JACOB'S LADDERS AND PROPERTIES OF COMPLETE ADDITIVITY AND COMPLETE MULTIPLICATIVITY IN THE SET OF REVERSE ITERATED INTEGRALS (ENERGIES)

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Abstract. New class of integral identities concerning constraints on behavior of the Riemann’s zeta function on the critical line is introduced in this paper. Namely, we have obtained new kind of $\sigma$-additivity and $\sigma$-multiplicativity in the class of reverse iterated integrals (energies).

1. Introduction

1.1. Let us remind we have proved the following theorem (see [3], (2.1) – (2.7)):
for every $L^2$-orthogonal system

$$\{f_n(t)\}_{n=1}^{\infty}, \ t \in [0, 2l], \ l = o\left(\frac{T}{\ln T}\right), \ T \to \infty$$

there is a continuum set of $L^2$-orthogonal systems

$$\{F_n(t; T, k, l)\}_{n=1}^{\infty} = \left\{ f_n(\varphi_1^k(t) - T) \prod_{r=0}^{k-1} |\tilde{Z}[\varphi_r^k(t)]| \right\}, \ t \in [T, T + 2l],$$

where

$$\varphi_1 \left[\frac{k}{T, T + 2l}\right] = \left[\frac{k-1}{T, T + 2l}\right], \ k = 1, \ldots, k_0,$$

$$\left[\frac{0}{T, T + 2l}\right] = \left[\frac{0}{T, T + 2l}\right], \ T \to \infty,$$

i. e. the following formula is valid

$$\int_{T}^{T+2l} f_m(\varphi_1^k(t) - T)f_n(\varphi_1^k(t) - T) \prod_{r=0}^{k-1} \tilde{Z}[\varphi_r^k(t)]dt =$$

$$= \left\{ \begin{array}{ll}
0, & m \neq n, \\
A_n, & m = n,
\end{array} \right.$$  

$$A_n = \int_{0}^{2l} f_n^2(t)dt.$$  

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Next, we have (see [2], (9.1), (9.2)) that
\[ \tilde{Z}(t) = \frac{d\varphi_1(t)}{dt} = \frac{Z^2(t)}{2\Phi'[\varphi(t)]} = \left| \zeta \left( \frac{1}{2} + it \right) \right|^2, \]
\[ \varphi_1(t) = \frac{1}{2} \varphi(t), \]
\[ \omega(t) = \left\{ 1 + O \left( \frac{\ln \ln t}{\ln t} \right) \right\} \ln t, \]
and (see [3], pp. 77, 329)
\[ Z(t) = e^{it} \zeta \left( \frac{1}{2} + it \right), \]
\[ \varphi(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma \left( \frac{1}{4} + \frac{it}{2} \right). \]

Hence, for the classical Fourier's orthogonal system, for example,
\[ \left\{ 1, \cos \frac{\pi t}{l}, \sin \frac{\pi t}{l}, \ldots, \cos \frac{n\pi t}{l}, \sin \frac{n\pi t}{l}, \ldots \right\}, \]
\[ t \in [0, 2l], \]
we have the following continuum set of orthogonal systems according to (1.1)
\[ \left\{ \prod_{r=0}^{k-1} \left| \frac{\zeta \left( \frac{1}{2} + i\varphi_1^r(t) \right)}{\sqrt{\omega[\varphi_1^r(t)]}} \right|, \ldots, \right. \]
\[ \left. \prod_{r=0}^{k-1} \left| \frac{\zeta \left( \frac{1}{2} + i\varphi_1^r(t) \right)}{\sqrt{\omega[\varphi_1^r(t)]}} \right| \cos \left( \frac{\pi}{l} n(\varphi_1^r(t) - T) \right), \right. \]
\[ \left. \prod_{r=0}^{k-1} \left| \frac{\zeta \left( \frac{1}{2} + i\varphi_1^r(t) \right)}{\sqrt{\omega[\varphi_1^r(t)]}} \right| \sin \left( \frac{\pi}{l} n(\varphi_1^r(t) - T) \right), \ldots \right\}, \]
\[ t \in \left[ kT, kT + 2l \right], \quad k = 1, \ldots, k_0, \]
\[ T \geq T[\varphi_1]. \]

1.2. We have already noticed in our paper [3] that the formula (1.1) can serve as a resource for new integral identities in the theory of the Riemann zeta-function.

