Killing Vector Fields and Superharmonic Field Theories

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Abstract

The harmonic action functional allows a natural generalisation to semi-Riemannian supergeometry, referred to as superharmonic action, which resembles the supersymmetric sigma models studied in high energy physics. We show that Killing vector fields are infinitesimal supersymmetries of the superharmonic action and prove three different Noether theorems in this context. En passant, we provide a homogeneous treatment of five characterisations of Killing vector fields on semi-Riemannian supermanifolds, thus filling a gap in the literature.

1 Introduction

Symmetries belong to the main ingredients of modern physical theories. In high energy physics, supersymmetry is a conjectured transition between the two types of elementary particles in nature: bosons and fermions, which differ in their statistics. The observed properties of these particles are, to a good agreement, described by quantum field theories, see e.g. [PS95] for a standard treatment. The quantum theories are, usually, based on classical field theories. In this context, bosons and fermions are described by even and odd fields, respectively. Free bosonic string theory is modelled on the well-known harmonic action [Jos01], while a class of supersymmetric extensions is given by so called supersymmetric sigma models [DF99].

In this article, we study a similar such extension which, from a mathematical point of view, is more natural and also more general in that it can be formulated for every pair of semi-Riemannian supermanifolds. We shall, therefore, refer to the resulting field theories as superharmonic. In contrast to the aforementioned sigma models, the superharmonic action always allows a "superspace formulation", meaning that all fields occurring can be included in a single superfield, a mathematical model of which is a map with flesh [Hé09].

Killing vector fields are known to be infinitesimal symmetries of the harmonic action. According to the Noether principle, every symmetry of a classical field theory should induce a conserved quantity. In the harmonic theory, this is indeed the case ([BE81], [Hé02]), while it is a priori not clear for the superharmonic action. Indeed, whereas certain elements of supergeometry, such as the divergence of a vector field, differ from the classical theory of manifolds merely by a number of signs, others are non-trivial extensions. Examples for the latter include integration theory and maps with flesh.
In the present article, we show that Killing vector fields on the domain as well as on the target space supermanifold are infinitesimal symmetries of the superharmonic action. As our main results, we formulate and prove three different Noether theorems in this context. This is the subject matter of Sec. 4. There are several equivalent definitions for a Killing vector field on a semi-Riemannian manifold, most of which have been generalised to supergeometry ([MSV96], [Goe08], [ACDS97]), while a homogeneous treatment of the subject is yet missing. Another aim of this article is to fill this gap. In Sec. 3, we prove equivalence of five characterisations of Killing vector fields. It turns out that the definition used in [ACDS97] is a non-equivalent variation. To start with, Sec. 2 reviews elements of the theory of semi-Riemannian supermanifolds needed later.

2 Semi-Riemannian Supermanifolds

Superharmonic field theories are based on semi-Riemannian supermanifolds. For later use, we briefly recall the relevant background here.

Throughout the article, we adopt the Berezin-Kostant-Leites definition of supermanifolds and their morphisms in terms of sheaves as in [Lei80]. In particular, a supermanifold is a ringed space \((M,\mathcal{O}_M)\), and a morphism \(\Phi : (M,\mathcal{O}_M) \to (N,\mathcal{O}_N)\) of supermanifolds consists of two parts \(\Phi = (\varphi, \phi)\). We shall occasionally abuse notation and write \(M\) instead of \((M,\mathcal{O}_M)\). Modern monographs on the general theory of supermanifolds include [Var04] and [CCF11] while aspects of Riemannian supergeometry are studied in [Goe08]. References for more specialised topics will be given at suitable positions throughout the text.

Following the conventions of [Gro11], we denote the (super) tangent sheaf, i.e. the sheaf of superderivations of \(\mathcal{O}_M\), by \(SM := \text{Der}(\mathcal{O}_M)\). The (super) tangent space at a point \(p \in M\) is defined by

\[
S_pM := \{ v : \mathcal{O}_M \to \mathbb{R} \mid \mathbb{R}-\text{linear}, \ v(fg) = v(f)\hat{g}(p) + (-1)^{|v||f|}\hat{f}(p)v(g) \}
\]

where \((\mathcal{O}_M)_p\) is the stalk of \(\mathcal{O}_M\) at \(p\) and tilde denotes the canonical projection \(\mathcal{O}_M \to \mathcal{O}_M^\ast\), \(f \to \hat{f}\) by evaluation \(\hat{f}(p) = ev|_p f\). Any vector field \(X \in SM\) gives rise to the tangent space valued map \(p \mapsto X(p) \in S_pM\) via

\[
X(p)(f) := \hat{X}(f)(p), \quad f \in (\mathcal{O}_M)_p
\]

Note the use of the shorthand notation \(X \in SM\), meaning \(X \in SM(U)\) for \(p \in U \subseteq M\). Similarly, we yield a canonical (classical) vector field \(\hat{X} \in \Gamma(TM)\) on \(M\) by setting \(\hat{X}(\hat{f}) := X(\hat{f})\). It is well-known that superfunctions \(f\) are not determined by their values \(\hat{f}(p)\) and, likewise, vector fields \(X\) are not determined by their values \(X(p)\). With respect to local coordinates \((\xi^1, \ldots, \xi^n, \xi^{n+1}, \ldots, \xi^{n+m})\) of \(M\), the tuple \(\left(\frac{\partial}{\partial \xi^1}, \ldots, \frac{\partial}{\partial \xi^{n+m}}\right)\) is a local \(\mathcal{O}_M\)-basis of \(SM\) which is adapted in the sense that the first \(n\) vector fields are even and the remaining \(m\) ones are odd. Likewise, the tuple \(\left(\frac{\partial}{\partial \xi^1}(p), \ldots, \frac{\partial}{\partial \xi^{n+m}}(p)\right)\) is an adapted basis of the super vector space \(S_pM\).

Tensor calculus on supermanifolds is based on superlinear algebra. For \(U \subseteq M\) sufficiently small, \(SM(U)\) is a free \(\mathcal{O}_M(U)\)-supermodule of rank \(n|m = \dim M\). In general, consider a supercommutative superalgebra \(A\) and free \(A\)-supermodules \(M\) and \(N\) of rank \(m|n\) and \(r|s\), respectively. With respect to adapted right bases \((f_1, \ldots, f_{n+m})\) and \((g_1, \ldots, g_{r+s})\) of \(M\) and \(N\), respectively, any superlinear map \(L : M \to N\) can...
be identified with a matrix $L \in \text{Mat}_{A}(m|n, r|s)$. Similarly we denote, for an even (super-)bilinear form $B \in \text{Hom}_{A}(M \otimes_{A} M, A)$, the corresponding matrix with entries $B_{jk} := B(f_{j}, f_{k})$ by the same symbol $B \in \text{Mat}(n|m, A)$.

Let $GL_{n|m}(A)$ denote the group of even and invertible $n|m$-matrices with entries in $A$. The orthosymplectic group of dimension $(t, s)|2m$ is defined as follows.

$$OSp_{(t,s)|2m}(A) := \{ L \in GL_{t+s|2m}(A) \mid \forall v, w \in A^{t+s|2m} : g_{0}(Lv, Lw) = g_{0}(v, w) \}$$

where $g_{0}$ denotes the standard supermetric which, with respect to the standard basis for $A^{t+s|2m}$, is given by the matrix

$$(1) \quad g_{0} := \begin{pmatrix} G_{t,s} & 0 \\ 0 & J_{2m} \end{pmatrix}$$

where

$$G_{t,s} := \begin{pmatrix} -1_{t \times t} & 0 \\ 0 & 1_{s \times s} \end{pmatrix}, \quad J_{2m} := \begin{pmatrix} J_{2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_{2} \end{pmatrix}, \quad J_{2} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The corresponding super Lie algebra is

$$osp_{(t,s)|2m}(A) := \{ L \in gl_{t+s|2m}(A) \mid g_{0}(Lv, w) = -(-1)^{|L||v|}g_{0}(v, Lw) \}$$

where $gl_{n|m}(A) := gl_{n|m} \otimes A$ is the super Lie algebra of all matrices with entries in $A$. For $A = \mathbb{R}$, we define $osp_{(t,s)|2m} := osp_{(t,s)|2m}(\mathbb{R})$. By means of choosing a basis $\{T^{j} \}$ of $osp_{(t,s)|2m}$, it is easy to see that $osp_{(t,s)|2m}(A) = osp_{(t,s)|2m}(\mathbb{R})$.

