Generalized BSDE With 2-Reflecting Barriers and Stochastic Quadratic Growth. Application to Dynkin Games

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Abstract

In this paper we study the existence of a solution for one-dimensional generalized backward stochastic differential equation (GBSDE for short) with two reflecting barriers under weak assumptions on the coefficients. In particular, we construct a maximal solution for such a GBSDE when the terminal condition \( \xi \) is only \( \mathcal{F}_T \)-measurable and the driver \( f \) is continuous with general growth with respect to the variable \( y \) and stochastic quadratic growth with respect to the variable \( z \) without assuming any \( P \)-integrability conditions. The proof of our main result is based on a comparison theorem, an exponential change and an approximation technique.

Finally, we give applications of our result to the Dynkin game problem as well as to the American game option. We prove the existence of a saddle-point for this game under weaker conditions in a general setting.

Keywords: Reflected backward stochastic differential equation; stochastic quadratic growth; comparison theorem; exponential transformation, Dynkin Game, American game option.

AMS Classification (1991): 60H10, 60H20.

1 Introduction

Backward stochastic differential equations (BSDEs for short) have been introduced long time ago by J. B. Bismut [6] both as the equations for the adjoint process in stochastic control as well as the model behind the Black and Scholes formula for the pricing and hedging of options in mathematical finance. However the first published paper on nonlinear BSDEs appeared only in 1990, by Pardoux and Peng [24]. A solution for such an equation is a couple of adapted processes \((Y,Z)\) with values in \(\mathbb{R} \times \mathbb{R}^d\) satisfying

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T.
\]  

(1.1)

In [24], the authors have proved the existence and uniqueness of the solution under conditions including basically the Lipschitz continuity of the generator \( f \). The relevance of BSDEs can be motivated by their

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many applications in mathematical finance, stochastic control and the second order PDE theory (see, for example, [11, 17, 23, 24, 7, 8, 20] and the references therein).

The notion of reflected BSDE has been introduced by El Karoui et al [13]. A solution of such an equation, associated with a coefficient $f$; a terminal value $\xi$ and a barrier $L$, is a triple of processes $(Y, Z, K)$ with values in $\mathbb{R} \times \mathbb{R}^{d} \times \mathbb{R}_{+}$ satisfying:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \quad Y_t \geq L_t \quad \forall t \leq T. \quad (1.2)$$

Here the additional process $K$ is continuous nondecreasing and its role is to push upwards the process $Y$ in order to keep it above the barrier $L$ and moreover it satisfies $\int_0^T (Y_s - L_s) dK_s = 0$, this means that the process $K$ acts only when the process reaches the barrier $L$. The authors of [13] have proved the existence and uniqueness of the solution when $\xi$ is square integrable, $f$ is uniformly Lipschitz with respect to $(y, z)$ and $L$ is square integrable and continuous. They also studied the relation with the obstacle problem for nonlinear parabolic PDE’s. In the Markov framework, the solution $Y$ of reflected BSDE provides a probabilistic formula for the unique viscosity solution of an obstacle problem for a parabolic partial differential equation. El Karoui, Pardoux and Quenez [14] found that the price of an American option is the solution of a RBSDE. Balliy et al [15] found that the price of an American option is the solution of a RBSDE. Balliy et al [15] found that the price of an American option is the solution of a RBSDE.

BSDE with two reflecting barriers has been first introduced by Civitanic and Karatzas [9]. A solution for such an equation, associated with a coefficient $f$; terminal value $\xi$ and two barriers $L$ and $U$, is a quadruple of processes $(Y, Z, K^+, K^-)$ with values in $\mathbb{R} \times \mathbb{R}^{d} \times \mathbb{R}_{+} \times \mathbb{R}_{+}$ satisfying:

$$\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_s dB_s, t \leq T, \\
(ii) & \quad \forall t \leq T, \; L_t \leq Y_t \leq U_t, \\
(iii) & \quad \int_0^T (Y_t - L_t) dK^+_t = \int_0^T (U_t - Y_t) dK^-_t = 0, \; a.s., \\
(iv) & \quad K^+_0 = K^-_0 = 0, \; K^+, K^-, \text{ are continuous nondecreasing.}
\end{align*} \quad (1.3)$$

Here the solution process $Y$ has to remain between $L$ and $U$ due to the cumulative action of processes $K^+$ and $K^-$. In the case of a uniformly Lipschitz coefficient $f$ and a square terminal condition $\xi$ the existence and uniqueness of a solution have been proved when the barriers $L$ and $U$ are either regular or satisfy Mokobodski’s condition which, roughly speaking, turns out into the existence of a difference of nonnegative supermartingales between $L$ and $U$. It has been shown also in [9] that the solution coincides with the value of a stochastic Dynkin game of optimal stopping. The link between obstacle PDEs and RBSDEs has been given in Hamadène and Hassani [19]. In Hamadène [16], applications of RBSDEs to Dynkin games theory as well as to American game option are given.

When the generator $f$ is only continuous there exists a solution to Equation (1.3) under one of the following group of conditions:

- $\xi$ is square integrable, $f$ has a uniform linear growth in $y$ and $z$, i.e. there exists a constant $C$ such that $|f(t, \omega, y, z)| \leq C(1 + |y| + |z|)$, and one of the barriers has to be regular, e.g. has to be semi-martingale (see Hamadène et al [13]).

- $\xi$ is bounded, $f$ has a general growth in $y$ and quadratic growth in $z$, i.e. there exist a constant $C$ and positive function $\phi$ which is bounded on compacts such that $|f(t, \omega, y, z)| \leq C(1 + \phi(|y|) + |z|^2)$, and the barriers satisfy the Mokobodski’s condition (see Bahlali et al [2]).
• $\xi$ is square integrable, $f$ has a uniform linear growth in $y$ and $z$ and the barriers are square integrable and completely separated i.e. $L_t < U_t, \forall t \in [0, T]$ (see Hamadène and Hassani [19]).

Compared to the previous works, the new features here are. First, we consider a more general equation: BSDE with two reflecting barriers $L$ and $U$ which involves integral with respect to a continuous and increasing process $A$. Second, the generator $f$ is continuous with general growth with respect to the variable $y$ and stochastic quadratic growth with respect to the variable $z$ of the form $C_s(\omega) | z |^2$ instead of $C | z |^2$ as usually done. Third, we do not assume the Mokobodski’s condition on the barriers $L$ and $U$, we suppose only that there exists a semimartingale between them. Firth, we do not make any assumptions on the $P$-integrability of the data. Finally, in the Dynkin game problem associated with $L, U, \xi$ and $Q$ with payoff:

$$J(\lambda, \sigma) = U\lambda 1_{\{\lambda < \sigma\}} + L\sigma 1_{\{\lambda > \sigma\}} + Q\sigma 1_{\{\sigma = \lambda < T\}} + \xi 1_{\{\sigma = \lambda = T\}},$$

we provide new results on the existence of a saddle-point for the game in a general setting. We prove also in the appendix a comparison theorem under general assumptions on the coefficients. Roughly speaking, we look for a quintuple of adapted processes $(Y, Z, K^+, K^-)$ satisfying:

$$
\begin{cases}
(i) \ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s)dA_s \\
+ \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_sdB_s, t \leq T, \\
(ii) \ \forall t \leq T, L_t \leq Y_t \leq U_t, \\
(iii) \ \int_0^T (Y_t - L_t)dK^+_t = \int_0^T (U_t - Y_t)dK^-_t = 0, a.s., \\
(iv) \ Y \in C, K^+, K^- \in K, Z \in L^{2,d}, \\
(v) \ dK^+ \perp dK^-,
\end{cases}
$$

under conditions including basically the continuity of the drivers $f$ and $g$ and without assuming any $P$-integrability conditions.

To prove our result we remark that Equation (1.3) can be transformed into a new one whose coefficients are more tractable. Roughly speaking, by using an exponential change, we obtain the following GBSDE:

$$
\begin{cases}
(i) \ \overline{Y}_t = \overline{\xi} + \int_t^T \overline{f}(s, \overline{Y}_s, \overline{Z}_s)ds + \int_t^T \overline{g}(s, \overline{Y}_s)d\overline{A}_s + \int_t^T d\overline{R}_s \\
+ \int_t^T d\overline{K}^+_s - \int_t^T d\overline{K}^-_s - \int_t^T \overline{Z}_sdB_s, t \leq T, \\
(ii) \ \forall t \leq T, \text{ } \overline{L}_t \leq \overline{Y}_t \leq \overline{U}_t, \\
(iii) \ \int_0^T (\overline{Y}_t - \overline{L}_t)d\overline{K}^+_t = \int_0^T (\overline{U}_t - \overline{Y}_t)d\overline{K}^-_t = 0, a.s., \\
(iv) \ \overline{dK}^+ \perp \overline{dK}^-,
\end{cases}
$$

where $\overline{f}$ is negative, $\overline{g}$ is negative and bounded, $\overline{\xi}, \overline{L}$ and $\overline{U}$ are bounded.

We remark first that Equations (1.4) and (1.5) are equivalent. Second, we show the existence and uniqueness of a solution for Equation (1.3) when $f$ and $g$ are bounded Lipschitz functions and the processes $L, U, \xi, A$ and $R$ are bounded. Third, we approximate the functions $\overline{f}$ and $\overline{g}$ by sequences of Lipschitz functions $(f_n)_n$ and $(g_n)_n$ and consider the stopped processes $\overline{dA}^*_n = 1_{\{s \leq \tau_n\}}d\overline{A}_s, n \in \mathbb{N}$ and $\overline{dR}^*_n = 1_{\{s \leq \tau_n\}}d\overline{R}_s, i \in \mathbb{N}$, where $(\tau_n)_{n \geq 0}$ is the family of stopping times defined by $\tau_n = \inf\{s \geq 0 :$
\(\overline{A} + \overline{R} + C + \int_0^T \eta, dr \geq n\} \land T\). We show, with the help of a comparison theorem, that the approximating process \((Y^{n,i}, Z^{n,i}, K^{n,i+}, K^{n,i-})\) converges, in some sense, to the processes \((\overline{Y}, \overline{Z}, \overline{K}^+, \overline{K}^-)\) which is the maximal solution of Equation (1.3). Finally, a logarithm transform leads to the solution of the initial problem.

As a first application we deal with the Dynkin game associated with \(L, U, \xi\) and \(Q\) with payoff:

\[
J(\lambda, \sigma) = U_\lambda 1_{\{\lambda < \sigma\}} + L_\sigma 1_{\{\lambda > \sigma\}} + \xi_1 1_{\{\sigma = \lambda = T\}}.
\]

The setting of this problem is the following. There are two players labelled player 1 and player 2. Player 1 chooses the stopping time \(\lambda\), player 2 chooses the stopping time \(\sigma\), and \(J(\lambda, \sigma)\) represents the amount paid by player 1 to player 2. It is the conditional expectation \(\mathbb{E} \left( F(J(\lambda, \sigma)) \mid \mathcal{F}_t \right)\) of this random payoff that player 1 tries to minimize and player 2 tries to maximize. The game stops when one player decides to stop, that is, at the stopping time \(\lambda \land \sigma\) before time \(T\) or at \(T\) if \(\lambda = \sigma = T\). The problem is to find a fair strategy of stopping times \((\lambda^*, \sigma^*)\) for player 1 and player 2 such that

\[
\mathbb{E} \left( F(J(\lambda^*, \sigma^*)) \right) \leq \mathbb{E} \left( F(J(\lambda, \sigma^*)) \right) \leq \mathbb{E} \left( F(J(\lambda, \sigma^*)) \right), \quad \text{for any stopping times } \lambda, \sigma.
\]

We show that this game is closely related to the notion of the solution of our BSDE as it is done in [9]. Moreover we prove the existence of this fair strategy under conditions out of the scope of the known results on the subject of the connection between RBSDE and Dynkin games, e.g. the barriers \(L\) and \(U\) are just \(L^1\)-integrable.

Our result is applied also in mathematical finance when we deal with American game option or a game contingent claim which is a contract between a seller \(A\) and a buyer \(B\) which can be terminated by \(A\) and exercised by \(B\) at any time \(t \in [0, T]\) up to a maturity date when the contract is terminated anyway. The buyer pays an initial amount which guarantees him a wealth \((L_t)_{t \leq T}\). The buyer can exercise when he wants before the maturity \(T\) of the option. If he decides to exercise at \(\sigma\) he gets an amount \(L_{\sigma}\). On the other hand, if the seller chooses \(\lambda\) as termination time he pays an amount \(U_{\lambda}\). Now if the seller and the buyer decide together to stop the contract at the same time \(\sigma\) then \(B\) gets a reward \(Q_\sigma 1_{\{\sigma < T\}} + \xi_1 1_{\{\sigma = T\}}\).

Let us describe our plan. First, most of the material used in this paper is defined in Section 2, an exponential transformation for our GBSDE with two reflecting barriers is also given. In Section 3, with the help of the comparison theorem and using an approximation technique, we prove the existence of a maximal solution for the transformed BSDE and then equivalently the existence of maximal solution for our GBSDE with two reflecting barriers. In section 4, we give applications to the Dynkin game as well as to the contingent claim game. Finally, an appendix is devoted to the proof of a comparison theorem for a general GBSDE with two reflecting barriers as well as the existence and uniqueness of a solution of Equation (1.3) when the coefficients \(f\) and \(g\) are bounded Lipschitz functions and the processes \(L, U, \xi, A\) and \(R\) are bounded.

## 2 Problem formulation, assumptions and exponential transformation for GBSDE

### 2.1 Assumptions and remarks

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, P)\) be a stochastic basis on which is defined a Brownian motion \((B_t)_{t \leq T}\) such that \((\mathcal{F}_t)_{t \leq T}\) is the natural filtration of \((B_t)_{t \leq T}\) and \(\mathcal{F}_0\) contains all \(P\)-null sets of \(\mathcal{F}\). Note that \((\mathcal{F}_t)_{t \leq T}\)
satisfies the usual conditions, i.e. it is right continuous and complete.

Let us now introduce the following notations:

- \( \mathcal{P} \) the sigma algebra of \( \mathcal{F}_t \)-progressively measurable sets on \( \Omega \times [0, T] \).
- \( \mathcal{C} \) the set of \( \mathbb{R} \)-valued \( \mathcal{P} \)-measurable continuous processes \((Y_t)_{t \leq T}\).
- \( \mathcal{L}^{2,d} \) the set of \( \mathbb{R}^d \)-valued and \( \mathcal{P} \)-measurable processes \((Z_t)_{t \leq T}\) such that
  \[
  \int_0^T |Z_s|^2 ds < \infty, \mathbb{P} \text{-a.s.}
  \]
- \( \mathcal{M}^{2,d} \) the set of \( \mathbb{R}^d \)-valued and \( \mathcal{P} \)-measurable processes \((Z_t)_{t \leq T}\) such that
  \[
  \mathbb{E} \int_0^T |Z_s|^2 ds < \infty.
  \]
- \( \mathcal{K} \) the set of \( \mathcal{P} \)-measurable continuous nondecreasing processes \((K_t)_{t \leq T}\) such that \( K_0 = 0 \) and \( K_T < +\infty, \mathbb{P} \)-a.s.
- \( \mathcal{K} - \mathcal{K} \) the set of \( \mathcal{P} \)-measurable and continuous processes \((V_t)_{t \leq T}\) such that there exist \( V^+, V^- \in \mathcal{K} \) satisfying: \( V = V^+ - V^- \).
- \( \mathcal{K}^2 \) the set of \( \mathcal{P} \)-measurable continuous nondecreasing processes \((K_t)_{t \leq T}\) such that \( K_0 = 0 \) and \( \mathbb{E}K_T^2 < +\infty \).

