THE MOTIVIC AND ÉTALE BECKER-GOTTlieb TRANSFER: THE CONSTRUCTION OF THE TRANSFER

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Abstract. The main goal of the paper is the construction of a variant of the Becker-Gottlieb transfer in the motivic and étale frameworks. This needs considerable work in equivariant motivic and étale homotopy theory, equivariant for the action of linear algebraic groups, which is discussed in the first half of the paper.

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2010 AMS Subject classification: 14F20, 14F42, 14L30.

The authors were supported by grants from the NSF at various stages on work on this paper. The second author would also like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme K-Theory, Algebraic Cycles and Motivic Homotopy Theory where part of the work on this paper was carried out. This work was also supported by EPSRC grant no EP/R014604/1 and by the Simons Foundation.
1. Introduction

We begin with the following example to motivate the discussion in the present paper. One knows from the work of Atiyah (see [At]) that for a compact manifold $M$ without boundary, the suspension spectrum of the Thom-space of the normal bundle $\nu$ to imbedding $M$ in a large dimensional Euclidean space is a Spanier-Whitehead dual to the suspension spectrum of $M$ modulo a certain shift. Now assume that the manifold $M$ is provided with the action of a compact Lie group $G$. Then one may find a $G$-equivariant imbedding of $M$ in a large dimensional Euclidean space $V = \mathbb{R}^N$ with a $G$-action. Now the normal bundle $\nu$ and its Thom-space inherit $G$-actions. The resulting Thom-Pontrjagin collapse map from the sphere $TP : S(V \oplus 1) \to \text{Th}(\nu)$ shows that the $G$-suspension spectrum of $\text{Th}(\nu)$ will be a Spanier-Whitehead dual to the $G$-suspension spectrum of $M$ (again modulo certain shifts). Observe that the map $TP$ followed by the diagonal map $\text{Th}(\nu) \to M \wedge \text{Th}(\nu)$ then provides a $G$-equivariant pre-transfer map which is then fed into the Borel construction to obtain the first step in the construction of the classical Becker-Gottlieb transfer. (See [BG75, section 3].)

The point we want to emphasize here is that the Spanier-Whitehead dual used in the above construction of the transfer is the $G$-suspension spectrum of $\text{Th}(\nu)$, so that on forgetting the $G$-action we obtain the Spanier-Whitehead dual in the non-equivariant setting. In other words, the Spanier-Whitehead dual used in the construction of the transfer is the same Spanier-Whitehead dual in the non-equivariant setting, but made equivariant simply by making the imbedding of $M$ in a large Euclidean space $G$-equivariant. One of our goals in this paper is to set up a suitable framework so that the Spanier-Whitehead dual of the suspension spectra of simplicial presheaves in the motivic and étale setting provided with $G$-action, reduce to the corresponding non-equivariant Spanier-Whitehead dual on forgetting the $G$-action. We also would like to point out that this approach seems essential, at least for now in the motivic setting, since no variant of Gabber’s refined alterations exists that is compatible with a group action.

At the same time, since no account of equivariant (stable) motivic homotopy theory for actions of all linear algebraic groups on smooth schemes exists yet in this generality, (analogous to the discussion in [LMS] for the topological situation), we need to set up the basic machinery in place. One may observe that when the groups involved are finite discrete groups, it is possible to incorporate the corresponding equivariant homotopy theory into the theory of motivic symmetric spectra: see [DRO2]. This approach fails when the groups are no longer finite.

Throughout the paper, we will work with smooth schemes of finite type over a given field $k$, which we refer to as the base field. The following is the basic framework we adopt throughout the paper:

**Basic framework adopted throughout the paper.**

**Basic assumptions on the base field.**

(1) A standing assumption throughout is that the base field $k$ is a perfect field of arbitrary characteristic.

(2) When considering actions by linear algebraic groups $G$ that are not special, we will also assume the base field is infinite to prevent certain unpleasant situations.

(3) On considering étale realizations of the transfer, it is important to assume that the base field $k$ has finite $\ell$–cohomological dimension, for $\ell \neq \text{char}(k)$ and satisfies the finiteness conditions that $H^n_{\text{ét}}(\text{Spec } k, \mathbb{Z}/\ell^n)$ is finitely generated in each degree $n$ and vanishes for all $n >> 0$, all $n > 0$.

(Such an assumption is not needed on dealing with the motivic transfer alone.)

One should be able to see that such an assumption is necessary to get any theory of Spanier-Whitehead duality on the étale site of $\text{Spec } k$.

**Basic assumptions on the linear algebraic groups considered:**

(1) we allow any linear algebraic group over $k$, irrespective of whether it is connected or not and

(2) we are not assuming it is special in the sense of Grothendieck (see [Ch]). This means, in particular, we allow groups such as all orthogonal groups and finite groups, which are all known to be non-special.
**Definition 1.1.** Let $M \in \mathcal{SH}(k)$ ($\mathcal{SH}(k_{et})$). For each prime number $\ell$, let $\mathbb{Z}_{(\ell)}$ denote the localization of the integers at the prime ideal $\ell$ and let $\mathbb{Z}_\ell = \lim_{\rightarrow \ell^n}$. Then we say $M$ is $\mathbb{Z}_{(\ell)}$-local ($\ell$-complete, $\ell$-primary torsion), if each $[S^{1,n} \wedge T^1 \wedge \Sigma^{\infty}_U, M]$ is a $\mathbb{Z}_{(\ell)}$-module ($\mathbb{Z}_\ell$-module, $\mathbb{Z}_\ell$-module which is torsion, respectively) as $U$ varies among the objects of the given site, and where $[S^{1,n} \wedge T^1 \wedge \Sigma^{\infty}_U, M]$ denotes $\text{Hom}$ in the stable homotopy category $\mathcal{SH}(k)$ ($\mathcal{SH}(k_{et})$, respectively).

Let $M \in \mathcal{SH}(k)$ ($\mathcal{SH}(k_{et})$). Then one may observe that if $\ell$ is a prime number, and $M$ is $\ell$-complete, then $M$ is $\mathbb{Z}_{(\ell)}$-local. This follows readily by observing that the natural map $\mathbb{Z} \to \mathbb{Z}_\ell$ factors through $\mathbb{Z}_{(\ell)}$ since every prime different from $\ell$ is inverted in $\mathbb{Z}_\ell$. One may also observe that if $E$ is a commutative ring spectrum which is $\mathbb{Z}_{(\ell)}$-local ($\ell$-complete), then any module spectrum $M$ over $E$ is also $\mathbb{Z}_{(\ell)}$-local ($\ell$-complete, respectively). $\ell$-completion in the motivic framework is discussed in some detail in [CJT2, Appendix].

We summarize the results on dualizability in the following Theorem.

**Theorem 1.2.** (Dualizability)

(i) Over a field $k$ of characteristic 0, the $T$-suspension spectrum of any smooth scheme of finite type over $k$ is dualizable in $\mathcal{SH}(k_{mot})$.

(ii) Assume the base field $k$ is of positive characteristic $p$. Let $E$ denote a motivic ring spectrum which is $\mathbb{Z}_{(\ell)}$-local as in Definition 1.1 for a prime $\ell$ different from $p$. Then $E \wedge X$ is dualizable in $\mathcal{SH}(k_{mot}, E)$, for any smooth scheme $X$ of finite type over $k$.

(iii) Assume the base field $k$ satisfies the finiteness conditions in (1.0.1). Let $\ell$ be a prime different from $\text{char}(k)$ and let $E$ denote a motivic ring spectrum whose homotopy groups are all $\ell$-primary torsion. Then $\varepsilon^* (E \wedge X)$ is dualizable in $\mathcal{SH}(k_{et}, \varepsilon^*(E))$ for any smooth scheme $X$ of finite type over $k$.

The following is an overview of our main results on the construction of the transfer. We start with an equivariant form of the pre-transfer which is defined using a theory of Spanier-Whitehead duality, which can be made to be compatible with a group action as discussed below. The next step in the construction of the transfer is to feed an equivariant form of the pre-transfer, such as in (8.2.3) into a Borel construction, which is discussed in section 8.3. Let $G$ denote a linear algebraic group. We need to carry out the construction and it needs to be carried out carefully so that if $X$ is a smooth scheme, one obtains the correct object. This construction is discussed in detail in section 8.3.

Let $X$ and $Y$ denote two simplicial presheaves provided with $G$-actions. We will consider the following three **basic contexts** for the transfer:

(a) $p : E \to B$ is a $G$-torsor for the action of a linear algebraic group $G$ with both $E$ and $B$ smooth quasi-projective schemes over $k$, with $B$ **connected** and

$$\pi_Y : E \times_G (Y \times X) \to E \times_G Y$$

1By default, whenever a group acts on a scheme or a simplicial presheaf, we will view it as a left action.
the induced map, where G acts diagonally on $Y \times X$. One may observe that, on taking $Y = \text{Spec } k$ with the trivial action of G, the map $\pi_Y$ becomes $\pi : E \times_G X \to B$ (the induced projection), which is an important special case.

(b) $BG^{gm,m}$ will denote the $m$-th degree approximation to the geometric classifying space of the linear algebraic group G as in [MV] (see also [To]), $p : E^{gm,m} \to BG^{gm,m}$ is the corresponding universal G-torsor and

$$\pi_Y : E^{gm,m} \times_G (Y \times X) \to E^{gm,m} \times_G Y$$

is the induced map.

(c) If $p_m (\pi_{Y,m})$ denotes the map denoted $p (\pi_Y)$ in (b), here we let $p = \lim_{m \to \infty} p_m$ and let

$$\pi_Y = \lim_{m \to \infty} \pi_{Y,m} : E^{gm} \times_G (Y \times X) = \lim_{m \to \infty} E^{gm,m} \times_G (Y \times X) \to \lim_{m \to \infty} E^{gm,m} \times_G Y = E^{gm} \times_G Y.$$

Strictly speaking, the above definitions apply only to the case where G is special. When G is not special, the above objects will in fact need to be replaced by the derived push-forward of the above objects viewed as sheaves on the big étale site of $k$ to the corresponding big Nisnevich site of $k$, as discussed in [3.3.0]. However, we will denote these new objects also by the same notation throughout, except when it is necessary to distinguish between them. Recall that, for G not special, we will assume the base field is also infinite to prevent certain unpleasant situations.

Throughout the following discussion, $E^G$ will denote any one of the G-equivariant spectra considered in [4.0.8], with $E$ denoting the corresponding non-equivariant spectrum: see Definition [4.13].

Then we obtain the following key result.

**Theorem 1.3.** Let $f : X \to X$ denote a G-equivariant map and let $\pi_Y : E \times_G (Y \times X) \to E \times_G Y$ denote any one of the maps considered in (a) through (c) above. Let $f_Y = \text{id}_Y \times f : Y \times X \to Y \times X$ denote the induced map.

Then in case (a), we obtain a map (called the transfer)

$$tr(f_Y) : \Sigma^\infty_T (E \times_G Y)_+ \to \Sigma^\infty_T (E \times_G (Y \times X))_+ \quad (tr(f_Y) : E \wedge (E \times_G Y)_+ \to E \wedge (E \times_G (Y \times X))_+)$$

in $SH(k)$ ($SH(k, \mathcal{E})$, respectively) if $\Sigma^\infty_T X_+$ is dualizable in $SH(k)$ (if $E \wedge X_+$ is dualizable in $SH(k, \mathcal{E})$, respectively) having the following properties.

(i) If $tr(f_Y)^m : \Sigma^\infty_T (E^{gm,m} \times_G Y)_+ \to \Sigma^\infty_T (E^{gm,m} \times_G (Y \times X))_+$ (tr$(f_Y)^m : E \wedge (E^{gm,m} \times_G Y)_+ \to E \wedge (E^{gm,m} \times_G (Y \times X))_+$) denotes the corresponding transfer maps in case (b), the maps $\{tr(f_Y)^m\}$ are compatible as $m$ varies. The corresponding induced map $\lim_{m \to \infty} tr(f_Y)^m$ will be denoted $tr(f_Y)$.

(ii) Assume the base field $k$ satisfies the finiteness conditions in [1.0.1]. Assume $E$ (which belongs to $Spt(k_{mot})$) is $\ell$-complete, in the sense of Definition [1.1] for some prime $\ell \neq \text{char}(k)$. Let $e^* : Spt(k_{mot}) \to Spt(k_{et})$ denote the functor induced by the obvious map sites from the étale site of $k$ to the Nisnevich site of $k$.

If $e^*(E \wedge X_+)$ is dualizable in $SH(k_{et}, e^*(\mathcal{E}))$, then there exists a transfer $tr(f_Y)$ in $SH(k_{et}, e^*(\mathcal{E}))$ satisfying similar properties.

**Remark 1.4.** Further properties of the transfer are discussed in [123] [22] Theorem 2.3].

We begin section 2, with a quick review of the basic model category framework for simplicial presheaves both in the motivic and étale settings. This is followed by a brief discussion of a model structure on the category of pointed simplicial presheaves provided with the action of a presheaf of groups. The next three sections discuss the categories of spectra used in the construction of the transfer. We let S denote either the base scheme or a fixed simplicial presheaf. Section 4 then starts with the category of equivariant spectra, denoted $Spt^G(S)$: such equivariant spectra will be indexed by the Thom spaces of finite dimensional representations over the given base S. $Spt(S)$ will denote the category of spectra that are indexed by the non-negative integers. To relate these two categories of spectra we also introduce intermediate categories of spectra denoted $Spt^G(S)$ and $Spt(S)$. The above categories of spectra are considered in both the motivic and étale contexts. Section 5 discusses various model structures on these categories. In section 6, we relate the model structures on the above categories of spectra.
The following is proven there.

**Theorem 1.5.** (See Proposition 6.3)

1. The categories of spectra $\text{Spt}(S)$ and $\text{Spt}(S)$ are related by adjoint functors $i^* : \text{Spt}(S) \to \text{Spt}(S)$ and $\mathbb{P} : \text{Spt}(S) \to \text{Spt}(S)$ which define a Quillen equivalence between the corresponding projective stable model structures.

2. The categories of spectra $\text{Spt}_G(S)$ and $\text{Spt}(S)$ are related by adjoint functors $j^* : \text{Spt}(S) \to \text{Spt}_G(S)$ and $\mathbb{P} : \text{Spt}_G(S) \to \text{Spt}(S)$ which define a Quillen equivalence between the corresponding projective stable model structures.

3. The functors $\mathbb{P}$ and $\mathbb{P}$ are strict monoidal functors.

In addition, there is an obvious forgetful functor $\mathbb{U} : \text{Spt}_G(S) \to \text{Spt}_G(S)$. It is shown in Proposition 4.14 that if $\mathcal{X}$ belongs to $\text{Spt}_G(S)$, then a functorial fibrant or cofibrant replacement of $\mathbb{U}(\mathcal{X})$ in fact belongs to $\text{Spt}_G(S)$. This observation, then enables us to show that one can define Spanier-Whitehead duals of spectra $\mathcal{X}$ in $\text{Spt}_G(S)$ so that $\mathbb{U}(\mathcal{X})$ are dualizable as objects in $\text{Spt}_G(S)$, and that then the Spanier-Whitehead duals in fact belong to $\text{Spt}_G(S)$. This is similar to the discussion in the first two paragraphs, on the Spanier-Whitehead dual in the topological setting of the suspension spectra of compact manifolds provided with group-actions and plays a key role in the construction of the transfer in section 8. Section 7 discusses Spanier-Whitehead duality in the motivic and étale settings and an appendix summarizes basic results on spherical fibrations and Thom spaces in the motivic and étale settings.

### 2. Basic Model category framework for simplicial presheaves

We will fix a *perfect* field $k$ as the base scheme, and then restrict to the category of smooth schemes of finite type over $k$. This category will be denoted $\text{Sm}_k$. This category will be provided with either the big Zariski, big Nisnevich or big étale topologies and the corresponding site will be denoted $\text{Sm}_k,\text{Zar}$, $\text{Sm}_k,\text{Nis}$ or $\text{Sm}_k,\text{et}$. (Observe as a result, that the objects of these categories are all smooth schemes of finite type over $k$ and hence have $\text{Spec} k$ as the terminal object, and the coverings of a given scheme will be either Zariski, Nisnevich or étale coverings.) If $k = \mathbb{C}$ is the field of complex numbers, one also considers the local homeomorphism topology. Here the coverings of an object $U$ are collections $\{U_i \to U(\mathbb{C})|i\}$, with each $U_i \to U(\mathbb{C})$ a quasi-finite map of topological spaces which are local homeomorphisms when $U(\mathbb{C})$ is provided with the transcendental topology. $\text{Sm}_{k,\text{th}}$ will denote the corresponding big site, where the objects are again smooth schemes over $k$.

The goal of this section is to establish a general framework for the rest of our work: though much of our work takes place in the motivic setting on the Nisnevich site, the étale and Betti realization functors make it essential that we state our results in this section so that they hold on any of the above sites. Results of a technical nature on the various model categories considered in this section will be discussed separately in later sections. Given the above choices for the categories of schemes, the following will be the main choice for a category of simplicial presheaves on it.

#### 2.1. Simplicial presheaves on $\text{Sm}_k$

The category of all unpointed simplicial presheaves on $\text{Sm}_k$ will be denoted $\text{Spc}(k)$, while the corresponding category of all pointed simplicial presheaves on this category will be denoted $\text{Spc}_*(k)$. Observe that the latter category is a symmetric monoidal category with the usual smash product of pointed simplicial presheaves as the product: this will be denoted $\wedge$.

Next one has several possible choices of *model structures* on the categories of simplicial presheaves on $\text{Spc}(k)$ and $\text{Spc}_*(k)$. For example, one has the *projective* model structure, where the fibrations and weak-equivalences are defined section-wise, with the cofibrations defined by the lifting property. One also has the *injective* model structure (which is also often called the object-wise model structure), where weak-equivalences and cofibrations are defined section-wise, with the fibrations defined by the lifting property. One of the main advantages of the injective model structure is that every object is cofibrant and every injective map of simplicial presheaves is a cofibration. All the model structures considered above are cofibrantly generated, and in fact combinatorial model categories: see, for example, [Lur] Proposition A.2.8.2. These are also *tractable* model structures, in the sense that the sources of the generating cofibrations and trivial...
cofibrations are cofibrant. The projective model structure is also cellular (see [Hirsch Definition 12.1.1]) and both model structures are left proper (see [Hirsch Definition 13.1.1]).

Next we let $S$ denote a fixed simplicial presheaf in $\Spc(k)$, which could be either the presheaf represented by an object of the site $\Sm_k$ or any simplicial presheaf in $\Spc(k)$. Then we let $(\Spc(k) \downarrow S)$ denote the category of objects over $S$ in $\Spc(k)$: an object in this category is an object $P \in \Spc(k)$ together with a map $p_P : P \to S$ and where a map from $(P, p_P)$ to $(Q, q_Q)$ is a map $f : P \to Q$ so that $p_{Q \circ f} = p_P$. Clearly there is a forgetful functor $U : (\Spc(k) \downarrow S) \to \Spc(k)$. It is shown in [Hirsch15 Theorem 1.5] that this model category is also a cofibrantly generated model category where a map $f : (P, p_P) \to (Q, p_Q)$ is a cofibration (fibration, weak-equivalence, generating cofibration, generating trivial cofibration) if and only if $U(f)$ is a cofibration (fibration, weak-equivalence, generating cofibration, generating trivial cofibration, respectively) in $\Spc(k)$. Moreover, it is shown in [Hirsch15 Theorem 1.7] that the model category $(\Spc(k) \downarrow S)$ is cellular and left proper, when the given model structure on $\Spc(k)$ is cellular and left proper.

2.1.1. Pointed simplicial presheaves. (i) The main choice for the category of simplicial presheaves, (which will be important in considering fiber-wise duality), will be the following. In general $S$ will denote a simplicial presheaf on $\Sm_k$ as in the last paragraph. We will restrict to the category of simplicial presheaves that are pointed over $S$, i.e., it is the category consisting of pairs $(P, p_P)$ in $(\Spc(k) \downarrow S)$ together with a section $s_P$ to $p_P$. A map $(P, p_P, s_P) \to (Q, p_Q, s_Q)$ will denote a map $f : P \to Q$ so that $p_Q \circ f = p_P$ and $f \circ s_P = s_Q$. This category will henceforth be denoted $\Spc(S)$. Therefore, $s_P$ sends $S$ isomorphically to a sub-object of $P$, which we denote by $s_P(S)$. We let the forgetful functor sending a triple $(P, p_P, s_P)$ to $P$ be denoted by $U$ again.

(An example of the case where $S$ is a simplicial presheaf appears in [S3.10]. (See also [S4.12].) In fact that is the reason for working in this generality.)

It is shown in [Hov99 Lemma 2.1.21] that one may define the structure of a cofibrantly generated model category on $\Spc(S)$ by defining a map $f : (P, p_P, s_P) \to (Q, p_Q, s_Q)$ to be a cofibration (fibration, weak-equivalence) if $U(f)$ is a cofibration (fibration, weak-equivalence, respectively) in $\Spc(k)$. Moreover the generating cofibrations (generating trivial cofibrations) for this model structure is given by

$$I_S = \{ i \sqcup S : A \sqcup S \to B \sqcup S | i : A \to B \in I, A, B, i \in (\Spc(k) \downarrow S) \},$$

$$J_S = \{ j \sqcup S : C \sqcup S \to D \sqcup S | j : C \to D \in J, C, D, j \in (\Spc(k) \downarrow S) \},$$

respectively.

if $I$ (J) denote the set of generating cofibrations (generating trivial cofibrations) for the model structure on $\Spc(k)$. It follows from [Hirsch15 Theorems 2.7 and 2.8] that the resulting model structure on $\Spc(S)$ is cellular and left proper when the model structure on $\Spc(k)$ that one starts with is. For an object $A \in (\Spc(k) \downarrow S)$, we will henceforth refer to $A \sqcup S$ as the object $A$ pointed by $S$ and denote it by $A_+$ for convenience.

We next define a monoidal structure on $\Spc_+(S)$ as follows. Let $P, Q \in \Spc_+(S)$. Then we let:

$$P \wedge^S Q = (P \times_S Q) / (s_P(S) \times S Q \cup P \times_S s_Q(S)).$$

It may be important to point out that the term on the right is the quotient over $S$, i.e. the pushout of: $S \leftarrow s_P(S) \times S Q \cup P \times S s_Q(S) \to P \times_S Q$. We skip the verification that $\Spc_+(S)$ with above smash product $\wedge^S$ is a closed symmetric monoidal category.

If the base object $S$ represents a point in the site, for example, is the spectrum of a field for the Zariski and Nisnevich sites and is the spectrum of a separably closed field for the étale site, then every simplicial presheaf has an obvious map to $S$, so that the above monoidal structure reduces to the familiar one. The main difference between the two cases is therefore, when $S$ is a general scheme or a chosen simplicial presheaf. In this case, the smash product $\wedge^S$ defines what corresponds to a fiber-wise smash product over $S$. The discussion of the transfer map in section 8 (see [S4.10] through [S4.17]) and Appendix A, Lemma 9.5 show that indeed the fiber-wise smash product is important for us.

Terminology 2.1. It is convenient for us to work with a general simplicial presheaf $S$ as the base for a considerable part of our discussion, in this introductory section. As a result we will let $S$ denote such a general simplicial presheaf for the most part in this section, except in those special cases where we need to require this to be the base field $k$. 
Further refinements of the above model structures. We need to modify these model structures, so that the resulting model structures satisfy the following basic requirements:

(i) the pushout-product axiom and the monoidal axiom with respect to the above monoidal structures hold.

(ii) Since one of the main focus is on motivic applications, we will always invert all maps of the form

\[ \{ pr : X \times A^1 \to X | X \} , \]

where \( X \) varies over all the objects in the given site. We will perform this localization even when considering the étale sites, since \( A^1 \) is acyclic in the étale topology only with respect to constant sheaves like \( \mathbb{Z}/\ell^\nu \), where \( \ell \) is different from the residue characteristics.

2.1.4. The motivic model structure on Nisnevich presheaves. We will accomplish this on the Nisnevich site as follows. One defines a presheaf \( P \in \text{Spc}_\ast(S) \) to be motivically fibrant if (i) \( P \in \text{Spc}_\ast(S) \) is fibrant, (ii) \( P \) sends an elementary distinguished square as in [MV] p. 96, Definition 1.3 whose component schemes when pointed by \( S \) belong to \( \text{Spc}_\ast(S) \) to a homotopy cartesian square and (iii) the obvious pullback \( \Gamma(U, P) \to \Gamma(U \times A^1, P) \) is a weak-equivalence, for all \( U \) in the site \( \text{Sm}_{k} \) when pointed by \( S \) belong to \( \text{Spc}_\ast(S) \). Then a map \( f : P \to Q \) in \( \text{Spc}_\ast(S) \) is a motivic weak-equivalence if the induced map \( \text{Map}(f, P) \) is a weak-equivalence for every motivically fibrant object \( P \), with \( \text{Map} \) denoting the simplicial mapping space. One then localizes such weak-equivalences. The resulting model structure will be denoted \( \text{Spc}_\ast(S_{\text{mot}}) \).

One may specify the generating trivial cofibrations for the localized model category considered above as in [DRO2 section 2].

An alternate approach that applies in general to any of the sites we consider is to localize by inverting hypercovers as in [DHI]. For the convenience of the reader, we will discuss a little bit of the background here. First one needs to fix a Grothendieck topology on \( \text{Sm}_{k} \), which we will assume is one of the topologies considered before, namely, the Zariski, the Nisnevich or the étale topologies. We will denote this site by \( \text{Sm}_{k} \).

Next one defines the notion of local fibrations for simplicial presheaves in \( \text{Spc}_\ast(S) \), which in particular, implies that the induced map on the stalks are all fibrations: see [DHI] section 3. A map of simplicial presheaves in \( \text{Spc}_\ast(S) \) is a local weak-equivalence if it induces a weak-equivalence at all the stalks. A map of simplicial presheaves in \( \text{Spc}_\ast(S) \) is a local acyclic fibration if it is a local fibration and a local weak-equivalence. Let \( U \in \text{Sm}_{k} \) so that \( U \sqcup S \) belongs to \( \text{Spc}_\ast(S) \). A hypercover \( U \sqcup S \to U \sqcup S \) is a simplicial presheaf in \( \text{Spc}_\ast(S) \) so that (i) each \( U \) is a co-product of representable objects from the site \( \text{Sm}_{k} \) with each \( U \sqcup S \) belonging to \( \text{Spc}_\ast(S) \), (ii) comes equipped with a map \( U \sqcup S \to U \sqcup S \) in \( \text{Spc}_\ast(S) \) that is a local acyclic fibration.

Next it is shown in [DHI] Proposition 6.4 that the class of all hypercovers contains a dense set of hypercovers so that any hypercover may be refined by one belonging to the above set. Following [DHI], a simplicial presheaf \( P \) has the descent property for all hypercovers if for all \( U \) in \( \text{Sm}_{k} \), and all hypercovers \( U \to U \), the induced map \( P(U \sqcup S) \to \text{holim}_{\Delta} \Gamma(U \sqcup S, P)[n] \) is a weak-equivalence.

By localizing with respect to maps of the form \( U \sqcup S \to U \sqcup S \) where \( U \) belongs to a dense set of hypercovers of of \( U \) and also maps of the form \( U \times A^1 \sqcup S \to U \sqcup S \), it is proven in [Dug Theorem 8.1] and [DHI] Example A. 10 that we obtain a model category which is Quillen equivalent to the model category \( \text{Spc}_\ast(S_{\text{mot}}) \) considered above. In this case one defines a generating set of trivial cofibrations as follows, where \( h_{X} \) denotes the simplicial presheaf associated to an object \( X \) in the site. Let

\[(2.1.5) \quad J' = \{ u : h_{U \times A^1} \sqcup S \to C_{U \sqcup S} \mid U \in \text{Sm}_{k} \} \cup \{ \sqcup_{\alpha} h_{U_{\alpha}} \sqcup S \to h_{U} \sqcup S \in \text{Sm}_{k}[U \text{ is the disjoint union of } U_{\alpha}] \}
\cup \{ h_{U_{\alpha}} \sqcup S \to \text{Cyl}(h_{U_{\alpha}} \to h_{U}) \sqcup S \mid U \in \text{Sm}_{k} \text{ and } U_{\alpha} \to U \text{ is a given hypercover } \}
\]

where we factor the obvious map \( h_{U \times A^1} \to h_{U} \) into a cofibration \( u : h_{U \times A^1} \to C_{U} \) followed by a simplicial homotopy equivalence \( C_{U} \to h_{U} \). (See [Isak] section 2.)

Though the resulting localized category is cellular and left-proper (see [Hirsch] Chapters 12 and 13]) it is unlikely to be weakly finitely generated: the main issue is that the hypercoverings, being simplicial objects, need not be small.
2.1.6. The injective model structures. In case one starts with the injective model structure on $\text{Spc}_*(S)$, one needs to modify the above set-up as follows. First the generating cofibrations for the injective model structure on $\text{Spc}(k)$ will be some set of injective maps $\{A_\alpha \to B_\alpha[n]\}$ and not the set $\{\delta [n] \times h_U \to \Delta[n] \times h_U \in \text{Spc}(k)|n \geq 0\}$. The resulting model structure will be a combinatorial model structure. Next in order to consider the left Bousfield localization for the motivic model structure, one needs to modify the set $J$ of the generating trivial cofibrations as follows: it will be the pushout product of maps of the form $f \Box g$, where $f$ denotes a map in the set $J'$, and $g : A_\alpha \to B_\alpha$ considered above.

Then the monoidal axiom and the pushout-product axiom may be verified readily as every object in this model structure is cofibrant. As a result it also provides the structure of a monoidal model structure.

