A Modified Spectral Phase Conjugation Algorithm for Quadratic Programming: Application to Water Resources Planning Problems

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Abstract. Water resources management system has been a major concern since it gives necessities of human-living as well as promoting economic development. Therefore, how to establish valid and accurate water management schemes for water managers to satisfy the demands from water users, is of great significance. In this study, a modified spectral phase conjugation algorithm for quadratic programming (MCQP) is developed for water resources management systems. The model is derived by modifying the search direction within spectral phase conjugation algorithm. MCQP can efficiently solve nonlinearity in the objective function and linearity in the constraint which lead to complexes in water resources management systems. Results show that reasonable solutions have been obtained to support the rational development of water resources management. They can help water managers to identify desired policies under various economic and environmental constraints. MCQP can be further applied to other resources management systems with mixed nonlinear and linear factors.

1. Introduction

In water-resources management systems, nonlinearities exist in numbers of system components, which may intensify the complexities in decision making process [1]. Many methods and models have been proposed to deal with nonlinearity in the optimization analysis. Among them, quadratic programming is a widely used method. For example, a fuzzy-queue-based inexact stochastic quadratic programming method was developed by Xu et al. for planning water resources management [2]. Kong et al. developed a duality theorem-based algorithm to solve problems of inexact quadratic programming [3]. An interval-fuzzy two-stage stochastic quadratic programming was proposed by Li et al. for analyzing the impact of changing water requirement and availability on irrigation [4]. Mariffio et al. concluded the application of quadratic optimization models in reservoir systems. It was found that the optimal release schedules of quadratic model are robust than those of simplified linear model [5].

This study is to develop a modified spectral phase conjugation algorithm for quadratic programming with linear constraint. MCQP will be applied a problem of water resources management. The solutions are suitable to real water resources planning problems.
2. Modified spectral phase conjugation algorithm for quadratic programming (MCQP)

Consider a quadratic programming with linear constraint problem:

$$\text{min } f(x)$$

s.t. $Ax = b$  \hspace{1cm} (1)

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously partially differentiable, $b \in \mathbb{R}^m$. $A \in \mathbb{R}^{mn \times n}$ is row full rank, can be decomposed as:

$$A = (B, N)$$  \hspace{1cm} (3)

where $B \in \mathbb{R}^{mn \times m}$ is nonsingular. Let $x = \left( x^b, x^N \right)^T$ for $x \in D$. So we have $x^b = B^{-1}b - B^{-1}Nx^N$.

Therefore, Model (1)-(2) can be transformed to an unconstrained optimization problem as follows:

$$\text{min } F(x^N) = f \left( x^b, x^N \right) = f \left( B^{-1}b - B^{-1}Nx^N, x^N \right)$$  \hspace{1cm} (4)

In this paper, a modified spectral conjugation algorithm is proposed to solve Model (4). The algorithm is developed as follows:

Step 1. Set constant $\varepsilon > 0$, point of feasible areas $x_0 \in D$ and $k = 0$.

Step 2. Let $x_k$ as current iteration point, the solution $\bar{d} = \left( \bar{d}^b, \bar{d}^N \right)^T$ of model (5)-(6) can be obtained.

$$\text{min } \nabla f(x_k)^T d$$

s.t. $Ad = 0, \|d^N\| \leq 1$  \hspace{1cm} (6)

Therefore, for any $d = \left( d^b, d^N \right)^T \in \mathbb{R}^n$ that satisfies $Ad = 0$, we have

$$\left( B, N \right) \begin{pmatrix} d^b \\ d^N \end{pmatrix} = 0$$  \hspace{1cm} (7)

$$d^b = -B^{-1}Nd^N.$$  \hspace{1cm} (8)

And the objective function of Model (5)-(6) are transformed to:

$$\nabla f(x_k)^T d = \left( \nabla_{x^b} f \left( x^b_k, x^N_k \right), \nabla_{x^N} f \left( x^b_k, x^N_k \right) \right)^T \begin{pmatrix} d^b \\ d^N \end{pmatrix} = \nabla F(x^N_k)d^N.$$  \hspace{1cm} (9)

