the proof generalises to a fibration without problems).}

We say a functor \( U: T \to V(\mathbb{E}) \) is monadic over \( \mathbb{E} \) if it is isomorphic to \( T \text{-Alg} \) where \( T \) over \( \text{cod}: V(\mathbb{E}) \to \mathbb{E} \). We then get the following.

Theorem 5.4.5. Let \((L_1, R_1)\) be an lawfs. Then there is an awfs \((L, R)\) algebraically free on \((L_1, R_1)\) if and only if the forgetful functor \( U: R_1 \text{-Alg} \to V(\mathbb{E}) \) is monadic over \( \text{cod}: V(\mathbb{E}) \to \mathbb{E} \).

Proof. By proposition 5.1.3 there is a monad \( R \) over \( \text{cod}^{op} \) and a morphism of pointed endofunctors \( \xi: R_1 \to R \) such that the corresponding map \( \bar{\xi}: R \text{-Alg} \to R_1 \text{-Alg} \) is an isomorphism. However, we can also view \( R \) as an rawfs \((L, R)\) and \( \bar{\xi} \) as a morphism of functorial factorisations \((L_1, R_1) \to (L, R)\).

We saw in lemma 5.4.2 that we can extend \( T \text{-Alg} \) to make a double category such that the forgetful functor to \( V(\mathbb{E}) \) is a double functor. However, \( \bar{\xi} \) is
an isomorphism between $R_1$-$Alg$ and $R$-$Alg$ over $E$. Hence we can also give $R$-$Alg$ the structure of a double category by passing back and forth across the isomorphism. We then use lemma 5.4.3 to assign a comultiplication to $R$, making it an awfs. By definition vertical composition is preserved by $\xi$, and so by lemma 5.4.4, $\xi$ must preserve comultiplication, making it an lawfs morphism $(L_1, R_1) \to (L, R)$. But we can now deduce that $(L, R)$ is algebraically free on $(L_1, R_1)$.

**Theorem 5.4.6.** Let $(L_1, R_1)$ be an lawfs. Write $U: R_1$-$Alg \to V(E)$ for the forgetful functor. Suppose we are given an initial object of $(f \downarrow U_Y)$ for every object $Y$ of $E$, and every vertical map $f: X \to Y$. Then there exists an algebraically free awfs on $(L_1, R_1)$.

**Proof.** By lemma 5.1.6 and theorem 5.4.5.

**Corollary 5.4.7.** Let $p: E \to B$ be a locally small bifibration. Fix a vertical map $m$. Suppose that for every $Y \in E$ and every vertical map $f: X \to Y$, the following pointed endofunctor on $E/Y$ has an initial algebra.

Send $g: Z \to Y$ to $(f, \rho_g): X \amalg R_1 g \to Y$, where $R_1$ is given by step-one. The point of the pointed endofunctor, at $g$ is the composition $Z \xrightarrow{\lambda g} R g \hookrightarrow X \amalg R g$.

Then the awfs cofibrantly generated by $m$ exists.

**Proof.** One can check that for each $Y \in E$ and each vertical map $f$, the category of algebras for the pointed endofunctor defined is isomorphic to $(f \downarrow U_Y)$. The result now follows from theorem 5.4.6.

### 5.5 Fibred and Strongly Fibred Algebraically Free Awfs’s

**Theorem 5.5.1.** Suppose that $p: E \to B$ is a fibration with dependent products (right adjoints to reindexing functors) and $(L_1, R_1)$ is a fibred lawfs. If the algebraically free awfs on $(L_1, R_1)$ exists (say $(L, R)$), then it is also fibred.

**Proof.** Same as theorem 5.3.7

**Theorem 5.5.2.** Suppose that $p: E \to B$ is a fibration with dependent products (right adjoints to reindexing functors), that each fibre category $E_I$ is locally cartesian closed and $(L_1, R_1)$ is a strongly fibred lawfs. If the algebraically free awfs on $(L_1, R_1)$ exists (say $(L, R)$), then it is also strongly fibred.

**Proof.** Same as theorem 5.3.8

### 6 Fibred Leibniz Construction

#### 6.1 Review of Monoidal Fibrations

We recall the following from [27, Section 12 and 13].

**Definition 6.1.1** (Shulman). A monoidal fibration is a Grothendieck fibration $p: E \to B$ together with monoidal products on $E$ and $B$ such that $p$ strictly preserves monoidal products and the monoidal product $\otimes$ on $E$ preserves cartesian maps.

A monoidal fibration is cartesian if ($B$ has products and) the monoidal product on $B$ is cartesian product $\times$. 

31
Proposition 6.1.2 (Shulman). Let \( p : E \to B \) be a cartesian monoidal fibration. Then for each \( I \in B \) we can define a monoidal product \( \otimes_I \) in each fibre category, and these are preserved (up to natural isomorphism) by reindexing. (Moreover this forms part of an equivalence of 2-categories between cartesian fibrations on \( B \) and pseudofunctors from \( B^{op} \) to monoidal categories - see [27, Theorem 12.7] for details.)

Example 6.1.3. If \( C \) is a category with pullbacks, then we can define a cartesian fibred monoidal product on the codomain fibration. We define the monoidal product on \( C^2 \) to just be the cartesian product on \( C^2 \), or in other words pointwise the cartesian product on \( C \). The product in each fibre category \( C/I \) is then just the cartesian product in \( C/I \), which is just pullback in \( C \).

Example 6.1.4. Let \( C \) be any category and \( \otimes \) a monoidal product on \( C \). We lift \( \otimes \) to a fibred monoidal product on the category indexed families fibration \( \text{Fan}(C) \to \text{Cat} \). Given \( X : A \to C \) and \( Y : B \to C \) we define \( X \otimes Y \) pointwise. That is, we define \( X \otimes Y : A \times B \to C \) by \( (X \otimes Y)(A, B) := X(A) \otimes Y(B) \). The resulting monoidal product \( \otimes_A \) in each fibre category \( [A, C] \) is of course also just defined pointwise.

Definition 6.1.5 (Shulman). Let \( p : E \to B \) be a cartesian monoidal fibration. We say that \( p \) is internally closed if each fibre \( E_I \) is closed monoidal (i.e. \( - \otimes I X \) and \( X \otimes I - \) have right adjoints for all \( X \)), and reindexing is closed monoidal (i.e. preserves these right adjoints).

Example 6.1.6. If \( C \) is locally cartesian closed then the codomain fibration with monoidal structure from example 6.1.6 is internally monoidal closed.

We will also use the following proposition.

Proposition 6.1.7. Let \( p: E \to B \) be an internally closed cartesian monoidal fibration. Suppose further that \( B \) has a terminal object, and that \( X \) is an object of \( E_1 \). For \( I \in B \), write \( \{Y, -\}_I \) for the right adjoint to the functor \( Y \otimes I - \).

Then there is an adjunction from \( E \) to \( E \) over \( B \) defined as follows. The left adjoint \( F \) is defined by \( F(Y) := I^*(X) \otimes_I Y \) for \( Y \in E_I \). The right adjoint \( G \) is defined by \( G(Z) := \{J^*(X), Z\} \) for \( Z \in E_J \).

Proof. It is straightforward to check that the definition given is an adjunction over \( B \). \( \square \)

6.2 Definition and Existence of the Fibred Leibniz Construction

The Leibniz construction is a standard construction in homotopical algebra (see e.g. [25, Construction 11.1.7]), which takes a monoidal closed structure on a category \( C \), and produces a monoidal closed structure on the arrow category \( C^2 \). It was applied to the semantics of homotopy type theory by Gambino and Sattler in [7] who showed how to construct dependent products for certain awfs’s cofibrantly generated by maps defined using pushout product, including as a special case the Kan fibrations in the CCHM cubical set model (from [5]).

In this section we show how to extend the Leibniz construction to monoidal fibrations, which will be used later in some of our examples.
Definition 6.2.1. Let \((\mathcal{C}, \otimes)\) be a monoidal category with pushouts. The pushout product is the monoidal product \(\hat{\otimes}\) defined on \(\mathcal{C}^2\) as follows. Given \(f: U \rightarrow V\) and \(g: X \rightarrow Y\), we define \(f \hat{\otimes} g\) as the map given by the universal property of the pushout below.

\[
\begin{array}{c}
U \otimes X \\
\downarrow U \otimes g \\
U \otimes Y \\
\downarrow f \otimes g \\
V \otimes Y
\end{array}
\rightarrow
\begin{array}{c}
V \otimes X \\
\downarrow V \otimes g \\
\phantom{V} \\
\phantom{f} \\
\phantom{\otimes g} \\
\phantom{U} \\
\phantom{g} \\
\phantom{\otimes} \\
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Proposition 6.2.2. Suppose we are given a monoidal fibration \(p: E \rightarrow B\) with fibred pushouts. Then pushout product restricts to a monoidal product on \(V(E)\) and this makes \(V(E) \rightarrow B\) into a monoidal fibration with the same monoidal product on \(B\).

Proof. First note that all the objects and maps in the pushout in the definition of pushout product lie in the fibre of \(p(f) \otimes p(g)\), and so \(f \hat{\otimes} g\) does too. Then using the fact that \(\otimes\) and pushouts preserve cartesian maps, pushout product must too. But this is enough to show we have a monoidal fibration.

Definition 6.2.4. Let \((\mathcal{C}, \otimes)\) be a monoidal category with pushouts and pullbacks. Suppose that for each \(X\), \(X \otimes -\) has a right adjoint \(\{X, -\}\) (referred to as cotensor). Then for each map \(f\), \(f \hat{\otimes} -\) has a right adjoint, \(\hat{\{f, -\}}\) referred to as pullback cotensor, which is defined explicitly as the map given by the universal property of the pullback below. Let \(f: U \rightarrow V\) and \(g: X \rightarrow Y\).

Diagram 6.2.4
We similarly can define a right adjoint to $- \hat{\otimes} f$ referred to as pullback hom.

**Proposition 6.2.5.** Suppose we are given an internally closed monoidal fibration $p: E \to B$ with fibred pushouts. Then $V(E) \to B$ is a internally closed monoidal fibration.

**Proof.** Construct the pullback cotensor in each fibre category $E_I$ to get a right adjoint to $f \hat{\otimes} I -$, which we'll write as $\{ f, - \}_I$. This is fibred since cotensor and pullback are both fibred.

**Proposition 6.2.6.** Let $p: E \to B$ be a monoidal closed fibration where $B$ has a terminal object. Suppose that $m_0$ is a vertical map $E$ in the fibre of the terminal object of $B$. Let $m$ be vertical over $I$ and $f$ vertical over $J$.

Solutions of the universal lifting problem from $I^*(m_0) \hat{\otimes} I m$ to $f$ correspond precisely to solutions of the universal lifting problem from $m$ to $\{ m_0, f \}$.

**Proof.** We apply the adjunction constructed in proposition 6.1.7 to pushout product and pullback cotensor. The result then follows from proposition 6.2.3 and the characterisation of adjoints of hom objects in lemma 4.3.3.

7 Examples

We now give several examples of Grothendieck fibrations together with explanations of what the general constructions look like in each instance.

A theme throughout these examples is that our general construction was defined in terms of hom objects and opcartesian maps, which are both unique up to isomorphism. Therefore we can characterise what step-one of the small argument looks like in each fibration by asking what are the hom objects and what are the opcartesian maps.