2.1.7. The motivic model structure on étale simplicial presheaves: $\text{Spc}_*(S_{\text{et}})$. Localizing with respect to maps of the form $U \sqcup S \to U \sqcup S$ where $U \sqcup S$ belongs to a dense set of hypercovers of $U \sqcup S$ and also maps of the form $U \times S \sqcup S \to U \sqcup S$, as $U$ varies among the objects of the site is the only possibility in this case. We discuss the model structure on $\text{Spc}(k_{\text{et}})$ which may be then modified as in (2.1.2) to define a corresponding model structure on $\text{Spc}_*(S_{\text{et}})$. In this case one defines a generating set of trivial cofibrations $J'$ just as in (2.1.5) by considering étale hypercovers in the place of Nisnevich hypercovers.

Let $J$ be the pushout product of maps of the form $f \Box g$, where $f$ denotes a map in the set $J'$, and $g : \delta [n] \to \Delta[n]$ denotes the obvious cofibration of simplicial sets. The set $J$ will then denote a generating set for the trivial cofibrations in the above motivic model structure.

Then we obtain the following result:

**Theorem 2.2.** On starting with the projective model structure on $\text{Spc}_*(S)$, the resulting model structure on $\text{Spc}_*(S_{\text{et}})$ is cofibrantly generated, cellular and left proper. On starting with the injective model structure on $\text{Spc}_*(S)$, the resulting model structure on $\text{Spc}_*(S_{\text{et}})$ is combinatorial and left proper. Both also satisfy the pushout-product axiom and the monoidal axiom so that they are monoidal model categories with respect to the smash product in (2.1.3).

**Proof.** The proof follows along the lines of the proof of the corresponding result for the Nisnevich simplicial presheaves as discussed in [DRO2, section 2]. The main difference is in the fact that we will be considering the set $\{h_{U*} \sqcup S \to \text{Cyl}(h_{U*} \to h_U) \sqcup S | U \in \text{Sm}_k \text{ that belongs to } \text{Spc}_*(S)\}$ in the place of the set $\{q : sq \to tq | q \text{ is an elementary distinguished square in } \text{Spc}_*(S)\}$. However, one may observe that if $\phi : U* \to U$ is a hypercover and $V$ is any object in the site, then $\phi \times V : U* \times V \to U \times V$ is also a hypercover and that moreover, the mapping cylinder $\text{Cyl}(h_{U* \times V} \to h_{U \times V}) = \text{Cyl}(h_{U*} \to h_U) \times V$. With this modification, the same arguments as in [DRO2, section 2] apply to complete the proof. □

**Terminology 2.3.** In order to carry out our discussions in as much generality as possible, we will adopt the following convention. Throughout the rest of the volume, we will let $\text{Spc}_*(S)$ denote one of the following, unless further clarified: (i) $\text{Spc}_*(S_{\text{mot}})$, (ii) $\text{Spc}_*(S_{\text{et}})$ or (iii) the category of pointed simplicial presheaves on $\text{Sm}_k$ pointed over a fixed simplicial presheaf $S$ as in (2.1.4) i.e., without further Bousfield localizations. Moreover, when $S = \text{Spec } k$, $\text{Spc}_*(S_{\text{mot}})$ ($\text{Spc}_*(S_{\text{et}})$) will be denoted $\text{Spc}_*(k_{\text{mot}})$ or $\text{Spc}_*(k)$ ($\text{Spc}_*(k_{\text{et}})$, respectively).

2.1.8. Pointed equivariant simplicial presheaves. We start with a linear algebraic group $G$ defined over $k$ (or equivalently a smooth affine group scheme of finite type over $k$). We will next do a base-extension to $S$, i.e., replace $G$ by $G_S = G \times_{\text{Spec } k} S$. But we will continue to denote $G_S$ by $G$. This way, we may assume, without loss of generality that $G \in \text{Spc}_*(S)$. In the following discussion, we will view $G$ as the corresponding presheaf of groups on the given site.

**Definition 2.4.** (i) Then

\begin{equation}
\text{Spc}^G_*(S)
\end{equation}

will denote the category of those presheaves $P \in \text{Spc}_*(S)$ provided with an action by the presheaf represented by $G$, with the morphisms being $G$-equivariant maps. (In particular, this means the group action preserves the base point of any pointed simplicial presheaf $P \in \text{Spc}^G_*(S)$.)

(ii) $\text{Spc}_*(S_{\text{mot}})$ ($\text{Spc}_*(S_{\text{et}})$) will denote the corresponding category of $G$-equivariant presheaves associated to presheaves in $\text{Spc}_*(S_{\text{mot}})$ ($\text{Spc}_*(S_{\text{et}})$, respectively).
Here it is important that the base scheme $S$ has trivial action by the group $G$ so that the maps $s : S \to P$ and $p : P \to S$ are $G$-equivariant. Then maps between such $G$-equivariant simplicial presheaves will be $G$-equivariant maps of simplicial presheaves, compatible with the structure maps $s$ and $p$.

Let $U : \mathbf{Spc}_*^G(S) \to \mathbf{Spc}_*^G(S)$ denote the forgetful functor forgetting the group action. Observe that if $P, Q \in \mathbf{Spc}_*^G(S)$, then $P \land^S Q$ defined above (i.e., with $P$ and $Q$ viewed as objects in $\mathbf{Spc}_*^G(S)$) has a natural induced $G$-action and therefore, defines an object in $\mathbf{Spc}_*^G(S)$. Therefore, we let the monoidal structure on $\mathbf{Spc}_*^G(S)$ be defined by $\land^S$ as in $(2.1.3)$, respectively. Similarly, if $P, Q \in \mathbf{Spc}_*^G(S)$, then the internal $\mathbf{Hom}_G(P, Q)$ in $\mathbf{Spc}_*^G(S)$ belongs to $\mathbf{Spc}_*^G(S)$. These basically prove:

\[(2.1.10) \quad U(P \land Q) = U(P) \land U(Q) \text{ and } U(\mathbf{Hom}_G(P, Q)) = \mathbf{Hom}_G(U(P), U(Q)), P, Q \in \mathbf{Spc}_*^G(S),\]

where $\mathbf{Hom}_G$ denotes the internal hom in $\mathbf{Spc}_*^G(S)$.

**Proposition 2.5.** Let $\mathbf{Spc}_*^G(S)$ be provided with one of the model structures defined above. Let $P \in \mathbf{Spc}_*^G(S)$.

(i) If $P' \to U(P)$ is a functorial cofibrant replacement in $\mathbf{Spc}_*^G(S)$, then $P' \in \mathbf{Spc}_*^G(S)$.

(ii) If $U(P) \to P''$ is a functorial fibrant replacement in $\mathbf{Spc}_*^G(S)$, then $P'' \in \mathbf{Spc}_*^G(S)$.

**Proof.** As the proof of (ii) is entirely similar to the proof of (i), we will explicitly consider only the proof of (i). Recall $G$ acts on $P$ as a presheaf, i.e., for each scheme $X$ in the given site, $G(X)$ is given an action on $P(X)$, compatible with restrictions for maps $U \to X$ in the site. The functoriality in the choice of the cofibrant replacement $P'$ shows that each $g_s \in G(X)$ then has an induced action on $P'(X)$, that the square

![Diagram](https://via.placeholder.com/150)

commutes, and that the corresponding squares for $U$ and $X$, for a map $U \to X$ in the given site are compatible. (See [Hov99] Definition 1.1.1] for details on functorial fibrant and cofibrant replacements.)

**3. Model structures on equivariant simplicial presheaves**

Here we will be making strong use of the model structures on simplicial presheaves in the non-equivariant setting, discussed already in the last section. One may recall from the discussion in the last section that the following are some of the main choices for the category of simplicial presheaves we consider. We will let $S$ denote either a scheme in $\text{Sm}_k$ or a fixed simplicial presheaf and then restrict to the category of simplicial presheaves on $\text{Sm}_k$ that are pointed over $S$, i.e., those simplicial presheaves $P$ on $\text{Sm}_k$ that come equipped with maps $p : P \to S$ and $s : S \to P$ so that $p \circ s = \text{id}_S$.

We will follow the generic notation adopted in Terminology $(2.1)$ so that $\mathbf{Spc}_*^G(S)$ will denote one of the three categories defined there, i.e. (i) $\mathbf{Spc}_*^G(S_{\text{mot}})$, (ii) $\mathbf{Spc}_*^G(S_{\text{st}})$ or (iii) the category of pointed simplicial presheaves on $\text{Sm}_k$ pointed over the fixed simplicial presheaf $S$ as in $(2.1.1)$, i.e., without any further Bousfield localizations.

Moreover, if $G$ denotes a linear algebraic group of finite type over $k$, we will assume that $G$ acts trivially on $S$ and $\mathbf{Spc}_*^G(S)$ will denote the corresponding category of $G$-equivariant simplicial presheaves pointed over $S$. Observe that $G$ itself identifies with the presheaf of groups represented by $G$, and therefore an action by $G$ on a simplicial presheaf has the obvious meaning. We let $G_S = G \times \text{Spec}_k S$.

We let $\mathbf{W}$ denote a family of subgroup-schemes of $G$ so that it has the following properties

(i) it is an inverse system ordered by inclusion,

(ii) if $H \in \mathbf{W}$, $H_G = \text{the core of } H$, i.e., the largest subgroup of $H$ that is normal in $G$, belongs to $\mathbf{W}$, and

(iii) if $H \in \mathbf{W}$ and $H' \supset H$ is a closed smooth subgroup-scheme of $G$, then $H' \in \mathbf{W}$.

In the case $G$ is a finite group, $\mathbf{W}$ will denote all subgroups of $G$. When $G$ is a smooth group-scheme, we will leave $\mathbf{W}$ unspecified, for now. Clearly the family of all closed smooth subgroup-schemes of a given smooth group-scheme satisfies all of the above properties, so that we will use this as a default choice of $\mathbf{W}$ when nothing else is specified.
For each subgroup-scheme \( H \in W \), let \( P^H \) denote the sub-presheaf of \( P \) of sections fixed by \( H \), i.e., \( \Gamma(U, P^H) = \Gamma(U, P)|^H \). If \( H \) is a normal subgroup-scheme of \( G \) and \( \bar{H} = G/H \), then
\[
\Gamma(U, P)|^H = \{ s \in \Gamma(U, P) \mid \text{the action of } G \text{ on } s \text{ factors through } \bar{H} \}.
\]

3.0.1. *The G-equivariant sheaves \( G/H_i \), \( H \in W \).* We let \( G/H \otimes S = \sqcup_{G/H} S \), i.e.,
\[
\Gamma(U, G/H \otimes S) = \sqcup_{\Gamma(U, G/H)} \Gamma(U, S).
\]
Then we let \( G/H_i = ((G/H) \otimes S) \sqcup S \). This will be viewed as an object of \( \text{Spc}_e^G(S) \) where the structure map \( p : G/H_i \to S \) sends all the summands \( S \) appearing in \( ((G/H) \otimes S) \sqcup S \) to \( S \) by the identity map of \( S \). The section to \( p \) is the map \( s \) that sends \( S \) by the identity to the outer summand \( S \) in \( ((G/H) \otimes S) \sqcup S \). We observe that we obtain the adjunction:
\[
\text{Spc}_e^G(S) \to \text{Spc}_e(S), Q \to Q^H \text{ as left adjoint the functor } P \to (G/H) \wedge^S P.
\]

**Proposition 3.1.**
(i) Let \( \phi : P' \to P \) denote a map of simplicial presheaves in \( \text{Spc}_e^G(S) \). Then \( \phi \) induces a map \( \phi^H : P'^H \to P^H \) for each subgroup \( H \in W \). The association \( \phi \to \phi^H \) is functorial in \( \phi \) in the sense that if \( \psi : P'' \to P' \) is another map, then the composition \( (\phi \circ \psi)^H = \phi^H \circ \psi^H \).

(ii) Let \( \{ Q_\alpha | \alpha \} \) denote a direct system of simplicial sub-presheaves of a simplicial presheaf \( Q \in \text{Spc}_e^G(S) \) indexed by a small filtered category. If \( K \) is any subgroup of \( G \), then \( \lim_\alpha (Q_\alpha)^K = \lim_\alpha Q_\alpha^K \).

**Proof.** (i) is clear. (ii) Observe that for a simplicial sub-presheaf \( Q' \) of \( Q \), with \( Q' \in \text{Spc}_e^G(S) \), \((Q')^K = Q' \cap Q^K\). Now each \( Q_\alpha \) maps injectively into \( Q \) and the structure maps of the direct system \( \{ Q_\alpha | \alpha \} \) are all injective maps. Therefore (ii) follows readily. \( \square \)

3.0.2. *Finitely presented objects.* Recall an object \( C \) in a category \( C \) is *finitely presented* or compact if \( \text{Hom}(C, -) \) commutes with all small filtered colimits in the second argument. Here \( \text{Hom}_C \) denotes the external hom in the category \( C \).

Next we define the following structure of a cofibrantly generated simplicial model category on \( \text{Spc}_e^G(S) \) starting with the projective model structure or the injective model structure on \( \text{Spc}_e(S) \): see 2.3. Let \( I \) (\( J \)) denote the generating cofibrations (generating trivial cofibrations) in \( \text{Spc}_e(S) \) for the corresponding model structure. Let \( \wedge^S \) denote the smash product defined in section 1, (2.1.3).

**Definition 3.2.** (The model structure) (i) The *generating cofibrations* are of the form
\[
I_G = \{(G/H)_+ \wedge^S i \mid i \in W, \quad H \in W \},
\]

(ii) the *generating trivial cofibrations* are of the form
\[
J_G = \{(G/H)_+ \wedge^S j \mid j \in J, H \subseteq G, \quad H \in W \} \quad \text{and}
\]

(iii) and the *weak-equivalences (fibrations) are maps* \( f : P' \to P \) in \( \text{Spc}_e^G(S) \) so that \( f^H : P'^H \to P^H \) is a weak-equivalence (fibration, respectively) in \( \text{Spc}_e(S) \) for all \( H \in W \).

**Theorem 3.3.**
(i) The above structure defines a cofibrantly generated simplicial model structure on \( \text{Spc}_e^G(S) \) that is proper.

(ii) The above model category is combinatorial and tractable (in the sense that it is locally presentable, and has sets of generating cofibrations and trivial cofibrations whose sources are also cofibrant.)

(iii) In addition, the smash product of pointed simplicial presheaves (defined as in 2.1.3) makes the above category symmetric monoidal and it satisfies the pushout-product axiom in both the injective and projective model structures. The unit for the smash product in \( \text{Spc}_e^G(S) \) is cofibrant in both the injective and projective model structures.

**Proof.** We first consider (i). A key observation is the following. Let \( H \subseteq G \) denote a subgroup belonging to \( W \). Then the functor \( P \to P^H, \text{Spc}_e^G(S) \to \text{Spc}_e(S) \) has a left-adjoint, namely, the functor
\[
(3.0.3) \quad Q \to (G/H)_+ \wedge^S Q.
\]
We now proceed to verify that the hypotheses of [Hov99, Theorem 2.1.19] are satisfied. It is obvious that the subcategory of weak-equivalences is closed under composition and retracts and has the two-out-of-three property. Next we proceed to verify that the domains of maps in I are small relative to I-cell (J-cell, respectively). It follows that the domains of the maps in I also have this property. The combined effect of the above two properties is that the fixed point functor \( Q \) defined above identify with \( J \). One may now observe using the adjunction that the fibrations with this property has already been observed above in general.) Observe that (i) The fixed point functor \( Q \to \mathbb{Q}_H \) preserves pushouts along any map of the form

\[
(id \wedge^S i) : (G/K)_+ \wedge^S A \to (G/K)_+ \wedge^S B, \quad \text{with} \ i \in I \text{ and}
\]

(ii) if \( \{P_\alpha|\alpha\} \) denotes a small filtered direct system of objects in \( \mathbf{Spc}_*^G(S) \) that are sub-objects of a given \( P \in \mathbf{Spc}_*^G(S) \), then by Proposition 3.1(ii), one obtains the identification

\[
(colim\alpha P_\alpha)^H \cong colim\alpha(P_\alpha)^H.
\]

We consider this first in the injective model structure. One may first recall that the projective model structure on \( \mathbf{Sp}^G(S) \) has as generating cofibrations: \( \{ (\delta[|n|_+] \wedge h_{X+}) \to (\Delta[|n|_+] \wedge h_{X+})|n\} \).

Let \( (G/H)_+ \wedge^S (\delta[|n|_+] \wedge^S h_{X+}) \to (G/H)_+ \wedge^S (\Delta[|n|_+] \wedge^S h_{X+}) \) denote a generating cofibration in \( I_G \). Suppose one is given a map \( f : (G/H)_+ \wedge^S (\delta[|n|_+] \wedge^S h_{X+}) \to colim\alpha P_\alpha \). By the adjunction in (3.0.3), this map corresponds to a map \( f^H : (\delta[|n|_+] \wedge^S h_{X+}) \to colim\alpha P_\alpha^H \). Since \( (\delta[|n|_+] \wedge^S h_{X+}) \) is small in \( \mathbf{Sp}^G(S) \), the map \( (\delta[|n|_+] \wedge^S h_{X+}) \to colim\alpha P_\alpha^H \) factors through some \( P_\alpha^H \). Therefore, its adjoint \( f : (G/H)_+ \wedge (\delta[|n|_+] \wedge^S h_{X+}) \to colim\alpha P_\alpha \) factors through \( P_\alpha \). This proves the domains of \( I_G \) are small relative to \( I_G \)-cell.

An entirely similar argument proves that the domains of \( J_G \) are small relative to \( J_G \)-cell. Observe that these two steps make use of the fact that the model structure on \( \mathbf{Sp}^G(S) \) is indeed the projective model structure.

Next we consider the injective model structure. In fact the following arguments work in both model structures. Let \( id \wedge^S i : (G/K)_+ \wedge^S A \to (G/K)_+ \wedge^S B \) denote a generating cofibration in \( I_G \), that is, the map \( i : A \to B \) is in \( I \). Here we make use of the observation that the fixed point functor \( Q \to \mathbb{Q}_H \) is cellular; that is, it has the following two properties for simplicial presheaves (see [Guill, Proposition 3.10]):

(i) The fixed point functor \( Q \to \mathbb{Q}_H \) preserves pushouts along any map of the form

\[
(id \wedge^S i) : (G/K)_+ \wedge^S A \to (G/K)_+ \wedge^S B, \quad \text{with} \ i \in I \text{ and}
\]

(ii) if \( \{P_\alpha|\alpha\} \) denotes a small filtered direct system of objects in \( \mathbf{Sp}^G_*(S) \) that are sub-objects of a

\[
P \in \mathbf{Sp}^G_*(S), \quad \text{one obtains the identification} \quad (colim\alpha P_\alpha)^H \cong colim\alpha(P_\alpha)^H.
\]

(The first property may be verified readily for the action of any group on presheaves of pointed sets. The last property has already been observed above in general.) Observe that \((id \wedge^S i)^H = \vee_{G/K \to G/K} i\) which is clearly a cofibration. The combined effect of the above two properties is that the fixed point functor \( Q \to \mathbb{Q}_H \) sends any \( I_G \)-cell (\( J_G \)-cell) to an \( I \)-cell (\( J \)-cell, respectively). Therefore since the domains of maps in \( I \) (\( J \)) are small relative to \( I \)-cell (\( J \)-cell, respectively), it follows that the domains of the maps in \( I_G \) (\( J_G \)) are also small relative \( I_G \)-cell (\( J_G \)-cell, respectively).

Recall that given a collection of maps \( J \) in a category, \( J \)-inj denotes those maps that have the right lifting property with respect to every map in \( J \). One may now observe using the adjunction that the fibrations defined above identify with \( J_G \)-inj and that the trivial fibrations (that is, the fibrations that are also weak-equivalences) identify with \( I_G \)-inj. Recall from [Hov99, 2.1.2] that \( I_G \)-cof (that is, the \( I_G \)-cofibrations) are the maps \( (I_G \)-inj) proj, that is, those maps that have the left lifting property with respect to every trivial fibration. We now observe that every map in \( J_G \)-cell is clearly a weak-equivalence. Therefore, suppose we
are given a commutative square in $\text{Spc}_G(S)$:

\[
\begin{array}{ccc}
(G/H)_+ \wedge^S A & \rightarrow & X \\
\downarrow u \wedge^S j & & \downarrow p \\
(G/H)_+ \wedge^S B & \rightarrow & Y
\end{array}
\]

with $j \in J$ and $p \in I_G \cap \text{inj}$. Then, by adjunction this corresponds to the commutative square:

\[
\begin{array}{ccc}
A & \rightarrow & X^H \\
\downarrow j & & \downarrow p^H \\
B & \rightarrow & Y^H
\end{array}
\]

in $\text{Spc}_G(S)$. Now $p^H$ is a trivial fibration and $j$ is a generating trivial cofibration, so that one obtains a lifting: $B \rightarrow X^H$ making the two triangles commute. By adjunction this lift corresponds to a lift $(G/H)_+ \wedge B \rightarrow X$ in the first diagram. This proves any map in $J_G - \text{cell}$ is in $I_G - \text{cof}$.

Since $I_G - \text{inj}$ corresponds to trivial fibrations, every map in $I_G - \text{inj}$ is a weak-equivalence and it is in $J_G - \text{inj}$ (which denote the fibrations). Since $J_G - \text{inj}$ denotes fibrations, it is clear that any map that is in $J_G - \text{inj}$ and is also a weak-equivalence is also in $I_G - \text{inj}$ (which denotes trivial fibrations). Therefore, we have verified all the hypotheses in [Hov99, Theorem 2.1.19] and therefore the first statement that the structures in Definition 3.2 define a cofibrantly generated model category in the theorem is proved.

The right properness may be established using the cellularity of the fixed point functors considered above. The right properness is clear since the fixed point functor preserves pull-backs.

Next observe that for any $G$, the objects of the form $(G/H)_+ \wedge^S (\Delta[n]_+ \wedge^S h_{U^+})$ as $U \in \text{Sm}_S$, $H \in W$ and $n \geq 0$ vary form a set of generators for $\text{Spc}_G(S)$. Therefore, every object in the above categories is a filtered colimits of objects $\{G_\alpha|\alpha\}$ obtained as finite colimits of the above generators. In the projective model structure, it is clear from the choices of the sets $I_G$ and $J_G$ that every object $(G/H)_+ \wedge^S h_{X^+}$, $X \in \text{Sm}/S$ is cofibrant. These follow from the fact that each object $h_{X^+}$ is cofibrant in the projective model structure on $\text{Spc}_G(S)$. In the injective model structure, all monomorphisms are cofibrations, so that the domains of the sets $I_G$ and $J_G$ are cofibrant. Therefore, it follows that these model structures are also tractable. These prove (ii).

Finally we prove that the model structures in Definition 3.2 define a symmetric monoidal model category structure on $\text{Spc}_G(S)$ with respect to the monoidal structure given by $\wedge^S$. Observe from [Hov03, Corollary 4.2.5] that in order to prove the pushout-product axiom holds in general, it suffices to prove that the pushout product of two generating cofibrations is a cofibration and that this pushout-product is also a weak-equivalence when one of the arguments is a generating trivial cofibration. Therefore, let $(G/H)_+ \wedge^S i : (G/H)_+ \wedge^S A \rightarrow (G/H)_+ \wedge^S B$ and let $(G/K)_+ \wedge^S j : (G/K)_+ \wedge^S X \rightarrow (G/K)_+ \wedge^S Y$ denote two generating cofibrations in $\text{Spc}_G(S)^G$. Then a key observation is that $(G/H)_+ \wedge^S (G/K)_+ \cong \vee (G/(H \cap K))_+$ where the $\vee$ is over the orbits of $G$ for the diagonal action of $G$ on $G/H \times G/K$. Therefore, it suffices to prove that for $E$ a fibrant object in $\text{Spc}_G(S)$, the induced map

\[
\text{Hom}((G/(H \cap K))_+ \wedge^S B \wedge^S Y, E) \rightarrow
\text{Hom}((G/(H \cap K))_+ \wedge^S A \wedge^S Y, E) \otimes_{\text{Hom}((G/(H \cap K))_+ \wedge^S A \wedge^S X, E)} \text{Hom}((G/(H \cap K))_+ \wedge^S B \wedge^S X, E)
\]

is a fibration in $\text{Spc}_G(S)$, which is a weak-equivalence if $i$ or $j$ is also weak-equivalence and where $\text{Hom}$ denotes the appropriate internal hom. The above map now identifies with

\[
\text{Hom}(B \wedge^S Y, E^{H \cap K}) \rightarrow \text{Hom}(A \wedge^S Y, E^{H \cap K}) \otimes_{\text{Hom}(A \wedge^S X, E^{H \cap K})} \text{Hom}(B \wedge^S X, E^{H \cap K})
\]

where $\text{Hom}$ now denotes the internal hom in $\text{Spc}_G(S)$. Therefore, the fact that the above map is a fibration and that it is a trivial fibration if $i$ or $j$ is also a weak-equivalence follows from the fact that the pushout-product axiom holds in $\text{Spc}_G(S)$. 


The unit for the smash-product $\wedge^S$ on $\operatorname{Spc}_G^G(S)$ defined in (2.1.3) is $S_+$ and this is cofibrant in both the injective and projective model structures on $\operatorname{Spc}_G^G(S)$.

Remark 3.4. With the above model structure on $\operatorname{Spc}_G^G(S)$, one may readily see that the forgetful functor $U : \operatorname{Spc}_G^G(S) \to \operatorname{Spc}_c(S)$ is a left Quillen functor, but an object $P \in \operatorname{Spc}_G^G(S)$ so that $U(P)$ is cofibrant in $\operatorname{Spc}_c(S)$ need not be cofibrant in $\operatorname{Spc}_G^G(S)$. An alternate way to put a model structure on the category $\operatorname{Spc}_G^G(S)$ is to transfer the model structure on $\operatorname{Spc}_c(S)$ by means of the underlying functor $U$ and a left adjoint to it. This adjoint is given by the functor sending a simplicial presheaf $P$ to $G \otimes P$ (which is defined by $(G \otimes P)(X) = \bigvee_{G(X)} P(X)$. Again an object $P \in \operatorname{Spc}_G^G(S)$ so that $U(P)$ is cofibrant as an object in $\operatorname{Spc}_c(S)$ need not be cofibrant in $\operatorname{Spc}_G^G(S)$.

3.0.6. A Key observation. As a result the composite functor $\mathcal{R}\operatorname{Hom}(\ ,
\ ) \circ U$ will be in general distinct from $U \circ \mathcal{R}\operatorname{Hom}(\ ,
\ )$, where $\mathcal{R}\operatorname{Hom}(\ ,
\ )$ denotes the internal derived Hom in $\operatorname{Spc}_c(S)$, $\operatorname{Spc}_G^G(S)$, respectively. Recall that the notion of Spanier-Whitehead duality we will need to use involves stable versions of the corresponding functors in the non-equivariant framework: hence the above approach is not helpful for us. (See the introductory paragraph to section 8 for more on this.) Therefore, we need to obtain an analogue of Proposition 2.5 for spectra: one of the goals of the discussion in the next section, is to accomplish this while at the same time setting up a category of spectra with group actions to be used throughout the rest of the paper.

4. Categories of spectra

Spectra play two distinct roles in our context:

(i) One may observe that the definition of the transfer is as a stable map of certain spectra, and its applications are to splitting maps of generalized cohomology theories defined with respect to spectra. Here spectra mean either motivic or étale spectra which are not necessarily equivariant. Moreover, the notion of Spanier-Whitehead dual that is needed for the transfer is essentially in the non-equivariant setting.

(ii) In contrast, the construction of the transfer as a stable map starts with a pre-transfer, which will have to be an equivariant map of equivariant spectra, which is then fed into the Borel-construction to obtain the transfer for generalized (Borel-style) equivariant cohomology theories. (Equivariant spectra are defined below.)

(iii) Thus, the spectra that enter into the construction of a pre-transfer (which have to be equivariant) are all equivariant spectra, though the transfer is applied to generalized cohomology theories that are defined with respect to spectra that need not be equivariant. This dual role of spectra, makes it necessary for us to proceed carefully and explaining how the two roles are related.

(iv) When the group $G$ is a finite group, the regular representation of $G$ will contain all the irreducible representations (at least in characteristic 0), so that one may define a suspension functor by taking the smash product with the Thom-space of the regular representation. As a result one can then define symmetric $G$-equivariant spectra readily as one does in the non-equivariant case. Since our interest is mainly when the group $G$ is a linear algebraic group of positive dimension, one cannot adopt this framework of symmetric spectra, which is why we have defined the category $\operatorname{Spt}^G(S)$ in the following discussion.

(v) As pointed out earlier, we consider actions by all linear algebraic groups on schemes both in the étale and motivic frameworks. Prior work in this area has been restricted to actions by special classes of algebraic groups, such as those that are special in the sense of [Ch] (as in [Lev8]) and linearly reductive groups such as in [HO], which in positive characteristics, are just products of tori and finite abelian groups with torsion prime to the characteristic. The need to consider actions by all linear algebraic groups is essential to obtain the full range of applications of the transfer, and this makes it necessary for us to consider equivariant unstable and stable homotopy theory in the étale and motivic framework. We do this as concisely and briefly as possible.

4.0.1. Equivariant spectra. Throughout the following discussion, we will adopt the following terminology: $G$ will denote a fixed linear algebraic group defined over the base scheme (which we assume again is a perfect
field) and $\mathcal{C}$ will denote the category $\text{Spc}_c(S)$, while $\mathcal{C}^G$ will denote the category $\text{Spc}^G(S)$. Here $S$ could be either the base scheme or a fixed simplicial presheaf, so that all the simplicial presheaves we consider will be pointed over $S$.

