Askey-Wilson relations and Leonard pairs

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Abstract

It is known that if \((A, A^*)\) is a Leonard pair, then the linear transformations \(A, A^*\) satisfy the Askey-Wilson relations

\[
A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^* = \gamma^* A^2 + \omega A + \eta I,
\]

\[
A^* A^2 - \beta A^* A A^* + A A^* + \gamma^* (A^* A + A A^*) - \delta^* A = \gamma^* A^2 + \omega^* A^* + \eta^* I,
\]

for some scalars \(\beta, \gamma, \gamma^*, \delta, \omega, \eta, \eta^*\). The problem of this paper is the following: given a pair of Askey-Wilson relations as above, how many Leonard pairs are there that satisfy those relations? It turns out that the answer is 5 in general. We give the generic number of Leonard pairs for each Askey-Wilson type of Askey-Wilson relations.

AMS 2000 MSC Classification: 05E35, 33D45, 33C45.

Keywords: Leonard pairs, Askey-Wilson relations.

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*Supported by the 21 Century COE Programme "Development of Dynamic Mathematics with High Functionality" of the Ministry of Education, Culture, Sports, Science and Technology of Japan.
1 Introduction

Throughout the paper, $K$ denotes an algebraically closed field. We assume the characteristic of $K$ is not equal to 2. Recall that a tridiagonal matrix is a square matrix which has nonzero entries only on the main diagonal, on the superdiagonal and the subdiagonal. A tridiagonal matrix is called irreducible whenever all entries on the superdiagonal and subdiagonal are nonzero.

Definition 1.1 Let $V$ be a vector space over $K$ with finite positive dimension. By a Leonard pair on $V$ we mean an ordered pair $(A, A^*)$, where $A : V \to V$ and $A^* : V \to V$ are linear transformations which satisfy the following two conditions:

(i) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is diagonal, and the matrix representing $A$ is irreducible tridiagonal.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A$ is diagonal, and the matrix representing $A^*$ is irreducible tridiagonal.

Remark 1.2 In this paper we do not use the conventional notation $A^*$ for the conjugate-transpose of $A$. In a Leonard pair $(A, A^*)$, the linear transformations $A$ and $A^*$ are arbitrary subject to the conditions (i) and (ii) above.

Definition 1.3 Let $V, W$ be vector spaces over $K$ with finite positive dimensions. Let $(A, A^*)$ denote a Leonard pair on $V$, and let $(B, B^*)$ denote a Leonard pair on $W$. By an isomorphism of Leonard pairs we mean an isomorphism of vector spaces $\sigma : V \to W$ such that $\sigma A \sigma^{-1} = B$ and $\sigma A^* \sigma^{-1} = B^*$. We say that $(A, A^*)$ and $(B, B^*)$ are isomorphic if there is an isomorphism of Leonard pairs from $(A, A^*)$ to $(B, B^*)$.

Leonard pairs occur in the theory of orthogonal polynomials, combinatorics, the representation theory of the Lie algebra $sl_2$ or the quantum group $U_q(sl_2)$. We refer to [Ter06] as a survey on Leonard pairs, and as a source of further references.

We have the following result [TV04, Theorem 1.5].

Theorem 1.4 Let $V$ denote a vector space over $K$ of finite positive dimension. Let $(A, A^*)$ be a Leonard pair on $V$. Then there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$, $\omega, \eta, \eta^*$ taken from $K$ such that

$$A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \varrho A^* = \gamma^* A^2 + \omega A + \eta I,$$

$$A^{*2} A - \beta A^* A A^* + AA^* 2 - \gamma^* (A^* A + AA^*) - \varrho^* A = \gamma A^{*2} + \omega A^* + \eta^* I.$$  

(1)  

(2)

The sequence is uniquely determined by the pair $(A, A^*)$ provided the dimension of $V$ is at least 4.

The equations (1)–(2) are called the Askey-Wilson relations. They first appeared in the work [Zhe91] of Zhedanov, where he showed that the Askey-Wilson polynomials give pairs of infinite-dimensional matrices which satisfy the Askey-Wilson relations. We
Table 1: Leonard pairs with fixed Askey-Wilson relations, if \( \dim V \geq 4 \)

| Askey-Wilson coefficients | Leonard pairs | Askey-Wilson type |
|---------------------------|--------------|-------------------|
| \( \beta \neq \pm 2, \gamma = \gamma^* = 0, \widehat{\alpha} \widehat{\alpha}^* \neq 0 \) | 5 | \( q \)-Racah |
| \( \beta \neq \pm 2, \gamma = \gamma^* = 0, \widehat{\alpha} = 0, \widehat{\alpha}^* \widehat{\alpha}^* \neq 0 \) | 4 | \( q \)-Hahn |
| \( \beta \neq \pm 2, \gamma = \gamma^* = 0, \widehat{\alpha}^* = \widehat{\alpha}^* \neq 0 \) | 4 | Dual \( q \)-Hahn |
| \( \beta \neq \pm 2, \gamma = \gamma^* = 0, \widehat{\alpha} = \widehat{\alpha}^* = \widehat{\alpha}^* \neq 0 \) | 1 | \( q \)-Krawtchouk |
| \( \beta \neq \pm 2, \gamma = \gamma^* = 0, \widehat{\alpha}^* = \widehat{\alpha}^* \neq 0 \) | 1 | Dual \( q \)-Krawtchouk |
| \( \beta \neq \pm 2, \gamma = \gamma^* = 0, \widehat{\alpha} = \widehat{\alpha}^* = \widehat{\alpha}^* \neq 0 \) | 3 | Quantum/affine \( q \)-Krawtchouk |
| \( \beta = 2, \gamma \gamma^* \neq 0, \widehat{\alpha} = \widehat{\alpha}^* = 0 \) | 4 | Racah |
| \( \beta = 2, \gamma = 0, \gamma^* \varrho \neq 0, \varrho = \omega = 0 \) | 3 | Hahn |
| \( \beta = 2, \gamma^* = 0, \gamma \varrho^* \neq 0, \varrho = \omega = 0 \) | 3 | Dual Hahn |
| \( \beta = 2, \gamma = \gamma^* = 0, \varrho \varrho^* \neq 0, \eta = \eta^* = 0 \) | 1 | Krawtchouk |
| \( \beta = -2, \gamma = \gamma^* = 0, \widehat{\alpha} \widehat{\alpha}^* \neq 0; \dim V \text{ odd} \) | 5 | Bannai-Ito |
| \( \beta = -2, \gamma = \gamma^* = 0, \widehat{\alpha} \widehat{\alpha}^* \neq 0; \dim V \text{ even} \) | 4 | Bannai-Ito |

denote this pair of equations by \( \text{AW}(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*) \). We refer to the 8 scalar parameters as the Askey-Wilson coefficients.

A natural question is the following: does a particular pair of Askey-Wilson relations determines a Leonard pair uniquely? An example in the next section shows that the answer is negative in general. One may ask then: if we fix the dimension of \( V \) and the 8 scalars \( \beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^* \), how many Leonard pairs are there which satisfy \( \text{AW}(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*) \)? This is the question that we consider in this paper.

It turns out that there may be up to 5 different Leonard pairs satisfying the same Askey-Wilson relations. As a preliminary check, let us consider the case \( \dim V = 1 \). Then we have 2 equations in 2 commuting unknowns \( A \) and \( A^* \). Computation of a Gröbner basis or a resultant shows that there are 5 solutions in general.

Table 1 represents our main results: the generic number of Leonard pairs, up to isomorphism, with the same Askey-Wilson relations for various sequences of the Askey-Wilson coefficients. We distinguish cases according to the classification of Askey-Wilson relations in [Vid06, Section 8], which mimics Terwilliger’s classification of parameter arrays representing Leonard pairs; see [Ter05] or [Ter06, Section 35] and Section 3 here. These results are valid if \( \dim V \geq 4 \).

The first column of Table 1 characterizes the distinguished cases in terms of the Askey-Wilson coefficients. The underlined expressions are not the defining conditions; they mean that the Askey-Wilson relations can be normalized by affine transformations

\[
(A, A^*) \mapsto (tA + c, t^*A^* + c^*), \quad \text{with } c, c^*, t, t^* \in \mathbb{K}; \, t, t^* \neq 0, \quad (3)
\]
into a form where the underlined expressions hold (provided that the preceding conditions are satisfied). Normalization of Askey-Wilson relations is adequately explained in [Vid06, Section 4]. Particularly, if \( \beta \neq 2 \) then the Askey-Wilson relations can be normalized so that \( \gamma = 0 \) and \( \gamma^* = 0 \). By \( \hat{\omega}, \hat{\omega}^*, \hat{\eta}, \hat{\eta}^* \) we denote other Askey-Wilson coefficients in such a normalization.

