Joint complete monotonicity of rational functions in two variables and toral $m$-isometric pairs

Akash Anand, Sameer Chavan and Rajkamal Nailwal

Dedicated to Professor Jan Stochel on the occasion of his 70th birthday

Abstract. We discuss the problem of classifying polynomials $p : \mathbb{R}_+^2 \to (0, \infty)$ for which $\frac{1}{p} = \frac{1}{\prod_{m,n \geq 0}}$ is joint completely monotone, where $p$ is a linear polynomial in $y$. We show that if $p(x, y) = a + bx + cy + dxy$ with $a > 0$ and $b, c, d \geq 0$, then $\frac{1}{p}$ is joint completely monotone if and only if $ad - bc \leq 0$. We also present an application to the Cauchy dual subnormality problem for toral 3-isometric weighted 2-shifts.

1. A two-dimensional Hausdorff moment problem

Let $\mathbb{Z}_+$ denote the set of nonnegative integers and let $\mathbb{R}_+$ denote the set of nonnegative real numbers. For a positive integer $n$ and a set $X$, let $X^n$ stand for the $n$-fold Cartesian product of $X$ with itself. Let $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$. Let $|\alpha|$ denote the sum $\alpha_1 + \cdots + \alpha_n$ and set $(\beta)_{\alpha} = \prod_{j=1}^n (\beta_j)_{\alpha_j}$, where $(\beta_j)_0 = 1$, $(\beta_j)_1 = \beta_j$ and

$$(\beta_j)_{\alpha_j} = \beta_j(\beta_j - 1) \cdots (\beta_j - \alpha_j + 1), \quad \alpha_j \geq 2, \ j = 1, \ldots, n.$$ 

We write $\alpha \preceq \beta$ if $\alpha_j \leq \beta_j$ for every $j = 1, \ldots, n$. For $\alpha \preceq \beta$, we let $\binom{\beta}{\alpha} = \prod_{j=1}^n (\beta_j)_{\alpha_j}^{-1}$.

For a net $\{a_{\alpha}\}_{\alpha \in \mathbb{Z}_+^n}$ and $j = 1, \ldots, n$, let $\Delta_j$ denote the forward difference operator given by

$$\Delta_j a_{\alpha} = a_{\alpha + \epsilon_j} - a_{\alpha}, \quad \alpha \in \mathbb{Z}_+^n,$$

where $\epsilon_j$ stands for the $n$-tuple with $j$th entry equal to 1 and 0 elsewhere.

Note that the linear operators $\Delta_1, \ldots, \Delta_n$ are mutually commuting. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, let $\Delta^\alpha$ denote the operator $\prod_{j=1}^n \Delta_{\alpha_j}$. For a function $f : \mathbb{R}_+^n \to (0, \infty)$ and $j = 1, \ldots, n$, let $\partial_j f$ be the $j$th partial derivative of $f$ whenever it exists. Note that the linear operators $\partial_1, \ldots, \partial_n$ are mutually commuting on the space of infinitely differentiable functions from $\mathbb{R}_+^n$ into $(0, \infty)$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, let $\partial^\alpha$ denote the operator $\prod_{j=1}^n \partial_{\alpha_j}$. Let $\mathbb{R} [x_1, \ldots, x_n]$ (for short, $\mathbb{R}[x]$) denote the ring of polynomials in the real variables $x_1, \ldots, x_n$.
variables \(x_1, \ldots, x_n\). A polynomial \(p \in \mathbb{R}[x]\) is said to be of multi-degree \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n\) if for each \(j = 1, \ldots, n\), \(\alpha_j\) is the largest integer for which \(\partial_x^j p \neq 0\). For \(\beta \in \mathbb{Z}_+^n\), we say that \(p\) is of multi-degree at most \(\beta\) if the multi-degree of \(p\) is \(\alpha\) and \(\alpha \leq \beta\). For a polynomial \(p \in \mathbb{R}[x_1]\), let \(\deg p\) denote the degree of \(p\). For the later purpose, we record the following fact, which is a consequence of [15] Proposition 2.1.

\[
\Delta^\beta p = 0 \text{ if } p \text{ is a polynomial of multi-degree } \gamma \text{ and } |\gamma| < |\beta|.
\]  

For a set \(X\) and a subset \(\Omega\) of \(X\), let \(1_\Omega\) denote the indicator function of \(\Omega\). For a point \(x \in X\), let \(\delta_x\) denote the Dirac delta measure with point mass at \(x\). Recall that the \(n\)-th moment of the multiplicative convolution \(\mu \odot \nu\) of finite signed Borel measures \(\mu\) and \(\nu\) is the product of the \(n\)-th moments of \(\mu\) and \(\nu\) (see [12] p. 944): For a Borel measurable subset \(\Omega\) of \([0,1]\),

\[
\int_{[0,1]} 1_\Omega(t)\mu \odot \nu(dt) = \int_{[0,1]} \int_{[0,1]} 1_\Omega(xy)\mu(dx)\nu(dy).
\]

We recall below some notions relevant for the present investigations:

(i) A net \(\mathbf{a} = \{a_\alpha\}_{\alpha \in \mathbb{Z}_+^n}\) is said to be joint completely monotone if

\[
(-1)^{\beta} \Delta^\beta a_\alpha \geq 0, \quad \beta \in \mathbb{Z}_+^n, \quad \alpha \in \mathbb{Z}_+^n.
\]

When \(n = 1\), we simply refer to \(\mathbf{a}\) as a completely monotone sequence. Following [7, 25], we say that a joint completely monotone net \(\mathbf{a}\) is minimal if for every \(j = 1, \ldots, n\) and for every \(\epsilon > 0\), \(\{a_{k\epsilon j} - \epsilon 1_{\{1\}}(k)\}_{k \in \mathbb{Z}_+}\) is not a completely monotone sequence.

(ii) An infinitely real differentiable function \(f : \mathbb{R}_+^n \to (0, \infty)\) is said to be joint completely monotone if

\[
(-1)^{\beta} (\partial^\beta f)(x) \geq 0, \quad \beta \in \mathbb{Z}_+^n, \quad x \in \mathbb{R}_+^n.
\]

When \(n = 1\), we simply refer to \(f\) as completely monotone. We say that \(f\) is separate completely monotone if for every \(x = (x_1, \ldots, x_n)\) in \(\mathbb{R}_+^n\) and \(j = 1, \ldots, n\), the function
\[
t \mapsto f(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)
\]
is completely monotone.

**Remark 1.1.** Let \(f : \mathbb{R}_+^n \to (0, \infty)\) be joint completely monotone. Note that for \(j = 1, \ldots, n\) and for fixed numbers \(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \in \mathbb{R}_+\),

\[
(-1)^m \partial_{x_j}^m f(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n) \geq 0, \quad y \in \mathbb{R}_+.
\]

Thus \(f(x_1, \ldots, x_{j-1}, \ldots, x_{j+1}, \ldots, x_n)\) is completely monotone, and since \(j\) is arbitrary, \(f\) is separate completely monotone. A similar remark applies to the joint completely monotone nets.

By the solution of the multi-dimensional Hausdorff moment problem (see [11] Proposition 4.6.11), \(\mathbf{a} = \{a_\alpha\}_{\alpha \in \mathbb{Z}_+^n}\) is a joint completely monotone net if and only if it is a Hausdorff moment net, that is, if there exists a finite positive regular Borel measure \(\mu\) concentrated on \([0,1]^n\) such that

\[
a_\alpha = \int_{[0,1]^n} t^\alpha \mu(dt), \quad \alpha \in \mathbb{Z}_+^n.
\]

If such a measure \(\mu\) exists, then it is unique. This is a consequence of the \(n\)-dimensional Weierstrass theorem and the Riesz representation theorem (see}
The investigations in this paper are motivated by the following multivariable analog of [4, Question 1.8]:

**Question 1.2.** Let \( n \) be a positive integer. For which polynomials \( p : \mathbb{R}^n_+ \to (0, \infty) \), the net \( \{ \frac{1}{p(\alpha)} \}_{\alpha \in \mathbb{Z}^n_+} \) is joint completely monotone (resp. minimal joint completely monotone)? If \( \{ \frac{1}{p(\alpha)} \}_{\alpha \in \mathbb{Z}^n_+} \) is joint completely monotone, then what is the representing measure of \( \{ \frac{1}{p(\alpha)} \}_{\alpha \in \mathbb{Z}^n_+} \)?

A motivation for this moment problem comes from the Cauchy dual subnormality problem for toral \( m \)-isometries (see [4, 5, 8, 9, 13, 18]; the reader is referred to [2, 3, 7, 8, 13, 15, 24] for the basic theory of toral \( m \)-isometries). Before we state the main result of this paper providing a partial answer to the aforementioned question in the case of \( n = 2 \), we discuss a couple of instructive nonexamples:

**Example 1.3.** Let \( p : \mathbb{R}^2_+ \to (0, \infty) \) be given polynomial. Let us see some situations where \( \frac{1}{p} \) fails to be joint completely monotone:

(i) Consider \( p(x, y) = a + bx + cx^2 + dy, x, y \in \mathbb{R}_+ \). A routine calculation shows that \( a > 0 \) and \( c, d \geq 0 \). If \( d \neq 0 \), then for some \( y_0 \in \mathbb{R}_+ \), \( p(y_0) := p(\cdot, y_0) \) has complex roots (since \( b^2 - 4(a + dy)c < 0 \) for large values of \( y \)). Hence, by [4, Proposition 4.3], \( \frac{1}{p} \) is never separate completely monotone, and hence, by Remark 1.1, it is not joint completely monotone.

(ii) Consider \( p(x, y) = 1 + y + xy, x, y \in \mathbb{R}_+ \). Since the reciprocal of any polynomial from \( \mathbb{R}_+ \) into \( (0, \infty) \) of degree 1 is completely monotone, \( \frac{1}{p} \) is separate completely monotone. Note that

\[
\frac{\partial_1 \partial_2 \left( \frac{1}{p} \right)(x, y) = \frac{yx + y - 1}{(1 + y + xy)^3}, \quad x, y \in \mathbb{R}_+.}
\]

Clearly, for \( x, y \in (0, 1/2) \), \( \partial_1 \partial_2 \left( \frac{1}{p} \right)(x, y) < 0 \), and hence \( \frac{1}{p} \) is not joint completely monotone.

Thus, the joint complete monotonicity of \( \frac{1}{p} \) may fail for a polynomial of multi-degree \((1, 1)\). \( \blacksquare \)

The above example raises the problem of describing all polynomials \( p : \mathbb{R}^2_+ \to (0, \infty) \) of bi-degree \((1, 1)\) for which the net \( \{ \frac{1}{p(\alpha)} \}_{\alpha \in \mathbb{Z}^2_+} \) is joint completely monotone. Needless to say, the following result leads to a complete solution to this intermediate problem (see Theorem 3.1).

