The uniqueness of symmetrizing measure and linear diffusions

Xing Fang, Jiangang Ying, Minzhi Zhao

2000 MR subject classification 60J45, 60J65

Key words. symmetrizing measure, linear diffusion, Dirichlet space, regular subspace

Abstract

In this short article, we shall study one-dimensional local Dirichlet spaces. One result, which has its independent interest, is to prove that irreducibility implies the uniqueness of symmetrizing measure for right Markov processes. The other result is to give a representation for any 1-dim local, irreducible and regular Dirichlet space and a necessary and sufficient condition for a Dirichlet space to be regular subspace of another Dirichlet space.

1 Introduction

Due to the pioneering works of Feller, one-dimensional diffusion has been a mature and very interesting topic in theory of Markov processes with its simplicity and clarity. There are a lot of literatures on this topic, e.g., Ito-McKean [8], Revuz-Yor [10], Rogers-Williams [9], among those most influential. As we shall see, one-dimensional irreducible diffusion is always symmetric. Thus it has no loss of generality that Dirichlet form approach is introduced to investigate one-dimensional diffusions. In this article, we shall discuss the properties of Dirichlet spaces associated with one-dimensional diffusions, and study one-dimensional diffusions by means of Dirichlet forms. At first a representation of the Dirichlet form associated with a one-dimensional diffusion will be formulated since we have not seen it explicitly in literature.

*The research of this author is supported in part by NSFC Grant No. 10671036
†The research of this author is supported in part by NSFC Grant No. 10601047
2 The uniqueness of symmetrizing measure

We first present a theorem which states a condition for uniqueness of symmetrizing measure and will be used later. This kind of results may be known in some other forms. We begin with a general right Markov process $X = (X_t, \mathbf{P}^x)$ on state space $E$ with semigroup $(\mathbf{P}_t)$ and resolvent $(U^\alpha)$. It is easy to see from the right continuity that for $x \in E$ and a finely open subset $D$, $\mathbf{P}^x(T_D < \infty) > 0$ if and only if $U^\alpha 1_D(x) > 0$. The process $X$ is called irreducible if $\mathbf{P}^x(T_D < \infty) > 0$ for any $x \in E$ and a finely open subset $D$, where $T_D$ is the hitting time of $D$.

Lemma 2.1 The following statements are equivalent.

1. $X$ is irreducible.
2. $U^\alpha 1_D$ is positive everywhere on $E$ for any non-empty finely open set $D$.
3. $U^\alpha 1_A$ is either identically zero or positive everywhere on $E$ for any Borel set $A$ or, in other words, $\{U^\alpha(x, \cdot) : x \in E\}$ are all mutually absolutely continuous.
4. All non-trivial excessive measures are mutually absolutely continuous.

Proof. The equivalence of (1) and (2) is easy. We shall prove that they are equivalent to (3). We may assume $\alpha = 0$. Suppose (1) is true. If $U_1A$ is not identically zero, then there exists $\delta > 0$ such that $D := \{U_1 > \delta\}$ is non-empty. Since $U_1A$ is excessive and thus finely continuous, $D$ is finely open and the fine closure of $D$ is contained in $\{U_1A \geq \delta\}$. Then

$$U_1A(x) \geq P_DU_1A(x) = \mathbf{E}^x \left(U_1A(X_{T_D})\right) \geq \delta \mathbf{P}^x(T_D < \infty) > 0.$$  

Conversely suppose (3) is true. Then for any finely open set $D$, by the right continuity of $X$, $U_1D(x) > 0$ for any $x \in D$. Therefore $U_1D$ is positive everywhere on $E$.

Let $\xi$ be an excessive measure. Since $\alpha \xi U^\alpha \leq \xi$, $\xi(A) = 0$ implies that $\xi U^\alpha(A) = 0$. However $\xi$ is non-trivial. Thus it follows from (3) that $U^\alpha 1_A \equiv 0$, i.e., $A$ is potential zero. Conversely if $A$ is potential zero, then $\xi(A) = 0$ for any excessive measure $\xi$. Therefore (3) implies (4).

Assume (4) holds. Since $U^\alpha(x, \cdot)$ is excessive for all $x$ and hence they are equivalent. This implies (3). \hfill $\Box$

A Borel set $A$ is called of potential zero if $U^\alpha 1_A$ is identically zero for some $\alpha \geq 0$ (thus for all $\alpha \geq 0$). A $\sigma$-finite measure $\mu$ on $E$ is said to be a symmetrizing measure of $X$ or $X$ is said to be $\mu$-symmetric if

$$(P_tu, v)_\mu = (u, P_tv)_\mu$$
for any measurable \( u, v \geq 0 \) and \( t > 0 \). It is easy to check that any symmetrizing measure is excessive and an excessive measure does not charge any set of potential zero.

