A partition of connected graphs

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Abstract

We define an algorithm $k$ which takes a connected graph $G$ on a totally ordered vertex set and returns an increasing tree $R$ (which is not necessarily a subtree of $G$). We characterize the set of graphs $G$ such that $k(G) = R$. Because this set has a simple structure (it is isomorphic to a product of non-empty power sets), it is easy to evaluate certain graph invariants in terms of increasing trees. In particular, we prove that, up to sign, the coefficient of $x^q$ in the chromatic polynomial $\chi_G(x)$ is the number of increasing forests with $q$ components that satisfy a condition that we call $G$-connectedness. We also find a bijection between increasing $G$-connected trees and broken circuit free subtrees of $G$.

We will work with finite labeled simple graphs. Usually we will identify a graph $G$ with its edge set; this should not cause any serious ambiguities. If the vertex set is $V$ then we say that $G$ is a graph on $V$. A (spanning) subgraph $Q$ of $G$ is a graph with the same vertex set as $G$ and a subset of the edges of $G$. The notation $Q \subseteq G$ means $Q$ is a subgraph of $G$. A rooted graph is a graph with a distinguished vertex called the root.

Define $\text{link}(v, S)$ to be the set of all possible edges joining $v$ to an element of $S$ (so if $v \notin S$, $\text{link}(v, S)$ has $|S|$ elements). If $G$ is a graph on $V$ and $S \subseteq V$, we define the restriction of $G$ to $S$, $G|_S$, to be the graph on $S$ whose edge set consists of all edges of $G$ with both ends in $S$.

We will use the symbols $\pi$ and $\sigma$ to denote set partitions. The notation $\pi \vdash S$ means $\pi$ is a set partition of the set $S$. The length (number of blocks) of $\pi$ is denoted by $\ell(\pi)$. A set partition $\sigma$ is called a refinement of a set partition $\pi$ if every block of $\sigma$ is contained in some block of $\pi$.

To each graph $G$ on $V$ there corresponds a set partition $s(G)$ such that two vertices $v, w \in V$ are in the same block of $s(G)$ if and only if there is a path in $G$ from $v$ to $w$. Equivalently, $s(G)$ is the maximal set partition of $V$ whose blocks are connected. The restriction of $G$ to a block of $s(G)$ is called a component of $G$.

If $G$ is a rooted connected graph on $V$ with root $r$, we will call the set partition $\pi = s(G|_{V - \{r\}})$ of $V - \{r\}$ the depth-first partition of $G$. To obtain a connected subgraph of a rooted connected graph $G$ on $V$, we can choose, for each block $\pi_i$ of $\pi$, a connected
A subgraph of $G|_{\pi}$, and a nonempty set of edges (in $G$) connecting $r$ to $\pi_i$. In fact, every connected subgraph of $G$ can be obtained in this way. Our Theorem 1 may be regarded as an iteration of this correspondence. The depth-first partition and this correspondence have been studied by Gessel [3].

A forest is a graph with no circuits. A tree is a connected forest. A basic property of trees is that there is a unique path (a sequence of distinct, adjacent vertices) between any two vertices. The distance between two vertices is defined to be the length of this path. In a rooted tree, the height of a vertex is defined to be its distance from the root. A vertex $w$ is called a descendant of a vertex $v$ (or $v$ is called an ancestor of $w$) if the heights of the vertices on the unique path from $v$ to $w$ are increasing (so in particular $v$ is always a descendant of itself). We define the join of $v$ and $w$ to be their unique common ancestor on the unique path between them.

Let $R$ be a rooted tree on the vertex set $V$, and let $v \in V$. We define $\text{des}(v, R) \subseteq V$ to be the set of descendants of $v$ (including $v$). If $v$ is not the root of $R$, we define $\text{parent}(v, R) \in V$ to be the closest vertex to $v$ in $R$ which is not a descendant of $v$. A rooted tree is increasing (according to a total order on $V$) if for each $v \in V$ and $w \in \text{des}(v, R)$ we have $v \preceq w$. Consequently, the root of an increasing tree must be the smallest element of $V$.

**Definition 1** Let $R$ be a rooted tree on the totally ordered vertex set $V$ with root $r$, and let $v \in V - \{r\}$. Define $J(v, R) = \text{link}(\text{parent}(v, R), \text{des}(v, R))$. If $G$ is a graph on $V$ and if for each $v \in V - \{r\}$ we have $J(v, R) \cap G \neq \emptyset$ then we say that $R$ is $G$-connected.

