Convergence of Deep Fictitious Play for Stochastic Differential Games

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Abstract

Stochastic differential games have been used extensively to model agents’ competitions in Finance, for instance, in P2P lending platforms from the Fintech industry, the banking system for systemic risk, and insurance markets. The recently proposed machine learning algorithm, deep fictitious play, provides a novel efficient tool for finding Markovian Nash equilibrium of large $N$-player asymmetric stochastic differential games [J. Han and R. Hu, Mathematical and Scientific Machine Learning Conference, 2020]. By incorporating the idea of fictitious play, the algorithm decouples the game into $N$ sub-optimization problems, and identifies each player’s optimal strategy with the deep backward stochastic differential equation (BSDE) method parallelly and repeatedly. In this paper, under appropriate conditions, we prove the convergence of deep fictitious play (DFP) to the true Nash equilibrium. We can also show that the strategy based on DFP forms an $\epsilon$-Nash equilibrium. We generalize the algorithm by proposing a new approach to decouple the games, and present numerical results of large population games showing the empirical convergence of the algorithm beyond the technical assumptions in the theorems.

Keywords: Deep fictitious play, convergence analysis, stochastic differential games, Markovian Nash equilibrium, backward stochastic differential equations.

1 Introduction

Deep neural network has become a popular and powerful tool in scientific computing, for its remarkable performance in approximating high-dimensional functions. Its success has brought natural applications in stochastic differential games, where high-dimensional optimization problems and/or stochastic differential equations are solved to provide the modeling and analysis of tactical interactions among multiple decision-makers in the context of a random dynamical system. These decision-makers, usually referred to as players or agents, can interact in a manner ranging from completely non-cooperative to completely cooperative. The nature of uncertainty makes stochastic differential games appropriate to be used for the study of competitions in Finance, e.g., in P2P lending platforms [61, 43] from the Fintech industry and insurance markets [63, 5, 15].

For non-cooperative stochastic differential games, a core problem is to compute the associated Nash equilibrium, which refers to a set of strategies so that when applied, no player will profit from unilaterally changing her own choice. When the games involve heterogeneous agents of moderate size, e.g., $5 \leq N \leq 100$, computing the Nash equilibrium becomes numerically challenging since conventional numerical algorithms lose their efficiency for $N$ beyond 5, and the mean-field framework has not started to work well with $N \leq 100$ asymmetric players.

To address the challenge, the authors have recently proposed the deep fictitious play (DFP) algorithms [27], providing a novel efficient tool for finding Markovian Nash equilibrium of large $N$-player asymmetric stochastic differential games. However, despite the efficient performance in simulation, the algorithm’s
theoretical foundation is still lacking, which will be the focus of this paper. In addition, we generalize the previous algorithms, and propose a general two-step scheme. The first step aims to recast the game into \( N \) sub-problems that will be repeatedly solved. The desired algorithm requires that, after the recast, the sub-problems are decoupled among different players given the previous stage’s solutions, and that their solutions converge to the true Nash equilibrium. Specifically, we propose two options for the first step:

I. Fictitious play. This approach was used in [27], assuming that players are myopic and will choose their best responses against others’ previous stage action at every subsequent stage. Therefore each player still faces a nonlinear optimization problem.

II. Policy update. This approach calculates the game values using all responses from the previous stage, and the current stage responses are determined as if they are the optimizers of the calculated game values.

The second step of the DFP algorithm aims to solve the sub-problems efficiently and accurately. Remark that, due to the large number of players and the high dimensionality of the controlled state process, each sub-problem may still be high-dimensional after the decoupling step. In [27], the Deep BSDE method was employed for each sub-problem, which presents excellent performance. It relies on the BSDE representations of semi-linear partial differential equations (PDEs) and deep learning approximations after discretizing the BSDE by an Euler scheme. It parametrizes the initial position of the backward process and the adjoint process by DNNs, then simulates both processes in a forward manner, aiming to minimize the discrepancy between the terminal value of the backward process and its network approximation. The analysis for the second step shall focus on this method. Meanwhile, we remark that other deep neural networks (DNNs) based algorithms, such as deep learning backward dynamic programming (DBDP) method [36] and deep Galerkin method [60], are also promising choices for solving sub-problems.

**Related literature.** The theoretical study of differential games was initiated by R. Isaacs in the early 1960s [38]. Later on, to better describe read world’s uncertainties, noises are added to the state of the system, and stochastic differential games now have been intensively used across many disciplines. Domains of applications include management science (e.g., operations management, marketing, finance, systemic risk), economics (e.g., industrial organization, environmental and macroeconomics, production of exhaustible resources), social science (e.g., networks, crowd behavior, congestion), biology (e.g., flocking), and military (e.g., cyber-attacks).

Fictitious play is well documented in the economics literature, as a learning process for finding Nash equilibria. It was firstly proposed by [9, 10] for normal-form games. Since then, there have been extensive studies on the convergence of fictitious play or its variation under different setting, for instance, see [47, 49, 42, 32, 6]. For stochastic differential games, besides [27], the most related work is [34], where fictitious play is used to design numerical algorithms for finding open-loop Nash equilibria. We remark that, the idea of fictitious play is not limited to study the games with a moderate number of heterogeneous players [34, 27], but has also been applied in mean-field games, e.g., some later works [11, 8, 19] in the community of financial mathematics.

The proposed policy update for the first step of the DFP algorithm closely follows policy iteration (PI) in spirit, which was initially introduced by Howard [33] for discounted Markovian decision problems (MDP). It consists of two steps: policy evaluation (obtaining the expected reward for a given policy) and policy improvement (updating the policy using the rewards for successor states). PI was later generalized to modified PI in [59], and has remained as the method of choice in designing reinforcement learning algorithms, e.g., see [24, 57] and the references therein.

The literature of using DNNs for learning high-dimensional function is rich, including methods for solving high-dimensional parabolic PDEs and BSDEs (e.g., the deep BSDE method [17, 28], the DBDP [36, 21], and many others [60, 3, 4, 56, 62, 39]). It also yields algorithms for solving the Schrödinger equation [31, 55, 30], stochastic control problems [26, 50], mean field games [14, 1] and nonlinear optimal stopping problems [35].

**Main contribution.** The contribution of this paper consists of the following: 1. We propose a general two-step scheme that extends the original deep fictitious play algorithm [27], and provide two options for solving the first step. The proposed algorithm can efficiently solve stochastic differential games with
heterogeneous agents of large size (e.g., $5 \leq N \leq 100$), and the presence of common noise. We provide the theoretical foundation for the proposed algorithms. In specific, with small time duration, we prove that the solutions to the decoupled sub-problems, if solved repeatedly and exactly at each stage, converge to the true Nash equilibrium; that the numerical solutions to each sub-problem tend to be exact as we refine the time step in the Euler scheme; and that the strategy based on numerical solutions forms an $\epsilon$-Nash equilibrium, after running sufficiently many stages and using sufficiently fine time step. We present numerical results showing empirical convergence even beyond the technical assumptions used in the theorems.

The rest of this paper is organized as follows. In Section 2, we give the mathematical formulation of general $N$-player asymmetric stochastic games in continuous time. The algorithms consisting of the decoupling step and sub-problem-solving step via deep learning are detailed in Section 3. Section 4 provides convergence analysis for the proposed algorithms, followed by numerical examples presented in Section 5. We make conclusive remarks in Section 6.

2 Mathematical Formulation

Throughout the paper, we shall use the following notations:

- A boldface character with a superscript $i$ refers to a collection of objects from all players;
- A regular character with a superscript $i$ refers to an objective from player $i$ (no matter a scalar or a vector) or the $i^{th}$ column of a vector;
- A boldface character with a superscript $-i$ refers to a collection of objects from all players except $i$;
- The state process $X_t$ introduced below is a common process to all players, and will always be in boldface.

We consider a general $N$-player non-zero-sum stochastic differential games. An $\mathbb{R}^n$-valued common state process $X_t$ is controlled by a Markovian strategy/policy\footnote{Hereafter, we shall use strategy and policy interchangeably.} $\alpha$:

$$dX_t^\alpha = b(t, X_t^\alpha, \alpha(t, X_t^\alpha)) \, dt + \Sigma(t, X_t^\alpha) \, dW_t, \quad X_0 = x_0,$$

where $\alpha = (\alpha^1, \ldots, \alpha^N)$ is the collection of all players’ $\mathcal{A}^i$-valued strategies. For simplicity, we assume $\mathcal{A}^i = \mathbb{R}^{d_{\alpha}}$ for $i = 1, 2, \ldots, N$. If not (e.g., some boundedness constraints are put on $\alpha^i$), we can assume there exist Lipschitz mappings $P_{\alpha}^i$ from $\mathbb{R}^{d_{\alpha}}$ to $\mathcal{A}^i$ so that $\mathcal{A}^i = P_{\alpha}^i(\mathbb{R}^{d_{\alpha}})$, and all the statements below hold easily with the help of the Lipschitz mappings. The drift and diffusion coefficients $b$ and $\Sigma$ are deterministic functions of the common state, $b: [0,T] \times \mathbb{R}^n \times \mathcal{A} \rightarrow \mathbb{R}^n$, $\Sigma: [0,T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times k}$, where $\mathcal{A} = \otimes_{i=1}^N \mathcal{A}^i = \mathbb{R}^{d_{\alpha}}$ is the space for the joint control $\alpha$, and $W$ is a $k$-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$.

Player $i$ aims at minimizing her expected total cost:

$$\inf_{\alpha^i \in \mathcal{A}^i} \mathbb{E} \left[ \int_0^T f^i(s, X_s^\alpha, \alpha(s, X_s^\alpha)) \, ds + g^i(X_T^\alpha) \right]$$

by choosing $\alpha^i$ among all admissible strategies $\mathcal{A}^i$:

$$\mathcal{A}^i = \{ \alpha^i(t, x) : \text{Borel measurable function } [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{d_{\alpha}} \},$$

where the running cost $f^i: [0,T] \times \mathbb{R}^n \times \mathcal{A} \rightarrow \mathbb{R}$ and the terminal cost $g^i: \mathbb{R}^n \rightarrow \mathbb{R}$ are deterministic measurable functions. Obviously, the quantity in (2) is also affected by other players’ strategies $\alpha^j$. To emphasize this dependence, we introduce the notation $J_t^i(\alpha^1, \ldots, \alpha^N)$ for the cost of player $i$ starting at $t$ when players choose their strategies $(\alpha^1, \ldots, \alpha^N)$:

$$J_t^i(\alpha^1, \ldots, \alpha^N) \equiv J_t^i(\alpha) := \mathbb{E} \left[ \int_t^T f^i(s, X_s^\alpha, \alpha(s, X_s^\alpha)) \, ds + g^i(X_T^\alpha) \right].$$
In the following sections, we shall present the algorithms for solving the above game and prove its theoretical convergence. In particular, we are interested in finding the Markovian Nash equilibrium (or the Markovian $\epsilon$-Nash equilibrium).

**Definition 2.1.** A Markovian $\epsilon$-Nash equilibrium is a tuple $\alpha^\epsilon = (\alpha^1, \ldots, \alpha^N, \epsilon) \in \mathcal{A}$, such that, for non-negative $\epsilon$,
\[
\forall i \in \mathcal{I}, \text{ and } \alpha^i \in \mathcal{A}^i, \quad J^i_0(\alpha^\epsilon) - \epsilon \leq J^i_0(\alpha^{1,\epsilon}, \ldots, \alpha^{i-1,\epsilon}, \alpha^i, \alpha^{i+1,\epsilon}, \ldots, \alpha^{N,\epsilon}).
\]

A Markovian Nash equilibrium, denoted by $\alpha^\epsilon$, is equivalent to an $\epsilon$-Nash equilibrium where $\epsilon = 0$. Here $\mathcal{A} = \bigotimes_{i=1}^N \mathcal{A}^i$ is the product space of $\mathcal{A}^i$, and $\mathcal{I} = \{1, 2, \ldots, N\}$ is the set of all players.

As discussed in [27], the formulation (1)–(2) is less restrictive than the usual case where player $i$ can only control her private state. Here, a common state $X^i_t$ is controlled by all agents, which is a common feature in economics literature (see e.g., [16, 58, 44]). Therefore, it is important to include it in our framework, although this will increase the coupling and make the problem harder to solve, both theoretically and numerically. Remark that the difficulty still persists in the limiting problem as $N \to \infty$ with indistinguishable players, when allowing $\alpha^i$ entering into others’ states. This is called the extended mean-field game and it has attracted certain attention recently (e.g., [22, 23, 12]). On the other hand, by choosing $b$ and $\Sigma$ in (1) properly, one can reduce the formulation (1) to the simpler case where each player controls her private state through $\alpha^i$. For instance, if each player’s private state is $d$-dimensional, we can let $n = dN$, $b = (b^1, \ldots, b^N)$ with $b^\ell \equiv b^\ell(t, x, \alpha^\ell)$ for $\ell = (i-1)d + 1, \ldots, id$, then the problem (1)–(2) is the standard modeling in financial mathematics literature, with the $i^{th}$ player’s $d$-dimensional private state $(X_t^{(i-1)d+1}, \ldots, X_t^{id})$ controlled by $\alpha^i$ only.

In the Markovian setting, the value function of player $i$ reads as:
\[
V^i(t, x) = \inf_{\alpha^i \in \mathcal{A}^i} \mathbb{E} \left[ \int_t^T f^i(s, X^\alpha_s, \alpha(s, X^\alpha_s)) \, ds + g^i(X^\alpha_T) \bigg| X^\alpha_t = x \right].
\]

Then, to compute the Markovian Nash equilibrium, we apply the dynamic programming principle and obtain a system of Hamilton-Jacobi-Bellman (HJB) equations:
\[
\begin{cases}
V^i_t + \inf_{\alpha^i \in \mathcal{A}^i} \left\{ b(t, x, \alpha) \cdot \nabla_x V^i + f^i(t, x, \alpha) \right\} + \frac{1}{2} \text{Tr}(\Sigma^T \text{Hess}_x V^i \Sigma) = 0, \\
V^i(T, x) = g^i(x), \quad i \in \mathcal{I},
\end{cases}
\tag{4}
\]

where $V^i_t$, $\nabla_x V^i$, $\text{Hess}_x V^i$, $f^i(t, x, \alpha)$ denote the derivative of $V^i$ with respect to $t$, the gradient and the Hessian of function $V^i$ with respect to $x$, and $\text{Tr}$ denotes the trace of a matrix. Note that the HJB system (4) is coupled, as each minimizer $\alpha^i$ depends on $V^i$ and the function $b(t, x, \alpha)$ in (4) depend on all minimizers $\alpha^\epsilon = (\alpha^1, \ldots, \alpha^N)$. Under appropriate conditions, the solution to (4) is related to BSDEs, using nonlinear Feynman-Kac formula (cf. [52, 18, 53]). To ease our notations of the BSDEs, we prescribe the following the relation on $b$ and $\Sigma$.

