Pure dephasing in a superconducting three-level system

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Abstract. The mathematical description of pure dephasing in a three-level system is not as straightforward as it is in a two-level system. Here we provide a detailed derivation of the Liouvillean for pure dephasing for a ladder three-level system, based on [J. Li et al., Phys. Rev. B 84, 104527 (2011)]. Numerical calculations based on this model give good fittings to the spectral line and Autler-Townes splitting observed in a superconducting phase qubit.

Josephson junction-based quantum circuits acting as qubits have attracted a great attention in the last decade. However, the inevitably existing higher quantum states in such systems have received less attention. Only recently phenomena such as Autler-Townes splitting, coherent population trapping, and electromagnetically induced transparency have been observed in superconducting multilevel systems [1, 2, 3, 4]. Recently the Autler-Townes effect was also studied experimentally in time-domain [5].

For two-level systems, it is known that pure dephasing is caused by longitudinal noises (i.e. along $\sigma_z$), which induces fluctuations in the transition frequency. For three-level systems we do not have a Bloch sphere picture to help our intuition. However, it is still possible to develop the theory by using the matrices $\sigma_{ij} = \langle i | j \rangle$, where $| i \rangle$ and $| j \rangle$ indicate energy levels of the system.

To get a better physical understanding of the results, let us start by clarifying the physical picture that underlies them. Imagine first that the three levels would fluctuate. It is natural to have the ground state as a reference; formally this comes about through the use of the completeness relation $\sigma_{00} = I - \sigma_{11} - \sigma_{22}$ ($I$ is Identity operator), which “propagates” the fluctuations of the ground state energy to the other two states. Indeed, this is a relatively standard procedure in the literature: for example, when discussing a model for the dephasing of the harmonic oscillator, Walls and Milburn [6] write the Liouvillean as (Eq. (15.46) p. 290)

$$\mathcal{L}_{dep} = \frac{\gamma}{2} \left[ 2 a^\dagger a \rho a^\dagger a - (a^\dagger a)^2 \rho - \rho (a^\dagger a)^2 \right].$$

(1)

Clearly there is no ground-state dephasing. Also, the expressions for dephasing in three-level atoms normally do not include the dephasing of the ground state (see Eq. (9) of [7] and Eq. (5) of [8]).

We consider the ladder three-level Hamiltonian studied in Appendix B of [10]

$$H_S = \hbar [\omega_{10}|1\rangle\langle 1| + (\omega_{10} + \omega_{21})|2\rangle\langle 2|],$$

(2)

with $| i \rangle \leftrightarrow | j \rangle$ transition frequency $\omega_{ij}$, and terms

$$H_F(t) = \hbar [\delta \omega_1(t)|1\rangle\langle 1| + \delta \omega_2(t)|2\rangle\langle 2|]$$

(3)
describing random fluctuations of transition frequencies. Written in matrix form, the total Hamiltonian reads

\[ H(t) = H_S + H_F(t) = \hbar \begin{bmatrix} 0 & 0 & 0 \\ 0 & \omega_{10} + \delta \omega_1(t) & 0 \\ 0 & 0 & \omega_{10} + \omega_{21} + \delta \omega_2(t) \end{bmatrix}. \tag{4} \]

In the interaction picture, the dynamics of this system’s density operator, \( \rho \), evolves according to the Schrödinger equation (a perturbation expansion to the second order in fluctuations)

\[ i \hbar \dot{\rho} = \left[ \hat{H}_F(t), \rho(t) \right] \approx \left[ \hat{H}_F(t), \rho(0) \right] + \frac{1}{\hbar} \int_0^t dt' \left[ \hat{H}_F(t'), \rho(t') \right], \tag{5} \]

where \( \hat{H}_F(t) = \exp(iH_S t/\hbar) \rho \exp(-iH_S t/\hbar) \).

Eq. (5) can be rewritten in the following form (an approximation \( \hat{\rho}_{jk}(t') \approx \hat{\rho}_{jk}(t) \) is made, see [9] or Appendix A of [10])

\[ \hat{\rho}(t) \approx -\frac{i}{\hbar} \left[ \hat{H}_F(t), \hat{\rho}(0) \right] - \frac{1}{2} \begin{bmatrix} 0 & \hat{\rho}_{01}(t) \Gamma_1^c & \hat{\rho}_{02}(t) \Gamma_2^c \\ \hat{\rho}_{10}(t) \Gamma_1^c & 0 & \hat{\rho}_{12}(t) \Gamma_1^c + \hat{\rho}_{21}(t) \Gamma_2^c \\ \hat{\rho}_{20}(t) \Gamma_2^c & 0 & \hat{\rho}_{22}(t) \end{bmatrix} \Gamma_1^c + \hat{\rho}_{12}(t) \Gamma_1^c + \hat{\rho}_{21}(t) \Gamma_2^c - S_{12} - S_{21}, \tag{6} \]

where the pure dephasing rate

\[ \Gamma_1^{c(2)} = \frac{2}{\hbar^2} \int_0^t dt'(\delta \omega_{1(2)}(t) \delta \omega_{1(2)}(t')), \tag{7} \]

and the cross spectral density

\[ S_{12} = \frac{2}{\hbar^2} \int_0^t dt'(\delta \omega_{1}(t) \delta \omega_{2}(t')) = S_{21}. \tag{8} \]

Note \( \varepsilon \equiv S_{12} + S_{21} \) is real, and the dephasing terms in Eq. (6) can be rewritten as

\[ L_{\text{dep}}[\rho] = \sum_{j=1,2} \frac{\Gamma_j^c}{2} (2\sigma_{jj}\rho \sigma_{jj} - \sigma_{jj}\rho - \rho \sigma_{jj}) + \varepsilon (\sigma_{11}\rho \sigma_{22} + \sigma_{22}\rho \sigma_{11}). \tag{9} \]

