Black Holes as Elementary Particles

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ABSTRACT

It is argued that the qualitative features of black holes, regarded as quantum mechanical objects, depend both on the parameters of the hole and on the microscopic theory in which it is embedded. A thermal description is inadequate for extremal holes. In particular, extreme holes of the charged dilaton family can have zero entropy but non-zero, and even (for $a > 1$) formally infinite, temperature. The existence of a tendency to radiate at the extreme, which threatens to overthrow any attempt to identify the entropy as available internal states and also to expose a naked singularity, is at first sight quite disturbing. However by analyzing the perturbations around the extreme holes we show that these holes are protected by mass gaps, or alternatively potential barriers, which remove them from thermal contact with the external world. We suggest that the behavior of these extreme dilaton black holes, which from the point of view of traditional black hole theory seems quite bizarre, can reasonably be interpreted as the holes doing their best to behave like normal elementary particles. The $a < 1$ holes behave qualitatively as extended objects.
1. Introduction and Summary

Is there a fundamental distinction between black holes and elementary particles? The use of concepts like entropy [1], temperature [2], and dissipative response [3] in the description of black hole interactions makes these objects seem very different from elementary particles. This has helped inspire some suspicion that the description of the holes may require a departure from the fundamental principles of quantum mechanics [4]. However a more conservative attitude is certainly not precluded. In the bulk of this paper, we shall analyze a particular class of black hole solutions (extremal dilaton black holes [5, 6]) in some detail, and argue that some of these do in fact appear to behave very much as elementary particles. First, though, let us briefly sketch a radically conservative interpretation of black hole physics in general – conservative, in the sense that it attempts to avoid the conclusion that these bodies have radically new and paradoxical properties, utterly unlike anything we have seen elsewhere in physics.

The most notable tangible qualitative physical properties associated with a black hole’s entropy, temperature, and dissipative response are probably:

1. Any projectile impinging within the geometrical radius is almost certainly absorbed.

2. The energy thus deposited is taken up by the hole and re-emitted only after a long time delay.

3. The form in which the energy is emitted has very little to do with the details of how it was deposited. Indeed, the emission may be regarded as thermal.

There are other instances in physics of entities with properties quite similar to those of black holes in these respects. The liquid droplet model of atomic nuclei springs to mind. Indeed any macroscopic body with many feebly interacting internal degrees of freedom, so large that the probability of a particle traversing the body without being slowed and captured is small, might be expected to exhibit these characteristic behaviors. Such a body will have many quasi-stable low-lying
states, corresponding to excitations of the internal degrees of freedom which only slowly dissipate. Almost regardless of its initial form, energy upon being absorbed will be distributed among the low-lying states in a statistical fashion (thermalized). This process of distribution will take a long time relative to the time an undeflected particle would take to pass through the body. Finally, the energy will escape at the surface in a form which has very little to do with how it was deposited: memory of the initial excitation has long since been obliterated. Thus, each of the main qualitative features of black hole interactions mentioned above is reproduced by this general, unmysterious class of physical objects.

There are a couple of obvious differences between black holes and liquid drop nuclei, however, which should prevent one from accepting the analogy too easily. First there is the fact that the gravitational interaction is universal, unlike the strong interaction. Thus conceivably there is something more fundamental about the thermalization process in the case of black holes, leading to irredeemable loss of information and breakdown of quantum mechanics (loss of unitary time evolution). Second there is the related fact that in the case of the compound nucleus we have a reasonably good understanding of how to derive the semi-phenomenological picture sketched above as an approximation to a more fundamental microscopic theory which is manifestly consistent with all general principles of physics, while in the black hole case there is no comparable understanding. The closest approach to such understanding at present is probably the “membrane paradigm” of black hole physics [3], which accurately summarizes many aspects of the low-energy interaction of black holes in terms of a dynamical surface theory coupled to the external world. It is plausible that quantization of a theory of this kind could give a more satisfactory microscopic understanding of the internal states of black holes, as states of the effective surface theory. (In this regard, it is most suggestive that the entropy is, to a first approximation, proportional to the surface area – that is, an extensive quantity in the effective theory.) The many feebly interacting degrees of freedom are the approximate normal modes of this surface theory; thus, they are collective oscillations of the gravitational field. Though much work will be
required to substantiate this picture, we feel there is no evident barrier to taking the proposed analogy quite literally.

From this point of view, the Schwarzschild black holes are very complicated mixtures of collective excitations of the gravitational field. Their finite temperature indicates that they are far from the ground state, and their large entropy indicates that they are embedded in a dense quasi-continuum of states – in particular, that there are many ($\sim e^{S} = \exp \frac{4\pi G M^2}{\hbar}$) states within the typical thermal energy interval $T = \hbar/(8\pi GM)$ sampled by the hole. It therefore seems reasonable to expect that a thermal description of the hole should be quite appropriate, and that deviations from it, in particular those due to quantum effects, should be small and very difficult to disentangle in practice. (Significant correlations between emission and absorption, or between successive emissions, should exist in the rare cases when such events are separated by small time intervals, of the order of at most the transit time.)

The situation is quite different for near-extremal black holes, i.e. holes with nearly vanishing temperature or entropy. As has recently been emphasized, and shall be reviewed below, the thermal description of such holes signals its own breakdown [7]. Emission of a single quantum changes the formal value of the temperature drastically. More fundamentally, the total specific heat of the hole (not the specific heat per unit volume) becomes small as the hole approaches extremality, which indicates that the number of distinct states available to the hole in its thermal energy interval becomes small, so that the foundation for a statistical description is removed. It is in this domain that the quantum theory of black holes should come into its own, and the question whether black holes behave like more familiar quantum objects or even elementary particles acquires a sharp edge.

The classic extremal black holes of the Kerr-Newman family have the character that they have finite entropy at zero temperature. According to the views outlined above (i.e. the conventional view of entropy as it appears elsewhere in physics), this
entropy should be viewed as an indication of massive unresolved degeneracy for these holes. This degeneracy is unusual, but not completely bizarre from a more microscopic point of view. After all the hole is capable of absorbing arbitrarily small amounts of energy (soft photons or gravitons); thus there are clearly many low-lying states, and in an approximate calculation \( i.e. \) the classical evaluation of black hole mass, which makes no reference to the state of the 2+1 dimensional membrane theory) these may appear exactly degenerate. The degeneracy at the classical level, even if it can be rationalized in this way, at a minimum makes the black holes assume more the character of complex extended objects rather than elementary particles.

Recently it has been discovered that the character of near-extremal black holes can be affected drastically by the field content of the world \([5, 6]\). In particular, inclusion of an additional scalar degree of freedom (the dilaton), whose existence has been considered on various ground and is suggested by superstring theory, with suitable simple couplings has been found to affect the properties of extremal charged (Reissner-Nordström) black holes drastically. As we shall review shortly, the relevant coupling of the dilaton is naturally parametrized by a numerical parameter \( a \). The classic Reissner-Nordström solution is unmodified for \( a = 0 \). For \( 0 < a < 1 \) the extremal black hole has zero entropy and zero temperature; for \( a = 1 \) it has zero entropy and finite temperature; for \( 1 < a \) it has zero entropy and formally infinite temperature.

In a previous paper \([7]\) a tentative interpretation of these results was put forward, that is entirely consistent with the general attitude described above. For \( 0 < a < 1 \) the interpretation is straightforward: zero entropy at zero temperature indicates that there is a non-degenerate ground state, which naturally does not radiate. For \( 1 \leq a \) the situation is more challenging. It was suggested that it could be understood only if the black hole has a mass gap – that is, if there are no arbitrarily low-energy excitations around the solution. For the entropy measures the density of available states, and with a non-zero temperature it becomes possible to sample states at any energies of order the temperature separation. Furthermore if
the formal temperature is non-zero while the entropy vanishes, we must anticipate that the mass gap is at least of order the temperature. If the formal temperature approaches infinity, as it does for the $1 < a$ holes as they approach extremality, then the mass gap must likewise approach infinity.

The main burden of the present paper is to demonstrate that these mass gaps do indeed exist. We shall perform a complete analysis of linear perturbations around the charged dilaton black hole solutions, to demonstrate this feature.

