Understanding Linchpin Variables in Markov Chain Monte Carlo

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Abstract

An introduction to the use of linchpin variables in Markov chain Monte Carlo (MCMC) is provided. Before the widespread adoption of MCMC methods, conditional sampling using linchpin variables was essentially the only practical approach for simulating from multivariate distributions. With the advent of MCMC, linchpin variables were largely ignored. However, there has been a resurgence of interest in using them in conjunction with MCMC methods and there are good reasons for doing so. A simple derivation of the method is provided, its validity, benefits, and limitations are discussed, and some examples in the research literature are presented.

1 Introduction

Modern statistical models are often sufficiently complicated so as to require the use of simulation for inference. Since the seminal work of Gelfand and Smith (1990), Markov chain Monte Carlo (MCMC) has become the default method for doing so, especially in the context of Bayesian inference. The Metropolis-Hastings (MH) algorithm (Hastings, 1970; Metropolis et al., 1953) is a commonly-used MCMC method due to its flexibility, ease of implementation, and theoretical validity under weak conditions. However, it is often challenging to develop effective MH algorithms, particularly when the target distribution is high-dimensional or has substantial correlation between components. A standard approach is to consider component-wise MCMC methods (Johnson et al., 2013; Jones et al., 2014) such as Gibbs samplers or conditional MH, also called Metropolis-within-Gibbs, perhaps using data augmentation (Hobert, 2011; Tanner and Wong, 1987). However, component-wise approaches can produce Markov chains that suffer from slow mixing (Béliele, 1998; Jonasson, 2017; Matthews, 1993).
The limitations of standard MCMC methods in modern applications has brought about a plethora of new approaches in specific statistical settings. However, our goal is to highlight an old and now under-appreciated technique using *linchpin variables*, that can often serve to simplify the sampling process and provides an organizing device for many of the novel sampling methods.

Before the widespread use of MCMC, the only potentially practical, general tool for sampling from multivariate joint distributions was the conditional sampling method (Devroye, 1986; Hörmann et al., 2004; Johnson, 1986). Let \( f(x,y) \) be a density function on \( \mathcal{X} \times \mathcal{Y} \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \) and \( f_{X|Y} \) be the density function of the conditional distribution of \( X \) given \( Y \). Let \( f_Y \) be the density function of the marginal distribution of \( Y \). If sampling from \( f_{X|Y} \) is straightforward, then \( Y \) is called a linchpin variable (Huber, 2016) since
\[
f(x,y) = f_{X|Y}(x|y) f_Y(y).
\]
Thus exact samples can be obtained by first simulating \( Y \sim f_Y \) followed by \( X \sim f_{X|Y} \). This idea is easily extended to the setting with more than two variables through the usual properties of joint probability functions.

**Example 1.** Consider the Rosenbrock (or banana) density on \( \mathbb{R}^2 \)
\[
f(x,y) \propto \exp \left\{ -\frac{1}{20} \left[ 100(x-y)^2 + (1-y)^2 \right] \right\}.
\]
This has become a popular and useful toy example for illustrating the performance of MCMC methods in highly correlated settings. In particular, because the contour plots resemble the shape of a banana, it can be a challenge to implement an effective MH algorithm. Notice that by inspection of the joint density, \( X|Y = y \sim N(y^2, 10^{-1}) \) and integrating \( f(x,y) \) with respect to \( x \) yields that \( Y \sim N(1,10) \). Hence \( Y \) is a linchpin variable and it is simple to implement conditional sampling.

Often the linchpin density, \( f_Y \), is complex enough to prevent direct sampling from it. When it is difficult to sample from \( f_Y \) directly, it is natural to turn to MCMC methods for doing so, yielding a so-called *linchpin variable sampler*. Our goal is to present advantages of using linchpin variable samplers, highlight some fundamental theoretical properties, and illustrate examples from the literature where they have been employed successfully.

