ON THE SIEGEL-WEIL FORMULA FOR CLASSICAL GROUPS OVER FUNCTION FIELDS

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Abstract. We establish a Siegel-Weil formula for classical groups over function fields, which asserts in many cases that the Siegel Eisenstein series is equal to a theta integral. This is a function-field analogue of the classical result proved by Weil in his 1965 Acta Math. paper.

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1. Introduction

Let \( k \) be a function field with \( \text{char}(k) \neq 2 \), and let \( D \) be a quadratic field extension of \( k \). Let \( G = U(n, n) \) be the quasi-split unitary group in \( 2n \) variables, and let \( H = U(V) \) be the unitary group associated to a non-degenerate hermitian space \( V \) over \( D \) of dimension \( m \) and of Witt index \( r \). In this paper, we study the Siegel-Weil formula for the pair \((G, H)\). Let \( X = V^n \cong M_{m \times n}(D) \). With the help of some extra data, there is Weil representation \( \omega \) of \( G(\mathbb{A}) \) on the space \( S(X(\mathbb{A})) \) of Schwartz-Bruhat functions on \( X(\mathbb{A}) \).

For \( \Phi \in S(X(\mathbb{A})) \), define the theta integral

\[
I(\Phi) = \int_{H(\mathbb{A})/H(k)} \sum_{x \in X(k)} \Phi(hx) \cdot dh,
\]

where \( dh \) is the Haar measure on \( H(\mathbb{A}) \) such that \( \text{vol}(H(\mathbb{A})/H(k)) = 1 \); and define the Siegel Eisenstein series

\[
E(\Phi) = \sum_{\gamma \in P(k) \setminus G(k)} \omega(\gamma)\Phi(0),
\]

where \( P \) is the Siegel parabolic subgroup of \( G \).

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In this paper, we will prove that the theta integral \( I(\Phi) \) is absolutely convergent whenever \( r = 0 \) or \( m-r > n \), and that the Siegel Eisenstein series \( E(\Phi) \) is absolutely convergent whenever \( m \geq 2n \). Moreover, when \( m > 2n \), we will prove the Siegel-Weil formula

\[
I(\Phi) = E(\Phi).
\]

These results are analogues of those in [Weil 1965]. We also consider the case where \( D \) is \( k \) or is a division quaternion algebra over \( k \), and we obtain similar results. We follow the methods in [Weil 1965] to prove these results; in particular, we make use of the reduction theory over function fields in [Harder 1969] and the theory of Eisenstein series over function fields in [Morris 1982].

The Siegel-Weil formulas over number fields have been well developed in recent years (see for example [Rallis 1987], [Kudla-Rallis 1988a], [Kudla-Rallis 1988b], [Kudla-Rallis 1994], [Ikeda 1996], [Ichino 2001], [Ichino 2004], [Ichino 2007], [Jiang-Soudry 2007], [Yamana 2011], [Yamana 2013a], and especially [Ji-Qiu-Takeda 2014] and the references cited therein). On the contrary, the Siegel-Weil formulas over function fields are only studied in some special cases ([Haris 1974], [Wei 2015], [Xiong 2017]).

We follow the strategy in [Weil 1965] to prove the Siegel-Weil formula. In particular, many results are proved following the methods in [Weil 1965] with some necessary changes from number-field situation to function-field situation.

The structure of this paper is as follows. In Section 2, we give various notation and conventions used in this paper. In Section 3, we give some results in reduction theory over function fields and prove some auxiliary lemmas. In Section 4, we study the Siegel Eisenstein series. In Section 5, we prove some uniqueness theorems analogous to those in [Weil 1965]. Finally, in Section 6, we give a convergence criterion for the theta integral, and prove the Siegel-Weil formula.

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2. Notation and conventions

Let \( q \) be a power of an odd prime number and let \( \mathbb{F}_q \) be the finite field of order \( q \). Let \( k \) be a function field in one variable over \( \mathbb{F}_q \), such that \( \mathbb{F}_q \) is algebraically closed in \( k \). Note that \( \text{char}(k) \neq 2 \) since \( q \) is odd.

Let \( \mathbb{F}_q[x] \) be the polynomial ring in one indeterminate over \( \mathbb{F}_q \), with fraction field \( \mathbb{F}_q(x) \), and regard \( k \) as a finite separable extension of \( \mathbb{F}_q(x) \). Let \( O_k \) be the ring of integers of \( k \), i.e. \( O_k \) is the integral closure of \( \mathbb{F}_q[x] \) in \( k \).

We fix a place \( v_0 \) of \( k \) which lies above the place \((x^{-1})\) of \( \mathbb{F}_q(x) \) and such that the residue field of \( k_{v_0} \) is \( \mathbb{F}_q \).

We fix an algebraically closed field extension \( \Omega \) of \( k \), called the universal domain.

Let \( D \) be a division algebra over \( k \), which is either \( k \), or a quadratic field extension of \( k \) or a division quaternion algebra over \( k \). Let \( \xi \rightarrow \bar{\xi} \) be the involution on \( D \) which is the identity
if \( D = k \), the nontrivial element in \( \text{Gal}(D/k) \) if \( D \) is a quadratic field extension of \( k \), and the main involution if \( D \) is a division quaternion algebra over \( k \).

Let \( N_{D/k} \) be the norm from \( D \) to \( k \), and let \( \text{tr}_{D/k} \) be the trace from \( D \) to \( k \). For a positive integer \( d \), let \( M_d(D) \) be the additive group of square matrices over \( D \) of order \( d \); for a matrix \( a \in M_d(D) \), let \( \nu(a) = N_{D/k}(\det a) \) and let \( \tau(a) = \text{tr}_{D/k}(\text{tr} a) \).

Let \( \eta = \pm 1 \). Let \( \delta \) be the dimension of \( D \) over \( k \), and let \( \delta' \) be the dimension over \( k \) of the space of elements \( \xi \) of \( D \) such that \( \bar{\xi} = \eta \xi \), and let \( \epsilon = \delta'/\delta \).

Then

\[
\epsilon = \begin{cases} 
0 & \text{if } D = k \text{ and } \eta = -1, \\
\frac{1}{4} & \text{if } D \text{ is a division quaternion algebra over } k \text{ and } \eta = 1, \\
\frac{1}{2} & \text{if } D \text{ is a quadratic field extension of } k, \\
\frac{3}{4} & \text{if } D \text{ is a division quaternion algebra over } k \text{ and } \eta = -1, \\
1 & \text{if } D = k \text{ and } \eta = 1.
\end{cases}
\]

Let \( m, n \) be two positive integers. For simplicity, we assume \( m \) is even if \( \epsilon = 1 \).

Let \( X = M_{m \times n}(D) \) be the additive group of \( m \times n \) matrices over \( D \).

For a matrix \( x \) over \( D \), let \( \bar{x} \) be the conjugate of \( x \), and let \( x^t \) be the transpose of \( x \). We often write \( x^* \) for the conjugate transpose of \( x \), i.e. \( x^* = \bar{x}^t \).

We write an element \( x \) of \( X \) in the form \( x = (x_1, \ldots, x_n) \), where each \( x_i \in M_{m \times 1}(D) \).

For \( 0 \leq r \leq n \), let \( X_r \) be the subspace of \( X \) consisting of elements of the form \( (x_1, \ldots, x_r, 0, \ldots, 0) \). In particular, \( X_0 = 0 \) and \( X_n = X \).

Let \( W = M_{1 \times 2n}(D) \), and equip it with an \((-\eta)\)-hermitian form \( \langle \ , \ \rangle : W \times W \to D \) given by

\[
\langle x, y \rangle = x \begin{pmatrix} 0 & 1_n \\ -\eta \cdot 1_n & 0 \end{pmatrix} y^*
\]

for \( x, y \in W \). In this case, we say \( W \) is a split \((-\eta)\)-hermitian space.

Let

\[
G = U(W) = \{ g \in GL_{2n}(D) : \langle xg, yg \rangle = \langle x, y \rangle, \forall x, y \in W \}.
\]

be the isometry group of \( W \).

Then

\[
G = \{ g \in \text{Res}_{D/k}GL_{2n} : g \begin{pmatrix} 0 & 1_n \\ -\eta \cdot 1_n & 0 \end{pmatrix} g^* = \begin{pmatrix} 0 & 1_n \\ -\eta \cdot 1_n & 0 \end{pmatrix} \},
\]

where \( \text{Res}_{D/k} \) is the Weil restriction from \( D\)-groups to \( k\)-groups.

We have the Siegel parabolic subgroup \( P = MN \) of \( G = U(W) \), where the Levi subgroup \( M \) is given by

\[
M = \{ m(a) = \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix} : a \in \text{Res}_{D/k}GL_n \},
\]

and the unipotent radical \( N \) is given by

\[
N = \{ n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} : b \in \text{Her}_n \},
\]
where
\[ \text{Her}_n(k) = \{ b \in M_n(D) : b^* = \eta \cdot b \} \]
is the space of \( \eta \)-hermitian matrices of order \( n \).

We have the Bruhat decomposition:
\[ G = \bigcup_{r=0}^{n} P w_r P = \bigcup_{r=0}^{n} P w_r N, \]
where
\[ w_r = \begin{pmatrix}
1_{n-r} & 0 & 0 & 1_r \\
0 & 1_{n-r} & -\eta \cdot 1_r & 0
\end{pmatrix}. \]

Let \( V = M_{m \times 1}(D) \), and equip it with an \( \eta \)-hermitian form \( (\ , \ ) : V \times V \to D \) given by
\[ (x, y) = x^* \cdot Q \cdot y, \]
where \( Q \) is an invertible element of \( M_m(D) \) such that \( Q^* = \eta \cdot Q \).

Denote the Witt index of \( V \) by \( r \), i.e. \( r \) is the dimension over \( D \) of a maximal totally isotropic subspace of the hermitian space \( V \).

Let
\[ U(V) = \{ h \in GL_m(D) : (hx, hy) = (x, y), \forall x, y \in V \}. \]
be the isometry group of \( V \).

Then
\[ U(V) = \{ h \in \text{Res}_{D/k}GL_m : h^* \cdot Q \cdot h = Q \}. \]

Let \( H \subset U(V) \) be the connected component of the identity in \( U(V) \).

The group \( U(V) \) acts on \( X \) via \( h \cdot (x_1, \ldots, x_n) = (h \cdot x_1, \ldots, h \cdot x_n) \), where \( x_i \in V = M_{m \times 1}(D) \).

Let \( W = V \otimes_D W \), and equip it with a symplectic form over \( k \) given by
\[ \langle \langle \ , \ \rangle \rangle = \kappa \cdot \text{tr}_{D/k}(\langle \ , \ \rangle \otimes \langle \ , \ \rangle), \]
where
\[ \kappa = \begin{cases}
2 & \text{if } D = k, \\
1 & \text{otherwise.}
\end{cases} \]

Note that the \( \kappa \) here is twice of that on [Kudla 1994, p. 364].

Let
\[ Sp(W) = \{ u \in GL_k(W) : \langle \langle xu, yu \rangle \rangle = \langle \langle x, y \rangle \rangle, \forall x, y \in W \} \]
be the symplectic group associated to the symplectic space \( W \).

Then \( (U(W), U(V)) \) form a reductive dual pair of type I in \( Sp(W) \) in the sense of [Howe 1979, n. 5].

There is a natural homomorphism
\[ \iota : G \to Sp(W) \]
given by \( (v \otimes w) \iota(g) = v \otimes wg \) for \( v \in V, w \in W \) and \( g \in G \).
Let \( i_X : X \to \text{Her}_n(k) \) be the mapping given by \( i_X(x) = (x, x) := ((x_1, x_2)) \) for \( x = (x_1, \ldots, x_n) \in X \) with each \( x_i \in V \).

Let \( \mathbb{A} \) be the ring of adeles of \( k \), and let \( \mathbb{A}^\times \) be the group of ideles of \( k \). For a place \( v \) of \( k \), let \( k_v \) be the completion of \( k \) at \( v \), let \( O_v \) be the ring of integers in \( k_v \), and let \( q_v \) be the order of the residue field of \( k_v \). Let \( O_{\mathbb{A}} = \prod_v O_v \), where \( v \) runs over the places of \( k \).

For an algebraic group \( \mathfrak{G} \) over \( k \) and for a \( k \)-algebra \( A \), we write \( \mathfrak{G}(A) \) (or \( \mathfrak{G}_A \)) for the group of \( A \)-points. In particular, for a place \( v \) of \( k \), we write \( \mathfrak{G}_v = \mathfrak{G}(k_v) \). For example, we write \( D_{\mathbb{A}} = D \otimes_k \mathbb{A} \) and \( X(\mathbb{A}) = X \otimes_k \mathbb{A} \).

For a place \( v \) of \( k \), let \( \cdot|_v \) be the absolute value on \( k_v \), and let \( \cdot|_\mathbb{A} = \prod_v \cdot|_v \) be the adelic absolute value on \( k \). Define the adelic absolute value on \( D_{\mathbb{A}} \) by \( |x|_{D_{\mathbb{A}}} = |N_{D/k}(x)|_\mathbb{A} \) for \( x \in D_{\mathbb{A}} \).

For the group \( G = U(W) \), there is an Iwasawa decomposition \( G(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A})G(O_{\mathbb{A}}) \).

For \( g \in G(\mathbb{A}) \), write \( g = m(a)n(b)g_1 \) with \( m(a) \in M(\mathbb{A}) \), \( n(b) \in N(\mathbb{A}) \) and \( g_1 \in G(O_{\mathbb{A}}) \), and define \( |a(g)| = |\det a|_{D_{\mathbb{A}}} \).

Fix a non-trivial additive character \( \psi : \mathbb{A}/k \to \mathbb{C}^\times \).

