An optimal boundedness result for weak solutions of double phase quasilinear parabolic equations

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Abstract

We obtain local boundedness of weak solutions of double phase quasilinear parabolic equations of the form

\[ u_t - \text{div} \left( |\nabla u|^{p-2} \nabla u + a(x,t)|\nabla u|^{q-2} \nabla u \right) = 0, \]

where, we have imposed the restrictions \( \frac{2N}{N+2} < p < \infty \), \( 0 \leq a(x,t) \leq M \) is measurable and \( q < p \frac{N+1}{N-1} \).

Keywords: boundedness, quasilinear parabolic equations, \((p,q)-growth\)

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1. Introduction

In this paper, we study weak solutions (see Definition 2.1) of

\[ u_t - \text{div} \mathcal{A}(x,t,\nabla u) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega_T), \]

where \( \mathcal{A}(x,t,\nabla u) \) is modelled after the quasilinear operator with \((p,q)\)-growth. More specifically, we assume that the nonlinear structure satisfies the following growth and coercivity conditions:
\((H1):\quad \frac{2N}{N+2} < p \leq q < \frac{p(N+1)}{N-1} < \infty.\)

\((H2):\quad \Lambda_0(|\nabla u|^p + a(x,t)|\nabla u|^q) \leq \langle A(x,t,\nabla u), \nabla u \rangle.\)

\((H3):\quad |A(x,t,\nabla u)| \leq \Lambda_1(|\nabla u|^{p-1} + a(x,t)|\nabla u|^{q-1}).\)

\((H4):\quad 0 \leq a(x,t) \leq M\) for some fixed number \(M\) is a measurable function.

**Remark 1.1.** In the elliptic case, the presence of Lavrentiev phenomenon automatically imposes certain restrictions on the relation between \(p\) and \(q\). Since we are only interested in boundedness of solutions, we refer to [7, 2, 4] for more details regarding this aspect.

In the parabolic situation, since we are interested in adapting the arguments from [7], we assume that the solution a priori has sufficiently integrability as in Definition 2.1 so that we can start with the weak formulation which gives the required energy estimates. This is done so as to avoid dealing with the technicalities that arise due to Lavrentiev phenomenon in the parabolic case. We refer the reader to [3] and references therein for some of these aspects.

The analysis of regularity for quasilinear parabolic equations typically requires a separate analysis of the singular case \((p < 2)\) and the degenerate case \((p > 2)\). In this paper, we use the techniques developed in [1] in order to combine the analysis for the full range \(\frac{2N}{N+2} < p < \infty\). Note that the standard techniques from [5] lead to the expected restriction on \(q\), i.e., \(q < p^\#\), where \(p^\# := \frac{p(N+2)}{N}\) is the parabolic Sobolev exponent. However, in this paper, we follow the ideas developed in [7] and employ Sobolev identity on the sphere to improve the restriction on \(q\) to \(q < \frac{p(N+1)}{N-1}\).

### 1.1. Notations

We begin by collecting the standard notation that will be used throughout the paper:

- We shall denote \(N\) to be the space dimension and the equation (1.1) will be studied locally in \(\mathbb{R}^{N+1}\). A point in \(\mathbb{R}^{N+1}\) will be denoted by \(z = (x,t)\).
- Let \(\Omega\) be a bounded domain in \(\mathbb{R}^N\) of boundary \(\partial \Omega\) and for \(0 < T < \infty\), let \(\Omega_T := \Omega \times (-T,T)\).
- We shall use the notation
  \[Q_{p,\theta} := B_p \times [-\theta, \theta],\]
  where \(B_p\) denotes the ball in \(\mathbb{R}^N\) centered at 0 with radius \(p\).
- We shall further use the notation \(\partial_p Q_{p,\theta} := \partial B_p \times (-\theta, \theta) \cup B_p \times \{-\theta\}\) to denote the parabolic boundary of the cylinder \(Q_{p,\theta}\).
- The maximum of two numbers \(a\) and \(b\) will be denoted by \(a \wedge b := \max(a,b)\).
\begin{itemize}
  \item Integration with respect to either space or time only will be denoted by a single integral $\int$ whereas integration with respect to both space and time will denoted by a double integral $\iint$.
  \item We use the notation $\mathcal{H}^{N-1}$ to denote the standard Hausdorff measure on the sphere contained in $\mathbb{R}^N$.
  \item We denote the parabolic Sobolev exponent on the ball by $p^\# := \frac{N + 2}{N}$ (see (2.1)) and Sobolev exponent on the sphere by $p_* := \frac{N + 1}{N - 1}$.
  \item The notation $a \lesssim b$ is shorthand for $a \leq Cb$ where $C$ is a universal constant which only depends on the dimension $N$, exponents $p, q$, and the numbers $\Lambda_0, \Lambda_1$ and $M$.
\end{itemize}

### 1.2. Main Result

We shall also require the solution $u$ to satisfy the following hypothesis:

