Fitness, Apprenticeship, and Polynomials

Bernd Sturmfels

Abstract This article discusses the design of the Apprenticeship Program at the Fields Institute, held 21 August–3 September 2016. Six themes from combinatorial algebraic geometry were selected for the two weeks: curves, surfaces, Grassmannians, convexity, abelian combinatorics, parameters and moduli. The activities were structured into fitness, research and scholarship. Combinatorics and concrete computations with polynomials (and theta functions) empowers young scholars in algebraic geometry, and it helps them to connect with the historic roots of their field. We illustrate our perspective for the threefold obtained by blowing up six points in \( \mathbb{P}^3 \).

1 Design

A thematic program on Combinatorial Algebraic Geometry took place at the Fields Institute, Toronto, Canada, during the Fall Semester 2016. The program organizers were David Cox, Megumi Harada, Diane Maclagan, Gregory Smith, and Ravi Vakil.

As part of this semester, the Clay Mathematics Institute funded the “Apprenticeship Weeks”, held 21 August–3 September 2016. This article discusses the design and mathematical scope of this fortnight. The structured activities took place in the mornings and afternoons on Monday, Wednesday, and Friday, as well as the mornings on Tuesday and Thursday. The posted schedule was identical for both weeks:

- MWF 9:00–9:30: Introduction to today’s theme
- MWF 9:30–11:15: Working on fitness problems
- MWF 11:15–12:15: Solutions to fitness problems
- MWF 14:00–14:30: Dividing into research teams
- MWF 14:30–17:00: Team work on projects

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The term “fitness” is an allusion to physical exercise. In order to improve physical fitness, many of us go to the gym. A personal trainer can greatly enhance that experience. The trainer develops your exercise plan and he pushes you beyond previously perceived limits. The trainer makes you sweat a lot, he ensures that you use exercise equipment correctly, and he helps you to feel good about yourself afterwards. In the context of team sports, the coach plays that role. She works towards the fitness of the entire team, where every player will contribute to the best of their abilities.

The six fitness sessions were designed to be as intense as those in sports. Ten problems were posted for each session, and these were available online two or three days in advance. By design, these demanding problems were open-ended and probed a different aspect of the theme. Section 3 of this article contains the complete list of problems, along with a brief discussion and references that contain some solutions.

The “apprentices” were about 40 early-career mathematicians, graduate students and postdocs, coming from a wide range of backgrounds. An essential feature of the Apprenticeship Weeks was the effort to build teams, and to promote collaboration as much as possible. This created an amazing sense of community within the group.

At 9:00am on each Monday, Wednesday or Friday, a brief introduction was given to each fitness question. We formed ten teams to work on the problems. At 11:15am we got together again, and one person from each team gave a brief presentation on what had been discussed and discovered. Working on a challenging problem, with a group of new collaborators, for less than two hours created a very intense and stimulating experience. A balanced selection process ensured that each participant had the opportunity to present for their team at least once.

At 2:00pm the entire group re-assembled and they discussed research-oriented problems for the afternoons. This was conducted in the style of the American Institute for Mathematics (AIM), whereby one of the participants serves as the discussion leader, and only that person is allowed to touch the blackboard. This led to an ample supply of excellent questions, some a direct continuation of the morning fitness problems, and others only vaguely inspired by these. Again, groups were formed for the afternoon, and they engaged in learning and research. Computations and literature search played a big role, and a lot of teaching went on in the groups.

Tuesday and Thursdays were discussion days. Here the aim was to create a sense of scholarship among the participants. The morning of these days involved studying various software packages, classical research papers from the 19-th and early 20-th centuries, and the diverse applications of combinatorial algebraic geometry. The prompts are given in Section 2. The afternoons on discussion days were unstructured to allow the participants time to ponder, probe, and write up their many new ideas.
2 Scholarship Prompts

Combinatorial algebraic geometry is a field that, by design, straddles mathematical boundaries. One aim is to study algebraic varieties with special combinatorial features. At its roots, this field is about systems of polynomial equations in several variables, and about symmetries and other special structures in their solution sets.

Section 5 offers a concrete illustration of this perspective for a system of polynomials in 32 variables. The objects of combinatorial algebraic geometry are amenable to a wide range of software tools, which are now used widely among the researchers.

Another point we discussed is the connection to problems outside of pure mathematics. A new field, Applied Algebraic Geometry, has arisen in the past decade. The techniques used there often connect back to 19th and early 20th century work in algebraic geometry, which is much more concrete and combinatorial than many recent developments. And, even for her study of current abstract theories, an apprentice may benefit from knowing the historic origins that have inspired the development of algebraic geometry. Understanding these aspects, by getting hands-on experiences and by studying original sources, was a focus in this part of the program.

In what follows we replicate the hand-outs for the four TuTh mornings. The common thread can be summarized as: back to the roots. These were given to the participants as prompts for explorations and discussions. For several of the participants, it was their first experience with software for algebraic geometry. For others, it offered a first opportunity to read an article that was published over 100 years ago.

Tuesday, August 23: Software

Which software tools are most useful for performing computations in Combinatorial Algebraic Geometry? Why?

Many of us are familiar with Macaulay2. Some of us are familiar with Singular. What are your favorite packages within these systems?

Lots of math is supported by general-purpose computer algebra systems such as Sage, Maple, Mathematica, or Magma. Do you use any of these regularly? For research or for teaching? How often and in which context?

Other packages that are useful for our community include Bertini, PHCpack, 4ti2, Polymake, Normaliz, GFan. What are these and what do they do? Who developed them and why?

Does visualization matter in algebraic geometry?

Have you tried software like Surfex?

Which software tool do you want to learn today?
Thursday, August 25: The 19th Century

Algebraic Geometry has a deep and distinguished history that goes back hundreds of years. Combinatorics entered the scene a bit more recently.

