Least energy solutions for affine $p$-Laplace equations involving subcritical and critical nonlinearities

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Abstract The paper is concerned with Lane-Emden and Brezis-Nirenberg problems involving the affine $p$-laplace nonlocal operator $\Delta_p^A$, which has been introduced in [21] driven by the affine $L^p$ energy $\mathcal{E}_{p,\Omega}$ from convex geometry due to Lutwak, Yang and Zhang [29]. We are particularly interested in the existence and nonexistence of positive $C^1$ solutions of least energy type. Part of the main difficulties are caused by the absence of convexity of $\mathcal{E}_{p,\Omega}$ and by the comparison $\mathcal{E}_{p,\Omega}(u) \leq \|u\|_{W^{1,p}_0(\Omega)}$ generally strict.

1 Introduction and main results

In two seminal works, Lutwak, Yang and Zhang established the famous sharp affine $L^p$ Sobolev inequality, namely, in [43] for $p = 1$ and in [29] for $1 < p < n$. Precisely, for any $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, it states that

$$
\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq K_{n,p} \mathcal{E}_p(u),
$$

where $p^* = \frac{np}{n-p}$ is the Sobolev critical exponent, $\mathcal{D}^{1,p}(\mathbb{R}^n)$ denotes the usual space of weakly differentiable functions in $\mathbb{R}^n$ endowed with the $L^p$ gradient norm and $\mathcal{E}_p(u)$ stands for the affine

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$L^p$ energy expressed for any $p \geq 1$ as

$$E_p(u) = \alpha_{n,p} \left( \frac{\int_{S^{n-1}} \left( \int_{\mathbb{R}^n} \left| \nabla_\xi u(x) \right|^p \, dx \right)^{-\frac{n}{p}} \, d\sigma(\xi) }{\int_{\mathbb{R}^n} \left| \nabla_\xi u(x) \right|^{p-1} \, dx} \right)^{-\frac{1}{p}}$$

with $\alpha_{n,p} = (2\omega_{n+p-2})^{-1/p} (n\omega_n \omega_{p-1})^{1/p} (n\omega_n)^{1/n}$. Here, $\nabla_\xi u(x)$ represents the directional derivative $\nabla u(x) \cdot \xi$ with respect to the direction $\xi \in S^{n-1}$ and $\omega_k$ is the volume of the unit Euclidean ball in $\mathbb{R}^k$.

The value of the optimal constant in (1) is well known and given for $p = 1$ by

$$K_{n,1} = \pi^{-\frac{n}{2}} n^{-1} \Gamma \left( \frac{n}{2} + 1 \right)^{\frac{1}{n}}$$

and for $1 < p < n$ by

$$K_{n,p} = \pi^{-\frac{n}{2}} n^{-\frac{1}{2}} \left( \frac{p-1}{n-p} \right)^{1-\frac{1}{p}} \left( \frac{\Gamma \left( \frac{n}{2} + 1 \right) \Gamma(n)}{\Gamma \left( -\frac{n}{p} + n + 1 \right) \Gamma \left( \frac{n}{p} \right)} \right)^{\frac{1}{n}}.$$

Moreover, the corresponding extremal functions are precisely $u(x) = a\chi_B (bA(x-x_0))$ for $p = 1$ (i.e. multiples of characteristics functions of ellipsoids) and

$$u(x) = a \left( 1 + b|A(x-x_0)|^{p-1} \right)^{1-\frac{1}{p}},$$

for $a \in \mathbb{R}$, $b > 0$, $x_0 \in \mathbb{R}^n$ and $A \in SL(n)$, where $B$ is the unit Euclidean ball in $\mathbb{R}^n$ and $SL(n)$ denotes the special linear group of $n \times n$ matrices with determinant equal to 1.

Ever since various improvements and new affine functional inequalities have emerged in a very comprehensive literature. For affine Sobolev type inequalities on the whole $\mathbb{R}^n$ we refer to [18, 19, 29, 43], for affine Sobolev type trace inequalities on the half space $\mathbb{R}^n_+$ to [11, 33] and for affine $L^p$ Poincaré-Sobolev inequalities on bounded domains to [26] for $p = 1$ and to [39] for $p = 2$.

For other affine functional inequalities we mention the references [6, 11, 17, 18, 19, 20, 22, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 42, 43].

More recently, Haddad, Jiménez and Montenegro [21] investigated affine $L^p$ Poincaré inequalities on bounded domains $\Omega$ and solved the associated Faber-Krahn problem for any $p \geq 1$. Its solution demanded the introduction of a differential operator $\Delta_p^A$ for $p > 1$ closely related to the affine $L^p$ energy on $W^{1,p}_0(\Omega)$ given by

$$\mathcal{E}_{p,\Omega}(u) = \alpha_{n,p} \left( \int_{S^{n-1}} \left( \int_{\Omega} \left| \nabla_\xi u(x) \right|^p \, dx \right)^{-\frac{n}{p}} \, d\sigma(\xi) \right)^{-\frac{1}{p}},$$

where $W^{1,p}_0(\Omega)$ denotes the completion of the space $C_0^\infty(\Omega)$ of smooth functions compactly supported in $\Omega$ with respect to the norm

$$\|u\|_{W^{1,p}_0(\Omega)} = \|\nabla u\|_{L^p(\Omega)} = \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{1}{p}}.$$
The affine $p$-Laplace operator $\Delta_p^A = \Delta_{p,\Omega}$ on $W^{1,p}_0(\Omega) \setminus \{0\}$ is defined as the nonlocal quasilinear operator in divergence form:
\[
\Delta_p^A u = -\text{div} \left( H_p^{-1}(\nabla u) \nabla H_p(\nabla u) \right),
\]
where
\[
H_p(\zeta) = \alpha_{n,p} \mathcal{E}^{n+p}_{p,\Omega}(u) \int_{S^{n-1}} \left( \int_{\Omega} |\nabla \xi u(x)|^p \, dx \right)^{-\frac{n+p}{p}} |\langle \xi, \zeta \rangle|^p \, d\sigma(\xi) \quad \text{for} \ \zeta \in \mathbb{R}^n.
\]
By using a key relation satisfied by $\mathcal{E}_{p,\Omega}$ which works specially in the case $p = 2$, we point out that $\Delta_2^A$ coincides with the affine Laplace operator introduced by Schindler and Tintarev in [39]. The name of $\Delta_p^A$ is inspired on two fundamental properties presented in Section 4 of [21]. For $p > 1$, firstly it coincides with the standard $p$-Laplace operator $\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$ for radial functions when $\Omega$ is a ball centered at the origin and, secondly, it verifies the affine invariance property $\Delta_p^A(u \circ T) = (\Delta_p^A u) \circ T$ on $T^{-1}(\Omega)$ for every $u \in W^{1,p}_0(\Omega)$ and $T \in SL(n)$.

