Tsallis entropy and entanglement constraints in multi-qubit systems

Jeong San Kim

Institute for Quantum Information Science, University of Calgary, Alberta T2N 1N4, Canada

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We show that the restricted sharability and distribution of multi-qubit entanglement can be characterized by Tsallis-q entropy. We first provide a class of bipartite entanglement measures named Tsallis-q entanglement, and provide its analytic formula in two-qubit systems for 1 ≤ q ≤ 4. For 2 ≤ q ≤ 3, we show a monogamy inequality of multi-qubit entanglement in terms of Tsallis-q entanglement, and we also provide a polygamy inequality using Tsallis-q entropy for 1 ≤ q ≤ 2 and 3 ≤ q ≤ 4.

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I. INTRODUCTION

Whereas classical correlations can be freely shared among parties in multi-party systems, quantum correlation especially quantum entanglement is known to have some restriction in its sharability and distribution. For example, in a tripartite system consisting of parties A, B and C, let us assume A is maximally entangled with both B and C simultaneously. Because maximal entanglement can be used to teleport an arbitrary unknown quantum state, A can teleport an unknown state ρ to B and C by using the simultaneous maximal entanglement. Now, each B and C has an identical copy of ρ, and this means cloning an unknown state ρ, which is impossible by no-cloning theorem [2]. In other words, the assumption of simultaneous maximal entanglement of A with B and C is quantum mechanically forbidden.

This restricted sharability of quantum entanglement is known as the Monogamy of Entanglement (MoE) [3], and it was also shown to play an important role in many applications of quantum information processing. For instance, in quantum cryptography, MoE can be used to restrict the possible correlation between authorized users and the eavesdropper, which is the basic concept of the security proof [4].

For three-qubit systems, MoE was first characterized in forms of a mathematical inequality using concurrence [5] as the bipartite entanglement measure. This characterization is known as CKW inequality named after its establishers, Coffman, Kundu and Wootters [6], and it was also generalized for multi-qubit systems later [7].

MoE in multi-qubit systems is mathematically well-characterized in terms of concurrence, it is however also known that CKW-type characterization for MoE is not generally true for other entanglement measures such as Entanglement of Formation (EoF) [8]: Even in multi-qubit systems, there exists an counterexample that violates CKW-type inequality in terms of EoF.

As bipartite entanglement measures, both concurrence and EoF of a bipartite pure state |ψ⟩_{AB} quantify the uncertainty of the subsystem ρ_A = tr_B |ψ⟩_{AB}⟨ψ| for a mixed state |ψ⟩_{AB}. For the case when |ψ⟩_{AB} is a two-qubit state, the uncertainty of ρ_A is completely determined by a single parameter. Furthermore, the extension of concurrence and that of EoF for a mixed state ρ_{AB} are based on the same method of convex-roof extension, which minimizes the average entanglement over all possible pure state decompositions of ρ_{AB}. In other words, concurrence and EoF for two-qubit states are essentially equivalent based on the same concept, the uncertainty of the subsystem. Moreover, it was also shown that these two measures are related by an monotone-increasing convex function [9].

However, these two equivalent measures for two-qubit systems show very different properties in multipartite systems in characterizing MoE, and this exposes the importance of having proper entanglement measures to characterize MoE even in multi-qubit systems. Moreover, for the study of general MoE in multipartite higher-dimensional quantum systems, having a proper bipartite entanglement measure is one of the most important and necessary things that must precede.

As generalizations of von Neumann entropy, there are two representative classes of entropies quantifying the uncertainty of quantum systems: One is quantum Rényi entropy [10, 11], and the other is quantum Tsallis entropy [12, 13]. Both of them are one-parameter classes parameterized by a nonnegative real number q, having von Neumann entropy as a special case when q → 1. Recently, it was shown that Rényi entropy can be used for CKW-type characterization of multi-qubit monogamy [13].

Here, we show that Tsallis entropy can characterize MoE in multi-qubit systems for a selective choice of the parameter q. Using quantum Tsallis entropy of order q (or Tsallis-q entropy), we first provide an one-parameter class of bipartite entanglement measures, Tsallis-q entanglement, and provide its analytic formula for arbitrary two-qubit states when 1 ≤ q ≤ 4. This class contains EoF as a special case when q → 1. Furthermore, we show the monogamy inequality of multi-qubit systems in terms of Tsallis-q entanglement for 2 ≤ q ≤ 3. For 1 ≤ q ≤ 2 or 3 ≤ q ≤ 4, we also provide a polygamy inequality of multi-qubit entanglement using Tsallis-q entropy.

This paper is organized as follows. In Section II A...
we recall the definition of Tsallis-$q$ entropy, and define Tsallis-$q$ entanglement and its dual quantity for bipartite quantum states. In Section IIIB we provide an analytic formula of Tsallis-$q$ entanglement for arbitrary two-qubit states when $1 \leq q \leq 4$. In Section IIC we derive a monogamy inequality of multi-qubit entanglement in terms of Tsallis-$q$ entanglement for $2 \leq q \leq 3$. We also provide a polygamy inequality of multi-qubit entanglement for $1 \leq q \leq 2$ or $3 \leq q \leq 4$. Finally, we summarize our results in Section IV.

