Quantum tilting modules over local rings

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Abstract
We show that tilting modules for quantum groups over local Noetherian domains of quantum characteristic 0 exist and the indecomposable tilting modules are parametrized by their highest weight. For this, we introduce a model category $\mathcal{X} = \mathcal{X}_{\mathcal{A}}(R)$ associated with a Noetherian $\mathbb{Z}[v, v^{-1}]$-domain $\mathcal{A}$ and a root system $R$. We show that if $\mathcal{A}$ is of quantum characteristic 0, the model category contains all $U_{\mathcal{A}}$-modules that admit a Weyl filtration. If $\mathcal{A}$ is in addition local, we study torsion phenomena in the model category. This leads to a construction of torsion free, or “maximal” objects in $\mathcal{X}$. We show that these correspond to tilting modules for the quantum group associated with $\mathcal{A}$ and $R$.

MSC 2020
20C20, 20G42 (primary)

1 | INTRODUCTION

Let $U = U_{\mathcal{A}}(R)$ be the quantum group associated with a (finite) root system $R$ and a $\mathbb{Z}[v, v^{-1}]$-algebra $\mathcal{A}$. We are interested in its category $\mathcal{O}$ and specifically in the subcategory of $\mathcal{O}$ that contains all objects that are finitely generated as $\mathcal{A}$-modules. If $\mathcal{A}$ is of quantum characteristic 0, that is, if the quantum numbers $[n]$ do not vanish in $\mathcal{A}$ for $n \neq 0$, then we can define for each dominant weight $\lambda$ the Weyl module $W(\lambda)$ in $\mathcal{O}$. By results of Andersen, Polo and Wen, $W(\lambda)$ is a free $\mathcal{A}$-module of finite rank and its character is given by Weyl’s character formula.

Recall that an object $T$ in $\mathcal{O}$ is called a “tilting module” if $T$ and its contravariant dual $dT$ admit a Weyl filtration, that is, a finite filtration with subquotients being isomorphic to Weyl modules.
In the case that $\mathcal{A} = \mathcal{K}$ is a field, it is known that tilting modules exist and that the indecomposable tilting modules are parametrized by their highest weight, which needs to be dominant (cf. appendix E in reference [6], where the modular case is treated). The existence of an indecomposable tilting module with highest (dominant) weight $\lambda$ is known in the case that $\mathcal{A}$ is a principal ideal domain (cf. lemma E.19 in reference [6]), and the uniqueness of these objects can be shown if $\mathcal{A}$ is a complete discrete valuation ring (proposition E.22 in [6]).

However, the case of $\mathcal{A} = \mathbb{F}_p$, the localization of $\mathbb{Z}[v, v^{-1}]$ at the kernel of the ring homomorphism $\mathbb{Z}[v, v^{-1}] \to \mathbb{F}_p$ that sends $v$ to 1, is of particular importance, as $U_\mathcal{A}$ is a natural (quantum) deformation of the hyperalgebra of the connected, simply connected, split reductive algebraic group with root system $\mathcal{R}$ over $\mathbb{F}_p$. In this case, $\mathcal{A}$ is a local ring of quantum characteristic 0, but not a principal ideal domain, and the arguments in [6] do not carry over. The main goal in this article is to fill this gap and prove that tilting modules exist over local rings of quantum characteristic 0, and that the indecomposable tilting modules are parametrized (up to isomorphism) by their (dominant) highest weight.

Our approach is very different from the approach of Jantzen, which is inspired by [3], which is in turn inspired by [9]. It involves a category $\mathcal{X} = \mathcal{X}_\mathcal{A}(\mathcal{R})$ that is defined as follows. Denote by $X$ the weight lattice of $\mathcal{R}$, and fix a set $\Pi \subset R$ of simple roots. Then the category $\mathcal{X}$ contains as objects $X$-graded $\mathcal{A}$-modules $M = \bigoplus_{\lambda \in X} M_\lambda$ that are endowed with $\mathcal{A}$-linear endomorphisms $E_{\alpha, n}$ and $F_{\alpha, n}$ of degree $+n_\alpha$ and $-n_\alpha$, respectively, for each $\alpha \in \Pi$ and $n \in \mathbb{Z}_{>0}$. We state three rather simple axioms, X1 (a boundedness condition on weights), X2 (a simple type $A_1$ commutation relation) and X3 (a replacement of the Serre relations) that ensure that $M$ carries a unique $U_\mathcal{A}$-module structure such that the $X$-grading is the weight decomposition, and the $E_{\alpha, n}$ and $F_{\alpha, n}$ are the action maps of the divided powers of the Serre generators. This construction gives rise to a fully faithful embedding of $\mathcal{X}$ into the category $\mathcal{O}$. This embedding is not an equivalence, but the image is big enough. In the case that $\mathcal{A}$ is of quantum characteristic 0, we show that the objects in $\mathcal{X}$ that are finitely generated over $\mathcal{A}$ correspond bijectively to the objects in $\mathcal{O}$ that admit a Weyl filtration.

Working with the category $\mathcal{X}$ has the advantage that one can construct objects locally, that is, weight space by weight space. Moreover, it provides a convenient framework to study torsion phenomena. In particular, to any object $M$ in $\mathcal{X}$ and any weight $\mu$ we associate a certain triple $M_{(\mu)} \subset M_{(\mu)} \subset M_{(\mu), \text{max}}$ of torsion free $\mathcal{A}$-modules with torsion quotients. We call an object $M$ in $\mathcal{X}$ minimal, if $M_{(\mu)} = M_{(\mu)}$, and maximal, if $M_{(\mu)} = M_{(\mu), \text{max}}$. Then we show that minimal as well as maximal objects exist, that the indecomposables are in both cases parametrized by their highest weight, which can be any (not necessarily dominant) weight $\lambda$. So we obtain two families $S_{\text{min}}(\lambda)$ and $S_{\text{max}}(\lambda)$ in $\mathcal{X}$ that we can consider, via the embedding above, as objects in $\mathcal{O}$. In the case that $\lambda$ is dominant, the object $S_{\text{min}}(\lambda)$ yields the Weyl module, and we show that the $S_{\text{max}}(\lambda)$ are actually indecomposable tilting modules. This settles the existence of tilting modules. The fact that two indecomposable tilting modules with the same highest weight are isomorphic is then a consequence of some properties of the maximal objects $S_{\text{max}}(\lambda)$.

2 | $X$-GRADED SPACES WITH OPERATORS

The main goal in this section is to define the category $\mathcal{X} = \mathcal{X}_\mathcal{A}(\mathcal{R})$ for a finite root system $\mathcal{R}$ and a unital Noetherian domain $\mathcal{A}$ that is a $\mathbb{Z}[v, v^{-1}]$-algebra.
2.1 | Quantum integers

Let \( v \) be an indeterminate and set \( \mathcal{Z} := \mathbb{Z}[v, v^{-1}] \). For \( n \in \mathbb{Z} \) and \( d > 0 \), we define the quantum integer

\[
[n]_d := \frac{v^d n - v^{-d} n}{v^d - v^{-d}} = \begin{cases} 
0, & \text{if } n = 0, \\
v^d(n-1) + v^d(n-3) + \ldots + v^d(-n+1), & \text{if } n > 0, \\
v^{-d}(-n-1) - v^{-d}(-n-3) - \ldots - v^{-d}(n+1), & \text{if } n < 0.
\end{cases}
\]

The quantum factorials are given by \([0]_d := 1\) \([1]_d := [2]_d \ldots [n]_d\) for \( n \geq 1 \). The quantum binomials are \( [n]_{\alpha, r} := [n]_d \cdot [n-r+1]_d \ldots [1]_d\) \( [r]_d\) for \( n \in \mathbb{Z} \) and \( r \geq 1 \). Note that under the ring homomorphism \( \mathcal{Z} \to \mathbb{Z} \) that sends \( v \) to 1, the quantum integer \([n]_d\) is sent to \( n \) for all \( n \in \mathbb{Z} \), independently of \( d \). Hence, \([n]_d\) is sent to \( n! \) and \([n]_{\alpha, r}\) to \( \binom{n}{r} \).

2.2 | Graded spaces with operators

We fix a finite root system \( R \) in a real vector space \( V \) and a basis \( \Pi \) of \( R \). The coroot for \( \alpha \in R \) is \( \alpha^\vee \in V^* \), and the weight lattice is \( X := \{ \lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \} \) for all \( \alpha \in R \). We denote by \( \leq \) the standard partial order on \( X \), that is, \( \mu \leq \lambda \) if and only if \( \lambda - \mu \) can be written as a sum of elements in \( \Pi \).

**Definition 2.1.**

1. A subset \( I \) of \( X \) is called ‘closed’ if \( \mu \in I \) and \( \mu \leq \lambda \) imply \( \lambda \in I \).
2. A subset \( S \) of \( X \) is called ‘quasi-bounded’ if for any \( \mu \in X \) the set \( \{ \lambda \in S \mid \mu \leq \lambda \} \) is finite.

Now let \( \mathcal{A} \) be a unital \( \mathcal{Z} \)-algebra that is a Noetherian domain. We denote by \( q \in \mathcal{A} \) the image of \( v \). Then \( q \) is invertible in \( \mathcal{A} \). Conversely, giving a \( \mathcal{Z} \)-algebra structure on a unital ring \( \mathcal{R} \) is the same as specifying an invertible element \( q \) of \( \mathcal{R} \). Let \( I \) be a closed subset of \( X \), and \( M = \bigoplus_{\mu \in I} M_\mu \) an \( I \)-graded \( \mathcal{A} \)-module. We say that \( \mu \) is a weight of \( M \) if \( M_\mu \neq \{0\} \). For any \( \mu \in I, \alpha \in \Pi \) and \( n > 0 \), we have \( \mu + n\alpha \in I \). Let

\[
F_{\mu, \alpha, n} : M_{\mu+n\alpha} \to M_\mu,
\]

\[
E_{\mu, \alpha, n} : M_\mu \to M_{\mu+n\alpha}
\]

be \( \mathcal{A} \)-linear homomorphisms. It is convenient to set \( E_{\mu, \alpha, 0} = F_{\mu, \alpha, 0} : = \text{id}_{M_\mu} \). In the following we often suppress the index “\( \mu \)” in the notation of the \( E \)- and \( F \)-maps if the source of the maps is clear from the context, but we sometimes also write \( E^{M}_{\alpha, n} \) and \( F^{M}_{\alpha, n} \) to specify the object \( M \) on which these homomorphisms are defined.

Now, we list some conditions on the above data. Denote by \( A = (\langle \alpha, \beta^\vee \rangle)_{\alpha, \beta \in \Pi} \) the Cartan matrix associated with the root system \( R \). Then there exists a vector \( d = (d_\alpha)_{\alpha \in \Pi} \) with entries in \( \{1, 2, 3\} \) such that \( (d_\alpha \langle \alpha, \beta^\vee \rangle)_{\alpha, \beta \in \Pi} \) is symmetric and each irreducible component of \( R \) contains some \( \alpha \) with \( d_\alpha = 1 \). The first two conditions are as follows.
(X1) The set of weights of $M$ is quasi-bounded and each $M_\mu$ is finitely generated as an $\mathcal{A}$-module.

(X2) For all $\mu \in I$, $\alpha, \beta \in \Pi$, $m, n > 0$ and $v \in M_{\mu+n\beta}$,

$$E_{\alpha,m}F_{\beta,n}(v) = \begin{cases} F_{\beta,n}E_{\alpha,m}(v), & \text{if } \alpha \neq \beta, \\ \min(m, n) \sum_{r=0}^{\min(m, n)} \left[ \langle \mu, \alpha^\vee \rangle + m + n \right] \frac{d_{\alpha}}{d_{\alpha}} F_{\alpha,n-r}E_{\alpha,m-r}(v), & \text{if } \alpha = \beta. \end{cases}$$

(The cautious reader may want to have a look at equation (a2) in section 6.5 of [7] to get an idea of where the second equation comes from.)

2.3 | Torsion subquotients

In order to formulate the third condition, we need some definitions. Suppose that $M$ satisfies (X1). Then for any $\mu \in I$, we can define

$$M_{\delta \mu} := \bigoplus_{\alpha \in \Pi, n > 0} M_{\mu+n\alpha}.$$ 

Note that, since the set of weights is quasi-bounded, only finitely many of the direct summands are non-zero. Let

$$E_\mu : M_\mu \to M_{\delta \mu},$$

$$F_\mu : M_{\delta \mu} \to M_\mu$$

be the column and the row vector with entries $E_{\mu,\alpha,n}$ and $F_{\mu,\alpha,n}$, respectively. We sometimes write $E_\mu^M$ and $F_\mu^M$ in order to specify the object $M$ on which $E_\mu$ and $F_\mu$ act. Set

$$M_{[\mu]} := E_\mu(\im F_\mu),$$

$$M_{(\mu)} := E_\mu(M_\mu).$$

So we have inclusions $M_{[\mu]} \subset M_{(\mu)} \subset M_{\delta \mu}$.

(X3) For all $\mu \in I$ the following holds:

(a) The restriction of $E_\mu : M_\mu \to M_{\delta \mu}$ to $\im F_\mu \subset M_\mu$ is injective and hence induces an isomorphism $\im F_\mu \cong M_{[\mu]}$.

(b) The quotient $M_{(\mu)}/M_{[\mu]}$ is a torsion $\mathcal{A}$-module.

(c) The quotient $M_\mu/\im F_\mu$ is a free $\mathcal{A}$-module.

Here is our first, rather easy, result.

**Lemma 2.2.** Suppose that our data satisfies (X1) and (X3). Then $M$ is a torsion free $\mathcal{A}$-module. In particular, the spaces $M_{\delta \mu}$, $M_{[\mu]}$ and $M_{(\mu)}$ are torsion free $\mathcal{A}$-modules for all $\mu \in I$. 
Proof. If $M$ is not torsion free, then assumption (X1) implies that there is a maximal weight $\mu$ of $M$ such that $M_{\mu}$ is not torsion free. By the maximality of $\mu$, the module $M_{\delta\mu}$ is torsion free. Hence so is its submodule $M_{[\mu]}$. From (X3a), it follows that $\im F_{\mu}$ is torsion free. By (X3c), the module $M_{\mu}/\im F_{\mu}$ is free, so $M_{\mu}$ must be torsion free and we have a contradiction.

The following results might shed some light on the assumption (X3).

**Lemma 2.3.** Suppose that the assumption (X3a) holds and let $\mu$ be an element in $I$. Then the following are equivalent.

1. $M_{\mu} = \ker E_{\mu} \oplus \im F_{\mu}$.
2. $M_{\{\mu\}} = M_{(\mu)}$.

Proof. If (1) holds, then $M_{(\mu)} = E_{\mu}(M_{\mu}) = E_{\mu}(\im F_{\mu}) = M_{[\mu]}$. Suppose that (2) holds, so $E_{\mu}(M_{\mu}) = E_{\mu}(\im F_{\mu})$. For each $m \in M_{\mu}$, there exists an element $\tilde{m} \in \im F_{\mu}$ such that $E_{\mu}(m) = E_{\mu}(\tilde{m})$, hence $m - \tilde{m} \in \ker E_{\mu}$. So $M_{\mu} = \ker E_{\mu} + \im F_{\mu}$. But condition (X3a) reads $\ker E_{\mu} \cap \im F_{\mu} = \{0\}$.

Hence, $M_{\mu} = \ker E_{\mu} \oplus \im F_{\mu}$. □

Denote by $H$ the quotient field of $A$. For an $A$-module $N$ let $N_X := N \otimes_A H$ be the associated $H$-module.

**Lemma 2.4.** Suppose that $M = \bigoplus_{\mu \in I} M_{\mu}$ satisfies condition (X1). Then condition (X3) is equivalent to the following set of conditions.

1. $M$ is a torsion-free $A$-module.
2. For all $\mu \in I$, we have $(M_{\mu})_X = (\ker E_{\mu})_X \oplus (\im F_{\mu})_X$.
3. Condition (X3c) holds: $M_{\mu}/\im F_{\mu}$ is a free $A$-module for all $\mu \in I$.

In particular, if $A = H$ is a field, then the conditions (X3abc) simplify to $M_{\mu} = \ker E_{\mu} \oplus \im F_{\mu}$ for all $\mu \in I$.

Proof. Suppose that (X3) is satisfied. We have already shown in Lemma 2.2 that (X1) and (X3) imply that $M$ is torsion free as an $A$-module. Moreover, (X3a) says that $\ker E_{\mu} \cap \im F_{\mu} = \{0\}$. Now, let $m \in M_{\mu}$. Then, by (X3b), there exists an element $\xi \in A$, $\xi \neq 0$ and $m' \in \im F_{\mu}$, such that $\xi E_{\mu}(m) = E_{\mu}(m')$. So $\xi \cdot m - m'$ is contained in the kernel of $E_{\mu}$ and we deduce $(M_{\mu})_X = (\ker E_{\mu})_X + (\im F_{\mu})_X$. The last two results say that $(M_{\mu})_X = (\ker E_{\mu})_X \oplus (\im F_{\mu})_X$. Hence, (1), (2) and (3) are satisfied.

Now assume that (1), (2) and (3) hold. As $M$ is torsion free, we can view it as a subspace in $M_X$. Hence, $(M_{\mu})_X = (\ker E_{\mu})_X \oplus (\im F_{\mu})_X$ implies that $E_{\mu}|_{\im F_{\mu}}$ is injective, that is, (X3a). It also implies that $E_{\mu}(\im F_{\mu})_X = E_{\mu}(M_{\mu})_X$, that is, $(M_{\mu})_X = (M_{\mu})_X$. Hence, the cokernel of the inclusion $M_{[\mu]} \subset M_{\{\mu\}}$ is a torsion module, so (X3b) holds. (X3c) holds is the assumption (3).

### 2.4 The category $\mathcal{X}'$

Let $I$ be a closed subset of $X$. 
Definition 2.5. The category \( \mathcal{X}_{I,\mathcal{A}} \) is defined as follows: Objects are \( I \)-graded \( \mathcal{A} \)-modules \( M = \bigoplus_{\mu \in I} M_{\mu} \) endowed with \( \mathcal{A} \)-linear homomorphisms \( F_{\mu,\alpha,n} : M_{\mu+n\alpha} \to M_{\mu} \) and \( E_{\mu,\alpha,n} : M_{\mu} \to M_{\mu+n\alpha} \) for all \( \mu \in I \), \( \alpha \in \Pi \) and \( n > 0 \), such that conditions (X1), (X2) and (X3) are satisfied. A morphism \( f : M \to N \) in \( \mathcal{X}_{I,\mathcal{A}} \) is a collection of \( \mathcal{A} \)-linear homomorphisms \( f_{\mu} : M_{\mu} \to N_{\mu} \) for all \( \mu \in I \), such that the diagrams commute for all \( \mu \in I, \alpha \in \Pi \) and \( n > 0 \).

If the ground ring is determined from the context, we write \( \mathcal{X}_I \) instead of \( \mathcal{X}_{I,\mathcal{A}} \). We also write \( \mathcal{X}' \) or \( \mathcal{X}_{I,\mathcal{A}}' \) for the “global” category \( \mathcal{X}_{X,\mathcal{A}} \).

If \( M \) and \( N \) are objects in \( \mathcal{X}_I \) and \( f = \{ f_{\mu} : M_{\mu} \to N_{\mu} \}_{\mu \in I} \) is a collection of homomorphisms, then we denote by \( f_{\delta} : M_{\delta} \to N_{\delta} \) the diagonal matrix with entries \( f_{\mu+n\alpha} \). Then \( f \) is a morphism in \( \mathcal{X}_I \) from \( M \) to \( N \) if and only if for all \( \mu \in I \) the diagrams commute.