In plain English, the mass gap means that the black holes have become universally repulsive rather than attractive for low energy perturbations. Thus when probed by scattering of low energy projectiles, they will appear as elementary particles. There will be no time delay, and the outgoing radiation will be strongly, obviously, and uniquely correlated with the incoming radiation. The $a > 1$ black holes will respond this way to arbitrarily high energy probes, at least semi-classically. The infinite mass gap also serves to turn off the Hawking radiation, despite the finite temperature, due to vanishing grey-body factors.

The contents of the remainder of this paper are as follows. In Sections 2 and 3 we briefly review the mathematical form of the dilaton black hole solutions and their (formal) thermodynamic interpretation, respectively. In Section 4 we set up the general perturbation problem for the fields intrinsic to the solutions, and show how it may be reduced to tractable form. That this is practical at all is a result of some remarkable and unexpected simplifications. (In passing, we also clarify and explain some “miraculous” aspects of the theory of perturbations about classic Reissner-Nordström holes.) In Sections 5 and 6 we discuss the physical results which can be inferred from the perturbation calculation, including quantitative results for grey-body factors and the mass gaps. The problem of a spectator scalar field is used to illustrate many of the qualitative results in a much simpler context. In Section 7 we briefly note the anomalous behavior of the Callan-Rubakov mode for scattering of minimally electrically charged chiral fermions off a magnetically charged hole, which apparently uniquely does not feel the repulsive barrier
surrounding $a > 1$ holes. Section 8 contains a few concluding observations.

2. Form of black holes

We shall be considering a class of theories involving coupled gravitational, electromagnetic and scalar (dilaton) fields. The action governing this class of theories is

$$S = \int d^4 x \sqrt{-g} \left( R - 2(\nabla \Phi)^2 + e^{-2a\Phi} F^2 \right). \quad (2.1)$$

We use the metric convention $(+ - - -)$. $a$ is a dimensionless parameter, which we may assume to be non-negative. For $a = 0$ (2.1) becomes the standard Einstein-Maxwell action with an extra free scalar field. In this case a solution of the coupled Einstein-Maxwell equations becomes a solution of the new theory, taking $\Phi = \text{const}$. The case $a = 1$ is suggested by superstring theory, treated to lowest order in the world-sheet and string loop expansion.

A complete set of equations of motions is readily derived from this action. They are Maxwell’s equations

$$\partial_\mu (e^{-2a\Phi} \sqrt{-g} F^{\mu\nu}) = 0,$$

$$F_{\mu\nu,\rho} + F_{\rho\mu,\nu} + F_{\nu\rho,\mu} = 0, \quad (2.2)$$

Einstein’s equation

$$R_{\alpha\beta} = e^{-2a\Phi} (-2F_{\alpha\nu} F^\nu_{\beta} + \frac{1}{2} F^2 g_{\alpha\beta}) + 2\partial_\alpha \Phi \partial_\beta \Phi, \quad (2.3)$$

and the dilaton equation

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) = \frac{1}{2} ae^{-2a\Phi} F^2. \quad (2.4)$$

Static, spherically symmetric solutions of these equations have been found, representing charged black holes [5, 6]. The solution for an electrically charged
dilaton black hole has the line element

\[ ds^2 = \lambda^2 dt^2 - \lambda^{-2}dr^2 - R^2d\theta^2 - R^2 \sin^2 \theta d\varphi^2 \]  \hspace{1cm} (2.5)

where

\[ \lambda^2 = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} \]  \hspace{1cm} (2.6)

and

\[ R^2 = r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}}. \]  \hspace{1cm} (2.7)

In these equations \( r_+ \) and \( r_- \) are the values of the parameter \( r \) at the outer and the inner horizon, respectively, where the coefficient of \( dt \) vanishes and the flow of global time \( t \) has no influence. \( r_+ \) and \( r_- \) are related to the mass and charge of the hole according to

\[ 2M = r_+ + \left(\frac{1-a^2}{1+a^2}\right)r_- \]  \hspace{1cm} (2.8)

and

\[ Q^2 = \frac{r_-r_+}{1+a^2}. \]  \hspace{1cm} (2.9)

Finally, the electric and dilaton fields are given by

\[ e^{2a\Phi} = \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}} \]  \hspace{1cm} (2.10)

and

\[ F_{tr} = \frac{e^{2a\Phi}Q}{R^2}. \]  \hspace{1cm} (2.11)

Note that \( R \), not \( r \), has the normal meaning of the radial variable, in the sense that the area of the sphere obtained by varying \( \theta \) and \( \phi \) at a fixed \( t \) and \( r \) (or \( R \)) is \( 4\pi R^2 \).
There are also magnetically charged solutions of the same general form. They are obtained from the electric solution by making the changes

\[ F'_{\mu\nu} = e^{-2a\phi} \frac{1}{2} \epsilon_{\mu\nu}^{\lambda\rho} F_{\lambda\rho} \]
\[ \phi' = -\phi . \quad (2.12) \]

Particularly interesting is the structure of horizons, and the structure of extremal holes.

For \( a = 0 \) there are both inner and outer horizons, at \( r = r_\pm \). The geometry is not singular at either of these. However for \( a > 0 \) the geometry does become singular at \( r_- \), because \( R \) vanishes there. In fact \( r = r_- \) is a spacelike surface of singularity, very similar to the singularity in the classic Schwarzschild metric.

This difference gives rise to a big difference in the structure of the extremal black holes (defined as those on the verge of exposing a naked singularity) in the two cases. In all cases the extremal holes occur when \( r_+ = r_- \), which occurs for

\[ M^2 = \frac{Q^2}{(1 + a^2)} \quad \text{(extremal)} . \quad (2.13) \]

For \( a = 0 \) the physical radius \( R (= r) \) of the horizon is finite, in fact equal to \( M \), at this point. Further increase of \( Q \) at fixed \( M \) induces \( r_\pm \) to wander off into the complex plane, exposing the pre-existing singularity at \( r = 0 \). For \( a > 0 \) the physical radius of the horizon vanishes for the extremal hole, and the geometry is singular there. Further increase of \( Q \) at fixed \( M \) leads to the singular “inner” horizon \( r_- \) extending outside the ordinary horizon \( r_+ \), which however both remain real (forever if \( a \geq 1 \); until \( Q^2 = M^2/(1 - a^2) \) if \( a < 1 \)).

The distance to the horizon is infinite for the classic extremal Reissner-Nordström hole, although it can be traversed in finite proper time. For \( a > 0 \)
the distance remains finite, that is
\[ \int_{r_+} \frac{dr}{\lambda} < \infty . \] (2.14)

For \( a > 1 \) the tortoise co-ordinate, which is the appropriate co-ordinate for analyzing the wave equation for external fields interacting with the hole, has a \textit{finite range}. That is,
\[ \int_{r_+} dr^* \equiv \int_{r_+} \frac{dr}{\lambda^2} \sim \int_{r_+} \frac{dr}{(r-r_+)^{(1+a^2)}} < \infty \ (a > 1) . \] (2.15)

The fact that the tortoise co-ordinate does not extend to \(-\infty\) for extremal holes when \( a > 1 \) has an important implication. The proof of the usual classical no-hair theorems rely crucially on the fact that for ordinary black holes the tortoise co-ordinate, which is the co-ordinate that governs the wave equation for perturbations, extends to \(-\infty\) at the horizon. It is this fact which prevents the existence of finite “Yukawa tails” starting at the horizon. Such arguments evidently do not apply to the extremal holes under discussion, and other considerations are required to address the question of whether classical hair exists for these holes. We shall see below that there are infinite potential barriers surrounding the hole, which forbid the existence of at least some forms of hair; but the answer in principle depends on non-universal aspects of the coupling of the hole to external fields. In any case, the fact that the horizon is in this profound sense only a finite distance away is certainly suggestive of the elementary particle interpretation of these holes we are developing.
3. Thermal interpretation

3.1. Temperature and entropy

The quickest way to find the temperature of a static black hole is to consider its behavior for imaginary values of the time \([8]\). As is well known, for the classic black hole the horizon then represents a conical singularity of the resulting solution of the Euclideanized Einstein equations, which can be removed if the imaginary time variable is taken to be periodic with just the right period. The nonsingular configuration can be regarded as a contribution to the partition function for temperature equal to the inverse of the imaginary time period. Alternatively, the periodicity shows that fields in the black hole geometry can be in thermal equilibrium with a heat bath at that temperature at spatial infinity. The same procedure works, without modification, for the dilaton holes. For the temperature, one finds

\[
T = \frac{1}{4\pi r_+} \left( \frac{r_+ - r_-}{r_+} \right) \frac{1-a^2}{1+a^2}.
\]  

(3.1)

The entropy may then be inferred from the second law of thermodynamics, or alternatively, following the argument sketched above, computed from the interpretation of the black hole as a saddle-point contribution to the partition function. Both methods give the same result, to wit

\[
S = \pi r_+^2 \left( \frac{r_+ - r_-}{r_+} \right)^2 \frac{2a^2}{1+a^2} = \frac{1}{4} A.
\]  

(3.2)

where \(A\) is the area of the event horizon.

