Using Information Theory to Measure Psychophysical Performance

James V Stone
Psychology Department, Sheffield University, England.
Email: j.v.stone@sheffield.ac.uk
File: combineRTbinary2021_v11a.tex.
December 14, 2021

Abstract

Most psychophysical experiments discard half the data collected. Specifically, experiments discard reaction time data, and use binary responses (e.g. yes/no) to measure performance. Here, Shannon’s information theory is used to define Shannon competence $s'$, which depends on the mutual information between stimulus strength (e.g. luminance) and a combination of reaction times and binary responses. Mutual information is the entropy of the joint distribution of responses minus the residual entropy after a model has been fitted to these responses. Here, this model is instantiated as a proportional rate diffusion model, with the additional innovation that the full covariance structure of responses is taken into account. Results suggest information associated with reaction times is independent of (i.e. additional to) information associated with binary responses, and that reaction time and binary responses together provide substantially more than the sum of their individual contributions (i.e. they act synergistically). Consequently, the additional information supplied by reaction times suggests that using combined reaction time and binary responses requires fewer stimulus presentations, without loss of precision in psychophysical parameters. Finally, because $s'$ takes account of both reaction time and binary responses, (and in contrast to $d'$) $s'$ is immune to speed-accuracy trade-offs, which vary between observers and experimental designs.

Two Sentence Summary

When presented with a stimulus, the observer’s binary responses (e.g. yes/no) and their associated reaction times depend on the stimulus strength (e.g. luminance). The amount of Shannon information gained by an observer estimated from a combination of binary responses and reaction times was substantially larger than the sum of information gains based on separate analyses of binary responses and reaction times, implying that binary responses and reaction times have a synergistic effect on the information gained by an observer.
Technical Summary

The method consists of estimating the Shannon information gained by an observer when presented with a stimulus. This is estimated as the mutual information between stimulus strength (e.g. luminance) and a combination of binary responses (e.g. yes/no) and their associated reaction times. Combined responses for a single observer were used to estimate a 2x2 covariance matrix $U$, which (under Gaussian assumptions) determines the entropy $H(U)$ of the joint distribution of binary responses and their reaction times. A proportional rate diffusion (PRD) model was fitted to each observer’s combined response data. The model’s residual noise defines a 2x2 covariance matrix $V$, which determines the entropy $H(V)$ of the joint conditional distribution of binary responses and their reaction times. The mutual information between stimulus strength and each observer’s mean responses is $I = H(U) - H(V)$ bits; further analysis was used to estimate the mutual information $I_{single}$ between stimulus strength and each observer’s individual responses. Model fitting was achieved by finding parameter values that maximise $I$. The rate at which an observer acquired information was estimated as $R = I_{single}/\overline{RT_{dec}}$ bits/s, where $\overline{RT_{dec}}$ is an observer’s mean decision time, which is a parameter of the fitted PRD model.
1 Introduction

In a typical psychophysical experiment, half the data collected from each observer is discarded. Specifically, binary responses (e.g. yes/no) are used to estimate parameters such as threshold, whereas reaction times are usually discarded. Here, we use Shannon’s information theory [Shannon and Weaver, 1949] to combine binary responses with reaction times to define Shannon competence $s’$. This has two key advantages: a) fewer stimulus presentations are required to achieve a given precision in estimates of psychophysical parameters, b) in contrast to conventional measures (e.g. $d’$), Shannon competence $s’$ is immune to the effects of observer-specific speed-accuracy trade-offs.

Here, we assume a two-alternative forced choice (2AFC) design, in which a pair of stimuli is shown to an observer, who then decides which stimulus is brighter (for example). The reference stimulus remains constant across trials, whereas the comparison stimulus varies (e.g. in brightness) between trials. Each presentation of a stimulus pair defines a single trial, and the (signed) difference between stimuli defines the stimulus strength $x_i$. The stimulus strength varies over $S$ different values, and each stimulus pair is presented $N_i$ times at each stimulus strength $x$. The observer’s response consists of a binary response $B$ and the reaction time $RT$ of that response.