An obvious potential benefit to the linchpin variable sampler is that it naturally reduces the dimension of the MCMC sampling problem since the target density is the marginal \( f_Y(y) \) instead of the joint \( f(x,y) \). Also, the linchpin variable sampler can be particularly effective when \( X \) and \( Y \) are heavily correlated (as demonstrated in a motivating example below); and finally, since information on \( X \) is not required to sample \( Y \), all post-processing (like thinning) can first be done on the linchpin variable, before sampling \( X \); see Owen (2017) for guidance on when thinning a Markov chain simulation might
be useful.

**Example 2.** Consider sampling from a $p$-variate normal distribution with mean $\mu$ and covariance $\Sigma$:

\[
\begin{pmatrix}
    X_1 \\
    X_2
\end{pmatrix}
\sim
\mathcal{N}_p
\begin{pmatrix}
    \mu_1 \\
    \mu_2
\end{pmatrix},
\Sigma =
\begin{pmatrix}
    \Sigma_{11} & \Sigma_{12} \\
    \Sigma_{21} & \Sigma_{22}
\end{pmatrix},
\]

where $\mu_1 \in \mathbb{R}^{p-r}$ and $\mu_2 \in \mathbb{R}^r$, $r < p$. The full conditional distributions are

\[
X_1 \mid X_2 = x_2 \sim \mathcal{N}_{p-r} (\mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_2), \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}) \quad \text{and} \quad
X_2 \mid X_1 = x_1 \sim \mathcal{N}_r (\mu_2 + \Sigma_{21} \Sigma_{11}^{-1} (x_1 - \mu_1), \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}).
\]

Let $p = 5$, $r = 1$, and $\Sigma$ be the $5 \times 5$ autocorrelation matrix with autocorrelation $\rho \in \{.5, .99\}$. MCMC algorithms are easily implemented in this example. For example the above full conditionals make it easy to implement a Gibbs sampler while a linchpin variable sampler with the linchpin variable being $X_2$ is also straightforward. Here, for the marginal of $X_2$, consider an MH algorithm with proposal $\text{Uniform}(x_2 - h, x_2 + h)$ with $h$ chosen to yield the approximate optimal scaling of \cite{Roberts97}.

Starting from the origin, both samplers are run for 5000 steps. The results are given in Figure 1. When $\rho = .5$, both methods perform similarly; however, as expected (cf. \cite{Raftery92}, when $\rho = .99$, the Gibbs sampler suffers from slow convergence. The linchpin variable sampler is unaffected by the higher correlation in the target distribution, as this correlation does not affect the marginal distribution for $X_2$.

![Figure 1: Trace plot for the last 1000 samples for $\rho = .50$ (left) and $\rho = .99$ (right)](image)

2 Linchpin variable sampler

Linchpin variable samplers yield valid MCMC algorithms and provide an organizing principle for seemingly disconnected Monte Carlo methods, but some basic MCMC concepts are required to get to that
2.1 Fundamentals of MCMC

A typical goal of using MCMC methods is to estimate features of a specified target density. For example, suppose $F$ is a probability distribution having support $\mathcal{Z}$ and density $f$. If $h : \mathcal{Z} \to \mathbb{R}$, the expectation

$$\mu := E_F[h(Z)] = \int_{\mathcal{Z}} h(z)f(z)\,dz,$$

may be of interest\(^1\). In many applications, the complexity of $f$ makes $\mu$ analytically intractable, forcing a turn to sampling methods for its estimation.

For any MCMC experiment, the practitioner is faced with two fundamental practical issues: (1) assessing when the sampling algorithm produces useful observations and (2) using the observations to reliably estimate $\mu$, that is, another MCMC experiment of the same run length will produce a similar estimate of $\mu$.

