Constraints in polysymplectic (covariant) Hamiltonian formalism

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Abstract. In the framework of polysymplectic Hamiltonian formalism, degenerate Lagrangian field systems are described as multi-Hamiltonian systems with Lagrangian constraints. The physically relevant case of degenerate quadratic Lagrangians is analyzed in detail, and the Koszul–Tate resolution of Lagrangian constraints is constructed in an explicit form. The particular case of Hamiltonian mechanics with time-dependent constraints is studied.

1 Introduction

Let \( Y \to X \) be a smooth fibre bundle of a classical field theory. We consider first order Lagrangian field systems whose configuration space is the first order jet manifold \( J^1Y \) of sections of \( Y \to X \). Polysymplectic Hamiltonian formalism enables us to describe these systems as constraint Hamiltonian systems on the Legendre bundle

\[
\Pi = \bigwedge^n T^*X \otimes V^*Y \otimes TX
\]

\[1, 3, 17, 19\]. Given fibred coordinates \((x^\lambda, y^i)\) on \( Y \), the Legendre bundle \( \Pi \) is provided with the holonomic coordinates \((x^\lambda, y^i, p^\lambda_i)\). Every Lagrangian

\[
L = L\omega : J^1Y \to \bigwedge^n T^*X, \quad \omega = dx^1 \wedge \cdots \wedge dx^n, \quad n = \dim X,
\]

on \( J^1Y \) yields the Legendre map

\[
\hat{L} : J^1Y \to \Pi, \quad p^\lambda_i \circ \hat{L} = \pi^\lambda_i = \partial_i^\lambda L.
\]

Lagrangian formalism on \( J^1Y \) and polysymplectic Hamiltonian formalism on \( \Pi \) are equivalent if a Lagrangian \( L \) is hyperregular, i.e., \( \hat{L} \) is a diffeomorphism. In Part I of the work, we study the case of almost regular Lagrangians \( L \) when: (i) the Lagrangian constraint space \( N_L = \hat{L}(J^1Y) \) is a closed imbedded subbundle \( i_N : N_L \to \Pi \) of the Legendre bundle \( \Pi \to Y \) and (ii) the Legendre map

\[
\hat{L} : J^1Y \to N_L
\]
is a fibred manifold with connected fibres. Lagrangians of the most of field models are of this type. From the mathematical viewpoint, this notion of degeneracy is particular appropriate in order to study Lagrangian constraints in polysymplectic Hamiltonian formalism. In this case, there are comprehensive relations between Euler–Lagrange and Cartan equations in Lagrangian formalism, Hamilton–De Donder equations in multisymplectic Hamiltonian formalism, covariant Hamilton equations and constrained Hamilton equations in polysymplectic Hamiltonian formalism (see Theorems 8, 9 and 12 below). The main peculiarity of these relations lies in the fact that a set of Hamiltonian forms is associated to a degenerate Lagrangian.

In Part II, we provide the detailed analysis of systems with degenerate quadratic Lagrangians, appropriate for application to many physical models. Such a Lagrangian $L$ yields splittings of the affine jet bundle $J^1Y \rightarrow Y$ and the Legendre bundle $\Pi \rightarrow Y$ (see Theorem 15 below). The corresponding projection operators enable us to construct the Koszul–Tate resolution of the Lagrangian constraints $N_L$ in an explicit form.

If $X = \mathbb{R}$, polysymplectic Hamiltonian formalism provides the adequate formulation of Hamiltonian time-dependent mechanics [10, 20]. Part III of the work is devoted to mechanical systems with time-dependent constraints. The key point lies in the fact that, in time-dependent mechanics, the canonical Poisson structure does not provide dynamic equations and the Poisson bracket of constraints with a Hamiltonian is ill-defined [12].

**PART I. Lagrangian constraints**

All maps throughout are smooth, while manifolds are real, finite-dimensional, Hausdorff, second-countable and connected. The $s$-order jet manifold $J^sY$ of a fibre bundle $Y \rightarrow X$ is endowed with the adapted coordinates $(x^\lambda, y^i_\Lambda)$, $0 \leq |\Lambda| \leq s$, where $\Lambda$ is a multi-index $(\lambda_k...\lambda_1)$, $|\Lambda| = k$. We denote by

$$h_0 : \phi_\lambda dx^\lambda + \phi_\Lambda^i dy^i_\Lambda \mapsto (\phi_\lambda + \phi_\Lambda^i y^i_\lambda + \Lambda) dx^\lambda$$

the exterior algebra homomorphism which sends exterior forms on $J^sY$ onto horizontal forms on $J^{s+1}Y \rightarrow X$ and vanishes on contact forms $\theta^i_A = dy_i^A - y^i_{\lambda+\Lambda} dx^\lambda$. Let $d_\lambda = \partial_\lambda + y^i_{\lambda+\Lambda} \partial_i^\Lambda$ be the total derivative and $d_H \phi = dx^\lambda \wedge d_\lambda \phi$ the horizontal differential such that $h_0 \circ d = d_H \circ h_0$. A connection on a fibre bundle $Y \rightarrow X$ is regarded as a global section

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)$$

of the affine jet bundle $\pi^0_1 : J^1Y \rightarrow Y$. Sections of the underlying vector bundle $T^*X \otimes VY \rightarrow Y$ are called soldering forms.
2 Lagrangian dynamics

This Section and Section 3 summarize the basic notions of Lagrangian and polysymplectic Hamiltonian formalisms (see [4, 5, 19] for a detailed exposition).

Given a Lagrangian \(L\) and its Lepagean equivalent \(H_L\), the first variational formula of the calculus of variations provides the canonical decomposition of the Lie derivative of \(L\) along a projectable vector field \(u\) on \(Y\):

\[
L_{\gamma^u}L = u_V | \mathcal{E}_L + dH_0 (u | H_L),
\]

where \(u_V = (u | \theta^i) \partial_i\) and

\[
\mathcal{E}_L = (\partial_i - d_0 \partial^\lambda_i) \mathcal{L} \theta^i \wedge \omega : J^2 Y \rightarrow T^* Y \wedge (\wedge^n T^* X)
\]

is the Euler–Lagrange operator. The kernel of \(\mathcal{E}_L\) is the Euler–Lagrange equations

\[
(\partial_i - d_0 \partial^\lambda_i) \mathcal{L} = 0.
\]

We will restrict our consideration to the Poincaré–Cartan form

\[
H_L = L + \pi^\lambda_i \theta^i \wedge \omega_\lambda, \quad \omega_\lambda = \partial_\lambda | \omega.
\]

It is a Lagrangian counterpart of Hamiltonian forms in polysymplectic Hamiltonian formalism. Being a Lepagean equivalent of the Lagrangian \(L = h_0 (H_L)\) on \(J^1 Y\), this is also a Lepagean equivalent of the Lagrangian

\[
\mathcal{T} = h_0 (H_L) = (\mathcal{L} + (\tilde{g}_\lambda^i - y_\lambda^i) \pi^\lambda_i) \omega, \quad \tilde{h}_0 (dy^i) = \tilde{g}_\lambda^i dx^\lambda,
\]

on the repeated jet manifold \(J^1 J^1 Y\) coordinated by \((x^\lambda, y^i, \tilde{g}_\lambda^i, y_\lambda^i, \tilde{y}_\lambda^i, y_{\lambda \mu}^i)\). The Euler–Lagrange operator for \(\mathcal{T}\) reads

\[
\mathcal{E}_T : J^1 J^1 Y \rightarrow T^* J^1 Y \wedge (\wedge^n T^* X),
\]

\[
\mathcal{E}_T = [(\partial_i \mathcal{L} - \tilde{d}_\lambda \pi^\lambda_i + \partial_i \pi^\lambda_j (\tilde{g}_\lambda^j - y_\lambda^j)) dy^i + \partial_i \pi^\mu_j (\tilde{g}_\mu^i - y_\mu^i) dy^j] \wedge \omega,
\]

where \(\tilde{d}_\lambda = \partial_\lambda + \tilde{g}_\lambda^i \partial_i + y_{\lambda \mu}^i \partial_\mu\). Its kernel \(\text{Ker} \mathcal{E}_T \subset J^1 J^1 Y\) is the Cartan equations

\[
\partial^\lambda_i \pi^\mu_j (\tilde{g}_\mu^i - y_\mu^i) = 0, \quad \partial_i \mathcal{L} - \tilde{d}_\lambda \pi^\lambda_i + (\tilde{g}_\lambda^j - y_\lambda^j) \partial_i \pi^\lambda_j = 0.
\]

On sections \(\pi : X \rightarrow J^1 Y\), the Cartan equations (\(\square\)) are equivalent to the condition

\[
\pi^* (u | dH_L) = 0
\]

for any vertical vector field \(u\) on \(J^1 Y \rightarrow X\). The Cartan equations are equivalent to the Euler–Lagrange equations on integrable sections \(\pi = J^1 s\) of \(J^1 Y \rightarrow X\).
The Poincaré–Cartan form $H_L$ (8) yields the Legendre morphism

$$\tilde{H}_L : J^1Y \rightarrow Z_Y, \quad (p_{\mu}^\nu, p) \circ \tilde{H}_L = (\pi_{\mu}^\nu, \mathcal{L} - \pi_{\nu}^\mu y_{\mu}),$$

of $J^1Y$ to the homogeneous Legendre bundle

$$Z_Y = T^*Y \wedge (\wedge^{n-1} T^*X)$$

(13)

equipped with the holonomic coordinates $(x^\lambda, y^i, p_i^\lambda, p)$. There is the 1-dimensional affine bundle

$$\pi_{Z\Pi} : Z_Y \rightarrow \Pi$$

(14)

modelled over the pull-back vector bundle $\Pi \times \wedge^n T^*X \rightarrow \Pi$. We have

$$\hat{L} = \pi_{Z\Pi} \circ \tilde{H}_L.$$  

(15)

Due to the monomorphism $Z_Y \hookrightarrow \wedge^n T^*Y$, the bundle $Z_Y$ is endowed with the pull-back

$$\Xi_Y = p\omega + p_i^\lambda dy^i \wedge \omega_\lambda$$

(16)

of the canonical form $\Theta$ on $\wedge^n T^*Y$ whose exterior differential $d\Theta$ is the $n$-multisymplectic form in the sense of Martin [15].

