HIGH TEMPERATURE EXPANSION OF STRING FREE ENERGY 
IN HYPERBOLIC SPACE

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Abstract

The high temperature behaviour of the open bosonic string free energy in the space $S^1 \otimes H^N$ with vanishingly small curvature is investigated. The leading term of the high temperature expansion of the one-loop free energy, near the Hagedorn instability, is obtained. The problem of infrared regularization of thermodynamical quantities is pointed out. For minimally coupling quantum fields related to the normal modes of strings, the results are similar to the ones valid for Rindler space. In the lower mass string states regime a connection with the quantum corrections to the black hole entropy is outlined.

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The behaviour of quantum fields in backgrounds with horizons has been actively investigated during the last years [1, 2, 3, 4, 5]. In quantum field theory, the black hole entropy, calculated near the horizon, diverges [3], and the interest for this long standing problem has been growing [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The presence of the horizon involves arbitrary high frequencies and it is argued that the string theory might be relevant for the description of physics at ultra-short distances [11, 19]. Indeed the high temperature behaviour of the string free energy can be extended in the presence of a black hole backgrounds [20, 21, 22]. As a result of modular invariance, the closed string correction to the entropy is ultraviolet finite. The calculation of the quantum correction to the entropy of a large black hole in (non)- critical string theory has been also performed in [23, 24, 25].

The technical difficulties in quantization are related to the construction of the string spectrum in black hole backgrounds. Conical backgrounds for strings, interpreted in the form of orbifolds [25], have been applied to the black hole physics in Ref. [21]. Different approaches to this problem give approximately the same behaviour for the free energy (entropy) [21, 22].

In this paper, we shall analyse the high temperature behaviour for the open bosonic string free energy in hyperbolic manifolds, using a new representation (valid for different types of open and closed strings), which has been introduced in Refs. [26, 27, 28, 29]. Our approximation will be carried out in space of vanishingly small curvature. For massless quantum fields, our techniques allows to obtain the leading term of the high temperature expansion, which are conformally related to the one valid in Rindler space.

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First of all, let us point out how the space-times with hyperbolic spatial section may be relevant in Rindler or black hole physics. To start with, we shall consider the effective action \( \Gamma[g, \beta] \) associated with a conformal scalar field in \( D \)-dimensional Rindler space, which approximates the geometry outside a very large black hole. The metric of the Euclidean Rindler space can be written as follows
\[
 ds^2 = \xi^2 d\tau^2 + d\xi^2 + \sum_{i=1}^{N-1} dy_i^2 ,
\]  
where \( \tau \) is the Euclidean time (periodically identified with period \( \beta \)), \( y_i \) are the \( (N-1) \)-transverse flat coordinates \( (D = N + 1) \). The lines \( \xi = \text{constant} \) correspond to observable undergoing constant acceleration \( \xi^{-1} \).