For example, we will see that we recover a definition due to Garner by applying our construction to fibrations of category indexed families on a category $C$. In these fibrations, opcartesian maps are described explicitly as left Kan extensions, which require cocompleteness of $C$ to construct. This gives an explanation for why cocompleteness of $C$ plays an important role in Garner’s small object argument, even for the relatively simple step-one part.

In contrast we will also look at codomain fibrations on a category $C$. In this case the opcartesian maps are simply given by composition, and so infinite colimits are not required for step-one. On the other hand hom objects are now given by local exponentials in slice categories, and so local cartesian closedness is a necessary condition for step-one.

7.1 Trivial Fibrations

Let $C$ be a category. Then the unique functor $C \to 1$ is a fibration. In this case a family of lifting problems is just a single lifting problem and a choice of diagonal fillers is just a diagonal filler. These fibrations are not locally small and in fact universal lifting problems do not exist.
7.2 Set Indexed Families

Set indexed families are the simplest nontrivial examples of Grothendieck fibrations that we will consider. Although they are simple, we can use them to illustrate the ideas that will turn up again in other definitions. We will give a fairly brief descriptions. See e.g. [12] for a more in depth reference on set indexed families fibrations.

Recall the definition of set indexed families fibrations from example 2.1.12. A family of lifting problems from \((F_i: U_i \to V_i)_{i \in I}\) to \((G_j: X_j \to Y_j)_{j \in J}\) consists of a set \(K\), together with maps \(f: K \to I\) and \(g: K \to J\) and for each \(k \in K\), a commutative square in \(C\) of the following form:

\[
\begin{array}{ccc}
U_{f(k)} & \xrightarrow{} & X_{g(k)} \\
\downarrow F_{f(k)} & & \downarrow G_{g(k)} \\
V_{f(k)} & \xrightarrow{} & Y_{g(k)}
\end{array}
\] (7.1)

A solution consists of a choice of diagonal filler for each such square.

If \(C\) is locally small, then so is \(p\), with \(\text{Hom}_{X,Y}(X,Y)\) given by the disjoint union of sets \(\coprod_{i \in I} \text{hom}(X_i,Y_i)\).

Hence the universal family of lifting problems is defined as follows. The indexing set (up to isomorphism) consists of triples \(\langle i, j, S \rangle\) where \(i \in I\), \(j \in J\) and \(S\) is a commutative square with left side equal to \(F_i\) and right side equal to \(G_j\). The square in \(C\) indexed at \(\langle i, j, S \rangle\) is just \(S\) itself.

7.3 Category Indexed Families

Recall from example 2.1.13 that \(p\) from the previous section can be extended to a fibration \(\text{Fam}(C) \to \text{Cat}\). An object of the larger \(\text{Fam}(C)\) consists of a small category \(C \in \text{Cat}\) together with a functor \(X: C \to C^{\times 2}\).

A vertical morphism over \(C\) is just an object of \([2, [C, C]]\), but since \([2, [C, C]] \cong [C \times 2, C] \cong [C, [2, C]]\), we can instead think of it as a functor from \(C\) to \(C^{\times 2}\).

A family of lifting problems from \(F: C \to C^{\times 2}\) to \(G: C \to C^{\times 2}\) consists of a commutative square for each object of the indexing category together with commutative cubes for each morphism. A choice of fillers consists of a choice of filler for each commutative square such that the resulting “diagonal squares” across each cube commute. We recover in this way Garner’s notion of lifting problem for \(C\) from [8].

\(p: \text{Fam}(C) \to \text{Cat}\) is locally small if \(C\) is locally small (in the usual category theoretic sense). In fact, given category indexed families \(X: A \to C\) and \(Y: B \to C\), \(\text{Hom}(X,Y)\) is simply the comma category \((X \downarrow Y)\). It is a bifibration if and only if \(C\) is cocomplete, with the opcartesian maps given by left Kan extensions. We will see that although the Beck-Chevalley condition does not hold in general for category indexed families we can still show that step-one is fibred.

We now work towards a proof that Garner’s definition of step-one of the small object argument is the same as the result of applying the general framework here to \(\text{Fam}(C)\). As a corollary of this we get an interesting new insight into Garner’s definition. The original definition of step-one is split into two pieces, first taking a certain colimit, and then a pushout. On the other hand in the new definition here, step-one is defined as a single opcartesian lift, or even just a single pushout.
in Fam(C). The key to seeing the link between the two definitions is lemma 7.3.4, where we proved that the single opcartesian lift over dom: V(Fam(C)) \to Fam(C) is equivalent to an opcartesian lift over Fam(C) \to \textbf{Cat}, followed by a pushout. Then for the special case where f is a vertical map over 1, we are taking the opcartesian lift along a map A \to 1, which is a colimit over a diagram of shape \mathcal{A}. This is now indeed of the same form as Garner's definition, although we still need to check that it is the same colimit, which will appear in lemma 7.3.4.

**Lemma 7.3.1.** Let \mathcal{C} be a locally small category. Let \mathcal{A}, \mathcal{B} and \mathcal{B}' be small categories. Suppose we are given \mathcal{X}, \mathcal{Y} and \mathcal{Y}' over \mathcal{A}, \mathcal{B}' and \mathcal{B} respectively. Suppose further that we have a map \chi: \mathcal{B}' \to \mathcal{B}. Write \pi_0 for the projection (\mathcal{X} \downarrow \mathcal{Y}) \to \mathcal{A} and \pi_1 for the projection (\mathcal{X} \downarrow \mathcal{Y}) \to \mathcal{B}. Write (\mathcal{X} \downarrow \chi) for the canonical map (\mathcal{X} \downarrow \chi^{*}(\mathcal{Y})) \to (\mathcal{X} \downarrow \mathcal{Y}) and \pi'_1 for the projection (\mathcal{X} \downarrow \chi^{*}(\mathcal{Y})) \to \mathcal{B}'. Then the canonical morphism \prod_{\pi'_1}(\mathcal{X} \downarrow \chi)^{*} \to \chi^{*}\prod_{\pi_1} is an isomorphism.

**Proof.** In general if we are given a pullback square where the lower map is an opfibration then Beck-Chevalley holds for that square. (This appears to be a folklore result, see e.g. [19] for a proof.) However it is easy to check that the projection (\mathcal{X} \downarrow \mathcal{Y}) \to \mathcal{B} is an opfibration.

**Lemma 7.3.2.** Step-one of the small object argument is fibred for category indexed families fibrations.

**Proof.** Note that V(Fam(C)) is isomorphic to Fam(C^2) over \textbf{Cat}, so is itself a category indexed family fibration. We then apply lemma 7.3.1 not to \mathcal{C} itself, but to C^2. We then note that for the proof of theorem 4.6.4 to hold, we don’t need the entire Beck-Chevalley condition, but only certain instances that are precisely covered by this case.

**Lemma 7.3.3.** Let \mathcal{C} be a category. Suppose that we are given an endofunctor \mathcal{T}_1: \mathcal{C} \to \mathcal{C}. Then there is a fibred endofunctor \mathcal{T} over Fam(\mathcal{C}) \to \textbf{Cat} with the property that \mathcal{T}(\mathcal{X}) = \mathcal{T}_1(\mathcal{X}) whenever \mathcal{X} \in \mathcal{C} \cong \{1, \mathcal{C}\} and \mathcal{T} is unique up to isomorphism. Furthermore, we can ensure that \mathcal{T} strictly preserves reindexing (i.e. that for all \sigma and \mathcal{X} we have \mathcal{T}(\sigma^{*}(\mathcal{X})) = \sigma^{*}(\mathcal{T}(\mathcal{X})) and \mathcal{T}(\sigma(\mathcal{X})) = \sigma(T(\mathcal{X}))), and is unique with this property.

**Proof.** We simply define \mathcal{T} pointwise. That is, given \mathcal{X} \in \mathcal{A}, \mathcal{C} we define \mathcal{T}(\mathcal{X}) = \mathcal{T}_1 \circ \mathcal{X}. It is easy to see that this extends to an endofunctor over Fam(\mathcal{C}) \to \mathcal{C}. We recall that we defined reindexing as composition, and it is clear that this is strictly preserved by \mathcal{T}.

It remains to show uniqueness up to isomorphism. Strict uniqueness when \mathcal{T} strictly preserves reindexing is similar but easier.

For \mathcal{Z} \in \mathcal{C}, write \overline{\mathcal{Z}} for the corresponding element of Fam(\mathcal{C}) over the fibre of 1 and for a morphism \mathcal{f} in \mathcal{C} write \overline{\mathcal{f}} for the corresponding vertical map over 1. Let \mathcal{T} be an endofunctor over Fam(\mathcal{C}) such that for all \mathcal{Z} in \mathcal{C} we have \mathcal{T}(\overline{\mathcal{Z}}) = \overline{\mathcal{T}(\mathcal{Z})} and for all \mathcal{f} in \mathcal{C} we have \mathcal{T}(\overline{\mathcal{f}}) = \overline{\mathcal{T}(\mathcal{f})}. Then to show that \mathcal{T} is isomorphic to the endofunctor described above, it suffices to find for each \mathcal{X}: \mathcal{A} \to \mathcal{C} in Fam(\mathcal{C}) and each \mathcal{A} \in \mathcal{A}, an isomorphism \alpha_{\mathcal{X}}^{\mathcal{A}}: \mathcal{T}(\mathcal{X}(\mathcal{A})) \to (\mathcal{T}(\mathcal{X}))(\mathcal{A}) such that \alpha_{\mathcal{X}}^{\mathcal{A}} is natural in both \mathcal{A} and \mathcal{X}.
Note that for any $X: A \to \mathbb{C}$ in $\text{Fam}(\mathbb{C})$ and for any object $A$ of $A$, we have a canonical cartesian map $X(A) \to X$ over the map $\gamma A: 1 \to A$. Applying $T$ gives us a cartesian map $T(X(A)) \to TX$ over $\gamma A$. However, such a map is exactly a map from the underlying object in $\mathbb{C}$ of $T(X(A))$ to $(TX)(A)$, which is an isomorphism, by cartesianness. We take this to be $\alpha^X_A$.