The second column indicates the generic number of Leonard pairs satisfying Askey-Wilson relations restricted by the conditions in the first column. The results are generic, so for some special values of the Askey-Wilson coefficients the number of distinct Leonard pairs may be smaller. In these special cases, one may either interpret missing Leonard pairs as degenerate, or one may argue that generically different Leonard pairs are isomorphic in the special case. This is explained in Remark 3.2 and demonstrated in Example 6.3 here below. If a sequence of Askey-Wilson coefficients satisfies neither condition set of the first column, there are no Leonard pairs satisfying those Askey-Wilson relations.

The third column gives the Askey-Wilson type of Askey-Wilson relations as defined in [Vid06, Section 8]. Leonard pairs have the same Askey-Wilson type as the Askey-Wilson relations that they satisfy, according to [Vid06, Theorem 8.1].

We use Terwilliger’s classification of parameter arrays representing Leonard pairs. Therefore in Section 3 we recall the definition of parameter arrays and classification terminology. In Section 4 we present normalized general parameter arrays and the Askey-Wilson relations for Leonard pairs represented by them. The results of Table 1 are proved in Section 5.

2 An example

Here we give an example of Askey-Wilson relations satisfied by different Leonard pairs. This example was observed by Curtin [Cur04] as well.

Let \( d \) be a non-negative integer, and let \( V \) be a vector space with dimension \( d + 1 \) over \( \mathbb{K} \). Let \( q \) denote a scalar which is not zero and not a root of unity. Set \( \beta = q^2 + q^{-2} \), Notice that \( \beta \neq \pm 2 \). We look for Leonard pairs on \( V \) which satisfy

\[
AW (\beta, 0, 0, 4 - \beta^2, 4 - \beta^2, 0, 0, 0). \tag{4}
\]

Existence of a Leonard pair satisfying these relations follows from [Cur04], where Terwilliger algebras for 2-homogeneous bipartite distance regular graphs are computed. The Terwilliger algebra is defined by two non-commuting generators and two relations. The relations differ from (4) by a scaling of the generators. The two generators can be represented as a Leonard pair \((A, A^*)\). The Leonard pair has the property that the tridiagonal forms for \( A \) and \( A^* \) of Definition 1.1 have only zero entries on the main diagonal. A rescaled version of \((A, A^*)\) must satisfy (4). Besides, Curtin [Cur04] computed “almost 2-homogeneous almost bipartite” Leonard pairs satisfying the same defining relations of the Terwilliger algebra. For these Leonard pairs, the tridiagonal forms of Definition 1.1 have precisely one nonzero entry on the main diagonal.
Here we present Leonard pairs of both kinds explicitly. They are scaled so that they satisfy (4). Let $A_1, A_1^*, A_2, A_2^*$ be the following matrices:

- **$A_1$** is tridiagonal, with zero entries on the main diagonal, the entries
  \[
  \sqrt{-1} \frac{q^{2d-2j} - q^{-2j-2d}}{q^{d-2j} + q^{2j-d}}, \quad \text{for } j = 0, \ldots, d-1,
  \]  
  on the superdiagonal, and the entries
  \[
  \sqrt{-1} \frac{q^{2j} - q^{-2j}}{q^{d-2j} + q^{2j-d}}, \quad \text{for } j = 1, \ldots, d,
  \]  
  on the subdiagonal.

- **$A_1^*$** is diagonal, with the entries
  \[
  \sqrt{-1} \left(q^{d-2j} - q^{-2j-d}\right), \quad \text{for } j = 0, \ldots, d,
  \]  
  on the main diagonal.

- **$A_2$** is tridiagonal, with the upper-left entry equal to \(\frac{q^{2d+2} - q^{-2d-2}}{q - q^{-1}}\), all other diagonal entries equal to zero, with the entries
  \[
  \frac{q^{2d-2j} - q^{2j-2d}}{q^{-2j-1} - q^{2j+1}}, \quad \text{for } j = 0, \ldots, d-1,
  \]  
  on the superdiagonal, and the entries
  \[
  \frac{q^{2d+2j+2} - q^{-2d-2j-2}}{q^{2j+1} - q^{-2j-1}}, \quad \text{for } j = 1, \ldots, d,
  \]  
  on the subdiagonal.

- **$A_2^*$** is diagonal, with the diagonal entries \(q^{2j+1} + q^{-2j-1}\), for \(j = 0, 1, \ldots, d\).

One can routinely check that the pairs \((A_1, A_1^*)\) and \((A_2, A_2^*)\) satisfy Askey-Wilson relations. Compared with the intersection arrays for 2-homogeneous bipartite distance regular graphs in [Cur01], we have replaced \(q \mapsto q^2\), and the matrices $A_1, A_1^*$ are multiplied by \(\sqrt{-1} \left(q^2 - q^{-2}\right) / (q^{d-2} + q^{2-d})\).

It is a routine computation to check that the matrix pairs \((A_1, A_1^*)\) and \((A_2, A_2^*)\) satisfy the Askey-Wilson relations. Since the matrices $A_1^*$ and $A_2^*$ have different sets of eigenvalues, the matrix pairs are not related by a conjugation. There are following ways to see that both \((A_1, A_1^*)\) and \((A_2, A_2^*)\) are Leonard pairs:

- Using Theorem 6.2 in [TV04]. For $i = 1, 2$, the sufficient conditions for \((A_i, A_i^*)\) to be a Leonard pair are the following:
  - There exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, q, q^*, \omega, \eta, \eta^*$ taken from $K$ such that the Askey-Wilson relations as in [11-22] hold.
\[ \tilde{q} \text{ is not a root of unity, where } \tilde{q} + \tilde{q}^{-1} = \beta. \]

- Both \( A_i \) and \( A_i^* \) are multiplicity free.
- \( V \) is irreducible as an \( A_i, A_i^* \) module.

- By using the classification of Leonard pairs [Ter06 Section 35]. Consider the most general \( q \)-Racah type:

\[
\begin{align*}
\theta_i &= \theta_0 + h \left( 1 - q^i \right) \left( 1 - s q^{i+1} \right) q^{-i}, \\
\theta_i^* &= \theta_0^* + h^* \left( 1 - q^i \right) \left( 1 - s^* q^{i+1} \right) q^{-i}, \\
\varphi_i &= \eta h h^* \left( 1 - q^i \right) \left( 1 - q^{-i-d-1} \right) \left( 1 - r_1 q^i \right) \left( 1 - r_2 q^i \right), \\
\phi_i &= \eta^* h h^* \left( 1 - q^i \right) \left( 1 - q^{-i-d-1} \right) (r_1 - s^* q^i) (r_2 - s^* q^i) / s^*. 
\end{align*}
\]

Here \( q \neq 0, \pm 1 \), the constants \( h, h^*, s, s^*, r_1, r_2 \) are nonzero and satisfy \( r_1 r_2 = s s^* q^{d+1} \), none of \( q, r_1 q^i, r_2 q^i, s q^i / r_1, s^* q^i / r_2 \) is equal to 1 for \( i = 1, \ldots, d \), and neither of \( s q^i, s^* q^i \) is equal to 1 for \( i = 2, \ldots, 2d \). To get the pair \( (A_1, A_1^*) \), we must replace \( q \mapsto q^2 \) and take

\[
\begin{align*}
\theta_0 &= \theta_0^* = \sqrt{-1} \left( q^d - q^{-d} \right), \\
h &= h^* = \sqrt{-1} q^d, \\
s &= s^* = -q^{-2d-2}, \\
r_1 &= -r_2 = \sqrt{-1} q^{-d-1},
\end{align*}
\]

and use explicit expressions in [Ter06 Section 27]. To get the pair \( (A_2, A_2^*) \), we must replace \( q \mapsto q^2 \) and take

\[
\begin{align*}
\theta_0 &= \theta_0^* = q + q^{-1}, \\
h &= h^* = q^{-1}, \\
s &= s^* = 1, \\
r_1 &= -1, \\
r_2 &= -q^{2d+2}.
\end{align*}
\]