Now let $g$ be an even (i.e. parity-preserving), nondegenerate and supersymmetric bilinear form (a supermetric for short). By an extension of the Gram-Schmidt procedure as detailed e.g. in Sec. 2.8 of [DeW84], there is an adapted basis $(e_{1}, \ldots, e_{t+s+2m})$ of $M$, such that $g = g_{0}$ on the level of matrices, which we shall call an $OSp_{(t,s)|2m}$-basis. In particular, the values of $t, s, m \in \mathbb{N}$ are uniquely determined by $g$. As in Sec. 3.5 of [Han12] with slightly different conventions, we introduce the even map $J$, which is defined with respect to an $OSp_{(t,s)|2m}$-basis $\{e_{i}\}$ as follows.

$$J\varepsilon_{k} := \begin{cases} -\varepsilon_{k} & k \leq t \\
\varepsilon_{k} & t < k \leq t + s \\
\varepsilon_{k+1} & k = t + s + 2l - 1 \\
-\varepsilon_{k-1} & k = t + s + 2l \end{cases}$$

This is such that $g(e_{k}, J\varepsilon_{j}) = (-1)^{|\varepsilon_{k}|}|\delta_{kj}$ and $J\varepsilon_{k} = (-1)^{|\varepsilon_{k}|}h_{km}e_{m}$ and, moreover, every $v \in M$ has the expansion

$$v = g(v, e_{j})J\varepsilon_{j} = (-1)^{|\varepsilon_{j}|}g(v, J\varepsilon_{j})e_{j}$$

g identifies any other bilinear form $K \in \text{Hom}_{A}(M \otimes_{A} M, A)$ with a superlinear map $\tilde{K} : M \rightarrow M$ of the same parity as $K$ via

$$g(\tilde{K}(v), w) = K(v, w)$$

The supertrace of $K$ with respect to $g$ is defined as

$$\text{str}_{g}K := \text{str} \tilde{K} = (-1)^{|f_{j}|(|\tilde{K}|+1)}\tilde{K}^{j}$$
where the second equation holds for any adapted (right) basis \( \{ f_j \} \) of \( M \) upon identifying \( K \) with a supermatrix. An explicit calculation then shows that, regardless of the parity of \( K \), the following formula holds for every \( OSp(t,s)_{2\mathbb{m}} \)-basis \( f_j = e_j \).

\[
\text{str}_y K = K (e_j, Je_j) = (-1)^{|e_j|} K (Je_j, e_j)
\]

We shall use the preceding superlinear algebra for the tensor calculus on a supermanifold \((M,\mathcal{O}_M)\). Let \( \text{End}_{\mathcal{O}_M}(SM) \) denote the sheaf of superlinear endomorphisms. As usual, we write \( E \in \text{End}_{\mathcal{O}_M}(SM) \) for a sheaf morphism \( E = \{ E_U \}_{U \in \mathcal{O}_M} \) which, by a slight abuse of notation, we shall call a section of the sheaf of endomorphisms. Let, similarly, \( \text{Hom}_{\mathcal{O}_M}(SM,\mathcal{O}_M) \) and \( \text{Hom}_{\mathcal{O}_M}(SM \otimes_{\mathcal{O}_M} SM,\mathcal{O}_M) \) denote the sheaves of superlinear maps and superbilinear maps, respectively. We identify the tensor product of two one-forms \( F,G \in \text{Hom}_{\mathcal{O}_M}(SM,\mathcal{O}_M) \) with a bilinear form via

\[
(F \otimes G)(X,Y) := (-1)^{[G][X]} F(X) \cdot G(Y)
\]

which can be taken as a definition \([\text{DM}99]\). Sections of any sheaf of multilinear forms will be commonly denoted as tensors or tensor fields. The differential \( df \) of a superfunction \( f \in \mathcal{O}_M \) is defined by the formula \( df[X] := (-1)^{|f||X|} X(f) \) for \( X \in SM \). In particular, we may consider the differential of coordinate functions \( \xi^i \). Our sign conventions are such that any even bilinear form \( B \in \text{Hom}_{\mathcal{O}_M}(SM \otimes_{\mathcal{O}_M} SM,\mathcal{O}_M) \) has the local form

\[
B = (-1)^{|\xi^i|+|\xi^j|} B_{ij} \cdot d\xi^i \otimes d\xi^j
\]

An even bilinear form \( g \) which is non-degenerate and (super-)symmetric is called a semi-Riemannian supermetric. Locally, the above treatment on superlinear algebra applies here. In particular, there exists an \( OSp(t,s)_{2\mathbb{m}} \)-basis \( \{ e_j \} \) with \( t,s,m \) being intrinsic invariants of \( g \).

Now consider a morphism \( \Phi = (\varphi,\phi) : (M,\mathcal{O}_M) \to (N,\mathcal{O}_N) \) of supermanifolds. Its differential is the morphism of sheaves

\[
d\Phi : \varphi_*SM \to \mathcal{S}\Phi, \quad d\Phi(Y) := Y \circ \phi
\]

where \( \mathcal{S}\Phi := \text{Der}(\mathcal{O}_N,\varphi_*\mathcal{O}_M) \) denotes the sheaf of derivations (vector fields) along \( \Phi \), which is locally free of rank the dimension of \( N \). The pullback of tensors on \( N \) is defined as follows. Following the conventions used in \([\text{Gro}11]\), let \( E \in \text{End}_{\mathcal{O}_N}(SN) \) and \( F \in \text{Hom}_{\mathcal{O}_N}(SN,\mathcal{O}_N) \) and \( B \in \text{Hom}_{\mathcal{O}_N}(SN \otimes_{\mathcal{O}_N} SN,\mathcal{O}_N) \) be tensor fields. Now, prescribing

\[
E_\Phi(\phi \circ Y) := \phi \circ E(Y), \quad F_\Phi(\phi \circ Y) := \phi \circ F(Y), \quad B_\Phi(\phi \circ Y, \phi \circ Z) := \phi \circ B(Y,Z)
\]

for \( Y,Z \in SN \), together with super(bi)linear extensions for general sections of \( \mathcal{S}\Phi \), yields well-defined sections \( E_\Phi \in \text{End}_{\varphi_*\mathcal{O}_M}(\mathcal{S}\Phi) \) and \( F_\Phi \in \text{Hom}_{\varphi_*\mathcal{O}_M}(\mathcal{S}\Phi,\varphi_*\mathcal{O}_M) \) and \( B_\Phi \in \text{Hom}_{\varphi_*\mathcal{O}_M}(\mathcal{S}\Phi \otimes_{\mathcal{O}_M} \mathcal{S}\Phi,\varphi_*\mathcal{O}_M) \), respectively.

A connection on \( N \) is an even \( \mathbb{R} \)-linear sheaf morphism \( \nabla : SN \to S^*N \otimes_{\mathcal{O}_N} SN \) satisfying the (graded) Leibniz rule. If \( (N,g) \) is a semi-Riemannian supermanifold, there is a unique connection which is (graded) metric and torsion-free, called the Levi-Civita connection \([\text{Goe}08]\). In general, for a connection \( \nabla \) on \( N \), there is a canonical pullback connection \( \nabla_\Phi : \mathcal{S}\Phi \to \varphi_*S^*M \otimes_{\varphi_*\mathcal{O}_M} \mathcal{S}\Phi \) (see \([\text{GW}12]\)), that, in the following, we shall denote simply by \( \nabla \). With respect to local coordinates \( \{ \eta^i \} \) on \( N \), it reads

\[
\nabla_X \left( (\phi \circ \partial_{\eta^j}) \cdot Y^j \right) = (-1)^{|X||\eta^j|} (\phi \circ \partial_{\eta^j}) \cdot X(Y^j) + X(\phi \circ \eta^j) \cdot (\phi \circ \nabla_{\eta^j} \partial_{\eta^j}) \cdot Y^j
\]
Lemma 2.1. Let \((N, g)\) be a semi-Riemannian supermanifold and \(\nabla\) a superconnection on \(N\) which is metric. Then the pullback \(\nabla_{\Phi}\) is metric in the following sense.

\[ X g_{\Phi} (Y, Z) = g_{\Phi} ((\nabla_{\Phi}) X Y, Z) + (-1)^{|X||Y|} g_{\Phi} (Y, (\nabla_{\Phi}) X Z) \]

holds true for every \(X \in \varphi_*SM\) as well as \(Y, Z \in S\Phi\).

Proof. This follows from a straightforward calculation in local coordinates using \((\nabla_{\Phi})\).

Moreover, one verifies that, in case \(\Phi\) is a diffeomorphism, it holds

\[ (\Phi^* F)(X, Y) := B_{\Phi} (d\Phi[X], d\Phi[Y]) \]

such that

\[ \Phi^*B \in \text{Hom}_{\varphi_*OM} (\varphi_*SM \otimes \varphi_*SM, \varphi_*OM) \cong \text{Hom}_{OM} (SM \otimes SM, OM) \]

and analogous for other tensors. The canonical identification with an (ordinary) tensor on \(M\) stated follows from the general theory of ringed spaces, e.g. from Thm. 4.4.14 of [Ten75] applied to the induced morphism \((\varphi, \text{id}) : (M, OM) \to (N, \varphi_*OM)\) of ringed spaces. A direct calculation yields

\[ \Phi^* (f \cdot F) = (\Phi^* f) \cdot (\Phi^* F) , \quad \Phi^* (F \otimes G) = \Phi^* F \otimes \Phi^* G \]

Moreover, one verifies that, in case \(\Phi\) is a diffeomorphism, it holds

\[ \Phi^* B (X, Y) = \phi \circ B \left((\phi^{-1}) \circ X \circ \phi, (\phi^{-1}) \circ Y \circ \phi\right) \]

which is the definition of the pullback in [Goe08]. The next lemma will be needed in calculations below.

