On Adler-Bell-Jackiw Anomaly in 3-brane Scenario

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Abstract

We investigate the ABJ anomaly in the framework of an effective field theory for a 3-brane scenario and show that the contribution from induced gravity on the brane depends on both the topological structure of the bulk space-time and the embedding of the brane in the bulk. This fact implies the existence of a non-trivial vacuum structure of bulk quantum gravity. Furthermore, we argue that this axial gravitational anomaly may not necessarily be cancelled by choosing the matter content on the brane since it could be considered as a possible effect from bulk quantum gravity.

1. Introduction

The idea that our observable space-time may be a (3+1)-dimensional topological defect of some higher dimensional quantum field theory has persisted over a number of years \cite{1,2}. In this scenario, the observed elementary particles are the light particles trapped on the (3+1)-dimensional defect and the Standard Model (SM), which is believed to be the correct theory characterizing the interactions among these elementary particles, appears as a low-energy effective theory of a more fundamental theory in higher dimensions. The exception is the graviton: it mediates the quantum gravitational interaction, and it can propagate in the whole bulk due to the equivalence between gravity and space-time geometry. Of course, it might be possible that some other particles such as heavy fermions – beyond the reach of present accelerators – also exist in the bulk space-time. It is remarkable that such exotic yet simple considerations can address some fundamental problems. For example, it provides an alternative mechanism for solving the hierarchy problem \cite{4}, in contrast to those that modify the SM itself such as technicolor and supersymmetric extensions. It also gives a natural explanation as to why gravitational interactions are much weaker than other forces \cite{3}, and it even gives an alternative means for addressing the cosmological problem \cite{2,5}.

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This scenario has gained support from an application of non-perturbative superstring theory. A physically realistic example is that an $N = 1$ $SU(5)$ supersymmetric gauge theory with three generations of chiral matter fields can indeed come from one sector of type-I string theory compactified on the $T^6/Z_3$ orientifold with five $D3$-branes placed at orientifold fixed points \[6\]. In general, a crude argument ignorant of the specific brane configuration is the following: the 3-branes provide a natural setting for the $(3+1)$-dimensional space-time, the massless modes of the open string attached to the branes lead to the observed gauge and matter fields, while the graviton comes from the low-lying excited states of the closed string in bulk space-time. The above situation is called the brane world scenario \[8\].

An explicit feature of the above brane scenario is that bulk gravity is an essential ingredient. Consequently quantum gravity in the bulk can affect physics on the brane. Concretely speaking, in the case that the extra dimensional space is compact, in addition to the massless graviton trapped on the brane, the Kaluza-Klein (KK) states represent bulk gravity propagating in the extra dimensions get involved in physical processes occurring on the brane, some typical examples of which include the emission of the KK graviton, the new scattering of SM particles from the exchange of KK states and graviton and some higher order corrections \[9\]-\[11\]. In the case of large extra dimensions, these new physical effects might be accessible to testing via accelerator experiments in the near future \[3\],\[11\]. Some bold attempts have been made within this framework to explain the recent measured deviation of the muon anomalous magnetic moment from the SM prediction \[12\].

However, as a reformed setting for describing elementary particle interactions, particularly the role played by quantum gravity, some dynamical features associated with gravity should be reconsidered in this scenario. One typical problem is the famous Adler-Bell-Jackiw (ABJ) anomaly for the axial vector current in a chiral gauge theory defined on the brane. It is well known that this anomaly can get a contribution in a background space-time with non-trivial topology \[14\],\[15\], which is usually called the axial gravitational anomaly in contrast to the pure gravitational anomalies such as the Einstein and Lorentz anomalies that arise in $D = 4n + 2$ dimensions \[17\],\[18\].

The novel physical features are that bulk quantum gravity (which at compact extra dimensions is effectively represented by the dynamics of graviton, vector, and scalar fields and the corresponding KK modes etc) becomes a dynamical field rather than a static background, and that brane fluctuations can occur. It is natural to ask whether or not these dynamical effects modify the axial gravitational anomaly. Furthermore, if such contributions do arise, what becomes of anomaly cancellation for a quantum field theory defined on the brane? To our knowledge these issues have not been explicitly addressed in the literature, and we consider them in this paper.

2. A $U(1)$ chiral gauge field model in 3-brane world

Let us start first from a simple model describing the low-energy dynamics of a 3-brane, which includes the $D$-dimensional bulk gravity and the chiral fermions confined on the brane interacting with a $U(1)$ gauge field and bulk gravity through the induced metric,

