The $\mathcal{N} = 1$ supersymmetric Wong equations and the non-abelian Landau problem

Michaël Fanuel$^1$, Jan Govaerts$^{1,2}$, Gabriel Y H Avossevou$^{3,4}$ and Anselme F Dossa$^3$

$^1$Centre for Cosmology, Particle Physics and Phenomenology (CP3), Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Chemin du Cyclotron 2, bte L7.01.01, B-1348, Louvain-la-Neuve, Belgium
$^2$International Chair in Mathematical Physics and Applications (ICMPA–UNESCO Chair), University of Abomey–Calavi, 072 B. P. 50, Cotonou, Republic of Benin
$^3$Unité de Recherche en Physique Théorique (URPT), Institut de Mathématiques et de Sciences Physiques (IMSP), 01 B. P. 613 Porto–Novo, Republic of Benin
$^4$Département de Physique, Université d’Abomey–Calavi (UAC), Republic of Benin

E-mail: Michael.Fanuel@uclouvain.be, Jan.Govaerts@uclouvain.be, gabavossevou@gmail.com and finedofas@yahoo.fr

Received 22 May 2014, revised 31 August 2014
Accepted for publication 11 September 2014
Published 30 October 2014

Abstract

A Lagrangian formulation is given extending to $\mathcal{N} = 1$ supersymmetry, or the motion of a charged point particle with spin in a non-abelian external field. The classical formulation is constructed for any external static non-abelian SU(N) gauge potential. As an illustration, a specific gauge is fixed, enabling canonical quantization and the study of the supersymmetric non-abelian Landau problem. The spectrum of the quantum Hamiltonian operator follows in accordance with the supersymmetric structure.

Keywords: supersymmetry, Landau problem, exact solution

PACS numbers: 12.60.Jv, 71.70.Di, 11.10.Nx

1. Introduction

The motion of a non-abelian charged particle in a classical non-abelian gauge field as described many years ago by the Wong equations [1] has been studied both for its theoretical interest [2–4] and in attempts to describe the phenomenology of strong interactions [5]. More recently a study of the non-abelian Landau problem [6], i.e., a quantum particle confined to a plane and subjected to a static and homogeneous perpendicular magnetic field [7], has shown...
that the effects of specific choices of non-abelian gauge potentials corresponding to homogeneous coloured magnetic fields could account for the presence of Rashba [8, 9] and Dresselhaus [10] spin–orbit interactions. Applications of similar models have been recently analyzed in the context of fractional quantum Hall effects [11], and non-abelian anyons [12, 13]. These results have revived interest in the study of the motion of coloured particles in non-abelian backgrounds. Undoubtedly the non-abelian Landau problem is a particular example of the motion of a coloured point particle in an external Yang–Mills gauge field. At the classical level this motion is governed by the Wong equations, generalizing ordinary cyclotron motion in the case of an electromagnetic field.

Inspired by the potential application of supersymmetry in condensed matter physics [14], the consequences of having in addition a supersymmetric invariant realization of the quantized system corresponding to the motion of a coloured particle in a classical external static non-abelian gauge field are studied in the present work. The authors of [15] addressed, in a classical formulation, the \( \mathcal{N} = 1 \) supersymmetric Wong equations for a spin 1/2 particle in an non-abelian background gauge field. The present work considers the case of a particle with arbitrary spin in a unitary (irreducible) representation of a compact gauge group. Furthermore, a canonical quantization of the classical formulation is constructed. Subsequently, as a particular illustration, the spectrum of the \( \mathcal{N} = 1 \) supersymmetric non-abelian Landau problem is obtained in the specific case of a spin 1/2 particle in a nontrivial static non-abelian background magnetic field. Finally, the inclusion of an electric potential term is discussed in the appendix. After this work was completed, our attention was called to similar superfield extensions of the Wong equations for \( \mathcal{N} = 1 \), \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) supersymmetry [16–19], including the coupling to external non-abelian gauge fields [20, 21].

