RIESZ MEANS OF THE DEDEKIND FUNCTION II

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Abstract. Let $\psi$ denote the Dedekind totient function defined by $\psi(n) = \sum_{d|n} \mu^2(n/d)$ with $\mu$ being the Möbius function. We shall consider the $k$-th Riesz mean of the arithmetical function $n/\psi(n)$ for any non-negative integer $k$ on the assumptions that the Riemann Hypothesis is true, and all the zeros $\rho$ on the critical line of the Riemann zeta function $\zeta$ are simple. Our result is an explicit representation of the error term in the formula obtained in a previous work of the second author and I. Kiuchi [3]. We also give an improvement on the error estimate under the assumption of the Gonek-Hejhal Hypothesis. And, we propose a proposition that is equivalent to the Riemann Hypothesis.

1. Statement of Results

Let $s = \sigma + it$ be a complex variable, where $\sigma$ and $t$ are real. Let $\varepsilon$ denote an arbitrarily small positive number, not necessarily the same ones at each occurrence. Let $\psi$ be the Dedekind totient function defined by $\psi(n) = \sum_{d|n} \mu^2(n/d)$, where $\mu$ is the Möbius function, and let $S_k(x)$ be the $k$-th Riesz mean of $n/\psi(n)$ defined by

$$S_k(x) = \frac{1}{k!} \sum_{l \leq x} \frac{l}{\psi(l)} \left( 1 - \frac{l}{x} \right)^k.$$ 

For any positive real number $x(\geq x_0)$ with $x_0$ being a sufficiently large positive number, the second author and I. Kiuchi [3] showed that the asymptotic relation

$$S_k(x) = \frac{\zeta(4)h(1)}{(k+1)!\zeta(2)} x + E_k(x)$$

holds with

$$h(s) := \prod_p \left( 1 + \frac{1}{p^{s+2}(1+1/p)} - \frac{1}{p^{2s+2}(1+1/p)} \right),$$

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where $E_k(x)$ is the error term. It was proved in [3] that

\begin{equation}
E_k(x) \ll \frac{x^{-\frac{1}{2}+\varepsilon}}{k}
\end{equation}

under the assumption that the Riemann Hypothesis is true.

Before the statement of our theorem, we define the function $h_n(s)$ by

\begin{equation}
\begin{aligned}
h_n(s) &:= \prod_p \left( 1 + \sum_{m=1}^{2^n-1} \frac{1}{p^{ns+m+1}} \frac{-1}{1 + \frac{1}{p}} - \frac{1}{p^{2^n s+2^n} \left( 1 + \frac{1}{p} \right)} \right),
\end{aligned}
\end{equation}

which is absolutely and uniformly convergent in any compact set in the half-plane $\text{Re } s > -1 + \frac{1}{2n}$ (see Lemma 2.1). The purpose of this paper is to obtain an explicit representation of $E_k(x)$ of the general $k$-th Riesz mean under the assumption that the Riemann Hypothesis is true, and all zeros of the Riemann zeta-function $\zeta(s)$ on the critical line is simple.

**Theorem 1.1.** Suppose that the Riemann Hypothesis is true, and all the zeros $\rho$ on the critical line of the Riemann zeta-function $\zeta(s)$ are simple. Then, for any positive integers $k \geq 2$ and $n > \frac{\log 4 \log x}{2 \log 2}$, there exists a number $T \left( x^4 \leq T \leq x^4 + 1 \right)$ such that

\begin{equation}
E_k(x) = Y_{k,n}(x,T)x^{-\frac{1}{2}} + O \left( x^{-1+\sqrt{\log x}} \left( \frac{\sqrt{\log x}}{(k-1)! + \frac{1}{k}} \right) \right)
\end{equation}

with an absolute constant $C > 0$, where

\begin{equation}
Y_{k,n}(x,T) := \text{Re} \sum_{0 < \gamma < T} \frac{\zeta \left( -\frac{1}{2} + i\gamma \right) \zeta(2^n-1 + 2ni\gamma) h_n \left( -\frac{1}{2} + i\gamma \right)}{\zeta' \left( \frac{1}{2} + i\gamma \right)} \times \frac{x^{i\gamma}}{(-\frac{1}{2} + i\gamma) (\frac{1}{2} + i\gamma) \cdots (k - \frac{1}{2} + i\gamma)}.
\end{equation}