- By exhibiting explicit transition matrices to a base mentioned in part (ii) of Definition [Ter06]. Entries of the transition matrices are \( q \)-hypergeometric series; see [Ter02 Section 16], [Ter04 Section 19] or [Ter06 Section 24]. Let \( P_i \) denote the \((d+1) \times (d+1)\) matrix with the \((i, j)\)-th entry equal to

\[
(-1)^{i} q^{2d j} (1 + q^j - 2d)^{(\frac{-q^{2d}; q^2}{q^2; q^2})_j (\frac{-q^{-2d}; q^2}{q^2; q^2})_j}_j \times 4_{\phi_3} \left( q^{-2i}, q^{-2i}, -q^{2i-2d}, -q^{-2i-2d}; q^{-2d}, \sqrt{-1} q^{-d}, -\sqrt{-1} q^{-1-d}; q^2; q^2 \right),
\]

and let \( P_2 \) denote the \((d+1) \times (d+1)\) matrix with the \((i, j)\)-th entry equal to

\[
\frac{(1 - q^{4j+2}) (\frac{-q^{2d+4}; q^2}{q^{2d+4}; q^2})_j (\frac{-q^{-2d}; q^2}{q^{-2d}; q^2})_j}_j 4_{\phi_3} \left( q^{-2j}, q^{-2j}, q^{2j+2}, q^{-2j+2}; q^{-2d}, -q^{2d+4}, -q^{2d}; q^2; q^2 \right).
\]

In these expressions, \( i, j \in \{0, 1, \ldots, d\} \). The \( q \)-hypergeometric \( 4_{\phi_3} \) series can be written as \( q \)-Racah polynomials; see [KS94 Section 3.2]. Using \( q \)-difference relations for \( q \)-Racah polynomials, we routinely check that \( A_i P_i = P_i A_i^* \) and \( A_i^* P_i = P_i A_i \) for \( i = 1, 2 \). This implies that conjugation by \( P_i \) converts the pair \( (A_i, A_i^*) \) to the matrix pair \( (A_i^*, A_i) \), and condition (ii) of Definition [Ter06] is satisfied.
The conclusion is that \((A_1, A_1^*)\) and \((A_2, A_2^*)\) are non-isomorphic Leonard pairs (in general), and they satisfy the same Askey-Wilson relations \([1]\). Both Leonard pairs are self-dual.

Table 1 predicts 5 Leonard pairs satisfying \([4]\). Indeed, the 5 Leonard pairs are

\[
(A_1, A_1^*), \quad (A_2, A_2^*), \quad (A_2, -A_2^*), \quad (-A_2, A_2^*), \quad (-A_2, -A_2^*). \tag{19}
\]

The last 4 Leonard pairs are non-isomorphic Leonard pairs related by affine transformations (which are affine scalings by \(-1\)). The same affine scalings of \((A_1, A_1^*)\) are isomorphic to \((A_1, A_1^*)\). Surely, the scalings leave the Askey-Wilson relations invariant.

The complex conjugation of \(\sqrt{-1}\) has the effect of multiplying both \(A_1\) and \(A_1^*\) by \(-1\). The same rescaling of \((A_2, A_2^*)\) is achieved by the substitution \(q \rightarrow -q\). The substitution \(q \mapsto 1/q\) preserves the Askey-Wilson relations as well; it has the mentioned affine rescaling action on \((A_1, A_1^*)\), and it leaves \((A_2, A_2^*)\) invariant.

### 3 Leonard pairs and parameter arrays

Leonard pairs are represented and classified by parameter arrays. More precisely, parameter arrays are in one-to-one correspondence with Leonard systems \([Ter06, Definition 3.2]\), and to each Leonard pair one associates 4 Leonard systems or parameter arrays. From now on, let \(d\) be a non-negative integer, and let \(V\) be a vector space with dimension \(d + 1\) over \(K\).

**Definition 3.1** By a parameter array over \(K\), of diameter \(d\), we mean a sequence

\[
(\theta_0, \theta_1, \ldots, \theta_d; \theta_0^*, \theta_1^*, \ldots, \theta_d^*; \varphi_1, \ldots, \varphi_d; \phi_1, \ldots, \phi_d) \tag{20}
\]

of scalars taken from \(K\), that satisfy the following conditions:

**PA1.** \(\theta_i \neq \theta_j\) and \(\theta_i^* \neq \theta_j^*\) if \(i \neq j\), for \(0 \leq i, j \leq d\).

**PA2.** \(\varphi_i \neq 0\) and \(\phi_i \neq 0\), for \(1 \leq i \leq d\).

**PA3.** \(\varphi_i = \phi_1 \sum_{j=0}^{i-1} \frac{\theta_j - \theta_{d-j}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*) (\theta_{i-1} - \theta_d), \) for \(1 \leq i \leq d\).

**PA4.** \(\phi_i = \phi_1 \sum_{j=0}^{i-1} \frac{\theta_j - \theta_{d-j}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*) (\theta_{d-i+1} - \theta_0), \) for \(1 \leq i \leq d\).

**PA5.** The expressions

\[
\frac{\theta_i - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_i^* - \theta_{i+1}}{\theta_{i-1}^* - \theta_i^*}
\]

are equal and independent of \(i\), for \(2 \leq i \leq d - 1\).
To get a Leonard pair from parameter array \((20)\), one must choose a basis for \(V\) and define the two linear transformations by the following matrices (with respect to that basis):

\[
\begin{pmatrix}
\theta_0 \\
1 & \theta_1 \\
1 & \theta_2 \\
\ddots & \ddots \\
1 & \theta_d
\end{pmatrix}, \quad \begin{pmatrix}
\theta^*_0 & \varphi_1 \\
\theta^*_1 & \varphi_2 \\
\theta^*_2 & \ddots \\
\ddots & \ddots \\
\theta^*_d & \varphi_d
\end{pmatrix}.
\tag{21}
\]

Alternatively, the following two matrices define an isomorphic Leonard pair on \(V\):

\[
\begin{pmatrix}
\theta_d \\
1 & \theta_{d-1} \\
1 & \theta_{d-2} \\
\ddots & \ddots \\
1 & \theta_0
\end{pmatrix}, \quad \begin{pmatrix}
\theta^*_0 & \phi_1 \\
\theta^*_1 & \phi_2 \\
\theta^*_2 & \ddots \\
\ddots & \ddots \\
\theta^*_d & \phi_d
\end{pmatrix}.
\tag{22}
\]

Conversely, if \((A, A^*)\) is a Leonard pair on \(V\), there exists \(1\)er06 Section 21\] a basis for \(V\) with respect to which the matrices for \(A, A^*\) have the bidiagonal forms in (21), respectively. There exists another basis for \(V\) with respect to which the matrices for \(A, A^*\) have the bidiagonal forms in (22), respectively, with the same scalars \(\theta_0, \theta_1, \ldots, \theta_d; \theta^*_0, \theta^*_1, \ldots, \theta^*_d\). Then the following 4 sequences are parameter arrays of diameter \(d\):

\[
(\theta_0, \theta_1, \ldots, \theta_d; \theta^*_0, \theta^*_1, \ldots, \theta^*_d; \varphi_1, \ldots, \varphi_d; \phi_1, \ldots, \phi_d),
\tag{23}
\]

\[
(\theta_0, \theta_1, \ldots, \theta_d; \theta^*_0, \theta^*_1, \ldots, \theta^*_d; \phi_1, \ldots, \phi_d; \varphi_1, \ldots, \varphi_d),
\tag{24}
\]

\[
(\theta_d, \ldots, \theta_1, \theta_0; \theta^*_d, \theta^*_1, \ldots, \theta^*_0; \varphi_d, \ldots, \varphi_1; \phi_d, \ldots, \phi_1),
\tag{25}
\]

\[
(\theta_d, \ldots, \theta_1, \theta_0; \theta^*_d, \theta^*_1, \ldots, \theta^*_0; \varphi_d, \ldots, \varphi_1; \phi_d, \ldots, \phi_1).
\tag{26}
\]

If we apply to any of these parameter arrays the construction above, we get back a Leonard pair isomorphic to \((A, A^*)\). These are all parameter arrays which correspond to \((A, A^*)\) in this way.

The parameter arrays in (23)–(26) are related by permutations. The permutation group is isomorphic to \(\mathbb{Z}_2 \times \mathbb{Z}_2\). The group action is without fixed points, since the eigenvalues \(\theta_i\)'s (or \(\theta^*_i\)'s) are distinct. Let \(\downarrow\) and \(\downarrow\downarrow\) denote the permutations which transform (23) into, respectively, (24) and (25). Observe that the composition \(\downarrow\downarrow\downarrow\) transforms (23) into (26). We refer to the permutations \(\downarrow\), \(\downarrow\downarrow\) and \(\downarrow\downarrow\downarrow\) as relation operators, because in \(1\)er06 Section 4\] the parameter arrays in (23)–(26) corresponding to \((A, A^*)\) and the 4 similar parameter arrays corresponding to the Leonard pair \((A^*, A)\) are called relatives of each other.
Parameter arrays are classified by Terwilliger in [Ter05]; alternatively, see [Ter06, Section 35]. For each parameter array, certain orthogonal polynomials naturally occur in entries of the transformation matrix between two bases characterized in Definition 4.1 for the corresponding Leonard pair. Terwilliger’s classification largely mimics the terminating branch of orthogonal polynomials in the Askey-Wilson scheme [KS94]. Specifically, the classification comprises Racah, Hahn, Krawtchouk polynomials and their q-versions, plus Bannai-Ito and orphan polynomials. Classes of parameter arrays can be identified by the type of corresponding orthogonal polynomials; we refer to them as Askey-Wilson types. The type of a parameter array is unambiguously defined if \( d \geq 3 \). We recapitulate Terwilliger’s classification in Section 4 by giving general normalized parameter arrays of each type.