Evidently the extremal holes, for which \(r_+ = r_-\), will have finite entropy in case \(a = 0\), but zero entropy otherwise. The temperature of the extremal holes is zero for \(a < 1\), finite and equal to \(1/(8\pi M)\) (the same value as for a Schwarzschild hole) for \(a = 1\), and infinite for \(a > 1\).
3.2. Grey-body factors

Assuming for the moment that a thermal description is appropriate, so that each black hole can be in equilibrium with a thermal bath at its characteristic temperature, there still remains the question of how well it couples thermally to the external world. In other words, to calculate how fast it exchanges energy in a given mode with the external world, and thereby to a first approximation how fast it will radiate that mode into empty space, one must compute the appropriate grey-body factor. This is essentially the square of the absorption amplitude for the mode. More precisely, rate of radiation is given by

\[
\frac{dM}{dt} = -\Sigma_{\text{modes}} \alpha \int \frac{d\sigma}{2\pi} B_{\alpha,\sigma} \left(1 - |R_{\alpha}(\sigma)|^2\right)\hbar \sigma ,
\]

(3.3)

where \(\sigma\) is the frequency of the mode in global time \(t\), and \(B\) is the Boltzmann occupancy factor. (Thus \(B = 1/(\exp(\hbar \sigma/T) \pm 1)\) for uncharged fermions and bosons, respectively. For charged modes, or for modes carrying angular momentum in the background of a Kerr hole, the Boltzmann factor would of course be modified by the appearance of appropriate chemical potentials.) \(R_{\alpha}\) is the reflection amplitude, that is the amplitude of the wave reflected back to \(r_* = -\infty\) relative to the amplitude of the wave that propagates to \(r_* = +\infty\), subject to the requirement that there is no amplitude for waves travelling in from infinity.

3.3. Breakdown of thermal description

There are very significant circumstances, in which even the approximate description of a black hole as a thermal body is not physically consistent [7]. If the emission of a single quantum of the “typical” radiation product, which of course changes the mass of the black hole, also changes the formal value of its temperature by an amount that is large relative to the temperature itself, then the notion of the Boltzmann factor for its emission, and concretely equation (3.3), becomes ambiguous. One does not know whether to use the temperature before emission,
the temperature after, or something in between. This would seem to be a rather clear signal that the effect of back-reaction on the black hole geometry cannot be neglected, even approximately, in describing radiation from the hole.

As was discussed extensively in [7], the circumstance just mentioned occurs, and the thermal description of the hole breaks down, whenever

\[ \left| \left( \frac{\partial T}{\partial M} \right)_{Q,J} \right| \geq 1. \] (3.4)

An alternative form of this condition is

\[ T \left( \frac{\partial S}{\partial T} \right)_{Q,J} \leq 1. \] (3.5)

The alternative form has a profound physical interpretation. It says that the thermal description must break down when the available entropy of the black hole, i.e. the number of states available to it within its thermal energy interval, is small. Indeed, if that happens clearly a statistical description of the physics, which then involves only a few degrees of freedom, becomes inappropriate.

Upon evaluating the left-hand side of (3.4) for near-extremal holes we find in the \( a = 0 \) case

\[ \left| \left( \frac{\partial T}{\partial M} \right)_{Q,J} \right| \to \frac{1}{2\pi M^2 (1 - \frac{Q^2}{M^2})^{-\frac{1}{2}}} \to \infty, \] (3.6)

as \( 1 - \frac{Q^2}{M^2} \to 0 \). And for \( a \neq 0 \) we find from (2.8) and (2.9) that

\[ r_+ - r_- \to \text{const.} \left( M^2 - \frac{Q^2}{1 + a^2} \right) \] (3.7)

as \( f \equiv 1 - \frac{Q^2}{M^2 (1 + a^2)} \to 0 \), where the constant is non-singular in this limit. From this we find that

\[ T \sim f^\frac{1-a^2}{1+a^2} \] (3.8)

and that \( \left( \frac{\partial T}{\partial M} \right) \) diverges, except in the special case \( a = 1 \). The sign of this diverging quantity is of some interest: it indicates that for \( a < 1 \) the black hole has a positive
specific heat (that is, it cools as it radiates) as it approaches extremality, whereas for $a > 1$ the specific heat is negative.

The physical reason for the breakdown of a thermal description, as we have said, is the sparsity of nearby states. Now ordinarily one would think that a black hole has a great variety of nearby states – after all, it seems it should be possible to dump arbitrarily small amounts of energy into the hole in different partial waves, ... . How could this expectation fail? Actually there are two different ways it could fail, which are realized for $a < 1$ and $a \geq 1$ (including $a = 1$) extremal holes respectively.

It can fail if the notion of what is a “nearby” state becomes very demanding – i.e. if the thermal interval becomes very small, or in plain English if the temperature vanishes. (It might seem that we have proved too much here, because thermodynamics has had some famous successes in describing low-temperature physics. However, as was mentioned in [7], it is true as a matter of principle that the domain of validity of a thermal description, for a body of fixed size, shrinks to zero as the temperature approaches zero. By passing to arbitrarily large bodies we create more and more low-energy states, and thermal states at lower and lower temperatures become meaningful. This is of course not possible for a black hole, which is a single object with a definite size.) This is what happens for $a < 1$.

It can also fail even if the measure of “nearby” is generous – if the temperature is finite or even formally divergent – if there is a sufficiently large mass gap for the hole. The existence of such a gap entails among other things that it must not be possible to dump arbitrarily small amounts of energy into the hole in many ways – that the black hole effectively repels low-energy perturbations! As we shall see, this is what happens for $a \geq 1$.

Another aspect of the extremal black holes is crucially related to the existence of the mass gap. That is the question whether they really are endpoints of Hawking radiation, or whether they continue to radiate. Of course if they did continue to radiate, they would expose a naked singularity. However the calculations that
follow make it very plausible that the radiation shuts off. For $a < 1$ of course there is nothing to show, because the temperature vanishes for the extremal hole. For $a > 1$ the temperature of the extremal hole is formally infinite, as is the mass gap. To elucidate the physics, one must consider how this limit is approached. We shall show that as the hole approaches extremality its grey-body factors kill all the radiation below a critical energy, so that the black-body emission spectrum is extremely distorted. The critical energy is always larger than the temperature, and in particular it becomes arbitrarily large – larger than the mass of the hole – as extremality is approached. Again, therefore, the radiation shuts off. The case $a = 1$ remains enigmatic.

It is important to note that in all cases the grey-body factor for low energy radiation is never quite zero for a non-extremal hole, so that the radiation never quite turns off. Thus the extremal hole is in fact approached, although slowly, as the radiation proceeds.

4. Formulation of perturbation equations

We turn now to the formulation of the equations satisfied by small perturbations of the fields present in the stationary configuration, that is the metric functions, the electromagnetic field and the dilaton field. For concreteness, we shall first study the case of an electrically charged dilaton black hole.

