Maximum Balanced Subgraph Problem
Parameterized Above Lower Bound *

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Abstract

We consider graphs without loops or parallel edges in which every edge is assigned + or −. Such a signed graph is balanced if its vertex set can be partitioned into parts \( V_1 \) and \( V_2 \) such that all edges between vertices in the same part have sign + and all edges between vertices of different parts have sign − (one of the parts may be empty). It is well-known that every connected signed graph with \( n \) vertices and \( m \) edges has a balanced subgraph with at least \( \frac{m}{2} + \frac{n-1}{4} \) edges and this bound is tight. We consider the following parameterized problem: given a connected signed graph \( G \) with \( n \) vertices and \( m \) edges, decide whether \( G \) has a balanced subgraph with at least \( \frac{m}{2} + \frac{n-1}{4} + k \) edges, where \( k \) is the parameter.

We obtain an algorithm for the problem of runtime \( 8^k (kn)^{O(1)} \). We also prove that for each instance \((G, k)\) of the problem, in polynomial time, we can either solve \((G, k)\) or produce an equivalent instance \((G', k')\) such that \( k' \leq k \) and \(|V(G')| = O(k^3)\). Our first result generalizes a result of Crowston, Jones and Mnich (ICALP 2012) on the corresponding parameterization of Max Cut (when every edge of \( G \) has sign −).

Our second result generalizes and significantly improves the corresponding result of Crowston, Jones and Mnich for MaxCut: they showed that \(|V(G')| = O(k^5)\).

1 Introduction

We consider undirected graphs with no parallel edges or loops and in which every edge is labelled by + or −. We call such graphs signed graphs, and edges, labelled by + and −, positive and negative edges, respectively. The labels + and − are the signs of the corresponding edges. Signed graphs are well-studied

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due to their various applications and interesting theoretical properties, see, e.g., [1, 5, 8, 9, 10, 11, 16]. Let $G = (V, E)$ be a signed graph and let $V = V_1 \cup V_2$ be a partition of the vertex set of $G$ (i.e., $V_1 \cap V_2 = \emptyset$). We say that $G$ is $(V_1, V_2)$-balanced if an edge with both endpoints in $V_1$, or both endpoints in $V_2$ is positive, and an edge with one endpoint in $V_1$ and one endpoint in $V_2$ is negative; $G$ is balanced if it is $(V_1, V_2)$-balanced for some partition $V_1, V_2$ of $V$ ($V_1$ or $V_2$ may be empty).

In some applications, we are interested in finding a maximum-size balanced subgraph of a signed graph [1, 5, 11, 16]. We will call this problem Signed Max Cut. This problem is a generalization of Max Cut and as such is NP-hard (Signed Max Cut is equivalent to Max Cut when all edges of $G$ are negative). Hüffner et al. [11] parameterized Signed Max Cut below a tight upper bound: decide whether $G = (V, E)$ contains a balanced subgraph with at least $|E| - k$ edges, where $k$ is the parameter. Hüffner et al. [11] showed that this parameterized problem is fixed-parameter tractable (FPT) using a simple reduction to the Edge Bipartization Problem: decide whether an unsigned graph can be made bipartite by deleting at most $k$ edges ($k$ is the parameter). Using this result and a number of heuristic reductions, Hüffner et al. [11] designed a nontrivial practical algorithm that allowed them to exactly solve several instances of Signed Max Cut that were previously solved only approximately by DasGupta et al. [5].

In this paper, we consider a different parameterization of Signed Max Cut: decide whether a connected signed graph $G$ with $n$ vertices and $m$ edges contains a subgraph with at least $m^2 + \frac{n - 1}{4} + \frac{k}{4}$ edges, where $k$ is the parameter. Note that $pt(G) = \frac{m^2}{2} + \frac{n - 1}{4} + \frac{k}{4}$ is a tight lower bound on the number of edges in a balanced subgraph of $G$ (this fact was first proved by Poljak and Turzík [14], for a different proof, see [2]). Thus, we will call this parameterized problem Signed Max Cut Above Tight Lower Bound or Signed Max Cut ATLB. Whilst the parameterization of Hüffner et al. of Max Cut ATLB is of interest when the maximum number of edges in a balanced subgraph $H$ of $G$ is close to the number of edges of $G$, Signed Max Cut ATLB is of interest when the maximum number of edges in $H$ is close to the minimum possible value in a signed graph on $n$ vertices and $m$ edges. Thus, the two parameterizations treat the opposite parts of the Signed Max Cut “spectrum.”

It appears that it is much harder to prove that Signed Max Cut ATLB is FPT than to show that the parameterization of Hüffner et al. of Signed Max Cut is. Indeed, Signed Max Cut ATLB is a generalization of the same parameterization of Max Cut (denoted by Max Cut ATLB) and the parameterized complexity of the latter was an open problem for many years (and was stated as an open problem in several papers) until Crowston et al. [4] developed a new approach for dealing with such parameterized problem. This approach was applied by Crowston et al. [5] to solve an open problem.
of Raman and Saurabh [13] on maximum-size acyclic subgraph of an oriented graph. Independently, this problem was also solved by Mnich et al. [12] who obtained the solution as a consequence of a meta-theorem which shows that several graph problems parameterized above a lower bound of Poljak and Turzík [14] are FPT under certain conditions.

While the meta-theorem is for both unlabeled and labeled graphs, all consequences of the meta-theorem in [12] are proved only for parameterized problems restricted to unlabelled graphs. A possible reason is that one of the conditions of the meta-theorem requires us to show that the problem under consideration is FPT on a special family of graphs, called almost forests of cliques. The meta-theorem is useful when it is relatively easy to find an FPT algorithm on almost forests of cliques. However, for Signed Max Cut ATLB it is not immediately clear what an FPT algorithm would be even on a clique.

Our attempts to check that Signed Max Cut ATLB is FPT on almost forests of cliques led us to reduction rules that are applicable not only to almost forests of cliques, but to arbitrary instances of Signed Max Cut ATLB. Thus, we found two alternatives to prove that Signed Max Cut ATLB is FPT: with and without the meta-theorem. Since the first alternative required stating the meta-theorem and all related notions and led us to a slightly slower algorithm than the second alternative, we decided to use the second alternative.

We reduce an arbitrary instance of Signed Max Cut ATLB to an instance which is an almost forest of cliques, but with an important additional property which allows us to make use of a slight modification of a dynamic programming algorithm of Crowston et al. [4] for Max Cut ATLB on almost forests of cliques.

Apart from showing that Max Cut ATLB is FPT, Crowston et al. [4] proved that the problem admits a kernel with $O(k^3)$ vertices. They also found a kernel with $O(k^3)$ vertices for a variation of Max Cut ATLB, where the lower bound used is weaker than the Poljak-Turzík bound. They conjectured that a kernel with $O(k^3)$ vertices exists for Max Cut ATLB as well. In the main result of this paper, we show that Signed Max Cut ATLB, which is a more general problem, also admits a polynomial-size kernel and, moreover, our kernel has $O(k^3)$ vertices. Despite considering a more general problem than in [4], we found a proof which is shorter and simpler than the one in [4]; in particular, we do not use the probabilistic method. An $O(k^3)$-vertex kernel for Signed Max Cut ATLB does not immediately imply an $O(k^3)$-vertex kernel for Max Cut ATLB, but the same argument as for Signed Max Cut ATLB shows that Max Cut ATLB admits an $O(k^3)$-vertex kernel. This confirms the conjecture above.

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4Forests of cliques are defined in the next section. An almost forest of cliques is obtained from a forest of cliques by adding to it a small graph together with some edges linking the small graph with the forest of cliques.
2 Terminology, Notation and Preliminaries

For a positive integer \( l \), \([l] = \{1, \ldots, l\} \). A cycle \( C \) in \( G \) is called positive (negative) if the number of negative edges in \( C \) is even (odd).\(^5\) The following characterization of balanced graphs is well-known.

**Theorem 1.** \([10]\) A signed graph \( G \) is balanced if and only if every cycle in \( G \) is positive.

Let \( G = (V, E) \) be a signed graph. For a subset \( W \) of \( V \), the \( W \)-switch of \( G \) is the signed graph \( G_W \) obtained from \( G \) by changing the signs of the edges between \( W \) and \( V \setminus W \). Note that a signed graph \( G \) is balanced if and only if there exists a subset \( W \) of \( V \) (\( W \) may coincide with \( V \)) such that \( G_W \) has no negative edges. Indeed, if \( G_W \) has no negative edges, \( G \) is \((W, V \setminus W)\)-balanced. If \( G \) is \((V_1, V_2)\)-balanced, then \( G_{V_1} \) has no negative edges.

Deciding whether a signed graph is balanced is polynomial-time solvable.

**Theorem 2.** \([3]\) Let \( G = (V, E) \) be a signed graph. Deciding whether \( G \) is balanced is polynomial-time solvable. Moreover, if \( G \) is balanced then, in polynomial time, we can find a subset \( W \) of \( V \) such that \( G_W \) has no negative edges.

