Entanglement Entropy of the Two-Dimensional ±J Ising Model on the Nishimori Line

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A classical analogue of the entanglement entropy $S$ is calculated on the system boundary of the two-dimensional Edwards-Anderson model, when the nearest-neighbor interaction is fixed to either $+J$ or $-J$ in the quenched random manner. The probability distribution function for the boundary spins is obtained by means of the time-evolving block decimation (TEBD) method. The random ensemble for the distribution function is given by the successive multiplications of position-dependent transfer matrices, whose width $N$ is up to 300. Identifying the distribution function as the quantum wave function, and performing the singular value decomposition on it, we obtain the corresponding classical analogue of the entanglement entropy $S$. The random average $\langle S \rangle$ is calculated along the Nishimori line, and its critical behavior around the Nishimori point is observed. The central charge for this boundary phenomena is estimated as $c = 0.397(2)$; this value is slightly smaller than those for the bulk critical phenomena reported so far.

I. INTRODUCTION

Effect of randomness on thermodynamic properties of magnetic systems has been one of the long issues in statistical physics. A well investigated theoretical model under the context is the Edwards-Anderson model [1], which was introduced for the analysis of the magnetic susceptibility of the Mn-Cu alloy [2]. A special case of this model is the $\pm J$ Ising model, where each bond randomly chooses either ferromagnetic coupling $-J < 0$ or antiferromagnetic one $J > 0$, and where $J$ represents the absolute value of the neighboring interaction. On the square lattice, the $\pm J$ Ising model exhibits either ferromagnetic or paramagnetic states [3, 4], when there are only short-range interactions. The spin glass state appears in higher dimensions, or under the presence of long-range interactions [5].

Analytic form of the averaged internal energy with respect to the randomness can be obtained for the $\pm J$ Ising model when the temperature $T$ and the probability $p$ of finding ferromagnetic bond satisfy so called the Nishimori condition [6, 7], which is represented by a curve in the phase diagram spanned by the parameters $p$ and $T$. The curve is customary called the Nishimori line. It should be noted that the internal energy on the curve does not show any critical singularity, even when the curve passes through the Nishimori point, which is the crossing point between Nishimori line and the boundary between the ferromagnetic and the paramagnetic phases. It means that some quantity other than the internal energy, such as free energy or thermodynamic entropy, should be observed for the detection of the phase boundary, as long as one observes the system along the Nishimori line. Importance of the Nishimori point can be found from the fact that the phase transition nature changes from the Ising universality to the percolation one at this point.

In this article, we focus on the probability distribution function for the boundary spins of the two-dimensional $\pm J$ Ising model on finite size lattices. It is expected that the distribution function reflects some aspects of the critical phenomena in the bulk part of the model. In particular, we consider rectangular systems, which are the stripes of width $N$, with open boundary conditions. The probability distribution function can be expressed by means of the transfer matrix formalism, which has been widely applied to the $\pm J$ Ising model [8, 12]. It is possible to interpret the probability distribution function as a quantum wave function of a one-dimensional quantum lattice model of length $N$. This is a kind of quantum-classical correspondence, which identifies a $d+1$-dimensional classical lattice system with a discrete path-integral representation of a $d$-dimensional quantum lattice system [13]. Successive multiplication of the transfer matrix, which contains position dependent randomness, with the distribution function is then interpreted as a kind of imaginary-time evolution under the presence of time-dependent randomness. We can thus effectively create a random ensemble for the corresponding one-dimensional quantum system.

It is possible to introduce the concept of quantum entanglement to the probability distribution function, through the quantum-classical correspondence. The bipartite entanglement entropy is a typical value that quantifies the strength of quantum entanglement. In uniform systems, the entanglement entropy is asymptotically proportional to the logarithm of the typical length scale, such as the correlation length [14, 15], and it exhibits singular behavior around the quantum criticality. In the following, we call the entanglement entropy defined for the probability distribution function ‘the classical analogue of the entanglement entropy’, and denote it by $S$. It is expected that this $S$ can also be used for the detec-
tion of criticality in classical random systems, typically, the $\pm J$ Ising model. Ohzeki and Jacobsen observed a quantity which is related to the change in $S$ with respect to the modification of boundary condition imposed for the $\pm J$ Ising model [10].