It may be interesting to study whether we can remove the $\varepsilon$-factor from the right-hand side of (1.3) or not. The second author and Kiuchi made use of the Gonek-Hejhal Hypothesis (S. M. Gonek [1] and D. Hejhal [2] independently conjectured), namely

\begin{equation}
J_{-\lambda}(T) := \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^{2\lambda}} \asymp T(\log T)^{(\lambda-1)^2}
\end{equation}

for real number $\lambda < \frac{3}{2}$ to improve the estimate (1.3) of the error term $E_k(x)$. In the present paper we have the following Theorem.
**Theorem 1.2.** If the function $J_{-1/2}(T)$ is estimated by $O(T^{1+\varepsilon})$ for any small fixed positive number $\varepsilon$, then the estimate $E_k(x) = O(x^{-\frac{1}{2}})$ holds for any positive integer $k \geq 2$.

Define

$$F(s) := \sum_{n=1}^{\infty} \frac{n}{\psi(n)n^s}$$

for $\text{Re } s > 1$. The above results will be proved by considering analytic continuation of $F(s)$. Also, the second author and Kiuchi studied the case of the Euler totient function by the same method in [4]. We also obtain the following results by analytic continuation of $F(s)$.

**Theorem 1.3.** Let $k \geq 0$, and $\Theta$ is supremum of real parts of non-trivial zeros of $\zeta(s)$. Then we have

$$E_k(x) = \Omega_{\pm}(x^{\Theta-1-\varepsilon}).$$

**Corollary 1.1.** The Riemann Hypothesis is equivalent to the statement that there exists an integer $k \geq 2$ such that $E_k(x) \ll x^{-\frac{1}{2}+\varepsilon}$.

**2. Some Lemmas**

In order to prove Theorems 1.1 and 1.2, we shall prepare the following lemmas.

**Lemma 2.1.** For any positive integer $n$ and $\text{Re } s > 1$, we have

$$F(s) = \frac{\zeta(s)\zeta(2^n s + 2^n)}{\zeta(s + 1)} h_n(s)$$

where $h_n(s)$ is defined by (1.4), and is absolutely and uniformly convergent in any compact set in the half-plane $\text{Re } s > -1 + \frac{1}{2n}.$

**Proof.** We use induction on $n$. The lemma is true for $n = 1$, which is Lemma 2.1 in [3]. Now, we assume that the statement (2.1) holds for $2, 3, \ldots, n - 1$. The induction assumption tells us

$$F(s) = \frac{\zeta(s)\zeta(2^{n-1} s + 2^{n-1})}{\zeta(s + 1)} h_{n-1}(s)$$

$$= \frac{\zeta(s)\zeta(2^n s + 2^n)}{\zeta(s + 1)} \prod_p \left(1 + \frac{1}{p^{2^{n-1} s + 2^{n-1}}}\right) h_{n-1}(s).$$
Now, for $\text{Re } s > 1$, we have

\[
\prod_p \left( 1 + \frac{1}{p^{2^{n-1}s+2^n-1}} \right) h_{n-1}(s)
\]

\[
= \prod_p \left( 1 + \frac{1}{p^{2^{n-1}s+2^n-1}} \right) \left( 1 + \sum_{m=1}^{2^{n-1}-1} \frac{1}{p^{m\delta+m+1}(1 + \frac{1}{p})} - \frac{1}{p^{2^{n-1}s+2^n-1}(1 + \frac{1}{p})} \right)
\]

\[
= \prod_p \left( 1 + \frac{2^{n-1}-1}{p^{m\delta+m+1}(1 + \frac{1}{p})} + \frac{1}{p^{2^{n-1}s+2^n-1}(1 + \frac{1}{p})} \right) + \sum_{m=1}^{2^{n-1}-1} \frac{1}{p^{(m+2^{n-1})s+(m+2^n-1)+1}(1 + \frac{1}{p})} - \frac{1}{p^{2^{n}+2^n}(1 + \frac{1}{p})}
\]

\[
= \prod_p \left( 1 + \frac{2^{n-1}-1}{p^{m\delta+m+1}(1 + \frac{1}{p})} - \frac{1}{p^{2^n+2^n}(1 + \frac{1}{p})} \right) = h_n(s),
\]

which completes the proof of Lemma 2.1. \qed

**Lemma 2.2.** Let $x$ be any sufficiently large real number, and let $\delta = 1/\sqrt{\log x}$. For any positive integer $n(> \frac{\log(2/\delta)}{\log 2} > 8)$, we have

\[
(2.2) \quad h_n(-1 + \delta + it) \ll \exp \left( \frac{C}{\delta} \right)
\]

with an absolute constant $C > 0$.