\textbf{(H5):} Let $\rho, \theta > 0$ and $B_\rho \subset \Omega$. We assume that the solution satisfies

$$u \in C^0(\Theta, \Gamma) \cap L^1(0, \rho; L^\infty(0, \rho; L^1(S^{N-1}))) \text{ and }$$

$$\sup_{-\Theta < t < \Theta} \int_{B_\rho} |u|^2 \, dx \geq \int_0^\rho \sup_{-\Theta < t < \Theta} \int_{S_ho} |u|^2 \, d\mathcal{H}^{N-1} \, dr.$$  \tag{1.2}

We shall prove the following boundedness theorem for the solution $u$ of (1.1).

\begin{theorem}
Let $2N \frac{N + 2}{N + 1} < p < \infty$, $p \leq q < \frac{p(N + 1)}{N - 1}$ and $\epsilon_0 := \frac{4}{N + 2}$ be given. Assume that hypotheses \textbf{(H1)-\textbf{(H5)}} hold. Furthermore, let $\sigma \in (0, 1)$ and $\rho, \theta \in (0, \infty)$ be given constants, then any non-negative weak solution $u \in L^p(-T, T; W^{1, p}(\Omega)) \cap L^q(-T, T; W^{1, q}(\Omega))$ of (1.1) satisfies

$$\sup_{\sigma \rho, \sigma \theta} u \leq \Gamma \hat{\mathcal{A}} \hat{\mathcal{B}} \left( 1 + C_3^{\max(\beta + \gamma, \kappa)} + C_3^{\max(\beta + \gamma, \kappa)} \right)^\frac{1}{\epsilon_0} b_1^{\frac{N}{N - 1}} \wedge 1.$$

Here, the constants $C_3 = \frac{\Gamma}{\sigma(N + 1)(1 + \kappa)}$ is from (4.7), $\hat{\mathcal{A}}, \hat{\mathcal{B}}$ is from (4.6), $\hat{\beta}, \hat{\gamma}, \kappa$ is from (3.22) and $\alpha, b_1$ is from (4.1). The expression for $\Gamma$ is taken to be

$$\Gamma := \left( \int_{Q_{\rho, \rho}} |u|^{p + \sigma} \, dz + \left( \int_{Q_{\rho, \rho}} |u|^{p + \sigma} \, dz \right) \left( \int_{Q_{\rho, \rho}} |\nabla u|^p \, dz + \frac{1}{|Q_{\rho, \rho}|} \sup_{-\Theta < t < \Theta} \int_{B_\rho} |u|^2 \, dx \right)^\kappa \right)^\frac{1}{\kappa + 1}$$  \tag{1.3}

\begin{remark}
We note that more general multiphase equations of the form

$$u_1 - \text{div} \left( |\nabla u|^{p - 2} \nabla u + \sum_{i=1}^L a_i(x, t) |\nabla u|^{q_i - 2} \nabla u \right) = 0,$$

with $q_i < \frac{N + 1}{N - 1}$ and $0 \leq a_i(x, t) \leq M$ is measurable can also be considered.
\end{remark}
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2. Preliminaries

In this section, we shall collect all the preliminary material required in the subsequent sections. We define the Steklov average as follows: let $h \in (0, 2T)$ be any positive number, then we define

$$
u_h(\cdot, t) := \begin{cases} \int_t^{t+h} u(\cdot, \tau) \, d\tau & t \in (-T, T-h), \\ 0 & \text{else.} \end{cases}$$

We shall first define the notion of weak solutions to (1.1).

**Definition 2.1** (Weak solution). We say that $u \in C^0(\mathbb{R}; L^2_{\text{loc}}(\Omega)) \cap L^p(\mathbb{R}; W_{\text{loc}}^{1,p}(\Omega)) \cap L^q(\mathbb{R}; W_{\text{loc}}^{1,q}(\Omega))$ is a weak solution of (1.1) if, for any $\phi \in C_c^\infty(\mathbb{R})$ and any $t \in (-T, T)$, the following holds:

$$\int_{\Omega \times \{t\}} \left\{ \frac{d|u|_h}{dt} \phi + \langle [A(x, t, \nabla u)]_h, \nabla \phi \rangle \right\} \, dx = 0 \quad \text{for any} \quad 0 < t < T - h.$$

The following energy estimate may be found for the weak solution $u$ of (1.1) by employing the test function $\phi = (u - k)_+ \zeta^q$. The proof is similar to that in [5, Proposition 3.1 from Section II].