Young scholars interested in algebraic geometry are strongly encouraged to familiarize themselves with the literature from the 19th century. Dig out papers from that period and read them! Go for the original sources. Some are in English. Do not be afraid of languages like French, German, Italian.

Today we form groups. Each group will explore the life and work of one mathematician, with focus on what he has done in algebraic geometry. Identify one key paper written by that author. Then present your findings.

Here are some suggestions, listed alphabetically:

- Alexander von Brill
- Arthur Cayley
- Michel Chasles
- Luigi Cremona
- Georges Halphen
- Otto Hesse
- Ernst Kummer
- Max Noether
- Julius Plücker
- Bernhard Riemann
- Friedrich Schottky
- Hermann Schubert
- Hieronymus Zeuthen

Tuesday, August 30: Applications

The recent years have seen a lot of interest in applications of algebraic geometry, outside of core pure mathematics. An influential event was a research year 2006-07 at the IMA in Minneapolis. Following a suggestion by Doug Arnold (then IMA director and SIAM president), it led to the creation of the SIAM activity group in Algebraic Geometry, and (ultimately) to the SIAM Journal on Applied Algebra and Geometry. The reader is referred to these resources for further information. These interactions with the sciences and engineering have been greatly enhanced by the interplay with Combinatorics and Computation seen here at the Fields Institute. However, the term “Algebraic Geometry” has to be understood now in a broad sense.

Today we form groups. Each group will get familiar with one field of application, and they will select one paper in Applied Algebraic Geometry that represent an interaction with that field. Read your paper and then present your findings. Here are some suggested fields, listed alphabetically:

- Approximation Theory
Thursday, September 1: The Early 20th Century

One week ago we examined the work of some algebraic geometers from the 19th century. Today, we move on to the early 20th century, to mathematics that was published prior to World War II. You are encouraged to familiarize yourselves with the literature from the period 1900-1939. Dig out papers from that period and read them! Go for the original sources. Some are written in English. Do not be afraid of languages like French, German, Italian, Russian.

Each group will explore the life and work of one mathematician, with focus on what (s)he has done in algebraic geometry during that period. Identify one key paper written by that author. Then present your findings.

Here are some suggestions, listed alphabetically:

- Eugenio Bertini
- Guido Castelnuovo
- Wei-Liang Chow
- Arthur B. Coble
- Wolfgang Gröbner
- William V.D. Hodge
- Wolfgang Krull
- Solomon Lefschetz
- Frank Morley
- Francis S. Macaulay
- Amalie Emmy Noether
- Ivan Georgievich Petrovsky
- Virginia Ragsdale
- Gaetano Scorza
- Francesco Severi
3 Fitness Prompts

This section presents the six worksheets for the morning sessions on Mondays, Wednesdays and Fridays. These prompts inspired most of the articles in this volume. Specific pointers to dates refer to events that took place at the Fields Institute. The next section contains notes for each problem, offering references and solutions.

Monday, August 22: Curves

1. Which genus can a smooth curve of degree 6 in $\mathbb{P}^3$ have? Give examples.
2. Let $f(x) = (x - 1)(x - 2)(x - 3)(x - 6)(x - 7)(x - 8)$ and consider the genus 2 curve $y^2 = f(x)$. Where is it in the moduli space $\mathcal{M}_2$? Compute the Igusa invariants. Draw the Berkovich skeleton for the field of 5-adic numbers.
3. The tact invariant of two plane conics is the polynomial of bidegree (6, 6) in the 6 + 6 coefficients which vanishes when the conics are tangent. Compute this invariant explicitly. How many terms does it have?
4. Bring’s curve lives in a hyperplane in $\mathbb{P}^4$. It is defined by $x_0^6 + x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$ for $i = 1, 2, 3$. What is its genus? Determine all tritangent planes of this curve.
5. Let $X$ be a curve of degree $d$ and genus $g$ in $\mathbb{P}^3$. The Chow form of $X$ defines a hypersurface in the Grassmannian $\text{Gr}(1, \mathbb{P}^3)$. Points are lines that meet $X$. Find the dimension and (bi)degree of its singular locus.
6. What are the equations of the secant varieties of elliptic normal curves?
7. Let $X_P$ be the toric variety defined by a 3-dimensional lattice polytope, as in Milena Hering’s July 18-22 course. Intersect $X_P$ with two general hyperplanes to get a curve. What is the degree and genus of that curve?
8. A 2009 article by Sean Keel and Jenia Tevelev presents Equations for $\mathcal{M}_{0,n}$. Write these equations in Macaulay2 format for $n = 5$ and $n = 6$. Can you see the $\psi$-classes (seen in Renzo Cavalieri’s July 18-22 course) in these coordinates?
9. Review the statement of Torelli’s Theorem for genus 3. Using Sage or Maple, compute the $3 \times 3$ Riemann matrix of the Fermat quartic $\{x^4 + y^4 + z^4 = 0\}$. How can you recover the curve from that matrix?
10. The moduli space $\mathcal{M}_7$ of genus 7 curves has dimension 18. What is the codimension of the locus of plane curves? Hint: Singularities are allowed.