2.5 | Base change

We now want to understand whether the conditions that define the category \( \mathcal{X} \) are stable under base change. So let \( \mathcal{A} \to \mathcal{B} \) be a homomorphism of unital \( \mathcal{Z} \)-algebras that are Noetherian domains. Let \( M \) be an object in \( \mathcal{X}_{I,\mathcal{A}} \). We define \( M_{\mathcal{B}} = \bigoplus_{\mu \in I} M_{\mathcal{B},\mu} \) by setting (as before) \( M_{\mathcal{B},\mu} := M_{\mu} \otimes_{\mathcal{A}} \mathcal{B} \). For \( \mu \in X, \alpha \in \Pi \) and \( n > 0 \), we have induced homomorphisms \( E_{\mu,\alpha,n}^{M_{\mathcal{B}}} = E_{\mu,\alpha,n} \otimes \text{id}_{\mathcal{B}} : M_{\mathcal{B},\mu} \to M_{\mathcal{B},\mu+n\alpha} \) and \( F_{\mu,\alpha,n}^{M_{\mathcal{B}}} = F_{\mu,\alpha,n} \otimes \text{id}_{\mathcal{B}} : M_{\mathcal{B},\mu+n\alpha} \to M_{\mathcal{B},\mu} \).

Lemma 2.6. Suppose that \( \mathcal{A} \to \mathcal{B} \) is a flat homomorphism. Then the object \( M_{\mathcal{B}} \) is contained in \( \mathcal{X}_{I,\mathcal{B}} \).

Proof. It is clear that the properties (X1) and (X2) are stable under arbitrary base change. Moreover, \( M_{\mathcal{B}(\mu)}, M_{\mathcal{B}[\mu]} \) and \( \text{im} F_{\mu}^{M_{\mathcal{B}}} \) are obtained from \( M_{(\mu)}, M_{[\mu]} \) and \( \text{im} F_{\mu}^{M} \) by flat base change for all \( \mu \in I \) by right exactness of base change. Hence, (X3b) and (X3c) also hold for \( M_{\mathcal{B}} \). Again by the flatness condition, the homomorphism \( E_{\mu}|_{\text{im} F_{\mu}} \) remains injective after base change. Hence, property (X3a) also holds. \( \square \)
EXTENDING MORPHISMS

We retain the notations of the previous section. Let $I' \subset I$ be closed subsets of $X$ and let $M$ be an object in $\mathcal{X}_I$. We define $M_{I'} := \bigoplus_{\mu \in I'} M_{\mu}$ and endow it with the homomorphisms $E_{\mu, \alpha, n}$ and $F_{\mu, \alpha, n}$ for all $\mu \in I'$. Then one easily checks that the properties (X1), (X2) and (X3) are preserved, so this defines an object $M_{I'}$ in $\mathcal{X}_{I'}$. For a morphism $f : M \to N$ in $\mathcal{X}_I$, we obtain a morphism $f_{I'} : M_{I'} \to N_{I'}$ by restriction, and this yields a functor

$$(\cdot)_{I'} : \mathcal{X}_I \to \mathcal{X}_{I'}$$

that we call the ‘restriction functor’.

3.1 Extensions of morphisms

The following proposition is a cornerstone of the approach outlined in this article. Its proof is not difficult, but lengthy.

**Proposition 3.1.** Let $I'$ be a closed subset of $X$ and suppose that $\mu \notin I'$ is such that $I := I' \cup \{\mu\}$ is also closed. Let $M$ and $N$ be objects in $\mathcal{X}_I$, and let $f' : M_{I'} \to N_{I'}$ be a morphism in $\mathcal{X}_{I'}$.

1. There exists a unique $\mathcal{A}$-linear homomorphism $\bar{f}_\mu : \text{im} F_{\mu}^M \to N_\mu$ such that the diagrams

$$
\xymatrix{
M_\delta \ar[r]^{f_\mu} \ar[d]^p_M & N_\delta \ar[d]^p_N \\
\text{im} F_{\mu}^M \ar[r]^{\bar{f}_\mu} & N_\mu
}
$$

commute. In particular, $f'_{\delta \mu}$ maps $M_{\mu}$ into $N_{\mu}$.

2. The following are equivalent.
   
   (a) There exists a morphism $f : M \to N$ in $\mathcal{X}_I$ such that $f_{I'} = f'$.
   
   (b) The homomorphism $f'_{\delta \mu} : M_{\delta \mu} \to N_{\delta \mu}$ maps $M_{\mu}$ into $N_{\mu}$.

**Proof.** First, we prove part (1). Set $\hat{M}_\mu := \bigoplus_{\beta \in \Pi, n > 0} M_{\mu+n\beta}$ and denote by $\hat{F}_{\beta, n} : M_{\mu+n\beta} \to \hat{M}_\mu$ the embedding of the corresponding direct summand. Define $\hat{E}_{\mu} : M_{\delta \mu} \to \hat{M}_\delta$ as the row vector with entries $\hat{E}_{\beta, n} \dagger$. For $\alpha \in \Pi$, $m > 0$ define an $\mathcal{A}$-linear map $\hat{P}_{\alpha, m} : \hat{M}_\mu \to M_{\mu+m\alpha}$ by additive extension of the following formulas. For $\beta \in \Pi$, $n > 0$ and $v \in M_{\mu+n\beta}$ set

$$
\hat{P}_{\alpha, m} \hat{F}_{\beta, n}(v) := \begin{cases} F_{\beta, n} E_{\alpha, m}(v), & \text{if } \alpha \neq \beta, \\
\sum_{0 \leq r < \min(m, n)} \left[ \langle \mu, \alpha \rangle + n + m \right]_{\alpha} d_{\alpha} F_{\alpha, n-r} E_{\alpha, m-r}(v), & \text{if } \alpha = \beta.
\end{cases}
$$

Let $\hat{E}_{\mu} : \hat{M}_\mu \to M_{\delta \mu}$ be the column vector with entries $\hat{E}_{\alpha, m}$.

\dagger The author is aware of the fact that this looks rather silly. There is a tautological identification $M_{\delta \mu} = \hat{M}_\mu$ that identifies $\hat{F}_{\mu}$ with the identity. However, $M_{\delta \mu}$ and $\hat{M}_\mu$ will play very different roles in the following.
Now define $\phi : \hat{M}_\mu \to M_\mu$ as the row vector with entries $F_{\beta, n}$. Obviously, the diagram

\[
\begin{array}{ccc}
\hat{M}_\mu & \xrightarrow{\phi} & M_\mu \\
\downarrow P_\mu & & \downarrow P_\mu \\
M_\delta \mu & & \end{array}
\]

commutes. As the $\hat{E}_{\alpha, m}$ and $\hat{F}_{\beta, n}$ maps satisfy the same commutation relations as the $E_{\alpha, m}$ and $F_{\beta, n}$ maps by (X2), and as $\hat{F}_\mu$ is surjective, also the diagram

\[
\begin{array}{ccc}
\hat{M}_\mu & \xrightarrow{\phi} & M_\mu \\
\downarrow E_\mu & & \downarrow E_\mu \\
M_\delta \mu & & \end{array}
\]

commutes. As $\hat{F}_\mu$ is surjective, we have $\text{im} \phi = \text{im} F_\mu$. As $E_\mu$ is injective when restricted to $\text{im} F_\mu$, we deduce that $\ker \phi = \ker E_\mu$, hence $\phi$ induces an isomorphism $\hat{M}_\mu / \ker \hat{E}_\mu \cong \text{im} F_\mu$.

Now let $\hat{f}_\mu : \hat{M}_\mu \to N_\mu$ be the row vector with entries $F_{N, n}^N \circ f'_{\mu + n\beta} : M_{\mu + n\beta} \to N_{\mu + n\beta} \to N_\mu$. Then the diagram

\[
\begin{array}{ccc}
M_\delta \mu & \xrightarrow{f'_{\mu}} & N_\delta \mu \\
\downarrow \hat{P}_\mu & & \downarrow p_\mu^{N} \\
\hat{M}_\mu & \xrightarrow{\hat{f}_\mu} & N_\mu \\
\end{array}
\]

commutes. By the same arguments as above, also the diagram

\[
\begin{array}{ccc}
M_\delta \mu & \xrightarrow{f'_{\mu}} & N_\delta \mu \\
\downarrow \hat{E}_\mu & & \downarrow \hat{E}_\mu \\
\hat{M}_\mu & \xrightarrow{\hat{f}_\mu} & N_\mu \\
\end{array}
\]

commutes. As $\hat{E}_\mu$ is surjective, the image of $\hat{f}_\mu$ is contained in $\text{im} F_\mu^N \subset N_\mu$. As $E_\mu^N$ is injective when restricted to $\text{im} F_\mu^N$, we deduce that $\hat{f}_\mu$ factors over the kernel of $\hat{E}_\mu$. But, as we have seen above, this is the kernel of $\phi$. We hence obtain an induced homomorphism $\tilde{f}_\mu : \text{im} F_\mu^M \cong \hat{M}_\mu / \ker \phi \to N_\mu$ such that the diagrams

\[
\begin{array}{ccc}
M_\delta \mu & \xrightarrow{f'_{\mu}} & N_\delta \mu \\
\downarrow P_\mu & & \downarrow P_\mu \\
\text{im} F_\mu^M & \xrightarrow{\tilde{f}_\mu} & N_\mu \\
\end{array}
\]

\[
\begin{array}{ccc}
M_\delta \mu & \xrightarrow{f'_{\mu}} & N_\delta \mu \\
\downarrow E_\mu & & \downarrow E_\mu \\
\text{im} F_\mu^M & \xrightarrow{\tilde{f}_\mu} & N_\mu \\
\end{array}
\]

are commutative.
commute. This shows the existence part of (1). The uniqueness is clear, as \( F^M_\mu : M_{\delta \mu} \to \text{im} F^M_\mu \) is surjective.

Now we show part (2). Assume that property (a) holds, that is, there exists a homomorphism \( f : M \to N \) that restricts to \( f' \). Then the diagram

\[
\begin{array}{ccc}
M_{\delta \mu} & \xrightarrow{f'_\mu} & N_{\delta \mu} \\
\uparrow F^M_\mu & & \uparrow E^N_\mu \\
M_\mu & \xrightarrow{f_\mu} & N_\mu
\end{array}
\]

commutes and hence \( f'_\delta \mu \) maps \( M_{(\mu)} = E^M_\mu (M_\mu) \) into \( N_{(\mu)} = E^N_\mu (N_\mu) \), so property (b) holds.

Now assume property (b) holds. We now need to construct an \( \mathcal{A} \)-linear map \( f_\mu : M_\mu \to N_\mu \) such that the diagrams

\[
\begin{array}{ccc}
M_{\delta \mu} & \xrightarrow{f'_\mu} & N_{\delta \mu} \\
\uparrow F^M_\mu & & \uparrow E^N_\mu \\
M_\mu & \xrightarrow{f_\mu} & N_\mu
\end{array}
\]

commute. By part (1), there exists a homomorphism \( \bar{f}_\mu : \text{im} F^M_\mu \to N_\mu \) such that the diagrams

\[
\begin{array}{ccc}
M_{\delta \mu} & \xrightarrow{f'_\mu} & N_{\delta \mu} \\
\uparrow F^M_\mu & & \uparrow E^N_\mu \\
\text{im} F^M_\mu & \xrightarrow{\bar{f}_\mu} & N_\mu
\end{array}
\]

commute. By assumption (X3c), the quotient \( M_\mu / \text{im} F^M_\mu \) is a free \( \mathcal{A} \)-module. We can hence fix a decomposition \( M_\mu = \text{im} F^M_\mu \oplus D \) with a free \( \mathcal{A} \)-module \( D \). We now construct a homomorphism \( \hat{f}_\mu : D \to N_\mu \) in such a way that \( \hat{f}_\mu = (\bar{f}_\mu, \hat{f}_\mu) \) serves our purpose. Note that no matter how we define \( \hat{f}_\mu \), we will always have \( f_\mu \circ F^M_\mu = F^N_\mu \circ f'_\delta \mu \) (cf. the left diagram of \((*)\)). So the only property that \( \hat{f}_\mu \) has to satisfy is that the diagram

\[
\begin{array}{ccc}
M_{\delta \mu} & \xrightarrow{f'_\mu} & N_{\delta \mu} \\
\uparrow F^M_\mu & & \uparrow E^N_\mu \\
D & \xrightarrow{\hat{f}_\mu} & N_\mu
\end{array}
\]

commutes. Since we assume that \( f'_\delta (E^M_\mu (M_\mu)) \) is contained in the image of \( E^N_\mu : N_\mu \to N_{\delta \mu} \), this also holds for \( f'_\delta (E^M_\mu (D)) \). As \( D \) is free, it is projective as an \( \mathcal{A} \)-module. So \( \hat{f}_\mu \) indeed exists. \( \square \)
Remark 3.2. In part (2) of the lemma above, the extension $f$ of $f'$ is in general not unique. In the notation of the proof of part (2), the $\mathcal{A}$-linear homomorphism $\hat{f}_\mu$ is in general not unique, nor is the decomposition $M_\mu = \text{im} F^M_\mu \oplus D$.

### 3.2  Minimal and maximal objects

Let $I$ be a closed subset of $X$ and let $M$ be an object in $\mathcal{X}_I$. Let $\mu \in I$. Define

$$M_{\{\mu\}, \text{max}} := \{m \in M_{\delta \mu} \mid \xi m \in M_{\{\mu\}} \text{ for some } \xi \in \mathcal{A}, \xi \neq 0\}.$$

So this is the preimage of the torsion part of $M_{\delta \mu}/M_{\{\mu\}}$ under the quotient map. Suppose that $N$ is a submodule of $M_{\delta \mu}$ that contains $M_{\{\mu\}}$. Then $N/M_{\{\mu\}}$ is a torsion module if and only if $N \subset M_{\{\mu\}, \text{max}}$. In particular, condition (X3b) now reads $M_{\{\mu\}} \subset M_{\{\mu\}, \text{max}}$. So we have inclusions

$$M_{\{\mu\}} \subset M_{\{\mu\}} \subset M_{\{\mu\}, \text{max}} \subset M_{\delta \mu}.$$

We now give the two extreme cases a name.

**Definition 3.3.** $M$ is called

1. **minimal**, if for all $\mu \in I$ we have $M_{\{\mu\}} = M_{\{\mu\}}$.
2. **maximal**, if for all $\mu \in I$ we have $M_{\{\mu\}} = M_{\{\mu\}, \text{max}}$.

In the two extreme cases we can extend morphisms according to the following result.

**Lemma 3.4.** Suppose that $I' \subset X$ is closed and that $\mu \notin I'$ is such that $I := I' \cup \{\mu\}$ is closed in $X$ as well. Let $M$ and $N$ be objects in $\mathcal{X}_I$ and suppose that $M_{\{\mu\}} = M_{\{\mu\}}$ or $N_{\{\mu\}} = N_{\{\mu\}, \text{max}}$. Then the functorial map

$$\text{Hom}_{\mathcal{X}_I}(M, N) \rightarrow \text{Hom}_{\mathcal{X}_{I'}}(M_{I'}, N_{I'})$$

is surjective.

**Proof.** Let $f' : M_{I'} \rightarrow N_{I'}$ be a morphism in $\mathcal{X}_{I'}$. By Proposition 3.1, there exists a (unique) $\tilde{f}_\mu : \text{im} F^M_\mu \rightarrow N_\mu$ such that the diagrams

$$\begin{array}{ccc}
M_{\delta \mu} & \xrightarrow{f'_{\mu}} & N_{\delta \mu} \\
\downarrow P^M_\mu & & \downarrow P^N_\mu \\
\text{im} F^M_\mu & \xrightarrow{\tilde{f}_\mu} & N_\mu
\end{array}$$

commute. This implies that $f'_{\delta \mu}$ maps $M_{\{\mu\}}$ into $N_{\{\mu\}}$ and hence $M_{\{\mu\}, \text{max}}$ into $N_{\{\mu\}, \text{max}}$. Either of the two assumptions in the statement of the lemma implies that $f'_{\delta \mu}$ maps $M_{\{\mu\}}$ into $N_{\{\mu\}}$, so the condition (2b) in Proposition 3.1 is satisfied. Hence there exists an extension $f : M \rightarrow N$ of $f'$. \[\square\]
THE MINIMAL EXTENSION FUNCTOR

4.1 The minimal extension object

In this section, we construct a ‘minimal extension’ in \( \mathcal{X}_I \) for any object in \( \mathcal{X}_{I'} \). In the next section, we show that the construction is actually functorial.

Proposition 4.1. Let \( I' \subset I \) be a pair of closed subsets of \( X \). Let \( M' \) be an object in \( \mathcal{X}_{I'} \).

(1) There exists an up to isomorphism unique object \( M \) in \( \mathcal{X}_I \) with the following properties.
   (a) The object \( M \) restricts to \( M' \), that is, \( M_{I'} \cong M' \).
   (b) For all objects \( N \) in \( \mathcal{X}_I \) the functorial homomorphism
       \[ \text{Hom}_{\mathcal{X}_I}(M, N) \to \text{Hom}_{\mathcal{X}_{I'}}(M_{I'}, N_{I'}) \]
       is an isomorphism.

(2) For the object \( M \) characterized in part (1) we have \( M_{\mu} = \text{im } F_{\mu} \) and hence \( M_{\{\mu\}} = M_{(\mu)} \) for all \( \mu \in I \setminus I' \), and \( M_{\mu} \neq 0 \) implies that there exists a weight \( \lambda \) of \( M' \) with \( \mu \leq \lambda \).

Proof. Note that the uniqueness statement in (1) follows directly from properties (1a) and (1b). So, in order to prove (1), we only need to show the existence of an object \( M \) satisfying (1a) and (1b). For this we give an explicit construction. Note that we can construct \( M \) in steps, that is, if we have closed subsets \( I' \subset I'' \subset I \) and we construct an object \( M'' \) in \( \mathcal{X}_{I'\cup S} \) satisfying properties (1a), (1b) and (2) with respect to \( M' \), and then we construct an object \( M \) in \( \mathcal{X}_I \) that satisfies properties (1a), (1b) and (2) with respect to \( M'' \), then \( M \) satisfies (1a), (1b) and (2) with respect to \( M' \).

So in a first step, we set \( M_{\nu} = M'_{\nu}, E_{\nu, \alpha, n} = E'_{\nu, \alpha, n} \) and \( F_{\nu, \alpha, n} = F'_{\nu, \alpha, n} \) for all \( \nu \in I', \alpha \in \Pi \) and \( n > 0 \) in order to make sure that (1a) is satisfied. Now, let \( S \) be the set of all \( \mu \in I \setminus I' \) that have the property that there exists no \( \lambda \) with \( \mu < \lambda \) such that \( M'_{\lambda} \neq \{0\} \). For all \( \mu \in S \) we set \( M_{\mu} = 0, E'_{\mu, \alpha, n} = 0 \) and \( F'_{\mu, \alpha, n} = 0 \). Note that \( I' \cup S \) is closed. Now we have constructed an object \( M \) in \( \mathcal{X}_{I'\cup S} \) that obviously satisfies (1a). For all \( \nu \in S \setminus I' \), we have \( M_{\nu} = 0, \) hence \( 0 = M_{\{\nu\}} = M_{(\nu)} \), hence (2) is satisfied as well. As \( \text{im } F_{\nu} = M_{\nu} = 0 \) for all these \( \nu \), property (1b) follows from part (1) in Proposition 3.1.

Hence, we have extended \( M' \) to an object in \( \mathcal{X}_{I'\cup S} \) in such a way that properties (1a), (1b) and (2) are satisfied. Now note that the set \( I \setminus (I' \cup S) \) is quasi-bounded because the set of weights of \( M' \) is. We can now proceed by induction and assume that \( I \setminus (I' \cup S) = \{\mu\} \) for a single element \( \mu \in I \).