Throughout the paper we assume that the following conditions hold true:

- \( L := \{L_t, 0 \leq t \leq T\} \) and \( U := \{U_t, 0 \leq t \leq T\} \) are two real valued barriers which are \( \mathcal{P} \)-measurable and continuous processes such that \( L_t \leq U_t, \forall t \in [0, T] \).
- \( \xi \) is an \( \mathcal{F}_T \)-measurable one dimensional random variable such that \( L_T \leq \xi \leq U_T \).
- \( f : \Omega \times [0, T] \times \mathbb{R}^{1+d} \rightarrow \mathbb{R} \) is a function which to \((t, \omega, y, z)\) associates \( f(t, \omega, y, z) \) which is continuous with respect to \((y, z)\) and \( \mathcal{P} \)-measurable.
- \( g : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is a function which to \((t, \omega, y)\) associates \( g(t, \omega, y) \) which is continuous with respect to \( y \) and \( \mathcal{P} \)-measurable.
- \( A \) and \( R \) are two processes in \( \mathcal{K} \) and \( \mathcal{K} - \mathcal{K} \) respectively.

Before giving the definition of our GBSDE with two reflecting barriers \( L \) and \( U \), we need to recall the following definition of two singular measures.

**Definition 2.1.** Let \( \mu_1 \) and \( \mu_2 \) be two positives measures defined on a measurable space \((\Lambda, \Sigma)\), we say that \( \mu_1 \) and \( \mu_2 \) are singular if there exist two disjoint sets \( A \) and \( B \) in \( \Sigma \) whose union is \( \Lambda \) such that \( \mu_1 \) is zero on all measurable subsets of \( B \) while \( \mu_2 \) is zero on all measurable subsets of \( A \). This is denoted by \( \mu_1 \perp \mu_2 \).

Let us now introduce the definition of our GBSDE with two reflecting obstacles \( L \) and \( U \).

**Definition 2.2.** We call \((Y, Z, K^+, K^-) := (Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}\) a solution of the GBSDE with two reflecting barriers, associated with coefficient \(fds + gdA_s + dR_s\); terminal value \(\xi\) and barriers \( L \) and \( U \).
$U$, if the following hold:

\[
\begin{aligned}
(i) \quad & Y_t = \xi + \int_0^T f(s, Y_s, Z_s)ds + \int_0^T dR_s + \int_0^T g(s, Y_s)dA_s \\
& + \int_s^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_s dB_s, t \leq T, \\
(ii) \quad & Y \text{ between } L \text{ and } U, \text{ i.e. } \forall t \leq T, L_t \leq Y_t \leq U_t, \\
(iii) \quad & \text{the Skorohod conditions hold:} \\
& \int_0^T (Y_t - L_t)dK^+_t = \int_0^T (U_t - Y_t)dK^-_t = 0, \text{ a.s.,} \\
(iv) \quad & Y \in \mathcal{C}, K^+, K^- \in \mathcal{K}, Z \in \mathcal{L}^{2,d}, \\
(v) \quad & dK^+ \perp dK^-.
\end{aligned}
\]  

(2.6)

Next, we are going to suppose weaker conditions on the data under which the GBSDE (2.6) has a solution. We shall need the following assumptions on $f$ and $g$:

(A.1) There exist two processes $\eta \in \mathcal{L}^0(\Omega, \mathcal{L}^1([0,T], \mathbb{R}_+))$ and $C \in \mathcal{C}$ such that:

\[\forall (s, \omega), \quad |f(s, \omega, y, z)| \leq \eta_s(\omega) + \frac{C_s(\omega)}{2} |z|^2, \forall y \in [L_s(\omega), U_s(\omega)], \forall z \in \mathbb{R}^d.\]

(A.2) For all $(s, \omega)$, $|g(s, \omega, y)| \leq 1$, $\forall y \in [L_s(\omega), U_s(\omega)]$.

For instance, Equation (2.6) may not have a solution. Take, for example, $L = U$ with $L$ not being a semi-martingale then obviously we can not find a 4-uple which satisfies $ii)$ of Equation (2.6). Therefore, in order to obtain a solution, we are led to assume:

(A.3) There exists a continuous semimartingale $S = S_0 + V^+_t - V^-_t + \int_0^T \alpha_s dB_s$, with $S_0 \in \mathbb{R}, V^+, V^- \in \mathcal{K}$ and $\alpha \in \mathcal{L}^{2,d}$, such that

\[L_t \leq S_t \leq U_t, \forall t \in [0,T].\]

(A.4) $L_t \leq 0 \leq U_t, \forall t \in [0,T]$.

Let us now give some remarks on the assumptions.

**Remark 2.1.** We have the following remarks.

1. It is not difficult to see that if $L$ or $U$ is a continuous semimartingale, then (A.3) holds true. Moreover if the barriers processes $L$ and $U$ are completely separated on $[0,T]$, i.e. $L_t < U_t, \forall t \in [0,T]$, then (A.3) holds also true. Indeed, let $\beta = \sup(\{L_s \mid + \mid U_s\})$. Since $L$ and $U$ are continuous then $\forall t \in [0,T], \left| \frac{U_t}{\beta_t} \right| \leq 1$ and $\left| \frac{L_t}{\beta_t} \right| \leq 1$. It follows then from Hamadène and Hassani [19] that there exists a continuous semimartingale $\overline{S}$ such that $\frac{L_t}{\beta_t} \leq \overline{S}_t \leq \frac{U_t}{\beta_t}, \forall t \in [0,T]$. Hence, the continuous semimartingale $\overline{S} \beta$ is between $L$ and $U$.

2. By taking $Y_t - S_t$ instead of $Y_t$ one can suppose, without loss of generality, that the semimartingale $S = 0$. Hence assumption (A.4) will be assumed instead of (A.3).
3. It should be noted that conditions (A.1) and (A.2) hold true if the functions $f$ and $g$ satisfy the following: \( \forall (s, \omega), \forall y \in [L_s(\omega), U_s(\omega)], \forall z \in \mathbb{R}^d,\)

\[
|f(s, \omega, y, z)| \leq \eta_s(\omega) + \phi(s, \omega, y) + \psi(s, \omega, y)|z|^2,
\]

and

\[
\forall (s, \omega), \ |g(s, \omega, y)| \leq \eta_s(\omega) + \varphi(s, \omega, y), \ \forall y \in [L_s(\omega), U_s(\omega)],
\]

where $\phi$, $\psi$ and $\varphi$ are continuous functions on \([0, T] \times \mathbb{R}\) and progressively measurable, \(\eta \in L^0(\Omega, L^1([0, T], ds, \mathbb{R}_+))\) and \(\tilde{\eta} \in L^0(\Omega, L^1([0, T], dA_s, \mathbb{R}_+)).\) To see that we just take, in condition (A.1), $\eta$ and $C$ as follows:

\[
\eta_t(\omega) = \tilde{\eta}_t(\omega) + \sup_{s \leq t} \sup_{\alpha \in [0, 1]} |\phi(s, \omega, \alpha L_s + (1 - \alpha)U_s)|, \\
C_t(\omega) = 2 \sup_{s \leq t} \sup_{\alpha \in [0, 1]} |\psi(s, \omega, \alpha L_s + (1 - \alpha)U_s)|.
\]

This means that the function $f$ can have, in particular, a general growth in $y$ and quadratic growth in $z$. Now suppose that the driver $g$ satisfies condition (A.2), then for all $(t, \omega)$ we have

\[
|g(t, \omega, y)| \leq \tilde{\eta}_t + \sup_{s \leq t} \sup_{\alpha \in [0, 1]} |\varphi(s, \omega, \alpha L_s + (1 - \alpha)U_s)| := \eta_t(\omega) \leq \tilde{\eta}_t(\omega) + 1.
\]

Now, if you take $\tilde{g}(t, y) = \frac{g(t, y)}{1 + \tilde{\eta}_t}$ and $(1 + \tilde{\eta}_t)dA_t$ instead of $g(t, y)$ and $dA_t$ respectively in equation (2.6), we have (A.2).

2.2 Exponential change for GBSDE with two reflecting obstacles

In this part, by using an exponential change, we transform the GBSDE into a new one whose coefficients are more tractable. This transformation allow us, in particular, to bound the terminal condition and the barriers associated with the transformed GBSDE.

Let $|R|$ be the total variation of the process $R$ and define the processes $m, \overline{\xi}, \overline{T}, \overline{\eta}, \overline{f}$ and $\overline{\bar{f}}$ as follows:

- $m_s = \sup_{r \leq s} |U_r| + 2 \sup_{r \leq s} |C_r| + |R_s| + A_s + 1.$
- $\overline{\xi} = e^{m_T(\xi - m_T)}, \overline{T_s} = e^{m_s(L_s - m_s)}, \overline{U_s} = e^{m_s(U_s - m_s)},$
- $\overline{g}(s, \overline{y}) = \overline{g}(s, (\overline{y} + \overline{T_s}) \wedge \overline{U_s}) - 4m_s,
\] with $\overline{g}(s, \overline{y}) = \overline{g}\left(m_s g(s, \frac{\ln(\overline{y})}{m_s} + m_s) dA_s, m_s dR_s - m_s \frac{\ln(\overline{y})}{m_s}\right), \overline{y} > 0,$
- $\overline{f}(s, \overline{y}, \overline{\tau}) = \overline{f}(s, (\overline{y} + \overline{T_s}) \wedge \overline{U_s}, \overline{\tau}) - \eta_s m_s,
\] with $\overline{f}(s, \overline{y}, \overline{\tau}) = \overline{f}\left(m_s f(s, \frac{\ln(\overline{y})}{m_s} + m_s, \overline{\tau} m_s) - \overline{\tau}^2 \overline{y}^2\right), \overline{y} > 0, \overline{\tau} \in \mathbb{R}^d,$
- $d\overline{A_s} = 8m_s dm_s$ and $d\overline{R_s} = 2d\overline{A_s} + \eta_s m_s ds.$

**Proposition 2.1.** Assume that assumptions (A.1), (A.2) and (A.4) hold. Then we have:

1. \( \forall t \in [0, T], \ \ 0 \leq \overline{T_t} \leq e^{-m^2_t} \leq \overline{\xi}_t \leq e^{-1} < 1 \) and $\overline{\xi}_T \leq \overline{T}_T.$
2. The function \( \widetilde{f} \) is \( \mathcal{P} \)-measurable and continuous with respect to \((y,z)\) satisfying: \( \forall (s,\omega) \in [0,T] \times \Omega, \forall \overline{y} \in [\overline{T}_s(\omega),\overline{U}_s(\omega)], \forall \overline{\sigma} \in \mathbb{R}^d \),

\[ -\eta_s m_s - \frac{[\overline{\sigma}]^2}{L_s} \leq \widetilde{f}(s,\omega,\overline{y},\overline{\sigma}) \leq \eta_s m_s. \quad (2.8) \]

3. For all \( s \in [0,T] \), \( \overline{y} \in \mathbb{R} \) and \( \overline{\sigma} \in \mathbb{R}^d \)

\[ -2\eta_s m_s - \frac{[\overline{\sigma}]^2}{L_s} \leq \overline{f}(s,\overline{y},\overline{\sigma}) \leq 0. \quad (2.9) \]

4. For all \((s,\omega) \in [0,T] \times \Omega \) and \( \overline{y} \in [\overline{T}_s(\omega),\overline{U}_s(\omega)] \)

\[ |\overline{g}(s,\overline{y})| \leq 4m_s, \quad \text{and} \quad -1 \leq \overline{g}(s,\overline{y}) \leq 0. \quad (2.10) \]

5. \( d\overline{R} \) is a positive measure.

Proof. Assertion 1. follows easily from assumption (A.4) and the fact that \( m_1 - 1 \geq U_1, \forall t \in [0,T] \).

Let us prove assertion 2. It is not difficult to see that \( \widetilde{f} \) is \( \mathcal{P} \)-measurable and continuous with respect to \((y,z)\) since \( f \) is. It remains to prove inequality (2.8). Let \((s,\omega) \in [0,T] \times \Omega, \overline{y} \in [\overline{T}_s(\omega),\overline{U}_s(\omega)] \) and \( \overline{\sigma} \in \mathbb{R}^d \), by condition (A.1), we have

\[ \widetilde{f}(s,\omega,\overline{y},\overline{\sigma}) \leq \overline{y} \left( m_s (\eta_s + \frac{C_s}{2m_s} \frac{[\overline{\sigma}]^2}{m_s[\overline{y}]^2}) - \frac{[\overline{\sigma}]^2}{2m_s[\overline{y}]} \right) \]

\[ \leq e^{-1} m_s \eta_s + \left( \frac{C_s}{2m_s} - \frac{1}{2} \right) \frac{[\overline{\sigma}]^2}{[\overline{y}]} \]

\[ \leq e^{-1} m_s \eta_s, \]

since \( \overline{y} \leq \overline{U}_s \leq e^{-1} \) and \( \frac{C_s}{2m_s} - \frac{1}{2} \leq 0, \) \( \forall s \in [0,T] \). On the other hand, by using condition (A.1), we get also that

\[ \overline{f}(s,\omega,\overline{y},\overline{\sigma}) \geq \overline{y} \left( m_s (-\eta_s - \frac{C_s}{2m_s} \frac{[\overline{\sigma}]^2}{m_s[\overline{y}]^2}) + \frac{[\overline{\sigma}]^2}{2m_s[\overline{y}]} \right) \]

\[ \geq -e^{-1} m_s \eta_s - \left( \frac{C_s}{2m_s} + \frac{1}{2} \right) \frac{[\overline{\sigma}]^2}{[\overline{y}]} \]

\[ \geq -e^{-1} m_s \eta_s - \frac{[\overline{\sigma}]^2}{L_s}, \]

since \( \overline{y} \geq \overline{T}_s > 0 \) and \( \frac{C_s}{2m_s} + \frac{1}{2} \leq 1, \) \( \forall s \in [0,T] \).

Inequality (2.8) follows easily from inequality (2.9) and then we get assertion 3. holds.

Assertions 4. and 5. follow immediately from assumption (A.2) and the definition of \( m \).

