Dynamics of the localized nonlinear waves in spin-1 Bose-Einstein condensates with time-space modulation

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Abstract We investigate the dynamics of the localized nonlinear matter wave in spin-1 Bose-Einstein condensates with trapping potentials and nonlinearities dependent on time and space. We solve the three coupled Gross-Pitaevskii equation by similarity transformation and obtain two families of exact matter wave solutions in terms of Jacobi elliptic functions and Mathieu equation. The localized states of the spinor matter wave describe the dynamics of vector breathing solitons, moving breathing solitons, quasibreathing solitons and resonant solitons. The results of stability show that one order vector breathing solitons, quasibreathing solitons, resonant solitons, and the moving breathing solitons $\psi_{\pm1}$ are all stable but the moving breathing solitons $\psi_{0}$ is unstable. We also present the experimental parameters to realize these phenomena in the future experiments.

PACS numbers: 03.75.Lm, 05.45.Yv, 42.65Tg

I. INTRODUCTION

For a decade, the experimental realization of Bose-Einstein condensations (BECs) at ultralow temperatures has attracted great interest in the atomic physics communication \cite{[1], [2]}. In

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recent years, one of the most important developments in BECs was the study of the spinor condensations. The idea of the spinor condensations was proposed by Ho and Ohmi [3, 4]. Stamper-Kurn et al. created the spinor condensations in experiment [5], which provides a new perspective to observe phenomena that are not present in single-component BECs. These included the formation of spin domains and spin textures [6, 7]. Later, the spinor BECs with $F > 1$ have also been studied theoretically [8, 9]. In contrast to single and two-component BECs, the spin-$F$ condensates described by macroscopic wave functions with $2F+1$ components have some new characteristics, including the vector character of the order parameter and the changed role of the spin relaxation collisions. Here we focus on BECs of alkali atoms in the $F=1$ hyperfine state, such as $^{23}$Na, $^{87}$Rb and $^7$Li [6, 10], restricted to one-dimensional space by purely optical means. In the absence of an external magnetic fields, the three internal states $m_F = 1, 0, -1$, with $m_F$ the magnetic quantum number, are generated, in which an $m_F = 1$ and an $m_F = -1$ atom can collide and produce two $m_F = 0$ atoms and vice versa. Under the mean-field approximation, the dynamics of the spinor condensates can be described by three-component Gross-Pitaevskii equation (GPE). Stimulated by the experiments done by JILA [11] and MIT [12] groups and the theoretical works of Ho, Ohmi and Machida [13, 14], many studies have been done [15-23]. While, there is few work on the spinor BECs with time-space modulation to present.

Matter waves as the natural outcomes of the mean-field descriptions have been observed experimentally and investigated theoretically [24, 25, 26]. For example, matter wave solitons in atom optics could be used for applications in atom laser, atom interferometry and coherent atom transport. Moreover, it is also helpful to the realization of quantum information processing and computation [27]. So it is interesting to develop a new technique for constructing particular solitons. One possible technique is to alter the interatomic interactions by means of external magnetic fields. Recent experiments show that the effective scattering length can be tuned by Feschbach resonance [28, 29, 30]. This brought about a good proposal for manipulation of the
nonlinear excitations and matter wave by controlling the time-dependent or space-dependent scattering strength [31-34]. In Refs. [35, 36], the matter wave solitons in BECs with time-dependent scattering length was investigated. Nonlinear matter wave in atomic-molecular BECs with space-modulated nonlinearity and in two-component BECs with time-space modulation nonlinearities have been studied [37, 38].

In this paper, we consider the spin-1 BECs with space and time dependent nonlinearities and trapping potentials, which can be described by a system of three-component GPEs. Different from the one-, two-component BECs, we can use an optically induced Feshbach resonance [39] or a confinement induced resonance [40] to tune the nonlinearities in the spinor BECs. Two kinds of localized nonlinear matter wave are given based on the Mathieu equation and Jacobi elliptic function, which take the form of vector solitons. We investigate in detail the vector breathing solitons, moving breathing solitons, quasibreathing solitons and resonant solitons. Dynamical stability of the obtained vector solitons are studied by means of the numerical simulations and the global stability of the different types of vector solitons are analysed. The results show that all the one order vector breathing solitons, quasibreathing solitons and resonant solitons are stable, as for the moving breathing solitons, the matter wave $\psi_1$ and $\psi_{-1}$ are stable but $\psi_0$ is unstable.

The paper is organized as follows. In Sec.II, the localized nonlinear wave solutions are presented based on the Mathieu equation and Jacobi elliptic function. Four kinds of vector solitons, including vector breathing solitons, moving breathing solitons, quasibreathing solitons and resonant solitons, are illustrated in Sec.III. Sec.IV analyse the dynamic stability of the vector soliton by the numerical simulations. An conclusion is given in the last section.

