Embedding the SU(3) sector of SO(8) supergravity in D = 11

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The SU(3)-invariant sector of maximal supergravity in four dimensions with an SO(8) gauging is uplifted to D = 11 supergravity. In order to do this, the SU(3)-neutral sector of the tensor and duality hierarchies of the D = 4, N = 8 supergravity is first worked out. The consistent D = 11 embedding of the full, dynamical SU(3) sector is then expressed at the level of the D = 11 metric and three-form gauge field in terms of these D = 4 tensors. The redundancies introduced by this approach are eliminated at the level of the D = 11 four-form field strength by making use of the D = 4 duality hierarchy. Our results encompass previously known truncations of D = 11 supergravity down to sectors of SO(8) supergravity with symmetry larger than SU(3), and include new ones. In particular, we obtain a new consistent truncation of D = 11 supergravity to minimal D = 4, N = 2 gauged supergravity.

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I. INTRODUCTION

Being complicated theories with large field contents, it proves useful for applications to truncate maximal gauged supergravities to smaller subsectors that are invariant under some symmetry group. In this paper, we will be interested in D = 4, N = 8 supergravity with an electric SO(8) gauging [1] and one of its most fruitful sectors: the one invariant under the SU(3) subgroup of SO(8). This sector preserves N = 2 supersymmetry and retains, along with the N = 2 gravity multiplet, a vector multiplet and a hypermultiplet with an Abelian gauging. The (AdS) vacuum structure in this sector has been completely charted [2] and the corresponding mass spectra within the full N = 8 theory determined [3,4]. Holographic duals have been established for some of these vacua as distinct superconformal phases [5,6] of the M2-brane field theory. Other interesting solutions of, for example, domain wall [7,8], defect [9], black hole [10] or Euclidean [11] type have been constructed in this sector that enjoy precise holographic interpretations [6,12].

The relevance for holography of D = 4, N = 8 SO(8)-gauged supergravity [1] is intimately linked to the fact that it can be obtained as a consistent truncation of D = 11 supergravity [13] on the seven-sphere, S7 [14,15].

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can be eliminated in $D = 4$ by imposing suitable duality relations among the field strengths of the tensor hierarchy [34]. Expressing the $D = 11$ embedding at the level of the four-form field strength and employing these $D = 4$ dualizations, redundancy-free uplifting formulas are obtained that contain only the dynamically independent fields (that is, the metric, the scalars and the electric vectors) that feature in the conventional $D = 4$ $\mathcal{N} = 8$ Lagrangian.

Some aspects of the SU(3)-invariant sector of SO(8)-gauged supergravity are summarized in Sec. II, and the SU(3)-invariant restricted tensor and duality hierarchies are constructed. Section III discusses the consistent uplift of the SU(3)-invariant sector into $D = 11$ supergravity following the tensor and duality hierarchy approach. Contact with the consistent uplift of previously known subsectors is made and a new $D = 11$ embedding of $D = 4$ $\mathcal{N} = 2$ pure gauged supergravity is established. Section IV further tests our formalism by recovering known AdS$_4$ solutions in $D = 11$ from uplift of critical points, and Sec. V concludes. Some technical details are contained in the Appendixes. Our conventions for $D = 11$ and $D = 4$ $\mathcal{N} = 8$ supergravity are those of [25].

II. THE SU(3)-IN Variant Sector of SO(8) Supergravity

Let us start by reviewing some aspects of the SU(3) sector of SO(8)-gauged supergravity. We choose a triangular, or Iwasawa, parametrization for the [SU(3)-invariant truncation of the] E$_{7(7)}/$SU(8) coset representative. Since previous literature often chooses the unitary gauge for the coset, we believe that our presentation has some intrinsic value even if the material that is covered (the Lagrangian in Sec. II A, the further subsectors in II C, and the vacuum structure in II D) is mostly review. The SU(3)-invariant, restricted tensor and duality hierarchies worked out in Sec. II B are new.

A. Field Content and Lagrangian

The SU(3)-invariant sector of SO(8)-gauged maximal four-dimensional supergravity [1] corresponds to an $\mathcal{N} = 2$ supergravity coupled to a vector and a hypermultiplet. In addition to the fields entering these $\mathcal{N} = 2$ multiplets, we wish to consider the SU(3)-singlets in the (restricted, in the sense of [25]) $\mathcal{N} = 8$ tensor hierarchy [32,33]. The relevant bosonic matter content thus includes

$$\begin{align*}
\text{the metric: } & ds^2_4, \\
\text{6 scalars: } & \varphi, \chi, \phi, a, \zeta, \bar{\zeta}, \\
\text{2 electric vectors and their magnetic duals: } & A^0, A^1, \tilde{A}_0, \tilde{A}_1, \\
\text{5 two-form potentials: } & B^0, B^2, B^{ab} = B^{(ab)}, \\
\text{4 three-form potentials: } & C^1, C^{ab} = C^{(ab)},
\end{align*}$$

\(u = 1, \ldots, 4\), (pseudo)scalars.\footnote{We will rarely need indices to label the scalars but, when needed, the local indices will be denoted $m = 1, \ldots, 6$, on the entire manifold \((2.2), \alpha = 1, 2\) on the first factor, and $u = 1, \ldots, 4$ on the second.}

The vectors gauge (electrically, in the usual symplectic frame), the U(1)$_7^2$, compact Cartan subgroup of the hypermultiplet isotropy group. In the Iwasawa parametrization of the scalar manifold \((2.2)\), the bosonic Lagrangian reads

$$\begin{align*}
\mathcal{L} = & R \text{vol}_4 + \frac{3}{2} (dp)^2 + \frac{3}{2} e^{2\varphi}(d\chi)^2 + 2(D\phi)^2 \\
& + \frac{1}{2} e^{\phi}(Da + \frac{1}{2}(\zeta D\bar{\zeta} - \bar{\zeta} D\zeta))^2 \\
& + \frac{1}{2} e^{2\phi}(D\zeta)^2 + \frac{1}{2} e^{2\phi}(D\bar{\zeta})^2 + \frac{1}{2} \mathcal{T}_{\zeta\bar{\zeta}} H^{(2)}_\zeta \wedge H^{(2)}_{\bar{\zeta}} \\
& + \frac{1}{2} R_{\zeta\bar{\zeta}} H^{(2)}_{(2)} \wedge H^{(2)}_{(2)} - V \text{vol}_4,
\end{align*}$$

with \((dp)^2 \equiv dp \wedge * dp\), etc. The covariant derivatives of the hyperscalars take on the form

\(u = 1, \ldots, 4\), (pseudo)scalars.\footnote{We will rarely need indices to label the scalars but, when needed, the local indices will be denoted $m = 1, \ldots, 6$, on the entire manifold \((2.2), \alpha = 1, 2\) on the first factor, and $u = 1, \ldots, 4$ on the second.}
\( D\phi = d\phi - gA^\alpha a, \quad Da = da + gA^0(1 + e^{-4\phi}(Z^2 - Y^2)), \)
\( D\zeta = d\zeta + gA^0e^{-2\phi}(\zeta Z - \zeta Y) - 3gA^1\zeta, \)
\( D\bar{\zeta} = d\bar{\zeta} + gA^0e^{-2\phi}(\bar{\zeta} Z + \bar{\zeta} Y) + 3gA^1\zeta, \)

where \( g \) is the gauge coupling constant. Following [31], here and throughout we have employed the shorthand definitions

\[
X \equiv 1 + e^{2\phi}\chi^2, \quad Y \equiv 1 + \frac{1}{4}e^{2\phi}(\zeta^2 + \bar{\zeta}^2), \quad Z \equiv e^{2\phi}a. \tag{2.5}
\]

The covariant derivatives (2.4) correspond to an electric gauging of the U(1)\(^2\) Cartan subgroup of SU(2) \times U(1) \subset SU(2, 1) generated by

\[
k_0 = \frac{1}{\sqrt{2}}(k[E_2] - k[F_2]), \quad k_1 = -k[H_2], \tag{2.6}
\]

where \( k[E_2] \), etc., are SU(2, 1) Killing vectors: see (A15) and (A16) for the explicit expressions for the Killing vectors of the scalar manifold (2.2) in our parametrization. The scalar potential \( V \) in (2.3) reads

\[
g^{-2}V = -12e^\phi - 6e^{-2\phi-e}\phi XY(e^{4\phi} + Y^2 + Z^2) - 12e\phi(Y - 1) \left(1 + Y - \frac{3}{2}XY\right)
+ 6e^{-2\phi-e}(Y - 1)(e^{4\phi} + Y^2 + Z^2)X^2 + e^{-3\phi}\left[\frac{1}{2}e^{-4\phi} + a^2 - 1 + \frac{1}{2}e^{4\phi}(1 + a^2)^2\right.
\]
\[
+ \left.\frac{1}{2}e^{-4\phi}(Y - 1)(1 + 2Z^2 - 2e^{4\phi} + Y(1 + 2e^{4\phi} + 2Z^2) + Y^2 + Y^3)\right]X^3, \tag{2.7}
\]

and derives from the following real superpotential (squared)

\[
W^2 = \frac{1}{32}g^2X \left[12e^{-\phi-2\phi}(X - 2)(Y - 2)(Y^2 + Z^2 + e^{4\phi}) + 36e\phi Y^2 + e^{-3\phi-4\phi}X^2(Y^2 + Z^2 + e^{4\phi})^2 - 16e^{-3\phi}X^2(Y - 1)
- 48e^{-\phi-2\phi}\sqrt{(X - 1)(Y - 1)(e^{4\phi} - Y^2 + Z^2)^2 + 4Y^2Z^2}\right], \tag{2.8}
\]

through the usual formula

\[
\frac{1}{4} V = 2G^{mn}\partial_mW\partial_nW - 3W^2. \tag{2.9}
\]

Here, \( G_{mn}, \ m = 1, \ldots, 6 \), denotes the nonlinear sigma model metric on (2.2), and \( G^{mn} \) its inverse, which can be read off from the scalar kinetic terms in the Lagrangian (2.3).

Finally, the gauge kinetic matrix is

\[
\mathcal{N}_{\Lambda\Sigma} = \mathcal{R}_{\Lambda\Sigma} + i\mathcal{I}_{\Lambda\Sigma} = \frac{1}{(2e^\phi\chi + i)}\left(\begin{array}{cc}
-\frac{e^\phi}{(e^\phi\chi + i)^2} & \frac{3e^\phi\chi}{(e^\phi\chi + i)} \\
\frac{3e^\phi\chi}{(e^\phi\chi + i)} & 3(e^\phi\chi^2 + e^{-\phi})
\end{array}\right), \tag{2.10}
\]

and the (electric) gauge two-form field strengths that appear in (2.3) are simply

\[
H^\Lambda_{(2)} = dA^\Lambda, \quad \Lambda = 0, 1. \tag{2.11}
\]

We have computed the SU(3)-invariant Lagrangian (2.3) and the quantities that define it using the \( D = 4, \mathcal{N} = 8 \) embedding tensor formalism [35] (see [36] for a recent review) with the conventions of [25] for the SO(8) gauging [1]. The superpotential (2.8) corresponds to one of the eigenvalues of the \( \mathcal{N} = 8 \) gravitino mass matrix restricted to the SU(3)-singlet space. See [4] for the \( \mathcal{N} = 2 \) special geometry of the model, in unitary gauge for the scalar coset. Superpotentials have previously appeared, also in unitary gauge, in [8,37].

**B. Restricted tensor and duality hierarchies**

Besides the electric gauge fields that enter the conventional supergravity Lagrangian, one may consider a set
of other gauge potentials in the so-called tensor hierarchy. The full $\mathcal{N}=8$ tensor hierarchy includes all vectors, both electric and magnetic, along with higher-rank (two-, three-, and four-form) gauge potentials, in representations of the duality group of the ungauged theory, $E_7$. The full tensor hierarchy corresponding to the $\mathcal{N}=2$ subsector at hand is obtained by retaining the singlets under the full tensor hierarchy corresponding to the restricted tensor hierarchy. Here, we are only interested in a subset of the full $E_7$-covariant fields in the hierarchy are necessary to describe the full $D=11$ embedding of $\mathcal{N}=8$ SO(8)-gauged supergravity, as argued in [25]. Only the vectors and some two- and three-form potentials in representations of the maximal SL(8, $\mathbb{R}$) subgroup of $E_7$ are relevant for this purpose. This subset was dubbed the restricted tensor hierarchy in [25]. Thus, the tensor fields that we want to consider are the singlets under SU(3) $\subset$ SL(8, $\mathbb{R}$) of the $\mathcal{N}=8$ restricted tensor hierarchy. The complete list is given in (2.1). See Appendix A for further details.

The field strengths of the SU(3)-invariant, restricted tensor hierarchy fields can be obtained by particularizing the $\mathcal{N}=8$ expressions given in [25], with the help of the expressions contained in Appendix A for their embedding into their $\mathcal{N}=8$ counterparts. The electric vector field strengths have already been given in (2.11), while the magnetic field strengths are

$$\tilde{H}_{(2)0} = dA_0 + gB^0, \quad \tilde{H}_{(2)1} = dA_1 - 2gB^2. \quad (2.12)$$

The three-form field strengths read, in turn,

$$H^0_{(3)} = dB^0, \quad H^2_{(3)} = dB^2, \quad H^{ab}_{(3)} = DB^{ab} + \frac{1}{4} (3A^0 \wedge dA_0 + 3A^0 \wedge dA^0 - A^1 \wedge dA_1 - A_1 \wedge dA^1)\delta^{ab}$$

$$+ 3gC^1 \delta^{ab} - 4gC^{ab} + \frac{1}{2} gC^c \delta^{ab}, \quad (2.13)$$

where $DB^{ab} = dB^{ab} + 2ge^{c(aA^0 \wedge B^b) c}$. Finally, the four-form field strengths are

$$H^1_{(4)} = dC^1 \wedge B^2, \quad H^{ab}_{(4)} = DC^{ab} + \frac{1}{2} H^0_{(2)} \wedge (e^{c(aB^b)c} + B^0 \delta^{ab}), \quad (2.14)$$

with $DC^{ab} = dC^{ab} + 2ge^{c(aA^0 \wedge C^b)c}$.

The field strengths (2.11)–(2.14) are subject to the Bianchi identities

$$dH^0_{(2)} = 0, \quad dH^1_{(2)} = 0, \quad d\tilde{H}_{(2)0} = gH^0_{(3)0}, \quad d\tilde{H}_{(2)1} = -2gH^0_{(3)2},$$

$$DH^{ab}_{(3)} = \left( \frac{3}{2} H^0_{(2)} \wedge \tilde{H}_{(2)0} - \frac{1}{2} H^2_{(2)} \wedge \tilde{H}_{(2)1} + 3gH^1_{(4)} + \frac{1}{2} gH^0_{(3)4} \right) \delta^{ab} - 4gH^0_{(4)},$$

$$dH^0_{(3)} = 0, \quad dH^2_{(3)} = 0, \quad dH^1_{(4)} = 0, \quad dH^{ab}_{(4)} = 0, \quad (2.15)$$

where we have defined $DH^{ab}_{(3)} = dH^{ab}_{(3)} - 2ge^{c(aA^0 \wedge H^{bc}_{(3)})c}$. These expressions particularize the Bianchi identities (14) of [25] to the present case.