Since the matrix dimension of the transfer matrix increases exponentially with respect to the system width $N$, direct numerical treatment is possible up to $N \sim 30$ at most. Merz and Chalker introduced bilinear fermionic operator representation of the transfer matrix, and extended the size up to $N = 256$, while most of the numerical data were obtained up to $N = 64$ [11]. The system size restriction is thus severe, since the random average should be taken over a large number of samples [8–12]. In order to overcome this system-size restriction, we employ the time-evolving block decimation (TEBD) method, which has many aspects in common with the density-matrix renormalization group (DMRG) method, which has many aspects in common with the classical analogue of the entanglement entropy $S$ is defined by means of the quantum-classical correspondence. In section IV, we show the numerical result for the average $\langle S \rangle$ obtained by the TEBD method. Conclusions are summarized in the last section, and we discuss improvements in numerical evaluations.

II. TRANSFER MATRIX FORMALISM

We consider the $\pm J$ Ising model on the square lattice. Leaving the discussion on the system boundary for the moment, we assume that the Hamiltonian for the bulk part of the system is written as

$$\mathcal{H} = \sum_{\ell,m} \left[ I_{\ell}^m \sigma_{\ell}^m \sigma_{\ell+1}^m + J_{\ell}^m \sigma_{\ell}^m \sigma_{\ell+1}^m \right],$$

(1)

where $\sigma^m_\ell = \pm 1$ denotes the Ising spin on the lattice point in the $\ell$-th column and in the $m$-th row. The Ising interaction in the vertical lattice direction between $\sigma^m_\ell$ and $\sigma^{m+1}_\ell$ is denoted by $I^m_\ell$, and in the horizontal lattice direction between $\sigma^m_\ell$ and $\sigma^m_{\ell+1}$ is denoted by $J^m_\ell$. We assume the presence of quenched randomness, where both $I^m_\ell$ and $J^m_\ell$ randomly take the value $-J < 0$ and $J > 0$, respectively, with the probability $p$ and $1 - p$ in the quenched manner. We also assume that there is no external magnetic field.

Figure 1 shows the phase diagram of this model with respect to $p$ and the temperature $T$ in the thermodynamic limit [3, 4, 10, 24]. There is a ferromagnetic region, where $T$ is sufficiently low and $p$ is close to unity. The dashed curve denotes the Nishimori line, which is specified by the Nishimori condition [6, 7]

$$\tanh \frac{J}{kT} = 2p - 1,$$

(2)

where $k$ denotes Boltzmann constant. On the curve, the random average of the thermal expectation value of the bond energy can be exactly evaluated as

$$\langle \varepsilon \rangle = -J \tanh \frac{J}{kT}.$$  

(3)

The dashed curve crosses the phase boundary at the Nishimori point $(p_c, T_c)$ shown by the black dot. It should be noted that $\langle \varepsilon \rangle$ in Eq. (3) shows no singularity at this point. Along the Nishimori line, the singularity in the thermodynamic free energy thus comes from the thermodynamic entropy only. The boundary between ferro- and paramagnetic phases changes its nature at the
For each face. The value of

\[ h^m_\ell = \frac{1}{2} \left[ I^m_\ell \sigma^m_\ell \sigma^{m+1}_{\ell+1} + I^m_{\ell+1} \sigma^{m+1}_{\ell} \sigma^m_{\ell+1} \\
+ J^m_\ell \sigma^m_\ell \sigma^{m+1}_{\ell+1} + J^m_{\ell+1} \sigma^{m+1}_{\ell} \sigma^m_{\ell+1} \right], \tag{4} \]

where the pre factor 1/2 reflects the fact that each bond is shared by two faces that are connected with each other. In the following, we consider a rectangular shaped finite size system of width \( N - 1 \) and height \( M - 1 \), where the size is counted by the number of faces. Thus the total number of the Ising spins in the rectangular system is \( NM \). For simplicity, we assume that \( N \) is even. Throughout this article we treat the finite-size system defined by the following Hamiltonian

\[ \mathcal{H}' = \sum_{m=1}^{M-1} \sum_{\ell=1}^{N-1} h^m_\ell, \tag{5} \]

with open boundary conditions. It should be noted that the interaction parameter on each bond at the system boundary is either \( J/2 \) or \( -J/2 \) by the definition of \( \mathcal{H}' \).