**Theorem 1.4 (Main theorem).** For \( a_j, b_j \in (0, \infty), j = 0, \ldots, k \), let

\[
a(x) = a_0 \prod_{j=1}^k (x + a_j), \quad b(x) = b_0 \prod_{j=1}^k (x + b_j), \quad x \in \mathbb{R}_+.
\]

(1.2)

The following statements are valid:
(i) \( \{ \frac{1}{b(m)+a(m)n} \}_{m,n \in \mathbb{Z}_+} \) is a minimal joint completely monotone net if
\[
b_1 \leq a_1 \leq b_2 \leq a_2 \leq \ldots \leq b_k \leq a_k,
\]
(1.3)
(ii) if \( \{ \frac{1}{b(m)+a(m)n} \}_{m,n \in \mathbb{Z}_+} \) is a joint completely monotone net, then
\[
\sum_{j=1}^{k} \frac{1}{a_j} \leq \sum_{j=1}^{k} \frac{1}{b_j},
\]
(1.4)
\[
\prod_{j=1}^{k} b_j \leq \prod_{j=1}^{k} a_j,
\]
(1.5)
\[
\sum_{j=1}^{k} b_j \leq \sum_{j=1}^{k} a_j.
\]
(1.6)

**Remark 1.5.** The condition (1.3) is not necessary for \( \{ \frac{1}{p(m,n)} \}_{m,n \in \mathbb{Z}_+} \) to be joint completely monotone. For example, for
\[
p(x, y) = (x+1)(x+4) + (x+2)(x+3)y,
\]
the net \( \{ \frac{1}{p(m,n)} \}_{m,n \in \mathbb{Z}_+} \) is joint completely monotone. This may be concluded from Theorem 3.3(i) (to be seen later). On the other hand, (1.4), (1.5) and (1.6) together are not sufficient to guarantee the joint complete monotonicity of \( \{ \frac{1}{p(m,n)} \}_{m,n \in \mathbb{Z}_+} \). Indeed, if
\[
p(x, y) = (x+1)(x+2) + (x+3)(x+4)y,
\]
then by [4, Proposition 4.3], the sequence \( \{ \frac{1}{p(m,n)} \}_{m \in \mathbb{Z}_+} \) is not completely monotone, and hence the net \( \{ \frac{1}{p(m,n)} \}_{m,n \in \mathbb{Z}_+} \) is not joint completely monotone (see Remark 1.1). Finally, note that if \( \{ \frac{1}{b(m)+a(m)n} \}_{m,n \in \mathbb{Z}_+} \) is a joint completely monotone net, then \( b_1 \leq a_k \). Indeed, if \( a_k < b_1 \), then
\[
\frac{1}{b_k} \leq \ldots \leq \frac{1}{b_1} < \frac{1}{a_k} \leq \ldots \leq \frac{1}{a_1},
\]
which contradicts the inequality in (1.4). ♦

**Plan of the paper.** A large portion of Section 2 occupies proof of Theorem 1.4. The proof begins with a generalization of Theorem 1.4(i) (see Theorem 2.1). We present two proofs of Theorem 2.1(a). The first of which exploits the partial fraction decomposition of rational functions in one variable (see [4, 18]) together with a theorem of Berg and Durán from [12]. The second proof is more elementary in approach and provides a method to compute the representing measures in question. We also present several consequences of Theorem 2.1. The first two of which are outcomes of the proof of Theorem 2.1 (see Corollary 2.6 and Corollary 2.7). We also generalize Theorem 1.4(i) providing a couple of families of joint completely monotone rational functions having nonconstant numerator (see Corollary 2.8).

In Section 3, we present answers to Question 1.2 in several subcases of lower bi-degree. The first result of this section describes all polynomials \( p : \mathbb{R}_+^2 \to (0, \infty) \) of bi-degree at most \((1, 1)\) for which \( \frac{1}{p} \) is joint completely...
monotone and also compute the representing measures in question (see Theorem 3.1). In the case of bi-degree \((2,1)\), we show that an answer to Question 1.2 boils down to two subcases (see Proposition 3.2). We then provide partial answers to Question 1.2 in these subcases (see Theorems 3.3 and 3.6). In particular, we improve Theorem 1.4(i) in the case of \(k=2\).

In Section 4, we provide a solution to the Cauchy dual subnormality problem for toral 3-isometric weighted 2-shifts which are separate 2-isometries (see Definition 4.1 and Theorem 4.9). The proof of Theorem 4.9 relies on Theorem 3.1 and characterizations of toral \(m\)-isometries (see Theorem 4.3) and of separate 2-isometries within the class of toral \(m\)-isometries (see Corollary 4.5). The operator tuple torally Cauchy dual to a toral 2-isometric weighted 2-shift is always jointly subnormal (see Corollary 4.11). We also exhibit a family of toral 3-isometric weighted 2-shift without jointly subnormal toral Cauchy dual (see Example 4.12).

We conclude the paper with a brief discussion on the role of the so-called coefficient-matrix in the moment problem addressed in Question 1.2. In particular, we employ the coefficient-matrix to reformulate some of the results of Section 3 (see Theorem 5.1 and (5.1)).

2. Proof of the main theorem and its consequences

The first step towards the proof of Theorem 1.4 is the following fact:

**Theorem 2.1.** Let \(a,b\) be as given in (1.2) and let \(\{c(m)\}_{m \in \mathbb{Z}^+}\) be a sequence of positive real numbers. Assume that

(A1) \(b_1 \leq a_1 \leq b_2 \leq a_2 \leq \ldots \leq b_k \leq a_k\),

(A2) \(\{\frac{c(m)}{a(m)}\}_{m \in \mathbb{Z}^+}\) is a completely monotone sequence.

Then the following statements are valid:

(a) the net \(\{\frac{c(m)}{b(m)+a(m)n}\}_{m,n \in \mathbb{Z}^+}\) is joint completely monotone,

(b) \(\{\frac{c(m)}{b(m)+a(m)n}\}_{m,n \in \mathbb{Z}^+}\) is a minimal joint completely monotone net provided \(\{\frac{c(m)}{a(m)}\}_{m \in \mathbb{Z}^+}\) is a minimal completely monotone sequence.

In the proof of Theorem 2.1 we need the following identity: For any real number \(x > 0\),

\[
\frac{(-1)^{l-1}}{(l-1)!} \int_0^1 (\log s)^{l-1} s^{x-1+n} \, ds = \frac{1}{(n+x)^l}, \quad l \geq 1, \quad n \geq 0
\]  

(cf. [4, Eqn (3.2)]). For the sake of completeness, we verify this identity by induction on \(l \geq 1\). Fix \(x > 0\). For \(l = 1\), we see that

\[
\frac{(-1)^{l-1}}{(l-1)!} \int_0^1 (\log s)^{l-1} s^{x-1+n} \, ds = \int_0^1 s^{x-1+n} \, ds = \frac{1}{n+x}, \quad n \geq 0,
\]
and hence, the equation (2.1) holds for \( l = 1 \). Now assuming (2.1) for \( l \geq 1 \), we note that for any nonnegative integer \( n \),

\[
\frac{(-1)^l}{l!} \int_0^1 (\log s)^l s^{-1+n} ds = \frac{(-1)^l}{l!} \left( \lim_{s \to 0} (\log s)^l s^{-1+n} \right) - \frac{l}{n + x} \int_0^1 (\log s)^l s^{-1+n} ds
\]

\[
= \frac{1}{n + x} \left( \frac{(-1)^l-1}{(l-1)!} \int_0^1 (\log s)^l-1 s^{-1+n} ds \right)
\]

\[
= \frac{1}{(n + x)^{l+1}}.
\]

This completes the verification of (2.1). In a proof of Theorem 2.1 we also need a special case of the following general fact.

**Lemma 2.2.** For \( b_0 > 0, m_i \in \mathbb{Z_+} \) and \( b_i \in (0, \infty) \), \( i = 1, \ldots, k \), let \( p : \mathbb{R}_+ \to (0, \infty) \) be any polynomial and \( q : \mathbb{R}_+ \to (0, \infty) \) be a non-constant polynomial given by \( q(x) = b_0 \prod_{i=1}^k (x + b_i)^m_i \). If \( \deg q \geq \deg p \), then the following statements are valid:

(i) there exist unique \( c_0 \in \mathbb{R}_+ \) and \( c_{ij} \in \mathbb{R} \) such that

\[
p(x) \quad q(x) = c_0 + \sum_{i=1}^k \sum_{j=1}^{m_i} \frac{c_{ij}}{(x + b_i)^j}, \quad x \in \mathbb{R}_+,
\]

(ii) if \( \sum_{i=1}^k \sum_{j=1}^{m_i} \frac{c_{ij}}{(j-1)!} (-\log s)^{j-1} s^{b_i} \leq 0 \) for every \( s \in (0, 1) \), then for every \( t \in (0, 1) \), \( \left\{ t^{\frac{p(n)}{q(n)}} \right\}_{n \in \mathbb{Z}_+} \) is a completely monotone sequence.

**Proof.** (i) If \( \deg p < \deg q \), then this fact is precisely [18, Proposition 2.1]. If \( \deg p = \deg q \), then by the polynomial long division (see [14, pg. 271, Example(2)]), there exists \( c_0 \in \mathbb{R} \) and a polynomial \( r \) such that \( \deg r < \deg q \) and

\[
p(x) \quad q(x) = c_0 + \frac{r(x)}{q(x)}, \quad x \in \mathbb{R}_+.
\]

Indeed, \( c_0 = \alpha_p/b_0 > 0 \), where \( \alpha_p \) denotes the leading coefficient of the polynomial \( p \). Another application of [18, Proposition 2.1] (applied to \( r \) and \( q \)) now yields the desired result.

(ii) Assume that

\[
\sum_{i=1}^k \sum_{j=1}^{m_i} \frac{c_{ij}}{(j-1)!} (-\log s)^{j-1} s^{b_i} \leq 0, \quad s \in (0, 1).
\]

(2.2)

For every \( n \in \mathbb{Z}_+ \), by (i), we have

\[
p(n) \quad q(n) = c_0 + \sum_{i=1}^k \sum_{j=1}^{m_i} \frac{c_{ij}}{(n + b_i)^j}
\]
the fact that the product of two completely monotone sequences is completely monotone (see \[22\]), since \(B\) problems.

Corollary 2.2, for every \(t\), this yields for every \((m, n)\) sequence of positive real numbers. Assume that \(|A(m)|\) is a Hausdorff moment sequence. In turn, in view of the assumption (A2), this combined with the assumption (2.2) shows that for every \(t\),

\[
\text{Proof.} \quad \text{Assume that (2.3) holds. Let } A = a/c \text{ and } B = b/c. \text{ Note that}
\]

\[
\frac{c(m)}{b(m) + a(m)n} = \frac{1}{B(m) + A(m)n} = \int_{[0,1]} t^n \frac{B(m)}{A(m)} dt, \quad m, n \in \mathbb{Z}_+.
\]

Since \(\frac{B}{A} = \frac{b}{a}\), it suffices to check that for every \(t \in (0, 1)\), \(|A(m)|\) is a Hausdorff moment sequence. In turn, in view of the assumption (A2) (which ensures that \(|A(m)|\) is a completely monotone sequence) and the fact that the product of two completely monotone sequences is completely monotone (see \[11\] Lemma 8.2.1(v)), this follows from \[2.3\].

Proof I of Theorem 2.1(a). In view of Lemma 2.3, it is sufficient to check that \[2.3\] holds. In case \(b_j = b_{j+1}\) for some \(j = 1, \ldots, k - 1\), then by the assumption (A1), \(a_j = b_j\). In this case, after cancelling the factors \(x + a_j\) and \(x + b_j\) from \(\frac{B}{A} = \frac{b}{a}\), we may assume that all \(b_j\)'s are distinct. Repeating the same argument, we may assume that \(a_j\)'s are also distinct.
By Lemma 2.2(i) (applied to $p = b$, $q = a$ and $m_i = 1$ for all $i$), there exist $c_0, c_1, \ldots, c_k \in \mathbb{R}$ such that
\[ \frac{B(m)}{A(m)} = \frac{b(m)}{a(m)} = c_0 + \sum_{j=1}^{k} \frac{c_j}{m + a_j}, \quad m \in \mathbb{Z}_+. \]  
(2.5)

Clearly, $c_0 = \frac{b_0}{a_0}$. Also, by [18 Proposition 2.1] (applied with all $b_i = 1$),
\[ c_j = \frac{b(-a_j)}{a_0 \prod_{1 \leq i \leq j < k} (a_l - a_j)}, \quad j = 1, \ldots, k. \]  
(2.6)

Since $b(x) = b_0 \prod_{i=1}^{k} (x + b_i)$,
\[ c_j = \frac{b_0}{a_0} \prod_{1 \leq i \leq j < k} (a_l - a_j) \]  
(2.7)
\[ = - \frac{b_0}{a_0} \prod_{1 \leq i \leq j} (a_j - b_i) \prod_{j+1 \leq i \leq k} (a_l - a_j) \]  
\[ = - \frac{b_0}{a_0} \prod_{1 \leq i \leq j} (a_j - b_i) \prod_{j+1 \leq i \leq k} (a_l - a_j), \quad j = 1, \ldots, k, \]
which is a non-positive real number in view of the assumption (A1). Thus $\sum_{j=1}^{k} c_j s^{a_j} \leq 0$ for every $s \in (0, 1)$. Hence, we get (2.3) from Lemma 2.2(ii) (applied to $p = b$ and $q = a$), completing the Proof I of Theorem 2.1(a). □

Although Lemma 2.2 provides an elegant proof of Theorem 2.1(a), a close examination of Proof I shows that an application of Lemma 2.2(ii) could be replaced by an argument based on multiplicative convolution of measures (cf. [12 Proof of Lemma 2.1]). Moreover, the following alternate proof of Theorem 2.1(a) provides an algorithm to compute the representing measure in question.