We identify \( \text{Her}_n(\mathbb{A}) \) with its Pontryagin dual via \( \psi \):
\[
[b_1, b_2] \mapsto \psi(\frac{1}{2}r(b_1 b_2)), \quad \text{for } b_1, b_2 \in \text{Her}_n(\mathbb{A}).
\]

For the adelic group \( \text{Sp}(\mathbb{W})(\mathbb{A}) \) and the local groups \( \text{Sp}(\mathbb{W})_v \), there are associated metaplectic groups \( \text{Mp}(\mathbb{W})_{\mathbb{A}} \) and \( \text{Mp}(\mathbb{W})_v \):
\[
1 \to \mathbb{C}^1 \to \text{Mp}(\mathbb{W})_{\mathbb{A}} \to \text{Sp}(\mathbb{W})(\mathbb{A}) \to 1,
\]
\[
1 \to \mathbb{C}^1 \to \text{Mp}(\mathbb{W})_v \to \text{Sp}(\mathbb{W})_v \to 1,
\]
here \( \mathbb{C}^1 = \{ z \in \mathbb{C} : |z| = 1 \} \). See [Weil 1964], [Rao 1993].

The metaplectic group \( \text{Mp}(\mathbb{W})_{\mathbb{A}} \) can be identified with the set \( \text{Sp}(\mathbb{W})_{\mathbb{A}} \times \mathbb{C}^1 \) equipped with a group multiplication
\[
(g_1, z_1) \cdot (g_2, z_2) = (g_1 g_2, z_1 z_2 \cdot c(g_1, g_2)),
\]
where \( c(g_1, g_2) = \prod_v c_v(g_1 v, g_2 v) \), and \( c_v(g_1 v, g_2 v) \) is Rao’s cocycle in [Rao 1993, Theorem 4.1]. Similarly \( \text{Mp}(\mathbb{W})_v \) can be identified with the set \( \text{Sp}(\mathbb{W})_v \times \mathbb{C}^1 \) by using Rao’s cocycle.

There is an associated Weil representation \( \omega_\psi \) of \( \text{Mp}(\mathbb{W})_{\mathbb{A}} \) on \( \mathcal{S}(X(\mathbb{A})) \) given by
\[
\omega_\psi(g, z) = z \cdot r_\psi(g),
\]
where
\[
r_\psi(g) = \otimes v r_\psi(v, g_v)
\]
and \( r_\psi(v, g_v) \) is defined in [Rao 1993, Theorem 3.5].

To construct Weil representation of \( G(\mathbb{A}) \), we fix a multiplicative adelic character \( \chi \) as follows.

If \( \epsilon = 0 \), let \( \chi \) be the quadratic character of \( \mathbb{A}^\times/k^\times \) given by \( \chi(x) = \prod_v (x_v, \det V)_v \), where \((,)_v \) is the Hilbert symbol for \( k_v \).

If \( \epsilon = \frac{1}{4} \), let \( \chi \) be the trivial character of \( \mathbb{A}^\times \).

If \( \epsilon = \frac{1}{2} \), let \( \chi \) be a character of \( D_{\mathbb{A}}^\times/D_k^\times \) with \( \chi|_{D_k^\times} = \epsilon_{E/F}^m \), where \( \epsilon_{E/F} \) is the quadratic Hecke character of \( k \) associated to \( D/k \) by class field theory.

If \( \epsilon = \frac{3}{4} \), let \( \chi \) be the quadratic character of \( \mathbb{A}^\times/k^\times \) given by \( \chi(x) = \prod_v (x_v, -1)^m \det V)_v \).
If $\epsilon = 1$, let $\chi$ be the trivial character of $\mathbb{A}^\times$.

With the help of $\chi$, there is a splitting

$$\tilde{i}_\chi : G(\mathbb{A}) \to Mp(\mathbb{W})_\mathbb{A},$$

$$g \mapsto (i(g), \beta_\chi(g)),$$

where $\beta_\chi(g) = \prod_v \beta_\chi_v(g_v) \in \mathbb{C}^1$ with $\beta_\chi_v(g_v)$ defined as in [Kudla 1994, Theorem 3.1].

Then the Weil representation $\omega$ of $G(\mathbb{A})$, associated to the data $(\psi, \chi)$, is defined to be

$$\omega = \omega_{\psi, \chi} := \tilde{i}_\chi \circ \omega.$$ 

It is given explicitly as follows:

$$\omega(m(a))\Phi(x) = \chi(a)|\nu(a)|_\mathbb{A}^{\frac{1}{2}} \Phi(xa),$$

$$\omega(n(b))\Phi(x) = \psi(q_b(x))\Phi(x),$$

$$\omega(w_n)\Phi(x) = \int_{X(\mathbb{A})} \Phi(y)\psi(\kappa \cdot \tau((y, x))) \, dy,$$

$$\omega(w_r)\Phi(x) = \int_{X''(\mathbb{A})} \Phi(x' + z)\psi(\kappa \cdot \tau((x'', z))) \, dz,$$

for $\Phi \in \mathcal{S}(X(\mathbb{A}))$, $m(a) \in M(\mathbb{A})$, $n(b) \in N(\mathbb{A})$, $x = x' + x'' \in X(\mathbb{A})$ with $x' \in X'(\mathbb{A})$ and $x'' \in X''(\mathbb{A})$, where $X' = \{(x_1, \ldots, x_{n-r}, 0, \ldots, 0)\}$ and $X'' = \{(0, \ldots, 0, x_{n-r+1}, \ldots, x_n)\}$.

Here

- $\chi(a) = \chi(\det a)$ if $[D : k] = 2$, and $\chi(a) = \chi(\nu(a))$ otherwise;

- $q_b(x) = \frac{1}{\tau((x, x)b)}$, where we recall $\kappa = 2$ if $D = k$ and $\kappa = 1$ if $D \neq k$;

- $(y, x) = ((y_i, x_j)) \in \text{Her}_n(\mathbb{A})$ for $y = (y_1, \ldots, y_n)$ and $x = (x_1, \ldots, x_n)$ in $X(\mathbb{A})$;

- $dy$ is the self-dual Haar measure on $X(\mathbb{A})$.

See [Kudla 1994, p. 400], [Kudla-Sweet 1997, p. 280] for the local case.

Let $G_m = GL_1$ be the multiplicative group in one variable over $k$.

For a locally compact group $G$, its modulus character $\delta_G(g)$ is the ratio of right and left Haar measures, i.e. $d_R g = \delta_G(g)d_L g$.

For an algebraic group $\mathfrak{G}$ over $k$, its algebraic module $\Delta_\mathfrak{G}$ is a $k$-rational character such that

$$\omega(a^{-1}xa) = \Delta_\mathfrak{G}(a)\omega(x),$$

where $\omega$ is any gauge on $\mathfrak{G}$. See [Weil 1965, p. 11].

### 3. Analytic preliminaries

In this section, we review Harder’s reduction theory over function fields ([Harder 1968], [Harder 1973]), and prove some auxiliary lemmas analogous to those in [Weil 1965].

Throughout this section, we let $G$ be either $G$ or $H$, where we recall $H$ is the connected component of the identity in $U(V)$. Note that $G = U(W)$ is quasi-split (containing a Borel
subgroup) and \( H \subset U(V) \) is connected, so we can apply the reduction theory over function fields to \( G \).

By changing the sign of \( \eta \), we can regard \( G \subset U(V) \) for a non-degenerate \( \eta \)-hermitian space \( V \) over \( D \), given as follows:

- \( G = U(V) \) when \( V \) is split,
- \( G \subset U(V) \) is the connected component of the identity when \( V \) is not split.

We assume that \( V \) is of dimension \( m \) and of Witt index \( r \).

Let \( T \cong (\mathbb{G}_m)^r \) be a maximal split torus in \( G \) given as follows. We may assume by choosing a suitable basis of \( V \) that the \( \eta \)-hermitian form on \( V \) is given by the matrix

\[
Q = \begin{pmatrix}
0 & 0 & 1_r \\
0 & Q_0 & 0 \\
\eta \cdot 1_r & 0 & 0
\end{pmatrix},
\]

where \( Q_0 \) is the matrix (of order \( m - 2r \)) of an anisotropic \( \eta \)-hermitian form. For \( t = (t_1, \ldots, t_r) \in (\mathbb{G}_m)^r \), denote by \( d(t) \) the diagonal matrix of order \( m \) whose diagonal elements are

\[
(t_1, \ldots, t_r, 1, \ldots, 1, t_1^{-1}, \ldots, t_r^{-1}).
\]

Then \( d \) is an isomorphism of \( (\mathbb{G}_m)^r \) onto a maximal split torus \( T \) of \( G \).

Recall we have fixed a place \( v_0 \) of \( k \) such that the residue field of \( k_{v_0} \) is \( \mathbb{F}_q \), and let \( \varpi \) be a uniformizer at \( v_0 \).

Motivated by the constructions on p. 19 of [Moeglin-Waldspurger 1995, § I.2.1.], we proceed as follows. For \( \tau \in \mathbb{Z} \), define an element \( a_\tau \in \mathbb{A}^\times \) whose component at \( v_0 \) is \( \varpi^\tau \) and the component at any other place is 1. Then let

\[
\Theta(T) = \{(a_{\tau_1}, \ldots, a_{\tau_r}) : \tau_i \in \mathbb{Z}\},
\]

and regard \( \Theta(T) \) as a subset of \( G(\mathbb{A}) \) via \((\mathbb{G}_m)^r \cong T\).

Let \( P_0 \) be a minimal \( k \)-parabolic subgroup of \( G \) which contains \( T \). Let

\[
P_0(\mathbb{A})^1 = \{p \in P_0(\mathbb{A}) : |\chi(p)| = 1, \forall \chi \in X_k(P_0)\},
\]

where \( X_k(P_0) \) is the group of \( k \)-characters of \( P_0 \).

We can define \( T(\mathbb{A})^1 \) and \( G(\mathbb{A})^1 \) similarly.

Then \( P_0(\mathbb{A})^1 / P_0(k) \) is compact and \( P_0(\mathbb{A}) / P_0(\mathbb{A})^1 \cong \Theta(T) \).

For \( c \in \mathbb{R} \), let

\[
\Theta(c) = \{\theta \in \Theta(T) : |\alpha(\theta)| \leq q^c, \forall \alpha \in \Delta\},
\]

where \( \Delta \) is the set of simple roots of \( G \) with respect to \( T \), which is given by

\[
\Delta = \{x_i - x_{i+1}, 2x_r : 1 \leq i \leq r - 1\}
\]

where \( x_i(t) = t_i \) for \( t = (t_1, \ldots, t_r) \in T \).

**Lemma 3.1.** For \( c \in \mathbb{R} \), we have

\[
\Theta(c) = \{(a_{\tau_1}, a_{\tau_2}, \ldots, a_{\tau_r}) \in \Theta(T) : -c/2 \leq \tau_r \leq \tau_{r-1} + c \leq \ldots \leq \tau_2 + (r - 2)c \leq \tau_1 + (r - 1)c\}.\]
In particular,

\[ \Theta(0) = \{(a_{\tau_1}, a_{\tau_2}, \ldots, a_{\tau_r}) \in \Theta(T) : 0 \leq \tau_1 \leq \tau_2 - 1 \leq \ldots \leq \tau_2 \leq \tau_1 \}. \]

Moreover, for any \( e \), we have

\[ \Theta(e) = (a_{-(2r-1)e/2}, a_{-(2r-3)e/2}, \ldots, a_{-e/2}, a_{-e/2}) \cdot \Theta(0). \]

For \( e \in \mathbb{R} \), let

\[ T(e) = \{ t \in T(\mathbb{A}) : |\alpha(t)| \leq q^e, \forall \alpha \in \Delta \}. \]

Then it is easy to see that \( T(e) = \Theta(e) \cdot T(\mathbb{A})^1 \).

For a compact subset \( C \) of \( G(\mathbb{A}) \) and a constant \( e \in \mathbb{R} \), the associated Siegel set is defined by

\[ \mathcal{S}(C; e) = C \cdot T(e) \cdot P_0(\mathbb{A})^1. \]

The fundamental set theorem in reduction theory claims that there is a Siegel set \( \mathcal{S}(C; e) \) in \( G(\mathbb{A}) \) such that

\[ \mathcal{G}(\mathbb{A}) = \mathcal{S}(C; e) \cdot G(k). \]

See [Springer 1994, pp. 211-212]. Note that \( \mathcal{G}(\mathbb{A})^1 = \mathcal{G}(\mathbb{A}) \).

We have the following results analogous to those in [Weil 1965, n. 10].

**Lemma 3.2.** (i) There exists a compact subset \( C \) of \( G(\mathbb{A}) \) such that

\[ \mathcal{G}(\mathbb{A}) = C \cdot \Theta(0) \cdot P_0(\mathbb{A})^1 \cdot G(k). \]

(ii) Suppose \( \mathcal{G} = U(W) \). Then there exists a compact subset \( C_1 \) of \( G(\mathbb{A}) \) such that

\[ \mathcal{G}(\mathbb{A}) = C_1 \cdot T(0) \cdot G(k). \]

**Proof.** (i) It is easy to check that

\[ T(e) \cdot P_0(\mathbb{A})^1 = \Theta(e) \cdot P_0(\mathbb{A})^1. \]

Thus it follows from the fundamental set theorem that there is a compact subset \( C \) of \( G(\mathbb{A}) \) and a constant \( e \) such that

\[ \mathcal{G}(\mathbb{A}) = C \cdot \Theta(e) \cdot P_0(\mathbb{A})^1 \cdot G(k). \]

Replacing \( C \) with \( C \cdot (a_{-(2r-1)e/2}, a_{-(2r-3)e/2}, \ldots, a_{-e/2}, a_{-e/2}) \), we may always assume that \( e = 0 \).