II. TSALLIS-$q$ ENTANGLEMENT

A. Definition

For any quantum state $\rho$, its Tsallis-$q$ entropy is defined as

$$T_q(\rho) = \frac{1}{q-1}(1-\text{tr} \rho^q),$$

for any $q > 0$ and $q \neq 1$. For the case when $\alpha$ tends to 1, $T_q(\rho)$ converges to the von Neumann entropy, that is

$$\lim_{q \to 1} T_q(\rho) = -\text{tr} \rho \log \rho = S(\rho).$$

In other words, Tsallis-$q$ entropy has a singularity at $q = 1$, and it can be replaced by von Neumann entropy. Throughout this paper, we will just consider $T_1(\rho) = S(\rho)$ for any quantum state $\rho$.

For a bipartite pure state $|\psi \rangle_{AB}$ and each $q > 0$, Tsallis-$q$ entanglement is

$$T_q(|\psi \rangle_{AB}) := T_q(\rho_A),$$

where $\rho_A = \text{tr}_B |\psi \rangle_{AB} \langle \psi |$ is the reduced density matrix onto subsystem $A$. For a mixed state $\rho_{AB}$, we define its Tsallis-$q$ entanglement via convex-roof extension, that is

$$T_q(\rho_{AB}) := \min_i \sum p_i T_q(|\psi_i \rangle_{AB}),$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i \rangle_{AB} \langle \psi_i |$.

As a dual quantity to Tsallis-$q$ entanglement, we also define Tsallis-$q$ entanglement of Assistance (TEoA) as

$$T_q^a(\rho_{AB}) := \max_i \sum p_i T_q(|\psi_i \rangle_{AB}),$$

where the maximum is taken over all possible pure state decompositions of $\rho_{AB}$.

Because Tsallis-$q$ entropy converges to von Neumann entropy when $q$ tends to 1, we have

$$\lim_{q \to 1} T_q(\rho_{AB}) = E_t(\rho_{AB}),$$

where $E_t(\rho_{AB})$ is the EoF of $\rho_{AB}$ defined as

$$E_t(\rho_{AB}) = \min_i \sum p_i S(\rho_A^i).$$

Here, the minimation is taken over all possible pure state decompositions of $\rho_{AB}$, such that

$$\rho_{AB} = \sum_i p_i |\phi_i \rangle_{AB} \langle \phi_i |,$$

with $\text{tr}_B |\phi_i \rangle_{AB} \langle \phi_i | = \rho_i^A$. In other words, Tsallis-$q$ entanglement is one-parameter generalization of EoF, and the singularity of $T_q(\rho_{AB})$ at $q = 1$ can be replaced by $E_t(\rho_{AB})$.

Similarly, we have

$$\lim_{q \to 1} T_q^a(\rho_{AB}) = E_a(\rho_{AB}),$$

where $E^a(\rho_{AB})$ is the Entanglement of Assistance (EoA) of $\rho_{AB}$ defined as

$$E_a(\rho_{AB}) = \max_i \sum p_i S(\rho_A^i).$$

Here, the maximum is taken over all possible pure state decompositions of $\rho_{AB}$, such that

$$\rho_{AB} = \sum_i p_i |\phi_i \rangle_{AB} \langle \phi_i |,$$

with $\text{tr}_B |\phi_i \rangle_{AB} \langle \phi_i | = \rho_i^A$.

B. Analytic formula for two-qubit states

Before we provide an analytic formula for Tsallis-$q$ entanglement in two-qubit systems, let us first recall the definition of concurrence and its functional relation with EoF in two-qubit systems.

For any bipartite pure state $|\psi \rangle_{AB}$, its concurrence, $C(|\psi \rangle_{AB})$ is defined as

$$C(|\psi \rangle_{AB}) = \sqrt{2(1-\rho_A^2)},$$

where $\rho_A = \text{tr}_B(|\psi \rangle_{AB} \langle \psi |)$. For a mixed state $\rho_{AB}$, its concurrence is defined as

$$C(\rho_{AB}) = \min_k \sum p_k C(|\psi_k \rangle_{AB}).$$

where the minimum is taken over all possible pure state decompositions, $\rho_{AB} = \sum_k p_k |\psi_k \rangle_{AB} \langle \psi_k |$.

For two-qubit systems, concurrence is known to have an analytic formula [3]; for any two-qubit state $\rho_{AB}$,

$$C(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$

where $\lambda_i$’s are the eigenvalues, in decreasing order, of $\sqrt{\rho_{AB} \rho_{AB}^\dagger}$ and $\rho_{AB} = \sigma_y \otimes \sigma_y \rho_{AB} \sigma_y ^\dagger \otimes \sigma_y$ with the Pauli operator $\sigma_y$. Furthermore, the relation between concurrence and EoF of a two-qubit mixed state $\rho_{AB}$ (or a pure state $|\psi \rangle_{AB} \in \mathbb{C}^2 \otimes \mathbb{C}^d$, $d \geq 2$), can be given as a monotone increasing, convex function [3], such that

$$E_t(\rho_{AB}) = E(C(\rho_{AB})).$$
\[ \mathcal{E}(x) = H\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - x^2}\right), \quad \text{for } 0 \leq x \leq 1, \]  
(16)

with the binary entropy function \( H(t) = -[t \log t + (1-t) \log(1-t)] \). In other words, the analytic formula of concurrence as well as its functional relation with EoF lead us to an analytic formula for EoF in two-qubit systems.