MCMC algorithms simulate realizations of a time-homogenous Markov chain, $Z_1, Z_2, Z_3, \ldots$, whose dynamics are given by a Markov transition kernel. Informally, the reader can think of the kernel as giving the probability of moving to a set $A$ in $n$ steps given that the current state is $z$, or

$$P^n(z, A) = \Pr(Z_{j+n} \in A \mid Z_j = z) \quad j, n \geq 1.$$

For simplicity of exposition we will restrict discussion to Markov chains that have a transition density\(^2\), that is, suppose $k: \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ so that

$$\int_{\mathcal{Z}} k(z' \mid z)\,dz' = 1,$$

and if $P^1 = P$, then

$$P(z, A) = \int_A k(z' \mid z)\,dz'.$$

To ensure that the simulation will eventually produce representative samples from $F$, $k$ should have invariant density $f$, that is, it satisfies

$$f(z') = \int_{\mathcal{Z}} f(z)k(z' \mid z)\,dz.$$

\(^1\)Often there are several expectations of interest, along with quantiles, marginal density plots, and so on, but the focus here is on estimating a single expectation; for the more general setting see Robertson et al. (2021) and Vats et al. (2019).

\(^2\)This should not be viewed as a real limitation since all of the arguments herein apply naturally, with the appropriate adjustments, much more generally.
This means that if \( Z_n \sim F \), then \( Z_{n+1} \sim F \) so the Markov chain is stationary. Of course, in most MCMC experiments it is difficult to make a draw directly from \( F \). A common way of ensuring that \( f \) is invariant for \( k \) is by constructing it so that it satisfies the detailed balance condition with respect to \( f \), that is,

\[
k(z' \mid z)f(z) = k(z \mid z')f(z') \quad \text{for all } z, z' \in \mathcal{Z}. \tag{4}
\]

It is easy to see by integrating both sides that detailed balance implies \( f \) is invariant for \( k \). Gibbs samplers have transition densities satisfying (3) and, in a formal sense, MH samplers satisfy (4).

Throughout the Markov kernel (chain) is assumed to be aperiodic, irreducible, and positive Harris recurrent (see [Meyn and Tweedie 2009](#) for definitions and a thorough treatment of the consequences). These assumptions are standard and are typically easily met in MCMC applications (see, e.g., [Roberts and Rosenthal 2006](#) [Tierney 1994](#)). Let \( \| \cdot \| \) be the total variation norm. The assumptions imply the Markov chain is ergodic

\[
\| P^n(z, \cdot) - F(\cdot) \| \to 0 \quad \text{as } n \to \infty. \tag{5}
\]

This implies that \( Z_n \overset{d}{\to} F \) as \( n \to \infty \), i.e., the marginal distribution of \( Z_n \) converges weakly to \( F \). Thus, as the Monte Carlo sample size increases, an MCMC simulation will produce a representative (albeit dependent) sample from \( F \) no matter the starting point, but the initial distribution can affect how long it takes for this to happen.

With a representative sample in hand, estimation of \( \mu \) is easy since the assumptions imply that the sample mean converges to \( \mu \). That is, as \( n \to \infty \), with probability 1,

\[
\bar{h}_n := \frac{1}{n} \sum_{i=1}^{n} h(Z_i) \to \mu. \tag{6}
\]

Note that if the Markov chain strong law holds for one initial distribution, it holds for every initial distribution, including point masses.

The rate of convergence plays a crucial role in the finite-time reliability of simulation experiments. Let \( M \) be a nonnegative function on \( \mathcal{Z} \) and \( \gamma \) a nonnegative function on \( \mathcal{Z}_+ \) such that

\[
\| P^n(z, \cdot) - F(\cdot) \| \leq M(z) \gamma(n). \tag{7}
\]

If \( \gamma(n) = \rho^n \) for some \( \rho < 1 \), then \( P \) is geometrically ergodic. Also, if \( P \) is geometrically ergodic with a bounded \( M \), \( P \) is uniformly ergodic. Constructive methods for establishing the existence of \( M \) and \( \gamma \) from equation (7) have been applied in many settings, but are beyond our scope; see [Jones and Hobert 2001](#) for an introduction.