Wednesday, August 24: Surfaces

1. A nondegenerate surface in $\mathbb{P}^n$ has degree at least $n - 1$. Prove this fact and determine all surfaces of degree $n - 1$. Give their equations.
2. How many lines lie on a surface obtained by intersecting two quadratic hypersurfaces in $\mathbb{P}^4$? Find an instance where all lines are defined over $\mathbb{Q}$.
3. What is the maximum number of singular points on an irreducible quartic surface in $\mathbb{P}^3$? Find a surface and compute its projective dual.
4. Given a general surface of degree $d$ in $\mathbb{P}^3$, the set of its bitangent lines is a surface in $\text{Gr}(1, \mathbb{P}^3)$. Determine the cohomology class (or bidegree) of that surface.
5. Pick two random circles $C_1$ and $C_2$ in $\mathbb{R}^3$. Compute their Minkowski sum $C_1 + C_2$ and their Hadamard product $C_1 \ast C_2$. Try other curves.
6. Let $X$ be the surface obtained by blowing up five general points in the plane. Compute the Cox ring of $X$. Which of its ideals describe points on $X$?
7. The incidences among the 27 lines on a cubic surface defines a 10-regular graph. Compute the complex of independent sets in this graph.
8. The Hilbert scheme of points on a smooth surface is smooth. Why? How many torus-fixed points are there on the Hilbert scheme of 20 points in $\mathbb{P}^2$? What can you say about the graph that connects them?
9. State the Hodge Index Theorem. Verify this theorem for cubic surfaces in $\mathbb{P}^3$, by explicitly computing the matrix for the intersection pairing.
10. List the equations of one Enriques surface. Verify its Hodge diamond.

**Friday, August 26: Grassmannians**

1. Find a point in $\text{Gr}(3, 6)$ with precisely 16 non-zero Plücker coordinates. As in June Huh’s July 18-22 course, determine the Chow ring of its matroid.
2. The coordinate ring of the Grassmannian $\text{Gr}(3, 6)$ is a cluster algebra of finite type. What are the cluster variables? List all the clusters.
3. Consider two general surfaces in $\mathbb{P}^3$ whose degrees are $d$ and $e$ respectively. How many lines in $\mathbb{P}^3$ are bitangent to both surfaces?
4. The rotation group $\text{SO}(n)$ is an affine variety in the space of real $n \times n$-matrices. Can you find a formula for the degree of this variety?
5. The complete flag variety for $\text{GL}(4)$ is a six-dimensional subvariety of $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$. Compute its ideal and determine its tropicalization.
6. Classify all toric ideals that arises as initial ideals for the flag variety above. For each such toric degeneration, compute the Newton-Okounkov body.
7. The Grassmannian $\text{Gr}(4, 7)$ has dimension 12. Four Schubert cycles of codimension 3 intersect in a finite number of points. How large can that number be? Exhibit explicit cycles whose intersection is reduced.
8. The affine Grassmannian and the Sato Grassmannian are two infinite-dimensional versions of the Grassmannian. How are they related?
9. The coordinate ring of the Grassmannian $\text{Gr}(2, 7)$ is $\mathbb{Z}^7$-graded. Determine the Hilbert series and the multidegree of $\text{Gr}(2, 7)$ for this grading.
10. The Lagrangian Grassmannian parametrizes $n$-dimensional isotropic subspaces in $\mathbb{C}^{2n}$. Find a Gröbner basis for its ideal. What is a ‘doset’?

**Monday, August 29: Convexity**

1. The set of nonnegative binary sextics is a closed full-dimensional convex cone in $\text{Sym}_6(\mathbb{R}^2) \simeq \mathbb{R}^7$. Determine the face poset of this convex cone.
2. Consider smooth projective toric fourfolds with eight invariant divisors. What is the maximal number of torus-fixed points of any such variety?

3. Choose three general ellipsoids in $\mathbb{R}^3$ and compute the convex hull of their union. Which algebraic surfaces contribute to the boundary?

4. Explain how the Alexandrov-Fenchel Inequalities (for convex bodies) can be derived from the Hodge Index Theorem (for algebraic surfaces).

5. The blow-up of $\mathbb{P}^3$ at six general points is a threefold that contains 32 special surfaces (exceptional classes). What are these surfaces? Which triples intersect? Hint: Find a 6-dimensional polytope that describes the combinatorics.

6. Prove that every face of a spectrahedron is an exposed face.

7. How many combinatorial types of reflexive polytopes are there in dimension 3? In dimension 4? Draw pictures of some extreme specimen.

8. A $4 \times 4$-matrix has six off-diagonal $2 \times 2$-minors. Their binomial ideal in 12 variables has a unique toric component. Determine the $f$-vector of the polytope (with 12 vertices) associated with this toric variety.

9. Consider the Plücker embedding of the real Grassmannian $\text{Gr}(2,5)$ in the unit sphere in $\mathbb{R}^{10}$. Describe its convex hull. Hint: Calibrations, Orbitopes.

10. Examine Minkowski sums of three tetrahedra in $\mathbb{R}^3$. What is the maximum number of vertices such a polytope can have? How to generalize?

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**Wednesday, August 31: Abelian Combinatorics**

1. The intersection of two quadratic surfaces in $\mathbb{P}^3$ is an elliptic curve. Explain its group structure in terms of geometric operations in $\mathbb{P}^3$.

2. A 2006 paper by Keiichi Gunji gives explicit equations for all abelian surfaces in $\mathbb{P}^8$. Verify his equations in Macaulay2. How to find the group law?

3. Experiment with Świerczewski’s Sage code for the numerical evaluation of the Riemann theta function $\theta(\tau; z)$. Verify the functional equation.

4. Theta functions with characteristics $\theta(\epsilon, \epsilon') (\tau; z)$ are indexed by two binary vectors $\epsilon, \epsilon' \in \{0, 1\}^g$. They are odd or even. How many each?

5. Fix the symplectic form $\langle x, y \rangle = x_1 y_4 + x_2 y_5 + x_3 y_6 + x_4 y_1 + x_5 y_2 + x_6 y_3$ on the 64-element vector space $(\mathbb{F}_2)^6$. Determine all isotropic subspaces.

6. Explain the combinatorics of the root system of type $E_7$. How would you choose coordinates? How many pairs of roots are orthogonal?

7. In 1879 Cayley published a paper in Crelle’s journal titled *Algorithms for ...* What did he do? How does it relate the previous two exercises?