Let $Z_L = \tilde{H}_L(J^1Y)$ be an imbedded subbundle $i_L : Z_L \hookrightarrow Z_Y$ of $Z_Y \rightarrow Y$. It is provided with the pull-back De Donder form $\Xi_L = i_L^* \Xi_Y$. We have

$$H_L = \tilde{H}_L^* \Xi_L = \tilde{H}_L^* (i_L^* \Xi_Y).$$

(17)

By analogy with the Cartan equations (12), the Hamilton–De Donder equations for sections $\mathfrak{s}$ of $Z_L \rightarrow X$ are written as

$$\mathfrak{s}'(u|d\Xi_L) = 0$$

(18)

where $u$ is an arbitrary vertical vector field on $Z_L \rightarrow X$.

**Theorem 1.** Let the Legendre morphism $\tilde{H}_L$ be a submersion. Then a section $\mathfrak{s}$ of $J^1Y \rightarrow X$ is a solution of the Cartan equations (12) iff $\tilde{H}_L \circ \mathfrak{s}$ is a solution of the Hamilton–De Donder equations (18) [7].
3 Polysymplectic Hamiltonian dynamics

The canonical polysymplectic form $\Omega$, Hamiltonian connections and Hamiltonian forms are the main ingredients in the covariant Hamiltonian dynamics on the Legendre bundle

$$\pi_{IX} = \pi \circ \pi_{IY} : \Pi \to Y \to X.$$ Let us consider the canonical bundle monomorphism

$$\theta = -p^\lambda_i dy^i \wedge \omega \otimes \partial_\lambda : \Pi \hookrightarrow T^*Y \otimes TX.$$ (19)

The polysymplectic form on $\Pi$ is defined as a unique $TX$-valued $(n+2)$-form

$$\Omega = dp^\lambda_i \wedge dy^i \wedge \omega \otimes \partial_\lambda$$ (20)

such that the relation $\Omega \rfloor \phi = -d(\theta \rfloor \phi)$ holds for any exterior 1-form $\phi$ on $X$. A connection

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma^i_\lambda \partial_i + \gamma^i_\lambda \gamma^j_\mu \partial_j \partial_\mu)$$

on $\Pi \to X$ is called a Hamiltonian connection if the exterior form $\gamma \rfloor \Omega$ is closed. A Hamiltonian form $H$ on $\Pi$ is defined as the pull-back

$$H = h^*\Xi_Y = p^\lambda_i dy^i \wedge \omega_\lambda - \mathcal{H}_\omega$$ (21)

of the canonical form $\Xi_Y$ (14) by a section $h$ of the affine bundle (14). Hamiltonian forms on $\Pi$ constitute an affine space modelled over the linear space of horizontal densities $\tilde{H} = \tilde{\mathcal{H}}_\omega$ on $\Pi \to X$.

**Theorem 2.** [4]. For every Hamiltonian form $H$ (21), there exists an associated Hamiltonian connection such that

$$\gamma \rfloor \Omega = dH, \quad \gamma^i_\lambda = \partial_\lambda \mathcal{H}, \quad \gamma^i_\lambda = -\partial_i \mathcal{H}.$$ (22)

Conversely, for any Hamiltonian connection $\gamma$, there exists a local Hamiltonian form $H$ on a neighbourhood of any point $q \in \Pi$ such that the relations (22) hold.

For instance, every connection $\Gamma$ on $Y \to X$ defines the section

$$h_\Gamma : dy^i \mapsto \Gamma^i_\lambda dx^\lambda$$

of the affine bundle $Z_\gamma \to \Pi$ and the corresponding Hamiltonian form

$$H_\Gamma = h^*_\Gamma \Xi_Y = p^\lambda_i dy^i \wedge \omega_\lambda - p^\lambda_i \Gamma^i_\lambda \omega.$$ (23)
As a consequence, every Hamiltonian form \( H \) admits the decomposition
\[
H = H_\Gamma - \tilde{H}_\Gamma = p^\lambda_i dy^i \wedge \omega_\lambda - p^\lambda_i \Gamma^i_\lambda \omega - \tilde{H}_\Gamma \omega. \tag{24}
\]

Any bundle morphism
\[
\Phi = dx^\lambda \otimes (\partial_\lambda + \Phi^i_\lambda \partial_i) : \Pi \to J^1Y, \tag{25}
\]
called a Hamiltonian map, defines the Hamiltonian form
\[
H_\Phi = \Phi \th e = p^\lambda_i dy^i \wedge \omega_\lambda - p^\lambda_i \Phi^i_\lambda \omega. \tag{26}
\]
Every Hamiltonian form \( H \) yields the Hamiltonian map \( \tilde{H} \) such that \( y^i_\lambda \circ \tilde{H} = \partial_\lambda H \). A Hamiltonian form \( H \) is called degenerate if the Hamiltonian map \( \tilde{H} \) is degenerate.

A Hamiltonian form \( H \) on \( \Pi \) can be seen as the Poincaré–Cartan form of the Lagrangian
\[
L_H = (p^\lambda_i y^i_\lambda - H) \omega \tag{27}
\]
on the jet manifold \( J^1\Pi \). The Euler–Lagrange operator \( \mathcal{E}_H \) for \( L_H \), called the Hamilton operator for \( H \), is
\[
\mathcal{E}_H : J^1\Pi \to T^*\Pi \wedge (\wedge^n T^*X),
\mathcal{E}_H = [(y^i_\lambda - \partial_\lambda^i \mathcal{H})dp^\lambda_i - (p^\lambda_i + \partial_\lambda \mathcal{H})dy^i] \wedge \omega. \tag{28}
\]
Its kernel is the covariant Hamilton equations
\[
y^i_\lambda = \partial^i_\lambda \mathcal{H}, \tag{29a}
\]
\[
p^\lambda_i = -\partial_\lambda \mathcal{H}. \tag{29b}
\]
It is readily observed that all Hamiltonian connections \( \mathcal{E}_{H} \) associated with a Hamiltonian form \( H \) live in the kernel of the Hamilton operator \( \mathcal{E}_H \). Consequently, every integral section \( J^1r = \gamma \circ r \) of a Hamiltonian connection \( \gamma \) associated with a Hamiltonian form \( H \) is a solution of the Hamilton equations \( \mathcal{E}_{H} \). Similarly to the Cartan equations \( \mathcal{E}_{\gamma} \), the Hamilton equations \( \mathcal{E}_{H} \) are equivalent to the condition
\[
r^*(u \wedge dH) = 0 \tag{30}
\]
for any vertical vector field \( u \) on \( \Pi \to X \).

**Remark 1.** Lagrangians \( \mathcal{L}_H \) play an important role in the path integral quantization of covariant Hamiltonian field theories \([18]\).
4 Degenerate systems

Let us state the relations between Lagrangian and polysymplectic Hamiltonian formalisms when a Lagrangian is degenerate (see \cite{4, 5} for a detailed exposition).

Given a Lagrangian $L$, a Hamiltonian form $H$ is said to be associated with $L$ if the relations

$$\hat{L} \circ \hat{H} \circ \hat{L} = \hat{L},$$  

$$(31a)$$

$$H = H \hat{H} + \hat{H}^* L$$  

$$(31b)$$

hold. A glance at the relation (31a) shows that $\hat{L} \circ \hat{H}$ is a projection

$$p^h_i(q) = \partial^\mu_i L(x^\mu, y^i, \partial^j_\lambda H(q)), \quad q \in N_L,$$  

$$(32)$$

of $\Pi$ onto the Lagrangian constraint space $N_L = \hat{L}(J^1 Y)$. Accordingly, $\hat{H} \circ \hat{L}$ is a projection of $J^1 Y$ onto $\hat{H}(N_L)$. A Hamiltonian form is called weakly associated with a Lagrangian $L$ if the condition (31b) holds on the Lagrangian constraint space $N_L$.

Given a Lagrangian $L$, one can construct associated and weakly associated Hamiltonian forms as follows.

Proposition 3. \cite{4}. If a Hamiltonian map $\Phi$ (25) obeys the relation (31a), then the Hamiltonian form $H = H_\phi + \Phi^* L$ is weakly associated with the Lagrangian $L$. If $\Phi = \hat{H}$, then $H$ is associated with $L$.

Hamiltonian forms weakly associated with a Lagrangian $L$ have the following common property \cite{4}.

Proposition 4. Restricted to the Lagrangian constraint space $N_L$, any Hamiltonian form $H$ weakly associated with a Lagrangian $L$ coincides with the pull-back

$$H \mid_{N_L} = \hat{H}^* H_L \mid_{N_L}$$

of the Poincaré–Cartan form $H_L$ \cite{8} by the Hamiltonian map $\hat{H}$.

Note that the essential difference between associated and weakly associated Hamiltonian forms lies in the fact that, as follows from the relation (31b), associated Hamiltonian forms are necessarily degenerate outside a Lagrangian constraint space. Further, we study weakly associated Hamiltonian forms.

Let us restrict our consideration to almost regular Lagrangians. In this case, Proposition 3 leads to the following criterion of the existence of weakly associated Hamiltonian forms.
Proposition 5. A Hamiltonian form weakly associated with an almost regular Lagrangian $L$ exists iff the fibred manifold $J^1Y \to N_L$ admits a global section.

The following property of almost regular Lagrangians plays an important role in the sequel [4].

Lemma 6. The Poincaré–Cartan form $H_L$ for an almost regular Lagrangian $L$ is constant on connected fibres of the fibred manifold $J^1Y \to N_L$.

Then we come to the following assertion [4].

Proposition 7. All Hamiltonian forms weakly associated with an almost regular Lagrangian $L$ coincide with each other on the Lagrangian constraint space $N_L$, and the Poincaré–Cartan form $H_L$ (33) for $L$ is the pull-back

$$H_L = \hat{L}^*H$$

of any such a Hamiltonian form $H$.

