THE FLAT STABLE MODULE CATEGORY OF A COHERENT RING

JAMES GILLESPIE

ABSTRACT. Let $R$ by a right coherent ring and $R$-Mod denote the category of left $R$-modules. We show that there is an abelian model structure on $R$-Mod whose cofibrant objects are precisely the Gorenstein flat modules. Employing a new method for constructing model structures, the key step is to show that a module is flat and cotorsion if and only if it is Gorenstein flat and Gorenstein cotorsion.

1. INTRODUCTION

Recall that a model structure on a category is a formal way of introducing a homotopy theory on that category. If the category we start with is abelian then the resulting homotopy theory is some variety of homological algebra, or relative homological algebra. Moreover the resulting homotopy category is triangulated. So a good way to construct a triangulated category is to construct an abelian model structure. The model structure then provides a tool to understand and study the triangulated homotopy category.

The first purpose of this paper is to construct a new abelian model category structure, the Gorenstein flat model structure, on the category of left $R$-modules where $R$ is any right coherent ring. The cofibrant objects in the model structure are precisely the Gorenstein flat modules which were introduced and studied by Enochs and several coauthors and subsequently studied by many other authors. In particular, see [EX96, EJ00, EJL04, Hol04, YL14], but there is certainly more literature on the subject. A second purpose of this paper is to illustrate a new method from [Gil14] for constructing abelian model structures. This powerful method works well to construct model structures in which the class $\mathcal{W}$ of trivial objects is not well understood, which is the case for the Gorenstein flat model structure. As we show at the end of the paper, $\mathcal{W}$ is well understood when $R$ is quasi-Frobenius, or more generally Gorenstein, or more generally a Ding-Chen ring. In this case, $\mathcal{W}$ is precisely the class of modules with finite flat dimension and the associated homotopy category is triangle equivalent to the stable module category $\text{Stmod}(R)$.

In fact, the homotopy category of the Gorenstein flat model structure is a generalization of the stable module category in the following sense. We will see that the full subcategory of left $R$-modules consisting of the modules that are both Gorenstein flat and cotorsion is a Frobenius category. The conflations, or short exact sequences, are the usual short exact sequences whose all three terms are Gorenstein flat and cotorsion. The injective-projective objects turn out to be the flat cotorsion...
modules. The homotopy category of the Gorenstein flat model structure is triangle equivalent to the stable module category of this Frobenius category.

This paper is brief for a few reasons. One reason is that the method of [Gil14] used to construct the model structure is doing much of the work. We explain and give a precise statement of this method in the next section. When pairing this theorem with Hovey’s one-to-one correspondence between cotorsion pairs and abelian model structures the problem reduces to showing that the flat modules and Gorenstein flat modules are each the left half of a complete hereditary cotorsion pair. But this is already known from the work of Enochs and coauthors. So the essential step is to show that the cores of these cotorsion pairs coincide. All of the terminology in this paragraph will now be explained in the next section.

2. preliminaries

We let $R$ be a ring and $R$-Mod denote the category of left $R$-modules. We assume the ring has an identity and that the modules are unital.

2.1. Cotorsion pairs and Hovey triples. For a class $S$ of $R$-modules, we let $S^\perp$ denote the class of all modules $N$ such that $\text{Ext}^1_R(S, N) = 0$ for all $S \in S$. On the other hand, $S^\perp$ denotes the class of all modules $M$ such that $\text{Ext}^1_R(M, S) = 0$ for all $S \in S$. By a cotorsion pair we mean a pair of classes of $R$-modules $(A, B)$ such that $B = A^\perp$ and $A = B^\perp$.

We call the cotorsion pair hereditary if the left class $A$ is resolving in the sense that whenever

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

is a short exact sequence with $A, A'' \in A$, then $A' \in A$ too. Since $R$-Mod has enough projectives and injectives this condition has been shown to be equivalent to the statement that $B$ is coresolving. See [Gar99].

