The Haag-Kastler Axioms for the $\mathcal{P}(\phi)_2$ Model on the De Sitter Space

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Abstract

We establish the Haag-Kastler axioms for a class of interacting quantum field theories on the two-dimensional de Sitter space, which satisfy finite speed of light. The $\mathcal{P}(\phi)_2$ model constructed in [3], describing massive scalar bosons with polynomial interactions, provides an example.

1 Introduction

The construction of interacting relativistic quantum field theories in $1+1$ dimensional Minkowski space exhibiting particle production, due to Glimm and Jaffe [14], is one of the crown jewels of mathematical physics. An interesting aspect of their construction is that the Haag-Kastler net for the interacting quantum field can be formulated on the Fock space for the free massive field. In fact, the von Neumann algebras associated to double cones whose base lie on the time-zero surface (i.e., the Cauchy surface) can be identified in both models. The spatial translations can as well be carried over from the model representing non-interacting particles. (Inspecting the Lie algebra relations of the Poincaré group, one realises that the Lorentz boosts can not be taken over from the free theory when the time evolution is modified. In other words: a deformation of the time-evolution requires a modification of the Lorentz boosts, while the spatial translations may remain unchanged.) Due to Haag’s theorem (see [25] for a formulation in an operator-algebraic language), the interacting vacuum state can not be given by a vector in Fock space (it is not normal with respect to the representation of the free massive field), but as long as the localization region of the observables one is interested in is bounded (in space-time) there are vectors in Fock space, which implement the interacting vacuum state locally. In other words, the interacting vacuum state is locally Fock.

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In de Sitter space, the situation is more favourable in two respects. Firstly, as the Cauchy surface is compact, Haag’s no-go theorem no longer applies (see again [26]) and any state which is locally Fock is normal with respect to the Fock representation. Secondly, if one replaces the spectrum condition (used in Minkowski space) by the geodesic KMS condition of Borchers and Buchholz [4] to ensure stability of the de Sitter vacuum state, the de Sitter vacuum is a thermal state for the Hawking temperature \((2\pi r)^{-1}\) for the unitary group of Lorentz boosts associated with any given wedge region (see [4]). This characterization of de Sitter vacua applies both to interacting and non-interacting theories.

Once one has constructed the family of von Neumann algebras describing the free massive scalar field, the facts just presented suggest that a look at (a generalisation [11] of) Araki’s perturbation theory of KMS states might be very useful: one can pick an arbitrary wedge (which we denote as \(W_1\)) and select a cyclic and separating vector \(\Omega\) in the natural positive cone associated to the von Neumann algebra \(A_\circ(W_1)\) for the wedge \(W_1\) (of the free massive field) and the Fock vacuum vector \(\Omega_\circ\). If this vector happens to be invariant under the rotations for the model representing non-interacting particles (leaving some geodesic Cauchy surface, which contains the edges of the wedge \(W_1\), invariant), one may hope that the candidates for the new boosts (i.e., implemented by the modular group for the pair \((A_\circ(W_1), \Omega)\)) together with the old rotations generate a new representation of \(SO_0(1,2)\). General criteria, which ensure that this is indeed the case, will be investigated elsewhere. However, for interesting examples, it is already known that the group closes: together with J. Barata the authors have shown in [3] that the interacting de Sitter vacuum vector for the \(\mathcal{P}(\phi)_2\) model (which lies in the natural positive cone for the pair \((A_\circ(W_1), \Omega_\circ)\)) can be used to define interacting boosts which, together with the rotations for the free field, generate a new unitary representation of the Lorentz group \(SO_0(1,2)\).

Now, if one already has an interacting representation of the Lorentz group acting on Fock space, it still remains to be shown that this new representation gives rise to a new family of von Neumann algebras satisfying the Haag-Kastler axioms. Our basic strategy to establish this fact is to identify the von Neumann algebras associated to the wedge \(W_1\) in the free and in the interacting theory. All other local algebras are then defined by applying the new representation of \(SO_0(1,2)\) (which represents the interacting dynamics) and taking intersections. This machinery ensures most of the Haag-Kastler axioms, but non-triviality of the double cone algebra\(^2\) has to be established separately. As the rotations are carried over from the model representing non-interacting particles, the von Neumann algebras associated to wedges whose bases lie on the

\(^1\)In fact, the authors would argue that there is a deep connection between Tomita-Takesaki modular theory and the representation theory of the group \(SL(2, \mathbb{R})\) (and, more general, non-compact Lie-algebras which contain \(SL(2, \mathbb{R})\) as a subgroup).

\(^2\)We note that non-triviality of the local algebras is difficult to establish in the approach pioneered by Gandalf Lechner, see [18, 19].
time-zero surface (i.e., the Cauchy surface) can be identified in both models, too. The same argument does not apply to the von Neumann algebras associated to double cones: a von Neumann algebra associated to a double cone $O$, whose base lies on the geodesic Cauchy surface, must be contained in the intersection of the von Neumann algebras associated to all wedges which contain $O$ (and not only those whose base lie on the Cauchy surface). Consequently, an additional condition is needed to ensure non-triviality of the double cone algebras. A sufficient condition is that the interacting representation satisfies the finite speed of light property introduced by Glimm and Jaffe [14]. As we will show in this work, the latter ensures that, just like in the Minkowski space case analyzed by Glimm and Jaffe, the von Neumann algebras associated to double cones whose base lie on the time-zero surface (i.e., the Cauchy surface) can be identified in both models. Finite speed of light has been verified for the $\mathcal{P}(\varphi)_2$ model on de Sitter space in [3].

2 One-particle space

The two-dimensional de Sitter space

\[ dS = \{ x \in \mathbb{R}^{1+2} \mid x_0^2 - x_1^2 - x_2^2 = -r^2 \} , \quad r > 0 , \]  

(1)

can be viewed as a one-sheeted hyperboloid, embedded in the (1+2)-dimensional Minkowski space $\mathbb{R}^{1+2}$. The embedding (1) is compatible with the metric and the causal structure, i.e., the de Sitter space $dS$ inherits its metric and the causal structure from the ambient Minkowski space.