Proposition 7 enables us to connect solutions of Euler–Lagrange and Cartan equations for an almost regular Lagrangian $L$ with solutions of Hamilton equations for Hamiltonian forms weakly associated with $L$.

Theorem 8. Let a section $r$ of $\Pi \to X$ be a solution of the Hamilton equations (29a) – (29b) for a Hamiltonian form $H$ weakly associated with an almost regular Lagrangian $L$. If $r$ lives in the Lagrangian constraint space $N_L$, the section $s = \pi_{1Y} \circ r$ of $Y \to X$ satisfies the Euler–Lagrange equations (7), while $\sigma = \tilde{H} \circ r$ obeys the Cartan equations (11).

Proof. The proof is based on the relation $\mathcal{L} = (J^1\hat{L})^*L_H$ where $\mathcal{L}$ is the Lagrangian (4) on $J^1J^1Y$ and $L_H$ is the Lagrangian (27) on $J^1\Pi$ [4, 8].

The converse assertion is more intricate [4, 9].

Theorem 9. Given an almost regular Lagrangian $L$, let a section $\sigma$ of the jet bundle $J^1Y \to X$ be a solution of the Cartan equations (11). Let $H$ be a Hamiltonian form weakly associated with $L$, and let $H$ satisfy the relation

$$\tilde{H} \circ \hat{L} \circ \sigma = J^1(\pi_0 \circ \sigma).$$

Then, the section $r = \hat{L} \circ \sigma$ of the Legendre bundle $\Pi \to X$ is a solution of the Hamilton equations (29a) – (29b) for $H$. 

8
Corollary 10. Theorems 8, 9 show that, if a solution $s$ of the Cartan equations provides a solution of the covariant Hamilton equations, its projection $\pi_1^0 \circ s$ onto $Y$ is a solution of the Euler–Lagrange equations.

Corollary 10 gives a solution of the so-called 'second order equation problem' in the case of almost regular Lagrangians.

We will say that a set of Hamiltonian forms $H$ weakly associated with an almost regular Lagrangian $L$ is complete if, for each solution $s$ of the Euler-Lagrange equations, there exists a solution $r$ of the Hamilton equations for a Hamiltonian form $H$ from this set such that $s = \pi_{HY} \circ r$. By virtue of Theorem 9, a set of weakly associated Hamiltonian forms is complete if, for every solution $s$ of the Euler–Lagrange equations for $L$, there is a Hamiltonian form $H$ from this set which fulfills the relation

$$\hat{H} \circ \hat{L} \circ J^1 s = J^1 s.$$ (35)

In accordance with Proposition 8, on an open neighbourhood in $\Pi$ of each point $q \in N_L$, there exists a complete set of local Hamiltonian forms weakly associated with an almost regular Lagrangian $L$.

One may conclude from Theorem 9 that the covariant Hamilton equations contain additional conditions in comparison with the Euler–Lagrange equations. In the case of an almost regular Lagrangian, one can introduce the constrained Hamilton equations which are weaker than the Hamilton equations restricted to the Lagrangian constraint space. Let the fibred manifold (4) admits a global section $\Psi$. We consider the pull-back

$$H_N = \Psi^* H_L,$$ (36)

called the constrained Hamiltonian form. By virtue of Lemma 6, it does not depend on the choice of a section $\Psi$ of the fibred manifold $J^1 Y \to N_L$, and so $H_L = \hat{L}^* H_N$. For sections $r$ of the fibre bundle $N_L \to X$, we can write the constrained Hamilton equations

$$r^*(u_N | dH_N) = 0,$$ (37)

where $u_N$ is an arbitrary vertical vector field on $N_L \to X$. These equations possess the following important properties [4, 5].

Theorem 11. For any Hamiltonian form $H$ weakly associated with an almost regular Lagrangian $L$, every solution $r$ of the Hamilton equations which lives in the Lagrangian constraint space $N_L$ is a solution of the constrained Hamilton equations (37).

Proof. The proof is based on the fact that such a Hamiltonian form $H$ defines the global section $\hat{H} \circ i_N$ of the fibred manifold (4), and $H_N = i_N^* H$. Then the constrained Hamilton equations can be written as

$$r^*(u_N | d\hat{i}_N^* H) = r^*(u_N | dH |_{N_L}) = 0.$$ (38)
They are obviously weaker than the Hamilton equations (30) restricted to $N_L$.

**Theorem 12.** The constrained Hamilton equations (37) are equivalent to the Hamilton–De Donder equations (18).

**Proof.** By virtue of the equality (15), the projection $\pi_{Z\Pi}$ (14) yields a surjection of $Z_L$ onto $N_L$. Given a section $\Psi$ of the fibred manifold (12), we have the morphism $\bar{H}_L \circ \Psi : N_L \to Z_L$. In accordance with Lemma (9), this is a surjection such that

$$\pi_{Z\Pi} \circ \bar{H}_L \circ \Psi = \text{Id}_{N_L}.$$

Hence, $\bar{H}_L \circ \Psi$ is a bundle isomorphism over $Y$ which is independent of the choice of a global section $\Psi$. Combining (17) and (36) gives $H_N = (\bar{H}_L \circ \Psi) \ast \Xi_L$ that leads to the desired equivalence.

Since $Z_L$ and $N_L$ are isomorphic, the Legendre morphism $H_L$ fulfills the conditions of Theorem 1. Then Theorems 1, 12 lead to the following assertion.

**Theorem 13.** Let $L$ be an almost regular Lagrangian such that the fibred manifold (4) has a global section. A section $s$ of the jet bundle $J^1Y \to X$ is a solution of the Cartan equations (12) iff $\hat{L} \circ s$ is a solution of the constrained Hamilton equations (37).

**Remark 2.** The constrained Hamiltonian form $H_N$ (36) defines the constrained Lagrangian

$$L_N = h_0(H_N) = (J^1i_N)^*L_H$$

(39)

on the jet manifold $J^1N_L$ of the fibre bundle $N_L \to X$. Then Theorem 13 follows from the relations

$$\bar{L} = (J^1\hat{L})^*L_N, \quad L_N = (J^1\Psi)^*\bar{L},$$

where $\bar{L}$ is the Lagrangian (9). The Euler–Lagrange equation for the constrained Lagrangian $L_N$ (39) are equivalent to the constrained Hamilton equations.

**PART II. Quadratic degenerate systems**

Let us study in detail the physically important case of almost regular quadratic Lagrangians.
5 Lagrangian constraints

Given a fibre bundle \( Y \to X \), let us consider a quadratic Lagrangian \( L \) which has the coordinate expression

\[
L = \frac{1}{2} a_{ij}^{\lambda \mu} y_{\lambda, j} y_{\mu, i} + b_i^\lambda y_i^\lambda + c,
\]

where \( a, b \) and \( c \) are local functions on \( Y \). This property is coordinate-independent. The associated Legendre map

\[
p_i^\lambda \circ \hat{L} = a_{ij}^{\lambda \mu} y_{\mu, j}^i + b_i^\lambda
\]

is an affine morphism over \( Y \). It defines the corresponding linear morphism

\[
\overline{L} : T^* X \otimes V Y \to \Pi, \quad p_i^\lambda \circ \overline{L} = a_{ij}^{\lambda \mu} \overline{y}_i^\mu,
\]

where \( \overline{y}_i^\mu \) are bundle coordinates on the vector bundle \( T^* X \otimes V Y \).

Let the Lagrangian \( L \) be almost regular, i.e., the matrix function \( a_{ij}^{\lambda \mu} \) is of constant rank. Then the Lagrangian constraint space \( N_L \) is an affine subbundle of the Legendre bundle \( \Pi \to Y \), modelled over the vector subbundle \( \overline{N_L} = \text{Im} \overline{L} \) of \( \Pi \to Y \). Hence, \( N_L \to Y \) has a global section \( r \). For the sake of simplicity, let us assume that \( r = 0 \) is the canonical zero section \( \hat{0}(Y) \) of \( \Pi \to Y \). Then \( \overline{N_L} = N_L \). Accordingly, the kernel of the Legendre map \( \Pi \) is an affine subbundle of the affine jet bundle \( J^1 Y \to Y \), modelled over the kernel of the linear morphism \( \overline{L} \).

Hence, there exists a connection

\[
\Gamma : Y \to \text{Ker} \hat{L} \subset J^1 Y,
\]

\[
a_{ij}^{\lambda \mu} \Gamma_{\mu}^j + b_i^\lambda = 0,
\]

on \( Y \to X \). Connections \( \Gamma \) constitute an affine space modelled over the linear space of soldering forms \( \phi \) on \( Y \to X \) satisfying the conditions

\[
a_{ij}^{\lambda \mu} \phi_{\mu}^j = 0, \quad \phi_{\lambda}^i b_i^\lambda = 0.
\]

Remark 3. In the general case of \( r \neq 0 \), one can consider connections \( \Gamma \) with values in \( \text{Ker} \hat{L}, \hat{L} \), i.e.,

\[
a_{ij}^{\lambda \mu} \Gamma_{\mu}^j + b_i^\lambda = r_i^\lambda,
\]

and replace \( b \) with \( b - r \) in all further constructions.

The following Theorem is the key point of our analysis of quadratic degenerate systems.
Theorem 14. There exists a linear bundle map

$$\sigma : \Pi \rightarrow T^*X \otimes VY, \quad \bar{y}_\lambda^i \circ \sigma = \sigma^{ij}_\mu p_{ij}^\mu,$$  \hspace{1cm} (46)

such that $T \circ \sigma \circ i_N = i_N$.