**Assumption 1.** There exists a measurable function $\phi$: $[0, T] \times \mathbb{R}^n \times \mathcal{A} \to \mathbb{R}^k$, so that $\Sigma(t, x, \alpha) \phi(t, x, \alpha) = b(t, x, \alpha)$ for any $(t, x, \alpha) \in [0, T] \times \mathbb{R}^n \times \mathcal{A}$.

Consequently, we can define the Hamiltonian function $H^i(t, x, \alpha, p^i): [0, T] \times \mathbb{R}^n \times \mathcal{A} \times \mathbb{R}^k \times \mathbb{R}^N \to \mathbb{R}^N$ by:
\[
H^i(t, x, \alpha, p^i) = \phi(t, x, \alpha) \cdot p^i + f^i(t, x, \alpha),
\tag{5}
\]

where $p^i$ denotes the $i^{th}$ column of $p$, and thus is an $\mathbb{R}^k$ vector. Using this notation, the HJB system can be rewritten as:
\[
V^i_t + \inf_{\alpha^i \in \mathcal{A}^i} H^i(t, x, \alpha, \Sigma^T \nabla_x V^i) + \frac{1}{2} \text{Tr}(\Sigma^T \text{Hess}_x V^i \Sigma) = 0, \quad \forall i \in \mathcal{I}.
\]
To better describe the optimal game policies, we define \( a(t, x, \alpha, p) : [0, T] \times \mathbb{R}^n \times A \times \mathbb{R}^{k \times N} \rightarrow \mathbb{R} \) by:

\[
a = (a^1, \ldots, a^N), \quad a^i(t, x, \alpha^{-i}, p) = \arg \min_{\alpha^i \in A^i} H^i(t, x, (\alpha^i, \alpha^{-i}), p^i), \quad \forall i \in I.
\]

In other words, \( a^i \) is the minimizer of the \( i^{th} \) Hamiltonian, with an emphasis of the dependence on the \( i^{th} \) player’s game value \( \Sigma^T \nabla x V^i \) and others’ strategies \( \alpha^{-i} \). Then, we define a function \( \alpha(t, x, p) \) as the fixed point of

\[
\alpha = a(t, x, \alpha, p).
\]

Note that, with the above notations \( a \) and \( \alpha \), we have assumed the minimizer in (6) exists and is unique, and (7) has a unique fixed point. Later in Assumption 2, we will detail explicit conditions on the model parameters, such that these assumptions are satisfied.

We now state the corresponding BSDE formulation of (4), which is the key component of the algorithm design in Section 3 and the convergence analysis in Section 4. Let \((X_t, Y_t, Z_t) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{k \times N}\) be the solution to the following BSDE:

\[
\begin{aligned}
X_t &= x_0 + \int_0^t \Sigma(s, X_s) \, dW_s, \\
Y_t &= g(X_T) + \int_t^T \dot{H}(s, X_s, Z_s) \, ds - \int_t^T Z_s^T \, dW_s,
\end{aligned}
\]

where \( \dot{H}(t, x, p) := H(t, x, \alpha(t, x, p), p) \) is the minimized Hamiltonian vector, and \( g(x) \equiv [g^1, \ldots, g^N]^T(x) \) is the vector form of all terminal costs. Then we have the relation:

\[
\begin{aligned}
Y_t &= [Y^1_t, \ldots, Y^N_t]^T, \quad Y^i_t = V^i(t, X_t), \\
Z_t &= [Z^1_t, \ldots, Z^N_t], \quad Z^i_t = \Sigma^T(t, X_t) \nabla x V^i(t, X_t),
\end{aligned}
\]

and the optimal game policy is expressed by

\[
\alpha^*_i = \alpha(t, X_t, Z_t).
\]

Using the relation (9), we notice \( \alpha^*_i = \alpha(t, X_t, \Sigma^T(t, X_t) \nabla x V(t, X_t))^2 \), and sometimes write \( \alpha^*_i = \alpha^*(t, X_t) \).

If solving directly, no matter which system ((4) or (8)), one will encounter computational difficulties due to the high dimensionality of \( X_t \) or the large number of agents. To overcome this, we propose a two-step scheme in Section 3, where we generalize the idea in [27] and offer two options for the first step. The convergence analysis with appropriate assumptions will be presented in Section 4.

3 Algorithm

The two-step scheme for solving Markovian Nash equilibrium works as follows. We first decouple the problem (1)-(2) into \( N \) independent sub-problems, for which we need to solve repeatedly and can solve in a parallel manner. Since each sub-problem may still be high-dimensional, we then solve them using deep neural networks with a reformulation in backward stochastic differential equations (BSDEs). Next, we describe the algorithms for each step in detail.

3.1 Step I: Decoupling

This step aims to decentralize the game, converting it into single-agent problems to be solved repeatedly. The algorithms start with an initial guess of the Nash equilibrium \( \alpha^0 = [\alpha^{1,0}, \ldots, \alpha^{N,0}] \) and produce a sequence of strategies afterward, which we denote by \( \alpha^1, \ldots, \alpha^m, \ldots \). The following two options at this step differ in how the sequence is determined. Notationwise, \( \alpha^m \) refers to the collection of all players’ policies at stage \( m \), and its \( i^{th} \) component \( a^{i,m} \) refers to player \( i \)’s choice.

\(^2\)We use \( \nabla x V \) as an \( n \times N \) matrix.
1. **Fictitious Play.** In this option of Step I, at each stage, each player faces an optimization problem (2) while assuming that others are using their strategies from the previous stage as fixed strategies. In other words, at stage $m + 1$, $\alpha^m$ is known to all players, and player $i$’s decision problem is

$$
\inf_{\alpha^i \in A^i} J^i_0(\alpha^i; \alpha^{-i,m}),
$$

where $J^i_0$ is defined in (3), and the state process $X_t$ follows (1) with $\alpha$ being replaced by $(\alpha^i; \alpha^{-i,m})$. Here $\alpha^{-i,m}$ represents the strategies of all players but player $i$ at stage $m$, and $(\alpha^i; \alpha^{-i,m})$ is a short notation of $(\alpha^1,m, \ldots, \alpha^{i-1,m}, \alpha^i, \alpha^{i+1,m}, \alpha^{N,m})$, which emphasis the parameter role of $\alpha^{-i,m}$.

Under the Markovian framework, we denote by $V^{i,m+1}$ the problem value of player $i$ at stage $m$. Following the idea of fictitious play, it is the solution of the following HJB system

$$
\begin{cases}
V_t^{i,m+1} + \inf_{\alpha^i \in A^i} H^i(t, x, (\alpha^i, \alpha^{-i,m})(t, x), \Sigma^T \nabla_x V^{i,m+1}) + \frac{1}{2} \text{Tr}(\Sigma^T \text{Hess}_x V^{i,m+1} \Sigma) = 0, \\
V^{i,m+1}(T, x) = g^i(x).
\end{cases}
$$

2. **Policy Update.** This is slightly different from fictitious play, where every player follows her strategy from the previous stage to update the problem value. In this case, it is no longer an optimization, but a linear problem for the value function induced by the fix strategy $\alpha^m$:

$$
\begin{cases}
V_t^{i,m+1} + H^i(t, x, \alpha^m(t, x), \Sigma^T \nabla_x V^{i,m+1}) + \frac{1}{2} \text{Tr}(\Sigma^T \text{Hess}_x V^{i,m+1} \Sigma) = 0, \\
V^{i,m+1}(T, x) = g^i(x).
\end{cases}
$$

After solving out the decoupled PDE (11) or (12), at the end of stage $m + 1$, a policy $\alpha^{i,m+1}$ is determined by

$$
\alpha^{i,m+1}(t, x) = \arg \min_{\alpha^i \in A^i} H^i(t, x, (\alpha^i, \alpha^{-i,m})(t, x), \Sigma^T \nabla_x V^{i,m+1}(t, x))
$$

and policies from all players together form $\alpha^{m+1}$.

Note that for fictitious play algorithm, $\alpha^{i,m+1}$ is indeed the optimal strategy of problem (10); while for policy update algorithm, the problem is linear, but we pretend that $V^{i,m+1}$ is the value of an optimization problem, and $\alpha^{i,m+1}$ is determined as if it is an optimizer.

### 3.2 Step II: Solving Each Sub-problem via BSDE

For each sub-problem, described by (11) or (12), we write down their BSDE counterpart:

$$
\begin{cases}
X_t = x_0 + \int_0^t \Sigma(s, X_s) \, dW_s, \\
Y_t^{i,m+1} = g^i(X_T) + \int_t^T \hat{H}^i(s, X_s, \alpha^{-i,m}(s, X_s), Z_s^{i,m+1}) \, ds - \int_t^T (Z_s^{i,m+1})^T \, dW_s,
\end{cases}
$$

where $\hat{H}^i$ is defined by

$$
\hat{H}^i(t, x, \alpha^{-i}, p^i) = H^i(t, x, (\alpha^i(t, x, \alpha^{-i}, p^i), \alpha^{-i}), p^i),
$$

or

$$
\begin{cases}
X_t = x_0 + \int_0^t \Sigma(s, X_s) \, dW_s, \\
Y_t^{i,m+1} = g^i(X_T) + \int_t^T H^i(s, X_s, \alpha^m(s, X_s), Z_s^{i,m+1}) \, ds - \int_t^T (Z_s^{i,m+1})^T \, dW_s.
\end{cases}
$$
Theorem 1. For the generic BSDE (17), we assume:

1. The functions $\mu$, $\Sigma$, $g$ and $F$ satisfy the following Lipschitz condition:

$$
|\mu(t,x_1) - \mu(t,x_2)|^2 + ||\Sigma(t,x_1) - \Sigma(t,x_2)||_F^2 + |F(t,x_1,p_1) - F(t,x_2,p_2)|^2
+ |g(x_1) - g(x_2)|^2 \leq L \left[ \|x_1 - x_2\|^2 + \|p_1 - p_2\|^2 \right],
$$

for a constant $L > 0$;
2. The functions $\mu$, $\Sigma$ and $h$ are all 1/2-Hölder continuous with respect to $t$. For simplicity, We also use $L$ for this Hölder constant.

3. The constant $L$ also denotes the upper bound of $|\mu(0,0)|^2$, $|\Sigma(0,0)||F(0,0,0)|^2$ and $|g(0)|^2$.

Then, we have the following two estimates:

$$\sup_{t \in [0,T]} \mathbb{E}|Y_t - Y^\pi_{\pi(t)}|^2 + \int_0^T \mathbb{E}\|Z_t - Z^\pi_{\pi(t)}\|_F^2 \, dt \leq C \left[ \|\pi\| + \mathbb{E}|g(X^\pi_T) - Y^\pi_T|^2 \right]$$  \hspace{1cm} (23)

$$\inf_{\psi_0 \in \mathcal{N}_0', \{\phi_k \in \mathcal{N}_k\}_{k=0}^{N_T}} \mathbb{E}|g(X^\pi_T) - Y^\pi_T|^2 \leq C \left[ \|\pi\| + \mathbb{E}|\tilde{Z}_{t_k} - \phi_k(X^\pi_{t_k})|^2 \Delta t_k \right]$$  \hspace{1cm} (24)

where $\|\pi\|$ and $\pi(t)$ are given in (18)--(19), $\tilde{Z}_{t_k} = (\Delta t_k)^{-1}\mathbb{E}[\int_{t_k}^{t_{k+1}} Z_t \, dt|X^\pi_{t_k}]$, and $C > 0$ is a constant only depending on $L$ and $T$.

Remark 1. Theorem 1 is a brief summary of Theorems 1 and 2 in [29]. The first inequality (23) shows that the distance between the true solution of BSDE (17) and the output of the deep BSDE method can be controlled by its loss function. In other words, in practice, the accuracy of the numerical solution is effectively indicated by the value of loss function. The second inequality (24) states that a small loss function of the deep BSDE method is attainable if the hypothesis spaces $(\mathcal{N}_0)$ and $(\{\mathcal{N}_k\}_{k=0}^{N_T})$ can approximate specific functions well. Beyond Theorems 1 and 2 in [29], there are still some theoretical issues remaining unresolved regarding the deep BSDE method, which are common in almost all the algorithms involving deep neural networks: First, it is unclear yet what types of hypothesis spaces can approximate the specific functions in the deep BSDE method without the curse of dimensionality (i.e., the number of parameters of neural networks grows at most polynomially both in dimension and the reciprocal of the approximation error). Second, even with suitable function spaces, it is hard to guarantee the optimization algorithm can find approximately the minimizer of the highly nonconvex loss function. We refer the interested readers to [17, 28, 29] for more detailed descriptions and theoretical justifications of the deep BSDE method. Details on the implementation in this paper are presented in Section 5.

4 Convergence Analysis

This section will provide the theoretical foundation for the deep fictitious play algorithm. Section 4.1 focuses on the decoupling step. Theorem 2 proves the convergence to the true Nash equilibrium, if the decoupled sub-problems are solved exactly and repeatedly. Section 4.2 focuses on the numerical error on the deep BSDE algorithm for solving each sub-problem. Theorem 3 presents a game version of Theorem 1. Section 4.3 combines the previous results, identifies the $\epsilon$-Nash equilibrium produced by deep fictitious play, and analyzes its numerical performance on the original game.

4.1 Convergence Analysis of the Decoupling Step

In this section, we will focus on the convergence of the decoupling step, i.e., how the systems defined by PDEs (11) (fictitious play) or (12) (policy update) converge to the system defined by PDEs (4), or equivalently, how the corresponding BSDE systems (14) (fictitious play) or (16) (policy update) converge to the BSDE system (8).