Our main goal is the study of the boundary problem
\[
\left\{ \begin{array}{ll}
\Delta_p^A u = u^{q-1} + \lambda u^{p-1} & \text{in} \quad \Omega, \\
u = 0 & \text{on} \quad \partial\Omega,
\end{array} \right.
\]
where $\Omega$ is a bounded open subset of $\mathbb{R}^n$ with smooth boundary (e.g. of $C^{2,\alpha}$ class).

Throughout this work, it is assumed that $1 < p < n$, $p < q \leq p^*$ and $\lambda$ is a real parameter. Particularly important cases are the Lane-Emden problem ($\lambda = 0$) and the Brezis-Nirenberg problem ($q = p^*$) regarding the affine $p$-Laplace operator.

A nonnegative function $u_0 \in W^{1,p}_0(\Omega) \setminus \{0\}$ is said to be a weak solution of (2), if
\[
\int_{\Omega} H_p^{-1}(\nabla u_0) \nabla H_p(\nabla u_0) \cdot \nabla \varphi \, dx = \int_{\Omega} u_0^{q-1} \varphi \, dx + \int_{\Omega} \lambda u_0^{p-1} \varphi \, dx
\]
for all $\varphi \in W^{1,p}_0(\Omega)$. If $u_0 \in C^1(\overline{\Omega})$ and $u_0 > 0$ in $\Omega$, we simply say that $u_0$ is a positive $C^1$ solution.

The left-hand side of (3) is precisely equal to
\[
\left. \frac{d}{dt} \mathcal{E}_{p,\Omega}^p(u_0 + t\varphi) \right|_{t=0},
\]
according to the computation of the directional derivative of $\mathcal{E}_{p,\Omega}^p(u)$ at $u_0 \in W^{1,p}_0(\Omega) \setminus \{0\}$ in the direction $\varphi \in W^{1,p}_0(\Omega)$ done in the proof of Theorem 10 of [21].

We are particularly interested in the existence of positive $C^1$ solution and nonexistence of nontrivial nonnegative weak solution that attains the minimum level.
$$c_A = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\mathcal{E}_{p,\Omega}(u) - \lambda \int_\Omega |u|^p \,dx}{(\int_\Omega |u|^q \,dx)^{\frac{p}{q}}}.$$  

Such a solution is called least energy type solution of (2).

In [21], the authors also showed that the operator $\Delta^A_p$ on $W^{1,p}_0(\Omega)$ possesses a positive principal eigenvalue $\lambda^A_{1,p} = \lambda^A_{1,p}(\Omega)$ characterized variationally by

$$\lambda^A_{1,p} = \inf\{\mathcal{E}_{p,\Omega}(u) : u \in W^{1,p}_0(\Omega), \|u\|_{L^p(\Omega)} = 1\}.$$  

Our main theorems are affine counterparts of pioneering results dealing with problems related to the $p$-Laplace operator.

**Theorem 1.1.** If $p < q < p^*$, then the affine problem (2) admits a positive $C^1$ least energy solution for any $\lambda < \lambda^A_{1,p}$.

An interesting consequence of this result occurs when $\lambda = 0$. In effect, for any $q \in [1, p^*$], the affine Sobolev inequality (4) yields the affine $L^q$ Poincaré-Sobolev inequality on $W^{1,p}_0(\Omega)$:

$$\mu^A_{p,q} \|u\|_{L^q(\Omega)} \leq \mathcal{E}_{p,\Omega}(u)$$

for an optimal constant $\mu^A_{p,q} = \mu^A_{p,q}(\Omega)$. A nonzero function $u_0 \in W^{1,p}_0(\Omega)$ is said to be an extremal for (1), if this inequality becomes equality. The existence of extremals has been established in [21] for $q = p$ and in [23] for $1 \leq q < p$. Using these results and Theorem 1.1, we readily derive

**Corollary 1.1.** The sharp affine inequality (11) admits extremal functions if, and only if, $1 \leq q < p^*$.

Clearly, the cases $p < q < p^*$ follow from the equality $\mu^A_{p,q} = c_A$ and Theorem 1.1. Already when $q = p^*$, exist no extremal as can easily be seen through usual arguments. In effect, a canonical rescaling immediately yields $\mu^A_{p,p^*} = K_{n,p}^{-1}$. On the other hand, as described in the introduction, the extremal functions for the sharp affine Sobolev inequality (11) doesn’t belong to $W^{1,p}_0(\Omega)$.

**Theorem 1.2.** If $q = p^*$ and $n \geq p^2$, then the affine problem (2) admits a positive $C^1$ least energy solution for any $0 < \lambda < \lambda^A_{1,p}$.

**Theorem 1.3.** If $q = p^*$ and $n < p^2$, then there exists a constant $\lambda_* > 0$ such that the affine problem (2) admits a positive $C^1$ least energy solution for any $\lambda_* < \lambda < \lambda^A_{1,p}$.

**Theorem 1.4.** If $p < q \leq p^*$, then the affine problem (2) admits no nontrivial least energy weak solution for any $\lambda \geq \lambda^A_{1,p}$.

**Theorem 1.5.** If $q = p^*$ and $\Omega$ is star-shaped, then the affine problem (2) admits no nontrivial nonnegative weak solution for any $\lambda \leq 0$.  

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The well-known versions of all theorems from 1.1 to 1.5 for the $p$-Laplace operator were established at various times in history. For works associated to Theorem 1.1, see [2, 15, 35], to Theorems 1.2 and 1.3, see [5] for $p = 2$ and [2, 3, 16] for $1 < p < n$ and to Theorem 1.5, see [36] for $p = 2$ and [10, 16, 37] for $1 < p < n$. Lastly, closely related to Theorem 1.4, we recall the results of nonexistence of positive $C^1$ solution for any $\lambda \geq \lambda_{1,p} = \lambda_{1,p}(\Delta_p)$ and of existence of nontrivial $C^1$ solution for any $\lambda > \lambda_{1,p}$ different from all min-max type eigenvalues of $\Delta_p$ on $W^{1,p}_0(\Omega)$, see [16] and [9], respectively. Whether these results are true or not when considering the operator $\Delta^A_p$ is an open question. For other related references, we refer to the works [1, 7, 8, 12, 13, 14, 24].

The proof of existence consists in finding minimizers for the quotient defining the minimum level $c_A$. The main obstacles are caused by the affine term $E_{p,\Omega}$. More precisely, the functional $u \in W^{1,p}_0(\Omega) \mapsto E_{p,\Omega}(u)$ is not convex and its geometry is non-coercive since there are unbounded sequences in $W^{1,p}_0(\Omega)$ with bounded affine $L^p$ energy. Examples of such sequences are constructed on the pages 17 and 18 of [21]. Already, the non-convexity follows from the reverse inequality of Proposition 4.1 which becomes strict for many functions in $W^{1,p}_0(\Omega)$. Despite the absence of an adequate variational structure, we prove in Theorem 2.1 that the referred energy functional is weakly lower semicontinuous on $W^{1,p}_0(\Omega)$.

