Computations of instanton invariants

Thomas Köppe

Abstract

Motivated by newly discovered properties of instantons on non-compact spaces, we realised that certain analytic invariants of vector bundles detect fine geometric properties. We present numerical algorithms, implemented in *Macaulay 2*, to compute these invariants.

Precisely, we obtain the direct image and first derived functor of the contraction map \( \pi: Z \to X \), where \( Z \) is the total space of a negative bundle over \( \mathbb{P}^1 \) and \( \pi \) contracts the zero section. We obtain two numerical invariants of a rank-2 vector bundle \( E \) on \( Z \), the width \( h^0(X; (\pi_* E)\vee/\pi_* E) \) and the height \( h^0(X; R^1\pi_* E) \), whose sum is the local holomorphic Euler characteristic \( \chi_{\text{loc}}(E) \).

Contents

1 Introduction 2

2 Definitions and background 3

3 Computation of \( h^0(X_k; (\pi_* E)\vee/\pi_* E) \) 4
  3.1 An example .......................................................... 5
  3.2 Description of the algorithm ........................................ 7
    3.2.1 Computation of \( M \) .............................................. 7
    3.2.2 Computation of \( M\vee \) and \( l(Q) \) .......................... 11
  3.3 Implementation of the algorithm ................................. 11

4 Computation of \( h^0(X_k; R^1\pi_* E) \) 14
  4.1 Description of the algorithm .................................... 15

5 Note on computing \( H^1(Z_k; \text{End } E) \) 18

6 Cancelling infinities: the computation of \( H^0(Z_k; \text{End } E) \) 18

7 Note on adapting the algorithms to other spaces 20

A Application: Rank-2 Bundles on \( Z_k \) and genericity 21
  A.1 The instanton case ........................................... 21
  A.2 Non-instanton bundles ........................................ 23

B Usage example 24
1 Introduction

In this paper we present effective algorithms, implemented in Macaulay 2, for the computation of two numerical invariants of locally free sheaves of rank 2 and with $c_1 = 0$ on a family of open complex surfaces $Z_k$ which contain a distinguished line $\ell$ of self-intersection $\ell^2 = -k$, $k > 0$. The interest in these sheaves arises from mathematical physics, since the Kobayashi-Hitchin correspondence identifies a certain subset of these sheaves with instantons on $Z_k$, and in this picture our two numerical invariants add up to the local charge of the instanton near the line $\ell$. However, the invariants are strictly finer than the charge, and they apply to a larger class of sheaves than just those which correspond to instantons, and they provide a way to stratify the moduli of $\mathfrak{sl}_2$-bundles on $Z_k$ into “nice” components.

The mathematical theory behind these sheaves and their relation to physics has been studied in [GKM08], and the study of their moduli is the subject of [BGK] and [BGK09]. The explicit computation of the numerical invariants has been an essential ingredient of several of the results in those papers, for the proof of which one used “direct computation”. It is the aim of this paper to describe general algorithms for these direct computations. A reference implementation in Macaulay 2 can be found on the author’s website at http://www.maths.ed.ac.uk/~s0571100/Instanton/.

Finally, we will see that the algorithms are actually easily adaptable to a larger class of computations of sheaf cohomology on more general spaces. One such adapted algorithm will be used on our upcoming paper [GK], where we study sheaves on local Calabi-Yau threefolds, such as $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and $\text{Tot}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$. Sheaves on Calabi-Yau threefolds are of interest to numerous mathematicians and physicists, in particular the aforementioned spaces appear in the context of brane theory in papers by Dijkgraaf–Vafa and G. Moore.

Outline. In §2 we give the basic definitions of the types of sheaves in whose computations we are interested in this paper, along with some background results. We define our two basic invariants, the width and the height. The algorithms for the explicit computation of the width and the height are described in §3 and §4, respectively. Finally, we describe how to compute similar invariants of endomorphism bundles in §5 and §6 and how to adapt the algorithms to other situations.

Acknowledgments. The algorithms in this paper are generalisations of the work of I. Swanson and E. Gasparim in [GS05]. The author is very grateful to D. Grayson and M. Stillman for their great work on creating Macaulay 2 and maintaining a friendly and active user community, and to E. Gasparim for the privilege of enjoyable collaboration and sage advise.
2 Definitions and background

Let $Z_k$ be the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-k)$ over $\mathbb{P}^1$ and $k > 0$. Denote by $\ell$ the zero section, so that $\ell^2 = -k$. Let $E$ be a holomorphic rank-2 vector bundle over $Z_k$ with $c_1(E) = 0$. It is known from [Gas97] that $E$ is an algebraic extension of algebraic line bundles,

$$0 \rightarrow \mathcal{O}(-j) \rightarrow E \rightarrow \mathcal{O}(j) \rightarrow 0, \quad (2.1)$$

where $\mathcal{O}(j)$ is the pull-back to $Z_k$ of $\mathcal{O}_{\mathbb{P}^1}(j)$ under the projection $Z_k \rightarrow \mathbb{P}^1$.

We fix once and for all coordinate charts $U = \{z, u\}$ and $V = \{w, v\}$ on $Z_k$, so that $w = z^{-1}$ and $v = z^ku$. (A more symmetric picture is given by weighted homogeneous coordinates $[x_0 : x_1 : x_2]$ of degrees $(1, -k, 1)$, but this is less useful for explicit computations.)

In these coordinates, an extension of the form (2.1) is given by the transition function from the $U$-chart to the $V$-chart that takes the form of the matrix

$$T = \begin{pmatrix} z^j & p(z, u) \\ 0 & z^{-j} \end{pmatrix}. \quad (2.2)$$

The integer $j \geq 0$ is called the splitting type of $E$, and the function $p$, which is holomorphic on $U \cap V$, represents an element of $\text{Ext}^1_{\mathcal{O}_{Z_k}}(\mathcal{O}(-j), \mathcal{O}(j))$. We write $[p]$ for the equivalence class of all functions that determine isomorphic extensions, and we say that the pair $(j, p)$ determines the bundle $E$ on $Z_k$. Hence $(j, q)$ defines isomorphic bundles for all $q \in [p]$.

It turns out that in our case the function $p$ is always a polynomial:

**Proposition 2.1 ([BGK]).** If $(j, p)$ determines a bundle of type (2.1) on $Z_k$, then we can choose $p$ to be a polynomial in $u$, $z$ and $z^{-1}$. Moreover, the polynomial can be chosen to have the form

$$p(z, u) = \sum_{r=1}^{2j-2} \sum_{s=kr-j+1}^{j-1} p_{rs}u^rz^s, \quad (2.3)$$

and if $u|p(z, u)$, the equivalence is given by simply setting all terms that do not appear in (2.3) to zero.

**Remark 2.2.** If $u$ does not divide $p$, then the extension (2.2) defines a bundle $E$ that is also an extension of line bundles of a lower splitting type. In that case the problem $(j, p)$ is ill-posed. This does not happen if $u$ divides $p$.

We want to compute explicitly from this data the so-called local holomorphic Euler characteristic

$$\chi^{\text{loc}}(E) := h^0(X_k; (\pi_*E)^{\vee \vee}/\pi_*E) + h^0(X_k; R^1\pi_*E),$$

where $X_k$ is obtained from $Z_k$ by contracting the zero-section via $\pi: Z_k \rightarrow X_k$. Since $\pi|_{Z_k \setminus \ell}$ is an isomorphism onto $X_k \setminus \{0\}$, the sheaves $(\pi_*E)^{\vee \vee}/\pi_*E$ and $R^1\pi_*E$ are supported over the single point $0 \in X_k$, and so their spaces of global sections are simply their values at 0. In symbols, we have

$$H^0(X_k; (\pi_*E)^{\vee \vee}/\pi_*E) = ((\pi_*E)^{\vee \vee}/\pi_*E)_0 =: Q_0, \text{ and}$$

$$H^0(R^1\pi_*E) = (R^1\pi_*E)_0.$$
To compute these stalks on $X_k$ we make heavy use of the Theorem on Formal Functions and instead compute sections of $E$ on $Z_k$. 

**Theorem on Formal Functions** (Grauert, Grothendieck). Let $\pi : Z \to X$ be a proper map of complex spaces and $F$ a coherent sheaf on $Z$. For $x \in X$ let $\ell := \pi^{-1}(x)$. Then

$$
(R^i \pi_\ast F)_x^\wedge \cong \lim_{n \to \infty} H^i(\ell^{(n)}; F|_{\ell^{(n)}}),
$$

where $\ell^{(n)}$ denotes the $n^{th}$ infinitesimal neighbourhood of $\ell$ in $Z$.

