Decentralized Approximate Newton Methods for In-Network Optimization

Hejie Wei, Zhihai Qu, Xuyang Wu, Hao Wang, and Jie Lu

Abstract—This paper proposes a set of Decentralized Approximate Newton (DEAN) methods for addressing in-network convex optimization, where nodes in a network seek for a consensus that minimizes the sum of their individual objective functions through local interactions only. The proposed DEAN algorithms allow each node to repeatedly take a local approximate Newton step, so that the nodes not only jointly emulate the (centralized) Newton method but also drive each other closer. Under weaker assumptions in comparison with most existing distributed Newton-type methods, the DEAN algorithms enable all the nodes to asymptotically reach a consensus that can be arbitrarily close to the optimum. Also, for a particular DEAN algorithm, the consensus error among the nodes vanishes at a linear rate and the iteration complexity to achieve any given accuracy in optimality is provided. Furthermore, when the optimization problem reduces to a quadratic program, the DEAN algorithms are guaranteed to linearly converge to the exact optimal solution.

Index Terms—Distributed optimization, decentralized algorithm, Newton method

I. INTRODUCTION

In many engineering applications such as learning by computer networks [1], coordination of multi-agent systems [2], estimation by sensor networks [3], and resource allocation in communication networks [4], nodes in a networked system often need to cooperate with each other in order to minimize the sum of their individual objective functions.

There have been a large number of decentralized/distributed algorithms for such in-network optimization problems, which allow nodes in the network to address the problem by means of interacting with their neighbors only. Most of these algorithms are first-order methods, where the nodes utilize subgradients/gradients of their local objectives to update (e.g., [3]–[23]). However, the first-order algorithms may suffer from slow convergence rate, especially when the problem is ill-conditioned. This motivates the development of decentralized second-order methods, where the Hessian matrices of the local objectives, if available, are involved in computing the iterates. Such second-order methods can be roughly classified into the following two categories:

The first category is the methods based on second-order approximations of certain dual-related objectives. For instance, the decentralized Exact Second-Order Method (ESOM) [24] considers a second-order approximation of an augmented Lagrangian function, and the Decentralized Quadratically Approximated ADMM (DQM) [25] introduces a quadratic approximation to a decentralized version of the Alternating Direction Method of Multipliers (ADMM).

The second category is the Newton-type methods, such as the distributed Broyden-Fletcher-Goldfarb-Shanno (D-BFGS) method [26], the Network-Newton (NN) method [27], the Distributed Quasi-Newton (DQN) method [28], and the Newton-Raphson Consensus (NRC) method [29]. Among these methods, D-BFGS, NN, and DQN relax the consensus constraint by adding a penalty to the objective function and approximate the Newton direction of the penalized objective in a decentralized manner. As a result, these methods are only guaranteed to converge to a suboptimal solution. NRC utilizes an average consensus scheme to approximate the Newton-Raphson direction in a distributed fashion. Although NRC may converge to the exact optimal solution, no explicit parameter condition to guarantee the convergence is provided, making it difficult to be implemented in practice.

In this paper, we propose a family of Decentralized Approximate Newton methods, referred to as DEAN, for solving in-network convex optimization. The DEAN algorithms are developed by letting every node execute a local Newton-like step at each iteration, where the inverse of the Hessian of its own objective function is involved and the gradient term in the conventional Newton step is replaced by the sum of the differences between the node and each of its neighbors, which are measured by the gradients of a class of locally strongly convex functions associated with the corresponding links. This intends to approximate the traditional (centralized) Newton method and in the meanwhile, drive all the nodes together. The DEAN algorithms are endowed with the following results and advantages:

1) The DEAN algorithms asymptotically drive all the nodes to a consensus that lies in an arbitrarily small neighborhood of the optimum. In addition, if the local objectives are positive definite quadratic functions, the nodes converge to the exact optimum at a linear rate.

2) With a particular choice of the functions associated with the links in DEAN, the disagreement among the nodes is shown to drop to zero at a linear rate. Further, for any given accuracy $\epsilon > 0$, we provide the iteration complexity (i.e., a bound on the number of iterations needed) to achieve $\epsilon$-suboptimality.

3) The above convergence results are established under the assumption that the local objectives of the nodes are
locally strongly convex, which is less restricted than the global strong convexity assumed by most existing second-order methods [24], [25], [27]–[29].

4) Compared to other Newton-type methods, DEAN has the lowest communication cost per iteration.

5) Simulation results illustrate the competitive performance of DEAN in comparison with several existing Newton-type methods.

The outline of this paper is as follows: Section II formulates the problem. Section III describes the proposed DEAN algorithms and Section IV is dedicated to the convergence analysis. Section V presents the simulation results. Concluding remarks are provided in Section VI. All the proofs are in the appendix.

A. Notation and preliminaries

Throughout this paper, we use \( \| \cdot \| \) to denote the Euclidean norm, \( \{ \cdot , \cdot \} \) the unordered pair, \( | \cdot | \) the absolute value of a real number or the cardinality of a set, and range(\( \cdot \)) the range of a matrix. In addition, \( 1_n \) is the \( n \)-dimensional all-one vector and \( I_n \) is the \( n \times n \) identity matrix. For any \( c_1, \ldots , c_n \in \mathbb{R} \),diag(\( c_1, \ldots , c_n \)) is the diagonal matrix whose diagonal entries are \( c_1, \ldots , c_n \). For any \( z_1, \ldots , z_n \in \mathbb{R}^n \), \( \mathbf{z} = [z_1; \ldots ; z_n] \in \mathbb{R}^{nN} \) is the vector obtained by stacking \( z_1, \ldots , z_n \). Given any \( r > 0 \) and \( x \in \mathbb{R}^n \), \( B(x; r) := \{ y \in \mathbb{R}^n : \| x - y \| \leq r \} \subset \mathbb{R}^n \) represents the closed ball with center \( x \) and radius \( r \). Also, for any set \( C \subset \mathbb{R}^n \), \( \text{conv}(C) \) is the convex hull of \( C \) and \( P_C(x) \) is the projection of \( x \in \mathbb{R}^n \) onto \( C \). For any differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \), \( \nabla f(x) \) denotes the gradient of \( f \) at \( x \in \mathbb{R}^n \) and, if \( f \) is twice differentiable, \( \nabla^2 f(x) \) represents the Hessian matrix of \( f \) at \( x \). For any \( A, B \in \mathbb{R}^{n \times n} \), \( A \succeq B \) means \( A - B \) is positive semidefinite and \( A \succ B \) means \( A - B \) is positive definite. For any symmetric positive semidefinite matrix \( H \in \mathbb{R}^n \), we use \( \lambda_i(H) \) to denote the \( i \)-th smallest eigenvalue of \( H \), \( \lambda_{\max}(H) \) the largest eigenvalue of \( H \), and \( H^\dagger \) the pseudoinverse of \( H \).

A differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \) is said to be locally strongly convex if for any convex and compact set \( C \), there exists \( \theta > 0 \) such that \( f(y) - f(x) - \nabla f(x)^T(y - x) \geq \theta \| y - x \|^2 / 2 \) for any \( x, y \in C \), where \( \theta \) is called the convexity parameter of \( f \) on \( C \). It is said to be (globally) strongly convex if there exists \( \theta > 0 \) such that \( f(y) - f(x) - \nabla f(x)^T(y - x) \geq \theta \| y - x \|^2 / 2 \) for any \( x, y \in \mathbb{R}^n \). A vector-valued or matrix-valued function \( h \) is said to be locally Lipschitz continuous if for any compact set \( C \) contained in the domain of \( h \), there exists \( L_C \geq 0 \) such that the Lipschitz condition \( \| h(x) - h(y) \| \leq L_C \| x - y \| \) holds for all \( x, y \in C \). Also, \( L_C \) is said to be the Lipschitz constant of \( h \) on \( C \).

II. PROBLEM FORMULATION

Consider an undirected, connected graph \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{1, 2, \ldots , N\} \), \( N \geq 2 \) is the node set and \( \mathcal{E} \subseteq \{ \{ i, j \} : i, j \in \mathcal{V}, i \neq j \} \) is the link set. For each node \( i \in \mathcal{V} \), the set of its neighbors is denoted by \( \mathcal{N}_i = \{ j \in \mathcal{V} : \{i, j \} \in \mathcal{E} \} \). The nodes are required to solve the following problem:

\[
\min_{x \in \mathbb{R}^n} \sum_{i \in \mathcal{V}} f_i(x), \tag{1}
\]

where each \( f_i : \mathbb{R}^n \to \mathbb{R} \) is the local objective function of node \( i \in \mathcal{V} \) and satisfies the following assumption:

**Assumption 1:** For each \( i \in \mathcal{V} \), \( f_i \) is twice continuously differentiable and locally strongly convex, and has a minimizer. In addition, \( \nabla^2 f_i \) is locally Lipschitz continuous.

Assumption 1 guarantees that each \( f_i \) has a unique minimizer \( x_i^* = \arg \min_{x \in \mathbb{R}^n} f_i(x) \in \mathbb{R}^n \), so that there is a unique optimal solution \( x^* \in \mathbb{R}^n \) to (1). In addition, given any convex and compact set \( C \subset \mathbb{R}^n \), there exist \( \theta_i, \Theta_i > 0 \) such that \( \theta_i I_n \preceq \nabla^2 f_i(x) \preceq \Theta_i I_n \text{ } \forall x \in C \). Another implication of Assumption 1 is that \( \nabla f_i \) is locally Lipschitz continuous.