$$S = S_{\text{B.G.}} + S_G + S_F$$

$$= \frac{\kappa^2}{8} \int d^D X \sqrt{-G} R - \frac{1}{4} \int d^4 x \epsilon^{X(x)} F_{\mu \nu} F^{\mu \nu}$$
We consider only the one-flavour case here and one may add a cosmological term for bulk gravity. The notation in the above action is standard, \(X^M = (x^\mu, y^r)\), \(M = 0, \cdots, D-1\), the local coordinate of bulk space-time, \(\mu = 0, \cdots, 3\) and \(r = 4, \cdots, D-1\) being the coordinates of the brane and extra dimensional space-time. To incorporate fermions on the brane, the induced vierbein \(e^a_\mu[X(x)]\) and its inverse \(e^\mu_a[X(x)]\) need some delicate consideration. As it is well known, fermions on the 3-brane are the spinorial representation of the local Lorentz group \(SO(1,3)\). To guarantee that this \(SO(1,3)\) group is identical to the appropriate subgroup of \(SO(1, D-1)\), the local Lorentz group of bulk space-time, one has to define \[3\]
\[
e^\mu_a[X(x)] \equiv R^a_A E^A_M(X) B^M_\mu, \quad B^M_\mu \equiv \partial_\mu X^M(x),
\]
where \(E^A_M(X)\) is the vierbein corresponding to bulk space-time metric \(G_{MN}(X) = \eta_{AB} E^A_M(X) E^B_N(X), A = (a, m)\) is the bulk Lorentz index, and \(a = 0, \cdots, 3\), \(m = 4, \cdots, D-1\). \(R(x)\) is actually an element of the bulk local \(SO(1, D-1)\) group depending only on the generator \(J^{am}\) \[3\],
\[
R(x) = \exp[i \theta_{am}(x) J^{am}],
\]
and must satisfy
\[
R^m_A E^A_M B^M_\mu = 0.
\]
(3)
The above two equations fix the requisite local Lorentz transformation to define the correct vierbein induced on the brane from the bulk metric. It has been shown that (2) and (3) indeed lead to the induced metric \[4\]
\[
g_{\mu\nu}[X(x)] = \eta_{ab} e^a_\mu[X(x)] e^b_\nu[X(x)] = G_{MN}(X) B^M_\mu B^N_\nu = \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^M}{\partial x^\nu} G_{MN}(X),
\]
and hence \(e[X(x)] = \sqrt{-g[X(x)]}\). Specifically the \(SO(1, D-1)\) transformation in the bulk, \(E^A_M(X) \rightarrow R^A_B(X) E^B_M\), will automatically lead to an \(SO(1,3)\) rotation on the induced vierbein, \(e^a_\mu \rightarrow r(x)^a_b e^b_\mu\). The fermions on the brane are just a spinor representation of \(r(x)^a_b\). In this sense, the spinor field on the brane is connected with the Lorentz symmetry in the bulk. Of course, the other possibility is that we can start directly from the representations of the Clifford group of the bulk space-time (i.e. the covering group of \(SO(1, D-1)\)) and then reduce it to the brane to get a spinor. However, since the irreducible representations of the Clifford group in higher dimensions depend heavily on the dimensionality, it seems to be impossible to get a unique chiral gauge theory. Thus we define the chiral fermions on the 3-brane from the representation of \(SO(1,3)\). For a \(U(1)\) gauge field and induced gravitational field on the brane, the operator \(D_\mu\) takes the usual form
\[
D_\mu = \partial_\mu - i Y A_\mu + \frac{1}{2} \omega_{\mu}^{ab} \sigma^{ab}, \quad \sigma^{ab} = \frac{1}{4} \{ \gamma^a, \gamma^b \}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]
where \(Y\) is the charge carried by chiral fermions and for simplicity we do not write out the gauge coupling explicitly. It should be emphasized that the concrete form of the induced
vierbein (2) and metric (4) depends on the dynamical behavior of the brane – either moving in the bulk or staying at a certain fixed point – but the physics is equivalent in these two cases.

It is easy to see that as for the usual 4-dimensional space-time case, the fermionic part has a brane coordinate-dependent vector and axial vector gauge transformation,

$$\psi(x) \rightarrow \exp[iY\theta(x)]\psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) \exp[-iY\theta(x)], \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \theta(x),$$

(5)

and

$$\psi(x) \rightarrow \exp[iY\gamma_5\vartheta(x)]\psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) \exp[iY\gamma_5\vartheta(x)], \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \vartheta(x),$$

(6)

as well as the reparametrization invariance of the brane, the infinitesimal version being [18]

$$x^\mu \rightarrow x^\mu - \xi^\mu(x), \quad \delta e^a_\mu(x) = e^a_\nu \nabla_\mu \xi^\nu + \xi^\nu \nabla_\nu e^a_\mu, \quad \delta e = \partial_\mu (\xi^\mu e),$$

$$\delta \omega^{ab}_\mu = \xi^\nu \partial_\nu \omega^{ab}_\mu + \omega^{ab}_\nu \partial_\mu \xi^\nu, \quad \delta \psi = \xi^\mu \partial_\mu \psi, \quad \delta \bar{\psi} = \xi^\mu \partial_\mu \bar{\psi}. \quad (7)$$

At the quantum level, the axial vector gauge symmetry cannot be simultaneously upheld with the vector gauge symmetry and the reparametrization invariance of the brane, and so becomes anomalous.

3. Quantization of 3-brane world and ABJ anomaly

Before turning to the axial gravitational anomaly, we briefly look at the quantization of the model. Even bypassing the renormalizability problem of bulk quantum gravity in the above effective field theory model, we still have no way to completely quantize the system with such a matter distribution. Since the brane world is somehow an effective field theory description, there are two perspectives one can adopt toward the quantization of such a physical system. The first perspective is that widely adopted in the literature: if the extra dimensions are compact, the bulk gravitational field in general admits an expansion in terms of the orthonormal modes living in the extra space and the KK modes on the world volume of 3-brane. In this framework, one can get an effective theory describing the interaction between the matter fields on the brane and the KK modes after integrating out the extra dimensions. The effects of the bulk gravitational field on the brane can then be detected by studying the quantization of this effective action. The second perspective is quite formal, but is universal to any brane world models regardless of what the extra space is like, either compact or having infinite size like the second class Randall-Sundrum model [20]. In this perspective (the one we shall adopt), one first quantizes the field theory on the brane, obtaining a quantum effective action relevant to bulk gravity, then subsequently considers the quantization of bulk gravity. These two versions of quantization of the 3-brane world should (at least qualitatively) lead to consistent results for quantum phenomena in the brane world when the extra dimensions are compact.

The second viewpoint shall shape our interpretation of the axial gravitational anomaly in 3-brane world. Note that the dynamics in the bulk is invariant under both diffeomorphisms and $SO(1, D-1)$ transformations, while on the brane, the theory has the reparametrization invariance, local $SO(1,3)$ symmetry and various gauge symmetries at the classical level.
One must choose gauge conditions to eliminate the redundant degrees of freedom connected with these symmetries of bulk gravity; gauge-fixing and the relevant ghost terms shall arise as usual for the gauge theory. The quantum theory of this system can be formally written out using the path integral

\[
Z = \int \prod_{M,N} \mathcal{D}H_{MN}(X) \prod_\mu \mathcal{D}_{\overline{A}_\mu}(x) \mathcal{D}\overline{\psi}(x) \mathcal{D}\psi(x) \exp i [S + \cdots]
\]

\[
= \int \prod_{M,N} \mathcal{D}H_{MN}(X) \left\{ \exp i \left[ \frac{\kappa^2}{8} \int d^D X \sqrt{-G} R + \cdots \right] \right\} \prod_\mu \mathcal{D}_{\overline{A}_\mu}(x) \mathcal{D}\overline{\psi}(x) \mathcal{D}\psi(x) 
\times \exp \left[ \int d^4 x e[X(x)] \left( e^\mu_a[X(x)] \overline{\psi} i \gamma^a \frac{1-\gamma^5}{2} D_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \cdots \right) \right] 
\]