The one-loop effective action (up to a contribution of a local functional measure) can be presented in the form
\[
 \Gamma[g, \beta] = \frac{1}{2} \, \Tr \ln A(\beta) ,
\]  
where \( A(\beta) \equiv \partial_\tau^2 + L_N \), and \( L_N \) is the standard Laplace-Beltrami operator acting in \( N \)-dimensional space. The entropy related to the free energy \( F[g, \beta] \equiv \beta^{-1} \Gamma[g, \beta] \) is
\[
 S = \beta^2 \frac{\partial}{\partial \beta} F[g, \beta] = \beta^2 \frac{\partial}{\partial \beta} F[g, \beta]_{\text{Ren}} ,
\]  
where the renormalization of the ultraviolet divergences of the free energy should be performed. The renormalized free energies \( F[g, \beta]_{\text{Ren}} \) for two conformally related static spaces \( g_{\mu\nu} = \exp (2\omega) \bar{g}_{\mu\nu} \) are related as follows
\[
 \Delta F[g, \omega] = F[g, \beta]_{\text{Ren}} - F[\bar{g}, \beta]_{\text{Ren}} .
\]  
In four-dimensional case, the explicit form of the term \( \Delta F[g, \omega] \) can be found in Refs. \[30, 31, 32\]. However, the difference \( \Delta F[g, \omega] \) for two conformally related theories in a static spacetime, is proportional to \( \beta \) and hence does not contribute to the entropy. Further, we may apply Eq.(4) to the particular case of the Rindler space. We have \( \omega = \frac{1}{2} \log \xi^2 \), and \( \bar{g}_{\mu\nu} = \exp (-2\omega) g_{\mu\nu} \) is the ultrastatic metric. As a result, conformally related fields in the manifold \( M \) of the form \( M = S^1 \otimes H^N \), where \( H^N \) is the simply connected real hyperbolic space, will be considered. The free energy can be presented in the form
\[
 F[\bar{g}, \beta] = \frac{1}{2} \, \Tr \ln \bar{A}(\beta) = -\frac{1}{2\beta} \zeta'(0) \bar{A}(\beta) ,
\]  
where \[33\]
\[
 \zeta(s|\bar{A}(\beta)) = \frac{\beta}{2} \frac{\Gamma(s - 1/2)}{\sqrt{\pi} \Gamma(s)} \zeta(s - 1/2|\bar{L}_N) 
 + \frac{1}{\sqrt{\pi} \Gamma(s) 2\pi i} \int_{\Re z = c} dz \zeta_R(z) \Gamma(\frac{z}{2}) \Gamma(\frac{z - 1}{2} + s) \zeta(\frac{z - 1}{2} + s|\bar{L}_N) (\frac{\beta}{2})^{-z-1} ,
\]  
\( c > N + 1 \), \( \zeta_R(z) \) is the Riemann zeta function, and the self-adjoint Laplace-Beltrami operator \( \bar{L}_N \) acting in the space \( H^N \). Finally, one can obtain (see Ref. \[33\])
\[
 F[g, \beta]_{\text{Ren}} = \frac{1}{2} \zeta(-\frac{1}{2}|\bar{L}_N)_{\text{Ren}} - \frac{1}{2\pi i} \int_{\Re z = c} dz \zeta_R(z) \Gamma(z - 1) \zeta(\frac{z - 1}{2}|\bar{L}_N) \beta^{-z} 
 \equiv F_0 + F(\beta) ,
\]
with
\[ \zeta(-\frac{1}{2}|\bar{L}_N)_{\text{Ren}} = FP\zeta(-\frac{1}{2}|\bar{L}_N) + (2 - 2\ln 2)Res_{z=\frac{1}{2}}\zeta(z|\bar{L}_N). \] (8)

Here \( F_0 \) is the vacuum energy, \( F(\beta) \) is the temperature dependent part of \( \mathcal{F}[\bar{g}, \beta]_{\text{Ren}} \), the symbols \( FP \) and \( \text{Res} \) denote the finite part and residue of the zeta function at the specified point, respectively. Note that though zeta function diverges, it does not depend on \( \beta \) and its contribution to the entropy vanishes. For the massive quantum fields one can also use conformal transformation techniques \([30, 31, 32]\) and construct a local zeta function \( \zeta(s, x|\bar{L}_N) \) (see for example \([16]\)).

Furthermore we shall be interested in massive quantum fields in even dimensional manifolds, i.e. \( N \) is odd, with spatial hyperbolic sections \( H^N \). Therefore the Laplace-Beltrami operator \( \bar{L}_N \) is acting in \( H^N \). At the end, we shall comment on the transition to the conformally invariant massless case. The zeta function associated with operator \( \bar{L}_N \) may be written in the form \([34, 35]\)

\[ \zeta(z|\bar{L}_N) = \frac{1}{(4\pi)^{N/2}} \frac{(N-1)/2}{\Gamma(N/2)} \sum_{k=1}^{\infty} \alpha_{k,N} b^{1+2k-2z} B(k + \frac{1}{2}, z - k - \frac{1}{2}), \] (9)

where \( B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y) \) is the Euler’s beta function, the coefficients \( \alpha_{k,N} \) are defined by expanding the products into polynomials in eigenvalues of \( \bar{L}_N \) (see for detail Ref.\([34, 35]\)) and \( b \) is known constant depending on the mass of field and on the curvature. Using the zeta function \([8]\), we obtain

\[ F(\beta) = -\frac{\sqrt{\pi}}{2\pi i (4\pi)^{\frac{N}{2} - 1}} \sum_{k=1}^{N-1} \alpha_{k,N} \Gamma(k + \frac{1}{2}) \times \int_{\text{Re} z=c} dz \zeta_R(z) \Gamma(\frac{\beta}{2}) \Gamma(\frac{\beta}{2} - k - 1) \frac{b^{2k-2z}}{\beta^{2k-2z}}. \] (10)

