SOME ENDMORPHISMS OF $II_1$ FACTORS

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Abstract. For any finite dimensional $C^*$-algebra $A$, we give an endomorphism $\Phi$ of the hyperfinite $II_1$ factor $R$ of finite Jones index such that:

$$\forall k \in \mathbb{N}, \Phi^k(R)' \cap R = \otimes^k A.$$ 

The Jones index $[R : \Phi(R)] = (\text{rank}(A))^2$, here $\text{rank}(A)$ is the dimension of the maximal abelian subalgebra of $A$.

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1. Introduction

In [12] R.Powers initiated the study of unital *-endomorphisms of the hyperfinite $II_1$ factor $R$. He defined a shift $\Phi$ on $R$ to be a unital *-endomorphism of $R$ such that $\bigcap_{k=1}^\infty \Phi^k(R) = C$ and defined the index of $\Phi$ as the Jones index $[R : \Phi(R)]$. He constructed a family of shifts, called binary shifts, on $R$. The index of a binary shift is 2. In [4] M.Choda generalized the construction and obtained a $\mathbb{N}$-parameterized family of shifts, called n-unitary shifts; the Jones index of a n-unitary shift is equal to $n$. In [10] G.Price constructed the first example of nonbinary shifts on $R$ of index 2. In [3] D.Bures and H.-S.Yin obtained an intrinsic characterization of such shifts, called group shifts: shifts on the twisted group von Neumann algebra.

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M.Enomoto, M.Nagisa, Y.Watatani, and H.Yoshida \[7\] calculated the relative commutant algebras $\Phi^k(R)' \cap R$ of binary shifts. Recently G.Price \[11\] obtained the complete classification of rational binary shifts. The entropy of a rational binary shift is $\frac{1}{2}\log 2$. On the other extreme of the spectrum, H.Narnhofer, W.Thirring, and E.Størmer utilized the so-called "power-irreducible" binary shifts and obtained surprising counterexamples for which the tensor product formula for entropy fails. In \[8\] V.Golodets and E.Størmer showed the relation of the center sequence of the binary shift and the entropy. If the center sequence grows faster than $0(n)$, then the mean entropy can take any value in $(\frac{1}{2}\log 2, \log 2]$.

In this paper we couple the notion of unitary shifts with the shift on $\bigotimes_{i=1}^{\infty} A_i$, for any finite dimensional $C^\ast$-algebra $A$. We then construct a family of shifts on $R$ of integer index and try to calculate the relative commutant algebras and the entropy. One of the most distinguished examples, $\Phi$ on $R$, gives:

$$\Phi^k(R)' \cap R = \bigotimes^k A.$$  

The Jones index $[R : \Phi(R)] = (\text{rank } A)^2$, here $\text{rank } A$ is the dimension of the maximal abelian subalgebra of $A$.

2. Preliminaries

Let $M$ be a $II_1$ factor with the canonical trace $\tau$. Denote the set of unital $*$-endomorphisms of $M$ by $\text{End}(M, \tau)$. Then $\Phi \in \text{End}(M, \tau)$ preserves the trace and $\Phi$ is injective. $\Phi(M)$ is a subfactor of $M$. If there exists a $\sigma \in \text{Aut}(M)$ with $\Phi_1 \cdot \sigma = \sigma \cdot \Phi_2$ for $\Phi_i \in \text{End}(M, \tau)$ ($i = 1, 2$) then $\Phi_1$ and $\Phi_2$ are said to be conjugate. If there exists a $\sigma \in \text{Aut}(M)$ and a unitary $u \in M$ such that $Adu \cdot \Phi_1 \cdot \sigma = \sigma \cdot \Phi_2$, then $\Phi_1$ and $\Phi_2$ are outer conjugate.

The Jones index $[M : \Phi(M)]$ is an outer-conjugacy invariant. We consider only the finite index case unless otherwise stated. In such case, there is a distinguished outer-conjugacy invariant: the tower of inclusions of finite dimensional $C^\ast$ algebras, $\{A_k = \Phi^k(M)' \cap M\}_{k=1}^{\infty}$.

Another well-known conjugacy invariant is the Connes-Størmer entropy.

**Lemma 1.** $A_k = \Phi^k(M)' \cap M$ contains an subalgebra that is isomorphic to $\bigotimes_{i=1}^{l} A_1$, the $k$-th tensor power of $A_1$, where $A_1 = \Phi(M)' \cap M$ as denoted.

**Proof.** We collect some facts here, for $1 \leq l < k$:

1. $\Phi^l(A_1)$ is isomorphic to $A_1$, since $\Phi^l$ is injective.
2. $\Phi^l(A_1) \cap A_1 \subseteq \Phi(M) \cap \Phi(M)' = C$. 


(3) \([A_1, \Phi^l(A_1)] = 0\) by the definition of \(A_1\).
(4) \(\Phi^l(A_1) \subseteq A_{i+1} \subseteq A_k\).

Put \(\{s_{i_1,j_1}\}_{i_1,j_1 \in F}\) to be a system of matrix units for \(A_1\). Similarly we have \(\{\Phi(s_{i_2,j_2})\}_{i_2,j_2 \in F}\), a system of matrix units for \(\Phi(A_1)\). Etc.