2. Lagrangian and supersymmetry

A typical problem is the motion of a charged particle in \( d \) spatial dimensions subjected to a generic static non-abelian gauge potential. Its classical dynamics are governed by

\[
L = \frac{1}{2} m \dot{x}_i^2 + \frac{1}{2} i \left( u_\alpha \dot{u}_\alpha \right) - u_\alpha \left( \dot{u}_\alpha + A^\alpha \dot{x}_i \right) + \dot{\chi}_j \bar{A}_j^\alpha \left( x_i \right) u_\alpha T^\alpha_{\dot{\alpha}} u_{\dot{\alpha}} - A^\alpha_0 \left( x_i \right) u_\alpha T^\alpha_{\dot{\alpha}} u_{\dot{\alpha}},
\]

where we have introduced additional complex variables \( u_\alpha \), that will be specified hereafter while \( A^\alpha_0 \left( x_i \right) \) and \( A^\alpha \left( x_i \right) \) are arbitrary time independent background fields (\( i = 1, \ldots, d \)). For simplicity we take a compact gauge group with hermitian generators, \( \left( T^\alpha \right)^\dagger = T^\alpha \), whose algebra is \( \left[ T^\alpha, T^\beta \right] = i f^{abc} T^c \). By construction the Lagrangian is invariant under local space-dependent gauge transformation, which includes a transformation of the background fields,

\[
u_\alpha \rightarrow g(x_i) \nu_{\dot{\alpha}} u_{\dot{\beta}},
\]

\[
A^\alpha_0 \left( x_i \right) T^\alpha \rightarrow g(x_i) A^\alpha_0 \left( x_i \right) T^\alpha g(x_i)^\dagger - i \partial_\theta g(x_i) g(x_i)^\dagger,
\]

\[
A^\alpha \left( x_i \right) T^\alpha \rightarrow g(x_i) A^\alpha \left( x_i \right) T^\alpha g(x_i)^\dagger,
\]

where \( g(x_i) \) is a group element with space dependent parameters \( \lambda^\alpha(x_i) \), obtained by exponentiation of the elements of the algebra: \( g(x_i) = \exp i \lambda^\alpha(x_i) T^\alpha \). In order to introduce the supersymmetric formulation we define the hermitian supercharge and superderivative

\[
Q = \partial_\theta + i \theta \partial_\theta, \quad D = \partial_\theta - i \theta \partial_\theta,
\]
which anticommute with each other while \( Q^2 = i \partial_i \) and \( D^2 = -i \partial_i \). The supersymmetric formulation requires the introduction of the real Grassmann odd coordinate \( \theta \). Furthermore it is customary to define the real supercoordinate \( X_i(t, \theta) = x_i(t) + i\theta \lambda_i(t) \) with \( i = 1, \ldots, d \) where the variable \( \lambda_i(t) \) is odd. The infinitesimal supersymmetric transformation of \( X_i \) of Grassmann odd parameter \( \epsilon \) is \( \delta X_i(t, \theta) = -i\epsilon QX_i(t, \theta) \), so that the components transform as

\[
\delta \epsilon x_i(t) = \epsilon \lambda_i(t), \quad \delta \epsilon \lambda_i(t) = i\epsilon \dot{x}_i(t).
\]

With the help of these definitions we readily find that the kinetic term \( m \dot{x}_i^2 / 2 \) is contained in the expression

\[
\int d\theta \left\{ -\frac{1}{2} m D^2 X_i \right\} = \frac{1}{2} m \dot{x}_i^2 - \frac{i}{2} m \dot{\lambda}_i \lambda_i.
\]

By construction, the last expression transforms by a total derivative under an infinitesimal SUSY transformation. We briefly recall the reason why this is so. As a general rule the Grassmann integral of a superfield \( \int \theta \theta \) selects \( \int \theta \theta \). The highest component \( L^{(2)} \) has the property of transforming by a total derivative under supersymmetric transformations. Indeed, the transformation

\[
\delta \epsilon L^{(2)} = -i\epsilon \partial_i L^{(1)}
\]

is such that we can identify the transformation rule \( \delta_\epsilon L^{(2)} = -\epsilon \partial_i L^{(1)} \). This construction produces a natural candidate for the supersymmetric extension of the original Lagrangian.

### 3. Bosonic degree of freedom

Henceforth we will consider \( u_a \) to be a Grassmann even complex variable, in a (complex) representation of the ‘gauge’ group. The case of a Grassmann odd variable can be developed in parallel with our analysis, as the reader is invited to verify. The complex supercoordinate containing \( u_a \) reads \( U_a(t, \theta) = u_a(t) + \partial u_a(t) \) whose components transform under supersymmetry as

\[
\delta u = -i\epsilon \omega, \quad \delta \omega = -\epsilon \dot{u}.
\]

