Cellular Automata vs. Quasisturmian Shifts

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Abstract. If \( L = \mathbb{Z}^D \) and \( A \) is a finite set, then \( A^L \) is a compact space; a cellular automaton (CA) is a continuous transformation \( \Phi : A^L \to A^L \) that commutes with all shift maps. A quasisturmian (QS) subshift is a shift-invariant subset obtained by mapping the trajectories of an irrational torus rotation through a partition of the torus. The image of a QS shift under a CA is again QS. We study the topological dynamical properties of CA restricted to QS shifts, and compare them to the properties of CA on the full shift \( A^L \). We investigate injectivity, surjectivity, transitivity, expansiveness, rigidity, fixed/periodic points, and invariant measures. We also study 'chopping': how iterating the CA fragments the partition generating the QS shift.

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Let \( D \geq 1 \), and let \( L = \mathbb{Z}^D \) be the \( D \)-dimensional lattice. If \( A \) is a (discretely topologized) finite set, then \( A^L \) is compact in the Tychonoff topology. For any \( v \in L \), let \( \sigma^v : A^L \to A^L \) be the shift map: \( \sigma^v(a) = [b_\ell]_{\ell \in L} \), where \( b_\ell = a_{\ell-v} \), for all \( \ell \in L \). In particular, if \( D = 1 \), let \( \sigma = \sigma^1 \) be the left-shift on \( A^\mathbb{Z} \). A cellular automaton (CA) is a continuous map \( \Phi : A^L \to A^L \) which commutes with all shifts: for any \( \ell \in L \), \( \sigma^\ell \circ \Phi = \Phi \circ \sigma^\ell \). A result of Curtis, Hedlund, and Lyndon \[26\] says any CA is determined by a local map \( \phi : A^B \to A \) (where \( B \subset L \) is some finite subset), such that, for all \( \ell \in L \) and all \( a \in A^L \), if we define \( a|_{\ell+B} = [a_{\ell+b}]_{b \in B} \in A^{\ell+B} \), then \( \Phi(a)_{\ell} = \phi(a|_{\ell+B}) \).

In \[11\], Hof and Knill studied the action of CA on 'circle shifts', a class of quasiperiodic subshifts similar to Sturmian shifts \[25, 33\]. They showed that the image of circle subshift under a CA is again a circle shift, and raised two questions:

- Empirically, iterating the CA fragments the partition generating the circle shift. Why?
• Is the action of a CA injective when restricted to a circle subshift?

In this paper, we generalize [11] by studying the action of CA upon quasisturmian systems. In [11] we introduce notation and terminology concerning torus rotations, measurable partitions of tori, and the Besicovitch metric $d_B$ [20]. We introduce $\mathcal{A}^T$, the space of measurable partitions of a torus $T$, endowed with the symmetric difference metric $d_\Delta$. In [2] we introduce quasisturmian (QS) shifts, which are natural generalizations of the classical Sturmian shift [25, 33], obtained by tracking the trajectories of a torus rotation system $(T, \varsigma)$ through a fixed open partition of $T$. Likewise, a QS measure is obtained by projecting $\varsigma$-trajectories through a measurable partition of $T$. We refer to both QS shifts and QS measures as quasisturmian systems. A QS sequence is (roughly speaking) a ‘typical’ element of a QS shift.

In [3], we examine the action of a CA on a QS system, and generalize the results of [11], to show that any CA $\Phi$ induces a natural transformation $\Phi_\varsigma$ on $\mathcal{A}^T$ (Theorem 3.1). The ‘induced’ dynamical system $(\mathcal{A}^T, d_\Delta, \Phi_\varsigma)$ is a topological dynamical system, because $\Phi_\varsigma$ is Lipschitz relative to $d_\Delta$ (Proposition 3.4). There is a natural conjugacy between a subsystem of $(\mathcal{A}^T, d_\Delta, \Phi)$ and the system $(QS_\varsigma, d_B, \Phi)$ (Proposition 3.5).

In §4 contains auxiliary technical results for §5 and §6. In §5, we address Hof and Knill’s first question, and show how a partition is ‘chopped’ into many small pieces under iteration of $\Phi_\varsigma$. In §6, we address Hof and Knill’s second question, and show that, ‘generically’, a CA restricted to a QS system is injective (Theorem 6.1). The map $\Phi_\varsigma : \mathcal{A}^T \rightarrow \mathcal{A}^T$ is not generally surjective, but does have a $d_\Delta$-dense image in $\mathcal{A}^T$ if $\Phi$ is surjective (Proposition 5.2). In §6 we study QS fixed points, periodic points, and travelling wave solutions for $\Phi$.

§10 and §11 contain auxiliary machinery for §12, §13 and §15. In §12 we give an example of a CA which acts expansively on $(QS_\varsigma, d_B)$ (Proposition 12.1), thereby refuting a plausible conjecture arising from [20]. In §13 we show that linear cellular automata are either niltropic or rigid when restricted to a quasisturmian shift (Proposition 13.2); in §11 we use this to show that most linear CA have no quasisturmian invariant measures (Proposition 14.5). §15 constructs a class of quasisturmian measures which are not asymptotically randomized by a simple linear CA.

Sections §1, §2, §3, §4, §5, §6, §7, §8, §9, §12, §13, §14, §15 and §16 are logically independent. §14 depends on §13.

1. Preliminaries and Notation

If $\ell, n \in \mathbb{Z}$, then $[\ell \ldots n] = \{ m \in \mathbb{Z} : \ell \leq m < n \}$. If $(X, d)$ is a metric space and $x, y \in X$, then “$x \sim y$” means $d(x, y) \leq \epsilon$. Let $\mathcal{M}(X)$ be the space of (Borel) probability measures on $X$. If $\lambda \in \mathcal{M}(X)$, then “$\lambda$-a.e.” means “$\lambda$-almost every.” Likewise, “$\lambda$-ae” means “$\lambda$-almost everywhere”. A meager subset of $X$ is a nowhere dense set, or any set obtained through the countable union or intersection of other...
meager sets; it is the topological analog of a ‘set of measure zero’.
A **coneager** set is the complement of a meager set.
A statement holds ‘topologically almost everywhere’ (top.æ) on \(X\) if it holds for all points in a coneager subset.

A **topological dynamical system** (TDS) is a triple \((X, d, \varphi)\) where \((X, d)\) is a metric space and \(\varphi : X \to X\) is a continuous transformation. If \((X, d, \varphi)\) is another TDS, then a TDS **epimorphism** is a continuous surjection \(f : X \to X\) such that \(f \circ \varphi = \varphi \circ f\); if \(f\) is a homeomorphism, then we say \(f\) is a TDS **isomorphism**.

A **measure-preserving dynamical system** (MPDS) is a triple \((X, \mu, \varphi)\) where \((X, \mu)\) is a probability space and \(\varphi : X \to X\) is a measurable transformation such that \(\varphi(\mu) = \mu\). If \((X, \pi, \tilde{\varphi})\) is another MPDS, then an MPDS **epimorphism** is a measure-preserving map \(f : X \to X\) such that \(f \circ \varphi = \tilde{\varphi} \circ f\); if \(f\) is bijective \((\mu, \pi)\)-wise, then \(f\) is an MPDS **isomorphism**.

Let \(L = \mathbb{Z}^D\) be a lattice. A **topological \(L\)-system** (TLS) is a triple \((X, d, \varsigma)\) where \((X, d)\) is a metric space and \(\varsigma\) is a continuous \(L\)-action on \(X\); if \(\ell \in L\), then we write the action of \(\ell\) as \(\varsigma^\ell\). For example, if \(\mathcal{A}\) is a finite alphabet and \(X = \mathcal{A}^L\) is the space of all \(L\)-indexed **configurations** of elements in \(\mathcal{A}\), then \(L\) acts by on \(\mathcal{A}^L\) by shifts; we indicate the shift action by \(\sigma\), and \((\mathcal{A}^L, d_c, \sigma)\) is a TLS (where \(d_c\) is the Cantor metric –see below). A **subshift** is a (Cantor)-closed, \(\sigma\)-invariant subset \(\Sigma \subset \mathcal{A}^\ell\); then \((X, d_c, \sigma)\) is also a TLS.

A **measure-preserving \(L\)-system** (MPLS) is a triple \((X, \mu, \varsigma)\) where \((X, \mu)\) is a probability space and \(\varsigma\) is a \(\mu\)-preserving \(L\)-action on \(X\). We define (epi/iso)morphisms of TLS and MPLS in the obvious way. A measurable subset \(U \subset X\) is \(\varsigma\)-**invariant** if \(\mu\left[U \Delta \varsigma^\ell(U)\right] = 0\) for all \(\ell \in L\), and \((X, \mu, \varsigma)\) is **ergodic** if, for any \(\varsigma\)-invariant set \(U\), either \(\mu[U] = 0\) or \(\mu[U] = 1\). For any \(N > 0\), let \(B(N) = [-N..N]^D \subset L\). Thus, \(|B(N)| = (2N)^D\).

**Generalized Ergodic Theorem:** If \((X, \mu, \varsigma)\) is ergodic, and \(U \subset X\) is measurable, then for all \(\mu\) \(x \in X\) are \((\mu, \varsigma)\)-generic for \(U\), meaning that

\[
\mu[U] = \lim_{N \to \infty} \frac{1}{(2N)^D} \sum_{b \in B(N)} \mathbb{1}_U \left(\varsigma^b(x)\right). \tag{10} \tag{38}
\]

For example, if \(\mu \in \mathcal{M}(\mathcal{A}^L)\) is \(\sigma\)-**invariant** (ie. \(\sigma^\ell(\mu) = \mu\) for all \(\ell \in L\)) then \((\mathcal{A}^L, \mu, \sigma)\) is a MPLS. An element \(a \in \mathcal{A}^L\) is **\(\mu\)-generic** if \(a\) is \((\mu, \sigma)\)-generic for all cylinder sets of \(\mathcal{A}^L\).

**The Cantor and Besicovitch Metrics:** The standard (Tychonoff) topology on \(\mathcal{A}^L\) is induced by the **Cantor metric**:

For any \(p, q \in \mathcal{A}^L\),

\[
d_C(p, q) := 2^{-D(p, q)},
\]

where

\[
D(p, q) := \min \{|\ell| : \ell \in L, p_\ell \neq q_\ell\}.
\]

The topological dynamics of CA in this metric were characterized in \([29, 19]\). For our purposes, however, it is more appropriate to use the **Besicovitch metric** \([20, 24]\) (see also \([18, 23]\)), defined as follows. If \(J \subset L\), then the **Cesàro density** of \(J\) is

\[
\lim_{N \to \infty} \frac{1}{(2N)^D} \sum_{b \in B(N)} \mathbb{1}_J \left(\varsigma^b(x)\right).
\]
Torus Rotation Systems: Let $T^1 := \mathbb{R}/\mathbb{Z}$, which we normally identify with $[0, 1)$. Fix $K \geq 1$ and let $T := T^K = T^1 \times \cdots \times T^1$ be the $K$-torus. For any $s \in T$, let $\rho^s : T \ni t \mapsto (t + s) \in T$ be the corresponding rotation map. Suppose $\tau : L \to T$ is a group monomorphism; for any $\ell \in L$, let $\zeta^\ell = \rho^{\tau(\ell)}$ denote the corresponding rotation of $T$. This defines a measure-preserving, topological $L$-system $(T, d, \lambda, \zeta)$ (where $d$ is the usual metric and $\lambda$ is the Lebesgue measure on $T$). We call this a torus rotation system.

**Proposition 1.1.** Suppose $\tau : L \to T$ is a monomorphism with dense image. Then:

(a) $(T, d, \zeta)$ is minimal (ie. every $t \in T$ has dense $\zeta$-orbit) and uniquely ergodic (ie. $\lambda$ is the only $\zeta$-invariant probability measure).

(b) Let $U \subset T$ be open, with $\lambda([0U]) = 0$. Then every $t \in T$ is $(\lambda, \zeta)$-generic for $U$.

**Proof.** (a) Minimal: Suppose $Z := \tau(L)$ is dense in $T$. Then for any $t \in T$, the $\zeta$-orbit $\{\zeta^\ell(t)\}_{\ell \in L} = t + Z$ is also dense. *Uniquely Ergodic:* If $\mu$ is a $\zeta$-invariant probability measure on $T$, then $\rho^z(\mu) = \mu$ for every $z \in Z$. But $Z$ is dense in $T$, so (by a weak*-convergence argument), we get $\rho^z(\mu) = \mu$ for all $t \in T$. But the Haar measure $\lambda$ is the only probability measure on $T$ that is $\rho^s$-invariant for all $s \in T$ [22] Thm 10.14, p.317); hence $\mu = \lambda$.

(b) For any $s \in T$ and $W \subset T$, let $I(s, W) := \{\ell \in L ; \zeta^\ell(s) \in W\}$, and let $D(s, W) := \text{density}(I(s, W))$. We want to show that $D(t, U) = D(U, t)$.

**Claim 1.** $\lambda(U) \leq D(t, U)$.

**Proof.** For any $\delta > 0$, let $U_\delta := \{u \in U ; d(u, U^\delta) > \delta\}$. Then $\bigcup_{\delta > 0} U_\delta = U$ (because if $u \in U$, but $u \notin U_\delta$ for any $\delta > 0$, then there is no $\delta$-ball around $u$ contained in $U$, contradicting that $U$ is open). Thus, $\lim_{\delta \to 0} \lambda(U_\delta) = \lambda(U)$. 

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Given $\epsilon > 0$, find $\delta > 0$ such that $\lambda[U_\delta] > \lambda[U] - \epsilon$. Let $G := \{ t \in T; t$ is generic for both $U$ and $U_\delta \}$. The Generalized Ergodic Theorem says that $\lambda[G] = 1$; hence $G$ is dense in $T$. Find $g \in G$ with $g \sim t$. Thus, for any $\ell \in L$, $\langle \zeta^\ell(t) \in U_\delta \rangle \Rightarrow \langle \zeta^\ell(t) \in U \rangle$ (because $\zeta^\ell$ is an isometry). Thus, $I(g, U_\delta) \subseteq I(t, U)$. Hence $D(t, U) \geq D(g, U_\delta) = \lambda[U_\delta] > \lambda[U] - \epsilon$. Since $\epsilon$ is arbitrary, we conclude that $D(t, U) \geq \lambda[U]$. $\Diamond$  

Claim 1

Now, let $V = \text{int}(U^c)$. Then $V$ is also open, and

$$
\lambda[U] \leq D(t, U) \leq 1 - D(t, V) \leq 1 - \lambda[V] \quad \text{(i)}
$$

Thus $D(t, U) = \lambda[U]$, as desired. $(\dagger)$ is by Claim 1. $(\ast)$ is because $I(t, V)$ is disjoint from $I(t, U)$. $(\dagger)$ is because Claim 1 (applied to $V$) yields $D(t, V) \geq \lambda[V]$.

$(\circ)$ is because $(V \cup U)^c = \partial U$, so $\lambda[V] + \lambda[U] = \lambda[V \cup U] = 1 - \lambda[\partial U] = 1 - 0 = 1$. $\Box$

**Note:** We assume throughout this paper that the hypothesis of Proposition 1.1 is satisfied.

**Example 1.2.** Let $D = 1$, so $L = Z$. Let $K = 1$, so $T = T^1$. Identify $T$ with $[0, 1)$ in the obvious way. Let $a \in (0, 1)$ be irrational; define $\tau : Z \to T^1$ by $\tau(z) = z \cdot a$ (mod 1). Thus, $\zeta^\tau(t) = t + za, \forall z \in Z, \forall t \in T$. Thus, $(T^1, \lambda, \zeta)$ is an irrational rotation of a circle. $\Diamond$

**Measurable partitions:** An $(A\text{-labelled})$ measurable partition of $T$ is a finite collection of disjoint measurable sets $P = \{ P_a \}_{a \in A}$ so that, if $P_* := \bigsqcup_{a \in A} P_a$, then $\lambda(P_*) = 1$.

We’ll often treat the partition $P$ as a measurable function $P : P_* \to A$, where $P^{-1}\{a\} := P_a$. Let $A^T$ be the set of all measurable $A$-labelled partitions of $T$, which we topologize with the symmetric difference metric, defined:

$$
d_\triangle(P, Q) = \sum_{a \in A} \lambda(P_a \triangle Q_a), \quad \text{for any } P, Q \in A^T. \hspace{1cm} (1)
$$

$(A^T, d_\triangle)$ is a complete and bounded metric space, but not compact (Proposition 1.1).

An $A$-labelled open partition of $T$ is a finite family of disjoint open sets $P = \{ P_a \}_{a \in A}$, such that $P_*$ is a dense open subset of $T$, and $\lambda(P_*) = 1$. If $\triangle A^T$ is the set of $A$-labelled open partitions of $T$, then $\triangle A^T \subset A^T$, and $\triangle A^T$ is $d_\triangle$-dense in $A^T$ (Corollary 1.2).

**2. Quasisturmian systems**

Quasisturmian systems are $\sigma$-invariant subsets or measures in $A^\mathbb{Z}$ which generalize the classical Sturmian shift of $\{25, 33\}$ (see also $3, 5, 6, 7, 17, 39$).

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QUASISTURMIAN SHIFTS: Let $\mathcal{P} = \{\mathcal{P}_a\}_{a \in \mathcal{A}}$ be an open partition of $T$. Let $\mathcal{P}_* = \bigsqcup_{a \in \mathcal{A}} \mathcal{P}_a$, and let $\mathcal{T} = \mathcal{T}(\mathcal{P}) := \bigcap_{t \in \mathcal{L}} \varsigma^t(\mathcal{P}_*)$; then $\mathcal{T}$ is a $\varsigma$-invariant, dense $G_\delta$ subset of $T$, and $\lambda(\mathcal{T}) = 1$. (We will write $\mathcal{T}(\mathcal{P})$ as $\mathcal{T}$ when the partition $\mathcal{P}$ is clear from context.)

Example 2.1. Let $K = 1$ so $T \cong [0, 1)$ as in Example 1.2. Let $\mathcal{A} = \{0, 1\}$, and let $\mathcal{P}_0 = (0, a)$ and $\mathcal{P}_1 = (a, 1)$. Then $\mathcal{P}_* \cong (0, 1) \setminus \{a\}$, and $\mathcal{T} \cong [0, 1) \setminus \{a \cdot \mod 1 ; z \in \mathbb{Z}\}$. \hfill $\Box$

Recall that $\mathcal{P} : \mathcal{T} \rightarrow \mathcal{A}$ is defined by $\mathcal{P}^{-1}\{a\} = \mathcal{P}_a$. For any $t \in \mathcal{T}$, let $\mathcal{P}_\varsigma(t) \in \mathcal{A}^L$ be the $\mathcal{P}$-trajectory of $t$. That is, for all $\ell \in \mathcal{L}$, $\mathcal{P}_\varsigma(t)_{\ell} = \mathcal{P}(\varsigma^\ell(t))$. This defines a function $\mathcal{P}_\varsigma : \mathcal{T} \rightarrow \mathcal{A}_\mathbb{Z}$.

**Proposition 2.2.** Let $\mathcal{P} \in \mathcal{A}_T^*$ be a nontrivial open partition of $T$. Then:

(a) $\mathcal{P}_\varsigma \circ \varsigma^\ell = \sigma^\ell \circ \mathcal{P}_\varsigma$ for all $\ell \in \mathcal{L}$.

(b) $\mathcal{P}_\varsigma : T \rightarrow \mathcal{A}^\mathbb{Z}$ is continuous with respect to both the $d_C$ and $d_B$ metrics on $\mathcal{A}^\mathbb{Z}$.

**Proof.** (a) is by definition of $\mathcal{P}_\varsigma$.

(b) $d_C$-continuous: Let $M \subset L$ be any finite subset; we want a neighbourhood $U$ around $t$ such that, if $t' \in U$, then $p_m = p'_m$ for all $m \in M$. Let $Q = \bigvee_{m \in M} \varsigma^m(\mathcal{P})$. The atoms of $Q$ are finite intersections of open sets, hence open. Let $U$ be the $Q$-atom containing $t$. If $t' \in U$ then for all $m \in M$, $\mathcal{P}(\varsigma^m(t)) = \mathcal{P}(\varsigma^m(t'))$, i.e. $p_m = p'_m$, as desired.

$d_B$-continuous: [See also Proposition 1.1(b)] Fix $\epsilon > 0$. We want $\delta > 0$ such that, if $t \sim_\delta t'$, then $d_B(p, p') < \epsilon$. For any $\delta > 0$ let $\mathcal{P}_\delta := \{p \in \mathcal{P}_* : d(p, (\mathcal{P}_a)^{\mathbb{Z}}) > \delta\}$. Find $\delta$ small enough that $\lambda(\mathcal{P}_\delta) > 1 - \epsilon$ (see second paragraph in Claim 1 of Proposition 1.1). If $t \sim_\delta t'$, then

$$d_B(p, p') = \text{density}(\ell \in \mathcal{L} : p_\ell \neq p'_\ell) \leq \lambda(\mathcal{P}_\delta)^{\mathbb{Z}} < \epsilon.$$  \hfill (\dagger)

($\star$) is by Proposition 1.1(b). (\dagger) is because, for any $\ell \in \mathcal{L}$, $p_\ell \neq p'_\ell$ $\implies$ $\varsigma^\ell(t) \notin \mathcal{P}_\delta$. To see this, suppose $p_\ell = a$ i.e. $\varsigma^\ell(t) \in \mathcal{P}_a$. If $\varsigma^\ell(t) \in \mathcal{P}_\delta$, then $d((\varsigma^\ell(t), (\mathcal{P}_a)^{\mathbb{Z}})) \leq \delta < d((\varsigma^\ell(t), (\mathcal{P}_a)^{\mathbb{Z}})) \leq d((\varsigma^\ell(t), (\mathcal{P}_a)^{\mathbb{Z}}))$. Hence $\varsigma^\ell(t') \in \mathcal{P}_a$ also; hence $p'_\ell = a = p_\ell$. By contradiction, if $p_\ell \neq p'_\ell$, then $\varsigma^\ell(t) \notin \mathcal{P}_\delta$. \hfill $\Box$

The $\varsigma$-quasisturmian (or $\mathcal{QS}_\varsigma$) shift induced by $\mathcal{P}$ is the $d_C$-closed, $\sigma$-invariant subset:

$$\Xi(\mathcal{P}) := d_C\text{-closure}\left(\mathcal{P}_\varsigma(\mathcal{T})\right) \subset \mathcal{A}^\mathbb{Z}.$$  

Example 2.2. If $K = 1$ and $\mathcal{P} = \{(0, a), (a, 1)\}$ as in Example 2.1 then $\Xi(\mathcal{P}) \subset \mathcal{A}^\mathbb{Z}$ is the classical Sturmian shift of $[25, 33]$. \hfill $\Box$
Proposition 2.4. Let \( \mathcal{A}_0^T = \left\{ \mathcal{P} \in \mathcal{A}^T : 0 \in \mathcal{T}(\mathcal{P}) \right\} \). For all \( \mathcal{P} \in \mathcal{A}_0^T \), let \( \xi_\mathcal{P} (\mathcal{P}) = \mathcal{P}_\mathcal{P}(0) \).

(a) \( \mathcal{A}_0^T \) is a comeager, \( \varsigma \)-invariant subset of \( \mathcal{A}^T \), and \( \xi_\mathcal{P} (\mathcal{A}_0^T) = \Omega \mathcal{S}_\varsigma \).

(b) \( \xi_\mathcal{P} : \mathcal{A}_0^T \to \Omega \mathcal{S}_\varsigma \) is a distance-halving isometry. That is, if \( \mathcal{P}, \mathcal{Q} \in \mathcal{A}_0^T \), then \( d_\Delta (\mathcal{P}, \mathcal{Q}) = 2 \cdot d_B (\xi_\mathcal{P} (\mathcal{P}), \xi_\mathcal{P} (\mathcal{Q})) \).

More generally, if \( \mathcal{P}, \mathcal{Q} \in \mathcal{A}_0^T \) and \( t \in \mathcal{T}(\mathcal{P}) \cap \mathcal{T}(\mathcal{Q}) \), then \( d_\Delta (\mathcal{P}, \mathcal{Q}) = 2 \cdot d_B \left( \mathcal{P}_\mathcal{P}(t), \mathcal{Q}_\mathcal{P}(t) \right) \).

(c) If \( \mathbf{p}, \mathbf{q} \in \Omega \mathcal{S}_\varsigma \), then \( d_B (\mathbf{p}, \mathbf{q}) = 0 \) \( \iff \mathbf{p} = \mathbf{q} \). Hence, \( d_B \) is a true metric when restricted to \( \Omega \mathcal{S}_\varsigma \).

(d) For any \( \ell \in \mathbb{L} \), \( \xi_\mathcal{P} \circ \varsigma^\ell = \sigma^\ell \circ \xi_\mathcal{P} \). Thus, \( \Omega \mathcal{S}_\varsigma \) is a \( \sigma \)-invariant subset of \( \mathcal{A}^T \).

(e) \( \xi_\mathcal{P} \) is a TDS isomorphism from \( \mathcal{A}_0^T \) to \( \Omega \mathcal{S}_\varsigma \times (\Omega \mathcal{S}_\varsigma, d_B, \sigma) \).

Proof. (a) It is clear that \( \mathcal{A}_0^T \) is a comeager and \( \varsigma \)-invariant set. Let \( \mathbf{q} \in \Omega \mathcal{S}_\varsigma \); hence there is some \( \mathcal{Q} \in \mathcal{A}_0^T \) and \( t \in \mathcal{T} \) such that \( \mathbf{q} = \mathcal{Q}_\mathcal{P}(t) \). Define partition \( \mathcal{P} \in \mathcal{A}_0^T \) such that \( \mathcal{P}_\mathcal{P} := \{ \mathbf{q} - t : \mathbf{q} \in \mathcal{Q}_\mathcal{P} \} \) for all \( \varsigma \in \mathcal{A} \). Then \( \mathcal{P} \in \mathcal{A}_0^T \), and \( \mathcal{P}_\mathcal{P}(0) = \mathcal{Q}_\mathcal{P}(t) = \mathbf{q} \).

(b) Let \( \mathbf{p} = \xi_\mathcal{P} (\mathcal{P}) \) and \( \mathbf{q} = \xi_\mathcal{P} (\mathcal{Q}) \). If \( a \in \mathcal{A} \), the Generalized Ergodic Theorem says:

\[
\lambda (\mathcal{P}_a \setminus Q_a) = \lim_{N \to \infty} \frac{\#\{ b \in \mathcal{B}(N) : p_b = a \neq q_b \}}{(2N)^D}.
\]

Thus,
\[
\sum_{a \in \mathcal{A}} \lambda (\mathcal{P}_a \setminus Q_a) = \sum_{a \in \mathcal{A}} \lim_{N \to \infty} \frac{\#\{ b \in \mathcal{B}(N) : p_b = a \neq q_b \}}{(2N)^D} = \lim_{N \to \infty} \frac{\#\{ b \in \mathcal{B}(N) : p_b \neq q_b \}}{(2N)^D} = d_B \left( \xi_\mathcal{P} (\mathcal{P}), \xi_\mathcal{P} (\mathcal{Q}) \right).
\]

Likewise,
\[
\sum_{a \in \mathcal{A}} \lambda (Q_a \setminus P_a) = d_B \left( \xi_\mathcal{P} (\mathcal{P}), \xi_\mathcal{P} (\mathcal{Q}) \right).
\]

Hence, \( d_\Delta (\mathcal{P}, \mathcal{Q}) = \sum_{a \in \mathcal{A}} \lambda (\mathcal{P}_a \setminus Q_a) + \sum_{a \in \mathcal{A}} \lambda (Q_a \setminus P_a) = 2 \cdot d_B \left( \xi_\mathcal{P} (\mathcal{P}), \xi_\mathcal{P} (\mathcal{Q}) \right) \).