By inspecting Terwilliger’s general parameter arrays [Ter06, Section 35], one can observe that the relation operators \( \downarrow \), \( \downarrow \downarrow \) do not change the Askey-Wilson type of a parameter array (but only the free parameters such as \( q, h, h^* \), \( s \) there), except that the \( \downarrow \) and \( \downarrow \downarrow \) relations mix up the quantum \( q \)-Krawtchouk and affine \( q \)-Krawtchouk types. Consequently, given a Leonard pair, all 4 associated parameter arrays have the same type, except when parameter arrays of the quantum \( q \)-Krawtchouk or affine \( q \)-Krawtchouk type occur. Therefore we can use the same classifying terminology for Leonard pairs, except that we have to merge the quantum \( q \)-Krawtchouk and affine \( q \)-Krawtchouk types.

Expressions for Askey-Wilson coefficients in terms of parameter arrays are given in [TV03, Theorem 4.5 and Theorem 5.3] and [Vid06, formulas (11)-(23)]. For example, we have:

\[
\beta + 1 = \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}, \quad \text{for } i = 2, \ldots, d - 1; \tag{27}
\]
\[
\gamma = \theta_{i-1} - \beta \theta_i + \theta_{i+1}, \quad \text{for } i = 1, \ldots, d - 1; \tag{28}
\]
\[
\gamma^* = \theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^*, \quad \text{for } i = 1, \ldots, d - 1; \tag{29}
\]
\[
\varrho = \theta_i^2 - \beta \theta_{i-1}^2 + \theta_{i+1}^2 - \gamma (\theta_i + \theta_{i-1}), \quad \text{for } i = 1, \ldots, d; \tag{30}
\]
\[
\varrho^* = \theta_i^2 - \beta \theta_{i-1}^2 + \theta_{i+1}^2 - \gamma^* (\theta_i^* + \theta_{i-1}^*), \quad \text{for } i = 1, \ldots, d. \tag{31}
\]

In principle, these equations can be used to compute parameter arrays (and consequently, Leonard pairs) satisfying fixed Askey-Wilson relations. For instance, one can use (28), (29) to eliminate consequently \( \theta_2, \ldots, \theta_d \) and \( \theta_{d-1}^*, \ldots, \theta_1^* \). Each solution of obtained equations represents a parameter array in general. Since we are interested in counting Leonard pairs rather than parameter arrays, we would get \( 5 \times 4 = 20 \) solutions in general. To get an equation system whose solutions correspond directly to Leonard pairs, one should find \( \downarrow \downarrow \)-invariant equations and rewrite them in terms of invariants of the \( \downarrow \downarrow \)-action. Examples of such invariants are, for \( i = 0, 1, \ldots, \left\lfloor \frac{d-1}{2} \right\rfloor \):

\[
\theta_i + \theta_{d-i}, \quad \theta_i \theta_{d-i}, \quad \varphi_i (\theta_{d-i+1} - \theta_{d-i}) + \varphi_{d-i+1} (\theta_i - \theta_{i-1}), \quad \theta_i^* + \theta_{d-i}^*.
\]

The Askey-Wilson coefficients are invariants as well. These direct equations can be investigated and solved if \( d \) is fixed and small. In general, it seems that one cannot avoid...
use of explicit solutions of recurrence relations such as (28)–(29), which basically leads to classification of parameter arrays. Therefore we openly use Terwilliger’s classification. In Section 4 we present general normalized parameter arrays and Askey-Wilson relations for them.

**Remark 3.2** For non-generic instances of Askey-Wilson relations, the number of distinct Leonard pairs is smaller than the respective generic number in Table 1. Within intersection theory (or moduli space) philosophy, there may be following “reasons” for this:

- Some solutions of a defining equation system represent “degenerate” objects rather than genuine Leonard pairs. In our situation, degenerate objects are represented by “parameter arrays” which do not satisfy the conditions PA1 and PA2 of Definition 3.1.
- General Leonard pairs in parametrized families are supposed to be generically different and non-isomorphic, but they may coincide or be isomorphic for special values of the parameters, or for special instances of Askey-Wilson relations. In these situations, one can assign a *multiplicity* to each solution so that multiplicities of all solutions add up to the generic number. Multiplicities should be defined by considering the defining equation system locally, or by an appropriate infinitesimal deformation of the parameters.

Example 6.3 here below presents instances of these situations. More generally, we may expect other two standard complications:

- Some “missing” solutions are at the “infinity” (or more technically, on a compactification of the “moduli space” of possible Leonard pairs). We do not need this interpretation within each Askey-Wilson type, unless we wish to have the most generic number of 5 Leonard pairs each time.

- A specialized defining equation system defines an algebraic variety of positive dimension. In this case we would have infinitely many solutions, continuous families of them. But this situation is not actual to us. (Lemma 4.1 in [Vid06] suggests this situation for the Askey-Wilson relations with \( \beta = 2, \gamma = 0, \gamma^* = 0, \omega^2 = \rho \rho^* \), but then all solutions are degenerate if \( d \geq 3 \).)

## 4 Normalized Leonard pairs

Let \((A, A^*)\) denote a Leonard pair, and let \(c, c^*, t, t^*\) denote scalars in \( \mathbb{K} \). It is easy to see that if \( t \) and \( t^* \) are nonzero, then \((tA + c, t^*A^* + c^*)\) is a Leonard pair again. We identify here affine transformations (3) acting on Leonard pairs. A corresponding action on parameter arrays is the following:

\[
\theta_i \mapsto t \theta_i + c, \quad \theta^*_i \mapsto t^* \theta^*_i + c^*, \quad \varphi_i \mapsto t t^* \varphi_i, \quad \phi_i \mapsto t t^* \phi_i.
\] (32)
Using affine transformations we can normalize a parameter array into a convenient form. We use the following normalizations.

**Lemma 4.1** The general parameter arrays in [Ter06] Examples 35.2–35.13] can be normalized by affine transformations \[2\] to the following forms:

- **The \(q\)-Racah case**: \(\theta_i = s q^{d-2i} + \frac{q^{2i-d}}{s} ,\quad \theta_i^* = s^* q^{d-2i} + \frac{q^{2i-d}}{s^*} \). 
  \[
  \varphi_i = \frac{q^{2d+2-4i}}{s s^* r} (1 - q^{2i}) (1 - q^{2i-2d-2}) (s s^* - r q^{2i-d}) (s^* r - q^{2i-d}) , \]
  \[
  \phi_i = \frac{q^{2d+2-4i}}{s s^* r} (1 - q^{2i}) (1 - q^{2i-2d-2}) (s^* r - q^{2i-d}) (s - s r q^{2i-d}) . \]

- **The \(q\)-Hahn case**: \(\theta_i = r q^{d-2i} ,\quad \theta_i^* = s^* q^{d-2i} + \frac{q^{2i-d}}{s} ,\)
  \[
  \varphi_i = \frac{q^{2d+2-4i}}{r} (1 - q^{2i}) (1 - q^{2i-2d-2}) (s^* r^2 - q^{2i-d}) , \]
  \[
  \phi_i = \frac{q^{d+1-2i}}{r s} (1 - q^{2i}) (1 - q^{2i-2d-2}) (s^* - r^2 q^{2i-d}) . \]

- **The dual \(q\)-Hahn case**: \(\theta_i = s q^{d-2i} + \frac{q^{2i-d}}{s} ,\quad \theta_i^* = r q^{d-2i} ,\)
  \[
  \varphi_i = \frac{q^{2d+2-4i}}{r s} (1 - q^{2i}) (1 - q^{2i-2d-2}) (s r^2 - q^{2i-d}) , \]
  \[
  \phi_i = \frac{q^{2d+2-4i}}{r s} (1 - q^{2i}) (1 - q^{2i-2d-2}) (r^2 - s q^{2i-d}) . \]