Throughout this paper, we shall use the following assumptions.

Assumption 2. We shall use $\|\cdot\|$ and $\|\cdot\|_F$ to denote the $L^2$-norm and Frobenius norm, respectively.

(1) The functions $\phi(t, x, \alpha) : [0, T] \times \mathbb{R}^n \times A \to \mathbb{R}^k$, $\Sigma(t, x) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times k}$, $f(t, x, \alpha) = (f_1, f_2, \ldots, f_N)^T(t, x, \alpha) : [0, T] \times \mathbb{R}^n \times A \to \mathbb{R}^N$ and $g(x) = [g_1, g_2, \ldots, g_N]^T(x) : \mathbb{R}^n \to \mathbb{R}^N$ are
Lipschitz with respect to $x$ and $\alpha$:

$$\|\phi(t, x_1, \alpha_1) - \phi(t, x_2, \alpha_2)\|^2 \leq \phi_x|x_1 - x_2|^2 + \phi_\alpha|\alpha_1 - \alpha_2|^2,$$

$$\|\Sigma(t, x_1) - \Sigma(t, x_2)\|^2 \leq \Sigma_x|x_1 - x_2|^2,$$

$$|f(t, x_1, \alpha_1) - f(t, x_2, \alpha_2)|^2 \leq f_x|x_1 - x_2|^2 + f_\alpha|\alpha_1 - \alpha_2|^2,$$

$$|g(x_1) - g(x_2)|^2 \leq g_x|x_1 - x_2|^2.$$

Here $\phi_x, \phi_\alpha, \Sigma_x, f_x, f_\alpha, g_x$ are all positive constants, and the same below.

(2) The function $a(t, x, \alpha, p)$ given in (6) is well-defined, and is Lipschitz with respect to $x$, $\alpha$ and $p$:

$$|a(t, x_1, \alpha_1, p_1) - a(t, x_2, \alpha_2, p_2)|^2 \leq a_x|x_1 - x_2|^2 + a_\alpha|\alpha_1 - \alpha_2|^2 + a_p\|p_1 - p_2\|^2_F,$$

with $a_\alpha < 1$. Notice that this also implies that $a(t, x, p)$ defined by (7) exists and is unique, which is Lipschitz with respect to $x$ and $p$:

$$|a(t, x_1, p_1) - a(t, x_2, p_2)|^2 \leq a_x|x_1 - x_2|^2 + a_p\|p_1 - p_2\|^2_F.$$

Here $a_x = a_x/(1 - a_\alpha)$ and $a_p = a_p/(1 - a_\alpha)$.

(3) The functions $\phi$ and $\Sigma$ are uniformly bound:

$$\|\Sigma(t, x)\|^2_F \leq M_\Sigma,$$

$$|\phi(t, x, \alpha)| \leq M_\phi.$$

(4) The functions $\phi$, $\Sigma$, $f$, $g$ and $a$ are all $1/2$-Hölder continuous with respect to $t$. We shall use $K$ as the upper bound of all the Hölder constants.

(5) The constant $K$ is also the upper bound of constants $|a(0, 0, 0, 0)|^2$, $|f(0, 0, 0)|^2$ and $|g(0)|^2$.

The first result in this section gives a uniform bound of the $Z$-component of the BSDEs, which is classical in the literature (cf. [66, Theorem 5.2.2]). We state and provide the proof of this lemma here to compute an exact upper bound for the convenience of later analysis.

**Lemma 1.** Let $(X_t, Y_t, Z_t)$, the adapted process taking value in $\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{k \times N}$, be the solution of the following BSDE system:

$$X_t = x_0 + \int_0^t \Sigma(s, X_s) \, dW_s,$$

$$Y_t = g(X_T) + \int_t^T F(s, X_s, Z_s) \, ds - \int_t^T (Z_s)^T \, dW_s,$$

where the coefficients $\Sigma$, $F$ and $g$ satisfy the following condition:

$$|g(x_1) - g(x_2)|^2 \leq g_x|x_1 - x_2|^2,$$

$$\|\Sigma(t, x_1) - \Sigma(t, x_2)\|^2_F \leq \Sigma_x|x_1 - x_2|^2,$$

$$|F(t, x_1, p_1) - F(t, x_2, p_2)|^2 \leq F_x|x_1 - x_2|^2 + F_p\|p_1 - p_2\|^2_F,$$

$$\|\Sigma(t, x)\|^2_F \leq M_\Sigma.$$

Then:

$$\|Z_t\|^2_S \leq M_2 e^{F_T} [g_x e^{\Sigma T} + F_x e^{\Sigma T} - 1] \, L \times \mathbb{P} \text{-a.s.},$$

where $\| \cdot \|_S$ denotes the spectral norm and $L$ is the Lebesgue measure on $[0, T]$.  

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Proof. Denote by $(X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)_{t \leq s \leq T}$ the solution of the following BSDE:

$$
\begin{aligned}
\begin{cases}
X^{t,x}_t &= x + \int_t^T \Sigma(u,X^{t,x}_u) \, dW_u, \\
Y^{t,x}_s &= g(X^{t,x}_T) + \int_s^T F(u,X^{t,x}_u,Z^{t,x}_u) \, du - \int_s^T (Z^{t,x}_u)^T \, dW_u,
\end{cases}
\end{aligned}
$$

for any $(t,x) \in [0,T] \times \mathbb{R}^n$. Then, for any $t_0 \in [0,T]$ and $x_1, x_2 \in \mathbb{R}^n$, let $(X^{t_0,j}_t, Y^{t_0,j}_t, Z^{t_0,j}_t)$ be the short notation\(^3\) for $(X^{t_0,x_1}_t, Y^{t_0,x_1}_t, Z^{t_0,x_1}_t)$, $j = 1, 2, t \in [t_0,T]$. The wellposedness of BSDEs (1) and (4.1) is classical in literature; see, e.g., [51].

We define $\delta X_t, \delta Y_t, \delta Z_t, \delta F_t$ and $\delta \Sigma_t$ as follows:

$$
\begin{aligned}
\delta X_t &= X^{1}_t - X^{2}_t, \quad \delta Y_t = Y^{1}_t - Y^{2}_t, \quad \delta Z_t = Z^{1}_t - Z^{2}_t, \\
\delta F_t &= F(t, X^{1}_t, Z^{1}_t) - F(t, X^{2}_t, Z^{2}_t), \quad \delta \Sigma_t = \Sigma(t, X^{1}_t) - \Sigma(t, X^{2}_t).
\end{aligned}
$$

Then, we have

$$
d\delta X_t = \delta \Sigma_t \, dW_t, \quad d\delta Y_t = -\delta F_t \, dt + (\delta Z_t)^T \, dW_t,
$$

and Itô’s lemma gives

$$
\begin{aligned}
d|\delta X| &= ||\delta \Sigma||_F^2 \, dt + 2(\delta X)^T \delta \Sigma \, dW_t, \\
|\delta Y|^2 &= \left[-2\delta F_t \cdot \delta Y_t + ||\delta Z||_F^2 \right] \, dt + 2(\delta Z_t \delta Y_t)^T \, dW_t.
\end{aligned}
$$

Taking the expectation on both sides yields

$$
\mathbb{E}|\delta X|^2 = |x_1 - x_2|^2 + \int_{t_0}^T \mathbb{E}||\delta \Sigma||_F^2 \, ds \leq |x_1 - x_2|^2 + \Sigma_x \int_{t_0}^T \mathbb{E}|\delta X|^2 \, ds,
$$

and by Grönwall’s inequality, we have $\mathbb{E}|\delta X|^2 \leq e^{\Sigma_x (t-t_0)} |x_1 - x_2|^2$. Similarly, we deduce that

$$
\mathbb{E}|\delta Y|^2 = \mathbb{E}|\delta Y_T|^2 + \int_t^T \mathbb{E}[2\delta F_s \cdot \delta Y_s - ||\delta Z||_F^2] \, ds
$$

$$
\leq g_x \mathbb{E}|\delta X|^2 + \int_t^T \left\{ F_x \mathbb{E}|\delta Y_s|^2 + F_x^{-1} \mathbb{E}|\delta F_s|^2 - \mathbb{E}||\delta Z||_F^2 \right\} \, ds
$$

$$
\leq g_x \mathbb{E}|\delta X|^2 + \int_t^T \left\{ F_x \mathbb{E}|\delta Y_s|^2 + F_x^{-1} \mathbb{E}|\delta X_s|^2 + F_x \mathbb{E}||\delta Z||_F^2 - \mathbb{E}||\delta Z||_F^2 \right\} \, ds
$$

$$
\leq \left[ g_x e^{\Sigma_x (T-t_0)} + F_x e^{\Sigma_x (T-t_0)} - F_x \Sigma_x \right] |x_1 - x_2|^2 + F_x \int_t^T \mathbb{E}|\delta Y_s|^2 \, ds
$$

$$
\leq \left[ g_x e^{\Sigma_x (T-t_0)} + F_x e^{\Sigma_x (T-t_0)} - F_x \Sigma_x \right] |x_1 - x_2|^2 + F_x \int_t^T \mathbb{E}|\delta Y_s|^2 \, ds
$$

and by Grönwall’s inequality, we have

$$
|\delta Y_{t_0}|^2 \leq e^{F_x (T-t_0)} \left[ g_x e^{\Sigma_x (T-t_0)} + F_x e^{\Sigma_x (T-t_0)} - F_x \Sigma_x \right] |x_1 - x_2|^2.
$$

Following the argument in [45, Theorem 3.1], we define $u(t,x) = Y^{t,x}_t$ and deduce $|u(t_0,x_1) - u(t_0,x_2)|^2 \leq L |x_1 - x_2|^2$ from (25), where $L = e^{F_x T} \left[ g_x e^{\Sigma_x T} + F_x e^{\Sigma_x T} - 1 \right]$. Therefore, we claim

$$
\|\nabla_x u(t,x)\|_3^2 \leq L \text{ a.s.} \text{ with the Lebesgue measure on } \mathbb{R}^n, \quad \forall t \in [0,T].
$$

---

\(^3\)The notation here is slightly inconsistent with the statement in Section 2: A boldface character with a superscript j denotes a solution with the jth initial condition $(t_0, x_j)$. 
Also noticing that \( Z_t = (\Sigma^T \nabla_x u)(t, X_t) \) \( \mathbb{P} \)-a.s. (cf. [45, Theorem 3.1]),
\[
\| (\Sigma^T \nabla_x u)(t, x) \|_S^2 \leq \| \Sigma(t, x) \|_S^3 \| \nabla_x u(t, x) \|_S^3 \leq \| \Sigma(t, x) \|_S^3 L \leq M_\Sigma L,
\]
and the law of \( X_t \) is absolute continuous with respect to the Lebesgue measure on \( \mathbb{R}^n \), we can get
\[
\| Z_t \|_S^3 \leq M_\Sigma L \quad \mathbb{P}\text{-a.s.,} \quad \forall t \in [0, T].
\]
Finally, since \( Z_t \) is measurable with respect to \( L([0, T]) \times \mathcal{F} \), where \( L([0, T]) \) denotes the collection of Lebesgue measurable sets on \([0, T]\), we obtain our conclusion. \( \square \)

Using the above lemma, we can prove that the solution to the BSDE system \((8)\) has a bounded \(Z\)-component, for sufficiently small \(T\). The standard estimate can not be applied directly here, as the driver \( \tilde{H} \) defined in \((8)\) is not global Lipschitz in \( p \). Thus we provide the following lemma.

**Lemma 2.** Under Assumptions 1 and 2, for sufficiently small time duration \( T \) (only depending on the constants involved in Assumption 2), there exists a unique adapted solution of the BSDE system \((8)\) such that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} (|X_t|^2 + |Y_t|^2) + \int_0^T \| Z_t \|_F^2 \, dt \right] < +\infty.
\]
Moreover,
\[
\| Z_t \|_F^2 \leq 2M_\Sigma g_x \quad \mathbb{L} \times \mathbb{P}\text{-a.s.}
\]

**Proof.** We first prove the existence. Fix \( M > 0 \), we use \( P_M(Z) \) to denote the projection from \( \mathbb{R}^{k \times N} \) to \( \{ Z \in \mathbb{R}^{k \times N} : \| Z \|_S^2 \leq M \} \), and use \( P_M(Z)^i \) for its \( i \)th column. Let \( \tilde{H} = [\tilde{H}^1, \ldots, \tilde{H}^N]^T \) with \(\tilde{H}^i(t, x, p) = \phi(t, x, \alpha(t, x, p)) \cdot P_M(p)^i + f^i(t, x, \alpha(t, x, p))\), its Lipschitz constants with respect to \( x \) and \( p \) are computed by
\[
\begin{align*}
\| \tilde{H}(t, x_1, p_1) - \tilde{H}(t, x_2, p_2) \|^2 &
\leq 3M\| \phi(t, x_1, \alpha(t, x_1, p_1)) - \phi(t, x_2, \alpha(t, x_2, p_2)) \|^2 + 3M\| P_M(p_1) - P_M(p_2) \|^2_F \\
&\quad + 3\| f(t, x_1, \alpha(t, x_1, p_1)) - f(t, x_2, \alpha(t, x_2, p_2)) \|^2 \\
&\leq 3(M\phi_x + f_x)\| x_1 - x_2 \|^2 + 3M\| p_1 - p_2 \|^2_F + 3(M\phi_x + f_x)\| \alpha(t, x_1, p_1) - \alpha(t, x_2, p_2) \|^2 \\
&\leq 3[M\phi_x + f_x + (M\phi_x + f_x)\alpha_x]\| x_1 - x_2 \|^2 + 3[M\phi + (M\phi_x + f_x)\alpha_p]\| p_1 - p_2 \|^2_F.
\end{align*}
\]

Now define \( M = 2M_\Sigma g_x \) and consider the solution \((X_t, \tilde{Y}_t, \tilde{Z}_t)\) to the following BSDE system
\[
\begin{align*}
X_t &= x_0 + \int_0^t \Sigma(s, X_s) \, dW_s, \\
\tilde{Y}_t &= g(X_T) + \int_t^T \tilde{H}(s, X_s, \tilde{Z}_s) \, ds - \int_t^T (\tilde{Z}_s)^T \, dW_s.
\end{align*}
\]

Following Lemma 1 and using the definition of \( M \), we have:
\[
\| \tilde{Z}_t \|_S^2 \leq M_\Sigma e^{3|M_\phi + (2M_\Sigma g_x \phi_x + f_x)\alpha_p|T} \\
&\quad \times \left[ g_x e^{\Sigma_T} + 3 \left( 2M_\Sigma g_x \phi_x + f_x + (2M_\Sigma g_x \phi_x + f_x)\alpha_x \right) \frac{e^{\Sigma_T} - 1}{3\Sigma_x [M_\phi + (2M_\Sigma g_x \phi_x + f_x)\alpha_p]} \right] \\
:= M(T).
\]

Therefore, if \( M(T) \leq 2M_\Sigma g_x \), \((X_t, \tilde{Y}_t, \tilde{Z}_t)\) is the desired solution to the BSDE system \((8)\) with \( \| \tilde{Z}_t \|_S^2 \leq 2M_\Sigma g_x \), which can be fulfilled if \( T \) is small enough.