The isometry group of $dS$ is the Lorentz group $O(1,2)$. The connected component containing the identity is $SO_0(1,2)$. This subgroup is generated by the rotations

\[ R_0(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} , \quad \alpha \in [0, 2\pi) , \]

and the Lorentz boosts

\[ \Lambda_1(t) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} , \quad t \in \mathbb{R} . \]

We will also need the rotated boosts

\[ \Lambda^{(\alpha)}(t) = R_0(\alpha)\Lambda_1(t)R_0(-\alpha) , \quad t \in \mathbb{R} . \]

(2)

According to our convention, the boosts $\Lambda_1(t) \equiv \Lambda^{(0)}(t)$ keep the $x_1$-axis invariant, and therefore correspond to boosts in the $x_2$-direction.
The circle
\[ S^1 = \{ x \in \mathbb{S} \mid x_0 = 0 \} \]
forms a Cauchy surface for \( dS \). For two points \( x = (0, r \sin \psi, r \cos \psi) \) and \( y = (0, r \sin \psi', r \cos \psi') \) on the circle \( S^1 \), the Wightman two-point function of a scalar free field (analysed in [8]) equals
\[ \mathcal{W}^{(2)}_\nu(x, y) = c_\nu \; p_{s^+}(-\cos(\psi - \psi')) . \] (3)
Here \( p_{s^+} \) is the Legendre function for the parameter
\[ s^+ = -\frac{1}{2} + i\nu , \] (4)
with
\[ \nu = \begin{cases} 
  i \sqrt{\frac{1}{4} - \zeta^2} & \text{if } 0 < \zeta < 1/2 , \\
  \sqrt{\zeta^2 - \frac{1}{4}} & \text{if } \zeta \geq 1/2 , 
\end{cases} \] (5)
and \( \zeta \) the eigenvalue of the Casimir operator of \( SO_0(1, 2) \). The two-point function (3) gives rise to a scalar product on \( C^\infty(S^1) \):
\[ \langle h, h' \rangle_{\mathcal{H}} = c_\nu \int_{S^1} r \, d\psi \int_{S^1} r \, d\psi' \; \overline{h(\psi)} \times p_{s^+}(-\cos(\psi - \psi')) h'(\psi') . \] (6)
The value of the positive normalisation constant \( c_\nu \) is
\[ c_\nu = \frac{1}{2 \sin(\pi s^+)} = \frac{1}{2 \cos(i\pi)} . \]
Note that the singularity for \( \psi = \psi' \) is integrable; see for instance p. 364 in [8]. The completion of \( C^\infty(S^1) \) w.r.t. this scalar product is a one-particle Hilbert space, which we denote by \( \mathcal{H} \). Clearly, the scalar product (6) depends on the value of the Casimir operator \( \zeta > 0 \), but we will suppress this dependence in the notation.

The Fourier coefficients of the Legendre function appearing in (3) and (6) were computed in the proof of Proposition 4.7.3 in [8]. The result can be casted in the following form:

**Lemma 2.1.** The scalar product (6) can be expressed as
\[ \langle h, h' \rangle_{\mathcal{H}} = \langle h, \omega h' \rangle_{L^2(S^1, rd\psi)} , \]
with \( \omega \) a strictly positive self-adjoint operator on \( L^2(S^1, rd\psi) \) with Fourier coefficients
\[ \tilde{\omega}(k) = r^{-1} (k + s^+) \frac{\Gamma \left( \frac{k+s^+}{2} \right) \Gamma \left( \frac{k+1-s^+}{2} \right)}{\Gamma \left( \frac{k-s^+}{2} \right) \Gamma \left( \frac{k+1+s^+}{2} \right)} , \; k \in \mathbb{Z} . \] (7)
We note that \( \tilde{\omega}(0) > 0 \) and \( \tilde{\omega}(-k) = \tilde{\omega}(k) \). The map \( \mathbb{R}^+ \ni k \mapsto \tilde{\omega}(k) \) is monotonically increasing and, asymptotically, one finds
\[
\lim_{k \to \pm \infty} \frac{r\tilde{\omega}(k)}{|k|} = 1,
\]
just like for the free massive scalar field on Minkowski space.

One of the key results in [3, Theorem 4.7.5] is that \( \mathcal{H} \) carries a representation of the Lorentz group:

**Theorem 2.2.** The rotations
\[
(u(R_0(\alpha))h)(\psi) = h(\psi - \alpha), \quad \alpha \in [0, 2\pi), \quad h \in \mathcal{H},
\]
and the boosts
\[
u(\Lambda_1(t)) = e^{it\omega \cos}, \quad t \in \mathbb{R}, \tag{8}
\]
generate a unitary irreducible representation of \( \text{SO}_0(1, 2) \) on \( \mathcal{H} \), which extends to an (anti-)unitary representation of \( \text{O}(1, 2) \). In particular, the reflection \( \Theta_{W_1} := P_1 T \) at the edge of the wedge \( W_1 \) is represented by the anti-linear map
\[
u(\Theta_{W_1})h = CP_1 h, \quad h \in \mathcal{H},
\]
where \( P_1 h(\psi) = h(\pi - \psi) \) and \( (Ch)(\psi) = \overline{h(\psi)} \) denotes the complex conjugation.

**Remarks 2.1.**

i.) The symbol “\( \hat{\cos} \)” in the exponent in (8) denotes the multiplication operator, mapping a function \( f(\psi) \) to \( \cos \psi \cdot f(\psi) \).

ii.) It is remarkable that both the representations of the principle series (those with eigenvalue \( \zeta \geq 1/2 \) of the Casimir operator) as well as the representations of the complementary series (those with \( 0 < \zeta < 1/2 \)) can be casted in the form given in Theorem 2.2. Note that both \( \mathcal{H} \) and \( \omega \) depend on \( \zeta \).

From the given unitary representation of the Lorentz group we now define a family of \( \mathbb{R} \)-linear subspaces of \( \mathcal{H} \) by a standard construction, called modular localization (see [9, 24]), which draws inspiration from the results [5, 6] of Bisognano and Wichmann.

**Definition 2.3.** (Modular Localization).

i.) For the wedge \( W_1 = \{ x \in dS \mid x_2 > |x_0| \} \), we set
\[
\mathcal{H}(W_1) = \{ h \in \mathcal{H}(u(\Lambda_1(\pi r))) \mid u(P_1 T)u(\Lambda_1(\pi r))h = h \},
\]
ii.) For an arbitrary wedge \( W = \Lambda W_1, \Lambda \in \text{SO}_0(1, 2) \), we set
\[
\mathcal{H}(W) = u(\Lambda)\mathcal{H}(W_1). \tag{9}
\]
For a causally complete, open and bounded region $\mathcal{O}$, we set
\[ H(\mathcal{O}) = \bigcap_{\mathcal{W} \subset \mathcal{O}} H(\mathcal{W}) . \tag{10} \]

Note that $H(\mathcal{W})$ is well-defined by (9) due to the following standard argument [9]: The only Lorentz transformations, which leave the wedge $\mathcal{W}$ invariant, are of the form $\Lambda \equiv \Lambda_1(t)$ for some $t \in \mathbb{R}$. But the representer $u(\Lambda_1(t))$ of such a boost commutes with both $u(P_1 T)$ and with $u(\Lambda_1(\text{i}\pi \tau))$ and thus leaves the $\mathbb{R}$-linear subspace $H(\mathcal{W}_1)$ invariant.

