Stochastic analysis on sub-Riemannian manifolds with transverse symmetries

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Dedicated, with admiration, to Donald Burkholder

Abstract

We prove a geometrically meaningful stochastic representation of the derivative of the heat semigroup on sub-Riemannian manifolds with transverse symmetries. This representation is obtained from the study of Bochner-Weitzenböck type formulas for sub-Laplacians on 1-forms. As a consequence, we prove new hypoelliptic heat semigroup gradient bounds under natural global geometric conditions. The results are new even in the case of the Heisenberg group which is the simplest example of a sub-Riemannian manifold with transverse symmetries.

Contents

1 Introduction 1
2 Sub-Riemannian manifolds with transverse symmetries 4
3 Bochner-Weitzenböck formulas for sub-Laplacians on one-forms 8
4 Gradient formulas and bounds for the heat semigroup 15
  4.1 Heat semigroup on one-forms 15
  4.2 Stochastic representation of the semigroup on one-forms 16
  4.3 Integration by parts formula 18
  4.4 Positive curvature and convergence to equilibrium 20

1 Introduction

As shown in the classical monographs by Hsu [21] and Stroock [32], stochastic analysis provide a set of powerful tools to study the geometry of manifolds. However, as of today,
most of the global applications are restricted to Riemannian geometry. The goal of the present work is to start developing global stochastic analysis tools in sub-Riemannian geometry.

A sub-Riemannian manifold is a smooth manifold $M$ equipped with a non-holonomic, or bracket generating, subbundle $\mathcal{H} \subset T\!M$ and a fiber inner product $g_{\mathcal{H}}$. This means that if we denote by $L(\mathcal{H})$ the Lie algebra of the vector fields generated by the global $C^\infty$ sections of $\mathcal{H}$, then $\text{span}\{X(x) \mid X \in L(\mathcal{H})\} = T_x(M)$ for every $x \in M$. Sub-Riemannian geometry takes its roots in very old problems related to isoperimetry but was internationally brought to the attention of mathematicians by E. Cartan’s pioneering address [11] at the Bologna International Congress of Mathematicians in 1928. Since then, it has been the focus of numerous studies by geometers. In particular, one should consult the monographs by Agrachev [1], Bellaïche [7], Gromov [19] and Montgomery [24] and the references therein.

For the last four decades, sub-Riemannian geometry has also been a center of interest for analysts because it is the natural geometry associated to subelliptic partial differential equations (see [25, 28]). Perhaps more surprisingly, sub-Riemannian geometry has also been widely studied by probabilists since the breakthrough [23] by Malliavin in 1976, where stochastic analysis and hypoellipticity theory merged together. The paper gave birth to the nowadays called Malliavin calculus, which has then be successfully applied in the study of hypoelliptic heat kernels. We mention in particular the works by Benarous [8], and Kusuoka-Stroock [22]. One may also consult the monograph [2] for further connections between probability theory and sub-Riemannian geometry.

Despite being an object of intensive studies, most of the developments in sub-Riemannian geometry to date are of a local nature. As a consequence, the theory presently lacks a body of results which, similarly to the Riemannian case, connect global properties of solutions of the relevant partial differential equations, or of the relevant stochastic processes to the geometry of the ambient manifold. However, in some special sub-Riemannian structures, a notion of Ricci lower bound has been made precise in several recent works [4, 5, 6]. Numerous new hypoelliptic functional inequalities were then obtained as a consequence. We mention in particular the subelliptic Li-Yau inequalities (see [6]), the subelliptic parabolic Harnack inequalities (see [6]), the Poincaré inequalities on balls (see [5]) and the log-Sobolev inequalities (see [4]).

In the present paper, by using probabilistic methods, we in particular reprove and actually largely improve under weaker conditions several inequalities that were obtained in [4] by using purely analytic methods. We also get new hypoelliptic inequalities which seem difficult to directly prove by analysis. We also should mention, that in our opinion, the probability method is in a sense more direct and overall simpler than the analytic methods that were built in [6]. More precisely, the results in Section 3 and 4 in [6] which were used to prove Hypothesis 1.4 may now be omitted, since Hypothesis 1.4 is a straightforward consequence of Corollary [4,6].
We now describe our main results. The paper is divided into two parts, a geometric part and a probabilistic part. The geometric part is devoted to the study of Bochner-Weitzenböck type formulas on sub-Riemannian manifolds with transverse symmetries (see Section 2 for the definitions). More precisely, our goal will be to introduce a natural family $\Box_\varepsilon$, $\varepsilon > 0$, of sub-Laplacians on one-forms that satisfy the intertwining

$$dL = \Box_\varepsilon d,$$

(1.1)

where $L$ is the sub-Laplacian and $d$ the exterior derivative. The operator $\Box_\varepsilon$ will be self-adjoint with respect to a Riemannian metric extension that contracts in the sense of Strichartz [30] to the sub-Riemannian metric when $\varepsilon \to 0$. Our main geometric result is then Theorem 3.3 where we prove that

$$\Box_\varepsilon = -(\nabla - \mathcal{F}^\varepsilon)^*(\nabla - \mathcal{F}^\varepsilon) + \frac{1}{2\varepsilon}J^*J - \mathfrak{Ric}_H,$$

and that for any smooth one-form $\eta$,

$$\frac{1}{2}L\|\eta\|_{2\varepsilon}^2 - \langle \Box_\varepsilon \eta, \eta \rangle_{2\varepsilon} = \sum_{i=1}^{d} \|\nabla_{X_i} \eta - \mathcal{F}^\varepsilon_{X_i} \eta\|^2_{2\varepsilon} + \mathfrak{Ric}_H - \frac{1}{2\varepsilon}J^*J.$$

The quantities $\mathcal{F}^\varepsilon$, $J^*J$ and $\mathfrak{Ric}_H$ are tensors that will be introduced in the text. We should mention that, to our knowledge, this Bochner-Weitzenböck formula is new even in the case of the Heisenberg group and it implies in a straightforward way the horizontal and the vertical Bochner’s identities proved in [6].

In the second part of the paper, we exploit the commutation (1.1) to give a probabilistic representation of the derivative $dP_t$ where $P_t$ is the semigroup generated by the sub-Laplacian. The representation actually follows from (1.1) by adapting in our case classical ideas by Bismut [9], Driver-Thalmaier [15], Elworthy [16, 17] and Thalmaier [33]. We deduce from this representation an integration by part formula in the spirit of Driver [13]. Several hypoelliptic heat semigroup gradient estimates are then obtained as a consequence.

To conclude, we should mention that the inequalities we obtain, in the spirit of [4], involve a vertical gradient. This, of course, does not mean that they are not geometrically meaningful, because we can see for instance that the gradient bound in Corollary 4.6 is actually equivalent to a global lower bound on the horizontal Ricci tensor of the sub-Riemannian connection. These hypoelliptic inequalities with vertical gradient have also been successfully been used in geometry, where real geometric theorems were proved as a consequence, like the subelliptic Bonnet-Myers [6] and in analysis where convergence to equilibrium for hypoelliptic kinetic Fokker-Planck equations were established [3, 34]. On the other hand, we have to say that the Driver-Melcher inequality [14] in the Heisenberg group, which only involves the horizontal gradient still remains a little mysterious for us, and that it would of course be extremely interesting to connect those type of inequalities to global geometric quantities.
2 Sub-Riemannian manifolds with transverse symmetries

The notion of sub-Riemannian manifold with transverse symmetries was introduced in [6]. We recall here the main geometric quantities and operators related to this structure that will be needed in the sequel and we refer to [6] for further details. We also introduce some new geometric invariants that shall later be needed.

Let $M$ be a smooth, connected manifold with dimension $d + h$. We assume that $M$ is equipped with a bracket generating distribution $\mathcal{H}$ of dimension $d$ and a fiberwise inner product $g_\mathcal{H}$ on that distribution. The distribution $\mathcal{H}$ is referred to as the set of horizontal directions. Sub-Riemannian geometry is the study of the geometry which is intrinsically associated to $(\mathcal{H}, g_\mathcal{H})$ (see [30]). In general, there is no canonical vertical complement of $\mathcal{H}$ in the tangent bundle $TM$, but in some cases the fiberwise inner product $g_\mathcal{H}$ determines one.

