GLOBAL KATO TYPE SMOOTHING ESTIMATES VIA LOCAL ONES FOR DISPERSIVE EQUATIONS

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ABSTRACT. In this paper we show that the local Kato type smoothing estimates are essentially equivalent to the global Kato type smoothing estimates for some class of dispersive equations including the Schrödinger equation. From this we immediately have two results as follows. One is that the known local Kato smoothing estimates are sharp. The sharp regularity ranges of the global Kato smoothing estimates are already known, but those of the local Kato smoothing estimates are not. Recently, Sun, Trélat, Zhang and Zhong [22] have shown it only in spacetime $\mathbb{R} \times \mathbb{R}$. Our result resolves this issue in higher dimensions. The other one is the sharp global-in-time maximal Schrödinger estimates. Recently, the pointwise convergence conjecture of the Schrödinger equation has been settled by Du–Guth–Li–Zhang [11] and Du–Zhang [12]. For this they proved related sharp local maximal Schrödinger estimates. By our result, these lead to the sharp global-in-time maximal Schrödinger estimates.

1. INTRODUCTION

Fix $n \geq 1$ and $m > 1$. Let $\Phi \in C^\infty(\mathbb{R}^n \setminus 0)$ be a real-valued function satisfying the following conditions:

\[
\begin{cases}
|\nabla \Phi(\xi)| \neq 0 & \text{for all } \xi \neq 0, \\
\Phi(\lambda \xi) = \lambda^m \Phi(\xi) & \text{for all } \lambda > 0 \text{ and } \xi \neq 0,
\end{cases}
\]

We are concerned with the solutions $u$ to equations

\[
\begin{cases}
i \partial_t u(t, x) + \Phi(D)u(t, x) = 0 & \text{in } I \times \mathbb{R}^n, \\
u(0, x) = f(x) & \text{in } \mathbb{R}^n,
\end{cases}
\]

where $\Phi(D)$ is the corresponding Fourier multiplier to the function $\Phi$, that is, $\Phi(D)$ is defined by $\Phi(D)f = (\Phi \hat{f})^{\vee}$.

For a function $f$ on $X \times Y$, we define the mixed norm $\|f\|_{L^q_X(L^r_Y)}$ by

\[
\|f\|_{L^q_X(L^r_Y)} := \left( \int_X \left( \int_Y |f(x, y)|^r \, dy \right)^{q/r} \, dx \right)^{1/q}.
\]

Let $B^n$ be the unit ball in $\mathbb{R}^n$ and $I = \{ t \in \mathbb{R} : 1/2 \leq t \leq 2 \}$ be an interval. For $1 \leq q, r \leq \infty$ and $\alpha \in \mathbb{R}$ we use the notations $K_{loc}(L^2_\alpha L^2; \alpha)$ and $K_{loc}(L^2_\alpha L^2; \alpha)$ to denote the local-in-time smoothing estimates

\[
\|\langle D \rangle^\alpha u\|_{L^q_\alpha(B^n; L^2_\alpha(I))} \leq C_{\alpha, m}\|f\|_{L^2(\mathbb{R}^n)}
\]

and

\[
\|\langle D \rangle^\alpha u\|_{L^q_\alpha(B^n; L^2_\alpha(I))} \leq C_{\alpha, m}\|f\|_{L^2(\mathbb{R}^n)}
\]

for all $f \in L^p(\mathbb{R}^n)$, respectively, where $(D)$ is the operator associated with a symbol $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. Similarly we use $K_{glob}(L^2_\alpha L^2; \alpha)$ and $K_{glob}(L^2_\alpha L^2; \alpha)$ to denote the global-in-time smoothing estimates

\[
\|\langle D \rangle^\alpha u\|_{L^q_\alpha(B^n; L^2_\alpha(\mathbb{R}))} \leq C_{\alpha, m}\|f\|_{L^2(\mathbb{R}^n)}
\]

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and
\[
\| (D)^\alpha u \|_{L^2_t(L^q_x)} \leq C_{\alpha,m} \| f \|_{L^2(\mathbb{R}^n)}
\]
for all \( f \in L^p(\mathbb{R}^n) \), respectively. For convenience we denote by \( K_{\text{loc}}(L^q_t; \alpha) := K_{\text{loc}}(L^q_t L^q_x; \alpha) \) and \( K_{\text{glob}}(L^q_t; \alpha) := K_{\text{glob}}(L^q_t L^q_x; \alpha) \).

In this paper we show that the local-in-time smoothing estimates \( K_{\text{loc}}(L^q_t L^q_x; \alpha) \), and \( K_{\text{loc}}(L^q_t L^q_x; \alpha) \) are essentially equivalent to the global-in-time smoothing estimates \( K_{\text{glob}}(L^q_t L^q_x; \alpha) \) and \( K_{\text{glob}}(L^q_t L^q_x; \alpha) \), respectively.