\[
= \int \prod_{M,N} \mathcal{D}H_{MN}(X) \left\{ \exp i \left[ \frac{\kappa^2}{8} \int d^D X \sqrt{-G} R + \cdots \right] \right\} \prod_\mu \mathcal{D}_{\overline{A}_\mu}(x) \exp \left[ i \int d^4 x e[X(x)] \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \cdots \right) \right] 
\times \det \left[ i e(X) e^\mu_a(X) \gamma^a \frac{1-\gamma^5}{2} D_\mu \right] 
\times \exp \left( iW[A, e(X)] \right) 
\]

\[
\text{(8)}
\]

where we write \( G_{MN}(X) = G^{(0)}_{MN}(X) + H_{MN}(X) \), \( A_\mu(x) = A^{(0)}_\mu(x) + \overline{A}_\mu(x) \) i.e., we adopt the general view that the bulk graviton is a spin-2 quantum field over certain background space-time \( G^{(0)}_{MN} \) and that there exists a vacuum configuration \( A^{(0)}_\mu \) for the gauge field. These may or may not be trivial \((G^{(0)}_{MN} = \eta_{MN}, A^{(0)}_\mu = 0)\), depending on the case under consideration. The ellipses denote the gauge-fixing and ghost terms for the diffeomorphism invariance of bulk space-time and the gauge symmetry on the brane as well as a possible cosmological term for bulk gravity. In particular, we write out the explicit dependence of the induced metric (or vierbein) on the bulk coordinate in order to show that the quantum effective action of the field theory is intimately related to brane dynamics in the bulk space-time.

With the setting \((8)\) for the quantization of brane world, we are now able to discuss the possible anomalies for the model \((\mathbb{I})\). Like the usual 4-dimensional chiral gauge theory in the gravitational and gauge field background, the effective action \(W[A, e(X)]\) cannot remain invariant under all the transformations given by \((\mathbb{I})\), \((\mathbb{II})\) and \((\mathbb{III})\). According to the definition, the effective action \(W[A, e(X)]\) of the model \((\mathbb{I})\) can be formally written as the sum of the Green functions of the current operators\(^1\).

\(^1\) It should be emphasized that at this stage both gauge and gravitational fields are purely background fields rather than the quantum ones.
\[ W[A, e(X)] \sim \sum \int \left[ \prod_{i} \left( d^4 x_i e[X(x_i)] \right) A_{\mu_i}(x_i) \prod_{j} \left( d^4 y_j e[X(y_j)] \right) A_{\nu_j}(y_j) \right. \\
\left. \times \prod_{k} \left( d^4 z_k e[X(z_k)] \right) g_{\lambda_k \rho_k}(z_k) \left\langle \prod_{i} \tilde{J}^{\mu_i}(x_i) \prod_{j} \tilde{\nu}_j(y_j) \prod_{k} \tilde{T}_{(F)}^{\lambda_k \rho_k}(z_k) \right\rangle \right] \text{connected}, \tag{9} \]

where the axial vector current \( J^5_{\mu} \), vector current \( J_{\mu} \) and the fermionic part of energy-momentum tensor \( T_{(F)\mu\nu} \) are, respectively,

\[
J_{\mu}(x) = Y \overline{\psi}(x) \gamma_{\mu} \psi(x), \\
J^5_{\mu}(x) = Y \overline{\psi}(x) \gamma_5 \gamma_{\mu} \psi(x), \\
T_{(F)\mu\nu} = \frac{2}{e[X(x)]} \delta g_{\mu\nu} = i \left[ \overline{\psi} \left( \gamma_{\mu} D_{\nu} + \gamma_{\nu} D_{\mu} \right) \frac{1 - \gamma^5}{2} \psi - g_{\mu\nu} \overline{\psi} \gamma_{\lambda} D_{\lambda} \frac{1 - \gamma^5}{2} \psi \right]. \tag{10} \]

It is well known that the contribution to \( W[A, e(X)] \) from the Green functions \( \langle \tilde{J}^5_{\mu}(x) \tilde{J}_\mu(y) \tilde{\nu}(z) \rangle \) and \( \langle \tilde{J}^5_{\mu}(x) \tilde{T}_{(F)\nu\rho}(y) \tilde{T}_{(F)\lambda\sigma}(z) \rangle \) cannot make the axial vector gauge symmetry compatible with both the vector gauge symmetry and the reparametrization invariance of the brane \( \square \).

\[
i \delta W[A, e(X)] = -\frac{i}{16} \int d^4 x d^4 y d^4 z e[X(x)] e[X(y)] e[Z(z)] \\
\times \left\{ \overline{\psi}(x) \left[ A_{\nu}(y) A_{\rho}(z) \left( \partial_\mu \tilde{J}^5_{\mu}(x) \tilde{\nu}(y) \tilde{\gamma}^\rho(z) \right) \\
g_{\nu\lambda}(y) g_{\rho\sigma}(z) \left( \partial_\mu \tilde{J}^5_{\mu}(x) \tilde{T}_{(F)\nu\rho}(y) \tilde{T}_{(F)\lambda\sigma}(z) \right) \right] \\
- 2 A_{\mu}(x) A_{\nu}(y) \theta(z) \left( \tilde{J}^5_{\mu}(x) \tilde{\nu}(y) \partial_\rho \tilde{\gamma}^\rho(z) \right) \\
- 2 A_{\mu}(x) g_{\nu\lambda}(y) \xi_{\sigma}(z) \left( \tilde{J}_{\mu}^5(x) \tilde{T}_{(F)\nu\rho}^{\lambda}(y) \nabla_\rho \tilde{T}_{(F)\lambda\sigma}(z) \right) \right\} \\
= i \int d^4 x e[X(x)] \left[ \frac{1}{2} \overline{\psi}(x) \partial_\mu \tilde{J}^5_{\mu} - \frac{1}{2} \theta(x) \partial_\mu \tilde{\gamma}^5_{\mu} - \xi_{\nu}(x) \nabla_\mu \tilde{T}^{\mu\nu}_{(F)} \right], \tag{11} \]

where \( \nabla_\mu \) being the covariant derivative defined with respect to Levi-Civita symbol of the induced metric. To carry out a concrete calculation of the anomaly, one usually makes a decomposition \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) and consider linearized gravity when \( |h_{\mu\nu}| \ll 1 \).\( \square \). As a consequence, one has \( g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \), and \( e[X(x)] = 1 + \frac{1}{2} h^{\mu\nu} h_{\mu\nu} \). The fermionic part of the classical action \( \square \) can be approximately written as