**Definition 2.1** It is said that $M$ is a sub-Riemannian manifold with transverse symmetries if there exists a $h$-dimensional Lie algebra $\mathcal{V}$ of sub-Riemannian Killing vector fields such that for every $x \in M$,

$$T_xM = \mathcal{H}(x) \oplus \mathcal{V}(x).$$

From now on, we assume that $M$ is a sub-Riemannian manifold with transverse symmetries. The distribution $\mathcal{V}$ is referred to as the set of vertical directions. The choice of an inner product $g_\mathcal{V}$ on the Lie algebra $\mathcal{V}$ naturally endows $M$ with a one-parameter family of Riemannian metrics that makes the decomposition $\mathcal{H} \oplus \mathcal{V}$ orthogonal:

$$g_\varepsilon = g_\mathcal{H} \oplus \varepsilon g_\mathcal{V}, \quad \varepsilon > 0.$$ 

Although $g_\varepsilon$ will be useful for the purpose of computations, the geometric objects that we are interested in, like the sub-Laplacian $L$ and its associated semigroup will of course not depend on $\varepsilon$.

The Riemannian volume measure of $(M, g_\varepsilon)$ is always a multiple of the Riemannian volume measure of $(M, g_1)$, therefore we will always use the Riemannian volume measure of $(M, g_1)$ which we will denote $\mu$. For notational convenience, we will often use the notation $\langle \cdot, \cdot \rangle_\varepsilon$ instead of $g_\varepsilon$.

At every point $x \in M$, we can find a local frame of vector fields $\{X_1, \cdots, X_d, Z_1, \cdots, Z_h\}$ such that on a neighborhood of $x$:

(a) $\{X_1, \cdots, X_d\}$ is a $g_\mathcal{H}$-orthonormal basis of $\mathcal{H}$;

(b) $\{Z_1, \cdots, Z_h\}$ is a $g_\mathcal{V}$-orthonormal basis of $\mathcal{V}$;

We observe that the following commutation relations hold:

$$[X_i, X_j] = \sum_{\ell=1}^d \omega_{ij}^\ell X_\ell + \sum_{m=1}^h \gamma_{ij}^m Z_m, \quad (2.2)$$
\[ [X_i, Z_m] = \sum_{\ell=1}^{d} \delta_{im}^{\ell} X_{\ell}, \]  

(2.3)

for smooth functions \(\omega_{ij}^{\ell}, \gamma_{ij}^{m}\) and \(\delta_{im}^{\ell}\) such that

\[ \delta_{im}^{\ell} = -\delta_{im}^{\ell}, \quad i, \ell = 1, \ldots, d, \text{ and } m = 1, \ldots, h. \]  

(2.4)

The property (2.4) follows from the property of \(Z_m\) being a sub-Riemannian Killing field. By convention, \(\omega_{ij}^{\ell} = -\omega_{ji}^{\ell}, \gamma_{ij}^{m} = -\gamma_{ji}^{m}\) and \(\delta_{im}^{\ell} = -\delta_{mi}^{\ell}\).

We define the horizontal gradient \(\nabla_H f\) of a function \(f\) as the projection of the Riemannian gradient of \(f\) on the horizontal bundle. Similarly, we define the vertical gradient \(\nabla_V f\) of a function \(f\) as the projection of the Riemannian gradient of \(f\) on the vertical bundle. In a local adapted frame, we have

\[ \nabla_H f = \sum_{i=1}^{d} (X_i f) X_i, \]

and

\[ \nabla_V f = \sum_{m=1}^{h} (Z_m f) Z_m. \]

The canonical sub-Laplacian in a sub-Riemannian manifold with transverse symmetries is the generator of the symmetric Dirichlet form

\[ \mathcal{E}_\varepsilon(f, g) = \int_{M} \langle \nabla_H f, \nabla_H g \rangle_\varepsilon d\mu. \]

It is a diffusion operator \(L\) on \(\mathbb{M}\) which is symmetric on \(C_0^\infty(\mathbb{M})\) with respect to the measure \(\mu\) and which does not depend on \(\varepsilon\).

Actually, it is readily seen that in an adapted frame, one has

\[ L = -\sum_{i=1}^{d} X_i^* X_i, \]

where \(X_i^*\) is the formal adjoint of \(X_i\). From the commutation relations in an adapted frame, we see that

\[ X_i^* = -X_i + \sum_{k=1}^{d} \omega_{ik}^{k}, \]

so that,

\[ L = \sum_{i=1}^{d} X_i^2 + X_0, \]

(2.5)
with
\[ X_0 = - \sum_{i,k=1}^{d} \omega^k_{i,k} X_i. \] (2.6)

On sub-Riemannian manifolds with transverse symmetries, there is a canonical connection.

**Proposition 2.2** (See [6]) There exists a unique connection \( \nabla \) on \( M \) satisfying the following properties:

(i) \( \nabla g_\varepsilon = 0, \varepsilon > 0; \)

(ii) If \( X \) and \( Y \) are horizontal vector fields, \( \nabla_X Y \) is horizontal;

(iii) If \( Z \in \mathcal{V} \), \( \nabla Z = 0; \)

(iv) If \( X, Y \) are horizontal vector fields and \( Z \in \mathcal{V} \), the torsion vector field \( T(X, Y) \) is vertical and \( T(X, Z) = 0. \)

Intuitively \( \nabla \) is the connection which coincides with the Levi-Civita connection of the Riemannian metric \( g_\varepsilon \) on the horizontal bundle \( \mathcal{H} \) and that parallelizes the Lie algebra \( \mathcal{V} \). Straightforward computations show that one has in a local adapted frame:

\[
\nabla_{X_i} X_j = \sum_{k=1}^{d} \frac{1}{2} \left( \omega^k_{ij} + \omega^j_{ki} + \omega^i_{kj} \right) X_k, \tag{2.7}
\]

\[
\nabla_{Z_m} X_i = - \sum_{\ell=1}^{d} \delta^\ell_{im} X_\ell, \tag{2.8}
\]

\[
\nabla Z_m = 0, \tag{2.9}
\]

and

\[
T(X_i, X_j) = - \sum_{m=1}^{h} \gamma^m_{ij} Z_m.
\]

We observe that, thanks to (2.5) and (2.6), in a local adapted frame we have

\[
L = \sum_{i=1}^{d} X_i^2 - \nabla_{X_i} X_i.
\]

To establish Bochner-Weitzenböck formulas, it will expedient to work in normal frames.

**Lemma 2.3** Let \( x \in M \). There exists a local adapted frame of vector fields

\[
\{X_1, \cdots, X_d, Z_1, \cdots, Z_h\}
\]

around \( x \), such that, at \( x \),

\[
\nabla_{X_i} X_j = 0.
\]

Such frame will be called an adapted normal frame around \( x \).
Proof. Since $\nabla$ coincides with a Levi-Civita connection on the horizontal bundle, the result essentially boils down to the existence of normal frames in Riemannian geometry. \hfill \Box

Observe that in a normal adapted frame, we have $\omega_{ij}^k = 0$. We now introduce some maps that will play an important role in the sequel. For $Z \in \mathcal{V}$, there is a unique skew-symmetric map $J_Z$ defined on the horizontal bundle $\mathcal{H}$ such that for every horizontal vector fields $X$ and $Y$,

$$g_{\mathcal{H}}(J_Z(X), Y) = g_{\mathcal{V}}(Z, T(X, Y)). \quad (2.10)$$

In a local adapted frame, we have

$$J_{Z_m}(X_i) = -\sum_{j=1}^{d} \gamma_{ij}^m X_j.$$ 

We finally recall the following definition that was introduced in [6]:

**Definition 2.4** The sub-Riemannian manifold $\mathcal{M}$ is said to be of Yang-Mills type, if for every horizontal vector field $X$, and any adapted local frame \(\{X_1, \cdots, X_d, Z_1, \cdots, Z_h\}\)

$$\sum_{\ell=1}^{d} (\nabla_{X_i} T)(X_{\ell}, X) = 0.$$ 

A quick computation shows that $\mathcal{M}$ is of Yang-Mills type if and only if for every $x \in \mathcal{M}$ and any adapted normal frame \(\{X_1, \cdots, X_d, Z_1, \cdots, Z_h\}\) around $x$, we have at $x$,

$$\sum_{i=1}^{d} X_i \gamma_{ij}^m = 0, \quad 1 \leq j \leq d, 1 \leq m \leq h.$$ 

There are many interesting examples of Yang-Mills sub-Riemannian manifolds with transverse symmetries (see [6]). An important class is given by the class of $K$-contact manifolds. A $K$-contact manifold is a contact manifold for which the Reeb vector $Z$ is a sub-Riemannian Killing vector field. Sasakian manifolds are examples of $K$-contact manifolds (see [12]) and in that case $J_Z$ is a complex structure on the horizontal bundle.