(ii) Since \( \mathcal{G} = U(W) \) is quasi-split, we can take \( P_0 \) to be a Borel subgroup \( B \) such that \( T \) is its Levi subgroup. The desired result follows from (i).

**Lemma 3.3.** (i) If \( C \) is a compact subset of \( P_0(\mathbb{A}) \), then the union of \( \theta C \theta^{-1} \) for \( \theta \in \Theta(0) \) is relatively compact in \( P_0(\mathbb{A}) \).

(ii) Let \( C_2 \) be a compact subset of \( P_0(\mathbb{A})^1 \) such that \( P_0(\mathbb{A})^1 = C_2 \cdot P_0(\mathbb{A}) \). Then there is a compact subset \( C_3 \) of \( P_0(\mathbb{A})^1 \) such that \( \theta C_2 \theta^{-1} \subset C_3 \) for all \( \theta \in \Theta(0) \).

**Proof.** (i) See [Godement 1963, Lemme 1, p. 217] for the number field case. The function field case can be proved similarly.

(ii) See [Weil 1963, pp. 18–19] for the number field case. In our case, we can similarly take \( C_3 \) to be the closure of the union of \( \theta C_2 \theta^{-1} \) for \( \theta \in \Theta(0) \).
Let $\varphi$ be a Haar measure on $G(\mathbb{A})$. To study the convergence of integrals on $G(\mathbb{A})/G(k)$, we will rely on the following lemma, which is an analogue of Lemma 4 in n. 11 of [Bourbaki 1963]. Let $\Delta_{P_0}$ be the modulus character of $P_0$. We write $\Theta^+ = \Theta(0)$ and $P_0(\mathbb{A}) = \Theta^+ \cdot P_0(\mathbb{A})^1$.

**Lemma 3.4.** Assume $C$ is a compact subset of $G(\mathbb{A})$ such that $G(\mathbb{A}) = C \cdot P_0(\mathbb{A})^+ \cdot G(k)$. Then there exists a compact subset $C_0$ of $G(\mathbb{A})$ and a constant $\gamma > 0$ such that

$$
\int_{G(\mathbb{A})/G(k)} |F(g)| \, dg \leq \gamma \int_{\Theta^+} F_0(\theta) \cdot |\Delta_{P_0}(\theta)|_{\mathbb{A}}^{-1} \, d\theta
$$

whenever $F, F_0$ are locally integrable functions on $G(\mathbb{A})/G(k)$ and on $\Theta^+$ respectively, such that $|F(\gamma \theta)| \leq F_0(\theta)$ for all $\gamma \in C_0$ and $\theta \in \Theta^+$.

**Proof.** Denote by $I$ the first member of (3.1), and by $\varphi_N$ the characteristic function of the set $N = C \cdot P_0(\mathbb{A})^+$. We have

$$
I \leq \int_{N/P_0(\mathbb{A})} |F(g)| \, dg = \int_{G(\mathbb{A})/P_0(\mathbb{A})} |F(g)| \varphi_N(g) \, dg.
$$

We will transform the last integral by means of the theory of quasi-invariant measures on homogeneous spaces (see [Bourbaki 1963], Chap. VII, §2, n. 5-8). According to this theory, we can construct a continuous function $h$ on $G(\mathbb{A})$, everywhere $> 0$, such that $h(gp) = h(g)|\Delta_{P_0}(p)|_{\mathbb{A}}$ for all $g \in G(\mathbb{A}), p \in P_0(\mathbb{A})$, and then a positive measure $\lambda$ on $G(\mathbb{A})/P_0(\mathbb{A})$ such that for every locally integrable function $f \geq 0$ on $G(\mathbb{A})/P_0(\mathbb{A})$, we have:

$$
\int_{G(\mathbb{A})/P_0(\mathbb{A})} f(g) \, dg = \int_{G(\mathbb{A})/P_0(\mathbb{A})} \left( h(g) \int_{P_0(\mathbb{A})/P_0(\mathbb{A})} f(gp) \, dp \right) \, d\lambda(g),
$$

where $\hat{g}$ is the image of $g \in G(\mathbb{A})$ in $G(\mathbb{A})/P_0(\mathbb{A})$ and $dp = |\Delta_{P_0}(\theta)|_{\mathbb{A}}^{-1} \, d\theta \, dp_1$ is the right invariant measure on $P_0(\mathbb{A}) = \Theta(T) \cdot P_0(\mathbb{A})^1$, where $d\theta, dp_1$ are Haar measures on $\Theta(T)$ and on $P_0(\mathbb{A})^1$ respectively. Applying this formula to the last member of (3.2), we obtain

$$
I \leq \int_{G(\mathbb{A})/P_0(\mathbb{A})} \Psi(\hat{g}) \, d\lambda(\hat{g}),
$$

where $\Psi$ is the function defined by

$$
\Psi(\hat{g}) = h(g) \int_{P_0(\mathbb{A})/P_0(\mathbb{A})} |F(gp)| \varphi_N(gp) \, dp.
$$

Since $G(\mathbb{A})/P_0(\mathbb{A})$ is compact, there is a compact subset $C_1$ of $G(\mathbb{A})$ such that $G(\mathbb{A}) = C_1 \cdot P(\mathbb{A})$. We can therefore assume that $g \in C_1$ in the second member of the above formula. But then we have $\varphi_N(gp) = 0$ when $p \notin C_1^{-1}N$. Put

$$
Q = C_1^{-1}N \cap P_0(\mathbb{A}) = (C_1^{-1}C \cap P_0(\mathbb{A})) \cdot P_0(\mathbb{A})^+ = (C_1^{-1}C \cap P_0(\mathbb{A})) \cdot \Theta^+ \cdot P(\mathbb{A})^1;
$$

let $\gamma_1$ be the supremum of $h$ on $C_1$, and let $F_1(p)$, for each $p \in P_0(\mathbb{A})$, be the supremum of $|F(gp)|$ for $g \in C_1$. Therefore

$$
\Psi(\hat{g}) \leq \gamma_1 \int_{Q/P_0(\mathbb{A})} |F(gp)| \, dp \leq \gamma_1 \int_{Q/P_0(\mathbb{A})} F_1(p) \, dp,
$$

and consequently, since $G(\mathbb{A})/P_0(\mathbb{A})$ is compact, we have

$$
I \leq \gamma_2 \int_{Q/P_0(\mathbb{A})} F_1(p) \, dp
$$

provided that the constant $\gamma_2$ is suitably chosen.
We identify $P_0(\mathcal{A})/P_0(\mathcal{A})^1$ with $\Theta(T)$; it is immediate that every compact subset of $\Theta(T)$ is contained in a set of the form $\theta_0\Theta^+$, with $\theta_0 \in \Theta(T)$; applying this remark to the image of $C_1^{-1}C \cap P_0(\mathcal{A})$ in $P_0(\mathcal{A})/P_0(\mathcal{A})^1 = \Theta(T)$, we conclude that there exists $\theta_0 \in \Theta(T)$ such that $Q$ is contained in $\theta_0\Theta^+ \cdot P_0(\mathcal{A})^1$. On the other hand, since $P_0(\mathcal{A})^1/P_0(k)$ is compact, there exists a compact subset $C_2$ of $P_0(\mathcal{A})^1$ such that $P_0(\mathcal{A})^1 = C_2 \cdot P_0(k)$, so we obtain $Q \subset \theta_0\Theta^+ \cdot C_2 \cdot P_0(k)$, and consequently

$$I \leq \gamma_2 \int_{\theta_0\Theta^+ \cdot C_2} F_1(p) \, dp.$$ 

Since $dp = |\Delta_{P_0(\theta)}|^{-1}_{\mathcal{A}} \, d\theta \, dp_0$, this can also be written as

$$I \leq \gamma_2 \int_{C_2} \left( \int_{\theta^+} F_1(\theta_0\theta p_0) \cdot |\Delta_{P_0(\theta_0\theta)}|^{-1}_{\mathcal{A}} \, d\theta \right) \, dp_0.$$ 

But by Lemma 3.3 there is a compact subset $C_3$ of $P_0(\mathcal{A})^1$ such that $\theta C_2 \theta^{-1} \subset C_3$, therefore $\theta_0\theta C_2 \subset \theta_0 C_3 \theta$ for any $\theta \in \Theta^+$. Thus if we denote by $F_2(\theta)$, for any $\theta \in \Theta^+$, the supremum of $F_1(p\theta)$ for $p \in \theta_0 C_3$, then we obtain

$$I \leq \gamma \int_{\theta^+} F_2(\theta) \cdot |\Delta_{P_0(\theta)}|^{-1}_{\mathcal{A}} \, d\theta$$

provided that the constant $\gamma$ is suitably chosen. It follows that the assertion of the lemma is verified if we take $C_0 = C_1\theta_0 C_3$. $\square$

Now let $X$ be an affine space on which $\mathcal{G}$ acts via a representation $\rho$ of $\mathcal{G}$ in $\text{Aut}(X)$. For every character $\lambda$ of $T$, we denote by $m_\lambda$ the dimension over $k$ of the space of vectors $a \in X_k$ such that $\rho(t) a = \lambda(t) a$ for any $t \in T$. The characters $\lambda$ of $T$ for which $m_\lambda > 0$ are the weights of the representation $\rho$; $m_\lambda$ is the multiplicity of the weight $\lambda$.

We have the following analogue of Lemme 5 in n.12 of Weil 1965.

**Lemma 3.5.** Let $C$ be a compact subset of $\mathcal{G}(\mathcal{A})$ such that $\mathcal{G}(\mathcal{A}) = C \cdot P_0(\mathcal{A})^+ \cdot \mathcal{G}(k)$; let $\rho$ be a representation of $\mathcal{G}$ in the group $\text{Aut}(X)$ of automorphisms of an affine space $X$. Then the integral

$$I(\Phi) = \int_{\mathcal{G}(\mathcal{A})/\mathcal{G}(k)} \sum_{\xi \in X(k)} \Phi(\rho(g)\xi) \cdot dg$$

is absolutely convergent for any function $\Phi \in \mathcal{S}(X(\mathcal{A}))$ whenever the integral

$$\int_{\theta^+} \prod_\lambda \sup(1, |\lambda(\theta)|^{-1}_{\mathcal{A}})^{m_\lambda} \cdot |\Delta_{P_0(\theta)}|^{-1}_{\mathcal{A}} \, d\theta$$

is convergent, where $\lambda$ runs over the weights of $\rho$; and when this is so, $I(\Phi)$ defines a positive tempered measure $I$.

**Proof.** If $I(\Phi)$ is absolutely convergent for any function $\Phi \in \mathcal{S}(X(\mathcal{A}))$, Lemme 5 of Weil 1964, n. 41] shows that this is uniformly on every compact subset of $\mathcal{S}(X(\mathcal{A}))$, whence it follows, according to Lemma 2 of Weil 1963, n. 2], that $I$ is a tempered distribution, therefore a positive tempered measure. Now let $C_0$ be a compact subset of $\mathcal{G}(\mathcal{A})$ with the property stated in Lemma 3.4 above. For $\Phi$ in $\mathcal{S}(X(\mathcal{A}))$, there exists, according to Lemma 5 in n. 41 of Weil 1964, a function $\Phi_1 \in \mathcal{S}(X(\mathcal{A}))$ such that

$$|\Phi(\rho(c)x)| \leq \Phi_1(x)$$
for all \( c \in C_0 \) and \( x \in X(k) \). Applying Lemma 3.4 to \( \Theta \) then shows that \( I(\Phi) \) is absolutely convergent provided that this is so for the integral

\[
I_1 = \int_{\Theta^+} \sum_{\xi \in X(k)} \Phi_1(\rho(\theta)\xi) \cdot |\Delta_{\rho_0}(\theta)|^{-1}_A \, d\theta.
\]

We write \( X(k) = X_{\nu_0} \times X' \), where \( X' \) is defined as \( X(\tilde{A}) \) but by means of places \( v \) of \( k \) for which \( v \neq v_0 \). Taking into account of the definition of \( S(X(\tilde{A})) \) (cf. [Weil 1964, n. 29]), we can assume that \( \Phi_1 \) is of the form

\[
\Phi_1(x) = \Phi_{v_0}(x_{v_0}) \Phi'(x'),
\]

where \( x_{v_0}, x' \) are the projections of \( x \in X(\tilde{A}) \) on \( X_{v_0} \) and on \( X' \), with \( \Phi_{v_0} \in S(X_{v_0}) \), \( \Phi' \) being the characteristic function of a compact open subgroup of \( X' \). Let \( L \) be the set of \( \xi \in X(k) \) whose projection on \( X' \) belongs to the support of \( \Phi' \). Then \( I_1 \) can be written as:

\[
I_1 = \int_{\Theta^+} \sum_{\xi \in L} \Phi_{v_0}(\rho(\theta)\xi) \cdot |\Delta_{\rho_0}(\theta)|^{-1}_A \, d\theta.
\]

For every weight \( \lambda \) of \( \rho \), let \( X_\lambda \) be the subspace of \( X_k \), of dimension \( m_\lambda \) over \( k \), formed of eigenvectors of the weight \( \lambda \), i.e. vectors \( a \) such that \( \rho(t)a = \lambda(t)a \) for \( t \in T \); \( X_k \) is the direct sum of \( X_\lambda \).

Let

\[
(a_{\lambda i})_{1 \leq i \leq m_\lambda}
\]

be a basis of \( X_\lambda \) over \( k \); replacing \( a_{\lambda i} \) by \( N^{-1}a_{\lambda i} \) if needed, where \( N \) is a suitable element in \( O_k \), we may assume that \( L \) is contained in the \( O_k \)-submodule of \( X_k \) generated by the set of \( a_{\lambda i} \).