\[
S_{(F)} = -\frac{1}{2} \int d^4 x e[X(x)] g^{\mu\nu} T_{(F)\mu\nu} \\
= \int d^4 x \left[ i \overline{\psi} \eta^{\mu\nu} \gamma_\mu D_\nu \frac{1 - \gamma^5}{2} \psi + \frac{1}{2} h^{\mu\nu} T_{(F)\mu\nu} \right], \tag{12} \]

where the subsidiary conditions \( \eta^{\mu\nu} \partial_\mu h_{\nu\rho}(x) = \eta^{\mu\nu} h_{\mu\nu}(x) = 0 \) are used. With the requirement of preserving vector gauge symmetry and the general covariance on the brane, i.e.,

\[2 \] A possible arising of Lorentz anomaly is ignored.
choosing $\partial_\mu \langle \tilde{J}^\mu \rangle = \nabla_\mu \langle \tilde{T}^{\mu\nu} \rangle = 0$, we have the anomaly for the axial vector current. It is a long-standing result that a direct calculation to the triangle and seagull diagrams of above three-point function gives \[14,15\]

$$
\partial_\mu \langle \tilde{J}^\mu \rangle = \frac{Y^3}{16\pi^2} \varepsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} + \frac{Y}{384\pi^2} \varepsilon^{\lambda\rho\sigma\delta} R^\mu_{\nu\lambda\rho} R^\nu_{\mu\sigma\delta}.
$$

(13)

4. Dependence of axial gravitational anomaly on embedding of 3-brane in a factorizable bulk space-time

In the 3-brane scenario, the first term of the chiral anomaly given in (13) comes from the instanton configuration of the gauge field confined on the brane, which is identical to the usual case since it is independent of the background space-time metric. The second term, contributed from the induced gravitational instanton background \[15\], should be relevant to the classical Euclidean configuration of the bulk gravitational field. To show this connection explicitly, let us recall briefly the submanifold theory in Riemannian Geometry \[21\]. For a $d$-dimensional submanifold $M$ of a $D$-dimensional Riemannian manifold $\mathcal{N}$ with a relation between their local coordinates, $X^M = X^M(x^\mu)$, $M = 0, 1, \cdots, D - 1$, $\mu = 0, 1, \cdots, d - 1$, one can define a quantity of rank $d$, $B^M_\mu \equiv \partial X^M / \partial x^\mu$, which plays a role of both a covariant vector in the submanifold and a contravariant vector in the bulk manifold. $B^M_\mu$ connects the local differential structure of the submanifold with that of the bulk manifold. The basis of their tangent spaces are related by

$$
\frac{\partial}{\partial x^\mu} = B^M_\mu \frac{\partial}{\partial X^M},
$$

and hence there exists a relation between the induced metric on the manifold and the bulk metric, $g_{\mu\nu} = G_{MN} B^M_\mu B^N_\nu$. A tangent space of the bulk manifold at a generic point can be decomposed into an orthogonal direct sum of the tangent space of the submanifold with a $n - m$-dimensional vector space equipped with orthonormal basis, $N_r$,

$$
\left< \frac{\partial}{\partial x^\mu}, N_r \right> = 0, \quad r = d - 1, \cdots, D - 1.
$$

This $D - d$-dimensional vector space is called normal space of the submanifold $M$. There exist the following relations according to the definition,

$$
G_{PQ} B^P_\mu N^Q_r = 0, \quad G_{PQ} N^P_r N^Q_s = \delta_{rs},
$$

where $N^P_r$ are the components of $N_r$ in bulk space-time. Assuming that $(B^P_\mu, N^Q_r)$ have the inverse $(\overline{B}^P_\mu, \overline{N}^Q_r)$ with respect to the bulk manifold indices $P, Q$, i.e.,

$$
B^P_\mu \overline{B}^\nu_P = \delta_\mu^\nu, \quad N^P_r \overline{N}^s_P = \delta^s_r,
$$

$$
B^P_\mu \overline{N}^r_P = N^r_P \overline{B}^P_\mu = 0,
$$

(14)

one can easily derive the following equations,
\[ \mathcal{B}_P^\mu g_{\nu\mu} = B_\mu^Q G_{QP}, \quad \mathcal{B}_P^\mu = G_{PQ} \mathcal{B}_Q^\mu, \quad B_\mu^P = G^{PQ} B_Q^\nu g_{\nu\mu}, \]
\[ g^{\mu\nu} = G^{PQ} \mathcal{B}_P^\mu \mathcal{B}_Q^\nu, \quad \mathcal{N}_P^r = \delta^r_B g^{Qs} G_{QP}, \quad N_r^P = G^{PQ} \mathcal{N}_Q^s \delta_{sr}, \] (15)

and
\[ \mathcal{B}_Q^\mu B_\mu^P + \mathcal{N}_Q^r N_r^P = \delta_Q^P, \quad g_{\mu\nu} \mathcal{B}_P^\mu \mathcal{B}_Q^\nu + \mathcal{N}_P^r \mathcal{N}_Q^s = G_{PQ}, \]
\[ \frac{1}{(D-d)!} \mathcal{N}_{P_1} \cdots \mathcal{N}_{P_{D-d}} \epsilon_{r_1 \cdots r_{D-d}} = \frac{1}{d!} \sqrt{-g} \epsilon_{\mu_1 \cdots \mu_D} B_\mu^Q, \cdots B_\mu^Q, \epsilon_{P_1 \cdots P_{D-d} Q_1 \cdots Q_d}. \] (16)

In addition, the covariant derivative \( \nabla_\mu B_\nu^P \) for fixed indices \( \mu, \nu \) is actually a normal vector of the submanifold \( M \),
\[ \nabla_\mu B_\nu^P = \partial_\nu B_\mu^P - \Gamma_{\mu\nu}^\xi B_\xi^P + B_\mu^R B_\nu^\xi \Gamma_{RS}^\mu M \]
\[ = \left( \partial_\nu B_\mu^Q + B_\mu^M B_\nu^N \Gamma_{MN}^Q \right) B_\mu^P + \delta_{rs} \mathcal{N}_P^r \mathcal{N}_Q^s, \]
\[ = K_{\mu\nu}^P = K_{\nu\mu}^P = K_{\mu\nu}^r N_r^P, \]
\[ K_{\mu\nu}^r = \left( \partial_\nu B_\mu^Q + B_\mu^M B_\nu^N \Gamma_{MN}^Q \right) B_\mu^P = 0, \] (17)

and the covariant derivative of \( N_r^P \) is given by the Weingarten formula,
\[ \nabla_\mu N_r^P = -\delta_{rs} \left( g^{\lambda\nu} K_{\mu\lambda}^s B_\nu^P - L_{st}^r N_t^P \right), \]
\[ L_{st}^r \equiv N_s^P \nabla_\mu N_r^P \] (18)