The 3-dimensional model spaces in $K$-contact geometry are Lie groups and we quickly describe them since they may serve as a guiding example and our results are already new in that case.

Given a number $\rho \in \mathbb{R}$, suppose that $\mathbb{G}(\rho)$ is a simply connected three-dimensional Lie group whose Lie algebra $\mathfrak{g}$ has a basis $\{X, Y, Z\}$ satisfying:

(i) $[X, Y] = Z$,

(ii) $[X, Z] = -\rho Y$,  

7
(iii) $[Y,Z] = \rho X$.

For instance, for $\rho = 0$, $G(\rho)$ is the Heisenberg group. For $\rho = 1$, $G(\rho)$ is $SU(2)$ and for $\rho = -1$, $G(\rho)$ is $SL(2)$. It is easy to see that if we consider the left-invariant distribution $\mathcal{H}$ generated by $\{X,Y\}$ and chose for $g_\mathcal{H}$ the left-invariant metric that makes $\{X,Y\}$ orthonormal then $(M,\mathcal{H},g_\mathcal{H})$ is a Yang-Mills sub-Riemannian manifold with transverse symmetry $Z$.

The sub-Laplacian on $G(\rho)$ is the left-invariant, second-order differential operator

$$L = X^2 + Y^2$$

and the connection $\nabla$ is given by

$$\nabla_X Y = \nabla_Y X = \nabla_X Z = \nabla_Y Z = 0$$

and

$$\nabla_Z X = -\rho Y, \quad \nabla_Z Y = \rho X.$$

### 3 Bochner-Weitzenböck formulas for sub-Laplacians on one-forms

From now on, in all the paper we consider a Yang-Mills sub-Riemannian manifold $M$ with transverse symmetries and adopt the notations of the previous section. In particular $L$ denotes the sub-Laplacian on $M$.

Obviously, there exist infinitely many second order differential operators $\mathcal{L}$ defined on one-forms such that for every smooth function $f$,

$$dL f = \mathcal{L} df,$$

where $d$ is the exterior derivative. In Riemannian geometry, a canonical $\mathcal{L}$ that satisfies the above commutation is the Hodge-de Rham Laplacian. On sub-Riemannian manifolds, even contact manifolds, there is no such canonical sub-Laplacian (see [27]) on one-forms. However, in our case, we will see in this section that there is a distinguished one-parameter family of sub-Laplacians on one-forms which are optimal when interested in Bochner-Weitzenböck’s type formulas and that satisfy the above commutation.

We first introduce some definitions and notations. A one-form $\eta$ will be said to be horizontal if for every $Z \in \mathcal{V}$, $\eta(Z) = 0$. If $\{X_1, \cdots, X_d, Z_1, \cdots, Z_h\}$ is a local adapted frame, then the corresponding coframe (dual basis) will be denoted by $\{\theta_1, \cdots, \theta_d, \nu_1, \cdots, \nu_h\}$

We will still denote by $L$ the covariant extension on one-forms of the sub-Laplacian. In a local adapted frame, we have thus

$$L = \sum_{i=1}^d \nabla_{X_i} \nabla_{X_i} - \nabla_{\nabla_{X_i} X_i}.$$
We define then $\text{Ric}_H$ as the fiberwise symmetric linear map on one forms such that for every smooth functions $f, g$,

$$\langle \text{Ric}_H(df), dg \rangle = \text{Ricci}(\nabla_H f, \nabla_H g),$$

where $\text{Ricci}$ is the Ricci curvature of the connection $\nabla$. Of course, $\text{Ric}_H$ does not depend on $\varepsilon$ because the above definition implies that $\text{Ric}_H$ is horizontal, that is transforms any one-form into an horizontal form. Actually, a computation shows that in a normal adapted frame around $x$, we have at $x$,

$$\text{Ric}_H(\eta) = \sum_{k, \ell=1}^{d} \frac{1}{2} (\rho_{k\ell} + \rho_{\ell k}) f_k \theta_{\ell},$$

where $\eta = \sum_{i=1}^{d} f_i \theta_i + \sum_{m=1}^{h} g_m \nu_m$ and

$$\rho_{k\ell} = \sum_{j=1}^{d} \sum_{m=1}^{h} \gamma_{k\ell}^m g_{j}^m + \sum_{j=1}^{d} X_{k}^j \omega_{\ell j}^j - X_{\ell}^j \omega_{k j}^j.$$

Finally, we consider the first order differential operator $\mathcal{J}$ defined in a local adapted frame by

$$\mathcal{J}(\eta) = \sum_{i,j=1}^{d} \sum_{m=1}^{h} \gamma_{ij}^m (X_j g_m) \theta_i,$$

where, again, $\eta = \sum_{i=1}^{d} f_i \theta_i + \sum_{m=1}^{h} g_m \nu_m$. More intrinsically, we can write

$$\mathcal{J} = \sum_{m=1}^{h} J_{Z_m}(d \iota Z_m)$$

where $J_{Z_m}$ is defined in (2.10) and $\iota$ is the interior product. This last expression shows that $\mathcal{J}$ does not depend on the choice on the local frame, and is therefore a globally defined first order differential operator on one-forms.

We are now in position to prove our first commutation result.

**Proposition 3.1** Let

$$\Box_\infty = L + 2\mathcal{J} - \text{Ric}_H.$$ 

Then, we have for every smooth function $f$,

$$dLf = \Box_\infty df. \quad (3.11)$$
Proof. Let $x \in \mathbb{M}$. It is enough to prove this commutation at $x$ in a local adapted normal frame $\{X_1, \cdots, X_d, Z_1, \cdots, Z_h\}$ around $x$. Observing that $L$ and $Z_m$ commute (see [6]), we have:

$$dLf = \sum_{i=1}^{d}(X_i Lf)\theta_i + \sum_{m=1}^{h}(Z_m Lf)\nu_m$$

$$= \sum_{i=1}^{d}(LX_i f)\theta_i + \sum_{m=1}^{h}(LZ_m f)\nu_m + \sum_{i=1}^{d}[X_i, L]f\theta_i$$

$$= Ldf + \sum_{i=1}^{d}([X_i, L]f)\theta_i.$$  

Keeping in mind that at the center of the frame $\omega^k_{ij} = 0$, and thanks to the Yang-Mills assumption

$$\sum_{i=1}^{d} X_i \gamma^m_{ij} = 0,$$

we now compute:

$$\sum_{i=1}^{d} ([X_i, L]f)\theta_i$$

$$= \sum_{i,j=1}^{d} ([X_i, X_j^2]f)\theta_i + \sum_{i=1}^{d} ([X_i, X_0]f)\theta_i$$

$$= \sum_{i=1}^{d} \left( [X_i, X_j]X_j f + X_j[X_i, X_j]f - \sum_{j,k=1}^{d} [X_i, \omega^k_{jk} X_j]f \right) \theta_i$$

$$= \sum_{i=1}^{d} \left( \sum_{j=1}^{d} \sum_{m=1}^{h} \gamma^m_{ij} (Z_m X_j f + X_j Z_m f) + \sum_{j,k=1}^{d} (X_j \omega^k_{ij} - X_i \omega^k_{jk}) X_k f \right) \theta_i$$

$$= \sum_{i=1}^{d} \left( 2 \sum_{j=1}^{d} \sum_{m=1}^{h} \gamma^m_{ij} (X_j Z_m f) - \sum_{j,k=1}^{d} \sum_{m=1}^{h} \gamma^m_{ij} \delta^k_{jm} X_k f + \sum_{j,k=1}^{d} (X_j \omega^k_{ij} - X_i \omega^k_{jk}) X_k f \right) \theta_i$$

It is now elementary to identify the terms in the above equality.  