Consider the following linear equation:
\[
\sum_{i_1,j_1, \ldots, i_k,j_k} a_{i_1,j_1, \ldots, i_k,j_k} s_{i_1,j_1} \Phi(s_{i_2,j_2}) \cdots \Phi^{k-1}(s_{i_k,j_k}) = 0.
\]

If we can conclude the coefficient \(a_{i_1,j_1, \ldots, i_k,j_k}\) is zero for every \(i_1, j_1, \ldots, i_k, j_k\), then the dimension of \(\bigvee_{l=1}^k \Phi^{l-1}(A_1)\) is equal to the \(k\)-th power of the dimension of \(A_1\). In other words, the linear independence is established.

Multiply the equation by \(s_{i_1,i_1} \Phi(s_{i_2,i_2}) \cdots \Phi^{k-1}(s_{i_k,i_k})\) at the left hand side and by \(s_{j_1,i_1} \Phi(s_{j_2,i_2}) \cdots \Phi^{k-1}(s_{j_k,i_k})\) at the right hand side, we get:
\[
a_{i_1,j_1, \ldots, i_k,j_k} s_{i_1,i_1} \Phi(s_{i_2,i_2}) \cdots \Phi^{k-1}(s_{i_k,i_k}) = 0.
\]

Note that \(s_{i_1,i_1} \Phi(s_{i_2,i_2}) \cdots \Phi^{k-1}(s_{i_k,i_k})\) is a projection and has nonzero trace, we can conclude: \(a_{i_1,j_1, \ldots, i_k,j_k}\) is zero for every \(i_1, j_1, \ldots, i_k, j_k\).

\[
\bigvee_{l=1}^k \Phi^{l-1}(A_1) \simeq \otimes_{l=1}^k \Phi^{l-1}(A_1) \simeq \otimes_{l=1}^k A_1.
\]

\(\square\)

Remark 1. The dimension of the relative commutant \(\Phi^k(M) \cap M\) is known to be bounded above by the Jones index \([M : \Phi(M)]^k\). Lemma 1 provides the lower bound for the growth estimate.

A good example is the canonical shift \([1]\) on the tower of higher relative commutants for a strongly amenable inclusion of \(II_1\) factors of finite index. The ascending union of higher relative commutants gives the hyperfinite \(II_1\) factor, and the canonical shift can be viewed as a *-endomorphism on the hyperfinite factor. Lemma 1 is nothing but the commutation relations in S.Pop'a’s \(\lambda\)-lattice axioms \([15]\).

A natural question arises with the above observation: for any finite dimensional C*-algebra \(A\), can we find a \(II_1\) factor \(M\) and a \(\Phi \in \text{End}(M, \tau)\) such that: for all \(k \in \mathbb{N}\),
\[
\Phi^k(M) \cap M \simeq \otimes_{i=1}^k A_i, \text{ where } A_i = A?
\]

The answer is positive and furthermore we can choose \(M\) to be the hyperfinite \(II_1\) factor \(R\). We give the construction in the next section. The main technical tool in the construction is \([12]\) R.Powers’ binary
shifts. We provide here the details of n-unitary shifts generalized by M.Choda for the convenience of the reader.

Let \( n \) be a positive integer. We treat a pair of sets \( Q \) and \( S \) of integers satisfying the following condition (\( \ast \)) for some integer \( m \):

\[
\begin{align*}
Q &= (i(1), i(2), \ldots, i(m)), \quad 0 \leq i(1) < i(2) < \cdots < i(m), \\
S &= (j(1), j(2), \ldots, j(m)), \quad j(l) = 1, 2, \ldots, n - 1, \\
& \quad \text{for } l = 1, 2, \ldots, m.
\end{align*}
\]

**Definition 1.** A unital \( \ast \)-endomorphism \( \Psi \) of \( R \) is called an \( n \)-unitary shift of \( R \) if there is a unitary \( u \in R \) satisfying the following:

1. \( u^n = 1 \);
2. \( R \) is generated by \( \{u, \Psi(u), \Psi^2(u), \ldots\} \);
3. \( \Psi^k(u)u = u\Psi^k(u) \) or \( \Psi^k(u)u = \gamma u\Psi^k(u) \) for all \( k = 1, 2, \ldots \), where \( \gamma = \exp(2\pi i^{-1}/n) \);
4. for each \( (Q, S) \) satisfying (\( \ast \)), there are an integer \( k(\geq 0) \) and a nontrivial \( \lambda \in T = \{\mu \in \mathbb{C}; |\mu| = 1\} \) such that

\[
\Psi^k(u)u(Q, S) = \lambda u(Q, S)\Psi^k(u),
\]

where \( u(Q, S) \) is defined by

\[
u(Q, S) = \Psi^{i(1)}(u)^{j(1)}\Psi^{i(2)}(u)^{j(2)} \cdots \Psi^{i(m)}(u)^{j(m)}.
\]

The unitary \( u \) is called a generator of \( \Psi \). Put \( S(\Psi; u) = \{k; \Psi^k(u)u = \gamma u\Psi^k(u)\} \). Note that the above condition (2) gives some rigidity on \( S(\Psi; u) \). The Jones index \( [R : \Psi(R)] \) is \( n \).

One interesting example of \( S_1 = S(\Psi_1; u_1) \) is \( \{1, 3, 6, 10, 15, \ldots, \frac{1}{2}l(l + 1), \ldots\} \), which corresponds to the n-stream \( \{0101001000100001000001\ldots\} \). It is pointed out that the relative commutant \( \Psi^k(R)' \cap R \) is always trivial for all \( k \). That is, our question for \( A = \mathbb{C} \) is answered by this example.