Then we can already define \( M_{\delta_{\mu}} := \bigoplus_{\beta \in \Pi, n > 0} M_{\mu+n\beta} \). For the construction of \( M_{\mu} \) and \( E_{\mu, \alpha, n} \) and \( F_{\mu, \alpha, n} \), we follow ideas that were already used in the proof of Proposition 3.1. So in a first step, we set \( \hat{M}_{\mu} := \bigoplus_{\beta \in \Pi, n > 0} M_{\mu+n\beta} \) and denote by \( \hat{F}_{\beta, n} : M_{\mu+n\beta} \to \hat{M}_{\mu} \) the canonical injection of a direct summand. We let \( \hat{F}_{\mu} : M_{\delta_{\mu}} \to \hat{M}_{\mu} \) be the row vector with entries \( \hat{F}_{\beta, n} \) (again, \( \hat{F}_{\mu} \) is the identity). For \( \alpha \in \Pi, m > 0 \) define an \( \mathcal{A} \) -linear map \( \hat{E}_{\alpha, m} : \hat{M}_{\mu} \to M_{\mu+ma} \) by additive extension.
of the following formulas. For $\beta \in \Pi$, $n > 0$ and $v \in M_{\mu+n\beta}$, set

$$\hat{E}_{\alpha,m} \hat{F}_{\beta,n}(v) := \begin{cases} F_{\beta,n}E_{\alpha,m}(v), & \text{if } \alpha \neq \beta, \\ \min(m,n) \sum_{r=0}^{n} \left[ \langle \mu, \alpha \rangle + m + n \right] d_{\alpha} F_{\alpha,n-r}E_{\alpha,m-r}(v), & \text{if } \alpha = \beta. \end{cases}$$

We denote by $\hat{E}_{\mu} : \hat{M}_{\mu} \to M_{\delta \mu}$ the column vector with entries $\hat{E}_{\alpha,m}$. Now define $M_{\mu} := \hat{M}_{\mu}/\ker \hat{E}_{\mu}$, and denote by $E_{\mu} : M_{\mu} \to M_{\delta \mu}$ and $F_{\mu} : M_{\delta \mu} \to M_{\mu}$ the homomorphisms induced by $\hat{E}_{\mu}$ and $\hat{F}_{\mu}$, respectively. Note that $F_{\mu}$ is surjective since $\hat{F}_{\mu}$ is. Denote by $E_{\mu,\alpha,n} : M_{\mu} \to M_{\mu+n\alpha}$ and by $F_{\mu,\alpha,n} : M_{\mu+n\alpha} \to M_{\mu}$ the entries of the row vector $E_{\mu}$ and the column vector $F_{\mu}$, respectively.

We claim that the above data yield an object in $\mathcal{X}$. Clearly, property (X1) is satisfied. Also, the commutation relations between the $E$- and $F$-maps follow from the relations between the respective maps on $M'$ and the construction of $E_{\mu}$ and $F_{\mu}$. Hence, (X2) is satisfied as well. The properties (X3) are satisfied for all weights $\nu$ with $\nu \neq \mu$, as they are satisfied for $M'$. For the weight $\mu$, however, we have $\ker E_{\mu} = \{0\}$, hence (X3a) is satisfied, and $M_{\mu} = \operatorname{im} F_{\mu}$, so $M_{\mu} = M_{\mu}$, which imply (X3b) and (X3c).

It remains to show that the object $M$ satisfies the properties (1a) and (1b). Part (1a) is clear from the construction. Part (1b) follows from $M_{\mu} = \operatorname{im} F_{\mu}$ and part (1) of Proposition 3.1. Hence, (1) is proven. Clearly $M_{\mu} \neq 0$ implies $M'_{\delta \mu} \neq 0$, hence there is some weight $\lambda$ of $M'$ with $\mu \leq \lambda$. Since $M_{\mu} = \operatorname{im} F_{\mu}$, we have $M_{\mu} = M_{\mu}$, hence (2).

### 4.2 A first example

Suppose that $R = \{ \pm \alpha \}$ is the root system of type $A_1$ with $\Pi = \{ \alpha \}$. We identify the weight lattice $X$ with $\mathbb{Z}$ by sending $\alpha \in X$ to $2 \in \mathbb{Z}$. Let $\lambda \in \mathbb{Z}$ and set $I' = \{ \lambda, \lambda + 2, \lambda + 4, \ldots \}$, $\mu = \lambda - 2$ and $I = I' \cup \{ \mu \}$. Define $M' := \bigoplus_{y \in I'} M'_{y}$ with $M'_{\lambda} = \mathcal{A}$ and $M'_{y} = 0$ for $y \neq \lambda$. All relevant $E$- and $F$-maps are zero, of course. In the notation of the proof of Proposition 4.1, we have $\hat{M}_{\mu} = M_{\delta \mu} = \mathcal{A}$ and $\hat{F}_{\mu} : M_{\delta \mu} \to \hat{M}_{\mu}$ is the identity. In the next step, we define the map $\hat{E}_{\mu} : \hat{M}_{\mu} \to M_{\delta \mu}$. The commutation relations force this map to be multiplication with the image of $[\mu+2] = [\lambda]$ in $\mathcal{A}$. So if $[\lambda]$ does not vanish in $\mathcal{A}$, the map $\hat{E}_{\mu}$ is injective, so $M_{\mu} = \mathcal{A}$ and $E_{\mu} : \mathcal{A} \to \mathcal{A}$ is multiplication with $[\lambda]$, while $F_{\mu} : \mathcal{A} \to \mathcal{A}$ is the identity. If $[\lambda]$ vanishes in $\mathcal{A}$, then $M_{\mu} = 0$.

Let us go one step further. Suppose that $[\lambda]$ did not vanish in $\mathcal{A}$. We would like to find an object $N$ that extends the object $M$ just defined to the weight $\mu := \lambda - 4$. We now have $N_{\delta \mu} = \hat{N}_{\mu} = \mathcal{A} \oplus \mathcal{A}$ and $\hat{F}_{\mu} : N_{\delta \mu} \to \hat{N}_{\mu}$ is again the identity. Now, let $v$ be a generator of $N_{\lambda} = M_{\lambda} = \mathcal{A}$. Then $v$ and $F_{1}v$ form a basis of $N_{\delta \mu}$, and $\hat{F}_{2}v, \hat{F}_{1}F_{1}v$ a basis of $\hat{N}_{\mu}$. Using the commutation relations we obtain

$$\hat{E}_{1}\hat{F}_{2}v = F_{2}E_{1}v + \left[ \begin{array}{c} \lambda - 4 + 1 + 2 \\ 1 \end{array} \right] F_{1}v = 0 + [\lambda - 1]F_{1}v,$$

$$\hat{E}_{1}\hat{F}_{1}F_{1}v = F_{1}E_{1}F_{1}v + \left[ \begin{array}{c} \lambda - 4 + 1 + 1 \\ 1 \end{array} \right] F_{1}v = [\lambda]F_{1}v + [\lambda - 2]F_{1}v.$$
\[ \hat{E}_2 \hat{F}_2 v = F_2 E_2 v + \begin{bmatrix} \lambda - 4 + 2 & 2 \\ 1 & 1 \end{bmatrix} F_1 E_1 v + \begin{bmatrix} \lambda - 4 + 2 & 2 \\ 1 & 1 \end{bmatrix} v \]
\[ = 0 + 0 + \begin{bmatrix} \lambda \\ 2 \end{bmatrix} v, \]
\[ \hat{E}_2 \hat{F}_1 F_1 v = F_1 E_2 F_1 v + \begin{bmatrix} \lambda - 4 + 2 & 1 \\ 1 & 1 \end{bmatrix} E_1 F_1 v \]
\[ = 0 + [\lambda - 1] [\lambda] v. \]

Hence, \( \hat{E}_\mu \) is given by the matrix \( \begin{pmatrix} [\lambda - 1] & [\lambda - 1] \\ [\lambda] + [\lambda - 2] & [\lambda] [\lambda - 1] \end{pmatrix} \). Suppose that \([2]\) is invertible in \( \mathcal{A} \).

A short calculation shows \([\lambda/2]([\lambda]) + [\lambda - 2]) = [\lambda][\lambda - 1]. \) So the second column is \([\lambda/2]\) times the first column, hence the kernel of \( \hat{E}_\mu \) has at least rank 1. It has rank 2 if \([\lambda - 1]\) vanishes in \( \mathcal{A} \). If the kernel has rank 1, then \( N_{\lambda - 4} \cong \mathcal{A} \) and \( E_{\lambda - 4} \) is given by the first column of the above matrix. If the rank of the kernel is 2, then \( N_{\lambda - 4} \) vanishes.

### 4.3 Functoriality

In this section, we show that the construction of the object \( M \) in Proposition 4.1 yields a functor \( E : \mathcal{X}_{I'} \to \mathcal{X}_I \).

**Proposition 4.2.** Let \( I' \subset I \) be a pair of closed subsets. Then there exists a functor \( E = E^I_{I'} : \mathcal{X}_{I'} \to \mathcal{X}_I \) with the following properties.

1. The functor \( E \) is left adjoint to the restriction functor \( (\cdot)_I' : \mathcal{X}_I \to \mathcal{X}_{I'} \).
2. The canonical transformation \( \text{id}_{\mathcal{X}_{I'}} \to (\cdot)_I' \circ E \) coming from the adjointness in (1) is an isomorphism of functors.
3. The functorial homomorphism \( \text{Hom}_{\mathcal{X}_{I'}}(M', N') \to \text{Hom}_{\mathcal{X}_I}(E(M'), E(N')) \) is an isomorphism.
4. For all \( \mu \in I \setminus I' \) and all \( M' \) in \( \mathcal{X}_{I'} \), we have \( E(M')_{\mu} = \text{im } F_{\mu} \) and hence \( E(M')_{\mu} \cong E(M')_{(\mu)} \) and \( E(M')_{\mu} \neq 0 \) implies that there exists a weight \( \lambda \) of \( M' \) with \( \mu \leq \lambda \).

Due to property (4), we call \( E^I_{I'}(M) \) the minimal extension of \( M \).

**Proof.** For any object \( M' \) in \( \mathcal{X}_{I'} \), we fix an object \( M \) satisfying the properties in Proposition 4.1. We also fix an (arbitrary) isomorphism \( \tau^{M'} : M' \sim M' \). Then we set \( E(M') = M \). Now, let \( f : M' \to N' \) be a homomorphism in \( \mathcal{X}_{I'} \). Then Proposition 4.1 shows that the homomorphism \( \text{Hom}_{\mathcal{X}_I}(E(M'), E(N')) \to \text{Hom}_{\mathcal{X}_{I'}}(E(M'), E(N')) \) that is given by the functor \( (\cdot)_{I'} \) is an isomorphism. The fixed identifications \( \tau^{M'} \) and \( \tau^{N'} \) allow us to obtain an isomorphism \( \text{Hom}_{\mathcal{X}_I}(E(M'), E(N')) \sim \text{Hom}_{\mathcal{X}_{I'}}(M', N') \). We define the homomorphism \( E(f) : E(M') \to E(N') \) as the preimage of \( f \). A moment’s thought shows that the map \( f \mapsto E(f) \) is compatible with compositions and respects the identity morphisms. Hence, we indeed obtain a functor \( E : \mathcal{X}_{I'} \to \mathcal{X}_I \).

We need to show that this functor has the required properties. By our construction, the fixed isomorphisms \( \tau^{M'} : M' \to M' \) yield a natural transformation \( \text{id}_{\mathcal{X}_{I'}} \to (\cdot)_{I'} \circ E \), hence (2). This natural
transformation induces a homomorphism
\[ \text{Hom}_{\mathcal{X}}(E(M'), N) \rightarrow \text{Hom}_{\mathcal{X}}(M', N_\prime) \]
that is functorial in both \( M' \) and \( N \). As \( r^{\prime}M' \) is an isomorphism, it follows from Proposition 4.1 that the above is a bijection, that is, \( E \) is left adjoint to \((\cdot)_{\prime} \), hence (1). Finally, (3) follows from the construction of the functor \( E \), and (4) is property (2) in Proposition 4.1.

4.4 The category of minimal objects

The properties of the minimal extension functors allow us now to classify all minimal objects in \( \mathcal{X} \).

**Proposition 4.3.** Suppose that \( \mathcal{A} \) is a unital Noetherian \( \mathcal{L} \)-domain.

1. For all \( \lambda \in X \), there exists an up to isomorphism unique object \( S_{\min}(\lambda) \) in \( \mathcal{X} \) with the following properties.
   a. \( S_{\min}(\lambda)_{\lambda} \) is free of rank 1 and \( S_{\min}(\lambda)_{\mu} \neq \{0\} \) implies \( \mu \leq \lambda \).
   b. \( S_{\min}(\lambda) \) is indecomposable and minimal.

   Moreover, the objects \( S_{\min}(\lambda) \) characterized in (1) have the following properties:

2. For all \( \lambda \in X \), we have \( \text{End}_{\mathcal{X}}(S_{\min}(\lambda)) = \mathcal{A} \cdot \text{id} \), and \( \text{Hom}_{\mathcal{X}}(S_{\min}(\lambda), S_{\min}(\mu)) = 0 \) for \( \lambda \neq \mu \).

3. Let \( S \) be a minimal object in \( \mathcal{X} \). Then there is an index set \( J \) and some elements \( \lambda_i \in X \) for \( i \in J \) such that \( S \cong \bigoplus_{i \in J} S_{\min}(\lambda_i) \).

Sometimes we will write \( S_{\min, \mathcal{A}}(\lambda) \) to incorporate the ground ring.

**Proof.** We start with proving that there exists an object \( S_{\min}(\lambda) \) satisfying the properties (1a) and (1b) as well as (2). We then show that (3) holds with this particular set of objects \( S_{\min}(\lambda) \), which then implies the remaining uniqueness statement in (1). So set \( I_\lambda := \{ \mu \in X \mid \lambda \leq \mu \} \). This is a closed subset of \( X \) that contains \( \lambda \) as a minimal element. Define \( S' = \bigoplus_{\mu \in I_\lambda} S_{\mu}' \) by setting \( S_\lambda' = \mathcal{A} \) and \( S_{\mu}' = \{0\} \) for \( \mu \in I_\lambda \setminus \{\lambda\} \). For any \( \mu \in I_\lambda, \alpha \in \Pi, n > 0 \), we have \( S'_{\mu + n\alpha} = 0 \), and so all the maps \( F_{\mu,\alpha,n} \) and \( E_{\mu,\alpha,n} \) are the zero homomorphisms. Then \( S' \) is an object in \( \mathcal{X}_{I_\lambda} \). We set \( S_{\min}(\lambda) := E^X_{I_\lambda}(S') \).

Proposition 4.2, (4) yields that \( S_{\min}(\lambda)_{\mu} = 0 \) unless \( \mu \leq \lambda \). As \( S_{\min}(\lambda)_{\lambda} = S_{\lambda}' = \mathcal{A} \), \( S_{\min}(\lambda) \) satisfies (1a). Proposition 4.2 yields \( \text{End}_{\mathcal{X}}(S_{\min}(\lambda)) = \text{End}_{\mathcal{X}_{I_\lambda}}(S') = \mathcal{A} \cdot \text{id} \), hence \( S_{\min}(\lambda) \) is indecomposable, and property (2) holds. The same proposition also implies that \( S_{\min}(\lambda)_{\mu} = S_{\min}(\lambda)_{\mu} \) for all \( \mu \in X \setminus I_\lambda \). One checks directly that this identity is also satisfied for all \( \mu \in I_\lambda \), hence \( S_{\min}(\lambda) \) is minimal.

We are now left with proving property (3), where we assume that the \( S_{\min}(\lambda) \) appearing in the statement are the objects we just constructed explicitly. So let \( S \) be a minimal object, and fix a maximal weight \( \lambda \) of \( S \) (which exists by (X1)). Then \( S_{\lambda} \) is a free \( \mathcal{A} \)-module by assumption (X3c). The maximality of \( \lambda \) implies that the restriction \( S_{\lambda} \) in \( \mathcal{X}_{I_\lambda} \) is isomorphic to a direct sum of copies of the object \( S' \) defined above. Since \( S \) and \( E^X_{I_\lambda}(\bigoplus S') \) are both minimal, Lemma 3.4 implies that the isomorphism \( S_{\lambda} \cong \bigoplus S' \) extends, so there are morphisms \( f : E^X_{I_\lambda}(\bigoplus S') \rightarrow S \) and

\[ f = \bigoplus_S f_S, \]

where \( f_S : E^X_{I_\lambda}(S') \rightarrow S \) is the natural projection for each \( S \).
\( g : S \rightarrow E^X_\mathcal{C} (\bigoplus S') \) such that \((g \circ f)|_\lambda \) is the identity. Now \( E^X_\mathcal{C} (\bigoplus S') \cong \bigoplus S_{\text{min}}(\lambda) \). Using the already proven property (2), we deduce that \( g \circ f \) is an automorphism. Hence, \( \bigoplus S_{\text{min}}(\lambda) \) is isomorphic to a direct summand of \( S \). By construction, \( \lambda \) is not a weight of a direct complement. From here, we can continue by induction to prove (3).

4.5  

Semisimplicity in the field case

Now assume for a moment that \( \mathcal{A} = \mathcal{K} \) is a field.

**Lemma 4.4.** Any object in \( \mathcal{X}_\mathcal{K} \) is minimal and maximal. Hence, any object in \( \mathcal{X}_\mathcal{K} \) is isomorphic to a direct sum of copies of the objects \( S(\lambda) := S_{\text{min}}(\lambda) \) for various \( \lambda \). In particular, \( \mathcal{X}_\mathcal{K} \) is a semisimple category.

**Proof.** As \( \mathcal{K} \) is a field, no torsion occurs, hence \( M_{(\mu)} = M_{(\mu)} = M_{(\mu),\text{max}} \), so \( M \) is both minimal and maximal. The claims now follow all from Proposition 4.3 and the fact that \( \text{End}_{\mathcal{X}_\mathcal{K}}(S_{\text{min}}(\lambda)) = \mathcal{K} \cdot \text{id} \).

5  

REPRESENTATIONS OF QUANTUM GROUPS

In this section, we show that the category \( \mathcal{X} \) has an interpretation in terms of quantum group representations. The main reasons for this are the uniqueness of the indecomposable minimal objects \( S_{\text{min}}(\lambda) \) and the semisimplicity statement in Lemma 4.4 in the field case.

5.1  

Quantum groups over \( \mathcal{Z} \)-algebras

We denote by \( U_\mathcal{Z} \) the quantum group over \( \mathcal{Z} = \mathbb{Z}[v, v^{-1}] \) (with divided powers) associated with the Cartan matrix \((\langle \alpha, \beta^\vee \rangle)_{\alpha, \beta \in \Pi} \) of \( R \). Its definition by generators and relations can be found in [7, sections 1.1–1.3]. We denote by \( e^{[n]}_\alpha, f^{[n]}_\alpha, k_\alpha, k^{-1}_\alpha \) for \( \alpha \in \Pi \) and \( n > 0 \) the standard generators of \( U_\mathcal{Z} \).

For \( \alpha \in R, n > 0 \) also the element

\[
\begin{bmatrix}
k_{\alpha} \\
n
\end{bmatrix}_\alpha := \prod_{s=1}^{n} k_{\alpha} v_{\alpha}^{-s+1} - k^{-1}_{\alpha} v_{\alpha}^{s-1} \\
v_{\alpha}^s - v_{\alpha}^{-s}
\]

is contained in \( U_\mathcal{Z} \) (where \( v_{\alpha} := v^{d_\alpha} \)). We let \( U_\mathcal{Z}^+, U_\mathcal{Z}^- \) and \( U_\mathcal{Z}^0 \) be the unital subalgebras of \( U_\mathcal{Z} \) that are generated by the sets \( \{e^{[n]}_\alpha\}, \{f^{[n]}_\alpha\} \) and \( \{k_{\alpha}, k^{-1}_\alpha, [k_{\alpha}]_n\} \), respectively. A remarkable fact, proven by Lusztig, is that each of these subalgebras is free over \( \mathcal{Z} \) and admits a PBW-type basis, and that the multiplication map \( U_\mathcal{Z}^- \otimes_\mathcal{Z} U_\mathcal{Z}^0 \otimes_\mathcal{Z} U_\mathcal{Z}^+ \rightarrow U_\mathcal{Z} \) is an isomorphism of \( \mathcal{Z} \)-modules (theorem 6.7 in [7]).