All of the fields in the restricted tensor hierarchy carry d.o.f., although not independent ones. They are instead subject to a duality hierarchy [34]. The magnetic two-form field strengths can be written as scalar-dependent combinations of the electric gauge field strengths and their Hodge duals:

$$\tilde{H}_{(2)0} = \frac{1}{X^2(4X - 3)} [-e^{\phi}(3X - 2) \ast H^0_{(2)} + 3e^{\phi}X(X - 1) \ast H^1_{(2)}] - 2e^{\phi}X^2 H^0_{(2)} + 3Xe^{\phi}X(2X - 1)H^1_{(2)},$$

$$\tilde{H}_{(2)1} = \frac{1}{X(4X - 3)} [3e^{\phi}(X - 1) \ast H^0_{(2)} - 3e^{-\phi}X^2 \ast H^1_{(2)} + 3Xe^{2\phi}(2X - 1)H^0_{(2)} + 6X^2H^1_{(2)}]. \quad (2.16)$$

The three-form field strengths are dual to scalar-dependent combinations of derivatives of scalars:
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\[ H^0_{(3)} = - \star \left[ \left( Y^2 - 2Y + Z^2 + e^{\phi} \left( Da + \frac{1}{2} (\zeta D_\zeta - \zeta D_\chi) \right) + Y(\zeta D_\zeta - \zeta D_\chi) + 2aDY - 4aYD\phi \right) \right], \]

\[ H^2_{(3)} = 3e^{2\phi} \star \left[ (Y - 1) \left( Da + \frac{1}{2} (\zeta D_\zeta - \zeta D_\chi) \right) + \frac{1}{2} (\zeta D_\zeta - \zeta D_\chi) \right], \]

\[ H^{77}_{(3)} = 2Z e^{2\phi} \left( Da + \frac{1}{2} (\zeta D_\zeta - \zeta D_\chi) \right) + 2DY - 4YD\phi + 3(d\phi - e^{2\phi} d\chi), \]

\[ H^{78}_{(3)} = \left( Y^2 - 2Y + Z^2 - e^{\phi} \left( Da + \frac{1}{2} (\zeta D_\zeta - \zeta D_\chi) \right) + Y(\zeta D_\zeta - \zeta D_\chi) + 2aDY - 4aYD\phi \right), \]

\[ H^{88}_{(3)} = -2Z e^{2\phi} \left( Da + \frac{1}{2} (\zeta D_\zeta - \zeta D_\chi) \right) + 2DY - 4YD\phi - 3(d\phi - e^{2\phi} d\chi). \]

Finally, the four-form field strengths correspond to the following scalar-dependent top forms on four-dimensional spacetime:

\[
\begin{align*}
H^{14}_{(4)} &= g(2e^{2\phi} Y(3X + 2Y - 3XY)) + e^{-\phi - 2\phi} X(X + Y - XY)(Y^2 + Z^2 + e^{\phi})|V_4, \\
H^{74}_{(4)} &= -gX[e^{-3\phi} X(2Y - 2Y + Z^2 + e^{4\phi}) + 6e^{-\phi + 2\phi}(XY - X - Y)]|V_4, \\
H^{78}_{(4)} &= -gXZ[e^{-3\phi - 2\phi} X^2(Y^2 + Z^2 + e^{4\phi}) + 6e^{-4\phi}(XY - X - Y)]|V_4, \\
H^{88}_{(4)} &= -gX[e^{-3\phi} X(2Y - 2Y + Z^2) + 6e^{-\phi - 2\phi}(XY - X - Y)(Y^2 + Z^2) + e^{-3\phi - 4\phi} X^2(Y^2 + Z^2)|V_4, \\
3g(2H^{14}_{(4)} - H^{74}_{(4)} - H^{88}_{(4)}) &= -k^a[H_0]\partial_a V|V_4, \\
2g(H^{74}_{(4)} - H^{88}_{(4)}) &= -k^a[H_1]\partial_a V|V_4, \\
4\sqrt{2}gH^{88}_{(4)} &= -(k^u[\varepsilon_2] + k^u[F_2])\partial_u V|V_4, \\
k^0_u\partial_u V &= 0, \quad k^u\partial_u V = 0. \tag{2.22}
\end{align*}
\]

In (2.21) and (2.22), \(Dq^u, u = 1, \ldots, 4\), collectively denote the hypermultiplet covariant derivatives (2.4); \(k_0\) and \(k_1\) are the hypermultiplet Killing vectors (2.6) along which the gauging is turned on; \(k[H_0]\) and \(k[H_1]\) are other Killing vectors [see (A15), (A16)] on each factor of the scalar manifold (2.2); and \(h_{\mu u}\) is the metric that can be read off from the hypermultiplet kinetic terms in the Lagrangian (2.3).

The last two identities in (2.22) reflect the invariance of the potential (2.7) under the gauged hypermultiplet isometries (2.6). These are the only symmetries of the SU(3)-invariant potential (2.7). The symmetry is enhanced in the subsectors that we now turn to discuss.

C. Some further subsectors

It is interesting to consider further subsectors contained in the SU(3)-invariant sector in the notation that we are using. A natural way to obtain those is to impose invariance under a subgroup \(G\) of SO(8) that contains SU(3). The relevant tensor hierarchy field strengths and their dualization conditions are obtained by bringing the \(G\)-invariant restrictions specified on a case-by-case basis below to
TABLE I. Number of bosonic tensor hierarchy fields in each subsector.

| Sector     | Scalars | Pseudoscalars | E&M vectors | Two-forms | Three-forms |
|------------|---------|---------------|-------------|-----------|-------------|
| SU(3)      | 3       | 3             | 4           | 5         | 4           |
| SU(3) × U(1)² | 1       | 1             | 4           | 1         | 2           |
| SU(3) × U(1)ₖ | 3       | 1             | 4           | 4         | 4           |
| SU(3) × U(1)ₙ | 1       | 3             | 4           | 2         | 2           |
| SU(3) × U(1)ₜ | 1       | 1             | 4           | 1         | 2           |
| SO(6)ₖ     | 3       | 0             | 2           | 4         | 4           |
| SO(4)ₖ     | 0       | 3             | 2           | 1         | 1           |
| SO(4)ₙ     | 0       | 0             | 2           | 1         | 1           |
| SO(7)ₖ     | 1       | 0             | 0           | 1         | 2           |
| SO(7)ₙ     | 0       | 1             | 0           | 0         | 1           |
| SO(7)ₜ     | 0       | 0             | 0           | 0         | 1           |
| G₂         | 1       | 1             | 0           | 1         | 2           |

(2.11)–(2.14) and (2.16)–(2.18). The field content in each of these subsectors is summarized for convenience in Table I.

An obvious yet still interesting sector is attained by requiring an additional invariance under the U(1)² with which SU(3) commutes inside SO(8). The resulting SU(3) × U(1)²-invariant sector throws out the hypermultiplet and sets identifications on the restricted tensor hierarchy,²

\[\phi = a = \zeta = \bar{\zeta} = 0,\]

\[B^0 = B^2 = B^{78} = 0, \quad B^{77} = B^{88},\]

\[C^{78} = 0, \quad C^{77} = C^{88}.\]  

(2.23)

This sector thus reduces to \(N = 2\) supergravity coupled to a vector multiplet with a Fayet-Iliopoulos gauging, namely, to the U(1)⁴-invariant sector (i.e., the gauged STU model) with all three vector multiplets identified, along with the relevant tensor hierarchy fields. Inserting (2.23) in (2.3), the Lagrangian indeed reduces to e.g., (6.28), (6.29) of [28] with the fields and coupling constants here and there identified as

\[e^{\varphi_{\text{here}}} = e^{-\varphi_{\text{there}} (1 + e^{2\varphi_{\text{here}}} x^2_{\text{here}})}, \quad \chi_{\text{here}} e^{\varphi_{\text{here}}} = \chi_{\text{there}} e^{\varphi_{\text{there}}},\]

\[A_{(1)}_{\text{there}} = -A^0_{\text{here}}, \quad A_{(1)}_{\text{here}} = A^1_{\text{there}}, \quad g_{\text{here}} = -g_{\text{there}}.\]  

(2.24)

The potential of the SU(3) × U(1)²-invariant sector, (2.7) with (2.23), acquires a symmetry under the compact generator, \(k[F_0] - k[F_0]\) in the notation of (A15), of the vector multiplet scalar manifold. The field redefinition in

the first line of (2.24) is a U(1) ⊂ SL(2, \(\mathbb{R}\)) transformation generated by this Killing vector, followed by a change of sign of \(\chi\).

One may also consider SU(3) × U(1)-invariant sectors, with U(1) chosen to be one of the three triality-invariant new factors with which SU(3) commutes inside SO(8). These invariant sectors are attained by setting

\[SU(3) × U(1)_v:\ \zeta = \bar{\zeta} = 0, \quad B^2 = 0,\]  

(2.25)

\[SU(3) × U(1)_c:\ e^{-2\phi} = 1 - \frac{1}{4} (\zeta^2 + \bar{\zeta}^2), \quad a = 0,\]

\[B^0 = -\frac{2}{3} B^2, \quad B^{78} = 0, \quad B^{77} = B^{88},\]

\[C^{78} = 0, \quad C^{77} = C^{88}.\]  

(2.26)

\[SU(3) × U(1)_t:\ \phi = a = \zeta = \bar{\zeta} = 0,\]

\[B^0 = B^2 = B^{78} = 0, \quad B^{77} = B^{88},\]

\[C^{78} = 0, \quad C^{77} = C^{88}.\]  

(2.27)

while retaining both vectors and their magnetic duals. Only the SU(3) × U(1)_v-invariant subtruncation is supersymmetric, and coincides with the SU(3) × U(1)² sector discussed above—in other words, invariance under U(1)_v cannot be enforced on top of SU(3) without also imposing U(1)_c invariance, but not the other way around. The other two subtruncations retain the would-be vector multiplet and “half” a hypermultiplet: either the scalars \(\phi, a\) in the SU(3) × U(1)_v sector, or the pseudoscalars \(\zeta, \bar{\zeta}\) in the SU(3) × U(1)_c sector, with \(\phi\) a function of the

²Curiously, \(B^0\) and \(B^2\) are allowed by group theory to be nonvanishing, but are set to \(B^0 = B^2 = 0\) by the duality relations (2.17) evaluated with the scalar restrictions (2.23). Similar comments apply to the condition \(B^2 = 0\) in (2.25) and \(B^0 = -\frac{1}{4} B^2\) in (2.26).

³Under triality, the representations \(8_v, 8_c, 8_s\) of SO(8) split under the subgroups SO(7)_i, SO(7)_j, SO(7)_k, as in e.g., (C.1) of [38], with labels \((v, s, c)\) there denoted \((v, s, c)\) here. We follow the spectrum conventions of e.g., [39] whereby, at the SO(8) vacuum, the (graviton, gravitini, vectors, spinors, scalars, pseudoscalars) of \(\mathcal{N} = 8\) supergravity lie in the \((1, 8_v, 28, 56, 35_c, 35_v)\) of SO(8).
pseudoscalars in the latter case. The covariant derivatives (2.4) simplify accordingly. In the SU(3) \times U(1)_{c} sector, \( \phi, a \) remain charged under \( A^{0} \) and no field is charged under \( A^{1} \). In the SU(3) \times U(1)_{c} sector the covariant derivatives reduce to

\[ D\zeta = d\zeta - g(A^{0} + 3A^{1})\zeta, \quad D\tilde{\zeta} = d\tilde{\zeta} + g(A^{0} + 3A^{1})\zeta, \]

(2.28)

showing that \( \zeta, \tilde{\zeta} \) become a doublet charged only under the combined gauge field \( A^{0} + 3A^{1} \).

It is possible to further truncate the SU(3) \times U(1)_{c} sector to a two-scalar model retaining \( (\varphi, \zeta) \) along with \( B^{77} = B^{88} \) and \( C^{1}, C^{77} = C^{88} \) by imposing (2.26) together with \( \chi = 0 \), \( \tilde{\zeta} = \zeta, A^{0} = A^{1} = 0 \), and \( B^{0} = -\frac{2}{3}B^{2} = 0 \). The Lagrangian is (2.3) with these identifications and the superpotential reduces, from (2.8), to

\[ W = \frac{1}{2\sqrt{2}} ge^{-3\varphi}(e^{2\varphi} - 3e^{2\varphi} + 2\varphi - 2), \]

(2.29)

where \( e^{2\varphi} \) is shorthand for the expression in terms of \( \zeta = \tilde{\zeta} \) that appears in (2.26). This is the model considered in [27]. The identifications

\[ e^{-\varphi_{\text{here}}} = \rho_{\text{there}}^{4}, \quad \zeta_{\text{here}}^{2} = \tilde{\zeta}_{\text{here}}^{2} = 2\tanh^{2}\chi_{\text{there}} \]

(2.30)

[the second equation implies \( e^{2\varphi_{\text{here}}} = \cosh^{2}\chi_{\text{there}} \) on (2.26)] indeed bring the superpotential (2.29) to (3.9) of [27], up to normalization.

The SU(3) \times U(1)_{c}-invariant sectors can be further reduced by imposing a larger SO(6) \sim SU(4) symmetry. The corresponding sectors are obtained by letting

SO(6), \( \zeta = \tilde{\zeta} = \chi = 0 \), \( A^{1} = \tilde{A}_{1} = 0 \), \( B^{2} = 0 \),

(2.31)

SU(4), \( e^{-2\varphi} = 1 - \frac{4}{3}(\zeta^{2} + \tilde{\zeta}^{2}) \), \( a = 0 \),

\[ e^{-2\varphi} = 1 - \chi^{2}, \quad A^{1} = A^{0} \equiv A, \]

\[ \tilde{A}_{1} = 3\tilde{A}_{0}, \quad B^{0} = -\frac{2}{3}B^{2}, \quad B^{ab} = 0, \]

\[ C^{1} = C^{77} = C^{88}, \quad C^{78} = 0, \]

(2.32)

SU(4), \( \phi = a = \zeta = \tilde{\zeta} = \varphi = \chi = 0 \),

\[ A^{1} = -A^{0}, \quad \tilde{A}_{1} = -3\tilde{A}_{0}, \quad B^{0} = \frac{2}{3}B^{2}, \quad B^{ab} = 0, \]

\[ C^{1} = C^{77} = C^{88}, \quad C^{78} = 0, \]

(2.33)

Again, only the SU(4)_{c}-invariant sector is supersymmetric: it truncates out the vector multiplet of the SU(3) \times U(1)_{c} sector, leading to minimal \( \mathcal{N} = 2 \) gauged supergravity.

Setting all scalars to zero as in (2.33), further setting consistently \( B^{0} = \frac{2}{3}B^{2} = 0 \), and rescaling for convenience the metric and the graviphoton as

\[ g_{\mu\nu} \equiv \frac{1}{4}g_{\mu\nu} \cdot \quad A^{1} = -A^{0} \equiv \frac{1}{4}\tilde{A}, \]

(2.34)

Eq. (2.3) reduces to the bosonic Lagrangian of pure \( \mathcal{N} = 2 \) gauged supergravity,

\[ \mathcal{L} = R_{\text{vol}} - \frac{1}{2} \tilde{F} \wedge \tilde{F} + 6g_{\text{vol}}^{4}, \]

(2.35)

with \( \tilde{F} = d\tilde{A} \). For later reference, we note that the only non-supersymmetric. Imposing invariance under SO(6), \( \phi, \varphi, a \) along with the gauge field \( A^{0} \), while invariance under SU(4), retains the pseudoscalars \( \chi, \zeta, \tilde{\zeta} \) along with \( A^{0} + A^{1} \). In the latter case, the scalars become functions of the pseudoscalars as indicated in (2.32).

It was noted in [4] that the SU(4),-invariant sector coincides with a truncation, considered in [40], of the \( D = 4 \mathcal{N} = 2 \) gauged supergravity obtained upon consistent truncation of M-theory on any (skew-whiffed) Sasaki-Einstein seven-manifold [41]. Indeed, using (2.32) and further identifying the pseudoscalars and vectors here and in [40] as

\[ \chi_{\text{here}} = h_{\text{there}}, \quad \zeta_{\text{here}} = -\sqrt{3}\text{Im}h_{\text{there}}, \quad \tilde{\zeta}_{\text{here}} = -\sqrt{3}\text{Re}h_{\text{there}}, \]

\[ A^{0}_{\text{here}} = A^{1}_{\text{there}}, \quad g_{\text{here}} = -(2L)^{1}_{\text{there}} \]

(2.37)

[which further imply \( \varphi_{\text{here}} = -2U_{\text{there}} - V_{\text{there}} \) and \( \phi_{\text{here}} = -3U_{\text{there}} \), with \( \varphi, \phi \) here subject to (2.32) and \( U, V \) there subject to their (4.1)], the Lagrangian (2.3) here reproduces (4.3) of [40]. Neither the SO(6), nor the SU(4), sectors admit a further truncation to the Einstein-Maxwell, bosonic Lagrangian (2.35) of minimal \( \mathcal{N} = 2 \) supergravity.