(c) follows from (b). (d) follows from the definitions, and (e) follows from (d). \( \square \)

Quasisturmian Measures: Let \( \lambda \) be the Lebesgue measure on \( \mathcal{T} \). Let \( \mathcal{P} \in \mathcal{A}^T \) be a measurable partition of \( \mathcal{T} \). Define \( \tilde{\mathcal{T}} \) as before, then \( \tilde{\mathcal{T}} \) is a \( \varsigma \)-invariant, measurable subset of \( \mathcal{T} \), and \( \lambda (\mathcal{T}) = 1 \). Define \( \mathcal{P}_\varsigma : \mathcal{T} \to \mathcal{A}^T \) as before; then \( \mathcal{P}_\varsigma \) is a measurable function defined \( \lambda \)-ae on \( \mathcal{T} \). Define \( \Upsilon_\varsigma : \mathcal{A}^T \to \mathcal{M}(\mathcal{A}^T) \) by
\( \Upsilon_\varsigma (\mathcal{P}) := \mathcal{P}_\varsigma (\lambda), \) for all \( \mathcal{P} \in \mathcal{A}_r \). Then \( \Upsilon_\varsigma (\mathcal{P}) \) is a \( \sigma \)-invariant measure on \( \mathcal{A}_r \), called the \( \varsigma \)-quasisturian (or QS) measure induced by \( \mathcal{P} \). If \( \mathcal{M}_{\varsigma}^{\text{qs}}(\mathcal{A}_r) \) is the set of \( \varsigma \)-quasisturian measures, then \( \mathcal{M}_{\varsigma}^{\text{qs}}(\mathcal{A}_r) \) is weak*-dense in the space of \( \sigma \)-ergodic probability measures on \( \mathcal{A}_r \) (Corollary 7.3).

Quasisturian measures, sequences, and shifts are related as follows: If \( \mathcal{P} \) is an open partition, then \( \text{supp} \left( \Upsilon_\varsigma (\mathcal{P}) \right) = \Xi_\varsigma (\mathcal{P}) \). Also, if \( \mathbf{t} \in \mathbf{T} \), then \( \Xi_\varsigma (\mathcal{P}_\varsigma (\mathbf{t})) \) is the \( \sigma \)-orbit closure of \( \mathcal{P}_\varsigma (\mathbf{t}) \), and \( \mathcal{P}_\varsigma (\mathbf{t}) \) is \( \sigma \)-generic for \( \Upsilon_\varsigma (\mathcal{P}) \).

**Proposition 2.5.** \( \Upsilon_\varsigma : \mathcal{A}_r \rightarrow \mathcal{M}_{\varsigma}^{\text{qs}}(\mathcal{A}_r) \) is continuous relative to \( d_\Delta \) and the weak* topology.

**Proof.** Suppose \( \{\mathcal{P}^{(n)}\}_{n=1}^\infty \subset \mathcal{A}_r \) is a sequence of partitions, and \( d_\Delta \lim_{n \to \infty} \mathcal{P}^{(n)} = \mathcal{P} \). Let \( \mu^{(n)} = \Upsilon_\varsigma (\mathcal{P}^{(n)}) \) for all \( n \), and let \( \mu = \Upsilon_\varsigma (\mathcal{P}) \); we claim that \( \lim_{n \to \infty} \mu^{(n)} = \mu \).

Suppose \( \mathcal{M} \subset \mathcal{L} \) is finite, and let \( \mathbf{w} \in \mathcal{A}_r \); we must show that \( \lim_{n \to \infty} \mu^{(n)}(\mathbf{w}) = \mu(\mathbf{w}) \). Let \( \mathbf{w} := [w_m]_{m \in \mathcal{M}} \). If \( \mathcal{P} = \{P_a\}_{a \in \mathcal{A}} \) and \( \mathcal{P}^{(n)} = \{P_a^{(n)}\}_{a \in \mathcal{A}} \) then

\[
\mu(\mathbf{w}) = \lambda \left[ \mathcal{P}_\varsigma^{-1}(\mathbf{w}) \right] = \lambda \left[ \bigcap_{m \in \mathcal{M}} \varsigma^m(P_{w_m}) \right].
\]

Likewise, \( \mu^{(n)}(\mathbf{w}) = \lambda \left[ \bigcap_{m \in \mathcal{M}} \varsigma^m(P_{w_m}^{(n)}) \right] \). Let \( M := \#(\mathcal{M}) \). If \( d_\Delta (\mathcal{P}, \mathcal{P}^{(n)}) < \epsilon \), then

\[
\lambda \left[ \bigcap_{m \in \mathcal{M}} \varsigma^m(P_{w_m}^{(n)}) \right] \to 2M \epsilon.
\]

this can be seen by setting \( J := 1 \) and \( K := M \) in Lemma 2.6 (c) below. \( \Box \)

The proof of Proposition 2.5 (and later, Theorem 3.4) uses the following lemma:

**Lemma 2.6.** Let \( \{P_i\}_{i=1}^J \) and \( \{O_i\}_{i=1}^J \) be measurable sets, with \( P_i \subset O_i \) and \( \lambda(O_i \setminus P_i) < \epsilon \), for all \( i \in [1..J] \).

(a) If \( \overline{P} = \bigcup_{i=1}^J P_i \) and \( \overline{O} = \bigcup_{i=1}^J O_i \), then \( \overline{P} \subset \overline{O} \), and \( \lambda(\overline{O} \setminus \overline{P}) \leq I \cdot \epsilon \).

(b) If \( \underline{P} = \bigcap_{i=1}^J P_i \) and \( \underline{O} = \bigcap_{i=1}^J O_i \), then \( \underline{P} \subset \underline{O} \), and \( \lambda(\underline{O} \setminus \underline{P}) \leq I \cdot \epsilon \).

(c) Let \( \{P_{j_k}^{(i)}\}_{i=1}^K \) and \( \{Q_{j_k}^{(i)}\}_{i=1}^K \) be measurable sets, with \( \lambda(Q_{j_k}^{(i)} \Delta P_{j_k}^{(i)}) < \epsilon \), for all \( j \in [1..J] \) and \( k \in [1..K] \).

If \( \underline{P} = \bigcup_{j=1}^J \bigcap_{k=1}^K P_{j_k}^{(i)} \) and \( \overline{Q} = \bigcup_{j=1}^J \bigcap_{k=1}^K Q_{j_k}^{(i)} \), then \( \lambda(\overline{Q} \Delta \underline{P}) < 2JK \cdot \epsilon \).

**Proof.** (a): \( \overline{O} \setminus \overline{P} = \bigcup_{i=1}^J (O_i \setminus P_i) \subset \bigcup_{i=1}^J (O_i \setminus P_i) \), so \( \lambda(\overline{O} \setminus \overline{P}) \leq \sum_{i=1}^J \lambda(O_i \setminus P_i) \leq I \cdot \epsilon \).

(b): Let \( O'_i := P'_i \) and \( P'_i := O'_i \) for \( i \in [1..J] \); then \( P'_i \subset O'_i \) and \( \lambda(O'_i \setminus P'_i) < \epsilon \). Now let \( \overline{P} := \bigcup_{i=1}^J P'_i \) and \( \overline{O} := \bigcup_{i=1}^J O'_i \). Then \( \overline{O} \setminus \overline{P} = \overline{P} \setminus \overline{O} \) and (a) implies \( \lambda(\overline{O} \setminus \overline{P}) < I \cdot \epsilon \).
(c) For all \( j \) and \( k \), let \( O_k^j := P_k^j \cup Q_k^j \). Thus, \( P_k^j \subset O_k^j \) and \( \lambda(O_k^j \setminus P_k^j) < \varepsilon \).
Likewise \( Q_k^j \subset O_k^j \) and \( \lambda(O_k^j \setminus Q_k^j) < \varepsilon \). Now, for each \( j \in [1..J] \), let
\[
O^j := \bigcap_{k=1}^K O_k^j, \quad P^j := \bigcap_{k=1}^K P_k^j, \quad \text{and} \quad Q^j := \bigcap_{k=1}^K Q_k^j.
\]
Thus, setting \( I = K \) in part (b) implies \( P^j \subset O^j \) and \( \lambda(O^j \setminus P^j) < K \cdot \varepsilon \).
Likewise, \( Q^j \subset O^j \) and \( \lambda(O^j \setminus Q^j) < K \cdot \varepsilon \).

Now let \( O := \bigcup_{j=1}^J O^j = \bigcup_{j=1}^J \bigcap_{k=1}^K O_k^j, \) and observe that \( P = \bigcup_{j=1}^J P^j \) and \( Q = \bigcup_{j=1}^J Q^j \).

Thus, setting \( I = J \) in part (a) implies that \( P \subset O \) and \( \lambda(O \setminus P) < JK \cdot \varepsilon \).
Likewise, \( Q \subset O \) and \( \lambda(O \setminus Q) < JK \cdot \varepsilon \). Thus, by the triangle inequality, \( \lambda(P \triangle Q) = 2JK\varepsilon \).

\[ \square \]

3. \( \text{CA on QS systems: induced dynamics on } \mathcal{A}^T \)

We begin by generalizing a result of Hof and Knill \( \text{[1]} \).

**Theorem 3.1.** Let \( \Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z} \) be a cellular automaton.

(a) \( \Phi(\Omega_S) \subseteq \Omega_S \). That is: if \( p \in \mathcal{A}^\mathbb{Z} \) is a QS sequence, then \( \Phi(p) \) is also a QS sequence.

(b) If \( \Psi \subset \mathcal{A}^\mathbb{Z} \) is a QS shift, then \( \Phi(\Psi) \) is also a QS shift.

(c) \( \Phi[\mathcal{M}^{\mathcal{QS}}(\mathcal{A}^\mathbb{Z})] \subseteq \mathcal{M}^{\mathcal{QS}}(\mathcal{A}^\mathbb{Z}) \). That is: if \( \mu \) is a QS measure, then \( \Phi(\mu) \) is also a QS measure.

\[ \square \]

To prove Theorem 3.1, suppose \( \Phi \) has local map \( \phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A} \) (where \( \mathbb{B} \subset \mathbb{L} \) is finite). Suppose \( \mathcal{P} \in \mathcal{A}^T \) is a measurable partition of \( T \). For each \( a \in \mathcal{A} \), define
\[
Q_a = \bigcup_{\phi(e) = a} \bigcap_{b \in \mathbb{B}} \zeta^{-b}(P_{eb}) \subset T.
\]

Now define measurable partition \( Q := \{ Q_a \}_{a \in \mathcal{A}} \). We write: \( Q = \Phi_\zeta(\mathcal{P}) \). Thus, \( \Phi \) induces a map \( \Phi_\zeta : \mathcal{A}^T \rightarrow \mathcal{A}^T \). It is easy to verify:

**Lemma 3.2.** Let \( \mathcal{P} \in \mathcal{A}^T \) and let \( Q = \Phi_\zeta(\mathcal{P}) \). Then \( \tilde{T}(\mathcal{P}) = \tilde{T}(Q) \). Also:

(a) If \( \mathcal{P} \) is open then \( \tilde{Q} \) is also open. If \( t \in \tilde{T} \), then \( \Phi_\zeta(\mathcal{P})(t) = Q_\zeta(t) \).

(b) \( \Phi_\zeta(\Xi_\zeta(\mathcal{P})) = \Xi_\zeta(\Phi_\zeta(\mathcal{P})) \).

(c) \( \Phi_\zeta(\Upsilon_\zeta(\mathcal{P})) = \Upsilon_\zeta(\Phi_\zeta(\mathcal{P})) \).

\[ \square \]

Theorem 3.1 follows: set \( p = \mathcal{P}(t), \ \Psi = \Xi_\zeta(\mathcal{P}), \) or \( \mu = \Upsilon_\zeta(\mathcal{P}) \) in Lemma 3.2.

**Example 3.3.** Linear Cellular Automata (see [1])

(a) Let \( \mathbb{L} = \mathbb{Z} \). Let \( \mathcal{A} = \mathbb{Z}/2 \) and let \( \Phi \) have local map \( \phi(a_0, a_1) = a_0 + a_1 \) (mod 2).

Then \( Q_1 = P_1 \triangle \zeta(P_1) \) and \( Q_0 = Q_1^0 = [P_0 \cap \zeta(P_0)] \cup [P_1 \cap \zeta(P_1)] \).

(See Figure 1).
More generally, let \( L = \mathbb{Z}^D \) and let \( B \subset L \) be finite. Let \( \mathcal{A} = \mathbb{Z}_2 \), and suppose \( \Phi \) has local map \( \phi : \mathcal{A}^B \to \mathcal{A} \) defined: \( \phi(a) := \sum_{b \in B} a_b (\text{mod } 2) \), for any \( a \in \mathcal{A}^B \). Then \( \Phi_\xi(\mathcal{P}) = \{Q_0, Q_1\} \), where \( Q_1 = \bigtriangleup \psi^B(\mathcal{P}_1) \), and \( Q_0 = T \setminus Q_1 \).

(c) Let \( n \in \mathbb{N} \) and let \( \mathcal{A} = \mathbb{Z}/n \). Let \( \varphi_b \in \mathbb{Z}/n \) be constants for all \( b \in B \). Suppose \( \Phi \) has local map \( \phi : \mathcal{A}^B \to \mathcal{A} \) defined \( \phi(a) := \sum_{b \in B} \varphi_b a_b (\text{mod } p) \), for any \( a \in \mathcal{A}^B \). Treat any \( \mathcal{P} \in \mathcal{A}^T \) as a function \( \mathcal{P} : T \to \mathcal{A} \). Then \( \Phi_\xi(\mathcal{P}) = \sum_{b \in B} \varphi_b \cdot \psi^B(\mathcal{P}) \).

If \( (X,d) \) is a metric space, recall that a function \( \varphi : X \to X \) is Lipschitz with constant \( K > 0 \) if \( \varphi \) is continuous, and furthermore, for any \( x, y \in X \), \( d\left(\varphi(x), \varphi(y)\right) < K \cdot d(x,y) \).

**Theorem 3.4.** \( (\mathcal{A}^T, d_\Delta, \Phi_\xi) \) is a topological dynamical system, and \( \Phi_\xi : \mathcal{A}^T \to \mathcal{A}^T \) is \( d_\Delta \)-Lipschitz.

**Proof.** Suppose \( \Phi \) has local map \( \phi : \mathcal{A}^B \to \mathcal{A} \). Let \( B = |B| \) and \( A = |A| \); hence \( |\mathcal{A}^B| = A^B \). We claim that \( \Phi_\xi \) has Lipschitz constant \( 2B \cdot A^B \). Suppose \( \mathcal{P}, \mathcal{P}' \in \mathcal{A}^T \), and \( d_\Delta(\mathcal{P}, \mathcal{P}') < \epsilon \). If \( \Phi_\xi(\mathcal{P}) = \mathcal{Q} \) and \( \Phi_\xi(\mathcal{P}') = \mathcal{Q}' \), then for all

\[ d\left(\varphi(x), \varphi(y)\right) < K \cdot d(x,y) \]
\(a \in A,\)

\[
Q_a = \bigcup_{e \in \mathcal{A}, \phi(e) = a} \bigcap_{b \in \mathcal{B}} \varsigma^{-b}(P_{e_b}) \quad \text{and} \quad Q'_a = \bigcup_{e \in \mathcal{A}, \phi(e) = a} \bigcap_{b \in \mathcal{B}} \varsigma^{-b}(P'_{e_b}).
\]

Set \(J := A^B\) and \(K := B\) in Lemma \(2.7\) to conclude \(d_\triangle (Q, Q') < 2BA^B \cdot \epsilon. \)

**Proposition 3.5.** Let \(\mathcal{A}_T^a\) and \(\xi : \mathcal{A}_T^a \rightarrow \Omega \Sigma_{\xi} \) be as in Proposition \(2.7\). Then \(\xi \circ \Phi_{\xi} = \Phi \circ \xi_{\xi}\). Thus, \(\Omega \Sigma_{\xi}\) is a \(\Phi\)-invariant subset of \(\mathcal{A}_T\), and \(\xi_{\xi}\) is an isomorphism from the topological dynamical system \((\mathcal{A}_T^a, d_\triangle, \Phi_{\xi})\) to the system \((\Omega \Sigma_{\xi}, d_B, \Phi)\).

**Proof.** Combine Lemma \(3.2\) with Proposition \(2.4\).

A topological dynamical system \((X, d, \varphi)\) is **equicontinuous** if, for every \(\epsilon > 0\), there is \(\delta > 0\) such that, for any \(x, y \in X\), \((d(x, y) < \delta) \implies (d(\varphi^n(x), \varphi^n(y)) < \epsilon \text{ for all } n \in \mathbb{N})\).

**Proposition 3.6.** If \((\mathcal{A}_T^a, d_C, \Phi)\) is equicontinuous, then \((\mathcal{A}_T^a, d_B, \Phi)\) is equicontinuous.

**Proof.** If \((\mathcal{A}_T^a, d_C, \Phi)\) is equicontinuous, then Proposition 7 of \(20\) says \((\mathcal{A}_T^a, d_B, \Phi)\) is equicontinuous. Thus, the subsystem \((\Omega \Sigma_{\xi}, d_B, \Phi)\) is also equicontinuous. Now apply Proposition \(3.5\).

**4. The space of measurable partitions†**

The next result is used to prove Theorem \(6.3\).

**Proposition 4.1.** \(\mathcal{A}_T^a\) is complete and bounded in the \(d_\triangle\) metric.

**Proof.** We’ll embed \(\mathcal{A}_T^a\) as a closed subset of \(L^2(T, \mathbb{C})\), so that the \(d_\triangle\) metric is equivalent to the (complete) \(L^2\) metric.

**Claim 1.** If \(P = \{P_a\}_{a \in A}\) and \(Q = \{Q_a\}_{a \in A}\), then \(d_\triangle(P, Q) = 2 \sum_{a,b \in A} \lambda[P_a \cap Q_b]\).

**Proof.** For any \(a \in A,\)

\[
\lambda[P_a \setminus Q_a] = \sum_{b \neq a} \lambda[P_a \cap Q_b].
\]

Thus,

\[
\sum_{a \in A} \lambda[P_a \setminus Q_a] = \sum_{a \in A} \sum_{b \neq a} \lambda[P_a \cap Q_b] = \sum_{a \neq b \in A} \lambda[P_a \cap Q_b].
\]

Thus, \(d_\triangle(P, Q) = \sum_{a \in A} \lambda[P_a \setminus Q_a] + \sum_{a \in A} \lambda[Q_a \setminus P_a] = 2 \sum_{a \neq b \in A} \lambda[P_a \cap Q_b].\)

\(\diamond \) Claim 1

† This section contains technical results which are used in \(4\), \(5\) and \(8\).

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Suppose $|A| = A$, and identify $A$ with the $A$th roots of unity in some arbitrary way:

$$A \equiv \{ e^{\frac{2\pi ki}{A}} : 0 \leq k < A \}.$$ 

Any $P \in A^T$ defines a function in $L^2(T;\mathbb{C})$ (also denoted by $P$), such that $P^{-1}\{a\} = P_a$ for any $a \in A$. We can then measure the distance between partitions in the $L^2$ metric: $\|P - Q\|_2 = \left( \int_T |P(t) - Q(t)|^2 \, d\lambda(t) \right)^{1/2}$. Let

$$m = \min_{a \neq b \in A} |a - b|^2$$

and $M = \max_{a \neq b \in A} |a - b|^2$.

**Claim 2.** For any $P, Q \in A^T$, $\sum_{t} d_\Delta(P, Q) < \|P - Q\|_2 < \frac{M}{2} d_\Delta(P, Q)$.

**Proof.**

$$\|P - Q\|_2^2 = \int_T |P(t) - Q(t)|^2 \, d\lambda(t) = \sum_{a \neq b \in A} |a - b|^2 \, d\lambda.$$

Thus, $\sum_{a \neq b \in A} |a - b|^2 \cdot \lambda(P_a \cap Q_b)$. Then,

$$\sum_{a \neq b \in A} m \cdot \lambda(P_a \cap Q_b) \leq \|P - Q\|_2^2 \leq \sum_{a \neq b \in A} M \cdot \lambda(P_a \cap Q_b).$$

Now apply Claim 1. \hfill $\diamond$ Claim 2

Thus, the $d_\Delta$ and $L^2$ metrics are equivalent, so $A^T$ is bounded and complete in $d_\Delta$ if and only if $A^T$ is bounded and complete in $L^2$. It remains to show:

**Claim 3.** $A^T$ is a closed, bounded subset of $L^2(T, \mathbb{C})$.

**Proof.** $A^T$ is bounded because it is a subset of the unit ball in $L^2$. To see that $A^T$ is closed, suppose \{ $P_n$ \}_{n=1}^\infty \subseteq A^T$ was a sequence of $A$-labelled partitions, and that $L^2 - \lim_{n \to \infty} P_n = P$, where $P \in L^2(T, \mathbb{C})$. We must show that $P$ is also an $A$-labelled partition—in other words, that the essential image of $P$ is $A \subseteq \mathbb{C}$.

Suppose not. Then there is some $\epsilon > 0$ and some subset $U \subseteq \mathbb{C}$ with $d(U, A) = \epsilon$ such that, if $U = P^{-1}(U) \subseteq T$, then $\lambda(U) > 0$. But then for any $n \in \mathbb{N}$,

$$\|P_n - P\|_2^2 = \int_T |P_n - P|^2 \, d\lambda \geq \int_U |P_n - P|^2 \, d\lambda \geq \int_U |\epsilon|^2 \, d\lambda = \epsilon^2 \cdot \lambda(U).$$

This contradicts the hypothesis that $\lim_{n \to \infty} \|P_n - P\|_2 = 0$. \hfill $\diamond$ Claim 3

**Remark.** Although $(A^T, d_\Delta)$ is complete and bounded, it is not compact. For example, let $A = \{0, 1\}$ and $T = [0, 1]$. Fix $n \in \mathbb{N}$, and for each $j \in \{1, 2^n\}$, let $I_n = (\frac{j-1}{2^n}, \frac{j}{2^n})$. Now define partition $P^{n} = \{ P_0^{(n)}, P_1^{(n)}\}$, where $P_0^{(n)} = \bigcup_{j=2}^{2^n} I_j$ and $P_1^{(n)} = \bigcup_{j=1}^{2^n-1} I_j$. It is easy to check that $d_\Delta(P^{(n)}, P^{(m)}) = 1$ for any $n \neq m$. Hence, \{ $P_n^{(n)}$ \}_{n=1}^\infty is an infinite sequence of partitions with no convergent subsequence, which would be impossible if $A^T$ were compact.

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4.1. **Dyadic partitions and the density of \(^{\text{a}}A^T\) in \(A^T\)** Identify \(T = [0,1]^K\) for some \(K \in \mathbb{N}\). For any \(n > 0\), an \(n\)-dyadic number is a number \(d = \frac{k}{2^n}\) for some \(k \in [0,2^n]\). An n-dyadic interval is a closed interval \(I = [d_1,d_2]\), where \(d_1, d_2\) are n-dyadic numbers; an n-dyadic cube in \(T\) is a set \(C = I_1 \times I_2 \times \ldots \times I_K\), where \(I_1,\ldots,I_K\) are n-dyadic intervals. An n-dyadic set is an open set \(D \subset T\) such that \(D = \text{int} \left( \bigcup_{j=1}^J C_j \right)\) for some collection \(\{C_1, \ldots, C_J\}\) of n-dyadic cubes. An \((A\)-labelled) n-dyadic partition is an open partition \(D = \{D_a\}_{a \in A}\) where \(D_a\) is n-dyadic for all \(a \in A\). A dyadic number (cube, set, partition, etc.) is one which is n-dyadic for some \(n > 0\). Let \(\hat{A}^T \subset \mathcal{A}^T\) be the set of all \(A\)-labelled dyadic partitions.

The next result is used to prove Lemma 4.3, Lemma 6.11 and Proposition 8.2.

**Proposition 4.2.** \(\hat{A}^T\) is \(d_\Delta\)-dense in \(A^T\). Thus, \(^{\text{a}}A^T\) is also \(d_\Delta\)-dense in \(A^T\).

Furthermore \(A^T\) is separable. \(\square\)

To prove Proposition 4.2 we use:

**Lemma 4.3.** (a) If \(D\) is an n-dyadic set, then \(\hat{D}\) is also \(N\)-dyadic for any \(N > n\).

(b) The union or intersection of any two \(N\)-dyadic sets is \(N\)-dyadic.

(c) If \(D \subset T\) is an \(N\)-dyadic set, then \(T \setminus \overline{D}\) is also \(N\)-dyadic.

(d) Let \(M \subset T\) be a measurable set. For any \(\delta > 0\), there is some \(N > 0\) and some \(N\)-dyadic set \(\hat{M} \subset T\) with \(\lambda(\hat{M} \Delta M) < \delta\).

(e) Let \(D \subset T\) be an \(N\)-dyadic set, and let \(M \subset D\) be a measurable subset. For any \(\delta > 0\), there is some \(N > n\) and some \(N\)-dyadic \(\hat{M} \subset D\) with \(\lambda(\hat{M} \Delta M) < \delta\).

(f) If \(M\) is measurable and \(D\) is an \(N\)-dyadic set, then \(\lambda(M \cap \overline{D}) = \lambda(M \cap D)\).

**Proof.** (a), (b) and (c) are immediate from the definition.

(d): Recall [22] Thm. 2.40(c), p.68, for example] that \(M\) can be \(\delta/2\)-approximated by a finite union of open cubes \(K_1,\ldots,K_J\) for some \(J > 0\). Next, for each \(j \in [1..J]\), there is some \(N_j\) such that \(K_j\) can be \((\delta/2J)\)-approximated by an \(N_j\)-dyadic set \(\hat{K}_j\). Let \(N = \max\{N_1,\ldots,N_J\}\). Then (a) says that \(\hat{K}_1,\ldots,\hat{K}_J\) are all \(N\)-dyadic sets. If \(D = \bigcup_{j=1}^J \hat{K}_j\), then \(\hat{D}\) is also an \(N\)-dyadic set (by (b)), and \(M \triangle \hat{D} = \bigcup_{j=1}^J \hat{K}_j \setminus \delta/2 D\).

(e): First use part (d) to find some \(N\)-dyadic set \(\hat{M} \subset T\) with \(\lambda(\hat{M} \Delta M) < \epsilon\). Now let \(\hat{M} = M \cap \hat{D}\). Then \(\hat{M}\) is also an \(N\)-dyadic set (by (a) and (b)), and since \(M \subset D\), we have \(\lambda(\hat{M} \Delta D) \leq \lambda(\hat{M} \Delta M) < \delta\).

(f): \(\partial D\) is a finite union of \((K-1)\)-dimensional hyperfaces, so \(\lambda(\partial D) = 0\). \(\square\)

**Proof of Proposition 4.2** Let \(P = \{P_a\}_{a \in A}\) be a measurable partition, and let \(\epsilon > 0\). We want a dyadic partition \(D = \{D_a\}_{a \in A}\) such that \(d_\Delta(P,D) < \epsilon\). For simplicity, let \(A := \{1,2,\ldots,A\}\) for some \(A \in \mathbb{N}\).

Fix \(\delta > 0\). Lemma 4.3(d) yields \(N_1 > 0\) and an \(N_1\)-dyadic set \(D_1 \subset T\) such that \(\lambda(P_1 \Delta D_1) < \delta\). Lemma 4.3(c) says that \(T' = T \setminus \overline{D_1}\) is also \(N_1\)-dyadic. For all \(a \in [3..A]\), let \(P'_a = P_a \cap T'\), while \(P'_2 = (P_3 \cap T') \cup (P_2 \cap T')\).

**Claim 1.** \(\lambda(P_a \Delta P'_a) < \delta\) for any \(a \in [3..A]\), while \(\lambda(P_2 \Delta P_2') < 2\delta\).
Lemma 4.5. \(P_a \subset P_a\) and \(P_a \setminus P_a' = P_a \cap D_1\), so that \(\lambda(P_a \triangle P_a') = \lambda(P_a \setminus P_a') = \lambda(P_a \cap D_1)\), where \(\ast\) is by Claim 1(f). But \(P_a \cap D_1 \subset D_1 \setminus P_1\), so \(\lambda(P_a \cap D_1) \leq \lambda(D_1 \setminus P_1) \leq \lambda(D_1 \setminus P_1) = \delta\). The proof for \(P_2\) is similar.