For a given spin configuration of the entire system, the Boltzmann weight for the whole system is expressed as

\[ e^{-\mathcal{H}'/kT} = \prod_{m=1}^{M-1} \prod_{\ell=1}^{N-1} \exp \left[ -\frac{h^m_\ell}{kT} \right], \tag{6} \]

which is the product of the local Boltzmann weight

\[ W^m_\ell = \exp \left[ -\frac{h^m_\ell}{kT} \right] \tag{7} \]

for each face. The value of \( W^m_\ell \) depends on the spins \( \sigma^m_\ell, \sigma^{m+1}_{\ell+1}, \) and \( \sigma^{m+1}_{\ell} \), and also on the interaction parameters \( I^m_\ell, I^m_{\ell+1}, J^m_\ell, \) and \( J^m_{\ell+1} \), that are either \( J \) or \( -J \). The partition function of the system is then represented as

\[ Z^M = \sum_{\text{conf.}} \prod_{m=1}^{M-1} \prod_{\ell=1}^{N-1} W^m_\ell, \tag{8} \]

where \( \text{conf.} \) denotes summation for all the possible spin configurations, which are \( 2^{NM} \) in total. We put index \( M \) on \( Z^M \) to clearly show the height of the system. We employ the transfer matrix formalism for the purpose of obtaining \( Z^M \) for various height \( M \) in a successive manner.

Let us introduce the notation \( \{\sigma^m\} \) which denotes the row of spins \( \sigma^m_1, \sigma^m_2, \ldots, \) and \( \sigma^m_N \), and define the transfer matrix from the \( m \)-th row to the \( (m+1) \)-th one as

\[ U^m(\{\sigma^m+1\} | \{\sigma^m\}) = \prod_{\ell=1}^{N-1} W^m_\ell, \tag{9} \]

where we have identified \( \{\sigma^m+1\} \) and \( \{\sigma^m\} \), respectively, as the left and the right matrix indices of \( U^m \). The matrix dimension is \( 2^N \). Figure 2 shows the position of spin variables in \( \{\sigma^m\} \) and \( \{\sigma^{m+1}\} \). Since \( U^m \) are dependent on \( m \), also \( U^m \) are. We can then simply express \( Z^M \) in the form

\[ Z^M = \sum_{\text{conf.}} \prod_{m=1}^{M-1} U^m(\{\sigma^m+1\} | \{\sigma^m\}). \tag{10} \]

By means of the transfer matrix, we can formally divide the configuration sum in Eqs. (9) and (10) into the configuration sum for each row of spins. We can perform the summation starting from the bottom \( \{\sigma^1\} \), to the top \( \{\sigma^M\} \). First we create a kind of partial sum

\[ V^2(\{\sigma^2\}) = \sum_{\{\sigma^1\}} U^1(\{\sigma^2\} | \{\sigma^1\}) \tag{11} \]

by taking configuration sum for all the spins contained in \( \{\sigma^1\} \). The obtained \( V^2(\{\sigma^2\}) \) can be regarded as the column vector of the dimension \( 2^N \). Note that this summation in Eq. (11) corresponds to the free boundary condition at the bottom of the system. For \( m \geq 2 \), we obtain the partial sum \( V^{m+1} \) from \( U^m \) and \( V^m \) recursively, in the manner as

\[ V^{m+1}(\{\sigma^{m+1}\}) = \sum_{\{\sigma^m\}} V^m(\{\sigma^m\}) U^m(\{\sigma^m+1\} | \{\sigma^m\}), \tag{12} \]

where the configuration sum is taken over all the spins in \( \{\sigma^m\} \). This summation can be regarded as the matrix multiplication between the transfer matrix \( U^m \) and the column vector \( V^m \). In short, we can express Eq. (12) as \( V^{m+1} = U^m V^m \). After the matrix multiplications of \( M - 2 \) times, we obtain \( V^M = U^{M-2} U^{M-4} \ldots U^2 V^2 \). Finally, we can express the partition function

\[ Z^M = \sum_{\{\sigma^M\}} V^M(\{\sigma^M\}), \tag{13} \]

by taking the configuration sum for \( \{\sigma^M\} \).
The probability of observing a particular spin configuration, which is specified by \(\{\sigma^M\}\) at the top of the finite size system, can be written by the ratio
\[
P(\{\sigma^M\}) = \frac{V^M(\{\sigma^M\})}{Z^M}.
\] (14)