II. LOCALIZED NONLINEAR MATTER WAVE SOLUTIONS

We consider the spin-1 BECs confined in the trapping potential $V_{ext} = \frac{m}{2} (\omega_x^2 x^2 + \omega_\perp (y^2 + z^2))$ with $m$ the mass of $^{23}Na$ atoms, $\omega_x$ and $\omega_\perp$ are the confining frequencies in the transverse and axial directions. The spin-1 BECs can be described by vectorial wave function $\Psi(x,t) = $
\((\psi_1(x, t), \psi_0(x, t), \psi_{-1}(x, t))^T\) with the components corresponding to the three values of the vertical spin projection \(m_F = +1, 0, -1\). When the temperature is lower than the critical temperature, the wave functions are governed by a set of three coupled dimensionless Gross-Pitaevskii equation \([3, 18, 41]\)

\[
\begin{align*}
\frac{i \partial \psi_1}{\partial t} &= (-\nabla^2 + g_n(|\psi_1|^2 + |\psi_0|^2 + |\psi_{-1}|^2) + g_s(|\psi_1|^2 + |\psi_0|^2 - |\psi_{-1}|^2) + V(x) + E_1)\psi_1 + g_s \psi^*_1 \psi_0^2, \\
\frac{i \partial \psi_0}{\partial t} &= (-\nabla^2 + g_n(|\psi_1|^2 + |\psi_0|^2 + |\psi_{-1}|^2) + g_s(|\psi_1|^2 + |\psi_{-1}|^2) + V(x) + E_0)\psi_0 + 2g_s \psi_{-1} \psi_0^2, \\
\frac{i \partial \psi_{-1}}{\partial t} &= (-\nabla^2 + g_n(|\psi_1|^2 + |\psi_0|^2 + |\psi_{-1}|^2) + g_s(|\psi_{-1}|^2 + |\psi_0|^2 - |\psi_1|^2) + V(x) + E^{-1})\psi_{-1} + g_s \psi_1^* \psi_0^2.
\end{align*}
\]

(1)

Here, \(\nabla^2 = \frac{\partial^2}{\partial x^2}\), \(V(x) = \frac{\omega x^2}{2}\) is the external trapping potential, \(E_j \in R\) is the dimensionless Zeeman energy of spin component \(m_F = -1, 0, 1\) and \(|\psi_1|^2 + |\psi_0|^2 + |\psi_{-1}|^2\) is the total condensate density. The strength of the interaction is given by \(g_n = \frac{4\pi \hbar^2 (a_0 + 2a_2)}{3m},\ g_s = \frac{4\pi \hbar^2 (a_2 - a_0)}{3m}\), where \(\hbar\) is the reduced Planck constant and \(a_0,\ a_2\) are the \(s\)-wave scattering lengths scattering channel of total hyperfine spin-0 and spin-2, respectively \([42]\). The length and time are measured in the unit of \(\sqrt{\frac{\hbar}{m a_\perp}}\) and \(w_\perp^{-1}\), respectively. Here we provide the experimental parameters for producing the spinor condensate composed of \(^{23}Na\) \([43, 44]\) with the total number \(N = 3 \times 10^6\). The external trapping potentials is given by \(V = \frac{\omega x^2}{2}\) with \(\omega = 2\pi \times 230Hz\). The scattering lengths \(a_0 = 50a_B\) and \(a_2 = 55a_B\) with Bohr radius \(a_B = 0.529\text{Å}\). In Refs. \([45, 46]\), the spinor BECS was illustrated systematically from experimental and theoretical progress. In our case, the scattering lengths depend on time and space, that is to say \(g_n = g_n(x, t),\ g_s = g_s(x, t)\), which can be realized by controlling the induced Feshchbach resonances optically or confinement induced resonances in the real BEC experiments.

In the following, we seek the exact localized solutions of \((1)\) for \(\lim_{|x| \to \infty} \psi_{\pm 1, 0} = 0\). To do this, the similarity transformation

\[
\begin{align*}
\Psi_1 &= \beta_1(x, t) U(X(x, t)) e^{i\alpha_1(x, t)}, \\
\Psi_{-1} &= \beta_{-1}(x, t) V(X(x, t)) e^{i\alpha_{-1}(x, t)}, \\
\Psi_0 &= \beta_0(x, t) W(X(x, t)) e^{\frac{i\alpha_1(x, t) + \alpha_{-1}(x, t)}{2}},
\end{align*}
\]

(2)
are taken to transform (1) to the ordinary differential equations (ODEs)

\[ U_{xx} + b_{11}U^3 + b_{12}UV^2 + b_{13}W^2U = 0, \]
\[ V_{xx} + b_{21}U^2V + b_{22}V^3 + b_{23}W^2V = 0, \]
\[ W_{xx} + b_{31}V^2W + b_{32}U^2W + b_{33}W^3 + b_{34}UVW = 0, \]

where \( b_{ij}, i = 1, 2, j = 1, 2, 3, 4 \) are constants. Substituting (2) into (1) and letting \( U(X), V(X), W(X) \) to satisfy (3), we obtain a set of partial differential equations (PDEs). Solving this set of PDEs, we have