**Proof.** We use (1.4) and the inequality $\pi(u) \leq C_1 \frac{u}{\log u}$ for any positive number $u \geq 2$ to obtain an upper bound for the function $h_n(-1 + \delta + it)$ for any positive integer $n(> \frac{\log(2/\delta)}{\log 2} > 8)$, namely

\[
|h_n(-1 + \delta + it)| = \prod_p \left( 1 + \sum_{m=1}^{2^n-1} \frac{1}{p^{m\delta+1}(1 + \frac{1}{p})} + \frac{1}{p^{2^n\delta}(1 + \frac{1}{p})} \right)
\]

\[
\leq \prod_p \left( 1 + \sum_{m=1}^{\infty} \frac{1}{p^{m\delta}(p+1)} + \frac{1}{p(p+1)} \right)
\]

\[
= \prod_p \left( 1 + \frac{1}{(p+1)(p^\delta - 1)} + \frac{1}{p(p+1)} \right)
\]
\[
= \exp \left( \sum_p \log \left( 1 + \frac{1}{(p+1)(p^\delta - 1)} + \frac{1}{p(p+1)} \right) \right)
\leq \exp \left( \sum_p \left( \frac{1}{(p+1)(p^\delta - 1)} + \frac{1}{p(p+1)} \right) \right) \ll \exp \left( \sum_p \frac{1}{(p+1)(p^\delta - 1)} \right)
\]
\[
= \exp \lim_{\tau \to \infty} \left( \frac{\pi(\tau)}{(\tau-1)(\tau^\delta - 1)} + \int_2^\tau \left( \frac{\pi(u)}{(u-1)^2(u^\delta - 1)} + \frac{\pi(u)\delta u^\delta}{(u-1)^2(u^\delta)^2} \right) \, du \right)
\leq \exp \left( C_1 \int_2^\infty \frac{du}{(u-1)^2(u^\delta - 1) \log u} + C_1 \delta \int_2^\infty \frac{du}{(u-1)^2(u^\delta - 1)^2 \log u} \right)
\]
\[
= \exp \left( \frac{2}{\delta} C_1 \int_2^\infty \frac{du}{(u-1)^2(u^\delta - 1)} + \frac{8}{\delta} C_1 \int_2^\infty \frac{du}{(u-1)^2(u^\delta - 1)^2 \log u} \right)
\ll \exp \left( \frac{C}{\delta} \right)
\]

with an absolute constant \( C > 0 \). This completes the proof of Lemma 2.2. \( \square \)

**Lemma 2.3.** Assume that the Riemann hypothesis is true. Then, there exists a number \( t \in [T, T+1] \) such that
\[
\zeta(\sigma + it) \ll t^\varepsilon \quad \text{and} \quad \frac{1}{\zeta(\sigma + it)} \ll t^\varepsilon
\]
for every \( 1/2 \leq \sigma \leq 2 \) and any sufficiently large real number \( T > 0 \).

*Proof.* The first assertion is given by (14.2.5) and (14.14.1), and the second assertion is given by (14.16.2) in the textbook [10], respectively. \( \square \)

### 3. Proof of Theorem 1.1

*Proof.* Suppose that the Riemann Hypothesis is true, and all the zeros \( \rho \) on the critical line of the Riemann zeta-function \( \zeta(s) \) are simple. Let \( x \) be any sufficiently large real number, and let \( \delta = 1/\sqrt{\log x} \). Let \( T \in [x^4, x^4 + 1] \) satisfy the condition Lemma 2.3 and \( n \) be a positive integer satisfying \( n > \frac{\log(2/\delta)}{\log 2} \). We make use of Lemma 2.1 and (5.19) in H. Montgomery and R. C. Vaughan [5] with \( \sigma_0 := 1 + \frac{1}{\log x} \) to obtain

\[
S_k(x) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} F(s) \frac{x^s}{s(s+1)(s+2) \cdots (s+k)} \, ds
\]

(3.1)

\[
= \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} F(s) \frac{x^s}{s(s+1)(s+2) \cdots (s+k)} \, ds + O\left( xT^{-k+\varepsilon} \right).
\]

(3.2)
Now, we move the line of integration to \( \text{Re} \, s = -1 + \delta \). In the rectangular contour formed by the line segments joining the points \( \sigma_0 - iT, \sigma_0 + iT, -1 + \delta + iT, -1 + \delta - iT \) and \( \sigma_0 - iT \) counter-clockwise, we observe that \( s = 1 \) is a simple pole, and \( s = -\frac{1}{2} + i\gamma \) is also a simple pole of the integrand. Thus we get the main term from the sum of the residues coming from the poles \( s = 1 \) and \( s = -\frac{1}{2} + i\gamma \).

