SCALABLE ARCHITECTURE FOR ADIABATIC QUANTUM COMPUTING OF NP-HARD PROBLEMS

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Abstract

We present a comprehensive review of past research into adiabatic quantum computation and then propose a scalable architecture for an adiabatic quantum computer that can treat NP-hard problems without requiring local coherent operations. Instead, computation can be performed entirely by adiabatically varying a magnetic field applied to all the qubits simultaneously. Local (incoherent) operations are needed only for: (1) switching on or off certain pairwise, nearest-neighbor inductive couplings in order to set the problem to be solved and (2) measuring some subset of the qubits in order to obtain the answer to the problem.

Keywords: quantum computing, combinatorial optimization, adiabatic, scalable implementation, NP

1. Introduction

Adiabatic quantum computation\(^1\) is a recently proposed, general approach to solving NP-hard combinatorial minimization problems.\(^2\) It consists of constructing a set of qubits with a time-dependent Hamiltonian \(\hat{H}(t)\) whose starting point \(\hat{H}_s\) has a ground state that is quickly reachable simply by cooling and whose final point \(\hat{H}_p\) has couplings that encode the cost scheme of a desired minimization problem. The name “adiabatic” comes from the fact that if the qubits are initialized in the ground state of \(\hat{H}_s\) and if \(\hat{H}(t)\) is varied slowly enough, then the qubits will overwhelmingly be in the ground state of \(\hat{H}(t)\) at all times \(t\), thus in principle completely bypassing the usual concern about local minima in \(\hat{H}_p\) confounding the search for the problem’s solution.
Sections 2 and 3 review the literature to date on adiabatic quantum computation. Section 4 presents the main result of this paper: a scalable architecture for an adiabatic quantum computer. Section 5 concludes.

2. Literature Review: The Time Complexity of Adiabatic Quantum Computation

The general time-complexity of adiabatic quantum computation is still an open problem. Intuitively, the key to fast, successful adiabatic computation is to ensure that $\hat{H}(t)$ always possesses a sizable spectral gap between its instantaneous ground and excited states, thus allowing one to vary $\hat{H}(t)$ quickly while still not ever providing enough energy to excite the system significantly out of its instantaneous ground state. Determining the asymptotic time complexity of adiabatic quantum computation remains an open problem because for $n$ qubits, $\hat{H}(t)$ is a $2^n \times 2^n$ matrix, and hence analytically obtaining lower bounds on the spectral gap is extremely difficult unless $\hat{H}(t)$ possesses significant symmetry.

Presently, there are three major results on the time complexity of adiabatic quantum computation. Together, they yield the general picture that for essentially unstructured NP-hard minimization problems like finding the ground state of a random-field magnet, adiabatic quantum computing should offer polynomial speedups—probably in the range of cube-root to sixth-root—over a Metropolis algorithm (simple cooling) approach to the same problems. Furthermore, for structured NP-hard minimization problems such as 3-SAT and MAX CLIQUE, adiabatic quantum computation can quite possibly offer exponential speedups over any classical algorithm (assuming $P \neq NP$)—at least on average.

More specifically, the first major time-complexity result is that adiabatic quantum computation can search an unsorted list of $N$ items for a single identifiable item in $O(\sqrt{N})$ time, thus matching the speedup of Grover’s algorithm. This, however, is a naïve way of characterizing the performance of adiabatic quantum computation on unstructured searches because an $n$th-root speedup still means that the minimum spectral gap in $\hat{H}(t)$ is exponentially shrinking, and practically one always needs to operate above some finite temperature limit.

It is therefore much more realistic when analyzing unstructured problems to abandon the idea that the system can always remain overwhelmingly in the instantaneous ground state of $\hat{H}(t)$ and instead view adiabatic quantum computing as an enhanced form of annealing. The potential for enhancement intuitively derives from two facts. One, quantum systems can tunnel through energy barriers instead of waiting for thermal excitation over them. Two, with a time dependent Hamiltonian
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$\hat{H}(t)$, one need not face the same barriers as those in energy landscape $H_p$ of one’s problem.