\(\diamond\) Claim 1

Lemma 14.3(e) yields some \(N_2 \geq N_1\) and an \(N_2\)-dyadic \(D_2 \subset T'\) such that \(\lambda(P_2 \triangle D_2) < \delta\). Hence, \(\lambda(P_2 \setminus D_2) \leq \lambda(P_2 \triangle D_2) < 2\delta + \delta = 3\delta\).

Now, Lemma 14.3(c) says \(T'' = T' \setminus \delta\) is an \(N_2\)-dyadic set. For all \(a \in \{4..A\}\), let \(P_a'' = P_a \cap T''\); hence \(P_a'' \subset P_a\) and \(\lambda(P_a \setminus P_a'') < \delta\), as in Claim 1. Also, let \(P_3'' = (P_2' \cap T'') \cup (P_3' \cap T'')\). Then \(\lambda(P_3'' \triangle P_3'') < 2\delta\) as in Claim 1.

Proceeding inductively, we obtain the triangle shown below:

\[
\begin{array}{ccccccc}
P_1 & \sim & D_1 & \sim & D_2 & \sim & D_3 & \sim \cdots \\
P_2 & \sim & D_2 & \sim & D_3 & \sim & \cdots \\
P_3 & \sim & D_3 & \sim & \cdots \\
P_4 & \sim & \cdots \\
P_A & \sim & \cdots \\
\end{array}
\]

Hence, \(P_a \sim (a+1)\delta\) \(D_a\) for any \(a \geq 2\). Now, \(D_1, \ldots, D_A\) are disjoint dyadic sets, and

\[
d_{\triangle}(P,D) < \lambda(P_1 \triangle D_1) + \sum_{a=2}^{A} \lambda(P_a \triangle D_a) < \delta + \sum_{a=2}^{A} (a+1)\delta = M \cdot \delta,
\]

where \(M := \frac{A(A+1)}{2} - 2\). So, choose \(\delta < \varepsilon/M\). \(\square\)

4.2 Simple Partitions and the Injectivity of \(P_\$\) If \(P \in \mathcal{A}_T\), and \(s \in T\), then \(s\) is a (translational) symmetry of \(P\) if \(\rho^s(P) = P\ (\lambda, \alpha)\). If \(P \in \mathcal{A}_T\), then we also require that \(\rho^s(P) = P\ (\text{top,} \alpha)\). The (translational) symmetries of \(P\) form a closed subgroup of \(T\); if this group is trivial, we say \(P\) is simple.

Example 4.4. Identify \(T^1 \cong [0,1)\). Let \(0 < \beta_1 < \beta_2 < \cdots < \beta_N < 1\) be irrational numbers. Then \(P := \{[0, \beta_1), [\beta_1, \beta_2), \ldots, [\beta_N, 1)\}\) is a simple partition of \(T^1\).

To see this, suppose \(s \in T^1\) is a symmetry. Then \(\rho^s[0, \beta_1) = [\beta_n, \beta_{n+1})\) for some \(n \in \{1, \ldots, N\}\). Hence, \(s = \beta_n\). But we can assume WLOG that \(s\) is a rational number (see Lemma 14.3). Hence, either \(\beta_n\) is rational (a contradiction) or \(s = 0\). \(\diamond\)

The next result is used to prove Theorem 4.1(a) and Lemma 14.3.

Lemma 4.5. If \(P \in \mathcal{A}_T\) is simple, then the map \(P_\$ : T \rightarrow \mathcal{A}_T\) is injective (\(\lambda, \alpha\)).
Proof. We must show that, for $\forall \lambda \ t_1, t_2 \in T$, if $t_2 \neq t_1$ then $P_\lambda(t_2) \neq P_\lambda(t_1)$. Let $s = t_2 - t_1$, and let $U = \{ t \in T : P(t) \neq P(t + s) \}$. Since $P$ has trivial symmetry group, it follows that $\lambda[U] > 0$. Let $U = \bigcup_{\ell \in L} A^\ell(U)$; then $\lambda[U] > 0$ and $U$ is $\zeta$-invariant. Since $\zeta$ is ergodic, it follows that $\lambda[U] = 1$. Thus, generically, $t_1 \in U$, which means that $t_1 \in A^\ell(U)$ for some $\ell \in L$. This means that $A^\ell(t_1) \in U$. Thus, $P_\lambda(t_1) = P(A^\ell(t_1)) \neq P(A^\ell(t_1) + s) = P(A^\ell(t_1 + s)) = P(A^\ell(t_2)) = P_\lambda(t_2)$. Hence $P_\lambda(t_1) \neq P_\lambda(t_2)$. \hfill $\Box$

In Proposition 5.12 we’ll need to replace a nonsimple partition with a simple ‘quotient’ partition.

**Lemma 4.6.** If $P \in ^0 A^T$ is not simple, let $S \subset T$ be the translational symmetry group of $P$.

1. $T = T/S$ is also a torus (possibly of lower dimension), and the quotient homomorphism $q : T \to T$ is continuous.
2. There is a unique $A$-valued, open partition $\overline{P} \in ^0 A^T$ such that $\overline{P} \circ q = P$.
3. There is a natural $L$-action on $\overline{T}$—denoted $\zeta$—such that for all $\ell \in L$, $\zeta^\ell \circ q = q \circ \zeta^\ell$.

Define $\overline{T}$ and $\overline{P}_\zeta : \overline{T} \to A^L$ in the obvious way. Then:

4. If $t \in \overline{T}$, and $\overline{t} = q(t)$, then $\overline{t} \in \overline{T}$, and $\overline{P}_\zeta(\overline{t}) = P_\zeta(t)$.
5. Hence, $\overline{P}_\zeta(\overline{T}) = P_\zeta(T)$, $\overline{\Xi}_\zeta(\overline{P}) = \Xi_\zeta(P)$, and $\overline{\Upsilon}_\zeta(\overline{P}) = \Upsilon_\zeta(P)$.
6. $\overline{P}_\zeta : \overline{T} \to A^L$ is injective $(\lambda\alpha)$.

**Proof.** (1) is because $S$ is a closed subgroup of $T$. (2) is because $S$ is a group of symmetries of $P$ (resp. $Q$). To see (3), suppose $\tau : L \to T$ is the homomorphism such that $A^\ell(t) = t + \tau(\ell)$ for all $t \in T$ and $\ell \in L$. Define $\overline{P} = q \circ \tau : L \to \overline{T}$, and then define $\overline{\zeta}^\ell(\overline{t}) = \overline{t} + \overline{\tau}(\ell)$ for all $\overline{t} \in \overline{T}$ and $\ell \in L$. (4) follows from the defining properties of $\overline{P}$ in (2) and $\zeta$ in (3), and (5) follows immediately from (4). To see (6), observe that the symmetry group of $\overline{P}$ is $q(S) = \{0\}$. Now apply Lemma 4.5. \hfill $\Box$

If $P$ is an open partition, we can strengthen Lemma 4.5 to get a homeomorphism: this is used to prove Proposition 5.12 and Theorem 6.1(b).

**Proposition 4.7.** Let $P \in ^0 A^T$ be a simple open partition. Let $\overline{Q} = P_\zeta(T) \subset A^L$. Then:

1. $P_\zeta : \overline{T} \to \overline{Q}$ is a uniform homeomorphism with respect to the $d_C$ metric on $\overline{Q}$.
2. $P_\zeta : \overline{T} \to \overline{Q}$ is a uniform homeomorphism with respect to the $d_B$ metric on $\overline{Q}$.
3. Thus, the Cantor topology and the Besicovitch topology agree on $\overline{Q}$.

---

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(d) Let $\mathfrak{p} = \Xi_c(P)$; then there is a continuous surjection $\xi : \mathfrak{p} \to T$ such that:

[i] For all $\ell \in L$, $\xi \circ \sigma^\ell = \xi^\ell \circ \xi$.

[ii] If $t \in T$, then $\xi \circ P_\xi(t) = t$. If $p \in \overline{\mathfrak{p}}$, then $P_\xi \circ \xi(p) = p$.

Proof. (a) $P_\xi$ is continuous by Proposition 2.2(b). To show $P_\xi$ is injective and uniformly open, fix $\epsilon > 0$. We can assume that the sequence

$$P = \lim_{n \to \infty} \mathfrak{p} \ni p_n$$

Suppose not; then for any $n \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that $P_n \in \mathfrak{p}$ and $d(t_n, u_n) > \epsilon$. Let $s_n = t_n - u_n$. By dropping to a subsequence if necessary, we can assume that the sequence $\{s_n\}_{n=1}^\infty$ converges to some $s \in T$, with $d(s, 0) > \epsilon$ (because $T$ is compact). We’ll show that $s$ is a symmetry of $P$ (thereby contradicting simplicity).

CLM 1. For any small enough $\delta > 0$, there is a $\delta$-dense subset $U_\delta \subset T$, such that $P(\rho^\delta(u)) = P(u)$ for all $u \in U_\delta$.

Proof. For any $t \in T$ and all $\ell \in L$ let $t^\ell := \xi^\ell(t)$. The torus rotation system $(T, d, \xi)$ is minimal and isometric, so if $n$ is large enough, then, for any $t \in T$, the set $\{t^b\}_{b \in \mathbb{B}(n)}$ is $\delta$-dense in $T$.

Recall (11) that $P^* := \bigcup_{a \in A} P_a$ is open and dense in $T$; thus $\partial P := \bigcup_{a \in A} \partial P_a$ is nowhere dense. Thus, $\rho^{-\delta}(\partial P)$ is nowhere dense (since $\rho^\delta$ is an isometry). So, if $\delta$ is small enough, the set $U_\delta := \{u^b_n ; b \in \mathbb{B}(n) \text{ and } d(\rho^\delta(u^b_n), \partial P) \geq \delta\}$ is $\delta$-dense in $T$.

If $n$ is large enough, then $s_n \sim_\delta s$. Thus, $t_n = u_n + s_n \sim_\delta u_n + s = \rho^\delta(u_n)$; thus, for all $b \in \mathbb{B}(n)$, $t^b_n \sim_\delta \rho^\delta(u^b_n)$. Thus, for any $u^b_n \in U_\delta$, $P(\rho^\delta(u^b_n)) = P(t^b_n) = P(\rho^\delta(u^b_n))$.

$(\ast)$ is because $t^b_n \sim_\delta \rho^\delta(u^b_n) \sim_\delta \partial P$; $(\ast)$ is because $P_\xi(t_n) = P_\xi(u_n) = P_\xi(s) = P_\xi(u_n)$.

Claim 1

Now, let $\{\delta_n\}_{n=1}^\infty$ be a sequence tending to zero, and for each $n \in \mathbb{N}$, let $U_{\delta_n}$ be as in Claim 1. Let $U := \bigcup_{n=1}^\infty U_{\delta_n}$; then $U$ is a dense subset of $T$ such that $P(\rho^\delta(u)) = P(u)$ for all $u \in U$. In other words, for each $a \in A$, $\rho^\delta(P_a) \cap U = P_a \cap U$, which means that $\rho^\delta(P_a) \cap U \cap \overline{P_a} = \emptyset$, where $\overline{P_a} := T \setminus P_a$. But $\rho^\delta(P_a) \cap \overline{P_a}$ is an open subset of $T$, so if it is disjoint from the dense set $U$, it must be empty. But if $\rho^\delta(P_a) \cap \overline{P_a} = \emptyset$, then $\rho^\delta(P_a) \subset \overline{P_a}$. By symmetric reasoning, $\rho^\delta(P_a) \supset P_a$; hence $\rho^\delta(P_a) = P_a$ (top.æ). This holds for all $a \in A$, so we conclude that $\rho^\delta(P) = P$ (top.æ). Thus, $s$ is a symmetry of $P$, contradicting the simplicity of $P$.

(b) We claim that, for any $\epsilon > 0$, there is some $\delta > 0$ such that, for any $t, u \in T$,

$$d(t, u) < \epsilon.$$

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The proof the same as (a): replace $\mathbb{B}(n)$ with $\mathcal{M}$, prove the appropriate version of Claim 1, deduce a symmetry, and derive a contradiction.

(c) Follows from (a) and (b).

(d) Recall that $\mathcal{P}$ is the $d_C$-closure of $\mathcal{P}^* = \mathcal{P}_c(\mathcal{T})$. Thus, if $p \in \mathcal{P}$, then there exists a sequence $\{t_n\}_{n=1}^\infty \subset \mathcal{T}$ such that $p = dC_{n \to \infty} \mathcal{P}_c(t_n)$. We can drop to a subsequence such that $\{t_n\}_{n=1}^\infty$ converges to some $t \in \mathcal{T}$ (because $\mathcal{T}$ is compact).

We define $\Psi(p) := t$.

$\Psi(p)$ is well-defined: Suppose $\{t'_n\}_{n=1}^\infty \subset \mathcal{T}$ was another sequence with $p = dC_{n \to \infty} \mathcal{P}_c(t'_n)$; we claim $\lim n \to \infty t'_n = \lim n \to \infty t_n$. Let $\epsilon > 0$, and let $M > 0$ be as in assertion 3. Find $N$ large enough that, if $n > N$, then $\mathcal{P}_c(t'_n)|_{\mathcal{B}(M)} = \mathcal{P}_c(t_n)|_{\mathcal{B}(M)}$. Then assertion 3 says that $t'_n - t_n$. Since $\epsilon$ is arbitrary, we conclude that $\lim n \to \infty t'_n = \lim n \to \infty t_n$.

Continuous: Fix $\epsilon > 0$. Let $M$ be as in assertion 3; and suppose $p|_{\mathcal{B}(M)} = p'|_{\mathcal{B}(M)}$. If $\{t_n\}_{n=1}^\infty \subset \mathcal{T}$ is such that $p = dC_{n \to \infty} \mathcal{P}_c(t_n)$ and $\{t'_n\}_{n=1}^\infty \subset \mathcal{T}$ is such that $p' = dC_{n \to \infty} \mathcal{P}_c(t'_n)$, then assertion 3 says that, for large $n$, we must have $t_n - t'_n \to 0$. Hence $\Psi(p) = dC_{n \to \infty} \mathcal{P}_c(t_n)$.

Surjection: Let $t \in \mathcal{T}$; find a sequence $\{t_n\}_{n=1}^\infty \subset \mathcal{T}$ such that $t = \lim n \to \infty t_n$. Let $p = dC_{n \to \infty} \mathcal{P}_c(t_n)$; then by definition, $t = \Psi(p)$.

[i]: If $p = dC_{n \to \infty} \mathcal{P}_c(t_n)$, then

$$\sigma^f(p) = dC_{n \to \infty} \sigma^f(\mathcal{P}_c(t_n)) = \mathcal{P}_c(\sigma^f(t_n)),$$

where $(*)$ is Proposition 2.2(a). Hence $\Psi(\sigma^f(p)) = \lim n \to \infty \sigma^f(t_n) = \sigma^f(\lim n \to \infty t_n) = \sigma^f(t)$.\[\square\]

Thus, in a sense, the topological L-system $(\mathcal{B}, d_C, \sigma)$ is ‘almost’ isomorphic to the system $(\mathcal{T}, d, \varsigma)$. The only caveat is that the function $\Psi$ is many-to-one on the elements of $\mathcal{T} \setminus \mathcal{T}$.

5. Boundary growth & Chopping

Suppose $\mathcal{T} = \mathcal{T}^1$, and $\mathcal{P}$ is an open partition of $\mathcal{T}$ such that each element of $\mathcal{P}$ is a finite collection of open intervals; then $\partial \mathcal{P} := \bigcup_{a \in \mathcal{A}} \partial \mathcal{P}_a$ is a finite set of points in $\mathcal{T}$. Let $\mathcal{P}$ be a cellular automaton. Hof and Knill [H] observed empirically that $\#(\partial (\Phi_n^\mathcal{P}))$ grows polynomially like $n^\alpha$ as $n \to \infty$, for some exponent $\alpha \leq D$ (where $\mathcal{L} = \mathbb{Z}^D$). They asked: is $\#(\partial (\Phi_n^\mathcal{P}))$ really growing? Is the growth polynomial? What’s the exact value of $\alpha$?
If \( \#(\partial (\Phi^n(P))) \) gets large as \( n \to \infty \), then each cell of \( \Phi^n(P) \) is ‘chopped’ into many tiny separate intervals: we say \( \Phi \) is chopping (we’ll make this precise later).

In this section, we investigate chopping, answer Hof and Knill’s questions, and generalize these ideas to \( T^K \).

**Partition boundary size in \( T^K \):** Suppose \( T = T^K \) for \( K \geq 1 \), and let \( P \in \mathcal{A}_T \). To characterize the growth of \( \partial (\Phi^n(P)) \), we first need a way to measure its size. Let \( \mathcal{C} = \{ C \subset T \mid C \text{ closed} \} \), and let \( \lceil \cdot \rceil : \mathcal{C} \to [0, \infty] \) be some ‘pseudomeasure’, satisfying:

(M1) **Monotonicity:** If \( C_1 \subset C_2 \), then \( \lceil C_1 \rceil \leq \lceil C_2 \rceil \).

(M2) **Additivity:** \( \lceil C_1 \cup C_2 \rceil = \lceil C_1 \rceil + \lceil C_2 \rceil \).

(M3) **Translation Invariance:** For any \( t \in T \), \( \lceil \rho^t(C) \rceil = \lceil C \rceil \).

(M4) **Nontriviality:** \( 0 < \lceil \partial P \rceil < \infty \), and \( 0 < \lceil (\Phi^n(P)) \rceil < \infty \) for all \( n \in \mathbb{N} \).

If \( T = T^1 \), then \( \partial P \) is usually a discrete subset of \( T \), and the obvious function satisfying (M1)-(M4) is \( \lceil C \rceil = \#(C) \) (modulo multiplication by some constant). However, if \( T = T^K \) for \( K \geq 2 \), then condition (M4) makes the choice of \( \lceil \cdot \rceil \) dependent on the geometry of \( P \):

- If \( \partial P \) is a union of piecewise smooth \((K - 1)\)-dimensional submanifolds of \( T^K \), then let \( \lceil \cdot \rceil_* \) be the \((K - 1)\)-dimensional Lebesgue measure. For example, if \( K = 1, 2, \) or \( 3 \), then \( \lceil \cdot \rceil_* \) measures cardinality, length, or surface area, respectively.

- If \( (K - 1) \leq \kappa < K \), and \( \partial P \) has Hausdorff dimension \( \kappa \), then let \( \lceil \cdot \rceil_\kappa \) be the \( \kappa \)-dimensional Hausdorff measure. If \( \kappa = (K - 1) \), then \( \lceil C \rceil_\kappa = \lceil C \rceil_* \).

- For any \( C \subset T \) and \( \epsilon > 0 \), let \( B(C, \epsilon) = \{ t \in T \mid d(t, c) < \epsilon \text{ for some } c \in C \} \). Define \( \lceil C \rceil_L = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \lambda(B(C, \epsilon)) \). If \( C \) is a \((K - 1)\)-dimensional submanifold, then \( \lceil C \rceil_L = \lceil C \rceil_* \). We call \( \lceil \cdot \rceil_L \) the Lipschitz pseudomeasure because of Proposition 5.11 below.

We say \( \Phi \) **chops** \( P \) **on average** if \( \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \lceil (\Phi^n(P)) \rceil = \infty \). Equivalently, there is a subset \( J \subset \mathbb{N} \) of Cesàro density \( 1 \) such that \( \lim_{J \to \infty} \lceil (\Phi^n(P)) \rceil = \infty \).

We say \( \Phi \) **chops** \( P \) **intermittently** if \( \limsup_{n \to \infty} \lceil (\Phi^n(P)) \rceil = \infty \). Equivalently, there is a (possibly zero-density) subset \( J \subset \mathbb{N} \) such that \( \lim_{J \to \infty} \lceil (\Phi^n(P)) \rceil = \infty \). Clearly, if \( \Phi \) chops \( P \) on average, then it does so intermittently (but not conversely).

Note that these definitions depend upon the pseudomeasure \( \lceil \cdot \rceil \). For a fixed choice of \( \lceil \cdot \rceil \), we say \( \Phi \) is \( \lceil \cdot \rceil **-chopping on average** (resp. intermittently) if, for any

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Let \( P \in \mathbb{A}^T \) with \( 0 < [\partial P] < \infty \), \( \Phi \) chops \( P \) on average (resp. intermittently) with respect to \([\ast]\).

Whenever \( \Phi \) chops \( P \), the growth rate of \( [\partial (\Phi^n_\ast P)] \) must be (sub)polynomial:

**Proposition 5.1.** Let \( \mathbb{L} = \mathbb{Z}^D \). Let \( \Phi : \mathbb{A}^D \to \mathbb{A}^D \) be a CA. There is a constant \( C > 0 \) such that, if \( P \in \mathbb{A}^T \) and \( n \in \mathbb{N} \), then \( [\partial (\Phi^n_\ast P)] \leq C \cdot n^D \cdot [\partial P] \).

**Proof.** Suppose \( \Phi \) has local map \( \phi : \mathbb{A}^D \to \mathbb{A}^D \) for some finite \( \mathbb{B} \subseteq \mathbb{L} \). It follows from eqn. 2 that \( \partial (\Phi_\ast P) \subseteq \bigcup_{b \in \mathbb{B}} b \cdot (\partial P) \). Hence, if \( B = \#(\mathbb{B}) \), then

\[
\left[ \partial (\Phi^n_\ast P) \right] \leq \left( \sum_{b \in \mathbb{B}} \left[ b \cdot (\partial P) \right] \right) \leq B \cdot [\partial P] = B \cdot [\partial P].
\]

Let \( \mathbb{B}_n = \{b_1 + \cdots + b_n \mid b_1, \ldots, b_n \in \mathbb{B} \} \). Then \( \Phi^n \) has a local map \( \phi^{(n)} : \mathbb{A}^{\mathbb{B}_n} \to \mathbb{A}^D \); hence, by reasoning similar to (4), we have:

\[
\left[ \partial (\Phi^{(n)}_\ast P) \right] \leq B_n \cdot [\partial P],
\]

where \( B_n = \#(\mathbb{B}_n) \). Find \( R > 0 \) such that \( \mathbb{B} \subseteq [-R \ldots R]^D \). Then \( \mathbb{B}_n \subseteq [-nR \ldots nR]^D \), so

\[
B_n \leq \#([-nR \ldots nR]^D) = C \cdot n^D,
\]

where \( C = (2R+1)^D \). Combine (4) and (6) to get:

\[
\left[ \partial (\Phi^{(n)}_\ast P) \right] \leq Cn^D \cdot [\partial P]. \quad \square
\]

**Chopping in Boolean Linear Cellular Automata:** Let \( \mathbb{A} = \mathbb{Z}_{/2} = \{0, 1\} \). A CA with local map \( \phi : \mathbb{A}^D \to \mathbb{A}^D \) is a **boolean linear cellular automaton** (BLCA) if

\[
\phi(a) = \sum_{b \in \mathbb{B}} a_b \pmod{2}, \quad \text{for any } a \in \mathbb{A}^D.
\]

We assume that \( \Phi \) is ‘nontrivial’ in the sense that \( \#(\mathbb{B}) > 1 \). Thus, Examples 3.3(a) and 3.3(b) were BLCA. We’ll show that BLCA are chopping on average. Then we’ll characterize the asymptotic growth rate of \( [\partial (\Phi^n_\ast P)] \) in a special case.

**Proposition 5.2.** Let \( P = \{P_0, P_1\} \) be an \( \mathbb{A} \)-indexed open partition of \( \mathbb{T} \). Then

\[
(a) \: \partial(P) = \partial(P_1) \quad \text{and} \quad (b) \: \bigcup_{b \in \mathbb{B}} b \cdot (\partial P) \subseteq \partial (\Phi_\ast P) \subseteq \bigcup_{b \in \mathbb{B}} b \cdot (\partial P).
\]

**Proof.** (a) is immediate. For (b), recall from Example 3.3(d) that \( \Phi_\ast (P) = \{Q_0, Q_1\} \), where \( Q_1 = \bigcup_{b \in \mathbb{B}} b \cdot (P_1) \), and \( Q_0 = \mathbb{T} \setminus Q_1 \). Now apply (a).

**Example 5.3.** Let \( \mathbb{L} = \mathbb{Z} \), \( \mathbb{T} = \mathbb{T}^1 \), and \( a \in (0, 1) \), as in Example 1.2. Assume \( a < \frac{1}{2} \).
(a) If \( \mathbf{P}_1 = (0, a) \) and \( \mathbf{P}_0 = (a, 1) \), then \( \partial \mathcal{P} = \{0, a\} \). If \( \Phi \) is as in Example 5.3(a) and \( \mathcal{Q} = \Phi_\beta(\mathcal{P}) \), then \( \mathcal{Q}_1 = (0, 2a) \) and \( \mathcal{Q}_0 = (2a, 1) \). Thus, \( \partial \mathcal{Q} = \{0, 2a\} = \{0, a\} \triangle \{a, 2a\} = \partial \mathcal{P} \triangle \varsigma^1(\partial \mathcal{P}) \). (See Figure 11 bottom left.)

(b) Let \( b < a \), and let \( \mathbf{P}_1 = (0, b) \) and \( \mathbf{P}_0 = (b, 1) \). Then \( \mathcal{Q}_1 = (0, b) \sqcup (a, b + a) \) and \( \mathcal{Q}_0 = (b, a) \sqcup (b + a, 1) \). Thus, \( \partial \mathcal{Q} = \{0, b, a, b + a\} = \{0, b\} \triangle \{a, b + a\} = \partial \mathcal{P} \triangle \varsigma^1(\partial \mathcal{P}) \). ♦

If \( t \in \mathbf{T} \), recall that the \( \varsigma \text{-orbit} \) of \( t \) is the set \( \Theta_t = \{ \varsigma^j(t) \}_{j \in \mathbb{L}} \). Let \( \mathcal{O} \) be the set of all \( \varsigma \) -orbits in \( \mathbf{T} \); then \( \mathbf{T} = \bigcup_{\Theta \in \mathcal{O}} \Theta \); so we can write \( \partial \mathcal{P} \) as a disjoint union:

\[
\partial \mathcal{P} = \bigcup_{\Theta \in \mathcal{O}} \partial_\Theta \mathcal{P}.
\]

where \( \partial_\Theta \mathcal{P} := \Theta \cap \partial \mathcal{P} \) for all \( \Theta \in \mathcal{O} \). The decomposition in eqn. (8) commutes with the action of \( \Phi_\beta \) on \( \partial \mathcal{P} \). That is, \( \partial \left( \Phi_\beta(\mathcal{P}) \right) = \bigcup_{\Theta \in \mathcal{O}} \partial_\Theta \left( \Phi_\beta(\mathcal{P}) \right) \), and Proposition 5.2(b) says:

For all \( \Theta \in \mathcal{O} \), \( \partial_\Theta \left( \Phi_\beta(\mathcal{P}) \right) \supseteq \bigtriangleup_{b \in \mathbf{A}} \beta^b(\partial_\Theta \mathcal{P}) \).