**Theorem 1.1.** Suppose that \( \Phi \) satisfies the condition (1.1). Then,

(i) If \( 2 \leq q \leq r \leq \infty \) then the local-in-time smoothing estimate \( K_{\text{loc}}(L^q_t; \alpha) \) implies the global-in-time smoothing estimate \( K_{\text{glob}}(L^q_t L^q_x; \alpha - \epsilon) \) for all \( \epsilon > 0 \).

(ii) If \( 1 \leq q \leq \infty \) and \( 2 \leq r \leq \infty \) then the local-in-time smoothing estimate \( K_{\text{loc}}(L^q_t L^q_x; \alpha) \) implies the global-in-time smoothing estimate \( K_{\text{glob}}(L^q_t L^q_x; \alpha - \epsilon) \) for all \( \epsilon > 0 \).

The most basic estimate for the solution \( u \) to (1.2) is the energy identity that for any \( t \in \mathbb{R} \),
\[
\| u(t) \|_{L^q(\mathbb{R}^n)} = C \| f \|_{L^2(\mathbb{R}^n)}
\]
for all \( f \in L^2(\mathbb{R}^n) \), where \( u(t) = u(t, \cdot) \). It implies that the solution \( u \) has as much regularity as the initial \( f \) in \( L^2 \)-space. Kato [13] first observed that an integration locally in spacetime makes the solution \( u \) smoother than the initial \( f \), and showed the local smoothing estimate \( K_{\text{loc}}(L^2_t; 1) \) for the KdV equation. Later, Constantin–Saut [8], Sjölin [21] and Vega [24] independently proved the local smoothing estimates \( K_{\text{loc}}(L^2_t; \alpha) \) for some class of dispersive equations including the Schrödinger equation.

In [24], Vega proved the global smoothing estimate \( K_{\text{glob}}(L^2_{t,x}; \alpha) \), \( \alpha < 1/2 \) for the Schrödinger equation in all dimensions, and the endpoint estimate \( K_{\text{glob}}(L^2_{t,x}; 1/2) \) was obtained by Ben-Artzi and Klainerman [1] for \( n \geq 3 \), Chihara [6] for \( n = 2 \) and Kenig–Ponce–Vega [14] for \( n = 1 \). For other dispersive equations, the global smoothing estimates \( K_{\text{glob}}(L^q_{t,x}; \alpha) \) are similarly obtained.

(For details, see e.g. [20] and the references therein.)

Theorem 1.1 not only gives another proof for the global smoothing estimates \( K_{\text{glob}}(L^q_{t,x}; \alpha) \) (except the endpoint), but also resolves the sharpness issue of the local smoothing estimates \( K_{\text{loc}}(L^q_{t,x}; \alpha) \). The sharpness of the known global smoothing estimates \( K_{\text{glob}}(L^q_{t,x}; \alpha) \) is already settled, but that of the local smoothing estimates \( K_{\text{loc}}(L^q_{t,x}; \alpha) \) is not. For instance, in the Schrödinger equation it is known that if \( \alpha > 1/2 \), the global smoothing \( K_{\text{glob}}(L^q_{t,x}; \alpha) \) fails (see, e.g., [20]), but the fact that if \( \alpha > 1/2 \) the local smoothing \( K_{\text{loc}}(L^q_{t,x}; \alpha) \) also fails was recently shown by Sun, Trélat, Zhang and Zhong [22] when \( n = 1 \). (In fact, Sun, Trélat, Zhang and Zhong considered some class of dispersive equations.) By Theorem 1.1 we can see that this fact holds in higher dimensions. Generally we have the following.

**Corollary 1.2.** Assume (1.1).

(i) Let \( 2 \leq q \leq r \leq \infty \). Suppose that there is an \( \alpha_0 \in \mathbb{R} \) such that if \( \alpha > \alpha_0 \) then the global-in-time smoothing estimate \( K_{\text{glob}}(L^q_t L^q_x; \alpha) \) fails. Then, for \( \alpha > \alpha_0 \) the local-in-time smoothing estimate \( K_{\text{loc}}(L^q_t L^q_x; \alpha) \) fails too.

(ii) Let \( 1 \leq q \leq \infty \), \( 2 \leq r \leq \infty \). In the statement (i) \( K_{\text{loc}}(L^q_t L^q_x; \alpha) \) and \( K_{\text{glob}}(L^q_t L^q_x; \alpha) \) can be replaced with \( K_{\text{loc}}(L^q_t L^q_x; \alpha) \) and \( K_{\text{glob}}(L^q_t L^q_x; \alpha) \) respectively.

