Exceptional points controlling oscillation death in coupled spintronic nano-oscillators

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The emergence of exceptional points (EPs) in the parameter space of a (2D) eigen-value problem is studied in a general sense in mathematical physics. In coupled systems, it gives rise to unique physical phenomena upon which novel approaches for the development of seminal types of highly sensitive sensors shall leverage their fascinating properties. Here, we demonstrate at room temperature the emergence of EPs in coupled spintronic nano-oscillators, coming along with the observation of amplitude death of self-oscillations and other complex dynamics. The main experimental features are properly described by the linearized theory of coupled system dynamics we develop. Interestingly, these spintronic nanoscale oscillators are deployment-ready in different applicational fields such as sensors or radiofrequency and wireless technology and, more recently, novel neuromorphic hardware solutions. Their unique and versatile properties, notably their large nonlinear behavior open up unprecedented perspectives in experiments as well as in theory on the physics of exceptional points.

Exceptional points (EPs) are singularities in the parameter space of a system corresponding to the coalescence of two or more eigenvalues and the associated eigenvectors [1–6]. They are a peculiar feature of nonconservative (open) systems that have both loss and gain and they emerge when these two effects compensate. From the fundamental point of view, EPs play an important role in the area of non-Hermitian quantum theory based on PT-symmetric Hamiltonians (with simultaneous parity-time invariance) [7]. In this context, they occur at the phase transitions between broken-unbroken PT-symmetry. While initially EPs were regarded as a mathematical-physics concept, in the last decade there has been a growing interest on EPs from the experimental point of view in such areas as atomic spectra measurements [8], microwave cavity experiments [9–13], chaotic optical microcavities [14] or optomechanical systems [15]. Exceptional points arise also in classical systems, such as coupled classical electric oscillators [16, 17], classical optical systems [18], classical spin dynamics [19, 20], and general dissipative classical systems [21].

From the applications point of view, the strong sensitivity of eigenvalues to perturbations near EPs has been used to devise new types of sensors with unprecedented sensitivity [11, 22, 23]. This was demonstrated in highly sensitive optical nanoparticle detection [24, 25], in laser gyroscopes [26], and in optically pumped semiconductor rings for temperature detection [27], and in coupled microcantilevers for ultrasensitive mass sensing [28].

In this study, we demonstrate both experimentally and theoretically how the concept of EPs can be used to control the oscillations in a system of two coupled spintronic nano-oscillators (STOs) (see fig. 1). These are typical spintronic nanoscale devices that convert magnetization dynamics in a thin layer into electrical microwaves [28]. They have both loss, associated with the damping of magnetization oscillations, and gain, provided by the spin-torque effect resulting from the spin-polarized currents injected into the device. In this respect, coupled STOs are archetypal systems to evidence the relevance of EPs in Spintronics. While maintaining...
the generality of the results, we use STOs based on the spin torque gyrotropic dynamics of a magnetic vortex core [26].

Very recently, there has been growing attention on EPs in the area of magnetism with theoretical works on spin dynamics [19–23] and spin-orbit systems [24]. Experimental results have been achieved in passive magnonic structures [11, 12] or opto-magnonic systems [15, 16]. However, the experimental study of EPs in coupled discrete nano-devices has been so far overlooked. In this respect, here we experimentally demonstrate that, by tuning the currents injected into the two STOs, it is possible to control the position of an EP in order to obtain the phenomenon of amplitude death [1]. This is the vanishing of the oscillation amplitudes of the coupled STOs, despite the increase of the spin torque (gain) in one oscillator (obtained through the increase of the injected dc current). Such amplitude death occurs for a certain specific range of current values as we will show later. This type of phenomena and the sensitiveness of the system to perturbations pave the road for the design of new types of field devices and/or current sensors based on STOs or potentially to devise novel hardware solutions for neuro-inspired computing schemes.

Our observations create an important connection between the physics of EPs and Spintronics, an area of research that has crucial technological implications for data storage and processing [18, 19], sensor technology [15, 16], wide-band high-frequency communication [13–15], and, more recently, bio-inspired networks for neuromorphic computing beyond CMOS [3, 4], to mention only a few. In this area, coupled STOs have been studied mainly with respect to the phenomenon of mutual synchronization [16, 20]. Our observation describes, up to our knowledge, the first experimental evidence to exploit EPs in coupled spintronics nano-devices.

