EXTENSION OF EULER LAGRANGE IDENTITY BY SUPERQUADRATIC POWER FUNCTIONS

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Abstract. Using convexity and superquadracity we extend in this paper Euler Lagrange identity, Bohr’s inequality and the triangle inequality.

1. Generalization of the triangle inequality via convexity

In [3] Theorem 1.1 inequalities related to the Euler Lagrange identity are proved on Banach space. Using the convexity of $x^p \geq 1, x \geq 0$ we prove in this section a generalization of this theorem for complex numbers, for which Bohr’s inequality is a special case. This gives us the tools to achieve the main result of Section 2. There we extend the result to the superquadratic functions $x^p \geq 2, x \geq 0$ and obtain the Euler Lagrange identity as a special case.

Theorem 1. Let $x$, $y$, $a$, $b$ be complex numbers and let $\mu$, $\nu$, $\lambda \in \mathbb{R} \setminus 0$ then

$$\frac{|x|^p}{\mu} + \frac{|y|^p}{\nu} \geq \frac{|ax + by|^p}{\lambda}$$

holds if

(i) $\mu > 0$, $\nu > 0$, $\lambda > 0$ and

$$|\lambda|^{1/(p-1)} \geq |\mu|^{1/(p-1)} |a|^q + |\nu|^{1/(p-1)} |b|^q,$$

(ii) $\mu < 0$, $\nu > 0$, $\lambda < 0$ and

$$|\lambda|^{1/(p-1)} \leq |\mu|^{1/(p-1)} |a|^q - |\nu|^{1/(p-1)} |b|^q,$$

(iii) $\mu > 0$, $\nu < 0$, $\lambda < 0$ and

$$|\lambda|^{1/(p-1)} \leq - |\mu|^{1/(p-1)} |a|^q + |\nu|^{1/(p-1)} |b|^q,$$

where $p > 1$ and $\frac{1}{s} + \frac{1}{t} = 1$.

Comment: Bohr’s inequality

$$sx^p + ty^p \geq \frac{1}{(s-1)s^{p-2}} ((s-1)x + y)^p \geq \frac{1}{2p-2} ((s-1)x + y)^p,$$

when $1 < s \leq 2$, $\frac{1}{s} + \frac{1}{t} = 1$, $p > 1$ is a special case of Theorem [3] for $a = s - 1$, $b = 1$, $\mu = \frac{1}{s}$, $\nu = \frac{1}{t}$, $\lambda = (s-1)s^{p-2}$ (see also [2]).

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We first prove a theorem similar to Theorem 1.1 in [3] but by dealing with a general integer \( n \) instead of \( n = 2 \). Our proof is completely different than the proof in [3]. It relies on the convexity of \( f(x) = x^p, \ p > 1, \ x \geq 0 \).

**Theorem 2.** Let \( x_i, \ a_i, \ i = 1, ..., n \) be complex numbers and \( p > 1, \ \frac{1}{q} + \frac{1}{p} = 1 \). Case (i): If \( \mu_1 > 0, \ i = 1, ..., n, \ \lambda > 0 \), then

\[
\left| \lambda \right|^{\frac{1}{p-1}} \geq \sum_{i=1}^{n} |\mu_i|^{\frac{1}{p-1}} |a_i|^q.
\]

Case (ii): If \( \mu_1 > 0, \ \mu_i < 0, \ i = 2, ..., n, \ \lambda > 0 \), then

\[
\left| \lambda \right|^{\frac{1}{p-1}} \leq |\mu_1|^{\frac{1}{p-1}} |a_1|^q - \sum_{i=2}^{n} |\mu_i|^{\frac{1}{p-1}} |a_i|^q.
\]

Case (iii): If \( \mu_1 < 0, \ \mu_i > 0, \ i = 2, ..., n, \ \lambda < 0 \), then

\[
\sum_{i=1}^{n} \left| x_i \right|^p \left| \frac{x_i}{\mu_i} \right| \geq \frac{\left| \sum a_i x_i \right|^p}{\lambda}
\]

where \( \lambda \) satisfies (1.4).