It is possible to enlarge the symmetry to the three different SO(7) subgroups of SO(8) by further imposing
SO(7): $\zeta = \bar{\zeta} = \chi = 0$, $\phi = \phi, a = 0,$

$A^0 = A^1 = \hat{A}_0 = \hat{A}_1 = 0,$

$B^0 = B^2 = B^{78} = 0$, $B^{88} = -7B^{77},$

$C^1 = C^{77}, C^{78} = 0,$ \hspace{1cm} (2.38)

SO(7): $e^{-2\phi} = 1 - \frac{1}{4}(\zeta^2 + \zeta^2) = 1 - \chi^2 = e^{-2\varphi},$

$\alpha = 0, \hspace{1cm} A^0 = A^1 = \hat{A}_0 = \hat{A}_1 = 0,$

$B^0 = B^2 = 0$, $B^{ab} = 0,$

$C^1 = C^{77} = C^{88}, C^{78} = 0,$ \hspace{1cm} (2.39)

SO(4): $e^{-2\phi} = 3$, $\chi = 0$, $e^{-2\phi} = 1 - \frac{1}{4}(\zeta^2 + \zeta^2) = \frac{2}{3}, \hspace{1cm} a = 0.$ \hspace{1cm} (2.43)

Second, identify the electric and magnetic vectors as

$A^0 = -3A^1 \equiv \frac{1}{2} \hat{A}, \hspace{1cm} \hat{A}_0 = -\frac{1}{9} \hat{A}_1 \equiv \frac{1}{6\sqrt{3}} \bar{\hat{A}}, \hspace{1cm} (2.44)$

turn off the two-form potentials, and retain an auxiliary three-form potential as

$B^0 = -\frac{2}{3} B^2 = B^{ab} = 0$, \hspace{1cm} $C^{78} = 0$, \hspace{1cm} $C^1 = C^{77} = C^{88}.$ \hspace{1cm} (2.45)

Finally, rescale the metric for convenience:

$g_{\mu\nu} \equiv \frac{1}{3\sqrt{3}} \tilde{g}_{\mu\nu}.$ \hspace{1cm} (2.46)

We have verified at the level of the bosonic field equations, including Einstein, that these identifications define a consistent truncation of the theory (2.3) to minimal $\mathcal{N} = 2$ gauged supergravity (2.35).

The identification of the electric vectors in (2.44) retains the SU(3) x U(1)$_c$-invariant vector [see (A17) with (A12)] that remains massless [see (2.28)] at the $\mathcal{N} = 2$ vacuum (2.43). For future reference, it is also interesting to keep track of the field strengths for this truncation. On (2.44), (2.45), the two-form potential contributions to the magnetic vector two-form field strengths (2.12) drop out, and the vector field strengths become

$H^0 = -\frac{3}{2} H^1 \equiv \frac{1}{2} \bar{\bar{F}}, \hspace{1cm} \bar{H}_0 = -\frac{1}{9} \bar{H}_1 \equiv \frac{1}{6\sqrt{3}} \bar{\bar{F}} = \frac{1}{6\sqrt{3}} \bar{\bar{F}}, \hspace{1cm} (2.47)$

with $\bar{\bar{F}} \equiv d\hat{A}$. The relations here for the magnetic field strengths are compatible with the vector duality relations (2.16) evaluated on the scalar vevs (2.43), and the last equality for the magnetic graviphoton field strength $\bar{\bar{F}}$ is fixed by $\bar{\bar{F}} = \partial \mathcal{L} / \partial \bar{\bar{F}}$, with $\mathcal{L}$ as in (2.35). Moving on to the three-form field strengths, we find that all of them are zero by bringing (2.44), (2.45) to their definitions (2.13) in terms of potentials. This was expected, as the three-form field strengths are dual to combinations (2.17) of (Hodge duals of) derivatives of scalars, and these have been frozen to their vevs (2.43). Finally, for the four-form field strengths
we obtain, from (2.14) with (2.45), $H_{(4)}^{78} = 0$, $H_{(4)}^{18} = H_{(4)}^{28} = dC^i$, expressions which are again compatible with the dualization conditions (2.18). Rescaling the volume form using (2.46), we find

$$H_{(4)}^{1} = H_{(4)}^{78} = H_{(4)}^{88} = \frac{1}{2\sqrt{3}} g_{\text{vol}} 4.$$  (2.48)

D. Vacuum structure

The list of vacua of $D = 4 \mathcal{N} = 8$ supergravity with an electric SO(8) gauging with at least SU(3) invariance, reproducing the results of [2] in our parametrization. For each point we give the residual supersymmetry $\mathcal{N}$ and bosonic symmetry $G_0$ for each vacuum, as well as its location in the scalar space (2.2) in the parametrization that we are using. The corresponding cosmological constant, the masses are given in units of the AdS radius, $L^2 = -6/V_0$. We have abbreviated $U(3) = SU(3) \times U(1)$.

| $\mathcal{N}$ | $G_0$ | $\chi$ | $e^{-\phi}$ | $e^{\phi}$ | $a$ | $\zeta$ | $\zeta_5$ | $g^{-2}V_0$ | $L^2M^2$ |
|---------------|-------|--------|-------------|-------------|-----|-----------|-----------|------------|------------|
| 8             | SO(8) | 0      | 1           | 1           | 0   | 0         | 0         | -24        | (-2, -2, -2, -2, -2) |
| 2             | U(3)  | 0      | $\sqrt{3}$ | $\sqrt{3}$ | 0   | $\sqrt{3}$ | $\sqrt{3}$ | -18$\sqrt{3}$ | (3 $\pm$ 17, 2, 2, 2, 0) |

III. D = 11 UPLIFT

We now switch gears and present the $D = 11$ embedding of the SU(3)-invariant sector considered in the previous section. We will use the consistent $S^7$ uplifting formulas given in [25]. It is a tedious, but otherwise mechanical, exercise to particularize the general $\mathcal{N} = 8$ uplifting formulas in that reference to the SU(3)-invariant sector at hand. Section III A contains the $D = 11$ uplift of the entire SU(3)-invariant sector while Sec. III B particularizes to some relevant subsectors and makes contact with previous literature. Section III C contains a new consistent truncation of $D = 11$ supergravity to minimal $D = 4 \mathcal{N} = 2$ gauged supergravity.

A. Uplift of the SU(3) sector

We first find it useful to present the result in terms of $\mathbb{R}^8$ “embedding coordinates” $\mu^A$, $A = 1, \ldots, 8$, in the $8_v$ of SO(8), that define the $S^7$ as the locus

$$\delta_{AB}H^A\mu^B = 1$$  (3.1)

in $\mathbb{R}^8$. Under SU(3), the $8_v$ of SO(8) breaks down as $8_v \rightarrow 3 + 3 + 1 + 1$. In maintaining an explicitly real notation, it is thus convenient to split $\mathbb{R}^8 = \mathbb{R}^6 \times \mathbb{R}^2$, and the indices as $A = (i, a)$, with $i = 1, \ldots, 6$ and $a = 7, 8$ respectively labeling the first and second factors. The $D = 11$ uplift of the SU(3)-invariant sector utilizes the tensors $\delta_{ij}$, $J_{ij}^{(6)}$ (real) and $\Omega_{ijk}^{(6)}$ (complex) that define the natural Calabi-Yau structure of $\mathbb{R}^6$. See (A6) for our conventions. Inside $\mathbb{R}^6$, these tensors are respectively invariant under SO(6) $\times SO(2)$, SU(3) $\times U(1)^2$ and SU(3) $\times U(1)$. where SO(2) rotates the $\mathbb{R}^2$ factor in $\mathbb{R}^6 = \mathbb{R}^6 \times \mathbb{R}^2$. Indices on $\mathbb{R}^6$ and $\mathbb{R}^2$ are raised and lowered with $\delta_{ij}$ and $\delta_{ab}$, respectively.

Only the $D = 4$ metric, the scalars, and the electric gauge fields in the SU(3)-invariant restricted duality
h = \begin{pmatrix} e^{2\phi} & Z \\ Z & e^{-2\phi}(Y^2 + Z^2) \end{pmatrix},

h^{-1} = Y^{-2} \begin{pmatrix} e^{-2\phi}(Y^2 + Z^2) & -Z \\ -Z & e^{2\phi} \end{pmatrix},

(3.2)

and the following combination of \( D = 4 \) scalars and constrained coordinates \( \mu^i, \mu^a \),

\[ \Delta_1 = e^{2\phi}Y_{\mu^i}\mu^i + X h_{ab}\mu^a\mu^b. \]

(3.3)

With these definitions, the embedding into the \( D = 11 \) metric reads

\[ ds^{2}_{11} = e^{-\varphi}X^{1/3} \Delta_1^{2/3}[ds^2_7 + g^{-2}e^{2\varphi}\Delta_1^{-1}(D\mu_iD\mu^i + e^{2\varphi}X^{-1}(Y - X)\Delta_1^{-3/2}(YJ_{ij}^{(6)}\mu^iD\mu^j + h_{ab}\epsilon^{bc}\mu^a\mu^b)]^2, \]

(3.4)

where \( e^{ab} \) is the totally antisymmetric symbol with two indices, and the covariant derivatives are defined as

\[ D\mu^i = d\mu^i - gA^1j^{(6)i}\mu_j, \quad D\mu^a = d\mu^a - gA^6\epsilon^{ab}\mu_b. \]

(3.5)

For generic values of the \( D = 4 \) scalars, the metric (3.4) enjoys an \( SU(3) \times U(1)_p \) isometry.

Moving on to the \( D = 11 \) three-form \( \tilde{A}_{(3)} \), all the \( D = 4 \) fields in the tensor hierarchy (2.1), except for the metric, enter its expression. A long calculation yields

\[ \tilde{F}_{(4)} = H_{(4)}^1\mu^i\mu^j + H_{(4)}^{ab}\mu^a\mu^b - \frac{1}{12}g^{-1}[H_{(3)a}^{a}\delta_{ij} + 4H_{(3)}^{\gamma\gamma}\epsilon^{ij}] \wedge \mu^i D\mu^j 

+ \frac{1}{2}g^{-1}[H_{ab}^{(3)} + H_{ab}^{(3)}] \wedge \mu_d D\mu^d + \frac{1}{6}g^{-2}\hat{H}_{(2)1}^{(6)} J_{ij}^{(6)} D\mu^i \wedge D\mu^j + \frac{1}{2}g^{-2}\hat{H}_{(2)0} \wedge \epsilon_{ab} D\mu^a \wedge D\mu^b 

+ \frac{1}{4}e^{2\phi}(v_1Re\Omega_{ij}^{(6)} - v_1Im\Omega_{ij}^{(6)})\mu^i D\mu^j \wedge D\mu^k \wedge H_{(2)}^0 

- \frac{1}{4}g^{-2}\Delta_1^{2/3}[2e^{2\phi}X^{-1}Y\mu^i D\mu^i \wedge D\mu^j + e^{2\phi}\epsilon_{ab} D\mu^a \wedge D\mu^b) 

- \frac{4}{3}e^{2\phi}\mu^i D\mu^k \wedge (Y_{ij}^{(6)} \mu^i D\mu^j + h_{ab}\epsilon_{ab}\mu^b) 

+ e^{2\phi}(v_2Re\Omega_{ij}^{(6)} - v_1Im\Omega_{ij}^{(6)})\mu^i D\mu^j \wedge D\mu^k \wedge H_{(2)}^1 + dA_{\text{scalars}}. \]

(3.10)
In this expression, $H_{(4)}^{1}, H_{(4)}^{2}$, etc., turn out to reproduce the $D = 4$ four-, three- and magnetic two-form field strengths (2.12)–(2.13) of the restricted tensor hierarchy (2.1). This provides a $D = 11$ crosscheck of the $D = 4$ calculation of Sec. II B. The terms that contain the electric two-form field strengths $H_{(2)}^{0}, H_{(2)}^{1}$, come from the vector contributions in the covariant derivatives $D \mu^{i}$ and $D \mu^{a}$ in (3.7). Finally, $dA_{\text{scalars}}$ contains two types of terms. The first type includes contributions of covariant derivatives of $D = 4$ scalars, wedged with three-forms on the internal $S^{7}$. The second type includes internal four-forms with coefficients that depend on the $D = 4$ scalars algebraically only. The presence in $A_{(3)}$ of $J_{ij}^{(2)}, \Omega^{(6)}_{ijk}$ and $h_{ab}$ breaks the symmetry of the full $D = 11$ configuration to SU(3), in agreement with the symmetry of the $D = 4$ model.

The above expressions give the complete embedding of the SU(3)-invariant, restricted tensor hierarchy (2.1) into $D = 11$ supergravity. As such, these expressions contain redundant $D = 4$ d.o.f. As argued in [25], these redundancies can be eliminated at the level of the $D = 11$ four-form field strength by making use of the $D = 4$ duality relations. Indeed, regarding the tensor field strengths in (3.10) as shorthand for the dualization conditions (2.16)–(2.18), Eqs. (3.4), (3.10) then express the embedding into $D = 11$ supergravity exclusively in terms of the dynamically independent (metric, electric-vector and scalar) d.o.f.

At a critical point, the terms in derivatives of the potential drop out and the Freund-Rubin term becomes proportional to the AdS$_4$ cosmological constant, in agreement with the general $N = 8$ discussion of [25]. See also [24] for a related discussion. All the Freund-Rubin terms that we write for the truncations to specific subsectors in Sec. III B

B. Uplift of some further subsectors

The uplifting formulas of Sec. III A simplify by imposing a symmetry enlargement, carried over to $D = 11$ by restricting the $D = 4$ fields as in Sec. II C. Introducing intrinsic $S^7$ angles by solving the constraint (3.1) is also facilitated in further subsectors, as some intrinsic angles are better suited than others to make the relevant symmetry apparent in $D = 11$. See Appendix B for some relevant geometric structures on $S^7$.

I. SU(3) x U(1)$^2$-invariant sector

For the SU(3) x U(1)$^2$-invariant sector (2.23), the embedding formulas for the $D = 11$ metric, (3.4), and three-form, (3.6), (3.7), become
The line element $d\xi^2(\mathbb{CP}^2)$ and the two-form $J^{(4)}$ respectively correspond to the Fubini-Study metric, normalized so that its Ricci tensor is six times the metric, and the Kähler form, with potential one-form $\sigma$ such that $d\sigma = 2J^{(4)}$, on the complex projective plane. Finally, $\Delta_1$, $\Delta_2$, and $\Delta_3$ are the following functions of the $S^7$ angle $\alpha$ and the SU(3) × U(1)$^2$-invariant, $D = 4$ vector multiplet scalars

\[
\begin{align*}
\Delta_1 &= X\sin^2\alpha + e^{2\varphi}\cos^2\alpha, \\
\Delta_2 &= e^{2\varphi}[\sin^4\alpha + (e^{2\varphi} + 2\chi^2 e^{2\varphi} + e^{-2\varphi}X^2)\sin^2\alpha\cos^2\alpha + \cos^4\alpha], \\
\Delta_3 &= [X^2 + \chi^2 e^{4\varphi}]\sin^2\alpha + e^{2\varphi}\cos^2\alpha.
\end{align*}
\]  

The function $\Delta_1$ here is simply the particularization of (3.3) to the present case. The four-form field strength corresponding to (3.13) can be computed to be

\[
\begin{align*}
\hat{F}^{(4)} &= 2g[2(e^{\varphi}\cos^2\alpha + e^{-\varphi}X\sin^2\alpha) + \widetilde{X}e^{-\varphi}]vol_4 + g^{-1}\sin 2\alpha(\ast d\varphi - e^{2\varphi}X d\chi) \wedge d\alpha \\
&\quad - \frac{1}{6} g^{-2}[\sin 2\alpha(\tilde{H}_1 + 3\tilde{H}_0) \wedge d\alpha \wedge D\psi_- - 2\tilde{H}_1 \wedge (\cos^2\alpha J^{(4)} - \sin \alpha \cos ada \wedge (D\tau_- + \sigma))] \\
&\quad + \frac{1}{2} g^{-2}X e^{\varphi} \sin 2\alpha \wedge (\tilde{H}_0 \wedge (D\tau_- + \sigma)) + (\tilde{H}_0 + \tilde{H}_1) \wedge D\psi_- \\
&\quad - 2\Delta_1^{-1}\cos^3\alpha(\tilde{H}_0 + \tilde{H}_1) \wedge J^{(4)} + 2\Delta_1^{-1}\cos^2\alpha \cos 2\alpha H_0 \wedge J^{(4)} \\
&\quad + g^{-3}\left\{ \frac{1}{2} g^{-2}X \sin \alpha \wedge (2\chi d\varphi - (X - 2)d\chi) \wedge d\alpha \wedge D\psi_- \wedge (D\tau_- + \sigma) \\
&\quad - e^{2\varphi}\Delta_1^{-2}\cos^4\alpha \sin 2\alpha \wedge d\alpha \wedge D\psi_- \wedge (D\tau_- + \sigma) \\
&\quad - e^{2\varphi}\Delta_1^{-2}\cos^2\alpha \sin 2\alpha \wedge d\alpha \wedge D\psi_- \wedge (D\tau_- + \sigma) \\
&\quad + \chi e^{2\varphi}X^{-1} \sin 2\alpha \wedge D\psi_- \wedge J^{(4)} - 2\chi e^{2\varphi}\Delta_1^{-1}\cos^4\alpha J^{(4)} \wedge J^{(4)} \\
&\quad + 2e^{2\varphi}\chi(\Delta_1 + X)\Delta_1^{-2}\sin \alpha \wedge J^{(4)} \\
&\quad + \frac{1}{2} e^{2\varphi}\Delta_1^{-2}\sin \alpha \wedge J^{(4)} \right\}.
\end{align*}
\]  

Here, we have explicitly made use of the dualization conditions (2.17), (2.18) for the three- and four-form field strengths, particularized to SU(3) × U(1)$^2$-invariant scalars via (2.23). The magnetic two-form field strengths $\tilde{H}_\Lambda$, $\Lambda = 0, 1$, stand for the dualized expressions (2.16).