Naturality in $X$ is straightforward, but naturality in $A$ is more difficult, so we now give a proof. Let $\sigma: A \to B$ in $A$. We need to verify that the following diagram commutes.

\[
\begin{array}{ccc}
T(X(A)) & \to & T(X(B)) \\
\alpha_X^A & \downarrow & \alpha_X^B \\
(TX)(A) & \to & (TX)(B)
\end{array}
\] (7.2)

The issue that we need to deal with is that, for $\sigma: A \to B$ in $\mathbb{C}$, the following diagram in $\text{Fam}(\mathbb{C})$ does not commute, since its image in $\text{Cat}$ does not commute.

\[
\begin{array}{ccc}
X(A) & \to & X \\
\alpha_X^A & \downarrow & \\
X(\sigma) & \to & X(B)
\end{array}
\]

For any map $g$ in $\mathbb{C}$, write $\tilde{g}$ for the corresponding object in $\text{Fam}(\mathbb{C})$ in the fibre of 2. Then similarly to before, we have for each $\sigma: A \to B$ in $A$ a canonical cartesian map $\tilde{X}(\sigma) \to X$ over the map $\gamma \sigma: 2 \to A$. As before, we have a canonical cartesian maps $T(\tilde{X}(\sigma)) \to TX$, which we'll write as $\beta_\sigma$. We can view $\beta_\sigma$ as a commutative square in $\mathbb{C}$ of the following form.

\[
\begin{array}{ccc}
\beta_{\sigma,0} & \downarrow & (TX)(A) \\
T(\tilde{X}(\sigma)) & \downarrow & (TX)(\sigma) \\
\beta_{\sigma,1} & \downarrow & (TX)(B)
\end{array}
\] (7.3)

Note that $\alpha^X_A$ factors through $\beta_{\sigma,0}$ and $\alpha^X_B$ factors through $\beta_{\sigma,1}$, by applying $T$ to the appropriate commutative diagram, which gives us the following, where each $\gamma_{\sigma,i}$ lies over the corresponding map $1 \to 2$ in $\text{Cat}$.

\[
\begin{array}{cc}
T(X(A)) & T(X(B)) \\
\gamma_{\sigma,0} & \beta_{\sigma,0} \\
\alpha^X_A & \alpha^X_B \\
\gamma_{\sigma,1} & \beta_{\sigma,1}
\end{array}
\] (7.4)

If we consider the special case where $A = B$ and $\sigma = 1_A$, then we have the
following.

\[
\beta_{1A,1} \circ T(1_{\overline{\sigma}(A)}) \circ \gamma_{1A,0} = \beta_{1A,0} \circ \gamma_{1A,0} = \alpha_{A}^X = \beta_{1A,1} \circ \gamma_{1A,1}
\]

Since \( \beta_{1A,1} \) is an isomorphism, we deduce that the following diagram commutes.

\[
\begin{array}{ccc}
T(X(A)) & \xrightarrow{\gamma_{1A,0}} & T(1_{\overline{\sigma}(A)}) \\
\downarrow{\gamma_{1A,1}} & & \downarrow{\gamma_{1A,1}} \\
T(1_{\overline{\sigma}(A)}) & & T(1_{\overline{\sigma}(A)})
\end{array}
\]

(7.5)

Next, suppose that we have a commutative square of the following form in \( C \).

\[
\begin{array}{ccc}
f & \xrightarrow{h} & k \\
g & & \\
\downarrow{h} & & \downarrow{k}
\end{array}
\]

Then we can view this as a vertical map in \( \text{Fam}(C) \) in the fibre of \( 2 \) from \( \tilde{f} \) to \( \tilde{g} \), which we write as \( (\tilde{h}, \tilde{k}) \). Applying \( T \) to \( (1_{X(A)}, X(\sigma)) \) gives us a commutative square of the following form.

\[
\begin{array}{ccc}
T(1_{\overline{\sigma}(A)}) & \xrightarrow{T(1_{\overline{\sigma}(A)})} & T(1_{\overline{\sigma}(A)}) \\
T(1_{\overline{\sigma}(A)}X(\sigma)) & & T(1_{\overline{\sigma}(A)}X(\sigma)) \\
T(\overline{\sigma}) & & T(\overline{\sigma})
\end{array}
\]

(7.6)

Furthermore such squares are compatible with \( \overline{h} \) and \( \overline{k} \), in the sense that we have the following commutative diagrams in \( \text{Fam}(C) \) over each of the corresponding maps \( 1 \rightarrow 2 \) in \( \text{Cat} \).

\[
\begin{array}{ccc}
\overline{h} & \xrightarrow{(\overline{h}, \overline{k})} & \overline{k} \\
\overline{g} & & \\
\overline{f} & & \overline{f}
\end{array}
\]

As before we apply \( T \) to a special case of the diagrams above, to get the following commutative diagrams.

\[
\begin{array}{ccc}
T(X(A)) & \xrightarrow{\gamma_{1A,0}} & T(1_{\overline{\sigma}(A)}) \\
\downarrow{\gamma_{\sigma,0}} & & \downarrow{\gamma_{\sigma,0}} \\
T(\overline{\sigma}) & & T(\overline{\sigma})
\end{array}
\quad
\begin{array}{ccc}
T(1_{\overline{\sigma}(A)}) & \xrightarrow{T(1_{\overline{\sigma}(A)})} & T(1_{\overline{\sigma}(A)}) \\
\downarrow{T(1_{\overline{\sigma}(A)}X(\sigma))} & & \downarrow{T(1_{\overline{\sigma}(A)}X(\sigma))} \\
T(\overline{\sigma}) & & T(\overline{\sigma})
\end{array}
\]

(7.7)
Finally, we can now calculate,

\[(TX)\circ \alpha_X^\Delta = (TX)\circ \beta_{\sigma,0} \circ \gamma_{\sigma,0} \]

by \((7.4)\)

\[= \beta_{\sigma,1} \circ T(X(\sigma)) \circ \gamma_{\sigma,0} \]

by \((7.5)\)

\[= \beta_{\sigma,1} \circ T(X(\sigma)) \circ T(1_X A, X(\sigma))_0 \circ \gamma_{1_A,0} \]

by \((7.3)\)

\[= \beta_{\sigma,1} \circ T(1_X A, X(\sigma))_1 \circ T(1_X A) \circ \gamma_{1_A,0} \]

by \((7.6)\)

\[= \beta_{\sigma,1} \circ T(1_X A, X(\sigma))_1 \circ \gamma_{1_A,1} \]

by \((7.7)\)

\[= \beta_{\sigma,1} \circ \gamma_{\sigma,1} \circ T(X(\sigma)) \]

by \((7.4)\)

\[= \alpha_{\beta}^\Delta \circ T(X(\sigma)) \]

by \((7.4)\)

But we can now deduce that the naturality square \((7.2)\) commutes, as required.

**Lemma 7.3.4.** The restriction of step-one to the fibre over \(1\) is isomorphic to Garner’s definition of step-one (the composition of the left adjoints to \(G_2\) and \(G_3\) in \([8, \text{Section 4}]\)).

**Proof.** Suppose we are given a vertical map \(m\) over a category \(A\). Write \(M\) for the corresponding functor \(A \to C^2\). Since we are just evaluating step-one on the fibre of \(1\), we suppose we are given a vertical map \(f\) over \(1\), which is just a map in \(C\), say \(f: X \to Y\).

Expanding the definition of step-one from section \([4.6]\) using the explicit description of hom objects in \(\text{Fam}(C)\), we see that \(L_1 f\) is constructed as follows. \(\text{Hom}(m, f)\) is the comma category \((M \downarrow f)\) and reindexing (i.e. composing) \(M\) along the first projection gives us a functor \(M': (M \downarrow f) \to C^2\). We have an opfibration \(\text{dom}: V(\text{Fam}(C)) \to \text{Fam}(C)\). \(L_1 f\) is then the opcartesian lift of \(M'\) along the image of the canonical map \(M' \to f\) under \(\text{dom}\). However, we have seen in lemma \([4.2.4]\) that this is equivalent to first taking the opcartesian lift of \(M'\) along the map \((M \downarrow f) \to 1\) followed by a vertical pushout over \(1\). The former is just the left Kan extension of \(M'\) along \((M \downarrow f) \to 1\), which is the colimit of \(M'\) regarded as a diagram in \(C^2\), or alternatively as the canonical map between the levelwise colimits in \(C\). The pushout over \(1\) is an ordinary pushout in \(C\). But this description now matches Garner’s.

**Remark 7.3.5.** Step 0 of the small object argument over \(1\) can be viewed as a left Kan extension in two different ways. Firstly, following our general construction we have already seen in the proof above that we take the left Kan extension of \(M'\) along \((M \downarrow f) \to 1\). This is just the colimit of the diagram \(M'\). Note, however that this diagram is, as observed by Garner, precisely the pointwise formula for the left Kan extension of \(M\) along itself at \(f\).

**Theorem 7.3.6.** Our definition of step-one of the small object argument is simply the pointwise lift of Garner’s definition from \([8]\) to \(\text{Fam}(C)\).

**Proof.** By lemmas \([7.3.4]\), \([7.3.2]\) and \([7.3.3]\).

**Theorem 7.3.7** (Garner). Let \(C\) be a category satisfying either one of the following two conditions.
1. For every $X \in C$ there is a regular cardinal $\alpha$ such that $C(X, -)$ preserves $\alpha$-filtered colimits.

2. $C$ possesses a proper, well-copowered strong factorisation system $(E, M)$, and for every object $X$ of $C$ there is $\alpha$ such that $C(X, -)$ preserves $\alpha$-filtered unions of $M$-subobjects.

Then for every vertical map $m$ in $\text{Fam}(C)$, the awfs cofibrantly generated by $m$ exists.

**Proof.** The well known result due to Garner [8, Theorem 4.4] is for ordinary awfs’s rather than for awfs’s fibred over $\text{Fam}(C)$, so we need to show how to extend the result to this case.

We saw in theorem 7.3.6 that our definition of step-one is isomorphic to the pointwise lift of Garner’s definition to $\text{Fam}(C)$. Garner’s construction of an algebraically free awfs clearly lifts pointwise to a fibred awfs over $\text{Fam}(C)$ which is clearly algebraically free on the pointwise lift of step-one.

**Remark 7.3.8.** Garner’s proof is not constructively valid as stated. We draw attention to two issues. Firstly for the small object argument to apply in many situations, such as categories of presheaves, we require the existence of uncountable regular ordinals. However Gitik has proved in [9] that this is independent of ZF. So this requires some form of the axiom of choice. On the other hand a very weak, and constructively acceptable form of choice such as AMC (as appears in [18]) should suffice.

Secondly the axiom of excluded middle is used in places. The main example is the assumption that every ordinal is either a limit or successor. The treatment of ordinals in [1, Section 9.4] suggests that with care the argument can be rephrased to work in constructive set theory, although it would be necessary to assume the existence of inaccessibles (as in [1, Chapter 18]), in addition to AMC.

Note that cofibrantly generated awfs’s are automatically fibred, since they are unique up to isomorphism, and we can always find a fibred one by lifting pointwise the cofibrantly generated awfs over 1. We can show something similar for strongly fibred.

**Theorem 7.3.9.** Suppose that $C$ is cocomplete and pullbacks exist and preserve all colimits. If $(L_1, R_1)$ is a strongly fibred lawfs and the algebraically free awfs on $(L_1, R_1)$ exists, say $(L, R)$. Then it is also strongly fibred.

**Proof.** Since $(L, R)$ exists, it has to be the pointwise lift of the explicit description given by Garner. Hence it suffices to show that Garner’s construction is preserved by pullbacks. However, this description is a colimit of iterations of $L_1$, which is preserved by pullbacks by the assumptions.

Finally, we remark that in this case the fibred pushout product construction from section 6.2 reduces down to the usual pushout product for monoidal categories.

Unfortunately, however when $C$ is monoidal closed, $\text{Fam}(C) \to \text{Cat}$ is not necessarily an internally closed monoidal fibration. Consider, for example, the case $C = \text{Set}$ with cartesian product. Then the cotensor over a small category $A$ is the exponential in $\text{Set}^A$, which is not preserved by reindexing.
7.4 Internal Category Indexed Families of Diagrams

Recall that we can define a notion of internal category, which generalises small categories by replacing the set of objects and the set of morphisms with objects in some (usually large) category. Furthermore, given a fibration \( p : E \to B \) and an internal category in \( B \), we can define a notion of diagram, which generalises functors \( A \to \mathcal{S}et \), where \( A \) is a small category. See e.g. [12] Chapter 7 for formal definitions of both of these.

Let \( p : E \to B \) be a fibration and let \( \mathcal{C} \) be an internal category in \( B \). Then we define a new fibration as follows. We define \( \textbf{Cat}_B \) to be the category of internal categories and internal functors in \( B \). We define \( \mathcal{E}_C \) to consist of pairs \( (D, X) \) where \( D \) is an internal category in \( B \) and \( X \) is a diagram of type \( C \times D \) in \( E \).