- **The \(q\)-Krawtchouk**: \(\theta_i = q^{d-2i} ,\quad \theta_i^* = s^* q^{d-2i} + \frac{q^{2i-d}}{s^*} ,\)
  \[
  \varphi_i = \frac{1}{s^*} (1 - q^{2i}) (1 - q^{2i-2d-2}) , \]
  \[
  \phi_i = \frac{1}{s^*} (1 - q^{2i}) (1 - q^{2i-2d-2}) . \]

- **The dual \(q\)-Krawtchouk**: \(\theta_i = s q^{d-2i} + \frac{q^{2i-d}}{s} ,\quad \theta_i^* = q^{d-2i} ,\)
  \[
  \varphi_i = s q^{2d+2-4i} (1 - q^{2i}) (1 - q^{2i-2d-2}) , \]
  \[
  \phi_i = \frac{q^{2d+2-4i}}{s} (1 - q^{2i}) (1 - q^{2i-2d-2}) . \]

- **The quantum \(q\)-Krawtchouk**: \(\theta_i = r q^{2i-d} ,\quad \theta_i^* = r q^{d-2i} ,\)
  \[
  \varphi_i = \frac{q^{d+1-2i}}{r} (1 - q^{2i}) (1 - q^{2i-2d-2}) , \]
  \[
  \phi_i = \frac{q^{2d+2-4i}}{r} (1 - q^{2i}) (1 - q^{2i-2d-2}) (r^3 - q^{2i-d}) . \]
• The affine $q$-Krawtchouk: $\theta_i = r q^{d-2i}$, $\theta_i^* = r q^{d-2i}$,
  \[ \varphi_i = \frac{q^{2d+2-4i}}{r} \left( 1 - q^{2i} \right) \left( 1 - q^{2i-d-2} \right) \left( \nu^3 - q^{2i-d-1} \right), \]
  \[ \phi_i = -\frac{q^{d+1-2i}}{r} \left( 1 - q^{2i-d-2} \right). \]

• The Racah case: $\theta_i = (i + u)(i + u + 1)$, $\theta_i^* = (i + u^*)(i + u^* + 1)$,
  \[ \varphi_i = i(i-d-1)(i + u + u^* - v)(i + u^* + d + 1 + v), \]
  \[ \phi_i = i(i-d-1)(i - u + u^* + v)(i - u + u^* - d - 1 - v). \]

• The Hahn case: $\theta_i = i + v - \frac{d}{2}$, $\theta_i^* = (i + u^*)(i + u^* + 1)$,
  \[ \varphi_i = i(i-d-1)(i + u^* + 2v), \]
  \[ \phi_i = -i(i-d-1)(i + u^* - 2v). \]

• The dual Hahn case: $\theta_i = (i + u)(i + u + 1)$, $\theta_i^* = i + v - \frac{d}{2}$,
  \[ \varphi_i = i(i-d-1)(i + u + 2v), \]
  \[ \phi_i = i(i-d-1)(i - u + 2v - d - 1). \]

• The Krawtchouk case: $\theta_i = i - \frac{d}{2}$, $\theta_i^* = i - \frac{d}{2}$,
  \[ \varphi_i = v i(i-d-1), \]
  \[ \phi_i = (v - 1)i(i-d-1). \]

• The Bannai-Ito case: $\theta_i = (-1)^i(i + u - \frac{d}{2})$, $\theta_i^* = (-1)^i(i + u^* - \frac{d}{2})$,
  \[ \varphi_i = \begin{cases} 
  -i(i + u + u^* - v - \frac{d+1}{2}), & \text{for } i \text{ even, } d \text{ even.} \\
  -(i-d-1)(i + u + u^* - v - \frac{d+1}{2}), & \text{for } i \text{ odd, } d \text{ even.} \\
  -i(i-d-1), & \text{for } i \text{ even, } d \text{ odd.} \\
  v^2(i + u + u^* - \frac{d+1}{2})^2, & \text{for } i \text{ odd, } d \text{ odd.} 
 \end{cases} \]
  \[ \phi_i = \begin{cases} 
  i(i - u + u^* - v - \frac{d+1}{2}), & \text{for } i \text{ even, } d \text{ even.} \\
  (i-d-1)(i - u + u^* + v - \frac{d+1}{2}), & \text{for } i \text{ odd, } d \text{ even.} \\
  -i(i-d-1), & \text{for } i \text{ even, } d \text{ odd.} \\
  v^2(i - u + u^* - \frac{d+1}{2})^2, & \text{for } i \text{ odd, } d \text{ odd.} 
 \end{cases} \]

In each case, $q, s, s^*, r$ are nonzero scalar parameters, or $u, u^*, v$ are scalar parameters, such that $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ for $0 \leq i < j \leq d$, and $\varphi_i \neq 0$, $\phi_i \neq 0$ for $1 \leq i \leq d$. 
Proof. These results are identical to the joint results of Lemma 6.1 and Lemma 7.1 in [Vid06]. (Compared with the parameter arrays in [Ter06], one notable substitution is $q \mapsto q^2$. For example, to get the normalized $q$-Racah parameter array from the general parameter array in (10)–(13), one may substitute $q \mapsto q^2$, $s \mapsto 1/s^2q^{2d+2}$, $s^* \mapsto 1/s^*q^{2d+2}$, $r \mapsto r/ss^*Q_{d+1}$ and adjust $\theta_0, \theta_0^*, h, h^*$ by affine scalings.) □

Affine transformations (3) act on Askey-Wilson relations as well. They do not change the number of Leonard pairs with the same Askey-Wilson relations. Hence it is enough to consider our problem for a set of normalized Askey-Wilson relations. Possible normalizations are discussed in [Vid06, Sections 4 and 8]. Askey-Wilson relations satisfied by at least one Leonard pair can be normalized as follows.

**Lemma 4.2** Let $\text{AW}(\beta, \gamma, \gamma^*, \varrho, \omega, \eta, \eta^*)$ denote a pair of Askey-Wilson relations satisfied by a Leonard pair. The relations can be uniquely normalized by affine translation $(A, A^*) \mapsto (A + c, A^* + c^*)$ as follows:

1. If $\beta \neq 2$, we can set $\gamma = 0, \gamma^* = 0$.
2. If $\beta = 2, \gamma \neq 0, \gamma^* \neq 0$, we can set $\varrho = 0, \varrho^* = 0$.
3. If $\beta = 2, \gamma = 0, \gamma^* \neq 0$, we can set $\varrho^* = 0, \omega = 0$.
4. If $\beta = 2, \gamma^* = 0, \gamma \neq 0$, we can set $\varrho = 0, \omega = 0$.
5. If $\beta = 2, \gamma = 0, \gamma^* = 0$ we can set $\eta = 0, \eta^* = 0$.

After the translation normalization, each of the two sequences

\[
(\gamma, \varrho, \eta, \eta^*) \quad \text{and} \quad (\gamma^*, \varrho^*, \eta^*, \eta)
\]

contains a nonzero Askey-Wilson coefficient. By affine scaling $(A, A^*) \mapsto (tA, tA^*)$ one can put the first nonzero coefficients in both sequences to any convenient nonzero values.

Proof. The normalization by affine translations follows from [Vid06, Lemma 4.1 and Part 3 of Theorem 8.1]. Note that parts 6 and 7 of [Vid06, Lemma 4.1] do not apply. Normalization by affine scaling follows from [Vid06, Lemma 5.2 (or Lemma 6.2) and Lemma 7.2]. Expression (33) is the same as [Vid06 formula (53)]. □

The Askey-Wilson relations for the parameter arrays of Lemma 4.1 are normalized according to the specifications of Lemma 4.2. The following lemma presents those Askey-Wilson relations. The first nonzero parameters in the two sequences (33) are normalized as in [Vid06 formula (54)], to the following values:

\[
\begin{align*}
\gamma, \gamma^* & : 2 \quad (\text{if } \beta = 2); \\
\varrho, \varrho^* & : \begin{cases} 
4 - \beta^2, & \text{if } \beta \neq \pm 2, \\
1, & \text{if } \beta = \pm 2;
\end{cases} \\
\eta, \eta^* & : \begin{cases} 
\sqrt{\beta + 2}(\beta - 2), & \text{if } \eta\eta^* \neq 0 \text{ or } \omega = 0, \\
\sqrt{\beta - 2}Q_{d+1}, & \text{if } \eta\eta^* = 0 \text{ and } \omega \neq 0.
\end{cases}
\end{align*}
\]
We should identify $\sqrt{\beta + 2} = q + 1/q$. This normalization of Askey-Wilson relations is not unique, and (in the $q$-cases) there may be two alternative normalizations with different signs of $\sqrt{\beta + 2}$; see [Vid06 Section 9].