Suppose now that Equations (2.6) has a solution \((Y,Z,K^+,K^-)\) and define the processes \( \overline{Y}, \overline{Z}, \overline{K}^+, \) and \( \overline{K}^- \) as follows:

\[ \overline{Y}_s = e^{m_s(Y_m - m_s)} \]

\[ \overline{Z}_s = m_s \overline{Y}_s \]

\[ d\overline{K}^+_s = m_s \overline{Y}_s \overline{K}^+_s \]

\[ d\overline{K}^-_s = m_s \overline{Y}_s \overline{K}^-_s \]

(2.11)
Then \((\bar{Y}, \bar{Z}, \bar{K}^+, \bar{K}^-)\) is satisfying the following GBSDE

\[
\begin{align*}
(i) \quad \bar{Y}_t &= \bar{x} + \int_t^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s)ds + \int_t^T \bar{g}(s, \bar{Y}_s)dA_s + \int_t^T d\bar{R}_s \\
& \quad + \int_t^T d\bar{K}_s^+ - \int_t^T d\bar{K}_s^- - \int_t^T \bar{Z}_s dB_s, \quad t \leq T, \\
(ii) \quad \forall t \leq T, \quad \bar{L}_t \leq \bar{Y}_t \leq \bar{U}_t, \\
(iii) \quad \int_0^T (\bar{Y}_t - \bar{L}_t)d\bar{K}_t^+ = \int_0^T (\bar{U}_t - \bar{Y}_t)d\bar{K}_t^- = 0, \text{ a.s.} \\
(iv) \quad \bar{Y} \in \mathcal{C}, \quad \bar{K}^+, \bar{K}^- \in \mathcal{K}, \quad \bar{Z} \in \mathcal{L}^{2, d}, \\
(v) \quad d\bar{K}^+ \perp d\bar{K}^-,
\end{align*}
\]

where \(\bar{x}, \bar{f}, \bar{g}, \bar{R}, \bar{L}\) and \(\bar{U}\) are given above. More precisely, we have the following.

**Proposition 2.2.** Equations \((2.6)\) and \((2.12)\) are equivalent, in the sense that if there exists a solution (resp. maximal solution) to one of them then there exists a solution (resp. maximal solution) for the other.

**Proof.** Suppose that Equation \((2.6)\) has a solution (resp. maximal solution), say \((Y, Z, K^+, K^-)\). It follows from Itô’s formula that

\[
e^{m_t(Y_t-m_t)} = e^{m_T(Y_T-m_T)} + \int_t^T m_s e^{m_s(Y_s-m_s)}(f(s, Y_s, Z_s)ds + dK_s^+ - dK_s^- - Z_s dB_s) + \int_t^T m_s e^{m_s(Y_s-m_s)}g(s, Y_s)dA_s + \int_t^T m_s e^{m_s(Y_s-m_s)}dm_s.
\]

Henceforth

\[
e^{m_t(Y_t-m_t)} = e^{m_T(Y_T-m_T)} + \int_t^T m_s e^{m_s(Y_s-m_s)}(f(s, Y_s, Z_s)ds - \frac{1}{2} |m_s Z_s|^2)ds \\
+ \int_t^T m_s e^{m_s(Y_s-m_s)}(2m_s - Y_s)dm_s + \int_t^T m_s e^{m_s(Y_s-m_s)}dK_s^+ \\
- \int_t^T m_s e^{m_s(Y_s-m_s)}dK_s^- - \int_t^T e^{m_s(Y_s-m_s)}m_s Z_s dB_s \\
+ \int_t^T \left(e^{m_s(Y_s-m_s)}m_s g(s, Y_s) \frac{dA_s}{dm_s}\right) dm_s + \int_t^T e^{m_s(Y_s-m_s)}m_s \frac{dR_s}{dm_s} dm_s.
\]

Then it is clear that \((\bar{Y}, \bar{Z}, \bar{K}^+, \bar{K}^-)\), defined by \((2.11)\) and associated with coefficient \(f ds + g dB + dA\), is a solution (resp. maximal solution) of Equation \((2.12)\). Conversely, Suppose that there exists a solution (resp. maximal solution) \((\bar{Y}, \bar{Z}, \bar{K}^+, \bar{K}^-)\) for Equation \((2.12)\). Hence, by setting, for all \(t \leq T\)

\[
Y_t = \frac{\ln(Y_t)}{m_t} + m_t, \quad Z_t = \frac{Z_t}{m_t Y_t}, \quad dK_t^+ = \frac{dR_t^+}{m_t Y_t},
\]

one can see that \((Y, Z, K^+, K^-)\) is a solution (resp. maximal solution) for Equation \((2.6)\).
Now, by taking advantage of Propositions 2.1 and 2.2, our problem is then reduced to find a maximal solution to the following GBSDE:

\[
\begin{cases}
(i) \quad Y_t = \xi + \int_0^T f(s, Y_s, Z_s) \, ds + \int_0^T g(s, Y_s) \, dA_s + \int_0^T dR_s \\
\quad \quad \quad \quad \quad \quad + \int_0^T (dK^+_s - dK^-_s) - \int_0^T Z_s \, dB_s, \quad t \leq T, \\
(ii) \quad \forall t \leq T, \quad L_t \leq Y_t \leq U_t, \\
(iii) \quad \int_0^T (Y_t - L_t) \, dK^+_t = \int_0^T (U_t - Y_t) \, dK^-_t = 0, \text{ a.s.} \\
(iv) \quad Y \in \mathcal{C}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \\
(v) \quad dK^+ \perp dK^-, 
\end{cases}
\]  
(2.13)

under the following assumptions:

(H.0) \( dR \geq 0 \), i.e. \( R \in \mathcal{K} \).

(H.1) There exist two processes \( \eta \in L^0(\Omega, L^1([0, T], \mathbb{R}^+)) \) and \( C \in \mathcal{C} \) such that:

\[
\forall (s, \omega), \quad -\eta_s(\omega) - \frac{C_s(\omega)}{2} \leq f(s, \omega, y, z) \leq 0, \quad \forall y \in \mathbb{R}, \quad \forall z \in \mathbb{R}^d. 
\]

(H.2) \( \forall (s, \omega), \quad -1 \leq g(s, \omega, y, z) \leq 0, \quad \forall y \in \mathbb{R} \).

(H.3) \( 0 < L_t \leq U_t < 1, \quad \forall t \in [0, T] \).

(H.4) There exists a continuous nondecreasing process \( S = S_0 - V \), with \( S_0 \in \mathbb{R}, V \in \mathcal{K} \), such that \( L_t \leq S_t \leq U_t, \quad \forall t \in [0, T] \).

We devote the next section to the existence of maximal solution for GBSDE (2.13) and then equivalently to the existence of maximal solution for GBSDE (2.16).

3  Existence of maximal solution for GBSDE (2.13)

Our objective now is to prove that, under assumptions (H.0)–(H.4), Equation (2.13) has a maximal solution \((Y_t, Z_t, K_t^+, K_t^-)_{t \leq T}\), in the sense that for any other \((Y'_t, Z'_t, K'_t^+, K'_t^-)_{t \leq T}\) of (2.13) we have for all \( t \leq T, Y_t \geq Y'_t, \) \( P \)-a.s.

3.1 Approximations

What we would like to do is to approximate the function \( f \) and \( g \) by sequences of functions \( f_n \) and \( g_n \) which satisfies properties 1 – 7 below. We also approximate the processes \( A \) and \( R \) by a sequences of processes \( A^n \) and \( R^n \). With the help of this double approximations, we can construct a maximal solution for Equation (2.13).

It is not difficult to prove the following lemma which gives an approximation of continuous functions by Lipschitz functions.

Lemma 3.1. Let \( f_n \) and \( g_n \) be two sequences of functions defined by

\[
f_n(t, y, z) = \sup_{p \in \mathbb{R}, q \in \mathbb{R}^d} \{ f(t, p, q) \vee (-n) - p - y - n|q - z| \},
\]

(3.14)
The following result follows easily from the Comparison theorem (Theorem A.1 in Appendix).

Assume that assumptions (H.0)–(H.4) hold. Then we have the following:

1. For all \((t, \omega, y, z, n) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{N}\),
   
   \[ f_0(t, y, z) = 0 \geq f_n(t, y, z) \geq f_{n+1}(t, y, z) \geq f(t, y, z) \geq -\eta - \frac{C_t}{2} |z|^2. \]

2. For all \((t, \omega, y, n) \in [0, T] \times \Omega \times \mathbb{R} \times \mathcal{N}\),
   
   \[ g_0(t, y) = 0 \geq g_n(t, y) \geq g_{n+1}(t, y) \geq g(t, y) \geq -1. \]

3. For all \((t, \omega, y, z, n) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{N}\),
   
   \[ -n \leq f_n(s, y, z) \leq 0. \]

4. \(f_n\) is uniformly \(n\)-Lipschitz with respect to \((y, z)\).

5. \(g_n\) is uniformly \(n\)-Lipschitz with respect to \(y\).

6. For all \((t, \omega) \in [0, T] \times \Omega\), \((f_n(t, y, z))_{n \geq 0}\) converges to \(f(t, y, z)\) as \(n\) goes to \(+\infty\) uniformly on every compact of \(\mathbb{R} \times \mathbb{R}^d\).

7. For all \((t, \omega) \in [0, T] \times \Omega\), \((g_n(t, y))_{n \geq 0}\) converges to \(g(t, y)\) as \(n\) goes to \(+\infty\) uniformly on every compact of \(\mathbb{R}\).

Let \((\tau_n)_{n \geq 0}\) be the family of stopping times defined by

\[
\tau_n = \inf\{s \geq 0 : A_s + R_s + C_s + \int_0^s \eta_s dr \geq n\} \wedge T. \tag{3.15}
\]

Set \(dA^n_s = 1_{\{s \leq \tau_n\}} dA_s, n \in \mathcal{N}\) and \(dR^i_s = 1_{\{s \leq \tau_i\}} dR_s, i \in \mathcal{N}\) and consider the following BSDE with two reflecting barriers

\[
\begin{cases}
(i) & Y^{n,i}_t = \xi + \int_0^T f_n(s, Y^{n,i}_s, Z^{n,i}_s)ds + \int_0^T g_n(s, Y^{n,i}_s)dA^n_s + \int_0^T dR^i_s \\
 & + \int_t^T dK^{n,i}_s - \int_t^T dK'^{n,i}_s - \int_t^T Z^{n,i}_s dB_s, t \leq T, \\
(ii) & \forall t \leq T, L_t \leq Y^{n,i}_t \leq U_t, \\
(iii) & \int_0^T (Y^{n,i}_t - L_t)dK^{n,i}_s = \int_0^T (U_t - Y^{n,i}_t)dK'^{n,i}_s = 0, \text{ a.s.} \\
(iv) & Y^{n,i}_T \in \mathcal{C}, K^{n,i}_T, K'^{n,i}_T \in \mathcal{K}, Z^{n,i} \in \mathbb{L}^{2,d}, \\
(v) & dK^{n,i}_s \perp dK'^{n,i}_s.
\end{cases} \tag{3.16}
\]

It follows from Theorem 3.1 (see Appendix) that Equation (3.17) has a unique solution. Moreover, for all \(n\) and \(i\)

\[
\mathbb{E} \sup_{t \leq T} |Y^{n,i}_t|^2 + \mathbb{E} \int_0^T |Z^{n,i}_s|^2 ds + \mathbb{E}(K^{n,i}_T)^2 < +\infty. \tag{3.18}
\]

The following result follows easily from the Comparison theorem (Theorem A.1 in Appendix).
Proposition 3.1. Let us suppose that assumptions (H.0)-(H.4) hold. Then we have the following.

i) Fix $n$, we get for all $i \geq 0$ and $t \leq T$
\[ L_t \leq Y_t^{n,i} \leq Y_t^{n,i+1} \leq U_t, \quad dK_t^{n,i+} \geq dK_t^{n,i+1} \quad \text{and} \quad dK_t^{n,i+1} \geq dK_t^{n,i-}. \]

ii) Fix $i$, we get for all $n \geq 0$ and $t \leq T$
\[ L_t \leq Y_t^{n+1,i} \leq Y_t^{n,i} \leq U_t, \quad dK_t^{n,i+} \leq dK_t^{n+1,i} \quad \text{and} \quad dK_t^{n+1,i} \leq dK_t^{n,i-}. \]

Proof. Since the family of stopping times $(\tau_i)_{i \geq 0}$ is increasing then, for all $i \geq 0$, $dR^i \leq dR^{i+1}$. The assertion i) follows by using inequality (3.18) and Theorem A.1 in Appendix. Assertion ii) follows also by using Lemma 3.1 and Theorem A.1 in Appendix.

3.2 The study of Equation (3.17) for $n$ fixed

What we want to do here is to study Equation (3.17) when $n$ is fixed. Let us set
• $Y^n = \sup_i Y_t^{n,i}$
• $dK^n = \sup dK_t^{n,i}$ which is a positive measure.
• $dK^n = \inf dK_t^{n,i}$ which is also a positive measure since $K_{T}^{n,0} < +\infty, \ P-a.s.$

Proposition 3.2. Assume that assumptions (H.0)-(H.4) hold. Then we have the following.