Among the existing distributed second-order methods [24]–[29], most of them assume the \( f_i \)’s to be (globally) strongly convex [24], [25], [27]–[29], which is more restricted than the local strong convexity in Assumption 1. One example of functions that are locally strongly convex but not (globally) strongly convex is the objective of logistic regression [1], i.e., \( \sum_{i=1}^n \sum_{j=1}^m \ln(1 + \exp(-v_i u_j^T x)) \) with \( v_i \in \{-1, 1\} \) and \( u_j \in \mathbb{R}^n \), which often arises in machine learning. The D-BFGS method [26] allows each \( f_i \) to be a general convex function, yet it requires, like other second-order algorithms in [24], [27]–[29], each \( \nabla f_i \) to be globally Lipschitz continuous, which is unnecessary for problem (1) under Assumption 1. Moreover, the local Lipschitz continuity of \( \nabla^2 f_i \) in Assumption 1 is weaker than the three times continuous differentiability of \( f_i \) in [29] and the global Lipschitz continuity of \( \nabla^2 f_i \) in [24], [25], [27].
for some constant $C > 0$. This indicates that once all the $x_i^k$’s become identical, the gradient sum $\sum_{i \in V} \nabla f_i(x_i^k)$ would remain constant. Further, if the initial gradient sum $\sum_{i \in V} \nabla f_i(x_i^0)$ is zero, by properly selecting the weights $\alpha_{i,j} \forall \{i,j\} \in E$, we may be able to keep $\|\sum_{i \in V} \nabla f_i(x_i^k)\| \forall k \geq 0$ very small, so that the $x_i^k$’s, once agreeing with each other, would be sufficiently close to the optimum $x^*$.

Based on the above observations, to make each $x_i^k$ approach the optimum $x^*$, we set each $x_i^0$ to the unique minimizer $x_i^*$ of $f_i$, i.e.,

$$x_i^0 = x_i^* := \arg \min_{x \in \mathbb{R}^n} f_i(x).$$

Hence, $\sum_{i \in V} \nabla f_i(x_i^0) = 0$. Note from this and (8) that if each $f_i$ is a positive definite quadratic function, i.e.,

$$f_i(x) = \frac{(x - b_i)^T B_i (x - b_i)}{2} \quad B_i = B_i^T > 0, \quad b_i \in \mathbb{R}^n, \quad (10)$$

then $\sum_{i \in V} \nabla f_i(x_i^k) = 0 \forall k \geq 0$. Thus, if all the $x_i^k$’s reach a consensus, the consensus is the optimum $x^*$.

The initialization (9) and the update (5) together yield a class of Decentralized Approximate Newton methods, referred to as DEAN algorithms. As is shown in Algorithm 1, the implementation of the DEAN algorithms is fully decentralized. The initialization (9) can be completed by each node on their own. The update (5) requires each node $i \in V$ to evaluate the inverse of the Hessian of its local objective $f_i$ at its current estimate $x_i^k$ and to exchange $x_i^k$ with its neighbors.

Algorithm 1 Decentralized Approximate Newton (DEAN) Method

1: **Initialization:**
2: 1: Each pair of neighboring nodes $i$ and $j$ agree on $\alpha_{i,j} > 0$ and $g_{i,j}$ satisfying Assumption 2.
3: 3: Each node $i \in V$ sets $x_i^0 = x_i^*$.
4: 4: for $k = 0, 1, \ldots$ do
5: 5: Each node $i \in V$ sends $x_i^k$ to every neighbor $j \in N_i$.
6: 6: Upon receiving $x_j^k \forall j \in N_i$, each node $i \in V$ updates $x_i^{k+1} = x_i^k + (\nabla^2 f_i(x_i^k))^{-1} \sum_{j \in N_i} \alpha_{i,j} (\nabla g_{i,j}(x_j^k) - \nabla g_{i,j}(x_i^k)).$
7: 7: end for

Prior to implementing DEAN, each pair of neighboring nodes $\{i,j\} \in E$ need to agree on the selection of the function $g_{i,j}$ satisfying Assumption 2. For the option of $g_{i,j}$ given by (6), the update (5) can be executed if each node $i$ shares its local objective $f_i$ with all its neighbors. However, this could be prohibitively costly in some cases. Instead, the nodes may adopt the following scheme to avoid exchanging the $f_i$’s. For every $k \geq 0$, each node $i \in V$ first sends $x_i^k$ and $\nabla f_i(x_i^k)$ to all its neighbors. Upon receiving $x_j^k$ and $\nabla f_j(x_j^k)$ from all its neighbors. Upon receiving $x_j^k$ and $\nabla f_j(x_j^k)$ from all its neighbors. Upon receiving $x_j^k$ and $\nabla f_j(x_j^k)$ from all its neighbors. Without exchanging its local objective $f_i$ with its neighbors. For another example of $g_{i,j}$ in (7), each pair of neighbors $\{i,j\} \in E$ only need to jointly determine $A_{i,j}$, which can be done at negligible communication cost (e.g., we may simply set $A_{i,j} = I_n \forall \{i,j\} \in E$).

Remark 1: If all the weights $\alpha_{i,j} \forall \{i,j\} \in E$ are identical, the DEAN algorithms (5) and (9) can be viewed as a
finite-difference discretization of the continuous-time ZGS algorithms in the earlier work [31], for which the ZGS manifold \( \{y_1, \ldots, y_n\} \in \mathbb{R}^{kN} : \sum_{i \in V} \nabla f_i(y_i) = 0 \) is guaranteed to be positive invariant. Nevertheless, here we allow each \( \alpha_{i,j} \) to be distinct and determined only by the neighboring nodes \( \{i, j\} \in \mathcal{E} \). Moreover, unlike the ZGS algorithms that require global strong convexity of the \( f_i \)'s to establish convergence, the DEAN algorithms relax this condition to local strong convexity. Furthermore, the discrete-time nature of DEAN requires significantly different tools for convergence analysis.

Finally, we compare DEAN with the existing decentralized Newton-type methods, including D-BFGS [26], NN-K, K ≥ 0 [27], DQN-K, K = 0, 1, 2 [28], and NRC [29], in respect of their communication costs at each iteration. Note from Algorithm 1 that DEAN essentially requires every node to transmit \( |N_i| \) vectors of dimension \( n \) per iteration. Every node at each iteration in D-BFGS needs to transmit 3\( |N_i| \) vectors of the same dimension. During one iteration of NN-K \( \forall K \geq 0 \) and DQN-K \( \forall K = 0, 1, 2, (K + 1)|N_i| \) transmissions of n-dimensional vectors are executed by each node. In addition, NRC needs every node to transmit 2\( |N_i| \) vectors in \( \mathbb{R}^n \) and 2\( |N_i| \) matrices in \( \mathbb{R}^{n \times n} \) at each iteration. Therefore, the communication cost of DEAN is the lowest among these Newton-type methods.

### IV. Convergence Analysis

In this section, we analyze the convergence performance of the DEAN algorithms.

To this end, we utilize the Lyapunov function candidate \( V : \mathbb{R}^{kN} \rightarrow \mathbb{R} \) given by

\[
V(x) = \sum_{i \in V} f_i(x^*) - f_i(x_i) - \nabla f_i(x_i)^T (x^* - x_i).
\]

Due to Assumption 1, \( V(x) \geq 0 \) \( \forall x \in \mathbb{R}^{kN} \) and the equality holds if and only if \( x = x^* = [x_1^*; \ldots; x_N^*] \). Hence, \( V \) can be viewed as a measure of the suboptimality of \( x \). Further, we introduce the following notations based on \( V \), which will be used to present the convergence results.

First of all, for each \( i \in V \), let

\[
C_i = \{x \in \mathbb{R}^n : f_i(x^*) - f_i(x) - \nabla f_i(x_i)^T (x^* - x_i) \leq V(x^*)\},
\]

where \( x^0 = [x_1^0; \ldots; x_N^0] \) is the initial state in the DEAN algorithms given by (9). Clearly, \( C_i \forall i \in V \) are compact. Thus, there exist \( \theta_i, \theta_j > 0 \) such that

\[
\nabla^2 f_i(x) \succeq \theta_i I_n, \quad \forall x \in \text{conv}(C_i),
\]

\[
\nabla^2 f_i(x) \succeq \theta_i I_n, \quad \forall x \in \text{conv}(\cup_{j \in \mathcal{V}C_j}).
\]

In addition, for each \( \{i, j\} \in \mathcal{E} \), define the compact set

\[
C_{i,j} = \text{conv}\left\{B \left(x_i^0, 2 \sqrt{\frac{2V(x^0)}{\theta_i}}\right) \cup B \left(x_j^0, 2 \sqrt{\frac{2V(x^0)}{\theta_j}}\right)\right\}.
\]

It follows from Assumption 2 that there exist \( \gamma_{i,j}, \Gamma_{i,j} > 0 \) such that

\[
\gamma_{i,j} \|x - y\|^2 \leq (\nabla g_{i,j}(x) - \nabla g_{i,j}(y))^T (x - y) \leq \Gamma_{i,j} \|x - y\|^2, \quad \forall x, y \in C_{i,j}.
\]

 Arbitrarily pick an \( \bar{\alpha} > 0 \) and suppose the weights \( \alpha_{i,j} \) \( \forall \{i, j\} \in \mathcal{E} \) are selected from the interval \( (0, \bar{\alpha}] \). Then, for each \( i \in V \), let

\[
\delta_i = \sqrt{2V(x^0)} \left[\frac{2}{\theta_i} + \frac{\bar{\alpha}}{\theta_j} \sum_{j \in N_i} \frac{1}{\sqrt{\theta_i}} + \frac{1}{\sqrt{\theta_j}}\right].
\]

Due again to Assumption 1 there exist \( \Theta_i > 0 \) and \( L_i \geq 0 \) such that \( \forall x, y \in B(x_i^0, \delta_i) \),

\[
(\nabla f_i(x) - \nabla f_i(y))^T (x - y) \leq \Theta_i \|x - y\|^2,
\]

\[
\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\| \leq L_i \|x - y\|.
\]

Moreover, we let \( \theta_i > 0 \) be such that

\[
\nabla^2 f_i(x) \succeq \theta_i I_n, \quad \forall x \in \text{conv}(C_i \cup B(x_i^0, \delta_i)).
\]

Note that \( \theta_i \leq \Theta_i \). For convenience, denote \( \theta = \min_{i \in \mathcal{V}} \Theta_i > 0 \). If \( f_i \) is (globally) strongly convex, then we can take \( \theta_i \) as well as the above \( \theta_i \) and \( \theta_i \) all equal to the convexity parameter of \( f_i \) over \( \mathbb{R}^n \).