For a unital \( \mathcal{Z} \)-algebra \( \mathcal{A} \), we set \( U_{\mathcal{A}} := U_\mathcal{Z} \otimes_\mathcal{Z} \mathcal{A} \) and \( U^{\ast}_{\mathcal{A}} := U^*_\mathcal{Z} \otimes_\mathcal{Z} \mathcal{A} \) for \( \ast = -, 0, + \). In this article, we consider \( U_{\mathcal{A}} \) only as an associative, unital algebra and forget about the Hopf algebra structure. For better readability, we write \( U, U^+, ... \) instead of \( U_{\mathcal{A}}, U^+_{\mathcal{A}}, ... \), once the \( \mathcal{Z} \)-algebra \( \mathcal{A} \) is fixed.
5.2 The category $\mathcal{O}$

By [2, lemma 1.1], every $\mu \in X$ yields a character

$$
\chi_{\mu} : U^0_{\mathcal{X}} \rightarrow \mathcal{Z}
$$

$$
k^{\pm 1}_\alpha \mapsto v^{\pm (\mu, \alpha^\vee)}_\alpha
$$

$$
\begin{bmatrix} k_x \\ r \end{bmatrix}_\alpha \mapsto \begin{bmatrix} \langle \mu, \alpha^\vee \rangle \\ r \end{bmatrix}_{d_\alpha} (\alpha \in \Pi, r \geq 0).
$$

We can extend the above character to a character $\chi_{\mu} : U^0_{\mathcal{A}} \rightarrow \mathcal{A}$. A $U_{\mathcal{A}}$-module $M$ is called a ‘weight module’ if $M = \bigoplus_{\mu \in X} M_{\mu}$, where

$$
M_{\mu} := \{ m \in M \mid H.m = \chi_{\mu}(H)m \text{ for all } H \in U^0_{\mathcal{A}} \}.
$$

Hence, all the weight modules that we consider in this article are “of type 1” (cf. [5, section 5.1]). An element $\mu \in X$ is called a weight of $M$ if $M_{\mu} \neq \{0\}$. Now suppose that $\mathcal{A}$ is also Noetherian.

**Definition 5.1.** Let $\mathcal{O} = \mathcal{O}_{\mathcal{A}}$ be the full subcategory of the category of $U_{\mathcal{A}}$-modules that contains all objects $M$ with the following properties.

1. $M$ is a weight module and its set of weights is quasi-bounded.
2. For each $\mu \in X$, the weight space $M_{\mu}$ is a finitely generated torsion-free $\mathcal{A}$-module.

**Remark 5.2.** Note that the definition above yields an $\mathcal{A}$-linear category, which in general is not abelian (due to the torsion freeness assumption). It is closed under taking subobjects (due to the fact that we assume that $\mathcal{A}$ is Noetherian). If $\mathcal{A} = \mathcal{K}$ is a field, then torsion freeness is always satisfied, and we obtain an abelian category.

Now, we establish a first, rather easy, link to the objects that we considered in the earlier chapters. Let us denote by $\mathcal{X}^{\text{pre}} = \mathcal{X}_{\mathcal{A}}^{\text{pre}}$ the category whose objects are $X$-graded $\mathcal{A}$-modules $M$ endowed with operators $E_{\alpha,n}$ and $F_{\alpha,n}$ as in Section 2 that satisfy conditions (X1) and (X2) (but not necessarily (X3)), and with morphisms being the $X$-graded $\mathcal{A}$-linear homomorphisms that commute with the $E$- and $F$-maps.

Let $M$ be an object in $\mathcal{O}$. Let us denote by $E_{\mu,\alpha,n} : M_{\mu} \rightarrow M_{\mu+n\alpha}$ and $F_{\mu,\alpha,n} : M_{\mu+n\alpha} \rightarrow M_{\mu}$ the homomorphisms given by the actions of $e_{\alpha}^{[n]}$ and $f_{\alpha}^{[n]}$, respectively. By forgetting structure, we now consider $M$ only as an $X$-graded space endowed with these operators.

**Lemma 5.3.** The above yields a fully faithful functor

$$
S : \mathcal{O} \rightarrow \mathcal{X}^{\text{pre}}.
$$

**Proof.** It is clear that the above construction is functorial. Let $M$ be an object in $\mathcal{O}$. We need to check that the graded space with operators that we obtain from $M$ satisfies the conditions (X1) and (X2). Condition (X1) is part of the definition of $\mathcal{O}$. Now we check condition (X2). The case
\( \alpha \neq \beta \) follows from the fact that \( e_\alpha^{[m]} \) and \( f_\beta^{[n]} \) commute. Now, we treat the case \( \alpha = \beta \). Set 
\[
\left[ k_\alpha; c \right]_\alpha = \prod_{s=1}^{r} \frac{k_\alpha v_\alpha^{c-s+1} - k_\alpha^{-1} v_\alpha^{-c+s-1}}{v_\alpha^{s} - v_\alpha^{-s}}.
\]
This element is contained in \( U^0 \) and acts as multiplication with 
\[
\prod_{s=1}^{r} \frac{v_\alpha^{c+s-1} - v_\alpha^{-c-s-1}}{v_\alpha^{s} - v_\alpha^{-s}}.
\]
on each vector of weight \( \nu \). By [7, Section 6.5] the following relations hold in \( U \) for all \( m, n > 0 \):
\[
e_\alpha^{[m]} f_\alpha^{[n]} = \sum_{r=0}^{\min(m,n)} f_\alpha^{[n-r]} \left[ k_\alpha; 2r - m - n \right]_\alpha e_\alpha^{[m-r]}.
\]
For \( \nu \in M_\mu \), we hence obtain 
\[
e_\alpha^{[m]} f_\alpha^{[n]}(\nu) = \sum_{r=0}^{\min(m,n)} f_\alpha^{[n-r]} \prod_{s=1}^{r} \frac{v_\alpha^{\zeta-s+1} - v_\alpha^{-\zeta+s-1}}{v_\alpha^{s} - v_\alpha^{-s}} e_\alpha^{[m-r]}(\nu),
\]
where \( \zeta = \langle \mu + (m-r)\alpha, \alpha^\vee \rangle + 2r - m - n = \langle \mu, \alpha^\vee \rangle + m - n \). In order to prove that condition (X2) holds, it remains to show that 
\[
\left[ \zeta \right]_\alpha = \prod_{s=1}^{r} \frac{v_\alpha^{\zeta-s+1} - v_\alpha^{-\zeta+s-1}}{v_\alpha^{s} - v_\alpha^{-s}},
\]
which is (almost) immediate from the definition. Hence, \( S \) is indeed a functor from \( \mathcal{O} \) to \( \mathcal{X}^{\text{pre}} \).

Now \( U \) is generated by the elements \( e_\alpha^{[n]}, f_\alpha^{[n]} \) for \( \alpha \in \Pi \) and \( n > 0 \) as an algebra over \( U^0 \) by the PBW-theorem. As the actions of the \( e^{[n]} \)- and \( f^{[n]} \)-elements are encoded by the \( E_n^- \) and \( F_n^- \)-homomorphisms, and as the action of \( U^0 \) is encoded by the \( X \)-grading, the functor \( S \) is fully faithful.

Note that we can consider \( \mathcal{X} \) as a full subcategory of \( \mathcal{X}^{\text{pre}} \). In the next section, we construct a functor \( R : \mathcal{X} \rightarrow \mathcal{O} \) that is right inverse to \( S \).

### 5.3 A functor from \( \mathcal{X} \) to \( \mathcal{O} \)

First, we suppose that \( \mathcal{A} = \mathcal{K} \) is a field. For each \( \lambda \), the character \( \chi_\lambda \) of \( U_0^0 \mathcal{K} \) can be uniquely extended to a character of \( U_0^0 \mathcal{K}^+ \) such that \( \chi_\lambda(e_\alpha^{[n]}) = 0 \) for all \( \alpha \in \Pi \), \( n > 0 \). So we obtain an \( U_0^0 \mathcal{K}^+ \)-module \( \mathcal{K}_\lambda \) of dimension 1. We denote by \( \Delta_{\mathcal{K}}(\lambda) := U_0^0 \mathcal{K}^+ \otimes_{U_0^0 \mathcal{K}^+} \mathcal{K}_\lambda \) the induced \( U_0^{\mathcal{K}^+} \)-module (this is the ‘Verma module’ with the highest weight \( \lambda \)). It has a unique irreducible quotient that we denote by \( L_{\mathcal{K}}(\lambda) \). Both are objects in \( \mathcal{O}_{\mathcal{K}} \). For more information on these objects, see [1], or, in the case \( q = 1 \), [4]. Recall that in the field case a minimal object is also maximal and we set \( S_{\mathcal{K}}(\lambda) := S_{\min, \mathcal{K}}(\lambda) \) (cf. Lemma 4.4).
Proposition 5.4. Suppose that $\mathcal{A} = \mathcal{K}$ is a field.

1. For all $\lambda \in X$, the object $S(L_{\mathcal{K}}(\lambda))$ is contained in the subcategory $\mathcal{X}_\mathcal{K}$ of $\mathcal{X}_{\mathcal{P}}$ and it is isomorphic to $S_{\mathcal{K}}(\lambda)$.

2. The functor $S$ induces an equivalence between the full subcategory $\mathcal{O}_{\mathcal{K}}^\text{ss}$ of semi-simple objects in $\mathcal{O}_\mathcal{K}$ and the category $\mathcal{X}_\mathcal{K}$.

Proof. We prove part (1). In view of Lemma 5.3, we need to check property (X3) in order to prove the first statement. Set $M = S(L_{\mathcal{K}}(\lambda))$. As $L_{\mathcal{K}}(\lambda)$ is an irreducible $U_{\mathcal{K}}$-module of highest weight $\lambda$, it is cyclic as a $U^-$-module, hence we have $\text{im} F_\mu = M_\mu$ for all $\mu \neq \lambda$, and $\text{im} F_\lambda = \{0\}$. On the other hand, there are no non-trivial primitive vectors in $L_{\mathcal{K}}(\lambda)$ of weight $\mu$ if $\mu \neq \lambda$. (A primitive vector is a vector annihilated by all $e_\alpha^{[n]}$.) So we have $\text{ker} E_\mu = \{0\}$ for $\mu \neq \lambda$ and $\text{ker} E_\lambda = M_\lambda$. In any case, we have $M_\mu = \text{ker} E_\mu \oplus \text{im} F_\mu$, which, by Lemma 2.4, is equivalent to the set of conditions (X3). Hence, $M$ is an object in $\mathcal{X}_\mathcal{K}$. Lemma 4.4 now yields that $M$ is isomorphic to a direct sum of various $S_{\mathcal{K}}(\mu)$’s. As $S$ is fully faithful, $M$ is indecomposable, and a comparison of weights shows $M \cong S_{\mathcal{K}}(\lambda)$.

Now (2) follows from the fact that $\mathcal{X}_\mathcal{K}$ is a semi-simple category with simple objects $S_{\mathcal{K}}(\lambda)$ by Lemma 4.4 and the fact that $S$ is fully faithful. □

Here is our ‘realization theorem’.

Theorem 5.5. Let $\mathcal{A}$ be a unital Noetherian domain that is a $\mathcal{K}$-algebra. Let $M$ be an object in $\mathcal{X} = \mathcal{X}_{\mathcal{A}}$. Then there exists a unique $U$-module structure on $M$ such that the following holds.

- The $X$-grading $M = \bigoplus_{\mu \in X} M_\mu$ is the weight decomposition.
- For all $\mu \in X$, $\alpha \in \Pi$, $n > 0$, the homomorphisms $E_{\mu, \alpha, n}$ and $F_{\mu, \alpha, n}$ are the action maps of $e_\alpha^{[n]}$ on $M_\mu$ and $f_\alpha^{[n]}$ on $M_{\mu + n \alpha}$, resp.

From this, we obtain a fully faithful functor $R : \mathcal{X} \to \mathcal{O}$, and we have $S \circ R \cong \text{id}_{\mathcal{X}}$.

Proof. Note that the uniqueness statement in the claim above follows immediately from the fact that $U$ is generated as an algebra by the elements $e_\alpha^{[n]}$, $f_\alpha^{[n]}$ and $k_\alpha$, $k_\alpha^{-1}$ for $\alpha \in \Pi$ and $n > 0$. We now prove the existence of a $U$-module structure on $M$ with the alleged properties.

First, suppose that $\mathcal{A} = \mathcal{K}$ is a field. Then every object in $\mathcal{X}_\mathcal{K}$ is isomorphic to a direct sum of various $S_{\mathcal{K}}(\lambda)$’s by Lemma 4.4. By Proposition 5.4, we have $S(L_{\mathcal{K}}(\lambda)) \cong S_{\mathcal{K}}(\lambda)$. Hence, any $S_{\mathcal{K}}(\lambda)$ carries the structure of an $U_{\mathcal{K}}$-module of the required kind (making it isomorphic to $L_{\mathcal{K}}(\lambda)$). So the result holds in the case that $\mathcal{A}$ is a field.

Now let $\mathcal{A}$ be a unital Noetherian domain. We denote by $\mathcal{K}$ its quotient field. For any object $M$ in $\mathcal{X} = \mathcal{X}_{\mathcal{A}}$, $M_{\mathcal{K}} = M \otimes_\mathcal{A} \mathcal{K}$ is an object in $\mathcal{X}_\mathcal{K}$ by Lemma 2.6. As $M$ is a torsion free $\mathcal{A}$-module by Lemma 2.4, we can view $M$ as an $\mathcal{A}$-submodule in $M_{\mathcal{K}}$. Now by the above, we can view $M_{\mathcal{K}}$ as an object in $\mathcal{O}_\mathcal{K}$. As $M$ is stable under the maps $E_{\alpha, n}$ and $F_{\alpha, n}$, it is stable under the action of $e_\alpha^{[n]}$ and $f_\alpha^{[n]}$. Moreover, it is clearly stable under the action of $k_\alpha$ and $k_\alpha^{-1}$. Hence it is stable under the action of $U_{\mathcal{A}} \subset U_{\mathcal{K}}$. So there is indeed a natural $U_{\mathcal{A}}$-module structure on $M$, and one immediately checks that this makes it into an object of category $\mathcal{O}_{\mathcal{A}}$.

Clearly the above $U_{\mathcal{A}}$-structure depends functorially on $M$, so we indeed obtain a functor $R$ from $\mathcal{X}_{\mathcal{A}}$ to $\mathcal{O}_{\mathcal{A}}$. It is clearly fully faithful and obviously $S \circ R$ is isomorphic to the identity on $\mathcal{X}_{\mathcal{A}}$. □
6 | OBJECTS ADMITTING A WEYL FILTRATION

The main goal of this section is to show that the functors $S$ and $R$ induce mutually inverse equivalences between the category $\mathcal{X}^{\text{fin}}$ of objects in $\mathcal{X}$ that are free of finite rank over $\mathcal{A}$, and the category $\mathcal{O}^W$ of objects in $\mathcal{O}$ that admit a (finite) Weyl filtration.

We need to assume that the quotient field of our ground ring $\mathcal{A}$ is generic, i.e. of ‘quantum characteristic’ 0 (see Definition 6.1). The finite dimensional representation theory of $U_\mathfrak{g}$ is semi-simple in this case. For the most part of this section we do not need $\mathcal{A}$ to be local.

6.1 | Generic algebras

Let $\mathcal{A}$ be a unital $\mathcal{L}$-algebra that is a Noetherian domain.

**Definition 6.1.** We say that $\mathcal{A}$ is ‘generic’ (or of quantum characteristic 0) if for all $n \neq 0$ and all $d > 0$ the image of the quantum integer $[n]_d$ in $\mathcal{A}$ is non-zero (i.e. invertible in the quotient field $\mathbb{K}$).

Recall that we denote by $q \in \mathcal{A}$ the image of $v$ under the structural homomorphism $\mathcal{L} \to \mathcal{A}$, $f \mapsto f \cdot 1_\mathcal{A}$. Note that if $\mathcal{A}$ is not generic, then $q$ is a root of unity in $\mathcal{A}$, as

$$[n]_d = \frac{v^{dn} - v^{-dn}}{v^d - v^{-d}} = \frac{v^{-dn+d} v^{2dn} - 1}{v^{2d} - 1}.$$ 

The converse is not true. For example, a field of characteristic 0 with $q = 1$ is generic. However, a field of positive characteristic and $q = 1$ is not generic, but the ring of $p$-adic integers $\mathbb{Z}_p$ with $q = 1$ is generic. If $\zeta \in \mathbb{C}$ is a root of unity of order $> 2$, then $\mathcal{A} = \mathbb{Q}(\zeta)$ with $q = \zeta$ is not generic.

Let $p$ be a prime number and denote by $\mathcal{L}_p$ the localization of $\mathcal{L}$ at the prime ideal

$$p := \{ g \in \mathcal{L} \mid g(1) \text{ is divisible by } p \} = \ker \left( \mathcal{L} \xrightarrow{v \mapsto 1} \mathbb{F}_p \right),$$

that is, $\mathcal{L}_p = \{ \frac{f}{g} \in \text{Quot } \mathcal{L} \mid g(1) \text{ is not divisible by } p \}$. So $\mathcal{L}_p$ with $q = v$ is a local and generic $\mathcal{L}$-algebra, and its residue field is $\mathbb{F}_p$ (with $q = 1$). Similarly, denote by $\sigma_l \in \mathbb{Q}[v]$ the $l$-th cyclotomic polynomial, and let $\mathbb{Q}[v]_{(\sigma_l)}$ be the localization at the prime ideal generated by $\sigma_l$. We obtain a local and generic $\mathcal{L}$-algebra with $q = v$, with residue field $\mathbb{Q}[\zeta_l]$, the $l$-th cyclotomic field (with $q = \zeta_l$).

The assumption that $\mathcal{A}$ is generic is not a decisive restriction even if one is interested in the theory of tilting modules for algebraic groups in positive characteristics or for quantum groups at roots of unity. As we will show in this article, these objects admit deformations to the generic and local algebras $\mathcal{L}_p$ and $\mathbb{Q}[v]_{(\sigma_l)}$ and one obtains information in the non-generic cases by specializing results from generic cases.
6.2 Modules admitting a Weyl filtration

Let us now assume that \( \mathcal{A} \) is generic. We denote by \( \mathcal{H} \) its quotient field. Recall that for any \( \lambda \in X \) we denote by \( L_{\mathcal{H}}(\lambda) \) the irreducible object in \( \mathcal{O}_{\mathcal{H}} \) with highest weight \( \lambda \). Then \( L_{\mathcal{H}}(\lambda) \) is finite dimensional if and only if \( \lambda \) is dominant, that is, is contained in the set \( X^+ = \{ \lambda \in X \mid \langle \lambda, \alpha^* \rangle \geq 0 \text{ for all } \alpha \in \Pi \} \) of dominant weights, cf. [1, theorem 2.3]. If \( \lambda \) is dominant, then we define \( W(\lambda) = W_{\mathcal{A}}(\lambda) \) as the \( \mathcal{H} \)-graded \( U_{\mathcal{A}} \)-submodule in \( L_{\mathcal{H}}(\lambda) \) generated by a non-zero element in \( L_{\mathcal{H}}(\lambda)_\lambda \). This is an object in \( \mathcal{O}_{\mathcal{H}} \) and it does not depend, up to isomorphism, on the choice of the element. It is called the ‘Weyl module’ with highest weight \( \lambda \).

**Proposition 6.2.** Let \( \mathcal{A} \) be local and generic, and let \( \lambda \in X \) be dominant. Then \( W(\lambda) \) is a free \( \mathcal{A} \)-module of finite rank and its character is given by Weyl’s character formula.

**Proof.** The above statement is proven for a special choice of \( \mathcal{A} \) in [2, proposition 1.22]. We claim that their proof carries over to the more general case above. As \( \mathcal{A} \) is generic, the character of \( L_{\mathcal{H}}(\lambda) = W_{\mathcal{A}}(\lambda) \otimes_{\mathcal{H}} \mathcal{H} \) is given by Weyl’s character formula. We denote by \( \mathcal{F} \) the residue field of \( \mathcal{A} \). As in loc.cit., it is now sufficient to show that for all weights \( \mu \) the \( \mathcal{F} \)-dimension of \( W_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathcal{F} \) is at most the \( \mathcal{H} \)-dimension of \( W_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathcal{H} \). But the \( \mathcal{U}_{\mathcal{F}} \)-module \( W_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathcal{F} \) is finite dimensional and of the highest weight \( \lambda \). Hence, (remark 4.2 in [10]) it is a quotient of the \( \mathcal{U}_{\mathcal{F}} \)-module \( D_{\mathcal{F}}(\lambda) \) that is obtained from the \( \mathcal{U}_{\mathcal{F}}^+ \)-module \( \mathcal{F}_\lambda \) by Joseph’s induction functor. But the character of \( D_{\mathcal{F}}(\lambda) \) is given by Weyl’s character formula as well (see [10, theorem 5.4, corollary 5.8]), hence the character of \( W_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathcal{F} \) is bounded from above by Weyl’s character formula. \( \square \)

**Definition 6.3.** We say that \( M \) admits a Weyl filtration if there is a finite filtration \( 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M \) and \( \lambda_1, \ldots, \lambda_n \in X^+ \) such that for each \( i = 1, \ldots, n \), the subquotient \( M_i/M_{i-1} \) is isomorphic to \( W(\lambda_i) \).