Since the background configuration is stationary and spherically symmetric, the most general metric that need be considered for first order perturbations will be nonstationary but may be assumed to be axially symmetric. According to [9], it can be written in the form:

$$g_{\mu\nu} = \begin{pmatrix}
 e^{2\nu} & \omega e^{2\psi} & 0 & 0 \\
 \omega e^{2\psi} & -e^{2\psi} & q_2 e^{2\psi} & q_3 e^{2\psi} \\
 0 & q_2 e^{2\psi} & -e^{2\mu_2} & 0 \\
 0 & q_3 e^{2\psi} & 0 & -e^{2\mu_3}
\end{pmatrix}.$$  \hspace{1cm} (4.1)
in the coordinate basis $x^\mu = (t, \varphi, r, \theta)$. It will prove convenient to express tensors in terms of a tetrad basis $e^{(a)}_\alpha$, chosen so that $e^{(a)}_\alpha e^{(b)}_\beta \eta_{(a)(b)} = g_{\alpha\beta}$ where $\eta_{\alpha\beta} = \text{diag}(+1, -1, -1, -1)$. The explicit form of the tetrads for the above metric is ([9], p. 81):

$$
e^{(0)}_\alpha = \left( e^\nu, 0, 0, 0 \right)$$
$$
e^{(1)}_\alpha = \left( -\omega e^\psi, e^\psi, -q_2 e^\psi, -q_3 e^\psi \right)$$
$$
e^{(2)}_\alpha = \left( 0, e^{\mu_2}, 0, 0 \right)$$
$$
e^{(3)}_\alpha = \left( 0, 0, e^{\mu_3}, 0 \right)$$

Henceforth, we will usually refer to tensors in their tetrad basis. To make this clear we will use Roman indices for general components and Arabic numbers for specific components, while continuing to use Greek indices and coordinate names to refer to tensor components in the coordinate basis. The transformation between the two different frames is achieved by the use of tetrads, so that we have for example $F_{\mu\nu} = F_{ab} e^{a}_\mu e^{b}_\nu$.

Given that the background configuration described in Section 2 consists of a diagonal metric, a non-zero dilaton field and $F_{02} = Q e^{2\Phi} R^{-2}$, the perturbed configuration is parametrized by small changes $\delta \nu, \delta \psi, \delta \mu_2, \delta \mu_3, \delta F_{02}$ and $\delta \Phi$ of the background fields, as well as small values of $\omega, q_2, q_3$ and $F_{ab}$ for $(ab) \neq (02)$. This yields a total inventory of fourteen functions. Fortunately, upon substituting these functions into the equations of motion and keeping only terms of first order in the perturbations, one observes that the linearized equations of motion naturally fall into two sets. One set relates $\delta F_{02}, F_{03}, F_{23}, \delta \nu, \delta \psi, \delta \mu_2, \delta \mu_3$ and $\delta \Phi$, while the other relates $F_{01}, F_{12}, F_{13}, \omega, q_2$ and $q_3$. The first set of equations describes polar perturbations, that preserve spherical symmetry, while the second one describes so-called axial perturbations. (Because nonzero $\omega, q_2$ and $q_3$ imply the loss of spherical symmetry, leaving only axial symmetry.) This standard terminology is taken over from [9], where the same separation was employed for the study of perturbations of classic Reissner-Nordström ($a = 0$) black holes. The polar equations prove to be much harder to analyze, even for the classic Reissner-Nordström holes. It is noteworthy that dilaton perturbations do not appear in the axial equations, which
can therefore be treated in the general case with little more difficulty than in the classic case.

4.1. The axial equations

The axial equations are obtained by linearizing the first Maxwell equation for \( \nu = \varphi \), the second Maxwell equation for \( (\mu\nu) = (\varphi tr) \) and \( (\varphi t\theta) \) and the Einstein equation for \( (ab) = (12) \) and \( (13) \). There are in fact two more axial equations but they turn out to be redundant. The analysis required to derive the axial equations in their explicit form precisely parallels the procedure outlined in [9], so we will restrict ourselves to a brief sketch. The partial differential equations in the variables \( t, r \) and \( \theta \) separate if we look for solutions with the time dependence \( e^{i\sigma t} \) and make the following ansatz for the radial and angular part (the angle \( \varphi \) does not enter, due to axial symmetry):

\[
(q_2, q_3) = \tilde{Q}(r) C_{l+2}^{-3/2} \frac{1}{R^2 \lambda^2 \sin^3 \theta},
\]

\[
F_{01} = B(r) \frac{dC_{l+2}^{-3/2}}{\sin^2 \theta \frac{d}{d\theta}},
\]

where \( C^\nu_n \) denotes the Gegenbauer function, which solves the equation

\[
\left( \frac{d}{d\theta} \sin^{2\nu} \theta \frac{d}{d\theta} + n(n + 2) \sin^{2\nu} \theta \right) C^\nu_n(\theta) = 0.
\]

With this substitution the five first order partial differential equations are reduced to two coupled ordinary second order differential equations:

\[
e^{2a\Phi} \left( e^{-2a\Phi} \lambda^2 (\lambda R B)_{,r} \right)_{,r} + \left( \sigma^2 R \lambda^{-1} - (\mu^2 + 2) \lambda R^{-1} - 4Q^2 \lambda e^{2a\Phi} R^{-3} \right) B
\]

\[
- Q e^{2a\Phi} R^{-4} \tilde{Q}(r) = 0
\]

\[
R^4 \left( \lambda^2 R^{-2} \tilde{Q}(r)_{,r} \right)_{,r} + (\sigma^2 R^2 \lambda^{-2} - \mu^2) \tilde{Q}(r) - 4\lambda Q \mu^2 R B = 0,
\]

where \( \mu^2 \equiv l(l+1) - 2 \). The first order derivatives can be eliminated by the change
of variables

\[ \tilde{Q}(r) = RH_2 \quad , \quad R\lambda B = -H_1 e^{a\Phi}/2\mu \quad \text{and} \quad dr = \lambda^2 dr^*. \]

We recall that the tortoise coordinate \( r^* \) tends for non-extremal black holes to \(-\infty\) as the horizon is approached, whereas it behaves like \( r + 2M \ln r \) for large radii.

Thus we arrive at two one-dimensional wave equations coupled by an interaction matrix:

\[
\left( \frac{d^2}{dr^*_*} + \sigma^2 \right) \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} V_1 & V_{12} \\ V_{12} & V_2 \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix},
\]

where

\[
V_1 = a^2(\Phi_{,r^*})^2 - a\Phi_{,r^*,r^*} + (\mu^2 + 2)\lambda^2 R^{-2} + 4Q^2 \lambda^2 e^{2a\Phi} R^{-4}
\]

\[
V_2 = 2R^{-2}(R_{,r^*})^2 - R^{-1} R_{,r^*,r^*} + \mu^2 \lambda^2 R^{-2}
\]

\[
V_{12} = -2Q\mu e^{a\Phi} \lambda^2 R^{-3}.
\]

The interaction matrix can in fact be diagonalized by a coordinate-independent similarity transformation so that two independent wave equations are obtained, with potentials \( U_1 \) and \( U_2 \) given by

\[
U_{1,2} = \frac{1}{2} \left( V_1 + V_2 \pm \sqrt{(V_1 - V_2)^2 + 4V_{12}^2} \right).
\]

These equations are roughly of the same form as those derived by Gilbert [10] in the special case \( a = 1 \), but the differences are crucial. Gilbert found that the potentials generically have negative portions, from which he concluded that charged dilaton black holes are unstable [11]. As we shall soon see, however, the potentials for non-extreme black holes go to zero both at the horizon and at large radii and are positive in the intermediate region.*

* The main error in Gilbert’s derivation seems to an algebraic slip: Substitution of his equations (G55) and (G56) into (G49) leads to (G57) with \( T_1 = - (\Phi_{,r^*,r^*} + \Phi_{,r^*})^2 \) instead of (G59). With this definition of \( T_1 \), (G62-66) agrees with (4.5)-(4.6) except for a sign change in \( \Phi \), which seems to indicate that he is effectively dealing with a magnetically charged hole.
Since the gravitational waves have spin two, it should not come as a surprise that the analytic form of the equations changes for \( l \leq 1 \). Equation (4.3) shows that \( \tilde{Q}(r) \) must be zero identically for \( l < 2 \) lest the metric functions \( q_2 \) or \( q_3 \) diverge at \( \sin \theta = 0 \). The two equations (4.4) then become inconsistent for \( l = 0 \) unless \( B \equiv 0 \), whereas for \( l = 1 \) the second equation of (4.4) is identically satisfied, leaving a wave equation for the electromagnetic mode with the potential \( V_1 \) having \( \mu^2 = 0 \).

