W$^2$\textit{p} ESTIMATES FOR ELLIPTIC EQUATIONS ON C$^{1,\alpha}$ DOMAINS

DONGSHENG LI, XUEMEI LI, AND KAI ZHANG

Abstract. In this paper, a new method is represented to investigate boundary W$^2$\textit{p} estimates for elliptic equations, which is, roughly speaking, to derive boundary W$^2$\textit{p} estimates from interior W$^2$\textit{p} estimates by Whitney decomposition. Using it, W$^2$\textit{p} estimates on C$^{1,\alpha}$ domains are obtained for nondivergence form linear elliptic equations and further more, fully nonlinear elliptic equations are also considered.

1. Introduction

In this paper, we represent a new method of investigating boundary W$^2$\textit{p} estimates for nondivergence form linear elliptic equations. By virtue of Whitney decomposition, we derive local boundary W$^2$\textit{p} estimates from interior W$^2$\textit{p} estimates. This is a new point of view for boundary estimates and from it, we give a new proof of W$^2$\textit{p} estimates on C$^{1,\alpha}$ domains as $1 - 1/p < \alpha \leq 1$ stated in [7]. It should be pointed out that our method can be extended to fully nonlinear elliptic equations as in Section 5. Our result for linear equations is the following theorem.

Theorem 1.1. Let $1 < p < \infty$, $1 - 1/p < \alpha \leq 1$ and $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a C$^{1,\alpha}$ boundary portion $T \subset \partial \Omega$. Let $L$ be an elliptic operator in nondivergence form:

$$Lu = a^{ij}D_{ij}u + b^iD_iu + cu$$

with coefficients satisfying for some positive constants $0 < \lambda \leq \Lambda < \infty$,

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n$$

and

$$||a^{ij}||_{C(\Omega)}, ||b^i||_{L^\infty(\Omega)}, ||c||_{L^\infty(\Omega)} \leq \Lambda.$$ 

Suppose that $u \in W^{2,p}(\Omega)$ is a strong solution of $Lu = f$ in $\Omega$ and $u = g$ on $T$ in the sense of $W^{1,p}(\Omega)$ with $f \in L^p(\Omega)$ and $g \in W^{2,p}(\Omega)$. Then, for any domain $\Omega' \subset\subset \Omega \cup T$,

$$||u||_{W^{2,p}(\Omega')} \leq C \left( ||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)} + ||g||_{W^{2,p}(\Omega)} \right),$$

where $C$ depends on $n, \lambda, \Lambda, \alpha, p, T, \Omega', \Omega$ and the moduli of continuity of $a^{ij}$.

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are invalid for the Fourier series technique to construct counterexamples showing that this is the case for any small ball $B_{\varepsilon}(x_0)$ and some suitably small $\varepsilon > 0$. By invoking the Sobolev embedding theorem, we may weaken the conditions on the lower order coefficients of $L$ to $b^i \in L^q(\Omega)$, where $q > n$ if $p \leq n$, $q = p$ if $p > n$, $r > n/2$ if $p \leq n/2$, $r = p$ if $p > n/2$.

$W^{2,p}$ regularity plays a vital role in the regularity theory of partial differential equations. Interior $W^{2,p}$ estimates for Poisson’s equation were first established in [3] by explicit representation formulas involving singular integral operators. Interior $W^{2,p}$ estimates for general nondivergence form elliptic equations are obtained on account of the fundamental observation that they can be treated locally as a perturbation of constant coefficient equations. Later, Wang [12] demonstrated a new proof via maximal function approach that is originated by Caffarelli [1].

Boundary $W^{2,p}$ estimates are first established on flat domains by Schwarz reflection principle and then on $C^{1,1}$ domains alongside a flattening argument, where the $C^{1,1}$ regularity of domains is needed since the second order derivatives of the flattening mapping appear in the transformed elliptic operators (cf. [5]).

Based on theory of Sobolev multipliers, Maz’ya and Shaposhnikova [9] relaxed $C^{1,1}$ regularity of domains to $M_p^{2-1/p}(\delta)$ (see Section 14.3.1 in [9] for its definition) for $1 < p \leq n$ and $W^{2-1/p,p}$ for $n < p < \infty$, where $\delta$ depends on the moduli of continuity of $a^{ij}$. They also proved that $1 - 1/p < \alpha \leq 1$, $C^{1,\alpha} \subset M_p^{2-1/p}(\delta)$ if $1 < p \leq n$ and $C^{1,\alpha} \subset W_p^{2-1/p}$ if $n < p < \infty$. Moreover, they constructed $C^{1,1-1/p}$ domains where no solutions exist in $W^{2,p}$, which implies that the condition $1 - 1/p < \alpha$ can not be weakened. For $p = 2$, Kondrat’ev and Èidel’mann [3] used the Fourier series technique to construct counterexamples showing that $W^{2,2}$ estimates are invalid for $C^{1,1/2}$ domains. We refer to Kondrat’ev and Oleinik [7] for a survey of the theory of boundary value problems in nonsmooth domains.

This paper investigates boundary $W^{2,p}$ estimates on $C^{1,\alpha}$ domains again not using singular integrals (and Sobolev multipliers). The main idea is to derive local boundary $W^{2,p}$ estimates on $C^{1,\alpha}$ domains from interior $W^{2,p}$ estimates by Whitney decomposition. Our approach is more direct and is applicable to both linear elliptic and fully nonlinear elliptic equations. The proof is built upon Whitney decomposition, which is an effective tool for obtaining boundary estimates from interior estimates. For instance, Cao, Li and Wang [4] utilized it to prove the optimal weighted $W^{2,p}$ estimates for elliptic equations with non-compatible conditions.

We illustrate our idea as follows. Let $\{Q_k\}_{k=1}^\infty$ be Whitney decomposition of $\Omega$ (Suppose $0 \in \partial \Omega$ and denote $\Omega_\tau = \Omega \cap B_\tau$) and $Q_k = Q \circ Q_k$ be $\frac{Q}{Q}$-dilation of $Q_k$ with respect to its center. We suppose $L = \Delta$ and consider

$$
\begin{cases}
\Delta u = f & \text{in } \Omega_1, \\
u = 0 & \text{on } (\partial \Omega)_1,
\end{cases}
$$

where $(\partial \Omega)_1 = \partial \Omega \cap B_1$. Deduce from interior $W^{2,p}$ estimates that

$$
||D^2 u||_{L_p(Q_k)}^p \leq C \left( d_k^{-2|\tilde{p}|} ||u - l||_{L_p(Q_k)}^p + ||f||_{L_p(Q_k)}^p \right)
$$

for some $\tilde{p} > 1$, some constant $C = C(n, p)$ and any affine function $l$, where $d_k$ denotes the diameter of $Q_k$. If $C^{1,\alpha}$ estimate holds up to the boundary, we can
take \( l \) such that
\[
|u(x) - l(x)| \leq C\text{dist}(x, \partial \Omega)^{1+\bar{\alpha}}, \forall x \in \Omega_{3/4}.
\]
That is,
\[
\|u - l\|_{L^p(\tilde{Q}_k)} \leq Cd_k^{(1+\bar{\alpha})\bar{p}+n}.
\]
It follows that
\[
\|D^2 u\|_{L^p(\tilde{Q}_k)}^p \leq C \left( d_k^{-\bar{\alpha}\bar{p}+n} + \|f\|_{L^p(\tilde{Q}_k)}^p \right).
\]
Take sum on both sides with respect to \( k \) and if \(-\hat{p} + \alpha\hat{p} + 1 > 0\), then \( \sum_k d_k^{-\bar{\alpha}\hat{p}+n} \) is convergent (cf. Lemma 2.5). Observe that here \( \alpha = \min\{\alpha, 1 - \frac{p}{n}\} \) if we assume that \( f \in L^p \) with \( p > n \), and that \( \hat{p} < \frac{p}{n} \) since \( 0 < -\hat{p} + \alpha\hat{p} + 1 \leq 1 - \frac{p}{n} \).

