Myerson on a Network

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Abstract

The auction of a single indivisible item is one of the most celebrated problems in mechanism design with transfers. Despite its simplicity, it provides arguably one of the cleanest and most insightful results in the literature. When the information of the auction is available to every participant, Myerson [17] provided a seminal result to characterize the incentive-compatible auctions along with revenue optimality. However, such a result does not hold in an auction on a network, where the information of the auction is spread via the agents, and they need incentives to forward the information. In recent times, a few auctions (e.g., [10, 15]) were designed that appropriately incentivize the intermediate nodes on the network to promulgate the information to potentially more valuable bidders. In this paper, we provide a Myerson-like characterization of incentive-compatible auctions on a network and show that the currently known auctions fall within this larger class of randomized auctions. We obtain the structure of the revenue optimal auction for i.i.d. bidders on arbitrary trees. We discuss the possibilities of addressing more general settings. Through experiments, we show that auctions following this characterization can provide a higher revenue than the currently known auctions on networks.

1 Introduction

Single indivisible item auction is a special setting of mechanism design with monetary transfers where multiple bidders contest to collect a single item. The true value of the item could be different for different agents and it is their private information, i.e., not known to the designer of a mechanism.¹ Despite its simplicity, single-item auction provides remarkable insights to the questions: (a) what is the structure of the mechanisms that reveal the agents’ true private information, (b) how to design mechanisms that maximize the expected revenue. In the world where the information that ‘an item is being auctioned’ is available to every possible bidder interested in this item, these two questions have been answered gracefully by Myerson in his seminal paper [17].

However, in various recent contexts of auctions, network of connections makes an important role in the information flow over the network. An agent diffuses the information into the network only if it finds it is beneficial to share. This setup has given rise to network auctions, where agents diffuse the information of the auction only if it (strictly or weakly) improves their utilities. This problem has given birth to the domain of diffusion auction design on networks and has received significant attention in the recent times [7, 10, 14, 15]. Because the information about the auction does not automatically reach every agent in this setup, the mechanism needs to incentivize the individuals to diffuse (or forward) the information. The Myerson [17] characterization does not follow here, and a fresh investigation is necessary to characterize the truthful and revenue maximizing diffusion auctions.

¹Since auctions are special cases of mechanisms, we will use these two terms interchangeably in this paper.
1.1 Our contributions

This paper provides a characterization of the incentive compatible (or truthful) diffusion auctions. In this setup, an agent may decide to forward or not forward the auction information in addition to reporting their valuations (which could be different from their true valuations).

1. Our first contribution is to update the classical definition of incentive compatibility to diffusion dominant strategy incentive compatibility (DDSIC, Def. 2), which is stricter than the IC definitions in the literature on network auction [14, e.g.] (Proposition 1). However, most of the known mechanisms in network auction satisfy this restrictive definition (§5).

2. We characterize DDSIC mechanisms (Theorem 1) and provide an example to show that this is a strict super-class of several truthful diffusion mechanisms known so far.

3. We propose a class of mechanisms called LbLEV that are DDSIC (Theorem 2) and individually rational (Theorem 3).

4. We find the revenue-optimal auction for i.i.d. bidders on a tree (Theorem 6) in §6.

5. When bidders are non-i.i.d., we experimentally exhibit that the parameters of LbLEV can be tuned, based on the prior information of the valuations and network structure, to yield a better revenue than the currently known truthful diffusion auctions (§7).

1.2 Related work

The area of diffusion auction design is relatively new, leading to a rather thin literature. Guo and Hao [7] provide a comprehensive survey of the domain. The first works on diffusion auction are due to Li et al. [10] and Lee [9]. In particular, [10] showed that the classical VCG mechanism [3, 6, 20] can be extended to the diffusion setting, but it may lead to a large deficit. They propose a new mechanism called IDM that mitigates this problem. In the following years, a few more diffusion auctions were proposed: CSM [11] for economic networks, MLDM for intermediary networks [13], TNM, CDM, WDM were on the unweighted and weighted networks [12, 15], FDM [21] and NRM [22] considered the money burning issue in network auction and proposed schemes to redistribute the money maintaining incentive compatibility. On the characterization results, Li et al. [14] provide a characterization for deterministic diffusion auctions and find optimal payments. Our approach, however, considers a broader approach to characterize all randomized diffusion auctions (which includes the deterministic auctions as a special case) and shows that better revenue-generating mechanisms can be found.

2 Basic Problem Setup

Consider a directed graph $G = (N \cup \{s\}, E)$, where $N = \{1, \ldots, n\}$ is the set of players involved in the auction of a single indivisible item and $s$ is a distinguished node called the seller. The set $E$ is the set of edges. Each edge $(i, j)$ denotes that if the node $i$ communicates an information, node $j$ will receive it. The typical examples of such graphs are online social networks where if an individual posts (or selectively communicates) an information, all its neighbors (or a selected subset of friends or followers) receive it. The direction signifies that almost all networks have asymmetric information flow (e.g., only followers receive the information from the followee on Twitter). The model, however, also includes other modes of broadcast (post on a webpage) or selective (e-mail) information diffusion.

In this network, we assume that there is a single seller $s$ who wants to sell the indivisible item. Every other node $i \in N$ is a potential buyer and the information about the auction flows only via the direction of each edge. The information cannot reach a node unless there is a directed path from $s$ to that node and each intermediate node decides to forward the information. An intermediate node may decide not to forward the information if it reduces its utility, which we define later.

This setup naturally brings up an auction-like information sharing game among the players. Each player $i \in N$ has a type $\theta_i = (v_i, r_i)$, where $v_i$ is the valuation of agent $i$ for the item, and $r_i$ is the set of her directed neighbors. The set of valuations and subsets of neighbors of player $i$ are denoted by $V_i$ and $R_i$, respectively. The type set of $i$, $\Theta_i$, is therefore, $V_i \times R_i$, and agent $i$ can report its type from this set. The information about the auction needs to reach via directed edges to player $i$ for her to participate in the auction. Therefore, the auction asks every agent to report her valuation for the item and to forward the information to its directed neighbors. In our model, this is captured
We consider auctions on this graph with randomized allocations to the agents. Formally, we define a
1. every agent’s utility is maximized by reporting her true valuation irrespective of the diffusing
2. for every true valuation, every agent’s utility is maximized by diffusing to all its neighbors
via their reported type \(\hat{\theta}_i = (\hat{v}_i, \hat{r}_i)\) for every agent \(i \in N\). We assume that the seller \(s\) is not a
strategic player in this auction, rather he wants to sell the object and always forwards the information
to his directed neighbors. The vector of the reported types of all the agents except \(i\) is denoted by
\(\hat{\theta}_{-i} = (\hat{v}_1, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n)\). We denote the set of all type profiles by \(\Theta := \prod_{i \in N} \Theta_i\).

Depending on the reported types of the agents, particularly, the reported \(\hat{r}_i\)’s, the auction may reach
only a subset of the agents in \(N\). Throughout this paper, we will use the notation \(r_i\) to denote the
true neighbor set of player \(i\) and assume that the reported \(\hat{r}_i\) can only be a subset of it. To denote the
reported valuation and directed neighbors on the subnetwork generated by \((\hat{\theta}_i, \hat{\theta}_{-i})\), we use a filter
function \(f^G\) for the graph \(G\), where \(f^G(\hat{\theta}_i, \hat{\theta}_{-i})\) denotes the reported valuation and directed neighbor
vector of the subgraph where each node has a directed path from \(s\) after the agents reported the type
profile \((\hat{\theta}_i, \hat{\theta}_{-i})\). In this setup, the auction design goal is to incentivize each node to truthfully reveal
its private valuation and forward regardless of others’ actions. It is known that such mechanism
exists [10]. One of the goals of this paper is to characterize all such mechanisms in an elegant
manner.

We consider auctions on this graph with randomized allocations to the agents. Formally, we define a
diffusion auction in this setup as follows.

**Definition 1 (Diffusion Auction)** A diffusion auction (DA) is given by the tuple \((g, p)\) where \(g\) and
\(p\) are the allocation and payment functions respectively. The allocation function \(g : \Theta \rightarrow \Delta_n\) is
such that its \(i\)-th component \(g_i(f^G(\hat{\theta}))\) denotes the probability of agent \(i\) winning the object, where
\(\Delta_n := \{ x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1 \}\). Similarly, the payment function \(p = (p_i)_{i \in N}\) is such that its \(i\)-th
component \(p_i : \Theta \rightarrow \mathbb{R}\) denotes the payment assigned to agent \(i\).

Note that \(g_i\) should operate on the subnetwork that remains connected to \(s\) after the agents choose
their actions \(\hat{\theta}\). Hence the notation \(g_i(f^G(\cdot))\) is used in the definition above. It is worth noting
that the notation generalizes the one used by Li et al. [10]. The action chosen by player \(i\) may
change the actions available to the other players and it is succinctly captured by the filter function
which also subsumes the definition in that paper. Also, note that DA is different from the classical
auction, because the types of each agent now contain both the valuation \((v_i)\) and the information
on forwarding \((r_i)\). The utility of agent \(i\) under DA is given by the standard quasi-linear model [19]:
\(v_i((\hat{\theta}_i, \hat{\theta}_{-i}); \theta_i) = v_i(g_i(f^G(\hat{\theta}_i, \hat{\theta}_{-i}))) - p_i(f^G(\hat{\theta}_i, \hat{\theta}_{-i}))\).

### 3 Design Desiderata

The first desirable property of an auction is truthfulness. However, in the context of auction on the
network, we need to ensure that the mechanism also incentivizes the agents to forward the information
in addition to being truthful about their valuations. The following definition captures both these
aspects.

**Definition 2 (Diffusion Dominant Strategy Incentive Compatibility)** A DA \((g, p)\) on a graph \(G\)
is diffusion dominant strategy incentive compatible (DDISC) if

1. every agent’s utility is maximized by reporting her true valuation irrespective of the diffusing
   status of herself and the other agents, i.e., for every \(i \in N\), \(\forall r_i, \hat{\theta}_{-i}\), the following holds
   \[ v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})), \forall v_i, v_i', r_i', r_i' \subseteq r_i, \text{ and,} \]

2. for every true valuation, every agent’s utility is maximized by diffusing to all its neighbors
   irrespective of the diffusing status of the other agents, i.e., for every \(i \in N\), \(\forall r_i, \hat{\theta}_{-i}\), the following holds
   \[ v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})), \forall v_i, \hat{\theta}_{-i}, r_i' \subseteq r_i. \]

We show that the above definition implies the following definition of incentive compatibility (restated
in our setting with the notation of this paper) given by Li et al. [14] and hence is stricter than that.

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3
However, we show that all the prominent mechanisms presented so far in the literature (e.g., IDM, TNN, etc.) follow this restrictive definition.

**Definition 3 (Incentive Compatibility [14])** A DA \((g, p)\) on a graph \(G\) is incentive-compatible (IC) if for every \(i \in N\), \(\forall r_i, \hat{\theta}_{-i}, v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v'_i, r'_i), \hat{\theta}_{-i}))\) and from condition 1 of Def. 2, we get

\[ v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) \]

\(\forall v_i, v'_i, r'_i \subseteq r_i, \forall i \in N\).

Our first claim is that the definition above is implied by our definition of DDSIC.

**Proposition 1** DDSIC implies IC.

**Proof:** In this proof, we will exhaustively list all the cases of manipulation under Def. 3 and show that each of the inequalities is implied by the conditions of DDSIC (Def. 2). Suppose, \((v_i, r_i)\) is the tuple of the true valuation and neighbor set of agent \(i\). The following cases of manipulation in \(v_i\) and \(r_i\) are exhaustive for Def. 3.

- **Case 1:** \((v'_i, r'_i)\), i.e., valuation is truthfully reported but diffusion is strategized. So, for \(v'_i = v_i \; r'_i \subseteq r_i\), the inequality of Def. 3 becomes

\[ v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) \]

\(\forall v_i, \hat{\theta}_{-i}, \forall i \in N\).

This is implied by condition 2 of Def. 2.

- **Case 2:** \((v'_i, r_i)\), i.e., valuation is manipulated but diffusion is not. Hence, with \(r'_i = r_i\), Def. 3 becomes

\[ v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v'_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v'_i, r_i), \hat{\theta}_{-i})) \]

\(\forall v_i, v'_i, \hat{\theta}_{-i}, \forall i \in N\).

This is implied by condition 1 of Def. 2.

- **Case 3:** \((v'_i, r'_i)\), i.e., both valuation and diffusion are strategized. Then the condition of Def. 3 becomes

\[ v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})), \]

\(\forall v_i, v'_i, \hat{\theta}_{-i}, \forall i \in N\).

Now from condition 2 of Def. 2, we get

\[ v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})), \]

\(\forall v_i, \hat{\theta}_{-i}, \forall i \in N\),

and from condition 1 of Def. 2, we get

\[ v_i g_i(f^G((v_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r'_i), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})), \]

\(\forall v_i, v'_i, \hat{\theta}_{-i}, \forall i \in N\).

Combining these two, we have

\[ v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})), \]

\(\forall v_i, v'_i, \hat{\theta}_{-i}, \forall i \in N\),

which is Eqn. (1).

This completes the proof.

The next desirable property deals with the participation guarantee of the agents.

