ANOTHER STATE ENTANGLEMENT MEASURE

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Given a state \(\omega\) of the (minimal \(C^*\)-) tensor product \(A \otimes B\) of unital \(C^*\)-algebras \(A\) and \(B\), its marginals are the states of \(A\) and \(B\) defined by

\[
\omega^A(a) = \omega(a \otimes 1_B), \quad a \in A, \quad \omega^B(b) = \omega(1_A \otimes b), \quad b \in B.
\]

Given a state \(\rho\) of \(A\) and a state \(\phi\) of \(B\), there is a unique state \(\omega\) of \(A \otimes B\) such that \(\omega(a \otimes b) = \rho(a)\phi(b)\) for all \(a \in A\) and all \(b \in B\); we denote this state by \(\rho \otimes \phi\). A \textbf{product-state} of \(A \otimes B\) is a state \(\omega\) of \(A \otimes B\) such that \(\omega = \omega^A \otimes \omega^B\). We write \(S_\pi(A \otimes B)\) for the product-states of \(A \otimes B\). The convex hull of \(S_\pi\), written \(\text{co}(S_\pi(A \otimes B))\), is the set of finite convex combinations of product states. The states of \(A \otimes B\) in the norm-closure of \(\text{co}(S_\pi(A \otimes B))\) are usually identified with the \textbf{separable} states of the composite system whose observables are described by \(A \otimes B\); the states which are not separable are termed \textbf{entangled}.

For a state \(\omega\) of a unital \(C^*\)-algebra \(A\), consider its finite convex decompositions: \(\omega = \sum_{j=1}^n \lambda_j \omega_j\), with \(0 \leq \lambda_j \leq 1\), \(\sum_{j=1}^n \lambda_j = 1\), and \(\omega_j\) a state of \(A\). Such a decompositon will be written \([\lambda_j, \omega_j]\) and \(\mathcal{D}_\omega\) denotes all such finite convex decompositions.

Consider the relative entropy \((\rho, \phi) \rightarrow S(\rho, \phi)\) for pairs of states \(\rho\) and \(\phi\) of a unital \(C^*\)-algebra. We use the original convention of Araki \cite{1}, which is also that used in \cite{2} which we use as a standard reference for the properties of relative entropy. We propose the following measure of entanglement

\begin{equation}
E(\omega) = \inf_{[\lambda_j, \omega_j] \in \mathcal{D}_\omega} \sum_{j=1}^n \lambda_j S(\omega_j, \omega_j^A \otimes \omega_j^B).
\end{equation}

We say a map \(\alpha\) from \(A \otimes B\) into \(C \otimes D\) \textbf{commutes with marginalization} if for every state \(\omega\) of \(C \otimes D\) one has \((\omega \circ \alpha)^A \otimes (\omega \circ \alpha)^B = (\omega^C \otimes \omega^D) \circ \alpha\).

We have the following result, whose proof will be provided in a forthcoming paper \cite{3}, along with result about a class of entanglement measures akin to (1):

1. \(0 \leq E(\omega) \leq S(\omega, \omega^A \otimes \omega^B)\) with equality in the right-hand side inequality if \(\omega\) is a pure state. \(E(\omega) = 0\) if \(\omega\) is a product-state.

2. \(E(\cdot)\) is convex (and in general not affine).

3. If \(\alpha\) and \(\beta\) are, respectively, \(\ast\)-isomorphisms of \(A\) onto \(C\) and of \(B\) onto \(D\) (\(A, B, C\) and \(D\) are unital \(C^*\)-algebras) then \(E(\omega \circ (\alpha \otimes \beta)) = E(\omega)\) for every state of \(C \otimes D\).

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\textsuperscript{3}If \(A\) is the algebra of bounded linear operators on a Hilbert space, then \(S(\rho, \phi) = Tr(D_\rho(log(D_\rho - log(D_\phi)))\), for normal states, where \(D_\rho\) (resp. \(D_\phi\)) is the density operator for which \(\rho(a) = Tr(D_\rho a), \quad a \in A\).
4. If $\gamma : A \otimes B \rightarrow C \otimes D$ is a unital, linear, continuous, Schwarz-positive map ($\gamma(z^*z) \geq \gamma(z)^*\gamma(z)$ for every $z \in A \otimes B$) which commutes with marginalization, then $E(\omega \circ \gamma) \leq E(\omega)$ for every state $\omega$ of $C \otimes D$.