Thus, we obtain a rough version \( W^{2,p} \) estimate up to the boundary (cf. Remark 3.6 and Theorem 5.1). To obtain an exquisite version \( W^{2,p} \) estimate up to the boundary (Theorem 1.1), we need further decompose \( u \) and for linear equations, this is possible. Actually, we set \( u = v + w \) such that \( v \) is a harmonic and \( w = \sum w_l \) with \( w_l \) satisfying
\[
\begin{aligned}
\Delta w_l &= f \chi_{Q_l} \quad \text{in } \Omega_1, \\
w_l &= 0 \quad \text{on } \partial \Omega_1.
\end{aligned}
\]
Since a large quantity of \( w_l \) are harmonic in \( Q_k \) and better boundary \( C^{1,\alpha} \) estimates hold for them, we can improve the above rough estimate (cf. Section 4).

The paper is organized as follows. In Section 2, Whitney decomposition and its relevant properties are concluded. In Section 3, we demonstrate some basic estimates for elliptic equations including \( W^{2,p} \) estimates for harmonic functions on \( C^{1,\alpha} \) domains. In Section 4, we show local boundary \( W^{2,p} \) estimates for Poisson’s equation on \( C^{1,\alpha} \) domains whose easy consequence is Theorem 4.11. In Section 5, \( W^{2,p} \) estimates for fully nonlinear elliptic equations on \( C^{1,\alpha} \) domains are considered.

We end this section by listing some notations.

**Notation.**
1. \( e_i = (0, \ldots, 0, 1, \ldots, 0) \) = \( i^{th} \) standard coordinate vector.
2. \( x' = (x^1, x^2, \ldots, x^{n-1}) \) and \( x = (x', x^n) \).
3. \( \mathbb{R}^+_n = \{ x \in \mathbb{R}^n : x^n > 0 \} \).
4. \( B_r(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < r \} \) and \( B^+_r(x_0) = B_r(x_0) \cap \mathbb{R}^+_n \).
5. \( B'_r = \{ x' \in \mathbb{R}^{n-1} : |x'| < r \} \) and \( T_r = \{ (x', 0) : x' \in B'_r \} \).
6. \( \Omega_r(x_0) = \Omega \cap B_r(x_0) \) and \( (\partial \Omega)_r(x_0) = \partial \Omega \cap B_r(x_0) \). We omit \( x_0 \) when \( x_0 = 0 \).
7. \( \text{diam}(E,F) = \text{distance from } E \text{ to } F, \forall E,F \subset \mathbb{R}^n \).
8. \( \text{dist}(E,F) = \text{distance from } E \text{ to } F, \forall E,F \subset \mathbb{R}^n \).
9. \( p' = p/(p-1) \) for \( 1 < p < \infty \).

2. **Whitney decomposition**

In what follows, by a cube we mean a closed cube in \( \mathbb{R}^n \), with sides parallel to the axes. We say two such cubes are disjoint if their interiors are disjoint.

**Lemma 2.1.** (Whitney decomposition) Let \( E \) be a non-empty closed set in \( \mathbb{R}^n \) and \( \Omega \) be its complement. Then there exists a sequence of cubes \( Q_k \) (called the Whitney cubes of \( \Omega \)) such that
\begin{enumerate}
\item \( \Omega = \bigcup_{k=1}^{\infty} Q_k; \)
\item \( \Omega \) is a disjoint union of \( Q_k; \)
\end{enumerate}
Lemma 2.4. For any \( \bar{Q}_k \) from Lemma 2.1 (iii) that

\[ \text{Since } 0 \in \partial \Omega_{r/3}, \text{ there exists a point } y \in \bar{Q}_k \text{ with } |y| \geq r. \]

Proof. If not, there exist a point \( x \in \Omega_{r/3} \) and a cube \( Q_k \) such that \( x \in Q_k \) but \( \bar{Q}_k \not\subset \Omega_{r/3} \). It follows that there exists a point \( y \in \bar{Q}_k \) with \( |y| \geq r \). Then we deduce from Lemma 2.1 (iii) that

\[ \text{dist}(Q_k, \partial \Omega_1) \geq \text{diam}Q_k = \frac{5}{6} \text{diam} \bar{Q}_k \geq \frac{5}{6}(|y| - |x|) \geq \frac{5}{9}r. \]

Since \( x \in Q_k \cap \Omega_{r/3} \) and \( 0 \in \partial \Omega_1 \),

\[ \text{dist}(Q_k, \partial \Omega_1) \leq |x| \leq r/3. \]

Thus we get a contradiction. \( \square \)

Lemma 2.2. Let \( Q_k \) be as in Lemma 2.1 and \( \tilde{Q}_k = \frac{5}{3}Q_k \) be \( \frac{5}{3} \)-dilation of \( Q_k \) with respect to its center. Then

(i) \( \Omega = \bigcup_{k=1}^{\infty} \tilde{Q}_k \);

Each point of \( \Omega \) is contained in at most \( 12^n \) of the cubes \( \tilde{Q}_k \).

Lemma 2.3.

\[ \Omega_{r/3} \subset \bigcup_{Q_k \subset \Omega_r} Q_k \] \quad (2.1)

Proof. For further calculation, we set \( d = \text{diam}Q_k \) and \( d_k = \text{diam}Q_k \).

Lemma 2.4. For any \( x_0 \in \Omega_{r/3} \) and \( r > 0 \) with \( \Omega_r(x_0) \subset \Omega_1 \),

\[ |\Omega_{r}(x_0) \cap \{ \text{dist}(x, (\partial \Omega_1) \leq d) \} | \leq Cr^{n-1}d \quad \text{for } d > 0. \] \quad (2.2)

where \( C \) depends only on \( n \) and \( K \).

Proof. Since \( 0 \in \partial \Omega_1 \), \( \Omega_1 = \{ x^n > \varphi(x') \} \cap B_1 \) and \( (\partial \Omega_1) = \{ x^n = \varphi(x') \} \cap B_1 \) with \( ||\varphi||_{C^{0,1}(B_1)} \leq K \), we have

\[ \Omega_r(x_0) \cap \{ \text{dist}(x, (\partial \Omega_1) \leq d) \} \subset \{ |x' - x_0'| \leq r, \varphi(x') \leq x^n \leq \varphi(x') + (K+1)d \}. \]

Since \( ||x' - x_0'| \leq r, \varphi(x') \leq x^n \leq \varphi(x') + (K+1)d \) \leq Cr^{n-1}d, \) we have

\[ |\Omega_{r}(x_0) \cap \{ \text{dist}(x, (\partial \Omega_1) \leq d) \} | \leq Cr^{n-1}d, \]

where \( C \) depends only on \( n \) and \( K \). \( \square \)
Lemma 2.5. If $q > n - 1$, then
\[ \sum_{Q_k \subset \Omega_{1/4}} d_k^q \leq C, \] (2.4)
where $C$ depends only on $n$, $q$ and $K$.

Proof. If $q \geq n$, (2.4) is obvious. In the following, we consider the case when $q < n$.

For any $Q_k \in F^s$, there exists $y_k \in (\partial \Omega_1)$ such that
\[ \text{dist}(Q_k, y_k) = \text{dist}(Q_k, \partial \Omega_1) \leq 4d_k \leq 2^{-s+2}, \]
where Lemma 2.4 (iii) is used. It follows from $d_k \leq 2^{-s}$ for $Q_k \in F^s$ that
\[ \text{dist}(x, (\partial \Omega)_1) \leq d_k + \text{dist}(Q_k, y_k) \leq 2^{-s} + 2^{-s+2} \leq 2^{-s+3}, \forall x \in Q_k \text{ and } Q_k \in F^s \]
and then
\[ F^s \subset \Omega_{1/4} \cap \{ \text{dist}(x, (\partial \Omega)_1) \leq 2^{-s+3} \}. \] (2.5)

By Lemma 2.4 we obtain
\[ |F^s| \leq C 2^{-s}, \] (2.6)
where $C$ depends only on $n$ and $K$.