The \( a_{\lambda i} \) also form a basis of \( X_{v_0} \) over \( k_{v_0} \); for \( x_{v_0} \in X_{v_0} \), we can thus write

\[
x_{v_0} = \sum_{\lambda, i} x_{\lambda i} a_{\lambda i}
\]

with \( x_{\lambda i} \in k_{v_0} \); then, if \( \alpha > 1 \), there is a constant \( C \) such that

\[
\Phi_{v_0}(x_{v_0}) \leq C \prod_{\lambda, i} (1 + |x_{\lambda i}|_{v_0}^\alpha)^{-1}.
\]

On the other hand, under these conditions, we have

\[
\rho(\theta)x_{v_0} = \sum_{\lambda, i} \lambda(\theta)x_{\lambda i} a_{\lambda i},
\]

where \( \lambda(\theta) \in k_{v_0}^* \); and, if \( x_{v_0} \) is the projection on \( X_{v_0} \) of an element \( \xi \) of \( L \), then all the \( x_{\lambda i} \) are elements in \( O_k \) by the choice of the bases \((a_{\lambda i})\). By the choice of \( v_0 \), we have \( |x_{\lambda i}|_{v_0} \geq 1 \). Thus we have

\[
\sum_{\xi \in L} \Phi_{v_0}(\rho(\theta)\xi) \leq C \prod_{\lambda} \left( \sum_{n \geq 0} \frac{1}{1 + |\lambda(\theta)|_{v_0}^\alpha n^\alpha} \right)^{m_\lambda}
\]

\[
\leq C' \prod_{\lambda} \sup(1, |\lambda(\theta)|_{v_0}^{-1})^{m_\lambda},
\]

where \( C' \) is a suitable constant. If we observe that \( |\lambda(\theta)|_A = |\lambda(\theta)|_{v_0} \) for any character \( \lambda \) of \( T \) and for any \( \theta \in \Theta^+ \), we see that this gives the announced conclusion. \( \Box \)
Finally, we have the following analogue of Lemme 6 of \cite{Weil 1965}, n. 13. Recall that we have fixed a place \( v_0 \) of \( k \) such that the residue field of \( k_{v_0} \) is \( \mathbb{F}_q \), and we denote by \( a_{\tau} \), for \( \tau \in \mathbb{Z} \), the idele \((a_v)\) given by \( a_v = \varpi^\tau \) for \( v = v_0 \), and \( a_v = 1 \) for any other place \( v \), where \( \varpi \) is a uniformizer of \( k_{v_0} \).

**Lemma 3.6.** Let \( (X^{(a)})_{1 \leq a \leq n} \) and \( Y \) be vector spaces over \( k \); let \( X = \prod_\alpha X^{(a)} \), and let \( p \) be a morphism of \( X \) into \( Y \), rational over \( k \) and such that \( p(0, x^{(2)}, \ldots, x^{(n)}) = 0 \) for any \( x^{(2)}, \ldots, x^{(n)} \). Let \( C_0 \) be a compact subset of \( S(X(\mathbb{A})) \), and let \( N \geq 0 \). Then there exists a function \( \Phi_0 \in S(X(\mathbb{A})) \) such that

\[
|a_{\tau_1}|^N |\Phi(a_{\tau_1} x^{(1)}, \ldots, a_{\tau_n} x^{(n)})| = q^{-\tau_1 N} |\Phi(a_{\tau_1} x^{(1)}, \ldots, a_{\tau_n} x^{(n)})| \leq \Phi_0(x)
\]

whenever \( \Phi \in C_0 \), \( \tau_1 \leq 0, \ldots, \tau_n \leq 0 \), \( x = (x^{(1)}, \ldots, x^{(n)}) \in X(\mathbb{A}) \), \( p(x) \in Y(k) \) and \( p(x) \neq 0 \).

**Proof.** Note that a morphism of an affine space into another is just a polynomial mapping. Thus if we choose bases of \( X \) and of \( Y \) over \( k \), then the coordinates of \( p(x) \) can be expressed as polynomials with coefficients in \( k \) by means of those of \( x \). We denote by \( d \) the largest degree of these polynomials. On the other hand, write \( X(\mathbb{A}) = X_{v_0} \times X' \), and likewise write \( X(\mathbb{A})^{(a)} = X_{v_0}^{(a)} \times X^{(a)} \) and \( Y(\mathbb{A}) = Y_{v_0} \times Y' \); \( p \) determines in an obvious way mappings of \( X_{v_0} \) into \( Y_{v_0} \) and of \( X' \) into \( Y' \). Choose bases of \( X_{v_0}^{(a)} \) and of \( Y_{v_0} \) over \( k_{v_0} \), and, for \( x_{v_0}^{(a)} \in X_{v_0}^{(a)} \) (resp. \( y_{v_0} \in Y_{v_0} \)), denote by \( r_\alpha(x_{v_0}^{(a)}) \) (resp. \( s(y_{v_0}) \)) the sum of the squares of the absolute value of the coordinates of \( x_{v_0}^{(a)} \) (resp. of \( y_{v_0} \)) with respect to these bases. For \( x_{v_0} = (x_{v_0}^{(1)}, \ldots, x_{v_0}^{(n)}) \in X_{v_0} \), put

\[
r'(x_{v_0}) = \sum_{\alpha \geq 2} r_\alpha(x_{v_0}^{(a)}), \quad r(x_{v_0}) = r_1(x_{v_0}^{(1)}) + r'(x_{v_0}).
\]

Since \( p(x) \) vanishes whenever \( x^{(1)} = 0 \), there is a constant \( C > 0 \) such that for any \( x_{v_0} \in X_{v_0} \):

\[
s(p(x_{v_0})) \leq C \cdot r_1(x_{v_0}^{(1)}) \cdot r(x_{v_0})^{d-1},
\]

and consequently, for \( t_1 \geq 1 \):

\[
s(p(x_{v_0})) \leq C t_1^{-2} (t_1^2 r_1(x_{v_0}^{(1)}) + r'(x_{v_0}))^d.
\]

For \( \tau = (\tau_1, \ldots, \tau_n) \), \( \tau_i \in \mathbb{Z} \), and \( x_{v_0} \in X_{v_0} \), let

\[
\varpi^\tau x_{v_0} = (\varpi^{\tau_1} x_{v_0}^{(1)}, \ldots, \varpi^{\tau_n} x_{v_0}^{(n)});
\]

the inequality which we have obtained shows that, whenever \( \tau_1 \leq 0, \ldots, \tau_n \leq 0 \):

\[
s(p(x_{v_0})) \leq C q^{-2 \tau_1} r(\varpi^\tau x_{v_0})^d.
\]

Now, if we apply Lemme 5 in n. 41 of \cite{Weil 1964}, then this shows that we can choose \( \Phi_1 \in S(X(\mathbb{A})) \) such that \( |\Phi(x)| \leq \Phi_1(x) \) for all \( \Phi \in C_0 \) and all \( x = (x_{v_0}, x') \in X(\mathbb{A}) \), and likewise we can assume that \( \Phi_1 \) is of the form

\[
\Phi_1(x) = \Phi_{v_0}(x_{v_0}) \Phi'(x'),
\]

where \( \Phi_{v_0} \in S(X_{v_0}) \) and \( \Phi' \) is the characteristic function of a compact open subgroup of \( X' \).

Let \( E \) be the set of points \( x = (x_{v_0}, x') \) of \( X(\mathbb{A}) \) such that \( p(x) \in Y(k) \), \( p(x) \neq 0 \) and \( \Phi'(x') \neq 0 \); we will show that, on \( E \), \( s(p(x_{v_0})) \) has an infimum \( \epsilon > 0 \). In fact, if it is not so, there will be a sequence of points \( x_\nu = (x_{v_\nu}, x'_\nu) \) of \( E \) such that the sequence \( p(x_{v_\nu}) \) tends to 0 in \( Y_{v_0} \). As the support of \( \Phi' \) is compact, we can assume at the same time that the sequence \( x'_{\nu} \) tends to a limit.
\( \bar{x}' \), therefore that \( p(x'_v) \) tends to \( p(\bar{x}') \). But then the sequence of points \( y_v = p(x_v) \) tends to a limit \( \bar{y} \) in \( Y(\mathbb{A}) \), for which we have \( \bar{y}_{v_0} = 0 \). As the \( y_v \) belong to \( Y(k) - \{0\} \), which is discrete in \( Y(\mathbb{A}) \), we have \( \bar{y} \in Y(k) \), \( \bar{y} \neq 0 \), therefore \( \bar{y}_{v_0} \neq 0 \) for any \( v \), whence the contradiction. Taking into account the inequality proved above, we thus have, for \( x \in E, \tau_1 \leq 0, \ldots, \tau_n \leq 0 \):

\[
q^{-\tau_1} \leq C' r(x_v y_{v_0})^{2/2}
\]

with \( C' = (C/\epsilon)^{1/2} \).

Now, for any \( i \geq 0 \), put

\[
a_i = \sup_{x_v \in X_{v_0}} (r(x_v y_{v_0})^i \Phi(x_v)).
\]

Let \( M \geq Nd/2 \) be an integer. According to Lemma 4 in n. 41 of [Veil 1964], there exists \( \varphi \in \mathcal{S}(\mathbb{R}) \) such that we have, for any \( r \in \mathbb{R} \):

\[
\varphi(x) \geq \inf_{i \geq 0} (a_{M+i}|r|^{-i}).
\]

For \( x \) and \( \tau \) as above, we thus have, for any \( i \geq 0 \):

\[
q^{-2\tau_1 M/d} \Phi_{v_0}(x_v y_{v_0}) \leq C'^{2M/d} r(x_v y_{v_0})^M \Phi_{v_0}(x_v y_{v_0})
\]

\[
\leq C'^{2M/d} a_{M+i} r(x_v y_{v_0})^{-i} \leq C'^{2M/d} a_{M+i} r(x_v y_{v_0})^{-i},
\]

and therefore

\[
q^{-\tau_1 N} \Phi_{v_0}(x_v y_{v_0}) \leq C'^{2M/d} \varphi(r(x_v y_{v_0})).
\]

Thus the conditions of the lemma will be satisfied by setting

\[
\Phi_0(x) = C'^{2M/d} \varphi(r(x_v y_{v_0})) \Phi'(x').
\]

\[\square\]

4. Siegel Eisenstein series

Recall \( G = U(W) \) and \( P \) is the Siegel parabolic subgroup of \( G \). For \( \Phi \in \mathcal{S}(X(\mathbb{A})), g \in G(\mathbb{A}) \) and \( s \in \mathbb{C} \), define the Siegel Eisenstein series by

\[
E(g, s, \Phi) = \sum_{\gamma \in P(F) \setminus G(F)} f_{\Phi}^{(s)}(\gamma g),
\]

where \( f_{\Phi}^{(s)}(g) = |a(g)|^{s - s_0} \omega(g) \Phi(0) \) and \( s_0 = (m + n + 1 - 2\epsilon)/2 \). Here \( |a(g)| \) is defined in Section 3.

We are interested in the behavior of \( E(g, s, \Phi) \) at \( s_0 = (m + n + 1 - 2\epsilon)/2 \).

Let \( P = MN \) be the Levi decomposition, where \( M \) is the Levi subgroup and \( N \) is the unipotent radical.

Let \( Z_M \) be the center of \( M \). Then \( Z_M \cong (\text{Res}_{D/k} \mathbb{G}_m)^n \). We write an element \( z \) of \( Z_M \) as the form \( z = (z_1, \ldots, z_n) \), which means that \( z = \text{diag}(z_1, \ldots, z_n) \in GL_n(D) \).

Let \( T \cong (\mathbb{G}_m)^n \) be the maximal split torus in \( G \) given by

\[
T = \{ t = (t_1, \ldots, t_n) \in Z_M : t_i \in \mathbb{G}_m \}.
\]
Lemma 4.1. For $z \in Z_M(\mathbb{A})$ and $g \in G(\mathbb{A})$, we have
\[ f_\beta^{(s)}(zg) = \lambda(z)f_\beta^{(s)}(g), \]
where
\[ \lambda(z) = \chi(z)|\nu(z)|_{\mathbb{A}}^{s+n+2(-1)/2} \]
for $z \in Z_M(\mathbb{A})$.

In particular, the real part $\text{Re}(\lambda)$ of $\lambda$ is given by
\[ \text{Re}(\lambda)(z) = |\nu(z)|_{\mathbb{A}}^{|(n+2\varepsilon-1)/2}. \]

Proof. This is just an application of the formulas of the Weil representation in Section \ref{sec:weil_representation}.

Lemma 4.2. The modulus character $\delta_P$ of $P(\mathbb{A})$ is given by
\[ \delta_P(p) := |\det ad(p)|^{1/2} = |a(p)|^{(n+2\varepsilon-1)/2}. \]
In particular,
\[ \delta_P(z) = |\nu(z)|_{\mathbb{A}}^{(n+2\varepsilon-1)/2} \]
for $z \in Z_M(\mathbb{A})$.

Lemma 4.3. Let $X_M(\mathbb{R})$ be the group of quasi-characters of $M(\mathbb{A})$ into $\mathbb{R}^+$. Then $X_M(\mathbb{R}) \cong \mathbb{R}^n$ via $\alpha(z) = |\nu(z_1)|_{\mathbb{R}}^{r_1} \cdots |\nu(z_n)|_{\mathbb{R}}^{r_n}$ for $z = (z_1, \ldots, z_n) \in Z_M(\mathbb{A})$, where $(r_1, \ldots, r_n) \in \mathbb{R}^n$.