**Proposition 2.4.** The subspaces introduced in Definition 2.3 have the following properties:

i.) (Wedge Duality). The $\mathbb{R}$-linear subspace $H(\mathcal{W}')$ for the opposite wedge
\[ \mathcal{W}' \equiv \{ x \in dS \mid x \text{ space-like separated from } \mathcal{W} \} \]
equals the symplectic complement
\[ H(\mathcal{W})' = \{ h \in H \mid \mathcal{H}(h, g) = 0 \quad \forall g \in H(\mathcal{W}) \} \]
of $H(\mathcal{W})$.

ii.) (Covariance). For $\Lambda \in SO_0(1, 2)$ and $\mathcal{O}$ a causally complete, open, connected and bounded region,
\[ \mathcal{H}(\Lambda \mathcal{O}) = u(\Lambda)H(\mathcal{O}) . \]

iii.) (Microcausality). For two space-like separated causally complete, open, bounded regions $\mathcal{O}_1$ and $\mathcal{O}_2$,
\[ \mathcal{H}(h_1, h_2)_\mathcal{H} = 0 \quad \forall h_i \in \mathcal{H}(\mathcal{O}_i) , \quad i = 1, 2 . \]

**Proof.** The items i.) and ii.) follow from Proposition 5.2 of [9], while iii.) follows from Theorem 5.4 of [9]. For the convenience of the reader, we indicate the arguments. As a consequence of the group relations, the anti-linear operator
\[ s_{\mathcal{W}} \equiv u(\Theta_{\mathcal{W}})u(\Lambda_1(\text{i}\pi \tau)) \]
is the Tomita operator for $\mathcal{H}(\mathcal{W}_1)$; namely, it is a densely defined involution (i.e., $s_{\mathcal{W}}^2 \subset 1$) and its eigenspace for eigenvalue $+1$ is just $\mathcal{H}(\mathcal{W}_1)$: from the group relations between $\Lambda_1(t)$, $R_0(\pi)$ and $\Theta_{\mathcal{W}} = P_1 T$ it follows that the operator $u(R_0(\pi))u(\Theta_{\mathcal{W}})$ commutes with the anti-linear operator $s_{\mathcal{W}}$ and thus leaves the $\mathbb{R}$-linear subspace $\mathcal{H}(\mathcal{W}_1)$ invariant. But this implies that
\[ u(\Theta_{\mathcal{W}})\mathcal{H}(\mathcal{W}_1) = \mathcal{H}(\mathcal{W}_1') , \tag{11} \]
since $R_0(\pi \mid W_1 = W_1'$. On the other hand, a general result on Tomita operators \cite[Prop. 2.3]{23} asserts that the anti-unitary part $u(\Theta W_1)$ in the polar decomposition of $s_{W_1}$ maps the eigenspace for the eigenvalue 1, namely $\mathcal{H}(W_1)$, onto its symplectic complement $\mathcal{H}(W_1)'$. Thus, Eq. (11) is just wedge duality (i.e., property i.) in the proposition) for $W_1$. For other wedges, duality follows from wedge covariance \cite[Prop. 6.5.5]{3}.

To prove property iii.), microcausality, pick a wedge $W$ such that $\partial_1 \subset W$ and $\partial_2 \subset W'$. Then

$$\mathcal{H}(\partial_1) \subset \mathcal{H}(W) = \mathcal{H}(W')' \subset \mathcal{H}(\partial_2)'$$

where we have used wedge duality.

Property ii.), covariance, follows from the definitions of $\mathcal{H}(W)$ and $\mathcal{H}(\partial)$, respectively. \hfill \Box

It will be useful to have an explicit formula for the real subspaces associated to a certain class of regions, namely double cones or wedges with base on $S^1$. For these regions, an alternative localization map (widely used in quantum field theory on Minkowski space) is available, which exploits the support properties of Cauchy data of solutions of the Klein-Gordon equation, by identifying $\Re h$ and $\omega^{-1} \Im h$, for $h \in \mathcal{H}$, with initial data of a solution of the Klein-Gordon equation: for an open interval in $S^1$, one defines the $\mathbb{R}$-linear subspaces

$$\mathcal{H}_I \doteq \{ h \in \mathcal{H} \mid \text{supp } \Re h \subset I, \text{ supp } \omega^{-1} \Im h \subset I \}.$$  \hspace{1cm} (12)

It has been shown in \cite[Prop. 6.5.5]{3} that

$$u(\Lambda_1(t))\mathcal{H}_I \subset \mathcal{H}_I, \quad I_1 \doteq \Gamma(\Lambda_1(t)I) \cap S^1,$$  \hspace{1cm} (13)

where $\Gamma(M)$ is the domain of dependence of a set $M$, i.e., the union of the future $\Gamma^+(M)$ and the past $\Gamma^-(M)$ of $M$. Equation (14) expresses the hyperbolic character of the Klein-Gordon equation in the Hilbert space context. It is common usage to refer to (13) as finite speed of light. A similar condition, which applies to interacting theories, will be presented in (27).

**Proposition 2.5.** The subspaces introduced in Definition \cite[2.3]{3} have the following properties:

i.) (Modular Localization $\Leftrightarrow$ Localization of Cauchy data). For a bounded open interval of length $|I| \leq \pi r$ in $S^1$ there holds

$$\mathcal{H}(\partial_1) = \mathcal{H}_I,$$  \hspace{1cm} (14)

where $\partial_1 = I''$ denotes the causal completion of the interval $I$ in $\partial S$.

ii.) (Additivity). For any open interval $I \subset I_+$, we have

$$\mathcal{H}(W_I) = \bigvee_{R_0(\alpha) \subset I_+} \mathcal{H}(\partial_R(\alpha)),$$  \hspace{1cm} (15)

where $\bigvee$ denotes the closure of the $\mathbb{R}$-linear span in $\mathcal{H}$.\Hfill \Box
iii.) (Standard Subspaces). The $\mathbb{R}$-linear subspaces $\mathcal{H}(O)$ are standard, i.e.,

$$\mathcal{H}(O) \cap i\mathcal{H}(O) = \{0\}, \quad \mathcal{H}(O) + i\mathcal{H}(O) = \mathcal{H},$$

(16)

for all double cones $O \subset dS$.

Proof. We prove property i.) first for the wedge $W_1$. Let $I_+$ be the open half-circle $I_+ = \{x \in S^1 \mid x_2 > 0\}$. It has been shown in [3, Prop. 6.4.3] that $\mathcal{H}_{I_+}$ is contained in $\mathcal{H}(W_1)$; see property iv.) of Definition A.7 in [3]. But $\mathcal{H}_{I_+}$ is invariant under the modular unitary group associated with $\mathcal{H}(W_1)$, namely $u(\Lambda(t))$, by the finite speed of light property (13). Further, $\mathcal{H}_{I_+}$ is also a standard subspace. Takesaki’s Theorem on standard subspaces (see, e.g., [20]) then asserts that

$$\mathcal{H}_{I_+} = \mathcal{H}(W_1),$$

(17)

as claimed. An alternative proof of (17) is provided by combining [3, Prop. 6.4.5] (which provides a unitary map between the covariant one-particle space and the canonical one) with [3, Theorem 6.4.6], which discusses restrictions of this unitary map to localised regions.