This is the probability distribution function we have mentioned so far. It should be noted that we can immediately consider the finite size system that contains an additional row of faces. Setting the interaction parameters \(I^M_\ell = \pm J\) for \(\ell = 1, 2, \ldots, N\) and \(J^{M+1}_\ell = \pm J\) for \(\ell = 1, 2, \ldots, N-1\) in the random manner as we have considered in Eq. (1), we can additionally define the transfer matrix \(U^M\) by Eq. (9). Considering Eq. (12) and substituting \(m = M\), we have \(V^{M+1} = U^M V^M\), and thus \(Z^{M+1}\) can be calculated from Eq. (13). It is straightforward that we can repeat this prolonging process to the arbitrary height \(M'\) of the system. As a result, we obtain the set of probability distribution function \(P(\{\sigma^{M'}\})\) for various heights \(M'\). If we consider those cases \(M' \gg N\), which means that the system height is sufficiently larger than width, this set can be regarded as an ensemble of boundary spin distribution function corresponding to a number of random patterns, since the \(\pm J\) Ising model has the self-averaging property.

III. CLASSICAL ANALOGUE OF THE ENTANGLEMENT ENTROPY

If one represents the imaginary time evolution of a \(d\)-dimensional quantum system via the path-integral formulation, we effectively obtain a \(d + 1\)-dimensional classical (or statistical) system. This relation is often called the quantum-classical correspondence. In our case, the multiplication of the transfer matrix \(U^M\) in Eq. (12) can be regarded as a step of the discrete (time-dependent) imaginary time evolution for a one-dimensional spin system that is expressed by the unnormalized wave function \(V^M\). For the purpose of clarifying the quantum-classical correspondence, let us introduce the following normalization
\[
\Psi^M(\{\sigma^M\}) = \frac{V^M(\{\sigma^M\})}{\sqrt{\sum_{\{\sigma^M\}} V^M(\{\sigma^M\})}},
\] (15)

where we regard \(\Psi^M(\{\sigma^M\})\) as the normalized wave function for the effective spin chain, which contains \(N\) spins.

Once we have the wave function and the corresponding quantum state, whatever they are, we can introduce the concept of entanglement, which represents quantum correlation among different parts of the system. In particular, we focus on the bipartite entanglement, and thus we divide the row of spin \(\{\sigma^M\}\) into the left half \(\{\sigma_L\} \equiv \sigma^M_1, \ldots, \sigma^M_{N/2}\) and the right half \(\{\sigma_R\} \equiv \sigma^M_{N/2+1}, \ldots, \sigma^M_N\). We apply the singular value decomposition (SVD) to the normalized wave function,
\[
\Psi^M(\{\sigma_L\}, \{\sigma_R\}) = \sum_\xi \lambda_\xi A_\xi(\{\sigma_L\}) B_\xi(\{\sigma_R\}),
\] (16)