We denote by \( \mathcal{O}^W = \mathcal{O}^W_{\mathcal{A}} \) the full subcategory of \( \mathcal{O} \) that contains all objects that admit a Weyl filtration. Note that if \( \mathcal{A} = \mathcal{H} \) is a generic field, then \( \mathcal{O}^W_{\mathcal{H}} \) is a semi-simple category (cf. theorem 5.15 and section 6.26 in [5]).

6.3 A criterion for Weyl filtrations

Suppose that \( \mathcal{A} \) is local and generic. Let \( M \) be a module for \( U = U_{\mathcal{A}} \).

**Lemma 6.4.** Suppose that \( M \) is finitely generated as an \( \mathcal{A} \)-module and that there exists a dominant element \( \lambda \in X \) such that the following holds.

1. The weight space \( M_\lambda \) generates \( M \) as a \( U^- \)-module.
2. The weight space \( M_\lambda \) is a free \( \mathcal{A} \)-module of finite rank \( r \).

Then \( M \) is isomorphic to a direct sum of \( r \) copies of \( W(\lambda) \).

**Proof.** First, let us show that \( M \) is free over \( \mathcal{A} \). Again we denote by \( \mathcal{H} \) and \( \mathcal{F} \) the quotient field and the residue field of \( \mathcal{A} \), respectively. Consider the \( \mathcal{H} \)-module \( M_{\mathcal{H}} \). It is of finite \( \mathcal{H} \)-dimension, and
generated by its \( \lambda \)-weight space, which is of \( \mathcal{F} \)-dimension \( r \). As \( \mathcal{A} \) is generic, \( M_{\mathcal{X}} \) is isomorphic to a direct sum of \( r \) copies of \( L_{\mathcal{X}}(\lambda) \). Now consider the \( U_{\mathcal{F}} \)-module \( M_{\mathcal{X}} \). Again, this is of finite \( \mathcal{F} \)-dimension and generated by its \( \lambda \)-weight space, which is of \( \mathcal{F} \)-dimension \( r \). As \( \mathcal{A} \) is generic, \( M_{\mathcal{X}} \) is isomorphic to a direct sum of \( r \) copies of \( L_{\mathcal{X}}(\lambda) \).

Hence \( M \) embeds into \( M_{\mathcal{X}} \cong L_{\mathcal{X}}(\lambda)^{\oplus r} \). As \( M \) is generated by \( M_{\lambda} \), which is an \( \mathcal{A} \)-lattice in \( (M_{\mathcal{X}})_{\lambda} \) of rank \( r \), \( M \) is isomorphic to a direct sum of \( r \) copies of the Weyl module \( W_{\mathcal{A}}(\lambda) \).

\[ \square \]

Let \( M \) be an object in \( \mathcal{O} \). For \( \lambda \in X \) define \( M[\lambda] \subset M \) as the \( U \)-submodule of \( M \) that is generated by all weight spaces \( B_{\mu} \) such that \( \mu \not\geq \lambda \). Then \( M/M[\lambda] \) is the largest quotient of \( M \) that has the property that all of its weights are smaller or equal to \( \lambda \). Set \( M[\lambda] = (M/M[\lambda])_{\lambda} \). Clearly, if \( N \subset M \) is a submodule, then \( N[\lambda] \) is a submodule of \( M[\lambda] \). Also, \( M[\lambda]_{\mu} = M_{\mu} \) for all \( \mu \not\leq \lambda \).

**Lemma 6.5.** For \( \nu \not\leq \lambda \), the inclusion \( M[\lambda][\nu] \subset M[\nu] \) is an isomorphism on the \( \nu \)-weight space.

**Proof.** For an object \( B \) in \( \mathcal{O} \) let us denote by \( B[\nu] \) the \( U \)-submodule of \( B \) that is generated by the weight spaces \( B_{\mu} \) with \( \mu > \nu \). Then \( B[\nu] \subset B[\nu'] \). The PBW theorem implies that \( B[\nu] \) is generated by \( \bigoplus_{\mu > \nu} B_{\mu} \) even over \( U^- \). Likewise, \( B[\nu] \) is generated by \( \bigoplus_{\mu \leq \nu} B_{\mu} \) over \( U^- \). Hence, \( B[\nu] = B[\nu'] \). Now \( M[\lambda][\nu] = M[\nu] \), as for any \( \mu > \nu \), we have \( \mu \not\leq \lambda \), so \( M[\lambda]_{\mu} = M_{\mu} \). Hence, \( M[\lambda][\nu] = M[\nu] \).

Now, we state and prove a criterion for the existence of Weyl filtrations.

**Proposition 6.6.** Suppose that \( \mathcal{A} \) is generic. Let \( M \) be an object in \( \mathcal{O} \). The following statements are equivalent.

1. The set of weights of \( M \) is finite and \( M[\nu] \) is a free \( \mathcal{A} \)-module of finite rank for all \( \nu \in X \).
2. \( M \) admits a Weyl filtration.

If either of the above holds, then the multiplicity of \( W(\mu) \) in a Weyl filtration equals the rank of \( M[\mu] \). In particular, \( M[\mu] \neq 0 \) implies that \( \mu \) is dominant.

**Proof.** Assume that (2) holds. Then the set of weights of \( M \) is finite. Standard arguments show that \( \text{Ext}^{1}_{\mathcal{O}}(W(\lambda), W(\mu)) = 0 \) if \( \mu \not\leq \lambda \) (cf. the proof of Lemma 6.4). Let \( \nu \in X \). The Ext-vanishing statement now implies that the subquotients of a given Weyl filtration of \( M \) be “rearranged” in such a way that we obtain a filtration \( 0 = M_0 \subset M_1 \subset ... \subset M_n = M \) and some \( 1 \leq r \leq s \leq n \) such that \( M_i/M_{i-1} \) has the highest weight \( \not\geq \nu \) if \( i \leq r \), has the highest weight \( \nu \) if \( r < i \leq s \), and has the highest weight \( < \nu \), if \( i > s \). Hence, \( M_i/M_r \) is a direct sum of copies of \( W(\nu) \). Moreover, in the notation of the paragraph preceding this proposition, we have \( M[\nu] = M_r \) and \( M[\nu] = (M/M_r)[\nu] = (M/M_r)_{\nu} \) (as \( (M/M_s)[\nu] = \{0\} \)). Hence, \( M[\nu] \) is free of finite rank \( s-r \) as an \( \mathcal{A} \)-module. So (2) implies (1).

Now assume that (1) holds. We can assume that \( M \neq 0 \). As the set of weights of \( M \) is finite there must be a minimal weight \( \lambda \) such that \( M[\lambda] \neq 0 \) (note that \( M[\gamma] = M_{\gamma} \) if \( \gamma \) is maximal among the weights of \( M \), so some \( M[\gamma] \) are non-zero). Let us fix such a minimal \( \lambda \). Set \( N := M[\lambda] \subset M \) and \( M' = M/N \), so by definition \( M'_{\lambda} = M_{\lambda} \). We claim the following.
(a) We have \( N[\nu] \cong M[\nu] \) for all \( \nu \neq \lambda \), and \( N[\lambda] = 0 \).
(b) \( M' \) is generated, as a \( U_{ad} \)-module, by its \( \lambda \)-weight space.
(c) \( M' \) is finitely generated as an \( \mathcal{A} \)-module, and \( \lambda \) is dominant.

If these statements are true, then we can prove that \( M \) admits a Weyl filtration as follows. From (a), we can deduce, by induction on the number of weights \( \nu \) with \( M[\nu] \neq 0 \), that \( N \) admits a Weyl filtration. Then from (b) and (c), we deduce, using the fact that \( M'_\lambda = M[\lambda] \) is free of finite rank (by assumption) and Lemma 6.4, that \( M' \) is isomorphic to a direct sum of \( \text{rk} M[\lambda] \) many copies of \( W(\lambda) \). So both \( N \) and \( M' = M/N \) admit a Weyl filtration, hence so does \( M \). The last statement in the proposition about the multiplicities follows by induction as well.

So let us prove (a). By definition, \( N \) is generated by its weight spaces \( N_\gamma \) with \( \gamma \neq \lambda \). For all \( \nu \leq \lambda \) we hence have \( N[\nu] = N \), so \( N[\nu] = 0 \). The minimality of \( \lambda \) implies \( M[\nu] = 0 \) for all \( \nu < \lambda \). Now suppose that \( \nu \neq \lambda \). Then \( N_\nu = M_\nu \) and \( N[\nu] = M[\nu] \) by Lemma 6.5, hence \( N[\nu] = (N/N[\nu])_\nu = (M/M[\nu])_\nu = M[\nu] \). Part (a) is proven.

Now we prove (b). Recall that all weights of \( M' \) are smaller or equal to \( \lambda \). If \( M' \) was not generated as a \( U_{ad} \)-module by its \( \lambda \)-weight space, then there would exist a weight \( \nu < \lambda \) such that \( M'_\nu \neq 0 \). Hence \( M' \) is not generated (over \( U_{ad} \)) by \( \bigoplus_{\gamma \neq \lambda} M'_\gamma \). As \( M' \) is quotient of \( M \), \( M \) cannot be generated by \( \bigoplus_{\gamma \neq \lambda} M'_\gamma \). Hence, \( M[\nu] \neq 0 \), which is a contradiction to the minimality of \( \lambda \).

We turn to statement (c). From assumption (1), it follows that \( M \) is finitely generated as an \( \mathcal{A} \)-module. Hence, so is its quotient \( M' \). By (b), \( M' \) is a quotient of a finite direct sum of copies of the Verma module \( \Delta(\lambda) \) (cf. the proof of Lemma 6.4). By Proposition 3.2 in [8], a simple highest weight module for \( U_{\mathcal{K}} \) is integrable only if \( \lambda \) is dominant. As \( M' \otimes_{ad} \mathcal{K} \) is finite dimensional, \( \lambda \) must be dominant. \( \square \)

### 6.4 Finitely generated objects in \( \mathcal{X} \)

Now we want to show that the objects in \( \mathcal{X} \) that are finitely generated as \( \mathcal{A} \)-modules, correspond, via the realization functor \( R \), to the objects in \( \mathcal{O} \) that admit a Weyl filtration. In a first step, we are interested in what happens if we apply the functor \( S \) to objects that admit a Weyl filtration.

**Proposition 6.7.** Suppose that \( \mathcal{A} \) is generic.

1. Let \( M \) be an object in \( \mathcal{O}^{ad} \). Then \( S(M) \) is an object in \( \mathcal{X} \).
2. For all dominant \( \lambda \), we have \( S(W(\lambda)) \cong S_{\min}(\lambda) \).

**Proof.** We prove claim (1). In view of Lemma 5.3, we need to show that the property (X3) is satisfied for \( S(M) \). So let \( \mu \in X \). As \( M_{\mathcal{K}} \) is semi-simple, we have \( (S(M)_\mu)_{\mathcal{K}} = (\text{im} F_\mu)_\mathcal{K} \oplus (\ker E_\mu)_{\mathcal{K}} \). As \( M \) admits a Weyl filtration, Proposition 6.2 implies that \( M \) is free as an \( \mathcal{A} \)-module. By Lemma 2.4, we now only need to show that property (X3c) holds. Let \( \{0\} = M_0 \subset M_1 \subset \ldots \subset M_n = M \) be a filtration such that \( M_{i+1}/M_i \) is isomorphic to \( W(\mu_i) \). As in the proof of Proposition 6.6, we can assume that there exists an integer \( r \) such that \( \mu < \mu_i \) implies \( i \leq r \). It follows that \( \text{im} F_\mu = (M_r)_\mu \subset M_\mu \). As the quotient \( M/M_r \) admits a filtration with subquotients isomorphic to Weyl modules with dominant highest weights, it is free as an \( \mathcal{A} \)-module by Proposition 6.2. In particular, its \( \mu \)-weight space is free. By the above, this identifies with \( M_\mu/\text{im} F_\mu \). Hence, property (X3c) holds, so \( S(M) \) is an object in \( \mathcal{X} \).
Now we prove claim (2). For $N = S(W(\lambda))$, we have $N_{\mu} = \im F_{\mu}$ for all $\mu \neq \lambda$ as $W(\lambda)$ is cyclic as a $U^-$-module and $U^-$ is generated by the elements $f^{[n]}_\alpha$ with $\alpha$ simple and $n > 0$. Hence $N$ is an indecomposable minimal object, so $N \cong S_{\min}(\nu)$ for some $\nu \in X$ by Proposition 4.3. A comparison of weights shows $\nu = \lambda$. □

Now we define the counterpart of $\mathcal{O}^W \subset \mathcal{O}$ in $\mathcal{X}$.

**Definition 6.8.** We denote by $\mathcal{X}^{\fin}$ the full subcategory of $\mathcal{X}$ that contains all objects $M$ that are finitely generated as $\mathcal{A}$-modules.

We will see in a moment that any object in $\mathcal{X}^{\fin}$ is automatically free as an $\mathcal{A}$-module. As we assume that each weight space of an object in $\mathcal{X}$ is finitely generated, the property in the definition above is equivalent to the set of weights being finite.

**Theorem 6.9.** Suppose that $\mathcal{A}$ is generic. Then the functors $S$ and $R$ restrict to mutually inverse equivalences between the categories $\mathcal{X}^{\fin}$ and $\mathcal{O}^W$.

**Proof.** In view of Proposition 6.7, we need to show that $S(M)$ is finitely generated as an $\mathcal{A}$-module for all objects $M$ in $\mathcal{O}$ that admit a Weyl filtration and, conversely, that $R(M)$ admits a Weyl filtration for all objects $M$ in $\mathcal{X}^{\fin}$. The first statement follows easily from the facts that the functor $S$ is the identity functor on the underlying $\mathcal{A}$-modules and that each Weyl module is free of finite rank as an $\mathcal{A}$-module.

Now suppose that $M$ is an object in $\mathcal{X}$ that is finitely generated as an $\mathcal{A}$-module. We already know from Theorem 5.5 that $R(M)$ is an object in $\mathcal{O}$. We want to employ Proposition 6.6, so we need to check that $R(M)$ has only finitely many weights and that $R(M)_{[\mu]}$ is a free $\mathcal{A}$-module of finite rank for all $\mu \in X$. The first statement is clear. For the second, note that we can canonically identify $R(M)_{[\mu]}$ with $M_{\mu}/\im F_{\mu}$. The latter is, by definition of the category $\mathcal{X}$, a free $\mathcal{A}$-module, and of finite rank as $M$ is of finite rank. □

From the above, we can deduce that each object in $\mathcal{X}^{\fin}$ is even ‘free’ of finite rank as an $\mathcal{A}$-module.

### 7 | THE MAXIMAL EXTENSION

Recall that we classified the subcategory of minimal objects in $\mathcal{X}$ in Proposition 4.3. In this section, we study the opposite extremal case, that is, we classify the ‘maximal’ objects in $\mathcal{X}$. As in the minimal case, for every $\lambda \in X$, there is an up to isomorphism unique indecomposable maximal object $S_{\max}(\lambda)$ with the highest weight $\lambda$. Using the results in the previous section, we show that $S_{\max}(\lambda)$ is a finitely generated, hence free, $\mathcal{A}$-module for all ‘dominant’ weights $\lambda$. In particular, $R(S_{\max}(\lambda))$ is an object in $\mathcal{O}$ that admits a Weyl filtration. It is plausible that $S_{\max}(\lambda)$ is free as an $\mathcal{A}$-module for arbitrary $\lambda$ (but not of finite rank if $\lambda$ is not dominant), but we cannot prove this.

In a subsequent section, we show that $S_{\max}(\lambda)$ admits a ‘non-degenerate contravariant symmetric bilinear form’ (cf. Definition 8.1) under the assumption that $\lambda$ is dominant (again, it is plausible that this restriction is not necessary). This fact is then used in the last section to show that $R(S_{\max}(\lambda))$ is an indecomposable tilting module.
For the construction of the maximal objects in \(\mathcal{X}\), we need to assume that 'projective covers' exist in the category of \(\mathcal{A}\)-modules. Hence, we assume that \(\mathcal{A}\) is a local ring. Here is a short reminder on projective covers. Let \(\mathcal{R}\) be a ring and \(M\) an \(\mathcal{R}\)-module. Recall that a 'projective cover' of \(M\) is a surjective homomorphism \(\phi : P \to M\) such that \(P\) is a projective \(\mathcal{R}\)-module and such that any submodule \(U \subset P\) with \(\phi(U) = M\) satisfies \(U = P\). If \(\mathcal{R}\) is a local ring, then projective covers exist for finitely generated \(\mathcal{R}\)-modules. They can be constructed as follows: Denote by \(\mathcal{F}\) the residue field of \(\mathcal{R}\). For an \(\mathcal{R}\)-module \(N\), we let \(N = N \otimes_{\mathcal{R}} \mathcal{F}\) be the associated \(\mathcal{F}\)-vector space.

### 7.1 The maximal extension

Let \(I' \subset I\) be a pair of closed subsets of \(X\). We now construct a maximal extension of an object \(M'\) in \(\mathcal{X}_{I'}\), that is, an object \(M\) in \(\mathcal{X}_I\) such that \(M_{I'}\) is isomorphic to \(M'\) and such that \(M_{(\mu)} = M_{[\mu],\text{max}}\) for all \(\mu \in I \setminus I'\). The following summarises the properties of this extension.

**Proposition 7.1.** Assume that \(\mathcal{A}\) is local. Let \(M'\) be an object in \(\mathcal{X}_{I'}\). Then there exists an up to isomorphism unique object \(M\) in \(\mathcal{X}_I\) with the following properties.

1. \(M_{I'}\) is isomorphic to \(M'\).
2. An endomorphism \(f : M \to M\) in \(\mathcal{X}_I\) is an automorphism if and only if \(f_{I'} : M_{I'} \to M_{I'}\) is an automorphism.
3. \(M_{(\mu)} = M_{[\mu],\text{max}}\) for all \(\mu \in I \setminus I'\).

We call \(M\) the 'maximal extension' of \(M'\).

**Proof.** As in the proof of Proposition 4.1, we argue that it is sufficient to consider the case \(I = I' \cup \{\mu\}\) for some \(\mu \notin I'\). So let us assume this in the course of the proof. Let us prove that an object \(M\) having the properties (1), (2) and (3) is unique. So suppose that \(M_1\) and \(M_2\) have these properties. Then, by (1), we have an isomorphism \(M_{1I'} \cong M_{2I'}\). From Proposition 3.1, we deduce that this isomorphism identifies \(M_{1(\mu)}\) with \(M_{2(\mu)}\) and hence \(M_{1[\mu],\text{max}}\) with \(M_{2[\mu],\text{max}}\), so \(M_{1(\mu)}\) with \(M_{2(\mu)}\) by property (3). So the condition in Lemma 3.4 is satisfied, so the chosen isomorphism extends to a homomorphism \(f : M_1 \to M_2\). Reversing the roles of \(M_2\) and \(M_1\) yields a homomorphism \(g : M_2 \to M_1\) in an analogous way. Now property (2) implies that \(gof\) and \(fog\) are automorphisms. Hence, \(f\) and \(g\) are isomorphisms.

It remains to show that an object \(M\) with properties (1), (2) and (3) exists. First, we consider the minimal extension \(\tilde{M} := E^I_{I'}(M)\). Then we can identify \(\tilde{M}_{[\mu],\text{max}}\) with \(M'_{[\mu]}\). We set \(Q := \tilde{M}_{[\mu],\text{max}} / \tilde{M}_{[\mu]}\), so this is a torsion \(\mathcal{A}\)-module. By property (XI), the \(\mathcal{A}\)-module \(\tilde{M}_{[\mu]}\) is finitely generated, hence so is its submodule \(\tilde{M}_{[\mu],\text{max}}\) (recall that we always assume that \(\mathcal{A}\) is Noetherian). Hence, \(Q\) is finitely generated. Now we fix a projective cover \(\tilde{\gamma} : D \to Q\) in the category of \(\mathcal{A}\)-modules, and we denote by \(\gamma : D \to \tilde{M}_{[\mu],\text{max}}\) a lift of \(\tilde{\gamma}\). We can also consider \(\gamma\) as a homomorphism from \(D\) to \(\tilde{M}_{[\mu]}\).