For a signed graph \( G \), \( \beta(G) \) will denote the maximum number of edges in a balanced subgraph of \( G \). Furthermore, for a signed graph \( G = (V, E) \), \( \text{pt}(G) \) denotes the Poljak-Turzík bound: \( \beta(G) \geq \text{pt}(G) \). If \( G \) is connected, then \( \text{pt}(G) = \frac{|E(G)|}{2} + \frac{|V(G)| - 1}{4} \), and if \( G \) has \( t \) components, then \( \text{pt}(G) = \frac{|E(G)|}{2} + \frac{|V(G)| - t}{4} \).

It is possible to find, in polynomial time, a balanced subgraph of \( G \) of size at least \( \text{pt}(G) \) \([14]\).

The following easy property will be very useful in later proofs. It follows from Theorem 1 by observing that for a signed graph the Poljak-Turzík bound does not depend on the signs of the edges and that, for any cycle in \( G \), the sign of the cycle in \( G \) and in \( G_W \) is the same.

**Corollary 1.** Let \( G = (V, E) \) be a signed graph and let \( W \subseteq V \). Then \( \text{pt}(G_W) = \text{pt}(G) \) and \( \beta(G_W) = \beta(G) \).

For a vertex set \( X \) in a graph \( G \), \( G[X] \) denotes the subgraph of \( G \) induced by \( X \). For disjoint vertex sets \( X, Y \) of graph \( G \), \( E(X, Y) \) denotes the set of edges between \( X \) and \( Y \). A **bridge** in a graph is an edge that, if deleted, increases the number of connected components of the graph. A **block** of a graph is either a maximal 2-connected subgraph or a connected component containing only one vertex.

For an edge set \( F \) of a signed graph \( G \), \( F^+ \) and \( F^- \) denote the set of positive and negative edges of \( F \), respectively. For a signed graph \( G = (V, E) \), the **dual** of \( G \) is the signed graph \( \bar{G} = (V, \bar{E}) \), where \( E^+ = E^- \) and \( E^- = E^+ \). A cycle in \( G \) is **dually positive (dually negative)** if the same cycle in \( \bar{G} \) is positive (negative).

For a graph \( G = (V, E) \), the **neighborhood** \( N_G(W) \) of \( W \subseteq V \) is defined as \( \{ v \in V : vw \in E, w \in W \} \setminus W \); the vertices in \( N_G(W) \) are called **neighbors of**

\(^5\)To obtain the sign of \( C \) simply compute the product of the signs of its edges.
W. If G is a signed graph, the positive neighbors of W ⊆ V are the neighbors of W in G⁺ = (V, E⁺); the set of positive neighbors is denoted N⁺(W). Similarly, for the negative neighbors N⁻(W).

The next theorem is the ‘dual’ of Theorem 1, in the sense that it is its equivalent formulation on the dual of a graph.

**Theorem 3.** Let G = (V, E) be a signed graph. Then the dual graph G is balanced if and only if G does not contain a dually negative cycle.

In the next sections, the notion of forest of cliques introduced in [4] plays a key role. A connected graph is a tree of cliques if the vertices of every cycle induce a clique. A forest of cliques is a graph whose components are trees of cliques. It follows from the definition that in a forest of cliques any block is a clique.

Note that a forest of cliques is a chordal graph, i.e., a graph in which every cycle has a chord, that is an edge between two vertices which are not adjacent in the cycle. The next lemma is a characterization of chordal graphs which have a balanced dual. A triangle is a cycle with three edges.

**Corollary 2.** Let G = (V, E) be a signed chordal graph. Then G is balanced if and only if G does not contain a positive triangle.

**Proof.** If G contains a positive triangle, then, by Theorem 3, G is not balanced.

Now suppose that G is not balanced. By Theorem 3, G contains a dually negative cycle, i.e., a cycle with odd number of positive edges, but all triangles in G are negative by hypothesis. Let C = v₁v₂...v₃v₁ be a dually negative cycle of minimum length and note that l > 3 as a dually negative triangle is positive. Since the graph is chordal, we can find three consecutive vertices of C that form a triangle T. Suppose T = v₁v₂v₃v₁. Recall that T is negative. So, if both v₁v₂ and v₂v₃ are positive edges (or negative edges), then v₁v₃ must be a negative edge; otherwise if one of the two edges is positive and the other negative, then v₁v₃ is a positive edge. In both cases, we conclude that C contains an odd number of positive edges if and only if C′ = v₁v₃v₄...v₃v₁ does, which is a contradiction since we supposed l to be the minimum length of a dually negative cycle.

**Corollary 3.** Let (G = (V, E), k) be an instance I of Signed Max Cut ATLB, let X ⊆ V(G) and let G[X] be a chordal graph which does not contain a positive triangle. Then there exists a set W ⊆ X, such that G = (G[X], k) is equivalent to I, and G[W][X] does not contain positive edges.

**Proof.** By Corollary 2, G[X] is balanced: hence, by definition of balanced graph, there exists W ⊆ X such that G[W][X] contains only positive edges, which means that G[W][X] contains only negative edges. By Corollary 1, (G[W], k) is an instance equivalent to the original one.

Lastly, the next lemmas describe useful properties of Max Cut ATLB which still hold for Signed Max Cut ATLB.
Lemma 1. Let \( G = (V, E) \) be a connected signed graph and let \( V = U \cup W \) such that \( U \cap W = \emptyset \), \( U \neq \emptyset \) and \( W \neq \emptyset \). Then \( \beta(G) \geq \beta(G[U]) + \beta(G[W]) + \frac{1}{4}|E(U, W)| \). In addition, if \( G[U] \) has \( c_1 \) components, \( G[W] \) has \( c_2 \) components, \( \beta(G[U]) \geq \text{pt}(G[U]) + \frac{k_1}{2} \) and \( \beta(G[W]) \geq \text{pt}(G[W]) + \frac{k_2}{2} \), then \( \beta(G) \geq \text{pt}(G) + \frac{k_1 + k_2 - (c_1 + c_2 - 1)}{4} \).

Proof. Let \( H (F) \) be a balanced subgraph of \( G[U] \) (\( G[W] \)) with maximum number of edges and let \( H (F) \) be \( (U_1, U_2) \)-balanced \( (W_1, W_2) \)-balanced. Let \( E_1 = E^+(U_1, W_1) \cup E^+(U_2, W_2) \cup E^-(U_1, W_2) \cup E^-(U_2, W_1) \) and \( E_2 = E(U, W) \setminus E_1 \). Observe that both \( E(H) \cup E(F) \cup E_1 \) and \( E(H) \cup E(F) \cup E_2 \) induce balanced subgraphs of \( G \) and the largest of them has at least \( \beta(G[U]) + \beta(G[W]) + \frac{1}{2}|E(U, W)| \) edges.

Now, observe that \( \text{pt}(G) = \text{pt}(G[U]) + \text{pt}(G[W]) + \frac{1}{2}|E(U, W)| + \frac{c_1 + c_2 - 1}{4} \). Hence \( \beta(G) \geq \text{pt}(G) + \frac{k_1 + k_2 - (c_1 + c_2 - 1)}{4} \). \( \square \)

Lemma 2. Let \( G = (V, E) \) be a signed graph, \( v \in V \) a cutvertex, \( Y \) a connected component of \( G - v \) and \( G' = G - Y \). Then \( \text{pt}(G) = \text{pt}(G[V \cup \{v\}]) + \text{pt}(G') \) and \( \beta(G) = \beta(G[V \cup \{v\}]) + \beta(G') \).

Proof. The first equality is easily verified. Concerning the other, let \( H_1 \) be a \( (V_1^1, V_2^1) \)-balanced subgraph of \( G[V \cup \{v\}] \) of size \( \beta(G[V \cup \{v\}]) \) and \( H_2 \) be a \( (V_1^2, V_2^2) \)-balanced subgraph of \( G' \) of size \( \beta(G') \). One may assume that \( v \in V_i \) for \( i = 1, 2 \). Therefore the balanced subgraph \( H \) of \( G \) induced by \( V_1 = V_1^1 \cup V_1^2 \) and \( V_2 = V_2^1 \cup V_2^2 \) is of size \( \beta(G[V \cup \{v\}]) + \beta(G') \), which means that \( \beta(G) \geq \beta(G[V \cup \{v\}]) + \beta(G') \). On the other hand, any balanced subgraph \( H \) of \( G \) induces balanced subgraphs of \( G[V \cup \{v\}] \) and \( G' \), which implies that \( \beta(G) \leq \beta(G[V \cup \{v\}]) + \beta(G') \). \( \square \)

3 Fixed-Parameter Tractability

In this section, we prove that \textsc{Signed Max Cut} ATLB is FPT by designing an algorithm of running time \( 2^k \text{OPT}(8^k) \). This algorithm is a generalization of the FPT algorithm obtained in [1] to solve \textsc{Max Cut} ATLB. Given an instance \((G = (V, E), k)\) of \textsc{Max Cut} ATLB, the algorithm presented in [3] applies some reduction rules that either answer \textsc{Yes} for \textsc{Max Cut} ATLB or produce a set \( S \) of at most \( 3k \) vertices such that \( G - S \) is a forest of cliques.