### 3 Computation of $h^0(X_k; (\pi_\ast E)^{\vee\vee}/\pi_\ast E)$

Let $E$ be a vector bundle on $Z_k$ determined by the data $(j, p)$ as described in the previous section. We want to compute the dimension of the vector space $Q_0 := ((\pi_\ast E)^{\vee\vee}/\pi_\ast E)_0$, which is the stalk at 0 of the skyscraper sheaf $Q$ defined by the exact sequence

$$0 \to \pi_\ast E \xrightarrow{ev} (\pi_\ast E)^{\vee\vee} \to Q \to 0.
$$

By definition, the $\mathcal{O}_{X_k}$-module structure on $\pi_\ast E$ is determined by the lifting map

$$
\overline{\pi} : \mathcal{O}_{X_k} \to \pi_\ast \mathcal{O}_{Z_k},
$$

which is an isomorphism away from 0 and whose stalk at 0 is, in $U$-coordinates, just $\overline{\pi}(w_i) = z^iu_i$, where

$$
\mathcal{O}_{X_k,0} =: S = \mathbb{C}[w_0, w_1, \ldots, w_k]/(w_iw_j - w_{i+1}w_{j-1}),
$$

where the ideal contains all the indices $i = 0, \ldots, k - 2$ and $j = i + 2, \ldots, k$.

Thus we are lead to compute the space of sections of $E$, first as $\mathbb{C}[z, u]$-module and then as an $S$-module. Since we are using the Theorem on Formal Functions for the computation, we will actually be computing the $\mathbb{C}[[z, u]]$- and $S^\wedge$-module structures. However, on any Noetherian locally ringed space $(X, \mathcal{O})$ the completion $\hat{\mathcal{O}}$ is a flat $\mathcal{O}$-module, and $\mathcal{F}_x^\wedge = \mathcal{F}_x \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ for any $\mathcal{O}$-sheaf $\mathcal{F}$ and all $x \in X$.

The computation of $\pi_\ast E$ proceeds in three steps: First we apply the Theorem on Formal Functions to express $(\pi_\ast E)_0$ in terms of the cohomology of $E$, i.e.

$$
(\pi_\ast E)_0^\wedge = \lim_{n \to \infty} H^0(\ell^{(n)}; E^{(n)}).
$$

In local coordinates, elements of $H^0(\ell^{(n)}; E^{(n)})$ are sections of $E$ of the form $\sigma = (a(z, u), b(z, u))$, where $a$ and $b$ are a priori power series in

$$
\mathcal{O}_{\ell^{(n)}}(U) = \bigoplus_{i=0}^n u^i.\mathbb{C}\{z\}.
$$

However, we require that the local section patch correctly onto the other chart, so that $T \sigma = (z^ja + pb, z^{-j}b)$ is a holomorphic section of $E^{(n)}(V)$, i.e. holomorphic in $(z^{-1}, z^ku)$. This shows that $a$ and $b$ are in fact polynomials, i.e. they contain only finitely many non-zero powers of $z$.
Remark 3.1. We have essentially demonstrated the GAGA correspondence for the projective schemes $\ell^{(n)}$: all holomorphic sheaves are algebraic; and even if we start with a holomorphic section of a sheaf, we are forced to conclude that it is algebraic. Consequently, it is irrelevant in our computations of cohomology whether we consider $\ell^{(n)}$ as an algebraic scheme (with the Zariski topology) or a complex analytic space (with the Euclidean topology).

For the second step, we have to show that we can compute the module structure of $(\pi_* E)^\wedge_0$ from a finite amount of data (essentially by only going up to a finite infinitesimal neighborhood, but see §3.2.1). To be slightly more precise, we will not compute $H^0(\ell^{(n)}; E^{(n)})$, but instead we will identify finitely many elements in $H^0(\ell^{(n)}; E^{(n)})$ that generate $(\pi_* E)^\wedge_0$ as a $\mathbb{C}[[z,u]]$-module. (The fact that we can do this depends crucially on the structure of the space $Z_k$ and the fact that the conormal bundle of $\ell \subset Z_k$ is ample.)

Finally, once we have computed $M := H^0(\ell^{(N)}; E^{(N)})$ (for some sufficiently large $N$) as a $\mathbb{C}[[z,u]]$-module, the third and final step is to find the $S^\wedge$-module structure on $M$ induced by the lifting map (3.1). Here we exploit the fact that $u$ is not a zero-divisor in $\mathbb{C}[[z,u]]$ and that every element in $\mathbb{C}[[z,u]]$ can be expressed in terms of $w_i = z^r u$ after multiplication by a sufficiently high power of $u$.

3.1 An example

An example is often more illuminating than a detailed theoretic description of an algorithm, so let us start with a typical one:

Example 3.2. Consider $Z_2$, the total space of $O_{\mathbb{P}^1}(-2)$, whose blow-down $X_2$ is a surface with an ordinary double point, which for convenience we give coordinates $x = w_0 = u$, $y = w_1 = z u$ and $w = w_2 = z^2 u$, where $x w = y^2$. Let the $E$ be the bundle on $Z_2$ determined by the extension $p(z,u) = u$ and of splitting type $j = 3$. To compute the module $M$ we are thus looking for sections $(a, b)$ of $E$ such that

\[
\begin{pmatrix}
 z^3 a + u b \\
 z^{-3} b
\end{pmatrix}
\]

is holomorphic in $z^{-1}$ and $z^2 u$. Since $a$ and $b$ are holomorphic in $(z,u)$, we can write

\[
a(z,u) = \sum_{r,s \geq 0} a_{rs} u^r z^s \quad \text{and} \quad b(z,u) = \sum_{r,s \geq 0} b_{rs} u^r z^s.
\]

The basic idea is to work “one infinitesimal neighbourhood at a time”, i.e. to deal with each power of $u$ separately, starting from $0$, until one has “enough” information. Thus, starting at $r = 0$, we see from the second entry of (3.2) that $z^{-3} b(z,u) \pmod u$ has to be holomorphic in $z^{-1}$, so that

\[
b(z,u) \equiv b_{00} + b_{01} z + b_{02} z^2 + b_{03} z^3 \pmod u.
\]
Next, there can be no holomorphic terms in $a$ with $r = 0$. Thus one continues at $r = 1$:

$$a(z, u) \equiv (a_{10}u + a_{11}uz + a_{12}uz^2 + \cdots) \pmod{u^2}.$$  

The term $z^3a_{10}u$ is not holomorphic in $(z^{-1}, z^2u)$, but it is matched by the subsequent $ub_{03}z^3$. Thus we have a relation: $a_{10} + b_{03} = 0$. There are no further terms in $a(z, u)$ for $r = 1$ that can be matched by $ub(z, u)$, so $a_{1s} = 0$ for $s \geq 1$.

We could now carry on to the next formal neighbourhood, find more terms for $b$

$$b_{10}u + b_{11}uz + b_{12}uz^2 + b_{13}uz^3 + b_{14}uz^4 + b_{15}uz^5,$$

and then calculate relations on $a_{2s}$. However, we shall see immediately that this adds no new information to the $S$-module of sections.

We must now find generators of the sections of $E$, but considered as a module over $S$, where $S$ is the ring

$$S = \mathbb{C}[x, y, w]/(xw - y^2).$$

We certainly have the following generators:

$$\beta_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 \\ z \end{pmatrix}, \quad \text{and} \quad \beta_2 = \begin{pmatrix} 0 \\ z^2 \end{pmatrix}.$$

But also, we must take into account the relation $a_{10} + b_{03} = 0$. Thus the last generator is

$$\gamma = \begin{pmatrix} -u \\ z^3 \end{pmatrix}.$$

Of course the module is not free over $S$, since we have the following relations:

$$y\beta_0 = x\beta_1 \quad \quad \quad y\beta_1 = x\beta_2 \quad \quad \quad w\beta_0 = y\beta_1 \quad \quad \quad w\beta_1 = y\beta_2.$$

The computation is actually complete now: Even if one were to consider higher generators, e.g. the section $\beta_4 = (0, u)$ coming from the $b_{10}$-term, it would just be a multiple of an existing generator (here $\beta_4 = x\beta_0$). Also, the $a_{20}$-term appears to provide a new, free generator $\alpha = (u^2, 0)$, since $z^3a_{20}u^2$ is actually holomorphic on $V$; however, we have $\alpha = y\beta_2 - x\gamma$.