In [3], D.Bures and H.-S.Yin introduced a notion of group shifts, constructed by realizing \( R \) as the twisted group von Neumann algebra on a discrete abelian group with a 2-cocycle. The special case of the group \( G = \oplus_{i=1}^{\infty} \mathbb{Z}_n(i) \) with a suitable 2-cocycle and the (right) 1-shift generalizes the above result by M.Choda. Put \( \{u_1, u_2, \ldots\} \) as the set of generators of \( G \). We can specify on the abelian group \( G \) any 2-cocycle \( \omega \) by its associated antisymmetric character \( \rho \) of \( G \),

\[
\rho(g \wedge h) := \omega(g, h)\overline{\omega(h, g)}
\]

\[
u_g \nu_h = \omega(g, h)\nu_{gh} = \omega(g, h)\overline{\omega(h, g)}\nu_h \nu_g = \rho(g \wedge h)\nu_h \nu_g.
\]
Define a character by:
\[
\rho(u_{2i-1} \wedge u_{2i}) = \gamma,
\rho(u_{2i-1} \wedge u_j) = 1, \text{ if } j \neq 2i,
\rho(u_{2i} \wedge u_{2i+2j}) = \gamma, \text{ if } j \in S_1,
\rho(u_{2i} \wedge u_{2i+2j}) = 1, \text{ if } j \notin S_1.
\]
The set \(S_1\) and \(\gamma\) is as mentioned above.

Take the (right) 2-shift \(\Psi_2\) on \(G; \Psi_2(u_i) = u_{i+2}\). This 2-shift is compatible with the given 2-cocycle. Proposition 1.2 \([3]\) tells us that \(\Psi_2\) can be extended to the twisted group von Neumann algebra, which is the hyperfinite II\(_1\) factor, with the following property:
\[
\Psi_2^k(R)' \cap R \simeq \mathbb{C}^{kn}; \quad [R : \Psi_2(R)] = n^2.
\]
In fact, \(\Psi_2^k(R)' \cap R\) is generated by \(\{u_1, u_3, \ldots, u_{2k-1}\}\). Observe that the von Neumann algebra generated by \(u_{2k-1}, u_{2k}\) is isomorphic to the \(n \times n\) matrix algebra. For our question, this construction realizes the case when \(A\) is abelian. The construction for the most general case is modeled on this special one.

### 3. Main Theorem

**Theorem 1.** For any finite dimensional \(C^*\)-algebra \(A\), there exists a \(\Phi \in \text{End}(R, \tau)\) such that the relative commutant \(\Phi^k(R)' \cap R\) is isomorphic to \(\otimes_{i=1}^k A(i)\). Here \(R\) is the hyperfinite II\(_1\) factor with the trace \(\tau\).

Since \(A\) is finite dimensional, then \(A\) can be decomposed as a direct sum of finitely many matrix algebras,
\[
A \simeq \bigoplus_{i=1}^j M_{a_i}(\mathbb{C}) \subseteq M_n(\mathbb{C}),
\]
where \(n = \sum_i a_i\).

For each \(i\), \(M_{a_i}(\mathbb{C}) \subset A\) (not a unital embedding) is generated by \(p_i, q_i \in \mathcal{U}(\mathbb{C}^{a_i})\) with:
\[
p_i^{a_i} = q_i^{a_i} = 1_{M_{a_i}(\mathbb{C})}; \quad \gamma_i = \exp(2\pi\sqrt{-1}/a_i), \quad p_i q_i = \gamma_i q_i p_i
\]
where \(p_i = [1 \gamma_i \gamma_i^2 \cdots \gamma_i^{a_i-1}]\) is the diagonal matrix in \(M_{a_i}(\mathbb{C})\), and \(q_i\) is the permutation matrix in \(M_{a_i}(\mathbb{C}), (1 \ 2 \ 3 \cdots a_i)\).

Denote by \(\mathcal{D}\) the diagonal algebra of \(A \subseteq M_n(\mathbb{C})\). \(\mathcal{D}\) is generated by \(\{p_i\}_{i=1}^j\). A simple observation is that \(M_n(\mathbb{C})\) is generated by \(\mathcal{D}\) and a unitary \(u\) with \(u^n = 1\). We can write \(u\) as the permutation matrix \((1 \ 2 \ 3 \cdots n)\). Every element \(x\) in \(M_n(\mathbb{C})\) is expressed as \(x = \sum_{i=0}^{n-1} x_i u^i\), where \(x_i \in \mathcal{D} \subseteq A\).
Note that $Adu$ defines a unital *-automorphism of $D$, yet not of $A$. $Adu$ doesn’t map $A$ into $A$. As a consequence, for $x$ in $D$ we have:
\[ u^i x = Adu^i(x)u^i; \quad u^i D = D u^i. \]

Define $v \in M_n(\mathbb{C})$ to be the permutation matrix:
\[ v = (a_1 (a_1 + a_2) (a_1 + a_2 + a_3) \cdots (a_1 + a_2 + \cdots + a_j)). \]
Then $v^j = 1$. Note that $Adv$ defines a unital *-automorphism of $D$. As a consequence, for $x$ in $D$ we have:
\[ v^i x = Adv^i(x)v^i; \quad v^i D = D v^i. \]

$D$ and $v$ don’t generate the full matrix algebra, $M_n(\mathbb{C})$. However $A$ and $v$ do generate $M_n(\mathbb{C})$. Therefore we have two ways to describe $M_n(\mathbb{C})$:
(1) via $D$ and $u$, or
(2) via $A$ and $v$ (or via $A$ and $r$ described below.)