Let now $I$ be an interval as in the proposition. As $O_I$ is causally complete,

$$\bigcap_{O_I \subset W} W = O_I = W(\alpha) \cap W(\beta)$$

for some fixed $\alpha, \beta \in [0, 2\pi)$, where

$$W(\alpha) = R_0(\alpha)W_1, \quad \alpha \in [0, 2\pi)$$

denotes a wedge whose edges lies on $S^1$. Inspecting the definitions and (17), we find that

$$\mathcal{H}_I = \mathcal{H}_{R_0(\alpha)I_+} \cap \mathcal{H}_{R_0(\beta)I_+} = \mathcal{H}(W(\alpha)) \cap \mathcal{H}(W(\beta)).$$

As both $W(\alpha)$ and $W(\beta)$ are wedges which contain $O_I$, we have

$$\mathcal{H}(O_I) \subseteq \mathcal{H}_I.$$

Next, we assume that $W$ is an arbitrary wedge which contains $O_I$. The opposite wedge $W'$ of $W$ is, like any wedge, of the form $\Lambda(\beta)(t)R_0(\alpha)W_1$ for suitable $\alpha, \beta$ and $t$. As a consequence of finite speed of light $^3$(13) we have

$$\mathcal{H}(W') = u(\Lambda(\beta)(t))\mathcal{H}(W(\alpha)) \subset \mathcal{H}_J$$

with $J = \Gamma(\Lambda(\beta)(t)R_0(\alpha)I_+) \cap S^1$, where $\Gamma(M)$ is the domain of dependence of a set $M$, i.e., the union of the future $\Gamma^+(M)$ and the past $\Gamma^-(M)$ of $M$. Note that

$^3$This property has been shown in [3, Prop. 6.5.5] for $\Lambda(t)$, but also holds for $\Lambda(\beta)(t)$ as $u(R_0(\alpha))\mathcal{H}_I = \{h \in \mathcal{H} \mid \text{supp} R_0(\alpha) h \subset R_0(\alpha)I, \text{supp} \omega^{-1} \text{supp} h \subset R_0(\alpha)I\}$ for all $\alpha \in [0, 2\pi)$. 

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a.) $\Gamma(\Lambda^{(\beta)}(t)\mathcal{R}_0(\alpha)I_{+}) = \Gamma(W')$;

b.) $W'$ is space-like to $I$, since $W$ contains $\mathcal{O}_1$.

Hence $\Gamma(W') \cap S^1$ is in the interior $I^c = S^1 \setminus \bar{I}$ of the complement of $I$ within $S^1$, and the same holds for $J$. Thus

$$\mathcal{H}(W') \subset \mathcal{H}_{I^c}.$$ 

Wedge duality now implies

$$\mathcal{H}(W) = \mathcal{H}(W')' \supseteq \mathcal{H}_I.$$ 

This verifies (14).

Property ii.). Additivity follows from the covariant formulation of the one-particle Hilbert space, which uses the Fourier-Helgason transformation to define $\mathbb{R}$-linear subspaces associated to bounded space-time regions, see [3]. However, it can also be verified directly, exploring ideas of [2, Sect. 5]. Using (14), the statement (15) is equivalent to the following one: for any open interval $I \subset I^+$,

$$\mathcal{H}_{I^+} = \bigvee_{R_0(\alpha)I \subset I^+} \mathcal{H}_{R_0(\alpha)I},$$

where $\bigvee$ denotes the closure of the $\mathbb{R}$-linear span in $\mathcal{H}$. Since the closure of $I_+$ is compact, it is sufficient to prove additivity for two overlapping open intervals $I_1, I_2 \subset I \equiv I_1 \cup I_2$. Inspecting (12), we see that we have to show that any $h \in \mathcal{H}_I$ can approximated by a sequence of functions $\{h_n\}_{n \in \mathbb{N}}$ such that

$$h_n = h_n^{(1)} + h_n^{(2)},$$

with

$$\text{supp } \Re h_n^{(i)} \subset I_i, \quad \text{supp } \Im \omega^{-1} h_n^{(i)} \subset I_i, \quad i = 1, 2.$$ 

Since $L^2(I, rd\psi)$ is dense in $\mathcal{H}_I$, we can choose $\Re h_n^{(i)} \in L^2(I_i, rd\psi)$. Additivity for the real part is then a consequence of the additivity of the relevant $L^2$-spaces. Since $\omega^{-1}$ is an injective bounded operator on $\mathcal{H}_I$, it is bijective onto its image, and thus every $h \in \mathcal{H}_I$ is of the form

$$h = \omega(\omega^{-1}h),$$

where $g := \omega^{-1}h \in \mathcal{H}^{1/2}(S^1)$ has support in $I$. Here $\mathcal{H}^{1/2}(S^1) \subset L^2(S^1, rd\psi)$ is the closure of $C^\infty(S^1)$ with respect to the norm

$$\|f\|_{\mathcal{H}^{1/2}(S^1)} := \sum_{k \in \mathbb{Z}} \tilde{\omega}(k)|f_k|^2.$$ 

Note that the map $\omega : \mathcal{H}^{1/2}(S^1) \to \mathcal{H}$ is unitary. Now, any $g$ in $L^2(I, rd\psi)$ can be decomposed in the form

$$g = \chi_1 g + \chi_2 g, \quad \chi_i \in C_0^\infty(I_i), \quad i = 1, 2.$$
The support properties are clearly satisfied. Hence additivity of the imaginary part $\mathcal{I}h$ of $h$ follows, once one has verified that the multiplication by a $C_0^\infty$-function defines a bounded operator on $H^{1/2}(S^1)$. This result is established in Lemma 2.6 below.

Property iii.), the standard property, follows from abstract reasons (Theorem 5.6 of [2], which is applicable since we are dealing with representations of the principal or complementary series). A direct argument can also be given: For $\mathcal{O} = \mathcal{W}_1$, the standard property is a general result about Tomita operators (see, e.g., [23]). The first identity in (16) then follows by covariance and isotony (since every double cone is contained in some wedge).

Adapting ideas of Borchers and Buchholz (see [4, Lemma 3.2]) to the one-particle space, one can show that the orthogonal complement of the complex linear span of $H$ is empty. Hence, the second identity in (16) follows too.

**Lemma 2.6.** The multiplication of $\chi \in C^\infty(S^1)$ on a vector of $H^{1/2}(S^1)$ is a bounded operator in $H^{1/2}(S^1)$.