Note that if we apply this with \( p : \text{Fam}(\text{Set}) \to \text{Set} \) the fibration of set indexed families of sets, we get the following. An internal category \( \mathcal{C} \) in \( \text{Set} \) is just a small category. Given another small category \( D \), a diagram of type \( C \times D \) is just a functor \( C \times D \to \text{Set} \). Then using the isomorphism \( [C \times D, \text{Set}] \cong [D, [C, \text{Set}]] \), we see that the resulting fibration is a special case of category indexed families from section 7.3 where \( C \) is the category \( [C, \text{Set}] \) of presheaves over \( C^{op} \).

**Theorem 7.4.1.** Suppose that \( p : E \to B \) is a locally small fibration, \( B \) is locally cartesian closed, and \( C \) is an internal category in \( B \). Then \( \mathcal{E}_C \to \text{Cat}_B \) is also locally small.

**Proof.** In [13] Lemma B2.3.15 (i) Johnstone shows how to do this for the base change of the fibration along the discrete category functor \( B \to \text{Cat}_B \). We will show how to extend this to the fibration over \( \text{Cat}_B \). The basic idea is to mimic internally the construction for category indexed families applied to a presheaf category. If we are given functors \( X, Y : A \to [C, \text{Set}] \), then the result cited above already gives us the internal version of the hom object in set indexed families over the objects of \( A \). We recall that the hom object for category indexed families is the small category where the set of objects is the hom object for set indexed families, which we recall consists of morphisms in \( [C, \text{Set}] \) (i.e. natural transformations), with morphisms being those morphisms in \( A \) that are compatible with the natural transformations.

Suppose we are given an internal category \( \mathcal{A} = A_1 \equiv A_0 \) in \( B \) together with diagrams \( X \) and \( Y \) over \( A \). We first need to define the object in \( \text{Cat}_B \) indexing the hom. That is, we need to define an internal category \( \text{Hom}_{A_1}(X, Y) \). Note that we can also view \( X \) and \( Y \) as objects in \( \mathcal{E}_C \) over \( A_0 \). By the result in loc. cit. we have a hom object \( h : \text{Hom}_{A_1}(X, Y) \to A_0 \) in \( B \). We will take this to be the object of objects in the internal category, so we just need to define the object of morphisms.

Let \( \mu : \partial_i^0(X) \to \partial_i^1(X) \) be the action of \( X \), and \( \nu : \partial_i^0(Y) \to \partial_i^1(Y) \) the action of \( Y \), which we view as vertical maps over \( \mathcal{E}_C \to B \) (strictly speaking these are morphisms over \( C_1 \times A_1 \), but we can view them as morphisms over \( C_0 \times A_1 \) by reindexing along the identity map for \( C \)). By lemma 5.1.9 (together with the result from loc. cit. and local cartesian closedness) we know that \( V(\mathcal{E}_C) \to B \) is locally small. We take the object of morphisms to be \( \text{Hom}_{A_1}(\mu, \nu) \), which comes equipped with a map \( k : \text{Hom}_{A_1}(\mu, \nu) \to A_1 \). By considering the domain and codomain of the universal square, we get canonical maps \( \partial_i^0, \partial_i^1 : \text{Hom}_{A_1}(\mu, \nu) \to \text{Hom}_{A_0}(X, Y) \) such that \( \partial_i \circ k = h \circ \partial_i^i \) for each \( i \). We can then define the identity
and multiplication by lifting those for $\mathcal{A}$. Note that the universal square from $\mu$ to $\nu$ is a vertical morphism between the diagrams $X$ and $Y$.

We now check that this is a hom object. Suppose that we are given an internal category $\mathcal{D}$ and an internal functor $F: \mathcal{D} \to \mathcal{A}$ together with a vertical map $F^* X \to F^* Y$. This is a vertical map $f: F^*_0(X) \to F^*_0(Y)$ over $E_C \to \mathbb{B}$ such that the following diagram commutes.

\[
\begin{array}{ccc}
F^*_0 \partial^*_0 X & \cong & \partial^*_0 F^*_0 X \\
\downarrow F^*_1 \mu & & \downarrow \partial^*_1 F^*_0 Y \\
F^*_1 \partial^*_1 X & \cong & \partial^*_1 F^*_0 X
\end{array}
\]

(7.8) together with the universal property of $\text{Hom}_{\mathbb{B}}$.

We now check that this does give an internal functor, and witnesses the universal property of the hom object.

**Theorem 7.4.2.** Suppose that $p: \mathcal{E} \to \mathbb{B}$ is a cocomplete fibration (i.e. has dependent coproducts satisfying Beck-Chevalley), and $\mathcal{C}$ is an internal category in $\mathbb{B}$. Then $E_C \to \text{Cat}_{\mathbb{B}}$ is a bifibration with all finite colimits.

**Proof.** Write $\tilde{E}$ for the category consisting of triples $(\mathcal{A}, X, \mu)$ where $\mathcal{A}$ is an internal category in $\mathbb{B}$ and $(X, \mu)$ is a diagram over $\mathcal{A}$. Then the projection $\tilde{E} \to \text{Cat}_{\mathbb{B}}$ is a fibration. Furthermore, for a small category $\mathcal{C}$, the fibration $E_C \to \text{Cat}_{\mathbb{B}}$ is the base change of $\tilde{E}$ along the functor $- \times \mathcal{C}$.

[13] Proposition B2.3.20] tells us that $E \to \text{Cat}_{\mathbb{B}}$ has finite colimits and dependent coproducts, and so is a bifibration. However, these properties are preserved by base change, so the same applies to $E_C \to \text{Cat}_{\mathbb{B}}$. 

**7.4.1 Presheaf Assemblies**

We recall some basic definitions in realizability. See e.g. [12 Section 1.2] for more details.

**Definition 7.4.3.** An assembly over $\mathcal{K}_1$, or $\omega$-set, consists of a pair $\langle X, E \rangle$ where $X$ is a set, and $E$ is a function $X \to \mathcal{P}^*(\mathbb{N})$ (non empty subsets of $\mathbb{N}$). We refer to $X$ as the *underlying set* and to $E$ as the *existence predicate* of the assembly.

We will also refer to these just as assemblies, since we won’t consider any other categories of assemblies in this paper.

**Definition 7.4.4.** Let $\langle X, E \rangle$ and $\langle X', E' \rangle$ be assemblies. We say a function $f: X \to X'$ is tracked, or computable if there exists $e \in \mathbb{N}$ satisfying the following. Write $\varphi_e$ for the $e$th computable function. For all $x \in X$ and all $n \in E(x)$, we have that $\varphi_e(n)$ is defined and $\varphi_e(n) \in E'(f(x))$.

**Definition 7.4.5.** Assemblies and computable functions form a category, which we denote $\text{Asm}$. 

42
Proposition 7.4.6. **Asm** has all finite limits and finite colimits and is locally cartesian closed.

Hence we see that the codomain fibration on **Asm** is locally small.

We now give explicit descriptions of internal category, internal functor and diagram in assemblies. These are all straightforward to prove by unfolding the general definition.

**Proposition 7.4.7.** An internal category over $\text{cod} : \text{Asm}^2 \to \text{Asm}$ is (up to equivalence) a small category $C = \langle C_1, C_0 \rangle$ together with existence predicates $E_0 : C_0 \to \mathcal{P}^*(\mathbb{N})$ and $E_1 : C_1 \to \mathcal{P}^*(\mathbb{N})$ such that the domain, codomain, identity and composition functions are all computable.

**Proposition 7.4.8.** An internal functor between internal categories $C$ and $D$ in **Asm** consists of a functor $F : C \to D$ between underlying small categories, such that its action on objects and action on morphisms are both computable functions.

**Proposition 7.4.9.** Let $C$ be an internal category in assemblies. A diagram over $C$ is (up to equivalence) a functor $X : C \to \text{Asm}$ which is uniformly computable, in the following sense. For each object $A$ of $C$, write $X_0(A)$ for the underlying set and $X_1(A)$ for the existence predicate of the assembly $X(A)$. Then there exists $e \in \mathbb{N}$, such that for all morphisms $s : A \to B$ in $C$, for all $n \in E_1(s)$, for all $x \in X_0(A)$, for all $m \in X_1(A)(x)$, $\varphi_e(n, m)$ is defined, with $\varphi_e(n, m) \in X_1(B)(X(s)(x))$.

Now applying the internal category indexed family construction to assemblies, we get the following description of lifting problems.

An object of $E_C$ consist of a functor $C \times D \to \text{Asm}$ that is uniformly computable. In particular, $E_{C,1}$ consists of presheaf assemblies over $C$. A vertical map can then be viewed as a uniformly computable functor $D \to [C, \text{Asm}]^2$ (where we define uniformly computable similarly to functors to **Asm**). A choice of fillers is then a choice of fillers from section 7.3 (i.e. a uniform choice of fillers in the sense of [8]) that satisfies the addition requirement of being uniformly computable.

In this way, we see it is easy to develop realizability variants of cofibrantly generated classes of maps in presheaf categories. To illustrate this, we show that Kan fibrations in simplicial assemblies, as defined by Stekelenburg in [28] are a cofibrantly generated class in our general sense.

We define the computable simplex category $\Delta$ to be the following internal category in **Asm**. We take the underlying category to be the usual simplex category, $\Delta$. We define the existence predicate on objects by $E_0([n]) = \{ n \}$. The existence predicate on morphisms is defined as follows. We can view $\sigma : [n] \to [m]$ as an order preserving function between finite sets. We define $E_1(\sigma)$ to be the set of natural numbers that track this function. (This is the same as constructing the simplex category in the internal logic of assemblies.) A simplicial assembly is then (up to equivalence) a functor $\Delta^{op} \to \text{Asm}$ which is uniformly computable.

We define a family of maps as follows. We take the underlying category $D$ to be the discrete category defined as follows. The underlying set consists of pairs $(n, k)$ where $k \leq n$. The existence predicate is defined taking $E((n, k))$ to be $\{ (n, k) \}$, where $(-, -)$ is a computable encoding for pairs. To define
a vertical map in the fibration over $D$ is to define a uniformly computable functor $\Delta^{op} \times D \to \text{Asm}^2$. We first note that the Yoneda embedding is clearly computable, and so we get a functor $D \to [\Delta, \text{Asm}]$ sending $(n, k)$ to $\Delta^n$. We can then make the $k$th horn $\Lambda_k^n$ into an assembly by taking the existence predicate to be the restriction of the existence predicate on $\Delta^n$. This then makes the functor sending $\langle n, k \rangle$ to the horn inclusion $\Lambda_k^n \hookrightarrow \Delta^n$ uniformly computable.

Finally, we can see that a map with the right lifting property against this functor is a computable variant of the usual notion of Kan fibration. By unfolding the definition of universal lifting problem for this case we can see that it is a morphism $f$ in simplicial assemblies with a Kan filling operator providing a filler for every lifting problem against a horn inclusion, which is uniformly computable. Here uniformly computable means uniform both in the choice of horn inclusion and the lifting problem. In other words, if we are given $n, k$ and natural numbers tracking the horizontal maps in a lifting problem of $\Lambda_k^n \hookrightarrow \Delta^n$ against $f$, then we can compute a number tracking the choice of diagonal filler.