A key idea of this section is that it is possible to extend these rules such that we include into \( S \) at least one vertex for every dually negative cycle of \( G \). As a result, Theorem [4] ensures that solving \textsc{Signed Max Cut} ATLB on \( G - S \) is equivalent to solving \textsc{Max Cut} ATLB. Therefore, it is possible to guess a partial solution on \( S \) and then solve \textsc{Max-Cut-With-Weighted-Vertices} on \( G - S \). Since a forest of cliques is a chordal graph, Corollary [5] implies that it is enough to put into \( S \) at least one vertex for every positive triangle in \( G \).

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\(^6\)In the \( O^* \)-notation widely used in parameterized algorithmics, we omit not only constants, but also polynomial factors.

\(^7\)This problem is defined just before Theorem [6]
(instead of every dually negative cycle). Our reduction rules are inspired by the rules used in \[4\], but our rules are more involved in order to deal with positive triangles.

The rules apply to an instance \((G, k)\) of Signed \textsc{Max Cut} \textsc{ATLB} and output an instance \((G', k')\) where \(G'\) is obtained by deleting some vertices of \(G\). In addition, the rules can mark some of the deleted vertices: marked vertices will form the set \(S\) such that \(G - S\) is a forest of cliques. Note that every time a rule marks some vertices, it also decreases the parameter \(k\).

The instance \((G', k')\) that the rules produce does not have to be equivalent to \((G, k)\), but it has the property that if it is a \textsc{Yes}-instance, then \((G, k)\) is a \textsc{Yes}-instance too. For this reason, these rules are called one-way reduction rules \[3\].

Note that in the description of the rules, \(G\) is a connected signed graph, and \(C\) and \(Y\) denote connected components of a signed graph such that \(C\) is a clique which does not contain a positive triangle.

**Reduction Rule 1.** If \(a\) \(b\) \(c\) is a positive triangle such that \(G - \{a, b, c\}\) is connected, then mark \(a, b, c\), delete them and set \(k' = k - 3\).

**Reduction Rule 2.** If \(a\) \(b\) \(c\) is a positive triangle such that \(G - \{a, b, c\}\) has two connected components \(C\) and \(Y\), then mark \(a, b, c\), delete them, delete \(C\), and set \(k' = k - 2\).

**Reduction Rule 3.** Let \(C\) be a connected component of \(G - v\) for some vertex \(v \notin G\). If there exist \(a, b \in V(C)\) such that \(G - \{a, b\}\) is connected and there is an edge \(ab\) but no edge \(bv\), then mark \(a\) and \(b\), delete them, and set \(k' = k - 2\).

**Reduction Rule 4.** Let \(C\) be a connected component of \(G - v\) for some vertex \(v \notin G\). If there exist \(a, b \in C\) such that \(G - \{a, b\}\) is connected and \(abv\) is a positive triangle, then mark \(a\) and \(b\), delete them, and set \(k' = k - 4\).

**Reduction Rule 5.** If there is a vertex \(v \in V(G)\) such that \(G - v\) has a connected component \(C\), \(G[V(C) \cup \{v\}]\) is a clique in \(G\), and \(G[V(C) \cup \{v\}]\) does not contain a positive triangle, then delete \(C\) and set \(k' = k\).

**Reduction Rule 6.** If \(a, b, c \in V(G)\), \(\{ab, bc\} \subseteq E(G)\) but \(ac \notin E(G)\), and \(G - \{a, b, c\}\) is connected, then mark \(a, b, c\), delete them, and set \(k' = k - 1\).

**Reduction Rule 7.** Let \(C, Y\) be the connected components of \(G - \{v, b\}\) for some vertices \(v, b \in V(G)\) such that \(vb \notin E(G)\). If \(G[V(C) \cup \{v\}]\) and \(G[V(C) \cup \{b\}]\) are cliques not containing any positive triangles, then mark \(v\) and \(b\), delete them, delete \(C\) and set \(k' = k - 1\).

**Definition 1.** A one-way reduction rule is safe if it does not transform a \textsc{No}-instance into a \textsc{Yes}-instance.

The intuitive understanding of how a one-way reduction rule works is that it removes a portion of the graph (while decreasing the parameter from \(k\) to \(k'\)) only if given any solution (i.e., a balanced subgraph) on the rest of the graph there is a way to extend it to the removed portion while always gaining an additional \(k - k'\) over the Poljak-Turzík bound.
Lemma 3. Let $G$ be a connected graph. If $C$ is a clique of $G$ such that $G - C$ is connected and if $C$ contains a positive triangle, then either Rule 1 or Rule 2 applies.

Proof. Let $abca$ be a positive triangle in $C$. Suppose Rule 1 does not apply. This means that $G - \{a, b, c\}$ is not connected: more precisely, $G - \{a, b, c\}$ has two components $G - C$ and $C - \{a, b, c\}$. Note that $C - \{a, b, c\}$ cannot contain a positive triangle, or otherwise Rule 1 would have applied. Therefore, Rule 2 applies.

Theorem 4. Rules 1-7 are safe.

Proof. Rule 1. Let $abca$ be a positive triangle as in the description of Rule 1. Suppose $\beta(G') \geq \text{pt}(G') + \frac{k'}{4}$, where $k' = k - 3$. Since $abca$ is a positive triangle, by Lemma 1 we obtain $\beta(G) \geq \text{pt}(G) + \frac{k - 3}{4} = \text{pt}(G) + \frac{k}{4}$.

Rule 2. Let $abca$ be a positive triangle such that $G - \{a, b, c\}$ has two components $C$ and $Y$. Suppose $\beta(Y) \geq \text{pt}(Y) + \frac{k'}{4}$, where $k' = k - 2$. We know that $\beta(C) \geq \text{pt}(C)$, and so by Lemma 1 we obtain $\beta(G[V\{Y\} \cup V(C)]) \geq \text{pt}(G[V\{Y\} \cup V(C)]) + \frac{k - 2}{4}$. Since $abca$ is a positive triangle, using Lemma 1 again we obtain $\beta(G) \geq \text{pt}(G) + \frac{k - 2}{4} = \text{pt}(G) + \frac{k}{4}$.

Rule 3. Let $v, a, b$ and $C$ be as in the description of Rule 3. Assume there exists a $(V_1', V_2')$-balanced subgraph $H'$ of $G'$ with at least $\text{pt}(G') + \frac{k'}{4}$ edges, where $k' = k - 2$. By Corollary 3 we may assume that all edges in $C$ are negative. In addition, we may assume that the edge $av$ is negative (the other case is similar). Lastly, without loss of generality assume that $v \in V_1'$. Now, consider the balanced subgraph $H$ of $G$ induced by $(V_1, V_2)$, where $V_1 = V_1' \cup \{b\}$ and $V_2 = V_2' \cup \{a\}$. Since $|E(a, V_1 \cap V(C)) \cup E(b, V_2 \cap V(C))| = |E(a, V_2 \cap V(C)) \cup E(b, V_1 \cap V(C))|$, it holds that $|E(H)| = |E(H')| + \frac{|E([a,b],[C-\{a,b\}])|}{2} + 2$. Moreover, $\text{pt}(G) = \text{pt}(G') + \frac{|E([a,b],[C-\{a,b\}])|}{2} + \frac{3}{2}$. Thus, $\beta(G) \geq \text{pt}(G) + \frac{k - 2}{4} = \text{pt}(G) + \frac{k}{4}$.

Rule 4. Let $v, a, b$ and $C$ be as in the description of Rule 4. Assume there exists a $(V_1', V_2')$-balanced subgraph $H'$ of $G'$ with at least $\text{pt}(G') + \frac{k'}{4}$ edges, where $k' = k - 4$. As in the proof for Rule 3, assume $C$ only contains negative edges, the edge $av$ is negative and $v \in V_1'$. Since $vabv$ is a positive triangle, this implies that the edge $bv$ is positive.

Now, consider the balanced subgraph $H$ of $G$ induced by $(V_1, V_2)$, where $V_1 = V_1' \cup \{b\}$ and $V_2 = V_2' \cup \{a\}$. As in the proof for Rule 3 it holds that $|E(a, V_1 \cap V(C)) \cup E(b, V_2 \cap V(C))| = |E(a, V_2 \cap V(C)) \cup E(b, V_1 \cap V(C))|$. Hence, $|E(H)| = |E(H')| + \frac{|E([a,b],[C-\{a,b\}])|}{2} + 3$, while $\text{pt}(G) = \text{pt}(G') + \frac{|E([a,b],[C-\{a,b\}])|}{2} + 2$. Thus, $\beta(G) \geq \text{pt}(G) + \frac{k - 4}{4} = \text{pt}(G) + \frac{k}{4}$.