If the observer chooses the comparison stimulus $n_i$ times at a given stimulus strength $x_i$ then the probability of choosing the comparison stimulus is estimated as the proportion

$$P_i = \frac{n_i}{N_i}. \quad (1)$$

The probability $P_i$ of choosing the comparison stimulus increases as the stimulus strength comparison increases, as shown in Figure 1a. For completeness, the proportion of correct

![Figure 1: The psychometric function (a) and chronometric function (b), from the face inversion experiment for one observer (see Figure 4). The width scaling factor $w$ applied to the comparison image is indicated on the abscissa. The stimulus strength is effectively $x = w - 1$. There were 21 discrete stimulus strengths, and the stimulus at each strength was presented $N_i = 20$ times.](image)

(a) Each dot represents the observed proportion of trials for which the observer chose the comparison stimulus, and the solid curve is the fitted psychometric function.

(b) Each dot represents the $RT$ of a single trial for the same responses as in Figure 1a ($RT$s greater than 2 seconds are not shown). The solid curve is the fitted chronometric function.
responses is

$$P^c_i = \frac{1}{1 + \exp[-|\log P_i/(1-P_i)|]}.$$  \(2\)

If the reaction time to \(j\)th presentation of the stimulus pair with signal amplitude \(x_i\) is \(\tau_j\), then the mean reaction time at \(x_i\) is

$$RT_i = \frac{1}{N_i} \sum_{j=1}^{N_i} \tau_j.$$  \(3\)

As the comparison stimulus strength increases, the mean reaction time \(RT\) increases until the stimulus strength of the comparison stimulus matches the stimulus strength of the reference stimulus, and then \(RT\) decreases again, as shown in Figure 1\(b\). Consequently, the mean \(RT\) increases until \(x\) equals zero, and then decreases again as \(x\) continues to increase.

The combined mean response of a single observer at stimulus strength \(x_i\) can be represented with the vector variable

$$y_i = (RT_i, P_i).$$  \(4\)

2 Measuring Mutual Information

Ultimately, performance is limited by the amount of Shannon information [Shannon and Weaver, 1949] \(I_{obs}(y; x)\) an observer gains when presented with stimuli with strength \(x\). Crucially, \(I_{obs}(y; x)\) cannot be less than the mutual information \(I(y; x)\) between stimulus strength and \(y\), the combined mean binary responses and their associated mean \(RT\)s,

$$I_{obs}(y; x) \geq I(y; x) \text{ bits}.$$  \(5\)

In other words, the observer gains an average of at least \(I(x; y)\) bits of information when presented with stimulus pairs at strength \(x\). We cannot measure \(I_{obs}(y; x)\), but we can measure \(I(x; y)\), which provides a lower bound for \(I_{obs}(y; x)\).

The mutual information can be obtained as the entropy \(H(y)\) of the observer’s responses minus the residual entropy in those responses conditioned on the stimulus strength \(x\),

$$I(x; y) = H(y) - H(y|x) \text{ bits}.$$  \(6\)

When expressed in terms of geometric areas in Figure 2, mutual information between stimulus strength \(x\) and an observer’s mean responses \(y\) is

$$a + b + c = (a + b + c + e + f + g) - (e + f + g) \text{ bits}.$$  \(7\)

Evaluating Unconditional Entropy \(H(y)\)

For a given observer, the grand mean reaction time is \(\overline{RT}\), and the overall proportion of trials on which the observer chooses the comparison stimulus is \(\overline{P}\). The observer’s grand mean response, taken over all trials and stimulus strengths, is

$$\bar{y} = (\overline{RT}, \overline{P}),$$  \(8\)

where \(\overline{P}\) is the grand mean value of \(P_i\) for a given observer

$$\overline{P} = \frac{1}{S} \sum_{i=1}^{S} P_i.$$  \(9\)
and $\overline{RT}$ is the grand mean value of $RT_i$ for a given observer

$$\overline{RT} = \frac{1}{S} \sum_{i=1}^{S} RT_i.$$  (10)

The noise in $RT$ at stimulus strength $x_i$ is

$$\eta_i^{RT} = RT_i - \overline{RT}.$$  (11)