**Lemma 4.3** As in the previous lemma, let $q, s, s^*, r$ denote nonzero scalar parameters, and $u, u^*, v$ denote scalar parameters. We use the following notations:

$$Q_j = q^j + q^{-j}, \quad Q_j^* = q^j - q^{-j}, \quad \text{for } j = 1, 2, \ldots,$$  

$$S = s + \frac{1}{s}, \quad S^* = s^* + \frac{1}{s^*}, \quad R = r + \frac{1}{r}.$$

The Askey-Wilson relations for the parameter arrays of Lemma 4.1 are:

- **For the** $q$-Racah case:

\[ AW(Q_2, 0, 0, -Q_2^* q^2, -Q_2^* q^2, -Q_1^* q^2 (S S^* + Q_{d+1} R)), \quad Q_1 Q_1^* (S R + Q_{d+1} S^*)^2, \quad Q_1 Q_1^* (S^* R + Q_{d+1} S^*). \]  

- **For the** $q$-Hahn case:

\[ AW(Q_2, 0, 0, 0, -Q_2^* q^2, -Q_2^* q^2, -Q_1^* q^2 (S^* r + Q_{d+1} r^{-1}), \quad Q_1 Q_1^* (S r^{-1} + Q_{d+1} r)^2). \]

- **For the dual** $q$-Hahn case:

\[ AW(Q_2, 0, 0, -Q_2^* q^2, 0, -Q_2^* q^2 (S r + Q_{d+1} r^{-1}), \quad Q_1 Q_1^* (S^* r^{-1} + Q_{d+1} r)^2, \quad Q_1 Q_1^*). \]

- **For the** $q$-Krawtchouk case:

\[ AW(Q_2, 0, 0, 0, -Q_2^* q^2, -Q_1^* q^2 S^*, 0, Q_1 Q_1^* Q_{d+1}). \]

- **For the dual** $q$-Krawtchouk case:

\[ AW(Q_2, 0, 0, -Q_2^* q^2, 0, -Q_1^* q^2 S^*, Q_1 Q_1^* Q_{d+1}, 0). \]

- **For the quantum** $q$-Krawtchouk and **affine** $q$-Krawtchouk cases:

\[ AW(Q_2, 0, 0, 0, -Q_1^* q^2 (r^2 + Q_{d+1} r^{-1}), Q_1 Q_1^* q^2, Q_1 Q_1^*). \]

- **For the** Racah case:

\[ AW(2, 2, 0, 0, -2u^2 - 2u^2 - 2v^2 - 2(d+1)(u + u^* + v) - 2d^2 - 4d, \quad 2 u (u + d + 1)(v - u^*) (v + u^* + d + 1), \quad 2 u^*(u^* + d + 1) (v - u) (v + u + d + 1). \]
• For the Hahn case:
\[ AW(2, 0, 2, 1, 0, 0, -(u^* + 1)(u^* + d) - 2v^2 - \frac{d^2}{4}, -4u^*(u^* + d + 1)v) \]. (44)

• For the dual Hahn case:
\[ AW(2, 2, 0, 0, 1, 0, -4u(u + d + 1)v, -(u + 1)(u + d) - 2v^2 - \frac{d^2}{4}, -4u^*(u + d + 1)v) \]. (45)

• For the Krawtchouk case:
\[ AW(2, 0, 0, 1, 1, 2v - 1, 0, 0). \] (46)

• For the Bannai-Ito case, if \( d \) is even:
\[ AW(-2, 0, 0, 1, 1, 4uu^* - 2(d+1)v, 2uv - (d+1)u^*, 2u^*v - (d+1)u) \]. (47)

• For the Bannai-Ito case, if \( d \) is odd:
\[ AW(-2, 0, 0, 1, 1, -2u^2 - 2u^*^2 + 2v^2 + \frac{(d+1)^2}{4}, -u^2 + u^*^2 - v^2 + \frac{(d+1)^2}{4}, u^2 - u^*^2 - v^2 + \frac{(d+1)^2}{4}). \] (48)

**Proof.** These results are identical to the joint results of Lemma 6.2 and Lemma 7.2 in [Vid06]. ⊤

The Askey-Wilson type can be defined for Askey-Wilson relations so that type nominations for Leonard pairs and Askey-Wilson relations are consistent; see [Vid06, Section 8]. The classification of Askey-Wilson relations is largely recapitulated by the first and third columns of Table [2].

An important question for us is the following. If we take concrete Askey-Wilson relations normalized according to Lemma 4.2 and formulas (34), are all Leonard pairs satisfying them representable by parameter arrays of Lemma 4.1? The following lemma settles this question.

**Lemma 4.4** 1. Any Leonard pair satisfying normalized Askey-Wilson relations can be represented by a normalized parameter array, except when the Askey-Wilson type is Bannai-Ito, and \( d \) is odd.

2. Suppose that \( d \) is odd. Let \((B, B^*)\) denote the Leonard pair represented by the parameter array of the Bannai-Ito type in Lemma 4.1. Then the following four Leonard pairs satisfy normalized Askey-Wilson relations of the Bannai-Ito type:
\[ (B, B^*), \ (-B, B^*), \ (B, -B^*), \ (-B, -B^*). \] (49)

Of these Leonard pairs, only \((B, B^*)\) can be represented by a normalized parameter array.

**Proof.** These are the results of Lemmas 9.6 in [Vid06]. The crucial observation is that the Bannai-Ito parameter array of Lemma 4.1 has the even-indexed \( \theta_i \)'s and the even indexed \( \theta_i^* \)'s in the increasing order; when \( d \) is odd, the relations operations \( \downarrow, \downarrow \) preserve this property, while affine scalings by \(-1\) reverse it. ⊤
5 Correctness of Table 1

Recall that we assume $d \geq 3$. By part 1 of [Vid06, Theorem 8.1], all Leonard pairs satisfy Askey-Wilson relations of their own Askey-Wilson type. Therefore, we prove correctness of Table 1 by considering Askey-Wilson relations of different types separately; in each case we look only for Leonard pairs of the same Askey-Wilson type.

As mentioned just before Lemma 4.2, it is enough to consider only normalized Askey-Wilson relations. By Lemma 4.4, all Leonard pairs satisfying normalized Askey-Wilson relations are representable by parameter arrays of Lemma 4.1, except when the Askey-Wilson type is Bannai-Ito and $d$ is odd. In all cases except the Bannai-Ito case with odd $d$, each Leonard pair solution of normalized Askey-Wilson relations is representable by a normalized parameter array. In these cases, we just assume free values of non-normalized coefficients in the Askey-Wilson relations of Lemma 4.3, equate those free values to the coefficient expressions in the free parameters (such as $s, s^*, r$ or $u, u^*, v$) of the corresponding general parameter array, and count solutions of obtained algebraic equations. We should take care of the fact that representation of normalized Leonard pairs by normalized parameter arrays is usually not unique.

If $\beta \neq \pm 2$, we have 4 possibilities for $q$. They are related by the substitutions $q \mapsto -q$, $q \mapsto 1/q$ and $q \mapsto -1/q$. We may consider $q$ fixed, because Tables 3 and 4 in [Vid06] show the following. If a Leonard pair is represented by a $q$-parameter array of Lemma 4.1 then it can be represented by a parameter array of Lemma 4.1 with $q$ replaced by $1/q$ as well, and such a replacement always yields an isomorphic Leonard pair. In the $q$-Racah and, for even $d$, the $q$-Krawtchouk and dual $q$-Krawtchouk cases, the same holds for the substitution $q \mapsto -q$. In the other $q$-cases, the substitution $q \mapsto -q$ leads to alternatively normalized Askey-Wilson relations (with the other sign of $\sqrt{\beta + 2}$). In any case, it is enough to count parameter arrays for one $q$-possibility.

Other transformations of normalized parameter arrays that preserve Leonard pairs are substitutions of their free parameters that leave the parameter arrays invariant, or realize the $\downarrow \downarrow$-relation operators. Discarding the substitutions which change $q$, these transformations are given in Table 2. The algebraic equations in the free parameters should be rewritten in invariants of these transformations. Examples of these invariants (for appropriate cases) are the expressions $S, S^*, R$ in (50).

In each Askey-Wilson case we ought to check whether solutions are generally non-degenerate. For this, one can check generic irreducibility (over the ring generated by free parameters) of the equation systems, or check that degenerate solutions form subvarieties with lower dimension. For fixed $\beta \neq \pm 2$, generically degenerate Leonard pairs occur only if $\beta = 2 \cos \pi / j$ for some $j \in \{1, 2, \ldots, d\}$, so that we have $q^{2j} = 1$.