where the vectors \(A_\xi\) and \(B_\xi\) satisfy the orthogonal relations
\[
\sum_{\{\sigma_L\}} A_\xi(\{\sigma_L\}) A_\xi'(\{\sigma_L\}) = \delta_{\xi\xi'},
\]
\[
\sum_{\{\sigma_R\}} B_\xi(\{\sigma_R\}) B_\xi'(\{\sigma_R\}) = \delta_{\xi\xi'}.
\] (17)
The singular values, \(\lambda_\xi\) in Eq. (16), are non-negative, and their square \(\lambda_\xi^2\) represent the eigenvalue of the reduced density matrix for either the left side of the system \(\rho_L(\{\sigma_L\} \mid \{\sigma_L\})\)
\[
= \sum_{\{\sigma_R\}} \Psi^M(\{\sigma_L\}, \{\sigma_R\}) \Psi^M(\{\sigma_L\}, \{\sigma_R\}),
\] (18)
or the right side of the system. Therefore, \(\lambda_\xi^2\) is the probability of observing the eigenstate of the density matrix \(\rho_L\) in the left side of the system. Under the normalization of \(\Psi^M\) in Eq. (15), we have \(\sum_\xi \lambda_\xi^2 = 1\), and thus the probability \(\lambda_\xi^2\) is normalized as well. According to this probability, we can define the von Neumann entropy
\[
S^M = - \sum_\xi \lambda_\xi^2 \ln \lambda_\xi^2,
\] (19)
which is the entanglement entropy we focus on in this article. Since we have been considering the \(\pm J\) Ising model, which is not a quantum system but a classical statistical lattice model, let us call \(S^M\) the classical analogue of the entanglement entropy; when there is no danger of confusion, we simply call \(S\) entanglement entropy, though. Because of the randomness in the system, \(S^M\) is dependent on the height \(M\). We introduce the average
\[
\langle S \rangle = \frac{1}{D} \sum_{M=M_0}^{M_0+D-1} S^M
\] (20)
over \(D\) samples, starting from an offset \(M_0 \gg N\). This value \(\langle S \rangle\) is obtained numerically in the following section.

IV. NUMERICAL RESULTS

For a uniform quantum system, the entanglement entropy is asymptotically proportional to the logarithm of the correlation length \([14, 15]\). Under the presence of randomness, the relation is not trivial. Thus we numerically investigate the behavior of \(\langle S \rangle\) in Eq. (20). In particular, we target the Nishimori point, which is essential in the phase diagram of the \(\pm J\) Ising model.
We numerically calculate the partial sum $V^M$ in Eqs. (11)-(14) with high numerical precision by means of the TEBD method [17, 18, 20], which has many aspects in common with the imaginary-time evolution by DMRG method [19]. The method allows us to represent $V^M$ in the form of orthogonal matrix product [21, 22]. Under the representation, the norm of $V^M$, which appears in the right-hand side of Eq. (15) as the denominator, can be obtained with small numerical effort, and also the wave function $\Psi^M$ can be. In each height $M$, the transfer matrix $U^M$ is implicitly prepared by creating $W^M_k$ for $\ell = 1$ to $N - \ell$ according to the stochastically created randomness for the bond interactions. The multiplication process $V^{M+1} = U^M V^M$ is performed by applying $W^M_k$ each by each to $V^M$ and taking configuration sum locally. It should be noted that $W^M_k$ does not represent local unitary evolution, and therefore just after the multiplication, $V^{M+1}$ is not represented as the canonical orthogonal matrix product [23]. Thus we adjust the matrices contained in $V^{M+1}$ into the correct canonical form by sweeping without any evolution in the TEBD method. After that, we calculate the singular values $\lambda^M_k$ in Eq. (16) at the center of the system to obtain $S^M$ by Eq. (19).

The necessary matrix dimension, $\chi$ in the matrix product representation, is dependent on the system width $N$, since long-range entanglement occasionally appears during the successive creation of $\Psi^M$. We treat the system widths up to $N = 300$, and therefore we check the convergence in $\langle S \rangle$ with respect to $\chi$ for the severest case, when $N = 300$ and at the Nishimori point. Performing trial calculations up to $\chi = 28$, we confirmed that $\chi$-dependence in $\langle S \rangle$ is negligible if $\chi \geq 22$ is satisfied. Thus we choose $\chi = 24$ in the following calculations, since $\chi = 28$ is numerically too time-consuming if the parameter sweeping on the Nishimori line is considered. The system size limitation chiefly comes from the computational time required for the random average, while the memory/storage requirement is not severe at all. Most of the numerical calculations are performed on K-computer, a parallelized super computer.