For any \( 2 \otimes d \) pure state \( |\psi\rangle_{AB} \) (especially a two-qubit pure state) with its Schmidt decomposition \( |\psi\rangle_{AB} = \sqrt{\lambda_0}|00\rangle_{AB} + \sqrt{\lambda_1}|11\rangle_{AB} \), its Tsallis-q entanglement is

\[ T_q(|\psi\rangle_{AB}) = T_q(\rho_A) = \frac{1}{q-1} (1 - \lambda_0^{q} - \lambda_1^{q}). \]  
(17)

Because the concurrence of \( |\psi\rangle_{AB} \) is

\[ C(|\psi\rangle_{AB}) = \sqrt{2(1 - \rho_{AB}^2)} = \sqrt{\lambda_0 \lambda_1}, \]  
(18)

it can be easily verified that

\[ T_q(|\psi\rangle_{AB}) = g_q(C(|\psi\rangle_{AB})), \]  
(19)

where \( g_q(x) \) is an analytic function defined as

\[ g_q(x) := \frac{1}{q-1} \left[ 1 - \left(\frac{1 + \sqrt{1 - x^2}}{2}\right)^q - \left(\frac{1 - \sqrt{1 - x^2}}{2}\right)^q \right] \]  
(20)

on \( 0 \leq x \leq 1 \). In other words, for any \( 2 \otimes d \) pure state \( |\psi\rangle_{AB} \), we have a functional relation between its concurrence and Tsallis-q entanglement for each \( q > 0 \). Note that \( g_q(x) \) converges to the function \( \mathcal{E}(x) \) in Eq. (16) for the case when \( q \) tends to 1.

It was shown that there exists an optimal decomposition for the concurrence of a two-qubit mixed state such that every pure state concurrence in the decomposition has the same value [2]. For any two-qubit state \( \rho_{AB} \), there exists a pure state decomposition \( \rho_{AB} = \sum_i p_i |\phi_i\rangle_{AB} \langle \phi_i| \) such that

\[ C(\rho_{AB}) = \sum_i p_i C(|\phi_i\rangle_{AB}), \]  
(21)

and

\[ C(|\phi_i\rangle_{AB}) = C(\rho_{AB}), \]  
(22)

for each \( i \). Based on this, one possible sufficient condition for the relation in Eq. (19) to be also true for two-qubit mixed states is that the function \( g_q(x) \) is monotonically increasing and convex [15]. In other words, we have

\[ T_q(\rho_{AB}) = g_q(C(\rho_{AB})) \]  
(23)

for any two-qubit mixed state \( \rho_{AB} \) provided that \( g_q(x) \) is monotonically increasing and convex. Moreover, for the range of \( q \) where \( g_q(x) \) is monotonically increasing and convex, Eq. (23) also implies an analytic formula of Tsallis-q entanglement for any two-qubit state.

Now, let us consider the monotonicity and convexity of \( g_q(x) \) in Eq. (21). Because \( g_q(x) \) is an analytic function on \( 0 \leq x \leq 1 \), its monotonicity and convexity follow from the nonnegativity of its first and second derivatives.

By taking the first derivative of \( g_q(x) \), we have

\[ \frac{dg_q(x)}{dx} = \frac{qx \left[ \left(1 + \sqrt{1 - x^2}\right)^{q-1} - \left(1 - \sqrt{1 - x^2}\right)^{q-1}\right]}{2^{q(q-1)} \sqrt{1 - x^2}}, \]  
(24)

which is always nonnegative on \( 0 \leq x \leq 1 \) for \( q > 0 \). It is also direct to check that Eq. (24) is strictly positive for \( 0 < x < 1 \). In other words, \( g_q(x) \) is a strictly monotone-increasing function for any \( q > 0 \).

For the second derivative of \( g_q(x) \), we have

\[ \frac{d^2g_q(x)}{dx^2} = \alpha \left[ \left(1 + \sqrt{1 - x^2}\right)^{q-2} - \frac{1 + \sqrt{1 - x^2}}{\sqrt{1 - x^2}} - x^2(q-1) \right] - \left(1 - \sqrt{1 - x^2}\right)^{q-2} \left(1 - \sqrt{1 - x^2}\right)^{q-2} \left(1 - \frac{x^2(q-1)}{\sqrt{1 - x^2}} + x^2(q-1) \right) \]  
(25)

where \( \alpha = \frac{q}{2^{2(q-1)}} \). Here, we first prove that \( g_q(x) \) is not convex for \( q \geq 5 \) by showing the existence of \( x_0 \) between 0 and 1 such that \( \frac{d^2g_q(x_0)}{dx^2} \) is negative. To see this, first note that the second term of the right-hand side in Eq. (25) is always negative for \( 0 < x < 1 \) if \( q > 1 \). Thus, it suffices to show that the first term of the right-hand side in Eq. (25) is nonpositive at \( x_0 \in (0,1) \) for \( q \geq 5 \). Furthermore, the only factor of the first term that can be negative is

\[ \left(1 + \sqrt{1 - x^2}\right)^{q-2} - x^2(q-1) \]  
(26)

since both \( \alpha \) and \( \frac{1 + \sqrt{1 - x^2}}{\sqrt{1 - x^2}} \) are always positive at
If \( q > 1 \), then a defining function such that

\[
h(x) = \frac{1 - \sqrt{1 - x^2}}{x^2 \sqrt{1 - x^2}} + 1, \tag{27}\]

the nonpositivity of Eq. (26) is equivalent to

\[
g(x) \leq h(x). \tag{28}\]