At this stage of our guiding example we record the $S$-module of sections that we just computed:

$$M := S[\beta_0, \beta_1, \beta_2, \gamma]/(y\beta_0 - x\beta_1, w\beta_0 - y\beta_1, y\beta_1 - x\beta_2, w\beta_1 - y\beta_2)$$

We now proceed to compute $M^\vee$, $M^{\vee\vee}$ and the quotient. The simplest, linearly independent elements of $M^\vee$ that we can write down are

$$\beta^\vee = \{ \beta_0 \mapsto x, \beta_1 \mapsto y, \beta_2 \mapsto w, \gamma \mapsto 0 \} \quad \text{and} \quad \gamma^\vee = \{ \beta_1 \mapsto 0, \gamma \mapsto 1 \}.$$

A moment’s reflection shows that all other possible maps are combinations or $S$-multiples of these two generators, and clearly there are no relations. Thus $M^\vee$ is the free $S$-module

$$M^\vee = S[\beta^\vee, \gamma^\vee].$$
Computations of instanton invariants

The bi-dual is now simply

\[ M^\vee\vee = S\left[ \beta^\vee\vee = \{ \beta^\vee \mapsto 1 \}, \gamma^\vee\vee = \{ \gamma^\vee \mapsto 1 \} \right]. \]

Evaluation on \( M \) yields:

\[
\begin{align*}
\text{ev}(\beta_0) &= \left\{ \begin{array}{l}
\beta^\vee \mapsto x \\
\gamma^\vee \mapsto 0
\end{array} \right\} = x\beta^\vee\vee \\
\text{ev}(\beta_1) &= \left\{ \begin{array}{l}
\beta^\vee \mapsto y \\
\gamma^\vee \mapsto 0
\end{array} \right\} = y\beta^\vee\vee \\
\text{ev}(\beta_2) &= \left\{ \begin{array}{l}
\beta^\vee \mapsto w \\
\gamma^\vee \mapsto 0
\end{array} \right\} = w\beta^\vee\vee \\
\text{ev}(\gamma) &= \left\{ \begin{array}{l}
\beta^\vee \mapsto 0 \\
\gamma^\vee \mapsto 1
\end{array} \right\} = \gamma^\vee\vee
\end{align*}
\]

So

\[ \text{coker}(\text{ev}) = \langle \beta^\vee\vee \rangle_{\mathbb{C}}, \]

which has dimension 1. So \( l(Q) = 1 \).

3.2 Description of the algorithm

The example of §3.1 suggests a general algorithm: We must consider two polynomials \( a \) and \( b \), use the condition that \( z^j a(z, u) + p(z, u) b(z, u) \) and \( z^{-j} b(z, u) \) be holomorphic in \( (z^{-1}, z^k u) \) to obtain relations on the coefficients \( a_{rs} \) and \( b_{rs} \), thence create the \( S \)-module \( M \), and finally compute the dimension of the quotient \( M^\vee\vee / M \).

The crucial consideration is that we only need consider finitely many terms in \( a(z, u) \) and \( b(z, u) \), and this will suffice to describe the module structure of \( M \). In other words, we guarantee that we can choose \textit{a priori} polynomials

\[
a(z, u) = \sum_{r=0}^{A_1} \sum_{s=0}^{A_2} a_{rs} z^s u^r \quad \text{and} \quad b(z, u) = \sum_{r=0}^{B_1} \sum_{s=0}^{B_2} b_{rs} z^s u^r,
\]

in which we treat the coefficients \( a_{rs} \) and \( b_{rs} \) as indeterminates, which together with the finitely many relations among them generate the module \( M \). The bounds \( A_1, A_2, B_1 \) and \( B_2 \) will only depend on \( k, j \) and \( p \), and they will be determined at the start of the algorithm. This is described in §3.2.1.

Following the computation of the relations among the coefficients, we require a small, technical routine to convert the \( \mathbb{C}[z, u] \)-module into an \( S \)-module. These technical algorithms are described at the end of §3.3. Finally, for the computation of the quotient \( M^\vee\vee / M \) we use the same computational method that was described in [GS05, Lemma 2.1 (iii)].

3.2.1 Computation of \( M \)

Intuitively, it is clear that to compute \( M \) one has to write down “enough” terms of \( a \) and \( b \), calculate \( f := z^j a + pb \) and set to zero all terms in \( f \) that are not holomorphic in \( z^{-1} \) and \( z^k u \). This gives a set of relations among the coefficients \( a_{rs} \) and \( b_{rs} \), which in turn determines a set of sections that generate \( M \). (In the example of §3.1, the relation \( a_{10} + b_{03} = 0 \) implied the generator \( \gamma = (-u, z^3)^T \).) In this section we give precise instructions on how to find the relations among coefficients and how to build from them a generating set of sections.
First let us fix some notation: To each coefficient \(a_{rs}\) and \(b_{rs}\), let us associate, respectively, “elementary” sections

\[
\sigma(a_{rs}) := \sigma^a_{rs} := \begin{pmatrix} z^s u^r \end{pmatrix} \quad \text{and} \quad \sigma(b_{rs}) := \sigma^b_{rs} := \begin{pmatrix} 0 \\ z^s u^r \end{pmatrix}.
\]

Then the generator associated to a relation \(R = a_{rs} + \sum_{il} R_{il} b_{il} = 0\), where \(R_{il}\) is non-zero for at least one \((i, l)\), is \(\sigma(R) := -\sigma^a_{rs} + \sum_{il} R_{il} \sigma^b_{il}\). We denote by \(\mathcal{R}\) the set of all such relations, so we may consider \(\mathcal{R}\) to be the “solution set” of the holomorphy condition \(\mathcal{T}\left( \frac{E}{\Gamma} \right) = 0\).

With this notation, \(M\) is generated as a \(\mathbb{C}[z, u]\)-module by the set \(G_\mathcal{R} := \{\sigma(R) : R \in \mathcal{R}\}\), and as an \(S\)-module by \(G'_\mathcal{R} := \{\pi_* \sigma(R) : R \in \mathcal{R}\}\).

There are two problems one faces when restricting oneself to a (finite) polynomial, which we turn into

**Objectives for the algorithm.**

1. One must find all generators of \(M\), i.e. one must ensure that \(G_\mathcal{R}\) generates \(M\). For example, on \(\mathbb{Z}_2\) with \(p = 0\) and \(j = 4\), the \(a_{20}\)-term contributes a free generator \((u^2, 0)\), which one could miss by only considering the \(r = 0\) and \(r = 1\) infinitesimal neighbourhoods for \(a\).

2. One must find all relations between \(b_{r's'}\) and \(a_{rs}\)-terms. Some \(b_{rs}\)-terms may appear to be free when one does not consider enough \(a_{rs}\)-terms. For example, on \(\mathbb{Z}_2\) with \(j = 5\) and \(p = u^2\), the term \(b_{05} z^5\) may erroneously seem to constitute the free generator \((0, z^5)\) if one does not include the second infinitesimal neighbourhood and finds \(a_{20} + b_{05} = 0\), so that the actual generator is \((-u^2, z^5)\).

There exists a precise bound on the number of infinitesimal neighbourhoods which one needs to consider. By including terms from a higher neighbourhoods into the polynomials \(a\) or \(b\), one may see new relations involving terms from lower neighbourhoods appear, but at the same time this will add new generating terms for which one might in turn be tempted to find new relations in even higher neighbourhoods. However, we have \textit{a priori} bounds on the terms in \(a\) and \(b\) that ensure that we compute the correct module structure on \(M\).

3. It is acceptable for \(\mathcal{R}\) to contain too many relations involving terms in \(a\). This happens when there are not enough terms in \(b\) to match. In [GS05] this was called a “fake relation”. However, if \(R \in \mathcal{R}\) is such a fake relation, and if by considering higher terms we would find the corresponding “real” relation to be \(R'\), then we can ensure that \(\sigma(R')\) is already contained in the module generated by \(G_\mathcal{R}\).

This will inevitably be the case when \(p\) contains several terms of different degree in \(u\): In that case one cannot possibly find all correct relations among a finite set of terms. The key is to allow high terms of \(a\) to be set to zero “erroneously”, rather than to miss a relation between a term \(b_{r's'}\) and a term \(a_{rs}\). (The latter would cause us to add a wrong generator, while the former only removes a potential generator – but we are careful to miss only multiples of earlier generators.)
We illustrate this important point by means of Example 3.2: Suppose we only considered \( b \) up to the neighbourhood \( r = 0 \), and \( a \) up to \( r = 2 \). Then we had to conclude the relation \( R : a_{22} = 0 \), which is a “fake relation”, whose corresponding “real” relation is \( R' : a_{22} + b_{15} = 0 \). However, \( \sigma(R') = uz^2\sigma(\gamma) = w_2\sigma(\gamma) \), so we do not need the generator \( \sigma(R') \).

The range of coefficients which one needs to consider depends on the extension \( p \):

**Definition 3.3.** Let \( p \in \mathbb{C}[z, z^{-1}, u] \). We define:

- \( \min_u := \) the minimal degree of \( u \) occurring in \( p \),
- \( \max_u := \) the maximal degree of \( u \) occurring in \( p \),
- \( \min_z := \) the minimal degree of \( z \) occurring in \( p \), and
- \( \max_z := \) the maximal degree of \( z \) occurring in \( p \).

- If \( p \equiv 0 \), then all the above values would be \(-\infty\); however, for this case we define \( \min_u := 0 \), which will later save us from having to consider this case separately.