8. The regular matroid $R_{10}$ defines a degeneration of abelian 5-folds. Describe its periodic tiling on $\mathbb{R}^3$ and secondary cone in the 2-nd Voronoi decomposition. Explain the application to Prym varieties due to Gwena.

9. Consider the Jacobian of the plane quartic curve defined over $\mathbb{Q}_2$ by
41x^4 + 1530x^3y + 3508x^3z + 1424x^2y^2 + 2490x^2yz \\
- 2274x^2z^2 + 470xy^3 + 680xy^2z - 930xyz^2 + 772x^3 \\
+ 535y^4 - 350y^3z - 1960y^2z^2 - 3090yz^3 - 2047z^4

Compute its limit in Alexeev’s moduli space for the 2-adic valuation.

10. Let $\Theta$ be the theta divisor on an abelian threefold $X$. Find $n = \dim H^0(X, k\Theta)$. What is the smallest integer $k$ such that $k\Theta$ is very ample? Can you compute (in Macaulay2) the ideal of the corresponding embedding $X \hookrightarrow \mathbb{P}^{n-1}$?

Friday, September 2: Parameters and Moduli

1. Write down (in Macaulay2 format) the two generators of the ring of invariants for ternary cubics. For which plane cubics do both invariants vanish?
2. Fix a $\mathbb{Z}$-grading on the polynomial ring $S = \mathbb{C}[a,b,c,d]$ defined by $\deg(a) = 1$, $\deg(b) = 4$, $\deg(c) = 5$, and $\deg(d) = 9$. Classify all homogeneous ideals $I$ such that $S/I$ has Hilbert function identically equal to 1.
3. Consider the Hilbert scheme of eight points in affine 4-space $\mathbb{A}^4$. Identify a point that is not in the main component. List its ideal generators.
4. Let $X$ be the set of all symmetric $4 \times 4$-matrices in $\mathbb{R}^{4 \times 4}$ that have an eigenvalue of multiplicity $\geq 2$. Compute the $\mathbb{C}$-Zariski closure of $X$.
5. Which cubic surfaces in $\mathbb{P}^3$ are stable? Which ones are semi-stable?
6. In his second lecture on August 15, Valery Alexeev used six lines in $\mathbb{P}^2$ to construct a certain moduli space of K3 surfaces with 15 singular points. List the most degenerate points in the boundary of that space.
7. Find the most singular point on the Hilbert scheme of 16 points in $\mathbb{A}^3$.
8. The polynomial ring $\mathbb{C}[x,y]$ is graded by the 2-element group $\mathbb{Z}/2\mathbb{Z}$ where $\deg(x) = 1$ and $\deg(y) = 1$. Classify all Hilbert functions of homogeneous ideals.
9. Consider all threefolds obtained by blowing up six general points in $\mathbb{P}^3$. Describe their Cox rings and Cox ideals. How can you compactify this moduli space?
10. The moduli space of tropical curves of genus 5 is a polyhedral space of dimension 12. Determine the number of $i$-faces for $i = 0, 1, 2, \ldots, 12$.

4 Notes, Solutions and References

Solutions to several of the sixty fitness problems can be found in the 16 articles of this volume. The articles are listed as the first 16 entries in our References. They will be published in the order in which they are cited in this section. In what follows we also offer references for other problems that did not lead to articles in this book.
Notes on Curves

1. Castelnuovo classified the degree and genus pairs \((d, g)\) for all smooth curves in \(\mathbb{P}^n\). This was extended to characteristic \(p\) by Ciliberto [25]. For \(n = 3, d = 6\), the possible genera are \(g = 0, 1, 2, 3, 4\). The Macaulay2 package RandomCurves can compute examples. The Hartshorne-Rao module [50] plays a key role.

2. See Section 2 in the article by Bolognese, Brandt and Chua [1]. The approach using Igusa invariants was developed by Helminck in [32].

3. The tact invariant has 3210 terms, by [57, Example 2.7].

4. See Section 2.1 in the article by Harris and Len [2]. The analogous problem for bitangents of plane quartics is discussed by Chan and Jiradilok [3].

5. This is solved in the article by Kohn, Nødland and Tripoli [4].

6. Following Fisher [29], elliptic normal curves are defined by the \(4 \times 4\)-subpfaffians of the Klein matrix, and their secant varieties are defined by its larger subpfaffians.

7. The degree of a projective toric variety \(X_P\) is the volume of its lattice polytope \(P\). The genus of a complete intersection in \(X_P\) was derived by Khovanskii in 1978. We recommend the tropical perspective offered by Steffens and Theobald in [53, §4.1].

8. See the article by Monin and Rana [5] for a solution up to \(n = 6\).

9. See [26] for how to compute the forward direction of the Torelli map of an arbitrary plane curve. For computing the backward direction in genus 3 see [61, §5.2].

10. Trinodal sextics form a 16-dimensional family; their codimension in \(\mathcal{M}_7\) is two. This is a result due to Severi, derived by Castryck and Voight in [24, Theorem 2.1].

Notes on Surfaces

1. This was solved by Del Pezzo in 1886. Eisenbud and Harris [27] give a beautiful introduction to the theory of varieties of minimal degree, including their equations.

2. This is a del Pezzo surface of degree 4. It has 16 lines. To make them rational, map \(\mathbb{P}^2\) into \(\mathbb{P}^3\) via a \(\mathbb{Q}\)-basis for the cubics that vanish at five rational points in \(\mathbb{P}^2\).