The map (46) is a solution of the pointwise algebraic equations

$$a \circ \sigma \circ a = a, \quad a^{ij}_\mu \sigma^{jk}_\alpha a^{\alpha \nu} = a^{\nu \nu}_b.$$  \hspace{1cm} (47)

Moreover, $\sigma \circ a = a \circ \sigma$ and $\sigma$ splits into the sum $\sigma = \sigma_0 + \sigma_1$ of two terms $\sigma_0$ and $\sigma_1$ satisfying the relations

$$\sigma_0 = \sigma_0 \circ a \circ \sigma_0, \quad a \circ \sigma_1 = \sigma_1 \circ a = 0.$$  \hspace{1cm} (48)

This splitting follows from the fact that the matrix $a$ in the Lagrangian (40) can be seen as a global section of constant rank of the tensor bundle

$$\wedge^2 T^*X \otimes [\wedge (TX \otimes V^*Y)] \rightarrow Y,$$

and there exists the bundle splitting

$$T^*X \otimes VY = \text{Ker } a \oplus E.$$  

Then $\sigma_0$ is a uniquely defined section of the fibre bundle $\wedge^n TX \otimes (\wedge^2 E) \rightarrow Y$, while $\sigma_1$ is an arbitrary sections of $\wedge^n TX \otimes (\wedge^2 \text{Ker } a) \rightarrow Y$.

Remark 4. In view of the relations (48), the above assumption that the Lagrangian constraint space $N_L \rightarrow Y$ admits a global zero section takes the form $b = (a \circ \sigma)b$.

Theorem 15. With the relations (44), (47) and (48), we obtain the decompositions

$$J^1Y = S(J^1Y) \oplus F(J^1Y) = \text{Ker } \tilde{L} \oplus \text{Im}(\sigma \circ \tilde{L}),$$  \hspace{1cm} (49a)

$$y^i_\lambda = S^i_\lambda + F^i_\lambda = [y^i_\lambda - \sigma^{jk}_\alpha (a^\alpha\mu_{kj} y^j_\mu + b^\alpha_k)] + [\sigma^{jk}_\alpha (a^\alpha\mu_{kj} y^j_\mu + b^\alpha_k)]$$  \hspace{1cm} (49b)

$$\Pi = R(\Pi) \oplus P(\Pi) = \text{Ker } \sigma_0 \oplus N_L,$$  \hspace{1cm} (50a)

$$p^\lambda_i = R^\lambda_i + P^\lambda_i = [p^\lambda_i - a^{ij}_\mu \sigma^{jk}_\alpha p^\alpha_k] + [a^{ij}_\mu \sigma^{jk}_\alpha p^\alpha_k].$$  \hspace{1cm} (50b)
With respect to the coordinates $\mathcal{S}_i^\lambda$, $\mathcal{F}_i^\lambda$ (49), and $\mathcal{R}_i^\lambda$, $\mathcal{P}_i^\lambda$ (50), the Lagrangian (40) reads
\[
\mathcal{L} = \frac{1}{2} a^{\lambda\mu}_{ij} \mathcal{F}_i^\lambda \mathcal{F}_j^\mu + c',
\] (51)
while the Lagrangian constraint space is given by the reducible constraints
\[
\mathcal{R}_i^\lambda = p_i^\lambda - a^{\lambda\mu}_{ij} \sigma_{\mu\alpha}^k p_k^\alpha = 0.
\] (52)

Note that, in gauge theory, we have the canonical splitting (49a) where $2\mathcal{F}$ is the strength tensor [4]. The Yang–Mills Lagrangian of gauge theory is exactly of the form (51) where $c' = 0$. The Lagrangian of Proca fields is also of the form (51) where $c'$ is the mass term. This is an example of a degenerate Lagrangian system without gauge symmetries.

Given the linear map $\sigma$ (46) and a connection $\Gamma$ (43), let us consider the affine Hamiltonian map
\[
\Phi_{\sigma\Gamma} = \Gamma \circ \Pi_Y + \sigma : \Pi \to J^1Y, \quad \Phi_{\sigma\Gamma}^i = \Gamma^i_\lambda + \sigma_{\lambda\mu}^k p_k^\mu.
\] (53)
It satisfies the relation (31a). Then the Hamiltonian form
\[
H_{\sigma\Gamma} = H_{\Phi_{\sigma\Gamma}} + \Phi_{\sigma\Gamma}^* \mathcal{L} = p_i^\lambda dy^i \wedge \omega_\lambda - [\Gamma^i_\lambda p_i^\lambda + \frac{1}{2} \sigma_{\lambda\mu}^k p_k^\mu p_j^\mu + \sigma_{\lambda\mu}^ij p_i^\lambda p_j^\mu - c'] \omega = (54)
\]
\[
(\mathcal{R}_i^\lambda + \mathcal{P}_i^\lambda) dy^i \wedge \omega_\lambda - [(\mathcal{R}_i^\lambda + \mathcal{P}_i^\lambda) \Gamma^i_\lambda + \frac{1}{2} \sigma_{\lambda\mu}^k p_i^\lambda \mathcal{P}_j^\mu + \sigma_{\lambda\mu}^ij \mathcal{R}_i^\lambda \mathcal{R}_j^\mu - c'] \omega,
\]
is weakly associated with the Lagrangian $L$ (40) in accordance with Proposition 3. The corresponding Lagrangian (27) reads
\[
L_H = [(\mathcal{R}_i^\lambda + \mathcal{P}_i^\lambda)(y_\lambda^i - \Gamma^i_\lambda) - \frac{1}{2} \sigma_{\lambda\mu}^k \mathcal{P}_i^\lambda \mathcal{P}_j^\mu - \sigma_{\lambda\mu}^ij \mathcal{R}_i^\lambda \mathcal{R}_j^\mu + c'] \omega.
\] (55)

**Theorem 16.** Given a linear map $\sigma$ (46), the Hamiltonian forms $H_{\sigma\Gamma}$ (54) parametrized by connections $\Gamma$ (43) constitute a complete set.

**Proof.** Let us consider the Hamilton equations (29a), written as the equality
\[
J^1(\pi_{HY} \circ r) = \tilde{H} \circ r
\] (56)
for a section $r$ of the Legendre bundle $\Pi \to X$. The Hamiltonian map $\tilde{H}_{\sigma\Gamma}$ reads
\[
\tilde{H}_{\sigma\Gamma} = \Phi_{\sigma\Gamma} + \frac{1}{2} \sigma_1 = \Gamma \circ \pi_{HY} + \sigma + \sigma_1.
\]
Due to the projections $S$, $F$ (19b), the Hamilton equations (56) break in two parts

\begin{align*}
S \circ J^1(\pi_{HY} \circ r) &= \Gamma, & (\text{57}) \\
(\delta - \sigma a)^{j\lambda}_i(\partial_\mu r^j - \Gamma^j_\mu) &= 0,
\end{align*}

\begin{align*}
F \circ J^1(\pi_{HY} \circ r) &= \sigma + \sigma_1, & (\text{58}) \\
(\sigma a)^{j\lambda}_i(\partial_\mu r^j - \Gamma^j_\mu) &= 0.
\end{align*}

Let $s$ be an arbitrary section of $Y \to X$, e.g., a solution of the Euler–Lagrange equations. There exists a connection $\Gamma$ (43) such that the relation (57) holds, namely, $\Gamma = S \circ \Gamma'$ where $\Gamma'$ is a connection on $Y \to X$ which has $s$ as an integral section. It is easily seen that, in this case, the Hamiltonian map (53) satisfies the relation (35) for $s$. Hence, the Hamiltonian forms (54) constitute a complete set.

We have different complete sets of Hamiltonian forms (54) for different $\sigma_1$. For instance, if $\sigma_1 = 0$, then $\Phi_{\sigma_1} = \tilde{H}_{\sigma_1}$ and the Hamiltonian forms (54) are associated with the Lagrangian (40). If $\sigma_1$ is non-degenerate, so are the Hamiltonian forms (54). Hamiltonian forms $H$ (54) of a complete set in Theorem 16 differ from each other in the term $\phi^i_\lambda R^\lambda_i$, where $\phi$ are the soldering forms (45). This term vanishes on the Lagrangian constraint space (52). Accordingly, the constrained Hamiltonian form reads

\[ H_N = i_N^* H_{\sigma_1} = \mathcal{P}_i^\lambda dy^i \wedge \omega_\lambda - \left[ \mathcal{P}_i^\lambda \Gamma^\lambda_\mu + \frac{1}{2} \sigma_{ij}^\mu \mathcal{P}_i^\lambda \mathcal{P}_j^\mu - c' \right], \]

and the constrained Hamilton equations (57) can be written. In the case of quadratic Lagrangians, we can improve Theorem 14 as follows [4, 5].

**Theorem 17.** For every Hamiltonian form $H_{\sigma_1}$ (54), the Hamilton equations (29b) and (58) restricted to the Lagrangian constraint space $N_L$ are equivalent to the constrained Hamilton equations.

It follows that, restricted to the Lagrangian constraint space, the Hamilton equations for different Hamiltonian forms (54) associated with the same quadratic Lagrangian (40) differ from each other in the equations (57). These equations are independent of momenta and play the role of gauge-type conditions.

Note that, in Hamiltonian gauge theory, the restricted Hamiltonian form and the restricted Hamilton equations are gauge invariant, while weakly associated Hamiltonian forms (54) and Lagrangians (53) contain gauge fixing terms. Moreover, one can find a complete set of non-degenerate Hamiltonian forms, that is essential for quantization.
6 Geometry of antighosts

Using the splitting (50a) and the corresponding projection operators
\[ P_{i\nu}^{\lambda k} = a_{ij}^{\lambda \mu} \sigma_{0 \mu \nu}^{jk}, \quad R_{i\nu}^{\lambda k} = (\delta_{i}^{k} \delta_{\nu}^{\lambda} - a_{ij}^{\lambda \mu} \sigma_{0 \mu \nu}^{jk}), \]
\[ P_{i\nu}^{\lambda k} R_{k}^{\nu} = 0, \quad R_{i\nu}^{\lambda k} R_{k}^{\nu} = R_{\lambda i}^{\nu}, \quad (59) \]
we can construct the Koszul–Tate resolution for the Lagrangian constraints (52) of a generic almost regular quadratic Lagrangian (40) in an explicit form. Since these constraints are reducible, one needs an infinite number of antighost fields in general [3, 9] (we follow the terminology of Ref. [9]). They are graded by the antighost number \( r \) and the Grassmann parity \( r \mod 2 \). Odd antighost fields are represented by elements of a simple graded manifold [13]. To describe even antighost fields, we should generalize the notion of a graded manifold to commutative graded algebras generated both by odd and even elements [11, 12, 13].