**Lemma 2.2.** Let $u \in C^0(\mathbb{R}; L^2_{\text{loc}}(\Omega)) \cap L^p(\mathbb{R}; W_{\text{loc}}^{1,p}(\Omega)) \cap L^q(\mathbb{R}; W_{\text{loc}}^{1,q}(\Omega))$ be a nonnegative, weak solution of (1.1) in the sense of Definition 2.1, then for any $k \in \mathbb{R}$, there exists a constant $C = C(N, p, q, \Lambda_0, \Lambda_1)$ such that

$$\sup_{-\theta < t < \theta} \int_{B_\rho} (u - k)_+^2 \zeta^q \, dx + \iint_{Q_{\rho, \theta}} |\nabla (u - k)_+|^p \zeta^q \, dz \leq C \iint_{Q_{\rho, \theta}} |(u - k)_+|^q |\nabla \zeta|^q \, dz + \iint_{Q_{\rho, \theta}} |(u - k)_+|^p |\nabla \zeta|^p \, dz$$

$$+ \iint_{Q_{\rho, \theta}} |(u - k)_+|^q \zeta^q - 1 |\partial_t \zeta| \, dz.$$ 

Here $\zeta \in C^\infty(Q_{\rho, \theta})$ is a cut-off function such that $\zeta = 0$ on $\partial_\rho Q_{\rho, \theta}$ for all $t \in (-\theta, \theta)$.

2.1. Well known Lemmas

The following parabolic Sobolev embedding holds true [5, Proposition 3.1 from Chapter I].

**Lemma 2.3.** Let $1 < s < \infty$, then for any $v \in V_0^{2,s}(\Omega_T)$, there exists a constant $C = C(N, s)$ such that

$$\iint_{\Omega_T} |v(x, t)|^s \, dz \leq C \left( \sup_{-T < t < T} \int_{\Omega} |v(x, t)|^2 \, dx \right)^{\frac{s}{2}} \left( \iint_{\Omega_T} |\nabla v(x, t)|^s \, dz \right),$$

where $s_\# := s \left( \frac{N}{2} + 2 \right)$ and $V_0^{2,s}(\Omega_T) := L^\infty(\mathbb{R}; L^2(\Omega)) \cap L^s(\mathbb{R}; W_0^{1,s}(\Omega))$.

Let us now recall the Sobolev embedding on the sphere $S_r^{N-1} := \partial B_r(0)$ which is a special case of the more general Sobolev embedding on compact manifolds, see [6, Theorem 2.6], see also [7, Equation (28)].
Lemma 2.4. For any $s \in [1, N-1)$, $r > 0$ and $v \in W^{1,s}(S_r^{N-1})$, there exists a constant $C = C(N, s)$ such that
\[
\left( \int_{S_r^{N-1}} |v(x)|^s \, d\mathcal{H}^{N-1} \right)^{\frac{1}{s}} \leq C \left( \int_{S_r^{N-1}} \nabla^T v(x)^s \, d\mathcal{H}^{N-1} + \frac{1}{r^s} \int_{S_r^{N-1}} |v(x)|^s \, d\mathcal{H}^{N-1} \right)^{\frac{1}{s}},
\]
where $\nabla^T$ denotes the tangential gradient on the sphere $S_r^{N-1}$ and $s_* := \frac{N-1}{N-1-s}$.

In order to apply Sobolev identity on the sphere to energy estimate (Lemma 2.2), a suitable test function is chosen in the following lemma which is from [7, Lemma 2.1].

Lemma 2.5. Let $N \geq 2$ be fixed, then for given $0 < \rho_1 < \rho_2 < \infty$ and any $v \in L^1(B_{\rho_2})$, let us consider the following minimization problem for some fixed $s > 1$:
\[
I(\rho_1, \rho_2, v) := \inf \left\{ \int_{B_{\rho_2}} |v|\nabla \eta|^s \, dx : \eta(x) \in C_0^1(B_{\rho_2}), 0 \leq \eta \leq 1, \eta \equiv 1 \text{ in } B_{\rho_1} \right\}.
\]
Denoting $S_r := \{x \in \mathbb{R}^N : |x| = r\}$, the following conclusion holds for every $\delta > 0$:
\[
I(\rho_1, \rho_2, v) \leq (\rho_2 - \rho_1)^{-(s-1+\frac{1}{s})} \left( \int_{\hat{r}_2}^{\rho_2} \left( \int_{S_r} |v| \, d\mathcal{H}^{N-1} \right)^{\delta} \, dr \right)^{\frac{1}{\delta}}.
\]

We now recall the following well known lemma concerning the geometric convergence of sequence of numbers (see [5, Lemma 4.1 from Section I] for the details):

Lemma 2.6. Let $\{Y_n\}$, $n = 0, 1, 2, \ldots$, be a sequence of positive number, satisfying the recursive inequalities
\[
Y_{n+1} \leq Cb^n Y_n^{1+\alpha},
\]
where $C > 1$, $b > 1$, and $\alpha > 0$ are given numbers. If
\[
Y_0 \leq C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha}},
\]
then $\{Y_n\}$ converges to zero as $n \to \infty$.