Theory

We first theoretically study the regime of small amplitudes oscillations of the two coupled STOs around their rest positions. From the linear equation governing these small oscillations, we find the condition for an EP to emerge. This leads to a formula that connects, at the EP, the relevant parameters of the coupled STOs: frequencies, gain/loss parameters, and coupling coefficient. By using this formula, the position of an EP in the parameter space can be controlled in order to determine the interval of injected current values in which amplitude death occurs.

More specifically, we study two coupled STOs (see fig. 1) with magnetization in a dynamical vortex state, hereafter referred to as STVOs (spin torque vortex oscillators). The gain mechanism, counteracting the natural damping and enabling self-sustained oscillations in each STVO, is provided by the spin-transfer torque that is proportional to the injected current. Self-oscillations set in when the injected current \( I \) is larger than a critical (threshold) current \( I_c \), which corresponds to the exact compensation of gain and loss.

Gyrations of the vortex core around the symmetry axis of each oscillator are modeled by the Thiele-like theory for which the overall state of the oscillators is given by the in-plane displacements \((\rho_{1x}, \rho_{2x})\) of the vortex cores from the center of the corresponding devices [53] (for details see Methods). The coupling between the two STOs, which is assumed to be symmetric, is obtained by feeding stripline antennas above each oscillator with the microwave voltage generated by the other, that in turn gives rise to a rf field (see fig. 1). For vortex core displacements sufficiently smaller than the radius of the devices, it is reasonable to assume a linear coupling [12, 13] between the STVOs (which is always the case for the range of injected currents used in this study). The coupling has both dissipative and conservative terms that are described by the coefficients \( k_c \) and \( k_d \), respectively.

The linearized Thiele equation governing the regime of small oscillations of the vortex cores around the rest position \( \rho_{1,2} = 0 \), written in terms of the complex state variables \( z_i = x_i + iy_i \) (with \( i = 1, 2 \)) associated to the \( i \)-th vortex core \( x \)- and \( y \)-axis position, reads

\[
\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = iA \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},
\]

with

\[
A = \begin{bmatrix} \omega_1 - i\beta_1 & k \\ k & \omega_2 - i\beta_2 \end{bmatrix},
\]

where \( k = k_c - ik_d \), \( \omega_i \) are the angular frequencies of vortex free oscillations, and \( \beta_i \) are the loss/gain parameters. These latter parameters are given by \( \beta_i = C_i l_i - d_i \omega_i \), where \( d_i \) are the damping constants and \( C_i \) are parameters determining the efficiency of the spin torque effect.

The matrix \( A \) in eq. (2) has the typical form for systems exhibiting EPs [12, 18]. In order to study the natural frequencies of the system \( A \), we assume a dependence of \( z_{1,2} \) on the time of the type \( e^{\nu t} \). The natural frequencies \( \nu_{1,2} \) are then obtained as eigenvalues of the matrix \( A \) and they are given by the following formula:

\[
\nu_{1,2} = \bar{\omega} \pm \sqrt{k^2 + \left(\bar{\omega} - i\bar{\beta}\right)^2},
\]

where

\[
\bar{\beta} = (\beta_1 + \beta_2)/2, \quad \bar{\omega} = (\omega_1 + \omega_2)/2,
\]

\[
\bar{\beta} = (\beta_1 - \beta_2)/2, \quad \bar{\omega} = (\omega_1 - \omega_2)/2.
\]

Stability of solutions is given by the following condition

\[
\text{Im}(\nu) \geq 0 \iff \text{Re}(\nu) \leq 0.
\]

EPs emerge when two natural frequencies coalesce along with the corresponding eigenvectors. This occurs when the square root term in eq. (3) is zero, leading to
the following condition on the parameters to obtain an EP:

\[ k_c - i k_d = \pm \left[ \frac{(\beta_1 - \beta_2)}{2} + i \frac{(\omega_1 - \omega_2)}{2} \right], \tag{5} \]

where we have explicitly expressed \( k, \tilde{\omega} \) and \( \tilde{\beta} \).