The affine context also affects strongly the study of existence of nonnegative weak solution in the subcritical and critical cases. In both ones, we make use of an affine Rellich-Kondrachov compactness theorem, established recently by Tintarev [40] for $1 < p < n$ and by the authors [26] for $p = 1$, which states that the affine ball $B^a_{p,\Omega} = \{u \in W^{1,p}_0(\Omega) : E_{p,\Omega}(u) \leq 1\}$ is compact in $L^q(\Omega)$ for every $1 \leq q < p^*$ (Theorem 2.2), and also of a quite useful consequence of its proof (Corollary 2.1). When $q = p^*$, it is well-known that the claim of compactness usually fails. However, using additional tools, we prove that minimizing sequences of $c_A$ are compact in $L^{p^*}(\Omega)$ for lower energy levels $c_A$, see Propositions 4.1, 4.2 and 4.3.

In Section 3, we focus on the $C^1$ regularity and positivity of weak solutions of (2) as well as a related Pohozaev type identity to be used in proof of Theorem 1.5.

It is worth mentioning that (iii) of Proposition 2.1 in the next section will play a fundamental role in the proof of the most ingredients quoted above.

## 2 Brief summary on the variational setting

For $1 < p < n$ and $p < q \leq p^*$, let $\Phi_A : W^{1,p}_0(\Omega) \to \mathbb{R}$ be the functional given by

$$\Phi_A(u) = E_{p,\Omega}(u) - \lambda \int_{\Omega} |u|^p \, dx$$

and $c_A = \inf_{u \in X} \Phi_A(u)$ be its least energy level on the set $X = \{u \in W^{1,p}_0(\Omega) : \|u\|_{L^q(\Omega)} = 1\}$. 

- $\Phi_A(u) = E_{p,\Omega}(u) - \lambda \int_{\Omega} |u|^p \, dx$
Clearly, $\Phi_A$ is well-defined and $c_A$ is always finite for any $\lambda \in \mathbb{R}$, since the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is continuous and the inequality for $p \geq 1$

$$\mathcal{E}_{p,\Omega}(u) \leq \|\nabla u\|_{L^p(\Omega)}$$

holds for any $u \in W^{1,p}_0(\Omega)$, see page 14 of [21].

Let $u_0 \in X$ be a nonnegative minimizer of $\Phi_A$ and assume that $c_A > 0$. Using the directional derivative of the affine term $\mathcal{E}_{p,\Omega}^a(u)$ described in the introduction, one easily checks that $u_0$ is a nonnegative weak solution of the problem

$$\begin{cases}
\Delta_A^a u &= c_A u^{q-1} + \lambda u^{p-1} \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega.
\end{cases}$$

(6)

Consequently, thanks to $(p-1)$-homogeneity of $\Delta_A^a$, a straightforward argument implies that $c_A^{\frac{q}{p}} u_0$ is a nonnegative weak solution of (2) of least energy type.

The first key point in the study of existence of minimizers for $c_A$ is the weak lower semi-continuity of the functional $u \in W^{1,p}_0(\Omega) \mapsto \mathcal{E}_{p,\Omega}(u)$. The property has been recently proved in [21] through an elegant argument based on its Theorem 9 and Lemma 1. We next provide an alternative elementary proof.

**Theorem 2.1.** If $u_k \rightharpoonup u_0$ weakly in $W^{1,p}_0(\Omega)$, then

$$\mathcal{E}_{p,\Omega}(u_0) \leq \liminf_{k \to \infty} \mathcal{E}_{p,\Omega}(u_k).$$

We make use of the following result which will also play a strategic role along the work:

**Proposition 2.1.** Let $u \in W^{1,p}_0(\Omega)$ with $p \geq 1$. The sentences are equivalent:

(i) $u = 0$;

(ii) $\mathcal{E}_{p,\Omega}(u) = 0$;

(iii) $\Psi_\xi(u) = 0$ for some $\xi \in \mathbb{S}^{n-1}$, where $\Psi_\xi(u) = \int_\Omega |\nabla_\xi u(x)|^p \, dx$.

**Proof.** Clearly, by (5), (i) implies (ii). If the claim (iii) occurs, then $\nabla_\xi u(x) = 0$ almost everywhere in $\Omega$ for some $\xi \in \mathbb{S}^{n-1}$, so $u$ is constant on line segments in $\Omega$ in the direction $\xi$. Then, since $u$ has zero trace on $\partial \Omega$, it follows the claim (i).

It remains to show that (ii) implies (iii). In fact, arguing by contradiction, assume that $\Psi_\xi(u) > 0$ for all $\xi \in \mathbb{S}^{n-1}$. Thanks to the continuity of the map $\xi \in \mathbb{S}^{n-1} \mapsto \Psi_\xi(u)$, there exists a constant $c > 0$ so that $\Psi_\xi(u) \geq c$ for all $\xi \in \mathbb{S}^{n-1}$. But the lower bound immediately yields $\mathcal{E}_{p,\Omega}(u) \geq c^{1/p} \alpha_{n,p}(n\omega_n)^{-1/n} > 0$. Thus, (ii) fails and we end the proof. \qed
Proof of Theorem 2.1. Let \( u_k \) be a sequence converging weakly to \( u_0 \) in \( W^{1,p}_0(\Omega) \). If \( u_0 = 0 \) then, by (ii) of Proposition 2.1, the statement follows trivially.

Assume \( u_0 \neq 0 \). Thanks to the convexity of the functional \( u \in W^{1,p}_0(\Omega) \mapsto \Psi_\xi(u) \), for any \( \xi \in \mathbb{S}^{n-1} \), we have

\[
\Psi_\xi(u_0) \leq \liminf_{k \to \infty} \Psi_\xi(u_k).
\]

We now ensure the existence of a constant \( c_0 > 0 \) and an integer \( k_0 \geq 1 \), both independent of \( \xi \in \mathbb{S}^{n-1} \), such that

\[
\Psi_\xi(u_k) \geq c_0
\]

for all \( k \geq k_0 \). Otherwise, module a renaming of indexes, we get a sequence \( \xi_k \in \mathbb{S}^{n-1} \) such that \( \xi_k \to \tilde{\xi} \) and \( \Psi_{\xi_k}(u_k) \leq k^{-1} \). Since \( u_k \) is bounded in \( W^{1,p}_0(\Omega) \), we find a constant \( C_1 > 0 \) such that

\[
\Psi_{\tilde{\xi}}(u_k) \leq C_1 \| \xi_k - \tilde{\xi} \| + 2^{p-1}k^{-1}.
\]

Letting \( k \to \infty \) and using (7), we get \( \Psi_{\tilde{\xi}}(u_0) = 0 \). But, by Proposition 2.1, we obtain the contradiction \( u_0 = 0 \), and so (8) is satisfied.