4.2. The polar equations

As in the axial case we will choose a subset of all the polar equations, namely the first Maxwell equation for \( \nu = r, \theta \), the second Maxwell equation for \( (\mu \nu \rho) = (\rho r \theta) \) the Einstein equation for \( (ab) = (02), (03), (23), (11), (22) \), and the dilaton equation. The other polar equations are redundant, as, in fact, is the \( R_{11} \)-equation which however we retain for later convenience. Following again closely the methods described in [9], we separate the partial differential equations by ascribing an \( e^{i\sigma t} \) time dependence to all perturbations \( \delta F_{02}, F_{03}, F_{23}, \delta \nu, \delta \psi, \delta \mu_2, \delta \mu_3 \) and \( \delta \Phi \) and making the following ansatz for their angular dependence:

\[
\begin{align*}
\delta \nu &= N(r)P_l(\theta) \\
\delta \psi &= T(r)P_l(\theta) + 2\mu^{-2}X(r)P_l(\theta), \theta \cot \theta \\
\delta \mu_2 &= L(r)P_l(\theta) \\
\delta \mu_3 &= T(r)P_l(\theta) + 2\mu^{-2}X(r)P_l(\theta), \theta \theta \\
\delta F_{02} &= -\frac{R^2\lambda^2}{2Q}B_{02}(r)P_l(\theta) \\
F_{03} &= \frac{R\lambda}{2Q}B_{03}P_l(\theta), \theta \\
F_{23} &= \frac{i\sigma R}{2Q\lambda}B_{23}P_l(\theta), \theta \\
\delta \Phi &= \phi P_l(\theta), \tag{4.7}
\end{align*}
\]

where \( P_l(\theta) \) are Legendre polynomials with index \( l \) which we write in terms of \( \mu^2 \equiv l(l+1) - 2 \). Substituting these functions into the equations of motion, using Chandrasekhar’s expressions for the \( R_{ab} \)’s and linearizing in the perturbations, the following equations are obtained. (They are listed in the order they were
The two algebraic equations can be substituted immediately into the other differential equations, leaving one second order and five first order differential equations (ignoring the redundant equation (4.14)) for the six functions \(N, L, X, B_{03}, B_{23}\) and \(\phi\).

\[
\chi^2 B_{02} - 4Q^2 e^{2a\Phi} R^{-4} (B_{23} - L - X - a\phi) + (\mu^2 + 2) R^{-2} B_{23} = 0 \tag{4.8}
\]

\[
B_{03} - e^{2a\Phi} R^{-2} (e^{-2a\Phi} R^2 B_{23})_{,r} = 0 \tag{4.9}
\]

\[
\sigma^2 R^2 \lambda^{-2} B_{23} + \left( \lambda^2 R^2 B_{03} \right)_{,r} + R^2 \lambda^2 B_{02} - 2Q^2 e^{2a\Phi} R^{-2} (N + L) = 0 \tag{4.10}
\]

\[
\frac{d}{dr} + (\ln R/\lambda)_{,r} (B_{23} - L - X) - (\ln R)_{,r} L + \Phi_{,r} \phi = 0 \tag{4.11}
\]

\[
B_{23} - T - L + 2\mu^{-2} X = 0 \tag{4.12}
\]

\[
(N - L)_{,r} - (\ln \lambda R)_{,r} L + (\ln \lambda R)_{,r} N - B_{03} + 2\Phi_{,r} \phi = 0 \tag{4.13}
\]

\[
X_{,rr} + 2(\ln R)_{,r} X_{,r} + \mu^2 (2\lambda^2 R^2)^{-1} (N + L) + \sigma^2 \lambda^{-4} X = 0 \tag{4.14}
\]

\[
2(\ln R)_{,r} N_{,r} + 2(\ln \lambda R)_{,r} (B_{23} - L - X)_{,r} - \mu^2 (\lambda R)^{-2} T - B_{02} + (\mu^2 + 2)(\lambda R)^{-2} N - 2(\ln R)_{,r} (\ln \lambda^2 R)_{,r} L + 2\Phi_{,r}^2 L + 2\sigma^2 \lambda^{-4} (B_{23} - L - X) - 2aQ^2 e^{2a\Phi} \lambda^{-2} R^{-4} \phi + 2\Phi_{,r} \phi_{,r} = 0 \tag{4.15}
\]

\[
R^{-2} (R^2 \lambda^2 \phi_{,r})_{,r} + \left( \sigma^2 \lambda^{-2} - (\mu^2 + 2) R^{-2} + 2aQ^2 e^{2a\Phi} R^{-4} \right) \phi + \lambda^2 \Phi_{,r} (N - 3L - 2X + 2B_{23})_{,r} - 2R^{-2} (R^2 \lambda^2 \Phi_{,r})_{,r} L + \lambda^2 a B_{02} = 0 \tag{4.16}
\]

* It is in fact more convenient to use the \(G_{22}\)-equation instead of the \(R_{22}\)-equation to obtain (4.15). For the \(R_{11}\)-equation, which is a sum of two terms with different angular dependences, both coefficients have to separately vanish. Only one of these two equations is displayed as (4.14).
4.3. **Particular Integral and Reduction of the Polar Equations**

Equations (4.8) through (4.15) reduce to the equations for the Reissner-Nordström perturbations given in Chandrasekhar ([9] page 232) in the limit \( a = 0 \). The subsequent procedure spelt out in [9] is somewhat mysterious and lacks a simple physical interpretation. It is described as a “remarkable fact that the system of equations of order five ... can be reduced to two independent equations of second order” (p. 233). This is achieved by defining *ad hoc* two nontrivial radius-dependent linear combinations of the perturbation functions that can be verified to reduce the system of differential equations. It was recognized that the origin of this reduction must lie in “some deeper fact at the base of the [differential] equations” (p. 149) and that it is connected with the existence of a particular solution to the set of equations. Some very clever work of Xanthapoulos [12] gives an algorithm which determines this particular integral and thereby enables the construction of the general solution, once the linear combinations which reduce the equations are known. It is not at all clear how to generalize this inspired guesswork to our more general case, so it was imperative for us to find a rational basis for the procedure.

We will argue that the particular integral is in fact a manifestation of the fact that the gauge has not yet been completely set. This fact can be exploited to find the particular integral and thereby to reduce the order of the equations, independent of any prior knowledge of the solution.

Coordinate transformations change the form of the metric without any physical effect. Perturbations which can be undone by a coordinate transformation therefore automatically satisfy the equations of motion. One might think that writing the metric in the form (4.1) eliminates all spurious degrees of freedom by specifying a unique coordinate system. This is not the case, however. There still are coordinate transformations which leave the metric in the form (4.1) but transform its individual components. Such transformations are necessarily independent of \( \varphi \), since the gauge choice (4.1) singles out \( \varphi \) as the angle corresponding to axial symmetry.
The infinitesimal coordinate transformation

\[ x^\mu = \bar{x}^\mu + \epsilon \delta x^\mu(\bar{x}^\mu) \]  

(4.17)

does not change the unperturbed metric \( g_{\mu\nu} \) to

\[ \bar{g}_{\mu\nu} = g_{\mu\nu} + \epsilon \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \delta x^\alpha + g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}. \]  

(4.18)

For the metric (2.5) we thereby obtain:

\[
\delta g_{\mu\nu} = \epsilon \begin{pmatrix}
2\lambda^2(\lambda^{-1}\lambda_x\delta r+\delta t, t) & 0 & \lambda^2\delta t, r-\lambda^{-2}\delta r, t & \lambda^2\delta t, \theta-R^2\delta \theta, t \\
0 & -2R^2\sin \theta^2(\cot \theta \delta \theta + R^{-1}R_r \delta r) & 0 & 0 \\
\lambda^2\delta t, r-\lambda^{-2}\delta r, t & 0 & -2\lambda^{-2}(\lambda_x \lambda^{-1}\delta r+\delta t, r) & -\lambda^{-2}\delta t, \theta+R^2\delta \theta, r \\
\lambda^2\delta t, \theta-R^2\delta \theta, t & 0 & -\lambda^{-2}\delta t, \theta+R^2\delta \theta, r & -2R^2(R^{-1}R_r \delta r+\delta \theta, r)
\end{pmatrix}
\]  

(4.19)

We must demand that the off-diagonal metric components vanish lest the gauge choice (4.1) be destroyed. The resulting three differential equations in \( \delta t, \delta r \) and \( \delta \theta \) are readily integrated to yield (taking again an \( e^{i\sigma t} \) time dependence):

\[
\delta t = \frac{R}{\lambda} c(\theta), \quad \delta r = \frac{\lambda^4}{i\sigma\delta r, r} \quad \text{and} \quad \delta \theta = \frac{\lambda^2}{i\sigma R^2 \delta t, \theta}. \]  