In above equations, \( \Gamma_{\mu\nu}^\lambda \) and \( \Gamma_{PQ}^M \) are the Christoffel symbols for the submanifold and the bulk manifold, respectively, and they have the following relation,
\[ \Gamma_{\mu\nu}^\lambda = \mathcal{B}_P^\lambda \left( \partial_\mu B_\nu^P + B_\mu^R B_\nu^S \Gamma_{RS}^P \right). \]

With above equations one can derive the Gauss, Codacci and Ricci equations,
\[ R^{\mu\rho}_{\nu\lambda} = B_\nu^N B_\lambda^P B_\rho^Q R_{PQ}^M \mathcal{B}_M^\mu + K_{\rho}^{\mu} K_{\lambda}^{P} R_{\lambda}^{Q} - K_{\lambda}^{\mu} K_{R}^{P} R_{\mu}^{Q} R_{\nu}^{R}, \]
\[ N_r^P B_\rho^Q B_\lambda^P R_{PQ}^M \mathcal{B}_M^\mu = \left( \nabla_\lambda K_{\mu}^{P} - \nabla_\mu K_{\lambda}^{P} \right) + \left( L_{\mu}^{s} K_{\lambda}^{P} - L_{\lambda}^{s} K_{\mu}^{P} \right), \]
\[ B_\mu^M B_\nu^N N_r^P B_\sigma^Q R_{PQ}^M = K_{\mu}^{\sigma} K_{\nu}^{s} - K_{\nu}^{\sigma} K_{\mu}^{s} + \nabla_\mu L_{\nu}^{s} - \nabla_\nu L_{\mu}^{s} \] (19)

The Gauss equation shows how the Riemannian curvature tensor of the submanifold is related to the bulk one, and the quantity \( K_{\rho}^{\mu} R_{\lambda}^{P} - K_{\lambda}^{\mu} R_{\mu}^{P} R_{\nu}^{R} \) is the extrinsic curvature tensor of the submanifold in the bulk manifold, which is completely composed the normal vectors of the submanifold. The Codacci and Ricci equations further gives how the Riemannian curvature tensor of bulk manifold is projected into the tangent and normal spaces at a generic point of the submanifold.

With above equations specializing to 4-dimensional submanifold case, we can rewrite the gravitational part of the chiral anomaly in terms of the bulk Riemannian tensor and the quantities characterizing the embedding of 3-brane in bulk space-time,
\[ \partial_\mu \langle \tilde{J}^\mu \rangle = \frac{Y}{384\pi^2} (A_1 + A_2 + A_3), \quad (20) \]

where

\[ A_1 = \left( B^\mu B_\mu N_2 \right) \left( B^\nu B_\nu N_3 \right) \epsilon^{\lambda\rho\sigma\delta} B_\lambda P_\rho B_\sigma P_\delta \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \::

The geometric meaning of above three terms are obvious. \( A_1 \) represents the topological invariant constructed from the projection of the bulk Riemannian curvature tensor into the tangent space of the 3-brane world volume, which can be equivalently described in terms of the bulk curvature tensor together with the normal vectors of the 3-brane world volume. \( A_2 \) gives a topological invariant constructed from extrinsic curvature tensor and the projection of bulk Riemannian curvature tensor into the tangent space, while \( A_3 \) is a topological invariant built purely from the extrinsic curvature tensor. In spite of the induced metric \( g_{\mu\nu} \) of the submanifold being a projection of the bulk metric into the tangent space, the Riemannian curvature tensor corresponding to the induced metric is not identical to the projection of bulk curvature tensor into the tangent space, since according to the Gauss equation there exists an extrinsic curvature tensor relevant to the normal space. Thus for a submanifold it is equivocal to speak of the topological meaning of the Pontrjagin class constructed from the Riemannian curvature tensor corresponding to the induced metric. It is necessary to pull the induced Riemannian curvature to the bulk manifold and discuss the topological meaning of the corresponding Pontrjagin class.

Now we convert above geometric objects into physics using the equivalence of Riemannian geometry and gravity. Eqs. (20) and (21) imply immediately that the axial gravitational anomaly observed in bulk space-time depends on both the topological structure of the bulk and the immersion of the brane, i.e., how the brane is geometrically located in bulk space-time. One may think that this conclusion does not make sense, since naive considerations suggest that since the axial vector current is confined to the brane, the gravitational anomaly should reside only in the topology of the brane. This would be true if the 3-brane were not embedded in a higher dimensional space-time and the fields \( X^M(x) \) describing the position of the brane were not dynamical fields. The dependence of the axial gravitational anomaly on the dynamics of the brane and bulk gravity can be further explained as follows. In general, there are two possibilities for the 3-brane in the bulk: the 3-brane either moves freely or sits at a fixed point of the extra dimension(s). In the former case, it is redundant to describe the position of the brane in terms of the bulk space-time coordinate \( X^M(x) \). One can eliminate this redundancy by choosing

\[ X^\mu(x) = x^\mu, \quad X^r(x) = \xi^r(x), \quad \mu = 0, \ldots, 3, \quad r = 4, \ldots, D. \]
The induced metric contains explicitly fields $\xi^\alpha(x)$ defined on the brane, which are called branons and describe the fluctuations of a 3-brane in bulk spacetime. In the latter case, there exists $X^M(x) = x^\mu \delta^M_\mu$.

The induced metric coincides with the bulk metric in the directions that the brane extends, $g_{\mu\nu} = G_{MN} \delta^M_\mu \delta^N_\nu$. However, in this case, the translation invariance of the extra dimensions is broken. The branons $\xi(x)$ will still arise as a Goldstone fields corresponding to the breaking of translational symmetry in the directions of extra dimensions. The dynamical effect of this kind of Goldstone boson was discussed in Ref. 22.

5. Brane world in non-factorizable bulk space-time: RS1 model

In the above, we considered the brane world to be of a type that the bulk space-time is factorizable, i.e., the position of the brane can be completely determined by the bulk coordinates as functions of the brane coordinates. However, there exist some brane worlds not separable from the extra dimension in the sense that the induced metric may depend on the coordinate of the extra dimension, which turns out to have more useful applications in particle physics phenomenology than the decomposable case. In the following, we discuss this case by considering a concrete model — the first Randall-Sundrum model (RS1) — and show explicitly how the axial gravitational anomaly presents in this kind of brane scenario.