Finally, combining (7), (8) and Fatou’s lemma, we derive

\[
\int_{\mathbb{S}^{n-1}} \Psi_\xi(u_0)^{-\frac{1}{p}} d\sigma(\xi) \geq \int_{\mathbb{S}^{n-1}} \limsup_{k \to \infty} \Psi_\xi(u_k)^{-\frac{1}{p}} d\sigma(\xi) \geq \limsup_{k \to \infty} \int_{\mathbb{S}^{n-1}} \Psi_\xi(u_k)^{-\frac{1}{p}} d\sigma(\xi),
\]

and hence

\[
\mathcal{E}_{p,\Omega}(u_0) = \alpha_{n,p} \left( \int_{\mathbb{S}^{n-1}} \Psi_\xi(u_0)^{-\frac{1}{p}} d\sigma(\xi) \right)^{-\frac{1}{p}}
\]

\[
\leq \liminf_{k \to \infty} \alpha_{n,p} \left( \int_{\mathbb{S}^{n-1}} \Psi_\xi(u_k)^{-\frac{1}{p}} d\sigma(\xi) \right)^{-\frac{1}{p}}
\]

\[
= \liminf_{k \to \infty} \mathcal{E}_{p,\Omega}(u_k).
\]

\[ \square \]

The second point was recently established by Tintarev for \( 1 < p < n \) (see Theorem 6.5.3 of [40]) and by the authors for \( p = 1 \) (see Theorem 4.1 of [26]) and is stated as follows.

**Theorem 2.2.** Let \( 1 \leq p < n \) and \( B^A_\Omega(\Omega) = \{ u \in W^{1,p}_0(\Omega) : \mathcal{E}_{p,\Omega}(u) \leq 1 \} \). The set \( B^A_\Omega(\Omega) \) is compact in \( L^q(\Omega) \) for every \( 1 \leq q < p^* \).
Given a sequence \( u_k \) in \( B^A_p(\Omega) \), the proof of this result involves the existence of matrices \( T_k \in SL(n) \) such that \( u_k \circ T_k \) is bounded in \( D^{1,p}(\mathbb{R}^n) \). If \( T_k \to \infty \) is proved there that \( u_k \to 0 \) in \( L^q(\Omega) \). This fact leads us to a simple consequence that deserves to be highlighted.

**Corollary 2.1.** Let \( u_k \) be a sequence in \( B^A_p(\Omega) \) such that \( u_k \to u_0 \) strongly in \( L^q(\Omega) \) for some \( 1 \leq q < p^* \). If \( u_0 \neq 0 \), then \( u_k \) is bounded in \( W^{1,p}_0(\Omega) \).

### 3 Some properties of weak solutions

We next present some important properties satisfied by weak solutions of (2). We begin with a result on \( C^1 \) regularity regarding critical nonlinearities.

**Proposition 3.1.** Let \( \Omega \) be a bounded domain with \( C^{2,\alpha} \) boundary and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be a \( C^1 \) function satisfying

\[
|f(x,t)| \leq b(x)(|t|^{p^*-1} + 1)
\]

for all \((x,t) \in \Omega \times \mathbb{R}\), where \( b \in L^\infty(\Omega) \). If \( u_0 \in W^{1,p}_0(\Omega) \setminus \{0\} \) is a weak solution of the problem

\[
\begin{align*}
\Delta^A_p u &= f(x,u) \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial\Omega,
\end{align*}
\]

then \( u_0 \in C^{1,\alpha}(\overline{\Omega}) \).

**Proof.** We first show that \( u_0 \in L^s(\Omega) \) for any \( s \geq 1 \). Note that \( u_0 \) can be seen as a weak solution of

\[
\begin{align*}
\Delta^A_p u &= g(x,u) \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial\Omega,
\end{align*}
\]

where \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is given by

\[
g(x,t) = \frac{f(x,u(x))}{|u_0(x)|^{p-1} + 1}(|t|^{p-1} + 1).
\]

Clearly, \( g \) is a Caratheodory function and satisfies \( |g(x,t)| \leq b_0(|t|^{p-1} + 1) \) for all \((x,t) \in \Omega \times \mathbb{R}\), where \( b_0 \) belongs to \( L^{n/p}(\Omega) \), since \( |f(x,u_0)| \leq b(x)(|u_0|^{p-1} + 1) \) and \( |u_0|^{p^*-p} \in L^{n/p}(\Omega) \).

Following now the proof of Proposition 1.2 of [16], we consider the test function \( \varphi = \psi_k(u_0) \) for each positive number \( k \), where \( \psi_k(t) = \int_0^t \eta_k(\theta)^p d\theta \) and \( \eta_k \) is the \( C^1 \) function given by \( \eta_k(\theta) = \text{sign}(\theta)|\theta|^{\frac{p}{p-1}} \) for \( |\theta| \leq k \) and by \( \eta_k(\theta) = \text{sign}(\theta)(|\theta|^{\frac{p}{p-1}} + (1 - \frac{\theta}{p})k^{\frac{p}{p-1}}) \) for \( |\theta| > k \). Then, we derive

\[
\int_{\Omega} \eta_k(u_0)^p H_{u_0}^{-1}(\nabla u_0) \nabla H_{u_0}(\nabla u_0) \cdot \nabla u_0 \, dx = \int_{\Omega} g(x,u_0) \psi_k(u_0) \, dx.
\]
On the other hand, using the 1-homogeneity of $H_{u_0}(\zeta)$ on $\zeta$ and (iii) of Proposition 3.2 (i.e., $u_0 \neq 0$), we obtain a constant $c_0 > 0$, depending on $u_0$, such that

\[
\int_{\Omega} \eta_k'(u_0)^p H_{u_0}^{p-1}(\nabla u_0) \nabla H_{u_0}(\nabla u_0) \cdot \nabla u_0 \, dx = \int_{\Omega} \eta_k'(u_0)^p H_{u_0}^{p} (\nabla u_0) \, dx \geq c_0 \int_{\Omega} \eta_k'(u_0)^p |\nabla u_0|^p \, dx.
\]

Thanks to this inequality, the same arguments as in [16], consisting in the well-known De Giorgi-Nash-Moser’s iterative scheme, can be applied and the first claim follows. Consequently, $f(x, u_0) \in L^s(\Omega)$ for any $s > n/p$ and thus Proposition 4 of [21] gives $u_0 \in C^{1,\alpha}(\overline{\Omega})$. □

The next result concerns with strong maximum principle and Hopf’s lemma for an equation associated to the operator $\Delta^A$.