A cotorsion pair is called complete if it has enough injectives and enough projectives. This means that for each $R$-module $M$ there exist short exact sequences $0 \rightarrow B \rightarrow A \rightarrow M \rightarrow 0$ and $0 \rightarrow M \rightarrow B' \rightarrow A' \rightarrow 0$ with $A, A' \in A$ and $B, B' \in B$. The books [EJ00] and [GT06] are standard references for cotorsion pairs.

By the main theorem of [Hov02] we know that an abelian model structure on $R$-Mod, in fact on any abelian category, is equivalent to a triple $(Q, W, R)$ of classes of objects for which $W$ is thick and $(Q \cap W, R)$ and $(Q, W \cap R)$ are each complete cotorsion pairs. By thick we mean that the class $W$ is closed under retracts (i.e., direct summands) and satisfies that whenever two out of three terms in a short exact sequence are in $W$ then so is the third. In this case, $Q$ is precisely the class of cofibrant objects of the model structure, $R$ are precisely the fibrant objects, and $W$ is the class of trivial objects. We hence denote an abelian model structure $M$ as a triple $M = (Q, W, R)$ and for short we will denote the two associated cotorsion pairs above by $(\bar{Q}, \bar{R})$ and $(\bar{Q}, \bar{R})$ where $\bar{Q} = Q \cap W$ is the class of trivially cofibrant objects and $\bar{R} = W \cap \bar{R}$ is the class of trivially fibrant objects. We say that $M$ is hereditary if both of these associated cotorsion pairs are hereditary. We will also call any abelian model structure $M = (Q, W, R)$ a Hovey triple. Besides [Hov02], the book [Hov99] is a standard reference for the theory of model categories.

By the core of an abelian model structure $M = (Q, W, R)$ we mean the class $Q \cap W \cap R$. This notion comes up in the following theorem giving a converse to Hovey’s main theorem in the case that we have hereditary cotorsion pairs.
Theorem 2.1 (How to construct a Hovey triple from two cotorsion pairs). Let 
\( (Q, \tilde{R}) \) and \( (\tilde{Q}, R) \) be two complete hereditary cotorsion pairs satisfying the two 
conditions below.

1. \( \tilde{R} \subseteq R \) and \( \tilde{Q} \subseteq Q \).
2. \( \tilde{Q} \cap R = Q \cap \tilde{R} \).

Then \( (Q, W, R) \) is a Hovey triple where the thick class \( W \) can be described in the 
two following ways:

\[
W = \{ X \in A \mid \exists \text{ a short exact sequence } X \twoheadrightarrow R \rightarrow Q \text{ with } R \in \tilde{R}, Q \in \tilde{Q} \}
\]

\[
= \{ X \in A \mid \exists \text{ a short exact sequence } R' \twoheadrightarrow Q' \rightarrow X \text{ with } R' \in \tilde{R}, Q' \in \tilde{Q} \}.
\]

Moreover, \( W \) is unique in the sense that if \( V \) is another thick class for which 
\( (Q, V, R) \) is a Hovey triple, then necessarily \( V = W \).

Theorem 2.1 just appeared in [Gil14]. It provides a powerful method for con-
structing abelian model structures and is the main tool used in this paper.

2.2. Gorenstein flat modules. We now recall the key definitions.

Definition 2.2. By a complete flat resolution we mean an exact chain complex 
\( F = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow F_{-2} \rightarrow \cdots \)

for which \( I \otimes_R F \) is also exact whenever \( I \) is an injective right \( R \)-module.

Definition 2.3. A left \( R \)-module \( M \) is called Gorenstein flat if \( M = Z_{-1}F \) (that 
is, \( \ker(F_{-1} \rightarrow F_{-2}) \)) for some complete flat resolution \( F \). We let \( GF \) denote 
the class of all Gorenstein flat modules and \( GC = GF^\perp \). We call the modules in \( GC \)
Gorenstein cotorsion.