We define \(M\) as follows. For all \(v \in I'\), \(\alpha \in \Pi\) and \(n > 0\) we set \(M_v := \tilde{M}_v\), \(E^M_{v,\alpha,n} := E^{\tilde{M}}_{v,\alpha,n}\) and \(F^M_{v,\alpha,n} := F^{\tilde{M}}_{v,\alpha,n}\). Then we set \(M_{[\mu]} := \tilde{M}_{[\mu]} \oplus D\) and define \(F^M_{\mu} := (F^{\tilde{M}}_{\mu}, 0)^T : M_{\delta[\mu]} = \tilde{M}_{\delta[\mu]} \to M_{\mu}\) and \(E^M_{\mu} := (E^{\tilde{M}}_{\mu}, \gamma) : M_{\mu} \to M_{\delta[\mu]} = \tilde{M}_{\delta[\mu]}\). We now show that \(M = \bigoplus_{v \in I'} M_v\) together with the
$E$- and $F$-maps above is an object in $\chi_I$. Condition (X1) is clearly satisfied. We now show that (X2) is also satisfied. Let $\nu \in I, \alpha, \beta \in \Pi, m, n > 0$ and $\nu \in M_{\gamma+n\beta}$. We need to show that

$$E_{\alpha,m}F_{\beta,n}(\nu) = \begin{cases} F_{\beta,n}E_{\alpha,m}(\nu), & \text{if } \alpha \neq \beta \\ \sum_{0 \leq r \leq \min(m,n)} \left[ \langle \nu, \alpha^\vee \rangle + m + n \right]_r F_{\alpha,n-r}E_{\alpha,m-r}(\nu), & \text{if } \alpha = \beta. \end{cases}$$

If $\nu \neq \mu$, then this follows immediately from the fact that (X2) is satisfied for $\tilde{M}$, and in the case $\nu = \mu$, it follows as $E_{\mu,\alpha,m}M$ coincides with $E_{\mu,\alpha,m}M$ on the image of $F_{\mu,\beta,n}$. We now check the condition (X3). It is satisfied for all $\nu \neq \mu$, as it is satisfied for $\tilde{M}$. In the case $\nu = \mu$, (X3a) follows from the corresponding condition for $\tilde{M}$ as the image of $F_{\mu,\beta,n}$ coincides with the image of $F_{\mu,\beta,n}$. We now check the condition (X3). It is satisfied for all $\nu \neq \mu$, as it is satisfied for $\tilde{M}$. In the case $\nu = \mu$, (X3a) follows from the corresponding condition for $\tilde{M}$ as the image of $F_{\mu,\beta,n}$ coincides with the image of $F_{\mu,\beta,n}$. We now check the condition (X3). It is satisfied for all $\nu \neq \mu$, as it is satisfied for $\tilde{M}$. In the case $\nu = \mu$, (X3a) follows from the corresponding condition for $\tilde{M}$ as the image of $F_{\mu,\beta,n}$ coincides with the image of $F_{\mu,\beta,n}$.

We need to check that $M$ satisfies the properties (1), (2) and (3). Clearly, $M_I = \tilde{M}_I \cong M'$ so (1) is satisfied. We have already observed that $M_{(\mu)} = M_{(\mu),\max}$ hence (3). Now let $f : M \to M$ be an endomorphism and suppose that $f_{I'} : M_{I'} \to M_{I'}$ is an automorphism, that is, $f_{I'} : M_{\gamma} \to M_{\gamma}$ is an automorphism for all $\nu \neq \mu$. Then $f_{\mu} : \tilde{M}_{\mu} \to \tilde{M}_{\mu}$ is an automorphism, and hence the restriction of $f_{\mu} : \text{im}F_{\mu} \to \text{im}F_{\mu}$ is an automorphism. Applying $E_{\mu}$ shows that $f_{(\mu)}$ is an automorphism of $M_{(\mu)}$. Hence, $f_{\mu}$ induces an automorphism of $M_{(\mu),\max}$ and we obtain an induced automorphism of the quotient $Q$ defined earlier in this proof. As $\gamma : D \to Q$ is a projective cover, also the induced endomorphism on $D$ must be an automorphism. Hence, $f_{\mu}$ is an automorphism, and hence so is $f$. Hence, (2) also holds.

### 7.2 The first example, continued

Recall the setup of the example discussed in Section 4.2. In the first step, we found that the homomorphism $E_{\mu} : \mathcal{A} \to \mathcal{A}$ is multiplication with $[\lambda]$. So $M_{(\mu)} = [\lambda]\mathcal{A} \subset M_{\tilde{\mu}} = \mathcal{A}$. Suppose that $[\lambda]$ does not vanish in $\mathcal{A}$. Then $M_{(\mu)} = [\lambda]\mathcal{A} \subset M_{(\mu),\max} = \mathcal{A}$. If $[\lambda]$ is not invertible in $\mathcal{A}$, then we have a strict embedding and a torsion quotient. In the notation of the above proposition, we need to choose a projective cover $\gamma : D \to M_{(\mu),\max}/M_{(\mu)}$ and obtain $M_{\mu} := \mathcal{A} \oplus D$ as the maximal extension.

### 7.3 An example in the $A_2$-case

Let $R = \{\pm \alpha, \pm \beta, \pm (\alpha + \beta)\}$ be the root system of type $A_2$ with $\Pi = \{\alpha, \beta\}$. Fix $\lambda \in X$ and set $\mu := \lambda - \alpha - \beta$. We set $I' := \{\nu \in X \mid \nu > \mu\}$ and $I := I' \cup \{\mu\}$. Then we define $M' := \bigoplus_{\nu \in I'} M'_{\nu}$, by $M'_{\lambda-\alpha} = M'_{\lambda-\beta} = M'_{\lambda} = \mathcal{A}$, and all other weight spaces are $\{0\}$. Let all $F$-maps between the non-zero weight spaces be the identity. Then the commutation relations force us to define $E_{\alpha,1} : M'_{\lambda-\alpha} \to M'_{\lambda}$ as multiplication with $[(\lambda, \alpha^\vee)]$ and $E_{\beta,1} : M'_{\lambda-\beta} \to M'_{\lambda}$ as multiplication with $[(\lambda, \beta^\vee)]$. All other $E$-maps are zero, of course. We assume that $[(\lambda, \alpha^\vee)]$ and $[(\lambda, \beta^\vee)]$ do not vanish in $\mathcal{A}$. 

We need to check that $M$ satisfies the properties (1), (2) and (3). Clearly, $M_I = \tilde{M}_I \cong M'$ so (1) is satisfied. We have already observed that $M_{(\mu)} = M_{(\mu),\max}$ hence (3). Now let $f : M \to M$ be an endomorphism and suppose that $f_{I'} : M_{I'} \to M_{I'}$ is an automorphism, that is, $f_{I'} : M_{\gamma} \to M_{\gamma}$ is an automorphism for all $\nu \neq \mu$. Then $f_{\mu} : M_{\mu} \to M_{\mu}$ is an automorphism, and hence the restriction of $f_{\mu} : \text{im}F_{\mu} \to \text{im}F_{\mu}$ is an automorphism. Applying $E_{\mu}$ shows that $f_{(\mu)}$ is an automorphism of $M_{(\mu)}$. Hence, $f_{\mu}$ induces an automorphism of $M_{(\mu),\max}$ and we obtain an induced automorphism of the quotient $Q$ defined earlier in this proof. As $\gamma : D \to Q$ is a projective cover, also the induced endomorphism on $D$ must be an automorphism. Hence, $f_{\mu}$ is an automorphism, and hence so is $f$. Hence, (2) also holds.

We need to check that $M$ satisfies the properties (1), (2) and (3). Clearly, $M_I = \tilde{M}_I \cong M'$ so (1) is satisfied. We have already observed that $M_{(\mu)} = M_{(\mu),\max}$ hence (3). Now let $f : M \to M$ be an endomorphism and suppose that $f_{I'} : M_{I'} \to M_{I'}$ is an automorphism, that is, $f_{I'} : M_{\gamma} \to M_{\gamma}$ is an automorphism for all $\nu \neq \mu$. Then $f_{\mu} : M_{\mu} \to M_{\mu}$ is an automorphism, and hence the restriction of $f_{\mu} : \text{im}F_{\mu} \to \text{im}F_{\mu}$ is an automorphism. Applying $E_{\mu}$ shows that $f_{(\mu)}$ is an automorphism of $M_{(\mu)}$. Hence, $f_{\mu}$ induces an automorphism of $M_{(\mu),\max}$ and we obtain an induced automorphism of the quotient $Q$ defined earlier in this proof. As $\gamma : D \to Q$ is a projective cover, also the induced endomorphism on $D$ must be an automorphism. Hence, $f_{\mu}$ is an automorphism, and hence so is $f$. Hence, (2) also holds.

We need to check that $M$ satisfies the properties (1), (2) and (3). Clearly, $M_I = \tilde{M}_I \cong M'$ so (1) is satisfied. We have already observed that $M_{(\mu)} = M_{(\mu),\max}$ hence (3). Now let $f : M \to M$ be an endomorphism and suppose that $f_{I'} : M_{I'} \to M_{I'}$ is an automorphism, that is, $f_{I'} : M_{\gamma} \to M_{\gamma}$ is an automorphism for all $\nu \neq \mu$. Then $f_{\mu} : M_{\mu} \to M_{\mu}$ is an automorphism, and hence the restriction of $f_{\mu} : \text{im}F_{\mu} \to \text{im}F_{\mu}$ is an automorphism. Applying $E_{\mu}$ shows that $f_{(\mu)}$ is an automorphism of $M_{(\mu)}$. Hence, $f_{\mu}$ induces an automorphism of $M_{(\mu),\max}$ and we obtain an induced automorphism of the quotient $Q$ defined earlier in this proof. As $\gamma : D \to Q$ is a projective cover, also the induced endomorphism on $D$ must be an automorphism. Hence, $f_{\mu}$ is an automorphism, and hence so is $f$. Hence, (2) also holds. 

\[ \square \]
Let $\tilde{M}$ be the minimal extension of $M'$ to the weight $\mu$. We then have $\tilde{M}_{\delta \mu} = M_{\mu + \alpha} \oplus M_{\mu + \beta} = \mathcal{A} \oplus \mathcal{A}$ with basis $(F_{\alpha,1}v, F_{\beta,1}v)$, where $v$ is a generator of $M'_\lambda$, and $\tilde{M}_\mu = \mathcal{A} \oplus \mathcal{A}$ with basis $(F_{\alpha,1}F_{\beta,1}v, F_{\beta,1}F_{\alpha,1}v)$. We calculate

$$E_{\alpha,1}F_{\alpha,1}F_{\beta,1}v = F_{\alpha,1}E_{\alpha,1}F_{\beta,1}v + [\langle \lambda - \beta, \alpha \rangle]F_{\beta,1}v$$

$$= [\langle \lambda - \beta, \alpha \rangle]F_{\beta,1}v,$$

and (symmetrically) for $E_{\beta,1}$. The homomorphism $E_\mu$ is hence given by the matrix

$$\left( \begin{array}{cc} [\langle \lambda, \alpha \rangle] & [\langle \lambda - \beta, \alpha \rangle] \\ [\langle \lambda - \beta, \beta \rangle] & [\langle \lambda, \beta \rangle] \end{array} \right).$$

Let us specialize this further. Suppose $\lambda = \rho$ and $p = 3$, and let $\mathcal{A} = \mathcal{X}_p$ be the localization of $\mathbb{Z}[v, v^{-1}]$ at the kernel of the homomorphism $\mathbb{Z}[v, v^{-1}] \to \mathbb{F}_3$ that sends $v$ to 1. The matrix then is

$$\left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right).$$

The element [2] is invertible in $\mathcal{X}_p$, and the matrix is congruent to

$$\left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right) = \left( \begin{array}{cc} -v^2 - 1 & v + v^{-1} \\ 0 & 1 \end{array} \right).$$

So its image has $v^2 + 1 + v^{-2} = v^{-2}(v^2 + v + 1)(v^2 - v + 1)$-torsion. Since $v^{-2}(v^2 - v + 1)$ is invertible in $\mathcal{X}_p$, this is $v^2 + v + 1$-torsion. (Note that $v^2 + v + 1$ is the third cyclotomic polynomial.) Hence, in this situation, the $\mathcal{A}$-module $M_{\rho - \alpha - \beta} = M_0$ is free of rank 3.

### 7.4 The category of maximal objects

Here, is an analogue of Proposition 4.3 in the maximal case.

**Proposition 7.2.** Suppose that $\mathcal{A}$ is local.

1. For all $\lambda \in X$, there exists an up to isomorphism unique object $S_{\text{max}}(\lambda)$ in $\mathcal{X}$ with the following properties.
   a. $S_{\text{max}}(\lambda)_\mu$ is free of rank 1 and $S_{\text{max}}(\lambda)_\mu \neq \{0\}$ implies $\mu \leq \lambda$.
   b. $S_{\text{max}}(\lambda)$ is indecomposable and maximal.

Moreover, the objects $S_{\text{max}}(\lambda)$ characterized in (1) have the following properties.

2. The endomorphism ring $\text{End}_{\mathcal{X}}(S_{\text{max}}(\lambda))$ is local for all $\lambda \in X$.
3. Let $S$ be a maximal object in $\mathcal{X}$. Then there is an index set $J$ and some elements $\lambda_i \in X$ for $i \in J$ such that $S \cong \bigoplus_{i \in J} S_{\text{max}}(\lambda_i)$.

Sometimes we will write $S_{\text{max}, \mathcal{A}}(\lambda)$ to incorporate the ground ring.

**Proof.** The proof follows closely the proof of the analogous proposition in the minimal case (Proposition 4.3). So we start with proving that there exists an object $S_{\text{max}}(\lambda)$ satisfying (1a), (1b) and (2) for all $\lambda \in X$. So fix $\lambda$ and set $I_\lambda := \{ \mu \in X \mid \lambda \leq \mu \}$. Let $S'$ be the object in $\mathcal{X}_{I_\lambda}$ constructed...
in Proposition 4.3 and denote by $S_{\text{max}}(\lambda)$ the maximal extension provided by Proposition 7.1. Then the construction of $S_{\text{max}}(\lambda)$ implies that its support is contained in $\{\mu \in X \mid \mu \leq \lambda\}$, and $S_{\text{max}}(\lambda)_{\lambda} = S^I_{\lambda} = \mathcal{A}$. So (1a) is satisfied. Proposition 7.1 also implies that an endomorphism $f$ of $S_{\text{max}}(\lambda)$ is an automorphism if and only if its restriction to $S_{\text{max}}(\lambda)_{I\lambda}$ is an automorphism. As $S_{\text{max}}(\lambda)$ is indecomposable. Finally, Proposition 7.1 implies that $S_{\text{max}}(\lambda)_{\mu} = S_{\text{max}}(\lambda)(\mu)$ for all $\mu \in X \setminus I_{\lambda}$. As the same identity also holds for all $\mu \in I_{\lambda}$ (as it holds for $S'$), we deduce that $S_{\text{max}}(\lambda)$ is maximal. Hence, (1b) is also satisfied.

We are now left with proving property (3) with the $S_{\text{max}}(\lambda)$ being the objects constructed above. Let $S$ be a maximal object and let $\lambda$ be a maximal weight of $S$. As in the proof of Proposition 4.3, we can fix an isomorphism $S_{I\lambda} \cong (S')^{\oplus n} \cong \tilde{S}_{I\lambda}$ with $\tilde{S} = S_{\text{max}}(\lambda)^{\oplus n}$. Since $S$ and $\tilde{S}$ are both maximal, Lemma 3.4 implies that this isomorphism extends, so there are morphisms $f : \tilde{S} \to S$ and $g : S \to \tilde{S}$ such that $(g \circ f)|_{I\lambda}$ is the identity. Using the already proven property (2) for $S_{\text{max}}(\lambda)$, we deduce that $g \circ f$ is an automorphism. Hence, $\tilde{S} = S_{\text{max}}(\lambda)^{\oplus n}$ is isomorphic to a direct summand of $S$. By construction, $\lambda$ is not a weight of a direct complement. From here, we can continue by induction to prove (3).

Note that the main ingredients in the existence result above are Propositions 4.2 and 7.1. The proofs of these propositions are constructive, that is, they can be read as an algorithm to construct the weight spaces of the objects $S_{\text{max}}(\lambda)$ inductively, starting with the highest weight space.

7.5 Objects generated by dominant weights

Let $M$ be an object in $\mathcal{A}_\mathfrak{A}$. Denote by $\tilde{M} \subset M$ the smallest $X$-graded $\mathfrak{A}$-submodule that is stable under all $F$-maps and contains $M_\nu$ for all dominant weights $\nu$. We say that $M$ is generated by dominant weights if $\tilde{M} = M$.

Lemma 7.3. Suppose that $\mathfrak{A}$ is generic and that $M$ is generated by dominant weights. Then $M$ is contained in $X^{\text{fin}}$. In particular, $R(M)$ admits a Weyl filtration.

Proof. Let $\mathcal{K}$ be the quotient field of $\mathfrak{A}$. As $M$ is generated by dominant weights, so is $M_{\mathcal{K}} \in X_{\mathcal{K}}$. By Lemma 4.4, $M_{\mathcal{K}}$ splits into a direct sum of various $S_{\mathcal{K}}(\nu_i)$ with dominant highest weights $\nu_i$. By (X1), the number of direct summands is finite. Now $S_{\mathcal{K}}(\nu_i) \cong S(L_{\mathcal{K}}(\nu_i))$ by Proposition 5.4. As $\nu_i$ is dominant, each $L_{\mathcal{K}}(\nu_i)$ is finite dimensional, hence so is each $S_{\mathcal{K}}(\nu_i)$. Hence, $M_{\mathcal{K}}$ is a finite dimensional vector space. In particular, its set of weights is finite. This equals the set of weights of $M$, so by the remark following Definition 6.8, $M$ is finitely generated as an $\mathfrak{A}$-module, hence is contained in $X^{\text{fin}}$. The last statement follows from Theorem 6.9. □

7.6 Minimal and maximal extensions and finite rank

Suppose that $\mathfrak{A}$ is generic and local.

Proposition 7.4. For all $\lambda \in X$, the following statements are equivalent.

1. $\lambda$ is dominant.
(2) $S_{\text{min}}(\lambda)$ is a finitely generated $A$-module.

(3) $S_{\text{max}}(\lambda)$ is a finitely generated $A$-module.

It follows from Theorem 6.9 that $S_{\text{min}}(\lambda)$ and $S_{\text{max}}(\lambda)$ (for dominant $\lambda$) yield, under the functor $R$, objects in $O$ that admit a Weyl filtration. In particular, they are free of finite rank as $A$-modules.

Proof. Again we denote by $K$ the quotient field of $A$. Now $S_{\text{min}}(\lambda)$ is finitely generated as an $A$-module if and only if its set of weights is finite, which is the case if and only if $S_{\text{min}}(\lambda)_K$ is a finite dimensional $K$-vector space. Under the functor $R$, the latter corresponds to $L_{\frac{\lambda}{\mu}}$ which is finite dimensional if and only if $\lambda$ is dominant. Hence, (1) and (2) are equivalent. If (3) holds, then $R(S_{\text{max}}(\lambda)_K)$ is a finite dimensional object in $O_K$. Hence, every simple subquotient has a dominant highest weight. In particular, its maximal weight $\lambda$ is dominant. Hence, (3) implies (1).

We are left with proving that $S := S_{\text{max}}(\lambda)$ is finitely generated as an $A$-module for dominant $\lambda$. So fix a dominant weight $\lambda$. We show that $S$ is generated by dominant weights in the sense defined in Section 7.5. Then our claim follows from Lemma 7.3. So let $\tilde{S} \subset S$ be the minimal $E$- and $F$-stable graded subspace that contains all $S_{\nu}$ for dominant weights $\nu$. If $\tilde{S} \neq S$, then there exists a weight $\mu$ with $\tilde{S}_\mu \neq S_\mu$. Let us choose a maximal such weight $\mu$. This $\mu$ cannot be dominant, but $\mu < \lambda$.