Obviously, $\Box_{\infty}$ is not the only operator that satisfies (3.11). Actually, since $d^2 = 0$, if $\Lambda$ is any fiberwise linear map from the space of two-forms into the space of one-forms, then we have

$$dLf = (\Box_{\infty} + \Lambda \circ df).$$

This raises the question of an optimal choice of $\Lambda$. The following proposition answers this question if optimality is understood in the sense of a corresponding Bochner-Weitzenböck’s formula.
Proposition 3.2 For any fiberwise linear map $\Lambda$ from the space of two-forms into the space of one-forms, and any $x \in \mathcal{M}$, we have

\[
\inf_{\eta, \|\eta(x)\| = 1} \left( \frac{1}{2} (L\|\eta\|_\varepsilon^2) (x) - \langle (\square_{\infty} + \Lambda \circ d)\eta(x), \eta(x) \rangle_\varepsilon \right)
\leq \inf_{\eta, \|\eta(x)\| = 1} \left( \frac{1}{2} (L\|\eta\|_\varepsilon^2) (x) - \left\langle \left( \square_{\infty} - \frac{1}{\varepsilon} T \circ d \right) \eta(x), \eta(x) \right\rangle_\varepsilon \right),
\]

where in the above notation, the torsion tensor $T$ is interpreted, by duality, as a fiberwise linear map from the space of two-forms into the space of one-forms.

Proof. Let $x \in \mathcal{M}$ and consider a normal adapted frame around $x$. The following computations are done at the center $x$ of the frame. Let us consider a smooth one-form

\[
\eta = \sum_{i=1}^d f_i \theta_i + \sum_{m=1}^h g_m \nu_m.
\]

We have,

\[
\frac{1}{2} (L\|\eta\|_\varepsilon^2) - \langle (\square_{\infty} + \Lambda \circ d)\eta, \eta \rangle_\varepsilon
= \sum_{i=1}^d \|\nabla H f_i\|_H^2 + \varepsilon \sum_{m=1}^h \|\nabla V g_m\|_V^2 - 2 \sum_{i,j=1}^d \sum_{m=1}^h \gamma_{ij}^m (X_j g_m) f_i - \langle \Lambda(d\eta), \eta \rangle_\varepsilon + \langle \text{Ric}_H \eta, \eta \rangle_H.
\]

(3.12)

On the other hand, the exterior derivative can be computed as follows:

\[
d\eta = \sum_{i,j=1}^d \left( X_i f_j - \frac{1}{2} \sum_{m=1}^h \gamma_{ij}^m g_m \right) \theta_i \wedge \theta_j + \sum_{j=1}^d \sum_{m=1}^h \left( X_j g_m - Z_m f_j - \sum_{i=1}^d \delta_{jm}^i f_i \right) \theta_j \wedge \nu_m
\]

\[
+ \sum_{m,\ell=1}^h Z_{\ell} g_m \nu_\ell \wedge \nu_m.
\]

Because of the vertical derivatives $Z_m f_i$ and $Z_\ell g_m$ that do not appear in (3.12), the quantity

\[
\inf_{\eta, \|\eta(x)\| = 1} \left( \frac{1}{2} (L\|\eta\|_\varepsilon^2) (x) - \langle (\square_{\infty} + \Lambda \circ d)\eta(x), \eta(x) \rangle_\varepsilon \right)
\]

is then finite if and only if $\Lambda(\nu_\ell \wedge \nu_m) = \Lambda(\theta_i \wedge \nu_m) = 0$, which we assume from now on. Also, clearly, every non zero term $\langle \Lambda(\theta_i \wedge \theta_j), \theta_k \rangle_H$ would decrease (3.13), so we can assume $\langle \Lambda(\theta_i \wedge \theta_j), \theta_k \rangle_H = 0$. Completing the squares in (3.12), we see then that the quantity to be maximized is

\[
\inf_{\eta, \|\eta(x)\| = 1} \left( -\frac{1}{4} \varepsilon^2 \sum_{i,j=1}^d \sum_{\ell=1}^h g_{\ell} \langle \Lambda(\theta_i \wedge \theta_j), \nu_\ell \rangle \nu_\ell + \frac{1}{2} \varepsilon \sum_{i,j=1}^d \sum_{m,\ell=1}^h \gamma_{ij}^m g_m g_\ell \langle \Lambda(\theta_i \wedge \theta_j), \nu_\ell \rangle \nu_\ell \right).
\]
We then easily see that the optimal choice of $\langle \Lambda(\theta_i \& \theta_j), \nu_\ell \rangle_V$ is given by

$$\langle \Lambda(\theta_i \& \theta_j), \nu_\ell \rangle_V = \frac{1}{\varepsilon} \gamma_{ij}. \quad \square$$

In the sequel, we shall denote

$$\square = \Box_{\infty} - \frac{1}{\varepsilon} T \circ d.$$

For our purpose, we will need to rewrite $\square$ in a sum of squares form, from which we will be able to deduce a stochastic representation of the semigroup $e^{\frac{1}{2}\Box_{\varepsilon}}$.

If $V$ is a horizontal vector field, we consider the fiberwise linear map from the space of one-forms into itself which is given by in a local adapted frame by

$$\mathcal{T}_V^\varepsilon \eta = - \sum_{j=1}^d \eta(T(V, X_j))\theta_j + \frac{1}{2\varepsilon} \sum_{m=1}^h \eta(JZ_m V)\nu_m.$$

We see that $\mathcal{T}_V^\varepsilon$ does not depend on the choice of the local adapted frame and thus, is a globally well-defined, smooth section. In a local adapted frame, if $\eta = \sum_{i=1}^d f_i \theta_i + \sum_{m=1}^h g_m \nu_m$, then we have

$$\mathcal{T}_V^\varepsilon \eta = \sum_{j=1}^d \sum_{\ell=1}^h \gamma_{ij}^\ell g_\ell \theta_j - \frac{1}{2\varepsilon} \sum_{j=1}^d \sum_{m=1}^h \gamma_{ij}^m f_j \nu_m.$$

**Theorem 3.3** In a local adapted frame, we have

$$\square = \sum_{i=1}^d (\nabla X_i - \mathcal{T}_V^\varepsilon X_i)^2 - (\nabla \nabla X_i X_i - \mathcal{T}_V^\varepsilon \nabla X_i X_i) + \frac{1}{2\varepsilon} \sum_{m=1}^h JZ_m^2 JZ_m - \mathcal{Ric}_H,$$

and for any smooth one-form $\eta$,

$$\frac{1}{2} \mathcal{L} \|\eta\|_2^2 - \langle \square \eta, \eta \rangle_{2\varepsilon} = \sum_{i=1}^d \| \nabla X_i \eta - \mathcal{T}_V^\varepsilon \nabla X_i \eta \|_2^2 + \mathcal{Ric}_H - \frac{1}{2\varepsilon} \sum_{m=1}^h JZ_m^2 JZ_m.$$  

**Proof.** It is enough to prove the two identities at the center of an adapted normal frame. From the definition of $\square$, at the center of the frame, we have for $\eta = \sum_{i=1}^d f_i \theta_i + \sum_{m=1}^h g_m \nu_m$,

$$\square = \sum_{i=1}^d \nabla X_i^\varepsilon \eta - 2 \sum_{i,j=1}^d \sum_{m=1}^h \gamma_{ij}^m (X_j g_m) \theta_i - \frac{1}{\varepsilon} \sum_{i,j=1}^d \left( X_i f_j - \frac{1}{2} \sum_{m=1}^h \gamma_{ij}^m g_m \right) \left( \sum_{m=1}^h \gamma_{ij}^m \nu_m \right) - \mathcal{Ric}_H.$$
On the other hand, still at the center of the frame, we compute
\[
(\nabla X_i - \mathcal{X}_X^\varepsilon)\eta = \sum_{j=1}^{d} \left( X_i f_j - \sum_{\ell=1}^{m} \gamma_{i\ell} f\ell \right) \theta_j + \sum_{m=1}^{h} \left( X_i g_m + \frac{1}{2\varepsilon} \sum_{j=1}^{d} \gamma_{ij}^m f_j \right) \nu_m.
\]
Keeping in mind that in a local adapted frame, we have
\[
J_{Z_m}(X_i) = -\sum_{j=1}^{d} \gamma_{ij}^m X_j,
\]
it is now an elementary exercise to check that
\[
\Box = \sum_{i=1}^{d} (\nabla X_i - \mathcal{X}_X^\varepsilon)^2 + \frac{1}{2\varepsilon} \sum_{m=1}^{h} J_{Z_m}^* J_{Z_m} - \text{Ric}_\mathcal{H}.
\]
The proof of the second identity follows the same lines as in the proof of Proposition 3.2. The details are let to the reader.