**Proposition 3.2.** Let $\Omega$ be a bounded domain with $C^{2,\alpha}$ boundary and $u_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$ be a weak solution of $\Delta^A u = \lambda|u|^{p-2}u + f(x)$ in $\Omega$, where $\lambda \leq 0$ and $f \in L^{\infty}(\Omega)$. If $f \geq 0$ in $\Omega$ and $f \neq 0$ in $\Omega$, then $u_0 > 0$ in $\Omega$ and $\frac{\partial u_0}{\partial \nu} < 0$ on $\partial\Omega$, where $\nu$ denotes the outward unit norm field to $\Omega$.

**Proof.** Note that the assumption $f \neq 0$ in $\Omega$ implies $u_0 \neq 0$ in $\Omega$. We first assert that $u_0 \geq 0$ in $\Omega$. Taking the test function $u_0^- = \min\{u_0, 0\} \in W_0^{1,p}(\Omega)$ in the equation and using the assumptions of proposition and the relation $H_{u_0}^{p-1}(\zeta) \nabla H_{u_0}(\zeta) \cdot \zeta = H_{u_0}^p(\zeta)$, we get

\[
\int_{\Omega} H_{u_0}(\nabla u_0^-) \, dx \leq 0,
\]

and since $u_0 \neq 0$, by Proposition 2.1, $u_0^- = 0$ in $\Omega$, in other words, $u_0$ is nonnegative.

Consider now the operator $\mathcal{L}_{p,u_0} := -\text{div} \mathcal{H}(\nabla u)$ on $W_0^{1,p}(\Omega)$, where $\mathcal{H}(\zeta) = H_{u_0}^{p-1}(\zeta) \nabla H_{u_0}(\zeta)$ for $\zeta \in \mathbb{R}^n$. Since $u_0$ is nonzero, by Proposition 2.1 and Cauchy-Schwartz inequality, there are constants $c_0, C_0 > 0$ such that $c_0 \leq \|\nabla \xi u_0\|_{L^p(\Omega)} \leq C_0$ for every $\xi \in \mathbb{R}^{n-1}$. Thanks to these inequalities, the operator $\mathcal{L}_{p,u_0}$ is uniformly elliptic, that is, there exist constants $c_1, C_1 > 0$ such that

(a) $\sum_{i,j=1}^n \frac{\partial \mathcal{H}_i(\zeta)}{\partial \zeta_j} \eta_i \eta_j \geq c_1 |\zeta|^{p-2} |\eta|^2$;

(b) $\sum_{i,j=1}^n \left| \frac{\partial \mathcal{H}_i(\zeta)}{\partial \zeta_j} \right| \leq C_1 |\zeta|^{p-2}$

for every $\zeta \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{R}^n$, where $\mathcal{H}_i(\zeta)$ is the $i$-th component of $\mathcal{H}(\zeta)$. For the computations of (a) and (b), see pages 25 and 26 of [21].

Since $f \in L^{\infty}(\Omega)$, by Proposition 3.1, we know that $u_0 \in C^1(\overline{\Omega})$. Then, since $u_0 \geq 0$ in $\Omega$ and $u_0 \neq 0$ in $\Omega$, evoking the strong maximum principle (Proposition 3.2.2 of [11]) and Hopf’s lemma (Proposition 3.2.1 of [11]) for $C^1$ super-solutions of quasilinear elliptic equations for operators satisfying (a) and (b) (see also [38]), we derive the two desired statements. □
Finally, we shall need a Pohozaev type identity satisfied by weak solutions of \( (2) \).

**Proposition 3.3.** Let \( \Omega \) be a bounded domain with \( C^{2,\alpha} \) boundary, \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( |f(t)| \leq b(|t|^{p-1} + 1) \) for all \( t \in \mathbb{R} \), where \( b > 0 \) is a constant, and \( F(t) = \int_0^t f(\theta) \, d\theta \). Then, the integral identity

\[
\left( \frac{1}{p} - 1 \right) \int_{\partial\Omega} H^p_{u_0}(\nabla u_0)(x \cdot \nu) \, d\sigma = \left( \frac{n}{p} - 1 \right) \int_\Omega u_0 f(u_0) \, dx - n \int_\Omega F(u_0) \, dx
\]

holds for any nontrivial weak solution \( u_0 \in W^{1,p}_0(\Omega) \) of

\[
\begin{cases}
\Delta_p u = f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

**Proof.** Let \( u_0 \in W^{1,p}_0(\Omega) \) be a nontrivial weak solution of \( (9) \). By Proposition 3.1, we know that \( u_0 \in C^1(\overline{\Omega}) \). Since \( \zeta \in \mathbb{R}^n \mapsto H^p_{u_0}(\zeta) \) is strictly convex, we can apply the Pohozaev identity established by Degiovanni, Musesti and Squassina for \( C^1 \) solutions of

\[
\begin{cases}
-\text{div} \left( \mathcal{H}(\nabla u) \right) = f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \( \mathcal{H}(\zeta) = H^p_{u_0}(\zeta) \nabla H^p_{u_0}(\zeta) \). Precisely, evoking Theorems 1 and 2 of [10] with the choice \( h(x) = x \) and \( a(x) = a \), where \( a \) is an arbitrary constant, we have

\[
\int_{\partial\Omega} \left( \frac{1}{p} H^p_{u_0}(\nabla u_0) - H^p_{u_0}(\nabla u_0) \nabla H^p_{u_0}(\nabla u_0) \cdot \nabla u_0 \right) (x \cdot \nu) \, d\sigma
\]

\[
= \frac{n}{p} \int_\Omega H^p_{u_0}(\nabla u_0) \, dx - \int_\Omega H^p_{u_0}(\nabla u_0) \nabla H^p_{u_0}(\nabla u_0) \cdot \nabla u_0 \, dx
\]

\[
- a \int_\Omega H^p_{u_0}(\nabla u_0) \nabla H^p_{u_0}(\nabla u_0) \cdot \nabla u_0 \, dx + \int_\Omega (x \cdot \nabla u_0 + au_0) f(u_0) \, dx.
\]