The significance of geometric ergodicity largely lies in the fact that it is a sufficient condition for
the existence of a central limit theorem (CLT); see Chan and Geyer (1994). More specifically, if $E_F[h^2+\delta(Z)] < \infty$ and the Markov chain is geometrically ergodic, then
\[ \sqrt{n}(\bar{h}_n - \mu) \overset{d}{\to} N(0,\sigma_h^2). \]

Now $\sigma_h^2$ is complicated since it accounts for the autocorrelation in the Markov chain. Fortunately, there are several methods to estimate it and then use the resulting confidence regions to assess the reliability of the estimates; (see e.g. Flegal et al., 2008; Flegal and Jones, 2010; Jones et al., 2006).

### 2.2 Linchpin variable MCMC sampler

Let $Z = X \times Y$ with $Z = (X,Y)$, so that $f(z) = f(x,y)$ where $x \in X$ and $y \in Y$. Assume that it is straightforward to sample $f_{X|Y}$, but MCMC is required to do so from $f_Y$. The linchpin variable MCMC sampler is given in Algorithm 1 where $k_Y$ is a Markov chain transition density that keeps $f_Y$ invariant.

**Algorithm 1** Linchpin variable sampler

1: Input: Current value $(X_j, Y_j)$
2: Draw $Y_{j+1} \sim k_Y(\cdot | Y_j)$.
3: Draw $X_{j+1} \sim f_{X|Y}(\cdot | Y_{j+1})$.
4: Set $j = j + 1$

In Algorithm 1, $k_Y : Y \times Y \to [0, \infty)$ so that it is a Markov transition density with invariant density $f_Y$. That is,
\[ \int_Y k_Y(y' | y) dy' = 1 \quad \text{and} \quad f_Y(y') = \int_Y f_Y(y) k_Y(y' | y) dy. \]

Consequently, the Markov transition density for the linchpin variable sampler $(x, y) \mapsto (x', y')$ is
\[ k(x', y' | x, y) = f_{X|Y}(x' | y') k_Y(y' | y). \tag{8} \]

Notice that $k$ leaves $f$ invariant since
\[
\int \int k(x', y' | x, y) f(x, y) dx \, dy = \int \int f_{X|Y}(x' | y') k_Y(y' | y) f(x, y) dy \, dx \\
= f_{X|Y}(x' | y') \int k_Y(y' | y) f_Y(y) \, dy \int f_{X|Y}(x | y) dx \, dy \\
= f_{X|Y}(x' | y') f_Y(y') \int k(y | y') dy \\
= f(x', y').
\]
Also, \( k_Y \) satisfies the detailed balance condition when
\[
k_Y(y' \mid y) f_Y(y) = k_Y(y \mid y') f_Y(y') \quad \text{for all } y, y'.
\] (9)

Transition kernel \( k_Y \) satisfies detailed balance with respect to \( f_Y \) if and only if the linchpin variable sampler satisfies detailed balance with respect to \( f \). That is, for all \( x', y', x, y \)
\[
k(x', y' \mid x, y) f(x, y) = k(x, y \mid x', y') f(x', y').
\] (10)

To see this, first assume (10). Let \( y, y' \in \mathcal{Y} \) and let \( x' \in \mathcal{X} \) be such that \( f_{X|Y}(x' \mid y') > 0 \). Then
\[
f_Y(y) k_Y(y' \mid y) = \int_{\mathcal{X}} f(x, y) k_Y(y' \mid y) \, dx
= \int_{\mathcal{X}} f(x, y) f_{X|Y}(x' \mid y') k_Y(y' \mid y) \, dx
= \int_{\mathcal{X}} f(x, y) k(x', y' \mid x, y) \, dx
= \int_{\mathcal{X}} f(x', y') f(x, y) k_Y(x \mid y) \, dx
= f_Y(y') k_Y(y \mid y'),
\]
and hence (9) holds. Now assume (9) so that,
\[
f(x, y) k(x', y' \mid x, y) = f(x, y) f_{X|Y}(x' \mid y') k_Y(y' \mid y)
= f_{X|Y}(x \mid y) f_{X|Y}(x' \mid y') f_Y(y') k_Y(y \mid y')
= f(x', y') f_{X|Y}(x \mid y) k_Y(y \mid y')
= f(x', y') k(x, y \mid x', y'),
\]
and hence (10) holds.