Our first result shows that \( V(x^k) \) is non-increasing in \( k \) and provides its drop at each iteration:

**Lemma 1 (Monotonicity of Lyapunov function):** Suppose Assumptions 1 and 2 hold. Let \( x^k = [x_1^k; \ldots; x_N^k] \) \( \forall k \geq 0 \) be generated by DEAN described in Algorithm 1 with \( 0 < \alpha_{i,j} \leq \bar{\alpha} \forall \{i, j\} \in \mathcal{E} \). If, in addition,

\[
\alpha_{i,j} < \frac{1}{2\Gamma_{i,j}} \min_{\theta_i} \left\{\frac{\theta_i^2}{|N_i|} \left(\Theta_i - \frac{L_i}{2} \sqrt{\frac{2V(x^0)}{\theta_i}}\right)^{-1}, \frac{\theta_i^2}{|N_j|} \left(\Theta_j - \frac{L_j}{2} \sqrt{\frac{2V(x^0)}{\theta_j}}\right)^{-1}\right\}, \forall \{i, j\} \in \mathcal{E},
\]

then for each \( k \geq 0 \),

\[
V(x^{k+1}) - V(x^k) \leq -\sum_{i \in \mathcal{V}} \sum_{j \in N_i} \gamma_{i,j} \alpha_{i,j} \|x_i^k - x_j^k\|^2 - \left[\frac{\theta_i}{2} - \Theta_i - \frac{L_i}{2} \sqrt{\frac{2V(x^0)}{\theta_i}}\right]|N_i| \left[\alpha_{i,j} + \frac{1}{2\Gamma_{i,j}}\right] \leq 0.
\]

**Proof:** See Appendix A.

**Remark 2:** In Lemma 1 as well as the statements in the rest of the paper, the constant \( \bar{\alpha} \) in the condition \( 0 < \alpha_{i,j} \leq \bar{\alpha} \forall \{i, j\} \in \mathcal{E} \) can be chosen as any positive scalar, which plays a role in \( \delta_i \) given by (16) and, thus, affects the values of \( \theta_i, \theta_i, L_i \forall i \in \mathcal{V} \).

Lemma 1 implies that \( V \) is a Lyapunov function which keeps strictly decreasing until \( x_i^k \forall i \in \mathcal{V} \) become identical. Also, since \( V(x^k) \) is bounded from below, \( \lim_{k \to \infty} V(x^k) \) exists. This leads to the theorem below, which says that all the nodes are able to reach a consensus:

**Theorem 1 (Asymptotic convergence to consensus):** Suppose Assumptions 1 and 2 hold. Let \( x^k = [x_1^k; \ldots; x_N^k] \) \( \forall k \geq 0 \) be generated by DEAN described in Algorithm 1 with \( 0 < \alpha_{i,j} \leq \bar{\alpha} \forall \{i, j\} \in \mathcal{E} \). Suppose (20) holds. Then,

\[
\lim_{k \to \infty} \|x_i^k - x_j^k\| = 0, \quad \forall i, j \in \mathcal{V}.
\]

**Proof:** See Appendix B.
(i.e., \( \lim_{k \to \infty} \|x^k - x^*\| \leq \epsilon \)), provided that the weights \( \alpha_{(i,j)} \) \( \forall \{i, j\} \in \mathcal{E} \) are properly related to \( \epsilon \):

**Theorem 2 (Asymptotic convergence to suboptimality):** Suppose Assumptions 1 and 2 hold. Let \( x^k = [x^k_1; \ldots; x^k_N] \) \( \forall k \geq 0 \) be generated by DEAN described in Algorithm 1 with \( 0 < \alpha_{(i,j)} \leq \bar{\alpha} \) \( \forall \{i, j\} \in \mathcal{E} \). For each \( i \in \mathcal{V} \), let \( \bar{\eta}_i = \|N_i|\nu_i\|N_i\|/\theta_i^2 \geq \bar{\eta}_0 \) and \( \bar{\eta}_i = \|N_i|\nu_i/\theta_i^2 \geq \bar{\eta}_0 \) \( 0 \) and \( \bar{\eta}_i = 2\|N_i|\nu_i/\theta_i^2 \geq \bar{\eta}_0 \). Given any \( \epsilon > 0 \), if

\[
\alpha_{(i,j)} < \frac{\epsilon}{\bar{\eta}_i + \bar{\eta}_j \epsilon} \min\left\{ \frac{1}{\bar{\eta}_i + \bar{\eta}_j \epsilon}, \frac{1}{\bar{\eta}_j + \bar{\eta}_i \epsilon} \right\}, \forall \{i, j\} \in \mathcal{E},
\]

then \( \lim_{k \to \infty} \|x^k - x^*\| \leq \epsilon \).

**Proof:** See Appendix C ■

Below, we explore the convergence rates of the DEAN algorithms. For simplicity, here we only consider DEAN with each \( g_{(i,j)}(x) = \frac{1}{2} x^T x, \) which indeed can be extended to more general cases. To present the convergence rate results, we consider the Laplacian matrix \( L_G \) of the graph \( G \):

\[
[L_G]_{ij} = \begin{cases} \|N_i|, & \text{if } i = j, \\ -1, & \text{if } \{i, j\} \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases}
\]

Observe that \( L_G \) is symmetric positive semidefinite. Also, since \( G \) is connected, \( L_G \) has only one eigenvalue at zero. Its second smallest eigenvalue (i.e., the algebraic connectivity of \( G \)) \( \lambda_2(L_G) > 0 \) and its largest eigenvalue \( \lambda_{\max}(L_G) \leq \min\{\|N_i|\nu_i\|N_i\|, \max_{\{i, j\} \in \mathcal{E}} \nu_i \nu_j \} \). The following theorem shows that the nodes achieve a consensus at a linear rate, which depends on \( \lambda_{\max}(L_G) \) and \( \lambda_2(L_G) \), and provides a bound on the distance between the consensus and the optimum \( x^* \):

**Theorem 3 (Rate of convergence):** Suppose Assumption 1 holds. Let \( x^k = [x^k_1; \ldots; x^k_N] \) \( \forall k \geq 0 \) be generated by DEAN described in Algorithm 1 with \( 0 < \alpha_{(i,j)} \leq \bar{\alpha} \) and \( g_{(i,j)}(x) = \frac{1}{2} x^T x, \forall \{i, j\} \in \mathcal{E} \). Suppose 2 holds and \( \alpha_{(i,j)} < \theta_i/\max_{\{i, j\} \in \mathcal{E}} \|N_i\| \forall \{i, j\} \in \mathcal{E} \). Then, there exists \( \bar{x} = [\bar{x}_1; \ldots; \bar{x}_N] \in \mathbb{R}^n \), \( \bar{x} \in \mathbb{R}^n \) such that

\[
\|x^k - \bar{x}\| \leq \frac{\max_{\{i, j\} \in \mathcal{E}} \alpha_{(i,j)} \lambda_{\max}(L_G)}{\theta_i(q - 1)} \|x^0\|q^k,
\]

where \( q = \max\{ \max_{\{i, j\} \in \mathcal{E}} \alpha_{(i,j)} \lambda_{\max}(L_G)/\theta_i - 1, 1 - \min_{\{i, j\} \in \mathcal{E}} \alpha_{(i,j)} \lambda_2(L_G)/\theta_i \} < 0 \). In addition,

\[
\|x^{k+1} - x^k\| \leq \max_{i \in \mathcal{V}} \frac{L_i}{\bar{\rho}_i} \sqrt{\nu_i \|x^0\|^2},
\]

where \( \bar{\rho}_i = \theta_i^2/\left(2\|N_i|\max_{j \in N_i} \alpha_{(i,j)}\right) \). If \( \bar{\rho}_i \) and \( \theta_i = \theta_i^2/\left(2\|N_i|\max_{j \in N_i} \alpha_{(i,j)}\right) \), then \( \bar{\rho}_i \) is defined by \( \bar{\eta}_i = \|N_i|\nu_i/\theta_i^2 \geq \bar{\eta}_0 \).