**Proof II of Theorem 2.1(a).** Note that by (2.5),
\[ \frac{B(m)}{A(m)} - 1 = \frac{b(m)}{a(m)} - t \sum_{j=1}^{k} \frac{c_j}{m + a_j} = \frac{b_0}{a_0} \prod_{j=1}^{k} \frac{c_j}{m + a_j}, \quad t > 0, \quad m \in \mathbb{Z}_+. \]  
(2.8)

For any $j = 1, \ldots, k$, $m \in \mathbb{Z}_+$ and $t > 0$, consider
\[ \frac{c_j}{t^{m+a_j}} = \sum_{l=0}^{\infty} \frac{(c_j \log t)^l}{l!(m + a_j)^l}. \]  
(2.9)
\[ \int_{[0,1]} s^m \delta_1(ds) + \sum_{l=1}^{\infty} \frac{(c_j \log t)^l}{(l-1)!!} \int_{[0,1]} (-\log s)^{l-1} s^{a_j-1+m} ds. \]

By the dominated convergence theorem, we obtain
\[ t^k \left( \frac{b_0}{a_0} - 1 \right) t^{c_j/m + a_j} = \int_{[0,1]} s^m \mu_{j,t}(ds), \quad m \in \mathbb{Z}_+, \quad j = 1, \ldots, k, \]  
(2.10)
where the measure $\mu_{j,t}$ is of the form
\[ \mu_{j,t}(ds) = t^k \left( \frac{b_0}{a_0} - 1 \right) \delta_1(ds) + w_j(s,t)ds \]  
(2.11)
with the weight function $w_j$ given by
\[ w_j(s,t) = t^k \left( \frac{b_0}{a_0} - 1 \right) s^{a_j-1} \sum_{l=1}^{\infty} \frac{(c_j \log t)^l(-\log s)^{l-1}}{(l-1)!!}, \quad s, t \in (0, 1). \]  
(2.12)
Clearly, $w_j$ integrable with respect to the Lebesgue measure on $[0,1]$ for every $j = 1,\ldots,k$. By the assumption (A2), there exists a positive finite Borel measure $\mu$ on $[0,1]$ such that
\[
\frac{c(m)}{a(m)} = \int_{[0,1]} s^m \mu(ds), \quad m \in \mathbb{Z}_+.
\]
Combining this with (2.8) and (2.10), we obtain
\[
\frac{b(m)-1}{A(m)} = \frac{c(m)}{a(m)} \prod_{j=1}^k t \left( \frac{a(m)}{b_0} - 1 \right) t^{s_j} \prod_{j=1}^k t^{s_j} = \int_{[0,1]} s^m \mu(ds) \prod_{j=1}^k \int_{[0,1]} s^m \mu_j, (d(s,t)) (2.13)
\]
where $\nu_t$ is the multiplicative convolution of $\mu$ and $\mu_j, t = 1,\ldots,k$. This combined with (2.4) yields
\[
\frac{c(m)}{b(m)+a(m)n} = \int_{[0,1]} \int_{[0,1]} s^m t^n \nu_t(ds)dt. (2.14)
\]
By (2.7) and the assumption (A1), $c_1,\ldots,c_k \leq 0$. It now follows from (2.12) that $w_j$ is nonnegative on $(0,1)$ for every $j = 1,\ldots,k$. Since the multiplicative convolution of positive measures is positive, this combined with (2.11) and (2.14) completes Proof II of (a). $\Box$

Remark 2.4. The assumption (A1) is used only in the last paragraph of Proof II of Theorem 2.1 (a) to conclude that $c_1,\ldots,c_k \leq 0$. $\diamond$

To prove Theorem 2.1 (b), we need an elementary fact pertaining to the convolution of finite signed measures.

**Lemma 2.5.** Let $\mu, \nu$ be finite signed Borel measures on $[0,1]$ and let $\mu \circ \nu$ be the multiplicative convolution of $\mu$ and $\nu$. If $\mu(\{0\}) = 0$ and $\nu(\{0\}) = 0$, then $\mu \circ \nu(\{0\}) = 0$.

**Proof.** Note that
\[
\mu \circ \nu(\{0\}) = \int_{[0,1]} \mathbb{1}_{\{0\}}(x) \mu \circ \nu(dx) \]
\[
= \int_{[0,1] \times [0,1]} \mathbb{1}_{\{0\}}(xy) \mu(dx) \nu(dy) \]
\[
= \int_{([0,1] \times \{0\}) \cup \{0\} \times [0,1]} \mathbb{1}_{\{0\}}(xy) \mu(dx) \nu(dy) \]
\[
= \int_{[0,1] \times \{0\}} \mu(dx) \nu(dy) + \int_{\{0\} \times [0,1]} \mu(dx) \nu(dy) \]
\[
= \mu([0,1]) \nu(\{0\}) + \mu(\{0\}) \nu((0,1]).
\]
This yields the desired conclusion. $\Box$
Proof of Theorem 2.1(b). Assume that \( \left\{ \frac{c(m)}{a(m)} \right\}_{m \in \mathbb{Z}_+} \) is a minimal completely monotone sequence with the representing measure \( \mu \). In view of [7] Proposition 5, it suffices to check that
\[
\eta([0,1] \times \{0\}) = 0, \quad \eta(\{0\} \times [0,1]) = 0,
\]
where \( \eta \) is the representing measure of \( \left\{ \frac{c(m)}{b(m)+a(m)n} \right\}_{m,n \in \mathbb{Z}_+} \). Let \( \nu_t \) be the multiplicative convolution of \( \mu \) and \( \mu_{j,t} \), \( j = 1, \ldots, k \) (see (2.11)). It follows from (2.14) that
\[
\eta([0,1] \times \{0\}) = \int_{[0,1]} 1_{[0,1]}(s,t) \nu_t(ds)dt = \int_{[0,1]} \left( \int_{[0,1]} 1_{\{0\}}(s) \nu_t(ds) \right) dt,
\]
which is clearly 0. Since \( \left\{ \frac{c(m)}{a(m)} \right\}_{m \in \mathbb{Z}_+} \) is a minimal completely monotone sequence, by [25] Theorem IV.14a, \( \mu(\{0\}) = 0 \). Further, since \( \mu_{j,t}(\{0\}) = 0 \), \( j = 1, \ldots, k \) (see (2.11)), repeated applications of Lemma 2.5 \((k-1)\) times) show that \( \nu_t(\{0\}) = 0 \) for any \( t \in (0,1) \). It now follows that
\[
\eta(\{0\} \times [0,1]) = \int_{[0,1]} 1_{\{0\} \times [0,1]}(s,t) \nu_t(ds)dt = \int_{[0,1]} \left( \int_{[0,1]} 1_{\{0\}}(s) \nu_t(ds) \right) dt = \int_{[0,1]} \nu_t(\{0\}) dt,
\]
which shows that \( \eta(\{0\} \times [0,1]) = 0 \). □

Before we present a variant of Theorem 1.4(ii) (in the case of constant sequence \( c \) with value 1), we record here the following consequence of [4] Theorem 3.1 and [25] Theorem IV.14a:

For a polynomial \( p : \mathbb{R}_+ \to (0,\infty) \) with all real roots, \( \left\{ \frac{1}{p(m)} \right\}_{m \in \mathbb{Z}_+} \) is a completely monotone sequence with the representing measure being a weighted Lebesgue measure. In particular, \( \left\{ \frac{1}{a(m)} \right\}_{m \in \mathbb{Z}_+} \) is a minimal completely monotone sequence.

Corollary 2.6. Let \( a, b \) be as given in (1.2). If \( \left\{ \frac{1}{b(m)+a(m)n} \right\}_{m,n \in \mathbb{Z}_+} \) is a joint completely monotone net, then it is a minimal joint completely monotone net.

Proof. An examination of the proof of Theorem 2.1(b) shows that (under the assumption that \( \left\{ \frac{c(m)}{b(m)+a(m)n} \right\}_{m,n \in \mathbb{Z}_+} \) is joint completely monotone) it depends only on (2.11), (2.13), (2.14) and Lemma 2.5. On the other hand, it is recorded in Remark 2.4 that the assumption (A1) is not required in the deduction of (2.10)-(2.14). Thus it suffices to check that

(i) \( \mu_{j,t}([0,1]) > 0 \), \( j = 1, \ldots, k, t \in (0,1) \), (so that Lemma 2.5 applies),
(ii) (A2) holds with \( c = 1 \).

The assertion (i) follows from (2.10), while (ii) is immediate from (2.15), completing the proof. □
Here is another consequence of the proof of Theorem 2.1.

**Corollary 2.7.** Let \( a, b \) be as given in \(#1.2\). Let \( j = 1, \ldots, k \) and \( t \in (0, 1) \). If \( (-1)^j b(-a_j) < 0 \), then \( \{ \frac{k}{t_j} b(-a_j) \}_{m \in \mathbb{Z}_+} \) is not a Hausdorff moment sequence, where \( d_j = a_0 \prod_{1 \leq l \leq j} (a_l - a_j) \) (we use here the convention that the product over empty set is 1).

**Proof.** Assume that \( (-1)^j b(-a_j) < 0 \) for some \( j = 1, \ldots, k \). By \(#2.6\),

\[
c_j = - \frac{(-1)^j b(-a_j)}{a_0 \prod_{1 \leq l \leq j-1} (a_j - a_l) \prod_{j+1 \leq l \leq k} (a_l - a_j)} > 0.
\]

In view of \(#2.10\), it suffices to check that for every \( t_0 \in (0, 1) \), \( w_j(\cdot; t_0) < 0 \) on some open subset of \((0, 1)\). For \( t_0 \in (0, 1) \), letting \( s_0 = e^{\frac{1}{j + \log t_0}} \) in \(#2.12\), we obtain

\[
w_j(s_0, t_0) = t_0^j \left( \frac{s_0}{s_0-1} \right)^{a_j-1} \sum_{l=1}^{\infty} \left( -\log s_0 \right)^{l-1} \left( l - 1 \right)! \left( c_j \log t_0 \right)^l
\]

\[
= c_j t_0^j \left( \frac{s_0}{s_0-1} \right)^{a_j-1} (\log t_0) \sum_{l=1}^{\infty} \left( -1 \right)^{l-1} \left( l - 1 \right)! \left( c_j \log t_0 \right)^l
\]

which is less than 0 by \(#2.10\). This shows that \( w_j(s_0, t_0) < 0 \). Since \( w_j \) is continuous at \((s_0, t_0)\), the proof is complete. \( \square \)

**Corollary 2.8.** Let \( a, b \) be as given in \(#1.2\) and assume that (A1) holds. Let \( F \) and \( G \) be two (possibly empty) subsets of \( \{1, \ldots, k\} \) and \( \{2, \ldots, k\} \), respectively. Let \( c \) be any one of the following choices:

\[
c(x) = \prod_{j \in F} (x + a_j), \quad c(x) = \prod_{j \in G} (x + b_j), \quad x \in \mathbb{R}_+.
\]

Then \( \{ \frac{c(m)}{a(m)} \}_{m \in \mathbb{Z}_+} \) is a minimal joint completely monotone net.