It is easy to see that any flat module is Gorenstein flat and conseque ntly any 
Gorenstein cotorsion module is cotorsion.

It is well known that \( (F, C) \), where \( F \) is the class of flat modules and \( C = F^\perp \) is 
the class of cotorsion modules, is a complete hereditary cotorsion pair. This was 
shown in [BBE01], and is also proved in [EJ00].

Definition 2.4. Call an \( R \)-module \( N \) flat cotorsion if it is both flat and cotorsion.

That is, if it belongs to the core \( F \cap C \) of the flat cotorsion pair \( (F, C) \).

Holm shows in [Hol04] that for a right coherent ring \( R \) the class \( GF \) is projectively 
resolving. This statement is encompassed in the following important theorem.

Theorem 2.5 (Enochs, Jenda, Lopez-Ramos [EJL04]). Let \( R \) be a right coher-
ent ring and \( GF \) the class of Gorenstein flat left \( R \)-modules. Then \( (GF, GC) \) is a 
complete hereditary cotorsion pair.

3. The Gorenstein flat model structure

Let \( R \) be a right coherent ring and \( R\text{-Mod} \) the category of left \( R \)-modules. Our 
goal is to construct an abelian model structure on \( R\text{-Mod} \) whose homotopy category 
is a generalization of the stable module category \( \text{Stmod}(R) \) in the case that \( R \) is 
Gorenstein. We also wish to have the class \( GF \) of Gorenstein flat modules as 
the cofibrant objects. The stable module category of a Gorenstein ring was first 
introduced by Hovey in [Hov02], and extended to Ding-Chen rings in [Gil10].
Lemma 3.1. The following are equivalent for an R-module N.

1. N is Gorenstein cotorsion.
2. N is cotorsion and the complex Hom$_R(F, N)$ is exact for all complete flat resolutions F.

Proof. First, note that for any exact complex E, we have an isomorphism
\[ \text{Ext}^1_{\text{ch}(R)}(E, S^0(N)) \cong \text{Ext}^1_R(Z_{-1}E, N). \]

Second, note that if N Gorenstein cotorsion then it is clearly cotorsion. Now for a cotorsion module N, we have Hom$_R(F, N) = \text{Hom}(F, S^0(N))$ is exact for all complete flat resolutions F if and only if Ext$_{d_1w}(F, S^0(N)) = 0$ for all complete flat resolutions F. (Here, Ext$_d(F, S^0(N))$ is the Yoneda Ext group of all degreewise split short exact sequences.) But since N is cotorsion this happens if and only if Ext$_1(F, S^0(N)) = 0$ for all complete flat resolutions F. Now the Lemma follows since the class of Gorenstein flat modules coincides with the class of all $\mathbb{Z}_nF$ where F is a complete flat resolution. \(\square\)

Proposition 3.2. Let R be a right coherent ring. Then the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$ and the Gorenstein flat cotorsion pair $(\mathcal{G}\mathcal{F}, \mathcal{GC})$ have the same core. That is, $\mathcal{G}\mathcal{F} \cap \mathcal{GC} = \mathcal{F} \cap \mathcal{C}$.

Proof. $(\subseteq)$ Say N $\in \mathcal{G}\mathcal{F} \cap \mathcal{GC}$. Since $\mathcal{GC} \subseteq \mathcal{C}$ we only need to argue that N is flat. We start by writing $N = Z_0F$ where F is some complete flat resolution, and we factor the differential $d_1 : F_1 \to F_0$ as $F_1 \xrightarrow{j} N \xrightarrow{1} F_0$. So $d_1 = i e$ where $e$ is an epimorphism and $i$ is a monomorphism. By Lemma 3.1 we know that Hom$_R(F, N)$ is exact. But since $e \in \ker d_2^*$, the exactness of Hom$_R(F, N)$ gives us $e \in \ker d_2^* = \text{Im} d_1^*$. This means we have a map $p \in \text{Hom}_R(F_0, N)$ such that $pd_1 = e$. So $pie = e$. So $pi = 1_N$. So N is a retract of the flat $F_0$, and so N too is flat.