**Theorem 2.1** Assume that \( X \) is irreducible. Then the symmetrizing measure of \( X \) is unique up to a constant. More precisely if both \( \mu \) and \( \nu \) are non-trivial symmetrizing measures of \( X \), then \( \nu = c\mu \) with a positive constant \( c \).

**Proof.** First of all there exists a measurable set \( H \) such that both \( \mu(H) \) and \( \nu(H) \) are positive and finite, because \( \mu \) and \( \nu \) are equivalent by Lemma 2.1. This is actually true when both measures are \( \sigma \)-finite and one is absolutely continuous with respect to another. Indeed, assume that \( \nu \ll \mu \). Since \( \nu \) is non-trivial and \( \sigma \)-finite, we may find a measurable set \( B \) such that \( 0 < \nu(B) < \infty \). Then \( \mu(B) > 0 \). Since \( \mu \) is \( \sigma \)-finite, there exist \( A_n \uparrow E \) such that \( 0 < \mu(A_n) < \infty \). Then \( \nu(A_n \cap B) \uparrow \nu(B) \) and \( \mu(A_n \cap B) \uparrow \mu(B) \). Hence there exists some \( n \) such that \( \nu(A_n \cap B) > 0 \). Take \( H = A_n \cap B \), which makes both \( \mu(H) \) and \( \nu(H) \) positive and finite.

Set \( c = \nu(H)/\mu(H) \). We may assume that \( c = 1 \) without loss of generality. Let \( m = \mu + \nu \). Then there is \( f_1, f_2 \geq 0 \) such \( \mu = f_1 \cdot m \) and \( \nu = f_2 \cdot m \). Let \( A = \{ f_1 > f_2 \} \), \( B = \{ f_1 = f_2 \} \) and \( C = \{ f_1 < f_2 \} \).

We shall show that \( \nu = \mu \). Otherwise \( \mu(A) > 0 \) or \( \nu(C) > 0 \). We assume that \( \mu(A) > 0 \) without loss of generality. Since \( \mu \) is \( \sigma \)-finite, there is \( A_n \in \mathcal{B}(E) \) such that \( A_n \subseteq A \), \( \mu(A_n) < \infty \) and \( A_n \uparrow A \). Let \( D = B \cup C \). For any integer \( n \) and \( \alpha > 0 \),

\[
(U^\alpha 1_{A_n}, 1_D)_{\mu} \leq (U^\alpha 1_{A_n}, 1_D)_{\nu} = (U^\alpha 1_D, 1_{A_n})_{\nu} \leq (U^\alpha 1_D, 1_{A_n})_{\mu}.
\]

Since \( (U^\alpha 1_{A_n}, 1_D)_{\mu} = (U^\alpha 1_D, 1_{A_n})_{\mu} \), it follows that \( (U^\alpha 1_D, 1_{A_n})_{\nu} = (U^\alpha 1_D, 1_{A_n})_{\mu} \).

Thus we have

\[
(U^\alpha 1_D, (1 - \frac{f_2}{f_1})1_{A_n})_{\mu} = (U^\alpha 1_D, 1_{A_n})_{\mu} - (U^\alpha 1_D, 1_{A_n})_{\nu} = 0.
\]

Since \( 1 - \frac{f_2}{f_1} > 0 \) on \( A \), let \( n \) go to infinity and by the monotone convergence theorem we get that \( (U^\alpha 1_D, 1_A)_{\mu} = 0 \). The irreducibility of \( X \) implies that \( U^\alpha 1_D = 0 \) identically or \( D \) is of potential zero. Therefore

\[
\mu(D) = \nu(D) = 0.
\]

Consequently,

\[
0 = \mu(H) - \nu(H) = \int_{H \cap A} (1 - \frac{f_2}{f_1}) d\mu.
\]
which leads to that $\mu(H \cap A) = 0$ and also $\mu(H) = 0$. The contradiction implies that $\nu = \mu$. \qed

The following example shows that the condition that any point may reach any finely open set is needed. Actually we may easily see that it is also necessary in the sense that if $X$ has a unique symmetrizing measure $m$, then $X$, restricted on the fine support of $m$, is irreducible.

Example: Let $J = \frac{1}{4}(\delta_1 + \delta_{-1} + \delta_{\sqrt{2}} + \delta_{-\sqrt{2}})$ defined on $\mathbb{R}$ and $\pi = \{\pi_t\}_{t > 0}$ the corresponding symmetric convolution semigroup; i.e., $\pi_t(x) = e^{-t\phi(x)}$ with
\[
\phi(x) = \int (1 - \cos xy)J(dy) = \frac{1}{2}(1 - \cos x) + \frac{1}{2}(1 - \cos \sqrt{2}x).
\]

Let $N = \{n + m\sqrt{2} : n, m \text{ are integers}\}$ and $\mu = \sum_{x \in N} \delta_x$. Then $\mu$ is $\sigma$-finite and also a symmetrizing measure. It is easy to check that any point may reach any open set but not any finely open set.