### 7.5 Codomain Fibrations

Let $C$ be a category with pullbacks. Then the codomain functor $\text{cod}: C^2 \to C$ is a fibration. We assume that $C$ has all finite limits. Opcartesian lifts in cod always exist, and are given by composition. The codomain fibration is locally small exactly when $C$ is locally cartesian closed: given $I$ in $C$, and $U \to I$ and $V \to I$ in $C/I$ we define $\text{Hom}_I(U, V)$ to be the exponential in $C/I$.

A family of maps, is just a morphism in a slice category. Suppose we are given families of maps $m$ and $f$ over $I$ and $J$ respectively as below:

\[
\begin{array}{ccc}
U & \xrightarrow{m} & V \\
\downarrow & & \downarrow \\
I & \leftarrow & \sigma^*(U)
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
J & \leftarrow & \sigma^*(V)
\end{array}
\]

A family of lifting problems from $m$ to $f$ over $K$ is a diagram of the following form, where both squares on the left are pullbacks.

\[
\begin{array}{ccc}
U & \xrightarrow{\sigma^*(U)} & X \\
\downarrow & & \downarrow \\
V & \xrightarrow{\sigma^*(V)} & Y \\
\downarrow & & \downarrow \\
I & \xrightarrow{\sigma} & K & \leftarrow & J
\end{array}
\]

From this description (and proposition 3.2.5), we easily get the following.

**Proposition 7.5.1.** Let $m$ be a vertical map over an object $I$ of $C$. Given a map $f$ in $C$, we view it as a vertical over $1$ (using the canonical isomorphism $C \cong C/1$). Then $f$ has the (fibred) right lifting property against $m$ if and only if it has the (ordinary) right lifting property against $\sigma^*(m)$ for every $K \in C$ and $\sigma: K \to I$.
The universal lifting problem is then defined by taking $K$ as below. Recall that we can define $\text{Hom}$ using the local exponential in a slice category (in this case $\mathcal{C}/I \times J$).

$$K := \text{Hom}(U, X) \times_{\text{Hom}(U, Y)} \text{Hom}(V, Y)$$

The right hand maps to $X$ and $Y$ are then the evident evaluation maps.

Over codomain fibrations we can easily show that lawfs’s and awfs’s are fibred, as we show below.

**Theorem 7.5.2.** Let $\mathcal{C}$ be locally cartesian closed category with pushouts. Let $m$ be a vertical map over $\text{cod}: \mathcal{C}^2 \to \mathcal{C}$.

1. Step-one of the small object argument on $m$ is well defined, giving us an lawfs $\langle L_1, R_1 \rangle$, where the $R_1$ algebra structures on a family of maps $f$ correspond precisely to solutions of the universal lifting problem of $f$ against $m$.

2. $(L_1, R_1)$ is a fibred lawfs.

3. If the awfs algebraically generated by $(L_1, R_1)$ exists, then it is also fibred.

**Proof.** First note that cod is a bifibration, with opcartesian maps given by “composition” and the Beck-Chevalley condition is an easy diagram chase. As explained above cod is also locally small. Local cartesian closedness is used again to get dependent products in $\mathcal{C}$. In particular reindexing preserves pullbacks (as long as they exist) since it has a right adjoint.

Hence we get an lawfs, and it is fibred by theorem 4.6.4. If the cofibrantly generated awfs exists then it is also fibred by theorem 5.5.1. □

Sometimes useful to use an alternative formulation of the universal lifting problem. When we gave the general definition of universal lifting problem we were motivated by thinking of a lifting problem from $U$ to $X$ and $Y$ together with a map $\beta: V \to Y$ satisfying the condition that $f \circ \alpha = \beta \circ m$. Using dependent products we can instead formalise an alternative idea. A lifting problem of $m$ against $f$ is a map $\beta: V \to Y$ together with a map $m^{-1}(\{v\}) \to f^{-1}(\{\beta(v)\})$ for each $v \in V$. We then no longer need to require an additional commutativity condition. If we were working internally in type theory, we might think of $U$ as a family of types indexed by $V$ and $X$ as a family of types indexed by $Y$. We would then express the the above using the type below

$$\Sigma_{v: V} \Sigma_{j: I} \Sigma_{\beta: V(i) \to Y(j)} \Pi_{v: V(i)} (U(i, v) \to X(j, \beta(v)))$$  \hspace{1cm} (7.9)

We can also formulate this idea in purely category theory terms using dependent products as follows.

First write $e: V \times_I \text{Hom}(V, Y) \to Y$ for the right hand map of the hom object (which is the evaluation map of the local exponential composed with the projection $\pi^*_Y(Y) \to Y$). Write $\tilde{m}$ for the map $\langle m, 1 \rangle: U \times_I \text{Hom}(V, Y) \to V \times_I \text{Hom}(V, Y)$. Write $q$ for the projection $V \times_I \text{Hom}(V, Y) \to \text{Hom}(V, Y)$.

We view the objects $U \times_I \text{Hom}(V, Y)$ and $e^*(X)$ as objects of $\mathcal{C}/(V \times_I \text{Hom}(V, Y))$ and then take the exponential in $\mathcal{C}/(V \times_I \text{Hom}(V, Y))$ from $U \times_I \text{Hom}(V, Y)$ to $e^*(X)$, corresponding to the term $U(v) \to Y(\beta(v))$ in (7.9). By the usual description of local exponentials in terms of dependent products, we can express this as $\Pi_{\tilde{m}} \tilde{m}^*(e^*(X))$. Note that this comes equipped with an
evaluation map $e': \tilde{m}^* \Pi_i \tilde{m}^* (e^*(X)) \rightarrow \tilde{m}^* e^*(X)$ over $V \times I \hom(V,Y)$, and we have $\tilde{m}^* \Pi_i \tilde{m}^* (e^*(X)) \cong U \times I \hom_{\mathcal{C}}(V,Y)$.

We then apply $\Pi_{q'}$ corresponding to the $\Pi_{q'}$ in (7.9) to get $\Pi_{q'} \Pi_i \tilde{m}^* (e^*(X))$ in $\mathcal{C}/\hom(V,Y)$. Write $q'$ for the projection $U \times I \hom(V,Y) \rightarrow \hom(V,Y)$. Then since $q' = q \circ \tilde{m}$, we have $\Pi_{q'} \Pi_i \tilde{m}^* (e^*(X)) \cong \Pi_{q'} \tilde{m}^* (e^*(X))$.

Finally note that we can view this as an object of $\mathcal{C}/I \times J$ by composing with the map $\hom(V,Y) \rightarrow I \times J$, corresponding to the $\Sigma_{g:V \rightarrow Y}$ in (7.9). We have now constructed the indexing object for the universal lifting problem. We summarise this as the lemma below.

**Lemma 7.5.3.** The universal lifting problem is isomorphic to a diagram of the following form, where $q'$, $\tilde{m}$ and $e$ are as above.

\[
\begin{array}{cccccc}
U & \rightarrow & U \times_I \Pi_{q'} \tilde{m}^* (e^*(X)) & \rightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
V & \rightarrow & V \times_I \Pi_{q'} \tilde{m}^* (e^*(X)) & \rightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
I & \leftarrow & \Pi_{q'} \tilde{m}^* (e^*(X)) & \rightarrow & J
\end{array}
\]

In type theoretic notation, $\Pi_{q'} \tilde{m}^* (e^*(X))$ is the type we started with in (7.9). Then $V \times \Pi_{q'} \tilde{m}^* (e^*(X))$ contains an element $v$ of $V(i)$. Similarly $U \times \Pi_{q'} \tilde{m}^* (e^*(X))$ contains an element of $U(i)$. The horizontal right hand maps are then given by evaluation.

**Proof.** The two descriptions are equivalent by the reasoning preceding the lemma.

Working in the internal logic of the category, one can either verify directly using the type theoretic description that this satisfies the universal property of the universal lifting problem, or show that the description matches the usual definition of the universal lifting problem.

We use the explicit description below to show that for an interesting class of generating family of left maps, the resulting laws is strongly fibred.

**Theorem 7.5.4.** Suppose that the map $V \rightarrow I$ is an isomorphism, then step 1 of the small object argument is strongly fibred.

Moreover, it can be described type theoretically as the type below.

\[
\Sigma_{j:J} \Sigma_{y:Y(j)} \Sigma_{i:I} (U(i) \rightarrow X(j,y))
\]

(7.11)

**Proof.** We first give a type theoretic description of the proof since, this is easier to follow. We use the description of the universal lifting problem in (7.9).

Since $V \rightarrow I$ is an isomorphism, we may assume that for any $i: I$, $V(i)$ is a singleton, that is, isomorphic to the unit type 1. Hence we can replace the term $\beta: V(i) \rightarrow Y(j)$ with $y: Y(j)$, and we can replace the term $\Pi_{e:V(i)} (U(i, v) \rightarrow X(j, \beta(v)))$ with $U(i) \rightarrow X(j, y)$. This gives the following simplification.

\[
\Sigma_{j:J} \Sigma_{y:Y(j)} \Sigma_{i:I} (U(i) \rightarrow X(j,y))
\]

(7.11)

But this is isomorphic to (7.11).
It’s then clear that pulling back along any map \( g: Y' \to Y \) over \( J \) is the same as substituting \( g(y) \) for \( y \), which is the same as pulling back \( X \) along \( g \), and then forming the type, so it is indeed stable under pullback.

For completeness we now also include a proof that doesn’t use any type theory.

Recall that over codomain fibrations \( \text{Hom}(V, Y) \) is \( \text{Hom}_{I \times J}(\pi_0(V), \pi_1(Y)) \) where \( \text{Hom}_{I \times J} \) is the exponential in \( C/I \times J \) and \( \pi_0 \) and \( \pi_1 \) are the projections from \( I \times J \) to \( I \) and \( J \). Since \( V \to I \) is an isomorphism we know that \( \pi_0(V) \) is the terminal object in \( C/I \times J \) and so \( \text{Hom}(V, Y) \) is isomorphic to \( \pi_1(Y) \) with evaluation map corresponding to the identity on \( \pi_1(Y) \), which is isomorphic to \( I \times Y \). So we can take \( e \) to be the projection \( I \times Y \to Y \).

For convenience we will now assume that in fact \( V = I \) and the map \( V \to I \) is the identity. We then have the further simplifications that \( U \times I \text{Hom}(V, Y) \) is \( U \times Y \), with \( q' \) equal to \( \langle m, 1_Y \rangle \) and \( V \times I \text{Hom}(V, Y) \) is \( I \times Y \), with \( \tilde{m} \) equal again to \( \langle m, 1_Y \rangle \). We will write both \( q \) and \( \tilde{m} \) as \( \tilde{m}_Y \).

In summary we can rewrite the universal lifting problem as below.

\[
\begin{array}{c}
U & \overset{m}{\longrightarrow} & U \times I \Pi \tilde{m}_Y (I \times X) & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
I & \quad \Pi \tilde{m}_Y (I \times X) & \longrightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
I & \quad \Pi \tilde{m}_Y (I \times X) & \longrightarrow & J
\end{array}
\]

Now suppose we are given \( f': X' \to Y' \) and maps making the following pullback square.

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow f \\
Y' & \longrightarrow & Y
\end{array}
\]

Consider the following cube, where the left and right faces are pullbacks by definition, and the upper back edge is given by the universal property of the pullback on the right.

\[
\begin{array}{ccc}
m_{Y'}(I \times X') & \longrightarrow & \tilde{m}_{Y'}(I \times X) \\
\downarrow & & \downarrow \\
I \times X' & \longrightarrow & I \times X \\
\downarrow & & \downarrow m_Y \\
U \times Y' & \longrightarrow & U \times Y \\
\downarrow & & \downarrow \tilde{m}_Y \\
I \times Y' & \longrightarrow & I \times Y
\end{array}
\]

The front face is a pullback by diagram chasing, and we deduce that the back face is also a pullback.

