Tempered Fractional Multistable Motion and Tempered Multifractional Stable Motion

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Abstract Tempered fractional multistable motion and tempered multifractional stable motion are extensions of fractional multistable motion and multifractional stable motion, respectively, by adding an exponential tempering to the integrands. This paper develops the basic properties of these processes, including scaling property, tail probabilities, absolute moment, sample path properties, pointwise Hölder exponent, Hölder continuity of quasi norm and (strong) localisability. In particular, tempered fractional multistable motion and tempered multifractional stable motion are of semi-long-range dependence structure. Therefore they provide two useful alternative models for data that exhibit strong dependence.

Keywords Stable processes · Multistable processes · Multifractional processes · Sample paths · Long-range dependence · Localisability

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1 Introduction

Linear fractional stable motion (LFSM) can be represented by the stochastic integral of a symmetric $\alpha$-stable random measure $dZ_\alpha(x)$, that is

$$X(t) = \int_{-\infty}^{\infty} \left[ (t-x)^{H-\frac{\alpha}{2}} - (-x)^{H-\frac{\alpha}{2}} \right] dZ_\alpha(x), \quad t \in \mathbb{R}, \quad (1.1)$$

where $0 < \alpha \leq 2, 0 < H < 1, (x)_+ = \max\{x, 0\}$ and $0^0 = 0$. See for example Samorodnitsky and Taqqu [15]. This stochastic process has two important features. It is self-similar with Hurst parameter $H$, i.e. for any $c > 0$, $t_1, ..., t_d \in \mathbb{R}$,

$$(X(ct_1), ..., X(ct_d)) \overset{d}{=} (c^H X(t_1), ..., c^H X(t_d)).$$
and it has stationary increments, i.e., for any $\tau \in \mathbb{R}$, $(X(t) - X(0), -\infty < t < \infty) \overset{d}{=} (X(\tau + t) - X(\tau), -\infty < t < \infty)$, where $\overset{d}{=}$ indicates equality in distribution. Because its increments can exhibit the heavy-tailed analog of long-range dependence (see Watkins et al. [18]), the model is useful in practice to model, for example, financial data, internet traffic, noise on telephone line, signal processing, and atmospheric noise, see Nolan [13] for many references.

There exist at least three extensions of LFSM, i.e., linear multifractional stable motion (LmFSM), linear fractional multistable motion (LFmSM) and linear tempered fractional stable motion (LTFSM). Stoev and Taqqu [16,17] first introduced LmFSM by replacing the self-similarity parameter $H$ in the integral representation of the LFSM by a time-varying function $H_t$. Stoev and Taqqu have examined the effect of the regularity of the function $H_t$ on the local structure of the process. They also showed that under certain Hölder regularity conditions on the function $H_t$, the LmFSM is locally equivalent to a LFSM, in the sense of finite-dimensional distributions. Thus LmFSM is a locally self-similar stochastic processes. Whereas the LFSM is always continuous in probability, this is not in general the case for LmFSM. Stoev and Taqqu have obtained necessary and sufficient conditions for the continuity in probability of the LmFSM. Falconer and Lévy Véhel [7] defined the second model extension of LFSM, called LFmSM. LFmSM behaves locally like linear fractional $\alpha(t)$-stable motion close to time $t$, in the sense that the local scaling limits are linear fractional $\alpha(t)$-stable motions, but where the stability index $\alpha(t)$ varies with $t$. This extension allows one to account for the fact that the nature of irregularity, including the stability level, may vary in time. See also Falconer and Liu [8] where the $\alpha$-stable random measure in [16] has been replaced by a time-varying $\alpha(t)$-multistable random measure. Recently, Meerschaert and Sabzikar [11] defined the third extension, termed LTFSM, by adding an exponential tempering to the power-law kernel in a LFSM. They showed that the LTFSM exhibits semi-long-range dependence, and therefore provides a useful alternative model for data that exhibit strong dependence.

In view of trying to combine the properties of both LFmSM and LTFSM, we define in this work a new stochastic process by adding an exponential tempering to the power-law kernel of LFmSM. Our linear tempered fractional multistable motion (LTFmSM) is thus an extension of LFmSM and LTFSM. In particular, linear tempered fractional multistable motion behaves locally like the linear fractional $\alpha(t)$-stable motion with stability index $\alpha(t)$ that varies in time $t$, and it exhibits semi-long-range dependence structure as LTFSM does. Similarly, to combine the properties of both LmFSM and LTFSM, we define another new stochastic process, called linear tempered multifractional stable motion (LTmFSM), by adding an exponential tempering to the power-law kernel of LmFSM. This new process is also of semi-long-range dependence structure. We also investigate the basic properties of the two new processes, including scaling properties, tail probabilities, absolute moment, sample path properties, pointwise Hölder exponent, Hölder continuity of quasi norm and (strong) localisability. Such properties are important and have been widely studied. For instance, Falconer and Liu [5] have investigated sample path properties, localisability and strong localisability of LFmSM; Le Guével and Lévy Véhel [9] have investigated the pointwise Hölder exponent of LFmSM; Ayache and Hamonier [2] have examined the fine path properties of LmFSM; Meerschaert and Sabzikar [11] have studied scaling properties, sample path properties and Hölder continuity of quasi norm of LTFSM.

Notice that in this work, we focus on the different properties between LTFmSM and LTmFSM, instead of the common properties. Thus we do not introduce linear tempered multifractional multistable motion (LTmFmSM). It is also worth noting that to understand
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these different properties between LTFmSM and LTmFSM will be helpful to the future study of LTmFmSM.

The remainder of this paper is organized as follows. In Section 2, we define the linear tempered fractional multistable motion and the linear tempered multifractional stable motion. In Section 3, we elucidate the dependence structure of the two stochastic processes. In Sections 4 - 8, we analyze their properties.

2 Definitions of TFmSM and TmFSM

Throughout this paper, for given $0 < a \leq b \leq 2$, the function $\alpha : \mathbb{R} \rightarrow [a, b]$ will be a Lebesgue measurable function that will play the role of a varying stability index. Define variable exponential Lebesgue space by

$$
F_\alpha := \{ f : f \text{ is measurable with } ||f||_\alpha < \infty \}
$$

where

$$
||f||_\alpha := \left\{ \lambda > 0 : \int_{-\infty}^{\infty} \frac{|f(x)|^{\alpha(x)}}{\lambda} dx = 1 \right\}.
$$

Then $|| \cdot ||_\alpha$ is a quasinorm.

Falconer and Liu [8] defined the multistable stochastic integral $I(f) := \int f(x) dM_\alpha(x)$, $f \in F_\alpha$, by specifying the finite-dimensional distribution of $I$. Here and after, $dM_\alpha(x)$ stands for the multistable measure, which is an independently scattered symmetric random measure. Assume $\alpha(x) \in [a, b] \subset (0, 2]$. Given $f_1, f_2, ..., f_d \in F_\alpha$, Falconer and Liu defined a probability distribution on the vector $(I(f_1), I(f_2), ..., I(f_d)) \in \mathbb{R}^d$ by the following characteristic function

$$
E\left[ e^{i\sum_{k=1}^{d} \theta_k I(f_k)} \right] = \exp \left( -\int_{-\infty}^{\infty} \left| \sum_{k=1}^{n} \theta_k f_k(x) \right|^{\alpha(x)} dx \right).
$$

The essential point is that $\alpha(x)$ may vary with $x$. With the definition of multistable stochastic integral, Falconer and Liu [8] (cf. Proposition 4.3 therein) defined linear fractional multistable motion (LFmSM)

$$
X(t) = \int_{-\infty}^{\infty} \left[ (t-x)^H \frac{\alpha(x)}{\pi(x)} - (-x)^H \frac{\alpha(x)}{\pi(x)} \right] dM_\alpha(x).
$$

They also investigated some basic properties of LFmSM, such as localisability and strong localisability.

By adding an exponential tempering to the power-law kernel in LFSM [11], that is

$$
X_{H,\alpha,\lambda}(t) := \int_{-\infty}^{\infty} \left[ e^{-\lambda(t-x)^H} (t-x)^H \frac{\alpha(x)}{\pi(x)} - e^{-\lambda(-x)^H} (-x)^H \frac{\alpha(x)}{\pi(x)} \right] dZ_\alpha(x),
$$

$\lambda > 0$, $0 < \alpha < 2$ and $0 < H < 1$, Meerschaert and Sabzikar [11] recently defined the so-called linear tempered fractional stable motion (LTFSM). They showed that the LTFSM exhibits semi-long-range dependence structure, and therefore it provides a useful alternative model for data that exhibit strong dependence.

Similarly, by adding an exponential tempering to the power-law kernel in a LFmSM [22], we define the following linear tempered fractional multistable motion. Such process is an extension of both LFmSM and LTFSM mentioned above.
**Definition 1** Let $\alpha(x) \in [a, b] \subset (0, 2]$ be a continuous function on $\mathbb{R}$. Given an independently scattered symmetric multistable random measure $dM_\alpha(x)$ on $\mathbb{R}$, the multistable stochastic integral

$$X_{H,\alpha}(x, t) := \int_{-\infty}^{\infty} \left[ e^{-\lambda(t-x)^+} (t-x)_+^{\frac{H-\frac{\alpha}{\alpha+\lambda}}{\alpha+\lambda}} - e^{-\lambda(-x)^+} (-x)_+^{\frac{H-\frac{\alpha}{\alpha+\lambda}}{\alpha+\lambda}} \right] dM_\alpha(x) \quad (2.4)$$

with $0 < H < 1, \lambda \geq 0, (x)_+ = \max\{x, 0\}$, and $\theta^0 = 0$ will be called a linear tempered fractional multistable motion ($LTF\text{mSM}$).

**Remark 1** $LTF\text{mSM}$ is a generalization of four known stochastic processes:

1. If $\alpha(x)$ is a constant function, the $LTF\text{mSM}$ is called $LTFSM$ mentioned above, defined and developed recently by Meerschaert and Sabzikar [11]. Moreover, if $\lambda = 0$, the $LTFSM$ reduces to linear fractional stable motion (LFSM), see the monograph of Samorodnitsky and Taqqu [15]. Furthermore, if $\alpha = 2$, the LFSM is the well-known Fractional Brownian motion (FBM), see Mandelbrot and Van Ness [12].

2. When $\lambda = 0$, the $LTF\text{mSM}$ becomes $LF\text{mSM}$, see Falconer and Liu [8].

3. With the exponential tempering, we can also define **multistable Yaglom noise**

$$Y_{H,\alpha}(x, t) = \int_{-\infty}^{\infty} \left[ e^{-\lambda(t-x)^+} (t-x)_+^{\frac{H-\frac{\alpha}{\alpha+\lambda}}{\alpha+\lambda}} - e^{-\lambda(-x)^+} (-x)_+^{\frac{H-\frac{\alpha}{\alpha+\lambda}}{\alpha+\lambda}} \right] dM_\alpha(x), \quad \lambda > 0. \quad (2.5)$$

In particular, when $\alpha(x) \equiv 1/H \in (0, 2]$, multistable Yaglom noise is known as Ornstein-Uhlenbeck process, see Example 3.6.3 of Samorodnitsky and Taqqu [15]. When $\alpha(x) \equiv \alpha$ for some constant $\alpha$, multistable Yaglom noise is called stable Yaglom noise, see Meerschaert and Sabzikar [11]. It is obvious that fractional multistable Yaglom noise is a multistable stochastic integral. It is also easy to see that

$$X_{H,\alpha}(x, t) = Y_{H,\alpha}(x, t) - Y_{H,\alpha}(x, 0), \quad \lambda > 0. \quad (2.6)$$

Denote by

$$G_{H,\alpha}(x, t, x) = e^{-\lambda(t-x)^+} (t-x)_+^{\frac{H-\frac{\alpha}{\alpha+\lambda}}{\alpha+\lambda}} - e^{-\lambda(-x)^+} (-x)_+^{\frac{H-\frac{\alpha}{\alpha+\lambda}}{\alpha+\lambda}}, \quad \lambda > 0. \quad (2.7)$$

It is easy to check that the function $G_{H,\alpha}(x, t, x)$ belong to $\mathcal{F}_\alpha$, so that $LTF\text{mSM}$ is well defined. Moreover, by the definition of multistable integral (cf. Falconer and Liu [8]), the characteristic function of $X_{H,\alpha}(x, \lambda) t$ is given as follows:

$$E \left[ e^{i \sum_{k=1}^{d} \theta_k X_{H,\alpha}(x, \lambda(t_k))} \right] = \exp \left\{ - \int_{-\infty}^{\infty} \sum_{k=1}^{d} \theta_k G_{H,\alpha}(x, \lambda(t_k, x))^{\alpha(x)} dx \right\}. \quad (2.6)$$

Similarly, when the Hurst parameter $H$ of (2.3) varies with time $t$, we have another extension of $LTFSM$.