Since \( h(x) \) is an analytic function on \( 0 < x < 1 \), it is direct to verify that it has a critical point at \( x_0 = \frac{\sqrt{2}}{2} \) with \( g_{q}(x_0) = 5 \), which is the global minimum. In other words, for \( q \geq 1 \), there always exists \( x_0 \in (0, 1) \) making Eq. (26) nonpositive, and thus \( g_{q}(x) \) is not convex for this region of \( q \).

For the region of \( q < 5 \), let us first consider the function \( g_{q}(x) \) of the integer value \( q \), that is \( q = 1, 2, 3 \) and \( 4 \). If \( q \rightarrow 1 \), \( g_{q}(x) \) converges to \( E(x) \) in Eq. (16), which is already known to be convex on \( 0 \leq x \leq 1 \). Furthermore, we have

\[
g_{2}(x) = \frac{x^2}{2}, \quad g_{3}(x) = \frac{3x^2}{8}, \quad g_{4}(x) = \frac{8x^2 - x^4}{24}, \tag{29}\]

which are convex polynomials on \( 0 \leq x \leq 1 \).

In fact, if we consider \( \frac{d^2 g_{q}(x)}{dx^2} \) in Eq. (25) as a function of \( x \) and \( q \)

\[
l(x, q) = \frac{d^2 g_{q}(x)}{dx^2}, \tag{30}\]

defined on the domain \( \mathcal{D} = \{(x, q)|0 \leq x \leq 1, 1 \leq q \leq 4\} \), it is tedious but also straightforward to check that \( l(x, q) \) does not have any vanishing gradient in the interior of \( \mathcal{D} \), and its function value on the boundary of \( \mathcal{D} \) is always nonnegative. Because \( l(x, q) \) is analytic in the interior of \( \mathcal{D} \), and continuous on the boundary, \( l(x, q) \) is nonnegative throughout the domain \( \mathcal{D} \), and this implies the convexity of \( g_{q}(x) \) for \( 1 \leq q \leq 4 \). Thus, we have the following theorem.

**Theorem 1.** For \( 1 \leq q \leq 4 \),

\[
g_{q}(x) = \frac{1}{q - 1} \left[ 1 - \left( \frac{1 + \sqrt{1 - x^2}}{2} \right)^q - \left( \frac{1 - \sqrt{1 - x^2}}{2} \right)^q \right]. \tag{31}\]

is a monotonically increasing convex function on \( 0 \leq x \leq 1 \). Furthermore, for this range of \( q \), any two-qubit state \( \rho_{AB} \) has an analytic formula for its Tsallis-\( q \) entanglement such that \( T_{q}(\rho_{AB}) = g_{q}(C(\rho_{AB})) \) where \( C(\rho_{AB}) \) is the concurrence of \( \rho_{AB} \).

Due to the continuity of \( g_{q}(x) \) with respect to \( q \), we can always assure the convexity of \( g_{q}(x) \) for some region of \( q \) slightly less than 1 or larger than 4. Furthermore, the continuity of \( l(x, q) \) in Eq. (30) also assures the existence of \( q_{0} \) between 4 and 5, at which the convexity of \( g_{q}(x) \) starts being violated. However, it is generally hard to get an algebraic solution of such \( q_{0} \) since \( \frac{d^2 g_{q}(x)}{dx^2} \) in Eq. (25) is not an algebraic function with respect to \( q \). Here, we have a numerical way of calculation to test various values of \( x \) and \( q \), and it is illustrated in Figure 1. According to Figure 1, \( g_{q}(x) \) is convex for the region \( 0.7 \leq q \leq 4.2 \), and thus the analytic formula of Tsallis-\( q \) entanglement for two-qubit states in Eq. (25) can also be claimed for this region of \( q \).

**III. MULTI-QUBIT ENTANGLEMENT CONSTRAINT IN TERMS OF TSALLIS-\( q \) ENTANGLEMENT**

Using concurrence as the bipartite entanglement measure, the monogamous property of a multi-qubit pure state \(|\psi\rangle_{A_{1}A_{2}...A_{n}} \rangle \) was shown to have a mathematical characterization as,

\[
C_{A_{1}(A_{2}...A_{n})}^{2} \leq C_{A_{1}A_{2}}^{2} + \cdots + C_{A_{1}A_{n}}^{2}, \tag{32}\]

where \( C_{A_{1}(A_{2}...A_{n})} = C(|\psi\rangle_{A_{1}(A_{2}...A_{n})}) \) is the concurrence of \(|\psi\rangle_{A_{1}(A_{2}...A_{n})} \rangle \) with respect to the bipartite cut between \( A_{1} \) and the others, and \( C_{A_{1}A_{i}} = C(\rho_{A_{1}A_{i}}) \) is the concurrence of the reduced density matrix \( \rho_{A_{1}A_{i}} \) for \( i = 2, \ldots, n \).