RS1 consists of two parallel 3-branes of opposite tension in the 5-dimensional bulk space-time, which turns out to be a slice of AdS$_5$ space. The extra dimension is a line segment, the orbifold $S^1/Z_2$ parametrized by the coordinate $y \in [-\pi, \pi]$. The two 3-branes localize at the orbifold fixed points of $Z_2$, $y = 0, \pi$, respectively. Contrary to the usual case with an extra dimension, the world-volume of the 3-brane and the extra dimension is non-factorizable. Despite having two 3-brane worlds, only the one with negative tension (localized at $y = \pi$ — called the visible brane) is the brane supporting the SM model; the other one (called the hidden brane) is just a necessary set-up to produce a warp factor in the 3-brane metric and generate the hierarchy. Since the extra space of the RS1 model is compact, it is convenient to use the first version stated above to discuss quantization. The classical solution to the bulk Einstein equation is the warped metric:

$$ds^2 = e^{-2kr_c|y|}[\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 dy^2].$$

where $k$ is the $AdS_5$ curvature, $M_{Pl}$ the 4-dimensional Planck scale and $r_c$ the radius of the extra dimension $S^1$. The bulk quantum gravity is described by the quantum fluctuation around the above warped metric:

$$ds^2 = G_{\mu\nu}(x, y)dx^\mu dx^\nu + r_c^2 dy^2,$$

$$G_{\mu\nu}(x, y) = e^{-2kr_c|y|}[\eta_{\mu\nu} + h_{\mu\nu}(x, y)]$$

The excitation of the modulus field (or radion) is not considered here and a mechanism is assumed to keep the metric component of the extra dimension frozen at $r_c$. 24.
Since the extra dimension is compact, the quantum fluctuations admit an orthonormal mode expansion \[23\],

\[
h_{\mu\nu}(x, y) = \sum_{n=0}^{\infty} h^{(n)}_{\mu\nu}(x) \frac{\chi_n(y)}{\sqrt{kr_c}},
\]

\[
\int_{-\pi}^{\pi} dy e^{-2kr_c y} \chi_m(y) \chi_n(y) = \delta_{mn},
\]

(23)

where \( h^{(m)}_{\mu\nu} \) and \( h^{(n)}_{\mu\nu} \) are the graviton and the K-K modes respectively. In addition, they satisfy the gauge conditions \( \eta^{\mu\nu} \partial_\mu h^{(n)}_{\nu\rho}(x) = 0 \) and \( \eta^{\mu\nu} h^{(n)}_{\mu\nu}(x) = 0 \). The four-dimensional effective Lagrangian density describing the interaction of KK modes with the matter fields on the visible 3-brane is \[23\]

\[
L = -\frac{1}{M^{5/2}} T^{\mu\nu}(x) h_{\mu\nu}(x, y)|_{y=\pi}
\]

\[
= -\frac{1}{M_{Pl}} T^{\mu\nu}_{(F)} \left[ h^{(0)}_{\mu\nu} + e^{kr_c \pi} \sum_{n=1}^{\infty} h^{(n)}_{\mu\nu} \right],
\]

(24)

where \( M \) is the 5-dimensional Planck scale and is related to its 4-dimensional counterpart on the visible 3-brane through the relation \( M^2_{Pl} = (1 - e^{-2kr_c \pi}) M^3 / k \) \[1\]. In situations where the energy-momentum tensor of the matter fields consists of chiral fermions as the model \[1\], if we consider gravity at linearized level, the effective Lagrangian (24) implies that the axial gravitational anomaly takes the following form,

\[
\partial_\mu \left( \hat{J}^{5\mu}_{\text{GRA}} \right) \sim \frac{1}{M^3} \epsilon^{\mu\nu\lambda\rho} R^\sigma_{\delta\mu\nu}(x, y) R^\delta_{\sigma\lambda\rho}(x, y)|_{y=\pi}
\]

\[
= e^{\mu\nu\lambda\rho} \left[ \frac{1}{M_{Pl}^2} R^{(0)\sigma}_{\delta\mu\nu} R^{(0)\delta}_{\sigma\lambda\rho} + \frac{2e^{kr_c \pi}}{M_{Pl}^2} R^{(0)\sigma}_{\delta\mu\nu} \sum_{n=1}^{\infty} R^{(n)\delta}_{\sigma\lambda\rho} 
\right.
\]

\[
+ \frac{e^{2kr_c \pi}}{M_{Pl}^2} \sum_{n,m=1}^{\infty} R^{(n)\sigma}_{\delta\mu\nu} R^{(m)\delta}_{\sigma\lambda\rho} \right],
\]

(25)

where \( R^{(0)\mu}_{\nu\lambda\rho} \) and \( R^{(n)\mu}_{\nu\lambda\rho} \) are the Riemannian curvatures corresponding to \( h^{(0)}_{\mu\nu} \) and \( h^{(n)}_{\mu\nu} \) respectively. We have employed the result that for linearized gravity there exists the expansion

\[
R^\sigma_{\delta\mu\nu}(x, y) = \sum_{n=0}^{\infty} R^{(n)\sigma}_{\delta\mu\nu} \frac{\chi^n(y)}{\sqrt{kr_c}}.
\]

The meaning of Eq. (25) needs some explanation. The chiral anomaly gets contributions from both the graviton on the brane and the KK modes. As it is well known, like the chiral anomaly in the gauge field background, the topological origin of the axial gravitational anomaly is the fermionic zero modes in the gravitational instanton background \[1\], a Euclidean solution to the Einstein equation with (anti-)self-dual Riemannian tensor. It is just the existence of this kind of Euclidean configuration in the gravitational field that leads to the difference of the chiral fermionic zero modes and thereby generates the axial gravitational anomaly. The existence of a gravitational instanton implies that the corresponding
quantum gravity theory must have a non-trivial vacuum structure because an instanton lies between two distinct vacua and produces a tunneling effect between them. Hence we naturally relate the $R^{(0)} \tilde{R}^{(0)}$ term in Eq. (24) to the gravitational instanton background on the 3-brane. The $R^{(0)} \tilde{R}^{(n)}$ and $R^{(n)} \tilde{R}^{(0)}$ terms are the contributions from the infinite tower of KK modes. Although one can similarly explain them in terms of an infinite tower of gravitational instanton-like objects, they are actually connected to the non-factorizable feature of the bulk space-time and to the topological structure of the extra space.