**Proof.** These are contrapositives of the statements in Theorem 1.1. Consider (i). if \( K_{\text{loc}}(L^q_t L^q_x; \alpha) \) holds for some \( \alpha > \alpha_0 \), then from (i) of Theorem 1.1 it follows that \( K_{\text{glob}}(L^q_t L^q_x; \alpha - \epsilon) \) holds for all \( \epsilon > 0 \). Since there is an \( \epsilon > 0 \) with \( \alpha - \epsilon > \alpha_0 \), it contradicts the supposition. We omit the proof of (ii) because it is almost identical. \(\square\)
Next we consider the local maximal estimates $K_{loc}(L^q_x L^\infty_t; \alpha)$. It is closely relevant to the pointwise convergence problem of the solution $u(t, x)$ as $t \to 0$. Carleson [5] posed the problem to determine the optimal range of $s$ for which the solution $u(t, x)$ to the Schrödinger equation converges to the initial $f(x)$ almost everywhere $x \in \mathbb{R}^n$ whenever $f \in H^s(\mathbb{R}^n)$. When $n = 1$, Carleson [5] proved the convergence for the sharp range $s \geq 1/4$ through the maximal estimate $K_{loc}(L^1_x L^{\infty}_t; -1/4)$. When $n = 2$, Bourgain [4], Moyua–Vargas–Vega [19], Tao–Vargas [23] and S. Lee [10] made improvements, and recently Du–Guth–Li [10] have obtained the convergence for the sharp range $s > 1/3$ by showing the local maximal $K_{loc}(L^2_x L^{\infty}_t; \alpha)$ for $\alpha < -1/3$. In higher dimensions, Sjölin [24], Vega [24], Bourgain [3] and Du–Guth–Li–Zhang [11] made progresses, and recently, Du–Zhang [12] have proven the convergence for the sharp range $s > n/2(n + 1)$ by showing the local maximal $K_{loc}(L^2_x L^{\infty}_t; \alpha)$ for $\alpha < -n/2(n + 1)$. (For sharpness of the regularity range $s$, see [2, 9, 17, 18].) (When $n = 1$ the maximal estimate $K_{loc}(L^2_x L^{\infty}_t; \alpha)$ for $\alpha \leq -1/4$ was already obtained by Kenig–Ruiz [15].)

By Kenig–Ponce–Vega [14], when $n = 1$, the global-in-time maximal estimate $K_{glob}(L^1_x L^{\infty}_t; \alpha)$, $\alpha \leq -1/4$ was proved for the Schrödinger equation, which also implies Carleson’s convergence result. As far as we know, the global maximal estimates $K_{glob}(L^2_x L^{\infty}_t; \alpha)$ have not been addressed in higher dimensions. By Theorem 1.1 from the local maximal estimates of Du–Guth–Li–Zhang [11] and Du–Zhang [12] we have the following:

**Corollary 1.3.** Let $u(t, x)$ be the solution to the free Schrödinger equation, i.e., $\Phi(\xi) = |\xi|^2$. Then,

- For $n = 2$, $K_{glob}(L^2_x L^{\infty}_t; \alpha)$ holds for all $\alpha < -1/3$,
- For $n \geq 1$, $K_{glob}(L^2_x L^{\infty}_t; \alpha)$ holds for all $\alpha < -\frac{n}{2(n+1)}$.

Moreover, the ranges of $\alpha$ in the above statements are sharp except the endpoint.

Very recently, for some class of dispersive equations including our cases [11], Cho and Ko [7] obtained the same local maximal estimates with those of Du–Guth–Li and Du-Zhang. Thus Corollary 1.3 holds when $\Phi$ satisfies (1.1), but in this case we do not know whether the above range of $\alpha$ is sharp or not except $m = 2$.

The paper is organized as follows. In next section we first reduce the solution $u$ to a frequency localized operator $U$ by using a standard Littlewood-Paley argument. Next, we define wave-packets and derive some properties of the wave-packet decomposition. Finally, we give the proof of Theorem 1.1.

## 2. FROM LOCAL-IN-TIME TO GLOBAL-IN-TIME

### 2.1. Reduction to frequency localized operators.

The solution $u$ has a representation of the form

$$u(t, x) = e^{i\Phi(D)} f(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i\Phi(\xi)} \hat{f}(\xi)d\xi$$

(2.1)

for all Schwartz functions $f$, where the Fourier transform $\hat{f}$ is defined as

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)dy.$$

We define a frequency localized operator $U$ by

$$U f(t, x) := \int e^{i(x\cdot\xi + t\Phi(\xi))} \hat{f}(\xi) \varphi(\xi)d\xi,$$

(2.2)

where $\varphi \in C_0^\infty$ is a bump function supported on

$$\Pi = \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2, |\xi/|\xi| - e_1| \leq \pi/4\},$$

where $e_1 = (1, 0, 0, \cdots, 0) \in \mathbb{R}^n$ is a standard unit vector. From a standard Littlewood-Paley argument we have the following lemma:
**Lemma 2.1.** Assume the second condition of (1.1). Let \( I_{Rm} := \{ t \in \mathbb{R} : R^m / 2 \leq t \leq 2R^m \} \).

(i) If \( K_{loc}(L^q_x L^r_t; \alpha) \) holds, then for any \( R \geq 1 \) the estimate
\[
\| Uf \|_{L^q(B_R; L^r(I_{Rm}))} \leq C_R \alpha R^{-\alpha + \frac{q}{q} + \frac{r}{r} - \frac{\beta}{2}} \| f \|_{L^2(\mathbb{R}^n)}
\]  
holds for all \( f \in L^p(\mathbb{R}^n) \). Inversely, if the estimate (2.3) holds for all \( R \geq 1 \) and all \( f \in L^p(\mathbb{R}^n) \), then \( K_{loc}(L^q_x L^r_t; \alpha - \epsilon) \) holds for all \( \epsilon > 0 \).