Before proving the reverse inclusion $(\supseteq)$, we recall that given a left (resp. right) R-module M, its character module is defined to be the right (resp. left) R-module $M^+ = \text{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$. It is a standard fact that M is flat if and only if $M^+$ is injective. A proof can be found in [EJ00], for example.

$(\supseteq)$ Now suppose N is flat cotorsion. This time we see N is clearly Gorenstein flat, so the point is to show that N must be Gorenstein cotorsion too. We again use Lemma 3.1 above. That is, let F be an arbitrary complete flat resolution. We will be done once we show Hom$_R(F, N)$ is also exact. First, note that the double character dual $N^{++}$ does have the property that Hom$_R(F, N^{++})$ is exact. Indeed, Hom$_R(F, N^{++})$ = Hom$_R(F, \text{Hom}_R(N^+, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_R(N^+ \otimes_R F, \mathbb{Q}/\mathbb{Z})$, and since N is flat we know $N^+$ is an injective (right) R-module. So this last complex must be exact just by assuming that N is flat.

Now that we have shown Hom$_R(F, N^{++})$ must be exact whenever N is flat, it is left to argue that Hom$_R(F, N)$ must also be exact when N is flat cotorsion. We will argue that Hom$_R(F, N)$ is in fact a retract of Hom$_R(F, N^{++})$; since exact complexes are closed under retracts this will complete the proof. Now N flat implies $N^+$ is injective. In particular, $N^+$ is absolutely pure and since R is (right) coherent we conclude from [Fie72 Theorem 2.2] that $N^{++}$ is also flat. Then from the proof of [EJ00 Proposition 5.3.9] we see that there is a pure exact sequence
\[ 0 \to N \to N^{++} \to N^{++}/N \to 0. \]
Since the class of flat modules is closed under pure quotients, we get that \( N^{++}/N \) is also flat. But by the assumption that \( N \) is also cotorsion, this means that the above sequence splits. Thus \( N \) is a retract of \( N^{++} \). Like all functors, \( \text{Hom}_R(F, -) \) must preserve retracts. So we conclude that \( \text{Hom}_R(F, N) \) must be an exact complex. \( \square \)

**Theorem 3.3.** Let \( R \) be a right coherent ring. Then the category \( \text{R-Mod} \) of left \( R \)-modules has an abelian model structure, the **Gorenstein flat model structure**, as follows:

- The cofibrant objects coincide with the class \( \mathcal{GF} \) of Gorenstein flat modules.
- The fibrant objects coincide with the class \( \mathcal{C} \) of cotorsion modules.
- The trivially cofibrant objects coincide with the class \( \mathcal{F} \) of flat modules.
- The trivially fibrant objects coincide with the class \( \mathcal{GC} \) of Gorenstein cotorsion modules.

An \( R \)-module \( M \) fits into a short exact sequence

\[
0 \to C \to F \to M \to 0
\]

with \( F \in \mathcal{F} \) and \( C \in \mathcal{GC} \) if and only if it fits into a short exact sequence

\[
0 \to M \to C' \to F' \to 0
\]

with \( F' \in \mathcal{F} \) and \( C' \in \mathcal{GC} \). Modules \( M \) with this property are precisely the trivial objects of the Gorenstein flat model structure.

