ONE-TO-ONE COMPOSANT MAPPINGS
OF \([0, \infty)\) AND \((-\infty, \infty)\)

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Abstract. Knaster continua and solenoids are well-known examples of indecomposable continua whose composants (maximal arcwise-connected subsets) are one-to-one images of lines. We show that essentially all non-trivial one-to-one composant images of (half-)lines are indecomposable. And if \(f\) is a one-to-one mapping of \([0, \infty)\) or \((-\infty, \infty)\), then there is an indecomposable continuum of which \(X := \text{ran}(f)\) is a composant if and only if \(f\) maps all final or initial segments densely and every non-closed sequence of arcs in \(X\) has a convergent subsequence in the hyperspace \(K(X) \cup \{X\}\). Accompanying the proofs are illustrations and examples.

1. Introduction

Throughout, \([0, \infty)\) denotes the half-line and \((-\infty, \infty)\) denotes the entire real line. Every mapping is assumed to be continuous; by image we shall always mean continuous image.\(^1\) All images of the (half-)line are assumed to be metrizable, and by a continuum (plural form continua) we mean a connected compact metrizable space. An arc is a homeomorphic copy of the interval \([0, 1]\).

A continuum \(Y\) is decomposable if there are two subcontinua \(H, K \subsetneq Y\) such that \(Y = H \cup K\); otherwise \(Y\) is indecomposable. We shall say, more generally, that a connected space \(X\) is indecomposable if \(X\) cannot be written as the union of two proper closed connected subsets. Equivalently, \(X\) is indecomposable if \(X\) is the only closed connected subset of \(X\) with non-void interior.

If \(Y\) is a continuum and \(x \in Y\), then \(X\) is the composant of \(x\) in \(Y\) means that

\[ X = \bigcup \{ K \subsetneq Y : K \text{ is a continuum and } x \in K \}. \]

More generally, \(X\) is a composant of \(Y\) if there exists \(x \in X\) such that \(X\) is the composant of \(x\) in \(Y\).

Given a continuum \(Y\), a line \(\ell \in \{ [0, \infty), (-\infty, \infty) \}\), and a mapping \(f : \ell \to Y\), one easily sees that \(\text{ran}(f) := f[\ell] \) (the range of \(f\)) is contained in a composant of \(Y\). The goal of this paper is to describe all one-to-one images of (half)-lines which are homeomorphic to composants of continua. Theorem I classifies all decomposable composant images, while Theorem II provides an internal characterization “composant-ness” which is independent of any particular embedding.

\(^1\)Being a one-to-one image of \([0, \infty)\) is the same as being the union of a strictly increasing sequence of arcs which share a common endpoint. Among locally connected, locally compact spaces, there are only 3 such images, and there are only 5 such images of \((-\infty, \infty)\) – Lelek & McAuley [11] and Nadler [14]. Much is also known about other types of images: compact [13, 2], confluent [15], aposyndetic [8], uniquely arcwise-connected [12], hereditarily unicoherent [1].

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Theorems. Let \( f : \ell \to X \) be a one-to-one mapping of \( \ell \in \{[0, \infty), (\infty, \infty)\} \) onto a metrizable space \( X \).

I. There is a decomposable continuum of which \( X \) is a composant if and only if

(A) \( X \) is compact (this is the only possibility if \( \ell = (\infty, \infty) \));
(B) \( X \cong [0, \infty) \), i.e. \( f \) is a homeomorphism; or
(C) \( \exists s \in (0, \infty) \) such that \( f[0, s) \) is open in \( X \) and \( f[s, \infty) \) is an indecomposable composant.

Moreover, if \( X \) is neither compact nor equal to the half-line, and \( Y \) is a decomposable continuum of which \( X \) is a composant, then there exists \( s \in (0, \infty) \) such that \( Y \setminus f[0, s) \) is an indecomposable continuum of which \( f[s, \infty) \) is a composant.

II. There is an indecomposable continuum of which \( X \) is a composant if and only if

1. \( \bigcup_{n<\omega} f[n, \infty) = X \) or \( \bigcup_{n<\omega} f[-\infty, -n] \cap \ell \) for every \( n < \omega \); and
2. \( \bigcup \{ A_n : n < \omega \} \in K(X) \cup \{X\} \) for every sequence of arcs \( (A_n) \in [K(X)]^\omega \) such that \( \bigcap \{ A_n : n < \omega \} \neq \emptyset \).

In condition (2) of Theorem II, \( K(X) \) is the set of non-empty compact subsets of \( X \), and \( c(X) = \{ (x_n) \in X^\omega : (\exists x \in X)(x_n \to x \text{ as } n \to \infty) \} \) is the set of convergent point sequences in \( X \).

In Section 7 we will prove two fairly general lemmas to obtain the following.

Theorem III. For every linear composant \( X \) there is a continuum \( Y \subseteq [0, 1]^3 \) such that \( \dim(Y) = 1 \) and \( X \) is a composant of \( Y \).

Here and elsewhere, the term linear is used to indicate a space which is a one-to-one image of the line or half-line.

We conclude in Sections 8 and 9 with several relevant examples and a list of important questions about composant embeddings, chainability, and indecomposability of first category plane images.

