An $su(1, 1)$ algebraic approach for the relativistic Kepler–Coulomb problem

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Abstract
We apply the Schrödinger factorization method to the radial second-order equation for the relativistic Kepler–Coulomb problem. From these operators we construct two sets of one-variable radial operators which are realizations for the $su(1, 1)$ Lie algebra. We use this algebraic structure to obtain the energy spectrum and the supersymmetric ground state for this system.

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1. Introduction

Compact and non-compact symmetries play a central role in the study of many properties of quantum systems because they form the basis for selection rules that forbid the existence of certain states and processes [1]. Also, from a given solution, symmetries allow us to obtain new solutions of the dynamical equations. Moreover, the conserved quantity associated with a given symmetry represents an integrability condition.

The generators of compact and non-compact algebras for a given Hamiltonian have not been constructed by a systematic method but intuitively found and forced to close an algebra [2–4]. In particular, the $su(1, 1)$–$so(2, 1)$ non-compact algebra admits infinite-dimensional representations [5, 6]. This symmetry has been successfully applied to formulate algebraic approaches to many non-relativistic quantum problems [7–17], where the corresponding realizations were given in terms of two variables. Also, some one-variable realizations for the $su(1, 1)$ algebra have been introduced [2]. Recently, it has been shown that the Schrödinger factorization operators can be used to construct the $su(1, 1)$ algebra generators for two- [18] and $N$-dimensional systems [19].
The relativistic Kepler–Coulomb problem is one of the few exactly solvable potentials in physics, and it has been studied in several ways: analytical [20–25] and factorization (algebraic) methods [26–28], shape-invariance [29] and SUSY QM for the first- [21, 30] and second-order equations [31]. The solvability of this problem is due to the conservation of the total angular momentum, the Dirac and Lippmann–Johnson operators [32]. The first two are due to the existence of spin, and the latter explains the degeneracy in the eigenvalues of the Dirac operator of the energy spectrum, and it reduces to the Runge–Lenz vector in the non-relativistic limit. Symmetries and SUSY QM are not independent. Indeed, it has been shown that supersymmetry is generated by the Lippmann–Johnson operator [32]. Also, the Dirac–Coulomb problem has been solved by using the Biedenharn–Temple operator [33].

In a series of papers, some realizations of compact and non-compact Lie algebras for the uncoupled second-order radial equations corresponding to the relativistic Kepler–Coulomb problem have been introduced [34–36]. However, in all these works, the origin of the generators was not given. Moreover, in order to close the $su(1, 1)$ Lie algebra, these generators were forced to depend on an extra variable which plays the role of a phase.

The aim of this paper is to study the relativistic Kepler–Coulomb problem from an $su(1, 1)$ algebraic approach by using the Schrödinger factorization and without the introduction of any additional variable.

In section 2, we obtain the uncoupled second-order differential equations satisfied by the radial components. Section 3 is divided into two parts. In the first by applying the Schrödinger factorization to an uncoupled second-order differential equation for the bound states, we obtain the corresponding $su(1, 1)$ algebra generators. In the second we use the theory of unitary representations to obtain the discrete energy spectrum for this system. In section 4, the SUSY ground state and the action of the $su(1, 1)$ algebra generators on the radial bound eigenstates are found. In section 5, for the continuum we obtain a set of operators which close the $su(1, 1)$ Lie algebra. Also, the explicit form for the scattering states is found.

Finally, we give some concluding remarks.