As noted in Sec. II C, the SU(3) × U(1)$^2$-invariant sector coincides with the gauged STU model with all three vector multiplets identified. This was embedded in $D = 11$ supergravity in [28] (see also [42]), along with the entire STU model. Our uplifting formulas (3.12), (3.16), obtained instead from the $D = 11$ embedding of the SU(3) sector, are in perfect agreement with (6.22)–(6.24) of [28]. This can be seen by using the definitions of the $D = 4$ redefinitions (2.24), which also imply $\tilde{H}_{\text{here}} = \tilde{R}_{\text{here}}$, $\hat{H}_{\text{there}} = -R_{\text{there}}$, along with the $S^7$ angle and one-form identifications

\[
\begin{align*}
\xi_{\text{there}} &= \alpha_{\text{here}} + \frac{\pi}{2} \phi_{\text{there}} = \psi_{\text{there}}, \\
\psi_{\text{there}} &= \psi_{\text{here}} + \tau_{\text{here}}, \\
B_{\text{there}} &= \sigma_{\text{here}}.
\end{align*}
\]  

or, in terms of the $\psi$, $\tau$ defined in Eq. (B1) of Appendix B, $\phi_{\text{there}} = -\psi$, $\psi_{\text{there}} = \tau$.

2. SU(4)-invariant sectors

While the deformations inflicted on the internal $S^7$ by the SU(3)-invariant $D = 4$ fields are inhomogeneous, enlarging the symmetry to SU(4), and SU(4), results in the deformations becoming homogeneous.

For the SU(4)$^3$-invariant $D = 4$ fields (2.32), the $D = 11$ embedding formulas (3.4), (3.6), (3.7) simplify to

\[
\begin{align*}
\tilde{d}\xi_{i1}^{(4)} &= e^{4\phi^2}\psi (\tilde{\xi}_{i1}^{2}) + g^{-2}[\psi \tilde{d}\xi_{i1}^{2}]^2 + e^{4\phi} \tilde{\psi} \tilde{d}\xi_{i1}^{2}, \\
\tilde{\alpha}_{(3)}^{(7)} &= C^1 + \frac{1}{2} \psi \tilde{d}\xi_{i1}^{2} \tilde{\psi} \tilde{d}\xi_{i1}^{2} + g^{-2} \tilde{\psi} \tilde{d}\xi_{i1}^{2} + \tilde{\psi} \tilde{d}\xi_{i1}^{2} \\
&\quad - g^{-3} \chi \tilde{d}\xi_{i1}^{2} \tilde{\psi} \tilde{d}\xi_{i1}^{2} - \frac{1}{2} \tilde{\psi} \tilde{d}\xi_{i1}^{2} \tilde{\psi} \tilde{d}\xi_{i1}^{2}.
\end{align*}
\]
where \( \phi, \varphi \) stand for the expressions in terms of \( \chi, \zeta, \bar{\zeta} \) given in (2.32). Here, \( ds^2(\mathbb{CP}^3) \) is the Fubini-Study metric on \( \mathbb{CP}^3 \) normalized so that the Ricci tensor is eight times the metric, and \( \eta^{(7)}, J_+^{(7)}, \Omega^{(7)}_+ \) are the homogeneous Sasaki-Einstein forms on \( S^6 \) defined in Appendix B. The four-form field strength corresponding to (3.19) reads

\[
\hat{F}_4 = -6ge^{\alpha\beta+3\varphi}\left[-1 + \chi^2 + \frac{1}{3}(\zeta^2 + \bar{\zeta}^2)\right]\text{vol}_4 + \frac{1}{2}g^{-1}e^{\alpha\beta}\ast(\zeta D\zeta - \bar{\zeta} D\bar{\zeta}) \wedge (\eta^{(7)} + gA) \\
+ \frac{g^{-2}(1-\chi^2)}{1 + 3\chi^2}\left[2\chi F - \sqrt{1 - \chi^2} \ast F\right] \wedge J_+^{(7)} \\
- g^{-3}\left[d\chi \wedge J_+^{(7)} \wedge (\eta^{(7)} + gA) - \frac{1}{2}D\zeta \wedge \text{Re}\Omega^{(7)}_+ - \frac{1}{2}D\bar{\zeta} \wedge \text{Im}\Omega^{(7)}_+ \right] \\
- 2g^{-3}\chi J_+^{(7)} \wedge J_+^{(7)} - 2g^{-3}(\zeta \text{Re}\Omega^{(7)}_+ - \bar{\zeta} \text{Im}\Omega^{(7)}_+) \wedge (\eta^{(7)} + gA),
\]

with, again, \( \phi, \varphi \) written in terms of \( \chi, \zeta, \bar{\zeta} \) as in (2.32). As noted in Sec. II C following [4], the SU(4)_c-invariant sector of SO(8) supergravity coincides with the model considered in [40]. Using the redefinitions (2.37) and straightforwardly identifying our Sasaki-Einstein structure with theirs, our uplifting formulas (3.18), (3.20) do indeed match (2.2), (2.3) of [40] when the identifications of their equation (4.1) are taken into account.

The SU(4)_c sector coincides with minimal \( \mathcal{N} = 2 \) gauged supergravity, (2.35). The \( D = 11 \) uplift of this sector can be achieved by bringing the restrictions (2.33) to the general formulas of Sec. III A or, equivalently, by further setting \( \varphi = \chi = 0, A^1 = -A^0 = \frac{1}{4}A \), and \( \tilde{A}_1 = -3\tilde{A}_0 \) in the uplifting formulas of Sec. III B 1. Using the rescaled fields (2.34) and the \( D = 4 \) field strengths (2.36), and combining the resulting expressions in terms of the Sasaki-Einstein forms \( J_+^{(7)}, \eta^{(7)} \) specified in Appendix B, the \( D = 11 \) uplift of the SU(4)_c sector can be written as

\[
d\hat{s}_{11}^2 = \frac{1}{4}d\hat{s}_4^2 + g^{-2}(d\hat{s}^2(\mathbb{CP}^3) + (\eta^{(7)} + \frac{1}{4}g\bar{A})^2),
\]

\[
\hat{F}_4 = \frac{3}{8}g\text{vol}_4 - \frac{1}{4}g^{-2}\bar{\chi} F \wedge J_+^{(7)}. \tag{3.21}
\]

This coincides with the consistent truncation of \( D = 11 \) supergravity down to minimal \( \mathcal{N} = 2 \) gauged supergravity obtained in [43], with straightforward identifications. An alternate \( D = 11 \) embedding of minimal \( \mathcal{N} = 2 \) supergravity will be given in Sec. III C.

### 3. \( G_2 \)-invariant sector

The \( D = 11 \) embedding formulas of Sec. III A particularized to the \( G_2 \)-invariant sector (2.41) become, in the relevant set of intrinsic coordinates described in Appendix B, the \( D = 11 \) uplift of the SU(4)_c sector can be written as

\[
d\hat{s}_{11}^2 = e^{-\varphi}X^{1/3}\Delta_1^{2/3}d\hat{s}_4^2 + g^{-2}X^{1/3}\Delta_1^{-1/3}(e^{2\varphi}X^{-3}\Delta_4d\beta^2 + \sin^2\beta ds^2(S^6)),
\]

\[
\tilde{A}_4 = C_1\sin^2\beta + C_{88}\cos^2\beta + 4g^{-1}\sin\beta \cos\beta B_{77} \wedge d\beta \\
+ g^{-3}\chi \Delta_1^{-1}\sin^2\beta(e^{2\varphi}X^{-3}\Delta_4 J \wedge d\beta + X^2 \sin\beta \cos\beta \text{Re}\Omega + e^{2\varphi}X^3 \sin^2\beta \text{Im}\Omega),
\]

where \( \beta \) is an angle on \( S^7 \), \( ds^2(S^6) \) is the round metric on \( S^6 \) normalized so that the Ricci tensor equals five times the metric, \( J \) and \( \Omega \) are the homogeneous nearly Kähler forms on \( S^6 \) and the function \( \Delta_1 \) is, from (3.3) with (B22),

\[
\Delta_1 = X(e^{-2\varphi}X^2 \cos^2\beta + e^{2\varphi}\sin^2\beta). \tag{3.23}
\]

The associated four-form field strength reads

\[
\hat{F}_4 = -ge^{-3\varphi}X^2[(X - 2)X^2 + e^{4\varphi}(X - 12]\sin^2\beta + e^{-4\varphi}X^2[X^3 + 7e^{4\varphi}(X - 2)\cos^2\beta]\text{vol}_4 \\
- 4g^{-1}\sin\beta \cos\beta( * d\varphi - e^{2\varphi} * d\chi) \wedge d\beta + g^{-3}e^{4\varphi}X^{-2}\sin^2\beta(2\varphi d\varphi - (X - 2)d\chi) \wedge J \wedge d\beta \\
+ 2g^{-3}\chi X\Delta_1 \sin^2\beta \cos\beta(\Delta_1 - 2e^{2\varphi}X\sin^2\beta)d\varphi \wedge \text{Re}\Omega + 4g^{-3}\chi X^3 \Delta_1^2 \sin^4\beta \cos^2\beta d\varphi \wedge \text{Im}\Omega \\
+ g^{-3}X^2 \Delta_1^{-1} \sin^2\beta \cos\beta(e^{2\varphi}(3X - 2)\sin^2\beta - e^{-2\varphi}X^2(X - 2)\cos^2\beta)d\chi \wedge \text{Re}\Omega \\
+ g^{-3}X^2 \Delta_1^{-1} \sin^4\beta(e^{2\varphi} \sin^2\beta - X(3X - 4)\cos^2\beta)d\chi \wedge \text{Im}\Omega.
\]
In order to obtain this expression, we have again made explicit use of the dualization conditions (2.17), (2.18) for the three- and four-form field strengths, particularized to the $G_2$-invariant sector (2.41). The $D = 11$ uplift of the various SO(7)-invariant sectors can be straightforwardly obtained by bringing (2.38)–(2.40) to (3.22)–(3.24). See [24] for a previous $D = 11$ uplift of the $G_2$-invariant sector.

C. Minimal $\mathcal{N} = 2$ gauged supergravity from $D = 11$

It was noted in Sec. II C that the SU(4)$_c$ sector coincides with minimal $\mathcal{N} = 2$ gauged supergravity. In Sec. III B 2, the corresponding $D = 11$ uplift was obtained and shown to coincide with the consistent embedding of [43]. It was also discussed at the end of Sec. II C that the SU(3) sector admits an alternative truncation to minimal $\mathcal{N} = 2$ supergravity, by fixing the scalars to their vevs (2.43) at the $\mathcal{N} = 2$, SU(3) $\times$ U(1)$_c$-invariant point and selecting the $\mathcal{N} = 2$ graviphoton as in (2.44). Brining these $D = 4$ identifications to the general SU(3)-invariant consistent uplifting formulas of Sec. III A, we obtain a new embedding of pure $\mathcal{N} = 2$ gauged supergravity into $D = 11$.

We find it convenient to present the result in local intrinsic $S^7$ coordinates $\psi'$, $\tau'$, $\alpha$, and in terms of a local five-dimensional Sasaki-Einstein structure $\mathfrak{q}'$, $\mathfrak{j}'$ and $\mathfrak{w}'$. The former are locally related to the global coordinates $\psi$, $\tau$, $\alpha$, defined in (B1), that are adapted to the topological description of $S^7$ as the join of $S^5$ and $S^1$, with $\alpha$ here identified with that in (B1) and

$$\psi = \psi', \quad \tau = \tau' - \frac{1}{3} \psi'.$$

The local five-dimensional Sasaki-Einstein structure forms $\mathfrak{q}'$, $\mathfrak{j}'$ and $\mathfrak{w}'$ are related to their globally defined counterparts $\mathfrak{q}^{(5)}$, $\mathfrak{j}^{(5)}$ and $\mathfrak{w}^{(5)}$ discussed in Appendix B and the global coordinate $\psi$ via

$$\mathfrak{q}' \equiv d\tau' + \sigma = \mathfrak{q}^{(5)} + \frac{1}{3} d\psi, \quad \mathfrak{j}' \equiv \mathfrak{j}^{(5)}, \quad \mathfrak{w}' \equiv e^{i\psi + i^\perp} \mathfrak{w}^{(5)}.$$

The real two-form $\mathfrak{j}'$ coincides with the Kähler form on $\mathbb{CP}^2$, $\sigma$ is a one-form on the latter such that $d\sigma = 2J'$ [given e.g., by (B1)] and the constant phase $e^{i\xi}$ in the complex two-form $\mathfrak{w}'$ has been chosen for convenience, in order to simplify the resulting expressions. The primed forms defined in (3.26) satisfy the Sasaki-Einstein conditions (B5) and (B6).

Bringing all these definitions, along with the $D = 4$ restrictions (2.43)–(2.46), to the uplifting formulas (3.4), (3.6), (3.7), we find a new consistent embedding of minimal $D = 4 \mathcal{N} = 2$ gauged supergravity (2.35) into the $D = 11$ metric and three-form:

$$d\tilde{s}^2_{11} = \frac{1}{3} \cdot 2^{-2/3} (1 + 2\sin^2 \alpha)^{-2/3} \left[d\tilde{s}^2_5 + g^{-2} \left[2d\alpha^2 + \frac{6\cos^2 \alpha}{1 + 2\sin^2 \alpha} d\tilde{s}^2(\mathbb{CP}^2)ight.ight.
$$

$$+ \frac{18\sin^2 \alpha \cos^2 \alpha - \eta^2}{1 + 8\sin^4 \alpha} \right] + \frac{1 + 8\sin^4 \alpha}{(1 + 2\sin^2 \alpha)^2} (d\psi' - \frac{3\cos^2 \alpha}{1 + 8\sin^4 \alpha} \eta')^2 \bigg]\right].$$

$$\hat{A}_{(3)} = C^1 - \frac{1}{2\sqrt{3}} g^{-2} \cos \alpha \hat{A} \wedge [\cos \alpha J' - \sin \alpha d\alpha \wedge \eta']$$

$$+ \frac{1}{\sqrt{3}} g^{-3} \cos^2 \alpha \left[d\alpha \wedge \text{Im}\mathfrak{w}' + \frac{\sin \alpha \cos \alpha}{1 + 2\sin^2 \alpha} (2d\psi' - 3\eta') \wedge \text{Re}\mathfrak{w}' \right].$$

These expressions depend explicitly on the dynamical $D = 4$ metric $d\tilde{s}^2_5$ and graviphoton $\hat{A}$. The former only features in $d\tilde{s}^2_{11}$ but not in $\hat{A}_{(3)}$. The latter appears both in $d\tilde{s}^2_{11}$ and in $\hat{A}_{(3)}$, but only through the gauge covariant derivative

$$D\psi' = d\psi' + \frac{1}{2} g \hat{A}. \quad (3.29)$$

This singles out $\psi'$ as the angle on the local $\mathcal{N} = 2$ “Reeb” direction and thus justifies the primed coordinates (3.25) that we chose to present the result. Two other $D = 4$ fields enter the consistent embedding through the three-form (3.28): the magnetic dual, $\hat{A}$, of the $D = 4$ graviphoton, and the auxiliary three-form potential $C^1$.
The four-form field strength corresponding to $\hat{A}_{(3)}$ in (3.27) can be computed with the help of (the primed version of) the Sasaki-Einstein conditions (B5), (B6). We find
\begin{equation}
\hat{F}_{(4)} = \frac{g}{2\sqrt{3}} \frac{\text{vol}}{\sqrt{3}} + \frac{g^3}{\sqrt{3}} \left[ -\frac{\cos^2 \alpha (7 - 10 \cos 2 \alpha + \cos 4 \alpha)}{(1 + 2 \sin^2 \alpha)^2} d\alpha \wedge D\psi' \wedge \text{Re} \Omega' \\
- \frac{6 \cos^4 \alpha}{(1 + 2 \sin^2 \alpha)^2} d\alpha \wedge \eta' \wedge \text{Re} \Omega' + \frac{6 \sin \cos^3 \alpha}{1 + 2 \sin^2 \alpha} D\psi' \wedge \eta' \wedge \text{Im} \Omega' \right] \\
+ \frac{g^2}{2\sqrt{3}} \left[ \frac{2 \sin \cos^3 \alpha}{1 + 2 \sin^2 \alpha} \hat{F} \wedge \text{Re} \Omega' + \cos \alpha \hat{F} \wedge (\cos \alpha \hat{J}' - \sin \alpha \hat{\alpha} \hat{J}) \right].
\end{equation}

Again, we have made use of appropriate dualization conditions, (2.47), (2.48) in this case, to express the result for the embedding (3.30) into the four-form only in terms of the independent $D = 4$ d.o.f. (the metric $d\hat{s}_4^2$, the graviphoton field strength $\hat{F} = d\hat{A}$ and its Hodge dual), that appear in the Lagrangian (2.25).