**Proof:** See Appendix D ■

Theorem 3 says that the consensus error among the nodes vanishes at a linear rate. In addition, the consensus \( \bar{x} \) can be sufficiently close to \( x^* \) if the weights \( \alpha_{(i,j)} \) \( \forall \{i, j\} \in \mathcal{E} \) are sufficiently small. Following Theorems 1 and 3 below we present the iteration complexity of DEAN, which states that \( \epsilon \)-accuracy can be reached within \( O(1 / \ln \rho) \) iterations:

**Theorem 4 (Iteration complexity):** Suppose Assumption 1 holds. Let \( x^k = [x^k_1; \ldots; x^k_N] \) \( \forall k \geq 0 \) be generated by DEAN described in Algorithm 1 with \( g_{(i,j)}(x) = \frac{1}{2} x^T x \forall \{i, j\} \in \mathcal{E} \). Given any \( \epsilon > 0 \), let

\[
\alpha_{(i,j)} = \frac{\epsilon}{\tilde{\zeta}_{(i,j)} + \zeta_{(i,j)}} \min_{\{i, j\} \in \mathcal{E}},
\]

then \( \|x^{k+1} - x^k\| \leq \epsilon \) for all \( k \geq K \), where

\[
K_e = \frac{\Theta}{\lambda_2(L_G)} \left( \max_{\{i, j\} \in \mathcal{E}} \zeta_{(i,j)} + \frac{\tilde{\zeta}_{(i,j)}}{\epsilon} \right) \ln \left( \frac{\max_{\{i, j\} \in \mathcal{E}} \zeta_{(i,j)}}{\epsilon} + \frac{\tilde{\zeta}_{(i,j)}}{\epsilon} \right).
\]

**Proof:** See Appendix E ■

Finally, recall from Section III that when each \( f_i \) is a positive definite quadratic function in the form of \( 10 \), we guarantee that \( \sum_{i \in \mathcal{V}} \nabla f_i(x^k) = 0 \forall k \geq 0 \). This, along with \( \bar{\rho} \) in Theorem 1 suggests that \( \lim_{k \to \infty} x^k = x^* \forall i \in \mathcal{V} \). Additionally, the rate of convergence to \( x^* \) is derived in the following proposition:

**Proposition 1:** Suppose Assumption 2 holds. For each \( i \in \mathcal{V} \), let \( f_i \) be given by \( \Theta_i = \lambda_1(B_i) \), and \( \Theta_i = \lambda_{\max}(B_i) \). Let \( x^k = [x^k_1; \ldots; x^k_N] \forall k \geq 0 \) be generated by DEAN described in Algorithm 1 with \( 0 < \alpha_{(i,j)} \leq \bar{\alpha} \) \( \forall \{i, j\} \in \mathcal{E} \). Also suppose 20 holds. Then for each \( k \geq 0 \),

\[

V(x^k) \leq (1 - \rho)^k V(x^0),
\]

\[
\sum_{i \in \mathcal{V}} \Theta_i \|x^0_i - x^i\|^2 \leq (1 - \rho)^k \sum_{i \in \mathcal{V}} \Theta_i \|x^0_i - x^\hat{i}\|^2,
\]

where \( \rho = \sup\{ \epsilon : \epsilon R \leq Q \} \in (0, 1) \), \( R = R^T \in \mathbb{R}^{n \times n} \) is a positive semidefinite matrix given by

\[
[R]_{ij} = \begin{cases} \frac{1}{2} - \frac{1}{N}, & \text{if } i = j, \\ -\Theta_i + \Theta_j, & \text{if } i \neq j, \end{cases}
\]

where \( Q = Q^T \in \mathbb{R}^{n \times n} \) is a positive semidefinite matrix given by

\[
[Q]_{ij} = \begin{cases} \nu_{ij} + \nu_{ji}, & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases}
\]

\[

\sum_{i \in \mathcal{V}} \nu_{ij} + \nu_{ji}, \quad \text{if } \{i, j\} \in \mathcal{E},
\]

and \( \Theta_i = \Theta_j = \frac{1}{\Theta_i + \Theta_j} + 1/\left(2\Gamma_{(i,j)}\right)\alpha_{(i,j)}, \forall \{i, j\} \in \mathcal{E} \).}

**Proof:** See Appendix F ■

V. **Numerical Examples**

In this section, we demonstrate the convergence performance of DEAN in comparison to several existing decentralized Newton-type methods via two numerical examples.
A. Decentralized logistic regression

We first consider a logistic regression problem [1] that often arises in machine learning.

Assume that the network is randomly generated with \( N \) interconnected servers with connectivity ratio \( r \), where \( r \) is defined as the ratio of the number of edges in the network to the number of all possible edges, i.e., \( N(N-1)/2 \). In addition, \( m_i \) training samples \((u_{ij}, v_{ij}) \in \mathbb{R}^n \times \{-1, +1\}, j = 1, \ldots, m_i \) are assigned to each server \( i \), where \( v_{ij} \in \mathbb{R}^n \) is the feature vector and \( u_{ij} \in \{-1, +1\} \) is the label. The goal is to predict the probability \( P(v = 1|u) = 1/(1 + \exp(-u^T x)) \) of having label \( v = 1 \) given a feature vector \( u \) whose class is unknown. The logistic regression problem is formulated as

\[
\min_{x \in \mathbb{R}^n} \frac{\lambda}{2} \|x\|^2 + \sum_{i=1}^N \sum_{j=1}^{m_i} \ln(1 + \exp(-u_{ij} u_{ij}^T x)), \quad (30)
\]

where the \( \ell_2 \)-regularization term is used to reduce over-fitting. Thus, the local objective \( f_i \) of server \( i \) is

\[
f_i(x) = \frac{\lambda}{2N} \|x\|^2 + \sum_{j=1}^{m_i} \ln(1 + \exp(-u_{ij} u_{ij}^T x)). \quad (31)
\]

In the simulations, we let \( N = 20, n = 3, r = 0.2, \) and \( m_i = 6 \ \forall i = 1, \ldots, N \). Each feature vector \( u_{ij} \) with label \( v_{ij} = 1 \) is generated from normal distribution with mean \( \delta \) and standard deviation \( \sigma_+ \), and each \( u_{ij} \) with \( v_{ij} = -1 \) is generated from normal distribution with mean \(-\delta\) and standard deviation \( \sigma_- \), where \( \delta = 15 \) and \( \sigma_+ = \sigma_- = 1 \).

We compare DEAN with the existing Newton-type methods NN-K [27] and DQN-K [28]. For DEAN, we set \( g_{(i,j)}(x) = \frac{1}{2} u_{ij} x \forall \{i, j\} \in \mathcal{E} \). For NN-K and DQN-K, we choose \( K \) to be 1 and 2, respectively, which lead to satisfactory convergence performance with mild computational and communication costs. In addition, the weight matrix \( W = W^T \) in NN and DQN is set as follows, \( \omega_{ij} = 1/\max(\{\mid N_i \mid, \mid N_j \mid \} + 2) \ \forall \{i, j\} \in \mathcal{E}, \omega_{ij} = 0 \ \forall \{i, j\} \notin \mathcal{E}, i \neq j, \) and \( \omega_{ii} = 1 - \sum_{j \in N_i} \omega_{ij} \forall i \in \mathcal{V} \). The remaining parameters of these algorithms are hand-optimized.

Since NN-K and DQN-K can only handle (globally) strongly convex problems, we choose \( \lambda = 1 \) in problem (30) for comparison. Fig. 1 plots the relative error \( \|x^k - x^*\|/\|x^0 - x^*\| \) versus the number \( k \) of iterations for each of the above algorithms, where the optimal solution \( x^* \) is calculated by YALMIP [32]. Observe that DEAN converges faster and achieves higher accuracy than NN-1, NN-2, DQN-1, and DQN-2. Moreover, recall from Section III that DEAN requires the fewest communications per iteration.

Subsequently, we consider problem (30) without the regularization term, i.e., \( \lambda = 0 \). In this case, each objective function \( f_i \) in (31) is only locally strongly convex. We illustrate the convergence performance of DEAN with different values of the weights \( \alpha_{(i,j)} \ \forall \{i, j\} \in \mathcal{E} \). For simplicity, we let all the weights be identical, i.e., \( \alpha_{(i,j)} = \alpha \ \forall \{i, j\} \in \mathcal{E} \). From Fig. 2, larger weights tend to yield faster convergence, yet lead to less accurate solutions to the problem.

B. Decentralized quadratic programming

We then consider a \( d \)-regular network consisting of \( N \) nodes, where \( d = 10 \) and \( N = 100 \). Suppose the nodes are required to solve the quadratic program

\[
\min_{x \in \mathbb{R}^n} \sum_{i=1}^N \frac{1}{2} x^T A_i x - b_i^T x, \quad (32)
\]

where the symmetric positive definite matrix \( A_i \in \mathbb{R}^{n \times n} \) and the vector \( b_i \in \mathbb{R}^n \) are associated with node \( i \). We let \( n = 3 \) and the eigenvalues of each \( A_i \) be randomly distributed between 1 and 10. The remaining simulation settings are the same as those in Section V-A.

Fig. 3 plots the evolution of DEAN, NN-1, NN-2, DQN-1, and DQN-2. Observe from Fig. 3 that NN-1, NN-2, DQN-1, and DQN-2 can only converge to a neighborhood of the optimum, while it seems that DEAN linearly converges to the exact optimum, which is consistent with Proposition 1.