**Proof.** Let $g$ be a vector in $H^{1/2}(S^1)$, i.e., $\sum_{k \in \mathbb{Z}} \tilde{\omega}(k)|g_k|^2 < \infty$, where

$$g_k := \frac{1}{\sqrt{2\pi}} \int_{S^1} r d\psi e^{ik\psi} f(\psi)$$

are the Fourier coefficients of $g$. We have to show that there is a constant $c > 0$ such that $\|\chi g\|_{H^{1/2}(S^1)} \leq c \|g\|_{H^{1/2}(S^1)}$. Note that $\chi \in C^\infty(S^1)$ implies that, for all $\ell \in \mathbb{N}$,

$$(1 + |n|)^\ell \chi_k \to 0 \quad \text{as} \quad |k| \to \infty.$$  \hfill (20)

In particular, $\chi \in H^{1/2}(S^1)$. Now using $(\chi \cdot h)_k = \sum_{k'} \chi_{k-k'} h_{k'}$, one gets

$$\|\chi h\|_{H^{1/2}(S^1)}^2 = \sum_{k \in \mathbb{Z}} \left| \tilde{\omega}(k) \right|^{1/2} \sum_{k' \in \mathbb{Z}} \chi_{k-k'} h_{k'} \right|^2 \leq \sum_{k,k' \in \mathbb{Z}} \tilde{\omega}(k+k') \left| \chi_k h_{k'} \right|^2.$$  

The facts that $\tilde{\omega}$ is monotonically increasing and $\tilde{\omega}(k)|k|^{-1} \to r^{-1}$ for large $k$ imply that there are positive constants $a, b$ such that $|k| \leq b\tilde{\omega}(k)$ and $\tilde{\omega}(k) \leq \tilde{\omega}(0) + a|k|$.

Thus,

$$\tilde{\omega}(0) \leq |\tilde{\omega}(k+k')| \leq \tilde{\omega}(0) + a|k| + a|k'| \quad \forall k,k' \in \mathbb{Z}$$

If (as we expect) the map $k \mapsto \tilde{\omega}(k)$ is convex, then one can set $a = b = 1.$
and
\[
\|x\|_{H^1/2(S^1)}^2 \leq \tilde{\omega}(0)\|x\|_{L^2}^2 + a\|x\|_{L^2}^2 \left( \sum_{k \in \mathbb{Z}} |k| |x_k|^2 \right) + a\|x\|_{L^2}^2 \left( \sum_{k' \in \mathbb{Z}} |k'| |h_{k'}|^2 \right)
\]
\[
\leq \left( 1 + ab \right)\|x\|_{L^2}^2 + \frac{ab}{\tilde{\omega}(0)}\|x\|_{H^1/2(S^1)}^2 \|h\|_{H^1/2(S^1)}^2 .
\] (21)

In the second inequality we used \[\tilde{\omega}(0)\|x\|_{L^2} \leq \|x\|_{H^1/2(S^1)}^2.\] \[\square\]

3 Nets of Local Algebras

The bosonic Fock space \[\mathcal{F} = \Gamma(\mathcal{H})\] over \(\mathcal{H}\) is defined as the direct sum of the \(n\)-particle spaces:

\[\Gamma(\mathcal{H}) \doteq \oplus_{n=0}^{\infty} \mathcal{H}^\otimes_n, \quad \mathcal{H}^\otimes_0 \doteq \mathbb{C},\]

with \(\mathcal{H}^\otimes_n\) the \(n\)-fold totally symmetric tensor product \(\otimes_s\) of \(\mathcal{H}\) with itself. The coherent vectors

\[\Gamma(h) \doteq \oplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} h \otimes_s \cdots \otimes_s h,\]

form a total set in \(\mathcal{F}\). The vector \(\Omega_0 = \Gamma(0)\) is called the Fock vacuum. One can also define\(^5\) second quantized operators:

**Lemma 3.1.** Let \(A\) be a closed, densely defined linear operator on \(\mathcal{H}\) with domain \(\mathcal{D}(A)\). Then

\[\Gamma(A) : \mathcal{F} \to \mathcal{F}\]

is the closure of the linear operator acting on the linear combinations of coherent vectors with exponent in \(\mathcal{D}(A)\) such that

\[\Gamma(A)\Gamma(h) = \Gamma(Ah) .\]

This exponentiation preserves self-adjointness, positivity and unitarity.

For \(h, g \in \mathcal{H}\), the relations

\[V(h)V(g) = e^{-i\langle h, g \rangle}V(h+g) ,\]

\[V(h)\Omega_0 = e^{-\frac{1}{2}\|h\|^2}V(ih) ,\]

\(^5\)To the best of our knowledge, this formulation first appeared in [15]. Our presentation follows closely [16].
define unitary operators, called the Weyl operators. They satisfy

\[ V^*(h) = V(-h) \quad \text{and} \quad V(0) = 1. \]

We use the Weyl operators to associate a von Neumann algebras acting on the Fock space \( \mathcal{F} \) to the wedge \( W_1 \): let \( \mathcal{A}_0(W_1) \) denote the von Neumann algebra generated by the Weyl operators \( \{ V(h) \mid h \in \mathcal{H}(W_1) \} \).

Given a representation \( \mathcal{U}(\Lambda), \Lambda \in \text{SO}_0(1, 2) \), of the Lorentz group acting on Fock space which acts kinematically on the Cauchy surface \( S^1 \) (i.e., the rotations leaving \( S^1 \) invariant are given by the pull-back acting on one-particle wave functions), we can define von Neumann algebras associated to arbitrary convex, causally complete, bounded regions. We proceed in steps, repeating the ideas which lie behind Definition 2.3 and starting from the free field algebra of the wedge \( W_1 \). Note that \( \Lambda W_1 \subset W_1 \Leftrightarrow \Lambda = \Lambda_1(t) \) for some \( t \in \mathbb{R} \).

**Definition 3.2.** Given a unitary representation \( \Lambda \mapsto \mathcal{U}(\Lambda) \) of the Lorentz group \( \text{SO}_0(1, 2) \) acting on Fock space \( \mathcal{F} \) and satisfying the condition

\[ \mathcal{U}(\Lambda_1(t)) \mathcal{A}_0(W_1) \mathcal{U}(\Lambda_1(t))^{-1} = \mathcal{A}_0(W_1), \quad t \in \mathbb{R}, \quad (22) \]

we define the following von Neumann algebras:

i.) For an arbitrary wedge \( W = \Lambda W_1, \Lambda \in \text{SO}_0(1, 2) \), we set

\[ \mathcal{A}(W) \doteq \mathcal{U}(\Lambda) \mathcal{A}_0(W_1) \mathcal{U}(\Lambda)^{-1}. \]

ii.) For an arbitrary bounded, causally complete, convex region (these are the de Sitter analogs of the double cones) \( \emptyset \subset dS \), we set

\[ \mathcal{A}(\emptyset) \doteq \bigcap_{W \supseteq \emptyset} \mathcal{A}(W). \]

The inclusion preserving map

\[ \emptyset \mapsto \mathcal{A}(\emptyset) \]

is called, in a slight abuse of the term, the net of local von Neumann algebras for the bosonic field on the de Sitter space \( dS \) transforming under \( \mathcal{U} \).