(4.20)

If we choose \( c(\theta) = P_l(\theta) \) and compare (4.19) with the definitions (4.7) of the radial functions, we obtain the following particular integral:

\[
N^{(0)} = -\sigma^2 R \lambda^{-1} + \lambda^3 \lambda_x (R \lambda^{-1})_r \\
L^{(0)} = \left( \lambda^4 (R \lambda^{-1})_r \right)_r - \lambda^3 \lambda_x (R \lambda^{-1})_r \\
T^{(0)} = \lambda^4 R^{-1} R_r (R \lambda^{-1})_r \\
X^{(0)} = \frac{\mu^2 \lambda}{2R} \\
B_{23}^{(0)} = T^{(0)} + L^{(0)} - 2\mu^{-2} X^{(0)} \\
\phi^{(0)} = \Phi_r P_l(\theta)^{-1}\delta r = \Phi_r \lambda^4 (R \lambda^{-1})_r
\]  

(4.21)

Given a particular solution to a system of differential equations the reduction
of the system can be done in a standard and, in principle, straightforward manner. We will sketch in the following a procedure to obtain three second order polar equations which keeps the algebra to a minimum, though even then use of the symbol manipulation program Mathematica proved invaluable.

It proves convenient to define a new function \( S \equiv B_{23} - L - X \) and write \( L \) in terms of \( S \). If we now write the perturbation functions in terms of the particular integral and the functions \( n, b, x, s \) and \( p \) as

\[
N = N^{(0)} s + n \\
B_{23} = B_{23}^{(0)} s + e^{\alpha \Phi} R^{-1} b \\
X = X^{(0)} s + R^{-1} x \\
S = S^{(0)} s \\
\phi = \phi^{(0)} s + R^{-1} p,
\]

and substitute them into the equations of motion, all terms in \( s \) vanish and (4.11) becomes an algebraic equation which can be solved for \( s, r \) easily. Furthermore, combining (4.13) and (4.15) so as to eliminate \( n, r \) gives an equation which can be solved for \( n \).

The polar equations can now finally be reduced to three second order differential equations in \( b, x \) and \( p \). Equations (4.14)\(^*\) and (4.16) are already second order equations in \( X \) and \( \phi \), respectively and substitution of (4.9) into (4.10) gives a second order equation for \( B_{23} \).

Thus substitution of (4.22), \( s, r, n \) and their derivatives into these second order equations results in a system of equations of the desired form:

\[
\left( \frac{d}{d r^*} + \sigma^2 \right) Y = V Y
\]

where \( Y \equiv (b, x, p)^T \) and \( V \) is a three by three interaction matrix. This is just a

\* We use (4.14) here for the first time, and purely for convenience. One could also obtain a much longer looking, though ultimately of course equivalent, second order equation in \( x \) from (4.13) and (4.15).
system of three coupled one-dimensional wave equations. The fact that no first order derivatives are present justifies the definition of \(b, x\) and \(p\) through (4.22). This definition was suggested by inspection of the second order differential equations in \(B_{23}, X\) and \(\phi\), which showed that if \(B_{23}, X\) and \(\phi\) were written as \(e^{a\Phi}R^{-1}\tilde{B}_{23}, R^{-1}\tilde{X}\) and \(R^{-1}\tilde{\phi}\), respectively, the equations contain no first order derivatives with respect to \(r^*\). It is, however, quite remarkable that after substitution of \(n, s, r\) and their derivatives — which do contain first order derivatives in these functions — the final equations do not contain first order derivatives.

Unfortunately, the interaction matrix is too complex to be displayed in explicit form. (It will be supplied in electronic form upon request.) Upon studying it for particular values of the parameters \(a\), mass and charge, we have found a most remarkable surprise: even though all components of the interaction matrix are complicated functions of the radius, it has radius-independent eigenvectors. This means that it can be brought to a diagonal form, and that its eigenvalues have the simple interpretation of being the potentials for three independent modes! Some plots of these potentials for specific values of the parameters are shown in Figure 1. The polar potentials, which we will study more carefully in the next chapter are of the same form as the axial potentials, in particular, they go to zero close to the horizon and for large radii, and they are positive in the intermediate region.

As was the case for the axial equations, also for the polar equations the modes with angular momentum less than two need to be considered separately. This time we expect to have two modes for \(l = 1\), since gravitational waves cannot participate, and only one mode for \(l = 0\), because electromagnetic waves are no longer possible either. That the gravitational mode disappears can be seen explicitly by realizing that the parametrization of the metric functions (4.7) is redundant for \(l = 1\), since in that case \(-P_1(\theta) = P_1(\theta)_{,\theta} = P_1(\theta)\cot \theta\) so that variations of \(X(r)\) can be absorbed by \(T(r)\), and independent on \(X(r)\) for \(l = 0\), where \(X(r)\) gets multiplied by \(0 = P_0(\theta)_{,\theta}\). For \(l = 0\) some of the details of the derivation of equations (4.8) through (4.16) have to be reviewed to realize that the equations (4.9), (4.10), (4.12), (4.13) and (4.14) are in fact satisfied identically because they are really
multiplied by \( P(\theta) \). Moreover, there is now a much larger family of coordinate transformation which don’t violate the gauge choice (4.1), namely arbitrary \( \delta r \) is allowed which can be used to set, for example, \( T = 0 \). The four remaining polar equations can then be reduced to a wave equation for the dilaton field alone.

4.4. Magnetically Charged Dilaton Black Holes

We will now show that the magnetic case can be treated in exactly the same manner as the electric case.

As mentioned in chapter 2, magnetically charged solutions are obtained from the electric solution through the duality transformation (2.12). If this transformation is regarded as a mere change of variables and substituted into the equations of motion, one finds that the form of the equations of motions is unaltered, except that unprimed quantities are replaced by primed ones. \( F' \) can therefore be regarded as an electromagnetic field tensor in its own right and will correspond to a uniform magnetic field if \( F \) corresponded to a uniform electric field. Conversely, starting with the magnetic solution and making the change of variables (2.12), the primed quantities will look like the electric solution, but of course they are still magnetic since a change of variables has no physical effect. The perturbation analysis in terms of the primed quantities is identical to our previous analysis of the electric case, and can be taken over without modification. At the end the unprimed physical quantities are restored by inverting the change of variables (2.12), which has as its consequences the interchange of electric and magnetic perturbations and change in the sign of the dilaton perturbation. For example, \( \delta F_{02} \), which transforms to \( e^{-2a\Phi'} F'_{13} \), will now appear in the axial perturbations.
4.5. Remark on stability

As a by-product of our analysis so far, we can address the question whether dilaton black holes are stable classically (i.e. ignoring Hawking radiation). From the fact that all potentials are positive outside the outer horizon, one may infer stability of the solution in this region using the same straightforward argument as employed by Chandrasekhar [9] for the special case of Reissner-Nordström black holes. Except for the classic case $a = 0$, $r_-$ is a curvature singularity. Thus subtleties concerning instability of the inner horizon, which occur for classic Reissner-Nordström black holes, generically do not arise.

5. Qualitative Features of the Potentials

In the previous chapter we have found that the equations governing the perturbations of the metric, electromagnetic, and dilaton fields can be reduced to five wave equations for five independent modes. These modes consist of various linear combinations of the original functions parametrizing the perturbations, whose direct physical meaning is not transparent. Since the potentials are too unwieldy to allow a useful description in closed form and analytical analysis, we will start by discussing the potential of a simpler, model problem. We shall consider a spectator scalar field propagating in the background of a charged dilaton black hole. As we shall see, this simple case seems to display the same main qualitative characteristics as the other, more intrinsic potentials.