Note that the sets $J(v, R)$ (as $v$ ranges over $V - \{r\}$) are disjoint. Also note that a $G$-connected tree need not be a subgraph of $G$ and that $G$ must be connected for any rooted tree to be $G$-connected.

**Definition 2** For each connected graph $G$ on a totally ordered vertex set $V$, define an increasing $G$-connected tree $k(G)$ by the following algorithm:

1. Let $H$ be an empty graph on $V$, and set $S = V$.
2. Let $\pi$ be the depth-first partition of $G|_{S}$ rooted at $r$=the smallest vertex in $S$. Add edges to $H$ connecting $r$ to the smallest vertex in each block of $\pi$.

3. For each block $\pi_i$ of $\pi$ with more than one element, return to step 2 with $S = \pi_i$.
4. Return $k(G) = H$.

**Example 1** The 6 increasing trees on $V = \{1, 2, 3, 4\}$ are listed vertically. To the right of each increasing tree $R$ are listed the subtrees $T$ of the complete graph on $V$ such that $k(T) = R$ (we have omitted the 22 connected subgraphs which are not trees). The breaks are indicated by dotted lines (see Theorem 3).
There is a different algorithm, called depth-first search, which produces subforests of $G$. Some enumerative applications of this algorithm have been studied by Gessel and Sagan [4]. A distinguishing difference between depth-first search and our algorithm is that depth-first search only follows the edges of $G$, whereas here we add edges connecting to the smallest vertex in each block of $\pi$ regardless of whether these are edges of $G$. The algorithms are related in that if $G$ is a connected graph and $R$ is a depth-first search subtree of $G$ then parts 2 and 3 of the next theorem hold (although the converse is not true).

**Theorem 1** Let $G$ be a connected graph on a totally ordered vertex set $V$, and let $R$ be an increasing $G$-connected tree on $V$. Then the following are equivalent:

1. $k(G) = R$

2. For each vertex $v \in V$, $G|_{\text{des}(v,R)}$ rooted at $v$ is connected and has the same depth-first partition as $R|_{\text{des}(v,R)}$ rooted at $v$.

3. For each non-root vertex $v \in V - \{r\}$ there is a nonempty set $E(v) \subseteq J(v,R)$ such that $G = \bigcup_{v \in V - \{r\}} E(v)$. 

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Proof. 1 $\Leftrightarrow$ 2 This follows easily from Definition 2.

2 $\Rightarrow$ 3 Let $E(v) = J(v, R) \cap G$. We need to show that every edge of $G$ lies in some $E(v)$. Let $e \in G$ and let $v < w$ be the vertices of $e$. We will show that $w$ is a descendant of $v$. Suppose this is false, and let $u$ be their join. Then $e \in G|_{\text{des}(u, R)}$, so $v$ and $w$ are in the same block of the depth first partition of $G|_{\text{des}(u, R)}$. This is a contradiction because they are in different blocks of the depth first partition of $R|_{\text{des}(u, R)}$. Now, since $w$ is a descendant of $v$, there is a unique vertex $z \in V$ (possibly equal to $w$) such that $\text{parent}(z) = v$ and $w \in \text{des}(z)$. Hence $e \in J(z, R) \cap G$.

3 $\Rightarrow$ 2 This is certainly true if $v$ (in part 2) is a leaf of $R$ (its only descendant is itself).

Let $v \in V$ and suppose it is true for all $w \in \text{des}(v, R) - f \text{r} g$. Let $v \in V$ be a leaf of $R$ because every edge of $J_G(w, R)$ (for any $w \in V$) connects a vertex to one of its descendants. This contradicts the fact that they are in different blocks of the depth-first partition of $R|_{\text{des}(v, R)}$. □

Remark 1 Actually the condition in Theorem 1 that $R$ be $G$-connected is not necessary because if $R$ is not $G$-connected then parts 1, 2 and 3 will be false.

Some algebraic invariants of graphs can be simply expressed in terms of connected subgraphs. We can use the algorithm $k$ to express such invariants in terms increasing trees. Moreover, Theorem 1 shows that the set $k^{-1}(R)$ has a simple structure, as illustrated by the next theorem.