For example, consider an Ising model of the form

$$\hat{H}_p = \sum_{i,j} J_{ij} \hat{\sigma}_z^{(i)} \hat{\sigma}_z^{(j)}$$  \hspace{1cm} (1)

where the couplings $\{J_{ij}\}$ are randomly drawn with a zero-mean Gaussian distribution from the interval $[-1, 1]$. If the Ising model cannot be drawn as a planar graph (with spins as vertices and nonzero couplings as edges), then finding the ground state of $\hat{H}_p$ is an NP-hard problem.\(^5\)

Quite general considerations of the distribution of low-lying excitations of any frustrated system such as that of Equ. (1) imply that simply cooling the system from $T = \infty$ to $T = 0$ in a time $\tau$ (physically, “$\infty$” means “sufficiently high to make all possible states essentially equiprobable”) leaves an average residual energy

$$\langle E(\tau) \rangle - E_{\text{ground}} \sim \ln(\tau)^{-\xi} \quad (\xi \leq 2)$$  \hspace{1cm} (2)

in the limit of large $\tau$.\(^6\) Monte Carlo simulations imply that $\xi$ can be as low as 1 for cooling to the ground states of some NP-hard random-field Ising models.\(^7\) In contrast, if instead one applies a transverse magnetic field to the system that decreases from $H = \infty$ to $H = 0$ in time $\tau$ (all done at $T = 0$), then according to a model\(^8\) that calculates the residual energy as the cumulative result of Landau-Zener type transitions\(^9\) at the avoided level crossings between the instantaneous ground state and first excited states of the resulting time-dependent Hamiltonian

$$\hat{H}(t) = \hat{H}_p + \Gamma(t) \sum_i \hat{\sigma}_x^{(i)} \quad (\Gamma(0) = \infty, \Gamma(\tau) = 0)$$  \hspace{1cm} (3)

one finds much improved performance

$$\langle E(\tau) \rangle - E_{\text{ground}} \sim \ln(\tau)^{-\xi_{QU}} \quad (\xi_{QU} \approx 6)$$  \hspace{1cm} (4)

The greater size of the exponent $\xi_{QU}$ in Equ. (4) over the exponent $\xi$ in Equ. (2) is the basis the second major time-complexity result—adiabatic quantum computation on a given unstructured NP problem should at least provide cube-root and perhaps as much as sixth-root speedups over a Metropolis algorithm (simple cooling) approach to the same problem.

The third, and perhaps most interesting, time-complexity result—the possibility of exponential speedup on almost all difficult instances of NP-hard problems—comes from explicit numerical integrations of Schrödinger’s equation to simulate adiabatic computation on sets of small instances of these problems ($\leq 26$ bits is roughly the limit of
current supercomputers). Since the class NP is a worst-case complexity measure, it is important to consider problem instances that are most likely to be truly difficult. For example, Hogg studied random 3-SAT, which is the problem of finding a string of \( n \) bits that completely satisfies \( m \) Boolean clauses, each of the form \( a \text{ OR } b \text{ OR } c \) where

1. the literals \( a, b, \) and \( c \) refer to 3 distinct bits chosen uniformly at random

2. each literal, with 50\% probability, contains a negation.

In the limit of large \( n \), random 3-SAT instances exhibit a first-order phase transition from being highly likely to have a satisfying assignment when \( m/n < 4.25 \) to being highly unlikely to have a satisfying assignment when \( m/n > 4.25 \), and it has been observed that the most difficult instances to solve appear to fall on the phase boundary. Hogg studied instances generated as near as possible to this phase boundary \( m/n = 4.25 \) and found that for \( n \leq 26 \) bits (the largest number that could be numerically analyzed) the median run time scaled only as \( O(n^3) \).

Random 3-SAT instances at the phase boundary still tend to have exponentially many satisfying solutions which, although they constitute an exponentially small fraction of the \( 2^n \) possible bit strings, one might worry constitute enough degeneracy to distort study of such small instances of \( \leq 26 \) bits. Hence, the earlier studies of Farhi et al. and Childs et al. focused on randomly generating NP-complete problem instances with unique satisfying assignments ("USA instances") rather than some ratio of constraints to bits. Median run times scaling only as \( O(n^2) \) were observed for both USA instances of random EXACT COVER up to 20 bits and random MAX CLIQUE up to 18 bits. (Random EXACT COVER is the problem of finding a string of \( n \) bits that satisfies \( m \) clauses of the form \( a + b + c = 1 \) where the literals \( a, b, c \) are distinct bits chosen uniformly at random. Random MAX CLIQUE is the problem of finding the largest subgraph for which every pair of points is connected by an edge within an \( n \)-vertex graph constructed such that each pair of vertices is connected by an edge with 50\% probability.)