We then prove the uniqueness. Let \((X_t, Y'_t, Z'_t)\) be another adapted solution of the BSDE system \((8)\) such that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} (|X_t|^2 + |Y'_t|^2) + \int_0^T \| Z'_t \|_F^2 \, dt \right] < +\infty.
\]
Define $\delta Y_t = Y'_t - Y_t$, $\delta Z_t = Z'_t - Z_t$ and $\delta H_t = H(t, X_t, Z'_t) - H(t, X_t, Z_t)$ and we can write
\[ d\delta Y_t = -\delta H_t \, dt + \delta Z_t \, dW_t. \]

Itô’s lemma gives
\[ d|\delta Y_t|^2 = [-2\delta Y_t \cdot \delta H_t + \|\delta Z_t\|^2] \, dt + 2(\delta Z_t \, \delta Y_t)^T \, dW_t. \]

Taking expectation on both side and using $\delta Y_T = 0$, we deduce that
\begin{equation}
\mathbb{E}[|\delta Y_t|^2] + \int_t^T \mathbb{E}[\|\delta Z_s\|^2] \, ds \leq 2 \int_t^T \mathbb{E}[\|\delta H_s\|] \cdot \mathbb{E}[|\delta Y_s|^2] \, ds + \lambda^{-1} \int_t^T \mathbb{E}[|\delta H_s|^2] \, ds, \tag{26} \end{equation}
for any $\lambda > 0$. By
\begin{align*}
\delta H_t &= \phi(t, X_t, \alpha(t, X_t, Z'_t)) \cdot Z'_t - \phi(t, X_t, \alpha(t, X_t, Z_t)) \cdot Z_t \\
& \quad + f(t, X_t, \alpha(t, X_t, Z'_t)) - f(t, X_t, \alpha(t, X_t, Z_t)) \\
& = \phi(t, X_t, \alpha(t, X_t, Z'_t)) \cdot (Z'_t - Z_t) + [\phi(t, X_t, \alpha(t, X_t, Z'_t)) - \phi(t, X_t, \alpha(t, X_t, Z_t))] \cdot Z_t \\
& \quad + f(t, X_t, \alpha(t, X_t, Z'_t)) - f(t, X_t, \alpha(t, X_t, Z_t)),
\end{align*}
we have $|\delta H_t|^2 \leq L_z \|\delta Z_t\|^2$ for any $L_z = 3[M_\varphi + 2M \gamma_2 \varphi \alpha_0 + f_0 \alpha_0]$. Taking $\lambda = L_z$ in (26), we deduce that
\[ \mathbb{E}[|\delta Y_t|^2] \leq L_z \int_t^T \mathbb{E}[|\delta Z_s|^2] \, ds \]
and therefore $Y'_t \equiv Y_t$ by Grönwall’s inequality. We then have $\int_0^T \mathbb{E}[|\delta Z_s|^2] \, dt = 0$ from the first equality in (26), which implies $Z'_t \equiv Z_t$. \hfill \Box

Recalling that $m$ is the index of the stage in the decoupling step, now we present the main result in this section regarding its convergence.

**Theorem 2.** Under Assumptions 1 and 2, for sufficiently small time duration $T$, there exist two constants $C > 0$ and $0 < q < 1$ such that
\begin{equation}
\sup_{0 \leq t \leq T} \mathbb{E}[|Y'_m - Y_t|^2] + \int_0^T \mathbb{E}[\|Z'_m - Z_t\|^2] \, dt + \int_0^T \mathbb{E}[|\alpha'_m - \alpha_t|^2] \, dt \leq Cq^m \int_0^T \mathbb{E} \left[ \sum_{i=1}^{N(m)} \alpha'_m - \alpha'_i \right] \, dt, \tag{27} \end{equation}
where $(Y'_m, Z'_m)$ is defined by
\[ Y'_m = [Y'_1, \ldots, Y'_N]^T, \quad Z'_m = [Z'_1, \ldots, Z'_N]^T \]
with $(Y'_i, Z'_i)$ from the BSDE systems (14) or (16), $(Y_t, Z_t)$ is defined in (8) , $\alpha'_m = \alpha'(t, X_t)$ and $\alpha'_i = \alpha'(t, X_t)$. Note that the choice of $T, C, q$ only depend on the constants involved in Assumption 2.

**Proof.** Theorem 2 states the convergence of both fictitious play (according to (14)) and policy update (according to (16)). The proofs of these two are very similar, and we shall focus on fictitious play method for brevity.

To perform convergence analysis, we first rewrite the BSDE systems to show the explicit dependence on the players’ strategies. For (8), it reads as
\begin{equation}
\begin{cases}
X_t = x_0 + \int_0^t \Sigma(s, X_s) \, dW_s, \\
Y_t = g(X_T) + \int_t^T H(s, X_s, \alpha'_s, Z_s) \, ds - \int_t^T (Z_s)^T \, dW_s, \\
\alpha'_t = a(t, X_t, \alpha'_t, Z_t),
\end{cases} \tag{28} \end{equation}
where $H, a$ are defined in (5) and (6). The rewritten system of (14) is
\begin{equation}
\begin{cases}
X_t = x_0 + \int_0^t \Sigma(s, X_s) \, dW_s, \\
Y_{m+1} = g(X_T) + \int_t^T \tilde{H}(s, X_s, \alpha^m, \alpha'^{m+1}, Z^{m+1}_s) \, ds - \int_t^T (Z^{m+1}_s)^T \, dW_s, \\
\alpha'^{m+1} = a(t, X_t, \alpha^m, Z^{m+1}_t),
\end{cases} \tag{29} \end{equation}

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where $\dot{H} = [\dot{H}_1, \ldots, \dot{H}_N]^T$, $\dot{H}^i(t, x, \beta, y, p) = H^i(t, x, (\gamma^i, \beta^{-i}), p^i)$ and $p^i$ stands for the $i$th column of $p$.

Note that this is slightly an abuse of notation with (15), to show the driver’s explicit dependence on $\alpha^{m+1}$.

Also note that the rewritten system (29) is simply a condensed form of (14), concatenating all $Y_t^{i,m}$ into $Y_t^m$, without changing its decoupled nature. This will also ease the notation in the following proof.

We now define $\delta H^m_t = H(t, X_t, \alpha_t^m, \alpha_t^{m+1}, Z_t^{m+1}) - H(t, X_t, \alpha_t^m, Z_t)$. According to Lemma 2, we have $\|Z_t\|_2^2 \leq 2M_\Sigma g_x$, and consequently,

$$
\begin{align*}
\delta H^m_t &= \phi(t, X_t, (\alpha_t^{i,m+1}, \alpha_t^{i,m})) \cdot Z_t^{i,m+1} + f^i(t, X_t, (\alpha_t^{i,m+1}, \alpha_t^{i,m})) \\
&\quad - \phi(t, X_t, \alpha^*_t) \cdot Z_t - f^i(t, X_t, \alpha^*_t)
\end{align*}
$$

$$
\begin{align*}
&= \phi(t, X_t, (\alpha_t^{i,m+1}, \alpha_t^{i,m})) \cdot (Z_t^{i,m+1} - Z_t) + [\phi(t, X_t, (\alpha_t^{i,m+1}, \alpha_t^{i,m})) - \phi(t, X_t, \alpha^*_t)] \cdot Z_t^i \\
&\quad + [f^i(t, X_t, \alpha^*_t) - f^i(t, X_t, \alpha^*_t)]
\end{align*}
$$

Therefore,

$$
|\delta H^m_t|^2 \leq 5M_\phi \|Z_t^{m+1} - Z_t\|^2_F + 5 \sum_{i=1}^N |Z_t^i| \phi_\alpha |\alpha_t^{i,m+1} - \alpha_t^{i,m}|^2 + 5 \|Z_t\|_2^2 \phi_\alpha |\alpha_t^m - \alpha_t^*_t|^2
$$

$$
+ 5 \sum_{i=1}^N f_\alpha |\alpha_t^{i,m+1} - \alpha_t^{i,m}|^2 + 5 f_\alpha |\alpha_t^m - \alpha_t^*_t|^2
$$

$$
\leq 5M_\phi \|Z_t^{m+1} - Z_t\|^2_F + (10 \phi_\alpha M_\Sigma g_x + 5 f_\alpha) \left[2|\alpha_t^m - \alpha_t^*_t|^2 + |\alpha_t^{m+1} - \alpha_t^*_t|^2\right].
$$

Next, we define $\delta Y_t^m = Y_t^m - Y_t$, $\delta Z_t^m = Z_t^m - Z_t$, $\delta \alpha_t^m = \alpha_t^m - \alpha_t^*_t$. Subtracting (28) from (29) gives

$$
d\delta Y_t^{m+1} = -\delta H_t^m \, dt + \delta Z_t^{m+1} \, dW_t.
$$

Using Itô’s lemma and taking expectation on both sides, we have

$$
\mathbb{E}|\delta Y_t^{m+1}|^2 + \int_t^T \mathbb{E}\|\delta Z_s^{m+1}\|^2_F \, ds = 2 \int_t^T \mathbb{E}[\delta H_s^m \cdot \delta Y_s^{m+1}] \, ds.
$$

By the Cauchy-Schwartz inequality, for any $\lambda > 0$, one has

$$
\begin{align*}
\mathbb{E}|\delta Y_t^{m+1}|^2 &\leq \int_t^T \mathbb{E}\|\delta Z_s^{m+1}\|^2_F \, ds \\
&\leq \lambda \int_t^T \mathbb{E}|\delta Y_s^{m+1}|^2 \, ds + \lambda^{-1} \int_t^T \mathbb{E}|\delta H_s^m|^2 \, ds
\end{align*}
$$

$$
\leq \lambda \int_t^T \mathbb{E}|\delta Y_s^{m+1}|^2 \, ds + \lambda^{-1} \left\{5M_\phi \|\delta Z_s^{m+1}\|^2_F + (10 \phi_\alpha M_\Sigma g_x + 5 f_\alpha) \left[2|\delta \alpha_s^m|^2 + |\delta \alpha_s^{m+1}|^2\right] \right\} \, ds.
$$

Choosing $\lambda = 5M_\phi$ in the above inequality yields

$$
\mathbb{E}|\delta Y_t^{m+1}|^2 \leq 5M_\phi \int_t^T \mathbb{E}|\delta Y_s^{m+1}|^2 \, ds + \frac{2\phi_\alpha M_\Sigma g_x + f_\alpha}{M_\phi} \int_t^T \left\{2\mathbb{E}|\delta \alpha_s^m|^2 + \mathbb{E}|\delta \alpha_s^{m+1}|^2\right\} \, ds.
$$

By Grönwall’s inequality, we obtain

$$
\begin{align*}
\sup_{0 \leq t \leq T} \mathbb{E}|\delta Y_t^{m+1}|^2 &\leq e^{5M_\phi T} \frac{2\phi_\alpha M_\Sigma g_x + f_\alpha}{M_\phi} \int_0^T \left\{2\mathbb{E}|\delta \alpha_t^m|^2 + \mathbb{E}|\delta \alpha_t^{m+1}|^2\right\} \, dt.
\end{align*}
$$
Choosing a constant $\lambda > 5M_\phi$ and using inequality (30) again produce

$$
\int_0^T \mathbb{E}[|\delta Z_t^{m+1}|^2 F_t] dt \\
\leq \frac{\lambda^2}{\lambda - 5M_\phi} \int_0^T \mathbb{E}[|\delta Y_t^{m+1}|^2] dt + \frac{10\phi_\alpha M \Sigma g_x + 5f_\alpha}{\lambda - 5M_\phi} \int_0^T \{2\mathbb{E}[|\delta \alpha_t^m|^2 + \mathbb{E}[|\delta \alpha_t^{m+1}|^2] \} dt \\
\leq \frac{10\phi_\alpha M \Sigma g_x + 5f_\alpha}{\lambda - 5M_\phi} \left[ \frac{\lambda^2 T e^{5M_s T}}{5M_\phi} + 1 \right] \int_0^T \{2\mathbb{E}[|\delta \alpha_t^m|^2 + \mathbb{E}[|\delta \alpha_t^{m+1}|^2] \} dt \\
:= C_2(\lambda) \int_0^T \{2\mathbb{E}[|\delta \alpha_t^m|^2 + \mathbb{E}[|\delta \alpha_t^{m+1}|^2] \} dt. \tag{32}
$$

Using the Lipschitz condition of the function $a$ and estimate (32) imply that

$$
\int_0^T \mathbb{E}[|\delta \alpha_t^{m+1}|^2] dt \leq \int_0^T \{a_\alpha \mathbb{E}[|\delta \alpha_t^m|^2 + a_p \mathbb{E}[|\delta Z_t^{m+1}|^2 F_t] \} dt \\
\leq (a_\alpha + 2a_p C_2(\lambda)) \int_0^T \mathbb{E}[|\delta \alpha_t^m|^2] dt + a_p C_2(\lambda) \int_0^T \mathbb{E}[|\delta Z_t^{m+1}|^2] dt. \tag{33}
$$

If $0 < \frac{a_\alpha + 2a_p C_2(\lambda)}{1 - a_\alpha F(\lambda)} < 1$, the claim (27) follows by the estimates (31), (32) and (33), and we are done with the proof. To this end, we only need to have $C_2(\lambda) < \min \{\frac{1}{a_\alpha}, \frac{1 - a_\alpha}{3a_p} \}$ as $a_\alpha < 1$. To this end, we can first choose $\lambda > 1$ such that $\frac{10\phi_\alpha M \Sigma g_x + 5f_\alpha}{\lambda - 5M_\phi} \ll 1$, and then choose $T \ll 1$ such that $\frac{\lambda^2 T e^{5M_s T}}{5M_\phi} \ll 1$, which leads to $C_2(\lambda) \ll 1$.