**Definition 4 (Individual Rationality)** A DA \((g, p)\) on a graph \(G\) is individually rational (IR) if

\[ v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geq 0, \forall v_i, r_i, \hat{\theta}_{-i}, \forall i \in N. \]
4 Characterization Results

Our first result is to characterize the **DDSIC** diffusion auctions. The result by Myerson [17] in the single indivisible item auction setup implicitly assumes that the knowledge of auction reaches all the players for free, and therefore, no additional incentive is required for the agents to diffuse the information of the auction into the network. But in our setting, the information reaches an agent on a network only if every predecessor in at least one path from the seller to that agent forwards this information. Our result, therefore, generalizes Myerson’s characterization result in network auctions.

For a given network \( G \), a diffusion diffusion auction (DA) \((g, p)\) is DDSIC if and only if the following two conditions hold:

1. the functions \( g_i(f^G((v_i, r_i), \hat{\theta} - i)) \) are monotone non-decreasing in \( v_i \), for all \( r_i, \hat{\theta} - i \), and \( i \in N \).
2. the payments are FFP (Def. 5).

**Discussions** The characterization result is much in the spirit of the result of Myerson [17]. However, there are the following important observations on these results.

- Condition 2 of the characterization result above refers to Def. 5 that has two conditions given by Eqns. (2) and (3). While Eqn. (3) is reminiscent of Myerson [17], the important difference here is in the VIPC terms. These terms in the payment formula of Myerson [17] were unrestricted for the characterization of DSIC. However, for DDSIC, Eqn. (2) puts additional constraints on the VIPC.

- Our result is unique since we provide a characterization of all randomized single indivisible item auctions. The closest characterization result to our knowledge applies to only deterministic auctions [14]. The example of the following DDSIC auction is not covered by the characterization of [14] but is covered under Theorem 1. We also provide a class of mechanisms later in this paper which subsumes many currently known mechanisms that are DDSIC.

**Example to illustrate the conditions of a randomized DDSIC auction.** The distinguishing factor of the truthfulness guarantee given by DDSIC is in the part where an agent may not diffuse the information to its neighbors. In this example, we will focus only on that part and illustrate the meaning of the conditions of Theorem 1. This example can be easily extended to a full-fledged randomized DDSIC auction. However, that needs the auction to be defined for every realized tree (or graph) and for every type profile \((v_i, r_i, \hat{\theta} - i)\), which will digress a reader from the main intuition of Theorem 1, and therefore, is skipped. For simplicity of exposition, we consider this auction only on a tree, where the true underlying network and the valuations are given by Fig. 1, and \( r_i \) can take values only in \( \{0, 1\} \), i.e., either forward to all its neighbors or not forwarding at all. We discuss the satisfaction of the conditions of Theorem 1, and consider the variation of \( v_i \) and \( r_i \) of each agent.
Figure 1: A randomized DDSIC auction.

In this example, agents $D$ and $E$ get the information of the auction only if $A$ forwards it at the first level of the tree. A DDSIC DA $(g, p)$ needs to decide the allocations $g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$, and the VIPC components $p_i(f^G((0, r_i), \hat{\theta}_{-i}))$ for $r_i = 0, 1$ and for all $i \in N$. For all agents $i \neq A$, $r_i$’s do not matter since they do not have any children in this tree. Therefore, $g_i$’s and VIPC$_i$’s, $i \neq A$, remain unchanged in this example auction when they set $r_i = 0$ or $r_i = 1$ given other agents’ reported types are fixed. Hence, Eqn. (2) is trivially satisfied for all agents except $A$. We discuss agent $A$’s satisfiability of Eqn. (2) separately later. In a nutshell, this example mechanism runs the modified residual claimant (RC) mechanism by Green and Laffont [5] (see below) in the first level of the tree. If agent $A$ forwards, then it divides $A$’s probability of allocation with its children and adjusts the payments according to Theorem 1. If $A$ does not forward, then it is just RC. Based on the forwarding decision of $A$, this example can be divided into two cases:

Case 1: $r_A = 0$: When agent $A$ does not forward the information, the auction stays limited to the agents $A$, $B$, and $C$. Let the auction give the object w.p. $2/3$ to the highest bidder, and w.p. $1/3$ to the second highest bidder. The payment of the highest bidder is $\frac{1}{3} \times$ the second highest bid. This payment is equally distributed among the non-winning agents, which, in this case, is the third highest bidder. This is the modified version of the residual claimant mechanism [5], which is DSIC (equivalent to DDSIC for a single level tree). Hence, under this case, the allocation probability of each agent is clearly monotone non-decreasing, since it increases from zero to $1/3$ when it becomes the second highest bidder, and from $1/3$ to $2/3$ when it becomes the highest bidder. The VIPC for each agent $i$ is given by $-\frac{1}{3} \times$ the second highest bid in the population except agent $i$. Therefore, VIPC$_A(r_A = 0) = -6/3 = -2, VIPC_B(r_A = 0) = -4/3, VIPC_C(r_A = 0) = -4/3$. The payments follow from Eqn. (3): $p_A(r_A = 0) = -2 + 0 + 0, p_B(r_A = 0) = -4/3 + 6 \times \frac{1}{3} - \frac{1}{3}(6 - 4) = 0, p_C(r_A = 0) = -4/3 + 9 \times \frac{2}{3} - \frac{1}{3}(6 - 4) - \frac{2}{3}(9 - 6) = 2$. The allocation probabilities are zero.

Figure 2: Allocation functions of the nodes in Fig. 1.
for every valuation of agents \(D\) and \(E\) and their VIPCs are zeros. Consequently, their payments are also zero in this case.

**Case 2:** \(r_A = 1\): Let the \(g_i\)'s be given by Fig. 2. Clearly, these are monotone non-decreasing. Let \(\text{VIPC}_A(r_A = 1) = -11/3, \text{VIPC}_B(r_A = 1) = -3, \text{VIPC}_C(r_A = 1) = -2, \text{VIPC}_D(r_A = 1) = 0, \text{VIPC}_E(r_A = 1) = 0\). The payments are given by Eqn. (3) as follows: \(p_A(r_A = 0) = -11/3 + 0 + 0, p_B(r_A = 0) = -3 + 0 + 0, p_C(r_A = 0) = -2 + 9 \times \frac{1}{3} - \frac{1}{3}(9 - 0) = 0, p_D(r_A = 0) = 0 + 0 + 0, p_E(r_A = 0) = 0 + 12 \times \frac{2}{3} - \frac{2}{3}(12 - 10) = 20/3\).

The satisfiability of Eqn. (2) for agent \(A\) warrants a separate discussion since it is the only agent which has a different VIPC when \(r_A = 0\) and \(r_A = 1\), keeping the other agents’ reported types fixed. However, we note that the LHS of Eqn. (2) for \(A\) is \(\text{VIPC}_A(r_A = 0) - \text{VIPC}_A(r_A = 1) = -2 + 11/3 = 5/3\) which is larger than the RHS which is zero since in both the cases above, agent \(A\) has no probability of winning the object. So, Eqn. (2) is satisfied for agent \(A\) too. Hence, this is a randomized DA that satisfies the conditions of Theorem 1 for \(\theta_{-i}\) given by Fig. 1. Extending this example for every \(\theta_{-i}\), we can show it to be DDSIC according to Theorem 1 that we prove next.

**Proof:** [of Theorem 1] \((\Rightarrow)\): In the forward direction of this proof, we begin with a DDSIC mechanism \((g, p)\) and show that the conditions of the theorem holds for this mechanism.

**DDSIC implies condition 1:** the monotonicity of allocation functions. Note that if the true valuations of agent \(i\) were \(v_i\) and \(v_i'\), then the utility the agent for diffusion type \(r_i\) would have been respectively as follows

\[
\begin{align*}
u_i((v_i, r_i), \hat{\theta}_{-i}) &= v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \\
u_i((v_i', r_i), \hat{\theta}_{-i}) &= v_i' g_i(f^G((v_i', r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i}))
\end{align*}
\]

Since \((g, p)\) is DDSIC, from point 1 of Def. 2, we have

\[
\begin{align*}v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) &\geq v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i}))) \\
&\Rightarrow v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \\
&\quad \geq v_i' g_i(f^G((v_i', r_i), \hat{\theta}_{-i}) + (v_i - v_i') g_i(f^G((v_i', r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i})))
\end{align*}
\]

The implication follows since we have added and subtracted the term \(v_i' g_i(f^G((v_i', r_i), \hat{\theta}_{-i})\) on the RHS. Reorganizing the terms on the RHS, we get

\[
\begin{align*}v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \\
&\quad \geq v_i' g_i(f^G((v_i', r_i), \hat{\theta}_{-i}) + (v_i - v_i') g_i(f^G((v_i', r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i}))) \\
&\quad \Rightarrow u_i((v_i, r_i), \hat{\theta}_{-i}) \geq u_i((v_i', r_i), \hat{\theta}_{-i}) + (v_i - v_i') g_i(f^G((v_i', r_i), \hat{\theta}_{-i}))
\end{align*}
\]

From convex analysis [18], we know that the above inequality implies that \(g_i(f^G((v_i', r_i), \hat{\theta}_{-i})\) is a sub-gradient of \(u_i\) at \(v_i'\), if \(u_i\) can be shown to be convex. In the following, we show that \(u_i\) is indeed convex.

For brevity, we use the shorthand \(h(v_i) := u_i((v_i, r_i), \hat{\theta}_{-i})\) and \(\phi(v_i') := g_i(f^G((v_i', r_i), \hat{\theta}_{-i}))\). Because \(v_i\) and \(v_i'\) were arbitrary in the above inequality, we can choose arbitrary \(x_i, z_i \in U_i\) and define \(y_i = \lambda x_i + (1 - \lambda) z_i\) where \(\lambda \in [0, 1]\). From the above inequality, we get

\[
\begin{align*}h(x_i) &\geq h(y_i) + \phi(y_i)(x_i - y_i) \\
h(z_i) &\geq h(y_i) + \phi(y_i)(z_i - y_i).
\end{align*}
\]

Multiplying Eqn. (6) by \(\lambda\) and Eqn. (7) by \((1 - \lambda)\) and adding, we get \(\lambda h(x_i) + (1 - \lambda) h(z_i) \geq h(y_i),\) which proves that \(h\) or the utility \(u_i\) is convex, and \(\phi\) or the allocation \(g_i\) is its sub-gradient. Since sub-gradient of a convex function is non-decreasing, we get the claimed implication.

**DDSIC implies condition 2:** the payment formulae. Again from convex analysis, we know that for any convex function \(h\) having subgradient \(\phi\), the following integral relation holds: \(h(y) = h(z) + \int_y^z \phi(t) dt\) for any \(y, z\) in the domain of \(h\). Therefore, using the same definitions of \(h\) and \(\phi\),
from the previous case, we get (when $i$ diffuses to $r_i$
\[ u_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = u_i(f^G((0, r_i), \hat{\theta}_{-i})) + \int_0^{V_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt \]
\[ \Rightarrow v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = -p_i(f^G((0, r_i), \hat{\theta}_{-i})) + \int_0^{V_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt \]
\[ \Rightarrow p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = p_i(f^G((0, r_i), \hat{\theta}_{-i})) + v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - \int_0^{V_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt \]

This is precisely Eqn. (3), which is one part of condition 2 of Theorem 1 (item (b) of Def. 5). To prove the other part (item (a) of Def. 5) of condition 2, we substitute the payment expressions derived above into point 2 of Def. 2, and get
\[ \int_0^{V_i} g_i(f^G((y, r_i), \hat{\theta}_{-i}))dy - p_i(f((0, r_i), \hat{\theta}_{-i})) \geq \int_0^{V_i} g_i(f^G((y, r_i'), \hat{\theta}_{-i}))dy - p_i(f((0, r_i'), \hat{\theta}_{-i})) \]
\[ \Rightarrow p_i(f((0, r_i'), \hat{\theta}_{-i})) - p_i(f((0, r_i), \hat{\theta}_{-i})) \geq \int_0^{V_i} \left( g_i(f^G((y, r_i'), \hat{\theta}_{-i})) - g_i(f^G((y, r_i), \hat{\theta}_{-i})) \right)dy. \]

The first inequality follows directly where the terms $v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ cancels out on the LHS and $v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))$ cancels out on the RHS. The second inequality follows by rearranging the first. Hence, this proves Eqn. (2), the other part of condition 2 of Theorem 1.

$(\Leftarrow)$: In the reverse direction of the proof, we start with the two conditions of Theorem 1 and show that they imply DDSIC (Def. 2). We first show that the condition 1 and part of condition 2 of Theorem 1, i.e., allocation monotonicity and Eqn. (3) of FFP (Def. 5), imply point 1 of DDSIC (Def. 2). Afterwards, we show that Eqns. (2) and (3) of FFP (Def. 5) implies point 2 of DDSIC.