5.1. Spectator Scalar Field

A massless, uncharged scalar field propagates in curved space according to the wave equation:

$$
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \eta \right) = 0.
$$

This equation reduces to a one-dimensional wave equation in $r^*$ if we make the ansatz $\eta = e^{i\lambda t} P_l(\theta) R^{-1} \tilde{\eta}$. The effective potential which appears in this equation
is \( V_\eta = R^{-1} R_{r^* r^*} + l(l + 1) \lambda^2 R^{-2} \). Upon substitution of the metric functions, this becomes:

\[
V_\eta = V_{\eta 1} V_{\eta 2},
\]

(5.2)

where

\[
V_{\eta 1} = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right) \frac{1 - 3a^2}{1 + a^2}
\]

\[
V_{\eta 2} = \frac{1}{r^2} \left[l(l + 1) + \frac{r_+ - r - r_+ (1 + a^2)^2 - (2 + a^2) r_- r_+}{(1 + a^2)^2 r^2} - \frac{a^4 r_- (1 - \frac{r_+}{r})}{(1 + a^2)^2 r (1 - \frac{r_-}{r})}\right]
\]

(5.3)

The most interesting features of this potential arise from the factor \( V_{\eta 1} \). \( V_{\eta 2} \) is positive definite and finite for all radii larger than the radius of the outer horizon \( r_+ \). This is the region we are mainly interested in, because the region inside the outer horizon is inaccessible to the outside world.

We are particularly interested in examining what happens as the black hole becomes extremally charged. For all non-extremal black holes, the potential vanishes at the horizon by virtue of the factor \( V_{\eta 1} \), and is finite outside the horizon. Considering even for a moment the extremal limit \( r_- = r_+ \), in which \( V_{\eta 1} \) becomes

\[
\left(1 - \frac{r_+}{r}\right) \frac{2 - 2a^2}{1 + a^2},
\]

it becomes clear that here too, as in the description of the thermal behavior, three cases must be distinguished: \( a < 1 \), \( a = 1 \) and \( a > 1 \).

\( a < 1 \): The potential is qualitatively the same as for a Reissner-Nordström black hole. For any charge not exceeding the extremal charge, it is zero at the outer horizon, has a finite maximum and goes back to zero for large radii.

\( a = 1 \): Now we have \( V_{\eta 1} = \frac{r_- r_+}{r^2 (r_+ - r_-)} \) and the potential is finite in the strictly extremal limit. Just before becoming extremal, the potential rises from zero at the horizon \( r_+ \) to its maximum value, which it attains at \( r - r_+ \sim r_+ - r_- \), with an ever increasing slope. The maximum of the potential therefore approaches the horizon as the black hole becomes extremally charged. This means that in
terms of the tortoise \( r^* \)-coordinate, which is after all the relevant coordinate for describing the propagation of waves, the potential becomes infinitely wide (we may recall that the horizon lies at \( r^* = -\infty \) for \( a \leq 1 \).) We therefore may say that a finite mass gap develops as the black hole becomes extremally charged. More precisely, excitations with less than a critical frequency are reflected with certainty.

\( a > 1 \): In this case, as we have seen above, the potential diverges at the horizon in the extremal limit. For non-extremal black holes the potential is, of course finite outside the horizon, but it has a maximum whose height grows as \((r_+ - r_-)^{-2a^2-1} \). It would be a little too hasty to conclude directly from this diverging behavior that the transmission coefficient goes to zero, since for \( a > 1 \) the horizon lies at a finite value of \( r^* \) in the extremal limit. Indeed, the potential decreases in width as it increases in height. An estimate of the WKB-integral \( \int \sqrt{V} \, dr^* \), however, yields a transmission amplitude which behaves as \( e^{-c \log |r_+ - r_-|} \), with \( c \) a positive constant, so that the transmission probability for any fixed frequency vanishes as the black hole approaches extremality. This result is substantiated by numerical integration of the wave-equation. So in this precise sense, there is an infinite mass gap for \( a > 1 \).

5.2. Axial and Polar potentials

Although a similar discussion of the qualitative features of the axial and polar potentials could be given at this point, it would be little more than several repetitions of the same story we have just told, in a considerably more cumbersome form. Instead we offer the plots of the axial and polar potentials in comparison with the spectator potentials displayed in Figure 1, for a variety of parameters. The potentials are seen to exhibit the same characteristics described in the previous section for the spectator field. Sampling the potentials of a few other partial waves confirms that the potentials increase, as might have been expected, with increasing angular momenta, but introduce no obvious qualitative changes. In particular, since
angular momentum enters the equations through the factor $\mu^2 \lambda^2 R^{-2}$, no infinite centrifugal barrier develops for $a \leq 1$ as the black hole becomes extremally charged, even though the area of the two-sphere located at the horizon tends towards zero (for $a > 1$ the potential diverges in all modes anyway).

6. Emission Spectrum

The rate of emission for a given bosonic mode is, according to Hawking [2], equal to

$$\Gamma_n \left( e^{\sigma/T} - 1 \right)^{-1}$$

where $\Gamma_n$ is the absorption coefficient for the mode $n$ and $T$ the temperature of the black hole. Given a wave equation with a potential barrier, the absorption coefficient is calculated in a straightforward manner: Imposing the boundary condition of incoming waves close enough to the horizon so that the potential is negligible, the wave equation is integrated outward to a region where the potential is again negligible and separated into incoming and reflected wave amplitudes. The absorption coefficient $\Gamma_n$ is then determined through $\Gamma_n = 1 - |A_{in}|^2 / |A_{ref}|^2$. Figure 2 displays an example of the dependence of the absorption coefficient on frequency. As expected the absorption coefficient drops rapidly to zero for frequencies below the potential barrier. Furthermore we display in Figure 3 the probability of emission of the spectator field in the $l = 2$ mode as obtained from (6.1) for various values of parameters on a semilogarithmic plot. Only the spectator spectrum is shown, but due to the similarities in the potentials, the spectrum for emission of gravitational, electromagnetic and dilaton waves look very similar.

It should be noted that compared to a black body, the spectrum of a black hole is seriously distorted in the low frequency regime and the total emission rate is reduced by several orders of magnitudes for all values of the parameters. This is because the potential barrier, even in the most favourable cases, is several times larger then the average available thermal energy $T$. This has the important physical
consequence, as discussed in [3] and further developed and exploited in [13], that it makes physical sense to think of the black hole in isolation as having a physical atmosphere at the thermal energy $T$ (or, more precisely, this divided by the value of $g_{00}^{1/2}$) which is only in weak thermal contact with the outside world.

In view of the issues discussed in Chapter 1, we again are particularly interested in the extremal limit. Once more, we must consider three different cases. Recalling the formula for the black hole temperature, we find:

$$T = \frac{1}{4\pi r_+} \left( \frac{r_+ - r_-}{r_+} \right)^{\frac{1-a^2}{1+a^2}}$$

$a < 1$: The temperature goes to zero as the black hole becomes extremal ($r_+ \to r_-$). The black hole therefore asymptotically approaches the extremal limit and the radiation switches off when it is reached. While the grey body factors drastically slow down the approach to extremality, they do not alter the physical behavior. The final state of the black hole is reasonably described as an extended object, similar to a liquid drop, for $a = 0$. For $0 < a < 1$ it very plausibly represents a unique, non-degenerate ground state, as indicated by its vanishing entropy. When an extremal hole does swallow a small amount of mass, its temperature will become much higher than the energetic distance back to the ground state, in view of (3.4). Nevertheless, the time delay for re-emission plausibly remains long. Indeed if we use the thermal picture at least seriously enough to use it to motivate qualitative bounds on the emission, we can argue as follows. The rate of emission can be bounded by that of a perfect black body, where to be generous we take the radius and temperature to be the values before the emission. (As we have mentioned before, both of these drop drastically with the emission of a single typical quantum.) Thus

$$\frac{dM}{dt} \sim A T^4 \sim (r_+ - r_-)^{\frac{4-2a^2}{1+a^2}} \sim (\delta M)^{\frac{4-2a^2}{1+a^2}}$$

(6.2)

using (3.1), (3.2), and (3.7). Thus for the time to re-emit the injected mass
δM we find

$$\delta t \gtrsim (\delta M)^{-3\left(\frac{1-a^2}{1+a^2}\right)}$$

which indeed diverges for small δM and a < 1. Thus the 0 < a < 1 holes behave physically as extended objects, similar to the classic a = 0 Reissner-Nordström holes. In a way it is gratifying that the massive degeneracy which appears in the classic case, and seems somewhat accidental, is lifted in the generic case.

a = 1: In this case, the temperature has a finite value as the extreme case is reached. As we have shown, simultaneously a finite mass gap develops. This heavily suppresses the radiation but doesn’t quite stop it. We have found nothing to prevent continued radiation, and formation of a naked singularity. One may also recall that the signature (3.4) for breakdown of a thermal description did not work for a = 1. Thus this case is quite enigmatic.

a > 1: The temperature now becomes formally infinite as the black approaches its extremal state. At the same time, however, the potentials grow at the same rate as the average thermal energy. This in itself would not be sufficient to turn off the radiation, but the radiation has to come to a halt nonetheless. We have already argued on general grounds in section 3.3. that the thermal description necessarily breaks down as the extremal state is approached because the emission of a quantum with typical thermal energy induces a large fractional change in the temperature, rendering the thermal description at least ambiguous. In the case at hand one can see what has to happen instead. The thermal description was derived by ignoring the back reaction of the radiation on the metric. Although it is not yet known how to incorporate the back reaction in a consistent manner, it certainly seems reasonable to anticipate that a black hole cannot possibly radiate matter with energies larger than the black hole’s mass. For quanta with energy less than the mass of the black hole, however, the rate of emission tends to zero by virtue of the development of an infinite mass gap. Hence the radiation slows down.
and comes to an end at the extremal limit, despite the infinite temperature. The infinite temperature however indicates that there will be no time delay in the scattering of low-energy quanta, again consistent with an elementary particle interpretation.