We may use the above formula, valid for the free energy of the quantum fields, in order to compute the one-loop free energy for the bosonic string. Strictly speaking, we should construct the string spectrum in the manifold with non-vanishing curvature. For the flat manifold, the spectrum, as an infinite sum of quantum fields present in normal modes of the string, is well-known. To simplify the calculations, one may assume that the curvature \( R = -N(N-1)a^{-2} \) is vanishingly small \( (a \to \infty) \). In addition, we shall be interested in the spectrum with almost linear Regge trajectories and the mass formula, for the open bosonic string, has the form \([36, 37, 38, 39]\)

\[ M^2 = 2 \left( \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} nN_n^i - 1 \right) + O(a^{-4}), \] (11)

where \( N_n^i \) are the occupation numbers of the transverse oscillators and the Regge slope parameter \( \alpha' \) has been chosen equal to one, for the sake of convenience. For the minimally coupled scalar field of mass \( m \), \( b^2 = \rho_N^2 + a^2 m^2 \), where \( \rho_N = (N-1)/2 \). In the limit \( a \to \infty \) the operator \( b^2 \), related to the quantum fields with mass given by Eq. \([11]\), has the form \( b^2 = a^2 M^2 + O(1) \). With the help of heat kernel representation, the trace of the complex power of this operator may be written in the form

\[ \text{Tr} b^{2k+2-z} = \frac{1}{\Gamma(\frac{\beta}{2} - k - 1)} \int_0^\infty dt t^{\frac{\beta}{2} - k - 2} \text{Tr} \exp \left[-t \left(a^2 M^2 + O(1)\right)\right]. \] (12)
Changing variables \( t \to t \pi a^{-2} \) and performing the trace over the entire Fock space we have

\[
\text{Tr} b^{2k+2z} = \frac{1}{\Gamma\left(\frac{3}{2} - k - 1\right)} \frac{(a^2)^{k+1-\frac{3}{2}}}{\pi} \int_0^\infty dt \frac{\pi^k}{(2k+1)!} \eta(it)^{-(N-1)} \left[ 1 + O(a^{-2}) \right],
\]

where \( \eta(\tau) = \exp\left(i\pi\tau/12\right) \prod_{n=1}^\infty \left[ 1 - \exp\left(-2\pi i\tau n\right) \right] \) is the Dedekind’s eta function. Hence, for the free energy we obtain

\[
F(\beta) = -\frac{\sqrt{\pi}}{2\pi i (4\pi)^{N/2+1}} \frac{(N-1)^{1/2}}{\Gamma\left(\frac{N}{2}\right)} \sum_{k=1}^{\infty} \alpha_{k,N} \Gamma(k + \frac{1}{2}) \left(\frac{a^2}{\pi}\right)^{k+1} \times \int_{\text{Re} z = c} dz \zeta_R(z) \frac{\beta^2 a^2}{4\pi} \eta^\prime(\tau) \left[ 1 + O(a^{-2}) \right],
\]

where the integral \( I_N(z) \) in Eq.(14), namely

\[
I_N(z) = \int_0^\infty dt t^{2k-2} \eta(it)^{-(N-1)},
\]

should be regularized. In the ultraviolet region, the integrand is not regular, since, for \( t \to 0 \), one has

\[
\eta(it)^{-(N-1)} = t^{(N-1)/2} e^{\pi(N-1)/12t} \left[ 1 + O\left(e^{-2\pi/t}\right) \right].
\]

The divergence in the infrared region \( (t \to \infty) \) is caused by the tachyon in the string spectrum. The tachyonic divergence signifies a vacuum instability which may be important to understand its role in string theory. In order to regularize the integral, it is more convenient to isolate the ultraviolet divergence and introduce cutoff parameters

\[
I_N(z)_{\text{Reg}} = \int_{\text{Re} z = c} dz \zeta_R(z) \frac{\beta^2 a^2}{4\pi} \eta^\prime(\tau) \left[ 1 + O(a^{-2}) \right],
\]

where the regularization in the infrared region of the analytical part \( G(\mu; \frac{3}{2} - k) \) (the parameter \( \mu \) provides the infrared cutoff) has been done. On the next stage of calculations the ultraviolet regularization will be removed \((\epsilon \to 0)\).