(ii) In the statement (i), \( K_{loc}(L^q_x L^r_t; \alpha) \) and the inequality (2.3) can be replaced with \( K_{glb}(L^q_x L^r_t; \alpha) \) and the inequality
\[
\| Uf \|_{L^q(B_R; L^r(I_{Rm}))} \leq C_R \alpha R^{-\alpha + \frac{q}{q} + \frac{r}{r} - \frac{\beta}{2}} \| f \|_{L^2(\mathbb{R}^n)},
\]
respectively.

(iii) In the statements (i) and (ii), \( L^q_x L^r_t \) can be replaced with \( W^{k} \).

**Proof.** The proofs of (i), (ii) and (iii) are almost identical. So, we will give the proof of (i) only. By commuting \( (D)^\alpha e^{it\Phi(D)} = e^{it\Phi(D)}(D)^\alpha \), we see that the estimate \( K_{loc}(2 \to (q, r); \alpha) \) is equivalent to the estimate
\[
\| u \|_{L^q(B^n; L^r(I_1))} \leq C_\alpha \| f \|_{H^{-\alpha}(\mathbb{R}^n)}.
\]
To show (2.3), we take an initial data \( f(x) = R^n(\tilde{\phi}v)^v(Rx) \) in the above inequality. Then after rescaling \( x \mapsto R^{-1}x \) and \( t \mapsto R^{-m}t \), we can obtain (2.3).

To show that the inequality (2.3) implies \( K_{loc}(2 \to (q, r); \alpha) \), let us introduce some necessary things. Let \( \psi_0 \) and \( \psi_k \) be smooth functions supported in \( \{ \xi \in \mathbb{R}^n : |\xi| \leq 2 \} \) and \( \{ \xi \in \mathbb{R}^n : 2^k - 1 \leq |\xi| \leq 2^k + 1 \} \) respectively such that \( 1 = \psi_0 + \sum_{k=1}^{\infty} \psi_k \). We define multipliers \( S_k \) by \( \tilde{S}_k f = \psi_k f \) for \( k = 0, 1, 2, \ldots \), and let \( \tilde{S}_k \) be multipliers given by a bump function adapted to \( \{ 2^k - 1 \leq |\xi| \leq 2^k + 1 \} \) such that \( \tilde{S}_k \approx \tilde{S}_k S_k \). If we define \( T_k f \) and \( f_k \) as \( T_k f(x, t) := e^{it\Phi(D)} \tilde{S}_k f(x) \) and \( f_k := \tilde{S}_k f \) respectively, one has
\[
e^{it\Phi(D)} f(x) = \sum_{k=0}^{\infty} T_k f_k.
\]
By the triangle inequality and the Cauchy–Schwarz inequality,
\[
\left\| \sum_{k=0}^{\infty} T_k f_k \right\|_{L^q(B^n; L^r(I_1))} \leq \left( \sum_{k=0}^{\infty} 2^{-2k} \right)^{1/2} \left( \sum_{k=0}^{\infty} 2^{2k} \| T_k f_k \|_{L^q(B^n; L^r(I_1))}^2 \right)^{1/2}
\leq C_\epsilon \left( \sum_{k=0}^{\infty} 2^{2k} \| T_k f_k \|_{L^q(B^n; L^r(I_1))}^2 \right)^{1/2}
\leq C_\epsilon \left( \sum_{k=0}^{\infty} 2^{2k} \| T_k f_k \|_{L^q(B^n; L^r(I_1))}^2 \right)^{1/2}
\]
for all \( \epsilon > 0 \).

It is easy to show that
\[
\| T_0 f_0 \|_{L^q(B^n; L^r(I_1))} \leq C_0 \| f_0 \|_{L^2(\mathbb{R}^n)}.
\]
Indeed, we have \( \| T_0 f_0 \|_{L^q(B^n; L^r(I_1))} \leq \| T_0 f_0 \|_{L^\infty(\mathbb{R}^n)} \leq \| f_0 \|_{L^1(\mathbb{R}^n)} \). Since \( f_0 \) is supported in the ball \( B(0, 2) \), by the Cauchy–Schwarz inequality this is bounded by \( C_0 \| f_0 \|_{L^2(\mathbb{R}^n)} \), which equals \( C_0 \| f_0 \|_{L^2(\mathbb{R}^n)} \) by Plancherel’s theorem.