Based on the Gauchy-Schwarz inequality, as to $\forall d = \begin{pmatrix} d^b \\ d^N \end{pmatrix} \in \mathbb{R}^n, \|d^N\| \leq 1$, we have

$$\nabla F(x^N_k)d^N \geq -\|\nabla F(x^N_k)\|d^N \geq -\|\nabla F(x^N_k)\|.$$  \hspace{1cm} (10)

Therefore, $\bar{d}_k = \left( \bar{d}^b_k, \bar{d}^N_k \right)^T$ can be got.

$$\bar{d}^N_k = -\frac{\nabla F(x^N_k)}{\|\nabla F(x^N_k)\|}$$  \hspace{1cm} (11)

$$\bar{d}^b_k = -B^{-1}N\bar{d}^N_k.$$  \hspace{1cm} (12)

Let $z_k = \nabla f(x_k)^T \bar{d}_k = \nabla F(x^N_k)^T \bar{d}^N_k = -\|\nabla F(x^N_k)\|$. $g_k = z_k \bar{d}_k$. If $z_k \leq \varepsilon$ is satisfied, the solution of Model (1)-(2) is $x_k$. Otherwise, go to Step 3.

Step 3. Determine a new search direction $d_k$ as follows:

$$d_k = \begin{cases} -g_k, & k = 0 \\ -\theta_k g_k + \beta_k d_{k-1}, & k \geq 1. \end{cases}$$  \hspace{1cm} (13)

where
\[ g_k = \left( -B^{-1}N, I \right)^T \nabla F \left( x_k^N \right) \]  
(14)

\[ \theta_k = 1 + \frac{\nabla f \left( x_k \right)^T d_{k-1}}{\left\| \nabla f \left( x_k \right) \right\|} \beta_k \]  
(15)

\[ \beta_k = \max \left\{ 0, \frac{z_k^2}{z_{k-1}^2} + \min \left\{ 0, \frac{z_k^T z_{k-1}}{z_{k-1}^T d_{k-1}} \right\} \right\} \]  
(16)

The determined search direction \( d_k \) is a feasible direction at \( x_k \).

Step 4. Determine \( \alpha_k \) through linear searching method to satisfy
\[ f \left( x_k + \alpha_k d_k \right) = \min f \left( x_k + \alpha d_k \right) \]. Let \( x_{k+1} = x_k + \alpha_k d_k \) and \( k = k + 1 \), then back to Step 2 to solve Model (5)-(6) until \( z_k \leq \varepsilon \) is satisfied.

3. Case study

In this study, the proposed MCQP method will be applied to a water-resources system planning problem. The studied system consists of a river and a storage reservoir. The functions of reservoir are flood control, irrigation, industrial and municipal water supplies. On one hand, flood diversion must be taken under the condition of overflows during flooding seasons. On the other hand, the decision makers want to know the amount of water they could allocate to water users. The benefits can be got when water are allocated as promised. However, if the promised water is not delivered, penalties are resulted to the local economy in accordance with the exceeded expenses which are functions of water shortage (shown in Figure 1).

![Figure 1. Penalties for water shortage](image)

In order to control floods and prevent waterlogging, a new model for water resources allocation based on MCQP can be designed as follows:

Maximize
\[ f = BX - \sum_{i=1}^{K} \left( \alpha Y_k + \beta \right) p_i Y_i - \sum_{i=1}^{K} p_i \left( F + VU \right) Z_i \]  
(17)

Subject to:

\[ ST_k = ST^0 + q_k - \left[ \frac{1}{2} \left( C_k + ST^0 \right) \right] - RE_k, \quad \forall k \]  
(18)

\[ (X - Y_k) \leq RE_k - U_k Z_k, \quad \forall k \]  
(19)

\[ RE_k - U_k Z_k \leq X, \quad \forall k \]  
(20)

\[ (X - Y_k) \geq U_k Z_k, \quad \forall k \]  
(21)

\[ ST_k \leq TR \]  
(22)
\[ ST_k \geq DR \]
\[ Y_k \leq X \]
\[ Z_k = \begin{cases} 
1, & \text{water diversion is undertaken} \\
0, & \text{otherwise} 
\end{cases} \forall k \]

where \( f \) is system benefit ($); \( k \) is index of scenarios for flow, where \( k = 1, 2, ..., K \); \( B \) is net benefit ($/m^3$); \( X \) is the target of water allocation (m\(^3\)); \( (\alpha Y_k + \beta) \) are penalties when water are not delivered ($/m^3$); \( Y_k \) is water shortage under scenario \( k \); \( q_k \) and \( p_k \) are available water resources (m\(^3\)) and the corresponding probability under scenario \( k \), respectively. \( F \) is fixed-charge cost to transfer the flooding ($); \( Z_k \) is whether or not the flooding diversion is needed to be undertaken under scenario \( k \); \( V \) and \( U_k \) are floated-charge cost for flooding diversion ($/m^3$) and the amount of flood to be diverted (m\(^3\)), respectively.