Lemma 2.2. Let \(f \in \mathcal{O}_N\) be a superfunction and \(Y \in SM\) be a super vector field. Then

\[ \Phi^* df[Y] = (-1)^{|f||Y|} d\Phi[Y](f) \]

Proof. The assertion holds by the following calculation in coordinates \(\{\eta^i\}\) on \(N\).

\[ \Phi^* df[Y] = df_{\Phi}(d\Phi[Y]) \]

\[ = df_{\Phi} \left( (\phi \circ \partial_{\eta^i}) d\Phi[Y]^i \right) \]

\[ = \phi \circ df[\partial_{\eta^i}] \cdot d\Phi[Y]^i \]

\[ = (-1)^{|f||\eta^i|} (\phi \circ \partial_{\eta^i})(f) \cdot d\Phi[Y]^i \]

\[ = (-1)^{|f||\eta^i|} (-1)^{|f||d\Phi[Y]^i|} \left((\phi \circ \partial_{\eta^i}) \cdot d\Phi[Y]^i\right)(f) \]

\[ = (-1)^{|f||\eta^i|} \Phi^* df[Y](f) \]

The matrix groups \(GL_{n|m}(A)\) and \(OSp_{(t,s)|2m}(A)\) give rise to super Lie groups \(GL_{n|m}\) and \(OSp_{(t,s)|2m}\), respectively. They are examples of matrix super Lie groups and, for the treatment of \(G\)-structures below, are most conveniently described in terms of \(S\)-points.
In general, to a supermanifold $M$ we can associate its functor of points $M(\cdot) : \text{SMan}^{\text{Op}} \to \text{Set}$ by sending any supermanifold $S$ to its $S$-point $M(S) := \text{Hom}(S,M)$ and any morphism $\Phi : T \to S$ to $M(\Phi) : M(S) \to M(T)$ through $m \mapsto m \circ \Phi$. By Yoneda’s lemma, morphisms $\psi : M \to N$ are in bijection with natural transformations $\psi(\cdot) : M(\cdot) \to N(\cdot)$. This induces a canonical embedding of the category SMan into the functor category $[\text{SMan}^{\text{Op}}, \text{Set}]$, elements of which are called generalised supermanifolds, and those in the image of the embedding are called representable. A generalised super Lie group is an object of the functor category $[\text{SMan}^{\text{Op}}, \text{Grp}]$ (which canonically embeds into $[\text{SMan}^{\text{Op}}, \text{Set}]$). The representing supermanifold, if existing, is called a super Lie group which, equivalently, can also be defined as a group object in the category of supermanifolds.

$GL_{n|m}$ is the functor which sends any supermanifold $S$ to the multiplicative group $GL_{n|m}(\mathcal{O}_S(S))$ of even and invertible $n|m$-matrices with values in $\mathcal{O}_S(S)$ and any morphism $T \to S$ to the induced map $GL_{n|m}(\mathcal{O}_S(S)) \to GL_{n|m}(\mathcal{O}_T(T))$. This functor is representable such that, on the level of $S$-points, multiplication is ordinary matrix multiplication. The same construction can be applied to natural subgroups of $GL_{n|m}(A)$. For example, the generalised super Lie group $OSp_{(t,s)|2m}$ is the functor defined through $S \mapsto OSp_{(t,s)|2m}(\mathcal{O}_S(S))$ and the corresponding morphism map. It is representable by the stabiliser condition $[\text{BCF09}]$. For our applications, it will be sufficient to consider all super Lie groups occuring as generalised. This is also the point of view of $[\text{ACDS97}]$.

We shall next describe frame fields and $G$-structures on a supermanifold, where $G$ is a super Lie group. In particular, we will see that supermetrics are equivalent to $OSp_{(t,s)|2m}$-structures.

A tuple $(X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+m})$ of $n$ even and $m$ odd vector fields on an open set $U \subseteq M$ is called a frame field, if the tuple $(X_1(p), \ldots, X_{n+m}(p))$ of tangent vectors is a basis of the supervector space $T_pM$ for every $p \in U$. If $U$ is sufficiently small (such that it lies in a coordinate chart), this condition is equivalent to $(X_1, \ldots, X_{n+m})$ being an adapted $\mathcal{O}_M(U)$-basis. The frame fields form a sheaf on $M$, denoted $U \mapsto \mathcal{F}(U)$.

The super Lie group $GL_{n|m}$ induces a sheaf of groups over $M$ via $GL_{n|m}(U) := GL_{n|m}(\mathcal{O}_M(U))$, where the latter corresponds to a morphism $U \to GL_{n|m}$ of supermanifolds. $GL_{n|m}(U)$ acts naturally from the right on the frame fields $\mathcal{F}(U)$. Explicitly,

$$\left( X_1, \ldots, X_{n+m} \right) \cdot A := \left( \sum_i X_i \cdot A_i, \ldots, \sum_i X_i \cdot A_m \right)$$

where $A_{jk}$ denotes the $jk$-th entry of the matrix of $A \in GL_{n|m}(U)$. If $U$ is sufficiently small, we see that this action is simply transitive.

A supermetric $g$ defines the subsheaf of $OSp_{(t,s)|2m}$-frames as follows. Denoting the standard basis of $A^{t+s|2m}$ by $\{e_i\}$, we set

$$\mathcal{F}_g(U) := \{ (X_1, \ldots) \in \mathcal{F}(U) \mid g(X_i, X_j) = g_0(e_i, e_j) \}$$

The orthosymplectic supergroup $OSp_{(t,s)|2m}$ acts on $\mathcal{F}_g$ via $\mathfrak{S}$. We already know that, given two frames $(X_1, \ldots), (Y_1, \ldots) \in \mathcal{F}_g(U)$ for $U$ sufficiently small, there is a unique $A \in GL_{t+s|2m}(U)$ such that $(X_1, \ldots) \cdot A = (Y_1, \ldots)$. But this is exactly the condition for $A \in OSp_{(t,s)|2m}$.

**Definition 2.3** ($[\text{ACDS97}]$). Let $G \subseteq GL_{n|m}$ be a Lie subgroup. A $G$-structure on a supermanifold $M$ is a sheaf $\mathcal{F}_G$ of subsets $\mathcal{F}_G \subseteq \mathcal{F}$ such that $G(U)$ acts on $\mathcal{F}_G(U)$ and for all points, there is a neighbourhood for which the action is simply transitive.
Thus, in particular, every supermetric \( g \) defines an \( \text{OSp}(t,s)2m \)-structure \( \mathcal{F}_g \). Conversely, assume that \( \mathcal{F}_{\text{OSp}} \) is an \( \text{OSp}(t,s)2m \)-structure. We construct a supermetric \( g \) as follows. Let \((X_1, \ldots) \in \mathcal{F}_{\text{OSp}}\) and define \( g \) by \( g(X_i, X_j) := g_0(e_i, e_j) \) and superbilinear extension. This definition does not depend on the chosen frame field in \( \mathcal{F}_{\text{OSp}} \) and is such that \( \mathcal{F}_g = \mathcal{F}_{\text{OSp}} \).

**Lemma 2.4.** There is a bijection between supermetrics and \( \text{OSp}(t,s)2m \)-structures.

An automorphism of \( M \) is an isomorphism \( \Phi : M \to M \) of supermanifolds. Its differential \( \Phi \) can be identified with a sheaf morphism \( d\Phi : \mathcal{S}M \to \varphi_s^{-1}\mathcal{S}M \) via \( d\Phi(Y) := \phi^{-1} \circ Y \circ \phi \). It induces a sheaf (iso)morphism \( d\Phi : \mathcal{F} \to \varphi_s^{-1}\mathcal{F} \), denoted by the same symbol.

**Definition 2.5.** Let \( \Phi : M \to M \) be an automorphism of a supermanifold \( M \). Let \( g \) be a semi-Riemannian supermetric and \( \mathcal{F}_G \) be a \( G \)-structure on \( M \).

(i) \( \Phi \) is called an automorphism of \( g \) if it is an isometry \( \Phi^*g = g \).

(ii) \( \Phi \) is called an automorphism of \( \mathcal{F}_G \) if \( d\Phi \mathcal{F}_G \subseteq \varphi_s^{-1}\mathcal{F}_G \).

**Lemma 2.6.** An automorphism \( \Phi : M \to M \) of \( M \) is an automorphism of \( g \) if and only if it is an automorphism of \( \mathcal{F}_g \).

**Proof.** Let \((X_1, \ldots) \in \mathcal{F}_g(U)\). By definition, \( g(X_i, X_j) = (\Phi^*g)(X_i, X_j) \) is equivalent to \( (d\Phi[X_1, \ldots]) \in \varphi_s^{-1}\mathcal{F}_g \), thus proving one direction. The converse follows from the same characterisation together with superbilinearity.

### 3 Killing Vector Fields

In this section, we shall give five equivalent characterisations of Killing vector fields on supermanifolds. While the building blocks of the theory are mostly available in the literature, a homogeneous treatment such as below is still missing, and there is sometimes some confusion about the concept. In particular, we will see how the notion of a Killing vector field as treated in [ACDS97] is different from the canonical one. We start with the Lie derivative of tensors.

#### 3.1 The Lie Derivative

The Lie derivative of a tensor on a manifold \( M \) with respect to a vector field \( X \in \Gamma(TM) \) is defined by means of the flow \( \varphi \) of \( X \) which has the defining properties \( \frac{d}{dt}\varphi_x(t) = X \circ \varphi_x(t) \) and \( \varphi_x(0) = x \). By a standard theorem [War83], every vector field \( X \) possesses a unique smooth flow \( \varphi : D(X) \to M \), which is defined on an open neighbourhood \( D(X) \subseteq \mathbb{R} \times M \) of \( \{0\} \times M \). \( X \) is called complete if \( D(X) = \mathbb{R} \times M \).

Now let \( X \) be a vector field on a supermanifold and \( \hat{X} \) be its canonical projection. We let \( D(X) \) be the open subsupermanifold of \( \mathbb{R}^{1|1} \times (M, \mathcal{O}_M) \) whose underlying smooth manifold is \( D(\hat{X}) \). If \( \hat{X} \) is complete, then \( D(\hat{X}) = \mathbb{R}^{1|1} \times (M, \mathcal{O}_M) \).