For example, in the case of the first function of the Fourier's system (1.3)
\[ f_1(t) = 1, \]
we have the following formula (see (1.1))
\[ \int_{T}^{T+2l} \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] \, dt = 2l, \quad k = 1, \ldots, k_0, \quad T \to \infty. \]

In this paper we shall interpret the formula (1.4) as new kind of unit operator as well as new kind of parametric integral. Consequently, we obtain new class of integral identities for the function
\[ Z^2(t) = \left| \zeta \left( \frac{1}{2} + it \right) \right|^2, \]
i.e. (see (1.2)) the constraints on a behavior of weakly-modulated function
\[ \left| \zeta \left( \frac{1}{2} + it \right) \right|^2. \]

Namely, we obtain new kind of \( \sigma \)-additivity and also \( \sigma \)-multiplicativity in the class of reversely iterated integrals (energies).

2. Theorem

2.1. Based on our formula (1.3) we give the following

**Theorem.** Let us denote by \( G(S) \) the class of functions
\[ g = g(u_1, \ldots, u_n), \]
\( (u_1, \ldots, u_n) \in S \subset \mathbb{R}^n \)
such that
\[ g \geq 0, \quad g = o \left( \frac{T}{\ln T} \right), \quad T \to \infty. \]

Then we have the following formula
\[ \forall g \in G(S) : \]
\[ \int_{T}^{T+g} \prod_{r=0}^{k-1} Z^2[\varphi^r_i(t)] dt = g, \quad k = 1, \ldots, k_0 \]
for every fixed \( k_0 \in \mathbb{N} \) and for every sufficiently big \( T > 0 \).

Let us remind the following properties connected with complicated structure of the formula (2.2). First of all we have (comp. [3], (2.5) – (2.7)) the following
\[ g = o \left( \frac{T}{\ln T} \right) \Rightarrow \]
\[ \left| [T, T + g] \right| = \frac{1}{k} \sum_{r=0}^{k-1} \left| [T, T + g] \right| \sim (1 - c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T}, \]
\[ \left| [T + g, T] \right| \sim \left| [T, T + g] \right| \Rightarrow \]
\[ \left| [T + g, T] \right| \sim \left| \frac{1}{k} \sum_{r=0}^{k-1} \left[ T, T + g \right] \right| \sim \cdots \sim \left[ T, T + g \right] \sim \cdots \]
where \( \pi(T) \) stands for the prime-counting function and \( c \) is the Euler’s constant.

**Remark 1.** Asymptotic behavior of the following disconnected set (see (2.3) – (2.5), comp. [3], (2.9))
\[ \Delta(T, k, g) = \bigcup_{r=0}^{k} [T, T + g] \]
is as follows: if \( T \to \infty \) then the components of the set (2.6) recede unboundedly each from other and all together are receding to infinity. Hence, if \( T \to \infty \), the set (2.6) behaves as one dimensional Friedmann-Hubble expanding universe.
Remark 2. For the arguments of functions in (2.2) we have: if
\[ t \in [kT, kT + g], \quad k = 1, \ldots, k_0 \]
then
\[ \phi^r_1(t) \in \left[ \frac{k-r}{k}, \frac{k-r}{k} T + g \right], \quad r = 0, 1, \ldots, k, \]
(comp. [3], (2.10)).

