Quantum open systems and turbulence

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Abstract

We show that the problem of non conservation of energy found in the spontaneous localization model developed by Ghirardi, Rimini and Weber [1] is very similar to the inconsistency between the stochastic models for turbulence and the Navier-Stokes equation. This sort of analogy may be useful in the development of both areas.

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I. INTRODUCTION

Some models have been developed to describe how classical behavior is regained from quantum mechanics, in an attempt to derive a unified description of physical phenomena at any level. Quantum mechanics allows the occurrence of linear superpositions of macroscopically distinguishable states, which are not observed in real life. A possible way to avoid them is by replacing the unitary evolution of quantum mechanics by a non-unitary equation. This was done for example by Ghirardi, Rimini and Weber (GRW) [1] and later improved by Ghirardi, Pearle and Rimini [2]. Here the Schrödinger equation is fundamentally modified having now a stochastic term, which causes the collapse of the wave function. Another possible modification of the von Neumann evolution equation is obtained by considering the interaction between the system of interest and a heat bath, as done in the decoherence models [3-5].

The master equation of the GRW model as well as the master equation of some old versions of the decoherence models [6] do not have a dissipation term, which leads to non-conservation of energy. Even though it is argued that this problem can be disregarded in the GRW model since the energy production is very small and probably unobservable [1,7] and under certain limits the dissipation term can in fact be neglected in the equation for open systems [8], we show here that the increase of energy found in these models is very similar to the inconsistency between the stochastic models for turbulence and the Navier-Stokes equation.

Our discussion is done in terms of the stochastic differential equations for position and momentum, because it allows an easy visualization of the problem and also a straightforward connection with the stochastic models for turbulence. It has been shown that the transport of energy, typical of turbulence, can only be obtained if a damping term is considered in the stochastic equations (and equivalently in the master equation - ME) [9]. The energy gained in the short length scale is actually transferred to smaller ones until it is finally dissipated.

A master equation with a dissipation term is characteristic of the newer versions of the decoherence model [3], which are usually associated with Brownian motion [10]. We show that its correspondent stochastic differential equations for position and momentum describe a Brownian motion only for long time, but turbulent diffusion is actually obtained for short time.

II. GRW MODEL AND DECOHERENCE MODEL

In the GRW/GPR model the master equation is now written as

\[ i\hbar \frac{\partial \rho(x, y, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - i\hbar \lambda (1 - e^{-\alpha/4(x-y)^2}) \right] \rho(x, y). \]  

The localization scale \(1/\sqrt{\alpha}\) and the frequency of collapse \(\lambda\) are parameters chosen in such a way that the new evolution equation does not give different results from the usual unitary evolution for microscopic systems with few degrees of freedom, but when a macroscopic system is described there is a fast decay of the macroscopic linear superpositions which are quickly transformed into statistical mixtures [11]. Apart from some attempts to associate
the new term to gravitation [11], it has been hard to explain its nature and origin and the
values of the parameters introduced are otherwise arbitrary.

Considering a free particle, the master equation obtained for the decoherence model in
the limit of high temperature is written as

\[
\frac{i \hbar}{2m} \frac{\partial^2 \rho(x, y, t)}{\partial x^2} - \frac{\partial^2 \rho(x, y, t)}{\partial y^2} - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right) - \frac{i}{\hbar} \gamma (x - y)^2 \rho(x, y, t),
\]

(2)

where \( \hbar \) is Planck’s constant, \( m \) is the particle mass, \( \gamma \) is the relaxation time, \( k \) is Boltzmann’s constant and \( T \) is the temperature of the heat bath.

Contrary to eq.(1) this master equation has a dissipation term, though it is also subjected
to some criticisms [13,14].

A. Stochastic differential equations for the GRW model

Describing a system with stochastic equations is very attractive, because of their clarity
and simplicity. In order to obtain the corresponding stochastic differential equations for the
GRW model, we first derive the Fokker-Planck equation (FPE) in phase space by using the
Wigner transform of the density matrix

\[
W_E(x, p, t) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dq \exp \left( -\frac{ipq}{\hbar} \right) \rho(x + q/2, x - q/2, t).
\]

(3)

Assuming \( (1 - e^{-\alpha(x-y)^2/4}) \sim \alpha(x-y)^2/4 \) and \( \epsilon = \hbar^2 \lambda \alpha/2m^2 \) we obtain

\[
\frac{\partial W_E}{\partial t} = -\frac{p}{m} \frac{\partial W_E}{\partial x} + \frac{\epsilon m^2}{2} \frac{\partial^2 W_E}{\partial p^2}.
\]

(4)

The same FPE is obtained for a conditional probability function \( W_C(x, p, t|x_0) \), where \( \int W_C(x, p, t|x_0) dx_0 = W_E(t, p|x) \) [3]. The stochastic process described by a conditional prob-
ability satisfying the FPE is equivalent to the following Itô stochastic differential equations
for momentum and position

\[
dp = \sqrt{m^2 \epsilon} \eta(t) dt, \tag{5}
\]

and

\[
dx = \frac{p}{m} dt, \tag{6}
\]

where the correlation function for the white noise is \( < \eta(t) \eta(t') > = \delta(t - t') \).