3. Local Iterative Estimates

Let $\sigma \in (0,1)$ and $\rho, \theta \in (0, \infty)$ be fixed numbers, then for $i \in \mathbb{N}$, we denote
\[
\rho_i \equiv \sigma \rho + \frac{1 - \sigma}{2^i} \rho \quad \text{and} \quad \theta_i \equiv \sigma \theta + \frac{1 - \sigma}{2^i} \theta.
\]
Corresponding to these $\rho_i$ and $\theta_i$, we define a sequence of nested cylinders
\[
Q_i := Q_{\rho_i, \theta_i} = B_i \times I_i \quad \text{with} \quad Q_0 := Q_{\rho, \theta} \quad \text{and} \quad Q_\infty := Q_{\rho, \sigma \theta}.
\]
Here, $B_i$ is shorthand for $B_{\rho_i}$ and $I_i$ denotes the interval $[-\theta_i, \theta_i]$. We also denote $\tilde{\rho}_i := \frac{\rho_i + \rho_{i+1}}{2}$ and $\tilde{\theta}_i := \frac{\theta_i + \theta_{i+1}}{2}$ which gives a sequence of intermediate cylinders satisfying $Q_{i+1} \subset \tilde{Q}_i := Q_{\tilde{\rho}_i, \tilde{\theta}_i} \subset Q_i$.

Corresponding to these cylinders, consider the following sequence of cut-off functions $\{\zeta_i\} = \eta_i(x)\xi_i(t)$ with $\zeta_i \in C_c^\infty(Q_i)$ satisfying $\zeta_i \equiv 1$ on $\tilde{Q}_i$ and $\zeta_i = 0$ on $\partial B Q_i$. Further, we choose the time derivative of the cut-off
functions to satisfy
\[ |\partial_t \xi_i| \leq \frac{2^{i+2}}{(1 - \sigma)\sigma}. \]  \tag{3.2}

Recalling \( p^\# := p \frac{N + 2}{N} \) which is the Sobolev exponent in Lemma 2.3, let \( \varepsilon_0 \) be a positive constant (to be eventually chosen) satisfying
\[ p^\# > p + \varepsilon_0 \quad \text{and} \quad p + \varepsilon_0 > 2. \]  \tag{3.3}

For some fixed \( k > 0 \) (a constant to be chosen later) and any \( i \in \mathbb{N} \), we set
\[ k_i := k - \frac{k}{2^i} \quad \text{with} \quad k_0 = 0 \quad \text{and} \quad k_\infty = k. \]  \tag{3.4}

Denoting the level sets
\[ A_{i+1} := \{ z \in Q_{i+1} : u(z) > k_{i+1} \}, \]
we see that the following lemma holds true (see [5, equation (7.2) of Chapter V] for the proof):

**Lemma 3.1.** For any \( s > 0 \), we have
\[ |A_{i+1}| \leq \frac{2^{i+1}}{k^s} \int_{Q_i} (u - k_i)^s_+ dz. \]  \tag{3.5}

3.1. Estimating the terms on the right hand side of Lemma 2.2

We shall apply Lemma 3.1 to the cylinders \( Q_i \) defined in (3.1) with \( k_i \) defined in (3.4) in order to estimate all the terms on the right hand side of the energy estimate in Lemma 2.2.

**Estimate for the first term on right hand side of Lemma 2.2:** For a specific choice of \( \eta_i(x) \) satisfying \( |\nabla \eta_i| \leq \frac{2^{i+1}}{(1 - \sigma)\rho} \), after applying Hölder’s inequality along with Lemma 3.1, we get
\[
\iint_{Q_i} (u - k_{i+1})^p_+ |\nabla \zeta_i|^p dz \leq \frac{2^{p(i+1)}}{(1 - \sigma)p\rho^p} \left( \int_{Q_i} (u - k_{i+1})^{p + \varepsilon_0} dz \right)^{\frac{p}{p + \varepsilon_0}} \left( \int_{Q_i} (u - k_{i+1})^s_+ dz \right)^{\frac{s}{p + \varepsilon_0}} \tag{3.5}
\leq \frac{2^{p(i+1)}}{(1 - \sigma)p\rho^p} \left( \int_{Q_i} (u - k_{i+1})^{p + \varepsilon_0} dz \right)^{\frac{p}{p + \varepsilon_0}} \left( \frac{2^{(p + \varepsilon_0)(i+1)}}{k^{p + \varepsilon_0}} \int_{Q_i} (u - k_{i+1})^{p + \varepsilon_0} dz \right)^{\frac{k^{p + \varepsilon_0}}{p + \varepsilon_0}} \tag{3.6}
\]

**Estimate for the second term on right hand side of Lemma 2.2:** We have the following sequence of inequalities:
\[
\iint_{Q_i} (u - k_{i+1})^2_+ |\zeta_i|^{q - 1} |\partial_t \zeta_i| dz \overset{(a)}{\leq} \frac{2^{i+2}}{(1 - \sigma)\sigma} \left( \int_{Q_i} (u - k_{i+1})^{p + \varepsilon_0} dz \right)^{\frac{p + \varepsilon_0}{p + \varepsilon_0}} |A_{i+1}|^{\frac{p + \varepsilon_0 - 2}{p + \varepsilon_0}} \tag{3.7}
\leq \frac{2^{i+2}}{(1 - \sigma)\sigma} \left( \frac{2^{(p + \varepsilon_0)(i+1)}}{k^{p + \varepsilon_0 - 2}} \int_{Q_i} (u - k_{i+1})^{p + \varepsilon_0} dz \right)^{\frac{k^{p + \varepsilon_0 - 2}}{p + \varepsilon_0}} \int_{Q_i} (u - k_i)^{p + \varepsilon_0} dz,
\]