Figure 2: Eigenvalues \( i\nu_{1,2} \) from eq. (4) when the EP is placed at \( I_1^* = I_{2,EP} = 7.1 \text{ mA} \). (a) Real and imaginary part of the eigenvalues as a function of the dc current \( I_2 \). (b) Eigenvalues in the complex plane (Re(\( i\nu \)), Im(\( i\nu \))). Black lines in (a) as well as black line and symbol in (b) refer to eigenvalues computed in the uncoupled case. The system parameters are: \( \Omega_{ms,1} = 2\pi \cdot 225 \text{ MHz} \), \( \Omega_{ms,2} = 2\pi \cdot 233 \text{ MHz} \), \( n_{Oe,1} = n_{Oe,2} = 2\pi \cdot 3 \text{ MHz/mA} \), \( d_1 = d_2 = 0.1 \), \( C_1 = 22.22 \text{ MHz/mA} \) and \( C_2 = 20.18 \text{ MHz/mA} \). The resulting coupling constant: \( k_{EP} = 9.76 - 25.13i \).

We consider now the case that will be also studied experimentally: we fix the current \( I_1 \) to a value \( I_1^* \) and the second current \( I_2 \) is swept from values below to values above \( I_{c,2} \). By using the condition (5), we can adjust the values of the parameters to have an EP at the desired value of the current \( I_2 \). The effect of the exceptional point on the eigenvalues of the matrix \( A \) is illustrated in fig. 2. The black curves in fig. 2(a) are the real and imaginary parts of the eigenvalues when there is no coupling \( (k_c = k_d = 0) \). When coupling is taken into account, an EP exists at \( I_{2,EP} = 7.1 \text{ mA} \), and it has the effect of attracting the two eigenvalues to one point in the complex plane. According to eq. (3), if at the EP \( \beta = (\beta_1 + \beta_2)/2 \) is negative (this occurs when \( \beta_2 < 0, |\beta_2| > |\beta_1| > 0 \)), then both eigenvalues \( i\nu \) have negative real parts. This is also visible in fig. 2(b) where the eigenvalues are plotted in the complex plane (Re(\( i\nu \)), Im(\( i\nu \))). In the proximity of the EP, both eigenvalues are in the plane Re(\( i\nu \)) < 0. This implies that the rest position of the two oscillators is stable, leading to the disappearance of both STOs’ oscillations. This phenomenon is called amplitude death and, as we have illustrated above, it can be controlled by the appropriate placing of the EP.

Since the condition for the onset of an EP is very sensitive to perturbations, it might happen that in experiments, the EP is not reached in a strict sense, as we discuss below on the experimental data. Nevertheless, if the parameters are such that the condition (5) is nearly verified, the amplitude death phenomenon is expected to be reliably observed as well.

It is important to note that, when the rest position is stable in terms of the linearized model, we find that this stability is also exhibited by the rest position in the full nonlinear equations (see Methods 2). Thus, for predicting the phenomenon of amplitude death, the linear theory is strictly appropriate. On the other hand, when the real part of the eigenvalue \( i\nu \) becomes positive – this happens at points in fig. 2 where Re(\( i\nu \)) crosses zero – the rest state becomes unstable. The regime that sets in after instability has an amplitude determined by the nonlinear saturation term in the Thiele equation and a frequency which slightly deviates from the Im(\( i\nu \)) at the aforementioned crossing. This phenomenon is referred to as a Hopf bifurcation. For values of parameters which correspond to a Hopf bifurcation point and no other bifurcations take place, the linear analysis can be used to estimate the frequency of the self-oscillating regimes by considering the imaginary parts of the natural frequency \( i\nu \) at the Hopf bifurcation. In the assessment of the experimental results, this concept is applied in order to identify the appropriate parameter values describing the amplitude death region as a function of the injected currents.

Experimental results

In this part, we describe the experimental results that demonstrate the emergence of EPs and the resulting amplitude death regions in our coupled STO system. All measurements have been conducted at room temperature \( T \sim 300 \text{ K} \). In the performed experiments, the current injected into the STVO 1 is kept constant to \( I_1^* \), while sweeping the current \( I_2 \) injected into STVO 2. Note that the onset for self-sustained oscillations in the uncoupled case is \( I_{c,1} \approx 6.95 \text{ mA} \) and \( I_{c,2} \approx 8 \text{ mA} \) for STVO 1 and 2, respectively. The evolution of their frequency with the applied current appears similar for the two uncoupled STOVs (see Methods 3).