**Proof.** In view of Theorem 2.1, it suffices to check that \( \{ \frac{c(m)}{a(m)} \}_{m \in \mathbb{Z}_+} \) is a minimal completely monotone sequence. If \( c(x) = \prod_{j \in F} (x + a_j) \), then \(#2.10\) (applied to \( p = a/c \)) shows that \( \{ \frac{c(m)}{a(m)} \}_{m \in \mathbb{Z}_+} \) is a minimal completely monotone sequence. To see the conclusion in the second case, let \( c(x) = \prod_{j \in G} (x + b_j) \). Write \( G = \{ i_1, \ldots, i_l \} \) and \( G' = \{ i_1 - 1, i_2 - 1, \ldots, i_l - 1 \} \) with \( i_1 \leq \ldots \leq i_l \). Note that

\[
c(m) = \frac{1}{a_0} \prod_{j=1}^l (m + b_{i_j}) = \frac{1}{d(m)} \prod_{j=1}^l (m + b_{i_j})
\]

where \( d(m) = a_0 \prod_{j \in G'} (m + a_j), m \in \mathbb{Z}_+ \). Consider

\[
\prod_{j=1}^l (m + b_{i_j}) = \prod_{j=1}^l (m + b_{i_j}) = \prod_{j=1}^l (m + a_{i_j - 1}), \quad m \in \mathbb{Z}_+.
\]
By the assumption (A1), we see that
\[ \sum_{j=1}^{p} b_{ij} \geq \sum_{j=1}^{p} a_{ij-1}, \quad p = 1, \ldots, l. \]

Hence, by [10] Theorem 1], \[ \prod_{j=1}^{n}(x+a_{ij}) \] is completely monotone function from \( \mathbb{R}_+ \) into \((0, \infty)\). Hence \( \{ \prod_{j=1}^{l}(m+b_{ij}) \}_{m \in \mathbb{Z}_+} \) is a minimal completely monotone sequence (see [25] Theorem IV.14b). □

We need the following general fact in the proof of Theorem 1.4 (ii):

**Lemma 2.9.** For polynomials \( a, b: \mathbb{R}_+ \to (0, \infty) \), let \( p(x, y) = b(x) + a(x)y \), \( x, y \in \mathbb{R}_+ \). If \( \frac{1}{p} \) is a joint completely monotone function, then
\[
\begin{align*}
d'(x)b(x) & \leq a(x)b'(x), \quad x \in \mathbb{R}_+, \quad (2.18) \\
a(x_2)b(x_1) & \leq a(x_1)b(x_2), \quad x_2 \geq x_1 \geq 0. \quad (2.19)
\end{align*}
\]

and \( \deg a \leq \deg b \).

**Proof.** Assume that \( \frac{1}{p} \) is a joint completely monotone function. A routine calculation using induction on \( n \geq 1 \) shows that
\[
\partial^n \left( \frac{1}{p} \right)(x, y) = \frac{(-1)^n n! a(x)^n}{(b(x) + a(x)y)^{n+1}}, \quad n \in \mathbb{Z}_+, \quad x, y \in \mathbb{R}_+.
\]

Thus, for any positive integer \( n \),
\[
\begin{align*}
(-1)^{n+1} \partial_{1} \partial_{2} \left( \frac{1}{p} \right)(x, y) &= -n! \partial_{1} \left( \frac{a(x)^n}{(b(x) + a(x)y)^{n+1}} \right) \\
&= -n! \left( \frac{n a(x)^{n-1} a'(x)}{(b(x) + a(x)y)^{n+1}} - (n + 1) \frac{a^n(x)(b'(x) + a'(x)y)}{(b(x) + a(x)y)^{n+2}} \right) \\
&= \frac{n! a(x)^{n-1}}{p(x, y)^{n+2}} \left( (n + 1)a(x)(b'(x) + a'(x)y) - na'(x)(b(x) + a(x)y) \right).
\end{align*}
\]

This together with the joint complete monotonicity of \( \frac{1}{p} \) yields
\[
(-1)^{n+1} \partial_{2} \partial_{1} \left( \frac{1}{p} \right)(x, y) \geq 0 \quad \text{implies}
\]
\[
(n + 1)a(x)(b'(x) + a'(x)y) \geq na'(x)(b(x) + a(x)y), \quad n \geq 1, \quad x, y \in \mathbb{R}_+.
\]

Letting \( y = 0 \) and dividing by \( n \), we get
\[
(1 + 1/n)a(x)b'(x) \geq a'(x)b(x), \quad n \geq 1, \quad x \in \mathbb{R}_+.
\]

We now let \( n \to \infty \) to get (2.18). To see (2.19), note that by (2.18),
\[
\frac{a'(x)}{a(x)} \leq \frac{b'(x)}{b(x)}, \quad x \in \mathbb{R}_+.
\]

After integrating over \([x_1, x_2]\) and taking exponential on both sides, we get (2.19). Letting \( x_1 = 0 \) and \( x_2 = m \) in (2.19), we obtain
\[
\frac{b(0)}{a(0)} \leq \frac{b(m)}{a(m)}, \quad m \geq 0.
\]
This combined with \( a(0) > 0 \) and \( b(0) > 0 \) yields \( \deg a \leq \deg b \).

\[ \text{□} \]

Remark 2.10. Assume that \( \frac{1}{p} \) is a joint completely monotone function. By (2.18), \( \left( \frac{a}{b} \right)' \leq 0 \) for every \( x \geq 0 \). This combined with (2.5) and (2.6) (with roles of \( a \) and \( b \) interchanged) shows that

\[ \sum_{j=1}^{k} \frac{\prod_{l=1}^{k}(a_l - b_j)}{\prod_{l \neq j \leq k}(b_l - b_j)(x + b_j)^2} \geq 0, \quad x \geq 0. \]  

(2.20)

In particular, by multiplying on the left hand side by \( x^2, x > 0 \), and letting \( x \to \infty \), we obtain

\[ \sum_{j=1}^{k} \frac{\prod_{l=1}^{k}(a_l - b_j)}{\prod_{l \neq j \leq k}(b_l - b_j)} \geq 0. \]

Moreover, we have

\[ \sum_{j=1}^{k} \frac{b_j \prod_{l=1}^{k}(a_l - b_j)}{\prod_{l \neq j \leq k}(b_l - b_j)} \geq 0. \]

This may be obtained by letting \( x = 0 \) in (2.20).

\[ \text{♯} \]

The following lemma justifies the fact that a minimal completely monotone function is interpolated by its natural extension to \( \mathbb{R}_+^2 \).

Lemma 2.11. Let \( a, b \) be as given in (1.2). Then there exists a finite Radon measure \( \nu \) on \( \mathbb{R}_+^2 \) such that

\[ \frac{1}{b(x) + a(x)y} = \int_{\mathbb{R}_+^2} e^{-(x_1 + y_2)} d\nu(t_1, t_2), \quad x, y \in \mathbb{R}_+. \]  

(2.22)

In particular, if the net \( \left\{ \frac{1}{b(m) + a(m)n} \right\}_{m,n \in \mathbb{Z}_+} \) is minimal joint completely monotone, then the function \( \frac{1}{b(x) + a(x)y} \) is joint completely monotone.

Proof. Note that a simple calculation as seen in (2.4) shows that

\[ \frac{1}{b(x) + a(x)y} = \int_{[0,1]} t^y \frac{\mu_0}{a(x)} \, dt, \quad x, y \in \mathbb{R}_+. \]  

(2.21)

Since \( \frac{\mu_0}{a(x)} \) is a finite product of completely monotone functions \( \frac{1}{(x + a_j)}, j = 1, \ldots, n \), we note that \( \frac{1}{a(x)} \) is joint completely monotone on \( \mathbb{R}_+ \). Now by [22, Theorem 1.1.4], there exists a unique Radon measure \( \mu \) on \( \mathbb{R}_+ \) such that

\[ \frac{1}{a(x)} = \int_{\mathbb{R}_+} e^{-xt} \mu(dt), \quad x \in \mathbb{R}_+. \]  

(2.22)

We argue as in the proof II of Theorem 2.11 (a) to see that,

\[ \frac{\mu_0}{t^{\frac{1}{a(x)}} - 1} = \prod_{j=1}^{k} t^{\frac{c_j}{t^{x+a_j}}}, \quad t > 0, \quad x \in \mathbb{R}_+, \]  

(2.23)

where \( c_1, \ldots, c_n \) are given by (2.7) and

\[ t^{\frac{1}{a(x)}} - 1 \]  

where \( m_0 \) and \( m_1 \) are given by (2.7).
with \( \mu_{j,t} \) given by (2.11). For every \( t \in (0,1) \), we now use the change of variable \( s = e^{-u} \) to see
\[
\frac{1}{b(x) + a(x)y} = \int_{[0,1]} \int_{\mathbb{R}_+} e^{-ux} \nu_t(du), \quad j, t
\]
where \( \nu_t(du) = \mu_{j,t}(-e^{-u}du) \). It now follows from (2.19) that
\[
\frac{1}{b(x) + a(x)y} = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} e^{-ux} e^{-vy}(e^{-v}) \nu_t(du), \quad x, y \in \mathbb{R}_+.
\]
This completes the proof of the first half.

Assume that the net \{1/(b(m) + a(m)n)\}_{m,n \in \mathbb{Z}_+} is minimal joint completely monotone. By [7, Proposition 6], there exists a joint completely monotone function \( f \) on \( \mathbb{R}_+^2 \) such that
\[
f(m, n) = \frac{1}{b(m) + a(m)n}, \quad m, n \in \mathbb{Z}_+.
\]
In view of the proof [7, Proposition 6], there exists a positive Radon measure \( \mu \) on \( \mathbb{R}_+^2 \) such that
\[
f(x, y) = \int_{\mathbb{R}_+^2} e^{-(x_1 + y_2)} d\mu(t_1, t_2), \quad x, y \in \mathbb{R}_+.
\]
By the first part, there exists a finite Radon measure \( \nu \) on \( \mathbb{R}_+^2 \) such that
\[
\frac{1}{b(x) + a(x)y} = \int_{\mathbb{R}_+^2} e^{-(x_1 + y_2)} d\nu(t_1, t_2), \quad x, y \in \mathbb{R}_+.
\]
It now follows from the uniqueness of the representing measure that \( \mu = \nu \), which completes the proof.

Proof of Theorem 1.4. (i) This is a special case of Corollary 2.8 (the case of \( F = \emptyset \)).

(ii) Assume that \{1/(b(m) + a(m)n)\}_{m,n \in \mathbb{Z}_+} is a joint completely monotone net. By Corollary 2.8, \{1/(b(m) + a(m)n)\}_{m,n \in \mathbb{Z}_+} is a minimal joint completely monotone net. Hence, by Lemma 2.11, \( \frac{1}{b(x) + a(x)y} \) is a completely monotone function, and consequently, Lemma 2.9 is applicable. To obtain (1.4), we let \( x = 0 \) in (2.18) and simplify the expression. By (2.19),
\[
\prod_{j=1}^{k}(x_1 + b_j) = \prod_{j=1}^{k}(x_2 + b_j) \leq \prod_{j=1}^{k}(x_2 + a_j), \quad x_2 \geq x_1 \geq 0.
\]
Taking \( x_2 \to \infty \), we get
\[
\prod_{j=1}^{k}(x_1 + b_j) \leq \prod_{j=1}^{k}(x_1 + a_j), \quad x_1 \geq 0.
\]
Letting \( x_1 = 0 \), we get (1.5). After cancelling \( x_1^k \) from both sides of (2.28), dividing by \( x_1^{k-1} \) and letting \( x_1 \to \infty \), we get (1.6).

\[ \square \]

3. Some cases of lower bi-degree

In this section, we present solutions to some special instances of Question 1.2 of lower bi-degree. Before we state and prove the first result in this direction, recall that for a positive real number \( \nu \), the Bessel function \( J_\nu(z) \) of the first kind of order \( \nu \) is given by

\[
J_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{z^2}{4})^k}{k!\Gamma(\nu + k + 1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0],
\]

where \( \Gamma \) denotes the Gamma function. The modified Bessel function \( I_\nu(z) \) of the first kind of order \( \nu \) is given by

\[
I_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{z^2}{4})^k}{k!\Gamma(\nu + k + 1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0],
\]

(see [1] Eqns 9.1.10 & 9.6.10). It is worth noting that the expression of \( w_j \), as given in (2.12), takes the following form: For any \( s, t \in (0, 1) \),

\[
w_j(s, t) = t^\frac{1}{2} \left( \frac{b_0}{b_0 - 1} \right) s^{\frac{1}{2}-1} \frac{c_j \log t}{\sqrt{-c_j \log s \log t}} I_1(2\sqrt{-c_j \log s \log t}).
\]

**Theorem 3.1 (Degree at most (1, 1)).** Let \( p : \mathbb{R}^2_+ \to (0, \infty) \) be a polynomial given by

\[
p(x, y) = a + bx + cy + dx y, \quad x, y \in \mathbb{R}^2_+,
\]

where \( a, b, c, d \in \mathbb{R} \). The following statements are equivalent:

(i) \( \frac{1}{p} \) is a joint completely monotone function,

(ii) \( M := bc - ad > 0 \),

(iii) \( \frac{1}{p(m, n)} \) is a joint completely monotone net,

(iv) \( \frac{1}{p(m, n)} \) is a minimal joint completely monotone net.