It is known that the fine topology is determined by the process and hard to identify usually. Hence it is hard to verify sometimes the irreducibility defined in the theorem. However under LSC, namely, assuming that $U^{\alpha}1_B$ is lower-semi-continuous for any Borel subset $B$ of $E$, the irreducibility is equivalent to the weaker one, which is easier to verify: $P^x(T_D < \infty) > 0$ for q.e. $x \in E$ and open subset $D \subset E$.

Remark As a remark, we would like to present a slight more general result which was provided by Masatoshi Fukushima in his comment to this theorem.

Suppose that $X$ is $\mu$-symmetric. The following two definitions refer to Definition 2.1.1 [2]. A Borel subset $A$ is called $(P_t)$-invariant if $1_A : P_t(1_A) < 0$ a.e. $\mu$ for all $t > 0$ and $f \in L^2(E, \mu)$, and $X$ is $\mu$-irreducible if any $(P_t)$-invariant set is trivial in the sense that either $\mu(A) = 0$ or $\mu(A^c) = 0$. Then the following statements are equivalent due to Theorem 3.5.6[2] and a similar proof of Lemma 2.1.

1. $X$ is $\mu$-irreducible;
2. If $D$ is finely open and $\mu(D) > 0$, then $P^x(T_D < \infty) > 0$ for q.e. $x \in E$;
3. $U^{\alpha}1_D > 0$ q.e. for any finely open $D$ with $\mu(D) > 0$;
4. $U^{\alpha}1_A$ is either 0 q.e. or positive q.e. for every Borel subset $A$.

It follows that if $X$ is $\mu$-irreducible, then all non-trivial excessive measures charging no $\mu$-polar sets are equivalent. Hence following the proof of Theorem 2.1 we have its Fukushima’s version.
**Theorem 2.2** Assume that a Borel right process $X$ is $\mu$-irreducible with respect to some non-trivial symmetrizing measure $\mu$ of $X$. If $\nu$ is a symmetrizing measure of $X$ charging no $\mu$-polar sets, then $\nu = c \cdot \mu$ for some constant $c \geq 0$.

### 3 Dirichlet forms on intervals

Let $I$ be an interval or a connected subset of $\mathbb{R}$ and $I^0$ its interior. Denote by $S(I)$ the totality of strictly increasing continuous functions on $I$. Let $s \in S(I)$. Let $m$ and $k$ two Radon measures on $I$ with $\text{supp}(m) = I$. Define a symmetric form $(\mathcal{E}_{(s,m,k)}, \mathcal{F}_{(s,m,k)})$ as follows:

$$\mathcal{F}_{(s,m,k)} = \{u \in L^2(I, m + k) : u \ll s \text{ and } \frac{du}{ds} \in L^2(I, ds) \}$$

$$\mathcal{E}_{(s,m,k)}(u, v) = \int_I \frac{du}{ds} \frac{dv}{ds} ds + \int_I u(x)v(x)k(dx), \text{ for } u, v \in \mathcal{F}_{(s,m,k)}.$$ 

It follows from [4] that $\mathcal{F}_{(s,m,k)}$ is the closure of the algebra generated by $s$ with respect to the norm $\sqrt{\mathcal{E}_{(s,m,k)}(\cdot, \cdot) + (\cdot, \cdot)_m}$. As in [5], if $I = (a_1, a_2)$, we call $a_1$ a regular boundary if $a_1 \notin I$, $s(a_1) > -\infty$ and $m((a_1, c)) + k((a_1, c)) < \infty$ for some $c \in I$. The regularity of $a_2$ is defined similarly. Define also

$$\mathcal{F}_{0}^{(s,m,k)} = \{u \in \mathcal{F}_{(s,m,k)} : u(a_i) = 0 \text{ if } a_i \text{ is regular boundary } \}$$

$$\mathcal{E}_{0}^{(s,m,k)}(u, v) = \mathcal{E}_{(s,m,k)}(u, v), \text{ for } u, v \in \mathcal{F}_{0}^{(s,m,k)}.$$ 

When $k = 0$, we write it as $(\mathcal{E}_{0}^{(s,m)}, \mathcal{F}_{0}^{(s,m)})$ for simplicity. The next lemma asserts that a Dirichlet form is built this way.

**Lemma 3.1** The form $(\mathcal{E}_{0}^{(s,m)}, \mathcal{F}_{0}^{(s,m)})$ is a local irreducible Dirichlet space on $L^2(I; m)$ regular on $I$ and it is strong local if and only if $k = 0$.

