Graviton exchange and complete 4–point functions in the AdS/CFT correspondence

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Abstract

The graviton exchange diagram for the correlation function of arbitrary scalar operators is evaluated in anti–de Sitter space, $AdS_{d+1}$. This enables us to complete the computation of the 4–point amplitudes of dilaton and axion fields in IIB supergravity on $AdS_5 \times S_5$. By the AdS/CFT correspondence, we obtain the 4–point functions of the marginal operators $\text{Tr}(F^2 + \ldots)$ and $\text{Tr}(\tilde{F}\tilde{F} + \ldots)$ in $\mathcal{N} = 4$, $d = 4$ $SU(N)$ SYM at large $N$, large $g_{YM}^2 N$. The short distance asymptotics of the amplitudes are studied. We find that in the direct channel the leading power singularity agrees with the expected contribution of the stress–energy tensor in a double OPE expansion. Logarithmic singularities occur in the complete 4–point functions at subleading orders.

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1 Introduction

The calculation of correlation functions is one useful way to test and explore the AdS/CFT correspondence [1, 2, 3], which relates $d$–dimensional conformal field theories with compactifications of string/M theory involving $AdS_{d+1}$. The simplest example of the correspondence is the duality between $\mathcal{N} = 4$, $d = 4$ $SU(N)$ SYM theory and type IIB string theory on $AdS_5 \times S_5$ with $N$ units of 5–form flux and compactification radius $R^2 = \alpha' (g^2_{YM} N)^{1/2}$. In the large $N$ limit with $\lambda = g^2_{YM} N$ fixed and large the supergravity approximation is valid. Correlators of gauge invariant local operators in the CFT at large $N$ and strong t’Hooft coupling $\lambda$ are related to supergravity amplitudes according to the prescription of [2, 3]. The 5–dimensional Newton constant $G_5 \sim R^3 / N^2$, so that the perturbative expansion in supergravity, if ultraviolet convergent, corresponds to the $1/N$ expansion in the CFT.

Broadly speaking, 2– and 3–point functions (see e.g. [4, 5, 6]) have provided evidence that the conjectured correspondence is correct, but 4–point functions are expected to contain more information about the non–perturbative dynamics of the CFT. Previous studies relevant to 4–point correlators include [7]–[19]. 4–point correlators for contact interactions of scalars in the bulk theory were the first to be studied [7, 8, 9] followed by diagrams with exchanged gauge bosons [10] and scalars [11, 12, 13]. (See also [14, 15] for a different approach). $\alpha'/R^2$ corrections are considered in [16], and there is an extensive literature on instanton contributions, see e.g. [17].

The simplest 4–point correlators that can be studied are those involving the marginal operators $O_\phi \sim Tr(F^2 + \ldots)$ and $O_C \sim Tr(F^\dagger F^\dagger + \ldots)$ corresponding to the dilaton and axion supergravity fields, as first stressed in [8]. To leading order in $N$, the amplitudes $\langle O_\phi O_\phi O_\phi O_\phi \rangle$, $\langle O_C O_C O_C O_C \rangle$ and $\langle O_\phi O_C O_\phi O_C \rangle$ factorize in products of 2–point functions (see Figures 1a and 3). Thanks to the non–renormalization theorem for the 2–point functions [20, 4], these disconnected contributions do not receive corrections in powers of $\alpha'/R^2 = 1/\lambda^{1/2}$. The next contribution to the 4–point amplitudes is thus a $1/N^2$ effect and involves tree–level, connected supergravity diagrams like the ones in Figure 2. The computation of $\langle O_\phi O_\phi O_\phi O_\phi \rangle$, $\langle O_C O_C O_C O_C \rangle$ and $\langle O_\phi O_C O_\phi O_C \rangle$ was started in [9] with the evaluation of the relevant quartic and scalar exchange diagrams (Figure 2s,u,q and Figure 4). Here we complete the computation by evaluating the remaining graviton exchange diagram (Figure 2t) and we initiate the analysis of the first realistic 4–point amplitude in the AdS/CFT correspondence.

We also present what we believe is a cross–checked and reliable calculation of the graviton exchange diagram between pairs of external scalars of arbitrary mass in $AdS_{d+1}$ for arbitrary $d$. The calculation was facilitated by the recently derived covariant form of the graviton propagator.
but it is still very complex compared to earlier work.

One theoretical framework to analyze results on 4–point functions in the operator product expansion (OPE) \cite{11,22}. The mere assumption of an OPE is quite restrictive and imposes constraints on the allowed form of the result. Let us assume a double “t–channel” OPE of the schematic form

\[
\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4) \rangle = \sum_{n,m} \frac{\alpha_n \langle O_n(x_1)O_m(x_2) \rangle \beta_m}{(x_1 - x_3)^{\Delta_1 + \Delta_3 - \Delta_m}(x_2 - x_4)^{\Delta_2 + \Delta_4 - \Delta_n}}
\]

containing the contribution of various primary operators \(O_p\) and their descendents \(\nabla^k O_p\) in the intermediate state. For simplicity we have assumed that these are scalars, but vector and tensor operators contribute in a similar way, each with a characteristic tensor structure. (For primary operators, \(\langle O_pO_{p'} \rangle\) vanishes unless \(\Delta_p = \Delta_{p'}\)).

Recognizing in the supergravity 4–point results a structure of the form (1.1) should allow to determine the operator content of the theory and its OPE structure in the large \(N\), large \(\lambda\) limit. Preliminary computations \cite{11,12} have indicated that the supergravity diagrams contain the expected contributions to (1.1) of chiral primary operators and their superconformal descendents. It is however clear that these contributions alone do not reproduce the supergravity result \cite{9}. A natural expectation is that appropriately defined normal–ordered products of chiral primaries and descendents also contribute to the OPE and form the full operator content of the theory in this limit. This set of operators has a dual interpretation in terms of multi–particle Kaluza–Klein states in supergravity. Massive string states are expected to decouple in this limit\footnote{Group–theoretic aspects of multi–particle and string states have been considered in \cite{28}.}. The computation of a complete realistic 4–point correlator presented here should allow to put these ideas to test.

An interesting issue raised in \cite{9} is the presence in the 4–point supergravity amplitudes of logarithms of the coordinate separation between two points in the limit when the points come close. Logarithmic singularities appear to be a generic feature of all the AdS processes studied so far \cite{10,12,13}, and we find the same situation for the graviton exchange. The question then is whether the logarithms cancel when the various contributions to a realistic correlator are assembled. If not, we should ask whether the logarithms can still be incorporated in the OPE framework. Here we find that logarithmic singularities do indeed occur in the complete 4–point functions.

As pointed out by Witten \cite{23}, logarithms can generically arise in the perturbative expansion of a CFT 4–point correlator as renormalization effects like mixings and corrections to the
dimensions of the exchanged operators. The perturbative parameter is in this case $1/N$, which is mapped by the correspondence to the gravitational coupling constant. The operators $\mathcal{O}_\phi$ and $\mathcal{O}_C$ are chiral and hence their dimensions are protected, but their OPE’s contain (besides chiral contributions like the stress–energy tensor) non–chiral composite operators like $\mathcal{O}_\phi \mathcal{O}_\phi$ that require a careful definition and can lead to renormalization effects [23]. (A somewhat different viewpoint has been described in a very recent paper [24], see also [25]).

It is an interesting subject for future work to analyze the constraints imposed by this interpretation on the allowed form of the logarithmic singularities and to assess the compatibility of these constraints with the supergravity results.

The paper is organized as follows.

In Section 2, we present the supergravity graphs that contribute to 4–point functions involving $\mathcal{O}_\phi, \mathcal{O}_C$, summarize our results for the amplitudes and make some remarks about their OPE interpretation.

In Section 3, we describe the general set–up for the calculation of the graviton exchange amplitude. We give a few geometric identities, summarize the results for the scalar and graviton propagators and present the integral associated with the graviton exchange graph.

In Section 4 and Section 5 we separately describe two independent computations of the graviton amplitude, for $\Delta = \Delta' = d = 4$ in Section 4 and for general $\Delta$, $\Delta'$ and $d$ in Section 5. Both computations reduce the graviton exchange amplitude to finite sums of scalar quartic graphs (see Figure 6). The two results are shown to precisely agree for $\Delta = \Delta' = d = 4$.

In Section 6, we develop integral representations and asymptotic series expansions for the quartic graphs (Figure 5), which are the basic building blocks of the answer. We find asymptotic serieses for the graviton exchange in terms of two conformally invariant variables.

Finally, in the Appendix we discuss some properties and mathematical identities of the quartic graphs.
2 4–point functions in the dilaton–axion sector

Following [8], we first discuss the dilaton–axion–graviton sector of IIB supergravity, dimensionally reduced on the classical background solution $AdS_5 \times S^5$ keeping only the constant modes on $S^5$. The relevant part of 5–dimensional action is

$$S = \frac{1}{2 \kappa^2} \int_{AdS_5} d^5z \sqrt{g} \left( -\mathcal{R} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} e^{2\phi} g^{\mu\nu} \partial_\mu C \partial_\nu C \right). \tag{2.1}$$

The 5–dimensional gravitational coupling $\kappa$ is related to the parameters of the compactification by $2\kappa^2 = \frac{15\pi^2 R^3}{N^2}$, where $N$ is the number of units of 5–form flux and $R$ the radius of the 5–sphere (equal to the $AdS_5$ scale, see equ.(3.16) below). We will usually set the $AdS_5$ scale $R \equiv 1$.

2.1 Witten diagrams

We wish to implement the prescription of [2, 3] to compute the CFT correlators $\langle O_\phi O_C O_\phi O_C \rangle$, $\langle O_\phi O_\phi O_\phi O_\phi \rangle$, $\langle O_C O_C O_C O_C \rangle$, where $O_\phi \sim \text{Tr}(F^2 + ...)$, $O_C \sim \text{Tr}(F \tilde{F} + ...)$ are the exactly marginal ($\Delta = 4$) SYM operators corresponding to the dilaton and axion fields [3].

Let us first consider $\langle O_\phi(x_1) O_C(x_2) O_\phi(x_3) O_C(x_4) \rangle$. The leading large $N$ contribution is given by the disconnected diagram in Figure 1a. This diagram, being the product of two 2–point functions, is proportional to $N^4/(x_{13}^8 x_{24}^8)$.

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2 The metric appearing in (2.1) is not the restriction of the original 10–dimensional metric to $AdS_5$, but it is related to it by a Weyl rescaling of the metric fluctuations [26, 3]. The fluctuation $h'_{\mu\nu}$ that gives the massless graviton in $AdS_5$ is given in terms of the original $h_{\mu\nu}$ by $h_{\mu\nu} = h'_{\mu\nu} - \frac{1}{3} \bar{g}_{\mu\nu} h^\alpha_\alpha$, where $\alpha$ is an index along $S_5$ and $\bar{g}_{\mu\nu}$ the background metric [26].

3 The precise structure of the composite operators $O_\phi$ and $O_C$ in terms of elementary SYM fields is in principle given by the variation of the on–shell $\mathcal{N} = 4$ lagrangian with respect to the marginal couplings $g_{YM}$ and $\theta$, or by supersymmetry transformations starting from the chiral primary $\text{Tr}X^i X^j$.
Figure 2: Connected $O(N^2)$ contributions to $\langle O \phi O C O \phi O C \rangle$.

The next contribution, of order $N^2$, comes from the diagrams in Figures 1b and 2. However, the one–loop diagrams in Figure 1b, thanks to the fact that the dimensions of the chiral operators $O \phi$, $O C$ are protected, only give a $1/N^2$ correction to the overall coefficient of the amplitude in Figure 1a. Among the diagrams in Figure 2, the sum s+u+q has been computed in [9].

Sections 4 and 5 of the paper are devoted to evaluation of the remaining graviton exchange diagram t.

Similarly, Figures 3 and 4 reproduce the relevant diagrams for $\langle O C(x) O C(x) O C(x) O C(x) \rangle$. The connected diagrams for $\langle O \phi(x) O \phi(x) O \phi(x) O \phi(x) \rangle$ involve only graviton exchanges. As shown in [9] the s,t,u scalar exchange diagrams in Figure 4 add up to zero. Hence, to this order,

$$\langle O \phi O \phi O \phi O \phi \rangle = \langle O C O C O C O C \rangle.$$  \hspace{1cm} (2.2)

\footnote{This correction precisely accounts for the fact the gauge group is $SU(N)$ rather than $U(N)$. Note that validity of the correspondence seems to require that there are no higher loop corrections in the supergravity 2–point functions.}
2.2 Summary of results

It turns out that upon integration over one of the bulk points, all 4-point AdS processes with external scalars, including the graviton exchange, reduce to a finite sum of scalar quartic graphs (see Figure 6). We denote quartic graphs of external conformal dimensions $\Delta_i$ with the symbol $D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1, x_3, x_2, x_4)$, as in Figure 5 (see equation (A.1) for the precise definition and the Appendix for a discussion of properties of these functions).

The final result for the graviton exchange graph in Figure 2t as sum of quartic graphs (for $\Delta = \Delta' = d = 4$), derived in Sections 4 and 5 below, is

$$I_{\text{grav}} = \left(\frac{6}{\pi^2}\right)^4 \left[ 16 x_{24}^2 \left( \frac{1}{2s} - 1 \right) D_{4455} + \frac{64}{9} \frac{x_{24}^2}{x_{13}^2} \frac{1}{s} D_{3355} + \frac{16}{3} \frac{x_{24}^2}{x_{13}^4} \frac{1}{s} D_{2255} \right] + 18 D_{4444} - \frac{46}{9} \frac{x_{24}^2}{x_{13}^2} D_{3344} - \frac{40}{9} \frac{x_{24}^4}{x_{13}^4} D_{2244} - \frac{8}{3} \frac{1}{x_{13}^6} D_{1144},$$

(2.3)

where we have introduced the conformally invariant variable

$$s \equiv \frac{1}{2} \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2}.$$

(2.4)
\[ D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} = \]

\begin{tikzpicture}
  \node (1) at (0,0) {$\Delta_1$};
  \node (2) at (2,0) {$\Delta_2$};
  \node (3) at (0,2) {$\Delta_3$};
  \node (4) at (2,2) {$\Delta_4$};
  \draw (1) -- (3);
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (4) -- (3);
  \draw (4) -- (2);
\end{tikzpicture}

Figure 5: Definition of \( D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} \).

See equations (5.23, 5.57–5.58) for the analogous result in the general case of arbitrary \( \Delta, \Delta' \), \( d \).

We also recall the result [9] for the sum of the amplitudes \( s, q, u \) in Figure 2

\[ I_s + I_u + I_q = \left( \frac{6}{\pi^2} \right)^4 \left[ 64 x_{24}^2 D_{4455} - 32 D_{4444} \right]. \quad (2.5) \]

The sum of (2.3) and (2.5) gives the connected order \( N^2 \) contribution to the correlator

\[ \langle O_\phi(x_1) O_C(x_2) O_\phi(x_3) O_C(x_4) \rangle. \]

The analogous result for \( \langle O_\phi O_\phi O_\phi O_C \rangle = \langle O_C O_C O_C O_C \rangle \) is obtained by cross–symmetrization of (2.3).

The functions \( D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} \) admit simple integral representations (see Section 6.1) and can all be obtained as derivatives with respect to \( x_{ij} \) of a single function (see Section A.3). In Section 6 we develop asymptotic series expansions for \( D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} \) in the conformally invariant variables \( s \) and \( t \),

\[ t = \frac{x_{12}^2 x_{23}^2 - x_{14}^2 x_{23}^2}{x_{12}^2 x_{24}^2 + x_{14}^2 x_{23}^2}. \quad (2.6) \]

We consider the “direct” or \( t \)–channel limit \( |x_{13}| \ll |x_{12}|, |x_{24}| \ll |x_{12}| \) which corresponds to \( s, t \to 0 \). The singular power terms in this limit are given by

\[ I_{grav} \bigg|_{\text{sing}} = \frac{2^{10}}{35 \pi^6} \frac{1}{x_{13}^6 x_{24}^6} \left[ s \left( 7 t^2 + 6 t^4 \right) + s^2 \left( -7 - +3 t^2 \right) - 8 s^3 \right]. \quad (2.7) \]

In addition, as in [9, 10, 12, 13] we find an infinite series of terms logarithmic in \( s \):

\[ I_{grav} \bigg|_{\text{log}} = \frac{3 \cdot 2^3}{\pi^6} \frac{\ln s}{x_{13}^8 x_{24}^8} \sum_{k=0}^{\infty} s^{4+k} \frac{\Gamma(k+4)}{\Gamma(k+1)} \left\{ -2(5k^2 + 20k + 16)(3k^2 + 15k + 22)a_{k+3}(t) \\
+(k+4)^2(15k^2 + 55k^2 + 42)a_{k+4}(t) \right\}, \quad (2.8) \]

where the functions \( a_k(t) \) are given by

\[ a_k(t) = \int_{-1}^{1} d\lambda \frac{(1 - \lambda^2)^k}{(1 + \lambda t)^{k+1}} = \sqrt{\pi} \frac{\Gamma(k+1)}{\Gamma \left( k + \frac{3}{2} \right)} F \left( k + 1; \frac{k}{2} + 1; \frac{k}{2} + \frac{3}{2}; t^2 \right). \quad (2.9) \]
As clear from the hypergeometric representation, \( a_k(t) \) admit power series expansions in \( t^2 \) with radius of convergence 1. Here we do not display the non–singular power terms in \( I_{\text{grav}} \) (see Section 6.2).

