ON CONCENTRATED TRAVELING VORTEX PAIRS WITH PRESCRIBED IMPULSE

GUODONG WANG

ABSTRACT. In this paper, we consider a constrained maximization problem related to planar vortex pairs with prescribed impulse. We prove existence, stability and asymptotic behavior for the maximizers, hence obtain a family of stable traveling vortex pairs approaching a pair of point vortices with equal magnitude and opposite signs. As a corollary, we get fine asymptotic estimates for Burton’s vortex pairs with large impulse. For the non-concentrated case, we prove a form of stability for the Chaplygin-Lamb dipole.

1. Introduction and main result

For a two-dimensional ideal fluid of unit density with vanishing velocity at infinity, the governing equations take the form

\[
\begin{aligned}
\partial_t \omega + v \cdot \nabla \omega &= 0, \\
v(t, x) &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t, y) \, dy,
\end{aligned}
\]

where \( v = (v_1, v_2) \) is the velocity field, \( \omega = \partial_{x_1} v_2 - \partial_{x_2} v_1 \) is the scalar vorticity, and \( \perp \) denotes the clockwise rotation through \( \pi/2 \), that is, \( (x_1, x_2) \perp = (x_2, -x_1) \). For initial vorticity \( \omega|_{t=0} = \omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2) \), global existence and uniqueness of weak solutions for the vorticity equation (1.1) was established by Yudovich [26]. A modern proof of Yudovich’s result can be found in [15] or [17].

In this paper, we are interested in a special class of solutions of Yudovich type, called traveling vortex pairs, that is, weak solutions exhibiting odd symmetry with respect to some line and traveling along the direction of the line at a constant speed without changing the profile. After a suitable translation and rotation, we can assume that \( L \) coincides with the \( x_1 \)-axis, then a vortex pair \( \omega \) traveling with speed \( b \) takes the form

\[
\omega(t, x) = \zeta(x_1 - bt, x_2) - \zeta(x_1 - bt, -x_2) \quad \forall \ x = (x_1, x_2) \in \mathbb{R}^2,
\]

where \( \zeta \) is an integrable and bounded function supported in the upper half-plane \( \Pi = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\} \). In the literature, it is often the case that \( \zeta \) has bounded support.

When studying vortex pairs, it is convenient to consider the following vorticity equation in the upper half-plane

\[
\begin{aligned}
\partial_t \omega + \nabla^\perp G \omega \cdot \nabla \omega &= 0, \\
G \omega(t, x) &= \frac{1}{2\pi} \int_{\Pi} \ln \frac{|x-y|}{|x-y|'} \omega(t, y) \, dy.
\end{aligned}
\]

Here \( \nabla^\perp = (\partial_{x_2}, -\partial_{x_1}) \), and \( \bar{y} \) denotes the reflection of \( y \) in the \( x_1 \)-axis, i.e., \( \bar{y} = (y_1, -y_2) \). Once we have obtained a traveling solution to (1.3), we can extend it to the whole plane.
by reflection in the $x_1$-axis to get a traveling vortex pair for \((1.1)\). The vorticity equation \((1.3)\) can also be regarded as the governing equation for an ideal fluid of unit density such that the velocity $v = (v_1, v_2)$ satisfies
\[
v_2 \equiv 0 \text{ on } \partial \Pi = \{x \in \mathbb{R}^2 | x_2 = 0\} \quad \text{and} \quad \lim_{|x| \to +\infty} |v(t, x)| = 0, \quad \forall t \geq 0.
\]

The study for traveling vortex pairs has lasted for more than one century. For experimental or numerical results, see \([10, 19, 20]\) for example. For rigorous existence results, see \([3, 5–7, 11, 13, 18, 21, 22, 24, 25]\) and the references therein. One famous traveling vortex pair is the Chaplygin-Lamb dipole, independently introduced by Chaplygin \([8]\) and Lamb \([13]\) in the early 20th century. For the Chaplygin-Lamb dipole, the stream function can be explicitly expressed in terms of the Bessel function of the first kind \((13, \text{p.245})\), and the support of the vorticity is exactly a disk (therefore the Chaplygin-Lamb dipole is also called Lamb's circular vortex pair). A form of orbital stability for the Chaplygin-Lamb dipole was recently proved by Abe-Choi \([1]\). In Section 5, we will consider the Chaplygin-Lamb dipole and prove a different kind of stability. Another typical example is a pair of point vortices with equal magnitude and opposite signs, taking the form
\[
\omega = \kappa \delta_{z(t)} - \kappa \delta_{\bar{z}(t)}, \quad z(t) = (bt, d), \quad (1.4)
\]
where $b$ represents the traveling speed, $d > 0$ is fixed, and $\delta_{z(t)}$ is the Dirac measure centered on $z(t)$. Note that \((1.4)\) is just a formal singular solution to the vorticity equation \((1.1)\), not in the sense of Yudovich. For a rigorous discussion on the relation between point vortices and the vorticity equation \((1.1)\), we refer the interested reader to Marchioro-Pulvirenti \([16]\). According to \([16]\), $z(t)$ in \((1.4)\) is supposed to satisfy the following Kirchhoff-Routh equation
\[
\frac{dz(t)}{dt} = \frac{\kappa}{2\pi} \frac{(z(t) - \bar{z}(t))^\perp}{|z(t) - \bar{z}(t)|^2}
\]
Therefore $\kappa, b, d$ necessarily satisfy the formula
\[
4\pi bd = \kappa. \quad (1.5)
\]

An important and interesting problem related to \((1.4)\) is the desingularization problem, that is, to find a family of regular solutions in the sense of Yudovich such that the vorticity is concentrated and localized near the two points $(0, d)$ and $(0, -d)$ and approximate the singular solution \((1.4)\). More precisely, one needs to construct a family of concentrated vortex pairs $\{\omega^\varepsilon\}$, where $\varepsilon > 0$ is a parameter, such that
\[
\omega^\varepsilon(t, x) = \zeta^\varepsilon(x_1 - bt, x_2) - \zeta^\varepsilon(x_1 - bt, -x_2), \quad \int_{\Pi} \zeta^\varepsilon(x)dx = \kappa, \quad \text{supp}(\zeta^\varepsilon) \subset B_{o(1)}(0, d),
\]
where $o(1) \to 0$ as $\varepsilon \to 0^+$. This problem can be studied via several methods, including the vorticity method (see \([6, 22]\)), the stream function method (see \([21, 25]\)), and the contour dynamics reformulation (see \([11]\)). Now that the Chaplygin-Lamb dipole is not a vortex pair of the above desingularization type since its support touches the $x_1$-axis.

Although there are various existence results on concentrated traveling vortex pairs in the literature, the corresponding stability problem is less well studied. In this paper, by
stability we mean Lyapunov type. More specifically, a vortex pair $\bar{\omega}$ is said to be stable if for any initial vorticity $\omega_0$ that is close to $\bar{\omega}$, the evolved vorticity $\omega_t$ remains close to $\bar{\omega}$ for all $t > 0$. Since a vortex pair translates along the $x_1$ direction at a constant speed, it is more reasonable to consider orbital stability, the precise meaning of which will be given in Theorem 1.2. It is worth mentioning that in most cases stability is a more sophisticated problem than existence, since it usually requires better estimates. To our limited knowledge, there are very few results on the stability of concentrated traveling vortex pairs in the literature.

Our main purpose in this paper is to prove the existence of a large class of concentrated traveling vortex pairs with certain stability. To make our main result concise, we introduce the concept of $L^p$-regular solutions. Throughout this paper, let $2 < p < +\infty$ be fixed.

Definition 1.1. An $L^p$-regular solution is a function $\omega \in C([0, +\infty); L^1 \cap L^p(\Pi))$ such that

(i) $\omega$ solves the vorticity equation (1.3) in the distributional sense, that is,

$$\int_\Pi \omega(0, x) \varphi(0, x) dx + \int_0^{+\infty} \int_\Pi \omega (\partial_t \varphi + \nabla^\perp G \omega \cdot \nabla \varphi) dx dt = 0 \quad \forall \varphi \in C^\infty_c (\Pi \times \mathbb{R});$$

(ii) the kinetic energy $E$ and the impulse (parallel to the $x_1$-axis) $I$, defined by

$$E(\omega(t, \cdot)) = \frac{1}{2} \int_\Pi \omega(t, x) G \omega(t, x) dx, \quad I(\omega(t, \cdot)) = \int_\Pi x_2 \omega(t, x) dx, \quad (1.6)$$

are conserved, that is, for all $t > 0$,

$$E(\omega(t, \cdot)) = E(\omega(0, \cdot)), \quad I(\omega(t, \cdot)) = I(\omega(0, \cdot)).$$

Note that for any initial vorticity $\omega_0 \in L^1 \cap L^p(\Pi)$ with bounded support, an $L^p$-regular solution always exists, but may not be unique, except in the case of $\omega_0$ being additionally bounded. See [3] for a detailed discussion on this issue.

We also need some notation. In the rest of this paper, let $\varrho \in L^p(\mathbb{R}^2)$ be given such that

(H1) $\varrho$ is nontrivial and nonnegative;

(H2) $\varrho$ has compact support;

(H3) $\varrho$ is radially symmetric and decreasing, that is,

$$\varrho(x) = \varrho(y) \quad \forall x, y \in \mathbb{R}^2 \text{ such that } |x| = |y|,$$

$$\varrho(x) \geq \varrho(y) \quad \forall x, y \in \mathbb{R}^2 \text{ such that } |x| \leq |y|.$$  

By (H1)-(H3), there exists some $r > 0$ such that

$$\{x \in \mathbb{R}^2 \mid \varrho(x) > 0 \} = B_r(0). \quad (1.7)$$

Here $0$ denotes the origin. Let $\kappa > 0$ be fixed such that

$$\kappa = \int_{\mathbb{R}^2} \varrho(x) dx. \quad (1.8)$$

For $\varepsilon > 0$, denote

$$\varrho^\varepsilon(x) = \frac{1}{\varepsilon^2} \varrho \left( \frac{x}{\varepsilon} \right), \quad x \in \mathbb{R}^2. \quad (1.9)$$
Let $\mathcal{R}(\varrho^\varepsilon)$ be the set of all equimeasurable rearrangements of $\varrho^\varepsilon$ on $\Pi$, that is,
\[ \mathcal{R}(\varrho^\varepsilon) = \{ v \in L^p(\Pi) \mid \mathcal{L}(\{ x \in \Pi \mid v(x) > s \}) = \mathcal{L}(\{ x \in \mathbb{R}^2 \mid \varrho^\varepsilon(x) > s \}) \ \forall \ s \in \mathbb{R} \}. \]

Here and henceforth, $\mathcal{L}$ denotes the two-dimensional Lebesgue measure.

For fixed $i_0 > 0$, define
\[ S_{\varepsilon,i_0} = \{ v \in \mathcal{R}(\varrho^\varepsilon) \mid I(v) = i_0 \}. \tag{1.10} \]

Consider the following maximization problem
\[ M_{\varepsilon,i_0} = \sup_{v \in S_{\varepsilon,i_0}} E(v). \tag{1.11} \]

Denote $\Gamma_{\varepsilon,i_0}$ the set of maximizers of $E$ over $S_{\varepsilon,i_0}$, that is,
\[ \Gamma_{\varepsilon,i_0} = \{ v \in S_{\varepsilon,i_0} \mid E(v) = M_{\varepsilon,i_0} \}. \tag{1.12} \]

Note that $\Gamma_{\varepsilon,i_0}$ might be empty.

Our main result can be summarized as follows.

**Theorem 1.2.** Let $p > 2$ and $i_0 > 0$ be given. Let $\varrho \in L^p(\Pi)$ satisfy (H1)-(H3). Let $\Gamma_{\varepsilon,i_0}$ be defined by (1.12). Then there exists some $\varepsilon_0 > 0$, depending only on $\varrho$ and $i_0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, the following conclusions hold.