2. The relativistic Kepler–Coulomb radial equation

The Dirac radial equation for the Kepler–Coulomb problem is [20, 30]

$$
\left( \frac{dG^{(1)}_k}{dr} + 1 \frac{k}{r} \right) G^{(1)}_k = \left( \begin{array}{cc} 0 & \alpha_1 \\ \alpha_2 & 0 \end{array} \right) \left( \begin{array}{c} G^{(1)}_k \\ G^{(2)}_k \end{array} \right),
$$

(1)

where $k$ is the eigenvalue of the Dirac operator $K = -\mathbf{\sigma} \cdot \mathbf{L} + 1$, $\gamma = \frac{Ze^2}{\hbar}$, $Z$ is the atomic number, $\alpha_1 = m + E$, $\alpha_2 = m - E$, $|k| = j + \frac{1}{2}$, $k = \pm 1, \pm 2, \pm 3, \ldots$, $\mathbf{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, $\mathbf{L}$ is the angular momentum operator and $\mathbf{I}$ is the $2 \times 2$ unit matrix.

By introducing the new variable $\rho = Er$, equation (1) leads to the pair of equations

$$
\left( \frac{k}{s} + \frac{m}{E} \right) F^{(2)}_k = \left( \frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s} \right) F^{(1)}_k,
$$

(2)

$$
\left( \frac{k}{s} - \frac{m}{E} \right) F^{(1)}_k = \left( -\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s} \right) F^{(2)}_k,
$$

(3)

where

$$
\frac{F^{(1)}_k}{F^{(2)}_k} = \left( \begin{array}{cc} k + s & -\gamma \\ -\gamma & k + s \end{array} \right) \frac{G^{(1)}_k}{G^{(2)}_k}
$$

(4)
and
\[ s = \sqrt{k^2 - \gamma^2}. \]  

From these relations we obtain the uncoupled second-order differential equations
\[
\left( -\frac{d^2}{d\rho^2} + \frac{s(s \pm 1)}{\rho^2} + \frac{\gamma^2}{s^2} - \frac{2\gamma}{\rho} \right) F_k^{(1,2)} = \left( \frac{k}{s} + \frac{m}{E} \right) \left( \frac{k}{s} - \frac{m}{E} \right) F_k^{(1,2)},
\]
where the superscript \((1, 2)\) corresponds to \((+,-)\) in the centrifugal term, respectively. Thus, for the bound states \((m^2 > E^2)\), equation (5) allows us to rewrite equation (6) as
\[
\left( -\rho^2 \frac{d^2}{d\rho^2} + s^2 \rho^2 - 2\gamma \rho \right) F_k^{(1)} = -s(s + 1) F_k^{(1)},
\]
\[
\left( -\rho^2 \frac{d^2}{d\rho^2} + s^2 \rho^2 - 2\gamma \rho \right) F_k^{(2)} = -s(s - 1) F_k^{(2)},
\]
where
\[ \xi^2 = \frac{m^2}{E^2} - 1. \]

It must be emphasized that equation (7) is formally obtained from equation (8) by making \(s \to s + 1\). Hence, by defining \(\psi_s \equiv F_k^{(2)}\), we conclude that
\[ F_k^{(1)} \propto \psi_{s+1}. \]

Thus, the solution for the Dirac equation in spinorial form is
\[
\Phi_k \equiv \left( F_k^{(1)}, F_k^{(2)} \right) = \left( \psi_{s+1}, \psi_s \right).
\]

3. The Schrödinger factorization and the \(su(1, 1)\) Lie algebra for bound states

In order to factorize the left-hand-side operator of equations (8), we apply the Schrödinger factorization [18, 37]. Thus, we propose a pair of first-order differential operators such that
\[
\left( \rho \frac{d}{d\rho} + a \rho + b \right) \left( -\rho \frac{d}{d\rho} + c \rho + f \right) \psi_s = g \psi_s,
\]
where \(a, b, c, f\) and \(g\) are constants to be determined. Expanding this expression and comparing it with equation (8) we obtain
\[ a = c = \pm \xi, \quad f = 1 + b = \mp \frac{\gamma}{\xi}, \quad g = b(b + 1) - s(s - 1). \]

Using these results, equation (8) is equivalent to
\[
(B_\pm + 1) B_\pm \psi_s = \left[ \frac{\gamma}{\xi} \pm \frac{1}{2} \right]^2 - \left( s - \frac{1}{2} \right)^2 \psi_s,
\]
where
\[ B_\pm = \mp \rho \frac{d}{d\rho} + \xi \rho - \frac{\gamma}{\xi}. \]