Similarly, the noise in $P_i$ is

$$\eta_i^P = P_i - \overline{P}.$$  (12)

The vector-valued noise in the mean responses at stimulus strength $x_i$ is then

$$\eta_i = (\eta_i^{RT}, \eta_i^P).$$  (13)

If the joint distribution of noise in $y$ is Gaussian then the probability (density) is

$$p(y_i) = \frac{1}{(2\pi)^{k/2}|U|^{1/2}} \exp \left( -\frac{1}{2} \eta_i U^{-1} \eta_i^T \right),$$  (14)

where $k = 2$ is the number of variables (i.e. $RT$ and $P$), as shown in Figure 3a, $T$ is the transpose operator, and $|U|$ is the determinant of the covariance matrix

$$U = \begin{pmatrix} \text{var}(\eta^{RT}) & \text{cov}(\eta^{RT}, \eta^P) \\ \text{cov}(\eta^{RT}, \eta^P) & \text{var}(\eta^P) \end{pmatrix}.$$  (15)

---

Figure 2: How the entropy $H(x)$ in stimulus strength $x$ is accounted for by the entropy $H(RT)$ in mean reaction time $RT$ and entropy $H(P)$ in the probability $P$ of a particular binary response. The entropies of $x$, $P$ and $RT$ are represented by the discs $X$, $Y$ and $Z$, respectively. The mutual information between $x$ and $P$ is $I(x; P) = (a + b)$, the mutual information between $x$ and $RT$ is $I(x; RT) = (a + c)$, and the mutual information between $x$ and the combined response $RT, P$ is $I(x; RT, P) = a + b + c$. 
where \( \text{cov} \) represents covariance

\[
\text{cov}(\eta^{RT}, \eta^P) = \text{cov}(RT, P) = \frac{1}{S} \sum_{i=1}^{S} (RT_i - \overline{RT})(P_i - \overline{P}),
\]

and \( \text{var} \) represents variance,

\[
\text{var}(\eta^{RT}) = \text{var}(RT) = \frac{1}{S} \sum_{i=1}^{S} (RT_i - \overline{RT})^2
\]

\[
\text{var}(\eta^P) = \text{var}(P) = \frac{1}{S} \sum_{i=1}^{S} (P_i - \overline{P})^2.
\]

The determinant \(|U|\) is a generalised measure of variance, which indicates the overall spread of the distribution of \( RT, P \) values, and is obtained as

\[
|U| = \text{var}(RT) \times \text{var}(P) - \text{cov}(RT, P).
\]

For example, a simple rotation of axes in Figure 3a ensures that the covariance terms become zero, and then it is more obvious that each variance term indicates the length of one axis of an ellipse, so the product of variances is proportional to the area (spread) of the ellipse.

Finally, the (differential) entropy of a Gaussian distribution with covariance matrix \( U \) is

\[
H(y) = 0.5 \log_2 (2\pi e)^2 |U| \text{ bits},
\]

where logarithms have base 2, which ensures that entropy is measured in bits.

**Figure 3:** Contour maps of fitted joint Gaussian distributions for a typical observer.

a) Joint distribution of observer’s data (Equation 14). Each dot represents the observer’s mean response \((RT_i, P_i)\) to one stimulus strength \(x_i\), where \(P_i\) is the probability of choosing the comparison stimulus and \(RT_i\) is the corresponding mean reaction time at stimulus strength \(x_i\).

b) Joint distribution of residual noise, after the model has been fitted to the observer’s data.
Evaluating Conditional Entropy $H(y|x)$

The term $H(y|x)$ in Equation 6 is the entropy of the joint distribution $p(RT, P|x)$, which corresponds to the area ($e + f + g$) in Figure 2. This is the average residual uncertainty in the value of $(RT, P)$ for a given stimulus strength. In order to evaluate $H(y|x)$, we follow the logic of the previous section. However, instead of estimating noise as the difference between the observer’s mean response $y_i$ and that observer’s grand mean, noise is estimated as the difference between $y_i$ and a model-based estimate $\hat{y}_i = (\hat{RT}_i, \hat{P}_i)$, where $\hat{RT}_i$ is the model’s estimate of the mean reaction time, and $\hat{P}_i$ is the model’s estimate of the proportion of trials on which the comparison stimulus was chosen. Accordingly, the model noise in the mean response at stimulus strength $x_i$ is

$$\varepsilon_i = (\varepsilon_{i}^{RT}, \varepsilon_{i}^{P}), \quad (22)$$

where

$$\varepsilon_{i}^{RT} = RT_i - \hat{RT}_i \quad (23)$$
$$\varepsilon_{i}^{P} = P_i - \hat{P}_i. \quad (24)$$