where to obtain (a), we made use of (3.2) along with Hölder’s inequality.
Estimate for the third term on right hand side of Lemma 2.2: In order to estimate this term, we follow the strategy from [7] as follows. Denoting

\[ q_* := \frac{N - 1}{N + 1} \quad \text{and} \quad \delta := \frac{N - 1}{q_* + N - 1}, \]

we get

\[ \int_{Q_i} (u - k_{i+1})^q \left| \nabla \zeta \right|^q \, dz \lesssim (\rho_i - \tilde{\rho}_i)^{-\frac{(q-1+\frac{2}{\delta})}{\delta}} \left( \int_{\tilde{\rho}_i}^{\rho_i} \left( \int_{-\theta_i}^{\theta_i} \int_{S_r} (u - k_{i+1})_{+}^q \, d\mathcal{H}^{N-1} \right)^{\frac{\delta}{\delta - 1}} \, dr \right)^{\frac{\delta - 1}{\delta}}. \quad (3.8) \]

Denoting \( q_0 = \frac{q_*(N - 1)}{N - 1 - q_*} \), we now estimate the integral on the right hand side of (3.8) by applying Hölder’s inequality to get

\[ \int_{\tilde{\rho}_i}^{\rho_i} \left( \int_{-\theta_i}^{\theta_i} \int_{S_r} (u - k_{i+1})_{+}^q \, d\mathcal{H}^{N-1} \right)^{\frac{\delta}{\delta - 1}} \, dr \leq \frac{1}{\min\{\sigma, \rho, 1\}^{\frac{\delta q_0}{\delta - 1}}} \int_{\tilde{\rho}_i}^{\rho_i} \left[ \left( \sup_{-\theta_i < t < \theta_i} \int_{S_r} (u - k_{i+1})_{+}^2 \, d\mathcal{H}^{N-1} \right)^{\frac{\delta q_0}{\delta - q_0}} \right]^{\frac{\delta - 1}{\delta}} \quad (3.9) \]

From Lemma 2.4, we see that

\[ \left( \int_{S_r} (u - k_{i+1})_{+}^{q_0} \, d\mathcal{H}^{N-1} \right)^{\frac{\delta q_0}{\delta - q_0}} \leq \int_{S_r} \left| \nabla (u - k_{i+1})_{+} \right|^{q_*} \, d\mathcal{H}^{N-1} + \frac{1}{\sigma^{q_*}} \int_{S_r} (u - k_{i+1})_{+}^{q_0} \, d\mathcal{H}^{N-1}. \quad (3.10) \]

Since \( \tilde{\rho}_i \leq \rho_i \) and \( \sigma \rho \leq \rho_i \leq \rho_i \leq \rho \), we can replace \( r^{\delta q_*} \) in (3.10) with \( (\sigma \rho)^{\delta q_*} \). Substituting (3.10) into (3.9), we get

\[ \int_{\tilde{\rho}_i}^{\rho_i} \left( \int_{-\theta_i}^{\theta_i} \int_{S_r} (u - k_{i+1})_{+}^q \, d\mathcal{H}^{N-1} \right)^{\frac{\delta}{\delta - 1}} \, dr \leq \frac{1}{\min\{\sigma, \rho, 1\}^{\frac{\delta q_0}{\delta - 1}}} \int_{\tilde{\rho}_i}^{\rho_i} \left[ \left( \sup_{-\theta_i < t < \theta_i} \int_{S_r} (u - k_{i+1})_{+}^2 \, d\mathcal{H}^{N-1} \right)^{\frac{\delta q_0}{\delta - q_0}} \right]^{\frac{\delta - 1}{\delta}} \quad (3.11) \]

where to obtain (a), we use Hölder’s inequality with respect to \( dr \) with exponents \( \frac{1}{\delta} \) and \( \frac{1}{1 - \frac{1}{\delta}} \), where we have used the fact that \( \frac{\delta q_0}{\delta - 1} = 1 - \delta \). Substituting (3.11) into (3.8), we get

\[ \int_{Q_i} (u - k_{i+1})_{+}^q \left| \nabla \zeta \right|^q \, dz \lesssim \frac{(\rho_i - \tilde{\rho}_i)^{-\frac{(q-1+\frac{2}{\delta})}{\delta}}}{\min\{\sigma, \rho, 1\}^{\frac{\delta q_0}{\delta - 1}}} \sup_{L_i} \int_{B_i} (u - k_{i+1})_{+}^2 \, dx \frac{1}{\min\{\sigma, \rho, 1\}^{\frac{\delta q_0}{\delta - 1}}} \int_{Q_i} \left( \left| \nabla (u - k_{i+1})_{+} \right|^{q_*} + \left| (u - k_{i+1})_{+} \right|^{q_*} \right) \, dz. \quad (3.12) \]