Let \( E = E_0 \oplus E_1 \to Z \) be the Whitney sum of vector bundles \( E_0 \to Z \) and \( E_1 \to Z \) over a manifold \( Z \). One can think of \( E \) as being a bundle of vector superspaces with a typical fibre \( V = V_0 \oplus V_1 \). Let us consider the exterior bundle
\[ \wedge E^* = \bigoplus_{k=0}^{\infty} (\wedge_{Z} E^*) \]
which is the tensor bundle \( \otimes E^* \) modulo elements
\[ e_0 e_0' - e_0' e_0, \quad e_1 e_1' + e_1' e_1, \quad e_0 e_1 - e_1 e_0, \quad e_0, e_0' \in E_0^*, \quad e_1, e_1' \in E_1^*, \quad z \in Z. \]

Global sections of \( \wedge E^* \) constitute a graded commutative algebra \( \mathcal{A}(Z) \) which is the product of the commutative algebra \( \mathcal{A}_0(Z) \) of global sections of the symmetric bundle \( \vee E_0^* \to Z \) and the graded algebra \( \mathcal{A}_1(Z) \) of global sections of the exterior bundle \( \wedge E_1^* \to Z \). The pair \((Z, \mathcal{A}(Z))\) is a (simple) graded manifold [11, 13]. For the sake of brevity, we agree to call \((Z, \mathcal{A}(Z))\) a graded commutative manifold. Accordingly, elements of \( \mathcal{A}(Z) \) are called graded commutative functions. Let \( \{\epsilon^a\} \) be the holonomic bases for \( E^* \to Z \) with respect to some bundle atlas \((z^A, \nu^i)\) of \( E \to Z \) with transition functions \( \{\rho^a_b\} \), i.e., \( \epsilon^a = \rho^a_b(z) \epsilon^b \). Then graded commutative functions read
\[ f = \sum_{k=0}^{\infty} \frac{1}{k!} f_{a_1 \ldots a_k} c^{a_1} \ldots c^{a_k}, \]
(61)
where \( f_{a_1 \ldots a_k} \) are local functions on \( Z \), and we omit the symbol of an exterior product of elements \( c \).
Let us introduce the differential calculus in these functions. We start from the \( \mathcal{A}(Z) \)-module \( \text{Der} \mathcal{A}(Z) \) of graded derivations of the graded commutative algebra \( \mathcal{A}(Z) \). They are defined as endomorphisms of \( \mathcal{A}(Z) \) such that

\[
  u(f f') = u(f) f' + (-1)^{|u||f|} f u(f')
\]

for homogeneous elements \( u \in \text{Der} \mathcal{A}(Z) \) and \( f, f' \in \mathcal{A}(Z) \). We use the notation \([\cdot]\) for the Grassmann parity. Due to the canonical splitting \( VE = E \times E \), the vertical tangent bundle \( VE \to E \) can be provided with the fibre bases \( \{ \partial_a \} \) dual of \( \{ c^a \} \). These are fibre bases for \( \text{pr}_2^* VE = E \). Then any derivation \( u \) of \( \mathcal{A}(U) \) on a trivialization domain \( U \) of \( E \) reads

\[
  u = u^A \partial_A + u^a \partial_a,
\]

where \( u^A, u^a \) are local graded commutative functions and \( u \) acts on \( f \in \mathcal{A}(U) \) by the rule

\[
  u(f a_1 \cdots a_k c^{a_1} \cdots c^{a_k}) = u^A \partial_A(f a_1 \cdots a_k)c^{a_1} \cdots c^{a_k} + u^a f a_1 \cdots a_k \partial_a(c^{a_1} \cdots c^{a_k}).
\]

This rule implies the corresponding coordinate transformation law

\[
  u^A = u^A, \quad u^a = \rho^a_j u^j + u^A \partial_A(\rho^a_j) c^j,
\]

of derivations (63). Let us consider the vector bundle \( \mathcal{V}_E \to Z \) which is locally isomorphic to the vector bundle

\[
  \mathcal{V}_E \mid_U \approx \bigwedge^* Z (\text{pr}_2^* VE \oplus T Z) \mid_U,
\]

and has the transition functions

\[
  z'_{i_1 \cdots i_k} = \rho^{-1}_{i_1} \cdots \rho^{-1}_{i_k} z_{a_1 \cdots a_k},
\]

\[
  v'^{j_1 \cdots j_k} = \rho^{-1}_{j_1} \cdots \rho^{-1}_{j_k} \left[ \rho^j_{j_1} v^{j_1 \cdots j_k} + \frac{k!}{(k-1)!} z^A_{b_1 \cdots b_{k-1}} \partial_A(\rho^j_{b_k}) \right]
\]

of the bundle coordinates \( (z^A_{a_1 \cdots a_k}, v^{b_1 \cdots b_k}) \), \( k = 0, \ldots \). It is readily observed that, for any trivialization domain \( U \), the \( \mathcal{A} \)-module \( \text{Der} \mathcal{A}(U) \) with the transition functions (64) is isomorphic to the \( \mathcal{A} \)-module of local sections of \( \mathcal{V}_E \mid_U \to U \). One can show that, if \( U' \subset U \) are open sets, there is the restriction morphism \( \text{Der} \mathcal{A}(U) \to \text{Der} \mathcal{A}(U') \). It follows that, restricted to an open subset \( U \), every derivation \( u \) of \( \mathcal{A}(Z) \) coincides with some local section \( u_U \) of \( \mathcal{V}_E \mid_U \to U \), whose collection \( \{ u_U, U \subset Z \} \) defines uniquely a global section of \( \mathcal{V}_E \to Z \), called a graded vector field on \( Z \).

The \( \bigwedge^* Z \)-dual \( \mathcal{V}^*_E \) of \( \mathcal{V}_E \) is a vector bundle over \( Z \) whose sections constitute the \( \mathcal{A}(Z) \)-module of graded 1-forms \( \phi = \phi_A dz^A + \phi_a dc^a \). Then the morphism \( \phi : u \to \mathcal{A}(Z) \) can be seen as the interior product

\[
  u \mid \phi = u^A \phi_A + (-1)^{|\phi_a|} u^a \phi_a.
\]

(65)
Graded $k$-forms $\phi$ are defined as sections of the graded exterior bundle $\wedge^k_Z V_E^*$ such that
\[ \phi \wedge \sigma = (-1)^{|\phi| + |\phi|} \sigma \wedge \phi, \]
where $|.|$ is the form degree. The interior product (65) is extended to higher graded forms by the rule
\[ u \downarrow (\phi \wedge \sigma) = (u \uparrow \phi) \wedge \sigma + (-1)^{|\phi| + |u|} \phi \wedge (u \downarrow \sigma). \]

The graded exterior differential $d$ of graded functions is introduced by the condition
\[ u \downarrow df = u(f) \]
for an arbitrary graded vector field $u$, and is extended uniquely to higher graded forms by the rules
\[ d(\phi \wedge \sigma) = (d\phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge (d\sigma), \quad d \circ d = 0. \]

7 The Koszul–Tate resolution

Let us turn to the splitting (50a) and the projection operators (59). To construct the vector bundle $E$ of antighosts, let us consider the vertical tangent bundle $V_{\Pi}$ of $\Pi \to Y$.

Let us choose the bundle $E$ as the Whitney sum of the bundles $E_0 \oplus E_1$ over $\Pi$ which are the infinite Whitney sum over $\Pi$ of the copies of $V_{\Pi}$. We have
\[ E = V_{\Pi} \oplus V_{\Pi} \oplus \cdots. \]

This bundle is provided with the holonomic coordinates $(x^\lambda, y^i, p_i^\lambda, \dot{p}_i^{\lambda(r)})$, $r = 0, 1, \ldots$, where $(x^\lambda, y^i, p_i^\lambda, \dot{p}_i^{\lambda(2l)})$ are coordinates on $E_0$, while $(x^\lambda, y^i, p_i^\lambda, \dot{p}_i^{\lambda(2l+1)})$ are those on $E_1$. By $r$ is meant the antighost number. The dual of $E \to \Pi$ is
\[ E^* = V_{\Pi}^* \oplus V_{\Pi}^* \oplus \cdots. \]

It is endowed with the associated fibre bases $\{c_i^{\lambda(r)}\}$, $r = 1, 2, \ldots$, such that $c_i^{\lambda(r)}$ have the same linear coordinate transformation law as the coordinates $p_i^\lambda$. The corresponding graded vector fields and graded forms are introduced on $\Pi$ as sections of the vector bundles $V_E$ and $V_E^*$, respectively.

The $C^\infty(\Pi)$-module $\mathcal{A}(\Pi)$ of graded functions is graded by the antighost number as
\[ \mathcal{A}(\Pi) = \bigoplus_{r=0}^{\infty} \mathcal{N}^r, \quad \mathcal{N}^0 = C^\infty(\Pi). \]

Its terms $\mathcal{N}^r$ constitute a complex
\[ 0 \leftarrow C^\infty(\Pi) \leftarrow \mathcal{N}^1 \leftarrow \cdots \]
with respect to the Koszul–Tate differential
\[ \delta : C^\infty(V^*Y) \rightarrow 0, \]
\[ \delta(c^\lambda_{i}^{(2l)}) = P^\lambda_{i\nu} c^{\nu(2l-1)}_k, \quad l > 0, \]
\[ \delta(c^\lambda_{i}^{(2l+1)}) = R^\lambda_{i\nu} c^{\nu(2l)}_k, \quad l > 0, \]
\[ \delta(c^\lambda_{i}^{(1)}) = R^\lambda_{i\nu} p^{\nu}_k. \]

The nilpotency property \( \delta \circ \delta = 0 \) of this differential is the corollary of the relations (60).