The $G$-spectra will be indexed not by the non-negative integers, but by the Thom-spaces of finite dimensional representations of the group-scheme $G$ (i.e., affine spaces over $k$ provided with a linear action by $G$). We will fix a set of finite dimensional representations of $G$, which contains all irreducible representations, and is closed under finite direct sums, just as is done in the topological framework in [LMS 1.2]. Henceforth, we will only consider representations that belong to this set. For each finite dimensional representation $V$ of $G$, we let $T^S_V = T_V \times_{\text{Spec} k} S$. Then we let $\text{Sph}_G^G$ denote the subcategory of $\mathcal{C}^G$ whose objects are $\{T^S_V|V\}$, and where $V$ varies over all finite dimensional representations of the group $G$ and $T_V$ denotes its Thom-space. We let the morphisms in this category be given by the maps $T^S_V \to T^S_{V \oplus W}$ induced by homothety classes of $k$-linear injective and $G$-equivariant maps $V \to V \oplus W$. One may observe that $T_V$ identifies with the quotient sheaf $\text{Proj}(V \oplus 1)/\text{Proj}(V)$, so that there is an injection $V \to T_V$ for every $G$-representation $V$.

Let $T = \mathbb{P}^1$ pointed by $\infty$. We also let $\text{Sph}_S$ denote the category whose objects are $\{T^S_S^n|n \geq 0\}$, but given the structure of $\mathcal{C}$-enriched category as follows. First, the morphisms in this category are given by the maps $T^S_S \to T^S_{S \oplus W}$ induced by homothety classes of $k$-linear injective maps $A^n \to A^{n+m}$. We will make $\text{Sph}_S^G(\text{Sph}_S)$ an enriched monoidal category, enriched over the category $\mathcal{C}^G(\mathcal{C}$, respectively) as follows. First let $S^0 = S_+ = S \sqcup S$. Then for $V, W$ that are $G$-representations, we let the $\mathcal{C}^G$-enriched internal hom in $\text{Sph}_S^G$ be defined by:

$$\text{Hom}_{\mathcal{C}^G}(T^S_V, T^S_{V \oplus W}) = (\bigcup_{\alpha:V \to V \oplus W} T^S_W) \sqcup S, W \neq \{0\}$$

Here the sum varies over all homotopy classes of $G$-equivariant and $k$-linear injective maps $\alpha: V \to V \oplus W$ and the summand $S$ denotes a base point added so that the above enriched homs are pointed simplicial presheaves over $S$. The base points in each of the summands $T^S_W$ correspond bijectively with the corresponding $\alpha$; similarly the unique $0$-simplex other than the base point in each of the summands $S^0$ corresponds bijectively with the corresponding $\alpha$. As a result, the $0$-simplices in $\text{Hom}_{\mathcal{C}^G}(T^S_V, T^S_{V \oplus W})$ correspond bijectively with the morphisms $T_V \to T_{V \oplus W}$ in the category underlying the enriched category $\text{Sph}_S^G$. One defines the $\mathcal{C}$-enriched internal hom in $\text{Sph}_S$ by a similar formula as in (4.0.2):

$$\text{Hom}_{\mathcal{C}}(T^S_S^n, T^S_{S \oplus m}) = (\bigcup_{\alpha:A^n \to A^{n+m}} T^S_{\alpha^m}) \sqcup S, m > 0$$

where now $\alpha$ varies over homothety classes of $k$-linear injective maps $A^n \to A^{n+m}$. In particular, when $m = 0$, the general linear group $GL_n$ acts on $T^n$.

**Proposition 4.1.** With the above definitions, the category $\text{Sph}_S^G$ is a symmetric monoidal $\mathcal{C}^G$-enriched category, where the monoidal structure is given by $T^S_V \wedge T^S_W = T^S_{V \oplus W}$. $\text{Sph}$ is a symmetric monoidal $\mathcal{C}$-enriched category, where the monoidal structure is given by $T^S_S \wedge T^S_S = T^S_{S + S}$. The forgetful functor $j: \text{Sph}_S^G \to \text{Sph}_S$ is an enriched functor of $\mathcal{C}$-enriched categories.

**Proof.** We first verify that $\text{Sph}_S^G$ is a $\mathcal{C}^G$-enriched category. To see this, observe that if $f: U \to U \oplus V$ is a $G$-equivariant injective linear map and $g: V \to V \oplus W$ is a $G$-equivariant injective linear map, the composition $(id \oplus g) \circ f: U \to U \oplus V \oplus W$ is an injective linear map that is also $G$-equivariant. The composition $\text{Hom}_{\mathcal{C}^G}(T^S_U, T^S_{U \oplus V}) \times \text{Hom}_{\mathcal{C}^G}(T^S_V, T^S_{V \oplus W}) \to \text{Hom}_{\mathcal{C}^G}(T^S_U, T^S_{U \oplus V \oplus W})$ sends the summand $T^S_V$ indexed by $f$ and the summand $T^S_W$ indexed by $g$ to the summand $T^S_{V \oplus W}$ indexed by $(id \oplus g) \circ f: U \to U \oplus V \oplus W$. One may now see readily that this pairing is associative and unital, so that $\text{Sph}_S^G$ is a $\mathcal{C}^G$-enriched category: see [Bor 6.2].

The monoidal structure sends $(T^S_U, T^S_V) \to T^S_U \wedge T^S_V = T^S_{U \oplus V}$. One may now observe that the associativity isomorphisms $(U \oplus V) \oplus W \cong U \oplus (V \oplus W)$ and the commutativity isomorphism $U \oplus V \cong V \oplus U$ are both $G$-equivariant maps. Therefore, one observes that the monoidal structure defined by the smash product
(T^S_1, T^S_2) \mapsto T^S_1 \wedge T^S_2 = T^S_{1 \vee 2} makes the category $\text{Sph}_G^S$ a symmetric monoidal category. One may then readily verify that the same pairing is a functor of $C^G$-enriched categories, which will prove $\text{Sph}_G^S$ is a symmetric monoidal $C^G$-enriched category.

The statements regarding the $C$-enriched category $\text{Sph}_S$ may be proven similarly. We skip the proof that $j$ is a simplicially enriched functor. 

\[\square\]

Next we proceed to define various categories of spectra as enriched functors. Given a symmetric monoidal category $A \to B$, [see \[Bor, 6.2, 6.3\].] (4.0.5)

\[H\]

Remark 4.3

\[\tilde{\text{Quillen}}\] equivalent to the latter category. In fact taking $G$ to be trivial will produce the category of spectra (The equivariant sphere spectrum and suspension spectra) $\text{Spt}$ category (4.0.4)

\[X \wedge Y\]

(4.0.5) The internal \(\text{Hom}(\mathcal{X}, \mathcal{Y})\) is defined by the corresponding end:

\[\text{Hom}(\mathcal{X}, \mathcal{Y})(T_v) = \int_{T_w \in \text{Ob}(\text{Sph}_G^S)} \text{Hom}_{C^G}(\mathcal{X}(T_w), \mathcal{Y}(T_v \oplus W)).\]

Remark 4.3. It may be important to point out that taking $G$ to be trivial does not define the familiar category $\text{Spt}(S)$ indexed by \(\{T^{\otimes n} | n \geq 0\}\), but a category of spectra which we show (see: section 6) is Quillen equivalent to the latter category. In fact taking $G$ to be trivial will produce the category of spectra denoted $\text{Spt}(S)$ defined below in Definition 4.9.

Definition 4.4. (The equivariant sphere spectrum and suspension spectra)

(i) The equivariant sphere spectrum $S^G_S$ will be defined to be the object in $\text{Spt}_G^G(S)$ given by the functor $\text{Sph}_G^S \to C^G$, that is, $S^G_S(T^S_1) = T^S_1$, $T^S_2 \in S^G$. When $G$ is trivial, this defines a sphere spectrum, which will be denoted $S^G_S$. (Observe that this defines an object in the category $\text{Spt}(S)$ defined in Definition 4.9)

(ii) On the Nisnevich site (étale site) of $\text{Spec} k$, $S^G_S$ will define the motivic sphere spectrum (the étale sphere spectrum), which will be denoted $S^G_{S, \text{mot}}$ or often simply $S^G_S$ ($S^G_{S, \text{et}}$, respectively).

(iii) When $S = \text{Spec} k$, we will denote $S^G_{S, \text{mot}}$ ($S^G_{S, \text{et}}$) by $S^G_k$ or simply $S^G_k$ ($S^G_{k, \text{et}}$, respectively).

Definition 4.5. (The equivariant motivic Eilenberg-Maclane spectrum)

(i) Let $\text{Spc}^{\text{tr}}_G(S)$ denote the category of all simplicial abelian presheaves with transfers on the Nisnevich site of $\text{Spec} k$ and pointed over $S$. Let $U : \text{Spc}^{\text{tr}}_G(S) \to \text{Spc}_G(S)$ denote the forgetful functor sending a simplicial abelian presheaf with transfers to the underlying pointed simplicial presheaf. For each representation $V$ of $G$, and a commutative Noetherian ring $R$ with $1$, we let $R^\otimes(V) = \text{cokernel}((Z^\otimes(\text{Proj}(V)) \otimes R) \to Z^\otimes(\text{Proj}(V) \oplus 1)) \otimes R)$. Now we let $\hat{H}(R)_S = \{U(R^\otimes(T_V)) | V\}$. The structure maps are given by:

(4.0.6) $T_W \wedge U(R^\otimes(T_V)) \to U(R^\otimes(T_W)) \wedge U(R^\otimes(T_V)) \to U(R^\otimes(T_W) \otimes^\otimes R^\otimes(T_V)) \cong U(R^\otimes(T_W \oplus V))$,

where $\otimes^\otimes$ denotes the monoidal structure on the category of simplicial abelian presheaves with transfers. (We skip the verification that this defines a ring spectrum in $\text{Spt}_G^G(k_{\text{mot}})$).

(ii) Taking $G$ to be trivial defines the motivic Eilenberg-Maclane spectrum, which will be denoted $\hat{H}(R)_S$ (and when $S = \text{Spec} k$ by $\hat{H}(R)_k$).
In case $\mathcal{E}^G$ is a commutative ring spectrum in $\text{Spt}^G(S)$, we will let $\text{Spt}^G(S, \mathcal{E}^G)$ denote the category consisting of module spectra over $\mathcal{E}^G$ and their maps. In this case, the smash product $\wedge$ will be replaced by $\wedge_{\mathcal{E}^G}$ which is defined as

$$(4.0.7) \quad M \wedge_{\mathcal{E}^G} N = \text{Coeq}(M \wedge\mathcal{E}^G \wedge N \to M \wedge N)$$

where the two maps above make use of the module structures on $M$ and $N$, respectively. The corresponding internal $\text{Hom}$ will be denoted $\text{Hom}_{\mathcal{E}^G}$.

The main $G$-equivariant ring spectra of interest to us, including the sphere spectrum $S^G$, will be the following:

(i) $S^G$, (ii) $S^G[p^{-1}]$ if the base scheme $S$ is a field of characteristic $p$, (iii) $S^G_S(\ell)$, which denotes the localization of $S^G_S$ at the prime ideal $(\ell)$, $\ell$ is a prime $\neq \text{char}(k)$,

$$(4.0.8) \quad \tilde{S}^G_S, \ell \text{ denotes the } \ell - \text{completed } G - \text{equivariant sphere spectrum as well as }$$

(iv) $\tilde{S}^G_{S, \ell}$, where $\ell$ is a prime $\neq \text{char}(k)$ and $\tilde{S}^G_{S, \ell}$ denotes the $\ell -$ completed $G -$ equivariant sphere spectrum as well as

(v) $H(\text{R})^G_S$ with $R = \mathbb{Z}/\ell^n$, with $\ell$ a prime $\neq \text{char}(k)$.

Here the completion at the prime $\ell$ is the Bousfield-Kan completion discussed in [CJ23-T2, Appendix].

**Definition 4.6.** (i) Let $\text{Spt}(S_{\text{mot}})$ denote the (usual) category of motivic spectra defined as follows. Its objects are $\mathcal{X} = \{X_n \in \text{Spc}_c(S_{\text{mot}}), \text{ along with structure maps } T^{\wedge m} \wedge X_n \to X_{n+m}|, m \in \mathbb{N}\}$. Morphisms between two such objects $\mathcal{X}$ and $\mathcal{Y}$ are defined as compatible collection of maps $X_n \to Y_n$, $n \in \mathbb{N}$ compatible with suspensions by $T^{\wedge m}$, $m \in \mathbb{N}$. When $S = \text{Spec }k$, this category will be denoted $\text{Spt}(k_{\text{mot}})$ of often simply $\text{Spt}(k)$.

(ii) The unit of this category is the motivic sphere spectrum denoted $S_k$. For a simplicial presheaf $P \in \text{Spc}_c(k_{\text{mot}})$, the suspension spectrum $S_k \wedge P$ will be denoted $\Sigma^\infty_+ P$.

(iii) $\text{Spt}(S_{\text{et}})$ will denote the corresponding category of $T$-spectra defined by starting with the category $\text{Spc}_c(S_{\text{et}})$ of pointed simplicial presheaves on the big étale site of $\text{Spec }k$ and pointed over $S$.

(iv) The unit of this category is the sphere spectrum denoted $S_{k_{\text{et}}}$.

**Remark 4.7.** We begin with the following remarks to motivate the constructions below. Here is a particularly tricky aspect of the construction of the pre-transfer. The Spanier-Whitehead duality one needs to invoke in the construction of the pre-transfer is in the setting of non-equivariant spectra and not in a corresponding category of equivariant spectra, such as the ones discussed above. There are several reasons for this choice, some of which are:

(i) Currently one does not have Spanier-Whitehead duality for algebraic varieties in the equivariant framework, since one does not yet have equivariant versions of Gabber’s refined alterations.

(ii) For the construction of the transfer in the context of Borel-style generalized equivariant cohomology theories this is all that is needed: see, for example, [BG75]. In more detail: all one needs in this context is Spanier-Whitehead duality in a non-equivariant setting, but applied to spectra with group actions.

(iii) On the other hand, we still need the Spanier-Whitehead dual of an object with a $G$-action to inherit a nice $G$-action and we need to use sphere-spectra which also have non-trivial $G$-actions. In fact, it is crucial that the source of the co-evaluation maps will have to be $G$-equivariant (sphere) spectra: otherwise the spectra showing up as the target of the co-evaluation maps will have no $G$-action: see Definition 1.4 and 2.2. In more detail: though we only need a non-equivariant form of Spanier-Whitehead duality, one needs to make all the constructions sufficiently equivariant so as to be able to feed them into the Borel construction.

(iv) In [BG75], the way these issues are resolved is by making sure the Thom-Pontrjagin collapse map (which plays the role of the coevaluation map) can be made equivariant. In our framework, the way we resolve these problems is as follows. First we use $G$-equivariant spectra to serve as the source of the coevaluation maps. Then we observe that for the underlying non-equivariant spectrum, associated to an equivariant spectrum, but viewed as an object in the category $\text{Spt}^G(S_{\text{mot}})$ (defined below), one can find functorial fibrant and cofibrant replacements in the latter category, and the functoriality implies that these objects come equipped with compatible $G$-actions. Further, we show in section [6]...
that the model category $\widetilde{\text{Spt}}^G(S_{\text{mot}})$ is Quillen equivalent to the usual category of non-equivariant spectra $\text{Spt}$ considered in Definition 4.10. Therefore, the dual we define will be making use of such functorial cofibrant and fibrant replacements of the underlying non-equivariant spectra in $\widetilde{\text{Spt}}^G(S_{\text{mot}})$ and therefore, though they correspond to duals in $\text{Spt}$, they still come equipped with nice $G$-actions. It is precisely these issues that make it necessary for us to introduce and work with the categories $\text{Spt}_{\text{mot}}^G(S_{\text{mot}})$ and $\text{Spt}(S_{\text{mot}})$ of spectra that come in between, and relate the category of equivariant spectra $\text{Spt}^G(S_{\text{mot}})$ with the category of spectra $\text{Spt}(S_{\text{mot}})$.

We now introduce the following intermediate categories, denoted $\text{Spt}^G(S)$, and $\text{Spt}(S)$, intermediate between $\text{Spt}^G(S)$ and $\text{Spt}(S)$ defined in Definition 4.10.

**Definition 4.8.** (i) The $\mathcal{C}$-enriched category $\text{Spt}^G(S) = [\text{Sph}_{\mathcal{G}}, \text{Spc}_{s}(S)]$. Therefore, the objects of this category are $\mathcal{C}$-enriched functors $\mathcal{X}': \text{Sph}_{\mathcal{G}} \to \mathcal{C}$, where $\mathcal{C} = \text{Spc}_{s}(S)$. One may observe that an object in this category is given by $\{\mathcal{X}^i(T^S_{S})|T^S_{S} \in \text{Sph}_{\mathcal{G}}\}$, provided with a compatible family of structure maps $T^S_{W} \wedge \mathcal{X}^j(T^S_{V}) \to \mathcal{X}^{i}(T^S_{W \oplus V})$ in $\text{Spc}_{s}(S)$, with $T^S_{W} = T^S_{W}$ as $\alpha$ varies over all homothety classes of $k$-linear $G$-equivariant injective maps $V \to V \oplus W$. However, the maps $T^S_{W} \wedge \mathcal{X}^j(T^S_{V}) \to \mathcal{X}^{i}(T^S_{W \oplus V})$ are no longer required to be $G$-equivariant.

(ii) The smash product and the internal hom of spectra in $\text{Spt}^G(S)$ are defined exactly as in the case of $\text{Spt}^G(S)$, but making use of the category $\text{Spc}_{s}(S)$ in the place of $\text{Spc}_{s}(S)$.

(iii) $\text{Spt}^G(S_{\text{mot}})$ and $\text{Spt}(S_{\text{mot}})$ will denote the corresponding category defined on the Nisnevich site by starting with $\text{Spc}_{s}(S_{\text{mot}})$ (on the étale site by starting with $\text{Spc}_{s}(S_{\text{mot}})$), respectively.

(iv) When $\mathcal{E}^G \in \text{Spt}^G(S)$ is a commutative ring spectrum, one defines the category $\text{Spt}^G(S, \mathcal{E}^G)$ similarly by replacing the pairings $T^S_{W} \wedge \mathcal{X}^j(T^S_{V}) \to \mathcal{X}^{i}(T^S_{W \oplus V})$ with the pairings: $\mathcal{E}^G(T^S_{W}) \wedge \mathcal{X}^j(T^S_{V}) \to \mathcal{X}^{i}(T^S_{W \oplus V})$.

Observe that there is a forgetful functor

$$\tilde{U} : \text{Spt}^G(S) \to \text{Spt}^G(S)$$

given by sending a $\mathcal{X} \in \text{Spt}^G(S)$ to $U \circ \mathcal{X}$, where $U : \text{Spc}_{s}(S) \to \text{Spc}_{s}(S)$ is the forgetful functor. When $\mathcal{E}^G \in \text{Spt}^G(S)$ is a commutative ring spectrum, one also obtains a forgetful functor $\tilde{U} : \text{Spt}^G(S, \mathcal{E}^G) \to \text{Spt}^G(S, \mathcal{E}^G)$.

**Definition 4.9.** (i) Let $\mathcal{C}$ denote the category $\text{Spc}_{s}(S)$. The $\mathcal{C}$-enriched category $\text{Spt}(S) = [\text{Sph}_{\mathcal{G}}, \mathcal{C}]$.

Therefore, the objects of this category are given by $\mathcal{C}$-enriched functors $\mathcal{X}' : \text{Sph}_{\mathcal{G}} \to \mathcal{C}$.

Again, paraphrasing this, such an object is given by $\{\mathcal{X}^i(T^S_{n})|n \geq 0\}$, provided with a compatible family of structure maps $T^S_{S, \alpha} \wedge \mathcal{X}^j(T^S_{S}) \to \mathcal{X}^{i}(T^S_{S, \alpha + j})$ in $\text{Spc}_{s}(S)$, with $T^S_{S, \alpha} = T^S_{S, \alpha}$ associated to each homothety class $\alpha$ of $k$-linear injective maps of $\mathbb{A}^n$ in $\mathbb{A}^{n+m}$, and the group of $k$-linear automorphisms of $\mathbb{A}^n$, (i.e., $GL_n$) acts on $\mathcal{X}^i(T^S_{S})$. (In this sense, the category $\text{Spt}(S)$ is similar to the category of what are called orthogonal spectra.)

Morphisms between two such objects $\{\mathcal{X}^i(T^S_{n})|n \geq 0\}$ and $\{\mathcal{X}^j(T^S_{n})|n \geq 0\}$ are given by compatible collections of maps $\{\mathcal{Y}^i(T^S_{n}) \to \mathcal{X}^j(T^S_{n})|n \geq 0\}$ which are compatible with the pairings: $T^S_{S, \alpha} \wedge \mathcal{Y}^j(T^S_{S}) \to \mathcal{Y}^i(T^S_{S, \alpha + j})$ and $T^S_{S, \alpha} \wedge \mathcal{X}^j(T^S_{S}) \to \mathcal{X}^{i}(T^S_{S, \alpha + j})$.

(ii) $\text{Spt}(S_{\text{mot}})$ and $\text{Spt}(S_{\text{mot}})$ will denote the corresponding category defined on the Nisnevich site (the étale site, respectively) of Spec $k$.

(iii) The smash product and the internal hom of spectra in $\text{Spt}(S)$ are defined again exactly as in the case of $\text{Spt}(S)$, but making use of the categories $\text{Sph}_{\mathcal{G}}$ and $\text{Spc}_{s}(S)$ in the place of $\text{Sph}_{\mathcal{G}}$ and $\text{Spc}_{s}(S)$.

(iv) Let $\tilde{U} : \text{Spt}^G(S) \to \text{Spt}^G(S)$ and $\tilde{P} : \text{Spt}^G(S) \to \text{Spt}(S)$ denote the functors considered in Definition 4.10.

When $\mathcal{E}^G \in \text{Spt}^G(S)$ is a commutative ring spectrum, one defines the category $\text{Spt}(S, \mathcal{E}^G)$ similarly by replacing the pairings $T^S_{S, \alpha} \wedge \mathcal{X}^j(T^S_{S}) \to \mathcal{X}^{i}(T^S_{S})$ with the pairings: $\mathcal{E}^G(T^S_{S}) \wedge \mathcal{X}^j(T^S_{S}) \to \mathcal{X}^{i}(T^S_{S})$. 

Proposition 4.10. Let $X, Y \in \text{Spt}^G(S)$. Then,
\[(4.0.11) \quad \tilde{U}(X \wedge Y) = \tilde{U}(X) \wedge \tilde{U}(Y) \text{ and } \tilde{U}(\text{Hom}_{\text{Spt}^G(S)}(X, Y)) = \text{Hom}_{\text{Spt}^G(S)}(\tilde{U}(X), \tilde{U}(Y)).\]

Corresponding results hold for the categories $\text{Spt}^G(S, E^G)$, $\tilde{\text{Spt}}^G(S, E^G)$.

Proof. The key observation is that the forgetful functor $U : \text{Spc}^G(S) \to \text{Spc}_e(S)$ is a strict monoidal functor in the sense $U(P \wedge^S Q) = U(P) \wedge U(Q)$ and $U(\text{Hom}^G(P, Q)) = \text{Hom}(U(P), U(Q))$ as already observed in (2.1.10). In addition one also observes that the same forgetful functor preserves and reflects all small colimits as well as all small limits. Therefore the definition of the smash product (the internal hom) in the category $\text{Spt}^G(S)$ as a co-end (end, respectively) in Definition 4.12 along with the definition of the corresponding functors in $\tilde{\text{Spt}}^G(S)$ completes the proof. □

Terminology 4.11. Model structures on the above categories of spectra: starting with the model structures on simplicial presheaves discussed in section 2.1 one may put various model structures on the above categories of spectra. This is discussed in detail in the next two sections. We do not discuss the specific details of these model structures here, as we believe that will take us away from our current discussion, except to point out that we choose to work with the category $\text{Spt}^G(S_{\text{mot}})$ and $\tilde{\text{Spt}}^G(S_{\text{mot}})$ provided with injective stable model structures as discussed in Proposition 6.2, observe that every object is cofibrant in this model structure. For the rest of the discussion in this section, we will implicitly make use of this model structure.

Definition 4.12. Let $F : C \to D$ denote a functor between monoidal categories. We say $F$ is a weakly monoidal functor, if for any pair of objects $X, Y$ in $C$, there is given a natural map
\[(4.0.12) \quad \mu : F(X) \otimes F(Y) \to F(X \otimes Y)\]
satisfying an associativity and unitality axiom as in [Bor, Definition 6.4.1]: note that there it is called a strong monoidal functor, if for any pair of objects $X, Y$ in $C$, there is given a natural map $\epsilon : e_D \to F(e_C)$, where $e_C (e_D)$ denotes the unit of the category $C (D)$, respectively. We say such a weakly monoidal functor is a strong monoidal functor (strict monoidal functor) if the map $\mu$ for all $X, Y$ in $C$ and the map $\epsilon$ are isomorphisms (are the identity morphisms, respectively).

Definition 4.13. (Passing from equivariant spectra to non-equivariant spectra) One starts with the forgetful functor $\tilde{U} : \text{Spt}^G(S) \to \tilde{\text{Spt}}^G(S)$. Since the indexing category for $\tilde{\text{Spt}}^G(S)$ is $\text{Sph}^G_S$, while the indexing category for $\text{Spt}(S)$ is $\text{Sph}_S$, the passage from $\tilde{\text{Spt}}^G(S)$ to $\text{Spt}(S)$ is more involved. As discussed in (6.0.5), this is carried out by a functor we denote by $\tilde{P}$.

Let $\text{Spt}(S)$ denote the (usual) category of spectra indexed by the non-negative integers as in Definition 4.6. Since $\text{Spt}(S)$ is indexed by the category $\text{Sph}_S$ which denote the Thom-spaces of all affine spaces $\{\mathbb{A}^n| n \geq 0\}$, there is an obvious functor $i^* : \tilde{\text{Spt}}^G(S) \to \text{Spt}(S)$; see (6.0.2). Thus the passage from $G$-equivariant spectra in $\text{Spt}^G(S)$ to the non-equivariant spectra $\text{Spt}(S)$ indexed by the non-negative integers is defined by the sequence of functors:
\[(4.0.13) \quad \text{Spt}^G(S) \xrightarrow{\tilde{U}} \tilde{\text{Spt}}^G(S) \xrightarrow{\tilde{P}} \text{Spt}(S) \xrightarrow{i^*} \text{Spt}(S).\]

Of these the first two functors $\tilde{U}$ and $\tilde{P}$ are strong monoidal functors (in fact $\tilde{U}$ is a strict monoidal functor), while the composition $i^* \circ P$ is the identity, where $P : \text{Spt}(S) \to \text{Spt}(S)$ is a functor left-adjoint to $i^*$ and which is also strong monoidal. (See Proposition 6.2.) Given a commutative ring spectrum $E^G_S \in \text{Spt}^G(S)$, we let $\mathcal{E}_S = i^*(\tilde{P}U(E^G))$, which is a commutative ring spectrum in $\text{Spt}(S)$. For example, the equivariant sphere spectrum $S^G_S$ provides $S_S = i^*(\tilde{P}U(S^G_S))$, the usual sphere spectrum. □

Of key importance is the observation that the $\tilde{U}(S^G_S)$ is the unit of the category $\tilde{\text{Spt}}^G(S)$ with respect to the smash product in $\tilde{\text{Spt}}^G(S)$. Similarly $\tilde{U}(E^G)$ is the unit of $\tilde{\text{Spt}}^G(S, E^G)$, $\tilde{P}(U(S^G_S))$ is the unit of the category $\text{Spt}(S)$, and $\tilde{P}(\tilde{U}(E^G))$ is the unit of $\text{Spt}(S, E^G)$, with respect to the corresponding smash products. In view of this, we will henceforth denote $\tilde{U}(S^G_S)$, $\tilde{P}(U(S^G_S))$ by $S^G_S$ and $\tilde{U}(E^G)$, $\tilde{P}(U(E^G))$ by $E^G$. 


Proposition 4.14. Let $X \in \text{Spt}^G(S)$ and let $\tilde{U}(X) \in \text{Spt}^G_{\text{mot}}(S)$ denote the forgetful functor $\tilde{U}$ (as in (4.0.9)) applied to $X$. If $\tilde{\alpha} : X'' \to \tilde{U}(X)$ ($\tilde{\beta} : \tilde{U}(X) \to X'$) is a cofibrant replacement (fibrant) replacement in the injective or projective stable model structure on $\text{Spt}^G_{\text{mot}}(S)$, then there exists $X'$ and $X''$ in $\text{Spt}^G(S)$, and maps $\alpha : X'' \to X$, $\beta : X' \to X'$ in $\text{Spt}^G(S)$ so that $\tilde{U}(\alpha) = \tilde{\alpha}$ and $\tilde{U}(\beta) = \tilde{\beta}$.