If \( V \) is a horizontal vector field, \( \mathcal{X}_V^\varepsilon \) is a skew-symmetric operator for the Riemannian metric \( g_{2\varepsilon} \), as a consequence, \( \Box \) is a symmetric operator for the metric \( g_{2\varepsilon} \) on the space of smooth and compactly supported one-forms. It is interesting that \( \Box \) is symmetric with respect to the metric \( g_{2\varepsilon} \) but not \( g_\varepsilon \) which is the one that was used to construct \( \Box \).

The operator \( \sum_{m=1}^{h} J_{Z_m}^* J_{Z_m} \) does not depend on the choice of the frame and shall concisely be denoted by \( J^*J \). We can note that in the case where \( M \) is a Sasakian manifold, like the Heisenberg group for instance, \( J^*J \) is the identity map on the horizontal distribution.

Similarly, the operator \( \sum_{i=1}^{d} (\nabla X_i - \mathcal{X}_X^\varepsilon)^2 - (\nabla \nabla X_i, X_i - \mathcal{X}_V^\varepsilon X_i) \) does not depend on the choice of the frame and shall concisely be denoted by \( -(\nabla - \mathcal{X}_V)^* (\nabla - \mathcal{X}_V) \). We therefore globally have
\[
\Box = -(\nabla - \mathcal{X}_V)^* (\nabla - \mathcal{X}_V) + \frac{1}{2\varepsilon} J^*J - \text{Ric}_\mathcal{H}.
\]
For later use, we record the following consequence of the computations done in the section.

**Proposition 3.4** Let \( \rho \in \mathbb{R} \), \( \kappa, \rho_2 \geq 0 \). Assume that for every horizontal one-form \( \eta \),
\[
\langle \text{Ric}_\mathcal{H}(\eta), \eta \rangle_{\mathcal{H}} \geq \rho \|\eta\|_{\mathcal{H}}^2, \quad \langle J^*J \eta, \eta \rangle_{\mathcal{H}} \leq \kappa \|\eta\|_{\mathcal{H}}^2,
\]
and that for every vertical one-form \( \eta \), and any horizontal coframe \( \{\theta_1, \ldots, \theta_d\} \),
\[
\frac{1}{4} \sum_{\ell,j=1}^{d} \langle T(\theta_\ell, \theta_j), \eta \rangle_{\mathcal{V}}^2 \geq \rho_2 \|\eta\|_{\mathcal{V}}^2,
\]

13
then we have, for any one-form \( \eta \),

\[
\frac{1}{2} L \| \eta \|_2^2 - \langle \Box_\varepsilon \eta, \eta \rangle_\varepsilon \geq \left( \rho - \frac{\kappa}{\varepsilon} \right) \| \eta_H \|_{\mathcal{H}}^2 + \rho_2 \| \eta_V \|_V^2
\]

and

\[
\frac{1}{2} L \| \eta \|_{2\varepsilon}^2 - \langle \Box_\varepsilon \eta, \eta \rangle_{2\varepsilon} \geq \left( \rho - \frac{\kappa}{2\varepsilon} \right) \| \eta_H \|_{\mathcal{H}}^2.
\]

To finish the section, we illustrate our formulas in the case of the model space \( G(\rho) \) that was introduced in Section 2. In that case, we have a basis of left invariant vector fields \( \{ X, Y, Z \} \) satisfying: \( [X, Y] = Z, [X, Z] = -\rho Y \), and \( [Y, Z] = \rho X \) and the sub-Laplacian is given by

\[
L = X^2 + Y^2.
\]

Every one-form can be written as \( \eta = f_1 \theta_1 + f_2 \theta_2 + g \nu \) where \( \{ \theta_1, \theta_2, \nu \} \) is the dual basis of \( \{ X, Y, Z \} \). We identify \( \eta \) with the column vector

\[
\eta = \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix}
\]

Elementary computations show then that

\[
\text{Ric}_H = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix},
\]

\[
\Box_\varepsilon = \begin{pmatrix} L - \rho & 0 & 2Y \\ 0 & L - \rho & -2X \\ -\frac{1}{\varepsilon} Y & -\frac{1}{\varepsilon} X & L - \frac{1}{\varepsilon} \end{pmatrix},
\]

\[
\mathcal{I}_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2\varepsilon} & 0 \end{pmatrix},
\]

\[
\mathcal{I}_Y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ \frac{1}{2\varepsilon} & 0 & 0 \end{pmatrix},
\]

and

\[
J^* J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
4 Gradient formulas and bounds for the heat semigroup

Throughout the section, we work under the same assumptions as the previous section and we moreover assume that for every horizontal one-form \( \eta \),

\[
\langle \text{Ric}_H(\eta), \eta \rangle_H \geq -K \| \eta \|_H^2, \quad \langle J^* J \eta, \eta \rangle_H \leq \kappa \| \eta \|_H^2,
\]

with \( K, \kappa \geq 0 \). We also assume that the manifold \( \mathbb{M} \) is metrically complete for the sub-Riemannian distance. Under these assumptions, it was proved in \([6]\) that the sub-Laplacian \( L \) is essentially self-adjoint on \( C_0^\infty(\mathbb{M}) \) and that the semigroup \( P_t = e^{\frac{1}{2} t L} \) is stochastically complete.

4.1 Heat semigroup on one-forms

We are interested in a stochastic representation of the semigroup on one-forms which is generated by \( \Box_\varepsilon \). This semigroup is well-defined by using the spectral theorem thanks to the following lemma.

**Lemma 4.1** The operator \( \Box_\varepsilon \) is essentially self-adjoint on the space of smooth and compactly supported one-forms for the Riemannian metric \( g_{2\varepsilon} \).

**Proof.** Since we assume \( \mathbb{M} \) to be metrically complete for the sub-Riemannian distance, it is also complete for the Riemannian distance associated to \( g_{2\varepsilon} \), because \( g_{2\varepsilon} \) is a Riemannian extension of \( g_H \). From \([29]\), there exists therefore a sequence \( h_n \in C_0^\infty(\mathbb{M}) \), such that \( 0 \leq h_n \leq 1 \) and \( \| \nabla_H h_n \|_\infty^2 + 2\varepsilon \| \nabla_V h_n \|_\infty^2 \to 0 \). In particular, note that we have \( \| \nabla_H h_n \|_\infty \to 0 \).