If the joint distribution of noise in $\hat{y}_i$ is Gaussian then

$$p(y_i|x_i) = \frac{1}{(2\pi)^{k/2}|V|^{1/2}} \exp\left(-\frac{1}{2} \varepsilon_i V^{-1} \varepsilon_i^T\right), \quad (25)$$

where $|V|$ is the determinant of the covariance matrix

$$V = \begin{pmatrix} \text{var}(\varepsilon^{RT}) & \text{cov}(\varepsilon^{RT}, \varepsilon^{P}) \\ \text{cov}(\varepsilon^{RT}, \varepsilon^{P}) & \text{var}(\varepsilon^{P}) \end{pmatrix}. \quad (26)$$

The (differential) entropy of a Gaussian distribution with covariance matrix $V$ is

$$H(V) = 0.5 \log_2 (2\pi e)^2 |V| \text{ bits}, \quad (27)$$

where the determinant of $V$ is

$$|V| = [\text{var}(\varepsilon^{RT}) \times \text{var}(\varepsilon^{P})] - \text{cov}(\varepsilon^{RT}, \varepsilon^{P}). \quad (28)$$

Finally, substituting Equations 21 and 27 into Equation 6 yields

$$I(x; y) = 0.5 \log_2 \left| \frac{U}{V} \right| \text{ bits}, \quad (29)$$

where $|U|/|V|$ is the ratio of elliptical areas in Figures 3a and 3b. Notice that the absolute range of reaction times and binary responses has no effect on the mutual information because this is based on the ratio variances before and after fitting a model to the data.

Information Gained from Responses Versus Mean Responses

So far we have derived an expression for the mutual information in the average response $y_i$ to a stimulus strength $x_i$, where this average is taken over $N_i$ trials. The average mutual information in the observer’s response in a single trial is

$$I(y_{\text{single}}; x) \geq \frac{1}{2} \log_2 \left[ 1 + \frac{2^{2I(x,y)} - 1}{N_i} \right] \text{ bits}, \quad (30)$$

with equality if distributions are Gaussian (see Appendix).
Estimating Mutual Information for RT

The observer entropy based only on reaction time is

$$H(\eta^{RT}) = 0.5 \log_2 2\pi e \text{var}(\eta^{RT}) \text{ bits.} \quad (31)$$

The model entropy based only on reaction time is

$$H(\varepsilon^{RT}) = 0.5 \log_2 2\pi e \text{var}(\varepsilon^{RT}) \text{ bits.} \quad (32)$$

The mutual information based on reaction time is then given by

$$I^{RT} = H(\eta^{RT}) - H(\varepsilon^{RT}) = 0.5 \log_2 \frac{\text{var}(\eta^{RT})}{\text{var}(\varepsilon^{RT})} \text{ bits.} \quad (33)$$

Estimating Mutual Information for P

By analogy, the mutual information based on binary responses is then given by

$$I^P = H(\eta^P) - H(\varepsilon^P) = 0.5 \log_2 \frac{\text{var}(\eta^P)}{\text{var}(\varepsilon^P)} \text{ bits.} \quad (34)$$