Proof. Recall that the linear algebraic group $G$ acts on a simplicial presheaf section-wise. Therefore, the functoriality of the cofibrant and fibrant replacements, shows as in (4.0.11) that one may find functorial cofibrant and fibrant replacements of objects in $\text{Spt}^G(S)$, the following squares commute for all $U$ in the site, for all $g \in \Gamma(U, G)$, all $T_W$ and $T_V \in \text{Sph}^G_S$:

\[
\begin{array}{c}
\Gamma(U, T_W^S) \wedge \Gamma(U, X'(T_V^S)) \\
\downarrow \quad \downarrow
\end{array}
\quad
\begin{array}{c}
\Gamma(U, T_W^S) \wedge \Gamma(U, X''(T_V^S)) \\
\downarrow \quad \downarrow
\end{array}
\quad
\begin{array}{c}
\Gamma(U, T_W^S) \wedge \Gamma(U, X'(T_V^S)) \\
\downarrow \quad \downarrow
\end{array}
\quad
\begin{array}{c}
\Gamma(U, T_W^S) \wedge \Gamma(U, X''(T_V^S))
\end{array}
\]

It follows that the functorial cofibrant replacement $X''$ of $\tilde{U}(X)$ in $\text{Spt}^G_{\text{mot}}(S)$ inherits a $G$-action from the $G$-action on $X$, making it belong to $\text{Spt}^G(S)$. A corresponding result holds for the functorial fibrant replacement as well. □

4.0.15. Derived functors of $\wedge$, the internal $\text{Hom}$ and the dual $D$ for equivariant spectra. Recall that the functor $\tilde{U} : \text{Spt}^G(S) \to \text{Spt}^G_{\text{mot}}(S)$ is a strict monoidal functor. Let $M, N \in \text{Spt}^G(S)$. The fact that one may find functorial cofibrant and fibrant replacements of objects in $\text{Spt}^G(S)$ shows that one may find a functorial cofibrant replacement $\tilde{\alpha} : \tilde{M}'' \to \tilde{U}(M)$ in $\text{Spt}^G_{\text{mot}}(S)$ and a functorial fibrant replacement $\tilde{\beta} : \tilde{U}(N) \to \tilde{N}'$ in $\text{Spt}^G_{\text{mot}}(S)$. The functoriality of the cofibrant and fibrant replacements, shows as in Proposition 4.14 that in fact that there exists $M'', N'$ and maps $\alpha : M'' \to M$, $\beta : N \to N'$ in $\text{Spt}^G(S)$, with $\tilde{U}(\alpha) = \tilde{\alpha}$ and $\tilde{U}(\beta) = \tilde{\beta}$. Therefore, we define

\[
(4.0.16) \quad M^L \wedge N = M'' \wedge N, \quad \mathcal{R}\text{Hom}(M, N) = \mathcal{H}\text{om}(M'', N'), \quad D(M) = \mathcal{R}\text{Hom}(M'', (S^G)')
\]

with $M^L \wedge N, \mathcal{R}\text{Hom}(M, N), D(M) \in \text{Spt}^G(S)$. (In fact, since we choose to work with the injective model structures, every object is cofibrant and therefore there is no need for any cofibrant replacements.) Similar conclusions will hold when $E^G \in \text{Spt}^G(S)$ is a commutative ring spectrum with the corresponding smash product $\wedge_{E^G}$ and $\mathcal{H}\text{om}_{E^G}$ defined in (4.0.7). (In this case the dual with respect to the ring spectrum $E^G$ will denoted $D_{E^G}$.)

Definition 4.15. (The associated homotopy categories). The homotopy category associated to $\text{Spt}^G(S_{\text{mot}})$ will be denoted $\mathcal{SH}^G(S_{\text{mot}})$, while the homotopy category associated to $\text{Spt}^G(S_{\text{mot}})$ will be denoted $\mathcal{SH}^G(S_{\text{mot}})$ and the homotopy category associated to $\text{Spt}(S_{\text{mot}})$ will be denoted $\mathcal{SH}(S_{\text{mot}})$. Similarly the homotopy category associated to $\text{Spt}(S_{\text{mot}})$ will be denoted $\mathcal{SH}(S_{\text{mot}})$, often denoted just $\mathcal{SH}(S)$. The corresponding étale variants will be denoted by the subscript et in the place of the subscript mot.
For a commutative ring spectrum $\mathcal{E}^G \in \text{Spt}^G(S_{\text{mot}})$, the homotopy category associated to $\text{Spt}^G(S_{\text{mot}}, \mathcal{E}^G)$ will be denoted $\mathcal{SH}^G(S_{\text{mot}}, \mathcal{E}^G)$ while the homotopy category associated to $\text{Spt}^G(S_{\text{mot}}, \mathcal{E}^G)$ will be denoted $\tilde{\mathcal{SH}}^G(S_{\text{mot}}, \mathcal{E}^G)$ and the homotopy category associated to $\text{Spt}(S_{\text{mot}}, \mathbb{P}(U)(\mathcal{E}^G))$ will be denoted $\mathcal{SH}(S_{\text{mot}}, \mathbb{P}(U)(\mathcal{E}^G))$. Similarly the homotopy category associated to $\text{Spt}(S_{\text{mot}}, \mathcal{E}_S)$ will be denoted $\mathcal{SH}(S_{\text{mot}}, \mathcal{E}_S)$ (or simply $\mathcal{SH}(S_{\text{mot}}, \mathcal{E}_S)$), where $\mathcal{E}_S = i^*(\mathbb{P}(U)(\mathcal{E}^G))$ is the associated non-equivariant spectrum.

5. Model structures for categories of spectra

One starts with the categories of spectra $\text{Spt}^G(S)$, $\tilde{\text{Spt}}^G(S)$ and $\tilde{\text{Spt}}(S)$ considered in Definitions 4.2 and 4.3. In case $\mathcal{E}^G$ is a commutative ring spectrum in $\text{Spt}^G(S)$ ($\tilde{\text{Spt}}^G(S)$) or if $\mathcal{E}$ is a commutative ring spectrum in $\tilde{\text{Spt}}(S)$ we will also consider the corresponding category $\text{Spt}^G(S, \mathcal{E}^G)$ ($\tilde{\text{Spt}}^G(S, \mathcal{E}^G)$) and $\tilde{\text{Spt}}(S, \mathcal{E})$ of module spectra.

5.1. Level-wise model structures. Throughout this discussion, we will assume the situation where $\text{Spc}_c(S)$ denotes the category of pointed simplicial presheaves, and $\text{Spc}_c^G(S)$ which is the category of pointed simplicial presheaves with $G$-action, both pointed over $S$ on either the big étale or the big Nisnevich or the big Zariski site over a fixed perfect field $k$.

$\text{Spc}_c(S)$ will be provided with a chosen model structure, namely the motivic model structure in the case of the Nisnevich site (see 2.1.4) and the étale model structure (see Theorem 2.2) in the case of the étale site which are both based on the projective model structures or the alternate injective model structures discussed in 2.1.0. Observe that every object in $\text{Spc}_c(S)$ is cofibrant in the injective model structure.

$\text{Spc}_c^G(S)$ will be provided with one of the model structures provided by Theorem 3.3. Since the generating cofibrations are defined as in Definition 3.2 not every object is cofibrant even in the corresponding injective model structure on $\text{Spc}_c^G(S)$.

5.1.1. The level-wise injective model structures. We will start with the injective model structures on $\text{Spc}_c(S)$. Here we make use of [Lur] Proposition A.3.3.2. The first observation is that the model categories $\text{Spt}_c(S_{\text{mot}})$ and $\text{Spt}_c(S_{\text{et}})$, when provided with the injective model structures, are excellent in the sense of [Lur] A.3.2.16: this means they are combinatorial, every monomorphism is a cofibration, cofibrations are stable under products, the weak-equivalences are stable under filtered colimits, the smash product $\wedge^S$ is a left Quillen functor and it satisfies the invertibility hypothesis. (The last may be deduced from the category of simplicial sets as in [Lur] Lemma A.3.2.20 by observing that the functor sending a simplicial set to the constant simplicial presheaf is a left Quillen functor as required.) Therefore, the required model structures follow from [Lur] Proposition A.3.3.2, and the discussion below should be just spelling out the details.

Here we define a map $f : \chi' \to \chi$ of spectra in $\tilde{\text{Spt}}^G(S)$ to be a level-wise injective cofibration (a level-wise injective weak-equivalence) if the induced map $f(T^S_V) : \chi'(T^S_V) \to \chi(T^S_V)$ is a cofibration (a weak-equivalence, respectively) for each $T^S_V \in \text{Sph}^G$. The level-wise injective fibrations are defined by the lifting property with respect to trivial cofibrations. One defines the level-wise injective model structure on the categories $\text{Spt}^G(S)$ and $\text{Spt}(S)$ similarly.

**Proposition 5.1.** (i) This defines a combinatorial (in fact, tractable) simplicial monoidal model structure on $\tilde{\text{Spt}}^G(S)$ that is left proper.

(ii) Every level-wise injective fibration is a level fibration, that is if $f : \chi' \to \chi$ is a fibration in the level-wise injective model structure, each of the induced maps $\chi'(T^S_V) \to \chi(T^S_V)$ is a fibration.

(iii) The cofibrations are the monomorphisms.

(iv) The unit of the monoidal structure on $\tilde{\text{Spt}}^G(S)$ and in fact every object in $\tilde{\text{Spt}}^G(S)$ is cofibrant in this model structure.

(v) The corresponding results hold for the categories $\text{Spt}^G(S)$ and $\tilde{\text{Spt}}(S)$.

**Proof.** We will only discuss the proofs for the category $\tilde{\text{Spt}}^G(S)$ since the proofs in the other two cases are quite similar. We start with the observation that the categories $\mathcal{C} = \text{Spc}_c(S)$, $\mathcal{C}^G = \text{Spc}_c^G(S)$ are simplicially enriched tractable simplicial model categories. The left-properness is obvious, since the cofibrations and
weak-equivalences are defined level-wise. The first conclusion follows now from [Lur, Proposition A.3.3.2]:
observe that the pushout-product axiom holds since cofibrations (weak-equivalences) are injective cofibrations
(weak-equivalences, respectively) and the pushout-product axiom holds in the monoidal model category \( \mathcal{C} \).
This proves the first statement for \( \mathcal{Spt}^G(S) \), as well as for \( \mathcal{Spt}(S) \).

The second statement follows from Proposition 5.2(iii) making use of the adjunction between the functors
\( \mathcal{E}val_{T^G_V} \) and \( \mathcal{F}_{T^G_V} \) discussed below. The third statement follows readily since we start with the injective model
structure on \( \mathcal{Spc}_*(S) \). Recall the unit of \( \mathcal{Spt}^G(S) \) is the functor \( \mathcal{Sph}^G_S \to \mathcal{C} \), which is the sphere spectrum \( \mathcal{C}^G \).
To prove it is cofibrant, all one has to observe is that \( \mathcal{Sph}^G_S(T^G_V) = T^G_V \) which is cofibrant in \( \mathcal{Spc}_*(S) \) for every
\( T^G_V \in \mathcal{Sph}^G_S \). It should be clear that the same arguments hold for the categories \( \mathcal{Spt}^G(S) \) and \( \mathcal{Spt}(S) \). □

5.1.2. The level-wise projective model structures. We will consider explicitly only \( \mathcal{Spt}^G(S) \). We will
start with the projective model structure on \( \mathcal{Spt}_*(S_{mot}) \). First we functorially replace every object \( T^G_V \) in
\( \mathcal{Sph}^G_S \) by an object that is cofibrant in \( \mathcal{C} = \mathcal{Spc}_*(S) \). The functoriality of the cofibrant replacement shows
that then, these functorial cofibrant replacements all come equipped with \( G \)-actions. Therefore, we will still
denote these cofibrant replacements by \( \{T^G_V|_V\} \). In the case of \( \mathcal{Spt}^G(S) \), we need to do the same with the
category \( \mathcal{C} \) replaced by \( G^G = \mathcal{Spc}^G(S) \).

We should also point out that the work of [DRO1, Theorems 4.2, 4.4] in fact provides such a model
structure on \( \mathcal{Spt}^G(S_{mot}) \), \( \mathcal{Spt}(S_{mot}) \) and on \( \mathcal{Spt}^G(S_{mot}) \) and that they extend readily to the corresponding
categories defined on the etale site. Therefore, the discussion below should be viewed as summarizing their
results in this case.

The weak-equivalences (fibrations) in the level-wise projective model structure are those maps of spectra
\( f : \mathcal{X} \to \mathcal{Y} \), for which each \( f(T^G_V) : \mathcal{X}(T^G_V) \to \mathcal{Y}(T^G_V), \) \( T^G_V \in \mathcal{Sph}^G_S \), are weak-equivalences (fibrations,
respectively) in \( \mathcal{C} = \mathcal{Spc}_*(S) \). The cofibrations in this model structure are defined by left-lifting property
with respect to the maps that are trivial fibrations in this model structure.

Next let \( \mathcal{F}_{T^G_V} \) denote the left-adjoint to the evaluation functor \( \mathcal{E}val_{T^G_V} \) sending a spectrum \( \mathcal{X} \in \mathcal{Spt}^G(S_{mot}) \)
(\( \mathcal{X} \in \mathcal{Spt}(S_{mot}), \mathcal{Spt}^G(S_{mot}) \)) to \( \mathcal{X}(T^G_V) \). One may observe that this is the spectrum defined by
\[
\mathcal{F}_{T^G_V}(C)(T^G_{V,W}) = \left( \bigcup_{\alpha : V \to V} C \cap T^G_{V,W} \right) \cup *, W \neq \{0\} = \left( \bigvee_{\alpha : V \to V} C \cup *, W = \{0\} \right)
\]
For each \( T^G_V \), let \( \mathcal{R}_{T^G_V} \) denote the \( \mathcal{Spc}_*(S_{mot}) \)-enriched functor defined by
\[
\mathcal{R}_{T^G_V}(P)(T^G_W) = \mathcal{Homs}_{\mathcal{Spc}_*(S)}((\bigcup_{\alpha : U \to T^G_U} P), P)
\]
when \( V = W \oplus U \), and the sum is indexed by homothety classes of \( k \)-linear injective maps \( U \to V \). (When \( V \) is not of the form \( U \oplus W \), we let \( \mathcal{R}_{T^G_V}(P)(T^G_W) = S^0 \).

Then \( \mathcal{R}_{T^G_V} : \mathcal{Spc}_*(S_{mot}) \to \mathcal{Spt}^G(S_{mot}) \) is right adjoint to the functor \( \mathcal{E}val_{T^G_V} \). One defines a right adjoint
\( \mathcal{R}_{T^G_V} \) to \( \mathcal{E}val_{T^G_V} : \mathcal{Spt}_*(S_{mot}) \to \mathcal{Spc}_*(S_{mot}) \) and to \( \mathcal{E}val_{T^G_V} : \mathcal{Spt}^G(S_{mot}) \to \mathcal{Spc}^G(S_{mot}) \) similarly.

Let \( I(J) \) denote the generating cofibrations (generating trivial cofibrations, respectively) of the model
category \( \mathcal{Spc}_*(S_{mot}) \). We define the generating cofibrations \( I_{\mathcal{Spt}^G(S_{mot})} \) (the generating trivial cofibrations
\( J_{\mathcal{Spc}^G(S_{mot})} \)) to be
\[
I_{\mathcal{Spt}^G(S_{mot})} = \bigcup_{T^G_V \in \mathcal{Sph}^G_S} \{ \mathcal{F}_{T^G_V}(i) | i \in I \} \quad J_{\mathcal{Spc}^G(S_{mot})} = \bigcup_{T^G_V \in \mathcal{Sph}^G_S} \{ \mathcal{F}_{T^G_V}(j) | j \in J \}.
\]
One defines the generating cofibrations \( I_{\mathcal{Spt}(S_{mot})} \) (\( I_{\mathcal{Spc}^G(S_{mot})} \)) of the level-wise projective model structure on \( \mathcal{Spt}(S_{mot}) \) (\( \mathcal{Spt}^G(S_{mot}) \)) similarly.

**Proposition 5.2.** (i) If \( A \in \mathcal{Spc}_*(S_{mot}) \) is small relative to the cofibrations (trivial cofibrations) in \( \mathcal{Spc}_*(S_{mot}) \),
then \( \mathcal{F}_{T^G_V}(A) \) is small relative \( I_{\mathcal{Spt}^G(S_{mot})} \).
A map \( f \) in \( \widetilde{\text{Spt}}^G(S_{\text{mot}}) \) is a level cofibration if and only if it has the left lifting property with respect to all maps of the form \( R_{T^U_V}^\heartsuit(g) \) where \( g \) is a trivial fibration (fibration, respectively) in \( \text{Spc}_*(S_{\text{mot}}) \). A map \( f \) in \( \widetilde{\text{Spt}}^G(S_{\text{mot}}) \) is a level trivial cofibration if and only if it has the left lifting property with respect to all maps of the form \( R_{T^U_V}^\heartsuit(g) \) where \( g \) is a fibration in \( \text{Spc}_*(S_{\text{mot}}) \).

Every map in \( I_{\text{Spt}}^a(S_{\text{mot}}) - \text{cof} \) is a level cofibration and every map in \( J_{\text{Spt}}^a(S_{\text{mot}}) - \text{cof} \) is a level trivial cofibration. (Here \( I_{\text{Spt}}^a(S_{\text{mot}}) - \text{cof} \) denotes the cofibrations generated by \( I_{\text{Spt}}^c(S_{\text{mot}}) \), respectively).

The domains of \( J_{\text{Spt}}^a(S_{\text{mot}}) \) are small relative to \( I_{\text{Spt}}^a(S_{\text{mot}}) - \text{cell} \) (\( J_{\text{Spt}}^c(S_{\text{mot}}) - \text{cell} \), respectively).

Corresponding results hold for \( \widetilde{\text{Spt}}(S_{\text{mot}}) \) and \( \text{Spt}^G(S_{\text{mot}}) \).

**Proof.** (i) The main point here is that the functor \( \mathcal{E}val_{T^U_V} \) being right adjoint to \( \mathcal{F}_{T^U_V} \) commutes with all small colimits.

(ii) Since \( R_{T^U_V}^\heartsuit \) is right adjoint to \( \mathcal{E}val_{T^U_V} \), \( f \) has the left lifting property with respect to \( R_{T^U_V}^\heartsuit(g) \) if and only if \( \mathcal{E}val_{T^U_V}(f) \) has the left-lifting property with respect to \( g \). (ii) follows readily from this observation.

(iii) Recall every object of \( \text{Spc}_*(S_{\text{mot}}) \) is assumed to be cofibrant in the injective model structure and that in the projective model structure we first replace every object \( T^U_V \in \text{Sph}^G \) functorially by a cofibrant replacement. Therefore, smashing with any \( T^U_V \) preserves cofibrations of \( \text{Spc}_*(S) \) with either the injective or the projective model structures. Therefore, every map in \( I_{\text{Spt}}^a(S_{\text{mot}}) \) is a level cofibration. (By (ii) this means \( R_{T^U_V}^\heartsuit(g) \in I_{\text{Spt}}^a(S_{\text{mot}}) - \text{cof} \) for all trivial fibrations \( g \) in \( \text{Spc}_*(S_{\text{mot}}) \). Recall every map in \( I_{\text{Spt}}^a(S) - \text{cof} \) has the left lifting property with respect to every map in \( I_{\text{Spt}}^c(S_{\text{mot}}) - \text{inj} \) and in particular with respect to every map \( R_{T^U_V}^\heartsuit(g) \), with \( g \) a trivial fibration in \( \text{Spc}_*(S_{\text{mot}}) \). Now the adjunction between \( \mathcal{E}val_{T^U_V} \) and \( R_{T^U_V}^\heartsuit \) completes the proof for \( I_{\text{Spt}}^a(S_{\text{mot}}) - \text{cof} \). The proof for \( J_{\text{Spt}}^a(S_{\text{mot}}) - \text{cof} \) and for \( \text{Spt}^G(S_{\text{mot}}) \) is similar.

(iv) follows readily in view of the adjunction between the free functor \( \mathcal{F}_{T^U_V} \) and \( \mathcal{E}val_{T^U_V} \).

**Proposition 5.3.** The projective cofibrations, the level fibrations and level equivalences define a cofibrantly generated model category structure on \( \text{Spt}^G(S_{\text{mot}}) \) with the generating cofibrations (generating trivial cofibrations) being \( I_{\text{Spt}}^a(S_{\text{mot}}) \) (\( J_{\text{Spt}}^a(S_{\text{mot}}) \), respectively). This model structure (called the level-wise projective model structure) has the following properties:

(i) Every projective cofibration (projective trivial cofibration) is a level cofibration (level trivial cofibration, respectively).

(ii) It is left-proper, right proper and is cellular.

(iii) The objects in \( \bigcup_{T^U_V \in \text{Sph}^G} \{ \mathcal{F}_{T^U_V}(\text{Sph}^G) \} \) are all finitely presented. The category \( \widetilde{\text{Spt}}^G(S_{\text{mot}}) \) is symmetric monoidal with the pairing defined in Definition 4.8(ii).

(iv) This category is locally presentable and hence is a tractable (and hence a combinatorial) model category.

(v) With the above structure, \( \text{Spt}^G(S_{\text{mot}}) \) is a symmetric monoidal model category satisfying the monoidal axiom.

(vi) Corresponding results hold for the level-wise projective model structure on \( \text{Spt}(S_{\text{mot}}) \) and \( \text{Spt}^G(S_{\text{mot}}) \) as well as on the corresponding categories defined on the étale sites.

**Proof.** First we sketch a proof showing the existence of a cofibrantly generated model category structure. The retract and two out of three axioms for level equivalences are immediate, as is the lifting axiom for a projective cofibration and a level trivial fibration. Clearly a map is a level trivial fibration if and only if it is in \( I_{\text{Spt}}^c(S_{\text{mot}}) - \text{inj} \) and a map is a projective cofibration if and only if it is in \( I_{\text{Spt}}^a(S_{\text{mot}}) - \text{cof} \). Now Proposition 5.2(iv) shows that \[ \text{Hov01 Theorem 2.1.14} \] applied to \( I_{\text{Spt}}^a(S_{\text{mot}}) \) then produces a functorial factorization of a map as the composition of a projective cofibration followed by a level trivial fibration.
By adjunction, a map is a level fibration if and only if it is in $J_{\text{Spt}^G(S_{\text{mot}})}$ -- cof is a level equivalence. Such maps have left-lifting property with respect to all level fibrations and hence with respect to all level trivial fibrations. Now Proposition 5.2(iii) shows that [Hov01 Theorem 2.1.14] applied to $J_{\text{Spt}^G(S_{\text{mot}})}$ then produces a functorial factorization of a map as the composition of a projective cofibration which is also a level equivalence followed by a level fibration.

Next we show that any projective cofibration and level equivalence $f$ is in $J_{\text{Spt}^G(S_{\text{mot}})}$ -- cof, and hence has the left lifting property with respect to level fibrations. To see this, we factor $f = pi$ where $i$ is in $J_{\text{Spt}^G(S_{\text{mot}})}$ -- cof and $p$ is in $J_{\text{Spt}^G(S_{\text{mot}})}$ -- inj. Then $p$ is a level fibration. Since $f$ and $i$ are both level equivalences, so is $p$. Therefore $p$ is a level trivial fibration and $f$ has the left lifting property with respect to $p$. This shows $f$ is a retract of $i$: see, for example, [Hov99 Lemma 1.1.9]. In particular $f$ belongs to $J_{\text{Spt}^G(S_{\text{mot}})}$ -- cof. These prove the existence of the projective model structure on $\text{Spt}^G(S_{\text{mot}})$. Clearly it is cofibrantly generated.

Statement (i) is essentially Proposition 5.2(iii). Since colimits and limits in $\text{Spt}^G(S_{\text{mot}})$ are taken levelwise, the statements in (ii) are clear. The first assertion in (iii) is clear since the objects in the subcategory $\text{Sph}^G$ are assumed to be finitely presented in $\mathcal{C}$. The assertions in (iii) on the monoidal structure follow from a theorem of Day: see [Day]. Statements (iv) and (v) follows from [DRO1 Theorems 4.2, 4.4].

5.1.5. Module spectra over a ring spectrum. Let $E^G \in \text{Spt}^G(S)$ denote a ring spectrum, and let $\tilde{U}(E^G) \in \text{Spt}^G(S)$ ($E = \tilde{U}(\tilde{U}(E^G)) \in \text{Spt}(S)$) denote the associated ring spectra. One then invokes the free $E^G$-module functor and the forgetful functor sending an $E^G$-module spectrum to its underlying spectrum along with [SSch Lemma 2.3, Theorem 4.1(2)] to obtain a corresponding cofibrantly generated model category structure on $\text{Spt}^G(S, E^G)$ ($\text{Spt}^G(S, \tilde{U}(E^G))$, $\text{Spt}^G(S, E^G)$, respectively). Observe that in this model structure the fibrations are those maps $f$ in $\text{Spt}^G(S, E^G)$ for which $f$ is a fibration in $\text{Spt}^G(S)$ and similarly for the other two model categories considered here.

5.2. The stable model structures on $\text{Spt}^G(S)$, $\text{Spt}^G(S, \tilde{U}(E^G))$, $\text{Spt}^G(S)$, $\text{Spt}^G(S, E^G)$ and on $\text{Spt}(S)$, $\text{Spt}(S, E^G)$. We proceed to define the stable model structure by applying a suitable Bousfield localization to the level-wise injective (projective)model structures considered above. This follows the approach in [Hov01 section 3]. We will explicitly consider only the case of $\text{Spt}^G(S)$, since essentially the same description applies to the categories $\text{Spt}(S)$, $\text{Spt}(S, E^G)$, $\text{Spt}^G(S, \tilde{U}(E^G))$, $\text{Spt}^G(S)$ and $\text{Spt}^G(S, E^G)$, with the only difference that while considering the last two categories $\text{Spt}^G(S)$ and $\text{Spt}^G(S, E^G)$, any reference to the category $\text{Spc}_c(S)$ will have to be replaced by the category $\text{Spc}_c(S)^G$. The corresponding model structure will be called the the injective (projective) stable model structure. (One may observe that the domains and co-domains of objects of the generating cofibrations are cofibrant, so that there is no need for a cofibrant replacement functor $Q$ as in [Hov01 section 3].)

Let $X \in \text{Spt}^G(S)$. Since $X$ is a $\text{Spc}_c(S)$-enriched functor $\text{Sph}_S^G \to \text{Spc}_c(S)$, we obtain a natural map

\begin{equation}
(\sqcup_{\alpha} T_W^S)/(, +) = \mathcal{H}om_{\text{Spc}_c(S)}(\mathcal{T}_V^S, T_W^S \wedge T_W^S) \to \mathcal{H}om_{\text{Spc}_c(S)}(X(T_V^S), \mathcal{X}(T_W^S \wedge T_W^S)),
\end{equation}

where $T_W^S(,)$ is a copy of $T_W^S$ indexed by $\alpha$, and where $\alpha$ varies over all homothety classes of $k$-linear injective and $G$-equivariant maps $V \to V \oplus W$.

Definition 5.4. ($\Omega$-spectra) A spectrum $\chi \in \text{Spt}^G(S)$ is an $\Omega$-spectrum if it is level-wise fibrant and each of the natural maps $\chi(T_V^S) \to \mathcal{H}om_{\text{Spc}_c(S)}(\mathcal{T}_V^S, \mathcal{T}_V^S \wedge \mathcal{T}_V^S)$, for each $\alpha$ as in (5.2.1) is an unstable weak-equivalence in the corresponding model structure on $\text{Spc}_c(S)$.

Let $\mathcal{F}_V^S$ denote the left-adjoint to the evaluation functor sending a spectrum $X \in \text{Spt}^G(S)$ to $\mathcal{X}(T_V^S)$: see (5.1.3). Let $C \in \text{Spc}_c(S)$ be an object that is cofibrant, and let $\chi \in \text{Spt}^G(S)$ be fibrant in the level-wise
injective (projective) model structure. Then
\[ \text{Map}(C, \chi(T^n_V)) = \text{Map}(C, \text{Eval}_{T^n_V}(\chi)) \simeq \text{Map}(F_{T^n_V}(C), \chi) \]
and
\[ \text{Map}(C, \text{Hom}_C(T^w_{S^n}, \chi(T^w_{S^n} \land T^w_W))) = \text{Map}(C, \text{Hom}_C(T^w_{S^n}, \text{Eval}_{T^w_{S^n}}(\chi))) \simeq \text{Map}(F_{T^w_{S^n}} \land T^w_{S^n}(C \land T^w_{S^n}), \chi). \]
Therefore, to convert \( \chi \) into an \( \Omega \)-spectrum, it suffices to invert the maps in \( S \), where
\[
(5.2.2) \quad S = \{ F_{T^w_{S^n}} \land T^w_{S^n}(C \land T^w_{S^n}) \to F_{T^w_{S^n}}(C) \mid C \in \text{Domains or Co-domains of } I, T_V, T_W \in \text{Sph}^G, \alpha \}
\]
corresponding to the above maps \( C \land T^w_{S^n} \to C \land \text{Hom}_{\text{Sph}^G}(T^w_{S^n}, T^w_{S^n} \land T^w_W) \) by adjunction, as \( \alpha \) varies over all homotopy classes of \( k \)-linear injective \( G \)-equivariant maps \( V \to V \oplus W \). (Here \( I \) denotes the generating cofibrations of \( \text{Spc}_*(S) \).) Similarly, for a commutative ring spectrum \( E^G \in \text{Spt}^G(S) \), one lets \( S_{\text{EC}} \) be defined using the corresponding free-functors for \( E^G \)-module-spectra. (See [Hov01, Proposition 3.2] that shows it suffices to consider the objects \( C \) that form the domains and co-domains of the generating cofibrations in \( \text{Spc}_*(S) \).)

The stable injective (projective) model structure on \( \widetilde{\text{Spt}}^G(S) \) \((\widetilde{\text{Spt}}^G(S, \text{U}(E^G)))\) is obtained by localizing the level-wise injective (projective) model structure with respect to the maps in \( S \) (\( S_{\text{EC}} \), respectively). The \( S \)-local weak-equivalences (\( S \)-local fibrations) will be referred to as the stable equivalences (stable fibrations, respectively). The cofibrations in the localized model structure are the cofibrations in the level-wise projective or injective model structures on \( \widetilde{\text{Spt}}^G(S) \) \((\widetilde{\text{Spt}}^G(S, \text{U}(E^G)))\), respectively.