Since dependent products in locally cartesian closed categories always satisfy
Beck-Chevalley, we can deduce that we also have the following pullback.

\[
\begin{array}{c}
\Pi_{m_Y}m_Y^*(I \times X') \\
\downarrow \\
I \times Y'
\end{array} \quad \begin{array}{c}
\Pi_{m_Y}m_Y^*(I \times X) \\
\downarrow \\
I \times Y
\end{array}
\]

We can now see that pulling back along \( Y' \to Y \) preserves the universal lifting problem, and so it also preserves step-one of the small object argument, which is just a pushout of the upper right square in the universal lifting problem (and pullbacks in locally cartesian closed categories always preserve pushouts since they are left adjoints). Since step-one is always fibred in a locally cartesian closed category, we can now deduce by lemma 4.4.7 that it is strongly fibred.

**Corollary 7.5.5.** Suppose that \( m \) is as above the map \( V \to I \) is an isomorphism. Then the cofibrantly generated awfs is strongly fibred if it exists.

**Proof.** By theorems 7.5.4 and 5.5.2.

**Remark 7.5.6.** One might expect fibred and strongly fibred to be equivalent for functorial factorisations over cod, since both involve pullbacks. However, this is not the case. A functorial factorisation takes as input a family of maps over an object \( J \) as below.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{} & \searrow & \downarrow{}
\end{array}
\]

It then factorises \( f \) as \( Lf \) followed by \( Rf \).

For the factorisation to be fibred says that it is stable under pullback along maps into \( J \). We have seen that for step-one this is always the case. On the other hand, strongly fibred says that the factorisation is stable under pullback along all maps into \( Y \). We have seen an interesting special case when step-one is strongly fibred, but it is not the case in general.

### 7.5.1 Individual Morphisms

We will illustrate how lifting problems work over codomain fibrations by first applying the definitions when we are just given individual maps as input (i.e. families of maps over the terminal object). This case can also be understood via enriched lifting problems where we view \( C \) as enriched over itself with cartesian product (as appears, for example in [25 Section 13.3]), but it illustrates similar ideas that turn up in the general case of families of maps in codomain fibrations, where it is not so clear how the definitions relate to existing notions in homotopical algebra.

Let \( i: U \to V \) and \( f: X \to Y \) be morphisms of \( C \). Then we can view them as morphisms in \( C^2 \) in the fibre of \( 1 \). A family of lifting problems consists of an object \( Z \) of \( C \) together with a commutative square of the following form in
Then, either using the adjunction between composition and pullback, or by checking directly, finding a solution to the family of lifting problems above is equivalent to finding a diagonal filler of the following square in \( C \) (or equivalently \( C/1 \)).

Recall that for codomain fibrations, \( \text{Hom}_{\mathcal{C}/I}(A, B) \) is just the local exponential in \( \mathcal{C}/I \). In particular \( \text{Hom}_{\mathcal{C}/1}(A, B) \) is just the exponential \( B^A \) in \( \mathcal{C} \).

For the example above, we know by proposition 5.2.5 that a filler for every family of lifting problems corresponds to a solution of the universal lifting problem and in turn to a coherent choice of solutions to lifting problems. Here coherence says that for any \( g: Z' \to Z \), the triangle in the middle of the diagram below commutes, where the diagonals are the choices of solutions.

The indexing object of the universal family is \( X^{U \times Y} Y^V \).

Then a solution to the universal lifting problem corresponds to a solution of the following lifting problem in \( \mathcal{C} \), where the top and bottom maps are given by evaluating the appropriate exponentials.

### 7.5.2 Strongly Fibred Cofibrations and Pushout Product

We now consider a general construction that generates a strongly fibred lawfs \( (C_1, F^1) \) a fibred (but usually not strongly fibred) lawfs \( (C^1, F_1) \) together with a morphism of lawfs \( \xi: (C_1, F^1) \to (C^1, F_1) \). We will show that this generalises certain constructions considered by Van den Berg and Frumin in [30] and by Pitts and Orton in [20].

Let \( \mathcal{C} \) be a locally cartesian closed category with pushouts and disjoint coproducts.
We suppose that we are given an interval object \( I \) with endpoints \( \delta_0, \delta_1 : 1 \to I \) together with a family of maps of the following form.

\[
\begin{array}{c}
1 \\
\downarrow^{i_0}
\end{array}
\xrightarrow{i_0}
\begin{array}{c}
I
\end{array}
\]

Write \( \Sigma \) for the class of maps that are pullbacks of \( i_0 \).

We first remark that we get the following explicit description of right maps over 1, as a special case of proposition 7.5.1.

**Proposition 7.5.7.** Let \( f \) be a morphism of \( C \). We also view \( f \) as a map in \( C/1 \). Then \( f \) has the (fibred) right lifting property against \( i_0 \) if and only if it has the (ordinary) right lifting property against \( m \) for every \( m \in \Sigma \).

As we saw in theorem 7.5.4, the fact that the right hand map is an identity means that we get a more explicit type theoretic description of the universal lifting problem, and that step 1 is strongly fibred.

Since the object on the left is the terminal object, we can in fact further reduce the type theoretic description even further in this case. We think of maps \( 1 \to U \) as terms of type \( \Sigma \) in the empty context. Since this map is also the display map for \( U \) over \( I \), we see that we can in fact think of this as a term \( i_0 \) of type \( I \) such that the family of types \( U(i) \) over \( i : I \) is defined by \( U(i) := i = i_0 \).

However, this implies that the top right horizontal map in the universal lifting problem (7.10) is an isomorphism. Explicitly, this map is the evaluation map \( \Sigma \) of \( U(i) \times (i = i_0 \to X(j, y)) \to X(j, y) \). Its inverse is then the map sending \( x : X(j, y) \) to \((i_0, \text{refl}, \lambda u.x)\).

But step-one of the small object argument is defined via pushout, and pushouts preserve isomorphisms, so we can deduce the following.

**Theorem 7.5.8.** Suppose we are given a family of maps as in (7.12). Then step 1 of the small object argument (which we refer to as \((C_1, F^1)\)) is isomorphic to the following type, and it is strongly fibred.

\[
\Sigma_{j:J} \Sigma_{y:Y(j)} (i = i_0 \to X(j, y))
\]

**Proposition 7.5.9.** The following are equivalent.

1. Every element of \( \Sigma \) is a pullback of \( i_0 \) in a unique way.
2. \( i_0 \) is the terminal object of the category with objects the elements of \( \Sigma \) and morphisms pullback squares.
3. The following "propositional extensionality" principle holds in the internal logic: \( \forall i, i' \in I \ (i = i_0 \Leftrightarrow i' = i_0) \Rightarrow i = i' \).

If \( C \) has a subobject classifier, \( \top : 1 \to \Omega \), then these are equivalent to \( I \) being a subobject of \( \Omega \), with \( i_0 \) the pullback of \( \top \) along the subobject inclusion.

**Definition 7.5.10.** If \( \Sigma \) satisfies one of the equivalent conditions in proposition 7.5.9 we say it is *extensional*.
Proposition 7.5.11. If $\Sigma$ is extensional then the $L_1$ coalgebra structure on a map $m$ is unique (if it exists).

Proof. This is straightforward by using the type theoretic definitions and then working in the internal logic of the category.

Theorem 7.5.12. If we are given a natural way to compose elements of $\Sigma$, then we can assign $(C_1, F^1)$ a multiplication map making it into an awfs.

Proof. By (the dual of) lemma 5.4.3

Example 7.5.13. Suppose that $\Sigma$ is closed under composition and extensional.

In this case we end up with an identical situation to the one considered by Bourke and Garner in [4, Section 4.4], and indeed the theorems above are minor variants of those considered by Bourke and Garner.

Gambino and Sattler proved in [7, Lemma 9.7] that if $\mathcal{C}$ is a presheaf category, then there is a suitable such $i_0$ for any class $\Sigma$ of monomorphisms closed under pullback and composition.

We now define the second laws $(C^1, F_1)$ to be the one generated by the coproduct of the two families of maps below.

\[
\begin{align*}
\begin{array}{ccc}
I + I & \xrightarrow{\delta_0 \times i_0} & I \\
I & \searrow & I \\
\downarrow & & \downarrow \\
I & \xrightarrow{\delta_1 \times i_0} & I \\
\end{array}
\end{align*}
\]

(7.13)

In many natural examples the interval comes equipped with a symmetry operation swapping the two endpoints. In this case, we clearly only need to use one of the diagrams above to get the same class of maps with right lifting property.

Again we get simple description of maps with the right lifting property.

Proposition 7.5.14. A map $f$ in $\mathcal{C}$, viewed as a map in $\mathcal{C}/1$ has the (fibred) right lifting property against the coproduct of the maps in (7.13) if and only if it has the (ordinary) right lifting property against $\delta_k \times m$ for all $m \in \Sigma$ and $k \in \{0, 1\}$.

Proof. By propositions 7.5.1 and 6.2.3

Example 7.5.15. Suppose that $i_0$ is terminal in the category of pullback squares and $\Sigma$ is closed under composition, as in example 7.5.13.

Suppose further that $\mathcal{C}$ is a topos, that the interval object comes equipped with connections, and the endpoint inclusions $\delta_0$ and $\delta_1$ are disjoint.

Suppose further that elements of $\Sigma$ are closed under finite union, and contain the map $[\delta_0, \delta_1] : 1 + 1 \to I$.

This is now the situation considered by Van den Berg and Frumin in [30].

By proposition 7.5.14 we see that the class of maps with the fibred right lifting property against $i_0$ are precisely the class of Kan fibrations that appear as [30, Definition 3.3].

We also recover the notion of filling structure due to Orton and Pitts in [20, Definition 4.2] as follows. Firstly, we only use one of the maps in (7.13), $\delta_0$. This means we are considering the universal lifting problem of $i_0 \times \delta_0$ against $f$. Then
one can show by working internally in type theory that solutions to the universal
lifting problem correspond precisely to filling structures. One might worry that
pushout product refers to pushout which requires quotients to define in type
theory, which don’t appear in the work of Orton and Pitts. An explanation
for this is that one can, by proposition 6.2.3, instead consider solutions of the
universal lifting problem of \( i_0 \) against \( \{ \delta_0, f \} \), which can be defined just using
pullback and exponentials and more closely matches the Pitts-Orton definition.

We will see later in section 8 that it is also possible to formulate the Pitts-
Orton definition of composition structure in a similar way.

Example 7.5.16 (van den Berg, Frumin). As a special case of the previous
example, we can take \( C \) to be the effective topos, the interval I to be \( \triangledown 2 \) and \( \Sigma \)
to be the class of all monomorphisms. This gives a nontrivial model structure
on a subcategory of the effective topos. See [30] for details.

Example 7.5.17. Suppose we are just given an interval object \( \delta_0, \delta_1 : 1 \to I \).
Then we can take \( I := 1 \) and \( i_0 := \delta_1 \).

7.5.3 Trivial Fibrations in 01-Substitution Sets

Recall that Pitts in [23] defined the category of 01-substitution sets as an equiv-
alent category to the category of cubical sets studied by Bezem Coquand and
Huber in [2]. See also [29] Section 1.2 for a description of the category and the
definition of Kan fibration. In a later paper [3], Bezem, Coquand and Huber
returned to this category of cubical sets and showed that it has a univalent
universe. One of the ideas that they developed in that paper was a notion of
cofibration and trivial fibration in the category of cubical sets.