Since there is no damping term in the FPE, the momentum has only a noise term which implies that

\[
< p^2 >_C = < p_0^2 >_E + m^2 \epsilon t \tag{7}
\]

\[
< x^2 >_C = < x_0^2 >_E + \frac{< p_0^2 >_E}{m^2} t^2 + \frac{\epsilon}{3} t^3. \tag{8}
\]
The integration for position becomes simple if we apply the identity \( \int_0^t dt' \int_0^{t'} dt'' = \int_0^t dt'' \int_0^t dt' \) \([14]\).

This cubic dependence for \( < x^2 >_C \), also obtained by GRW \([1]\), indicates turbulence, though they had not realized it. This analogy has only been made recently \([12,16,17]\). Had they noticed this and the development of these models might have followed a different direction, probably trying to justify the origin of the added noise with a chaotic medium instead of gravitation. However GRW did notice the violation of energy caused by the time dependence of \( < p^2 >_C \), though they consider this problem negligible since the chosen parameters lead to a very small increase of energy. Notice that old versions of decoherence \([6]\) did not have a damping term either and even though we cannot talk of non conservation of energy, since the system in this case is open, its energy does increase. We will see in the next subsection that the addition of a damping term in the stochastic equations (and similarly in the ME) can in fact be negligible in the short time limit of open systems, but it is necessary for a consistent and generic model and is essential in the energy transport typical of turbulence.

**B. Stochastic models for turbulence**

In a completely different field, fluid mechanics, there were attempts to construct stochastic models of turbulence that departed from equations similar to the differential equations (5) and (6). Differently from the Brownian motion, which has a linear dependence on time for \( < x^2 > \), turbulence has a cubic dependence, but a linear dependence on time reappears for \( < p^2 > \). Because of this, the first attempts to build stochastic models for turbulence tried to see it as a Brownian diffusion in the momentum space \([18]\) and therefore just considered a noise term in the stochastic equation for momentum as in eq. (5). These models failed to describe the main feature of turbulence, which is the energy transport between the motions of various scales \([9,19]\). The only way to recover this transport, and therefore to not contradict the Navier-Stokes equation (the fundamental equation of turbulence, equivalent in importance to the Schrödinger equation in quantum mechanics), is by considering a damping term in the FPE or equivalently, by adding a momentum dependent term in its stochastic equation. We should then work with

\[
\frac{dp}{dt} = a(p)dt + \sqrt{2m\varepsilon(t)} dt,
\]

\[
dx = \frac{p}{m} dt,
\]

where \( a(p) \) is a function of momentum chosen as \(-\gamma p\). The justification for this choice will be clear soon when we discuss the energy transport.

These equations lead to the following new FPE

\[
\frac{\partial W_C}{\partial t} = -\frac{p}{m} \frac{\partial W_C}{\partial x} + \gamma \frac{\partial (pW_C)}{\partial p} + \frac{m^2 \varepsilon}{2} \frac{\partial^2 W_C}{\partial p^2}.
\]

This is exactly the equation we obtain by doing the Wigner transform to the decoherence master equation (2) and assuming \( \varepsilon = 2\gamma kT/m \). This FPE is usually associated with
Brownian motion, but this is so only for a large time scale, on a short time scale we actually have turbulence.

From (9) and (10) we have

\[ p(t) = p_0 e^{-\gamma t} + \sqrt{m^2 \varepsilon} \int_0^t e^{-\gamma(t-t')} \xi(t') dt', \] (12)

\[ x(t) = x_0 + \frac{1}{m} \int_0^t p(t') dt', \] (13)

which lead to the quadratic averages

\[ < p^2(t) >_C = < p_0^2 >_E e^{-2\gamma t} + \frac{m^2 \varepsilon}{2\gamma} (1 - e^{-2\gamma t}), \] (14)

\[ < x^2(t) >_C = < x_0^2 >_E + \frac{< p_0^2 >_E}{m^2 \gamma^2} (1 - e^{-\gamma t})^2 + \frac{\varepsilon}{2 \gamma^3} [2\gamma t - 3 + 4e^{-\gamma t} - e^{-2\gamma t}]. \] (15)

On a large time scale the system attains equilibrium, \(< p^2(t) >_C \rightarrow m^2 \varepsilon/(2\gamma)\), showing that energy is conserved, and \(< x^2(t) >_C \rightarrow \varepsilon t/\gamma^2\), which is the typical result for the Brownian motion.