The truncation (3.27), (3.30) of $D = 11$ supergravity down to pure $D = 4 \mathcal{N} = 2$ gauged supergravity (2.35) is consistent by construction. As a check on our formalism, we have explicitly verified consistency at the level of the Bianchi identities and equations of motion for the $D = 11$ four-form: its field equations are indeed satisfied, provided the $D = 4$ Bianchi, $d\hat{F} = 0$, and equation of motion, $d\hat{F} = 0$, of the $D = 4$ graviphoton are imposed. Some details can be found in Appendix C. Moreover, these local uplifting formulas are still valid if, more generally, $\eta', J', \Omega$ are taken to be the defining forms of any Sasaki-Einstein five-manifold, and $d\hat{s}_2^2(\mathbb{CP}^2)$ is replaced with the metric on the corresponding local Kähler-Einstein base.

**IV. RECOVERING $D=11$ AdS$_4$ SOLUTIONS**

Setting the scalars to the vevs at each critical point with at least SU(3) invariance that were recorded in Table II, and turning off the relevant tensor hierarchy fields, the consistent embedding formulas of Sec. III produce AdS$_4$ solutions of $D = 11$ supergravity. All these $D = 11$ solutions are known, so our presentation must necessarily be brief. Our main motivation to work out these solutions is rather to test the consistency of the uplifting formulas of [25] [and their particularization to an explicit, SU(3)-invariant, subsector].

Except for the more involved $D = 11$ Einstein equation, we have indeed verified that the metrics and four-forms that we write below do indeed solve the eleven-dimensional field equations. Please refer to Appendix D for details.

We present the solutions in the appropriate intrinsic $S^7$ angles defined in Appendix B. These have already been employed in Sec. III B to write the consistent $D = 11$ embedding of various further subsectors. Also, AdS$_4$ is always taken to be unit radius (so that the Ricci tensor equals $-3$ times the metric). As a consequence, the metric $d\hat{s}_4^2(\text{AdS}_4)$ that appears in the expressions below is related to the metric $d\bar{s}_4^2$ that appears in the $D = 4$ Lagrangian (2.3) and $D = 11$ embedding (3.4) by a rescaling
\begin{equation}
ds_4^2 = -6V_0^{-1} d^2(\text{AdS}_4),
\end{equation}
where $V_0$ is the cosmological constant at each critical point given in Table II. The Freund-Rubin term is rescaled accordingly with respect to (3.11).

Let us first discuss the supersymmetric solutions. The $\mathcal{N} = 8, \text{SO}(8)$ point uplifts to the Freund-Rubin solution [44] for which the internal four-form vanishes and the internal metric is the round, Einstein metric $d\bar{s}_4^2(S^7)$, given in e.g., (B3) or (B17). The $\mathcal{N} = 2, \text{SU}(3) \times \text{U}(1)_c$ critical point uplifts to the $D = 11$ CPW solution [27]. A local form of this solution can be obtained from the expressions in Sec. III C by turning off the $D = 4$ graviphoton, $\hat{A} = 0$, $\hat{F} = 0$, and fixing the metric to $d\bar{s}_4^2 = g^{-2} d^2(\text{AdS}_4)$. As a check, we have verified that the solution in $\mathbb{R}^8$ embedding coordinates $\mu^4$, directly obtained from the formulas in Sec. III A, perfectly agrees with the CPW solution as given in [45]. Finally, the $\mathcal{N} = 1$ $G_2$-invariant solution can be written, using the results and the notation of Sec. III B 3, in terms of the homogeneous nearly Kähler structure of the $S^6$ inside $S^7$ as
\begin{equation}
d\bar{s}_4^2 = g^{-2} \left( \frac{25}{12} \right)^{1/6} \left( 2 + \cos 2\beta \right)^{2/3} \left[ \frac{5}{24} d\bar{s}_4^2(\text{AdS}_4) + \frac{1}{3} d\beta^2 + \frac{\sin^2 \beta}{2 + \cos 2\beta} d\bar{s}_4^2(S^6) \right],
\end{equation}
\begin{equation}
\hat{F}_{(4)} = \frac{1}{8} \left( \frac{25}{12} \right)^{5/4} g^{-3} \text{vol}(\text{AdS}_4) + \frac{\sqrt{2} g^{-3} \sin^2 \beta}{3^{1/4}(2 + \cos 2\beta)^2} \left[ \sqrt{2} \sin^2 \beta \text{Re} \Omega \wedge d\beta \\
- \sin \beta \cos \beta (5 + \cos 2\beta) \text{Im} \Omega \wedge d\beta - \sin^2 \beta (2 + \cos 2\beta) J \wedge J \right],
\end{equation}
with internal three-form potential
This solution was first obtained by de Wit, Nicolai and Warner [15].

Turning to the nonsupersymmetric solutions, the SO(7) critical points can again be uplifted using the results and conventions of Sec. III B 3. The SO(7)\(_c\) solution uplifts to a solution first written by de Wit and Nicolai [46]. In our conventions, we get

\[
ds^2_{11} = 5^{-5/6} g^{-2} (3 + 2 \cos 2\beta)^{2/3} \left[ \frac{3}{4} ds^2(\text{AdS}_4) + d\beta^2 + \frac{5 \sin^2 \beta}{3 + 2 \cos 2\beta} ds^2(S^7) \right],
\]

\[
\hat{F}_{(4)} = \frac{9}{8} \cdot 5^{-3/4} g^{-3} \text{vol}(\text{AdS}_4),
\]

while the SO(7)\(_c\) point uplifts to Englert’s solution [47]

\[
ds^2_{11} = g^{-2} \left( \frac{4}{5} \right)^{1/3} \left[ \frac{3}{10} ds^2(\text{AdS}_4) + ds^2(S^7) \right],
\]

\[
\hat{F}_{(4)} = \frac{18}{25 \sqrt{5} g^3} \text{vol}(\text{AdS}_4) + \frac{4 \sin^2 \beta}{\sqrt{5} g^3} \left[ \text{Re}\Omega \wedge d\beta - \cot \beta \text{Im}\Omega \wedge d\beta - \frac{1}{2} \mathcal{J} \wedge \mathcal{J} \right],
\]

with internal three-form

\[
A = \frac{\sin^2 \beta}{2 \sqrt{5} g^3} \left[ 2 \sin^2 \beta \text{Im}\Omega + 2 \mathcal{J} \wedge d\beta + \sin 2\beta \text{Re}\Omega \right].
\]

In the SO(7)\(_c\) solution, \(ds^2(S^7)\) is, as always, the round, SO(8)-invariant metric. It should be understood in this context as the sine-cone form (B23). Since SO(7)\(_c\) \(\supset\) SU(4)\(_c\), this solution can also be reobtained from the SU(4)\(_c\)-invariant truncation of Sec. III B 2 and written in terms of the homogeneous Sasaki-Einstein structure on \(S^7\). The \(D = 11\) metric is the same appearing in (4.5) with \(ds^2(S^7)\) now understood as the Hopf fibration (B17), and the four-form is given by

\[
\hat{F}_{(4)} = \frac{18}{25 \sqrt{5} g^3} \text{vol}(\text{AdS}_4)
\]

\[
+ \frac{2}{\sqrt{5} g^3} [2 \text{Re}\Omega_{+}^{(7)} \wedge \eta_{+}^{(7)} - J_{+}^{(7)} \wedge J_{+}^{(7)}],
\]

with internal three-form

\[
A = -\frac{1}{\sqrt{5} g^3} [J_{+}^{(7)} \wedge \eta_{+}^{(7)} + \text{Im}\Omega_{+}^{(7)}].
\]

The metric in (4.5) and four-form (4.7) for the SO(7)\(_c\) solution coincide with (3.11) of [40] upon using the redefinitions (2.37), and making an appropriate choice for the phase of the complex scalar \(\chi_{\text{here}} \equiv -\frac{1}{\sqrt{3}} ([\zeta_{\text{here}} + i \zeta_{\text{here}}],\) which is again unfixed at the critical point. We obtain perfect agreement with [40] upon shifting that phase by \(\pi\).

Finally, the SU(4)\(_c\)-invariant point gives rise to the Pope-Warner solution [48] in eleven dimensions. Using the results of Sec. III B 2, this solution can also be written in terms of the homogeneous Sasaki-Einstein structure on \(S^7\) as

\[
ds^2_{11} = \frac{1}{21/3} g^{2} \left[ \frac{3}{8} ds^2(\text{AdS}_4) + ds^2(\mathbb{C}P^3_{+}) + 2 \eta_{+}^{(7)} \otimes \eta_{+}^{(7)} \right],
\]

\[
\hat{F}_{(4)} = \frac{9}{32 g^3} \text{vol}(\text{AdS}_4) - \frac{2}{g^3} [\text{Re}\Omega_{+}^{(7)} \wedge \eta_{+}^{(7)} - \text{Im}\Omega_{+}^{(7)} \wedge \eta_{+}^{(7)}],
\]

where the internal three-form potential is now

\[
A = \frac{1}{2} g^{-3} [\text{Re}\Omega_{+}^{(7)} + \text{Im}\Omega_{+}^{(7)}].
\]

We again find agreement with [40]: (4.9) coincides with (3.8) of that reference when the identifications (2.37) are taken into account and the phase of \(\chi_{\text{here}} \equiv -\frac{1}{\sqrt{3}} ([\zeta_{\text{here}} + i \zeta_{\text{here}}],\) which is again unfixed at the critical point, is shifted by \(\pi\).

V. DISCUSSION

The main goal of this paper was to test the formulas of [25] for the consistent truncation [14] of \(D = 11\) supergravity [13] on \(S^7\) down to \(D = 4\). The SO(8)-gauged supergravity [1]. We have done so by particularizing these formulas to the SU(3)-invariant sector of the \(D = 4\) supergravity, using an explicit parametrization. When further restricted appropriately, our results correctly reproduce
previously known consistent embeddings of sectors that preserve symmetries larger than SU(3). Our formalism thus extends previous literature and provides a unified $D = 11$ embedding of the full SU(3)-invariant sector of SO(8) supergravity including all dynamical (bosonic) fields. It does so systematically, by using the restricted tensor hierarchy approach of [25].

As another crosscheck on the formulas of [25], we have rederived the known AdS$_4$ solutions of $D = 11$ supergravity that arise upon consistent uplift of the critical points of SO(8) supergravity with at least SU(3) symmetry [2]. Again, we have found perfect agreement with the existing literature. As a further test, we have checked that the $D = 11$ field equations are indeed verified on these AdS$_4$ solutions. Moreover, we have done this in a unified way for all of them; please refer to Appendix D for the details. This should again be regarded as a stringent test on the consistency of our formalism. Although we have not explicitly verified the $D = 11$ Einstein equation due to its more involved structure, we have reproduced known solutions, like the ones presented in [40], for which the Einstein equation has been verified.

We have also obtained new embeddings of minimal $D = 4 \mathcal{N} = 2$ gauged supergravity both into its parent $D = 4 \mathcal{N} = 8$ SO(8)-gauged supergravity and into $D = 11$ supergravity. A previously known embedding is obtained by fixing the scalars to their vs at the SO(8) point and then selecting the graviphoton $\tilde{A}$ as an appropriate combination of the two SU(3)-invariant vectors $A^\Lambda$, $\Lambda = 0, 1$. The resulting $D = 11$ consistent uplift coincides with a previously known one, constructed in Sec. 2 of [43], that is in fact valid for any Sasaki-Einstein seven-manifold. The consistency of this truncation, at least within $D = 4$ theories, is guaranteed by symmetry principles. This is because this embedding of minimal $\mathcal{N} = 2$ supergravity into $\mathcal{N} = 8$ coincides with the SU(4)$_s$-invariant sector of the latter.

More interestingly, we have shown $\mathcal{N} = 8$ SO(8) supergravity to admit an alternative truncation to minimal $\mathcal{N} = 2$ supergravity by similarly fixing the scalars to their vs at, now, Warner’s $\mathcal{N} = 2$ SU(3) $\times$ U(1)$_c$ point [2] and again selecting the graviphoton $\tilde{A}$ appropriately. Although this alternative truncation is not driven by any apparent symmetry principle, it is nevertheless consistent. We have explicitly verified this at the level of the $D = 4$ equations of motion that follow from the Lagrangian (2.3), including Einstein. Using our formalism, we have then uplifted this minimal $\mathcal{N} = 2$ supergravity to $D = 11$ in Sec. III C. Again, we have explicitly verified the consistency of the $D = 11$ embedding; see Appendix C. Thus, we have constructed the consistent truncation of $D = 11$ supergravity on the $\mathcal{N} = 2$ AdS$_4$ solution of CPW [27] down to minimal $D = 4 \mathcal{N} = 2$ gauged supergravity, predicted to exist by the general conjecture of [43].