**Proposition 5.5.**  
(i) The corresponding stable model structure on \( \widetilde{\text{Spt}}^G(S) \) \((\widetilde{\text{Spt}}^G(S, E^G))\) is cofibrantly generated and left proper. It is also locally presentable, and hence combinatorial (tractable).
(ii) The fibrant objects in the stable model structure on \( \widetilde{\text{Spt}}^G(S) \) \((\widetilde{\text{Spt}}^G(S, E^G))\) are the \( \Omega \)-spectra defined above.
(iii) The category \( \widetilde{\text{Spt}}^G(S) \) \((\widetilde{\text{Spt}}^G(S, E^G))\) is a symmetric monoidal model category (i.e., satisfies the pushout-product axiom: see [SSch, Definition 3.1]) in the injective stable model structures with the monoidal structure being the same in both the model structures. In the injective model structure, the unit is cofibrant and the monoid axiom (see [SSch, Definition 3.3]) is also satisfied.
(iv) The first two statements also hold for the categories \( \text{Spt}^G(S) \) \((\text{Spt}^G(S, E^G))\), while all three statements hold also for \( \text{Spt}(S) \) and \( \text{Spt}(S, E) \).

**Proof.** The proof of the first statement in (i) is entirely similar to the proof of [Hov01 Theorem 3.4] and is therefore skipped. (Since \( \text{Spc}_*(S) \) is left proper (cellular), so is the projective model structure on \( \widetilde{\text{Spt}}^G(S) \) (which is a category of \( \text{Spc}_*(S) \)-enriched functors \( \text{Sph}^G \to \text{Spc}_*(S) \) as proved above. It is shown in [Hirsch Proposition 3.4.4 and Theorem 4.1.1] that then the localization of the projective model structure is also left proper and cellular.) The fact it is locally presentable and hence combinatorial follows from the corresponding property of the unstable model categories. Our hypotheses show that the domains of the maps in \( I \) and \( J \) are cofibrant. These prove all the statements in (i). (ii) is clear.

Clearly, since the monoidal structure is the same as in the unstable setting, the above category of spectra is clearly symmetric monoidal. Now to prove it is a monoidal model category, it suffices to prove that the pushout-product axiom holds. Moreover, since an enriched functor \( X : \text{Sph}^G \to \text{Spc}_*(S) \) is cofibrant if \( X(T^n_V) \) is cofibrant in \( \text{Spc}_*(S) \) for every \( T^n_V \in \text{Sph}^G \) and every object of \( \text{Spc}_*(S) \) is assumed to be cofibrant, it follows that every such functor is cofibrant in the injective stable model structure. Thus every object is cofibrant in the injective model structure, and therefore the pushout-product axiom and the monoidal axiom are satisfied. It is clear that the unit is cofibrant in the injective model structure. This proves (iii).

\[ \square \]

6. Comparison of Several Model Categories of Spectra

We adopt the framework of [2.1.4] and [2.1.7]. Let \( \text{Spt}(S_{\text{mot}}) \) denote the (usual) category of motivic spectra defined as in Definition 1.6. Recall its objects are
\[
\mathcal{X} = \{ X_n \in \text{Spc}_*(S_{\text{mot}}), \text{ along with structure maps } T^{\land m} \land X_n \to X_{n+m} \mid n, m \in \mathbb{N} \}.
\]
Morphisms between two such objects $X$ and $Y$ are defined as compatible collection of maps $X_n \to Y_n$, $n \in \mathbb{N}$ compatible with suspensions by $T^\wedge n$, $m \in \mathbb{N}$. $\text{Spt}(S_{\text{et}})$ will denote the corresponding category of $T$-spectra defined on the big étale site of $\text{Spec} \; k$.

We proceed to relate the category $\text{Spt}(S_{\text{mot}})$ with $\tilde{\text{Spt}}(S_{\text{mot}})$ and the category $\text{Spt}(S_{\text{et}})$ with $\tilde{\text{Spt}}(S_{\text{et}})$, which were defined in Definition 4.0.3. In order to handle both at the same time, we will let $\text{Spt}(S)$ denote either $\text{Spt}(S_{\text{mot}})$ or $\text{Spt}(S_{\text{et}})$ and $\tilde{\text{Spt}}(S)$ will denote either $\tilde{\text{Spt}}(S_{\text{mot}})$ or $\tilde{\text{Spt}}(S_{\text{et}})$.

We will now identify $\mathbb{N}$ with the $\text{Spc}_r(S)$-enriched subcategory of $\text{Sph}_S$ consisting of the objects $\{T_S^{\wedge n} | n \geq 0\}$ and where

\[(6.0.1) \quad \text{Hom}_N(T^{\wedge n}, T_S^{\wedge m}) = T_S^{\wedge m}, \text{if } m > 0 \]
\[= S_0, \text{if } m = 0.\]

Next we define a faithful functor of $\text{Spc}_r(S)$-enriched categories $i : \mathbb{N} \to \text{Sph}_S$ as follows. We send each object $T^{\wedge n}$ to itself. We send the $T_S^{\wedge m}$ on the right hand side of (6.0.1) to the $T^{\wedge m}$ on the right-hand-side of (4.0.3) indexed by the imbedding of $\mathbb{A}^n$ in $\mathbb{A}^{n+m}$ as the first $n$-factors when $m > 0$, while we also send the $S_0$ on the right-hand-side of (4.0.3) to the summand $S_0$ indexed by the identity map $\mathbb{A}^n \to \mathbb{A}^n$ appearing in the right-hand-side of (4.0.3), when $m = 0$. Thus, we obtain a $\text{Spc}_r(S)$-enriched faithful functor $i : \mathbb{N} \to \text{Sph}_S$.

The functor $i^*$ defines a simplicially enriched functor $\tilde{\text{Spt}}(S) \to \text{Spt}(S)$. The functor $i^*$ admits a left adjoint, which we denote by

\[(6.0.2) \quad F : \text{Spt}(S) \to \tilde{\text{Spt}}(S).\]

One defines both a projective, as well as an injective model structure on the category $\text{Spt}(S)$, both level-wise and stably: this may be done just as in the last section and therefore we skip the details. Though for the most part we will only work with the injective model structures, the projective model structures seem helpful for comparing the model categories $\text{Spt}(S)$ and $\tilde{\text{Spt}}(S)$.

The free functor $\text{Spc}_r(S) \to \text{Spt}(S)$ left adjoint to the evaluation functor $\text{Eval}_{T^{\wedge n}} : \text{Spt}(S) \to \text{Spc}_r(S)$, sending $X \mapsto X(T^{\wedge n})$ will be denoted $F_{T^{\wedge n}}$. Let $I$ denote the set of generating cofibrations of the model category $\text{Spc}_r(S)$ provided with the projective model structure. The stable model structure on $\text{Spt}(S)$ will be obtained by inverting maps in

\[(6.0.3) \quad S_N = \{F_{T^{\wedge n}}(C \wedge T^{\wedge m}) \to F_{T^{\wedge n}}(C) \mid C \in \text{Domains or Co-domains of } I, m, n \in \mathbb{N}\}.\]

The free functor $\text{Spc}_r(S) \to \tilde{\text{Spt}}(S)$ left adjoint to the evaluation functor $\text{Eval}_{T^{\wedge n}} : \tilde{\text{Spt}}(S) \to \text{Spc}_r(S)$, sending $X \mapsto X(T^{\wedge n})$ will be denoted $F_{T^{\wedge n}}$. Let $I$ again denote the set of generating cofibrations of the model category $\text{Spc}_r(S)$ provided with the projective model structure. The stable model structure on $\tilde{\text{Spt}}(S)$ will be obtained by inverting maps in

\[(6.0.4) \quad S_N = \{F_{T^{\wedge n}}(C \wedge T^{\wedge m}) \to F_{T^{\wedge n}}(C) \mid C \in \text{Domains or Co-domains of } I, m, n \in \mathbb{N}\}.\]

We will provide both $\text{Spt}(S)$ and $\tilde{\text{Spt}}(S)$ with the projective level-wise and the corresponding projective stable model structures.

Remark 6.1. Observe that a key difference between the two categories $\text{Sph}_S$ and $\mathbb{N}$ is that the simplicially enriched hom in the category $\text{Sph}_S$ has many symmetries making $\text{Sph}_S$ a symmetric monoidal category and much bigger than the indexing category $\mathbb{N}$ for $\text{Spt}(S)$. Nevertheless we proceed to show that at the homotopy category level, the categories $\tilde{\text{Spt}}(S)$ and $\text{Spt}(S)$ are equivalent. (This should be viewed as the analogue of the equivalence between the homotopy categories of orthogonal spectra and spectra in the topological setting.)

Next we proceed to relate the categories $\tilde{\text{Spt}}^G(S)$ and $\tilde{\text{Spt}}(S)$. At this point the reader may want to consult section 4.0.3. One may then recall that the indexing category for the category $\tilde{\text{Spt}}^G(S)$ is $\text{Sph}_S^G$, which is the $\text{Spc}_r(S)$-enriched category whose objects are the Thom-spaces $\{T^{\wedge n}_S|V\}$ of all finite dimensional representations of the given linear algebraic group $G$, while the indexing category for the category $\text{Spt}(S)$ is $\{T^{\wedge n}_S | n \geq 0\}$, and where the $\text{Spc}_r(S)$-enriched hom between $T^{\wedge n}_S$ and $T^{\wedge n+m}_S$ is defined in Definition 4.0.3.
To relate the $\mathbf{Spc}_c(S)$-enriched categories, $\tilde{\mathbf{Spt}}(S)$ and $\tilde{\mathbf{Spt}}^G(S)$, one first observes that there is a forgetful functor $j : \mathbf{Sph}_G^G \to \mathbf{Sph}_G$ that sends the Thom-space, $\mathbf{T}_G^n$, of a $G$-representation $V$ to $\mathbf{T}_G^n$ but viewing $V$ as just a $k$-vector space, forgetting the $G$-action. Therefore, pull-back by $j$ defines the $\mathbf{Spc}_c(S)$-enriched functor $j^* : \tilde{\mathbf{Spt}}(S) \to \tilde{\mathbf{Spt}}^G(S)$. One defines a functor
\begin{equation} \tilde{\mathbb{P}} : \tilde{\mathbf{Spt}}^G(S) \to \tilde{\mathbf{Spt}}(S) \end{equation}
as the left-adjoint to $j^*$. For a $\mathcal{X} \in \tilde{\mathbf{Spt}}^G(S)$, $\tilde{\mathbb{P}}(\mathcal{X})$ is defined as a $\mathbf{Spc}_c(S)$-enriched left Kan-extension along the functor $j : \mathbf{Sph}_G^G \to \mathbf{Sph}_G$. Moreover, the stable projective model structure on $\tilde{\mathbf{Spt}}^G(S)$ is obtained from the level-wise projective model structure by inverting maps in the collection $\mathbf{S}$ defined in (5.1.4).

We proceed to show that the $\mathbf{Spc}_c(S)$-enriched stable model categories $\tilde{\mathbf{Spt}}^G(S)$, $\tilde{\mathbf{Spt}}(S)$ and $\mathbf{Spt}(S)$ are Quillen equivalent. The proof will compare both $\tilde{\mathbf{Spt}}^G(S)$ and $\mathbf{Spt}(S)$ with $\mathbf{Spt}(S)$. Since both these comparisons proceed similarly, we deal with them both in the following proposition.

**Proposition 6.2.**

(i) The functors $\mathbb{P}$ and $i^*$ define a Quillen adjunction between the projective stable model structures on $\mathbf{Spt}(S)$ and $\mathbf{Spt}(S)$. This is, in fact, a Quillen equivalence.

(ii) The functors $\tilde{\mathbb{P}}$ and $j^*$ define a Quillen equivalence between the stable projective model structures on $\tilde{\mathbf{Spt}}(S)$ and $\tilde{\mathbf{Spt}}^G(S)$.

(iii) The functors $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are strong monoidal functors.

**Proof.** It should be clear that $i^*(j^*)$ preserves fibrations and weak-equivalences in the level-wise projective model structures. Therefore, its left adjoint $\mathbb{P}$ ($\tilde{\mathbb{P}}$) preserves the cofibrations and trivial cofibrations in the level-wise projective model structures. It is also clear that $i^*$ sends $\Omega$-spectra in $\mathbf{Spt}(S)$ to $\Omega$-spectra in $\mathbf{Spt}(S)$ and that $j^*$ sends $\Omega$-spectra in $\tilde{\mathbf{Spt}}(S)$ to $\Omega$-spectra in $\tilde{\mathbf{Spt}}^G(S)$. Therefore, the functors $\mathbb{P}$ and $\tilde{\mathbb{P}}$ preserve stable weak-equivalences between cofibrant objects.

Next we consider the generating trivial cofibrations for the projective stable model structure on $\tilde{\mathbf{Spt}}^G(S)$. For this one starts with the generating trivial cofibrations for the level-wise projective model structure defined in (6.0.1). Then one replaces each of the maps in the set $\mathbf{S}$ defined in (5.1.4) by the corresponding simplicial mapping cylinder and adds these maps to the generating trivial cofibrations for the level-wise projective model structure. One may denote the resulting set by $\tilde{\mathbf{S}}^G$. Finally one takes the pushout-products of the maps in $\tilde{\mathbf{S}}^G$ with $\delta \Delta[n]_+ \to \Delta[n]_+, n \geq 0$. This will be the set of generating trivial cofibrations for the stable projective model structure on $\tilde{\mathbf{Spt}}^G(S)$.

By replacing the set $\mathbf{S}$ with $\tilde{\mathbf{S}}$ defined in (6.0.1) ($\mathbf{S}_N$ defined in (6.0.3)) and the generating trivial cofibrations for the level-wise projective model structure with the corresponding generating trivial cofibrations for the level-wise projective model structure for the category $\tilde{\mathbf{Spt}}(S)$ ($\mathbf{Spt}(S)$, respectively), one obtains the generating trivial cofibrations for the category $\mathbf{Spt}(S)$ ($\mathbf{Spt}(S)$, respectively).

Next, the adjunction between the free functors and the evaluation functors provides the identification:
\begin{equation} \mathbb{P}(F_{\mathbf{T}_G^n}) = F_{\mathbf{T}_G^n} \text{ and } \tilde{\mathbb{P}}(F_{\mathbf{T}_G^n}) = F_{j^*(\mathbf{T}_G^n)}. \end{equation}
(This follows readily from the identifications $\text{Eval}_{\mathbf{T}_G^n}(i^*(\mathcal{X})) = \text{Eval}_{\mathbf{T}_G^n}(\mathcal{X})$ and $\text{Eval}_{\tilde{\mathbf{Spt}}^G}(j^*(\mathcal{X})) = \text{Eval}_{\mathbf{Spt}}(\mathcal{X}).$) Therefore, it follows that $\mathbb{P}$ ($\tilde{\mathbb{P}}$) sends the generating trivial cofibrations in the projective stable model structure on $\mathbf{Spt}(S)$ ($\mathbf{Spt}^G(S)$) to the generating trivial cofibrations in the projective stable model structure on $\mathbf{Spt}(S)$ ($\mathbf{Spt}(S)$, respectively). Since the functor $\mathbb{P}$ ($\tilde{\mathbb{P}}$) also preserves pushouts and filtered colimits, it follows that it preserves trivial cofibrations in the stable projective model structure. Since the cofibrations in the projective stable model structure are the same as in the projective level-wise model structure, it follows that $\mathbb{P}$ ($\tilde{\mathbb{P}}$) also preserves these, thereby proving that the functors $(\mathbb{P}, i^*)$ ($\tilde{\mathbb{P}}, j^*$)) are Quillen equivalent, which may be proven in the usual manner. 

---

\footnote{We skip the proof that the injective and projective stable model structures appearing below are Quillen equivalent, which may be proven in the usual manner.}
define a Quillen adjunction of the projective stable model structures on \( \text{Spt}(S) \) and \( \tilde{\text{Spt}}(S) \) (\( \text{Spt}^G(S) \) and \( \text{Spt}(S) \), respectively).

Next observe that the functor \( i^* \) \((j^*)\) being a right Quillen functor preserves trivial fibrations in the stable model structure, and therefore, (by Ken Brown’s lemma: see [Hov99 Lemma 1.1.12]), it preserves all stable weak-equivalences between stably fibrant objects. In fact a stable weak-equivalence between stably fibrant objects is a level-wise weak-equivalence and \( i^* \) \((j^*)\) clearly preserves these. Next we already saw that \( i^* \) \((j^*)\) preserves \( \Omega \)-spectra and therefore all stably fibrant objects. Therefore, suppose the map \( V \) evaluating a spectrum at each object \( T \) as one varies over the generating trivial cofibrations. Since the above pushout and the colimit are taken after objects is a level-wise weak-equivalence and weak-equivalences between stably fibrant objects. In fact a stable weak-equivalence between stably fibrant model structure, and therefore, (by Ken Brown’s lemma: see [Hov99, Lemma 1.1.12]), it preserves all stable fibrant objects.

One may see the first by evaluating both sides at \( T \) as one varies over the generating trivial cofibrations. Since the above pushout and the colimit are taken after \( \text{Sph}_S \) are also just finite dimensional \( k \)-vector spaces (i.e. without any \( G \)-action), it follows that \( f \) itself is a level-wise weak-equivalence of spectra and therefore also a stable weak-equivalence in \( \text{Spt}(S) \). Stated another way, this shows that the functor \( i^* \) both detects and preserves stable weak-equivalences between fibrant objects. An entirely similar argument proves that \( j^* \) both preserves and detects stable weak-equivalences between fibrant objects.

Next we make the following observation:

\[
(6.0.7) \quad i^*(F_{T_V}) = F_{T_{S^n}}, \quad \text{where} \quad n = \text{dim}(V) \quad \text{and} \quad j^*(F_{j(T_V)}) = F_{T_{S^n}}.
\]

One may see the first by evaluating both sides at \( T \) for every \( n \in \mathbb{N} \) and the second by evaluating both sides at \( T_{S^n} \in \text{Sph}_S \).

The next step is to show the following holds: let \( Q \) denote the fibrant replacement functor in the projective stable model category structures on any one of the model categories \( \text{Spt}^G_S \), \( \tilde{\text{Spt}}(S) \) and \( \text{Spt}(S) \). Then the functors \( i^* \) and \( j^* \) strictly commute with \( Q \) in the sense

\[
(6.0.8) \quad Q \circ i^* = i^* \circ Q \quad \text{and} \quad Q \circ j^* = j^* \circ Q.
\]

We will only provide a proof for the first equality, since the second equality may be proved in a similar manner. To see this, one needs to recall how a functorial fibrant replacement is constructed making use of the small object argument: see [Hov99, Theorem 2.1.14]. We will consider this for an object \( X \in \text{Spt}(S) \). It is defined as the transfinite colimit of a filtered direct system of spectra \( X_n \in \text{Spt}(S) \), starting with \( X_0 = X \).

In order to obtain \( X_{n+1} \) from \( X_n \), we consider all commutative squares of the form

\[
\begin{array}{ccc}
A_n & \longrightarrow & X_n \\
\downarrow & & \downarrow \\
B_n & \longrightarrow & *
\end{array}
\]

with \( A_n \to B_n \) one of the generating trivial cofibrations in the projective stable model structure. Then we let \( X_{n+1} \) be defined as the corresponding pushout, after having replaced \( A_n \to B_n \) by the sum of all such maps as one varies over the generating trivial cofibrations. Since the above pushout and the colimit are taken after evaluating a spectrum at each object \( T_V \), it should be clear that the functor \( i^* \) commutes with such colimits and pushouts. Moreover, (6.0.7) shows that the functor \( i^* \) sends the generating trivial cofibrations for the stable projective model structure on \( \text{Spt}(S) \) to the generating trivial cofibrations of the stable projective model structure on \( \text{Spt} \) and that every generating trivial cofibration in this model structure on \( \text{Spt}(S) \) may be obtained by applying the functor \( i^* \) to a generating trivial cofibration in the above model structure on \( \text{Spt}(S) \).

Finally, we now observe from [HSS, Lemma 4.1.7], that it suffices to prove that for any object \( X \in \text{Spt}(S) \) \((Y \in \tilde{\text{Spt}}^G(S) \)\), which is cofibrant in the projective stable model structure there, the composite map \( X \to i^*F(X) \to i^*Q(\mathcal{P}(X)) \) \((Y \to j^*Q(\mathcal{P}(Y)) \to j^*Q(\mathcal{P}(\mathcal{P}(Y))) \) is a stable weak-equivalence. In view of (6.0.8), we obtain the identification \( i^*Q(\mathcal{P}(X)) = Q(i^*(\mathcal{P}(X))) = Q(j^*(\mathcal{P}(Y))) \), respectively. Clearly the map \( i^*\mathcal{P}(X) \to Q(i^*(\mathcal{P}(X))) \) \((j^*\mathcal{P}(Y) \to Q(j^*(\mathcal{P}(Y))) \) is a stable weak-equivalence, since \( Q(i^*(\mathcal{P}(X))) \) \((Q(j^*(\mathcal{P}(Y))) \) is a stably fibrant replacement of \( i^*(\mathcal{P}(X)) \) \((j^*(\mathcal{P}(Y))) \), respectively.)
Next one recalls how a functorial cofibrant replacement is constructed making use of the small object argument: see [Hov99, Theorem 2.1.14]. We will consider this for an object $\mathcal{X} \in \text{Spt}(S)$. It is defined as the transfinite colimit of a filtered direct system of spectra $\mathcal{X}_\alpha \in \text{Spt}(S)$, starting with $\mathcal{X}_0 = *$. In order to obtain $\mathcal{X}_{\alpha+1}$ from $\mathcal{X}_\alpha$, we consider all commutative squares of the form

$$
\begin{array}{ccc}
A_{\alpha} & \rightarrow & X_{\alpha} \\
\downarrow & & \downarrow \\
B_{\alpha} & \rightarrow & X
\end{array}
$$

with $A_{\alpha} \rightarrow B_{\alpha}$ one of the generating cofibrations in the projective stable model structure. Then we let $\mathcal{X}_{\alpha+1}$ be defined as the corresponding pushout, after having replaced $A_{\alpha} \rightarrow B_{\alpha}$ by the sum of all such maps as one varies over the generating cofibrations. Since the above pushout and the colimit are taken after evaluating a spectrum at each object $T^{\otimes n}$, it should be clear that the functor $i^*$ commutes with such colimits and pushouts. One may also see that the functor $P$ (being left-adjoint to $i^*$) commutes with such colimits.

Moreover the map $\mathcal{X} \leftarrow c\mathcal{X}$ is a cofibrant replacement of $\mathcal{X}$. Since $P$ is a left Quillen functor of the stable projective model structures, it clearly preserves stable weak-equivalences between cofibrant objects. Moreover, all the maps in the composition

$$(6.0.9) \quad \mathcal{X} \leftarrow c\mathcal{X} \rightarrow i^*(P(c\mathcal{X}))$$

are stable weak-equivalence, when $c\mathcal{X}$ is a cofibrant replacement of $\mathcal{X}$. Since $P$ is a left Quillen functor of the stable projective model structures, it clearly preserves stable weak-equivalences between cofibrant objects. Therefore, the conclusion from the above discussion is that both the maps

$$(6.0.10) \quad i^*(P(c\mathcal{X})) \rightarrow Q(i^*(P(c\mathcal{X}))) \simeq i^*(Q(P(c\mathcal{X}))) \rightarrow i^*(Q(P(\mathcal{X})))$$

are weak-equivalences. The first map is one since $Q$ is the fibrant replacement functor considered above. The weak-equivalence after that comes from the fact that the functor $Q$ commutes with the functor $i^*$ as proven in (6.0.8). The fact that last map is a weak-equivalence comes from the fact that the functor $i^*$ preserves stable weak-equivalences between stably fibrant objects. Therefore, the commutative diagram

$$
\begin{array}{ccc}
c\mathcal{X} & \xrightarrow{\simeq} & i^*P(c\mathcal{X}) \\
\downarrow & & \downarrow \\
c\mathcal{X} & \xrightarrow{\simeq} & i^*Q(\mathcal{X})
\end{array}
$$

shows that composition of the maps in the bottom row is a weak-equivalence for $\mathcal{X}$ stably cofibrant in $\text{Spt}(S)$. One may similarly prove that composition of the maps in the diagram

$$(6.0.11) \quad \mathcal{Y} \rightarrow j^*\tilde{P}(\mathcal{Y}) \rightarrow Q(j^*\tilde{P}(\mathcal{Y})) \simeq j^*Q(\tilde{P}(\mathcal{Y})) \rightarrow j^*Q(\tilde{P}(\mathcal{Y}))$$

is a stable weak-equivalence for $\mathcal{Y} \in \text{Spt}(S)$ which is stably cofibrant.

These complete the proof of the first two statements. Observe that both the functors $P$ and $\tilde{P}$ are left-Kan extensions and therefore, commute with the smash-products of spectra, which are also left-Kan extensions. This completes the proof of the proposition. □

Given a commutative ring spectrum $\mathcal{E}^G \in \text{Spt}^G$, we let $\mathcal{E} = i^*(\tilde{P})(\mathcal{E}^G)$, which is a commutative ring spectrum in $\text{Spt}(S)$. For example, the equivariant sphere spectrum $S^G$ provides $S = i^*(\tilde{P})(U(S^G))$, the usual sphere spectrum. Then one readily proves the existence of a Quillen equivalence between the stable model categories $\text{Spt}^G(S, \tilde{U}(\mathcal{E}^G))$ and $\text{Spt}(S, \tilde{P})(\mathcal{E}^G))$ as well as between the stable model categories $\text{Spt}(S, \tilde{P})(\mathcal{E}^G))$ and $\text{Spt}(S, \mathcal{E})$, just as in Proposition 6.72.
7. **Spanier-Whitehead duality in the motivic and étale setting**

We begin with the following result on the Spanier-Whitehead dual in the motivic and étale setting.

**Theorem 7.1.** Let \( k \) denote a perfect field of characteristic \( p \geq 0 \) and let \( X \) denote a smooth quasi-projective scheme over \( k \). Let \( \ell \) denote a fixed prime different from \( \text{char}(k) \) and let \( \mathcal{E} \) denote a commutative motivic ring spectrum which is \( \mathbb{Z}/(\ell) \)-local. Then \( \mathcal{E} \wedge X_+ \) is dualizable in the category \( \text{Spt}(k_{\text{mot}}, \mathcal{E}) \) of module spectra over \( \mathcal{E} \) with the same conclusion holding with no conditions on the spectrum \( \mathcal{E} \) if \( X \) is projective and smooth. In particular, this holds for ring spectra \( \mathcal{E} \) of the form \( K \wedge \mathbb{H}(\mathbb{Z}/\ell^n) \), where \( K \) is a commutative motivic ring spectrum, \( \ell \) is a prime different from \( p \) and \( n \geq 1 \). Here \( \mathbb{H}(\mathbb{Z}/\ell^n) \) denotes the usual \( \mathbb{Z}/\ell^n \)-motivic Eilenberg-Maclane spectrum, and \( \wedge \) is the derived smash product.

**Proof.** The proof makes strong use of Gabber’s refined alterations. Though the arguments below are now rather well-known (see, for example, [K] or [HKO, 2.5]), it is necessary for us to sketch the relevant arguments in some detail, so as to show that they indeed carry through under étale realization and change of base fields.

We will give two somewhat different proofs of this result, one of which holds only when \( \mathcal{E} \) admits weak traces in the sense of [K] and the other holds more generally making use of [Ri13]. Next one may observe that to prove the spectrum \( \mathcal{E} \wedge X_+ \) is dualizable in \( \text{Spt}(k_{\text{mot}}, \mathcal{E}) \), it suffices to prove that the natural maps

\[
(7.0.1) \quad \eta_{\mathcal{E}}^X : P \wedge L \mathcal{E} \text{Hom}_\mathcal{E}(\mathcal{E} \wedge X_+, \mathcal{E}) \to \text{Hom}_\mathcal{E}(\mathcal{E} \wedge X_+, P), \quad \mathcal{E} \wedge X_+ \to D_{\mathcal{E}}(D_{\mathcal{E}}(\mathcal{E} \wedge X_+))
\]

are weak-equivalences for every \( \mathcal{E} \)-module spectrum \( P \).

In case \( X \) is projective and smooth, it is well-known (see the remarks following Appendix A: Definition 9.8) that the Thom-space of the virtual normal bundle (defined as in Appendix A: Definition 9.8) over \( X \) desuspended a finite number of times is a (Spanier-Whitehead) dual of \( \Sigma^n X_+ \). Therefore, the above Thom-space de-suspended a finite number of times and smashed with \( \mathcal{E} \) will be a (Spanier-Whitehead) dual of \( \mathcal{E} \wedge X_+ \) in the category of \( \mathcal{E} \)-module spectra.

Recall \( \mathcal{SH}(k, \mathcal{E}) \) denote the motivic stable homotopy category of \( \mathcal{E} \)-module spectra, i.e., the homotopy category associated to \( \text{Spt}(k_{\text{mot}}, \mathcal{E}) \). Let \( \mathcal{SH}_d(k, \mathcal{E}) \) denote the localizing subcategory of \( \mathcal{SH}(k, \mathcal{E}) \) which is generated by the shifted \( \mathcal{E} \)-suspension spectra of smooth connected schemes of dimension \( \leq d \). In general one proceeds by ascending induction on the dimension of \( X \) to prove that the maps in (7.0.1) are weak-equivalences, the case of dimension 0 reducing to the case \( X \) is projective and smooth. When \( X \) is quasi-projective of dimension \( d \), one may assume \( j : X \to Y \) is an open immersion in a projective scheme \( Y \) and let \( f : Y' \to Y \) denote the map given by Gabber’s refined alteration so that \( X' = f^{-1}(X) \) is the complement of a divisor with strict normal crossings. Let \( U \subseteq X \) denote the open subscheme over which \( f \) restricts to an \( f_{\text{ps}} \ell \)-cover \( g : V = f^{-1}(U) \to U \).