3. The winner, with 16 singular points, is the Kummer surface [34]. It is self-dual.

4. This is solved in the article by Kohn, Nødland and Tripoli [4].

5. See Section 5 in the article by Friedenberg, Oneto and Williams [6].

6. This is the del Pezzo surface in Problem 2. Its Cox ring is a polynomial ring in 16 variables modulo an ideal generated by 20 quadrics. Ideal generators that are universal over the base \(\mathbb{M}_{0,5}\) are listed in [47, Proposition 2.1]. Ideals of points on the surface are torus translates of the toric ideal of the 5-dimensional demicube \(D_5\). For six points in \(\mathbb{P}^2\) we refer to Bernal, Corey, Donten-Bury, Fujita and Merz [7].

7. This is the clique complex of the Schl"{a}fli graph. The \(f\)-vector of this simplicial complex is \((27, 216, 720, 1080, 648, 72)\). The Schl"{a}fli graph is the edge graph of the \(E_8\)-polytope, denoted \(2_{21}\), which is a cross section of the Mori cone of the surface.

8. The torus-fixed points on \(\text{Hilb}_{20}(\mathbb{P}^2)\) are indexed by ordered triples of partitions.
(λ₁, λ₂, λ₃) with |λ₁| + |λ₂| + |λ₃| = 20. The number of such triples equals 341,649. The graph that connects them is a variant of the graph for the Hilbert scheme of points in the affine plane. The latter was studied by Hering and Maclagan in [33].

9. The signature of the intersection pairing is \((1, r - 1)\) where \(r\) is the rank of the Picard group. This is \(r = 7\) for the cubic surface. From the analysis in Problem 7, we can get various symmetric matrices that represent the intersection pairing.

10. See the article by Bolognese, Harris and Jelisiejew [8].

Notes on Grassmannians

1. See the article by Wiltshire-Gordon, Woo and Zajackowska [9].

2. In addition to the 20 Plücker coordinates \(p_{ijk}\), one needs two more functions, namely \(p_{123} p_{456} - p_{124} p_{356}\) and \(p_{234} p_{561} - p_{235} p_{461}\). The six boundary Plücker coordinates \(p_{123}, p_{234}, p_{345}, p_{456}, p_{561}, p_{612}\) are frozen. The other 16 coordinates are the cluster variables for Gr\((3, 6)\). This was derived by Scott in [51, Theorem 6].

3. This is worked out in the article by Kohn, Nødland and Tripoli [4].

4. This is the main result of Brandt, Bruce, Brysiewicz, Krone and Robeva [10].

5. See the article by Bossinger, Lamboglia, Mincheva and Mohammadi [11].

6. See the article by Bossinger, Lamboglia, Mincheva, Mohammadi [11].

7. The maximum number is 8. This is obtained by taking the partition \((2, 1)\) four times. For this problem, and many other Schubert problems, instances exist where all solutions are real. See the works of Sottile, specifically [52, Theorem 3.9 (iv)].

8. The Sato Grassmannian is more general than the affine Grassmannian. These are studied, respectively, in integrable systems and in geometric representation theory.

9. A formula for the \(\mathbb{Z}^n\)-graded Hilbert series of Gr\((2, n)\) is given by Witaszek [63, §3.3]. For an introduction to multidegrees see [40, §8.5]. Try the Macaulay2 commands Grassmannian and multidegree. Escobar and Knutson [12] determine the multidegree of a variety that is important in computer vision.

10. The coordinate ring of the Lagrangian Grassmannian is an algebra with straightening law over a doset. See the exposition in [48, §3].

Notes on Convexity

1. The face lattice of the cone of non-negative binary forms of degree \(d\) is described in Barvinok’s textbook [20, §II.11]. In more variables this is much more difficult.

2. This seems to be an open problem. For seven invariant divisors, this was resolved by Gretenkort et al. [30]. Note the conjecture stated in the last line of that paper.

3. We refer to Nash, Pir, Sottile and Ying [13] and to the YouTube video The Convex Hull of Ellipsoids by Nicola Gisermann, Michael Hemmer, and Elmar Schömer.

4. We refer to Ewald’s textbook, specifically [28, §IV.5 and §VII.6].
5. The relevant polytope is the 6-dimensional demicube; its 32 vertices correspond to the 32 special divisors. See the notes for Problem 9 in Parameters and Moduli.

6. This was first proved by Ramana and Goldman in [43, Corollary 1].

7. Kreuzer and Skarke [37] classified such reflexive polytopes up to lattice isomorphism. There are 4319 in dimension 3, and there are 473800776 in dimension 4. Lars Kastner classified the list of 4319 into combinatorial types. He found that there are 558 combinatorial types of reflexive 3-polytopes. They have up to 14 vertices.

8. This 6-dimensional polytope is obtained from the direct product of two identical regular tetrahedra by removing the four pairs of corresponding vertices. It is the convex hull of the points $e_i \oplus e_j$ in $\mathbb{R}^4 \oplus \mathbb{R}^4$ where $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Using the software PolyMAKE, we find its f-vector to be $(12, 54, 110, 108, 52, 12)$.

9. The faces of the Grassmann orbitopes $\text{conv}(\text{Gr}(2, n))$ for $n \geq 5$ are described in [49, Theorem 7.3]. It is best to start with the easier case $n = 4$ in [49, Example 7.1].

10. The maximum number of vertices is 38, by the formula of Karavelas et al. in [35, §6.1, equation (49)]. A definitive solution to the problem of characterizing face numbers of Minkowski sums of polytopes was given by Adiprasito and Sanyal [17].

Notes on Abelian Combinatorics

1. A beautiful solution was written up by Qiaochu Yuan when he was a high school student; see [62]. The idea is to simultaneously diagonalize the two quadrics, then project their intersection curve into the plane, thereby obtaining an Edwards curve.

2. This is a system of 9 quadrics and 3 cubics, derived from Coble’s cubic as in [45, Theorem 3.2]. Using theta functions as in [45, Lemma 3.3], one gets the group law.