VI. CONCLUSION

In this paper, we develop a class of novel decentralized approximate Newton (DEAN) methods for in-network optimization. With appropriate algorithm parameters, the DEAN algorithms allow nodes in a network to reach a consensus at a linear rate, which can be arbitrarily close to the optimal solution. In addition, when the problem reduces to a quadratic program, linear convergence to the exact optimum can be
achieved. Compared with the existing decentralized Newton-type methods, the DEAN algorithms relax the global strong convexity of the objective functions to local strong convexity and require lower communication costs.

ACKNOWLEDGMENT

We would like to thank Prof. Choon Yik Tang with the School of Electrical and Computer Engineering at the University of Oklahoma for his valuable comments and suggestions that help improve the quality of this manuscript.

APPENDIX

A. Proof of Lemma 7

We first introduce the following notations: For each $k \geq 0$ and each $\{i, j\} \in \mathcal{E}$, let

$$\phi_{ij}^k = \nabla g_{i,j}(x_j^k) - \nabla g_{i,j}(x_i^k).$$

Observe that $\phi_{ij}^k = -\phi_{ij}^k$. In addition, for each $i \in \mathcal{V}$, let

$$\delta_i^k = \tilde{\alpha} \sum_{j \in \mathcal{N}_i} \|\phi_{ij}^k\| \cdot \|\nabla^2 f_i(x_i^k)\|^{-1}$$

and

$$\tilde{\delta}_i = \sqrt{2V(x^0)\tilde{\alpha}} \sum_{j \in \mathcal{N}_i} \sum_{i,j} \Gamma_{i,j} \left( \frac{1}{\sqrt{\delta_i}} + \frac{1}{\sqrt{\delta_j}} \right).$$

Clearly, $\tilde{\delta}_i \leq \delta_i$, where $\delta_i$ is given by (16).

For each $k \geq 0$, define $\Delta V(x^k) = V(x^{k+1}) - V(x^k)$. Then,

$$\Delta V(x^k) = \sum_{i \in \mathcal{V}} f_i(x_i^k) - f_i(x_i^{k+1}) + \nabla f_i(x_i^k)^T (x_i^{k+1} - x_i^k)$$

$$+ \left[ \nabla f_i(x_i^{k+1}) - \nabla f_i(x_i^k) \right]^T (x_i^{k+1} - x_i^k),$$

in which

$$\left[ \nabla f_i(x_i^{k+1}) - \nabla f_i(x_i^k) \right]^T (x_i^{k+1} - x_i^k)$$

$$= \left[ \nabla f_i(x_i^{k+1}) - \nabla f_i(x_i^k) \right]^T (x_i^{k+1} - x_i^k)$$

$$+ \left[ \nabla f_i(x_i^{k+1}) - \nabla f_i(x_i^k) \right]^T (x_i^k - x_i^*), \quad \forall i \in \mathcal{V}. \quad (35)$$

To bound the second term on the right-hand side of (35), applying the Fundamental Theorem of Calculus to $\nabla f_i$ yields

$$\nabla f_i(x_i^{k+1}) = \nabla f_i(x_i^k) + \int_0^1 \nabla^2 f_i(x_i^k + s(x_i^{k+1} - x_i^k))(x_i^{k+1} - x_i^k)ds.$$ This leads to

$$\nabla^2 f_i(x_i^k)^T (x_i^{k+1} - x_i^k)$$

$$=(x_i^k - x_i^*)^T \nabla^2 f_i(x_i^{k+1})(x_i^{k+1} - x_i^k)$$

$$+ \int_0^1 (x_i^k - x_i^*)^T [\nabla^2 f_i(x_i^k + s(x_i^{k+1} - x_i^k))$$

$$- \nabla^2 f_i(x_i^k)](x_i^{k+1} - x_i^k)ds.$$

(36)

Note from (5) and (33) that the first quadratic term on the right-hand side of (36) can be written as $(x_i^k - x_i^*)^T \sum_{j \in \mathcal{N}_i} \alpha(i,j) \phi_{ij}^k$. Also note from $\phi_{ij}^k = -\phi_{ji}^k$ and $\alpha(i,j) = \alpha(j,i)$ that $\sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \alpha(i,j) \phi_{ij}^k = 0$. It follows that

$$\sum_{i \in \mathcal{V}} (x_i^k - x_i^*)^T (\nabla f_i(x_i^{k+1}) - \nabla f_i(x_i^k))$$

$$= \sum_{i \in \mathcal{V}} (x_i^k - x_i^*)^T \sum_{j \in \mathcal{N}_i} \alpha(i,j) \phi_{ij}^k + \int_0^1 \int_0^1 (x_i^k - x_i^*)^T$$

$$\cdot [\nabla^2 f_i(x_i^k + s(x_i^{k+1} - x_i^k)) - \nabla^2 f_i(x_i^k)](x_i^{k+1} - x_i^k)ds. \quad (37)$$

Also, since $\phi_{ij}^k = -\phi_{ji}^k$,

$$\sum_{i \in \mathcal{V}} (x_i^k)^T \sum_{j \in \mathcal{N}_i} \alpha(i,j) \phi_{ij}^k = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} 2\alpha(i,j) (x_i^k - x_j^k)^T \phi_{ij}^k.$$

This, along with (37), implies that

$$\sum_{i \in \mathcal{V}} (x_i^k - x_i^*)^T (\nabla f_i(x_i^{k+1}) - \nabla f_i(x_i^k))$$

$$\leq \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} 2\alpha(i,j) |x_i^k - x_j^k|^2 \phi_{ij}^k + \sum_{i \in \mathcal{V}} \int_0^1 \int_0^1 \int_0^1 (x_i^k - x_i^*)^T$$

$$\cdot [\nabla^2 f_i(x_i^k + s(x_i^{k+1} - x_i^k)) - \nabla^2 f_i(x_i^k)](x_i^{k+1} - x_i^k)ds. \quad (38)$$

By substituting the above inequality into (35) and combining the resulting inequality with (34), we obtain

$$\Delta V(x^k) \leq \sum_{i \in \mathcal{V}} f_i(x_i^k) - f_i(x_i^{k+1}) + \nabla f_i(x_i^k)^T (x_i^{k+1} - x_i^k)$$

$$+ \left[ \nabla f_i(x_i^{k+1}) - \nabla f_i(x_i^k) \right]^T (x_i^{k+1} - x_i^k)$$

$$= \sum_{i \in \mathcal{V}} \left( \left\| \nabla^2 f_i(x_i^k) \right\|^{-1} \right) \cdot \sum_{j \in \mathcal{N}_i} \left\| \phi_{ij}^k \right\|$$

$$\leq \frac{\tilde{\alpha}}{\delta_i} \left( \sum_{j \in \mathcal{N}_i} \left\| \phi_{ij}^k \right\| \right).$$

(39)

and

$$\|x_i^{k+1} - x_i^k\|^2 \leq \left( \sum_{j \in \mathcal{N}_i} \left\| \phi_{ij}^k \right\|^2 \right) \|x_i^k - x_i^*\|^2.$$ (40)

Based on the above, we now prove by induction that (21) holds for each $k \geq 0$. When $k = 0$, because $x_0^i \in C_i$, $\forall i \in \mathcal{V}$ and because of (12), we have $||\nabla^2 f_i(x_i^0)||^{-1} \leq \frac{1}{\delta_i}$ and $\|x_i^0 - x_i^*\| \leq \sqrt{\frac{2V(x^0)}{\delta_i}}$. Also, from (35), (36), and (15),

$$\delta_i^k \leq \frac{\tilde{\alpha}}{\delta_i} \sum_{j \in \mathcal{N}_i} \Gamma_{i,j} \left( \|x_j^0 - x_i^*\| + \|x_i^0 - x_i^*\| \right)$$

This completes the proof of Lemma 7.
\[ \Delta V(x^0) \leq \sum_{i \in V} \Gamma_{i,j} \sqrt{2V(x^0)} \left( \frac{1}{\theta_j} + \frac{1}{\theta_i} \right) = \delta_i. \]

This, along with (16) and (39), gives \( x_i^l \in B(x_0^0, \delta_i) \subseteq B(x_0^0, \delta_i). \) By using this relation and applying (17), (18), and (19) to (38), we obtain

\[ \Delta V(x^0) \leq \sum_{i \in V} \left( \frac{\theta_i}{2} - \Theta_i - L_i \|x_i^0 - x^*\| \right) \int_0^1 sds \|x_i^1 - x_i^0\|^2 \]

\[ - \sum_{j \in N_i} \frac{1}{2} \alpha_{i,j} (x_i^0 - x_j^0)^T \phi_{i,j}. \]

Further, by (42), (40), and (19),

\[ \|x_i^{l+1} - x_i^l\|^2 \leq \frac{N_i}{\theta_i} \sum_{j \in N_i} \alpha_{i,j}^2 \|\phi_{i,j}\|^2. \]

It follows from (43) and (44) that

\[ \Delta V(x^l) \leq \sum_{i \in V} \left( \frac{\theta_i}{2} - \Theta_i - L_i \sqrt{2V(x_i^0)} \right) \frac{N_i}{\theta_i} \|\alpha_{i,j}\|^2 \]

\[ + \frac{1}{2} \Gamma_{i,j} \alpha_{i,j} \|\phi_{i,j}\|^2. \]