**Proof.** We only prove the first statement. The second is clear. Let $J = s(I)$ and define a regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(J, m\circ s^{-1})$ (refer to [5] Example 1.2.2) for a proof) as follows:

$$\mathcal{F} = \{u \in L^2(J, (m + k)\circ s^{-1}) : u \text{ is absolutely continuous and } u' \in L^2(J) \}$$

$$\mathcal{E}(u, v) = \int_J u'(x)v'(x)dx + \int_J u(x)v(x)(k\circ s^{-1})(dx), \text{ for } u, v \in \mathcal{F}.$$ 

Then $(\mathcal{E}_{0}^{(s,m,k)}, \mathcal{F}_{0}^{(s,m,k)})$ is a state-space transform of $(\mathcal{E}, \mathcal{F})$ induced by the function $s^{-1}$. It shows that $(\mathcal{E}_{0}^{(s,m,k)}, \mathcal{F}_{0}^{(s,m,k)})$ is a Dirichlet form on $L^2(I, m)$ by [3] lemma 3.1. The regularity follows from the fact that $u\circ s^{-1} \in \mathcal{F}_{0}^{(s,m,k)} \cap C_c(I)$ whenever $u \in \mathcal{F} \cap C_c(J)$. The local property of $(\mathcal{F}_{0}^{(s,m,k)}, \mathcal{E}_{0}^{(s,m,k)})$ is obvious. \qed
4 Representation of one-dimensional local Dirichlet space

Let $I$ be an interval or a connected subset of $\mathbb{R}$ and $I^\circ$ its interior.

**Definition 4.1** A diffusion $X = (X_t, P^x)$ with life time $\zeta$ on $I$ is a Hunt process on $I$ with continuous sample paths on $[0, \zeta)$. A diffusion $X$ is called irreducible if for any $x, y \in I$, $P^x(T_y < \infty) > 0$, where $T_y$ denotes the hitting time of $y$.

The irreducibility defined here implies the regularity in [10] and [9]. The reason we use irreducibility is that $I$ is the state space of $X$, while in [10] and [9], $I$ may contain a trap, thus not a real state space. Another thing which needs to be noted is that a diffusion defined this way is allowed being 'killed' inside $I$, while in some literature it is not allowed. A diffusion not allowed being killed inside $I$ is called locally conservative.

The local conservativeness is equivalent to the following property: for any $x \in I^\circ$, there exist $a, b \in I$ with $a < b$ and $x \in (a, b)$ such that $P^x(T_a \land T_b < \infty) = 1$; if $x$ is the right (resp. left) end-point of $I$ included in $I$ and finite, then there exists $a \in I$ and $a < x$ (resp. $a > x$) such that $P^x(T_a < \infty) = 1$. For any regular diffusion $X$, we shall obtain a process $X'$ through the well-known Ikeda-Nagasawa-Watanabe piecing together procedure. It is easy to show that $X'$ is a locally conservative regular diffusion on $I$, and $X$ is obtained by killing $X'$ at a rate given by a PCAF. We say that $X'$ is a resurrected process of $X$ and $X$ is a subprocess of $X'$. As VII(3.2) in [10] or (46.12) in [9], a locally conservative regular diffusion $X$ on $I$ has so-called scale function, namely, there exists a continuous, strictly increasing function $s$ on $I$ such that for any $a, b, x \in I$ with $a < b$ and $a \leq x \leq b$,

$$P^x(T_b < T_a) = \frac{s(x) - s(a)}{s(b) - s(a)}. \quad (4.1)$$

The function $s$ is unique up to a linear transformation. This function $s$ is called a scale function of $X$. A diffusion with scale function $s(x) = x$ is said to be in natural scale. It is easy to check that if $s$ is a scale function of $X$, then $s(X)$ is a diffusion on $s(I)$ in natural scale. A Brownian motion on $I$ is a diffusion on $I$ which moves like Brownian motion inside $I$ and is reflected at any end-point which is finite and in $I$ and get absorbed at any end point which is finite but not in $I$. Clearly Brownian motion on $I$ is clearly in natural scale. Thus Blumenthal-Getoor-McKean’s theorem (Theorem 5.5.1 [11]) implies that a diffusion on $I$ in natural scale is identical in law with a time change of Brownian motion on $I$. More precisely, let $X$ be a locally conservative regular diffusion in natural scale. Then there exists a measure $\xi$ on $\mathbb{R}$,
fully supported on $I$, and a Brownian motion $B = (B_t)$ on $I$ such that $X$ is equivalent in law to $(B_{\tau_t})$ where $\tau = (\tau_t)$ is the continuous inverse of the PCAF $A = (A_t)$ of $B$ with Revuz measure $\xi$. The measure $\xi$ is called the speed measure of $X$. Obviously $X$ is symmetric with respect to $\xi$.