3 The CEPRD Model

We require a model $M$ which has a ‘response’ to a stimulus strength $x_i$, where this response is a mean reaction time $\hat{RT}$ and a probability $\hat{P}_i$

$$\hat{y}_i = (\hat{RT}_i, \hat{P}_i) = M(\theta, x_i), \quad (35)$$

where $\theta$ is a vector of parameters, defined below. The model $M$ can be instantiated as a proportional rate diffusion (PRD) model [Palmer et al., 2005]. Here, we use of the extended PRD (EPRD) model [Stone, 2014]; this differs from the PRD model by making use of the probability $P$ that an observer chooses a comparison stimulus, and by incorporating a point-of-subjective-equality parameter $x_{PSE}$. The model’s mean $RT$ at stimulus strength $x_i$ defines the chronometric function

$$\hat{RT}_i = \hat{RT}_{\text{dec,}i} + \hat{RT}_m \text{ seconds,} \quad (36)$$

where $\hat{RT}_m$ is the component of reaction time required for physical movement once a decision has been made, and where the mean time taken to decide which response to make is

$$\hat{RT}_{\text{dec,}i} = \frac{A}{K(x_i - x_{\text{PSE}})} \tanh(AK(x_i - x_{\text{PSE}})) \text{ seconds.} \quad (37)$$

The perceived stimulus strength is

$$x_i - x_{\text{PSE}}, \quad (38)$$

where $x_{\text{PSE}}$ is the signal strength $x$ at which the reference and comparison stimuli are perceived as being the same. The model response probability $\hat{P}_i$ at stimulus strength $x_i$ defines the psychometric function

$$\hat{P}_i = \frac{\tanh(AK(x_i - x_{\text{PSE}}))}{2} + 1/2. \quad (39)$$
Solving for \( \tanh(AK(x_i - x_{PSE})) \) and substituting in Equation 37, the relation between the psychometric and chronometric functions is

\[
\hat{RT}_{dec,i} = \frac{A}{K} \frac{2\hat{P} - 1}{x_i - x_{PSE}} + \hat{RT}_m \text{ seconds,}
\]

(40)

From Equation 39, the probability that an observer chooses the comparison stimulus depends on the product \( AK \). In contrast, (from Equation 40) decision time depends on the ratio \( A/K \) and on the product \( AK \).

The model parameters for a single observer can be represented as the vector

\[
\theta = (A, K, RT_m, x_{PSE}).
\]

(41)

Model parameters are estimated using Equation 25 to define the log likelihood

\[
L(\theta) = \sum_{i=1}^{S} \log_2 p(y_i|x_i).
\]

(42)

Values of the parameters \( \theta \) that maximise \( L \) are found using the simplex search algorithm [Press et al., 1989]. After fitting the model to an observer’s data, Equation 36 provides a value of \( RT_i \), and Equation 39 provides a value \( \hat{P}_i \), for a given a stimulus strength \( x_i \). These equations were used to plot the solid curves in Figures 1a and 1b.

Note that estimating model parameters by maximising \( L(\theta) \) represents an improvement on the methods reported in [Palmer et al., 2005] and [Stone, 2014]. In those papers, model parameters were estimated by minimising the product \( \text{var}(\varepsilon_P) \times \text{var}(\varepsilon_{RT}) \), which amounts to ignoring the covariance between \( P \) and \( RT \). To differentiate between these models, the current version is called the covariant EPRD (CEPRD) model.

4 Results

Demonstration Experiment: Fat-Face Thin

The CEPRD model described above was used to estimate the parameters \( \theta \) for a simple demonstration experiment. On each trial, the observer was presented with a coloured picture of an upright face and an inverted face (see Figure 4) on a computer screen, and was required to indicate which face appeared to be wider by pressing a left/right computer key. The faces remained visible until a response was made, and there was an interval of 0.5s between trials. For half of the

![Image of faces](http://illusionoftheyear.com/2010/the-fat-face-thin-fft-illusion).

Figure 4: Schematic illustration of typical stimulus shown to observer on a single trial. The observer has to choose the face that looks wider. The stimulus used in the experiment was a picture of the actor James Corden’s face, with all background details removed (the illusion can be seen at [http://illusionoftheyear.com/2010/the-fat-face-thin-fft-illusion]).
trials, the reference stimulus was an upright face, and the comparison stimulus was an inverted version of the same face, and these were swapped for the other half of the trials. The width of the comparison image was determined by one of 21 stretch factors \( s = 0.90, 0.91, ..., 1.10 \) (the heights of both stimuli were the same, and constant throughout the experiment). The stimulus strength was defined to be \( x = s - 1 \), so that \( x \) varied between -0.1 and 0.1. For a given value of \( s_i \), the observer was presented with the same stimulus pair for a total of \( N_i = 20 \) trials. Stimuli were shown in random order, and the left/right position of reference/comparison stimuli was counterbalanced across trials.