To prove that \( \Box_\varepsilon \) is essentially self-adjoint it is enough (see \([29]\)) to prove that for some \( \lambda > 0 \),

\[
\lambda \int_M h_n^2 \| \eta \|_{2\varepsilon}^2 = \int_M \langle h_n^2 \eta, \Box_\varepsilon \eta \rangle_{2\varepsilon} \\
= -\int_M \langle \nabla(h_n^2 \eta) - \Box_\varepsilon (h_n^2 \eta), \nabla \eta - \Box_\varepsilon \eta \rangle_{2\varepsilon} + \int_M h_n^2 \left\langle \left( \frac{1}{2\varepsilon} J^* J - \text{Ric}_H \right) (\eta), \eta \right\rangle_{2\varepsilon} \\
= -\int_M h_n^2 \| \nabla \eta - \Box_\varepsilon \eta \|_{2\varepsilon}^2 - 2 \int_M h_n \langle \eta, \nabla \nabla_H h_n \eta \rangle_{2\varepsilon} + \int_M h_n^2 \left\langle \left( \frac{1}{2\varepsilon} J^* J - \text{Ric}_H \right) (\eta), \eta \right\rangle_{2\varepsilon}.
\]

From our assumptions, the symmetric tensor \( \frac{1}{2\varepsilon} J^* J - \text{Ric}_H \) is bounded from above, thus by choosing \( \lambda \) big enough, we have

\[
\int_M h_n^2 \| \nabla \eta - \Box_\varepsilon \eta \|_{2\varepsilon}^2 + 2 \int_M h_n \langle \eta, \nabla \nabla_H h_n \eta \rangle_{2\varepsilon} \leq 0.
\]
By letting \( n \to \infty \), we easily deduce that \( \| \nabla \eta - \xi \eta \|_{L^2}^2 = 0 \) which implies \( \nabla \eta - \xi \eta = 0 \). If we come back to the equation \( \square \xi \eta = \lambda \eta \) and the expression of \( \square \xi \), we see that it implies:

\[
\left( \frac{1}{2\xi} J^* J - \mathcal{R} \mathcal{H} \right) (\eta) = \lambda \eta.
\]

Our choice of \( \eta \) forces then \( \eta = 0 \). \( \square \)

Since \( \frac{1}{2} \square \xi \) is essentially self-adjoint, it admits a unique self-adjoint extension which generates through the spectral theorem a semigroup \( Q_t^\xi = e^{\frac{1}{2} \square \xi t} \). As already mentioned, we will denote by \( P_t = e^{\frac{1}{2} t L} \) the semigroup generated by \( \frac{1}{2} L \). We have the following commutation property:

**Lemma 4.2** If \( f \in C^\infty_0 (M) \), then for every \( t \geq 0 \),

\[ dP_t f = Q_t^\xi df. \]

**Proof.** Let \( \eta_t = Q_t^\xi df \). It is the unique solution in \( L^2 \) of the heat equation

\[ \frac{\partial \eta}{\partial t} = \frac{1}{2} \square \xi \eta, \]

with initial condition \( \eta_0 = df \). From [6], we have that \( \alpha_t = dP_t f \) is in \( L^2 \), and from the fact that

\[ dL = \square \xi d, \]

we see that \( \alpha \) solves the heat equation

\[ \frac{\partial \alpha}{\partial t} = \frac{1}{2} \square \xi \alpha \]

with the same initial condition \( \alpha_0 = df \). We conclude thus \( \alpha = \eta \). \( \square \)

### 4.2 Stochastic representation of the semigroup on one-forms

We now turn to the stochastic representation of \( Q_t^\xi \). We denote by \( (X_t)_{t \geq 0} \) the symmetric diffusion process generated by \( \frac{1}{2} L \). Since \( P_t \) is stochastically complete, \( (X_t)_{t \geq 0} \) has an infinite lifetime.

Consider the process \( \tau_t^\xi : T^*_X M \to T^*_X M \) to be the solution of the following covariant Stratonovitch stochastic differential equation:

\[
d [\tau_t^\xi \alpha(X_t)] = \tau_t^\xi \left( \nabla_{\odot X_t} - \xi \odot X_t + \frac{1}{2} \left( \frac{1}{2\xi} J^* J - \mathcal{R} \mathcal{H} \right) dt \right) \alpha(X_t), \quad \tau_0^\xi = \text{Id}, \quad (4.14)
\]

where \( \alpha \) is any smooth one-form. We have the following key estimate:

**Lemma 4.3** For every \( t \geq 0 \), we have almost surely,

\[ \| \tau_t^\xi \alpha(X_t) \|_{L^2} \leq e^{\frac{1}{2} (K + \frac{1}{2\xi}) t} \| \alpha(X_0) \|_{L^2}. \]
Proof. The estimate stems from the fact that $\xi^\varepsilon$ is skew-symmetric for the Riemannian metric $g_{2\varepsilon}$, which implies that the connection $\nabla - \xi^\varepsilon$ is metric. The deterministic upper bound on $\tau^\varepsilon$ is therefore a consequence of the pointwise lower bound on $\text{Ric}_H - \frac{1}{2\varepsilon}J^*J$ and Gronwall’s lemma.

More precisely, consider the process $\Theta^\varepsilon_t : T^*_X M \to T^*_X M$ to be the solution of the following covariant Stratonovitch stochastic differential equation:

$$d[\Theta^\varepsilon_t \alpha(X_t)] = \Theta^\varepsilon_t \left( \nabla_{odX_t} - \xi^\varepsilon_{odX_t} \right) \alpha(X_t), \quad \tau^\varepsilon_0 = \text{Id}, \quad (4.15)$$

where $\alpha$ is any smooth one-form. Since $\xi^\varepsilon$ is skew-symmetric, $\Theta^\varepsilon_t$ is an isometry for the Riemannian metric $g_{2\varepsilon}$. Consider now the multiplicative functional $(M^\varepsilon_t)_{t \geq 0}$, solution of the equation

$$\frac{dM^\varepsilon_t}{dt} = \frac{1}{2} M_t \Theta^\varepsilon_t \left( \frac{1}{2\varepsilon} J^*J - \text{Ric}_H \right) \left( \Theta^\varepsilon_t \right)^{-1}, \quad M^\varepsilon_0 = \text{Id}.$$

With the previous notations, we of course have $\tau^\varepsilon_t = M^\varepsilon_t \Theta^\varepsilon_t$. Thus, the upper bound on $\tau^\varepsilon$ boils down to an upper bound on $M^\varepsilon$ which is obtained as a consequence of Gronwall’s inequality.

**Theorem 4.4** Let $\eta$ be a smooth and compactly supported one-form. Then for every $t \geq 0$, and $x \in M$,

$$(Q^\varepsilon_t \eta)(x) = \mathbb{E}_x (\tau^\varepsilon_t \eta(X_t)).$$

**Proof.** It is basically a consequence of the definition of $\tau^\varepsilon$ and Itô’s formula which implies that for every $t \geq 0$ the process

$$N_s = \tau^\varepsilon_s (Q^\varepsilon_{t-s} \eta)(X_s),$$

is a martingale. □

Combining Lemma 4.2 with Theorem 4.4 we get therefore the following representation for the derivative of the semigroup:

**Corollary 4.5** Let $f \in C^\infty_0(M)$. Then for every $t \geq 0$, and $x \in M$,

$$dP_t f(x) = \mathbb{E}_x (\tau^\varepsilon_t df(X_t)).$$

This eventually leads to a neat gradient bound for the semigroup $P_t$.

**Corollary 4.6** For every $f \in C^\infty_0(M)$, $\varepsilon > 0$, $t \geq 0$,

$$\sqrt{\|\nabla_H P_t f\|^2_H + 2\varepsilon \|\nabla_V P_t f\|^2_V} \leq e^\frac{1}{2} (K + \frac{\kappa}{2\varepsilon}) t P_t \left( \sqrt{\|\nabla_H f\|^2_H + 2\varepsilon \|\nabla_V f\|^2_V} \right).$$

We remark that this gradient bound is new in our framework and is stronger than similar gradient bounds in [4]. It also immediately implies that Hypothesis 1.4 of [6] is satisfied on Yang-Mills sub-Riemannian manifolds with transverse symmetries.
4.3 Integration by parts formula

As before, we denote by \((X_t)_{t \geq 0}\) the \(L\)-diffusion process. The stochastic parallel transport for the connection \(\nabla\) along the paths of \((X_t)_{t \geq 0}\) will be denoted by \(//_0.t\). Since the connection \(\nabla\) is horizontal, the map \(//_0.t : T_{X_0}M \to T_{X_0}M\) is an isometry that preserves the horizontal bundle, that is, if \(u \in \mathcal{H}_{X_0}\), then \(//_0.t u \in \mathcal{H}_{X_t}\). We see then that the anti-development of \((X_t)_{t \geq 0}\),

\[
B_t = \int_0^t //_{0,s}^{-1} \circ dX_s,
\]

is a Brownian motion in the horizontal space \(\mathcal{H}_{X_0}\). The following integration by parts formula will play an important role in the sequel.