Rule 5. Let $v$ and $C$ be as in the description of Rule 5. Suppose $\beta(G') \geq \text{pt}(G') + \frac{k}{4}$. We know that $\beta(G[V(C) \cup \{v\}]) \geq \text{pt}(G[V(C) \cup \{v\}])$. Then, by Lemma 2 $\beta(G) \geq \text{pt}(G) + \frac{k}{4}$.

Rule 6. Let $a, b, c$ be as in the description of Rule 6 and let $P = G[\{a, b, c\}]$. Note that $\text{pt}(P) = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$ and, whatever the signs of its edges, $P$ is a
balanced graph by Theorem 1. Therefore, \( \beta(P) = 2 = \text{pt}(P) + \frac{1}{4} \). Suppose \( \beta(G') \geq \text{pt}(G') + \frac{k'}{4} \), where \( k' = k - 1 \). Then by Lemma 1 \( \beta(G) \geq \text{pt}(G) + \frac{k' + 2 - 1}{4} = \text{pt}(G) + \frac{k}{4} \) edges.

**Rule 7.** Let \( v, b \) and \( C, Y \) be as in the description of Rule 7. Suppose \( \beta(Y) \geq \text{pt}(Y) + \frac{k}{4} \), where \( k = k - 1 \). We claim that \( \beta(G - Y) \geq \text{pt}(G - Y) + \frac{1}{4} \). If this holds, using Lemma 1 we obtain that \( \beta(G) \geq \text{pt}(G) + \frac{k + 1}{4} = \text{pt}(G) + \frac{k}{4} \).

Let \( G'' = G - Y \) and \( V(C) = \{v_1, \ldots, v_n\} \). Note that \( \text{pt}(G'') = \frac{n(n - 1)}{4} + n + \frac{n}{4} \).

By Corollary 3 we may assume that \( G'' \) contains only negative edges. If \( n \) is even, consider the partition \((V_1, V_2)\) of \( V(G'') \) where \( V_1 = \{v, b, v_1, \ldots, v_{\frac{n-1}{2}}\} \) and \( V_2 = \{v_{\frac{n+1}{2}}, \ldots, v_n\} \). The balanced subgraph induced by this partition contains \( (\frac{n}{2} + 1)^2 = \text{pt}(G'') + \frac{3}{4} \) edges. On the other hand, if \( n \) is odd, consider the partition \((V_1, V_2)\) of \( V(G'') \) where \( V_1 = \{v, b, v_1, \ldots, v_{\frac{n+1}{2}}\} \) and \( V_2 = \{v_{\frac{n+1}{2}}, \ldots, v_n\} \). The balanced subgraph induced by this partition contains \( n + \frac{n}{4} \). \( \frac{n}{4} \) edges.

We now show that the reduction rules preserve connectedness and that there is always one of them which applies to a graph with at least one edge. To show this, we use the following lemma, based on a result in [3] but first expressed in the following form in [3].

**Lemma 4.** For any connected graph \( Q \), at least one of the following properties holds:

A. There exist \( v \in V(Q) \) and \( X \subseteq V(Q) \) such that \( G[X] \) is a connected component of \( Q - v \) and \( G[X] \) is a clique;

B. There exist \( a, b, c \in V(Q) \) such that \( Q[\{a, b, c\}] \) is isomorphic to \( P_3 \) and \( Q - \{a, b, c\} \) is connected;

C. There exist \( v, b \in V(Q) \) such that \( vb \notin E(Q) \), \( Q - \{v, b\} \) is disconnected, and for all connected components \( G[X] \) of \( Q - \{v, b\} \), except possibly one, \( G[X \cup \{v\}] \) and \( G[X \cup \{b\}] \) are cliques.

**Lemma 5.** For a connected graph \( G \) with at least one edge, at least one of Rules 2, 4 applies. In addition, the graph \( G' \) which is produced is connected.

**Proof.** It is not difficult to see that the graph \( G' \) is connected, since when it is not obvious, its connectedness is part of the conditions for the rule to apply.

If property A of Lemma 3 holds, and \( G[X] \) contains a positive triangle \( abc \), then by Lemma 3 either Rule 1 or Rule 2 applies. If \( 2 \leq |N_G(v) \cap X| \leq |X| - 1 \), then Rule 3 applies. If \( |N_G(v) \cap X| = |X| \) and there exist \( a, b \in X \) such that \( vabo \) is a positive triangle, Rule 4 applies; otherwise, \( G[X \cup \{v\}] \) contains no positive triangles, and Rule 5 applies. Finally, if \( N_G(v) \cap X = \{x\} \), Rule 6 applies for \( x \) with clique \( G[X \setminus \{x\}] \).

If property B of Lemma 4 holds, then Rule 6 applies. If property C of Lemma 4 holds, consider the case when \( G - \{v, b\} \) has two connected components. Let
be the other connected component. If \( Z \) is connected to only one of \( v \) or \( b \), then property A holds. Otherwise, if \( G[X \cup \{x\}] \) contains a positive triangle, where \( x \in \{v, b\} \), then by Lemma 5 either Rule 1 or Rule 2 applies. So we may assume that \( G[X \cup \{b, v\}] \) contains no positive triangles, in which case Rule 7 applies.

If \( G - \{v, b\} \) has at least three connected components, at least two of them, \( X_1, X_2 \), form cliques with both \( v \) and \( b \) and possibly one component \( Y \) does not. Assume without loss of generality that \( Y \) has an edge to \( v \). Then Rule 6 applies for the path \( x_1bx_2 \), where \( x_1 \in X_1 \), \( x_2 \in X_2 \).

The following lemma gives structural results on \( S \) and \( G - S \). Note that from now on, \((G = (V, E), k)\) denotes the original instance of Signed Max Cut ATLB and \((G' = (V', E'), k')\) denotes the instance obtained by applying Rules 1-7 exhaustively. The set \( S \subseteq V \) denotes the set of vertices which are marked by the rules.

**Lemma 6.** Given a connected graph \( G \), if we apply Rules 1-7 exhaustively, either the set \( S \) of marked vertices has cardinality at most \( 3k \), or \( k' \leq 0 \). In addition, \( G - S \) is a forest of cliques that does not contain a positive triangle.

**Proof.** Observe that for every reduction rule where some vertices are marked, at most 3 vertices are marked, and the parameter decreases by at least 1. This means that if \( k' > 0 \), then the reduction rules cannot have marked more than \( 3k \) vertices.

To show that \( G - S \) is a forest of cliques that does not contain a positive triangle, proceed by induction. It is trivially true that the empty graph and the graph with only one vertex are forests of cliques that do not contain positive triangles. Now suppose that we apply one of the rules, transforming a graph \( G_1 \) into a graph \( G_2 \); suppose in addition that \( G_2 - (S \cap V(G_2)) \) is a forest of cliques that does not contain a positive triangle: we claim that \( G_1 - (S \cap V(G_1)) \) is a forest of cliques that does not contain a positive triangle, too. In the case of Rules 1, 3, 4 and 6, \( G_1 - (S \cap V(G_1)) \) is equal to \( G_2 - (S \cap V(G_2)) \), therefore the claim is trivially true. For Rule 5, note that \( G_1 - (S \cap V(G_1)) \) is obtained from \( G_2 - (S \cap V(G_2)) \) by either adding a disjoint clique not containing a positive triangle if \( v \in S \), or adding a clique not containing a positive triangle and identifying one of its vertices with \( v \) (where \( v \) is a cutvertex as in the description of Rule 5). Finally, for Rules 2 and 7, \( G_1 - (S \cap V(G_1)) \) is obtained from \( G_2 - (S \cap V(G_2)) \) by adding one disjoint clique not containing a positive triangle.

Finally, it is possible to prove that Signed Max Cut ATLB is FPT. First we state Max-Cut-with-Weighted-Vertices as in [4].
Theorem 5. Signed Max Cut ATLB can be solved in time $O^*(8^k)$.

Proof. Let $(G = (V,E), k)$ be an instance of Signed Max Cut ATLB. Apply Rules \[1\]-\[17\] exhaustively, producing an instance $(G' = (V', E'), k')$ and a set $S \subseteq V$ of marked vertices. If $k' \leq 0$, $(G', k')$ is a trivial Yes-instance. Since the rules are safe, it follows that $(G, k)$ is a Yes-instance, too.

Otherwise, $k' > 0$. Note that by Lemma \[6\], $|S| \leq 3k$ and $G - S$ is a forest of cliques, which is a chordal graph without positive triangles. Hence, by Corollary \[3\] we may assume that $G - S$ does not contain positive edges.