We recall results concerning the flow of super vector fields from [MSV93]. To integrate also odd vector fields, the derivation \( \partial_t \) needs to be endowed by an odd part to a (super-)derivation \( D \). As argued in [MSV93], one is naturally let to one of three integration models \( D^{(1)} = \partial_t + \partial_\tau, D^{(2)} = \partial_t + \partial_\tau + \tau \partial_t \) and \( D^{(3)} = \partial_t + \tau \partial_t + \partial_\tau \), corresponding
to the three different super Lie algebra structures on $\mathbb{R}^{1|1}$.

Due to the ev-morphism, all of the following does, however, not depend on the choice of the integration model and, for brevity, we shall denote either derivation by $D \in \mathcal{S}(\mathbb{R}^{1|1})$. To simplify notation, we shall also denote its lift to $D(X)$ by the same symbol $D$.

**Definition 3.1.** Let $X$ be a super vector field. Its flow is a morphism $\Phi = (\varphi, \phi) : D(X) \to M$ such that the following equations hold

$$
ev|_{t=0} \circ D \circ \phi = \ev|_{t=0} \circ \phi \circ X, \quad \ev|_{t=0} \circ \phi = \id$$

which are referred to as the flow condition and initial condition, respectively.

In calculations, only the homogeneous part of $D$ with the same parity as $X$ occurs since the flow equation splits into two equations (according to the $\mathbb{Z}_2$-decomposition of $X$ and $D$), of which one vanishes if $X$ is homogeneous. By a slight abuse of notation, we may thus write $|D| = |X|$.

**Theorem 3.2** ([MSV93]). Every super vector field possesses a unique flow.

For the strong flow condition $D \circ \phi = \phi \circ X$ without the ev morphism to hold, $X$ must satisfy certain conditions as shown in [MSV93].

**Definition 3.3.** Let $f \in \mathcal{O}_M$ be a superfunction. Its Lie derivative along $X$ is

$$L_X f := \ev|_{t=0} D \circ \Phi^* f$$

where $\Phi : D(X) \to M$ is the flow of $X$.

**Definition 3.4.** Let $B \in \text{Hom}_{\mathcal{O}_M}(SM \otimes \ldots \otimes SM, \mathcal{O}_M)$ be a multilinear form. We define its Lie derivative along $X$ as

$$L_X B := \ev|_{t=0} D \circ \Phi^* B \in \text{Hom}_{\mathcal{O}_M}(SM \otimes \ldots \otimes SM, \mathcal{O}_M)$$

where, on the right hand side, the pullback $\Phi^* B$ is implicitly understood to be restricted to vector fields in $SM \subseteq \mathcal{S}(D(X))$.

**Lemma 3.5.** Let $f \in \mathcal{O}_M$ be a superfunction and $Y \in SM$ be a super vector field. Then, for $X$, $Y$ and $f$ of homogeneous parity,

$$L_X f = X(f), \quad L_X df[Y] = (-1)^{|Y||(|f|+|X|)} Y \circ X(f)$$

**Proof.** Using the defining properties of the flow, we calculate

$$L_X f = \ev|_{t=0} D \circ \phi(f) = \ev|_{t=0} \phi \circ X(f) = X(f)$$

The second assertion holds by the following calculation.

$$L_X df[Y] = \ev|_{t=0} D \circ \Phi^* df[Y]$$

$$= (-1)^{|f||Y|} \ev|_{t=0} D \circ d\Phi[Y](f)$$

$$= (-1)^{|f||Y|} \ev|_{t=0} D \circ Y \circ \phi(f)$$

$$= (-1)^{|f||Y|} (-1)^{|Y||D|} Y \circ \ev|_{t=0} D \circ \phi(f)$$

$$= (-1)^{|f||Y|} (-1)^{|Y||X|} Y \circ \ev|_{t=0} \phi \circ X(f)$$

$$= (-1)^{|Y||(|f|+|X|)} Y \circ X(f)$$

Here, we used Lem. [2.2] and $t, \tau$-independence of $Y$.  

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Lemma 3.6. Let $\Phi : \mathcal{D}(X) \to M$ be the flow of a vector field $X$. Then the initial condition implies the generalisation

$$\text{ev}|_{t=0} \Phi^* B = B$$

for multilinear forms $B$.

Proof. We prove the statement for a one-form $F$, the general case is analogous. Let $Y \in \mathcal{SM}$ be a vector field. We let $\eta^1, \ldots, \eta^{n+m}$ denote local coordinates on $M$ and $\xi^1, \ldots, \xi^{n+m+2}$ the corresponding coordinates on $\mathcal{D}(X)$ such that $\xi^i = \eta^i$ for $i \in \{1, \ldots, n+m\}$ and $\xi^{n+m+1} = t$ and $\xi^{n+m+2} = \tau$. Then

$$\text{ev}|_{t=0} \Phi^* F[Y] = \text{ev}|_{t=0} F_\Phi (d\Phi[Y])$$

$$= \text{ev}|_{t=0} F_\Phi \left( (\phi \circ \partial_{\eta^i}) \cdot d\Phi[Y]^i \right)$$

$$= (-1)^{|\xi|+|\eta^i|} |\eta^i| \text{ev}|_{t=0} \left( \phi \circ F(\partial_{\eta^i}) \cdot \frac{\partial}{\partial \xi^j} \phi(\eta^i) \cdot Y^j \right)$$

$$= (-1)^{|\xi|+|\eta^i|} |\eta^i| \text{ev}|_{t=0} \left( \phi \circ F(\partial_{\eta^i}) \cdot \text{ev}|_{t=0} \left( \frac{\partial}{\partial \xi^j} \phi(\eta^i) \cdot Y^j \right) \right)$$

By assumption, $Y^i = Y^\tau = 0$ such that we may replace $\frac{\partial}{\partial \xi^j}$ by $\frac{\partial}{\partial \eta^j}$, which commutes with ev, such that

$$\text{ev}|_{t=0} \Phi^* F[Y] = (-1)^{|\eta^i|+|\eta^i|} |\eta^i| \cdot \frac{\partial}{\partial \eta^j} \text{ev}|_{t=0} \phi(\eta^i) \cdot Y^j$$

$$= (-1)^{|\eta^i|+|\eta^i|} |\eta^i| \cdot \frac{\partial}{\partial \eta^j} \phi(\eta^i) \cdot Y^j$$

$$= F(Y) \quad \Box$$

Lemma 3.7. Let $f \in \mathcal{O}_M$ be a superfunction and $F, G \in \text{Hom}_{\mathcal{O}_M}(\mathcal{SM}, \mathcal{O}_M)$ be one forms. Then

$$L_X (f \cdot F) = L_X f \cdot F + (-1)^{|f||X|} f \cdot L_X F$$

$$L_X (F \otimes G) = L_X F \otimes G + (-1)^{|X||F|} F \otimes L_X G$$

Proof. The first assertion follows from a straightforward calculation, using (7), the derivation property of $D$ as well as $\text{ev}|_{t=0} (f \cdot g) = (\text{ev}|_{t=0} f) \cdot (\text{ev}|_{t=0} g)$ and Lem. 3.6 and Lem. 3.5. The second assertion is shown analogous. \Box

Any multilinear form can be (locally) written as the tensor product of one-forms of the form $df$ for a superfunction, multiplied with a superfunction. By Lem. 3.7, the Lie derivative of these building blocks is independent of the integration model chosen (i.e. independent of $D$). By Lem. 3.7, the Lie derivative of a general multilinear form is uniquely determined by the building blocks and thus also independent of the integration model. In particular, for an even bilinear form, one starts with

$$L_X B(Y, Z) = (-1)^{|\xi|+|\eta|+|\xi^i|} L_X \left( B_{ij} \cdot d\xi^i \otimes d\xi^j \right) (Y, Z)$$

as in (3). An explicit calculation yields the following result.
Lemma 3.8. Let $B \in \text{Hom}_{\text{O}_M}(SM \otimes SM, O_M)$ be an even bilinear form. Then

$$L_X B(Y, Z) = XB(Y, Z) - B([X, Y], Z) - (-1)^{|X||Y|}B(Y, [X, Z])$$

The right hand side of Lem. 3.8 is taken as the definition of the Lie derivative in [Kli03] (up to a global sign). For semi-Riemannian supermetrics, we have the following characterisation, which naturally generalises the analogous formula in the classical case. The proof is based on $\nabla$ being metric and torsion-free and (up to signs) the same as in the classical case, thus omitted.

Lemma 3.9. Let $g$ be a semi-Riemannian supermetric and $\nabla$ the Levi-Civita superconnection. Then the Lie derivative of $g$ can be written

$$L_X g(Y, Z) = (-1)^{|X||Y|} \left( g(\nabla_Y X, Z) + (-1)^{|X||Y|+|X||Z|+|Y||Z|} g(\nabla_Z X, Y) \right)$$

3.2 Killing Vector Fields

Definition 3.10. Let $((M, \mathcal{O}_M), g)$ be a semi-Riemannian supermanifold. A Killing vector field is a vector field $X$ such that $L_X g = 0$.

For the proof of the following characterisation theorem, we shall need the flow equation for superbilinear forms as stated in the next lemma, which is easily generalised to general multilinear forms. A similar result is stated in [MSV93] as Prop. 4.3.

Lemma 3.11. Let $B$ be a superbilinear form. Then the flow equation holds:

$$\text{ev}_{|t=t_0} D \circ \Phi^* B = \text{ev}_{|t=t_0} \Phi^* L_X B$$

Moreover, if $X$ possesses a strong flow s.th. $D \circ \phi = \phi \circ X$, then the flow equation is satisfied without the ev morphism.