**Conditions 1 and 2 $\Rightarrow$ point 1 of DDSIC.** We are given that $g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ is monotone non-decreasing in $v_i$, for a diffusion type $r_i$ (i.e., when agent $i$ diffuses to neighbour set $r_i$) and payment is given by Eqn. (3), for all $i \in N$. Assuming $v_i$ to be the true valuation of agent $i$, the utility of agent $i$ when she is truthful is given by
\[ v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \]
\[ = v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) - v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) + \int_0^{V_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt, \tag{8} \]
and the utility when she misreports to $v_i'$ is given by
\[ v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i})) \]
\[ = v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) - v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) + \int_0^{V_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt. \tag{9} \]

Subtracting Eqn. (9) from Eqn. (8), we get
\[ v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = [v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i}))] \]
\[ = (v_i' - v_i) g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) + \int_{v_i'}^{V_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt. \tag{10} \]

Since $g_i$ is monotone non-decreasing and non-negative, the RHS of Eqn. (10) is always non-negative. Hence, we have the point 1 of DDSIC.

**Condition 2 $\Rightarrow$ point 2 of DDSIC.** Here we have Eqn. (2) satisfied. Given that we also have the expression of the payment given by Eqn. (3) satisfied, adding and subtracting $v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ on the LHS of Eqn. (2) and adding and subtracting $v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))$ on the RHS and then rearranging, we get point 2 of DDSIC.

Together with the forward and reverse direction of this proof, we get the theorem.

In the following section, we present a few examples of DDSIC auctions and demonstrate how they differ from the classical single item auction without forwarding constraints.
5 Examples of DDSIC mechanisms

We begin with a novel and nontrivial mechanism that we call Level-by-Level Exponential Valuation (LbLEV) mechanism. This mechanism assigns some exponents to each of the agents and considers an arborescence of the graph generated from the reported $r_i$’s, keeping $s$ as the root.\footnote{An arborescence is a directed graph in which, for a vertex $u$ called the root and any other vertex $v$, there is exactly one directed path from $u$ to $v$ [4]. Since this is a tree with a specific node as the root, we will use the term arborescence and tree interchangeably in this paper.}

5.1 Level-by-Level Exponential Valuation (LbLEV) mechanism

We first present the high-level idea of our mechanism before formalizing it in Alg. 1. As the name suggests, the mechanism is run at every level of an arborescence $\hat{T}$ from the root towards the leaves. The mechanism LbLEV is parametrized by a vector $t \in \mathbb{R}_{\geq 0}^n$, where $t_i$ denotes the exponent of agent $i$. Different choices of $t$ and $\hat{T}$ for the same input instance create a class of mechanisms, and we call each of them LbLEV. Let $T_i$ be the subtree rooted at node $i$ including $i$. At each level of this tree, the mechanism sets an offset for the parent node(s) of that level (each parent node having a disjoint set of children), and finds the maximum valuation $v_i^{\text{max}}$ in $T_i$, for every $i$ at that level. It deducts the offset from $v_i^{\text{max}}$ to calculate $i$’s effective valuation $\rho_i$, and decides which $T_i$’s “stay in the game”. For the $i$’s that stay, the mechanism considers the largest $\rho_i$ and tentatively sets it as the “winning subtree” and calculates its “actual payment”. The mechanism repeats at every next level treating the current root of the tentatively winning subtree as the parent with an updated offset. Alg. 1 details out the description of this mechanism in an algorithmic manner.

In the arborescence $\hat{T}$, every agent receives the difference between the payment that their children in $\hat{T}$ give to that agent and the payment she makes to her parent. The algorithm terminates either at some leaf node or at a node that has large enough offset such that none of its children “stay in the game”. We call that node the winner of LbLEV.

Since the preprocessing step (line 1) picks an arbitrary arborescence, it is important to ask whether an agent can manipulate the arborescence in her favor. The arbitrary arborescence is independent of the agents’ reported valuations. It depends only on the reported $r_i$’s of the agents. Therefore, an agent $i$ can only manipulate by not inviting another agent $j$ in this setup, and if $i$ were the only agent inviting $j$ in this network, agent $j$ would not appear in any arbitrary arborescence chosen by the mechanism. When we fix an arborescence, that is arbitrarily picked by the mechanism under this setup, it is equivalent to another setup with the same arborescence where all the actions of the agents except $i$ and $j$ remain the same as the current setup and $i$ does not diffuses the information to $j$. In our next theorem, we prove that agent $i$ does not benefit by not diffusing the information for every realized arborescence. Hence, the manipulation of $r_i$ by an agent in the preprocessing step is not beneficial.

Illustration of LbLEV through an example. Suppose the arborescence $\hat{T}$ generated from the reported graph $G$ is as shown in Fig. 3 for 11 nodes (named $A, B, \ldots, K$ in the figure) in addition to the seller $S$. The tuple next to agent $i$ denotes $(v_i, t_i)$, for all $i = A, B, \ldots, K$, where $v_i$ is the reported valuation and $t_i$ is the exponent set by the mechanism. At level $1$ of $\hat{T}$, from Alg. 1 we get offset= 0. We find the effective valuations to be $\rho_A = 750$, $\rho_B = 6$, $\rho_C = 9$. Now, we observe that $\rho_A^{1A} = 750$ is the highest among the $\rho_i^{1i}$’s of that level. Hence, by line 16, agent $A$ is set as the tentative winner and agent $B$ and $C$ and their subtrees are set as non-winners. Also, the effective payment of agent $A$ to $S$ is $\rho_B^{1B/1A} = 9^{1/1} = 729$. From line 19, the actual payment of $A$ becomes the same as its effective payment since at this level offset= 0. Also, for the next iteration, i.e., for level $= 2$, parent $= A$ and offset $= 729$ are set.

At level $= 2$, $\rho_D = 735 - 729 = 6$, $\rho_E = 750 - 729 = 21$ and $\rho_F = 4 - 729 = -725$. As $\rho_F < 0$, $F$ and its subtree is removed (line 8). Line 14 stands false, since $v_A = 730 \not\in [729 + 6^{1/2}] = \text{offset} + \rho_i^{1i/1i}$. Hence, $E$ is the tentative winner, and it pays $729 + 6^{1/2} = 731.45$. Again from line 19, the next level (level $= 3$) details are updated, i.e., parent $= E$ and offset $= 731.45$. 

Algorithm 1: LbLEV

Input: reported types $\hat{\theta}_i = (\hat{v}_i, \hat{r}_i), \hat{r}_i \subseteq r_i, \forall i \in N$
Output: winner of the auction (which can be $\emptyset$), payments of each agent

1 Preprocessing: Let $\hat{G}$ be the subgraph induced from $\hat{r}_i, i \in N$, on the set of vertices. Let $\hat{T}$ be an arborescence of $\hat{G}$ rooted at the seller $s$. Pick $t \in \mathbb{R}_{>0}^n$ independent of the input
2 if $\hat{v}_i = 0, \forall i \in N$ then
3 Item is not sold and payment is set to zero for all agents, STOP
4 Initialization: all agents are non-winners and their actual payments are zeros, set offset = 0, level = 1, parent = $s$, $v_{parent} = 0$
5 In this level of $\hat{T}$:
6 for each node $i \in \text{children}(\text{parent})$ do
7 Set effective valuation $\rho_i := \max\{\hat{v}_j : j \in \hat{T}_i\} - \text{offset}$
8 Remove the nodes that have $\rho_i < 0$, denote the rest of the agents with $N_{\text{remain}}$
9 if $N_{\text{remain}} \neq \emptyset$ then
10 Sort the nodes in decreasing order of $\rho_{i^*}$
11 Compute $z := \rho_{\ell^*}/\ell^*$, where $i^*$ is the highest in this order and $\ell$ is the second highest node in the decreasing $\rho_{i^*}$ order
12 else
13 Set $z = 0$
14 if $v_{\text{parent}} \geq \text{offset} + z$ then
15 STOP and go to Step 23
16 Set the highest node $i^*$ in this order as the tentative winner and its effective payment to be $z$
17 All nodes and their subtrees except $i^*$ are declared non-winners
18 The actual payment of $i^*$ to parent = effective payment + offset
19 parent = $i^*$, offset = actual payment of $i^*$
20 level = level + 1
21 Repeat Steps 5 to 20 with the updated parent and offset for the new level
22 STOP when no agent $i$ has $\rho_i \geq 0$ OR the leaf nodes are reached
23 Set tentative winner as final winner; final payments are the actual payments that are paid to the respective parents of $\hat{T}$

Figure 3: An example instance of LbLEV.

At level = 3, $\rho_J = 745 - 731.45 = 13.55$, $\rho_K = 750 - 731.45 = 18.55$, Agent $K$ becomes the tentative winner as line 14 returns false. The actual payment of $K$ is $731.45 + 13.55^{1/2} = 735.13$.

As agent $K$ is a leaf node, the algorithm stops via line 22 and $K$ is declared as the final winner (line 23). Agent $A$ gets the difference between the amounts it receives from its children and pays to its parent, i.e., $731.45 - 729 = 2.45$ (one can think of this amount as the commission to forward the information, which offsets its payoff when it manipulates and does not forward). Similarly, agent $E$
We show that the VIPC when agent \( \hat{w} \) is defined for a cleaner presentation of the proof. These are:

- **offset**(\( i \)) for the children of \( i \), i.e., at a level where \( i \) is the auctioneer.
- **children**(\( i \)) and **parent**(\( i \)) are the children and parent of \( i \) respectively in \( \hat{T} \), and is defined in the usual way for standard trees. Therefore, **offset**(**parent**(\( i \))) will denote the offset set at one level before agent \( i \) by the **parent** of \( i \), and will refer to the previous iteration of the LbLEV mechanism. Similarly, **children**(**parent**(\( i \))) denotes the siblings of \( i \) including herself.
- **winner**(\( i \)) := \( \text{argmax}_{j \in \text{children}(i)} \rho_j^i \), and **runnerup**(\( i \)) := \( \text{argmax}_{j \in \text{children}(i) \setminus \text{winner}(i)} \rho_j^i \) denote the winner and runnerup respectively of the auction at a level where \( i \) is the auctioneer. Ties are broken arbitrarily in both these cases.

In the extreme case where the reported valuation of every agent is zero, the allocation probability is zero for every agent and so are their payments (according to line 2 of Alg. 1). Here it trivially satisfies all the three conditions of Theorem 1. So, in the rest of the proof, we will assume that at least one agent has a positive reported valuation. Therefore, if some agent reports her valuation to be zero, she is not allocated the object.

**Part 1: LbLEV satisfies Condition 1 of Theorem 1**: To show that the allocation function under LbLEV satisfies monotonicity w.r.t. the valuation of every agent, we need to show that for every pair \( v_i, \tilde{v}_i \) s.t. \( \tilde{v}_i > v_i \), the allocation probability at \( v_i \) is at least as much as at \( \tilde{v}_i \), for all \( i \) (when agent \( i \) diffuses to \( r_i \)), and for all \( i \in N \). Since the LbLEV mechanism is defined w.r.t. the effective valuations \( \rho_i \)'s, and not \( v_i \)'s, showing this is non-trivial. Based on the fact that a typical agent \( i \) can belong to one of the three classes after the outcome of the mechanism is chosen, we have the following cases when agent \( i \) reports a valuation of \( v_i \).
Case 1: agent \( i \) is a not-on-path non-winner: in this case, \( g_i(f^G(v_i, r_i), \hat{\theta}_{-i}) = 0 \). From the description of \( \text{LbLEV} \), it is clear that for \( v'_i > v_i \), either agent \( i \) can remain a not-on-path non-winner, or it can become a winner. It cannot become an on-path non-winner because it would imply that there was another agent in \( i \)'s subtree that had a maximum valuation in this network at agent \( i \)'s original valuation \( v_i \), and then agent \( i \) could not be a not-on-path non-winner. In both the cases where agent \( i \) is not-on-path non-winner or winner, \( g_i(f^G(v'_i, r_i), \hat{\theta}_{-i}) \geq 0 \), hence it is monotone non-decreasing.

Case 2: agent \( i \) is an on-path non-winner: the allocation for agent \( i \) is \( g_i(f^G(v_i, r_i), \hat{\theta}_{-i}) = 0 \) in this case as well. This agent is on-path non-winner with bid \( v_i \) implies that there is an agent \( T_i \) that has reported the winning bid. Now, if agent \( i \) bids \( v'_i \) which is higher than \( v_i \), it can either continue to be an on-path non-winner or may become the new winner at a sufficiently high bid. In both these cases, the allocation probability is monotone non-decreasing.

Case 3: agent \( i \) is the winner: here \( g_i(f^G(v_i, r_i), \hat{\theta}_{-i}) = 1 \). We need only to show that for all \( v'_i > v_i \), agent \( i \) continues to be the winner. This is fairly easy to see from Alg. 1. An agent can be the winner either when it is the parent node in line 14 or line 22.

In line 14, since agent \( i \) is the parent and it satisfies the if condition of that line, an increase in its valuation will continue to hold that condition true and \( i \) will continue to be the winner.

In line 22, agent \( i \) is the auctioneer whose offset is higher than the valuations of all agents in its subtree. The offset is not a function of agent \( i \)'s valuation – it is determined by \( \hat{\rho}^{i,i}_k \), where \( k \) is the second highest node in the decreasing \( \hat{\rho}^{i,i}_k \) order (line 13). Hence, an increase in \( v_i \) will continue keeping \( i \) to be the winner.

These three cases together prove this part of the proof.