The analogous result for the sum of the graphs s+u+q in Figure 2 is

\[
I_s + I_u + I_q \bigg|_{\log} = \frac{2^6 \cdot 3 \cdot 5}{\pi^6} \sum_{k=0}^{\infty} \ln s \cdot \frac{x_{13}^8 x_{24}^8}{s^{k+4}} \left\{ (k+1)^2 (k+2)^2 (k+3)^2 (3k+4) a_{k+3}(t) \right\}.
\]

(2.10)

The contribution \( I_s + I_u + I_q \) has no power singularities.

We now turn to discuss some physical implications of these results.

### 2.3 OPE interpretation

Let us compare the singular power terms of (2.7) with those expected from the OPE (1.1). In the direct channel limit \( |x_{13}| \ll |x_{12}|, |x_{24}| \ll |x_{12}| \) the leading terms of the variables \( s \) and \( t \) are

\[
s \sim \frac{x_{13}^2 x_{24}^2}{4 x_{12}^4} \quad t \sim -\frac{x_{13} \cdot J(x_{12}) \cdot x_{24}}{x_{12}^2}
\]

(2.11)

where \( J_{ij} = \delta_{ij} - 2y_i y_j / y^2 \) is the well–known Jacobian tensor of the conformal inversion \( y'_{i} = y_{i} / y^2 \). The leading term of (2.7) can then be written as

\[
|_{\text{sing}} \left. I_{\text{grav}} \right| = \frac{2^6}{5 \pi^6} \sum_{k=0}^{\infty} \ln s \frac{x_{13}^8 x_{24}^8}{s^{k+4}} \left\{ 4 \left( x_{13} \cdot J(x_{12}) \cdot x_{24} \right)^2 - x_{13}^2 x_{24}^2 \right\} + \ldots
\]

(2.12)

with subleading terms suppressed by powers of \( |x_{13}|/|x_{12}| \) and \( |x_{24}|/|x_{12}| \). We note from (1.1) that (2.12) describes the contribution to the OPE of an operator \( O_p \) of dimension \( \Delta = 4 \). We show below that the tensorial structure agrees with the the expected contribution of the stress–energy tensor of the boundary theory. It is worth mentioning first that various subcontributions to the amplitude \( I_{\text{grav}} \) (some of the \( D \) functions in (2.3)) have leading power \( 1/(x_{13}^6 x_{24}^6 x_{12}^4) \) indicative of a scalar operator of dimension \( \Delta = 2 \), which would not be expected in the graviton exchange process. The fact that this term cancels in the full amplitude is then an important check of the calculation.

Let us consider a scalar operator \( O_\Delta \) of scale–dimension \( \Delta \) in \( d \)–dimensional space–time. The contribution of the conserved traceless stress–tensor \( T_{ij} \) to the OPE of \( O_\Delta(x_1)O_\Delta(x_3) \) is

\[
O_\Delta(x_1)O_\Delta(x_3) \sim k \frac{x_{13}^i x_{13}^j}{x_{13}^{2\Delta+2-d}} T_{ij}(x_1)
\]

(2.13)

and the 2–point function of the stress tensor is

\[
\langle T_{ij}(x_1)T_{kl}(x_2) \rangle = \frac{c}{2} J_{ik}(x_{12})J_{jl}(x_{12}) + J_{il}(x_{12})J_{jk}(x_{12}) - \frac{2}{d} \delta_{ij} \delta_{kl}
\]

(2.14)
which is conserved and traceless in any dimension. Note that \( J_{ik}(y)J_{kj}(y) = \delta_{ij} \). We thus see that the stress tensor contribution to the general scalar double OPE is

\[
\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta'(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta'(x_4) \rangle \sim \frac{kck'}{d} \frac{d(x_{13} \cdot J(x_{12}) \cdot x_{24})^2 - x_{13}^2 x_{24}^2}{x_{13}^{2\Delta+2-d} x_{24}^{2\Delta'+2-d} x_{12}^{2\Delta+2\Delta'}}.
\]

This form is in perfect agreement with (2.12). Further relevant information on 2– and 3–point functions of the stress–energy tensor can be found in [27].

Let us now consider the logarithmic terms. We see from the sum of (2.8) and (2.10) that an infinite series of terms logarithmic in \( s \) occurs in the direct channel expansion of \( \langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle \).

Since the serieses (2.8) and (2.10) have a rather different structure, this conclusion appears quite robust. (In particular, it is insensitive to the relative normalization of \( I_{\text{grav}} \) and \( I_s + I_u + I_q \)). We plead exhaustion and excuse ourselves from carrying a similar analysis for the crossed channel limit of \( \langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle \) and for \( \langle \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \rangle \). The reader can find the necessary ingredients in Section 6.2. As mentioned in the Introduction, one should be able to interpret these logarithmic terms as \( 1/N^2 \) renormalization effects related to the contribution of composite operators to the OPE [11][23]. For example, the leading logarithmic term in the direct channel limit,

\[
\frac{1}{(x_{12})^{16}} \log \left( \frac{x_{13} x_{24}}{x_{12}^4} \right),
\]

could be related to the presence in (1.1) of the non–chiral composite operators : \( \mathcal{O}_\phi \mathcal{O}_\phi \) and : \( \mathcal{O}_C \mathcal{O}_C \). It is an interesting topic for future research to precisely identify the contributions of various composite operators, and the patterns of their renormalization and mixing, in the intricate series structures (2.8), (2.10). A detailed OPE interpretation of these supergravity results should provide us with new non–perturbative information about the \( \mathcal{N} = 4 \) SYM theory.

3 General set–up

As in most previous work on correlation functions, we work on the Euclidean continuation of \( \text{AdS}_{d+1} \), viewed as the upper half space in \( z_\mu \in \mathbb{R}^{d+1} \), with \( z_0 > 0 \). The metric \( g_{\mu\nu} \) and Christoffel symbols \( \Gamma^\kappa_{\mu\nu} \) are given by

\[
ds^2 = \sum_{\mu,\nu=0}^{d} g_{\mu\nu}dz_\mu dz_\nu = \frac{R^2}{z_0^2} (dz_0^2 + \sum_{i=1}^{d} dz_i^2)
\]

\[
\Gamma^\kappa_{\mu\nu} = \frac{1}{R z_0^2} (\delta_0^\kappa \delta_{\mu\nu} - \delta_\mu^\kappa \delta_{\nu0} - \delta_\nu^\kappa \delta_{\mu0})
\]

and the curvature scalar is \( R = -d(d+1)/R^2 \). We henceforth set the AdS scale \( R \equiv 1 \). This space is a maximally symmetric solution of the gravitational action

\[
S_g = -\frac{1}{2\kappa^2} \int dz \sqrt{g} (R - \Lambda)
\]
with \( \Lambda = -d(d-1) \).

It is well known that invariant bi–scalar functions on \( \text{AdS}_{d+1} \), such as scalar field propagators, are most simply expressed in terms of the chordal distance variable \( u \), defined by

\[
u \equiv \frac{(z-w)^2}{2z_0w_0}
\]

(3.19)

where \( (z-w)^2 = \delta_{\mu\nu}(z-w)_\mu(z-w)_\nu \) is the “flat Euclidean distance”. Invariant tensor functions, such as the gauge or the graviton propagator, may be expanded in terms of bases of invariant bi-tensors, which are derivatives of \( u \). For example, for rank 1, we have \((\partial_\mu = \partial/\partial z^\mu \) and \(\partial_{\nu'} = \partial/\partial w'^{\nu'} \))

\[
\partial_\mu u = \frac{1}{z_0} \left( \frac{(z-w)_\mu}{w_0} - u\delta_\mu 0 \right)
\]

(3.20)

\[
\partial_{\nu'} u = \frac{1}{w_0} \left( \frac{(w-z)_{\nu'}}{z_0} - u\delta_{\nu'0} \right)
\]

(3.21)

and for rank 2, there is \( \partial_\mu u \partial_{\nu'} u \) as well as

\[
\partial_\mu \partial_{\nu'} u = -\frac{1}{z_0 w_0} [\delta_{\mu\nu'} + \frac{1}{w_0} (z-w)_\mu \delta_{\nu'0} + \frac{1}{z_0} (w-z)_{\nu'} \delta_{\mu0} - u\delta_{\mu0} \delta_{\nu'0}].
\]

(3.22)

Throughout this paper, we shall also make use of differentiation and contraction relations between these basis tensors, which we list here,

\[
\Box u = D^\nu \partial_\nu u = (d+1)(1 + u)
\]

(3.23)

\[
D^\mu u \partial_\mu u = u(2 + u)
\]

(3.24)

\[
D_\mu \partial_{\nu'} u = g_{\mu\nu'}(1 + u)
\]

(3.25)

\[
(D^\mu u) (D_\mu \partial_{\nu'} u) = \partial_{\nu'} u \partial_\nu u
\]

(3.26)

\[
(D^\mu u) (\partial_\mu \partial_{\nu'} u) = (1 + u) \partial_{\nu'} u
\]

(3.27)

\[
(D^\mu \partial_{\mu'} u) (\partial_{\mu} \partial_{\nu'} u) = g_{\mu\nu'} + \partial_{\mu} u \partial_{\mu'} u.
\]

(3.28)

\[
\Box F(u) = u(u+2)F''(u) + (d+1)(1 + u)F'(u)
\]

(3.29)

These relations may be derived using (3.21), (3.22) and the metric and Christoffel symbols of (3.17) for \( \text{AdS}_{d+1} \).

### 3.1 Scalar and graviton propagators

The bulk–to–boundary propagator (or Poisson kernel) for a scalar field of mass \( m^2 = \Delta(\Delta - d) \) is well–known \[3, 4\] and given by

\[
K_\Delta(z, \vec{x}) = C_\Delta \tilde{K}_\Delta(z, \vec{x}) = C_\Delta \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta
\]

(3.30)
with the following normalization
\[ C_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})}. \]  

(3.31)

Bulk–to–bulk propagators for scalar fields of dimension \( \Delta \), with mass \( m^2 = \Delta(\Delta - d) \), were derived in [29]. They can be expressed as hypergeometric functions in several equivalent ways. The expression which appears best suited for the integrals which occur in exchange diagrams [12, 13] is to use a hypergeometric function whose argument is \( \xi \)
\[ \xi \equiv \frac{1}{1 + u} = \frac{2z_0w_0}{(z_0^2 + w_0^2 + (z - w)^2)}. \]  

(3.32)

The propagator is then given by
\[ G_{\Delta}(u) = 2^\Delta \tilde{C}_\Delta \xi^\Delta \text{F}\left(\frac{\Delta}{2}, \frac{\Delta}{2} + 1; \Delta - \frac{d}{2} + 1; \xi^2\right). \]  

(3.33)

The propagator for massless scalars, with \( \Delta = d \), is relevant for the graviton. When \( \Delta = d \) is an even integer, the hypergeometric expression (3.33) can be rewritten [21] in terms of elementary functions. In particular, for \( d = 4 \), we have
\[ G_4(u) = -\frac{1}{8\pi^2} \left\{ \frac{2(1 + u)}{\sqrt{u(2 + u)}} - \frac{1 + u}{\sqrt{u(2 + u)^3}} - 2 \right\}. \]  

(3.35)

The graviton propagator [21] can be expressed as a superposition of 5 independent fourth rank bi–tensors, of which 2 are gauge independent and 3 are gauge artifacts. The gauge terms represent pure diffeomorphisms, and their contribution to the integrals in the exchange diagram vanishes because the stress tensor is conserved. The physical part of the propagator involves two scalar functions \( G(u) \) and \( H(u) \), and is given by
\[ G_{\mu\nu\mu'\nu'}(z, w) = (\partial_{\mu} \partial_{\nu'} u \partial_{\nu} \partial_{\mu'} u + \partial_{\mu} \partial_{\nu'} u \partial_{\nu} \partial_{\mu'} u) G(u) + g_{\mu\nu} g_{\mu'\nu'} H(u). \]  

(3.36)

The function \( G(u) \) is equal to the massless scalar propagator \( G_d \).

A representation of \( H(u) \) as a hypergeometric function was given in [21]. It was also expressed in terms of \( G(u) \) and its first integral \( \bar{G}(u) \), defined by \( \bar{G}(u)' = G(u) \) and the boundary condition \( \bar{G}(\infty) = 0 \), which is a more useful form, given by
\[ -(d - 1)H(u) = 2(1 + u)^2 G(u) + 2(d - 2)(1 + u)\bar{G}(u). \]  

(3.37)
Again, when $d$ is even, $H(u)$ admits an elementary expression; in particular, when $d = 4$, we have

$$H(u) = -\frac{1}{12\pi^2}(-6(1+u)^4 + 9(1+u)^2 - 2) \frac{1+u}{(u(2+u))^2} - \frac{1}{2\pi^2}(1+u)^2. \quad (3.38)$$

### 3.2 Structure of the graviton exchange amplitude

The graviton exchange amplitude associated with the Wittendiagram of Figure 2t is given by

$$I_{grav} = \frac{1}{4} \int dz \sqrt{g} \int dw \sqrt{g} T_{13}^{\mu\nu}(z) G_{\mu\nu\mu'\nu'}(z, w) T_{24}^{\mu'\nu'}(w) \quad (3.39)$$

where $G_{\mu\nu\mu'\nu'}$ is the graviton propagator (3.36). The vertex factor $T_{13}^{\mu\nu}(z)$ is given by

$$T_{13}^{\mu\nu}(z) = D^{\mu}K_\Delta(z, x_1)D^{\nu}K_\Delta(z, x_3) + D^{\nu}K_\Delta(z, x_1)D^{\mu}K_\Delta(z, x_3)$$

$$-g^{\mu\nu}[\partial_\rho K_\Delta(z, x_1)D^{\rho}K_\Delta(z, x_3) + m^2K_\Delta(z, x_1)K_\Delta(z, x_3)]. \quad (3.40)$$

The combination $T_{24}^{\mu'\nu'}(w)$ is obtained from (3.40) by replacing $x_1 \to x_2$, $x_3 \to x_4$, $\Delta \to \Delta'$ $z \to w$. The stress–energy tensor $T_{\mu\nu}$ is conserved, $D_\mu T_{13}^{\mu\nu} = D_\mu T_{24}^{\mu'\nu'} = 0$ thanks to the propagator equations $(\Box - m^2)K_\Delta = (\Box - m'^2)K_{\Delta'} = 0$.

It is the high tensorial rank of the propagator and vertex factors that make this amplitude more difficult than previously studied exchanges. The calculation is made tractable by splitting the amplitude into several terms and using partial integration of derivatives. There are several ways to organize this process, and what we have done and will present are complete calculations by two different methods which are then compared and shown to give identical results for the special case $d = \Delta = \Delta' = 4$, i.e. axions and dilatons in the type IIB theory. The two methods are separately presented in Sections 4 and 5.

### 4 The graviton exchange graph for $\Delta = \Delta' = d = 4$

The graviton propagator involves non–trivial tensorial structures. Nevertheless, it turns out that it is possible to reduce the graviton exchange graph to the sum of purely scalar amplitudes, with a peculiar pattern of bulk–to–bulk and bulk–to–boundary scalar propagators. We describe this reduction in Section 4.1.

