ORIGINAL GRAPHS OF LINK GRAPHS

BIN JIA

Department of Mathematics and Statistics
The University of Melbourne
Victoria 3010, Australia.
Email: jiabinqq@gmail.com Mobile: +61 404639816

Abstract. Let \( \ell \geq 0 \) be an integer, and \( G \) be a graph without loops. An \( \ell \)-link of \( G \) is a walk of length \( \ell \) in which consecutive edges are different. We identify an \( \ell \)-link with its reverse sequence. The \( \ell \)-link graph \( L_\ell(G) \) of \( G \) is defined to have vertices the \( \ell \)-links of \( G \), such that two vertices of \( L_\ell(G) \) are adjacent if their corresponding \( \ell \)-links are the initial and final subsequences of an \( (\ell + 1) \)-link of \( G \). A graph \( G \) is called an \( \ell \)-root of a graph \( H \) if \( L_\ell(G) \cong H \). For example, \( L_0(G) \cong G \). And the 1-link graph of a simple graph is the line graph of that graph. Moreover, let \( H \) be a finite connected simple graph. Whitney’s isomorphism theorem (1932) states if \( H \) has two connected nonnull simple 1-roots, then \( H \cong K_3 \), and the two 1-roots are isomorphic to \( K_3 \) and \( K_{1,3} \) respectively.

This paper investigates the \( \ell \)-roots of finite graphs. We show that every \( \ell \)-root is a certain combination of a finite minimal \( \ell \)-root and trees of bounded diameter. This transfers the study of \( \ell \)-roots into that of finite minimal \( \ell \)-roots. As a qualitative generalisation of Whitney’s isomorphism theorem, we bound from above the number, size, order and maximum degree of minimal \( \ell \)-roots of a finite graph. This work forms the basis for solving the recognition and determination problems for \( \ell \)-link graphs in our future papers. As a byproduct, we characterise the \( \ell \)-roots of some special graphs including cycles.

Similar results are obtained for path graphs introduced by Broersma and Hoede (1989). \( G \) is an \( \ell \)-path root of a graph \( H \) if \( H \) is isomorphic to the \( \ell \)-path graph of \( G \). We bound from above the number, size and order of minimal \( \ell \)-path roots of a finite graph.

Keywords. link graph; root; path graph; path root.

1. Introduction and main results

As a generalisation of line graphs \([17]\) and path graphs \([5]\), the link graph of a graph \( G \) was introduced by Jia and Wood \([11]\) who studied the connectedness, chromatic number and minors of the link graph based on the structure of \( G \). This

Bin Jia gratefully acknowledges scholarships provided by The University of Melbourne.
paper deals with the reverse; that is, for an integer \( \ell \geq 0 \) and a finite graph \( H \), we study the graphs whose \( \ell \)-link graphs are isomorphic to \( H \).

Unless stated otherwise, all graphs are undirected and loopless. A graph may be finite or infinite, and may be simple or contain parallel edges. In particular, \( H \) always denotes a finite graph. The order and size of \( H \) are \( n(H) := |V(H)| \) and \( m(H) := |E(H)| \) respectively. Throughout this paper, \( \ell \geq 0 \) is an integer. An \( \ell \)-link is a walk of length \( \ell \) in which consecutive edges are different. We identify an \( \ell \)-link with its reverse sequence. An \( \ell \)-path is an \( \ell \)-link without repeated vertices.

The \( \ell \)-link graph \( L_\ell(G) \) of a graph \( G \) is defined to have vertices the \( \ell \)-links of \( G \), and two vertices are adjacent if their corresponding \( \ell \)-links are the initial and final subsequences of an \( (\ell + 1) \)-link of \( G \). Furthermore, if \( G \) contains parallel edges, then by [11, Observation 3.4], two \( \ell \)-links of \( G \) may be the initial and final subsequences of each of \( \mu \geq 2 \) different \( (\ell + 1) \)-links of \( G \). In this case, there are \( \mu \) edges between the two corresponding vertices in \( L_\ell(G) \). Rigorous definitions are given in Section 2.

A graph \( G \) is called an \( \ell \)-root of \( H \) if \( L_\ell(G) \cong H \). Let \( \mathbb{R}_\ell(H) \) be the set of minimal (up to the subgraph relation) \( \ell \)-roots of \( H \). The line graph \( L(G) \) of \( G \) is the simple graph with vertex set \( E(G) \), in which two vertices are adjacent if their corresponding edges share a common end vertex in \( G \). By definition \( L_1(G) = L(G) \) if and only if \( G \) is simple. So Whitney’s isomorphism theorem [17] can be restated as this: Let \( H \) be a connected nonnull simple graph. Then \( |\mathbb{R}_1(H)| \leq 2 \), with equality holds if and only \( H \cong K_3 \) and \( \mathbb{R}_1(H) = \{K_3, K_{1,3}\} \). Moreover, it is not difficult to see that every 1-root of \( K_3 \) is the disjoint union of one of \( K_3 \) and \( K_{1,3} \), and zero or more isolated vertices. Motivated by Whitney’s isomorphism theorem, this paper aims to answer the following three general questions for a finite graph \( H \) and an integer \( \ell \) to some extent:

- How many minimal \( \ell \)-roots can \( H \) have?
- How large can a minimal \( \ell \)-root of \( H \) be?
- How to construct all \( \ell \)-roots of \( H \) from minimal \( \ell \)-roots of \( H \)?

We answer the first two questions by the following theorem, which is a qualitative generalisation of Whitney’s isomorphism theorem.

**Theorem 1.1.** Let \( \ell \geq 0 \) be an integer, and \( H \) be a finite graph. Then the maximum degree, order, size, and total number of minimal \( \ell \)-roots of \( H \) are finite and bounded by functions of \( \ell \) and \( |V(H)| \).

**Remark.** The order and size of minimal \( \ell \)-roots of \( H \) are bounded in Lemma 6.3. Then other parameters are trivially bounded. But we are able to improve the remaining bounds by further investigating the structure of link graphs. In particular, \( |\mathbb{R}_\ell(H)| \) is bounded in Lemma 6.4. And the maximum degree of a minimal \( \ell \)-root is bounded in Corollary 6.6 and Lemma 6.7.

Before continuing, we give more motivations for the first question. First of all, for some infinite families of graphs \( H \), \( |\mathbb{R}_\ell(H)| \) is bounded by a constant number.
For example, Lemma 3.2 characterises all minimal $\ell$-roots of a cycle, which implies that the number of minimal $\ell$-roots of a cycle is one or two. However, $|\mathbb{R}_\ell(H)|$ is not always bounded by a constant number. For instance, Lemma 3.1 finds all minimal $\ell$-roots of $2K_1$, which indicates that the number of minimal $\ell$-roots of $2K_1$ increases with $\ell$. On the other hand, in Section 3 we exemplify that, for fixed $\ell \geq 4$ and any given number $k$, there exists a connected graph $H$ with $k$ or more minimal connected $\ell$-roots. The last two examples show that in Theorem 1.1, it is necessary that the number of minimal $\ell$-roots of $H$ are bounded by functions of $\ell$ and $|V(H)|$.

Denote by $X \subseteq Y$, $X \subset Y$, $X \leq Y$, and $X < Y$ that $X$ is isomorphic to a subgraph, proper subgraph, induced subgraph and proper induced subgraph of a graph $Y$ respectively. A graph is $\ell$-finite if its $\ell$-link graph is finite. So all finite graphs are $\ell$-finite, but not vice versa. For example, let $T'$ be the tree obtained by pasting the middle vertex of a 4-path at the center of a star $K_{1,t}$. Then $T'$ is infinite, and is 4-finite since its 4-link graph is $K_1$.

Two $\ell$-finite graphs $X$ and $Y$ are $\ell$-equivalent, write $X \sim_\ell Y$, if there exists a graph $Z \subseteq X,Y$ such that $L_\ell(Z) \cong L_\ell(X) \cong L_\ell(Y)$. For example, for every pair of integers $i,j \geq 0$, we have $T^i \sim_4 T^j$ since $L_4(T^i) \cong L_4(T^j) \cong L_4(T^0) \cong K_1$. An $\ell$-finite graph $X$ is $\ell$-minimal if $X$ is null or $L_\ell(Y) \subset L_\ell(X)$ for every $Y \subset X$. For instance, an $\ell$-path is $\ell$-minimal. By definitions, a graph is $\ell$-minimal if and only if it is a minimal $\ell$-root of a finite graph.

Let $\mathbb{R}_\ell[H]$ be the set of $\ell$-roots of $H$. By the analysis above, $\mathbb{R}_4[K_1]$ is an equivalence class with equivalence relation $\sim_4$. Moreover, $\mathbb{R}_1[K_3]$ is the union of two equivalence classes with equivalence relation $\sim_1$. And the two classes contain $K_3$ and $K_{1,3}$ respectively. The lemma below is proved in Section 3. Together with Theorem 1.1 it says that $\sim_\ell$ divides $\mathbb{R}_\ell[H]$ into finitely many equivalence classes.

**Lemma 1.2.** For each integer $\ell \geq 0$, $\sim_\ell$ is an equivalence relation on $\ell$-finite graphs, such that each $\ell$-equivalence class contains a unique (up to isomorphism) $\ell$-minimal graph. And this graph is isomorphic to an induced subgraph of every graph in its class.

The *distance* $\text{dist}_G(u,v)$ between $u,v \in V(G)$ is $+\infty$ if $u,v$ are in different components of $G$, and the minimum length of a path of $G$ between $u,v$ otherwise. The *eccentricity* of $v \in V(G)$ is $\text{ecc}_G(v) := \sup\{\text{dist}_G(u,v) | u \in V(G)\}$. The *diameter* of $G$ is $\text{diam}(G) := \sup\{\text{ecc}_G(v) | v \in V(G)\}$.

Lemma 1.3 answers the third general question of this paper, which is also proved in Section 4. It says that an $\ell$-root of a finite graph is a certain combination of a minimal $\ell$-root and trees of bounded diameter. This transfers the study of $\ell$-roots into that of minimal $\ell$-roots. In Section 3 we first obtain all minimal $\ell$-roots of a cycle. Then we characterise all $\ell$-roots of a cycle by applying the lemma below.
Lemma 1.3. Let $G$ be the minimal graph of an $\ell$-equivalence class. Then a graph belongs to this class if and only if it can be obtained from $G$ in two steps:

1. For each acyclic component $T$ of $G$ of diameter within $[\ell, 2\ell - 4]$, and every vertex $u$ of eccentricity $s$ in $T$ such that $[\ell/2] \leq s \leq \ell - 2$, paste to $u$ the root of a rooted tree of height at most $\ell - s - 1$.