From here we consider all normalized Askey-Wilson relations case by case. We use the notation of Lemma 4.3. Also, we denote

$$U = \left( u + \frac{d+1}{2} \right)^2, \quad U^* = \left( u^* + \frac{d+1}{2} \right)^2, \quad V = \left( v + \frac{d+1}{2} \right)^2.$$  \hspace{1cm} (50)

In the $q$-Racah case, we introduce the following indeterminants:

$$x = \frac{S}{Q_{d+1}}, \quad y = \frac{S^*}{Q_{d+1}}, \quad z = \frac{R}{Q_{d+1}}.$$  \hspace{1cm} (51)
Askey-Wilson type | Parameter array stays invariant | Conversion to relatives
--- | --- | ---
$q$-Racah | $r \mapsto 1/r$ | $s \mapsto 1/s$ | $s^* \mapsto 1/s^*$
$q$-Hahn | — | — | $s^* \mapsto 1/s^*$
Dual $q$-Hahn | — | $s \mapsto 1/s$ | —
$q$-Krawtchouk | — | — | $s^* \mapsto 1/s^*$
Dual $q$-Krawtchouk | — | $s \mapsto 1/s$ | —
Racah | $v \mapsto -v - d - 1$ | $u \mapsto -u - d - 1$ | $u^* \mapsto -u^* - d - 1$
Hahn | — | — | —
Dual Hahn | — | $u \mapsto -u - d - 1$ | —
Bannai-Ito, $d$ odd | $v \mapsto -v$ | $u \mapsto -u$ | $u^* \mapsto -u^*$

Table 2: Reparametrizations preserving Leonard pairs

They are invariant under the relevant transformations of Table 2. Equating the non-normalized Askey-Wilson coefficients gives the equations

$$\begin{align*}
xy + z &= C_1, \\
xz + y &= C_2, \\
yz + x &= C_3,
\end{align*}$$

where

$$C_1 = -\frac{\omega}{Q_1^2 Q_{d+1}^2}, \quad C_2 = \frac{\eta}{Q_1 Q_1^2 Q_{d+1}^2}, \quad C_3 = \frac{\eta^*}{Q_1 Q_1^2 Q_{d+1}^2}.$$  

Elimination of $y, z$ from (52) gives the degree 5 equation

$$(x - C_3)(x^2 - 1)^2 + C_1 C_2 (x^2 - 1) - (C_1^2 + C_2^2) x = 0. \quad (53)$$

Each solution gives exactly one Leonard pair satisfying the normalized Askey-Wilson relations $AW(q + q^{-1}, 0, 0, -Q_2^2, -Q_2'^2, \omega, \eta, \eta^*)$. There are more solutions in terms of $(s, s^*, r)$, but distinct Leonard pairs come from distinct $(x, y, z)$. The polynomial in (53) does not have multiple roots (in $x$) in general. Hence the generic number of Leonard pairs is 5.

In the $q$-Hahn case, invariant variables are $S^*$, $r$, and free Askey-Wilson coefficients are $\omega, \eta^*$. Elimination of $S^*$ gives a polynomial of degree 4 in $r$, without multiple roots in general. The generic number of Leonard pairs is 4. The dual $q$-Hahn case is similar.

In the $q$-Krawtchouk case, we have the equation $\omega = -Q_1^2 S^*$ which obviously has one solution in $S^*$. The dual $q$-Krawtchouk case is similar.

For Askey-Wilson relations of the quantum/affine $q$-Krawtchouk case, we have a cubic equation in $r$. The solutions represent 3 Leonard pairs of the same type. The
Leonard pairs can be represented by parameter arrays of the quantum $q$-Krawtchouk type, or the affine $q$-Krawtchouk type.

In the Racah case, we use (50) and rewrite the equations as

\[
\omega = -2U - 2U^* - 2V - \frac{(d-1)(d+3)}{4},
\eta = 2\left(U - \frac{(d+1)^2}{4}\right)(V - U^*),
\eta^* = 2\left(U^* - \frac{(d+1)^2}{4}\right)(V - U).
\]

Here $U$, $U^*$, $V$ are invariants by Table 2. The degree of equations suggests the generic number $4 = 1 \cdot 2 \cdot 2$ of solutions. Elimination of two of the three invariants confirms this generic number.

In the Hahn case, we have the equations

\[
\eta = -U^* - 2v^2 - \frac{d^2 + 2d - 1}{4},
\eta^* = -4v\left(U^* - \frac{(d+1)^2}{4}\right).
\]

The invariants are $U^*$ and $v$. Elimination of $U^*$ gives a cubic equation in $v$:

\[
v^3 + \left(\frac{d}{2} + \frac{(d+2)}{4}\right)v - \eta^* = 0.
\]

Hence there are 3 Leonard pairs in general. The dual Hahn case is similar.

In the Krawtchouk case, we obviously have one solution.

In the Bannai-Ito case for even $d$, after setting

\[
x = -\frac{2u}{d+1}, \quad y = -\frac{2u^*}{d+1}, \quad z = -\frac{2v}{d+1},
\]

we arrive at an equation system of the same form as in (52), so the generic number of solutions is 5 as well.

In the Bannai-Ito case for odd $d$ we have to keep in mind part 2 of Lemma 4.4. The invariants under relevant transformations of Table 2 are $u^2$, $u^{*2}$, $v^2$. The expressions for $\omega$, $\eta$, $\eta^*$ in Lemma 4.3 are linear in these invariants, so there is only one solution representable by a parameter array of Lemma 4.1. But part 2 of Lemma 4.4 asserts that there are 4 Leonard pairs in total.

Correctness of Table 1 is proved.

6 More examples

First we reconsider the example in Section 2. The Askey-Wilson relations in (4) have the $q$-Racah type, so looking for normalized Leonard pairs satisfying them leads us to the equation system (52) with $\omega = 0$, $\eta = 0$, $\eta^* = 0$. The equation system has the following solutions:

\[
(x, y, z) \in \{(0, 0, 0), (1, 1, -1), (1, -1, 1), (-1, 1, 1), (-1, -1, -1)\}.
\]
These solutions correspond to the Leonard pairs in (19), respectively. Indeed, the Leonard pairs \((A_1, A_1^*)\) and \((A_2, A_2^*)\) can be obtained from the \(q\)-Racah parameter array of Lemma 4.1 by specializing, respectively,

\[ s = s^* = r = \sqrt{-1}, \quad \text{and} \quad s = s^* = q^{-d-1}, \quad r = -q^{-d-1}. \]  

With this identification, the Leonard pair \((A_1, A_1^*)\) is in the \([d^*0^*0d]\) basis in the terminology of [Ter02], while the Leonard pair \((A_2, A_2^*)\) is in the \([0^*d^*0d]\) basis.

**Example 6.1** Consider the Askey-Wilson relations \(AW(-2, 0, 0, 1, 0, 0, 0)\). Leonard pairs satisfying it have the Bannai-Ito type.

For even \(d\), we have the same solutions in terms of (57) as in (58). The solution \((0, 0, 0)\) corresponds to the Leonard pair \((B_1, B_1^*)\) defined by the following matrices.

The matrix for \(B_1\) is diagonal, with the following sequence of diagonal entries:

\[ -\frac{d}{2}, \quad \frac{d}{2} - 1, \quad 2 - \frac{d}{2}, \quad \frac{d}{2} - 3, \quad \ldots, \quad 1 - \frac{d}{2}, \quad \frac{d}{2}. \]  

The matrix for \(B_1^*\) is tridiagonal:

\[
F_1^* = \begin{pmatrix}
0 & \frac{d}{2} & 0 & \frac{d-1}{2} & \ldots & 1 & \frac{d-3}{2} & \ldots \\
\frac{1}{2} & 0 & \frac{d-1}{2} & 1 & \ldots & \frac{d-2}{2} & \ldots \\
& 1 & 0 & \frac{d-2}{2} & \ldots & \frac{d-3}{2} & \ldots \\
& & 1 & 0 & \frac{d-3}{2} & \ldots & \frac{d-4}{2} & \ldots \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & 1 & 0 & \frac{d-3}{2} & \ldots \\
& & & & & \frac{1}{2} & 0 & \frac{d-2}{2} \\
& & & & & & \frac{d}{2} & 0
\end{pmatrix}.
\]  

This matrix looks familiar from representation theory of the Lie algebra \(sl_2\). Let \(B_1^\wedge\) denote the diagonal matrix with the same set of diagonal entries as \(B_1\), but arranged in the increasing order. Then one can check that \((B_1^\wedge, B_1^*)\) is a Leonard pair of the Krawtchouk type. Up to scaling, this Leonard pair (with any \(d\)) occurs in [Go02].

For any \(d\), let \((B_2, B_2^*)\) denote the Leonard pair defined by the following matrices.