The above MAX CLIQUE result may be seen to carry extra significance because it is widely conjectured there does not exist any classical polynomial-time algorithm that can identify cliques in such \( n \)-vertex random graphs of size \( > (1 + \epsilon) \log_2(n) \) for any \( \epsilon > 0 \). In contrast, the maximum clique of an \( n \)-vertex random graph is almost always \( > \log_2(n) \) and on average is \( 2 \log_2(n) \). This discrepancy is in fact used as the basis for a cryptographic scheme.
3. Literature Review: Robustness of Adiabatic Quantum Computation versus Decoherence

Beyond the potential for significant speedups over classical computing, which is an attraction that adiabatic quantum computing shares with other proposed methods of quantum computing, adiabatic quantum computing is especially attractive for being intrinsically robust against environmental noise.\textsuperscript{14} Firstly, since an adiabatic algorithm aims to keep its qubits in the instantaneous ground state of the computational Hamiltonian $\hat{H}(t)$, dissipation is not intrinsically harmful, but potentially helpful. (Of course, one cannot depend wholly on dissipation because then classical thermal annealing could have solved the problem with equal efficiency.) Secondly, if indeed the spectral gap of $\hat{H}(t)$ is large enough so that the system can be kept overwhelmingly in just one energy eigenstate (namely, the ground state), then the dephasing problem can be essentially circumvented. This is because so long as $\hat{H}(t)$ dominates over the environmental couplings to the qubits, the energy eigenstates of $\hat{H}(t)$ will be the preferred basis for dephasing, meaning that an adiabatic quantum computer based on $\hat{H}(t)$ would never present the environment with a superposition of states that it could dephase.

Such dominance is easy to achieve in the canonical spin-boson model with an Ohmic environment having an exponential cutoff after a frequency $\omega_c$.\textsuperscript{15} That is, if $\Delta_o$ denotes the bare energy splitting in the qubits due to coherent tunnelling between their limiting, classical states in the absence of coupling to the environment and $\alpha$ represents the (dimensionless) slope of the environmental spectral density at $\omega = 0$, then one can conclude non-perturbatively that the environmentally corrected tunnel splitting $\Delta$ is

$$
\Delta = \Delta_o \left( \frac{\Delta_o}{\omega_c} \right)^{\alpha/(1-\alpha)} \\
= \Delta_o \left[ 1 - \alpha \ln \left( \frac{\omega_c}{\Delta_o} \right) + O(\alpha^2) \right] \quad (5)
$$

Hence, if $\alpha \ll 1$ and $\omega_c$ is only a few orders of magnitude above $\Delta_o$, then $\Delta \approx \Delta_o$ meaning that the qubits’ Hamiltonian does indeed dominate over the environmental couplings and its energy eigenbasis is the preferred basis for dephasing.

This desirable situation should not be changed by the addition of $1/f$ or other low-frequency-dominated noise to the Ohmic environment. Components in the environmental spectral density that are of much lower frequency than the bare tunnel splitting $\Delta_o$ denote environmental dynamics that are much slower than that of the qubits. Therefore, essen-
tially regardless of the qubit-environment coupling, these components 
can only become entangled to time-averaged properties of the qubits 
such as their energy.\textsuperscript{16} While low-frequency components could thus signi-
ificantly change dephasing rates, they should not change the fact that 
the environment respects the energy eigenbasis of the qubits’ Hamilton-
ian as the preferred basis for dephasing, meaning that adiabatic algo-
rithms would still essentially circumvent the dephasing problem.