Observe that
\[ \bigcup_{Q_k \subset \Omega_{1/4}} \bigcup_{s=2} \bigcup_{Q_k \in F^s} Q_k. \]

If $q > n - 1$, we derive from (2.8) and (2.6) that
\[ \sum_{\tilde{Q}_k \subset \Omega_{1/4}} d_k^q \leq \sum_{s=2} \left\{ \sum_{Q_k \in F^s} \left( d_k^{q-n} \cdot d_k^n \right) \right\} \leq \sum_{s=2} \left\{ 2^{-s(q-n)} \cdot \sum_{Q_k \in F^s} d_k^n \right\} \leq C \sum_{s=2} 2^{-s(q-n)} |F^s| \leq C \sum_{s=2} 2^{-s(q-n) + 1} \leq C, \]
where $C$ depends only on $n$, $q$ and $K$. \qed

Fix $s_0 \geq 2$ and a cube $Q_k \in F^{s_0}$. We classify the cubes $Q_l \in F^s$ according to
\[ \text{dist}(Q_l, Q_k) \]
\[ F^{s,j}_{Q_k} = \begin{cases} \bigcup_{l} \{ Q_l \in F^s, \text{dist}(Q_l, Q_k) \leq 2^{-s_0+5} \}, & j = 0, \\ \bigcup_{l} \{ Q_l \in F^s, 2^{-s_0+j+4} < \text{dist}(Q_l, Q_k) \leq 2^{-s_0+j+5} \}, & j \geq 1. \end{cases} \] (2.7)

First, we specify range of indexes $j$ and $s$ such that $F^{s,j}_{Q_k} = \emptyset$. Since $F^{s,j}_{Q_k} \subset F^s \subset \Omega_{1/4}$, we see that $F^{s,j}_{Q_k} = \emptyset$ for $j > s_0$. For any $Q_l \in F^{s,j}_{Q_k}$, we have $Q_l \in F^s$ and then
\[ 2^{-s-1} < d_l \leq \text{dist}(Q_l, (\partial \Omega)_1) \leq \text{dist}(Q_l, Q_k) + \text{dist}(Q_k, (\partial \Omega)_1) + \text{diam}Q_k \]
\[ = \text{dist}(Q_l, Q_k) + \text{dist}(Q_k, (\partial \Omega)_1) + d_k \leq 2^{-s_0+j+5} + 2^{-s_0+j+3} + 2^{-s_0} \leq 2^{-s_0+j+6}, \]
where Lemma 2.4 (iii) is used. Hence, $F^{s,j}_{Q_k} = \emptyset$ for $s < s_0 - j - 6$. In conclusion, for any fixed $s_0 \geq 2$ and $Q_k \in F^{s_0}$,
\[ F^{s,j}_{Q_k} = \emptyset \text{ for } j > s_0 \text{ or } s < s_0 - j - 6. \] (2.8)
and by (2.1),
\[ \Omega_{1/12} \subset \bigcup_{Q_l \subset \Omega_{1/4}} Q_l = \bigcup_{j=0}^{s_0} \bigcup_{s=s_0-j-6}^{\infty} F_{Q_k}^{s,j}. \] (2.9)

**Lemma 2.6.** Fix \( s_0 \geq 2 \) and a cube \( Q_k \in F^{s_0} \). There exists a constant \( C \) depending only on \( n \) and \( K \) such that
\[ |F_{Q_k}^{s,j}| \leq C2^{2(-s_0+j)(n-1)-s}. \] (2.10)

**Proof.** By (2.8), we only consider the case when \( 0 \leq j \leq s_0 \) and \( s \geq s_0 - j - 6 \). Since \( Q_k \in F^{s_0} \subset \Omega_{1/4} \), there exists \( y_k \in (\partial \Omega)_1 \) such that
\[ \text{dist}(Q_k, y_k) = \text{dist}(Q_k, \partial \Omega_1) \leq 4d_k \leq 2^{-s_0+2}, \]
where Lemma 2.4 (iii) is used. It follows that for any \( Q_l \in F_{Q_k}^{s,j} \),
\[ \text{dist}(Q_l, y_k) \leq \text{dist}(Q_l, Q_k) + \text{dist}(Q_k, y_k) + \text{diam}Q_k \]
\[ = \text{dist}(Q_l, Q_k) + \text{dist}(Q_k, y_k) + d_k \leq 2^{-s_0+j+5} + 2^{-s_0+2} + 2^{-s_0} \leq 2^{-s_0+j+6} \]
and by \( d_l \leq 2^{-s} \leq 2^{-s_0+j+6} \),
\[ \text{dist}(x, y_k) \leq d_l + \text{dist}(Q_l, y_k) \leq 2^{-s_0+j+6} + 2^{-s_0+j+6} \leq 2^{-s_0+j+7}, \forall x \in Q_l. \]
Combining the above estimate and (2.5), we obtain
\[ F_{Q_k}^{s,j} \subset \Omega_{2^{-s_0+j+7}}(y_k) \cap \{\text{dist}(x, (\partial \Omega)_1) \leq 2^{-s+3}\}. \]
(2.10) then follows by Lemma 2.4. \(\square\)

**Lemma 2.7.** Fix a cube \( Q_l \in F^s \) and change \( Q_k \in F^{s_0} \). Then there exist at most \( C2^{j(n-1)} \) cubes \( Q_k \in F^{s_0} \) such that \( Q_l \in F_{Q_k}^{s,j} \), where \( C \) depends only on \( n \) and \( K \).

**Proof.** By (2.8), we only consider the case when \( 0 \leq j \leq s_0 \) and \( s \geq s_0 - j - 6 \). If \( Q_l \in F_{Q_k}^{s,j} \) then \( \text{dist}(Q_l, Q_k) \leq 2^{-s_0+j+5} \). It then follows from \( d_k \leq 2^{-s_0} \) and \( d_l \leq 2^{-s} \leq 2^{-s_0+j+6} \) that
\[ \text{dist}(x, x_i) \leq d_k + \text{dist}(Q_l, Q_k) + d_l \leq 2^{-s_0+j+7}, \forall x \in Q_k, \]
where \( x_i \) is the center of \( Q_l \). We deduce from (2.5) and the above estimate that
\[ \bigcup_k \{Q_k : Q_k \in F^{s_0} \text{ s.t. } Q_l \in F_{Q_k}^{s,j}\} \subset \{ |x' - x_i| \leq 2^{-s_0+j+7}, \text{dist}(x, (\partial \Omega)_1) \leq 2^{-s_0+3}\} \]
and then by Lemma 2.4
\[ \left| \bigcup_k \{Q_k : Q_k \in F^{s_0} \text{ s.t. } Q_l \in F_{Q_k}^{s,j}\} \right| \leq C2^{2(-s_0+j)(n-1)-s_0} = C2^{j(n-1)-ns_0}, \]
where \( C \) is a constant depending only on \( n \) and \( K \). Since for any \( Q_k \in F^{s_0} \), \( |Q_k| = c_n d_k^n \geq c_n 2^{-ns_0-n} \) for some constant \( c_n \) depending only on \( n \), we have
\[ \bigcup_k \{Q_k : Q_k \in F^{s_0} \text{ s.t. } Q_l \in F_{Q_k}^{s,j}\} \]
contains at most \( \frac{2^n C}{c_n} 2^{j(n-1)} \) cubes in \( F^{s_0} \). The lemma is thus proved. \(\square\)
3. Preliminary results

We start with the following \(L^p\) estimates.