It is readily observed that the complex (66) with respect to the differential (67) has the homology groups
\[ H_{k>0} = 0, \quad H_0 = C^\infty(\Pi)/I_{NL} = C^\infty(N_L), \]
where \( I_{NL} \) is an ideal of smooth functions on \( \Pi \) which vanish on the Lagrangian constraint space \( N_L \). Thus, this is a desired Koszul–Tate resolution of the Lagrangian constraints (52).

Note that, in different particular cases of the degenerate quadratic Lagrangian (40), the complex (66) may have a subcomplex, which is also the Koszul–Tate resolution. For instance, if the matrix \( a \) is diagonal with respect to some adapted coordinates on \( J^1Y \), the constraints (52) are irreducible and the complex (66) contains a subcomplex which consists only of the antighosts \( c^\lambda_{i}^{(1)} \).

**PART III. Constraints in time-dependent mechanics**

If \( X = R \), polysymplectic Hamiltonian formalism provides the adequate Hamiltonian formulation of time-dependent mechanics \( [10, 20] \). Here, we study holonomic time-dependent constraints \( [12] \).

Note that, in contrast with the existent formulations of time-dependent mechanics, we do not imply any preliminary splitting of its momentum phase space \( \Pi = R \times Z \). From the physical viewpoint, this splitting characterizes a certain reference frame, and is violated by time-dependent transformations. Given such a splitting, \( \Pi \) is endowed with the product of the zero Poisson structure on \( R \) and the Poisson structure on \( Z \). A Hamiltonian \( H \) is defined as a real function on \( \Pi \). The corresponding Hamiltonian vector field \( \partial_H \) on \( \Pi \) is vertical with respect to the fibration \( \Pi \rightarrow R \). Due to the natural imbedding \( \Pi \times_R TR \rightarrow T\Pi \) one introduces the vector field \( \gamma_H = \partial_t + \partial_H \), where \( \partial_t \) is the standard vector field on \( R \). The Hamilton equations are equations for the integral
curves of the vector field $\gamma_H$, while the evolution equation on the Poisson algebra $C^\infty(\Pi)$ of smooth functions on $\Pi$ is given by the Lie derivative

$$L_{\gamma_H} f = \partial_t f + \{H, f\}.$$  

However, the splitting in the right-hand side of this expression is violated by time-dependent transformations, and a Hamiltonian $H$ is not scalar under these transformations. Its Poisson bracket with functions $f \in C^\infty(\Pi)$ is ill-defined, and is not maintained under time-dependent transformations. This fact is the key point of the study of constraints in Hamiltonian time-dependent mechanics.

8 Hamiltonian time-dependent mechanics

Let us consider time-dependent mechanics on a configuration bundle $Q \to \mathbb{R}$.

Remark 5. The following peculiarities of fibre bundles over $\mathbb{R}$ should be emphasized. Their base $\mathbb{R}$ is parametrized by the Cartesian coordinates $t$ with the transition functions $t' = t + \text{const.}$, and is provided with the standard vector field $\partial_t$ and the standard 1-form $dt$. A vector field $u$ on a fibre bundle $Y \to \mathbb{R}$ is said to be projectable if $u \rfloor dt$ is constant. From now on, by vector fields on fibre bundles over $\mathbb{R}$ are meant only projectable vector fields. Let $Y \to \mathbb{R}$ be a fibre bundle coordinated by $(t, y^A)$ and $J^1Y$ its first order jet manifold, equipped with the adapted coordinates $(t, y^A, y^A_t)$. There is the canonical imbedding $J^1Y \to TY$ over $Y$ whose image is the affine subbundle of elements $v \in TY$ such that $v \rfloor dt = 1$. This subbundle is modelled over the vertical tangent bundle $VY \to Y$. As a consequence, there is one-to-one correspondence between the connections on the fibre bundle $Y \to \mathbb{R}$ and the vector fields $\Gamma$ on $Y$ such that $\Gamma \rfloor dt = 1$. A connection $\Gamma$ on $Y \to \mathbb{R}$ yields a 1-dimensional distribution on $Y$, transversal to the fibration $Y \to \mathbb{R}$. As a consequence, it defines an atlas of local constant trivializations of $Y \to \mathbb{R}$ whose transition functions are independent of $t$ and $\Gamma = \partial_t$. Conversely, every atlas of local constant trivializations of a fibre bundle $Y \to \mathbb{R}$ sets a connection on $Y \to \mathbb{R}$ which is $\partial_t$ relative to this atlas. In particular, every trivialization of $Y \to \mathbb{R}$ yields a complete connection $\Gamma$ on $Y$, and vice versa.

The momentum phase space of time-dependent mechanics is the vertical cotangent bundle

$$V^*Q \xrightarrow{\pi_Q} Q \xrightarrow{\pi} \mathbb{R}$$

endowed with holonomic coordinates $(t, q^i, p_i)$. The homogeneous Legendre bundle $Z_Q$ (13) is the cotangent bundle $T^*Q$. The $V^*Q$ is provided with the canonical Poisson structure $\{,\}_V$ such that

$$\zeta^* \{f, g\}_V = \{\zeta^* f, \zeta^* g\}_T, \quad f, g \in C^\infty(V^*Q),$$

(68)
where \( \zeta = \pi_{\Pi} \) (14) is the natural fibration
\[
\zeta : T^*Q \to V^*Q,
\]
and \( \{ , \} \) is the canonical Poisson structure on the cotangent bundle \( T^*Q \) provided with the symplectic form \( d\Xi \). The characteristic distribution of \( \{ , \} \) coincides with the vertical tangent bundle \( VV^*Q \) of \( V^*Q \to \mathbb{R} \).

Given a section \( h \) of the fibre bundle (69), let us consider the pull-back forms
\[
\Theta = h^*(\Xi \wedge dt), \quad \Omega = h^*(d\Xi \wedge dt)
\]
on \( V^*Q \). It is readily observed that these forms are independent of \( h \), and are canonical on \( V^*Q \). Then a Hamiltonian vector field \( \vartheta_f \) for a function \( f \) on \( V^*Q \) is given by the relation
\[
\vartheta_f \rfloor \Omega = -\partial f \wedge dt,
\]
while the Poisson bracket (68) is written as
\[
\{ f, g \}_V dt = \vartheta_g \rfloor \vartheta_f \rfloor \Omega.
\]

Thus, the 3-form \( \Omega \) (70) provides \( V^*Q \) with the Poisson structure \( \{ , \}_V \) in an equivalent way. Furthermore, holonomic coordinates on \( V^*Q \) are canonical for the Poisson structure (68) such that
\[
\Omega = dp_i \wedge dq^i \wedge dt,
\]
\[
\{ f, g \}_V = \partial_i f \partial_{i} g - \partial_i g \partial_{i} f, \quad f, g \in C^\infty(V^*Q).
\]

**Lemma 18.** [10, 20]. A vector field \( u \) on \( V^*Q \) is canonical for the Poisson structure \( \{ , \}_V \) iff the form \( u \rfloor \Omega \) is closed. The closed form \( u \rfloor \Omega \) is exact.

With respect to the Poisson bracket (71), the Hamiltonian vector field \( \vartheta_f \) for a function \( f \) on the Legendre bundle \( V^*Q \) is
\[
\vartheta_f = \partial^i f \partial_i - \partial_i f \partial^i.
\]
It is vertical. Conversely, one can show that every vertical canonical vector field on the Legendre bundle \( V^*Q \to \mathbb{R} \) is locally a Hamiltonian vector field.

**Proposition 19.** Let a connection \( \gamma \) on the Legendre bundle \( V^*Q \to \mathbb{R} \) be a canonical vector field for the Poisson structure \( \{ , \}_V \). Then \( \gamma \rfloor \Omega = dH \) where \( H \) is locally a Hamiltonian form. Conversely, any Hamiltonian form
\[
H = h^*\Xi = p_idq^i - \mathcal{H} dt
\]
on the momentum phase space $V^*Q$ admits a unique Hamiltonian connection
\[
\gamma_H = \partial_t + \partial^iH\partial_i - \partial_iH\partial^i.
\] (73)

Remark 6. A glance at the expression (72) shows that, given a trivialization of the configuration bundle $Q \to \mathbb{R}$, the Hamiltonian form $H$ (72) is the well-known integral invariant of Poincaré–Cartan where $H$ plays the role of a Hamiltonian.

Hamiltonian forms in time-dependent mechanics constitute an affine space modelled over the vector space of horizontal densities $fdt$ on $V^*Q \to \mathbb{R}$, i.e., over $C^\infty(V^*Q)$. Accordingly, Hamiltonian connections $\gamma_H$ make up an affine space modelled over the vector space of Hamiltonian vector fields. Every Hamiltonian form yields the Hamiltonian map
\[
\hat{H} = J^1\pi_Q \circ \gamma_H : V^*Q \to J^1Q, \quad q^i_t \circ \hat{H} = \partial^iH.
\] (74)

In particular, let $\Gamma$ be a connection on $Q \to \mathbb{R}$. It characterizes a reference frame in time-dependent mechanics [2, 10, 16, 20], and defines the frame Hamiltonian form
\[
H_\Gamma = p_i dq^i - p_i \Gamma^i dt.
\]

The corresponding Hamiltonian connection is the canonical lift
\[
V^*\Gamma = \partial_t + \Gamma^i\partial_i - p_i\partial_j \Gamma^i \partial^j
\]
of $\Gamma$ onto $V^*Q \to \mathbb{R}$. Then any Hamiltonian form $H$ on $V^*Q$ admits the splittings
\[
H = H_\Gamma - \tilde{H}_\Gamma dt, \quad H = p_i \Gamma^i + \tilde{H}_\Gamma,
\] (75)

where $\tilde{H}_\Gamma$ is the energy function with respect to the reference frame $\Gamma$ [11, 20].