For each \( \Theta \in \mathcal{O} \), fix a representative \( t_\Theta \in \Theta \), and define \( \beta_\Theta : \mathbf{A}^T \rightarrow \mathbf{A}^\mathbf{L} \) as follows: for any \( \mathcal{P} \in \mathbf{A}^T \), let \( \beta_\Theta(\mathcal{P}) := b \), where \( (b_{\ell} = 1) \iff \left( \varsigma^j(t_\Theta) \in \partial_\Theta(\mathcal{P}) \right) \). Then eqn. (9) implies:

\[
\beta_\Theta \left( \Phi_\beta(\mathcal{P}) \right) \supseteq \Phi \left( \beta_\Theta(\mathcal{P}) \right) \text{ (componentwise).}
\]

Example 5.4. Let \( \Phi, \mathcal{P} \) and \( \mathcal{Q} = \Phi_\beta(\mathcal{P}) \) be as in Example 5.3(a); then all elements of \( \partial \mathcal{P} = \{0, a\} \) and \( \partial \mathcal{Q} = \{0, 2a\} \) belong to the orbit \( \mathcal{O}_0 \) of zero, and

\[
\beta_0(\mathcal{P}) = [\ldots 0 0 0 1 0 0 0 \ldots] \\
\text{while} \quad \beta_0(\mathcal{Q}) = [\ldots 0 0 0 1 0 1 0 0 \ldots] = \Phi(\beta_0(\mathcal{P})).
\]

(wherethe zeroth element of each sequence is underlined) ♦

If \( b \in \mathbf{A}^\mathbf{L} \), let \( \text{supp}(b) := \{ \ell \in \mathbf{L} ; b_{\ell} = 1 \} \). Thus,

\[
\partial_\Theta \mathcal{P} = \{ \varsigma^j(t_\Theta) ; \ell \in \text{supp}(\beta_\Theta(\mathcal{P})) \}.
\]

Hence, the growth of partition boundaries under the action of \( \Phi_\beta \) is directly related to the growth in the support of boolean configurations under the action of \( \Phi \).

**Proposition 5.5.** Let \( \Phi \) be any BLCA.

(a) For all \( n \in \mathbb{N} \), let \( \mathbb{B}_n \subseteq \mathbb{L} \) be such that \( \Phi^n \) has local map \( \phi_n(a) = \sum_{b \in \mathbb{B}_n} a_b \) (mod 2). There is a subset \( \mathcal{J} \subseteq \mathbb{N} \) with density (\( \mathcal{J} \)) = 1 such that \( \lim_{\mathcal{J} \rightarrow \infty} \#(\mathcal{B}_j) = \infty \).

(b) For all \( a \in \mathbf{A}^\mathbf{L} \) with finite support, there is a subset \( \mathcal{J} \subseteq \mathbb{N} \) with density (\( \mathcal{J} \)) = 1 such that \( \lim_{\mathcal{J} \rightarrow \infty} \#(\text{supp}(\Phi^j(a))) = \infty \).
Proof. See Theorem 15 of [35].

Corollary 5.6. If \( T = T^1 \) and \( \Phi \) is any nontrivial BLCA, then \( \Phi_\zeta \) is \#-chopping on average.

Proof. Let \( P \in \mathcal{A}^T \), and suppose \( \partial P \) is finite. For each \( O \in \mathcal{D} \), let \( b_o := \beta_o(P) \); then \( \text{supp}(b_o) \) is finite, and is nontrivial for only finitely many \( O \in \mathcal{D} \). Thus, for any \( j \in \mathbb{N} \),

\[
\# \left( \partial (\Phi_{\zeta}^j [P]) \right) \geq \sum_{O \in \mathcal{D}} \# \left( \partial (\Phi_{\zeta}^j [P]) \right) \equiv \sum_{O \in \mathcal{D}} \# \left( \text{supp}(\beta_o(\Phi_{\zeta}^j [P])) \right),
\]

where \((i)\) is by eqn. (11); \((\dagger)\) is by eqn. (10).

For each \( O \in \mathcal{D} \), Proposition 5.6 yields some \( J_o \subseteq \mathbb{N} \) such that \( \text{density}(J_o) = 1 \) and \( \lim_{J_o \to \infty} \# \left( \text{supp}(\Phi_{\zeta}^j(b_o)) \right) = \infty \). Let \( J := \bigcup_{O \in \mathcal{D}} J_o \); then density \( J = 1 \), and eqn. (12) implies that \( \lim_{J_o \to \infty} \# \left( \partial (\Phi_{\zeta}^j [P]) \right) = \infty \); hence \( \Phi_{\zeta} \) chops \( P \) on average.

To generalize Corollary 5.6 to \( T^K \) \((K > 1)\), we need some notation. If \( S \subseteq T \) is some subset, then many points in \( S \) may share the same \( \zeta \)-orbit. Define

\[
\zeta(S) := S \setminus \bigcup_{0 \neq \ell \in \mathbb{L}} \zeta^\ell(S) = \{ s \in S : O_s \cap S = \{ s \} \}.
\]

Let \( \mathcal{A}^T := \{ P \in \mathcal{A}^T : \zeta^1(\partial P) > 0 \} \). For example, if \( T = T^1 \) and \( [\bullet] = \#(\bullet) \), then

\[
(P \in \mathcal{A}^T) \iff (\zeta^1(\partial P) \neq \emptyset) \iff \text{(There is some } s \in \partial P \text{ which is not in the orbit of any other } t \in \partial P \text{)}.
\]

Lemma 5.7. If \( \Phi \) is the BLCA \( [\bullet] \), then \( \#(\mathbb{B}) \cdot [\zeta^1(\partial P)] \leq [\partial (\Phi_{\zeta}(P))] \leq \#(\mathbb{B}) \cdot [\partial P] \).

Proof. Let \( S = \zeta^1(\partial P) \) and let \( U = \partial P \setminus S \). By definition of \( S \), the sets \( \{ \zeta^b(S) \}_{b \in \mathbb{B}} \) are disjoint both from one another and from the set \( \bigcup_{b \in \mathbb{B}} \zeta^b(U) \). Thus, Proposition 5.2(b) says:

\[
\bigcup_{b \in \mathbb{B}} \zeta^b(S) \subseteq \bigtriangleup_{b \in \mathbb{B}} \zeta^b(\partial P) \subseteq \partial (\Phi_{\zeta}(P)) \subseteq \bigcup_{b \in \mathbb{B}} \zeta^b(\partial P).
\]

Thus, \( \#(\mathbb{B}) \cdot \#(S) \equiv \sum_{b \in \mathbb{B}} [\zeta^b(S)] \equiv \left[ \bigcup_{b \in \mathbb{B}} \zeta^b(S) \right] \leq \left[ \partial (\Phi_{\zeta}(P)) \right] \leq \sum_{b \in \mathbb{B}} [\zeta^b(\partial P)] \equiv \#(\mathbb{B}) \cdot [\partial P].\)

The (M1) inequalities follow from eqn. (13) and property (M1) of \([\bullet]\).
Proposition 5.8. Let $T = T^K$. If $\mathcal{P} \in \mathcal{A}^T$, then any nontrivial BLCA chops $\mathcal{P}$ on average.

Proof. Let $\Phi$ be a BLCA. For all $n \in \mathbb{N}$, let $B_n \subset \mathbb{L}$ be such that $\Phi^n$ has local map $\phi_n(a) = \sum_{b \in B_n} a_b \pmod{2}$. Thus, Lemma 5.9 says that $\lceil \partial(\Phi^n(\mathcal{P})) \rceil \geq \#(B_n) \cdot \lceil \zeta^{-1}(\partial \mathcal{P}) \rceil$. Recall $\mathcal{P} \in \mathcal{A}^T$, so $\lceil \zeta^{-1}(\partial \mathcal{P}) \rceil > 0$. Proposition 5.8(a) yields a subset $J \subset \mathbb{N}$ of density one such that $\lim_{j \to \infty} \#(B_j) = \infty$. Thus,

$$\lim_{j \to \infty} \lceil \partial(\Phi^n(\mathcal{P})) \rceil = \infty.$$ 

Let $\mathcal{A}^T := \left\{ \mathcal{P} \in \mathcal{A}^T; \lceil \zeta^{-1}(\partial \mathcal{P}) \rceil = \lceil \partial \mathcal{P} \rceil \right\}$. For example, if $T = T^1$ and $[\cdot] = \#(\cdot)$, then

$$\left( \mathcal{P} \in \mathcal{A}^T \right) \iff \left( \zeta^{-1}(\partial \mathcal{P}) = \partial \mathcal{P} \right) \iff \left( \text{Every element of } \partial \mathcal{P} \text{ occupies a distinct } \zeta\text{-orbit} \right).$$

Lemma 5.9. $\mathcal{A}^T$ is a $d_\Delta$-dense subset of $\mathcal{A}^T$.

Proof. Let $\frac{1}{2}\mathcal{A}^T$ be the set of dyadic open partitions of $T$ (see §4.1). Proposition 4.2 says that $\frac{1}{2}\mathcal{A}^T$ is a $d_\Delta$-dense subset of $\mathcal{A}^T$. Thus, it suffices to show that $\frac{1}{2}\mathcal{A}^T \subseteq \mathcal{A}^T$.

To see this, suppose $\mathcal{D} \in \frac{1}{2}\mathcal{A}^T$ is an $n$-dyadic partition for some $n \in \mathbb{N}$. Identify $T \cong [0,1]^K$ as usual. Let $E = \{ \frac{1}{2}^k \}_{k=0}^{\infty}$ be the set of $n$-dyadic numbers, and for each $k \in [1..K]$, let $C_k = [0,1]^{K-1} \times E \times [0,1]^{K-k}$. If $C = \bigcup_{k=1}^{K} C_k$, then $\partial \mathcal{D} \subset C$.

For any nonzero $\ell \in \mathbb{N}$, $[\zeta^\ell(C) \cap C] = 0$, because $\zeta^\ell(C)$ and $C$ intersect transversely. But $\partial \mathcal{D} \subset C$, so $[\zeta^\ell(\partial \mathcal{D}) \cap (\partial \mathcal{D})] = 0$ also. Thus, $[\zeta^\ell(\partial \mathcal{D})] = [\partial \mathcal{D}]$. Hence $\mathcal{D} \in \mathcal{A}^T$.

Since this holds for any $\mathcal{D} \in \frac{1}{2}\mathcal{A}^T$, we conclude that $\frac{1}{2}\mathcal{A}^T \subseteq \mathcal{A}^T$, as desired. $\square$

We’ll now precisely characterize the growth rate of $\lceil \partial(\Phi^n(\mathcal{P})) \rceil$ for a particular BLCA.

Proposition 5.10. Let $\Phi: \mathcal{A}^T \to \mathcal{A}^T$ be the BLCA of Example 3.3(a). Let $\zeta$ be a $\mathbb{Z}$-action on $T = T^K$, and let $\mathcal{P} \in \mathcal{A}^T$. Then, as $n \to \infty$...

(a) ...the maximum of $\lceil \partial(\Phi^n(\mathcal{P})) \rceil$ grows linearly. That is:

$$0 < \left\lceil \zeta^{-1}(\partial \mathcal{P}) \right\rceil \leq \limsup_{n \to \infty} \frac{1}{n} \left\lceil \partial(\Phi^n(\mathcal{P})) \right\rceil \leq \lceil \partial \mathcal{P} \rceil.$$

(b) ...the minimum of $\lceil \partial(\Phi^n(\mathcal{P})) \rceil$ remains constant:

$$\liminf_{n \to \infty} \left\lceil \partial(\Phi^n(\mathcal{P})) \right\rceil \leq 2 \lceil \partial \mathcal{P} \rceil.$$
Proof. For any $n \in \mathbb{N}$, let $\mathcal{B}_n \subset \mathbb{L}$ be such that $\Phi^n$ has local map $\phi_n(a) = \sum_{b \in \mathcal{B}_n} a_b \pmod{2}$. Let $\nu(n)$ be the number of 1’s in the binary expansion of $n$.

Claim 1. For any $n \in \mathbb{N}$, \[ 2^{\nu(n)} \left\lceil \frac{1}{n} \right\rceil \leq \left\lceil \partial (\Phi^n[\mathcal{P}]) \right\rceil \leq 2^{\nu(n)} \left\lfloor \partial \right\rfloor, \]
with equality when $\mathcal{P} \in \mathcal{T}$. \hfill $\diamond$ claim 1

(a) To see that $\limsup_{n \to \infty} \frac{1}{n} \left\lceil \partial (\Phi^n[\mathcal{P}]) \right\rceil \leq \left\lceil \partial \right\rceil$, observe that $\mathcal{B}_n \subset \{0..n\}$. Thus, $\#(\mathcal{B}_n) \leq \#(0..n) = n + 1$. Thus,
\[ \limsup_{n \to \infty} \frac{1}{n} \left\lceil \partial (\Phi^n[\mathcal{P}]) \right\rceil \leq \limsup_{n \to \infty} \frac{1}{n} \#(\mathcal{B}_n) \cdot \left\lceil \partial \right\rceil \leq \lim_{n \to \infty} \frac{n + 1}{n} \cdot \left\lceil \partial \right\rceil, \]
where $(\ast)$ is by Lemma 5.7.

To see that $\limsup_{n \to \infty} \frac{1}{n} \left\lceil \partial (\Phi^n[\mathcal{P}]) \right\rceil \geq \left\lceil \partial \right\rceil$, let $n = 2^m - 1$ for some $m \in \mathbb{N}$. Then $\nu(n) = m$, so Claim 1 says $\left\lceil \partial (\Phi^n[\mathcal{P}]) \right\rceil \geq 2^\nu(n) \cdot \left\lceil \partial \right\rceil = 2^m \cdot \left\lceil \partial \right\rceil$. Thus,
\[ \limsup_{n \to \infty} \frac{\left\lceil \partial (\Phi^n[\mathcal{P}]) \right\rceil}{n} \geq \limsup_{m \to \infty} \frac{\left\lceil \partial \Phi^{(2^m-1)}(\mathcal{P}) \right\rceil}{2^m - 1} \geq \lim_{m \to \infty} \frac{2^m \cdot \left\lceil \partial \right\rceil}{2^m - 1} = \left\lceil \partial \right\rceil. \]

(b) If $n = 2^m$ for some $m \in \mathbb{N}$, then $\nu(n) = 1$, so Claim 1 says $\left\lceil \partial (\Phi^n[\mathcal{P}]) \right\rceil \leq 2^\nu(n) \cdot \left\lceil \partial \right\rceil = 2 \cdot \left\lceil \partial \right\rceil$. Thus, $\liminf_{n \to \infty} \left\lceil \partial (\Phi^n[\mathcal{P}]) \right\rceil \leq \liminf_{m \to \infty} \left\lceil \partial \Phi^{(2^m)}(\mathcal{P}) \right\rceil = 2 \cdot \left\lceil \partial \right\rceil$.

(d) follows from the second part of Claim 1

(c) For any $n \in \mathbb{N}$, let $f(n) = 2^\nu(n)$, and for any $N \in \mathbb{N}$, let $\tilde{A}(N) = \frac{1}{N} \sum_{n=0}^{N-1} f(n)$. Thus, Claim 1 implies that $\tilde{A}(N) \cdot \left\lceil \partial \right\rceil \leq A(N) \leq \tilde{A}(N) \cdot \left\lceil \partial \right\rceil$. Hence,
\[ 1 = \lim_{N \to \infty} 1 + \frac{\log \left\lceil \partial \right\rceil}{\log(A(N))} \leq \lim_{N \to \infty} \frac{\log(A(N))}{\log(A(N))} \leq \lim_{N \to \infty} 1 + \frac{\log \left\lceil \partial \right\rceil}{\log(A(N))} = 1. \]
Thus, it suffices to examine the asymptotics of $\tilde{A}(N)$.

Suppose $N = 2^{M_0} + 2^{M_1} + \cdots + 2^{M_J}$ for some $M_0 > M_1 > \cdots > M_J$.

**Claim 2.** $\tilde{A}(N) = \frac{1}{N} \sum_{j=0}^{J} 2^j \cdot 3^{M_j}$

**Proof.** Let $n$ be a random element of $[0..N)$ (with uniform distribution). Then $f(n)$ is also a random variable, and $A(N) = \mathbb{E}(f(n))$ is the expected value of $f(n)$. Let $I_0 := [0..2^{M_0})$, $I_1 := [2^{M_0}..2^{M_0}+2^{M_1})$, and in general,

$$I_j := \left[ \sum_{i=0}^{j-1} 2^{M_i} \ldots \sum_{i=0}^{j} 2^{M_i} \right], \quad \text{for all } j \in [0..J].$$

Then $[0..N) = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_J$, and for all $j \in [0..J]$, $\mathbb{P}(n \in I_j) = 2^{M_j}/N$.

**Claim 2.1.** Suppose $n \in I_0$. Then $\mathbb{E}(f(n) \mid n \in I_0) = \left(\frac{3}{2}\right)^{M_0}$.

**Proof.** Write $n$ in binary notation. Then the $M_0$ binary digits of $n$ are independent, equidistributed boolean random variables, so $\nu(n)$ is a random variable with a binomial distribution:

$$\mathbb{P}(\nu(n) = m) = \frac{1}{2^{M_0}} \binom{M_0}{m} \quad \text{for any } m \in [0..M_0]. \quad (14)$$

Thus, $\mathbb{E}(f(n)) = \sum_{m=0}^{M_0} 2^m \cdot \mathbb{P}(f(n) = 2^m) = \sum_{m=0}^{M_0} 2^m \cdot \mathbb{P}(\nu(n) = m)$

$$= \left(\frac{3}{2}\right)^{M_0} \sum_{m=0}^{M_0} 2^m \cdot \binom{M_0}{m} = \left(\frac{3}{2}\right)^{M_0},$$

where $(\ast)$ is by eqn. (14), and (B) is the Binomial Theorem. \hfill $\nabla$ Claim 2.1

**Claim 2.2.** For any $j \in [0..J]$, $\mathbb{E}(f(n) \mid n \in I_j) = 2^j \cdot \left(\frac{3}{2}\right)^{M_j}$.

**Proof.** If $j = 0$, this is Claim 2.1. Let $j \geq 1$. If $n \in I_j$, then $n = 2^{M_0} + \cdots + 2^{M_{j-1}} + n_1$, for some $n_1 \in [0..2^{M_j})$. Thus, $\nu(n) = j + \nu(n_1)$, so that $f(n) = 2^j \cdot f(n_1)$. Thus

$$\mathbb{E}(f(n) \mid n \in I_j) = 2^j \cdot \mathbb{E}(f(n_1) \mid n_1 \in [0..2^{M_j}]) = 2^j \cdot \left(\frac{3}{2}\right)^{M_j},$$

where $(\ast)$ is like Claim 2.1. \hfill $\nabla$ Claim 2.2

It follows that $\tilde{A}(N) = \mathbb{E}(f(n)) = \sum_{j=0}^{J} \mathbb{E}(f(n) \mid n \in I_j) \cdot \mathbb{P}(n \in I_j)$

$$\equiv \sum_{j=0}^{J} 2^j \cdot \left(\frac{3}{2}\right)^{M_j} \cdot \frac{2^{M_j}}{N} = \frac{1}{N} \sum_{j=0}^{J} 2^j \cdot 3^{M_j},$$

where $(\ast)$ is by Claim 2.2. \hfill $\Diamond$ Claim 2

**Claim 3.** $\tilde{A}(N) > \frac{1}{3} N^\alpha$. 

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Proof. \[ \frac{1}{N} \sum_{j=0}^{J} 2^j \cdot 3M_j \geq \frac{3M_0}{N} \geq \frac{3}{2} \frac{M_0}{N} \cdot \frac{1}{3} \frac{2^j}{2M_0+1} = \frac{1}{3} 2^{\alpha(M_0+1)} \geq \frac{1}{3} N^\alpha. \]

({\text{C2}}) is by Claim 2 and the (*) inequalities are because \( N < 2^M_0 + 1 \). \( \diamond \) claim 3

Claim 4. \( \tilde{A}(N) \leq 3 \cdot N^\alpha. \)

Proof. Since \( M_0 > M_1 > \cdots > M_J \), we know that \( M_j \leq M_0 - j \) for all \( j \in \{1..J\} \).

Thus,

\[
\tilde{A}(N) \leq \frac{1}{N} \sum_{j=0}^{J} 2^j \cdot 3M_j \leq \frac{1}{N} \sum_{j=0}^{J} 2^j \cdot 3M_0 - j = \frac{3M_0}{N} \sum_{j=0}^{J} \left( \frac{2}{3} \right)^j
= \frac{3M_0}{N} \frac{1 - \left( \frac{2}{3} \right)^{J+1}}{1 - \frac{2}{3}} \leq \frac{3M_0}{N} \left( 1 - \left( \frac{2}{3} \right)^{J+1} \right) \leq \frac{3M_0 + 1}{N}
= 3 \cdot \frac{3M_0}{N} \leq \frac{3 \cdot 3M_0}{2M_0} = 3 \cdot 2^\alpha M_0 \leq 3 \cdot N^\alpha
\]

({\text{C2}}) is by Claim 2 and the (*) inequalities are because \( 2^M_0 \leq N \). \( \diamond \) claim 4

Combining Claims 3 and 4 yields \( \frac{1}{3} N^\alpha < \tilde{A}(N) \leq 3 \cdot N^\alpha \).

Hence, \( \alpha \log(N) - \log(3) < \log(\tilde{A}(N)) \leq \alpha \log(N) + \log(3) \);

Hence, \( \alpha - \frac{\log(3)}{\log(N)} < \frac{\log(\tilde{A}(N))}{\log(N)} \leq \alpha + \frac{\log(3)}{\log(N)}. \)

Taking the limit as \( N \to \infty \), we conclude that \( \lim_{N \to \infty} \frac{\log(\tilde{A}(N))}{\log(N)} = \alpha \), as desired. \( \square \)

Remarks. (i) Proposition 5.10(b) shows that, in general, we can only expect chopping to occur along a subset of \( \mathbb{J} \subset \mathbb{N} \) of density one.

(ii) We proved Proposition 5.10 for the very simple BLCA of Example 3.3. Similar results are probably true for arbitrary BLCA, but the appropriate version of Claim 4 will be much more complex in general, leading to more complex formulae in parts (a), (b) and (c).

The Lipschitz pseudomeasure: Recall that \( [S]_L = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \lambda(B(S, \epsilon)) \).

This is an increasing limit, because for any \( \epsilon > 0 \) and \( n \in \mathbb{N} \), \( \lambda (B(S, \epsilon)) \leq n \cdot \lambda (B(S, 1/n)) \).

Thus:

\[
\text{For all } \epsilon > 0, \quad \lambda (B(S, \epsilon)) \leq 2\epsilon \cdot [S]_L. \quad (15)
\]

If \( P \in A^\ast \), then \( [\partial P]_L \) affects the continuity properties of \( P_\epsilon \) as follows:
Proposition 5.11. Let $\mathcal{P} \subseteq \mathcal{A}^T$, and endow $\mathcal{A}^L$ with the Besicovitch metric. Then:

(a) $\mathcal{P}_* : \overline{T} \to \mathcal{A}^L$ is Lipschitz, with Lipschitz constant $\|\partial \mathcal{P}\|_L$. That is, for any $s, t \in \overline{T}$, 
$$d_B\left(\mathcal{P}_*(s), \mathcal{P}_*(t)\right) \leq \|\partial \mathcal{P}\|_L \cdot d(s, t).$$

(b) Let $T = T^1$ (so $|\mathcal{P}|_L = \#(\mathcal{P})$), and let $\langle \mathcal{P} \rangle = \min \left\{ d(b_1, b_2) : b_1, b_2 \in \partial \mathcal{P} \text{ distinct} \right\}$.
For any $s, t \in \overline{T}$, 
$$\left( d(s, t) \leq \langle \mathcal{P} \rangle \right) \implies \left( d_B\left(\mathcal{P}_*(s), \mathcal{P}_*(t)\right) = \#(\partial \mathcal{P}) \cdot d(s, t) \right).$$

Proof. (a) Identify $T \cong [0, 1)^K$. For simplicity assume $s = 0$, and let $t = (t_1, \ldots, t_K)$, where $t_k \in [0, 1]$ for all $k \in [1..K]$. Assuming that $\|t_1, \ldots, t_K\| \leq ||(1-t_1), \ldots, (1-t_K)||$, the shortest path from 0 to $t$ is along the line segment $I = \{ \langle rt_1, \ldots, rt_K \rangle : r \in [0, 1] \}$. Thus, if $d(0, t) = \epsilon$, then length $|I| = \epsilon$. Let $m = (\frac{1}{2}t_1, \ldots, \frac{1}{2}t_K)$ be the midpoint of $I$, and let $B(m, \frac{1}{2})$ be the ball of radius $\epsilon/2$ about $m$. Then, for any $\ell \in \mathbb{L}$,

$$\left( \mathcal{P}_*(0) \neq \mathcal{P}_*(t) \right) \iff \left( \mathcal{P}(\zeta^\ell(0)) \neq \mathcal{P}(\zeta^\ell(t)) \right) \iff \left( \partial \mathcal{P} \cap B(\zeta^\ell(m), \frac{1}{2}) \neq \emptyset \right) \iff \left( \zeta^\ell(m) \in B(\partial \mathcal{P}, \frac{1}{2}) \right).$$

(*) is because $I \subseteq B(m, \frac{1}{2})$. If $K = 1$, then $I = B(m, \frac{1}{2})$, and the “$\iff$” is actually a “$\iff$”: if also $\epsilon < \langle \mathcal{P} \rangle$, then the “$\iff$” is also a “$\iff$”. Thus,
$$d_B\left(\mathcal{P}_*(0), \mathcal{P}_*(t)\right) = \text{density}(\ell \in \mathbb{L} ; \mathcal{P}_*(0) \neq \mathcal{P}_*(t)) \leq \text{density}(\ell \in \mathbb{L} ; \zeta^\ell(m) \in B(\partial \mathcal{P}, \frac{1}{2})) \equiv \lambda\left[B(\partial \mathcal{P}, \frac{1}{2})\right].$$

(*) is by (16), and is an equality if $K = 1$ and $\epsilon < \langle \mathcal{P} \rangle$. (†) is by Proposition 5.11(b). Thus,
$$d_B\left(\mathcal{P}_*(0), \mathcal{P}_*(t)\right) \overset{(\cdot)}{\leq} \lambda\left[B(\partial \mathcal{P}, \frac{1}{2})\right] \overset{(\cdot)}{\leq} \epsilon \cdot |\partial \mathcal{P}|_L,$$

where (†) is by eqn (16). If $K = 1$ and $\epsilon < \langle \mathcal{P} \rangle$, then (*) is again an equality.

(b) Suppose $K = 1$ and $\partial \mathcal{P}$ is finite. If $\epsilon < \langle \mathcal{P} \rangle$, then $B(\partial \mathcal{P}, \frac{1}{2}) = \bigsqcup_{b \in \partial \mathcal{P}} B(b, \frac{1}{2})$.
Thus, $d_B\left(\mathcal{P}_*(0), \mathcal{P}_*(t)\right) \overset{(\cdot)}{=} \lambda\left[B(\partial \mathcal{P}, \frac{1}{2})\right] = \sum_{b \in \partial \mathcal{P}} \lambda\left[B(b, \frac{1}{2})\right] = \sum_{b \in \partial \mathcal{P}} \epsilon = \epsilon \cdot |\partial \mathcal{P}|. \quad \square$

Note: The equality in Proposition 5.11(b) fails if $d(s, t) > \langle \mathcal{P} \rangle$. For example, let $A = \{0, 1\}$, and let $\mathcal{P}_0 = \{0, \frac{1}{2}\} \cup \left(\frac{2}{3}, \frac{1}{2}\right)$, while $\mathcal{P}_1 = \left(\frac{1}{4}, \frac{3}{4}\right) \cup \left(\frac{1}{2}, 1\right)$. Thus, $|\partial \mathcal{P}| = 4$. However, $\langle \mathcal{P} \rangle = \frac{249}{1000} < \frac{1}{2}$, and if $d(s, t) = \frac{1}{2} > \langle \mathcal{P} \rangle$, then $d_B\left(\mathcal{P}_*(s), \mathcal{P}_*(t)\right) = \frac{1}{1000} \neq 4 \cdot d(s, t).$
Sensitivity and $[\bullet]_L$-Chopping Let $(X, d, \varphi)$ be a topological dynamical system, and let $\xi > 0$. If $x, y \in X$, then $(x, y)$ is a $\xi$-expansive pair if $d(\varphi^n(y), \varphi^n(y_1)) > \xi$ for some $n \in \mathbb{N}$. We say $(X, d, \varphi)$ is $\xi$-expansive (see [12] if $(x, y)$ is $\xi$-expansive for every $x, y \in X$ with $x \neq y$. If $Y \subseteq X$ is a subset (not necessarily $\varphi$-invariant), then $\varphi|_Y$ is $\xi$-sensitive if for all $y \in Y$, and all $\delta > 0$, there some $y_1 \in Y$ with $d(y, y_1) < \delta$ such that $(y, y_1)$ is a $\xi$-expansive pair. Thus, if $(X, d, \varphi)$ is $\xi$-expansive, then $\phi|_Y$ is $\xi$-sensitive for any $Y \subseteq X$.