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APPENDIX A: DETAILS ON THE SU(3) SECTOR

Let $t_A^B$, $t_{ABCD}$, with $A = 1, \ldots, 8$ indices in the fundamental of SL(8, $\mathbb{R}$), be the $E_{7(7)}$ generators in the SL(8, $\mathbb{R}$) basis, in the conventions of Appendix C of [31]. The SO(8) $\subset$ SL(8, $\mathbb{R}$) $\subset$ $E_{7(7)}$ subgroup is generated by $T_{AB} = 2t_A^C \delta_{BC}$. The generators of SU(3) $\subset$ SO(8) can then be taken to be $\tilde{\lambda}_\alpha$, $\alpha = 1, \ldots, 8$, defined as

$$
\tilde{\lambda}_1 = T_{14} - T_{23}, \quad \tilde{\lambda}_2 = -T_{13} - T_{24}, \quad \tilde{\lambda}_3 = T_{12} - T_{34}, \quad \tilde{\lambda}_4 = T_{16} - T_{25}, \quad \\
\tilde{\lambda}_5 = -T_{15} - T_{26}, \quad \tilde{\lambda}_6 = T_{36} - T_{45}, \quad \tilde{\lambda}_7 = -T_{35} - T_{46}, \quad \tilde{\lambda}_8 = \frac{1}{\sqrt{3}}(T_{12} + T_{34} - 2T_{56}).
$$

(A1)

These generators indeed close into the SU(3) commutation relations

$$
[\tilde{\lambda}_\alpha, \tilde{\lambda}_\beta] = 2 f_{\alpha \beta \gamma} \tilde{\lambda}_\gamma,
$$

(A2)

with $f_{\alpha \beta \gamma} = f_{[\alpha \beta \gamma]}$ Gell-Mann’s structure constants,

$$
f_{123} = 1, \quad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}.
$$

(A3)

Inside $E_{7(7)}$, the SU(3) generated by (A1) commutes with SL(2, $\mathbb{R}$) $\times$ SU(2, 1), with the first factor generated by

$$
H_0 = -\frac{1}{2}(t_i^i - 3t_a^a), \quad E_0 = 3J^{(6)ij}e^{ahl}t_{ijab}, \quad F_0 = \frac{3}{2}J^{(6)ij}f^{(6)khh}t_{ijkh},
$$

(A4)
and the second factor by

\[
H_1 = -t_7^3 + t_8^8, \quad H_2 = f_j^{(6)i} t_i^j,
\]

\[
E_{1i} = -\sqrt{2} \text{Im} \Omega^{(6)ij} t_{ijk8}, \quad E_{12} = -\sqrt{2} \text{Re} \Omega^{(6)ij} t_{ijk8}, \quad E_2 = -\sqrt{2} t_8^8,
\]

\[
F_{1i} = \sqrt{2} \text{Re} \Omega^{(6)ij} t_{ijk7}, \quad F_{12} = -\sqrt{2} \text{Im} \Omega^{(6)ij} t_{ijk7}, \quad F_2 = -\sqrt{2} t_7^8. \quad (A5)
\]

These are the numerator groups in the scalar manifold (2.2). In (A4) and (A5) we have split the indices as \(A = (i, a)\), with \(i = 1, \ldots, 6\) in the fundamental of \(\text{SO(6)}\), and \(a = 7, 8\), by effectively identifying the fundamental of \(\text{SL}(8, \mathbb{R})\) with the \(8_p\) of \(\text{SO(8)}\). We have employed the SU(3)-invariant Calabi-Yau (1,1) and (3,0) forms

\[
J^{(6)} = e^{12} + e^{34} + e^{56}, \quad \Omega^{(6)} = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6), \quad (A6)
\]
on \(\mathbb{R}^6 \subset \mathbb{R}^8\), with \(e^{12} \equiv dx^1 \wedge dx^2\), etc., and \(x^i\) the \(\mathbb{R}^6\) Cartesian coordinates. We have also introduced the Levi-Civita tensor \(\epsilon_{ab}\) in the \(\mathbb{R}^2 \subset \mathbb{R}^8\) plane spanned by the 7,8 directions. Indices \(i, j\) and \(a, b\) are raised and lowered with \(\delta_{ij}\) and \(\delta_{ab}\). The generators (A4) and (A5) indeed commute with each other and respectively close into the SL(2,\(\mathbb{R}\)),

\[
[H_0, E_0] = 2E_0, \quad [H_0, F_0] = -2F_0, \quad [E_0, F_0] = H_0, \quad (A7)
\]

and SU(2,1) commutation relations,

\[
[H_1, H_2] = 0, \quad [H_1, E_{1i}] = E_{1i}, \quad [H_2, E_{1i}] = -3\epsilon_{ij} E_{1j}, \quad [H_1, E_2] = 2E_2, \quad [H_2, E_2] = 0, \quad [H_1, F_{1i}] = -E_{1i}, \quad [H_2, F_{1i}] = -3\epsilon_{ij} F_{1j}, \quad [H_1, F_2] = -2F_2, \quad [H_2, F_2] = 0, \quad [E_{1i}, E_{2i}] = -\sqrt{2}E_2, \quad [F_{1i}, F_{2i}] = \sqrt{2}E_2, \quad [F_{1i}, F_2] = 0, \quad [E_{1i}, F_{1j}] = \delta_{ij}H_1 + \epsilon_{ij} H_2, \quad [E_{1i}, F_2] = \sqrt{2}\epsilon_{ij} F_{1j}, \quad [E_2, F_{1i}] = \sqrt{2}\epsilon_{ij} E_{1j}, \quad [E_2, F_2] = 2H_1, \quad (A8)
\]

with, here and only here, \(i = 1, 2\). The generators of the maximal compact subgroup of SU(2,1) are

\[
K_0 \equiv E_2 - F_2 - \frac{\sqrt{2}}{3} H_2, \quad K_1 \equiv \frac{1}{\sqrt{8}} (E_{11} - F_{11}), \quad K_2 \equiv \frac{1}{\sqrt{8}} (E_{12} - F_{12}), \quad K_3 \equiv -\frac{1}{4\sqrt{2}} (E_2 - F_2) - \frac{1}{4} H_2, \quad (A9)
\]

and close into the SU(2) \(\times\) U(1) commutation relations

\[
[K_0, K_x] = 0, \quad [K_x, K_y] = \epsilon_{xyz} K_z, \quad x = 1, 2, 3. \quad (A10)
\]

It is also interesting to note that the three different U(1)’s with which SU(3) commutes inside the SO(8) subgroups SO(6)_c, SU(4)_c and SU(4)_s are respectively generated by

\[
\text{U(1)}_c : -J_j^{(6)i} t_i^j, \quad (A11)
\]

\[
\text{U(1)}_s : -J_j^{(6)i} t_i^j + 3\epsilon_{ab} t_a^b, \quad \quad (A12)
\]

\[
\text{U(1)}_c : -\lambda J_j^{(6)i} t_i^j + 3\epsilon_{ab} t_a^b, \quad \text{with} \quad \lambda \in \mathbb{R}, \quad \lambda \neq 1. \quad (A13)
\]

With these details, the SU(3)-invariant bosonic field content and its interactions described in Sec. II can be constructed from the parent \(\mathcal{N} = 8\) supergravity. Per the analysis above, the SU(3)-invariant scalar manifold is (2.2). A coset representative is
\[
\mathcal{V} = e^{-\chi E_0} e^{-\phi H_0} e^{\frac{1}{2\sqrt{2}}(aE_2-E_{11}-\tilde{E}_{12})} e^{-\phi H_1},
\]

and the quadratic scalar matrix that enters the bosonic Lagrangian is \( M = \mathcal{V} \mathcal{V}^T \). The metric on (2.2) that determines the scalar kinetic terms in the Lagrangian (2.3) is then reproduced through \(-\frac{1}{36} \mathcal{D} \mathcal{M} \wedge * \mathcal{D} \mathcal{M}^{-1}\). For reference, the \( SL(2, \mathbb{R}) \times SU(2, 1) \) Killing vectors of this metric, normalized to obey the commutation relations (A7), (A8), are

\[
k[H_0] = 2\partial_\varphi - 2\chi \partial_\varphi, \quad k[E_0] = \partial_\varphi, \quad k[F_0] = 2\partial_\varphi + (e^{-2\varphi} - \chi^2) \partial_\chi,
\]

and

\[
k[H_1] = \partial_\phi - 2a\partial_\mu - \zeta \partial_\xi - \tilde{\zeta} \partial_\xi, \quad k[H_2] = 3\tilde{\zeta} \partial_\xi - 3\zeta \partial_\xi,
\]

\[
k[E_{11}] = \frac{1}{\sqrt{2}} (\zeta \partial_\alpha - 2\partial_\xi), \quad k[E_{12}] = \frac{1}{\sqrt{2}} (\zeta \partial_\alpha + 2\partial_\xi), \quad k[E_2] = \sqrt{2} \partial_\alpha,
\]

\[
k[F_2] = \sqrt{2}(a \partial_\phi - e^{-\phi}(Z^2 - Y^2) \partial_\alpha - (a\zeta - e^{-2\phi} \tilde{\zeta} Y) \partial_\xi - e^{-2\phi}(\zeta Z + \tilde{\zeta} Y) \partial_\chi),
\]

\[
k[F_{11}] = \frac{1}{\sqrt{2}} \left( -\zeta \partial_\phi + (a\zeta - e^{-2\phi} \tilde{\zeta} Y) \partial_\alpha - \frac{1}{2} (4e^{-2\phi} - \zeta^2 + 3\tilde{\zeta}^2) \partial_\xi + 2(a + \zeta \tilde{\zeta}) \partial_\chi \right),
\]

\[
k[F_{12}] = \frac{1}{\sqrt{2}} \left( \tilde{\zeta} \partial_\phi - (a\tilde{\zeta} + e^{-2\phi} \tilde{\zeta} Y) \partial_\alpha + 2(a - \zeta \tilde{\zeta}) \partial_\chi + \frac{1}{2} (4e^{-2\phi} + 3\zeta^2 - \tilde{\zeta}^2) \partial_\chi \right).
\]

Moving on, we need to specify how the SU(3)-invariant tensor fields in (2.1) are embedded into their \( N = 8 \) counterparts. Recall that the restricted \( \tilde{N} = 8 \) tensor hierarchy contains 28 electric vectors \( \mathcal{A}^{AB} \), 28 magnetic vectors \( \mathcal{A}_{AB} \), 63 two-forms \( B_A^B \) and 36 three-forms \( C^{AB} \), in representations of \( SL(8, \mathbb{R}) \) [25]. In order to determine the embedding of the SU(3)-invariant vectors \( \mathcal{A}^A, \tilde{\mathcal{A}}_A, \Lambda = 0, 1, \) into their \( \tilde{N} = 8 \) counterparts, we note that SU(3) commutes inside SO(8) \( \subset E_7(7) \) with the \( U(1) \) \( \tilde{U}(1) \) generated, in the notation of (A5), by \( (E_2 - F_2) \) and \( H_2 \) or, equivalently, by \( K^0 \) and \( K^3 \) defined in (A9). These are the Cartan generators of the maximal compact subgroup \( SU(2) \times U(1) \) of the hypermultiplet scalar manifold. Splitting again the \( \tilde{N} = 8 \) index as below (A5), \( A = (i, a) \), and fixing the normalizations for convenience we have the following embedding into the \( N = 8 \) vectors:

\[
\mathcal{A}_i = \mathcal{A}_i^A J^{(6)i}_A, \quad \mathcal{A}_a = \epsilon^{ab} A^0, \quad \tilde{\mathcal{A}}_i = \frac{1}{3} \tilde{\mathcal{A}}_i J^{(6)i}_A, \quad \tilde{\mathcal{A}}_a = \tilde{\mathcal{A}}_a \epsilon_{ab}.
\]

Similarly, for the two-form potentials we define

\[
B_i = -\frac{1}{12} B_a^a \delta_i^j + \frac{1}{3} B^2 J^{(6)i}_j, \quad B_a^b = \frac{1}{2} B_a^b - \frac{1}{2} B^0 e_a^b.
\]

and for the three-form potentials,

\[
C_i = C^i \delta i, \quad C^a = C_a^b.
\]

The field strengths and couplings brought to Sec. II can be obtained by inserting these expressions into the \( N = 8 \) equations given in [25]. For example, the gauge covariant derivative acting on the scalars reduce to \( D = d + \frac{1}{\sqrt{2}} g(k[E_2] - k[F_2]) A^0 - g k[H_2] A^1 \), and this in turn reproduces (2.4) upon use of the relevant Killing vectors in (A16).

**APPENDIX B: INTRINSIC COORDINATES AND GEOMETRIC STRUCTURES ON S^7**

There are various sets of intrinsic coordinates that prove useful in our context, each of them adapted to different geometric structures on \( S^7 \). The expressions below have been used to particularize the general SU(3)-invariant consistent embedding formulas of Sec. III A to the further subsectors of Sec. III B and the AdS\(_4\) solutions of Sec. IV.

1. \( S^7 \) as the join of \( S^4 \) and a Sasaki-Einstein \( S^5 \)

The first set of coordinates solves the constraint (3.1) by splitting \( \mu^A, \; A = 1, \ldots, 8, \) as

\[
\mu^1 = \cos \alpha \tilde{\mu}^1, \quad i = 1, \ldots, 6, \quad \mu^7 = \sin \alpha \cos \psi, \quad \mu^8 = \sin \alpha \sin \psi.
\]

with \( 0 \leq \alpha \leq \pi/2, \; 0 \leq \psi < 2\pi, \) and \( \tilde{\mu}^i, \; i = 1, \ldots, 6, \) defining in turn an \( S^5 \), i.e., subject to the constraint \( \delta_i \tilde{\mu}^i \tilde{\mu}^i = 1 \). The intrinsic coordinates (B1) are adapted to the topological description of \( S^7 \) as the join of \( S^5 \) and \( S^4 \), for which the round, Einstein, SO(8)-invariant metric.
with $ds^2(S^5) = \delta_{AB}d\mu^A d\mu^B$, the Kähler form on $\mathbb{CP}^2$, so that $\eta^{(5)} \equiv d\tau + \sigma$ and $J^{(5)} = J^{(4)}$. For completeness, we note that $ds^2(\mathbb{CP}^2)$ can be written in terms of complex projective coordinates $\xi^i$, $i = 1, 2, 3$, as
\[
d s^2(\mathbb{CP}^2) = \frac{d\xi_i d\xi^i}{1 + \xi_i^2} - \frac{(\xi_i d\xi^i)(d\xi^j)}{(1 + \xi_i^2)(1 + \xi_j^2)}.
\]
by introducing complex coordinates on $\mathbb{R}^6 = \mathbb{C}^3$ through
\[
\bar{\mu}^1 + i\bar{\mu}^2 = \frac{1}{\sqrt{1 + \xi_i^2}} e^{i\tau} \xi^1, \quad \bar{\mu}^3 + i\bar{\mu}^4 = \frac{1}{\sqrt{1 + \xi_i^2}} e^{i\tau} \xi^2.
\]
In these coordinates, the one-form $\sigma$ in (B8) reads
\[
\sigma = \frac{i}{2} \frac{\xi^i d\xi_i - \bar{\xi}_i d\bar{\xi}_i}{1 + \xi_i^2}.
\]

2. $S^7$ with its homogeneous Sasaki-Einstein structure

A second set of intrinsic coordinates on $S^7$ can be chosen that adapt themselves to its two natural, homogeneous seven-dimensional Sasaki-Einstein structures. These descend on $S^7$ from the Calabi-Yau forms $J^{(8)}$, $\Omega^{(8)}$ on $\mathbb{R}^8$,
\[
J^{(8)} = J^{(6)} \pm e^{78} = e^{12} \pm e^{34} + e^{66} \pm e^{78},
\]
\[
\Omega^{(8)} = \Omega^{(6)} \wedge (e^7 \pm ie^8) = (e^1 + ie^2) \wedge (e^3 + ie^4)
\]
\[
\wedge (e^5 + ie^6) \wedge (e^7 \pm ie^8),
\]
that are invariant under $SU(4)_\pm$ for the $+$ sign and $SU(4)_-$ for the $-$ sign. In terms of the constrained coordinates $\mu^A$, $A = 1, \ldots, 8$, that define $S^7$ as the locus (3.1) in $\mathbb{R}^8$, the Sasaki-Einstein structure forms induced on $S^7$ are
\[
\eta^{(7)} = J^{(8)} \pm \mu^A d\mu^B, \quad J^{(7)} = \frac{1}{2} J^{(8)} d\mu^A \wedge d\mu^B,
\]
\[
\Omega^{(7)} = \frac{1}{6} \Omega^{(8)} \wedge (\Omega^{(7)} \wedge d\mu^C \wedge d\mu^D).
\]
These are subject to
\[
J^{(7)} \wedge \Omega^{(7)} = 0,
\]
\[
J^{(7)} \wedge J^{(7)} \wedge \eta^{(7)} = \frac{3}{4} \Omega^{(7)} \wedge \Omega^{(7)} \wedge \eta^{(7)} = \mp 6\text{vol}(S^7),
\]
and
\[
ds^2(\mathbb{CP}^2) = 2J^{(7)}, \quad d\Omega^{(7)} = 4i\eta^{(7)} \wedge \Omega^{(7)}.
\]
The seven-dimensional Sasaki-Einstein structure (B13) is related to its five-dimensional counterpart (B4) and the angles (B1) through
\[
\begin{align*}
\eta^{(7)}_\pm &= \cos^2 \alpha \eta^{(5)} \pm \sin^2 \alpha d\psi, \\
J^{(7)}_\pm &= \cos^2 \alpha J^{(5)} \pm \sin \alpha \cos \alpha \, d\psi \wedge (d\psi \mp \eta^{(5)}), \\
\Omega^{(7)}_\pm &= \pm \psi^2 \cos^2 \alpha [d\alpha \pm \iota \cos \alpha \sin \alpha (d\psi \mp \eta^{(5)})] \wedge \Omega^{(5)}.
\end{align*}
\] (B16)