For a given bundle \( E \) on \( Z_k \) determined by \((j, p)\), there are immediate bounds on the number of degrees of \( z \) that need to be considered for each fixed \( r \):

**Proposition 3.4.** For any degree \( r \) of \( u \) and independent of \( p \), only the terms

\[
    b_{r0}u^r + \cdots + b_{r,kr+j}u^rz^{kr+j}
\]

occur in \( b \).

**Proof.** We require that \( z^{-j}b(z, u) \) be holomorphic in \( z^{-1} \) and \( z^ku \). By multiplying \( \sum_{s=0}^{\infty} b_{rs}z^{s}u^r \) by \( z^{-j} \) we see that the only terms that are holomorphic in \( z^{-1} \) and \( z^ku \) are those claimed.

**Proposition 3.5.** For any \( r < \min_u \), the terms in \( a \) of degree \( r \) in \( u \), if any, are

\[
    a_{r0}u^r + \cdots + a_{r,kr+j}u^rz^{kr+j}.
\]

**Proof.** Since \( r < \min_u \), no term \( a_{rs}u^rz^s \) can be combined with any term in \( pb \), so the problem reduces to making \( z^ja_{rs}u^rz^s \) holomorphic in \( z^ku \), which results precisely in those terms stated.

**Proposition 3.6.** For \( r \geq \min_u \), the only terms \( a_{rs}u^rz^s \) that can possibly be non-zero satisfy

\[
    0 \leq s \leq \max\{k(r - \min_u) + j + \max\{\max_z, 0\}, kr - j\}.
\]

**Proof.** Consider all terms in \( z^ja_{rs}u^rz^s \) that are not holomorphic in \( z^{-1} \) and \( z^ku \): They must vanish unless they can be matched by a term in \( pb \). The only terms in \( pb \) that have degree \( r \) in \( u \) are of the form \( b_{r,s}u^{r'}z^{s'} \), where \( r - \max_u \leq r' \leq r - \min_u \). Since the terms in \( b \) are as in Proposition 3.4, \( s \) has to run at least up to \( kr_{\max} + j = k(r - \min_u) + j \), but the multiplication
Computations of instanton invariants

pb may have shifted the term matching $a_{rs}$ by up to $\max(0, \max_z)$ places up, which explains the first term in the statement.

Secondly, terms up to $s = kr - j$ are automatically holomorphic in the expression $z^ja$, so if $kr - j$ is greater than the previous expression, all terms up to $kr - j$ must be considered, and all the coefficients are free.

Finally, we must turn the Objectives 1–3 into ranges for $r$ that we choose to consider.

**Proposition 3.7.** By considering only a truncated generic section

\[
\begin{pmatrix} a \\ b \end{pmatrix} = \sum_{r=0}^{\min_u-1} \sum_{s=0}^{kr-j} a_{rs} \binom{z^su^r}{0} + \sum_{r=\min_u}^{\alpha} \sum_{s=0}^{\beta} a_{rs} \binom{z^su^r}{0} + \sum_{r=0}^{\gamma} \sum_{s=0}^{kr+j} b_{rs} \binom{0}{z^su^r},
\]  

(3.3)

where

\[
\alpha := \max\{\lceil j/k \rceil, \max_u\} + \min_u, \\
\beta := \max\{kr - j, k(r - \min_u) + j + \max\{\max_z, 0\}\}, \text{ and} \\
\gamma := \max\{\lceil j/k \rceil, \max_u\},
\]

one finds enough generators to compute the $S$-module $M$.

**Remark 3.8.** This statement contains two facts: First, we claim that our choice of polynomials $a$ and $b$ gives enough coefficients from which we form the generators $G_R$ of $M$. Secondly, we claim that the set $R$ of relations is correct in the following sense: If we set $A = B = \infty$ and if $R_\infty$ denotes the associated set of relations, then one of two things happens for each $R \in R_\infty$: Either $R$ is already a relation in $R$, or $\sigma(R)$ is an $S$-multiple of $\sigma(R')$ for some $R' \in R$. (This case was illustrated in Objective 3.)

**Proof.** Let us denote the three big sums in (3.3) by $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ respectively from left to right. Since the section $(a, b)$ is holomorphic on $U$, we must have $s \geq 0$, and the upper bounds for $s$ in each of the three sums is given respectively by Propositions 3.5, 3.6 and 3.4. To justify the choice of the remaining bounds, consider the condition

\[
T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} z^ja + p(z, u)b \\ z^{-j}b \end{pmatrix} \in \Gamma(E; V).
\]

The term $\Sigma_1$ is seen to contribute free generators of $M$, since no term in $T\Sigma_1$ can be matched by any term from $T\Sigma_3$.

The important choice is that of the bound $\alpha$. Once this has been chosen, we will only consider $b$ up to $u$-degree $\alpha - \min_u$, such that $p(z, u)b$ is matched with $a$ (this justifies Objective 3 above). This will justify the choice of $\gamma = \alpha - \min_u$. Moreover this ensures that there cannot be any generators coming from $a$ that are erroneously considered as free. It remains to prove that our choice of the bound $\alpha$ leads to correct computation of the module $M$.

By construction, all the generators we get from $a$ are correct, while the generators coming from $b$ are either correct or fake. We have to show two things: (i) all fake relations are multiples of genuine relations, and (ii) any relation of $M$ is a multiple of a relation that we have already found. But both (i) and (ii) follow directly from the choice of $\alpha$. 

3.2.2 Computation of $M^\vee\vee$ and $l(Q)$

This last section is merely included for completeness. It is no computational obstacle to compute the dual and bi-dual of $M$:

$$ M^\vee := \text{Hom}_S(M, S) \quad \text{and} \quad M^{\vee\vee} := \text{Hom}_S(M^\vee, S). $$

The evaluation map $ev: M \hookrightarrow M^\vee\vee$ is the natural map given by

$$ ev(a): \phi \mapsto \phi(a) \quad \text{for all } a \in M, \phi \in M^\vee. $$

Lastly, note that dimension is invariant under completion, i.e. $\dim Q^\wedge_0 = \dim Q^\wedge$, so we have $l(Q) \dim(\text{coker}(ev))$.

3.3 Implementation of the algorithm

Our reference implementation of the algorithm is written in Macaulay 2 [M2], a computer algebra package for commutative algebra. The technical aspects of this implementation are specific to that language, and the Macaulay 2-code is available from the author’s website at http://www.maths.ed.ac.uk/~s0571100/Instanton/. Here we present the generic part of the algorithm in pseudo-code.

Input and output. The algorithm takes as input the data $(k, p, j)$, where $k > 0$ and $j \geq 0$ are integers and $p$ is a polynomial in $(z^{\pm 1}, u)$. The main function $iWidth(k, p, j)$ computes the width $l(Q)$ for the bundle $E$ on $Z_k$ determined by $(j, p)$.

Auxiliary functions. The main function $iWidth(k, p, j)$ calls several auxiliary functions: The function $\text{makeSectionsAndRing}(k, p, j)$ creates the polynomials $a(z, u)$ and $b(z, u)$ according to Propositions 3.4, 3.5 and 3.7. The function $\text{getRelations}(k, fTv)$ computes the relations among the coefficients of $a$ and $b$, where $fTv = z^j a(z, u) + p(z, u)b(z, u)$. (Note that $fTv$ contains all the necessary information.) The function $\text{makeModule}$ constructs the $S$-module $M$ from the data $aPoly$ and $bPoly$, which arise respectively from $a(z, u)$ and $b(z, u)$ by applying all the relations. (For example, if $a_{20} + b_{05} = 0$ is a relation, then we substitute $a_{20} \rightarrow b_{05}$ in $a$.) Finally, $\text{qLength}(M)$ computes $l(Q)$ from the module $M$; for its implementation we refer to [GS05].

The main function. Name: $iWidth$. Input: $(k, p, j)$. Output: the instanton width $l(Q)$.

Pseudo code.

```plaintext
{aPoly, bPoly, allVars} := makeSectionsAndRing(k, p, j)
fTv := z^j * aPoly + p * bPoly
relRes := getRelations(k, fTv)
apply substitutions from relRes to aPoly and bPoly
M := makeModule(k, aPoly, bPoly, allVars)
return qLength(M)
```
**Auxiliary function.** Name: `makeSectionsAndRing`. Input: \((k, p, j)\). Output: the polynomials \(a(z, u)\) and \(b(z, u)\), and `allVars`, a collection of all coefficients occurring in \(a(z, u)\) and \(b(z, u)\).