Figure 3: Measured frequency spectra of the coupled system vs. current \( I_2 \) for \( I_1^* = 8 \text{ mA} \) (a). The labeling corresponds to the STVO in which the oscillation mode is mainly localized. (b) Corresponding theoretically determined evolution of eigenvalues. The system parameters are: \( \Omega_{ms,1} = 2\pi \cdot 225 \text{ MHz} \), \( \Omega_{ms,2} = 2\pi \cdot 233 \text{ MHz} \), \( n_{Oe,1} = n_{Oe,2} = 2\pi \cdot 3 \text{ MHz/mA} \), \( d_1 = d_2 = 0.1 \), \( C_1 = 22.22 \text{ MHz/mA} \) and \( C_2 = 20.18 \text{ MHz/mA} \). The resulting coupling constant: \( k_{EP} = 10.68 - 25.13i \).
In fig. 3a, we display the frequency spectra measured at $I_1^* = 8$ mA while $I_2$ is changed. No amplitude death is observed, however a frequency branching for $I_2 \geq 8$ mA is present. We ascribe this phenomenon to the presence of an EP in the linearized model. Therefore, we use formula (5) collocating the EP at $(I_{1,EP}^*, I_{2,EP}) = (8, 8)$ mA. Based on this identification, the linear coupling constant can be determined and the theory parameters adjusted in order to compute the eigenvalues $i\nu_{1,2}$ as a function of the current $I_2$ (fig. 3b). From the general point of view, the eigenvalues $i\nu_{1,2}$ give information about the linear dynamics around the rest position. Their imaginary parts show a branching similar to the experimentally observed oscillation frequencies (fig. 3a). Indeed, the theoretical linear approach provides a good access to the analysis of the intrinsically nonlinear regime of the experimental self-oscillations. In the Methods section, we show the consistency of the linear model with numerical computations of the mutually coupled nonlinear dynamics based on the Thiele equations. In fig. 2c, the real part confirms that the rest-position is unstable over the entire range $I_2$. The decrease of $\text{Re}(i\nu)$ in the proximity of the EP reasons the experimental linewidth broadening in the range $I_2 \in [7.2; 8]$ mA in fig. 3a where noise induced fluctuations become important due to the vicinity of the stability axis ($\text{Re}(i\nu) = 0$).

From the prediction of our model, we expect that the decreasing of the gain effect together with the attraction of the eigenvalues around the EP makes the amplitude death phenomenon observable. To confirm this behavior, in fig. 4a - 4d we perform measurements of the coupled system for smaller $I_1^*$ for which the eigenvalue real part can explicitly become negative due to the EP and hence, amplitude death occurs in this regime. With respect to the critical currents of the uncoupled STVOs, STVO 1 is undercritical in fig. 4a and overcritical in fig. 4b - 4d. The overall range of oscillation death evolves with $I_1^*$ (fig. 4b, 4d), whereas rather the lower value $I_2$ of the interval in which oscillations are suppressed is affected than the upper one which remains quasi constant. Increasing $I_1^*$ tends to stabilize the oscillation of STVO 1 and in consequence counteracts the occurrence of the amplitude death. This leads to a decrease of the current range in which no oscillation is detected (see fig. 4).

Furthermore, for $I_1^* < 7.8$ mA and $I_2 > 8$ mA, the oscillations from STVO 1 show a lower output power together with a larger linewidth than it would be expected for self-sustained oscillations. When the current $I_1^*$ is close to 8 mA, in the vicinity of the EP, thermal noise can induce stochastic transitions between the oscillatory regime and the rest state corresponding to amplitude death (clearly visible in fig. 4d). For currents $I_1^* \geq 8$ mA (see fig. 3a for $I_1^* = 8$ mA), oscillation death is no more occurring, however, the linewidth of the oscillation is clearly enhanced in a small range $I_2 \in [7; 8]$ mA. This range however decreases with increasing currents $I_1^*$. At even larger currents $I_1^* \geq 9$ mA (see Methods 3), the two STVOs tend to mutually synchronize, a phenomenon that is commonly known for STOs [40, 49] and which refers to the strongly nonlinear characteristics of the oscillator, far from the Hopf bifurcation point.