Because \( x_i^l \in C_i \forall i \in V \) and because of (15) and (33), we have \( \|\phi_{i,j}\|^2 \geq \gamma_{i,j}^2 \|x_i^l - x_j^l\|^2. \) Thus, if \( \alpha_{i,j} \in \mathcal{E} \) satisfy (20), (21) holds for \( k = l. \) By induction, (21) holds for all \( k \geq 0. \)

**B. Proof of Theorem 1**

Note from Lemma 1 that \( V(x^k) \) is non-increasing. In addition, since \( V(x^0) \) is bounded from below by 0, \( \lim_{k \to \infty} V(x^k) \) exists. This, along with (21), implies that \( \sum_{k=0}^\infty \sum_{i \in C_i} \|x_i^k - x_i^\ast\|^2 \) is finite. It then follows that for any \( (i, j) \in \mathcal{E} \) satisfy (20), (21) holds for \( k = l. \) By induction, (21) holds for all \( k \geq 0. \)

**C. Proof of Theorem 2**

To prove the theorem, we first introduce the following notations: For each \( i \in V, \) let

\[ \rho_i = \frac{\theta_i^2}{2N_i \max_{j \in N_i} \{\alpha_{i,j}\} \Gamma_{i,j}} - \Theta_i - L_i \sqrt{2V(x_i^0)} \frac{\theta_i}{\theta_i}. \]

(45)

Note from (20) that \( \rho_i \geq 0. \) Let \( \Theta_i \) be such that \( \|\nabla^2 f_i(x)\| \leq \Theta_i, \) \( \forall x \in \text{conv}\{\cup_{j \in C_i}\}. \)

(46)

Also, let \( \bar{x}^k = \frac{1}{N} \sum_{i \in V} x_i^k \) and \( \tilde{x}^k = [\bar{x}^k; \ldots; \bar{x}^k] \in \mathbb{R}^{nN}. \)

It can be shown that (23) guarantees (20) and therefore, the conclusion of Lemma 1 holds. This leads to \( x_i^k \in C_i \forall i \in V. \) From (19), \( \|\nabla^2 f_i(x_i^k)\| \leq \frac{1}{\theta_i}. \) It then follows from (5) that

\[ \|x_i^{k+1} - x_i^k\|^2 \leq \frac{N_i}{\theta_i} \sum_{j \in N_i} \alpha_{i,j} \|\phi_{i,j}\|^2. \]

(47)

This, along with (44), implies that

\[ \sum_{i \in V} \sum_{j \in N_i} \frac{\alpha_{i,j}}{2} (x_i^k - x_j^k)^T \phi_{i,j} \leq - \sum_{i \in V} \frac{\alpha_{i,j}}{2} \|\phi_{i,j}\|^2 \]

\[ \leq - \sum_{i \in V} \frac{\theta_i^2}{2N_i \max_{j \in N_i} \{\alpha_{i,j}\} \Gamma_{i,j}} \frac{N_i}{\theta_i} \sum_{j \in N_i} \alpha_{i,j} \|\phi_{i,j}\|^2 \]

\[ \leq - \sum_{i \in V} \frac{2N_i \max_{j \in N_i} \{\alpha_{i,j}\}}{\theta_i} \frac{\theta_i^2}{2N_i \max_{j \in N_i} \{\alpha_{i,j}\} \Gamma_{i,j}} \|x_i^{k+1} - x_i^k\|^2. \]

By substituting (47) into (43), we obtain

\[ V(x^k) - V(x^{k+1}) \geq \sum_{i \in V} \left( \frac{\theta_i}{2} - \Theta_i - L_i \sqrt{2V(x_i^0)} \frac{\theta_i}{\theta_i} \right) \]

\[ \|x_i^{k+1} - x_i^k\|^2 \leq \frac{N_i}{\theta_i} \sum_{j \in N_i} \alpha_{i,j} \|\phi_{i,j}\|^2. \]
It then follows from (45) that
\[
\sum_{i \in V} \rho_i \|x^{k+1}_i - x^k_i\|^2 \leq V(x^k) - V(x^{k+1}).
\] (48)

We now attempt to bound \(\|\sum_{i \in V} \nabla f_i(x^k_i)\| \forall k \geq 0\). Using (8) and (13), we obtain
\[
\|\sum_{i \in V} \left( \nabla f_i(x^{k+1}_i) - \nabla f_i(x^k_i) \right) \| \leq \sum_{i \in V} \frac{L_i}{2} \|x^{k+1}_i - x^k_i\|^2.
\]
Combining this with (9) and (13) gives
\[
\|\sum_{i \in V} \nabla f_i(x^k_i)\| = \|\sum_{i \in V} \left( \nabla f_i(x^k_i) - \nabla f_i(x^0_i) \right)\| \\
\leq \sum_{i \in V} \frac{L_i}{2} \|x^{k+1}_i - x^k_i\|^2 \\
\leq \sum_{i \in V} \frac{L_i}{2\rho_i} \sum_{i \in V} \rho_i \|x^{k+1}_i - x^k_i\|^2 \\
\leq \sum_{i \in V} \frac{L_i}{2\rho_i} (V(x^0) - V(x^k)) \leq \max_{i \in V} \frac{L_i}{2\rho_i} V(x^0).
\] (49)

Below, for each \(k \geq 0\), we provide a bound on \(\|x^k - x^*\|\).

Recall from Theorem 1 that \(\|x^k - x^*\| \to 0\) as \(k \to \infty\). Also, note from (23) that \(\max_{i \in V} \frac{1}{\rho_i} \cdot \frac{N}{\sum_{i \in V} \theta_i} V(x^0) \leq \epsilon\). Consequently, \(\lim_{k \to \infty} \|x^k - x^*\| \leq \epsilon\).

D. Proof of Theorem

Since \(\alpha_{(i,j)}(x) = 1/2\alpha^T x \in \mathcal{E}, \) (55) becomes
\[
x^{k+1}_i = x^k_i - (\nabla^2 f_i(x^k_i))^{-1} \sum_{j \in N_i} \alpha_{(i,j)}(x^{k}_i - x^j_i), \quad \forall i \in V,
\]
or equivalently,
\[
x^{k+1} = x^k - (\nabla^2 F(x^k))^{-1} (H_G \otimes I_n)x^k.
\] (53)

Let \(y^k = (H_G^\top \otimes I_n)x^k\). Multiplying \((H_G^\top \otimes I_n)\) on both sides of (53) yields
\[
y^{k+1} = y^k - (H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n)y^k).
\]

Moreover, since \(y^k \in S^\perp\), we have \(y^k = (H_G H_G^\top \otimes I_n)y^k\). It follows that
\[
y^{k+1} = [(H_G H_G^\top \otimes I_n)
\quad - (H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n))](y^k).
\] (55)

Below, we bound the term in the bracket of (55). For any \(x \in \mathbb{R}^{N \times N}\), let \(z = P_{S^\perp}(x)\). Due to (54),
\[
x^T \left[(H_G H_G^\top \otimes I_n) - (H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n))\right]x
\]
\[= z^T \left[I_N \otimes I_n - (H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n))\right]z.
\] (56)

Further, since \((H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n))\) is positive semidefinite and \(z \in S^\perp\), we have
\[
(1 - \lambda_{\max}((H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n)))\|z\|^2 \leq z^T \left[I_N \otimes I_n - (H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n))\right]z \leq (1 - \lambda_2((H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n))))\|z\|^2.
\] (57)

Also note that \(\|z\| \leq \|x\|\). It then follows from (56) and (57) that for any \(x \in \mathbb{R}^{N \times N}\),
\[
\|x^T \left[(H_G H_G^\top \otimes I_n) - (H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n))\right]x\| \\
\leq \max \{ \lambda_{\max}((H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n)) - 1, \\
1 - \lambda_2((H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n)))\} \|x\|^2.
\]

Therefore, for each \(k \geq 0\),
\[
\|((H_G H_G^\top \otimes I_n) - (H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n)))x\| \\
\leq \max \{ \lambda_{\max}((H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n)) - 1, \\
1 - \lambda_2((H_G^\top \otimes I_n)((\nabla^2 F(x^k))^{-1}(H_G^\top \otimes I_n)))\} \|x\|^2.
\]
where the last inequality comes from $x_i^k \in B(x_i^0; \delta_i) \forall i \in V$. Furthermore, note that $\lambda_2(H_G)$ and $\lambda_{max}(H_G)$ in the above inequality can be bounded as follows:

$$\lambda_2(H_G) \geq \min_{\{i,j\} \in E} \alpha_{(i,j)} \lambda_2(L_G),$$

$$\lambda_{max}(H_G) \leq \max_{\{i,j\} \in E} \alpha_{(i,j)} \lambda_{max}(L_G),$$

where $L_G = L_G^T \in \mathbb{R}^{N \times N}$ is given by (59). This, along with (58), leads to

$$\|(H_G H_G^\dagger \otimes I_n) - (H_G^\dagger \otimes I_n)(\nabla^2 F(x^k))^{-1}(H_G^\dagger \otimes I_n)\| \leq \epsilon.$$ 

In addition, because of the condition $\alpha_{(i,j)} < \theta/(\max_{i \in V}[|N_i|]) \forall \{i,j\} \in E$ and because $\lambda_{max}(L_G) \leq \min\{N, 2 \max_{i \in V}[|N_i|]\}$ we have $\max_{\{i,j\} \in E} \alpha_{(i,j)} \lambda_{max}(L_G)/\theta < 2$ and, thus, $0 < q < 1$. Combining (62) with (59) implies that $\|y^{k+1}\| \leq q\|y^k\|$ and $\|y^k\| \leq q^k\|y^0\|$.