**Lemma 3.1.** Let \(\Omega\) be a Lipschitz domain in \(\mathbb{R}^n\) and \(u\) satisfy
\[
\begin{aligned}
\Delta u &= f \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]
where \(f \in L^p(\Omega)\) for \(1 < p < \infty\) and \(\text{supp} f \subset D \subset \Omega\).

Then we have for any measurable set \(E \subset \Omega\),
\[
|u|_{L^p(E)} \leq C|E|^{\frac{1}{np}}|D|^{\frac{1}{2p'}}||f||_{L^p(D)},
\]
(3.1)
where \(C\) depends only on \(n\) and \(p\).

**Proof.** Let \(G = G(x, y)\) be the (Dirichlet) Green's function of the operator \(-\Delta\) on the domain \(\Omega\). By Green's representation formula, we have
\[
u(x) = \int_{\Omega} G(x, y) f(y) dy, \quad \forall x \in \Omega.
\]

Let \(\Gamma = \Gamma(x, y)\) be the normalized fundamental solution of Laplace's equation. For \(n \geq 3\), by comparison principle,
\[
0 \leq G(x, y) \leq \Gamma(x, y) = C_n |x-y|^{2-n}, \quad \forall x, y \in \Omega,
\]
where \(C_n\) depends only on \(n\). Hence,
\[
|u(x)| \leq C_n \int_D |x-y|^{2-n} |f(y)| dy, \quad \forall x \in \Omega.
\]

By Hölder's inequality,
\[
\int_D |x-y|^{2-n} |f(y)| dy = \int_D |x-y|^{\frac{2-n}{p'}} |x-y|^{\frac{2}{np}} |f(y)| dy
\]
\[
\leq \left( \int_D |x-y|^{2-n} dy \right)^{\frac{1}{p'}} \left( \int_D |x-y|^{2-n} |f(y)|^p dy \right)^{\frac{1}{p}}.
\]

Choose \(R > 0\) such that \(|D| = |B_R(x)|\) and we deduce that
\[
\int_D |x-y|^{2-n} dy \leq \int_{B_R(x)} |x-y|^{2-n} dy \leq C_n R^2 \leq C_n |D|^\frac{2}{p'}, \quad \forall x \in \mathbb{R}^n.
\]

Therefore,
\[
|u(x)| \leq C|D|^{\frac{1}{p'}} \left( \int_D |x-y|^{2-n} |f(y)|^p dy \right)^{\frac{1}{p}},
\]
where \(C\) depends only on \(n\) and \(p\). It follows that
\[
\int_E |u(x)|^p dx \leq C|D|^{\frac{2}{p'}} \int_E \int_D |x-y|^{2-n} |f(y)|^p dy dx
\]
\[
\leq C|D|^{\frac{2}{p'}} \int_D |f(y)|^p dy \sup_{y \in D} \int_E |x-y|^{2-n} dx.
\]

Similar to the derivation of (3.2), we have
\[
\sup_{y \in D} \int_E |x-y|^{2-n} dx \leq C|E|^\frac{2}{p'}
\]
and then
\[ \int_E |u(x)|^p \, dx \leq C|E|^{\frac{2}{p}}|D|^{\frac{2n}{p}} \int_D |f(y)|^p \, dy, \]
which implies \(E_\alpha\) for \(n \geq 3\).

The proof for \(n = 2\) is similar and we omit it here. \(\square\)

The next lemma concerns pointwise boundary \(C^{1,\alpha}\) estimates and we refer to Theorem 1.6 in [8] for its proof.

**Lemma 3.2.** Assume \(0 \in \partial\Omega\) and there exists \(\varphi \in C^{1,\alpha}(B_\epsilon')\) such that
\[ \Omega_1 = B_1 \cap \{x^n > \varphi(x')\} \quad \text{and} \quad (\partial\Omega)_1 = B_1 \cap \{x^n = \varphi(x')\} \]
for \(0 < \alpha < 1\). Let \(u\) satisfy
\[ \begin{cases} 
\Delta u = f & \text{in} \quad \Omega_1, \\
u = g & \text{on} \quad (\partial\Omega)_1,
\end{cases} \]
where \(g \in C^{1,\alpha}(0)\) and \(f \in L^p(\Omega_1)\) such that
\[ ||f||_{L^p(\Omega_1)} \leq K_f r^{-\alpha}, \quad \forall 0 < r < 1 \]
for some constant \(K_f\). Then \(u \in C^{1,\alpha}(0)\), i.e., there exists an affine function \(l\) such that
\[ |u(x) - l(x)| \leq C|x|^1 + (||u||_{L^\infty(\Omega_1)} + K_f + ||g||_{C^{1,\alpha}(0)}) , \quad \forall x \in \Omega_{r_0} \]
and
\[ |Dl| \leq C(||u||_{L^\infty(\Omega_1)} + K_f + ||g||_{C^{1,\alpha}(0)}), \]
where \(C\) and \(r_0\) depends on \(n, \alpha\) and \(||\varphi||_{C^{1,\alpha}(B_\epsilon')}\).

**Remark 3.3.** If \(f \in L^p(\Omega_1)\) for \(p > n\), then by Hölder’s inequality, we have
\[ ||f||_{L^p(\Omega_1)} \leq C_n r^{1-n/p}, \quad \forall 0 < r < 1, \]
where \(C_n\) depends only on \(n\). From Lemma 3.2, we obtain pointwise boundary \(C^{1,\min\{\alpha, 1-n/p\}}\) regularity, which is optimal by Sobolev embedding theorem.

**Corollary 3.4.** Let \(u\) satisfy
\[ \begin{cases} 
\Delta u = 0 & \text{in} \quad \Omega_r, \\
u = 0 & \text{on} \quad (\partial\Omega)_r,
\end{cases} \]
with \(0 < r \leq 1\). Then \(u\) is \(C^{1,\alpha}\) at \(x_0\) for any \(x_0 \in (\partial\Omega)_{r/2}\), i.e., there exists an affine function \(l_{x_0}\) such that for any \(1 < p < \infty\),
\[ |u(x) - l_{x_0}(x)| \leq C r^{-1(1+\alpha+n/p)}|x - x_0|^{1+\alpha}||u||_{L^p(\Omega_r)}, \quad \forall x \in \Omega_{3r/4} \quad (3.3) \]
and
\[ |Dl_{x_0}| \leq C r^{1-n/p}||u||_{L^p(\Omega_r)}, \quad (3.4) \]
where \(C\) depends on \(n, \alpha, p\) and \(||\varphi||_{C^{1,\alpha}(B_\epsilon')}\).

**Proof.** We may assume that \(r = 1\). By boundary local maximum principle (see Theorem 9.26 in [5]), we have \(||u||_{L^\infty(\Omega_{3r/4})} \leq C||u||_{L^p(\Omega_1)}\), where \(C\) depends only on \(n\) and \(p\). Then \((3.3)\) and \((3.4)\) follow from Lemma 3.2 and standard scaling arguments. \(\square\)
We end this section by the following local boundary $W^{2,p}$ estimates for harmonic functions on $C^{1,\alpha}$ domains.

**Theorem 3.5.** Let $1 < p < \infty$ and $1 - 1/p < \alpha \leq 1$. Assume that $0 \in \partial\Omega$ and there exists $\varphi \in C^{1,\alpha}(B_1^1)$ such that

$$\Omega_1 = B_1 \cap \{x^n > \varphi(x')\} \quad \text{and} \quad (\partial\Omega)_1 = B_1 \cap \{x^n = \varphi(x')\}.$$ 

If $u \in W^{2,p}(\Omega_1)$ satisfies

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega_1, \\
u = 0 & \text{on } (\partial\Omega)_1,
\end{cases}
\]

then we have

$$||D^2u||_{L^p(\Omega_{1/12})} \leq C||u||_{L^p(\Omega_1)},$$

where $C$ depends on $n, \alpha, p$ and $||\varphi||_{C^{1,\alpha}(B_1^1)}$.