1. $\Gamma_{\varepsilon,i_0} \neq \emptyset$.
2. Every $\zeta \in \Gamma_{\varepsilon,i_0}$ has bounded support, and is Steiner-symmetric with respect to the line $x_1 = c$ for some $c \in \mathbb{R}$.
3. For every $\zeta \in \Gamma_{\varepsilon,i_0}$, there exists some positive number $\lambda_\zeta$ and some increasing function $f_\zeta : \mathbb{R} \to \mathbb{R} \cup \{ \pm \infty \}$ such that
   \[ \zeta = f_\zeta(\mathcal{G}\zeta - \lambda_\zeta x_2) \text{ a.e. in } \Pi, \tag{1.13} \]
   \[ \zeta = 0 \text{ a.e. in } \{ x \in \Pi \mid \mathcal{G}\zeta(x) - \lambda_\zeta x_2 \leq 0 \}. \tag{1.14} \]
4. For every $\zeta \in \Gamma_{\varepsilon,i_0}$, $\omega(t,x) = \zeta(x_1 - \lambda_\zeta t, x_2) - \zeta(x_1 - \lambda_\zeta t, -x_2)$ is an $L^p$-regular solution to the vorticity (1.3), thus yielding a traveling vortex pair with speed $\lambda_\zeta$.
5. Define
   \[ \|v\|_{X_p} = \|I(v)\| + \|v\|_{L^1(\Pi)} + \|v\|_{L^p(\Pi)}. \tag{1.15} \]
   Then $\Gamma_{\varepsilon,i_0}$ is orbitally stable in the $X_p$ norm, that is, for any $\varepsilon > 0$, there exists some $\delta > 0$, such that for any $L^p$-regular solution $\omega$ satisfying $\omega(0, \cdot)$ is nonnegative with bounded support,
   \[ \inf_{\zeta \in \Gamma_{\varepsilon,i_0}} \|\omega(0, \cdot) - \zeta\|_{X_p} < \delta, \]
   it holds that
   \[ \inf_{\zeta \in \Gamma_{\varepsilon,i_0}} \|\omega(t, \cdot) - \zeta\|_{X_p} < \epsilon \ \forall \ t \geq 0. \]
(6) For every $\zeta \in \Gamma_{\varepsilon,i_0}$, denote $V_\zeta = \{ x \in \Pi \mid \zeta(x) > 0 \}$. Then there is some $C > 0$, depending only on $\varrho$ and $i_0$, such that

$$\text{diam}(V_\zeta) \leq C \varepsilon \quad \forall \zeta \in \Gamma_{\varepsilon,i_0},$$

where $\text{diam}(V_\zeta)$ is the diameter of $V_\zeta$ given by

$$\text{diam}(V_\zeta) = \sup_{x,y \in V_\zeta} |x - y|.$$

(7) As $\varepsilon \to 0^+$, $\lambda_\zeta \to \kappa^2/(4\pi i_0)$ uniformly for $\zeta \in \Gamma_{\varepsilon,i_0}$. More precisely, for any $\varepsilon > 0$, there exists some $\varepsilon_2 \in (0, \varepsilon_0)$, depending only on $\varrho, i_0$ and $\varepsilon$, such that for any $\varepsilon \in (0, \varepsilon_2)$, it holds that

$$\left| \lambda_\zeta - \frac{\kappa^2}{4\pi i_0} \right| < \varepsilon \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}.$$

(8) Define

$$\Gamma_{0,\varepsilon,i_0}^0 = \left\{ \zeta \in \Gamma_{\varepsilon,i_0} \mid \int_{\Pi} x_1 \zeta(x) dx = 0 \right\}. \quad (1.16)$$

For $\zeta \in \Gamma_{0,\varepsilon,i_0}^0$, extend $\zeta$ to $\mathbb{R}^2$ such that $\zeta \equiv 0$ in the lower half-plane. Define

$$\nu^{0, \varepsilon}(x) = \varepsilon^2 \zeta(\varepsilon x + \hat{x}), \quad \hat{x} = (0, i_0/\kappa).$$

Then $\nu^{0, \varepsilon} \to \varrho$ in $L^p(\mathbb{R}^2)$ uniformly as $\varepsilon \to 0^+$. More precisely, for any $\varepsilon > 0$, there exists some $\varepsilon_3 \in (0, \varepsilon_0)$, depending only on $\varrho, i_0, p$ and $\varepsilon$, such that for any $\varepsilon \in (0, \varepsilon_3)$, it holds that

$$\| \nu^{0, \varepsilon} - \varrho \|_{L^p(\mathbb{R}^2)} < \varepsilon \quad \forall \zeta \in \Gamma_{0,\varepsilon,i_0}^0.$$

Remark 1.3. Since $E$ and $I$ are both invariant under any translation parallel to the $x_1$-axis, it is clear that

$$\Gamma_{\varepsilon,i_0} = \{ \zeta(\cdot + ce_1) \mid \zeta \in \Gamma_{0,\varepsilon,i_0}^0, \ c \in \mathbb{R} \}, \quad (1.17)$$

where $e_1 = (1,0)$. Therefore $\Gamma_{0,\varepsilon,i_0}^0$ is exactly the set of maximizers that are Steiner-symmetric in the $x_2$-axis. As a result, by (1.17) and item (8) in Theorem 1.2 we see that after some suitable translation and scaling, any $\zeta \in \Gamma_{\varepsilon,i_0}$ converges uniformly to $\varrho$ in $L^p(\mathbb{R}^2)$.

Remark 1.4. By the definition of $\Gamma_{0,\varepsilon,i_0}^0$ and the fact that $I(\zeta) = i_0$ for any $\zeta \in \Gamma_{\varepsilon,i_0}$, we see that

$$\frac{1}{\kappa} \int_{\Pi} x_1 \zeta(x) dx = \left(0, \frac{i_0}{\kappa}\right) = \hat{x}.$$
Therefore Theorem 1.2 in fact provides a family of desingularized solutions for the following pair of point vortices

\[
\omega = \kappa \delta_{z(t)} - \kappa \delta_{\bar{z}(t)}, \quad z(t) = \left( \frac{\kappa^2 t}{4\pi i_0}, \frac{i_0}{\kappa} \right).
\]  

The proofs of items (1)-(5) in Theorem 1.2 follow from Burton’s results in [6] and the scaling properties of \( S_{\varepsilon,i_0} \) and \( \Gamma_{\varepsilon,i_0} \). Our focus in this paper is the asymptotic behavior (6)-(8). To prove (6)-(8), we adapt the energy method established by Turkington in [22]. However, unlike the bounded domain case in [22], in this paper the unboundedness of the upper half-plane causes great trouble since the regular part of the Green function is no longer bounded from below. To overcome this difficulty, we need to show the uniform boundedness of the vortex core. This is achieved based on the idea developed by the author in [23]. Of course, due to the different variational nature, some new technical difficulties appear during this process, most of which are caused by the traveling speed. In contrast to [23], the vortex pairs in this paper have unknown traveling speed that depends on the parameter \( \varepsilon \). To apply the method in [23], some proper uniform estimates for the traveling speed should be deduced at the beginning. To achieve this, we use a Pohozaev type identity proved by Burton in [3] and the fact that the kinetic energy on the vortex core is uniformly bounded. As to the limit of the traveling speed as \( \varepsilon \to 0^+ \), we derive a useful formula (see Lemma 3.20) such that the traveling speed can be expressed in terms of an integral involving the vorticity. Except for the traveling speed, some specific estimates are also different from the ones in [23] and need to be taken seriously.

Below we recall some closely related results in the literature and give some comments on Theorem 1.2. In [22], p.1061, Turkington proposed a possible method to construct concentrated vortex pairs with prescribed impulse, that is, to maximize \( E \) over

\[
J_{a,\lambda} = \left\{ v \in L^\infty(\Pi) \mid \|v\|_{L^1(\Pi)} = \kappa, \ I(v) = i_0, \ 0 \leq v \leq \lambda, \ \text{supp}(v) \subset [-a,a] \times [0,a] \right\},
\]

where \( \kappa, i_0 > 0 \) are fixed, \( a > i_0/\kappa \), and \( \lambda > 0 \) is a parameter. Following Turkington’s idea in [22], it is not hard to prove that for sufficiently large \( \lambda \), depending on \( \kappa, i_0 \) and \( a \), there exists a maximizer, and any maximizer \( \zeta \) satisfies

\[
\zeta = \lambda 1_{\Omega_\zeta},
\]  

for some bounded open set \( \Omega_\zeta \subset \Pi \) depending on \( \zeta \). Here \( 1 \) denotes the characteristic function. Moreover, similar asymptotic estimates as in Theorem 1.2 hold as \( \lambda \to +\infty \). Since any maximizer has the patch form (1.19), maximizing \( E \) relative to \( J_{a,\lambda} \) is actually equivalent to maximizing \( E \) relative to

\[
K_{a,\lambda} = \left\{ v = \lambda 1_\Omega \mid \Omega \subset \Pi, \ \lambda C(\Omega) = \kappa, \ I(v) = i_0, \ \text{supp}(v) \subset [-a,a] \times [0,a] \right\}.
\]

This is nearly one special case of our maximization problem (1.11) if \( \phi \) is chosen to be a suitable patch function, except for an extra constraint on the supports of the admissible functions. A natural question is whether this maximization problem results in the same solutions as (1.11) does (up to a translation in the \( x_1 \) direction). To answer this question, one needs to show that the parameter \( a \) has no impact on the maximization problem. Unfortunately, according to Turkington’s method in [22], the asymptotic estimates for the
maximizers inevitably depend on $a$, and it might even happen that the diameter of the supports the maximizers go to infinity as $a \to +\infty$. One consequence of our Theorem 1.2 is that it provides a positive answer to this question. In fact, since all the maximizers of the maximization problem (1.11) have vanishing supports near $\bar{x} = (0, i_0/\kappa)$, the constraint $\text{supp}(v) \subset [-a, a] \times [0, a]$ does not make any impact if $a > i_0/\kappa$. As a corollary, the vortex patch pairs obtained by Turkington’s method are orbitally stable in the sense of item (5) in Theorem 1.2.

Another related work is [3], in which Burton considered a maximization problem related to traveling vortex pairs with prescribed rearrangement and large impulse. More specifically, Burton studied the maximization of $E$ subject to

$$\mathcal{F}_{a, \alpha} = \{ v \in \mathcal{S}_{1, \alpha} \mid \text{supp}(v) \subset [-a, a] \times [0, a] \},$$

where $a, \alpha$ are positive constants, and $\mathcal{S}_{1, \alpha}$ is defined in analogous to (1.10), that is,

$$\mathcal{S}_{1, \alpha} = \{ v \in \mathcal{R}(\varrho) \mid I(v) = \alpha \}. \quad (1.20)$$

Burton proved that for sufficiently large $\alpha$ depending only on $\varrho$, and sufficiently large $a$ depending on $\varrho$ and $\alpha$, there exists a maximizer, and any maximizer corresponds to a traveling vortex pair with unknown speed depending on $\alpha$ and $a$. Since $\mathcal{F}_{a, \alpha}$ is not an invariant class of the vorticity equation (1.3), stability analysis for the vortex pairs obtained in this way is quite complicated. To obtain stable traveling vortex pairs, Burton [6] considered the maximization of $E$ subject to $\mathcal{S}_{1, \alpha}$ directly. Therein, existence and a form of orbital stability were obtained based on Douglas’ work [9] and a concentrated-compactness argument. The precise statements of these results will be presented in Lemma 2.1, Section 2. An interesting problem related to Burton’s vortex pairs in [6] is to study the asymptotic behavior as $\alpha \to +\infty$, including what these vortex pairs look like and where they are located. As we will see in Section 4, this problem can be easily solved as a consequence Theorem 1.2.

In addition to the vortex pairs mentioned above, which have prescribed impulse, in [3] and [22] the authors also studied traveling vortex pairs with prescribed speed by studying a similar variational problem. A stability criterion for this kind of vortex pairs was proved in [5]. Based on the results in [3] and inspired by the method in [22], recently the author in [23] proved the existence of a family of concentrated stable vortex pairs with prescribed traveling speed. It is worth mentioning that whether these two kinds of vortex pairs (that is, one with prescribed impulse and the other with prescribed speed) are the same is still open. See [6], p. 548 for a brief discussion.

In Theorem 1.2 existence and stability are obtained for small $\varepsilon$. For the general case, related questions are less well understood. However, we will show in Section 5 that for some specially chosen rearrangement class and impulse, existence and stability also hold, and the maximizers are exactly the Chaplygin-Lamb dipole together with its translations parallel to the $x_1$-axis.