**Definition 2** Let $H_t \in [a, b] \subset (0, 1)$ be a continuous function on $\mathbb{R}$. Given an independent scattered SoS stable random measure $dZ_\alpha(x)$ on $\mathbb{R}$ with control measure $dx$, the stable stochastic integral

$$X_{H_t,\alpha}(x, t) := \int_{-\infty}^{\infty} \left[ e^{-\lambda(t-x)^+} (t-x)_+^{\frac{H_t-\frac{\alpha}{\alpha+\lambda}}{\alpha+\lambda}} - e^{-\lambda(-x)^+} (-x)_+^{\frac{H_t-\frac{\alpha}{\alpha+\lambda}}{\alpha+\lambda}} \right] dZ_\alpha(x) \quad (2.7)$$

with $0 < \alpha \leq 2, \lambda \geq 0, (x)_+ = \max\{x, 0\}$, and $\theta^0 = 0$ will be called a linear tempered multifractional stable motion ($LT\text{mFSM}$).
Remark 2 LTmFSM is a generalization of six known stochastic processes:

1. When $\alpha = 2$, $\lambda = 0$ and $H_t \equiv H$ for a constant $H$, the LTmFSM is the well-known Fractional Brownian motion (FBM), see Mandelbrot and Van Ness [12].
2. If $\lambda = 0$ and $H_t \equiv H$ for a constant $H$, the LTmFSM becomes the known linear fractional stable motion (LFSM), see the monograph of Samorodnitsky and Taqqu [15].
3. When $\alpha = 2$ and $\lambda = 0$, the LTmFSM is a multifractional Brownian motion (MfBM), see Peltier and Lévy Véhel [14]. The MfBM is an extension of FBM to multifractional case.
4. If $\lambda = 0$, the LTmFSM is called linear multifractional stable motion (LmFSM), see Stoew and Taqqu [16,17].
5. If $\alpha = 2$ and $H_t \equiv H$ for a constant $H$, the LTmFSM reduces to a tempered fractional Brownian motion (TFBM), see Meerschaert and Sabzikar [10].
6. If $H_t \equiv H$ for a constant $H$, the LTmFSM is called a linear tempered fractional stable motion (LTFSM) defined and developed recently by Meerschaert and Sabzikar [11].

Denote $G_{H_t, \alpha, \lambda}(t, x) = e^{-\lambda(t-x)} + (t-x)^{H_t-\frac{\alpha}{\lambda}} - e^{\lambda(-x)} + (-x)^{H_t-\frac{\alpha}{\lambda}}$, $\lambda \geq 0$. By the definition of stable integral (cf. Samorodnitsky and Taqqu [15]), the characteristic function of $X_{H_t, \alpha, \lambda}(t)$ is given as follows:

$$
E\left[e^{i\sum_{k=1}^{n} \theta_k X_{H_t, \alpha, \lambda}(t_k)}\right] = \exp\left\{- \int_{-\infty}^{\infty} \sum_{k=1}^{n} \theta_k G_{H_t, \alpha, \lambda}(t_k, x) |x|^\alpha dx\right\}.
$$

The characteristic function of $X_{H_t, \alpha, \lambda}(t)$ is given as follows:

$$
E\left[e^{i\sum_{k=1}^{n} \theta_k X_{H_t, \alpha, \lambda}(t_k)}\right] = \exp\left\{- \int_{-\infty}^{\infty} \sum_{k=1}^{n} \theta_k G_{H_t, \alpha, \lambda}(t_k, x) |x|^\alpha dx\right\}.
$$

3 Dependence structure of LTFmSM and LTmFSM

In this section, we study the behaviour of increments of LTFmSM and LTmFSM, usually termed the “noise” of these processes.

Denote by

$$
Y(t) = X(t+1) - X(t) \quad \text{for integers} \quad -\infty < t < \infty
$$

the noise of the processes $X$. Astrauskas et al. [1] studied the dependence structure of linear fractional stable motion using the following nonparametric measure of dependence (see also Meerschaert and Sabzikar [11]). Define

$$
R_{t_1}(t) = R(\theta_1, \theta_2, t_1, t_1 + t) := E\left[e^{i\theta_1 Y(t_1)} + \theta_2 Y(t_1 + t)\right] - E\left[e^{i\theta_1 Y(t_1)}\right] E\left[e^{i\theta_2 Y(t_1 + t)}\right] \quad (3.9)
$$

for $t_1, t, \theta_1, \theta_2 \in \mathbb{R}$. If we also define

$$
I(\theta_1, \theta_2, t_1, t_1 + t) = \log \left(E\left[e^{i\theta_1 Y(t_1)}\right]\right) \log \left(E\left[e^{i\theta_2 Y(t_1 + t)}\right]\right) - \log \left(E\left[e^{i\theta_1 Y(t_1 + t)} + \theta_2 Y(t_1 + t)\right]\right),
$$

then we have

$$
R_{t_1}(t) = K(\theta_1, \theta_2, t_1, t_1 + t) \left(e^{-I(\theta_1, \theta_2, t_1, t_1 + t)} - 1\right), \quad (3.10)
$$
where
\[ K(\theta_1, \theta_2, t_1, t_1 + t) = \mathbb{E}[e^{i\theta_1 Y(t_1)}]\mathbb{E}[e^{i\theta_2 Y(t_1 + t)}]. \]

In particular, for stationary processes, \( R_{t_1}(t) \) does not depend on \( t_1 \), see Meerschaert and Sabzikar [11]. In this case, we denote \( R_{t_1}(t) \) by \( R(t) \) for simplicity. Note however that the increments of the two processes that we define in this work are not stationary in general.

We first recall the dependence structure of LTFSM. Given two real-valued functions \( f(t), g(t) \) on \( \mathbb{R} \), we will write
\[ f(t) \preceq g(t) \]
if \(|f(t)/g(t)| \leq C_1 \) for all \( t > 0 \) sufficiently large and some \( 0 < C_1 < \infty \). In particular, if \( f(t) \preceq g(t) \) and \( g(t) \preceq f(t) \), we will write
\[ f(t) \asymp g(t). \]

Thus \( f(t) \asymp g(t) \) is equivalent to \( C_1 \leq |f(t)/g(t)| \leq C_2 \) for all \( t > 0 \) sufficiently large and some \( 0 < C_1 < C_2 < \infty \). With these notations, Meerschaert and Sabzikar [11] recently proved that if \( \lambda > 0 \) and \( 0 < \alpha \leq 1 \), then TFSN has the following property
\[ R(t) \asymp e^{-\lambda \alpha t H^{\alpha - 1}}, \quad t \to \infty, \quad (3.11) \]
for \( \theta_1 \theta_2 \neq 0 \). Meerschaert and Sabzikar [11] also proved that if \( \lambda > 0 \), \( 1 < \alpha < 2 \) and \( \frac{1}{\alpha} < H \), then TFSN has the following property
\[ R(t) \asymp e^{-\lambda t^{H - \frac{1}{\alpha}}}, \quad t \to \infty, \quad (3.12) \]
for \( \theta_1 \theta_2 \neq 0 \).

3.1 Dependence structure of LTFmSM

The following two theorems show that LTFmSM and LTFSM share the similar dependence structure.

**Definition 3** Given an LTFmSM defined by (2.4), we define the tempered fractional multistable noise (TFmSN)
\[ Y_{H, \alpha(x), \lambda}(t) := X_{H, \alpha(x), \lambda}(t + 1) - X_{H, \alpha(x), \lambda}(t) \quad (3.13) \]
for integers \( -\infty < t < \infty \).

In particular, if \( \alpha(x) \equiv \alpha \) for a constant \( \alpha \in (0, 2) \), then the TFmSN reduces to the tempered fractional stable noise, see Meerschaert and Sabzikar [11].

**Proposition 1** Let \( \alpha(x) \in [a, b] \subset (0, 1) \) be a continuous function on \( \mathbb{R} \). Let \( Y_{H, \alpha(x), \lambda}(t) \) be the tempered fractional multistable noise (3.13). Assume \( \lambda > 0 \). Then
\[ e^{-\lambda t^{H^{\alpha - 1}}} \preceq R_{t_1}(t) \preceq e^{-\lambda \alpha t^{H^{\alpha - 1}}}, \quad t \to \infty, \quad (3.14) \]
for any \( t_1 \in \mathbb{R} \) and \( \theta_1 \theta_2 \neq 0 \).
Define $g_t(x) = e^{-\lambda(t-x)+x}H_{-\frac{1}{\alpha a}}(t-x)$ for $t \in \mathbb{R}$ and write

$$I(t_1, t_2, t_1, t_1 + t) = \int_{-\infty}^{\infty} \theta_1 [g_{t_1+1}(x) - g_{t_1}(x)] \, dx + \int_{-\infty}^{\infty} \theta_2 [g_{t_1+t+1}(x) - g_{t_1+t}(x)] \, dx - \int_{-\infty}^{\infty} \theta_1 [g_{t_1+1}(x) - g_{t_1}(x)] \, dx - \int_{-\infty}^{\infty} \theta_2 [g_{t_1+t+1}(x) - g_{t_1+t}(x)] \, dx$$

where

$$I_1(t) = \int_{-\infty}^{t} \theta_1 [g_{t_1+1}(x) - g_{t_1}(x)] \, dx$$

and

$$I_2(t) = \int_{t}^{t+1} \left( \int_{-\infty}^{\infty} \theta_1 [g_{t_1+1}(x) - g_{t_1+t}(x)] \, dx - \int_{-\infty}^{\infty} \theta_2 [g_{t_1+t+1}(x) - g_{t_1+t}(x)] \, dx \right) \, dx.$$
Thus
\[ |I_1(t)| e^{\lambda at} t^{1-Hb} \leq 2(\theta_2^a + |\theta_2|^b) \int_{-\infty}^{t_1} F_\lambda(x)dx \leq C_1(\theta_2^a + |\theta_2|^b), \]
where \( C_1 > 0 \) depends only on \( a, b, H \) and \( \lambda \). Hence
\[ |I_1(t)| \leq C_1(\theta_2^a + |\theta_2|^b)e^{-\lambda at} t^{Hb-1}. \]  (3.19)

Next for \( I_2(t) \), we have the following estimation. Using inequality (3.17) again, we obtain
\[ |I_2(t)| \leq 2 \int_{t_1}^{t+1} \theta_2|g_{t_1+t+1}(x) - g_{t_1}(x)| \alpha(x) dx. \]  (3.20)

Applying the mean value theorem to see that for \( t \geq 2 \) and any \( x \in (t_1, t_1 + 1) \), we have
\[ \left| g_{t_1+t+1}(x) - g_{t_1}(x) \right| \leq \left| -\lambda e^{-\lambda(u-x)}(u-x)H - \frac{1}{\alpha(x)} \right| + (H - \frac{1}{\alpha(x)})e^{-\lambda(u-x)}(u-x)H - \frac{1}{\alpha(x)} - 1 \]
\[ \leq e^{-\lambda(t-1)}(\lambda(t-1)H - \frac{1}{\alpha(x)} - H)(t-1)H - \frac{1}{\alpha(x)} \]
\[ \leq e^{-\lambda(t-1)}\left( \frac{1}{\alpha(x)} - H + \lambda \right)(t-1)H - \frac{1}{\alpha(x)}, \]
where \( u \in (t_1 + t, t_1 + t + 1) \). Returning to (3.20), we get
\[ |I_2(t)| \leq 2(\theta_2^a + |\theta_2|^b) \int_{t_1}^{t+1} e^{-\lambda(t-1)}\left( \frac{1}{\alpha(x)} - H + \lambda \right)(t-1)H - \frac{1}{\alpha(x)} \alpha(x) dx \leq C_2(\theta_2^a + |\theta_2|^b)e^{-\lambda at} t^{Hb-1} \] (3.21)
for large \( t \), where \( C_2 > 0 \) depends only on \( a, b, H \) and \( \lambda \). Combining the inequalities (3.16), (3.18), (3.19) and (3.21) together, we obtain
\[ 0 \leq I(\theta_1, \theta_2, t_1, t_1 + t) \leq C_3 e^{-\lambda at} t^{Hb-1} \] as \( t \to \infty \), (3.22)
where \( C_3 \) does not depend on \( t \). Using the following equality
\[ |x_1|^\alpha + |x_2|^\alpha - |x_1 + x_2|^\alpha = |x_2|^\alpha - \frac{\alpha}{|x_1 + \theta x_2|^{1-\alpha}} |x_2| \]
for all \( x_1, x_2 \neq 0, 0 < \alpha < 1 \) and some \( |\theta| \leq 1 \), we obtain
\[ |x_1|^\alpha + |x_2|^\alpha - |x_1 + x_2|^\alpha \sim |x_2|^\alpha \] (3.23)
for all \( x_1 \neq 0, x_2 \to 0 \) and \( 0 < \alpha < 1 \). Thus for \( t \to \infty \),
\[ |I_1(t)| \sim \int_{-\infty}^{t_1} \theta_2|g_{t_1+t+1}(x) - g_{t_1+t}(x)| \alpha(x) dx. \] (3.24)
Applying the dominated convergence theorem yields
\[\lim_{t \to \infty} \left| I_1(t) e^{-\lambda t} t^{1-H} \right| \geq \lim_{t \to \infty} \int_{-\infty}^{t_1} \left| \theta_2 [g_{t_1+j} - g_{t_1}] e^{-\lambda (t_1-x)} (1-e^{-\lambda}) \right|^\alpha dx \]
(3.25)

Then (3.16), (3.18) and (3.25) implies that for large \( t \)
\[I(\theta_1, \theta_2, t_1, t_1 + t) \geq -I_1(t) \geq \frac{1}{2} C_2 e^{-\lambda t} t^{H-1}, \]
(3.26)

where \( C_2 = \int_{-\infty}^{t_1} \left| \theta_2 [e^{-\lambda (t_1-x)} (1-e^{-\lambda})] \right|^\alpha dx \) does not depend on \( t \). Combining (3.22) and (3.26) together, we have
\[e^{-\lambda t} t^{H-1} \leq I(\theta_1, \theta_2, t_1, t_1 + t) \leq e^{-\lambda t} t^{H-1} \]
(3.27)

for \( \theta_1 \theta_2 \neq 0 \). It is easy to see that

\[K(\theta_1, \theta_2, t_1, t_1 + t) = \exp \left\{ - \int_{-\infty}^{t} \left[ \theta_2 [e^{-\lambda (1-u)} (1-u)^H - e^{-\lambda u} (1-u)^H] \right]^\alpha du \times \exp \left\{ - \int_{-\infty}^{t} \left[ \theta_2 [e^{-\lambda (1-u)} (1-u)^H - e^{-\lambda u} (1-u)^H] \right]^\alpha du \right\} \geq \exp \left\{ -2 \left( |\theta_1|^a + |\theta_2|^b \right)^t \int_{-\infty}^{t} M_\lambda(u) du \right\} \]

where
\[M_\lambda(u) = e^{-\lambda (1-u)} (1-u)^H - e^{-\lambda u} (1-u)^H + e^{-\lambda u} (1-u)^H - e^{-\lambda (1-u)} (1-u)^H \]
is integrable on \( (-\infty, 1] \) with respect to \( u \), and that
\[K(\theta_1, \theta_2, t_1, t_1 + t) \leq 1. \]

Since \( I(\theta_1, \theta_2, t_1, t_1 + t) \to 0 \) as \( t \to \infty \), it follows from (3.10) that \( R_{t_1} (t) \sim -K(\theta_1, \theta_2, t_1, t_1 + t) I(\theta_1, \theta_2, t_1, t_1 + t) \). Hence (3.14) follows by (3.27).