**Proof.** Let $\{Q_k\}_{k=1}^\infty$ be Whitney decomposition of $\Omega_1$ and $\tilde{Q}_k = \frac{4}{5}Q_k$. For any $\tilde{Q}_k \subset \Omega_{1/4}$, we let $y_k \in (\partial\Omega)_{1/2}$ and $\tilde{x}_k \in \partial\tilde{Q}_k$ such that

$$|\tilde{x}_k - y_k| = \text{dist}(\tilde{Q}_k, \partial\Omega_1) < \text{dist}(Q_k, \partial\Omega_1) \leq 4d_k,$$

where Lemma 2.1 (iii) is used in the last inequality. Consequently, we see that

$$|x - y_k| \leq |x - \tilde{x}_k| + |\tilde{x}_k - y_k| \leq 6d_k, \quad \forall x \in \tilde{Q}_k.$$ 

It then follows from Corollary 3.4 that $u \in C^{1,\alpha}(y_k)$ and there exists an affine function $l_{y_k}$ (written by $l$ for simplicity in the following) such that

$$|(u - l)(x)| \leq C|x - y_k|^{1+\alpha}||u||_{L^p(\Omega_1)} \leq Cd_k^{1+\alpha}||u||_{L^p(\Omega_1)}, \quad \forall x \in \tilde{Q}_k,$$

where $C$ depends on $n, \alpha, p$ and $||\varphi||_{C^{1,\alpha}(B_1^1)}$.

Since $u - l$ satisfies $\Delta(u - l) = 0$ in $\tilde{Q}_k$, we deduce from interior $W^{2,p}$ estimate and the above pointwise $C^{1,\alpha}$ estimate that

$$||D^2(u - l)||_{L^p(Q_k)} \leq C\sup_{Q_k}||u||_{L^p(\Omega_1)} \leq Cd_k^{\alpha/p - 2}||u - l||_{L^p(\tilde{Q}_k)} \leq Cd_k^{\alpha/p + \alpha - 1}||u||_{L^p(\Omega_1)},$$

where $C$ depends on $n, \alpha, p$ and $||\varphi||_{C^{1,\alpha}(B_1^1)}$.

In conclusion, since $n - (1 - \alpha)p > n - 1$ as $1/(1 - p) < \alpha \leq 1$, we infer from Lemma 2.3 (3.6) and Lemma 2.5 that

$$||D^2u||_{L^p(\Omega_{1/12})}^p \leq \sum_{\tilde{Q}_k \subset \Omega_{1/4}} ||D^2u||_{L^p(Q_k)}^p \leq C||u||_{L^p(\Omega_1)}^p \sum_{\tilde{Q}_k \subset \Omega_{1/4}} d_k^{\alpha - (1-\alpha)p} \leq C||u||_{L^p(\Omega_1)}^p,$$

where $C$ depends on $n, \alpha, p$ and $||\varphi||_{C^{1,\alpha}(B_1^1)}$. \qed

**Remark 3.6.** Theorem 3.4 follows from interior $W^{2,p}$ estimate, boundary $C^{1,\alpha}$ estimate (Corollary 3.4) and Whitney decomposition (Lemma 2.1, 2.2 and 2.7). If we apply this argument to non-homogeneous equation $\Delta u = f$, then $p > n$ is needed and we can only arrive at

$$||D^2u||_{L^p(\Omega_{1/12})} \leq C(||u||_{L^p(\Omega_1)} + ||f||_{L^p(\Omega_1)}) \quad \text{with} \quad 1 \leq p_0 < \min\{1/(1-\alpha), p/n\}.$$
since only \( u \in C^{1,\min\{\alpha,1-\frac{p}{n}\}} \) can be obtained as \( p > n \) (cf. Remark \( \PageIndex{3.3} \)). To improve the above estimate and remove the restriction \( p > n \), we need to decompose \( u \) according to Whitney decomposition (see more details in Section 4).

4. \( W^{2,p} \) estimate for Poisson’s equation on \( C^{1,\alpha} \) domain

By considering \( u - g \) and using the technique of perturbation from the constant coefficient case, Theorem \( \PageIndex{4.1} \) follows easily from the following \( W^{2,p} \) estimates for Poisson’s equation.

**Theorem 4.1.** Let \( 1 < p < \infty \) and \( 1 - 1/p < \alpha \leq 1 \). Assume \( 0 \in \partial \Omega \) and there exists \( \varphi \in C^{1,\alpha}(B_{1}') \) such that
\[
\Omega_1 = B_1 \cap \{x^n > \varphi(x')\} \quad \text{and} \quad (\partial \Omega)_1 = B_1 \cap \{x^n = \varphi(x')\}.
\]
If \( u \in W^{2,p}(\Omega_1) \) and \( f \in L^p(\Omega_1) \) such that
\[
\begin{align*}
\Delta u &= f & \text{in } \Omega_1, \\
u &= 0 & \text{on } (\partial \Omega)_1,
\end{align*}
\]
then we have
\[
||D^2u||_{L^p(\Omega_{1/2})} \leq C(||u||_{L^p(\Omega_1)} + ||f||_{L^p(\Omega_1)}),
\]
where \( C \) depends on \( n, \alpha, p \) and \( ||\varphi||_{C^{1,\alpha}(B_{1}')} \).

**Proof.** Let \( \{Q_l\}_{l=1}^{\infty} \) be Whitney decomposition of \( \Omega_1 \) and \( \tilde{Q}_l = \frac{1}{2}Q_l \). We separate \( u \) to be
\[
u = w
\]
such that
\[
\begin{align*}
\Delta v &= f^\chi_{\cup Q_l \subset \Omega_{1/4}} & \text{in } \Omega_1, \\
v &= u & \text{on } \partial \Omega_1
\end{align*}
\]
and
\[
\begin{align*}
\Delta w &= f^\chi_{\cup Q_l \subset \Omega_{1/4}} & \text{in } \Omega_1, \\
w &= 0 & \text{on } \partial \Omega_1.
\end{align*}
\]

Since, by Lemma \( \PageIndex{2.3} \), \( \Omega_{1/2} \subset \bigcup Q_k \subset \Omega_{1/4} \), we have \( v \) is harmonic in \( \Omega_{1/2} \). And then by Theorem \( \PageIndex{4.3} \)
\[
||D^2v||_{L^p(\Omega_{1/2})} \leq C(||v||_{L^p(\Omega_{1/2})} \leq C(||u||_{L^p(\Omega_1)} + ||f||_{L^p(\Omega_1)}).
\]
Our sequent work is devoted to prove the following estimate:
\[
||D^2w||_{L^p(\Omega_{1/2})} \leq C||f||_{L^p(\Omega_1)}
\]
where \( C \) depends on \( n, \alpha, p \) and \( ||\varphi||_{C^{1,\alpha}(B_{1}')} \).

For this purpose, we decompose \( w \) according to Whitney decomposition as follows. Set
\[
\mathcal{F}^s = \bigcup_{k} \{Q_k : 2^{-s-1} < d_k \leq 2^{-s}, \tilde{Q}_k \subset \Omega_{1/4}\}, \quad s = 2, 3, ...
\]
as in \( \PageIndex{2.3} \). Fix \( s_0 \geq 2 \) and \( Q_k \in \mathcal{F}^{s_0} \). Let
\[
\mathcal{F}^{s_0}_j = \begin{cases} 
\bigcup_{l} \{Q_l \in \mathcal{F}^s, \text{dist}(Q_l, Q_k) \leq 2^{-s_0+5}\}, & j = 0, \\
\bigcup_{l} \{Q_l \in \mathcal{F}^s, 2^{-s_0+j+4} < \text{dist}(Q_l, Q_k) \leq 2^{-s_0+j+5}\}, & j \geq 1
\end{cases}
\]
as in (2.7) and \( w^{s,j}_k \) satisfy
\[
\begin{align*}
\Delta w^{s,j}_k &= f_{\mathcal{F}^s_{Q_k}} \quad \text{in} \quad \Omega_1, \\
\quad w^{s,j}_k &= 0 \quad \text{on} \quad \partial \Omega_1.
\end{align*}
\] (4.5)
Recall (2.40), that is,
\[
\Omega_{1/2} \subset \bigcup_{Q_k \subset \Omega_{1/4}} Q_k = \bigcup_{j=0}^{s_0} \bigcup_{s=-j-6}^{\infty} \mathcal{F}^s_{\tilde{Q}_k}.
\]
It then follows that
\[
w = \sum_{j=0}^{s_0} \sum_{s=-j-6}^{\infty} w^{s,j}_k \quad \text{in} \quad \Omega_1.
\] (4.6)
We need the following estimate of \( w^{s,j}_k \).