3. See [61] and compare with Problem 9 in Curves.

4. For the $2^{2g}$ pairs $(\varepsilon, \varepsilon')$, we check whether $\varepsilon \cdot \varepsilon'$ is even or odd. There are $2^{2g-1}(2^g + 1)$ even theta characteristics and $2^{2g-1}(2^g - 1)$ odd theta characteristics.

5. The number of isotropic subspaces of $(\mathbb{F}_2)^6$ is 63 of dimension 1, it is 315 in dimension 2, and it is 135 in dimension 3. The latter are the Lagrangians [46, §6].

6. The root system of type $E_7$ has 63 positive roots. They are discussed in [46, §6].

7. Cayley gives a bijection between the 63 positive roots of $E_7$ with the 63 non-zero vectors in $(\mathbb{F}_2)^6$. Two roots have inner product zero if and only if the corresponding vectors in $(\mathbb{F}_2)^6 \setminus \{0\}$ are orthogonal in the setting of Problem 5. See [46, Table 1].

8. This refers to Gwena’s article [31]. Since the matroid $R_{10}$ is not co-graphic, the corresponding tropical abelian varieties are not in the Schottky locus of Jacobians.

9. This fitness problem is solved in the article by Bolognese, Brandt and Chua [1] Chan and Jiradilok [3] study an important special family of plane quartics.

10. The divisor $k\Theta$ is very ample for $k = 3$. This embeds any abelian threefold into $\mathbb{P}^{26}$. For products of three cubic curves, each in $\mathbb{P}^2$, this gives the Segre embedding.
Notes on Parameters and Moduli

1. The solution can be found, for instance, on the website

   http://math.stanford.edu/~notzeb/aronhold.html

The two generators have degree 4 and 6. The quartic invariant is known as the Aronhold invariant and it vanishes when the ternary cubic is a sum of three cubes of linear forms. Both invariants vanish when the cubic curve has a cusp.

2. This refers to extra irreducible components in toric Hilbert schemes [42]. These schemes were first introduced by Arnold [19], who coined the term A-graded algebras. Theorem 10.4 in [54] established the existence of an extra component for $A = (1347)$. We ask to verify the second entry in Table 10-1 on page 88 of [54].

3. Cartwright et al. [22] showed that the Hilbert scheme of eight points in $\mathbb{A}^4$ has two irreducible components. An explicit point in the non-smoothable component is given in the article by Douvropoulos, Jelisiejew, Nødland and Teitler [14].

4. At first, it is surprising that $X$ has codimension 2. The point is that we work over the real numbers $\mathbb{R}$. The analogous set over $\mathbb{C}$ is the hypersurface of a sum-of-squares polynomial. The $\mathbb{C}$-Zariski closure of $X$ is a nice variety of codimension 2. The defining ideal and its Hilbert-Burch resolution are explained in [56, §7.1].

5. This is an exercise in Geometric Invariant Theory [41]. A cubic surface is stable if and only if it has at most ordinary double points (A1 singularities). For semi-stable surfaces, A2 singularities are allowed. For an exposition see [44, Theorem 3.6]; this is E. Reinecke’s Bachelor thesis, written under the supervision of D. Huybrechts.

6. This is the moduli space of stable hyperplane arrangements [18], here for the case of six lines in $\mathbb{P}^2$. The precise space depends on a choice of parameters [18, §5.7]. For some natural parameters, this is the tropical compactification associated with the tropical Grassmannian $Gr(3, 6)$, so the most degenerate points correspond to the seven generic types of tropical planes in 5-space, shown in [38, Figure 5.4.1].

7. See [55, Theorem 2.3].

8. For each partition, representing a monomial ideal in $\mathbb{C}[x, y]$, we count the odd and even boxes in its Young diagram. The resulting Hilbert functions $h : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{N}$ are $(h(\text{even}), h(\text{odd})) = (k^2 + m, k(k + 1) + m)$ or $((k + 1)^2 + m, k(k + 1) + m)$, where $k, m \in \mathbb{N}$. This was contributed by Dori Bejleri. For more details see [21, §1.3].

9. The blow-up of $\mathbb{P}^{n-3}$ at $n$ points is a Mori dream space. Its Cox ring has $2^{n-1}$ generators, constructed explicitly by Castravet and Tevelev in [23]. These form a Khovanskii basis [36], by [60, Theorem 7.10]. The Cox ideal is studied in [59]. Each point on its variety represents a rank two stable quasiparabolic vector bundle on $\mathbb{P}^1$ with $n$ marked points. The relevant moduli space is $\mathcal{M}_{0,n}$.

10. The moduli space of tropical curves of genus 5 serves as the first example in the article by Lin and Ulirsch [15]. The article by Kastner, Shaw and Winz [16] discusses state-of-the-art software tools for computing with such polyhedral spaces.
5 Polynomials

The author of this article holds the firm belief that algebraic geometry concerns the study of solution sets to systems of polynomial equations. Historically, geometers explored curves and surfaces that are zero sets of polynomials. It is the insights gained from these basic figures that have led, over the course of centuries, to the profound depth and remarkable breadth of contemporary algebraic geometry. However, many of the current theories are now far removed from explicit varieties, and polynomials are nowhere in sight. What we are advocating is for algebraic geometry to take an outward-looking perspective. Our readers should be aware of the wealth of applications in the sciences and engineering, and be open to a “back to the basics” approach in both teaching and scholarship. From this perspective, the interaction with combinatorics can be particularly valuable. Indeed, combinatorics is known to some as the “nanotechnology of mathematics”. It is all about explicit objects, those that can be counted, enumerated, and dissected with laser precision. And, these objects include some beautiful polynomials and the ideals they generate.