**Pseudo code.**

1. \(\text{minU} := \text{minimal } u\text{-degree of } p\)
2. \(\text{maxU} := \text{maximal } u\text{-degree of } p\)
3. \(\text{minZ} := \text{minimal } z\text{-degree of } p\)
4. \(\text{maxZ} := \text{maximal } z\text{-degree of } p\)

\[
\begin{align*}
\text{aMax} & := \max(\text{ceiling}(j/k), \text{maxU}) + \text{minU} \\
\text{bMax} & := \text{aMax} - \text{minU} \\
\text{if } p = 0 \text{ then ( } \text{minU} = 0; \text{ bMax} = 0; \text{ aMax} = \text{ceiling}(j/k) )
\end{align*}
\]

- \(a_{rs}\) such that \(r = 0, \ldots, \text{minU} - 1\) and \(s = 0, \ldots, kr - j\);
- \(a_{rs}\) such that \(r = \text{minU}, \ldots, \text{aMax}\) and \(s = 0, \ldots, \max\{kr - j, k(r - \text{minU}) + j + \text{max}\{\text{maxZ}, 0\}\}\);
- \(b_{rs}\) such that \(r = 0, \ldots, \text{bMax}\) and \(s = 0, \ldots, kr + j\).

\[
\begin{align*}
\text{aPoly} & := \sum(a_{(r,s)} z^s u^r) \\
\text{bPoly} & := \sum(b_{(r,s)} z^s u^r) \\
\text{allVars} & := \text{collection of all coefficients } a_{(r,s)} \text{ and } b_{(r,s)}
\end{align*}
\]

**return \{ aPoly, bPoly, allVars \}**

**Main algorithm.** Name: `getRelations`. Input: \((k, fTv)\), where \(fTv = z^j a(z, u) + p(z, u)b(z, u)\). Output: A collection of relations like \(\{a_{20} + b_{05}, a_{31} + b_{14} + b_{06}\}\).

**Synopsis.** Each relation is the coefficient of a monomial \(z^s u^r\) in \(fTv\) for which \(s > kr\).

**Pseudo code.**

\[
\begin{align*}
\text{rels} & := \{ \} \\
\text{expSet} & := \text{set of exponents } (r, s) \text{ appearing in } fTv
\end{align*}
\]

for each \((r, s)\) in expSet do

- if \(s \leq k \times r\) then continue
- \(\text{term} := \text{the } z^s u^r\text{-term in } fTv\)
- \(\text{rel} := \text{the coefficient of } \text{term}, \text{scaled to be monic}\)
- \(\text{rels} := \text{rels} + \{ \text{rel} \}\)

end for

**return rels**
Computations of instanton invariants

Auxiliary function. Name: makeModule. Input: \((k, \text{aPoly}, \text{bPoly}, \text{allVars})\). Here \(\text{aPoly}\) and \(\text{bPoly}\) are the results of substituting all the relations into the original \(a(z, u)\) and \(b(z, u)\). Output: the \(S\)-module \(M\) (e.g. its presentation matrix over \(S\)).

Synopsis. Iterating over each coefficient in \text{allVars}, we set this coefficient to 1 and all others to 0 to get a section \((a, b)\) of \(E\). By multiplying with a high power of \(u\) (called \(uexp\)), we can express \((u^N a, u^N b)\) as a section of \(\pi_* E\), and those sections generate \(M\).

Pseudo code.

\[
\begin{align*}
S &:= \text{makeRing}(k) \\
\text{Smodule} &:= \text{image}(0_0): S^1 \to S^2 \\
N &:= \text{the maximum of } s - kr \text{ over all monomial terms } z^s u^r \text{ in } \text{aPoly} \text{ and } \text{bPoly} \\
uexp &:= \text{ceiling}(N/k) \\
\text{aPoly} &:= \text{aPoly} \ast u^{uexp} \\
\text{bPoly} &:= \text{bPoly} \ast u^{uexp} \\
\end{align*}
\]

for each coefficient \(c\) in \text{allVars} do
\[
\begin{align*}
\text{a} &:= \text{aPoly} \text{ with } c = 1 \text{ and all other coefficients } = 0 \\
\text{b} &:= \text{aPoly} \text{ with } c = 1 \text{ and all other coefficients } = 0 \\
\text{Smodule} &:= \text{Smodule} + \text{image}(\piStar^{(\text{a}, S)}): S^1 \to S^2 \\
\end{align*}
\]
end do

return \(\text{Smodule}\)

This function calls two further auxiliary functions, \text{makeRing}(k) and \text{piStar}. The first one, \text{makeRing}(k), returns the quotient ring \(S := \mathbb{C}[w_0, \ldots, w_k]/(w_i w_j - w_i+1 w_j-1)\) for \(i = 0, \ldots, k - 2\) and \(j = i + 2, \ldots, k\). The second function, \text{piStar}, converts monomials \(u^r z^s\) into monomials \(\prod_i w_i^{n_i}\) in \(S\), where \(\sum_i n_i = r\) and \(\sum_i in_i = s\). This is possible because we multiplied every term by the sufficiently high power \(u^{uexp}\).

Auxiliary function. Name: \text{piStar}. Input: \((p, S)\), where \(p\) is some polynomial in \(u\) and \(z\) in which each term is of sufficiently high degree in \(u\), and \(S\) is the target ring. Output: The polynomial \(p\) expressed in \(w_i\)-coordinates, where \(w_i = z^i u\).

Pseudo code.

\[
\begin{align*}
\text{res} &:= 0 \quad // \text{this will store the result} \\
k &:= \text{the number } k \text{ if } S = \mathbb{C}[w_0, \ldots, w_k]/(w_i w_j - w_{i+1} w_{j-1}) \\
&\quad // \text{we have variables } w_0, \ldots, w_k \\
\text{for each term } t \text{ in } p \text{ do} \\
\text{degU} &:= u\text{-degree of } t \\
\text{degZ} &:= z\text{-degree of } t \\
\text{fctr} &:= 1 \\
\end{align*}
\]

if \(\text{degZ} > k \ast \text{degU}\) then
\[
\text{error: this term is not convertible!}
\]
end if

13
Computations of instanton invariants

\text{diff} := k
while (\text{diff} \neq 0) do
    \text{fctr} := \text{fctr} \times w_{\text{diff}}^{(\deg Z/\text{diff})} // \deg Z/\text{diff} \text{ is integer division}
    \deg U := \deg U - (\deg Z/\text{diff})
    \deg Z := \deg Z \mod \text{diff}
    \text{diff} := \text{diff} - 1
end do

\text{fctr} := \text{fctr} \times w_0^{\deg U}
\text{res} := \text{res} + \text{fctr} \times (\text{coefficient of } t)

return \text{res}

At last, we need to compute the length of the module \( Q \), which equals the dimension of \( M^\vee / M \) as a \( \mathbb{C} \)-vector space. The computation is performed by the function \texttt{qLength} using a presentation matrix for \( M \); the actual algorithm is precisely the one described in [GS05, Lemma 2.1 (iii)].

4 Computation of \( h^0(X_k; R^1\pi_*E) \)

Let \( E \) be a bundle on \( Z_k \) of type (2.1) determined by \( (j, p) \). The sheaf \( R^1\pi_*E \) is supported at the origin, since \( \pi \) is an isomorphism everywhere else. Therefore \( H^0(X_k; R^1\pi_*E) \cong (R^1\pi_*E)_0 \).

The Theorem of Formal Function gives

\[
(R^1\pi_*E)^\wedge_0 = \lim_{n \to \infty} H^1(\ell(n); E^{(n)})
\]

However, this limit stabilises at a finite \( n \), and so we may simply compute the finite-dimensional vector space \( H^1(Z_k; E) \); then its dimension is the height of \( E \).

In this section we present an algorithm that produces a basis for \( H^1(Z_k; E) \). For this we use the \v{C}ech description

\[
H^1(Z_k; E) = \frac{\Gamma(E; U \cap V)}{\Gamma(E; U) \oplus \Gamma(E; V)},
\]

so we are looking for sections of \( E \) on the overlap \( U \cap V \) modulo sections on either \( U \) or \( V \). We recall that \( U \) and \( V \) are affine, and we may consider our sheaves either as analytic sheaves over complex spaces or as sheaves over algebraic schemes; both points of view give the same results.

\textbf{Proposition 4.1 ([BGK, Lemma 2.9])}. Let \( E \) be determined by \( (j, p) \). Then every 1-cocycle in \( H^1(Z_k; E) \) can be represented locally over \( U \) as

\[
\sum_{r=0}^{\left\lfloor \frac{j-2}{k} \right\rfloor} \sum_{s=kr-j+1}^{-1} \binom{a_{rs}}{0} z^s u^r.
\]
Computations of instanton invariants

The idea is the following: The vector space \( H^1(Z_k; E) \) is certainly spanned by all the monomial cocycles \( c_{rs} := (a_{rs}z^su^r, 0)^T \) from Equation (4.1), so we need to identify which linear combinations of the \( c_{rs} \) vanish in cohomology. But \( c_{rs} \) vanishes in cohomology precisely if there is a function \( b \) holomorphic on \( U \) such that

\[
T \left( a_{rs}z^su^r \begin{pmatrix} b \\ z^{-j}b \end{pmatrix} \right) = (a_{rs}z^{s+j}u^r + pbz^{-j}b)
\]

is holomorphic on \( V \). (Here \( T \) is the transition matrix for \( E \) from Equation (2.2).) Since \( p \) is a polynomial, only finitely many terms in \( b \) need to be considered, and we obtain an algorithm.