### 4.2 Numerical Error Analysis

This section is dedicated to analyzing the numerical error introduced by the deep BSDE method when solving each sub-problem. Specifically, we aim to control the distance between $(X_t, Y_t, Z_t)$ defined in (8) and the discrete system $(\hat{X}_t, \hat{Y}_t, \hat{Z}_t)$ satisfying:

$$
\begin{align*}
X_{t+k+1}^\pi &= X_t^\pi + \Sigma(t_k, X_t^\pi) \Delta W_k, \quad X_0^\pi = x_0, \\
Y_{t+k+1}^{\pi,m+1} &= Y_{t_k}^{\pi,m} - h^m(t_k, X_{t_k}^\pi, Z_{t_k}^{\pi,m+1}) \Delta t_k + (Z_{t_k}^{\pi,m+1})^T \Delta W_k,
\end{align*}
$$

(34)

where $h^m$ is either $[h_1^m, \ldots, h_N^m]^T$ with $h_i^m(t, x, p) =$ inf$_{\alpha', \pi \in \mathcal{H}} H^i(t, x, (\alpha', \pi, \alpha' - \pi, m)(t, x), p)$ when decoupled through fictitious play, or $h_i^m(t, x, p) = H^i(t, x, \alpha^m, \pi)(t, x), p)$ when decoupled through policy update. As stated in Section 3.2, $Y_0^{\pi,m+1}$ and $Z_{0}^{\pi,m+1}$ are parameterized by neural networks,

$$
Y_0^{\pi,m+1} = \psi_0^{m+1}(X_0^\pi), \quad Z_{0}^{\pi,m+1} = \phi_k^{m+1}(X_0^\pi),
$$

where $\psi_0^{m}$ and $\{\phi_k^{m}\}_{k=0}^{N_\pi - 1}$ are the optimal deterministic maps determined at stage $m$ that belongs to the hypothesis spaces (cf. Section 3.2). Afterwards, the $(m+1)$th-stage policies defined on $T \times \mathbb{R}^n$ are updated by:

$$
\alpha_{t+k+1}^{\pi,m+1}(t, x) = a(t, x, \alpha_t^{\pi,m}(t, x), \phi_t^{m+1}(t, x)), \quad \forall (t, x) \in T \times \mathbb{R}^n \tag{35}
$$

where $\phi^m(t_k, x) = \hat{\phi}^m(t_k, x)$. Note that the above notation is simply a vector form of the deep BSDE method applied to system (14) or (16). It does not change the decoupling nature of the deep fictitious play algorithm, i.e., each entry $(Y_\pi^{\pi,m}, Z_\pi^{\pi,m})$ in $(Y^{\pi,m}, Z^{\pi,m})$ still solves its own problem.

Initially, we hope to apply Theorem 1 to the BSDE system (14) and (16). By the game feature and the decoupling scheme, stage $m + 1$’s estimates rely on the regularity of stage $m$’s policy $\alpha^m(t, x)$ (see definition in (13)). Specifically, it requires the following condition, in addition to Assumption 2:

$$
|\alpha^m(t_1, x_1) - \alpha^m(t_2, x_2)|^2 \leq L(t_1 - t_2) + |x_1 - x_2|^2.
$$

However, in general, this property is not inherited from stage to stage. To circumvent this issue, we introduce a projection operator $P_{X^{\pi,m}}$, which needs to be applied at the end of each stage. Along this line, we need the following assumption.
Assumption 3. The optimal policy $\alpha^*$ as a function on $[0,T] \times \mathbb{R}^n$ is Lipschitz with respect to $x$ and 1/2-Hölder continuous with respect to $t$, i.e.,

$$|\alpha^*(t_1, x_1) - \alpha^*(t_2, x_2)|^2 \leq L(|t_1 - t_2| + |x_1 - x_2|^2).$$

We also assume that $|\alpha^*(t, x)|^2 \leq L(1 + |x|^2)$ for any $(t, x) \in [0, T] \times \mathbb{R}^n$.

Recall the set $\mathcal{T}$ containing all endpoints of the partition $\pi$ on $[0,T]$ from (19), we define a Hilbert space on $\mathcal{T} \times \mathbb{R}^n$:

$$\mathcal{H}^\pi = \left\{ \alpha: \text{measurable functions from } \mathcal{T} \times \mathbb{R}^n \to \mathbb{R}^{N_{\pi} \times n}, \sum_{k=0}^{N_{\pi} - 1} \mathbb{E}|\alpha(t_k, X_{t_k}^\pi)|^2 \Delta t_k < +\infty \right\},$$

with norm $||\alpha||^2_{\mathcal{H}^\pi} := \sum_{k=0}^{N_{\pi} - 1} \mathbb{E}|\alpha(t_k, X_{t_k}^\pi)|^2 \Delta t_k$, and a subset

$$\mathcal{N}^\pi = \left\{ \alpha: \mathcal{T} \times \mathbb{R}^n \to \mathbb{R}^{N_{\pi} \times n}, |\alpha(t_1, x_1) - \alpha(t_2, x_2)|^2 \leq L'||t_1 - t_2| + |x_1 - x_2|^2|, |\alpha(t, x)|^2 \leq L'(1 + |x|^2) \right\}^4$$

with a constant $L' \geq L$. We claim that $\mathcal{N}^\pi$ is a closed convex subset of $\mathcal{H}^\pi$, so the projection $P_{\mathcal{N}^\pi}$ from $\mathcal{H}^\pi$ to $\mathcal{N}^\pi$ exists and does not increase distance (cf., [7, Chapter 5]):

$$\|P_{\mathcal{N}^\pi} f_1 - P_{\mathcal{N}^\pi} f_2\|_{\mathcal{H}^\pi} \leq \|f_1 - f_2\|_{\mathcal{H}^\pi} \quad \forall f_1, f_2 \in \mathcal{H}^\pi.$$  \hspace{1cm} (36)

The convexity of $\mathcal{N}^\pi$ is straightforward. To see the closedness, let $\{\alpha_j\}_{j \geq 1}$ be a convergent sequence in $\mathcal{N}^\pi$, then there exists a subsequence of $\{\alpha_j\}_{j \geq 1}$, denoted by $\{\alpha_{j_k}\}_{k \geq 1}$, that converges to $\alpha_\infty$ a.s.. Since

$$|\alpha_{j_k}(t_1, x_1) - \alpha_{j_k}(t_2, x_2)|^2 \leq L'||t_1 - t_2| + |x_1 - x_2|^2|, \quad |\alpha_{j_k}(t, x)|^2 \leq L'(1 + |x|^2),$$

and let $k \to +\infty$, we obtain

$$|\alpha_\infty(t_1, x_1) - \alpha_\infty(t_2, x_2)|^2 \leq L'||t_1 - t_2| + |x_1 - x_2|^2|, \quad |\alpha_\infty(t, x)|^2 \leq L'(1 + |x|^2), \text{ a.s..}$$

Noticing that the functions in $\mathcal{N}^\pi$ and $\mathcal{H}^\pi$ are identical if they agree almost everywhere, we can conclude $\alpha_\infty \in \mathcal{N}^\pi$ and $\mathcal{N}^\pi$ is closed.

Therefore, we are able to apply the projection operator $P_{\mathcal{N}^\pi}$ at the end of each stage, i.e., we change equation (35) to

$$\begin{align*}
\alpha^{\pi,m+1}(t, x) &= a(t, x, \alpha^{\pi,m}(t, x), \phi^{m+1}(t, x)), \\
\alpha^{\pi,m+1} &= P_{\mathcal{N}^\pi}(\tilde{\alpha}^{\pi,m+1}).
\end{align*}$$

The main theorem in this section is as follows.

Theorem 3. Under Assumptions 1–3, let $\alpha^{\pi,0}$ be a measurable function from $\mathcal{T} \times \mathbb{R}^n$ to $A$ satisfying:

$$|\alpha^{\pi,0}(t_1, x_1) - \alpha^{\pi,0}(t_2, x_2)|^2 \leq L'||t_1 - t_2| + |x_1 - x_2|^2|, \quad |\alpha^{\pi,0}(t, x)|^2 \leq L'(1 + |x|^2). \hspace{1cm} (38)$$

Then, if $T$ is sufficiently small, we have

$$\sup_{t \in [0,T]} \mathbb{E}|Y_t - Y_{\pi(t)}^{\pi,m}|^2 + \int_0^T \mathbb{E}||Z_t - Z_{\pi(t)}^{\pi,m}(t)||_{\mathcal{F}_t}^2 \, dt + \int_0^T \mathbb{E}|\alpha_t - \alpha_{\pi(t)}^{\pi,m}(t)|^2 \, dt \leq C \left[ \|\pi\| + q m \int_0^T \mathbb{E}|\alpha_t - \alpha_{\pi(t)}^{\pi,0}(t)|^2 \, dt + \sum_{j=1}^m q^{m-j} \mathbb{E}|g(X_{T_j}^\pi) - Y_T^{\pi,j}|^2 \right]. \hspace{1cm} (39)$$

where $(X_{T_k}^\pi, Y_{T_k}^{\pi,m}, Z_{T_k}^{\pi,m})$ is defined in (34), $\alpha_{\pi(t)}^{\pi,m} = \alpha_{\pi(t)}^{\pi,m}(t_k, X_{t_k}^\pi)$ for $t \in [t_k, t_{k+1})$, and $C > 0$, $0 < q < 1$ are two constants depending only on $T$, $L'$ and the constants involved in Assumption 2, which

---

Note that here $t_1$ and $t_2$ refer to two arbitrary points in $\mathcal{T}$, but not the first and second endpoints in the partition $\pi$. Later, depending on the contexts, the notation $t_1$ and $t_2$ in $|t_1 - t_2|$ may also refer to arbitrary points in $[0,T]$.
may vary from line to line in the proof. Here \((X^\pi_{t_k}, Y^\pi_{t_k}, Z^\pi_{t_k})\) represents either the discrete BSDE system using fictitious play or policy update in the decoupling step, depending on the definition of \(h\) in (34).

Moreover, if we define \((Y^m_t, Z^m_t)\) as

\[
Y^m_t = [Y^{1,m}_t, \ldots, Y^{N,m}_t]^T, \quad Z^m_t = [Z^{1,m}_t, \ldots, Z^{N,m}_t]
\]

with \((Y^{i,m}_t, Z^{i,m}_t)\) from the BSDE systems (14) in the setting of fictitious play or (16) in the setting of policy update, in which the previous stage policy is given by the extension of the numerical approximation in time

\[
\alpha^m(t, x) = \inf_{t' \in T} [\alpha^{\pi,m}(t', x) + L|t' - t|],
\]

we have

\[
\inf_{\phi^m_0 \in N_0^\pi, (\phi^m_k \in N_k)_{k=0}^{N_T-1}} \mathbb{E}[g(X^\pi_T) - Y^m_T] \leq C \left[ \|\pi\| + \inf_{\psi^m_0 \in N_0^\pi, (\psi^m_k \in N_k)_{k=0}^{N_T-1}} \left\{ \mathbb{E}[Y^m_0 - \psi^m_0(x_0)]^2 + \sum_{k=0}^{N_T-1} \mathbb{E}[Z^m_k - \phi^m_k(X^\pi_k)]^2 \right\} \right],
\]

where \(Z^m_k = (\Delta t_k)^{-1} \mathbb{E} \int_{t_k}^{t_{k+1}} Z^m_t \, dt \, X^\pi_{t_k}\).

**Remark 2.** We have the following remarks regarding Theorem 3:

1. The interpretation of Theorem 3 is similar to that of Theorem 1. The first inequality (39) shows that the distance between the true solution of BSDE (8) and the output of the deep BSDE method at stage \(m\) can be controlled together by the mesh size, the error of the initial policy and the loss functions achieved at all the previous stages. The second inequality (41) states that the loss function of deep BSDE method at each stage is small if the approximation capability of the parametric function spaces \((N_0^\pi)\) and \((N_k)_{k=0}^{N_T-1}\) is high. The overall message conveyed in Theorem 3 is that, if the deep BSDE method can solve each sub-problem accurately enough, the deep fictitious play method will produce a strategy close to the Nash equilibrium.

2. Note that there is a slight abuse of notation in the statement of Theorem 3, since \((Y^m_t, Z^m_t)\) and \(\alpha^m\) have already been introduced in Sections 4.1 and 3.1, as the theoretical solution from the decoupling step at stage \(m\). In this section, to avoid introducing further complicated notations, we still refer \((Y^m_t, Z^m_t)\) as the theoretical solution depending on \(\alpha^m\), but \(\alpha^m\) is the interpolation (40) of the deep BSDE solution \(\alpha^{\pi,m} - \alpha^m\) at stage \(m - 1\). Nevertheless, some estimates, in particular, equations (31)–(33), derived in Theorem 2 in Section 4.1 still applies, with an argument similar to the original proof. This new definition is indeed needed to apply Theorem 1 on \((Y^\pi_{t_k}^{m,m}, Z^\pi_{t_k}^{m,m})\) and \((Y^m_t, Z^m_t)\), and the particular form (40) ensures that the Hölder continuity, as a prerequisite of Theorem 1, is preserved after the extension (cf. [46]).

3. Also we still refer \(N_0^\pi\) and \(N_k\) as the hypothesis spaces for \(\psi^m_0 : \mathbb{R}^n \to \mathbb{R}^N, \phi^m_k : \mathbb{R}^n \to \mathbb{R}^{k \times N}\), without introducing superscript \(m\) to indicate the stage.