Not surprisingly there is a relation between $u$ and $v$;
\[ u = v_1 v_2 v_3 \cdots v_j v, \text{ where } v_i = q_i + 1 - 1_{M_{a_i}}(\mathbb{C}). \]

Define $r := svs$, while
\[ s = [0 0 \cdots 0 1 a_1 0 0 \cdots 0 1_{a_1+a_2} \cdots 0 0 \cdots 0 1_{a_1+a_2+\cdots+a_j}] \in D. \]
Thus $A$ and $r$ generates $M_n(\mathbb{C})$.

On the other hand, define $w = \sum_{i=1}^j \gamma^{i-1} 1_{M_{a_i}}(\mathbb{C})$, where $\gamma = \exp(2\pi \sqrt{-1}/j)$ and $\gamma^j = 1$. Note that $w$ is in the center of $A$. Two simple yet important observations are that:
(1) $Adw$ acts trivially on $A$, which contains $D$.
(2) $Adw(r) = \gamma r$.

We now construct a tower of inclusion of finite dimensional $C^*$-algebras $M_k$ with a trace $\tau$. The ascending union $M = \cup_{k \in \mathbb{N}} M_k$ contains infinite copies of $M_n(\mathbb{C})$, and thus of $A$. Number them respectively by $D_1, r_1, A_1, w_1, D_2, r_2, A_2, w_2, D_3, r_3, A_3, w_3, \cdots$.

We endow on this algebra the following properties:
\[ [A_l, A_m] = 0, \text{ if } l \neq m; \]
\[ [r_l, A_m] = 0, \text{ if } l \neq m; \]
\[ r_l r_m = \gamma r_m r_l, \text{ if } |l-m| \in S_1 = \{1, 3, 6, 10, 15, \cdots\}, \]
\[ r_l r_m = r_m r_l, \text{ otherwise.} \]

Here $\gamma = \exp(2\pi \sqrt{-1}/j)$ as above.

The construction is an induction process. We have handy the embedding $A_1 \subseteq M_n(\mathbb{C}) = M_1$, which is isomorphic to the inclusion
of \( A \otimes 1_{M_n(\mathbb{C})} \) inside \( M_n(\mathbb{C}) \otimes 1_{M_n(\mathbb{C})} \). Identify \( A_2 \) in \( \otimes^2 M_n(\mathbb{C}) \) by \( 1_{M_n(\mathbb{C})} \otimes A \).

Consider the *-automorphism on \( \otimes^2 M_n(\mathbb{C}) \), \( \text{Ad}(w \otimes v) \). We have the following results:

1. \( \text{Ad}(w \otimes v) = \text{Ad}(1_{M_n(\mathbb{C})} \otimes v) \) when restricted on \( 1_{M_n(\mathbb{C})} \otimes A = A_2 \).
2. \( \text{Ad}(w \otimes v) \) acts trivially on \( A \otimes 1_{M_n(\mathbb{C})} = A_1 \).
3. \( \text{Ad}(w \otimes v)(r_1 \otimes 1_{M_n(\mathbb{C})}) = \gamma r_1 \otimes 1_{M_n(\mathbb{C})} \).

Take \( v_2 = w \otimes v \in \otimes^2 M_n(\mathbb{C}) \), \( v_2' = 1 \). Define \( r_2 := s_2v_2s_2 = w \otimes r \), where \( s_2 \) is as above a projection in \( D_2 \subset A_2 \). We have the following properties:

1. \( \langle A_2, r_2 \rangle \approx M_n(\mathbb{C}) \) by the existence of a system of matrix units;
2. \( [A_1, A_2] = 0, [A_1, r_2] = 0; \)
3. \( \text{Adv}(r_1) = \gamma r_1 \), i.e., \( r_1r_2 = \gamma r_2r_1 \);
4. The trace \( \tau \) on \( M_2 = \langle A_1, r_1, A_2, r_2 \rangle \subset \otimes^2 M_n(\mathbb{C}) \) is an extension of the normalized trace \( \tau = \frac{1}{n} Tr \) on \( M_n(\mathbb{C}) = M_1 = \langle A_1, r_1 \rangle \).

**Lemma 2.** In the above construction, \( M_2 = \langle A_1, r_1, A_2, r_2 \rangle \) is equal to \( \otimes^2 M_n(\mathbb{C}) \) as a concrete \( C^* \)-algebra.

Assume we have obtained \( M_k = \langle A_1, r_1, A_2, r_2, \ldots, A_k, r_k \rangle \) equal to \( \otimes^k M_n(\mathbb{C}) \) with the trace \( \tau \). We identify \( M_k \) as \( M_k \otimes 1_{M_n(\mathbb{C})} \) by sending \( x \in M_k \) to \( x \otimes 1_{M_n(\mathbb{C})} \), and similarly \( A_{k+1} \) as \( 1_{M_k} \otimes A \).