**Proof.** As noted in Section 2 we already know that \( (\mathcal{F}, \mathcal{C}) \) is a complete hereditary cotorsion pair. Since \( R \) is right coherent we also have from Theorem 2.5 that \( (\mathcal{GF}, \mathcal{GC}) \) is a complete hereditary cotorsion pair. Proposition 3.2 says that the cores of these cotorsion pairs are equal. Clearly \( \mathcal{F} \subseteq \mathcal{GF} \), and this is equivalent to \( \mathcal{GC} \subseteq \mathcal{C} \). So Theorem 2.1 produces a model structure exactly as described. \( \square \)

Recall that a **Frobenius category** is an exact category with enough injectives and projectives and in which the projective and injective objects coincide. Given a Frobenius category \( \mathcal{F} \), we can form the stable category \( \mathcal{F}/\sim \), where \( f \sim g \) if \( g - f \) factors through a projective-injective. The main fact about Frobenius categories is that the stable category has the structure of a triangulated category. See \[\text{Hap88}\]. On the other hand, the homotopy category of an abelian model structure is naturally a triangulated category as well \[\text{Hov99}, \text{Chapters 6 and 7}\], and equivalent to a Frobenius category whenever it is hereditary by \[\text{Gil11}\]. In particular, we get the following corollary.

**Corollary 3.4.** Let \( R \) be a right coherent ring. Then the full subcategory \( \mathcal{GF} \cap \mathcal{C} \) of \( \text{R-Mod} \) consisting of the Gorenstein flat and cotorsion modules is a Frobenius category with respect to its inherited exact structure. The projective-injective objects are precisely the flat cotorsion modules. Moreover, the homotopy category of the Gorenstein flat model structure is triangle equivalent to the stable category

\[
(\mathcal{GF} \cap \mathcal{C})/\sim
\]

where \( f \sim g \) if and only if \( g - f \) factors through a flat cotorsion module.

**Proof.** By the inherited exact structure we mean that the short exact sequences (or conflations) are the usual short exact sequences but with all three terms in \( \mathcal{GF} \cap \mathcal{C} \). Since we have a hereditary Hovey triple \( (\mathcal{GF}, \mathcal{W}, \mathcal{C}) \) whose core \( \mathcal{GF} \cap \mathcal{W} \cap \mathcal{C} \) equals the class of flat cotorsion modules, the result follows from \[\text{Gil11}, \text{Sections 4 and 5}\]. \( \square \)
We end by pointing out that the homotopy category of the Gorenstein flat model structure is equivalent to the usual stable module category $\text{Stmod}(R)$ in the case that $R$ is a Ding-Chen ring in the sense of [Gil10]. These are the coherent analog of Gorenstein rings. In particular, the class of Ding-Chen rings includes Gorenstein rings and hence quasi-Frobenius rings.

**Corollary 3.5.** Let $R$ be a Ding-Chen ring. That is, a (left and right) coherent ring which has finite FP-injective dimension as both a left and right module over itself. Then a module is trivial in the Gorenstein flat model structure if and only if it has finite flat dimension, or equivalently, if and only if it has finite FP-injective dimension. In the case that $R$ is Gorenstein this is equivalent to the module having finite injective dimension and also equivalent to it having finite projective dimension.

**Proof.** Apply [Gil10, Theorem 4.10]. This theorem says that when $R$ is a Ding-Chen ring, we have another potential Hovey triple $(\mathcal{GF}, \mathcal{V}, \mathcal{C})$ where $\mathcal{V}$ are the trivial objects described in the corollary. It follows from the uniqueness property of Theorem 2.1 that the class $\mathcal{V}$ of trivial objects must coincide with $\mathcal{W}$ from Theorem □

Other generalizations of $\text{Stmod}(R)$, whose cofibrant (resp. fibrant) objects are based on Gorenstein projective modules (resp. Gorenstein injective modules) appear in [BGH12]. When $R$ is Gorenstein, or even Ding-Chen, all three of these model structures produce equivalent homotopy categories. However, for a general coherent ring it is not clear when these different generalizations are equivalent. In particular we would like to know if the trivial objects of the Gorenstein flat model structure coincide with the trivial objects of the Gorenstein AC-projective model structure when $R$ is coherent.

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Ramapo College of New Jersey, School of Theoretical and Applied Science, 505 Ramapo Valley Road, Mahwah, NJ 07430

E-mail address, Jim Gillespie: jgillesp@ramapo.edu

URL: http://pages.ramapo.edu/~jgillesp/