**Proof.** Since \(\alpha^{\pi,m} \in N_0^\pi\), with the condition (38) and (40), we can obtain (cf. [46])

\[
|\alpha^m(t_1, x_1) - \alpha^m(t_2, x_2)|^2 \leq L^2|t_1 - t_2| + |x_1 - x_2|^2, \quad |\alpha^m(t, x)| \leq L^2(1 + \sqrt{\|\pi\| + |x|^2}).
\]

Thus, the inequality (41) follows from Theorem 1.

We next prove the inequality (39). As before, we will focus on proving the case of fictitious play in the sequel, and we claim that the statements also hold for policy update using a similar argument.

We first need error estimates on the BSDE (8)’s Euler-type scheme. To this end, we recall the following result (see [64, 48]) on the generic BSDE (17) for notation brevity:

\[
\begin{align*}
X^\pi_{t_{k+1}} &= X^\pi_{t_k} + \mu(t_k, X^\pi_{t_k}) \Delta t_k + \Sigma(t_k, X^\pi_{t_k}) \Delta W_k, \quad X^\pi_0 = x, \\
Y^\pi_t &= \mathbb{E}[Y^\pi_{t_{k+1}} | X^\pi_{t_k}] + F(t_k, X^\pi_{t_k}, Z^\pi_{t_k}) \Delta t_k, \quad Y^\pi_T = g(X^\pi_T), \\
Z^\pi_{t_k} &= \frac{1}{\Delta t_k} \mathbb{E}[(Y^\pi_{t_{k+1}})^T \Delta W_k | X^\pi_{t_k}], \quad \forall k = 0, 1, \ldots, N_T - 1.
\end{align*}
\]
With the same assumptions stated in Theorem 1, one has the following estimate of the discretization error:

\[
\sup_{t \in [0,T]} \mathbb{E}|X_t - X^\pi_{\pi(t)}|^2 + \mathbb{E}|Y_t - Y^\pi_{\pi(t)}|^2 + \int_0^T \mathbb{E}\|Z_t - Z^\pi_{\pi(t)}\|_F^2 \, dt \leq C\|\pi\|,
\]

where \((X_t, Y_t, Z_t)\) solves (17) and \(C\) is a constant depending only on \(L\) appearing in Theorem 1.

In the sequel, we use \(\{Y^\pi_{\pi(t)}\}_{0 \leq k \leq N_T}, \{Z^\pi_{\pi(t)}\}_{0 \leq k \leq N_T - 1}\) to denote the above Euler scheme for BSDE (8). The discretized \(X\)-component coincides with \(\{X^\pi_{\pi(t)}\}_{0 \leq k \leq N_T}\), defined in (34), so we will stick to this notation. It is straightforward to see that, under Assumptions 2–3, the BSDE (8) satisfies the assumptions in Theorem 1, so one has:

\[
\sup_{t \in [0,T]} \mathbb{E}|X_t - X^\pi_{\pi(t)}|^2 \leq C\|\pi\|, \tag{43}
\]

\[
\sup_{t \in [0,T]} \mathbb{E}|Y_t - Y^\pi_{\pi(t)}|^2 + \int_0^T \mathbb{E}\|Z_t - Z^\pi_{\pi(t)}\|_F^2 \, dt \leq C\|\pi\|. \tag{44}
\]

For the \(Z\)-part error, we decompose it into two terms by the AM-GM inequality, for any \(\lambda > 0\):

\[
\int_0^T \mathbb{E}\|Z_t - Z^\pi_{\pi(t)}\|_F^2 \, dt \leq (1 + \lambda^{-1}) \int_0^T \mathbb{E}\|Z_t - Z^{\pi,m+1}_{\pi(t)}\|_F^2 \, dt + (1 + \lambda) \int_0^T \mathbb{E}\|Z^{\pi,m+1}_{\pi(t)} - Z^{\pi,m+1}_{\pi(t)}\|_F^2 \, dt. \tag{45}
\]

A similar inequality can be written on the \(Y\)-part error. For both of them, the second term is taken care by applying Theorem 1 to \(\{Y^m, Z^m\}\). More precisely, under Assumptions 2–3 and (42), one has:

\[
\sup_{t \in [0,T]} \mathbb{E}|Y^{\pi,m+1}_{\pi(t)} - Y_{\pi(t)}\|^2 + \int_0^T \mathbb{E}\|Z^{\pi,m+1}_{\pi(t)} - Z^\pi_{\pi(t)}\|_F^2 \, dt \leq C\left[\|\pi\| + \mathbb{E}|Y^\pi_{\pi(t)} - g(X^\pi_{\pi(t)})|^2\right]. \tag{46}
\]

where \((Y^\pi_{\pi(t)}), Z^\pi_{\pi(t)}\) is defined in (34). For the first term in (44), we recall (32) and (33) and deduce

\[
\int_0^T \mathbb{E}\|Z_t - Z^{\pi,m+1}_{\pi(t)}\|_F^2 \, dt \leq C\left[\left(2 + a_{\alpha} + 2 a_p c_Z(\lambda)\right) \left[2 + a_{\alpha} + 2 a_p c_Z(\lambda)\right] \right] \int_0^T \mathbb{E}|\alpha^m_t - \alpha^m(t, X_t)|^2 \, dt
\]

\[
\leq (1 + \lambda^{-1}) C\left[\left(2 + a_{\alpha} + 2 a_p c_Z(\lambda)\right) \left[2 + a_{\alpha} + 2 a_p c_Z(\lambda)\right] \right] \int_0^T \mathbb{E}|\alpha^m_t - \alpha^m_{\pi(t)}|^2 \, dt + C\|\pi\|, \tag{47}
\]

for \(C_Z(\lambda) < a_p^{-1}\), where we have used

\[
\int_0^T \mathbb{E}|\alpha^m_{\pi(t)} - \alpha^m_t|^2 \, dt \leq C\|\pi\| \tag{47}
\]

as a consequence of (42) and (43), and \(C_\lambda\) is a constant depending on \(T\) and the constants involved in Assumption 2 and \(\lambda\). Combining (44), (45) and (46), we claim that the \(Z\)-part error in (39) is obtained if

\[
(1 + \lambda^{-1}) C\left[\left(2 + a_{\alpha} + 2 a_p c_Z(\lambda)\right) \left[2 + a_{\alpha} + 2 a_p c_Z(\lambda)\right] \right] < 1.
\]

A similar conclusion on \(Y\)-part is obtained using equations (45), (31), (33), and (47).

We next derive an estimate that is useful in controlling the \(\alpha\)-part error. For any constant \(\lambda > 0\), we compute:

\[
\sum_{k=0}^{N_T-1} \mathbb{E}\|Z^\pi_{\Delta t_k} - Z^\pi_{\Delta t_k}\|_F^2 \, \Delta t_k = \int_0^T \mathbb{E}\|Z^\pi_{\pi(t)} - Z^\pi_{\pi(t)}\|_F^2 \, dt
\]

\[
\leq (1 + \lambda^{-1}) \int_0^T \mathbb{E}\|Z_t - Z^{\pi,m+1}_{\pi(t)}\|_F^2 \, dt + 2(1 + \lambda) \int_0^T \mathbb{E}\|Z_t - Z^\pi_{\pi(t)}\|_F^2 + \mathbb{E}\|Z^{\pi,m+1}_{\pi(t)} - Z^{\pi,m+1}_{\pi(t)}\|_F^2 \, dt
\]

\[
\leq (1 + \lambda^{-1}) \int_0^T \mathbb{E}\|Z_t - Z^{\pi,m+1}_{\pi(t)}\|_F^2 \, dt + C\|\pi\| + \mathbb{E}|Y^\pi_{\pi(t)} - g(X^\pi_{\pi(t)})|^2
\]

\[
\leq (1 + \lambda^{-1}) C\left[\left(2 + a_{\alpha} + 2 a_p c_Z(\lambda)\right) \left[2 + a_{\alpha} + 2 a_p c_Z(\lambda)\right] \right] \int_0^T \mathbb{E}|\alpha^m_t - \alpha^m_{\pi(t)}|^2 \, dt
\]

\[
+ C\|\pi\| + \mathbb{E}|Y^\pi_{\pi(t)} - g(X^\pi_{\pi(t)})|^2, \tag{48}
\]

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where we have used the AM-GM inequality and (46). Here $C_λ$ is a constant depending on $T$, $L'$, $L_α$, the constants involved in Assumption 2 and $λ$, which may vary from line to line in the following derivations.

For the $α$-part error $∫_0^T E|α_t^* - α_t^{*, m}|^2 dt$ in (39), we plan to

1. express $I := \sum_{k=0}^{N_λ} \frac{T}{k} E[α^* - α^{*, m} + 1]^2(t_k, X_{t_k}^\pi) Δt_k$ in terms of $∫_0^T E|α_t^* - α_t^{*, m}|^2 dt$;

2. obtain the estimate of $II := \sum_{k=0}^{N_λ} \frac{T}{k} E[α^* - α^{*, m} + 1]^2(t_k, X_{t_k}^\pi) Δt_k \leq 1$ by the property (36) of $P_N^λ$;

3. take care the difference between the $α$-part error and $II$:

$$III := \int_0^T E|α_t^* - α_t^*(π(t), X_{π(t)}^\pi)|^2 dt \leq C∥π∥. \quad (49)$$

Step (2) follows from the fact that $α^{*, m} + 1$ is defined as the projection of $α^{*, m} + 1$ into $N^λ$, and that $α^* ∈ N^λ$ if viewed as a function on $T × \mathbb{R}^n$. Step (3) is a consequence of Assumption 3 and (43). So it remains to address step (1).

To this end, we define $α_t^{*, m} = α(t_k, X_{t_k}^\pi, Z_{t_k}^\pi)$, then $α_t^{*, m} = α(t_k, X_{t_k}^\pi, α_{t_k}^{*, m}, Z_{t_k}^\pi)$, and $α_t^{*, m} = α^{*, m}(t_k, X_{t_k}^\pi)$, then $α_t^{*, m} = α(t_k, X_{t_k}^\pi, α_{t_k}^{*, m}, Z_{t_k}^{*, m})$. Thus,

$$I \leq (1 + λ^{-1}) \sum_{k=0}^{N_λ} E[α_{t_k}^{*, m} - α_{t_k}^{*, m + 1}]^2 Δt_k + C_λ \sum_{k=0}^{N_λ} E[α_{t_k}^{*, m} - α^*(t_k, X_{t_k}^\pi)]^2 Δt_k$$

$$:= (1 + λ^{-1})I^{(1)} + C_λI^{(2)}. \quad (50)$$

For term $I^{(1)}$, using (48) and the Lipschitz condition of $α$ in (37), we obtain

$$I^{(1)} \leq \sum_{k=0}^{N_λ} \left[ a_p E||Z_{t_k}^\pi - Z_{t_k}^{*, m + 1}||_F^2 + a_α E|α_{t_k}^{*, m} - α_{t_k}^{*, m}|^2 \right] Δt_k$$

$$\leq \left[ a_p (1 + λ^{-1})^2 C_z(λ) \left[ 2 + \frac{a_α + 2a_p C_z(λ)}{1 - α_p C_z(λ)} \right] + a_α (1 + λ^{-1}) \right] \int_0^T E|α_t^* - α_t^{*, m}|^2 dt$$

$$+ C_λ \left[ ||π|| + E|Y_{T}^{*, m + 1} - g(X_{T}^\pi)|^2 \right],$$

where we have also used

$$\int_0^T E|α_t^* - α_t^{*, m}|^2 dt = \int_0^T E|α(t, X_t, Z_t) - α(π(t), X_{π(t)}^\pi, Z_{π(t)}^\pi)|^2 dt \leq C||π||.$$
Here are some remarks regarding Theorem 3 on its implication for numerical algorithms. The primary concern is how we can implement the projection mapping in practice if wished. Note that we choose $1/2$-Hölder continuity in time in Assumption 3 for the generality of the result, although numerically it is challenging to guarantee the Hölder continuity. If we replace that with the Lipschitz continuity in time, as a more restrictive condition, and instead consider the projection onto the space with the Lipschitz continuity, the estimates still hold. Accordingly, there are some practical approaches in the literature on ensuring the Lipschitz continuity of deep neural networks that can be introduced in our algorithms. For instance, [20] gives an efficient and accurate estimation of Lipschitz constants for deep Neural networks, and [54] further extends it for robust training with regularization to keep the Lipschitz constant of neural networks small.

In practice, Wasserstein GAN [2, 25] has shown remarkable performance when using weight clipping as a loose but efficient way to impose the Lipschitz constraint. Therefore we can leverage similar techniques to keep the Lipschitz regularity during the training of the deep fictitious play. Also, notice that in the above, we define a single projection $N^\pi$ from the space of all players’ strategies $\alpha^{\pi,m}$, in consideration of the simplicity of the statement. One can also use the projection of $\alpha^{\pi,m}$ for each player with possibly easier numerical implementation and the same theoretical guarantee.

4.3 On the $\epsilon$-Nash Equilibrium

This section combines the previous analysis, identifies the $\epsilon$-Nash equilibrium produced by the deep fictitious play, and evaluates its performance on the original game.

**Theorem 4.** Under Assumptions 1–3, if $\hat{\alpha}$ is a policy function on $[0, T] \times \mathbb{R}^n$ and Lipschitz in $x$, and

$$\int_0^T \mathbb{E}|\alpha^*-\hat{\alpha}|^2(t, X_t) \, dt \leq \epsilon,$$

where $X_t$ is the forward component of (8), then

1. Given $\hat{\alpha}$, the game values produced by $\hat{\alpha}$ are near the Nash equilibrium, i.e.,

$$|\hat{J}_0(\hat{\alpha}) - J_0(\alpha^*)|^2 \leq C \epsilon,$$

where $\hat{J}_0(\hat{\alpha}) = [\hat{J}_0(\hat{\alpha}), \ldots, \hat{J}_0^N(\hat{\alpha})]$ with $\hat{J}_0(\hat{\alpha}) := \inf_{\beta^i \in A^i} J_0^i(\beta^i, \hat{\alpha}^{-i})$, $J_0(\hat{\alpha}) = [J_0^i(\hat{\alpha}), \ldots, J_0^N(\hat{\alpha})]$ with $J_0^i$ defined in (3). Thus, there exists $0 < \epsilon_1 \ll 1$ such that $\sum_{i=1}^N \epsilon_i^2 \leq C \epsilon$ and

$$J_0^i(\beta^i, \hat{\alpha}^{-i}) \geq J_0^i(\hat{\alpha}) - \epsilon_1, \quad \forall \beta^i \in A^i \text{ and } i \in I.$$

Here $C$ is a constant depending on the coefficients involved in Assumptions 2–3 and $T$, which may vary from line to line in the proof.