The round metric on \(S^7\) adapted to seven-dimensional Sasaki-Einstein structure reads, similarly to (B8),
\[
d^2(S^7) = d^2(CP^3) + (d\psi \pm \sigma)^2,
\] (B17)
where \(d^2(CP^3)\) is the Fubini-Study metric, normalized so that the Ricci tensor equals eight times the metric. The \(\pm\) refers to two different embeddings of \(CP^3\) into \(S^7\), with isometry group SU(4) \(\subset SO(8)\) for the \(+\) sign and SU(4) \(\subset SO(8)\) for the \(-\) sign. The angles \(\psi \pm\) have period \(2\pi\) and the one-forms \(\sigma \pm\) in (B17) obey \(d\sigma_\pm = 2J^{(7)}_\pm\), so that \(\eta^{(7)}_\pm = d\psi_\pm + \sigma_\pm\). It is also useful to make manifest the \(CP^2\) that resides inside \(CP^3\), which is equipped with the complex projective coordinates \(\xi^i\), \(i = 1, 2\), that appear in (B10) and the metric (B9). This can be achieved by writing
\[
\begin{align*}
\mu^1 + i\mu^2 &= \frac{1}{\sqrt{1 + \xi^1 \bar{\xi}^1}} \cos \alpha e^{i(\psi_+ + \tau_\pm)} \xi^1, \\
\mu^3 + i\mu^4 &= \frac{1}{\sqrt{1 + \xi^2 \bar{\xi}^2}} \cos \alpha e^{i(\psi_+ + \tau_\pm)} \xi^2, \\
\mu^5 + i\mu^6 &= \frac{1}{\sqrt{1 + \xi^3 \bar{\xi}^3}} \cos \alpha e^{i(\psi_+ + \tau_\pm)} \xi^3, \\
\mu^7 + i\mu^8 &= \sin \alpha e^{i\psi_\pm},
\end{align*}
\] (B18)
where \(\tau_\pm\) are angles of period \(2\pi\). The metrics \(d^2(CP^3)\) and one-forms \(\sigma_\pm\) inside the round \(S^7\) metric (B17) can be written in terms of the coordinates (B18) as
\[
d^2(CP^3) = d\alpha^2 + \cos^2 \alpha d\tau_\pm + \sigma^2,
\] (B19)
and
\[
\sigma_\pm = \cos^2 \alpha (d\tau_\pm + \sigma),
\] (B20)
with \(d^2(CP^2)\) and \(\sigma\) respectively given by (B9) and (B11).

3. \(S^7\) as the sine-cone over a nearly Kähler \(S^6\)

A third and final set of intrinsic angles on \(S^7\) is better suited to describe the solutions with at least \(G_2\) symmetry. First split the \(\mu^4\), \(A = 1, \ldots, 8\), as \(\mu^A = (\mu^4, \mu^8)\), with \(I = 1, \ldots, 7\), and then let
\[
\mu^4 = \sin \beta \nu^I, \quad \mu^8 = \cos \beta
\] (B22)
where \(0 \leq \beta \leq \pi\), and \(\nu^I, I = 1, \ldots, 7\), define an \(S^6\) through the constraint \(\delta_{IJ} \nu^I \nu^J = 1\). In these coordinates, the round metric (B2) takes on the local sine-cone form
\[
d^2(S^7) = d\beta^2 + \sin^2 \beta d^2(S^6),
\] (B23)
where \(d^2(S^6) = \delta_{IJ} \nu^I \nu^J\) is the round, Einstein metric on \(S^6\) normalized so that the Ricci tensor equals five times the metric. This \(S^6\) is naturally endowed with the homogeneous nearly Kähler structure \((J, \Omega)\) inherited from the closed associative and co-associative forms,
\[
\psi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},
\] (B24)
\[
\nu = e^{1234} + e^{1256} + e^{3456} + e^{1367} + e^{1457} + e^{2357} - e^{2467},
\] (B25)
on the \(\mathbb{R}^7\) factor of \(\mathbb{R}^8 = \mathbb{R}^7 \times \mathbb{R}\) in which \(S^6\) is embedded:
\[
J = \frac{1}{2} \psi_{IJ} \nu^I \nu^J \wedge d\nu^K,
\]
\[
\Omega = \frac{1}{6} (\psi_{IKL} - i
\bar{\psi}_{IKL} \nu^I) \nu^J \wedge d\nu^K \wedge d\nu^L.
\] (B26)
The nearly Kähler forms are subject to
\[
J \wedge \Omega = 0, \quad \Omega \wedge \bar{\Omega} = -\frac{4i}{3} J \wedge J = -8i \text{vol}(S^6),
\] (B27)
and
\[
dJ = 3\text{Re} \Omega, \quad d\text{Im} \Omega = -2J \wedge J.
\] (B28)
It is also useful to note the following relations between the associative and co-associative forms \(\psi, \nu\) written in constrained \(\mathbb{R}^8\) coordinates \(\mu^A = (\mu^4, \mu^8)\), the \(S^7\) coordinate \(\beta\) in (B22), and the nearly Kähler forms (B26):

\footnote{The typography we use for the nearly Kähler forms on \(S^6\) differentiates them from the Calabi-Yau forms (A6) on \(\mathbb{R}^6\). For that reason, we omit labels (6) for the former. Similarly, we omit labels (7) for the associative and co-associative forms on \(\mathbb{R}^7\).}
\[
\frac{1}{2} \psi_{IKL} d\mu^I \wedge d\mu^K \wedge d\mu^L = -\sin^4 \beta \mathcal{J} \wedge d\beta,
\]
\[
\frac{1}{6} \psi_{IKM} d\mu^I \wedge d\mu^J \wedge d\mu^K = \sin^3 \beta \text{Re}\Omega + \sin^2 \beta \cos \beta \mathcal{J} \wedge d\beta,
\]
\[
\frac{1}{6} \psi_{IKL} d\mu^I \wedge d\mu^K \wedge d\mu^L = -\sin^4 \beta \text{Im}\Omega,
\]
\[
\frac{1}{24} \psi_{IKL} d\mu^I \wedge d\mu^J \wedge d\mu^K \wedge d\mu^L = \frac{1}{2} \sin^4 \beta \mathcal{J} \wedge \mathcal{J} + \sin^3 \beta \cos \beta \text{Im}\Omega \wedge d\beta.
\]

Finally, the following relations hold between the associative and co-associative forms on \( \mathbb{R}^8 = \mathbb{R}^7 \times \mathbb{R} \) and the Calabi-Yau forms \( \mathbb{R}^8 = \mathbb{R}^6 \times \mathbb{R}^2 \):

\[
\frac{1}{2} \psi_{IKL} d\mu^I \wedge d\mu^K \wedge d\mu^L = J_{ij}^{(6)} \mu^i \wedge d\mu^j + \frac{1}{2} (J_{ik}^{(6)} \mu^j + \text{Re} \Omega_{ijk} \mu^j) d\mu^i \wedge d\mu^k,
\]
\[
\frac{1}{6} \psi_{IKM} d\mu^I \wedge d\mu^J \wedge d\mu^K = \frac{1}{6} \text{Re} \Omega_{ijk} d\mu^i \wedge d\mu^j \wedge d\mu^k + \frac{1}{2} J_{ij}^{(6)} d\mu^l \wedge d\mu^i \wedge d\mu^l + \frac{1}{2} J_{ik}^{(6)} d\mu^l \wedge d\mu^l \wedge d\mu^l,
\]
\[
\frac{1}{6} \psi_{IKL} d\mu^I \wedge d\mu^K \wedge d\mu^L = -\frac{1}{6} \text{Im} \Omega_{ijk} d\mu^i \wedge d\mu^j \wedge d\mu^k + \frac{1}{2} J_{ij}^{(6)} d\mu^l \wedge d\mu^l \wedge d\mu^l
\]
\[+ \frac{1}{2} \text{Im} \Omega_{ijk} d\mu^i \wedge d\mu^j \wedge d\mu^k.
\]

These expressions come handy to derive the \( G_2 \)-invariant consistent uplifting formulas of Sec. III B 3 from the general expressions of Sec. III A. They are also useful to rewrite the solutions (4.2)–(4.6) with at least \( G_2 \) symmetry in the form (D1)–(D7), in order to verify that they satisfy the equations of motion.

**APPENDIX C: CONSISTENCY OF THE MINIMAL \( \mathcal{N} = 2 \) TRUNCATION**

We have explicitly verified at the level of the \( D = 4 \) field equations that the restrictions (2.43)–(2.48) define a consistent truncation of the SU(3)-invariant theory (2.3) to minimal \( \mathcal{N} = 2 \) gauged supergravity (2.35). In turn, the consistency of the \( D = 11 \) embedding of the entire SU(3) sector described in Sec. III A guarantees the consistency of the new uplift of minimal \( \mathcal{N} = 2 \) supergravity given in Sec. III C. We have nevertheless checked consistency explicitly at the level of the Bianchi identity and the equation of motion of the \( D = 11 \) four-form

\[
\hat{F}^{(4)} = d\hat{A}^{(3)},
\]

\[
d\hat{F}^{(4)} = 0, \quad d^\ast \hat{F}^{(4)} + \frac{1}{2} \hat{F}^{(4)} \wedge \hat{F}^{(4)} = 0. \tag{C1}
\]

The configuration (3.27), (3.30) does solve the \( D = 11 \) field equations (C1) provided the Bianchi identity and the Maxwell equation for the \( D = 4 \) graviphoton,

\[
d\hat{F} = 0, \quad d^\ast \hat{F} = 0, \tag{C2}
\]

are imposed.

It is straightforward to see that the \( D = 11 \) Bianchi identity is satisfied. HITting (3.30) with the differential operator we obtain, after using (C1) and the algebraic and differential conditions for the local five-dimensional Sasaki-Einstein structure (3.26) [that is, (B5), (B6) written for the primed forms \( \eta' \), \( J' \) and \( \Omega' \)],

\[
d\hat{F}^{(4)} = g^3 \left[ \cos^2 \alpha \left( 7 - 10 \cos 2\alpha + \cos 4\alpha \right) \right] \frac{d\alpha}{(1 + 2 \sin^2 \alpha)^2} \wedge \left( \frac{g}{2} \hat{F} \wedge \text{Re} \Omega' + 3 d\eta' \wedge \text{Im} \Omega' \wedge \eta' \right)
\]
\[+ 6 \partial \alpha \left( \sin \cos^3 \alpha \right) \frac{d\alpha}{1 + 2 \sin^2 \alpha} \wedge d\eta' \wedge \eta' \wedge \text{Im} \Omega' + \frac{3 \sin \cos^3 \alpha}{1 + 2 \sin^2 \alpha} \hat{F} \wedge \eta' \wedge \text{Im} \Omega'
\]
\[+ \frac{g^2}{2 \sqrt{3}} \left[ 2 \partial \alpha \left( \sin \cos^3 \alpha \right) \right] d\alpha \wedge \hat{F} \wedge \text{Re} \Omega' - \frac{6 \sin \cos^3 \alpha}{1 + 2 \sin^2 \alpha} \hat{F} \wedge \text{Im} \Omega' \wedge \eta'. \tag{C3}
\]

Terms with the same form dependence cancel each other, thus leading to \( d\hat{F}^{(4)} = 0 \).
Moving on to the equation of motion, we find it useful for the calculation to introduce the obvious frame that can be read off from (3.27),

\[ \hat{e}^\mu = \frac{(1 + 2\sin^2\alpha)^{1/3}}{2^{1/3}\sqrt{3}} \hat{e}^\nu, \quad \text{with} \quad \hat{e}^\mu \text{ a vierbein for } ds_4^2, \]

\[ \hat{e}^\rho = \frac{2^{1/6}\cos \alpha}{g(1 + 2\sin^2\alpha)^{1/6}} e^\rho, \quad \text{with} \quad e^\rho \text{ a vierbein for } ds^2(\mathbb{CP}^2), \]

\[ \hat{e}^8 = \frac{2^{1/6}(1 + 2\sin^2\alpha)^{1/3}}{\sqrt{3}g} d\alpha, \]

\[ \hat{e}^9 = \frac{2^{1/6}\sqrt{3} \sin \alpha \cos \alpha(1 + 2\sin^2\alpha)^{1/3}}{g(1 + 8\sin^4\alpha)^{1/2}} \eta', \]

\[ \hat{e}^{10} = \frac{(1 + 8\sin^4\alpha)^{1/2}}{2^{1/3}\sqrt{3}g(1 + 2\sin^2\alpha)^{2/3}} \left( D\eta' - \frac{3\cos^2\alpha}{1 + 8\sin^4\alpha} \eta' \right), \]

with \( \alpha = 0, 1, 2, 3 \) and \( p = 4, 5, 6, 7 \). Using this frame, the Hodge dual of \( \hat{F}_{(4)} \) reads

\[ \hat{\hat{F}}_{(4)} = -\frac{3^{3/2}\cos^4\alpha}{g(1 + 2\sin^2\alpha)^2} \hat{e}^{8910} \wedge J' \wedge J' - \frac{(1 + 2\sin^2\alpha)^{2/3}\cos^2\alpha}{2^{1/6}\cdot 3^{3/2} g} \text{vol}_4 \wedge \hat{e}^8 \wedge \text{Im} \Omega' \]

\[ + \frac{\cos^2\alpha(7 - 10\cos 2\alpha + \cos 4\alpha)}{2^{7/6}\cdot 3^{3/2}g(1 + 2\sin^2\alpha)^{1/3}(1 + 8\sin^4\alpha)^{1/2}} \text{vol}_4 \wedge \hat{e}^9 \wedge \text{Re} \Omega' \]

\[ - \frac{\cos^3\alpha(7 - 10\cos 2\alpha + \cos 4\alpha)}{3^{3/2}\cdot 2^{5/6}\cdot 3^{3/2} g \sin \alpha(1 + 2\sin^2\alpha)^{1/3}(1 + 8\sin^4\alpha)^{1/2}} \text{vol}_4 \wedge \hat{e}^{10} \wedge \text{Re} \Omega' \]

\[ + \frac{\sin \alpha \cos^3\alpha}{\sqrt{3}g^2(1 + 2\sin^2\alpha)} \hat{F} \wedge \hat{e}^{8910} \wedge \text{Re} \Omega' - \frac{\cos^2\alpha}{2\sqrt{3}g^2} \hat{F} \wedge \hat{e}^{8910} \wedge J' \]

\[ + \frac{(1 + 8\sin^4\alpha)^{1/2}\cos^4\alpha}{2^{5/6}\cdot 3^{3/2} g^4(1 + 2\sin^2\alpha)^{3/3}} \hat{F} \wedge \hat{e}^{10} \wedge J' \wedge J', \]

where \( \hat{e}^{8910} = \hat{e}^8 \wedge \hat{e}^9 \wedge \hat{e}^{10} \). Computing the differential of (C5) with the help of the Sasaki-Einstein conditions satisfied by \( \eta', J' \) and \( \Omega' \), as well as \( \hat{F}_{(4)} \wedge \hat{F}_{(4)} \) from (3.30) and putting everything together, we find that the \( D = 11 \) equation of motion in (C1) is indeed satisfied on the \( D = 4 \) field equations (C2).

**APPENDIX D: D = 11 EQUATIONS OF MOTION ON THE AdS_4 SOLUTIONS**

The AdS_4 solutions that we brought to Sec. IV of the main text are obtained from the consistent uplifting formulas of Sec. IIIA by turning off the relevant tensor hierarchy fields, fixing the \( D = 4 \) scalars to the vevs recorded in Table II, and fixing the \( \mathbb{R}^8 \) embedding coordinates \( \mu^A, A = 1, \ldots, 8 \), in terms of various sets of intrinsic angles on \( S^7 \) discussed in Appendix B. The particular choice of intrinsic coordinates for each solution was made on a case-by-case basis, as specific sets of coordinates are more suitable than others to highlight the specific symmetry of a solution. While this is obviously the best approach for the sake of presentation, it is definitely inconvenient to check the \( D = 11 \) equations of motion, as one would also need to proceed on a case-by-case basis for each solution.