By the second condition (1.1) we have
\[
Uf(t, x) = T_k [(f * \varphi^\nu)(2^k .)](2^{-mk}t, 2^{-k}x).
\]
Thus by using a proper finite partitioning and (2.3),
\[
\| T_k f_k \|_{L^q(B^n; L^r(I_1))} \leq C_\alpha 2^{-k\alpha} \| f_k \|_{L^2(\mathbb{R}^n)}
\]
for \( k = 1, 2, 3, \ldots \). We insert these estimates into (2.4). Then,
\[
\left\| \sum_{k=0}^{\infty} T_k f_k \right\|_{L^q(\mathbb{R}^n: L^r(\mathbb{I}))} \leq C_{\alpha, \epsilon} \left( \sum_k 2^{-2k(\alpha-\epsilon)} \|f_k\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2} = C_{\alpha, \epsilon} \|f_k\|_{H^{\alpha-\epsilon}(\mathbb{R}^n)}.
\]
Therefore, by (2.4) we have \( K_{\text{loc}}(2 \to (q, r); \alpha - \epsilon) \) for all \( \epsilon > 0 \). \( \square \)

2.2. Wave packet decomposition. Now we introduce a wave packet decomposition. We first define some functions for partitioning. Let \( \phi \) be a bump function supported in \( B(0, 3/2) \) such that
\[
\sum_{j \in \mathbb{Z}^n} \phi^2(x - j) = 1. \tag{2.7}
\]
Let \( \psi \) be a Schwartz function whose Fourier transform is supported in \( B(0, 2/3) \) such that
\[
\sum_{j \in \mathbb{Z}^n} \psi^2(\xi - j) = 1. \tag{2.8}
\]
For \( R \geq 1 \), let \( \mathcal{P}_R := R\mathbb{Z}^n \) and \( \mathcal{V}_{R^{-1}} := R^{-1} \mathbb{Z}^n \) be lattice sets. For each \( (l, v) \in \mathcal{P}_R \times \mathcal{V}_{R^{-1}} \) we define \( \phi_l \) and \( \psi_v \) as \( \phi_l(x) := \phi(\frac{x}{R}) \) and \( \psi_v(\xi) := \psi(R(\xi - v)) \) respectively, and using these we define a function \( f_{(l,v)} \) by
\[
f_{(l,v)} := m_v(\phi_l f), \tag{2.9}
\]
where \( m_v \) is a multiplier defined by \( \hat{m}_v f(\xi) = \psi_v(\xi) \hat{f}(\xi) \). We see that \( f_{(l,v)} \) is supported in \( B(l, CR) \) and its Fourier transform is essentially supported in \( B(v, \frac{1}{CR}) \). We have a following decomposition of \( f \) as
\[
f = \sum_{(l,v) \in \mathcal{P}_R \times \mathcal{V}_{R^{-1}}} f_{(l,v)}. \tag{2.10}
\]
The \( f_{(l,v)} \) have the following properties:

**Lemma 2.2.** Let \( R \geq 1 \), and for each \( (l, v) \in \mathcal{P}_R \times \mathcal{V}_{R^{-1}} \) let \( f_{(l,v)} \) be defined as in (2.9). Then,
\begin{enumerate}
  \item For \( (t, x) \in I_{R^m} \times \mathbb{R}^n \),
  \[
  |U f_{(l,v)}(t, x)| \leq C_M R^{-n/2} (1 + R^{-1} |(x - l) + t \nabla \Phi(v)|) M \|f_{(l,v)}\|_{L^2(\mathbb{R}^n)}
  \]
  for all \( M \geq 1 \).
  \item \[
  \left( \sum_{(l,v) \in \mathcal{P}_R \times \mathcal{V}_{R^{-1}}} \|f_{(l,v)}\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2} = \|f\|_{L^2(\mathbb{R}^n)}.
  \]
  \item For any sub-collection \( \tilde{\mathcal{P}} \subset \mathcal{P}_R \) and \( \tilde{\mathcal{V}} \subset \mathcal{V}_{R^{-1}} \),
  \[
  \left\| \sum_{(l,v) \in \tilde{\mathcal{P}} \times \tilde{\mathcal{V}}} f_{(l,v)} \right\|_{L^2(\mathbb{R}^n)} \leq C \left( \sum_{(l,v) \in \mathcal{P} \times \mathcal{V}} \|f_{(l,v)}\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2}.
  \]
\end{enumerate}

**Remark.** For each \( (l, v) \in \mathcal{P}_R \times \mathcal{V}_{R^{-1}} \) we define a set \( T_{(l,v)} \) by
\[
T_{(l,v)} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : |(x - l) + t \nabla \Phi(v)| \leq R\},
\]
which is the \( R \)-neighborhood of the line that is passing through \( (0, l) \in \mathbb{R} \times \mathbb{R}^n \) and parallel to \((-1, \nabla \Phi(v))\). The property (i) implies that \( U f_{(l,v)} \) is essentially supported in \( T_{(l,v)} \) in \( I_{R^m} \times \mathbb{R}^n \).
Proof. Consider Property (i). We write as

$$Uf(l,v)(t,x) = \int K_v(t,x-y)f(l,v)(y)dy,$$

where the kernel $K_v$ is defined by

$$K_v(t,x) = \int e^{i(x \cdot \xi + t\Phi(\xi))} \tilde{\chi}_v \varphi(\xi) d\xi \quad (2.11)$$

where $\tilde{\chi}_v$ is a smooth function supported in a small neighborhood of the ball $B(v, \frac{2}{3R})$ such that $\tilde{\chi}_v = 1$ on the ball $B(v, \frac{2}{3R})$. Using a stationary phase method we can obtain that for $(t,x) \in I_{Rm} \times \mathbb{R}^n$,

$$|K_v(t,x)| \leq C_M R^{-n}(1 + R^{-1}|x + t\nabla \Phi(v)|)^{-M}, \quad \forall M \geq 1.$$