Given a Hamiltonian form $H$ (72) and the associated Hamiltonian connection $\gamma_H$ (73), the kernel of the covariant differential $D_{\gamma_H}$ defines the Hamilton equations
\[
q^i_t = \partial^iH, \quad p_i = -\partial_iH.
\] (76)

A Hamiltonian form $H$ (72) is the Poincaré–Cartan form for the Lagrangian
\[
L_H = h_0(H) = (p_i q^i_t - H)dt
\]
on the jet manifold $J^1V^*Q$. This Lagrangian is a convenient tool in order to apply the standard Lagrangian technique to Hamiltonian time-dependent mechanics. As in the polysymplectic case, the Hamilton equations (76) for $H$ are exactly the Lagrange equations for $L_H$. Furthermore, given a function $f \in C^\infty(V^*Q)$ and its pull-back onto $J^1V^*Q$, let us consider the bracket
\[
(f, L_H) = \delta^i f \delta_i L_H - \delta_i f \delta^i L_H = L_{\gamma_H} f - dt f.
\] (77)
where $\delta^i, \delta_i$ are variational derivatives (in the spirit of the Batalin–Vilkovisky antibracket). Then the equation $(f, L_H) = 0$ is the evolution equation
\[ df = L_H f = \partial_t f + \{ H, f \}_V \] (77)
in time-dependent mechanics. Note that, taken separately, the terms in its right-hand side are ill-behaved objects under reference frame transformations. With the splitting (74), the evolution equation (77) is brought into the frame-covariant form
\[ L_H f = V^* \Gamma H + \{ \tilde{H}_V, f \}_V, \]
but its right-hand side does not reduce to a Poisson bracket.

The following construction enables us to represent the right-hand side of the evolution equation (77) as a pure Poisson bracket. Given a Hamiltonian form $H = h^* \Xi$, let us consider its pull-back $\zeta^* H$ onto the cotangent bundle $T^* Q$. It is readily observed that the difference $\Xi - \zeta^* H$ is a horizontal 1-form on $T^* Q \to \mathbb{R}$, while
\[ H^* = \partial_t (\Xi - \zeta^* H) = p + H \] (78)
is a function on $T^* Q$. Then the relation
\[ \zeta^* (L_H f) = \{ H^*, \zeta^* f \}_T \] (79)
holds for any function $f \in C^\infty (V^* Q)$. In particular, $f$ is an integral of motion iff its bracket (79) vanishes. Note that $\gamma_H = T \zeta (\partial_{H^*})$ where $\partial_{H^*}$ be the Hamiltonian vector field for the function $H^*$ (78) with respect to the canonical Poisson structure $\{ , \}_T$ on $T^* Q$.

9 Time-dependent constraints

With the Poisson bracket $\{ , \}_V$, an algebra of time-dependent constraints can be described similarly to that in conservative Hamiltonian mechanics, but we should use the relation (74) in order to extend the constraint algorithm to time-dependent constraints.

Let $N$ be a closed imbedded subbundle $i_N : N \hookrightarrow V^* Q$ of the Legendre bundle $V^* Q \to \mathbb{R}$, treated as a constraint space. Note that $N$ is neither Lagrangian nor symplectic submanifold with respect to the Poisson structure $\{ , \}_V$. Let us consider the ideal $I_N \subset C^\infty (V^* Q)$ of functions $f$ on $V^* Q$ which vanish on $N$, i.e., $i_N^* f = 0$. Its elements are said to be constraints. There is the isomorphism
\[ C^\infty (V^* Q)/I_N \cong C^\infty (N) \] (80)
of associative commutative algebras. By the normalize $\mathcal{T}_N$ of the ideal $I_N$ is meant the subset of functions of $C^\infty(V^*Q)$ whose Hamiltonian vector fields restrict to vector fields on $N$ \[ \{ f \in C^\infty(V^*Q) : \{ f, g \}_V \in I_N, \, \forall g \in I_N \} \quad \text{(81)} \]

It follows from the Jacobi identity that the normalizer (81) is a Poisson subalgebra of $C^\infty(V^*Q)$. Put \[ I'_N = I_N \cap \mathcal{T}_N \quad \text{(82)} \]

This is also a Poisson subalgebra of $\mathcal{T}_N$. Its elements are called the first class constraints, while the remaining elements of $I_N$ are the second class constraints. It is readily observed that $I^2_N \subset I'_N$.

**Remark 7.** Let $N$ be a coisotropic submanifold of $V^*Q$. Then $I_N \subset \mathcal{T}_N$ and $I_N = I'_N$, i.e., all constraints are of first class.

Let $H$ be a Hamiltonian form on the momentum phase space $V^*Q$. In accordance with the relation (82), a constraint $f \in I_N$ is preserved with respect to a Hamiltonian form $H$ if the bracket (79) vanishes on the constraint space. It follows that solutions of the Hamilton equations (76) do not leave the constraint space $N$ if \[ \{ \mathcal{H}^*, \zeta^* I_N \}_T \subset \zeta^* I_N \quad \text{(83)} \]

If this relation does not hold, let us introduce secondary constraints \( \{ \mathcal{H}^*, \zeta^* f \}_T, f \in I_N \), which belong to $\zeta^*(C^\infty(V^*Q))$. If the set of primary and secondary constraints is not closed with respect to the relation (83), one can add the tertiary constraints \( \{ \mathcal{H}^*, \{ \mathcal{H}^*, \zeta^* f_0 \}_T \}_T \), and so on.

Let us assume that $N$ is a final constraint space for a Hamiltonian form $H$. If $H$ satisfies the relation (83), so is a Hamiltonian form \[ H_f = H - f dt \quad \text{(84)} \]

where $f \in I'_N$ is a first class constraint. Though Hamiltonian forms $H$ and $H_f$ coincide with each other on the constraint space $N$, the corresponding Hamilton equations have different solutions in $N$ because $dH \vert_N \neq dH_f \vert_N$. At the same time, $d(i_N^* H) = d(i_N^* H_f)$. Therefore, let us consider the constrained Hamiltonian form \[ H_N = i_N^* H_f \quad \text{(85)} \]

which is the same for all $f \in I'_N$. Note that $H_N$ (85) is not a true Hamiltonian form on $N \to \mathbb{R}$ in general. On sections $r$ of the bundle $N \to \mathbb{R}$, we can write the constrained Hamilton equations \[ r^*(u_N \vert dH_N) = 0, \quad \text{(86)} \]
where \( u_N \) is an arbitrary vertical vector field on \( N \to \mathbb{R} \). It is readily observed that, for any Hamiltonian form \( H_f \), every solution of the Hamilton equations which lives in the constraint space \( N \) is a solution of the constrained Hamilton equations (86).

Let us mention the problem of constructing a generalized Hamiltonian system, similar to that for a Dirac constraint system in conservative mechanics. Let \( H \) satisfy the condition \( \{ \mathcal{H}^*, \zeta^* I_N \}_T \subset I_N \), whereas \( \{ \mathcal{H}^*, \zeta^* I_N \}_T \not\subset I_N \). The goal is to find a constraint \( f \in I_N \) such that the modified Hamiltonian \( H - f dt \) would satisfy the condition

\[
\{ \mathcal{H}^* + \zeta^* f, \zeta^* I_N \}_T \subset \zeta^* I_N.
\]

This is an equation for a second-class constraint \( f \).

The above construction, except the isomorphism (80), can be applied to any ideal \( I \) of \( C^\infty(V^*Q) \), treated as an ideal of constraints [9]. In particular, an ideal \( I \) is said to be coisotropic if it is a Poisson algebra. In this case, \( I \) is a Poisson subalgebra of the normalize \( \mathcal{T} \) (81), and coincides with \( I' \) (82).

Note that, since \( \zeta^*(\mathcal{L}_\partial, H) \neq \{ \zeta^* f, \mathcal{H}^* \}_T \), the constraints \( f \in I_N \) preserved with respect to a Hamiltonian form \( H \) (i.e. \( \{ \zeta^* f, \mathcal{H}^* \}_T \in I_N \)) are not generators of gauge symmetries of \( H \) in general. At the same time, the generators of gauge symmetries of a Hamiltonian form \( H \) define an ideal of constraints as follows. Let \( \mathcal{A} \) be a Lie algebra of generators \( u \) of gauge symmetries of a Hamiltonian form \( H \). The corresponding symmetry currents \( J_u = u| H \) on \( V^*Q \) constitute a Lie algebra with respect to the Poisson bracket

\[
\{ J_u, J_{u'} \} = J_{[u, u']}
\]

on \( V^*Q \). Let \( I_A \) denotes the ideal of \( C^\infty(V^*Q) \) generated by these symmetry currents. It is readily observed that this ideal is coisotropic. Then one can think of \( I_A \) as being an ideal of first class constraints compatible with the Hamiltonian form \( H \), i.e.,

\[
\{ \mathcal{H}^*, \zeta^* I_A \}_T \subset \zeta^* I_A.
\]

Note that any Hamiltonian form \( H_u = H - J_\theta dt, \ u \in \mathcal{A} \), obeys the same relation (87), but other currents \( J_{u'} \) are not conserved with respect to \( H_u \), unless \( [u, u'] = 0 \).