Part 2: \( \text{LbLEV} \) satisfies Eqn. (3) (one part of Condition 2) of Theorem 1: To show this part of the proof, we need to show that the payments given by Alg. 1 for each agent matches Eqn. (3). Note that, after a monotone non-decreasing allocation rule has been picked (as seen in the previous case of this proof) by the algorithm, the only variable quantity in the payment formula is the VIPC term. The other two terms, i.e., the second and third term on the RHS of Eqn. (3) are already fixed given the allocation. So, in order to complete the proof, we need to find an appropriate VIPC such that the payment given by \( \text{LbLEV} \) exactly matches the sum of those two terms and the VIPC.

We will denote the payment for agent \( i \) given by the mechanism as \( p_i^{\text{LbLEV}}(v_i, r'_i, \hat{\theta}_{-i}) \) when the reported type of agent \( i \) is \((v_i, r'_i)\) and that for the other agents are \( \hat{\theta}_{-i} \).

Case 1: agent \( i \) is a not-on-path non-winner: note that, in this case, \( g_i(f^G(v_i, r'_i), \hat{\theta}_{-i}) = 0 \). Hence, the last two terms in the RHS of Eqn. (3) vanishes. We need to set that the VIPC term which is exactly equal to \( p_i^{\text{LbLEV}}(v_i, r'_i, \hat{\theta}_{-i}) \), the actual payment of agent \( i \) under \( \text{LbLEV} \), and show that it is indeed independent of \( v_i \) for it to be qualified as a VIPC. From the algorithm, we see that \( p_i^{\text{LbLEV}}(v_i, r'_i, \hat{\theta}_{-i}) = 0 \), since \( i \) is a not-on-path non-winner agent. Hence, \( \text{VIPC}_i = 0 \), and it matches Condition 2 of Theorem 1.

Case 2: agent \( i \) is an on-path non-winner: for this case as well, the situation is similar to the previous case: \( g_i(f^G(v_i, r'_i), \hat{\theta}_{-i}) = 0 \), and hence, the last two terms in the RHS of Eqn. (3) vanishes. We need to calculate \( p_i^{\text{LbLEV}}(v_i, r'_i, \hat{\theta}_{-i}) \), the actual payment of agent \( i \) under \( \text{LbLEV} \), and show that it is indeed independent of \( v_i \) for it to be qualified as VIPC.

From the \( \text{LbLEV} \) algorithm, we see that the net payment of a on-path non-winner agent \( i \) has the following simple structure:

\[
p_i^{\text{LbLEV}}(v_i, r'_i, \hat{\theta}_{-i}) = \pi(\hat{T}_i) - \sum_{j \in \text{children}(i)} \pi(\hat{T}_j) =: \pi(\hat{T}_i) - R_i, \; \forall r'_i \subseteq r_i \tag{11}
\]

Where \( \pi(\hat{T}_k) \) is payment made by the subtree rooted at \( k \) to its parent (called the actual payment in Alg. 1). Therefore, \( \sum_{j \in \text{children}(i)} \pi(\hat{T}_j) \) is the net payment received by agent \( i \) from the subtrees

\[3\text{Since we discuss the values of the VIPCs at a specific instance, i.e., after the variables } r'_i, \hat{\theta}_{-i}, \text{ have realized, the arguments of those VIPC terms are clear from the context. Therefore, we will omit the arguments of the VIPC terms for brevity in the rest of the paper.}]}
rooted at its children nodes. Therefore, the payment of agent \( i \) is just the difference between the payment it makes to parent\((i)\) minus the sum of the payments it receives from children\((i)\) (according to line 23). We use the shorthand \( R_i \) to denote \( \sum_{j \in \text{children}(i)} \pi(T_j) \).

Now, we need to show that the RHS of Eqn. (11) is independent of \( v_i \) and then we are done claiming it to be VIPC\(_i\). From the algorithm, we find that the payment received by \( i \) has a rather simpler form:

\[
R_i = \text{offset}(i) + \rho_{\ell}^{t_i/t_k}, \text{ where } k = \text{winner}(i), \ell = \text{runnerup}(i).
\]  

(12)

The above equation means that the payment received by \( i \) is the actual payment the winner of this level, \( k \), makes to its parent \( i \), and it is the sum of two terms: (a) the offset set by \( i \) while auctioning at that level and (b) the \( \rho \) of the second highest bidder in the decreasing order of \( \rho_i^{t_i/t_k} \) raised to an appropriate exponent.

From line 19, we find that \( \text{offset}(i) = \pi(T_i) \), i.e., the payment the winning agent \( i \) of a level makes to its parent is set as the \( \text{offset} \) in the next level. Note that, there must be at least one next level since it is an \textit{on-path non-winner}. Therefore, the RHS of Eqn. (11) becomes \(-\rho_{\ell}^{t_i/t_k}\). We claim that this is independent of \( v_i \). The exponents are constants and independent of \( v_i \). The term \( \rho_{\ell} = v_k - \text{offset}(i) \) is also independent of \( v_i \). This is because the first term \( v_k \) is independent of \( v_i \). The second term \( \text{offset}(i) \) is the payment \( \pi(T_i) \), that \( i \) made to its parent which is a function of the \( \text{offset(parent}(i)) \) and the valuations of the agents other than agent \( i \) in the level where parent\((i)\) was the auctioneer and \( i \) was a participant of that auction.

\[
\pi(T_i) = \text{offset(parent}(i)) + \rho_j^{t_j/t_k}, \text{ where } j = \text{runnerup(parent}(i))\).
\]

Since, both these terms are independent of \( v_i \), the RHS of Eqn. (11) is independent of \( v_i \) and hence it is a valid VIPC\(_i\).

Case 3: agent \( i \) is the \textit{winner}: unlike the previous two cases, here \( g_i(f^G(v_i, r'_i), \hat{\theta}_{-i}) = 1 \). Therefore, the payment given by Eqn. (3) has the last two terms on the RHS that are non-zero. The first term of the RHS, i.e., the VIPC, is the payment of the agent when it reports 0 instead of \( v_i \). From the algorithm, we observe that an agent \( i \) can become a winner in two possible ways: (i) if \( i \)'s valuation is larger than the maximum payment it can extract from its children\((i)\) in \( T_i \) (line 14), or (ii) if \( i \)'s offset is so high that none of its children has a positive \textit{effective} valuation \( \rho \) or \( i \) is a leaf node (line 22). In the first case, if agent \( i \) reports a valuation of 0, then it becomes a \textit{on-path non-winner}, and in the second, it becomes a \textit{not-on-path non-winner}. In the following, we consider these two cases separately and show that \( p_i^{\text{LbLEV}}((v_i, r'_i), \hat{\theta}_{-i}) \) indeed matches the expression of Eqn. (3).

• Case 3(i): bidding 0 makes agent \( i \) a \textit{on-path non-winner}: when agent \( i \) is an \textit{on-path non-winner}, its payment is \( \text{VIPC}_i = -\rho_{\ell}^{t_i/t_k} \), where \( k = \text{winner}(i), \ell = \text{runnerup}(i) \) (from Case 2 above).

Given the allocation function is already fixed via the LbLEV mechanism, the last two terms in Eqn. (3) can be written w.r.t. that allocation as:

\[
g_i(f^G((v_i, r'_i), \hat{\theta}_{-i}))v_i - \int_0^{v_i} g_i(f^G((v, r'_i), \hat{\theta}_{-i}))dv \]

\[
= g_i(f^G((v_i, r'_i), \hat{\theta}_{-i}))v_i - \int_0^{l_i} g_i(f^G((v, r'_i), \hat{\theta}_{-i}))dv - \int_{l_i}^{v_i} g_i(f^G((v, r'_i), \hat{\theta}_{-i}))dv \]

\[
= 1 \cdot v_i - \int_0^{l_i} 0 \cdot dv - \int_{l_i}^{v_i} 1 \cdot dv = l_i,
\]

where \( l_i \) is the threshold after which agent \( i \) starts becoming the winner. The \textit{winner} of the LbLEV mechanism is deterministic and this agent starts becoming the \textit{winner} when its valuation crosses a threshold point that we define to be \( l_i \), which is guaranteed to exist. Now, we see that agent \( i \), which was an \textit{on-path non-winner}, starts becoming a \textit{winner} only when it is the parent in line 14 of the algorithm. Hence the threshold \( l_i \) will be as follows.

\[
l_i = \text{offset}(i) + \rho_{\ell}^{t_i/t_k}, \text{ where } k = \text{winner}(i), \ell = \text{runnerup}(i).
\]

Therefore, the entire payment given by Eqn. (3) is \( \text{VIPC}_i + l_i = \text{offset}(i) \). From the algorithm, we notice that the \textit{winner} pays its own \textit{offset} to its parent. Therefore,

\[
p_i^{\text{LbLEV}}((v_i, r'_i), \hat{\theta}_{-i}) = \text{offset}(i)
\]

Hence, we have the equality of the LbLEV payment with that of Eqn. (3).
• **Case 3(ii):** bidding 0 makes agent $i$ a not-on-path non-winner: when agent $i$ is a not-on-path non-winner, its payment is $\text{VIPC}_i = 0$ (from Case 1 above). Given the allocation function is already fixed via the LbLEV mechanism, the last two terms in Eqn. (3) can be written w.r.t. that allocation as

$$g_i(f^G((v, v'), \hat{\theta}_{\ldots i}))v_i - \int_0^{v_i} g_i(f^G((v, v'), \hat{\theta}_{\ldots i}))dv = g_i(f^G((v, v'), \hat{\theta}_{\ldots i}))v_i - \int_0^{v_i} g_i(f^G((v, v'), \hat{\theta}_{\ldots i}))dv - \int_{v_i}^{v'} g_i(f^G((v, v'), \hat{\theta}_{\ldots i}))dv$$

$$= 1 \cdot v_i - \int_0^{k_i} 0 \cdot dv - \int_{k_i}^{v_i} 1 \cdot dv = k_i,$$

where $k_i$ is the threshold after which agent $i$ starts becoming the winner. The winner of the LbLEV mechanism is deterministic and this agent starts becoming the winner when its value crosses a threshold point that we define to be $k_i$, which is guaranteed to exist. Since, in this case, $i$ was a not-on-path non-winner till its value reached $k_i$, this critical valuation is given by

$$k_i = \text{offset}(\text{parent}(i)) + \rho_i^{t_i/t_i},$$

where $j = \text{runnerup}(\text{parent}(i))$.

If $v_i$ crosses the value on the RHS above, then $\rho_i$ becomes the maximum among the children of $\text{parent}(i)$, and hence it becomes the winner. On the other hand, under this case, $i$ becomes the winner because line 22 of the LbLEV mechanism becomes effective. Hence, we have

$$\rho_i^{\text{LEB}}((v_i, v'), \hat{\theta}_{\ldots i}) = \text{offset}(\text{parent}(i)) + \rho_i^{t_i/t_i},$$

where $j = \text{runnerup}(\text{parent}(i))$.

Since $\text{VIPC}_i = 0$ in this case, we have the equality of the LbLEV payment with that of Eqn. (3).

**Part 3: LbLEV satisfies Eqn. (2) (the other part of Condition 2) of Theorem 1:** In this condition, we need to compare VIPC, between the cases when $i$ forwards to its complete neighbor set ($r_i$) versus a subset of its neighbors ($r'_i \subseteq r_i$). By reporting a diffusion type, agent $i$ can be in one of the three classes: not-on-path non-winner, on-path non-winner, or winner. We handle these cases one by one.

**Case 1:** agent $i$ is a not-on-path non-winner when it diffuses to complete neighbor set $r'_i$: then either all nodes from $i$ to the root $s$ on $T$ were never tentative winners, or some node in that path is the winner. In both cases, if agent $i$ does not forward the information to $r_i$ and instead diffuses to $r'_i$, s.t. $r'_i \subseteq r_i$, the winner does not change. Hence, $\text{VIPC}_i = 0$ and $g_i = 0$ for any diffusion type. Therefore, Eqn. (2) is trivially satisfied.

**Case 2:** agent $i$ is the winner node when it diffuses to complete neighbor set $r_i$: then LbLEV already treats $i$ as if it does not forward and calculates VIPC. So, VIPC of agent $i$ at diffusion type $r_i$ is the same as when $i$ forwards to $r'_i$, some subset of $r_i$. Also, $i$ will continue to be the winner for any diffusion type. Hence $g_i = 1$ for any diffusion type reported by agent $i$. Therefore, similar to Case 1, Eqn. (2) is trivially satisfied.

**Case 3:** agent $i$ is an on-path non-winner node when it diffuses to entire neighbor set $r_i$: this is a non-trivial case, since by strategic forwarding, agent $i$ may change the winner. By partial or not forwarding, an on-path non-winner agent $i$ can either become a not-on-path non-winner or a winner.

• When $i$ becomes a not-on-path non-winner by diffusing to $r'_i \subseteq r_i$, its utility becomes 0 (see Case 1 of Part 2 above). However, an on-path non-winner draws utility $R_i - \pi(\hat{T}_i)$ where $R_i = \text{offset}(i) + \rho_i^{t_i/t_i} = \pi(\hat{T}_i) + \rho_i^{t_i/t_i}$ (see Case 2 of Part 2 above). As $\rho_i \geq 0$, hence utility being an on-path non-winner is non-negative and makes diffusion to $r_i$ a weakly better option for $i$.