Proof. By a direct calculation along the lines of the proofs of Lem. 3.5 and Lem. 3.6, one verifies the (weak) flow equations

$$\text{ev}_{|t=t_0} D \circ \Phi^* f = \text{ev}_{|t=t_0} \Phi^* L_X f , \quad \text{ev}_{|t=t_0} D \circ \Phi^* df = \text{ev}_{|t=t_0} \Phi^* L_X df$$

for a superfunction $f$ as well as $df$. Writing $B$ in local form as in (3), a further calculation using these building blocks as well as Lem. 3.7 and (1) yields the assertion. It is clear that all steps can be done without ev provided that the strong flow condition holds.

Let $g$ be a semi-Riemannian supermetric on $M$ with associated $OSp(t,s)|_{2m}$-structure $\mathcal{F}_g = \mathcal{F}_{OSp(t,s)|_{2m}}$ as in Lem. 2.1. Let $U \subseteq M$ be open. We write $E \in \mathcal{F}_g(U)$ as $E = (X_1, \ldots, X_{t+s+2m})$ such that $g(X_i, X_j) = \delta_0(e_i, e_j)$. Any supermatrix with entries in $\mathcal{O}_M(U)$ acts on $E$ via $[S]$. Following [ACDS97], we call $U$ ”small” if it is such that the action of $OSp(t,s)|_{2m}(U)$ on $\mathcal{F}_g(U)$ is simply transitive. Moreover, we set $L_X E := ([X, X_1], \ldots, [X, X_{t+s+2m}])$ for a vector field $X \in SM$. With this notation, our characterisation theorem can be stated as follows.

Theorem 3.12. Let $X \in SM$ be a vector field. Then the following conditions are equivalent.

(i) $X$ is Killing, i.e. $L_X g = 0$. 


(ii) For all \(Y, Z \in SM, g(\nabla_Y X, Z) + (-1)^{|X||Y|+|X||Z|+|Y||Z|}g(\nabla_Z X, Y) = 0\).

(iii) The metric \(\Phi^*g = g\) is preserved by the flow \(\Phi\) of \(X\).

(iv) \(d\Phi_F \subseteq \varphi^{-1}_*F\) for the flow \(\Phi\) of \(X\).

(v) \(L_X E \in E \cdot (\mathfrak{osp}(t,s)_{\geq 2m} \otimes \mathcal{O}_M(U))\) for all "small" \(U \subseteq M\) and \(E \in F_g(U)\).

Proof. (i) \(\iff\) (ii) is immediate by Lem. 3.9.

(iii) \(\implies\) (i): Assume that \(\Phi^*g = g\). Then, by \(t, \tau\)-independence of \(g\), we yield

\[
L_X g = \text{ev}|_{t=0} D \circ \Phi^* g = \text{ev}|_{t=0} D \circ g = 0
\]

(i) \(\implies\) (iii): Let \(X\) be a Killing vector field, we want to show that \(\Phi^*g = g\) follows. Let \(Y, Z\) be vector fields of pure parity on \(M\) and consider the superfunction

\[
f := \Phi^* g(Y, Z) \in \mathcal{O}_{D(X)}, \quad f = f^0 + \tau \cdot f^1, \quad f^0, f^1 \in \mathcal{O}_M
\]

Since \(g\) is purely even and \(\Phi\), being a morphism, is even, \(f\) has a fixed parity and thus \(f^0\) and \(f^1\) are of opposite parity (because of \(\tau\)). By assumption, \(L_X g = 0\) vanishes. Therefore, by the flow equation of Lem. 3.11 we have

\[
0 = \text{ev}|_{t=0} D(f) = \text{ev}|_{t=0} (\partial_t + \partial_\tau)(f^0 + \tau \cdot f^1) = \text{ev}|_{t=0} (\partial_t f^0 + \tau \cdot \partial_t f^1 + f^1)
\]

The first and second terms are of opposite parity (as stated above). Therefore, each summand vanishes individually such that \(f^1 = 0\) and \(\partial_t f^0 = 0\). Consider the local representation \(f^0 = \sum J f^0_\theta J\) with odd coordinates \(\theta^i\) on \(M\) and multiindices \(J\), for which we yield \(\partial_t f^0_J = 0\) (an equation for ordinary functions). Now, by Lem. 3.6 we have the initial condition

\[
f^0(t = 0) = \text{ev}|_{t=0} f = \text{ev}|_{t=0} g(Y, Z) = g(Y, Z)
\]

where \(f^0(t = 0) = \sum J f^0_\theta J(t = 0)\) (the right hand side can be expanded into an analogous sum). Therefore, we conclude

\[
\Phi^* g(Y, Z) = f = f^0 = g(Y, Z)
\]

for all vector fields \(Y, Z\) of fixed parity. Therefore, \(\Phi^* g = g\) which was to be shown.

(iii) \(\iff\) (iv): The proof of Lem. 2.6 applies verbatim.

(i) \(\iff\) (v): Let \(E = (X_1, \ldots) \in F_g(U)\) and \(X \in SM\). There is a supermatrix \(L \in \mathfrak{g}_l \otimes \mathcal{O}_M(U)\) such that \([X, X_i] = X_m \cdot L_{mi}\). By Lem. 3.8 we have

\[
L_X g(X_i, X_j) = -g([X, X_i], X_j) - (-1)^{|X||X_i|} g(X_i, [X, X_j])
\]

\[
= -g(X_m \cdot L_{mi}, X_j) - (-1)^{|X||X_i|} g(X_i, X_m \cdot L_{mj})
\]

\[
= -g_0(L \cdot e_i, e_j) - (-1)^{|g||e_i|} g_0(e_i, L \cdot e_j)
\]

where \(e_i\) is the standard basis of \(\mathcal{O}_M(U)_{t+s} \otimes \mathcal{O}_M(U)\). It follows immediately that \(L_X g = 0\) is equivalent to \(L \in \mathfrak{osp}(t,s)_{\geq 2m} \otimes \mathcal{O}_M(U)\).
Finally, we consider Killing vector fields on spinor supermanifolds, the example of supermanifolds as considered in [ACDS97]. Let \((M, g)\) be a spin manifold and consider a parallel non-degenerate suitable bilinear form \(g_1\) on the spinor bundle \(S\). We also assume that \(g_1\) is skew-symmetric (consult [Har90] for a classification of such forms) such that \(g + g_1\) induces a Riemannian supermanifold on the split supermanifold \((M, \Gamma(\Lambda S))\). There is a canonical monomorphism \(\iota: \Gamma(TM) \oplus \Gamma(S^*) \to SM\) of sheaves which induces an isomorphism \(\iota : T_pM \oplus S^*_p \to S_pM\) for every \(p \in M\). It follows that \(\{\iota(X_1), \iota(s^*_j)\}\) is a (local) basis for \(SM\) if \(\{X_1\}\) is a basis of \(\Gamma(TM)\) and \(\{s^*_j\}\) is a basis of \(S^*\). We identify sections \(s^* \in \Gamma(S^*)\) with \(s \in \Gamma(S)\) via \(g_1\).

**Lemma 3.13.** Let \(s^* \in \Gamma(S^*)\). Then the super vector field \(\iota(s^*)\) is Killing if and only if \(s^*\) is a parallel spinor.

In particular, we obtain existence results for Killing vector fields on spinor supermanifolds from the classification of spin manifolds admitting parallel spinors [MS00].

The classical harmonic action functional for maps \(\varphi : M \to N\) between semi-Riemannian manifolds \((M, h)\) and \((N, g)\) reads

\[
\mathcal{A}(\varphi) = \frac{1}{2} \int_M \text{dvol}_h \text{tr}_h(\varphi^* g) = \frac{1}{2} \int_M \text{dvol}_h h^{ij}(x)(\varphi^* g)_{ij} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)
\]

Critical points of this functional are called harmonic maps. For an exhaustive treatment of the Riemannian case, consult [Xin96]. In this section, we study a natural generalisation of (9) for semi-Riemannian supermanifolds \((M, h)\) and \((N, g)\) and prove three Noether theorems in this context.

Throughout, \(M\) is assumed to be compact and superoriented as explained below after introducing some terminology. For a superfunction \(f \in \mathcal{O}_M\), we write \(f > 0\) if \(\hat{f}(p) > 0\) for all \(p\). Moreover, we set \(|f| := m \cdot f\), where \(m : M \to \{\pm 1\}\) is such that \(|f| > 0\). It can be shown that, in case \(f\) is even and \(f > 0\), there is a unique square root \(\sqrt{f}\) which is constructed using Taylor-like expansion in odd coordinates.

Now, \(M\) is called superoriented if it has an atlas of coordinate charts such that, for every coordinate transformation \(\Phi = (\varphi, \phi) : \mathbb{R}^{|m|} \to \mathbb{R}^{|n|}\), both \(\text{det}(d\varphi) > 0\) and \(\text{sdet}(d\Phi) > 0\) hold [Sha88]. Here, the first condition (classical orientedness) is needed to make the integral over densities (sections of the superdeterminant sheaf, see Chp. 3 of
Moreover, it is supersymmetric, i.e. \( B \) has the parity of \( X \). With this notation, \( \text{dvol}_h \) can be defined by the local expression

\[
\text{dvol}_h = [d^n x d^m \theta] \cdot \sqrt{|\text{sdet} h|}
\]

which is independent of the coordinates used. A treatment of Riemannian volume forms in a slightly different style can be found in Sec. 3.5 of [Han12].