Therefore, to solve Signed Max Cut ATLB on $G$, we can guess a balanced subgraph of $G[S]$, induced by a partition $(V_1, V_2)$, and then solve Max-Cut-with-Weighted-Vertices for $G - S$. The weight of a vertex $v \in V(G - S)$ is defined in the following way: let $n_1^+(v)$ be the number of positive neighbors of $v$ in $V_i$ and $n_1^-(v)$ be the number of negative neighbors of $v$ in $V_i$; then $w_1(v) = n_1^+(v) + n_2^+(v)$ and $w_2(v) = n_1^-(v) + n_2^-(v)$.

Since Max-Cut-with-Weighted-Vertices is solvable in polynomial time on a forest of cliques (see Lemma 9 in \[4\]) and the number of possible partitions of $S$ is bounded by $2^{3k}$, this gives an $O^*(8^k)$-algorithm to solve Signed Max Cut ATLB.

4 Kernelization

In this section, we show that Signed Max Cut ATLB admits a kernel with $O(k^3)$ vertices. The proof of Theorem 5 implies the following key result for our kernelization.

Corollary 4. Let $(G = (V,E), k)$ be an instance of Signed Max Cut ATLB. In polynomial time, either we can conclude that $(G, k)$ is a Yes-instance or we can find a set $S$ of at most $3k$ vertices for which we may assume that $G - S$ is a forest of cliques without positive edges.

The kernel is obtained via the application of a new set of reduction rules and using structural results that bound the size of No-instances $(G, k)$. First, we need some additional terminology. For a block $C$ in $G - S$, let $C_{\text{int}} = \{x \in V(C) : N_{G-S}(x) \subseteq V(C)\}$ be the interior of $C$, and let $C_{\text{ext}} = V(C) \setminus C_{\text{int}}$ be the exterior of $C$. If a block $C$ is such that $C_{\text{int}} \cap N_G(S) \neq \emptyset$, $C$ is a special block. We say a block $C$ is a path block if $|V(C)| = 2 = |C_{\text{ext}}|$. A path vertex is a vertex which is contained only in path blocks. A block $C$ in $G - S$ is a leaf block if $|C_{\text{ext}}| \leq 1$. 
The following reduction rules are two-way reduction rules: they apply to an instance \((G, k)\) and produce an equivalent instance \((G', k')\).

**Reduction Rule 8.** Let \(C\) be a block in \(G - S\). If there exists \(X \subseteq C_{\text{int}}\) such that \(|X| > \frac{|V(C)| + |N_G(X) \cap S|}{2} \geq 1\), then delete two arbitrary vertices \(x_1, x_2 \in X\) and set \(k' = k\).

**Reduction Rule 9.** Let \(C\) be a block in \(G - S\). If \(|V(C)|\) is even and there exists \(X \subseteq C_{\text{int}}\) such that \(|X| = \frac{|V(C)|}{2}\) and \(N_G(X) \cap S = \emptyset\), then delete a vertex \(x \in X\) and set \(k' = k - 1\).

**Reduction Rule 10.** Let \(C\) be a block in \(G - S\) with vertex set \(\{x, y, u\}\), such that \(N_G(u) = \{x, y\}\). If the edge \(xy\) is a bridge in \(G - \{u\}\), delete \(C\), add a new vertex \(z\), positive edges \(\{v \in N_{C-u}(\{x, y\})\}\), negative edges \(\{zv : v \in N_{C-u}(\{x, y\})\}\) and set \(k' = k\). Otherwise, delete \(u\) and the edge \(xy\) and set \(k' = k - 1\).

**Reduction Rule 11.** Let \(T\) be a connected component of \(G - S\) only adjacent to a vertex \(s \in S\). Form a Max-Cut-with-Weighted-Vertices instance on \(T\) by defining \(w_1(x) = 1\) if \(x \in N_G^+(s) \cap T\) (\(w_1(x) = 0\) otherwise) and \(w_2(y) = 1\) if \(y \in N_G^-(s) \cap T\) (\(w_2(y) = 0\) otherwise). Let \(\beta(G[V(T) \cup \{s\}]) = \min\{G[V(T) \cup \{s\}] + \frac{p\cdot |V(T)|}{|V(T)| + 1}\}. Then delete \(T\) and set \(k' = k - p\).

Note that the value of \(p\) in Rule 11 can be found in polynomial time by solving Max-Cut-with-Weighted-Vertices on \(T\).

A two-way reduction rule is valid if it transforms Yes-instances into Yes-instances and No-instances into No-instances. Theorem 6 shows that Rules 8, 9, and 11 are valid. To prove Theorem 6, we need the following two lemmas.

**Lemma 7.** Let \(C\) be a block in \(G - S\). If there exists \(X \subseteq C_{\text{int}}\) such that \(|X| \geq \frac{|V(C)|}{2}\), then there exists a \((V_1, V_2)\)-balanced subgraph \(H\) of \(G\) with \(\beta(G)\) edges such that at least one of the following inequalities holds:

- \(|V_2 \cap V(C)| \leq |V_1 \cap V(C)| \leq |N_G(X) \cap S| + |V_2 \cap V(C)|\);
- \(|V_2 \cap V(C)| \leq |V_1 \cap V(C)| \leq |V_2 \cap V(C)| + 1\).

**Proof.** We may assume that \(|V_1 \cap V(C)| > |V_2 \cap V(C)|\). Note that if \(|V_1 \cap V(C)| > |V_2 \cap V(C)|\), then \(X \cap V_1 \neq \emptyset\) (because \(|X| \geq \frac{|V(C)|}{2}\)).

First, if \(N_G(X) \cap S = \emptyset\) and \(|V_1 \cap V(C)| \geq |V_2 \cap V(C)| + 2\), then, for any \(x \in X \cap V_1\), the subgraph induced by the partition \((V_1 \setminus \{x\}, V_2 \cup \{x\})\) has more edges than the subgraph induced by \((V_1, V_2)\), which is a contradiction.

Now, suppose that \(N_G(X) \cap S \neq \emptyset\) and suppose also that \(|V_1 \cap V(C)| - |V_2 \cap V(C)|\) is minimal. If \(|V_1 \cap V(C)| \leq |V_2 \cap V(C)| + 1\) we are done, so suppose \(|V_1 \cap V(C)| \geq |V_2 \cap V(C)| + 2\). Consider the partition \(V_1' = V_1 \setminus \{x\}, V_2' = V_2 \cup \{x\}\), where \(x \in V_1 \cap X\), and the balanced subgraph \(H'\) induced by this partition. Then \(|E(H')| \geq |E(H)| + |E(V_1 \setminus \{x\}, x)| - |E(V_2, x)| \geq |E(H)| + \frac{|E(C) + E(V_2, x)|}{2}\).
where $t$.

Lemma 8. Let $C$ be a block in $G - S$. If there exists $X \subseteq C$ such that $|X| > \frac{\text{Vol}(C)}{2} + |N_G(X) \cap S|$, then, for any $x_1, x_2 \in X$, there exists a $(V_1, V_2)$-balanced subgraph $H$ of $G$ with $\beta(G)$ edges such that $x_1 \in V_1$ and $x_2 \in V_2$.

Proof. First, we claim that there exist vertices $x_1, x_2 \in X$ for which the result holds. Let $H$ be a $(V_1, V_2)$-balanced subgraph of $G$ with $\beta(G)$ edges as given by Lemma 7.

Suppose $N_G(X) \cap S = \emptyset$. Then, by Lemma 7 it holds that $|V_2 \cap V(C)| \leq |V_1 \cap V(C)| \leq |V_2 \cap V(C)| + 1$; in addition, $|X| > \frac{|V(C)|}{2}$. Hence, either we can find $x_1$ and $x_2$ as required, or $X = V_1 \cap V(C)$ and $|V_1 \cap V(C)| = |V_2 \cap V(C)| + 1$. In the second case, pick a vertex $x \in V_1$ and form the partition $V_1' = V_1 \setminus \{x\}$ and $V_2' = V_2 \cup \{x\}$. Consider the balanced subgraph $H'$ induced by this partition. Observe that $|E(H')| = |E(H)| - |E(x, V_2)| + |E(x, V_1 \setminus \{x\})| = |E(H)| - |V_2 \cap V(C)| + |V_1 \cap V(C)| - 1 = |E(H)|$, so $H'$ is a maximum balanced subgraph for which we can find $x_1$ and $x_2$ as required.

Now, suppose $N_G(X) \cap S \neq \emptyset$. Then, by Lemma 7 it holds that $|V_2 \cap V(C)| \leq |V_1 \cap V(C)| \leq |N_G(X) \cap S| + |V_2 \cap V(C)|$. For the sake of contradiction, suppose $X \subseteq V_1 \cap V(C)$. In both cases, this means that $|X| \leq |V_1 \cap V(C)|$. Note that $|V(C)| = |V_1 \cap V(C)| + |V_2 \cap V(C)| = 2|V_2 \cap V(C)| + t$, where $t \leq |N_G(X) \cap S|$. Hence, $|V_1 \cap V(C)| \geq |X| \leq \frac{|V(C)|}{2} + \frac{|N_G(X) \cap S|}{2} \geq |V_2 \cap V(C)| + t = |V_1 \cap V(C)|$, which is a contradiction.