• In the other case, when $i$ becomes a winner by strategic forwarding, its utility becomes $v_i - \pi(\hat{T}_i)$. Note that $i$ is an on-path non-winner because it failed the 1st condition in line 14, hence,

$$v_i < \text{offset}(i) + \rho_i^{t_i/t_i}, \text{ where } k = \text{winner}(i), \ell = \text{runnerup}(i). \tag{13}$$

Now, as an on-path non-winner, $i$'s utility is $R_i - \pi(\hat{T}_i)$ (since the allocation probability of $i$ is zero by Eqn. (11)). However, the actual payment by children$(i)$ to $i$ is

$$R_i = \text{offset}(i) + \rho_i^{t_i/t_i}, \text{ where } k = \text{winner}(i), \ell = \text{runnerup}(i). \tag{14}$$
Therefore, from Eqns. (13) and (14) we get, $v_i < R_i$. Agent $i$ pays $\pi(\hat{T}_i)$ to its parent regardless of whether it forwards to $r_i$ or not. Therefore, if agent $i$ forwards to $r_i$, it gets an utility of $R_i - \pi(\hat{T}_i) = \rho_i^{l_i} > 0$ which is larger than the utility $v_i - \pi(\hat{T}_i)$ of $i$ when it diffuses to some $r_i' \subset r_i$ and becomes the winner. Therefore, in this case as well, forwarding to $r_i$ is better than any partial forwarding to $r_i' \subset r_i$ for agent $i$.

Since agent $i$ and $r_i' \subset r_i$ is arbitrary, we conclude that forwarding to the entire neighbor set $r_i$ is a weakly dominant strategy than forwarding to a subset $r_i'$ for every $i$ in each of the three cases above. This gives,

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geq v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))$$

Since, we have already shown in the previous part of this proof that the payment expression is given by Eqn. (3), expanding $p_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ and $p_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))$ according to that expression in the above inequality, we get

$$p_i(f^G((0, r_i'), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) \geq \int_0^y \left( g_i(f^G((y, r_i'), \hat{\theta}_{-i})) - g_i(f^G((y, r_i), \hat{\theta}_{-i})) \right) dy.$$  

Hence, Eqn. (2) holds for LbLEV.

Hence, combining the Parts 1 to 3, we conclude that LbLEV is DDSIC.

**Theorem 3** LbLEV is IR.

**Proof:** We need to show that for every agent $i \in N$, the net utility $v_i - p_i^{LbLEV}((v_i, r_i), \hat{\theta}_{-i}) \geq 0$.

We know that for a reported type profile $((v_i, r_i), \hat{\theta}_{-i})$, agent $i$ can be one of the following.

- **Agent $i$ is not-on-path non-winner:** in this case, $i$ has an allocation probability of zero, since it never becomes a tentative winner. Also, by Alg. 1, its payment remains zero throughout. Therefore, such an agent satisfies the IR condition (Def. 4) trivially.

- **Agent $i$ is on-path non-winner:** in this case, $i$’s allocation probability is still zero, and it makes a payment of $\pi(\hat{T}_i) - R_i$ as given by Eqn. (11). In the discussion following that equation, we see that $R_i = \text{offset}(i) + \rho_i^{l_i}/t_i$, where $k = \text{winner}(i)$, $\ell = \text{runnerup}(i)$, (Eqn. (12)) and that $\text{offset}(i) = \pi(\hat{T}_i)$. Therefore, the utility of agent $i$ in this case is $0 - \pi(\hat{T}_i) + R_i = \rho_i^{l_i}/t_i > 0$.

- **Agent $i$ is winner:** from the algorithm, we observe that an agent $i$ can become a winner in two possible ways: (i) if $i$’s valuation is larger than the maximum payment it can extract from $\text{children}(i)$ in $\hat{T}_i$ (line 14), or (ii) if $i$’s offset is so high that none of its children has a positive effective valuation $\rho$ or $i$ is a leaf node (line 22). However, in both the cases, the payment of the agent is given by $\text{offset}(\text{parent}(i)) + \rho_j^{l_j}/t_j$, where $j = \text{runnerup}(\text{parent}(i))$.

Also, agent $i$ is the winner implies that its $\rho_i$ is such that $\rho_i^{l_i}$ is the largest among all its siblings, i.e., $\text{children}(\text{parent}(i))$. Therefore, we can write

$$\rho_j^{l_j} \geq \rho_i^{l_i}, \text{ where } j = \text{runnerup}(\text{parent}(i)) \Rightarrow \rho_i \geq \rho_j^{l_j}/t_j, \text{ since } \rho_i, t_i > 0 \Rightarrow v_i \geq \text{offset}(\text{parent}(i)) + \rho_j^{l_j}/t_j = p_i^{LbLEV}((v_i, r_i), \hat{\theta}_{-i}) \Rightarrow v_i - p_i^{LbLEV}((v_i, r_i), \hat{\theta}_{-i}) \geq 0.$$  

The second implication follows from the fact that LbLEV sets $\rho_i$ to be the difference between the reported valuation of $i$ and the offset set by its parent node. Therefore, the RHS becomes the payment of agent $i$.

Considering the three cases above, we conclude that LbLEV is IR.

**5.2 Information Diffusion Mechanism (IDM) [10]**

The Information Diffusion Mechanism (IDM), proposed by Li et al. [10], has a similar structure as LbLEV, however, it does not have any exponents for each agent. Consider the maximum bidder $m$ in the network. IDM defines the sequence of cut vertices that disconnects $m$ from the source as the
critical diffusion sequence (CDS). The mechanism finds the winner by identifying an agent in the CDS, that if did not forward, would have been the node with the highest valuation in the rest of the network. Every agent in this sequence is critical for information diffusion, i.e., every node in this sequence can choke the information flow to the nodes that come after it in that sequence. We refer the reader to [10] for the full details of the formal definitions as it is impossible to include all the details in this proof. Since the structure is similar to LbLEV, we will use the same terminologies: (i) winner, (ii) on-path non-winner, and (iii) not-on-path non-winner, as we used earlier. IDM is Incentive Compatible [10]. Since DDSIC is stricter than IC (Proposition 1), we need to show that IDM’s allocation and payment rules abide by the two conditions of Theorem 1.

Part 1: IDM satisfies Condition 1 of Theorem 1: We need to show that the allocation probability of every agent in IDM is monotone non-decreasing in its revealed valuation, for any diffusion status of the agent. Formally, we need to prove that for all \(i \in N\), for each \(v_i, v'_i \) s.t. \(v'_i > v_i\),

\[
g_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) \geq g_i(f^G((v_i, r'_i), \hat{\theta}_{-i})), \quad \text{for all } r'_i \subseteq r_i. \tag{15}
\]

In IDM, a node \(i\) can have \(g_i\) to be either 0 or 1, meaning either it does not win (not-on-path non-winner or on-path non-winner) or is a winner respectively. IDM allocates the object to the agent with the lowest index in the critical diffusion sequence (CDS) [10]. With a slight abuse of notation, we renumber the nodes such that \(i\) and \(i + 1\) become the consecutive nodes on the CDS, in order to simplify the notation. Also, as defined in [10], let \(d_i\) denote the set of nodes (including \(i\) itself) for which \(i\) is the critical node for information diffusion. Hence, in IDM, an agent \(i\) can become the winner in two ways: (a) if \(i\) is the highest bidder in the entire network, or (b) if \(i\) is the \(i\)-th node in the CDS such that \(v_i = v^*_{d_i+1}\) where \(v^*_{d_i+1}\) is defined as the highest bid in the network when the subnetwork after the \((i+1)\)-th node in the CDS has been removed.

If we consider \(v'_i > v_i\), at \(v_i\), agent \(i\) can either be the winner or a non-winner. If it is the winner at \(v_i\), then it is clearly going to remain the winner since both cases (a) and (b) in the previous paragraph will continue to hold at \(v'_i\).

If \(i\) is a not-on-path non-winner at \(v_i\), and the current winner is a critical diffusion node of \(i\) (i.e., if the winner does not forward, node \(i\) does not get the information), then for any \(v'_i\) it will remain a non-winner. But this maintains the inequality of Eqn. (15).

If \(i\) is a not-on-path non-winner at \(v_i\), and the current winner is not a critical diffusion node of \(i\), then at \(v'_i\) it may continue to be a non-winner or may become the winner via mode (a) explained above. In both cases, Eqn. (15) is satisfied.

If \(i\) is a on-path non-winner at \(v_i\), then it lives on the critical path from the seller to the current winner. At \(v'_i\), it can either stay on-path non-winner, or can become the winner via mode (b) explained above. In both cases, Eqn. (15) is satisfied.

Since, the chosen \(v_i\) and \(v'_i\) are arbitrary, we conclude that IDM satisfies condition 1 of Theorem 1.

Part 2: IDM satisfies Eqn. (3) (one part of Condition 2) of Theorem 1: Given that the allocation rule is fixed, we need to show that the payment under IDM satisfies Eqn. (3). To show this, we need to set the VIPC terms for every agent. We set VIPC\(_i\) = 0, for all \(i\) that are not-on-path non-winner agents. Apart from these nodes, all other nodes live on the CDS. Hence, for each \(i\) on the CDS, i.e., the on-path non-winner nodes and the winner, VIPC\(_i\) = \(v^*_{d_i} - v^*_{d_i+1}\)\(\setminus\{i\}\). Note that, \(v^*_{d_i}\) is defined as the highest bid in the network when the diffusion subnetwork starting from \(i\) is removed and \(v^*_{d_i+1}\)\(\setminus\{i\}\) is the highest bid in the network excluding agent \(i\) and the diffusion subnetwork starting from the agent next to \(i\) in the CDS. Both these terms are independent of the valuation of \(i\).

The allocation probability of all agents except the winner is zero. Therefore, we need to show that for the on-path non-winner and not-on-path non-winner agents the payments under IDM equal the corresponding VIPCs.

For every not-on-path non-winner agent \(i\), this is straightforward, since \(p_i^{\text{IDM}} = 0\) and VIPC\(_i\) = 0.

For each on-path non-winner agent \(i\), \(p_i^{\text{IDM}} = v^*_{d_i} - v^*_{d_i+1}\). Since \(i\) is not the winner, \(v_i \neq v^*_{d_i+1}\), which implies \(v^*_{d_i+1} = v^*_{(-d_i+1)\setminus\{i\}}\). Hence, \(p_i^{\text{IDM}} = v^*_{d_i} - v^*_{(-d_i+1)\setminus\{i\}} = \text{VIPC}_i\).
For the winner, \( p_i^{\text{IDM}} = v^*_d \). From the payment expression of DDSIC auctions (Eqn. (3)) and our choice of VIPC, we have:

\[
p_i(f^G((v_i, r_i), \hat{\theta}_i)) = v_i g_i(f^G((v_i, r_i), \hat{\theta}_i)) - \int_0^{v_i} g_i(f^G((y, r_i), \hat{\theta}_i)) dy
\]

\[
= \text{VIPC}_i + v_i g_i(f^G((v_i, r_i), \hat{\theta}_i)) - \int_0^{v_i} g_i(f^G((y, r_i), \hat{\theta}_i)) dy
\]

\[
= \text{VIPC}_i + v_i g_i(f^G((v_i, r_i), \hat{\theta}_i)) - \int_0^{k_i} g_i(f^G((y, r_i), \hat{\theta}_i)) dy - \int_{k_i}^{v_i} g_i(f^G((y, r_i), \hat{\theta}_i)) dy
\]

where \( k_i \) is the critical bid where \( i \) starts becoming the winner,

\[
= \text{VIPC}_i + v_i \cdot 1 - \int_0^{k_i} 0 \cdot dy - \int_{k_i}^{v_i} 1 \cdot dy = \text{VIPC}_i + k_i
\]

\[
= v^*_d - v^*_{(-d_i+1) \setminus \{i\}} + k_i
\]

substituting the expression of VIPC,

\[
= v^*_d.
\]

The last equality holds because an agent \( i \) in the CDS wins only when its valuation crosses \( v^*_{(-d_i+1) \setminus \{i\}} \).

Hence, for all three types of agents, we have proved that the payment under IDM follows Eqn. (3).

**Part 3: IDM satisfies Eqn. (2) (the other part of Condition 2) of Theorem 1:** The chosen VIPCs in the previous section should also satisfy Eqn. (2) to make IDM DDSIC. We show that in this section for each of the three types of agents in IDM.

Observe that in IDM, the *not-on-path non-winner* agents with their current bids, remain *not-on-path non-winner* agents whether they choose to diffuse or not. Also, their payments remain zero under both conditions and these payments are equal to their VIPCs (which are zeros) as proved in the previous section. Therefore, the terms \( p_i(f^G((0, r'_i), \hat{\theta}_{-i})) \) and \( p_i(f^G((0, r_i), \hat{\theta}_{-i})) \) are zeros for each *not-on-path non-winner* agent \( i \). Hence, Eqn. (2) is trivially satisfied for these agents.

