Intersecting delocalized p-branes

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Abstract

A model considered in the paper generalizes supergravity type model to the case of delocalized membrane sources. A generalization of intersecting p-brane solution with delocalized membranes is presented.

1 Introduction

Membrane theories consider membranes as mapping $x$ from manifold $V$ of dimension $n$ to (pseudo)Riemannian manifold $M$ of dimension $D > n$. The manifold $M$ is interpreted as space-time (“bulk”). The image $x(V)$ is interpreted as membrane. Membrane appears in bulk equation in the form of singular sources localized at $x(V)$. A procedure of regularization can be considered as replacement of common membrane, which has zero thickness by some sort of thick membrane, which corresponds to continuous distribution of infinitely light membranes.

A covariant method of regularization of closed membrane theories (“$M \to F$-approach”) was presented in the papers. ($M \to F$)-approach replaces mapping $x : V \to M$ by mapping $\varphi : M \to F$, where $F$ is a manifold of dimension $D - n$. If for some point $\phi \in F$ inverse image $\varphi^{-1}(\phi)$ is submanifold of dimension $n$, then it has to be considered as infinitely light membrane. The similar ideas were suggested before in papers (see also references in paper).

The generalization of p-brane solution with delocalized membranes was also presented in the papers. A natural generalization of extremal intersecting p-brane solution with delocalized membranes is constructed in the present paper (on intersecting p-brane solutions see and references in these papers).

2 Notation

$D$-dimensional space-time, i.e. smooth (pseudo)Riemannian manifold $M$ with metric $ds^2 = g_{MN} dx^M dx^N$, $M, N, \ldots = 0, \ldots, D - 1$ is considered. In further calculations $M$ is considered to be a finite region in $\mathbb{R}^D$ with smooth boundary $\partial M$.

For differential forms $A$ and $B$ with components $A_{M_1 \ldots M_q}$ and $B_{N_1 \ldots N_p}$ the following tensor is introduced

$$(A, B)^{(k)}_{M_{k+1} \ldots M_q, N_{k+1} \ldots N_p} = \frac{1}{k!} g^{M_1 N_1} \ldots g^{M_k N_k} A_{M_1 \ldots M_q} B_{N_1 \ldots N_p}.$$
Index \((k)\) indicates the number of indices to contract. If it can not lead to ambiguity, \((k)\) is skipped.

For differential form of power \(q\) it is convenient to introduce two norms \(\|A\|_2 = \Sigma(A, A)^{(q)}\), where \(\Sigma = \pm 1\), and Hodge duality operation \(*A = (\Omega, A)^{(q)}\), where \(\Omega = \sqrt{|g|} d^D X\) is form of volume. Here and below \(g = \det(g_{MN})\), \(\sigma = \text{sgn}(g)\).

### 3 Action and equations of motion

Let us consider the action of the following form

\[
S = \int d^D X \sqrt{|g|} \left( \frac{R}{2\kappa^2} - \frac{1}{2} \|\partial \phi\|^2 - \sum_I \frac{1}{2} \beta_I e^{-\alpha_I \cdot \phi} \|F^{(I)}\|^2 + \sum_a \lambda_{(I,a)} e^{\frac{1}{2} \alpha_{I,a} \cdot \phi} \|J^{(I,a)}\| \Sigma_{(I,a)} \sigma + s_{(I,a)} \lambda_{(I,a)}^2 \phi_{(I,a)} J^{(I,a)} \right) .
\]  

(1)

Here \(\phi = (\phi^1, \ldots, \phi^n)\) is a set of scalar fields (dilatons), \(\alpha_I \cdot \phi = \sum_{i=1}^n \alpha^i_I \phi^i\), \(F^{(I)} = dA^{(I)}\) are exact \((q_I + 1)\)-forms, \(J^{(I,a)} = d\phi_{(I,a)} \wedge \ldots \wedge d\phi_{(I,a)}^n \wedge \Sigma_{(I,a)} = \pm 1\), \(s_{(I,a)} = \pm 1\), \(\alpha^I_I, \lambda_{(I,a)}^2\) and \(h_{(I,a)}\) are real constants.