Definition 3 Let $G$ be a connected graph on $V$. Define

$$\eta^G(t) = \sum_{Q \subseteq G \text{ connected}} t^{|Q|}$$

where $|Q|$ denotes the number of edges in $Q$.

Theorem 2

$$\eta^G(t) = \sum_{\substack{R \text{ increasing} \\ G-\text{connected}}} \prod_{v \in V - \{r\}} [(1 + t)^{|J(v, R) \cap G|} - 1]$$

Proof. We have

$$\eta^G(t) = \sum_{\substack{R \text{ increasing} \\ G-\text{connected}}} \sum_{Q \subseteq G} t^{|Q|}$$
Now, the generating function for the cardinality of nonempty subsets of a set $S$ is

$$f_S(x) = \sum_{\emptyset \neq T \subseteq S} x^{|T|} = (1 + x)^{|S|} - 1$$

Hence from Theorem 1 part 3,

$$\sum_{Q \subseteq G} t^{Q|} = \sum_{Q = \bigcup_{v \in V-R} E(v) \atop \emptyset \neq E(v) \subseteq J(v,R) \cap G} t^{Q|} = \prod_{v \in V-R} f_{J(v,R) \cap G}(t)$$

from which the result follows. □

The chromatic polynomial $\chi_G(x)$ of a graph $G$ is a polynomial which evaluates to the number of proper colorings of $G$ with $x$ colors. The subgraph expansion of $\chi_G(x)$ is

$$\chi_G(x) = \sum_{Q \subseteq G} (-1)^{|Q|} x^{c(Q)}$$

where $c(Q)$ is the number of components of $Q$. See [1] for background on the chromatic polynomial.

We define an increasing $G$-connected forest $R$ to be a forest where each component $R|_{s(R)}$ is an increasing $G|_{s(R)}$-connected tree. For a graph $G$, let $t(G)$ be the (integer) partition whose parts are the sizes of the blocks of $s(G)$. For background on the chromatic symmetric function $X_G = X_G(x_1, x_2, \ldots)$ of a graph $G$, see [5] and [6]. For background on the chromatic symmetric function in non-commuting variables $Y_G = Y_G(x_1, x_2, \ldots)$, see [2].

**Corollary 1** Let $G$ be a graph on a totally ordered vertex set $V$ with $|V| = n$.

1. The coefficient of $(-1)^{n-1}x$ in the chromatic polynomial $\chi_G(x)$ is the number of increasing $G$-connected trees.

2. The coefficient of $(-1)^{n-q}x^q$ in the chromatic polynomial $\chi_G(x)$ is the number of increasing $G$-connected forests with $q$ components (or, equivalently, with $n-q$ edges).

3. The coefficient of $(-1)^{n-t(\lambda)} p_{\lambda}$ in the chromatic symmetric function $X_G$ is the number of increasing $G$-connected forests $R$ such that $t(R) = \lambda$.

4. The coefficient of $(-1)^{n-\ell(\pi)} p_{\pi}$ in the chromatic symmetric function in non-commuting variables $Y_G$ is the number of increasing $G$-connected forests $R$ such that $s(R) = \pi$.

**Proof.** 1. Let $a^G$ be the coefficient of $x$ in $\chi_G(x)$. From the subgraph expansion we have

$$a^G = \sum_{Q \subseteq G \text{ connected}} (-1)^{|Q|} = \eta^G(-1) = \sum_{R \text{ increasing } G\text{-connected}} \prod_{v \in V-R} (-1)$$

where $\eta^G(-1)$ is the number of increasing $G$-connected forests $R$ such that $s(R) = \pi$. See [2].
We don’t need to worry about $0^0$ because the $G$-connectedness of $R$ implies that $J(v, R) \cap G$ is never empty.

4. We will prove part 4, the others being simple specializations. Let $H^G_{\pi}$ be the number of increasing $G$-connected forests $R$ such that $s(R) = \pi$, and let $H^G$ be the number of increasing $G$-connected trees. Then using part 1 we have

$$H^G_{\pi} = \prod_{i=1}^{\ell(\pi)} H^G|_{s_i} = (-1)^{n-\ell(\pi)} \prod_{i=1}^{\ell(\pi)} \sum_{Q \subseteq G|_{s_i}} (-1)^{|Q|}$$

(1)