2. Recurrence

Suppose \( f \) is a mapping of \( \ell \in \{[0, \infty), (\infty, \infty)\} \). Let \( X = \text{ran}(f) \). If \( \ell = [0, \infty) \), then we say \( f \) is recurrent if \( \overline{f[n, \infty)} = X \) for each \( n < \omega \). If \( \ell = (\infty, \infty) \), then \( f \) is:

- positively-recurrent if \( \overline{f[n, \infty)} = X \) for each \( n < \omega \);
- negatively-recurrent if \( \overline{f(-\infty, -n]} = X \) for each \( n < \omega \);
- recurrent if \( f \) is positively or negatively recurrent; and
- bi-recurrent if \( f \) is both positively and negatively recurrent.

Remark. According to definitions, condition (1) in Theorem II says “\( f \) is recurrent”.

Proposition 1. If \( f \) is one-to-one and recurrent, then \( X \) is of the first category of Baire.

Proof. By the hypotheses, if \( \ell = [0, \infty) \) then each \( f[0, n] \) is nowhere dense in \( X \). Likewise, if \( \ell = (\infty, \infty) \) then each \( f[-n, n] \) is nowhere dense in \( X \). \( \square \)

Proposition 2. If \( X \) is non-degenerate and indecomposable, then \( f \) is recurrent.

Proof. By contraposition. Suppose \( f \) is not recurrent.

Case 1: \( \ell = [0, \infty) \). Then there exists \( n < \omega \) such that \( \overline{f[n, \infty)} \neq X \). If \( f[0, n] \neq X \) then \( X \) is the union of the two proper closed connected sets \( f[0, n] \) and \( f[n, \infty) \), whence
X is decomposable. On the other hand, if \( f[0, n] = X \) then X is locally connected by the Hahn-Mazurkiewicz Theorem. Then either X is either degenerate or decomposable.

**Case 2:** \( \ell = (\infty, \infty) \). Then there exists \( n < \omega \) such that \( (-\infty, -n] \neq X \neq [n, \infty) \).

The goal is to show X is decomposable, so we may assume each of the connected sets \( f(-\infty, -n] \) and \( f[n, \infty) \) is nowhere dense in X. Then \( f[-n, n] \) has non-void interior in X, so if \( f[-n, n] \neq X \) then X is automatically decomposable. Otherwise X is locally connected and X is either degenerate or decomposable. \( \square \)

**Proposition 3.** Suppose \( f \) is one-to-one and \( Y \) is a continuum of which \( X \) is a composant. Then \( f \) is recurrent if and only if \( Y \) is indecomposable.

**Proof.** In a decomposable continuum every composant has non-void interior. So by Proposition 1 and the Baire Category Theorem, if \( f \) is recurrent then it must be that \( Y \) is indecomposable. Conversely, if \( Y \) is indecomposable then \( X \) is indecomposable by \( \overline{X} = Y \) (composants are dense). Then \( f \) is recurrent by Proposition 2. \( \square \)

### 3. Proof of Theorem I

Let \( f : \ell \rightarrow X \) be a one-to-one mapping of \( \ell \in \{[0, \infty), (-\infty, \infty)\} \) onto X.

Any of \( (A) \) through \( (C) \) is sufficient:

If \( (A) \), then X is a decomposable continuum (Propositions 1 & 2) with composant equal to itself. If \( (B) \), then X is the composant of \( f(0) \) in the one-point compactification of X, just as \([0, 1]\) is the composant of 0 in the unit interval \([0, 1]\).\(^2\) If \( (C) \), and \( Y \) is an indecomposable continuum of which \( f[s, \infty) \) is a composant, then X is the composant of \( f(0) \) in the decomposable continuum \( f(0, s) \cup Y \).

One of \( (A) \) through \( (C) \) is necessary:

Suppose \( Y \) is a decomposable continuum of which \( X \) is a composant.

**Case 1:** \( \ell = (-\infty, \infty) \).

We show \( (A) \). Well, suppose for a contradiction X is non-compact. Then at least one of \( f(-\infty, 0] \) and \( f[0, \infty) \) is non-compact. Without loss of generality, \( f[0, \infty) \) is non-compact. Let \( r \in (-\infty, \infty) \) be such that X is the composant of \( f(r) \), and let \( n < \omega \). We have \( f[r, \infty) = X \) by maximality of X, so \( f(-\infty, r) \subseteq f[n, \infty) \). Then \( f(r) \in f[n, \infty) \). As before, \( f[n, \infty) = X \). Since \( n < \omega \) was arbitrary, \( f \) is (positively-)recurrent. By Proposition 3, this contradicts decomposability of \( Y \).

**Case 2:** \( \ell = [0, \infty) \).

Suppose neither \( (A) \) nor \( (B) \) holds. We show \( (C) \).