Instead of plain morphisms \( \Phi = (\varphi, \phi) : M \to N \) of supermanifolds, we consider morphisms \( \Phi : M \times \mathbb{R}^{0|L} \to N \) in order to obtain odd "component fields" (for simplicity, think of terms of \( \phi \)) as models for fermions. Following [Hel09], we call such morphisms maps with flesh, while the same concept occurs with several names in the literature, see [DF99] and [Khe07]. For the following treatment, it suffices to consider a fixed value of \( L \in \mathbb{N} \) which is large enough to make the calculations consistent, cf. the discussion in [Hel09]. For a functorial point of view (considering all values of \( L \) simultaneously), we refer to [Han12]. In the following, we shall simply write \( \Phi : M \to N \) for maps with flesh, leaving the superpoint \( \mathbb{R}^{0|L} \) implicit. The differential \( d\Phi \) is then implicitly restricted to the tangent sheaf of \( M \) tensored with the algebra \( \wedge \mathbb{R}^L \) of superfunctions of \( \mathbb{R}^{0|L} \). Tensors on \( SM \) are similarly endowed to that sheaf by \( \wedge \mathbb{R}^L \)-multilinear extension. For details, consult [Gro11].

With these preparations, we define the superharmonic action functional as

\[
\mathcal{A}(\Phi) := \frac{1}{2} \int_M \text{dvol}_h \text{str}_h(\Phi^* g) = \frac{1}{2} \int_M \text{dvol}_h g_{\Phi}(d\Phi[e_j], d\Phi[J e_j])
\]

with \( \text{str}_h \) as in [2], and where \( \{e_j\} \) is a local \( OSp(t,s)2m \)-frame on \( (M, h) \).

**Definition 4.1.** Let \( \Phi : (M, h) \to (N, g) \) be a morphism. Then

\[
B_{X,Y}(\Phi) := \langle \nabla_X d\Phi \rangle [Y] = \nabla_X (d\Phi[Y]) - d\Phi[\nabla_X Y]
\]

is called 2nd fundamental form.

**Lemma 4.2.** The 2nd fundamental form is a tensor \( B_{.,.}(\Phi) \in \text{Hom}(SM \otimes_M SM, SM) \). Moreover, it is supersymmetric, i.e. \( B_{X,Y} = (-1)^{|X||Y|} B_{Y,X} \).

**Proof.** \( B \) is a tensor since \( B_{fX,Y}(\Phi) = (-1)^{|X||f|} B_{fX,fY}(\Phi) = f B_{X,Y}(\Phi) \) is satisfied. To show supersymmetry, it thus suffices to consider coordinate vector fields. One verifies that the expression of \( B_{\partial_{\xi^i},\partial_{\theta^j}}(\Phi) \) in terms of coordinates \( \{\xi^i\} \) on \( M \) is supersymmetric in \( i \leftrightarrow j \).

**Definition 4.3.** The super trace of the second fundamental form is called tension field.

\[
\tau(\Phi) := \text{str}_h B = \text{str}_h (\nabla d\Phi)[.] = (\nabla_{e_j} d\Phi)[J e_j]
\]

**Theorem 4.4** ([Han12], Thm. 6.29). \( \Phi \) is a critical point of the action functional (10) if and only if the Euler-Lagrange equation \( \tau(\Phi) = 0 \) holds.

The corresponding theorem in [Han12] is formulated for a special case (in particular, \( h \) is Riemannian there), but the proof provided there applies to the general case.

We also need the divergence of a super vector field. Classically, it can be defined as \( \text{tr}(\nabla X) \). In the super case, an additional sign occurs since the map \( Y \mapsto \nabla Y \cdot X \) (which has the parity of \( X \)) is not a superlinear map in case \( X \) is odd.
Definition 4.5. We set $\text{div} X := \text{str} \left( Y \mapsto (-1)^{|X||Y|} \nabla_Y X \right) = (-1)^{|e_j||X|} g \left( \nabla_X e_j, J e_j \right)$.

For the characterisation in a local $OSp(t,s)|2m$-frame, beware that the supertrace

$$\text{div} X = (-1)^{|e_j||X|+1} \left( Y \mapsto (-1)^{|X||Y|} \nabla_Y X \right)$$

is defined with respect to right coordinates. We define next the analogon for a vector field along $\Phi$. Consider the super-bilinear form $(X,Y) \mapsto (-1)^{|X||\xi|} g_{\Phi} \left( \nabla_X \xi, \Phi \right)$. Again, the sign is necessary to make it a super-bilinear form.

Definition 4.6. Let $\xi \in S\Phi$ be a vector field along $\Phi$. We define its divergence to be

$$\text{div} \xi := \text{str}_h \left( (X,Y) \mapsto (-1)^{|X||\xi|} g_{\Phi} \left( \nabla_X \xi, \Phi \right) \right) = (-1)^{|e_i||\xi|} g_{\Phi} \left( \nabla_X e_i, \Phi \right)$$

Lemma 4.7. Let $\xi \in S\Phi$ and set $W_{\xi} := g_{\Phi} \left( \xi, \Phi \right) J e_j$ which has the parity of $\xi$.

Then $\text{div} W_{\xi} = \text{div} \xi + g_{\Phi} \left( \xi, \tau(\Phi) \right)$.

Proof. The assertion is shown by the following calculation, using $|W_{\xi}| = |\xi|$ as well as the metric property of both $h$ and $g_{\Phi}$.

$$\text{div} W_{\xi} = (-1)^{|e_i|(|1+|\xi|)} h \left( \nabla_X e_i g_{\Phi} \left( \xi, \Phi \right) J e_j, e_i \right)$$

$$= (-1)^{|e_i|(|1+|\xi|)} J e_j g_{\Phi} \left( \xi, \Phi \right) h \left( J e_j, e_i \right) + (-1)^{|e_i|(|1+|\xi|)} J e_j g_{\Phi} \left( \xi, \Phi \right) \left( \nabla_X e_i \right) = (-1)^{|e_i|(|1+|\xi|)} J e_j g_{\Phi} \left( \xi, \Phi \right) \left( \nabla_X e_i + (-1)^{|e_j| g_{\Phi} \left( \nabla e_j, \Phi \right) \left( \nabla e_j, e_j \right) - g_{\Phi} \left( \nabla e_j, \Phi \right) \left( \nabla e_j, e_j \right) \right) = \text{div} \xi + g_{\Phi} \left( \xi, \tau(\Phi) \right)$$

\[ \square \]

4.1 Target Space Symmetries

Killing vector fields $\xi \in \Gamma(TN)$ on the target space are infinitesimal symmetries of the harmonic action $[\text{II}$. This can be seen as follows. Consider the flow $F_t$ of $\xi$. We alter $\varphi$ by moving along the flow lines via $\varphi_t(x) := F_t(\varphi(x))$. In physicists’ notation, this means that, infinitesimally, $\varphi(x) \to \varphi(x) + \xi(\varphi(x)) dt$. The infinitesimal change of $\mathcal{A}(\varphi)$ by $\xi$ thus becomes

$$\frac{d}{dt}|_0 \mathcal{A}(F_t \circ \varphi) = \frac{1}{2} \frac{d}{dt}|_0 \int_M \text{dvol}_h \text{tr}_h \left( \left( F_t \circ \varphi \right)^* g \right) = \frac{1}{2} \int_M \text{dvol}_h \text{tr}_h \left( \varphi^* \frac{\partial}{\partial t}|_0 F_t^* g \right)$$

$$= \frac{1}{2} \int_M \text{dvol}_h \text{tr}_h (\varphi^* L_{\xi} g)$$

Consider now the context of the superharmonic action $[\text{II}$. We let $\xi \in S\mathcal{N}$ be a vector field on $\mathcal{N}$ and denote by $F : \mathcal{D}(\xi) \times \mathcal{N} \to \mathcal{N}$ its flow as in Def. $[\text{II}$. Note that $F$ is a plain morphism of supermanifolds while $\Phi$ is a map with flesh. The analogon of $F_t(\varphi(x))$ is $F \circ \Phi := F \circ (\text{id} \times \Phi) : \mathcal{D}(\xi) \times M \to \mathcal{N},$ and the (finite) change of $\mathcal{A}(\Phi)$ by $F$ reads

$$\mathcal{A}(F \circ \Phi) = \frac{1}{2} \int_M \text{dvol}_h \text{str}_h \left( (F \circ \Phi)^* g \right) = \frac{1}{2} \int_M \text{dvol}_h (F \circ \Phi)^* g \left( e_j, J e_j \right)$$
Definition and Lemma 4.8. The infinitesimal change of $A$ by $\xi \in SN$ is

$$\ev|_{t=0}D_A(F \circ \Phi) = \frac{1}{2} \int_m \text{dvol}_h \text{str}_h (\Phi^*(L_{\xi} g))$$

Proof. By compactness of $M$, we may interchange integration and differentiation, such that

$$\ev|_{t=0}D_A(F \circ \Phi) = \frac{1}{2} \int_M \text{dvol}_h \ev|_{t=0}D((F \circ \Phi)^* g) (e_j, \gev_j)$$

Choosing local coordinates

$$\Phi \ast \Phi (\gev_j) = \frac{1}{2} \int_M \text{dvol}_h \Phi^* (L_{\xi} g) (e_j, \gev_j)$$

In this calculation, the third equation holds since only $F^* g$ depends on the flow coordinates on $D(\xi)$. It is proved by a straightforward calculation in local coordinates. \(\square\)

It follows that, again, Killing vector fields on $N$ are infinitesimal symmetries! According to the Noether principle, there should be an induced conserved quantity. We will show next that this is indeed the case. The need the following analogon of Lem.