If (ii) holds, then for every positive integer \( l \), \( \frac{1}{p(m, n)} \) is a minimal joint completely monotone net with the representing measure given by

\[
\begin{align*}
\left\{ \frac{s^{\frac{1}{2}}-1}{d(l-1)!} \left( \log t \log s \right)^{l-1} \frac{d}{M} \right\} & \int_{0}^{\infty} \frac{1}{I_{l-1}} \left( \frac{2}{d} \sqrt{M \log s \log t} \right) dsdt, \quad M > 0, \quad d \neq 0, \\
\left\{ \frac{s^{\frac{1}{2}}-1}{d(l-1)!} \left( \log t \log s \right)^{l-1} \frac{d}{M} \right\} & \int_{0}^{\infty} \frac{1}{(l-1)!} \frac{d}{M} \log(s)^{l-1} s^{\frac{1}{2}-1} ds d\delta_{0}(t) ds, \quad M = 0, \quad d \neq 0, \quad (3.2)
\end{align*}
\]

Moreover, if \( d \neq 0, M \geq 0 \) and \( \omega_{M,l} \) denotes the weight function of the representing measure of \( \frac{1}{p(m, n)} \) \( m, n \in \mathbb{Z}_+ \), then

\[
\lim_{M \to 0} \omega_{M,l} = \omega_{0,l}, \quad l \geq 1.
\]

**Proof.** Since \( p(0, 0) = a \) and the image of \( p \) is contained in \((0, \infty)\), \( a \) is positive. After dividing \( p(x, 0) \) by \( x \) and letting \( x \to \infty \), we conclude that \( b \)
is nonnegative. By symmetry, \( c \) is also nonnegative. Dividing \( p(x, x) \) by \( x^2 \) and letting \( x \to \infty \), we see that \( d \) is nonnegative. Thus

\[
a > 0, \ b \geq 0, \ c \geq 0, \ d \geq 0.
\]

(3.4)

To see the equivalence of (i)-(iv), we show that

\[(iv) \implies (iii) \implies (ii) \implies (iv) \implies (i) \implies (ii).\]

(i) \( \Rightarrow \) (ii): Let \( a(x) = c + dx \), \( b(x) = a + bx \). By Lemma 2.9, \( a'(0)b(0) \leq a(0)b'(0) \) or equivalently, \( M \geq 0 \).

(iv) \( \Rightarrow \) (i): This follows from [7, Proposition 6].

(ii) \( \Rightarrow \) (iv): We first consider the case of \( d \neq 0 \). Since \( ad - bc \leq 0 \), by (3.4), \( b \) and \( c \) are positive. For \( x, y \in \mathbb{R}_+ \), we write

\[
p(x, y) = a + bx + cy + dxy = b(x + a/b) + d(x + c/d)y.
\]

Since \( \{ \frac{1}{m + c/d} \}_{m \in \mathbb{Z}_+} \) is a completely monotone sequence and \( a/b \leq c/d \), an application of Theorem 1.3(i) (to \( a(m) = c + dm \) and \( b(m) = a + bm \), \( m \in \mathbb{Z}_+ \)) shows that \( \{ \frac{1}{p(m, n)} \}_{m,n \in \mathbb{Z}_+} \) is a minimal joint completely monotone net. We now consider the case of \( d = 0 \). If both \( b \) and \( c \) are zero, then \( \frac{1}{p} \), being a constant polynomial, is a joint completely monotone net. Now consider the case when either \( b \neq 0 \) or \( c \neq 0 \). By the symmetry, we may assume that \( b \neq 0 \). Note that

\[
\frac{1}{(a + bm + cn)^l} = \frac{(-1)^l}{b(l-1)!} \int_{[0,1]} (\log s)^{l-1} s^{-a/b-1} m ds, \ m, n \in \mathbb{Z}_+.
\]

Since \( s^{\frac{a}{b+c}} = \int_0^1 t^n d\delta_{a/b+c}(t) \) for \( n \geq 0 \) and \( s \in (0, 1) \), in case of \( d \neq 0 \), the representing measure of \( \{ \frac{1}{p(m, n)} \}_{m,n \in \mathbb{Z}_+} \) is given by (3.2). It now immediate from [7, Proposition 5] that \( \{ \frac{1}{p(m, n)} \}_{m,n \in \mathbb{Z}_+} \) is a minimal joint completely monotone net.

(iv) \( \Rightarrow \) (iii): Trivial.

Before we prove (iii) \( \Rightarrow \) (ii), let us see some general facts under the assumption that \( d \neq 0 \). Let \( l \) be a positive integer. Since \( \{ \frac{1}{p(m, n)} \}_{m,n \in \mathbb{Z}_+} \) is joint completely monotone, so is the net \( \{ \frac{1}{p(m, n)} \}_{m,n \in \mathbb{Z}_+} \) (see [11, Lemma 8.2.1(v)]). We now find the representing measure of \( \{ \frac{1}{p(m, n)} \}_{m,n \in \mathbb{Z}_+} \). Note that for \( m, n \in \mathbb{Z}_+ \),

\[
\frac{1}{p(m, n)^l} = \frac{1}{(b + dn)^l (a + cn + m)^l} = \frac{1}{(b + dn)^l (l - 1)!} \int_{[0,1]} (\log s)^{l-1} s^{-a/b-1} m ds
\]

\[
= \int_{[0,1]} s^m \left( \frac{1}{(b + dn)^l} \left( \log s \right)^{l-1} s^{\frac{a}{b+c} - 1} \right) ds. \quad (3.5)
\]

Further, for any \( s > 0 \) and \( n \in \mathbb{Z}_+ \),

\[
\frac{1}{(b + dn)^l} s^{\frac{a}{b+c}}.
\]
\[ \begin{align*}
&= \frac{1}{(b + dn)^{a - \frac{d}{dn}}} e^{\frac{a - \frac{d}{dn}}{b + dn} \log s} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(a - \frac{d}{dn})^k}{(b + dn)^k} \log s^k \\
&= \frac{\sum_{k=0}^{\infty} \frac{(a - \frac{d}{dn})^k}{d^k + k!} (\log s)^k}{\sum_{k=0}^{\infty} \frac{\log s^k}{k!} (-\log t)^{k+1}} \\
&= \int_{[0,1]} t^n \left( \sum_{k=0}^{\infty} \frac{(a - \frac{d}{dn})^k}{d^k + k!} (\log s)^k (-\log t)^{k+1-1} \right) dt,
\end{align*} \]

where we can interchange the series and the integral by the dominated convergence theorem. This combined with (3.2) shows that

\[ \frac{1}{p(m, n)} = \int_0^1 \int_{[0,1]} s^m t^n \omega_1(s,t) dt ds, \quad m, n \in \mathbb{Z}_+, \]

where \( \omega_1 \) is as given in (3.2).

\[ \omega_1(s,t) = \frac{s^{\frac{c}{2}} t^{\frac{d}{2}}}{d(l-1)!} \sum_{k=0}^{\infty} \left( \frac{M}{d^2} \right)^{l} (\log t \log s)^{l+1} \frac{d^k}{k!} (k+1)! (t)^{k+1-1}, \quad s, t \in (0,1). \] (3.6)

If \( M > 0 \), then the above discussion together with (3.1) yields (3.2) and (3.3). (iii)\( \Rightarrow \) (ii): Assume that \( M = bc - ad < 0 \). By (3.4), \( d > 0 \). Hence, by (3.6),

\[ \omega_1(s,t) = \frac{s^{\frac{c}{2}} t^{\frac{d}{2}}}{d} \sum_{k=0}^{\infty} \left( \frac{M}{d^2} \right)^{k} (\log t \log s)^{k} \frac{d^k}{(k!)^2} \]

However, the Bessel function \( J_0 \) takes negative values on some open interval in \( (0, \infty) \) (see [1], Eqn 9.1.18), and hence \( \omega_1 \) takes negative values on some open subset of \( (0,1) \times (0,1) \). It follows that \( \{ \frac{1}{p(m,n)} \}_{m,n \in \mathbb{Z}_+} \) is not a joint completely monotone net. This yields (iii)\( \Rightarrow \) (ii), completing the proof. \( \square \)

**Case of bi-degree** \((2,1)\). A nonconstant polynomial \( p: \mathbb{R}_+^2 \to (0, \infty) \) of bi-degree \((2,1)\) is given by \( p(x,y) = b(x) + a(x)y \). Note that \( p(x,0) = b(x) \), and hence \( b \) is a mapping from \( \mathbb{R}_+ \) into \( (0, \infty) \). If \( a(x_0) < 0 \) for some \( x_0 \in \mathbb{R}_+ \), then for large value of \( y \), \( p(x_0, y) < 0 \), and hence \( a \) maps \( \mathbb{R}_+ \) into \( \mathbb{R}_+ \). Assume that \( \frac{1}{p} \) is joint completely monotone. By Example [1, (1)], \( a \) is never constant.

Also, by Lemma [2.9], \( \deg a \leq \deg b \). Furthermore, we have the following:

- If \( b \) has a complex root, then \( \frac{1}{p(x,y)} \) is not completely monotone (see [4], Proposition 4.3), and hence \( b \) has negative real roots.
- If possible, assume that \( a \) has a complex root. Then \( a \) maps \( \mathbb{R} \) into \( (0, \infty) \), and hence for any \( x \in \mathbb{R} \), we have \( b(x) + a(x)y > 0 \) for sufficiently large \( y \). This shows \( b + ay \) has a complex root, and by another application of [4], Proposition 4.3], \( \frac{1}{p} \) is not separately monotone. This together with Remark [1,1] leads to a contradiction.
that \( \frac{1}{p} \) is not joint completely monotone. Hence \( a \) has nonpositive real roots.

We summarize below the discussion in the preceding paragraph:

**Proposition 3.2.** Consider a nonconstant polynomial \( p : \mathbb{R}_+^2 \to (0, \infty) \) of bi-degree \((2,1)\) given by \( p(x, y) = b(x) + a(x)y \). Assume that \( \frac{1}{p} \) is joint completely monotone. Then the following statements are valid:

- (i) \( a \) maps \( \mathbb{R}_+ \) into \( \mathbb{R}_+ \), \( 1 \leq \deg a \leq 2 \) and \( a \) has nonpositive real roots,
- (ii) \( b \) maps \( \mathbb{R}_+ \) into \( (0, \infty) \), \( \deg b = 2 \) and \( b \) has negative real roots.

In particular, the following possibilities occur:

- (a) \( p(x, y) = b_0(x + b_1)(x + b_2) + a_0(x + a_1)(x + a_2)y \),
- (b) \( p(x, y) = b_0(x + b_1)(x + b_2) + a_0(x + a_1)y \),

where \( a_0, b_0, b_1, b_2 > 0 \) and \( a_1, a_2 \geq 0 \).

We discuss below the first sub-case of bi-degree \((2,1)\), which is an outcome of a careful examination of the proof Theorem 2.1.