The experimentally observed amplitude death is very well reproduced by our modelling of the coupled STVOs. In fig. 4e - 4h we present the corresponding real and imaginary parts of the eigenvalues $i\nu_{1,2}$ as a function of the current $I_2$. Except for the value of the coupling constant, which in principle depends on the electric interface between STVOs as well as on their dynamical state, the modelling parameters are the same as those used in fig. 3b. We find that by only rotating the coupling constant $k_{EP} \rightarrow k_{EP} e^{i\phi_k}$ in the complex plane $(k_c, k_d)$ without changing its modulus, the amplitude death phenomena can be completely described. The rotation angle for the two cases where the amplitude death is evident at $I_1^* = 7.1$ and 7.5 mA (figs. 4b & 4c) is $\phi_k = 40$ and 45°, respectively. In fig. 4b - 4d the amplitude death current ranges can be recognized by looking at where the condition $\text{Re}(i\nu_{1,2}) \leq 0$ is satisfied. Then, at the upper current value $I_2$ of the amplitude death regime, the real part of one eigenvalue crosses the real axis and the corresponding mode becomes unstable. This situation corresponds to a Hopf bifurcation which brings the system to self-oscillations. Such consideration permits to rigorously justify the presence of the upper branch in the measured spectra. The discussed Hopf bifurcation point does not significantly change its position while the square root like upper branch of $\text{Re}(i\nu)$ at lower currents $I_2$ implies a strong dependence of the amplitude death range’s lower boundary on the fixed current $I_1^*$, as also found experimentally. For larger current $I_2$, in the case $I_1^* = 7.5$ mA, also the real part of the other eigenvalue (blue curve in $\text{Re}(i\nu)$ becomes positive, but it stays close to the real axis. In the experiments, which are subject to thermal fluctuations, this manifests as the described linewidth broadening of STVO 1’s oscillation at relative smaller power. Similar situation occurs for $I_1^* = 6.9$ mA and $I_1^* = 7.1$ mA. In both cases the value of the rotation angle is set to $\phi_k = 40^\circ$. The main difference with the $I_1^* = 7.5$ mA case is that only the real part of one eigenvalue crosses the real axis. The other stays close to it. Similar to before, thermal fluctuations however permit oscillations of large linewidth in the experiments. For $I_1^* = 7.8$ mA, the measured spectra are similar as for $I_1^* = 8$ mA and hence we set $\phi_k = 0^\circ$. The oscillations death for this case is experimentally observed (see fig. 4d), but is not described by the linear theory. However, the stochasticity of the transitions between oscillation regime and rest state suggests that thermal fluctuations play in this case a dominant role in determining the stability of the oscillators. Indeed, the main characteristics of the coupled system can be accessed by the developed linearized theory. The study of the eigenvalues as a function of the current permits to unravel the key features of the coupled STVO system’s frequency response. It is important to stress that the eigenvalues refer to the eigenvectors of the ma-
Figure 4: Measured frequency spectra of the coupled system vs. current $I_2$ in STVO 2 for different currents $I_1^*$ in STVO 1 (a-d). (e-h) Real and imaginary part of $i\nu_1$ (red) and $i\nu_2$ (blue) fixing current $I_1$ and changing current $I_2$. Gray lines refer to the same quantity evaluated when the coupling constant is set to 0. Experimental and theoretical graphs in the same column correspond to the same parameters.

Outlook

One of the interesting specificities of the spintronic nano-oscillators is their strong nonlinearity which makes them a promising candidate for various applications and leads to a tremendous manifold of physical phenomena unified in these little devices. The emergence of an EP in a nanoscale nonlinear system is of fundamental interest. Beyond the already mentioned implications for the development of novel types of sensors operating at exceptional points [11] [22], these systems are anticipated to unravel fascinating physics. This includes phenomena such as chaos, complex bifurcations, or the emergence of topological operations around the EP [13]. Complex dynamics and as well the demonstrated occurrence of stochastic stability might furthermore complement the field of hardware based neuromorphic computing that recently gained attention in the context of spintronics [23], for instance as stochastic spiking neurons. Moreover, higher dimensionally coupled systems have been realized with STOs [24] that are anticipated to facilitate the emergence of higher order exceptional points [13] [25] [22]. All these different aspects are still to be explored and potentially lead to intriguing findings in nonlinear non-Hermitian systems.