---

6Technically speaking, this family of algebras is not a “net”, since in de Sitter space not every pair of double cones is contained in a double cone.
Remarks 3.1.

i.) In case $U \equiv U_o$,

$$U_o(\Lambda) = \Gamma(u(\Lambda)), \quad \Lambda \in \text{SO}_0(1, 2),$$

we will denote the generator of one-parameter unitary group $t \mapsto U_o(\Lambda^t)$ by $L_o$ and the local algebra by $\mathcal{A}_o(\emptyset)$. It follows from a result by Araki [7, Theorem 1] and Proposition 2.4 ii.) that

$$\mathcal{A}_o(\emptyset) = \{ V(h) \mid h \in \mathcal{H}, \text{ supp } (\Re h, \omega^{-1} \Im h) \subset I \times I \}'';$$

just as one might have expected.

ii.) For the $\Lambda(\phi)_2$ model on the de Sitter space, the representation $U$ of $\text{SO}_0(1, 2)$ in Definition 3.2 is generated by the rotations $U(R^o(\alpha))$, $\alpha \in [0, 2\pi)$, and the boosts $U(\Lambda^t_1)$, $t \in \mathbb{R}$. The latter are defined as follows:

a.) The rotations leaving the Cauchy surface $S^1$ invariant are the free ones,

$$U(R^o(\alpha)) = \Gamma(u(R^o(\alpha))), \quad \alpha \in [0, 2\pi); \quad (25)$$

b.) The generator of the one-parameter unitary group $t \mapsto U(\Lambda^t_1)$ can be expressed in terms of canonical fields and canonical momenta (see [3] for details):

$$L = L_o + \lim_{\epsilon \to 0} \int_{S^1} r \cos \psi \, d\psi : \mathcal{P}(\varphi(\delta_{\epsilon}(-\psi))),$$  (26)

where $\mathcal{P}$ is a real-valued polynomial, bounded from below, $\varphi(h)$ is the generator of the one-parameter unitary group $s \mapsto V(sh)$, and $\delta_{\epsilon}$ approximates the Dirac delta function as $\epsilon \to 0$. As usual, the $: : \cdot \cdot$ indicates normal ordering.

iii.) The representations presented in i.) and ii.) extend to representations of $O(1, 2)$ by adding the reflections $\Gamma(u(P_1))$ and $\Gamma(u(T))$. Note that in the interacting case ii.) the reflection

$$U(\Theta_W) = U(\Lambda) \Gamma(u(\Theta_W)) U^{-1}(\Lambda)$$

associated to a wedge $W = \Lambda W_1$, $\Lambda \in \text{SO}_0(1, 2)$, whose edges do not lie on the Cauchy surface $S^1$ will in general not coincide with $\Gamma(u(\Theta_W))$.

In the general case, we will need a criterion for the representation of the Lorentz group which ensures that the intersection in (24) is not trivial. A sufficient condition is the following:
**Definition 3.3.** Assume the von Neumann algebras are defined as in Definition 3.2. The net of local algebras is said to satisfy finite speed of light, if for any wedge \( W \), the algebra \( \mathcal{A}(W) \) is contained in the time-zero Weyl algebra

\[
\{ V(h) \mid h \in \mathcal{H}, \text{supp } \Re h \subset J, \text{supp } \omega^{-1} \Im h \subset J \}'',
\]

where \( J = \Gamma(W) \cap S^1 \).

**Remarks 3.2.**

i.) If we associate von Neumann algebras to intervals \( I \) by setting

\[
\mathcal{R}(I) = \mathcal{A}(0_I), \quad \mathcal{R}_\circ(I) = \mathcal{A}_\circ(0_I), \quad I \subset S^1,
\]

then (27) may be formulated in the same way as finite speed of light was originally defined by Glimm and Jaffe in [14] see Theorem 6.7 for the free case and Theorem 8.1 for the interacting case, namely by requesting that

\[
U(\Lambda)\mathcal{R}(I)U^{-1}(\Lambda) \subset \mathcal{R}_\circ(\Gamma(\Lambda I) \cap S^1).
\]

Of course, on Minkowski space one would use a time translations by some \( |t| < \delta \) instead of the boosts, and then \( \Gamma(\Lambda I) \cap S^1 \) would just be equal to \( I + [-\delta, \delta] \).

ii.) For the \( \mathcal{P}(\varphi)_2 \) model on de Sitter space, property (27), which encodes finite speed of light, was established in Theorem 10.1.1 in [3]. As in [14, Theorem 8.1] the key property, which implies finite speed of light, is the additivity of the integral in the second term in (26).

We can now state a key result of our investigation.

**Theorem 3.4.** Assume the net of local algebras satisfies finite speed of light (in the sense of Definition 3.3). Then the local algebras associated to an interval \( I \subset S^1 \) on the Cauchy surface coincide with those of the free theory, i.e.,

\[
\mathcal{A}(0_I) = \mathcal{A}_\circ(0_I), \quad I \subset S^1.
\]

**Proof.** The proof follows the ideas exposed in the proof of Proposition 2.4. Thus the key step is to show that for any wedge \( W \) which contains \( 0_I \), we have

\[
\mathcal{A}(W) \subset \{ V(h) \mid h \in \mathcal{H}, \text{supp } \Re h \subset I^c, \text{supp } \omega^{-1} \Im h \subset I^c \}''
\]

where \( I^c = S^1 \setminus \overline{I} \). As the edges of \( W \) are necessarily space- or light-like to \( 0_I \), this inclusion follows from finite speed of light as expressed in (27). By duality,

\[
\mathcal{A}(W) \supset \{ V(h) \mid h \in \mathcal{H}, \text{supp } \Re h \subset I, \text{supp } \omega^{-1} \Im h \subset I \}''
\]

whenever \( W \) includes \( 0_I \). \( \square \)
Remark 3.1. The circle $S^1$, which we use to identify the free field and the interacting field, could be replaced by any space-like geodesic $\Lambda S^1$, $\Lambda \in SO_0(1,2)$. The Fock space simply carries two (in fact, infinitely many if one just varies the coupling constants) nets of local algebras, namely $\mathcal{O} \mapsto \mathcal{A}_\circ(\mathcal{O})$ and $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, and one may identify them on any of the space-like geodesic $\Lambda S^1$, $\Lambda \in SO_0(1,2)$.

4 The Haag-Kastler Axioms

Assume we have defined a net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ of local algebras according to Definition 3.2, with the rotations $R_0(\alpha)$, leaving the Cauchy surface $S^1$ invariant, implemented by the kinematic representation $U(R_0(\alpha)) = \Gamma(u(R_0(\alpha)))$, $\alpha \in [0,2\pi)$, and the boosts $\Lambda_1(t)$, leaving the wedge $W_1$ invariant, implemented by the modular group for the pair $(\mathcal{A}_\circ(W_1), \Omega)$, with $\Omega$ a cyclic and separating for $\mathcal{A}_\circ(W_1)$.