In order to check that the \( D = 11 \) equations of motion hold it is more convenient to proceed differently. First, leave the \( D = 4 \) scalars as temporarily unfixed constants, and make a choice of intrinsic \( S^7 \) coordinates (regardless of whether they would be well adapted to specific sectors). For this purpose, we have chosen the intrinsic coordinates (B1). The \( D = 11 \) metric and four-form then get expanded in terms of the global five-dimensional Sasaki-Einstein structure \( \eta^{(5)}, J^{(5)}, \Omega^{(5)} \) specified in Appendix B, with coefficients that depend on the \( D = 4 \) scalars along with the \( S^7 \) angles \( \psi \) and \( \psi ' \). Second, plug these expressions into the \( D = 11 \) field equations (C1) and obtain, with the help of the Sasaki-Einstein relations (B5), (B6), the set of equations that the coefficients must obey for the \( D = 11 \) equations to hold. Finally, verify that these equations are satisfied when the \( D = 4 \) scalars are fixed to the critical points recorded in Table II.
Proceeding this way, we find that the \( D = 11 \) metric (3.4) can be written in terms of the intrinsic angles (B1) as

\[
\begin{align*}
    ds_{11}^2 &= \Delta^{-1} ds_{4}^2 + ds_{7}^2, \\
    ds_{4}^2 &= G_5 da^2 + G_7 d\psi^2 + 2G_6 da d\psi + G_4 ds^2(\mathbb{CP}^2) + (G_3 + G_4)(\eta^{(5)})^2 - 2(G_5 da + G_2 d\psi)\eta^{(5)},
\end{align*}
\]

where both the warp factor,

\[
\Delta^{-1} = e^{-\frac{\varphi}{3}} \Delta^{2/3},
\]

given by \( \Delta_1 \) in (3.3) with (B1), and the coefficients of the internal metric \( ds_{7}^2 \) depend on the \( S^7 \) angles \( \alpha, \psi \) and the \( D = 4 \) scalars:

\[
\begin{align*}
    G_1 &= \frac{\Delta^2}{g^2} \left[ -\frac{1}{2} e^{-2\varphi} \sin \alpha \cos^2 \alpha (X - Y)(2ae^{4\varphi} \cos 2\psi - \sin 2\psi(-Y^2 - Z^2 + e^{4\varphi})) \right], \\
    G_2 &= \frac{\Delta^2}{g^2} \left[ e^{-2\varphi} \sin^2 \alpha \cos^2 \alpha (X - Y)(ae^{4\varphi} \sin 2\psi + \sin^2 \psi(Y^2 + Z^2) + e^{4\varphi} \cos^2 \psi) \right], \\
    G_3 &= \frac{\Delta^2}{g^2} \left[ Y \cos^4 \alpha (Y - X) \right], \\
    G_4 &= \frac{\Delta^2}{g^2} \left[ X^2 \sin^2 \alpha \cos^2 \alpha (ae^{4\varphi} \sin 2\psi + \sin^2 \psi(Y^2 + Z^2) + e^{4\varphi} \cos^2 \psi) + XY \cos^4 \alpha \right], \\
    G_5 &= \frac{\Delta^2}{g^2} \left[ Y X \sin^2 \alpha \cos^2 \alpha - \frac{1}{64} \sin^2(2\alpha)(e^{2\varphi} (\zeta^2 + \bar{\zeta}^2) + 4)(e^{2\varphi} (\zeta^2 + \bar{\zeta}^2) - 4e^{2\varphi} \chi^2) \\
    &\quad + X^2 \sin^4 \alpha e^{-2(\varphi + \varphi)} (ae^{4\varphi} \sin 2\psi + \sin^2 \psi(Y^2 + Z^2) + e^{4\varphi} \cos^2 \psi) \\
    &\quad + e^{-4\varphi} \cos^2 \alpha [-2ae^{4\varphi} \sin \psi \cos \psi + \sin^2 \psi(Y^2 + Z^2) + e^{4\varphi} \sin^2 \psi] \\
    &\quad \times \sin^2 \alpha (ae^{4\varphi} \sin 2\psi + \sin^2 \psi(Y^2 + Z^2) + e^{4\varphi} \cos^2 \psi) + \cos^2 \alpha e^{2(\varphi + \varphi)} \right], \\
    G_6 &= \frac{\Delta^2}{g^2} \left[ e^{-4\varphi} \sin \alpha \cos \alpha (-ae^{4\varphi} \cos 2\psi + \sin \psi \cos \psi(-Y^2 - Z^2 + e^{4\varphi})) \right] \\
    &\quad \times \sin^2 \alpha (ae^{4\varphi} \sin 2\psi + \sin^2 \psi(Y^2 + Z^2) + e^{4\varphi} \cos^2 \psi) + \cos^2 \alpha e^{2(\varphi + \varphi)}], \\
    G_7 &= \frac{\Delta^2}{g^2} \left[ e^{-4\varphi} \sin^2 \alpha (ae^{4\varphi} \sin 2\psi + \sin^2 \psi(Y^2 + Z^2) + e^{4\varphi} \cos^2 \psi) \right] \\
    &\quad \times \sin^2 \alpha (ae^{4\varphi} \sin 2\psi + \sin^2 \psi(Y^2 + Z^2) + e^{4\varphi} \cos^2 \psi) + \cos^2 \alpha e^{2(\varphi + \varphi)}].
\end{align*}
\]

Turning off the \( D = 4 \) tensor hierarchy fields (except for the local three-form \( C_{FR} \equiv C^1 = C^7 = C^{88} \) whose role is merely to serve as a local potential for the Freund-Rubin term) in the three form (3.6), its pull-back on \( S^7 \) induced by (B1) reads

\[
\begin{align*}
    \hat{A}_{(3)} &= (L_2 da + L_3 d\psi) \wedge J^{(5)} + (L_4 da + L_5 d\psi) \wedge \text{Re} \Omega^{(5)} + (L_6 da + L_7 d\psi) \wedge \text{Im} \Omega^{(5)} \\
    &\quad + (L_8 \text{Im} \Omega^{(5)} + L_9 \text{Re} \Omega^{(5)} + L_{10} J^{(5)}) \wedge \eta^{(5)} + L_1 da \wedge d\psi \wedge \eta^{(5)} + C_{FR}. 
\end{align*}
\]

The coefficients here are given by
\[ L_1 = \frac{\Delta^3}{8g} \left( \frac{1}{2} \chi \sin acos^2 \alpha \cos^{\psi - 4\phi} \right) \left[ \sin \alpha \sin 2\alpha (X - Y)e^{2(\psi + \phi)}(e^{2\phi}X^2 - Y^2 + 1)(ae^{4\phi} \sin 2\psi + \sin^2\psi(Y^2 + Z^2) + e^{4\phi} \cos^2\psi) \\
- 2(X^2 \sin^2\alpha \cos^{4\phi} \sin 2\psi + \sin^2\psi(Y^2 + Z^2) + e^{4\phi} \cos^2\psi) + Y^2 \cos^2\alpha e^{2(\psi + \phi)} \right] \\
\times \left( \cos \alpha e^{2(\psi + \phi)} \sin \alpha (\sin 2\psi(Y^2 + Z^2) + Z e^{2\phi} \sin 2\psi + e^{4\phi} \cos^2\psi) \right), \]

\[ L_2 = \frac{\Delta^3}{g} \left[ -\chi e^{-\psi - 4\phi} X \sin acos^3\alpha (\sin \psi \cos \psi (Y^2 - Z^2 + e^{4\phi}) - Z e^{2\phi} \cos 2\psi) \right] \]

\[ L_3 = -\frac{\tan \alpha \sin \psi (Y^2 + Z^2 + 2Z e^{2\phi} \cos \psi + e^{4\phi} \cos^2 \psi)}{Z e^{2\phi} \cos 2\psi - \sin \psi \cos \psi (Y^2 - Z^2 + e^{4\phi})} L_2, \]

\[ L_4 = \frac{\Delta^3}{g} \left( \frac{1}{2} X \cos^2 \alpha \cos^{3\psi - 2\phi} \right)[X \sin^2 \alpha (ae^{4\phi} \sin 2\psi + \sin^2\psi(Y^2 + Z^2) + e^{4\phi} \cos^2\psi) + Y \cos^2 \alpha e^{2(\psi + \phi)} \]

\[ \times [X \sin^2 \alpha (\zeta e^{2\phi} \cos \psi + \sin \psi(\zeta Y + \zeta Z)) + e^{2\phi} \cos^2 \alpha (\zeta e^{2\phi} \sin \psi + \cos \psi(\zeta Y - \zeta Z))], \]

\[ L_5 = -\frac{e^{2\phi} (\zeta e^{2\phi} \cos \psi + \sin \psi(\zeta Z - \zeta Y))}{X (\sin 2\psi (Y^2 - Z^2 + e^{4\phi}) - 2Z e^{2\phi} \cos 2\psi)} L_2, \]

\[ L_6 = \frac{2}{\sin 2\alpha} \left( e^{-2\phi} X \sin^2 \alpha + \frac{\cos^2 \alpha (\cos \psi(\zeta Y + \zeta Z) - \zeta e^{2\phi} \sin \psi)}{\zeta e^{2\phi} \cos \psi + \sin \psi(\zeta Z - \zeta Y)} \right) L_5, \]

\[ L_7 = \frac{-\zeta e^{2\phi} \cos \psi + \zeta Y \sin \psi + \zeta Z \sin \psi}{\zeta e^{2\phi} \cos \psi - \zeta Y \sin \psi + \zeta Z \sin \psi} L_5, \]

\[ L_8 = -e^{-2\phi} X L_7, \]

\[ L_9 = -e^{-2\phi} X L_5, \]

\[ L_{10} = \frac{\Delta^3}{g} (-e^{\psi} X \cos^2 \alpha) [X^2 \sin^2 \alpha \cos^2 \alpha e^{-2(\psi + \phi)}(ae^{4\phi} \sin 2\psi + \sin^2\psi(Y^2 + Z^2) + e^{4\phi} \cos^2\psi) + XY \cos^2 \alpha]. \quad (D5) \]

Finally, the \( D = 11 \) four-form \( \hat{F}_4 = d\hat{A}_{(3)} \) is

\[ \hat{F}_4 = U \text{vol}_4 + da \wedge d\psi \wedge (f_1 J^{(5)} + f_2 \text{Re} \Omega^{(5)} + f_3 \text{Im} \Omega^{(5)}) + f_4 J^{(5)} \wedge J^{(5)} \]

\[ + [(f_4 da + f_5 d\psi) \wedge \text{Re} \Omega^{(5)} + (f_6 da + f_7 d\psi) \wedge \text{Im} \Omega^{(5)} + (f_8 da + f_9 d\psi) \wedge J^{(5)}] \wedge \eta^{(5)}, \quad (D6) \]

where the Freund Rubin term is given by \( U \text{vol}_4 = H_{(4)}^{\mu} \mu^4 + H_{(4)}^{\nu} \mu^\nu \) evaluated on \( B_1 \) and on the \( D = 4 \) dualization conditions \((2.18)\). The functional coefficients in \( (D6) \) can be written in terms of the coefficients \( (D5) \) of the three form \( (D4) \) as

\[ f_1 = 2L_1 + \partial_\alpha L_3 - \partial_\psi L_2, \quad f_6 = 3L_4 + \partial_\alpha L_8, \]

\[ f_2 = \partial_\alpha L_5 - \partial_\psi L_4, \quad f_7 = 3L_5 + \partial_\psi L_8, \]

\[ f_3 = \partial_\alpha L_7 - \partial_\psi L_6, \quad f_8 = \partial_\alpha L_{10}, \]

\[ f_4 = -3L_6 + \partial_\psi L_9, \quad f_9 = \partial_\alpha L_{10}, \]

\[ f_5 = -3L_7 + \partial_\psi L_9, \quad f_{10} = 2L_{10}. \quad (D7) \]

The Bianchi identity \( d\hat{F}_4 = 0 \) amounts to verifying the following relations:

\[ 3f_3 + \partial_\alpha f_5 - \partial_\psi f_4 = 0, \quad -3f_2 + \partial_\alpha f_7 - \partial_\psi f_6 = 0, \]

\[ \partial_\alpha f_{10} - 2f_8 = 0, \quad \partial_\alpha f_9 - \partial_\psi f_8 = 0, \quad \partial_\psi f_{10} - 2f_9 = 0. \quad (D8) \]

Of course, these conditions are automatically satisfied by construction for all values of the \( D = 4 \) scalars upon using \( (D7) \).
We next compute the Hodge dual of $\tilde{F}_{(4)}$ given in (D6) with respect to the $D = 11$ metric (D1). We obtain

$$
\tilde{F}_{(4)} = \Delta^2 \text{vol}_7 + \Delta^{-2} \text{vol}_4 \wedge [(p_1 \rho d\alpha + p_2 \rho d\psi) \wedge J^{(5)} + (p_4 \rho d\alpha + p_5 \rho d\psi) \wedge \text{Re} \Omega^{(5)} + (p_8 \rho d\alpha + p_9 \rho d\psi) \wedge \text{Im} \Omega^{(5)}$$

$$
+ (p_{10} \rho d\alpha + p_{11} \rho d\psi) \wedge \eta^{(5)}],
$$

with coefficients

$$
p_1 = \frac{1}{\Delta G_V} [f_1 G_4 - f_0 G_5 + f_5 G_6], \quad p_6 = \frac{1}{\Delta^2 G_V} [f_5 G_1 - f_4 G_2 - f_2 (G_3 + G_4)],$$

$$
p_2 = \frac{1}{\Delta^2 G_V} [f_1 G_2 - f_0 G_6 + f_6 G_7], \quad p_7 = \frac{1}{\Delta^2 G_V} [f_3 G_1 - f_7 G_5 + f_6 G_6],$$

$$
p_3 = \frac{1}{\Delta^2 G_V} [f_9 G_1 - f_8 G_2 - f_1 (G_3 + G_4)], \quad p_8 = \frac{1}{\Delta^2 G_V} [f_3 G_2 - f_7 G_6 + f_6 G_7],$$

$$
p_4 = \frac{1}{\Delta^2 G_V} [f_2 G_4 - f_3 G_5 + f_6 G_6], \quad p_9 = \frac{1}{\Delta^2 G_V} [f_7 G_1 - f_6 G_2 - f_3 (G_3 + G_4)],$$

$$
p_5 = \frac{1}{\Delta^2 G_V} [f_2 G_2 - f_3 G_6 + f_6 G_7], \quad p_{10} = -\frac{2}{\Delta^2} G_V G_V^2 f_{10}.
$$

Here,

$$
G_V = \sqrt{-G_3 G_7^3 + 2G_2 G_6 G_4 - G_3 G_6^2 - G_3 G_5 G_7 - G_3 G_5 G_6 G_7 + G_3 G_5 G_6 G_7 + G_3 G_5 G_6 G_7}
$$

is related to the volume element corresponding to the internal metric $d\xi^2$ in (D1). With these definitions, the equation of motion in (C1) for the $D = 11$ four-form becomes equivalent to the following conditions:

$$
U f_1 + \partial_\alpha p_2 - \partial_\psi p_1 + 2p_{10} = 0, \quad U f_6 + \partial_\alpha p_9 + 3p_4 = 0,
$$

$$
U f_2 + \partial_\alpha p_3 - \partial_\psi p_4 = 0, \quad U f_7 + \partial_\alpha p_9 + 3p_5 = 0,
$$

$$
U f_3 + \partial_\alpha p_8 - \partial_\psi p_7 = 0, \quad U f_8 + \partial_\alpha p_3 = 0,
$$

$$
U f_4 + \partial_\alpha p_6 - 3p_7 = 0, \quad U f_9 + \partial_\psi p_3 = 0,
$$

$$
U f_5 + \partial_\psi p_6 - 3p_8 = 0, \quad U f_{10} + 2p_3 = 0.
$$

We have verified that equations (D12) hold when the $D = 4$ scalars are evaluated at any of the critical points collected in Table II. We have also checked that all the metric and four-forms for the explicit AdS$_4$ solutions written in Sec. IV can be brought to the form (D1)–(D7), with the help of the relations given in Appendix B. Thus, the explicit AdS$_4$ configurations of Sec. IV do indeed solve the $D = 11$ field equations (C1).
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