For an arbitrary *on-path non-winner* agent \( i \), we consider its utility when it forwards to all its neighbors

\[
v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = -\text{VIPC}_i = v^*_{-d_i+1} - v^*_d \tag{16}
\]

**Case (a):** If \( i \) becomes the *winner* without diffusing to all its neighbors, then according to IDM’s allocation and payment policy, its utility becomes

\[
v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = 1 \cdot v_i - v^*_d \tag{17}
\]

Note that \( v_i < v^*_d \), since otherwise \( i \) would have become the *winner* even while diffusing. Hence from Eqns. (16) and (17), we get

\[
v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) > v_i g_i(f^G((v_i, r'_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r'_i), \hat{\theta}_{-i}))). \tag{18}
\]

Expanding the payments according to Eqn. (3) on both the sides of the above equation, we get

\[
p_i(f^G((0, r'_i), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) > \int_0^{v_i} (g_i(f^G((y, r'_i), \hat{\theta}_{-i})) - g_i(f^G((y, r_i), \hat{\theta}_{-i}))) dy \tag{19}
\]

which satisfies Eqn. (2).

**Case (b):** The other possibility is that an *on-path non-winner* agent \( i \) may be excluded from the CDS if it decides to not diffuse to all its neighbors. This can happen when \( i \)’s own bid is not high enough to become the highest in the rest of the network. Therefore, then \( i \) becomes a *not-on-path non-winner* agent and its utility becomes zero. However, it is easy to see that in such a case as well Eqns. (18) and (19) follow.
With such similarity, it is easy to see that \( TNM \) satisfies the criteria that it is the highest bidder when its ‘threshold neighbourhood’ and all the generalizes is almost similar to \( TNM \). Similarly, \( TNM \) generalizes \( IDM \) via expanding the set \( d_i \) to \( z_i \), which is defined as the set of nodes that come next to agent \( i \) in all possible paths from \( i \) to \( i + 1 \) where \( i \) and \( i + 1 \) are two sequential nodes in CDS. The formal definition is available in [15]. Similarly, \( d_{z_i} \) is defined as the set containing \( z_i \) and all the descendants of \( z_i \), i.e., the set of all such nodes who would not have participated in the auction had none in \( z_i \) diffused the information.

In \( TNM \), a node \( i \) in the CDS is declared winner when its valuation \( v_i = v^*_{z_i} \). The winner in \( TNM \) satisfies the criteria that it is the highest bidder when its ‘threshold neighbourhood’ and all the descendant nodes of that threshold neighbourhood are not participating. Hence, similar to \( IDM \), the winner’s payment becomes \( v^*_{z_i} \), while that for \( IDM \) is \( v^*_{d_i+1} \).

With such similarity, it is easy to see that \( TNM \) can be proved to be DDSIC by adopting a similar proof technique of Section 5.2 that proves \( IDM \) is DDSIC. All the arguments follow for \( TNM \) by replacing all the \( d_{i+1} \) terms by \( d_{z_i} \). We omit the details since it has already been presented in Section 5.2.

### 6 Optimal Auction on a Tree

The optimal auction is the one that maximizes the expected revenue. This is done assuming that the prior of the valuations are known to the auctioneer, which is a common assumption in classical auction literature [17, e.g.]. In this section, we consider the revenue-optimal auction where the prior distribution over \( (v_i, v_{-i}) \) is given by \( P \) and is a common knowledge. We also assume that the underlying network connecting the agents is a directed tree with the seller \( s \) as the root. We discuss how the results on this special type of network can be extended to arbitrary directed graphs. However, the formal analysis of such a setup is left as a future exercise. First, we define the notion of truthfulness in the prior-based setup.

**Definition 6 (Diffusion Bayesian Incentive Compatibility)** A DA on a graph \( G \) is diffusion Bayesian incentive compatible (DBIC) if

1. every agent’s expected utility is maximized by reporting her true valuation irrespective of the diffusion status of herself and the other agents, i.e., for every \( i \in N \), \( \forall r_i, \hat{r}_{-i} \), the following holds

\[
\mathbb{E}_{v_{-i}} | v_i g_i \left( f^G((v_i, r_i'), (v_{-i}, \hat{r}_{-i})) \right) - p_i \left( f^G((v_i, r_i'), (v_{-i}, \hat{r}_{-i})) \right) | \\
\geq \mathbb{E}_{v_{-i}} | v_i g_i \left( f^G((v_i', r_i'), (v_{-i}, \hat{r}_{-i})) \right) - p_i \left( f^G((v_i', r_i'), (v_{-i}, \hat{r}_{-i})) \right) | , \quad \forall v_i, v_i' \subseteq r_i, \text{ and,}
\]

2. for every true valuation, every agent’s expected utility is maximized by diffusing to all its neighbors irrespective of the diffusion status of the other agents, i.e., for every \( i \in N \), \( \forall r_i, \hat{r}_{-i} \), the following holds

\[
\mathbb{E}_{v_{-i}} | v_i g_i \left( f^G((v_i, r_i), (v_{-i}, \hat{r}_{-i})) \right) - p_i \left( f^G((v_i, r_i), (v_{-i}, \hat{r}_{-i})) \right) | \\
\geq \mathbb{E}_{v_{-i}} | v_i g_i \left( f^G((v_i', r_i'), (v_{-i}, \hat{r}_{-i})) \right) - p_i \left( f^G((v_i', r_i'), (v_{-i}, \hat{r}_{-i})) \right) | , \quad \forall v_i, v_i' \subseteq r_i.
\]

It is easy to see that DDSIC implies DBIC since DBIC requires Conditions 1 and 2 of Def. 2 to hold only in expectation.

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4This assumption is primarily due to two reasons: (a) for prior-free auctions, the worst-case revenue can be arbitrarily bad, hence revenue maximization does not yield any useful result, and (b) in practice, the prior on the users’ valuation can be estimated from the historical data.
6.1 Characterization of DBIC Mechanisms

Our first result is to characterize the DBIC auctions. For convenience, we define the expected allocation and payments with the short notation as described below.

\[ \alpha_i((v_i, r_i'), \hat{r}_{-i}) = \mathbb{E}_{v_{-i}|v_i} [g_i(f^G((v_i, r_i'), (v_{-i}, \hat{r}_{-i})))] \quad \text{(allocation)} \tag{20} \]

\[ \text{pay}_i((v_i, r_i'), \hat{r}_{-i}) = \mathbb{E}_{v_{-i}|v_i} [p_i(f^G((v_i, r_i'), (v_{-i}, \hat{r}_{-i})))] \quad \text{(payment)} \tag{21} \]

In the Bayesian setup, the notion of participation guarantee is also weakened to interim individual rationality (IIR) where the expected utility of a player to join the mechanism is non-negative after she learns her own type.

Definition 7 (Interim Individual Rationality) A DA \((g, p)\) on a graph \(G\) is interim individually rational (IIR) if \(v_i \alpha_i((v_i, r_i), \hat{r}_{-i}) - \text{pay}_i((v_i, r_i), \hat{r}_{-i}) \geq 0\), \(\forall v_i, \theta_{-i}, r_i, \forall i \in N\), where \(r_i\) is the true neighbor set of \(i\).

Similar to DBIC, it is easy to see that IR implies IIR since IIR requires the conditions of Def. 4 to hold only in expectation. Next, we define the following payment structure to succinctly characterize the DBIC auctions.

Definition 8 (Forwarding-Friendly Expected Payments) For a given network \(G\), a DA \((g, p)\) has forwarding-friendly expected payments (FFEP) \(\pi\) if, for the given allocation function \(\alpha\), the payment \(\text{pay}_i\), for each player \(i \in N\) is such that, for every \(v_i, r_i\), and \(\hat{r}_{-i}\), the following two conditions hold.

(a) The values of \(\text{pay}_i((0, r_i'), \hat{r}_{-i})\) and \(\text{pay}_i((0, r_i), \hat{r}_{-i})\) are arbitrary real numbers that satisfies the following inequality for every \(r_i' \subseteq r_i\).

\[ \text{pay}_i((0, r_i'), \hat{r}_{-i}) - \text{pay}_i((0, r_i), \hat{r}_{-i}) \geq \int_0^{v_i} (\alpha_i((y, r_i'), \hat{r}_{-i}) - \alpha_i((y, r_i), \hat{r}_{-i})) \, dy \tag{22} \]

(b) For every \(r_i' \subseteq r_i\), the following payment payment formula is satisfied.

\[ \text{pay}_i((v_i, r_i'), \hat{r}_{-i}) = \text{pay}_i((0, r_i'), \hat{r}_{-i}) + v_i \alpha_i((v_i, r_i'), \hat{r}_{-i}) - \int_0^{v_i} \alpha_i((y, r_i'), \hat{r}_{-i})) \, dy \tag{23} \]

Theorem 4 (DBIC Characterization) A DA \((\alpha, \pi)\) is DBIC if and only if the following two conditions hold:

1. for every \(i \in N\) and \(r_i, \hat{r}_{-i}\), the functions \(\alpha_i((v_i, r_i'), \hat{r}_{-i})\) is non-decreasing in \(v_i\), for every \(r_i' \subseteq r_i\).

2. the payments are FFEP (Def. 8).

Proof sketch: The proof of this theorem is identical to that of Theorem 1 with the allocations and payments, \(g_i\) and \(p_i\), replaced with their expected versions, \(\alpha_i\) and \(\text{pay}_i\) (Eqns. (20) and (21)), respectively. We skip rewriting the identical steps with the above-mentioned substitution.

6.2 A superclass of LbLEV

We can naturally extend the mechanism LbLEV to a more general structure where it retains the level-by-level property, but determines the tentative winner at every level through a general deterministic allocation rule \(g\) which is monotone non-decreasing and runs only on the agents at a given level. Define the corresponding payment as

\[ \rho_i(\rho_{-i}) = \rho_i g_i(\rho_{-i}) - \int_0^{\rho_i} g_i(y, \rho_{-i}) \, dy \tag{24} \]

We note that the payment formula in Eqn. (24) is the same as the payment formula in the classical result of Myerson [17] with the VIPC term being zero. For LbLEV, at each level, we were using a specific choice of \(g\) and the corresponding payment with VIPC term being zero. We call this class of mechanisms LbLEV-Gen, which is also described algorithmically in Alg. 2. An instance of this class is represented by the choice of the allocation function \(g\), which naturally subsumes LbLEV.
Algorithm 2: LbL-Gen

Input: reported types $\hat{\theta}_i = (\hat{v}_i, \hat{r}_i), \hat{r}_i \subseteq r_i, \forall i \in N$
Output: winner of the auction (which can be $\emptyset$), payments of each agent

1 Preprocessing: Let $G$ be the subgraph induced from $\hat{r}_i, i \in N$, on the set of vertices. Let $\hat{T}$ be an arborescence of $G$ rooted at the seller $s$. Pick an arbitrary monotone non-decreasing deterministic allocation $g$
2 if $\hat{v}_i = 0, \forall i \in N$ then
  Item is not sold and payment is set to zero for all agents, STOP
4 Initialization: all agents are non-winners and their actual payments are zeros, set $\text{offset} = 0$, level = 1, parent = $s$, $v_{parent} = 0$
5 In this level of $\hat{T}$:
6 for each node $i \in \text{children(parent)}$ do
7   Set effective valuation $\rho_i := \max\{\hat{v}_j : j \in \hat{T}_i\} - \text{offset}$
8   Remove the nodes that have $\rho_i < 0$, denote the rest of the agents with $N_{\text{remain}}$
9   if $N_{\text{remain}} \neq \emptyset$ then
10      Find $i^*$ where $g_{i^*}(\rho_i, \rho_{N_{\text{remain}} \setminus \{i^*\}}) = 1$
11      Compute $z := g_{i^*}(\rho_i, \rho_{N_{\text{remain}} \setminus \{i^*\}})$, given by Eqn. (24)
12   else
13      Set $z = 0$
14     if $v_{\text{parent}} \geq \text{offset} + z$ then
15        STOP and go to Step 23
16    Set agent $i^*$ as the tentative winner and its effective payment to be $z$
17    All nodes and their subtrees except $i^*$ are declared non-winners
18    The actual payment of $i^*$ to parent = effective payment + offset
19    parent = $i^*$, offset = actual payment of $i^*$
20    level = level + 1
21 Repeat Steps 5 to 20 with the updated parent and offset for the new level
22 STOP when no agent $i$ has $\rho_i \geq 0$ OR the leaf nodes are reached
23 Set tentative winner as final winner; final payments are the actual payments that are paid to the respective parents of $\hat{T}$

Is LbL-Gen too restrictive? In the world without networks, i.e., when the information of the auction is available to all the agents to begin with, we know that the revenue-optimal auction picks the agent with maximum virtual valuation (a term that is formally defined in the following section) and the payments have a specific integral formula [17]. However, in a network auction, the fundamental difference is that the agents who are critical for information diffusion should be adequately incentivized to diffuse the information. The auction, therefore, has to give ‘commissions’ (or ‘fees’) to such forwarding nodes to ‘match-up’ to their utilities when they do not forward, and can only earn the residual payment as revenue. This observation intuitively hints that the revenue to the seller will be determined by the residual payments coming to the seller from its immediate neighbors (irrespective of how the commissions are distributed among the intermediate nodes). The LbL-Gen class precisely provides that minimal commission to the forwarding nodes when the underlying network is a tree. The intuition that only the first-level of nodes are relevant for revenue is also formalized in the next section. Hence, we feel that the class of LbL-Gen is quite general for revenue maximization on trees with i.i.d. valuations, though we leave a formal proof of this intuition as a future exercise.