Theorem 4.1. In case the net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ respects finite speed of light in the sense of Definition 3.3, it will also satisfy the following Haag-Kastler axioms:

i.) (Isotony). The local algebras satisfy
$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2.$$ Here $\mathcal{O}_1$ and $\mathcal{O}_2$ are either double cones or wedges (but the result extends to arbitrary regions once (28) has been established).

ii.) (Locality). The local algebras satisfy
$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}^{\prime}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2^\prime.$$ Here $\mathcal{O}_2^\prime$ denotes the space-like complement of $\mathcal{O}$ in $dS$ and $\mathcal{A}(\mathcal{O})^\prime$ is the commutant of $\mathcal{A}(\mathcal{O})$ in $B(F)$.

iii.) (Covariance). The representation $U: \Lambda \mapsto U(\Lambda)$ acts geometrically, i.e.,
$$U(\Lambda)\mathcal{A}(\mathcal{O})U(\Lambda)^{-1} = \mathcal{A}(\Lambda\mathcal{O}) \quad \Lambda \in SO_0(1,2).$$

iv.) (Existence and Uniqueness of the Vacuum $\Omega$). There exists a unique (up to a phase$^7$) unit vector in $F$, namely $\Omega$, which

a.) is invariant under the action of $U(SO_0(1,2));$

b.) satisfies the geodesic KMS condition: for every wedge $W = \Lambda W_1$, $\Lambda \in SO_0(1,2)$, the partial state
$$\omega_{\mid \mathcal{A}(W)}(\Lambda) \equiv \langle \Omega, \Lambda \Omega \rangle, \quad \Lambda \in \mathcal{A}(W),$$ satisfies the KMS-condition at inverse temperature $\beta = 2\pi r$ with respect to the one-parameter group $t \mapsto U(\Lambda W(t/r))$, $t \in \mathbb{R}$.

$^7$The phase is uniquely fixed, if one insists that the vector $\Omega$ lies in the natural positive cone $P^\circ(\mathcal{A}_\circ(W_1), \Omega_\circ)$.  

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v.) (Additivity). For a double cone or a wedge, there holds

\[ \mathcal{A}(X) = \bigvee_{O \subset X} \mathcal{A}(O) \, . \]  

(28)

The right hand side denotes the von Neumann algebra generated by the local algebras associated to double cones \(O\) contained in \(X\). (It thus makes sense to define \(\mathcal{A}(X)\) for arbitrary regions \(X\) by Eq. (28).)

v′.) (Weak additivity). For each double cone \(O \subset dS\) there holds

\[ \bigvee_{\Lambda \in SO_0(1,2)} \mathcal{A}(\Lambda O) = \mathcal{A}(dS) \quad (= \mathcal{B}(\mathcal{F})) \, . \]

vi.) (Time-slice axiom [10]). Let \(I\) be an interval on a geodesic Cauchy surface and let \(I''\) be its causal completion. Let \(\Xi \subset I''\) be a neighbourhood of \(I\). Then

\[ \mathcal{A}(\Xi) = \mathcal{A}(I'') \, , \]

where both algebras are defined via Eq. (28). In particular, the algebra of observables located within an arbitrary small time-slice coincides with the algebra of all observables.

Proof. Property i.), isotony, follows directly from the definition given in (24).

Next, let us establish property ii.), locality. If \(O_1\) and \(O_2\) are two space-like separated causally complete, open and bounded regions, then there exists a wedge \(W = \Lambda W\) such that

\[ O_1 \subset W \quad \text{and} \quad O_2 \subset W' . \]

Now the interacting net inherits wedge duality

\[ \mathcal{A}(W') = \mathcal{A}(W)' \]

from the identity \(\mathcal{A}(W'_1) = \mathcal{A}(W_1)'\) using (23). These facts imply locality.

Now, let us prove property iii.), covariance. Let \(\Lambda \in SO_0(1,2)\) be fixed. By construction, the set of all wedges equals \(\{\Lambda W_1 \mid \Lambda \in SO_0(1,2)\}\). Thus

\[ \mathcal{A}(\Lambda O) = \bigcap_{\Lambda O \subset \Lambda W} \mathcal{A}(\Lambda W) = \bigcap_{O \subset W} U(\Lambda) \mathcal{A}(W) U(\Lambda)^{-1} \]

\[ = U(\Lambda) \left( \bigcap_{O \subset W} \mathcal{A}(W) \right) U(\Lambda)^{-1} = U(\Lambda) \mathcal{A}(\emptyset) U(\Lambda)^{-1} , \]

proving covariance.

Property iv.) is established next: existence of the de Sitter vacuum is guaranteed by construction, as the state induced by the vector \(\Omega\) is a thermal state for the Hawking temperature \((2\pi r)^{-1}\) with respect to modular group for the
pair \((A_o(W_1), \Omega)\). By covariance, this property extends to arbitrary wedges. *Uniqueness of the de Sitter vacuum state* was established in [3, Theorem 9.3.7]. It follows from the fact that for the free field the KMS state for the algebra on \(A_o(W_1)\) is unique, hence \(A(W_1) = A_o(W_1)\) (see Definition [3.2] is a factor, and uniqueness of the interacting KMS state now is a direct consequence of [7, Proposition 5.3.29].

Next, we will establish property \(v\). The inclusion

\[
\mathcal{A}(X) \supset \bigvee_{O \subset X} \mathcal{A}(O).
\]

is a consequence of isotony. Moreover, if \(X\) is a double cone, then \(X\) itself is among the double cones on the right hand side, so the inclusion \(\subset\) automatically holds. It remains to prove the inclusion \(\subset\) if \(X\) is a wedge. If \(X = W_1\), then \(A(X) = A_o(X)\), for which the additivity property for the one-particle space (15) implies

\[
\mathcal{A}_o(X) = \bigvee_{R_o(\alpha) \subset I} \mathcal{A}_o(R_o(\alpha)I), \quad \alpha \in [0, 2\pi),
\]

where \(I\) is any (arbitrarily small) interval contained in \(I_+ = W_1 \cap S^1\), whose causal completion \(\partial I = I''\). Thus,

\[
\mathcal{A}(W_1) = \bigvee_{R_o(\alpha) \subset I_+} \mathcal{A}(R_o(\alpha)I) \subset \bigvee_{O \subset X} \mathcal{A}(O).
\]

Thus, for \(X = W_1\) the inclusion \(\subset\) holds. By covariance, it also holds if \(X\) is any other wedge.