**Proof.** Case (i): It is obvious that it is enough to prove this case of the theorem for \( a_i, \ x_i \geq 0, \ i = 1, ..., n \) and show that here

\[
\sum_{i=1}^{n} \left| x_i \right|^p \left| \frac{x_i}{\mu_i} \right| \geq \frac{\left| \sum a_i x_i \right|^p}{\lambda}
\]

holds if

\[
\left| \lambda \right|^{\frac{1}{p-1}} \geq \sum_{i=1}^{n} \left| a_i \right|^q \mu_i.
\]

Let us consider first a more general inequality than (1.5) where instead of the function \( f(x) = x^p, \ p > 1, \ x \geq 0 \), we deal with a positive strictly increasing convex function \( f \) on \((0, \infty)\) which satisfies \( f^{-1}(AB) \geq f^{-1}(A) f^{-1}(B) \), \( A, B > 0 \). In this case we write

\[
\sum_{i=1}^{n} \frac{f(x_i)}{\mu_i} = \sum_{i=1}^{n} Q_i f \left( f^{-1} \left( \frac{f(x_i)}{\mu_i Q_i} \right) \right),
\]
and then by the convexity of \( f \) we get

\[
\sum_{i=1}^{n} Q_i f \left( f^{-1} \left( \frac{f(x_i)}{\mu_i Q_i} \right) \right) \geq \left( \sum_{j=1}^{n} Q_j \right) f \left( \frac{\sum_{i=1}^{n} Q_i f^{-1} \left( \frac{f(x_i)}{\mu_i Q_i} \right)}{\sum_{j=1}^{n} Q_j} \right).
\]

As \( f^{-1}(AB) \geq f^{-1}(A) f^{-1}(B) \) and \( f \) is increasing we get that

\[
\left( \sum_{j=1}^{n} Q_j \right) f \left( \frac{\sum_{i=1}^{n} Q_i f^{-1} \left( \frac{f(x_i)}{\mu_i Q_i} \right)}{\sum_{j=1}^{n} Q_j} \right) \geq \left( \sum_{j=1}^{n} Q_j \right) f \left( \frac{\sum_{i=1}^{n} \mu_i Q_i f^{-1} \left( \frac{1}{\mu_i Q_i} \right)}{\sum_{j=1}^{n} Q_j} \right).
\]

Therefore, from (1.7), (1.8) and (1.9) it is enough to solve the equality

\[
\left( \sum_{j=1}^{n} Q_j \right) f \left( \frac{\sum_{i=1}^{n} Q_i f^{-1} \left( \frac{1}{\mu_i Q_i} \right)}{\sum_{j=1}^{n} Q_j} \right) = f \left( \sum_{i=1}^{n} a_i x_i \right),
\]

in other words to solve

\[
Q_i f^{-1} \left( \frac{1}{\mu_i Q_i} \right) \geq a_i, \quad i = 1, \ldots, n
\]

and then insert

\[
\lambda = \left( \sum_{j=1}^{n} Q_j \right)^{-1}
\]

in order for \( \lambda \) to satisfy for given \( \mu_i > 0 \) and \( a_i \geq 0, i = 1, \ldots, n \) the inequality

\[
\sum_{i=1}^{n} f \left( \frac{x_i}{\mu_i Q_i} \right) \geq f \left( \sum_{i=1}^{n} a_i x_i \right).
\]

Repeating \( \lambda \) by

\[
\lambda > \lambda = \left( \sum_{j=1}^{n} Q_j \right)^{-1}
\]

inequality (1.12) holds too.

Now we return to deal with our function \( f(x) = x^p, p > 1, x \geq 0 \). This is a nonnegative increasing convex function for \( x \geq 0 \) and it satisfies \( f^{-1}(AB) = f^{-1}(A) f^{-1}(B) \) for \( A, B > 0 \).

Returning to the proof of (1.5) under the condition (1.6) we obtain from (1.10) that

\[
Q_i (\mu_i Q_i)^{-\frac{1}{p}} \left( \sum_{j=1}^{n} Q_j \right)^{-1} = a_i, \quad i = 1, \ldots, n.
\]
Solving (1.14) we get that

\[ Q_i = \frac{\mu_i^{\frac{1}{p-1}} a_i^q}{\left( \sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q \right)^{p-1}}, \quad i = 1, \ldots, n, \tag{1.15} \]

and from (1.11) that

\[ \lambda = \left( \sum_{i=1}^{n} Q_i \right)^{-1} = \left( \sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q \right)^{p-1}. \tag{1.16} \]

Hence from (1.13), (1.5) and (1.6) are proved when \( a_i, x_i \geq 0, i = 1, \ldots, n \) and therefore (1.1) and (1.2) are proved for the complex numbers \( x_i, a_i, i = 1, \ldots, n \).