First note that if \( p = 0 \), then there can be no relations among the \( c_{rs} \), and \( H^1(Z_k; E) = \langle \{c_{rs}\} \rangle \).

**Proposition 4.2.** If \( p \neq 0 \), let \( \min_u \) be the smallest degree of \( u \) appearing in \( p \). To obtain \( H^1(Z_k; E) \), it suffices to check Equation (4.2) for polynomials of the form

\[
b(z, u) = \sum_{r=0}^{\left\lfloor \frac{j^2-k}{j} \right\rfloor} \sum_{s=-j}^{kr} b_{rs}z^su^r.
\]

**Proof.** This is immediate from the form of \( p \) in Proposition 2.1 and Equation (4.2). \(\square\)

### 4.1 Description of the algorithm

The algorithm itself consists of two parts: The first part computes all the linear relations between the generators \( c_{rs} \); it returns a list of all basis elements for \( H^1(Z_k; E) \) and a set of relations, which may contain lots of redundant information.

The second part of the algorithm takes these sets of generators and relations and reduces them to a minimal set of generators and relations. From this new data, we compute the dimension of \( H^1(Z_k; E) \) as the minimal number of generators minus the minimal number of relations.

**First part: finding relations**

1. Let \( b \) be as in Proposition 4.2, treating all the coefficients as indeterminates.

2. For each monomial \( z^su^r \) for \( (r, s) \) in \( \{(r, s) : r = 0, \ldots, \left\lfloor \frac{j^2-k}{j} \right\rfloor \} \) and \( s = kr - j + 1, \ldots, -1 \), do the following:
   
   (a) Let \( S \) be the set of all terms in \( pb \) with degree \( (r, s) \) in \( (u, z) \).
   
   (b) If \( S = \emptyset \), then \( c_{rs} := (a_{rs}z^su^r, 0)^T \) is an independent generator of \( H^1(Z_k; E) \).
   
   (c) Otherwise, if \( S \) is non-empty, let \( B \) be the set of all coefficients \( b_{il} \) appearing in \( S \), and let \( b' = \sum_{b_{il} \in B} b_{il}z^{l}u^{i} \). Note that \( z^su^r \) is proportional to at least one term of \( pb' \) by construction.
   
   (d) Remove from \( pb' \) all terms that are proportional to \( z^su^r \) and all terms which are holomorphic on \( U \); call the result \( q \).
Finally, let $Q$ be the set of terms in $q$ that is not holomorphic on $V$. If $Q = \emptyset$, then the cycle $c_{rs}$ vanishes in cohomology, otherwise we keep $c_{rs}$ as a non-trivial generator and obtain the relation $z^u u^r + \sum_{t \in Q} t = 0$.

**Implementation.** Name: iHeight. Input: $(k, p, j)$ corresponding to the bundle $E$ determined by $(j, p)$ on $Z_k$. Output: a pair $(G, R)$, where $G$ is a set of monomials $t$ such that the cycles $(t, 0)^T$ span a vector space $V$, and $R$ is a set of linear relations on $V$ (involving the coefficients $b_{rs}$) such that $H^1(Z_k; E) = V / R$.

**Pseudo code.**

```plaintext
minU := minimal u-degree occurring in p
if p = 0 then minU = 0
bMax := floor((j-2)/k) - minU
make all indices $b_{(r,s)}$ for $r = 0, \ldots, bMax$ and $s = -j, \ldots, k \ast r$
bPoly := $\sum_{r,s} b_{(r,s)} \ast u^r \ast z^s$
pb := bPoly \ast p
alist := list of terms $u^r \ast z^s$ for $r = 0, \ldots, \text{floor((j - 2)/k)}$ and
$s = k \ast r - j + 1, \ldots, -1$
aNonTrivials := {} // These two variables
aRelations := {} // store the final result
for each aCycle in alist do
    // To begin, find all terms in pb that cancel aCycle
    pbpruned := all terms from pb with the same (z, u)-degree as aCycle

    if (pbpruned = {}) then // nothing can cancel aCycle
        aNonTrivials := aNonTrivials + {aCycle}
        continue
    else
        leftb := all terms in bPoly that contain coefficients in pbpruned
        leftpb := leftb \ast p
        aCycleR := (all terms in leftpb that are proportional to aCycle)
        leftpb := leftpb - aCycleR
        leftpb := leftpb - (all terms holomorphic in (z, u))

        leftovers := those terms of $z^j \ast leftpb$ that are not holomorphic in $(z^{-1}, z^k u)$

        if leftovers = 0 then
            aNonTrivials := aNonTrivials + {aCycle}
            aRelations := aRelations + {aCycleR + leftovers}
        end if
    end if
end for
```
Computations of instanton invariants

return (aNonTrivials, aRelations)

Second part: reducing to minimal generators and relations  The first part of the
algorithm produces two sets of data: a set $G$ of generating monomials of for form $z^s u^r$ (i.e. the
cocycle $(z^s u^r, 0)^T$ is non-trivial in $H^1(Z_k; E)$, and a set $R$ of relations which are polynomials
in $z, z^{-1}, u$ with coefficients $b_{ij}$. Let $C$ be the set of all coefficients $b_{ij}$ that can appear; $C$ is
determined by Proposition 4.2. To find minimal generators and relations, proceed as follows:

- Build a new set $R''$ of relations without indeterminates as follows: For each relation
  $r \in R$, for each $\beta \in C$, set $\beta = 1$ and all other coefficients in $C \setminus \{\beta\}$ to zero; add the
  relation $r|_{\beta=1, C\setminus \{\beta\}=0}$ to $R''$.

- Build a new set of generators $G'$ and a new set of relations $R'$ by starting with $G' = G$
  and $R' = R''$ as follows: Let $N$ be the set of monomial relations in $R'$, i.e. relations
  of the form $z^s u^r = 0$. For each $r \in N$, remove $r$ from $G'$ and substitute $r = 0$ into
  every relation in $R'$. Let $N$ be the new set of monomial relations in $R'$ and repeat until
  $N = \emptyset$.

- The final set $G'$ is a minimal set of generators, and the final set $R'$ is a minimal set of
  relations.

Implementation in pseudo code.  Name: fixHeightRelations. Input: $(G, R)$, the sets
of generators and relations which the iHeight algorithm produced. Output: a new pair
$(G', R')$, where $G'$ is a minimal set of generators for the vector space $H^1(Z_k; E)$, and $R'$
is a new set of linear relations, usually empty. Thus $|G'| - |R'|$ is the actual value of the height
of $E$ (this number is also returned in the actual implementation).

Pseudo code.

```pseudo
if (G = {} or R = {}) then
  return (G, R)
end if

rels := {}  // this stores the result R
allvars := the set of coefficients b_{r,s}

for each term t in G do
  for each v in allvars do
    l1 := t with v=1 and all other variables set to zero
    if l1 != 0 then rels = rels + \{l1\}
  end for
end for

prunednontrivs := G
prunedrels := rels
nullguys := the set of one-term relations (e.g. $z^3 u^2 = 0$) in prunedrels
```

17
while (nullguys != {}) do
    for each term t in nullguys do
        replace \((R + t)\) by \((R)\) in prunedrels
        replace \((R + t)\) by \((R)\) in prunednontrivs
    end for
    nullguys := the set of one-term relations in prunedrels
end while

return (prunednontrivs, prunedrels)

5 Note on computing \(H^1(Z_k; \text{End} \, E)\)

In the next two sections we compute invariants of the endomorphism bundle \(\text{End}(E) = E \otimes E^\vee\). This bundle plays a fundamental role in the deformation theory of the sheaf \(E\): \(H^1(Z_k; \text{End} \, E)\) is precisely the tangent space at \(E\) of the moduli of holomorphic bundles diffeomorphic to \(E\). This follows for example from [DK90, Proposition 6.4.3], as the Kuranishi map vanishes on \(Z_k\) (since \(H^i(Z_k; F) = 0\) for all \(i \geq 2\) and every coherent sheaf \(F\)).