\[ \alpha_{\pm 1}(x, t) = -\frac{\lambda(t)x^2}{2\lambda(t)} + \xi_{\pm 1}(t) - \frac{\delta(t)x}{\lambda(t)}, \quad \beta_j(x, t) = \theta_j(t)\lambda(t) - \frac{1}{2}e^{\frac{\lambda(t)x^2}{2\lambda(t)}}x, \quad X = \frac{1}{2}\sqrt{\pi}\text{erf}(\lambda(t)x + \delta(t)), \]

where \( \text{erf}(s) = \frac{2}{\sqrt{\pi}}\int_0^s e^{-t^2}dt \) is called an error function, the undetermined functions \( \xi_{\pm 1}(t) \) and \( \theta_j(t), \ (j = -1, 0, 1) \) satisfy

\[ \lambda\lambda_{tt} - 2\lambda_t^2 - (3\omega^2 + \omega_0^2)\lambda^2 + \lambda_0^6 = 0, \]
\[ -4\lambda_t\delta_t + 2\delta_{tt}\lambda + 2\lambda_0^5\delta = 0, \]
\[ -\delta_t^2 + \delta^2\lambda^4 - 2\lambda^2\xi_{1t} - 2E_1\lambda^2 + \lambda^4 = 0, \]
\[ -\delta_t^2 + \delta^2\lambda^4 - 2\lambda^2\xi_{-1t} - 2E_{-1}\lambda^2 + \lambda^4 = 0, \]
\[ -\delta_t^2 + \delta^2\lambda^4 - \lambda^2\xi_{1t} - \lambda^2\xi_{-1t} - 2E_0\lambda^2 + \lambda^4 = 0, \]
\[ \theta_{jt}\lambda - \theta_j\lambda_t = 0, \quad j = -1, 0, 1. \]

To obtain explicit solutions, we choose \( \omega^2 \) as the form

\[ \omega^2 = \omega_0^2 + \epsilon\cos(\varpi t), \]

where \( \epsilon \in (-1, 1) \) and \( \omega_0, \varpi \in \mathbb{R} \). Now under the condition \( E_1 + E_{-1} = E_0 \), solving (5) gives

\[ \lambda(t) = (\text{Asin}^2\omega t + \text{Bcos}^2\omega t + \frac{2\sqrt{\omega^2(AB\omega^2 - 1)\sin\omega t\cos\omega t}}{\omega^2})^{-\frac{1}{2}}, \]
\[ \theta(t)_{jj} = c_j\lambda, \quad j = -1, 0, 1, \]
\[ \xi(t)_{\pm 1} = \frac{1}{2}\int \frac{\lambda_t^4 + \lambda^4 - \delta_t^2}{\lambda^4} dt - E_{\pm 1}t, \]
\[ \delta(t) = c_2e^{i\lambda^2dt} + c_3e^{-i\lambda^2dt}, \]

\[ E_1 = \frac{2\sqrt{\omega^2(AB\omega^2 - 1)\sin\omega t\cos\omega t}}{\lambda^2}, \]
\[ E_{-1} = \frac{2\sqrt{\omega^2(AB\omega^2 - 1)\sin\omega t\cos\omega t}}{-\lambda^2}, \]
\[ E_0 = \frac{2\sqrt{\omega^2(AB\omega^2 - 1)\sin\omega t\cos\omega t}}{\omega^2}. \]
where \( c_j, (j = -1, 0, 1, 2, 3) \) are arbitrary constants, and \( A, B \) are constants satisfying \( AB - C^2 = \frac{1}{\xi_1\xi_2 - \xi_2\xi_3} \) with \( (\xi_1, \xi_2) \) being two linear independent solutions of the Mathieu equation

\[
\xi_{tt} + \omega^2 \xi = 0. \tag{8}
\]

When the interactions are attractive, the ODEs \[3\] have two families of exact solutions

\[
U^{(1)} = -\sqrt{-c^2_{12} + 2c^2_{1}u^2_{12}} cn(\mu_1 X, \frac{\sqrt{2}}{2})
\]

\[
V^{(1)} = \sqrt{-c^2_{12} + 2c^2_{1}u^2_{12}} cn(\mu_1 X, \frac{\sqrt{2}}{2})
\]

\[
W^{(1)} = cn(\mu_1 X, \frac{\sqrt{2}}{2})
\]

and

\[
U^{(2)} = -\sqrt{-c^2_{12} + 2c^2_{1}u^2_{12}} sd(\mu_2 X, \frac{\sqrt{2}}{2})
\]

\[
V^{(2)} = \sqrt{-c^2_{12} + 2c^2_{1}u^2_{12}} sd(\mu_2 X, \frac{\sqrt{2}}{2})
\]

\[
W^{(2)} = sd(\mu_2 X, \frac{\sqrt{2}}{2})
\]

Here, \( \mu_1, \mu_2 \) are arbitrary constants, \( cn, sd = sn/dn \) are Jacobi elliptic functions. When the interactions are repulsive, the solutions of the ODEs \[3\] are similar to \(9\) and \(10\).