From equation (8) we obtain the operator
\[
\Sigma_3 \psi_s = \frac{1}{2\xi} \left( -\rho \frac{d^2}{d\rho^2} + \xi^2 \rho + \frac{s(s - 1)}{\rho} \right) \psi_s = \frac{\gamma}{\xi} \psi_s.
\]

Therefore, using equation (15) we define a new pair of operators
\[ \Sigma_3 \equiv \pm \rho \frac{d}{d\rho} + \xi \rho - \Sigma_3. \]
3.1. $su(1, 1)$ Lie algebra

By direct calculation it is immediate to show that the operators $\Sigma_\pm$ and $\Sigma_3$ close the $su(1, 1)$ Lie algebra

$$\left[ \Sigma_\pm, \Sigma_3 \right] = \mp \Sigma_\pm,$$

$$\left[ \Sigma_+, \Sigma_- \right] = -2 \Sigma_3,$$

with the quadratic Casimir operator

$$\Sigma^2 = -\Sigma_3 \Sigma_+ + \Sigma_3^2 - \Sigma_3.$$  \hspace{1cm} (20)

From equations (16) and (17), we find that the eigenvalue equation for this operator is

$$\Sigma^2 \psi_s = s(s-1) \psi_s.$$ \hspace{1cm} (21)

As was emphasized above, by performing the change $s \to s+1$ to $\psi^s_{ns}$, we obtain the upper component of the spinor $\Phi_1 \psi^s_{s+1}$. In this way, its corresponding differential operators are

$$\Xi_3 = \frac{1}{2\xi} \left( -\rho \frac{d^2}{d\rho^2} + \xi^2 \rho + \frac{s(s+1)}{\rho} \right),$$ \hspace{1cm} (22)

$$\Xi_\pm = \mp \rho \frac{d}{d\rho} + \xi \rho - \Xi_3,$$ \hspace{1cm} (23)

and by direct calculation we show that they satisfy the $su(1, 1)$ Lie algebra

$$\left[ \Xi_\pm, \Xi_3 \right] = \mp \Xi_\pm,$$ \hspace{1cm} (24)

$$\left[ \Xi_+, \Xi_- \right] = -2 \Xi_3.$$ \hspace{1cm} (25)

3.2. $su(1, 1)$ unitary representation

In order to establish the properties of the operators $\Sigma_3$ and $\Sigma_\pm$, we define the inner product on the Hilbert space spanned by the radial eigenfunctions of the relativistic Kepler–Coulomb problem as \cite{38}

$$(\phi, \zeta) \equiv \int_0^\infty \phi^*(\rho) \zeta(\rho) \rho^{-1} d\rho.$$ \hspace{1cm} (26)

Thus, it follows that the operator $\Sigma_3$ is Hermitian. Moreover, using equations (17) and (26) we prove that the operators $\Sigma_\pm$ are the Hermitian conjugates,

$$\Sigma_\pm = \Sigma^\dagger_\mp.$$ \hspace{1cm} (27)

The theory of unitary irreducible representations of the $su(1, 1)$ Lie algebra has been studied in several works \cite{38, 39} and it is based on the equations

$$C^2|\mu\nu\rangle = \mu(\mu+1)|\mu\nu\rangle,$$ \hspace{1cm} (28)

$$C_3|\mu\nu\rangle = v|\mu\nu\rangle,$$ \hspace{1cm} (29)

$$C_\pm|\mu\nu\rangle = [(v \mp \mu)(v \pm \mu \pm 1)]^{1/2}|\mu\nu \pm 1\rangle,$$ \hspace{1cm} (30)

where $C^2$ is the quadratic Casimir operator, $v = \mu + q + 1$, $q = 0, 1, 2, \ldots$ and $\mu > -1$. The last relation means that $C_+(C_-)$ are the raising (lowering) operators for $v$. 4
Therefore, from equations (21) and (28), and (16) and (29), we find
\[
\mu_s = s - 1,
\]
\[
\nu_s = n_s + s = \frac{\gamma}{\xi},
\]
respectively, with \(n_s = 0, 1, 2, \ldots\).