Furthermore, upon integration over one of the two bulk variables, which we carry out in Section 4.2, each effective scalar exchange can be expressed a sum of quartic graphs with appropriate external dimensions. The final answer for the graviton exchange in terms of these basic
building blocks (see Figure 6) is given in equation (4.38). The quartic graphs admit asymptotic series expansion which we describe in Section 6. It is also worth mentioning at this point that each quartic graph can be obtained by taking successive derivatives of a single basic function, see section A.3.

### 4.1 Reduction to scalar exchanges

We need to compute the graviton exchange amplitude (3.39) for $m^2 = m'^2 = 0$. Using the form (3.36) for the graviton propagator, we have:

\[
I_{\text{grav}} = (C_4)^4 (I_H + I_G)
\]

\[
I_H = \int [dz] [dw] \left[ \partial^\mu \tilde{K}_4(z,x_1) \partial^\nu \tilde{K}_4(z,x_3) - \frac{1}{2} g^\mu_\nu \partial_\lambda \tilde{K}_4(z,x_1) \partial^\lambda \tilde{K}_4(z,x_3) \right] \cdot g_{\mu\nu} g_{\mu'\nu'} H(u) \left[ \partial^\mu' \tilde{K}_4(w,x_2) \partial^\nu' \tilde{K}_4(w,x_4) - \frac{1}{2} g^{\mu'\nu'} \partial_\lambda' \tilde{K}_4(w,x_2) \partial^\lambda' \tilde{K}_4(w,x_4) \right]
\]

\[
I_G = \int [dz] [dw] \left[ \partial^\mu \tilde{K}_4(z,x_1) \partial^\nu \tilde{K}_4(z,x_3) - \frac{1}{2} g^\mu_\nu \partial_\lambda \tilde{K}_4(z,x_1) \partial^\lambda \tilde{K}_4(z,x_3) \right] \cdot (\partial_\mu \partial_\nu u \partial_\nu' u + \partial_\mu \partial_\nu u \partial_\nu' u) G(u) \cdot \left[ \partial^\mu' \tilde{K}_4(w,x_2) \partial^\nu' \tilde{K}_4(w,x_4) - \frac{1}{2} g^{\mu'\nu'} \partial_\lambda' \tilde{K}_4(w,x_2) \partial^\lambda' \tilde{K}_4(w,x_4) \right]
\]

where $C_4 = \frac{6}{\pi^2}$ is the normalization factor (3.31) of the bulk–to–boundary propagator. The tensorial structures in $I_H$ immediately trivialize:

\[
I_H = \left(1 - \frac{5}{2}\right)^2 \int [dz] [dw] \partial_\mu \tilde{K}_4 \partial^\mu \tilde{K}_4 H(u) \partial_\mu' \tilde{K}_4 \partial^\mu' \tilde{K}_4
\]

\[
= \left(\frac{9}{4}\right) \int [dz] [dw] \tilde{K}_4 \tilde{K}_4 \frac{1}{4} \Box^2 H(u) \tilde{K}_4 \tilde{K}_4
\]

\footnote{We introduce the notation $[dz] \equiv \sqrt{g} d^5z$.}
where we have used integration by parts and the equation of motion \( \Box \tilde{K}_4 = 0 \) to eliminate the derivatives on the \( \tilde{K} \)'s.

Now we consider \( I_G \), and it is useful to split into 4 parts:

\[
I_G = I_G^1 + I_G^2 + I_G^3 + I_G^4
\]

\[
I_G^1 = \int [dz][dw] \partial^\mu \tilde{K}_4(z, x_1) \partial^\nu \tilde{K}_4(z, x_2) \cdot (\partial_\mu \partial_\nu u \partial_\rho \partial_\sigma u + \partial_\rho \partial_\sigma u \partial_\mu \partial_\nu u) G(u) \partial^{\mu'} \tilde{K}_4(w, x_2) \partial^{\nu'} \tilde{K}_4(w, x_4)
\]

\[
I_G^2 = \int [dz][dw] \partial^\mu \tilde{K}_4(z, x_1) \partial^\nu \tilde{K}_4(z, x_3) \cdot \left( \partial_\mu \partial_\nu u \partial_\rho \partial_\sigma u + \partial_\rho \partial_\sigma u \partial_\mu \partial_\nu u \right) G(u) \left( -\frac{1}{2} g^{\mu \nu'} \partial_\lambda \tilde{K}_4(w, x_2) \partial^\lambda \tilde{K}_4(w, x_4) \right)
\]

\[
I_G^3 = \int [dz][dw] \left( -\frac{1}{2} g^{\mu \nu} \partial_\lambda \tilde{K}_4(z, x_1) \partial^\lambda \tilde{K}_4(z, x_3) \right) \cdot \left( \partial_\mu \partial_\nu u \partial_\rho \partial_\sigma u + \partial_\rho \partial_\sigma u \partial_\mu \partial_\nu u \right) G(u) \left( -\frac{1}{2} g^{\mu' \nu'} \partial_\lambda \tilde{K}_4(w, x_2) \partial^\lambda \tilde{K}_4(w, x_4) \right)
\]

\[
I_G^4 = \int [dz][dw] \left( -\frac{1}{2} g^{\mu \nu} \partial_\lambda \tilde{K}_4(z, x_1) \partial^\lambda \tilde{K}_4(z, x_3) \right) \cdot \left( \partial_\mu \partial_\nu u \partial_\rho \partial_\sigma u + \partial_\rho \partial_\sigma u \partial_\mu \partial_\nu u \right) G(u) \left( -\frac{1}{2} g^{\mu' \nu'} \partial_\lambda \tilde{K}_4(w, x_2) \partial^\lambda \tilde{K}_4(w, x_4) \right)
\]

We wish to eliminate all the tensor indices and all the derivatives, so that the graviton exchange is reduced to a sum of effective scalar graphs. With this program in mind, we observe a few pretty identities. First:

\[
\partial^\mu \tilde{K}_\Delta(z, x_1) \partial_\mu \partial_\nu u \partial^{\nu'} \tilde{K}_\Delta(w, x_2) = \\
\Delta^2 \left[ -\tilde{K}_\Delta(z, x_1) \tilde{K}_{\Delta+1}(w, x_2) - \tilde{K}_\Delta(z, x_1) \tilde{K}_{\Delta-1}(w, x_2) - \tilde{K}_{\Delta+1}(z, x_1) \tilde{K}_\Delta(w, x_2) \tilde{K}_{\Delta-1}(w, x_1) \\
+ 2x_{12}^2 \tilde{K}_{\Delta+1}(z, x_1) \tilde{K}_{\Delta+1}(w, x_2) + (1 + u) \tilde{K}_\Delta(z, x_1) \tilde{K}_\Delta(w, x_2) \right]
\]

(4.10)

It is simplest to verify this identity by the methods described in [4], where one uses conformal transformations to go to a coordinate system where point \( x_1 \) is mapped to infinity and point \( x_2 \) to zero. Further:

\[
\tilde{K}_{\Delta+1}(z, x_1) \tilde{K}_{\Delta-1}(w, x_2) = \frac{1}{\Delta} \partial^\mu \tilde{K}_\Delta(z, x_1) \partial_\mu u + (1 + u) \tilde{K}_\Delta(z, x_1)
\]

(4.11)

Inserting twice (4.11) into (4.10) we get:

\[
\partial^\mu \tilde{K}_\Delta(z, x_1) \partial_\mu \partial_\nu u \partial^{\nu'} \tilde{K}_\Delta(w, x_2) = \\
\Delta^2 \left[ -\frac{1}{\Delta} \tilde{K}_\Delta(z, x_1) \partial^{\mu'} \tilde{K}_\Delta(w, x_2) \partial_\mu u - \frac{1}{\Delta} \partial^\mu \tilde{K}_\Delta(z, x_1) \partial_\mu u \tilde{K}_\Delta(w, x_2) \\
+ 2x_{12}^2 \tilde{K}_{\Delta+1}(z, x_1) \tilde{K}_{\Delta+1}(w, x_2) - (1 + u) \tilde{K}_\Delta(z, x_1) \tilde{K}_\Delta(w, x_2) \right]
\]

(4.12)

We now evaluate (4.6-4.9) one by one.
Writing (4.6) as

\[ I_1^G = \int [dz][dw] \left( \partial^\mu \bar{K}_4(z, x_1) \partial_\mu \partial_{\mu'} u \partial^{\mu'} \bar{K}_4(w, x_2) \right) G(u) \times \frac{\partial^\nu \bar{K}_4(z, x_3)}{\partial_\nu} \partial_{\nu'} \partial_{\nu'} u \partial^{\nu'} \bar{K}_4(w, x_4) + \{ x_1 \leftrightarrow x_3 \} \tag{4.13} \]

and inserting twice (4.12) for \( \Delta = 4 \) we obtain 16 + 16 terms many of which are related by a simple symmetrization. Below we present the manipulations performed on the inequivalent terms. We often suppress the coordinate labels, and give the expressions with the propagators in the following order: \((z, x_1), (z, x_3), (w, x_2), (w, x_1)\) unless stated otherwise. Referring to the terms in (4.12) we get:

**I \times I:**

\[
I \times I = 4^2 \int [dz\, dw] \bar{K}_4 \bar{K}_4 G \partial^\mu \bar{K}_4 \partial_\mu u \partial^{\mu'} \bar{K}_4 \partial_{\nu'} u = 4^2 \int [dz\, dw] \bar{K}_4 \bar{K}_4 \left[ D_{\mu'} \bar{K}_4 \int \int u G - g_{\mu'\nu'}(1 + u) \int u G \right] \times \left( T^{\nu\nu'} + \frac{1}{2} g^{\mu'\nu'}(1 + u) \int u G \right) = 4^2 \int [dz\, dw] \bar{K}_4 \bar{K}_4 \left[ \frac{1}{4} \Box^2 \int \int u G - \frac{1}{2} \Box \left\{ (1 + u) \int u G \right\} \right] \bar{K}_4 \bar{K}_4 \tag{4.14} \]

Here we have used\(^6\)

\[ G(u) \partial_{\mu'} \partial_{\nu'} u = [D_{\mu'} \partial_{\nu'} \int \int u G - g_{\mu'\nu'}(1 + u) \int u G] \]

thanks to (3.25) and we also used the conservation of the stress-energy tensor integrating by parts to get the last equality.

**I \times II:**

\[
I \times II = 4^2 \int [dz\, dw] \bar{K}_4 \partial^\mu \bar{K}_4 \partial_\mu u G \partial^{\mu'} \bar{K}_4 \partial_{\mu'} u \partial^{\nu'} \bar{K}_4 \partial_{\nu'} u = \frac{1}{4} \Box^2 \int \int u G \bar{K}_4 \bar{K}_4 \tag{4.15} \]

Using \( \partial_{\mu'} u \partial^{\mu'} u = \partial_\mu \int u G \) we get by integration by parts:

\[
I \times II = -4^2 \int [dz\, dw] \bar{K}_4 \partial^\mu \bar{K}_4 \partial_\mu u \int u G \partial^{\mu'} \bar{K}_4 \partial_{\mu'} \bar{K}_4 \tag{4.16} \]

\[
-I \times II = -4^2 \int [dz\, dw] \bar{K}_4 \partial^{\mu'} \bar{K}_4 \partial_{\mu'} u \int u G \partial_\mu \partial_{\mu'} \bar{K}_4 \bar{K}_4 \]

where we have used \( \Box \bar{K}_4 = 0 \) in the bulk. The first term in (4.16) can be easily processed to give

\[
4^2 \int [dz\, dw] \bar{K}_4 \bar{K}_4 \frac{1}{4} \Box^2 \int \int u G \bar{K}_4 \bar{K}_4 \tag{4.17} \]

\(^6\)Here our convention is that \( \int u F = \int_a^u F(u) \, dw \), where \( a \) is chosen to ensure the fastest possible falloff of \( \int u F \) in the \( u \to \infty \) limit.
The second term in (4.16) is handled by inserting again the identity (4.12) with \( x_1 \leftrightarrow x_3 \) and going through by now familiar manipulations. It gives

\[
\int [dz \, dw] \bar{K}_4 \bar{K}_4 \left[ -4^3 \Box \int \int G + 4^4(1 + u) \int G \right] \bar{K}_4 \bar{K}_4 - 2 \cdot 4^4 x_{34}^2 \int [dz \, dw] \bar{K}_4 \bar{K}_5 \int \int G \bar{K}_5 \bar{K}_4
\]

(4.18)

I \times III: Upon integration by parts,

\[
I \times III = 2 \cdot 4^3 x_{34}^2 \int [dz \, dw] \bar{K}_4 \bar{K}_5 \int G \, \partial^\mu \bar{K}_4 \, \partial_{\mu} \bar{K}_5
\]

(4.19)

III \times III:

\[
4 \cdot 4^4 x_{12}^2 x_{34}^2 \int [dz \, dw] \bar{K}_5 \bar{K}_5 G \bar{K}_5 \bar{K}_5
\]

(4.21)

III \times IV:

\[
-2 \cdot 4^4 x_{12}^2 \int [dz \, dw] \bar{K}_5 \bar{K}_4 G(1 + u) \bar{K}_5 \bar{K}_4
\]

(4.22)

IV \times IV:

\[
4^4 \int [dz \, dw] \bar{K}_4 \bar{K}_4 G(1 + u)^2 \bar{K}_4 \bar{K}_4
\]

(4.23)

4.1.2 \( I_G^2, I_G^3 \) and \( I_G^4 \)

Using (8.24), after some similar algebra we arrive at

\[
I_G^2 = I_G^3 = - \int [dz \, dw] \bar{K}_4 \bar{K}_4 \frac{1}{4} \Box^2 \left[ G + \frac{3}{2} \left(1 + u\right) \int G + \frac{1}{2} u(1 + u) G \right] \bar{K}_4 \bar{K}_4
\]

(4.24)

\[
I_G^4 = \frac{1}{2} \int [dz \, dw] \bar{K}_4 \bar{K}_4 \frac{1}{4} \Box^2 \left[ 5G + u(1 + u) G \right] \bar{K}_4 \bar{K}_4
\]

(4.25)

4.1.3 The graviton amplitude in terms of scalar exchanges

Adding all the terms above with the appropriate symmetrizations we get the complete graviton graph in terms of effective scalar exchanges:

\[
\frac{I_{\text{grav}}}{(C_4)^4} = \int [dz \, dw] \bar{K}_4 \bar{K}_4 \left[ \Box^2 \left( \frac{9}{16} H + 2 \cdot 4^2 \int \int G + \frac{1}{8} G - \frac{3}{4} \left(1 + u\right) \int G - \frac{1}{8} u(1 + u) G \right) \right]
\]

16
$$\begin{align*}
+ \Box \left( -4^4 \int^u G + \frac{1}{2} 4^3 (1 + u) \int^u G - 4^4 \int^u G \right) \\
+ 4^5 (1 + u) \int^u G + 2 \cdot 4^4 (1 + u)^2 G \right] \tilde{K}_4 \tilde{K}_4 \\
+ x_{34}^2 \int [dz\, dw] \tilde{K}_4 \tilde{K}_5 \left[ -18 \cdot 4^3 \int^u G + 2 \cdot 4^3 \int^u G - 2 \cdot 4^4 G (1 + u) \right] \tilde{K}_4 \tilde{K}_5 \\
+ \{3 \, \text{perms}\} \\
+ (x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2) \int [dz\, dw] \tilde{K}_5 \tilde{K}_5 \quad 4 \cdot 4^4 G \tilde{K}_5 \tilde{K}_5
\end{align*}$$

(4.26)

where the 3 permutations of the second integral are obtained by exchanging \((x_1, x_2) \leftrightarrow (x_3, x_4)\) and \(x_1 \leftrightarrow x_3\). The formula above can be simplified by explicit application of Laplace operator \((3.29)\) and using the equations obeyed by \(G\) and \(H\) given in Section 3.1. We get

$$\begin{align*}
\frac{L_{\text{grav}}}{(C_4)^4} &= \int [dz\, dw] \tilde{K}_4 \tilde{K}_5 \left[ -72 u^2 - 144 u + 168 \right] G + 168 (u + 1) \int^u G \right] \tilde{K}_4 \tilde{K}_4 \\
+ x_{34}^2 \int [dz\, dw] \tilde{K}_4 \tilde{K}_5 \left[ -768 \int^u G - 256 G (1 + u) \right] \tilde{K}_4 \tilde{K}_5 \\
+ \{3 \, \text{perms}\} \\
+ (x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2) \int [dz\, dw] \tilde{K}_5 \tilde{K}_5 \quad 1024 G \tilde{K}_5 \tilde{K}_5 \\
+ 10 \int [dw] \tilde{K}_4 \tilde{K}_4 \tilde{K}_4 \tilde{K}_4 - 16 x_{24}^2 \int [dw] \tilde{K}_4 \tilde{K}_4 \quad (4.27)
\end{align*}$$

(4.27)

The last two terms in this expression arise from delta functions generated by the application of the Laplace operator\(\Box\). In particular the last term comes from:

$$\begin{align*}
\int [dz\, dw] \tilde{K}_4 \tilde{K}_4 \Box \delta(z, w) \tilde{K}_4 \tilde{K}_4 \\
= 2 \int [dw] \tilde{K}_4 \tilde{K}_4 \partial_{\mu} \tilde{K}_4 \partial^{\mu} \tilde{K}_4 = 2 \int [dw] \tilde{K}_4 \tilde{K}_4 \left( 16 \tilde{K}_4 \tilde{K}_4 - 32 x_{24}^2 \tilde{K}_5 \tilde{K}_5 \right)
\end{align*}$$

(4.28)

where in the last equality we have used \((A.5)\).