This paper is organized as follows. In Section 2, we provide some preliminaries for later use. In Section 3, we give the proof of Theorem 1.2. In Section 4, we study the asymptotic
behavior of Burton’s vortex pairs in \cite{6} based on Theorem 1.2. In Section 5, we discuss a special non-concentrated case and prove a form of stability for the Chaplygin-Lamb dipole.

2. Preliminaries

In this section, we present some preliminaries that will be used in Section 3.

The first lemma summarizes what Burton proved in \cite{6} regarding the maximization of $E$ relative to $S_{1,\alpha}$, which is defined by (1.20) in Section 1.

**Lemma 2.1.** Let $\rho$ satisfy (H1)-(H3) in Section 1, $\alpha > 0$ be a parameter, and $\Sigma_{\alpha}$ be the set of maximizers of $E$ relative to $S_{1,\alpha}$. Then there exists some $\alpha_0 > 0$, depending only on $\rho$, such that for any $\alpha > \alpha_0$ the following assertions hold.

(i) $\Sigma_{\alpha} \neq \emptyset$.

(ii) Every $\zeta \in \Sigma_{\alpha}$ has bounded support, and is Steiner-symmetric with respect to some line $x_1 = c$ for some $c \in \mathbb{R}$.

(iii) For every $\zeta \in \Sigma_{\alpha}$, there exist some positive number $q_{\zeta} > 0$ and some increasing function $\phi_{\zeta} : \mathbb{R} \to \mathbb{R} \cup \{\pm \infty\}$ such that

$$
\zeta = \phi_{\zeta}(G\zeta - q_{\zeta}x_2) \text{ a.e. in } \Pi,
$$

$$
\zeta = 0 \text{ a.e. in } \{x \in \Pi \mid G\zeta(x) - q_{\zeta}x_2 \leq 0\}.
$$

(iv) For every $\zeta \in \Sigma_{\alpha}$, $\omega(t,x) = \zeta(x_1 - q_{\zeta}t, x_2) - \zeta(x_1 - q_{\zeta}t, -x_2)$ is an $L^p$-regular solution to the vorticity (1.3).

(v) $\Sigma_{\alpha}$ is orbitally stable in the $X_p$ norm in the following sense: for any $\epsilon > 0$, there exists some $\delta > 0$, depending only on $\rho, \alpha, p$ and $\epsilon$, such that for any $L^p$ regular solution $\omega$ satisfying

$$
\omega(0, \cdot) \text{ is nonnegative with bounded support}, \quad \inf_{\zeta \in \Sigma_{\alpha}} \|\omega(0, \cdot) - \zeta\|_{X_p} < \delta,
$$

it holds that

$$
\inf_{\zeta \in \Sigma_{\alpha}} \|\omega(t, \cdot) - \zeta\|_{X_p} < \epsilon \quad \forall \ t \geq 0.
$$

**Proof.** The items (i)-(iii) follow from Lemmas 9 and 10 in \cite{6}, (iv) follows from Section 5 in \cite{3}, and (v) follows from Theorem 2 in \cite{6}.

**Remark 2.2.** In \cite{6}, Burton in fact proved that there exist $A_{\alpha}, B_{\alpha} > 0$ such that

$$
\text{supp}(\zeta) \subset [-A_{\alpha}, A_{\alpha}] \times [0, B_{\alpha}] \quad \forall \zeta \in \Sigma_{\alpha}.
$$

However, how $A_{\alpha}$ and $B_{\alpha}$ depend on $\alpha$ is not specified therein. As we will see in Section 3, this issue is crucial for the proof of Theorem 1.2.

**Lemma 2.3.** Let $1 < s < +\infty$ and $0 < \theta < 1$. Then there are positive numbers $c_1, c_2, c_3$, depending only on $s$ and $\theta$, such that for any $v$ satisfying

$$
v \in L^s(\Pi), \quad v(x) \geq 0 \text{ a.e. } x \in \Pi, \quad I(v) < +\infty,
$$

it holds that

$$
\mathcal{G}v(x) \leq x_2^{-1}(c_1 \ln x_2 + c_2)I(v) + c_3x_2^{-\theta}\|v\|_{L^s(\Pi)}^{1-\theta}I(v)^{\theta} \quad \forall x = (x_1, x_2) \in \Pi, \ x_2 \geq 1.
$$
Proof. It follows from Lemma 5 in [6].

**Lemma 2.4.** Let $2 < s < +\infty$. Then there exists a positive number $K$, depending only on $s$, such that for any $v$ satisfying

$$v \in L^1 \cap L^s(\Pi), \quad v(x) \geq 0 \text{ a.e. } x \in \Pi, \quad v \text{ is Steiner-symmetric in the } x_2\text{-axis},$$

it holds for any $x = (x_1, x_2) \in \Pi$ with $|x_1| \geq 1$ that

$$Gv(x) \leq K \left( I(v) + \|v\|_{L^1(\Pi)} + \|v\|_{L^s(\Pi)} \right) x_2 |x_1|^{-1/(2s)}.$$ 

Proof. It follows from Lemma 5 in [6].

The following lemma presents two rearrangements inequalities that will be used later.

**Lemma 2.5.** Let $u, v, w$ be nonnegative Lebesgue measurable functions on $\mathbb{R}^2$. Then

$$\int_{\mathbb{R}^2} uvdx \leq \int_{\mathbb{R}^2} u^*v^*dx, \quad (2.1)$$

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(x)v(x-y)w(y)dxdy \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u^*(x)v^*(x-y)w^*(y)dxdy. \quad (2.2)$$

Here $u^*$ denote the symmetric-decreasing rearrangement of $u$.

Proof. See §3.4 and §3.7 in Lieb–Loss’s book [14].

We also need the following lemma proved by Burchard–Guo.

**Lemma 2.6.** Let $\{u_n\}_{n=1}^{+\infty} \subset L^2(\mathbb{R}^2)$ such that for each $n$

$$u_n(x) \geq 0 \text{ a.e. } x \in \mathbb{R}^2, \quad \int_{\mathbb{R}^2} xu_n(x)dx = 0, \quad \text{supp}(u_n) \subset B_\alpha(0) \quad (2.3)$$

for some $\alpha > 0$. If $u_n \rightharpoonup u$ and $u_n^* \rightharpoonup v$ for some $u, v \in L^2(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} u(x)u(y)dxdy \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} v(x)v(y)dxdy.$$ 

Moreover, the equality holds if and only if $u = v$.

Proof. It is a special case of Lemma 3.2 in [2].

3. **Proof of Theorem 1.2**

In this section, we give the proof of Theorem 1.2. For clarity, we divide the proof into three subsections.
3.1. **Existence and stability.** The aim of this subsection is to prove items (1)-(4) in Theorem 1.2 based on Lemma 2.1.

To begin with, we prove the following scaling properties of $S_{\varepsilon,i_0}$ and $\Gamma_{\varepsilon,i_0}$ (see Section 1 for their definitions).

**Lemma 3.1.** For any function $v : \Pi \rightarrow \mathbb{R}$, we have

(i) $v \in S_{1,i_0/\varepsilon}$ if and only if $v^\varepsilon \in S_{\varepsilon,i_0}$;

(ii) $v \in \Gamma_{1,i_0/\varepsilon}$ if and only if $v^\varepsilon \in \Gamma_{\varepsilon,i_0}$.

**Proof.** By definition, $v \in S_{1,i_0/\varepsilon}$ if and only if $v \in \mathcal{R}(\hat{\varepsilon})$ and $I(v) = i_0/\varepsilon$. It is clear that $v \in \mathcal{R}(\hat{\varepsilon})$ if and only if $v^\varepsilon \in \mathcal{R}(\hat{\varepsilon})$. Besides, since $I(v^\varepsilon) = \int_{\Pi} x^2 v^\varepsilon(x) dx = \frac{1}{\varepsilon^2} \int_{\Pi} x^2 v \left( \frac{x}{\varepsilon} \right) dx = \varepsilon I(v)$,

we see that $I(v) = i_0/\varepsilon$ if and only if $I(v^\varepsilon) = i_0$. Hence (i) is proved.

To prove (ii), we first claim that $E(v^\varepsilon) = E(v) \forall v \in S_{1,i_0/\varepsilon}$. (3.1)

In fact, by change of variables we have

$$E(v^\varepsilon) = \frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \ln \left| \frac{x - \bar{y}}{x - y} \right| \frac{1}{\varepsilon^2} v \left( \frac{x}{\varepsilon} \right) \frac{1}{\varepsilon^2} v \left( \frac{y}{\varepsilon} \right) dy \, dx$$

$$E(v^\varepsilon) = \frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \ln \left| \frac{x - \bar{y}}{x - y} \right| v(x) v(y) dy \, dx$$

$$E(v^\varepsilon) = E(v).$$

Now (ii) is an easy consequence of (i) and (3.1). □

**Proposition 3.2.** There exists some $\varepsilon_0 > 0$, depending only on $\phi$ and $i_0$, such that for any $0 < \varepsilon < \varepsilon_0$, the assertions (1)-(5) in Theorem 1.2 hold.

**Proof.** By choosing $\alpha = i_0/\varepsilon$ in Lemma 2.1, we see that for sufficiently small $\varepsilon$ the items (i)-(v) in Lemma 2.1 hold with $\Sigma_\alpha$ replaced by $\Gamma_{1,i_0/\varepsilon}$. Then the assertions (1)-(5) in Theorem 1.2 follows immediately from (ii) in Lemma 3.1. □

From now on, we always assume that $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0$ determined in Proposition 3.2.

By Proposition 3.2, for any $\zeta \in \Gamma_{\varepsilon,i_0}$ there exist some $\lambda_\zeta > 0$ and some increasing function $f_\zeta : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm \infty\}$ such that

$$\zeta = f_\zeta(\mathcal{G}_\zeta - \lambda_\zeta x_2) \text{ a.e. in } \Pi,$$

$$\zeta = 0 \text{ a.e. in } \{x \in \Pi \mid \mathcal{G}_\zeta(x) - \lambda_\zeta x_2 \leq 0\}.$$ 

From (3.3), we get

$$f_\zeta(s) = 0 \forall s \in (-\infty, 0].$$ 

(3.4)
Define
\[ \sigma_\zeta = \sup_{x \in \Pi} (G_\zeta(x) - \lambda \zeta x_2). \] (3.5)
Then it is easy to see that \( f_\zeta(s) \in \mathbb{R} \) if \( s < \sigma_\zeta \). But it may happen that
\[ \lim_{s \to \sigma_\zeta^-} f_\zeta(s) = +\infty. \] (3.6)

Define the Lagrangian multiplier \( \mu_\zeta \) related to \( \zeta \) as follows
\[ \mu_\zeta = \inf \{ s \in \mathbb{R} \mid f_\zeta(s) > 0 \}. \] (3.7)
Then by (3.4) it holds that
\[ \mu_\zeta \geq 0. \] (3.8)

Following the argument as in [23], Lemma 3.2, it is easy to check that
\[ V_\zeta = \{ x \in \Pi \mid G_\zeta(x) - \lambda \zeta x_2 > \mu_\zeta \}. \] (3.9)
Here \( V_\zeta \) is the vortex core related to \( \zeta \) defined in Theorem 1.2.

3.2. Uniform estimates. Now we deduce some uniform estimates independent of the parameter \( \varepsilon \) for the maximizers obtained in Proposition 3.2. In particular, we will show that all the maximizers are supported in some bounded region not depending on \( \varepsilon \).

For convenience, below we use \( C_1, C_2, C_3, \ldots \) to denote positive numbers depending on \( \rho \) and \( i_0 \), but not on \( \varepsilon \).

We begin with the following basic energy estimate.

**Lemma 3.3.** There exists some \( C_1 > 0 \) such that
\[ E(\zeta) \geq -\frac{\kappa^2}{4\pi} \ln \varepsilon - C_1 \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}. \] (3.10)

**Proof.** Without loss of generality assume that
\[ 0 < \varepsilon < \frac{i_0}{4r\kappa}, \] (3.11)
where \( r \) is the positive number given in (1.7). Define
\[ \bar{v}(x) = \rho^\varepsilon(x - \hat{x}), \]
where \( \hat{x} = (0, i_0/\kappa) \) as in Theorem 1.2. By (3.11), it is easy to check that \( \bar{v}1_\Pi \in \mathcal{R}(\rho^\varepsilon) \).