**Lemma 4.2.** Fix \( s_0 \geq 2 \) and \( Q_k \in \mathcal{F}^{s_0} \). Let
\[
m = s - s_0 \quad \text{and} \quad \beta = \alpha + \frac{n}{p} \cdot \left( \frac{2}{n+p} \right) - 1.
\] (4.7)
Then
\[
||D^2 w^{s,j}_k||_{L^p(Q_k)} \leq C 2^{-j \beta \frac{2np}{n+p}} ||f||_{L^p(\mathcal{F}^s_{\tilde{Q}_k})},
\] (4.8)
where \( C \) depends on \( n, \alpha, p \) and \( ||\varphi||_{C^{1,\alpha}(B_1)} \).

**Proof.** We divide the proof of (4.8) into two cases: \( j = 0 \) and \( j \geq 1 \).

(i) As \( j = 0 \), by (4.5) and Lemma 3.1, we have
\[
||w^{s,j}_k||_{L^p(Q_k)} \leq C ||\tilde{Q}_k||^{\frac{2}{np}} ||f||_{L^p(\mathcal{F}^s_{\tilde{Q}_k})}.
\]
Since \( Q_k \in \mathcal{F}^{s_0} \), \( |\tilde{Q}_k| \leq 2^{-s_0 n} \). By Lemma 3.1, \( |\mathcal{F}^{s_0}_{\tilde{Q}_k}| \leq C 2^{-s_0(n-1)-s} \). Thus,
\[
||w^{s,j}_k||_{L^p(Q_k)} \leq C 2^{-2s_0 \frac{2np}{n+p}} ||f||_{L^p(\mathcal{F}^s_{\tilde{Q}_k})}.
\]
In view of \( \Delta w^{s,0}_k = f_{\mathcal{F}^s_{\tilde{Q}_k}} \) in \( \tilde{Q}_k \), we deduce from interior \( W^{2,p} \) estimate that
\[
||D^2 w^{s,0}_k||_{L^p(Q_k)} \leq C \left( d_k^{-2} ||w^{s,0}_k||_{L^p(\tilde{Q}_k)} + ||f||_{L^p(\mathcal{F}^s_{\tilde{Q}_k})} \right)
\] (4.9)
\[
\leq C \left( 2^{-2s_0 \frac{2np}{n+p}} ||f||_{L^p(\mathcal{F}^s_{\tilde{Q}_k})} + ||f||_{L^p(\mathcal{F}^{s_0}_{\tilde{Q}_k})} \right).
\]
If \( m \leq -6 \), we infer from (2.8) that \( \mathcal{F}^{s,j}_{Q_k} = \emptyset \). Hence we only need consider \( m \geq -6 \). As \(-6 \leq m < 5 \), (4.8) follows easily from (4.9). As \( m \geq 5 \), we claim \( \mathcal{F}^{s,0}_{Q_k} \cap \tilde{Q}_k = \emptyset \). Actually, for any \( Q_l \in \mathcal{F}^{s_0}_{Q_k} \), we have
\[
dist(x, \partial \Omega_1) \leq d_l + \dist(Q_l, \partial \Omega_1) \leq 5d_l \leq 2^{-s+3} = 2^{-m-s_0+3} \leq 2^{-s_0-2}, \quad \forall x \in Q_l.
\]
However, since \( Q_k \in \mathcal{F}^{s_0} \),
\[
dist(\tilde{Q}_k, \partial \Omega_1) \geq \dist(Q_k, \partial \Omega_1) - d_k/5 \geq 4d_k/5 > 2^{-s_0-2}.
\]
Hence \( \mathcal{F}^{s,0}_{Q_k} \cap \tilde{Q}_k = \emptyset \) and then we derive (4.8) from (4.9).
Combining it with (4.10), we get
$$F \text{ then } x$$
From Corollary 3.4, it follows that there exists an affine function $l$ such that for any $x \in \Omega_{2^{-s_0+j+3}}(y_k)$,
$$|w^s,j_l(x)| \leq C2^{-(j-s_0)(1+\alpha+n/p)}|x-y_k|^{1+\alpha}||w^s,j||_{L^p(\Omega_{2^{-s_0+j+3}}(y_k))}.$$  

From (4.15) and Lemma 3.3, we deduce that
$$||w^s,j_l||_{L^p(\Omega_{2^{-s_0+j+3}}(y_k))} \leq C|B_{2^{-s_0+j+3}}|\frac{2^j}{p^r}||w^s,j||_{L^p(F_{Q_k}^s,j)} \leq C2^{-2(s_0-j-\frac{2-n\alpha}{np})}||f||_{L^p(F_{Q_k}^s,j)},$$
where $|F_{Q_k}^s,j| \leq C2^{(s_0+j)(n-1)-s}$ is used that is given by Lemma 2.6.

By (4.10), $\tilde{Q}_k \subset \Omega_{2^{-s_0+j+3}}(y_k)$ and then for any $x \in \tilde{Q}_k$,
$$|x-y_k| \leq 2^{-s_0+3}.$$

Combining above estimates, we obtain
$$||w^s,j_l||_{L^\infty(\tilde{Q}_k)} \leq C2^{-(j-s_0)(1+\alpha+n/p)-s_0(1+\alpha)}||w^s,j||_{L^p(\Omega_{2^{-s_0+j+3}}(y_k))} \leq C2^{-s_0(2-\beta)-j\beta-\frac{2n}{np}}||f||_{L^p(F_{Q_k}^s,j)},$$
where $\beta = \alpha + \frac{\alpha}{p} + \frac{2}{np^r}$ is defined by (4.17).

In view of (4.10) and (4.11),
$$\Delta(w^s,j_l) = 0 \text{ in } \tilde{Q}_k.$$

Using interior $W^{2,p}$ estimate,
$$||D^2w^s,j_l||_{L^p(Q_k)} \leq C\delta^{n/p-2}||w^s,j||_{L^\infty(\tilde{Q}_k)} \leq C2^{-j\beta-\frac{2n}{np}}||f||_{L^p(F_{Q_k}^s,j)},$$
where $C$ depends on $n, \alpha, p$ and $||\varphi||_{C^{1,\alpha}(B_{r/2}^j)}.$
Now we continue the proof of Theorem 4.1.