### 4.2 Reduction to quartic graphs

We first observe the identity

$$\int^u G = -G_3 + (1 + u) G,$$

(4.29)

where \(G_3\) is a scalar propagator of \(m^2 = -3\), corresponding to a boundary conformal dimension \(\Delta = 3\):

$$- (\Box + 3) G_3 = \delta(z, w).$$

(4.30)

\(^7\)The coordinate dependence of the \(K\)'s is: \((w, x_1), (w, x_3), (w, x_2), (w, x_4)\).
Using (4.29), we see that the complete graviton graph (4.27) involves effective scalar exchanges of the form

\[ I_{\Delta_1,\Delta_3,\Delta_2,\Delta_4}^{\Delta_5,p} = \int [dz \, dw] \bar{K}_{\Delta_1} \bar{K}_{\Delta_3} (1 + u)^p G_{\Delta_5} \bar{K}_{\Delta_2} \bar{K}_{\Delta_4}, \quad (4.31) \]

plus some quartic interactions (last line of (4.27)).

We now proceed to derive a general formula to perform the \( z \) integration in (4.31), following the methods developed in [13]. Quite remarkably, upon integration over \( z \), (4.31) reduces to a finite sum of effective quartic graphs, see Figure 6.

Translating \( x_1 \to 0 \) and performing conformal inversion (see [3] for a detailed account), we can write

\[ I_{\Delta_1,\Delta_3,\Delta_2,\Delta_4}^{\Delta_5,p} = |x_{31}|^{-2\Delta_1} |x_{21}|^{-2\Delta_2} |x_{41}|^{-2\Delta_4} \int [dw] R(w - x'_{31}) \bar{K}_{\Delta_2}(w, x'_{21}) \bar{K}_{\Delta_4}(w, x'_{41}), \quad (4.32) \]

where \( x' \equiv \vec{x}/x^2 \) and

\[ R_{\Delta_1,\Delta_3}^{\Delta_5,p}(w) = \int [dz] z^{\Delta_1} \bar{K}_{\Delta_3}(z)(1 + u)^p G_{\Delta_5}(u). \quad (4.33) \]

As compared to [13] we allow the bulk propagator to be multiplied by \((1 + u)^p\) (see (3.3–3.4) in [13]). We now use the hypergeometric series expansion (3.33). Inserting this series into the expression for \( R_{\Delta_1,\Delta_3}^{\Delta_5,p} \), we can perform the \( z \) integral term by term by a standard Feynman parameterization, and resum the resulting series. We get

\[ R_{\Delta_1,\Delta_3}^{\Delta_5,p}(w) = 2^{\Delta_5-p+1} C_{\Delta_5, \pi^{d/2}} \frac{\Gamma[\frac{1}{2}(\Delta_5 - p + \Delta_3 - \Delta_1)] \Gamma[\frac{1}{2}(\Delta_5 - p + \Delta_1 + \Delta_3 - d)]}{\Gamma[\Delta - p] \Gamma[\Delta_3]} \times \left( \frac{w_0}{w^2} \right)^{\frac{\Delta_1}{w^{\Delta_1 - \Delta_3}}} \int_0^1 d\gamma \frac{(1 - \gamma)^{\Delta_1 - 1} \gamma^{\Delta_1 - \frac{1}{2}(\Delta_5 - p - \Delta_1 - \Delta_3) - 1}}{\left( \frac{w_0}{w^2} + \gamma - \frac{w_0}{w^2} \right)^{\Delta_1}} \right. \\
\times 4 F_3 \left( \Delta_5, \Delta_5 + \frac{1}{2}, \Delta_5 - p; \Delta_1, \Delta_5 - p + \Delta_1, \Delta_5 + \Delta_3 - d; \Delta_5 - \frac{d}{2} + 1, \Delta_5 - p, \Delta_5 - p + \frac{1}{2}; \gamma \right) \quad (4.34) \]

For \( p = 0 \) we recover equation (3.11) in [13]. It turns out that for the cases relevant to the graviton amplitude, the hypergeometric function \( 4 F_3 \) is elementary and the Feynman parameter integral can be explicitly done. The result is always a simple binomial in \( w_0 \) and \( w_0/w^2 \). The relevant cases are:

\[ R_{4,4}^{4,0} = \frac{1}{36} w_0^3 \left( \frac{w_0}{w^2} \right)^3 + \frac{1}{48} w_0^2 \left( \frac{w_0}{w^2} \right)^2 \]
\[ R_{4,4}^{4,2} = \frac{1}{36} w_0^3 \left( \frac{w_0}{w^2} \right)^3 + \frac{7}{288} w_0^2 \left( \frac{w_0}{w^2} \right)^2 + \frac{1}{48} w_0 \left( \frac{w_0}{w^2} \right) \]
\[ R_{4,4}^{3,1} = \frac{1}{36} w_0^3 \left( \frac{w_0}{w^2} \right)^3 + \frac{1}{36} w_0^2 \left( \frac{w_0}{w^2} \right)^2 + \frac{1}{36} w_0 \left( \frac{w_0}{w^2} \right) \quad (4.35) \]
We see that each term in $R^{3,0}_{4,5}$ is a product of bulk–to–boundary propagators. Indeed, $w_0^\Delta$ corresponds in this inverted frame to a propagator at $\vec{x}' = \infty$, likewise $(w_0/w^2)\Delta$ corresponds to a propagator at $\vec{x}' = 0$. Inserting each such term in the expression for $I^{\Delta p}_{\Delta_1\Delta_2\Delta_4}$ (equ. (4.32)), and going back from the inverted variables $\vec{x}'_i$ to the original variables $\vec{x}_i$, we recognize the integral defining a quartic graph. For example

$$
|x_{31}|^{-2\Delta_1} |x_{21}|^{-2\Delta_2} |x_{41}|^{-2\Delta_4} \int [dw] w_0^{2\Delta} \left( \frac{w_0}{(w - x_{31})^2} \right)^{\bar{\Delta}} \bar{K}_{\Delta_2}(w, x_{21}) \bar{K}_{\Delta_4}(w, x_{41})
$$

$$
= \int [dw] \left( \frac{w_0}{(w - x_1)^2} \right)^\Delta \left( \frac{w_0}{(w - x_3)^2} \right)^\Delta \left( \frac{w_0}{(w - x_4)^2} \right)^\Delta \left( \frac{w_0}{(w - x_2)^2} \right)^\Delta
$$

$$
\equiv D_{\Delta\Delta\Delta_2\Delta_4}(x_1, x_3, x_2, x_4),
$$
(4.36)

where in the last line we have used the important notation for quartic graphs (see Figure 5) introduced in (A.1). We can finally write the full graviton amplitude as a sum of quartic graphs:

$$
I_{\text{grav}} = \left( \frac{6}{\pi^2} \right)^4 \left[ 16 \left( x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2 - x_{24}^2 \right) D_{4455} + \frac{128}{9} x_{13}^4 \left( x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2 \right) D_{3355} \right.
$$

$$
+ \frac{32}{3} x_{13}^6 \left( x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2 \right) D_{2255} + 10 D_{4444} + \frac{14}{3} x_{13}^4 D_{3344} + \frac{8}{3} x_{13}^4 D_{2244} \left.
$$

$$
- \frac{8}{3} x_{13}^4 D_{1144} - \frac{16}{3} x_{13}^4 \left( x_{12}^2 D_{4354} + x_{14}^2 D_{4345} + x_{34}^2 D_{3445} + x_{23}^2 D_{3454} \right)
$$

$$
- \frac{32}{9} x_{13}^4 \left( x_{12}^2 D_{3254} + x_{14}^2 D_{3245} + x_{34}^2 D_{2345} + x_{23}^2 D_{2354} \right) \right].
$$
(4.37)

The graviton amplitude (5.39) is symmetric under $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_4$. These symmetries are explicit in the final expression for $I_{\text{grav}}$, indeed some of the $D$ functions (of the form $D_{\Delta\Delta\Delta\Delta}$) are symmetric by themselves, while asymmetric $D$ functions appear in all the symmetric permutations. It turns out that thanks to the remarkable properties of the $D$ functions (see equ. (A.11)), the answer can be rewritten in terms of $D_{\Delta\Delta\Delta\Delta}$’s alone. Introducing the conformal invariant variable $s \equiv \frac{1}{2} x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2$, we get

$$
I_{\text{grav}} = \left( \frac{6}{\pi^2} \right)^4 \left[ 16 x_{24}^2 \left( \frac{1}{2s} - 1 \right) D_{4455} + \frac{64 x_{24}^2}{9} \frac{1}{x_{13}^2} s D_{3355} + \frac{16 x_{24}^2}{3} \frac{1}{x_{13}^2} s D_{2255} \right].
$$
(4.38)
The graviton amplitude (3.39) is, for the case \( \Delta = \Delta' = 4 \) that we are considering, also symmetric under \((x_1, x_3) \leftrightarrow (x_2, x_4)\). Although not immediately manifest in the expression above, this symmetry is actually present thanks to the identity (A.1) obeyed by the \( D \) functions.

5 General graviton exchange graph

We expect that the amplitudes for graviton exchange between massive scalars will be useful in general studies of the AdS/CFT correspondence. As in past work [10, 13] we therefore assume initially that \( d, \Delta, \) and \( \Delta' \) are arbitrary, constrained only by the unitarity bound \( \Delta, \Delta' \geq d/2 \). We will assume integer values at the point where this step simplifies the calculation, and specialize still later to the case \( d = \Delta = \Delta' = 4 \) to present detailed asymptotic formulas for dilatons and axions in the type IIB supergravity.

The first step in the evaluation of the amplitude (3.39) is to split it into contributions from the terms in \( H(u) \) and \( G(u) \) in the graviton propagator, and to split the latter into a term proportional to the metric \( g^{\mu\nu} \) in \( T_{24}^{\mu\nu}(z) \) of (3.40) plus the remaining term, viz.

\[
I_{\text{grav}} = \frac{1}{4} A_{\text{grav}} = \frac{1}{4} (A^H + A_S^G + A_T^G). \tag{5.1}
\]

The three contributions are then given by

\[
A_S^G = \int dz \sqrt{g} \int dw \sqrt{g} [\partial_\rho K(1) D^\rho K(3) + m^2 K(1) K(3)](z) I_{\mu'\nu'}(z, w) T_{24}^{\mu'\nu'}(w) \tag{5.2}
\]

\[
A_T^G = 2 \int dz \sqrt{g} \int dw \sqrt{g} \partial_\mu K(1) \partial_\nu K(3) D^\mu \partial_{\mu'} u D^\nu \partial_{\nu'} u G(u) T_{24}^{\mu'\nu'}(w) + (1 \leftrightarrow 3) \tag{5.3}
\]

\[
A^H = \int dz \sqrt{g} \int dw \sqrt{g} g \cdot T_{13}(z) H(u) g \cdot T_{24}(w) \tag{5.4}
\]

where we use the abbreviation \( g \cdot T = g_{\mu\nu} T^{\mu\nu} \), and

\[
I_{\mu'\nu'} \equiv -g_{\mu\nu} G(u)[D^\mu \partial_{\mu'} u D^\nu \partial_{\nu'} u + D^\nu \partial_{\nu'} u D^\mu \partial_{\mu'} u] \]

\[
= -2G(u) (g_{\mu'\nu'} + \partial_{\mu'} u \partial_{\nu'} u) \tag{5.5}
\]

where (3.27) is used to obtain the second line in (5.5). The symmetrization in 1 \( \leftrightarrow \) 3 in (5.3) will be useful for later steps.

The \( w \)-integral in \( A_S^G \) that involves the tensor \( \partial_{\mu'} u \partial_{\nu'} u \) of (5.5) may be simplified by using \( \partial_{\mu'} u G(u) = \partial_{\mu'} \tilde{G}(u) \), integrating by parts in \( w \) and using the covariant conservation of \( T_{24} \),

\[
\int dw \sqrt{g} G(u) \partial_{\mu'} u \partial_{\nu'} u T_{24}^{\mu'\nu'}(w) = -\int dw \sqrt{g} \tilde{G}(u) D_{\mu'} \partial_{\nu'} u T_{24}^{\mu'\nu'}(w)
\]
Putting together this rearrangement of the \( A^G_S \) part, we have

\[
A^G_S = \int dz \sqrt{g} \int dw \sqrt{|g|} \partial_{\mu} K(1) D^\mu K(3) + m^2 K(1)K(3)](z) \
\times \{-2G(u) + 2(1 + u)\bar{G}(u)\} g \cdot T_{24}(w)
\]  

Next, we use the propagator equations \((\Box - m^2)K(1) = (\Box - m^2)K(3) = 0\) to obtain the following identity

\[
[\partial_{\mu} K(1) D^\mu K(3) + m^2 K(1)K(3)](z) = \frac{1}{2} \Box_z \{K(1)K(3)\}(z)
\]  

Substituting this identity into \( A^G_S \), integrating by parts the operator \( \Box_z \), neglecting vanishing boundary terms and using (3.29), we find

\[
\Box_z \{(1 + u)\bar{G}(u)\} = -2G(u) + 4(1 + u)^2G(u) + 2d(1 + u)\bar{G}(u)
\]  

which then gives

\[
A^G_S = \int dz \sqrt{g} \int dw \sqrt{|g|} K(1)K(3)\{-\Box_z G(u) - 2G(u) + 4(1 + u)^2G(u) + 2d(1 + u)\bar{G}(u)\} g \cdot T_{24}(w)
\]  

Before simplifying the \( g \cdot T_{24} \) factor in the integrand, we first treat \( A^G_T \) and \( A^H \) in a similar manner. For \( A^H \), we use again (5.8) to simplify the \( z \)-integration and to cast it in the following form

\[
A^H = \int dz \sqrt{g} \int dw \sqrt{|g|} K(1)K(3)\{-\frac{1}{2}(d - 1)\Box_z H(u) - 2m^2 H(u)\} g \cdot T_{24}(w)
\]  

To simplify \( A^G_T \), we begin with partial integration of \( \partial_{\nu} \) in the \( z \)-integral in (5.3), and split \( A^G_T \) as follows

\[
A^G_T = -2A^G_{T1} - 2A^G_{T2}
\]  

where

\[
A^G_{T1} = \int dz \sqrt{g} \int dw \sqrt{|g|} D_{\nu}[D^\mu \partial_{\mu'} u D^\nu \partial_{\nu'} u G(u)]T_{24}^{\mu_{\nu'}}(w)
\]  

\[
A^G_{T2} = \int dz \sqrt{g} \int dw \sqrt{|g|} D_{\nu} \partial_{\mu} K(1)K(3)[D^\mu \partial_{\nu'} u D^\nu \partial_{\nu'} u G(u)]T_{24}^{\mu_{\nu'}}(w) + (1 \leftrightarrow 3).
\]  

Now, \( A^G_{T1} \) may be simplified by working out the tensor algebra using (3.23–3.27) and again \( \partial_{\nu'} u G(u) = \partial_{\nu'} \bar{G}(u) \) to obtain

\[
D_{\nu}[D^\mu \partial_{\mu'} u D^\nu \partial_{\nu'} u G(u)] = D_{\mu'}(\ldots) + D_{\nu'}(\ldots) - D^\mu u g_{\mu\nu'} J(u)
\]

\[
J(u) = (1 + u)G(u) + (d + 1)\bar{G}(u)
\]
The terms with $D_{\mu'}$ and $D_{\nu'}$ cancel by partial integration in (5.13) by conservation of $T_{24}$. Finally, integrating by parts once more in $\partial_{\mu}$ and using $D^\mu u J(u) = D^\mu \int^{u} J$, we get the following simple result for $A^{G}_{T_{1}}$.

$$A^{G}_{T_{1}} = -\int dz \sqrt{g} \int dw \sqrt{g} \partial_{\mu} \{K(1)K(3)\} D^\mu u J(u) g \cdot T_{24}(w)$$

$$= \int dz \sqrt{g} \int dw \sqrt{g} K(1)K(3)\{u(2 + u)J'(u) + (d + 1)(1 + u)J(u)\} g \cdot T_{24}(w)$$

(5.16)

It is more difficult to deal with $A^{G}_{T_{2}}$. To simplify the integral representation in (5.14), it is very convenient to set $x_{1} = 0$ in the first term and then perform an inversion transformation of the integral (in $z$ and $w$) as explained in [4]. The symmetric step in $1 \leftrightarrow 3$ is done later. It is now easy to evaluate the double covariant derivative of the inverted bulk-to-boundary propagator $K(1') = C_{\Delta} z_{0}^{\Delta}$,

$$D_{\nu} \partial_{\mu} K(1') = -\Delta K(1') g_{\mu\nu} + \Delta(\Delta + 1) K(1') z_{0}^{2} g_{\mu 0} g_{\nu 0}$$

(5.17)

The contribution of the first term in (5.17) is proportional to the metric $g_{\mu\nu}$, and may be treated by the same technique used for $A^{G}_{G}$. It acquires an “effective scalar propagator” proportional to the term in {\ldots} in (5.7). We thus find for this contribution to $A^{G}_{T_{2}}$ the term

$$-|x_{21}'|^{2\Delta'} |x_{31}'|^{2\Delta} |x_{41}'|^{2\Delta} \int dz \sqrt{g} \int dw \sqrt{g} K(1')K(3')\{G(u)-(1+u)\bar{G}(u)\} g \cdot T_{24}(w) + (1 \leftrightarrow 3).$$

(5.18)

Note that the prefactor contains the scale factors from the inversion.