If we consider that the operators \(\Sigma_{\pm}\) leave fixed the quantum number \(\mu_s\), then from equation (31), the values of \(s\) remain fixed. This is because the change \(\nu_s \rightarrow \nu_s \pm 1\) induced by the operators \(\Sigma_{\pm}\) on the basis vectors \(|\mu_s \nu_s\rangle\) implies \(n_s \rightarrow n_s \pm 1\). Thus, by setting \(|\mu_s \nu_s\rangle \rightarrow \psi_{n_s s}\) and from equations (30), (31) and (32), we find
\[
\Sigma_{\pm} \psi_{n_s s} = Q_{n_s \pm 1} \psi_{n_s \pm 1},
\]
with \(Q_{n_s \pm 1} = [(n_s + s + 1 \mp s)(n_s + s + 1 \pm s \pm 1)]^{1/2}\) a real number.

Using equations (9) and (32) we find that the energy spectrum for the lower component of the spinor given in equation (11), \(\psi_{n_s s}\), is
\[
E_s = m \left[1 + \frac{\gamma^2}{(n_s + s)^2}\right]^{-1/2},
\]
From equation (33) we obtain that the action of the ladder operators \(\Sigma_{\pm}\) on the functions \(\psi_{n_s s+1}\) is
\[
\Sigma_{\pm} \psi_{n_s s+1} = Q_{n_s \pm 1} \psi_{n_s \pm 1}.
\]
Hence, from equation (34) we find that the energy spectrum for the function \(\psi_{n_s s+1}\) is
\[
E_{s+1} = m \left[1 + \frac{\gamma^2}{(n_s + s + 1)^2}\right]^{-1/2}.
\]
Since \(\psi_{n_s s+1}\) and \(\psi_{n_s s}\) are the components for the spinor \(\Phi_s\), they must have the same energy. This means that \(E_s = E_{s+1}\). Therefore, from equations (34) and (36) we obtain
\[
n \equiv n_s = n_{s+1} + 1,
\]
where \(n = 0, 1, 2, 3, \ldots\) is the radial quantum number. Thus, the energy spectrum for the relativistic Kepler–Coulomb problem is
\[
E = m \left[1 + \frac{\gamma^2}{(n + s)^2}\right]^{-1/2},
\]
where the positiveness of \(E\) ensures that the components of the spinor given in (11) are quadratically integrable [34]. In this way, \(\Phi_n^\mu\) is given by
\[
\Phi_n^\mu = \left( \begin{array}{c} F_n^{(1)} \\ F_n^{(2)} \end{array} \right) = \left( \begin{array}{c} \psi_{s+1}^{n-1} \\ \psi_s^{n+1} \end{array} \right).
\]
This result, which has been obtained from the theory of unitary representations, can be deduced from an analytical approach as it is shown in the next section.

4. Schrödinger and SUSY QM ground states

For \(n = 0\) and from equations (33) and (37), we find that only the state
\[
\psi_s^0 = n \rho^s e^{-\gamma \rho / s}
\]
is normalizable with respect to the inner product defined in expression (26) and satisfies the differential equation \( \Sigma_\omega \psi'_0 = 0 \). Also, for \( n = 0 \) and from equation (37), we find \( n_{s+1} = -1 \). For these values, \( \mathcal{Q}^{(-1,s+1)} \) results in a complex number. Thus, from the theory of unitary representations and equation (35), the function \( \psi'_{s+1} \) is non-normalizable [39]. Since this function is not a physically acceptable solution, the spinor corresponding to \( n = 0 \) is

\[
\Phi_k^0 = \begin{pmatrix} 0 \\ \rho^s e^{-\gamma/\rho} \end{pmatrix},
\]

which we denote as the Schrödinger ground state. Note that \( \Phi_k^0 \) is equal to the SUSY ground state for the relativistic Kepler–Coulomb problem found in [30, 32].