Let $\Delta$ be the set of simple roots of $G$ with respect to $T$. Then
\[ \Delta = \{x_i - x_{i+1} : 1 \leq i \leq n-1\} \cup \{2x_n\}, \]
where $x_i(t) = t_i, (x_i - x_{i+1})(t) = t_i t_{i+1}^{-1}$, and $(2x_n)(t) = t_n^2$ for $t = (t_1, \ldots, t_n) \in T$

Lemma 4.4. $P$ is defined by $\Theta := \Delta - \{2x_n\}$. In other words, the set $\Delta(P)$ of simple roots of $P$ with respect to the maximal split torus $T_0$ is given by
\[ \Delta(P) := \Delta - \Theta = \{2x_n\}. \]

Recall that for a character $\alpha \in X(T)$ and a cocharacter $\beta \in Y(T)$, there is a pairing $(\alpha, \beta) \in \mathbb{Z}$ defined by
\[ \beta(\alpha(t)) = t^{(\alpha, \beta)}, \quad \forall t \in T. \]

For a root $\alpha$, the corresponding coroot $\alpha^\vee$ is defined by $(\alpha, \alpha^\vee) = 2$.

Recall the Weyl chamber $C_P$ associated to $P$ is given by
\[ C_P = \{\beta \in X_M(\mathbb{R}) : (\beta, \alpha^\vee) \geq 0, \forall \alpha \in \Delta(P)\}, \]
where $\alpha^\vee$ is the coroot corresponding to $\alpha$.

Lemma 4.5. The Weyl chamber $C_P$ associated to $P$ is given by
\[ C_P = \{\beta = (r_1, \ldots, r_n) \in \mathbb{R}^n : r_n \geq 0\}, \]
where we write an element $\beta$ of $X_M(\mathbb{R})$ as the form $\beta = (r_1, \ldots, r_n) \in \mathbb{R}^n$ via $X_M(\mathbb{R}) \cong \mathbb{R}^n$.

Proof. For $\alpha = 2x_n$, $\alpha^\vee$ is given by $\alpha^\vee(t) = (1, \ldots, 1, t)$, since $\alpha^\vee$ satisfies $(\alpha, \alpha^\vee) = 2$, i.e. $\alpha(\alpha^\vee(t)) = t^2$. For $\beta = (r_1, \ldots, r_n) \in X_M(\mathbb{A}) \cong \mathbb{R}^n$, $(\beta, \alpha^\vee) = r_n$, since $\beta(\alpha^\vee(t)) = \beta(1, \ldots, 1, t) = t^{r_n}$. \qed
Theorem 4.6. If \( \Re(s) \geq (n + 2\epsilon - 1)/2 \), then for any \( g \in G(\mathbb{A}) \), the series \( E(g, s, \Phi) \) is absolutely convergent for all \( \Phi \in \mathcal{S}(X(\mathbb{A})) \), and uniformly in \( \Phi \) on every compact subset of \( \mathcal{S}(X(\mathbb{A})) \). In particular, if \( m \geq 2n + 4\epsilon - 2 \), then \( E(g, s, \Phi) \) is holomorphic at \( s_0 = (m - n + 1 - 2\epsilon)/2 \).

Proof. This follows from Godement’s convergence criterion (see Théorème 3 on p. 125 of [Godement 1967] for the number field case, and see Lemma 2.2 on p. 118 of [Morris 1982] for the function field case), which asserts in our case that the series \( \sum_{\gamma \in P(F)\backslash G(F)} F^{\Phi}_{\gamma}(g) \) converges uniformly on subsets of \( G(\mathbb{A}) \) compact modulo \( Z(\mathbb{A}) \) provided \( \Re(\lambda) - 2\delta_P \in C_P \), where \( Z \) is the center of \( G \).

Note that \( \Re(\lambda) - 2\delta_P \in C_P \) if and only if \( \Re(s) + (n + 2\epsilon - 1)/2 - (n + 2\epsilon - 1) \geq 0 \), i.e. \( \Re(s) \geq (n + 2\epsilon - 1)/2 \). \( \square \)

From now on, we write
\[
E(\Phi) = E(1, s_0, \Phi)
\]
for \( \Phi \in \mathcal{S}(X(\mathbb{A})) \). Then
\[
E(\Phi) = \sum_{\gamma \in P(F)\backslash G(F)} \omega(\gamma)\Phi(0),
\]
the series on the right side being absolutely convergent whenever \( m \geq 2n + 4\epsilon - 2 \).

By Bruhat decomposition \( G(k) = \cup_{r=0}^n P(k)w_r P(k) = \cup_{r=0}^n P(k)w_r N(k) \), we have
\[
E(\Phi) = \sum_{r=0}^n \sum_{b \in \text{Her}_r(k)} \omega(w_r n(b))\Phi(0).
\]
The term \( \omega(w_r n(b))\Phi(0) \) is given by
\[
\omega(w_r n(b))\Phi(0) = \int_{X_r(\mathbb{A})} \omega(n(b))\Phi(x) \, dx = \int_{X_r(\mathbb{A})} \Phi(x)\psi(q_b(x)) \, dx,
\]
where \( X_r = \{(x_1, \ldots, x_r, 0, \ldots, 0)\} \subset X \).

In particular, the term \( \omega(w_r n(b))\Phi(0) \) is given by
\[
\omega(w_n n(b))\Phi(0) = \int_{X(\mathbb{A})} \omega(n(b))\Phi(x) \, dx = \int_{X(\mathbb{A})} \Phi(x)\psi(q_b(x)) \, dx.
\]

For \( 0 \leq r \leq n \), let
\[
E_{X_r}(\Phi) = \sum_{b \in \text{Her}_r(k)} \omega(w_r n(b))\Phi(0).
\]
Then
\[
E(\Phi) = \sum_{0 \leq r \leq n} E_{X_r}(\Phi) = E_X(\Phi) + \sum_{0 \leq r \leq n-1} E_{X_r}(\Phi).
\]

We assume \( m \geq 2n + 4\epsilon - 2 \) in the rest of this section. Then by Theorem 4.6 the above series (4.1) and (4.2) are absolutely convergent, uniformly in \( \Phi \) on every compact subset of \( \mathcal{S}(X(\mathbb{A})) \).

For \( b \in \text{Her}_n(k) \), let
\[
F^*_\Phi(b) = \int_{X(\mathbb{A})} \Phi(x)\psi(q_b(x)) \, dx.
\]
Then

\[ E_X(\Phi) = \sum_{b \in \text{Her}_n(k)} F_b^\ast(b), \]

and it follows that this series is absolutely convergent, uniformly in \( \Phi \) on every compact subset of \( S(X(\Lambda)) \). In other words, if we substitute \( X(\Lambda), \text{Her}_n(\Lambda), \text{Her}_n(k) \) and \( i_X \) for \( X, G, \Gamma \) and \( f \), respectively, in Proposition 2 of n. 2 of [Weil 1964], where \( i_X : X \to \text{Her}_n \) is the mapping given by \( i_X(x) = (x, x) := ((x_i, x_j)) \) for \( x = (x_1, \ldots, x_n) \) in \( X \) with each \( x_i \in V \), then the condition (\( B_1 \)) at the end of that section is satisfied. As, by the results of [Weil 1964], the condition (\( B_0 \)) is also satisfied, it follows that the condition (B) of Proposition 2 in n. 2 of [Weil 1964] is satisfied. So we have, by this proposition, that

\[ E_X(\Phi) = \sum_{b \in \text{Her}_n(k)} F_{\Phi}(b), \]

where \( F_{\Phi} \) is the Fourier transform of \( F_b^\ast \); moreover, this Fourier transform is given, for each \( b \in \text{Her}_n(\Lambda) \), by the formula

\[ F_{\Phi}(b) = \int \Phi(x) \, d\mu_b(x), \]

where \( \mu_b \) is a positive tempered measure on \( X(\Lambda) \), of support contained in \( i_X^{-1}(\{b\}) \); and \( F_{\Phi} \) and \( F_b^\ast \) are continuous and integrable functions on \( \text{Her}_n(\Lambda) \). Finally, Proposition 2 in n. 2 of [Weil 1965] shows that the second member of (4.3) is absolutely convergent; as \( \mu_b \) are positive measures, we conclude, by Lemme 5 in n. 41 of [Weil 1964], that the second member converges uniformly on every compact subset of \( S(X(\Lambda)) \). According to Lemme 2 in n. 2 of [Weil 1965], this shows that \( E_X \) is a positive tempered measure, given by

\[ E_X = \sum_{b \in \text{Her}_n(k)} \mu_b, \]

where \( \mu_b(\Phi) = \int \Phi \, d\mu_b = F_{\Phi}(b) \). Finally, we conclude similarly from (4.2) that \( E \) is a positive tempered measure, given by the sum of the measures \( E_X \).

Taking for \( \Phi \) a function of the form

\[ \Phi(x) = \prod_v \Phi_v(x_v) \quad (x = (x_v) \in X(\Lambda)), \]

where the product is over all the places \( v \) of \( k \), where \( \Phi_v \) belongs to \( S(X_v) \) for any \( v \), and where for almost all \( v \), \( \Phi_v \) is the characteristic function of \( X_v^0 := X(O_v) \). The function \( \Phi \) being thus chosen, denote by \( F_v \) and \( F_v^\ast \), for each \( v \), the functions defined on \( \text{Her}_n(k_v) \) by the formulas

\[ F_v(b) = \int_{U_v(b)} \Phi_v(x) \, |\theta_b(x)|_v, \quad F_v^\ast(b) = \int_{X_v} \Phi_v(x) \psi_v(q_b(x)) \, dx|_v; \]

here we write \( U_v(b) \) for the variety formed by points of \( i_X^{-1}(\{i\}) \) of maximal rank in \( X_v \), and \( \theta_b \) the gauge defined on this variety by the formula (29) in n. 37 of [Weil 1965]. According to Proposition 6 in n. 37 of [Weil 1965], \( F_v \) and \( F_v^\ast \) are continuous, integrable, and are Fourier transforms of each other. By the hypotheses made on \( \Phi \), we see immediately that \( F_v^\ast \) takes constant value 1 on \( \text{Her}_n(k_v)^o \) for almost all \( v \); here, \( \text{Her}_n(k_v)^o \) denotes the lattice in \( \text{Her}_n(k_v) \) generated by an arbitrarily chosen basis \( \text{Her}_n(k_v)^o \) of \( \text{Her}_n(k) \) over \( k \).

It is then immediate that, for any \( b = (b_v) \in \text{Her}_n(\Lambda) \), we have

\[ F_{\Phi}(b) = \prod_v F_v^\ast(b_v), \]
where almost all the factors of the second member are of value 1. We deduce that

\[ \int |F_\Phi(b)| \cdot |db|_\Lambda = \prod_v \int |F_\Phi^*(b_v)| \cdot |db|_v. \]

In this relation, the first member is \(< +\infty\); it is \(\neq 0\) unless \(F_\Phi^* = 0\); besides, we can always modify a finite number of the functions \(\Phi_v\) so as to have \(F_\Phi^* \neq 0\), for example by taking \(\Phi_v \geq 0\) and \(\Phi_v \neq 0\) for any \(v\), which implies that \(F_\Phi \neq 0\) and consequently \(F_\Phi^* \neq 0\). As almost all the factors of the second member are \(\geq 1\), it follows that the second member is absolutely convergent (in the sense defined in note (1) of n. 4 of [Weil 1965]). We conclude easily that the Fourier transform \(F_\Phi^*\) of \(F_\Phi^*\) is the product of the Fourier transforms \(F_\Phi\) of \(F_\Phi^*\), that is, for any \(b = (b_v) \in \text{Her}_n(\Lambda)\), we have

\[ F_\Phi(b) = \prod_v F_\Phi(b_v), \]

where the product of the second member is absolutely convergent.

If we denote by \(\mu_v\) the tempered measure on \(\text{Her}_n(k_v)\) determined by the measure \(|\theta_{b_v}|_v\) on \(U_v(b_v)\), then \(F_\Phi(b_v)\) is just \(\mu_v(X_v^0)\) whenever \(\Phi_v\) is the characteristic function of \(X_v^0\). The above formula thus shows that the product of \(\mu_v(X_v^0)\) is absolutely convergent, and that the measure \(\mu_\Phi\) which appears in the above expression of \(F_\Phi\) is just \(\prod \mu_v\).

When \(b\) belongs to \(\text{Her}_n(k)\), the set \(i_X^{-1}(\{b\})\), on the universal domain \(\Omega\), is a \(k\)-closed subset of \(X(\Omega)\). We denote by \(U(b)\) the set of points of maximal rank of this set; it is a \(k\)-open subset of \(i_X^{-1}(\{b\})\); according to Proposition 3 in n. 22 of [Weil 1965], when \(U(b)\) is not empty, it is an orbit of the group \(U(V)\), taking also on the universal domain. We conclude easily from Lemme 8 of n. 17 of [Weil 1965] that, if \(K \supset k\) is a field containing \(k\), then the set \(U(b)_K\) of points of \(U(b)\) which are rational over \(K\) is just the set of points of \(i_X^{-1}(\{b\})\) in \(X(K)\) which are of maximal rank in \(X(K)\). In particular, for \(K = k_v\), we see that \(U(b)_v\) is just the set \(U_v(b)\).

For \(b \in \text{Her}_n(k)\), let \(\theta_b\) denote the gauge on the variety \(U(b)\) defined by the formula

\[ \theta_b(x) = \left( \frac{dx}{di_X(x)} \right)_b, \]

in the sense in [Weil 1965] n. 6).

We have the following analogue of Lemme 19 of [Weil 1965] n. 43], whose proof is similar and relies on the results in [Weil 1982]. We omit the proof.