1. There exists a process $Z^n \in \mathcal{L}^{2,d}$ such that, for all $j \in \mathbb{N}$,
\[ \mathbb{E} \int_0^{t_j} |Z_s^{n,i} - Z_s^{n}|^2 ds \to 0, \quad \text{as } i \text{ goes to infinity}. \]

2. The process $(Y^n, Z^n, K^{n+}, K^{n-})$ is the unique solution of the following GBSDE with two reflecting barriers
\[
\begin{cases}
(i) & Y_t^n = \xi + \int_0^T f_n(s, Y_s^n, Z_s^n)ds + \int_0^T g_n(s, Y_s^n)dA_s^n + \int_0^T dR_s \\
 & \quad + \int_0^T dK_t^{n+} - \int_0^T dK_t^{n-} - \int_0^T Z_s^n dB_s, \ t \leq T, \\
(ii) & \forall t \leq T, \ L_t \leq Y_t^n \leq U_t, \\
(iii) & \int_0^T (Y_t^n - L_t)dK_t^{n+} = \int_0^T (U_t - Y_t^n)dK_t^{n-} = 0, \ a.s., \\
(iv) & Y^n \in \mathcal{C} \quad K^{n+}, K^{n-} \in \mathcal{K} \quad Z^n \in \mathcal{L}^{2,d}, \\
(v) & dK^{n+} \perp dK^{n-}.
\end{cases}
\]

Proof. 1. Let $j, i, i' \in \mathbb{N}$ such that $j \leq i \leq i'$ and $t \in [0, \tau_j]$ where $\tau_j$ is defined in (3.16). Since the family of stopping times $(\tau_j)_{j \geq 0}$ is increasing, we have
\[ \int_t^{\tau_j} (dR^i_s - dR^{i'}_s) = \int_t^{\tau_j \wedge \tau_i} dR_s - \int_t^{\tau_j \wedge \tau_{i'}} dR_s = \int_t^{\tau_j} dR_s - \int_t^{\tau_j} dR_s = 0 \]
Moreover, by taking advantage of the fact that $f_n$ and $g_n$ are $n$–Lipschitz, we get

\[
Y_t^{n,i} - Y_t^{n,i'} = Y_{\tau_j}^{n,i} - Y_{\tau_j}^{n,i'} + \int_{\tau_j}^t (f_n(s, Y_s^{n,i}, Z_s^{n,i}) - f_n(s, Y_s^{n,i'}, Z_s^{n,i'}))ds + \int_{\tau_j}^t (g_n(s, Y_s^{n,i}) - g_n(s, Y_s^{n,i'}))dA_s + \int_{\tau_j}^t (dK_s^{n,i} - dK_s^{n,i'}) - \int_{\tau_j}^t (Z_s^{n,i} - Z_s^{n,i'})dB_s,
\]

where $C^{n,i,i'}, \alpha^{n,i,i'}$ and $\beta^{n,i,i'}$ are bounded by $n$ and $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^d$. Set $e_t^n := e^{2(A_t^n - (2n^2 + 2n)t)}$. Applying Itô’s formula to $(Y_t^{n,i} - Y_t^{n,i'})^2 e_t^n$, we get

\[
(Y_t^{n,i} - Y_t^{n,i'})^2 e_t^n + \int_{\tau_j}^t e_s^n|Z_s^{n,i} - Z_s^{n,i'}|^2 ds
\]

\[
+ \int_{\tau_j}^t e_s^n (2n(Y_s^{n,i} - Y_s^{n,i'})^2 dA_s + (2n^2 + 2n)(Y_s^{n,i} - Y_s^{n,i'})^2 ds)
\]

\[
= (Y_{\tau_j}^{n,i} - Y_{\tau_j}^{n,i'})^2 e_{\tau_j}^n + 2 \int_{\tau_j}^t e_s^n(Y_s^{n,i} - Y_s^{n,i'})\left(\alpha_s^{n,i,i'}(Y_s^{n,i} - Y_s^{n,i'}) + \langle \beta_s^{n,i,i'}, Z_s^{n,i} - Z_s^{n,i'} \rangle \right) ds
\]

\[
+ 2 \int_{\tau_j}^t C_s^{n,i,i'} e_s^n(Y_s^{n,i} - Y_s^{n,i'})^2 dA_s + \int_{\tau_j}^t e_s^n(Y_s^{n,i} - Y_s^{n,i'})(dK_s^{n,i} - dK_s^{n,i'})
\]

\[
- \int_{\tau_j}^t e_s^n(Y_s^{n,i} - Y_s^{n,i'})dK_s^{n,i} - dK_s^{n,i'} - 2 \int_{\tau_j}^t e_s^n(Y_s^{n,i} - Y_s^{n,i'})(Z_s^{n,i} - Z_s^{n,i'}, dB_s).
\]

In force of Proposition 3.3, we may conclude that for $t \in [0, \tau_j]$,

\[
\int_{\tau_j}^t e_s^n(Y_s^{n,i} - Y_s^{n,i'})(dK_s^{n,i} - dK_s^{n,i'}) \leq 0 \quad \text{and} \quad -\int_{\tau_j}^t e_s^n(Y_s^{n,i} - Y_s^{n,i'})(dK_s^{n,i} - dK_s^{n,i'}) \leq 0.
\]

and

\[
2 \int_{\tau_j}^t e_s^n(Y_s^{n,i} - Y_s^{n,i'})(\langle \beta_s^{n,i,i'}, Z_s^{n,i} - Z_s^{n,i'} \rangle ds
\]

\[
\leq 2n^2 \int_{\tau_j}^t e_s^n(Y_s^{n,i} - Y_s^{n,i'})^2 ds + \frac{1}{2} \int_{\tau_j}^t e_s^n|Z_s^{n,i} - Z_s^{n,i'}|^2 ds.
\]

Hence it follows that

\[
(Y_{\tau_j}^{n,i} - Y_{\tau_j}^{n,i'})^2 e_{\tau_j}^n + \frac{1}{2} \int_{\tau_j}^t e_s^n|Z_s^{n,i} - Z_s^{n,i'}|^2 ds
\]

\[
\leq (Y_{\tau_j}^{n,i} - Y_{\tau_j}^{n,i'})^2 e_{\tau_j}^n - 2 \int_{\tau_j}^t e_s^n(Y_s^{n,i} - Y_s^{n,i'})(Z_s^{n,i} - Z_s^{n,i'}, dB_s).
\]

According to Bulkholder-Davis-Gundy inequality, there exists a constant $C_n > 0$ such that

\[
\mathbb{E} \left[ \sup_{t \leq \tau_j} (Y_s^{n,i} - Y_s^{n,i'})^2 + \frac{1}{2} \int_0^{\tau_j} |Z_s^{n,i} - Z_s^{n,i'}|^2 ds \right] \leq C_n \mathbb{E} (Y_{\tau_j}^{n,i} - Y_{\tau_j}^{n,i'})^2.
\]
In view of assumption $(H.3)$ and Lebesgue’s dominated convergence theorem we have $\forall j \in \mathbb{N}$,

$$\mathbb{E} \left[ Y_{\tau_j}^{n,i} - Y_{\tau_j}^{n,i'} \right]^2 \to 0, \quad \text{as } i, i' \text{ goes to } +\infty.$$ 

As a consequence, there exists $Z^n \in \mathcal{L}^{2,d}$ such that $\forall j \in \mathbb{N}$

$$\mathbb{E} \int_0^{\tau_j} |Z_s^{n,i} - Z_s^{n,i'}|^2 ds \to 0, \quad \text{as } i \text{ goes to } +\infty.$$ 

Assertion 1. is then proved.

2. Let us first prove assertion $(iv)$ of Equation $(3.19)$. In view of passing to the limit in Equation $(3.20)$ we get

$$\mathbb{E} \left[ \sup_{t \leq \tau_j} (Y_{t}^{n,i} - Y_{t}^{n,i'})^2 \right] \to 0, \quad \text{as } i \text{ goes to } +\infty,$$

and then we can conclude that $Y^n$ is continuous, i.e. $Y^n \in \mathcal{C}$.

It is clear, from Proposition $(3.1)$, that $K^{n,i+}$ converges to the continuous and increasing process $K^{n,+}$. Moreover, $\mathbb{E}(K_T^{n,+})^2 \leq \mathbb{E}(K_T^{n,0,+})^2 < +\infty$, for all $n$. Therefore $K^{n,+} \in \mathcal{K}$.

Now, passing to the limit in Equation $(3.17)(i)$ on $[0, \tau_j]$, we get also that $\mathbb{E}(K_T^{n,j} - K_T^{n,j-}) < +\infty$, $\forall j,n$. Since $P(\bigcup_{j=1}^{\infty}(\tau_j = T]) = 1$, we get $K_T^{n,j} < +\infty$, $\forall n$ P-a.s. Then $K^{n,j} \in \mathcal{K}$. Consequently, assertion $(iv)$ of Equation $(3.19)$ is proved.

Let us now show that $(Y^n, Z^n, K^{n,+}, K^{n,-})$ satisfies $(i)$. In view of passing to the limit, as $i$ goes to infinity, in the following equation

$$Y_t^{n,i} = Y_{\tau_j}^{n,i} + \int_t^{\tau_j} f_n(s, Y_s^{n,i}, Z_s^{n,i}) ds + \int_t^{\tau_j} g_n(s, Y_s^{n,i}) dA_s^n + \int_t^{\tau_j} dR_s^n + \int_t^{\tau_j} dK^{n,i+}_s - \int_t^{\tau_j} dK^{n,i-}_s - \int_t^{\tau_j} Z_s^{n,i} dB_s,$$

We obtain, $P-$a.s.

$$Y_t^n = Y_{\tau_j}^{n} + \int_t^{\tau_j} f_n(s, Y_s^n, Z_s^n) ds + \int_t^{\tau_j} g_n(s, Y_s^n) dA_s^n + \int_t^{\tau_j} dR_s^n + \int_t^{\tau_j} dK^{n,+}_s - \int_t^{\tau_j} dK^{n,-}_s - \int_t^{\tau_j} Z_s^n dB_s,$$

Since $\tau_j$ is a stationary stopping time we get $P-$a.s.

$$Y_t^n = \xi + \int_0^t f_n(s, Y_s^n, Z_s^n) ds + \int_0^t g_n(s, Y_s^n) dA_s^n + \int_0^t dR_s^n + \int_0^t dK^{n,+}_s - \int_0^t dK^{n,-}_s - \int_0^t Z_s^n dB_s,$$

Hence the process $(Y^n, Z^n, K^{n,+}, K^{n,-})$ satisfies $(i)$ of Equation $(3.19)$.

We now prove that the Skorohod conditions $(iii)$ of Equation $(3.19)$ is satisfied. Since $Y^n = \sup_i Y_i^n$, we have

$$0 \leq \int_0^T (U_t - Y_t^n)dK_t^{n,i-} \leq \int_0^T (U_t - Y_t^{n,i})dK_t^{n,i-} = 0.$$ 

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Therefore
\[ \int_0^T (U_t - Y_t^n) dK_t^{n,i} = 0. \]

It follows then from Fatou’s lemma that
\[ 0 \leq \int_0^T (U_t - Y_t^n) dK_t^n \leq \liminf_i \int_0^T (U_t - Y_t^n) dK_t^{n,i} = 0. \]

Consequently
\[ \int_0^T (U_t - Y_t^n) dK_t^n = 0. \]

On the other hand
\[ 0 \leq \int_0^T (Y_t^{n,i} - L_t) dK_t^{n,i} \leq \int_0^T (Y_t^{n,i} - L_t) dK_t^{n,i} = 0. \]

Hence
\[ \int_0^T (Y_t^{n,i} - L_t) dK_t^{n,i} = 0. \]

Applying Fatou’s lemma we obtain
\[ 0 \leq \int_0^T (Y_t^n - L_t) dK_t^n \leq \liminf_i \int_0^T (Y_t^{n,i} - L_t) dK_t^{n,i} = 0. \]

Henceforth
\[ \int_0^T (Y_t^n - L_t) dK_t^n = 0. \]

Now, since \( dK^{n,+} = \inf_i dK^{n,i,+} \), \( dK^{n,-} = \sup_i dK^{n,i,-} \) and the measures \( dK^{n,i,+} \) and \( dK^{n,i,-} \) are singular, it follows that \( dK^{n,+} \) and \( dK^{n,-} \) are also singular. The proof of Proposition 3.2 is finished.

\[ \square \]

### 3.3 The study of the limit Equation (3.19)

Let \((Y^n, Z^n, K^{n,+}, K^{n,-})\) be the process given in Proposition 3.2 the unique solution of the GBSDE with two reflecting barriers (3.19).

We should recall here that by using standard techniques of BSDE, we get that for all \( j \) and \( n \),
\[ \mathbb{E}(K_T^{n,+})^2 + \mathbb{E} \int_0^T |Z^n_s|^2 \, ds < +\infty \quad (3.21) \]
and then \( \mathbb{E}(K_T^{n,-})^2 < +\infty \), for all \( j \) and \( n \).

In view of passing to the limit in Proposition 3.1, we get the following.

**Proposition 3.3.** For all \( n \geq 0 \), we obtain
\[ Y^{n+1} \leq Y^n. \]

Moreover, \( \forall n \geq 0 \)
\[ dK_t^{n,+} \leq dK_t^{n+1,+} \quad \text{and} \quad dK_t^{n,+} \leq dK_t^{n,-}. \]
In order to study Equation (3.19), let us set

- \( Y = \inf Y^n \).
- \( dK^+_n = \sup dK^{n+} \), which is also a positive measure.
- \( dK^-_n = \inf dK^{n-} \), which is also a positive measure since \( K_T^n < +\infty, P - a.s. \)  

The following result states the convergence of the process \( Z^n \) in \( L^2([0, T] \times \Omega) \).

**Proposition 3.4.** Assume that assumptions (H.0)–(H.4) hold. Then there exists a process \( Z \in L^2 \) such that, for all \( j \),

\[
\mathbb{E} \int_0^{\tau_j} |Z^n_t - Z_t|^2 ds \rightarrow 0, \quad \text{as } n \text{ goes to infinity.}
\]

**Proof.** For \( s \in [0, 1] \) and \( j \in \mathbb{N} \), let us set \( \psi(s) = \frac{e^{12js} - 1}{12j} - s \). We mention that \( \psi \) satisfies the following for all \( s \in [0, 1] \),

\[
\psi'(s) = e^{12js} - 1, \quad \psi''(s) = 12je^{12js} = 12j\psi'(s) + 12j
\]

\[
0 \leq 12js \leq \psi'(s) \leq 12je^{12js} \leq 12je^{12j}.
\]

Recall that \( dA^n_t = 1_{\{s \leq \tau_n\}} dA_s \), for \( n \in \mathbb{N} \). Applying Itô’s formula to \( \psi(Y^n - Y^m) \), we get for \( m \geq n \) and \( t \leq \tau_j \),

\[
\psi(Y^n_t - Y^m_t) = \psi(Y^n_{\tau_j} - Y^m_{\tau_j}) - \int_{\tau_j}^t g_m(s, Y^m_s)\psi'(Y^n_s - Y^m_s)1_{\{\tau_n \leq s \leq \tau_m\}} dA_s
\]

\[
+ \int_{\tau_j}^t (\psi'(Y^n_s - Y^m_s)g_m(s, Y^m_s) - g_m(s, Y^m_s)\psi'(Y^n_s - Y^m_s))1_{\{\tau_n \leq s \leq \tau_m\}} dA_s
\]

\[
+ \int_{\tau_j}^t (f_m(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s))\psi'(Y^n_s - Y^m_s) ds
\]

\[
+ \int_{\tau_j}^t \psi'(Y^n_s - Y^m_s)(K^n_s - K^m_s) ds
\]

\[
= \int_{\tau_j}^t \psi'(Y^n_s - Y^m_s)(Z^n_s - Z^m_s) dB_s - \frac{1}{2} \int_{\tau_j}^t \psi'(Y^n_s - Y^m_s) |Z^n_s - Z^m_s|^2 ds.
\]

Since, \( K^{n+} \) (resp. \( K^{m+} \)) moves only when \( Y^n \) (resp. \( Y^m \)) reaches the obstacles \( L, Y^m \leq Y^n \) and \( \psi'(0) = 0 \), we have

\[
\int_{\tau_j}^t \psi'(Y^n_s - Y^m_s)(K^n_s - K^m_s) ds = - \int_{\tau_j}^t \psi'(Y^n_s - L_s) dK_s^{n+}.
\]

By the same way we get also that

\[
\int_{\tau_j}^t \psi'(Y^n_s - Y^m_s)(K^n_s - K^m_s) ds = \int_{\tau_j}^t \psi'(Y^n_s - Y^m_s) dK_s^{n-}.
\]
Henceforth

\[
\psi(Y^n_t - Y^m_t) = \psi(Y^n_{r_j} - Y^m_{r_j}) - \int_{r_j}^{T_j} g_m(s, Y^n_s)^2 \psi'(Y^n_s - Y^m_s) \mathbb{1}_{\{s \leq r_m\}} ds + \int_{r_j}^{T_j} \left( g_n(s, Y^n_s) - g_m(s, Y^m_s) \right) \psi'(Y^n_s - Y^m_s) \mathbb{1}_{\{s \leq r_m\}} ds
\]

where \( n, m, j \) are positive measures depending on \( n, m \) and \( j \).