2. The generated game paths $X_t^{\hat{\alpha}}$ are close to the paths $X_t^{\alpha^*}$ associated to the Nash equilibrium:

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| X_t^{\alpha^*} - X_t^{\hat{\alpha}} \right|^2 \right] \leq C_\lambda \epsilon^\lambda,$$

where $X_t^{\alpha^*}$ and $X_t^{\hat{\alpha}}$ follow (1) with the true Nash equilibrium strategy $\alpha^*$ and $\hat{\alpha}$. Here $\lambda$ is an arbitrary constant in $(0, 1)$, and $C_\lambda$ is a constant depending on the coefficients involved in Assumptions 2–3, $T$ and $\lambda$.

Immediately, we have the following corollary.

**Corollary 1.** Under Assumptions 1–3, assuming the sub-problems (34) are solved accurate enough at all stages, i.e.,

$$\mathbb{E}|g(X_T^j) - Y_T^{\pi,j}|^2 \leq C \epsilon^2, \quad \forall j \leq m,$$

here $C$ is a constant depending only on the coefficients involved in Assumptions 2–3 and $T$. Then, for sufficiently large $m$ and small mesh size $\|\pi\|$, the strategy $\alpha^m$ defined in (40), as an interpolated policy based on the deep fictitious play, forms an $\epsilon$-Nash equilibrium.
Proof. This follows from (53) in Theorem 4, with the assumptions satisfied according to equations (39), (42) and (47).

Remark 3. As mentioned in Remark 1, there are still some theoretical issues unsolved regarding the approximation error and optimization of the deep BSDE method. The analysis of the deep fictitious play method has similar issues that remain open. To circumvent these issues and have a rigorous statement for \( \epsilon \)-Nash equilibrium, we introduce assumption (54). In practice, an observable proxy of (54) is the training loss of the deep BSDE method evaluated by its Monte Carlo counterpart.

Proof of Theorem 4. The proof of item (1) relies on the estimates of BSDEs presented previously. Let \((X_t, Y_t^{i,FP}, Z_t^{i,FP})\) solves (14) with \( \alpha^{-i,m} \) replaced by \( \hat{\alpha}^{-i} \). By the nonlinear Feynman-Kac formula (cf. [52, 18, 53]) and the associated HJB equation, we have \( \mathbb{E}[Y_0^{i,FP}] = J_0^i(\hat{\alpha}) \). Therefore, we have

\[
|\tilde{J}_0(\hat{\alpha}) - J_0(\alpha^*)|^2 = |\mathbb{E}[Y_0^{FP}] - \mathbb{E}[Y_0]|^2 \leq \mathbb{E}[Y_0^{FP} - Y_0]^2.
\]  

To bound the above term, we claim a stronger result:

\[
\sup_{0 \leq t \leq T} \mathbb{E}|Y_t^{FP} - Y_t|^2 + \int_0^T \mathbb{E}\|Z_t^{FP} - Z_t\|_F^2 \, dt \leq C\epsilon,
\]  

where \((Y_t, Z_t)\) solves (8), \(Y_t^{FP} = [Y_t^{1,FP}, \ldots, Y_t^{N,FP}]^T\), and \(Z_t^{FP} = [Z_t^{1,FP}, \ldots, Z_t^{N,FP}]\).

The proof of (56) essentially follow the same argument as in Theorem 2 with replacing \( \alpha^m \) by \( \hat{\alpha}_t \), \( \alpha_t^{m+1} \) by \( \beta_t := \alpha(t, X_t, \hat{\alpha}, Z_t^{FP}) \), \( Y_t^{m+1} \) by \( Y_t^{FP} \) and \( Z_t^{m+1} \) by \( Z_t^{FP} \). More precisely, following the derivation of (31) and (32), with \( \lambda > 5M \), one has

\[
\sup_{0 \leq t \leq T} \mathbb{E}\|Y_t^{FP} - Y_t\|^2 + \int_0^T \mathbb{E}\|Z_t^{FP} - Z_t\|_F^2 \, dt \leq \left( e^{5M T} \frac{2\theta_\alpha M \Sigma g_x + f_\alpha}{M_\phi} + C_\varphi(\lambda) \right) \int_0^T \mathbb{E}\|\beta_t - \alpha^*\|^2 + 2\mathbb{E}|\hat{\alpha}_t - \alpha_t^*|^2 \, dt,
\]

where \( C_\varphi(\lambda) = \frac{10\theta_\alpha M \Sigma g_x + 5f_\alpha}{\lambda - 5M_\phi} \left( \frac{\lambda^2 T e^{\lambda M_\phi T}}{5M_\phi} + 1 \right) \). Similarly, from (33), we deduce

\[
\int_0^T \mathbb{E}|\beta_t - \alpha^*|^2 \, dt \leq \frac{a_\alpha + 2a_p C_\varphi(\lambda)}{1 - a_p C_\varphi(\lambda)} \int_0^T \mathbb{E}|\hat{\alpha}_t - \alpha_t^*|^2 \, dt.
\]

Combing the above two equations with (51) gives (56). By (55) and (56), we have the first equality in (52).

If we let \((X_t, Y_t^{i,PU}, Z_t^{i,PU})\) solve (16) with \( \alpha^m \) replaced by \( \hat{\alpha} \), an argument similar to (55) and (56) can give the second inequality in (52). Then (53) is obtained by observing

\[
J_0^i(\hat{\alpha}, \hat{\alpha}) \geq J_0^i(\hat{\alpha}), \quad \forall \beta^i \in A_i,
\]

and \( |\tilde{J}_0(\hat{\alpha}) - J_0(\alpha^*)| \leq C\epsilon \).

We now prove item (2). Under the standing assumptions, we first observe that \( b^1(t, x) := b(t, x, \alpha^*(t, x)) = \Sigma(t, x)\phi(t, x, \alpha^*(t, x)) \) and \( b^2(t, x) := b(t, x, \hat{\alpha}(t, x)) = \Sigma(t, x)\phi(t, x, \hat{\alpha}(t, x)) \) are Lipschitz in \( x \). Thus \( X_t^\alpha \) is well-defined, and the standard estimates in SDE gives (cf. [65, Theorem 3.2.4])

\[
\mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \left| X_t^{\alpha^*} - X_t^{\hat{\alpha}} \right| \right)^2 \right] \leq C \mathbb{E} \left[ \left( \int_0^T b^1(t, X_t^{\alpha^*}) - b^2(\tilde{t}, X_t^{\hat{\alpha}}) \, dt \right)^2 \right].
\]  

To bound the right-hand side above with the condition (51), let us define a new probability measure \( \mathbb{Q} \), and denote by \( Z \) the Radon-Nikodym derivative:

\[
d\mathbb{Q} = d\mathbb{P} = Z := \exp \left\{ - \int_0^T \phi_t^{\alpha^*} \cdot dW_t - \frac{1}{2} \int_0^T |\phi_t^{\alpha^*}|^2 \, dt \right\},
\]

20
where \( \phi_{t}^{\alpha} := \phi(t, X_{t}^{\alpha}, \alpha^{*}(t, X_{t}^{\alpha})) \). By Assumptions 2, the Novikov’s condition is fulfilled. Thus \( \mathbb{Q} \sim \mathbb{P} \), and \( W_{t}^{\mathbb{Q}} := W + \int_{0}^{t} \phi_{s}^{\alpha} \, ds \) is a standard Brownian motion under \( \mathbb{Q} \). In particular, the process \( X_{t}^{\alpha^{*}} \) can be rewritten as \( X_{t}^{\alpha^{*}} = x_{0} + \int_{0}^{t} \Sigma(s, X_{s}^{\alpha^{*}}) \, dW_{s}^{\mathbb{Q}} \), and immediately from (51) we have

\[
\int_{0}^{T} \mathbb{E}_{\mathbb{Q}}|\alpha^{*} - \hat{\alpha}|^{2}(t, X_{t}^{\alpha^{*}}) \, dt \leq c,
\]  
(58)

where we denote by \( \mathbb{E}_{\mathbb{Q}} \) the expectation under measure \( \mathbb{Q} \). We next compute a bound for \( Z^{-\gamma} \) under \( \mathbb{Q} \), for \( \gamma > 1 \):

\[
\mathbb{E}_{\mathbb{Q}}[Z^{-\gamma}] = \mathbb{E}_{\mathbb{Q}}\left[ \exp \left\{ \gamma \int_{0}^{T} \phi_{t}^{\alpha^{*}} \cdot dW_{t}^{\mathbb{Q}} - \frac{\gamma}{2} \int_{0}^{T} |\phi_{t}^{\alpha^{*}}|^{2} \, dt \right\} \right]
\]
\[
\leq \mathbb{E}_{\mathbb{Q}}^{1/2} \left[ \exp \left\{ 2\gamma \int_{0}^{T} \phi_{t}^{\alpha^{*}} \cdot dW_{t}^{\mathbb{Q}} - 2\gamma^{2} \int_{0}^{T} |\phi_{t}^{\alpha^{*}}|^{2} \, dt \right\} \right] \times \mathbb{E}_{\mathbb{Q}}^{1/2} \left[ \exp \left\{ (2\gamma^{2} - \gamma) \int_{0}^{T} |\phi_{t}^{\alpha^{*}}|^{2} \, dt \right\} \right]
\]
\[
\leq e^{CT(\gamma^{2} - \frac{1}{2})},
\]
where \( \mathbb{E}_{\mathbb{Q}}^{p} \) denote \( (\mathbb{E}_{\mathbb{Q}}[\cdot])^{p} \), and we have used the Cauchy-Schwartz inequality and the boundedness of \( \phi \) in Assumptions 2. Therefore, we have

\[
\mathbb{E} \left[ \left( \int_{0}^{T} b^{1}(t, X_{t}^{\alpha^{*}}) - b^{2}(t, X_{t}^{\alpha^{*}}) \, dt \right)^{2} \right]
\]
\[
\leq \mathbb{E}_{\mathbb{Q}}^{1 - \frac{1}{p}} \left[ \left( \int_{0}^{T} b^{1}(t, X_{t}^{\alpha^{*}}) - b^{2}(t, X_{t}^{\alpha^{*}}) \, dt \right)^{2} \right] \mathbb{E}_{\mathbb{Q}}^{\frac{1}{p}}[Z^{-\gamma}]
\]
\[
\leq C_{\gamma} \mathbb{E}_{\mathbb{Q}}^{1 - \frac{1}{p}} \left[ \left( \int_{0}^{T} b^{1}(t, X_{t}^{\alpha^{*}}) - b^{2}(t, X_{t}^{\alpha^{*}}) \, dt \right)^{2} \right] \mathbb{E}_{\mathbb{Q}}^{\frac{1}{p}} \left[ \left( \int_{0}^{T} b^{1}(t, X_{t}^{\alpha^{*}}) - b^{2}(t, X_{t}^{\alpha^{*}}) \, dt \right)^{2} \right]^{\frac{1}{2}}
\]
\[
\leq C_{\gamma} \mathbb{E}_{\mathbb{Q}}^{1 - \frac{1}{p}} \left[ \int_{0}^{T} |\alpha^{*} - \hat{\alpha}|^{2}(t, X_{t}^{\alpha^{*}}) \, dt \right] \leq C_{\gamma} \epsilon^{1 - \frac{1}{p}},
\]
where we have consecutively used Hölder’s inequality, the estimate of \( \mathbb{E}_{\mathbb{Q}}[Z^{-\gamma}] \), the Lipschitz property of \( \phi(t, x, \alpha) \), the boundedness of \( \Sigma \) and \( b \), and the estimate (58). Here \( C_{\gamma} \) is a constant depending on the constants involved in Assumptions 1–2 in addition to \( T \) and \( \gamma \), which may vary from line to line. With (57) and noticing \( 0 < 1 - \frac{1}{p} < 1 \) we conclude.

In practice, the game is play on \( T \), but not \([0, T]\). Therefore, we will define a discrete version of the stochastic differential game (1)–(2) and evaluate the performance of \( \alpha^{\pi, m} \) in section 4.2 on the discrete game. To be precise, given a policy function \( \alpha^{\pi} \) on \( T \times \mathbb{R}^{n} \), we define the discrete state process \( X_{t}^{\pi, \alpha^{\pi}} \) and discrete individual cost functional \( J_{0}^{\pi, i}(\alpha^{\pi}) \) as follows

\[
X_{t_{k+1}}^{\pi, \alpha^{\pi}} = x_{0} + b(k, X_{t_{k}}^{\pi, \alpha^{\pi}}, \alpha^{\pi}(t_{k}, X_{t_{k}}^{\pi, \alpha^{\pi}}))\Delta t_{k} + \Sigma(t_{k}, X_{t_{k}}^{\pi, \alpha^{\pi}}) \, dW_{k}, \quad X_{0}^{\pi, \alpha^{\pi}} = x_{0},
\]  
(59)

\[
J_{0}^{\pi, i}(\alpha^{\pi}) = \mathbb{E} \left[ \sum_{j=0}^{\pi - 1} f^{i}(t_{k}, X_{t_{k}}^{\pi, \alpha^{\pi}}, \alpha^{\pi}(t_{k}, X_{t_{k}}^{\pi, \alpha^{\pi}}))\Delta t_{k} + g^{i}(X_{T}^{\pi, \alpha^{\pi}}) \right].
\]

Note that when there are both \( \pi \) and \( \alpha \) in the superscript of \( X \), it refers to the (discrete) process of the original state (1), and when there is only \( \pi \) in the superscript, it refers to the (discrete) process \( X_{t} = x_{0} + \int_{0}^{t} \Sigma(s, X_{s}) \, dW_{s} \). We then state a discrete version of Theorem 4.