2. Add to $G$ zero or more acyclic components of diameter at most $\ell - 1$.

Introduced by Broersma and Hoede [5], the $\ell$-path graph $\mathbb{P}_\ell(G)$ is the simple graph with vertices the $\ell$-paths of $G$, where two vertices are adjacent if the union of their corresponding paths forms a path of length $\ell + 1$ or a cycle of length $\ell + 1$ in $G$. Let $\ell \geq 2$. It follows from the definition that $\mathbb{P}_\ell(G)$ is a subgraph of $\mathbb{L}_\ell(G)$. And the two graphs are isomorphic if and only if $girth(G) > \ell$. For each $\ell \geq 0$, we say $G$ is an $\ell$-path root of $H$ if $\mathbb{P}_\ell(G) \cong H$. Let $Q_\ell(H)$ be the set of minimal (up to the subgraph relation) $\ell$-path roots of $H$. Li [12] proved that $H$ has at most one simple 2-path root of minimum degree at least 3. Prisner [15] showed that $Q_\ell(H)$ contains at most one simple graph of minimum degree greater than $\ell$. By Li and Liu [13], if $H$ is connected and nonnull, then $Q_2(H)$ contains at most two simple graphs. In fact, the finite graphs having exactly two minimal simple 2-path roots have been characterised by Aldred, Ellingham, Hemminger and Jipsen [2]. Some results about $\ell$-roots can be proved, with slight variations, for $\ell$-path roots:

Theorem 1.4. Let $\ell \geq 0$ be an integer, and $H$ be a finite graph. Then the order, size, and total number of minimal $\ell$-path roots of $H$ are finite and bounded by functions of $\ell$ and $|V(H)|$.

2. Terminology

This section presents some definitions and simple facts. The reader is referred to [8] for notation and terminology on finite and infinite graphs. A graph is said to be cyclic if it contains a cycle, and acyclic otherwise. Let $G$ be a graph, and $c(G)$ (respectively, $o(G)$, $a(G)$) be the cardinality of the set of (respectively, cyclic, acyclic) connected components of $G$. A ray is an infinite graph with vertex set $\{v_0, v_1, \ldots\}$ and edges $e_i$ between $v_{i-1}$ and $v_i$, for $i \geq 1$. The radius $\text{rad}(G)$ of $G$ is $+\infty$ if $G$ is disconnected, and $\min\{\text{ecc}_G(v) | v \in V(G)\}$ otherwise. For each tree $T$, Wu and Chao [18] proved that $\text{rad}(T) = \lceil\text{diam}(T)/2\rceil$.

Denote by $K_t$ the complete graph on $t \geq 0$ vertices. In particular, $K_0$ is the null graph. Let $tG$ be the disjoint union of $t \geq 0$ copies of $G$. For $t \geq 1$, $tK_1$ is called the empty graph on $t$ vertices. For $s \geq 1$, the $s$-subdivision $G^{(s)}$ of $G$ is the graph obtained by replacing every edge of $G$ with an $s$-path. So $G^{(1)} = G$. Let $e$ be an edge of a tree $T$ with end vertices $u$ and $v$. Let $T_e^u$ be the component of $T - e$ containing $u$. Let $T_e^v := T_e^v \cup \{e\}$. A unit is a vertex or an edge. The subgraph of $G$ induced by $U \subseteq V(G)$ is the maximal subgraph of $G$ with vertex
set $U$. For $\emptyset \neq F \subseteq E(G)$, the subgraph of $G$ induced by $F \cup U$ is the minimal subgraph of $G$ with edge set $F$, and vertex set including $U$.

An $\ell$-arc (or $*$-arc if we ignore the length) is a sequence $\vec{L} := (v_0, e_1, \ldots, e_\ell, v_\ell)$, where $e_i$ is an edge with end vertices $v_{i-1}$ and $v_i$ such that $e_j \neq e_{j+1}$ for $i \in [\ell] := \{1, 2, \ldots, \ell\}$ and $j \in [\ell - 1]$. Note that $\vec{L}$ is different from $-\vec{L} := (v_\ell, e_\ell, \ldots, e_1, v_0)$ unless $\ell = 0$. For each $i \in [\ell]$, $\vec{e}_i := (v_{i-1}, e_i, v_i)$ is called an arc for short. $v_0, v_\ell, \vec{e}_1$ and $\vec{e}_\ell$ are the tail vertex, head vertex, tail arc and head arc of $\vec{L}$ respectively. The $\ell$-link (or $*$-link if we ignore the length) $L := [v_0, e_1, \ldots, e_\ell, v_\ell] = [v_\ell, e_\ell, \ldots, e_1, v_0]$ is obtained by taking $\vec{L}$ and $-\vec{L}$ as a single object; that is, $L := \{\vec{L}, -\vec{L}\}$. For example, a 0-link is a vertex, and a 1-link can be identified with an edge. For $0 \leq i \leq j \leq \ell$, $\vec{R} := \vec{L}(i, j) := (v_i, e_{i+1}, \ldots, e_j, v_j)$ is called a $(j - i)$-arc (or a subsequence for short) of $\vec{L}$, and $\vec{L}[i, j] := R$ is a $(j - i)$-link (or a subsequence for short) of $L$. For $\ell \geq 1$, we say $L$ is formed by the $(\ell - 1)$-links $\vec{L}[0, \ell - 1]$ and $\vec{L}[1, \ell]$. An $\ell$-dipath is an $\ell$-arc without repeated vertices. We say $\vec{L}$ is an $\ell$-dicycle if $v_0 = v_\ell$ and $\vec{L}(0, \ell - 1)$ is a dipath. And in this case, $L$ is an $\ell$-cycle. For $\ell \geq 2$, we usually use $\vec{C}_\ell$ to denote an $\ell$-dicycle, and use $C_\ell$ to denote an $\ell$-cycle. An $\ell$-path is an $\ell$-link without repeated vertices. We use $\vec{L}_\ell(G), \mathcal{L}_\ell(G)$, and $\mathcal{R}_\ell(G)$ to denote the sets of $\ell$-arcs, $\ell$-links, and $\ell$-paths of $G$ respectively.

Let $\vec{L} := (v_0, e_1, \ldots, e_\ell, v_\ell) \in \vec{L}_\ell(G)$, and $\vec{R} := (u_0, f_1, \ldots, f_s, u_s) \in \vec{L}_s(G)$ such that $v_\ell = u_0$ and $e_\ell \neq f_1$. The conjunctions of $\vec{L}$ and $\vec{R}$ are $Q := (\vec{L}, \vec{R}) := (v_0, e_1, \ldots, e_\ell, v_\ell = u_0, f_1, \ldots, f_s, u_s) \in \vec{L}_{\ell+s}(G)$ and $[\vec{L}, \vec{R}] := Q \in \mathcal{L}_{\ell+s}(G)$. For $Q \in \mathcal{L}_{\ell+s}(G)$, let $\vec{L}_i := Q(i, \ell + i)$, and $\vec{Q}_j := Q(j - 1, \ell + j)$ for $i \in [0, s]$ and $j \in [s]$. By definition $Q_j \in \mathcal{L}_{\ell+1}(G)$ yields an edge $Q_j^\ell := [L_{j-1}, Q_j, L_j]$ of $\mathcal{L}_\ell(G)$. So $Q^\ell := [L_0, Q_1, L_1, \ldots, L_{s-1}, Q_s, L_s]$ can be seen as an $s$-link, while $Q^\ell := (L_0, Q_1, L_1, \ldots, L_{s-1}, Q_s, L_s)$ is an $s$-arc of $\mathcal{L}_\ell(G)$. We say that $L_0$ can be shunted to $L_s$ through $\vec{Q}$. $Q^\ell := \{L_0, L_1, \ldots, L_s\}$ and $Q^\ell := \{\vec{L}_0, \vec{L}_1, \ldots, \vec{L}_s\}$ are the sets of images of $L_0$ and $\vec{L}_0$ respectively during this shunting. More generally, for $R, R' \in \vec{L}_\ell(G)$, we say $R$ can be shunted to $R'$ if there are $\ell$-links $R = R_0, R_1, \ldots, R_s = R'$, and $*$-arcs $\vec{P}_1, \ldots, \vec{P}_s$ of $G$ such that $R_{i-1}$ can be shunted to $R_i$ through $\vec{P}_i$ for $i \in [s]$.

3. Examples and basis

We begin with some examples and basic analysis that help to build some general impressions on $\ell$-roots, and explain some of our motivations.

First of all, we characterise the minimal $\ell$-roots of $2K_1$.

Lemma 3.1. Let $P := [v_0, \ldots, v_\ell]$ be an $\ell$-path, and $T_i$ be obtained from $P$ by pasting $v_i$ at an end vertex of another $i$-path, where $i \in [0, \ell]$. Then $\vec{R}_\ell(2K_1) = \{2P, T_i | 1 \leq i \leq \lceil \frac{\ell}{2} \rceil\}$. Further, $|\vec{R}_\ell(2K_1)|$ is 1 if $\ell = 0$, and is $\lceil \frac{\ell + 1}{2} \rceil$ if $\ell \geq 1$. 

Proof. Clearly, for $1 \leq i \leq \left\lceil \frac{\ell - 1}{2} \right\rceil$, $L_\ell(2P) \cong L_\ell(T_i) \cong 2K_1$. If $G \in \mathbb{R}_\ell(2K_1)$ contains a cycle $C$, then $L_\ell(G)$ contains a cycle $L_\ell(C)$, which is impossible. Thus $G$ is a forest containing exactly one $\ell$-path $Q$ other than $P$. If $P$ and $Q$ are vertex disjoint, then $G = P \cup Q \cong 2P$ because of the minimality. Otherwise, assign directions such that $\vec{P} = (\vec{P}_1, \vec{R}, \vec{P}_2)$ and $\vec{Q} = (\vec{Q}_1, \vec{R}, \vec{Q}_2)$, where $R$ is the maximal common path of $P$ and $Q$. $P_i \in \mathcal{P}_{s_i}(G)$ and $Q_i \in \mathcal{P}_{t_i}(G)$ for $i \in \{1, 2\}$. Since $P \neq Q$, without loss of generality, $\vec{P}_1 \neq \vec{Q}_1$ and $s_1 \geq t_1$. Then $s_2 \leq t_2$, and $\vec{L} := (\vec{P}_1, \vec{R}, \vec{Q}_2) \in \mathcal{L}_{t_2-s_2}(G) \setminus \{\vec{Q}\}$. Since $2K_1$ contains no edge, $\mathcal{L}_{t+1}(G) = \emptyset$ and so $s_2 = t_2$. Thus $s_1 = t_1 \geq 1$, and $L \in \mathcal{P}_{t}(G)$. So $\vec{L} = \vec{P}$ since otherwise, $G$ contains three pairwise different $\ell$-paths $L$, $P$ and $Q$. Note that $[\vec{P}_1, - \vec{Q}_1] \in \mathcal{P}_{2s_1}(G) \setminus \{P, Q\}$. So $2s_1 < \ell$ and the lemma follows.