Further, by (54),

$$\frac{\|x^{k+1} - x^k\|}{\|(\nabla^2 F(x^k))^{-1}(H_G^\dagger \otimes I_n)y^k\|} \leq \frac{\lambda_{max}(H_G^\dagger)}{\theta} \frac{\lambda_{max}(H_G)}{\theta} \|x^0\| q^k \|y^k\| \leq \epsilon.$$ 

Hence, given any $\epsilon > 0$, $\forall k_1 \geq k_2 \geq K_\epsilon$ with $K_\epsilon = \lceil \log_q \frac{\lambda_{max}(H_G^\dagger)/\|x^0\|}{\epsilon} \rceil$, we have

$$\|x^{k_1} - x^{k_2}\| \leq \sum_{l=k_2}^{k_1-1} \|x^{l+1} - x^l\| \leq \frac{\lambda_{max}(H_G^\dagger)}{\theta} \|x^0\| \sum_{l=k_2}^{k_1-1} q^l \leq \frac{\lambda_{max}(H_G^\dagger)}{\theta(1-q)} \|x^0\| q^{K_\epsilon} \leq \epsilon.$$ 

Therefore, the Cauchy sequence $(x^k)_{k=0}^{\infty} \subset \mathbb{R}^n$ converges to some $\tilde{x} = [\tilde{x}, \ldots, \tilde{x}] \in \mathbb{R}^n$. Moreover, for each $k \geq 0$, from (61) and (63),

$$\|\tilde{x} - x^k\| = \|\lim_{t \to \infty} (x^t - x^k)\| = \lim_{t \to \infty} \sum_{l=k}^{t-1} \|x^{l+1} - x^l\| \leq \sum_{l=k}^{\infty} \|x^{l+1} - x^l\| \leq \frac{\lambda_{max}(H_G^\dagger)}{\theta} \|x^0\| \sum_{l=k}^{\infty} q^l \leq \frac{\max_{\{i,j\} \in E} \alpha_{(i,j)} \lambda_{max}(L_G)}{\theta(1-q)} \|x^0\| q^k.$$

Finally, we provide a bound on $\|\tilde{x} - x^k\|$. Since $\lim_{k \to \infty} x^k = \tilde{x}$, we have $\lim_{k \to \infty} \|x^k - \tilde{x}\| = 0$, where $\tilde{x} = [\tilde{x}, \ldots, \tilde{x}] \in \mathbb{R}^n$ and $\tilde{x} = \frac{1}{N} \sum_{i \in V} x_i^k$. It then follows from (62) that

$$\|\tilde{x} - x^*\| \leq \max_{i \in V} \frac{L_i}{\rho_i} \frac{\sqrt{N}}{2 \sum_{i \in V} \theta_i} V(\tilde{x}),$$

where $\rho_i > 0 \forall i \in V$ are given by (53). Moreover, for each $\{i,j\} \in E$, since $g_{(i,j)} = \frac{1}{N} \Gamma_{(i,j)} \Gamma_{(i,j)} = 1$. As a result, $\rho_i = \tilde{\rho}_i \forall i \in V$ and this completes the proof.

E. Proof of Theorem 4

Let $\hat{q} = 1 - \min_{\{i,j\} \in E} \alpha_{(i,j)} \lambda_2(L_G)/\Theta$ and $\Lambda_\theta = \text{diag}(\theta_1, \ldots, \theta_N)$. Note from Theorem 2 that $\eta_1 \geq 2 \max_{i \in V}[|N_i|] \big| (\nabla^2 F(x^0))^{-1} \big| \theta_2^2 \geq |N_i| \theta_2 \forall i \in V$. Due to (25), (20) in Lemma 4 holds. This leads to $x_i^k \in C_i$. It follows from (19) that $\|\nabla^2 F(x^k)\|^{-1} \leq \frac{1}{\theta}$. Thus, $\|\nabla^2 F(x^k)\|^{-1} \leq (\Lambda_\theta \otimes I_n)^{-1}$. Further, from (25), for each $\{i,j\} \in E$,

$$\alpha_{(i,j)} \leq \frac{1}{\frac{1}{2} \max_{\{i,j\} \in E}(\hat{q} \frac{1}{\theta})} \leq \min\{\frac{\theta_i}{2|N_i|}, \frac{\theta_j}{2|N_j|}\}.$$ 

Using the Gershgorin circle theorem, every eigenvalue of $H_G - \Lambda_\theta$ lies within at least one of the intervals $[a_{ii} - r_i, a_{ii} + r_i] \forall i \in V$, where $a_{ii} = \sum_{j \in N_i} \alpha_{(i,j)} - \theta_j$ and $r_i = \sum_{j \in N_i} \alpha_{(i,j)}$. Notice from (60) that $a_{ii} + r_i = 2 \sum_{j \in N_i} \alpha_{(i,j)} - \theta_i \leq 0$. Thus, $[a_{ii} - r_i, a_{ii} + r_i] \subseteq [-\theta_i, 0] \forall i \in V$ and $H_G - \Lambda_\theta \leq 0$. This leads to $\Lambda_\theta \otimes I_n \geq (H_G^\dagger \otimes I_n)(H_G^\dagger \otimes I_n)$. Hence, from the Schur complement condition,

$$\begin{bmatrix} I_{nN} & H_G^\dagger \otimes I_n \\ H_G^\dagger \otimes I_n & \Lambda_\theta \otimes I_n \end{bmatrix} \succeq 0,$$

which is equivalent to

$$\begin{bmatrix} I_{nN} & H_G^\dagger \otimes I_n \\ H_G^\dagger \otimes I_n & I_{nN} \end{bmatrix} \succeq 0.$$ 

Using the Schur complement condition again, we obtain $I_{nN} - (H_G^\dagger \otimes I_n)(\Lambda_\theta \otimes I_n)^{-1}(H_G^\dagger \otimes I_n) \succeq 0$. Also, since $\|\nabla^2 F(x^k)\|^{-1} \leq (\Lambda_\theta \otimes I_n)$, we have $(H_G^\dagger \otimes I_n)(\nabla^2 F(x^k))^{-1}(H_G^\dagger \otimes I_n) \preceq (H_G^\dagger \otimes I_n)(\Lambda_\theta \otimes I_n)^{-1}(H_G^\dagger \otimes I_n) \preceq I_{nN}$. As a result,

$$\lambda_{max}(H_G^\dagger \otimes I_n)(\nabla^2 F(x^k))^{-1}(H_G^\dagger \otimes I_n) \leq 1.$$ 

This, along with (68) and (60), implies that

$$\|(H_G H_G^\dagger \otimes I_n) - (H_G^\dagger \otimes I_n)(\nabla^2 F(x^k))^{-1}(H_G^\dagger \otimes I_n)\| \leq 1 - \lambda_2((H_G^\dagger \otimes I_n)(\nabla^2 F(x^k))^{-1}(H_G^\dagger \otimes I_n)) \leq 1 - \lambda_2(H_G^\dagger) \Theta.$$ 

By following the idea in the proof of Theorem 3 and replacing $\hat{q}$ with $\hat{q}$ in (64), we obtain

$$\|\tilde{x} - x^k\| \leq \Theta \max_{\{i,j\} \in E} \alpha_{(i,j)} \lambda_{max}(L_G) \|x^0\| \left(1 - \min_{\{i,j\} \in E} \alpha_{(i,j)} \lambda_2(L_G)/\Theta \right)^k.$$ 

Note that for any $y \in [0, 1]$, $1 - y \leq \exp(-y)$. Then, for any positive integer $k$, $(1 - y^k) \leq \exp(-ky)$. This leads to

$$\|\tilde{x} - x^k\| \leq \Theta \max_{\{i,j\} \in E} \alpha_{(i,j)} \lambda_{max}(L_G) \|x^0\| \exp(-k \min_{\{i,j\} \in E} \alpha_{(i,j)} \lambda_2(L_G)/\Theta).$$
Given an arbitrary $\epsilon > 0$, for any $k \geq K_{\epsilon}$ with
\[
K_{\epsilon} = \left[ \theta \left( \min_{i,j \in V} \alpha_{i,j} \lambda_2(L_g) \right) \cdot \ln \left( \frac{2 \lambda_{\max}(L_g) \|x^0\| \Theta \max_{i,j \in V} \alpha_{i,j}}{\epsilon \theta \lambda_2(L_g) \min_{i,j \in V} \alpha_{i,j}} \right) \right],
\]
(67)
it can be shown that $\|x^k - \tilde{x}\| \leq \frac{\epsilon}{2}$. From (25), we have $\alpha_{i,j} < \frac{\epsilon}{2 \lambda_2(L_g)} \forall i, j \in E$. Incorporating this inequality and (25) into (67) yields
\[
K_{\epsilon} < K_{\epsilon} = \left[ \frac{\Theta}{\lambda_2(L_g)} \left( \frac{\max_{i,j \in E} \tilde{\zeta}_{i,j}}{\epsilon} + \frac{\max_{i,j \in E} \tilde{\zeta}_{i,j}}{\epsilon} \right) \right] \cdot \ln \left( \frac{2 \lambda_{\max}(L_g) \|x^0\| \Theta \max_{i,j \in V} \alpha_{i,j}}{\epsilon \theta \lambda_2(L_g) \min_{i,j \in V} \alpha_{i,j}} \right),
\]
which, for any $k \geq K_{\epsilon}$, $\|x^k - \tilde{x}\| \leq \frac{\epsilon}{2}$. Further, by combining (25) and (45) with (65), we obtain $\|x - x^*\| \leq \frac{\epsilon}{2}$. Consequently, $\|x^k - x^*\| \leq \epsilon \forall k \geq K_{\epsilon}$.