Consider the *-automorphism on \( M_k \otimes M_n(\mathbb{C}), \text{Ad}((w_1^{b_1}w_2^{b_2} \cdots w_k^{b_k}) \otimes v), \forall l \leq k: \)

\[
b_l = 1, \text{ if } |k + 1 - l| \in S_1; b_l = 0, \text{ otherwise.}
\]

We have:

1. \( v_{k+1}^l = 1 \), where \( v_{k+1} \) is a unitary;
2. \( r_{k+1} := s_{k+1}v_{k+1}s_{k+1} = (w_1^{b_1}w_2^{b_2} \cdots w_k^{b_k}) \otimes r \) and the algebra generated by \( \langle A_{k+1}, r_{k+1} \rangle \) is isomorphic to \( M_n(\mathbb{C}) \);
3. \( [A_{k+1}, M_k] = 0; \)
4. \( [r_{k+1}, A_l] = 0, \text{ for } l \leq k; \)
5. \( \text{Adv}_{k+1}(r_l) = \gamma r_l \), if \( |k + 1 - l| \notin S_1; \text{Ad}_{k+1}(r_l) = r_l \), otherwise;
6. There exists an extending trace \( \tau \) on the finite dimensional \( C^* \)-algebra \( M_{k+1} = \langle M_k, A_{k+1}, r_{k+1} \rangle \).

**Lemma 3.** \( M_k \) is equal to \( \otimes^k M_n(\mathbb{C}) \).

By induction we have constructed the ascending tower of finite dimensional \( C^* \)-algebras with the desired properties.

We now explore some useful properties of the finite dimensional \( C^* \)-algebra, \( M_k \).
Lemma 4. For all $k$, $M_k$ is the linear span of the words, $x_1 \cdot x_2 \cdot x_3 \cdot \cdots \cdot x_k$, where $x_i \in M_n(\mathbb{C})_i = \langle A_i, r_i \rangle$.

Proof. It suffices to prove $x_l x_i$ is in $\langle A_i, r_i \rangle \cdot \langle A_l, r_l \rangle$, where $i < l$. $x_l \in (A_l r_l + A_l)^n$ and $x_i \in (A_i r_i + A_i)^n$ through the decomposition of $M_n(\mathbb{C})_i$ and $M_n(\mathbb{C})_l$ by $D_i, u_i, D_l, u_l$. Thus it suffices to prove $(A_l r_l + A_l) \cdot (A_i r_i + A_i) \subseteq (A_l r_l + A_l) \cdot (A_i r_i + A_i)$.

□

Lemma 5. In fact, $\langle A, r \rangle \simeq M_n(\mathbb{C})$ is of the form:

$$A + ArA + Ar^2A + \cdots + Ar^{j-1}A.$$

Proof. It suffices to observe that

\[
\begin{align*}
 r &= sv = vs, \quad [s, v] = [s, r] = 0 \\
 r^* &= su^* = v^* s, \quad rr^* = r^* r = s \\
 rAr &= rsAr = r^2r^*Ar = r^2s(v^*Av)s \subset r^2A \\
 r^* &= r^{j-1}, \quad r^j = s
\end{align*}
\]

□

Lemma 6. Consider the pair $(M, \tau)$ as described above and the GNS-construction. Identify everything mentioned above as its image. We $M''$ is the hyperfinite $\text{II}_1$ factor.

Proof. There is one and only one tracial state on $M_k$ for all $k \in \mathbb{N}$. Hence the tracial state on $M$ is unique. Therefore $M''$ is the hyperfinite $\text{II}_1$ factor, $R$.

□

Define a unital *-endomorphism, $\Phi$, on $R$ to be the (right) one-shift: i.e., sending $A_k$ to $A_{k+1}$, and sending $r_k$ to $r_{k+1}$. We observe that $\Phi(R)$ is a $\text{II}_1$ factor and

$$[R : \Phi(R)] = n^2.$$

Lemma 7. The relative commutant $\Phi^k(R)' \cap R$ is exactly $\otimes_{i=1}^k A$. 

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Proof. Because of our decomposition in Lemma 4 and Lemma 5, \( R \) can be written as

\[
(\sum_{i=0}^{j} A_1 r_1^i A_1) \cdot (\sum_{i=0}^{j} A_2 r_2^i A_2) \cdots \cdot (\sum_{i=0}^{j} A_k r_k^i A_k) \cdot \Phi^k(R).
\]

Assume \( x \in R \cap \Phi^k(R)' \). \( x \) can be written, as in Lemma 4, of the following form:

\[
x = \sum_{\vec{\alpha} \in \{0, 1, \ldots, j-1\}^k} y_1^{\vec{\alpha}} r_1^{c_1} z_1^{\vec{\alpha}} y_2^{\vec{\alpha}} r_2^{c_2} z_2^{\vec{\alpha}} \cdots y_k^{\vec{\alpha}} r_k^{c_k} z_k^{\vec{\alpha}} \cdot \Phi^k(R),
\]

where \( \vec{\alpha} = (c_1, c_2, \ldots, c_k) \) is a multi-index and \( y^{\vec{\alpha}} \) is in \( \Phi^k(R) \). Note that \( \Phi^k(R) \) is the weak closure of \( \{ \Phi^k(M_i) \}_{i=1}^{\infty} \).