As a dual value to concurrence, **Concurrence of Assistance (CoA)** \([16]\) of a bipartite state \( \rho_{AB} \) is defined as

\[
C^{\alpha}(\rho_{AB}) = \max \sum_{k} p_{k} C(|\psi_{k}\rangle_{AB}), \tag{33}\]

where the maximum is taken over all possible pure state decompositions of \( \rho_{AB} = \sum_{k} p_{k}|\psi_{k}\rangle_{AB}\langle\psi_{k}|. \) Furthermore, it was also shown that there exists a polygamy (or dual monogamy) relation of multi-qubit entanglement in terms of CoA \([17]\). For any multi-qubit pure state \(|\psi\rangle_{A_{1}...A_{n}} \rangle \), we have

\[
C_{A_{1}(A_{2}...A_{n})}^{2} \leq (C_{A_{1}A_{2}}^{\alpha})^{2} + \cdots + (C_{A_{1}A_{n}}^{\alpha})^{2}, \tag{34}\]
where $C_{\rho,\alpha}$ is the CoA of the reduced density matrix $\rho_{A_i A_i}$ for $i = 2, \ldots, n$.

Here, we show that this monogamous and polygamous property of multi-qubit entanglement can also be characterized in terms of Tsallis-$q$ entanglement and TEOA. Before this, we provide an important property of the function $g_\alpha(x)$ in Eq. (20) for the proof of multi-qubit monogamy and polygamy relations.

For each $q > 0$, let us define a two-variable function $m_q(x, y)$,

$$m_q(x, y) := g_q \left( \sqrt{x^2 + y^2} \right) - g_q(x) - g_q(y),$$

for an analytic function on the domain $D = \{(x, y) | 0 \leq x, y, x^2 + y^2 \leq 1 \}$. Since $m_q(x, y)$ is continuous on the domain $D$ and analytic in the interior, its maximum or minimum values can arise only at the critical points or on the boundary of $D$. By taking the first-order partial derivatives of $m_q(x, y)$, we have its gradient

$$\nabla m_p(x, y) = \left( \frac{\partial m_p(x, y)}{\partial x}, \frac{\partial m_p(x, y)}{\partial y} \right)$$

where

$$\begin{align*}
\frac{\partial m_q(x, y)}{\partial x} &= \alpha x \left[ \left(1 + \sqrt{1 - x^2 - y^2} \right)^{q-1} - \left(1 + \sqrt{1 - x^2 - y^2} \right)^{q-1} \right] \\
\frac{\partial m_q(x, y)}{\partial y} &= \alpha y \left[ \left(1 + \sqrt{1 - x^2 - y^2} \right)^{q-1} - \left(1 + \sqrt{1 - x^2 - y^2} \right)^{q-1} \right]
\end{align*}$$

$$\begin{align*}
\frac{\partial m_q(x, y)}{\partial x} &= \frac{\partial m_q(x, y)}{\partial y} = 0 \text{ for an analytic function}
\end{align*}$$

for $0 < t < 1$. Furthermore, it is straightforward to see that $\frac{\partial m_q(t)}{\partial t} < 0$ for $q > 1$. In other words, $n_q(t)$ is a strictly monotone-decreasing function with respect to $t$ for $q > 1$; therefore Eq. (38) implies $x_0 = y_0$.

Now, let us consider the function values of $m_q(x, y)$ on the boundary of $D$. If $x = 0$ or $y = 0$, it is clear that $m_q(x, y) = 0$. For the case when $x^2 + y^2 = 1$, $m_q(x, y) = 0$ becomes a single variable function

$$b_q(x) = \beta \left[ \left(1 + \sqrt{1 - x^2} \right)^q + \left(1 - \sqrt{1 - x^2} \right)^q \right]$$

$$+ \beta \left[ \left(1 + x^2\right)^q + \left(1 - x^2\right)^q - 2 - 2x^q \right]$$

with $\beta = \frac{1}{q-1}$. For the case when $q = 1$, it is clear form Eq. (20) that $m_q(x, y) = 0$, and thus $b_q(x) = 0$. If $q$ is neither 2 nor 3, $b_q(x)$ has only one critical point at $x = \frac{1}{\sqrt{2}}$ for any $q > 1$. Because $b_q(0) = b_q(1) = 0$, which are the function values at the boundary, the signs of the function values of $b_q(x)$ are totally determined by that of $b_q \left( \frac{1}{\sqrt{2}} \right)$, which is the function value at the critical point.

Now, we have

$$b_q \left( \frac{1}{\sqrt{2}} \right) = \frac{2}{(q-1)2^q} \left[ \left(1 + \frac{1}{\sqrt{2}} \right)^q + \left(1 - \frac{1}{\sqrt{2}} \right)^q \right]$$

$$- \frac{1}{(q-1)2^q} \left(2 + 2x^q\right),$$

whose function value with respect to $q$ is illustrated in Figure 2.