Let now $X$ be an irreducible diffusion on $I$ and $X'$ the resurrected process of $X$ with scale function $s$. Then $s(X')$ is symmetric with respect to its speed measure $\xi$ and therefore $X'$ is symmetric with respect to $\xi \circ s$. The diffusion $X$, the subprocess of $X'$, is certainly still symmetric to $\xi \circ s$. An $m$-symmetric Markov process on state space $E$ always determines a Dirichlet form on $L^2(E, m)$. A standard reference for theory of Dirichlet form is \[5\], to which we refer for terminologies, notations and results. By results in theory of Dirichlet form, the Dirichlet form associated with $X'$ is strongly local, irreducible and regular on $I$. It follows then that the Dirichlet form associated with $X$ is local, irreducible and regular on $I$. Conversely, given a local, irreducible and regular Dirichlet form on $L^2(I, m)$ with a fully supported Radon measure $m$ on $I$, it is easily seen that the corresponding Markov process must be an irreducible diffusion on $I$. Therefore one-dimensional irreducible diffusions are in one-to-one correspondence with one-dimensional local, irreducible and regular Dirichlet forms. This illustrates that no generality will be lost if we start from such a Dirichlet form as we shall do in the following sections. In §2, we shall present a sufficient condition for the uniqueness of symmetrizing measure. Actually, this condition is almost necessary too. In §3 we will give a representation for any 1-dim local, irreducible and regular Dirichlet space. In §4, we will give a necessary and sufficient condition for a Dirichlet space to be regular subspace of another Dirichlet space, which generalizes the main result in \[3\]. As application, two examples is presented to illustrate that Brownian motion has not only regular extensions and but also non-conservative regular subspaces.

Fixing an interval $I$ and given a fully-supported Radon measure $m$ on $I$, we shall consider in this section the representation of a local, irreducible and regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(I, m)$ in terms of the scale function of the associated diffusion. The form $(\mathcal{E}, \mathcal{F})$ is assumed to be irreducible, i.e., the associated semigroup is $m$-invariant. Let $X = (X_t, \mathbf{P}^x)$ be the diffusion process on $I$ associated with $(\mathcal{E}, \mathcal{F})$. It is well known that the process $X = (X_t, \mathbf{P}^x)$ associated with a local irreducible regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(I, m)$ is an irreducible $m$-symmetric diffusion on $I$. In addition $(\mathcal{E}, \mathcal{F})$ is strong local if and only if $(X_t, \mathbf{P}^x)$ is locally conservative.

Next we give the representation theorem of one-dimensional local, irreducible and regular Dirichlet space.

**Theorem 4.1** Let $I = (a_1, a_2)$ be any interval and $m$ a Radon measure on $I$ with
$\text{supp}(m) = I$. If $(\mathcal{E}, \mathcal{F})$ be a local irreducible regular Dirichlet space on $L^2(I, m)$, then

$$(\mathcal{E}, \mathcal{F}) = (\mathcal{E}_0^{(s,m,k)}, \mathcal{F}_0^{(s,m,k)})$$

where $k$ is a Radon measure on $I$ and $s \in \mathcal{S}(I)$. Furthermore $s$ is a scale function for $(X_t, P^x)$ which is the diffusion associated with $(\mathcal{E}, \mathcal{F})$.

Proof. We shall first assume that $(\mathcal{E}, \mathcal{F})$ is strongly local. Let $s$ be a scale function of $X = (X_t, P^x)$ associated with $(\mathcal{E}, \mathcal{F})$, and $Y = (Y_t, Q^x), x \in I$ be the diffusion associated with Dirichlet space $(\mathcal{F}_0^{(s,m)}, \mathcal{E}_0^{(s,m)})$. Then $X$ and $Y$ have the same scale function and thus the same hitting distributions. It follows from Blumenthal-Getoor-McKean Theorem that there exists a strictly increasing continuous additive functional $A_t$ of $X$ such that $(Y_t, Q^x), x \in I$ and $(\tilde{X}_t, P^x), x \in I$ are equivalent, where $\tilde{X}_t = X_{\tau_t}$, and $(\tau_t)$ is the inverse of $(A_t)$.

Note that $(\tilde{X}_t, P^x), x \in I$ is $\xi$-symmetric, where $\xi$ is the Revuz measure of $A$ with respect to $m$, and also $m$-symmetric since it is equivalent to $(Y_t, Q^x), x \in I$. By Theorem 2.1, $\xi$ is a multiple of $m$ or $A_t = ct$ for some positive constant $c$. It shows that $\tilde{X}_t = X_{\xi_t}$. Therefore

$$\mathcal{F} = \mathcal{F}_0^{(s,m)}, \mathcal{E} = c \cdot \mathcal{E}_0^{(s,m)}$$

by (1.3.15) and (1.3.17) in [5].