Moreover, by a simple calculation we have
\[ I(\bar{v}1_\Pi) = \int_\Pi x_2 \rho^\varepsilon(x - \hat{x})dx = \int_\Pi (x_2 + \hat{x}_2) \rho^\varepsilon(x)dx = \hat{x}_2 \int_\Pi \rho^\varepsilon(x)dx = i_0. \]
Here we used the fact that \( \int_\Pi x_2 \rho^\varepsilon(x)dx = 0 \) due to the radial symmetry of \( \rho^\varepsilon \). Therefore we have obtained \( \bar{v}1_\Pi \in \mathcal{S}_{\varepsilon,i_0} \), which implies
\[ E(\zeta) \geq E(\bar{v}1_\Pi) \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}. \]
To finish the proof, it suffices to take into account the following estimate for \( E(\bar{v}1_\Pi) \):
\[ E(\bar{v}1_\Pi) = \frac{1}{4\pi} \int_\Pi \int_\Pi \ln \frac{|x - \hat{y}|}{|x - y|} \rho^\varepsilon(x - \hat{x}) \rho^\varepsilon(y - \hat{x})dxdy \] (3.12)
\[
\frac{1}{4\pi} \int_{B_{r\varepsilon}(\hat{x})} \int_{B_{r\varepsilon}(\hat{x})} \ln \frac{|x - y|}{|x - \bar{y}|} \tilde{g}(x - \hat{x}) \tilde{g}(y - \hat{x}) \, dx \, dy
\]
(3.13)

\[
\geq \frac{1}{4\pi} \int_{B_{r\varepsilon}(\hat{x})} \int_{B_{r\varepsilon}(\hat{x})} \ln \frac{i_0}{2r \kappa \varepsilon} \tilde{g}(x - \hat{x}) \tilde{g}(y - \hat{x}) \, dx \, dy
\]
(3.14)

\[
= - \frac{\kappa^2}{4\pi} \ln \varepsilon - \frac{\kappa^2}{4\pi} \ln \frac{r}{i_0}.
\]
(3.15)

Note that in (3.14) we used the following two facts:

\[
|x - y| \leq 2r \varepsilon \quad \forall x, y \in B_{r\varepsilon}(\hat{x}),
\]

\[
|x - \bar{y}| \geq |y - \bar{y}| - |x - y| \geq 2 \left( \frac{i_0}{\kappa} - r \varepsilon \right) - 2r \varepsilon = \frac{2i_0}{\kappa} - 2r \varepsilon \geq \frac{i_0}{\kappa} \quad \forall x, y \in B_{r\varepsilon}(\hat{x}).
\]

\[\square\]

To proceed, for \(\zeta \in \Gamma_{\varepsilon, i_0}\) we define

\[
T_\zeta = \frac{1}{2} \int_\Pi \zeta (G\zeta - \lambda \zeta x_2 - \mu \zeta) \, dx.
\]
(3.16)

**Lemma 3.4.** There exists some \(C_2 > 0\) such that

\[
T_\zeta \leq C_2 \quad \forall \zeta \in \Gamma_{\varepsilon, i_0}.
\]
(3.17)

**Proof.** Fix \(\zeta \in \Gamma_{\varepsilon, i_0}\). Denote \(\phi = G\zeta - \lambda \zeta x_2 - \mu \zeta\) and \(\phi^+ = \max\{\phi, 0\}\). By (3.9), we have

\[
V_\zeta = \{x \in \Pi \mid \phi^+(x) > 0\},
\]

from which we deduce that

\[
T_\zeta = \frac{1}{2} \int_\Pi \zeta \phi \, dx = \frac{1}{2} \int_\Pi \zeta \phi^+ \, dx.
\]

By (3.8), \(\phi^+\) vanishes on \(\partial \Pi\). Therefore we can apply integration by parts to get

\[
T_\zeta = \frac{1}{2} \int_\Pi \zeta \phi^+ \, dx = \frac{1}{2} \int_\Pi |\nabla \phi^+|^2 \, dx.
\]
(3.18)

On the other hand, by Hölder’s inequality we have

\[
2T_\zeta = \int_\Pi \zeta \phi^+ \, dx
\]

\[
\leq \|\zeta\|_{L^p(\Pi)} \|\phi^+\|_{L^q(\Pi)}
\]
\[
= \|\tilde{g}\|_{L^p(\mathbb{R}^2)} \|\phi^+\|_{L^q(\Pi)}
\]
\[
= \varepsilon^{-\frac{2}{q}} \|\tilde{g}\|_{L^p(\mathbb{R}^2)} \|\phi^+\|_{L^q(\Pi)}.
\]
(3.19)

Here \(q = p/(p - 1)\) is the Hölder conjugate exponent of \(p\). Since \(p > 2\), we have \(q < 2\). We estimate \(\|\phi^+\|_{L^q(\Pi)}\) as follows

\[
\|\phi^+\|_{L^q(\Pi)} \leq \|1_{V_\zeta}\|_{L^\frac{2q}{q-2}(\Pi)} \|\phi^+\|_{L^2(\Pi)}
\]
(3.20)
\[
\leq S \|1_{V_\zeta} \|_{L^{2q}(\Pi)} \| \nabla \phi^+ \|_{L^1(\Pi)} \tag{3.21}
\]
\[
\leq S \|1_{V_\zeta} \|_{L^{2q}(\Pi)} \| 1_{V_\zeta} \|_{L^2(\Pi)} \| \nabla \phi^+ \|_{L^2(\Pi)} \tag{3.22}
\]
\[
= S \mathcal{L}^{1/q}(V_\zeta) \| \nabla \phi^+ \|_{L^2(\Pi)} \tag{3.23}
\]
\[
= S \pi^{1/4} r^{2/4} \varepsilon^{2/4} \| \nabla \phi^+ \|_{L^2(\Pi)} \tag{3.24}
\]

Here \(S\) is a generic positive constant. Note that we used Hölder’s inequality in (3.20) and (3.22), Sobolev’s inequality (see Theorem 4.8 in [12] for example) in (3.21), and the following fact in (3.24)
\[
\mathcal{L}(V_\zeta) = \mathcal{L} \left( \{ x \in \mathbb{R}^2 \mid \varrho(x) > 0 \} \right) = \varepsilon^2 \mathcal{L} \left( \{ x \in \mathbb{R}^2 \mid \varrho(x) > 0 \} \right) = \pi r^2 \varepsilon^2.
\]

Therefore we obtain
\[
T_\zeta \leq \frac{1}{2} S \pi^{1/4} r^{2/4} \varepsilon^{2/4} \| \nabla \phi^+ \|_{L^2(\Pi)}. \tag{3.25}
\]

The desired result follows from (3.18) and (3.25) immediately. \(\square\)

**Remark 3.5.** From (3.18), it is easy to see that
\[
T_\zeta = \frac{1}{2} \int_{V_\zeta} | \nabla G_\zeta |^2 \, dx.
\]

Therefore \(T_\zeta\) in fact represents the kinetic energy of the fluid on the vortex core \(V_\zeta\).

**Lemma 3.6.** There exists some \(C_3 \geq 0\) such that
\[
\mu_\zeta \geq -\frac{\kappa}{2\pi} \ln \varepsilon - \frac{\lambda_\zeta i_0}{\kappa} - C_3 \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}. \tag{3.26}
\]

**Proof.** By the definition of \(T_\zeta\) (3.16), it holds that
\[
T_\zeta = E(\zeta) - \frac{1}{2} \lambda_\zeta i_0 - \frac{1}{2} \kappa \mu_\zeta.
\]

Then (3.26) follows from (3.10) and (3.17). \(\square\)

**Lemma 3.7.** There exists some \(C_4 > 0\) such that
\[
G_\zeta(x) \geq -\frac{\kappa}{2\pi} \ln \varepsilon + \lambda_\zeta x_2 - \frac{\lambda_\zeta i_0}{\kappa} - C_4 \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}, \ x \in V_\zeta. \tag{3.27}
\]

**Proof.** For any \(\zeta \in \Gamma_{\varepsilon,i_0}\) and \(x \in V_\zeta\), recalling (3.9), we have
\[
G_\zeta(x) - \lambda_\zeta x_2 > \mu_\zeta.
\]

Then (3.27) follows from (3.26). \(\square\)

For \(\zeta \in \Gamma_{\varepsilon,i_0}\), define
\[
F_\zeta(s) = \int_{-\infty}^s f_\zeta(\tau) \, d\tau = \int_{\mu_\zeta}^s f_\zeta(\tau) \, d\tau, \tag{3.28}
\]
where \( f_\zeta \) is determined by Proposition 3.2. Then \( F_\zeta \) is locally Lipschitz continuous in \((-\infty, \sigma_\zeta)\) and
\[
F_\zeta'(s) = f_\zeta(s) \text{ a.e. } s \in (-\infty, \sigma_\zeta).
\]
Recall that \( \sigma_\zeta \) is defined by (3.5). Moreover, since \( F_\zeta \) vanishes on \((-\infty, \mu_\zeta]\), taking into account (3.9), we have
\[
\{ x \in \Pi \mid F_\zeta(G_\zeta(x) - \lambda_\zeta x_2) > 0 \} \subset \{ x \in \Pi \mid G_\zeta(x) - \lambda_\zeta x_2 > \mu_\zeta \} = V_\zeta.
\]
Consequently \( F_\zeta(G_\zeta - \lambda_\zeta x_2) \) has bounded support.

Our next step is to prove the uniform boundedness from above for the traveling speed \( \lambda_\zeta \). To this end, we need the following Pohozaev type identity, which is a straightforward consequence of Lemma 9 in [3]. See also (13) in [6], p.565.

**Lemma 3.8.** For any \( \zeta \in \Gamma_{\varepsilon,i_0} \), the following identity holds
\[
\int_{\Pi} F_\zeta(G_\zeta - \lambda_\zeta x_2) dx = \frac{\lambda_\zeta i_0}{2}.
\]

Based on Lemmas 3.4 and 3.8 we are able to prove

**Lemma 3.9.** There exists some \( C_5 > 0 \) such that
\[
\lambda_\zeta \leq C_5 \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}.
\]

**Proof.** Fix \( \zeta \in \Gamma_{\varepsilon,i_0} \). Since \( f_\zeta \) is increasing, we have
\[
F_\zeta(s) = \int_{\mu_\zeta}^s f_\zeta(\tau) d\tau \leq \int_{\mu_\zeta}^s f_\zeta(s) d\tau = f_\zeta(s)(s - \mu_\zeta) \quad \forall s \in \mathbb{R}.
\]
Therefore
\[
\int_{\Pi} F_\zeta(G_\zeta - \lambda_\zeta x_2) dx \leq \int_{\Pi} f_\zeta(G_\zeta - \lambda_\zeta x_2)(G_\zeta - \lambda_\zeta x_2 - \mu_\zeta) dx
\]
\[
= \int_{\Pi} \zeta(G_\zeta - \lambda_\zeta x_2 - \mu_\zeta) dx
\]
\[
= 2T_\zeta.
\]
The desired estimate (3.31) follows immediately from (3.17) and (3.30).

With Lemma 3.9 we can improve Lemma 3.7 as follows.

**Lemma 3.10.** There exists some \( C_6 > 0 \) such that
\[
G_\zeta(x) \geq -\frac{\kappa}{2\pi} \ln \varepsilon + \lambda_\zeta x_2 - C_6 \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}, \ x \in V_\zeta.
\]

Consequently (recall \( \lambda_\zeta > 0 \))
\[
G_\zeta(x) \geq -\frac{\kappa}{2\pi} \ln \varepsilon - C_6 \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}, \ x \in V_\zeta.
\]

Now we turn to estimating the size of the vortex core \( V_\zeta \). As a preliminary, we need the following a priori upper bound for \( G_\zeta \) based on Lemma 2.3.
Lemma 3.11. There exists some $C_7 > 0$ such that
\[ \mathcal{G}_\zeta(x) \leq C_7 x_2 \varepsilon^{-1/2} \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}, \ x_2 \geq 1. \] (3.35)

Proof. Fix $\zeta \in \Gamma_{\varepsilon,i_0}$. Choosing $s = 2$ and $\theta = 1/2$ in Lemma 2.3 it holds for any $x = (x_1, x_2) \in \Pi$ with $x_2 \geq 1$ that
\[
\mathcal{G}_\zeta(x) \leq x_2^{-1} (c_1 \ln x_2 + c_2) I(\zeta) + c_3 x_2^{-1/2} \|\zeta\|_{L^2(\Pi)}^{1/2} I(\zeta)^{1/2} \\
= x_2^{-1} (c_1 \ln x_2 + c_2) i_0 + c_3 x_2^{-1/2} \|\tilde{\varphi}\|_{L^2(\Pi)}^{1/2} \varphi_0^{1/2} \\
= \frac{i_0 (c_1 \ln x_2 + c_2)}{x_2} + \frac{c_3 \|\varphi\|_{L^2(\Pi)}^{1/2} \varphi_0^{1/2}}{\sqrt{x_2 \varepsilon}}.
\]

Here we used the fact that $\|\tilde{\varphi}\|_{L^2(\Pi)} = \varepsilon^{-1} \|\varphi\|_{L^2(\Pi)}$. This obviously implies (3.35).
\[ \square \]

The following lemma provides a rough estimate for the size of $V_\zeta$ in the $x_2$ direction.