By variation of action (1) over metric \(g_{MN}\), scalar fields \(\phi\), \(q_I\)-forms \(A^{(I)}\) and membrane potentials \(\phi_{(I,a)}\) one can find equations of motion. Einstein equations, found by variation of metric have the form

\[
R_{MN} - \frac{1}{2} R g_{MN} = \kappa^2 T_{MN},
\]

(2)

where energy-momentum tensor is defined by the following formulae

\[
T_{MN}^{(\phi)} = \sum_I (T_{MN}^{(I)} + \sum_a T_{MN}^{(I,a)}),
\]

(3)

\[
T_{MN}^{(\phi)} = \partial_M \phi \cdot \partial_N \phi - \frac{1}{2} \|\partial \phi\|^2 g_{MN},
\]

(4)

\[
T_{MN}^{(I)} = \beta_I e^{-\alpha_I \cdot \phi} \left( (F^{(I)}, F^{(I)})_{MN} - \frac{1}{2} \|F^{(I)}\|^2 g_{MN} \right),
\]

(5)

\[
T_{MN}^{(I,a)} = -\lambda_{(I,a)} e^{\frac{1}{2} \alpha_{I,a} \cdot \phi} \|J^{(I,a)}\| \Sigma_{(I,a)} \sigma \left( g_{MN} - \frac{(J^{(I,a)}, J^{(I,a)})_{MN}}{\|J^{(I,a)}\|^2} \right).
\]

(6)

Equations, found by variation of forms \(A^{(I)}\), have the form

\[
\beta \delta \left( e^{-\alpha_I \cdot \phi} F^{(I)} \right) = \sigma (-1)^{q_I + 1} Q^{(I)},
\]

(7)

where the source \(Q^{(I)}\) defined by

\[
Q^{(I)} = - \sum_a \lambda_{(I,a)} s_{(I,a)} \frac{1}{h_{(I,a)}} J^{(I,a)}.
\]

(8)

Equations, found by variation of scalar fields, have the form

\[
\Box \phi = - \sum_I (Q_{\phi I} + Q_{\phi I,a}),
\]

(9)
where the sources are defined by
\[
Q_{\phi I} = \frac{1}{2} \alpha_I \beta_I e^{-\alpha_I \phi} \| F^{(I)} \|^2, \\
Q_{\phi I s} = -\frac{\alpha_I}{2} e^{\frac{1}{2} \alpha_I \phi} \sum_a \lambda_{(I,a)} \| J^{(I,a)} \|_{\Sigma(I,a)\sigma}.
\]
Equations, found by variation of membrane potentials, have the form
\[
\left( \delta \left[ \Sigma(I,a)\sigma \frac{e^{-\alpha_I \phi} J(I,a)}{\| J(I,a) \|_{\Sigma(I,a)\sigma}} \right] + (-1)^{q_I(D-q_I)} \frac{s(I,a) \ast A(I,a)}{h(I,a)} \right) d\varphi^{\alpha_I} \wedge \ldots \wedge d\varphi^{\alpha_D-q_I-1} = 0.
\]

4 Delocalized p-brane solutions

The goal of the paper is to present special solutions of equations (2), (7), (9) and (12).

One can introduce a set of orthonormal projectors \( P^{(I,a)}_{M} \), \( \bar{P}^{(I,a)}_{M} = \delta^M_N - P^{(I,a)}_{M} \) numerated by index \((I,a)\). In the given coordinate system
\[
\text{tr} \bar{P}^{(I,a)} = \bar{P}^{(I,a)}_M = q_I; \quad P^{(I,a)}_N = 0, \quad M \neq N,
\]
i.e. \( \bar{P}^{(I,a)}_N = \Delta_{(I,a)M} \delta_{MN} \) (there is no summation over \( M \)), \( \Delta_{(I,a)M} \in \{0, 1\} \).