#### 3.1. Non-abelian transformations and supersymmetric action

Respecting gauge invariance requires particular care. Here we consider a background with vanishing electric potential while the appendix shows how to generalize the formulation to include an electric potential, \( A_0^a(x) \neq 0 \). Inspired by the supersymmetrization in the Abelian Landau problem and references \[22, 23\], we define the natural candidate \( V = i\partial \left( X_\mu, DX_\mu \right) = iDX_\mu A_\mu^a(X) T^a \), for the analogue of the vector superfield. The components of \( V \) are

\[
V = -i\dot{\lambda}_i A_i(x_i) + \theta \left\{ \dot{x}_i A_i(x_i) + i \dot{\lambda}_i \dot{\lambda}_j \partial_j A_j(x_i) \right\}
\]

where \( A_i(x_i) = A_\mu^a(x_i) T^a \) for \( i, j = 1, \ldots, d \). Henceforth the gauge index of the representation is kept implicit. We still need to investigate the gauge transformation properties of the vector superfield in order to define an invariant action. In the case of a \( U(1) \) gauge group \[24\], the gauge transformation of the vector superfield is \( V \to V - iDA \) where the superfield \( A \) is real.
and reads
\[ \Lambda(X_i) = A^a(x_i)T^a + i\partial \lambda_i \partial A^a(x_i)T^a \]  
so that
\[ iDA(X_i) = \left\{ -\lambda_i \partial A^a(x_i) + \theta \delta_A \right\} T^a. \]  
It is straightforward to extend the transformation for the case of the SU(N) gauge group. The non-abelian transformation of the vector superfield
\[ V \rightarrow e^{iA}Ve^{-iA} + e^{iD}De^{-iD}, \]  
when written in its infinitesimal form, reads
\[ \delta V = -iDA + [iA; V] \]
\[ = \lambda_i(x_i)\left( \partial \Lambda A(x_i) + f^{abc}A^a(x_i)A^b(x_i)T^c \right) \]
\[ - \theta \left\{ \lambda_i \left( \partial \Lambda + f^{abc}A^a(x_i)A^b(x_i)T^c \right) \right. \]
\[ + i\delta \lambda_j \partial \left( \partial \Lambda A(x_i) + f^{abc}A^a(x_i)A^b(x_i)T^c \right) \],
providing the desired result. Besides, the gauge transformation of the variable \( u \) is
\[ u \rightarrow \exp\left(i\lambda^a(x_i)T^a\right)u \]  
and therefore the supersymmetric extension is easily guessed. The gauge transformation of parameter \( \Lambda(X) = \Lambda(X)^i \) valued in the Lie algebra representation, of the superfields \( U \) and \( U^\dagger \) is
\[ U \rightarrow e^{iU}, \quad U^\dagger \rightarrow U^{\dagger e^{-iU}}. \]  
The supersymmetric generalization of the second and third term of the Lagrangian (1)
\[ L_3 = \int d\theta U^{\dagger}(-D + V)U \]  
is straightforwardly written in components
\[ L_3 = \left\{ iu^i\lambda_i + u^i\lambda_i A_i(x_i)u + iu^i\lambda_i \lambda_j \partial A_i(x_i)u \right. \]
\[ - u^i A_0(x_i)u + w^i w + w^i \lambda_i A_i(x_i)u + u^i \lambda_i A_i(x_i)w \right\}. \]  
Furthermore the complete expression whose first term provides the supersymmetric variation of \( L_3 \) is \( U^{\dagger}(-D + V)U = (-u^i w - w^i \lambda_i A_i u) + \partial L_3 \), and, according to (8), gives
\[ \delta L_3 = -\epsilon \partial \left(-u^i w - w^i \lambda_i A_i u\right). \]  
However the variables \( w \) and \( w^i \) have no dynamics and hence are auxiliary. The elimination of these variables by their equations of motion gives
\[ L_3 = u^i \left(i\partial_i \lambda_i \lambda_j \partial A_i(x_i) - \partial A_i(x_i) - i\left[A_j(x_i) A_i(x_i)\right]\right)u, \]  
which is explicitly invariant under a non-abelian gauge transformation provided the background field \( A_i(x_i) \) is transformed as well according to (3). Finally we notice that the results obtained are not explicitly real but we provide a remedy by considering instead
\[ L_3 = \int d\theta \frac{1}{2} \left\{ (-D + V)U^{\dagger}U + U^{\dagger}(-D + V)U \right\} \]  
so that the supersymmetric variation becomes \( \delta_L L_3 = -\epsilon \partial\left(-\frac{1}{2}u^i w - \frac{1}{2}w^i u - u^i \lambda_i A_i u\right) \), vanishing when the equations of motion are satisfied. In conclusion, the supersymmetric
action

\[ S = \int dt d\theta \left\{ -\frac{1}{2} m D^2 X_i DX_i - L_3 \right\} = \int dt L_{\text{SUSY}} \]  