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Methods

1. Theory

Thiele’s equations. The model equations for the two coupled STVOs are the following:

\[
\frac{d\rho_i}{dt} = \Omega_i(\rho_i, I_i) n_i \times \rho_i + [C_i I_i - d_i \cdot \Omega_i(\rho_i, I_i)] \rho_i + K \cdot \rho_j ,
\]

where \( i, j \in \{1, 2\}, i \neq j \), denoting STVO 1 or 2. The symbol \( n_i \) denotes the unit vector along the symmetry axis of the oscillator, \( I_i \) is the injected current, the variable \( \rho_i \) is the in-plane vortex core position normalized to the radius \( R_i \) of the device, \( \Omega_i \) is the conservative oscillations angular frequency, \( d_i \) a dimensionless damping constant and \( C_i \) is a parameter determining the efficiency of the spin torque effect. This latter parameter determines also the critical current \( I_{c,i} \) needed for exciting self-sustained oscillations in the uncoupled case. The angular frequency \( \Omega_i \) is a function of \( I_i \) and of the magnitude \( \rho_i = |\rho_i| \): \( \Omega(\rho, I) = \Omega_{ms} (1/\Gamma - (\rho/2)^2) + n_{oe} I (1 - \rho^2/2) \), where \( \Omega_{ms} \) and \( n_{oe} \) take into account the influence on the frequency of the magnetostatic field generated by the magnetic vortex state and of the Oersted field generated by the injected current, respectively. The coupling between STVO \( i \) and \( j \) is described by a \( 2 \times 2 \) matrix:

\[
K \cdot \rho_j = k_d \rho_j + k_s n_j \times \rho_j ,
\]

where \( k_s \) and \( k_d \) are conservative and dissipative coupling coefficients, respectively.

\( P\mathcal{T}\)-symmetry. It is interesting to notice that at perfect gain compensation, \( \beta_1 = -\beta_2 = \beta \), and the same frequencies of the uncoupled oscillators, \( \omega_1 = \omega_2 = \omega \), the natural frequencies are given by the simplified formula: \( \nu_{1,2} = \omega \pm \sqrt{k^2 - \beta^2} \). Thus, the natural frequencies are both real when \( \beta < k \), while for \( \beta > k \) an imaginary part appears. The system has in this case a \( P\mathcal{T}\)-symmetry when the frequencies are real and a broken \( P\mathcal{T}\)-symmetry when the frequencies are complex. The symmetry-breaking bifurcation occurs when \( \beta = k \) and this condition corresponds to an EP. Representing an interesting case, this situation however requires a very fine tuning of the two oscillators’ properties.

2. Nonlinear dynamics

The results presented in section Experiments have their theoretical foundation in the linearized theory of the coupled system dynamics although the measured spectra refer to phenomena (e.g self-oscillation regimes) which can only be rigorously explained with nonlinear theory arguments. It is therefore important to show the consistency of the linear model with computations of nonlinear dynamics of the coupled STVOs.

Numerical model. Based on the single STVO measured spectra shown in figs. S2 the parameters of the Thiele equation for each oscillator have been determined. In particular, from the critical current \( I_{c,i} \) and the corresponding self-oscillation frequency, the value of the constants \( \Omega_{ms,i}, C_i \) assuming certain values for \( n_{oe} \) and \( d_i \) can be estimated. The nonlinear current dependence of the measured self-oscillation frequency is reproduced by assuming a polynomial dependence on the vortex core position of the damping term \( d_i(\rho_i) \). We remark that this modeling procedure is for the single STVO uncoupled from the other one. The coupling effect is estimated as described in the section Experiments.

Simulations. In fig. S1 the computed spectra from time integration of the nonlinear dynamics of the coupled STVOs are shown. Fixing the current \( I_1 \), the dynamics of the coupled oscillators is simulated for each value of the current \( I_2 \) in the range \((3,10,5) \) mA, taking an ensemble of \( N \) initial conditions randomly distributed in a disk around the origin of radius \( \rho/R = 0.01 \). Then from the self-oscillations regime we estimate the power as: \( P = 1/N \log_{10} (\sum_i \|x_{i,t}\|^2) \), where \( x = FFT[x] \), and the notation \( x_{i,t} \) refers to the x coordinate of the vortex core of the STVO-n.