**Lemma 4.7.** For each \(b \in \text{Her}_n(k)\), 1 is a system of convergence factors for \(U(b)\), and we have \(\mu_\Phi = |\theta_\Phi|_\Lambda\).

In summary, we have shown the following results.

**Theorem 4.8.** Assume that \(m \geq 2n + 4\epsilon - 2\). Put, for \(\Phi \in \mathcal{S}(X(\Lambda))\):

\[ E_X(\Phi) = \sum_{b \in \text{Her}_n(k)} \int_{X(\Lambda)} \Phi(x) \psi(q_b(x)) \, dx. \]

Then the series of the second member is absolutely convergent, and \(E_X\) is a positive tempered measure. Moreover, for each \(b \in \text{Her}_n(k)\), 1 is a system of convergence factors for the variety \(U(b)\) of points of \(i_X^{-1}(\{b\})\) of maximal rank; and, if \(\theta_b\) denotes the gauge on this variety defined by the formula

\[ \theta_b(x) = \left( \frac{dx}{di_X(x)} \right)_b, \]
then the measure $|θ_b|_λ$ on $U(b)_λ$ defines a positive tempered measure $μ_b$ on $X(Λ)$, and we have

$$E_X = \sum_{b ∈ H(\mathfrak{a})} μ_b.$$  

**Theorem 4.9.** Assume that $m ≥ 2n + 4ε - 2$. Put, for $Φ ∈ S(X(Λ))$:

$$E(Φ) = \sum_{γ ∈ P(F) \backslash G(F)} ω(γ)Φ(0).$$

Then the series of the second member is absolutely convergent; $E$ is a positive tempered measure; and we have

$$E = \sum_{r=0}^{n} E_X,$$

where $E_X$ is defined as in Theorem 4.3.

5. **Uniqueness theorems**

In this section, we will prove some results analogous to those in [Weil 1965, Chap. V], which will be used in the proof of the Siegel-Weil formula. We assume that $m > 2n + 4ε - 2$ in this section. We identify $G(k)$ with its image in $Mp(\mathbb{W})_λ$ by means of the Weil representation $ω$, and we say a tempered measure on $X(Λ)$ is invariant under $G(k)$ when it is invariant under $ω(g)$ for any $g ∈ G(k)$; we also say that a measure (tempered or not) on $X(Λ)$ is invariant under an element $h$ of $H(Λ)$ if it is so under the mapping $x ↦ hx$ of $X(Λ)$ onto itself. Recall that, by the corollary to Proposition 9 of n. 51 of [Weil 1964], the automorphisms $Φ ↦ ω(S)Φ$ and $Φ(x) ↦ Φ(hx)$ of $S(X(Λ))$, for $S ∈ Mp(\mathbb{W})_λ$ and $h ∈ H(Λ)$, are permutable; it is also the same for the corresponding automorphisms of the space of tempered distributions on $X(Λ)$.

Let $E$ be a tempered measure on $X(Λ)$, invariant under $G(k)$, and let $Φ ∈ S(X(Λ))$; then $S ↦ E(ω(S)Φ)$ is a continuous function on $Mp(\mathbb{W})_λ$, left invariant under $G(k)$. We will give conditions for this function to be bounded on $Mp(\mathbb{W})_λ$, uniformly in $Φ$ on every compact subset of $S(X(Λ))$; for this, we apply the results of reduction theory to the group $G$.

For $x ∈ X(k) = M_{m×n}(D)$, denote by $x_1, \ldots, x_n$ the columns of the matrix $x$, so that $x_α ∈ M_{m×1}(D)$ for $1 ≤ α ≤ n$, and write $x = (x_1, \ldots, x_n)$. Let $t = (t_1, \ldots, t_n)$ be an element of $(\mathbb{G}_m)^n$; we denote by $λ_t$ the automorphism of $X$ defined by the diagonal matrix whose diagonal elements are $t_1, \ldots, t_n$; it can also be written as

$$x = (x_1, \ldots, x_n) ↦ xλ_t = (x_1t_1, \ldots, x_nt_n).$$

For $t ∈ (\mathbb{G}_m)^n$, denote by $\tilde{λ}_t$ the automorphism of $\text{Her}_n$ determined by the automorphism $λ_t$ of $X$, which is given by

$$b = (b_αβ) ↦ b\tilde{λ}_t = (b_αβt_αt_β).$$

Then the determinants of $λ_t$ and of $\tilde{λ}_t$, with respect to bases of $X(k)$ and of $\text{Her}_n(k)$ over $k$, are respectively

$$\det(λ_t) = (t_1 \ldots t_n)^m, \quad \det(\tilde{λ}_t) = (t_1 \ldots t_n)^{(n+2ε-1)δ},$$

where we recall $δ$ is the dimension of $D$ over $k$. We conclude that the gauge $θ_b(x)$ on $U(b)$, defined in Theorem 4.3, is transformed by $λ_t$ to the gauge

$$θ_b(xλ_t^{-1}) = (t_1 \ldots t_n)^{(-m+n+2ε-1)δ}θ_{b′}(x)$$  

(5.1)
on $U(b')$, with $b' = b\lambda_t$.

In particular, for $t \in (A^\times)^n$, $\lambda_t$ and $\lambda_t$ are automorphisms of $X(A)$ and of $\text{Her}_n(A)$, respectively. If we put $|t|_A = |t_1, \ldots, t_n|_A$, and regard $\lambda_t = \text{diag}(t_1, \ldots, t_n) \in GL_n(A)$, then we have, for $\Phi \in \mathcal{S}(X(A))$, $x = (x_1, \ldots, x_n)$:

$$\omega(m(\lambda_t))\Phi(x) = \chi(\lambda_t)|t|_A^{n\delta/2}\Phi(x_1t_1, \ldots, x_n t_n).$$

(5.2)

We denote by $T$ the image of $(G_m)^n$ in $G$ under $t \mapsto m(\lambda_t) = \begin{pmatrix} \lambda_t & 0 \\ 0 & (\lambda_t^*)^{-1} \end{pmatrix}$, where we regard $\lambda_t = \text{diag}(t_1, \ldots, t_n) \in GL_n(D)$; then $T$ is a maximal split torus of $G$. The strictly positive roots of $G$ with respect to $T$ are $x_\alpha - x_\beta$ and $x_\alpha + x_\beta$ for $1 \leq \alpha < \beta \leq n$, together with $2x_\alpha$ for $1 \leq \alpha \leq n$ in the case where $\epsilon > 0$ (see [Weil 1965 n. 47]). Recall we have defined $\Theta(T)$ and $\Theta^+ = \Theta(0)$ in Section 3. Let $T(\hat{\lambda})^+ = \Theta^+ \cdot T(\hat{\lambda})^1$. By Lemma 5.2 there is a compact subset $C_1$ of $G(A)$ such that $G(A) = C_1 \cdot T(\hat{\lambda})^+ \cdot G(k)$. Let $T(\hat{\lambda})'$ be the subset of $T(\hat{\lambda})$ formed of elements $m(\lambda_t)$ of $T(\hat{\lambda})$ for which

$$|t_1|_A \geq \ldots \geq |t_n|_A \geq 1.$$

For $\epsilon > 0$, we verify easily that $T(\hat{\lambda})^+ = T(\hat{\lambda})'^{-1}$, so that by putting $C = C_1^{-1}$ we have $G(A) = G(k) \cdot T(\hat{\lambda})' \cdot C$. If $\epsilon = 0$, we verify easily that $T(\hat{\lambda})^+$ is the union of $T(\hat{\lambda})'^{-1}$ and $s_1^{-1}T(\hat{\lambda})'^{-1}s_1$, where

$$s_1 = \begin{pmatrix} 1_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1_{n-1} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

so that by putting $C = C_1^{-1} \cup s_1C_1^{-1}$ we have $G(A) = G(k) \cdot T(\hat{\lambda})' \cdot C$.

In what follows we will identify $(A^\times)^n$ with $T(A)$ by means of the isomorphism $t \mapsto m(\lambda_t)$, and also with its image in $\text{Mp}(\mathbb{W})_A$ by means of the isomorphism $t \mapsto (i(m(\lambda_t)), 1)$. With this convention, we can thus write $\text{Mp}(\mathbb{W})_A = G(k) \cdot T(\hat{\lambda})' \cdot \pi^{-1}(C)$, where $\pi$ is the canonical projection of $\text{Mp}(\mathbb{W})_A$ onto $G(A)$; $\pi^{-1}(C)$ is then a compact subset of $\text{Mp}(\mathbb{W})_A$. This implies the following lemma, which is an analogue of Lemma 20 of [Weil 1965, 2].

**Lemma 5.1.** Let $\hat{E}$ be a tempered measure on $X(A)$, invariant under $G(k)$; let $T(\hat{\lambda})''$ be a subset of $T(\hat{\lambda})'$ such that $T(\hat{\lambda})' \subset T(k) \cdot T(\hat{\lambda})'' \cdot C'$, where $C'$ is a compact subset of $T(\hat{\lambda})$. Then, for the function $S \mapsto \hat{E}(\omega(S)\Phi)$ to be bounded on $\text{Mp}(\mathbb{W})_A$, uniformly in $\Phi$ on every compact subset of $\mathcal{S}(X(A))$, it is necessary and sufficient that it is so on $T(\hat{\lambda})''$.

For each $b \in \text{Her}_n(k)$, denote by $b_1, \ldots, b_n$ the columns of the matrix $b$, and write $b = (b_1, \ldots, b_n)$. We thus have $b_\alpha \in M_{n \times 1}(D)$ for $1 \leq \alpha \leq n$. If $b = i_X(x) = (x, x)$ for some $x \in X$, then $b_\alpha = x^* \cdot Q \cdot x$. For $0 \leq \alpha \leq n$, we denote by $\text{Her}_n^{(\alpha)}(k)$ the set of elements $b = (b_1, \ldots, b_n)$ of $\text{Her}_n(k)$ such that $b_1 = \ldots = b_\alpha = 0$ and $b_{\alpha+1} \neq 0$; $\text{Her}_n(k)$ is thus the disjoint union of $\text{Her}_n^{(\alpha)}(k)$ for $0 \leq \alpha \leq n$.

We have the following analogue of Lemma 21 of [Weil 1965 n. 48].

**Lemma 5.2.** Let $\hat{E}$ be a positive tempered measure on $X(A)$, invariant under $T(k)$, whose support is contained in the union of $\text{i}_X^{-1}(\{b\})$ for $b \in \text{Her}_n^{(0)}(k)$. Then the function $S \mapsto \hat{E}(\omega(S)\Phi)$ is bounded on $T(\hat{\lambda})'$, uniformly in $\Phi$ on every compact subset of $\mathcal{S}(X(A))$. 
Proof. As before, we denote by \( \Theta(T) \) the set of elements of \( T(\mathbb{A}) \) of the form \((a_{\tau_1}, \ldots, a_{\tau_n})\), with \( \tau_n \in \mathbb{Z} \) for \( 1 \leq \alpha \leq n \); put \( \Theta' = \Theta(T) \cap T(\mathbb{A})' \); \( \Theta' \) is the set of elements of the above form for which \( \tau_1 \leq \ldots \leq \tau_n \leq 0 \). There is a compact subset \( C' \) of \( T(\mathbb{A})' \) such that \( T(\mathbb{A})' = T(\mathbb{A}) \cdot \Theta' \cdot C' \).

Let \( C_0 \) be a compact subset of \( \mathcal{S}(X(\mathbb{A})) \); let \( C_0' \) be the set of \( \omega(\theta)\Phi \) for \( \theta \in C' \), \( \Phi \in C_0 \). Applying Lemma 3.4 to the spaces \( X = M_{m \times n}(D), X^{(n)} = M_{m \times 1}(D) \) for \( 1 \leq \alpha \leq n, Y = M_{n \times 1}(D) \), and to the morphism \( x \mapsto p(x) = x^* \cdot Q \cdot x_1 \) of \( X \) into \( Y \), we conclude that there exists \( \Phi_0 \in \mathcal{S}(X(\mathbb{A})) \) such that \( |\omega(\theta)\Phi| \leq \Phi_0 \) on the support of \( E \) for all \( \theta \in \Theta' \), \( \Phi \in C_0' \). The conclusion of the lemma follows.

Let \( b \in \text{Her}_n(k) \); let \( j \) be the canonical injection of \( U(b) \) into \( X \); \( j \) then determines an injective mapping \( j_\lambda \) of \( U(b)_\lambda \) into \( X(\mathbb{A}) \), and more precisely into \( i_X^{-1}(\{b\}) \). Following [Weil 1965, n. 49], we say a measure on \( X(\mathbb{A}) \) is carried by (portée par) \( U(b)_\lambda \) if it is the image under \( j_\lambda \) of a measure on \( U(b)_\lambda \). For example, this is so for the measure \( \mu_\lambda \) which by definition is the image of \( |\theta|_\lambda \) under \( j_\lambda \), and which appears in Theorem 3.8. When \( b \) is a non-degenerate element of \( \text{Her}_n(k) \), it results from the remarks in n. 25 of [Weil 1967] that \( j_\lambda \) is an isomorphism of \( U(b)_\lambda \) onto \( i_X^{-1}(\{b\}) \); in this case, every measure of support contained in \( i_X^{-1}(\{b\}) \) is carried by \( U(b)_\lambda \).

On the other hand, in the following, a place \( v \) of \( k \) will be given once and for all, and we write \( X(\mathbb{A}) = X_v \times X' \). For \( x \in X(\mathbb{A}) \), we write \( x = (x_v, x') \), where \( x_v \) and \( x' \) are the projections of \( x \) onto \( X_v \) and onto \( X' \) respectively. We write similarly \( U(b)_\lambda = U(b)_v \times U(b)' \).

We have the following analogue of Lemma 22 in n. 49 of [Weil 1965].