When imposing the bounded condition \( \lim_{|x| \to \infty} \psi_{\pm 1,0}(x) = 0 \), we have \( \mu_1 = \frac{2(2n+1)K(\frac{\sqrt{2}}{2})}{\sqrt{n}} \) and \( \mu_2 = \frac{4mK(\frac{\sqrt{2}}{2})}{\sqrt{n}} \), where the natural number \( n \) and \( m \) are the order of the solitons, and

\[
K(\frac{\sqrt{2}}{2}) = \int_{0}^{\pi/2} [1 - \frac{1}{2} \sin^2 \xi]^{-1/2} d\xi.
\]

Based on \(4\), \(7\)-\(10\), we work out two families of exact solutions of the dimensionless Gross-Pitaevskii equation \[1\]

\[
\psi^{(j)}_1 = c_1 \sqrt{\lambda(t)} e^{\frac{\delta(t)^2}{2}} e^{\gamma(x,t)+i\alpha_1(x,t)} U^{(j)}(X),
\]

\[
\psi^{(j)}_{-1} = c_{-1} \sqrt{\lambda(t)} e^{\frac{\delta(t)^2}{2}} e^{\gamma(x,t)+i\alpha_{-1}(x,t)} V^{(j)}(X),
\]

\[
\psi^{(j)}_0 = c_0 \sqrt{\lambda(t)} e^{\frac{\delta(t)^2}{2}} e^{\gamma(x,t)+\frac{(i\alpha_1(x,t)+i\alpha_{-1}(x,t))}{2}} W^{(j)}(X),
\]

where \( \delta(t), \alpha_{\pm 1}(x,t) \) are given in \(7\), and \( U^{(j)}, V^{(j)}, W^{(j)} (j = 1, 2) \) are given in \(9\)-\(10\). The significance of each quantity is: \( \gamma(x,t) = \lambda(t)x(\lambda(t)x + 2\delta(t)) \), coordinate for observing soliton’s envelope; \( \alpha_1(x,t) \), coordinate for observing soliton’s carrier waves; \( \sqrt{\lambda(t)} e^{\frac{\delta(t)^2}{2}} \), amplitude of the solitons. It is easy to see that \( \lim_{|x| \to \infty} \psi_{\pm 1,0}(x) = 0 \) by direct computation, so these two families of solutions are localized nonlinear wave solutions.
III. DYNAMICS OF THE LOCALIZED NONLINEAR MATTER WAVE

The localized matter waves given by (11) feature soliton properties: they propagate undistorted and undergo quasielastic collisions. According to different choices of $\omega$ and $\epsilon$, different types of behaviors can be classified as:

(a) Breathing solitons, when $\epsilon = 0$.

(b) Quasiperiodic soliton, when $\epsilon \neq 0$ and the two linear independent solutions $\xi_1$, $\xi_2$ belong to the stability domain of (8).

(c) Resonant soliton, when $\epsilon \neq 0$ and $\xi_1$, $\xi_2$ are in the instability domain of (8).

In this section, we will consider the dynamics of the localized nonlinear matter wave and propose how to control them by the external trapping potentials and the space-time inhomogeneous s-wave scattering lengths in future experiments. In the following, we take $^{23}Na$ condensate containing $3 \times 10^6$ atoms and the parameters are all taken as $c_{-1} = 1$, $c_0 = 0.5$, $c_1 = 1$, $b_{12} = 0.5$, $A = 5$, $B = 2$.

A. Breathing solitons

Here, we take $\epsilon = 0$. Now we study how the space- and time- dependent nonlinearities $g_n$ and $g_s$ control the dynamics of the localized nonlinear waves. In this case, the interactions $g_n = -g_s = -\frac{b_{12}}{4\sqrt{\lambda(t)}} e^{-\frac{1}{\lambda(t)}} e^{-(\lambda(t)x+\delta(t))^2}$ are all space- and time- dependent, which can be realized by controlling the optically induced Feshbach resonance or a confinement induced resonance in the real BEC experiments. The corresponding localized nonlinear wave can be obtained from (11). These solutions show different features according to the choice of the parameter $\delta(t)$.

Case 1. $\delta(t) = 0$. In this case, we take the frequencies $\omega_x = 20\pi Hz$ and $\omega_\perp = 50\pi Hz$, the ratio of the confining frequency $\omega = 0.4$. In Fig.1, we show the evolution of density profiles for the one-order wave function $\psi_1^{(1)}$ and $\psi_1^{(2)}$ with the above parameters. The density profiles for $\psi_2^{(1)}$ and $\psi_2^{(2)}$ are the same to that of $\psi_1^{(1)}$ and $\psi_1^{(2)}$. It can be observed that the density wave packets are localized in space and periodically oscillating in time, which are called breathing solitons. Here $\sqrt{\lambda(t)}$ and $1/\lambda(t)$ are the amplitude and width of the matter wave, respectively.
Fig.1 (a) to Fig.1 (c) describe the density profiles of the one-order wave function $\psi_1^{(1)}$, $\psi_{-1}^{(2)}$, and $\psi_0^{(2)}$, respectively. Fig.1 (d) demonstrates the total density distribution $|\psi_1^{(1)}|^2 + |\psi_{-1}^{(2)}|^2 + |\psi_0^{(2)}|^2$ for the spinor BECs. Fig.1 (e) demonstrates the amplitude $\sqrt{\lambda(t)}$ and the width $1/\lambda(t)$ of the wave functions. It is observed that the amplitude and the width of the localized nonlinear waves vary periodically with the increasing time. In all figures of this paper, the units of space length and time are $1.38\mu m$ and $0.7ms$.