Case (ii): If \( \mu_1 > 0, \mu_i < 0, i = 2, \ldots, n \) and \( \lambda > 0 \) we rewrite (1.3) as

\[ \left| \sum_{i=2}^{n} a_i x_i \right|^p |\lambda|^{-1} + \sum_{i=1}^{n} |x_i|^p |\mu_i| \geq |x_1|^p |\mu_1|. \tag{1.17} \]

Let us make the substitutions

\[ |\mu_i| = \nu_i, \quad i = 2, \ldots, n, \quad |\mu_1| = \Lambda, \quad |\lambda| = \nu, \]

\[ z_1 = \sum_{i=1}^{n} a_i x_i, \quad z_i = x_i, \quad i = 2, \ldots, n, \]

and

\[ x_1 = \frac{1}{a_1} z_1 + \sum_{i=2}^{n} \left( -\frac{a_i}{a_1} \right) z_i = \sum_{i=1}^{n} C_i z_i. \]

Inequality (1.17) becomes

\[ \sum_{i=1}^{n} \frac{|z_i|^p}{\nu_i} \geq \frac{|\sum_{i=1}^{n} C_i z_i|^p}{\Lambda}. \]

Therefore from Case (i) we get that

\[ \Lambda^{p-1} \geq \sum_{i=1}^{n} \nu_i^{\frac{1}{p-1}} |C_i|^q. \]

In other words (1.3) holds when

\[ |\mu_1|^{\frac{1}{p-1}} \geq |\lambda|^{\frac{1}{p-1}} |a_1|^q + \sum_{i=2}^{n} |\mu_i|^{\frac{1}{p-1}} \left| \frac{a_i}{a_1} \right|^q, \]

which is the same as (1.4).

The proof of Case (iii) follows immediately from Case (ii).

This completes the proof of the theorem. \( \square \)

**Corollary 1.** For \( n = 2 \) we get Theorem 4 which is Theorem 1.1 in [3] for complex numbers \( x_i, a_i, i = 1, \ldots, n \).
2. Extension of Euler Lagrange Type Identity

Now we extend the Euler Lagrange type inequalities by introducing the set of superquadratic functions and its basic properties. Euler Lagrange identity is a special case of this extension.

A function \( f : [0, b) \to \mathbb{R} \) is superquadratic provided that for all \( x \in [0, b) \) there exists a constant \( C_f(x) \in \mathbb{R} \) such that the inequality

\[
(2.1) \quad f(y) \geq f(x) + C_f(x)(y - x) + f(|y - x|),
\]

holds for all \( y \in [0, b) \). ([1, Definition 2.1]) The function \( f : [0, b) \to \mathbb{R} \) is subquadratic if \( -f \) is superquadratic.

According to [1, Theorem 2.2] the inequality

\[
(2.2) \quad f \left( \int h(s) d\mu(s) \right) \leq \int \left( f(h(s)) - f \left( \left| h(s) - \int h(s) d\mu(s) \right| \right) \right) d\mu(s)
\]

holds for all probability measures \( \mu \) and all nonnegative \( \mu \)-integrable \( h \), if and only if \( f \) is superquadratic.

The discrete version of (2.2) is

\[
(2.3) \quad f \left( \sum_{i=1}^{n} \alpha_i x_i \right) \leq \sum_{i=1}^{n} \alpha_i \left( f(x_i) - f \left( \left| x_i - \sum_{j=1}^{n} \alpha_j x_j \right| \right) \right),
\]

\( x_i \in [0, b), \quad \alpha_i \geq 0, \quad 1 = i, \ldots, n, \quad \sum_{i=1}^{n} \alpha_i = 1. \)

The power functions \( f(x) = x^p, x \geq 0 \), are convex and superquadratic for \( p \geq 2 \), and convex and subquadratic for \( 1 \leq p \leq 2 \). Inequalities (2.1), (2.2) and (2.3) reduce to inequalities for the function \( f(x) = x^2 \).

Now we use (2.3) in order to get the Euler Lagrange type inequality.

**Theorem 3.** Let \( x_i \geq 0, \alpha_i \geq 0, \mu_i > 0, i = 1, \ldots, n, p \geq 2, \frac{1}{p} + \frac{1}{q} = 1. \) Then

\[
(2.4) \quad \sum_{i=1}^{n} \frac{x_i^p}{\mu_i} \geq \left( \sum_{i=1}^{n} \alpha_i x_i \right)^p \left( \sum_{j=1}^{n} \frac{\mu_j^{-1} a_j^q}{\mu_j^{-1} a_j^q} \right)^{p-1} + \sum_{i=1}^{n} \frac{\mu_i^{-1} a_i^q}{\mu_i^{-1} a_i^q} \left( \left( \frac{1}{\alpha_i \mu_i} \right)^{\frac{1}{p-1}} \left( \sum_{j=1}^{n} \frac{\mu_j^{-1} a_j^q}{\mu_j^{-1} a_j^q} \right)^{\frac{1}{p-1}} \left( \sum_{j=1}^{n} \mu_j^{-1} a_j^q \right)^{1-p} \left( \sum_{j=1}^{n} \alpha_j \right)^p \right)^{p-1}.
\]