**Claim 3.1.** \( X \) is the composant of \( f(0) \) in \( Y \).

**Proof of Claim 3.1.** Let \( t \in [0, \infty) \) be such that \( X \) is the composant of \( f(t) \) in \( Y \). Let \( P \) be the composant of \( f(0) \) in \( Y \). Apparently \( X \subseteq P \) because for each \( x \in X \) the arc \( f[0, f^{-1}(x)] \) is a proper subset of \( Y \) (by \( \neg(A) \)). Now let \( y \in P \). There is a continuum \( K \subseteq Y \) with \( \{f(0), y\} \subseteq K \). If \( f[0, t] \cup K \neq Y \), then clearly \( y \in X \). If \( f[0, t] \cup K = Y \), then \( f(t, \infty) \subseteq K \), whence \( f(t) \in K \). Again \( y \in X \). Thus \( P \subseteq X \). Combining both inclusions, we have \( P = X \). \( \square \)

**Claim 3.2.** There exists \( t > 0 \) such that \( f[0, t) \) is open in \( Y \).

\(^2\)Moreover, if \( Y \) is any compactification of \( X \simeq [0, \infty) \), then \( X \) is the composant of \( f(0) \) in \( Y \).
POOF OF CLAIM 3.2. For otherwise \( f(0) \in \overline{f[n, \infty)} \) for every \( n < \omega \). Let \( n < \omega \) such that \( \overline{f[n, \infty)} \neq X \) (Proposition 3). Then \( \overline{f[n, \infty)} \) is a proper subcontinuum of \( Y \) containing \( f(0) \) and meeting \( Y \setminus X \) (by \( \neg(A) \)). In light of Claim 3.1, this contradicts maximality of \( X \). \( \square \)

Let \( s = \sup \{ t \in (0, \infty) : f[0, t) \text{ is open in } X \} \). Observe that \( s > 0 \) by Claim 3.2, and \( s < \infty \) by \( \neg(B) \). Also, \( f[0, s) = \bigcup \{ f[0, t) : t \in (0, \infty) \text{ and } f[0, t) \text{ is open in } X \} \) is open in \( X \), and \( f[s, \infty) \) is the composant of \( f(s) \) in \( \overline{f[s, \infty)} \).

Claim 3.3. \( \overline{f[s, \infty)} \) is indecomposable.

POOF OF CLAIM 3.3. By Proposition 3, it suffices to show \( f \upharpoonright [s, \infty) \) is recurrent. Suppose to the contrary that there exists \( m < \omega \) such that \( f[s + m, \infty) \) is not dense in \( f[s, \infty) \). Then \( f(s, s + m) \) has non-empty interior in \( Y \). By definition of \( s \) we have \( s \in \overline{f[s + m, \infty)} \). Then \( \overline{f[s + m, \infty)} \) is a proper subcontinuum of \( Y \) containing \( f(s) \) and meeting \( Y \setminus X \) (by \( \neg(A) \)). Then \( f[0, s] \cup \overline{f[s + m, \infty)} \) is a proper subcontinuum of \( Y \) that violates maximality of \( X \). \( \square \)

This concludes the proof of Theorem I.

4. PROOF OF THEOREM II (NECESSITY)

Suppose \( Y \) is an indecomposable continuum of which \( X \) is a composant. Then (1) is true by Proposition 3. Towards proving (2), let \( (A_n) \in [K(X)]^\omega \) such that \( c(X) \cap \prod \{ A_n : n < \omega \} \neq \emptyset \). Let \( x \) be the limit point of an element of \( c(X) \cap \prod \{ A_n : n < \omega \} \). Supposing \( \bigcup \{ A_n : n < \omega \} \notin K(X) \), there exists \( y \in \bigcup \{ A_n : n < \omega \} \setminus X \). Let \( K \) be the component of \( y \) in \( \bigcup \{ A_n : n < \omega \} \). Observe that \( x \in K \). The composants of an indecomposable continuum are pairwise disjoint, so \( X \) is the composant of \( x \) in \( Y \). Thus \( K = Y \). It follows that \( \bigcup \{ A_n : n < \omega \} = X \).

5. ARCS IN ran(\( f \))

Before proving the opposite direction in Theorem II, we need two more propositions (this section) and a key lemma (next section).

Proposition 4 is classic.

Proposition 4. Let \( f : [0, \infty) \to X \) be a continuous bijection onto a non-compact space \( X \). For any continuum \( K \in K(X) \) there are two numbers \( a, b \in [0, \infty) \) such that \( a \leq b \) and \( f[a, b] = K \). In particular, \( K \) is a point or an arc.

PROOF. No tail of \( [0, \infty) \) maps into \( K \). For if \( f[n, \infty) \subseteq K \), then \( X = f[0, n] \cup K \) is compact. So there is an unbounded sequence of numbers \( r_0 < r_1 < \ldots \) in \( (0, \infty) \setminus f^{-1}[K] \). Since \( [0, \infty) \setminus f^{-1}[K] \) is open, there are two additional sequences \( (a_k) \) and \( (b_k) \) such that \( 0 = a_0 < b_k < r_k < a_{k+1} < b_{k+1} \) for each \( k < \omega \) and \( (b_k, a_{k+1}) \cap f^{-1}[K] = \emptyset \). Then \( K \) is covered by the disjoint compacta \( f[a_k, b_k], k < \omega \). By Sierpiński’s Theorem (stated in Section 6), there exists \( k^* \) such that \( K \subseteq f[a_{k^*}, b_{k^*}] \). Since \( f \upharpoonright [a_{k^*}, b_{k^*}] \) is a homeomorphism, we can find the desired numbers \( a \leq b \) in \( [a_{k^*}, b_{k^*}] \). If \( a = b \) then \( K \) is a point; otherwise \( K \) is an arc. \( \square \)

By nearly identical arguments, Proposition 4 is also true when \( \ell = (-\infty, \infty) \) and \( f \) is bi-recurrent.
Remark. Let \( f : \ell \to X \) be a continuous bijection onto a composant space \( X \), where \( \ell \in \{ [0, \infty), (-\infty, \infty) \} \). If \( \ell = [0, \infty) \) and \( f \) is recurrent, or \( \ell = (-\infty, \infty) \) and \( f \) is bi-recurrent, then the following are true. By the composant property every proper closed connected subset of \( X \) is compact. So by Proposition 4, \( f \) is confluent and every non-degenerate proper closed connected subset of \( X \) is an arc. In particular, \( X \) is hereditarily unicoherent. Since neither end of \( X \) terminates to form a circle, \( X \) is also uniquely arcwise-connected.