Since \( Y' \) is smooth and projective, \( \mathcal{E} \wedge Y' \) is dualizable in the category of \( \mathcal{E} \)-module spectra. One may next observe that if

\[
(7.0.2) \quad A' \to A \to A'' \to A[1] \text{ is a stable cofiber sequence in } \text{Spt}(k_{\text{mot}}, \mathcal{E}) \text{ and if two of the three terms } A', A \text{ and } A'' \text{ are dualizable in } \text{Spt}(k_{\text{mot}}, \mathcal{E}), \text{ so is the third term }.
\]

Therefore, by homotopy-purity, induction on the number of irreducible components of \( Y' - X' \) one observes that \( \mathcal{E} \wedge X'_+ \) is also dualizable in the same category. Now one considers the stable cofiber sequences:

\[
(7.0.3) \quad \mathcal{E} \wedge V_+ \to \mathcal{E} \wedge X'_+ \to \mathcal{E} \wedge X'/V, \quad \mathcal{E} \wedge U_+ \to \mathcal{E} \wedge X_+ \to \mathcal{E} \wedge X/U.
\]

By an argument as in [RO Lemma 66] (see also [HKO, 2.5]), both \( \mathcal{E} \wedge X'/V \) and \( \mathcal{E} \wedge X/U \) belong to \( \mathcal{SH}_{d-1}(k, \mathcal{E}) \). We will provide some details on this argument, for the convenience of the reader. In case the complement \( Z = X' - V \) is also smooth, the homotopy purity Theorem [MV] Theorem 3.2.33 shows that \( X'/V \) is weakly equivalent to the Thom-space of the normal bundle \( N \) associated to the closed immersion \( Z \to X' \). Ascending induction on the number of open sets in a Zariski open covering over which the normal bundle \( N \) trivializes reduces it to the case when \( N \) is trivial. In general, this case the conclusion is clear. In general, since the base field is assumed to be perfect, one can stratify \( Z \) by a finite number of locally closed subschemes.
that are smooth. This gives rise to a sequence of motivic spaces filtering $X'/V$, so that the homotopy cofiber of two successive terms will be of the form considered earlier. An entirely similar argument applies to $X/U$.

Therefore, by the induction hypotheses, both the maps $\eta_{X'/V}$ and $\eta_{X/U}$ are weak-equivalences, where $\eta_{X'/V}$ ($\eta_{X/U}$) denotes the map corresponding to $\eta_X$ in (7.0.1) when $X$ is replaced by $X'/V$ ($X/U$, respectively). It follows therefore, by (7.0.2), that the map $\eta_{X}'$ is also a weak-equivalence. Now we make the key observation proven below that $\mathcal{E} \wedge U_+$ is a retract of $\mathcal{E} \wedge V_+$ at least when $U$ is a sufficiently small Zariski open subscheme and that, therefore, the map $\eta_{X}'$ is also a weak-equivalence. Now the second stable cofiber sequence in (7.0.3) together with another application of (7.0.2) proves that the map $\eta_{X}'$ is also a weak-equivalence. One may prove the second map in (7.0.4) is a weak-equivalence by a similar argument.

It follows straight from the definition that the spectra $K_{\mathcal{S}_k}^L \mathbb{H}(Z/\ell^n)$ and $\mathbb{H}(Z/\ell^n)$ are $Z(\ell)$-local. (Observe also that $\mathbb{H}(Z/\ell^n)$ admits weak-traces and that $K_{\mathcal{S}_k}^L \mathbb{H}(Z/\ell^n)$ admits weak traces when $K$ admits weak-traces.)

Lemma 7.2. (i) Let $V$, $U$ denote two smooth schemes over $k$ and let $g : V \to U$ denote an fpst-$\ell'$-cover, where $\ell$ is a fixed prime different from $\text{char}(k)$. If $E$ is a commutative ring spectrum which is $Z(\ell)$-local and which admits weak traces as in $[K]$, then the map $\text{id}_E \wedge g_+ : E \wedge V_+ \to E \wedge U_+$ has a section.

(ii) More generally the same conclusion holds if $U$ is a sufficiently small Zariski open subscheme and for any commutative motivic ring spectrum $E$ that is $Z(\ell)$-local.

Proof. (i) The first observation is that it suffices to show the induced natural transformation:

$$[E \wedge U_+, \cdot]_{\mathcal{E}_k^L} [E \wedge V_+, \cdot]$$

has a splitting, where $[K, L]$ denotes homotopy classes of maps in $\text{Spt}(k_{\text{mot}}, \mathcal{E})$, with $K$ cofibrant and $L$ fibrant. Denoting the structure map $U \to \text{Spec} k$ by $a$, one may identify $[E \wedge U_+, F] ([E \wedge V_+, F])$ with $[E, R_\alpha a^*(F)] ([E, R_\alpha R_\gamma g^* a^*(F)])$, respectively for any fibrant $\mathcal{E}$-module spectrum $F$. Since $E$ has a structure of traces, so does $F$. Therefore, the natural map $R_\alpha a^*(F) \to R_\alpha R_\gamma g^* a^*(F)$ has a splitting provided by the map $d^{-1}\text{Tr}(g)$, where $d$ is the degree of the map $g$ and $\text{Tr}(g)$ denotes the trace associated to $g$.

(ii) The proof of (ii) is essentially worked out in [11].

We proceed to show that the notion of dualizability is preserved by various standard operations, like change of base fields, or change of sites. Recall that we have already assumed the base scheme is a perfect field $k$ satisfying the hypothesis (1.0.1). Let $\bar{k}$ denote an algebraic closure of $k$. Then we obtain the following functors (which in fact denote the corresponding left-derived functors):

$$\epsilon^* : \text{Spt}(k_{\text{mot}}) \to \text{Spt}(\bar{k}_{\text{et}}), \epsilon^* : \text{Spt}(\bar{k}_{\text{mot}}) \to \text{Spt}(\bar{k}_{\text{et}}) \text{ and } \eta^* : \text{Spt}(k_{\text{et}}) \to \text{Spt}(\bar{k}_{\text{et}}).$$

Since étale cohomology is well-behaved only with torsion coefficients prime to the characteristic, one will need to also consider the functors $\theta : \text{Spt}(\bar{k}_{\text{et}}) \to \text{Spt}(\bar{k}_{\text{et}})$ sending commutative ring spectra $\mathcal{E}$ to $\mathcal{E} \wedge_{\mathcal{S}_k^L} \mathbb{H}(Z/\ell^n)$ where $\mathbb{H}(Z/\ell^n)$ denotes the mod-$\ell^n$ Eilenberg-Maclane spectrum in $\text{Spt}(\bar{k}_{\text{et}}, \mathcal{E})$, with $\ell$ a fixed prime different from $\text{char}(k)$. Again, if $\ell$ is a fixed prime different from $\text{char}(k)$, and $\mathcal{E}$ is a commutative ring spectrum in $\text{Spt}(k_{\text{et}}, \mathcal{E})$, we will also consider the functor sending spectra $M \in \text{Spt}(k_{\text{et}}, \mathcal{E})$ to $M \wedge_\mathcal{E} \mathcal{E}(\ell^n)$: we will denote this functor by $\phi_\mathcal{E}$. We will adopt the convention that the above functors in fact denote their corresponding left derived functors.

Proposition 7.3. Let $\ell$ denote a fixed prime different from $\text{char}(k)$, where $k$ is assumed to be a perfect field satisfying the hypothesis (1.0.1). Let $n$ denote a positive integer. If $\mathcal{E}$ is a commutative motivic ring spectrum so that it is $\ell$-primary torsion as in Definition (7.1), then the functors $\epsilon^*, \epsilon^*, \eta^*$ send the dualizable objects of the form $\mathcal{E} \wedge X_+$ appearing in Theorem (7.1) to dualizable objects.

The same conclusion holds for the functors $\theta$ and $\phi_\mathcal{E}$ if $\mathcal{E}$ is a motivic ring spectrum that is $\ell$-complete.
Proof. One may make use of the fact that the base field is perfect to see that base-change to the algebraic closure of the base field sends projective smooth schemes to projective smooth schemes and preserves strict normal crossings divisors.

Observe that the functor $\epsilon^*$ sends motivic spectra which are $\ell$-primary torsion for a fixed prime different from $\text{char}(k)$ to étale spectra which are $\ell$-primary torsion and preserves all split maps. It also sends (motivic) spectra with traces to spectra with traces. Therefore, the same argument making use of the stable cofiber sequences in (1.0.3) carries over to prove that $\epsilon^*$ and $\bar{\epsilon}^*$ send dualizable objects in Theorem 7.1 to dualizable objects, when the spectrum $\mathcal{E}$ is $\ell$-primary torsion. One may prove similarly that the functor $\eta^*$ sends dualizable objects appearing in Theorem 7.1 to dualizable objects. The conclusion that the functors $\theta$ and $\phi_\ell$ send dualizable objects to dualizable objects should be straight-forward. This completes the proof of the Proposition. \hfill \square

8. Construction of the transfer

In this section, we proceed to obtain transfer maps for torsors for linear algebraic groups, i.e., when $p : E \to B$ is a smooth map of smooth quasi-projective schemes that is a $G$-torsor for a linear algebraic group $G$. We adopt the framework discussed in Theorem 1.3.

Next, recall the definition of weakly monoidal functors from Definition 4.12. Let $\text{Spt}'$ and $\text{Spt}$ denote two symmetric monoidal stable model categories. We say a weakly monoidal functor $F : \text{Spt}' \to \text{Spt}$ is a monoidal functor if the map $\mu : F(X') \otimes F(Y') \to F(X' \otimes Y')$ in (4.0.12) is a weak-equivalence for all objects $X'$ and $Y'$ in $\text{Spt}'$ and if the given map $\epsilon : A \to F(A')$ is a weak-equivalence, where $A'$ (A) denotes the unit of the category $\text{Spt}'$ (Spt, respectively).

Proposition 8.1. (See [DP] 2.2 Theorem and [DP] 2.4 Corollary.) Assume that the functor $F$ is monoidal, induces a functor of the corresponding homotopy categories, that the object $X' \in \text{Spt}'$ is dualizable, and $A'$ is the unit of $\text{Spt}'$. Then $F(X') \in \text{Spt}$ is dualizable and $F(D(X')) \simeq \text{Hom}(F(X'), F(A'))$, where $\text{Hom}$ again denotes the derived (internal) $\text{Hom}$ in Spt.

At this point we make implicit use of the chain of equivalences of stable model category structures on $\text{Spt}^G(k_{\text{mot}})$, $\text{Spt}(k_{\text{mot}})$ and $\text{Spt}(k_{\text{mot}})$ proven in Proposition 6.2 which are in fact given by monoidal functors. Therefore, Proposition 8.1 shows that the theory of Spanier-Whitehead duality currently known in $\text{Spt}(k_{\text{mot}})$ carries over to $\text{Spt}^G(k_{\text{mot}})$.

8.1. Construction of the transfer in a general framework. Assume that $\text{Spt}$ denotes a symmetric monoidal stable model category where the monoidal structure is denoted $\wedge$ and where the unit of the monoidal structure is denoted $\mathbb{S}$. We will further assume that the object $X$ in Spt comes equipped with a diagonal map $\Delta : X \to X \wedge X$ and a co-unit map $\kappa : X \to \mathbb{S}$ so that $\Delta$ provides $X$ with the structure of a co-algebra: see [DP] section 5.

Definition 8.2. (i) Now one may define the trace associated to any self-map $f : X \to X$ of an object that is dualizable as follows. Recall that we have denoted the evaluation map $D\!X \wedge X \to \mathbb{S}$ by $e$. The dual of this map is the co-evaluation map $\epsilon : \mathbb{S} \to X \wedge D\!X$. Now the trace of $f$ (denoted $\tau_X(f)$ or often just $\tau(f)$) is the composition (in $SH$)

$$\mathbb{S} \xrightarrow{\epsilon} X \wedge D\!X \xrightarrow{\epsilon \otimes \text{id}} D\!X \wedge X \wedge \mathbb{S} \xrightarrow{\text{id} \otimes \Delta} D\!X \wedge \mathbb{S} \xrightarrow{\epsilon} \mathbb{S},$$

where $\tau$ is the map interchanging the two factors.

(ii) Then, assuming $X$ comes equipped with a diagonal map $\Delta : X \to X \wedge X$ so that $X$ has the structure of a co-algebra, we define the transfer as the composition in $SH$ :

$$\text{tr}(f) : \mathbb{S} \xrightarrow{\epsilon} X \wedge D\!X \xrightarrow{\epsilon \otimes \text{id}} D\!X \wedge X \wedge \mathbb{S} \xrightarrow{\text{id} \otimes \Delta} D\!X \wedge X \wedge \mathbb{S} \xrightarrow{\epsilon \otimes \delta} X \wedge X \otimes \mathbb{S} \wedge X = X'$$

(iii) If $Y \in \text{Spt}$ is another object, we will also consider the following variant $\text{tr}(f_Y) = Y \wedge \text{tr}(f) : Y \wedge \mathbb{S} \to Y \wedge X$. 

The composition \(DX \wedge X^{id \Delta} DX \wedge X^\Delta \to DX \wedge X \wedge X\) will often be denoted \(id \wedge \Delta(f)\).

Assume in addition to the above situation that \(A\) denote a commutative ring object in the symmetric monoidal model category \(\text{Spt}\). Let \(\text{Spt}_A\) denote the subcategory of \(\text{Spt}\) consisting of objects \(M\) provided with an associative and commutative pairing \(A \wedge M \to M;\) see, for example, \([SSch]\). The category \(\text{Spt}_A\) will be provided with the monoidal structure defined by \(M \wedge_A N\) defined as the co-equalizer: \(M \wedge A \wedge N \to M \wedge N\), where the two arrows denote the multiplication by \(A\) on the right on \(M\) and on the left on \(N\).

We will denote this category by \(\text{Spt}_A\). This will be provided with the model structure defined in \([SSch]\) Theorem 4.1] so that it is also a stable symmetric monoidal model category.

**Proposition 8.3.** Assume the above situation.

(i) Then the functor \(\text{Spt} \to \text{Spt}_A\), given by \(X \mapsto A \wedge X\) is a monoidal functor.

(ii) Let \(X \in \text{Spt}\) and let \(f : X \to X\) denote any map in \(\text{Spt}\). Then \(A \wedge \tau_X(f) = \tau_{A \wedge X}(id \wedge f)\).

**Proof.** (i) is clear. To see (ii), first observe that \(A \wedge D(X) = A \wedge RH\text{om}(X, S) \simeq RH\text{om}_A(A \wedge X, A) = D_A(A \wedge X)\), where \(RH\text{om}\,(RH\text{om}_A)\) denotes the derived internal \(\text{Hom}\) in \(\text{Spt}\) (\(\text{Spt}_A\), respectively). Now the definition of the trace \(\tau_{A \wedge X}(id \wedge f)\) shows that it identifies with \(A \wedge \tau_X(f)\). This completes the proof of (ii). \(\square\)

8.2. **The G-equivalent pre-transfer.** Let \(X\) denote a smooth quasi-projective scheme, or more generally an unpointed simplicial presheaf defined on \(\text{Sm}_k\), subject to the requirement that \(\Sigma^*_T X_+ \in \text{Spt}(k_{\text{mot}})\) be dualizable. Corresponding results will hold if \(E \times X_+\) is dualizable in \(\text{Spt}(k_{\text{mot}}, E)\) where \(E^G \in \text{Spt}_{k_{\text{mot}}}^{G}(k_{\text{mot}})\) is a commutative ring spectrum, with \(E = i^*(\mathcal{P}(\mathcal{E}^G)) \in \text{Spt}(k_{\text{mot}})\) denoting the corresponding non-equivariant ring spectrum. Then the equivariant sphere spectrum \(S^G\) will be replaced by \(E^G\) everywhere in the construction discussed below.

We will further assume \(X\) is provided with an action by the linear algebraic group \(G\). Associated to any \(G\)-equivariant self-map \(f : X \to X\) over the base field \(k\), we will presently define a pre-transfer map following roughly the definition given in \([DP]\). The main improvement we need is to make all the maps that enter into the definition of the pre-transfer \(G\)-equivariant. We will define the \(G\)-equivariant pre-transfer as the composition of a sequence of maps in \(\text{Spt}^{G}_{(k_{\text{mot}})}\) which are all \(G\)-equivariant. Throughout the following definition we will often abbreviate \(S^G \times X_+\) to just \(X_+\) and taking the dual will mean as in \([4.0.16]\).

**Definition 8.4.** (i) Accordingly we proceed to first define a \(G\)-equivariant co-evaluation map, where the source is the \(G\)-sphere spectrum \(S^G\). We start with the evaluation map \(e : D(X_+) \wedge X_+ \to S^G\). On taking its dual in \(\widetilde{\text{Spt}}^{G}_{(k_{\text{mot}})}\), we obtain the map

\[
(8.2.1) \quad e : S^G \simeq D(S^G) \to D(D(X_+) \wedge X_+) \to D(D(X_+) \wedge X_+) \to D(D(X_+) \wedge X_+) \to D(D(X_+) \wedge X_+) \to D(D(X_+) \wedge X_+)
\]

The above composition will be the co-evaluation map \(c\). Observe that all the maps above are \(G\)-equivariant and the maps going in the wrong-direction are in fact weak-equivalences.

(ii) Next we consider the map:

\[
(8.2.2) \quad X_+ \wedge D(X_+) \xrightarrow{\tau} D(X_+) \wedge X_+ \xrightarrow{id \Delta} D(X_+) \wedge X_+ \wedge X_+ \wedge X_+ \wedge X_+ \xrightarrow{id \Delta} D(X_+) \wedge X_+ \wedge X_+ \wedge X_+ \xrightarrow{id \Delta} D(X_+) \wedge X_+ \wedge X_+ \wedge X_+ \wedge X_+.
\]

Observe that the above diagram is in fact a diagram in \(\widetilde{\text{Spt}}^{G}_{(k_{\text{mot}})}\) where all the spectra and the maps are \(G\)-equivariant. \(\square\) Now we may compose the co-evaluation map in \((8.2.1)\) with the map in \((8.2.2)\) to define the \(G\)-equivariant pre-transfer, denoted \(tr^G(f)\). Therefore, this will be the following composition:

\[
(8.2.3) \quad tr^G(f_+) : S^G \simeq D(S^G) \to D(D(X_+) \wedge X_+) \to D(D(X_+) \wedge X_+) \to D(D(X_+) \wedge X_+) \to S^G \wedge X_+.
\]

(iii) Given \(Y\), which is another smooth quasi-projective scheme, or more generally an unpointed simplicial presheaf defined on \(\text{Sm}_k\), provided with an action by \(G\), we define \(tr^G(f_+ : Y \times S^G \to Y \times (S^G \wedge X_+)\) to be \(id_Y \times tr^G(f_+)\).

\(^3\)One can put in a slightly more general form of the diagonal map \(\Delta\), which will in fact be important for establishing the localization or Mayer-Vietoris properties of the pre-transfer. This is discussed in \([1P23]\) Definition 2.2.)
(iv) We define the trace, \( \tau_X(f_\mathcal{X}) \) to be the composition of the pre-transfer with the map \( S^G \wedge X_+ \to S^G \) collapsing all of \( X_+ \) to \( \text{Spec} \ k_+ \). Similarly we define \( \tau_X(f_{\mathcal{Y}+}) \) to be the composition of the pre-transfer \( tr^G(f_{\mathcal{Y}+}) \) with the map \( Y \times (S^G \wedge X_+) \to Y \times S^G \). For the most part we will suppress the superscript \( G \) and denote the above traces as \( \tau_X(f_\mathcal{X}) \) or \( \tau_X(f_{\mathcal{Y}+}) \).

(v) If \( \mathcal{E} \in \text{Spt}^G(k_{\text{mot}}) \) is a commutative ring spectrum, \( \mathcal{E} = i^*\tilde{p}(\tilde{U}(E^G)) \) is the associated ring spectrum in \( \text{Spt}^G(k_{\text{mot}}) \), and \( \mathcal{E} \times X_+ \in \text{Spt}^G(k_{\text{mot}}) \) is dualizable, one defines co-evaluation, pre-transfer and trace maps similarly by replacing \( S^G \wedge X_+ \) by \( E^G \wedge X_+ \) (\( E^G \), respectively).

The next goal is to define a transfer map that will define a wrong-way map in generalized cohomology for a \( G \)-torsor \( p : E \to B \), as well as in Borel-style equivariant generalized motivic (and étale) cohomology associated to actions of linear algebraic groups. Our approach follows closely the construction in [BG75, section 3], in spirit.

8.2.4. Convention. Let \( G \) denote a linear algebraic group. We need to carry out the construction of the transfer in two distinct contexts: (i) when the group \( G \) is special in Grothendieck’s terminology: see [Ch]. For example, \( G \) could be a \( \text{GL}_n \) for some \( n \) or a finite product of \( \text{GL}_n \)s and (ii) when \( G \) is not necessarily special. In the first case, every \( G \)-torsor is locally trivial on the Zariski (and hence the Nisnevich) topology, while in the second case \( G \)-torsors are locally trivial only in the étale topology.

In both cases, we will let \( \text{BG}^g_{m,m} \) denote the \( m \)-th degree approximation to the classifying space of the group \( G \) (its principal \( G \)-bundle, respectively) as in [MV] (see also [Tot]). These are, in general, quasi-projective smooth schemes over \( k \). It is important for us to observe that each \( \text{EG}^g_{m,m} \), with \( m \) sufficiently large has \( k \)-rational points, where \( k \) is the base field. (This will imply that \( \text{BG}^g_{m,m} \), with \( m \) sufficiently large also has \( k \)-rational points.)

Next we start with a \( G \)-torsor \( E \to B \), with both \( E \) and \( B \) smooth quasi-projective schemes over \( k \). We will further assume that \( B \) is always connected. Next, we will find affine replacements for these schemes. One may first find an affine replacement \( \tilde{B} \) for \( B \) (\( \text{BG}^g_{m,m} \) for \( \text{BG}^g_{m,m} \)) by applying the well-known construction of Jouanolou (see [Joun]) and then define \( \tilde{E} (\text{EG}^g_{m,m}) \) as the pull-back:

\[
\tilde{E} = \tilde{B} E \times_B \tilde{p} : \tilde{E} \to \tilde{B}, \quad (\text{EG}^g_{m,m} = \text{BG}^g_{m,m} \times_{\text{BG}^g_{m,m}} \text{EG}^g_{m,m}, \tilde{p} : \text{EG}^g_{m,m} \to \text{BG}^g_{m,m})
\]

\( (8.2.5) \)

\[
\pi_Y : \tilde{E} \times_Y X = \tilde{E} \times_X (Y \times X) \to \tilde{E} \times_Y Y = \tilde{E}_Y, \quad (\pi_{Y,m} : \text{EG}^g_{m,m} \times_Y (Y \times X) \to \text{EG}^g_{m,m} \times_Y Y)
\]

\[
\pi : \tilde{E}_Y = \tilde{E} \times_Y Y \to \tilde{B}, \quad (\pi_m : \tilde{E}_{m,Y} = \text{EG}^g_{m,m} \times_Y Y \to \text{BG}^g_{m,m} = \tilde{B}_m)
\]

8.3. The Borel construction applied to simplicial presheaves with \( G \)-action. We break this discussion into two cases, depending on whether the group \( G \) is special in Grothendieck’s classification (see [Ch]).

In both cases, \( \text{Spc}^G(k_{\text{mot}}) \) will denote the category of pointed \( G \)-equivariant presheaves on the big Nisnevich site of \( k \) as in Definition [2.4].

Case 1: when \( G \) is special. Recall this includes all the linear algebraic groups \( \text{GL}_n, \text{SL}_n, \text{Sp}_{2n}, n \geq 1 \). In this case, we start with the construction (i.e., the functor):

\[
\text{Spc}^G(k_{\text{mot}}) \to \text{Spc}^G(\tilde{B}), X \mapsto \tilde{E} \times_G X
\]

where the quotient construction is explained below. (If we start with an unpointed simplicial presheaf \( X \), we let \( X = X_+ \) and we will always assume that the action by \( G \) on \( X \) preserves the base point. Therefore, there is a canonical section \( \tilde{B} \to \tilde{E} \times_X X \).) Clearly this extends to a functor:

\[
\text{Spt}^G(k_{\text{mot}}) \to \text{Spt}^G(\tilde{B}), X \mapsto \tilde{E} \times_G X.
\]

where \( \text{Spt}^G(\tilde{B}) = [\text{Sph}^G, \text{Spc}^G(\tilde{B})] \).

In [8.3.1], one cannot view the product \( \tilde{E} \times X \) as a presheaf on the big Nisnevich site and take the quotient by the action of \( G \), with \( G \) again viewed as a Nisnevich presheaf: though such a quotient will be a presheaf on the big Nisnevich site, this will not be the presheaf represented by the scheme (or algebraic space) \( \tilde{E} \times_G X \), when \( X \) is a scheme. In order to get this latter presheaf, when \( G \) is special, one needs to start with a Zariski
open cover \{U_i|i\} of \widehat{B} over which \widehat{E} is trivial, and then glue together the sheaves \(U_i \times X\) making use of the gluing data provided by the torsor \(E \to B\).

A nice way to view this construction is as follows, at least when \(X\) is a Nisnevich sheaf: one needs to in fact take the quotient sheaf associated to the presheaf quotient of \(\widetilde{E} \times X\) by the \(G\)-action on the big Nisnevich site. Then this produces the right object.

Denoting by \((\widetilde{E} \times_G X)|_{U_i}\) the restriction of \(\widetilde{E}\) to \(U_i\), it is clear that \((\widetilde{E} \times_G X)|_{U_i}\) identifies with \(U_i \times X\). Therefore, it is clear that the construction \((8.3.2)\) sends a \(G\)-equivariant map \(\alpha : X \to Y\) so that \(\hat{U}(\alpha)\) is a (stable) weak-equivalence in \(\text{Spt}^G_{(k_{mot})}\) to a (stable) weak-equivalence in \(\text{Spt}^G_{(\widehat{B})}\).

Case 2: Next assume that \(G\) is not necessarily special, in which case we will assume the base field \(k\) is infinite. \((8.3.4)\) or consider the diagram:

\[
\begin{array}{ccc}
BG & \xleftarrow{p_1} & EG \times_G EG^{gm} \\
& & \\
& & \downarrow{p_2} \\
& & BG^{gm}.
\end{array}
\]

Then, one may observe that the fibers of both maps \(p_1\) and \(p_2\) over a strictly Hensel ring are acyclic: the fibers of \(p_1\) are acyclic because we have inverted \(\mathbb{A}^1\) (and therefore, \(EG^{gm}\) is acyclic), and the fibers of \(p_2\) are acyclic because they are the simplicial \(EG\). Thus \(p_1\) and \(p_2\) induce weak-equivalences of the corresponding simplicial sheaves. \(\text{(See [J22, Theorem 1.5] for a similar argument at the level of equivariant derived categories.) Let }\epsilon : \text{Sm}_{k,et} \to \text{Sm}_{k,Nis}\text{ denote the map of sites from the big étale site of }S = \text{Spec }k\text{ to the big Nisnevich site of }S.\text{ It follows therefore that one obtains the identification}\)

\[
(8.3.5) \quad R\epsilon_*(BG) \simeq \lim_{m \to \infty} BG_{et}^{gm,m}
\]

in \(\text{Spc}_*(k_{mot})\). (Here we will use the injective model structure on simplicial presheaves prior to \(\mathbb{A}^1\)-localization: see \([2.1.0]\).) In this case, the construction \((8.3.1)\) is replaced by:

\[
(8.3.6) \quad X \mapsto R\epsilon_*(\widehat{E} \times_G (\alpha \circ \epsilon^*)(X)), \quad \text{Spc}^G_{(k_{mot})} \xrightarrow{a \circ \epsilon^*} \text{Spc}^G_{(k_{et})} \xrightarrow{\widehat{E} \times_G (-)} \text{Spc}^G_{(k_{et})} \xrightarrow{R\epsilon_*} \text{Spc}_*(R\epsilon_*(\widehat{B}_{et})).
\]

Here we have adopted the following conventions: the functor \(\epsilon^*\) sends a simplicial presheaf on the big Nisnevich site \(\text{Sm}_{k,Nis}\) to a simplicial presheaf on the big étale site \(\text{Sm}_{k,et}\), and the functor \(a\) sends a simplicial presheaf on the big étale site \(\text{Sm}_{k,et}\) to its associated sheaf on the same site, and the superscript \(et\) denotes the fact we are taking quotient sheaves on the étale site. \(\text{Spc}_*(R\epsilon_*(\widehat{B}_{et}))\) denotes the category of simplicial presheaves on \(\text{Sm}_{k,Nis}\) pointed over the simplicial presheaf \(R\epsilon_*(\widehat{B}_{et})\).

Clearly this extends to a functor:

\[
(8.3.7) \quad X \mapsto R\epsilon_*(\widehat{E} \times_G (\alpha \circ \epsilon^*)(X)), \quad \text{Spt}^G_{(k_{mot})} \xrightarrow{a \circ \epsilon^*} \text{Spt}^G_{(k_{et})} \xrightarrow{R\epsilon_*} \text{Spt}^G_{(R\epsilon_*(\widehat{B}_{et}))}
\]

where \(\text{Spt}^G_{(R\epsilon_*(\widehat{B}_{et}))} = [\text{Sph}^G, \text{Spc}_*(R\epsilon_*(\widehat{B}_{et}))]\).