Let us define fields by the following equations
\[
A^{(I)} = \sum_a h^{(I,a)} \omega^{(I,a)},
\]
where
\[
\omega^{(I,a)} = e^{W^{(I,a)}} \bigwedge_{\Delta_{(I,a)M}=1} dX^M,
\]
h^{(I,a)} are constants, \( H^{(I,a)} \) are smooth positive functions (parametrizing functions), \( W^{(I,a)} \) are smooth functions, which satisfy conditions \( (P^{(I,a)} \partial)_M W^{(I,a)} = 0 \).

\[
ds^2 = \left( \prod_{I,a} H^{(I,a)} \right)^{\frac{1}{2}} \sum_{(I,a)} \beta_I h^{(I,a)} \alpha_I \ln H^{(I,a)} \right) \eta_{MN} dX^M dX^N,
\]
where \( \eta_{MN} = \text{diag}(\pm 1, \ldots, \pm 1) \).

\[
\phi = \frac{1}{2} \sum_{I,a} \Sigma(I,a)\beta_I h^{2(I,a)} \alpha_I \ln H^{(I,a)}.
\]

\[
J^{(I,a)} = J^{(I,a)} \ast \omega^{(I,a)}.
\]

Theorem on p-brane solutions with sources. If fields defined by equations (14), (16), (17), (18), satisfy the following conditions
\[
|J^{(I,a)}| = \frac{\Sigma(I,a)\beta_I h^{2(I,a)} \alpha_I}{H^{2(I,a)}} e^{-\frac{1}{2} \alpha_I \phi} \frac{\Box H^{(I,a)}}{H^{2(I,a)}},
\]
\[
\text{sgn} \| \omega^{(I,a)} \|^2 = \Sigma(I,a),
\]
\[
(\bar{P}^{(I,a)} \partial)_M H^{(I,a)} = 0,
\]
\[(P^{(I,a)} \partial)_M H^{(I,a)} (\bar{P}^{(I,a)} \partial)_N \left( \kappa^2 \sum_{J,b} \Sigma_{(J,b)} \beta_J h^2_{(J,b)} \ln H_{(J,b)} \right) 2 \Delta_{(I,a)M} + W_{(I,a)} \) = 0, \tag{22}

\ln H^{(I,a)} = \kappa^2 \sum_{J,b} \Sigma_{(J,b)} \beta_J h^2_{(J,b)} \ln H_{(J,b)} \left[ \frac{q_I q_J}{D-2} - \text{tr}(\bar{P}^{(I,a)} \bar{P}^{(J,b)}) - \frac{\alpha_I \cdot \alpha_J}{4 \kappa^2} \right] + W_{(I,a)}, \tag{23}

\text{tr}(\bar{P}^{(I,b)} \bar{P}^{(I,c)}) \neq q_I - 1, \tag{24}

s_{(I,a)} = -\Sigma_{(I,a)} \beta_J h^2_{(J,b)} \ln H_{(J,b)} \left[ q_I - 1 \right], \tag{25}

Then fields (14), (16), (17), (18) satisfy equations of motion (2), (7), (9), (12).

If \( \Box H^{(I,a)} = 0 \) for all \( (I,a) \), then membrane fields \( J^{(I,a)} \) vanish and the solution is a regular intersecting extremal p-brane solution (see [5, 6] and references in these papers for examples). Vice-versa, one can introduce non-trivial membranes fields to intersecting extremal electric-type p-brane solution by setting \( \Box H^{(I,a)} \neq 0 \).

5 Examples

As a simplest example, let us consider solution of Einstein-Maxwell equations in the presence dust cloud (with zero pressure) with charge density equal to mass density.

The action has the following form

\[ S = \int d^4X \sqrt{-g} \left( \frac{R}{2} - \frac{\|F\|^2}{2} - \|J\| - \frac{1}{\sqrt{2}} (\ast J, A) \right), \]

where \( F = dA \) is electromagnetic field, \( A \) is 4-vector potential, \( J = d\varphi^1 \wedge d\varphi^2 \wedge d\varphi^3 \).