(19)

reads in components

\[ S = \int dt \left\{ \frac{1}{2} m \dot{x}_i^2 - \frac{i}{2} m \dot{\lambda}_i \lambda_i + \frac{i}{2} u \dot{u} - \frac{i}{2} \dot{u}' u \\
+ u \dot{x}_i A_i (x_i) u + \frac{1}{2} i u' \dot{\lambda}_i \lambda_j G_{ji} u \right\}. \]  

(20)

The non-abelian field strength \( G_{ji} = \partial_j A_i (x_i) - \partial_i A_j (x_i) - i[A_j (x_i); A_i (x_i)] \) transforms under gauge transformation by a conjugation, so that \( u^i G_{ij} u \) is a gauge invariant expression.

The variation of this Lagrangian under supersymmetry

\[ \delta_\lambda L_{\text{SUSY}} = \partial_\lambda \left( \frac{1}{2} m \dot{x}_i \lambda_i \right) \]  

(21)

gives rise to a surface term in the action, accounting for the fact that the auxiliary variables \( w \) and \( w^\dagger \) have been removed. The supersymmetry charge is readily found to be \( Q_{\text{SUSY}} = m \dot{x}_i \lambda_i \).

3.2. Hamiltonian formulation

The introduction of additional variables \( p_i \) for \( i = 1, \ldots, d \) provides an alternative formulation of the action (20)

\[ S = \int dt \left\{ \dot{x}_i p_i - \frac{i}{2} m \dot{\lambda}_i \lambda_i + \frac{i}{2} u \dot{u} - \frac{i}{2} \dot{u}' u \\
- \frac{1}{2m} (p_i - u^\dagger A_i u)^2 + \frac{i}{2} \dot{\lambda}_i \lambda_j u \dot{G}_{ij} u \right\}. \]  

(22)

The last action being already in the so-called first order Hamiltonian form we take advantage of its convenient form to read off the Hamiltonian

\[ H = \frac{1}{2m} (p_i - u^\dagger A_i u)^2 + \frac{i}{2} \dot{\lambda}_i \lambda_j u \dot{G}_{ij} u \]  

(23)

as well as the Grassmann graded Poisson (or Dirac) brackets

\[ \{ x_i; p_j \} = \delta_{ij}, \quad \{ u; u^\dagger \}_D = -i, \quad \{ \dot{x}_i; \lambda_j \}_D = -i \frac{m}{2} \delta_{ij}. \]  

(24)

A straightforward analysis of constraints following the Dirac algorithm [25] reaches the same conclusions. In the Hamiltonian formalism the supercharge \( Q_{\text{SUSY}} = (p_i - u^\dagger A_i T^a u) \dot{\lambda}_i \) generates the infinitesimal supersymmetry transformations through the Poisson brackets.

The transformations

\[ \{ e Q_{\text{SUSY}}; x_i \} = -\delta_{ij} e \lambda_i, \quad \{ e Q_{\text{SUSY}}; \lambda_j \} = -i \delta_{ij} e \lambda_i, \]  

(25)

\[ \{ e Q_{\text{SUSY}}; u \} = -ie A_i^a (x_i) T^a \dot{\lambda}_i u = cw, \]  

(26)

correspond up to a sign to the ones in (6) and (9). As is very well known the supercharge is a square root of the Hamiltonian: \( \{ Q_{\text{SUSY}}, Q_{\text{SUSY}} \} = -2iH \), where the curly brackets denote Poisson brackets.
4. Canonical quantization

In accordance with the correspondence principle we proceed to canonical quantization by introducing the following commutators and anti-commutators:

\[
\left[ \hat{x}_i; \hat{p}_j \right] = i\hbar \delta_{ij}, \quad \left[ u; u^\dagger \right] = \hbar, \quad \{ \lambda_i; \lambda_j \} = \frac{\hbar}{m} \delta_{ij}.
\]  
(27)

Henceforth all curly brackets \{;\} will denote anticommutators. In addition we introduce conveniently ‘normalized’ operators \(\lambda_i = \sqrt{\hbar/2m} \tilde{\gamma}_i\), satisfying a Clifford algebra \([\gamma_i; \gamma_j] = 2\delta_{ij}\) in Euclidian signature. The algebra of the ‘non-abelian charges’:

\[
[I^a; P^b] = i\hbar \delta^{abc} I^c,
\]
where \(I^a = u^\dagger T^a u\), contains an additional factor \(\hbar\) with respect to the Lie algebra considered at the beginning. The Hamiltonian operator easily follows

\[
\hat{H} = \frac{1}{2m} \left( \hat{p}_i - A_i^a (\hat{x}_i) I^a \right)^2 - \frac{\hbar}{4m} \tilde{\gamma}_i f_i G_i^a (\hat{x}_i) I^a.
\]  
(28)