From an analytical approach, to determine \( \Phi_k^0 \) for higher levels we solve the differential equation (8) by proposing

\[
F_k = \rho^s e^{-\xi \rho} f(\rho).
\]

Thus, \( f(\rho) \) must satisfy

\[
\left[ \frac{d^2}{dy^2} + \left( 2s - y \right) \frac{d}{dy} + \frac{y}{\xi} - s \right] f(y/2\xi) = 0,
\]

where \( y = 2\xi \rho \). The solution for this equation is the confluent hypergeometric function \( _1F_1(\cdot, \cdot; z) \). Thus, we get

\[
F_k = \eta_2 \rho^s e^{-\xi \rho} _1F_1(-n, 2s; 2\xi \rho).
\]

In a similar way, the solution for equation (7) is

\[
F_k = \eta_1 \rho^s e^{-\xi \rho} _1F_1(-n + 1, 2s + 2; 2\xi \rho).
\]

These equations are in agreement with our results obtained from the theory of unitary representations, equation (39). Moreover, it is known that \( _1F_1(0, b; z) = 1 \), whereas \( _1F_1(a, b; z) \) diverges for \( a > 0 \). Thus, for \( n = 0 \),

\[
F_k = \eta_1 \rho^s e^{-\xi \rho},
\]

while the function \( F_k^{(1)} \) is not square-integrable, and it must be taken as the zero function. This result is in accordance with equation (41) which has been obtained from an algebraic approach.

From definition (37), equations (33) and (35) imply

\[
\Sigma_\pm \psi^n_s \propto \psi^{n\pm 1}_s,
\]

\[
\Sigma_\pm \psi^{n-1}_{s+1} \propto \psi^{n-1\pm 1}_{s+1}.
\]

It must be noted that equation (48) is valid for any value of the radial quantum number except for \( n = 0 \). This is because the upper component of the spinor for \( n = 1 \) cannot be obtained from the action of the operator \( \Sigma_\omega \) on the upper component of the Schrödinger ground state (equation (41)). These results are shown in figure 1. We illustrate the action of the SUSY operators \( A^\pm \) on the eigenstates of the relativistic Kepler–Coulomb problem [30]. Also, it is shown that the lower components for the states corresponding to \( n = 0 \) are annihilated by the operators \( \Sigma_\omega \) and \( A^- \). This result and the fact that the upper component of the SUSY [30] and the Schrödinger ground states are zero imply the equality of these states.

Equations (16), (47) and (48) allow us to show that the action of the Schrödinger operators on the states \( \psi^n_s \) is

\[
B_{\pm} \psi^n_s \propto \psi^{n\pm 1}_s.
\]

A similar result can be obtained from equation (37) and the operators \( \Sigma_\omega \). These results imply that the action of the Schrödinger operators on the components of the spinor \( \Phi_k^0 \) is to change only the radial quantum number \( n \) leaving fixed the Dirac quantum number \( k = k(s) \).
5. Continuum states and $su(1,1)$ algebra

For the scattering states ($m^2 < E^2$), the second-order radial equations are

\[
\begin{align*}
\left(-\rho^2 \frac{d^2}{d\rho^2} - \zeta^2 \rho^2 - 2\gamma \rho\right) R^{(1)} &= -s(s+1)R^{(1)}, \\
\left(-\rho^2 \frac{d^2}{d\rho^2} - \zeta^2 \rho^2 - 2\gamma \rho\right) R^{(2)} &= -s(s-1)R^{(2)},
\end{align*}
\]

where

\[
\zeta^2 = 1 - \frac{m^2}{E^2}. \tag{52}
\]