In other words, the function $m_q(x, y)$ in Eq. (35) has no vanishing gradient in the domain $D$ for $q > 1$, and its function values at the boundary of $D$ is always nonpositive for $1 \leq q < 2$ and $3 < q \leq 4$, whereas $m_q(x, y)$ is always nonnegative for $2 < q < 3$. Thus, we have

$$g_q \left( \sqrt{x^2 + y^2} \right) \leq g_q(x) + g_q(y) \quad \text{for } 1 < q < 2 \text{ and } 3 < q < 4,$$

and

$$g_q \left( \sqrt{x^2 + y^2} \right) \geq g_q(x) + g_q(y) \quad \text{for } 2 < q < 3.$$
and the last inequality is by definition of Tsallis-q \( g_{\rho} \) for any two-qubit mixed state or 2-qubit pure state \( \rho \).

Theorem 2. For a multi-qubit state \( \rho_{A_1 \cdots A_n} \) and \( 2 \leq q \leq 3 \), we have

\[
T_q(\rho_{A_1(A_2 \cdots A_n)}) \geq T_q(\rho_{A_1A_2}) + \cdots + T_q(\rho_{A_1A_n})
\]

(45)

where \( T_q(\rho_{A_1(A_2 \cdots A_n)}) \) is the Tsallis-q entanglement of \( \rho_{A_1(A_2 \cdots A_n)} \) with respect to the bipartite cut between \( A_1 \) and \( A_2 \cdots A_n \), and \( T_q(\rho_{A_1A_i}) \) is the Tsallis-q entanglement of the reduced density matrix \( \rho_{A_1A_i} \) for \( i = 2, \ldots, n \).

Proof. For the case when \( q = 2 \) or 3, Eq. (20) implies

\[
T_2(\rho_{AB}) = \frac{C_{AB}}{2}, \quad T_3(\rho_{AB}) = \frac{3}{2} C_{AB}^2,
\]

(46)

for any two-qubit mixed state or 2\( \otimes d \) pure state \( \rho_{AB} \) and its concurrence \( C_{AB} \). Thus, the monogamy inequality in Eq. (15) follows from Eqs. (32) and (46).

For \( 2 < q < 3 \), we first prove the theorem for \( n \)-qubit pure state \( |\psi\rangle_{A_1 \cdots A_n} \). Note that Eq. (32) is equivalent to

\[
C_{A_1(A_2 \cdots A_n)} \geq \sqrt{C_{A_1A_2}^2 + \cdots + C_{A_1A_n}^2},
\]

(47)

for any \( n \)-qubit pure state \( |\psi\rangle_{A_1(A_2 \cdots A_n)} \). Thus, from Eq. (13) together with Eq. (17), we have

\[
T_q(|\psi\rangle_{A_1(A_2 \cdots A_n)}) = g_q(C_{A_1(A_2 \cdots A_n)})
\]

\[
\geq g_q(\sqrt{C_{A_1A_2}^2 + \cdots + C_{A_1A_n}^2})
\]

\[
\geq g_q(C_{A_1A_2}) + \cdots + g_q(C_{A_1A_n})
\]

\[
= T_q(\rho_{A_1A_2}) + \cdots + T_q(\rho_{A_1A_n})
\]

(48)

where the first equality is by the functional relation between the concurrence and the Tsallis-q entanglement for 2\( \otimes d \) pure states, the first inequality is by the monotonicity of \( g_q(x) \), the other inequalities are by iterative use of Eq. (13), and the last equality is by Theorem 1.

For a \( n \)-qubit mixed state \( \rho_{A_1(A_2 \cdots A_n)} \), let \( \rho_{A_1(A_2 \cdots A_n)} = \sum_j p_j |\psi_j\rangle_{A_1(A_2 \cdots A_n)} \langle \psi_j| \) be an optimal decomposition such that

\[
T_q(\rho_{A_1(A_2 \cdots A_n)}) = \sum_j p_j T_q(|\psi_j\rangle_{A_1(A_2 \cdots A_n)}).
\]

Because each \( |\psi_j\rangle_{A_1(A_2 \cdots A_n)} \) in the decomposition is an \( n \)-qubit pure state, we have

\[
T_q(\rho_{A_1(A_2 \cdots A_n)}) = \sum_j p_j T_q(|\psi_j\rangle_{A_1(A_2 \cdots A_n)})
\]

\[
\geq \sum_j p_j \left( T_q(\rho_{A_1A_2}) + \cdots + T_q(\rho_{A_1A_n}) \right)
\]

\[
= \sum_j p_j T_q(\rho_{A_1A_2}) + \cdots + \sum_j p_j T_q(\rho_{A_1A_n})
\]

\[
\geq T_q(\rho_{A_1A_2}) + \cdots + T_q(\rho_{A_1A_n}),
\]

(49)

Now, let us consider the polygamy of multi-qubit entanglement using Tsallis-q entropy. We first note that the function \( g_{\rho}(x) \) in Eq. (20) can also relate CoA and TEOA of a two-qubit state \( \rho_{AB} \). By letting \( \rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i| \) be an optimal decomposition for its
CoA, that is,
\[
C^a(\rho_{AB}) = \sum_i p_i C(|\psi_i\rangle_{AB}),
\]
we have
\[
g_q(C^a(\rho_{AB})) = g_q \left( \sum_i p_i C(|\psi_i\rangle_{AB}) \right) \\
\leq \sum_i p_i g_q(C(|\psi_i\rangle_{AB})) \\
= \sum_i p_i T_q(|\psi_i\rangle_{AB}) \\
\leq T^q_a(\rho_{AB})
\]
where the first inequality can be assured by the convexity of \(g_q(x)\) and the last inequality is by the definition of TEOa. Because \(g_q(x)\) is convex for \(1 \leq q \leq 4\), Eq. (51) is thus true for this region of \(q\). Furthermore, \(g_q(x)\) satisfies the property of Eq. (42) for \(1 \leq q \leq 2\) or \(3 \leq q \leq 4\). Thus, we have the following theorem of the polygamy inequality in multi-qubit systems.