![Figure 1. (1) $\mathbb{R}_1(C_3)$ (2) $\mathbb{R}_3(C_4)$ (3) $\mathbb{R}_\ell(C_5)$ (4) $\mathbb{R}_2(C_6)$](image)

Whitney [17] proved that $\mathbb{R}_1(K_3) = \{K_3, K_{1,3}\}$ (Figure 1(1)). As a generalisation, Broersma and Hoede [5] pointed out that a 6-cycle is the 2-path (and hence 2-link) graph of $K_{1,3}^{(2)}$ and itself (Figure 1(4)). Figure 1(2) are the minimal 3-roots of $C_4$. Figure 1(3) is the minimal $\ell$-root of $C_5$, which is $C_5$ itself. More generally, we now characterise the minimal $\ell$-roots of all cycles. Clearly, for a given $\ell \geq 0$, every cycle has a unique cyclic minimal $\ell$-root which is isomorphic to itself. So we only need to consider acyclic minimal $\ell$-roots.

**Lemma 3.2.** Let $T$ be a minimal acyclic $\ell$-root of a $t$-cycle. Then $\ell \geq 1$, and either $t = 3\ell$ and $T \cong K_{1,3}^{(\ell)}$, or there is $s \geq 1$ such that $t = 4s$, $\ell \geq 2s + 1$, and $T$ is obtained by joining the middle vertices of two 2s-paths by an $(\ell - s)$-path.

Lemma 3.2 is proved in Section 6. Together with Lemma 1.3, it gives all $\ell$-roots $G$ of a $t$-cycle as follows: If $G$ is cyclic, it is the disjoint union of a $t$-cycle and zero or more trees of diameter at most $\ell - 1$. Otherwise, $G$ is a forest and
\( \ell \geq 1 \). In this situation, if \( t = 3\ell \), since \( \text{diam}(K_{1,\ell}^{(\ell)}) = 2\ell \), then \( G \) is the disjoint union of \( K_{1,\ell}^{(\ell)} \) and trees of diameter at most \( \ell - 1 \). In the final case of Lemma 3.2, \( \text{diam}(T) = \ell + s \leq 2\ell - 2 \). Let \([v_0, \ldots, v_{\ell-s}]\) be the path of \( T \) between the middle vertices of the two 2s-paths. Then \( \text{ecc}_T(v_i) = \max\{i, \ell - i\} \) for \( i \in [\ell - s] \). So \( G \) is obtained from \( T \) by first pasting to each \( v_i \), where \( i \in \{2, 3, \ldots, \ell - s - 2\} \), the root of a rooted tree of height less than \( \min\{i, \ell - s - i\} \), and then adding acyclic components of diameter less than \( \ell \).

In comparison with Lemma 3.2, a 4s-cycle has at least three minimal \((2s+1)\)-path roots, two of which are cyclic. Let \( t \geq 1 \) and \( \ell \geq s + 1 \) be integers. Let \( G \) be the graph formed by connecting two \((s+1)\)-cycles with an \((\ell - s)\)-path. One can easily check that \( G \) is a minimal \( \ell \)-path root of a 4s-cycle.

Broersma and Hoede [5] asked that, for \( \ell = 2 \), whether there exist three pairwise non-isomorphic simple connected graphs whose \( \ell \)-path graphs are isomorphic to the same connected nonnull graph. A negative answer was given by Li and Liu [13]. We now give examples of graphs \( H \) for which \( |R_3(H)| \geq 3 \), and for each \( \ell \geq 4 \), there are graphs \( H \) such that \( |R_3(H)| \) is unbounded. The following construction will be useful.

![Figure 2. The 3-link graph of \( T(v, 3) \) is \( T \)](image)

Let \( T \) be a finite tree, and \( v \) be a vertex of degree \( d \) in \( T \). Assign to \( v \) an integer \( t_v \) as follows: If \( d \geq 2 \), then \( t_v := \text{diam}(T) \). If \( d \leq 1 \) and \( T \) is a path, then \( t_v := -1 \). Otherwise, \( d = 1 \), and there exists a path \([v, \ldots, e, u]\) of minimum length such that \( \deg_T(u) \geq 3 \). In this case, let \( t_v := \text{diam}(T_u) \). Denote by \( T(v, \ell) \) the tree obtained by pasting an end vertex of an extra \( \ell \)-path to \( v \). For example, in Figure 2, \( T \) is a 2-path with middle vertex \( v \), and \( T(v, 3) \) is the tree obtained from \( T \) by pasting an end vertex of a 3-path at \( v \). It is not difficult to see that the 3-link graph \( T(v, 3) \) is isomorphic to \( T \).

Figure 2 is just a special case of the following lemma which says that every tree \( T \) is an \( \ell \)-link graph for each \( \ell > \text{diam}(T) \).

**Lemma 3.3.** Let \( T \) be a tree, and \( \ell \geq t_v + 1 \). Then \( T \cong \mathbb{L}_\ell(T(v, \ell)) = \mathbb{P}_\ell(T(v, \ell)) \).

**Proof.** Let \( \bar{L} \) be the \( \ell \)-arc of head vertex \( v \) such that \( L \) is the extra path. Consider the shunting of \( L \) in \( G := T(v, \ell) \). One can check that the mapping \( \bar{L}'[\ell, \ell] \mapsto L' \), for every image \( \bar{L}' \) of \( \bar{L} \), is an isomorphism from \( T \) to \( \mathbb{L}_\ell(G) \). \( \blacksquare \)
An orbit of a graph $G$ is a maximal subset $U \subseteq V(G)$ such that for every pair of vertices in $U$, one can be mapped to the other by an automorphism of $G$ (see [3] for more about algebraic graph theory). For each $\ell \geq 1$, the number of non-isomorphic trees $T(v, \ell)$, over all $v \in V(T)$, equals the number of orbits of $T$. Two vertices in the same orbit have the same eccentricity. So the number of orbits of $T$ is at least the radius plus one, which is $\lfloor \text{diam}(T)/2 \rfloor + 1$, with equality holds if and only if the set of leaves is an orbit of $T$.

As explained below, Lemma 3.3 implies that there are infinitely many trees $T$ of diameter 3 such that $Q_{\ell}(T) \cap R_{\ell}(T)$ contains at least four trees, where $\ell \geq 3$.

Let $T$ be obtained by adding an edge between the centers of $K_{1,p}$ and $K_{1,q}$, where $p > q \geq 1$. Then $\text{diam}(T) = 3$, and $T$ has four orbits. So there are exactly four non-isomorphic $T(v, \ell)$’s for each $\ell \geq 3$.

Let $\ell \geq s \geq 4$ and $k \geq 1$ be given integers. Lemma 3.3 also implies that there exists a tree $T$ of diameter $s$, such that $Q_{\ell}(T) \cap R_{\ell}(T)$ contains at least $k$ trees: Let $T(k)$ be obtained by pasting a leaf of each star $K_{1,i+1}$, where $i \in [k]$, at the same end vertex of an $(s-2)$-path. Then $\text{diam}(T(s)) = s$, and the number of orbits of $T(k)$ is $\lfloor \frac{s}{2} \rfloor + 1$ if $k = 1$, and $s + 2k - 1$ if $k \geq 2$.

4. Constructing $\ell$-equivalence classes

In this section, we explain the process of constructing $\ell$-roots from minimal $\ell$-roots, which allows us to concentrate on the latter in our future study.

4.1. Incidence units. Two $*$-links of a graph $G$ are incident if one is a subsequence of the other. A $*$-link is said to be $\ell$-incident if it is incident to an $\ell$-link. It follows from the definitions immediately that every $\ell$-link is $\ell$-incident, and every $\ell$-incident $*$-link is $s$-incident, for $s \leq \ell$. Conversely, a $t$-link is not $\ell$-incident if and only if it is not $s$-incident for each $s \geq \ell$. And if this is the case, then $\ell \geq t + 1$. Let $T$ be a tree obtained by joining the centres of two stars such that each star contains at least two edges. Then all units of $T$ are 3-incident. However, every 2-path of a star is not 3-incident in $T$. The fact below allows us to focus on incidence units of trees of finite diameter.

**Lemma 4.1.** Let $G$ be a connected nonnull graph. Then $G$ contains a cycle or a ray if and only if for every $\ell \geq 0$, all units of $G$ are $\ell$-incident.

**Proof.** ($\Leftarrow$) Suppose not. Then $G$ is a tree of finite diameter $s$. Then no unit of $G$ is $(s+1)$-incident. ($\Rightarrow$) Let $X$ be a cycle or a ray in $G$. Clearly, every unit of $X$ is $\ell$-incident. So we only need to show that every $e \in E(G) \setminus E(X)$ is $\ell$-incident. Since $G$ is connected, there exists a dipath $\tilde{P}$ of minimum length with tail edge $e$ and head vertex $x \in V(X)$. Clearly, $X$ contains an $\ell$-arc $\tilde{R}$ starting from $x$. Then $L := (\tilde{P}, \tilde{R})[0, \ell]$ is an $\ell$-link of $G$ incident to $e$. 

\[ \blacksquare \]
Let $T$ be a tree. By definitions, if $\text{diam}(T) \leq \ell - 1$, then none unit of $T$ is $\ell$-incident. The following statement says that, to study $\ell$-incidence units of $T$, we only need to consider the case of $\ell \leq \text{diam}(T) \leq 2\ell - 4$.

**Lemma 4.2.** Let $T$ be a tree of diameter at least $\max\{\ell, 2\ell - 3\}$. Then all units of $T$ are $\ell$-incident.

**Proof.** Let $s := \text{diam}(T)$, and $Q$ be an $s$-path of $T$. Since $s \geq \ell$, every unit of $Q$ is $\ell$-incident. So we only need to show that each $e \in E(T) \setminus E(Q)$ is $\ell$-incident in $T$. Let $\bar{P}$ be a dipath of minimum length $t$ with tail edge $e$ and head vertex some $u \in V(Q)$. Then $t \geq 1$, and $P$ and $Q$ are edge disjoint. Clearly, $Q$ has a dipath $\bar{R}$ of length at least $\lceil s/2 \rceil \geq \ell - 1$ from $u$. So $e$ is incident to the path $[\bar{P}, \bar{R}]$ of length at least $t + \ell - 1 \geq \ell$. □

Wu et al. [18] presented a linear time algorithm computing the eccentricity of a vertex of a finite tree. Based on this work, the following observation provides a linear time algorithm testing if a vertex is $\ell$-incident in a finite tree.

**Observation 4.3.** Let $T$ be a tree and $\ell \geq 0$ be an integer. Then $u \in V(T)$ is $\ell$-incident in $T$ if and only if either $u$ is a leaf and $\text{ecc}_T(u) \geq \ell$, or there exist different $e, f \in E(T)$ incident to $u$, such that $\text{ecc}_{T_u}(u) + \text{ecc}_{T_f}(u) \geq \ell$.

Based on Observation 4.3, the lemma below can be formalised into a linear time algorithm for testing if an edge of a finite tree is $\ell$-incident.

**Lemma 4.4.** Let $\ell \geq 0$, and $P$ be a path of a tree $T$. Then all units of $P$ are $\ell$-incident in $T$ if and only if both end vertices of $P$ are $\ell$-incident in $T$.