If \( P^y_n(y, \cdot) \) denotes the conditional distribution of the marginal Markov chain after \( n \) steps, and \( F_Y \) denotes the distribution associated with \( f_Y \), then under the regularity conditions discussed in Section 2.1 as \( n \to \infty \),
\[
\| P^y_n(y, \cdot) - F_Y(\cdot) \| \to 0.
\]
The linchpin construction encourages the intuition that the dynamics of $k_Y$ transfers to the dynamics of $k$; this is indeed the case. Let $L$ denote conditional distribution, then it is clear from the construction of the linchpin variable sampler that for $j \geq 1$

$$L(X_j, Y_j \mid X_0, Y_0, Y_j) = L(X_j, Y_j \mid Y_j).$$

That is, $\{Y_j\}$ is de-initializing for $\{(X_j, Y_j)\}$. From Corollary 2 in [Roberts and Rosenthal (2001)](https://dx.doi.org/10.1080/01621459.2001.10476698), the Markov chains converge to their respective invariant distributions at the same rate. That is,

$$\|P^n((x, y), \cdot) - F(\cdot)\| = \|P^n_{Y}(y, \cdot) - F_Y(\cdot)\|.$$ (11)

As discussed in Section 2.1, the convergence rate is important for ensuring reliable simulation efforts. The significance of (11) is that one only needs to study the convergence of $P_Y$, that is, the Markov chain targeting $f_Y$; a concrete example is presented in Section 3.1.

The point of this section boils down to the fact that, by construction, $P_Y$ is a valid Markov kernel for $F_Y$ if and only if the linchpin variable sampler is valid for $F$. Moreover, the dynamics of both Markov chains are determined by the dynamics of $P_Y$.

## 3 Linchpin in the literature

Given the historic relevance of conditional sampling methods, linchpin variable samplers have been employed in a variety of scenarios. Their success in the examples below is typically due to either (1) superior mixing in the lower-dimensional space, (2) de-correlation of components via the linchpin variables or (3) lower post-processing costs.

The following presents three examples from the literature where linchpin variable samplers have been employed successfully. In addition to the models described here, linchpin variable samplers have been employed by [Bezener et al. (2018)](https://doi.org/10.1111/1467-9868.12656), [Blei et al. (2003)](https://doi.org/10.1198/016214503777474029), [Norton et al. (2017)](https://doi.org/10.1080/01621459.2017.1258261), and [West et al. (2014)](https://doi.org/10.1111/1467-9868.12450).

### 3.1 Collapsed Gibbs sampler in Bayesian vector autoregression

For $p > 0$, let $X_1, X_2, \ldots$ be a $p$-vector of predictors, and let $\epsilon_1, \epsilon_2, \ldots$ be independent and identically distributed according to a $N(0, \Sigma)$, where $\Sigma$ is an $r \times r$ positive-definite matrix. Let $B \in \mathbb{R}^{r \times r}$ and for some $q \geq 1$ and $i = 1, \ldots, q$ let $A_i \in \mathbb{R}^{r \times r}$ such that,

$$Y_t = \sum_{i=1}^{q} A_i^T Y_{t-1} + B^T X_t + \epsilon_t.$$
The process \( \{X_t\} \) is independent of \( \{\epsilon_t\} \). Let \( A = [A_1^T, A_2^T, \ldots, A_q^T]^T \in \mathbb{R}^{qr \times r} \), and let all the data observed until time \( K \) be \( D = \{(X_1, Y_1), (X_2, Y_2), \ldots, (X_K, Y_K)\} \). Ekvall and Jones (2021) consider prior specifications for \((A, B, \Sigma)\). Specifically, for fixed hyper-parameters \( m \in \mathbb{R}^{qr^2} \), \( C \), a \( qr^2 \times qr^2 \) positive-definite matrix, and \( D \), an \( r \times r \) positive-definite matrix, the three parameters are given the following independent priors:

\[
\begin{align*}
  f(\text{vec}(A)) & \propto \exp \left\{ -\frac{1}{2} [\text{vec}(A) - m]^T C [\text{vec}(A) - m] \right\} , \\
  f(\Sigma) & \propto \det(\Sigma)^{-a/2} \exp \left\{ \text{tr} \left( -\frac{1}{2} D \Sigma^{-1} \right) \right\} \quad \text{and} , \\
  f(B) & \propto 1 .
\end{align*}
\]