The spectra computed from the nonlinear model confirm the validity of the approach used in the Experiments section to estimate the coupling constant \( k_{EP} \). The similarity of fig. S3 and fig. S1B is a strong indication of the fact that the branching of the frequency in the measured spectra is due to the passage in the proximity of an EP. The agreement between experimental results and numerical simulations in terms of frequency value and frequency gap when the spectra become double peaked is obtained also for the other cases where the current \( I_1 \) is changed in the range \([6;9;8] \) mA. Finally, the agreement on the amplitude death current ranges is not surprising since the use of the linear theory for their estimation is a rigorous result of the nonlinear system theory.

3. Experimental description

Device fabrication. The studied STVO devices are magnetic tunnel junctions containing a pinned layer made of a conventional synthetic antiferromagnetic stack (SAF), a MgO tunnel barrier and an NiFe free layer in a magnetic vortex configuration. The magnetoresistive ratio related to the tunnel magnetoresistance effect (TMR) lies around 110% at room temperature and the area resistance product is \( RA \approx 21 \Omega \mu m^2 \). In detail, the SAF is composed of \( IrMn(60)/Co_{70}Fe_{30}(2.6)/Ru(0.85)/Co_{40}Fe_{60}B_{20}(2.6) \) and the total layer stack is \( Ta(5)/CuN(50)/Ta(5)/Ru(5)/SAF/MgO(1)/Co_{40}Fe_{60}B_{20}(2)/Ta(0.2)/Ni_{80}Fe_{20}(7)/Ta(10)/CuN(30)/Ru(5) \), with the nanometer layer thickness in brackets. The growth of the amorphous NiFe free layer is decoupled from the lower CoFeB-layer by a 0.2 nm Ta-layer. This structure permits to exploit the high tunnel magnetoresistance (TMR) ratio of the crystalline CoFeB-junction and the magnetically softer NiFe for the vortex dynamics. The layers are deposited...
on high resistivity SiO₂ substrates by ultrahigh vacuum magnetron sputtering and subsequently annealed for 2h at \( T = 330^\circ\text{C} \) at an applied magnetic field of 1T along the SAF’s easy axis. The patterning of the circular tunnel junctions is conducted using e-beam lithography and Ar ion etching. They have an actual diameter of \( 2R = 370\,\text{nm} \) and the microwave field line of \( 1\,\mu\text{m} \times 300\,\text{nm} \) is lithographed 300 nm above the nanopillar.

**Measurements.** Measurements are conducted under an applied out-of-plane field of \( \mu_0H_\perp = 360\,\text{mT} \). It tilts the in-plane magnetization of the SAF slightly into the perpendicular direction. That induces an out-of-plane spin current polarization, necessary for an efficient spin transfer torque (STT) \(^{[54]}\). The STT provides the gain mechanism in the STVOs and is generated by injected dc currents, which are separately controlled for the two STVOs by two dc current sources. The TMR effect converts the vortex magnetization dynamics into an electrical rf signal. The electrical rf output signal of each oscillator is amplified by 30 dB and injected into the field line located above the other STVO (see Fig.1) in order to implement a symmetric coupling scheme through the generated rf Oersted fields. In the circuit, the dc and rf current parts are separated through a bias tee and the dc electrical properties are monitored by a voltmeter. The emitted rf signals of both coupled STVOs are combined and recorded by a spectrum analyzer.

**Frequency evolution of the STVOs in the uncoupled case.** In order to reveal the emergence of an exceptional point and complex dynamics in the coupled STVO system, the uncoupled characteristics of the oscillators should be sufficiently similar for realizing a symmetric situation with reciprocal coupling.

In fig. **S2** the power spectra for the two uncoupled STVOs are measured separately. They show that in the vicinity of the threshold current for self-oscillations the frequency characteristics of the two STVOs are very similar. The region close to the onset of oscillations is the one of interest for the study of EPs.

**Synchronization at larger current densities.** In Fig. **S3** we show the power spectra recorded for larger current densities in which the effect of mutual synchronization is observed between \( I_2 = 8.2 \) and 8.8 mA. Such synchronization implies an increase of the emitted power and a decrease of the spectral linewidth.

![Figure S2: Frequency spectra of the two independent STVOs.](image)

![Figure S3: Experimental characterization in the regime of mutual synchronization of the two coupled STVOs: \( I_1^* = 9.1\,\text{mA}, \mu_0H_\perp = 360\,\text{mT}, T = 300\,\text{K} \).](image)

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