For completeness, the mean parameter values are \( A = 0.848 \), \( K = 31.3 \), \( x_{PSE} = 1.00 \), which are consistent with values reported for other experiments using this type of model [Palmer et al., 2005]. The average reaction time across all observers is \( \overline{RT} = 1.130 \text{s} \), and the average decision time is \( \overline{RT}_{dec} = 0.632 \text{s} \).

The main results are summarised in Figure 5. The following estimates are means, where each mean is taken over all five observers, as shown in Figure 5. The mean mutual information per trial between combined reaction time, binary response and stimulus strength is

\[
I(y_{\text{single}}; x) = 1.62 \text{ bits.} \quad (43)
\]

Combined with the mean decision time of 0.632 seconds, this implies the average observer acquires information at the rate of

\[
R = \frac{I(y_{\text{single}}; x)}{\overline{RT}_{\text{dec}}} = \frac{1.62}{0.632} = 2.56 \text{ bits/s.} \quad (44)
\]

The mean mutual information between reaction time per trial and stimulus strength is

\[
I(\tau; x) = 0.075 \text{ bits.} \quad (45)
\]

![Figure 5](image-url)

Figure 5: Mutual information between response per trial and stimulus strength for 5 observers, obtained (Equation 30). For each observer, reading from left to right, each bar represents (a, blue bar) mutual information \( I(\tau; x) \) between reaction time and stimulus strength \( x \). (b, green bar) mutual information \( I(r; x) \) between binary response and \( x \). (c, yellow bar) total mutual information \( I(\tau, r; x) \).
The mean mutual information between binary responses and stimulus strength is

\[ I(r; x) = 0.970 \text{ bits}. \] (46)

If reaction time and binary responses represented independent sources then the total mutual information would be

\[ I_{\text{sum}} = I(\tau; x) + I(r; x) = 0.75 + 0.970 = 1.045 \text{ bits}. \] (47)

The combination of reaction times and binary responses provides more information \( I(y_{\text{single}}; x) = 1.62 \text{ bits} \) about \( x \) than the sum of their individual contributions \( I_{\text{sum}} = 1.045 \text{ bits} \). This, in turn, suggests that there is synergy between reaction time and binary responses. Even if this were not true, the finding that reaction times provide additional information means that making use of both reaction times and binary responses should increase the precision of parameter estimates, or (equivalently) substantially reduce the number experimental trials required.

**Shannon Competence and D-prime**

For a signal to noise ratio \( SNR \), mutual Information obeys the relation \[ I \leq 1/2 \log_2(1 + SNR) \text{ bits}, \] (48)

with equality if distributions are Gaussian, as is assumed here. The signal to noise ratio \( SNR = S/N \), where \( S \) and \( N \) are standard symbols for signal and noise variance, respectively.

To relate mutual information to more conventional measures of performance such as the discriminability measure \( d' \) (d-prime), we note that \( d'^2 = S/N \) \[ \text{Rieke et al., 1997} \], so that

\[ I(r; x) = 1/2 \log_2(1 + d'^2) \text{ bits}. \] (49)

Solving Equation (49) for \( d' \) yields

\[ d' = (2^{2I(r; x)} - 1)^{1/2}. \] (50)

Given that \( I(r; x) = 0.970 \), it follows that, if only binary responses were used to measure performance then this would have yielded a value of \( d' = 1.68 \).