In view of Lemma 3.1 (assertions 1 and 2) and Inequality (3.23), we get

\[
\int_{r_j}^{T_j} \left( g_n(s, Y^n_s) - g_m(s, Y^m_s) \right) \psi'(Y^n_s - Y^m_s) \mathbb{1}_{\{s \leq r_m\}} ds \\
\leq - \int_{r_j}^{T_j} g_m(s, Y^m_s)^2 \psi'(Y^n_s - Y^m_s) \mathbb{1}_{\{s \leq r_m\}} ds \\
\leq \int_{r_j}^{T_j} \psi'(Y^n_s - Y^m_s) \mathbb{1}_{\{s \leq r_m\}} ds \\
\leq 12 j \int_{r_j}^{T_j} e^{12j}(Y^n_s - Y^m_s) dA_s,
\]

and

\[
\int_{r_j}^{T_j} \left( f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s) \right) \psi'(Y^n_s - Y^m_s) ds \\
\leq - \int_{r_j}^{T_j} f_m(s, Y^m_s, Z^m_s) \psi'(Y^n_s - Y^m_s) \mathbb{1}_{\{s \leq r_m\}} ds \\
\leq \int_{r_j}^{T_j} \left( \eta_s + \frac{C_s}{2} |Z_s|^2 \right) \psi'(Y^n_s - Y^m_s) \mathbb{1}_{\{s \leq r_m\}} ds \\
\leq 12 j \int_{r_j}^{T_j} e^{12j} \eta_s (Y^n_s - Y^m_s) ds + \frac{1}{2} \int_{r_j}^{T_j} |Z_s|^2 \psi'(Y^n_s - Y^m_s) ds.
\]

It follows then that

\[
\psi(Y^n_t - Y^m_t) + 12 j \int_{r_j}^{T_j} (Y^n_t - L_s) dK^{m+}_s + 12 j \int_{r_j}^{T_j} (U_s - Y^m_s) dK^{n-}_s \\
+ 12 j \int_{r_j}^{T_j} e^{12j}(Y^n_s - Y^m_s) dA_s + 12 j \int_{r_j}^{T_j} e^{12j} \eta_s (Y^n_s - Y^m_s) ds - \int_{r_j}^{T_j} dR^{m,j}_s
\]

where \( dR^{m,j} \) is a positive measure depending on \( n, m \) and \( j \). In order to understand the terms of \( dR^{m,j} \), let us give an example. Since \( 12 j s \leq \psi'(s) \), the term

\[
- \int_{r_j}^{T_j} \psi'(Y^n_s - L_s) dK^{m+}_s,
\]
Now, there exist a subsequence \( (m_k^l) \) of \( m \) and a process \( \tilde{Z}^i \in L^2(\Omega, L^2([0, \tau_j]; \mathcal{F}^d)) \) such that
\[
Z_{s_{\tau_j}}^{m_k^l} 1_{\{s \leq \tau_j\}} \quad \text{converges weakly in } L^2(\Omega, L^2([0, \tau_j]; \mathcal{F}^d)) \quad \text{to the process } \tilde{Z}^i 1_{\{s \leq \tau_j\}} \quad \text{as } k \text{ goes to infinity}
\]
and
\[
\sqrt{\psi(Y_s^{n_k} - Y_s^{m_k}) (Z_s^n - Z_s^{m_k})} 1_{\{s \leq \tau_j\}} \quad \text{converges weakly in } L^2(\Omega, L^2([0, \tau_j]; \mathcal{F}^d)) \quad \text{to } \sqrt{\psi(Y_s^n - Y_s)(Z_s^n - \tilde{Z}^i_s)} 1_{\{s \leq \tau_j\}} \quad \text{as } k \text{ goes to infinity, since } \sqrt{\psi(Y_s^n - Y_s^{m_k})} \text{ converges strongly to } \sqrt{\psi(Y_s^n - Y_s)}.
\]
Now, since
\[ 4jE \int_0^{\tau_j} \psi(Y^n_s - Y_s) |Z^n_s - \hat{Z}_j|^2 ds + 6jE \int_0^{\tau_j} |Z^n_s - \hat{Z}_j|^2 ds \]
\[ \leq \lim \inf_k \left[ 4jE \int_0^{\tau_j} \psi(Y^n_s - Y_s^m_k) |Z^n_s - Z_s^m_k|^2 ds + 6jE \int_0^{\tau_j} |Z^n_s - Z_s^m_k|^2 ds \right], \]

By taking \( m = m_k, k \geq n \), in Equation (3.25) and tending \( k \) to infinity, we get
\[ E \psi(Y^n_t - Y_t) + 4jE \int_0^{\tau_j} \psi(Y^n_s - Y_s) |Z^n_s - \hat{Z}_j|^2 ds + 6jE \int_0^{\tau_j} |Z^n_s - \hat{Z}_j|^2 ds \]
\[ \leq E \left( \psi(Y^n_{\tau_j} - Y_t) \right) + 12je^{12j}E \int_0^{\tau_j} ((Y^n_s - Y_s) + 1_{\{s > \tau_n\}}) dA_s \]
\[ + 12jE \int_0^{\tau_j} e^{12j} \eta_n(Y^n_s - Y_s) ds + 4jE \int_0^{\tau_j} \psi(Y^n_s - Y_s) |Z^n_s - \hat{Z}_j|^2 ds \]
\[ + 4jE \int_0^{\tau_j} \psi(Y^n_s - Y_s) |\hat{Z}_j|^2 ds. \]

Hence
\[ \lim_{n \to +\infty} E \int_0^{\tau_j} |Z^n_s - \hat{Z}_j|^2 ds = 0. \]

By the uniqueness of the limit we obtain that
\[ \hat{Z}_j(\omega)1_{\{0 \leq s \leq \tau_j(\omega)\}} = \hat{Z}_{j+1}(\omega)1_{\{0 \leq s \leq \tau_j(\omega)\}}, \quad P(d\omega) ds - a.e. \]

For \( s \in [0, T] \), let us set \( Z_s(\omega) = \lim_j \hat{Z}_j1_{\{s \leq \tau_j\}} = \hat{Z}_J(\omega)(\omega) \), where \( j(\omega) \) is such that \( \tau_j(\omega) = T \). Then, for all \( j \in \mathbb{N} \)
\[ E \int_0^{\tau_j} |Z_s|^2 ds < +\infty. \]
Hence, \( \int_0^T |Z_s|^2 ds < +\infty, P - a.s. \) Moreover for all \( j \in \mathbb{N} \), we have
\[ \lim_{n \to +\infty} E \int_0^{\tau_j} |Z^n_s - Z_s|^2 ds = 0. \]

Proposition 3.3 proved.

### 3.4 Main Result

Now we are ready to give the main result of this paper.

**Theorem 3.1.** Assume that assumptions (H.0) – (H.4) hold. Then the process \( Y, Z, K^+, K^- \), defined by (3.22) and Proposition 3.4, is the maximal solution for Equation (2.13).

**Proof.** Let us now prove that the process \( Y, Z, K^+, K^- \) is the maximal solution for Equation (2.13). To begin with, let us show that the process \( Y \) is continuous. From Equation (3.25) and according to Burkholder-Davis-Gundy inequality, there exists a constant \( C > 0 \) such that
\[ E \sup_{s \leq \tau_j} \psi(Y^n_s - Y_s) \]
\[ \leq E \left( \psi(Y^n_{\tau_j} - Y_{\tau_j}) \right) + 12je^{12j}E \int_0^{\tau_j} ((Y^n_s - Y_s) + 1_{\{s > \tau_n\}}) dA_s \]
\[ + 12jE \int_0^{\tau_j} e^{12j} \eta_n(Y^n_s - Y_s) ds + 4jE \int_0^{\tau_j} \psi(Y^n_s - Y_s) |Z_s|^2 ds \]
\[ + C E \left( \int_0^{\tau_j} |\psi(Y^n_s - Y_s)|^2 |Z^n_s - Z_s|^2 ds \right). \]
Hence
\[
\lim_n E \sup_{s \leq \tau_j} \psi(Y^n_s - Y_s) = 0,
\]
and then
\[
\lim_n E \sup_{s \leq \tau_j} |Y^n_s - Y_s| = 0.
\]
It follows that \(Y\) is continuous, since \(P[\bigcup_{j=1}^{\infty} (\tau_j = T)] = 1\).

Now, in view of (3.27) there exists a subsequence \(n^i\) of \(n\) such that:

1. \(E \int_0^{\tau_j} |Z^{n_i}_s - Z_s|^2 ds \leq \frac{1}{2^k}\) and \(E \int_0^{\tau_j} \sum_{k=0}^{+\infty} |Z^{n_i}_s - Z_s|^2 ds \leq 2\),

2. \(Z^{n_i}_s(\omega) \rightarrow Z_s(\omega)\), a.e. \((s, \omega) \in [0, \tau_j] \times \Omega\), and \(|Z^{n_i}_s(\omega)| \leq h^i_s\), a.e. \((s, \omega) \in [0, \tau_j] \times \Omega\), where
\[
h^i_s = 1_{(s \leq \tau_j)} \left(2|Z|^2 + 2 \sum_{k=0}^{+\infty} |Z^{n_i}_s - Z_s|^2 \right)^{\frac{1}{2}}.
\]

Hence, in view of Lemma 3.1, we obtain
\[
E \int_0^{\tau_j} |f_{n^i}(s, Y^{n_i}_s, Z^{n_i}_s) - f(s, Y^{n_i}_s, Z^{n_i}_s)| ds
\]
\[
= E \int_0^{\tau_j} f_{n^i}(s, Y^{n_i}_s, Z^{n_i}_s) - f(s, Y^{n_i}_s, Z^{n_i}_s) ds
\]
\[
= E \int_0^{\tau_j} \left( f_{n^i}(s, Y^{n_i}_s, Z^{n_i}_s) - f(s, Y^{n_i}_s, Z^{n_i}_s) \right) 1_{\{|Z^{n_i}_s - Z_s| \leq 1\}} ds
\]
\[
+ E \int_0^{\tau_j} \left( f_{n^i}(s, Y^{n_i}_s, Z^{n_i}_s) - f(s, Y^{n_i}_s, Z^{n_i}_s) \right) 1_{\{|Z^{n_i}_s - Z_s| \geq 1\}} ds
\]
\[
\leq E \int_0^{\tau_j} |f_{n^i}(s, y, z) - f(s, y, z)| ds
\]
\[
+ E \int_0^{\tau_j} \sup_{(y, z) \in [0, 1] \times B(Z, 1)} \left( f_{n^i}(s, y, z) - f(s, y, z) \right) ds - E \int_0^{\tau_j} f(s, Y^{n_i}_s, Z^{n_i}_s) 1_{\{|Z^{n_i}_s - Z_s| \geq 1\}} ds
\]
\[
\leq E \int_0^{\tau_j} \left( |\eta + j| Z^{n_i}_s - Z_s |^2 + j | Z_s |^2 \right) 1_{\{|Z^{n_i}_s - Z_s| \geq 1\}} ds,
\]
where \(B(Z, 1)\) is the closed ball of center \(Z\) and radius 1.

But, taking account of Lemma 3.1 and using Lebesgue’s dominated convergence theorem, we get
\[
\lim_k E \int_0^{\tau_j} \sup_{(y, z) \in [0, 1] \times B(Z, 1)} \left( f_{n^i}(s, y, z) - f(s, y, z) \right) ds = 0,
\]
and
\[
\lim_k E \int_0^{\tau_j} \left( |\eta + j| Z^{n_i}_s - Z_s |^2 + j | Z_s |^2 \right) 1_{\{|Z^{n_i}_s - Z_s| \geq 1\}} ds = 0.
\]
Therefore
\[
\lim_k E \int_0^{\tau_j} \left( f_{n^i}(s, Y^{n_i}_s, Z^{n_i}_s) - f(s, Y^{n_i}_s, Z^{n_i}_s) \right) ds = 0.
\]
It follows then from Lebesgue’s dominated convergence theorem that for all \(j \in N\)
\[
\lim_k E \int_0^{\tau_j} |f(s, Y^{n_i}_s, Z^{n_i}_s) - f(s, Y_s, Z_s)| ds = 0.
\]
Hence for all \( j \in \mathbb{N} \)
\[
\lim_{k} \mathbb{E} \int_{0}^{\tau_j} | f_{n_k}(s, Y_{s}^{n_k}, Z_{s}^{n_k}) - f(s, Y_s, Z_s) | \, ds = 0.
\]
Since the above limit doesn’t depend on the choice of the subsequence \((n_k)^{j}\) we have for all \( j \in \mathbb{N} \)
\[
\lim_{n} \mathbb{E} \int_{0}^{\tau_j} | f_n(s, Y_{s}^{n}, Z_{s}^{n}) - f(s, Y_s, Z_s) | \, ds = 0.
\]
It not difficult also to prove that for all \( j \in \mathbb{N} \)
\[
\lim_{n} \mathbb{E} \int_{0}^{\tau_j} | g_n(s, Y_{s}^{n}) - g(s, Y_s) | \, dA_s = 0.
\]
From Equation (3.19)(i) we obtain, \( \forall j, \sup_{n} \mathbb{E} K_{\tau_j}^{n+} < +\infty \). It then follows from Fatou’s lemma that for any \( j \in \mathbb{N} \), \( \mathbb{E} K_{\tau_j}^{+} < +\infty \). Henceforth \( K_{\tau_j}^{+} < +\infty \), \( P \)-a.s.
Now let us prove the minimality conditions. We have
\[
\int_{0}^{T} (U_t - Y_t^{n})dK_t^{-} = 0.
\]
Hence, since \( dK^{-} = \inf_{n} dK_{n}^{n-} \), we get
\[
\int_{0}^{T} (U_t - Y_t^{n})dK_t^{-} = 0.
\]
It follows then from Fatou’s lemma that
\[
\int_{0}^{T} (U_t - Y_t)dK_t^{-} = 0.
\]
On the other hand
\[
\int_{0}^{T} (Y_t^{n} - L_t)dK_t^{n+} = 0.
\]
Hence, since \( Y = \inf_{n} Y^{n} \), we obtain
\[
\int_{0}^{T} (Y_t - L_t)dK_t^{n+} = 0.
\]
Applying Fatou’s lemma we obtain
\[
\int_{0}^{T} (Y_t - L_t)dK_t^{+} = 0.
\]
Now, since \( dK^{+} = \sup_{n} dK_{n}^{n+} \), \( dK^{-} = \inf_{n} dK_{n}^{n-} \) and the measures \( dK^{n+} \) and \( dK^{n-} \) are singular, it follows that \( dK^{+} \) and \( dK^{-} \) are singular.
Now it is not difficult to see that the process \((Y, Z, K^{+}, K^{-})\) satisfies Equation (2.13). It remains to prove \((Y, Z, K^{+}, K^{-})\) is maximal. Let \((Y', Z', K'^{+}, K'^{-})\) be an another solution to Equation (2.13). By comparison theorem we have that \( Y' \leq Y^{n} \) and then \( Y' \leq Y \). The proof of Theorem 3.4 is then finished.
3.5 Existence of maximal solution for Equation (2.6)

Theorem 3.2. Let assumptions (A.1)-(A.3) hold true. Then there exists a maximal solution for Equation (2.6).