Let us make this point clear by recalling the origin of KK modes. In general, the appearance of gravitational KK modes comes from a decomposition of the bulk space-time into a physical space-time and a compact extra space. Any bulk field $H(X)$, $X = (x, y)$, has a mode expansion, $H(x, y) = \sum_n h^{(n)}(x)\chi_n(y)$. The topological structure of the extra space determines the orthonormal modes $\chi_n(y)$ and thereby the KK modes. In particular, the isometry group of the extra dimensions becomes a global symmetry of the effective theory defined in the physical space-time and even a dynamical symmetry if this global symmetry can be gauged. In this sense, the physical property of KK modes depends intimately on the topological structure of the extra dimension. This is consistent with our inference that the axial gravitational anomaly is relevant to the bulk space-time. An explicit investigation of how the geometry of large but compact extra dimensions in a factorizable bulk space-time affects the field theory on the 3-brane through KK mode-emission was carried out in Ref. [25], and it was shown there that the low-level KK modes can distinguish the topology of the extra space, whereas the high energy modes seem not to be sensitive to it. However, RS1 model is special in the sense that the bulk space-time is not factorizable. The induced metric is thereby identical to the warped metric [16] and has a dependence on the coordinate of extra dimension. Consequently, the induced metric admits a KK modes expansion which leads to the anomaly of the form (25). From a phenomenological consideration, the result (24) implies that a physical process associated with the above chiral anomaly can receive contributions from every levels of KK modes. In particular, Eq. (24) shows that the contributions from the graviton and KK modes are graded by the energy scales $M_{pl}$ and $e^{-kr_c}M_{pl}$. With an appropriate choice on the size of the extra dimension, one can make $e^{-kr_c}M_{pl}$ be at the order of the weak scale. Thus it is possible to probe the existence of such an extra dimension with the physical process described by anomalous diagrams.

Can we express the axial gravitational anomaly in terms of the bulk Riemannian tensor and the quantities describing the embedding of 3-brane and understand the corresponding topological meaning in the same way as the case of decomposable bulk space-time? In general, it is no clear how to rigorously define a submanifold theory for this case. The reason is that the induced metric is dependent on the extra dimension(s), $g_{\mu\nu} = g_{\mu\nu}(x, y)$. This implies that the bulk coordinates characterizing the embedding of the brane depend on the extra dimension, $X^M = X^M(x, y)$. As a consequence, this may imply that the rank of $B_\mu^M(x, y)$ is less than $d$, making it impossible to define the normal vectors and normal space of the submanifold. However, in some special cases such as the RS1 model, the brane is fixed in certain point of the extra dimension and the normal vectors can be well defined. Thus we can write the Riemannian curvature tensors of the brane, which include all the curvatures for the zero and KK modes in the case that the extra dimensions are compact, in terms of the corresponding bulk Riemannian curvature tensors and extrinsic curvature tensors as the decomposable bulk space-time case.
6. Anomaly cancellation and possible origin from bulk quantum gravity

Turning next to the anomaly cancellation problem, the gauge field contribution, i.e., \( F \tilde{F} \) must be cancelled as usual. Thus there is no inconsistency for the \( SU(2)_L \times U(1)_Y \) SM defined on the brane, since the requirement \( \sum_i Y_i^3 = 0 \) for the cancellation of the anomaly contributed from the gauge field is equivalent to the condition \( \sum_i Y_i = 0 \) for the gravitational anomaly cancellation for the SM fermions, the index \( i \) denoting the flavours of the fermionic particles. This can be easily verified with the Gellman-Nishijima formula, \( Y = 2(Q - T_3) \), and the fact that the electromagnetic current is anomaly free as a vector current \([26]\), \( Q \) and \( T_3 \) being the electric charge and the remaining generator of \( SU(2)_L \) after spontaneous breaking. However, there exist some extensions of the SM that are free of gauge field part but not of gravitational field contributions \([17]\). If this occurs, it is not necessary to cancel the gravitational anomaly part of the brane world, in contrast to the usual case. One straightforward observation is that the topological number density

\[
e^\mu\nu\lambda\rho R^\sigma_{\delta\mu\nu} R^\delta_{\sigma\lambda\rho} = \nabla_\mu K^\mu,
\]

\( K^\mu = 4e^{\mu\nu\lambda\rho} \left( \Gamma^\nu_{\nu\delta} \partial_\lambda \Gamma^\delta_{\rho\delta} + \frac{2}{3} \Gamma^\nu_{\nu\delta} \Gamma^\delta_{\lambda\alpha} \Gamma^\alpha_{\rho\delta} \right) \),

(26)

\( \Gamma^\lambda_{\mu\nu} \) being the Christoffel symbols with respect to the induced metric on the brane. Thus one can make a shift on the quantum effective action

\[
\mathcal{W}[A,e(X)] = \mathcal{W}[A,e(X)] + \frac{1}{384\pi^2} \int d^4xe[X(x)]A_\mu(x)K^\mu[X(x)].
\]

(27)

\( \mathcal{W}[A,e(X)] \) is invariant under the gauge transformation \( \delta A_\mu = \partial_\mu \vartheta(x) \) but violates the reparametrization invariance given in (7), since \( K^\mu \) is not invariant under general covariant coordinate transformations \([7]\). However, the breaking of general covariance on the brane is not a serious problem, since this can be considered as an effect of bulk quantum gravity. This can be easily observed as follows. Under the infinitesimal coordinate transformation given in (7), the induced metric vary as \( \delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \), we have

\[
\delta \mathcal{W}[A,e(X)] = \int d^4xe[X(x)] \frac{\delta \mathcal{W}[A,e]}{\delta g_{\mu\nu}(x)} \left( \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \right)
\]

\[
= -\int d^4xe[X(x)]\xi^\nu \nabla_\mu (T_{\mu\nu}).
\]

(28)

Eq. (28) relates directly the origin of the anomalous general covariance on the brane to \( K^\mu \). However, as shown in (26), \( K^\mu \) is a functional of the induced metric on the brane. General covariance of the bulk space-time remains unaffected by the redefinition (28) of the effective action since the induced metric is a scalar with respect to general coordinate transformations of the bulk space-time. There is no reason to exclude the possibility that the \( K^\mu \) term can be generated from bulk quantum gravity since it is completely consistent with the bulk space-time symmetry. Of course, the concrete physical process of generating such a term is not clear since we know little about bulk quantum gravity. In brane-world scenarios, the general covariance of bulk space-time is the most fundamental symmetry. Reparametrization
invariance on the brane, represented by the induced metric, is dominated by the general covariance of bulk space-time and the way in which the brane is embedded in the bulk. Based on above considerations we can impute the anomalous breaking of general covariance on the brane to the quantum effects of bulk gravity.