By transforming it back to the Schrödinger picture and taking the energy relaxation into account, the modified Liouvillian reads (see also Eq. (29) of [10])

\[ L[\rho] = \frac{1}{2} \begin{bmatrix} 2\Gamma_{10}\rho_{11} & -\lambda_{10}\rho_{01} & -\lambda_{21}\rho_{02} \\ -\lambda_{10}\rho_{10} & -2\Gamma_{10}\rho_{11} + 2\Gamma_{21}\rho_{22} & -(\lambda_{10} + \lambda_{21} - \varepsilon)\rho_{12} \\ -\lambda_{21}\rho_{20} & -(\lambda_{10} + \lambda_{21} - \varepsilon)\rho_{21} & -2\Gamma_{21}\rho_{22} \end{bmatrix}, \tag{10} \]

where \( \Gamma_{ij} \) denotes interlevel relaxation rate, and \( \lambda_{ij} = \Gamma_{ij} + \Gamma_i^c \).
Figure 1. Numerical fitting of $|1\rangle \rightarrow |2\rangle$ spectral line (from FIG. 1(c) of [2]) with different values of $\varepsilon$. $\Gamma_{10} = 2\pi \times 7$ MHz, $\Gamma_{21} = 2\pi \times 11$ MHz, $\Gamma_{1}^\phi = 2\pi \times 7$ MHz and $\Gamma_{2}^\phi = 2\pi \times 16$ MHz are used.

Figure 2. Numerical fitting of one Autler-Townes splitting (from FIG. 3(a) of [2]) with different values of $\varepsilon$. $\Gamma_{10} = 2\pi \times 7$ MHz, $\Gamma_{21} = 2\pi \times 11$ MHz, $\Gamma_{1}^\phi = 2\pi \times 7$ MHz and $\Gamma_{2}^\phi = 2\pi \times 16$ MHz are used.
Eq. (10) is very similar to Eq. (5) of [8], except that there are extra terms with $\varepsilon$. Then one question to ask is whether we can neglect these terms. In the case of uncorrelated fluctuations $\langle \delta \omega_1(t) \delta \omega_2(t') \rangle = 0$, $\varepsilon$ vanishes due to its definition (for this reason let us call the terms with $\varepsilon$ correlated dephasing terms). For nonvanishing $\varepsilon$, we try to estimate its effect by fitting the experimental data numerically with various values of $\varepsilon$ (dotted curves in Fig. 1 and Fig. 2).

The numerical fittings indicate that the effect of correlated dephasing is small, although finite $\varepsilon$ seems to provide certain improvement. This is because $\lambda_{10} + \lambda_{21}$ is much larger than $\varepsilon$. Even in the (absurdly) worst-case scenario for our model, in which $\delta \omega_1$ and $\delta \omega_2$ would be perfectly correlated, $\varepsilon = 2\sqrt{\Gamma_1 \Gamma_2}$, and $\lambda_{10} + \lambda_{21} - \varepsilon$ is still larger than the sum of energy relaxation rates $\Gamma_{10} + \Gamma_{21}$.

In conclusion, for a three-level system it should be safe to neglect the correlated dephasing terms for simplicity, and only use

$$L_{dep}[\rho] \approx \sum_{j=1,2} \frac{\Gamma_j}{2} (2\sigma_{jj}\rho\sigma_{jj} - \sigma_{jj}\rho - \rho\sigma_{jj}) ,$$

(11)

as pure dephasing Liouvillian. For superconducting phase qubit, the correlated fluctuations can be due to variations of a qubit parameter, and the first suspect is the noise in dc flux bias [11]. Reducing these fluctuations is largely an experimental problem: it depends on the filtering of biasing lines, on the magnetic field isolation of the sample holder, etc. In contrast, uncorrelated fluctuations are due to random Stark shifts, which are energy-conserving virtual processes [9]. These processes cannot be easily reduced by better control of the electronics.

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References
[1] Baur M, Filipp S, Bianchetti R, Fink J M, Göppl M, Steffen L, Leek P J, Blais A and Wallraff A 2009 Phys. Rev. Lett. 102 243602
[2] Sillanpää M A, Li J, Cicak K, Altomare F, Park J I, Simmonds R W, Paraoanu G S and Hakonen P J 2009 Phys. Rev. Lett. 103 193601
[3] Kelly W R, Dutton Z, Schlafer J, Mookerji B, Ohki T A, Kline J S and Pappas D P 2010 Phys. Rev. Lett. 104 163601
[4] Abdulmalikov Jr A A, Astafiev O, Zagoskin A M, Pashkin Y A, Nakamura Y and Tsai J S 2010 Phys. Rev. Lett. 104 193601
[5] Li J, Paraoanu G S, Cicak K, Altomare F, Park J I, Simmonds R W, Sillanpää M A and Hakonen P J 2012 Sci. Rep. 2 645
[6] Walls D F and Milburn G J 2008 Quantum Optics (Berlin: Springer-Verlag)
[7] Fleischhauer M, Imamoglu A and Marangos J P 2005 Rev. Mod. Phys. 77 633-673
[8] Raitzsch U, Heidemann R, Weimer H, Butscher B, Kollmann P, Löw R, Büchler H P and Pfau T 2009 New J. Phys. 11 055014
[9] Carmichael H 1993 An Open Systems Approach to Quantum Optics (Berlin: Springer-Verlag)
[10] Li J, Paraoanu G S, Cicak K, Altomare F, Park J I, Simmonds R W, Sillanpää M A and Hakonen P J 2011 Phys. Rev. B 84, 104527
[11] Martinis J M, Nam S, Aumentado J and Lang K M 2003 Phys. Rev. B 67 094510