\[ Y_k, U_k \geq 0, \ \forall k \]

**Table 1.** Parameter related to water users and economic data

| Parameters          | Water user |
|---------------------|------------|
| \( X \) (10\(^6\)m\(^3\)) | 3800       |
| \( B \) ($/m^3$)    | 0.0205     |
| \( F \) ($/m^3$)    | 0.0037     |
| \( V \) ($/m^3$)    | 0.0225     |

**Table 2.** Streamflow data

| Streamflow level         | Probability and stream inflow |
|--------------------------|-------------------------------|
|                          | \( p_k \) | \( q_k \) (10\(^6\)m\(^3\)) |
| Low (L)                  | 0.15     | 4300                           |
| Low-Medium (LM)          | 0.30     | 5500                           |
| Medium-High (MH)         | 0.30     | 6800                           |
| High (H)                 | 0.25     | 9200                           |

Water shortage and flood diversion of the water resources system are in Figure 2. Water shortages under streamflow levels L and L-M are 1492 (10\(^6\)m\(^3\)) and 292 (10\(^6\)m\(^3\)), respectively. There are no water shortages under streamflow levels H and M-H. Flood diversions under streamflow levels H and M-H are 720.5 (10\(^6\)m\(^3\)) and 3120.5 (10\(^6\)m\(^3\)), respectively. There are no flood diversions under streamflow levels L and L-M. It indicates that the amount of water shortage decreases when the streamflow increases, and the amount of flood diversion increases when the streamflow decreases.
Figure 2. Water shortage and flood diversion

Figure 3. Release flow from reservoir

Figure 3 shows the release flow from reservoir under scenario. Release flow under streamflow levels L, L-M, M-H and H are 4308 (10^6 m^3), 5508 (10^6 m^3), 6520.5 (10^6 m^3) and 8920.5 (10^6 m^3), respectively. The amount of release flow increases when the streamflow increases. Therefore, schemes for water shortage, flood diversion and release flow should be designed under various streamflow levels and storage capacities. The optimal system benefit is 88.23 (10^6 $).

4. Conclusion
A modified spectral phase conjugation algorithm for quadratic programming (MCQP) is proposed to obtain optimal water-allocation strategies for water resources planning systems. Results show that (1) MCQP is capable for dealing with problems with nonlinear objective function and linear constraints. (2) Optimal water-allocation strategies generated by MCQP can provide information for decision makers. MCQP can be applied to theatrical and practical applications in water resources management systems.

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Reference
[1] A. Sarhadi, D.H. Burn, F. Johnson, et al. Water Resources Climate Change Projections using
Supervised Nonlinear and Multivariate Soft Computing Techniques. *Journal of Hydrology*, 2016, 536: 119-132.

[2] C. Xu, Y.P. Li, G.H. Huang, et al. Development of A Fuzzy-queue-based Stochastic Quadratic Program for Water Resources Planning. *Stochastic Environmental Research and Risk Assessment*, 2014, 28(6): 1613-1627.

[3] X.M. Kong, G.H. Huang, Y.R. Fan, Y.P. Li, A Duality Theorem-based Algorithm for Inexact Quadratic Programming Problems: Application to Waste Management under Uncertainty. *Engineering Optimization*, 2015, 48: 1-20.

[4] M. Li, P. Guo, V.P. Singh, et al. Irrigation Water Allocation Using an Inexact Two-Stage Quadratic Programming with Fuzzy Input under Climate Change. *Journal of the American Water Resources Association*, 2016, 52(3): B667-684.

[5] M.A. Mariffo, H.A. Loaiciga, Quadratic Model for Reservoir Management: Application to the Central Valley Project, 1985.