That is,

\[
\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} F(s) \frac{x^s}{s(s+1)(s+2)\cdots(s+k)} ds = \frac{1}{2\pi i} \left\{ \int_{-1 + \delta + iT}^{\sigma_0 + iT} + \int_{-1 + \delta - iT}^{-1 + \delta + iT} \right\} F(s) \frac{x^s}{s(s+1)\cdots(s+k)} ds + \sum_{0 < |\gamma| < T} \text{Res}_{s=\frac{1}{2} + i\gamma} \left( F(s) \frac{x^s}{s(s+1)\cdots(s+k)} \right) + \text{Res}_{s=1} \left( F(s) \frac{x^s}{s(s+1)\cdots(s+k)} \right).
\]

The last term on the right-hand side of (3.3) are evaluated by the second author and Kiuchi [3], who have calculated that

\[
\text{Res}_{s=1} \left( \frac{F(s)x^s}{s(s+1)\cdots(s+k)} \right) = \frac{\zeta(4)}{(k+1)!\zeta(2)} \prod_p \left( 1 + \frac{1}{p^2(p+1)} - \frac{1}{p^2(p+1)} \right) x.
\]

Furthermore, we have

\[
\sum_{0 < |\gamma| < T} \text{Res}_{s=\frac{1}{2} + i\gamma} \left( F(s) \frac{x^s}{s(s+1)\cdots(s+k)} \right) = \sum_{0 < |\gamma| < T} \text{Re} \left( \frac{\zeta \left( \frac{1}{2} + i\gamma \right) \zeta(2n-1 + 2n i\gamma)}{\zeta' \left( \frac{1}{2} + i\gamma \right)} h_n \left( -\frac{1}{2} + i\gamma \right) \times \frac{x^{i\gamma}}{\left( \frac{1}{2} + i\gamma \right) \left( \frac{3}{2} + i\gamma \right) \cdots \left( k - \frac{1}{2} + i\gamma \right)} \right) x^{-\frac{1}{2}}
\]

by the assumptions.

Let \( T \geq T_0 \), where \( T_0 \) is a sufficiently large real number. Denote by \( Q_k(x) \) the second term (the left vertical line segment) of the integral on the right-hand side
of (3.3). Using (2.2), we have

\[ Q_k(x) := \frac{1}{2\pi} \int_{-T}^{T} F(-1+\delta+it)x^{-1+\delta+it} dt \]

\[ = \frac{x^{-1+\delta}}{2\pi} \left( \int_{|t| \leq T_0} + \int_{T_0 \leq |t| \leq T} \right) \frac{\zeta(-1+\delta+it)\zeta(2n\delta+2niT)}{\zeta(\delta+it)} \times \]

\[ h_n(-1+\delta+it)x^it \]

\[ F(s) \frac{\zeta(s+1+iT)\zeta(2n(s+1)+2niT)}{\zeta(1+s+iT)T^{k+1}} ds \]

\[ \ll \frac{x^{-1+\delta}}{\delta(k-1)!} \exp \left( \frac{C}{\delta} \right) \]

\[ + x^{-1+\delta} \exp \left( \frac{C}{\delta} \right) \int_{T_0 \leq |t| \leq T} \frac{t^{\frac{3}{2}-\delta}\zeta(2-\delta-it)}{t^{1-\delta}\zeta(1-\delta-it)t^{k+1}} dt, \]

where \( C > 0 \) is an absolute constant. Using Lemma 2.3 we have

\[ Q_k(x) \ll x^{-1+\frac{C}{\log x}} \left( \frac{\sqrt{\log x}}{(k-1)!} + \frac{1}{k} \right) \]

for any positive integer \( k \geq 2 \).