However, the mutual information \( I(y_{\text{single}}; x) \) takes account of both \( RT \) and \( P \), which allows a more general measure of performance to be defined by making use of the relation,

\[ I(y; x) = 1/2 \log_2(1 + s'^2) \text{ bits}. \] (51)

Solving for \( s' \) allows a new measure, Shannon competence, to be defined as

\[ s' = (2^{2I(y_{\text{single}}; x)} - 1)^{1/2}. \] (52)

The value of \( I(y_{\text{single}}; x) = 1.62 \text{ bits} \) implies a Shannon competence of \( s' = 2.91 \). Comparing this to \( d' = 1.68 \) suggests that taking account of both \( RT \) and \( P \) almost doubles the estimated discriminability of stimuli in this experiment.
5 Discussion

Early attempts to incorporate both reaction time and binary responses into measures of performance tended to be ad hoc (see [Vandierendonck, 2017] and [Stafford et al., 2020] for reviews). For example, the inverse efficiency score (IES) [Townsend and Ashby, 1978] is the ratio IES = RT/(percent correct), which has not been found not to be justified [Bruyer and Brysbaert, 2011]. Specifically, the sub-title of the paper [Bruyer and Brysbaert, 2011] asks the question: Is the inverse efficiency score (IES) a better dependent variable than the mean reaction time (RT) and the percentage of errors (PE)? According to that paper, the answer seems to be no.

Even though more recent attempts are sophisticated in many respects, they ignore the covariance between RT and binary responses (e.g. Equation 17 in [Bogacz et al., 2006], and Equation 14 in [Bogacz et al., 2010]). Similarly, PRD models explicitly ignore such interactions, as explained in the next paragraph.

In summary, this paper presents four innovations.

1. The covariant extended proportional rate diffusion (CEPRD) model was fitted by taking into account the full covariance structure of reaction times and binary responses. This contrasts with both Palmer et al’s PRD model [Palmer et al., 2005] and Stone’s EPRD model [Stone, 2014], where model parameter values were estimated by minimising the product of reaction time variance and binary response variance. Thus, both of these previous models implicitly assume that the covariance between reaction times and binary responses is zero, an assumption which is unwarranted from the results reported here.

2. On a related theme, this represents an improvement on Stone’s EPRD model [Stone, 2014], which used independent estimates of information based either on RT or binary responses (but not both) to estimate lower and upper bounds of mutual information $I_{single}$.

3. By estimating the entropy of the joint distribution of reaction times and binary responses and comparing this to the residual entropy of the joint distribution after responses have been fitted to a model, we were able to estimate the mutual information between responses and stimulus strength. When divided by observer decision time, the information rate in bits/s was estimated.

4. This mutual information allows a particular observer-specific signal-to-noise ratio to be calculated, which can be used to estimate $s'$, a measure of discriminability that is exactly analogous to the conventional discriminability measure $d'$.

Finally, note that the general strategy outlined in Section 2 does not depend on any particular model (e.g. EPRD). For example, model-free estimates the unconditional entropy $H(U)$ and of the conditional entropy $H(V)$ could be obtained from of a three-dimensional table, in which the axes are reaction times, binary responses and stimulus strength.
A Appendix

A.1 The Shannon Information of a Single Response

We have derived expressions for the Shannon information implicit in the average reaction time $RT_i$ and also in the average binary response, which is summarised as the proportion $P_i$ of comparison responses, for a stimulus strength $x_i$. Here, we derive an expression for the Shannon information associated with a single trial; first for reaction time, and then for binary responses.

As the number of trials at each stimulus strength is increased, so the variance in each mean reaction time $RT$ decreases, and the central limit theorem (CLM) ensures that the distribution of errors becomes increasingly Gaussian. The mutual information between two variables (e.g. reaction time and stimulus strength) depends on the signal to noise ratio $SNR$,

$$I \leq \frac{1}{2} \log_2(1 + SNR),$$

(53)

where $SNR$ is the signal variance expressed as a fraction of the noise variance in the measurement [Shannon and Weaver, 1949].