**Proposition 2** Let \( \alpha(x) \in [a, b] \subset (1, 2] \) be a continuous function on \( \mathbb{R} \). Let \( Y_{H, \alpha(x), \lambda}(t) \) be the tempered fractional multistable noise (3.13). Assume \( \lambda > 0 \) and \( 1/a < H < 1 \). Then
\[e^{-\lambda t} t^{H-\frac{1}{a}} \leq R_{t_1} (t) \leq e^{-\lambda t} t^{H-\frac{1}{b}}, \quad t \to \infty, \]
(3.28)
for any \( t_1 \in \mathbb{R} \) and \( \theta_1 \theta_2 \neq 0 \).
Proof. Recall $I_1(t)$ and $I_2(t)$ defined by (3.10). First we consider the case

$$
\theta_1 \theta_2 |g_{t_1+1}(x) - g_t(x)| |g_{t_1+t_1+1}(x) - g_{t_1+t}(x)| \geq 0.
$$

Using the inequality

$$
0 \leq |x_1 + x_2|^\alpha - |x_1|^\alpha - |x_2|^\alpha \leq \alpha |x_1| |x_2|^\alpha - 1
$$

for all $x_1 x_2 \geq 0$ and $1 < \alpha \leq 2$, we have

$$
I_1(t) \geq 0 \quad \text{and} \quad I_2(t) \geq 0.
$$

First, we give an estimation for $I_1(t)$. By (3.29), we obtain

$$
I_1(t) \leq \int_{-\infty}^{t_1} \alpha(x) \theta_2 |g_{t_1+t_1+1}(x) - g_{t_1+t}(x)| \theta_1 |g_{t_1+1}(x) - g_{t_1}(x)| dx
$$

$$
\leq 2|\theta_2| \max \{ |\theta_1|^{\alpha-1}, |\theta_1|^{-1} \} \int_{-\infty}^{t_1} |g_{t_1+t_1+1}(x) - g_{t_1+t}(x)| \theta_1 |g_{t_1+1}(x) - g_{t_1}(x)|^{\alpha(x)-1} dx.
$$

It is easy to see that for large $t$ and $x \leq t_1$,

$$
\left| \theta_1 |g_{t_1+1}(x) - g_{t_1}(x)| e^{\lambda t H - H} \right|
$$

$$
= e^{-\lambda(t_1-x)} \left( e^{-\lambda} \left( 1 + \frac{1 + t_1 - x}{t} \right)^{H - \frac{1}{\alpha(x)}} - \left( 1 + \frac{t_1 - x}{t} \right)^{H - \frac{1}{\alpha(x)}} \right)
$$

$$
\leq e^{-\lambda(t_1-x)} (1 + e^{-\lambda}) \left( 1 + \frac{1 + t_1 - x}{t} \right)^{H - \frac{1}{\alpha(x)}} + \left( 1 + \frac{t_1 - x}{t} \right)^{H - \frac{1}{\alpha(x)}}
$$

$$
\leq e^{-\lambda(t_1-x)} (1 + e^{-\lambda})(2 + t_1 - x)^{H - \frac{1}{\alpha(x)}}.
$$

(3.30)

Therefore, for large $t$ and $x \leq t_1$,

$$
I_1(t) \leq 2|\theta_2| \max \{ |\theta_1|^{\alpha-1}, |\theta_1|^{-1} \}
$$

$$
\times \int_{-\infty}^{t_1} e^{-\lambda t H - \frac{1}{\alpha(x)}} (1 + e^{-\lambda}) \left( 2 + t_1 - x \right)^{H - \frac{1}{\alpha(x)}} e^{-\lambda(t_1-x)} |g_{t_1+1}(x) - g_{t_1}(x)|^{\alpha(x)-1} dx
$$

$$
\leq 2|\theta_2| \max \{ |\theta_1|^{\alpha-1}, |\theta_1|^{-1} \} e^{-\lambda t H - \frac{1}{\alpha(x)}}
$$

$$
\times \int_{-\infty}^{t_1} (1 + e^{-\lambda}) \left( 2 + t_1 - x \right)^{H - \frac{1}{\alpha(x)}} e^{-\lambda(t_1-x)} |g_{t_1+1}(x) - g_{t_1}(x)|^{\alpha(x)-1} dx.
$$

(3.31)

Recall $g_t(x) = e^{-\lambda(t-x)}(t-x)_+^{H - \frac{1}{\alpha(x)}}$, and that

$$
|g_{t_1+1}(x) - g_{t_1}(x)|^{\alpha(x)-1} \leq |g_{t_1+1}(x)|^{\alpha(x)-1} + |g_{t_1}(x)|^{\alpha(x)-1}
$$

(cf. (3.17) for the last inequality). Since $\alpha(x) - 1 \leq b - 1 < 1$ and $H > \frac{1}{\alpha(x)} \geq \frac{1}{\alpha}(x)$, from (3.31), we obtain

$$
I_1(t) \leq C_1 |\theta_2| \max \{ |\theta_1|^{\alpha-1}, |\theta_1|^{-1} \} e^{-\lambda t H - \frac{1}{\alpha(x)}},
$$

(3.32)
where $C_1$ does not depend on $t$. Next, we give an estimation for $I_2(t)$. Using (3.29) again, we obtain

$$I_2(t) = \int_{t_1}^{t_1+1} \left[ \theta_1 g_{t_1+1}(x) + \theta_2 g_{t_1+1}(x) - g_{t_1+1}(x) \right]^{\alpha(x)} \left[ \theta_1 g_{t_1+1}(x) - g_{t_1+1}(x) \right]^{\alpha(x)} \, dx$$

$$\leq \int_{t_1}^{t_1+1} \alpha(x) \left[ \theta_2 [g_{t_1+1}(x) - g_{t_1+t}(x)] \right]^{\alpha(x)-1} \, dx$$

$$\leq 2 |\theta_2| \max \left\{ |\theta_1|^{a-1}, |\theta_1|^{b-1} \right\} \int_{t_1}^{t_1+1} \left| g_{t_1+t+1}(x) - g_{t_1+t}(x) \right| \left| g_{t_1+t+1}(x) - g_{t_1+t}(x) \right|^{\alpha(x)-1} \, dx.$$  

By (3.30), it follows that for large $t$,

$$I_2(t) \leq 2 |\theta_2| \max \left\{ |\theta_1|^{a-1}, |\theta_1|^{b-1} \right\} e^{-\lambda t (H^{-1} + \frac{1}{t})} \left( 1 + e^{-\lambda} \right) \left( 1 + \frac{t_1 - x}{t} \right)^{H - \frac{1}{\alpha(x)}} \left| g_{t_1+t+1}(x) \right|^{\alpha(x)-1} \, dx$$

$$\leq C_2 |\theta_2| \max \left\{ |\theta_1|^{a-1}, |\theta_1|^{b-1} \right\} e^{-\lambda t (H^{-1} + \frac{1}{t})},$$

where $C_2$ does not depend on $t$. Therefore, from (3.32) and (3.33), for large $t$,

$$I(\theta_1, \theta_2, t_1, t_1 + t) \leq C_3 e^{-\lambda t (H^{-1} + \frac{1}{t})}.$$  

(3.34)

where $C_3$ does not depend on $t$.

Notice that

$$|x_1 + x_2|^\alpha - |x_1|^\alpha - |x_2|^\alpha \sim \alpha |x_1| |x_2|^\alpha-1$$

for all $x_2 \neq 0, x_1 \to 0$ and $1 < \alpha \leq 2$. Then we have

$$I_1(t) \sim \int_{-\infty}^{t_1} \alpha(x) |\theta_1 g_{t_1+t+1}(x) - g_{t_1+t}(x)| \left| g_{t_1+1}(x) - g_{t_1}(x) \right|^{\alpha(x)-1} \, dx.$$  

(3.35)

It is easy to see that for large $t$ and $x \leq t_1$,

$$\lim_{t \to \infty} \left| g_{t_1+t+1}(x) - g_{t_1+t}(x) \right| e^{\lambda t (H^{-1} + \frac{1}{t})}$$

$$= \lim_{t \to \infty} \left| e^{-\lambda (t_1-x)} \left( 1 + \frac{t_1 - x}{t} \right)^{H - \frac{1}{\alpha(x)}} \right|$$

$$= e^{-\lambda (t_1-x)} \left( 1 - e^{-\lambda} \right).$$

(3.36)

Applying the dominated convergence theorem yields

$$\lim_{t \to \infty} I_1(t) e^{\lambda t (H^{-1} + \frac{1}{t})}$$

$$\geq \lim_{t \to \infty} |\theta_2| \min \left\{ |\theta_1|^{a-1}, |\theta_1|^{b-1} \right\}$$

$$\times \int_{-\infty}^{t_1} \left| g_{t_1+t+1}(x) - g_{t_1+t}(x) \right| e^{\lambda t (H^{-1} + \frac{1}{t})} \left| g_{t_1+1}(x) - g_{t_1}(x) \right|^{\alpha(x)-1} \, dx$$

$$= |\theta_2| \min \left\{ |\theta_1|^{a-1}, |\theta_1|^{b-1} \right\} \int_{-\infty}^{t_1} \left| g_{t_1+1}(x) - g_{t_1}(x) \right|^{\alpha(x)-1} \, dx.$$  

(3.37)
Thus
\[ I_1(t) \geq e^{-\lambda t H_{-1}} \]  
(3.37)
for \( \theta_1 \theta_2 \neq 0 \). Combining (3.34) and (3.37) together, we have
\[ e^{-\lambda t H_{-1}} \preceq I_1(t) \preceq I(\theta_1, \theta_2, t_1, t_1 + t) \preceq e^{-\lambda t H_{-1}} \]  
(3.38)
for \( \theta_1 \theta_2 \neq 0 \).

For the case \( \theta_1 \theta_2 \neq 0 \), by the inequality
\[ - (\alpha + 1) |x_1| |x_2|^{\alpha - 1} \leq |x_1 + x_2|^\alpha - |x_1|^\alpha - |x_2|^\alpha \leq 0 \]  
(3.39)
for all \( x_1 x_2 < 0 \) such that \( |x_1| \leq |x_2| \) and \( 1 < \alpha \leq 2 \), we have
\[ I_1(t) \leq 0 \quad \text{and} \quad I_2(t) \leq 0. \]

By a similar argument, we can also obtain (3.38).

Since \( I(\theta_1, \theta_2, t_1, t_1 + t) \rightarrow 0 \) as \( t \rightarrow \infty \), it follows from (3.10) that \( R_{t_1}(t) \sim -K(\theta_1, \theta_2, t_1, t_1 + t) I(\theta_1, \theta_2, t_1, t_1 + t) \); hence (3.28) holds.

It is easy to see that when \( a = b = \alpha \), the results of Propositions 1 and 2 reduce to the results of Meerschaert and Sabzikar [11]. Thus our results can be regarded as generalizations of the results of Meerschaert and Sabzikar.

3.2 Dependence structure of LTmFSM

In this section, we consider the increment of LTmFSM. The following two theorems extend the dependence structure of LTFSM to the case of LTmFSM.

**Definition 4** Given an LTmFSM defined by (2.7), we define the tempered multifractional stable noise (TmFSN)
\[ Y_{H_t, \alpha, \lambda}(t) := X_{H_{t+1}, \alpha, \lambda}(t+1) - X_{H_t, \alpha, \lambda}(t) \]  
(3.40)
for integers \(-\infty < t < \infty\).

In particular, if \( H_t \equiv H \) for a constant \( H \in (0, 1) \), then the TmFSN reduces to the tempered fractional stable noise. The next theorem shows that LTmFSM has a dependence structure more general than that of LTFSM.