**FIG. 1.** Dynamics of the breathing solitons in the spin-1 BEC with spatiotemporally modulated nonlinearities. (a) to (d) exhibit the evolution of the density distribution $|\psi_1^{(1)}|^2$, $|\psi_{-1}^{(2)}|^2$, $|\psi_0^{(2)}|^2$ and the total density distribution $|\psi_1^{(1)}|^2 + |\psi_{-1}^{(2)}|^2 + |\psi_0^{(2)}|^2$, respectively. (e) demonstrates the amplitude (red one) and the width (blue one). The ratio of the confining frequency is taken as $\omega = 0.4$.

**Case 2.** $\delta(t) \neq 0$ and it is given by $[\overline{7}]$. In this situation, the tapping potential is still time-independent, but the interactions $g_n$, $g_s$ given by $[\overline{7}]$ and $[\overline{8}]$ become more complex, the amplitude of the nonlinear matter wave becomes $\sqrt{\lambda(t)}e^{-\frac{\delta(t)^2}{2}}$, and the center of mass of the solitons move time periodically with non-zero velocity. So the nonlinear matter waves are called moving breathing. For convenience, we assume the ratio of the confining frequencies $\omega$ is time...
independent to illustrate the dynamics of the moving breathing soliton. Here we still take \( \omega = 0.4 \) and \( \varepsilon = 0 \). Fig. 2 (a)-(c) describe the time evolution of the density profiles of one-order wave function \( \psi_1^{(1)} \), \( \psi_{-1}^{(2)} \) and \( \psi_0^{(2)} \), respectively. Fig. 2 (d) demonstrates the total density distribution \( |\psi_1^{(1)}|^2 + |\psi_{-1}^{(2)}|^2 + |\psi_0^{(2)}|^2 \) for the spinor BECs. It can be observed the nonlinear matter waves are space localized and moving periodically with respect to time. Fig. 2 (e) describes the amplitude of the breathing solitons (red one) and the moving breathing solitons (blue one). It can be found that the amplitudes of the nonlinear waves vary periodically versus \( t \), and the amplitudes of the moving breathing solitons are higher than the breathing solitons.

**FIG. 2.** Dynamics of the moving breathing solitons in the spin-1 BEC with spatiotemporally modulated nonlinearities. (a) to (d) display the time evolution of the density distribution \( |\psi_1^{(1)}|^2 \), \( |\psi_{-1}^{(2)}|^2 \), \( |\psi_0^{(2)}|^2 \) and \( |\psi_1^{(1)}|^2 + |\psi_{-1}^{(2)}|^2 + |\psi_0^{(2)}|^2 \), respectively. (e) reveals the amplitude of the breathing solitons (red one) and the moving breathing solitons (blue one). The ratio of the confining frequency is still taken as \( \omega = 0.4 \) and the other parameters are taken as \( c_2 = c_3 = 1/3 \).

**B. Quasibreathing solitons**

Now, we consider the case of \( \epsilon \neq 0 \). In order to give an example of quasibreathing solitons, we take \( \omega_0 = 0.4 \), \( \overline{\omega} = 2 \) and \( \epsilon = 0.1 \) in (6), i.e. the ratio of the confining frequency \( \omega = 1/3 \).
$\sqrt{0.16 + 0.03\cos(0.2t)}$ is time dependent function, which ensure that two linear independent solutions $\xi_1$ and $\xi_2$ of the Mathieu equation [6] belong to its stability region. So (6) has two incommensurable frequencies. In this way, the localized matter wave given by the solutions [11] exhibit quasiperiodic behaviors and the trapping potential and interactions are still space and time dependent.

**FIG. 3.** Dynamics of the quasibreathing solitons in the spin-1 BEC with spatiotemporally modulated nonlinearities. (a1) and (a2) express the evolution of the density and contour distribution $|\psi^{(1)}_1|^2$, respectively. (b1) and (b2) exhibit the evolution of the density and contour distribution $|\psi^{(2)}_1|^2$, respectively. (c1) and (c2) display the evolution of the density and contour distribution $|\psi^{(2)}_0|^2$, respectively. (d1) and (d2) illustrate the evolution of the density and contour distribution $|\psi^{(1)}_1|^2 + |\psi^{(2)}_1|^2 + |\psi^{(2)}_0|^2$. (e) demonstrates the amplitude (red one) and the width (blue one). The parameters are taken as $\omega_1 = 0.03\cos(2t)$, $\omega_2 = 0.2$, $\delta = 0$. 
Fig. 3 shows an example of the quasiperiodic behavior. The first column to the fourth column describe the time evolution of the density and contour profiles of the one-order wave function \( \psi_1^{(1)}, \psi_{-1}^{(2)}, \psi_0^{(2)} \) and the total density distribution \(|\psi_1^{(1)}|^2 + |\psi_{-1}^{(2)}|^2 + |\psi_0^{(2)}|^2\), respectively. Fig. 3 (e) describes the amplitude (red one) and the width (blue one) of the quasibreathing solitons. It can be observed the nonlinear matter waves are space localized and quasiperiodic with respect to time and it can also be fond that the amplitude and width of the nonlinear waves are quasiperiodically versus \( t \).