Lemma 3.12. There exists some $C_8 > 0$ such that
\[ x_2 \leq C_8 \varepsilon^{-1} \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}, \ x \in V_\zeta. \] (3.36)

Proof. Without loss of generality, we assume that $\varepsilon_0$ is small enough such that
\[ -\frac{\kappa}{2\pi} \ln \varepsilon - C_6 \geq 1 \quad \forall \varepsilon \in (0, \varepsilon_0). \]

Hence by (3.34) we have
\[ \mathcal{G}_\zeta(x) \geq 1 \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}, \ x \in V_\zeta. \] (3.37)

Then the desired result follows from (3.35).
\[ \square \]

Remark 3.13. In the proof of Lemma 3.11 we could have chosen other $p$ and $\theta$ to get a better estimate than (3.35), hence a better estimate than (3.36). However, as we will see, this makes no difference to the subsequent arguments.

To estimate the size of $V_\zeta$ in the $x_1$ direction, we need the following lemma similar to Lemma 3.11. Recall the definition of $\Gamma_{\varepsilon,i_0}^0$ given by (1.16).

Lemma 3.14. There exists some $C_9 > 0$ such that
\[ \mathcal{G}_\zeta(x) \leq C_9 \varepsilon^{2/p-3} |x_1|^{-1/(2p)} \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}^0, \ x \in V_\zeta, \ |x_1| \geq 1. \] (3.38)

Proof. Fix $\zeta \in \Gamma_{\varepsilon,i_0}^0$. By Remark 1.3 $\zeta$ is Steiner-symmetric in the $x_2$-axis. Choosing $s = p$ in Lemma 2.4 it holds for any $x = (x_1, x_2) \in \Pi$ with $|x_1| \geq 1$ that
\[
\mathcal{G}_\zeta(x) \leq K \left( i_0 + \|\zeta\|_{L^1(\Pi)} + \|\zeta\|_{L^p(\Pi)} \right) x_2 |x_1|^{-1/(2p)} \\
= K \left( i_0 + \kappa + \varepsilon^{2/p-2} \|\varphi\|_{L^p(\Pi)} \right) x_2 |x_1|^{-1/(2p)} \\
\leq KC_8 \varepsilon^{-1} \left( i_0 + \kappa + \varepsilon^{2/p-2} \|\varphi\|_{L^p(\Pi)} \right) |x_1|^{-1/(2p)} \\
= KC_8 \varepsilon^{2/p-3} \left( (i_0 + \kappa) \varepsilon^{2-2/p} + \|\varphi\|_{L^p(\Pi)} \right) |x_1|^{-1/(2p)}
\]
\[ \leq KC_8 \varepsilon^{2/p-3} \left((i_0 + \kappa)\varepsilon_0^{2-2/p} + \|\varrho\|_{L^p(\Pi)} \right) |x_1|^{-1/(2p)}. \]

Here we used Lemma 3.12 and the following facts:

\[ \|\zeta\|_{L^1(\Pi)} = \|\varrho\|_{L^1(\Pi)} = \kappa, \]
\[ \|\zeta\|_{L^p(\Pi)} = \|\varrho\|_{L^p(\Pi)} = \varepsilon^{2/p-2}\|\varrho\|_{L^p(\Pi)}. \]

Hence the proof is finished by choosing \[ C_9 = KC_8 \left((i_0 + \kappa)\varepsilon_0^{2-2/p} + \|\varrho\|_{L^p(\Pi)} \right). \]

\[ \square \]

Based on Lemma 3.10 and Lemma 3.14, we are able to provide a rough estimate for the size of \( V_\zeta \) in the \( x_1 \) direction.

**Lemma 3.15.** There exists some \( C_{10} > 0 \) such that

\[ |x_1| \leq C_{10} \varepsilon^{4-6p} \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}^0, \ x \in V_\zeta. \]  \hspace{1cm} (3.39)

**Proof.** Fix \( \zeta \in \Gamma_{\varepsilon,i_0}^0 \). By (3.31) and (3.38), for any \( x \in V_\zeta \) such that \( |x_1| \geq 1 \), we have

\[ -\frac{\kappa}{2\pi} \ln \varepsilon - C_6 \leq C_9 \varepsilon^{2/p-3} |x_1|^{-1/(2p)}, \]

which implies

\[ -\frac{\kappa}{2\pi} \ln \varepsilon_0 - C_6 \leq C_9 \varepsilon^{2/p-3} |x_1|^{-1/(2p)}. \]

Without loss of generality, assume that \( \varepsilon_0 \) is small enough such that

\[ -\frac{\kappa}{2\pi} \ln \varepsilon_0 - C_6 > 0. \]

Then

\[ |x_1| \leq C_9^{2p} \left(-\frac{\kappa}{2\pi} \ln \varepsilon_0 - C_6\right)^{-2p} \varepsilon^{4-6p}. \]

Hence for any \( x \in V_\zeta \) we have

\[ |x_1| \leq C_9^{2p} \left(-\frac{\kappa}{2\pi} \ln \varepsilon_0 - C_6\right)^{-2p} \varepsilon^{4-6p} + 1 \]
\[ \leq C_9^{2p} \left(-\frac{\kappa}{2\pi} \ln \varepsilon_0 - C_6\right)^{-2p} \varepsilon^{4-6p} + \varepsilon_0^{6p-4} \varepsilon^{4-6p}. \]

The proof is completed by choosing

\[ C_{10} = C_9^{2p} \left(-\frac{\kappa}{2\pi} \ln \varepsilon_0 - C_6\right)^{-2p} + \varepsilon_0^{6p-4}. \]

\[ \square \]
For $\zeta \in \Gamma_{\varepsilon,i_0}$, define

$$l_\zeta = \sup_{x,y \in V_\zeta} |x - y|.$$  

Since $V_\zeta$ is contained in the upper half-plane, it is easy to check that

$$\text{diam}(V_\zeta) \leq l_\zeta. \quad (3.40)$$

We have the following rough estimate for $l_\zeta$.

**Lemma 3.16.** There exists some $C_{11} > 0$ such that

$$l_\zeta \leq C_{11} \varepsilon^{4-6p} \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}. \quad (3.41)$$

**Proof.** Fix $\zeta \in \Gamma_{\varepsilon,i_0}$. Without loss of generality, assume that $\zeta \in \Gamma_{\varepsilon,i_0}^0$. Combining Lemma 3.12 and Lemma 3.15, it holds for any $x,y \in V_\zeta$ that

$$|x - y|^2 = (x_1 - y_1)^2 + (x_2 + y_2)^2 \leq 2 (x_1^2 + y_1^2) + 2 (x_2^2 + y_2^2) \leq 4 \left( C_{10}^2 \varepsilon^{8-12p} + C_8^2 \varepsilon^{-2} \right) \leq 4 \left( C_{10}^2 + C_8^2 \varepsilon_0^{12p-10} \right) \varepsilon^{8-12p}. \quad (3.45)$$

The proof is finished by choosing

$$C_{11} = 2 \left( C_{10}^2 + C_8^2 \varepsilon_0^{12p-10} \right)^{1/2}. \quad \Box$$

Having made enough preparations, we are ready to deduce the most important uniform estimate in this paper, that is, the uniform boundedness of $l_\zeta$, from which the uniform boundedness of $V_\zeta$ follows immediately due to (3.40)

**Proposition 3.17.** There exists some $C_{12} > 0$ such that

$$l_\zeta \leq C_{12} \quad \zeta \in \Gamma_{\varepsilon,i_0}. \quad (3.42)$$

**Proof.** We prove this proposition by contradiction. Assume that there exist $\{\varepsilon_n\}_{n=1}^{+\infty} \subset (0,\varepsilon_0)$ and $\{\zeta_n\}_{n=1}^{+\infty} \subset \Gamma_{\varepsilon_n,i_0}$ such that

$$\lim_{n \to +\infty} l_{\zeta_n} = +\infty. \quad (3.43)$$

By Lemma 3.16 we must have

$$\lim_{n \to +\infty} \varepsilon_n = 0. \quad (3.44)$$

Without loss of generality, we assume that $\{\zeta_n\}_{n=1}^{+\infty} \subset \Gamma_{\varepsilon_0,i_0}^0$.

Below for simplicity denote $l_n = l_{\zeta_n}, V_n = V_{\zeta_n}$. For any $x \in V_n$, by (3.34) it holds that

$$G_{\zeta_n}(x) \geq -\frac{\kappa}{2\pi} \ln \varepsilon_n - C_6,$$

which can be equivalently written as

$$\int_{\Pi} \ln \frac{\varepsilon_n}{|x - y|} \zeta_n(y) dy \geq -\int_{V_n} \ln |x - y| \zeta_n(y) dy - 2\pi C_6. \quad (3.45)$$
Since $|x - y| \leq l_n$ for any $x, y \in V_n$, we further get
\[
\int_{\Pi} \ln \frac{\varepsilon_n}{|x - y|} \zeta_n(y) dy \geq -\kappa \ln l_n - 2\pi C_6. \tag{3.46}
\]
Divide the integral in (3.46) into two parts
\[
\int_{\Pi} \ln \frac{\varepsilon_n}{|x - y|} \zeta_n(y) dy = \int_{|x-y| \leq \frac{l_n}{60}} \ln \frac{\varepsilon_n}{|x - y|} \zeta_n(y) dy + \int_{|x-y| \geq \frac{l_n}{60}} \ln \frac{\varepsilon_n}{|x - y|} \zeta_n(y) dy
\]
where $N$ is a fixed large positive integer such that
\[
N > 60p - 30. \tag{3.47}
\]
Note that since $p > 2$, we have
\[
N > 90. \tag{3.48}
\]
For $I_1$, it is easy to check by change of variables and the rearrangement inequality (2.1) that
\[
\int_{|x-y| \leq \frac{l_n}{60}} \ln \frac{\varepsilon_n}{|x - y|} \zeta_n(y) dy \leq \int_{\Pi} \ln \frac{\varepsilon_n}{|x - y|} \zeta_n(y) dy \leq \int_{\mathbb{R}^2} \ln \frac{1}{|y|} \varrho(y) dy \leq K_1 \tag{3.49}
\]
for some positive number $K_1$ depending only on $\varrho$. For $I_2$, we have
\[
\int_{|x-y| \geq \frac{l_n}{60}} \ln \frac{\varepsilon_n}{|x - y|} \zeta_n(y) dy \leq \ln \frac{N\varepsilon_n}{l_n} \int_{|x-y| \geq \frac{l_n}{60}} \zeta_n(y) dy. \tag{3.50}
\]
From (3.46), (3.49) and (3.50) we can get
\[
\ln \frac{N\varepsilon_n}{l_n} \int_{|x-y| \geq \frac{l_n}{60}} \zeta_n(y) dy \geq -\kappa \ln l_n - 2\pi C_6 - K_1. \tag{3.51}
\]
Recalling (3.43) and (3.44), it holds that for sufficiently large $n$ that
\[
\frac{N\varepsilon_n}{l_n} < 1,
\]
therefore (3.51) is equivalent to
\[
\int_{|x-y| \geq \frac{l_n}{60}} \zeta_n(y) dy \leq \frac{\kappa \ln l_n + 2\pi C_6 + K_1}{\ln l_n - \ln \varepsilon_n - \ln N}. \tag{3.52}
\]
Define
\[
h(s) = \frac{\kappa s + 2\pi C_6 + K_1}{s - \ln \varepsilon_n - \ln N}.
\]
It is easy to check that $h$ is increasing in $(0, +\infty)$ if $n$ is large enough. Combining Lemma 3.16 we get from (3.52) that
\[
\int_{|x-y| \geq \frac{l_n}{60}} \zeta_n(y) dy \leq \frac{\kappa \ln (C_1\varepsilon_n^{4-6p}) + 2\pi C_6 + K_1}{\ln (C_1\varepsilon_n^{4-6p}) - \ln \varepsilon_n - \ln N}. \tag{3.53}
\]
Due to (3.44), we have
\[
\lim_{n \to +\infty} \frac{\kappa \ln (C_{11} \varepsilon_n^{4-6p}) + 2\pi C_6 + K_1}{\ln (C_{11} \varepsilon_n^{4-6p}) - \ln \varepsilon_n - \ln N} = \frac{6p - 4}{6p - 3} \kappa.
\] (3.54)

Recalling (3.47), it is easy to check that
\[
\frac{6p - 4}{6p - 3} < 1 - \frac{10}{N}.
\] (3.55)

Therefore by (3.53)-(3.55) we have
\[
\int_{|x-y| \geq \frac{l_n}{N}} \zeta_n(y)dy < \left(1 - \frac{10}{N}\right) \kappa,
\]
provided that \(n\) is large enough. Taking into account the fact that \(\|\zeta_n\|_{L^1(\Pi)} = 1 \forall n\), we get for sufficiently large \(n\) that
\[
\int_{|x-y| \leq \frac{l_n}{N}} \zeta_n(y)dy > \frac{10}{N} \kappa.
\] (3.56)

Note that (3.56) holds for arbitrary \(x \in V_n\).