However scale functions of a linear diffusion could differ by a linear transform. When the scale function is properly chosen, the constant $c$ above could be 1 (and shall be taken to be 1 in the sequel). For example $s' = s/c \in \mathcal{S}(I)$ is also a scale function for $(X_t, P^x)$ and we have

$$\mathcal{F} = \mathcal{F}_0^{(s',m)}, \mathcal{E} = \mathcal{E}_0^{(s',m)}.$$  

In general, when $(\mathcal{E}, \mathcal{F})$ is local, we have the following Beurling-Deny decomposition by [5, Theorem 3.2.1]

$$\mathcal{E}(u, v) = \mathcal{E}^c(u, v) + \int_I u(x)v(x)k(dx), u, v \in \mathcal{F} \cap C_0(I),$$

where $\mathcal{E}^c$ is the strongly local part of $\mathcal{E}$. Define a new symmetric form $(\mathcal{E}', \mathcal{F}')$ on $L^2(I, m + k)$:

$$\mathcal{F}' = \mathcal{F}, \mathcal{E}' = \mathcal{E}^c.$$  

Then $(\mathcal{E}', \mathcal{F}')$ is a strongly local irreducible regular Dirichlet space on $L^2(I, m + k)$. By the conclusion in the first part, it follows that

$$\mathcal{E}^c = \mathcal{E}_0^{(s,m)}, \mathcal{F} = \mathcal{F}' = \mathcal{F}_0^{(s,m+k)} = \mathcal{F}_0^{(s,m,k)}.$$
The proof is completed. □

Remark. After reading the result above, Professor Fukushima also provides a more intrinsic proof. Here “intrinsic” means a proof without using big theorems developed above but only using a very profound analysis on the one-dimensional diffusion presented in classical books K. Ito[7], [6] and Ito-McKean[8]. We shall outline the proof here which is quoted from Professor Fukushima’s e-mail.

1. Given a diffusion $X$ on a one-dimensional interval $I$, its scale function $s$ and speed measure $m$ are already defined. As you know, $m$ is defined simply by using the concave property of the mean exit time from a sub-interval of $I$ when $X$ is locally conservative.

2. ......

5 Regular subspaces

Let $(\mathcal{E}', \mathcal{F}')$ and $(\mathcal{E}, \mathcal{F})$ be two irreducible regular Dirichlet spaces on $L^2(I, m)$. The space $(\mathcal{E}', \mathcal{F}')$ is called a regular subspace of $(\mathcal{E}, \mathcal{F})$ if $\mathcal{F}' \subset \mathcal{F}$ and $\mathcal{E}(u, v) = \mathcal{E}'(u, v)$ for any $u, v \in \mathcal{F}'$. All non-trivial regular subspaces of linear Brownian motion is characterized clearly in [3]. In this section we shall further give a necessary and sufficient condition for $(\mathcal{E}', \mathcal{F}')$ to be a regular Dirichlet subspace of $(\mathcal{E}, \mathcal{F})$, which extends the result in [3].

Using the representation in §3, we have

$$\begin{align*}
(\mathcal{E}, \mathcal{F}) &= (\mathcal{E}_0^{[s_1, m, k_1]}, \mathcal{F}_0^{[s_1, m, k_1]}), \\
(\mathcal{E}', \mathcal{F}') &= (\mathcal{E}_0^{[s_2, m, k_2]}, \mathcal{F}_0^{[s_2, m, k_2]}),
\end{align*}$$

where $s_1, s_2 \in \mathbb{S}(I)$ and $k_1, k_2$ are two Radon measures on $I$. Now comes our main result.

**Theorem 5.1** Let $(\mathcal{E}', \mathcal{F}')$ and $(\mathcal{E}, \mathcal{F})$ be two local irreducible regular Dirichlet spaces on $L^2(I, m)$. Then $(\mathcal{E}', \mathcal{F}')$ is a regular subspace of $(\mathcal{E}, \mathcal{F})$ if and only if

1. $k_1 = k_2$,

2. $d\mathcal{S}_2$ is absolutely continuous with respect to $d\mathcal{S}_1$ and the density $d\mathcal{S}_2/d\mathcal{S}_1$ is either 1 or 0 a.e. $d\mathcal{S}_1$. 