Ekvall and Jones (2021) provide conditions which yield that \((A, B, \Sigma)|D\) has a proper distribution. All the full conditionals are available in closed form, and thus a three variable Gibbs sampler is possible. However, Ekvall and Jones (2021) note that since \( A|\Sigma, D \) and \( \Sigma|D \) are available to sample from, a collapsed Gibbs sampler can be constructed which transitions from \((A, B, \Sigma) \mapsto (A', B', \Sigma')\) as

\[
k(A', B', \Sigma' \mid A, B, \Sigma) = f(B' \mid A', \Sigma', D) k_L(A', \Sigma' \mid A, \Sigma) \]

\[
:= f(B' \mid A', \Sigma', D) f(A' \mid \Sigma', D) f(\Sigma' \mid A, D) .
\]

Thus the linchpin variable is \((A, \Sigma)\) and a Gibbs sampler is employed for the marginal sampling. As explained in Ekvall and Jones (2021), the relatively simple form of the transition density here makes it easier to analyze the rate of convergence of the Markov chain, relative to the three-variable Gibbs sampler.

Collapsed Gibbs samplers have been employed in a variety of models; see Blei and Lafferty (2009), Chatterji and Pachter (2005), Koop et al. (2009), Kuo and Yang (2006), and Papaspiliopoulos et al. (2020) for some examples.

### 3.2 Bayesian linear models

Let \( Y \in \mathbb{R}^n \), \( \beta \in \mathbb{R}^p \), \( u \in \mathbb{R}^k \), \( X \) be a known \( n \times p \) full column rank design matrix, and \( Z \) a known \( n \times k \) full column rank matrix. Also assume that \( \max\{p, k\} < n \). Then a Bayesian linear model is given by
the following hierarchy

\[ Y | \beta, u, \lambda_E, \lambda_R \sim N_n \left( X\beta + Zu, \lambda_E^{-1}I_n \right) \]
\[ g(\beta) \propto 1 \]
\[ u | \lambda_E, \lambda_R \sim N_k \left( 0, \lambda_R^{-1}I_k \right) \]
\[ \lambda_E \sim \text{Gamma}(e_1, e_2) \]
\[ \lambda_R \sim \text{Gamma}(r_1, r_2) . \]

(12)

Assume that \( e_1, e_2, r_1, r_2 > 0 \) are known hyper-parameters. Sun et al. (2001) show that this hierarchy results in a proper posterior. Let \( \xi = (\beta^T, u^T)^T \), \( \lambda = (\lambda_E, \lambda_R)^T \) and let \( y \) denote all of the data. Then the posterior density satisfies

\[ f(\beta, u, \lambda_E, \lambda_R | y) = f(\xi, \lambda | y) = f_{\xi|\lambda}(\xi | \lambda, y) f_{\lambda}(\lambda | y) . \]

(13)

It is easy to sample the conditional \( \xi | \lambda, y \) and hence \( \lambda \) is a linchpin variable for \( f(\xi, \lambda | y) \). Acosta (2015) employs a random walk MH algorithm on the linchpin variable \( \lambda \). Note that \( (\xi, \lambda) \) is of dimension \( p + k + 2 \), but the linchpin variable \( \lambda \) is only 2-dimensional. The dramatic reduction in the state-space from \( (p + k + 2) \)-dimensions to 2-dimension yields a much more pliable MCMC procedure. Acosta (2015) also notes that an accept-reject sampler that draws independent samples from \( f_{\lambda}(\lambda | y) \) is possible. However, the accept-reject sampler is computationally expensive. So, although a complete Monte Carlo procedure using accept-reject sampling is impractical, starting from stationarity by drawing the initial state of the Markov chain from \( f_{\lambda}(\lambda | y) \) is certainly possible in this linchpin variable sampler.