For every \( \epsilon > 0 \), there exists an integer \( i \in \mathbb{N} \) such that

\[
\forall \vec{\alpha}, \quad \| x - \sum_{\vec{\alpha} \in \{0, 1, \ldots, j-1\}^k} y_1^{\vec{\alpha}} r_1^{c_1} z_1^{\vec{\alpha}} y_2^{\vec{\alpha}} r_2^{c_2} z_2^{\vec{\alpha}} \cdots y_k^{\vec{\alpha}} r_k^{c_k} z_k^{\vec{\alpha}} \cdot \Phi^k(M_i) \|_2 < \delta
\]

where \( \delta = \left( \frac{j}{n} \right)^k \epsilon \).

Put \( L = l(l + 1)/2 + 1 \) for some integer \( l > k + 1 \). We have the following properties:

\[
[r_L, A_1] = [r_L, A_2] = \cdots = [r_L, A_{k+i}] = 0
\]
\[
[r_L, r_2] = [r_L, r_3] = \cdots = [r_L, r_{k+i}] = 0
\]
\[
r_L r_1 = \gamma r_1 r_L, \quad r_L r_1^{c_1} = \gamma^{c_1} r_1 r_L
\]
\[
r_L r_1^s = r_L^s r_L = s_L
\]
\[
r_L r_1 r_1^s = \gamma r_1 s_L, \quad r_L r_1^{c_1} r_1^s = \gamma^{c_1} r_1 s_L
\]
for \( 0 \leq m \leq j - 1 \), \( r_L^m r_1^{c_1} r_1^s = \gamma^{c_1 m} r_1 s_L \)
\[
[s_L, A_1] = [s_L, A_2] = \cdots = [s_L, A_{k+i}] = 0
\]
\[
[s_L, r_1] = [s_L, r_2] = \cdots = [s_L, r_{k+i}] = 0
\]
Therefore we claim:

\[ \| (x - \sum_{\vec{\alpha} \in \{0, \ldots, j-1\}^k} \cdot y_{1}^{\vec{\alpha}} r_{1}^{\vec{\alpha}} z_{1}^{\vec{\alpha}} y_{2}^{\vec{\alpha}} r_{2}^{\vec{\alpha}} z_{2}^{\vec{\alpha}} \cdot \cdot \cdot y_{k}^{\vec{\alpha}} r_{k}^{\vec{\alpha}} z_{k}^{\vec{\alpha}} \cdot z_{L} \cdot s_{L} \|_{2,\tau} = \]

\[ \| (x - \sum_{\vec{\alpha} \in \{0, \ldots, j-1\}^k} \cdot y_{1}^{\vec{\alpha}} r_{1}^{\vec{\alpha}} z_{1}^{\vec{\alpha}} y_{2}^{\vec{\alpha}} r_{2}^{\vec{\alpha}} z_{2}^{\vec{\alpha}} \cdot \cdot \cdot y_{k}^{\vec{\alpha}} r_{k}^{\vec{\alpha}} z_{k}^{\vec{\alpha}} \cdot z_{L} \cdot s_{L} \|_{2,\tau} = \]

\[ \| \sum_{\vec{\alpha} \in \{0, \ldots, j-1\}^k} \frac{1}{j} \sum_{m=0}^{j-1} (r_{L}^{m} x r_{L}^{* m} - y_{1}^{\vec{\alpha}} r_{1}^{\vec{\alpha}} (r_{L}^{* m})_{L} z_{1}^{\vec{\alpha}} y_{2}^{\vec{\alpha}} r_{2}^{\vec{\alpha}} z_{2}^{\vec{\alpha}} \cdot \cdot \cdot y_{k}^{\vec{\alpha}} r_{k}^{\vec{\alpha}} z_{k}^{\vec{\alpha}} \cdot z_{L} \cdot s_{L} \|_{2,\tau} = \]

\[ \| \sum_{\vec{\alpha} \in \{0, \ldots, j-1\}^k, c_{1} = 0} \sum_{m=0}^{j-1} (x - y_{1}^{\vec{\alpha}} r_{1}^{\vec{\alpha}} z_{1}^{\vec{\alpha}} y_{2}^{\vec{\alpha}} r_{2}^{\vec{\alpha}} z_{2}^{\vec{\alpha}} \cdot \cdot \cdot y_{k}^{\vec{\alpha}} r_{k}^{\vec{\alpha}} z_{k}^{\vec{\alpha}} \cdot z_{L} \cdot s_{L} \|_{2,\tau} = \]

\[ \sqrt{\frac{j}{n}} \| x - \sum_{\vec{\alpha} \in \{0, \ldots, j-1\}^k, c_{1} = 0} y_{1}^{\vec{\alpha}} z_{1}^{\vec{\alpha}} y_{2}^{\vec{\alpha}} r_{2}^{\vec{\alpha}} z_{2}^{\vec{\alpha}} \cdot \cdot \cdot y_{k}^{\vec{\alpha}} r_{k}^{\vec{\alpha}} z_{k}^{\vec{\alpha}} \cdot z_{L} \cdot s_{L} \|_{2,\tau} = \]

since \( \{ x, y_{1}^{\vec{\alpha}}, z_{1}^{\vec{\alpha}}, y_{2}^{\vec{\alpha}}, r_{2}^{\vec{\alpha}}, z_{2}^{\vec{\alpha}}, \cdot \cdot \cdot, y_{k}^{\vec{\alpha}}, r_{k}^{\vec{\alpha}}, z_{k}^{\vec{\alpha}} \} \) \( \subset \) \( \{ s_{L}, r_{L}, A_{L} \} \)

and \( \text{Tr}(s_{L}) = \frac{j}{n} \).