A mapping \( f \) of the line or half-line \( \ell \) is \textit{arc-complete} provided for every three sequences \( a, b, c \in \ell^\omega \) such that \( a_n < b_n \) and \( c_n \in [a_n, b_n] \), if \( f(c) \) converges in \( X := \text{ran}(f) \) then \( \bigcup \{ f[a_n, b_n] : n < \omega \} \) is compact or equal to \( X \).

We now give a subsequence criterion for arc-completeness. The topology on \( K(X) \) is the Vietoris topology (equals the topology generated by a Hausdorff metric).

**Proposition 5.** Let \( f : [0, \infty) \to X \) be a continuous bijection. The following are equivalent.

(i) \( f \) is arc-complete;

(ii) condition (2) in Theorem II;

(iii) for every sequence of arcs \( (A_n) \in [K(X)]^\omega \), if \( c(X) \cap \prod \{ A_n : n < \omega \} \neq \emptyset \) and \( \bigcup \{ A_n : n < \omega \} \neq X \) then a subsequence of \( (A_n) \) converges in \( K(X) \).

**Proof.** (i) \( \iff \) (ii) is an immediate consequence of Proposition 4.

We will now prove (ii) \( \iff \) (iii). For this purpose, let \( Y \) be any metric compactum in which \( X \) is densely embedded.

Suppose (ii). Let \( (A_n) \in [K(X)]^\omega \) be such that \( c(X) \cap \prod \{ A_n : n < \omega \} \neq \emptyset \) and \( \bigcup \{ A_n : n < \omega \} \neq X \). Then \( \bigcup \{ A_n : n < \omega \} \) is compact by hypothesis. By compactness of \( K(Y) \), a subsequence \( (A_{n_k}) \) converges to a point \( K \in K(Y) \). Then

\[
K \subseteq \bigcup \{ A_{n_k} : k < \omega \} = \bigcup \{ A_{n_k} : k < \omega \},
\]

whence \( K \in K(X) \). This proves (iii).

Conversely, suppose (iii). Let \( (A_n) \in [K(X)]^\omega \) be such that \( c(X) \cap \prod \{ A_n : n < \omega \} \neq \emptyset \) and \( \bigcup \{ A_n : n < \omega \} \neq \{ X \} \). We show \( \bigcup \{ A_n : n < \omega \} \in K(X) \). To that end, let \( y \in \bigcup \{ A_n : n < \omega \} \) and show \( y \in X \). There exists \( (y_k) \in \bigcup \{ A_n : n < \omega \}^\omega \) such that \( y_k \to y \).

For each \( k < \omega \), let \( n_k \) be such that \( y_k \in A_{n_k} \). By compactness of each \( A_n \), we may assume that \( \{ n_k : k < \omega \} \) is infinite and in strictly increasing order. Applying the hypothesis to \( (A_{n_k}) \), we find that a subsequence of \( (A_{n_k}) \) converges to a point \( K \in K(X) \). Then \( y \in K \subseteq X \). Since \( y \) was arbitrary, we have

\[
\bigcup \{ A_n : n < \omega \} = \bigcup \{ A_n : n < \omega \} \in K(X).
\]

This proves (ii).

Proposition 5 is also true with \( (-\infty, \infty) \) in the place of \( [0, \infty) \), but proving (i) \( \implies \) (ii) requires some care. Suppose (i), and assume \( X \) is non-compact. Let \( (A_n) \in [K(X)]^\omega \). If for every \( n < \omega \), \( (-\infty, \infty) \setminus f^{-1}[A_n] \) is unbounded in the positive and negative directions, then the pre-image of each arc is a closed and bounded interval (Proposition 4 arguments), and (ii) follows. If, on the other hand, an initial or final segment of \( (-\infty, \infty) \) maps into an arc \( A_m \), then every other \( A_n \) \( (n \neq m) \) has a pre-image \([a_n, b_n]\). Applying (i) to these intervals will show (ii).
6. Zero-dimensional collections of arcs

The result in this section is based on ‘The Sierpiński Theorem’ – [3] 6.1.27.

Theorem (Sierpiński). If the continuum $X$ has a countable cover \( \{X_i : i < \omega\} \) by pairwise disjoint closed subsets, then at most one of the sets $X_i$ is non-empty.

That statement is false with ‘connected space’ in the place of ‘continuum’; Figure 1 shows a connected space $\mathcal{E}_\omega$ which is the union of $\omega$-many disjoint arcs: $\mathcal{E}_\omega = \bigcup\{C_n : n < \omega\}$. In $\mathcal{E}_\omega$, observe that for each $m < \omega$ there is a subsequence $(C_{n_k})$ and a sequence of points $(x_k) \in \prod\{C_{n_k} : k < \omega\}$ such that $(x_k)$ converges to a point of $C_m$ and $\bigcup\{C_{n_k} : k < \omega\}$ is non-compact.