3.3 Bayesian variable selection

Let \( y \) be a response vector in \( \mathbb{R}^n \), \( X \) be an \( n \times p \) matrix of predictors, and \( \beta \in \mathbb{R}^p \) be the vector of coefficients. Narisetty and He (2014) introduced the following Bayesian shrinkage and diffusing priors model:

\[ y | X, \beta, \sigma^2 \sim N(X\beta, \sigma^2 I_n) \]
\[ \beta | \sigma^2, Z_i \sim \mathbb{I}(Z_i = 0) N(0, \sigma^2 \tau_{0,n}^2) + \mathbb{I}(Z_i = 1) N(0, \sigma^2 \tau_{1,n}^2) \]
\[ P(Z_i = 1) = 1 - P(Z_i = 0) = q_n \quad \text{and} \quad \sigma^2 \sim IG(\alpha_1, \alpha_2) , \]

(14)

where \( \tau_{0,n}^2 \) and \( \tau_{1,n}^2 \) are positive functions of \( n \) and \( \alpha_1, \alpha_2 > 0 \). The latent variables \( Z_i \) are indicators of whether the \( i \)th variable is active or not. The resulting posterior is proper and Narisetty and He (2014) proposed a Gibbs sampler to sample from it using a three-variable Gibbs sampler since the full
conditional distribution $\beta | Z, \sigma^2, \sigma^2 | \beta, Z,$ and $Z | \sigma^2, \beta$ are all available in closed form.

Alternatively, one could use a linchpin variable sampler (see e.g. Yang et al. 2016; Zhou et al. 2022). The joint posterior distribution of $(\beta, \sigma^2, Z)$ admits the following decomposition.

$$f(\beta, \sigma^2, Z | y) = f(\beta, \sigma^2 | Z, y) f(Z | y).$$

As it turns out, $\beta, \sigma^2 | Z, y$ is available to sample from and thus, $Z$ is a linchpin variable.

The reduction in dimension here is only from $2p+1$ to $p+1$, which is not too advantageous. However, one can run any choice of MH algorithm to sample from $f(Z | y)$. The state space for $Z$ is finite, and thus an irreducible Markov chain on the linchpin variable is automatically uniformly ergodic, yielding a uniformly ergodic Markov chain for the joint posterior. In fact, the finite state space allows for a detailed study of mixing time of the algorithm or the use of locally informed proposals which can lead to superior mixing properties on discrete state-spaces (Liang et al. 2022; Yang et al. 2016; Zanella 2020; Zhou et al. 2022).

4 Discussion

Conditional sampling algorithms were common before the advent of MCMC. Although they are still used, as highlighted in the prequel, their discussion is somehow commonly absent in the toolkit of all MCMC algorithms.

The linchpin variable sampler provides a unifying framework for several MCMC algorithms based on the well-established idea of conditional sampling. The linchpin variable sample is most effective when the dimension of the linchpin variable is significantly smaller than the dimension of the joint variable or when the joint target distribution exhibits significant correlation, and the linchpin variable does not retain that correlation structure.

There are, of course, cases when the linchpin variable sampler presents little advantage over other samplers. For example, consider the Bayesian lasso model with a posterior in $p+1$ dimensions; here $p$ is the number of regression covariates. A linchpin variable sampler is possible to implement here, but the linchpin variable is $p$-dimensional, yielding little advantage. In fact, introducing a $p$-dimensional auxiliary variable in the Bayesian lasso model yields an efficient Gibbs sampler (Park and Casella 2008).

Given any sampling problem, it is worthwhile to assess whether using a linchpin variable construction is feasible, before continuing on to more complex algorithms.
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