**Proof.** We only need to consider ($\Leftarrow$) with the length of $P$ at least 1. The case of $\ell \leq 3$ follows from Lemma 4.2. Now let $\ell \geq 4$. For a contradiction, let $P$ be a minimal counterexample such that its ends $u, v$ are contained in two $\ell$-paths $Q_u$ and $Q_v$ respectively. Clearly, $Q_u$ contains a sub-path $L_u$ starting from $u$ and of length $s_u \geq \lceil \ell/2 \rceil$. By the minimality of $P$, none inner vertex of $P$ belongs to $Q_u$ or $Q_v$. So the union of $L_u, P$ and $L_v$ forms a path of length at least $s_u + s_v + 1 > \ell$ in $T$, contradicting that $P$ is not $\ell$-incident. □

### 4.2. Incidence subgraphs.

The $\ell$-incidence subgraph $G[\ell]$ of a graph $G$ is the graph induced by the $\ell$-incidence units of $G$. By definition, $G[\ell] = G$ if $\ell = 0$ or $G$ is null. And for each $\ell \in \{1, 2, 3\}$, $G[\ell]$ can be obtained from $G$ by deleting all acyclic components of diameter less than $\ell$. For each $G \in \mathbb{R}_\ell[K_1]$, $G[\ell]$ is an $\ell$-path. The statements below, follow from the definitions and Lemma 4.1, allow us to concentrate on incidence subgraphs of trees of finite diameter.

**Corollary 4.5.** Let $G$ be a graph, and $s \geq \ell \geq 0$ be integers. Then every $s$-link of $G$ belongs to $G[\ell]$. If further $G$ is nonnull and connected, then $G$ contains a cycle or a ray if and only if for every $\ell \geq 0$, $G = G[\ell]$. 

A rough structure of $T[\ell]$ can be derived from Lemma 4.4.

**Corollary 4.6.** Let $T$ be a tree of finite diameter, and $\ell \leq \text{diam}(T)$. Then $T[\ell]$ is an induced subtree of $T$. And each leaf of $T[\ell]$ is a leaf of $T$.

**Proof.** By Lemma 4.4, $T[\ell] \subseteq T$. Let $v$ be a leaf of $T[\ell]$. By Corollary 4.5, there is an $\ell$-path $L$ of $T[\ell]$ with an end $v$. Suppose for a contradiction that $v$ is not a leaf of $T$. Then there exists $e \in E(T) \setminus E(T[\ell])$ incident to $v$. Then the union of $e$ and $L$ forms an $(\ell+1)$-path of $T$, contradicting that $e$ is not $\ell$-incident. 

Let $u \in V(T)$ and $X$ be a subtree of $T$. Denote by $T_X^u$ the component of $T - E(X)$ containing $u$. An accurate structure of $T[\ell]$ is given as follows.

**Lemma 4.7.** Let $T$ be a tree of finite diameter, $\ell \leq \text{diam}(T)$, and $X$ be a subtree of $T$. Then $X = T[\ell]$ if and only if $X = X[\ell]$, and for each $u \in V(X)$, either

1. $\text{ecc}_X(u) \geq \ell - 1$, and $T^u := T_X^u$ is a single vertex $u$. Or
2. $[\ell/2] \leq \text{ecc}_X(u) \leq \ell - 2$, and $\text{ecc}_X(u) + \text{ecc}_{T^u}(u) \leq \ell - 1$.

**Proof.** If $\ell \leq 3$, then by Lemma 4.2, $T = T[\ell]$ and the lemma follows. Now let $\ell \geq 4$. ($\Rightarrow$) By Corollary 4.5, we have $s := \text{diam}(X) = \text{diam}(T) \geq \ell$, and every $u \in V(X)$ is $\ell$-incident in $X$. So $X[\ell] = X$ is nonnull, and $s \geq \text{ecc}_X(u) \geq \text{radi}(X) = \lceil s/2 \rceil \geq \lceil \ell/2 \rceil$. By Corollary 4.6, $T^u$ is a maximal subtree of $T$, of which the only unit that is $\ell$-incident in $T$ is the vertex $u$. Let $v \in V(T^u)$ such that $t := \text{dist}(u,v) = \text{ecc}_{T^u}(u)$. Then $T^u$ is not a single vertex and if only if $t \geq 1$. If this is the case, then $\ell - 1 \geq \text{ecc}_{T^u}(v) \geq t + \text{ecc}_X(u)$, and the statement follows.

($\Leftarrow$) Since $X = X[\ell] \subseteq T[\ell]$, we only need to show that $T[\ell] \subseteq X$. Suppose not. Then there exists some $P := (v_0, e_1, \ldots, e_\ell, v_\ell) \in \mathcal{L}_\ell(T)$, and a maximum $s \in [\ell]$ such that $\bar{P}[0, s]$ belongs to $T^u$, where $u := v_s$.

According to (2), $\text{radi}(T^u) \leq \text{ecc}_{T^u}(u) \leq \ell - 1 - \text{ecc}_X(u) \leq \ell - 1 - \lceil \ell/2 \rceil = \lceil \ell/2 \rceil - 1$. So $\text{diam}(T^u) \leq 2 \text{radi}(T^u) \leq \ell - 2$. Hence there is a maximum $t \geq s + 1$ such that $t \leq \ell$ and $\bar{R} := \bar{P}[s, t]$ belongs to $X$. Since $\text{ecc}_X(u) + \text{ecc}_{T^u}(u) \leq \ell - 1$, we have $t \leq \ell - 1$. Since $X = X[\ell]$ is nonnull, there exists $\bar{L} := (\bar{L}_1, \bar{P}(s_1, t_1), \bar{L}_2) \in \mathcal{L}_\ell(X)$, where $s \leq s_1 < t_1 \leq t$, and $\bar{L}_1$ and $\bar{L}_2$ are edge disjoint with $P$.

Since $\text{ecc}_X(u) + \text{ecc}_{T^u}(u) < \ell$, $(\bar{P}(0, t_1), \bar{L}_2)$ is a dipath of length less than $\ell$. So $\bar{L}_2$ is of length less than $\ell - t_1$. Since $\text{ecc}_X(v_t) + \text{ecc}_{T^u}(v_t) < \ell$, we have $(\bar{L}_1, \bar{P}(s_1, t))$ is a dipath of length less than $\ell$. Thus $\bar{L}_1$ is of length less than $s_1$. But then $\bar{L}$ is of length less than $s_1 + t_1 - s_1 + \ell - t_1 = \ell$, a contradiction.

### 4.3. Equivalence classes.

In this subsection we build the relationships among $\ell$-minimal graphs, $\ell$-incidence subgraphs, and $\ell$-equivalence classes.

**Lemma 4.8.** Each $\ell$-finite graph $G$ is $\ell$-equivalent to $G[\ell]$.

**Proof.** By Corollary 4.5, all $\ell$-links and $(\ell + 1)$-links of $G$ belong to $G[\ell]$. So $\mathbb{L}_\ell(G) = \mathbb{L}_\ell(G[\ell])$. Note that $G[\ell] \subseteq G$. Thus the lemma follows.
The following lemma links \( \ell \)-incidence units with \( \ell \)-minimal graphs.

**Lemma 4.9.** An \( \ell \)-finite graph \( G \) is \( \ell \)-minimal if and only if \( G = G[\ell] \).

**Proof.** By Corollary [4.5], every unit of \( G[\ell] \) is \( \ell \)-incident in \( G[\ell] \). So deleting a unit from \( G[\ell] \) erases at least one \( \ell \)-link from \( G[\ell] \). So \( G[\ell] \) is \( \ell \)-minimal. Conversely, if \( G[\ell] \subset G \), then \( G \) is not \( \ell \)-minimal since, by Lemma 4.8, \( \mathbb{L}_\ell(G[\ell]) = \mathbb{L}_\ell(G) \). \( \blacksquare \)

Below we connect \( \ell \)-equivalence relation and \( \ell \)-incidence graphs.

**Lemma 4.10.** Given \( \ell \)-equivalence graphs \( X, Y \), \( X \sim_\ell Y \) if and only if \( X[\ell] \cong Y[\ell] \).

**Proof.** (\( \Leftarrow \)) Let \( Z := X[\ell] \subseteq X, Y \). By Lemma 4.8, \( \mathbb{L}_\ell(X) \cong \mathbb{L}_\ell(Z) \cong \mathbb{L}_\ell(Y) \). So \( X \sim_\ell Y \). (\( \Rightarrow \)) By definition, there exists an \( \ell \)-minimal graph \( Z \subseteq X, Y \) such that \( \mathbb{L}_\ell(Z) \cong \mathbb{L}_\ell(Y) \cong \mathbb{L}_\ell(Z) \). By Lemma 4.9, \( Z = Z[\ell] \subseteq X[\ell] \) since \( Z \subseteq X \). But by Lemma 4.8, \( \mathbb{L}_\ell(Z) = \mathbb{L}_\ell(X) = \mathbb{L}_\ell(X[\ell]) \). So \( Z \cong X[\ell] \) since, by Lemma 4.9, \( X[\ell] \) is \( \ell \)-minimal. Similarly, \( Y[\ell] \cong Z \) and the lemma follows. \( \blacksquare \)

We show in the following that \( \sim_\ell \) is an equivalence relation.

**Proof of Lemma 1.2.** The reflexivity and symmetry of \( \sim_\ell \) follow from the definition. To show the transitivity, let \( X \sim_\ell Y \) and \( Y \sim_\ell Z \). Then by Lemma 4.10, \( X[\ell] \cong Y[\ell] \cong Z[\ell] \), and hence \( X \sim_\ell Z \). The uniqueness of the \( \ell \)-minimal graph in its class follows from Lemmas 4.9 and 4.10. The fact that \( G[\ell] \) is an induced subgraph of \( G \) follows from Corollaries 4.5 and 4.6. \( \blacksquare \)

Now we construct \( \ell \)-roots from minimal \( \ell \)-roots.

**Proof of Lemma 1.3.** Let \( Z \sim_\ell G \), and \( Y \) be a component of \( Z \). Since \( Z, Y \) and \( G \) are \( \ell \)-finite, they do not contain rays. If \( Y \) contains a cycle, then \( Y = Y[\ell] \) is a component of \( G \) by Corollary 4.5. Now let \( Y \) be a tree. If \( \text{diam}(Y) \leq \ell - 1 \), then \( Y[\ell] \cong K_0 \). In this case, there can be arbitrarily many such \( Y \). If \( \text{diam}(Y) \geq \max\{\ell, 2\ell - 3\} \), then \( Y = Y[\ell] \) is a component of \( G \) by Lemma 4.2. The case of \( \ell \leq \text{diam}(Y) \leq 2\ell - 4 \) follows from Lemma 1.7. \( \blacksquare \)

5. Partitioned \( \ell \)-link graphs

A sufficient and necessary condition for an \( \ell \)-link graph to be connected was given by Jia and Wood [11]. In this section, we study the cyclic components of \( \ell \)-roots. The investigation helps to further understand the structure of \( \ell \)-link graphs, and bound the parameters of minimal \( \ell \)-roots.

5.1. Definitions and basis. Let \( H \) be a graph admitting a partition \( \mathcal{V} \) of \( V(G) \) and a partition \( \mathcal{E} \) of \( E(G) \). Then \( \tilde{H} := (H, \mathcal{V}, \mathcal{E}) \) is called a partitioned graph. For each graph \( G \), let \( \mathcal{V}_0(G) := \{v\} \subseteq V(G) \), and \( \mathcal{E}_0(G) := \{e\} \subseteq E(G) \).