2.2. Let us notice the following about the interpretation of the formula (2.2).

Remark 3. New type of operator
\[ (2.7) \quad \hat{H}O = \int_{T}^{kT+O} \prod_{r=0}^{k-1} Z^2[\phi^r_1(t)] dt, \quad k = 1, \ldots, k_0 \]
is defined by the our formula (2.2). This operator acts via the upper limit of the reversely iterated integral as follows
\[ \hat{H}g = \left( \int_{T}^{kT+O} \prod_{r=0}^{k-1} Z^2[\phi^r_1(t)] dt \right) g = \int_{T}^{kT+g} \prod_{r=0}^{k-1} Z^2[\phi^r_1(t)] dt. \]
We have, of course, that (see (2.2))
\[ \hat{H}g = g, \quad \forall g \in G(S), \]
i.e. \( \hat{H} \) is the unit operator.

Remark 4. A kind of closed analytic cycle on \( G(S) \) is defined by our formula (2.2). Namely, we have
\[ g \rightarrow [T, T + g] \rightarrow \left[ \frac{k}{k}, \frac{k}{k} T + g \right] \rightarrow \int_{T}^{kT+g} \prod_{r=0}^{k-1} Z^2[\phi^r_1(t)] dt = g, \quad k = 1, \ldots, k_0. \]

3. Complete additivity of the reversely iterated integrals (energies). Comparison with the \( \sigma \)-additivity

3.1. First of all, the following corollary holds true.

Corollary 1. Let
\[ 0 \leq g_l(u_1, \ldots, u_n) = g(l), \quad (u_1, \ldots, u_n) \in S \]
and
\[ g = \sum_{l=1}^{\infty} g_l \in G(S). \]
Then

\[
\int_{k}^{k+T} \sum_{r=0}^{k} g(l) \prod_{r=0}^{k-l-1} Z^2[\varphi_1(t)] \, dt = \\
= \sum_{l=1}^{\infty} \int_{k+T}^{k+T+g(l)} \prod_{r=0}^{k-l} Z^2[\varphi_1(t)] \, dt,
\]

(3.3)

\[k, k(l) = 1, \ldots, k_0, \quad T \to \infty.\]

**Remark 5.** We shall call the property (3.2) as the complete additivity of the reversely iterated integrals (energies).

3.2. Next, we give the following

**Example 1.** If

\[k = 17, \quad l = 1, 2\]

then we obtain from (3.2) by using the mean-value theorem (comp. [3], (4.3) – (4.5)) that

\[
\int_{k}^{17} \sum_{r=0}^{16} \left| \frac{1}{2} + i \varphi_1(t) \right|^2 \, dt \sim \\
\sim \ln 16 \, T \cdot \int_{1}^{17} \left| \frac{1}{2} + it \right|^2 \, dt + \\
+ \ln 10 \, T \cdot \int_{7}^{17} \prod_{r=0}^{6} \left| \frac{1}{2} + i \varphi_1(t) \right|^2 \, dt, \quad T \to \infty
\]

together with other formulae of the complete finite set defined by the condition (3.3).

3.3. Furthermore, we define following planar figures (comp. [4], (3.1) – (3.3)):

\[
P_k(T, g) = \left\{ (t, y) : t \in [T, T + \sum_{l} g(l)], \ y \in [0, \prod_{r=0}^{k-1} Z^2[\varphi_1(t)]] \right\},
\]

(3.5)

\[k = 1, \ldots, k_0, \]

\[
P_{k(l)}(T, g(l)) = \left\{ (t, y) : t \in [T, T + g(l)], \ y \in [0, \prod_{r=0}^{k(l)-1} Z^2[\varphi_1(t)]] \right\},
\]

(3.6)

\[k = 1, \ldots, k_0, \quad l = 1, 2, \ldots.\]

**Remark 6.** Consequently, we have (see [5.2]) for measures of these planar figures the following

\[
m\{P_k\} = \sum_{l=1}^{\infty} m\{P_{k(l)}\},
\]

(3.7)
where
\[ \{ l_1 \neq l_2 & \text{& } k(l_1) = k(l_2) \} \Rightarrow P_{k(l_1)}^l \bigcap P_{k(l_2)}^l = \emptyset. \]

Of course, the sequence
\[ \{ k(l) \}_{l=1}^{\infty} \]
contains an infinite set for which
\[ k(l_1) = k(l_2) \]
(see (3.3); \( k_0 \) being fixed).