To conclude the proof, notice that for a $(V_1, V_2)$-balanced subgraph $H$ of $G$ with $\beta(G)$ edges and vertices $x_1, x_2 \in X$ such that $x_1 \in V_1$ and $x_2 \in V_2$, we have $|E(H)| = |E(H')|$, where $H'$ is a balanced subgraph induced by $V_1' = V_1 \setminus \{x_1\} \cup \{x_2\}$ and $V_2' = V_2 \setminus \{x_2\} \cup \{x_1\}$; this is true because $N_G^+(x_1) \cap N_G^-(x_2) = N_G^+(x_2) \cap N_G^-(x_1)$ and $N_G^+(x_1) \cap S = N_G^+(x_2) \cap S$.

Theorem 6. Rules [S][T] are valid.

Proof. Rule [S]: Let $C, X$ be as in the description of Rule [S]. Let $x_1, x_2 \in X$. By Lemma [S] there exists a $(V_1, V_2)$-balanced subgraph $H$ of $G$ with $\beta(G)$ edges such that $x_1 \in V_1$ and $x_2 \in V_2$. Now, let $G' = G - \{x_1, x_2\}$ and $H' = H - \{x_1, x_2\}$. Since $N_G^+(x_1) \cap S = N_G^+(x_2) \cap S$ and $N_G^-(x_1) \cap S = N_G^-(x_2) \cap S$, it holds that $|E(H)| = |E(H')| + \frac{|E(G,\{x_1, x_2\})|}{2} + 1$, and so $\beta(G') \leq \beta(G) + \frac{|E(G,\{x_1, x_2\})|}{2} + 1 \geq \beta(G)$.

Conversely, by Lemma [T] $\beta(G) \geq \beta(G') + \frac{|E(G,\{x_1, x_2\})|}{2} + 1$. Finally, observe that $pt(G) = pt(G') + \frac{|E(G,\{x_1, x_2\})|}{2}$ if and only if $G'$ admits a balanced subgraph of size $pt(G') + \frac{k}{4}$, which implies that $\beta(G) = pt(G') + \frac{k}{4}$. Hence, $G$ admits a balanced subgraph of size $pt(G) + \frac{k}{4}$ if and only if $G'$ admits a balanced subgraph of size $pt(G') + \frac{k}{4}$.

Rule [T]: Let $C, X$ and $x \in X$ be as in the description of Rule [T]. By Lemma 7 there exists a $(V_1, V_2)$-balanced subgraph $H$ of $G$ with $\beta(G)$ edges, such that
|V_1 \cap V(C)| = |V_2 \cap V(C)|. Consider the graph \( G' = G - \{x\} \) formed by the application of the rule and the balanced subgraph \( H' = H - \{x\} \). Then \( |E(H)| = |E(H')| + \frac{|V(C)|}{2} \), and thus \( \beta(G') \geq \beta(G) - \frac{|V(C)|}{2} \). Conversely, by Lemma 1 \( \beta(G) \geq \beta(G') + \frac{|V(C)|}{2} \). However, \( pt(G) = pt(G') + \frac{|V(C)|}{2} - \frac{1}{4} \). Hence, \( \beta(G) - pt(G) = \beta(G') - pt(G') + \frac{1}{4} \). Therefore, \( G \) admits a balanced subgraph of size \( pt(G) + \frac{1}{4} \) if and only if \( G' \) admits a balanced subgraph of size \( pt(G') + \frac{3}{4} - \frac{1}{4} \).

**Rule 10.** Let \( C \) and \( \{x, y, u\} \) be as in the description of Rule 10. Firstly consider the case when \( xy \) is a bridge in \( G - \{u\} \). For any maximal balanced subgraph \( H \) of \( G \), without loss of generality one may assume that \( xu, yu \in E(H) \) and \( xy \notin E(H) \). Suppose \( H \) is induced by a partition \((V_1, V_2)\) and \( x, y \in V_1 \). Form a balanced subgraph of \( G' \) from \( H - \{x, y, u\} \) by placing \( z \) in \( V_1 \). Therefore, \( \beta(G) = \beta(G') + 2 \). Since \( pt(G) = pt(G') + \frac{4}{1} + \frac{4}{1} = pt(G') + 2 \), it follows that \( \beta(G) = pt(G') + \frac{1}{4} \) if and only if \( \beta(G') = pt(G') + \frac{1}{4} \).

Now consider the case when \( xy \) is not a bridge in \( G - \{u\} \). Then the graph \( G' \) formed by deleting the vertex \( u \) and the edge \( xy \) is connected. Furthermore, regardless of whether \( x \) and \( y \) are in the same partition that induces a balanced subgraph \( H' \) of \( G' \), \( H' \) can be extended to a balanced subgraph \( H \) of \( G \) such that \( |E(H)| = |E(H')| + 2 \). This means that, as before, \( \beta(G) = \beta(G') + 2 \). But in this case \( pt(G) = pt(G') + \frac{4}{1} \) and thus \( \beta(G) = pt(G') + \frac{1}{4} \) if and only if \( \beta(G') = pt(G') + \frac{1}{4} \).

**Rule 11.** Let \( T \) and \( s \in S \) be as in the description of Rule 11. Since \( \beta(G[V(T) \cup \{s\}]) = pt(G[V(T) \cup \{s\}]) + \frac{4}{1} \), by Lemma 2 \( \beta(G) = \beta(G - T) + pt(G[V(T) \cup \{s\}]) + \frac{4}{1} \). Also, by Lemma 2 \( pt(G) = pt(G - T) + pt(G[V(T) \cup \{s\}]) \). Hence \( \beta(G) - pt(G) = \beta(G - T) - pt(G - T) + \frac{4}{1} \), which implies that \( G \) admits a balanced subgraph of size \( pt(G - T) + \frac{1}{4} \) if and only if \( G - T \) admits a balanced subgraph of size \( pt(G - T) + \frac{4}{1} \).

To show the existence of a kernel with \( O(k^3) \) vertices, it is enough to give a bound on the number of non-path blocks, the number of vertices in these blocks and the number of path vertices. This is done by Corollaries 5 and 7 and Lemma 13.

While Lemma 14 applies to any graph reduced by Rule 8, the proofs of Corollaries 6 and 7 rely on Lemma 13 which gives a general structural result on forest of cliques with a bounded number of special blocks and bounded path length. Corollary 5 and Lemma 11 provide sufficient conditions for a reduced instance to be a Yes-instance, thus producing a bound on the number of special blocks and the path length of No-instances. Lastly, Theorem 7 puts the results together to show the existence of the kernel.

Henceforth, we assume that the instance \((G, k)\) is such that \( G \) is reduced by Rules 8 and 11, \( G - S \) is a forest of cliques which does not contain a positive edge and \( |S| \leq 3k \).

**Lemma 9.** Let \( T \) be a connected component of \( G - S \). Then for every leaf block \( C \) of \( T \), \( N_G(C_{ml}) \cap S \neq \emptyset \). Furthermore, if \( |N_G(S) \cap V(T)| = 1 \), then \( T \) consists of a single vertex.
Proof. We start by proving the first claim. Note that if \( T = C \) consists of a single vertex, then \( N_G(C_{\text{int}}) \cap S \neq \emptyset \) since \( G \) is connected. So assume that \( C \) has at least two vertices. Suppose that \( N_G(C_{\text{int}}) \cap S = \emptyset \) and let \( X = C_{\text{int}} \). Then if \( |C_{\text{int}}| > |C_{\text{ext}}| \), Rule \( \text{X} \) applies. If \( |C_{\text{int}}| = |C_{\text{ext}}| \) then Rule \( \text{Y} \) applies. Otherwise, \( |C_{\text{int}}| < |C_{\text{ext}}| \) and since \( |C_{\text{ext}}| \leq 1 \) (as \( C \) is a leaf block), \( C \) has only one vertex, which contradicts our assumption above. For the second claim, first note that since \( |N_G(S) \cap V(T)| = 1 \), \( Q \) has one leaf block and so \( T \) consists of a single block. Let \( N_G(S) \cap V(T) = \{v\} \) and \( X = V(T) - \{v\} \). If \( |X| > 1 \), Rule \( \text{X} \) applies. If \( |X| = 1 \), Rule \( \text{Y} \) applies. Hence \( V(T) = \{v\} \).

Let \( B \) be the set of non-path blocks.

**Lemma 10.** If there exists a vertex \( s \in S \) such that \( \sum_{C \in B} |N_G(C_{\text{int}}) \cap \{s\}| \geq 2(|S| - 1 + k) \), then \((G, k)\) is a Yes-instance.

Proof. Form \( T \subseteq N_G(s) \) by picking a vertex from each block \( C \) for which \( |N_G(C_{\text{int}}) \cap \{s\}| = 1 \). If there exists a vertex \( x \in C_{\text{int}} \) such that \( N_G(x) \cap S = \{s\} \), pick this, otherwise pick \( x \in C_{\text{int}} \) arbitrarily. Let \( U = T \cup \{s\} \) and \( W = V \setminus U \).

