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Capra-convexity, convex factorization and variational formulations for the \(\ell_0\) pseudonorm.
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Summary: The so-called \(\ell_0\) pseudonorm, or cardinality function, counts the number of nonzero components of a vector. In this paper, we analyze the \(\ell_0\) pseudonorm by means of so-called Capra (constant along primal rays) conjugacies, for which the underlying source norm and its dual norm are both orthant-strictly monotonic (a notion that we formally introduce and that encompasses the \(\ell_p\)-norms, but for the extreme ones). We obtain three main results. First, we show that the \(\ell_0\) pseudonorm is equal to its Capra-biconjugate, that is, is a Capra-convex function. Second, we deduce an unexpected consequence, that we call convex factorization: the \(\ell_0\) pseudonorm coincides, on the unit sphere of the source norm, with a proper convex lower semicontinuous function. Third, we establish a variational formulation for the \(\ell_0\) pseudonorm by means of generalized top-\(k\) dual norms and \(k\)-support dual norms (that we formally introduce).

MSC:
46N10 Applications of functional analysis in optimization, convex analysis, mathematical programming, economics
49N15 Duality theory (optimization)
46B99 Normed linear spaces and Banach spaces; Banach lattices
52A41 Convex functions and convex programs in convex geometry
90C46 Optimality conditions and duality in mathematical programming

Keywords:
\(\ell_0\) pseudonorm; orthant-strictly monotonic norm; Fenchel-Moreau conjugacy; generalized \(k\)-support dual norm; sparse optimization

Full Text: DOI

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