**Lemma 5.3.** Let \( b \in \text{Her}_n(k) \). Let \( \mu \) be a positive tempered measure carried by \( U(b)_\lambda \) and invariant under \( H_v \). Then, to every function \( \Phi' \in \mathcal{S}(X') \), there corresponds a constant \( c(\Phi') \) such that for any \( \Phi_v \in \mathcal{S}(X_v) \) we have:

\[
\int \Phi_v(x_v)\Phi'(x')d\mu((x_v, x')) = c(\Phi') \int_{U(b)_v} \Phi_v \cdot |\theta|_v.
\]

**Proof.** By hypothesis, \( \mu \) is the image of a measure \( \nu \) on \( U(b)_\lambda \), i.e. the first member of 3.8 is the integral of \( \Phi_v(x_v)\Phi'(x') \) on \( U(b)_v \times U(b)' \) with respect to \( \nu \). Assume first \( \Phi' \geq 0 \). By hypothesis, the integral in question is finite whenever \( \Phi_v \geq 0 \) and belongs to \( \mathcal{S}(X_v) \), thus also whenever \( \Phi_v \) is continuous and of compact support on \( X_v \), and especially whenever \( \Phi_v \) is continuous and of compact support on \( U(b)_v \). It can thus be written as \( \int \Phi_v d\nu_v \), where \( \nu_v \) is a positive measure on \( U(b)_v \). According to the hypothesis made on \( \mu \), \( \nu_v \) is invariant under \( H_v \). We can then identify \( U(b)_v \) with the homogeneous space determined by \( H_v \) and the stabilizer of one of its points in \( H_v \). But the definition of the gauge \( \theta_b \) in Theorem 4.8 shows that it is invariant under \( H \), up to a factor \( \pm 1 \); consequently, the measure \( |\theta|_v \) is invariant under \( H_v \). The theorems on the uniqueness of the invariant measure on homogeneous spaces (cf. [Bourbaki 1963], Chap. VII, §2, n. 6) then show that \( \nu_v \) only differs from \( |\theta|_v \) by a scalar factor \( c(\Phi') \). The general case can be reduced to the special case \( \Phi' \geq 0 \) and thus follows immediately.

Now we can prove the following analogue of Lemma 23 in n. 49 of [Weil 1967].

**Lemma 5.4.** Let \( \hat{\Phi} \) be a positive tempered measure, invariant under \( T(k) \) and is the sum of measures \( \hat{\mu}_b \) respectively carried by \( U(b)_\lambda \) for \( b \in \text{Her}_n(k) \). Assume that there exists a place \( v \) of
$k$ such that $\hat{E}$ is invariant under $H_v$. Then the function $S \mapsto \hat{E}(\omega(S)\Phi)$ is bounded on $T(\mathbb{A})'$, uniformly in $\Phi$ on every compact subset of $\mathcal{S}(X(\mathbb{A}))$.

Proof. Let $\hat{E}_\alpha$, for $0 \leq \alpha \leq n$ be the sum of $\hat{\mu}_b$ for $b \in \text{Her}_n^{(\alpha)}(k)$; we will have $\hat{E} = \hat{E}_0 + \ldots + \hat{E}_n$. If $t \in T(k)$, then $\lambda_t$ determines a permutation on each of the sets $\text{Her}_n^{(\alpha)}(k)$, so that each of the measures $\hat{E}_\alpha$ is invariant under $T(k)$. On the other hand, $H(\mathbb{A})$ leaves invariant each of the sets $i^{-1}_X\{b\}$; with the hypotheses of the statement, it follows that $H_v$ leaves invariant each of the measures $\hat{\mu}_b$, thus also each of the $\hat{E}_\alpha$; this satisfies thus the same hypotheses as $\hat{E}$, and we are reduced to dealing with $\hat{E}_\alpha$.

Thus let $\alpha$ be such that $0 \leq \alpha \leq n$. There is a constant $q_1$, equal to 1 if $v = v_0$ and to $q_v$ if $v \neq v_0$, such that there exists, for every $\tau \in \mathbb{Z}$, an element $y$ of $k$, satisfying $q^{-\tau} \leq |y|_v \leq q_1 q^{-\tau}$; and there exists a compact subset $C$ of $\mathbb{A}^\times$ such that every element $t$ of $\mathbb{A}^\times$ satisfying $1 \geq |t|_\mathbb{A} \geq q_1^{-1}$ can be written as the form $\rho c$ with $\rho \in k$, $c \in C$; we denote by $C^n$ the compact subset of $T(\mathbb{A})$ formed of the elements $(c_1, \ldots, c_n)$ with $c_\beta \in C$ for $1 \leq \beta \leq n$. Let $C_0$ be a compact subset of $\mathcal{S}(X(\mathbb{A}))$, and we apply Lemma 3.6 to the space $X = M_{m \times n}(D)$ considered as a product of the spaces

$$X^{(1)} = M_{m \times (\alpha+1)}(D), \quad X^{(2)} = \ldots = X^{(n-\alpha)} = M_{m \times 1}(D)$$

in such a way that the projections of $x = (x_1, \ldots, x_n)$ on these spaces are respectively $(x_1, \ldots, x_{\alpha+1})$, $(x_{\alpha+2}, \ldots, x_n)$; we take $Y = M_{n \times (\alpha+1)}(D)$, and $p$ the morphism of $X$ into $Y$ given by

$$p(x) = x^* \cdot q \cdot (x_1, \ldots, x_{\alpha+1}).$$

It is concluded that there exists $\Phi_0 \in \mathcal{S}(X(\mathbb{A}))$ such that we have

$$|\omega(m(\lambda_\beta))\omega(m(\lambda_\alpha))\Phi(x)| \leq \Phi_0(x)$$

for all $x \in X(\mathbb{A})$, $i_x(x) \in \text{Her}_n^{(\alpha)}(k)$, $c \in C^n$, $\Phi \in C_0$, and $\theta$ belonging to the set $\Theta'_\alpha$ of elements $(a_{\tau_1}, \ldots, a_{\tau_n})$ of $\Theta(T)$ which satisfy the condition

$$\tau_1 = \ldots = \tau_{\alpha+1} \leq \ldots \leq \tau_n \leq 0.$$

Furthermore, we can assume that $\Phi_0$ has been taken in the form $\Phi(x_v)\Psi(x')$, with

$$\Phi_v \in \mathcal{S}(X_v), \quad \Psi' \in \mathcal{S}(X').$$

Now let $t = (t_1, \ldots, t_n)$ be an element of $T(\mathbb{A})'$. For $1 \leq \beta \leq \alpha$, let $y_\beta \in k_v$ be such that $|y_\beta|_v$ is between $|t_\beta t^{-1}_{\alpha+1}|_\mathbb{A}$ and $q_1 |t_\beta t_{\alpha+1}|_\mathbb{A}$; let $y_\beta = 1$ for $\beta \geq \alpha + 1$; we will have $|y_\beta|_v \geq 1$ for $1 \leq \beta \leq n$. On the other hand, for $\beta \geq \alpha + 1$, let $\tau_\beta \in \mathbb{Z}$ be such that $|a_{\tau_\beta}|_\mathbb{A} = |t_\beta|_\mathbb{A}$, and let $\tau_\beta = \tau_{\alpha+1}$ for $1 \leq \beta \leq \alpha$. For every $\beta$, we will have

$$1 \geq |t_\beta y_\beta a_{\tau_\beta}^{-1}|_\mathbb{A} \geq q_1^{-1},$$

so that we can write $t_\beta = \rho_\beta y_\beta a_{\tau_\beta} c_\beta$, with $\rho_\beta \in k$, $c_\beta \in C$, for $1 \leq \beta \leq n$. By putting

$$y = (y_1, \ldots, y_n), \quad \theta = (a_{\tau_1}, \ldots, a_{\tau_n}),$$

we will thus have $t = \rho y \theta c$ with $\rho \in T(k)$, $y \in T_v$, $\theta \in \Theta'_\alpha$ and $c \in C^n$. As $\hat{E}_\alpha$ is invariant under $T(k)$, it follows that we have

$$|\hat{E}_\alpha(\omega(m(\lambda_\beta))\Phi)| \leq \hat{E}_\alpha(\omega(m(\lambda_\beta))\Phi_0)$$

for all $\Phi \in C_0$, $\Phi_0$ being chosen as above.
To evaluate the second member of this integral, we apply Lemma 5.3 to each of the measures \( \hat{\mu}_b \) for \( b \in \text{Her}_n^{(\alpha)}(k) \); denoting by \( c_b(\Phi') \) the constant which appears in that lemma when we substitute \( \hat{\mu}_b \) with \( \mu \), we obtain

\[
\hat{E}_\alpha(\omega(m(\lambda_y)))\Phi_0) = \sum_{b \in \text{Her}_n^{(\alpha)}(k)} c_b(\Phi') \int_{U(b)_v} \omega(m(\lambda_y))\Phi_v \cdot |\theta_b|_v.
\]

As we have \( y_\beta = 1 \) for \( \beta \geq \alpha + 1 \), it results from the definition of \( \text{Her}_n^{(\alpha)}(k) \) that the automorphism \( \tilde{\lambda}_y \) of \( \text{Her}_n(k_v) \) determined by \( \lambda_y \) leaves invariant all the elements of \( \text{Her}_n^{(\alpha)}(k) \). Note that we have

\[
\hat{E}_\alpha(\omega(m(\lambda_y)))\Phi_0) = |y_1 \cdots y_\alpha|_v^{(-m+2n+4\epsilon-2)\delta/2} \hat{E}_\alpha(\Phi_0).
\]

As we have assumed that \( m > 2n + 4\epsilon - 2 \), the exponent of the second member is \( < 0 \). As we have \( |y_\beta|_v \geq 1 \) for all \( \beta \), we obtain

\[
|\hat{E}_\alpha(\omega(m(\lambda_y)))\Phi_0)| \leq \hat{E}_\alpha(\Phi_0),
\]

this inequality being valid for all \( t \in T(\mathbb{A})' \) and \( \Phi \in C_0 \). This completes the proof.

Now we can prove the main result of this section, which is an analogue of Théorème 4 of [Weil 1965 n. 4]. We denote by \( E \) the positive tempered measure on \( X(\mathbb{A}) \) given by the Siegel Eisenstein series \( E(\Phi) \) whenever it converges absolutely.

**Theorem 5.5.** Assume that \( m > 2n + 4\epsilon - 2 \). Let \( v \) be a place of \( k \) such that \( U(0)_v \) is not empty. Let \( E' \) be a positive tempered measure on \( X(\mathbb{A}) \), invariant under \( G(k) \) and under \( H_v \), and such that \( E' - E \) is a sum of measures carried by \( U(b)_\mathbb{A} \) for \( b \in \text{Her}_n(k) \). Then we have \( E' = E \).

**Proof.** With the notations of Theorems 4.3 and 1.1, we have \( E = \sum_{0 \leq r \leq n} E_X_r ; E_X \) is the sum of the measures \( |\theta_b|_\mathbb{A} \) respectively carried by \( U(b)_\mathbb{A} \), while \( E_X \), has its support contained in \( X_r(\mathbb{A}) \) for any \( r < n \). On the universal domain, let \( U \) be the set of points of \( X \) of maximal rank; it is \( k \)-open; it is an orbit for the group \( \text{Aut}(V) \); for any \( b \in \text{Her}_n(k) \), \( U(b) \) is a subvariety of \( U \) and is thus \( k \)-closed in \( U \). Let \( F = X - U \); it is a \( k \)-closed subset of \( X \), invariant under the group \( \text{Aut}(V) \) and especially under \( H \subset U(V) \), which contains \( X_r \) whenever \( r < n \). Consequently, \( F(\mathbb{A}) \) is a closed subset of \( X(\mathbb{A}) \), invariant under \( H(\mathbb{A}) \) and obviously also under \( \text{Aut}(X_k) \), which contains \( X_r(\mathbb{A}) \) for \( r < n \) and has no common point with \( U(b)_\mathbb{A} \) for any \( b \in \text{Her}_n(k) \). It follows that \( E_X \) is the restriction of \( E \) to the open set \( X(\mathbb{A}) - F(\mathbb{A}) \), and that the sum \( \sum E_X_r \) over \( 0 \leq r < n \) is the restriction of \( E \) to \( F(\mathbb{A}) \). The hypothesis made on \( E' \) then implies that the restriction \( \hat{E} \) of \( E' \) to \( X(\mathbb{A}) - F(\mathbb{A}) \) is the sum of the measures \( \hat{\mu}_b \) respectively carried by the \( U(b)_\mathbb{A} \) for \( b \in \text{Her}_n(k) \), and that the restriction of \( E' \) to \( F(\mathbb{A}) \) is the same as that of \( E \), so that we have \( E' - \hat{E} = E - E_X \); furthermore, as \( E' \) and \( F(\mathbb{A}) \) are invariant under \( H_v \) and under \( \text{Aut}(X_k) \), it is the same for \( \hat{E} \), which thus satisfies the hypotheses of Lemma 5.4.

According to that lemma, the function \( S \mapsto \hat{E}(\omega(S)\Phi) \) is bounded on \( T(\mathbb{A})' \), uniformly in \( \Phi \) on every compact subset of \( S(X(\mathbb{A})) \). This conclusion can be applied in particular to \( E_X \), which is deduced from \( E \) as \( \hat{E} \) is from \( E' \); it can be applied also to the tempered measure \( E'' \) given by

\[
E'' = E' - E = \hat{E} - E_X.
\]
But this measure is invariant under $G(k)$, since $E$ and $E'$ are so; we can thus apply Lemma 3.1, which shows that the function $S \mapsto E''(\omega(S)\Phi)$ is bounded on $Mp(W)_\lambda$, for any $\Phi \in S(X(\mathbb{A}))$. For every $\Phi \in S(X(\mathbb{A}))$, we denote by $M(\Phi)$ the supremum of $|E''(\omega(S)\Phi)|$ for $S \in Mp(W)_\lambda$; we have $M(\omega(S)\Phi) = M(\Phi)$ for all $S \in Mp(W)_\lambda$.