If \((j, p)\) determines a bundle \(E\) on \(Z_k\) as before, with transition function \(T\) given by Equation (2.2), then the endomorphism bundle \(\text{End} \, E = E \otimes E^\vee\) is a rank-4 bundle whose transition function is given, after a convenient change of coordinates

\[
P := \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} = P^{-1},
\]

by

\[
S := \begin{pmatrix}
1 & z^{-j}p & z^{j}p & p^2 \\
0 & z^{2j} & 0 & z^{j}p \\
0 & 0 & z^{-2j} & z^{-j}p \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \text{so} \quad S^{-1} = \begin{pmatrix}
1 & -z^{-j}p & -z^{j}p & p^2 \\
0 & z^{-2j} & 0 & -z^{-j}p \\
0 & 0 & z^{2j} & -z^{j}p \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The computation of \(H^1(Z_k; \text{End} \, E)\) is essentially the same as for \(H^1(Z_k; E)\), in the sense that there exists a simple canonical representative for every 1-cocycle, and the first part of the algorithm from §4 translates almost literally, while the second part remains unchanged. An implementation of this algorithm, named hiend(k,p,j), is contained in our Macaulay 2 code at http://www.maths.ed.ac.uk/~s0571100/Instanton/.

6 Cancelling infinities: the computation of \(H^0(Z_k; \text{End} \, E)\)

On the other hand, the space \(H^0(Z_k; \text{End} \, E)\) is infinite-dimensional, so we cannot compute it directly. However, if we are given two different bundles \((j, p_1)\) and \((j, p_2)\), we can compute
Computations of instanton invariants

a “relative dimension” of $H^0$-spaces as follows: For each $n \geq 0$, the space $H^0(ℓ^n; E^{(n)})$ is finite-dimensional, so we can compute the difference $Δ_n(p_1,p_2) := h^0(ℓ^{(n)}; E(p_1)^{(n)}) − h^0(ℓ^{(n)}; E(p_2)^{(n)})$. Since $p_1, p_2$ are polynomials, $Δ_n$ is constant for $n \gg 0$. Finally, we can define a function

$$Δ(E) \equiv h(j,p) := \lim_{n→∞} Δ_n(0,p) ,$$

which is non-negative since the split bundle given by $p = 0$ has the largest amount of sections on each infinitesimal neighbourhood $ℓ^{(n)}$. (See [BGK] for a discussion of this non-trivial fact.)

In the remainder of this section we describe an algorithm to compute $h^0(ℓ^{(n)}; End E^{(n)})$ from the input data $(j,p)$ and $n$. This amounts to finding the most general section $σ ∈ Γ(End E; U)$ such that $Sσ ∈ Γ(End E; V)$. Let $σ = (a,b,c,d)^T$, where we write a typical component as $a(z,u) = \sum_{r,s≥0} a_{rs} z^r u^s$. We have

$$S \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a + z^j pb + z^{-j} pc + p^2 d \\ z^{2j} b + z^{j} pd \\ z^{-2j} c + z^{-j} pd \\ d \end{pmatrix}. \tag{6.1}$$

For computing $H^0$, we see that $d$ and we know that $z^{-j} p$ is holomorphic on $V$, thus so is $z^{-j} pd$. This implies immediately that

$$d(z,u) = \sum_{r≥0} \sum_{s = 0}^{kr} d_{rs} z^r u^s \quad \text{and} \quad c(z,u) = \sum_{r≥0} \sum_{s = 0}^{kr+2j} c_{rs} z^r u^s .$$

Next, in the second entry of (6.1), $z^{j} pd$ contains powers $z^r u^s$ for $s ≤ kr + 2j − 1$, so that we may take $b(z,u) = \sum_{r≥0} \sum_{s = 0}^{kr−1} (we could have taken into account the fact that $u$ divides $p$ for an even sharper bound, but choose not to). Finally, similar considerations show that we may take $a(z,u) = \sum_{r≥0} \sum_{s = 0}^{kr+2j−1} a_{rs} z^r u^s$. Here we assume that $j > 0$ and that $p$ is in canonical form (2.3); if $j = 0$, there exists only one bundle anyway.

We now describe an algorithm that computes for a given $n \geq 0$ the vector space $H^0(ℓ^{(n)}; E^{(n)})$. The algorithm generates a set of linear relations on a larger vector space spanned by monomials, and in our reference implementation we use a built-in function from Macaulay 2 to compute the dimension of the resulting space directly.

Implementation. Name: $h0end$. Input: $(k,p,j,n)$, where $(k,p,j)$ determine as before a bundle $E$ on $Z_k$, and $n$ is the infinitesimal neighbourhood. Output: $h^0(ℓ^{(n)}; E^{(n)})$.

Pseudo code.

- $aPoly := \sum_{r = 0}^{n} \sum_{s = 0}^{kr+2j} a_r(r,s) z^r u^s$
- $bPoly := \sum_{r = 0}^{n} \sum_{s = 0}^{kr} b_r(r,s) z^r u^s$
- $cPoly := \sum_{r = 0}^{n} \sum_{s = 0}^{kr+2j} c_r(r,s) z^r u^s$
- $dPoly := \sum_{r = 0}^{n} \sum_{s = 0}^{kr} d_r(r,s) z^r u^s$
Computations of instanton invariants

\[ e_1 := a_{\text{Poly}} + z^{-j} \cdot p \cdot b_{\text{Poly}} + z^{-(-j)} \cdot p \cdot c_{\text{Poly}} + p^2 \cdot d_{\text{Poly}} \]
\[ e_2 := z^{-2j} \cdot b_{\text{Poly}} + z^{-j} \cdot p \cdot d_{\text{Poly}} \]
\[ e_3 := z^{-2(-j)} \cdot c_{\text{Poly}} + z^{-(-j)} \cdot p \cdot d_{\text{Poly}} \]
\[ e_4 := d_{\text{Poly}} \]

relations := {} for each polynomial pol in \{e_1, e_2, e_3, e_4\} do
  s := z-degree(t)
  r := u-degree(t)
  for each term t in pol do
    if s <= k \cdot r then
      continue
    end if
  end for
  badterms := terms in pol of the same degree as t
  badcoefs := badterms / z^s \cdot u^r
  relations := relations + \{badcoefs = 0\}
end for
return (the dimension of \langle a_{rs}, b_{rs}, c_{rs}, d_{rs} \rangle / \text{relations})

7 Note on adapting the algorithms to other spaces

To conclude this paper we outline how to adapt the algorithms from this paper to any space \( X = \text{Tot} \left( \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i) \right) \) for which we have a GAGA principle and in which we can contract the zero section. We refer to [BGK] and infer that \( X \) satisfies GAGA at least when we have \( a_i < 0 \) for all \( i \), in which case all vector bundles are algebraically filtered, and in particular every rank-2 bundle is an algebraic extension of algebraic line bundles.

The changes in the algorithm are basically as follows: We now need coordinate charts
\[ U = \{(z, u_1, \ldots, u_n)\} \quad \text{and} \quad V = \{(w, v_1, \ldots, v_n)\} \]
which patch together via \( w = z^{-1} \) and \( v_i = z^{-a_i} u_i \). If \( E \) is a bundle on \( X \) of type (2.1), then its transition function is determined by the splitting type \( j \) and a polynomial \( p \in \mathbb{C}[z^\pm, u_1, \ldots, u_n] \), which can be put into a canonical form.

**Example.** On the Calabi-Yau threefold \( W_1 := \text{Tot} \left( \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \right) \), the polynomial \( p \) has the canonical form
\[ p(z, u, v) = \sum_{t=\epsilon}^{2j-2} \sum_{r=1-\epsilon}^{2j-2-t} \sum_{s=r+1-t-j+1}^{j-1} p_{trs} z^a u_1^r u_2^s, \quad \epsilon \in \{1, 2\} . \]

The putative local sections of \( E \) and \( \text{End} E \) that one uses in the computation of \( l(Q) \) and \( h^i(X; \text{End} E) \) are now also polynomials in \( z, u_1, \ldots, u_n \), but otherwise the algorithms are
essentially identical. The adapted algorithms for the flop space $W_1$ are available together with the previous algorithms on the author’s website.

We wish to state a last result, on which the author hit only after adapting the algorithm to $W_1$ and always computing zero.

**Proposition 7.1 (cf. [GK]).** Let $X$ be as above, with contractible zero section isomorphic to $\mathbb{P}^1$, and let $\pi: X \to X'$ be the contraction. If $\dim X > 2$, then for any locally free sheaf $E$ on $X$, the width of $E$ vanishes, i.e.

$$H^0(X'; (\pi_\ast E)^\vee / (\pi_\ast E)) = 0.$$ 

### A Application: Rank-2 Bundles on $Z_k$ and genericity

In this section we tabulate some results that were computed with the algorithms from this paper. We will consider bundles of rank 2 on the surfaces $Z_k := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ of splitting type $j$ which are determined by a polynomial $p$ according to Equation (2.2). The moduli of such bundles has been studied in [BGK], and here we provide concrete examples of elements of the moduli and their numerical invariants.