10 BRST charge for Lagrangian constraints

Lagrangian constraints in time-dependent mechanics are described in the same manner as in the general polysymplectic case [12]. At the same time, the canonical Poisson structure on the momentum phase space \( V^*Q \) enables us to construct the BRST charge for the Koszul-Tate differential.
In time-dependent mechanics, the vector bundle $E$ of antighosts for Lagrangian constraints is the infinite Whitney sum

$$E = V_Q(V^*Q) \oplus V_Q(V^*Q) \oplus \cdots$$

over $V^*Q$ of the copies of $V_Q(V^*Q)$. This bundle is provided with the holonomic coordinates $(t, q^i, p_i, \dot{q}_i^{(r)}, \dot{p}_i^{(r)})$, $r = 0, 1, \ldots$, where $(t, q^i, p_i, \dot{q}_i^{(2l)})$ are coordinates on $E_0$, while $(t, q^i, p_i, \dot{q}_i^{(2l+1)})$ are those on $E_1$. The dual of $E \to V^*Q$ is

$$E^* = V_Q^*(V^*Q) \oplus V_Q^*(V^*Q) \oplus \cdots.$$ 

It is endowed with the associated fibre bases $\{c_i^{(r)}\}$, $r = 1, 2, \ldots$, such that $c_i^{(r)}$ have the same linear coordinate transformation law as the coordinates $p_i$. The corresponding graded vector fields and graded forms are introduced on $V^*Q$ as sections of the vector bundles $V_E$ and $V_E^*$, respectively. The $C^\infty(V^*Q)$-module $A(V^*Q)$ of graded commutative functions is graded by the antighost number $r$. Its terms $N^r$ constitute the Koszul–Tate resolution (66) with respect to the Koszul–Tate differential

$$\delta : C^\infty(V^*Q) \to 0,$$

$$\delta(c_i^{(2l)}) = a_{ij}^k \sigma^j_0 c_k^{(2l-1)}, \quad l > 0,$$

$$\delta(c_i^{(2l+1)}) = (\delta^k_i - a_{ij}^k \sigma^j_0) c_k^{(2l)}, \quad l > 0,$$

$$\delta(c_i^{(1)}) = (\delta^k_i - a_{ij}^k \sigma^j_0) p_k.$$

Let us construct the BRST charge $Q$ such that

$$\delta(f) = \{Q, f\}, \quad f \in A(V^*Q),$$

with respect to some Poisson bracket. The problem is to find the Poisson bracket such that $\{f, g\} = 0$ for all $f, g \in C^\infty(V^*Q)$.

To overcome this difficulty, one can consider the vertical extension of Hamiltonian formalism onto the configuration bundle $VQ \to \mathbb{R}$ [3, 10, 12]. The corresponding Legendre bundle $V^*(VQ)$ is isomorphic to $V(V^*Q)$, and is provided with the holonomic coordinates $(t, q^i, p_i, \dot{q}_i, \dot{p}_i)$ such that $(q^i, \dot{p}_i)$ and $(\dot{q}_i, p_i)$ are conjugate pairs of canonical coordinates. The momentum phase space $V(V^*Q)$ is endowed with the canonical exterior 3-form

$$\Omega_V = \partial_V \Omega = [dp_i \wedge dq^i + dp_i \wedge dq_i^{(l)}] \wedge dt,$$

where we use the compact notation

$$\dot{t} = \frac{\partial}{\partial q_i}, \quad \dot{q}_i = \frac{\partial}{\partial p_i}, \quad \partial_V = \dot{q}_i \partial_i + \dot{p}_i \partial^i.$$
The corresponding Poisson bracket on \( V(V^*Q) \) reads

\[
\{ f, g \}_{VV} = \partial^i f \partial_i g + \partial^i f \dot{\partial}_i g - \partial^i g \dot{\partial}_i f - \dot{\partial}^i g \dot{\partial}_i f.
\]

To extend this bracket to graded functions, let us consider the following graded extension of Hamiltonian formalism \([8, 10, 13, 14]\). We will assume that \( Q \to \mathbb{R} \) is a vector bundle, and will further denote \( \Pi = V^*Q \).

Let us consider the vertical tangent bundle \( VV \Pi \). It admits the canonical decomposition

\[
VV \Pi = V \Pi \oplus R V \Pi \to V \Pi.
\] (89)

Let choose the bundle \( E \) as the Whitney sum of the bundles \( E_0 \oplus E_1 \) over \( V \Pi \) which are the infinite Whitney sum over \( V \Pi \) of the copies of \( VV \Pi \). In view of the decomposition (89), we have

\[
E = V \Pi \oplus V \Pi \oplus \cdots \oplus V \Pi.
\]

This bundle is provided with the holonomic coordinates \((t, q^i, p_i, \dot{q}_{(r)}^i, \dot{p}_i^{(r)})\), \( r = 0, 1, \ldots \), where \((t, q^i, p_i, \dot{q}_{(2l)}^i, \dot{p}_i^{(2l)})\) are coordinates on \( E_0 \) and \((t, q^i, p_i, \dot{q}_{(2l+1)}^i, \dot{p}_i^{(2l+1)})\) are those on \( E_1 \). The dual of \( E \to V \Pi \) is

\[
E^* = V \Pi \oplus V \Pi^* \oplus \cdots.
\]

It is endowed with the associated fibre bases \( \{ c_i^{(r)}, c_i^{(r)}, c_i^{(r)}, c_i^{(r)} \} \), \( r = 1, \ldots \). The corresponding graded vector fields and graded forms are introduced on \( V \Pi \) as sections of the vector bundles \( V V \Pi \) and \( V^* V \Pi \), respectively. Let us complexify these bundles as \( C \otimes V V \Pi \) and \( C \otimes V^* V \Pi \).

The BRST extension of the form (88) on \( V^*Q \) is the 3-form

\[
\Omega_S = \Omega_V + i \sum_{r=1}^{\infty} (dc_{(r)}^i \wedge dc_{(r)}^i - dc_{(r)}^i \wedge \overline{dc}_{(r)}^i) \wedge dt
\]

The corresponding bracket of graded functions on \( V^*Q \) reads

\[
\{ f, g \}_S = \{ f, g \}_{VV} - i \sum_{r=1}^{\infty} (-1)^{r|f|} \left[ \frac{\partial f}{\partial c_{(r)}^i} \frac{\partial g}{\partial c_{(r)}^i} - (-1)^r \frac{\partial f}{\partial \overline{c}_{(r)}^i} \frac{\partial g}{\partial \overline{c}_{(r)}^i} \right] - \left( -1 \right)^r \frac{\partial f}{\partial \overline{c}_{(r)}^i} \frac{\partial g}{\partial c_{(r)}^i}.
\] (90)
It satisfies the condition \( \{ f, g \}_S = -(-1)^{|f||g|}\{ g, f \}_S \). Then the desired BRST charge takes the form
\[
Q = i[c_i^j(1) (\delta^k_i - a_{ij} \sigma^j_0) p_k + \sum_{l=1}^{\infty} (c_{(2l+1)}^i a_{ij} \sigma^j_0 c_{(2l-1)}^k + c_{(2l+1)}^i (\delta^k_i - a_{ij} \sigma^j_0) c_{(2l)}^k)].
\]

Due to the bracket (90), one can use this charge in order to obtain the BRST complex for antighosts \( c_i^{(r)} \) and ghosts \( \overline{c}_{(r)}^j \) such that
\[
\overline{c}_{(2l-1)}^i \mapsto a_{kj} \sigma^j_0 \overline{c}_{(2l)}^j, \quad \overline{c}_{(2l)}^i \mapsto - (\delta^i_k - a_{ij} \sigma^j_0) \overline{c}_{(2l+1)}^k, \quad l > 0.
\]

References

[1] C. Bartocci, U. Bruzzo and D. Hernández Ruipérez, The Geometry of Supermanifolds (Kluwer Academic Publ., Dordrecht, 1991).

[2] A. Echeverría-Enríquez, M. Muñoz-Lecanda and N. Román-Roy, Non-standard connections in classical mechanics, J. Phys. A 28 (1995) 5553.

[3] F. Fisch, M. Henneaux, J. Stasheff and C. Teitelboim, Existence, uniqueness and cohomology of the classical BRST charge with ghosts of ghosts, Commun. Math. Phys. 120 (1989) 379.

[4] G. Giachetta, L. Mangiarotti and G. Sardanashvily, New Lagrangian and Hamiltonian Methods in Field Theory (World Scientific, Singapore, 1997).

[5] G. Giachetta, L. Mangiarotti and G. Sardanashvily, Covariant Hamilton equations for field theory, J. Phys. A 32 (1999) 6629.

[6] G. Giachetta, L. Mangiarotti and G. Sardanashvily, Nonholonomic constraints in time-dependent mechanics, J. Math. Phys. 40 (1999) 1376.

[7] M. Gotay, A multisymplectic framework for classical field theory and the calculus of variations. I. Covariant Hamiltonian formalism, in Mechanics, Analysis and Geometry: 200 Years after Lagrange, ed. M. Francaviglia (North Holland, Amsterdam, 1991), p. 203.

[8] E. Gozzi, M. Reuter and W. Thacker, Phys. Rev. D40 (1989) 3363; D46 (1992) 757; E. Gozzi and M. Reuter, Int. J. Mod. Phys. 9 (1994) 2191.

[9] T. Kimura, Generalized classical BRST cohomology and reduction of Poisson manifolds, Commun. Math. Phys. 151 (1993) 155.
[10] L. Mangiarotti and G. Sardanashvily, *Gauge Mechanics* (World Scientific, Singapore, 1998).

[11] L. Mangiarotti and G. Sardanashvily, The Koszul–Tate cohomology in covariant Hamiltonian formalism, *Modern Phys. Lett.* **14** (1999) 2201.

[12] L. Mangiarotti and G. Sardanashvily, Constraints in Hamiltonian time-dependent mechanics, *J. Math. Phys.* **41** (2000) 2858.

[13] L. Mangiarotti and G. Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).

[14] L. Mangiarotti and G. Sardanashvily, SUSY-extended field theory, *Int. J. Mod. Phys.* **15** (2000) (appear).

[15] G. Martin, A Darboux theorem for multi-symplectic manifolds, *Lett. Math. Phys.* **16** (1988) 133.

[16] E. Massa and E. Pagani, Jet bundle geometry, dynamical connections and the inverse problem of Lagrangian mechanics, *Ann. Inst. Henri Poincaré* **61** (1994) 17.

[17] G. Sardanashvily, Constraint field systems in multimomentum canonical variables, *J. Math. Phys.* **35** (1994) 6584.

[18] G. Sardanashvily, Multimomentum Hamiltonian formalism in quantum field theory, *Int. J. Theor. Phys.* **33** (1994) 2373.

[19] G. Sardanashvily, *Generalized Hamiltonian Formalism for Field Theory. Constraint systems* (World Scientific, Singapore, 1995).

[20] G. Sardanashvily, Hamiltonian time-dependent mechanics, *J. Math. Phys.* **39** (1998) 2714.