The integral of the second term in (5.17) contains the factor.

$$z_{0}^{2} g_{\mu 0} g_{\nu 0} D^{\mu} \partial_{\mu} u \ D^{\nu} \partial_{\nu} u = (z_{0} g_{\mu \nu} + \partial_{\mu} u)(z_{0} g_{\nu \nu} + \partial_{\nu} u)$$

(5.19)

Integration in $w$ against $G(u)T_{24}^{\mu \nu'}(w)$ gives rise to three types of terms

$$\int dw \sqrt{g}(z_{0} g_{\mu \nu} + \partial_{\mu} u)(z_{0} g_{\nu \nu} + \partial_{\nu} u)G(u)T_{24}^{\mu \nu'}(w)$$

$$= z_{0}^{2} \int dw \sqrt{g} G(u)T_{24}(w)_{\nu\nu} + 2z_{0} \int dw \sqrt{g} g_{\nu \nu'} \partial_{\nu} u G(u) T_{24}^{\mu \nu'}(w)$$

$$- \int dw \sqrt{g} (1 + u)\bar{G}(u) g \cdot T_{24}(w).$$

(5.20)

The second integral on the right hand side may be further simplified by using once more $\partial_{\nu} u G(u) = \partial_{\nu} \bar{G}(u)$, integrating by parts, using conservation of $T_{24}$ and being careful to taking into account the fact that the integral is the $0'$ component of a vector instead of a scalar. Thus there is a non-vanishing contribution of Christoffel symbols, which gives

$$\int dw \sqrt{g} g_{\nu \nu'} \partial_{\nu} u G(u) T_{24}^{\mu \nu'}(w) = \int dw \sqrt{g} \frac{1}{w_{0}} \bar{G}(u) g \cdot T_{24}(w)$$

(5.21)
We now combine (5.18), (5.20) and (5.21) to write an expression for \( A_{T^2}^G \), namely

\[
A_{T^2}^G = \int_{x_{21}} dx' \int_{x_{31}} dx' \int_{x_{41}} dx' \left\{ \left[ -\Delta G(u) - \Delta^2 (1 + u) G(u) + \Delta (\Delta + 1) \frac{2\zeta_0}{w_0} G(u) \right] g \cdot T_{24}(w) \right. \\
+ \Delta (\Delta + 1) \frac{2\zeta_0}{w_0} G(u) T_{24}(w)_{o,o'} \left. \right\} + (1 \leftrightarrow 3)
\]

(5.22)

### 5.1 Final simplified form

We are now in a position to assemble all contributions to the graviton exchange diagram by combining results for \( A^t \), \( A^G \), \( A^T \) and \( A_{T^2}^G \). The \( z \)-integrals are easiest to carry out after inversion, so we apply inversion to all contributions and rewrite \( A_{\text{grav}} \) with a universal conformal factor extracted, viz.

\[
A_{\text{grav}} = |x'_{21}|^{2\Delta} |x'_{31}|^{2\Delta} |x'_{41}|^{2\Delta'} (B^{tt} + B^{dd} + B^{dd}) + (1 \leftrightarrow 3)
\]

(5.23)

where the reduced amplitudes \( B \) are given by

\[
B^{tt} = \int dz \sqrt{g} \int dw \sqrt{g} K(1') K(3') P(u) g \cdot T_{24}(w)
\]

(5.24)

\[
B^{dd} = -4\Delta (\Delta + 1) \int dz \sqrt{g} \int dw \sqrt{g} \frac{z_0}{w_0} K(1') K(3') \tilde{G}(u) g \cdot T_{24}(w)
\]

(5.25)

\[
B^{dd} = -2\Delta (\Delta + 1) \int dz \sqrt{g} \int dw \sqrt{g} \frac{z_0^2}{w_0} K(1') K(3') \tilde{G}(u) T_{24}(w)_{o,o'}
\]

(5.26)

The function \( P(u) \) is gotten by combining all contributions involving \( g \cdot T_{24} \) (except that from \( A_{T^2}^G \)) and is given by

\[
P(u) = -\frac{1}{2} \Box_z G - \frac{1}{4} (d - 1) \Box_z H - G + 2(1 + u)^2 G + d(1 + u) \tilde{G}(u) - m^2 H \\
- u(2 + u) J' - (d + 1)(1 + u) J + 2\Delta G + 2\Delta^2 (1 + u) \tilde{G}(u)
\]

(5.27)

The relation between \( H(u) \) and \( G(u) \) was given in (5.37) and may be used to further simplify the form of \( P(u) \). While both \( \Box_z G \) and \( \Box_z H \) have a term proportional to \( \delta(z,w) \), the relative coefficients of both terms are such that this \( \delta \)-functions cancels out of the full \( P(u) \), and we are left with

\[
P(u) = 2\Delta G - 2u(2 + u) G + 2(\Delta^2 - d - 1) (1 + u) \tilde{G}(u) - m^2 H(u)
\]

\[
= 2\left\{ \Delta + 1 + \frac{m^2 - d + 1}{d - 1} (1 + u)^2 \right\} G(u) + 2\left\{ \Delta^2 - d - 1 + \frac{m^2(d - 2)}{d - 1} \right\} (1 + u) \tilde{G}(u)
\]

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Finally, the expression $T_{24}(w)_{0'0'}$ may be worked out explicitly,

$$
T_{24}(w)_{0'0'} = (\Delta')^2 K_{\Delta'2'} K_{\Delta'4'} \left\{ \left(1 - \frac{(m')^2}{(\Delta')^2}\right) \frac{1}{w_0^2} - \frac{4}{(w - x')^2} \right\}
$$

and we can use an identity similar to (5.8) to obtain a covariant expression for $g \cdot T_{24}(w)$, namely

$$
g \cdot T_{24}(w) = \left( -\frac{1}{2}(d - 1) \Box w - 2m^2 \right) \{ K(2')K(4') \}.
$$

5.2 General integrals over interaction points

We shall use the following strategy for the calculation of the integrals over the interaction points $z$ and $w$ in the reduced amplitudes of (5.24–5.26). First, we shift both $z$ and $w$ by $x'_{31}$; by translation invariance, the integrals depend only upon the new variables $x \equiv x'_{41} - x'_{31}$ and $y \equiv x'_{21} - x'_{31}$. The $z$-integrations then only depend upon the variable $w$, and may be carried out explicitly in terms of elementary functions by methods similar to the ones used in [10] and [13]. Only after the $z$-integrals are carried out are the explicit forms of $g \cdot T_{24}$ and $T_{24}(w)_{0'0'}$ required and used. The remaining $w$-integrals may be recast as integral representations that admit simple asymptotic expansions.

To prepare for the $z$-integrations, we note that $P(u)$ in (5.24)) and (5.28) involves the invariant function $G(u)$ and its first integral $\bar{G}(u)$, and the same functions appear in (5.25,5.26). To apply the methods of [10] and [13] we need the series expansions of $G(u)$ and $\bar{G}(u)$ in the variable $\xi$ of (3.32). For $G(u)$ this is just the hypergeometric series for $G_d(u)$ in (3.33) and we obtain the series for $\bar{G}(u)$ by direct integration. These expansions are given by

$$
G(u) = \frac{1}{2} C_G \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{d}{2} + \frac{1}{2})}{\Gamma(\frac{d}{2} + \frac{1}{2})} \frac{1}{k!} \xi^{2k+d}
$$

$$
\bar{G}(u) = -\frac{1}{4} C_G \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{d}{2} - \frac{1}{2})}{\Gamma(\frac{d}{2} + \frac{1}{2})} \frac{1}{k!} \xi^{2k+d-1}
$$

These series expansions are uniformly convergent inside any disc $|\xi| < 1$. The normalization constant may be read off from (3.34) and (3.31) for $\Delta = d$, and we find $C_G = 2^d d \bar{C}_d$.

There are five independent $z$-integrals required to evaluate the graviton exchange amplitudes. They are as follows,

$$
Z_1(w) = \int dz \sqrt{g} K(1')K(3')G(u)
$$
In terms of these integrals, the original amplitudes are given by

\[ B_{\mu} = \int dw \sqrt{g} \left\{ 2(\Delta + 1)Z_1(w) + 2\frac{m^2 - d + 1}{d - 1}Z_2(w) \right\} + 2(\Delta^2 - d - 1 + \frac{m^2(d - 2)}{d - 1})Z_3(w) \} g \cdot T_{24}(w) \]

\[ B_{dd} = \int dw \sqrt{g} \left\{ -4\Delta(\Delta + 1)Z_4(w) \} g \cdot T_{24}(w) \right\} \]

\[ B_{00} = \int dw \sqrt{g} \left\{ -2\Delta(\Delta + 1)Z_5(w) \} w_0^2T_{24}(w)\right\} \]

It remains to evaluate the \( z \)-integrals.

### 5.2.1 Performing the \( z \)-integrals

The \( z \)-integrations are carried out term by term on the series expansions of (5.30), and all the integrals we need in (5.24–5.26) are of the following form (with \( 2a, \ 2b = 0, 1 \) or 2)

\[
\int_{0}^{\infty} d\zeta_0 \int_{R^d} d\bar{z}_0 \zeta_0^{2\Delta + 2a - d - 1} \left( \frac{2\zeta_0 w_0}{\zeta_0^2 + w_0^2 + (\bar{z} - \bar{w})^2} \right)^{2k + d - 2b} \\
= \pi^{d/2} \Gamma\left( \frac{1}{2} \right) \Gamma(\Delta + k + a - b) \Gamma(k + \frac{d}{2} - a - b) \Gamma(\Delta) \Gamma(k + \frac{d}{2} - b + \frac{1}{2}) \frac{w_0^{2a}}{(\alpha w_0^2 + (1 - \alpha)w^2)^{k + \frac{d}{2} - a - b}} \\
\times \int_{0}^{1} d\alpha \alpha^{2a - 1}(1 - \alpha)^{\Delta - 1} \left( \frac{\alpha w_0^2}{(\alpha w_0^2 + (1 - \alpha)w^2) \right)^{k + \frac{d}{2} - a - b} \] (5.39)

In the integrals \( Z_j(w) \) of (5.31–5.35), the values taken by \( (a, b) \) are \((0, 0), (0, 1), (0, 1), (\frac{1}{2}, \frac{1}{2})\) and \((1, 0)\) for \( j = 1, 2, 3, 4, 5 \) respectively. The calculation of the \( z \)-integrals is slightly involved, but is essentially the same for each of the \( Z_j \)-integrals. Here, we shall present in detail only the calculation for \( Z_1 \), and restrict to presenting the final results for the remaining 4 integrals.

To compute \( Z_1(w) \), we use the expansion of (5.30) for the function \( G(u) \) and integrate term by term in \( z \) using the integral formula of (5.39), here with \( a = b = 0 \). Assembling these results, we notice that the factors \( \Gamma(k + \frac{d}{2} + \frac{1}{2}) \) and \( \Gamma(k + \frac{d}{2}) \) cancel between numerators and denominators. Also, interchanging the order of the \( \alpha \)-integration of (5.39) and the \( k \)-sum of
(5.30), we are left with the following result

\[
Z_1(w) = \frac{\pi^\frac{d}{2} \Gamma\left(\frac{d}{2}\right)}{2 \Gamma\left(\frac{d}{2} + \frac{1}{2}\right)} C_G C_\Delta^2 \int_0^1 \frac{d\alpha}{\alpha} (1 - \alpha)^{\Delta - 1} f_{\Delta, \alpha} \left(\frac{\alpha w_0^2}{\alpha w_0^2 + (1 - \alpha)w^2}\right)
\]

\[
f_{\Delta, p}(\zeta) = \frac{\Gamma(k + \Delta)}{\Gamma(\Delta) k!} \frac{\zeta^{k+p}}{k+p}
\]

Assuming that \( d \) is even and \( d \geq 4 \) throughout, we have \( p > 1 \) and the function \( f_{\Delta, p} \) may be easily evaluated in terms of elementary functions. We begin by noticing that

\[
f_{\Delta, p}(\zeta) = \zeta^p \left(\frac{d}{d\zeta}\right)^{p-1} \sum_{k=0}^{\infty} \frac{\Gamma(k + \Delta)}{\Gamma(\Delta) k!} \zeta^{k+p-1}
\]

In view of the presence of the multiple derivative operation in front, we are free to add into the sum the terms with \( k = -p + 1, -p + 2, \cdots, -1 \). Then, we shift \( k \to k - p \) and obtain

\[
f_{\Delta, p}(\zeta) = \zeta^p \left(\frac{d}{d\zeta}\right)^{p-1} \sum_{k=0}^{\infty} \frac{\Gamma(k + \Delta - p)}{\Gamma(\Delta) k!} \zeta^{k-1}
\]

The infinite sum is proportional to \( \zeta^{-1}[(1 - \zeta)^{-\Delta + p} - 1] \) and the multiple differentiations may be carried out explicitly. The final result is

\[
f_{\Delta, p}(\zeta) = (-)^p \frac{\Gamma(p)}{\Gamma(\Delta)} \left[ \Gamma(\Delta - p) \sum_{\ell=0}^{p-1} (-)^\ell \frac{\Gamma(\Delta - p + \ell)}{\ell!} \frac{\zeta^\ell}{(1 - \zeta)^{\Delta - p + \ell}} \right]
\]

Upon substituting the value \( \zeta = \alpha w_0^2 / (\alpha w_0^2 + (1 - \alpha)w^2) \), and using the binomial expansion for the (positive) powers of the combination \( \alpha w_0^2 + (1 - \alpha)w^2 \), we find

\[
f_{\Delta, p}(\zeta) = (-)^p \frac{\Gamma(p)}{\Gamma(\Delta)} \sum_{k=0}^{\Delta - 2} \sum_{\ell=0}^{p-1} (-)^\ell \frac{\Gamma(\Delta - p + \ell) \Gamma(\Delta - p + 1)}{\ell! \Gamma(\Delta - p + \ell - k) \Gamma(k + 1)} \left(\frac{\alpha w_0^2}{(1 - \alpha)w^2}\right)^{k+1}
\]