**Lemma 1.** Let $X = \bigcup\{A_n : n < \omega\}$ be the union of a countable sequence of disjoint continua. If the closure $\overline{\bigcup\{A_{n_k} : k < \omega\}}$ is compact for every subsequence $(A_{n_k})$ such that $c(X) \cap \prod\{A_{n_k} : k < \omega\} \neq \emptyset$, then the decomposition $\tilde{X} := \{A_n : n < \omega\}$ is zero-dimensional.

**Proof.** It suffices to show $\tilde{X}$ is regular because every countable regular space has dimension zero. (Hint: Every regular Lindelöf space is normal so Urysohn’s Lemma can be applied.)

To that end, let $m < \omega$ and let $C$ be a closed subset of $X$ that misses $A_m$ and is a union of constituents; $C = \bigcup\{A_n : n \in I\}$ for some non-empty $I \subseteq \omega \setminus \{m\}$. Assume that $m = 0$. We find disjoint $X$-open sets $U_0$ and $U_1$, each of which is a union of continua from $\{A_n : n < \omega\}$ and such that $A_0 \subseteq U_0$ and $C \subseteq U_1$.

Recursively define two sequences of open sets $(U_0^n)$ and $(U_1^n)$ as follows.

**Step 0:** There exists $\epsilon_0 > 0$ such that
\[
d(A_{n_0}, A_0) + d(A_{n_0}, C) \geq 4\epsilon_0
\]
for all $n < \omega$.\(^3\) Otherwise, there is a sequence of arcs $(A_{n_k})$ such that $d(A_{n_k}, A_0) + d(A_{n_k}, C) \to 0$ as $k \to \infty$. There exists $x \in A_0$ and a sequence of points $x_j \in \bigcup\{A_{n_k} : k < \omega\}$ such that $x_j \to x$ as $j \to \infty$. Eventually $n_k \neq 0$ because $d(A_0, C) > 0$. Thus $d(A_{n_k}, A_0) > 0$ for sufficiently large $k$, so that
\[
\{k < \omega : (\exists j < \omega)(x_j \in A_{n_k})\}
\]
is infinite. By hypothesis, a subsequence of $(A_{n_k})$ converges to a point $K \in K(X)$ (consult Proposition 5), which is necessarily a continuum. Then $x \in K$ and $d(K, C) = 0$. As $d(A_0, C) > 0$, this means $K \cap A_l \neq \emptyset$ for some $l \neq 0$, contradicting Sierpiński’s Theorem.

Put $A_{-1} = C$, $\epsilon_{-1} = \epsilon_0$, $N^0_0 = \{0\}$, $N^0_{-1} = \{-1\}$,
\[
U^0_0 = B(A_0, 2\epsilon_0), \quad \text{and} \quad U^0_1 = B(A_{-1}, 2\epsilon_{-1}).
\]
This completes the base step.

\(^3\)Here $d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$ for $A, B \subseteq X$. 

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**Figure 1.** $\mathcal{E}_\omega$
Step $n$: Suppose $N_i^{n-1} \subseteq \omega \cup \{-1\}$ and open sets $U_i^{n-1} \subseteq X$ have been defined for each $i < 2$, and that numbers $\epsilon_k > 0$ ($k \in N_i^{n-1} \cup N_i^{n-1}$) have been chosen, so that:

$$B(A_k, \epsilon_k[3 - \sum_{j=0}^{n-1} 2^{-j}]) \subseteq U_i^{n-1} \text{ for each } k \in N_i^{n-1}; \text{ and}$$

$$A_k \cap U_i^{n-1} = \emptyset \text{ or } A_k \cap U_i^{n-1} = \emptyset \text{ for each } k < \omega.$$  \hfill (6.1)

Let $k^* < \omega \setminus (N_i^{n-1} \cup N_i^{n-1})$ be least such that $A_k^* \cap U_i^{n-1} \neq \emptyset$ for some (unique) $i < 2$ (if there is no such $k^*$, then end the recursion and put $U_i^l = U_i^{n-1}$ and $N_i^l = N_i^{n-1}$ for each $l \geq n$ and $i < 2$).

There exists $\epsilon_k^*$ such that no constituent within $2\epsilon_k^*$ of $A_k^*$ also meets

$$U_i^{n-1} := \bigcup \{B(A_k, \epsilon_k[3 - \sum_{j=0}^{n-1} 2^{-j}]) : k \in N_i^{n-1}\}.$$  Otherwise, there exists $k \in N_i^{n-1}$ such that continua meeting $B(A_k, \epsilon_k[3 - \sum_{j=0}^{n-1} 2^{-j}])$ get arbitrarily close to $A_k^*$. As in the base step, a subsequence of continua converges to a continuum $K \in K(X)$ with

$$K \cap A_k^* \neq \emptyset \quad \text{and} \quad K \cap D(A_k, \epsilon_k[3 - \sum_{j=0}^{n-1} 2^{-j}]) \neq \emptyset.$$  By (6.1) we have $A_k^* \cap D(A_k, \epsilon_k[3 - \sum_{j=0}^{n-1} 2^{-j}]) = \emptyset$. Thus $K \cap A_l \neq \emptyset$ for some $l \neq k^*$, contradicting Sierpiński’s Theorem.