A main result of \([MV]\) Proposition 2.6, p. 135 is that the term on the right is weakly-equivalent to \(\epsilon_* \left( \lim_{m \to \infty} BG_{et}^{gm,m} \right) = \lim_{m \to \infty} \epsilon_*(BG_{et}^{gm,m})\).
If \( \{U_i \mid i \in I\} \) is an étale cover of \( \overline{B} \) over which \( \overline{E} \) is trivial, the same argument as above shows that

\[
(\overline{E} \times_G \mathcal{X} ((a \circ \epsilon^*)(\mathcal{X})))_{|U_i} = U_i \times (a \circ \epsilon^*)(\mathcal{X}),
\]

so that the functor \( \mathcal{X} \mapsto \overline{E} \times_G \mathcal{X} ((a \circ \epsilon^*)(\mathcal{X})) \) sends a \( G \)-equivariant map \( \alpha : \mathcal{X} \to \mathcal{Y} \) for which \( \overline{U}(\alpha) \) is a (stable) weak-equivalence in \( \text{Spt}_G(\text{k_{mot}}) \) to a (stable) weak-equivalence in \( \text{Spt}_G(\overline{B}_{et}) \). Therefore, the functor \( \mathcal{X} \mapsto \text{Re}_\epsilon(\overline{E} \times_G \mathcal{X} ((a_\text{et}(\mathcal{X}))) \) sends a \( G \)-equivariant map \( \alpha : \mathcal{X} \to \mathcal{Y} \) for which \( \overline{U}(\alpha) \) is a (stable) weak-equivalence in \( \text{Spt}_G(\text{k_{mot}}) \) to a (stable) weak-equivalence in \( \text{Spt}_G(\text{Re}_\epsilon(\overline{B}_{et})) \).

In case \( X \) is already a sheaf on the big étale site, \( (a \circ \epsilon^*)(X) = X \) and therefore, we may replace \( (a \circ \epsilon^*)(X) \) in \( \text{(8.3.6)} \) by just \( X \) in the definition of the Borel construction. (This applies to the case where \( X = X \) is a scheme.)

**Terminology 8.5.** Throughout the remainder of the paper, we will abbreviate the functor in \( \text{(8.3.6)} \) (\( \text{8.3.7} \)) by

\[
X \mapsto \text{Re}_\epsilon(\overline{E} \times_G \mathcal{X} X), \quad X \in \text{Spc}_G(\text{k_{mot}}), \quad (\mathcal{X} \mapsto \overline{E} \times_G \mathcal{X} X), \quad \mathcal{X} \in \text{Spt}_G(\text{k_{mot}}), \quad \text{respectively}.
\]

Though there is a discussion of the classifying spaces of linear algebraic groups in [MV, 4.2], it lacks a corresponding discussion on the Borel construction \( E^{gm,m}_G \times G X \) for \( X \) a smooth scheme. We complete our discussion, by providing a comparison of \( E^{gm,m}_G \times G X \) with \( \text{Re}_\epsilon(EG^{gm,m}_G \times_G X) \) when \( X \) is a smooth scheme. We first replace \( \lim_{m \to \infty} E^{gm,m}_G \times_G X \) and \( \lim_{m \to \infty} E^{gm,m}_G \times_G X \) by fibrant simplicial presheaves \( \overline{BG}_{et} \) and \( EG^{et}_G \times_G X \), so that the induced map \( EG^{et}_G \times_G X \to \overline{BG}_{et} \) is a fibration with fiber \( \hat{X} \), which is a fibrant replacement for \( X \). Let \( U_\infty = \lim_{m \to \infty} EG^{gm,m}_G \times_G X \). Now one forms the cartesian square in \( \text{Spc}_a(\text{k}_{et}) \):

\[
\begin{array}{ccc}
\text{E}(U_\infty, G)_{et} \times_G^{\mathcal{X}} \hat{X} & \longrightarrow & EG^{et}_G \times_G^{\mathcal{X}} \hat{X} \\
\downarrow & & \downarrow \\
B(U_\infty, G)_{et} & \longrightarrow & \overline{BG}_{et}.
\end{array}
\]

Here \( \text{E}(U_\infty, G)_{et} \) is the étale simplicial presheaf given in degree \( n \) by \( U_\infty^{n+1} \), and with the structure maps provided by the projections of \( U_\infty^{n} \) to the various factors \( U_\infty^n \) and by the diagonal maps \( U_\infty \to U_\infty^n \). This structure square remains a cartesian square on applying the push-forward \( \epsilon_\alpha \) to the Nisnevich site. [MV] Lemma 2.5, 4.2 shows that the resulting map in the bottom row is an isomorphism in \( \text{Spc}_a(\text{k}_{mot}) \), so that is the resulting map in the top row. Finally an argument exactly as on [MV] p. 136 shows that one obtains an identification \( \epsilon_{\alpha}(E(U_\infty,G)_{et} \times_G^{\mathcal{X}} \hat{X}) \simeq \epsilon_{\alpha}(E^{gm,m}_G \times_G \hat{X}) = \epsilon_{\alpha}(\lim_{m \to \infty} EG^{gm,m}_G \times_G \hat{X}) \).

Therefore, we obtain the identification for a smooth scheme \( X \):

\[
\begin{align*}
\text{Re}_\epsilon(\lim_{m \to \infty} EG^{gm,m}_G \times_G^{\mathcal{X}} X) & \simeq \text{Re}_\epsilon(\lim_{m \to \infty} EG^{gm,m}_G \times_G^{\mathcal{X}} X) = \epsilon_{\alpha}(EG^{et}_G \times_G^{\mathcal{X}} X) \simeq \epsilon_{\alpha}(\lim_{m \to \infty} (EG^{gm,m}_G \times_G^{\mathcal{X}} X)).
\end{align*}
\]

Finally, for convenience in the following steps, we will denote both the Borel constructions given in \( \text{(8.3.2)} \) and \( \text{(8.3.7)} \) by the notation \( \mathcal{X} \mapsto \overline{E} \times_G \mathcal{X} \). Moreover, we will denote by \( \overline{B} \), the object denoted by this symbol in \( \text{(8.2.5)} \) when \( G \) is special, and the object \( \text{Re}_\epsilon(\overline{B}_{et}) \) considered in \( \text{(8.3.6)} \) when \( G \) is not special.

### 8.4. Construction of the transfer.

Next we proceed to construct the transfer as a stable map, i.e. a map in \( \mathcal{SH}(k) \), when \( \Sigma_T^{\infty} X_+ \) is dualizable in \( \mathcal{SH}(k) \) and \( G \) is special (and a variant of this map when \( G \) is non-special):

\[
\begin{align*}
\text{tr}(f_Y) : \Sigma_T^{\infty}(\overline{E} \times G Y)_+ & \to \Sigma_T^{\infty}(\overline{E} \times G (Y \times X))_+ \quad \text{(tr}(f_Y) : (\Sigma_T^{\infty}(\overline{E}^{gm,m}_G \times_G Y)_+) \to \Sigma_T^{\infty}(\overline{E}^{gm,m}_G \times_G (Y \times X)_+). \\
\end{align*}
\]

This will be constructed as a composition of several maps in \( \text{Spt}(\text{k}_{mot}) \), with some of the maps going the wrong-way, and these wrong-way maps will all be weak-equivalences in \( \text{Spt}(\text{k}_{mot}) \). In case \( E \times X_+ \in \text{Spt}(\text{k}_{mot}, E) \) is dualizable for a commutative ring spectrum \( E^G \in \text{Spt}^G(\text{k}_{mot}) \) with \( E = i^*(\overline{P}(\overline{E}^G)) \), \( E \times X_+ \in \text{Spt}(\text{k}_{et}, E) \) is dualizable for a commutative ring spectrum \( E^{G} \in \text{Spt}^G(\text{k}_{et}) \), so that \( E \) is \( \ell \)-complete for some...
prime \( \ell \neq \text{char}(k) \), respectively) the transfer we obtain will be of the following form when \( G \) is special (and a variant of this map when \( G \) is non-special):

\[
(8.4.2) \quad \text{tr}(f_Y) : \mathcal{E} \wedge (\tilde{E} \times G Y)_+ \to \mathcal{E} \wedge (\tilde{E} \times G (Y \times X))_+ \quad (\text{tr}(f_Y) : \mathcal{E} \wedge (\tilde{E} G^{gm,m} \times G Y)_+ \to \mathcal{E} \wedge (\tilde{E} G^{gm,m} \times G (Y \times X))_+.)
\]

**Remark 8.6.** The following remarks may provide some insight and motivation to the construction of the transfer discussed in Steps 1 through 5 below. We have tried to define a transfer that depends only on the \( G \)-object \( X \) and the \( G \)-equivariant self-map \( f \) and which does not depend on any further choices. This makes it necessary to start with the \( G \)-equivariant pre-transfer as in \((8.2.3)\). As a result, we are forced to make use of the framework of the category \( \text{Spt} \ (k_{\text{mot}}) \). However, if one chooses to replace the \( G \)-equivariant sphere spectrum \( S^G \) by just the suspension spectrum of the Thom-space \( T_V \), for a fixed (but large enough) representation \( V \) of \( G \), then the use of the category \( \text{Spt} \ (k_{\text{mot}}) \) could be circumvented by just using a variant of Proposition \( 2.3 \) valid for suspension spectra. The construction of the transfer in \([BG75]\) in fact adopts this latter approach: in their framework, the co-evaluation map corresponds to a Thom-Pontrjagin collapse map associated to the Thom-space of a fixed \( G \)-representation. Such an approach does not seem to work generally in the motivic context, though it could be made to work when \( X \) denotes a projective smooth scheme, provided one makes use of the Voevodsky collapse (see Appendix A, Definition \( 9.8 \) in the place of the classical Thom-Pontrjagin collapse.

**Step 1.** As the next step in the construction of the transfer map \( \text{tr}(f_Y) \), we start with the \( G \)-equivariant pre-transfer \( \text{tr}^G(f_Y) \) in \((8.2.3)\) to obtain the stable map over \( \tilde{E}_Y \), i.e., as a composition of several maps in \( \text{Spt}^G (\tilde{E}_Y) \), where the wrong-way maps are all weak-equivalences.

\[
(8.4.3) \quad \tilde{E} \times_G (Y \times S^G)^{id \times G \text{tr}^G(f_Y)} \to \tilde{E} \times_G (Y \times (S^G \wedge X_+)).
\]

(Here we are making use of the observation that the above Borel construction preserves weak-equivalences as observed in the discussion on the Borel construction, so that we can suppress the fact that the above map is in fact a composition of several maps, some of which go the wrong-way as observed in \((8.2.3)\).) On applying the construction \( \tilde{E} \times_G (\cdot) \) with a \( G \)-equivariant ring spectrum \( \mathcal{E}^G \) (as in \((1.0.3)\)) in the place of \( S^G \), the resulting stable map takes on the form:

\[
(8.4.4) \quad \tilde{E} \times_G (Y \times \mathcal{E}^G)^{id \times G \text{tr}^G(f_Y)} \to \tilde{E} \times_G (Y \times (\mathcal{E}^G \wedge X_+)).
\]

**Remark 8.7.** The remaining steps in the construction of the transfer may be easily explained by fact that the sphere spectrum \( S^G \) and the ring spectrum \( \mathcal{E}^G \) appearing above have non-trivial actions by \( G \), so that neither the source nor the target of the maps in \((8.4.3)\) and \((8.4.4)\) will become suspension spectra of \( \tilde{E}_Y \) or \( \tilde{E} \times_G (Y \times X)_+ \) without the considerable efforts in the remaining steps. We will discuss the remaining steps in detail only for the sphere spectrum \( S^G \). This suffices, since the only other ring spectra \( \mathcal{E}^G \) we consider will be restricted to those appearing in the list in \([1.0.3]\).

**Step 2.** Next let \( V \) denote a fixed (but arbitrary) finite dimensional representation of the group \( G \)\(^5\). At this point we need to briefly consider two cases, (a) where \( G \) is special and (b) where it is not. In case (a), it should be clear that

\[
(8.4.5) \quad \tilde{E} \times_G V \text{ is a vector bundle } \xi^V \text{ on the affine scheme } \tilde{B},
\]

where the quotient construction is done as in \((3.3.1)\), that is on the Zariski site. In case (b), one considers instead:

\[
(8.4.6) \quad \tilde{E} \times_G^\ell V,
\]

where the quotient is taken on the étale topology. Apriori, this is a vector bundle that is locally trivial on the étale topology of \( \tilde{B} \). But any such vector bundle corresponds to a \( GL_n \)-torsor on the étale topology of \( \tilde{B} \).
\(\tilde{B}\), and hence (by Hilbert’s theorem 90: see [Mil, Chapter III, Proposition 4.9]), is in fact locally trivial on the Zariski topology of \(B\). We will denote this vector bundle also by \(\xi^V\).

Since \(\tilde{B}\) is an affine scheme over \(\text{Spec} k\), we can find a complimentary vector bundle \(\eta^V\) on \(\tilde{B}\) so that
\[(8.4.7)\quad \xi^V \oplus \eta^V\text{ is a trivial bundle over }\tilde{B}\text{ and of rank } N, \text{ for some integer } N.

For the remainder of this step, we will consider the case when \(\tilde{E} = E G^{gm,m}\) and \(\tilde{B} = B G^{gm,m}\), for a fixed integer \(m\). We will denote the first by \(\mathcal{E}_m\) and the latter by \(\mathcal{B}_m\). We will denote the vector bundle \(EG^{gm,m} \times_G V (EG^{gm,m} \times_G V)\) on the affine scheme \(\mathcal{B}_m = B G^{gm,m}\) by \(\xi^V_m\). The complimentary vector bundle \(\eta^V\) chosen above will now denote \(\eta^V_m\). We proceed to show that we can choose the integer \(N\) independent of \(m\), so that a single choice of \(N\) will work for all \(m\). Since the map \(EG^{gm,m} \to BG^{gm,m}\) is affine, one can readily see that the scheme \(EG^{gm,m}\) is also an affine scheme. Let \(R_m\) denote the co-ordinate ring of \(EG^{gm,m}\) and let \(R = \varinjlim R_m\). Under the correspondence between projective modules over \(R_m\) and vector bundles over \(\text{Spec } R_m\), \((R_m \otimes V)^G\) corresponds to \(EG^{gm,m} \times_G V = \xi^V_m\).

We proceed to show that
\[(8.4.8)\quad (R \otimes V)^G_k\]
is a finitely generated projective module over the ring \(R^G\). To see this, we proceed as follows. Let \(I_m\) be the ideal defining \(BG^{gm,m}\) as a closed subscheme in \(\text{Spec } (R^G)\). Then, \((R \otimes V)^G_k/(I_m \otimes (R \otimes V)^G_k)\) corresponds to the vector bundle \(\mathcal{E}^V_m\), and therefore, is a finitely generated projective \(R^G/I_m\)-module. In fact, if \(\mathfrak{M}\) denotes a maximal ideal in the ring \(R^G\) and \(I_m\) denotes the image of the ideal \(I_m\) in the local ring \(R^G_{\mathfrak{m}}\), then one can see that the ranks of the inverse system of free modules \(\{I_m \otimes (R \otimes V)^G_k/(R \otimes V)^G_k\mid \mathfrak{m}\}\) are the same finite integer given by the rank of \(V\). Therefore, their inverse limit, which identifies with \((R \otimes V)^G_k\) is a free \(R^G_{\mathfrak{m}}\)-module of rank equal to the rank of \(V\). It follows that, \((R \otimes V)^G_k\) is a finitely generated projective module over the ring \(R^G\).

Therefore, there exists some finitely generated free \(R^G\)-module \(F\) (of rank \(N\)) and a split surjection
\[(8.4.9)\quad \xi : F \twoheadrightarrow (R \otimes V)^G_k.

Then one sees that the induced maps
\[(8.4.10)\quad \xi/I_m : F/(I_m \otimes F) \twoheadrightarrow (R \otimes V)^G_k/(I_m \otimes (R \otimes V)^G_k)

are also split surjections for each \(m\), and these splittings are in fact compatible, as they are all induced by the splitting to the map in \((8.4.3)\). Therefore, we obtain a compatible collection of complements to the inverse system of bundles \(\xi^V_m\) in the trivial bundle of rank \(N\) over \(BG^{gm,m}\), compatible as \(m\) varies. We denote the complement to \(\xi^V_m\) in the trivial bundle of rank \(N\) over \(BG^{gm,m}\) as \(\eta^V_m\).

Next we will consider the case the group \(G\) is special, in which case the arguments in the following paragraph hold. Denoting by \(T_V\) the Thom-space of the representation \(V\), the bundle \(EG^{gm,m} \times_G T_V\) is a sphere-bundle over \(B_m\), which will be denoted \(S(\xi^V_m \oplus 1)\) in the terminology of Appendix, \((9.2)\). Similarly \(S(\eta^V_m \oplus 1)\) denotes the corresponding sphere bundle over \(B_m\). Now Lemma \((9.7)\) (see the Appendix) shows that one obtains the identification:
\[(8.4.11)\quad S(\xi^V_m \oplus 1) \wedge^{B_m} S(\eta^V_m \oplus 1) \simeq S(\xi^V_m \oplus \eta^V_m \oplus 1).

Observe that there is a canonical section \(s_V : B_m \to EG^{gm,m} \times_G T_V = S(\xi^V_m \oplus 1)\), and a canonical section \(s_{\eta^V} : B_m \to S(\eta^V_m \oplus 1)\), which together define a section \(s_m : B_m \to S(\xi^V_m \oplus 1) \wedge^{B_m} S(\eta^V_m \oplus 1)\) of pointed simplicial presheaves over \(B_m = B G^{gm,m}\). Then the quotient \((S(\xi^V_m \oplus 1) \wedge^{B_m} S(\eta^V_m \oplus 1))/s(B_m)\) identifies with the Thom-space of the bundle \(\xi^V_m \oplus \eta^V_m\). Since \(\eta^V_m\) was chosen to be a vector bundle complimentary to \(\xi^V_m\), \(\xi^V_m \oplus \eta^V_m\) is a trivial bundle (of rank \(N\)) so that the above Thom-space identifies with \(T^\wedge N(BG^{gm,m})_+\). Moreover, this holds independent of \(m\).
In case the group G is not special, one has to replace $\widetilde{\text{EG}}^{gm,m} \times_G T_V$ by $\widetilde{\text{EG}}^{gm,m} \times_G \epsilon^* \tau(T_V)$. This identifies with $S(\epsilon^*(\xi_V^m \oplus 1))$. Now one has to take the smash product of the above object with $S(\epsilon^*(\eta_m^V \oplus 1))$ over $\epsilon^*(B_m)$. This will identify with $S(\epsilon^*(\xi_m^V \oplus \eta_m^V \oplus 1))$. Since $\eta^V$ was chosen to be complementary to $\xi^V$, it follows that the bundle $\xi^V \oplus \eta^V$ is a trivial vector bundle of rank $N$, so that $S(\epsilon^*(\xi^V_m \oplus \eta^V_m \oplus 1)) \cong \epsilon^*(T^{\wedge N}) \times B_m$. Then one applies $R_{\epsilon^*}$ to the resulting object to obtain a pointed simplicial presheaf over $R_{\epsilon^*}(B_m)$. This identifies with $R_{\epsilon^*}(\epsilon^*(T^{\wedge N})) \times R_{\epsilon^*}(\epsilon^*(\widetilde{\text{BG}}^{gm,m}))$. Finally one has to collapse the corresponding section to obtain $R_{\epsilon^*}(\epsilon^*(T^{\wedge N})) \wedge (R_{\epsilon^*}(\widetilde{\text{BG}}^{gm,m}))_+$. Let $\pi_Y$ denote either of the two projections $\tilde{E} \times_G (Y \times X) \to \tilde{E} \times_G (Y)$ or $\epsilon_m = \epsilon(\tilde{\text{BG}}^{gm,m}) \times_G (Y \times X) \to \text{EG}^{gm,m} \times_G Y = B_m$. Since the second case is subsumed by the first, we will only discuss the first case explicitly in steps 3 through 5.

**Step 3.** First we will again assume that the group-scheme G is special. Now observe that the sphere bundle $\tilde{E} \times_G (Y \times X)$ identifies with the pullback $S(\pi_Y^* \tau(\xi^V) \oplus 1) = \pi_Y^* S(\xi^V) \wedge \pi_Y^* S(\eta^V) \parallel \pi_Y^* \tau$. Since $\epsilon^V$ was chosen to be complementary to $\xi^V$, it follows that the bundle $\pi_Y^* \tau(\xi^V \oplus \eta^V)$ is trivial, so that the resulting Thom-space identifies with $T^{\wedge N}(\tilde{E} \times_G (Y \times X))_+$. When the group-scheme G is not special, one adopts an argument as in the last paragraph of Step 2 to obtain a corresponding result.

**Step 4.** Observe that there is section $t' : \tilde{E}_Y \to \tilde{E} \times_G (Y \times (X_+ \wedge T_V))$. Combining that with the canonical section $\tilde{E}_Y \to S(\pi_Y^* \tau(\eta^V) \oplus 1)$ defines a section $t : \tilde{E}_Y \to (\tilde{E} \times_G (Y \times (X_+ \wedge T_V))) \wedge \pi_Y^* S(\pi_Y^* \tau(\eta^V) \oplus 1)$. Now a key observation is that $(\tilde{E} \times_G (Y \times (X_+ \wedge T_V))) \wedge \pi_Y^* S(\pi_Y^* \tau(\eta^V) \oplus 1)$ is an object defined over $E_Y$ and that collapsing the section $t$ identifies the resulting object with $(S(\pi_Y^* \tau(\xi^V) \oplus 1)) \wedge \pi_Y^* S(\pi_Y^* \tau(\eta^V) \oplus 1)\parallel \pi_Y^* \tau(\eta^V) \parallel \pi_Y^* \tau(\xi^V) \parallel \pi_Y^* \tau(\eta^V) \parallel \pi_Y^* \tau(\xi^V) \parallel \pi_Y^* \tau(\eta^V) \parallel \pi_Y^* \tau(\xi^V) \parallel \pi_Y^* \tau(\eta^V)$ is the canonical section. (See [BG95] (3.7) and (3.8) for the classical case.)

One may see this as follows, first under the assumption that the group-scheme G is special. Assume that $\{U_i|\}$ is a Zariski open cover of $B$ over which the G-torsor $p : \tilde{E} \to B$ trivializes. $(S(\pi_Y^* \tau(\xi^V) \oplus 1))_{U_i}$ now is of the form: $U_i \times (Y \times X \times T_V) \to U_i \times Y \times X$. We may assume that the vector bundle $\eta^V$ also trivializes over the cover $\{U_i|\}$. Then $(S(\pi_Y^* \tau(\xi^V) \oplus 1)) \wedge \pi_Y^* S(\pi_Y^* \tau(\eta^V) \oplus 1))_{U_i} = U_i \times ((Y \times X) \times (T_V \wedge T_W))$, where $W$ corresponds to the fibers of the vector bundle $\eta^V$. The section $\sigma_{U_i} : \tilde{E}_{Y \times X|U_i} \to (S(\pi_Y^* \tau(\xi^V) \oplus 1)) \wedge \pi_Y^* S(\pi_Y^* \tau(\eta^V) \oplus 1))_{U_i}$ now corresponds to the canonical section $U_i \times Y \times X \to U_i \times ((Y \times X) \times (T_V \wedge T_W))$. Intermediate to collapsing the section $\sigma$ is to take the pushout of

\[(8.4.11) \quad \tilde{E}_Y \leftarrow \tilde{E} \times_G (Y \times X) \to S(\pi_Y^* \tau(\xi^V) \oplus 1)) \wedge \pi_Y^* S(\pi_Y^* \tau(\eta^V) \oplus 1)).\]

Over $U_i$, this corresponds to taking the pushout of $U_i \times Y \leftarrow U_i \times (Y \times X) \to U_i \times ((Y \times X) \times (T_V \wedge T_W))$. The resulting pushout then identifies with $U_i \times Y \times (X_+ \wedge (T_V \wedge T_W))$, which in fact identifies with $\tilde{E} \times_G (Y \times (X_+ \wedge T_V)) \wedge \pi_Y^* S(\eta^V \oplus 1))_{U_i}$. Observe that collapsing the section $\sigma$ can be done in two stages, by first taking the pushout in (8.4.11) and then by collapsing the resulting section from $E_Y$. These complete the verification of the observation in Step 4, at least in the case the group-scheme G is special. When G is not special, one adopts a similar argument using an étale cover $\{U_i|\in I\}$ of $B$ over which $\tilde{E} \to B$ is trivial.
Step 5. Let \( s : \overset{\to}{E}Y \to \overset{\to}{E} \times_G (Y \times T_V) \wedge \overset{\to}{E}V S(\pi^*(\eta^V \oplus 1)) \) denote the canonical section. Then we proceed to show that the sections \( s \) and \( t \) are compatible in the sense that the diagram

\[
\begin{array}{ccc}
\overset{\to}{E}Y & \xrightarrow{s} & \overset{\to}{E} \times_G (Y \times T_V) \wedge \overset{\to}{E}V S(\pi^*(\eta^V \oplus 1)) \\
\downarrow t & & \downarrow (id \times_G tr_G(f_Y))(T_V) \wedge \overset{\to}{E}V id \\
\overset{\to}{E} \times_G (Y \times (X_+ \wedge T_V)) & \wedge \overset{\to}{E}V S(\pi^*(\eta^V \oplus 1))
\end{array}
\]

commutes, that is, in the sense discussed next. Here \( (id \times_G tr_G(f_Y))(T_V) \) is the component of the map of spectra \( id \times_G tr_G(f_Y)' \) indexed by \( T_V \). One may break this map into a sequence of maps

\[
\begin{array}{cccc}
Y_0(T_V) = \overset{\to}{E} \times_G (Y \times T_V) & \to & Y_1(T_V) = \overset{\to}{E} \times_G (Y \times \mathcal{X}_1(T_V)) & \leftarrow Y_2(T_V) = \overset{\to}{E} \times_G (Y \times \mathcal{X}_2(T_V)) \\
\to & & \to & \\
Y_3(T_V) = \overset{\to}{E} \times_G (Y \times (X_+ \wedge T_V))
\end{array}
\]

where the maps \( \{ T_V \to \mathcal{X}_1(T_V) \leftarrow \mathcal{X}_2(T_V) \to X_+ \wedge T_V \} \) define the \( G \)-equivariant pre-transfer considered in \((8.2.3)\). Observe that each of the objects in \((8.4.13)\) is pointed over \( \overset{\to}{E}V \). (When \( G \) is non-special, the quotient sheaves in the diagram \((8.4.13)\), and in the discussion below, are all taken in the \( \acute{e}tale \) topology on \( \overset{\to}{E}V \) and one will have to replace the diagram in \((8.4.12)\) with \( R_X \) applied to all the terms there.) One may observe that the corresponding sections from \( \overset{\to}{E}V \) are all compatible as the group action leaves the base points of \( T_V, \mathcal{X}_1(T_V), \mathcal{X}_2(T_V) \) and \( X_+ \wedge T_V \) fixed. This results in the following commutative diagram over \( \overset{\to}{E}V \):

\[
\begin{array}{cccc}
Y_0(T_V) \wedge \overset{\to}{E}V S(\pi^*(\eta^V \oplus 1)) & \xrightarrow{y_0 = s} & Y_1(T_V) \wedge \overset{\to}{E}V S(\pi^*(\eta^V \oplus 1)) & \xleftarrow{y_1} \xrightarrow{y_2} Y_2(T_V) \wedge \overset{\to}{E}V S(\pi^*(\eta^V \oplus 1)) & \xrightarrow{y_3 = t} Y_3(T_V) \wedge \overset{\to}{E}V S(\pi^*(\eta^V \oplus 1))
\end{array}
\]

By the commutativity of the triangle in \((8.4.12)\), we mean the commutativity of all the corresponding triangles that make up the diagram in \((8.4.14)\) and this is now clear in view of the above observations. When \( \overset{\to}{E}Y = \mathcal{E}_{m,Y} = EG gm,m \times_G Y \), one may again observe that the corresponding sections from \( \mathcal{E}_{m,Y} \) are all compatible as the group action leaves the base points of \( T_V, \mathcal{X}_1(T_V), \mathcal{X}_2(T_V) \) and \( X_+ \wedge T_V \) fixed. This results in a corresponding diagram over each \( \mathcal{E}_{m,Y} \) and the arguments in Step 2 above show that such commutative triangles are compatible as \( m \) varies.