The following example serves as an illustration. We work in a polynomial ring $\mathbb{Q}[p]$ in 32 variables, one for each subset of $\{1, 2, 3, 4, 5, 6\}$ whose cardinality is odd:

$$P_1, P_2, \ldots, P_6, P_{123}, P_{124}, P_{125}, \ldots, P_{356}, P_{456}, P_{12345}, P_{12346}, \ldots, P_{23456}.$$  

The polynomial ring $\mathbb{Q}[p]$ is $\mathbb{Z}_7$-graded by setting $\deg(p_\sigma) = e_0 + \sum_{i \in \sigma} e_i$, where $e_0, e_1, \ldots, e_6$ is the standard basis of $\mathbb{Z}_7$. Let $X$ be a $5 \times 6$-matrix of variables, and let $I$ be the kernel of the ring map $\mathbb{Q}[p] \to \mathbb{Q}[X]$ that takes the variables $p_\sigma$ to the determinant of the submatrix of $X$ with column indices $\sigma$ and row indices $1, 2, \ldots, |\sigma|$.

The ideal $I$ is prime and $\mathbb{Z}_7$-graded. It has multiple geometric interpretations. First of all, it describes the partial flag variety of points in 2-planes in hyperplanes in $\mathbb{P}^5$. This flag variety lives in $\mathbb{P}^5 \times \mathbb{P}^{19} \times \mathbb{P}^5$, thanks to the Plücker embedding. Its projection into the factor $\mathbb{P}^{19}$ is the Grassmannian $\text{Gr}(3, 6)$ of 2-planes in $\mathbb{P}^5$. Flag varieties are studied by Bossinger, Lamboglia, Mincheva and Mohammadi in [11].

But, let the allure of polynomials now speak for itself. Our ideal $I$ has 66 minimal quadratic generators. Sixty generators are unique up to scaling in their degree:

| degree | ideal generator |
|--------|-----------------|
| $(2, 0, 0, 1, 1, 1)$ | $P_{3}P_{4}P_{56} - P_{4}P_{356} + P_{5}P_{346} - P_{6}P_{345}$ |
| $(2, 0, 1, 0, 1, 1)$ | $P_{2}P_{4}P_{6} - P_{4}P_{256} + P_{5}P_{246} - P_{6}P_{245}$ |
| ... | ... |
| $(2, 1, 1, 1, 1, 0, 0)$ | $P_{1}P_{2}P_{3} - P_{2}P_{134} + P_{3}P_{124} - P_{4}P_{123}$ |
| $(2, 0, 1, 1, 1, 1, 2)$ | $P_{256}P_{346} - P_{246}P_{356} + P_{236}P_{456}$ |
| ... | ... |
| $(2, 2, 1, 1, 1, 1, 0)$ | $P_{12}P_{3}P_{14} - P_{12}P_{135} + P_{123}P_{145}$ |
| $(2, 1, 1, 1, 1, 2, 2)$ | $P_{156}P_{23456} - P_{256}P_{13456} + P_{356}P_{12456} - P_{456}P_{12356}$ |
| ... | ... |
| $(2, 2, 1, 1, 1, 1)$ | $P_{123}P_{12456} - P_{124}P_{12356} + P_{125}P_{12346} - P_{126}P_{12345}$ |
The other six minimal generators live in degree \((2, 1, 1, 1, 1, 1, 1)\). These are the 4-term Grassmann-Pücker relations, like \(p_{126}p_{345} - p_{125}p_{346} + p_{124}p_{356} - p_{123}p_{456}\).

Here is an alternate interpretation of the ideal \(I\). It defines a variety of dimension \(15 = \binom{6}{2}\) in \(\mathbb{P}^{31}\) known as the *spinor variety*. In this guise, \(I\) encodes the algebraic relations among the principal subpfaffians of a skew-symmetric \(6 \times 6\)-matrix. Such subpfaffians are indexed with the subsets of \(\{1, 2, 3, 4, 5, 6\}\) of even cardinality. The trick is to fix a natural bijection between even and odd subsets. This variety is similar to the *Lagrangian Grassmannian* seen in fitness problem # 10 on Grassmannians.

At this point, readers who like combinatorics and computations may study Cox ideal by duplicating the ideal of the spinor variety:

\[
I = (p_{126}p_{345} - p_{125}p_{346} + p_{124}p_{356} - p_{123}p_{456})
\]

We now come to a third, and even more interesting, geometric interpretation of our 66 polynomials. It has to do with *Cox rings*, and their Khovanskii bases, similar to those in the article by Bernal, Corey, Donten-Bury, Fujita and Merz. We begin by replacing the generic \(5 \times 6\)-matrix \(X\) by one that has the special form in [23, (1.2)]:

\[
X = \begin{pmatrix}
  u_1^2x_1 & u_2^2x_2 & u_3^2x_3 & u_4^2x_4 & u_5^2x_5 & u_6^2x_6 \\
  u_1y_1 & u_2y_2 & u_3y_3 & u_4y_4 & u_5y_5 & u_6y_6 \\
  u_1v_1x_1 & u_2v_2x_2 & u_3v_3x_3 & u_4v_4x_4 & u_5v_5x_5 & u_6v_6x_6 \\
  v_1y_1 & v_2y_2 & v_3y_3 & v_4y_4 & v_5y_5 & v_6y_6 \\
  v_1^2x_1 & v_2^2x_2 & v_3^2x_3 & v_4^2x_4 & v_5^2x_5 & v_6^2x_6
\end{pmatrix}.
\]

Now, the polynomial ring \(\mathbb{Q}[X]\) gets replaced by \(k[x_1, \ldots, x_6, y_1, \ldots, y_6]\) where \(k\) is the field extension of \(\mathbb{Q}\) generated by the entries of a \(2 \times 6\)-matrix of scalars:

\[
U = \begin{pmatrix}
  u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
  v_1 & v_2 & v_3 & v_4 & v_5 & v_6
\end{pmatrix}.
\]

We assume that the \(2 \times 2\)-minors of \(U\) are non-zero. Let \(J\) denote the kernel of the odd-minors map \(k[p] \to k[X]\) as before. The ideal \(J\) is also \(\mathbb{Z}^7\)-graded and it strictly contains the ideal \(I\). Castravet and Tevelev [23, Theorem 1.1] proved that \(k[p]/J\) is the *Cox ring* of the blow-up of \(\mathbb{P}^2_k\) at six points. These points are Gale dual to \(U\).