Based on (3.56), we deduce a contradiction as follows. Denote
\[
s_n = \sup_{x \in V_n} x_2 \quad t_n = \sup_{x \in V_n} |x_1|.
\]
Since we have assumed \(\{\zeta_n\}_{n=1}^{+\infty} \subset \Gamma_{\varepsilon_0}^0\), is easy to check that
\[
l_n \leq 2(s_n + t_n) \quad \forall n,
\] (3.57)
which yields either \(s_n \geq l_n/4\) or \(t_n \geq l_n/4\) for each \(n\). Therefore at least one of the following two assertions holds.

(A1) There exists a subsequence \(\{s_{n_j}\}_{j=1}^{+\infty}\) such that
\[
s_{n_j} \geq \frac{l_{n_j}}{4} \quad \forall j.
\]

(A2) There exists a subsequence \(\{t_{n_j}\}_{j=1}^{+\infty}\) such that
\[
t_{n_j} \geq \frac{l_{n_j}}{4} \quad \forall j.
\]

First we show that (A1) is impossible. In fact, if (A1) holds, then we can choose a sequence \(\{x_j\}_{j=1}^{+\infty} \subset V_n\) such that
\[
x_{j,2} \geq \frac{l_{n_j}}{8} \quad \forall j.
\] (3.58)

Here \(x_j = (x_{j,1}, x_{j,2})\). Using (3.56), we have for sufficiently large \(n\) that
\[
\int_{\Pi} y_2 \zeta_{n_j}(y)dy \geq \int_{|x_j-y| \leq \frac{l_{n_j}}{N}} y_2 \zeta_{n_j}(y)dy
\]
\[ \geq \int_{|x_n - y| \leq \frac{t_{n_j}}{N}} (x_{j,2} - |x_{j,2} - y_2|) \zeta_{n_j}(y) dy \]
\[ \geq \int_{|x_n - y| \leq \frac{t_{n_j}}{N}} \left( \frac{t_{n_j}}{8} - \frac{l_{n_j}}{N} \right) \zeta_{n_j}(y) dy \]
\[ \geq \left( \frac{1}{8} - \frac{1}{N} \right) \frac{10}{N} \kappa t_{n_j}, \]
which goes to \(+\infty\) as \(n \rightarrow +\infty\). This contradicts the fact that \(I(\zeta_n) = i_0\). Therefore (A1) is impossible.

To finish the proof, it suffices to show that (A2) cannot possibly hold, either. In fact, if (A2) holds, then for any \(x = (x_1, x_2) \in V_{n_j}\) it holds that
\[ \int_{|x - y| \leq \frac{t_{n_j}}{N}} \zeta_{n_j}(y) dy \leq \int_{|x_{1} - y_{1}| \leq \frac{l_{n_j}}{N}} \zeta_{n_j}(y) dy \leq \int_{|x_{1} - y_{1}| \leq \frac{4t_{n_j}}{N}} \zeta_{n_j}(y) dy. \quad (3.59) \]
This together with (3.56) yields for sufficiently large \(j\) that
\[ \int_{|x_{1} - y_{1}| \leq \frac{4t_{n_j}}{N}} \zeta_{n_j}(y) dy > \frac{10}{N} \kappa \quad \forall x \in V_{n_j}. \quad (3.60) \]

Below fix a sufficient large \(j\) such that (3.60) holds. Using the fact that each \(\zeta_{n_j}\) is Steiner-symmetric in the \(x_2\)-axis, we obtain
\[ \int_{|a - y_{1}| \leq \frac{4t_{n_j}}{N}} \zeta_{n_j}(y) dy > \frac{10}{N} \kappa \quad \forall a \in (-t_{n_j}, t_{n_j}). \quad (3.61) \]
For any \(a \in \mathbb{R}\), denote by \(\lceil a \rceil\) the smallest integer not less than \(a\). Define
\[ a_k = \frac{4(2k - 1)t_{n_j}}{N}, \quad k = 1, \ldots, \left\lceil \frac{N}{10} \right\rceil. \]
Then for any \(1 \leq k \leq \left\lceil \frac{N}{10} \right\rceil\), it holds that
\[ 0 < a_k \leq \frac{8}{N} \frac{\lceil N \rceil}{10} - 4t_{n_j} \leq \frac{8}{N} \left( \frac{N}{10} + 1 \right) - 4t_{n_j} = \left( \frac{4}{5} + \frac{4}{N} \right) t_{n_j} < t_{n_j}. \]
Here we used (3.48) in the last inequality. Choosing \(a = a_k\) in (3.61) and summing over \(k = 1, \ldots, \left\lceil \frac{N}{10} \right\rceil\), we get
\[ \int_{\Pi} \zeta_{n_j}(y) dy \geq \sum_{k=1}^{\left\lceil \frac{N}{10} \right\rceil} \int_{|y_{1} - a_{k}| \leq \frac{4t_{n_j}}{N}} \zeta_{n_j}(y) dy > \sum_{k=1}^{\left\lceil \frac{N}{10} \right\rceil} \frac{10}{N} \kappa \geq \kappa, \quad (3.62) \]
a contradiction to the fact that \(\|\zeta_{n_j}\|_{L^1(\Pi)} = \kappa\). Hence the proof is finished. \(\square\)
3.3. **Asymptotic behavior.** In this section, we study the asymptotic behavior of the maximizers as $\varepsilon \to 0^+$. With Proposition 3.17 at hand, the unboundedness of the upper half-plane does not cause trouble anymore, hence the subsequent proofs are analogous to the ones in [22].

First we prove a better bound for the size of the vortex core.

**Proposition 3.18.** There exists some $C_{13} > 0$ such that

$$\text{diam}(V_\zeta) \leq C_{13} \varepsilon \quad \forall \zeta \in \Gamma_{\varepsilon,i_0}.$$ 

**Proof.** Fix $\zeta \in \Gamma_{\varepsilon,i_0}$. By (3.34) it holds for any $x \in V_\zeta$ that

$$G_\zeta(x) \geq -\frac{\kappa}{2\pi} \ln \varepsilon - C_6,$$

or equivalently,

$$\int_{\Pi} \ln \frac{\varepsilon}{|x-y|} \zeta(y) dy \geq -\int_{V_\zeta} \ln |x-\bar{y}| \zeta(y) dy - 2\pi C_6.$$ (3.63)

Taking into account Proposition 3.17, we have

$$\int_{\Pi} \ln \frac{\varepsilon}{|x-y|} \zeta(y) dy \geq -\kappa \ln C_{12} - 2\pi C_6.$$ (3.64)

As in the proof of Proposition 3.17 we divide the integral in (3.64) into two parts

$$\int_{|x-y| \leq R\varepsilon} \ln \frac{\varepsilon}{|x-y|} \zeta(y) dy = \int_{|x-y| \leq R\varepsilon} \ln \frac{\varepsilon}{|x-y|} \zeta(y) dy + \int_{|x-y| \geq R\varepsilon} \ln \frac{\varepsilon}{|x-y|} \zeta(y) dy,$$

where $R > 1$ is to be determined later. As in (3.49), for the first integral it holds that

$$\int_{|x-y| \leq R\varepsilon} \ln \frac{\varepsilon}{|x-y|} \zeta(y) dy \leq \int_{\Pi} \ln \frac{\varepsilon}{|x-y|} \zeta(y) dy \leq \int_{\mathbb{R}^2} \ln \frac{1}{|y|} g(y) dy \leq K_1,$$ (3.65)

where $K_1 > 0$ depends only on $g$. For the second integral, we have

$$\int_{|x-y| \geq R\varepsilon} \ln \frac{\varepsilon}{|x-y|} \zeta(y) dy \leq -\ln R \int_{|x-y| \geq R\varepsilon} \zeta(y) dy.$$ (3.66)

Combining (3.64), (3.65) and (3.66) we obtain

$$\int_{|x-y| \geq R\varepsilon} \zeta(y) dy \leq \frac{\kappa \ln C_{12} + 2\pi C_6 + K_1}{\ln R}.$$ (3.67)

Choose $R$ to be sufficiently large, depending only on $g$ and $i_0$, such that

$$\frac{\kappa \ln C_{12} + 2\pi C_6 + K_1}{\ln R} \leq \frac{\kappa}{3}.$$  

Then we get from (3.67) that

$$\int_{|x-y| \geq R\varepsilon} \zeta(y) dy \leq \frac{\kappa}{3},$$
Lemma 3.20. For any $\Theta$, on the other hand, by a direct calculation we have

$$
\int_{|x-y|\leq R_\varepsilon} \zeta(y)dy \geq \frac{2}{3}\kappa.
$$

Since (3.68) holds for arbitrary $x \in V_\zeta$, we can easily show that $\text{diam}(V_\zeta) \leq 2R_\varepsilon$. In fact, suppose otherwise $\text{diam}(V_\zeta) > 2R_\varepsilon$. Then we can choose two points $x_1, x_2 \in V_\zeta$ such that $|x_1 - x_2| > 2R_\varepsilon$. It is clear that $B_{R_\varepsilon}(x_1) \cap B_{R_\varepsilon}(x_2) = \emptyset$, therefore

$$
\int_{|x-y|\leq R_\varepsilon} \zeta(y)dy \geq \int_{|x_1-y|\leq R_\varepsilon} \zeta(y)dy + \int_{|x_2-y|\leq R_\varepsilon} \zeta(y)dy \geq \frac{4}{3}\kappa,
$$

a contradiction to the fact that $\|\zeta\|_{L^1(\Pi)} = \kappa$. The proof is finished by choosing $C_{13} = 2R$. □

For any $\zeta \in \Gamma_{0, \xi_0}$ its center is $\hat{x} = (0, i_0/\kappa)$. From this fact and Proposition 3.18 we immediately obtain

**Lemma 3.19.** Any $\zeta \in \Gamma_{0, \xi_0}$ converges uniformly to $\kappa \delta_\hat{x}$ in the distributional sense as $\varepsilon \to 0^+$. More precisely, for any $\varphi \in C(\Pi)$ and any $\varepsilon > 0$, there exists some $\delta > 0$, depending only on $\varphi, i_0, \varphi$ and $\varepsilon$, such that

$$
\left| \int_{\Pi} \varphi \zeta dx - \kappa \varphi(\hat{x}) \right| < \varepsilon \quad \forall \zeta \in \Gamma_{0, \xi_0}.
$$

To study the limit of the traveling speed $\lambda_\zeta$ as $\varepsilon \to 0^+$, we prove a useful formula below.