On a short time scale we recover the time dependence typical of turbulence

\[ < p^2(t) >_C \rightarrow < p_0^2 >_E + m^2 \varepsilon t, \] (16)

\[ < x^2(t) >_C \rightarrow < x_0^2 >_E + \frac{< p_0^2 >_E}{m^2} t^2 + \frac{\varepsilon}{3} t^3. \] (17)

Notice that from the derivative of eq. (14) we can calculate the critical damping value. If \( \gamma > m^2 \varepsilon/(2 < p_0^2 >_E) \) the system loses energy and when \( \gamma < m^2 \varepsilon/(2 < p_0^2 >_E) \) the system gains energy. On the short time scale the absence of the parameter \( \gamma \) in the equations above confirms that in this limit the dissipation term can be neglected. In any case, in the large time limit \( t \gg \gamma^{-1} \), energy is conserved, as remarked above.

The relaxation term added to the stochastic equation for momentum is responsible for the transportation of the injected energy to a much smaller scale, where it is dissipated. In other words, energy always ends up being dissipated in the environment. In the language of turbulence \( \varepsilon \) is the energy injected into the fluid per unit time and unit mass, in an intermediary length scale - the inertial regime - it is transferred to smaller scales and is finally dissipated in a much smaller length scale, called Kolmogorov dissipation scale [20]. Considering that we are in the inertial regime, even though the dissipation exists, we cannot see it, but we are able to verify the transport of energy.

The moment of third order of velocity \( v \) represents the energy transport between the motions of various scales. Its relation with \( \varepsilon \) was derived by Kolmogorov from the Navier-Stokes equation and is given by his classical expression [21]

\[ < v^3 >_E = -c_1 \varepsilon x. \] (18)
Multiplying the new FPE for $W_E(t,p|x)$ by $p^2$ and integrating over $p$ we have

$$\frac{\partial < p^2 >_E}{\partial t} + \frac{1}{m} \frac{\partial < p^3 >_E}{\partial x} - 2 < pa >_E = \varepsilon m^2. \tag{19}$$

Using Kolmogorov’s law and the fact that the first term of (19) is zero for statistically stationary turbulence [1] (notice that $< p^2 >_E$ does not depend on time in statistically stationary turbulence, but $< p^2 >_C$ does), we arrive at

$$-c_1 m^2 \varepsilon - 2 < pa >_E = m^2 \varepsilon,$$ \tag{20}

which can only be satisfied if $a$ depends on $p$ with a negative coefficient, such as $a = -\gamma p$ and $< p^2 >_E = c_2 \gamma^{-1} m^2 \varepsilon$, with $2c_2 - c_1 = 1$. We can see that Kolmogorov’s law justifies the added relaxation term in eq. (3).

It is now clear that the FPE (11) describes turbulence and Brownian motion. It is the time scale that determines which kind of diffusion we actually have. When dealing with quantum open systems and the transition quantum/classic we should keep in mind two time scales: the decoherence time and the relaxation time and any proposed experiment should take them into account.

### III. CONCLUSIONS

The connection between these two different fields is very rich and there is certainly a lot more to be done. They can give important contributions to each other. From the study of turbulence we have shown that the energy gained or lost by an open system is always dissipated in the environment. It should also be mentioned that eq. (2) can also be obtained by a more generic method developed by Kusnezov, Bulgac and Do Dang (KBD) [12], where they try to give a better justification to the origin of irreversibility in the master equation. Both system and environment have a dynamics and the system evolves in a chaotic background, whose dynamic evolution is described by the random matrix theory. This may indicate that the technique of random matrices can be useful in the study of turbulence, a field whose main equation has always been the Navier-Stokes equation, a purely phenomenological equation. Departing from the master equation obtained with the KBD method, we can interpret the corresponding stochastic differential equations as describing a tracer immersed in a chaotic medium. We intend to extend this analogy between turbulence and the KBD method in a future publication, where we also intend to look for the stochastic equations associated with the fractional Fokker-Planck equation obtained with this method [21].

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