By Lemma 2.3, we deduce that

\[ \Omega_{1/12} \subset \bigcup_{\tilde{Q}_k \subset \Omega_{1/4}} Q_k = \bigcup_{s_0 = 2}^{\infty} \bigcup_{Q_k \in F^{s_0}} Q_k \]

and then

\[ \| D^2 w \|_{L^p(\Omega_{1/12})}^p \leq \sum_{Q_k \subset \Omega_{1/4}} \sum_{s_0 = 2}^{\infty} \| D^2 w \|_{L^p(Q_k)}^p = \sum_{s_0 = 2}^{\infty} \sum_{Q_k \in F^{s_0}} \| D^2 w \|_{L^p(Q_k)}^p. \]

From (4.6) and Minkowski’s inequality, it follows that

\[ \| D^2 w \|_{L^p(Q_k)} \leq \sum_{s = s_0 - j - 6}^{s_0} \| D^2 w^{s,j} \|_{L^p(Q_k)} \]

and then

\[ \| D^2 w \|_{L^p(\Omega_{1/12})}^p \leq \sum_{j = 0}^{\infty} \sum_{s = s_0 - j - 6}^{s_0} \left( \sum_{j = 0}^{\infty} \sum_{s = s_0 - j - 6}^{s_0} \| D^2 w^{s,j} \|_{L^p(Q_k)} \right)^p. \] (4.12)

Let \( \tau > 0 \) (depending on \( n, \alpha \) and \( p \)) to be determined later and by Hölder’s inequality,

\[
\left( \sum_{j = 0}^{s_0} \sum_{s = s_0 - j - 6}^{\infty} \| D^2 w^{s,j} \|_{L^p(Q_k)} \right)^p \leq C \sum_{j = 0}^{s_0} \left( \sum_{s = s_0 - j - 6}^{\infty} \| D^2 w^{s,j} \|_{L^p(Q_k)} \right)^p \]

\[
\leq C \sum_{j = 0}^{s_0} 2^{j \tau p} \left( \sum_{s = s_0 - j - 6}^{s_0} \| D^2 w^{s,j} \|_{L^p(Q_k)} \right)^p + 2^{j \tau p} \left( \sum_{s = s_0 + 1}^{\infty} \| D^2 w^{s,j} \|_{L^p(Q_k)} \right)^p. \]

Using Hölder’s inequality again,

\[
\left( \sum_{s = s_0 - j - 6}^{\infty} \| D^2 w^{s,j} \|_{L^p(Q_k)} \right)^p \leq C \sum_{s = s_0 - j - 6}^{\infty} 2^{(s - j - 6) \tau p} \| D^2 w^{s,j} \|_{L^p(Q_k)}^p
\]

and

\[
\left( \sum_{s = s_0 + 1}^{\infty} \| D^2 w^{s,j} \|_{L^p(Q_k)} \right)^p \leq C \sum_{s = s_0 + 1}^{\infty} 2^{(s - s_0 - j - 6) \tau p} \| D^2 w^{s,j} \|_{L^p(Q_k)}^p.
\]

Recall \( m = s - s_0 \) given by (4.7). We derive from the above estimates that

\[
\left( \sum_{j = 0}^{s_0} \sum_{s = s_0 - j - 6}^{\infty} \| D^2 w^{s,j} \|_{L^p(Q_k)} \right)^p \leq C \sum_{j = 0}^{s_0} \sum_{s = s_0 - j - 6}^{\infty} 2^{j |m| \tau p} \| D^2 w^{s,j} \|_{L^p(Q_k)}^p.
\]

Substitute it into (4.12) and consequently,

\[ \| D^2 w \|_{L^p(\Omega_{1/12})} \leq C \sum_{s_0 = 2}^{\infty} \sum_{Q_k \in F^{s_0}} \sum_{j = 0}^{s_0} \sum_{s = s_0 - j - 6}^{\infty} 2^{j |m| \tau p} \| D^2 w^{s,j} \|_{L^p(Q_k)}^p. \]

By exchanging summation order,

\[ \| D^2 w \|_{L^p(\Omega_{1/12})}^p \leq C \sum_{s = -4}^{\infty} \sum_{j = 0}^{s + j + 6} \sum_{s_0 = 2}^{\infty} \sum_{Q_k \in F^{s_0}} 2^{j |m| \tau p} \| D^2 w^{s,j} \|_{L^p(Q_k)}^p. \]
From Lemma 2.7, it follows that
\[
\|D^2w\|^{p}_{L^p(Q_1/2)} \leq C \sum_{s=-4}^{\infty} \sum_{j=0}^{s+j+6} \sum_{s_0=2}^{s+j+6} 2^{(j+6)|m|} \tau p - j\beta p - \frac{2np}{mp} + j(n-1) \|f\|^{p}_{L^p(F_{Q_k}^t)}.
\]
Since, by Lemma 2.7 for any fixed \( s, j \) and \( s_0 \),
\[
\sum_{Q_k \in F^{t_{s_0}}} \|f\|^{p}_{L^p(F_{Q_k}^t)} \leq \sum_{Q_k \in F^{t_{s_0}}} \sum_{Q_l \in F_{Q_k}^t} \|f\|^{p}_{L^p(Q_l)} \leq C 2^{j(n-1)} \|f\|^{p}_{L^p(F^t)},
\]
we have
\[
\|D^2w\|^{p}_{L^p(Q_1/2)} \leq C \sum_{s=-4}^{\infty} \sum_{j=0}^{s+j+6} \sum_{s_0=2}^{s+j+6} 2^{(j+6)|m|} \tau p - j\beta p - \frac{2np}{mp} + j(n-1) \|f\|^{p}_{L^p(F^t)}.
\]
(4.13)

Now we choose \( \tau \) as follows. Recall that \( \beta = \alpha + \frac{n}{p} + \frac{2}{np'} - 1 \). Since \( \alpha > 1 - 1/p \),
we have \( \beta - \frac{2}{np'} = \alpha + \frac{n}{p} - 1 > \frac{np}{n} \) and then take \( \tau > 0 \) small enough such that
\[
\beta - \frac{2}{np'} - 2\tau > \frac{n-1}{p} \quad \text{and} \quad \frac{2}{np'} > \tau.
\]
Set
\[
\sigma = \min\{p(\beta - \frac{2}{np'} - 2\tau) - (n-1), \ p(\frac{2}{np'} - \tau)\} > 0.
\]
As \( 2 \leq s_0 \leq s \), we have \( |m| = s - s_0 = m \) and then
\[
2^{(j+6)|m|} \tau p - j\beta p - \frac{2np}{mp} + j(n-1) \leq 2^{-j(p(\beta - \tau) - (n-1)) - mp(\frac{2}{np} - \tau)} \leq 2^{-(j+m)\sigma}.
\]
As \( s + 1 \leq s_0 \leq s + j + 6 \), we have \( |m| = s_0 - s = -m \leq j + 6 \) and then
\[
2^{(j+6)|m|} \tau p - j\beta p - \frac{2np}{mp} + j(n-1) \leq 2^{2(j+6)p - j\beta p + \frac{2(j+6)p}{np} + j(n-1)} \leq 2^{-j\sigma + 6\tau p + \frac{np}{n}} \leq C 2^{-j\sigma},
\]
where \( C \) depends on \( n \) and \( p \).
From the above two estimates, it follows that,
\[
\sum_{j=0}^{\infty} \sum_{s_0=2}^{s+j+6} 2^{(j+|m|)\tau p - j\beta p - \frac{2np}{mp} + j(n-1)} \leq \sum_{j=0}^{\infty} \sum_{s_0=2}^{s+j+6} 2^{-(j+m)\sigma} \leq C (j+6)2^{-j\sigma} \leq C.
\]
Substitute it into (4.13) and then
\[
\|D^2w\|^{p}_{L^p(Q_1/2)} \leq C \sum_{s=-4}^{\infty} \|f\|^{p}_{L^p(F^t)} \leq C \|f\|^{p}_{L^p(Q_1)},
\]
where \( C \) depends on \( n, \alpha, \beta \) and \( \|\varphi\|^{p}_{C^{1,\alpha}(B_1')} \). Thus, (4.1) holds. Combining (4.3) and (4.4), we conclude (4.2). \( \square \)
5. Fully nonlinear elliptic equation

In this section, we will exploit our method to fully nonlinear elliptic equations and the main result is the following theorem.