Proof. Let \((\bar{Y}_t, Z_t, \bar{K}^+_t, \bar{K}^-_t)_{t \leq T}\) be the maximal solution of Equation (2.13) then, for any \(t \leq T\), we have

\[
\begin{align*}
(i) \quad \bar{Y}_t &= \xi + \int_0^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s)ds + \int_0^T g(s, \bar{Y}_s)dA_s + \int_0^T dB_s \\
&\quad + \int_0^T d\bar{K}^+_s - \int_0^T d\bar{K}^-_s - \int_0^T \bar{Z}_s dB_s, \ t \leq T, \\
(ii) \quad \bar{Y}_t \text{ between } L \text{ and } U, \ i.e. \forall t \leq T, \ L_t \leq \bar{Y}_t \leq U_t, \\
(iii) \quad \text{the Skorohod conditions hold :} \\
\int_0^T (\bar{Y}_t - L_t)d\bar{K}^+_t = \int_0^T (U_t - \bar{Y}_t)d\bar{K}^-_t = 0, \ a.s. \\
(iv) \quad \bar{Y} \in C \ \bar{K}^+, \bar{K}^- \in \mathcal{K} \ \bar{Z} \in \mathcal{L}^{2,d}, \\
(v) \quad d\bar{K}^+ \perp d\bar{K}^-.
\end{align*}
\]

Now, for all \(t \leq T\), let us set

\[
Y_t = \frac{\ln(\bar{Y}_t)}{m_t} + m_t, \quad Z_t = \frac{\bar{Z}_t}{m_Y}, \quad d\bar{K}^\pm_t = d\bar{K}^\pm_t.
\]

By using Itô’s formula to \(\frac{\ln(\bar{Y}_t)}{m_t} + m_t\), we have

\[
Y_t = \xi + \int_0^T f(s, Y_s, Z_s)ds + \int_0^T dK^+_s - \int_0^T dK^-_s - \int_0^T Z_s dB_s, \ t \leq T.
\]

Therefore it is not difficult to prove that \((Y, Z, K^+, K^-)\) is a maximal solution for the GBSDE with two reflecting barriers (2.6). This completes the proof. 

4 Applications

4.1 Application to the game theory

Our purpose in this section is to show that the existence of a solution \((Y, Z, K^+, K^-)\) to the GBSDE implies that \(Y\) is the value of a certain stochastic game of stopping. First introduced by Dynkin and Yushkevich [10] and later studied, in different contexts, by several authors, including Neveu [23], Bensoussan and Friedman [4], Bismut [5], Morimoto [22], Alario-Nazaret, Lepeltier and Marchal [1], Lepeltier and Maingueneau [21], Cvitanic and Karatzas [9] and others, such stochastic games are known as Dynkin games.

Let \(\xi, L, U\) be as in the beginning. Let \(Q\) be a process such that, \(\forall t \in [0, T] \ L_t \leq Q_t \leq U_t, \ P - a.s\) and assume that assumption (A.3) holds true. Let \(F : \mathbb{R} \rightarrow \mathbb{R}\) be a continuous nondecreasing function such that: for every semimartingale \(S\) such that \(L \leq S \leq U\), \(F(S)\) is also a semimartingale.

Consider the payoff

\[
J(\lambda, \sigma) = U\mathbb{1}_{\{\lambda < \sigma\}} + L\mathbb{1}_{\{\lambda > \sigma\}} + Q\mathbb{1}_{\{\sigma = \lambda < T\}} + \xi\mathbb{1}_{\{\sigma = \lambda = T\}}
\]

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The setting of our problem of Dynkin game is the following. There are two players labelled player 1 and player 2. Player 1 chooses the stopping time $\lambda$, player 2 chooses the stopping time $\sigma$, and $J(\lambda, \sigma)$ represents the amount paid by player 1 to player 2. It is the conditional expectation $\mathbb{E}\left(F(J(\lambda, \sigma)) \mid \mathcal{F}_t\right)$ of this random payoff that player 1 tries to minimize and player 2 tries to maximize. The game stops when one player decides to stop, that is, at the stopping time $\lambda \land \sigma$ before time $T$ or at $T$ if $\lambda = \sigma = T$.

Our objective is to show existence of a fair strategy, to be precise a saddle-point, for the game and to characterize it. We show that, a RBSDE with two reflecting barriers is associated. This RBSDE gives the value function of the game and allows us to construct a saddle-point.

Consider the following RBSDE

$$\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_s dB_s , t \leq T, \\
(ii) & \quad \forall t \leq T, F(L_t) \leq Y_t \leq F(U_t), \\
(iv) & \quad \int_0^T (Y_t - F(L_t))dK^+_t = \int_0^T (F(U_t) - Y_t)dK^-_t = 0, \text{ a.s.,} \\
(v) & \quad dK^+ \perp dK^-.
\end{align*}$$

(4.30)

Let $\lambda_t^*$ and $\sigma_t^*$ be the stopping times defined as follows:

$$\lambda_t^* = \inf\{s \geq t : Y_s = F(U_s)\} \land T \quad \text{and} \quad \sigma_t^* = \inf\{s \geq t : Y_s = F(L_s)\} \land T.$$

We have the following.

**Theorem 4.1.** Assume the following assumptions:

j) $\mathbb{E}F(L_0)^- < +\infty$, for all stopping time $0 \leq \sigma \leq T$, where $F(L)^- = \sup(-F(L), 0)$.

jj) $\lim_{t \to +\infty} \mathbb{P}\left[\sup_{s \leq T} F(U_s)^+ > t\right] = 0$, where $F(L)^+ = \sup(F(L), 0)$.

jjj) $\lim_{t \to +\infty} \mathbb{P}\left[\sup_{s \leq T} F(L_s)^- > t\right] = 0$.

Then

$$Y_t = \mathbb{E}\left(F(J(\lambda_t^*, \sigma_t^*)) \mid \mathcal{F}_t\right)$$

$$= \sup_{\sigma \in \mathcal{T}_t} \mathbb{E}\left(F(J(\lambda_t^*, \sigma)) \mid \mathcal{F}_t\right)$$

$$= \inf_{\lambda \in \mathcal{T}_t} \mathbb{E}\left(F(J(\lambda, \sigma_t^*)) \mid \mathcal{F}_t\right)$$

$$= \inf_{\lambda \in \mathcal{T}_t} \sup_{\sigma \in \mathcal{T}_t} \mathbb{E}\left(F(J(\lambda, \sigma)) \mid \mathcal{F}_t\right)$$

$$= \sup_{\lambda \in \mathcal{T}_t} \inf_{\sigma \in \mathcal{T}_t} \mathbb{E}\left(F(J(\lambda, \sigma)) \mid \mathcal{F}_t\right),$$

(4.31)

where $\mathcal{T}_t$ is the set of stopping times valued between $t$ and $T$. $(Y_t)_{t \leq T}$ can be interpreted as the value of the game and $(\lambda_0^*, \sigma_0^*)$ as the fair strategy for the two players, i.e. for any stopping times $\lambda$ and $\sigma$,

$$\mathbb{E}\left(F(J(\lambda_0^*, \sigma))\right) \leq \mathbb{E}\left(F(J(\lambda_0^*, \sigma_0^*))\right) = Y_0 \leq \mathbb{E}\left(F(J(\lambda, \sigma_0^*))\right).$$
**Proof.** Let \((t^+_n)_n\) and \((t^-_n)_n\) be two nondecreasing sequences such that
\[
\liminf_{n \to +\infty} t^+_n P\left( \sup_{s \leq t} F(U_s) > t^+_n \right) = 0, \quad \liminf_{n \to +\infty} t^-_n P\left( \sup_{s \leq t} F(L_s)^- > t^-_n \right) = 0. \tag{4.32}
\]

Let also \((\alpha_i)_{i \geq 0}\) and \((\tau^\pm_i)_{i \geq 0}\) be families of stopping times defined by
\[
\alpha_i = \inf\{s \geq t : \int_t^s |Z_r|^2 dr \geq i\} \land T, \quad \tau^+_i = \inf\{s \geq t : Y^+_s > t^+_i\} \land T.
\]

It follows from Equation (4.30) that for every stopping time \(\sigma \in \mathcal{T}_t\)
\[
Y_t \geq IE\left(Y^+_\lambda_t \land \sigma \land \tau^-_m \mid \mathcal{F}_t\right) = IE\left(Y^+_\lambda_t \land \sigma \land \tau^-_m \mid \mathcal{F}_t\right) - IE\left(Y^-_{\lambda_t \land \sigma} \land \tau^-_m \mid \mathcal{F}_t\right).
\]

In view of passing to the limit on \(i\) and \(n\) respectively and using Fatou’s lemma, we have
\[
Y_t \geq IE\left(Y^+_\lambda_t \land \sigma \mid \mathcal{F}_t\right) - \liminf_m IE\left(Y^-_{\lambda_t \land \sigma} \land \tau^-_m \mid \mathcal{F}_t\right).
\]

Now tending \(m\) to infinity we get
\[
Y_t \geq IE\left(Y^+_\lambda_t \land \sigma \mid \mathcal{F}_t\right) - \liminf_m IE\left(Y^-_{\lambda_t \land \sigma} \land \tau^-_m \mid \mathcal{F}_t\right).
\]

But
\[
\liminf_m IE\left(Y^-_{\lambda_t \land \sigma} \land \tau^-_m \mid \mathcal{F}_t\right) = \liminf_m \left[IE\left(Y^-_{\lambda_t \land \sigma} \land \lambda_t \land \tau^-_m \mid \mathcal{F}_t\right) + t^-_m IE\left(1_{\lambda_t \land \sigma \land \tau^-_m} \mid \mathcal{F}_t\right)\right] = IE\left(Y^-_{\lambda_t \land \sigma} \mid \mathcal{F}_t\right) + \liminf_m t^-_m IE\left(1_{\lambda_t \land \sigma \land \tau^-_m} \mid \mathcal{F}_t\right) \leq IE\left(Y^-_{\lambda_t \land \sigma} \mid \mathcal{F}_t\right) + \liminf_m t^-_m IE\left(1_{\sup_{s \leq T} F(L_s)^- > t^-_m} \mid \mathcal{F}_t\right) = IE\left(Y^-_{\lambda_t \land \sigma} \mid \mathcal{F}_t\right),
\]

where we have used the limit appeared in (4.32).

It follows then that for all \(\sigma \in \mathcal{T}_t\),
\[
Y_t \geq IE\left(Y^+_\lambda_t \land \sigma \mid \mathcal{F}_t\right) - IE\left(Y^-_{\lambda_t \land \sigma} \mid \mathcal{F}_t\right) = IE\left(Y^-_{\lambda_t \land \sigma} \mid \mathcal{F}_t\right) \geq IE\left(F(U_{\lambda_t}) 1_{\lambda_t < \sigma} + F(L_{\sigma}) 1_{\lambda_t > \sigma} + F(Q_{\sigma}) 1_{\sigma = \lambda_t < T} + F(\xi) 1_{\sigma = \lambda_t = T} \mid \mathcal{F}_t\right) \tag{4.33}
\]
\[
= IE\left(F(J(\lambda_t, \sigma)) \mid \mathcal{F}_t\right).
\]
Now, for every stopping time \( \lambda \in \mathcal{T}_t \), we obtain

\[
Y_t \leq \mathbb{E}\left( Y_{\lambda \wedge \sigma_t} \mid \mathcal{F}_t \right)
= \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^+ \mid \mathcal{F}_t \right) - \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^- \mid \mathcal{F}_t \right)
\leq \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^+ \mid \mathcal{F}_t \right) - \limsup_{m \to \infty} \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^- \mid \mathcal{F}_t \right)
\leq \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^+ \mid \mathcal{F}_t \right) - \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^- \mid \mathcal{F}_t \right).
\]

Hence

\[
Y_t + \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^- \mid \mathcal{F}_t \right) \leq \liminf_{n \to \infty} \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^+ \mid \mathcal{F}_t \right) - \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^- \mid \mathcal{F}_t \right) \leq \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^- \mid \mathcal{F}_t \right) + \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^+ \mid \mathcal{F}_t \right).
\]

It follows that

\[
Y_t \leq -\mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^- \mid \mathcal{F}_t \right) + \mathbb{E}\left( Y_{\lambda \wedge \sigma_t}^+ \mid \mathcal{F}_t \right)
= \mathbb{E}\left( Y_{\lambda \wedge \sigma_t} \mid \mathcal{F}_t \right)
\leq E\left( F(U_\lambda)1_{\{\lambda < \sigma^*\}} + F(L_{\sigma^*})1_{\{\lambda > \sigma^*\}} + F(Q_{\sigma^*})1_{\{\sigma^* = \lambda < T\}} + F(\xi)1_{\{\sigma^* = \lambda = T\}} \right)
= \mathbb{E}\left( F(J(\lambda, \sigma^*)) \mid \mathcal{F}_t \right).
\]

In force of inequalities (4.33) and (4.34) we obtain that for all \( \sigma, \lambda \in \mathcal{T}_t \)

\[
\mathbb{E}\left( F(J(\lambda^*_t, \sigma)) \mid \mathcal{F}_t \right) \leq Y_t \leq \mathbb{E}\left( F(J(\lambda, \sigma^*_t)) \mid \mathcal{F}_t \right).
\]

Henceforth

\[
\inf_{\lambda \in \mathcal{T}_t} \sup_{\sigma \in \mathcal{T}_t} \mathbb{E}\left( F(J(\lambda^*_t, \sigma)) \mid \mathcal{F}_t \right)
\leq \sup_{\sigma \in \mathcal{T}_t} \mathbb{E}\left( F(J(\lambda^*_t, \sigma)) \mid \mathcal{F}_t \right)
\leq Y_t
\leq \inf_{\lambda \in \mathcal{T}_t} \mathbb{E}\left( F(J(\lambda, \sigma^*_t)) \mid \mathcal{F}_t \right)
\leq \sup_{\sigma \in \mathcal{T}_t} \inf_{\lambda \in \mathcal{T}_t} \mathbb{E}\left( F(J(\lambda, \sigma^*_t)) \mid \mathcal{F}_t \right),
\]

and then equality (4.31) follows.
Definition 4.2. The fair price of a contingent claim game is the smallest initial endowment for which the hedging strategy exists. It is defined by
\[ L_t \leq \min_{x \geq 0} \mathbb{E}_x [X] \]

Remark 4.1. We should remark here that:
1. If \( \mathbb{E} \sup_{t \leq T} (|F(U_t| + |F(L_t)|) < +\infty \) then assumptions of Theorem 4.1 hold true and then we have proved the existence of a fair strategy for the game under weak assumptions on the data and in a general setting since the processes \( L \) and \( U \) are assumed, in general, to be bounded or square integrable and the function \( F \) is of polynomial or logarithmic or exponential type.
2. If we suppose that \( F(x) = e^{-\alpha x} \), we have a utility function which is of exponential type and then our result can give, in particular, a solution to the existence a saddle point for the risk-sensitive problem (see [12] for more details).