**Proposition 5.12.** (a) If $P \in \mathcal{A}^T$, and $\mathcal{P} = \mathcal{P}_\xi(\mathcal{T})$, then

$$\big( (\mathcal{P}, d_B, \Phi) \text{ is sensitive} \big) \Rightarrow \big( \Phi_\xi \text{ intermittently } [\bullet]_L \text{-chops } P \big).$$

(b) Thus, $\big( (\Omega \mathcal{E}_\xi, d_B, \Phi) \text{ is expansive} \big) \Rightarrow \big( \Phi_\xi \text{ is intermittently } [\bullet]_L \text{-chopping} \big)$.

**Proof.** (b) follows from (a), so we'll prove (a).

**Case 1:** $(P$ is simple) Let $p_0 = P_\xi(0)$, and fix $\epsilon > 0$. Proposition [17](b) says $P_\xi$ is a homeomorphism from $(T, d)$ to $(\mathcal{P}, d_B)$. Thus, there is some $\delta > 0$ such that, for any $t \in T$, $\big( d_B(P_\xi(0), P_\xi(t)) < \delta \big) \Rightarrow \big( d(0, t) < \epsilon \big)$.

Suppose $(\mathcal{P}, d_B, \Phi)$ is $\xi$-sensitive for some $\xi > 0$. Then there is some $p_0 \in \mathcal{P}$ with $d_B(p_0, p_1) < \delta$, but $d_B \big( \Phi^n(p_0), \Phi^n(p_1) \big) > \xi$ for some $n \in \mathbb{N}$.

Suppose $p_1 = P_\xi(t_1)$, where $t_1 \in T$ and $d(0, t_1) < \epsilon$. Let $P^{(n)} = \Phi^n(P)$. Thus, we have

$$\frac{\xi}{\epsilon} \leq \frac{d_B \big( P^{(n)}(0), P^{(n)}(t_1) \big)}{d(0, t_1)} < \left[ \partial P^{(n)} \right]_L,$$

where $(\ast)$ is by Proposition [12](a). But $\epsilon$ can be made arbitrarily small. Hence, $\left[ \partial P^{(n)} \right]_L$ can become arbitrarily large as $n \to \infty$.

**Case 2:** $(P$ is not simple) Use Lemma [1.8](b) to replace $P$ with a ‘quotient’ partition $\mathcal{P}$ on a quotient torus $\mathcal{T}$, with a quotient $L$-action $\xi$, such that $\mathcal{T}$ is simple.

**Claim 1.** Let $S$ be the symmetry group of $P$, and let $q : T \to \mathcal{T}$ be the quotient map, as in Lemma [1.16](b). Let

$$^0\mathcal{A}^T_S := \{ Q \in ^0\mathcal{A}^T ; \text{ all elements of } S \text{ are symmetries of } Q \}.$$

(a) For any $Q \in ^0\mathcal{A}^T_S$, there is a unique partition $\mathcal{Q} \subseteq ^0\mathcal{A}^T$ such that $\mathcal{Q} \circ q = Q$.

(b) $\Phi_\xi \big( ^0\mathcal{A}^T_S \big) \subseteq ^0\mathcal{A}^T_S$. In particular, if $Q = \Phi_\xi(P)$, then $Q \in ^0\mathcal{A}^T_S$.

(c) Let $\Phi_\xi : ^0\mathcal{A}^T \to ^0\mathcal{A}^T$ be the induced map on partitions of $\mathcal{T}$. Then $\Phi_\xi(\mathcal{P}) = \mathcal{P}$.

(d) There is a constant $C > 0$ such that, for any $Q \in ^0\mathcal{A}^T_S$, $\left[ \partial Q \right]_L = C \cdot \left[ \partial \mathcal{Q} \right]_L$.

(e) Hence, $\big( \Phi_\xi \text{ intermittently chops } P \big) \iff \big( \Phi_\xi \text{ intermittently chops } \mathcal{P} \big)$.
Suppose Proposition 6.2. (Theorem 6.1.) We will prove: To prove Theorem 6.1, recall that \( P \in A \) is injective when it is restricted to a quasisturmian shift? We will prove:

\[
\text{(a)} \quad \text{Let } \nu = \Upsilon_\zeta(P), \text{ then either } \Phi \text{ is constant } (\mu,\omega), \text{ or } \Phi \text{ is injective } (\mu,\omega).
\]

\[
\text{(b)} \quad \text{Let } \nu = \Upsilon_\zeta(P), \text{ then } \Phi|_{\mathcal{Q}} \text{ is constant, or } \Phi|_{\mathcal{Q}} \text{ is injective.}
\]

To prove Theorem 6.1, recall that \( \mathcal{P} \) is \textit{simple} if \( \mathcal{P} \) has no translational symmetries (6.2).

Proposition 6.2. Suppose \( \mathcal{P} \) and \( \mathcal{Q} = \Phi_\zeta(\mathcal{P}) \) are both simple.

\[
\text{(a)} \quad \text{If } \mathcal{P} \in \mathcal{T} \text{ and } \mu = \Upsilon_\zeta(\mathcal{P}), \text{ then } \Phi \text{ is injective } (\mu,\omega).
\]

\[
\text{(b)} \quad \text{If } \mathcal{P} \in \mathcal{T} \text{ and } \mathcal{Q} = \Xi_\zeta(\mathcal{P}), \text{ then } \Phi|_{\mathcal{Q}} \text{ is injective.}
\]

Proof. (a) Let \( \nu = \Upsilon_\zeta(\mathcal{Q}) \). Then Lemma 6.2(c) says that \( \nu = \Phi(\mu) \). Let 
\[
\mathcal{M}_\Phi := \{ q \in \mathcal{A}^\zeta : q \text{ has multiple } \Phi \text{-preimages in } \mathcal{P} \}.
\]

We must show that \( \nu[\mathcal{M}_\Phi] = 0 \).

Let \( \mathcal{M}_{\mathcal{Q}_{\phi}} := \{ q \in \mathcal{A}^\zeta : q \text{ has multiple } \mathcal{Q}_{\phi} \text{-preimages in } \mathcal{T} \} \). Lemma 4.5 says \( \nu[\mathcal{M}_{\mathcal{Q}_{\phi}}] = 0 \).
We claim that \( \mathcal{M}_\Phi \subset \mathcal{M}_Q \). To see this, let \( q \in \mathcal{M}_\Phi \). Thus, there are \( p_1, p_2 \in \mathcal{P} \) with \( \Phi(p_1) = q = \Phi(p_2) \). Let \( t_1 := \mathcal{P}^{-1}_\sigma(p_1) \) and \( t_2 := \mathcal{P}^{-1}_\sigma(p_2) \). Now, \( p_1 \neq p_2 \), so Lemma 3.2 says \( t_1 \neq t_2 \) (\( \lambda \)-a.s.). But Lemma 4.5(a) says \( Q_\sigma(t_1) = q = Q_\sigma(t_2) \). Thus \( q \in \mathcal{M}_Q \).

Thus, \( \mathcal{M}_\Phi \subset \mathcal{M}_Q \), so \( \nu[\mathcal{M}_\Phi] = 0 \).

(b) Let \( p_1, p_2 \in \mathcal{P} \), and suppose \( \Phi(p_1) = \Phi(p_2) \). Let \( \xi : \mathcal{P} \rightarrow \mathcal{T} \) be as in Proposition 4.7(d). Let \( t_j = \xi(p_j) \) for for \( j = 1, 2 \), and let \( s = t_2 - t_1 \). If \( U := T \cap \rho^{-s}(T) \), then \( \lambda[U] = 1 \), because \( \lambda[T] = 1 = \lambda[\rho^{-s}(T)] \). Thus, if \( Q := \mathcal{P}_\epsilon(U) \), then \( \mu[Q] = 1 \). We’ll show that \( \Phi \) is many-to-one on \( Q \), thus contradicting (a).

If \( q_1 \in Q \), then \( q_1 = \mathcal{P}_\epsilon(u_1) \) for some \( u_1 \in U \). Let \( u_2 := u_1 + s \); then \( u_2 \in T \). Let \( q_2 := \mathcal{P}_\epsilon(u_2) \). Then \( q_2 \neq q_1 \) (since \( \mathcal{P}_\epsilon \) is injective by Proposition 4.7), but we claim \( \Phi(q_1) = \Phi(q_2) \).

To see this, use the minimality of \( (\mathcal{P}, d_C, \sigma) \) to get \( \{\ell_n\}_{n=1}^\infty \subset \mathbb{L} \) with \( d_C \lim_{n \to \infty} \sigma^{\ell_n}(p_1) = q_1 \). Now, \( (\mathcal{P}, d_C) \) is compact, so drop to a subsequence such that \( \{\sigma^{\ell_n}(p_2)\}_{n=1}^\infty \) converges in \( \mathcal{P} \).

**Claim 1.** \( d_{C^-} \lim_{n \to \infty} \sigma^{\ell_n}(p_2) = q_2 \).

**Proof.** Let \( q'_2 := d_{C^-} \lim_{n \to \infty} \sigma^{\ell_n}(p_2) \). To see that \( q'_2 = q_2 \), first note that

\[
\lim_{n \to \infty} \sigma^{\ell_n}(t_1) = \lim_{n \to \infty} \sigma^{\ell_n}(\xi(p_1)) \overset{(*)}{=} \lim_{n \to \infty} \xi(\sigma^{\ell_n}(p_1)) = \xi(\lim_{n \to \infty} \sigma^{\ell_n}(p_1)) = \xi(q_1) = u_1, \tag{18}
\]

where (\( * \)) is Proposition 4.7(d)[ii]; (\( \dagger \)) is because \( \xi \) is continuous; and (\( \ddagger \)) is Proposition 4.7(d)[iii].

If \( u'_2 := \xi(q'_2) \), then by similar reasoning, \( \lim_{n \to \infty} \sigma^{\ell_n}(t_2) = u'_2 \). But \( t_2 = t_1 + s \), so \( \lim_{n \to \infty} \sigma^{\ell_n}(t_2) = \lim_{n \to \infty} \sigma^{\ell_n}(t_1) + s \overset{\text{[18]}}{=} u_1 + s = u_2 \). Thus, \( u'_2 = u_2 \). Thus, \( q'_2 = \mathcal{P}_\epsilon(u'_2) = \mathcal{P}_\epsilon(u_2) = q_2 \); where (\( * \)) is Proposition 4.7(d)[ii]. \( \diamond \) claim 1

Thus, \( \Phi(q_1) = \xi(\lim_{n \to \infty} \sigma^{\ell_n}(p_1)) \overset{(*)}{=} \lim_{n \to \infty} d_{C^-} \lim_{n \to \infty} \sigma^{\ell_n}(\Phi(p_1)) = \lim_{n \to \infty} d_{C^-} \lim_{n \to \infty} \sigma^{\ell_n}(\Phi(p_2)) \overset{(*)}{=} \Phi(d_{C^-} \lim_{n \to \infty} \sigma^{\ell_n}(p_2)) = \Phi(q_2) \), where (\( \dagger \)) is because \( \lim_{n \to \infty} \sigma^{\ell_n}(p_1) = q_1 \); (\( * \)) is because \( \Phi \) is continuous and \( \sigma \)-commuting; (\( \ddagger \)) is because \( \Phi(p_1) = \Phi(p_2) \) by hypothesis; and (\( \circ \)) is by Claim 1. Thus \( \Phi(q_1) = \Phi(q_2) \).

This works for any \( q_1 \in Q \). Thus, \( \Phi|_Q \) is noninjective, but \( \mu[Q] = 1 \), which contradicts (a). \( \square \)

When are the hypotheses of Proposition 4.2 satisfied? An element \( t = (t_1, \ldots, t_K) \in T \cong [0,1)^K \) is called **totally rational** if \( t_1, \ldots, t_K \) are all rational
numbers. It follows:

\[ n \cdot t = 0 \text{ for some } n \in \mathbb{N} \quad \iff \quad t \text{ is totally rational} \]  

(19)

Let \( T_Q \subset T \) be the set of all nonzero totally rational elements.

**Lemma 6.3.** If \( P \in \mathcal{A}^T \) is a nonsimple partition, then it has a nontrivial symmetry in \( T_Q \).

**Proof.** Suppose \( T = T^K \). Let \( S \) be the translational symmetry group of \( P \); then \( S \) is a closed subgroup of \( T \), so there is a topological group isomorphism \( \phi : T^J \times S_0 \to S \) (where \( 0 \leq J \leq K \) and \( S_0 \) is a finite group). If \( J > 0 \), let \( F \subset T^J \) be a nontrivial finite subgroup; if \( J = 0 \), then \( S_0 \) is nontrivial, so let \( F = S_0 \). Let \( F' = \phi(F) \subset S \). Then \( F' \) is a finite subgroup of \( T \), and therefore, by (19), can only contain totally rational elements. \( \square \)

Let \( s^t A_T \) be the set of simple partitions; let \( t^s A_T \) be the set of trivial partitions (so that \( |t^s A_T| = |A| \) is finite), and let \( s^t A_T := s^t A_T \cup t^s A_T = \{ P \in A_T : P \) is either simple or trivial\}.

**Corollary 6.4.** \( s^t A_T \) is a dense \( G_δ \) subset of \( A_T \).

**Proof.** If \( q \in T_Q \), let \( q^s A_T \subset A_T \) be all \( q \)-symmetric partitions.\( q^s A_T \) is closed in \( A_T \): If \( \{P_1, P_2, \ldots \} \) is a sequence of \( q \)-symmetric partitions converging to some \( d_\Delta \)-limit \( P \), then \( P \) is also \( q \)-symmetric.\( q^s A_T \) is nowhere dense in \( A_T \): Observe that the \( q \)-symmetry of any \( P \in q^s A_T \) can be disrupted by a small ‘perturbation’ —ie. by removing a small piece from \( P_a \) and adding it to \( P_b \) for some \( a, b \in A \).

Thus, \( (A_T \setminus q^s A_T) \) is open and dense in \( A_T \). Now, Lemma \ref{lemma:q-symmetric-set} implies that 
\[ q^s A_T = \bigcap_{q \in T_Q} (A_T \setminus q^s A_T). \]  
But \( T_Q \) is countable, so \( q^s A_T \) is a countable intersection of open dense sets, thus, dense \( G_δ \).

Since \( s^t A_T \) is finite, it is also \( G_δ \); hence \( s^t A_T \) is dense and \( G_δ \). \( \square \)

Corollary \ref{cor:q-symmetric-set} and Proposition \ref{prop:q-symmetric-set} do not suffice to prove Theorem \ref{thm:simple-partitions} even if \( P \) is simple, \( \Phi_t(P) \) may not be. We need conditions to ensure that both \( P \) and \( \Phi_t(P) \) are simple.

If \( B \) is a Boolean algebra of subsets of \( T \), then \( B \) is **totally simple** if no nontrivial set in \( B \) maps to itself under any nontrivial totally rational translation. That is: for any \( B \in B \), with \( 0 < \lambda(B) < 1 \), and any \( q \in T_Q \), \( \rho^q(B) \neq B \).

**Example 6.5.** Let \( a \in [0, 1) \cong T^1 \) be irrational, and let \( B \) be the Boolean algebra consisting of all finite unions of subintervals of \( T^1 \) of the form \( (na, ma) \) for some \( n, m \in \mathbb{Z} \) (where \( na \) and \( ma \) are interpreted mod 1). Then \( B \) is
totally simple. To see this, suppose \( B \in \mathcal{B} \); then \( B = \bigcup_{j=1}^{J}(n_ja, m_ja) \) for some \( n_1, \ldots, n_J, m_1, \ldots, m_J \in \mathbb{Z} \). Assume WLOG that \( n_1 a \) is a boundary point of \( B \). If \( \rho^j(B) = B \) for some \( q \in \mathbb{Q} \), then \( \rho^j(n_1 a) = n_j a \) for some \( j \in [1..J] \), which means \( q = n_j a - n_1 a = (n_j - n_1) a \) is an integer multiple of \( a \). But \( q \) is rational and \( a \) is irrational, so we must have \( n_j - n_1 = 0 \); hence \( q = 0 \).

If \( Q = \{ Q_a \}_{a \in A} \) is a partition, we write “\( Q \prec \mathcal{B} \)” if \( Q_a \in \mathcal{B} \) for all \( a \in A \).

If \( \mathcal{P} = \{ P_a \}_{a \in A} \) is a partition, let \( \zeta^\mathcal{P}(\mathcal{P}) \) be the Boolean algebra generated by the set \( \{ \zeta^\mathcal{P}(P_a) ; \ell \in L, a \in A \} \) under all finite unions and intersections.

**Lemma 6.6.** (a) If \( \mathcal{B} \) is a totally simple boolean algebra, and \( Q \prec \mathcal{B} \) is a nontrivial partition, then \( Q \) is simple.

(b) If \( \Phi \) is any cellular automaton, then \( \Phi^\mathcal{P}(\mathcal{P}) \prec \zeta^\mathcal{P}(\mathcal{P}) \).

(c) Suppose \( \zeta^\mathcal{P}(\mathcal{P}) \) is totally simple. If \( \Phi^\mathcal{P}(\mathcal{P}) \) is nontrivial, then \( \Phi^\mathcal{P}(\mathcal{P}) \) is simple.

**Proof.** (a) \( Q \) is nontrivial, and \( Q \prec \mathcal{B} \), so there is some \( a \in A \) such that \( 0 < \lambda(Q_a) < 1 \), and \( Q_a \in \mathcal{B} \). If \( Q \) is nonsimple, then Lemma 6.3 yields a nontrivial symmetry \( t \in T_Q \) with \( \rho^\mathcal{P}(Q) = Q \). Hence \( \rho^\mathcal{P}(Q_a) = Q_a \), which contradicts the total simplicity of \( \mathcal{B} \).

(b) follows from eqn. 12 defining \( \Phi^\mathcal{P}(\mathcal{P}) \). (c) then follows from (a) and (b) \( \square \)

**Example 6.7.** Let \( T = T^1 \cong [0, 1) \) and \( L = \mathbb{Z} \). Let \( a \in [0, 1) \) be irrational, and let \( \mathbb{Z} \) act by \( \zeta^T(t) = t + za \) as in Example 2.2. Let \( A = \{ 0, 1 \} \) and let \( \mathcal{P} = \{ (0, a), (a, 1) \} \) as in Example 2.1. Then \( \zeta^\mathcal{P}(\mathcal{P}) \) is the totally simple Boolean algebra from Example 6.6. Thus, if \( \Phi \) is any CA, and \( \Phi^\mathcal{P}(\mathcal{P}) \) is nontrivial [eg. Example 3.3(a)], then \( \Phi^\mathcal{P}(\mathcal{P}) \) is simple. \( \diamond \)

When is \( \zeta^\mathcal{P}(\mathcal{P}) \) totally simple? If \( \mathcal{P} = \{ P_a \}_{a \in A} \) is an open partition of \( T \), and \( \mathcal{S} \) act by \( \zeta^\mathcal{S}(t) = t + za \) as in Example 6.7. Let \( A = \{ 0, 1 \} \) and let \( \mathcal{P} = \{ (0, a), (a, 1) \} \) as in Example 2.1. Then \( \zeta^\mathcal{P}(\mathcal{P}) \) is the totally simple Boolean algebra from Example 6.6. Thus, if \( \Phi \) is any CA, and \( \Phi^\mathcal{P}(\mathcal{P}) \) is nontrivial [eg. Example 3.3(a)], then \( \Phi^\mathcal{P}(\mathcal{P}) \) is simple.

When is \( \zeta^\mathcal{P}(\mathcal{P}) \) totally simple? If \( \mathcal{P} = \{ P_a \}_{a \in A} \) is an open partition of \( T \), and \( \mathcal{S} \) act by \( \zeta^\mathcal{S}(t) = t + za \) as in Example 6.7. Then \( \mathcal{P} \) has \( \mathcal{S} \)-local symmetry if there are \( a, b \in A \) and open sets \( \mathcal{O}, \mathcal{O}' \subset T \) so that \( (\partial P_a) \cap \mathcal{O} \neq \emptyset \neq (\partial P_b) \cap \mathcal{O}' \), and \( \rho^\mathcal{P}((\partial P_a) \cap \mathcal{O}) = (\partial P_b) \cap \mathcal{O}' \). We say \( \mathcal{P} \) is primitive if \( \mathcal{P} \) has no nontrivial \( \mathcal{S} \)-local symmetries for any \( \mathcal{S} \in \mathcal{L}_{\mathcal{Q}} := \{ \zeta^\mathcal{P}(q) ; \ell \in L, q \in T_Q \} \).

**Example 6.8.** Let \( \zeta \) and \( \mathcal{P} \) be as in Example 6.7. Then \( \mathcal{P} \) has \( \mathcal{A} \)-local symmetry, but is still primitive, because \( \mathcal{A} \notin \mathcal{L}_{\mathcal{Q}} := \{ za + q ; z \in \mathbb{Z}, q \in T_Q \} \) (because \( 0 \notin T_Q \)). \( \diamond \)

**Lemma 6.9.** If \( \mathcal{P} \in \mathcal{A}^T \) is primitive, then \( \zeta^\mathcal{P}(\mathcal{P}) \) is totally simple.

**Proof.** For any \( a \in A \) and \( \ell \in L \), let \( P_a^\ell = \zeta^\mathcal{P}(P_a) \). Then any \( U \in \zeta^\mathcal{P}(\mathcal{P}) \) has the form

\[
U = \bigcup_{j=1}^{J} \bigcap_{k=1}^{K} P_{a_{jk}}^\ell,
\]

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for some \( J, K > 0 \) and some collections \( \{a_{jk}\}_{j=1}^{J} \subset \mathcal{A} \) and \( \{\ell_{jk}\}_{j=1}^{J} \subset \mathbb{L} \). Suppose \( \rho^3(U) = U \) for some \( q \in T_{Q} \). Then \( \rho^3(\partial U) = \partial U \). But
\[
\partial U \subset \bigcup_{j=1}^{J} \partial \left( \bigcap_{k=1}^{K} \mathcal{P}_{a_{jk}} \right),
\]
and, for all \( j \in \{1..J\} \),
\[
\partial \left( \bigcap_{k=1}^{K} \mathcal{P}_{a_{jk}} \right) = \bigcup_{k=1}^{K} \left( \partial \mathcal{P}_{a_{jk}} \cap \bigcap_{k \neq k=1}^{K} \mathcal{P}_{a_{jk}} \right).
\]
Hence, we must have open sets \( V, V' \subset T \) such that
\[
\rho^3 \left( \bigcap_{j=1}^{J} \partial \mathcal{P}_{a_{jk}} \cap \bigcap_{k \neq k=1}^{K} \mathcal{P}_{a_{jk}} \right) = \bigcup_{k=1}^{K} \left( \partial \mathcal{P}_{a_{jk}} \cap \bigcap_{k \neq k=1}^{K} \mathcal{P}_{a_{jk}} \right).
\]

\[
(20)
\]
for some \( j, j' \in \{1..J\} \) and \( k_+, k'_+ \in \{1..K\} \). Let \( a := a_{j_+} \) and \( a' := a_{j'_+} \); let \( \ell := \ell_{j_+} \) and \( \ell' := \ell'_{j'_+} \). If \( U := V \cap \bigcap_{k \neq k=1}^{K} \mathcal{P}_{a_{jk}} \) and \( U' := V' \cap \bigcap_{k \neq k=1}^{K} \mathcal{P}_{a_{jk}} \), then \( U \) and \( U' \) are open sets, and (20) becomes:
\[
\rho^3 \left( (\partial \mathcal{P}_a \cap U) \cap \bigcap_{k \neq k=1}^{K} \mathcal{P}_{a_{jk}} \right) = \bigcup_{k=1}^{K} \left( \partial \mathcal{P}_{a_{jk}} \cap \bigcap_{k \neq k=1}^{K} \mathcal{P}_{a_{jk}} \right).
\]
Equivalently,
\[
\rho^3 (\varsigma(\partial \mathcal{P}_a) \cap U) = \varsigma' (\partial \mathcal{P}_{a'}) \cap U'.
\]
Equivalently,
\[
\rho^3 \circ \varsigma' (\partial \mathcal{P}_a) \cap O = (\partial \mathcal{P}_{a'}) \cap O',
\]
where \( O := \varsigma^{-\ell}(U) \) and \( O' := \varsigma'^{-\ell'}(U') \). Equivalently, \( \rho^3 ((\partial \mathcal{P}_a) \cap O) = (\partial \mathcal{P}_{a'}) \cap O', \) where \( s := \varsigma^{-\ell'}(q) \in LT_Q \).
Thus, \( \mathcal{P} \) has an \( s \)-local symmetry, contradicting our hypothesis.

\[
\square
\]

**Corollary 6.10.** Let \( \mathcal{P}^T \subset \mathcal{A}^T \) be the set of primitive partitions. Then \( \mathcal{P}^T \subset \Phi^{-1} (\text{st} \mathcal{A}^T) \).

**Proof.** If \( \mathcal{P} \in \mathcal{P}^T \), then Lemma 6.9 says \( \varsigma \big\{ \mathcal{P} \big\} \) is totally simple. Hence, Lemma 6.9(c) says that either \( \Phi (\mathcal{P}) \) is trivial or \( \Phi (\mathcal{P}) \) is simple. Hence, \( \Phi (\mathcal{P}) \subset \text{st} \mathcal{A}^T \).

\[
\square
\]

**Lemma 6.11.** \( \mathcal{P}^T \) is \( d_{\Delta} \)-dense in \( \mathcal{A}^T \).

**Proof.** Let \( \mathcal{M} \in \mathcal{A}^T \) be a measurable partition and let \( \epsilon > 0 \). Lemma 4.2 says \( \mathcal{P}^T \) is \( d_{\Delta} \)-dense in \( \mathcal{A}^T \), so there is a dyadic partition \( \mathcal{D} \sim \mathcal{M} \). We claim there is a primitive partition \( \mathcal{P} \sim \mathcal{D} \). Thus, \( \mathcal{P} \sim \mathcal{D} \sim \mathcal{M} \), so \( \mathcal{P} \sim \mathcal{M} \), so we’re done.

To construct the primitive partition \( \mathcal{P} \sim \mathcal{D} \), suppose \( T = T^K \), and consider two cases:

- **Case \( K = 1 \)**: \( T = T_1 = [0,1] \), so \( \partial \mathcal{D} \) is a countable set of points. By slightly perturbing these points, we can construct \( \mathcal{P} \in \mathcal{A}^T \) with \( \mathcal{P} \sim \mathcal{D} \) such that, for any \( b_1, b_2 \in \partial \mathcal{P} \), \( b_1 - b_2 \notin LT_Q \). Hence, \( \mathcal{P} \) is primitive.

- **Case \( K > 1 \)**: First, observe that, if \( s \in T \) and \( \mathcal{D} \) has a \( s \)-local symmetry, then \( s \) must be a dyadic element of \( T \). Thus, it suffices to ‘perturb’ \( \mathcal{D} \) so as to disrupt all dyadic local symmetries, without introducing any other local symmetries.
If $D_a \in D$, then $\partial D_a$ is a finite disjoint union of $(K - 1)$-dimensional dyadic cubic faces. Let $K_a$ be the set of these faces. Let $K = \bigcup_{a \in A} K_a$; then $K$ has a finite collection of dyadic cubic faces. Now, let $\{\alpha_\kappa\}_{\kappa \in K} \in [0, 1]$ be distinct frequencies. If $D$ is $n$-dyadic, then let $\delta < \frac{1}{2^{n+1}}$. For each $a \in A$, let $P_a$ be the set obtained as follows: begin with $D_a$, and ‘corrugate’ each face $\kappa \in K_a$, with a sine wave of frequency $\alpha_\kappa$ and amplitude $\delta$ (Fig. 2).