Using the relation \( H^p_{u_0}(\zeta) \nabla H^p_{u_0}(\zeta) \cdot \zeta = H^p_{u_0}(\zeta) \) (which comes from the 1-homogenenity of \( H^p_{u_0}(\zeta) \)) and that \( u_0 \) solves \( (10) \), the above identity can be placed into the simpler form

\[
\left( \frac{1}{p} - 1 \right) \int_{\partial\Omega} H^p_{u_0}(\nabla u_0)(x \cdot \nu) \, d\sigma = \left( \frac{n}{p} - 1 - a \right) \int_\Omega H^p_{u_0}(\nabla u_0) \, dx + \int_\Omega (x \cdot \nabla u_0 + au_0) f(u_0) \, dx
\]

\[
= \left( \frac{n}{p} - 1 \right) \int_\Omega u_0 f(u_0) \, dx + \int_\Omega (x \cdot \nabla u_0) f(u_0) \, dx.
\]

On the other hand, the divergence theorem and the condition \( u_0 = 0 \) on \( \partial\Omega \) lead to
\[
\int_{\Omega} (x \cdot \nabla u_0) f(u_0) \, dx = \frac{1}{2} \int_{\Omega} \nabla(|x|^2) \cdot \nabla F(u_0) \, dx \\
= -\frac{1}{2} \int_{\Omega} \Delta(|x|^2) F(u_0) \, dx + \int_{\partial \Omega} F(u_0)(x \cdot \nu) \, d\sigma \\
= -n \int_{\Omega} F(u_0) \, dx.
\]

Replacing this equality in the previous one, one gets the wished identity for weak solutions of (9).

\section{Proof of existence theorems}

We prove Theorems 1.1, 1.2 and 1.3 by applying the direct method to the functional $\Phi_{\mathcal{A}}$ constrained to the set $X$. According to Section 2, it suffices to show that the least energy level $c_{\mathcal{A}} = \inf_{u \in X} \Phi_{\mathcal{A}}(u)$ is positive and is achieved for some positive $C^1$ function in $X$.

We begin with the subcritical case, whose main ingredients are Theorems 2.1 and 2.2, Corollary 2.1 and Propositions 3.1 and 3.2.

\textit{Proof of Theorem 1.1.} Let $u_k$ be a minimizing sequence of $\Phi_{\mathcal{A}}$ in $X$. Since $q > p$, by Hölder’s inequality, $u_k$ is bounded in $L^p(\Omega)$, so the affine energy $E_{p,\Omega}(u_k)$ is bounded too. Then, by Theorem 2.2, there exists $u_0 \in W^{1,p}_0(\Omega)$ such that $u_k \rightharpoonup u_0$ strongly in $L^p(\Omega)$ and in $L^q(\Omega)$ once $q < p^*$. In particular, $u_0 \in X$ and thus, by Corollary 2.1, $u_k$ is bounded in $W^{1,p}_0(\Omega)$. Passing to a subsequence, if necessary, one may assume that $u_k \rightharpoonup u_0$ weakly in $W^{1,p}_0(\Omega)$. Then, by Theorem 2.1, we derive

$$
\Phi_{\mathcal{A}}(u_0) \leq \liminf_{k \to \infty} \Phi_{\mathcal{A}}(u_k) = c_{\mathcal{A}},
$$

and thus $u_0$ minimizes $\Phi_{\mathcal{A}}$ in $X$. Moreover, we can assume that $u_0$ is nonnegative, since $|u_0| \in X$ and $\Phi_{\mathcal{A}}(|u_0|) = \Phi_{\mathcal{A}}(u_0)$.

Using now the assumption $\lambda < \lambda^A_{1,p}$ and the sharp affine $L^p$ Poincaré inequality (Theorem 4 of [21]), we get

$$
c_{\mathcal{A}} = \Phi_{\mathcal{A}}(u_0) = E^p_{p,\Omega}(u_0) - \lambda \int_{\Omega} |u_0|^p \, dx \geq (\lambda^A_{1,p} - \lambda) \int_{\Omega} |u_0|^p \, dx > 0.
$$

Therefore, $u_0$ is a nontrivial nonnegative weak solution of (6). Finally, by Propositions 3.1 and 3.2, $u_0$ is a positive $C^1$ solution of (6).

The existence of positive $C^1$ solutions to critical problems requires three more results. The first one considers the truncation for $h > 0$:

$$
T_h(s) = \min(\max(s, -h), h) \quad \text{and} \quad R_h(s) = s - T_h(s).
$$
A simple computation gives \( \| \nabla u \|_{L^p(\Omega)}^p = \| \nabla T_h u \|_{L^p(\Omega)}^p + \| \nabla R_h u \|_{L^p(\Omega)}^p \) for every \( u \in W_0^{1,p}(\Omega) \). Unfortunately, the equality is not valid within the affine setting, but it is still possible to guarantee an inequality thanks to (iii) of Proposition 2.1.

**Proposition 4.1.** For any \( u \in W_0^{1,p}(\Omega) \), we have

\[
\mathcal{E}_{p,\Omega}(u) \geq \mathcal{E}_{p,\Omega}(T_h u) + \mathcal{E}_{p,\Omega}(R_h u).
\]

**Proof.** From the definition of \( T_h(s) \), we have \( T_h u, R_h u \in W_0^{1,p}(\Omega) \) and \( \Psi_\xi(u) = \Psi_\xi(T_h u) + \Psi_\xi(R_h u) \) for all \( \xi \in S^{n-1} \). Note that the decomposition implies \( \mathcal{E}_{p,\Omega}(u) \geq \mathcal{E}_{p,\Omega}(T_h u) \) and \( \mathcal{E}_{p,\Omega}(u) \geq \mathcal{E}_{p,\Omega}(R_h u) \). So, the statement follows in the case that \( \mathcal{E}_{p,\Omega}(T_h u) = 0 \) or \( \mathcal{E}_{p,\Omega}(R_h u) = 0 \).

Assume now that \( \mathcal{E}_{p,\Omega}(T_h u) \) and \( \mathcal{E}_{p,\Omega}(R_h u) \) are nonzero. By (iii) of Proposition 2.1 we have \( \Psi_\xi(T_h u), \Psi_\xi(R_h u) > 0 \) for all \( \xi \in S^{n-1} \). Then, by the reverse Minkowski inequality for negative exponents, we get

\[
\mathcal{E}_{p,\Omega}(u) = \alpha_{n,p} \left( \frac{1}{S^{n-1}} \int_{S^{n-1}} (\Psi_\xi(T_h u) + \Psi_\xi(R_h u))^{-\frac{p}{p-1}} d\sigma(\xi) \right)^{-\frac{p}{p-1}}
\]

\[
\geq \alpha_{n,p} \left( \frac{1}{S^{n-1}} \int_{S^{n-1}} \Psi_\xi(T_h u)^{-\frac{p}{p-1}} d\sigma(\xi) \right)^{-\frac{p}{p-1}} + \alpha_{n,p} \left( \frac{1}{S^{n-1}} \int_{S^{n-1}} \Psi_\xi(R_h u)^{-\frac{p}{p-1}} d\sigma(\xi) \right)^{-\frac{p}{p-1}}
\]

\[
= \mathcal{E}_{p,\Omega}(T_h u) + \mathcal{E}_{p,\Omega}(R_h u)
\]

for every \( u \in W_0^{1,p}(\Omega) \). \( \square \)

**Proposition 4.2.** Let \( q = p^* \) and assume that \( 0 < c_A < K_{n,p}^{-p} \), where \( K_{n,p} \) is the best constant for the sharp affine \( L^p \) Sobolev inequality \( \textbf{[1]} \). Then, \( \Phi_A \) admits a minimizer \( u_0 \) in \( X \).