Moreover, the commutativity of the diagram \((8.4.14)\) shows that there is an induced map on the quotients by the sections \( y_i, \ i = 0, 1, 2, 3 \). Observe that on taking smash product over \( \overset{\to}{E}V \) with \( \overset{\to}{E} \times_G ((Y \times X_+) \wedge T_W) \wedge \overset{\to}{E}V S(\pi^*(\eta^W \oplus 1)) \), one obtains a map of the diagram in \((8.4.14)\) to the corresponding diagram with \( V \oplus W \) in the place of \( V \). This observation shows that if we define spectra \( Z_i, \ i = 0, 1, 2, 3 \) in \( Spt_G \) by

\[
(8.4.15) \quad Z_{i,N_V} = (Y_i(T_V) \wedge \overset{\to}{E}V S(\pi^*(\eta^V \oplus 1)))/y_i(\overset{\to}{E}V), N_V = \dim(V) + \rank(\eta^V)
\]

and if \( N_W = \dim(W) + \rank(\eta^V) \), the smash product pairings \( T W_n \wedge Z_{i,N_V} \to Z_{i,N_V \oplus W} \) are compatible with the maps between the \( Z_i \) considered above. (Note that these spectra are indexed by the integers \( \{ N_V \} \) and not by all the non-negative integers. However, since \( \{ N_V \} \) is cofinal in \( \mathbb{N} \), this suffices.) One may also observe that the wrong-way map \( Z_2 \to Z_1 \) is a stable equivalence. Therefore, collapsing out the sections \( y_i, \ i = 0, 3 \), then provides the stable map (which in fact is a composition of several maps, with the ones going in the wrong direction being stable weak-equivalences)

\[
(8.4.16) \quad tr(f_Y) : \Sigma_T^{\infty}(\overset{\to}{E} \times_G Y)_+ \to \Sigma_T^{\infty}(\overset{\to}{E} \times_G (Y \times X))_+, \quad tr(f_Y) : \Sigma_T^{\infty}(EG gm,m_\times G Y)_+ \to \Sigma_T^{\infty}(EG gm,m_\times G (Y \times X))_+
\]
in case the group $G$ is special, and the following stable map (which in fact is a composition of several maps, with the ones going in the wrong direction being stable weak-equivalences) in case $G$ is not special:

\[(8.4.17)\]

\[\begin{align*}
tr(f_Y) &: \text{Re}_+ (e^* S_k) \wedge \text{Re}_+ (\overline{E} \times G Y)_+ \to \text{Re}_+ (e^* S_k) \wedge \text{Re}_+ (\overline{E} \times G (Y \times X))_+, \\
tr(f_Y)^m &: \text{Re}_+ (e^* S_k) \wedge \text{Re}_+ (E^{gm,m} \times G Y)_+ \to \text{Re}_+ (e^* S_k) \wedge \text{Re}_+ (E^{gm,m} \times G (Y \times X))_+.
\end{align*}\]

These maps are also compatible as $m$ varies, as observed above and in Step 2. The pairings $\mathcal{T} \wedge \text{Re}_+ (e^*(\mathcal{T}^{gm})) \to \text{Re}_+ (\mathcal{T}) \wedge \text{Re}_+ (e^*(\mathcal{T}^{gm}))$ shows that $\text{Re}_+ (e^* S_k)$ is indeed a motivic spectrum.

**Definition 8.8.** (The transfer.) Therefore, taking the colimit over $m \to \infty$, one obtains the following stable transfer map (in $\mathcal{SH}(k)$) on the Borel construction when $G$ is special:

\[(8.4.18)\]

\[tr(f_Y) : \Sigma_T^\infty (\overline{E} \times G Y)_+ \to \Sigma_T^\infty (\overline{E} \times G (Y \times X))_+,
tr(f_Y)^m : \Sigma_T^\infty (E^{gm,m} \times G Y)_+ \to \Sigma_T^\infty (E^{gm,m} \times G (Y \times X))_+.
\]

Next we consider the when the group $G$ is non-special. One may observe from the commutative diagram \[(8.4.11)\] that all the maps involved in the definition of the transfer maps $\tilde{tr}(f_Y)$ and $\tilde{tr}(f_Y)^m$ in \[(8.4.17)\] are maps of module spectra over the motivic ring spectrum $\text{Re}_+ (e^* S_k)$. Therefore, we will now define the transfer maps, when $G$ is non-special to be

\[(8.4.19)\]

\[tr(f_Y) = \lim_{m \to \infty} tr(f_Y)^m : \text{Re}_+ (e^* S_k) \wedge \text{Re}_+ (E^{gm,m} \times G Y)_+ \to \text{Re}_+ (e^* S_k) \wedge \text{Re}_+ (E^{gm,m} \times G (Y \times X))_+.
\]

Henceforth we will let $E^{gm} \times G Y = \lim_{m \to \infty} \text{Re}_+ E^{gm,m} \times G Y$ and $E^{gm} \times G (Y \times X) = \lim_{m \to \infty} \text{Re}_+ E^{gm,m} \times G (Y \times X)$.

If $E^G$ denotes a commutative $G$-equivariant ring spectrum as in \[(4.0.8)\], $E^i = i^* (\mathcal{P}^\natural (E^G))$ is the corresponding ring spectrum in $\mathbf{Spt}$, and $E^i \wedge X_+$ is dualizable in $\mathcal{SH}(k, E)$, the same constructions applied to the $G$-equivariant pre-transfer \[(4.2.3)\] and making use of smashing with $S$ provides us with the transfer map (in $\mathcal{SH}(k, E)$):

\[(8.4.20)\]

\[\begin{align*}
tr(f_Y)_E &: E \wedge (\overline{E} \times G Y)_+ \to E \wedge (\overline{E} \times G (Y \times X))_+, \\
tr(f_Y)_E &: E \wedge (E^{gm} \times G Y)_+ \to E \wedge (E^{gm} \times G (Y \times X))_+, \text{ when } G \text{ is special, and} \\
tr(f_Y)_E &: \text{Re}_+ e^* (E) \wedge \text{Re}_+ (E^{gm} \times G Y)_+ \to \text{Re}_+ e^* (E) \wedge \text{Re}_+ (E^{gm} \times G (Y \times X))_+,
\end{align*}\]

\[tr(f_Y)_E : \text{Re}_+ e^* (E) \wedge \text{Re}_+ (E^{gm} \times G (Y \times X))_+ \to \text{Re}_+ e^* (E) \wedge \text{Re}_+ (E^{gm} \times G (Y \times X))_+, \text{ when } G \text{ is non-special.}
\]

\[\square\]

**Remark 8.9.** Suppose $X = G/H$ for a closed linear algebraic subgroup and $Y = \text{Spec } k$. Then the identification

\[\text{Re}_+ (E^{gm} \times G Y) = \lim_{m \to \infty} \text{Re}_+ (E^{gm} \times G Y)_+ \to \text{Re}_+ (E^{gm} \times G (Y \times X))_+ \]

shows that in this case the target of the transfer map in \[(8.4.18)\] is $\Sigma_T^\infty (E^{gm})_+$ and the target of the transfer map in \[(8.4.20)\] is $E \wedge (E^{gm})_+$ or $\text{Re}_+ (e^* (E) \wedge BH^{gm})_+$ depending on whether $G$ and $H$ are special or not.

**Example 8.10.** The following provides a notable class of examples of such a transfer. Let $i : H \to G$ be as above and let $Y$ denote a smooth quasi-projective scheme over $k$ with an action by $G$. Assume further that $E$ is a commutative ring spectrum in $\mathbf{Spt}(k_{mot})$ and $\ell$ is a prime different from $\text{char}(k)$ so that $E$ is $\mathbb{Z}_{(\ell)}$-local.

Then, the $G$-scheme $G \times_H Y$ identifies as a $G$-scheme with $G/H \times Y$ (provided with the diagonal action by $G$). Clearly $G/H$ is dualizable in $\mathbf{Spt}(k_{mot})$ in case $\text{char}(k) = 0$ and $E \wedge G/H_+$ is dualizable in $\mathbf{Spt}(k_{mot}, E)$ in case $\text{char}(k) = \ell$ and $G$ is special, is then the stable map

\[\begin{align*}
tr(id_Y) : \Sigma_T^\infty (E^{gm,m} \times G Y)_+ &\to \Sigma_T^\infty (E^{gm,m} \times G (G \times_H Y))_+ \simeq \Sigma_T^\infty (EH^{gm,m} \times_H Y)_+.
\end{align*}\]
in the first case and the map
\[ tr(id_Y) : \mathcal{E} \land (E G^{gm,m} \times_G Y)_+ \to \mathcal{E} \land (E G^{gm,m} \times_G (G \times_H Y))_+ \cong \mathcal{E} \land (E H^{gm,m} \times_H Y)_+ \]
in the second case. In case the groups G and H are not special, one obtains corresponding stable transfer maps involving \( R \mathcal{E}_* \) as in \([8.4.17]\) and \([8.4.20]\).

9. Appendix: Spherical fibrations and Thom-spaces in the motivic and étale setting

The main goal of this section is to collect together various basic results on Thom spaces of algebraic vector bundles and relate them to Spanier-Whitehead duality in the both the motivic and étale framework. Throughout the following discussion we will let \( S \) denote a Noetherian affine smooth scheme defined and of finite type over a given perfect field \( k \).

Throughout the following discussion we will let \( S \) denote a Noetherian affine smooth scheme defined and of finite type over \( k \).

9.1. Basic results on Thom-spaces. We begin with the following basic observation on vector bundles over affine schemes. \( \text{Spc}_*(\text{Sm}_{/S}) \) will denote the category of pointed simplicial presheaves on the Nisnevich site of \( S \) defined as in \([2.1.3]\).

**Proposition 9.1.**
1. Let \( X \) denote any affine scheme. Then any vector bundle \( \mathcal{E} \) on \( X \) has a complement, i.e., there exists another vector bundle \( \mathcal{E}^1 \) so that \( \mathcal{E} \oplus \mathcal{E}^1 \) is a trivial vector bundle.
2. Assume \( X \) is again an affine scheme. Then, if \( \mathcal{E} \) and \( \mathcal{F} \) are two vector bundles on \( X \), then they represent the same class in the Grothendieck group \( K^0(X) \) if and only if they are stably isomorphic, i.e., isomorphic after the addition of some trivial vector bundles.
3. Let \( X \) denote a quasi-projective scheme, i.e., locally closed in some projective space over an affine base scheme \( S \). Then there exists an affine scheme \( \tilde{X} \) together with a surjective map \( \tilde{X} \to X \) so that \( \tilde{X} \) is an affine-space bundle over \( X \). In particular, the map \( X \to \tilde{X} \) is an \( \mathbb{A}^1 \)-equivalence.

**Proof.** (i) is clear from the fact that vector bundles on affine schemes correspond to projective modules over the corresponding coordinate ring. (ii) is discussed in \([\text{Joum}]\) Lemma 1.5 and often referred to as the Jouanolou trick. \( \square \)

We will next summarize some well-known facts about Thom-spaces in \( \text{Spc}_*(\text{Sm}_{/S}) \). If \( \alpha \) is a vector bundle over a smooth scheme \( X \) over the (base) scheme \( S \), then one needs to define the Thom-space of \( \alpha \) to be the following canonical homotopy pushout:

\[
\begin{array}{ccc}
E(\alpha) - X & \longrightarrow & E(\alpha) \\
\downarrow & & \downarrow \\
S & \longrightarrow & \text{Th}(\alpha)
\end{array}
\]

where \( E(\alpha) \) denotes the total space of the vector bundle \( \alpha \). Since \( E(\alpha) \) and \( E(\alpha) - X \) map to \( X \) and then to \( S \), \( \text{Th}(\alpha) \) maps to \( S \). The map \( S \to \text{Th}(\alpha) \) provides a section to the induced map \( \text{Th}(\alpha) \to S \), so that \( \text{Th}(\alpha) \) is pointed over \( S \) and hence is an object in \( \text{Spc}_*(\text{Sm}_{/S}) \). (We may often assume that injective maps are cofibrations, in which case the map in the top row is a cofibration, and therefore, it suffices to take the ordinary pushout, in the place of the homotopy pushout.)

When we view \( E(\alpha) \) and \( E(\alpha) - X \) as sheaves on the big étale site, the corresponding pushout of sheaves on the big étale site of \( S \) will be denoted \( \text{Th}(\alpha)_{\text{ét}} \).

**Proposition 9.2.** Let \( \alpha \) denote a vector bundle over the scheme \( X \).

1. Viewing \( P(\alpha \oplus e^1) = \text{Proj}_X(\alpha \oplus e^1) \) and \( P(\alpha) = \text{Proj}_X(\alpha) \) as simplicial presheaves over the base scheme \( S \) and taking the quotient presheaf, \( P(\alpha \oplus e^1)/P(\alpha) \cong \text{Th}(\alpha) \) where \( P(\beta) \) denotes the projective space bundle associated to a vector bundle \( \beta \) and \( e^1 \) denotes a trivial bundle of rank 1. Viewing \( P(\alpha \oplus e^1) \) and \( P(\alpha) \) as simplicial presheaves over \( X \) and taking the quotient presheaf over \( X \), \( P(\alpha \oplus e^1)/X P(\alpha) \cong S(\alpha \oplus e^1) \), a sphere bundle over \( X \). (This may be called the “one-point compactification of the vector bundle \( \alpha \).”)

The obvious projection \( S(\alpha \oplus e^1) \to X \) has a section \( s \) that sends a point in \( X \) to the point at \( \infty \) in the fiber over that point. Now \( S(\alpha \oplus e^1)/s(X) \cong \text{Th}(\alpha) \).

2. If \( X \to Y \) is a closed immersion of smooth schemes with \( N \) denoting the corresponding normal bundle, then \( \text{Th}(N) \cong X/X - Y \).
(iii) Let \( g : S' \to S \) denote a map of smooth schemes and let \( g^*(\alpha) \) denote the induced vector bundle on \( X' = X \times S' \). Then \( g \) induces a map \( \text{Th}(g^*(\alpha)) \to \text{Th}(\alpha) \) compatible with the given map \( g : S' \to S \).

Moreover, the induced map \( \text{Th}(g^*(\alpha)) \to \text{Th}(\alpha) \) is natural in \( g \) and \( \alpha \).

Proof. We skip the proof as the above statements are rather well-known. \( \square \)

Notation 9.3. When we view \( P(\alpha) \) and \( P(\alpha \oplus e^1) \) as sheaves on the big étale site of \( X \) the corresponding quotient \( P(\alpha \oplus e^1)_{\text{et}/X} / P(\alpha)_{\text{et}} \) will be denoted \( S(\alpha \oplus 1)_{\text{et}} \).

9.2. The fiber-wise join of simplicial presheaves (spectra) fibering over another simplicial presheaf (spectrum). Given maps of simplicial presheaves \( Y \to X \) and \( Z \to X \), the fiber-wise join \( Y \ast_X Z \) is the simplicial presheaf defined as the (canonical) homotopy pushout

\[
\begin{array}{ccc}
Y \times Z & \longrightarrow & Z \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y \ast_X Z 
\end{array}
\]

One may readily verify that the above construction extends readily to spectra. We elaborate a bit on the above construction and its application to Thom-spaces. First we show that the fiber-wise join indeed does what it is supposed to do.

Lemma 9.4. Assume \( X, Y \) and \( Z \) are simplicial presheaves as above. Then:

(i) There is an induced map \( Y \ast_X Z \to X \).

(ii) If \( Y_x, Z_x \) denote the fibers over \( x \in X \), \((Y \ast_X Z)_x \simeq S^1 \wedge (Y_x \wedge Z_x)\).

(iii) Therefore, if \( X \) denotes the simplicial presheaf represented by a smooth scheme, and \( Y, Z \) are pointed simplicial presheaves over \( X \), \( Y \ast_X Z \simeq (S^1 \times X) \wedge^X (Y \wedge^X Z) \simeq Y \wedge^X ((S^1 \times X) \wedge^X Z) \).

Proof. This is a well-known result. See for example, [CS, Lemma 2.1]. \( \square \)

If \( \beta \) is a vector bundle over the smooth scheme \( X \), we let \( \bar{\beta} \to X \) denote the associated bundle \( \beta - 0 \to X \). Now with \( \bar{\beta} = \beta \cup X \), one obtains \((S^1 \times X) \wedge^X \bar{\beta} \simeq (\beta \cup X) \cong S(\beta + 1)\). The last \( \cong \) is an isomorphism as simplicial presheaves over \( X \) while the \( \simeq \) is a weak-equivalence of such simplicial presheaves. The last isomorphism may be seen by working locally on \( X \), so that \( \beta \) is trivial. The \( \simeq \) follows from the observation that the fibers of \( \beta \) are acyclic so that \((S^1 \times X) \wedge^X \bar{\beta} \simeq (\beta \cup X) \) as simplicial presheaves over \( X \).

Lemma 9.5. Let \( \alpha \) and \( \beta \) denote two vector bundles over the scheme \( X \). Then we obtain the identifications:

(i) \( S(\alpha \oplus e^1) \ast_X \bar{\beta} \simeq S(\alpha \oplus e^1) \wedge^X S(\beta \oplus e^1) \simeq S(\alpha \oplus \beta \oplus e^1) \) where \( \bar{\beta} \) denotes \( \beta - 0 \to X \), the associated sphere bundle.

(ii) The map \( S(\alpha \oplus \beta \oplus e^1) \to X \) has a section \( s \) sending each point of \( X \) to the point at \( \infty \) in the fiber over that point. Then the quotient \( S(\alpha \oplus \beta \oplus e^1)/S(X) = \text{Th}(\alpha \oplus \beta) \), which is the Thom-space of \( \alpha \oplus \beta \).

Proof. Since (ii) is rather straightforward, we will discuss only (i). In view of the weak-equivalences above between \((S^1 \times X) \wedge^X \bar{\beta} \) and \( S(\beta \oplus e^1) \), and the observation that \( \wedge^X \) is a homotopy pushout of simplicial presheaves over \( X \) (in the injective model structure), it follows that \( S(\alpha \oplus e^1) \ast_X \bar{\beta} \simeq S(\alpha \oplus e^1) \wedge^X ((S^1 \times X) \wedge^X \bar{\beta}) \simeq S(\alpha \oplus e^1) \wedge^X S(\beta \oplus e^1) \). Since the last fibers over \( X \), one may work locally on \( X \) and show readily that it identifies with \( S(\alpha \oplus \beta \oplus e^1) \). \( \square \)

Lemma 9.6. Let \( B = \text{Spec} \, k \) denote the base field. Let \( G \) denote a linear algebraic group defined over \( B \) and acting on the simplicial presheaves \( E \) and \( X \) over the base scheme \( B \). Assume that \( E \) is in fact a smooth scheme of finite type over \( B \) so that the (geometric) quotient \( E/G \) exists and is in fact a scheme of finite type over \( B \). Let \( P \) denote a pointed simplicial presheaf in \( \text{Spc}_c(B) \) together with a \( G \)-action that leaves the base point of \( P \) fixed. Then \( E \times_X (P \wedge X) \cong P_{E/G} \wedge^{E/G} (E \times_X X) \), where \( P_{E/G} = P \times_G E \).

Proof. The proof is skipped as one may readily verify the above conclusions. \( \square \)
9.3. **Motivic Atiyah duality.** The rest of this section will be devoted to summarizing a version of Atiyah-duality (see [At]) that applies to the motivic and also the étale context: accordingly, we will assume that for any smooth projective scheme $X$ over a perfect field $k$, there exists a vector bundle over the scheme $X$ (which we call the virtual normal bundle) so that the $T$-suspension spectrum of its Thom-space is a Spanier-Whitehead dual of the suspension spectrum $\Sigma \tau X_+$ in the category $\text{Spt}(k_{mot})$. The idea of the proof may be summarized as follows: the Voevodsky collapse considered in Definition 9.8 (see below), provides a co-evaluation map $\eta : \Sigma \tau X_+ \to \text{Spt}(k_{mot})$. Here $\text{Spt}$ is the virtual normal bundle considered in Definition 9.3. One defines an evaluation map dual to this and with these, one shows that $\text{Th}(\alpha)$ is in fact a Spanier-Whitehead dual of $\Sigma \tau X_+$ modulo certain shifts.

Under the assumption that the base scheme $S = k$ is a perfect field satisfying the finiteness hypothesis as in (1.0.1), we may readily observe that the pullback functors $\epsilon^* : \text{Spt}(k_{mot}) \to \text{Spt}(k_{et})$, $\epsilon^* : \text{Spt}(k_{mot}) \to \text{Spt}(k_{mot})$, and $\eta^* : \text{Spt}(k_{et}) \to \text{Spt}(k_{mot})$ considered in (7.0.3) as well as the functors $\theta$ and $\phi$ (discussed in the paragraph below (7.0.4)) send suspension spectra of the Thom-spaces in the framework of the source, to suspension spectra of the corresponding Thom-spaces in the framework of the target, are compatible with the smash-products and internal Hom-s in these categories and also send maps that are homotopic to the identity to maps that are homotopic to the identity. Therefore, the discussion below carries over from the framework of $\text{Spt}(k_{mot})$ to all of the other frameworks (at least after inverting $A^1$ in all these frameworks).

Thus, the construction of a Spanier-Whitehead dual from the Thom-space of a vector bundle worked out below in the motivic framework carries over to the étale setting after smashing with an $\ell$-complete spectrum, $\ell$ being prime to the characteristic.

Over algebraically closed fields of arbitrary characteristic, there is already a different construction valid in the étale setting and making strong of use of étale tubular neighborhoods: see [JS6] and [JS7].

**Definition 9.7.** (The diagonal map.) Next we consider the following diagonal map. Let $\alpha, \beta$ denote two vector bundles on the scheme $X$. Then there is a diagonal map $\text{Th}(\alpha \oplus \beta) \to \text{Th}(\alpha) \land \text{Th}(\beta)$. This map is induced by the map $E(\alpha \oplus \beta) \to E(\alpha) \times E(\beta)$ lying over the diagonal map $X \to X \times X$. In this case, one may verify that $E(\alpha \oplus \beta) - \{0\}$ maps to $(E(\alpha) - \{0\}) \times E(\beta) \cup E(\alpha) \times (E(\beta) - \{0\})$. Taking $\alpha$ to be a zero-dimensional bundle, one obtains the diagonal map

$$\Delta : \text{Th}(\beta) \to X_+ \land \text{Th}(\beta).$$

One may interpret the above diagonal map in terms of the associated disk and sphere bundles as follows:

$$\Delta' : \text{Th}(\beta) = P(\beta \oplus e^1)/P(\beta) \to P(\beta \oplus e^1)/P(\beta) \times P(\beta \oplus e^1)/P(\beta) = (P(\beta \oplus e^1) \times P(\beta \oplus e^1))/P(\beta \oplus e^1) \times P(\beta)$$

Now one composes with the projection $P(\beta \oplus e^1) \to X$ to define the diagonal map in (9.3.1).

9.4. **Basic framework: the projective case.** Assume next that $X$ and $Y$ are smooth projective schemes with $X$ provided with a closed immersion into $Y$ over $k$. $Y$ will usually denote a projective space over $k$, but we denote it by $Y$ for simplicity of notation. Let $\tau_X (\tau_Y, N)$ denote the tangent bundle to $X$ (the tangent bundle to $Y$ and the normal bundle associated to the imbedding of $X$ in $Y$, respectively). Then one obtains the short exact sequence

$$0 \to \tau_X \to \tau_Y|X \to N \to 0.$$  

Let $\pi_Y : \hat{Y} \to Y$ denote the affine replacement provided by Jouanolou’s construction. Let $\hat{X} = X \times Y \hat{Y}$ and let $\pi_X : \hat{X} \to X$ denote the induced map. Then the following are proven in [Voev] Proposition 2.7 through Theorem 2.11:

1. There exists a vector bundle $V$ on $Y$ so that $\pi_Y^* (V) \oplus \pi_Y^* (\tau_Y)$ is stably isomorphic to a trivial vector bundle. So we will assume that $\pi_Y^* (V) \oplus \pi_Y^* (\tau_Y) \oplus e^m \cong e^n$ for some $m$ and $n$. We will replace $V$ by $V \oplus e^m$ so that $\pi_Y^* (V) \oplus \pi_Y^* (\tau_Y) \oplus e^m \cong e^n$.

2. There exists a collapse map $V : T^n \to \text{Th}(V)$. See [Voev] Lemma 2.10 and Theorem 2.11.

One may observe that $\pi_X^* (N \oplus V|X) \oplus \pi_X^* (\tau_X)$ is also stably trivial. If $\pi_X^* (N \oplus V|X) \oplus \pi_X^* (\tau_X) \oplus e^m \cong e^n$ for some $m$ and $N$, we will replace $V$ by $V \oplus e^m$ and we will make the following definition.
Definition 9.8. (Virtual normal bundle in the projective case, the Voevodsky collapse and the corresponding co-evaluation map) We let $\nu_X = \mathcal{N} \oplus V_X$ and call it the virtual normal bundle to $X$ in $Y$. Taking $Y = X$, we see that $\nu_Y$ has the property that $\pi^*_Y(\nu_Y)$ is a complement to $\pi^*_Y(\tau_Y)$ in some trivial bundle over $\hat{Y}$.

Clearly $\text{Th}(\nu_X) \simeq V/V - X$ where $X$ is imbedded in $V$ by the composite imbedding $X \to Y^{0-\text{section}} E(V)$. Therefore, one obtains a collapse map $\text{Th}(V) = V/V - Y \to V/V - X \simeq \text{Th}(\nu_X)$. Composing with the collapse $V : T^n \to \text{Th}(V)$ one obtains the collapse $V_X : T^n \to \text{Th}(\nu_X)$. Composing with the diagonal map $\Delta$ (considered above), one obtains a map $c : T^n \to X \times \text{Th}(\nu_X)$. The main result we need is that this map is indeed a co-evaluation map in the sense of [DP, 1.3 Theorem], so that $\Sigma^{\infty} \wedge T^n \wedge \text{Th}(\nu_X)$ is indeed a Spanier-Whitehead dual of $\Sigma^{\infty} T X_+$. This is rather well-known by now, as discussed for example in [Hu-Kr].

References

[At] M. F. Atiyah, Thom complexes, Proc. London Math. Soc. (3), 11, (1961), 291–310.
[BG75] J. Becker and D. Gottlieb, The transfer map and fiber bundles, Topology, 14, (1975), 1-12.
[Bor] F. Borceux, Handbook of categorical algebra. II: Categories and structures, Cambridge University Press, Cambridge, (1994).
[CE] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, (1956).
[Ch] Séminaire C. Chevalley, 2 année, Annuaires de Chow et applications, Paris: Secrétariat mathématique, (1958).
[CJ23-T2] G. Carlsson and R. Joshua, The motivic and étale Becker-Gottlieb transfer and splittings, Preprint, (2023).
[CS] D. Shatur and J. Scherer, The motivic and étale Becker-Gottlieb transfer and splittings, Preprint, (2023).
[Hov99] M. Hovey, Model categories, AMS, (1999).
[Hov03] M. Hovey, Monoidal model categories, Math AT/9803002, (2003).
[HSS] M. Hovey, B. Shipley and J. Smith, Symmetric spectra, JAMS, 13, (2000), no.1, 149-208.
[Hu-Kr] P. Hu and I. Kriz, Appendix A: On the Picard group of the stable $\mathbb{A}^1$ homotopy category, Topology, 44 (2005), 669-640.
[Isak] D. C. Isaksen, Étale realization on the $\mathbb{A}^1$-homotopy theory of schemes, Advances in Math, 184, (2004), 37-63.
[Joum] J. P. Jouanolou, Une suite exacte de Mayer-Vietoris en $K$-Théorie algébrique, Lect. Notes in Math., 341, 293-316, Springer, (1973).
[J22] R. Joshua, Equivariant Derived Categories for Toroidal Group Imbeddings, Transformation Groups, issue 1, (113-162), (2022).
[JP23] R. Joshua and P. Pelaye, Additivity of motivic trace and the motivic Euler-characteristic, Advances In Math, 429 (2023), 109184.
[J01] R. Joshua, Mod-$l$ algebraic $K$-theory and Higher Chow groups of linear varieties, Camb. Phil. Soc. Proc., 130, (2001), 37-60.
[J02] R. Joshua, Derived functors for maps of simplicial spaces, JPAA, 171, (2002), 219-248.
[JT] R. Joshua, Mod-$l$ Spanier-Whitehead duality and the Becker-Gottlieb transfer in étale homotopy and applications, Ph. D thesis, Northwestern University, (1984).
[J86] R. Joshua, Mod-$l$ Spanier-Whitehead duality in étale homotopy, Transactions of the AMS, 296, 151-166, (1986).
[J87] R. Joshua, Becker-Gottlieb transfer in étale homotopy theory, Amer. J. Math., 107, 453-498, (1987).
[K] S. Kelley, Triangulated categories of motives in positive characteristics, Ph. D Thesis, Université de Paris XIII, arXiv:1305.5549v1, (2013).
[Lev] M. Levine, The homotopy coneive tower, Journal of Topology, (2008), 1, 217-267.
[Lev18] M. Levine, *Motivic Euler Characteristics and Witt-valued Characteristic classes*, Nagoya Math Journal,(2019). DOI:https://doi.org/10.1017/nmj.2019.6.

[LMS] L. G. Lewis, J. P. May, and M. Steinberger, *Equivariant Stable Homotopy Theory*, Lect. Notes in Mathematics, 1213, Springer, (1985).

[Lur] J. Lurie, *Higher Topos Theory*, Annals of Math Study, 170, Princeton University Press, (2009).

[Mi] J. Milne, *Etale Cohomology*, Princeton University Press, (1980).

[MV] F. Morel and V. Voevodsky, *A1-homotopy theory of schemes*, I. H. E. S Publ. Math., 90, (1999), 45–143 (2001).

[ncatlab] https://www.ncatlab.org/nlab/show/monoidal functor.

[RO] O. Rondigs and P. Ostvæer, *Modules over motivic cohomology*, Advances in Math., 219, (2008), 689-727.

[Ri05] J. Riou, *Dualité de Spanier-Whitehead en Géométrie Algébriques*, C. R. Math. Acad. Sci. Paris, 340, (2005), no. 6, 431-436.

[Ri13] J. Riou, *Ét Alterations and Dualizability*, Appendix B to Algebraic Elliptic Cohomology Theory and Flops I, by M. Levine, Y. Yang and G. Zhao, Preprint, (2013).

[SSch] B. Shipley and S. Schwede, *Algebras and modules in monoidal model categories*, Proc. London Math Soc, 80, 491-511, (1998).

[SpWh58] E. Spanier and J. H. C. Whitehead, *Duality in relative homotopy theory*, Ann. of Math. (2), 67, (1958), 203–238.

[Tot] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), 249-281, Proc. Symposia in Pure Math, 67, AMS, Providence, (1999).

[Voev] Vladimir Voevodsky: *Motivic cohomology with Z/2-coefficients*, Publ. Math. Inst. Hautes Études Sci. No. 98 (2003), 59-104.