Proof. It suffices to prove it for the case that both \((\mathcal{E}', \mathcal{F}')\) and \((\mathcal{E}, \mathcal{F})\) are strongly local. Assume that \(\mathcal{F}' \subseteq \mathcal{F}\) and let \((X_t, \mathbf{P}_x)\) and \((X'_t, \mathbf{P}'_x)\) be the diffusion processes associated with \((\mathcal{E}, \mathcal{F})\) and \((\mathcal{E}', \mathcal{F}')\), respectively. For any \(a < c < x_0 < d < b\), define

\[ u^x_{(c,d)}(x) := \mathbf{P}'_x(T_{x_0} < T_{(c,d)}). \]

We have \(u^x_{(c,d)}(x) \in \mathcal{F}' \subseteq \mathcal{F}\), and it shows that \(u^x_{(c,d)}(x)\) is absolutely continuous with respect to \(s_1\), while \(u^x_{(c,d)}\) is a linear transformation of \(s_2\) on \((c, x_0)\). It follows that \(ds_2\) is absolutely continuous with respect to \(ds_1\) on \((c, x_0)\). Similarly it is also true on \((x_0, d)\). Taking \((c, d) \uparrow (a, b)\), it follows that \(ds_2\) is absolutely continuous with respect to \(ds_1\). Let \(f := ds_2/ds_1\). Then we have

\[
\mathcal{E}'(u, v) = \int_I \frac{du}{ds_2} \frac{dv}{ds_2} ds_2; \\
\mathcal{E}(u, v) = \int_I \frac{du}{ds_1} \frac{dv}{ds_1} ds_1 \\
= \int_I \frac{du}{ds_2} \frac{dv}{ds_2} f^2 ds_1 \\
= \int_I \frac{du}{ds_2} \frac{dv}{ds_2} f ds_2
\]

for any \(u, v \in \mathcal{F}'\). It follows then that \(f ds_1 = f^2 ds_1\) and that either \(f = 0\) or \(f = 1\) a.e. with respect to \(ds_1\). Since \(s_1\) and \(s_2\) are continuous and strictly increasing, \(f\) has the property that for any \(x, y \in I\) with \(x < y\),

\[
\int_x^y 1_{\{f = 1\}} ds_1 > 0. \quad (5.1)
\]

The converse is obvious from the above discussion. \(\square\)

Let now

\[(\mathcal{E}, \mathcal{F}) = (\mathcal{E}_0^{(s,m,k)}, \mathcal{F}_0^{(s,m,k)})\]

be a local irreducible regular Dirichlet spaces on \(L^2(I, m)\). Take a Borel set \(A\) having property that for any \(x, y \in I\) with \(x < y\),

\[
\int_x^y 1_A \cdot ds > 0. \quad (5.2)
\]

Define \(ds_0 = 1_A \cdot ds\). Then \(s_0 \in S(I)\) and \((\mathcal{E}_0^{(s_0,m,k)}, \mathcal{F}_0^{(s_0,m,k)})\) is a regular subspace of \((\mathcal{E}, \mathcal{F})\). It is easy to check that

\[
\mathcal{F}_0^{(s_0,m,k)} = \{u \in \mathcal{F}: du/ds = 0 \text{ a.e. with respect to } ds \text{ on } A\}.
\]

Hence we have a corollary.
Corollary 5.1 For any Borel set $A$ satisfying (5.2),

$$\mathcal{F}^A = \{ u \in \mathcal{F} : \frac{du}{ds} = 0 \text{ a.e. with respect to } ds \text{ on } A \}$$

is a regular subspace of $(\mathcal{E}, \mathcal{F})$. Conversely any regular subspace of $(\mathcal{E}, \mathcal{F})$ is induced by such a set.

Finally, we shall give two interesting examples. The first example is a local irreducible and regular Dirichlet space which takes the Dirichlet space $(H^1([0,1]), \frac{1}{2}D)$ of reflected Brownian motion on $[0,1]$ as a proper regular subspace.

Example 1. Let $c(x)$ be the standard Cantor function on $[0,1]$ and let $s(x) := x + c(x)$. Take $m$ to be the Lebesgue measure on $[0,1]$. Then the Dirichlet space $(H^1([0,1]), \frac{1}{2}D)$, corresponding to Brownian motion on $[0,1]$, is a regular subspace of $(\mathcal{F}^{(a,m)}, \frac{1}{2}E^{(a,m)})$ by the theorem above and $H^1([0,1])$ is properly contained in $\mathcal{F}^{(a,m)}$.

The second example shows that 1-dim Brownian motion has a non-conservative regular subspace. For this we state a criterion for irreducible one-dimensional diffusions to be conservative (see [9]). Let

$$(\mathcal{E}, \mathcal{F}) = (\mathcal{E}_0^{(a,m,k)}, \mathcal{F}_0^{(a,m,k)})$$

where $k$ is a Radon measure on $I$ and $s \in \mathcal{S}(I)$, be a local, irreducible and regular Dirichlet space on $L^2(I, m)$ and $X = (X_t, P_x)$ the associated diffusion. In this case it is either recurrent or transient. We call the left endpoint $a$ of $I$ is

(1) of the first class if $a$ is finite and $a \in I$;

(2) of the second class if $a \notin I$ and $s(a) = -\infty$;

(3) of the third class if $a \notin I$ and $s(a) > -\infty$.