**Proof.** Let \( u_k \) be a minimizing sequence of \( \Phi_A \) in \( X \). Proceeding as in the proof of Theorem 1.1 by Theorem 2.2, \( u_k \to u_0 \) strongly in \( L^p(\Omega) \), modulo a subsequence. One may also assume that \( u_k \to u_0 \) almost everywhere in \( \Omega \) and \( T_h u_k \to T_h u_0 \) weakly in \( L^{p^*}(\Omega) \).

Using the affine Sobolev inequality on \( W_0^{1,p}(\Omega) \),

\[
K_{n,p}^{-p} \left( \int_{\Omega} |u|^{p^*} \, dx \right)^{\frac{p}{p^*}} \leq \mathcal{E}_{p,\Omega}(u),
\]

we get

\[
c_A = \lim_{k \to \infty} \left( \mathcal{E}_{p,\Omega}(u_k) - \lambda \int_{\Omega} |u_k|^{p^*} \, dx \right) \geq K_{n,p}^{-p} - \lambda \int_{\Omega} |u_0|^{p^*} \, dx,
\]

so the condition \( c_A < K_{n,p}^{-p} \) implies that \( u_0 \neq 0 \). Hence, by Corollary 2.1 and Theorem 2.1 we have \( u_k \to u_0 \) weakly in \( W_0^{1,p}(\Omega) \) and \( \Phi_A(u_0) \leq c_A \). It only remains to show that \( u_0 \in X \).

Evoking Proposition 4.1 we easily deduce that
\[
c_A = \lim_{k \to \infty} \Phi_A(u_k) \\
\geq \lim_{k \to \infty} (\Phi_A(T_h u_k) + \Phi_A(R_h u_k)) \\
\geq c_A \lim_{k \to \infty} \left( \|T_h u_k\|_{L^p(\Omega)}^p + \|R_h u_k\|_{L^p(\Omega)}^p \right).
\]
Applying now Lemma 3.1 of [4], we derive
\[
c_A \geq c_A \left[ \|T_h u_0\|_{L^p(\Omega)}^p + \left( 1 + \|R_h u_0\|_{L^p(\Omega)} - \|u_0\|_{L^p(\Omega)} \right)^{\frac{p}{n}} \right].
\]
Using the condition \(c_A > 0\) and letting \(h \to \infty\), we obtain
\[
1 \geq \left( \|u_0\|_{L^p(\Omega)} \right)^{\frac{p}{n}} + \left( 1 - \|u_0\|_{L^p(\Omega)} \right)^{\frac{p}{n}}
\]
and thus \(u_0 \in X\) because \(u_0 \neq 0\).

\[\square\]

**Proposition 4.3.** The equality \(E_p(u) = \|\nabla u\|_{L^p(\mathbb{R}^n)}\) is valid for every radial function \(u \in D_1^1(\mathbb{R}^n)\).

**Proof.** It suffices to prove the result for \(u \neq 0\). By (iii) of Proposition [2,1] we recall that \(0 < c \leq \Psi_\xi(u) \leq C\) for all \(\xi \in S^{n-1}\), where \(c\) and \(C\) are positive constants.

We remark that the proof of the inequality \(E_p(u) \leq \|\nabla u\|_{L^p(\mathbb{R}^n)}\) involves two independent steps. The first one consists in applying Hölder’s inequality as follows:

\[
n_{\omega_n} = \int_{S^{n-1}} 1 d\sigma(\xi) = \int_{S^{n-1}} \Psi_\xi(u)^{\frac{n}{n+p}} \Psi_\xi(u)^{-\frac{n}{n+p}} d\sigma(\xi) \\
\leq \left( \int_{S^{n-1}} \Psi_\xi(u) d\sigma(\xi) \right)^{\frac{n}{n+p}} \left( \int_{S^{n-1}} \Psi_\xi(u)^{-\frac{n}{p}} d\sigma(\xi) \right)^{-\frac{p}{n+p}},
\]
which yields
\[
(n_{\omega_n})^{\frac{n+p}{n}} \left( \int_{S^{n-1}} \Psi_\xi(u)^{-\frac{n}{p}} d\sigma(\xi) \right)^{-\frac{p}{n}} \leq \int_{S^{n-1}} \Psi_\xi(u) d\sigma(\xi).
\]
The second step makes use of the Fubini’s theorem on the above right-hand side, so we get
\[
\int_{S^{n-1}} \Psi_\xi(u) d\sigma(\xi) = \int_{S^{n-1}} \int_{\mathbb{R}^n} |\nabla u(x) \cdot \xi|^p dx d\sigma(\xi) \\
= \int_{\mathbb{R}^n} \int_{S^{n-1}} |\nabla u(x) \cdot \xi|^p d\sigma(\xi) dx \\
= \left( \int_{S^{n-1}} |\xi_0 \cdot \xi|^p d\sigma(\xi) \right) \left( \int_{\mathbb{R}^n} |\nabla u(x)|^p dx \right),
\]
where \(\xi_0\) is any fixed point \(\xi_0\) in \(S^{n-1}\).
On the other hand, we know from [29] that
\[ \alpha_{n,p} = (n\omega_n)^{\frac{n+p}{np}} \left( \int_{S^{n-1}} |\xi_0 \cdot \xi|^p d\sigma(\xi) \right)^{\frac{1}{p}}. \]

Hence, joining (11) and (12), we obtain \( \mathcal{E}_p(u) \leq \|\nabla u\|_{L^p(\mathbb{R}^n)} \). Moreover, equality holds if, and only if, the step of the Hölder’s inequality becomes equality. But the latter is equivalent to the function \( \xi \in S^{n-1} \mapsto \Psi_\xi(u) \) to be constant.

Finally, for any radial function \( u \in D^{1,p}(\mathbb{R}^n) \), it easily follows that
\[ \Psi_\xi(u) = \int_{\mathbb{R}^n} |\nabla u(x) \cdot \xi|^p dx = \int_{\mathbb{R}^n} r^{-p}|u'(r)|^p|x \cdot \xi|^p dx = \int_{\mathbb{R}^n} r^{-p}|u'(r)|^p|x \cdot \xi|^p dx \]
for all \( \xi \in S^{n-1} \). In other words, \( \Psi_\xi(u) \) doest’n depend on \( \xi \) and this concludes the proof. \( \square \)

Proof of Theorems 1.2 and 1.3. Arguing as in the proof of Theorem 1.1 with the aid of Propositions 3.1 and 3.2, it suffices to establish the existence of a minimizer of \( \Phi_A \) in \( X \).