By induction,

\[ \| x - \sum_{\vec{\alpha} \in \{0, \ldots, j-1\}^k, c_{1} = 0} \cdot y_{1}^{\vec{\alpha}} z_{1}^{\vec{\alpha}} y_{2}^{\vec{\alpha}} r_{2}^{\vec{\alpha}} z_{2}^{\vec{\alpha}} \cdot \cdot \cdot y_{k}^{\vec{\alpha}} r_{k}^{\vec{\alpha}} z_{k}^{\vec{\alpha}} \cdot z_{L} \cdot s_{L} \|_{2,\tau} < \sqrt{\frac{n}{j} \delta} \]

\[ \| x - \sum_{\vec{\alpha} \in \{0, \ldots, j-1\}^k, c_{1} = c_{2} = 0} \cdot y_{1}^{\vec{\alpha}} z_{1}^{\vec{\alpha}} y_{2}^{\vec{\alpha}} r_{2}^{\vec{\alpha}} z_{2}^{\vec{\alpha}} \cdot \cdot \cdot y_{k}^{\vec{\alpha}} r_{k}^{\vec{\alpha}} z_{k}^{\vec{\alpha}} \cdot z_{L} \cdot s_{L} \|_{2,\tau} < \left( \sqrt{\frac{n}{j} \delta} \right)^{k} \]

\[ \| x - \sum_{\vec{\alpha} \in \{0\}^k} \cdot y_{1}^{\vec{\alpha}} z_{1}^{\vec{\alpha}} y_{2}^{\vec{\alpha}} r_{2}^{\vec{\alpha}} z_{2}^{\vec{\alpha}} \cdot \cdot \cdot y_{k}^{\vec{\alpha}} r_{k}^{\vec{\alpha}} z_{k}^{\vec{\alpha}} \cdot z_{L} \cdot s_{L} \|_{2,\tau} < \left( \sqrt{\frac{n}{j} \delta} \right)^{k} \]

Note that the von Neumann algebra \( \{ x, y_{1}^{\vec{\alpha}}, z_{1}^{\vec{\alpha}}, y_{2}^{\vec{\alpha}}, z_{2}^{\vec{\alpha}}, \cdot \cdot \cdot y_{k}^{\vec{\alpha}}, z_{k}^{\vec{\alpha}} \} \) commutes with \( \Phi^{k}(M) \), which is a \( II_{1} \) factor. Any element in the former von Neumann algebra has a scalar conditional expectation onto \( \Phi^{k}(M) \). In short, the former von Neumann algebra and \( \Phi^{k}(M) \) are mutually orthogonal.
According to [13], we have:

\[
\|x - \sum_{\tilde{a} \in \{0\}^k} y_1^{\tilde{a}_1} y_2^{\tilde{a}_2} \cdots y_k^{\tilde{a}_k} \tau(z^{\tilde{a}})\|_{2,\tau} < \epsilon
\]

\[
\sum_{\tilde{a} \in \{0\}^k} y_1^{\tilde{a}_1} y_2^{\tilde{a}_2} \cdots y_k^{\tilde{a}_k} \tau(z^{\tilde{a}}) \in A_1 \cdot A_2 \cdots A_k = \otimes^k A
\]

\[
\square
\]

4. Discussion

The construction of \( \Phi \) depends on the choice of the anticommutation set \( S_1 \). In the case of \( n \)-unitary shifts, different choices of \( S(\Psi; u) \) give uncountably many nonconjugate shifts and at least a countably infinite family of shifts that are pairwise not outer conjugate. Not to mention in [5] the existence of uncountably many non-outer-conjugate nonbinary shifts, exploiting different 2-cocycles on the group \( G = \oplus_i \mathbb{Z}_2^{(i)} \). Each of the above has a counterpart in our construction.

Given a finite dimensional \( C^* \)-algebra \( A \) with \( \text{rank}(A) = n \) and an \( n \)-unitary shift with the anticommutation set \( S(\Psi; u) \) satisfies the following condition:

\[
S(\Psi; u) = \{k_i \mid k_{i+1} > k_i, \text{ and } k_{i+2} - k_{i+1} > k_{i+1} - k_i, \forall i \in \mathbb{N}\}
\]

Denote by \( M(\Psi; u) \) to be the ascending union of \( M_k(\Psi; u) = \langle A_1(\Psi; u), r_1(\Psi; u), A_2(\Psi; u), r_2(\Psi; u), \cdots, A_k(\Psi; u), r_k(\Psi; u) \rangle \) with the trace \( \tau(\Psi; u) \). We have

\[
A \simeq A_1(\Psi; u) \simeq A_2(\Psi; u) \cdots
\]

Similarly, consider the pair \( (M(\Psi; u), \tau(\Psi; u)) \) as described above and the GNS construction. Identify everything mentioned above as its image. The weak closure \( M(\Psi; u)^\prime \) is the hyperfinite \( II_1 \) factor, \( R \).

Define a unital \(*\)-endomorphism, \( \Psi \), on \( R \) to be the (right) one-shift: i.e., sending \( A_k(\Psi; u) \) to \( A_{k+1}(\Psi; u) \), and sending \( r_k(\Psi; u) \) to \( r_{k+1}(\Psi; u) \). We observe that \( \Psi(R) \) is a \( II_1 \) factor and \([R : \Psi(R)] = n^2\).

\[
\Psi^k(R) \cap R = \otimes^k A.
\]

We calculate the Connes-Størmer entropy [16] in the following paragraph.