C. Resonant solitons

Here, we still consider the case of \( \epsilon \neq 0 \), i.e., the ratio of the confining frequency \( \omega \) is time dependent. In this case, we choose \( \omega_0 = 0.44, \ \bar{\omega} = 32 \) and \( \epsilon = 0.003 \) in Mathieu equation (6), which ensure that two linear independent solutions \( \xi_1 \) and \( \xi_2 \) of (6) belong to its instability region. Thus, the localized matter waves given by the solutions (11) show the resonant solitons behaviors and the trapping potential and interactions are still space and time dependent. In Fig. 4, we show the evolution of the density and contour profiles for the resonant solitons. The first to fourth column in Fig.4 demonstrate the evolution of the density and contour profiles of the one order wave function \( \psi_1^{(1)}, \psi_{-1}^{(2)}, \psi_0^{(2)} \), and \(|\psi_1^{(1)}|^2 + |\psi_{-1}^{(2)}|^2 + |\psi_0^{(2)}|^2\), respectively. It can be observed the nonlinear matter waves are space localized and time resonant. Fig. 4 (e) shows the amplitude and width versus time for the resonant solitons. It can be observed that the amplitude of the resonant solitons is low and its width are large at beginning. After a while, the amplitudes become higher but the widths become small. This phenomenon appear gradually as time goes. So this nonlinear matter wave demonstrate the resonant soliton behavior. This resonant behaviors comes from the cooperation of the spatiotemporal inhomogeneous interactions and trapping potential.
FIG. 4. Dynamics of the Resonant breathing solitons in the spin-1 BEC with spatiotemporally modulated nonlinearities. (a1) and (a2) explain the evolution of the density and contour $|\psi_1|^2$, respectively. (b1) and (b2) exhibit the evolution of the density and contour $|\psi_{-1}|^2$, respectively. (c1) and (c2) express the evolution of the density and contour $|\psi_0|^2$, respectively. (d1) and (d2) demonstrate the evolution of the density and contour $|\psi_1^{(1)}|^2 + |\psi_{-1}^{(2)}|^2 + |\psi_0^{(2)}|^2$, respectively. (e) shows the amplitude of the breathing solitons (red one) and the moving breathing solitons (blue one). Here the ratio of the confining frequency $\omega = \sqrt{0.19 + 0.03 \cos 32t}$.

IV. STABILITY ANALYSIS

Now we study the dynamical stability of the localized nonlinear wave solutions [11] by performing some numerical simulations. Here we run the numerical simulations by use of the
split-step Fourier transformation. The domain is composed of $N = 512$ grids point and the step sized of time integration is $\tau = 0.0001$. We take $\psi_j(x, 0)$ ($j = -1, 0, 1$) as an initial values and the simulations are lasting up to $t = 200$. The simulation results show that:

(a) When $\delta = 0$, the exact localized nonlinear wave solutions (11) is dynamically stable for $j = 1$ and $n = 0$, that is to say the one order breathing soliton, quasibreathing soliton and resonant soliton are all stable.

(b) When $\delta \neq 0$, the moving solitons $\psi^{(1)}_1$ and $\psi^{(1)}_0$ are dynamically stable, while $\psi^{(1)}_{-1}$ is unstable.

Fig. 5 show the time evolution of the one-order breathing solitons for $\delta = 0$ and $\varepsilon = 0$ with $g_n(x, t) = -g_s(x, t) = \frac{1}{8} \nu(t) - \frac{1}{2} e^{-\frac{x^2}{\nu(t)}}$, $\nu(t) = 5 \sin\left(\frac{\pi}{5}t\right)^2 + \sqrt{15} \sin\left(\frac{2\pi}{5}t\right) \cos\left(\frac{2\pi}{5}t\right)$. It can be observed that the one-order breathing solitons $\psi^{(1)}_1$, $\psi^{(1)}_0$ and $\psi^{(1)}_{-1}$ are dynamical stable.

\[ \begin{align*}
\text{FIG. 5.} \quad &\text{Evolution of the one-order breathing solitons for } \delta = 0, \varepsilon = 0 \text{ and the nonlinearity } g_n(x, t) = \\
&\quad \frac{1}{8} \nu(t) - \frac{1}{2} e^{-\frac{x^2}{\nu(t)}}, \quad \nu(t) = 5 \sin\left(\frac{\pi}{5}t\right)^2 + \sqrt{15} \sin\left(\frac{2\pi}{5}t\right) \cos\left(\frac{2\pi}{5}t\right). \quad \text{The other parameters are the same as used in Fig1.}
\end{align*} \]