**Theorem 3.3** (Subcase \((a)\) of bi-degree \((2,1)\)). For \( a_j, b_j \in (0, \infty) \), \( j = 0, 1, 2 \), let \( a(x) = a_0(x + a_1)(x + a_2) \) and \( b(x) = b_0(x + b_1)(x + b_2), x \in \mathbb{R}_+ \) with \( a_1 \leq a_2 \) and \( b_1 \leq b_2 \). Then the following statements are valid:

- (i) If \( \left\{ \frac{1}{b(m) + a(m)n} \right\}_{m,n \in \mathbb{Z}_+} \) is a minimal joint completely monotone net provided \( (b_1 \leq a_1 \leq b_2 \text{ or } b_1 \leq a_2 \leq b_3) \) and \( b_1 + b_2 \leq a_1 + a_2 \), \( (3.7) \)
- (ii) If \( \left\{ \frac{1}{b(m) + a(m)n} \right\}_{m,n \in \mathbb{Z}_+} \) is joint completely monotone, then
  \[
  \frac{1}{a_1} + \frac{1}{a_2} \leq \frac{1}{b_1} + \frac{1}{b_2}, \quad b_1 + b_2 \leq a_1 + a_2.
  \]

We need a lemma in the proof of Theorem 3.3.

**Lemma 3.4.** Let \( c_1, c_2, b_1, b_2 \in \mathbb{R} \) with \( b_1 < b_2 \). The following statements are equivalent:

- (i) For every \( s \in (0, 1) \), \( c_1 s^{b_1} + c_2 s^{b_2} \leq 0 \),
- (ii) \( c_1 \leq 0 \) and \( c_1 + c_2 \leq 0 \).

**Proof.** For \( s \in (0, 1) \), note that
  \[
  c_1 s^{b_1} + c_2 s^{b_2} = s^{b_1} (c_1 + c_2 s^{b_2-b_1}). \tag{3.8}
  \]

(i) \( \Rightarrow \) (ii) Since for every \( s \in (0, 1) \), \( s^{b_1} > 0 \) and \( c_1 s^{b_1} + c_2 s^{b_2} \leq 0 \), we must have \( c_1 + c_2 s^{b_2-b_1} \leq 0 \) for every \( s \in (0, 1) \). Since \( b_2 - b_1 \) \( > 0 \), letting \( s \to 0 \), we obtain \( c_1 \leq 0 \), and letting \( s \to 1 \), we obtain \( c_1 + c_2 \leq 0 \).

(ii) \( \Rightarrow \) (i) If \( c_2 \leq 0 \) then clearly for every \( s \in (0, 1) \), \( c_1 s^{b_1} + c_2 s^{b_2} \leq 0 \). So we may assume that \( c_2 > 0 \). In view of (3.8), it suffices to check that \( c_1 + c_2 s^{b_2-b_1} \leq 0 \) for every \( s \in (0, 1) \). Indeed, since \( b_2 - b_1 > 0 \),
  \[
  c_1 + c_2 s^{b_2-b_1} \leq c_1 + c_2 \leq 0, \quad s \in (0, 1).
  \]

This completes the proof. \( \square \)
This is easily seen to be equivalent to

\[ b_1 \leq a_2 \leq b_2 \text{ and } b_1 + b_2 \leq a_1 + a_2, \]

then \( b_1 \leq a_1 \leq b_2 \). Indeed, since \( a_1 \leq a_2 \leq b_2 \), if \( a_1 < b_1 \), then \( a_1 + a_2 < b_1 + b_2 \). Hence, without loss of generality, we may assume that \( b_1 \leq a_1 \leq b_2 \).

If \( b_1 = b_2 \), then \( a_1 = b_1 \) and \( b_2 \leq a_2 \). The desired conclusion in this case now follows from Theorem \( 1.4(i) \). Hence, we may assume that \( b_1 < b_2 \). We imitate the proof of Theorem \( 2.1 \).

Indeed, in view the discussion following (2.8), it suffices to check that

\[ c_1 s^{b_1} + c_2 s^{b_2} \leq 0, \quad s \in (0, 1), \]

where \( c_1 \) and \( c_2 \) are given by (2.7). We consider two cases:

- **Case 1**: \( a_1 < a_2 \): In view of Lemma \( 3.1 \) it suffices to check that \( c_1 \leq 0 \) and \( c_1 + c_2 \leq 0 \). By (2.7),

\[ c_1 = \frac{b_1 (b_1 - a_1)(b_2 - a_1)}{a_0 (a_2 - a_1)}, \quad c_2 = -\frac{b_0 (b_1 - a_2)(b_2 - a_2)}{a_0 (a_2 - a_1)}. \]

Hence, \( c_1 \leq 0 \) and \( c_1 + c_2 \leq 0 \) if and only if

\[ (b_1 - a_1)(b_2 - a_1) \leq 0, \quad (b_1 - a_1)(b_2 - a_1) \leq (b_1 - a_2)(b_2 - a_2). \]

This is easily seen to be equivalent to

\[ (b_1 - a_1)(b_2 - a_1) \leq 0, \quad (b_1 + b_2)(a_2 - a_1) \leq (a_1 + a_2)(a_2 - a_1). \]

Both these inequalities follow at once from the assumption (3.7).

- **Case 2**: \( a_1 = a_2 \): By the partial fraction,

\[ \frac{(m + b_2)(m + b_2)}{(m + a_1)^2} = 1 + \frac{c_1}{m + a_1} + \frac{c_2}{(m + a_1)^2}, \quad m \in \mathbb{Z}_+, \]

where \( c_1 = b_1 + b_2 - 2a_1 \) and \( c_2 = (b_1 - a_1)(b_2 - a_1) \). By (3.7), \( c_1 \leq 0 \) and \( c_2 \leq 0 \), and hence we get (3.9). The fact that \( \{ \frac{1}{b(m) + a(m)n} \}_{m,n \in \mathbb{Z}_+} \) is a minimal joint completely monotone net now follows from Corollary 2.6.

Part (ii) is a special case of Theorem 1.4(ii). \( \Box \)

**Remark 3.5.** Assume that (3.7) holds. It follows from Theorem 3.3(i) that \( \{ \frac{1}{b(m) + a(m)n} \}_{m,n \in \mathbb{Z}_+} \) is a minimal joint completely monotone net. This combined with [7, Proposition 6] shows that \( \frac{1}{p} \) is a joint completely monotone function. \( \diamond \)

Here is the second sub-case of bi-degree \( (2, 1) \).

**Theorem 3.6 (Subcase (b) of bi-degree \( (2, 1) \)).** For \( a_0, a_1, b_0, b_1, b_2 \in (0, \infty) \), let \( a(x) = a_0(x + a_1) \) and \( b(x) = b_0(x + b_1)(x + b_2) \), \( x \in \mathbb{R}_+ \). If \( b_1 \leq a_1 \leq b_2 \), then \( \{ \frac{1}{b(m) + a(m)n} \}_{m,n \in \mathbb{Z}_+} \) is joint completely monotone. Conversely, if \( \{ \frac{1}{b(m) + a(m)n} \}_{m,n \in \mathbb{Z}_+} \) is joint completely monotone, then

\[ \frac{1}{a_1} \leq \frac{1}{b_1} + \frac{1}{b_2}. \]

**Proof.** Assume that \( b_1 \leq a_1 \leq b_2 \). If \( b_1 = a_1 \), then

\[ \frac{1}{b(m) + a(m)n} = \frac{1}{m + a_1 b_0 m + a_0 n + b_0 b_2}, \quad m, n \in \mathbb{Z}_+, \]

and hence the desired conclusion in this case follows from Theorem 3.1 and [11, Lemma 8.2.1(v)]. The same argument shows that \( \{ \frac{1}{b(m) + a(m)n} \}_{m,n \in \mathbb{Z}_+} \) is joint completely monotone if \( a_1 = b_2 \).
To complete the proof of the necessity part, let \( c_0 = \frac{b_0}{a_0} \) and note that
\[
\frac{b(x)}{a(x)} = \frac{c_0(x + b_1)(x + b_2)}{(x + a_1)} = c_0(x + b_1 + b_2 - a_1) + c_0 \frac{(b_1 - a_1)(b_2 - a_1)}{x + a_1}, \quad x \in \mathbb{R}_+.
\]
It follows that
\[
\frac{b(m)}{a(m)} = \frac{c_0 m t^{c_0(b_1+b_2-a_1)} t^{c_0(b_1-a_1)(b_2-a_1)}}{m + a_1}, \quad m \in \mathbb{Z}_+.
\]
Now assume that \( b_1 < a_1 < b_2 \) and note that \( (b_1 - a_1)(b_2 - a_1) < 0 \). One may now argue as in the proof II of Theorem 2.11 (a) (see (2.10)-(2.12)) to show that for every \( t \in (0,1) \), \( \{ t^{c_0(b_1-a_1)(b_2-a_1)} \} \) is a Hausdorff moment sequence. Since \( \{ t^{c_0(m)} \} \) is a Hausdorff moment sequence for every \( t \in (0,1) \), Lemma 2.3 (see (2.15)) together with (3.10) completes the proof of the sufficiency part.

Let \( c_1 = c_0(b_1-a_1)(b_2-a_1) \). Arguing as in (2.9), we conclude that for \( t \in (0,1) \), we have
\[
\frac{t^{c_1}}{(m+a_1)} = \sum_{l=0}^{\infty} \frac{(c_1 \log t)^l}{l!(m+a_1)^{l+1}} \sum_{l=1}^{\infty} \frac{(c_1 \log t)^l}{a_0(l)!^2} \int_{[0,1]} (-\log s)^l s^{a_1+1+m} ds.
\]
This combined with (3.10) yields
\[
\frac{b(m)}{a(m)} = \frac{c_0 m t^{c_0(b_1+b_2-a_1)} \sum_{l=1}^{\infty} \frac{(c_1 \log t)^l}{a_0(l)!^2} \int_{[0,1]} (-\log s)^l s^{a_1+1+m} ds, \quad m \in \mathbb{Z}_+, \ t \in (0,1).
\]
Thus, for some integrable function \( w(s,t) \) (by an application of the dominated convergence theorem), we have
\[
\frac{b(m)}{a(m)} = \int_0^1 t^{c_0 m} s^{c_0} w(s,t) ds = \int_0^1 (t^{c_0} s)^m w(s,t) ds, \quad m \in \mathbb{Z}_+, \ t \in (0,1).
\]
Now using change of variable \( st^{c_0} = s_1 \) and applying (2.4), we may conclude that the representing measure of \( \{ \frac{1}{b(m)+a(m)n} \} \) is a weighted Lebesgue measure. One may now argue as in the proof of Lemma 2.11 to see that \( \frac{1}{b(x)+a(x)y} \) is a completely monotone function. The desired conclusion now follows from Lemma 2.9 by letting \( x = 0 \) in (2.18). \( \square \)

4. The Cauchy dual subnormality problem for toral 3-isometric shifts

Let \( n \) be a positive integer and let \( H \) be a complex Hilbert space. We say that \( T \) is commuting \( n \)-tuple on \( H \) if \( T_1, \ldots, T_n \) are bounded linear operators on \( H \) such that \( T_i T_j = T_j T_i \) for every \( 1 \leq i \neq j \leq n \). Let \( m \) be a positive
Following \[2, 7, 19\], we say that a commuting \(n\)-tuple \(T\) is a **toral \(m\)-isometry** if

\[
\sum_{\alpha \in \mathbb{Z}_n^+} (-1)^{\lvert \alpha \rvert} \binom{\beta}{\alpha} T^{\alpha^*}T^\alpha = 0, \quad \beta \in \mathbb{Z}_n^+, \quad \lvert \beta \rvert = m,
\]

where \(T^\alpha\) stands for the bounded linear operator \(\prod_{j=1}^n T_j^{\alpha_j}\) and \(T^{\alpha^*}\) denotes the Hilbert space adjoint of \(T^\alpha\).

**Definition 4.1.** We say that a commuting \(n\)-tuple \(T\) is a **separate \(m\)-isometry** if \(T_1, \ldots, T_n\) are \(m\)-isometries.

**Remark 4.2.** For any commuting \(n\)-tuple \(T\),

\[
\Delta^\beta(T^{\alpha_1^*}T^{\alpha_2})\bigg|_{\gamma=0} = \sum_{\alpha \in \mathbb{Z}_n^+} (-1)^{\lvert \alpha \rvert + \lvert \beta \rvert} \binom{\beta}{\alpha} T^{\alpha_1^*}T^{\alpha_2}, \quad \beta \in \mathbb{Z}_n^+, \quad \lvert \beta \rvert = m.
\]

The verification of this identity is similar to that of \[15\, Eqn (2.1)]. Clearly, a toral \(m\)-isometry is a separate \(m\)-isometry. Indeed, for any \(j = 1, \ldots, n\), letting \(\beta = m\epsilon_j\) in (4.1) shows that \(T_j\) is an \(m\)-isometry. In general, a separate \(m\)-isometry is not a toral \(m\)-isometry (see Remark 4.7 below). ◯

The following is a well known fact for a single operator (see \[3\, Equation (1.3)], \[15\, Corollary 3.5]).