We first assert that the condition \( \lambda < \lambda^A_{1,p} \) implies that \( c_A > 0 \) in the critical case too. Indeed, since \( q = p^* \), the sharp affine \( L^p \) Poincaré and Sobolev inequalities lead to
\[ \Phi_A(u) = \mathcal{E}_{p,\Omega}(u) - \lambda \int_{\Omega} |u|^p dx \geq \begin{cases} \mathcal{E}_{p,\Omega}(u) \geq K_{n,p}^{-p}, & \text{if } \lambda \leq 0, \\ (1 - \frac{\lambda}{\lambda^A_{1,p}})\mathcal{E}_{p,\Omega}(u) \geq (1 - \frac{\lambda}{\lambda^A_{1,p}})K_{n,p}^{-p}, & \text{if } \lambda > 0 \end{cases} \]
for every \( u \in X \). Therefore, \( c_A > 0 \) in any situation.

We now show that \( c_A < K_{n,p}^{-p} \) provided that \( \lambda > 0 \). By Proposition 4.3, the best constant \( K_{n,p} \) for the affine inequality (1) coincides with the corresponding one for the classical \( L^p \) Sobolev inequality.

For \( n \geq p^2 \), we evoke the well-known construction by Azorero and Peral [2] of a function \( w_0 \in X \) such that
\[ c_A \leq \Phi_A(w_0) = \mathcal{E}_{p,\Omega}(w_0) - \lambda \int_{\Omega} |w_0|^p dx \leq \int_{\Omega} |\nabla w_0|^p dx - \lambda \int_{\Omega} |w_0|^p dx < K_{n,p}^{-p}. \]
Here it was used (5).

For \( n < p^2 \), we take a principal eigenfunction \( \varphi_{1,p}^A \) of \( \Delta_p^A \) on \( W_0^{1,p}(\Omega) \) with \( \|\varphi_{1,p}^A\|_{L^p(\Omega)} = 1 \), whose existence is ensured by Theorem 4 of [21], so \( \varphi_{1,p}^A \in X \). Set \( \lambda_* = \lambda_{1,p} - K_{n,p}^{-p}\|\varphi_{1,p}^A\|_{L^p(\Omega)}^{-p} \). Note that \( \lambda_* > 0 \), because
\[ \lambda_{1,p}^A \|\varphi_{1,p}^A\|_{L^p(\Omega)}^p = \mathcal{E}_{p,\Omega}(\varphi_{1,p}^A) > K_{n,p}^{-p}, \]
where the strict inequality follows from the characterization of extremals as quoted in the introduction. For \( \lambda > \lambda_* \) and \( w_0 = \varphi_{1,p}^A \), we obtain
\[ c_A \leq \Phi_A(w_0) = \mathcal{E}^p_{p,\Omega}(w_0) - \lambda \int_{\Omega} |w_0|^p \, dx \]
\[ \leq \lambda^{1/p}_1 \|\varphi_{1,p}^A\|_{L^p(\Omega)}^p - \lambda \int_{\Omega} |w_0|^p \, dx \]
\[ < \lambda^{1/p}_1 \|\varphi_{1,p}^A\|_{L^p(\Omega)}^p - \lambda \|\varphi_{1,p}^A\|_{L^p(\Omega)} = K_{n,p}^- \]

Finally, under the assumptions of Theorems 1.2 and 1.3, we deduce that \( 0 < c_A < K_{n,p}^- \) and hence, by Proposition 4.2, we complete the proof. \( \square \)

5 Proof of nonexistence theorems

We prove Theorems 1.4 and 1.5 by using Propositions 3.1, 3.2 and 3.3.

**Proof of Theorem 1.4.** Let \( p < q \leq p^* \) and \( \lambda \geq \lambda^{1/p}_1 \). Assume that the problem (2) admits a nontrivial least energy weak solution \( u_0 \in W^{1,p}_0(\Omega) \). Then,
\[ c_A = \frac{\mathcal{E}^p_{p,\Omega}(u_0) - \lambda \int_{\Omega} |u_0|^p \, dx}{\left( \int_{\Omega} |u_0|^q \, dx \right)^{\frac{p}{q}}} = \frac{\int_{\Omega} |u_0|^p \, dx}{\left( \int_{\Omega} |u_0|^q \, dx \right)^{\frac{p}{q}}} > 0. \]

On the other hand, for a principal eigenfunction \( \varphi_{1,p}^A \) of \( \Delta_p^A \) on \( W^{1,p}_0(\Omega) \) with \( \|\varphi_{1,p}^A\|_{L^q(\Omega)} = 1 \), we get
\[ c_A \leq \frac{\mathcal{E}^p_{p,\Omega}(\varphi_{1,p}^A) - \lambda \int_{\Omega} (\varphi_{1,p}^A)^p \, dx}{\left( \int_{\Omega} |\varphi_{1,p}^A|^q \, dx \right)^{\frac{p}{q}}} = (\lambda^{1/p}_1 - \lambda) \int_{\Omega} (\varphi_{1,p}^A)^p \, dx \leq 0, \]
and thus we derive a contradiction. \( \square \)

**Proof of Theorem 1.5.** Let \( u_0 \in W^{1,p}_0(\Omega) \) be a nontrivial nonnegative weak solution of (2). By Propositions 3.1 and 3.2, \( u_0 \) is a positive \( C^1 \) function that satisfies \( \nabla u_0 \neq 0 \) on \( \partial \Omega \). Assume without loss of generality that \( \Omega \) is star-shaped with respect to the origin. Thus, we have \( x \cdot \nu > 0 \) on \( \partial \Omega \). Since \( q = p^* \), applying Proposition 3.3 we get the contradiction
\[ 0 > \left( \frac{1}{p} - 1 \right) \int_{\partial \Omega} H_{u_0}^p(\nabla u_0)(x \cdot \nu) \, d\sigma = \left( \frac{n}{p} - 1 \right) \int_{\Omega} u_0^q + \lambda u_0^p \, dx - n \int_{\Omega} \frac{1}{q} u_0^q + \frac{\lambda}{p} u_0^p \, dx \]
\[ = \left( \frac{n - p}{p} - \frac{n}{q} \right) \int_{\Omega} u_0^q - \lambda \int_{\Omega} u_0^p \, dx \]
\[ = -\lambda \int_{\Omega} u_0^p \, dx \geq 0 \]
for every \( \lambda \leq 0 \). \( \square \)

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