Now let $I := \{ \nu \in X \mid \mu < \nu \} \cup \{ \nu \in X \mid \nu \notin \lambda \}$. This is a closed subset of $X$ and it does not contain $\mu$. Moreover, $X \setminus I \subset \{ \leq \lambda \}$ is bounded from above. Now set $C := E_I^X(S_{\text{max}}(\lambda)_I)$. By the maximality of $\mu$, $S_{\text{max}}(\lambda)_I$ and hence $C$ is generated by dominant weights. So Lemma 7.3 implies that $R(C)$ admits a Weyl filtration. In particular, $\text{im} F^C_{\mu}$ and $C_{\delta \mu}$ are free $A$-modules (of finite rank).

Now $\tilde{S}_{\mu} \neq S_{\mu}$ implies, by the construction of $S = S_{\text{max}}(\lambda)$, that the cokernel of $E_{\mu}^S \lim F^S_{\mu} : \text{im} F^S_{\mu} \to S_{\tilde{S}_{\mu}}$ has non-vanishing torsion. But this homomorphism coincides with $E_{\mu}^C \lim F^C_{\mu} : \text{im} F^C_{\mu} \to C_{\delta \mu}$ as $C_I = S_I$. This implies that $E_{\mu}^C \lim F^C_{\mu} : \text{im} F^C_{\mu} \to C_{\delta \mu}$ is not injective over the residue field $F$ of $A$. From this, we deduce that there exists a primitive vector of weight $\mu$ in the $U_F$-module $R(C)$. As the $U$-$A$-module $R(C)$ admits a Weyl filtration, there must be a Weyl module having a primitive vector of weight $\mu$ when reduced to $F$. But this implies that $\mu$ is dominant, which contradicts our assumption.

\textbf{8 \quad CONTRAVARIANT FORMS}

In order to connect maximal objects in $X$ to tilting modules in $O$, we need to add another ingredient to the theory: contravariant forms. The main result of this section is that there exists a ‘non-degenerate’ contravariant form on the maximal object $S_{\text{max}}(\lambda)$, provided that each weight space $S_{\text{max}}(\lambda)_\mu$ is a free $A$-module. Using the results of the previous section, we know that this is the case if $\lambda$ is a dominant weight (probably this restriction is not necessary). In the next section, we show that the existence of a non-degenerate contravariant form on $S_{\text{max}}(\lambda)$ implies that $R(S_{\text{max}}(\lambda))$ is a tilting module.

\textbf{8.1 \quad Contravariant forms}

Let $A$ be a Noetherian unital $\mathcal{X}$-algebra that is a domain. Let $I$ be a closed subset of $X$ and $M$ an object in $X_I$. Let $b : M \times M \to A$ be an $A$-bilinear form.
Definition 8.1. We say that $b$ is a symmetric contravariant form on $M$ if the following holds.

1. $b$ is symmetric.
2. $b(m, n) = 0$ if $m \in M_{\mu}$ and $n \in M_{\nu}$ and $\mu \neq \nu$.
3. For all $\mu \in I$, $\alpha \in \Pi$, $n > 0$, $x \in M_{\mu}$, $y \in M_{\mu+n\alpha}$, we have
   \[ b(E_{\mu,\alpha,n}(x), y) = b(x, F_{\mu,\alpha,n}(y)). \]

Property (2) implies that every symmetric contravariant form $b$ splits into the direct sum $\bigoplus_{\mu \in I} b_{\mu}$ of its weight components. We write $b_{\delta \mu}$ for the restriction of $b$ to $M_{\delta \mu} \times M_{\delta \mu}$. Note that if $b$ satisfies (1) and (2), then property (3) is equivalent to $b_{\mu}(x, F_{\mu}(y)) = b_{\delta \mu}(E_{\mu}(x), y)$ for all $\mu \in I$, $x \in M_{\mu}$, $y \in M_{\delta \mu}$.

8.2 | Non-degeneracy and torsion vanishing

Recall that a symmetric bilinear form $b : N \times N \to \mathcal{A}$ on a finitely generated $\mathcal{A}$-module $N$ is called 'non-degenerate' if the induced homomorphism $N \to N^* := \text{Hom}_{\mathcal{A}}(N, \mathcal{A})$, $n \mapsto b(n, \cdot)$, is an isomorphism. If $\mathcal{A}$ is a local algebra with residue field $\mathcal{F}$, and $N$ is a free $\mathcal{A}$-module, then $b$ is non-degenerate if and only if its specialization $\overline{b}$ on $\overline{N} := N \otimes_\mathcal{A} \mathcal{F}$ is non-degenerate.

A symmetric contravariant bilinear form $b$ on an object $M$ in $\mathcal{X}_I$ is non-degenerate if and only if the weight components $b_{\mu}$ are non-degenerate for all $\mu \in I$. We denote by $\text{rad } b = \{ n \in M \mid b(n, m) = 0 \text{ for all } m \in M \}$ the radical of $b$. Note that $\text{rad } b = 0$ if $b$ is non-degenerate, but the converse statement is not true in general. If $\mathcal{A} = \mathcal{X}$ is a field, and each $M_{\mu}$ is finite dimensional, then non-degeneracy is equivalent to the vanishing of the radical.

In order to study contravariant forms, the following quite general result will be helpful for us. It holds for any commutative ring $\mathcal{A}$.

Lemma 8.2. Let $S$ and $T$ be $\mathcal{A}$-modules and assume that $T$ is projective as an $\mathcal{A}$-module. Let $F : S \to T$ and $E : T \to S$ be homomorphisms. Suppose that $b_S : S \times S \to \mathcal{A}$ and $b_T : T \times T \to \mathcal{A}$ are symmetric, non-degenerate bilinear forms such that
   \[ b_T(F(s), t) = b_S(s, E(t)) \]
for all $s \in S$ and $t \in T$. If the inclusion $i_F : F(S) \subset T$ splits, then the inclusion $i_E : E(T) \subset S$ splits as well.

Proof. For $s \in \text{ker } F$, we have $0 = b_T(F(s), t) = b_S(s, E(t)) = 0$ for all $t \in T$. Hence, we can define a homomorphism $\phi : E(T) \to F(S)^*$ by setting $\phi(E(t))(F(s)) = b_S(s, E(t))$ (i.e. this definition does not depend on the choice of $s$ in the preimage of $F(s)$). Then the right hand side of the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{E} & E(T) & \xrightarrow{i_E} & S \\
\downarrow{b_T} & & \downarrow{\phi} & & \downarrow{b_S} \\
T^* & \xrightarrow{i_F^*} & F(S)^* & \xrightarrow{F^*} & S^*
\end{array}
\]
commutes. By the adjointness property, \( \phi(E(t))(F(s)) = b_T(F(s), t) \) for all \( s \in S \) and \( t \in T \), hence also the left hand side commutes. The vertical homomorphisms on the left and the right are isomorphisms (as \( b_S \) and \( b_T \) are supposed to be non-degenerate). As \( i_E \) is injective, \( \phi \) is injective. Suppose that \( i_F : F(S) \subseteq T \) splits. Then the dual homomorphism \( i_F^* : T^* \rightarrow F(S)^* \) is surjective. Hence, \( \phi \) is surjective, so it is an isomorphism. As \( F(S) \) is projective (it is a direct summand of \( T \)), the surjective homomorphism \( F : S \rightarrow F(S) \) splits, and hence \( F^* : F(S)^* \rightarrow S^* \) splits. Hence, the inclusion \( i_E : E(T) \rightarrow S \) splits. \( \square \)

The following is our main application of Lemma 8.2. Let \( I \) be a closed subset of \( X \). We assume that \( \mathcal{A} \) is a unital Noetherian \( \mathcal{Z} \)-domain.

**Proposition 8.3.** Let \( M \) be an object in \( \mathcal{X}_I \) that is free as an \( \mathcal{A} \)-module. Suppose that there exists a non-degenerate symmetric contravariant form on \( M \). Then \( M \) is maximal.

**Proof.** We need to show that \( M_{(\mu)} = M_{(\mu)} \max \) for all \( \mu \in X \). This is equivalent to \( M_{\delta \mu}/M_{(\mu)} \) being torsion free for any \( \mu \). In the following, we will show that the inclusion \( M_{(\mu)} \subseteq M_{\delta \mu} \) splits for any \( \mu \), so the torsion freeness of \( M_{\delta \mu} \) implies that the quotient \( M_{\delta \mu}/M_{(\mu)} \) is torsion free as well.

Let \( b : M \times M \rightarrow \mathcal{A} \) be a non-degenerate symmetric contravariant form on \( M \). For all \( \mu \in X \), it induces symmetric, non-degenerate bilinear forms \( b_{\delta \mu} \) and \( b_{\mu} \) on the \( \mathcal{A} \)-modules \( M_{\delta \mu} \) and \( M_{\mu} \), respectively, with

\[
b_{\mu}(F_{\mu}(v), w) = b_{\delta \mu}(v, E_{\mu}(w))
\]

for all \( v \in M_{\delta \mu} \) and \( w \in M_{\mu} \). By assumption, \( M_{\mu} \) and \( M_{\delta \mu} \) are free \( \mathcal{A} \)-modules of finite rank. Moreover, by axiom (X3), the quotient \( M_{\mu}/F_{\mu}(M_{\delta \mu}) \) is a free \( \mathcal{A} \)-module. Hence, the inclusion \( F_{\mu}(M_{\delta \mu}) \subseteq M_{\mu} \) splits and we can apply Lemma 8.2 and deduce that the inclusion \( M_{(\mu)} = E_{\mu}(M_{\mu}) \subseteq M_{\delta \mu} \) splits. \( \square \)

### 8.3 Extension of contravariant forms

Let \( I' \subset I \) be closed subsets of \( X \). Suppose that \( M \) is an object in \( \mathcal{X}_I \) and \( b : M \times M \rightarrow \mathcal{A} \) is a contravariant form on \( M \). Then the restriction \( b' \) of \( b \) to \( M' \times M' \) is a contravariant form on \( M' \). We now show that contravariant forms extend uniquely to minimal extensions of objects.

**Lemma 8.4.** Let \( I' \subset I \) be a pair of closed subsets of \( X \). Let \( M' \) be an object in \( \mathcal{X}_{I'} \) and let \( b' \) be a symmetric contravariant form on \( M' \). Let \( M := E_{I'}M' \) be the minimal extension, and fix an isomorphism \( M' \cong M' \). Then there exists a unique symmetric contravariant form \( b \) on \( M \) that restricts to \( b' \) under the chosen isomorphism. Moreover, the following holds:

1. If \( \text{rad} b' = 0 \), then \( \text{rad} b = 0 \).
2. If \( \mathcal{A} = \mathcal{K} \) is a field and \( b' \) is non-degenerate, then \( b \) is non-degenerate as well.

**Proof.** Again we can assume that \( I = I' \cup \{\mu\} \) for some \( \mu \not\in I' \) by an inductive argument. As in the proof of Proposition 4.1, we construct the \( \mathcal{A} \)-module \( \hat{M}_{\mu} \) together with the homomorphisms \( \hat{F}_{\mu} : M_{\delta \mu} \rightarrow \hat{M}_{\mu} \) and \( \hat{E}_{\mu} : \hat{M}_{\mu} \rightarrow M_{\delta \mu} \). Then, by construction, \( M_{\mu} = \hat{M}_{\mu}/\ker \hat{E}_{\mu} \), and \( E_{\mu} \) and \( F_{\mu} \) are the homomorphisms induced by \( \hat{E}_{\mu} \) and \( \hat{F}_{\mu} \). Recall that \( \hat{F}_{\mu} \) is the direct sum of the
homomorphisms $\hat{F}_{\alpha,m} : M_{\mu+m\alpha} \to \hat{M}_\mu$ and that it is an isomorphism. So we can define a bilinear form $\hat{b}_\mu$ on $\hat{M}_\mu$ by setting

$$\hat{b}_\mu(\hat{F}_{\alpha,m}(x), \hat{F}_{\beta,n}(y)) := b'(x, \hat{E}_{\alpha,m}\hat{F}_{\beta,n}(y))$$

for $\alpha, \beta \in \Pi, m, n > 0, x \in M_{\mu+m\alpha}$ and $y \in M_{\mu+n\beta}$.

Let us show now that this bilinear form is symmetric. This amounts to showing that $b'(x, \hat{E}_{\alpha,m}\hat{F}_{\beta,n}(y)) = b'(y, \hat{E}_{\beta,n}\hat{F}_{\alpha,m}(x))$. We now use the definition of the homomorphism $\hat{E}_{\beta,n}$ that the reader finds in the proof of Proposition 4.1. Suppose that $\alpha \neq \beta$. Then

$$b'(x, \hat{E}_{\alpha,m}\hat{F}_{\beta,n}(y)) = b'(x, F_{\beta,n}E_{\alpha,m}(y))$$

(by definition of $\hat{E}_{\alpha,m}$)

$$= b'(F_{\alpha,m}E_{\beta,n}(x), y)$$

(by contravariance of $b'$)

$$= b'(\hat{E}_{\beta,n}\hat{F}_{\alpha,m}(x), y)$$

(by definition of $\hat{E}_{\beta,n}$)

$$= b'(y, \hat{E}_{\beta,n}\hat{F}_{\alpha,m}(x))$$

(as $b'$ is symmetric).

Now suppose that $\alpha = \beta$. Then we can write $\hat{E}_{\alpha,m}\hat{F}_{\alpha,n}(y) = \sum_{r \geq 0} c_r F_{\alpha,n-r}E_{\alpha,m-r}(y)$ with $c_r = \frac{\langle \mu, \alpha \rangle + m + n}{d_\alpha}$. It will be important later that $c_r$ depends only on $\mu, \alpha, r$ and the sum $m + n$. The contravariance of $b'$ then yields

$$b'(x, \hat{E}_{\alpha,m}\hat{F}_{\alpha,n}(y)) = b'(x, \sum_{r \geq 0} c_r F_{\alpha,n-r}E_{\alpha,m-r}(y))$$

$$= b'(\sum_{r \geq 0} c_r F_{\alpha,m-r}E_{\alpha,n-r}(x), y).$$

Now $\sum_{r \geq 0} c_r F_{\alpha,m-r}E_{\alpha,n-r}(x) = \hat{E}_{\alpha,n}\hat{F}_{\alpha,m}(x)$, again by definition (note that the coefficient $c_r$ is the same if we replace the triple $(y, m, n)$ with $(x, n, m)$!), so

$$b'(x, \hat{E}_{\alpha,m}\hat{F}_{\alpha,n}(y)) = b'(\hat{E}_{\alpha,n}\hat{F}_{\alpha,m}(x), y).$$

Using the symmetry of $b'$ one last time yields the required result. So $\hat{b}_\mu$ is symmetric.

We now need to show that $\hat{b}_\mu : \hat{M}_\mu \times \hat{M}_\mu \to A$ descends to a bilinear form on $M_\mu = \hat{M}_\mu / \ker \hat{E}_\mu$. So suppose that $y \in \ker \hat{E}_\mu$. Then $\hat{b}_\mu(\hat{F}_\mu(x), y) = b'(x, \hat{E}_\mu(y)) = 0$ for all $x \in M_{\beta_\mu}$.

As $\hat{F}_\mu$ is surjective, $y$ is contained in the radical of $\hat{b}_\mu$, and hence we obtain an induced bilinear form $b_\mu$ on the quotient $M_\mu = \hat{M}_\mu / \ker \hat{E}_\mu$. We now define the bilinear form $b$ on $M = M' \oplus M_\mu$ as the (orthogonal) direct sum of $b'$ and $b_\mu$.

Now we show that $b$ is contravariant. We need to check that

$$b(F_{\nu,\alpha,m}(x), y) = b(x, E_{\nu,\alpha,m}(y))$$

for all $\nu \in I, \alpha \in \Pi, m > 0, x \in M_{\nu+m\alpha}$, $y \in M_{\nu}$. For $\nu \neq \mu$, this follows directly from the contravariance of $b'$. In the case $\mu = \nu$, this is a direct consequence of the definition of $\hat{b}_\mu$ and the fact that $E_{\mu,\alpha,m}$ and $F_{\mu,\alpha,m}$ are induced by $\hat{E}_{\mu,\alpha,m}$ and $\hat{F}_{\mu,\alpha,m}$. 
We now show that the form $b$ is unique. But note that the contravariance forces us to have $b_\mu(F_{\alpha,m}(x), F_{\beta,n}(y)) = b'(x, E_{\alpha,m} F_{\beta,n}(y))$. As the homomorphism $F_\mu : M_{\delta \mu} \rightarrow M_\mu$ is surjective (as $M$ is the minimal extension), this shows that there is at most one extension of $b'$ to a contravariant form on $M$.

Now we prove (1). Suppose that the radical of $b'$ vanishes. Suppose that $x$ is in the radical of $b$. Then $x \in M_\mu$. The equation $0 = b_\mu(x, F_\mu(y)) = b'_\mu(E_\mu(x), y)$ shows that $E_\mu(x)$ is in the radical of $b'_\delta \mu$. The non-degeneracy of $b'_\delta \mu$ implies that $E_\mu(x) = 0$. As $F_\mu : M_{\delta \mu} \rightarrow M_\mu$ is surjective and as $E_\mu$ is injective on the image of $F_\mu$, we deduce that $x = 0$. Hence, $b_\mu$, and hence $b$, has vanishing radical.

Now note that if $\mathcal{A} = \mathcal{K}$ is a field, then a symmetric bilinear form on a finite dimensional $\mathcal{K}$-vector space is non-degenerate if and only if its radical vanishes. Hence, (1) implies that $b_\mu$ is non-degenerate if $b'$ is non-degenerate. So we deduce (2).

### 8.4 Non-degenerate extensions

Suppose that we are in the situation of Lemma 8.4 (with $I = I' \cup \{\mu\}$) and assume that the form $b'$ on $M'$ is non-degenerate. Its unique extension $b$ on $E^I_\mu(M')$ possibly is degenerate. The next result shows that in this case, we can extend $E^I_\mu(M')$ further in such a way that $b'$ admits a non-degenerate contravariant extension $b$. However, we have to assume that $E^I_\mu(M')_\mu$ is a free $\mathcal{A}$-module and that $\mathcal{A}$ is local.

**Proposition 8.5.** Suppose that $\mathcal{A}$ is local. Let $I'$ be a closed subset of $X$ and let $\mu \in X$ be such that $I := I' \cup \{\mu\}$ is also closed. Let $M'$ be an object in $\mathcal{X}_{I'}$, and let $b'$ a non-degenerate symmetric contravariant bilinear form on $M'$. Suppose that the $\mathcal{A}$-module $E^I_\mu(M')_\mu$ is free. Then there exists an object $M$ in $\mathcal{X}_I$ and a non-degenerate symmetric contravariant form $b$ on $M$ with the following properties.

1. There is an isomorphism $M_I \cong M'$ that identifies $b_I$ with $b'$.
2. An endomorphism $f$ of $M$ is an automorphism if and only if its restriction to $M_I$ is an automorphism.

**Proof.** Denote by $\tilde{b}$ the unique symmetric contravariant form on $\tilde{M} := E^I_\mu(M')$ that extends $b'$ (Lemma 8.4). If $\tilde{b}$ is non-degenerate, we set $M := \tilde{M}$ and $b := \tilde{b}$. Then property (1) holds by construction, and property (2) follows from Proposition 4.2.