**Theorem 4.3.** A commuting \(n\)-tuple \(T = (T_1, \ldots, T_n)\) on \(H\) is a toral \(m\)-isometry if and only if

\[
T^{\alpha_1^*}T^{\alpha_2} = \sum_{k=0}^{m-1} \sum_{\beta \in \mathbb{Z}_n^+} \frac{\Delta^\beta(T^{\alpha_1^*}T^{\alpha_2})|_{\alpha=0}}{\beta!} (\alpha)_\beta, \quad \alpha \in \mathbb{Z}_n^+.
\]

**Proof.** Define \(m(\alpha) = T^{\alpha^*}T^\alpha, \alpha \in \mathbb{Z}_n^+\). By (1.1),

\(T\) is a toral \(m\)-isometry \iff \(\Delta^\beta m = 0, \beta \in \mathbb{Z}_n^+, \lvert \beta \rvert \geq m\).

By the Newton’s Interpolation Formula in several variables (see \[7\, Remark 2, Equation (G)]), for every \(h \in H\),

\[
\langle m(\alpha)h, h \rangle = \sum_{k=0}^{\infty} \sum_{\beta \in \mathbb{Z}_n^+} \frac{\Delta^\beta(\langle m(\alpha)h, h \rangle)|_{\alpha=0}}{\beta!} (\alpha)_\beta, \quad \alpha \in \mathbb{Z}_n^+.
\]

This combined with

\[
\Delta^\beta(\langle m(\cdot)h, h \rangle) = \langle \Delta^\beta m(\cdot)h, h \rangle, \quad \beta \in \mathbb{Z}_n^+, \quad h \in H,
\]

yields the following identity:

\[
\langle m(\alpha)h, h \rangle = \sum_{k=0}^{\infty} \sum_{\beta \in \mathbb{Z}_n^+} \frac{\Delta^\beta(\langle m(\alpha)h, h \rangle)|_{\alpha=0}}{\beta!} (\alpha)_\beta, \quad \alpha \in \mathbb{Z}_n^+, \quad h \in H.
\]

This together with (4.2) gives the necessity part. The sufficiency part follows from (1.1) and (4.2). ◯
The following, a special case of [13, Proposition 4.1(iii)], is immediate from Theorem 4.3.

**Corollary 4.4.** A commuting n-tuple \( T = (T_1, \ldots, T_n) \) on \( H \) is a toral 2-isometry if and only if
\[
T^{\pi\alpha}T^\alpha = I + \sum_{j=1}^{n} \alpha_j (T_j^* T_j - I), \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_n^2.
\]

The next result characterizes separate 2-isometries within the class of toral \( m \)-isometric pairs.

**Corollary 4.5.** Let \( m \) be an integer bigger than 1. A toral \( m \)-isometry pair \( T = (T_1, T_2) \) on \( H \) is a separate 2-isometry if and only if
\[
T^{\pi\alpha}T^\alpha = I + \alpha_1 A + \alpha_2 (B + \alpha_1 C), \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_n^2.
\]
where \( A = T_1 T_1^* - I, B = T_1 T_2 - I \) and \( C = T_1^* B T_1 - B \). If (4.3) holds, then \( T \) is a toral 3-isometry.

**Proof.** If \( m = 2 \), then the desired equivalence follows from Corollary 4.3 (the identity (4.3) holds with \( C = 0 \)). Hence, we may assume that \( m \geq 3 \). Suppose that \( T \) is a separate 2-isometry. Define \( m(\alpha) = T^{\pi\alpha}T^\alpha \), \( \alpha \in \mathbb{Z}_n^2 \). Since \( T_2 \) is a 2-isometry, by (4.2) (applied to \( m = 2 \) and \( n = 1 \)), \( \Delta_j^2(T_2^{\pi\alpha_2}T_2^{\alpha_2}) = 0 \) for every \( j \geq 2 \). This combined with Theorem 4.3 yields
\[
m(\alpha) = \sum_{j=0}^{m-1} \sum_{\beta \in \mathbb{Z}_n^2, \beta_1 + \beta_2 = j} \frac{\Delta_1^{\beta_1} \Delta_2^{\beta_2}(m(\alpha))|_{\alpha=0}}{\beta_1! \beta_2!} \alpha_1^{\beta_1} \alpha_2^{\beta_2} \quad (4.4)
\]
for every \( \alpha \in \mathbb{Z}_n^2 \). However, since \( T_1 \) is a 2-isometry, once again by (4.2),
\[
\Delta_j^1(m(\alpha)|_{\alpha=0}) = \Delta_j^1(T_1^{\pi\alpha_1}T_1^{\alpha_1})|_{\alpha_1=0} = 0, \quad j \geq 2,
\]
\[
\Delta_j^{-1} \Delta_2(m(\alpha)|_{\alpha=0}) = \Delta_j^{-1}(T_2^{\pi\alpha_1}(T_2 T_2 - I)T_1^{\alpha_1})|_{\alpha_1=0} = 0, \quad j \geq 3.
\]
This combined with (4.3) yields
\[
m(\alpha) = \sum_{j=0}^{1} \frac{\Delta_j^1(T_1^{\pi\alpha_1}T_1^{\alpha_1})|_{\alpha_1=0}}{j!} \alpha_1^{j} j!
\]
\[
+ \alpha_2 \sum_{j=0}^{1} \frac{\Delta_j^1(T_1^{\pi\alpha_1}(T_2 T_2 - I)T_1^{\alpha_1})|_{\alpha_1=0}}{j!} \alpha_1^{j} j!
\]
\[
= I + \alpha_1 \Delta_1(T_1^{\pi\alpha_1})|_{\alpha_1=0} + \alpha_2 (B + \alpha_1 \Delta_1(T_1^{\pi\alpha_1} B T_1^{\alpha_1})|_{\alpha_1=0}).
\]
This gives the necessity part. The sufficiency part follows from applications of Corollary 4.4 separately to \( T_1 \) and \( T_2 \). The remaining part now follows from Theorem 4.3. \( \square \)

Let \( w = \{w_\alpha^{(j)}: j = 1, \ldots, n, \alpha \in \mathbb{Z}_n^2\} \) be a set of nonzero complex numbers. Let \( \mathcal{H} \) be a complex separable Hilbert space with orthonormal
basis $\mathcal{E} = \{e_{\alpha}\}_{\alpha \in \mathbb{Z}^n_+}$. A weighted $n$-shift $\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_n)$ with respect to $\mathcal{E}$ is defined by

$$\mathcal{W}_je_{\alpha} := w^{(j)}_\alpha e_{\alpha + \varepsilon_j}, \quad j = 1, \ldots, n,$$

where $\varepsilon_j$ is the $m$-tuple with 1 in the $j$th place and zeros elsewhere. Clearly, $\mathcal{W}_1, \ldots, \mathcal{W}_n$ extend to densely defined linear operators on the linear span of $\{e_{\alpha}\}_{\alpha \in \mathbb{Z}^n_+}$. Note that $\mathcal{W}_1, \ldots, \mathcal{W}_n$ extend boundedly to $\mathcal{H}$ if and only if $\sup_{\alpha \in \mathbb{Z}^n_+} |w^{(j)}_\alpha| < \infty$ for every $j = 1, \ldots, n$. It is easy to see that for $i, j = 1, \ldots, n$,

$$\mathcal{W}_i\mathcal{W}_j = \mathcal{W}_j\mathcal{W}_i \iff w^{(i)}_\alpha w^{(j)}_{\alpha + \varepsilon_i} = w^{(j)}_\alpha w^{(i)}_{\alpha + \varepsilon_j}, \quad \alpha \in \mathbb{Z}^n_+. \quad (4.5)$$

The reader is referred to [17] for the basic theory of weighted multishifts.

In what follows, we always assume that the weight multi-sequence $\mathbf{W}$ consists of positive numbers, $\mathcal{W}$ extends boundedly to $\mathcal{H}$ and that $\mathcal{W}$ is a commuting $n$-tuple. We indicate the weighted $n$-shift $\mathcal{W}$ with weight multi-sequence $\mathbf{W}$ by $\mathcal{W} : \{w^{(j)}_\alpha\}$.

**Proposition 4.6.** For a weighted 2-shift $\mathcal{W} : \{w^{(j)}_\alpha\}$, the following statements are valid:

(i) $\mathcal{W}$ is a total $m$-isometry if and only if

$$\|\mathcal{W}^\alpha e_0\|^2 = \sum_{k=0}^{m-1} \sum_{\beta \in \mathbb{Z}^n_+} |w^{(\beta)}_\alpha|^2 |(\beta)|_\alpha = 0 \quad (\alpha, \beta) \in \mathbb{Z}^n_+, \quad (4.6)$$

(ii) if (4.6) holds, then $\mathcal{W}$ is a separate 2-isometry if and only if

$$\|\mathcal{W}^\alpha e_0\|^2 = 1 + \alpha_1 b + \alpha_2 (c + \alpha_1 d), \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+, \quad (4.7)$$

where $b, c, d$ are given by

$$b = (w^{(1)}_0)^2 - 1, \quad c = (w^{(2)}_0)^2 - 1, \quad d = 1 - (w^{(1)}_0)^2 - (w^{(2)}_0)^2 + (w^{(1)}_0)^2 (w^{(2)}_0)^2. \quad (4.8)$$

If (4.7) holds, then $\mathcal{W}$ is a total 3-isometry.

**Proof.** (i) The necessity part follows from Theorem 1.3. To see the sufficiency part, assume that (4.6) holds. Note that for any $\beta \in \mathbb{Z}^n_+$, there exists a positive scalar $m(\beta)$ such that

$$\mathcal{W}^\beta e_0 = m(\beta) e_\beta. \quad (4.9)$$

It now follows that for some real scalars $b_{\beta, \delta}$

$$\|\mathcal{W}^\alpha e_\beta\|^2 = \frac{1}{m(\beta)^2} \|\mathcal{W}^{\alpha + \beta} e_0\|^2 \overset{(4.5)}{=} \sum_{k=0}^{m-1} \sum_{\delta \in \mathbb{Z}^n_+} b_{\beta, \delta} |(\alpha + \beta)|_\delta, \quad \alpha, \beta \in \mathbb{Z}^n_+. \quad (4.10)$$

This combined with (1.1) yields

$$\sum_{\alpha \in \mathbb{Z}^n_+} (-1)^{|\alpha|} \binom{\gamma}{\alpha} \|\mathcal{W}^\alpha e_\beta\|^2 = 0, \quad \beta \in \mathbb{Z}^2_+, \gamma \in \mathbb{Z}^2_+, \quad |\gamma| = m."
Since \( \{W^\alpha e_\beta\}_{\beta \in \mathbb{Z}_+^2} \) is an orthogonal set for every \( \alpha \in \mathbb{Z}_+^2 \), we conclude that
\[
\sum_{\alpha \in \mathbb{Z}_+^2, 0 \leq \alpha \leq \gamma} (-1)^{|\alpha|} \binom{\gamma}{\alpha} \|W^\alpha h\|^2 = 0, \quad h \in H, \gamma \in \mathbb{Z}_+^2, \quad |\gamma| = m.
\]

This shows that \( W \) is a toral \( m \)-isometry.

(ii) Assume that (4.10) holds. Similar to the verification of (i), this may be deduced from (4.9) and Corollary 4.5.