The convenient notation \(\sigma_{ij} = \frac{i}{2} [\gamma_i; \gamma_j]\) renders manifest the fact that we deal with a spinor representation of \(SO(d)\). The operator corresponding to the supercharge

\[
\hat{Q}_{\text{suy}} = (\hat{p}_i - u^\dagger A_i^a (\hat{x}_i) T^a u) \lambda_i / \sqrt{\hbar}
\]

is also the square root of \(\hat{H}\), as readily shown after a short algebra: \(\hat{Q}_{\text{suy}}^2 = \hat{H}\).

4.1. Constant non-abelian magnetic field

As an illustration we discuss the particular case corresponding to the problem of a particle confined in a plane in a non-abelian homogeneous magnetic field. For simplicity we restrict ourselves to the \(U(1) \times SU(2)\) gauge group (with \(a = 0, 1, 2, 3\)) and consider the fundamental representation of \(SU(2)\). Representing the \(\lambda_i\) sector with the help of Pauli matrices \([\sigma_{ij}] = 2i\sigma_{12} = 2i\sigma_3\), we find that the Hamiltonian takes the suggestive form

\[
\hat{H} = \frac{1}{2m} \left( \hat{p}_i - A_i^a (\hat{x}_i) I^a \right)^2 + \frac{\hbar}{2m} \tilde{\gamma}_i f_i G_i^a (\hat{x}_i) I^a.
\]  
(30)

In order to study the corresponding Landau problem [6], let us choose the non-abelian vector potential \(A_i (\hat{x}_i) = -(\frac{1}{2} B_e x_i + \beta \sum_{a=1}^3 e_a I^a)\), leading to the non-abelian magnetic field \(G_{12} = BI + \hbar^2 I^3\). Consider then the Fock operators (see for example [26])

\[
a_i = \frac{1}{\sqrt{2}} \sqrt{\frac{m_0 c}{\hbar}} \left( \hat{x}_i + \frac{2i}{m_0 c} \hat{p}_i \right), \quad a_i^\dagger = \frac{1}{\sqrt{2}} \sqrt{\frac{m_0 c}{\hbar}} \left( \hat{x}_i - \frac{2i}{m_0 c} \hat{p}_i \right),
\]

\[\text{with the cyclotron frequency } \omega_c = B/m. \]

We introduce the ‘chiral’ oscillators \((a_+, a_-)\) which verify the Fock algebra \([a_\pm; a_\mp] = 1\), defined through the expressions

\[
a_\pm = \frac{1}{\sqrt{2}} \left( a_1 \mp i a_2 \right), \quad a_\mp = \frac{1}{\sqrt{2}} \left( a_1^\dagger \pm i a_2^\dagger \right).
\]

Therefore the phase space operators \((\hat{x}_i, \hat{p}_i)\) can be expressed in terms of \((a_\pm, a_\mp)\). These notations prove themselves to be appropriate in order to rewrite the quantum supercharge

\[
\sqrt{2m} \hat{Q}_{\text{suy}} = -i \left( \sqrt{2} \hbar B (a_+^\dagger \Sigma_+ + 2i \Sigma_+ b) + i \left( \sqrt{2} \hbar B (a_-^\dagger + 2i \Sigma_-) b^\dagger \right) \right),
\]

\[\text{where we defined the ladder operators } \Sigma_\pm = (\hat{x}_1 \pm i \hat{x}_2) / 2 \text{satisfying } [\Sigma_+; \Sigma_-] = \pm 2i \Sigma_3. \] The fermionic Fock operators used above are defined by \(b = (\gamma_1 - i \gamma_2) / 2\) and \(b^\dagger = (\gamma_1 + i \gamma_2) / 2\).
and satisfy \( \{ h; b^\dagger \} = 1 \). The particular representation of the \((\Sigma_+, \Sigma_1)\) algebra considered here is that of ‘spin’ \( 1/2 \): \( \Sigma_+ |\mp \rangle = h |\pm \rangle \), and \( \Sigma_1 |\pm \rangle = \pm h |\pm \rangle \). The \( 2 \times 2 \) matrix representation of the fermionic Fock algebra enables us to recover the structure of supersymmetric quantum mechanics

\[
\hat{Q}_{\text{susy}} = \begin{pmatrix} 0 & A' \\ A & 0 \end{pmatrix}
\]

where

\[
A = i \left( \sqrt{\frac{\beta B}{m}} a_- + \frac{\beta}{\sqrt{2}m} \Sigma_- \right), \quad A' = -i \left( \frac{\beta B}{m} a_-^\dagger + \frac{\beta}{\sqrt{2}m} \Sigma_- \right).
\]