Next we estimate the contributions coming from the upper horizontal line (the lower horizontal line is similar). First we split the integral as

\[ \int_{-\frac{1}{2}+iT}^{\sigma_0+iT} F(s) \frac{x^s}{s(s+1)\cdots(s+k)} ds \]

\[ = \left( \int_{\frac{1}{2}+iT}^{\sigma_0+iT} + \int_{-\frac{1}{2}+iT}^{-1+\delta+iT} \right) F(s) \frac{x^s}{s(s+1)\cdots(s+k)} ds \]

\[ =: I_1 + I_2 + I_3, \]

say. We use the functional equation for the Riemann zeta-function, Lemmas 2.1 and 2.3 to obtain

\[ |I_1| \leq \int_{\frac{1}{2}}^{\sigma_0} \left| \frac{\zeta(s+it)\zeta(2n(s+iT)+2niT)}{\zeta(1+s+iT)T^{k+1}} \right| ds \]

\[ \ll \int_{\frac{1}{2}}^{\sigma_0} \frac{T^{\varepsilon x}}{T^{k+1}} d\sigma \ll xT^{-k+\varepsilon}. \]
Similarly, we have
\[ |I_2| \ll \int_{-\frac{1}{2}}^{\frac{1}{2}} |T^{\frac{1}{2} - \sigma} \zeta(1 - \sigma - iT) T^n h_n(\sigma + iT) \frac{x^\sigma}{T^{k+1}} | d\sigma \]
\[ \ll \int_{-\frac{1}{2}}^{\frac{1}{2}} T^{2\varepsilon} \frac{x^{\varepsilon}}{T^{k+1}} d\sigma \ll x^{\varepsilon} T^{-k+2}, \]
and
\[ |I_3| \ll \int_{-1+\delta}^{-\frac{1}{2}} T^{\frac{1}{2} - \sigma} \zeta(1 - \sigma - iT) x^\sigma d\sigma \]
\[ \ll \int_{-1+\delta}^{-\frac{1}{2}} T^{\frac{1}{2} - \delta} T^\varepsilon x^{\frac{1}{2}} \frac{1}{T^{k+1}} \ll x^{-\frac{1}{2}} T^{\frac{1}{2} - \delta + \varepsilon - k} \]
for a positive number \( T \) (\( x^4 \leq T \leq x^4 + 1 \)).

Hence using horizontal lines defined by \( \text{Re} = \pm T \) to move the line of integration in (3.3), we find that the total contribution of the horizontal lines in absolute value is
\[ \ll x T^{-k+\varepsilon}. \]

Applying the relation (3.3) and the error estimates (3.5), (3.6) to (3.2), we obtain, for \( k \geq 2 \),
\[ S_k(x) = \frac{\zeta(4) h(1)}{(k+1)! \zeta(2)} x + Y_{k,n}(x,T) x^{-\frac{1}{2}} \]
\[ + O \left( x^{-1+\frac{c}{\sqrt{\log x}}} \left( \frac{\sqrt{\log x}}{(k-1)!} + \frac{1}{k} \right) \right) \]
with a positive number \( T \) (\( x^4 \leq T \leq x^4 + 1 \)), where
\[ Y_{k,n}(x,T) := \text{Re} \sum_{0 < \gamma < T} \frac{\zeta(-\frac{1}{2} + i\gamma) \zeta(2^n - 1 + 2^n i\gamma) h_n \left( -\frac{1}{2} + i\gamma \right)}{\zeta'(\frac{1}{2} + i\gamma)} \]
\[ \times \frac{x^{i\gamma}}{(-\frac{1}{2} + i\gamma) \left( \frac{1}{2} + i\gamma \right) \cdots (k - \frac{1}{2} + i\gamma)}, \]
which completes the proof of the identity (1.5). \( \square \)
4. Proof of Theorem 1.2

We use (3.8), Lemma 2.3, the functional equation for the Riemann zeta-function and Stirling’s formula for \( \chi(s) \) to obtain

\[
\sum_{0<\gamma<T} \Re \left( \frac{\chi\left(-\frac{1}{2}+i\gamma\right) \zeta\left(\frac{3}{2}+i\gamma\right) \zeta(2n-1+2ni\gamma) h_n \left(-\frac{1}{2}+i\gamma\right)}{\zeta'\left(\frac{1}{2}+i\gamma\right) x^{i\gamma} \left(-\frac{1}{2}+i\gamma\right) \left(\frac{1}{2}+i\gamma\right) \cdots \left(k-\frac{1}{2}+i\gamma\right)} \right)
\]

\[\ll \sum_{0<\gamma<T} \frac{1}{\gamma^{k-\frac{1}{2}-\varepsilon}|\zeta'(\frac{1}{2}+i\gamma)|}.\]