4.2 Application to American game option
Following the same idea as in Hamadène [16], we discuss, in this section, American game option pricing problems in finance and their relationship with RBSDEs. Consider a security market \( \mathcal{M} \) that contains, say, one bond and one stock. Suppose that their prices are subject to the following system of stochastic differential equations:
\[
\begin{cases}
    dS^0_t = rS^0_t dt, & S^0_0 > 0 \\
    dS_t = S_t(b dt + \delta dB_t), & S_0 > 0
\end{cases}
\]

Let \( X \) an \( \mathcal{F}_t \)-measurable random variable such that \( X \geq 0 \). A self-financing portfolio after \( t \) with endowment at time \( t \) is \( X \), a \( \mathcal{P} \)-measurable process \( \pi = (\beta_s, \gamma_s)_{t \leq \lambda \leq T} \) with values in \( \mathbb{R}^2 \) such that:
(i) \( \int^T_t (|\beta_s| + (\gamma_s S_s)^2) ds < \infty \).

(ii) If \( Z^\pi_s = \beta_s S^0_s + \gamma_s S_s, \quad s \leq T \), then \( Z^\pi_s = X + \int^\lambda_s \beta_u dS^0_u + \int^\lambda_s \gamma_u dS_u, \quad \forall s \leq T \).

Let \( \xi, L, U \) be as in the beginning such that \( 0 \leq L \leq U \). Let \( Q \) be a process such that, \( \forall t \in [0,T] \), \( L_t \leq Q_t \leq U_t, \ P \text{-a.s.} \), and assume that assumption (A.3) holds true.

Definition 4.1. A hedge against the game with payoff
\[ J(s, \lambda) := U\lambda 1_{\{\lambda < s\}} + L_s 1_{\{s < \lambda\}} + Q_s 1_{\{s=\lambda<T\}} + \xi 1_{\{s=\lambda=T\}} \]

after \( t \) whose endowment at \( t \) is \( X \) is a pair \((\pi, \lambda)\), where \( \pi \) is self-financing portfolio after \( t \) whose endowment at \( t \) is \( X \) and a stopping time \( \lambda \geq t \), satisfying: \( P \text{-a.s.} \forall s \in [t, T] \),
\[ Z^\pi_{s \wedge \lambda} \geq J(s, \lambda) := U\lambda 1_{\{\lambda < s\}} + L_s 1_{\{s < \lambda\}} + Q_s 1_{\{s=\lambda<T\}} + \xi 1_{\{s=\lambda=T\}} \]

Definition 4.2. The fair price of a contingent claim game is the smallest initial endowment for which the hedging strategy exists. It is defined by
\[ V_t := \inf \{ X \geq 0, \exists (\pi, \lambda) \text{ such that } Z^\pi_{s \wedge \lambda} \geq J(s, \lambda), \forall t \leq s \leq T, \ P \text{-a.s.} \} \]

Now, let \( P^* \) be the probability on \((\Omega, \mathcal{F})\) under which the actualized price of the asset is a martingale, i.e.
\[ \frac{dP^*}{dP} := \exp \left( -\delta^{-1} (b-r) B_t - \frac{1}{2} (\delta^{-1} (b-r))^2 t \right), \quad t \leq T \]
Hence the process $W_t = B_t + \delta^{-1} (b-r)t$ is an $(F_t, P^*)$-Brownian motion. Let consider, on the probability space $(\Omega, \mathcal{F}, P^*)$, the following RBSDE

$$
\left\{
\begin{array}{ll}
(i) & Y_t = e^{-rT}\xi + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_s dW_s, t \leq T, \\
(ii) & \forall t \leq T, e^{-rt}L_t \leq Y_t \leq e^{-rt}U_t, \\
(iii) & \int_0^T (Y_t - e^{-rt}L_t) dK^+_t = \int_0^T (e^{-rt}U_t - Y_t) dK^-_t = 0, \text{ a.s.,} \\
(iv) & \forall Y \in \mathcal{C}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in L^2,d,
\end{array}
\right.
\tag{4.35}
$$

Let $\lambda^*_t$ and $\sigma^*_t$ the stopping times defined as follows

$$
\lambda^*_t = \inf \{ s \geq t : Y_s = e^{-rt}U_s \} \wedge T \quad \text{and} \quad \sigma^*_t = \inf \{ s \geq t : Y_s = e^{-rt}L_s \} \wedge T.
$$

It follows from the previous section that for all $\sigma, \lambda \in \mathcal{T}_t$

$$
\mathbb{E} \left( J(\lambda^*_t, \sigma) \mid F_t \right) \leq Y_t = \mathbb{E} \left( J(\lambda^*_t, \sigma^*_t) \mid F_t \right) \leq \mathbb{E} \left( J(\lambda, \sigma^*_t) \mid F_t \right),
$$

where

$$
J(\lambda, \sigma) = e^{-r\lambda}U_1 \{ 1 < \sigma \} + e^{-r\sigma}L_1 \{ \lambda > \sigma \} + e^{-r\lambda}Q_s \{ \sigma = 1 \} + e^{-rT}1 \{ \sigma = T \}.
$$

We have the following.

**Theorem 4.2.** Assume that $\liminf_{t \to \infty} t P^* \{ |U_s| > t \} = 0$. Then, for any $t \leq T$, $V_t = e^{rt}Y_t$. Moreover there exists a hedge against the option given by:

$$
\gamma_s = \frac{e^{rs}Z_s}{\delta S_s} 1_{\{ s \leq \lambda^*_t \}} \quad \text{and} \quad \beta_s = \left( e^{rs}(Y_t + \int_t^s Z_u dW_u) - \gamma_sS_s \right)(S_0^0)^{-1}, \forall s \in [t, T].
$$

**Proof.** Let $(\pi, \lambda)$ a hedge against the option. Therefore $\lambda \geq t$ and $\pi = (\beta_s, \gamma_s)_{t \leq s \leq T}$ is a self-financing portfolio whose value at $t$ is $X$ satisfying $Z_{s \wedge \lambda}^\pi \geq J(s, \lambda), \forall t \leq s \leq T$. But

$$
e^{-r(s \wedge \lambda)}Z_{s \wedge \lambda}^\pi = e^{-rt}X + \delta \int_t^{s \wedge \lambda} \gamma_uS_u e^{-ru}dW_u \geq e^{-r(s \wedge \lambda)}J(s, \lambda), \forall t \leq s \leq T.
$$

Let $\sigma$ a stopping time $\geq t$. Putting $s = \sigma$ and taking the conditional expectation we obtain

$$
e^{-rt}X \geq \mathbb{E}^* \left( e^{-r(s \wedge \lambda)}J(\sigma, \lambda) \mid F_t \right).
$$

Then we have

$$
e^{-rt}X \geq \sup_{t \leq \sigma \leq T} \mathbb{E}^* \left( e^{-r(s \wedge \lambda)}J(\sigma, \lambda) \mid F_t \right) \geq \inf_{t \leq \sigma \leq T} \sup_{s \geq t} \mathbb{E}^* \left( e^{-r(s \wedge \lambda)} \mid J(\sigma, \lambda) \mid F_t \right) = Y_t.
$$

Henceforth $V_t \geq e^{rt}Y_t$. Let us now prove the reverse inequality. It is not difficult to see that

$$
Y_t + \int_t^{s \wedge \lambda^*_t} Z_u dW_u \leq e^{-r(s \wedge \lambda^*_t)}J(s, \lambda^*_t), \forall t \leq s \leq T.
$$

Now if we put for all $s \in [t, T]$ $\gamma_s = \frac{e^{rs}Z_s}{\delta S_s} 1_{\{ s \leq \lambda^*_t \}}$, and $\beta_s = \left( e^{rs}(Y_t + \int_t^s Z_u dW_u) - \gamma_sS_s \right)(S_0^0)^{-1}$. Hence $(\beta_s, \gamma_s)_{t \leq s \leq T}$ is a self-financing portfolio whose value at $t$ is $e^{rt}Y_t$. On other hand $e^{r(s \wedge \lambda^*_t)}(Y_t + \int_t^{s \wedge \lambda^*_t} Z_u dW_u) \geq J(s, \lambda^*_t), \forall s \in [t, T]$. Hence $((\beta_s, \gamma_s)_{t \leq s \leq T}, \lambda_t^*)$ is a hedge against the game option. Then $e^{rt}Y_t \geq V_t$. Henceforth $e^{rt}Y_t = V_t$.  

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5 Appendix

A Comparison theorem

The comparison theorem for real-valued BSDEs turns out to be one of the classic results of the theory of BSDE. It allows to compare the solutions of two real-value BSDEs whenever we can compare the terminal conditions and the generators. This section is devoted to show a comparison theorem for the following GSBE with generator $hdA_t$:

$$\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T h(s, Y_s, Z_s) dA_s + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s, \quad t \leq T; \\
(ii) & \quad \forall t \leq T, ~ L_t \leq Y_t \leq U_t, \\
(iii) & \quad \int_0^T (Y_t - L_t) dK_t^+ = \int_0^T (U_t - Y_t) dK_t^- = 0, \text{ a.s.} \\
(iv) & \quad Y \in \mathcal{C} \quad K^+, K^- \in \mathcal{K} \quad Z \in L^{2,d}, \\
(v) & \quad dK^+ \perp dK^-.
\end{align*}$$

(A.36)

Let $(Y^i, Z^i, K^{i+}, K^{i-})$ $(i = 1, 2)$ be two solutions of Equation (A.36) associated respectively with $(\xi^1, h^1, A^1, L^1, U^1)$ and $(\xi^2, h^2, A^2, L^2, U^2)$, such that, for $(i=1,2)$ the following assumptions are satisfied:

(D.1) $L^i : [0, T] \times \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ and $U^i : [0, T] \times \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ are two continuous barriers processes satisfying $L^1_t \leq L^2_t$ and $U^1_t \leq U^2_t, \forall t \in [0, T]$.

(D.2) $\xi^1 \leq \xi^2$.

(D.3) $A^i \in \mathcal{K}$ and for all $(s, \omega)$, $h^1(s, Y^1_s, Z^1_s) dA^1_s \leq h^2(s, Y^1_s, Z^1_s) dA^2_s$.

(D.4) There exist two processes $\alpha \in \mathcal{K}$ and $b \in L^{2,1}$ such that :

$$\begin{align*}
(h^2(s, Y^1_s, Z^1_s) - h^2(s, Y^2_s, Z^2_s)) dA^2_s \leq |Y^1_s - Y^2_s| \, da_s + |Z^1_s - Z^2_s| \, b \, ds.
\end{align*}$$

Set $\Gamma_t = e^{\tilde{\alpha}_t + \int_0^t \tilde{b}_s \, dB_s - \frac{1}{2} \int_0^t |\tilde{b}_s|^2 \, ds}$, where the processes $\tilde{\alpha}$ and $\tilde{b}$ are defined as follows :

$$\tilde{\alpha}_s = \int_0^s \frac{Y^1_t - Y^2_t}{Y^1_t - Y^2_t} 1(y^1_t \neq y^2_t) \, da_t \quad \text{and} \quad \tilde{b}_t = b_t \frac{Z^1_t - Z^2_t}{Z^1_t - Z^2_t} 1(z^1_t \neq z^2_t).$$

We have the following.

Theorem A.1. (Comparison theorem) Assume that assumptions (D.1) – (D.4) hold true.

i) If $\liminf \limits_{r \to +\infty} \sup \limits_{0 \leq s \leq T} \Gamma_s (Y^1_s - Y^2_s)^+ > r$, we have $Y^1_t \leq Y^2_t$, for $t \in [0, T]$, a.s.

ii) If $Y^1_t \leq Y^2_t$, for $t \in [0, T]$, a.s., then :

$$\begin{align*}
1_{\{U^1_t = U^2_t\}} dK^1_t^- \leq dK^2_t^- \quad \text{and} \quad 1_{\{L^1_t = L^2_t\}} dK^2_t^+ \leq dK^1_t^+.
\end{align*}$$

Before giving the proof of Theorem A.1 let us give the following remarks.

Remark A.1. It should be noted that:
1. If \( \mathbb{E} \sup_{t \leq T} \Gamma_t (Y_t^1 - Y_t^2)^+ < +\infty \) then \( \lim_{r \to +\infty} r P \left[ \sup_{0 \leq s \leq T} \Gamma_s (Y_s^1 - Y_s^2)^+ > r \right] = 0. \)

2. If there exist constants \( C > 0 \) and \( p > 1 \) such that \( \alpha_T + \int_0^T |b_s|^2 \, ds \leq C \) and \( \mathbb{E} \sup_{t \leq T} ((Y_t^1 - Y_t^2)^+)^p < +\infty \), then we obtain \( \mathbb{E} \sup_{t \leq T} \Gamma_t (Y_t^1 - Y_t^2)^+ < +\infty. \)

3. If \( U^i \equiv +\infty \), then \( dK^{i-} \equiv 0 \), for \( i = 1, 2. \)

4. If \( L^i \equiv -\infty \), then \( dK^{i+} \equiv 0 \), for \( i = 1, 2. \)

**Proof of Theorem (A.1)** It follows from assumptions (D.3) and (D.4) that
\[
dR^1 := h^1(s, Y_s^1, Z_s^1) dA_s^1 - Y_s^1 \, d\tilde{\alpha}_s - (Z_s^1, \tilde{b}_s) \, ds \leq dR^2 := h^2(s, Y_s^2, Z_s^2) dA_s^2 - Y_s^2 \, d\tilde{\alpha}_s - (Z_s^2, \tilde{b}_s) \, ds. \quad (A.37)
\]

By Itô’s formula we have, for \( i = 1, 2 \)
\[
\Gamma_t Y_t^i = \Gamma_T \xi^i + \int_t^T \Gamma_s \left( h^i(s, Y_s^i, Z_s^i) dA_s^i + dK_s^{i+} - dK_s^{i-} - Z_s^i \, dB_s \right) - \int_t^T Y_s^i \left( \Gamma_s \, d\tilde{\alpha}_s + \Gamma_s \left( \tilde{b}_s, dB_s \right) \right) - \int_t^T \Gamma_s \left( \tilde{b}_s, Z_s^i \right) \, ds.
\]