As a result the quantum Hamiltonian

\[
\hat{H} = \begin{pmatrix} A'A & 0 \\ 0 & AA' \end{pmatrix} = \begin{pmatrix} \hat{H}_B & 0 \\ 0 & \hat{H}_F \end{pmatrix}
\]

manifests a more tractable structure that we elaborate further. It is the customary to name the operators \( A'A \) and \( AA' \) the bosonic and fermionic Hamiltonians, respectively. Considering the eigenvalue problem for the bosonic and fermionic Hamiltonian, it is well known that an eigenstate \( |\psi_{(a_0)} \rangle \) of non-vanishing eigenvalue for the bosonic Hamiltonian has a superpartner eigenstate of the fermionic Hamiltonian \( |\psi_{(b)} \rangle = A |\psi_{(a_0)} \rangle \) associated to the same eigenvalue. As a matter of fact the case of an eigenstate with vanishing eigenvalue needs to be considered separately. The bosonic Hamiltonian \( A'A \) possesses a zero-energy eigenstate provided by the condition \( A |\psi_{(a_0)} \rangle = 0 \):

\[
|\psi_{(a_0)} \rangle = |n_-, 0, n_+ \rangle
\]

with \( \{ n_- = 0, n_+, + \} = |n_-, \rangle \otimes |n_+ \rangle \otimes |\pm \rangle \) where \( |n_\pm \rangle \) are normalized chiral Fock states. Nevertheless the fermionic Hamiltonian \( AA' \) does not have a zero-energy eigenstate so that \( |\psi_{(a_0)} \rangle \) has no superpartner. In consequence the reminder of the spectrum is found among the nonvanishing eigenvalues of the bosonic Hamiltonian

\[
\hat{H}_B = \hbar \omega_0 a_- a_- + \sqrt{\frac{2}{m}} \beta B \left( a_-^\dagger \Sigma_- + a_- \Sigma_- \right) + \frac{2\beta^2}{m} \Sigma_+ \Sigma_-.
\]

It is a straightforward exercise to obtain the nonvanishing eigenvalues of \( \hat{H}_B \):

\[
E_{(n_- \pm)}^B = \left( \hbar \omega_0 \left( n_- + \frac{1}{2} \right) + \hbar^2 \beta^2 \right) \left\{ 1 \pm \frac{1}{\sqrt{1 - \frac{\beta \omega_0}{m}}} \right\} \left( n_- \frac{1}{2} + \frac{\beta \omega_0}{m} \right),
\]

for \( n_- \in \mathbb{N} \), associated to the eigenstates

\[
\frac{1}{N_{(n_- \pm)}^B} \left| E_{(n_- \pm)}^B \right| = \sqrt{2\beta B \left( n_- + 1 \right)} \frac{\hbar \omega_0}{m} |n_-, n_+, + \rangle
\]

\[
+ \left( E_{(n_- \pm)}^B - 2 \frac{\beta^2 \omega_0^2}{m} - \hbar \omega_0 n_- \right) |n_- + 1, n_+, - \rangle
\]

where \( N_{(n_- \pm)}^B \) is an appropriate normalization. Finally the eigenvectors of \( \hat{H}_F \) given by \( A |E_{(n_- \pm)}^B \rangle \) correspond to the same nonzero eigenvalues, as is expected because of
supersymmetry. Note how each of these energy eigenvalues are doubly infinitely degenerate, corresponding to the supersymmetric extension of the ordinary Landau levels, except for the lowest Landau level which is $\mathcal{N} = 1$ supersymmetry invariant and of zero energy eigenvalue.

5. Conclusion

Given the present construction of the $\mathcal{N} = 1$ supersymmetric generalization of the Wong equations, within the restriction to the $U(1) \times SU(2)$ gauge group we discussed the supersymmetric structure of the spectrum of a particular case: the non-abelian Landau problem. The relevance of this model to low dimensional materials exhibiting spin–orbit interactions may be investigated. Besides its phenomenological interest the Landau problem illustrates important modern mathematical developments, such as non-commutative geometry. As a perspective the projection of the dynamics on selected Landau levels and an investigation of the ensuing consequences of supersymmetry on the non-commutative planar Euclidian geometry may be carried out.