**Proposition 3** Let \( H_t \in [a, b] \subset (0, 1) \) be a continuous function on \( \mathbb{R} \). Let \( Y_{H_t, \alpha, \lambda}(t) \) be a tempered multifractional stable noise (3.40) for some \( 0 < \alpha < 1 \). Assume \( \lambda > 0 \). Then
\[ R_{t_1}(t) \sim e^{-\lambda t \alpha H_t - 1}, \quad t \rightarrow \infty, \]  
(3.41)
for \( \theta_1 \theta_2 \neq 0 \).
\textbf{Proof.} By the definition (2.1), TmFSN has the following representation
\begin{equation}
Y_{H_t, \alpha, \lambda}(t) = \int_{-\infty}^{\infty} \left[ e^{-\lambda(t+1-x)} H_{t+1} - e^{-\lambda(t-x)} H_t \right] dZ_t(x). \quad (3.42)
\end{equation}
Define \( h_t(x) = (t - x) H_{t+1} e^{-\lambda(t-x)} \) for \( t \in \mathbb{R} \) and write
\begin{equation}
I(\theta_1, \theta_2, t_1, t_1 + t) = \int_{-\infty}^{\infty} \left[ \theta_1[h_{t_1+1}(x) - h_{t_1}(x)] + \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \right]^\alpha dx
- \int_{-\infty}^{\infty} \left[ \theta_1[h_{t_1+1}(x) - h_{t_1}(x)] \right]^\alpha dx + \int_{-\infty}^{\infty} \left[ \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \right]^\alpha dx
= I_3(t) + I_4(t), \quad (3.43)
\end{equation}
where
\begin{align*}
I_3(t) &= \int_{-\infty}^{t_1} \left[ \theta_1[h_{t_1+1}(x) - h_{t_1}(x)] + \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \right]^\alpha dx \\
&\quad - \int_{-\infty}^{t_1} \left[ \theta_1[h_{t_1+1}(x) - h_{t_1}(x)] \right]^\alpha dx - \int_{-\infty}^{t_1} \left[ \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \right]^\alpha dx \\
I_4(t) &= \int_{t_1}^{t_1+1} \left[ \theta_1[h_{t_1+1}(x) + \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \right]^\alpha \\
&\quad - \left[ \theta_1[h_{t_1+1}(x)] \right]^\alpha - \left[ \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \right]^\alpha \right] dx.
\end{align*}
Using (3.17) again, we obtain
\begin{equation}
I_3(t) \leq 0 \quad \text{and} \quad I_4(t) \leq 0. \quad (3.44)
\end{equation}
First, we give an estimation for \( I_3(t) \). For large \( t \),
\begin{align*}
|I_3(t)| &\leq 2 \int_{-\infty}^{t_1} \left[ \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \right]^\alpha dx \\
&\leq 2 \theta_2^\alpha e^{-\lambda t_1 H_{t+1}} \int_{-\infty}^{t_1} \left[ h_{t_1+t+1}(x) - h_{t_1+t}(x) \right] e^{\lambda t H_{t+1}} H_t^\alpha dx.
\end{align*}
Notice that \( H_t \leq 1 \). It is easy to see that for \( x \leq t_1 \) and \( t > 1 \),
\begin{align*}
\left[ h_{t_1+t+1}(x) - h_{t_1+t}(x) \right] e^{\lambda t H_{t+1}} H_t^\alpha &\leq e^{-\lambda(t_1-x)} \left[ 1 + \frac{t_1 - x}{t} \right]^{H_t H_{t+1}} - \left( 1 + \frac{t_1 - x}{t} \right)^{H_t H_{t+1}} \\
&\leq e^{-\lambda(t_1-x)} \left[ 1 + e^{-\lambda} \right] \left[ 1 + \frac{t_1 - x}{t} H_{t+1} \right]^{H_{t+1} \alpha} + \left( 1 + \frac{t_1 - x}{t} \right)^{H_{t+1} \alpha-1} \\
&\leq 2 e^{-\lambda(t_1-x)} \left( 1 + e^{-\lambda} \right)^\alpha.
\end{align*}
Thus
\[ |I_3(t)| e^{\lambda t} t^{1-\alpha H_t} \leq 4|\theta_2|^\alpha \int_{-\infty}^{t_1} e^{-\lambda (t_1-x)} (1 + e^{-\lambda})^\alpha dx \]
\[ \leq C_1 |\theta_2|^\alpha, \]
where \( C_1 > 0 \) depends only on \( \alpha \) and \( \lambda \). Hence
\[ |I_3(t)| \leq C_1 |\theta_2|^\alpha e^{-\lambda t t^{\alpha H_t - 1}}. \] (3.45)

Next for \( I_4(t) \), we have the following estimation. Using inequality (3.17) again, we obtain
\[ |I_4(t)| \leq 2 \int_{t_1}^{t_1+1} |\theta_2| |h_{t_1+t+1}(x) - h_{t_1+t}(x)|^\alpha dx. \] (3.46)

Applying the mean value theorem to see that for \( t \geq 2 \) and any \( x \in (t_1, t_1 + 1) \), we have
\[ |h_{t_1+t+1}(x) - h_{t_1+t}(x)| \]
\[ \leq \left| -\lambda e^{-\lambda(u-x)}(u-x)^{H_t-\frac{1}{\alpha}} + (H_t - \frac{1}{\alpha})e^{-\lambda(u-x)}(u-x)^{H_t-\frac{1}{\alpha}-1} \right| \]
\[ \leq e^{-\lambda(t-1)} \left( \lambda(t-1)^{H_t-\frac{1}{\alpha}} + \frac{1}{\alpha} - H_t \right)(t-1)^{H_t-\frac{1}{\alpha}-1} \]
\[ \leq e^{-\lambda(t-1)} \left( \frac{1}{\alpha} - H_t + \lambda \right)(t-1)^{H_t-\frac{1}{\alpha}}, \] (3.47)

where \( u \in (t_1 + t, t_1 + t + 1) \). Returning to (3.40), we get
\[ |I_4(t)| \leq 2|\theta_2|^\alpha \int_{t_1}^{t_1+1} e^{-\lambda(t-1)} \left( \frac{1}{\alpha} - H_t + \lambda \right)t^{H_t-\frac{1}{\alpha}} \right|^\alpha dx \]
\[ \leq C_2 |\theta_2|^\alpha e^{-\lambda t t^{\alpha H_t - 1}} \] (3.48)

for large \( t \), where \( C_2 > 0 \) depends only on \( \alpha \) and \( \lambda \). Combining the inequalities (3.45), (3.47), (3.48) together, we obtain
\[ 0 \leq -I(\theta_1, \theta_2, t_1, t_1 + t) \leq C_3 |\theta_2|^\alpha e^{-\lambda t t^{\alpha H_t - 1}} \] (3.49)

for large \( t \), where \( C_3 \) does not depend on \( t \). By (3.24), it holds for \( t \to \infty \),
\[ |I_3(t)| \sim \int_{-\infty}^{t_1} |\theta_2| |h_{t_1+t+1}(x) - h_{t_1+t}(x)|^\alpha dx. \] (3.50)

It is easy to see that for \( x \leq t_1 \) and \( t > 1 \),
\[ \lim_{t \to \infty} \left| h_{t_1+t+1}(x) - h_{t_1+t}(x) \right|^{\alpha H_t} \]
\[ \leq \lim_{t \to \infty} e^{-\lambda(t_1-x)} \left( e^{-\lambda(t_1-x)(1 + \frac{t_1 + t - x}{t})^{H_t-\frac{1}{\alpha}}} - (1 + \frac{t_1 - x}{t})^{H_t-\frac{1}{\alpha}} \right)^\alpha \]
\[ \leq e^{-\lambda(t_1-x)(1 - e^{-\lambda})} \right|^\alpha. \]

Notice that \( H_t(1 - e^{-\lambda}) < 1 \). Applying the dominated convergence theorem yields
\[ \lim_{t \to \infty} |I_3(t)| e^{\lambda t t^{1-H_t}} \geq \lim_{t \to \infty} \int_{-\infty}^{t_1} |\theta_2| |h_{t_1+t+1}(x) - h_{t_1+t}(x)|^{\alpha H_t} dx \]
\[ \int_{-\infty}^{t_1} |\theta_2| e^{-\lambda(t_1-x)} (1 - e^{-\lambda})^\alpha dx. \] (3.51)
Then (3.34), (3.44) and (3.51) implies that for large $t$

$$-I(\theta_1, \theta_2, t_1, t_1 + t) \geq -I_3(t) \geq \frac{1}{2} C_3 e^{-\lambda a t} H_\alpha^{-1},$$

(3.52)

where $C_3 = \int_{-\infty}^1 |\theta_2 e^{-\lambda t(x-t)} (1 - e^{-\lambda})|\alpha dx$ does not depend on $t$. Combining (3.40) and (3.52) together, we have

$$I(\theta_1, \theta_2, t_1, t_1 + t) \approx e^{-\lambda a t} H_\alpha^{-1}$$

(3.53)

for $\theta_1 \theta_2 \neq 0$. It is easy to see that

$$K(\theta_1, \theta_2, t_1, t_1 + t) = \exp\left\{-\int_{-\infty}^1 \lambda(1-u)+ (1-u)^{H_\alpha+1-\frac{1}{\alpha}} \right\}$$

$$\times \exp\left\{-\int_{-\infty}^1 \lambda(1-u)+ (1-u)^{H_\alpha+1-\frac{1}{\alpha}} \right\}$$

$$\geq \exp\left\{-2|\theta_1|^\alpha + |\theta_2|^\alpha\right\} T(u)du,$$

(3.54)

where

$$T(u) := e^{-\lambda a (1-u)+ (1-u)^{\alpha a-1} + (1-u)^{\alpha b-1}}$$

is integrable on $(-\infty, 1]$ with respect to $u$, and that $|K(\theta_1, \theta_2, t_1, t_1 + t)| \leq 1$. Since

$I(\theta_1, \theta_2, t_1, t_1 + t) \to 0$ as $t \to \infty$, it follows that

$$R_{t_1}(t) \asymp -K(\theta_1, \theta_2, t_1, t_1 + t) I(\theta_1, \theta_2, t_1, t_1 + t);$$

hence (3.31) follows by (3.55).

**Proposition 4** Let $H_t \in [a, b] \subset (1, 2]$ be a continuous function on $\mathbf{R}$. Let $Y_{H_t, \alpha, \lambda}(t)$ be a tempered multifractional stable noise (3.40). Assume $\lambda > 0$, $1 < \alpha \leq 2$ and $1/\alpha < H_t < 1$. Then

$$R_{t_1}(t) \approx e^{-\lambda a t} H_\alpha^{-1}, \quad t \to \infty,$$

(3.55)

for $\theta_1 \theta_2 \neq 0$.

**Proof.** Recall $I_3(t)$ and $I_4(t)$ defined by (3.34). We only prove the case

$$\theta_1 \theta_2 [h_{t_1+1}(x) - h_{t_1}(x)][h_{t_1} + h_{t_1+1}(x) - h_{t_1+t}(x)] \geq 0.$$

For the case $\theta_1 \theta_2 [h_{t_1+1}(x) - h_{t_1}(x)][h_{t_1} + h_{t_1+1}(x) - h_{t_1+t}(x)] < 0$, the proof is similar. Using inequality (3.29), we have

$$I_3(t) \geq 0 \quad \text{and} \quad I_4(t) \geq 0.$$

(3.56)

First, we give an estimation for $I_3(t)$. Using the inequality (3.29), we obtain

$$I_3(t) \leq \int_{-\infty}^{t_1} \alpha |\theta_2 [h_{t_1+1}(x) - h_{t_1+t}(x)]| \theta_1 [h_{t_1} + h_{t_1+1}(x) - h_{t_1+t}(x)] dx$$

$$\leq 2|\theta_2| \alpha^{-1} \int_{-\infty}^{t_1} \left|h_{t_1+1}(x) - h_{t_1+t}(x)\right|h_{t_1} + h_{t_1+1}(x) - h_{t_1+t}(x) dx.$$
It is easy to see that for large $t$ and $x \leq t_1$,

\[
\left| h_{t_1+t+1}(x) - h_{t_1+t}(x) \right| e^{\lambda t - H_t} = \left| e^{-\lambda(t_1-x)} \left( e^{-\lambda} \left( 1 + \frac{1 + t_1 - x}{t} \right)^{H_t - \frac{B}{\alpha}} - \left( 1 + \frac{1 - t}{t} \right)^{H_t - \frac{B}{\alpha}} \right) \right| \\
\leq e^{-\lambda(t_1-x)} \left( 1 + e^{-\lambda} \right) \left( 1 + \frac{1 + t_1 - x}{t} \right)^{H_t - \frac{B}{\alpha}} + \left( 1 + \frac{1 - t}{t} \right)^{H_t - \frac{B}{\alpha}}.
\]

(3.57)

Therefore, for large $t$ and $x \leq t_1$,

\[
I_3(t) \leq 2|2_2||\theta_1|^{\alpha-1} e^{-\lambda t H_t - \frac{B}{\alpha}} \\
\times \int_{-\infty}^{t_1} e^{-\lambda t} e^{-\frac{B}{\alpha}} (1 + e^{-\lambda} (2 + t_1 - x)^{H_t - \frac{B}{\alpha}} e^{-\lambda/(t_1-x)} \left| h_{t_1+1}(x) - h_{t_1}(x) \right|^\alpha dx
\]

\leq 2|2_2||\theta_1|^{\alpha-1} e^{-\lambda t H_t - \frac{B}{\alpha}} \\
\times \int_{-\infty}^{t_1} (1 + e^{-\lambda} (2 + t_1 - x)^{H_t - \frac{B}{\alpha}} e^{-\lambda/(t_1-x)} \left| h_{t_1+1}(x) - h_{t_1}(x) \right|^\alpha dx.
\]

(3.58)

From (3.58), we obtain

\[
I_3(t) \leq C_1|2_2||\theta_1|^{\alpha-1} e^{-\lambda t H_t - \frac{B}{\alpha}},
\]

(3.59)

where $C_1$ does not depend on $t$.