From this it follows that for $(t,x) \in I_{Rm} \times \mathbb{R}^n$,

$$|Uf(l,v)(t,x)| \leq C_M R^{-n}(1 + R^{-1}|x - l + t\nabla \Phi(v)|)^{-M} \int |f(l,v)(y)|dy$$

$$\leq C_M R^{-n/2}(1 + R^{-1}|x - l + t\nabla \Phi(v)|)^{-M} \|f(l,v)\|_{L^2(\mathbb{R}^n)} \quad (2.12)$$

for any $M \geq 1$, where the Cauchy-Schwarz inequality is used in the last line. Thus, Property (i) is obtained.

Consider Property (ii). By Plancherel’s theorem and (2.8),

$$\sum_l \sum_v \int |f(l,v; R)|^2 = \sum_l \sum_v \int |\psi_v \hat{\varphi_l f}|^2 = \sum_l \int |\hat{\varphi_l f}|^2.$$

We use Plancherel’s theorem again, and by (2.7) the above equation equals

$$\sum_l \int |\varphi_l f|^2 = \int |f|^2.$$

Thus we have Property (ii).

Consider Property (iii). By Plancherel’s theorem and a partition of unity $\{\psi_v\}$,

$$\int \sum_{v \in V} \sum_{l \in P} |f(l,v)|^2 \leq C \sum_{v \in V} \int |\psi_v \sum_{l \in P} \hat{\varphi_l f}|^2.$$

By Plancherel’s theorem, the right side of the above equation equals

$$C \sum_{v \in V} \int |\sum_{l \in P} \psi_v^* \varphi_l f|^2 = C \sum_{v \in V} \int |\sum_{l \in P} f(l,v)|^2.$$

Since $f(l,v)$ is supported in $B(l,CR)$, the above equation is

$$\leq C \sum_{v \in V} \sum_{l \in P} \int |f(l,v)|^2.$$

Thus, we have Property (iii).
2.3. Proof of Theorem 1.1. To prove our theorem we use the following geometric observation in [10] Lemma 2.3. Let \( \Gamma := \{ (\Phi(\xi), \xi) \} \) be a surface in \( \mathbb{R} \times \mathbb{R}^n \), where \( \Phi \) satisfies the first condition of (1.1). Then, for any \( \xi \in \{ \xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2 \} \) the angle between the normal vector \(-1, \nabla \Phi(\xi)\) to \( \Gamma \) and the \( t\)-axis is larger than some positive constant.

Now, let us consider (i). By \( K_{loc}(L^2_I; \alpha) \) and (i) of Lemma 2.1 we have that for any \( R \geq 1, \)

\[
\|Uf\|_{L^q_0(B_R; L^q_I(I_{RM}))} \leq C_\alpha R^{-\alpha + \frac{n}{q} + \frac{n}{q} - \frac{n}{q}} \|f\|_{L^2_\Gamma(\mathbb{R}^n)}. \tag{2.13}
\]

Let \( C_0 > 1 \) be a large constant. To show \( K_{gh}(L^2_I; \alpha - \varepsilon) \) for all \( \varepsilon > 0 \) it suffices to prove that for any \( R \geq C_0, \)

\[
\|Uf\|_{L^q_0(B_R; L^q_I(I_{RM}))} \leq C_{\alpha, \varepsilon} R^\varepsilon R^{-\alpha + \frac{n}{q} + \frac{n}{q} - \frac{n}{q}} \|f\|_{L^2(\mathbb{R}^n)} \tag{2.14}
\]

for all \( f \in L^p(\mathbb{R}^n) \) and all \( \varepsilon > 0 \), where \( \mathbb{R}^+ \) denotes the set of nonnegative real numbers. Indeed, if \( 1 \leq R \leq C_0 \), we have (2.14) by some trivial estimates.

To show (2.14) it suffices to show that for any \( k \geq 1, \)

\[
\|Uf\|_{L^q_0(B_R; L^q_I(I_{RM}))} \leq R^\varepsilon A(R) \|f\|_{L^2(\mathbb{R}^n)} + C_M R^{-M} \|f\|_{L^2(\mathbb{R}^n)} \tag{2.15}
\]

for all \( f \in L^p(\mathbb{R}^n) \), all \( \varepsilon > 0 \) and all \( M \geq 1 \), where

\[
A(R) := C_{\alpha, \varepsilon} R^{-\alpha + \frac{n}{q} + \frac{n}{q} - \frac{n}{q}}. \tag{2.16}
\]

Indeed, we can see that the estimate (2.15) is translation invariant. So, after translating \( I_{RMk} \) to \([0, \frac{1}{2} R^{n-k}]\), we take a limit \( k \to \infty \). Since the right side of (2.15) is independent of \( k \), we obtain (2.14).