In this section we will define the corresponding notion of trivial fibration in
01-substitution sets, and in fact define it as a cofibrantly generated class over
the codomain fibration. We will assume the reader is familiar with nominal sets.
See [22] for a general introduction.

We first define a 01-substitution set \( P_{\text{fin}}(A) + 1 \) as follows. The underlying
nominal set is the usual definition of \( P_{\text{fin}}(A) + 1 \) in nominal sets (recalling
\( P_{\text{fin}}(A) \) is the nominal set of finite subsets of \( A \) with the pointwise action). We
write the unique element of 1 as \( \top \). We define the action of substitutions by
\( x(a := i) := \top \) for all \( x \in V, a \in A \) and \( i \in 2 \). We also write \( \top \) for the coproduct
inclusion \( 1 \to P_{\text{fin}}(A) + 1 \) (which we note is a morphism in \( \text{01Sub} \), not just in
nominal sets).

We consider the laws cofibrantly generated by the following family of maps:

\[
\begin{array}{ccc}
1 & \xrightarrow{\top} & P_{\text{fin}}(A) + 1 \\
\downarrow & & \downarrow \\
\top & \xrightarrow{\dashv} & P_{\text{fin}}(A) + 1
\end{array}
\]

Proposition 7.5.18. \( P_{\text{fin}}(A) + 1 \) is extensional (in the sense of definition
7.5.10).

Proof. We use the condition that “propositional extensionality” holds.

Suppose we are given \( A, B \in P_{\text{fin}}(A) + 1 \) such that \( A = \top \iff B = \top \) holds in
the internal logic. First note that this clearly rules out \( A = \top \) and \( B \in P_{\text{fin}}(A) \).
Similarly for $A \in \mathcal{P}_{\text{fin}}(\mathbb{A})$ and $B = \top$. When $A = \top$ and $B = \top$ we vacuously have $A = B$.

It only remains to consider the case where both $A$ and $B$ are elements of $\mathcal{P}_{\text{fin}}(\mathbb{A})$. For all $a \in \mathbb{A}$ we have that $A(a := 0) = \top$ if and only if $B(a := 0) = \top$. But this implies $a \in A$ if and only if $a \in B$, and so $A = B$ as required. $\square$

Let $\square_A$ be the image of a representable under Pitts’ equivalence between 01-substitution sets and cubical sets, as defined in [29, Section 5.1]. Note that maps $\square_A \to \mathcal{P}_{\text{fin}}(\mathbb{A}) + 1$ correspond precisely to elements of $\mathcal{P}_{\text{fin}}(\mathbb{A}) + 1$ for which $A$ is a support. Such an element is either $\top$, or of the form $A_1$ where $A_1 \subset A$. Write $A_2$ for $A \setminus A_1$. Then we have $\square_A \cong \square_{A_1} \otimes \square_{A_2}$. Write $\partial \square_{A_1}$ for the subobject of $\square_{A_1}$ with elements $\sigma: A_1 \to 2$ such that $\sigma(a) \in 2$ for some $a \in A_1$. Then the pullback of $\top$ is of the following form.

\[
\begin{array}{ccc}
\partial \square_{A_1} \otimes \square_{A_2} & \to & 1 \\
\square_{A_1} \otimes \square_{A_2} & \to & \mathcal{P}_{\text{fin}}(\mathbb{A}) + 1
\end{array}
\]  

(7.14)

Suppose we are given a map $\sigma: \square_{A_1} \to \square_A$. We say it is non degenerate if the map $\square_{A_1} \to \mathcal{P}_{\text{fin}}(\mathbb{A}) + 1$ is non trivial. In this case we must have $\square_{A_1} \cong \square_{A_1} \otimes \square_{A_2}$ for some $A_2$ where $\sigma$ is an automorphism of $\square_{A_1}$. We then have a pullback of the following form.

\[
\begin{array}{ccc}
\partial \square_{A_1} \otimes \square_{A_2} & \to & \partial \square_{A_1} \otimes \square_{A_2} \\
\square_{A_1} \otimes \square_{A_2} & \to & \square_{A_1} \otimes \square_{A_2}
\end{array}
\]  

(7.15)

Then we see that the fibred right lifting property against $\top$ gives us a choice of filler for the (ordinary) lifting problem against each map on the left of (7.14), subject to the compatibility condition given by (7.15). However, since the category of 01-substitution sets is equivalent to a presheaf category, the converse also holds, by a similar argument to [11, Theorem 9.1].

By the same reasoning as in [11, Remark 3.9] or [29, Section 5] we see that maps $f: X \to Y$ with this property against such maps correspond precisely with those with a boundary filling operator, defined as follows.

**Definition 7.5.19.** Let $f: X \to Y$. A boundary (or tube) over a finite set $A \subseteq \mathbb{A}$ consists of an element $y$ of $Y$ together with a function $u: A \times 2 \to X$ satisfying the following conditions for all $a \in A$ and $i \in 2$.

1. $a \# u(a, i)$
2. $f(u(a, i)) = y(a := i)$
3. For $a' \in A$ such that $a' \neq a$ and for $i' \in 2$, $u(a, i)(a' := i') = u(a', i')(a := i)$

A filler of a boundary $(u, y)$ consists of an element $x$ of $X$ such that $f(x) = y$ and for all $a \in A$ and $i \in 2$, $u(a, i) = x(a := i)$.

A boundary filling operator on $f$ consists of a choice of filler $\uparrow(u, y)$ for every boundary satisfying the following.
1. For all \( \pi \in \text{Perm}(A) \), \( \pi \cdot (\uparrow(u, y)) = \uparrow(\pi \cdot u, \pi \cdot y) \).

2. For all \( a \in A \setminus A \) and \( i \in 2 \), we have \( \uparrow(u, y)(a := i) = \uparrow(u(a := i), y(a := i)) \).

### 7.5.4 Trivial Fibrations and Fibrations in CCHM Cubical Sets

We will define two classes of maps in the Cohen-Coquand-Huber-Mörtburg category of cubical sets from [5]. Following Gambino and Sattler in [7], Van den Berg and Frumin in [30] and Orton and Pitts in [20], we do this as a special case of the construction in section 7.5.2.

We first recall the definition of CCHM cubical sets from [5].

**Definition 7.5.20.** A de Morgan algebra is a bounded distributive lattice \( \langle L, \land, \lor, \neg \rangle \) with top element, 1, and bottom element 0, and an involution \( \neg : L \to L \) satisfying the following for all \( r, s \in L \):

- \( \neg 1 = 0 \)
- \( \neg 0 = 1 \)
- \( \neg(r \lor s) = (\neg r) \land (\neg s) \)
- \( \neg(r \land s) = (\neg r) \lor (\neg s) \)

The forgetful functor from de Morgan algebras to \( \text{Set} \) is monadic, so we can alternatively view de Morgan algebras as algebras over a monad on \( \text{Set} \) which we denote \( dM \).

**Definition 7.5.21.** Fix a countably infinite set, \( A \). (When working constructively also assume that \( A \) has decidable equality.)

The category of cubes, \( C \) is the full (small) subcategory of the Kleisli category of \( dM \) on finite subsets of \( A \).

The category of CCHM cubical sets is the functor category \( \text{Set}^C \).

**Definition 7.5.22.** We define the interval object \( I \) as the cubical set defined by \( I(A) := dM(A) \). Alternatively, this is the canonical map from the Kleisli category to \( dM\text{-Alg} \) composed with the forgetful functor \( dM\text{-Alg} \to \text{Set} \).

Alternatively again, this is also isomorphic to the representable \( y\{a\} \) for \( a \in A \).

The endpoints \( \delta_0, \delta_1 : 1 \to I \) are given by 0 and 1 respectively in the de Morgan algebras.

We now just need to define the classifying map for the cofibrations. This is what Cohen, Coquand, Huber and Mörtburg refer to as the face lattice. We will denote it \( 1 \to F \), and give three alternative definitions. Two abstract, and one more concrete (based on two definitions from [5]).

First, note that one might be tempted to take \( F = \mathbb{1} \), and \( 1 \to F \) to be be one of the endpoint inclusions, say \( \delta_1 : 1 \to I \). This does generate a pair of lawf's (as we saw in example 7.5.17), but it is not extensional (in the sense of definition 7.5.10). To see this note that for all \( \sigma : \{a\} \to dM\{a\} \), we have \( \sigma(a \land \neg a) \neq 1 \) if and only if \( 0 = 1 \), but \( a \land \neg a \neq 0 \). Hence, one might motivate the definition of face map, by “making \( \delta_1 \) extensional efficiently as possible.”

**Definition 7.5.23.** (This appears at the bottom of [5 Section 8.1]). We define \( 1 \to F \) as follows. Let \( \chi : I \to \Omega \) be the classifying map for the monomorphism \( \delta_1 \). Let \( I \to F \to \Omega \) be the image factorisation of \( \chi \). Define the classifying map \( 1 \to F \) to be the pullback of \( \top : 1 \to \Omega \) along the inclusion \( F \to \Omega \).
The construction above requires the existence of a subobject classifier. This is sometimes rejected in constructive mathematics for predicativity reasons (see [18]). Hence we give the following alternative definition, which works more generally in any Π-pretopos (but is equivalent to the definition above in a topos).

**Definition 7.5.24.** We define an equivalence relation, ∼ on 1 using the following definition in the internal logic. For i, i' ∈ 1, we set a ∼ b if (a = 1) ⇔ (b = 1).

We define F to be the quotient 1/∼. We define 1 → F to be the composition

1 ↦ Δ 1 ↦ 1/∼.

Finally, we recall from [5, Section 4.1], that we can also give the following more concrete, syntactic definition of the face lattice.

**Definition 7.5.25.** Given a finite set A, we define F(A) to be the distributive lattice generated by A + A, subject to the following relation. Write the elements of A + A as (a = 0) and (a = 1) for a ∈ A. The relation is (a = 0) ∧ (a = 1) = 0.

Now similarly to [4, Example 9.3], [30, Example 3.1(2)] and [20], we can characterise the CCHM notion of Kan filling operator as follows. Given a morphism f: X → Y in cubical sets, a Kan filling operator is a solution to the universal lifting problem to f from the following family of maps.

\[
\begin{array}{ccc}
I + 1 & F & I \times F \\
\downarrow & \delta_1 \times \top & \downarrow \\
F & \downarrow \\
& F
\end{array}
\]

In fact, we can now characterise Kan filling operators as cofibrantly generated in two senses, since, as Gambino and Sattler show in [7], they are also algebraically cofibrantly generated in Garner’s sense.

### 8 A Further Generalisation: Lifting Problems for Squares

In this section we consider more general notion of lifting problem due to Sattler [26, Section 6] and show how the earlier results in this paper can be adapted to also work with this definition.

#### 8.1 Definition

We first give a fibred version of Sattler’s definition. Throughout, we assume that we are given a fibration p: E → B. We will recover Sattler’s definition by applying this to a category indexed family fibration.