The subgraph expansion of $Y_G$ is

$$Y_G = \sum_{Q \subseteq G} (-1)^{|Q|} p_s(Q)$$

Hence

$$Y_G = \sum_{\pi \vdash V} p_{\pi} \sum_{Q \subseteq G \ s(Q) = \pi} (-1)^{|Q|} = \sum_{\pi \vdash V} p_{\pi} \prod_{i=1}^{\ell(\pi)} \sum_{Q \subseteq G|_{s_i}} (-1)^{|Q|}$$

Substituting (1), we obtain the desired result. □

If $G$ is a graph on a totally ordered vertex set $V$, we extend the ordering of the vertices to an ordering of the edges lexicographically. A broken circuit of $H \subseteq G$ is a set of edges $B \subseteq H$ such that there is some edge $e \in G$, smaller than every edge of $B$, such that $B \cup e$ is a circuit. Note that $B$ being a broken circuit of $H$ depends both on $H$ and $G$. If $H \subseteq G$ contains no broken circuits then it is called broken circuit free. Note that if $H$ contains a circuit then it also contains a broken circuit. Consequently, a broken circuit free subgraph is always a forest. If $T \subseteq G$ is a subtree of $G$ and the edge $e \in G$, $e \notin T$ is the smallest edge in the unique circuit in $T \cup \{e\}$ then we will call $e$ a break in $T$. Hence the set of breaks in a subtree $T$ is in bijection with the set of broken circuits of $T$.

Whitney’s Broken Circuit Theorem [7] shows that if $G$ is a connected graph with $n$ vertices, the coefficient of $(-1)^{n-1}x$ in $\chi_G(x)$ is the number of broken circuit free subtrees of $G$. Hence there should be a bijection between broken circuit free subtrees and increasing $G$-connected trees.

**Theorem 3** Let $V$ be a totally ordered vertex set with smallest element $r$, and let $G$ be a connected graph on $V$. Let $T \subseteq G$ be a subtree of $G$, and let $R = k(T)$. Let $E(v)$ for $v \in V - \{r\}$ be as in Theorem 1 part 3. Then $E(v)$ contains only one element $e(v)$ (otherwise $T$ would have more than $|V|-1$ edges so it could not be a tree). For $v \in V - \{r\}$, let $d(v)$ be the set of elements of $J(v, R) \cap G$ which are smaller than $e(v)$. Then the set of breaks in $T$ is

$$\bigcup_{v \in V - \{r\}} d(v)$$
Proof. Let \( J = \bigcup_{v \in V - \{v\}} J(v, R) \cap G \). Since \( k(G) \) may be different from \( R \), \( J \) may be different from \( G \). We will first show that if \( e \in G \) but \( e \notin J \) then \( e \) is not a break. Let \( v < w \in V \) be the vertices of \( e \). Then \( w \) is not a descendant of \( v \) because otherwise we would have \( e \in J \). Let \( u \in V \) be the join of \( v \) and \( w \) in \( R \). Then Theorem 1 part 2 implies that \( u \) is also the join of \( v \) and \( w \) in \( T|_{\text{des}(v)} \) (rooted at \( u \)). Therefore, the cycle created by adding \( e \) to \( T \) contains an edge connected to \( u \). Since \( u < v < w \), \( e \) cannot be a break.

Now suppose \( e \notin J(v, R) \cap G \) is smaller than \( e(v) \). We will show that \( e \) is a break. Let \( H = T|_{\text{des}(v)} \cap \text{parent}(v) \). Then \( \text{parent}(v) \) is the smallest vertex in the vertex set of \( H \). Therefore, \( e \) is smaller than any other edge in \( H \). Since \( H \) is a tree, adding \( e \) would create a unique circuit in \( H \). Hence \( e \) is a break.

Now suppose \( e \in J(v, R) \cap G \) is larger than \( e(v) \). Then, letting \( H \) be as before, we see that \( e(v) \) must belong to the circuit which \( e \) creates. But \( e(v) \) is smaller than \( e \), so \( e \) cannot be a break. \( \square \)

**Corollary 2** The function

\[
f(R) = \bigcup_{v \in V - \{v\}} \min(J(v, R) \cap G)
\]

is a bijection between increasing \( G \)-connected trees and broken circuit free subtrees, and \( f^{-1}(T) = k(T) \).

Of course, this bijection generalizes to a bijection between increasing \( G \)-connected forests with \( q \) components and broken circuit free subforests of \( G \) with \( q \) components.

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