3.9

Lemma 4.9. Let $\nabla = \nabla_\Phi$ denote the pullback connection as in (20). Then

$$\Phi^*(L_{\xi} g) (Y, Z) = (-1)^{\xi || Y ||} g_\Phi (\nabla_Y (\phi \circ \xi), d\Phi [Z]) + (-1)^{\xi || Y || + \xi || Z ||} g_\Phi (d\Phi [Y], \nabla_Z (\phi \circ \xi))$$

Proof. Choosing local coordinates $\{\xi = \eta^i\}$ on $N$, the assertion is reduced to Lem. 3.9 as follows.

$$\Phi^*(L_{\xi} g) (Y, Z) = (-1)^{\xi || Y ||} g_\Phi ((\phi \circ \partial_i, d\Phi [Y]^i, (\phi \circ \partial_j) d\Phi [Z]^j)$$

$$= (-1)^{\xi || Y || + \xi || Z ||} d\Phi [Y]^i \phi \circ L_{\xi} g (\partial_i, \partial_j) \cdot d\Phi [Z]^j$$

$$= (-1)^{\xi || Y || + \xi || Z ||} d\Phi [Y]^i \cdot \phi \circ (g (\nabla_{\partial_i \xi}, \partial_j) + (-1)^{\xi || Y || + \xi || Z ||} g_\Phi (\nabla_{\partial_j \xi}, \partial_i) \cdot d\Phi [Z]^j$$

$$= (-1)^{\xi || Y ||} g_\Phi (\nabla_Y (\phi \circ \xi), d\Phi [Z]) + (-1)^{\xi || Y || + \xi || Z ||} g_\Phi (d\Phi [Y], \nabla_Z (\phi \circ \xi))$$

\(\square\)

Theorem 4.10 (Noether). Let $\xi \in SN$ be a Killing vector field $(L_{\xi} g = 0)$. Then the divergence $\text{div} (\phi \circ \xi) = 0$ vanishes. If, moreover, $\Phi$ is a superharmonic map (solution of the Euler-Lagrange equation $\tau (\Phi) = 0$), then $\text{div} W_{\phi \circ \xi} = 0$ vanishes, too, where $W_{\phi \circ \xi} := g_\Phi (\phi \circ \xi, d\Phi [e_j]) J e_j$. 15
Theorem 4.12

Proof. $L_\xi g = 0$ implies

\[
0 = \Phi^*(L_\xi g) (e_j, Je_j)
= (-1)^{|\xi||e_j|} g_\Phi (\nabla_{e_j}(\phi \circ \xi), d\Phi[Je_j]) + g_\Phi (d\Phi[e_j], \nabla_{Je_j}(\phi \circ \xi))
= (-1)^{|\xi||e_j|} (g_\Phi (\nabla_{e_j}(\phi \circ \xi), d\Phi[Je_j]) + (-1)^{|e_j|} g_\Phi (\nabla_{Je_j}(\phi \circ \xi), d\Phi[e_j]))
= 2\text{div}(\phi \circ \xi)
\]

using Lem. 4.9. The second statement now follows immediately from Lem. 4.7.

4.2 Domain Space Symmetries

We have seen that the infinitesimal change of the superharmonic action \([10]\) by a Killing vector field on the target space vanishes. Let us now consider the corresponding infinitesimal change by a vector field $\xi \in SM$ on the domain space with flow $F : D(\xi) \times M \to M$.

Definition and Lemma 4.11. The infinitesimal change of $A$ by $\xi \in SM$ is

\[
ev|_{t=0}DA(\Phi \circ F) = \frac{1}{2} \int_M \text{dsvol}_h \text{str}(L_\xi(\Phi^*g))
\]

Proof. Analogous to the proof of Lem. 4.8 we calculate

\[
ev|_{t=0}DA(\Phi \circ F) = \frac{1}{2} \int_M \text{dsvol}_h \text{ev}|_{t=0}D((\Phi \circ F)^*g) (e_j, Je_j)
= \frac{1}{2} \int_M \text{dsvol}_h \text{ev}|_{t=0}DF^*(\Phi^*g) (e_j, Je_j)
= \frac{1}{2} \int_M \text{dsvol}_h L_\xi(\Phi^*g) (e_j, Je_j)
= \frac{1}{2} \int_M \text{dsvol}_h \text{str}(L_\xi(\Phi^*g))
\]

As usual, $\xi$ is called an infinitesimal symmetry if this expression vanishes for all morphisms $\Phi$. Opposed to the target space situation, the vanishing of the Lie derivative $L_\xi(\Phi^*g)$ (for all $\Phi$) is harder to achieve. In case symmetry is present, it is usually only such that $L_\xi(\Phi^*g)$ is some exact term depending on $\Phi$ (but integrated over to zero).

Theorem 4.12 (Noether). Let $\xi \in SM$ be a $\Phi$-Killing vector field, i.e. such that $L_\xi(\Phi^*g) = 0$. Then $\text{div}(d\Phi[\xi]) = \text{div}(\xi \circ \phi) = 0$ vanishes. If, moreover, $\Phi$ is a superharmonic map, then $\text{div}W_{d\Phi[\xi]} = 0$ vanishes, where $W_{d\Phi[\xi]} = \Phi^*g (\xi, e_j) Je_j$.

Proof. Here, $\Phi^*g$ is a supermetric. However, Lem. 3.3 is not directly applicable since it would lead to the Levi-Civita connection $\nabla^{\Phi^*g}$ of this supermetric rather than to the Levi-Civita connection $\nabla = \nabla^h$ of $h$. We thus step back and use Lem. 3.3 as well as torsion-freeness of $\nabla$ to obtain

\[
0 = L_\xi(\Phi^*g) (e_j, Je_j)
= \xi \Phi^*g (e_j, Je_j) - \Phi^*g ([\xi, e_j], Je_j) - (-1)^{|\xi||e_j|} \Phi^*g (e_j, [\xi, Je_j])
= \xi \Phi^*g (e_j, Je_j) - \Phi^*g (\nabla e_j, Je_j) + (-1)^{|\xi||e_j|} \Phi^*g (\nabla e_j, Je_j)
- (-1)^{|\xi||e_j|} \Phi^*g (e_j, \nabla (Je_j)) + \Phi^*g (e_j, \nabla Je_j)
\]
Now, by Lem. 2.1, \( g_\Phi \) is metric, and we thus obtain

\[
0 = g_\Phi \left( \nabla_\xi (d\Phi[e_j], d\Phi[J e_j]) \right) + (-1)^{|\xi|} g_\Phi \left( d\Phi[e_j], \nabla_\xi (d\Phi[J e_j]) \right)
- g_\Phi \left( d\Phi[\nabla_\xi e_j], d\Phi[J e_j] \right) + (-1)^{|\xi|} g_\Phi \left( d\Phi[\nabla_\xi \xi], d\Phi[J e_j] \right)
- (-1)^{|\xi|} g_\Phi \left( d\Phi[e_j], d\Phi[\nabla_\xi (J e_j)] \right) + g_\Phi \left( d\Phi[e_j], d\Phi[\nabla_\xi e_j] \right)
\]

We combine the first and third and the second and fifth terms, respectively, such that

\[
0 = g_\Phi \left( (\nabla_\xi d\Phi)[e_j], d\Phi[J e_j] \right) + (-1)^{|\xi|} g_\Phi \left( d\Phi[e_j], (\nabla_\xi d\Phi)[J e_j] \right)
+ (-1)^{|\xi|} g_\Phi \left( d\Phi[\nabla_\xi e_j], d\Phi[J e_j] \right) - g_\Phi \left( (\nabla_\xi d\Phi)[\xi], d\Phi[J e_j] \right)
+ (g_\Phi \left( d\Phi[e_j], \nabla_\xi e_j\right) d\Phi[\xi]) - g_\Phi \left( d\Phi[e_j], (\nabla_\xi d\Phi)[\xi] \right)
\]

We combine the first and fourth and second and sixth terms, respectively, and use Lem. 1.2 such that

\[
0 = g_\Phi \left( B_{\xi,e_j}(\Phi) - (-1)^{|\xi|} B_{\xi,e_j}(\Phi) \right)
+ g_\Phi \left( d\Phi[e_j], \nabla_\xi (d\Phi)[\xi] \right) + (-1)^{|\xi|} g_\Phi \left( \nabla_\xi d\Phi[\xi], d\Phi[J e_j] \right)
= g_\Phi \left( d\Phi[e_j], \nabla_\xi e_j\right) d\Phi[\xi] \right) + (-1)^{|\xi|} g_\Phi \left( \nabla_\xi d\Phi[\xi], d\Phi[J e_j] \right)
= (-1)^{|\xi|} B_{\xi,e_j}(\Phi) \right) + (-1)^{|\xi|} g_\Phi \left( \nabla_\xi d\Phi[\xi], d\Phi[J e_j] \right)
= 2\text{div}(d\Phi[\xi])
\]

The second statement now follows immediately from Lem. 4.7.