Observe that \( G[U] \) is balanced by Theorem \( \text{I} \) as \( G[U] \) is a tree. Thus \( \beta(G[U]) = |T| = \frac{|T|}{4} + \frac{|T|}{4} = \text{pt}(G[U]) + \frac{|T|}{4} \).

Consider a connected component \( Q \) of \( G - S \). By Rule \( \text{II} \) \( |N_G(Q) \cap S| \geq 2 \) and by Lemma \( \text{J} \) if \( |N_G(S) \cap V(Q)| = 1 \) then \( Q \) consists of a single vertex. Otherwise, either \( (N_G(S) \setminus N_G(s)) \cap V(Q) \neq \emptyset \), or \( Q \) has at least two vertices in \( T \). Moreover, note that the removal of interior vertices does not disconnect the component itself. Hence \( G[W] \) has at most \(|S| - 1 + \frac{|T|}{4}\) connected components.

Applying Lemma \( \text{J} \) \( \beta(G) \geq \text{pt}(G) + \frac{|T|}{4} - \frac{|S| - 1 + |T|}{4} = \text{pt}(G) + \frac{|T|}{4} - \frac{|S| - 1}{4} \).

Hence if \( |T| \geq 2(|S| - 1 + k) \), then \((G, k)\) is a Yes-instance.

**Corollary 5.** If \( \sum_{C \in B} |N_G(C_{\text{int}}) \cap S| \geq |S|(|S| - 3 + 2k) + 1, \) the instance is a Yes-instance. Otherwise, \( \sum_{C \in B} |N_G(C_{\text{int}}) \cap S| \leq 3k(|S| - 3) \).

Proof. If \( \sum_{C \in B} |N_G(C_{\text{int}}) \cap S| \geq |S|(|S| - 3 + 2k) + 1, \) then for some \( s \in S \) we have \( \sum_{C \in B} |N_G(C_{\text{int}}) \cap \{s\}| \geq 2|S| - 3 + 2k + 1/|S| \) and, since the sum is integral, \( \sum_{C \in B} |N_G(C_{\text{int}}) \cap \{s\}| \geq 2(|S| - 1 + k) \). Thus, \((G, k)\), by Lemma \( \text{II} \) is a Yes-instance. The second inequality of the corollary follows from the fact that \(|S| \leq 3k \).

**Lemma 11.** If in \( G - S \) there exist vertices \( U = \{u_1, u_2, \ldots, u_p\} \) such that \( N_{G - S}(u_i) = \{u_{i-1}, u_{i+1}\} \) for \( 2 \leq i \leq p - 1 \), and \( p \geq |S| + k + 1 \), then \((G, k)\) is a Yes-instance. Otherwise, \( p \leq 4k \).

Proof. Observe that \( G[U] \) is balanced by Theorem \( \text{I} \). Thus \( \beta(G[U]) = p - 1 = \text{pt}(G[U]) \). Let \( W = V \setminus U \) and observe that \( G[W] \) has at most \(|S|\) components, since, by Lemma \( \text{J} \) every vertex in \( G - U \) has a path to a vertex in \( S \). Applying Lemma \( \text{J} \) \( \beta(G) \geq \text{pt}(G) + \frac{|W|}{4} - \frac{|S|}{4} \). Hence if \( p - 1 - |S| \geq k, (G, k) \) is a Yes-instance. \( \square \)
Lemma 12. A block \( C \) in \( G - S \) such that \( |C_{\text{ext}}| = 2 \) is either special or it is a path block.

Proof. Suppose \( C \) is not special. If \( |V(C)| \geq 5 \), then Reduction Rule 8 would apply. If \( |V(C)| = 4 \), then Reduction Rule 9 would apply. If \( |V(C)| = 3 \), then Reduction Rule 10 would apply. Hence \( |V(C)| = 2 \) and it is a path block.

In \( G - S \), a pure path is a path consisting exclusively of path vertices. Note that every path vertex belongs to a unique pure path.

Lemma 13. Suppose \( G - S \) has at most \( l \) special blocks and the number of vertices in each pure path is bounded by \( p \). Then \( G - S \) contains at most \( 2l \) non-path blocks and \( 2pl \) path vertices.

Proof. It suffices to prove that if every connected component \( T \) of \( G - S \) has at most \( l_T \) special blocks, then \( T \) contains at most \( 2l_T \) non-path blocks and \( 2pl_T \) path vertices. So, we may assume that \( T = G - S \) is connected. Pick an arbitrary non-path block \( C_R \) as the ‘root’ node. Define the distance \( d(C_R, C) \) as the number of non-path blocks different from \( C_R \) visited in a path from a vertex in \( C_R \) to a vertex in \( C \). For every non-path block \( C \) in \( T \), the parent block \( C' \) is the unique non-path block such that \( C' \) contains an edge of any path from \( C_R \) to \( C \) and \( d(C_R, C) - d(C_R, C') = 1 \). In addition, \( C_R \) is the parent of every block \( C \) such that \( d(C_R, C) = 1 \).

Consider the tree \( F \) that contains a vertex for every non-path block of \( T \) and such that there is an edge between two vertices if and only if one of the corresponding blocks is the parent of the other. Observe that given a vertex \( v \in F \) which corresponds to a block \( C \) of \( T \), it holds that \( d_F(v) \geq |C_{\text{ext}}| \). In addition, by Lemma 11 every leaf in \( F \) corresponds to a special block.

Now, we know that in a tree the number of vertices of degree greater or equal to three is bounded by the number of leaves. Moreover, by Lemma 12 if a block \( C \) is such that \( |C_{\text{ext}}| = 2 \), then it is either special or a path block. Thus, the number of non-path blocks is bounded by \( 2l \).

Furthermore, note that the number of pure paths in \( T \) is bounded by the number of edges in \( F \), which is bounded by \( 2l - 1 \). Since every pure path contains at most \( p \) path vertices, the number of path vertices is bounded by \( (2l - 1)p < 2pl \).

Corollary 6. \( G - S \) contains at most \( 6k(8k-3) \) non-path blocks and \( 24k^2(8k-3) \) path vertices.

Proof. By Corollary 5, \( G - S \) contains at most \( 3k(8k-3) \) special blocks and by Lemma 11 the length of every pure path is bounded by \( 4k \). Thus, Lemma 13 implies that \( G - S \) contains at most \( 6k(8k-3) \) non-path blocks and \( 24k^2(8k-3) \) path vertices.

Corollary 7. \( G - S \) contains at most \( 12k(8k-3) \) vertices in the exteriors of non-path blocks.
Proof. For any component of $T$ and any vertex $v$ in $C_{ext}$, $v$ corresponds to an edge of $F$. Furthermore, for any edge of $F$ there are at most two exterior vertices in $T$ that correspond to it. Therefore, $|\bigcup_{C \in B} C_{ext}| \leq 2|B| \leq 12k(8k - 3)$. □

Lemma 14. For a block if $|V(C)| \geq 2|C_{ext}| + |N_G(C_{int}) \cap S|[2|S| + 2k + 1], then $(G, k)$ is a YES-instance. Otherwise, $|V(C)| \leq 2|C_{ext}| + |N_G(C_{int}) \cap S|[2|S| + 2k + 1].$

Proof. Consider a fixed $s \in N_G(C_{int}) \cap S$. We will show that we may assume that either $|N_G^{+}(s) \cap C_{int}| \leq \frac{k+|S|}{2}$ or $|N_G^{-}(s) \cap C_{int}| \geq |C_{int}| - \frac{k+|S|}{2}$, because otherwise $(G, k)$ is a YES-instance.

Indeed, suppose $\left|\frac{k+|S|}{2}\right| \leq |N_G^{+}(s) \cap C_{int}| \leq |C_{int}| - \frac{k+|S|}{2}$. Let $U_1 \subseteq N_G^{+}(s) \cap C_{int}$, $|U_1| = \left|\frac{k+|S|}{2}\right|$, and let $U_2 \subseteq C_{int} \setminus N_G^{+}(s)$, $|U_2| = \left|\frac{k+|S|}{2}\right|$. Let $U = U_1 \cup U_2 \cup \{s\}$ and consider the subgraph $H$ of $G[U]$ induced by the edges $E(U_1, U_2) \cup E(s, (U_1 \cap N_G^{-}(s))) \cup E(s, (U_2 \cap N_G^{-}(s)))$. Observe that $H$ is $(U_1 \cup \{s\}, U_2)$-balanced and so $\beta(G[U]) \geq |U_1|^2 + |U_1 \cap N_G^{+}(s)| + |U_2 \cap N_G^{-}(s)|$. Furthermore, $pt(G[U]) = |U_1|^2 + \frac{|U_1 \cap N_G^{+}(s)| + |U_2 \cap N_G^{-}(s)|}{2}$, and hence $\beta(G[U]) \geq pt(G[U]) + \frac{k+|S|}{4}$.