The above arguments reveal a remarkable feature of brane world scenarios: viewed from the brane perspective, quantum bulk gravitational effects can in principle render a chiral gauge theory on the brane quantum mechanically inconsistent, breaking either general covariance or chiral gauge symmetry. Roughly speaking, quantum fluctuations of bulk gravity described by $G_{MN}(X)$ can affect the induced metric $g_{\mu\nu}(x)$ through the relation (2) by generating the term $K^\mu$, which couples with the dynamical gauge field $A_\mu(x)$ on the brane. The quantum effective action for the brane system is then described by $\mathcal{W}[A, e(X)]$, which is invariant under general coordinate transformation of bulk space-time, but not with respect to the reparametrization of the brane. To preserve general covariance on 3-brane, one must redefine the quantum effective action as

$$W[A, e(X)] = \mathcal{W}[A, e(X)] - \frac{1}{384\pi^2} \int d^4x e[X(x)] A_\mu(x) K^\mu[X(x)].$$

It should be emphasized that viewed from the bulk perspective the above redefinition of the quantum effective action is a finite renormalization, and both $W$ and $\mathcal{W}$ characterize the same bulk physics. However, $W$ and $\mathcal{W}$ describe distinct quantum phenomena for the brane system: for a physical process represented by $W[A, e(X)]$, general covariance is preserved but the chiral symmetry is violated, while the converse takes place for $\mathcal{W}$.

There are two perspectives one can adopt based on the preceding considerations. The first is that both general covariance and chiral symmetry must be preserved on the brane. In this case the second term in Eq. (27) must vanish, constraining the possible form of the bulk quantum theory of gravity. However, an alternative perspective is to regard Eq. (27) as providing a means to probe quantum gravity in the bulk. The term $\int d^4x e[X(x)] A_\mu(x) K^\mu[X(x)]$ will be deduced from observation of physical processes on the 3-brane: the axial vector (or chiral) current on the brane in a gravitational background definitely receives a gravitational anomalous breaking either in chiral symmetry or in general covariance of the brane system. Keeping the brane theory gauge anomaly-free (as indicated by present-day experiments), observations of apparent quantum inconsistencies as described by Eqs. (27) or (29) could be regarded as providing signature effects of bulk quantum gravity. Since the term responsible for breakdown of general covariance of the brane system is actually allowed by the symmetry of the full bulk theory, such a viewpoint cannot be ruled out.

How viable is such a perspective? Although a concrete physical process leading to $K^\mu[X(x)]$ is not yet clear, the dynamical phenomenon of anomaly inflow (relevant to defects such as strings, domain walls and p-branes) [27,28] provides further support for this viewpoint. If defects like strings and domain walls exist in a given space-time, there will in general be chiral fermionic zero modes trapped on the defect, even if one starts with a vector type gauge theory (i.e., the matter fields are Dirac fermions). According to the Atiyah-Singer index theorem [29], a chiral anomaly localized on the defect must arise. This means that the gauge charges (in case of a gauge anomaly) or the energy and momentum (in case of a gravitational anomaly) carried by the chiral fermionic zero modes trapped on the defect are not conserved. Equivalently the fermionic determinant contributed from the
chiral fermionic zero modes localized on the defect is not gauge invariant (or not generally covariant). However the full bulk theory must be anomaly-free since one began with a vector gauge theory. Thus the gauge charges (for a gauge anomaly) or energy and momentum (for gravitational anomaly) carried by the massive fermionic modes away from the defects will also not be conserved and must “flow” into the defect to compensate for the non-conservation due to the anomaly localized on the defect. In other words the fermionic determinant contributed by the massive fermions away from the defect must be also not gauge invariant, counteracting the anomaly in the fermionic zero mode defect determinant to ensure chiral gauge symmetry. From a brane-world perspective this dynamical mechanism could be called anomaly “outflow”. It entails a loss of energy and momentum conservation on the brane because the anomaly permits energy and momentum transfer from brane to bulk (or vice-versa) via the anomaly. In the scenario we have considered, the chiral fermions on the 3-brane are put in arbitrarily by hand and the bulk theory is pure gravity. However, we can still envision an extension of the bulk theory that can provide the requisite anomaly inflow mechanism: chiral fermions in the 3-brane can be realized as chiral fermionic zero modes of this more fundamental theory trapped on the (3 + 1)-dimensional defect [30]. In this manner the possibility that a breakdown of general covariance on the brane due to the presence of an axial gravitational anomaly is not ruled out, and could be induced by quantum effects of bulk gravity.

An obvious objection to this second viewpoint is that the quantum field theory localized on the brane will not be renormalizable. We do not regard this objection as being fatal to the second perspective described above. Faith in renormalization of a quantum field theory is in part based on our ignorance of quantum gravity. In the process of performing renormalization, ultraviolet divergences are absorbed into redefinitions of bare parameters such as mass and coupling constants. The presumed rationale behind this approach to render a theory well defined is that one is neglecting effects from quantum gravity at very short-distances, i.e. one is just transferring ultraviolet divergences to this regime. Hence in a model containing quantum gravity, non-renormalizability due to gravitational effects from chiral anomalies need not be considered as firm criteria for judging consistency of a theory before we understand quantum gravity completely.

7. Summary

We have shown that due to the presence of dynamical bulk gravity fields and the dynamics of the 3-brane itself, the axial gravitational anomaly depends both on both the topological structure and the embedding of the 3-brane in the bulk. We have found that in this brane scenario, the gravitational part of the ABJ anomaly can be converted into an effective action for bulk gravity without breaking local gauge symmetry on the 3-brane. It is obvious that this effective action respects both diffeomorphism and local Lorentz symmetries of the bulk space-time since the induced metric is a scalar with respect to both of these symmetries. The axial gravitational anomaly on the brane can either imply constraints on the full bulk theory of quantum gravity or it can be regarded as providing an observational signature of the non-trivial vacuum structure of bulk quantum gravity via the transfer of energy and momentum from brane to bulk. We contend that either viewpoint is acceptable based on our current knowledge of quantum gravity.
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