We show that each member of LbL-Gen also follows the desirable properties like LbLEV. Since the proof is quite similar to that of LbLEV, we provide the sketch to show exactly the places where the proof differs.

Theorem 5 LbL-Gen is DDSIC and IR.

Proof sketch: The proof follows an identical line of arguments as Theorems 2 and 3, with a few changes. We follow the same definitions of on-path non-winner, not-on-path non-winner, and winner for the different types of agents, and use the terms offset, children, and parent as defined there.
The winner function is updated as \( \text{winner}(i) = \arg_{j \in \text{children}(i)} \{ q_j(\rho_j, \rho_{\text{remains}} \setminus \{j\}) = 1 \} \), and there is no runnerup function.

**Part 1: LbL-Gen satisfies Condition 1 of Theorem 1:** This part follows by the same arguments and the fact that \( q \) is monotone non-decreasing.

**Part 2 and 3: LbL-Gen satisfies Eqs. (2) and (3) (Condition 2) of Theorem 1:** These two arguments ensure the payment formula and the condition on the VIPC terms. The same conditions can be obtained with the same set of arguments by replacing \( \rho^{t_i/t_k}_i \) with \( \inf \{ \rho_i \in \mathbb{R} : q_i(\rho_i, \rho_{\text{remains}} \setminus \{i\}) = 1 \} \) at every level of the arborescence. Since \( \rho_i \) is obtained by subtracting \( \text{offset}(\text{parent}(i)) \) from agent \( i \)'s valuation and that agent is removed if this number is negative, \( \rho_i \)'s are non-negative by design. Hence, the number \( \inf \{ \rho_i \in \mathbb{R} : q_i(\rho_i, \rho_{\text{remains}} \setminus \{i\}) = 1 \} \) is also non-negative. Therefore, this number follows every argument that \( \rho^{t_i/t_k}_i \) followed at every level of the proof of Theorem 2 in an identical way.

Collecting these three parts, we prove that LbL-Gen is DDSIC.

Similarly, the IR proof follows an identical set of arguments follow with the same substitution of \( \rho^{t_i/t_k}_i \) with \( \inf \{ \rho_i \in \mathbb{R} : q_i(\rho_i, \rho_{\text{remains}} \setminus \{i\}) = 1 \} \) at every level of the arborescence. We skip rewriting the identical steps with the above-mentioned substitution.

Since LbL-Gen is DDSIC and IR, it is DBIC and IIR.

In what follows, we will find a mechanism in the LbL-Gen class of mechanisms which maximizes the expected revenue of the seller when the underlying graph is a tree and the valuations of the buyers are i.i.d. The high-level idea of our proof is the following. We observe from Eqn. (24) that the revenue of the seller in any mechanism in LbL-Gen is simply the sum of the payments made by the buyers at the first level, when their valuations are replaced with their effective valuations, i.e., the maximum valuation in their subtree. This allows us to "replace" each buyer at the first level, with the buyer having a maximum valuation in its subtree. We formalize this idea in the proof of Theorem 6 and Lemma 1. It seems to us that the idea may be formalized for general directed networks (not necessarily trees). We discuss this point in §6.4.

### 6.3 Optimal mechanism for i.i.d. valuations

In the objective of finding the revenue-optimal mechanism on a network, we address the problem in steps. We first consider the mechanisms in the class LbL-Gen, assuming that the priors on the valuations are known to the designer and that all \( v_i \)'s are i.i.d. with distribution \( F \). We also assume that this distribution follows the monotone hazard rate (MHR) condition, i.e., \( f(x)/(1 - F(x)) \) is non-decreasing in \( x \).\(^5\) Since the underlying graph is a directed tree, the arborescence \( \hat{T} \) generated by LbL-Gen from the reported \( r_i \)'s is a unique subtree of the original tree. We assume that the sellers has a directed edge from itself to all its neighbors in this arborescence, i.e., seller always forwards to all its neighbors. In this section, we find the optimal mechanism for this arborescence. To do that, first, we need to define a transformed auction (TA) of an LbL-Gen as follows.

**Definition 9 (Transformed Auction)** A transformed auction (TA) of an LbL-Gen on an arborescence \( \hat{T} \) is the mechanism where each subtree \( \hat{T}_i, i \in \text{children}(s) \) is replaced with a node with a valuation of \( \max_{j \in \hat{T}_i} v_j \), and the allocation and payments are given by \( (g_i, \rho_i), i \in \text{children}(s) \).

Note that a TA does not specify the allocations beyond the first level of the arborescence. This is because, we will only be interested in the revenue generated by a TA, and every TA, regardless of how it allocates the object and extracts payments in the subsequent levels, will earn the same revenue, as shown formally in the following result.

**Lemma 1** The revenue earned by an LbL-Gen auction is identical to its TA.

**Proof:** Note that in LbL-Gen, the net payment the seller \( s \) receives comes directly from the nodes in the first level of the arborescence. The offset is zero, and the payment is calculated based on the

\(\text{footnote continued}\)

\(^5\)The intuitive meaning of this condition is that the distribution is not heavy-tailed. Many distributions, e.g., uniform and exponential, follow the MHR condition [1].
maximum valuation in the subtree of the agents in the first level. The rest of the payments in the tree are internally adjusted within the nodes and does not reach the seller. Therefore, the total revenue earned by an LbL-Gen auction can be simulated by transforming every first-level nodes with their valuations replaced with the maximum valuation of their subtree and applying \((g, \rho)\) on those nodes. Hence, we have the lemma.

Given the above lemma, we can, WLOG, look only at the TA mechanisms for revenue maximization. In the TA of a given LbL-Gen, the revenue maximization problem is restricted to the first level of the arborescence. However, the nodes of this restricted arborescence are the transformed nodes whose valuations are the maximum valuations of their respective subtrees. For notational simplicity, we use a fresh index \(\ell\) to denote these transformed nodes at the first level, i.e., for \(\text{children}(s)\). The transformed valuation of \(\ell\) is denoted by \(v_\ell := \max_{j \in \ell} v_j\). Again, to reduce notational complexity, the set of the players in this TA is represented by \(\hat{N} := \text{children}(s)\). In the following, we state the fact that the \(v_\ell\)’s also follow the MHR property.

**Fact 1** If the distribution of a finite number of i.i.d. random variables satisfies MHR condition, then the distribution of those random variables also satisfies MHR condition.

Proof: Suppose, there are \(n\) i.i.d. random variables given by \(X_1, X_2, \ldots, X_n\), and their distribution is given by \(F\). Let \(Y := \max\{X_1, X_2, \ldots, X_n\}\). It is given that \(F\) satisfies MHR condition, i.e., \(f(x)/(1 - F(x))\) is monotone non-decreasing. Denote the distribution and density of \(Y\) by \(\hat{F}\) and \(\hat{f}\) respectively. Now,

\[
\hat{F}(x) = P(Y \leq x) = P(\max\{X_1, X_2, \ldots, X_n\} \leq x) \\
= P(\cap_{i=1}^{n}\{X_i \leq x\}) = \prod_{i=1}^{n} P(X_i \leq x) = F^n(x).
\]

Hence, \(\hat{f}(x) = nF^{n-1}(x)f(x)\).

Therefore,

\[
\frac{\hat{f}(x)}{1 - \hat{F}(x)} = \frac{nF^{n-1}(x)f(x)}{1 - F^n(x)} = \frac{f(x)}{1 - F(x)} \cdot \left(1 + \frac{1}{F(x)} + \frac{1}{F^2(x)} + \ldots + \frac{1}{F^{n-1}(x)}\right).
\]

Since \(F(x)\) is non-decreasing, \(1/F(x)\) is non-increasing. Hence, the denominator of the last term in the last expression is also non-increasing, leading the expression to be non-decreasing. Since \(\frac{f(x)}{1 - F(x)}\) is non-decreasing as well, we conclude that \(\frac{f(x)}{1 - F(x)}\) is also non-decreasing and hence \(Y\) satisfies MHR.

We now focus on the revenue maximization problem. Note that, \(g\) and \(\rho\) are particular choices of the allocations \(g\) and \(\rho\) respectively. Therefore, the expected allocations and payments are given by Eqns. (20) and (21) with \(g\) and \(\rho\) replaced with \(g\) and \(\rho\) respectively. In particular, the VIPC term in Eqn. (21) is zero for the nodes in the TA since the payment \(\rho\) sets it to zero for the nodes in the first level of the class LbL-Gen. Also, in the TA, the offset is zero. Therefore, \(\rho_\ell = v_\ell\), \(\forall \ell \in \hat{N}\).

The neighbor component of the types \(r_\ell\) are no longer relevant since the mechanism is restricted to the first level in the TA. Hence, we can reduce the arguments of \(\text{pay}_\ell\) and \(\alpha_\ell\) to only \(v_\ell\) in Eqns. (20) and (21). Since, the only variable parameter in the payment of the agents is the allocation function \(g\), the optimization problem for revenue maximization in the LbL-Gen class is given by

\[
\max \sum_{\ell \in \hat{N}} \int_{v_\ell=0}^{b_\ell} \text{pay}_\ell(v_\ell) f_\ell(v_\ell) \, dv_\ell \\
\text{s.t. } g = g, \text{ where } g \text{ is monotone non-decreasing and deterministic}
\]

In the above equation, \(f_\ell\) is the density of \(v_\ell\), which is assumed to have a bounded support of \([0, b_\ell]\). We will denote the corresponding distribution with \(F_\ell\). This optimization problem now reduces to

\[\text{Fact 1}\] The VIPC term needs to be non-positive for the auction to be IIR, and since our objective is to maximize revenue, it must be zero. This is ensured by \(\rho_\ell\).
When we migrate away from i.i.d. valuations, it is not clear if the nodes in the TA satisfy MHR or a relatively weaker condition of regularity (which only requires the virtual valuations to be non-decreasing). Hence, the revenue maximization problem becomes far more challenging.

The classic single item auction setting of Myerson [17]. Following that analysis, we find that the individual terms in the sum of the objective function of Eqn. (25) can be written as follows

\[
\int_{v_\ell=0}^{b_\ell} \text{pay}_\ell(v_\ell) f_\ell(v_\ell) \, dv_\ell
\]

\[
= \int_{0}^{b_\ell} w_\ell(v_\ell) \alpha_\ell(v_\ell) f_\ell(v_\ell) \, dv_\ell
\]

\[
= \int_{0}^{b_\ell} w_\ell(v_\ell) \left( \int_{v_{\ell-}} g_\ell(v_\ell, v_{\ell-}) f_{\ell-}(v_{\ell-}) \, dv_{\ell-} \right) f_\ell(v_\ell) \, dv_\ell
\]

\[
= \int_{v} w_\ell(v_\ell) g_\ell(v_\ell, v_{\ell-}) f(v) \, dv.
\]

The expression \( w_\ell(x) := x - (1 - F_\ell(x))/f_\ell(x) \) is defined as the virtual valuation of agent \( \ell \) and for completeness, the derivation of the first equality is provided in the appendix. The second equality holds after expanding \( \alpha_\ell(v_\ell) \) from Eqn. (20). The last equality holds since the valuations are independent (but may not be identically distributed as the number of nodes in the subtree of \( \ell \) can be different from that of \( \ell' \)), and \( f \) denotes the joint probability density of \( (v_\ell, v_{\ell-}) \).

The objective function of Eqn. (25) can therefore be written as

\[
\int_{v} \left( \sum_{\ell \in \hat{N}} w_\ell(v_\ell) g_\ell(v_\ell, v_{\ell-}) \right) f(v) \, dv.
\]

The solution to the unconstrained version of the optimization problem given by Eqn. (25) is rather simple.

\[
\text{if } w_\ell(v_\ell) < 0, \forall \ell \in \hat{N}, \text{ then } f_\ell(v_\ell) = 0, \forall \ell \in \hat{N}
\]

\[
\text{else } g_\ell(v_\ell, v_{\ell-}) = \begin{cases} 
1 & \text{if } w_\ell(v_\ell) \geq w_k(v_k), \forall k \in \hat{N} \\
0 & \text{otherwise}
\end{cases}
\]

(26)

The ties in \( w_\ell(v_\ell) \) are broken arbitrarily. Since the distributions of \( v_\ell, \ell \in \hat{N} \) satisfy MHR, the virtual valuations, \( w_\ell \), are monotone non-decreasing. Also, since this mechanism breaks the tie arbitrarily in favor of an agent, the allocation is also deterministic. Therefore, the optimal solution of the unconstrained problem of Eqn. (25) also happens to be the optimal solution of the constrained problem. We find the payments of the winner from Eqn. (24) as follows.

\[
\text{define } \kappa^*_\ell(v_{\ell-}) = \inf \{ y : g_\ell(y, v_{\ell-}) = 1 \},
\]

\[
\rho_\ell(v_\ell, v_{\ell-}) = \kappa^*_\ell(v_{\ell-}) \cdot g_\ell(v_\ell, v_{\ell-}),
\]

(27)

where \( \kappa^*_\ell(v_{\ell-}) \) is the minimum valuation of agent \( \ell \) to become the winner. Formally, we define the auction as follows.