**Remark 4.7.** Note that \( W \) is a toral 2-isometry if and only if there exist unique nonnegative numbers \( b, c \) such that
\[
\|W^\alpha e_0\|^2 = 1 + b\alpha_1 + c\alpha_2, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2.
\]
Moreover, \( b = (w_0^{(1)})^2 - 1 \) and \( c = (w_0^{(2)})^2 - 1 \). These observations were implicitly recorded in [8 Remark 3]. Thus, for any \( \alpha \in \mathbb{Z}_+^2 \),
\[
w_\alpha^{(1)} = \frac{\|W^{\alpha+e_1} e_0\|}{\|W^\alpha e_0\|} = \sqrt{\frac{1 + ((w_0^{(1)})^2 - 1)(\alpha_1 + 1) + ((w_0^{(1)})^2 - 1)\alpha_2}{1 + ((w_0^{(1)})^2 - 1)\alpha_1 + ((w_0^{(1)})^2 - 1)\alpha_2}}.
\]

Similarly, one can see that
\[
w_\alpha^{(2)} = \sqrt{\frac{1 + ((w_0^{(1)})^2 - 1)\alpha_1 + ((w_0^{(1)})^2 - 1)\alpha_2}{1 + ((w_0^{(1)})^2 - 1)\alpha_1 + ((w_0^{(1)})^2 - 1)\alpha_2}}, \quad \alpha \in \mathbb{Z}_+^2.
\]

This provides 2-variable counterpart of [16 Lemma 6.1(ii)].

For a polynomial \( p(\alpha) = 1 + b\alpha_1 + c\alpha_2 + d\alpha_1\alpha_2 \) with nonnegative real numbers \( b, c, d, \) consider the weighted 2-shift \( W_p \) with weights
\[
w_\alpha^{(j)} = \sqrt{\frac{p(\alpha + e_j)}{p(\alpha)}}, \quad \alpha \in \mathbb{Z}_+^2, \quad j = 1, 2.
\] (4.10)

By Proposition 4.6 \( W_p \) is a separate 2-isometry. If \( d > 0 \), then \( W_p \) is never a toral 2-isometry (see Corollary 4.1).

Let \( T = (T_1, \ldots, T_n) \) be a commuting \( n \)-tuple on \( H \). We say that \( T \) is *jointly subnormal* if there exist a Hilbert space \( K \) containing \( H \) and a commuting \( n \)-tuple \( N \) of normal operators \( N_1, \ldots, N_n \) on \( K \) such that
\[
T_j = N_j|_H, \quad j = 1, \ldots, n.
\]

Assume that \( T_j^* T_j \) is invertible for every \( j = 1, \ldots, n \). Following [24, 13], we refer to the \( m \)-tuple \( T^4 := (T_1^4, \ldots, T_n^4) \) as the *operator tuple torally Cauchy dual* to \( T \), where \( T_j^4 := T_j (T_j^* T_j)^{-1} \), \( j = 1, \ldots, n \).

**Remark 4.8.** Let \( T = (T_1, \ldots, T_n) \) be a separate 2-isometry on \( H \). By Richter’s lemma (see [19 Lemma 1]),
\[
T_j^* T_j \geq I, \quad j = 1, \ldots, n.
\] (4.11)

It follows that the operator \( n \)-tuple \( T^4 \) torally Cauchy dual to \( T \) exists and
\[
(T_j^4)^* T_j^4 \leq I, \quad j = 1, \ldots, n.
\] (4.12)

Note that the operator \( n \)-tuple torally Cauchy dual to \( T^4 \) exists and it is equal to \( T \).
Let \( \mathcal{W} : \{w^{(j)}_\alpha\} \) be a weighted \( n \)-shift such that \( \mathcal{W}^* \mathcal{W}_j \) is invertible for every \( j = 1, \ldots, n \). The operator tuple \( \mathcal{W}^t \) torally Cauchy dual to the weighted \( n \)-shift \( \mathcal{W} \) satisfies

\[
\mathcal{W}_j^t e_\alpha = \frac{1}{w^{(j)}_\alpha} e_{\alpha + \epsilon_j}, \quad j = 1, \ldots, n.
\]

(4.13)

Hence, by (4.5), \( \mathcal{W}^t \) is commuting:

\[
\mathcal{W}_i^t \mathcal{W}_j^t = \mathcal{W}_j^t \mathcal{W}_i^t, \quad 1 \leq i \neq j \leq n.
\]

Moreover, by (4.13),

\[
\|((\mathcal{W}^t)^\alpha e_0\| = \frac{1}{\|\mathcal{W}^\alpha e_0\|^2}, \quad \alpha \in \mathbb{Z}^n_+.
\]

(4.14)

We now present a complete solution to the Cauchy dual subnormality problem for the class of toral \( 3 \)-isometric weighted 2-shifts, which are separate 2-isometries.

**Theorem 4.9.** Let \( \mathcal{W} : \{w^{(j)}_\alpha\} \) be a toral \( 3 \)-isometric weighted 2-shift. If \( \mathcal{W} \) is a separate 2-isometry, then the operator tuple \( \mathcal{W}^t \) torally Cauchy dual to \( \mathcal{W} \) exists, and the following statements are equivalent:

(i) the operator tuple \( \mathcal{W}^t \) is jointly subnormal,

(ii) either \( 1 - (w^{(1)}_0)^2 - (w^{(2)}_0)^2 + (w^{(1)}_1)^2(w^{(2)}_2)^2 = 0 \) or \( w^{(2)}_{e_1} \leq w^{(2)}_0 \),

(iii) either \( \mathcal{W} \) is a toral 2-isometry or \( w^{(2)}_{e_1} \leq w^{(2)}_0 \).

**Proof.** Assume that \( \mathcal{W} \) is a separate 2-isometry. By Remark 4.8, the operator tuple \( \mathcal{W}^t \) torally Cauchy dual to \( \mathcal{W} \) exists. Also, by Proposition 4.6 ii) and 4.14,

\[
\|((\mathcal{W}^t)^\alpha e_0\| = \frac{1}{1 + \alpha_1 b + \alpha_2 (c + \alpha_1 d)}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+.
\]

(4.15)

where \( b, c, d \) are given by (4.8). Note that by (4.11), \( b \geq 0 \) and \( c \geq 0 \). One may now apply [6] Theorem 4.4] together with (4.9) and (4.12) to see that \( \mathcal{W}^t \) is jointly subnormal if and only if \( \|((\mathcal{W}^t)^\alpha e_0\|_{\alpha \in \mathbb{Z}^2_+} \) is a joint completely monotone net (see also the discussion prior to [7] Eqn (E)). On the other hand, by Theorem 4.4 and 4.14,

\[
\|((\mathcal{W}^t)^\alpha e_0\|_{\alpha \in \mathbb{Z}^2_+} \) is joint completely monotone \iff d \leq bc.
\]

(4.16)

(4.17)

To get the equivalence of (i) and (ii), note that if \( d \neq 0 \), then by (4.8), \( d \leq bc \) if and only if

\[
(w^{(1)}_0)^2((w^{(2)}_{e_1})^2 - 1) \leq c(b + 1) = ((w^{(2)}_0)^2 - 1)(w^{(1)}_0)^2,
\]

which is equivalent to

\[
(w^{(1)}_0)^2((w^{(2)}_0)^2 - (w^{(2)}_{e_1})^2) \geq 0 \iff w^{(2)}_{e_1} \leq w^{(2)}_0.
\]

The equivalence of (i) and (ii) now follows from (4.8), (4.10) and (4.17). Finally, since a separate 2-isometry \( \mathcal{W} \) is a toral 2-isometry if and only if \( d = 0 \) (see Proposition 4.6 ii)), the equivalence of (ii) and (iii) is immediate. \( \square \)

**Remark 4.10.** By (4.5), \( w^{(2)}_{e_1} \leq w^{(2)}_0 \) if and only if \( w^{(1)}_{e_2} \leq w^{(1)}_0 \). \( \diamond \)
The first part of the following corollary may also be deduced from \cite{B19} Proposition 6).

**Corollary 4.11.** Under the hypotheses of Theorem 4.9, the following statements are valid:

(i) if $\mathcal{W}$ is a toral 2-isometry, $\mathcal{W}^t$ is jointly subnormal,

(ii) if $\mathcal{W}$ is not a toral 2-isometry, then $\mathcal{W}^t$ is jointly subnormal if and only if $w_{z_1}^{(2)} \leq w_0^{(2)}$.

We conclude the section with an example of a toral 3-isometry for which the operator tuple torally Cauchy dual is not jointly subnormal.

**Example 4.12.** For real numbers $a, b, c, d > 0$, consider the polynomial $p : \mathbb{R}_+^2 \to (0, \infty)$ given by

$$p(a) = 1 + a_1 b + a_2 (c + a_1 d).$$

Let $\mathcal{W}_p$ be the weighted 2-shift with weights given by \cite{B19}. By Proposition 4.6, $\mathcal{W}_p$ is a joint 3-isometry, which is also a separate 2-isometry. Further, by the discussion following \cite{B19}, $w_{z_1}^{(2)} \leq w_0^{(2)}$ if and only if $d \leq bc$. It is now clear from Theorem 4.9 that there exist joint 3-isometries $\mathcal{W}$ for which $\mathcal{W}^t$ is not jointly subnormal (for example, let $b = 1$, $c = 2$ and $d = 3$).

5. Joint complete monotonicity and a coefficient-matrix

A solution to the Cauchy dual subnormality problem for toral 3-isometric weighted 2-shifts requires characterization of polynomials $p : \mathbb{R}_+^2 \to (0, \infty)$ of bi-degree at most $(2, 2)$ for which $\{ \frac{1}{p(m,n)} \}_{m,n \in \mathbb{Z}_+}$ is joint completely monotone. We have already seen one special instance of this problem (see Theorem 4.9). In the remaining part of this paper, we briefly discuss role of the coefficient-matrix in the classification of the polynomials $p : \mathbb{R}_+^2 \to (0, \infty)$ for which $\frac{1}{p}$ is joint completely monotone. For a positive integer $N$ and $p_N(x, y) = \sum_{m,n=0}^N a_{m,n} x^m y^n$, consider the coefficient-matrix $A_{p_N} = (a_{m,n})_{m,n=0}^N$ associated with $p_N$. The equivalence of (i) and (ii) of Theorem 3.1 can be rephrased as follows:

**Theorem 5.1.** The function $\frac{1}{p_1}$ is joint completely monotone if and only if $\det A_{p_1} \leq 0$.

Consider a polynomial $p_2$ of the form $(x+b_1)(x+b_2)+(x+a_1)(x+a_2)y$, where $0 < a_1 \leq a_2$ and $0 < b_1 \leq b_2$. Note that

$$A_{p_2} = \begin{pmatrix}
 b_1 b_2 & a_1 a_2 & 0 \\
 b_1 + b_2 & a_1 + a_2 & 0 \\
 1 & 1 & 0
\end{pmatrix}.$$  

Clearly, $\det A_{p_2} = 0$. If $B_{p_2}$ denotes the minor of $A_{p_2}$ obtained after excluding the third row and the third column, then

$$\det B_{p_2} = b_1 b_2 (a_1 + a_2) - a_1 a_2 (b_1 + b_2).$$

Note that the condition $\det B_{p_2} \leq 0$ is precisely the condition \cite{B19} with $k = 2$. Hence, by Lemma 2.9, $\det B_{p_2} \leq 0$ is a necessary condition for $\frac{1}{p_2}$ to be joint completely monotone. On the other hand, by Remark 3.5, $\frac{1}{p_2}$ is
joint completely monotone provided (3.7) holds. Thus the condition (3.7) implies that \( \det B_{p_2} \leq 0 \). Summarizing the discussion above, we have

\[
(3.7) \implies \frac{1}{p_2} \text{ is joint completely monotone} \implies \det B_{p_2} \leq 0.
\] 

(5.1)

Interestingly, \( \det B_{p_2} \leq 0 \) neither ensures (3.7) nor the joint complete monotonicity of \( \frac{1}{p_2} \) (for the first assertion, consider \( a_1 = 6, a_2 = 9, b_1 = 1 \) and \( b_2 = 5 \), and for the second one, see Example 4.3(i)). Needless to say, the problem of finding same set of sufficient and necessary conditions for \( \frac{1}{p_2} \) to be joint completely monotone remains unresolved.

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Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, India

Email address: akasha@iitk.ac.in
Email address: chavan@iitk.ac.in
Email address: rnailwal@iitk.ac.in