Next, we give an estimation for $I_4(t)$. Using inequality (5.20) again, we obtain

\[
I_4(t) \leq \int_{t_1}^{t_1+1} \alpha \left| \theta_2 \left| h_{t_1+t+1}(x) - h_{t_1+t}(x) \right| \theta_1 \left| h_{t_1+1}(x) \right|^\alpha dx
\]

\leq 2|2_2||\theta_1|^{\alpha-1} C_2 \int_{t_1}^{t_1+1} \left| h_{t_1+t+1}(x) - h_{t_1+t}(x) \right| \left| h_{t_1+1}(x) \right|^\alpha dx.
\]

(3.60)

By (3.57) and $H_t \leq b$, it follows that for large $t$,

\[
I_4(t) \leq 2|2_2||\theta_1|^{\alpha-1} e^{-\lambda t H_t - \frac{B}{\alpha}} \\
\times \int_{t_1}^{t_1+1} (1 + e^{-\lambda} (2 + t_1 - x)^{H_t - \frac{B}{\alpha}} e^{-\lambda/(t_1-x)} \left| h_{t_1+1}(x) \right|^\alpha dx
\]

\leq C_2|2_2||\theta_1|^{\alpha-1} e^{-\lambda t H_t - \frac{B}{\alpha}},
\]

(3.60)

where $C_2$ does not depend on $t$. Therefore, from (3.59) and (3.60), for large $t$,

\[
I(\theta_1, \theta_2, t_1, t_1 + t) \leq C_3 e^{-\lambda t H_t - \frac{B}{\alpha}}.
\]

(3.61)

where $C_3$ does not depend on $t$. By (3.35) we have

\[
I_4(t) \sim \int_{-\infty}^{t_1} \alpha \left| \theta_2 \left| h_{t_2+t+1}(x) - h_{t_2+t}(x) \right| \theta_1 \left| h_{t_2+1}(x) - h_{t_2}(x) \right|^\alpha dx.
\]
Applying the dominated convergence theorem yields
\[
\lim_{t \to \infty} \left| \left[ h_{t_1+t+1}(x) - h_{t_1+t}(x) \right] e^{\lambda t} t^{\frac{1}{H_t}} \right| \\
= \lim_{t \to \infty} \left| e^{-\lambda (t_1-x)} \left( e^{-x} \left( 1 + \frac{1 + t_1 - x}{t} \right)^{H_t - \frac{1}{2}} - \left( 1 + \frac{t_1 - x}{t} \right)^{H_t - \frac{1}{2}} \right) \right| \\
= e^{-\lambda (t_1-x)} \left( 1 - e^{-\lambda} \right).
\]
(3.62)

Applying the dominated convergence theorem yields
\[
\lim_{t \to \infty} I_4(t) e^{\lambda t} t^{\frac{1}{H_t}} \\
\geq \lim_{t \to \infty} \alpha |\theta_2| |\theta_1|^{-1} \int_{-\infty}^{t_1} \left| \left[ h_{t_1+t+1}(x) - h_{t_1+t}(x) \right] e^{\lambda t} t^{\frac{1}{H_t}} \right| \left| h_{t_1+1}(x) - h_{t_1}(x) \right|^{-1} dx \\
\geq \alpha |\theta_2| |\theta_1|^{-1} \int_{-\infty}^{t_1} e^{-\lambda (t_1-x)} \left( 1 - e^{-\lambda} \right) \left| h_{t_1+1}(x) - h_{t_1}(x) \right|^{-1} dx.
\]

Thus
\[
I_4(t) \geq e^{-\lambda t} t^{\frac{1}{H_t}}.
\]
(3.63)

for \( \theta_1 \theta_2 \neq 0 \). Combining (3.56), (3.61) and (3.63) together, we have
\[
e^{-\lambda t} t^{\frac{1}{H_t}} \leq I_4(t) \leq I(\theta_1, \theta_2, t_1, t_1 + t) \leq e^{-\lambda t} t^{\frac{1}{H_t}}
\]
(3.64)

for \( \theta_1 \theta_2 \neq 0 \). Since \( I(\theta_1, \theta_2, t_1, t_1 + t) \to 0 \) as \( t \to \infty \), it follows from (3.10) that \( R_{t_1}(t) \sim -K(\theta_1, \theta_2, t_1, t_1 + t) I(\theta_1, \theta_2, t_1, t_1 + t) \); hence (3.56) holds. \( \square \)

It is easy to see that when \( H_t \equiv H \), Propositions 3 and 4 also reduce to the results of Meerschaert and Sabzikar [11]. Thus these properties also can be regarded as generalizations of the results of Meerschaert and Sabzikar [11].

Remark 3 One says that a symmetric \( \alpha \)-stable process \( X(t) \) exhibits long-range dependence if for any \( t_1 \in \mathbb{R} \),
\[
\sum_{n=0}^{\infty} \left| R_{t_1}(n) \right| = \infty,
\]
(3.65)

where \( R_{t_1}(t) \) is defined by (3.10). It is obvious that LTFmSM and LTmFSM are not long-range dependent, but they exhibit semi-long-range dependence, that is, for \( \lambda > 0 \) sufficiently small, the sum (3.65) is large, and it tends to infinity as \( \lambda \to 0 \). Therefore, LTFmSM and LTmFSM provide two useful alternative models for data that exhibit strong dependence.

4 Scaling property and tail probabilities

The following result shows that LTmFSM \((2.7)\) has a nice scaling property, involving both the time scale and the tempering. Denote by \( f_{\text{fdd}} \) equality in the sense of finite dimensional distributions.

**Proposition 5** For any scale factor \( c > 0 \), it holds
\[
\left\{ X_{H_{t_1} \alpha} (ct) \right\}_{t \in \mathbb{R}} \overset{f_{\text{fdd}}}{=} \left\{ e^{H_{t_1} \alpha} X_{H_{t_1} \alpha} c \lambda (t) \right\}_{t \in \mathbb{R}}.
\]
(4.66)
Proof. It is easy to see that

\[ G_{H_t, \alpha, \lambda}(c t, c x) = c^{H_t - \frac{t}{\lambda}} G_{H_t, \alpha, \lambda}(t, x). \]

Notice that \( dZ_\alpha(c x) \) has control measure \( c^{\frac{t}{\lambda}} dx \). Given \( t_1 < t_2 < \ldots < t_n \), a change of variable \( x = cx' \) then yields

\[
(X_{H_{t_i}, \alpha, \lambda}(c t_i) : i = 1, \ldots, n) = \left( \int_{-\infty}^{\infty} G_{H_{t_i}, \alpha, \lambda}(c t_i, cx) dZ_\alpha(x) : i = 1, \ldots, n \right)
\]

\[
= \left( \int_{-\infty}^{\infty} G_{H_{t_i}, \alpha, \lambda}(c t_i, cx') dZ_\alpha(cx') : i = 1, \ldots, n \right)
\]

\[
\overset{d}{=} \left( \int_{-\infty}^{\infty} c^{H_{t_i} - \frac{t_i}{\lambda}} G_{H_{t_i}, \alpha, \lambda}(t_i, x') c^{\frac{t_i}{\lambda}} dZ_\alpha(x') : i = 1, \ldots, n \right)
\]

\[
= \left( c^{H_{t_i}} \int_{-\infty}^{\infty} G_{H_{t_i}, \alpha, \lambda}(t_i, x') dZ_\alpha(x') : i = 1, \ldots, n \right)
\]

\[
= \left( c^{H_{t_i}} X_{H_{t_i}, \alpha, \lambda}(t_i) : i = 1, \ldots, n \right),
\]

where \( \overset{d}{=} \) indicates equality in distribution. So that (4.66) holds. \( \square \)

We say that a stochastic process \( X(t), t \in I \), is stochastic H"older continuous of exponent \( \beta \in (0, \infty) \) if it holds

\[ \limsup_{t,v \in I, |t-v| \to 0} P(|X(t) - X(v)| \geq C|t-v|^{\beta}) = 0 \]

for a positive constant \( C \). It is obvious that if \( X(u) \) is stochastic H"older continuous of exponent \( \beta_1 \), then \( X(u) \) is stochastic H"older continuous of exponent \( \beta_2 \in (0, \beta_1] \).

The following proposition shows that LTFmSM is stochastic H"older continuous.

Proposition 6. There is a number \( C \), depending only on \( a, b, \lambda \) and \( H \), such that for all \( t, v \in \mathbb{R} \) and any \( y > 0 \),

\[ P \left( \left| X_{H, \alpha}(z, \lambda)(t) - X_{H, \alpha}(z, \lambda)(v) \right| \geq y \right) \leq \frac{C}{y^\alpha + y^\beta} (|t - v|^H_\alpha + |t - v|^H_\beta) \quad (4.67) \]

In particular, (4.67) implies that for any \( \beta \in (0, \alpha) \) and all \( t, v \) satisfying \( |t - v| \leq 1 \),

\[ P \left( \left| X_{H, \alpha}(z, \lambda)(t) - X_{H, \alpha}(z, \lambda)(v) \right| \geq |t - v|^{\beta} \right) \leq C |t - v|^{H(a-\beta)}, \quad (4.68) \]

which implies that \( X_{H, \alpha}(z, \lambda)(t) \) is stochastic H"older continuous of exponent \( \beta \in (0, \alpha) \).

Proof. By Proposition 2.3 of Falconer and Liu [3], it follows that for any \( y > 0 \),

\[ P \left( \left| X_{H, \alpha}(z, \lambda)(t) - X_{H, \alpha}(z, \lambda)(v) \right| \geq y \right) \]

\[ \leq C_1 \int_{-\infty}^{\infty} \left| G_{H, \alpha}(z, \lambda)(t, x) - G_{H, \alpha}(z, \lambda)(v, x) \right|^{\alpha(x)} dx \]

\[ \leq \frac{C_1}{y^{\alpha} + y^\beta} \int_{-\infty}^{\infty} \left| G_{H, \alpha}(z, \lambda)(t, x) - G_{H, \alpha}(z, \lambda)(v, x) \right|^{\alpha(x)} dx, \quad (4.69) \]

where \( G_{H, \alpha}(z, \lambda)(t, x) \) is defined by (2.2). Without loss of generality, we assume that \( t \geq v \). Then

\[ \int_{-\infty}^{\infty} \left| G_{H, \alpha}(z, \lambda)(t, x) - G_{H, \alpha}(z, \lambda)(v, x) \right|^{\alpha(x)} dx = I_1 + I_2, \quad (4.70) \]
Next, we estimate

\[ I_1 = \int_{-\infty}^{t} \left| G_{H,\alpha(x),\lambda}(t, x) - G_{H,\alpha(x),\lambda}(v, x) \right|^{\alpha(x)} dx, \]

\[ I_2 = \int_{v}^{t} e^{-\lambda(x)(t-x)} (t-x)^{H\alpha(x)-1} dx. \]

Using the inequality \(|x+y|^\alpha \leq 2^\alpha (|x|^\alpha + |y|^\alpha)\) for all \(x, y \in \mathbb{R}\) and any \(\alpha > 0\), we have

\[ I_1 \leq 4(I_{11} + I_{12}), \]

where

\[ I_{11} = \int_{-\infty}^{0} \left| e^{-\lambda(t-x)}(t-x)^{H-\frac{1}{\alpha(x)}} - e^{-\lambda(t-x)}(v-x)^{H-\frac{1}{\alpha(x)}} \right|^{\alpha(x)} dx, \]

\[ I_{12} = \int_{-\infty}^{0} \left| e^{-\lambda(t-x)}(v-x)^{H-\frac{1}{\alpha(x)}} - e^{-\lambda(v-x)}(v-x)^{H-\frac{1}{\alpha(x)}} \right|^{\alpha(x)} dx. \]

Let \(h = t - v\). We deduce the following estimation of \(I_{11}\):

\[
I_{11} \leq \int_{-\infty}^{0} \left| (t-x)^{H-\frac{1}{\alpha(x)}} - (v-x)^{H-\frac{1}{\alpha(x)}} \right|^{\alpha(x)} dx \\
\leq \int_{-\infty}^{0} \left| (h + v - x)^{H-\frac{1}{\alpha(x)}} - (v-x)^{H-\frac{1}{\alpha(x)}} \right|^{\alpha(x)} dx \\
= \int_{-\infty}^{0} \left| \left(1 + \frac{v - x}{h}\right)^{H-\frac{1}{\alpha(x)}} - \left(1 + \frac{v - x}{h}\right)^{H-\frac{1}{\alpha(x)}} \right|^{\alpha(x)} h^{H\alpha(x)-1} dx \\
= \int_{0}^{\infty} \left| (1 + u)^{H-\frac{1}{\alpha(\lambda u)}} - u^{H-\frac{1}{\alpha(\lambda u)}} \right|^{\alpha(v-hu)} h^{H\alpha(x)-1} du \\
\leq \int_{0}^{\infty} \left| (1 + u)^{H-\frac{1}{\alpha(\lambda u)}} - u^{H-\frac{1}{\alpha(\lambda u)}} \right|^{\alpha(v-hu)} du \left( h^{Ha} + h^{Hb} \right). \tag{4.71}
\]

Next, we estimate \(I_{12}\). Notice that \(|e^{-x} - e^{-y}| \leq |x - y|\) for \(x, y > 0\). Substitute \(u = v - x\) to see that for \(\lambda > 0\),

\[
I_{12} = \int_{-\infty}^{v} \frac{1}{(\lambda \alpha(x))^{H\alpha(x)-1}} (\lambda \alpha(x)(v-x))^{H\alpha(x)-1} e^{-\lambda \alpha(x)(v-x)} |e^{-\lambda(t-v)} - 1|^{\alpha(x)} dx \\
\leq C_{12} \int_{-\infty}^{v} (\lambda \alpha(x)(v-x))^{H\alpha(x)-1} e^{-\lambda \alpha(x)(v-x)} \min \left\{ (t-v)^{\alpha(x)}, 1 \right\} dx \\
\leq C_{12} \min \left\{ |t-v|^{\alpha}, 1 \right\} \int_{0}^{v} (\lambda \alpha(v-u)u)^{H\alpha(v-u)-1} e^{-\lambda \alpha(v-u)u} du \\
\leq C_{12} \min \left\{ |t-v|^{\alpha}, 1 \right\} \int_{0}^{v} \max_{\alpha \in [a,b]} \left\{ (\lambda \alpha u)^{H\alpha-1} e^{-\lambda \alpha u} \right\} du \\
\leq C_{13} \min \left\{ |t-v|^{\alpha}, 1 \right\}. \tag{4.72}
\]
It is obvious that if $\lambda = 0$, then $I_{12} = 0$, and thus (4.72) holds obviously for all $\lambda \geq 0$. By simple calculations, we get

$$I_2 \leq \int_0^t (t-x)^{H_\beta-1} \, dx$$

$$\leq \begin{cases} \int_0^t (t-x)^{H_\beta-1} \, dx & \text{if } t-v \leq 1 \\ \int_v^{t-1} (t-x)^{H_\beta-1} \, dx + \int_v^1 (t-x)^{H_\beta-1} \, dx & \text{if } t-v > 1 \end{cases}$$

$$\leq C(t^{|t-v|^{H_\beta}} + |t-v|^{H_\beta}).$$

Returning to (4.70), we obtain

$$\int_\infty^\infty |G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x)|^{\alpha(x)} \, dx$$

$$\leq C_5 \left( |t-v|^{H_\alpha} + |t-v|^{H_\beta} + \min \left\{ |t-v|^{\alpha}, 1 \right\} \right)$$

$$\leq C_6 \left( |t-v|^{H_\alpha} + |t-v|^{H_\beta} \right).$$

Hence, for $y > 0$,

$$P \left( \left| X_{H,\alpha(x),\lambda}(t) - X_{H,\alpha(x),\lambda}(v) \right| \geq y \right) \leq \frac{C_7}{y} \left| t-v \right|^{H_\alpha} + |t-v|^{H_\beta}. \quad (4.74)$$

This completes the proof of Proposition 6.