4. Main Result: A Scalable Architecture for 
Adiabatic Quantum Computing

Another special advantage of adiabatic quantum computing is the 
main point of this paper: there exists a simple, scalable architecture 
for adiabatic quantum computation that can handle NP-hard problems 
without requiring local coherent manipulations of the qubits. The start-
ing point for the architecture is the fact that it is NP-hard to calcu-
late the ground state of a 2D antiferromagnetically coupled Ising model 
placed in a magnetic field even if each Ising spin is coupled to no more 
than 3 others.\textsuperscript{5} In other words, if \( G = (V, E) \) is a planar graph with 
vertices \( V \) and edges \( E \) and even if every vertex in \( V \) belongs to \( \leq 3 \) 
edges in \( E \), then it is NP-hard to find the ground state of

\[
\hat{H} = \sum_{i \in V} \hat{\sigma}_z^{(i)} + \sum_{(i,j) \in E} \hat{\sigma}_z^{(i)} \hat{\sigma}_z^{(j)} 
\]  

(6)

which is an Ising model with spins at the vertices \( V \), equal strength anti-
ferromagnetic couplings along the edges \( E \), and a homogeneously applied 
field giving each spin a Zeeman splitting equal to the antiferromagnetic 
coupling strength. (The direction of this applied field sets the “\( z \)-axis.”)

This proof of NP-hardness isomorphically maps the above Ising model 
onto the MAX INDEPENDENT SET problem, which is the problem of 
finding for a graph \( G = (V, E) \) the largest subset \( S \) of the vertices \( V \) 
such that no two members of \( S \) are joined by an edge from \( E \). Solv-
ing MAX INDEPENDENT SET for a given graph \( G = (V, E) \) is com-
pletely equivalent to solving MAX CLIQUE for the graph’s complement 
\( G^c \equiv (V, (V \times V)/E) \). (In other words, \( G^c \) connects with edges all the 
pairs of vertices that were unconnected in \( G \) and disconnects all the pairs 
were connected.) Therefore, the MAX CLIQUE results\textsuperscript{12,13} referenced 
above imply that MAX INDEPENDENT SET is an equally interesting 
problem for treatment by adiabatic quantum computation, likely 
admitting not even a rough approximation in polynomial time classi-
cally, yet apparently admitting efficient solution by quantum adiabatic 
evolution—at least in small instances (and thus, hopefully, for almost 
al, if not all, larger instances).
There exists a simple, regular architecture that can be programmed to pose the above NP-hard Ising model ground state problem for any instance with \( \leq n \) spins. One way to see this is to recall that any planar graph \( G \) with \( \leq 3 \) edges per vertex admits a topologically equivalent embedding \( Q \) in a square grid of \( \leq 9n^2 \) vertices.\(^{17}\) In other words, there exists a map \( Q \) that (1) maps each vertex \( v \) of \( G \) onto a distinct point \( Q(v) \) of the grid and (2) inserts “dummy vertices” so as to map an edge \( e \) connecting vertices \( u \) and \( v \) of \( G \) into a path \( Q(e) \) on the square grid such that \( Q(u) \) and \( Q(v) \) are its endpoints and such that if \( e \) and \( f \) are two distinct edges of \( G \), then \( Q(e) \) and \( Q(f) \) do not intersect (except, obviously, if \( e \) and \( f \) share an endpoint, in which case \( Q(e) \) and \( Q(f) \) will also share an endpoint).

Thus, one simple, regular architecture would be of the type depicted below in Fig. 1: two stacked square grids of ferromagnetic and antiferromagnetic couplings, respectively, having at the center of each square a qubit with a ferromagnetic coupling to, say, the inductive couplings

\[ \approx \]

\[ \approx \]

Figure 1. (A) A regular 4-vertex graph for an Ising model. Vertices denote location of qubits, and edges denote antiferromagnetic couplings. (B) Topologically equivalent square grid embedding. (C) Possible implementation. Boldface lines denote couplings switched on, dashed lines denote couplings switched off.
meeting at lower right corner of the square. Since a useful adiabatic scheme would be one like Equ. (3) which merely adds a homogeneous, steadily decreasing transverse magnetic field to the problem Hamiltonian, the coherent evolution of the qubits could be affected simply with a globally applied magnetic field. This avoids the need both for any local coherent manipulations as well as for tunable rather than merely switchable couplings. Furthermore, as the the switching of the couplings is necessary only for programming the desired instance to be solved, it can be done “offline,” before any coherent manipulation of the qubits begins, thus obviating the need for “quiet” switches.