**Lemma 3.20.** For any $\zeta \in \Gamma_{\varepsilon, i_0}$, it holds that

$$
\lambda_\zeta = \frac{1}{2\pi\kappa} \int_{\Pi} \int_{\Pi} \frac{x_2 + y_2}{|x-y|^2} \zeta(x)\zeta(y)dxdy.
$$

**Proof.** Fix $\zeta \in \Gamma_{\varepsilon, i_0}$. Recall that $F_\zeta$ is defined by (3.28). By an almost identical argument as in the proof of Lemma 9 in [3], we can show that $F_\zeta(\mathcal{G}_\zeta - \lambda_\zeta x_2) \in W^{1,p}(\Pi)$ and, moreover, the following chain rule holds true

$$
\partial_{x_2} (F_\zeta(\mathcal{G}_\zeta - \lambda_\zeta x_2)) = f_\zeta(\mathcal{G}_\zeta - \lambda_\zeta x_2) \partial_{x_2} (\mathcal{G}_\zeta - \lambda_\zeta x_2) = (\partial_{x_2} \mathcal{G}_\zeta - \lambda_\zeta). \tag{3.69}
$$

Integrating both sides of (3.69) over $\Pi$, and taking into account the fact that $F_\zeta(\mathcal{G}_\zeta - \lambda_\zeta x_2)$ has bounded support, we get

$$
\lambda_\zeta = \frac{1}{\kappa} \int_{\Pi} \zeta \partial_{x_2} \mathcal{G}_\zeta dx. \tag{3.70}
$$

On the other hand, by a direct calculation we have

$$
\partial_{x_2} \mathcal{G}_\zeta(x) = \frac{1}{2\pi} \int_{\Pi} \frac{x_2 + y_2}{|x-y|^2} \zeta(y)dy + \frac{1}{2\pi} \int_{\Pi} \frac{y_2 - x_2}{|x-y|^2} \zeta(y)dy. \tag{3.71}
$$

Inserting (3.71) into (3.70) we obtain

$$
\lambda_\zeta = \frac{1}{2\pi\kappa} \int_{\Pi} \int_{\Pi} \frac{x_2 + y_2}{|x-y|^2} \zeta(x)\zeta(y)dxdy + \frac{1}{2\pi\kappa} \int_{\Pi} \int_{\Pi} \frac{y_2 - x_2}{|x-y|^2} \zeta(x)\zeta(y)dxdy. \tag{3.72}
$$

Observe that the second integral on the right-hand side of (3.72) is exactly zero due to the anti-symmetry of the integrand. Hence the proof is finished. □
Now we are ready to estimate the traveling speed $\lambda_\zeta$.

**Proposition 3.21.** As $\varepsilon \to 0^+$, $\lambda_\zeta \to \kappa^2/(4\pi i_0)$ uniformly with respect to the choice of $\zeta \in \Gamma_{\varepsilon,i_0}$.

**Proof.** Using Lemma 3.19 and Lemma 3.20, we have

$$
\lim_{\varepsilon \to 0^+} \lambda_\zeta = \lim_{\varepsilon \to 0^+} \frac{\kappa}{2\pi} \left( \frac{x_2 + y_2}{|x - y|^2} \zeta(x)\zeta(y) \right)
= \frac{\kappa^2}{4\pi i_0}.
$$

Moreover, the convergence is uniform with respect to the choice of $\zeta \in \Gamma_{\varepsilon,i_0}$ since the convergence $\zeta \to \kappa\delta_0$ in Lemma 3.19 is uniform. $\Box$

As in Theorem 1.2, for $\zeta \in \Gamma_{0,\varepsilon,i_0}$, extend $\zeta$ to $\mathbb{R}^2$ such that $\zeta = 0$ in the lower half-plane and define $\nu_{\zeta,\varepsilon}(x) = \varepsilon^2 \zeta(\varepsilon x + \hat{x})$.

**Proposition 3.22.** As $\varepsilon \to 0^+$, $\nu_{\zeta,\varepsilon} \to \varrho$ in $L^p(\mathbb{R}^2)$, uniformly with respect to the choice of $\zeta \in \Gamma_{0,\varepsilon,i_0}$.

**Proof.** It suffices to show that for any sequence $\{\varepsilon_n\}_{n=1}^{+\infty} \subset (0,\varepsilon_0)$ such that $\varepsilon_n \to 0^+$ as $n \to +\infty$, and any sequence $\{\zeta_n\}_{n=1}^{+\infty} \subset \Gamma_{0,\varepsilon,i_0}$, it holds that $\nu_{\zeta_n,\varepsilon_n} \to \varrho$ in $L^p(\mathbb{R}^2)$ as $n \to +\infty$.

For simplicity, below we denote $\nu_n = \nu_{\zeta_n,\varepsilon_n}$. It is easy to check that for each $n$

$$\nu_n \geq 0 \text{ a.e. in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} x \nu_n(x)dx = 0, \quad \text{supp}(\nu_n) \subset B_{C_{13}}(0). \quad (3.73)$$

Besides, it is also clear that

$$\nu_n^* = \varrho \quad \forall n. \quad (3.74)$$

Here $\nu_n^*$ denote the symmetric-decreasing rearrangement of $\nu_n$. Up to a subsequence, we assume that

$$\nu_n \rightharpoonup \nu \text{ in } L^p(\mathbb{R}^2) \text{ as } n \to +\infty. \quad (3.75)$$

Then it is clear that

$$\int_{\mathbb{R}^2} x \nu(x)dx = 0, \quad \text{supp}(\nu) \subset B_{C_{13}}(0). \quad (3.76)$$

Besides, observing that $\{|\nu_n|\}_{n=1}^{+\infty}$ is bounded in $L^2(D)$, we deduce from (3.75) that

$$\nu_n \to \nu \text{ in } L^2(\mathbb{R}^2) \text{ as } n \to +\infty. \quad (3.77)$$

Define

$$v_n(x) = \frac{1}{\varepsilon_n^2} \varrho \left( \frac{x - \hat{x}}{\varepsilon_n} \right). \quad (3.78)$$
As in the proof of Lemma 3.11, it is easy to check that \( v_n \in S_{\varepsilon_n, i_0} \) for each \( n \). Therefore, \( E(\zeta_n) \geq E(v_n) \), that is,
\[
\frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \ln \frac{|x - \hat{y}|}{|x - y|} \zeta_n(x) \zeta_n(y) dxdy \geq \frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \ln \frac{|x - \hat{y}|}{|x - y|} v_n(x) v_n(y) dxdy, \quad (3.79)
\]
or equivalently,
\[
\frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \ln \frac{1}{|x - y|} \zeta_n(x) \zeta_n(y) dxdy + \frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \ln |x - \hat{y}| \zeta_n(x) \zeta_n(y) dxdy \geq \frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \ln |x - \hat{y}| v_n(x) v_n(y) dxdy. \quad (3.80)
\]
By the definition of \( v_n \), it is easy to check that as \( n \to +\infty \)
\[
\frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \ln |x - \hat{y}| v_n(x) v_n(y) dxdy \to \frac{\kappa^2}{4\pi} \ln \left( \frac{2i_0}{\kappa} \right). \quad (3.81)
\]
By Lemma 3.19 it is also clear that as \( n \to +\infty \)
\[
\frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \ln |x - \hat{y}| \zeta_n(x) \zeta_n(y) dxdy \to \frac{\kappa^2}{4\pi} \ln \left( \frac{2i_0}{\kappa} \right). \quad (3.82)
\]
From (3.81) and (3.82), we can write (3.80) as
\[
\int_{\Pi} \int_{\Pi} \ln \frac{1}{|x - y|} \zeta_n(x) \zeta_n(y) dxdy \geq \int_{\Pi} \int_{\Pi} \ln \frac{1}{|x - y|} v_n(x) v_n(y) dxdy + \gamma_n, \quad (3.83)
\]
where \( \{\gamma_n\}_{n=1}^{+\infty} \) satisfies \( \gamma_n \to 0 \) as \( n \to +\infty \). Taking into account the relation
\[
\zeta_n(x) = \frac{1}{\varepsilon_n^2} \nu_n \left( \frac{x - \hat{x}}{\varepsilon_n} \right),
\]
and the fact that \( \nu_n \rightharpoonup \nu \) in \( L^2(\mathbb{R}^2) \) as \( n \to +\infty \), we can pass to the limit \( n \to +\infty \) in (3.83) to get
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x - y|} \nu(x) \nu(y) dxdy \geq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x - y|} \varrho(x) \varrho(y) dxdy. \quad (3.84)
\]
Now we can apply Lemma 2.6 by choosing \( u_n = \nu_n \) and \( u_n^* = \varrho \) (this is possible by (3.74)) to get \( \nu = \varrho \). Note that the condition (2.3) in Lemma 2.6 is satisfied by (3.73).

To conclude, we have obtained as \( n \to +\infty \)
\[
\nu_n \rightharpoonup \varrho \text{ in } L^p(\mathbb{R}^2). \quad (3.85)
\]
On the other hand, in view of (3.74) we have
\[
\|\nu_n\|_{L^p(\mathbb{R}^2)} = \|\nu_n^*\|_{L^p(\mathbb{R}^2)} = \|\varrho\|_{L^p(\mathbb{R}^2)} \quad \forall n. \quad (3.86)
\]
From (3.85) and (3.86), by uniform convexity we deduce that
\[
\nu_n \rightharpoonup \varrho \text{ in } L^p(\mathbb{R}^2) \text{ as } n \to +\infty. \quad (3.87)
\]
Hence the proposition has been proved.

\[\square\]
Proof of Theorem 1.2. Items (1)-(5) have been proved in Proposition 3.2. Items (6)-(8) follow from Propositions 3.18, 3.21 and 3.22, respectively. □

4. Fine asymptotic estimates for Burton’s vortex pairs

As an application of Theorem 1.2, we are able to figure out the asymptotic behavior for Burton’s vortex pairs in [6] as the impulse goes to infinity.

Recall that \( \Sigma_\alpha \) is the set of maximizers of \( E \) relative to \( S_{1,\alpha} \). See Lemma 2.1. Denote \( \Sigma_0^\alpha \) the set of maximizers that are Steiner-symmetric in the \( x_1 \)-axis, or equivalently,

\[
\Sigma_0^\alpha = \left\{ \zeta \in \Sigma_\alpha \mid \int_{\Pi} x_1 \zeta(x) dx = 0 \right\}. \tag{4.1}
\]

The main result of this section is the following theorem.

Theorem 4.1. Let \( \alpha_0 > 0 \) be determined in Lemma 2.1. Then for any \( \alpha > \alpha_0 \), the following assertions for \( \Sigma_\alpha \) hold true.

(a) There exists some \( C > 0 \), depending only on \( \varrho \), such that

\[
\text{diam}(V_{\zeta}) \leq C \quad \forall \zeta \in \Sigma_\alpha.
\]

(b) Let \( q_{\zeta} \) be determined by (iii) in Lemma 2.1, then

\[
\alpha q_{\zeta} \to \frac{\kappa^2}{4\pi} \quad \text{as } \alpha \to +\infty,
\]

uniformly for \( \zeta \in \Sigma_0^\alpha \). More precisely, for any \( \epsilon > 0 \), there exists some \( \alpha_1 > \alpha_0 \), depending only on \( \varrho \) and \( \epsilon \), such that for any \( \alpha > \alpha_1 \), it holds that

\[
\left| \alpha q_{\zeta} - \frac{\kappa^2}{4\pi} \right| < \epsilon \quad \forall \zeta \in \Sigma_0^\alpha.
\]

(c) For \( \zeta \in \Sigma_0^\alpha \), extend \( \zeta \) to \( \mathbb{R}^2 \) such that \( \zeta = 0 \) in the lower half-plane and define

\[
\nu^{\zeta,\alpha}(x) = \zeta(x + \tilde{x}_\alpha), \quad \tilde{x}_\alpha = \left( 0, \frac{\alpha}{\kappa} \right).
\]

Then \( \nu^{\zeta,\alpha} \to \varrho \) in \( L^p(\mathbb{R}) \) as \( \alpha \to +\infty \), uniformly for \( \zeta \in \Sigma_0^\alpha \). More precisely, for any \( \epsilon > 0 \), there exists some \( \alpha_2 > \alpha_0 \), such that for any \( \alpha > \alpha_2 \), it holds that

\[
\| \nu^{\zeta,\alpha} - \varrho \|_{L^p(\mathbb{R}^2)} < \epsilon \quad \forall \zeta \in \Sigma_0^\alpha.
\]