Put $U_i^n := U_i^{n-1} \cup B(A_k^*, 2\epsilon_k^*)$, $N_i^n := N_i^{n-1} \cup \{k^*\}$, and $N_i^{n-1} := N_i^{n-1}$. The conditions in (6.1) are now satisfied with all instances of $n-1$ replaced by $n$. This completes the recursive step. Let

$$N_0 = \bigcup_{n<\omega} N_i^n;$$

$$N_1 = \left( I \cup \bigcup_{n<\omega} N_i^n \right) \setminus \{-1\}; \text{ and}$$

$$U_i = \bigcup_{k \in N_i} B(A_k, \epsilon_k) \text{ for each } i < 2.$$  

Obviously $U_0$ and $U_1$ are open sets, $A_0 \subseteq U_0$, and $C \subseteq U_1$. By construction we also have $U_0 \cap U_1 = \emptyset$, and $U_i$ is the union of $\{A_k : k \in N_i\}$ for $i < 2$. \hfill $\Box$
Suppose $f$ is recurrent \& arc-complete.

Let $\{d_i : i < \omega\}$ be a dense subset of $X$, and assume $\text{diam}(X) > 2$.

**Claim 7.1.** For each $i \in \{1, 2, 3, \ldots\}$ there is a sequence $(A_n)$ of disjoint continua in $X \setminus B(d_i, \frac{1}{i+1})$ such that

1. $X \setminus B(d_i, \frac{1}{i+1}) \subseteq X_i := \bigcup \{A_n : n < \omega\}$;
2. if $n \neq 0$ then $A_n$ is an arc; and
3. $\overline{X_i} = X_i$.

**Proof of Claim 7.1.**

Case 1: $\ell = [0, \infty)$.

By recurrence of $f$ there is a sequence increasing sequence $(r_k) \in [0, \infty)^\omega$ such that $f(r_k) \in B(d_i, \frac{1}{i+1})$ and $r_k \to \infty$ as $k \to \infty$. For each $z \in X \setminus B(d_i, \frac{1}{i+1})$ there is a unique $k < \omega$ such that $f^{-1}(z) \in (r_k, r_{k+1})$; let $M_z$ be the component of $f^{-1}(z)$ in $(r_k, r_{k+1}) \setminus f^{-1}[B(d_i, \frac{1}{i+1})]$. Each $M_z$ is a non-degenerate closed and bounded interval, and $\{M_z : z \in X \setminus B(d_i, \frac{1}{i+1})\}$ is countable. Let $\{A_n : n < \omega\}$ be an enumeration of $\{f[M_z] : z \in X \setminus B(d_i, \frac{1}{i+1})\}$.

Properties (i) and (ii) are clear. We need to prove $\overline{X_i} \subseteq X_i$ for (iii). Well, let $y \in \overline{X_i}$. There is a sub-sequence of arcs $(A_{n_i})$ such that $y \in \bigcup \{A_{n_i} : k < \omega\}$. By the arc-complete property, a subsequence of $(A_{m_i})$ converges to a continuum $K \in \mathcal{K}(X \setminus B(d_i, \frac{1}{i+1}))$. Then $y \in K$ and $K \cap X \setminus B(d_i, \frac{1}{i+1}) \neq \emptyset$. Let $z \in K \cap X \setminus B(d_i, \frac{1}{i+1})$. By Proposition 4 and maximality of $M_z$ we have $y \in K \subseteq f[M_z] \subseteq X_i$.

Case 2: $\ell = (-\infty, \infty)$.

Assume $f$ is positively recurrent. By the proof of Case 1 we may further assume that $f(\infty, r] \cap B(d_i, \frac{1}{i+1}) = \emptyset$ for some $r \in (-\infty, \infty)$. Let

$$ s = \sup \{r \in (-\infty, \infty) : f(\infty, r] \subseteq X \setminus B(d_i, \frac{1}{i+1})\}, $$

and put $R = f(\infty, s]$. As in Case 1 there is a sequence $(B_m)$ of disjoint arcs in $X \setminus B(d_i, \frac{1}{i+1})$ such that

$$ f(s, \infty) \setminus B(d_i, \frac{1}{i+1}) \subseteq \bigcup \{B_m : m < \omega\} $$

and $B_m \setminus B(d_i, \frac{1}{i+1}) \neq \emptyset$ for each $m < \omega$.

There is at most one $m < \omega$ such that $\overline{R} \cap B_m \neq \emptyset$. For suppose otherwise that $\overline{R} \cap B_m \neq \emptyset$ and $\overline{R} \cap B_1 \neq \emptyset$. Then there are sequences $(r_n), (s_n) \in (-\infty, s]^\omega$ such that $s_{n+1} < r_n < s_n$ for every $n < \omega$, $r_n \to -\infty$ as $n \to \infty$, $(f(r_n))$ converges to a point in $B_0$, and $(f(s_n))$ converges to a point in $B_1$. Let

$$ X' := B_0 \cup B_1 \cup \bigcup \{f[r_n, s_n] : n < \omega\}. $$

By Lemma 1 there is an $X'$-clopen set $C$ such that $B_0 \subseteq C$ and $C \cap B_1 = \emptyset$. There exists $n < \omega$ such that $f(r_n) \in C$ and $f(s_n) \in X' \setminus C$, contradicting the fact that $f[r_n, s_n]$ is connected.