Remarkably, upon including the factor of \( \alpha^{-1}(1 - \alpha)^{\Delta - 1} \) of the integral in (5.40), the integrand is polynomial in \( \alpha \) and may be carried out term by term in (5.44). The final result for this calculation as well as for that of the remaining \( Z_j \) may be expressed in the following final form

\[
Z_j(w) = \sum_{k=0}^{\Delta - 2} Z_j^{(k)} \left(\frac{w_0^2}{w^2}\right)^{k+1} j = 1, \cdots, 5
\]

with the coefficients \( Z_j^{(k)} \) dependent only on \( \Delta \) and \( d \) and given as follows

\[
Z_j^{(k)} = (-)^\frac{d}{2} \frac{\pi \frac{d}{2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d}{2} + \frac{1}{2}\right)}{2 \Gamma\left(\frac{d}{2} + \frac{1}{2}\right) \Gamma(\Delta)^2} C_G C_\Delta^2 \tilde{Z}_j^{(k)}
\]

\[
= (-)^\frac{d}{2} \frac{\Gamma\left(\frac{d}{2}\right) (\Delta - \frac{d}{2})^2}{4 \pi d \Gamma(\Delta - \frac{d}{2} + 1)} \tilde{Z}_j^{(k)},
\]
The action of the Laplace operator on the various powers of $w$ also uses

\[
\hat{Z}_1^{(k)} = \sum_{\ell=0}^{\frac{d-1}{2}} (-1)^{\ell} \frac{\Gamma(\Delta + \ell - \frac{d}{2}) \Gamma(\Delta - k - 1) \Gamma(k + 1)}{\ell! \Gamma(k - \ell + 2) \Gamma(\Delta - k + \ell - \frac{d}{2})}
\]

(5.48)

\[
\hat{Z}_2^{(k)} = \frac{1}{2} \sum_{\ell=0}^{\frac{d-1}{2}} (-1)^{\ell} \frac{\Gamma(\Delta + \ell - \frac{d}{2} - 1) \Gamma(\Delta - k - 1) \Gamma(k + 1)}{\ell! \Gamma(k - \ell + 3) \Gamma(\Delta - k + \ell - \frac{d}{2} - 1)}
\]

(5.49)

\[
\hat{Z}_3^{(k)} = \frac{1}{2} \sum_{\ell=0}^{\frac{d-1}{2}} (-1)^{\ell} \frac{\Gamma(\Delta + \ell - \frac{d}{2} - 1) \Gamma(\Delta - k - 1) \Gamma(k + 1)}{\ell! \Gamma(k - \ell + 3) \Gamma(\Delta - k + \ell - \frac{d}{2} - 1)}
\]

(5.50)

\[
\hat{Z}_4^{(k)} = \frac{2(\Delta - \frac{d}{2} + 2)(\Delta - \frac{d}{2} + 1)}{d\Delta(\Delta + 1)} \times \sum_{\ell=0}^{\frac{d-2}{2}} (-1)^{\ell} \frac{\Gamma(\Delta + \ell - \frac{d}{2} - 1) \Gamma(\Delta - k - 1) \Gamma(k + 3)}{\ell! \Gamma(k - \ell + 3) \Gamma(\Delta - k + \ell - \frac{d}{2} + 1)}
\]

(5.51)

We conclude by noticing that the relation between $Z_j^{(k)}$ and $\hat{Z}_j^{(k)}$ simplifies considerably upon using the explicit forms for $C_G$ and $C_\Delta$, as was done in (5.47).

### 5.2.2 Reduction to $w$-integrals

Our purpose here is to express the $w$–integrals in $B^{it}$, $B^{dd}$, and $B^{00}$ of (5.36–5.38) in terms of the following standard integral

\[
W_k^{\Delta'}(a, b) \equiv \int dw \sqrt{g} \frac{w_0^{2\Delta'+2a+2k}}{w^{2k}} \frac{1}{(w-x)^{2\Delta'}} \frac{1}{(w-y)^{2\Delta'+2b}}
\]

(5.53)

We also use $\tilde{W}_k^{\Delta'}(a, b)$ which represents $W_k^{\Delta'}(a, b)$ with $x \leftrightarrow y$. Introducing the constants

\[
Z_k^{(k)} = 2(\Delta + 1)Z_1^{(k)} + 2\frac{m^2 - d + 1}{d - 1}Z_2^{(k)} + 2(\Delta^2 - d - 1 + \frac{m^2(d-2)}{d-1})Z_3^{(k)} - 4\Delta(\Delta + 1)Z_4^{(k)}
\]

(5.54)

we find the following expression for $B^{it} + B^{dd}$, after partial integration of $\Box_w$,

\[
B^{it} + B^{dd} = \sum_{k=0}^{\Delta-2} Z_k^{(k)} \int dw \sqrt{g} \left( -\frac{1}{2}(d-1)\Box_w - 2m^2 \right) \frac{w_0^{2k}}{w^{2k}} K(2') K(4')
\]

(5.55)

The action of the Laplace operator on the various powers of $\frac{w_0^2}{w^2}$ is easily evaluated with the help of the following formula

\[
\Box_w \left( \frac{w_0^2}{w^2} \right)^k = 2k(2k - d) \left( \frac{w_0^2}{w^2} \right)^k - 4k^2 \left( \frac{w_0^2}{w^2} \right)^{k+1}
\]

(5.56)
and we obtain the following expression for the amplitude in terms of $W$ functions

\[ B^{tt} + B^{dd} = c^2 \sum_{k=0}^{\Delta - 2} Z^{(k)} \left\{ -(d - 1)(k + 1)(2k + 2 - d) - 2m^2 \right\} W_k^{\Delta'}(0, 0) \]

\[ + 2(d - 1)(k + 1)^2 W_k^{\Delta'}(0, 0) \] (5.57)

Proceeding analogously for the contribution of $B^{00}$ with the help of (5.28) and (5.38), we find

\[ B^{00} = -2\Delta(\Delta + 1)(\Delta')^2 c^2 \sum_{k=0}^{\Delta - 2} Z^{(k)} \left\{ (1 - \frac{m^2}{\Delta^2}) W_k^{\Delta'}(0, 0) - 4W_k^{\Delta'}(1, 1) - 4\tilde{W}_k^{\Delta'}(1, 1) \right\} \]

\[ + 8W_k^{\Delta'+1}(0, 0) + 2(x - y)^2 W_k^{\Delta'+1}(0, 0) \] (5.58)

As in the special case $\Delta = \Delta' = d = 4$ already discussed in Section 3, we recognize that the general graviton exchange amplitude is a finite sum of quartic graphs. In fact, each $W_k^{\Delta'}(a, b)$ is the amplitude of a 4–point contact diagram evaluated in the inverted coordinates (with appropriate inversion prefactors omitted). The scale dimension of the external propagators are $\Delta_1 = k + 2a - b, \Delta_3 = k, \Delta_2 = \Delta' + b$ and $\Delta_4 = \Delta'$ (see equ.(A.3)).

### 5.3 Graviton exchange graph for $d = \Delta = \Delta' = 4$

For $\Delta = \Delta' = d = 4$, the masses of the scalars vanish $m = m' = 0$, and the $k$ and $\ell$-sums in the results for the $z$-integral functions $I_j$ truncate after just a few terms. We need the $z$-integral functions $Z_j(w), j = 1, \cdots, 5$, which may be read off from (5.45) and (5.48–5.52) with $\Delta = d = 4$,

\[ Z_1(w) = \frac{1}{2\pi^4}( + \frac{3w_0^4}{2w^4} + \frac{2w_0^6}{w^6}) \] (5.59)

\[ Z_2(w) = \frac{1}{2\pi^4}(\frac{3w_0^2}{2w^2} + \frac{7w_0^4}{4w^4} + \frac{2w_0^6}{w^6}) \] (5.60)

\[ Z_3(w) = \frac{1}{2\pi^4}(\frac{w_0^2}{2w^2} - \frac{w_0^4}{4w^4}) \] (5.61)

\[ Z_4(w) = \frac{1}{2\pi^4}(\frac{3w_0^2}{8w^2} - \frac{w_0^4}{8w^4}) \] (5.62)

\[ Z_5(w) = \frac{3}{10\pi^4}(\frac{w_0^2}{w^2} + \frac{w_0^4}{w^4} + \frac{w_0^6}{w^6}) \] (5.63)

Using these integrals, the expressions for $B^{tt} + B^{dd}$ and $B^{00}$ become quite simple and are given as follows,

\[ B^{tt} + B^{dd} = \frac{8}{\pi^4} \int dw \sqrt{g} \left\{ \frac{w_0^2}{w^2} + \frac{w_0^4}{w^4} + \frac{w_0^6}{w^6} \right\} g \cdot T_{24}(w) \] (5.64)
\[ B^{00} = -\frac{12}{\pi^4} \int dw \sqrt{g} \left\{ w_0^2 \left( \frac{w_0^6}{w^4} + \frac{w_0^6}{w^4} \right) w_0^2 T_{24} (w) \right\} \]  
\[ (5.65) \]

When \( m' = 0 \) and \( d = 4 \), the combination \( g \cdot T_{24} \) in (5.29) simplifies. Upon integration by parts, and making use of the differentiation formula (5.56), we obtain the following expression

\[ B^{tt} + B^{dd} = \frac{2^6 \cdot 3^3}{\pi^8} \int dw \sqrt{g} \left\{ \frac{w_0^2}{w^2} + \frac{w_0^4}{w^4} + \frac{w_0^6}{w^6} + 9 \frac{w_0^8}{w^8} \right\} \frac{w_0^8}{(w-x)^8(w-y)^8} \]

\[ = \frac{2^6 \cdot 3^3}{\pi^8} \left\{ W_1^4(0,0) + W_2^4(0,0) + W_3^4(0,0) + 9W_4^4(0,0) \right\} \]  
\[ (5.66) \]

The expression for \( B^{00} \) may be obtained in an analogously, using (5.28) for \( m' = 0 \), \( \Delta = 4 \).

This directly gives

\[ B^{00} = -\frac{2^9 \cdot 3^3}{\pi^8} \sum_{p=1}^3 \left\{ W_p^4(0,0) - 4W_p^4(1,1) - 4\tilde{W}_p^4(1,1) + 8W_p^5(1,0) + 2(x-y)^2W_p^5(0,0) \right\} \]  
\[ (5.67) \]

Using the expression for \( W_p^4(1,1) \) + \( \tilde{W}_p^4(1,1) \) in terms of \( W(0,0) \) to be derived in (6.5), this formula may be recast in terms of \( W(0,0) \) and \( W(1,0) \) only, viz.

\[ B^{00} = -\frac{2^9 \cdot 3^3}{\pi^8} \left[ 3W_4^4(0,0) + \sum_{p=1}^3 \left\{ -2W_p^4(0,0) + 8W_p^5(1,0) + 2(x-y)^2W_p^5(0,0) \right\} \right] \]  
\[ (5.68) \]

Adding the contributions of \( B^{tt} + B^{dd} \) and \( B^{00} \), we finally obtain the expression for the full \( B \) in terms of \( W \)-functions and we have

\[ B = -\frac{2^6 \cdot 3^3}{\pi^8} \left[ 15W_4^4(0,0) + \sum_{p=1}^3 \left\{ -17W_p^4(0,0) + 64W_p^5(1,0) + 16(x-y)^2W_p^5(0,0) \right\} \right] \]  
\[ (5.69) \]

The full graviton amplitude \( I_{grav} \) is obtained by multiplying \( B \) by the appropriate kinematic factors and symmetrizing under \( 1 \leftrightarrow 3 \) (see (5.1), (5.23)).

### 5.4 Equivalence with the result in Section 3

We now make contact with the result obtained in Section 4. We recall that \( W_{\Delta}(a, b) \) are just scalar quartic graphs in the inverted coordinates (with some kinematic factors omitted), see equ.(A.3). One can easily convert (5.69) and (5.28) into the notations Section 4, and get a sum of \( D \)-functions. The representation of the graviton exchange graph that is obtained in this way does not at first appear to coincide with the result (4.38). In particular, terms of the form \( x_1^2x_4^2D_{p+2p,p+55} + x_2^2x_3^2D_{p,p+2p + 255} \) arise from \( W_p^5(1,0) \) in (5.69) and its symmetrization in \( 1 \leftrightarrow 3 \). Thanks to the many identities that connect the \( D \) functions (see the Appendix), the
two representations of the answer are in fact exactly equal. We first use (A.19) to eliminate the “asymmetric” $D$’s in the result of Section 5. We get

$$I_{\text{grav}} = \left( \frac{6}{\pi^2} \right)^4 \left[ 16 x_{24}^2 \left( \frac{1}{2s} - 1 \right) D_{4455} + \frac{32}{3} x_{24}^2 \left( -1 + \frac{2}{3s} \right) \frac{x_{24}^2}{x_{13}^2} D_{3355} \right] (5.70)$$

\[ + \frac{32}{3} x_{24}^2 \left( \frac{1}{2s} - 1 \right) D_{2255} - \frac{32}{3} x_{24}^2 \frac{x_{24}^2}{x_{13}^2} D_{1155} + 24 D_{4444} \]

\[ + \frac{8}{9} x_{24}^2 D_{3344} + \frac{14}{9} x_{13}^4 D_{2244} + \frac{10}{3} x_{13}^6 D_{1144} \].

Now (4.38), (5.70) are both in terms of $D$–functions of the form $D_{\Delta \Delta \Delta \Delta}$. By repeated application of (A.9) one can convert one representation into the other. We regard this non–trivial match as a strong check of our result.

6. Asymptotic expansions

We have seen that the graviton exchange amplitude (and generically all AdS 4–point processes with external scalars) can be expressed as a finite sum of quartic graphs, see (4.38), (5.57–5.58), (5.70). In this Section we develop asymptotic series expansions for the scalar quartic graphs (Figure 5) in terms of conformally invariant variables. This series expansions allow to analyze the supergravity results in terms of the expected double OPE (1.1). In Section 3 and 4 we have used slightly different notations for the quartic graphs, namely $D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4}$ and $W_{\Delta}^\Delta(a, b)$. The connection between the two is given in (A.3). Here the expansions are performed for the $W_{\Delta}^\Delta(a, b)$ representation of the quartic graph.

In Section 6.3 we assemble the series expansions of the $W$’s that appear in the representation (5.23,5.69) of the graviton exchange for $\Delta = \Delta' = d = 4$. We concentrate on the direct channel and display explicitly the singular terms and all the logarithmic contributions. The complete expansions, in both direct and crossed channels, can be easily obtained from the formulas in Section 6.2.

6.1 Integral representations of $W_{\Delta}^{\Delta'}(a, b)$

To evaluate $W_{\Delta}^{\Delta'}(a, b)$, we follow the methods of [10] and [13]. We introduce a first Feynman parameter $\alpha$ for the denominators $w^2$ and $(w - x)^2$ and a second Feynman parameter $\beta$ for the resulting denominator and $(w - y)^2$. The $\bar{w}$ and $w_0$ integrals may then be carried out using
standard formulas, and we find

\[ W_k^{\Delta'}(a, b) = \frac{\pi \frac{d}{2}}{2} \frac{\Gamma(k + \Delta' + a - \frac{d}{2})\Gamma(\Delta' + b - a)}{\Gamma(k)\Gamma(\Delta')\Gamma(\Delta' + b)} \times \int_0^1 d\alpha \int_0^1 d\beta \frac{\alpha^{\Delta' - 1}(1 - \alpha)^{k-1}\beta^{\Delta' + b - 1}(1 - \beta)^{k+a-b-1}}{[\beta(y - \alpha x)^2 + \alpha(1 - \alpha) x^2]^{\Delta' - a + b}} \]  

(6.1)

Upon performing the following change of variables familiar from [10] and [13],

\[ \alpha = \frac{1}{1 + u}, \quad \beta = \frac{u}{u + v + uv} \]

(6.2)

we obtain an integral representation similar that of [10] and [13],

\[ W_k^{\Delta'}(a, b) = \frac{\pi \frac{d}{2}}{2} \frac{\Gamma(k + \Delta' + a - \frac{d}{2})\Gamma(\Delta' + b - a)}{\Gamma(k)\Gamma(\Delta')\Gamma(\Delta' + b)} \int_0^\infty du \int_0^\infty dv \]

\[ \times \frac{u^{k+a-1}v^{k+a-b-1}}{(u + v + uv)^{k+2} \left[ (x - y)^2 + uy^2 + vx^2 \right]^{\Delta' - a + b}} \]

(6.3)

Now the function \( W \) with \( b \neq 0 \) only enters the calculation of \( B^{00} \) (equ.(5.67)), and appears there only in the form of the sum \( W_k^{\Delta'}(1, 1) + \bar{W}_k^{\Delta'}(1, 1) \). This particular combination may be re-expressed in terms of \( W \)-functions with \( b = 0 \) only. This would be difficult to see from the \( w \)-integral definition (5.53), but is manifest from the integral representation (6.3), by using the following relation

\[ \frac{u}{(u + v + uv)^{k+1}} + \frac{v}{(u + v + uv)^{k+1}} = \frac{1}{(u + v + uv)^k} - \frac{uv}{(u + v + uv)^{k+1}} \]  

(6.4)

Taking normalization factors into account properly, we find

\[ W_k^{\Delta'}(1, 1) + \bar{W}_k^{\Delta'}(1, 1) = \frac{k + \Delta' - \frac{d}{2}}{\Delta'} W_k^{\Delta'}(0, 0) - \frac{k}{\Delta'} W_{k+1}^{\Delta'}(0, 0) \]

(6.5)

As a result of this identity, there will be only two classes of \( w \)-integral functions entering into the graviton exchange amplitudes : \( W_k^{\Delta'}(0, 0) \) and \( W_k^{\Delta'}(1, 0) \).