Since all the frequencies are distinct, there can by no local symmetry from any face to any other; hence $P_a$ is primitive. If $\delta$ is small enough, then $P \approx \varepsilon/2 D$. 

**Corollary 6.12.** For any $n \in \mathbb{N}$, $\Phi_\varsigma^{-n} (\text{st} A^T)$ is a dense $G_\delta$ subset of $(A^T, d_{\Delta})$.

**Proof.** $G_\delta$: Corollary 6.4 says $\text{st} A^T$ is a $G_\delta$ subset of $A^T$. Proposition 6.1 says $\Phi_\varsigma^n$ is $d_{\Delta}$-continuous, so the $\Phi_\varsigma^n$-preimage of any open set is open; hence, $\Phi_\varsigma^{-n} (\text{st} A^T)$ is also $G_\delta$.

$Dense$: Corollary 6.10 says $\nu A^T \subset \Phi_\varsigma^{-n} (\text{st} A^T)$, while Lemma 6.11 says that $\nu A^T$ is dense in $A^T$; hence $\Phi_\varsigma^{-n} (\text{st} A^T)$ is also dense in $A^T$.

**Proof of Theorem 6.1** First, recall that $(A^T, d_{\Delta})$ is a complete metric space (Proposition 4.1), and thus, a Baire space [42, Corollary 25.4(b), p.186].

Now, let $\ast A^T := \bigcap_{n=1}^{\infty} \Phi_\varsigma^{-n} (\text{st} A^T)$. Then $\ast A^T$ is a countable intersection of dense $G_\delta$ subsets of $A^T$ (Corollary 6.12), and thus, is itself a dense $G_\delta$ subset (because $A^T$ is Baire). By construction, $\Phi_\varsigma (\ast A^T) \supseteq \ast A^T$.

If $P \in \ast A^T$ is nontrivial, then $P$ is simple, and either $\Phi_\varsigma (P)$ is trivial (ie. $\Phi|_P$ is constant) or $\Phi_\varsigma (P)$ is also simple. Now apply Proposition 6.2(a) and (b) respectively to get Theorem 6.1(a) and (b) respectively. 

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7. **Customized Quasisturmian Systems†**

Let $B(N) := [-N..N]^D \subset \mathbb{L}$. Fix a word $w \in A_{B(N)}$ and a letter $s \in A$. If $x \in A^\mathbb{L}$, and $0 < \epsilon < 1$, then we say $x$ is $\epsilon$-tiled by $w$ with spacer $s$ if there is a subset $J \subset \mathbb{L}$ (the **skeleton** of the tiling) such that, as shown in Figure 3:

\[(T1) \ (j_1 + B(N)) \cap (j_2 + B(N)) = \emptyset, \text{ for any distinct } j_1, j_2 \in J.\]

\[(T2) \ \text{density}(J) > \frac{1 - \epsilon}{(2N)^D}. \quad \text{Thus, } \text{density}(B(N) + J) = (2N)^D \cdot \text{density}(J) > 1 - \epsilon.\]

\[(T3) \ \text{For every } j \in J, \ x |_{j + B(N)} = w.\]

\[(T4) \ \text{For any } \ell \notin (B(N) + J), \ x_\ell = s.\]

![Figure 3. a is $\epsilon$-tilled with tile $w$, with skeleton $J$ and spacer $s$.](image)

**Proposition 7.1.** Let $N \in \mathbb{N}$ and $\epsilon > 0$. For any tile $w \in A_{B(N)}$ and ‘spacer’ $s \in A$, there is an open partition $P \in \mathcal{O}(\mathbb{T})$ such that every element of $\Xi_\epsilon(P)$ is $\epsilon$-tiled by $w$ with spacer $s$. 

To prove Proposition 7.1 we use a $\mathbb{Z}^D$-action version of the Rokhlin-Kakutani-Halmos Lemma:

**Lemma 7.2.**[9] For any $N \in \mathbb{N}$ and $\epsilon > 0$, there exists an open subset $J \subset \mathbb{T}$ such that:

\[(a) \quad \text{The sets } \{\zeta^b(J)\}_{b \in B(N)} \text{ are disjoint}.\]

\[(b) \quad \lambda(J) > \frac{1 - \epsilon}{(2N)^D}, \text{ and thus, } \lambda \left( \bigcup_{b \in B(N)} \zeta^b(J) \right) > 1 - \epsilon.\]

† This section contains technical results which are used in §8 and §9.

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If $L = \mathbb{Z}$, then $\mathcal{B}(N) = [-N..N]$, and $\{c^b(J)\}_{b \in \mathcal{B}(N)}$ is called an $(\epsilon, N)$-Rokhlin tower of height $2N$, with base $J$; the sets $c^b(J)$ are the levels of the tower. If $L = \mathbb{Z}^D$ for $D \geq 2$, then we call the structure $\{c^b(J)\}_{b \in \mathcal{B}(N)}$ an $(\epsilon, N)$-Rokhlin city, with base $J$. The sets $c^b(J)$ are the houses of the city.

Think of the elements of $\mathcal{A}$ as ‘colours’; then we can build an $\mathcal{A}$-labelled partition by ‘painting’ the houses of the Rokhlin city in different $\mathcal{A}$-colours, as follows. Fix a word $w \in \mathcal{A}^{\mathbb{B}(N)}$, and a ‘spacer’ letter $s \in \mathcal{A}$. Let $S := \text{int}(T \setminus \bigcup_{b \in \mathcal{B}(N)} c^b(J))$.

If $w = [w_b]_{b \in \mathcal{B}(N)}$, then for any $a \in \mathcal{A}$, let $P_a := \{b \in \mathcal{B}(N) : w_b = a\}$. Define partition $\mathcal{P} := \{P_a\}_{a \in \mathcal{A}}$, where

$$P_a := S \cup \bigcup_{p \in \mathcal{P}_a} c^p(J), \quad \text{and, for any } a \in \mathcal{A} \setminus \{s\}, \quad P_a := \bigcup_{p \in \mathcal{P}_a} c^p(J).$$

We say $\mathcal{P}$ is obtained by painting Rokhlin city $\{c^b(J)\}_{b \in \mathcal{B}(N)}$ with word $w$ and spacer $s$.

**Lemma 7.3.** Let $N \in \mathbb{N}$ and $\epsilon > 0$, and let $\{c^b(J)\}_{b \in \mathcal{B}(N)}$ be an $(\epsilon, N)$-Rokhlin city. Suppose $\mathcal{P}$ is obtained by painting $\{c^b(J)\}_{b \in \mathcal{B}(N)}$ with $w$ and $s$, as in eqn. (21).

If $t \in T$ and $p = \mathcal{P}_c(t)$, then $p$ is $\epsilon$-tiled by $w$ with skeleton $J = \{\ell \in L : c^\ell(t) \in J\}$.

**Proof.** The Generalized Ergodic Theorem says density $(J) = \lambda(J) > \frac{1}{(2D)^{\epsilon}}$. Also, $\forall j \in J$, $p|_{j + \mathcal{B}(N)} = w$. Finally, $(\ell \notin J + \mathcal{B}(N)) \Rightarrow (c^\ell(t) \in S) \Rightarrow (\ell = s)$. \hfill $\square$

**Proof of Proposition 7.4.** Let $\{c^b(J)\}_{b \in \mathcal{B}(N)}$ be an $(\epsilon, N)$-Rokhlin city, provided by Lemma 7.3. Now paint the city with $w$ and $s$, and apply Lemma 7.2. \hfill $\square$

**Corollary 7.4.** $\mathcal{Q} \mathcal{S}_c$ is dense in $\mathcal{A}^L$ in the Cantor topology.

**Proof.** Fix $a \in \mathcal{A}^L$ and $\epsilon > 0$. Let $t \in T$; we’ll build a partition $Q \in \mathcal{A}^T$ so that $q = \mathcal{Q}_c(t)$ is $\epsilon$-close to $a$ in the Cantor metric. Let $N > -\log_2(\epsilon)$, and let $w = a|_{\mathcal{B}(N)}$. Use Proposition 7.1 to find $\mathcal{P} \in \mathcal{A}^T$ such that $p = \mathcal{P}_c(t)$ is tiled by copies of $w$. Thus, there is some $\ell \in L$ such that $p|_{\ell + \mathcal{B}(N)} = w$. Let $Q = c^\ell(\mathcal{P})$; thus, if $q = \mathcal{Q}_c(t)$, then $q = c^\ell(p)$ by Proposition 7.2(a), so that $q|_{\mathcal{B}(N)} = w = a|_{\mathcal{B}(N)}$, so that $d_C(q, a) < 2^{-N} < \epsilon$. \hfill $\square$

Let $\mathcal{M}^{\text{erg}}(\mathcal{A}^L)$ be the space of $\sigma$-ergodic probability measures on $\mathcal{A}^L$, with the weak* topology induced by convergence along cylinder sets.

**Corollary 7.5.** $\mathcal{M}^{\text{erg}}(\mathcal{A}^L)$ is weak* dense in $\mathcal{M}^{\text{erg}}(\mathcal{A}^L)$. 

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Proof. Let \( \nu \in \mathcal{M}_{\text{erg}}(\mathcal{A}_1) \). Fix \( N > 0 \); let \( c_1, \ldots, c_J \in \mathcal{A}_{\mathbb{B}(N)} \), and for each \( j \in [1..J], \) let \( \mathcal{C}_j := \{ a \in \mathcal{A}_1 \mid a|_{\mathbb{B}(N)} = c_j \} \) be the corresponding cylinder sets. For any \( \epsilon > 0 \), we’ll construct a quasisturmian measure \( \mu \in \mathcal{M}_{\epsilon}(\mathcal{A}_1) \) such that \( \mu[\mathcal{C}_j] \sim \nu[\mathcal{C}_j] \) for all \( j \in [1..J]. \)

The Generalized Ergodic Theorem yields some \( a \in \mathcal{A}_1 \) which is \((\sigma, \nu)\)-generic for \( \mathcal{C}_1, \ldots, \mathcal{C}_J \). Thus, if \( M \) is large enough, then

\[
\text{For all } j \in [1..J], \quad \nu[\mathcal{C}_j] \sim \frac{1}{(2M)^D} \sum_{b \in \mathbb{B}(M)} \mathbb{I}_j \left( \sigma^b(a) \right),
\]

where \( \mathbb{I}_j \) is the characteristic function of \( \mathcal{C}_j \). Now, \( \mathbb{I}_j(a) \) is a function only of \( a|_{\mathbb{B}(N)} \), so the sum (22) is a function only of \( w := a|_{\mathbb{B}(N+\delta)} \). Use Proposition 7.1 to find some \( P \in \mathcal{Q}(\mathcal{A}_1) \) such that \( P \in \Xi_\delta(P) \) is \( \epsilon \)-tiled with \( w \). If \( \mu = \Upsilon_\delta(P) \), then \( P \) is \((\sigma, \mu)\)-generic, so for all \( j \in [1..J], \)

\[
\mu[\mathcal{C}_j] = \lim_{K \to \infty} \frac{1}{(2K)^D} \sum_{b \in \mathbb{B}(N)} \mathbb{I}_j \left( \sigma^b(p) \right) \sim \frac{1}{(2M)^D} \sum_{b \in \mathbb{B}(M)} \mathbb{I}_j \left( \sigma^b(w) \right)
\]

\[
= \frac{1}{(2M)^D} \sum_{b \in \mathbb{B}(M)} \mathbb{I}_j \left( \sigma^b(a) \right) \sim \nu[\mathcal{C}_j].
\]

\( \square \)

Lemma 7.6. Let \( x, x' \in \mathcal{A}_1 \) with \( d_B(x, x') \leq \delta \). Suppose \( N > 0 \), and that \( x \) can be \( \epsilon \)-tiled by some \( w \in \mathcal{A}_{\mathbb{B}(N)} \), with skeleton \( J \subset \mathbb{L} \). Let \( \delta < (1 - \epsilon)/(2N)^D \), so \( \epsilon' := (1 - \epsilon)/(2N)^D \). Then \( x' \) can be \( \epsilon' \)-tiled by \( w \), with skeleton \( J' \subset \mathbb{L} \), such that density \( (J \cap J') \geq \text{density } (J) - \delta \).

Proof. Let \( J' = \{ \ell \in \mathbb{L} \mid x'|_{\ell+\mathbb{B}(N)} = w \} \), and let \( J_\Delta = J \setminus J' \). Thus, \( J \cap J' = J \setminus J_\Delta \).

We’ll show that density \( (J_\Delta) \leq \delta \). Thus, density \( (J \cap J') = \text{density } (J) - \delta \).

Let \( K = \{ j \in J \mid x_j \neq x_j' \} \). Then density \( (K) = d_B(x, x') = \delta \), and for any \( j \in J_\Delta \),

\[\begin{align*}
(j \in J_\Delta) & \iff (x'|_{j+\mathbb{B}(N)} \neq w) \iff (x'|_{j+\mathbb{B}(N)} \neq x|_{j+\mathbb{B}(N)}) \\
& \iff (x'_{j+b} \neq x_{j+b}, \text{ for some } b \in \mathbb{B}(N)) \\
& \iff (j + b \in K, \text{ for some } b \in \mathbb{B}(N)).
\end{align*}\]

Thus, we can define a function \( \beta : J_\Delta \to \mathbb{B}(N) \) such that \( j + \beta(j) \in K \) for all \( j \in J_\Delta \). This defines a function \( \kappa : J_\Delta \to K \) by \( \kappa(j) = j + \beta(j) \).

Note that \( \kappa \) is an injection, because for any distinct \( j_1, j_2 \in J_\Delta \subset J \), tiling condition \( (T1) \) says that \( (j_1 + \mathbb{B}(N)) \cap (j_2 + \mathbb{B}(N)) = \emptyset \); hence \( j_1 + \beta(j_1) \neq j_2 + \beta(j_2) \).

Claim 1. Let \( J_\kappa = \kappa(J_\Delta) \subset K \). Then density \( (J_\kappa) = \text{density } (J_\Delta) \).
First let $P \not \in A$.

A natural conjecture: Surjectivity and image density.

Unfortunately, this is false. For example, suppose surjective. For any $P \in A$.

Proof. For any $M > 0$, every element of $J_\Delta \cap B(M)$ must go to some element of $J_\kappa \cap B(M + N)$ under $\kappa$. Likewise, every element of $J_\kappa \cap B(M - N)$ must ‘come from’ $J_\Delta \cap B(M)$. Thus, $\# [J_\kappa \cap B(M - N)] \leq (\text{(i)}) \# [J_\Delta \cap B(M)] \leq (\text{(ii)}) \# [J_\kappa \cap B(M + N)]$.

Thus, density $(J_\kappa) = \lim_{M \to \infty} \frac{\# [J_\kappa \cap B(M - N)]}{(2M - 2N)^D}$

$= \lim_{M \to \infty} \frac{(2M)^D}{(2M - 2N)^D} \cdot \left( \lim_{M \to \infty} \frac{\# [J_\kappa \cap B(M - N)]}{(2M)^D} \right)$

$= \lim_{M \to \infty} \frac{\# [J_\kappa \cap B(M - N)]}{(2M)^D} \leq (\text{(A)}) \lim_{M \to \infty} \frac{\# [J_\Delta \cap B(M)]}{(2M)^D}$

$= \text{density}(J_\Delta)$

$\leq \lim_{M \to \infty} \frac{\# [J_\kappa \cap B(M + N)]}{(2M)^D}$

$= \left( \lim_{M \to \infty} \frac{(2M + 2N)^D}{(2M)^D} \right) \cdot \left( \lim_{M \to \infty} \frac{\# [J_\kappa \cap B(M + N)]}{(2M + 2N)^D} \right)$

$= \text{density}(J_\kappa)$,

where (A) is by (a) and (B) is by (b).

Thus, Claim 1 implies density $(J_\Delta) = \text{density}(J_\kappa) \leq \text{density}(K) = \delta$, as desired. $\Box$

**Corollary 7.7.** If $X \subset QS_\kappa$ is $\sigma$-invariant and $d_B$-dense in $QS_\kappa$, then $X$ is also $d_C$-dense in $QS_\kappa$.

**Proof.** Let $q \in QS_\kappa$ and $N > 0$; we want $x \in X$ such that $x|_{B(N)} = q|_{B(N)}$. Let $w := q|_{B(N)}$. If $\epsilon > 0$, then Proposition 7.1 yields some $q' \in QS_\kappa$ which is $\epsilon$-tiled by $w$. Let $\delta < (1 - \epsilon)/(2N)^D$, and use the $d_B$-density of $QS_\kappa$ to find $x' \in X$ with $d_B(x', q') < \delta$. Then Lemma 7.1 says that $x'$ can be $\epsilon$-tiled by $w$. Thus, there is some $\ell \in L$ such that $x'|_{B(N)} + \ell = w$. If $x = \sigma^{-\ell}(x')$, then $x|_{B(N)} = w$, and $x \in X$ because $X$ is $\sigma$-invariant. $\Box$

**8. Surjectivity and image density**

A natural conjecture: If $\Phi : A^I \to A^I$ is surjective, then $\Phi_\kappa : A^T \to A^T$ is also surjective. Unfortunately, this is false. For example, suppose $A = \{0, 1\} = \mathbb{Z}/2$. For any $P \in A^T$, if $P = \{P_0, P_1\}$, then let $\overline{P} := \{\overline{P}_0, \overline{P}_1\}$, where $\overline{P}_0 := P_1$ and $\overline{P}_1 := P_0$.

**Proposition 8.1.** Let $\Phi$ be as in Example 3.3(a). If $P \in \Phi_\kappa(A^T)$, then $\overline{P} \not \in \Phi_\kappa(A^T)$.

**Proof.** First let $U \in \mathcal{A}^T$ be the ‘unity’ partition $U \equiv 1$, ie: $U_0 = 0, \ U_1 = T$. We claim $U \not \in \Phi_\kappa(A^T)$. To see this, suppose $Q \in A^T$ and $U = \Phi_\kappa(Q)$. Treating $U$ and

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Q as functions from T to A, we have: \( U(t) = Q(t) + Q(\tau(t)) \pmod{2} \) for all \( t \in T \)—i.e. \( U = Q + Q \circ \tau \pmod{2} \). But \( U \equiv 1 \), so \( 1 = Q + Q \circ \tau \), which means \( Q \circ \tau = 1 - Q \). Thus, \( Q \circ \tau = (1 - Q) \circ \tau = 1 - (1 - Q) = Q \). Thus, \( Q \) is \( \mathbb{Z}^2 \)-invariant. But \( \tau \) is totally ergodic, so this means that \( Q \) is a constant —either \( Q \equiv 1 \) or \( Q \equiv 0 \). Neither of these partitions maps to \( U \) under \( \Phi \), so \( U \not\in \Phi(\mathcal{A}^T) \).

Now, let \( P \in \mathcal{A}^T \) and suppose \( P = \Phi(\mathcal{Q}) \) and \( \mathcal{P} = \Phi(\mathcal{Q}') \) for some \( Q, Q' \in \mathcal{A}^T \). Note that \( P + \mathcal{P} \equiv U \). Let \( Q^\dagger := Q + Q' \); then \( \Phi(Q^\dagger) = \Phi(Q) + \Phi(Q') = P + \mathcal{P} = U \), where \((*)\) is because \( \Phi \) is \( \mathbb{Z}/2\)-linear. But we know that \( U \not\in \Phi(\mathcal{A}^T) \). Contradiction.

Thus, \( \Phi \) is not surjective: \( \Phi(\mathcal{A}^T) \) fills at most ‘half’ of \( \mathcal{A}^T \). Nevertheless, we will prove:

**Theorem 8.2.** Let \( \Phi : \mathcal{A}^Z \rightarrow \mathcal{A}^Z \) be any CA. The following are equivalent:

(a) \( \Phi \) is surjective onto \( \mathcal{A}^Z \).

(b) \( \Phi(\mathcal{A}^T) \) is \( d_A \)-dense in \( \mathcal{A}^T \); and \( \Phi(\mathcal{A}^T) \) is \( d_A \)-dense in \( \mathcal{A}^T \).

(c) \( \Phi(\mathcal{O}_\mathcal{S}_\mathcal{C}) \) is \( d_B \)-dense in \( \mathcal{O}_\mathcal{S}_\mathcal{C} \).

(d) \( \Phi(\mathcal{O}_\mathcal{S}_\mathcal{C}) \) is \( d_C \)-dense in \( \mathcal{O}_\mathcal{S}_\mathcal{C} \).

If \( L = \mathbb{Z} \) (as in Theorem 8.2), then there is some irrational \( a \in T \) such that \( \zeta(t) = t + za \) for any \( z \in \mathbb{Z} \) and \( t \in T \). The system \((T, \zeta, \lambda)\) is called an *irrational rotation*. To prove Theorem 8.2 we’ll use the *rank one* property of irrational rotations.

**Theorem 8.3.** (del Junco) \[^{[8]}\]

Any irrational rotation is *topologically rank one*. That is, there is a sequence \( \{J_i\}_{i=1}^\infty \) of open subsets of \( T \) such that:

(a) \( J_i \) is the base of an \((\epsilon_i, N_i)\)-Rokhlin tower (see \[^{[7]}\], where \( \epsilon_i \rightarrow 0 \) and \( N_i \rightarrow \infty \).

(b) Any measurable subset \( W \subset T \) can be approximated arbitrarily well by a disjoint union of tower levels. That is: for any \( \delta > 0 \), there is some \( i \in \mathbb{N} \) and some subset \( M \subset \mathbb{B}(N_i) \) such that, if \( W = \bigcup_{m \in M} \zeta^m(J_i) \), then \( \lambda(W \Delta \tilde{W}) < \delta \). \( \Box \)

**Proof of Proposition 8.2.** \[^{[8]}\]

**“(b) \implies (c)”** First note that \( \Phi_\zeta(\mathcal{A}_0^T) \) is dense in \( \mathcal{A}_0^T \), because Proposition 8.2(a) says \( \mathcal{A}_0^T \) is dense in \( \mathcal{A}_T \), so \( \Phi_\zeta(\mathcal{A}_0^T) \) is dense in \( \mathcal{A}_T \) (and thus, in \( \mathcal{A}_1^T \)). Now apply Proposition 8.3.

**“(c) \implies (d)”** follows from Corollary 7.7.

**“(d) \implies (a)”** Corollary 7.7 implies \( \Phi(\mathcal{O}_\mathcal{S}_\mathcal{C}) \) is \( d_C \)-dense in \( \mathcal{A}_1 \); thus, \( \Phi(\mathcal{A}_1) \) is \( d_C \)-dense in \( \mathcal{A}_1 \). But \( \Phi \) is continuous and \( \mathcal{A}_1 \) is \( d_C \)-compact, so \( \Phi(\mathcal{A}_1) \) is also \( d_C \)-compact, thus, \( d_C \)-closed. Hence, \( \Phi(\mathcal{A}_1) = \mathcal{A}_1 \).

**“(a) \implies (b)”** We’ll show that \( \Phi_\zeta(\mathcal{A}_1^T) \) is dense in \( \mathcal{A}_1^T \). It follows that \( \Phi_\zeta(\mathcal{A}^T) \) is dense in \( \mathcal{A}^T \), because Proposition 8.2 says \( \mathcal{A}_1^T \) is dense in \( \mathcal{A}^T \).

Let \( P \in \mathcal{A}^T \) and \( \epsilon > 0 \). We’ll construct \( Q' \in \mathcal{A}^T \) such that \( d(\Phi_\zeta(Q'), P) < \epsilon \).
Let $\delta := \varepsilon/5$. Let $\{J_i, \delta_i, N_i\}_{i=1}^\infty$ be as in Theorem 8. For any $i \in \mathbb{N}$ and any
$b \in \mathbb{B}(N_i)$, let $J_i^b := \zeta^b(J_i)$. Then let $S_i := \text{int}(T \setminus \bigcup_{b \in \mathbb{B}(N_i)} J_i^b)$. Fix $s \in A$.

**Claim 1.** There is $i \in \mathbb{N}$ and a word $w \in A^{B(N_i)}$ such that, if $\tilde{\mathcal{P}}$ is the partition
obtained by painting city $\{J_i^b\}_{b \in \mathbb{B}(N_i)}$ with $w$ and spacer $s$, then $d_{\Delta}(\mathcal{P}, \tilde{\mathcal{P}}) < \delta$.

**Proof.** Let $A := \#(A)$. For each $a \in A$, use Theorem 8.2(b) to find some
set $\tilde{P}_a$ (a union of levels in $\{J_i^b\}_{b \in \mathbb{B}(N_i)}$) such that
$\lambda(\tilde{P}_a \Delta \tilde{P}_a) < \frac{\delta}{2A}$. Assume
$\{\tilde{P}_a\}_{a \in A}$ are disjoint. Enlarge $\tilde{P}_s$ by adjoining $S_i$ to it. If $i$ is large enough,
then $\lambda(S_i) < \frac{\delta}{2}$. Thus,

\[d_{\Delta}(\mathcal{P}, \tilde{\mathcal{P}}) = \lambda(\tilde{P}_s \Delta \tilde{P}_s) + \sum_{s \neq a \in A} \lambda(\tilde{P}_a \Delta \tilde{P}_a) \leq \frac{\delta}{2} + \frac{\delta}{2A} + \sum_{s \neq a \in A} \frac{\delta}{2A} = \delta.
\]

We define $w$: for any $a \in A$ and $b \in \mathbb{B}(N_i)$, $(w_b = a) \iff (J_i^b \subseteq \tilde{P}_a)$. Then
$\tilde{\mathcal{P}}$ results from painting $\{J_i^b\}_{b \in \mathbb{B}(N_i)}$ with $w$ and spacer $s$, as in eqn. (24) of [27].

\[\diamondsuit\text{ Claim 1}\]

Fix $t \in \tilde{T} \cap J_i$ and let $\tilde{p} := \tilde{\mathcal{P}}(t)$. Thus, $\tilde{p}|_{\mathbb{B}(N_i)} = w$. Now, $\Phi$ is surjective on $A^1$, so find $\tilde{q} \in A^1$ so that $\Phi(\tilde{q}) = \tilde{p}$. Suppose $\Phi$ has local map $\phi: A^{B(n)} \rightarrow A$, and let $N_i' := N_i - n$. If $v = \tilde{q}|_{\mathbb{B}(N_i')}$, then $\Phi(v) = \Phi(\tilde{q})|_{\mathbb{B}(N_i')} = \tilde{p}|_{\mathbb{B}(N_i')} = w|_{\mathbb{B}(N_i')}$. Build a partition $Q' = \{Q'_a\}_{a \in A}$ by painting $\{J_i^b\}_{b \in \mathbb{B}(N_i)}$ with $v$, and then let $q' := Q'_a(t)$. Let $p' := \Phi(q')$.

**Claim 2.** If $i$ is made large enough, then $d_B(p', \tilde{p}) < 2\delta$.

**Proof.** Let $J_i = \{\ell \in \mathbb{L} : \zeta^0(0) \in J_i\}$. For all $j \in J_i$, Lemma 8.3 says $\tilde{p}|_{j+B(N_i')} = w$ and $q'|_{j+B(N_i')} = v$. Thus, $p'|_{j+B(N_i')} = \phi(v) = w|_{B(N_i')} = \tilde{p}|_{j+B(N_i')}$. Thus,

\[p'|_{J_i+B(N_i')} = \tilde{p}|_{J_i+B(N_i')}.
\]

Hence, $d_B(p', \tilde{p}) = \text{density}(\ell \in \mathbb{L} : p'^\ell \neq \tilde{p}_t) \leq \text{density}(L \setminus (J_i + B(N_i')))$

\[= 1 - \text{density}(J_i + B(N_i')) = 1 - (2N_i')^D \cdot \lambda(J_i) \leq 1 - \left(\frac{N_i'}{N_i}\right)^D \cdot (1 - \delta_i)
\]

\[(\text{23})
\]

is by eqn. (23); (†) is the Generalized Ergodic Theorem, and (*) is because $J_i$ is the base of a $(\delta_i, N_i)$-Rokhlin tower.