**F. Proof of Proposition 7**

We first provide a quadratic bound on $V(x^k) - V(x^{k+1})$. For each $i \in V$ and each $j \in N_i$, note from (20) that
\[
\frac{\theta_i}{2} - \Theta_i \frac{\mathcal{N}_i(\alpha_{i,j})}{\theta_i^2} + \frac{1}{2 \Gamma_{i,j}} \geq 0,
\]
where $Q$ is defined in (22). Next, we provide another quadratic bound on $V(x^k) \forall k \geq 0$. Since $x^*$ is the optimal solution, we have
\[
\frac{1}{N} \sum_{i \in V} x_i^k \leq \frac{1}{N} \sum_{i \in V} x_i^k.
\]
Because $f_i \forall i \in V$ are quadratic and because of (8) and (9),
\[
\sum_{i \in V} \nabla f_i(x_i^k) = 0, \quad \forall k \geq 0.
\]
It follows from (11), (69), and (70) that
\[
V(x^k) \leq \sum_{i \in V} \frac{1}{2} \|x_i^k - x^k_i\|^2 - \sum_{i \in V} \frac{1}{N} \sum_{j \in V} x_j^k \left( R \otimes I_n \right) x_i^k,
\]
where $R$ is defined in (28). According to [31, Theorem 2], we can show that $R$ and $Q$ are positive semidefinite with only one eigenvalue at 0 corresponding to an eigenvector $v_1$, and thus $\rho = \sup \{ \epsilon \in \mathbb{R} : \epsilon R \preceq Q \} \in (0, \infty)$. From (68) and (71), we have $\rho V(x^{k+1}) \leq V(x^k) - V(x^{k+1}) \forall k \geq 0$, i.e.,
\[
V(x^{k+1}) \leq (1 - \rho) V(x^k), \quad \forall k \geq 0.
\]
This, along with $V(x^k) \geq 0 \forall k \geq 0$, implies that $0 < \rho < 1$. Therefore, (26) holds. This further leads to
\[
\sum_{i \in V} \frac{\theta_i}{2} \|x_i^k - x^*\|^2 \leq V(x^k) \leq (1 - \rho)^k \sum_{i \in V} \frac{\Theta_i}{2} \|x_i^k - x^*\|^2,
\]
i.e., (27) holds.

**REFERENCES**

[1] F. Bach, “Adaptivity of averaged stochastic gradient descent to local strong convexity for logistic regression,” *Journal of Machine Learning Research*, vol. 15, pp. 595–627, 2014.

[2] Y. Cao, W. Yu, W. Ren, and G. Chen, “An overview of recent progress in the study of distributed multi-agent coordination,” *IEEE Transactions on Industrial Informatics*, vol. 9, no. 1, pp. 427–438, 2013.

[3] M. G. Rabbat and R. D. Nowak, “Distributed optimization in sensor networks,” in *Proc. International Symposium on Information Processing in Sensor Networks*, Berkeley, CA, 2004, pp. 20–27.

[4] A. Beck, A. Nedić, A. Ozdaglar, and M. Teboulle, “An $O(1/k)$ gradient method for network resource allocation problems,” *IEEE Transactions on Control of Network Systems*, vol. 20, no. 1, pp. 64–73, 2014.

[5] A. Nedić and A. Ozdaglar, “Distributed subgradient methods for multi-agent optimization,” *IEEE Transactions on Automatic Control*, vol. 54, no. 1, pp. 48–61, 2009.

[6] B. Johansson, M. Rabi, and M. Johansson, “A randomized incremental subgradient method for distributed optimization in networked systems,” *SIAM Journal on Optimization*, vol. 20, no. 3, pp. 1157–1170, 2009.

[7] A. Nedić, A. Ozdaglar, and P. A. Parrilo, “Constrained consensus and optimization in multi-agent networks,” *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 922–938, 2010.

[8] J. Duchi, A. Agarwal, and M. Wainwright, “Dual averaging for distributed optimization: Convergence and network scaling,” *IEEE Transactions on Automatic Control*, vol. 57, no. 3, pp. 592–606, 2012.

[9] M. Zhu and S. Martinez, “On distributed convex optimization under inequality and equality constraints,” *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 151–164, 2012.

[10] D. Jakovetić, J. Xavier, and J. Moura, “Fast distributed gradient methods,” *IEEE Transactions on Automatic Control*, vol. 59, no. 5, pp. 1311–1316, 2014.

[11] A. Nedić and A. Olshovsky, “Distributed optimization over time-varying directed graphs,” *IEEE Transactions on Automatic Control*, vol. 60, no. 3, pp. 601–615, 2015.

[12] W. Shi, Q. Ling, G. Wu, and W. Yin, “EXTRA: an exact first-order algorithm for decentralized consensus optimization,” *SIAM Journal on Optimization*, vol. 25, no. 2, pp. 944–966, 2015.

[13] ———, “A proximal gradient algorithm for decentralized composite optimization,” *IEEE Transactions on Signal Processing*, vol. 63, no. 22, pp. 6013–6023, 2015.

[14] P. Lin, W. Ren, and Y. Song, “Distributed multi-agent optimization subject to nonidentical constraints and communication delays,” *Automatica*, vol. 65, pp. 120–131, 2016.

[15] P. Bianchi, W. Hachem, and F. Fuehrer, “A coordinate descent primal-dual algorithm and application to distributed asynchronous optimization,” *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 2947–2957, 2016.

[16] A. Nedić and A. Olshovsky, “Stochastic gradient-push for strongly convex functions on time-varying directed graphs,” *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 3936–3947, 2016.

[17] G. Qu and N. Li, “Harnessing smoothness to accelerate distributed optimization,” *IEEE Transactions on Control of Network Systems*, 2017.

[18] C. Xi and U. Khan, “Dextra: A fast algorithm for optimization over directed graphs,” *IEEE Transactions on Automatic Control*, vol. 62, no. 10, pp. 4980–4993, 2017.

[19] M. Mahdouni and A. Ozdaglar, “Convergence rate of distributed admm over networks,” *IEEE Transactions on Automatic Control*, vol. 62, no. 10, pp. 5082–5095, 2017.
[20] C. Xi, V. Mai, R. Xin, E. Abed, , and U. Khan, “Linear convergence in optimization over directed graphs with row-stochastic matrices,” *IEEE Transactions on Automatic Control*, vol. 63, no. 10, pp. 3558–3565, 2018.
[21] C. Xi, R. Xin, , and U. Khan, “ADD-OPT: Accelerated distributed directed optimization,” *IEEE Transactions on Automatic Control*, vol. 63, no. 5, pp. 1329–1339, 2018.
[22] C. Shi and G. Yang, “Augmented lagrange algorithms for distributed optimization over multi-agent networks via edge-based method,” *Automatica*, vol. 94, pp. 55–62, 2018.
[23] D. Jakovetić, “A unification and generalization of exact distributed first order methods,” accepted to *IEEE Transactions on Signal and Information Processing over Networks*, 2018.
[24] A. Mokhtari, W. Shi, Q. Ling, and A. Ribeiro, “A decentralized second-order method with exact linear convergence rate for consensus optimization,” *IEEE Transactions on Signal & Information Processing Over Networks*, vol. 2, no. 4, pp. 507–522, 2016.
[25] , “Dqm: Decentralized quadratically approximated alternating direction method of multipliers,” *IEEE Transactions on Signal & Information Processing*, vol. 64, no. 19, pp. 5158–5173, 2016.
[26] M. Eisen, A. Mokhtari, and A. Ribeiro, “Decentralized quasi-newton methods,” *IEEE Transactions on Signal Processing*, vol. 65, no. 10, pp. 2613–2628, 2017.
[27] A. Mokhtari, Q. Ling, and A. Ribeiro, “Network newton distributed optimization methods,” *IEEE Transactions on Signal Processing*, vol. 65, no. 1, pp. 146–161, 2017.
[28] D. Bajović, D. Jakovetić, N. Krejić, and N. Jerinkić, “Newton-like method with diagonal correction for distributed optimization,” *SIAM Journal on Optimization*, vol. 27, no. 2, pp. 1171–1203, 2017.
[29] D. Varagnolo, F. Zanella, A. Cenedese, G. Pillonetto, and L. Schenato, “Newton-raphson consensus for distributed convex optimization,” *IEEE Transactions on Automatic Control*, vol. 61, no. 4, pp. 994–1009, 2016.
[30] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.
[31] J. Lu and C. Y. Tang, “Zero-gradient-sum algorithms for distributed convex optimization: The continuous-time case,” *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2348–2354, 2012.
[32] J. Löfberg, “Yalmip: A toolbox for modeling and optimization in matlab,” in Proc. *IEEE International Symposium on Computer Aided Control Systems Design*, Taipei, Taiwan, 2004, pp. 284–289.
[33] Y. Nesterov, *Introductory lectures on Convex Optimization: A Basic Course*. Norwell, MA: Kluwer Academic Publishers, 2004.