**Theorem 5.** Under Assumptions 1–3, if \( \hat{\alpha}^{\pi} \) is a policy function on \( T \times \mathbb{R}^{n} \), Lipschitz in \( x \) and Hölder continuous with \( t \):

\[
|\hat{\alpha}^{\pi}(t_{1}, x_{1}) - \hat{\alpha}^{\pi}(t_{2}, x_{2})|^{2} \leq L'[|t_{1} - t_{2}| + |x_{1} - x_{2}|],
\]

Proof. Let \( \hat{\alpha} \) be obtained by (43), (60) and (61), we have
\[
\int_0^T \mathbb{E}[\alpha^*(t, X_t) - \hat{\alpha}^*(\pi(t), X_{\pi(t)}^\pi)]^2 \, dt \leq \epsilon,
\] (60)
then

(1) The value of the discrete game produced by \( \alpha^* \) is close to the one associated to the Nash equilibrium of the continuous game, i.e.,
\[
|J_0^*(\hat{\alpha}^* - J_0(\alpha^*)|^2 \leq C[\epsilon + \|\pi\|],
\]
where \( J_0^*(\hat{\alpha}^*) = [J_0^{\pi,1}(\hat{\alpha}^*), \ldots, J_0^{\pi,N}(\hat{\alpha}^*)] \). Moreover, there exists \( 0 < \epsilon_i < 1 \) such that \( \sum_{i=1}^N \epsilon_i^2 \leq C[\epsilon + \|\pi\|] \) and
\[
J_i^0(\beta^g, \hat{\alpha}^* - \epsilon) = J_i^0(\hat{\alpha}^*) \geq \epsilon_i, \quad \forall \beta^g \in \mathcal{A} \text{ and } i \in I.
\]
Here \( C \) is a constant depending on the coefficients involved in Assumptions 2–3, \( L' \) and \( T \), which may vary from line to line in the proof.

(2) The generated game paths \( X_{t_k}^{\pi_i, \hat{\alpha}^*} \) are close to the paths \( X_{t_k}^{\pi_i, \alpha^*} \) associated to the Nash equilibrium:
\[
\mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \left| X_t^{\alpha^*} - X_{\pi(t)}^{\pi_i, \hat{\alpha}^*} \right| \right)^2 \right] \leq C_\lambda[\epsilon + \|\pi\|],
\]
where \( X_t^{\alpha^*} \) follow (1) with the true Nash equilibrium strategy \( \alpha^* \) and \( X_{t_k}^{\pi_i, \hat{\alpha}^*} \) follow (59). Here \( \lambda \) is an arbitrary constant in \( (0, 1) \), and \( C_\lambda \) is a constant depending on the coefficients involved in Assumptions 2–3, \( L' \) and \( \lambda \).

Proof. Let \( \hat{\alpha}(t, x) = \inf_{t' \in T} [\hat{\alpha}^* (t', x) + L'|t' - t|] \), then with an argument similar to that in Theorem 3, \( \hat{\alpha} \) satisfies:
\[
|\hat{\alpha}(t_1, x_1) - \hat{\alpha}(t_2, x_2)|^2 \leq L'(|t_1 - t_2| + |x_1 - x_2|)^2.
\] (61)
By (43), (60) and (61), we have
\[
\begin{align*}
\int_0^T \mathbb{E}[\alpha^* - \hat{\alpha}]^2(t, X_t) \, dt \\
\leq 2 \left[ \int_0^T \mathbb{E}[\alpha^*(t, X_t) - \hat{\alpha}^*(\pi(t), X_{\pi(t)}^\pi)]^2 \, dt \\ + \int_0^T \mathbb{E}|\hat{\alpha}(t, X_t) - \hat{\alpha}^*(\pi(t), X_{\pi(t)}^\pi)]^2 \, dt \right] \\
\leq C[\|\pi\| + \epsilon].
\end{align*}
\] (62)
By the regularity of \( \hat{\alpha} \) (c.f. (61)) and the standard estimates of the Euler Scheme of SDE (c.f. [41]), we can obtain
\[
\mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \left| X_t^{\hat{\alpha}} - X_{\pi(t)}^{\pi_i, \hat{\alpha}^*} \right| \right)^2 \right] \leq C\|\pi\|.
\] (63)
Observing that
\[
J_0^0(\hat{\alpha}^*) - J_0(\hat{\alpha})
= \mathbb{E} \left[ \int_0^T \left[ f(\pi(t), X_{\pi(t)}^{\pi_i, \hat{\alpha}^*}, \hat{\alpha}^*(\pi(t), X_{\pi(t)}^{\pi_i, \hat{\alpha}^*})) - f(t, X_t^{\hat{\alpha}}, \hat{\alpha}(t, X_t^{\hat{\alpha}})) \right] dt \right.
\]
\[
+ \left. g(X_t^{\pi_i, \hat{\alpha}^*}) - g(X_t^{\hat{\alpha}}) \right],
\]
with (61), (63) and Assumption 2, one has
\[
|J_0^0(\hat{\alpha}^*) - J_0(\hat{\alpha})|^2 \leq C[\|\pi\|].
\] (64)
Finally, with (62), (63), (64) and Theorem 4, we reach all the conclusions of this theorem.
\[ \square \]
5 Numerical Examples

We supplement our theoretical analysis by numerical examples. We shall mainly focus on how deep BSDE performs when combined with policy update strategy in the decoupling step, i.e., when solving (16). We refer readers to [27] for the numerical performance when the deep BSDE method is used to solve the sub-problems derived from fictitious play. The example we present here is an inter-bank game concerning the systemic risk [13]. Assume an inter-bank market with \( N \) banks, and denote by \( X_i \in \mathbb{R} \) the log-monetary reserves of bank \( i \) at time \( t \). Its dynamics are modeled as the following diffusion processes,

\[
\begin{align*}
    dX_i^t &= \left[a(X_i^t - X_i^0_t) + \alpha_i^t\right]dt + \sigma \left( \rho dW_i^0 + \sqrt{1 - \rho^2} dW_i^i \right) , \\
    X_t &= \frac{1}{N} \sum_{i=1}^{N} X_i^t , \quad i \in \mathcal{I}.
\end{align*}
\]

Here \( a(X_i^t - X_i^0_t) \) represents the rate at which bank \( i \) borrows from or lends to other banks in the lending market, while \( \alpha_i^t \) denotes its control rate of cash flows to a central bank. The standard Brownian motions \( \{W_i^0\}_{i=0}^{N} \) are independent, in which \( \{W_i^i, i \geq 1\} \) stands for the idiosyncratic noises and \( W_i^0 \) denotes the systemic shock, or so-called common noise in the general context. To describe the model in the form of (1), we concatenate the log-monetary reserves \( X_i^t \) of \( N \) banks to form \( X^P = [X_1^t, \ldots, X_N^t]^T \). The associated drift term and diffusion term are defined as

\[
\begin{align*}
    b(t, x, \alpha) &= [a(\bar{x} - x^1) + \alpha^1, \ldots, a(\bar{x} - x^N) + \alpha^N]^T \in \mathbb{R}^{N \times 1} , \\
    \bar{x} &= \frac{1}{N} \sum_{i=1}^{N} x^i , \\
    \Sigma(t, x) &= \begin{bmatrix}
    \sigma \rho & \sigma \sqrt{1 - \rho^2} & 0 & \cdots & 0 \\
    \sigma \rho & 0 & \sigma \sqrt{1 - \rho^2} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    \sigma \rho & 0 & 0 & \cdots & \sigma \sqrt{1 - \rho^2}
\end{bmatrix} \in \mathbb{R}^{N \times (N+1)},
\end{align*}
\]

and \( W_i = (W_i^0, \ldots, W_i^N) \) is \((N+1)\)-dimensional. The cost functional (3) that player \( i \) wishes to minimize has the form

\[
f^i(t, x, \alpha) = \frac{1}{2}(\alpha^i)^2 - q\alpha^i(\bar{x} - x^i) + \frac{c}{2}(\bar{x} - x^i)^2 , \quad g^i(x) = \frac{c}{2}(\bar{x} - x^i)^2.
\]

Under such specifications, the solution of this game admits a quadratic form whose coefficient functions can be solved from a Riccati equation. We direct the interested readers to [13, 27] for the detailed interpretation of this model and the explicit characterization of the solution.

In our numerical computation, we choose \( N = 10, T = 1, \) and

\[
a = 0.1, \quad q = 0.1, \quad c = 0.5, \quad \epsilon = 0.5, \quad \rho = 0.2, \quad \sigma = 1.
\]

We discretize the time \([0, T]\) into \( N_T = 40 \) intervals and specify the hypothesis spaces \( \mathcal{N}_0^i \) and \( \{\mathcal{N}_k^i\}_{k=0}^{N_T-1} \) for each player \( i \) as follows. We parametrize \( V^i(0, x) \) (the superscript \( m \) is dropped again for simplicity) with a neural network, denoted by \( \text{Net}_V(x) \), as the space \( \mathcal{N}_0^i \) of \( Y_0^i \). We also parametrize \( \nabla_{x} V^i(t, x) \) with another network, denoted by \( \text{Net}_{V_V}(t, x) \), as the space of \( \{\mathcal{N}_k^i\}_{k=0}^{N_T-1} \), in which the timestamp \( t_k \) is provided as another dimension of the input. This choice is in consistence with our theoretical analysis in Theorem 3 involving the linear interpolation of the strategy in time.

In this numerical example, we use fully-connected feedforward networks to instantiate both \( \text{Net}_V(x) \) and \( \text{Net}_{V_V}(t, x) \). Since the problem is homogeneous among all the players, we let two networks share the same parameters among all the players and only solve player 1’s problem for updating the parameters. Both networks consist of three hidden layers with a width of 40. The activation function is hyperbolic tangent, and the technique of batch normalization [37] is adopted right after each linear transformation and before activation. For simplicity, we do not impose the projection procedure discussed in Section 4.2.

Regarding the optimization, the loss function in Deep BSDE is differentiable with respect to the network parameters. We can use backpropagation to derive the gradient of the loss function with respect to all the parameters in the neural networks and employ stochastic gradient descent (SGD) to optimize all the parameters. In this work, we use Adam optimizer [40] to optimize network parameters with constant learning rate 5e-4 and batch size 256. The parameters are updated by 30000 steps in total.
Figure 1: A sample path for each player of the inter-bank game when $N = 10$, obtained from decoupling the problem by fictitious play and solving the sub-problems with the Deep BSDE method. Top: the optimal state process $X^i_t$ (solid lines) and its approximation $\hat{X}^i_t$ (circles) provided by the optimized neural networks, under the same realized path of Brownian motion. Bottom: comparisons of the strategies $\alpha^i_t$ and $\hat{\alpha}^i_t$ (dashed lines).

To implement the algorithm, we also need to specify the distribution of the initial state $x_0$ in (14) or (16). We follow the same way as in [27]. Each component of $x_0$, as the initial state of each player, is sampled independently from the uniform distribution on $[-\delta_0, \delta_0]$. $\delta_0$ is chosen such that in the following process driven by the optimal policy $\alpha^*$,

$$dX^\alpha_t = b(t, X^\alpha_t, \alpha^*(t, X^\alpha_t)) dt + \Sigma(t, X^\alpha_t) dW_t, \quad X_0 = x_0,$$

the standard deviation of $\{X_t\}_{t=0}^T$ is approximately $\delta_0$. In other words, $\delta_0$ is determined as a fixed-point. The rationale for such a procedure is to make sure the data generated for the learning is representative enough in the whole state space.

Note that our technical assumptions are not strictly satisfied in this example, since $f, g$ are not Lipschitz continuous, $\phi$ is not uniform bounded, and $T$ is not sufficiently small. Nevertheless, our numerical results show that the deep BSDE method can solve this game when combined with policy update. In particular, we compute the relative error of controls (proportional to the gradient of value function):

$$\text{RSE} = \frac{\sum_{0 \leq k \leq N_T-1} \left( \nabla_x V^1(t_k, x^{(j)}_{t_k}) - \nabla_x \hat{V}^1(t_k, x^{(j)}_{t_k}) \right)^2}{\sum_{0 \leq k \leq N_T-1} \left( \nabla_x V^1(t_k, x^{(j)}_{t_k}) - \nabla_x \hat{V}^1 \right)^2},$$

where $V^1$ is the true solution (of player 1), $\hat{V}^1$ is the prediction from the neural networks, and $\hat{V}^1$ (resp. $\nabla_x V^1$) is the average of $V^1$ (resp. $\nabla_x V^1$) evaluated at all the indices $j, k$. To compute the relative error, we generate $J = 256$ ground truth sample paths $\{x^{(j)}_{t_k}\}_{k=0}^{N_T-1}$ using Euler scheme based on (16) and the true
optimal strategy. Note that the superscript \((j)\) here does not mean the player index, but the \(j^{th}\) path for all players. The final RSE for the Deep BSDE method is 0.27\%. Figures 1 presents one sample path for each player of the optimal state process \(X^i_t\) and the optimal control \(\alpha^i_t\) vs. their approximations \(\hat{X}^i_t, \hat{\alpha}^i_t\), with good agreement.

6 Conclusion

In this paper, we established the theoretical foundation for the deep fictitious play algorithm for finding Markovian Nash equilibrium proposed in [27]. Specifically, we proved the following three things: 1. The solutions of the decoupled sub-problems, if solved exactly and repeatedly, converge to the true Nash equilibrium; 2. The numerical error of each sub-problem, if solved by deep BSDE individually and repeatedly, converges to zero subject to the universal approximation capacity of neural networks; 3. The interpolated strategy based on the deep fictitious play algorithm forms a \(\epsilon\)-Nash equilibrium, after sufficiently many stages \(m\) and with sufficiently small mesh \(\|\pi\|\). We also generalize the algorithm by proposing a new approach to decouple the games, and present a numerical example in the end to show the empirical convergence beyond the technical assumptions used in the theorems. In the future, with this solidly established theory of deep fictitious play, we aim to study the competitions in Finance, including P2P lending platforms from the Fintech industry and insurance markets. We also plan to generalize the theory and algorithm to stochastic differential games with delays.

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