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In order to estimate $J$ from (3.12), we proceed as follows: Since $q_\ast < p$, we get
\begin{align*}
J \leq & \left( \iint_{Q_i} \left| \nabla (u - k_{i+1}) \right|^p + (u - k_{i+1})^p \right) dz \\
& \leq \frac{2^{(p-1)}(p+1)}{k^p \varepsilon_0} \left( \int_{Q_i} (u - k_i)^p dz \right)^{\frac{p}{p-\varepsilon_0}} \\
& \leq \frac{2^{(p-1)}(p+1)}{k^p \varepsilon_0} \left( \iint_{Q_i} \left| \nabla (u - k_i) \right|^p + (u - k_i)^p \right) dz.
\end{align*}

Substituting (3.13) into (3.12) and making use of (3.7) and (3.6) into the energy estimate Lemma 2.2, we get
\begin{align*}
&\sup_{-\delta_i < t < \delta_i} \int_{B_{\delta_i}} (u - k_{i+1})^2 dx + \iint_{Q_i} \left| \nabla (u - k_{i+1}) \right|^p dz \\
&\leq \left( \frac{2^{(i+2)}(p+1)}{(1-\sigma)^p \rho_0} + \frac{2^{(p+\varepsilon_0-2)(i+1)}}{(1-\sigma)\rho_0} \right) \left( \int_{Q_i} (u - k_i)^p dz \right)^{\frac{p}{p-\varepsilon_0}} \\
&\quad + \frac{2^{(i+2)(q-1+\frac{1}{2})}}{(1-\sigma)^{(q-1+\frac{1}{2})} \rho(q-1+\frac{1}{2})} \frac{2^{(p+\varepsilon_0)(i+1)}}{k^{p-q} - \varepsilon_0} \left( \sup_{-\delta_i < t < \delta_i} \int_{B_{\delta_i}} (u - k_i)^2 dx \right)^{\frac{1}{p-\varepsilon_0}} \\
&\quad + \frac{2^{(i+2)(q-1+\frac{1}{2})}}{(1-\sigma)^{(q-1+\frac{1}{2})} \rho(q-1+\frac{1}{2})} \frac{2^{(p+\varepsilon_0)(i+1)}}{k^{p-q}} \left( \sup_{-\delta_i < t < \delta_i} \int_{B_{\delta_i}} (u - k_i)^2 dx \right)^{\frac{1}{p-\varepsilon_0}} \left( \iint_{Q_i} \nabla (u - k_i)^2 dz \right).
\end{align*}

**Definition 3.2.** We define the following quantities:
\begin{align*}
Y_i &:= \iint_{Q_i} (u - k_i)^p dz, \\
Z_i &:= \iint_{Q_i} \left| \nabla (u - k_i) \right|^p dz.
\end{align*}

Setting
\begin{align*}
A_1 &= \frac{1}{\min\{\sigma, \rho, 1\}^{q-1+\frac{1}{2}}} \frac{1}{(1-\sigma)^{(q-1+\frac{1}{2})} \rho(q-1+\frac{1}{2})}, \\
A_2 &= p + \varepsilon_0 + \frac{3q}{N+1} = p - q_\ast + \varepsilon_0 + q - 1 + \frac{1}{\delta}, \\
M_i &= Y_i + Y_i Z_i^{\frac{1}{p-\varepsilon_0}} + Z_i^{1+\frac{1}{p-\varepsilon_0}},
\end{align*}
we can now rewrite (3.14) as follows:
\begin{align*}
&\sup_{-\delta_i < t < \delta_i} \int_{B_{\delta_i}} (u - k_{i+1})^2 dx + \iint_{Q_i} \left| \nabla (u - k_{i+1}) \right|^p dz \\
&\leq 2A_2^{(i+2)} \left( \frac{1}{\rho \varepsilon_0} + \frac{1}{(1-\sigma)^p} \right) + A_1 |Q_{\rho,0}|^{\frac{q}{p-\varepsilon_0}} \left( \frac{1}{k^{p-q_\ast}} + \frac{1}{k^{p-q_\ast+\varepsilon_0}} \right) |Q_i|M_i \\
&= bA_k |Q_i|M_i,
\end{align*}
where we have set
\begin{align*}
b &:= 2A_2 \quad \text{and} \quad A_k := C4A_2 \left( \frac{1}{\rho \varepsilon_0} + \frac{1}{(1-\sigma)^p} \right) + A_1 |Q_{\rho,0}|^{\frac{q}{p-\varepsilon_0}} \left( \frac{1}{k^{p-q_\ast}} + \frac{1}{k^{p-q_\ast+\varepsilon_0}} \right).
\end{align*}

Here $C$ is the constant appearing in (3.16).
3.2. Applying Sobolev embedding

Recalling $p^\# = \frac{p(N+2)}{N}$, we have the following sequence of estimates:

$$Y_{i+1} \leq \iint_{\tilde{Q}_i} (u - k_{i+1})_+^{p+\varepsilon_0} \tilde{\zeta}_i^{p^\#} \, dz$$