Let \( \ell \geq 1 \). For \( R \in \mathcal{L}_{\ell-1}(G) \), let \( \mathcal{L}_\ell(R) \) be the set of \( \ell \)-links of \( G \) of middle subsequence \( R \), \( \mathcal{L}[\ell](R) := \{Q[\ell]Q \in \mathcal{L}_{\ell+1}(R)\} \), and \( \mathcal{E}_\ell(G) := \{\mathcal{L}[\ell](R) \neq \emptyset|R \in \mathcal{L}_{\ell-1}(G)\} \). Let \( E_G(u, v) \) be the set of edges of \( G \) between \( u, v \in V(G) \), and
Lemma 5.3. Let \( \ell \geq 2 \), let \( V_{\ell}(G) := \{E_{G}(u,v) \neq \emptyset | u,v \in V(G)\} \). For \( \ell \geq 2 \), let \( \mathcal{V}_{\ell}(G) := \{ \mathcal{L}_{\ell}(R) \neq \emptyset | R \in \mathcal{L}_{\ell-2}(G) \} \). By [11] Lemma 4.1, for \( \ell \neq 1 \), \( V_{\ell}(G) \) consists of independent sets of \( \mathbb{L}_{\ell}(G) \). For \( \ell \geq 0 \), \( \mathbb{L}_{\ell}(G) := (\mathcal{L}_{\ell}(G), \mathcal{V}_{\ell}(G), \mathcal{E}_{\ell}(G)) \) is a partitioned graph, and called a partitioned \( \ell \)-link graph of \( G \). \( (V_{\ell}(G), \mathcal{E}_{\ell}(G)) \) is called an \( \ell \)-link partition of \( H \). \( G \) is an \( \ell \)-root of \( H \) if \( \mathbb{L}_{\ell}(G) \cong H \). Denote by \( \mathbb{R}_{\ell}[H] \) and \( \mathbb{R}_{\ell}[\tilde{H}] \) respectively the sets of all \( \ell \)-roots and minimal \( \ell \)-roots of \( \tilde{H} \). The statement below follows from definitions.

**Proposition 5.1.** A graph is an \( \ell \)-link graph if and only if it admits an \( \ell \)-link partition. Moreover, \( \mathbb{R}_{\ell}(H) \) (respectively, \( \mathbb{R}_{\ell}[H] \)) is the union of \( \mathbb{R}_{\ell}(\tilde{H}) \) (respectively, \( \mathbb{R}_{\ell}[\tilde{H}] \)) over all partitioned graphs \( \tilde{H} \) of \( H \).

An \( \ell \)-link (respectively, \( \ell \)-arc) of \( \tilde{H} \) is an \( \ell \)-link (respectively, \( \ell \)-arc) of \( H \) whose consecutive edges are in different edge parts of \( \tilde{H} \). The lemma below indicates that every \( s \)-link of \( \mathbb{L}_{\ell}(G) \) arises from an \((\ell + s)\)-link of \( G \).

**Lemma 5.2.** Let \( G \) be a graph, and \( L \) be an \( s \)-link of \( \mathbb{L}_{\ell}(G) \). Then \( L \) is an \( s \)-link of \( \mathbb{L}_{\ell}(G) \) if and only if there exists an \((\ell + s)\)-link \( R \) of \( G \) such that \( L = R^{[\ell]} \).

**Proof.** It is trivial for \( \ell = 0 \) or \( s \leq 1 \). Now let \( \ell \geq 1 \), \( s \geq 2 \), and \( L := [L_{0}, Q_{1}, \ldots, Q_{s}, L_{s}] \). (\( \Leftarrow \)) Let \( L = R^{[\ell]} \), and \( P_{i} \) be the middle \((\ell - 1)\)-link of \( Q_{i} \), where \( i \in [s] \). By definition \( Q_{i}^{[\ell]} = [L_{i-1}, Q_{i}, L_{i}] \in \mathcal{L}_{\ell}(P_{i}) \in \mathcal{E}_{\ell}(G) \). By [11] Observation 3.3, \( P_{i} \neq P_{i+1} \) for \( i \in [s - 1] \). So \( Q_{i}^{[\ell]} \) and \( Q_{i+1}^{[\ell]} \) are in different edge parts of \( \mathcal{E}_{\ell}(G) \). Thus \( L \) is an \( s \)-link of \( \mathbb{L}_{\ell}(G) \).

(\( \Rightarrow \)) Let \( L \) be an \( s \)-link of \( \mathbb{L}_{\ell}(G) \), where \( L_{i} := [v_{i}, e_{i+1}, \ldots, v_{i+\ell}] \in \mathcal{L}_{\ell}(G) \) for \( i \in \{0, 1\} \), and \( Q_{1} := [v_{0}, e_{1}, \ldots, e_{\ell+1}, v_{\ell+1}] \in \mathcal{L}_{\ell+1}(G) \). Suppose \( Q_{2} \) has the form \([u_{0}, f_{1}, v_{1}, \ldots, e_{\ell+1}, v_{\ell+1}] \). Then \( Q_{2} \) and \( Q_{1} \) have the same middle \((\ell - 1)\)-link \( P := [v_{1}, e_{2}, \ldots, e_{\ell}, v_{\ell}] \), and hence are in the same part \( \mathcal{L}_{\ell} = \mathcal{E}_{\ell}(G) \), contradicting that \( L \) is an \( s \)-link of \( \mathbb{L}_{\ell}(G) \). So \( Q_{2} \) has the form \([v_{1}, e_{2}, \ldots, e_{\ell+2}, v_{\ell+2}] \), and \( L_{2} = [v_{2}, e_{3}, \ldots, e_{\ell+2}, v_{\ell+2}] \). Continue this analysis, we have that \( L_{i} := [v_{i}, e_{i+1}, \ldots, v_{i+\ell}] \in \mathcal{L}_{\ell}(G) \), and \( Q_{i} := [v_{i-1}, e_{i}, \ldots, v_{i+\ell}] \in \mathcal{L}_{\ell+1}(G) \). Then \( R := [v_{0}, e_{1}, \ldots, v_{s+\ell}] \in \mathcal{L}_{\ell+1}(G) \) and \( L = R^{[\ell]} \).

As shown below, each closed \( s \)-link of \( \mathbb{L}_{\ell}(G) \) stems from a closed \( s \)-link of \( G \).

**Lemma 5.3.** Let \( \ell \geq 0 \) and \( s \geq 2 \) be integers. Let \( G \) be a graph, and \( \tilde{R} \) be an \((\ell + s)\)-arc of \( G \). Then \( R^{[\ell]} \) is a closed \( s \)-link of \( \mathbb{L}_{\ell}(G) \) if and only if \( \tilde{R}(0, \ell) = \tilde{R}(s, \ell + s) \).

**Proof.** By Lemma 5.2 \([L_{0}, Q_{1}, \ldots, Q_{s}, L_{s}] := R^{[\ell]} \) is an \( s \)-link of \( \mathbb{L}_{\ell}(G) \). Clearly, \( R^{[\ell]} \) is closed if and only if \( \tilde{R}(0, \ell) = \tilde{R}(s, \ell + s) \). Let \( \tilde{R} := (v_{0}, e_{1}, \ldots, v_{s+\ell}) \). Suppose \( \tilde{R}(0, \ell) = \tilde{R}(s, \ell + s) \); that is, \((v_{0}, e_{1}, \ldots, e_{\ell}, v_{\ell}) = (v_{\ell+1}, e_{\ell+2}, \ldots, e_{s+1}, v_{s}) \). Then \( Q_{1} = [v_{0}, e_{1}, \ldots, e_{\ell+1}, v_{\ell+1}] \) and \( Q_{s} = [v_{s-1}, e_{s}, \ldots, e_{\ell+s}, v_{\ell+s}] \) have the same middle \((\ell - 1)\)-link \( P := [v_{1}, e_{2}, \ldots, e_{\ell}, v_{\ell}] = [v_{\ell+1}, e_{\ell+2}, \ldots, e_{s+1}, v_{s}] \). So \( Q_{1}, Q_{s} \)
correspond to edges of $\tilde{\mathcal{L}}_\ell(G)$ from the same edge part $\mathcal{L}^{[\ell]}(P)$, contradicting that $\bar{R}^{[\ell]}$ is an $s$-link of $\tilde{\mathcal{L}}_\ell(G)$. Thus $\bar{R}(0, \ell) = \bar{R}(s, \ell + s)$. 

5.2. Cycles in partitioned graphs. A cycle of $\tilde{H}$ is a cycle of $H$ whose the consecutive edges are in different edge parts. $\tilde{H}$ and its partition are cyclic if $\tilde{H}$ contains a cycle, and acyclic otherwise. For example, for each $t$-cycle $C$, $\tilde{\mathcal{L}}_\ell(C)$ is cyclic. When $t \geq 3$ is divisible by 3 or 4, by Lemma 3.2 there exists a tree $T$ and an integer $\ell$ such that $C \cong \mathcal{L}_\ell(T)$ can be organised into an acyclic partitioned graph $\tilde{\mathcal{L}}_\ell(T)$. Each component $X$ of $H$ corresponds to a partitioned subgraph $\tilde{X}$ of $\tilde{H}$. $\tilde{X}$ is called a component of $\tilde{H}$. Let $o(H)$ and $a(H)$ be the cardinalities of the sets of cyclic and acyclic components of $\tilde{H}$ respectively. The following lemma says that the number of cyclic components is invariant (see, for example, [14]) under the partitioned $\ell$-link graph construction.

Lemma 5.4. Let $G$ be a graph, and $\ell \geq 0$. Then $o(G) = o(\tilde{\mathcal{L}}_\ell(G))$.

Proof. Let $\tilde{H} := \tilde{\mathcal{L}}_\ell(G)$, $X$ be a component of $G$ containing a cycle $C$, $\tilde{X}_\ell$ be the component of $\tilde{H}$ containing $\tilde{\mathcal{L}}_\ell(C)$. We only need to show that $\varphi : X \mapsto \tilde{X}_\ell$ is a bijection from the cyclic components of $G$ to that of $\tilde{H}$. For $i \in \{1, 2\}$, let $C_i$ be a closed $s_i$-link of $H$. By Lemma 5.3, $G$ contains an $(\ell + s_i)$-arc $\bar{R}_i$ such that $C_i = \bar{R}_i^{[\ell]}$, and that $\bar{R}_i((0, s_i)$ is a closed $s_i$-arc $\bar{O}_i$ of $G$.

First we show that $\varphi$ is a well defined surjection. Assume that $R_1$ and $R_2$ are $*$-links of $X$. Note that $\bar{R}_1[0, \ell]$ can be shunted into $O_1$ of $X$. Thus by [11, Lemma 3.7], $\bar{R}_1[0, \ell]$ can be shunted to $\bar{R}_2[0, \ell]$ in $X$, and the images of the shunting form a $*$-link from $C_1$ to $C_2$ in $\tilde{X}_\ell$.