4.3 Domain Space Symmetries II

In the previous subsection, we have considered \( \Phi \)-Killing vector fields \( \xi \in SM \). We will next prove a Noether theorem for the more common vector fields \( \xi \in SM \) which are Killing with respect to the metric \( h \), thus generalising a classical result due to Baird and Eells [BE81]. We denote the super energy of \( \Phi \) by

\[
e(\Phi) := \frac{1}{2} \text{str}_h \Phi^* g = \frac{1}{2} \Phi^* g (e_j, J e_j)
\]

and define the stress-energy tensor by

\[
S_\Phi := e(\Phi) h - \Phi^* g \in \text{Hom}_{O_M}(SM \otimes O_M, SM, O_M)
\]

Moreover, for any tensor \( S \in \text{Hom}_{O_M}(SM \otimes O_M, SM, O_M) \), we define

\[
(\nabla_X S)(Y, Z) := XS_\Phi(Y, Z) - S(\nabla_X Y, Z) - (-1)^{|X||Y|} S(Y, \nabla_X Z)
\]

and

\[
\text{div}S[\xi] := \text{str}_h \left( (X, Z) \mapsto (-1)^{|X||\xi|}(\nabla_X S)(\xi, Z) \right) = (-1)^{|\xi|} \langle \nabla_\xi S \rangle(\xi, J e_i)
\]

where \( \xi \in SM \) is a vector field. As for the signs, cf. the discussion in the context of Def. 4.5.
Lemma 4.13. Let $\xi \in SM$. Then

$$\text{div} S_\Phi[\xi] = -g_\Phi (d\Phi[\xi], \tau(\Phi))$$

Proof. We calculate

$$\text{div} S_\Phi[\xi] = (-1)^{|e_i|}e_i S_\Phi(\xi, J e_i) - (-1)^{|e_i|}S_\Phi(\nabla e_i \xi, J e_i) - S_\Phi(\xi, \nabla e_i (J e_i))$$

$$= (-1)^{|e_i|}e_i \left( \frac{1}{2}g_\Phi (d\Phi[e_j], d\Phi[J e_j]) h(\xi, J e_i) - g_\Phi (d\Phi[\xi], d\Phi[J e_i]) \right)$$

$$- (-1)^{|e_i|}e(\Phi) h(\nabla e_i \xi, J e_i) - e(\Phi) h(\xi, \nabla e_i (J e_i))$$

$$+ (-1)^{|e_i|}g_\Phi (d\Phi[\nabla e_i \xi], d\Phi[J e_i]) + g_\Phi (d\Phi[\xi], d\Phi[\nabla e_i (J e_i)])$$

$$= \frac{1}{2} g_\Phi (d\Phi[e_j], d\Phi[J e_j]) + (-1)^{|e_i|}e(\Phi) e_i h(\xi, J e_i)$$

$$- (-1)^{|e_i|}g_\Phi (d\Phi[\xi], d\Phi[J e_i])$$

$$- (-1)^{|e_i|}e(\Phi) h(\nabla e_i \xi, J e_i) - e(\Phi) h(\xi, \nabla e_i (J e_i))$$

$$+ (-1)^{|e_i|}g_\Phi (d\Phi[\nabla e_i \xi], d\Phi[J e_i]) + g_\Phi (d\Phi[\xi], d\Phi[\nabla e_i (J e_i)])$$

Here, the second term cancels with the fourth and fifth such that

$$\text{div} S_\Phi[\xi] = \frac{1}{2} g_\Phi (d\Phi[e_j], d\Phi[J e_j]) - (-1)^{|e_i|}e_i \circ g_\Phi (d\Phi[\xi], d\Phi[J e_i])$$

$$+ (-1)^{|e_i|}g_\Phi (d\Phi[\nabla e_i \xi], d\Phi[J e_i]) + g_\Phi (d\Phi[\xi], d\Phi[\nabla e_i (J e_i)])$$

$$= \frac{1}{2} g_\Phi (\nabla e_i d\Phi[e_j], d\Phi[J e_j]) + \frac{1}{2}(-1)^{|e_i|}g_\Phi (d\Phi[e_j], \nabla e_i d\Phi[J e_j])$$

$$- (-1)^{|e_i|}g_\Phi (\nabla e_i d\Phi[\xi], d\Phi[J e_i]) - g_\Phi (d\Phi[\xi], \nabla e_i d\Phi[J e_i])$$

$$+ (-1)^{|e_i|}g_\Phi (d\Phi[\nabla e_i \xi], d\Phi[J e_i]) + g_\Phi (d\Phi[\xi], d\Phi[\nabla e_i (J e_i)])$$

By supersymmetry of $g_\Phi$ and $h$, we see that the first two terms coincide. Moreover, we combine the third and fifth and the fourth and sixth terms, respectively, such that

$$\text{div} S_\Phi[\xi] = g_\Phi \left( \nabla e_i d\Phi[e_j], d\Phi[J e_j] \right) - (-1)^{|e_i|}g_\Phi ((\nabla e_i d\Phi)[\xi], d\Phi[J e_i])$$

$$- g_\Phi (d\Phi[\xi], (\nabla e_i d\Phi)[J e_i])$$

$$= g_\Phi \left( \left( \nabla e_i d\Phi[e_j] - (-1)^{|e_i|} (\nabla e_i d\Phi)[\xi] + d\Phi [\nabla e_j \xi], d\Phi[J e_j] \right) \right)$$

$$- g_\Phi (d\Phi[\xi], \tau(\Phi))$$

$$= g_\Phi (d\Phi[\nabla e_j \xi], d\Phi[J e_j]) - g_\Phi (d\Phi[\xi], \tau(\Phi))$$

using Lem. 4.12. Here, the first term vanishes by symmetry considerations. \qed

Lemma 4.14. Let $\xi \in SM$. Then, for $Y_\xi := S_\Phi(\xi, e_i) J e_i$, we have

$$\text{div} Y_\xi = \text{div} S_\Phi[\xi] + \frac{1}{2} (-1)^{|e_i|} L_\xi h(e_i, J e_j) S_\Phi(e_j, J e_i)$$

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Proof. We calculate

\[
\text{div} Y_{\xi} = (-1)^{|e_j||1+|\xi||} h \left( \nabla_{Je_j} [S_{\Phi}(\xi, e_{i}), Je_i], e_j \right)
\]

\[
= (-1)^{|e_j||1+|\xi||} Je_j S_{\Phi}(\xi, e_i) \cdot h (Je_i, e_j)
\]

\[
+ (-1)^{|e_i|+|e_j|}|e_i| S_{\Phi}(\xi, e_i) h \left( \nabla_{Je_j}(Je_i), e_j \right)
\]

\[
= (-1)^{|e_i||1+|\xi||} Je_i S_{\Phi}(\xi, e_i) + (-1)^{|e_j||1+|\xi||} e_{i} h (\nabla_{Je_j}(Je_i), e_j)
\]

\[
= (-1)^{|e_i||1+|\xi||} (\nabla_{Je_i} S_{\Phi})(\xi, e_i) + (-1)^{|e_i||1+|\xi||} S_{\Phi}(\nabla_{Je_i}, e_i)
\]

\[
+ (-1)^{|e_i|} S_{\Phi}(\xi, \nabla_{Je_i} e_i) - (-1)^{|e_i|} S_{\Phi}(\xi, e_i) h (Je_i, \nabla_{Je_j} e_j)
\]

\[
= (-1)^{|e_i||1+|\xi||} (\nabla_{Je_i} S_{\Phi})(\xi, e_i) + (-1)^{|e_i||1+|\xi||} S_{\Phi}(\nabla_{Je_i}, e_i)
\]

\[
= \text{div} S_{\Phi}[\xi] + (-1)^{|e_i||1+|\xi||} S_{\Phi}(\nabla_{Je_i}, e_i)
\]

The second term can be transformed as follows. We use relabelling of summation indices as well as exchange of \(e_k\) and \(Je_k\) with appropriate sign such that

\[
(2) = (-1)^{|e_i||1+|\xi||} S_{\Phi}(e_j \cdot h (Je_j, \nabla_{e_i} \xi), Je_i)
\]

\[
= (-1)^{|e_i||1+|\xi||+|e_j||+|e_i||+|e_j|||\xi||} h (Je_j, \nabla_{e_i} \xi) S_{\Phi}(e_j, Je_i)
\]

\[
= (-1)^{|e_i||1+|\xi||+|e_j||+|e_i||+|e_j|||\xi||} \frac{1}{2} \left( (-1)^{|e_j|} h (Je_j, \nabla_{e_i} \xi) S_{\Phi}(e_j, Je_i)
\right)
\]

\[
+ (-1)^{|e_i|+|e_j||\xi||} h (e_i, \nabla_{Je_j} \xi) S_{\Phi}(e_j, Je_i)
\]

\[
= \frac{1}{2} (-1)^{|e_j|} \left( (-1)^{|e_i||\xi||} h (\nabla_{e_i} \xi, Je_j) + (-1)^{|e_j||\xi||+|e_i||} h (\nabla_{Je_j} \xi, e_i) \right) S_{\Phi}(e_j, Je_i)
\]

\[
= \frac{1}{2} (-1)^{|e_j|} L_{\xi} h (e_i, Je_j) S_{\Phi}(e_j, Je_i)
\]

The last equation holds by Lem. \ref{lem:spinor}. □

**Theorem 4.15** (Noether). Let \(\xi \in SM\) be a Killing vector field and \(\Phi\) be superharmonic. Then \(\text{div} Y_{\xi} = 0\) vanishes.

**Proof.** By Lem. \ref{lem:spinor} and the Killing property of \(\xi\), we have \(\text{div} Y_{\xi} = \text{div} S_{\Phi}[\xi]\). The statement now follows directly from Lem. \ref{lem:spinor}. □

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