Proof. Take \( \alpha = i_0/\varepsilon \). By Lemma 3.1, it holds that

\[
\Sigma_\alpha = \Gamma_{1, i_0/\varepsilon} = \{ v \in L^\infty(\Pi) \mid v^\varepsilon \in \Gamma_{\varepsilon, i_0} \}.
\]

Therefore (6)-(8) in Theorem 1.2 can be equivalently expressed as follows:

(6)' \( \text{diam}(V_{\zeta}) \leq C\varepsilon \) for any \( \zeta \in \Sigma_\alpha \), where \( C > 0 \) does not depend on \( \alpha \) (or \( \varepsilon \));

(7)' \( \lambda_{\zeta} \to \kappa^2/(4\pi \varrho i_0) \) uniformly for any \( \zeta \in \Sigma_\alpha \), where \( \lambda_{\zeta} \) is determined by item (3) in Theorem 1.2;

(8)' as \( \varepsilon \to 0^+, \varepsilon^2 \zeta^\varepsilon(\varepsilon \cdot +\tilde{x}) \to \varrho \) in \( L^p(\mathbb{R}^2) \), uniformly for any \( \zeta \in \Sigma_\alpha \).
To prove (a), it suffices to use (6)' and the following fact
\[ V_{ζε} = εV_ζ := \{ εx | x ∈ V_ζ \}. \]

To prove (b), observe that
\[ ζε = f_ζ(\mathcal{G}ζε - λζεx_2) \text{ a.e. } x ∈ Π \iff ζ = ε^2f_ζ(\mathcal{G}ζ - ελζεx_2) \text{ a.e. } x ∈ Π. \] (4.2)

Here we used the fact that
\[ Gζε(x) = Gζ(\frac{x}{ε}) \quad ∀ x ∈ Π. \]

On the other hand, by item (iii) in Lemma 3.1, it holds that
\[ ζ = φ_ζ(\mathcal{G}ζ - q_ζx_2) \text{ a.e. } x ∈ Π. \] (4.3)

Comparing (4.2) and (4.3) we get
\[ φ_ζ = ε^2f_ζ, \quad q_ζ = ελζε, \]
which together with (7)' yields (b). For (c), observe that
\[ ε^2ζε(εx + ˆx) = ζ(x + ε^{-1} ˆx) = ζ(x + ˜x_α) \quad ∀ x ∈ \mathbb{R}^2. \]

Therefore (c) follows from (8)' immediately. □

5. Stability of Chaplygin-Lamb dipole

First we recall the explicit expression of the Chaplygin-Lamb dipole. Define
\[ ψ_c(x) = \begin{cases} x_2 - \frac{2J_1(|x|)}{J_1(a)}x_2, & x ∈ Π, \quad |x| ≤ a, \\ \frac{a}{|x|^2}x_2, & x ∈ Π, \quad |x| ≥ a, \end{cases} \] (5.1)

where \( J_k, k = 0, 1, \) is the \( k \)-th order Bessel function of the first kind, and \( a \) is the first positive zero of \( J_1 \). Then \( ψ_c ∈ W^{2,s}_{\text{loc}}(Π) \) and satisfies
\[ -Δψ = (ψ - x_2)^+ \quad \text{a.e. in } Π. \] (5.2)

Define
\[ ζ_c = -Δψ_c. \]

In view of (5.2), it is easy to check that \( ζ_c(x_1 - t, x_2) \) solves the vorticity (1.3), thus corresponds to a traveling vortex pair with unit speed, generally referred to as the Chaplygin-Lamb dipole. Note that
\[ I(ζ_c) = πa^2, \quad ||ζ_c||^2_{L^2(Π)} = πa^2. \] (5.3)

See [4], Lemma 6 for example.

Our purpose in this section is to prove the following theorem concerning the stability of the Chaplygin-Lamb dipole.
Theorem 5.1. Let $2 < s < +\infty$ be fixed. Define
\[ C = \{ \zeta_c(x_1 - \beta, x_2) \mid \beta \in \mathbb{R} \}. \] (5.4)
Then $C$ is stable in the following sense: for any $\epsilon > 0$, there exists some $\delta > 0$, such that for any $L^s$-regular solution $\omega$ to the vorticity equation (1.3) satisfying $\omega(0, \cdot)$ is nonnegative with bounded support, $\inf_{\zeta \in C} \| \omega(0, \cdot) - \zeta \|_{X^s} < \delta$,
it holds that $\inf_{\zeta \in C} \| \omega(t, \cdot) - \zeta \|_{X^s} < \epsilon$ $\forall t \geq 0$.

Here the norm $\| \cdot \|_{X^s}$ is defined as in Theorem 1.2.

Remark 5.2. Abe-Choi [1] recently proved a similar stability result based on a new variational characterization of the Chaplygin-Lamb dipole. However, the initial perturbation class and the norm of measuring stability adopted by Abe-Choi are different from the ones in Theorem 5.1.

The proof of Theorem 5.1 is based on a stability theorem proved by Burton in [6] and a variational characterization proved by Burton in [4]. They are stated as follows.

Theorem 5.3 ([6], Theorem 2). Let $2 < s < +\infty$, $v_0 \in L^p(\Pi)$ be nonnegative with bounded support, and $\gamma > 0$ be fixed. Let $\mathcal{R}(v_0)$ be the rearrangement class of $v_0$ in $\Pi$, and $\overline{\mathcal{R}}^W(v_0)$ be the closure of $\mathcal{R}(v_0)$ in the weak topology of $L^s(\Pi)$. Denote by $\Sigma$ the set of maximizers of $E$ subject to
\[ \left\{ v \in \overline{\mathcal{R}}^W(v_0) \mid I(v) = \gamma \right\}. \]
If $\emptyset \neq \Sigma \subset \mathcal{R}(v_0)$, then $\Sigma$ is stable as in Theorem 5.1.

Theorem 5.4 ([4], p. 76). The functional $E - I$ attains it maximum value subject to
\[ \left\{ v \in L^2(\Pi) \mid v \geq 0 \text{ a.e., } I(v) < +\infty, \|v\|_{L^2(\Pi)} \leq \pi^{1/2} a \right\}, \]
and any maximizer must be of the form $\alpha \zeta_c(\alpha(x_1 - \beta), \alpha x_2)$ for some $\alpha \geq 0$ and some $\beta \in \mathbb{R}$.

Note that Theorem 5.4 did not appear as a theorem in [4], but can be found at the end of Section 3, p. 76.

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. By Theorem 5.3 it suffices to show that $C$ is exactly the set of maximizers of $E$ subject to
\[ \mathcal{A} := \left\{ v \in \overline{\mathcal{R}}^W(\zeta_c) \mid I(v) = \pi a^2 \right\}. \]
To this end, define
\[ \mathcal{B} := \left\{ v \in L^2(\Pi) \mid v \geq 0 \text{ a.e., } I(v) = \pi a^2, \|v\|_{L^2(\Pi)} \leq \pi^{1/2} a \right\}. \]
In view of (5.3) and Theorem 5.4, it is easy to see that \( \mathcal{C} \) is exactly the set of maximizers of \( E - I \) over \( \mathcal{B} \). Therefore \( \mathcal{C} \) is exactly the set of maximizers of \( E \) over \( \mathcal{B} \). Taking into account the fact that \( \mathcal{C} \subset \mathcal{A} \subset \mathcal{B} \), we immediately deduce that \( \mathcal{C} \) is exactly the set of maximizers of \( E \) over \( \mathcal{A} \). Hence the proof is finished.

\( \square \)

**Remark 5.5.** By the proof of Theorem 5.1 \( \mathcal{C} \) is in fact the set of maximizers of \( E \) subject to
\[
\{ v \in \mathcal{R}(\zeta_c) \mid I(v) = \pi a^2 \}.
\]
This means that for some special rearrangement class and impulse, existence and stability in Theorem 1.2 also holds true, even if \( \varepsilon \) is not small. However, for general rearrangement class and impulse the theory is far from completed.

**Acknowledgements:** G. Wang was supported by the National Natural Science Foundation of China grants (12001135, 12071098), and the China Postdoctoral Science Foundation grants (2019M661261, 2021T140163).

**References**

[1] K. Abe and K. Choi, Stability of Lamb Dipoles. *Arch. Ration. Mech. Anal.*, 2022(244), 877–917.
[2] A. Burchard and Y. Guo, Compactness via symmetrization, *J. Funct. Anal.*, 214(1)(2004), 40–73.
[3] G. R. Burton, Steady symmetric vortex pairs and rearrangements. *Proc. Roy. Soc. Edinburgh Sect. A*, 108(1988), 269–290.
[4] G. R. Burton, Isoperimetric properties of Lamb’s circular vortex-pair. *J. Math. Fluid Mech.*, 7(2005), 68–80.
[5] G. R. Burton, H.J. Nussenzveig Lopes, M.C. Lopes Filho, Nonlinear stability for steady vortex pairs, *Comm. Math. Phys.*, 324(2013), 445–463.
[6] G. R. Burton, Compactness and stability for planar vortex-pairs with prescribed impulse. *J. Differential Equations*, 270(2021), 547–572.
[7] D. Cao, S. Lai and W. Zhan, Traveling vortex pairs for 2D incompressible Euler equations. *Calc. Var. Partial Differential Equations*, 60(2021), Paper No. 75, 16 pp.
[8] S. A. Chaplygin, One case of vortex motion in fluid, *Trudy Otd. Fiz. Nauk Imper. Mosk. Obshch. Lyub. Estest.*, 1903(11), 11–14.
[9] R. J. Douglas, Rearrangements of functions on unbounded domains, *Proc. Roy. Soc. Edinburgh Sect. A*, 124(1994), 621–644.
[10] J. Duc and J. Sommeria, Experimental characterization of steady two dimensional vortex couples, *J.Fluid Mech.*, 192(1988), 175–192.
[11] T. Hmidi and T. Mateu, Existence of corotating and counter-rotating vortex pairs for active scalar equations. *Comm. Math. Phys.*, 350(2017), 699–747.
[12] L. C. Evans and R. Gariepy, Measure theory and fine properties of functions. Revised edition. *Textbooks in Mathematics*. CRC Press, Boca Raton, FL, 2015.
[13] H. Lamb, Hydrodynamics, 6th edition, Cambridge: Cambridge University Press, 1932.
[14] E. H. Lieb and M. Loss, Analysis, Second edition, Graduate Studies in Mathematics, Vol. 14. American Mathematical Society, Providence, RI (2001).
[15] A. J. Majda and A. L. Bertozzi, Vorticity and incompressible flow, Cambridge Texts in Applied Mathematics, Vol. 27. Cambridge University Press, 2002.
[16] C. Marchioro and M. Pulvirenti, Vortices and localization in Euler flows, Comm. Math. Phys., 154(1993), 49–61.
[17] C. Marchioro and M. Pulvirenti, Mathematical theory of incompressible noviscous fluids, Springer-Verlag, 1994.
[18] J. Norbury, Steady planar vortex pairs in an ideal fluid. Comm. Pure Appl. Math., 28(1975), 679–700.
[19] E. Overman and N. Zabusky, Coaxial scattering of Euler equation translating V–states via contour dynamics, J. Fluid Mech., 125(1982), 187–202.
[20] D. Pullin, Contour dynamics methods, Annual Review of Fluid Mechanics, 24(1992), 89–115.
[21] D. Smets and J. Van Schaftingen, Desingulariation of vortices for the Euler equation, Arch. Ration. Mech. Anal., 198(2010), 869–925.
[22] B. Turkington, On steady vortex flow in two dimensions. I, II, Comm. Partial Differential Equations, 8(1983), 999–1030, 1031–1071.
[23] G. Wang, Asymptotic estimates for concentrated vortex pairs, arXiv:2203.11800
[24] J. Yang, On the existence of steady planar vortices. Ann. Univ. Ferrara Sez. VII (N. S.), 37(1991), 111–129.
[25] J. Yang, T. Kubota, The steady motion of a symmetric, finite core size, counterrotating vortex pair. SIAM J. Appl. Math., 54(1994), 14–25.
[26] V. I. Yudovich, Non-stationary flow of an ideal incompressible fluid, USSR Comp. Math. & Math.Phys., 3(1963), 1407–1456 [English].

Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin 150001, P.R. China
Email address: wangguodong@hit.edu.cn