We still need to show that $\varphi$ is injective. Assume $C_1$ and $C_2$ are joined by a $*$-link $Q$ of $X$ between, say, $\bar{R}_1[0, \ell] \in V(C_1)$ and $\bar{R}_2[0, \ell] \in V(C_2)$. Then $\bar{R}_i[0, \ell] \in \mathcal{L}_\ell(G)$, where $i \in \{1, 2\}$, can be shunted to each other in $G$, with images corresponding to the vertices of $Q$ in $X$. 

Every graph is a disjoint union of its connected components. But the relationship between a partitioned graph $\tilde{H}$ and its components is more complicated. For different components $\tilde{X}$ and $\tilde{Y}$ of $\tilde{H}$, it is possible that a vertex part $U$ of $\tilde{X}$ and a vertex part $V$ of $\tilde{Y}$ are two disjoint subsets of a vertex part $W$ of $\tilde{H}$. Lemma 5.3 leads to a rough process of building $\tilde{\mathcal{L}}_\ell(G)$ from its components. We can first fix all cyclic components such that no two units from different components are in the same part of $\tilde{H}$. And then for each acyclic component, we either set it as an independent fixed component, or merge some of its vertex parts with that of a unique fixed graph to get a larger fixed graph.

By definition, $o(\tilde{H}) \leq o(H)$, $a(H) \leq a(\tilde{H})$, and $a(H) + o(H) = c(H) = c(\tilde{H}) = a(\tilde{H}) + o(\tilde{H})$. As a consequence of Lemma 5.4, we have:
Corollary 5.5. For each integer $\ell \geq 0$, a graph $G$ is acyclic if and only if $a(\mathbb{L}_\ell(G)) = c(\mathbb{L}_\ell(G))$.

5.3. Computing cyclic components. We explain how to decide whether a component of $\tilde{H} := (H, V, E)$ is cyclic, and compute $o(\tilde{H})$ and $a(\tilde{H})$ in quadratic time. Let $E(u)$ be the edge parts in $E$ incident to $u \in V(H)$, and $r(E) := \max\{|E(u)| \mid u \in V(H)\}$.

Definition 5.6. Let $\tilde{H} := (H, V, E)$ be a partitioned graph, and $\tilde{H}$ be the digraph with vertices $(u, E)$, where $E \in E(u)$, such that there is an arc from $(u, E)$ to $(v, F)$ if $E \neq F$, and there is $e \in E$ between $u$ and $v$.

In the following, we transfer the problem of computing $o(\tilde{H})$ to that of detecting if a component of $\tilde{H}$ contains a dicycle.

Lemma 5.7. In Definition 5.6, for each component $\tilde{X}$ of $\tilde{H}$, we have that $\tilde{X}$ is the disjoint union of some components of $\tilde{H}$. And $\tilde{H}$ is the disjoint union of $\tilde{X}$ over all components $\tilde{X}$ of $\tilde{H}$. Moreover, $\tilde{X}$ is cyclic if and only if $\tilde{X}$ contains a dicycle.

Proof. The first two statements follow from definitions. We now prove the last statement. Let $s \geq 2$, and $C := [v_0, e_1, \ldots, e_s, v_0]$ be a cycle of $X$. For $i \in [s]$, let $e_i \in E_i \in E$. Let $e_{s+1} := e_1$ and $E_{s+1} := E_1$. By definition, $C$ is a cycle of $\tilde{X}$ if and only if $E_i \neq E_{i+1}$ for $i \in [s]$, which is equivalent to saying that $u_i := (v_i, e_i)$ is a vertex of $\tilde{H}$, while $f_i := (u_i-1, u_i)$ is an arc of $\tilde{H}$ for $i \in [s + 1]$; that is, $(u_0, f_1, \ldots, f_s, u_s = u_0)$ is a dicycle of $\tilde{X}$. \hfill \blacksquare

Remark. Let $n := n(H)$, $m := m(H)$ and $r := r(E)$. Then we have $n(\tilde{H}) = \sum_{u \in V(H)} |E(u)| \leq \sum_{u \in V(H)} \deg_H(u) = 2m$. Every arc $(u, e, v)$ of $H$ with $e \in E \in E$ corresponds to $|E(v)| - 1$ arcs of $\tilde{H}$; that is, $((u, E), (v, F))$ for $F \in E(v) \setminus \{E\}$. So $m(\tilde{H}) \leq \sum_{(u, e, v) \in \tilde{E}(H)} (|E(v)| - 1) \leq \sum_{u \in V(H)} \deg_H(u)(|E(v)| - 1) \leq 2m(r - 1)$.

An $O(n + m)$-time algorithm for dividing $H$ into connected components was given by Hopcroft and Tarjan [10]. For each component $\tilde{X}$ of $\tilde{H}$, Tarjan’s algorithm [10], with time complexity $O(n(\tilde{X}) + m(\tilde{X})) = O(r m(X))$, can be used to detect the existence of dicycles in $\tilde{X}$, and hence that of cycles in $\tilde{X}$ by Lemma 5.7. So the time complexity for computing $o(\tilde{H})$ is $O(m^2 + n)$ in general situations, and is $O(m + n)$ if $r$ is bounded. By [11, Lemma 4.1], $r(E_\ell(G)) \leq 2$ for $\ell \geq 1$. So it requires $O(m + n)$-time to compute the number of cyclic components of a finite partitioned $\ell$-link graph for each $\ell \geq 0$.

6. Bounding the number of minimal roots

We bound in this section the order, size, maximum degree and total number of minimal $\ell$-roots of a finite graph. This lays a basis for solving the recognition and determination problems for $\ell$-link graphs in our future work.
6.1. **Incidence pairs.** For \( s, \ell \geq 0 \), and \( L \in \mathcal{L}_\ell(G) \), let \( \mathcal{I}_G(L, s) \) be the set of pairs \((L,R)\) such that \( R \in \mathcal{L}_s(G) \) is incident to \( L \). Let \( i_G(L, s) := |\mathcal{I}_G(L, s)| \).

Define girth\((L)\) to be \(+\infty\) if \( L \) is a path, and the minimum length of a sub cycle of \( L \) otherwise. To dodge confusions, denote by \( \hat{L} \) the graph induced by the units of \( L \). Then girth\((\hat{L})\) \(\leq\) girth\((L)\) and the inequality may hold. For example, let \( L := [v_0, e_1, v_1, e_2, v_2, v_3, v_0, e_4, v_1] \). Then girth\((\hat{L})\) = 2 and girth\((L)\) = 3.

**Lemma 6.1.** Let \( \ell \geq s \geq 0 \), \( L \in \mathcal{L}_\ell(G) \), and \( g := \text{girth}(L^{[s]}) \). Then \( \min\{g, \ell - s + 1\} \leq i_G(L, s) \leq \ell - s + 1 \). Further, \( i_G(L, s) = \ell - s + 1 \) if and only if \( g = +\infty \) and only if \( L^{[s]} \) is an \((\ell - s)\)-path if and only if \( g \geq \ell - s + 1 \). Otherwise, if \( g \leq \ell - s \), then \( i_G(L, s) = g \) if and only if \( \hat{L}^{[s]} \) is a \( g \)-cycle.

**Proof.** \( L^{[s]} \) is an \((\ell - s)\)-link of \( H := \mathcal{L}_s(G) \). So \( t := i_G(L, s) = |L^{[s]}| \leq \ell - s + 1 \), with the last equality holds if and only if \( L^{[s]} \) is a path. If \( g \leq \ell - s \), then \( L^{[s]} \) contains a sub \( g \)-cycle on which every pair of different vertices of \( H \) corresponding to a pair of different sub \( s \)-links of \( L \). So \( t \geq g \), with equality holds if and only if all units of \( H \) on \( L^{[s]} \) belong to the \( g \)-cycle.  

Let \( \mathcal{I}_G(\ell, s) := \bigcup_{L \in \mathcal{L}_\ell(G)} \mathcal{I}_G(L, s) \), and \( i_G(\ell, s) := |\mathcal{I}_G(\ell, s)| \). Then \( i_G(\ell, s) = \sum_{L \in \mathcal{L}_\ell(G)} i_G(L, s) \) can be bounded as follows:

**Corollary 6.2.** Let \( \ell \geq s \geq 0 \) be integers, and \( G \) be an \( \ell \)-finite graph of girth \( g \). Then

\[
\min\{g, \ell - s + 1\}|\mathcal{L}_\ell(G)| \leq i_G(\ell, s) \leq (\ell - s + 1)|\mathcal{L}_\ell(G)|.
\]

Further, \( i_G(\ell, s) = (\ell - s + 1)|\mathcal{L}_\ell(G)| \) if and only if \( g \geq \ell - s + 1 \). If \( g \leq \ell - s \), then \( i_G(\ell, s) = g|\mathcal{L}_\ell(G)| \) if and only if \( G \) is a disjoint union of \( g \)-cycles.

**Proof.** Let \( t := \min\{\text{girth}(L^{[s]})|L \in \mathcal{L}_\ell(G)\} \). Note that \( g \geq \ell - s + 1 \) if and only if \( t = +\infty \). And \( g \leq \ell - s \) if and only if \( t = g \). So the statements can be verified by summing the results in Lemma 6.1 over \( L \in \mathcal{L}_\ell(G) \).

The order and size of minimal \( \ell \)-roots are bounded as follows.

**Lemma 6.3.** Let \( \ell \geq 1 \) and \( G \) be an \( \ell \)-minimal graph. Then both \( G \) and \( \tilde{H} := \mathcal{L}_\ell(G) \) are finite. Further, \( m(G) \leq \ell n(H) \), and \( n(G) \leq \ell n(H) + a(\tilde{H}) \).