(3.18)

$$\leq C \left( \iint_{\tilde{Q}_i} (u - k_{i+1})_+^{p^\#} \tilde{\zeta}_i^{p^\#} \left( \frac{|A_{i+1}|}{|Q_i|} \right)^{\frac{p^\# - p - \varepsilon_0}{p^\#}} \right)^{\frac{p^\# - p - \varepsilon_0}{p^\#}},$$

(3.5)

$$\leq C \left( \iint_{\tilde{Q}_i} (u - k_{i+1})_+^{p^\#} \tilde{\zeta}_i^{p^\#} \left( \frac{2^{(i+1)(p+\varepsilon_0)}}{k^{p+\varepsilon_0}} Y_i \right)^{\frac{p^\# - p - \varepsilon_0}{p^\#}} \right)^{\frac{p^\# - p - \varepsilon_0}{p^\#}},$$

where (a) follows from Hölder’s inequality which may be applied since $p^\# > p + \varepsilon_0$ and $\tilde{\zeta}_i \in C_c(\tilde{Q}_i)$ satisfying $\tilde{\zeta}_i = 1$ on $Q_{i+1}$ and $\tilde{\zeta}_i = 0$ on $\partial_p \tilde{Q}_i$.

By the parabolic Sobolev embedding in Lemma 2.3, we have

$$\iint_{\tilde{Q}_i} |u - k_{i+1}|_+^{p^\#} \tilde{\zeta}_i^{p^\#} \, dz \lesssim \left( \sup_{-\tilde{\delta}_i < t < \tilde{\delta}_i} \int_{B_{\tilde{\delta}_i}} |(u - k_{i+1})_+|^2 \, dx \right)^{\frac{p^\#}{2}} \times \left( \iint_{\tilde{Q}_i} |\nabla (u - k_{i+1})_+|^p \, dz + \iint_{\tilde{Q}_i} |(u - k_{i+1})_+|^p |\nabla \tilde{\zeta}_i|^p \, dz \right).$$

(3.19)

The first two terms appearing on the right hand side of (3.19) are estimated using (3.16) and the last term is estimated using (3.6) to get

$$\iint_{\tilde{Q}_i} |u - k_{i+1}|_+^{p^\#} \tilde{\zeta}_i^{p^\#} \, dz \lesssim (b^i A_k |Q_i| M_i)^{\frac{p^\#}{2}} \left( b^i A_k |Q_i| M_i + \frac{2^{(p+\varepsilon_0)(i+1)}}{(1-\sigma)^p} \rho^p |k^{\varepsilon_0}| |Q_i| Y_i \right)^{\frac{p^\# - p - \varepsilon_0}{p^\#}},$$

(3.20)

where we have used the fact that $2^{p+\varepsilon_0} \leq b_i \frac{1}{(1-\sigma)^p} \rho^p k^{\varepsilon_0} \lesssim A_k$ and $Y_i \leq M_i$.

Combining (3.20) with (3.18), we get

$$Y_{i+1} \leq C 2(b^i A_k |Q_i| M_i)^{\left(1 + \frac{p}{N}\right)} \frac{2^{(p+\varepsilon_0)(i+1)}}{k^{p+\varepsilon_0}} \left( \frac{p^\# - p - \varepsilon_0}{p^\#} \right)^{\frac{p^\# - p - \varepsilon_0}{p^\#}} Y_i.$$

(3.21)

Denoting the constants

$$\kappa := \frac{q}{N + 1}, \quad 1 + \beta := \left(1 + \frac{p}{N}\right) \left(1 + \frac{p + \varepsilon_0}{p^\#}\right) \quad \text{and} \quad \gamma := \frac{p^\# - p - \varepsilon_0}{p^\#},$$

(3.22)

we can rewrite (3.21) (after redefining $A_k$ given in (3.17) to include the constant $C$ appearing in (3.21)) as

$$Y_{i+1} \leq b^i (1+\beta+\gamma) A_k^{1+\beta} |Q_i|^{1+\beta} M_i^{1+\beta} Y_i^\gamma.$$

(3.23)

Dividing (3.16) by $|Q_i|$ and redefining $A_k$ given in (3.17) to include additional universal constants coming from $|Q_{i+1}| \leq |Q_i| \leq 2^{N+1}|Q_{i+1}|$, we get

$$Z_{i+1} \leq b^i A_k M_i.$$

(3.24)
4. Proof of Theorem 1.2

Choosing $\varepsilon_0 := \frac{4}{N+2}$, we see that the restrictions in (3.3) are satisfied and hence the estimates from the previous section are applicable. Set

$$\alpha := \min\{\beta + \gamma, \kappa\}, \quad b_1 := \max\{b^{1+\beta+\gamma}, b^{1+\kappa}, b^\kappa\} \quad \text{and} \quad \tau := \min\{\varepsilon_0, p + \varepsilon_0 - 2, p - q_*, p - q_* + \varepsilon_0\}, \quad (4.1)$$

where after restricting $k \geq 1$ to be eventually chosen and recalling the restriction $q_* < p$, we see that

$$A_k \leq \frac{A_{1+\beta}}{k_\tau} := \frac{C4^A \left( \frac{1}{\theta_p} + \frac{1}{\sigma} \right) \frac{1}{1-\sigma} + 2A_1|Q_{\rho, \theta}|^{\frac{\gamma}{\gamma+1}}}{k_\tau}. \quad (4.2)$$