The following proposition shows that LTmFSM is also stochastic Hölder continuous.

**Proposition 7** Let $\lambda > 0$. There is a number $C$ depending only on $a, b, \alpha$ and $\lambda$, such that for all $z > 0$,

$$P \left( \left| X_{H,\alpha(x),\lambda}(t) - X_{H,\alpha(x),\lambda}(s) \right| \geq z \right) \leq \frac{C}{z^\alpha} \left| t-s \right|^{H_\alpha_h} + \left| H_\alpha_h - H_\alpha \right|^\alpha \quad (4.75)$$

for all $t, s \in \mathbb{R}$ satisfying $t \geq s$. In particular, if $H_\alpha$ is $\gamma$-Hölder continuous, that is $\left| H_\alpha - H_\alpha \right| \leq C |t-s|^{\gamma}$, then (4.73) implies that for any $\beta \in (0, \min\{a, \gamma\})$ and all $t, s \in \mathbb{R}$ satisfying $|t-s| \leq 1$,

$$P \left( \left| X_{H,\alpha}(t) - X_{H,\alpha}(s) \right| \geq |t-s|^{\beta} \right) \leq C \left( |t-s|^{\alpha(a-\beta)} + |t-s|^{\alpha(\gamma-\beta)} \right). \quad (4.76)$$

which implies that $X_{H,\alpha}(t)$ is stochastic Hölder continuous of exponent $\beta \in (0, \min\{a, \gamma\})$.

**Proof.** By the Billingsley inequality (cf. p. 47 of [3]), it follows that for $z > 0$,

$$P \left( \left| X_{H,\alpha}(t) - X_{H,\alpha}(s) \right| \geq z \right)$$

$$\leq \frac{z}{2} \int_{-2z}^{2z} \left( 1 - \mathbb{E} \left[ e^{yz(X_{H,\alpha}(t) - X_{H,\alpha}(s))} \right] \right) dy$$

$$= \frac{z}{2} \int_{-2z}^{2z} \left( 1 - \exp \left\{ - \int_\infty^\infty \left| y \left( G_{H,\alpha}(t,x) - G_{H,\alpha}(s,x) \right) \right|^{\alpha} dx \right\} \right) dy$$

$$\leq \frac{z}{2} \int_{-2z}^{2z} \int_\infty^\infty \left| y \left( G_{H,\alpha}(t,x) - G_{H,\alpha}(s,x) \right) \right|^{\alpha} dx \, dy$$
Using the inequality for any $\alpha > 0$, 

$$|x + y + z|^\alpha \leq 3^\alpha (|x|^\alpha + |y|^\alpha + |z|^\alpha), \quad x, y, z \in \mathbb{R},$$

we have

$$\int_{-\infty}^{\infty} |G_{H_t,\alpha,\lambda}(t, x) - G_{H_t,\alpha,\lambda}(s, x)|^\alpha dx \leq 3^\alpha (I_1 + I_2 + I_3), \quad (4.78)$$

where

$$I_1 = \int_{-\infty}^{\infty} \left| e^{-\lambda(t-x)}+ (t - x)^{H_t - \frac{1}{\alpha}} - e^{-\lambda(s-x)+ (s - x)^{H_t - \frac{1}{\alpha}}} \right|^\alpha dx,$$

$$I_2 = \int_{-\infty}^{\infty} \left| e^{-\lambda(s-x)+ (s - x)^{H_t - \frac{1}{\alpha}}} - e^{-\lambda(s-x)+ (s - x)^{H_t - \frac{1}{\alpha}}} \right|^\alpha dx,$$

$$I_3 = \int_{-\infty}^{\infty} \left| e^{-\lambda(-x)+ (-x)^{H_t - \frac{1}{\alpha}}} - e^{-\lambda(-x)+ (-x)^{H_t - \frac{1}{\alpha}}} \right|^\alpha dx.$$ 

It is easy to see that

$$I_1 \leq 2^\alpha (I_{11} + I_{12}), \quad (4.79)$$

where

$$I_{11} = \int_{-\infty}^{\infty} \left| e^{-\lambda(t-x)}+ (t - x)^{H_t - \frac{1}{\alpha}} - e^{-\lambda(t-x)+ (s - x)^{H_t - \frac{1}{\alpha}}} \right|^\alpha dx,$$

$$I_{12} = \int_{-\infty}^{\infty} \left| e^{-\lambda(t-x)}+ (s - x)^{H_t - \frac{1}{\alpha}} - e^{-\lambda(s-x)+ (s - x)^{H_t - \frac{1}{\alpha}}} \right|^\alpha dx.$$ 

Let $h = t - s > 0$. Notice that $\left| (1 + u)^{H_t - \frac{1}{\alpha}} - u^{H_t - \frac{1}{\alpha}} \right| \leq 2\beta u^{H_t - \frac{1}{\alpha} - 1}, u \to \infty$. Then we deduce the following estimation of $I_{11}$:

$$I_{11} = \int_{-\infty}^{t} e^{-\lambda(t-x)}\left| (t - x)^{H_t - \frac{1}{\alpha}} - (s - x)^{H_t - \frac{1}{\alpha}} \right|^\alpha dx$$

$$= \int_{-\infty}^{t} e^{-\lambda(t-x)}\left| (1 + \frac{s - x}{h})^{H_t - \frac{1}{\alpha}} - (\frac{s - x}{h})^{H_t - \frac{1}{\alpha}} \right|^\alpha h^{H_t - \frac{1}{\alpha} - 1} dx$$

$$= \int_{-1}^{\infty} e^{-\lambda h(1+u)}\left| (1 + u)^{H_t - \frac{1}{\alpha}} - u^{H_t - \frac{1}{\alpha}} \right|^\alpha h^{H_t - \frac{1}{\alpha}} du$$

$$\leq \int_{-1}^{\infty} e^{-\lambda h(1+u)}\left| (1 + u)^{H_t - \frac{1}{\alpha}} - u^{H_t - \frac{1}{\alpha}} \right|^\alpha du \left| h^{H_t - \frac{1}{\alpha}} \right|$$

$$\leq C_{11} h^{H_t - \frac{1}{\alpha}} = C_{11} |t - s|^{H_t - \frac{1}{\alpha}}. \quad (4.80)$$
Next, consider the item $I_{12}$. Substitute $u = s - x$ and then $w = \lambda$ to see that for $\lambda > 0$,

$$I_{12} = \int_{-\infty}^{0} (s - x)^{\alpha H_t - 1} e^{-\lambda (s - x)} \left| e^{-\lambda t (t-s)} - 1 \right| dx \leq \int_{-\infty}^{0} (s - x)^{\alpha H_t - 1} e^{-\lambda (s - x)} dx \min \left\{ (t-s)^{\alpha}, 1 \right\}$$

$$= \int_{0}^{\infty} u^{\alpha H_t - 1} e^{-\lambda u} du \min \left\{ (t-s)^{\alpha}, 1 \right\} \leq C_{12} \min \left\{ |t-s|^{\alpha}, 1 \right\}, \tag{4.81}$$

where the second line of the last inequalities follows by the inequality $|e^{-x} - e^{-y}| \leq |x - y|$ for all $x, y \geq 0$. It is obvious that if $\lambda = 0$, then $I_{12} = 0$. Thus (4.81) also holds for $\lambda = 0$. Combining (4.79), (4.80) and (4.81) together, we get

$$I_1 \leq C_1 \left( |t-s|^{\alpha H_t} + \min \left\{ |t-s|^{\alpha}, 1 \right\} \right). \tag{4.82}$$

In the sequel, we give the estimations of $I_2$ and $I_3$. Without loss of generality, we assume that $H_t \geq H_s$. By some simple calculations, we get

$$I_2 = \int_{-\infty}^{\infty} e^{-\lambda (s-x)} \left| (s-x)^{H_t - \frac{1}{2}} - (s-x)^{H_s - \frac{1}{2}} \right| dx$$

$$\leq \int_{0}^{\infty} e^{-\lambda u} \left| u^{H_t - u^{H_s}} - 1 \right| du \leq C_2 |H_t - H_s|^\alpha, \tag{4.83}$$

where $H_\theta \in [H_s, H_t]$. Similarly, we have

$$I_3 \leq C_3 |H_t - H_s|^\alpha. \tag{4.84}$$

Combining the inequalities (4.78), (4.82), (4.83) and (4.84) together, we obtain

$$\int_{-\infty}^{\infty} \left| G_{H_t, \alpha, \lambda}(t, x) - G_{H_s, \alpha, \lambda}(s, x) \right|^\alpha dx \leq C_4 \left( |t-s|^{\alpha H_t} + \min \left\{ |t-s|^{\alpha}, 1 \right\} + |H_t - H_s|^\alpha \right)$$

$$\leq C_5 \left( |t-s|^{\alpha H_t} + |H_t - H_s|^\alpha \right). \tag{4.85}$$

Returning to (4.77), we get for $z > 0$,

$$P \left( \left| X_{H_t, \alpha, \lambda}(t) - X_{H_s, \alpha, \lambda}(s) \right| \geq z \right) \leq \frac{C_6}{z^\alpha} \left( |t-s|^{\alpha H_t} + |H_t - H_s|^\alpha \right).$$

This completes the proof of Proposition 4. \qed
5 Absolute moments

We estimate the absolute (incremental) moments of the LTFmSM.

**Proposition 8** If $0 < p < a$, then there is a number $C_1$, depending only on $p, a, b, \lambda$ and $H$, such that for all $t, v \in \mathbb{R}$,

$$E\left[|X_{H,\alpha}(t) - X_{H,\alpha}(v)|^p\right] \leq C_1 \left(|t - v|^{Ha} + |t - v|^{Hb}\right). \quad (5.86)$$

Moreover, it holds

$$\lim_{p \to a^-} \left(a - p\right)E\left[|X_{H,\alpha}(t) - X_{H,\alpha}(v)|^p\right] \leq C_2 \left(|t - v|^{Ha} + |t - v|^{Hb}\right), \quad (5.87)$$

where the number $C_2$ does not depend on $p$.

**Proof.** Using Proposition 6, we deduce that

$$E\left[|X_{H,\alpha}(t) - X_{H,\alpha}(v)|^p\right] = p \int_0^\infty y^{p-1}P\left(|X_{H,\alpha}(t) - X_{H,\alpha}(v)| \geq y\right)dy \leq C_1 p \left(\int_0^1 y^{p-1}dy + \int_1^\infty y^{p-1-a}dy\right) \left(|t - v|^{Ha} + |t - v|^{Hb}\right) \leq C_2 \left(1 + \frac{p}{a - p}\right) \left(|t - v|^{Ha} + |t - v|^{Hb}\right). \quad (5.88)$$

Moreover, the last inequality implies that

$$\lim_{p \to a^-} \left(a - p\right)E\left[|X_{H,\alpha}(t) - X_{H,\alpha}(v)|^p\right] \leq aC_2 \left(|t - v|^{Ha} + |t - v|^{Hb}\right). \quad (5.89)$$

This completes the proof of Proposition 8.

The next proposition gives an estimate for the absolute (incremental) moment of the LTmFSM.

**Proposition 9** If $0 < p < \alpha$, then there is a number $C$ depending only on $p, a, b$ and $H$, such that

$$E\left[|X_{H,\alpha}(t) - X_{H,\alpha}(s)|^p\right] \leq C \left(|t - s|^{\alpha H_t} + |H_t - H_s|^\alpha\right) \quad (5.90)$$

for all $t, s \in \mathbb{R}$ satisfying $t \geq s$. Moreover, it holds

$$\lim_{p \to a^-} \left(a - p\right)E\left[|X_{H,\alpha}(t) - X_{H,\alpha}(s)|^p\right] \leq C \left(|t - s|^{\alpha H_t} + |H_t - H_s|^\alpha\right) \quad (5.91)$$

for all $t, s \in \mathbb{R}$ satisfying $t \geq s$.