We will prove (2.14) by induction. When \( k = 1 \), it follows from (2.13). We assume that for some \( k \geq 1 \) the estimate (2.15) holds for all \( f \in L^p(\mathbb{R}^n) \), all \( \varepsilon > 0 \) and all \( M \geq 1 \). Let \( \{ t_j \} \) be a maximal \( R^{n-k} \)-separated subset in the interval \( I_{RMk+1} \), and let \( I_j \) be the interval of length \( R^{n-k} \) with a center at \( t_j \). Since \( q \leq r \), we have

\[
\|Uf\|_{L^q_0(B_R; L^q_I(I_{RMk+1}))} \leq \left( \int_{B_R} \left( \sum_j |Uf|_{L^q_I(I_j)} \right)^{q/r} dx \right)^{1/q} \leq \left( \sum_j \|Uf\|_{L^q_I(I_j)}^q \right)^{1/q} = \left( \sum_j \|f\|_{L^q(B_R; L^q_I(I_j))}^q \right)^{1/q}. \tag{2.17}
\]

Let \( \varepsilon > 0 \) be given, and let \( \kappa \in (0, 1) \) be a small constant which will be chosen later. Let \( \Delta_j := I_j \times B_{R^{n-k}} \) be a \( R^{n-k} \)-cube, and let \( K \Delta_j \) denote the \( K \)-dilation of \( \Delta_j \) with the center of dilation at its center for \( K > 0 \). We decompose \( f \) into \( \{ f_{l,v} : (l, v) \in P_{R^{n-k}} \times V_{R^{-n-k}} \} \).

By using (i) of Lemma 2.2,

\[
\|Uf\|_{L^q_0(B_R; L^q_I(I_{RMk+1}))} \leq \left( \sum_j \sum_{(l,v) \in W_j(KR^n)} |Uf_{l,v}|_{L^q_0(B_R; L^q_I(I_j))}^q \right)^{1/q} + C_{M, \kappa} R^{-M} \|f\|_{L^2(\mathbb{R}^n)}
\]

for any \( M \geq 1 \), where

\[
W_j(K) := \{ (l, v) \in P_{R^{n-k}} \times V_{R^{-n-k}} : T_{l,v} \cap K \Delta_j \neq \emptyset \}. \tag{2.18}
\]

Since (2.15) is translation invariant, by the induction hypothesis and embedding \( \ell^2 \subset \ell^q, \)

\[
\left( \sum_j \sum_{(l,v) \in W_j(KR^n)} \|Uf_{l,v}\|_{L^q_0(B_R; L^q_I(I_j))}^q \right)^{1/q} \leq R^\varepsilon A(R) \left( \sum_j \sum_{(l,v) \in W_j(KR^n)} \|f_{l,v}\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2}.
\]
By (iii) in Lemma 2.2,

\[ \sum_j \left\| \sum_{(l,v) \in W_j(\kappa R^n)} f(l,v) \right\|_{L^2(\mathbb{R}^n)}^2 \leq C \sum_j \sum_{l \in T(l,v) \cap \kappa R^n} \sum_{\Delta_j \neq \emptyset} \|f(l,v)\|_{L^2(\mathbb{R}^n)}^2. \]

By rearranging the summations, the right side of the above inequality equals

\[ \sum_j \sum_{(l,v) \in W_j(\kappa R^n)} \sum_{T(l,v) \cap \kappa R^n} \sum_{\Delta_j \neq \emptyset} \|f(l,v)\|_{L^2(\mathbb{R}^n)}^2. \]

From the first condition of (1.1) it follows that the angle formed by the vector \((-1, \nabla \Phi(\xi))\) and the \(t\)-axis is larger than some positive constant. Thus, if \(l\) and \(v\) are given then the number of \(j\) with \(T(l,v) \cap \Delta_j \neq \emptyset\) is \(O(1)\), see Figure 1. By this observation the above equation is

\[ \leq C \kappa^C R^C \sum_j \sum_{l \in T(l,v)} \sum_{\Delta_j \neq \emptyset} \|f(l,v)\|_{L^2(\mathbb{R}^n)}^2, \]

and by (ii) of Lemma 2.2 it equals

\[ C \kappa^C R^C \|f\|_{L^2(\mathbb{R}^n)}^2. \]

From combining all the above equations it follows that

\[ \|Uf\|_{L^q(R^n)} \leq C \kappa^C R^C A(R) \|f\|_{L^2(\mathbb{R}^n)}. \]

If we choose \(\kappa \in (0, 1)\) with \(C \kappa^C \leq 1\), then (2.15) is obtained.