**Theorem 3.** For any multi-qubit state \(\rho_{A_1\ldots A_n}\) and \(1 \leq q \leq 2\) or \(3 \leq q \leq 4\), we have
\[
T_q(\rho_{A_1(A_2\ldots A_n)}) \leq T^a_q(\rho_{A_1A_2}) + \cdots + T^a_q(\rho_{A_1A_n})
\]
where \(T_q(\rho_{A_1(A_2\ldots A_n)})\) is the Tsallis-\(q\) entanglement of \(|\psi\rangle_{A_1(A_2\ldots A_n)}\) with respect to the bipartite cut between \(A_1\) and \(A_2\ldots A_n\), and \(T^a_q(\rho_{A_1A_i})\) is the TEOa of the reduced density matrix \(\rho_{A_1A_i}\) for \(i = 2,\ldots, n\).

**Proof.** We first prove the theorem for a \(n\)-qubit pure state, and generalize it into mixed states.

For the case when \(q\) tends to 1, Tsallis-\(q\) entanglement converges to EoA in Eq. (10). It was shown that the polygamy inequality of multi-qubit systems can be shown in terms of EoA [13]. For the case when \(q = 2\) or 3, it is also straightforward from Eqs. (29) and (34).

For a \(n\)-qubit pure state \(|\psi\rangle_{A_1(A_2\ldots A_n)}\) and \(1 < q < 2\) or \(3 < q < 4\), let us first assume that \((C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_n})^2 \leq 1\) in Eq. (41). Then we have
\[
T_q(|\psi\rangle_{A_1(A_2\ldots A_n)}) = g_q(C^a_{A_1(A_2\ldots A_n)}) \\
\leq g_q(\sqrt{(C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_n})^2}) \\
\leq g_q(C^a_{A_1A_2}) \nonumber
\]
\[
+ g_q(\sqrt{(C^a_{A_1A_3})^2 + \cdots + (C^a_{A_1A_n})^2}) \\
\vdots \\
\leq g_q(C^a_{A_1A_2}) + \cdots + g_q(C^a_{A_1A_n}) \\
\leq T^a_q(\rho_{A_1A_2}) + \cdots + T^a_q(\rho_{A_1A_n}),
\]
where the first inequality is due to the monotonicity of the function \(g_q(x)\), the second and third inequalities are obtained by iterative use of Eq. (42), and the last inequality is by Eq. (41).

Now, let us assume that \((C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_n})^2 > 1\). Due to the monotonicity of \(g_q(x)\), we first note that
\[
T_q(|\psi\rangle_{A_1(A_2\ldots A_n)}) = g_q \left( C(|\psi\rangle_{A_1A_2\ldots A_n}) \right) \\
\leq g_q(1) \\
= \frac{1}{q-1} \left( 1 - \frac{1}{2^{q-1}} \right)
\]
for any multi-qubit pure state \(|\psi\rangle_{A_1(A_2\ldots A_n)}\), and \(q > 1\). By letting \(\gamma = \frac{1}{q-1} \left( 1 - \frac{1}{2^{q-1}} \right)\), it is thus enough to show that \(T^a_q(\rho_{A_1A_2}) + \cdots + T^a_q(\rho_{A_1A_n}) \geq \gamma\).

Here, we note that there exists \(k \in \{2,\ldots, n-1\}\) such that
\[
(C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_k})^2 \leq 1, \\
(C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_{k+1}})^2 > 1.
\]
If we let
\[
T := (C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_{k+1}})^2 - 1,
\]
we have
\[
\gamma = g_q(1) \\
= g_q \left( \sqrt{(C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_{k+1}})^2 - T} \right) \\
\leq g_q \left( \sqrt{(C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_k})^2} \right) \\
+ g_q \left( \sqrt{(C^a_{A_1A_{k+1}})^2 - T} \right) \\
\leq g_q(C^a_{A_1A_k}) + \cdots + g_q(C^a_{A_1A_{k+1}}) + g_q(C^a_{A_1A_{k+2}}) \\
\leq T^a_q(\rho_{A_1A_2}) + \cdots + T^a_q(\rho_{A_1A_n}),
\]
where the first inequality is by using Eq. (42) with respect to \((C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_k})^2\) and \((C^a_{A_1A_{k+1}})^2 - T\), the second inequality is by iterative use of Eq. (42) on \((C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_k})^2\), and the last inequality is by Eq. (41).