This section is divided into two cases, first the *instanton case* for which $j = nk$ for some positive integer $n$, followed by non-instanton bundles. This terminology arises because it was shown in [GKM08] and [BGK] that a holomorphic bundle of the type which we consider corresponds to an instanton on $Z_k$ precisely when $j = nk$ (via a version of the Kobayashi-Hitchin correspondence).

#### A.1 The instanton case

In the case that $j = nk$ for some $n \in \mathbb{N}$, the bundle $E$ determined by $(j, p)$ corresponds to an instanton on $Z_k$. In this case it was shown that the split bundle is the unique one with the highest invariants $(h, w)$ (the height and width), while the generic bundles are those with the lowest invariants $(h, w)$. Our computations indicate that the condition of being generic or split also corresponds to $h^1(\text{End} E)$ being respectively minimal or maximal.

| $k$ | $j$ | $p$ | $h^1(\text{End} E)$ | $\Delta$ | $h^1 - \Delta$ | $(w, h)$ |
|-----|-----|-----|---------------------|----------|-----------------|----------|
| 1   | 2   | $u, zu$ | 4                   | 2        | 2               | (1,1)    |
| 1   | 2   | $zu^2$  | 5                   | 1        | 4               | (2,1)    |
| 1   | 2   | 0       | 6                   | 0        | 6               | (3,1)    |
| 1   | 3   | $z^{-1}u, z^2u$ | 11               | 4        | 7               | (3,2)    |
| 1   | 3   | $u, zu$  | 9                   | 6        | 3               | (1,2)    |
| 1   | 3   | $z^{-1}u + z^2u$ | 9               | 6        | 3               | (1,2)    |
| 1   | 3   | $u^2, z^2u^2$ | 12                | 3        | 9               | (3,3)    |
| 1   | 3   | $zu^2$  | 11                | 4        | 7               | (2,3)    |

*Continued on next page*
In fact, the numerical evidence leads us to conjecture one and discover another relation between the invariants \((h, w)\) and \((h^1, \Delta)\).
Conjecture: \( w + h = c = (h^1 - \Delta - j)/2 + j/k. \)

Proposition A.1. \( \Delta + h^1 = h^1(\text{End(split)}) \), or equivalently
\[
h^1(\text{End } E) - h^0(\text{End } E|E(m)) = h^1(\text{End}(\text{split})) - h^0(\text{End}(\text{split}|E(m)))
\]
for all large \( m \).

Proof. We can express the statement in terms of the Hilbert polynomial
\[
\phi_{E,m}(n) := \chi(\text{End } E^{(m)}(n)) = h^1(\text{End } E|E(m)) - h^0(\text{End } E|E(m)) \ ;
\]
then the statement is \( \phi_{E,m}(0) = \phi_{\text{split},m}(0) \). But in fact we have \( \phi_{E,m}(n) = \phi_{\text{split},m}(n) \) for any \( n \) and \( m \), since the Hilbert polynomial cannot distinguish extensions on \( E(m) \subset Z_k \), as one sees from direct computation. We retain the clause “for all large \( m \)” since \( h^1(\text{End } E|E(m)) \) stabilises to \( h^1(\text{End } E) \) eventually. \( \square \)

From [BGK] and the Conjecture we get \( h^1 + \Delta = n(2nk + k - 2) \) for \( j = nk \). We also get that for the generic bundle \( p = zu \), we have \( h^1 - \Delta = (3k - 2)n - 2 \) for \( k > 1 \) and \( h^1 - \Delta = n \) for \( k = 1 \).

Corollary: Assuming the Conjecture, we have \( \Delta = n^2k - c \), \( h^1 = kn(n + 1) - 2n + c \).

A.2 Non-instanton bundles
Here \( j \not\equiv 0 \pmod{k} \).

\[
\begin{array}{c|c|c|c|c|c|c}
 k & j & p & h^1(\text{End } E) & \Delta & h^1 - \Delta & (w, h) \\
\hline
 2 & 3 & u & 7 & 2 & 5 & (1, 2) \\
 2 & 3 & zu & 7 & 2 & 5 & (0, 2) \\
 2 & 3 & z^2u & 7 & 2 & 5 & (1, 2) \\
 2 & 3 & u + z^2u & 7 & 2 & 5 & (0, 2) \\
 2 & 3 & z^2u^2 & 8 & 1 & 7 & (2, 2) \\
 2 & 3 & 0 & 9 & 0 & 9 & (2, 2) \\
\hline
 3 & 4 & u & 10 & 2 & 8 & (1, 3) \\
 3 & 4 & zu, z^2u & 10 & 2 & 8 & (0, 3) \\
 3 & 4 & z^3u & 10 & 2 & 8 & (1, 3) \\
 3 & 4 & u + z^3u & 10 & 2 & 8 & (0, 3) \\
 3 & 4 & z^3u^2 & 11 & 1 & 10 & (2, 3) \\
 3 & 4 & 0 & 12 & 0 & 12 & (2, 3) \\
\hline
 3 & 5 & z^{-1}u & 16 & 2 & 14 & (2, 4) \\
 3 & 5 & u & 15 & 3 & 12 & (1, 4) \\
 3 & 5 & zu, z^2u & 14 & 4 & 10 & (0, 4)
\end{array}
\]

Continued on next page
Computations of instanton invariants

| $k$ | $j$ | $p$     | $h^1(\text{End } E)$ | $\Delta$ | $h^1 - \Delta$ | $(w, h)$ |
|-----|-----|---------|----------------------|----------|----------------|----------|
| 3   | 5   | $z^3u$  | 15                   | 3        | 12             | (1, 4)   |
| 3   | 5   | $z^4u$  | 16                   | 2        | 14             | (2, 4)   |
| 3   | 5   | $u + z^4u$ | 14             | 4        | 10             | (0, 4)   |
| 3   | 5   | $z^{-1}u + z^4u$ | 15           | 3        | 12             | (1, 4)   |
| 3   | 5   | $z^2u^2$ | 17                   | 1        | 16             | (2, 5)   |
| 3   | 5   | $z^3u^2$ | 17                   | 1        | 16             | (2, 5)   |
| 3   | 5   | $z^4u^2$ | 17                   | 1        | 16             | (2, 5)   |
| 3   | 5   | 0       | 18                   | 0        | 0              | (3, 5)   |

We see that the invariants $(w, h)$ are now required to distinguish the generic bundles (those with lowest $(w, h)$), whereas the split bundle is no longer the only one with the highest values of $(w, h)$ (see [BGK] for details). However, $h^1(\text{End } E)$ still distinguishes the split bundle. The physical interpretation of these “non-instanton” bundles invites further exploration.

B Usage example

The algorithms that we described in this paper are implemented in Macaulay 2 version 1.1, the code is contained in the file InstantonInvariants2.m2. Suppose we want to study the bundle on $Z_2$ of splitting type 7 given by the polynomial $p = z^{-1}u + zu^2$. We set Macaulay up as follows:

```macaulay2
$ M2 InstantonInvariants2.m2
Macaulay 2, version 1.1
with packages: ... 

i1 : p = z*u^2+z^-1*u;
```

Now we compute the width and height, and $h^1(Z_2; \text{End } E)$. When calling `iWidth`, we use the option `Verbose=>false` to suppress additional output.

```macaulay2
i2 : iWidth(2,p,7,Verbose=>false)
```

We find that the width is 2, the height 6, and $h^1(Z_2; \text{End } E) = 33$.

References

[BGK] Eduardo Ballico, Elizabeth Gasparim, and Thomas Köppe, *Vector bundles near negative curves: moduli and local Euler characteristic*, to appear in Comm. Algebra.
Computations of instanton invariants

[BGK09] Eduardo Ballico, Elizabeth Gasparim, and Thomas Köppe, *Local moduli of holomorphic bundles*, J. Pure Appl. Algebra 213 (2009), 397–408.

[DK90] Simon K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1990.

[Gas97] Elizabeth Gasparim, *Holomorphic bundles on $\mathcal{O}(-k)$ are algebraic*, Comm. Algebra 25 (1997), no. 9, 3001–3009.

[GK] Elizabeth Gasparim and Thomas Köppe, *Moduli of bundles on Calabi-Yau threefolds*, work in progress.

[GKM08] Elizabeth Gasparim, Thomas Köppe, and Pushan Majumdar, *Local holomorphic Euler characteristic and instanton decay*, Pure Appl. Math. Q. 4 (2008), no. 2, 161–179, Special Issue: In honor of Fedya Bogomolov, Part 1.

[GS05] Elizabeth Gasparim and Irena Swanson, *Computing instanton numbers of curve singularities*, J. Symbolic Computation 40 (2005), no. 2, 965–978.

[M2] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.

Thomas Köppe
School of Mathematics, The University of Edinburgh
James Clerk Maxwell Building, The King’s Buildings
Mayfield Road, Edinburgh, UK, EH9 3JZ
E-mail: t.koepe@ed.ac.uk