Consider (ii). The proof is similar to that of (i). The condition \(q \leq r\) in (i) is not needed in (ii). This condition is only used in (2.16), but we can have

\[ \|Uf\|_{L^r(I_R^{mk+1}; L^q(B_R))} \leq \left( \sum_j \|Uf\|_{L^q(I_j; L^q(B_R))} \right)^{1/r} \]

without the condition \(q \leq r\). We omit the rests because it is almost identical.
References

[1] Matania Ben-Artzi and Sergiu Klainerman, Decay and regularity for the Schrödinger equation, Journal d'Analyse Mathématique 58 (1992), no. 1, 25–37.

[2] Jean Bourgain, A note on the Schrödinger maximal function, Journal d'Analyse Mathématique 130 (2016), no. 1, 393–396.

[3] Jean Bourgain, On the Schrödinger maximal function in higher dimension, Proceedings of the Steklov Institute of Mathematics 280 (2013), no. 1, 46–60.

[4] Jean Bourgain, Some new estimates on oscillatory integrals, Essays on Fourier Analysis in Honor of Elias. M. Stein (Princeton, NJ, 1991), Princeton Math. Ser., vol. 42, Princeton University Press, 1995, pp. 83–112.

[5] Lennart Carleson, Some analytic problems related to statistical mechanics, Euclidean harmonic analysis, Lecture Notes in Math. 779, pp. 5–45.

[6] Hiroyuki Chihara, Smoothing effects of dispersive pseudodifferential equations, Communications in Partial Differential Equations 27 (2002), 1953–2005.

[7] Chu-hee Cho and Hyerim Ko, A note on maximal estimates of generalized Schrödinger equation, arXiv preprint arXiv:1809.03246 (2018).

[8] Peter Constantin and Jean-Claude Saut, Local smoothing properties of dispersive equations, Journal of the American Mathematical Society 1 (1988), no. 2, 413–439.

[9] Bjorn E.J. Dahlberg and Carlos E. Kenig, A note on the almost everywhere behavior of solutions to the Schrödinger equation, Harmonic analysis, Lecture Notes in Math. 908, pp. 205–209.

[10] Xiumin Du, Larry Guth, and XiaoChun Li, A sharp Schrödinger maximal estimate in $\mathbb{R}^2$, Annals of Mathematics 186 (2017), no. 2, 607–640.

[11] Xiumin Du, Larry Guth, XiaoChun Li, and Ruixiang Zhang, Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimate, Forum of Mathematics, Sigma 6 (2018), e14. doi:10.1017/fms.2018.11.

[12] Xiumin Du and Ruixiang Zhang, Sharp $L^2$ estimate of Schrödinger maximal function in higher dimensions, arXiv preprint arXiv:1805.02775 (2018).

[13] Tosio Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Studies in Appl. Math. Adv. in Math. Suppl. Stud. 8 (1983), 93–128.

[14] Carlos E Kenig, Gustavo Ponce, and Luis Vega, Oscillatory integrals and regularity of dispersive equations, Indiana University Mathematics Journal 40 (1991), no. 1, 33–69.

[15] Carlos E Kenig and Alberto Ruiz, A strong type $(2, 2)$ estimate for a maximal operator associated to the Schrödinger equation, Transactions of the American Mathematical Society 280 (1983), no. 1, 239–246.

[16] Sanghyuk Lee, On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$, International Mathematics Research Notices 2006 (2006), 1–21.

[17] Renato Lucà and Keith M Rogers, Coherence on fractals versus pointwise convergence for the Schrödinger equation, Communications in Mathematical Physics 351 (2017), no. 1, 341–359.

[18] Renato Lucà and Keith M Rogers, A note on pointwise convergence for the Schrödinger equation, Mathematical Proceedings of the Cambridge Philosophical Society (2017), 1–10. doi:10.1017/S0305004117000743.

[19] Adela Moyua, Ana Vargas, and Luis Vega, Schrödinger maximal function and restriction properties of the Fourier transform, International Mathematics Research Notices 1996 (1996), no. 16, 793–815.

[20] Michael Ruzhansky and Mitsuru Sugimoto, Smoothing properties of evolution equations via canonical transforms and comparison principle, Proceedings of the London Mathematical Society 105 (2012), no. 2, 393–423.

[21] Per Sjölin, Regularity of solutions to the Schrödinger equation, Duke mathematical journal 55 (1987), no. 3, 699–715.

[22] Shu-Ming Sun, Emmanuel Trélat, Bing-Yu Zhang, and Ning Zhong, On sharpness of the local Kato-smoothing property for dispersive wave equations, Proceedings of the American Mathematical Society 145 (2017), no. 2, 653–664.

[23] T. Tao and A. Vargas, A bilinear approach to cone multipliers II. Applications, Geometric and Functional Analysis 10 (2000), no. 1, 216–258.

[24] Luis Vega, Schrödinger equations: pointwise convergence to the initial data, Proceedings of the American Mathematical Society 102 (1988), no. 4, 874–878.