The measure $E''$ is the sum of the measures $\mu''_\mu = \hat{\mu}_b - \mu_b$, where $\mu_b$ denotes once again the measure $|\theta_b|_\lambda$ carried by $U(b)_\lambda$. We thus have, for $\Phi \in S(X(\mathbb{A}))$:

$$E''(\Phi) = \sum_{b \in \text{Her}_n(k)} \int \Phi \, d\mu''_b;$$

in this formula, the series of the second member is absolutely convergent, uniformly in $\Phi$ on every compact subset of $S(X(\mathbb{A}))$, since it is obviously also the series similarly formed by means of the positive measures $\hat{\mu}_b$ and $\mu_b$. Let $b^* \in \text{Her}_n(\mathbb{A})$, we then have, for $\Phi \in S(X(\mathbb{A}))$:

$$\omega(n(b^*))\Phi(x) = \Phi(x)\psi(q_b(x));$$

and consequently

$$E''(\omega(n(b^*))\Phi) = \sum_{b \in \text{Her}_n(k)} \psi(\frac{K}{2} \tau(bb^*)) \int \Phi \, d\mu''_b.$$ 

We can consider this formula as giving the expansion of the first member into Fourier series on the compact group $\text{Her}_n(\mathbb{A})/\text{Her}_n(k)$. As the first member, in absolute value, is $\leq M(\Phi)$, we have, by virtue of the Fourier formulas, that

$$|\int \Phi \, d\mu''_b| \leq M(\Phi),$$

and consequently, replacing $\Phi$ by $\omega(S)\Phi$,

$$(5.4) \quad |\int \omega(S)\Phi \cdot d\mu''_b| \leq M(\Phi),$$

this integral being valid for all $S \in Mp(W)_\lambda$, $b \in \text{Her}_n(k)$ and $\Phi \in S(X(\mathbb{A}))$.

Taking for $\Phi$ the form $\Phi_v(x) = \Phi'(x')$, with $\Phi_v \in S(X_v)$, $\Phi' \in S(X')$. By the hypotheses made on $E'$, the measures $\hat{\mu}_b$ are invariant under $H_v$, and the same is true for $\mu_b$; we can thus apply Lemma 3.3 to them. Consequently, we can write

$$\int \Phi \, d\mu''_b = c_b(\Phi') \int_{U(b)_v} \Phi_v \cdot |\theta_b|_v.$$ 

We now replace $\Phi$ by $\omega(m(\lambda_t))\Phi$ with $t \in T_v$ in this formula; this is equivalent to not changing $\Phi'$ but replacing $\Phi_v$ by $\omega(m(\lambda_t))\Phi_v$, the later function being given by the formula analogous to (5.2). If we put $b' = b\lambda_t$, then this gives, by (5.1):

$$(5.5) \quad \int \omega(m(\lambda_t))\Phi \cdot d\mu''_b = c_b(\Phi') |t_1 \ldots t_n|_v^{(m-2n+4\epsilon-2)/2} \int_{U(b')_v} \Phi_v \cdot |\theta_{b'}|_v.$$ 

Denote by $F(b')$ the integral which appears in the second member; then Proposition 6 of n.37 of Weil 1963 shows that it is a continuous function of $b' \in \text{Her}_n(k_v)$, so that $F(b')$ tends to $F(0)$ when all the $|t_n|_v$ tend to 0. As the exponent of $|t_1 \ldots t_n|_v$ in the second member of (5.5) is $< 0$ by the assumption $m > 2n + 4\epsilon - 2$, and the second member must remain bounded for all $t \in T_v$, we conclude that $c_b(\Phi')F(0) = 0$. But $F(0)$ is given by

$$F(0) = \int_{U(0)_v} \Phi_v \cdot |\theta_0|_v,$$
and, by hypothesis, \( U(0)_v \) is not empty; we can thus choose \( \Phi_v \) in such a way that \( F(0) \) is not zero. We thus have \( c_0(\Phi') = 0 \), and consequently \( \int \Phi d\mu'' = 0 \) whenever \( \Phi \) is of the form \( \Phi_v(x_v)\Phi'(x') \). This implies obviously \( \mu''_b = 0 \). As this is so for any \( b \in \text{Her}_n(k_v) \), we thus have \( E'' = 0 \), i.e. \( E' = E \).

\[ \square \]

Observe that Theorem 5.3 provides a characterization of the measure \( E_X \), by induction on the rank \( n \) of \( X \) over \( M_m(D) \), from \( E_0 = \delta_0 \).

6. The Siegel-Weil Formula

In this section, we will prove the Siegel-Weil formula, which is an equality relating the Siegel Eisenstein series \( E(\Phi) \) with the theta integral \( I(\Phi) \).

For \( \Phi \in S(X(\mathbb{A})) \), define the theta integral by

\[ I(\Phi) = \int_{H(\mathbb{A})/H(k)} \sum_{\xi \in X(k)} \Phi(h\xi) \cdot dh, \]

where \( dh \) is the Haar measure on \( H(\mathbb{A})/H(k) \) such that \( \text{vol}(H(\mathbb{A})/H(k)) = 1 \). Recall \( H \subset U(V) \) is the connected component of the identity.

We have the following convergence criterion for the theta integral \( I(\Phi) \), which is an analogue of Prop. 8 of n. 51 of [Weil 1965]. Recall \( r \) is the Witt index of the hermitian space \( V \).

**Proposition 6.1.** The theta integral \( I(\Phi) \) is absolutely convergent for any \( \Phi \in S(X(\mathbb{A})) \) whenever \( r = 0 \) or \( m - r > n + 2\epsilon - 1 \).

**Proof.** If \( r = 0 \), then \( H(\mathbb{A})/H(k) \) is compact and the theta integral is absolutely convergent.

Now assume \( r > 0 \). Then we can choose a basis of \( V \) for which the \( \eta \)-hermitian form is given by a matrix of the form

\[ Q = \begin{pmatrix} 0 & 0 & 1_r \\ 0 & Q_0 & 0 \\ \eta \cdot 1_r & 0 & 0 \end{pmatrix}, \]

where \( Q_0 \) is the matrix (of order \( m - 2r \)) of an anisotropic \( \eta \)-hermitian form. Let \( T \cong (\mathbb{G}_m)^r \) be the maximal split torus in \( H \) consisting of diagonal matrices of order \( m \) whose diagonal elements are

\[ (t_1, \ldots, t_r, 1, \ldots, 1, t_1^{-1}, \ldots, t_r^{-1}), \]

with each \( t_i \in \mathbb{G}_m \). Let \( P_0 \) be a minimal parabolic subgroup of \( H \) which contains \( T \).

Let \( \rho : H \to \text{Aut}(X) \) be the representation given by \( \rho(h)x = hx \). For each character \( \lambda \) of \( T \), let \( m_\lambda \) be the multiplicity of \( \lambda \). The weights of \( \rho \) are \( x_i \) and \( -x_i \) for \( 1 \leq i \leq r \), each with multiplicity \( \delta n \).

By Lemma 3.3, it suffices to show that

\[ \int_{\Theta} \prod_{\lambda} \sup(1, |\lambda(\theta)|^{-1}_k)^{m_\lambda} \cdot |\Delta_{P_0}(\theta)|^{-1}_k \, d\theta \]

is convergent whenever \( m - r > n + 2\epsilon - 1 \).
Note that
\[ \Theta^+ := \Theta(0) = \{(a_{\tau_1}, \ldots, a_{\tau_r}) : 0 \leq \tau_r \leq \ldots \leq \tau_1\}. \]
For \( \theta = (a_{\tau_1}, \ldots, a_{\tau_r}) \in \Theta^+ \), we have
\[ \Delta_{\theta}(\theta)^{-1} = \prod_{1 \leq i \leq r} a_{\tau_i}^{\delta(m-2i+2-2\epsilon)} \]
and hence
\[ |\Delta_{\theta}(\theta)|^{-1} = \prod_{1 \leq i \leq r} q^{-\delta\tau_i(m-2i+2-2\epsilon)}. \]

Thus we have
\[
\int_{\Theta^+} \prod_{\lambda} \sup(1, |\lambda(\theta)|^{-1})^{m_\lambda} \cdot |\Delta_{\theta}(\theta)|^{-1} d\theta \\
= \sum_{0 \leq \tau_r \leq \ldots \leq \tau_1} \prod_{1 \leq i \leq r} q^{-\delta\tau_i(m-n-2i+2-2\epsilon)} \\
= c_1 \ldots c_{r-1} \sum_{\tau_r \geq 0} q^{-r\delta\tau_i(m-n-r+1-2\epsilon)},
\]
where \( c_j = (1 - q^{-\delta\tau_i(m-n-j+1-2\epsilon)})^{-1} \).

Note that the above multiple series converges if and only if \( m - n - r + 1 - 2\epsilon > 0 \), i.e. \( m - r > n + 2\epsilon - 1 \). The desired conclusion follows. \( \square \)

Now we can show the Siegel-Weil formula. We follow the proof of Théorème 5 in n. 52 of [Weil 1965].

**Theorem 6.2.** Assume that \( m > 2n + 4\epsilon - 2 \). Then we have
\[ I(\Phi) = E(\Phi), \]
and for every \( b \in \text{Her}_n(k) \) we have
\[
(6.1) \quad \int_{H(\mathbb{A})/H(k)} \sum_{\xi \in U(b)_{\mathbb{A}}} \Phi(h\xi) \cdot dh = \int \Phi \, d\mu_b,
\]
where \( \mu_b \) is the measure \( |\theta_b|_{\mathbb{A}} \) determined on \( U(b)_{\mathbb{A}} \) by the gauge \( \theta_b \) defined in Theorem 4.8.

**Proof.** We proceed by induction on \( n \).

For \( n = 0 \), the assertion of the theorem is reduced to \( I = \delta_0 \), which is an obvious consequence of the hypothesis \( \text{vol}(H(\mathbb{A})/H(k)) = 1 \), here \( \delta_0 \) is the measure given by \( \delta_0(\Phi) = \Phi(0) \). We proceed by induction on \( n \), and suppose \( n \geq 1 \). Since by hypothesis \( I(\Phi) \) is convergent for any \( \Phi \in \mathcal{S}(X(\mathbb{A})) \), Lemma 2 in n. 2 of [Weil 1965] joint with Lemma 5 in n. 41 of [Weil 1964] show immediately that \( I \) is a positive tempered measure. Now Théorème 6 in n. 41 of [Weil 1964] and Prop. 9 in n.51 of [Weil 1964] show that \( I \) is invariant under \( G(F) \); it is also obviously invariant under \( H_n \). Similarly, if we denote by \( I_b(\Phi) \) the first member of \( (6.1) \), then \( I_b \) is a positive tempered measure. Let \( I_X \) be the sum of \( I_b \) for \( b \in \text{Her}_n(k) \); we can consider \( I_X \) as defined by the integral similar to that which defines \( I \), but where the summation is restricted to the elements \( \xi \) of \( X(k) \) which are of maximal rank in \( X(k) \). Similarly, for the submodule \( X_r \) of \( X \), where \( 0 \leq r \leq n - 1 \), denote by \( I_{X_r} \) the positive tempered measure defined by the integral similar to that which defines \( I_X \), but where the summation is restricted to the elements \( \xi \) of \( X_r(k) \) which are of maximal rank in \( X_r(k) \). Taking into account of Theorem 4.8, we see that...
Theorem 6.2 for $X$ implies that $I_X = E_X$; as a result, the induction hypothesis implies that $I_{X_r} = E_{X_r}$ for the submodule $X_r$ of $X$ whenever $r < n$. Thus we have, by this hypothesis:

\[
I = \sum_{b \in \text{Her}_n(k)} I_b + \sum_{0 \leq r \leq n-1} E_{X_r}.
\]

According to Theorem 5.5, the second sum of the second member is just $E - E_X$. On the other hand, according to Prop. 3 in n. 22 of [Weil 1965], those of $U(b)_k$ which are nonempty are orbits of $U(V)(k)$ in $X(k)$; then formula (11) in n. 7 of [Weil 1965] shows that the measures $I_b$ are respectively carried by $U(b)_k$. Consequently, $I$ satisfies all the hypotheses of Theorem 5.5, thus $I = E$, and $I_X = E_X$ by (6.2). As $I_b$ and $\mu_b$ are the restrictions of $I_X$ and of $E_X$ to the set $i_X^{-1}(\{b\})$ respectively, it follows that $I_b = \mu_b$ for any $b \in \text{Her}_n(k)$.

This completes the proof of Theorem 6.2. □

For $\Phi \in S(X(\mathbb{A}))$ and $g \in G(\mathbb{A})$, let

\[
I(g, \Phi) = I(\omega(g)\Phi).
\]

Then the theta integral $I(g, \Phi)$ is absolutely convergent whenever $r = 0$ or $m - r > n + 2\epsilon - 1$.

**Corollary 6.3.** Assume $m > 2n + 4\epsilon - 2$. Then for all $\Phi \in S(X(\mathbb{A}))$ and $g \in G(\mathbb{A})$,

(i) $I(g, \Phi)$ is absolutely convergent, $E(g, s, \Phi)$ is holomorphic at $s = s_0$, where $s_0 = (m - n + 1 - 2\epsilon)/2$;

(ii) moreover, we have

\[
I(g, \Phi) = E(g, s_0, \Phi).
\]

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