5. If $\omega$ is separable then $E(\omega) = 0$.

6. $E(\omega) = 0$ iff $\omega$ lies in the $w^*$-closure of $co(S_\pi(A \otimes B))$.

7. For $n$ ($n \geq 1$) states $\omega_1, \omega_2, \ldots, \omega_n$ of $A \otimes B$,

\begin{equation}
E((\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n) \circ \zeta_n) \leq \sum_{j=1}^{n} E(\omega_j),
\end{equation}

where $\zeta_n$ is the $*$-isomorphism

\begin{equation}
\left\{\underbrace{A \otimes A \otimes \cdots \otimes A}_n\right\} \otimes \left\{\underbrace{B \otimes B \otimes \cdots \otimes B}_n\right\} \xrightarrow{\zeta_n} \left(\underbrace{A \otimes B}_n\right) \otimes \left(\underbrace{A \otimes B}_n\right) \otimes \cdots \otimes \left(\underbrace{A \otimes B}_n\right),
\end{equation}

given by $\zeta_n((a_1 \otimes a_2 \otimes \cdots \otimes a_n) \otimes (b_1 \otimes b_2 \otimes \cdots \otimes b_n)) = (a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \otimes \cdots \otimes (a_n \otimes b_n)$.

One has,

\begin{equation}
\lim_{n \rightarrow \infty} n^{-1}E((\omega \otimes \omega \otimes \cdots \otimes \omega) \circ \zeta_n) \leq E(\omega).
\end{equation}

In both (2) and (4), the left-hand side is computed with respect to marginalization with respect to the two factors in $\{\}$-brackets in (3).

8. If $A$ or $B$ is abelian then $E \equiv 0$.

9. Let $\mathcal{M}_\omega$ be the (Radon)-measures on the state space with barycenter $\omega$, then

$$E(\omega) = \inf_{\{\mu \in \mathcal{M}_\omega\}} \int \mu(d\phi)S(\phi, \phi^A \otimes \phi^B),$$

and there exists $\mu_\omega \in \mathcal{M}_\omega$ such that

$$E(\omega) = \int \mu_\omega(d\phi)S(\phi, \phi^A \otimes \phi^B).$$

The crucial condition of “commutation with marginalization” involved in property 4. of $E$ is met by the “LQCC” maps considered in [4]. “LQCC” means “local quantum operations” with “classical communication”, and these are the relevant maps in the games that Alice and Bob play.

Like most known entanglement measures (see e.g., [4,5]), except that devised by Vidal and Werner [6], the calculation of $E$ involves an infimum over a rather unmanageable set. Using Kosaki’s variational expression ([7]) for the relative entropy, one obtains a lower bound on $E$ which can be possibly used to devise a strategy to show that $E(\omega) > 0$ for a specific state $\omega$. 

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One can replace the relative entropy in the definition of $E$ by other, suitable functions, e.g. $\| \phi - \phi^A \otimes \phi^B \|$, without losing the basic properties of $E$; this is studied in [3].

In a previous version of this announcement [8], we claimed that $E$ was additive, that is, equality holds in (2). We withdraw this claim because we have found a mistake in our “proof”.

REFERENCES

1. H. Araki: Relative entropy for states of von Neumann algebras I [& II]. Publ. RIMS Kyoto Univ. 11, 809-833 (1976); [& 13: 173-192 (1977)].

2. M. Ohya, and D. Petz: Quantum Entropy and Its Use. Springer-Verlag Berlin Heidelberg 1993.

3. O.A. Nagel, and G.A. Raggio: A family of state entanglement measures. Work in progress.

4. M.J. Donald, M. Horodecki, and O. Rudolph: The uniqueness theorem for entanglement measures. J. Math. Phys. 43, 4252-4272 (2002).

5. V. Vedral: The role of relative entropy in quantum information theory. Rev. Mod. Phys. 74, 197-234 (2002).

6. G. Vidal, and R.F. Werner: A computable measure of entanglement. Phys. Rev. A 65, 032314 (2002).

7. H. Kosaki: Relative entropy for states: a variational expression. J. Operator Th. 16, 335-348 (1986).

8. O.A. Nagel and G.A. Raggio: Another state entanglement measure. quant-ph/0306024 v2.