Property \(v'\), *weak additivity*, follows form a similar argument: for each double cone \(O \subset dS\) there exists a Lorentz transformation \(\Lambda_0 \in SO_0(1, 2)\) such that \(\partial = \Lambda_0 \partial I\) for some open interval \(I \subset S^1\). Now

\[
\bigvee_{\Lambda \in SO_0(2,1)} \mathcal{A}(\Lambda O) = \bigvee_{\Lambda \in SO_0(2,1)} \mathcal{A}(\Lambda \Lambda_0 O_1)
\]

\[
= \bigvee_{\Lambda \in SO_0(2,1)} \mathcal{A}(\Lambda O_1) \supset \bigvee_{\alpha \in [0, 2\pi)} \mathcal{A}(R_o(\alpha)O_1)
\]

\[
= \bigvee_{\alpha \in [0, 2\pi)} \mathcal{A}_o(R_o(\alpha)O) = \mathcal{B}(F).
\]

Again, the last equality relies on the additivity property for the one-particle space. Hence, property \(v'\), *weak additivity*, is established.

Let us prove property \(vi\), the *time-slice axiom*. As the geodesic Cauchy surfaces are precisely the Lorentz transforms of intervals on the equator \(S^1\) (see, e.g., [22]), it is sufficient to consider an interval \(I \subset S^1\) and a neighbourhood \(\Sigma \subset I''\) of \(I\). If the length of \(I\) is less than \(\pi r\), then \(I''\) is a double cone and
\( \mathcal{A}(I^\prime) = \mathcal{A}_0(I^\prime) \). Pick any double cone \( \mathcal{O} \subset \Xi \) with base on \( S^1 \). Then the additivity property of the free net implies that

\[
\mathcal{A}_0(I^\prime) = \bigvee_{R_0(\alpha) \subset \Xi} \mathcal{A}_0(R_0(\alpha)O), \quad \alpha \in [0, 2\pi).
\]

Now \( \mathcal{A}_0(R_0(\alpha)O) \) coincides with \( \mathcal{A}(R_0(\alpha)O) \), and hence the above identity implies \( \mathcal{A}(I^\prime) \subset \mathcal{A}(\Xi) \). The other inclusion follows from isotony.

If the length of \( I \) is at least \( \pi r \), then additivity implies that \( \mathcal{A}(I^\prime) \) is generated by the wedge algebras \( \mathcal{A}(R_0(\alpha)W_1) = \mathcal{A}_0(R_0(\alpha)W_1), \ R_0(\alpha)W_1 \subset I^\prime \). Hence, as before, \( \mathcal{A}(I^\prime) = \mathcal{A}_0(I^\prime) \). As before, the additivity property of the free net implies that

\[
\mathcal{A}(I^\prime) = \bigvee_{R_0(\alpha) \subset \Xi} \mathcal{A}_0(R_0(\alpha)O), \quad \alpha \in [0, 2\pi).
\]

Now, just as before, \( \mathcal{A}_0(R_0(\alpha)O) \) coincides with \( \mathcal{A}(R_0(\alpha)O) \), and hence the above identity implies \( \mathcal{A}(I^\prime) \subset \mathcal{A}(\Xi) \). The other inclusion follows from isotony.

The local algebras \( \mathcal{A}_0(\mathcal{O}) \) and \( \mathcal{A}_0(\mathcal{W}) \), for double cones and wedges, respectively, are hyper-finite type III factors; see [13].

**Corollary 4.2.** Let \( X \) be either a double cone \( \mathcal{O} \) or a wedge \( W \). It follows that \( \mathcal{A}(X) \) is a hyperfinite type III factor.

**Proof.** For wedges and double cones with base in \( S^1 \) the interacting algebras coincide with the free ones, for which the factor property is known as stated before the Corollary. Any other wedge or double cone can be mapped by a Lorentz transformation to one with base in \( S^1 \), and the claim follows by covariance.

**Remarks 4.1.**

i.) Finite speed of light was used to establish the additivity properties \( v, v' \) as well as \( v' \). The properties i.) to iv.) did not require this property of the representation \( U \) of \( SO_0(1,2) \).

ii.) It has been shown by Borchers and Buchholz that if one assumes the geodesic KMS condition to hold for some \( \beta > 0 \), then automatically \( \beta = 2\pi r \); see [4, Theorem 6.2].

**Theorem 4.3** (Borchers & Buchholz [4]). The following properties hold for any algebraic quantum theory \( \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \) on the de Sitter space, which satisfies the Haag-Kastler axioms i.) - v.) stated in Theorem 4.1.

i.) (The Reeh-Schlieder property). For any open region \( \mathcal{O} \subset dS \) there holds

\[
\mathcal{A}(\mathcal{O})\Omega = \Omega.
\]
ii.) (Wedge duality). For any wedge $W \subset dS$ there holds

$$J_W \mathcal{A}(W) J_W = \mathcal{A}(W)' = \mathcal{A}(W').$$

Here $J_W$ denotes the modular conjugation associated to $(\mathcal{A}(W), \Omega)$.

iii.) (The PCT symmetry). Let $T$ denote the time reflection and $P_1$ the reflection across the $x_0$-$x_1$-plane. Then the modular conjugation $J_{W_1}$ for the wedge $W_1$ is an anti-unitary representer of the reflection

$$TP_1 \in O(1,2),$$

which induces the corresponding action on $U(\text{SO}_0(1,2))$ and $\mathcal{A}(dS)$. A similar result holds for an arbitrary wedge $\Lambda W_1, \Lambda \in \text{SO}_0(1,2)$; the relevant reflection is then $\Lambda TP_1 \Lambda^{-1} \in O(1,2)$.

Proof. i.) is Theorem 3.1 in [4]; ii.) is Proposition 6.1 in [4]; iii.) is Theorem 6.3 in [4].

Remark 4.1. Another result one might want to mention is that uniqueness of the interacting de Sitter vacuum implies that it is weakly mixing with respect to the action of the boosts [4, Corollary 4.4].

5 Summary

In a truly pioneering work [12], published in 1975, Figari, Høegh-Krohn and Nappi constructed the Wightman $n$-point functions of the $\mathcal{P}(\phi)^2$ model in a wedge of de Sitter space-time from the Schwinger functions on the Euclidean sphere. More recently, the authors provided an elementary construction (which ignores domain questions) of this model in its canonical formulation [17]. The main objective of this work was to show that once the $\mathcal{P}(\phi)^2$ model on the de Sitter model has been established in its canonical formulation, a covariant formulation is readily accessible using the machinery developed by Brunetti, Guido and Longo [9]. As the interacting representation of the Lorentz group satisfies finite speed of light, the Haag-Kastler type axioms of Borchers and Buchholz [4] can easily be verified. To the best of our knowledge, this is the first time that a non-perturbative, covariant, interacting quantum field theory satisfying the basic properties encoded in the axioms has been established on a curved space-time.

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Footnote: It was shown in [3] that the operator sum in [26], which involves two operators that are neither bounded from below nor from above, is essentially self-adjoint on its natural domain.
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