If there exists such an $m$, call it $m^*$ and let $A_0 = \overline{R} \cup B_{m^*}$, $A_n = B_{n-1}$ for $1 \leq n \leq m^*$, and $A_n = B_m$ for $n > m^*$. Otherwise set $A_0 = \overline{R}$ and $A_n = B_{n-1}$ for each $n \geq 1$. The sequence $(A_n)$ is as desired.

Each $\tilde{X}_i$ (the set of continua components of $X_i$) is zero-dimensional by Lemma 1. $\tilde{X}_i$ is also separable and metrizable, and thus has a basis of clopen sets $\{C_{i,j} : j < \omega\}$ (cf. §46 V Theorem 3 [10]). For each $i < \omega$ let $\varphi_i : X_i \to \tilde{X}_i$ be the canonical epimorphism.
Since $X_i$ is closed in $X$, Urysohn’s Lemma provides for each $\langle i, j \rangle \in \omega^2$ a mapping $f_{i,j} : X \to [0, 1]$ such that
\[
f_{i,j}[\varphi_i^{-1}[C_{i,j}]] = 0 \quad \text{and} \quad f_{i,j}[X_i \setminus \varphi_i^{-1}[C_{i,j}]] = 1.
\]
Let $h : X \hookrightarrow [0, 1]^\omega$ be a homeomorphic embedding of $X$ into the Hilbert cube such that for every $\langle i, j \rangle \in \omega^2$ there exists $n < \omega$ such that $\pi_n \circ h = f_{i,j}$.

Then $Y := \overline{h[X]}$ is a metrizable continuum in which $X$ is densely embedded, and
\[
(7.1) \quad \varphi_i^{-1}[C_{i,j}] \cap X_i \setminus \varphi_i^{-1}[C_{i,j}] = \emptyset \quad \text{for each } \langle i, j \rangle \in \omega^2.
\]

**Claim 7.2.** $X$ is a composant of $Y$.

**Proof of Claim 7.2.** It suffices to show $X$ contains every proper subcontinuum of $Y$ that meets $X$. Well, suppose $K$ is a compact proper subset of $Y$ that contains points $x \in X$ and $y \in Y \setminus X$. There exists $i < \omega$ such that $\overline{B(d_i, \frac{1}{4})} \cap K = \emptyset$, so that $K \subseteq \overline{X_i}$. Denote by $\{A_n : n < \omega\}$ the set of constituents of $X_i$.

Let $m < \omega$ be the unique integer such that $x \in A_m$. There is a sequence $(x_k) \in (X_i \setminus A_m)^\omega$ such that $x_k \to y$ as $k \to \infty$. Let $n_k < \omega$ such that $x_k \in A_{n_k}$. By arc-completeness $\bigcup \{A_{n_k} : k < \omega\}$ is closed in $X$. So there exists $j < \omega$ such that $A_m \in C_{i,j}$ and $C_{i,j} \cap \{A_{n_k} : k < \omega\} = \emptyset$. In addition to (7.1) we have
\[
x \in \varphi_i^{-1}[C_{i,j}];
\]
\[
p \in X_i \setminus \varphi_i^{-1}[C_{i,j}]; \quad \text{and}
\]
\[
K \subseteq \overline{X_i} = \varphi_i^{-1}[C_{i,j}] \cup X_i \setminus \varphi_i^{-1}[C_{i,j}].
\]
Therefore $K$ is not connected. \hfill $\square$

This completes our proof of Theorem II.

8. Dimension-Preserving Compactifications

Throughout this section, assume $X$ is a connected separable metric space.

By a compactification of $X$ we shall mean a compact metrizable space in which $X$ is densely embedded. If $\xi X$ and $\gamma X$ are two compactifications of $X$, then write $\xi X \geq \gamma X$ if there is a continuous surjection $\hat{f} : \xi X \to \gamma X$ such that $\hat{f} \upharpoonright X$ is the inclusion $X \hookrightarrow \gamma X$. More precisely, $\hat{f} \upharpoonright \xi[X] = \gamma \circ \xi^{-1}$, where $\xi : X \hookrightarrow \xi X$ and $\gamma : X \hookrightarrow \gamma X$ are dense homeomorphic embeddings.

**Lemma 2.** For every compactification $\gamma X$ there is a compactification $\xi X \geq \gamma X$ such that $\dim(\xi X) = \dim(X)$.