Acknowledgements

The work of MF is supported by the National Fund for Scientific Research (F R S –FNRS, Belgium) through an Aspirant research fellowship. This work is supported by the Belgian Federal Office for Scientific, Technical and Cultural Affairs through the Interuniversity Attraction Pole P6/11.

Appendix. Electric potential

As a generalization, it is possible to include in the Lagrangian a scalar potential term, $-u^i \Lambda_1^\dagger A_0^a (x) T^a u$, where we restrict ourselves to the case of a bosonic $u$. The chromo-electric potential $A_0$ is valued in the Lie algebra of the gauge group. The results obtained below can probably be found by considering the nonrelativistic limit of the model for a spinning particle in a background field with $D = 1$ supergravity, as studied in [27]. In the present work we obtain the supersymmetric action from the superspace formalism. To do so we define in addition the real Grassmann odd superfields $Y = y_0 + \theta y_1$ and $\Lambda = \Lambda_0 + \theta \Lambda_1$, where $y_0$ and $\Lambda_0$ are real fermions, and $y_1, \Lambda_1$ are real bosons. With the help of these supercoordinates, we build

$$\Pi_0 = -YU^\dagger A_0(X) U + \Lambda (DY - 1)$$  \hspace{1cm} (A.1)

$$= \Lambda_0 (y_1 - 1) - y_0 u^i A_0^a (x_i) u + \theta \left( i \Lambda_0 \tilde{y}_0 + \Lambda_1 (y_1 - 1) - y_1 u^i A_0 (x_i) u \right. \left. + y_0 u^i A_0 (x_i) w + w^i A_0 (x_i) u y_0 + i u^i y_0 \partial_i A_0 (x_i) u \right),$$ \hspace{1cm} (A.2)

whose highest component $L_0 = \int d\theta \Pi_0$ contains the term $-u^i A_0^a (x_i) u$ after solving the equation of motion for $\Lambda_1$. The result is

1 This $\Lambda$ should not be confused with the $\Lambda$ superfield gauge parameter in (11).
The role of the variable $A_0$ is now clear, it is the conjugate momentum of $y_0$. The complete Lagrangian obtained from the highest component of the superfield

$$
-\frac{1}{2}mD^2X_iDX_i + \frac{1}{2}\left\{ (-D + V) U^\dagger U + U^\dagger (-D + V) U \right\}
$$

where $V = i(DX_i)A_i(X_i)$, and takes the following form in components

$$
L_T = \frac{1}{2}m\dot{x}_i^2 - \frac{i}{2}m\dot{\lambda}_i\dot{\lambda}_i + iA_0\dot{y}_0 + u^\dagger i\partial_j + \dot{x}_iA_i(x_i))u - u^\dagger A_0(x_i)u
$$

$$
+ w^\dagger w + w^\dagger \lambda_iA_i(x_i)u + u^\dagger \lambda_iA_i(x_i)w + u^\dagger y_0A_0(x_i)w + w^\dagger y_0A_0(x_i)u
$$

$$
+ u^\dagger \dot{y}_0\dot{\lambda}_i\partial A_0(x_i)u + u^\dagger \dot{\lambda}_i\dot{\lambda}_j\partial A_i(x_i)u.
$$

In order to make the expression explicitly invariant under time independent gauge transformations we solve the equations of motions for $w$ and $w^\dagger$, and we obtain

$$
L_T = \frac{1}{2}m\dot{x}_i^2 - \frac{i}{2}m\dot{\lambda}_i\dot{\lambda}_i + iA_0\dot{y}_0 + u^\dagger (i\partial_j + \dot{x}_iA_i(x_i))u - u^\dagger A_0(x_i)u
$$

$$
- iy_0\dot{\lambda}_iG_{00}u - \frac{i}{2}\dot{\lambda}_i\dot{\lambda}_jG_{ij}u,
$$

with the ‘chromo-electric’ field $G_{00} = -\partial A_0(x_i) - i[A_0; A_i](x_i)$. Within the Hamiltonian analysis, the Dirac bracket $\{y_0; A_0\}_D = -i$ is straightforwardly obtained and the classical supercharge is then given by the expression

$$
Q_T = \left( p_i - u^\dagger A_i u \right) \dot{\lambda}_i + A_0 + y_0u^\dagger A_0u,
$$

which verifies $\{Q_T; Q_T\} = -2iH_T$ with

$$
H_T = \frac{1}{2m}\left( p_i - u^\dagger A_i u \right)^2 + \frac{i}{2}\dot{\lambda}_i\dot{\lambda}_jG_{ij}u + iy_0\dot{\lambda}_iG_{00}u + u^\dagger A_0u.
$$