We call $a$ is dissipative if $a$ is of the third class and

$$\int_a^c (s(x) - s(a)) m(dx) < \infty$$

for some $c \in I$, and hence for all $c \in I$. Obviously, the finiteness (5.4) is independent of the choice of the scale function $s$ and the point $c$. If $a$ is not dissipative, we call it conservative. The dissipativeness and conservativeness for the right endpoint may be defined similarly. Fix a point $c > a$, define $M(x) := m((x, c))$ for $a < x < c$. 

11
Lemma 5.1 The left end-point $a$ is dissipative if and only if $a$ is of the third class and
\[
\int_a^c M(x)ds(x) < \infty.
\tag{5.5}
\]
If $a$ is dissipative, $\lim_{x \to a} M(x)s(x) = 0$. Similar conclusions hold for the right end-point.

Theorem 5.2 The Dirichlet space $(\mathcal{E}, \mathcal{F})$ (or $X$) is

(1) recurrent if and only if $k = 0$ and both endpoints are of the first class or the second class;

(2) conservative if and only if $k = 0$ and both endpoints are conservative.

We now give an example which illustrates that the Dirichlet space $(\frac{1}{2}D, H^1_0(\mathbb{R}))$ of Brownian motion on the real line $\mathbb{R}$ has non-conservative regular subspaces, comparing an example in [3] which shows Brownian motion has transient regular subspaces.

Example 2. Define a local irreducible and regular Dirichlet space $(\mathcal{E}^{(s,m)}_0, \mathcal{F}^{(s,m)}_0)$ on $L^2(\mathbb{R}, m)$, where $m$ is the usual Lebesgue measure, by giving a scale function
\[
s(x) = \int_0^x 1_G(y)dy, \quad x \in \mathbb{R},
\]
where
\[
G = \bigcup_{r_n \in Q} \left(r_n - \frac{1}{2^{n+1}}, r_n + \frac{1}{2^{n+1}}\right),
\tag{5.6}
\]
where $Q$ is the set of positive rational numbers. We choose an order on $Q$ as follows: if $a, b \in Q$, and $a = \frac{q_1}{p_1}$, $b = \frac{q_2}{p_2}$ take the simplest form, we define
\[
a < b \iff \text{either } p_1 + q_1 < p_2 + q_2 \text{ or } p_1 + q_1 = p_2 + q_2 \text{ and } q_1 < q_2.
\]
Then the order $<$ makes $Q$ a sequence $\{r_n\}$ in (5.6). Clearly $r_n \leq n$. Thus
\[
\int_0^\infty xs(x) \leq \sum_n \int_{(r_n - \frac{1}{2^{n+1}}, r_n + \frac{1}{2^{n+1}})} xdx = \sum_n \frac{r_n}{2^n} \leq \sum_n \frac{n}{2^n} < \infty.
\]
This shows the right endpoint is dissipative. Therefore the associated process is not conservative.

Acknowledgements: The authors would like to thank Professor M. Fukushima for his helpful suggestions.
References

[1] R. Blumental and R. K. Getoor, MARKOV PROCESSES AND POTENTIAL THEORY, Academic Press, New York, 1968

[2] Z. Q. Chen and T. Fukushima, SYMMETRIC MARKOV PROCESSES, TIME CHANGE AND BOUNDARY THEORY, available at http://www.math.washington.edu/~zchen/CF/cfbook-PUP32.pdf

[3] X. Fang, M. Fukushima and J. Ying, On regular Dirichlet subspaces of $H^1(I)$ and associated diffusions, Osaka, J. Math., 42(2005), 27-41

[4] X. Fang, P. He, J. Ying, Algebraic structure on Dirichlet spaces, Acta Math. Sin. (Engl. Ser.) 22 (2006), no. 3, 723–728

[5] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet forms and symmetric Markov processes, Walter de Gruyter, 1994

[6] K. Ito, Essentials of stochastic processes, AMS, 2006, (translated from Japanese version of 1957)

[7] K. Ito, LECTURES TO STOCHASTIC PROCESSES, Tata institute, 1971

[8] K. Ito, H.P. Mckean, DIFFUSION PROCESSES AND THEIR SAMPLE PATHS, Springer, Berlin Heidelberg New York Tokyo, 1965

[9] L. C. G. Rogers and D. Williams, DIFFUSIONS, MARKOV PROCESSES AND MARTINGALES, Volume 2, Cambridge University Press, 2000

[10] D. Revuz, M. Yor, CONTINUOUS MARTINGALES AND BROWNIAN MOTION, Springer-Verlag, 1991

Addresses:

**X. Fang**: Department of Mathematics, Fudan University, Shanghai, China. 
Email: fangxing@fudan.edu.cn

**J. Ying**: Department of Mathematics, Fudan University, Shanghai, China. 
Email: jgying@fudan.edu.cn

**M. Zhao**: Department of Mathematics, Zhejiang University, Hangzhou, China 
e-mail: zhaomz@zju.edu.cn