**Proof.** Using Proposition 7, we deduce that for all $t, s \in \mathbb{R}$ satisfying $t \geq s$,

$$E\left[|X_{H,\alpha}(t) - X_{H,\alpha}(s)|^p\right] = p \int_0^\infty y^{p-1}P\left(|X_{H,\alpha}(t) - X_{H,\alpha}(s)| \geq y\right)dy \leq C_1 p \left(\int_0^1 y^{p-1}dy + \int_1^\infty y^{p-1-a}dy\right) \left(|t - s|^{\alpha H_t} + |H_t - H_s|^\alpha\right) \leq C_2 \left(1 + \frac{p}{a - p}\right) \left(|t - s|^{\alpha H_t} + |H_t - H_s|^\alpha\right). \quad (5.92)$$
which gives the desired inequalities. This completes the proof of Proposition 9.

For LFmSM, Le Guével and Lévy Véhel [9] have investigated the asymptotic behaviour of $E[|X(t + r) - X(t)|^\gamma]$, $r \to 0$, for some positive constant $\eta > 0$. The following proposition gives a result similar to the one of Le Guével and Lévy Véhel for LFmSM.

**Proposition 10** For each $t \in \mathbb{R}$ satisfying $H\alpha(t) \neq 1$ and all $\gamma \in (0, a)$, it holds

$$\lim_{r \to 0^+} \frac{E[|X_{H,\alpha}(x,\lambda)(t + r) - X_{H,\alpha}(x,\lambda)(t)|^\gamma]}{r^H} = F(\gamma, t),$$

where

$$F(\gamma, t) = \left( \int_{-\infty}^{\infty} \left( (1 - x)^{H - \frac{\alpha(t)}{\gamma}} - (-x)^{H - \frac{\alpha(t)}{\gamma}} \right) \alpha(t) \gamma d\gamma \right) \frac{2^{\gamma - 1} \Gamma \left( \frac{1 - \gamma}{\alpha(t)} \right)}{\gamma \int_0^\infty u^{-\gamma - 1} \sin^2(u)du}$$

and $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ is the gamma function.

**Proof.** Notice that for all $\gamma \in (0, a)$ and all $u \in [0, 1]$,

$$E\left[ \frac{X_{H,\alpha}(x,\lambda)(t + r) - X_{H,\alpha}(x,\lambda)(t)}{r^H} \right]^\gamma = \gamma \int_0^\infty z^{\gamma - 1} P \left( \left| \frac{X_{H,\alpha}(x,\lambda)(t + r) - X_{H,\alpha}(x,\lambda)(t)}{r^H} \right| \geq z \right) dz. \quad (5.93)$$

Notice that $X_{H,\alpha}(x,\lambda)(t)$ is localisable at $t$ to $X(t)$ defined by (8.108) (cf. Proposition 19 whose proof does not involve Proposition 10). Thus

$$P \left( \left| \frac{X_{H,\alpha}(x,\lambda)(t + r) - X_{H,\alpha}(x,\lambda)(t)}{r^H} \right| \geq z \right) \to P \left( |X(1)| \geq z \right), \quad r \to 0.$$ 

By Proposition 8 for $z$ large enough,

$$P \left( \left| \frac{X_{H,\alpha}(x,\lambda)(t + r) - X_{H,\alpha}(x,\lambda)(t)}{r^H} \right| \geq z \right) \leq C \frac{1}{z^a}.$$

Hence, by the Lebesgue dominated convergence theorem, we have

$$\lim_{r \to 0^+} E\left[ \frac{X_{H,\alpha}(x,\lambda)(t + r) - X_{H,\alpha}(x,\lambda)(t)}{r^H} \right]^\gamma = \gamma \int_0^\infty z^{\gamma - 1} P \left( |X(1)| \geq z \right) dz = E[|X(1)|^\gamma] = \left( \int_{-\infty}^{\infty} \left( (1 - x)^{H - \frac{\alpha(t)}{\gamma}} - (-x)^{H - \frac{\alpha(t)}{\gamma}} \right) \alpha(t) \gamma d\gamma \right) \frac{2^{\gamma - 1} \Gamma \left( \frac{1 - \gamma}{\alpha(t)} \right)}{\gamma \int_0^\infty u^{-\gamma - 1} \sin^2(u)du},$$

which gives the desired equality. We refer to Property 1.2.17 of Samorodnitsky and Taqqu [15] for the last line of the last equality.

\[\square\]
6 Sample path properties

When $H_a > 1$, the following proposition implies that every LTmFSM process has an a.s. Hölder continuous version.

**Proposition 11** If $H_a > 1$, then $X_{H, \alpha(x), \lambda}(t)$ has a continuous version such that its paths are almost surely $\beta$–Hölder continuous for any $0 < \beta < (H_a - 1)/a$.

**Proof.** By Proposition 8, we have for any $0 < p < a$ and all $t, v$ satisfying $|t - v| \leq 1$,

$$E\left[\left|X_{H, \alpha(x), \lambda}(t) - X_{H, \alpha(x), \lambda}(v)\right|^p\right] \leq C|t - v|^{H_a}.$$  \hspace{1cm} (6.94)

The Kolmogorov continuity theorem implies that $X_{H, \alpha(x), \lambda}(t)$ has a continuous version such that its paths are almost surely $\beta$–Hölder continuous for any $0 < \beta < (H_a - 1)/p$. Letting $p$ tend to $a$ completes the proof. \qed

Recall that a stochastic process $X(t), t \in T$, on a probability space $(\Omega, \mathcal{F}, P)$ is called separable if there is a countable set $T^* \subset T$ and an even $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 0$, such that for any closed set $F \subset \mathbb{R}$ we have

$$\{\omega : X(t) \in F, \, \forall t \in T^*\} \setminus \{\omega : X(t) \in F, \, \forall t \in T\} \subset \Omega_0.$$

See Chapter 9 of Samorodnitsky and Taqqu [15] for more details.

When $H_t \alpha < 1$ and $\lambda > 0$, the following proposition shows that every separable version of LTmFSM process has unbounded paths.

**Proposition 12** If $H_t \alpha < 1$ and $\lambda > 0$, then for any separable version of the TmFSM process, we have

$$P\left(\{\omega : \sup_{t \in [c, d]} |X_{H_t, \alpha, \lambda}(t)| = \infty\}\right) = 1.$$  \hspace{1cm} (6.95)

**Proof.** Consider the countable set $T^* := Q \cap [c, d]$, where $Q$ denotes the set of rational numbers. Since $T^*$ is dense in $[c, d]$, there exists a sequence of numbers $\{t_n\}_{n \in \mathbb{N}} \in T^*$, such that for any $x \in [c, d]$, $t_n \to x$ as $n \to \infty$. Therefore, it holds

$$f^*(T^*; x) := \sup_{t \in T^*} G_{H_t, \alpha, \lambda}(t, x) \geq \sup_{t_n \in T^*} G_{H_t, \alpha, \lambda}(t_n, x) =: f^*_n(T^*; x) = \infty, \quad n \to \infty.$$

Thus $\int_c^d f^*(T^*; x)dx = \infty$, and this contradicts Condition (10.2.14) of Theorem 10.2.3 in Samorodnitsky and Taqqu [15]. Therefore, the stochastic process $\{X_{H_t, \alpha, \lambda}\}$ does not have a version with bounded paths on the interval $(c, d)$, and this completes the proof. \qed

For LTmFSM process with $H_t \alpha > 1$, we have the following proposition.

**Proposition 13** Assume that $H_t$ is $\gamma$–Hölder continuous, $\gamma > 1/\alpha$, that is

$$|H_t - H_s| \leq C|t - s|^{\gamma}\hspace{1cm} (6.96)$$

for $t, s \in \mathbb{R}$ satisfying $|t - s| \leq 1$. If $\alpha \min\{H_t, \gamma\} > 1$, then $X_{H_t, \alpha, \lambda}(t)$ has a continuous version, such that its paths are almost surely $\beta$–Hölder continuous for any $0 < \beta < \min\{H_t, \gamma\} - 1/\alpha$. 

Proof. By Proposition 11 and (6.96), we have for any $0 < p < \alpha$ and all $t, s$ satisfying $|t - s| \leq 1$,
\[
E \left[ |X_{Ht,\alpha,\lambda}(t) - X_{Hs,\alpha,\lambda}(s)|^p \right] \leq C_1 \left( |t - s|^\alpha H_t + |H_t - H_s|^\alpha \right) \\
\leq C_2 \left( |t - s|^\alpha H_t + |t - s|^\alpha \gamma \right).
\]
The Kolmogorov continuity theorem implies that $X_{Ht,\alpha,\lambda}(t)$ has a continuous version, such that its paths are almost surely $\gamma$–Hölder continuous for any $0 < \beta < (\alpha \min\{H_t, \gamma\} - 1)/p$.
Let $p \to \alpha$. We completes the proof of Proposition 11. \qed

Denote by
\[
\tilde{H}_t(\omega) = \sup \left\{ \gamma : \lim_{r \to 0} \frac{|X_{Ht,\alpha,\lambda}(t + r, \omega) - X_{Ht,\alpha,\lambda}(t, \omega)|}{|r|^\gamma} = 0 \right\}
\]
the Hölder exponent of the LTmFSM $X_{Ht,\alpha,\lambda}(\cdot)$ at $t$.

Proposition 14 If $Ha > 1$, then $\tilde{H}_t(\omega) \geq H - 1/\alpha$.

Proof. It follows by Proposition 11. \qed

Let
\[
\tilde{H}_t(\omega) = \sup \left\{ \gamma : \lim_{r \to 0} \frac{|X_{Ht,\alpha,\lambda}(t + r, \omega) - X_{Ht,\alpha,\lambda}(t, \omega)|}{|r|^\gamma} = 0 \right\}
\]
be the Hölder exponent of the LTmFSM $X_{Ht,\alpha,\lambda}(\cdot)$ at $t$.

Proposition 15 Assume that $H_t$ is $\gamma$–Hölder continuous, $\gamma > 1/\alpha$, that is
\[
|H_t - H_s| \leq C|t - s|^\gamma
\]
for $t, s \in \mathbb{R}$ satisfying $|t - s| \leq 1$. If $\alpha \min\{H_t, \gamma\} > 1$, then $\tilde{H}_t(\omega) \geq \min\{H_t, \gamma\} - 1/\alpha$.

Proof. It follows by Proposition 13. \qed

7 Hölder continuity of quasi norm

Denote by
\[
\|X_{Ht,\alpha,\lambda}(t)\|_\alpha := \left\{ y > 0 : \frac{\int_{-\infty}^{\infty} \left| G_{Ht,\alpha,\lambda}(t, x) \right|^\alpha y dx}{y} = 1 \right\}
\]
for $t \in \mathbb{R}$. Then $\| \cdot \|_\alpha$ is a quasi norm. In particular, if $\alpha(x) \equiv p \geq 1$ for a constant $p$, then $\|X_{H,\alpha,\lambda}(t)\|_p$ is the $L^p(\mathbb{R})$ norm of $G_{H,\alpha,\lambda}(t, x)$. Moreover, when $\alpha(x) \equiv \alpha$ for a constant $\alpha \in (0, 2]$, then it holds
\[
\|X_{Ht,\alpha,\lambda}(t)\|_\alpha = \left( - \log E[e^{iX_{Ht,\alpha,\lambda}(t)}] \right)^{1/\alpha} = \left( \int_{-\infty}^{\infty} \left| G_{Ht,\alpha,\lambda}(t, x) \right|^\alpha dx \right)^{1/\alpha},
\]
see Meerschaert and Sabzikar 11.

The next proposition implies that the quasi norm of LTmFSM process is Hölder continuous in time $t$.\[\]
Proposition 16 There are two positive numbers \(c\) and \(C\), depending only on \(a, b, \lambda\) and \(H\), such that

\[
c |t - v|^{Hb/a} \leq \left\| X_{H, \alpha(x), \lambda}(t) - X_{H, \alpha(x), \lambda}(v) \right\| |_{\alpha} \leq C |t - v|^{Hb/b} \tag{7.98}
\]

for all \(t, v \in \mathbb{R}\) satisfying \(|t - v| \leq 1\).

Proof. Denote by \(\rho = \left\| X_{H, \alpha(x), \lambda}(t) - X_{H, \alpha(x), \lambda}(v) \right\| |_{\alpha}\). Assume that \(t > v\), and write

\[
\int_{-\infty}^{\infty} \left| G_{H, \alpha(x), \lambda}(t, x) - G_{H, \alpha(x), \lambda}(v, x) \right|^\alpha dx \\
\geq \int_{-\infty}^{t} e^{-\lambda \alpha(x)(t-x)}(t-x)^{H\alpha(x)-1} dx \\
\geq e^{-\lambda b(t-v)} \int_{v}^{t} (t-x)^{H\alpha(x)-1} dx \\
\geq e^{-\lambda b(t-v)} \int_{v}^{t} (t-x)^{Hb-1} dx \\
\geq e^{-\lambda b} \frac{1}{Hb} (t-v)^{Hb} \
\tag{7.99}
\]

uniformly for all \(t, v \in \mathbb{R}\) satisfying \(|t - v| \leq 1\). Therefore, we have

\[
1 = \int_{-\infty}^{\infty} \left| G_{H, \alpha(x), \lambda}(t, x) - G_{H, \alpha(x), \lambda}(v, x) \right|^\alpha dx \\
\geq \int_{-\infty}^{\infty} \left| G_{H, \alpha(x), \lambda}(t, x) - G_{H, \alpha(x), \lambda}(v, x) \right|^\alpha dx \min \left\{ \frac{1}{\rho a}, \frac{1}{\rho^b} \right\} \\
\geq e^{-\lambda b} \frac{1}{Hb} (t-v)^{Hb} \min \left\{ \frac{1}{\rho a}, \frac{1}{\rho^b} \right\}.
\]

The last inequality implies the lower bound of \(\rho\). By (7.73), we have

\[
\int_{-\infty}^{\infty} \left| G_{H, \alpha(x), \lambda}(t, x) - G_{H, \alpha(x), \lambda}(v, x) \right|^\alpha dx \leq C_1 |t - v|^{H a} \tag{7.100}
\]

uniformly for all \(t, v \in \mathbb{R}\) satisfying \(|t - v| \leq 1\). Then

\[
1 = \int_{-\infty}^{\infty} \left| G_{H, \alpha(x), \lambda}(t, x) - G_{H, \alpha(x), \lambda}(v, x) \right|^\alpha dx \\
\leq \int_{-\infty}^{\infty} \left| G_{H, \alpha(x), \lambda}(t, x) - G_{H, \alpha(x), \lambda}(v, x) \right|^\alpha dx \max \left\{ \frac{1}{\rho a}, \frac{1}{\rho^b} \right\} \\
\leq C_1 |t - v|^{H a} \max \left\{ \frac{1}{\rho a}, \frac{1}{\rho^b} \right\} \tag{7.101}
\]

whenever \(|t - v| \leq 1\). Inequality (7.101) implies the upper bound of \(\rho\). \(\square\)

When \(a = b\) and \(1/\alpha < H < 1\), Proposition 16 reduces to Lemma 4.2 of Meerschaert and Sabzikar [11]. Hence Proposition 16 can be regarded as a generalization of this lemma.