References

[1] Wong S K 1970 Field and particle equations for the classical Yang-Mills field and particles with isotopic spin Nuovo Cimento. A 65 689

[2] Balachandran A P, Salomonson P, Skagerstam B-S and Winnberg J-O 1977 Classical description of particle interacting with nonabelian gauge field Phys. Rev. D 15 2308

[3] Wipf A W 1985 Nonrelativistic Yang-Mills particles in a spherically symmetric monopole J. Phys. A: Math. Gen. 18 2379

[4] Horvathy P A 1982 An action principle for isospin (unpublished)

[5] Jalilian-Marian J, Jeon S and Venugopalan R 2001 Wong’s equations and the small $x$ effective action in QCD Phys. Rev. D 63 036004

[6] Estienne B, Haaker S M and Schoutens K 2011 Particles in non-abelian gauge potentials—Landau problem and insertion of non-abelian flux New. J. Phys. 13 045012

[7] Landau L and Lifshitz E M 2003 Quantum Mechanics (Non-Relativistic Theory) 3rd edn (Course on Theoretical Physics vol 3) (London: Heinemann)

[8] Rashba E I 1960 Fiz. Tverd. Tela 2 1224–38
Rashba E I 1960 Sov. Phys.—Solid State 2 1109–22
[9] Rashba E I 1960 Sov. Phys.–Solid State 2 1109 (Engl. transl.)
[10] Dresselhaus G 1955 Spin-orbit coupling effects in zinc blende structures Phys. Rev. 100 580
[11] Palmer R N and Pachos J K 2011 Fractional quantum Hall effect in a $U(1) \times SU(2)$ gauge field New J. Phys. 13 065002
[12] Burrello M and Trombettoni A 2010 Non-abelian anyons from degenerate Landau levels of ultracold atoms in artificial gauge potentials Phys. Rev. Lett. 105 125304
[13] Burrello M and Trombettoni A 2011 Ultracold atoms in $U(2)$ non-abelian gauge potentials preserving the Landau levels Phys. Rev. A 84 043625
[14] Ezawa M 2008 Supersymmetry and unconventional quantum Hall effect in graphene Phys. Lett. A 372 924
[15] Linden N, Macfarlane A J and van Holten J W 1995 Particle motion in a Yang-Mills field: Wong’s equations and spin 1/2 analogs Czech. J. Phys. 46 209
[16] Fedoruk S, Ivanov E and Lechtenfeld O 2009 Supersymmetric Calogero models by gauging Phys. Rev. D 79 105015
[17] Fedoruk S, Ivanov E and Lechtenfeld O 2009 $OSp(4|2)$ Superconformal mechanics J. High Energy Phys. JHEP08(2009)081
[18] Fedoruk S, Ivanov E and Lechtenfeld O 2010 New $D(2, 1; \alpha)$ mechanics with spin variables J. High Energy Phys. JHEP04(2010)129
[19] Fedoruk S, Ivanov E and Lechtenfeld O 2012 Superconformal mechanics J. Phys. A: Math. Theor. 45 173001
[20] Ivanov E, Konyushikhin M A and Smilga A V 2010 SQM with non-abelian self-dual fields: harmonic superspace description J. High Energy Phys. JHEP05(2010)033
[21] Ivanov E and Konyushikhin M A 2010 Supersymmetric quantum mechanics in non-abelian monopole background Phys. Rev. D 82 085014
[22] Rietdijk R H 1992 Classical and quantum mechanics of a spinning particle interacting with antisymmetric non-Abelian tensor fields Mod. Phys. Lett. A 7 881
[23] Rietdijk R H 1992 Classical and quantum mechanics of a spinning particle interacting with background fields Class. Quant. Grav. 9 1395
[24] Ben Geloun J, Govaerts J and Scholtz F G 2009 The $N = 1$ supersymmetric Landau problem and its supersymmetric Landau level projections: the $N = 1$ supersymmetric Moyal–Voros superplane J. Phys. A: Math. Theor. 42 495203
[25] Govaerts J 1991 Hamiltonian Quantisation and Constrained Dynamics (Leuven Notes in Mathematical and Theoretical Physics vol B4) (Leuven: Leuven University Press)
[26] Govaerts J, Hounkonnou M N and Mweene H V 2009 Variations on the planar Landau problem: canonical transformations, a purely linear potential and the half-plane J. Phys. A: Math. Theor. 42 485209
[27] van Holten J W 1996 $D = 1$ supergravity and spinning particles (Field Theory to Quantum Groups) ed B Jancewicz and B Sobczyk (Singapore: World Scientific) p 173