**Definition 10 (Maximum Virtual Valuation Auction (maxViVa))** The maximum virtual valuation auction is a subclass of LbL-Gen, where the TAs of that subclass follow the allocation and payments given by Eqns. (26) and (27) respectively.

We consolidate the arguments above in the form of the following theorem.

**Theorem 6** For a tree with agents having i.i.d. MHR valuations, the revenue-optimal LbL-Gen auction is maxViVa.

Note that neither IDM nor LbLEV is maxViVa because they do not use any priors. Therefore, the revenue-maximizing auction in this setting is a new class of mechanisms that has not been explored in the literature.

### 6.4 Extension to non-i.i.d. agents and general graphs

When we migrate away from i.i.d. valuations, it is not clear if the nodes in the TA satisfy MHR or a relatively weaker condition of regularity (which only requires the virtual valuations to be non-decreasing). Hence, the revenue maximization problem becomes far more challenging.
We provide an experimental study in the next section that shows that if the i.i.d. assumption does not hold, a special auction from the LbLEV class can yield more revenue than the currently known network auctions.

To extend our results to general graphs with i.i.d. valuations, the difficulty is to find the optimal arborescence, since LbL-Gen does not specify any arborescence when there are multiple possibilities. The optimal revenue problem given by Eqn. (26) will have an arborescence $T$ as an argument for the payment and the density functions and an additional constraint that the arborescence $T$ is obtained from a given directed graph $G$. Since there are many possible $T$ from a given $G$, the one that yields the optimal revenue is non-trivial and may not be computationally tractable. We address this problem in our future work.

7 Designing LbLEV for Improving Revenue

In the previous section, we have investigated the case of i.i.d. valuations. In this section, we show that if the valuations are non-i.i.d., the revenue maximizing auction does not have the same structure as $\text{max} V_i V_a$.

LbLEV generates a class of mechanisms (Alg. 1) for different choices of the exponents $t \in \mathbb{R}^n_{>0}$ and arborescences $\hat{T}$. While each mechanism in that class is DDSIC and IR, we can anticipate that certain choices may lead to a higher revenue collected by the auction. In this section, we test that hypothesis and find out how the exponents $t$ can be designed such that it improves the revenue over the other DDSIC and IR auctions known in the literature. The revenues are compared for the same arborescence $\hat{T}$ for all competing auctions.

Revenue maximization in auction is typically done in a prior-based approach [8, 16, 17]. Such approaches assume that the distribution of the types are known by the mechanism designer from its past interactions with the agents. We adopt a similar prior-based approach. In the context of network auctions, the types of the agents consist of the valuations and their neighbors’ set and we assume a prior on both.

Other DDSIC mechanisms to compare. Guo and Hao [7] provide a comprehensive survey of the auctions on the network. TNM and CDM reduce to IDM when restricted to a tree. The objective of FDM and NRM is to redistribute the revenue so that there is little surplus – hence, they are unsuitable for a revenue comparison. Other mechanisms like MLDM, CSM, and WDM apply to very specific settings, e.g., distribution markets with intermediaries, economic networks, and weighted networks respectively. Hence, we find that IDM is the sole candidate to compare with LbLEV.

Two-stage tree generation model. For the experiments, we need to iterate over randomly generated trees connecting the agents, and also equip the mechanism designer with some prior information about the connections model. To do so, we adopt a two-stage tree generation model, where the first stage is observable by the designer, but the second stage is not. It resembles the situation: the designer knows who can probably be the children of which nodes, but cannot deterministically observe it while designing the mechanism.

The mechanism LbLEV extracts an arborescence $\hat{T}$ from the graph generated through the reported $r_i$’s. For our experiments, we create the $r_i$’s in two stages. First, we generate a base tree in the following way. We fix the number of nodes $n$ in this tree. Starting from the seller, which is not a part of the agent set, a random children set is picked for each node at every level of the tree where the size of the children set is drawn uniformly at random between 1 and $\lfloor n/3 \rfloor$ from the rest of agents without replacement. This process is continued until all the $n$ nodes are exhausted (the last parent node gets the remaining number of nodes when it draws more than that remaining number).

The second stage probabilistically activates the edges on this base tree. The activated set of edges provides a sub-tree of the base tree, and this is considered to be the final tree $\hat{T}$. More concretely, once the base tree is realized, the actual set $r_i$ is generated by tossing a coin with probability drawn from Beta(5, 1) for each edge from $i$ to its children. We assume that the second stage of randomization is not part of the prior information available to the mechanism designer, and hence, the choice of the exponent vector $t$ cannot depend on it. The second stage, therefore, helps us to cross-validate the mechanism designed from the prior information.
Valuation generation model. The valuations are drawn independently from $\mathcal{N}(\mu, \sigma^2)$. We assume that there are three classes of agents: high, medium, and low, having $\mu$ to be 100, 70, and 50 respectively, and the same $\sigma$. We will see the effect of $\sigma$ on the revenue in our experiments.

The prior information for the designer consists of the first stage of the tree generation process and the valuation generation process.

Setting the exponent vector $t$. Suppose, we knew that the nodes $i^*$ and $\ell$ are the first level nodes whose subtrees are the winner and runnerup respectively in LbLEV. From Alg. 1, we know that the effective valuations $\rho_i$ of a first level node $i$ in the tree $\hat{T}$ is the maximum valuation of the nodes in the subtree of $i$, i.e., $\hat{T}_i$. Note that, the revenue generated in LbLEV is $\rho_i^{t_i/t_i^*}$. Hence, a larger exponent ratio $t_i/t_i^*$ yields a better revenue as long as $\rho_i^{t_i/t_i^*} \leq \rho_{i^*}$. This is the driving philosophy of the following choice of $t$ with only the prior information.

On the base tree, we replace the nodes’ valuations with the means, which is a prior information, and call the node having the highest and second highest $\rho_i$’s in the first level of the base tree to be the expected winner and runnerup respectively. Suppose, these two mean valuations are $w_{winner}^e$ and $w_{runnerup}^e$ respectively. We set the exponent of the first level expected runnerup is set to $(1 - \lambda) \cdot 1 + \lambda \cdot \frac{\log w_{winner}^e}{\log w_{runnerup}^e}$, with $\lambda$ being a parameter chosen by the mechanism. This is a convex combination between 1, which is the exponent ratio for IDM, and the other extreme $\frac{\log w_{winner}^e}{\log w_{runnerup}^e}$. If the true winner and the runnerup would have indeed come from the subtrees of the expected winner and runnerup, then the exponent ratio $\frac{\log w_{winner}}{\log w_{runnerup}}$ would have extracted the maximum revenue in LbLEV. This is the intuition of using this factor as a candidate for the expected runnerup’s exponent.

The exponents of all other agents (including the expected winner) are set to 1. Note that $t$ is decided based on the first stage of the tree generation process and the prior of the valuation. It is independent of the second level of the tree generation process, and therefore, is agnostic of the actual tree $\hat{T}$. Such a $t$ is independent of the agents’ actions and is consistent with Alg. 1. Indeed there is a possibility that a probabilistic draw of the second stage of the tree generation process may have a different winner and runnerup than their expected ones, which makes this choice of $t$ sub-optimal than IDM for revenue.

First set of experiments. In this set of experiments, we find the effect of the three parameters, $n$, $\sigma$, and $\lambda$, on the revenue of the two auctions: (1) LbLEV with the chosen exponent vector $t$ as above and (2) IDM. We consider one agent from class high, $\lceil (n - 1)/2 \rceil$ agents from class medium, and $\lceil (n - 1)/2 \rceil$ agents from class low. This choice is to observe how the exponents $t$ make a difference in the revenue earned. If there are many agents of class high, then it is highly probable that both the expected winner and the runnerup in the first level of the base tree has the same mean valuation, which makes the optimal exponent to be unity – same as IDM. In the first experiment, we consider different values for $\sigma$ and $n$, and compare the revenue earned by LbLEV and IDM with varying $\lambda$. The results are shown in Fig. 4. For every $\lambda$, the base graph and the edge-activation probability (drawn from Beta(5,1)) generation have been repeated 100 times, and for each of such instances the edge activation and valuation generation for all agents have been repeated 100 times. The plot shows the mean percentage improvement of the revenue of LbLEV over IDM, with the standard error around it. Observe that, for every pair of $\sigma$ and $n$, there is an optimal convex combination (say $\lambda^*$) for which

![Figure 4: Percentage increase in revenue for LbLEV over IDM w.r.t. $\lambda$ for different $\sigma$ and $n$.](image)
Figure 5: Percentage increase in revenue for LbLEV over IDM for different $\sigma$ and learned $\lambda^*$ (and hence $t$).

the revenue gap between LbLEV and IDM reaches a maxima. Recall that this $\lambda^*$ also determines the exponent of the expected runnerup in the base tree.

**Second set of experiments.** The first set of experiments gives us the insight that the optimal convex combination factor $\lambda^*$ depends on $n$ and $\sigma$. Motivated by this observation, in this set of experiments, we run a regression model to learn the optimal $\lambda^*(n, \sigma)$ from several such instances of $(n, \sigma, \lambda^*)$.

We used the random forest regressor [2] with the parameter of number of decision trees set to 100 to learn the $\lambda^*$ function. We chose the random forest regressor for two reasons: (a) from the examples, the function seems non-linear and instead of choosing a fixed non-linear function, an ensemble regressor could perform better, and (b) random forest gave the best performance among the few other ensemble regressors we tested with (e.g., ADABOOST, GradientBoosting). We find that with the learned $\lambda^*$, which yields the exponents $t$, LbLEV performs better than IDM. For certain choices of $(\sigma, n)$, particularly when both $n$ and $\sigma$ are large, the learned exponents are close to 1 for all agents, almost reducing LbLEV to IDM. Fig. 5 shows the results.

8 Summary and Plans of Extension

In this paper, we provided a characterization of randomized truthful single indivisible item auctions on a network. Our results are the network counterpart of the Myerson’s result [17]. We obtained the detailed description of the revenue optimal mechanism in the special setting of trees with i.i.d. MHR valuations in a fairly general class. When i.i.d. assumption does not hold, we provided a mechanism from our characterized class to experimentally show an improvement in the revenue from the currently known diffusion mechanisms. The question of finding the revenue optimal mechanism for a general network is still open and we want to pursue that as a future work.

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Appendix

A Derivation of the virtual valuation

We need to show that

$$\int_0^{b_\ell} \text{pay}_\ell(v_\ell)f_\ell(v_\ell)\,dv_\ell = \int_0^{b_\ell} w_\ell(v_\ell)\alpha_\ell(v_\ell)f_\ell(v_\ell)\,dv_\ell,$$  

(28)

where,

$$\text{pay}_\ell(v_\ell) = v_\ell\alpha_\ell(v_\ell) - \int_0^{v_\ell} \alpha_\ell(y)\,dy,$$  

(29)

and,

$$w_\ell(v_\ell) = v_\ell - \frac{1 - F_\ell(v_\ell)}{f_\ell(v_\ell)}.$$  

(30)

Substituting Eqn. (29) in the LHS of Eqn. (28), we get

$$\int_0^{b_\ell} \text{pay}_\ell(v_\ell)f_\ell(v_\ell)\,dv_\ell$$
\[ \begin{align*}
&= \int_0^{b_\ell} v_\ell \alpha_\ell(v_\ell) f_\ell(v_\ell) \, dv_\ell - \int_0^{b_\ell} \int_0^{v_\ell} \alpha_\ell(y) f_\ell(v_\ell) \, dy \, dv_\ell \\
&= \int_0^{b_\ell} v_\ell \alpha_\ell(v_\ell) f_\ell(v_\ell) \, dv_\ell - \int_0^{b_\ell} \int_y^{b_\ell} \alpha_\ell(y) f_\ell(v_\ell) \, dv_\ell \, dy \\
&= \int_0^{b_\ell} v_\ell \alpha_\ell(v_\ell) f_\ell(v_\ell) \, dv_\ell - \int_0^{b_\ell} \alpha_\ell(y) \left( \int_y^{b_\ell} f_\ell(v_\ell) \, dv_\ell \right) \, dy \\
&= \int_0^{b_\ell} v_\ell \alpha_\ell(v_\ell) f_\ell(v_\ell) \, dv_\ell - \int_0^{b_\ell} \alpha_\ell(y)(1 - F_\ell(y)) \, dy \\
&= \int_0^{b_\ell} \left( v_\ell - \frac{1 - F_\ell(v_\ell)}{f_\ell(v_\ell)} \right) \alpha_\ell(v_\ell) f_\ell(v_\ell) \, dv_\ell.
\end{align*} \]

In the second equality, we interchange the order of integration. The integrable space has the following order: \( y \) varying from \( 0 \rightarrow v_\ell \) and thereafter \( v_\ell \) varying from \( 0 \rightarrow b_\ell \) which is equivalent to \( v_\ell \) varying from \( y \rightarrow b_\ell \) and thereafter \( y \) varying from \( 0 \rightarrow b_\ell \). The third equality holds by taking the \( \alpha_\ell(y) \) term outside the inner integral since it is independent of \( v_\ell \). The next equality holds since the distribution of \( v_\ell \) has the support of \( [0, b_\ell] \), hence at \( b_\ell \), the value of \( F_\ell \) is unity. The last equality is obtained by using the same integration variable for both integrals and rearranging them.