3.4. Let us remind that the Lebesgue measure is countable additive, or \( \sigma \)-additive, i.e. if
\[ A_1, A_2, \ldots, A_n, \ldots \]
are measurable sets and if they are pairwise disjoint
\[ p \neq q \Rightarrow A_p \bigcap A_q = \emptyset, \]
then
\[ m\{A\} = m\left( \bigcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} m\{A_k\}. \]

Remark 7. Hence, we see (comp. (3.8), (3.9)) that the complete additivity in our using differs from the \( \sigma \)-additivity. In our case (3.7) the Jordan’s measure would be sufficient.

4. COMPLETE MULTIPLICATIVITY OF THE REVERSELY ITERATED INTEGRALS (ENERGIES)

4.1. Next, the following corollary holds true.

**Corollary 2.** Let
\[ g_l > 0; \quad g_l, \prod_{l=1}^{\infty} g_l \in G(S), \]
then
\[ \int_{0}^{\infty} T+g(l) k^{-1} \prod_{r=0}^{k-1} \tilde{Z}^2 [\varphi_1^r(t)] dt = \prod_{l=1}^{\infty} \int_{0}^{T+g(l) k^{-1} \prod_{r=0}^{k-1} \tilde{Z}^2 [\varphi_1^r(t)] } dt, \]
\[ k, k(l) = 1, \ldots, k_0, \quad T \rightarrow \infty. \]

Remark 8. We shall call the property (4.1) as the complete multiplicativity of the reversely iterated integrals (energies).

\[ ^1 \text{Of course} \]
\[ g(l_1)g(l_2) \in G(S) \]
does not imply
\[ g(l_1) \in G(S) \text{ & } g(l_2) \in G(S). \]
Remark 9. We obtain from (3.5), (3.6) and (4.1) that

\[ m \{ Q_k \} = \prod_{l=1}^{\infty} m \{ P_{k(l)}^l \}, \]

where in (3.5)

\[ P_k \left( \sum \{ g(l) \} \right) \rightarrow \prod \{ g(l) \} = Q_k, \]

and the property (3.8) holds true.

Remark 10. Formulae (3.2), (4.1) (i.e. (3.7), (4.2)) are not accessible by the current methods in the theory of the Riemann zeta-function.

4.2. Now we give two examples.

Example 2. If

\[ k = 17, \quad l = 1, 2 \]

then we obtain from (4.1) by the usual way (comp. Example 1) that

\[ \int \frac{1}{T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \ dt \sim \]

\[ \sim \frac{1}{\ln^{23} T} \int_{T}^{17} \prod_{r=0}^{16} \sum_{\alpha} \left| \varphi_1^\alpha(t) \right|^2 dt \cdot \int_{T}^{7} \prod_{r=0}^{6} \left| \varphi_1^\alpha(t) \right|^2 dt, \]

\[ T \to \infty. \]

This formula represents the complete finite (for fixed \( T, g(1), g(2) \)) set of formulae defined by the condition (4.3).

Example 3. Canonical arithmetic formula

\[ n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} \in G(S), \quad n \in \mathbb{N}; \quad a_s = a(s) \]

corresponds to the following formula

\[ (\ln T) \sum \left[ k(l) - k \right] \int_{T}^{k} \prod_{r=0}^{k-1} \left| \varphi_1^\alpha r(t) \right|^2 dt \sim \]

\[ \sim \prod_{l=1}^{s} \left\{ \int_{l(l)}^{k(l)} \prod_{r=0}^{k(l)-1} \left| \varphi_1^\alpha r(t) \right|^2 dt \right\}^{\alpha(l)} \quad \text{as} \quad T \to \infty. \]