Fig. 6 show the evolution of the one-order moving breathing solitons with $\delta = 1$, $\varepsilon = 0$ and $g_n(x, t) = -g_s(x, t) = \frac{1}{8} \nu(t) - \frac{1}{2} e^{-\frac{x^2}{\nu(t)}}$. It can be seen that the moving breathing solitons $\psi^{(1)}_1$ and $\psi^{(1)}_0$ are dynamical stable, while $\psi^{(1)}_{-1}$ is unstable.
FIG. 6. Evolution of the one-order moving breathing solitons with $\delta = 1$, $\varepsilon = 0$ and the nonlinearity $g_n(x, t) = -g_s(x, t) = \frac{1}{8} \nu(t)^{-\frac{1}{2}} e^{\left(1 - \frac{x}{\nu(t)}\right)^2}$. The other parameters are the same as used in Fig2.

V. CONCLUSION

In this paper, we study the dynamics of the localized nonlinear matter wave solutions of the three-component GPEs with time- and space- dependent nonlinearities for $F=1$ spinor BECs. These solutions are derived in terms of Jacobi elliptic functions and Mathieu equation by similarity transformation. Further, we illustrate that the localization of the nonlinear matter wave takes the form of the vector breathing solitons, moving breathing solitons, quasibreathing solitons and resonant solitons. The dynamical stability of the all kinds of vector solitons are analysed by numerical stimulation. The results show that the one-order breathing solitons, quasibreathing solitons, resonant soltons and the moving breathing solitons except for the matter wave $\psi_0$ are all stable.

We take the sodium atom $^{23}Na$ with the total number $N = 3 \times 10^6$ as an example to show how to create various soliton phenomena under the condition that the Zeeman energy $E_j$ satisfies $E_1 + E_{-1} = E_0$. When the confining frequencies in the transverse and axial directions are taken as $\omega_x = 20\pi Hz$ and $\omega_\perp = 50\pi Hz$, respectively, the breathing solitons can be observed with the interactions $g_n(x, t) = -g_s(x, t) = \frac{1}{8} \nu(t)^{-\frac{1}{2}} e^{-\frac{x^2}{\nu(t)}}$, the moving breathing solitons can be observed when $g_n(x, t) = -g_s(x, t) = \frac{1}{8} \nu(t)^{-\frac{1}{2}} e^{\left(1 - \frac{x}{\nu(t)}\right)^2}$. When the confining frequencies in the transverse are the functions of $t$, the quasibreathing soliton and resonant solitons may
be observed. For example, the interactions \( g_n(x, t) = -g_s(x, t) = -e^{(\lambda(t)x+\delta(t))^2/(8\lambda(t))} \) and the confining frequencies \( \omega_x = (16 + 3\cos0.2t)\pi Hz \) and \( \omega_\perp = 100\pi Hz \), the quasibreathing soliton can appear. And the resonant soliton can appear with the confining frequencies \( \omega_x = (19 + 0.03\cos32t)\pi Hz \) and \( \omega_\perp = 100\pi Hz \). We hope that these dynamics behaviors of the spin-1 BECs with spatiotemporal nonlinearities can be realized in the future experiment and help us to understand these phenomena further.

**Acknowledgements**

This work was supported by the NKRDP under grants Nos. 2016YFA0301500, NSFC under grants Nos. 11434015, 61227902, 61378017, KZ201610005011, SKLQOQOD under grants No. KF201403, SPRPCAS under grants No. XDB01020300, XDB21030300.

**References**

[1] C. J. Pethick, and H. Smith, Bose-Einstein condensation in dilute gases. Cambridge: Cambridge University Press; 2002.

[2] R. Ozeri, N. Katz. J. Steinhauer, and N. Davidson, Rev. Mod. Phys. **77**, 187 (2005).

[3] T. L. Ho and V. B. Shenoy, Phys. Rev. Lett. **77**, 3276 (1996); T. L. Ho, Phys. Rev. Lett. **81**, 742 (1998).

[4] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. **67**, 1822 (1998); T. Isoshima, K. Machida and T. Ohmi, Phys. Rev. A **60**, 4857 (1999).

[5] D. M. Stamper-Kurn, M. R. Andrews, A. P. Chikkatur, S. Inouye, H.-J. Miesner, J. Stenger, and W. Ketterle, Phys. Rev. Lett. **80**, 2027 (1998).

[6] J. Stenger, S. Inouye, D. Stamper-Kurn, H. J. Miesner, A. Chikkatur and W. Ketterle, Nature **396**, 345 (1998).
[7] M. Vengalattore, S. R. Leslie, J. Guzman and D. M. Stamper-Kurn, Phys. Rev. Lett. 100, 170403 (2008).

[8] H. Schmaljohann, M. Erhard, J. Kronjager, M. Kottke, S. van Staa, L. Cacciapuoti, J. J. Arlt, K. Bongs and K. Sengstock, Phys. Rev. Lett. 92, 040402 (2004).

[9] H. Saito and M. Ueda, Phys. Rev. A 72, 053628 (2005).

[10] H. J. Miesner, D. M. Stamper-Kurn, J. Stenger, S. Inouye, A. P. Chikkatur and W. Ketterle, Phys. Rev. Lett. 82, 2228 (1999).