**Proof.** Assume $\gamma X \subseteq [0, 1]^\omega$, and let $\pi_n : [0, 1]^\omega \to [0, 1]$ be the $n$-th coordinate projections. According to 1.7.C in [5] (also [4]), there is a compactification $\xi X$ such that $\dim(\xi X) = \dim(X)$ and each mapping $f_n := \pi_n \circ \gamma \circ \xi^{-1} : \xi X \to [0, 1]$ continuously extends to $\xi X$. Let $\hat{f}_n : \xi X \to [0, 1]$ be the extension of $f_n$, and define $\hat{f} : \xi X \to [0, 1]^\omega$ by $\pi_n \circ \hat{f} = \hat{f}_n$. Then $\hat{f}$ maps onto $\gamma X$ and witnesses $\xi X \geq \gamma X$. \hfill $\square$

**Lemma 3.** If $X$ is a composant of $\gamma X \leq \xi X$, then $X$ is a composant of $\xi X$.
Proof. Let $\hat{f}$ witness $\xi X \supseteq \gamma X$. Since $\hat{f} | \xi X = \gamma \circ \xi^{-1}$ is a homeomorphism onto $\gamma | X$, and $\gamma | X$ is dense in $\gamma X$, we have
\begin{equation}
\hat{f}^{-1}[\gamma | X] = \xi | X.
\end{equation}
For the remainder of the proof let us identify $X$, $\gamma | X$, and $\xi | X$.

Suppose $X$ is a composant of $\gamma X$. Let $x \in X$ be such that $X$ is the composant of $x$ in $\gamma X$. Let $P$ be the composant of $x$ in $\xi X$. Apparently, $X \subseteq P$. On the other hand, if $z \in P \setminus X$, and $Z \supseteq \{x, z\}$ is a proper subcontinuum of $\xi X$, then by (9.1) the subcontinuum $\hat{f}[Z]$ is proper and violates maximality of $X$ in $\gamma X$. Thus $P \subseteq X$. Combining the two inclusions, we have $P = X$.

Lemmas 2 and 3 directly imply Theorem III (stated in Section 1).

9. Examples in the Plane

Figures 3 through 5 show seven non-homeomorphic indecomposable plane sets. All are one-to-one images of $[0, \infty)$, except for $X_5$, $X_5$ and $X_6$, which are one-to-one images of $(-\infty, \infty)$. Examples $X_0$, $X_3$, $X_4$ and $X_5$ are composants; $X_1$, $X_2$ and $X_6$ are not.

- $X_0$ is the well-known visible composant of the bucket-handle continuum. It is a one-to-one recurrent image of $[0, \infty)$.
- $X_1$ and $X_2$ are a one-to-one recurrent images of $[0, \infty)$ which fail to be composants. $X_1$ is not arc-complete, as there is a sequence vertical arcs in $X_1$ whose endpoints limit to both $(0, 0) \not\in X_1$ and $(0, 1) \in X_1$. $X_2$ is not arc-complete; consider the horizontal arcs which limit to both $\langle \frac{1}{3}, \frac{1}{3} \rangle$ and $\langle \frac{2}{3}, \frac{1}{3} \rangle$. $X_2$ also has a disconnected
proper quasi-component. The two intervals $P := \left\{ \frac{1}{3} \right\} \times [0, \frac{1}{3}]$ and $Q := \left( \frac{2}{3}, \frac{1}{3} \right] \times [0, \frac{1}{3}]$ form a quasi-component of the proper closed subset $X_2 \cap ([0, 1] \times [0, \frac{1}{3}])$. This contrasts with $X_1$, whose only proper quasi-components are arcs. In any compactification of $X_2$, $P$ and $Q$ must be joined by a proper subcontinuum that goes outside of $X_2$.

- $X_3$ is a composant image of $(-\infty, \infty)$ which is recurrent but not bi-recurrent. The invisible composants of the bucket-handle, as well as the composants of the solenoid, are bi-recurrent.

![Figure 4. $X_4$ $X_5$](image1.png)

- $X_4$ is a one-to-one recurrent composant image of $[0, \infty)$. $X_5$ and $X_6$ are one-to-one bi-recurrent images of $(-\infty, \infty)$. $X_5$ is a composant; $X_6$ is not. Observe that $X_4$, $X_5$, and $X_6$ have no $\mathbb{Q} \times (-1, 1)$-neighborhoods at their red points.

![Figure 5. $X_6$](image2.png)

10. Questions

**Question 1.** Does every one-to-one composant image of $[0, \infty)$ embed into the plane?

**Question 2.** If $X$ is a one-to-one recurrent image of $[0, \infty)$, and $Y$ is a continuum of which $X$ is a composant, then is $Y$ necessarily chainable?

Compact one-to-one images of $[0, \infty)$ embed into the plane [13]. So by Theorem I and the fact that chainable continua are planar, a positive answer to Question 2 implies a positive answer to Question 1.

These examples derive from the quinary double bucket-handle continuum (the inverse limit of arcs with $N$-shaped bonding map). That continuum has two accessible composants, each of which is a one-to-one image of $[0, \infty)$; $X_5$ is obtained by gluing together the endpoints of these two composants. $X_6$ is a one-to-one continuous image of $X_5$. 

\[\text{Figure 5. $X_6$}\]
Question 3. Is every linear indecomposable composant equal to a composant of a continuum each of whose composants is linear?

The final two questions are motivated by a result of F. Burton Jones [7, 8]: Every locally connected one-to-one plane image of the (half-)line is locally compact.

Question 4. Is every recurrent one-to-one plane image of $[0, \infty)$ indecomposable?

Question 5. Is every bi-recurrent one-to-one plane image of $(-\infty, \infty)$ indecomposable?

Without restricting to the plane, the answers to Questions 5 and 6 are no. There is a one-to-one image of $[0, \infty)$ which is both locally connected and dense in Euclidean 3-space [7]. By similar methods, one obtains a locally connected bi-recurrent one-to-one image of $(-\infty, \infty)$.

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