**Proof.** Since \( G \) is \( \ell \)-finite, \( \tilde{H} \) is finite. By Corollary 6.2, \( i_G(\ell, 1) \leq \ell n(H) \) is finite. By Lemma 4.3, for each \( e \in E(G) \), \( i_G(e, \ell) \geq 1 \). Summing this inequality over \( e \in E(G) \), we have \( m(G) \leq i_G(\ell, 1) \) is finite. By Lemma 5.4, \( o(G) = o(\tilde{H}) \), and hence \( a(G) = c(G) - o(G) \leq c(\tilde{H}) - o(\tilde{H}) = a(\tilde{H}) \). So \( n(G) \leq m(G) + a(G) \leq \ell n(H) + a(\tilde{H}) \) is finite.
6.2. The number of minimal roots. We have known that $|\mathbb{R}_0(G)| = 1$, and $|\mathbb{R}_t(K_s)| = 1$ for $s \in \{0,1,2\}$. By Lemma 3.1, $|\mathbb{R}_t(2K_1)| = \lfloor \frac{3t^2}{12} \rfloor$ for $t \geq 1$. Let $\tilde{a}_t(H)$ be the maximum $a(H)$ over all partitioned $t$-link graphs $H := (H, V, E)$. Clearly, $\tilde{a}_t(H) \leq c(H)$. Denote by $\psi(p, q)$ the number of nonisomorphic graphs with $p$ vertices and $q$ edges. Then $\psi(p, q)$ is 0 if $p \leq 1$ and $q \geq 1$; is 1 if $p = 2$ or $q = 0$; is 1 if $p \geq 2$ and $q \leq 1$; is less than $\binom{p}{2}^q$ if $p \geq 3$ and $q \geq 1$; and by Hardy [9], is the nearest integer to $\frac{(p+3)^2}{12}$ if $p = 3$. Further, when $p \geq 3$ and $q \geq 1$, we have that $\binom{p}{2}^q \geq 3$, and $\frac{(p-1)!}{(p-2)!} < e\frac{p}{2}$. Thus $\sum_{i=0}^{p} \sum_{j=0}^{q} \psi(i, j) \leq p + q + \sum_{i=3}^{p} \sum_{j=1}^{q} \binom{i}{2}^j < p + q + \frac{3}{2} \sum_{i=3}^{p} \binom{i}{2}^q < p + q + \frac{3}{2} (p/2)^q [1 - e^{-\frac{3p}{2}}]^{-1}.$

Lemma 6.4. Let $\ell \geq 1$ be an integer, and $H$ be a finite graph. Let $n := n(H) \geq 2$, and $a := \tilde{a}_\ell(H)$. Then $H$ has at most $(\ell n + a)^{2\ell n}$ minimal $\ell$-roots, of which at most $\frac{3}{2}(\ell n + 1)^{\ell n}$ are trees, and at most $\frac{3}{2}a(a + 1)(\ell n + a)^{\ell n}$ are forests.

Proof. Let $G \in \mathbb{R}_\ell(H)$. By Lemma 6.3, $m(G) \leq \ell n$, and $n(G) \leq \ell n + a \leq 2\ell n$. By the analysis above, $|\mathbb{R}_\ell(H)| < 2\ell n + a + \frac{3}{2} (\ell n + a)^{\ell n} < (\ell n + a)^{2\ell n}$.

Cayley’s formula [4, 6] states that there are $p^{p-2}$ unequal trees on vertex set $[p]$. So the number of trees in $\mathbb{R}_\ell(H)$ is at most $\sum_{p=1}^{\ell n+1} p^{p-2} < \frac{3}{2}(\ell n + 1)^{\ell n}$.

By Aigner and Ziegler [1], the number of unequal forests on vertex set $[p]$ of $k$ components is $k p^{p-k-1}$. So the number of forests in $\mathbb{R}_\ell(H)$ is at most $\sum_{k=1}^a \sum_{p=a}^{\ell n+k} k p^{p-k-1} < \frac{3}{2} \sum_{k=1}^a k(\ell n + k)^{\ell n-1} < \frac{3}{4} a(a + 1)(\ell n + a)^{\ell n-1}.$

6.3. The maximum degree of minimal roots. We have proved that minimal $\ell$-roots of a finite graph are finite. So in this subsection, we only deal with finite graphs. Let $E$ be a partition of $E(H)$. We may identify $E \in E$ with the subgraph of $H$ induced by $E$. Let $D(E)$ be the set of $\deg_E(v)$ over $E \in E$ incident to $v \in V(H)$. Let $D(G) := \{\deg_G(v) - 1 \geq 1 | v \in V(G)\}$. Clearly, $D(G) = \emptyset$ if and only if $\Delta(G) \leq 1$ if and only if for all $\ell \geq 1$, $D(E_\ell(G)) = \emptyset$.

Lemma 6.5. Let $\ell \geq 1$, and $G$ be a finite connected graph of $\Delta(G) \geq 2$. Let $D := D(G)$, and $D_\ell := D(E_\ell)$. If $G$ is cyclic, then $D = D_1$. Otherwise, $G$ is a tree of diameter $s \geq 2$, and $D = D_1 \supseteq \ldots \supseteq D_{s-1} \supseteq \emptyset = D_s = D_{s+1} = \ldots$.

Proof. Let $\ell, d \geq 1$. By definition, $d \in D_\ell$ if and only if there is an $\ell$-arc $\bar{L}$ starting from some $v \in V(G)$ with $\deg_G(v) = d + 1$. So $D = D_1$. When $\ell \geq 2$, $\bar{L}(0, \ell - 1)$ is an $(\ell - 1)$-arc starting from $v$. So $d \in D_{\ell-1}$. In another word, $D_{\ell-1} \supseteq D_\ell$ for all $\ell \geq 2$.

On the one hand, let $C$ be a cycle of $G$. For each $v \in V(G)$, since $G$ is connected, there is a dipath $\bar{P}$ of minimum length from $v$ to some $u \in V(C)$. Clearly, there is an $\ell$-arc $\bar{R}$ of $C$ starting from $u$. Then $\bar{L} := (\bar{P}, \bar{R})(0, \ell)$ is an $\ell$-arc of $G$ starting from $v$. Thus $D_\ell = D$ by the analysis above.
On the other hand, let $G$ be a tree of diameter $s \geq 2$. By the definitions, $\ell \geq s$ if and only if $E(\mathbb{L}_\ell(G)) = \emptyset$ if and only if $D_\ell = \emptyset$. \hfill \blacksquare

Define $\Delta(\mathcal{E})$ to be 0 if $\mathcal{E} = \emptyset$, and the maximum of $D(\mathcal{E})$ otherwise. A lower bound of $\Delta(G)$ for $G \in \mathbb{R}_\ell[^2 \hat{H}]$ follows from Lemma 6.5.

**Corollary 6.6.** Let $\ell \geq 1$ be an integer, and $G$ be a finite graph. Let $\mathcal{E} := \mathcal{E}_\ell(G)$. Then $\Delta(G) = \Delta(\mathcal{E}) = 0$ if and only if $E(G) = \emptyset$. And $E(G) \neq \emptyset$ if and only if $\Delta(G) \geq \Delta(\mathcal{E}) + 1$. Further assume that $G$ is connected. Then $G$ contains a cycle if and only if for all $\ell \geq 1$, we have $\Delta(G) = \Delta(\mathcal{E}) + 1 \leq \Delta(\mathbb{L}_d(G))$.

Below we display a connection between the degrees of an $\ell$-minimal tree and the number of components of the $\ell$-link graph of the tree.

**Lemma 6.7.** Let $\ell \geq 1$ be an integer, $T$ be a finite $\ell$-minimal tree, and $v$ be a vertex of eccentricity less than $\ell$ in $T$. Then $\deg_T(v) \leq c(\mathbb{L}_\ell(T)) + 1$.

**Proof.** Since $T$ is $\ell$-minimal, so $\text{diam}(T) \geq \ell \geq 1$. Thus $s := \text{ecc}_T(v) \geq 1$ and $\ell \geq s + 1 \geq 2$. Let $d := \deg_T(v)$. If $d \leq 1$, there is nothing to show.

Now let $d \geq 2$. For $i \in [d]$, let $e_i := (v, e_i, u_i)$ be the arcs of $T$ starting from $v$. Then there exists $\vec{R} \in \mathcal{L}_\ell(T)$ starting from, say, $\vec{e}_d$. Since $T$ is $\ell$-minimal, $e_i$ is $\ell$-incident in $T$. For $i \in [d - 1]$, by Lemma 4.4, $t_i := \text{ecc}_{\mathbb{L}_\ell(T)}(u_i) \geq \ell - s - 1 \geq 0$. So there is a $t_i$-arc $Q_i$ from $u_i$ to some $v_i$ in $T_{u_i}^v$. Obviously, $L_i := [Q_i(\ell - s - 1, 0), \vec{e}_i, \vec{R}]$, for $i \in [d - 1]$, are $d - 1$ different $\ell$-paths containing $v$ in $T$.

Suppose for a contradiction that $L_i$ can be shunted to $L_j$ for some $1 \leq i < j \leq d - 1$. Since $e_i$ separates $v$ from $v_i$, so $v$ is an image of $v_i$ during the shunting. But then $\text{ecc}_T(v) \geq \ell$, a contradiction. So $L_i$ and $L_j$ correspond to vertices in different components of $H := \mathbb{L}_\ell(T)$. Hence $d - 1 \leq c(H)$. \hfill \blacksquare

As an application of Lemma 6.7, we characterise the minimal roots of a cycle.

**Proof of Lemma 3.2** Since $C := \mathbb{L}_\ell(T)$ is connected and $T$ is minimal, so $T$ is connected and hence is a tree. For $u, v \in V(T)$, define $D(u, v) := \deg_T(u) + \deg_T(v)$. Since $C$ is 2-regular, $D(u, v) = 4$ if $\text{dist}_T(u, v) = \ell$.

We claim that, if $\text{dist}_T(u, v) \geq \ell$, then $D(u, v) \leq 4$. Suppose for a contradiction that $D(u, v) \geq 5$. Without loss of generality, assume $\deg_T(u) \geq 3$. Since $\text{dist}_T(u, v) \geq \ell$, there exists some vertex $w$ on the path of $T$ from $u$ to $v$ such that $\text{dist}_T(u, w) = \ell$. Then $\deg_T(w) \geq 2$. Thus $D(u, w) = 5 > 4$, a contradiction.

By the minimality of $T$, for each leaf $w$ of $T$, there exists some $v \in V(T)$ such that $\text{dist}_T(v, w) = \ell$ and $\deg_T(v) = 3$. By Lemma 6.7, for each $v \in V(T)$ with $\text{ecc}_T(v) < \ell$, we have $\deg_T(v) \leq c(C) + 1 = 2$. So $\deg_T(v) \in \{1, 2, 3\}$ for $v \in V(T)$. Let $k$ be the number of degree-3 vertices in $T$. Then $k \geq 1$ since $T$ contains degree-1 vertices.

If $k = 1$, then $T$ contains exactly three leaves ([7, Page 67]). Since $T$ is minimal, each leaf $u$ of $T$ is the end vertex of some $\ell$-path $P$ of $T$. Let $v$ be the other end of $P$. Then $\deg(v) = 3$ since $\text{dist}_T(u, v) = \ell$. So each leaf of $T$ is at
distance $\ell$ from the unique degree-3 vertex $v$ of $T$. As a consequence, $T \cong K_{1,3}^{(\ell)}$ and $t = 3\ell$.

Now assume that $k \geq 2$. Let $\bar{Q} := (v_0, \ldots, v_q)$ be a dipath in $T$ such that $v_0$ and $v_q$ are degree-3 vertices at maximum distance in $T$. Since $D(v_0, v_q) = 6$, we have $q = \text{dist}_T(v_0, v_q) < \ell$.