The next step is to obtain an iterative estimate in terms of $M_{i+1}$ (see (3.15)). In order to do this, we shall get an iterative estimate for each of the constituent terms of $M_{i+1}$ as follows:

**Iterative estimate for $Y_{i+1}$:** From (3.23), we have

$$Y_{i+1} \leq b_1^{i+\beta}|Q_{\rho, \theta}|^{1+\beta} M_i^{1+\beta+\gamma}. \quad (4.3)$$

**Iterative estimate for $Y_{i+1} Z_{i+1}^{\kappa}$:** From (3.24), we see that

$$Z_{i+1}^{\kappa} \leq b_1^{i+\beta}|Q_{\rho, \theta}|^{1+\beta} M_i^{1+\beta+\kappa},$$

using which, we get

$$Y_{i+1} Z_{i+1}^{\kappa} \leq b_1^{i+\beta}|Q_{\rho, \theta}|^{1+\beta} M_i^{1+\beta+\gamma+\kappa}. \quad (4.4)$$

**Iterative estimate for $Z_{i+1}^{1+\kappa}$:** From (3.24), we see that

$$Z_{i+1}^{1+\kappa} \leq b_1^{i+\beta}|Q_{\rho, \theta}|^{1+\beta} M_i^{1+\kappa}. \quad (4.5)$$

Combining (4.2), (4.3) and (4.4), we get

$$M_{i+1} \leq b_1^{2i} \frac{\hat{A}}{k_\tau} \left( M_i^{1+\beta+\gamma} + M_i^{1+\beta+\gamma+\kappa} + M_i^{1+\kappa} \right), \quad (4.6)$$

where we have set (recall $\tau$ is defined in (4.1))

$$\hat{A} := \max\{1, |Q_{\rho, \theta}|^{1+\beta}\} \max\{A^{1+\beta}, A^{1+\beta+\kappa}, A^{1+\kappa}\},$$

$$\hat{\tau} := \min\{\tau(1 + \beta) + (p + \varepsilon_0)\gamma, \tau(1 + \kappa)\}. \quad (4.7)$$

Recalling (1.3), it is easy to see that

$$M_i \leq \frac{\Gamma}{\sigma(N+1)(1+\kappa)} := C_3.$$
Making use of (4.7) into (4.5), we get
\[
M_{i+1} \leq b_1^2 \frac{A}{k^2} \left( \left( \frac{M_i}{C_3} \right)^{1+\beta+\gamma} C_3^{1+\beta+\gamma} + \left( \frac{M_i}{C_3} \right)^{1+\beta+\gamma+\kappa} C_3^{1+\beta+\gamma+\kappa} + \left( \frac{M_i}{C_3} \right)^{1+\kappa} C_3^{1+\kappa} \right)
\]
\[
\leq b_1^2 \frac{A}{k^2} \left( \left( \frac{M_i}{C_3} \right)^{1+\min\{\beta+\gamma,\kappa\}} \right) \left( C_3^{1+\beta+\gamma} + C_3^{1+\beta+\gamma+\kappa} + C_3^{1+\kappa} \right)
\]
\[
= b_1^2 \frac{A}{k^2} M_i^{1+\min\{\beta+\gamma,\kappa\}} \left( 1 + C_3^{\max\{\beta+\gamma,\kappa\} - \min\{\beta+\gamma,\kappa\}} + C_3^{\max\{\beta+\gamma,\kappa\}} \right).
\]

We can now apply Lemma 2.6 to the sequence \( M_i \) to get \( M_\infty = 0 \) (which implies \( u \leq k \)) provided
\[
\Gamma = M_0 \leq \left( \frac{A}{k^2} \left( 1 + C_3^{\max\{\beta+\gamma,\kappa\} - \min\{\beta+\gamma,\kappa\}} + C_3^{\max\{\beta+\gamma,\kappa\}} \right) \right)^{\frac{1}{\gamma}} b_1^{\frac{1}{\gamma}} \leq \frac{1}{\alpha} b_1 \beta^\gamma.
\]

From this, we can choose \( k \) as follows:
\[
k = \Gamma^{\frac{1}{\gamma}} \left( 1 + C_3^{\max\{\beta+\gamma,\kappa\} - \min\{\beta+\gamma,\kappa\}} + C_3^{\max\{\beta+\gamma,\kappa\}} \right)^{\frac{1}{\gamma}} b_1^{\frac{1}{\gamma}} \wedge 1.
\]

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