7. The Callan-Rubakov Mode

There is a special feature of the interaction of elementary particle magnetic monopoles with minimally charged fermions, which is quite relevant to the circle of ideas discussed above. We have been much concerned with the question, whether various particles – the particles in the basic, or spectators – can reach the singular (for \(a \neq 0\)) horizon. It is a famous fact [14, 15] that minimally charged fermions in the total angular momentum zero mode are focused right to the core of particle magnetic monopoles. The reasons for this are connected to anomalies and topology [16, 17], and are therefore such as might be expected to apply even to black holes. Do they?

The Dirac equation in curved space is written most conveniently in terms of vierbeins and a connection determined by (see e.g.,[18], p. 85ff.):

\[
\gamma^a \nabla_a \psi = 0, \tag{7.1}
\]

where

\[
\nabla_a = e^\mu_a (\partial_\mu + ieA_\mu + \Gamma_\mu) \tag{7.2}
\]

and

\[
\Gamma_\mu = \frac{1}{8} [\gamma^a, \gamma^b] e^\nu_a (\partial_\mu e_{b\nu} - \Gamma^\rho_{\mu\nu} e_{b\rho}). \tag{7.3}
\]

For a magnetically charged black hole of minimal charge \(1/2e\) the only non-zero component of the electromagnetic field tensor is \(F_{\theta\phi} = \sin \theta/2e\). This may be obtained from the vector potential \(A_\mu\) whose only non-zero component
is $A_\varphi = -\cos \theta/2e$. The explicit Dirac equation for a fermion in the background of a minimally charged magnetic black hole simplifies by using, as usual, the tortoise coordinate $r^*$ and performing the change of variable $\xi = \psi/R\sqrt{\lambda}$. The equation is then

$$\left(\gamma^0 \partial_t + \frac{\gamma^1 \lambda}{R \sin \theta} \partial_\varphi + \frac{\gamma^2 \lambda}{R} \partial_{r^*} + \frac{\gamma^3 \lambda}{2R} (-i \gamma^1 + \gamma^3) \right) \xi = 0. \quad (7.4)$$

In the chiral representation of the $\gamma$-matrices, $\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$, the equation for massless fermions separates immediately into two sets of equations for two-spinors. If for convenience we identify $(\sigma^1, \sigma^2, \sigma^3)$ with the standard Pauli-matrices $(\sigma_y, \sigma_z, \sigma_x)$ and write the two-spinor as $(u \atop v)$, these equations become:

$$\begin{align*}
(\pm \partial_t + \partial_{r^*}) u_\pm + \frac{\lambda}{R} \left( \frac{i}{\sin \theta} \partial_\varphi + \partial_\theta + \cot \theta \right) v_\pm &= 0 \\
(\pm \partial_t - \partial_{r^*}) v_\pm + \frac{\lambda}{R} \left( -\frac{i}{\sin \theta} \partial_\varphi + \partial_\theta \right) u_\pm &= 0,
\end{align*} \quad (7.5)$$

where $\pm$ distinguishes fermions of opposite chirality. Acting with

$$\frac{1}{\sin^2 \theta} (i \partial_\varphi + \sin \theta \partial_\theta) \sin \theta$$

from the left on the second equation and substituting into the first equation, when the angular dependence of $u$ is taken to be $P_l(\theta)$, we arrive at the following equation:

$$\begin{align*}
(\pm \partial_t - \partial_{r^*}) \frac{R}{\lambda} (\pm \partial_t + \partial_{r^*}) u_\pm + \frac{\lambda}{R} l(l+1) u_\pm &= 0.
\end{align*} \quad (7.6)$$

We see that fermions are in general subject to a frequency dependent potential. This occurs also for Kerr black holes, and is a sign of the underlying time-reversal asymmetry of the problem. (Magnetic charge is $T$ odd.)
More significantly, we see that the characteristic behavior of the Callan-Rubakov mode survives the transition from standard to black hole monopoles. For zero angular momentum no potential at all is present – regardless of the black hole parameters! The possibility of modes without any potential is particularly significant for dilaton black holes, especially those with $a > 1$. This mode is the only one that can penetrate to the core of an extremally charged black hole in this regime. In view of the formally infinite temperature of the black hole, it would seem at the classical level that catastrophic radiation would ensue. We believe that a proper quantization will identify a stable ground state and a discrete spectrum of dyonic excitations around it, similar to what occurs for particle monopoles in flat space – but this is a problem for the future.

8. Final Comments

Although falling well short of a derivation, we believe that the calculations and arguments given above provide substantial evidence for the consistency of the physical picture outlined in the introductory section. An important qualitative conclusion is that extremal black holes with the same quantum numbers can, for different field contents of the world, represent rather different physical entities. The extremal charged dilaton black holes appear to be extended spherical objects for $a < 1$, and elementary point objects for $a > 1$. Unfortunately the case suggested by superstring theory, $a = 1$, is enigmatic. However, we can hardly refrain from observing that a string is the intermediate case between a spherical membrane and a point!

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FIGURE CAPTIONS

1) The effective potentials in units $M^{-2}$ versus tortoise $r^*$ in units of $M$ for $l = 2$ partial waves, $a = 0, 1$ and 2 and various values of the black hole charge measured in units of the extremal charge $M\sqrt{1 + a^2}$, where $M$ is the black hole mass. For each set of parameters there are one spectator, two axial and three polar potentials, which we display on different plots. Figure 1a ($a = 0$) shows the classical Reissner-Nordström black hole which is representative for dilaton black holes with $0 \leq a < 1$. For $a = 1$ (Fig. 1b) the broadening of the potential signals the emergence of a mass gap as the black hole becomes extremally charged. $a = 2$ (Fig. 1c) shows the potentials to increase without bound in height while decreasing in width as the extremal limit is approached, which is characteristic for dilaton black holes with $a > 1$. Note the changes in scale for the different values of $a$.

2) A typical plot for the dependence of the absorption coefficient on the frequency of incident waves measured in units of $M^{-1}$. The parameters chosen are $a = 2$, angular momentum $l = 2$ and charge $Q = 0, 0.8$ and $0.99$ measured in units of the extremal charge $M\sqrt{1 + a^2}$ and the plot is for scattering of spectator fields off a dilaton black hole background.

3) A semilogarithmic plot (base 10) of the probability of emission versus frequency measured in units of $M^{-1}$ of a spectator scalar field in the $l = 2$ mode for $a = 0, 1$ and 2 and various values of the black hole charge measured in units of the extremal charge $M\sqrt{1 + a^2}$, where $M$ is the black hole mass. In all cases the spectrum is seen to be dramatically distorted in the low frequency regime due to grey-body factors. While the plot for $a = 0$, which is the classic Reissner-Nordström case representative for dilaton black holes with $0 \leq a < 1$, merely shows a general decrease in emission when the black hole becomes extremally charged — in accordance with the decreasing temperature —, for $a = 1$ the development of a mass gap at frequency $5/(4M)$ predicted from the widening of the potential at a value of $3/(2M)$. 38
is apparent. For $a = 2$ (last plot), the peak of the spectrum reaches after an initial decrease a constant level, which however shifts to higher frequencies, as extremality is approached. As discussed in the text, the spectrum must necessarily be inappropriate for frequencies larger than the black hole mass, so that the emission turns off when the black hole is extremally charged.