**Theorem 5.1.** Let $1 < p < \infty$ and $0 < \alpha_0 \leq \alpha \leq 1$. Assume that $\Omega$ is of class $C^{1,\alpha}$ with $0 \in \partial \Omega$ and $u$ is a solution of the following elliptic equation

$$F(D^2u, x) = f(x) \text{ in } \Omega_1 \text{ with } f \in L^p(\Omega_1). \quad (5.1)$$

Suppose $F$ satisfies interior $W^{2,p}$ estimate with constant $c_e$, that is, for any solution $v$ of (5.1) and any $B_r(x_0) \subset \Omega_1$,

$$\|D^2v\|_{L^p(B_{r/2}(x_0))} \leq c_e \left( r^{n/p-2} \|v\|_{L^\infty(B_r(x_0))} + \|f\|_{L^p(B_r(x_0))} \right) \quad (5.2)$$

and $u$ satisfies pointwise boundary $C^{1,\alpha_0}$ estimate with constant $c_b$, that is, for any $x_0 \in (\partial \Omega)_{1/2}$, there exists an affine function $l_{x_0}$ such that

$$|(u - l_{x_0})(x)| \leq c_b |x - x_0|^{1+\alpha_0} \text{ and } |Dl_{x_0}| \leq c_b. \quad (5.3)$$

Then we have the following two estimates:

(i) If $\alpha_0 > 1 - 1/p$, then

$$\|D^2u\|_{L^p(\Omega_{1/12})} \leq C \left( 1 + \|f\|_{L^p(\Omega_1)} \right), \quad (5.4)$$

where $C$ depends on $n$, $\alpha_0$, $p$, $c_e$, $c_b$, and $\Omega$.

(ii) If $\alpha_0 \leq 1 - 1/p$, then for any $1 \leq p_0 < 1/(1 - \alpha_0)$,

$$\|D^2u\|_{L^{p_0}(\Omega_{1/12})} \leq C \left( 1 + \|f\|_{L^p(\Omega_1)} \right), \quad (5.5)$$

where $C$ depends on $n$, $\alpha_0$, $p_0$, $c_e$, $c_b$ and $\Omega$.

**Proof.** Let $\{Q_k\}_{k=1}^\infty$ be Whitney decomposition of $\Omega_1$, $\tilde{Q}_k = 2^k Q_k$ and we first prove that for any $Q_k \subset \tilde{Q}_k \subset \Omega_{1/4}$,

$$\|D^2u\|_{L^p(Q_k)} \leq C(\delta_k^{n/p+\alpha_0-1} + \|f\|_{L^p(\tilde{Q}_k)}), \quad (5.6)$$

where $C$ depends on $c_e$ and $c_b$.

Indeed, since $\tilde{Q}_k \subset \Omega_{1/4}$, there exist two points $y_k \in (\partial \Omega)_{1/2}$ and $\tilde{x}_k \in \partial \tilde{Q}_k$ such that

$$|\tilde{x}_k - y_k| = \text{dist}(\tilde{Q}_k, \partial \Omega_1) < \text{dist}(Q_k, \partial \Omega_1) \leq 4\delta_k,$$

where Lemma 2.1 (iii) is used in the last inequality. Consequently,

$$|x - y_k| \leq |x - \tilde{x}_k| + |\tilde{x}_k - y_k| \leq 6\delta_k, \quad \forall x \in \tilde{Q}_k.$$

By (5.3), there exists an affine function $l_{y_k}$ (written as $l$ for simplicity in the following) such that

$$|(u - l)(x)| \leq c_b |x - y_k|^{1+\alpha_0} \leq c_b (6\delta_k)^{1+\alpha_0}, \quad \forall x \in \tilde{Q}_k.$$

Since $u - l$ still satisfies $F(D^2(u - l), x) = f(x)$, we have, by (5.2),

$$\|D^2(u - l)\|_{L^p(Q_k)} \leq c_e (\delta_k^{n/p-2} \|u - l\|_{L^\infty(Q_k)} + \|f\|_{L^p(\tilde{Q}_k)}),$$

$$\leq C(\delta_k^{n/p+\alpha_0-1} + \|f\|_{L^p(\tilde{Q}_k)}),$$

where $C$ depends on $c_e$ and $c_b$. Thus, (5.3) holds.
For any $1 \le q \le p$, by Hölder’s inequality, (5.6) and Young’s inequality, we deduce
\[
\int_{Q_k} |D^2u|^q \, dx \le C d_k^{n-1} (1/q) \left( \int_{Q_k} |D^2u|^p \, dx \right)^{q/p} 
\le C \left\{ d_k^{n-1} (1-\alpha_0)q + d_k^{n-qn/p} \left( \int_{\tilde{Q}_k} |f|^p \, dx \right)^{q/p} \right\}
\le C \left( d_k^{n-1} (1-\alpha_0)q + d_k^n + \int_{\tilde{Q}_k} |f|^p \, dx \right),
\]
where $C$ depends on $n, \alpha_0, p$ and $\Omega$. Set $q = p$ in (5.7) and it follows that
\[
\int_{Q_k} |D^2u|^p \, dx \le C \left( d_k^{n-1} (1-\alpha_0)p + d_k^n + \int_{\tilde{Q}_k} |f|^p \, dx \right).
\]
Since \(\Omega_{1/4} \subset \bigcup_{Q_k \subset \Omega_{1/4}} Q_k\), we deduce from the above two estimates that
\[
\int_{\Omega_{1/12}} |D^2u|^p \, dx \le \sum_{\tilde{Q}_k \subset \Omega_{1/4}} \int_{Q_k} |D^2u|^p \, dx 
\le C \sum_{\tilde{Q}_k \subset \Omega_{1/4}} \left( d_k^{n-1} (1-\alpha_0)p + d_k^n + \int_{\tilde{Q}_k} |f|^p \, dx \right) \le C \left( 1 + \int_{\Omega_{1/4}} |f|^p \, dx \right),
\]
where $C$ depends on $n, \alpha_0, p, \Omega$. This gives (5.4).

If $\alpha_0 \le 1 - 1/p$, then $p \ge 1/(1-\alpha_0)$. For any $1 \le p_0 < 1/(1-\alpha_0)$, we have $n - (1-\alpha_0)p_0 > n - 1$ and then by Lemma 2.5,
\[
\sum_{\tilde{Q}_k \subset \Omega_{1/4}} d_k^{n-1} (1-\alpha_0)p_0 \le C,
\]
where $C$ depends on $n, \alpha_0, p_0$ and $\Omega$. Since now $p_0 < p$, set $q = p_0$ in (5.7) and we obtain
\[
\int_{Q_k} |D^2u|^{p_0} \, dx \le C \left( d_k^{n-1} (1-\alpha_0)p_0 + d_k^n + \int_{\tilde{Q}_k} |f|^{p_0} \, dx \right).
\]
From \(\Omega_{1/12} \subset \bigcup_{Q_k \subset \Omega_{1/4}} Q_k\) and the above two estimates, we deduce that
\[
\int_{\Omega_{1/12}} |D^2u|^{p_0} \, dx \le \sum_{\tilde{Q}_k \subset \Omega_{1/4}} \int_{Q_k} |D^2u|^{p_0} \, dx 
\le C \sum_{\tilde{Q}_k \subset \Omega_{1/4}} \left( d_k^{n-1} (1-\alpha_0)p_0 + d_k^n + \int_{\tilde{Q}_k} |f|^{p_0} \, dx \right) \le C \left( 1 + \int_{\Omega_{1/4}} |f|^{p_0} \, dx \right),
\]
where $C$ depends on $n, \alpha_0, p, p_0, c_c, c_b$ and $\Omega$. Hence (5.5) holds. \qed
Remark 5.2. From Theorem 5.1, we see again that by Whitney decomposition, local boundary $W^{2,p}$ estimate follows from interior $W^{2,p}$ estimate and boundary $C^{1,\alpha}$ estimate which are assumed. As for interior $W^{2,p}$ estimate, we refer to [1] and Theorem 7.1 in [2]; as for boundary $C^{1,\alpha}$ estimate, we refer to [8] and [10].

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