We always set $M_0$ in Eq. (20), the number of discarded samples, more than 10 times larger than $N$, in order to avoid the effect of open boundary condition at the bottom of the system. The number of samples is chosen from $D = 5 \times 10^4$ to $D = 2.5 \times 10^6$, depending on the system size $N$ and also on the purpose of obtaining $\langle S \rangle$. In case the system is critical, the effective number of independent samples is roughly estimated as $D/N$. Thus we divide the $D$ numbers of $S^M$ into bins, each of that contain 1000 steps, and calculate the sub averages of $S^M$ in each bin. Assuming that these sub averages obey the Gaussian distribution, we estimate the standard deviation in $\langle S \rangle$ [23], and two times the standard deviation is considered as the numerical error in $\langle S \rangle$.

Throughout this section, we choose the parameter $J$ as the unit of energy, and set $k = 1$. All the calculations are performed for the parameter set $(p, T)$ that satisfies the Nishimori condition in Eq. (2), within the range $0.3 \leq T \leq 2.0$. We choose temperature $T$ as an independent parameter, and regard the probability $p$ as a function of $T$. The Nishimori line starts from $(p, T) = (1, 0)$, where the corresponding quantum state is the superposition of all-up and all-down ferromagnetic states according to the free boundary condition, and $\langle S \rangle$ is equal to $\ln 2$ regardless of the system size $N$. The value $\ln 2$ is nothing but the entanglement entropy of the Greenberger-Horne-Zeilinger (GHZ) state [28].

Figure 3 shows the calculated entanglement entropy $\langle S \rangle$ with respect to $T$. Each plot is obtained from $D = 5 \times 10^4$ samples. When the temperature $T$ is sufficiently larger than $J/k = 1$, where $p$ is close to $1/2$, $\langle S \rangle$ is a decreasing function of $T$ and converges to zero in high temperature limit, as it is naturally expected. When the system size is relatively large, a peak appears in $\langle S \rangle$ in the neighborhood of $T \sim 1.0$, where the peak height increases with $N$. The peak position is also an increasing function of $N$.

In order to observe the peak structure of $\langle S \rangle$ in detail, we calculate it within the narrow temperature window $0.92 \leq T \leq 0.98$ as shown in Fig. 4, where $D = 1.5 \times 10^6$.
samples are taken for the cases $N = 150, 200,$ and $300$, according to Eq. (21).

For the purpose of determining the critical temperature $T_c$ in the thermodynamic limit $N \to \infty$, we employ the Bayesian inference method by Harada [29], the method which was successfully applied to random systems by Nakamura and Shirakura [30], and by Dupont et al [31]. The temperature region $0.92 \leq T \leq 0.98$ is considered for the cases $N = 150$ and $200$, and $0.93 \leq T \leq 0.98$ is considered for $N = 300$ in this FSS analysis.

Figure 5 shows the obtained scaling plot. From this best-fit result, the critical temperature is estimated as $T_c = 0.9564(3)$, where the corresponding probability is $p_c = 0.89004(6)$; the number in the parenthesis represents the standard deviation in the last digit. The value is slightly smaller than previously reported ones $p_c = 0.8906 \sim 0.8908$ [4, 12, 32, 33]. As the estimation for the critical exponent $\nu$, we obtain $\nu = 1.59(4)$ through the scaling analysis. This value is larger than $\nu = 1.33$ reported by Picco et al [12]. The central charge is estimated as $c = 0.397(2)$. The estimated $\nu$ through the scaling analysis would contain relatively large error, since they tend to be affected by the slight change of the temperature window $0.92 \leq T \leq 0.98$, which is considered in the current study, while the value of $T_c$ and $c$ is relatively stable.

We perform an additional calculation at the estimated $T_c$ in order to confirm the critical behavior in $S$, collecting $D = 1.5 \times 10^6$ samples for $N = 80, 120, 160, 240,$ and $320$. Figure 6 shows $\langle S \rangle$ at $T = 0.9564$ with respect to $\ln N$. The linear dependence $\langle S \rangle \propto \ln N$ is clearly observed for larger $N$. If we assume the conformal invariance at criticality, where the leading term of the entropy is proportional to $\frac{c}{6} \ln N$, the relation which is implicitly shown in Eq. (21), we obtain the central charge $c = 0.404(3)$ if we use all the plotted data, and $c = 0.395(4)$ if we consider the cases $N = 160, 240,$ and $320$ only. The latter estimate is consistent with $c = 0.397(2)$ obtained from the finite size scaling in Fig. 5. These estimated values of $c$ are smaller than $c = 0.464$ by Picco et al [12] and $c = 0.463$ by de Queiroz et al [32].