The matrix for \(B_2\) is diagonal, with the following sequence of diagonal entries:

\[ \frac{1}{2}, \quad -\frac{3}{2}, \quad \frac{5}{2}, \quad -\frac{7}{2}, \quad \ldots, \quad (-1)^d(d+\frac{1}{2}). \]  

The matrix for \(B_2^*\) is tridiagonal, with exactly one nonzero entry on the main diagonal:

\[
\begin{pmatrix}
\frac{d+1}{2} & \frac{d}{2} & 0 & \frac{d-1}{2} & \ldots & 1 & \frac{d-3}{2} & \ldots \\
\frac{d+2}{2} & 0 & \frac{d-1}{2} & \ldots & \frac{d-2}{2} & \ldots \\
\frac{d+3}{2} & 0 & \ldots & \frac{d-3}{2} & \ldots \\
\frac{d+4}{2} & 0 & \ldots & \frac{d-4}{2} & \ldots \\
\frac{d+5}{2} & 0 & \ldots & \frac{d-5}{2} & \ldots \\
\frac{d+6}{2} & 0 & \ldots & \frac{d-6}{2} & \ldots \\
\frac{d+7}{2} & 0 & \ldots & \frac{d-7}{2} & 1 & \frac{d-5}{2} & \ldots \\
\frac{d+8}{2} & 0 & \ldots & \frac{d-8}{2} & \frac{d}{2} & 0
\end{pmatrix}.
\]
It turns out that \((B_2, B_2^*)\), \((-B_2, B_2^*)\), \((-B_2, -B_2^*)\) satisfy the Askey-Wilson relations under consideration. For even \(d\), these Leonard pairs correspond to the other 4 solutions in \((63)\). To see the sign-flipping relation between corresponding parameter arrays of Lemma 4.1, one has to apply the \(\ddagger\)-operations. For odd \(d\), the solution representable by the parameter array in Lemma 4.1 is \((B_2, B_2^*)\), and then we should take into account part 2 of Lemma 4.4.

**Example 6.2** Here we consider Askey-Wilson relations of the Racah type with \(\omega = 0\), \(\eta = 0\), \(\eta^* = 0\). According to \((63)\), there are 4 normalized Leonard pairs satisfying these relations:

\[
(U, U^*, V) \in \left\{ \left( \begin{array}{ccc} (d+1)^2 \quad (d+1)^2 \quad 1-6d-3d^2 \\ 4 & 4 & \frac{1-6d-3d^2}{4} \\ (d+1)^2 & 1-6d-3d^2 & (d+1)^2 \end{array} \right), \right.
\]

\[
\left. \left( \begin{array}{ccc} 1-6d-3d^2 \quad (d+1)^2 \quad (d+1)^2 \\ 4 & 4 & 4 \end{array} \right), \right.
\]

\[
\left. \left( \begin{array}{ccc} -(d-1)(d+3) \quad -(d-1)(d+3) \\ 12 & 12 & 12 \end{array} \right), \right.
\]

\[
\left. \left( \begin{array}{ccc} -(d-1)(d+3) \quad -d(d+1) \\ 12 & 12 & 12 \end{array} \right) \right\}. \quad (64)
\]

Explicit diagonal-tridiagonal forms can be obtained as follows. (They are not necessarily normalized to standard diagonal-tridiagonal forms.) Let \(F_1\) be the diagonal matrix with the following diagonal entries:

\[
0, \quad 2, \quad 6, \quad 12, \quad 20, \quad \ldots, \quad d(d+1). \quad (65)
\]

Let \(F_2\) denote the tridiagonal matrix with the following entries on the superdiagonal, the main diagonal and the subdiagonal, respectively:

\[
\frac{i-d-1}{2} \left( j^2 + (d+1)j + d^2 + 2d \right), \quad j = 1, \ldots, d; \quad (66)
\]

\[
-1, \quad -3, \quad -6, \quad -10, \quad \ldots, \quad -\frac{d(d+1)}{2}, \quad (67)
\]

\[
\frac{j+d+1}{2} \left( j^2 - (d+1)j + d^2 + 2d \right), \quad j = 1, \ldots, d. \quad (68)
\]

Let \(F_3^*\) denote the matrix with the same entries as \(F_2\), except that the entry in the upper-left corner is multiplied by \(-1\). Let \(\bar{u} = -\frac{d+1}{2} + \frac{1}{2}\sqrt{-\frac{(d-1)(d+3)}{3}}\). Let \(F_3\) be the diagonal matrix with the diagonal entries \((j + \bar{u})(j + \bar{u} + 1), j = 0, \ldots, 1\), and let \(F_4\) be the tridiagonal matrix with the following entries on the superdiagonal, the main diagonal and the subdiagonal, respectively:

\[
\frac{(j - d - 1)(j + 2\bar{u} + 3\bar{u} + d + 1)}{2(2j + 2\bar{u} - 1)}, \quad j = 1, \ldots, d; \quad (69)
\]

\[
-\frac{1}{2} (j + \bar{u})(j + \bar{u} + 1), \quad j = 0, \ldots, d; \quad (70)
\]

\[
\frac{j(j + 2\bar{u} + d + 1)(j - \bar{u} - d - 1)}{2(2j + 2\bar{u} + 1)}, \quad j = 1, \ldots, d. \quad (71)
\]

Then \((F_1, F_2), (F_3, F_2^*), (F_2^*, F_1), (F_3, F_4)\) are matrix pairs representing the 4 Leonard pairs.
Example 6.3 Here we consider Askey-Wilson relations of the Hahn type with \( \eta^* = 0 \). There must be solutions with \( v = 0 \) and with \( U^* = \frac{(d+1)^2}{4} \). We want all entries in the representing matrices to be in \( Q \), so we must have \( (u^* + \frac{d+1}{2})^2 - 2v^2 = \frac{(d+1)^2}{4} \).

Rational solutions of this equation can be parametrized with \( v = \frac{(d+1)t^2}{t^2 - 2} \), which gives the following family of Askey-Wilson relations:

\[
AW \left( 2, 0, 2, 1, 0, 0, \frac{1}{2} - \frac{(d+1)^2(t^4 + 4)}{2(t^2 - 2)^2}, 0 \right). 
\] (72)

The 3 Leonard pairs can represented by parameter arrays of Lemma 4.1 with \( (u^*, v) \in \left\{ \left( \frac{2(d+1)}{t^2 - 2}, 0 \right), \left( 0, \frac{(d+1)t}{t^2 - 2} \right), \left( 0, -\frac{(d+1)t}{t^2 - 2} \right) \right\} \). (73)

For \( t = 1 \) we have 3 solutions, as expected. They are representable (after the \( \downarrow \) operation) by

\[
(u^*, v) \in \{ (d + 1, 0), (0, d + 1), (0, -d - 1) \}. 
\] (74)

For \( t = 3 \), we have

\[
u^* \pm 2v \in \left\{ \frac{2(d+1)}{7}, \frac{9(d+1)}{7}, \frac{6(d+1)}{7}, \frac{13(d+1)}{7}, \frac{6(d+1)}{7}, \frac{(d+1)}{7} \right\}. 
\] (75)

If \( d + 1 \) is divisible by 7, then we have only one Leonard pair solution, because two other solutions have \( \varphi_i \varphi_i = 0 \) for \( i = \frac{(d+1)}{7} \) or \( i = \frac{6(d+1)}{7} \), so they are degenerate. A similar statement holds for \( t = 4 \).

For \( t = 0 \), all three solutions in (73) give the Leonard pair representable by the parameter array of Lemma 4.1 with \( (u^*, v) = (0, 0) \). So we have just one solution “of multiplicity 3”.

Example 6.4 Suppose that \( \xi \in \mathbb{C} \) satisfies \( \xi^{d+1} = 2 \), and consider the Askey-Wilson relations

\[
AW \left( \xi^2 + \xi^{-2}, 0, 0, 0, 0, -\frac{21(\xi - \xi^{-1})^2}{4}, (\xi + \xi^{-1})(\xi - \xi^{-1})^2, (\xi + \xi^{-1})(\xi - \xi^{-1})^2 \right). 
\]

Leonard pairs satisfying these relations can be represented by parameter arrays of the quantum \( q \)-Krawtchouk of the affine \( q \)-Krawtchouk types. There are 3 such Leonard pairs. To get affine \( q \)-Krawtchouk parameter arrays, one may take \( q = \xi \) so that \( Q_{d+1} = \frac{5}{2} \). The cubic equation is then \( r^3 + \frac{5}{2} = \frac{21}{4}r \). The solutions have \( r \in \{ 2, \frac{1}{2}, -\frac{3}{2} \} \).

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