**Proposition 4.7** Let \(x \in \mathbb{M}\). For any \(C^1\) adapted process \(\gamma : \mathbb{R}_{\geq 0} \to \mathcal{H}_x\) such that \(\mathbb{E}_x \left( \int_0^{+\infty} \|\gamma'(s)\|_{\mathcal{H}}^2 ds \right) < +\infty\) and any \(f \in C^1_0(\mathbb{M})\), \(t \geq 0\),

\[
\mathbb{E}_x \left( f(X_t) \int_0^t \langle \gamma'(s), d\mathcal{B}_s \rangle_{\mathcal{H}} \right) = \mathbb{E}_x \left( \left\langle \tau_{s}^\varepsilon f(X_t), \int_0^t (\tau_{s}^\varepsilon)^{-1} //_{0,s} \gamma'(s) ds \right\rangle_{2\varepsilon} \right).
\]

**Proof.** We fix \(t \geq 0\) and denote

\[
N_s = \tau_{s}^\varepsilon (dP_{1-s}) f(X_s).
\]

It is a martingale process. We have then for \(f \in C^1_0(\mathbb{M})\),

\[
\mathbb{E}_x \left( f(X_t) \int_0^t \langle \gamma'(s), d\mathcal{B}_s \rangle_{\mathcal{H}} \right) = \mathbb{E}_x \left( f(X_t) \int_0^t //_{0,s} \gamma'(s), //_{0,s} d\mathcal{B}_s \rangle_{\mathcal{H}} \right)
\]

\[
= \mathbb{E}_x \left( (f(X_t) - \mathbb{E}_x (f(X_t))) \int_0^t //_{0,s} \gamma'(s), //_{0,s} d\mathcal{B}_s \rangle_{\mathcal{H}} \right)
\]

\[
= \mathbb{E}_x \left( \int_0^t \langle dP_{1-s} f(X_s), //_{0,s} d\mathcal{B}_s \rangle_{\mathcal{H}} \int_0^t //_{0,s} \gamma'(s), //_{0,s} d\mathcal{B}_s \rangle_{\mathcal{H}} \right)
\]

\[
= \mathbb{E}_x \left( \int_0^t \langle dP_{1-s} f(X_s), //_{0,s} \gamma'(s) \rangle_{\mathcal{H}} ds \right)
\]

\[
= \mathbb{E}_x \left( \int_0^t \langle \tau_{s}^\varepsilon dP_{1-s} f(X_s), (\tau_{s}^\varepsilon)^{-1} //_{0,s} \gamma'(s) \rangle_{2\varepsilon} ds \right)
\]

\[
= \mathbb{E}_x \left( \int_0^t \langle N_s, (\tau_{s}^\varepsilon)^{-1} //_{0,s} \gamma'(s) \rangle_{2\varepsilon} ds \right)
\]

\[
= \mathbb{E}_x \left( \left\langle N_t, \int_0^t (\tau_{s}^\varepsilon)^{-1} //_{0,s} \gamma'(s) ds \right\rangle_{2\varepsilon} \right),
\]

where we integrated by parts in the last equality. \(\square\)

Let us observe that we can reinterpret the integration by parts formula of Proposition 4.7 in a slightly different way.

18
Corollary 4.8 Let \( x \in \mathbb{M} \). For any \( C^1 \) adapted process \( \gamma : \mathbb{R}_{\geq 0} \rightarrow \mathcal{H}_x \) such that 
\[
\mathbb{E}_x \left( \int_0^{+\infty} \| \gamma'(s) \|_H^2 \, ds \right) < +\infty
\]
and any \( f \in C_0^\infty(\mathbb{M}) \), \( t \geq 0 \),
\[
\mathbb{E}_x \left( \left\langle df(X_t) \big| \big|_{0,t} \varepsilon \right\rangle_{2\varepsilon} \right) = \mathbb{E}_x \left( \left\langle f(X_t) \big| \big|_0 t \gamma'(s) \, dB_s \right\rangle_{\mathcal{H}} \right),
\]
where \( v \) is the solution of the Stratonovitch stochastic differential equation in \( T_x \mathbb{M} \),:
\[
\begin{cases}
    dv(t) = \mathbb{E}_0^x \left( T_0^1 T_0^s \left( \left\langle \tau_0^1, \gamma'(s) \right\rangle_0 \mathbb{E}_x \left( \tau_0^s \, df(X_t) \big| F_s \right) \right) \right) /_{0,t} v(t) + \gamma'(t) dt \\
v(0) = 0.
\end{cases}
\]

Proof. It is a consequence of Itô’s formula that
\[
v(t) = \int_0^t \mathbb{E}_0^x \left( \left\langle \tau_0^1, \gamma'(s) \right\rangle_0 /_{0,s} \varepsilon \right) \, ds\]
is the solution of the above stochastic differential equation. We conclude then with Proposition 4.7. \( \square \)

As an immediate consequence of the integration by parts formula, we obtain the following Clark-Ocone type representation.

Proposition 4.9 Let \( X_0 = x \in \mathbb{M} \). For every \( f \in C_0^\infty(\mathbb{M}) \), and every \( t \geq 0 \),
\[
f(X_t) = P_t f(x) + \int_0^t \mathbb{E}_x \left( \left\langle \tau_0^1, \gamma'(s) \right\rangle_0 /_{0,s} \varepsilon \right) \, dB_s \big| \mathcal{F}_s \big| \big|_{0,t} \gamma'(s) \, ds
\]
where \( (\mathcal{F}_t)_{t \geq 0} \) is the natural filtration of \( (B_t)_{t \geq 0} \).

Proof. Let \( t \geq 0 \). From Itô’s integral representation theorem, we can write
\[
f(X_t) = P_t f(x) + \int_0^t \left\langle a_s, dB_s \right\rangle_{\mathcal{H}},
\]
for some adapted and square integrable \( (a_s)_{0 \leq s \leq t} \). Using the Proposition 4.7, we obtain therefore,
\[
\mathbb{E}_x \left( \int_0^t \left\langle \gamma'(s), a_s \right\rangle_{\mathcal{H}} ds \right) = \mathbb{E}_x \left( \left\langle \tau_0^t, \int_0^t \left\langle \tau_0^s, \gamma'(s) \right\rangle_0 /_{0,s} \varepsilon \right\rangle_0 \right).
\]
Since \( \gamma' \) is arbitrary, we obtain that
\[
a_s = \mathbb{E}_x \left( \left\langle \tau_0^1, \gamma'(s) \right\rangle_0 /_{0,s} \varepsilon \right) \, dB_s \big| \mathcal{F}_s \big| \big|_{0,t} \gamma'(s) \, ds.
\]

\( \square \)

We deduce first the following Poincaré inequality for the heat kernel measure.
Proposition 4.10 For every \( f \in C_0^\infty(M) \), \( t \geq 0 \), \( x \in M \), \( \varepsilon > 0 \),
\[
P_t(f^2)(x) - (P_t f)^2(x) \leq \frac{e^{(K + \frac{\kappa}{2})t} - 1}{K + \frac{\kappa}{2\varepsilon}} \left[ P_t(\|\nabla_H f\|^2)(x) + 2\varepsilon P_t(\|\nabla_V f\|^2)(x) \right]
\]

Proof. From the previous proposition and Lemma 4.3 we have
\[
\mathbb{E}_x ( (f(X_t) - P_t f(x))^2 ) \leq \int_0^t e^{(K + \frac{\kappa}{2\varepsilon})(t-s)} ds P_t(\|df\|_2^2)(x).
\]

We also get the log-Sobolev inequality for the heat kernel measure.