**Lemma 8.** \( H(\Psi) \geq \ln n = \ln[R : \Psi(R)] \) no matter of the choice of the anticommutation set \( S(\Psi, u) \).
Proof.

\[ H(\Psi) = \lim_{j \rightarrow \infty} H(M_j, \Psi) \geq \lim_{j \rightarrow \infty} H(\otimes^j A, \Psi) \]

\[ = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} H(\otimes^j A, \Psi(\otimes^j A), \cdots, \Psi^{k-1}(\otimes^j A)) \]

\[ = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} H(\otimes^{j+k-1} A) = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{n^{j+k-1}} - \frac{1}{n^{j+k-1}} \ln\left(\frac{1}{n^{j+k-1}}\right) \]

\[ = \lim_{j \rightarrow \infty} \ln n = \ln n \]

\[ \square \]

Lemma 9. \( H(\Psi) \leq \ln n = \ln[R : \Psi(R)] \) no matter of the choice of the anticommutation set \( S(\Psi, u) \).

Proof.

\[ H(\Psi) = \lim_{j \rightarrow \infty} H(M_j, \Psi) = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} H(M_j, \Psi(M_j), \cdots, \Psi^{k-1}(M_j)) \]

\[ \leq \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} H(M_{j+k-1}) = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} H(\otimes^{j+k-1} M_n(\mathbb{C})) \]

\[ = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} (j + k - 1) \ln n = \ln n \]

\[ \square \]

Therefore, \( H(\Psi) = \ln n = \ln[R : \Psi(R)] \) no matter of the choice of the anticommutation set \( S(\Psi, u) \).

5. Digression

In this section, we construct inclusions of non-hyperfinite II_1-factors via free products with amalgamation.

Theorem 2. For any finite dimensional C*-algebra \( A \), there exists a tower of inclusions of II_1-factors, \( M \subset M_1 \subset M_2 \subset M_3 \subset \cdots \), with the trace \( \tau \) such that

\[ M' \cap M_k = \otimes^k_{i=1} A. \]

The main tool is the relative commutant theorem by S.Popa.

Lemma 10. Let \( (P_1, \tau_1), (P_2, \tau_2) \) be two finite von Neumann algebras with a common von Neumann subalgebra \( B \subset P_1, B \subset P_2 \), such that \( P_1 = \overline{Q \otimes B} \) where \( Q \) is a nonatomic finite von Neumann algebra. If \( (P, \tau) \) denotes the amalgamated free product \( (P_1, \tau_1) \ast_B (P_2, \tau_2) \) then \( Q_0' \cap P = (Q_0' \cap Q) \otimes B \) for any nonatomic von Neumann subalgebra \( Q_0 \subset Q \).
Assume that $L \subset P$ is a von Neumann subalgebra satisfying the properties:

1. $Q_0 \subset L$.
2. $L \cap P_2$ contains an element $y \neq 0$ orthogonal to $B$, i.e. $E_B(y) = 0$ and with $E_B(y^*y) \in \mathbb{C}^1$.

Then $L' \cap P = L' \cap B$. If in addition $L \cap B = \mathbb{C}$ then $L$ is a type $II_1$ factor.

As before, we can embed $A$ in the full matrix algebra $M_n(\mathbb{C})$. Consider the tensor product of $M_n(\mathbb{C})$ and $M_2(\mathbb{C})$. Identify $A$ as its image in the tensor product. Take the element $y$ in $M_2(\mathbb{C})$:

$$y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

In $M_{2n}(\mathbb{C})$, $y \neq 0$ is an element orthogonal to $A$, i.e., $E_A(y) = 0$ and with $E_A(y^*y) = 1$.

Take $M$ to be any $II_1$-factor. We construct $M_1$ via the following map $\Gamma$:

$$M_1 = \Gamma(M) = (M \otimes A) * A M_{2n}(\mathbb{C}).$$

The trace $\tau$ on $M$ can be extended to $M_1$.

Apply the above lemma.

$$P_1 = M \otimes A;$$

$$P_2 = M_{2n}(\mathbb{C});$$

$$L = P = M_1.$$ 

We get:

$$M_1' \cap M_1 = M_1' \cap A \subseteq M_{2n}(\mathbb{C})' \cap A = \mathbb{C}.$$ 

That is, $M_1$ is a nonhyperfinite $II_1$ factor.

The relative commutant

$$M' \cap M_1 = (M' \cap M) \otimes A = A$$

It also gives that $M_1' \cap M_1 = \mathbb{C}$.

Viewing $\Gamma$ as a machine producing $II_1$ factors, we get an ascending towers of $II_1$ factors:

$$M \subset \Gamma(M) = M_1 \subset \cdots M_i \subset \Gamma(M_i) = M_{i+1} \cdots.$$ 

There is a unique trace $\tau$ associated to each $M_i$.

We calculate the relative commutant algebra $M' \cap M_k$ by induction. Let us assume

$$M' \cap M_k = \otimes^k A.$$
By the above lemma,

\[ M' \cap M_{k+1} = (M' \cap M_k) \otimes A = (\otimes^k A) \otimes A = \otimes^{k+1} A. \]

In the end, we would boldly suggest an analogy between binary shifts and free products with amalgamation as in [2].

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