By variation over fields \( g_{MN}, A_M, \varphi^\alpha \) one finds the following equations of motion

\[ R_{MN} - \frac{1}{2} R g_{MN} = (F,F)_{MN} + \left(\frac{J,J}{\|J\|}\right)_{MN} - \left( \frac{1}{2} \|F\|^2 + \|J\| \right) g_{MN}, \]

\[ \left( \delta \left( \frac{J}{\|J\|} - \frac{\ast A}{\sqrt{2}} \right), d\varphi^\alpha \wedge d\varphi^\beta \right) = 0, \quad \delta F = - \frac{\ast J}{\sqrt{2}} . \]

The fields defined by the following equations solve the equations of motion

\[ ds^2 = -H^{-2} dX^0 dX^0 + H^2 \delta_{\alpha\beta} dX^\alpha dX^\beta, \]

\[ A = \frac{\sqrt{2}}{H} dX^0, \quad J = -2 \triangle H dX^1 \wedge dX^2 \wedge dX^3. \]

Here \( H \) is smooth positive function of space coordinates \( X^\alpha, \alpha, \beta = 1, 2, 3, \triangle = \partial_1^2 + \partial_2^2 + \partial_3^2 \). Calculating \( \|J\| \) one has to choose a branch of square root, which gives \( \|J\| \triangle H < 0 \).

The following example describes intersecting delocalized 2-branes in 11d supergravity.

The action has the following form

\[ S = \int d^{11}X \sqrt{-g} \left( \frac{R}{2} - \frac{\|F\|^2}{2} - \|J^{(1)}\| - \|J^{(2)}\| + \sqrt{2} \left( \ast J^{(1)} + \ast J^{(2)}, A \right) \right) , \]
where $F = dA$, $A$ is form of power 3, $J^{(a)} = d\varphi^{a}_1 \wedge \ldots \wedge d\varphi^8_a$, $a = 1, 2$.

By variation over fields $g_{MN}$, $A_{MNK}$, $\varphi^a_\alpha$ one finds the following equations of motion

$$R_{MN} - \frac{1}{2} R g_{MN} = (F, F)_{MN} + \left( \frac{1}{\|J^{(1)}\|} \frac{(J^{(1)}, J^{(1)})_{MN}}{\|J^{(1)}\|} + \left( \frac{1}{2} \|F\|^2 + \|J^{(1)}\| + \|J^{(2)}\| \right) g_{MN},
$$

$$\left( \delta \left( \frac{J^{(a)}}{\|J^{(a)}\|} - \sqrt{2} \ast A \right), d\varphi^{a}_1 \wedge \ldots \wedge d\varphi^8_a \right) = 0, \quad \delta F = -\sqrt{2}(\ast J^{(1)} + \ast J^{(2)}).$$

The fields defined by the following equations solve the equations of motion

$$ds^2 = -H_1^{-2/3}H_2^{-2/3}dX^0dX^0 + H_1^{-2/3}H_2^{1/3}(dX^1dX^1 + dX^2dX^2) + H_1^{1/3}H_2^{-2/3}(dX^3dX^3 + dX^4dX^4) + H_1^{1/3}H_2^{1/3} \sum_{\alpha=5}^{10} dX^\alpha dX^\alpha,$$

$$A = \frac{1}{\sqrt{2}H_1} dX^0 \wedge dX^1 \wedge dX^2 + \frac{1}{\sqrt{2}H_2} dX^0 \wedge dX^3 \wedge dX^4,$$

$$J^{(1)} = -\frac{1}{2} \triangle H_1 dX^3 \wedge \ldots \wedge dX^{10},$$

$$J^{(2)} = -\frac{1}{2} \triangle H_2 dX^1 \wedge dX^2 \wedge dX^5 \wedge dX^6 \wedge \ldots \wedge dX^{10}.$$

Here $H_a$ are smooth positive function of space coordinates $X^\alpha$, $\alpha, \beta = 5, 6, 7, 8, 9, 10$, $\triangle = \sum_{\alpha=5}^{10} \partial^2_\alpha$. Calculating $\|J^{(a)}\|$ one has to choose a branch of square root, which gives $\|J^{(a)}\|\triangle H_a < 0$.

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