Proposition 4.11 For every \( f \in C_0^\infty(M) \), \( t \geq 0 \), \( x \in M \), \( \varepsilon > 0 \),
\[
P_t(f^2 \ln f^2)(x) - P_t(f^2)(x) \ln P_t(f^2)(x) \leq 2 e^{(K + \frac{\kappa}{2\varepsilon})t} \left[ P_t(\|\nabla_H f\|^2)(x) + 2\varepsilon P_t(\|\nabla_V f\|^2)(x) \right]
\]

Proof. The method for proving the log-Sobolev inequality from a representation theorem like Proposition 4.9 is due to [10] and the argument is easy to reproduce in our setting. Denote \( G = f(X_t)^2 \) and consider the martingale \( N_s = \mathbb{E}(G|\mathcal{F}_s) \). Applying now Itô’s formula to \( N_s \ln N_s \) and taking expectation yields
\[
\mathbb{E}_x ( (f(X_t) - P_t f(x))^2 ) \leq \int_0^t \left[ P_t(\|\nabla_H f\|^2)(x) + 2\varepsilon P_t(\|\nabla_V f\|^2)(x) \right]
\]

Thus we have from Cauchy-Schwarz inequality
\[
\mathbb{E}_x (N_t \ln N_t) - \mathbb{E}_x (N_0 \ln N_0) = \frac{1}{2} \mathbb{E}_x \left( \int_0^t \frac{d[N]_s}{N_s} \right),
\]
where \([N]\) is the quadratic variation of \( N \). From Proposition 4.9 applied with \( f^2 \), we have
\[
dN_s = 2 \langle \mathbb{E} (f(X_t)(\tau_s^-)^{-1}\tau_i^- df(X_t) | \mathcal{F}_s), \|0,s dB_s \rangle \rangle_H.
\]

Thus we have from Cauchy-Schwarz inequality
\[
\mathbb{E}_x (N_t \ln N_t) - \mathbb{E}_x (N_0 \ln N_0) \leq 2 \mathbb{E}_x \left( \int_0^t \frac{\|\mathbb{E} (f(X_t)(\tau_s^-)^{-1}\tau_i^- df(X_t) | \mathcal{F}_s) \|_{2\varepsilon}^2}{N_s} ds \right)
\]
\[
\leq 2 \int_0^t e^{(K + \frac{\kappa}{2\varepsilon})(t-s)} ds P_t(\|df\|_2^2)(x).
\]

4.4 Positive curvature and convergence to equilibrium

In this final section we prove that if the tensor \( \text{Ric}_H \) is bounded from below by a positive constant on the horizontal bundle, then by exploiting a further geometric quantity we can prove convergence of the semigroup when \( t \to +\infty \) and get sharp quantitative estimates.
in the form of a Poincaré and a log-Sobolev inequality with an exponential decay for the heat kernel measure.

So, we assume throughout the section that for every horizontal one-form $\eta$,
\[
\langle \mathfrak{Ric}_H(\eta), \eta \rangle_H \geq \rho_1\|\eta\|_H^2, \quad \langle \mathcal{J}^* J \eta, \eta \rangle_H \leq \kappa \|\eta\|_H^2,
\]
and that for every vertical one-form $\eta$, and any horizontal coframe $\{\theta_1, \cdots, \theta_d\}$,
\[
\frac{1}{4} \sum_{i,j=1}^d (T(\theta_i, \theta_j), \eta)^2 \geq \rho_2\|\eta\|_V^2,
\]
where $\rho_1, \rho_2 > 0$ and $\kappa \geq 0$. As proved in Proposition 3.4 for any one-form $\eta$,
\[
\frac{1}{2}L\|\eta\|^2_\varepsilon - \langle \Box_\varepsilon \eta, \eta \rangle_\varepsilon \geq \left( \rho_1 - \frac{\kappa}{\varepsilon} \right) \|\eta\|_H^2 + \rho_2\|\eta\|_V^2.
\]
This implies of course
\[
\frac{1}{2}L\|\eta\|^2_\varepsilon - \langle \Box_\varepsilon \eta, \eta \rangle_\varepsilon \geq \inf \left( \rho_1 - \frac{\kappa}{\varepsilon}, \frac{\rho_2}{\varepsilon} \right) \|\eta\|^2_\varepsilon.
\]
The constant $\inf \left( \rho_1 - \frac{\kappa}{\varepsilon}, \frac{\rho_2}{\varepsilon} \right)$ is maximal when $\rho_1 - \frac{\kappa}{\varepsilon} = \frac{\rho_2}{\varepsilon}$, that is $\varepsilon = \frac{\rho_1 \rho_2}{\rho_1 + \rho_2}$. For this choice of $\varepsilon$, we have then
\[
\inf \left( \rho_1 - \frac{\kappa}{\varepsilon}, \frac{\rho_2}{\varepsilon} \right) = \frac{\rho_1 \rho_2}{\kappa + \rho_2}.
\]
We have then following estimate which is obtained by analyzing the Itô-Stratonovitch correction term in the stochastic differential equation (4.15).

**Lemma 4.12** Let $\varepsilon = \frac{\rho_1 + \rho_2}{\rho_1}$. For every $t \geq 0$,
\[
\mathbb{E}\left( \|\tau_t^\varepsilon \alpha(X_t)\|_\varepsilon^2 \right) \leq e^{-\frac{\rho_1 \rho_2}{\kappa + \rho_2} t} \mathbb{E}\left( \|\alpha(X_0)\|_\varepsilon^2 \right).
\]

Arguing then as before, we obtain the following Bakry-Émery, Poincaré and log-Sobolev inequalities.

**Proposition 4.13** For every $f \in C^\infty_0(M)$, $t \geq 0$, $x \in M$,
\[
\|\nabla_H P_t f\|^2 + \frac{\kappa + \rho_2}{\rho_1} \|\nabla_V P_t f\|^2 \leq e^{-\frac{\rho_1 \rho_2}{\kappa + \rho_2}} t \left[ P_t(\|\nabla_H f\|^2)(x) + \frac{\kappa + \rho_2}{\rho_1} P_t(\|\nabla_V f\|^2)(x) \right]
\]
\[
P_t(f^2)(x) - (P_t f)^2(x) \leq \frac{\kappa + \rho_2}{\rho_1 \rho_2} \left( 1 - e^{-\frac{\rho_1 \rho_2}{\kappa + \rho_2} t} \right) \left[ P_t(\|\nabla_H f\|^2)(x) + \frac{\kappa + \rho_2}{\rho_1} P_t(\|\nabla_V f\|^2)(x) \right],
\]
and
\[
P_t(f^2 \ln f^2)(x) - P_t(f^2)(x) \ln P_t(f^2)(x)
\]
\[
\leq 2 \frac{\kappa + \rho_2}{\rho_1 \rho_2} \left( 1 - e^{-\frac{\rho_1 \rho_2}{\kappa + \rho_2} t} \right) \left[ P_t(\|\nabla_H f\|^2)(x) + \frac{\kappa + \rho_2}{\rho_1} P_t(\|\nabla_V f\|^2)(x) \right].
\]
The first of the above inequality was already proved in [4] by completely different methods and implies $\mu(M) < +\infty$ and also that when $t \to +\infty$, $P_t f \to \frac{1}{\mu(M)}$. It is worth pointing out that in the present framework, the two above Poincaré and log-Sobolev inequalities are new but, by taking the limit when $t \to \infty$ we get the two inequalities

$$\int_M f^2 d\mu - \left( \int_M f d\mu \right)^2 \leq \frac{\kappa + \rho_2}{\rho_1 \rho_2} \left[ \int_M \|\nabla H f\|^2 d\mu + \frac{\kappa + \rho_2}{\rho_1} \int_M \|\nabla V f\|^2 d\mu \right]$$

and

$$\int_M f^2 \ln f^2 d\mu - \int_M f^2 d\mu \ln \int_M f^2 d\mu \leq \frac{2(\kappa + \rho_2)}{\rho_1 \rho_2} \left[ \int_M \|\nabla H f\|^2 d\mu + \frac{\kappa + \rho_2}{\rho_1} \int_M \|\nabla V f\|^2 d\mu \right].$$

which were also already proved in [4] with the very same constants.

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