Note that equations (50) and (51) are obtained from (7) and (8) by making the substitution $\xi \rightarrow -i\zeta$. This fact allows us to write equation (51) as

\[
\left(B^+_\pm 1\right) B^+_\pm R^{(2)} = \left[\left(\frac{i\gamma}{\zeta} \pm \frac{1}{2}\right)^2 - \left(s - \frac{1}{2}\right)^2\right] R^{(2)},
\]

where

\[
B^+_\pm = \mp \rho \frac{d}{d\rho} - i\zeta \rho - \frac{i\gamma}{\zeta}. \tag{54}
\]
From equation (53) it is immediately concluded that there is not a state $R(0)$ such that $B'_0 R(0) = 0$. Also, the change above in equations (16) and (17) leads to

$$
\Sigma'_i = \frac{i}{2\xi} \left( -\rho \frac{d^2}{d\rho^2} - \frac{s(s-1)}{\rho} \right),
$$

(55)

$$
\Sigma'_h \equiv \mp \rho \frac{d}{d\rho} - i \xi \rho - \Sigma'_3.
$$

(56)

Moreover, a direct calculation shows that these operators satisfy the $su(1,1)$ Lie algebra

$$
[\Sigma'_+ \mp , \Sigma'_3 ] = -2 \Sigma'_1.
$$

(57)

By using the inner product (26), we prove that the operators (55) and (56) satisfy $(\Sigma'_3)^\dagger = -\Sigma'_3$ and $(\Sigma'_h)^\dagger = -\Sigma'_h$ which implies that these operators are anti-Hermitian. Therefore, their eigenvalues are purely imaginary, and consequently the theory of unitary representations cannot be used to obtain similar results to those for discrete states. Analogous results are obtained for equation (51).

Nevertheless, from the analytical approach, the explicit form for the scattering states can be found. Considering that $n = \gamma/\xi - s$ and substituting $\xi \to -i \zeta$ into equations (44) and (45), we obtain

$$
R^{(1)} = \rho^{s+1} e^{i\xi\rho} _1 F_1 (s + i\gamma/\xi + 1, 2s + 2; -2i\zeta\rho),
$$

(58)

$$
R^{(2)} = \rho^s e^{i\xi\rho} _1 F_1 (s - i\gamma/\xi, 2s; -2i\zeta\rho),
$$

(59)

which are similar to those reported in [40].

6. Concluding remarks

We have shown that the algebraic treatment for the radial equations of the relativistic Kepler–Coulomb problem is reduced to finding the non-compact symmetries for equation (7) or (8). For the bound states and by applying the Schrödinger factorization, we found the one-variable generators for the $su(1,1)$ Lie algebra. From the theory of unitary representations and the relation between the components of the spinor $\Phi^n_k$, we found the energy spectrum for this system in a purely algebraic way. Moreover, equations (47) and (48) imply that the $su(1,1)$ algebra generators $\Xi'_\pm$, $\Xi'_3$, and $\Sigma'_\pm$, $\Sigma'_3$, are represented by infinite-dimensional Hilbert subspaces of the radial quantum states. The non-compact nature of the $su(1,1)$ algebra for this problem reflects that for a fixed Dirac quantum number, the radial quantum number is bounded from below and unbounded from above. We showed that the Schrödinger ground state, which corresponds to the lowest value of the radial quantum number $n = 0$, is equal to the SUSY ground state [30, 32].

We noted that the radial equations for scattering states can be obtained from those for bound states via the substitution $\xi \to -i \zeta$. Therefore, the same treatment we applied to the bound states can be used to obtain a set of operators which close the $su(1,1)$ algebra for the scattering states. However, the anti-Hermiticity of these operators do not allow us to use the theory of unitary representations. As a consequence, following our technique, an algebraic approach cannot be performed for this case. Nevertheless, the explicit form of the scattering states was found from an analytical approach.

Finally, we emphasize that our treatment does not introduce an extra variable and shows the origin of the $su(1,1)$ Lie algebra generators.
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