For a \(n\)-qubit mixed state \(\rho_{A_1(A_2\ldots A_n)}\), let \(\rho_{A_1(A_2\ldots A_n)} = \sum_j p_j |\psi_j\rangle_{A_1(A_2\ldots A_n)} \langle \psi_j|\) be an optimal decomposition for TEOa such that
\[
T^a_q(\rho_{A_1(A_2\ldots A_n)}) = \sum_j p_j T^a_q \left( |\psi_j\rangle_{A_1(A_2\ldots A_n)} \langle \psi_j| \right).
\]
Because each \(|\psi_j\rangle_{A_1(A_2\ldots A_n)}\) in the decomposition is an \(n\)-qubit pure state, we have
\[ T_q^a (\rho_{A_1(A_2\cdots A_n)}) = \sum_j p_j T_q^a (|\psi_j\rangle_{A_1(A_2\cdots A_n)}) \]
\[ \leq \sum_j p_j \left( T_q^a (\rho_{A_1A_2}) + \cdots + T_q^a (\rho_{A_1A_n}) \right) \]
\[ = \sum_j p_j T_q^a (\rho_j^{A_1A_2}) + \cdots + \sum_j p_j T_q^a (\rho_j^{A_1A_n}) \]
\[ \leq T_q^a (\rho_{A_1A_2}) + \cdots + T_q^a (\rho_{A_1A_n}), \quad (58) \]

where \( \rho_j^{A_1A_i} \) is the reduced density matrix of \( |\psi_j\rangle_{A_1(A_2\cdots A_n)} \) onto subsystem \( A_1A_i \) for each \( i = 2, \cdots, n \) and the last inequality is by definition of TEoA for each \( \rho_{A_1A_i} \).

Although Theorem 3 provides the polygamy inequality of multi-qubit entanglement in terms of TEoA for \( 1 \leq q \leq 2 \) or \( 3 \leq q \leq 4 \), it is also clear that Eq. (52) is also true for \( q \) slightly larger than 4 or less than 1 due to its continuity with respect to \( q \).

IV. CONCLUSION

Using Tsallis-\( q \) entropy, we have established a class of bipartite entanglement measures, Tsallis-\( q \) entanglement, and provided its analytic formula in two-qubit systems for \( 1 \leq q \leq 4 \). Based on the functional relation between concurrence and Tsallis-\( q \) entanglement, we have shown that the monogamy of multi-qubit entanglement can be mathematically characterized in terms of Tsallis-\( q \) entanglement for \( 2 \leq q \leq 3 \). We have also provided a polygamy inequality of multi-qubit entanglement in terms of TEoA for \( 1 \leq q \leq 2 \) and \( 3 \leq q \leq 4 \).

The class of monogamy and polygamy inequalities of multi-qubit entanglement we provided here consists of infinitely many inequalities parameterized by \( q \). We believe that our result will provide useful tools and strong candidates for general monogamy and polygamy relations of entanglement in multipartite higher-dimensional quantum systems, which is one of the most important and necessary topics in the study of multipartite quantum entanglement.

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[1] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[2] W. K. Wootters and W. H. Zurek, Nature 299, 802 (1982).
[3] B. M. Terhal, IBM J. Research and Development 48, 71 (2004).
[4] L. Masanes, Phys. Rev. Lett. 102, 140501 (2009).
[5] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[6] V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
[7] T. Osborne and F. Verstraete, Phys. Rev. Lett. 96, 220503 (2006).
[8] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[9] A. Rényi, Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability (University of California Press, Berkeley, 1960) I, p. 547-561.
[10] R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A 210, 377 (1996).
[11] C. Tsallis, J. Stat. Phys. 52, 479 (1988).
[12] P. T. Landsberg and V. Vedral, Phys. Lett. A 247, 211 (1998).
[13] J. S. Kim and B. C. Sanders, arXiv.org:0911.5180 (2009).
[14] O. Cohen, Phys. Rev. Lett. 80, 2493 (1998).
[15] Due to the existence of the decomposition satisfying Eqs (21) and (22), we have

\[ g_q (C(\rho_{AB})) = g_q \left( \sum_i p_i C(|\phi_i\rangle_{AB}) \right) \]
\[ = \sum_i p_i g_q \left( C(|\phi_i\rangle_{AB}) \right) \]
\[ = \sum_i p_i T_q (|\phi_i\rangle_{AB}) \]
\[ \geq T_q (\rho_{AB}). \]

Conversely, the existence of the optimal decomposition of \( \rho_{AB} = \sum_i q_i |\mu_i\rangle_{AB} \langle \mu_i| \) for Tsallis-\( q \) entanglement leads
us to
\[ T_q(\rho_{AB}) = \sum_j q_j T_q(|\mu_j\rangle_{AB}) \]
\[ = \sum_j q_j g_q(C(|\mu_j\rangle_{AB})) \]
\[ \geq g_q \left( \sum_j q_j C(|\mu_j\rangle_{AB}) \right) \]
\[ \geq g_q(C(\rho_{AB})), \]

where the first and second inequalities are due to the convexity and monotonicity of \( g_q(x) \).

[16] T. Laustsen, F. Verstraete and S. J. van Enk, Quantum Inf. Comput. 3, 64 (2003).
[17] G. Gour, S. Bandyopadhay and B. C. Sanders, J. Math. Phys. 48, 012108 (2007).
[18] F. Buscemi, G. Gour and J. S. Kim, Phys. Rev. A 80, 012324 (2009).