V. CONCLUSIONS AND DISCUSSIONS

The entanglement entropy $\langle S \rangle$ of the probability distribution function of the $\pm J$ Ising model is numerically analyzed on the system boundary, by means of the transfer matrix formalism. The distribution function is expressed by the orthogonal matrix product, where the transfer-matrix multiplication is performed by the TEBD method. Non-analytic behavior is observed in $\langle S \rangle$ around the Nishimori point. The estimated critical temperature $T_c = 0.9564$ is slightly higher than the previously reported values, as long as it is determined from the data shown in Fig. 4.

The estimated central charge $c = 0.397(2)$ by finite size scaling and $c = 0.395(4)$ from Fig. 6 are smaller than those reported values $c = 0.463 \sim 0.464$ so far [12, 32]. The discrepancy could be attributed by the fact that we have observed the boundary spin distribution, where the statistical properties could be qualitatively different from those of the bulk part. It should be noted that the definition of the classical analogue of the entanglement entropy in the bulk part is not trivial when there is randomness. This is because there is no (imaginary-) time reversal symmetry in the transfer matrix formalism, and that the spin distribution functions can be defined independently from upper and lower halves of the system. Another possible scenario is that the correction to the scaling, which comes from the sub-leading terms to the scaling form of the free energy, affects the estimation of the central charge $c$, in particular when $N$ is small.
A numerical challenge is to obtain better estimates for $T_c$ and central charge $c$. Further extensive calculations for larger system sizes $N$ would be required for this aim. To increase $N$ seems to be straightforward in TEBD method, but spontaneous symmetry breaking, which is caused by a tiny numerical round-off error, frequently occurs when $N$ is larger than 300. This is partially because the artificial energy gap decreases with respect to $N$. This problem could be solved by always breaking the spin inversion symmetry, introducing a weak external magnetic field $h$ or imposing ferromagnetic boundary condition.

In the phase diagram of the $\pm J$ Ising model, there are wide possible choices of the parameter set $(p, T)$ other than those on the Nishimori line: we have performed trial calculations. At the phase boundary between the Nishimori point and the transition point of the pure Ising model with $p = 1$, the TEBD method can be easily applied, and the transition nature agrees with the Ising universality. On the other hand, the analysis at the phase boundary below the Nishimori point is not straightforward, since the spontaneous symmetry breaking we have discussed easily occur when the ferromagnetic bonds are accidentally concentrated. We have to modify the numerical procedures so that the GHZ-like state can be treated in a stable manner.

The partial sum $V^M$ obtained by the TEBD method can be used for the estimation of various quantities, such as internal energy and correlation functions. From the normalization factor of $V^M$ it is also possible to obtain the free energy of the system, and thus one can obtain the thermodynamic entropy by subtracting the internal energy from the free energy, whereas the numerical cost is relatively large.

Our further interest is in a spatial distribution of the entanglement entropy at the system boundary. Such analysis has been carried out for one-dimensional random-bond quantum spin chains, which have a layered structure in entangled pair [34, 35]. In the case of the $\pm J$ Ising model we have treated, the randomness is present in both horizontal (or spatial) and vertical (or time) directions of the lattice. A method of treating such an ‘isotropic’ disorder is the tensor renormalization group (TRG) [36, 37], which was once applied to $\pm J$ Ising model [38, 39]. From the viewpoint of the modern tensor network renormalization (TNR) formalisms [40, 42], to capture the entanglement structure contained in the system is important for numerically precise renormalization-group transformations. What would be the appropriate, or adaptive, tensor-network structure under such randomness?

The $\pm J$ Ising model on the square lattice does not possess the spin glass phase, and therefore, it is not possible to observe singular behaviors of the entanglement entropy around the spin-glass transition. Such a study can be performed on the cubic lattice, where the application of the TEBD method would require much more extensive computation. To observe Rény entropy is another choice, which can be detected by means of the Monte-Carlo simulations.

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