[11] D. S. Hall, M. R. Matthews, C. E. Wieman and E. A. Cornell, Phys. Rev. Lett. 81, 1543 (1998).

[12] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur and W. Ketterle, Nature 396, 345 (1998).

[13] T. L. Ho, Phys. Rev. Lett. 81, 742 (1998).

[14] T. Ohmi, K. Machida, J. Phys. Soc. Jpn. 67, 1822 (1998).

[15] C. V. Ciobanu, S.-K. Yip, and T.-L. Ho, Phys. Rev. A 61, 033607 (2000).

[16] M. Koashi and M. Ueda, Phys. Rev. Lett. 84, 1066 (2000).

[17] D. S. Wang, Y. Q. Ma and X. G. Li, Commun. Nolinear Sci Numer Simulat. 19, 3556 (2014).

[18] A. C. Ji, W. M. Liu, J. L. Song and F. Zhou, Phys. Rev. Lett. 101, 010402 (2008).

[19] J. Ieda, T. Miyakawa and M. Wadati, J. Phys. Soc. Jpn. 73, 2996 (2004).

[20] M. Uchiyama, J. Ieda, and M. Wadati, J. Phys. Soc. Jpn. 75, 064002 (2006).

[21] L. Li, Z. Li, B. A. Malomed, D. Mihalache and W. M. Liu, Phys. Rev. A 72, 033611 (2005).
[22] B. J. Dabrowska-Wuster, E. A. Ostrovskaya, T. J. Alexander and Y. S. Kivshar, Phys. Rev. A 75, 023617 (2007).

[23] L. Zhou, H. Pu, H. Y. Ling, K. Y. Zhang and W. P. Zhang, Phys. Rev. A 81, 063641 (2010).

[24] Z. Dutton, M. Budde, C. Slowe and Lene Vestergaard Hau, Science 293, 663 (2001).

[25] K. E. Strecker, G. B. Partridge, A. G. Truscott and R. G. Hulet, Nature 417, 150 (2002).

[26] B. Eiermann, Th. Anker, M. Albiez, M. Taglieber, P. Treutlein, K.-P. Marzlin and M. K. Oberthaler, Phys. Rev. Lett. 92, 230401 (2004).

[27] P. Meystre, Atom Optics (Springer-Verlag, New York, 2001).

[28] S. Inouye, M. R. Andrews, J. Stenger, H.-J. Miesner, D. M. Stamper-Kurn and W. Ketterle, Nature 392, 151 (1998).

[29] P. G. Kevrekidis, G. Theocharis, D. J. Frantzeskakis, and Boris A. Malomed, Phys. Rev. Lett. 90, 230401 (2003).

[30] M. Theis, G. Thalhammer, K. Winkler, M. Hellwig, G. Ruff, R. Grimm and J. H. Denschlag, Phys. Rev. Lett. 93, 123001 (2004).

[31] J. Belmonte-Beitia, V. M. Perez-Garcia, V. Vekslerchik and P. J. Torres, Phys. Rev. Lett. 98, 064102 (2007).

[32] R. Yamazaki, S. Taie, S. Sugawa and Y. Takahashi, Phys. Rev. Lett. 105, 050405 (2010).

[33] V. M. Perez-Garcia, Vladimir V. Konotop and Valeriy A. Brazhnyi, Phys. Rev. Lett. 92, 220403 (2004).

[34] D. S. Wang, X. H. Hu and W. M. Liu, Phys. Rev. A 82, 023612 (2010).

[35] Z. X. Liang, Z. D. Zhang and W. M. Liu, Phys. Rev. Lett. 94, 050402 (2005).

[36] E. Kengne, The European Physical Journal B 94, 050402 (2005).
[37] Y. Q. Yao, J. Li, W. Han and W. M. Liu, Sci. Rep. 89, 78 (2016).

[38] T. J. Alexander, K. Heenan, M. Salerno and E. A. Ostrovskaya, Phys. Rev. A 85, 063626 (2012).

[39] J. M. Gerton, B. J. Frew and R. G. Hulet, Phys. Rev. A 64, 053410 (2001).

[40] T. Bergeman, M. G. Moore and M. Olshanii, Phys. Rev. Lett. 91, 163201 (2003).

[41] C. K. Law, H. Pu, and N. P. Bigelow, Phys. Rev. Lett. 81, 5257 (1998).

[42] K. W. Madison, F. Chevy, V. Bretin and J. Dalibard, Phys. Rev. Lett. 86, 4443 (2001).

[43] N. N. Klausen, J. L. Bohn and C. H. Greene. Phys. Rev. A 64, 053602 (2001).

[44] B. J. Dabrowska-Wuster, E. A. Ostrovskaya, T. J. Alexander and Y. S. Kivshar. Phys. Rev. A 75, 023617 (2007).

[45] Dan M. Stamper-Kurn and M. Ueda, Rev. Mod Phys 85, 1192 (2013).

[46] Y. Kawaguchi and M. Ueda, Phys. Reports 520, 253(2012).