*If \( 1 < p \leq 2 \) the inverse of (2.4) holds.*

**Proof.** In Theorem 2 we showed that for \( x_i \geq 0, \alpha_i \geq 0, \mu_i > 0, i = 1, \ldots, n. \) inequalities (1.5) and (1.6) hold. There

\[
(2.5) \quad \sum_{i=1}^{n} Q_i (A_i)^p = \sum_{i=1}^{n} \left| \frac{x_i}{\mu_i} \right|^p
\]
where

\[(2.6) \quad Q_i = \frac{\mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q \right)^p}, \quad i = 1, \ldots, n,\]

\[(2.7) \quad A_i = \frac{1}{(a_i \mu_i)^{\frac{1}{p-1}}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q \right) x_i, \quad i = 1, \ldots, n\]

and

\[(2.8) \quad \frac{\sum_{i=1}^{n} Q_i A_i}{\sum_{j=1}^{n} Q_j} = \sum_{i=1}^{n} a_i x_i.\]

Therefore, as \(f(x) = x^p, \quad p \geq 2, \quad x \geq 0\) is superquadratic, (2.3) becomes by inserting (2.6)-(2.8)

\[(2.9) \quad \sum_{i=1}^{n} Q_i (A_i)^p \geq \frac{\left(\sum_{i=1}^{n} a_i x_i\right)^p}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q \right)^{p-1}} \left(\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q \right)^p x_i - \sum_{i=1}^{n} a_j x_j \right)^p.\]

Hence from (2.5) and (2.9) we get that (2.4) holds.

If \(1 < p \leq 2\) then \(f(x) = x^p, \quad x \geq 0\) is a subquadratic function, therefore the reverse of (2.4) holds. \(\Box\)

Corollary 2. In case \(n=2\) we get that

\[(2.10) \quad \frac{x^p}{\mu} + \frac{y^p}{\nu} \geq \frac{(ax + by)^p}{\left(\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q \right)^{p-1}} \left(\left(\frac{1}{a \mu}\right)^{\frac{1}{p-1}} x - \frac{ax + by}{\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q} \right)^p \right. + \left. \left(\frac{1}{\nu b}\right)^{\frac{1}{p-1}} y - \frac{ax + by}{\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q} \right)^p.\]

In particular if \(f(x) = x^2, \quad n = 2\) as Inequality (2.4) reduces to equality we get from (2.10) that

\[\frac{x^2}{\mu} + \frac{y^2}{\nu} = \frac{(ax + by)^2}{\mu^2 + \nu^2} + \frac{(\nu x - a \mu y)^2}{\mu \nu (\mu^2 + \nu^2)}.\]
which is Euler Lagrange type identity.

**Corollary 3.** From Theorem 3 as $f(x) = x^p$, $1 < p \leq 2$ is both subquadratic and convex, we get that

$$
0 \leq \sum_{i=1}^{n} \frac{x_i^p}{\mu_i} - \left( \frac{\sum a_i x_i}{\sum a_i} \right)^p \\
\leq \sum_{i=1}^{n} \frac{\frac{1}{\mu_i} a_i^q}{\frac{1}{\mu_i} a_i^q} \left( \left( \frac{1}{a_i \mu_i} \right)^{\frac{1}{p-1}} \left( \sum_{j=1}^{n} \frac{\mu_j^{\frac{1}{p-1}} a_j^q}{\mu_j^{\frac{1}{p-1}} a_j^q} \right)^{\frac{1}{p-1}} \left( x_i - \sum_{j=1}^{n} a_j x_j \right)^p. 
$$

**References**

[1] S. Abramovich, G. Jameson and G. Sinnamon, *Refining Jensen’s inequality*, Bull. Math. Sc. Math. Roum. 47 (2004), 3-14

[2] S. Abramovich, J. Baric, and J. Pecaric, *Superquadracity, Bohr’s inequality and deviation from a Mean Value*, Australian Journal of Mathematical Analysis and Applications, 7 (1), (2009), Article 1.

[3] Sin-Ei Takahasi, J. M. Rassias, S. Saitoh, and Y. Takahashi, *Refined generalization of the triangle inequality on Banach spaces*, Journal of Mathematical Inequalities and Applications, 2010.

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