Similarly, a relation exists expressing \( W_k^{\Delta'}(1, 0) \) in terms of \( W(0, 0) \)-functions. This may be established by using the fact that the quantity

\[ \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left( \frac{(uv)^{k-1}}{(u + v + uv)^k \left[ (x - y)^2 + wy^2 + vx^2 \right]^{\Delta'}} \right) \]

has vanishing integral in \( u \) and \( v \), and by carrying out the derivatives explicitly and regrouping the result in terms of \( W \)-functions. The final result is

\[ 2(k + 1)(\Delta')^2 W_k^{\Delta'}(1, 0) = k(k + \Delta' - \frac{d}{2})(k + \Delta' - \frac{d}{2} - 1)W_k^{\Delta' - 1}(0, 0) \]

\[ -k(2k + 1)(k + \Delta' - \frac{d}{2})W_{k+1}^{\Delta' - 1}(0, 0) \]

\[ +k(k + 1)^2 W_{k+2}^{\Delta' - 1}(0, 0) - k(\Delta')^2(2k + 1)(x^2 + y^2)W_{k+1}(0, 0) \]

(6.7)
The \( \Delta - \)integrals \( W^{\Delta'(0,0)}_\Delta \) and \( W^{\Delta'(1,0)}_\Delta \) may each be expressed in terms of derivatives on two universal functions. To show this, we proceed as in [10] and [13], where analogous results were obtained for the scalar and gauge exchange graphs. We begin by introducing the conformal invariants

\[
\begin{align*}
\frac{s}{1} &= \frac{1}{2} \frac{(x - y)^2}{x^2 + y^2} = \frac{1}{2} \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2} \quad (6.8) \\
\frac{t}{1} &= \frac{x^2 - y^2}{x^2 + y^2} = \frac{x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2} \quad (6.9)
\end{align*}
\]

whose ranges are \( 0 \leq s \leq 1 \) and \( -1 \leq t \leq 1 \). Next, we perform a change of variables

\[
\begin{align*}
u &= 2 \rho (1 - \lambda) \\
\Delta' - a
\end{align*}
\]

under which we have

\[
W^{\Delta'(a,0)}_\Delta = \pi \frac{\Gamma(k + \Delta' + a - \frac{d}{2}) \Gamma(\Delta' - a)}{2^{\Delta' + a} \Gamma(k) \Gamma(\Delta')^2 (x^2 + y^2)^{\Delta' - a}} \int_0^\infty d\rho \int_{-1}^1 d\lambda \frac{(1 - \lambda^2)^{k+2a-1}}{1 + \rho (1 - \lambda^2)^{k+2a-1}} \frac{1}{(s + \rho \lambda t)^{\Delta' - a}} \quad (6.11)
\]

It is now possible to write the right hand side as a derivative with respect to \( s \) of order \( \Delta' - a - 1 \) of an integral in which the denominator involving \( s \) appears to degree 1, using

\[
\begin{align*}
\frac{1}{(s + \omega)^p} = \left( -\right)^{p+1} \frac{\Gamma(p)}{\Gamma(p+1)} \left( \frac{\partial}{\partial s} \right)^{p-1} \left( \frac{1}{s + \omega} \right)
\end{align*}
\]

Next, we change variables to \( \rho = s/\mu \) and recognize that the new integral is a derivative with respect of \( s \) of order \( k - 1 + 2a \). Putting all together, we obtain

\[
W^{\Delta'(a,0)}_\Delta = \pi \frac{1}{2^{\Delta' + a}} \frac{\Gamma(k + \Delta' + a - \frac{d}{2}) \Gamma(\Delta' - a)}{\Gamma(k) \Gamma(\Delta')^2 (x^2 + y^2)^{\Delta' - a}} \times \frac{\partial}{\partial s}^{\Delta' - a - 1} \left\{ s^{k-1} \left( \frac{\partial}{\partial s} \right)^{k-1+2a} I_a(s,t) \right\} \quad (6.13)
\]

where the universal functions \( I_a(s,t) \) are given by the following integral representations

\[
I_a(s,t) = \frac{s^{2a}}{2} \int_0^\infty d\mu \int_{-1}^1 d\lambda \frac{(1 - \lambda^2)^a}{\mu + s(1 - \lambda^2)} \frac{1}{1 + \mu + \lambda t}
\]

\[
= \frac{s^{2a}}{2} \int_{-1}^1 d\lambda \frac{(1 - \lambda^2)^a}{1 + \lambda t - s(1 - \lambda^2)} \ln \frac{1 + \lambda t}{s(1 - \lambda^2)} \quad (6.14)
\]

The integrals \( I_a(s,t) \) are perfectly convergent and produce analytic functions in \( s \) and \( t \), with logarithmic singularities in \( s \) and \( t \).
6.2 Series expansions of $W_k^\Delta'(a, b)$

Series expansions of the functions $W_k^\Delta(a, b)$ may be obtained easily from the series expansions of the universal functions $I_a(s, t)$. There are two different regions in which the expansion will be needed:

a) The direct channel ("t–channel") limit $|x_{13}| \ll |x_{12}|, |x_{24}| \ll |x_{12}|$, which corresponds to $s, t \to 0$.

b) The two crossed channels; one ("s–channel") is the limit $|x_{12}| \ll |x_{13}|, |x_{34}| \ll |x_{12}|$, which corresponds to $s \to 1/2, t \to -1$, and the other ("u–channel") is $|x_{23}| \ll |x_{34}|, |x_{14}| \ll |x_{34}|$ in which $s \to 1/2, t \to 1$.

We shall now discuss each limit in turn.

(a) Direct channel series expansion

The direct channel limit is given by $s, t \to 0$, and the expansions of the functions $I_a(s, t)$ are given by

$$I_0(s, t) = \sum_{k=0}^{\infty} \{- \ln a_k(t) + b_k(t)\} s^k \quad (6.15)$$

$$I_1(s, t) = \sum_{k=0}^{\infty} \{- \ln a_k(t) + \hat{b}_k(t)\} s^{k+2} \quad (6.16)$$

where the coefficient functions are given by

$$a_k(t) = \int_{-1}^{1} \frac{(1 - \lambda^2)^k}{(1 + \lambda t)^{k+1}} d\lambda \quad b_k(t) = \int_{-1}^{1} \frac{(1 - \lambda^2)^k}{(1 + \lambda t)^{k+1}} \ln \frac{1 + \lambda t}{1 - \lambda^2} \quad (6.17)$$

$$\hat{a}_k(t) = \int_{-1}^{1} \frac{(1 - \lambda^2)^k}{(1 + \lambda t)^{k+1}} d\lambda \quad \hat{b}_k(t) = \int_{-1}^{1} \frac{(1 - \lambda^2)^k}{(1 + \lambda t)^{k+1}} \ln \frac{1 + \lambda t}{1 - \lambda^2} \quad (6.18)$$

The coefficient functions admit Taylor series expansions in powers of $t$ with radius of convergence 1. Actually, in view of (6.7), we have the following relations between these functions

$$(k + 2)\hat{a}_k(t) = (k + 1)(2a_k(t) - a_{k+1}(t)) \quad (6.19)$$

$$(k + 2)^2 \hat{b}_k(t) = (k + 1)(k + 2)(2b_k(t) - b_{k+1}(t)) - 2a_k(t) + a_{k+1}(t) \quad (6.20)$$

From (6.13) and (6.15, 6.16), we obtain the series expansions of $W_k^\Delta'(0, 0)$ and $W_k^\Delta'(1, 0)$ using the following differentiation formulas

$$s^p \left( \frac{\partial}{\partial s} \right)^p s^k = \frac{\Gamma(k+1)}{\Gamma(k-p+1)} s^k \quad (6.21)$$

$$s^p \left( \frac{\partial}{\partial s} \right)^p \{s^k \ln s\} = \frac{\Gamma(k+1)}{\Gamma(k-p+1)} s^k \{\ln s + \psi(k+1) - \psi(k-p+1)\} \quad (6.22)$$
We find
\[
W_p^\Delta(0,0) = \frac{(-)^{\Delta+p} \pi^{\frac{d}{2}} \Gamma(p + \Delta - \frac{d}{2})}{2^\Delta \Gamma(p) \Gamma(\Delta)^2 (x^2 + y^2)^\Delta} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)^2 s^{k-\Delta+1}}{\Gamma(k-p+2) \Gamma(k-\Delta+2)} \left\{ b_k(t) - a_k(t) \ln s + 2\psi(k+1) - \psi(k-\Delta+2) - \psi(k-p+2) \right\}
\] (6.23)
and
\[
W_p^{\Delta}(1,0) = \frac{(-)^{\Delta+p+1} \pi^{\frac{d}{2}} \Gamma(p + \Delta - \frac{d}{2} + 1)}{2^{\Delta+1} \Gamma(p) \Gamma(p+2) \Gamma(\Delta)^2 (x^2 + y^2)^{\Delta-1}} \sum_{k=0}^{\infty} \frac{\Gamma(k+1) \Gamma(k+3) s^{k-\Delta+2}}{\Gamma(k-p+2) \Gamma(k-\Delta+3)} \left\{ \tilde{b}_k(t) - \tilde{a}_k(t) \ln s + \psi(k+1) + \psi(k+3) - \psi(k-\Delta+3) - \psi(k-p+2) \right\}
\] (6.24)

The presentation of these series expansions is slightly formal in the sense that for $k \leq \Delta - 2$, the $\Gamma(k-\Delta+2)$ function in the denominator produces a zero, while the $\psi(k-\Delta+2)$ term produces a pole, which together yield a finite result, which amounts to a pole term in $s$. Its coefficient can be obtained from the formula $\lim_{x \to 0} \psi(x-q)/\Gamma(x-q) = (-)^{q+1}\Gamma(q+1)$ for any non-negative integer $q$.

(b) Crossed channel series expansion

The crossed channel asymptotics is given by $s \to \frac{1}{2}$ and $t \to \pm 1$, and may also be obtained from the series expansion of the functions $I_0(s,t)$, with $a = 0, 1$. Actually, it suffices to obtain the expansion of $I_0(s,t)$ and thus of $W_p^{\Delta'}(0,0)$ in this limit and then to compute the series expansion of $W_p^{\Delta'}(1,0)$ by using the relation (6.7). This is useful in this case, since the expansion of $I_1(s,t)$ appears more involved than that of $I_0(s,t)$.

We start from the definition of $I_0(s,t)$ in (6.14) as a double integral and consecutively perform the following changes of variables $\mu = (1 + \lambda t)\sigma$ and $\tau = (1 + \sigma)^{-1}$, so that
\[
I_0(s,t) = \int_0^1 d\tau \int_{-1}^1 d\lambda \frac{1}{(1 - \tau)(1 + \lambda t) + \tau s(1 - \lambda^2)}
\] (6.25)

This form of the universal function $I_0(s,t)$ is now precisely of the form studied in [13], and the $\lambda$-integral may be performed explicitly in an elementary way. We obtain, as in [13]
\[
I_0(s,t) = I_0^{\log}(s,t) + I_0^{\reg}(s,t)
\] (6.26)

where
\[
I_0^{\log}(s,t) = - \ln(1 - t^2) \int_0^1 d\tau \frac{1}{\sqrt{\omega^2 - \tau^2(1 - t^2)}}
\] (6.27)
\[
I_0^{\reg}(s,t) = 2 \int_0^1 d\tau \frac{1}{\sqrt{\omega^2 - \tau^2(1 - t^2)}} \ln \left\{ \frac{\omega}{\tau} + \sqrt{\frac{\omega^2}{\tau^2} - (1 - t^2)} \right\}
\] (6.28)
where the composite variable \( \omega \) is defined by \( \omega = 1 - (1 - 2s)(1 - \tau) \). In the neighborhood of \( s = \frac{1}{2} \) and \( t = \pm 1 \), we have \( \omega \sim 1 \) and \( 1 - t^2 \sim 0 \), so that the integrals in \((6.27, 6.28)\) are both uniformly convergent, and may be Taylor expanded in powers of \((2s - 1)\) and \((1 - t^2)\). Thus, \( I_0^{\text{reg}}(s, t) \) is analytic in both \( s \) and \( t \) in the neighborhood of \( s = \frac{1}{2} \) and \( t = \pm 1 \), and all non-analyticity is contained in the factor \( \ln(1 - t^2) \) of \( I_0^{\text{log}}(s, t) \). The integral admits a double Taylor expansion given by

\[
I_0^{\text{log}}(s, t) = -\ln(1 - t^2) \sum_{k=0}^{\infty} (1 - 2s)^k \alpha_k(t)
\]

\[
\alpha_k(t) = \frac{1}{k + 1} \frac{\Gamma\left(\frac{1}{2}, \frac{k + 1}{2}; \frac{k + 3}{2}; 1 - t^2\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{k!} \frac{\Gamma\left(1 - \frac{1}{2}k\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{(1 - t^2)^k}{2 \ell + k + 1} \tag{6.29}
\]

This expansion may be used to evaluate the logarithmic part of \( W_p^{\Delta'}(0, 0) \) and we obtain the following result

\[
W_p^{\Delta'}(0, 0) \bigg|_{\log} = -2^{p-2} \pi^{\frac{d}{2}} \ln(1 - t^2) \frac{\Gamma(p + \Delta' - \frac{d}{2})}{\Gamma(p) \Gamma(\Delta')(x^2 + y^2)^{\Delta'}} \sum_{\ell=0}^{\Delta'-1} \sum_{k=0}^{\infty} \frac{(-2)^{-\ell} \Gamma(k + 1)}{\Gamma(\Delta' - \ell) \Gamma(p - \ell) \ell!} \frac{1}{1} \alpha_k(t) \tag{6.30}
\]

Notice that in the crossed channel, no power singularities arise.