Invoking the CLM, we assume that the distribution of differences $\Delta \tau$ between individual reaction times $\tau$ and the mean reaction time (at one stimulus strength) is Gaussian. Because the mutual information is defined as the entropy of $\tau$ minus the entropy of the noise $\Delta \tau$ in $\tau$, we can assume equality in Equation 53 [Rieke et al., 1997]. In fact, we do not need to rely on the central limit theorem here, because even if the perturbing noise is not Gaussian, Shannon’s Theorem 18 [Shannon and Weaver, 1949] implies equality in Equation 53 so that

$$I = \frac{1}{2} \log_2(1 + SNR) \text{ bits.}$$

(54)

We already have a value for the mutual information $I(RT; x)$, so we can re-arrange Equation 53 to find the SNR associated with $RT$:

$$SNR_{RT} = 2^{2I(x,RT)} - 1 \text{ bits.}$$

(55)

However, the mutual information $I(x, RT)$ obtained tells us how much average Shannon information each mean reaction time provides about stimulus strength, whereas we want to know how much average information each individual reaction time provides about stimulus strength. Because the value of SNR in Equation 53 is based on mean RTs, each of which involves $N_i$ trials, the variance of the measurement noise has been reduced by a factor of $N_i$ relative to the noise in the reaction time of a single trial (provided this noise is iid). This implies that the value of SNR for a single trial is

$$SNR_{\tau} = SNR_{RT}/N_i$$

$$= (2^{2I(RT;x)} - 1)/N_i \text{ bits.}$$

(56)

If we substitute $SNR_{\tau}$ into Equation 53 then we obtain an estimate of the average Shannon information $I(x, \tau)$ implicit in the observer’s reaction time in a single trial

$$I(\tau;x) = \frac{1}{2} \log_2 \left[ 1 + \frac{(2^{2I(RT;x)} - 1)}{N_i} \right] \text{ bits.}$$

(58)

By analogy, the average Shannon information $I(x, r)$ implicit in the observer’s binary response $r$ in a single trial is

$$I(r;x) = \frac{1}{2} \log_2 \left[ 1 + \frac{(2^{2I(P;x)} - 1)}{N_i} \right] \text{ bits.}$$

(59)
References

[Bogacz et al., 2006] Bogacz, R., Brown, E., Moehlis, J., Holmes, P., and Cohen, J. (2006). The physics of optimal decision making: a formal analysis of models of performance in two-alternative forced-choice tasks. *Psychological review*, 113(4):700.

[Bogacz et al., 2010] Bogacz, R., Hu, P., Holmes, P., and Cohen, J. (2010). Do humans produce the speed–accuracy trade-off that maximizes reward rate? *The Quarterly Journal of Experimental Psychology*, 63(5):863–891.

[Bruyer and Brysbaert, 2011] Bruyer, R. and Brysbaert, M. (2011). Combining speed and accuracy in cognitive psychology: Is the inverse efficiency score (ies) a better dependent variable than the mean reaction time (rt) and the percentage of errors (pe)? *Psychologica Belgica*, 51(1):5–13.

[Palmer et al., 2005] Palmer, J., Huk, A., and Shadlen, M. (2005). The effect of stimulus strength on the speed and accuracy of a perceptual decision. *J Vision*, 5:376–404.

[Press et al., 1989] Press, W., Flannery, B., Teukolsky, S., and Vetterling, W. (1989). *Numerical Recipes in C*. Cambridge University Press.

[Rieke et al., 1997] Rieke, F., Warland, D., de Ruyter van Steveninck, R., and Bialek, W. (1997). *Spikes: Exploring the Neural Code*. MIT Press, Cambridge, MA.

[Shannon and Weaver, 1949] Shannon, C. and Weaver, W. (1949). *The Mathematical Theory of Communication*. University of Illinois Press.

[Stafford et al., 2020] Stafford, T., Pirrone, A., Croucher, M., and Krystalli, A. (2020). Quantifying the benefits of using decision models with response time and accuracy data. *Behavior research methods*, 52(5):2142–2155.

[Stone, 2015] Stone, J. V. (2015). *Information Theory: A Tutorial Introduction*.

[Stone, 2014] Stone, J. V. (2014). Using reaction times and binary responses to estimate psychophysical performance: An information theoretic analysis. *Frontiers in Neuroscience*, 8(35).

[Townsend and Ashby, 1978] Townsend, J. and Ashby, F. (1978). Methods of modeling capacity in simple processing systems. *Cognitive theory*, 3:199–139.

[Vandierendonck, 2017] Vandierendonck, A. (2017). A comparison of methods to combine speed and accuracy measures of performance: A rejoinder on the binning procedure. *Behavior research methods*, 49(2):653–673.