If $k = 2$, then $T$ contains exactly four leaves \[7, \text{Page 67}\]. Consequently, $T$ is the union of two paths $[\bar{L}_i, \bar{Q}, \bar{R}_i]$, where $i \in \{1, 2\}$, and $\bar{L}_i, \bar{R}_i$ are four internally disjoint paths of length $\ell_i$, $\ell_i \geq 1$ respectively. By the analysis above, there is an $\ell$-path between each leaf and one of $v_0$ and $v_q$. Let $(w_0, w_1, \ldots, w_{\ell_1} = v_0) := \bar{L}_1$. If $\text{dist}_T(w_0, v_0) = \ell$, then $\text{dist}_T(w_1, v_q) = \ell + 1 + q \geq \ell$, and $D(w_1, v_q) > 4$, a contradiction. So $\text{dist}_T(w_0, v_q) = \ell$, and $\ell_1 = \ell - q$. Let $s := \ell - q$. By symmetry, $\ell_1 = \ell_2 = s_1 = s_2 = s$. Now let $\bar{P}_{ij} := (\bar{L}_i, \bar{Q}, \bar{R}_j)$ for $i, j \in \{1, 2\}$. Then the images during the shunting of $L$ through $\bar{P}_{11}, \bar{P}_{21}, \bar{P}_{22}, \bar{P}_{12}$ form a 4-cycle $C'$ of $\bar{L}_\ell(T)$. Hence $C' = C$, and all $\ell$-links of $T$ are aforementioned images of $L$. Thus $[\bar{L}_1, \bar{L}_2] \in \mathcal{P}_{2\ell}(T)$ contains no $\ell$-link; that is, $\ell \geq 2s + 1$.

Assume for a contradiction that $k \geq 3$. Since $q$ is maximum, there exists some $p \in [q - 1]$ such that $\deg_T(v_p) = 3$. By Lemma 6.7, $\text{ecc}_T(v_p) \geq \ell$. So there exists an $\ell$-dipath $\bar{L} := (u_0, \ldots, u_t = v_p)$ of $T$ such that $v_0, v_q$ are separated from $u_0$ by $v_p$. Then $\text{dist}_T(v_0, u_1) = \ell + 1 + p \geq \ell$, and $D(v_0, u_1) > 4$, which is a contradiction.

We now bound the maximum degree of a finite tree in terms of $\bar{L}_\ell(T)$.

**Corollary 6.8.** Let $\ell \geq 1$, $T$ be a finite tree, $\bar{H} := (H, \mathcal{V}, \mathcal{E}) := \bar{L}_\ell(T)$ and $s := \max\{\text{ecc}_T(v)| \deg_T(v) = \Delta(T)\}$. Then

1. If $s \geq \ell + 1$, then $\Delta(T) = \Delta(\mathcal{E}) + 1 \leq \Delta(\bar{H})$.
2. If $\ell = s$, then either $s = \Delta(T) = 1$, and $\Delta(\bar{H}) = 0$; or $s \geq 2$ and $\Delta(T) = \Delta(\mathcal{E}) + 1 \leq \Delta(\bar{H}) + 1$.
3. If $\ell \geq s + 1$ and $T$ is $\ell$-minimal, then $\Delta(T) \leq \alpha(\bar{H}) + 1 \leq c(\bar{H}) + 1$.

**Proof.** (1) and (2) are implied by the proof of Lemma 6.5. (3) follows from Lemma 6.7 and Corollary 5.5.

The maximum degree of minimal $\ell$-roots $G$ of $\bar{H}$ or $H$. The results in turn help to bound the parameters of $s$-link graphs of $G$. Clearly, for $s \geq 1$, a graph of $q \geq 2$ edges contains at most $q(q - 1)^{s-1}$ $s$-links, with equality holds if and only if all these edges are between the same pair of vertices.

**Lemma 6.9.** Let $\ell$ and $s$ be positive integers, and $H$ be a finite graph. Let $n := n(H) \geq 2$, and $b := b(\bar{H})$. Let $G \in \mathbb{R}_s(\bar{H})$, and $(X, \mathcal{Y}, \mathcal{E}) := \bar{L}_s(G)$. Then $\Delta(G) \leq b + 1$, $\Delta(\mathcal{E}) \leq b$, $\Delta(X) \leq 2b$, $\max\{|V|, |E| | V \in \mathcal{Y}, E \in \mathcal{E}\} \leq b^2$, $m(X) \leq \ell n(\ell n - 1)^s$, and $n(X) \leq \ell n(\ell n - 1)^{s-1}$. 
Proof. Let \( Y, Z \) be the subgraphs induced by the cyclic and acyclic components of \( G \) respectively. Let \( (H, \mathcal{V}, \mathcal{E}) := \tilde{H} \). By Corollary 6.6 we have \( \Delta(Y) \leq \Delta(\mathcal{E}) + 1 \). By Corollary 6.8 we have \( \Delta(Z) \leq a(\tilde{H}) + 1 \). So \( \Delta(G) = \max\{\Delta(Y), \Delta(Z)\} \leq b + 1 \). The rest of the lemma follows from the analysis above.

7. Path graphs

Some ideas and techniques used in the investigation of \( \ell \)-link graphs can be applied to the study of \( \ell \)-path graphs. We end this paper by bounding the parameters of minimal \( \ell \)-path roots.

7.1. Quantitative analysis. A graph is \( \ell \)-path finite if its \( \ell \)-path graph is finite. Note that an \( \ell \)-finite graph is \( \ell \)-path finite, but not vice versa. For example, the disjoint union of infinitely many \( \ell \)-cycles is \( \ell \)-path finite but not \( \ell \)-finite. Two \( \ell \)-path finite graphs \( X \) and \( Y \) are \( \ell \)-path equivalent, write \( X \sim_{\ell} Y \), if there exists some graph \( Z \subseteq X, Y \) such that \( P_{\ell}(X) \cong P_{\ell}(Y) \cong P_{\ell}(Z) \). An \( \ell \)-path finite graph \( G \) is said to be \( \ell \)-path minimal if either \( G \) is null, or for each \( X \subset G \), \( P_{\ell}(X) \subset P_{\ell}(G) \). Similar with Lemma 1.2, we have:

Proposition 7.1. \( \sim_{\ell} \) defines an equivalence relation on \( \ell \)-path finite graphs. Further, each \( \ell \)-path equivalence class contains a unique (up to isomorphism) minimal graph, which is \( \ell \)-path minimal. Moreover, an \( \ell \)-path minimal graph is a subgraph of every graph in its \( \ell \)-path equivalence class.

In the following, we exemplify that an \( \ell \)-minimal graph may not be \( \ell \)-path minimal. And an \( \ell \)-path minimal graph may not be an induced subgraph of another graph in its \( \ell \)-path equivalence class.

Example 7.2. Let \( G \) be a graph obtained from a path \( P = [v_0, \ldots, v_4] \) by adding an edge \( e \) between \( v_1 \) and \( v_3 \). By Lemmas 4.1 and 4.9, \( G \) and \( P \) are 4-minimal. Since \( P_4(G) = P_4(P) \cong K_1, G \) is not \( \ell \)-path minimal. Moreover, \( P \) is 4-path minimal and is a subgraph but not an induced subgraph of \( G \).

The example below indicates that, a graph can be both \( \ell \)-minimal and \( \ell \)-path minimal. But even so its \( \ell \)-link and \( \ell \)-path graphs may not be isomorphic.

Example 7.3. The complete graph \( K_{\ell+1} \) is both \( \ell \)-minimal and \( \ell \)-path minimal. Clearly, for \( \ell = 0, 1 \) and 2, \( L_{\ell}(K_{\ell+1}) = P_{\ell}(K_{\ell+1}) \) is isomorphic to \( K_1, K_1 \) and \( K_3 \) respectively. Now let \( \ell \geq 3 \). By [11, Theorem 1.2], \( L_{\ell}(K_{\ell+1}) \) contains a \( K_{\ell+1} \)-minor. However, \( P_{\ell}(K_{\ell+1}) \) consists of \( \ell! / 2 \) disjoint cycles of length \( \ell + 1 \).

By definition \( Q_0(H) = \{H\} \) if \( H \) is simple. Moreover \( Q_\ell(K_0) = \{K_0\} \), and \( Q_\ell(K_1) \) consists of an \( \ell \)-path. \( Q_1(K_2) \) consists of a 2-path and a 2-cycle. When \( \ell \geq 2 \), \( Q_\ell(K_2) \) consists of an \( (\ell + 1) \)-path. Similar with Lemma 6.3, the order, size and the total number of minimal \( \ell \)-path roots of a finite graph are bounded.
Lemma 7.4. Let $\ell \geq 1$ be an integer, and $H$ be a finite graph such that $n := n(H) \geq 2$. Let $c := c(H)$, and $G \in Q_\ell(H)$. Then $n(G) \leq \ell n + c$ and $m(G) \leq \ell n$, with each equality holds if and only if $G$ is a disjoint union of $\ell$-paths. Moreover, $Q_\ell(H)$ contains at most $(\ell n + c)^{2\ell n}$ graphs, in which at most $\frac{3}{2}(\ell n + 1)^{\ell n - 1}$ are trees, and at most $\frac{3}{4}c(c + 1)(\ell n + c)^{\ell n - 1}$ are forests.

References

[1] Martin Aigner and Günter M. Ziegler. Proofs from The Book. Springer-Verlag, fourth edition, 2010.
[2] Robert E. L. Aldred, Mark N. Ellingham, Robert L. Hemminger, and Peter Jipsen. $P_3$-isomorphisms for graphs. J. Graph Theory, 26(1):35–51, 1997.
[3] Norman Biggs. Algebraic graph theory. Cambridge University Press, second edition, 1993.
[4] Karl Wilhelm Borchardt. Über eine Interpolationsformel für eine Art symmetrischer Funktionen und über deren Anwendung. Königl. Akad. d. Wiss., 1860.
[5] Hajo Broersma and Cornelis Hoede. Path graphs. J. Graph Theory, 13(4):427–444, 1989.
[6] Arthur Cayley. A theorem on trees. Collected Mathematical Papers, Volume 13, page 26, 1909.
[7] Gary Chartrand and Ortrud R. Oellermann. Applied and algorithmic graph theory. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., 1993.
[8] Reinhard Diestel. Graph Theory, volume 173 of Graduate Texts in Mathematics. Springer, fourth edition, 2010.
[9] Godfrey H. Hardy. Some famous problems of the theory of numbers. Clarendon Press, 1920.
[10] John Hopcroft and Robert Tarjan. Algorithm 447 Efficient algorithms for graph manipulation [H]. Communications of the ACM, 16(6):372 – 378, 1973.
[11] Bin Jia and David R. Wood. Hadwiger’s conjecture for $\ell$-link graphs. 2014. arXiv:1402.7235.
[12] Xueliang Li. Isomorphisms of $P_3$-graphs. J. Graph Theory, 21(1):81–85, 1996.
[13] Xueliang Li and Yan Liu. Nonexistence of triples of nonisomorphic connected graphs with isomorphic connected $P_3$-graphs. Electron. J. Combin., 15(1):31, 8, 2008.
[14] Peter J. Olver. Classical invariant theory, volume 44 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1999.
[15] Erich Prisner. Recognizing $k$-path graphs. Discrete Appl. Math., 99(1-3):169–181, 2000.
[16] Robert Tarjan. Depth-first search and linear graph algorithms. SIAM J. Comput., 1(2):146–160, 1972.
[17] Hassler Whitney. Congruent graphs and the connectivity of graphs. Amer. J. Math., 54(1):150–168, 1932.
[18] Bang Ye Wu and Kun-Mao Chao. Spanning trees and optimization problems. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2004.