### 6.3 Asymptotic expansion for the graviton exchange

We now turn to the direct channel asymptotic expansion of the graviton exchange graph for \( \Delta = \Delta' = d = 4 \). The power singularity terms may be read off directly from the general asymptotic expansion formula \((6.23)\) restricted to \( d = 4 \), and we have

\[
W_p^{\Delta}(0, 0) \sim \frac{\pi^2 \Gamma(p + \Delta - 2)}{2 \Delta \Gamma(p)^2 \Gamma(\Delta)^2} \frac{(-)^{p-1} \Gamma(k + 1)^2 \Gamma(\Delta - k - 1)}{\Gamma(k - p + 2)} \frac{a_k(t)}{s^{\Delta - 1 - k}} \tag{6.31}
\]

Similarly, we have from \((6.24)\)

\[
W_p^{\Delta}(1, 0) \sim \frac{\pi^2 \Gamma(p + \Delta - 1)}{2 \Delta + 1 \Gamma(p) \Gamma(p + 2) \Gamma(\Delta)^2} \frac{(-)^{p-1} \Gamma(k + 1) \Gamma(k + 3) \Gamma(\Delta - k - 2)}{\Gamma(k - p + 2)} \frac{\hat{a}_k(t)}{s^{\Delta - 2 - k}} \tag{6.32}
\]

The full singular power part of the amplitude is now easily obtained by working out the asymptotics above in the cases \( W_p^4(0, 0), W_p^5(0, 0) \) and \( W_p^5(1, 0) \) with \( p = 1, 2, 3 \). The function
\(W^4_{p}(0,0)\) has no power singularities and does not contribute here. Putting all together, we have

\[
B_{\text{sing}} = -\frac{48}{\pi^6} \frac{1}{(x^2 + y^2)^4} \left[-2 \left( \frac{a_0(t)}{s^3} + \frac{a_1(t)}{s^2} + \frac{a_2(t)}{s} \right) + 3 \left( \frac{\hat{a}_0(t)}{s^3} + \frac{\hat{a}_1(t)}{s^2} + \frac{\hat{a}_2(t)}{s} \right) \right] \tag{6.33}
\]

Using the series expansions of the functions \(a_k(t)\) and \(\hat{a}_k(t)\) to low orders, taking into account that generically, \(s\) vanishes like \(t^2\),

\[
\begin{align*}
a_0(t) &= 2 + \frac{2}{3} t^2 + \frac{2}{5} t^4 & a_1(t) &= \frac{4}{3} + \frac{4}{5} t^2 & a_2(t) &= \frac{16}{15} \\
\hat{a}_0(t) &= \frac{4}{3} + \frac{4}{15} t^2 + \frac{4}{35} t^4 & \hat{a}_1(t) &= \frac{16}{15} + \frac{48}{105} t^2 & \hat{a}_2(t) &= \frac{32}{35} \tag{6.34}
\end{align*}
\]

The final result for the singular part of \(B\) is

\[
B_{\text{sing}} = -\frac{2^7}{35 \pi^6} \frac{1}{(x^2 + y^2)^4} \left[ \frac{1}{s^3} (-7 t^2 - 6 t^4) + \frac{1}{s^2} (7 - 3 t^2) + \frac{8}{s} \right] \tag{6.35}
\]

Repristinating the overall kinematic factors we get the final result for the singular terms in the direct channel of the graviton amplitude

\[
I_{\text{grav}} \mid_{\text{sing}} = \frac{2^{10}}{35 \pi^6} \frac{1}{x_{13}^2 x_{24}^2} \left[ s (7 t^2 + 6 t^4) + s^2 (-7 + 3 t^2) - 8 s^3 \right] \tag{6.36}
\]

Notice that the leading singularity \(x_{13}^{-6}\) cancels between the various tensor contributions to the amplitude. The physical interpretation of this singular expansion is discussed in Section 2.3.

The logarithmic singularities may be read off directly from the asymptotic expansion formulas of (6.23, 6.24), and we have

\[
\begin{align*}
W^4_{p}(0,0) &= (-)^{p+1} \frac{\pi^2}{2^6 3^2} \frac{\Gamma(p + 2)}{(x^2 + y^2)^4} \ln s \frac{s \Gamma(k + 4)^2 s^k}{\Gamma(k + 5 - p) \Gamma(k + 1)} a_{k+3}(t) \\
W^5_{p}(0,0) &= (-)^{p} \frac{\pi^2}{2^{11} 3^2} \frac{\Gamma(p + 3)}{(x^2 + y^2)^5} \ln s \frac{s \Gamma(k + 5)^2 s^k}{\Gamma(k + 6 - p) \Gamma(k + 1)} a_{k+4}(t) \tag{6.37}
\end{align*}
\]

\[
W^5_{p}(1,0) = (-)^{p+1} \frac{\pi^2}{2^{12} 3^2} \frac{\Gamma(p + 4)}{\Gamma(p) \Gamma(p + 2)} \ln s \frac{s \Gamma(k + 6)^2 s^k}{(x^2 + y^2)^4} \frac{s \Gamma(k + 5 - p) \Gamma(k + 1)}{\Gamma(k + 1)} \hat{a}_{k+3}(t)
\]

Assembling these contributions to the logarithmic singularity and expressing the coefficient functions \(\hat{a}_k(t)\) in terms of \(a_k(t)\) using (6.19) we get

\[
I_{\text{grav}} \mid_{\text{log}} = \frac{3 \cdot 2^3}{\pi^6} \frac{\ln s}{x_{13}^2 x_{24}^2} \sum_{k=0}^{\infty} s^{4+k} \frac{s \Gamma(k + 4)}{\Gamma(k + 1)} \left\{ -2 (5 k^2 + 20 k + 16) (3k^2 + 15k + 22) a_{k+3}(t) \\
+ (k + 4)^2 (15k^2 + 55k^2 + 42) a_{k+4}(t) \right\} \tag{6.38}
\]
A Appendix

Properties of \( D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} \)

We have seen that a basic building block in expressing the 4–point functions is the quantity \( D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} \), defined by

\[
D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4}(x_1, x_3, x_2, x_4) = \int \frac{d^{d+1}z}{z_0^{d+1}} \tilde{K}_{\Delta_1}(z, x_1) \tilde{K}_{\Delta_3}(z, x_3) \tilde{K}_{\Delta_2}(z, x_2) \tilde{K}_{\Delta_4}(z, x_4) \tag{A.1}
\]

where \( \tilde{K}_{\Delta}(z, x) \) is

\[
\tilde{K}_{\Delta}(z, x) = \left( \frac{z_0}{z_0 + (\vec{z} - \vec{x})^2} \right)^\Delta. \tag{A.2}
\]

(note the different normalization from \( K(z, x) \), equ.(3.31)). Thus \( D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} \) corresponds to a quartic interaction between scalars of dimension \( \Delta_i \), with a simple non–derivative interaction vertex, see Figure 5. Note that sometimes we suppress the explicit coordinate dependence of the \( D \) functions. Coordinate labels are always understood to be in the order \((x_1, x_3, x_2, x_4)\).

While the result of the computation of the graviton exchange graph gives a sum of many different \( D \) functions, in fact all these functions are closely related to each other. We show that one can relate \( D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} \) to \( D_{\Delta_1-1 \Delta_3-1 \Delta_2 \Delta_4} \) and \( D_{\Delta_1 \Delta_3 \Delta_2+1 \Delta_4+1} \) (see for example (A.9)). Further, all the \( D \) functions can be obtained from differentiating one single expression (which can be obtained in closed form) with respect to the variables \( x_{ij}^2 \). This is shown in section A.3. Using this latter fact we show how for example how \( D_{\Delta \Delta+1 \Delta \Delta+1} \) (symmetrizing permutations) can be related easily to expressions of the form \( D_{\Delta \Delta \Delta \Delta} \) (see (A.16)). These relations are useful to arrive at the two simplified forms of the graviton amplitude (4.38), (5.70) given in the text and to show their equivalence.

A.1 Relation between \( D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} \) and \( W^\Delta_k(a, b) \)

The standard integral introduced in (5.53) is just a quartic graph evaluated in the inverted frame, with some kinematic factors omitted. The precise relation with \( D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} \) is

\[
W^\Delta_k(a, b) = x_{13}^{2k} x_{14}^{2\Delta'} x_{12}^{2(\Delta'+b)} D_{2a-b+k, \Delta, \Delta'} \tag{A.3}
\]

A.2 Derivative vertices

The first thing we note is that if we have a quartic interaction with derivatives, given by a coupling

\[
\phi_{\Delta_1}(z) \phi_{\Delta_3}(z) \frac{\partial}{\partial z_{\mu}} \phi_{\Delta_2}(z) \frac{\partial}{\partial z_{\nu}} \phi_{\Delta_4}(z) \eta^{\mu\nu}, \tag{A.4}
\]
then the computation of the 4–point function with such an interaction can again be reduced to a sum of terms of the form (A.1). This is done with the identity [9]

\[ g^\mu\nu \frac{\partial}{\partial z^\mu} \tilde{K}(z, x_1) \frac{\partial}{\partial z^\nu} \tilde{K}(z, x_2) = \Delta_1 \Delta_2 \left[ \tilde{K}_{\Delta_1} (z, x_1) \tilde{K}_{\Delta_2} (z, x_2) - 2 x_{12}^2 \tilde{K}_{\Delta_1+1} (z, x_1) \tilde{K}_{\Delta_2+1} (z, x_2) \right]. \] 

(A.5)

Thus

\[ D_{\Delta_1, \Delta_2, \Delta_3, \Delta_4} \equiv \int \frac{dz}{z_{01}^{d+1}} \tilde{K}_{\Delta_1} (z, x_1) \tilde{K}_{\Delta_3} (z, x_3) \frac{\partial}{\partial z^\mu} \tilde{K}_{\Delta_2} (z, x_2) z_0^2 \frac{\partial}{\partial z^\mu} \tilde{K}_{\Delta_4} (z, x_4) \]

(A.6)

\[ = \Delta_2 \Delta_4 \left( D_{\Delta_1, \Delta_3, \Delta_2, \Delta_4} - 2 x_{24}^2 D_{\Delta_1, \Delta_3, \Delta_2+1, \Delta_4+1} \right). \]

A.3 Lowering and raising \( \Delta_i \)

Not only does the identity (A.6) allow us to remove derivatives from the quartic vertex, it is also useful to relate various \( D \) functions to each other. Let us rewrite the l.h.s. in (A.6) as

\[ D_{\Delta_1, \Delta_3, \partial_2, \partial_4} = \frac{1}{2} \int \frac{dz}{z_{01}^{d+1}} \tilde{K}_{\Delta_1} (z, x_1) \tilde{K}_{\Delta_3} (z, x_3) \frac{\partial}{\partial z^\mu} \tilde{K}_{\Delta_2} (z, x_2) z_0^2 \frac{\partial}{\partial z^\mu} \tilde{K}_{\Delta_4} (z, x_4) \]

(A.7)

where \( m_\Delta^2 \equiv \Delta (\Delta - d) \). Upon integrating by parts of the first term in (A.7) we get

\[ \frac{1}{2} \int [dz] \square_z \left( \tilde{K}_{\Delta_1} \tilde{K}_{\Delta_3} \right) \tilde{K}_{\Delta_2} \tilde{K}_{\Delta_4} = \int [dz] \frac{\partial}{\partial z^\mu} \tilde{K}_{\Delta_1} z_0^2 z_0^2 \frac{\partial}{\partial z^\mu} \tilde{K}_{\Delta_3} \tilde{K}_{\Delta_2} \tilde{K}_{\Delta_4} \]

(A.8)

\[ + \frac{1}{2} \left( m_{\Delta_1}^2 + m_{\Delta_3}^2 \right) D_{\Delta_1, \Delta_3, \Delta_2, \Delta_4} \]

Putting relations (A.6, A.7, A.8) together, we find in particular, for \( \Delta_1 = \Delta_3 = \Delta, \Delta_2 = \Delta_4 = \tilde{\Delta} \):

\[ \tilde{\Delta}^2 x_{24}^2 D_{\Delta \tilde{\Delta} + 1, \tilde{\Delta} + 1} = \Delta^2 x_{13}^2 D_{\Delta + 1, \tilde{\Delta} + 1} + \frac{1}{2} \left( \tilde{\Delta}^2 - \Delta^2 + m_{\Delta}^2 - m_{\tilde{\Delta}}^2 \right) D_{\Delta \tilde{\Delta} + 1, \tilde{\Delta} + 1} \]

(A.9)

A special case is \( \Delta = \tilde{\Delta} \), which implies:

\[ x_{24}^2 D_{\Delta \Delta + 1, \tilde{\Delta} + 1} = x_{13}^2 D_{\Delta + 1, \Delta + 1} \Delta \Delta. \]

(A.10)

Iteration of (A.9) allows one to prove that more generally

\[ (x_{24}^2)^n D_{\Delta \Delta + n \Delta + n} = (x_{13}^2)^n D_{\Delta + n \Delta + n} \Delta \Delta. \]

(A.11)
A.4 Obtaining $D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4}$ in closed form

By using a Schwinger parameterization and performing the $z$ integrals, one finds [30] (and references therein):

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_3, x_2, x_4) = \frac{\pi^d \Gamma(\frac{\Sigma-d}{2}) \Gamma(\frac{\Sigma}{2})}{2 \prod_i \Gamma(\Delta_i)} \int \prod_j d\alpha_j \frac{(\sum_i \alpha_i - 1) \delta(\sum_k \alpha_k \alpha_l x_{kl}^2)^{\frac{\Sigma}{2}}}{\left(\sum_k \alpha_k \alpha_l x_{kl}^2\right)^{\frac{\Sigma}{2}}}$$ (A.12)

where

$$\Sigma \equiv \sum_i \Delta_i.$$ (A.13)

We observe that any $D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4}$ can be obtained by differentiating an appropriate number of times in the variables $x_{ij}$ the basic function

$$B(x_{ij}) = \int \prod_j d\alpha_j \frac{\delta(\sum_i \alpha_i - 1)}\left(\sum_k \alpha_k \alpha_l x_{kl}^2\right)^{\frac{\Sigma}{2}}.$$ (A.14)

$B(x_{ij})$ is given in closed form in [30]. From the integral representation (A.12) we immediately find

$$\frac{\partial}{\partial x_{13}} D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4} = -\frac{2\Delta_1 \Delta_3}{\Sigma - d} D_{\Delta_1 + 1 \Delta_3 + 1 \Delta_2 \Delta_4}$$ (A.15)

A.5 Symmetrizing identities

Equation (A.15) can be used to show that a sum of $D$ functions which is symmetric under $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_4$ can always be rewritten in a basis in which each individual term shares this symmetry, i.e. each term is of the form $D_{\Delta \Delta \tilde{\Delta} \tilde{\Delta}}$. For example:

$$x_{12}^2 D_{\Delta+1 \Delta+1 \Delta+1} + x_{14}^2 D_{\Delta+1 \Delta \Delta+1} = \frac{\Sigma - d}{2\Delta} D_{\Delta \Delta \tilde{\Delta} \tilde{\Delta}} - \frac{\Delta}{\Delta} x_{13}^2 D_{\Delta+1 \Delta+1 \Delta+1 \Delta+1}$$ (A.16)

where $\Sigma \equiv 2\Delta + 2\tilde{\Delta}$. Let us see how to derive this identity. It follows from conformal invariance that

$$D_{\Delta_1 \Delta_2 \Delta_4} = \left(\prod_{i<j} (x_{ij}^2)^{-\frac{\Delta_i + \Delta_j}{2} + \frac{\Sigma}{6}}\right) E_{\Delta_1 \Delta_2 \Delta_4}(\xi, \eta)$$ (A.17)

where $\xi \equiv x_{12}^2 x_{34}^2, \eta \equiv x_{12}^2 x_{14}^2 x_{24}^2$ are conformal cross ratios. From simple chain rule manipulations we then get

$$\left(x_{12}^2 \frac{\partial}{\partial x_{12}^2} + x_{13}^2 \frac{\partial}{\partial x_{13}^2} + x_{14}^2 \frac{\partial}{\partial x_{14}^2}\right) E_{\Delta_1 \Delta_2 \Delta_4}(\xi, \eta) = 0.$$ (A.18)
Using (A.15), the last equation is tantamount to (A.16) for \( \Delta_1 = \Delta_3 = \Delta, \Delta_2 = \Delta_4 = \tilde{\Delta} \). Similar arguments lead to the more complicated identity

\[
x_{12}^2 x_{14}^2 D_{\Delta+2 \Delta \Delta + 1 \tilde{\Delta} + 1} + x_{23}^2 x_{34}^2 D_{\Delta \Delta + 2 \tilde{\Delta} + 1 \tilde{\Delta} + 1} = \]

\[
\begin{align*}
&-\frac{\Delta}{\Delta + 1} (x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2) D_{\Delta+1 \Delta+1 \tilde{\Delta} + 1} + \frac{\Delta(\Sigma - d) (\Sigma + 2 - d)}{4(\Delta + 1) \Delta^2} D_{\Delta \Delta \Delta} \\
&- \frac{\Delta(2\Delta + 1)(\Sigma + 2 - d)}{2(\Delta + 1) \Delta^2} x_{13}^2 D_{\Delta+1 \Delta+1 \tilde{\Delta} + 1} + \frac{\Delta(\Delta + 1)}{\Delta^2} x_{13}^4 D_{\Delta+2 \Delta+2 \tilde{\Delta} + 1},
\end{align*}
\]

where \( \Sigma = 2\Delta + 2\tilde{\Delta} \).

A.6 Series expansion of \( D_{\Delta \Delta \Delta \tilde{\Delta}} \)

From (A.3) and (6.23):

\[
D_{\Delta \Delta \tilde{\Delta} \tilde{\Delta}}(x_1, x_3, x_2, x_4) = \left\{ b_k(t) - a_k(t) \right\} \ln s + 2\psi(k + 1) - \psi(k - \Delta + 2) - \psi(k - \tilde{\Delta} + 2) \right\}.
\]

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