**Definition 8.1.1.** A family of squares over I ∈ B is a commutative square in E_I.

**Proposition 8.1.2.** Equivalently, a family of squares is a commutative square in E where all maps are vertical, or a vertical map in V(E), or an object of V(V(E)).
**Definition 8.1.3.** Suppose we are given a family of squares over $I \in \mathcal{B}$ and a family of squares over $J \in \mathcal{B}$, as below.

```
              m  n
                ↓  ↓
I ----> ---+----> J
```

A family of lifting problems from $(m, n)$ to $(f, g)$ is an object $K$ of $\mathcal{B}$ together with maps $\sigma: K \to I$ and $\tau: K \to J$ and a map from $\sigma^*(n)$ to $\tau^*(f)$ in $V(\mathcal{E})$, or equivalently, the middle square in the diagram below in $E_K$. A solution, or family of fillers of the lifting problem is the dotted diagonal map in the diagram below making two commutative triangles.

```
              σ^*(m)  σ^*(n)
                ↓  ↓
I ----> ---+----> J
```

Note that a lifting problem of the square $(m, n)$ against the square $(f, g)$ is exactly a lifting problem of $n$ against $f$.

**Definition 8.1.4.** A lifting problem of $(m, n)$ against $(f, g)$ is a universal lifting problem if it is universal as a lifting problem of $n$ against $f$.

Since this is a special case of our earlier definition, we immediately see that universal lifting problem is unique up to isomorphism and that the universal lifting problem exists whenever $p$ is locally small and $\mathcal{B}$ has all finite limits.

We easily get the counterpart to proposition 3.2.5 as below.

**Proposition 8.1.5.** Let $\mathcal{B}$ have finite limits, let $p$ be a locally small fibration and let $(m, n)$ and $(f, g)$ be families of squares. Then the following are equivalent.

1. Every family of lifting problems from $(m, n)$ to $(f, g)$ has a solution.
2. The universal family of lifting problems from $(m, n)$ to $(f, g)$ has a solution.
3. There is a coherent choice of solutions to all families of lifting problems from $(m, n)$ to $(f, g)$.

Just as in loc. cit. we note that every family of maps can be viewed as a family of squares. We can show this succinctly by working over $V(\mathcal{E})$: every object $f$ of $V(\mathcal{E})$ gives us a vertical map $1_f$ of $V(\mathcal{E})$. Again, following Sattler, we will focus on lifting problems where the right hand map is of this form. This is most useful in practice since usually what we are interested in is algebraic structure on maps cofibrantly generated by squares.

We can carry out step-one of the small object argument as follows. This time the construction only gives a functorial factorisation rather than a lawfs in general.
As before, it is easiest to work in $V(E)$, over the composition of bifibrations, $V(E) \xrightarrow{\text{dom}} E \xrightarrow{p} \mathcal{B}$.

We are given a generating square $m \to n$ in $V(E)$ over $I \in \mathcal{B}$ and an object $g$ of $V(E)$ over $J \in \mathcal{B}$. We first form the universal lifting problem. Say that $h: \text{Hom}(n, g) \to I$. This gives us a diagram of the form below.

\[
\begin{array}{ccc}
m & \to & n \\
\downarrow & & \downarrow \\
h^*(m) & \to & h^*(n)
\end{array}
\]

We then factorise the map $h^*(m) \to g$ as an opcartesian map over dom followed by a vertical map over dom. By lemma 4.2.7 this is the same as taking the following pushout in $E$ (where the right hand square is the universal lifting problem). In particular we know that the pushout exists and can also be expressed as a levelwise opcartesian map followed by a vertical pushout.

\[
\begin{array}{ccc}
m & \to & n & \to & g \\
\downarrow & & \downarrow & \uparrow & \downarrow \\
h^*(m) & \to & h^*(n) & \to & \text{pushout}
\end{array}
\]

However, in this form it is clear that we get a factorisation of $g$, which in fact gives a functorial factorisation such that algebras over the corresponding pointed endofunctor correspond precisely to solutions of the universal lifting problem.

### 8.2 Squares over a Codomain Fibration

We now specialise to codomain fibrations. In this case a family of squares indexed by $I$ is a diagram of the following form.

\[
\begin{array}{ccc}
U_0 & \to & U_1 \\
\downarrow & \downarrow & \downarrow \\
V_0 & \to & V_1
\end{array}
\]

\[ (8.1) \]

**Lemma 8.2.1.** Suppose that the map $V_1 \to I$ in (8.1) is an isomorphism. Then,

1. The functorial factorisation on (8.1) is strongly fibred.

2. If the algebraically free rawfs on the functorial factorisation exists, then it is also strongly fibred.

**Proof.** By an easy argument in the internal logic similar to that of theorem 7.5.3 and then applying theorem 5.3.8 for showing the awfs is also fibred (when it exists).
Example 8.2.2. As Sattler shows in [26, Section 6], this notion of lifting problem can be used to define Kan composition in CCHM cubical sets. Combining this with our earlier remarks, we see that the category of maps with Kan composition operator is cofibrantly generated by the following family of squares.

\[
\begin{array}{c}
1 \\
\downarrow \tau \\
F \\
\downarrow (\delta_1, 1_F) \\
\downarrow \tau \times \delta_0 \\
\downarrow \delta_0 \\
\downarrow \downarrow \\
\downarrow F \\
\end{array}
\]

Example 8.2.3. Again working in CCHM cubical sets, we define a weak fibration to be a map with the right lifting property against the following family of squares. The intuition is that we define a weaker notion of Kan filling operator in which instead of requiring a diagonal filler for all lifting problems of \(m \times \delta_0\) against a map \(f: X \to Y\), we only require it for those where the map \(I \times \Sigma \to Y\) factors (necessarily uniquely) through the projection \(\Sigma \times I \to \Sigma\).

\[
\begin{array}{c}
\downarrow \tau \times \delta_0 \\
\downarrow \delta_0 \\
\downarrow \downarrow \\
\downarrow F \\
\end{array}
\]

Since the lower right object is terminal in \(C/\Sigma\), we see that the cofibrantly generated lawfs is strongly fibred, and the cofibrantly generated awfs is too, if it exists. This construction may be useful for developing an abstract version of the approach to the semantics of higher inductive types in [5].

We can also combine this with the previous example to get a weak version of Kan composition operator:

\[
\begin{array}{c}
1 \\
\downarrow \tau \\
\downarrow \delta_0 \\
\downarrow \downarrow \\
\downarrow F \\
\end{array}
\]

9 Conclusion and Directions for Future Work

9.1 Cofibrantly Generated Awfs’s in \(\Pi W\)-Pretoposes with WISC

In this paper we saw a new fibred variation of the definition of cofibrantly generated awfs. We also saw that Garner’s small object argument tells us that for
certain categories, cofibrantly generated awfs’s over the category indexed families fibration always exist. However, so far we have not seen any corresponding result for cofibrantly generated awfs’s over codomain fibrations. One possible way to approach this would be to carry out a transfinite construction similar to Garner’s small object argument. However, this approach has the drawback that it requires infinite colimits. This results in natural examples (such as internal presheaves in realizability toposes) where one can define step-one of the small object argument, but where the construction of cofibrantly generated awfs’s does not work. The author will instead develop a new, alternative approach in a separate paper. Roughly speaking the idea is as follows. A category has $W$-types if certain endofunctors, referred to as polynomial endofunctors admit initial algebras [17]. The author will develop a new generalisation of $W$-types in which one instead uses initial algebras of certain pointed endofunctors, and that these initial algebras can be constructed from $W$-types provided that a weak choice axiom known as weakly initial set of covers (WISC) holds. For codomain fibrations, the pointed endofunctor in corollary 5.4.7 will be an example of such a pointed endofunctor, and so we will deduce that cofibrantly generated awfs’s over the codomain fibration exist in this case.

9.2 Applications to Realizability

One of the main aims of this work was to develop a definition of cofibrantly generated that is suitable for use in realizability toposes, categories of assemblies and variants, which at the same time can be used in proofs that are easy generalisations of existing work in homotopical algebra. The main issue is that these categories are not cocomplete. This makes it difficult to apply some standard arguments in homotopical algebra, such as the small object argument.

A natural way to develop a realizability variant of CCHM cubical sets is to construct the category of cubical sets internally in a category of assemblies. Within our general framework we have now seen two different approaches to defining classes of maps within these categories. The most promising approach is to work over the codomain fibration as in section 7.5.2. However, the approach of internal category indexed families of presheaves from section 7.4.1 is in some ways more flexible and may also be useful, for instance when working with realizability variants of BCH cubical sets, where separated product is not fibred with respect to the codomain fibration.

A somewhat surprising fact, first observed by Van Oosten in [33] is that the effective topos itself admits nontrivial homotopical structure. We have shown (in example 7.5.16) that a more recent variant by Van den Berg and Frumin fits within our general framework. Another realizability topos that promises to have rich (but apparently as yet unexplored) homotopical structure is the function realizability topos, and its relative the Kleene-Vesley topos (as defined in [32, Section 4.5]).

9.3 The BCH Cubical Set Model

The Bezem-Coquand-Huber cubical set model [2] was the first example of a constructively valid “homotopical” model of type theory. Since then, many authors have focused on the newer Cohen-Coquand-Huber-Mörtberg cubical set model. However, the original BCH model remains an interesting topic. Bezem,
Coquand and Huber have shown in a more recent paper [3] that the univalence axiom holds in BCH cubical sets, confirming that this approach does indeed give a model of homotopy type theory.

We have seen here in section 7.5.3 that the acyclic fibrations appearing in that paper can be characterised elegantly as cofibrantly generated with respect to the codomain fibration (for the equivalent category of 01-substitution sets). Kan fibrations in BCH cubical sets are part of an awfs cofibrantly generated with respect to the category indexed family fibration, and moreover this can be done constructively [29], but the question remains whether there is a more elegant definition similar to the case for acyclic fibrations, or Kan fibrations in CCHM cubical sets.

The main obstacle is that the definition used by Bezem, Coquand and Huber requires a particular monoidal product, called separated product to state. Since it is unclear how to extend separated product to a fibred monoidal product over \( \text{01Sub}^2 \rightarrow \text{01Sub} \), we cannot readily use the fibred Leibniz construction to get the BCH definition of Kan fibration. A partial answer to this has been provided by Alex Simpson, who has shown (in currently unpublished work) that if instead of \( \text{01Sub}^2 \), one works in a suitable subcategory (of so called independent squares), and the restriction of cod to this subcategory, then one does obtain a cartesian monoidal fibration whose restriction to the fibre over the terminal object is the monoidal category of \( \text{01Sub} \) with separated product.

9.4 Relation to Other Generalised Notions of Lifting Problem

In this paper we have seen a generalised notion of cofibrantly generated awfs’s. However, it is not the only such generalisation.

For example, as Riehl explains in [25, Section 13.3], given a monoidal category, one can define a notion of enriched lifting property. There is an overlap between the examples considered here and enriched lifting properties. Namely, the fibred lifting property between maps over the terminal object in a codomain fibration can also be viewed as the enriched lifting property (over the cartesian monoidal product) between functors from the trivial enriched category. However, neither approach seems to be more general than the other. One can give a rough intuition for the relation between fibred and enriched lifting problems as follows. In an enriched category, for any two objects \( X \) and \( Y \), \( \text{hom}(X, Y) \) is an object of a certain category and so can be manipulated via the internal logic of that category. However, to talk about a collection of objects, or maps with different domains/codomains we still need to use some external notion of set. On the other hand, when working over a fibration, even more can be done internally, requiring very little from the ambient set theory. It is natural to ask whether it is possible to combine the definitions together to get something that subsumes both notions. Such a combination would likely involve Shulman’s work on monoidal fibrations from [27].

In [4, Section 6], Bourke and Garner consider a notion of lifting problem between double categories. Once again, it seems that this is neither more general, nor a special case of the framework we consider here. Therefore it’s natural to ask whether there is another more general notion that includes both.
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61
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