Now consider $W = V \setminus U$. Any connected component of $G - S$ is connected to two vertices in $S$, hence $G[W]$ has at most $|S| - 1$ components adjacent to vertices in $S \setminus \{s\}$ and one component corresponding to the block $C$. Applying Lemma 13, $\beta(G) \geq pt(G) + \frac{k+|S|}{4}$, which means that $(G, k)$ is a YES-instance.

Similarly, we can show that we may assume that either $|N_G^{-}(s) \cap C_{int}| \leq \frac{k+|S|}{2}$ or $|N_G^{-}(s) \cap C_{int}| \geq |C_{int}| - \frac{k+|S|}{2}$, because otherwise $(G, k)$ is a YES-instance.

Let $S_1^{+} = \{s \in S : 0 \leq |N_G^{+}(s) \cap C_{int}| \leq \frac{k+|S|}{2}\}$, $S_2^{+} = (N_G^{+}(C_{int}) \cap S) \setminus S_1^{+}$ and $X^+ = \{v \in C_{int} \setminus N_G^{+}(S_1^{+}) : \forall s \in S_2^{+}\}$. Observe that for all $s \in S_1^{+}$, $|N_G^{+}(s) \cap C_{int}| \geq |C_{int}| - \frac{k+|S|}{2}$, which means that $|X^+| \geq |C_{int} \setminus N_G^{+}(S_1^{+})| - |S_2^{+}| \geq \frac{k+|S|}{2}$. In addition, $|N_G^{+}(S_1^{+}) \cap C_{int}| \leq |S_1^{+}| \frac{k+|S|}{2}$, hence $|C_{int} \setminus N_G^{+}(S_1^{+})| \geq |C_{int}| - |S_2^{+}| \frac{k+|S|}{2}$. Therefore, $|X^+| \geq |C_{int}| - (|S_2^{+}| + |S_2^{+}|) \frac{k+|S|}{2} = |C_{int}| - |N_G^{-}(C_{int}) \cap S| \frac{k+|S|}{2} \geq |C_{int}| - |N_G^{-}(C_{int}) \cap S| \frac{k+|S|}{2}$.

With similar definitions and the same argument we obtain $|X^-| \geq |C_{int} - |N_G(C_{int}) \cap S| \frac{k+|S|}{2}|$. Now let $X = X^+ \cap X^-$ and observe that and $|X| \geq |C_{int} - |N_G(C_{int}) \cap S|\{k + |S|\}$.

However, by Rule 3 $|X| \leq \frac{|V(C)| + |N_G(C_{int}) \cap S|}{2}$. So, $|C_{int}| \leq |N_G(C_{int}) \cap S|\{k + |S|\}$, and so $|V(C)| \leq 2|C_{ext}| + |N_G(C_{int}) \cap S|[2|S| + 2k + 1]$ as claimed. □

Theorem 7. **Signed Max Cut ATLB** has a kernel with $O(k^3)$ vertices.

Proof. Let $(G = (V, E), k)$ be an instance of **Signed Max Cut ATLB**. As in Theorem 5 apply Rules 1 exhaustively: either the instance is a YES-instance, or there exists $S \subseteq V$ such that $|S| \leq 3k$ and $G - S$ is a forest of cliques which does not contain a positive edge.
Now, apply Rules 8–11 exhaustively to \((G, k)\) to obtain a new instance \((G', k')\). If \(k' \leq 0\), then \((G, k)\) is a Yes-instance since Rules 8–11 are valid. Now let \(G = G', k = k'\). Check whether \((G, k)\) is a Yes-instance due to Corollary 5, Lemma 11 or Lemma 14. If this is not the case, by Corollary 6, \(G - S\) contains at most \(6k(8k - 3)\) non-path blocks and \(24k^2(8k - 3)\) path vertices. Hence, by Lemma 14, \(|V(G)|\) is at most

\[
|S| + 24k^2(8k - 3) + \sum_{C \in \mathcal{B}} |V(C)| \leq O(k^3) + 2 \sum_{C \in \mathcal{B}} |C_{\text{ext}}| + (8k + 1) \sum_{C \in \mathcal{B}} |N_G(C_{\text{int}}) \cap S|
\]

Now, applying Corollary 5 and Corollary 7, we obtain:

\[
|V(G)| \leq O(k^3) + 48k(8k - 3) + 3k(8k - 3)(8k + 1) = O(k^3).
\]

It is not hard to verify that no reduction rule of this paper increases the number of positive edges. Thus, considering an input \(G\) of \textsc{Max Cut ATLB} as an input of \textsc{Signed Max Cut ATLB} by assigning minus to each edge of \(G\), we have the following:

**Corollary 8.** \textsc{Max Cut ATLB} has a kernel with \(O(k^3)\) vertices.

5 Extensions and Open Questions

In the previous sections, the input of \textsc{Signed Max Cut ATLB} is a signed graph without parallel edges. However, in some applications (cf. [8, 9]), signed graphs may have parallel edges of opposite signs. We may easily extend inputs of \textsc{Signed Max Cut ATLB} to such graphs. Indeed, if \(G\) is such a graph we may remove all pairs of parallel edges from \(G\) and obtain an equivalent instance of \textsc{Signed Max Cut ATLB}.

In fact, the Poljak-Turzík bound can be extended to edge-weighted graphs [13]. Let \(G\) be a connected signed graph in which each edge \(e\) is assigned a positive weight \(w(e)\). The weight \(w(Q)\) of an edge-weighted graph \(Q\) is the sum of weights of its edges. Then \(G\) contains a balanced subgraph with weight at least \(w(G)/2 + w(T)/4\), where \(T\) is a spanning tree of \(G\) of minimum weight [14]. It would be interesting to establish parameterized complexities of \textsc{Max Cut ATLB} and \textsc{Signed Max Cut ATLB} extended to edge-weighted graphs using the Poljak-Turzík bound above.

References

[1] C. Chiang, A. B. Kahng, S. Sinha, X. Xu, and A. Z. Zelikovsky. Fast and efficient bright-field AAPSM conflict detection and correction. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 26(1):11–126, 2007.
[2] R. Crowston, M. Fellows, G. Gutin, M. Jones, F. Rosamond, S. Thomassé and A. Yeo. Simultaneously Satisfying Linear Equations Over $\mathbb{F}_2$: MaxLin2 and Max-$r$-Lin2 Parameterized Above Average. In FSTTCS 2011, LIPICS 13, 229–240, 2011.

[3] R. Crowston, G. Gutin and M. Jones, Directed Acyclic Subgraph Problem Parameterized above the Poljak-Turzík Bound. In FSTTCS 2012, LIPICS 18, 400–411, 2012.

[4] R. Crowston, M. Jones, and M. Mnich. Max-Cut Parameterized Above the Edwards-Erdős Bound. arXiv:1112.3506v2. A preliminary version published in ICALP 2012, Part I, Lect. Notes Comput. Sci. 7391 (2012) 242–253.

[5] B. DasGupta, G. A. Enciso, E. D. Sontag, and Y. Zhang. Algorithmic and complexity results for decompositions of biological networks into monotone subsystems. Biosystems 90(1):161–178, 2007.

[6] R. G. Downey and M. R. Fellows, Parameterized Complexity. Springer-Verlag, 1999.

[7] J. Flum and M. Grohe, Parameterized Complexity Theory, Springer-Verlag, 2006.

[8] N. Gülşınar, G. Gutin, G. Mitra and A. Zverovitch, Extracting Pure Network Submatrices in Linear Programs Using Signed Graphs. Discrete Applied Mathematics 137 (2004) 359–372.

[9] G. Gutin and A. Zverovitch, Extracting pure network submatrices in linear programs using signed graphs, Part 2. Communications of DQM 6 (2003) 58–65.

[10] F. Harary. On the notion of balance of a signed graph. Michigan Math. J., 2(2):143–146, 1953.

[11] F. Hüffner, N. Betzler, and R. Niedermeier. Optimal edge deletions for signed graph balancing. In 6th international Conf. on Experimental Algorithms (WEA’07), 297–310, 2007.

[12] M. Mnich, G. Philip, S. Saurabh, and O. Suchý, Beyond Max-Cut: $\lambda$-Extendible Properties Parameterized Above the Poljak-Turzík Bound. In FSTTCS 2012, LIPICS 18, 412–423, 2012.

[13] R. Niedermeier. Invitation to Fixed-Parameter Algorithms, Oxford UP, 2006.

[14] S. Poljak and D. Turzík, A polynomial time heuristic for certain subgraph optimization problems with guaranteed worst case bound. Discrete Mathematics, 58 (1) (1986) 99–104.
[15] V. Raman and S. Saurabh, Parameterized algorithms for feedback set problems and their duals in tournaments. Theor. Comput. Sci., 351 (3) (2006) 446–458.

[16] T. Zaslavsky. Bibliography of signed and gain graphs. Electronic Journal of Combinatorics, DS8, 1998.