Make $i$ large enough that $\delta_i < \delta$, and also $N_i > \frac{n}{1 - \sqrt[n]{1 - \delta}}$, so that

\[\left(\frac{N_i'}{N_i}\right)^D > (1 - \delta).
\]

Hence $1 - \left(\frac{N_i'}{N_i}\right)^D \cdot (1 - \delta_i) \leq 1 - (1 - \delta)^2 = \delta(2 - \delta) < 2\delta$.

\[\diamondsuit\text{ Claim 2}\]
If $P^r := \Phi_\ell(Q^r)$, then
\begin{equation}
\mathcal{P}^r_\ell(t) \overset{\circ}{\Rightarrow} \Phi(\mathcal{Q}^r(t)) = \Phi(q^r) = p^r. \tag{24}
\end{equation}
Thus, $d_\Delta(P^r, \bar{P}) \overset{\circ}{\Rightarrow} 2d_B\left(\mathcal{P}^r_\ell(t), \bar{\mathcal{P}}_\ell(t)\right) \overset{\circ}{\Rightarrow} 2d_B(p^r, \bar{p}) < 4\delta. \tag{25}$
Thus, $d_\Delta(P^r, P) \leq d_\Delta(P^r, \bar{P}) + d_\Delta(\bar{P}, P) < 4\delta + \delta = 5\delta < \epsilon$.

Here, (\textcircled{a}) is Lemma 3.2(a); (\textstar) is by Proposition 2.4(b); (\textdagger) is by eqn.(24); (C2) is Claim 2, and (\dagger) is by eqn.(25) and Claim 4.

Thus, $d_\Delta(\Phi_\ell(Q^r), P) < \epsilon$. But $\epsilon$ is arbitrary. So $\Phi_\ell(\mathcal{A}^T)$ is dense in $\mathcal{A}^T$. \hfill $\square$

Remarks. (a) If $\Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ is surjective, and $\Phi_\ell(\mathcal{A}^T)$ is a closed subset of $\mathcal{A}^T$, then Theorem 8.2(b) implies that $\Phi_\ell$ is surjective. When (if ever) is $\Phi_\ell(\mathcal{A}^T)$ closed?

(b) Extending Theorem 8.2 to $L = \mathbb{Z}^D$ would immediately extend Theorem 8.2 to $L = \mathbb{Z}^D$.

9. Fixed points and Periodic Solutions

If $\Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ is a cellular automaton, let $\text{Fix}[\Phi] := \{a \in \mathcal{A}^\mathbb{Z} : \Phi(a) = a\}$ be the set of all fixed points of $\Phi$. If $p \in \mathbb{N}$, then a $p$-periodic point for $\Phi$ is an element of $\text{Fix}[\Phi^p]$. If $v \in \mathbb{L}$, then a $p$-periodic travelling wave with velocity $v$ is an element of $\text{Fix}[\Phi^p \circ \sigma^v]$. When does $\Phi$ have quasisturmian fixed/periodic points? First, it is easy to verify:

**Proposition 9.1.** $\text{Fix}[\Phi], \text{Fix}[\Phi^p]$ and $\text{Fix}[\Phi^p \circ \sigma^v]$ are subshifts of finite type (SFTs). \hfill $\square$

(See [28] 31 for an introduction to subshifts of finite type). Thus, we ask: if $\mathfrak{F} \subset \mathcal{A}^\mathbb{Z}$ is an SFT, when is $\mathfrak{F} \cap \mathfrak{Q} \mathfrak{S} \subset \mathcal{A}^\mathbb{Z}$ nonempty? Let $\mathfrak{F} \subset \mathcal{A}^\mathbb{Z}$ be an SFT, and let $a \in \mathcal{A}$. We say $a$ is an inert state for $\mathfrak{F}$ if $c \in \mathfrak{F}$, where $c$ is the constant sequence such that $c_\ell = a$ for all $\ell \in \mathbb{L}$.

**Example 9.2.** Let $\mathcal{A} = \{0, 1\}$.

\textcircled{a} Let $\mathbb{B} = [-1...1]^D$, so $#(\mathbb{B}) = 3^D$. The **Voter Model** [40] has local map:

$$\phi(a) := \begin{cases} 1 & \text{if } \mathfrak{R}(a) > 3^D/2; \\ 0 & \text{if } \mathfrak{R}(a) < 3^D/2. \end{cases}$$

where $\mathfrak{R}(a) := \sum_{b \in \mathbb{B}} a_b$.

Both 0 and 1 are inert states for $\text{Fix}[\Phi]$.

\textcircled{b} Let $\mathbb{B} \subseteq \mathbb{Z}^2$, and let $0 \leq s_0 \leq b_0 \leq b_1 \leq s_1 \leq #(\mathbb{B})$. A **Larger than Life** (LtL) CA [13] 14 [15] has local map

$$\phi(a) := \begin{cases} 1 & \text{if } a_0 = 1 \text{ and } s_0 \leq \mathfrak{R}(a) \leq s_1; \\ 1 & \text{if } a_0 = 0 \text{ and } b_0 \leq \mathfrak{R}(a) \leq b_1; \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathfrak{R}(a) := \sum_{b \in \mathbb{B}} a_b$. 

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For example, J.H. Conway’s *Game of Life* \[12\] is an LtL CA, with \(B = [-1..1]^2\), \(s_0 = b_0 = b_1 = 3\), and \(s_1 = 4\). LtL CA have many fixed points, periodic points, and travelling waves with various velocities (for example, the well-known *gliders* and *fish of Life*). The corresponding SFTs all have 0 as an inert state. \(\diamondsuit\)

Let \(V > 0\). An SFT \(\mathfrak{F}\) with inert state 0 is *Dirichlet* with *valence* \(V\) if, for any \(r > V\), and any \(\mathfrak{F}\)-admissible configuration \(a \in A^{B(r)}\), there exists \(b \in \mathfrak{F}\) such that \(b|_{B(r-V)} = a|_{B(r-V)}\) but \(b_\ell = 0\) for all \(\ell \in \mathbb{L} \setminus B(r+V)\). We call \(b\) a *Dirichlet extension* of \(a\).

**Proposition 9.3.** Suppose \(\mathfrak{F}\) is a Dirichlet SFT. If \(\varsigma\) is any ergodic \(\mathbb{L}\)-action on \(T\), then \(\mathfrak{F} \cap \mathbb{Q}\overline{S}_\varsigma \neq \emptyset\) and contains nonconstant elements.

**Proof.** Let \(t \in T\). We must construct an open partition \(P \in \mathcal{O}A^\mathbb{T}\) such that \(P_\varsigma(t) \in \mathfrak{F}\). We will do this by painting a Rokhlin city (see \(\S 7\)). Let \(r > V\), and let \(V\) be the Dirichlet valence of \(\mathfrak{F}\). Let \(N := r + V\), and let \(\mathcal{O}_B(N)\) be a Rokhlin city in \(T\). Let \(a \in A^{B(r)}\) be some \(\mathfrak{F}\)-admissible sequence, and let \(b \in A^L\) be a Dirichlet extension of \(a\). Let \(w := b|_{B(N)}\), let \(\epsilon > 0\); then Corollary 7.1 yields an open partition \(P \in \mathcal{O}A^\mathbb{T}\) so that every element of \(\Xi_\varsigma(P)\) is \(\epsilon\)-tiled by \(w\) with spacer 0. It follows that every element of \(\Xi_\varsigma(P)\) is \(\mathfrak{F}\)-admissible. \(\Box\)

For example, the fixed point SFT of the Voter Model is Dirichlet, as are the SFTs of fixed points, periodic points, and travelling waves for any LtL CA. Thus, \(\mathbb{Q}\overline{S}_\varsigma\) contains nonconstant fixed points for the Voter Model, and nonconstant travelling waves for LtL CA.

10. **Background on Linear Cellular Automata**

Suppose that \(p\) is prime and \(A = \mathbb{Z}/p = [0..p)\) is a cyclic group; then \(A^L\) is also an abelian group under componentwise addition. We say \(\Phi\) is a *linear cellular automaton* (LCA) if \(\Phi\) has a local map \(\phi : A^B \to A\) of the form
\[
\phi(a) = \sum_{b \in B} \varphi_b a_b \pmod{p}, \quad \text{for any } a \in A^B, \tag{26}
\]
where \(B \subset L\) is finite, and \(\varphi_b \in [1..p)\) are constants for all \(b \in B\). Equivalently,
\[
\Phi(a) = \sum_{b \in B} \varphi_b \cdot \sigma^b(a), \quad \text{for any } a \in A^L. \tag{27}
\]
If \(p = 2\), then \(A = \mathbb{Z}/2\), and \(\Phi\) is called a *boolean linear cellular automaton* (BLCA), and eqns. (26) and (27) become:
\[
\phi(a) = \sum_{b \in B} a_b, \quad \text{for any } a \in A^B, \quad \text{and} \quad \Phi(a) = \sum_{b \in B} \sigma^b(a), \quad \text{for any } a \in A^L. \tag{28}
\]
\(\dagger\) This section contains technical results which are used in \(\S\ 12\), \(\S\ 13\), and \(\S\ 15\).

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We call $\mathcal{B}$ the *neighbourhood* of $\Phi$. A BLCA is entirely determined by its neighbourhood.

The advantage of the ‘polynomial of shifts’ notation in equations (27) and (28) is that iteration of $\Phi$ corresponds to multiplying the polynomial by itself. For example, if $\Phi = 1 + \sigma$, then the Binomial Theorem says $\Phi^n = \sum_{n=0}^{N} \binom{N}{n} \sigma^n$.

For any $N \in \mathbb{N}$, let $[N^{(i)}]_{i=0}^{\infty}$ be the $p$-ary expansion of $N$, so that $N = \sum_{i=0}^{\infty} N^{(i)} p^i$. Let $\mathcal{L}(N) := \{ n \in [0..N] : n^{(i)} \leq N^{(i)}, \text{for } \forall i \in \mathbb{N} \}$. To get binomial coefficients mod $p$, we use:

**Lucas’ Theorem** [22]: $\left[ \begin{array}{c} N \\ n \end{array} \right]_p = \prod_{i=0}^{\infty} \left[ \begin{array}{c} N^{(i)} \\ n^{(i)} \end{array} \right]_p$, where we define $\left[ \begin{array}{c} N^{(i)} \\ n^{(i)} \end{array} \right]_p := 0$ if $n^{(i)} > N^{(i)}$, and $\left[ \begin{array}{c} 0 \\ 0 \end{array} \right]_p := 1$. Thus, $\left[ \begin{array}{c} N \\ n \end{array} \right]_p \neq 0$ iff $n \in \mathcal{L}(N)$.

For example, if $p = 2$ and $\Phi = 1 + \sigma$, then Lucas’ Theorem says that $\Phi^N = \sum_{n \in \mathcal{L}(N)} \sigma^n$. More generally, Lucas’ Theorem and Fermat’s Theorem [11] §6, Thm.1] together imply:

**Lemma 10.1.** Let $p$ be prime, let $\mathcal{A} = \mathbb{Z}/p$, and let $\Phi = \sum_{b \in \mathcal{B}} \phi_b \cdot \sigma^b$, as in eqn. [27]. Then for any $m \in \mathbb{N}$, if $P := p^m$, then $\Phi^P = \sum_{b \in \mathcal{B}} \phi_b \cdot \sigma^{p^m b}$.

11. **Background on Torus Rotation Systems**

Let $\tau : \mathbb{L} \rightarrow \mathbb{T}$ be a group monomorphism with dense image, and for any $\ell \in \mathbb{L}$, let $\zeta^{\ell} = \rho^{\tau(\ell)}$ denote the corresponding rotation of $\mathbb{T}$. This defines an ergodic torus rotation system $(\mathbb{T}, d, \zeta)$. Torus rotation systems are *minimal* in the sense that every $t \in \mathbb{T}$ has dense $\zeta$-orbit in $\mathbb{T}$. For the ‘generic’ torus rotation system, an even stronger property holds. Suppose $\mathbb{L} = \mathbb{Z}$. If $p \in \mathbb{N}$; then the system $(\mathbb{T}, d, \zeta)$ is *minimal along powers of $p$* if, for any $t \in \mathbb{T}$, and any $\epsilon > 0$, there is some $m \in \mathbb{N}$ such that $\zeta^{(p^m)}(0) \in \epsilon \cdot t$.

More generally, suppose $\mathbb{L} = \mathbb{Z}^D$. For each $d \in [1..D]$, let $\mathbf{e}_d := (0, \ldots, 0, 1, \ldots, 0) \in \mathbb{L}$. If $p \in \mathbb{N}$, then the system $(\mathbb{T}, d, \zeta)$ is *minimal along powers of $p$* if, for any $\mathbf{t}_1, \ldots, \mathbf{t}_D \in \mathbb{T}$, and any $\epsilon > 0$, there is some $m \in \mathbb{N}$ such that, for all $d \in [1..D]$, $\zeta^{(p^m \mathbf{e}_d)}(0) \in \epsilon \cdot \mathbf{t}_d$.

We’ll show that the ‘generic’ torus rotation is minimal along powers of $p$. First, note that for any monomorphism $\tau : \mathbb{L} \rightarrow \mathbb{T}$, there are unique $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_D \in \mathbb{T}$ such that $\tau(\ell) = \ell_1 \mathbf{a}_1 + \ell_2 \mathbf{a}_2 + \cdots + \ell_D \mathbf{a}_D$ for any $\ell = (\ell_1, \ldots, \ell_D) \in \mathbb{L}$. Treat

† This section contains technical results which are used in [12] and [13].

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(a₁,...,a_D) as an element of the Cartesian power (T)^D, and let λ^D be the product Lebesgue measure on (T)^D.

**Proposition 11.1.** Fix p \in \mathbb{N}. For \forall \lambda, \nu (a₁,...,a_D) \in (T)^D the system (T,d,ξ) is minimal along powers of p.

**Proof.** Let T = T^K \cong [0,1]^K. For each d ∈ [1..D], let a_d = (a_d₁,t_d₂,⋯,a_dₖ,t_dₖ). and let t_d = (t_d₁,t_d₂,⋯,t_dₖ). For each k ∈ [1..K], suppose a_d and t_d have p-ary expansions:

\[ a_d = \sum_{i=1}^{\infty} \frac{a_{di}}{p^i}, \quad \text{and} \quad t_d = \sum_{i=1}^{\infty} \frac{t_{di}}{p^i}, \]

where \( a_{di} \in \{0,p\} \) and \( t_{di} \in \{0,p\} \) for all \( i \in \{1,2,3,\ldots\} \). Let \( L := \lfloor \log_p(\epsilon/K) \rfloor \).

Let \( A = \{0,p\} \). Fix \( d \in [1..D] \) and \( k \in [1..K] \). For \( \forall \lambda\ a_d \in T \), the p-ary sequence \( a_d = (a_d₁,a_d₂,\ldots,a_dₖ) \) is generic for the \( \left( \frac{1}{p},\frac{1}{p} \right) \)-Bernoulli measure on \( A^K \). Thus, the word \( t_d := [t_{d₁},t_{d₂},\ldots,t_{dₖ}] \) occurs in \( a_d \) with frequency \( p^{-L} \). Furthermore, for \( \forall \lambda, \nu (a₁,\ldots,a_D) \in (T)^D \) the collection of sequences \( \{a_d\}_{k=1}^{K} \in A^K \) are independent. Hence, the words \( t_d \) occur simultaneously in \( a_d \) for all \( d \in [1..D] \) and \( k \in [1..K] \) with frequency \( p^{-L-K} > 0 \). Thus, there is some \( m \in \mathbb{N} \) such that \( [a_{d₁}m₊₁,a_{d₂}m₊₂,\ldots,a_{dₖ}m₊L] = [t_{d₁},t_{d₂},\ldots,t_{dₖ}], \)

for all \( k \in [1..K] \) and \( d \in [1..D] \). Thus, \( p^m \cdot a_d \overset{K/p^L}{\sim} t_d \), for all \( k \in [1..K] \) and \( d \in [1..D] \). Thus, for all \( d \in [1..D] \), \( \zeta(p^m) \sim 0 \) and \( p^m a_d \overset{K/p^L}{\sim} t_d \), and

\[ K/p^L \leq \frac{K}{K} \leq \epsilon. \]

12. **Expansiveness**

Let \( \Phi : A^K \to A^K \) be a cellular automaton, and let \( \xi > 0 \). If \( p \in \mathcal{O}_{\mathfrak{S}_\xi} \), we say that the topological dynamical system \( (\mathcal{O}_{\mathfrak{S}_\xi},d_B,\Phi) \) is (positively) \( \xi \)-expansive at \( p \) if, for any \( q \in \mathcal{O}_{\mathfrak{S}_\xi} \) with \( q \neq p \), there is some \( n \in \mathbb{N} \) such that \( d_B(\Phi^n(p),\Phi^n(q)) > \xi \). We say \( (\mathcal{O}_{\mathfrak{S}_\xi},d_B,\Phi) \) is (positively) \( \xi \)-expansive if \( (\mathcal{O}_{\mathfrak{S}_\xi},d_B,\Phi) \) is \( \xi \)-expansive at every \( p \in \mathcal{O}_{\mathfrak{S}_\xi} \).

**Proposition 13 of [20]** states that a cellular automaton \( \Phi \) is never expansive in the Besicovitch topology. This is proved by constructing a configuration \( a \in A^K \) that is ‘nonexpansive’ for \( \Phi \). However, \( a \notin \mathcal{O}_{\mathfrak{S}_\xi} \), so the proof of Proposition 13 in [20] does not apply to \( (\mathcal{O}_{\mathfrak{S}_\xi},d_B,\Phi) \).

Expansiveness is the ‘opposite’ of equicontinuity. Torus rotation systems are equicontinuous; hence, any shift map acts equicontinuously on \( \mathcal{O}_{\mathfrak{S}_\xi} \). It is natural to conjecture that all CA act equicontinuously on \( \mathcal{O}_{\mathfrak{S}_\xi} \), especially in light of Proposition 13 in [20]. We’ll refute this by showing that the boolean linear CA of Example 3.3(a) is expansive on \( \mathcal{O}_{\mathfrak{S}_\xi} \).

**Proposition 12.1.** Fix \( a \in T \), and suppose \( L = \mathbb{Z} \) acts on \( T \) by \( \zeta(t) = t + z \cdot a \) for any \( t \in T \) and \( z \in \mathbb{Z} \). Let \( A = \mathbb{Z}/2 = \{0,1\} \), and let \( \Phi = 1 + \sigma \) be the BLCA of Example 3.3(a). For \( \forall \lambda \ a \in T \), and for any \( \xi < 1 \), the system \( (\mathcal{O}_{\mathfrak{S}_\xi},d_B,\Phi) \) is \( \xi \)-expansive.
Proof. Let \( \mathbf{o} = (\ldots, 0, 0, 0, \ldots) \in \mathcal{A}^\mathbb{Z} \). Then \( \Phi(\mathbf{o}) = \mathbf{o} \), because \( \Phi \) is a linear CA. We will first show that \( (\Omega\mathcal{S}_\varsigma, d_B, \Phi) \) is \( \xi \)-expansive at \( \mathbf{o} \).

Let \( \mathbf{p} \in \Omega\mathcal{S}_\varsigma \), so that \( \mathbf{p} = \mathcal{P}_\varsigma(t) \) for some \( \mathcal{P} \in \mathcal{A}\mathcal{T} \) and \( t \in T \). Let \( \mathcal{O} \) be the constant zero partition (i.e. \( \mathcal{O} = \{O_0, O_1\} \), where \( O_0 := T \) and \( O_1 := \emptyset \)). Thus, \( \mathbf{o} = \mathcal{O}_\varsigma(t) \), and Proposition 2.3(a) and Lemma 3.2(a) imply that \( d_B(\Phi^\mathbf{n}(\mathbf{p}), \mathbf{o}) = \frac{1}{2}d_\Delta(\Phi^\mathbf{n}(\mathcal{P}), \mathcal{O}) \). Thus, we seek \( n \in \mathbb{N} \) such that \( d_\Delta(\Phi^\mathbf{n}(\mathcal{P}), \mathcal{O}) > 2\xi \).

**Claim 1.** Let \( \mathcal{P} \subset \mathcal{T} \) be measurable, with \( 0 < \lambda[\mathcal{P}] < 1 \). There is some \( t \in \mathcal{T} \) such that \( \lambda[\mathcal{P} \cap \rho^t(\mathcal{P})] < \lambda[\mathcal{P}]^2 \).

**Proof.** First note that
\[
\int_T \lambda[\mathcal{P} \cap \rho^t(\mathcal{P})] \, d\lambda[t] = \int_T \int_{\mathcal{P}} \mathbb{I}_{\rho^t(\mathcal{P})}(s) \, d\lambda[s] \, d\lambda[t] 
\]

\[
\Rightarrow \int_{\mathcal{P}} \int_T \mathbb{I}_{\rho^{-t}(\mathcal{P})}(-t) \, d\lambda[t] \, d\lambda[s] = \int_{\mathcal{P}} \lambda[\rho^{-s}(\mathcal{P})] \, d\lambda[s] 
\]

\[
\Rightarrow \int_{\mathcal{P}} \lambda[\mathcal{P}] \, d\lambda[s] = \lambda[\mathcal{P}] \cdot \lambda[\mathcal{P}] = \lambda[\mathcal{P}]^2. \tag{29}
\]

(*) is because \( \mathbb{I}_{\rho^t(\mathcal{P})}(s) = \mathbb{I}_{\mathcal{P}}(s-t) = \mathbb{I}_{\rho^{-t}(\mathcal{P})}(-t) \) for any \( t, s \in \mathcal{T} \). (H) is because \( \lambda \) is the Haar measure on \( \mathcal{T} \).

But \( \lambda[\mathcal{P} \cap \rho^0(\mathcal{P})] = \lambda[\mathcal{P}] > \lambda[\mathcal{P}]^2 \). Thus, eqn. 29 implies there must be some \( t \neq 0 \) such that \( \lambda[\mathcal{P} \cap \rho^t(\mathcal{P})] < \lambda[\mathcal{P}]^2 \). \( \diamond \) Claim 1

**Claim 2.** Let \( \mathcal{P} \subset \mathcal{T} \) be measurable, with \( 0 < \lambda[\mathcal{P}] < 1 \). For all \( \lambda, \mu \in \mathcal{T} \), there is some \( m \in \mathbb{N} \) such that \( \lambda[\mathcal{P} \cap \mathcal{Q}^\mathcal{P}(\mathcal{Q}^\mathcal{P})] > \left( 2 - \lambda[\mathcal{P}] \right) \cdot \lambda[\mathcal{P}] \).

**Proof.** This follows from Claim 1 and Proposition 1.1. \( \diamond \) Claim 2

**Claim 3.** There is an increasing sequence \( \{m_k\}_{k=1}^\infty \subset \mathbb{N} \), such that, if \( \mathcal{P}^0 := \mathcal{P} \), and for all \( k \in \mathbb{N} \), \( \mathcal{P}^k := \Phi_\varsigma^{(2m_k)}(\mathcal{P}^{k-1}) \), then for all \( k \in \mathbb{N} \),
\[
\lambda[\mathcal{P}^{k+1}] > \left( 2 - \lambda[\mathcal{P}^k] \right) \cdot \lambda[\mathcal{P}^k].
\]

**Proof.** Suppose \( \mathcal{P} = \{\mathcal{P}_0, \mathcal{P}_1\} \). Let \( m_1 \) be the result of setting \( \mathcal{P} = \mathcal{P}_{1} \) in Claim 2. Lemma 1.1 says that \( \Phi(2m) = 1 + \sigma(2m) \), so \( \Phi_\varsigma^{(2m)} = 1 + \varsigma(2m) \). Thus, if \( \mathcal{P}^1 := \Phi_\varsigma^{(2m)}(\mathcal{P}) \), then \( \mathcal{P}^1 = \mathcal{P} \Delta \varsigma^{2m}(\mathcal{P}_1) \); hence Claim 2 says \( \lambda[\mathcal{P}^1] > \left( 2 - \lambda[\mathcal{P}_1] \right) \cdot \lambda[\mathcal{P}_1] \).

Inductively, suppose we have \( m_k \in \mathbb{N} \) and partition \( \mathcal{P}^k := \Phi_\varsigma^{(2m_k)}(\mathcal{P}^{k-1}) \). Let \( m_{k+1} \) be the result of setting \( \mathcal{P} = \mathcal{P}_{k+1} \) in Claim 2. Then \( \Phi_\varsigma^{(2m_{k+1})} = 1 + \sigma(2m_{k+1}) \), so \( \Phi_\varsigma^{(2m_{k+1})} = 1 + \varsigma(2m_{k+1}) \). Thus, if \( \mathcal{P}^{k+1} := \Phi_\varsigma^{(2m_{k+1})}(\mathcal{P}) \), then \( \mathcal{P}^{k+1} = \mathcal{P} \Delta \varsigma^{2m_{k+1}}(\mathcal{P}_{k+1}) \); hence Claim 2 says \( \lambda[\mathcal{P}^{k+1}] > \left( 2 - \lambda[\mathcal{P}_{k+1}] \right) \cdot \lambda[\mathcal{P}_{k+1}] \). \( \diamond \) Claim 3

**Claim 4.** If \( \{r_k\}_{k=1}^\infty \subset (0, 1) \), and \( r_{k+1} > (2 - r_k) \cdot r_k \) for all \( k \in \mathbb{N} \), then \( \lim_{k \to \infty} r_k = 1 \).
Proof. The sequence is increasing (because $r_k < 1$, so $(2 - r_k) > (2 - 1) = 1$, so $r_{k+1} > (2 - r_k) \cdot r_k > r_k$, for all $k \in \mathbb{N}$). We claim that sup $r_k = 1$.

Suppose not; then $\exists y < 1$ such that $r_k < y$, for all $k \in \mathbb{N}$. But then $r_{k+1} = (2 - r_k) \cdot r_k > (2 - y) \cdot r_k$, for all $k$; hence, $r_k > (2 - y)^k \cdot r_1$. Thus,

$$\lim_{k \to \infty} r_k \geq r_1 \cdot \lim_{k \to \infty} (2 - y)^k = \infty.$$  Contradiction. \hfill $\diamond$ claim 4

Now, for any $k \in \mathbb{N}$, $d_\Delta(O, P^k) = 2 \cdot \lambda [O \Delta P^k] = 2 \cdot \lambda [P^k]$. Thus, combining Claims 3 and 4 we conclude that $\lim_{k \to \infty} d_\Delta(O, P^k) = 2 \cdot \lim_{k \to \infty} \lambda [P^k] = 2$.

But observe that

$$P^k = \Phi^{(2m_k)}(P_{k-1}) = \ldots = \Phi^{(2m_k)} \circ \Phi^{(2m_{k-1})} \circ \ldots \circ \Phi^{(2m_1)}(P) = \Phi^{mk}(P),$$

where $n_k = 2^{m_k + 2^{m_{k-1}} + \ldots + 2^{m_1}}$.

Hence, $\lim_{k \to \infty} d_B(o, \Phi^{nk}(p)) = \frac{1}{2} \lim_{k \to \infty} d_\Delta(O, P^k) = 1 > \xi$, where $(\ast)$ is by Proposition 2.4(b) and Lemma 2.2(a).