5. On a set of asymptotically equivalent integrals connected with the length of the Riemann’s curve

5.1. Let us denote the roots of the equations

\[ Z(t) = 0, \quad Z'(t) = 0 \]

by the symbols

\[ \{ \gamma \}, \{ t_0 \}; \quad \gamma \neq t_0 \]

correspondingly.
Remark 11. On the Riemann hypothesis the points of sequences \( \{ \gamma \} \) and \( \{ t_0 \} \) are separated each from other (see [3], Cor. 3), i.e. in this case we have
\[
\gamma' < t_0 < \gamma'',
\]
where \( \gamma', \gamma'' \) are neighbouring points of the sequence \( \{ \gamma \} \). Of course, the value \( Z(t_0) \) is then locally extremal value of the function \( Z(t) \) located at the point \( t = t_0 \).

Next, we have proved (see [5], (1.5)) the following asymptotic formula. On the Riemann hypothesis we have
\[
\int_T^{T+H} \sqrt{1 + \{Z'(t)\}^2} dt = 2 \cdot \sum_{T \leq t_0 \leq T+H} |Z(t_0)| + \Theta H + O\left(\frac{H}{\ln T}\right),
\]
(5.1)
\[\Theta = \Theta(T, H) \in (0, 1), \quad H = T^\epsilon, \quad T \to \infty,\]
for every small and fixed \( \epsilon > 0 \).

Remark 12. Geometric meaning of the formula (5.1) is as follows: the length of the Riemann’s curve
\[
y = Z(t), \quad t \in [T, T+H]
\]
is asymptotically equal to the double of the sum of local maxima of the function
\[|Z(t)|, \quad t \in [T, T+H].\]

5.2. Since we have by Remark 11 that
\[
\sum_{T \leq t_0 \leq T+H} 1 = O(H \ln T),
\]
and (see [9], p. 237)
\[
Z(t) = O \left(\frac{H}{\ln T}\right),
\]
then (comp. (2.1))
\[
\sum_{T \leq t_0 \leq T+H} |Z(t_0)| = O \left(HT^{\frac{\epsilon}{1+\epsilon}} \ln T\right) = O \left(T^{x+\frac{\epsilon}{1+\epsilon}} \ln T\right) = O(T^{2\epsilon}) = o \left(\frac{T}{\ln T}\right).
\]
Consequently, we have the following (see also the formula [5], (A.1))

Corollary 3. On the Riemann hypothesis we have
\[
\int_T^{T+H} \sqrt{1 + \{Z'(t)\}^2} dt = \int_T^{T+H} \sqrt{1 + \{Z'_{\varphi_1}(t)\}^2} dt \sim \int_T^{T+H} \frac{1}{t} \prod_{r=0}^{k-1} Z^2[\varphi_1^r(t)] dt \sim \int_T^{T+H} \frac{1}{t} \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt \sim \sum_{T \leq t_0 \leq T+H} \int_{t_0}^{T+H} \frac{1}{t} \prod_{r=0}^{k(t_0)-1} Z^2[\varphi_1^r(t)] dt, \quad k, k(t_0) = 1, \ldots, k_0, \quad T \to \infty.
\]
(5.2)
Remark 13. The formula \[5.2\] contains asymptotic expressions of irrational integrals in \(Z'\) by rational integrals in \(\tilde{Z}\).

Remark 14. We emphasise the following simple formula

\[
\int_{T}^{T+H} \sqrt{1 + \{Z'(t)\}^2} dt \sim \sum_{T \leq t_0 \leq T+H} \int_{T}^{T+2|Z(t_0)|} \tilde{Z}^2(t) dt, \quad T \to \infty,
\]

as well as the formula

\[
\int_{T}^{T+H} \sqrt{1 + \{Z'(t)\}^2} dt \sim \frac{1}{\ln T} \sum_{T \leq t_0 \leq T+H} \int_{T}^{T+2|Z(t_0)|} \zeta\left(\frac{1}{2} + it\right)^2 dt, \quad T \to \infty.
\]

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