It suffices to show that \( Y_{k,n}(x,T) \) converges for \( k = 2 \). Using (1.7) and partial summation we obtain

\[
\sum_{0<\gamma<T} \frac{1}{\gamma^{\frac{1}{2}}|\zeta'\left(\frac{1}{2}+i\gamma\right)|} \ll \left[ \frac{J_{-\frac{1}{2}}(t)}{t^{\frac{1}{2}-\varepsilon}} \right]_T^1 + \int_{14}^T \frac{J_{-\frac{1}{2}}(t)}{t^{\frac{1}{2}-\varepsilon}} dt \ll 1,
\]

which implies the estimate \( E_k(x) = O\left(x^{-\frac{3}{2}}\right) \) for any positive integer \( k \geq 2 \). \( \square \)

5. Proof of Theorem 1.3 and Corollary 1.1

Proof of Theorem 1.3. We use reductio ad absurdum. Namely, we suppose that there exists a number \( \frac{1}{5} > \varepsilon_0 > 0 \) such that

\[
S_k(x) - \frac{\zeta(4)h(1)}{(k+1)!\zeta(2)} x \leq x^{\Theta-1-\varepsilon_0} \quad (x > K(\varepsilon_0)).
\]

Define \( G(s) \) by

\[
G(s) := \int_{14}^\infty \frac{S_k(x) - \frac{\zeta(4)h(1)}{(k+1)!\zeta(2)} x}{x^{s+1}} dx
\]

for \( \sigma > 1 \). We use the Mellin transform, and we have

\[
G(s) = \frac{\zeta(s)\zeta(4s+4)}{\zeta(s+1)s\cdots(s+k)} h_2(s) - \frac{\zeta(4)h(1)}{(k+1)!\zeta(2)} \frac{1}{s-1} - \frac{1}{s+1-\Theta+\varepsilon_0}
\]

\[\ll \sum_{0<\gamma<T} \frac{1}{\gamma^{k-\frac{1}{2}-\varepsilon}|\zeta'(\frac{1}{2}+i\gamma)|}.\]
by (3.1). So, we have
\[
\int_{1}^{\infty} S_k(x) - \frac{\zeta(4)h(1)}{(k+1)!\zeta(2)} x - x^{\Theta-1-\varepsilon_0} dx
\]
\[
= \frac{\zeta(s)\zeta(4s+4)}{\zeta(s+1)s\cdots(s+k)} h_2(s) - \frac{\zeta(4)h(1)}{(k+1)!\zeta(2)} \frac{1}{s-1} - \frac{1}{s+1-\Theta+\varepsilon_0}
\]
for \( \sigma > 1 \). Notice that the integral term on the left-hand side is absolutely convergent for \( \sigma > -1+\Theta-\varepsilon_0 \) since the right-hand side is a definite value when \( s = \sigma > -1+\Theta-\varepsilon_0 \). However, there is a zero of \( \zeta(s+1) \) from the definition of \( \Theta \), and so
\[
\frac{\zeta(s)\zeta(2s+2)}{\zeta(s+1)s\cdots(s+k)} h(s) - \frac{\zeta(4)h(1)}{(k+1)!\zeta(2)} \frac{1}{s-1} - \frac{1}{s+1-\Theta+\varepsilon_0}
\]
is not regular for \( \sigma > -1+\Theta-\varepsilon_0 \), which is a contradiction. Therefore
\[
E_k(x) = S_k(x) - \frac{\zeta(4)h(1)}{(k+1)!\zeta(2)} x = \Omega_+(x^{\Theta-1-\varepsilon})
\]
holds for any positive \( \varepsilon \).

Similarly, we have
\[
E_k(x) = S_k(x) - \frac{\zeta(4)h(1)}{(k+1)!\zeta(2)} x = \Omega_-(x^{\Theta-1-\varepsilon}).
\]

Proof of Corollary 1.1. The Riemann Hypothesis is sufficient for the statement that there exists an integer \( k \geq 2 \) such that \( E_k(x) \ll x^{-\frac{1}{2}+\varepsilon} \) by (1.3). We also see that the inverse assertion holds by Theorem 1.3. \( \square \)

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