A final consideration for scalability is to calculate how much imprecision can exist in the parameters of the qubits, the couplings, and the applied field before one loses the ability to encode reliably in the above architecture any desired $n$-bit instance of MAX INDEPENDENT SET. To answer this, first recall that Ref. 5 proved that the maximum independent sets of a graph $G = (V,E)$ are in 1-to-1 correspondence with the ground state(s) of the Hamiltonian in Equ. (6), i.e., an Ising model with spins at the vertices $V$, antiferromagnetic couplings all of equal strength along the edges $E$, and a homogeneous applied magnetic field giving every spin a Zeeman splitting equal to the antiferromagnetic coupling strength. However, so long as one is satisfied with obtaining any one of the maximum independent sets of $G$ (which is reasonable since obtaining any one is NP-hard), one does not need to calibrate so precisely the couplings among themselves and to the applied field because the ground state(s) of any Ising model from the family

$$\hat{H} = \sum_{i \in V} J_i \hat{\sigma}^z_i + \sum_{(i,j) \in E} J_i J_j \hat{\sigma}^z_i \hat{\sigma}^z_j$$ (7) $$

where \{J_i\} are arbitrary positive scalars will also be ground state(s) of the Equ. (6) Ising model and hence maximum independent set(s) of $G$. Hence, in regard to programming the computer so that its final ground state has a large overlap with at least one solution of a desired instance of MAX INDEPENDENT SET, the proposed architecture is robust against arbitrary variance in the magnetic moments of the qubits. (Note, however, that while variance in \{J_i\} poses no problem in programming the computer, it might still cause a problem in ensuring that the computer produces accurate solutions quickly. Specifically, as the variance becomes larger, the problem Hamiltonian (7) looks more and more like random field magnet than a “structured” NP problem, likely lessening the chances that the adiabatic scheme $\hat{H}(t)$ will keep a large spectral gap and thus permit a running time exponentially faster than any classical algorithm.)
More general errors can come from inhomogeneities in both the magnitude and the direction of the applied magnetic field and from inductive couplings beyond the desired ones between nearest neighbors. Thus, to Equ. (7) one can add the general perturbation

\[
\hat{H}' = \sum_{k \in \{x,y,z\}} \sum_{i \in V} \epsilon_k^{(i)} \sigma_k^{(i)} + \sum_{k,l \in \{x,y,z\}} \sum_{i>j} \delta_{kl}^{(ij)} \sigma_k^{(i)} \sigma_l^{(j)}
\]

Note that Pauli operator terms such as \(\sigma_k^{(i)}\) or \(\sigma_k^{(i)} \sigma_l^{(j)}\) merely prescribe bit and/or phase flips in the eigenbasis of the Equ. (7) problem Hamiltonian, meaning that each of the perturbing terms in \(\hat{H}'\) couples an unperturbed ground state to only one other unperturbed eigenstate. Therefore, if \(n\) denotes the number of qubits, \(\lambda\) denotes the largest of the scalars \(\{\epsilon_k^{(i)}, \delta_{kl}^{(ij)}\}\) in Equ. (8), and \(\Delta\) denotes the minimum increase in energy caused by flipping 1 or 2 qubits of the ground state (thus, \(\Delta \approx \min_i |2J_i|\)), then \(\lambda n^2 / \Delta\) serves as a simple upper bound on the magnitude of the first order correction to the ground state wavefunction. Higher order corrections should be well-behaved since performing 1 or 2 bit flips on any excited state also typically entails an energy change of at least \(\Delta\), implying that the entire perturbation series correction should essentially converge like the geometric series \(\sum_k (\lambda n^2 / \Delta)^k\). Hence, the architecture’s tolerance for general imprecision in the qubit couplings and inhomogeneity in the applied field decreases quadratically in the number of qubits. (If unwanted inductive couplings can be limited to a constant number of “next-nearest” neighbors, then the tolerance shall decrease only linearly.)

5. Conclusion

We have reviewed the theory of adiabatic quantum computation regarding its potential speedup over classical computation and its greater robustness toward noise over other forms of quantum computation. We then proposed a scalable architecture for adiabatic quantum computation based on the NP-hardness of calculating the ground state of a planar antiferromagnetically coupled Ising model placed in a magnetic field. Clearly, such an architecture could be used with any qubit whose states have opposite magnetic dipole moments. Future work will present a detailed proposal for its implementation using superconducting persistent-current qubits, which constitute a current, promising approach to lithographable solid-state qubits.
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