We refer to \(J\) as the *Cox ideal* of that rational threefold whose Picard group \(\mathbb{Z}^7\) furnishes the grading. The affine variety in \(A^{32}_k\) defined by \(J\) is 10-dimensional (it is the *universal torsor*). Quotienting by a 7-dimensional torus action yields our threefold. The same story for blowing up five points in \(\mathbb{P}^2_k\) is problem # 6 on Surfaces.

In [59] we construct the Cox ideal by duplicating the ideal of the spinor variety:
Here \( \mathbf{u} \) is a vector in \((K^*)^3\) that is derived from \(U\). The ideal \( \mathbf{u} \ast I \) is obtained from \(I\) by scaling the variables \(f_\sigma\) with the coordinates of \(\mathbf{u}\). In particular, the Cox ideal \(J\) is minimally generated by 132 quadrics. Now, there are two generators in each of the sixty \(\mathbb{Z}_7\)-degrees in our table, and there are 12 generators in degree \((2, 1, 1, 1, 1, 1)\).

Following [60, Example 7.6], we fix the rational function field \(k = \mathbb{Q}(t)\) and set

\[
U = \begin{pmatrix}
1 & t & t^2 & t^3 & t^4 & t^5 \\
1 & t^3 & t^2 & t & 1
\end{pmatrix}.
\]

The ring map \(k[p] \to k[X]\) now maps the variables \(p_\sigma\) like this:

\[
\begin{array}{c}
p_1 \mapsto x_1 \\
p_{123} \mapsto x_1y_2x_3t^6 - (x_1x_2y_3 + x_1x_2x_3)t^7 + (x_1x_2x_3 + x_1x_3y_3)t^9 - x_1y_2x_3t^{10} \\
p_{12345} \mapsto x_1y_2x_3y_4x_5t^{10} - (x_1y_2x_3y_4x_5 + x_1y_2x_3x_4y_5 + \cdots + x_1y_2x_3x_4x_5)t^{11} + \cdots
\end{array}
\]

Here is a typical example of a \(\mathbb{Z}_7\)-degree with two minimal ideal generators:

\[
(2, 1, 1, 1, 1, 0, 0) \quad p_1p_{234} - p_2p_{134} + p_3p_{124} - p_4p_{123} \\
(2, 1, 1, 1, 0, 0) \quad t^4p_1p_{234} - t^2p_2p_{134} + t^2p_1p_{234} + p_4p_{123}
\]

The algebra generators \(p_\sigma\) form a Khovanskii basis for \(k[p]/J\) with respect to the \(t\)-adic valuation. The toric algebra resulting from this flat family is generated by the underlined monomials. Its toric ideal \(\text{in}(J)\) is generated by 132 binomial quadrics:

| degree     | pair of binomial generators for \(\text{in}(J)\) |
|------------|-----------------------------------------------|
| \((2, 0, 0, 1, 1, 1)\) | \(p_3p_{456} - p_4p_{356}\) \(p_5p_{346} - p_6p_{345}\) |
| \((2, 0, 1, 0, 1, 1)\) | \(p_2p_{456} - p_4p_{256}\) \(p_5p_{246} - p_6p_{245}\) |
| \((2, 1, 1, 1, 0, 0)\) | \(p_1p_{234} - p_2p_{134}\) \(p_3p_{124} - p_4p_{123}\) |
| \((2, 0, 1, 1, 1, 2)\) | \(p_6p_{23456} - p_{236}p_{456}\) \(p_{236}p_{346} - p_{256}p_{346}\) |
| \((2, 2, 1, 1, 1, 0)\) | \(p_{123}p_{12345} - p_{123}p_{1345}\) \(p_{124}p_{135} - p_{125}p_{134}\) |
| \((2, 1, 1, 2, 2, 2)\) | \(p_{156}p_{23456} - p_{256}p_{13456}\) \(p_{356}p_{12456} - p_{456}p_{12356}\) |
| \((2, 2, 2, 1, 1, 1)\) | \(p_{123}p_{12345} - p_{124}p_{12356}\) \(p_{125}p_{12346} - p_{126}p_{12345}\) |

These 132 binomials define a toric variety that is a degeneration of our universal torsor. The ideal \(\text{in}(J)\) is relevant in both biology and physics. It represents the Jukes-Cantor model in phylogenetics [58] and the Wess-Zumino-Witten model in conformal field theory [39]. Beautiful polynomials can bring the sciences together.

Let us turn to another fitness problem. The past three pages offered a capoeira approach to #9 in Parameters and Moduli. The compactification is that given by the tropical variety of the universal Cox ideal, to be computed as in [45, 47]. The base space is \(\mathcal{M}_{0,6}\), with points represented by \(2 \times 6\)-matrices \(U\) as in (1). We encoun-
tered several themes that are featured in other articles in this book: flag varieties, Grassmannians, $\mathbb{Z}_n$-gradings, Cox rings, Khovanskii bases, and toric ideals. The connection to spinor varieties was developed in the article [59] with Mauricio Velasco. The formula (2) is derived in [59, Theorem 7.4] for the blow-up of $\mathbb{P}^{n-3}$ at $n$ points when $n \leq 8$. It is still a conjecture for $n \geq 9$. On your trail towards solving such open problems, fill your backpack with polynomials. They will guide you.

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