Then there exists \( \tilde{Z}^i \in \mathcal{L}^{2, d} \) for \( i = 1, 2 \), such that
\[
\tilde{Y}_t^i = \xi^i + \int_t^T \Gamma_s dR^i + \int_t^T \Gamma_s dK^{i+} - \int_t^T \Gamma_s dK^{i-} - \int_t^T \tilde{Z}_s^i dB_s, \quad (A.38)
\]

where \( \xi^i = \Gamma_T \xi^i \) and \( dR^i \) is defined in Inequality (A.37).

i) Let \( \tau_n \geq 0 \) and \( \sigma_n \geq 0 \) be two families of stopping times defined by
\[
\tau_n = \inf \{ s \geq 0 : \int_0^s |\tilde{Z}_u^i - \tilde{Z}_u^j|^2 \, du \geq n \} \land T, \quad \sigma_n = \inf \{ s \geq 0 : \Gamma_s (Y_s^1 - Y_s^2)^+ > t_n \} \land T,
\]

where \( (t_n)_n \) is a sequence tending to \( +\infty \) and satisfying \( \lim_{n \to +\infty} t_n P \left[ \sup_{0 \leq s \leq T} \Gamma_s (Y_s^1 - Y_s^2)^+ > t_n \right] = 0. \)

By Itô-Tanaka formula we get
\[
\Gamma_{t \land \tau_n \land \sigma_m} (Y_{t \land \tau_n \land \sigma_m}^1 - Y_{t \land \tau_n \land \sigma_m}^2)^+ = \Gamma_{\tau_n \land \sigma_m} (Y_{\tau_n \land \sigma_m}^1 - Y_{\tau_n \land \sigma_m}^2)^+ + \int_{t \land \tau_n \land \sigma_m}^{\tau_n \land \sigma_m} \Gamma_s \left( 1_{Y_s^2 > Y_s^1} dK_s^{i+} - 1_{Y_s^2 > Y_s^1} dK_s^{i-} \right)_{0}^{\infty} + \int_{t \land \tau_n \land \sigma_m}^{\tau_n \land \sigma_m} \Gamma_s 1_{Y_s^2 > Y_s^1} (dR_s^1 - dR_s^2)_{\leq 0}^{\infty} - \int_{t \land \tau_n \land \sigma_m}^{\tau_n \land \sigma_m} \Gamma_s 1_{Y_s^2 > Y_s^1} (dK_s)_{\leq 0}^{\infty} \land T - \int_{t \land \tau_n \land \sigma_m}^{\tau_n \land \sigma_m} 1_{Y_s^2 > Y_s^1} (\tilde{Z}_s^1 - \tilde{Z}_s^2) dB_s - \int_{t \land \tau_n \land \sigma_m}^{\tau_n \land \sigma_m} dh_r,
\]

where \( (b_t)_{t \in [0, T]} \) is continuous and nondecreasing process. Taking expectation in both sides, it follows that
\[
\mathbb{E} \Gamma_{t \land \tau_n \land \sigma_m} (Y_{t \land \tau_n \land \sigma_m}^1 - Y_{t \land \tau_n \land \sigma_m}^2)^+ \leq \mathbb{E} \Gamma_{\tau_n \land \sigma_m} (Y_{\tau_n \land \sigma_m}^1 - Y_{\tau_n \land \sigma_m}^2)^+.
\]
Using Lebesgue’s convergence dominated theorem we obtain
\[
\mathbb{E} \Gamma_t^{\sigma_m} (Y^1_{t, \sigma_m} - Y^2_{t, \sigma_m})^+ \\
\leq \mathbb{E} \Gamma_{\sigma_m} (Y^1_{\sigma_m} - Y^2_{\sigma_m})^+ 1_{\{\sigma_m < T\}} = t_m P(\sigma_m < T) = t_m P \left[ \sup_{0 \leq s \leq T} \Gamma_s (Y^1_s - Y^2_s)^+ > t_m \right].
\]
Taking the limit in both sides, as \( m \) goes to \( +\infty \), of the above inequality we have
\[
\mathbb{E} (\Gamma_t (Y^1_t - Y^2_t)^+) = 0.
\]
Hence \( Y^1_t \leq Y^2_t \), for \( t \in [0, T] \), a.s.

\textit{ii)} Assume that \( Y^1_t = Y^2_t \), for \( t \in [0, T] \), a.s., which is equivalent to \( Y^1_t \leq Y^2_t \), for \( t \in [0, T] \), a.s. Coming back now to Equation (A.38) and applying Itô-Tanaka formula to \( (\tilde{Y}^2_t - \tilde{Y}^1_t)^+ = \tilde{Y}^2_t - \tilde{Y}^1_t \) we get
\[
1_{\{\tilde{Y}^1_t = \tilde{Y}^2_t\}} \left( \Gamma_s (dK^{2+}_s - dK^{1+}_s) - \Gamma_s (dK^{2-}_s - dK^{1-}_s) \right) = -1_{\{\tilde{Y}^1_t = \tilde{Y}^2_t\}} d\Gamma_t,
\]
where \( (\tilde{Y}_t)_{t \in [0, T]} \) is continuous and nondecreasing process.
Since \( dK^{1+} \perp dK^{-1} \) and \( dK^{2+} \perp dK^{-2} \), we obtain
\[
1_{\{\tilde{Y}^1_t = \tilde{Y}^2_t\}} \Gamma_s dK^{2+}_s \leq 1_{\{\tilde{Y}^1_t = \tilde{Y}^2_t\}} \Gamma_s dK^{1+}_s,
\]
and
\[
1_{\{\tilde{Y}^1_t = \tilde{Y}^2_t\}} \Gamma_s dK^{1-}_s \leq 1_{\{\tilde{Y}^1_t = \tilde{Y}^2_t\}} \Gamma_s dK^{2-}_s.
\]
Consequently
\[
1_{\{\tilde{L}_t = \tilde{L}_t\}} dK^{2+}_t = 1_{\{\tilde{Y}^2_t = \tilde{Y}^1_t\}} dK^{2+}_t \leq dK^{1+}_t,
\]
and
\[
1_{\{\tilde{U}_t = \tilde{U}_t\}} dK^{1-}_t \leq dK^{2-}_t.
\]
This completes the proof of Theorem [A.1].

\section*{B Existence and uniqueness of solutions for GBSDE under strong assumptions on the coefficients}

The goal of this section is to study the existence and uniqueness of the solution of GBSDE (2.6) under strong assumptions on the coefficients.

We assume the following assumptions:

\textbf{(C.1)} (i) \( f \) is uniformly Lipschitz with respect to \((y, z)\), i.e., there exists a constant \( 0 < C_1 < \infty \) such that for any \( y, y', z, z' \in \mathbb{R} \),
\[
|f(\omega, t, y, z) - f(\omega, t, y', z')| \leq C_1 (|y - y'| + |z - z'|).
\]

(ii) There exists a constant \( C_2 > 0 \) such that for all \( y, y', z, z' \in \mathbb{R} \), \( -C_2 \leq f(\omega, t, y, z) \leq 0 \).

\textbf{(C.2)} (i) \( g \) is uniformly Lipschitz with respect to \( y \), i.e., there exists a constant \( 0 < C_3 < \infty \) such that for any \( y, y' \in \mathbb{R} \),
\[
|g(\omega, t, y) - g(\omega, t, y')| \leq C_3 |y - y'|.
\]

(ii) For all \( t \in [0, T] \), \( y \in \mathbb{R} \), \( -1 \leq g(t, y) \leq 0 \).

\textbf{(C.3)} For all \( t \in [0, T] \), \( 0 \leq L_t \leq U_t < 1 \).

\textbf{(C.4)} There exist constants \( C_4, C_5 > 0 \) such that \( A_T \leq C_4 \) and \( |R|_T \leq C_5 \).
**Theorem B.1.** Let assumptions (C.1) – (C.4) and (H.4) hold true. Then there exists a unique solution for Equation (2.6). Moreover,

\[
\mathbb{E} \int_0^T | Z_t |^2 \, dt + \mathbb{E}(K_T^2) < +\infty.
\]

**Proof.** Uniqueness follows directly from comparison theorem. Let us focus on the existence. Since the assumptions made in this part are classical, we just give the main steps of the existence proof.

**Step 1.** Assume that \( g \equiv 0 \). Let \( \tilde{Z} \in L^2([0, T] \times \Omega; \mathbb{R}^d) \) such that

\[
R_T - V_T = \mathbb{E}(R_T - V_T) + \int_0^T \tilde{Z}_s dB_s,
\]

where \( V \) is the process in (H.4).

Define \( \mathcal{L} \) and \( \mathcal{U} \) as follows:

\[
\mathcal{L}_t = L_t + R_t - \mathbb{E}(R_T - V_T | \mathcal{F}_t), \quad \mathcal{U}_t = U_t + R_t - \mathbb{E}(R_T - V_T | \mathcal{F}_t),
\]

Let us also define two nonnegative supermartingales \( \theta \) and \( \zeta \) by

\[
\theta_t = S_0 + \mathbb{E}(V_T + R_T^- | \mathcal{F}_t) - V_t - R_t^-, \quad \zeta_t = \mathbb{E}(R_T^+ | \mathcal{F}_t) - R_t^+.
\]

It is not difficult to see that

\[
\mathcal{L}_t \leq \theta_t - \zeta_t \leq \mathcal{U}_t,
\]

and then \( (\mathcal{L}, \mathcal{U}) \) satisfies Mokobodski’s condition. It follows then from Cvitanic and Karatzas [9] or Bahlali et al. [2] that there exists a unique solution to the following GBSDE with two reflecting barriers

\[
\begin{align*}
(i) \quad & Y_t = \xi + V_T + \int_t^T f(s, Y_s, Z_s) \, ds + \mathbb{E}(R_T - V_T | \mathcal{F}_s), Z_s + \tilde{Z}_s) \, ds \nonumber \\
& \quad + \int_t^T dK_s^+ - \int_t^T dK^-_s - \int_t^T Z_s dB_s, \quad t \leq T, \\
(ii) \quad & \forall t \leq T, \mathcal{L}_t \leq \mathcal{Y}_t \leq \mathcal{U}_t, \\
(iii) \quad & \int_0^T (\mathcal{Y}_t - \mathcal{L}_t) dK^+_t = \int_0^T (\mathcal{U}_t - \mathcal{Y}_t) dK^-_t = 0, \text{ a.s.} \\
(iv) \quad & \mathcal{Y} \in \mathcal{C} \quad K^+, K^- \in \mathcal{K} \quad Z \in \mathcal{M}^{2,d}.
\end{align*}
\]

Hence \( (Y_t := \mathcal{Y}_t - R_t + \mathbb{E}(R_T - V_T | \mathcal{F}_t), Z := \mathcal{Z}_t + \tilde{Z}_t, K^+, K^-) \) is the unique solution of the following GBSDE

\[
\begin{align*}
(i) \quad & Y_t = \xi + \int_0^T f(s, Y_s, Z_s) \, ds + \int_t^T dR_s + \int_t^T dK^+_s - \int_t^T dK^-_s \nonumber \\
& \quad - \int_t^T Z_s dB_s, \quad t \leq T, \\
(ii) \quad & \forall t \leq T, L_t \leq Y_t \leq U_t, \\
(iii) \quad & \int_0^T (Y_t - L_t) dK^+_t = \int_0^T (U_t - Y_t) dK^-_t = 0, \text{ a.s.}, \\
(iv) \quad & Y \in \mathcal{C} \quad K^+, K^- \in \mathcal{K} \quad Z \in \mathcal{M}^{2,d}, \\
(v) \quad & dK^+ \perp dK^-.
\end{align*}
\]
Step 2. Assume now that \( g \neq 0 \). We prove existence of solutions by using a Picard approximation. Let 
\( (Y^n, Z^n, K^+, K^-) = (S, 0, 0, 0) \) and consider the following BSDE with two reflecting barriers

\[
\begin{align*}
(i) & \quad Y_t^{n+1} = \xi + \int_t^T f(s, Y_s^{n+1}, Z_s^{n+1})ds + \int_t^T g(s, Y_s^n) dA_s + \int_t^T dR_s \\
& \quad \quad + \int_t^T dK_t^{+, n+1} - \int_t^T dK_t^{-, n+1} - \int_t^T Z_s^{n+1} dB_s, t \leq T, \\
(ii) & \quad \forall t \leq T, \quad L_t \leq Y_t^{n+1} \leq U_t, \\
(iii) & \quad \text{the Skorohod conditions hold:} \\
& \quad \int_0^T (Y_t^{n+1} - L_t) dK_t^{+, n+1} = \int_0^T (U_t - Y_t^{n+1}) dK_t^{-, n+1} = 0, \text{ a.s.} \\
(iv) & \quad Y_t^{n+1} \in C, \quad K_t^{+, n+1}, K_t^{-, n+1} \in K^2, \quad Z_t^{n+1} \in \mathcal{M}^{2, d}, \\
(v) & \quad dK_t^{+, n+1} \perp dK_t^{-, n+1},
\end{align*}
\]

which admits a solution due to Step 1.

By using standard calculations for BSDEs one can prove that there exists a process \( Y \in L^2(\Omega, C[0, T]) \) such that

\[
\lim_{n \to +\infty} \mathbb{E} \sup_{t \leq T} (Y_t^n - Y_t)^2 = 0.
\]

Let \( (\overline{Y}, K^+, K^-, Z) \) be the solution, which exists according to Step 1, of the following GBSDE with two reflecting barriers

\[
\begin{align*}
(i) & \quad \overline{Y}_t = \xi + \int_t^T f(s, \overline{Y}_s, Z_t)ds + \int_t^T g(s, Y_s) dA_s + \int_t^T dR_s \\
& \quad \quad + \int_t^T dK_t^+ - \int_t^T dK_t^- - \int_t^T Z_s dB_s, t \leq T, \\
(ii) & \quad \forall t \leq T, \quad L_t \leq \overline{Y}_t \leq U_t, \\
(iii) & \quad \int_0^T (\overline{Y}_t - L_t) dK_t^+ = \int_0^T (U_t - \overline{Y}_t) dK_t^- = 0, \text{ a.s.} \\
(iv) & \quad \overline{Y}_t \in C, \quad K_t^+, K_t^- \in K^2, \quad Z \in \mathcal{M}^{2, d}, \\
(v) & \quad dK_t^+ \perp dK_t^-.
\end{align*}
\]

It is not difficult to prove that there exists a constant \( C > 0 \) such that

\[
\mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - \overline{Y}_t)^2 \leq C \mathbb{E} \sup_{t \leq T} (Y_t^n - Y_t)^2.
\]

Hence

\[
\lim_{n \to +\infty} \mathbb{E} \sup_{t \leq T} (Y_t^n - \overline{Y}_t)^2 = 0.
\]

It follows that for all \( s \leq T \)

\[
\mathbb{E} \sup_{s \leq T} | Y_s - \overline{Y}_s |^2 = 0,
\]

and then for all \( s \leq T \), \( Y_s = \overline{Y}_s, P\text{-a.s.} \) The proof of existence is then finished.

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