It is convenient to introduce the volume \( V_{D-2} = a^{D-2} V(\mathcal{F}_{D-2}) \), where the constant factor \( V(\mathcal{F}_{D-2}) \) can be referred to as the finite volume of the fundamental domain associated with a compact hyperbolic manifold. In the limit \( a \to \infty \), the leading term in the sum (14) is the term with \( k = (N-1)/2 \), and we have (here we have restored the \( \alpha' \) dependence)

\[
F(\beta) = -\frac{V_{D-2} a^2}{4\pi i V(\mathcal{F}_{D-2})} \frac{1}{(4\pi^2 \alpha')^{13}} \int_{\text{Re} z = c} dz \zeta_R(z) \Gamma\left(\frac{3}{2}\right) \frac{\beta^2 a^2}{4\pi \alpha'} \zeta^\prime(\zeta) \frac{1}{\zeta^2} \left[ 1 + O(a^{-2}) \right],
\]

The free energy (18) can be present in terms of Laurent series in inverse power of \( \beta \) [27]. In this approach, the inverse critical Hagedorn temperature \( \beta_c = \pi \sqrt{8\alpha'} \) arises, for the flat
background, as the convergence condition of the corresponding Laurent series. The integrand
(18) is a meromorphic function which has first order poles at $z = 2n$, $n \in \mathbb{N}$. Moving the line of
integration $\text{Re} z = c$ to the left (and, therefore, crossing the poles) one can obtain for the free
energy

$$
F(\beta) = -\frac{\pi V_{D-2}}{12(4\pi^2\alpha')^{12}V(F_{D-2})}\beta^{-2}\left[1 + \left(144 - \beta^{-2}\frac{4\pi^4}{15}\right)\frac{\alpha'}{a^2}\right] + O\left(\left(\frac{\alpha'}{a^2}\right)^2\right). \tag{19}
$$

We conclude with some remarks. In this letter we have obtained the leading term in the high
temperature expansion for the free energy of open bosonic string. The corresponding entropy
$S$ can be easily obtained using Eq. (18). Our analysis have been carried out for the open string
in manifold $S^1 \otimes H^N$, with vanishingly small curvature of the spatial section. By the way, the
asymptotic behaviour for the closed string free energy can be also calculated with the help of
the techniques presented here. From the physical point of view, two main types of theories, open
and closed strings, admit a similar high temperature behaviour. For flat spaces ($a = \infty$), Eq.
(13) gives the first term of the high temperature expansion of the free energy (see, for example
[27]).

Furthermore, we would like to comment on the results we have obtained in connection with
the quantum corrections to the black hole entropy. Let us recall that the energy density of a gas
of free bosonic strings must be small enough to be unafflicted by the Jeans instability and large
enough to contain many degrees of freedom, i.e. $a^2 \gg \alpha'$. Near the Hagedorn temperature, the
energy density behaves as $(\alpha')^{-13}$. Thus, both the two conditions are satisfied if and only if
$G(\alpha')^{-12} \ll 1 \ [10, 11]$, where the Newton constant $G$ defines the Plank length. For the lower
mass states, $M^2 \propto (\alpha'/a^2)^2 \ [13]$ and as a consequence, the mass operator formally is vani-
shingly small. In this regime, our result may be conformally related to the result obtained for massless
quantum fields in Rindler space or near black hole horizon.

The $\beta$-behaviour of the free energy (19) has the dependence on temperature near the Hage-
dorn transition which is similar to the one found in Ref. [10]. Although the thermal dependence
in Eq. (19), corresponds to quantum fields in two dimensions, such a result can be interpreted as
an indication of a vast reduction of the fundamental degrees of freedom in string theory [10, 22]. We
point out that similar leading high temperature expansion has been obtained in [21, 22]. In fact, for the free energy in orbifold string theory, considered in Ref. [21], the large $N$ limit
($N = 2\pi\beta^{-1}$) means that the one loop string amplitude $A_N = -\beta F$ behaves like $\sim \beta^{-1}\log \beta$ and
therefore the leading power $\beta$-dependence is similar to the behaviour obtained in our formalism.

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