Now suppose that $b$ is degenerate. Let $\mathcal{F}$ be the residue field of $\mathcal{A}$ and consider the bilinear form $\tilde{b}$ on $\overline{\tilde{M}} := \tilde{M} \otimes_{\mathcal{A}} \mathcal{F}$. This is a symmetric bilinear contravariant form on $\overline{\tilde{M}}$ and it is degenerate. As $b'$ is non-degenerate, the radical $\overline{R}$ of $\tilde{b}$ is contained in $\overline{\tilde{M}}_\mu$. Now let us fix a vector space complement to $\overline{R}$ in $\overline{\tilde{M}}_\mu$, that is, a decomposition $\overline{\tilde{M}}_\mu = \overline{D} \oplus \overline{R}$. Then the restriction of $\tilde{b}$ to $\overline{D} \times \overline{D}$ is non-degenerate. By assumption, $\overline{\tilde{M}}_\mu$ is a free $\mathcal{A}$-module, so we can choose a lift $\overline{\tilde{M}}_\mu = D \oplus R$ of the former decomposition with free $\mathcal{A}$-modules $D$ and $R$ (of finite rank, as $\overline{\tilde{M}}_\mu$ is of finite rank). Now let $S$ be a free $\mathcal{A}$-module of the same rank as $R$, and fix a non-degenerate bilinear pairing $c : R \times S \rightarrow \mathcal{A}$. Let $b_\mu$ be the bilinear form on $\overline{\tilde{M}}_\mu \oplus S = D \oplus R \oplus S$ defined by

$$b_\mu(d_1 + r_1 + s_1, d_2 + r_2 + s_2) = \tilde{b}_\mu(d_1 + r_1, d_2 + r_2) + c(r_1, s_2) + c(r_2, s_1).$$
where $d_i \in D$, $r_i \in R$, $s_i \in S$ for $i = 1, 2$. This is a symmetric bilinear form. We claim that it is non-degenerate. Let $\bar{b}_\mu$ be the specialization of $b_\mu$ to $\bar{D} \oplus \bar{R} \oplus \bar{S}$. It agrees with $\bar{b}_\mu$ on $\bar{D} \oplus \bar{R}$. Recall that $\bar{R}$ is the radical of $\bar{b}$, so $\bar{b}_\mu(r, d) = 0$ for $r \in \bar{R}$ and $d \in \bar{D}$. Hence, we have

$$\bar{b}_\mu(d_1 + r_1 + s_1, d_2 + r_2 + s_2) = \bar{b}_\mu(d_1, d_2) + \bar{c}(r_1, s_2) + \bar{c}(r_2, s_2)$$

for $d_i \in \bar{D}$, $r_i \in \bar{R}$, $s_i \in \bar{S}$ for $i = 1, 2$. Hence, $\bar{b}_\mu$ splits into the non-degenerate bilinear form $\bar{b}_\mu$ on $\bar{D} \times \bar{D}$ and the non-degenerate bilinear form on $(\bar{R} \oplus \bar{S}) \times (\bar{R} \oplus \bar{S})$ induced by the non-degenerate pairing $\bar{c} : \bar{R} \times \bar{S} \to \mathcal{F}$. Hence, $\bar{b}_\mu$, and hence $b_\mu$, is non-degenerate.

Now set $M := \tilde{M} \oplus \tilde{S}$ and let $M_\mu := \tilde{M}_\mu \oplus \tilde{S}$ be its $\mu$-component. Define $F_\mu : M_\delta \mu \to M_\mu$ as the composition of $F\tilde{M}_\mu : \tilde{M}_\delta \mu = M_\delta \mu \to \tilde{M}_\mu$ and the inclusion $\tilde{M}_\mu \subset M_\mu$ of the direct summand. Define a homomorphism $E'_{\mu} : S \to M_{\delta \mu}$ in the following way. For $s \in S$, let $E'_{\mu}(s) \in M_{\delta \mu}$ be the element such that $b'(E'_{\mu}(s), y) = b_{\mu}(s, F_{\mu}(y))$ for all $y \in M_{\delta \mu}$.

We obtain an $\mathcal{A}$-linear homomorphism $E'_{\mu} : S \to M_{\delta \mu}$ and can define $E_{\mu} : = (E_{\tilde{M}_\mu} \oplus E'_{\mu})^T : M_{\mu} = \tilde{M}_\mu \oplus S \to M_{\delta \mu}$.

We now claim that the object $M$, together with the homomorphisms $F_{\mu}$ and $E_{\mu}$ that we just constructed, is an object in $\mathcal{X}_\mu$. The axiom (X1) is immediate. The axiom (X2) only involves the action of $E_{\mu}$ on the image of $F_{\mu}$. As (X2) holds for $\tilde{M}$, it also holds for $M$. The same argument implies that (X3a) holds. By construction, $M_{\mu}/\text{im} F_{\mu} = S$ is a free $\mathcal{A}$-module. Hence, we are left with axiom (X3b), that is, we have to show that the quotient $E_{\mu}(M_{\mu})/E_{\mu}(\text{im} F_{\mu})$ is a torsion $\mathcal{A}$-module. This means that we have to show that $E_{\mu}(M_{\mu})$ is contained in $M_{(\mu), \text{max}}$. As $\text{im} F_{\mu} = \text{im} F_{\tilde{M}} = \tilde{M}_{\mu} \subset M_{\mu}$, we have $M_{(\mu), \text{max}} = \tilde{M}_{(\mu), \text{max}}$. As $E_{\mu}(M_{\mu}) = E_{\tilde{M}}(\tilde{M}_{\mu}) \subset \tilde{M}_{(\mu), \text{max}}$, it is sufficient to show that $E'_{\mu}(s)$ is contained in $\tilde{M}_{(\mu), \text{max}}$ for all $s \in S$. This means that there exists some $\xi \in \mathcal{A}$ and an element $z \in \tilde{M}_{\mu} = \text{im} F_{\tilde{M}}$ such that $\xi E'(s) = E_{\mu}(z)$. Let $\mathcal{K}$ be the quotient field of $\mathcal{A}$. We denote by $\bar{b}_{\mathcal{K}}$ the induced bilinear form on $\tilde{M}_{\mathcal{K}} = \tilde{M} \otimes_{\mathcal{A}} \mathcal{K}$. Part (3) of Lemma 8.4 shows that the form $\bar{b}_{\mathcal{K}}$ is non-degenerate. Hence, there exists an element $x \in (\tilde{M}_{\mu})_{\mathcal{K}}$ with the property that

$$\bar{b}_{\mathcal{K}}(x, F_{\mu}(y)) = b'_{\mathcal{K}}(E'_{\mu}(s), y)$$

for all $y \in M_{\delta \mu}$ (note that by definition of $E'_{\mu}$, the right hand side vanishes for $y$ in the kernel of $F_{\mu}$). A comparison with equation $(\ast)$ shows that $E_{\mu}(x) = E'_{\mu}(s)$. Now we kill denominators. Let $\xi \in \mathcal{A} \setminus \{0\}$ be such that $z := \xi x \in \tilde{M}_{\mu} \subset (\tilde{M}_{\mu})_{\mathcal{K}}$. Then $\xi E'(s) = E_{\mu}(z)$, which proves our claim.

Finally, we need to check that the bilinear form $b$ is a non-degenerate symmetric contravariant form on $M$. We have already checked that $b$ is symmetric and non-degenerate. For the contravariance, it suffices to check that $b_{\mu}(x, F_{\mu}(y)) = b_{\delta \mu}(E_{\mu}(x), y)$, as $b$ is an extension of $b'$. For $x \in \tilde{M}_{\mu}$, this follows from the contravariance of $b$, and for $x \in S$, this is ensured by the definition of $E'_{\mu}$.

So we have now constructed an object $M$ and a non-degenerate symmetric contravariant form $b$ on $M$. From the construction, property (1) immediately follows. Now we show that (2) holds. First, we make the following observation. Let $x \in \tilde{M}_{\mu}$. Then $b(x, F_{\mu}(y)) = \bar{b}(\bar{E}_{\mu}(x), y)$ and hence
the non-degeneracy of \( b \) implies that \( \overline{E}_\mu(x) = 0 \) if and only if \( x \in (\text{im} \overline{F}_\mu)^\perp \). By construction of \( b \), we have \( (\text{im} \overline{F}_\mu)^\perp = \overline{R} \subset \text{im} \overline{F}_\mu \). Now let \( f \) be an endomorphism of \( M \). If \( f \) is an automorphism, then its restriction \( f_{\mu'} \) to \( M_{\mu'} \cong M' \) is an automorphism as well. Conversely, suppose that \( f_{\mu'} \) is an automorphism. We want to show that this implies that \( f \) is an automorphism. For this, we have to show that \( f_\mu \) is an automorphism. As \( \mathcal{A} \) is local, we can equivalently show that \( \overline{f}_\mu \) is an automorphism. As \( \overline{M}_\mu \) is an \( \mathcal{F} \)-vector space of finite dimension, it suffices to show that \( \overline{f}_\mu \) is injective. So suppose that \( x \in \overline{M}_\mu \) is such that \( \overline{f}_\mu(x) = 0 \). Then \( 0 = \overline{E}_\mu \overline{f}_\mu(x) = \overline{f}_{\delta\mu} \overline{E}_\mu(x) \). Our assumption implies that \( \overline{f}_{\delta\mu} \) is an automorphism, so we deduce that \( \overline{E}_\mu(x) = 0 \). By the above, this implies that \( x \) is contained in the image of \( \overline{F}_\mu \). As \( \overline{f}_{\delta\mu} \) is an automorphism, \( \overline{f}_\mu \) restricts to an automorphism on the image of \( \overline{F}_\mu \) in \( \overline{M}_\mu \). Hence, \( x = 0 \), so \( \overline{f}_\mu \) is injective. \( \Box \)

### 8.5 Contravariant forms on maximal objects

In this section, we use Proposition 8.5 to show that there exists a non-degenerate contravariant symmetric form on \( S_{\text{max}}(\lambda) \) provided that each weight space of \( S_{\text{max}}(\lambda) \) is a free \( \mathcal{A} \)-module.

**Proposition 8.6.** Suppose that \( \mathcal{A} \) is a local ring. Let \( \lambda \in X \) and suppose that \( S_{\text{max}}(\lambda)_\mu \) is a free \( \mathcal{A} \)-module for all \( \mu \in X \). Then there exists a non-degenerate symmetric contravariant form on \( S_{\text{max}}(\lambda) \).

**Remark 8.7.** If \( \lambda \) is dominant, then the assumption of the proposition is satisfied due to the remark following Proposition 7.4.

**Proof.** For notational convenience, we set \( S := S_{\text{max}}(\lambda) \). We construct a symmetric bilinear form \( b_\mu \) on \( S_\mu \) such that the direct sum \( b := \bigoplus_{\mu \in X} b_\mu \) is a symmetric contravariant bilinear form on \( S \). First, we set \( b_\mu = 0 \) for all \( \mu \notin \lambda \) (we have \( S_\mu = 0 \) in these cases). Then we fix an arbitrary non-degenerate symmetric form on the free \( \mathcal{A} \)-module \( S_\lambda \) of rank 1. Now suppose that we have already constructed a non-degenerate symmetric contravariant form \( b' \) on \( S_{I'} \) for some closed subset \( I' \), and suppose that \( \mu \notin I' \) is such that \( I := I' \cup \{ \mu \} \) is closed as well. By construction of the maximal extension, \( E_{\mu'}(S_{I'})_\mu \) identifies with \( \text{im} F_{\mu'} \subset S_\mu \). By (X3), \( \text{im} F_{\mu'} \) is a direct summand in \( S_\mu \) and by assumption, \( S_\mu \) is a free \( \mathcal{A} \)-module. Hence, we can apply Proposition 8.5 and obtain an object \( M \) in \( \mathcal{O}_I \) and a non-degenerate contravariant form \( b \) on \( M \) such that \( M_{\mu'} \cong S_{I'} \) and \( b_{I'} \cong b' \). Part (2) of Proposition 8.5 and the fact that \( S_{I'} \) is indecomposable imply that \( M \) is indecomposable. By Proposition 8.3, \( M \) is maximal, hence it must be isomorphic to \( S_{\text{max}}(\mu)_I \) for some \( \mu \in I \). As \( M_{\mu'} \cong S_{\text{max}}(\lambda)_{I'} \), we have \( M \cong S_{\text{max}}(\lambda)_I \). So we obtain that there exists a non-degenerate symmetric contravariant form on \( S_{\text{max}}(\lambda)_I \). We continue by induction. \( \Box \)

### 9 TILTING MODULES IN \( \mathcal{O}_\mathcal{A} \)

There is also the notion of a contravariant form on representations of a quantum group. Before we come to its definition, recall that there is an antiautomorphism \( \tau \) of order 2 on \( U_\mathcal{A} \) that maps \( e_\alpha \) to \( f_\alpha \) and \( k_\alpha^{\pm1} \) to \( k_\alpha^{\pm1} \) (this is an immediate consequence of the definition of \( U_\mathcal{A} \) by generators and relations). The contravariant dual of an object \( M \) in \( \mathcal{O}_\mathcal{A} \) is given by \( dM = \bigoplus_{\mu \in X} M^{\ast}_{\mu'} \subset M^{\ast} \).
with the action of $U_{\mathcal{A}}$ twisted by the anti-automorphism $\tau$. A homomorphism $M \to dM$ is hence the same as an $\mathcal{A}$-bilinear form $b : M \times M \to \mathcal{A}$ that satisfies

- $b(x.m, n) = b(m, \tau(x).n)$ for all $x \in U_{\mathcal{A}}$, $m, n \in M$.
- $b(m, n) = 0$ if $m \in M_{\mu}$, $n \in M_{\nu}$ and $\mu \neq \nu$.

Such a form is also called a ‘contravariant form on $M$’.

Suppose that $\mathcal{A}$ is a generic $\mathcal{Z}$-algebra. Recall that a ‘tilting module’ in $\Theta_{\mathcal{A}}$ is an object $T$ such that $T$ and its dual $dT$ admit a Weyl filtration. We denote by $\Theta_{\mathcal{A}}^{\text{tilt}}$ the full subcategory of $\Theta_{\mathcal{A}}$ that contains all tilting modules.

**Theorem 9.1.** Assume that $\mathcal{A}$ is local and generic.

1. For any dominant weight $\lambda$, the object $T(\lambda) := R(S_{\max}(\lambda))$ is an indecomposable self-dual tilting module in $\Theta$, and its endomorphism ring is local.
2. The functors $R$ and $S$ induce inverse equivalences between the category of maximal objects in $\mathcal{X}^{\text{fin}}_{\mathcal{A}}$ and $\Theta^{\text{tilt}}_{\mathcal{A}}$.
3. If $T$ is a tilting module in $\Theta_{\mathcal{A}}$, then there are $\lambda_1, \ldots, \lambda_l \in X^+$ such that $T \cong T(\lambda_1) \oplus \cdots \oplus T(\lambda_l)$.

**Proof.** Proposition 8.6 shows that $S_{\max}(\lambda)$ admits a non-degenerate contravariant form, hence $R(S_{\max}(\lambda))$ admits a non-degenerate contravariant form. In particular, $R(S_{\max}(\lambda))$ is self-dual. As $\lambda$ is supposed to be dominant, $S_{\max}(\lambda)$ is an object in $\mathcal{X}^{\text{fin}}$ (cf. Proposition 7.4). Hence, $T(\lambda) = R(S_{\max}(\lambda))$ admits a Weyl filtration. As it is self-dual, it is a tilting module. Its endomorphism ring is isomorphic to the endomorphism ring of $S_{\max}(\lambda)$ as $\mathcal{A}$ is a fully faithful functor. In particular, this endomorphism ring is local, and hence $T(\lambda)$ is indecomposable. Hence, (1).

As each maximal object in $\mathcal{X}^{\text{fin}}$ is isomorphic to a direct sum of various $S_{\max}(\lambda)$ with $\lambda$ dominant (by Proposition 7.4), (1) implies that the functor $R$ maps each maximal object in $\mathcal{X}^{\text{fin}}$ to a tilting module. Conversely, let $T$ be an indecomposable tilting module in $\Theta_{\mathcal{A}}$. As $T$ admits a Weyl filtration, $S(T)$ is an object in $\mathcal{X}^{\text{fin}}$ by Theorem 6.9. Let $\lambda$ be a maximal weight of $T$. Then $T_{\lambda}$ is a free module of finite rank. Let us fix a direct sum decomposition $S(T)_{\lambda} = A \oplus B$, where $A$ is free of rank 1. By the maximality of $S_{\max}(\lambda)$ we obtain from Lemma 3.4 a morphism $\bar{f} : S(T) \to S_{\max}(\lambda)$ that maps $A$ isomorphically onto $S_{\max}(\lambda)_{\lambda}$ and $B$ to 0. Likewise, we can find a morphism $\bar{g} : S(dT) \to S_{\max}(\lambda)$ that maps $dA$ isomorphically onto $S_{\max}(\lambda)_{\lambda}$ and $dB$ to 0. Applying the functor $R$, we obtain homomorphisms $f : T \to T(\lambda)$ and $g : dT \to T(\lambda)$. We now consider the dual homomorphism $d\bar{g} : dT(\lambda) \to T$, and we fix an isomorphism $h : T(\lambda) \cong dT(\lambda)$ (this is possible by (1)). Then the composition $f \circ d \circ g \circ h$ is an endomorphism of $T(\lambda)$ that is an automorphism on the highest weight space. As the endomorphism ring of $T(\lambda)$ is local, this composition is an automorphism. Hence, $T(\lambda)$ is isomorphic to a direct summand of $T$. As $T$ was supposed to be indecomposable, we obtain $T \cong T(\lambda)$. As each direct summand of a tilting module is a tilting module again, we obtain (3) by induction. Now (3) and (1), together with the fact that $R$ and $S$ are inverse equivalences between $\mathcal{X}^{\text{fin}}$ and $\Theta^W$, yield (2). □

**A List of Notations**

The ring $\mathcal{A}$ is always supposed to be a $\mathcal{Z} := \mathbb{Z}[\nu, \nu^{-1}]$-module that is unital, Noetherian and a domain. In some parts of the article, we assume, in addition, that $\mathcal{A}$ is a local ring, or ‘generic’ (see Definition 6.1), or both. The basic datum is an $X$-graded $\mathcal{A}$-module $M = \bigoplus_{\mu \in X} M_{\mu}$ together...
with $\mathfrak{A}$-linear homomorphisms $E_{\mu,\alpha,n} : M_\mu \to M_{\mu+n\alpha}$ and $F_{\mu,\alpha,n} : M_{\mu+n\alpha} \to M_\mu$ for each simple root $\alpha$ and each positive integer $n$. Then we define the following for each $\mu \in X$:

| Symbol           | Description                                                                 |
|------------------|-----------------------------------------------------------------------------|
| $M_{\delta\mu}$  | $= \bigoplus_{\alpha \in \Pi, n > 0} M_{\mu+n\alpha}$                    |
| $E_\mu : M_\mu \to M_{\delta\mu}$ | the column vector with entries $E_{\mu,\alpha,n}$                           |
| $F_\mu : M_{\delta\mu} \to M_\mu$ | the row vector with entries $F_{\mu,\alpha,n}$                             |
| $M_{(\mu)} \subset M_{\delta\mu}$ | the image of $E_\mu : M_\mu \to M_{\delta\mu}$                           |
| $M_{(\mu)} \subset M_{(\mu)}$ | the image of $E_\mu \circ F_\mu : M_{\delta\mu} \to M_{\delta\mu}$       |
| $M_{(\mu),\text{max}} \subset M_{\delta\mu}$ | the preimage of the torsion part of $M_{\delta\mu}/M_{(\mu)}$. |

ACKNOWLEDGEMENTS
Open Access funding enabled and organized by Projekt DEAL.

JOURNAL INFORMATION
The Journal of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES
1. H. H. Andersen, BGG categories in prime characteristics, Math. Z. 301 (2022), 1481–1505.
2. H. H. Andersen, P. Polo, and K. X. Wen, Representations of quantum algebras, Invent. Math. 104 (1991), no. 1, 1–59.
3. S. Donkin, On tilting modules for algebraic groups, Math. Z. 212 (1993), 39–60.
4. P. Fiebig, Periodicity for subquotients of the modular category $\mathcal{O}$, to appear in Transform. Groups (2022).
5. J. C. Jantzen, Lectures on quantum groups, Graduate Studies in Mathematics, 6. American Mathematical Society, Providence, RI, 1996.
6. J. C. Jantzen, Representations of algebraic groups, Mathematical Surveys and Monographs 107, 2nd ed., American Mathematical Society, Providence, RI, 2003.
7. G. Lusztig, Quantum groups at roots of 1, Geom. Dedicata 35 (1990), 89–114.
8. G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, Adv. Math 70 (1998), 237–249.
9. C. M. Ringel, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, Math. Z. 208 (1991), 209–225.
10. S. Ryom-Hansen, A $q$-analogue of Kempf’s vanishing theorem, Mosc. Math. J. 3 (2003), no. 1, 173–187.