It follows that $(\Phi, \OmegaS, d_B)$ is $\xi$-expansive at $o$. To see that $(\Phi, \OmegaS, d_B)$ is $\xi$-expansive everywhere, let $p, q \in \OmegaS$; we must find some $n \in \mathbb{N}$ such that $d_B(\Phi^n(p), \Phi^n(q)) > \xi$. Let $r = p - q$; find $n \in \mathbb{N}$ such that $d_B(\Phi^n(r), o) = d_B(\Phi^n(r), \Phi^n(o)) > \xi$. Thus, $d_B(\Phi^n(p), \Phi^n(q)) = d_B(\Phi^n(p) - \Phi^n(q), o) \geq d_B(\Phi^n(r), o) > \xi$, where $(L)$ is because $\Phi$ is linear. \hfill $\Box$

13. Niltropism and Rigidity

Let $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$. The system $(\OmegaS, d_B, \Phi)$ is rigid along $\{n_k\}_{k=1}^{\infty}$ if, for all $q \in \OmegaS$, $d_B \lim_{k \to \infty} \Phi^{n_k}(q) = q$. Suppose $A$ is an abelian group with identity element 0. Let $o \in A^I$ be the constant zero configuration (i.e. $o_T = 0$, for all $T \in \mathbb{N}$). Then $(\OmegaS, d_B, \Phi)$ is niltropic along $\{n_k\}_{k=1}^{\infty}$ if, for all $q \in \OmegaS$, $d_B \lim_{k \to \infty} \Phi^{n_k}(q) = o$.

Likewise, $(A^T, d_\Delta, \Phi^T)$ is rigid along $\{n_k\}_{k=1}^{\infty}$ if, for all $Q \in A^T$, $d_\Delta \lim_{k \to \infty} \Phi^{nk}(Q) = O$, and $(A^T, d_\Delta, \Phi^T)$ is niltropic along $\{n_k\}_{k=1}^{\infty}$ if for all $Q \in A^T$, $d_\Delta \lim_{k \to \infty} \Phi^{nk}(Q) = O$, where $O \in A^T$ is the trivial partition (i.e. $O_0 := T$ and $O_a := \emptyset$ if $a \neq 0$).

Example 13.1. Let $L = \mathbb{Z}$, and let $a \in L$ be such that $\zeta(t) = t + z \cdot a$ for all $z \in L$ and $t \in T$. Then $(\OmegaS, d_B, \sigma)$ is rigid. To see this, find a sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ such that $\lim_{k \to \infty} n_k \cdot a = 0$ in $T$. Thus, if $t \in T$, then

$$\lim_{k \to \infty} \zeta^{n_k}(t) = \lim_{k \to \infty} n_k a + t = t. \quad (30)$$

Now, for any $p \in \OmegaS$, there is some $P \in \OmegaS$ and some $t \in T$ such that $p = P_0(t)$; hence, $\lim_{k \to \infty} \sigma^{n_k}(p) = \lim_{k \to \infty} \sigma^{n_k}(P_0(t)) = \lim_{k \to \infty} P_0(\zeta^{n_k}(t)) = P_0(t) = p$. Here, $(\ast)$ is by Proposition 2.2(a), and $(\dagger)$ is by Proposition 2.2(b) and eqn. (30). \hfill $\diamond$

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If \( A = \mathbb{Z}/p \) and \( \Phi \) is the LCA of eqn. \( 27 \) in \( 10 \) we define \( \text{trace}[\Phi] := \sum_{b \in B} \varphi_b \) (mod \( p \)). Suppose \( L = \mathbb{Z}^D \). Let \( \tau : L \to T \) be a monomorphism such that \( \zeta^{(t)}(\ell) = \rho^{(t)}(\ell) \) for all \( \ell \in L \) and \( t \in T \). Then there are unique \( a_1, a_2, \ldots, a_D \in T \) such that \( \tau(\ell) = \ell_1 a_1 + \ell_2 a_2 + \cdots + \ell_D a_D \) for any \( \ell = (\ell_1, \ldots, \ell_D) \in L \).

**Theorem 13.2.** Let \( A = \mathbb{Z}/p \) (\( p \) prime). For \( \forall \ a_1, \ldots, a_D \in T \), there is a sequence \( \{ m_j \}_{j=1}^\infty \in \mathbb{N} \) such that, if \( n_j := p^m_j \) for all \( j \in \mathbb{N} \), then for any linear cellular automaton \( \Phi \),

\[
(\text{a}) \quad \text{If trace}[\Phi] \equiv 0, \text{ then } (A^T, d_\Delta, \Phi_c) \text{ and } (Q\mathcal{S}_L, d_B, \Phi) \text{ are niltropic along } \{ n_k \}_{k=1}^\infty.
\]

\[
(\text{b}) \quad \text{If Trace}[\Phi] \neq 0, \text{ then } (A^T, d_\Delta, \Phi_c) \text{ and } (Q\mathcal{S}_L, d_B, \Phi) \text{ are rigid along } \{ (p-1) \cdot n_k \}_{k=1}^\infty.
\]

**Proof.** We’ll show rigidity/niltropism for \( (A^T, d_\Delta, \Phi_c) \); rigidity/niltropism for \( (Q\mathcal{S}_L, d_B, \Phi) \) follows from Proposition \( 35 \). Let \( \{ \epsilon_j \}_{j=1}^\infty \) be a sequence decreasing to zero. For all \( j > 0 \), Proposition \( 11.1 \) yields some \( m_j \in \mathbb{N} \) such that \( d(p^{m_j} \cdot a_d, 0) < \epsilon_j \) for all \( d \in [1..D] \). Let \( n_j := p^{m_j} \). Let \( \Phi \) be the LCA of eqn. \( 27 \) in \( 10 \) and let \( M := \max |b| \). Let \( \mathcal{P} \in A^T \) and fix \( \delta > 0 \).

**Claim 1.** There exists \( J \in \mathbb{N} \) such that, if \( j > J \), then for all \( b \in B \), \( d_\Delta \left( \sigma^{(n_j \cdot b)}(\mathcal{P}), \mathcal{P} \right) < \delta \).

**Proof.** The function \( T \ni t \mapsto \rho^t(\mathcal{P}) \in A^T \) is \( d_\Delta \)-continuous \( 22 \) Prop 8.5, p.229. Thus, find some \( \epsilon_j > 0 \) such that,

\[
\text{For any } t \in T, \quad \left( d(t, 0) < M \cdot \epsilon_j \right) \implies \left( d_\Delta \left( \rho^t(\mathcal{P}), \mathcal{P} \right) < \delta \right).
\]

If \( b = (b_1, \ldots, b_D) \), then \( \zeta^{(n_j \cdot b)} = \rho^{(b_1 n_j a_1 + \cdots + b_D n_j a_D)} \). If \( j > J \), then \( d((b_1 n_j a_1 + \cdots + b_D n_j a_D), 0) < \epsilon_j \) for all \( d \in [1..D] \). Hence, \( d([b_1 n_j a_1 + \cdots + b_D n_j a_D], 0) < (|b_1| + \cdots + |b_D|) \cdot \epsilon_j = |b| \cdot \epsilon_j < M \epsilon_j \), so \( d_\Delta \left( \zeta^{(n_j \cdot b)}(\mathcal{P}), \mathcal{P} \right) = d_\Delta \left( \rho^{(b_1 n_j a_1 + \cdots + b_D n_j a_D)}(\mathcal{P}), \mathcal{P} \right) < \delta \). \( \diamond \) Claim 1

Let \( \Phi \) be as in eqn. \( 27 \), and let \( R := \text{trace}[\Phi] \). Let \( B := \#(B) \). If \( j > 0 \), then \( \Phi^{n_j}(\mathcal{P}) = \sum_{b \in B} \varphi_b \zeta^{(n_j \cdot b)}(\mathcal{P}) \). If \( R = 0 \), then \( d_\Delta \left( \Phi^{n_j}(\mathcal{P}), \mathcal{O} \right) = 0 \).

This works for any \( \delta > 0 \). Thus, if \( R = 0 \), then \( \lim_{j \to \infty} d_\Delta \left( \Phi^{n_j}(\mathcal{P}), \mathcal{O} \right) = 0 \).

If \( R \neq 0 \), then \( \Phi^{(p-1) n_j}(\mathcal{P}) \sim R^{p-1} \cdot \mathcal{P} \), where \( \delta_1 := (p-1)B \cdot \delta \), and \( * \) is by Fermat’s Theorem \( 14 \) §6, Thm.1. Thus, \( \lim_{j \to \infty} d_\Delta \left( \Phi^{(p-1) n_j}(\mathcal{P}), \mathcal{P} \right) = 0 \). \( \Box \)

**Example 13.3.** Let \( L = \mathbb{Z} \).

\( (a) \) If \( A = \mathbb{Z}/2 \), then \( 1 + \sigma \) is niltropic, but \( 1 + \sigma + \sigma^2 \) is rigid.

\( (b) \) If \( A = \mathbb{Z}/3 \), then \( 1 + \sigma \) is rigid, while \( 1 + \sigma + \sigma^2 \) and \( 1 + 2 \sigma \) are niltropic. \( \diamond \)
CA-Invariant Quasisturmian Measures

If $\Phi : A^I \to A^I$ is a CA, are there any $\Phi$-invariant quasisturmian measures on $A^I$? The answer is reminiscent of J. King’s Weak Closure Theorem \cite{27}.

**Theorem 14.1.** Let $\Phi : A^I \to A^I$ be a CA. If $P \in A^T$ and $\mu = \Upsilon_\varsigma (P)$, then

$$\mu \text{ is } \Phi\text{-invariant} \iff \left( \Phi_\varsigma (P) = \rho^t (P) \text{ for some } t \in \mathbf{T} \right).$$

To prove this, we must first characterize the set $M_\varsigma^{qs} (A^I)$ of $\varsigma$-quasisturmian measures.

**Proposition 14.2.** Let $\mu \in M(A^I)$. Then $\left( \mu \text{ is quasisturmian} \right)$

$$\iff \text{\text{The MP\text{-}LS} (A^I, \sigma, \mu) \text{ is isomorphic to a torus rotation}.}$$

To prove Proposition 14.2 in turn, we use the following lemma:

**Lemma 14.3.** Let $P \in A^T$ and let $\varsigma$ be an $\mathbb{L}$-action on $\mathbf{T}$. Let $\mu = \Upsilon_\varsigma (P)$.

(a) $P_\varsigma : \mathbf{T} \to A^I$ is an (MP\text{-}LS) epimorphism from $(\mathbf{T}, \varsigma, \lambda)$ to $(A^I, \sigma, \mu)$.

(b) If $P$ is simple (see 14.2), then $P_\varsigma$ is an isomorphism from $(\mathbf{T}, \varsigma, \lambda)$ to $(A^I, \sigma, \mu)$.

**Proof.** (a) $P_\varsigma : \mathbf{T} \to A^I$ is a measurable function, and Proposition 2.2(a) says $\sigma^\ell \circ P_\varsigma = P_\varsigma \circ \varsigma^\ell$ for any $\ell \in \mathbb{L}$. Hence $P_\varsigma$ is an epimorphism from $(\mathbf{T}, \varsigma, \lambda)$ to $(A^I, \sigma, \mu)$.

(b) If $P$ is simple then Lemma 14.3 says that $P_\varsigma$ is injective (\lambda-ae). Thus, $P_\varsigma$ is an isomorphism from $(\mathbf{T}, \varsigma, \lambda)$ to $(A^I, \sigma, \mu)$. □

**Proof of Proposition 14.2** \text{‘$\implies$’} Let $P \in A^T$, let $\varsigma$ be an $\mathbb{L}$-action on $\mathbf{T}$, and let $\mu = \Upsilon_\varsigma (P)$. If $P$ is simple then Lemma 14.3(b) says $P_\varsigma$ is an MP\text{-}LS isomorphism from $(\mathbf{T}, \varsigma, \lambda)$ to $(A^I, \sigma, \mu)$. If $P$ is not simple, then Lemma 14.3(e) yields a simple partition $\overline{P} \in A^T$ with $\mu = \Upsilon_\varsigma (\overline{P})$; then Lemma 14.3(b) says $P_\varsigma$ is an isomorphism from $(\mathbf{T}, \varsigma, \lambda)$ to $(A^I, \sigma, \mu)$.

\text{‘$\iff$’} Suppose $\Psi : \mathbf{T} \to A^I$ is an isomorphism from $(\mathbf{T}, \varsigma, \lambda)$ to $(A^I, \sigma, \mu)$. We claim that $\Psi = P_\varsigma$ for some measurable partition $P \in A^T$. For any $\ell \in \mathbb{L}$, let $\Pr_\ell : A^I \to A$ be projection onto the $\ell$th coordinate —ie. $\Pr_\ell (a) := a_\ell$, for any $a \in A^I$. Define $P := \Pr_0 \circ \Psi : \mathbf{T} \to A$. Then $P$ is a measurable $A$-valued function —ie. an $A$-labelled partition —on $\mathbf{T}$. Observe that, for $\forall_\lambda t \in \mathbf{T}$ and all $\ell \in \mathbb{L}$,

$$P_\varsigma (t)_\ell = \Pr_\ell \circ \varsigma^\ell (t) = \Pr_0 \circ \varsigma^\ell (t) = \Pr_0 \circ \varsigma^\ell \circ \Psi (t) = \Pr_\ell \circ \Psi (t) = \Psi (t)_\ell.$$

Hence $P_\varsigma (t) = \Psi (t)$. This holds for $\forall_\lambda t \in \mathbf{T}$, so $P_\varsigma = \Psi$ (\lambda-ae). Hence, $\mu = \Psi (\lambda) = P_\varsigma (\lambda) = \Upsilon_\varsigma (P)$, so $\mu$ is quasisturmian. □
Proof of Theorem 14.1. “⇐⇒” is obvious. We must prove “⇒”.

Case 1: (P is simple) If Q = Φξ(P), then Υξ(Q) = Υξ(Φξ(P)) = Φ(μ) = μ, where (ς) is Lemma 14.3(c), and (†) is because μ is Φ-invariant. Thus it suffices to show:

Claim 1. If Q ∈ AT and Υξ(Q) = μ, then Q = ρα(P) for some t ∈ T.

Proof. P is simple, so Lemma 14.3(b) says P, is an MPLS isomorphism from (T, ζ, β) to (AT, σ, μ). Q may not be simple, but Lemma 14.3(a) says that Qξ : T → AT is an MPLS epimorphism. Thus, ψ := P−1 ◦ Qξ : T → T is an endomorphism from (T, ζ, β) to itself, (ie. ψ : T → T and ψ ◦ ψ = ψ). In other words, Q = ρα(P).

Claim 1.1. There is some a ∈ T such that ψ = ρα.

Proof. Define α : T → T by α(t) := ψ(t) − t. We claim that α is constant (λ-ς).

Why? α is measurable because ψ is measurable. α is ς-invariant, because for any ρ ∈ L and t ∈ T, ω(t) := ψ(t) − t = ψ(t) − α(t) = ψ(t) − α(t) = (ψ(t) − α(t)). But ς is ergodic, so α must be constant (λ-ς).

Hence, ψ = ρα, where a is the constant value of α. □ Claim 1.1

But then (P−1 ◦ Qξ = ρα) ⇐⇒ (Qξ = P, ◦ ρα) ⇐⇒ (Q = P ◦ ρα, (λ-ς)). In other words, Q = ρα(P). □ Claim 1

Case 2: (P is not simple) Lemma 14.0(c) yields a simple partition P such that μ = Υξ(P). When applied to P, Case 1 implies that Φξ(P) = ρα(P) for some t ∈ T. Let q : T → T be the quotient map, and t ∈ q−1(t); it follows that Φξ(P) = ρα(P). □

Example 14.4. Suppose q ∈ QT is a quasisturmian travelling wave for Φ, with period 1 (see §1). Thus, q = Qξ(t), for some Q ∈ QT and t ∈ T. Hence, μ := Υξ(Q) is a Φ-invariant QS measure. If q has velocity v ∈ L, then Φξ(Q) = ψ Q.

Let L = ZD, and let a1, ..., aD ∈ T be as defined prior to Theorem 14.2 in §1.

Proposition 14.5. Let A = Z/p (p prime); let Φ be a linear CA with trace [Φ] = 0 (see §1). Then for any a1, ..., aD ∈ T, there are no nontrivial Φ-invariant measures in M(Th(A)).

Proof. Suppose μ = Υξ(P) for some P ∈ AT. By Lemma 14.3(c), we can assume P is simple. If μ is Φ-invariant, then Corollary 14.1 says that Φξ(P) = ρα(P) for some t ∈ T. But trace [Φ] = 0 (mod p), so Theorem 14.2(a) yields a sequence {nj}j=1 ∞ such that

\[ \lim_{j \to \infty} \rho^{n_j}(P) = \lim_{j \to \infty} \Phi^{n_j}(P) = O, \]

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where \( \mathcal{O} \) is the trivial partition. However \( \rho^t \) acts isometrically on \( \mathcal{A}^T \), so for any \( j \in \mathbb{N} \)
\[
 d(\rho^{n_j}(\mathcal{P}), \mathcal{O}) = d(\mathcal{P}, \rho^{-n_j}(\mathcal{O})) = d(\mathcal{P}, \mathcal{O}). \tag{33}
\]
Combining equations \( \text{(32)} \) and \( \text{(33)} \) we conclude that \( d(\mathcal{P}, \mathcal{O}) = 0 \). Thus \( \mathcal{P} = \mathcal{O} \).

**Example 14.6.** Let \( L := \mathbb{Z} \), and recall Example \([13.3(a)](\text{a})\).

If \( A = \mathbb{Z}/2 \), then Example \([13.3(b)](\text{b})\) implies that \( 1 + \sigma \) has no QS invariant measures.

If \( A = \mathbb{Z}/3 \), then Example \([13.3(b)](\text{b})\) implies that \( 1 + 2\sigma \) has no QS invariant measures.

\[\square\]

15. **Asymptotic Nonrandomization**

Let \( A = \mathbb{Z}/2 \), and let \( \eta \) be the \((\frac{1}{2}, \frac{1}{2})\) Bernoulli measure on \( \mathcal{A}^\mathbb{Z} \). If \( \mu \in \mathcal{M}(\mathcal{A}^\mathbb{Z}) \) and \( \Phi \) is a CA, then \( \Phi \) asymptotically randomizes \( \mu \) if there is a set \( J \subset \mathbb{N} \) of density 1 such that \( \lim_{J \ni j \to \infty} \Phi_j(\mu) = \eta \). Linear cellular automata asymptotically randomize a wide range of measures, including most Bernoulli measures \([2, 30]\) and Markov chains \([35, 34]\), and also Markov random fields supported on the full shift, subshifts of finite type, or sofic shifts \([37, 36]\). Indeed, the current literature has no examples of nonperiodic measures on \( \mathbb{Z}/2^\mathbb{Z} \) that are not asymptotically randomized by LCA. We will now show that a broad class of quasisturmian measures are not asymptotically randomized by the linear CA \( \Phi = 1 + \sigma \).

A measure \( \mu \in \mathcal{M}(\mathcal{A}^\mathbb{Z}) \) is dyadically recurrent if there is a sequence \( n_k \to \infty \) and constants \( \epsilon, \delta > 0 \) such that, if \( k \in \mathbb{N} \), then \( \mu[0...2^n_k] > \epsilon \), where we define
\[
\mathcal{R}_k := \{ a \in \mathcal{A}^\mathbb{Z} ; a_{\ell} = a_{\ell + 2^n_k} \text{ for all } \ell \in [0...2^n_k) \}, \tag{34}
\]
Thus, if \( N_k := 2^n_k \), then \( a_{[0...\delta N_k)} = a_{(N_k...N_k+\delta N_k)} \) for all \( a \in \mathcal{A}^\mathbb{Z} \) in a subset of measure \( \epsilon \).

**Proposition 15.1.** If \( \mu \) is dyadically recurrent, and \( \Phi := 1 + \sigma \), then \( \Phi \) cannot asymptotically randomize \( \mu \).

**Proof.** Let \( E := [- \log_2(\epsilon)] + 1 \), such that \( 2^{-E} \leq \frac{\epsilon}{2} \). Find \( K \in \mathbb{N} \) such that, if \( k \geq K \), then \( \delta 2^{n_k-1} > E \). Let \( J := \{ 2^n_k + j ; k \geq K \text{ and } j \in [0...\delta 2^{n_k-1}) \} \).

Then
\[
\text{density}(J) \geq \lim_{k \to \infty} \frac{\# [2^n_k...2^n_k+\delta 2^{n_k-1})]}{\# [0...2^{n_k+1})} = \lim_{k \to \infty} \frac{\delta 2^{n_k-1}}{2^{n_k+1}} = \frac{\delta}{4} > 0.
\]

If \( J = (2^n_k + j) \in J \), then Lucas’ Theorem \([10]\) implies that \( \Phi^J = \sum_{\ell \in E(J)} \sigma^{\ell} + \)
Proposition 15.2. Let \( t \in \bigcup_{n=0}^{\infty} \), then for all \( e \in [0, E) \),

\[
\Phi^j(a)_e = \sum_{\ell \in L(j)} a_{e+\ell} + \sum_{\ell \in L(j)} a_{e+\ell+2^{n_k}} = \sum_{\ell \in L(j)} \left( a_{e+\ell} + a_{e+\ell+2^{n_k}} \right)
\]

\( \equiv (,) \): if \( \ell \in [0, j] \), then \( e + \ell < E + j < 2 \cdot \delta 2^{n_k} - 1 \). Thus, if \( a_{e+\ell} = a_{e+\ell+2^{n_k}} \), by eqn. [34].

Thus, if \( \Omega := \{ b \in A^2 : b_{[0, E]} = [0, 0] \} \), then \( \Phi^j(a) \in \Omega \) for all \( a \in R_k \). Hence \( \Phi^j(\mu)[\Omega] \geq \mu[\Omega_k] > \epsilon > \epsilon/2 \geq 2^{-E} = \eta[\Omega] \). Thus, \( \Phi^j(\mu)[\Omega] \) cannot converge to \( \eta[\Omega] \) along elements in \( \mathbb{J} \). But density \( (\mathbb{J}) > 0 \); thus, \( \Phi^j(\mu) \) cannot weak\(^*\) converge to \( \eta \) along a set of density 1.

Now we'll construct a dyadically recurrent quasisturmian measure. If \( L = \mathbb{Z} \), then a \( \mathbb{Z} \)-action \( \varsigma \) is dyadically recurrent if there is a sequence \( n_k \rightarrow k \rightarrow \infty \) such that, for all \( t \in T \), \( d(\varsigma^{2^{n_k}}(t), t) < 2^{-n_k} \). Recall the definition of the Lipschitz pseudomeasure \([*]_L\) from \( \mathbb{J} \).

**Proposition 15.2.** Let \( \mathcal{P} \in \mathcal{A}^T \), with \( [\partial \mathcal{P}]_L < \infty \). If \( \varsigma \) is dyadically recurrent, then \( \mu = \mathcal{Y}_\varsigma(\mathcal{P}) \) is dyadically recurrent.

**Proof.** Fix \( \delta < \frac{1}{2 \cdot |\mathcal{P}|_L} \); then \( \epsilon := 2 \delta \cdot |\mathcal{P}|_L < 1 \), so that \( \epsilon := 1 - \epsilon' > 0 \).

If \( C = B(\partial \mathcal{P}, 2^{-n_k}) \), then \( |C| \leq 2^{-n_k} \cdot |\mathcal{P}|_L \) by eqn. [10] in \( \mathbb{J} \). Observe that, for any \( t \in T \), \( (\mathcal{P}(t) \neq \mathcal{P}(\varsigma^{2^{n_k}}(t))) \implies (t \in C) \). Now, for any \( \ell \in [0..2^{-n_k-m}] \), define

\[
B_\ell := \{ t \in T : (\mathcal{P}(t_\ell) \neq \mathcal{P}(t_{2^{n_k}+\ell}) \} = \{ t \in T : (\varsigma^{\ell}(t) \neq \mathcal{P}(\varsigma^{\ell+2^{n_k}}(t))) \}
\]

\( \subset \{ t \in T : \varsigma^{\ell}(t) \in C \} = \varsigma^{-\ell}(C) \).

Thus, \( |B_\ell| < |C| \leq 2^{-n_k} \cdot |\mathcal{P}|_L \). So if \( B := \bigcup_{\ell=0}^{2^{n_k}} B_\ell \), then \( |B| \leq 2 \delta \cdot |\mathcal{P}|_L = \epsilon' < 1 \). If \( R_k := T \setminus B \) and \( \mathcal{R}_k := \mathcal{P}_\varsigma(R_k) \), then \( \mu[\mathcal{R}_k] = \lambda[R_k] \geq 1 - \epsilon' = \epsilon \).

If \( t \in R_k \), then \( \mathcal{P}(t_\ell) = \mathcal{P}(t_{\ell+2^{n_k}}) \), for all \( \ell \in [0..2^{n_k}) \). So if \( a \in \mathcal{R}_k \), then \( a_\ell = a_{\ell+2^{n_k}} \), for all \( \ell \in [0..2^{n_k}) \). \( \Box \)

When is a \( \mathbb{Z} \)-action \( \varsigma \) dyadically recurrent? Let \( T := T^1 \); then there is some \( r \in T \) such that \( \varsigma^z(t) = t + zn \) for all \( t \in T \) and \( z \in \mathbb{Z} \). Identify \( T^1 \cong [0, 1) \), and suppose that \( r \) has binary expansion \( r = 0.r_1r_2r_3 \ldots \). Say that \( r \) is dyadically recurrent if there is a sequence \( n_k \rightarrow k \rightarrow \infty \) such that \( r_j = 0 \) for all \( j \in (n_k..2n_k) \).
For example, let $n_1 = 1$, and inductively define $n_{k+1} = 2n_k + 1$. This gives the sequence $\{1, 3, 7, 15, 31, \ldots \}$. Then the number
\[
  r = \sum_{k=1}^{\infty} 2^{n_k} = 0.100000000000\ldots
\]
is dyadically recurrent.

**Proposition 15.3.** Let $R \subset T^1$ be the set of dyadically recurrent elements.

(a) If $r \in R$, then $\varsigma^z(t) := t + zn_k$ is a dyadically recurrent $Z$-action.

(b) $R$ is a dense $G_\delta$ subset of $T^1$, but $\lambda[R] = 0$.

**Proof.** (a) For any $k \in \mathbb{N}$, observe that
\[
  2^{n_k} \cdot r = 0.r_{(n_k+1)}r_{(n_k+2)}\ldots = 0.000\ldots r_{(2n_k+1)}r_{(2n_k+2)}\ldots < 2^{-n_k}.
\]
Thus, for any $t \in T$, $d(\varsigma^{2n_k}(t), t) = d(2^{n_k}r + t, t) = |2^{n_k} : r| < 2^{-n_k}$.

(b) For all $N \in \mathbb{N}$, let $I_N := \left(0, \frac{1}{22N}\right)$, and let $U_N := \bigcup_{n=0}^{2N} \left(\frac{N}{22N} + I_N\right)$. Thus, if $u \in U_N$ and $u := 0.u_1u_2u_3\ldots$, then $u_n = 0$ for $n \in [N,2N)$. Now $U_N$ is open (it’s a union of open intervals), and $U_N$ is $(\frac{1}{22N})$-dense in $T^1$. Thus, for any $N \in \mathbb{N}$, $B_N := \bigcup_{n=N}^{\infty} U_n$ is open and dense in $T^1$. But $R = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} U_n = \bigcap_{N=1}^{\infty} B_N$, so $R$ is dense and $G_\delta$. Finally, $\lambda[R] = 0$ by the Borel-Cantelli Lemma, because
\[
  \sum_{n=1}^{\infty} \lambda[U_n] = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.
\]

**Questions:**
(a) Proposition [15.1] says there must be other weak* cluster points of the set $\{\sum_{n=1}^{N} \Phi^n(\mu)\}_{N=1}^{\infty}$ besides $\eta$. What are they? Any cluster point $\mu_\infty$ will be a $\Phi$-invariant measure, so if $\mu_\infty$ has nonzero entropy then $\mu_\infty = \eta$ by [4] Theorem 4.1]. Thus, we know that $\mu_\infty$ has zero entropy. However, Example [14.6] says that $\mu_\infty$ cannot be quasisturmian.

(b) Propositions [15.1] [15.2] and [15.3] together imply asymptotic nonrandomization for a class of irrational rotations which is comeager but of measure zero. Is there any quasisturmian measure which is asymptotically randomized by any cellular automaton?

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