The next proposition implies that the quasi norm of LTmFSM process is Hölder continuous in time \(t\).
Proposition 17 There exist two positive numbers $c$ and $C$, depending only on $a, b, \lambda$ and $\alpha$, such that
\[
|t-s|^H_i \leq \left| X_{H_i,\alpha,\lambda}(t) - X_{H_i,\alpha,\lambda}(s) \right| \leq C \left( |t-s|^H_i + |H_t - H_s| \right) \tag{7.102}
\]
for all $t, s$ satisfying $0 \leq s \leq t \leq s + 1$.

Proof. From the proof of Proposition 7 we have
\[
\int_{-\infty}^{\infty} \left| G_{H,\alpha,\lambda}(t, x) - G_{H,\alpha,\lambda}(v, x) \right|^\alpha dx \leq C_1 \left( |t-s|^{\alpha H_i} + |H_t - H_s|^\alpha \right)
\]
\[
\leq 2C_1 \max \left\{ |t-s|^{\alpha H_i}, \ |H_t - H_s|^\alpha \right\}.
\]
Hence,
\[
\left| X_{H_i,\alpha,\lambda}(t) - X_{H_i,\alpha,\lambda}(s) \right| \leq (2C_1)^{1/\alpha} \left( |t-s|^{H_i} + |H_t - H_s| \right),
\]
which gives the desired upper bound in (7.102).

Next, consider the lower bound of $\left| X_{H_i,\alpha,\lambda}(t) - X_{H_i,\alpha,\lambda}(s) \right| \alpha$. Write
\[
\int_{-\infty}^{\infty} \left| G_{H,\alpha,\lambda}(t, x) - G_{H,\alpha,\lambda}(s, x) \right|^\alpha dx \geq \int_s^t \left| e^{-\lambda(t-x)}(t-x)^{H_i - \frac{1}{2}} \right|^\alpha dx
\]
\[
\geq e^{-\lambda(t-s)} \int_s^t (t-x)^{H_i - 1} dx
\]
\[
\geq e^{-\lambda(t-s)} \int_s^t (t-x)^{H_i - 1} dx
\]
\[
\geq e^{-\lambda(t-s)} \int_s^t (t-x)^{H_i - 1} dx \tag{7.103}
\]
uniformly for all $t, s$ satisfying $s \leq t \leq s + 1$. Therefore, we have
\[
\left| X_{H_i,\alpha,\lambda}(t) - X_{H_i,\alpha,\lambda}(s) \right| \geq e^{-\lambda \left( \frac{1}{b\alpha} \right)^{1/\alpha}} |t-s|^H_i,
\]
which gives the desired lower bound in (7.102). \qed

For $\alpha \in (0, 1]$, the next proposition shows that the upper bound of (7.102) is also exact.

Proposition 18 Assume $\alpha \in (0, 1]$ and $t_0 > 0$. Then there is a positive number $c$, depending only on $a, b, \lambda, t_0$ and $\alpha$, such that
\[
\left| X_{H_i,\alpha,\lambda}(t) - X_{H_i,\alpha,\lambda}(s) \right| \alpha \geq c \left( |t-s|^H_i + |H_t - H_s| \right) \tag{7.104}
\]
for all $t, s$ satisfying $t_0 \leq s \leq t \leq s + 1$.

Proof. If $|t-s|^H_i \geq c_1 |H_t - H_s|$ for some $c_1 > 0$, depending only on $a, b, \lambda, t_0$ and $\alpha$, then (7.102) implies (7.104). Thus we only need to consider the case that
\[
|H_t - H_s| \geq c_1 |t-s|^H_i \tag{7.105}
\]
for any $c_1 > 0$ depending only on $a, b, \lambda, t_0$ and $\alpha$. Applying the inequality
\[
|x|^\alpha - |y|^\alpha \leq |x-y|^\alpha, \quad x, y \in \mathbb{R} \text{ and } \alpha \in (0, 1],
\]
we have
\[
\int_{-\infty}^{\infty} \left| G_{H_1,\alpha}(t, x) - G_{H_1,\alpha}(s, x) \right|^\alpha dx \\
\geq \int_0^s \left| e^{-\lambda(t-x)}(t-x)^{H_1-\frac{\alpha}{\lambda}} - e^{-\lambda(s-x)}(s-x)^{H_1-\frac{\alpha}{\lambda}} \right|^\alpha dx \\
= \int_0^s \left| e^{-\lambda(t-x)}(t-x)^{H_1-\frac{\alpha}{\lambda}} - e^{-\lambda(s-x)}(s-x)^{H_1-\frac{\alpha}{\lambda}} + e^{-\lambda(s-x)}(s-x)^{H_1-\frac{\alpha}{\lambda}} - e^{-\lambda(s-x)}(s-x)^{H_1-\frac{\alpha}{\lambda}} \right|^\alpha dx \\
\geq \int_0^s \left| e^{-\lambda(t-x)}(t-x)^{H_1-\frac{\alpha}{\lambda}} - e^{-\lambda(s-x)}(s-x)^{H_1-\frac{\alpha}{\lambda}} \right|^\alpha dx \\
- \int_0^s \left| e^{-\lambda(t-x)}(t-x)^{H_1-\frac{\alpha}{\lambda}} - e^{-\lambda(s-x)}(s-x)^{H_1-\frac{\alpha}{\lambda}} \right|^\alpha dx.
\] (7.106)

By the mean value theorem and (4.82), the last inequality implies that for \( \alpha \in (0, 1] \) and all \( t, s \) satisfying \( 0 \leq s \leq t \leq s + 1 \),
\[
\int_{-\infty}^{\infty} \left| G_{H_1,\alpha}(t, x) - G_{H_1,\alpha}(s, x) \right|^\alpha dx \\
\geq \int_0^s e^{-\lambda(s-x)(s-x)^{\alpha H_1-1}}|\log(s-x)|^\alpha dx \left| H_t - H_s \right|^\alpha - C_{11}|t-s|^{\alpha H_1} \\
\geq c_00 \left| H_t - H_s \right|^\alpha - C_{11}|t-s|^{\alpha H_1},
\]
where \( c_00, C_{11} > 0 \) depending only on \( a, b, \lambda, t_0 \) and \( \alpha \). By (7.105), it follows that for \( \alpha \in (0, 1] \),
\[
\int_{-\infty}^{\infty} \left| G_{H_1,\alpha}(t, x) - G_{H_1,\alpha}(s, x) \right|^\alpha dx \geq c \left| H_t - H_s \right|^\alpha,
\]
where \( c > 0 \) depending only on \( a, b, \lambda, t_0 \) and \( \alpha \). Therefore (7.104) holds for \( \alpha \in (0, 1] \). \( \Box \)

8 Localisability and strong localisability

Recall that a stochastic process \( X(t), t \in \mathbb{R} \), is said to be \( h \)-localisable at \( u \) (cf. Falconer [4, 5]), with \( h > 0 \), if there exists a non-trivial process \( X'_u \), called the tangent process of \( X \) at \( u \), such that
\[
\lim_{r \to 0} X(u + rv) - X(u) \overset{fdd}{\to} X'_u(v),
\] (8.107)

where \( \overset{fdd}{\to} \) stands for convergence in finite-dimensional distributions.

The following proposition shows that LTFmSM is \( H \)-localisable.

**Proposition 19** For each \( u \in \mathbb{R} \) satisfying \( H\alpha(u) \neq 1 \), the LTFmSM process \( X_{H,\alpha}(x,t) \) is \( H \)-localisable at \( u \) with local form
\[
X(t) := \int_{-\infty}^{\infty} \left[ (t-x)^{H-\frac{\alpha(u)}{\lambda}} - (-x)^{H-\frac{\alpha(u)}{\lambda}} \right] dZ_{\alpha(u)}(x),
\] (8.108)

where \( dZ_{\alpha(u)}(x) \) is a symmetric \( \alpha(u) \)-stable random measure.
Proof. Given $u_1 < u_2 < \ldots < u_d$, denote

$$S_r(u_k) = \frac{X_{H,\alpha,\lambda}(t + r u_k) - X_{H,\alpha,\lambda}(t)}{r^H} \quad (8.109)$$

for $r > 0$ and $k = 1, \ldots, d$. Then

$$\mathbb{E}\left[e^i \sum_{k=1}^d \theta_k S_r(u_k)\right] = \exp \left\{ - \int_{-\infty}^{\infty} \left| \sum_{k=1}^d \theta_k r^{-H} \left( G_{H,\alpha(x),\lambda}(t + r u_k, x) - G_{H,\alpha(x),\lambda}(t, x) \right) \right|^{\alpha(x)} dx \right\}.$$

Let $x = t + rz$. It follows that

$$\int_{-\infty}^{\infty} \left| \sum_{k=1}^d \theta_k r^{-H} \left( G_{H,\alpha(x),\lambda}(t + r u_k, x) - G_{H,\alpha(x),\lambda}(t, x) \right) \right|^{\alpha(x)} dx$$

$$= \int_{-\infty}^{\infty} \left| \sum_{k=1}^d \theta_k \left( e^{-\lambda r (u_k - z)} (u_k - z)^{H-\frac{1}{\alpha(1+r)}} - e^{-\lambda r (-z)} (-z)^{H-\frac{1}{\alpha(1+r)}} \right) \right|^{\alpha(1+r)} dz.$$

Recall that $\alpha(x) \in [a, b]$ is a continuous function on $\mathbb{R}$. Thus

$$\lim_{r \to 0} \left| \sum_{k=1}^d \theta_k \left( (u_k - z)^{H-\frac{1}{\alpha(1+r)}} - (-z)^{H-\frac{1}{\alpha(1+r)}} \right) \right|^{\alpha(1)} = \left| \sum_{k=1}^d \theta_k \left( (u_k - z)^{H-\frac{1}{\alpha(1)}} - (-z)^{H-\frac{1}{\alpha(1)}} \right) \right|^{\alpha(1)}.$$

It is obvious that

$$\left| \sum_{k=1}^d \theta_k \left( e^{-\lambda r (u_k - z)} (u_k - z)^{H-\frac{1}{\alpha(1+r)}} - e^{-\lambda r (-z)} (-z)^{H-\frac{1}{\alpha(1+r)}} \right) \right|^{\alpha(1+r)}$$

$$\leq \sum_{k=1}^d |\theta_k|^{\alpha(1+r)} \left( (u_k - z)^{H\alpha(1+r)-1} - (-z)^{H\alpha(1+r)-1} \right)$$

$$\leq \sum_{k=1}^d \sup_{\alpha \in [a, b]} |\theta_k|^{\alpha} \left( (u_k - z)^{H\alpha-1} - (-z)^{H\alpha-1} \right)$$

is integrable on $\mathbb{R}$ with respect to $z$. The dominated convergence theorem implies that

$$\lim_{r \to 0} \mathbb{E}\left[e^i \sum_{k=1}^d \theta_k S_r(u_k)\right] = \exp \left\{ - \int_{-\infty}^{\infty} \left| \sum_{k=1}^d \theta_k \left( (u_k - z)^{H-\frac{1}{\alpha(1)}} - (-z)^{H-\frac{1}{\alpha(1)}} \right) \right|^{\alpha(1)} dz \right\}$$

$$= \mathbb{E}\left[e^i \sum_{k=1}^d \theta_k X(u_k)\right],$$

where $X(t)$ is defined by (8.108). By Lévy’s continuous theorem, we have

$$\lim_{r \to 0} S_r(u_k) \overset{fdd}{=} X(u_k).$$
Thus \(X_{H,\alpha_0}(x),\lambda(t), t \in \mathbb{R}\), is \(H\)-localisable to \(X(\cdot)\) defined by (8.108).

When \(\lambda = 0\) and \(1/a - 1/b < H < 1 + 1/b - 1/a\), Falconer and Liu proved that \(X_{H,\alpha_0}(0,t), t \in \mathbb{R}\), is \(H\)-localisable, see Proposition 4.3 of [5]. Notice that Proposition [9] does not assume that \(1/a - 1/b < H < 1 + 1/b - 1/a\), and it holds for any \(\lambda \geq 0\). Thus Proposition [10] extends the result of Falconer and Liu.

Recall that \(X(t), t \in \mathbb{R}\), is said to be \(h\)-strongly localisable at \(u\) to \(X'_{h}(v)\) with \(h > 0\) (cf. Falconer and Liu [8]), if the convergence in (8.107) occurs in distribution with respect to the metric of uniform convergence on bounded intervals, and \(X\) and \(X'_{h}\) have versions in \(C(R)\) (the space of continuous function on \(R\)).

The next proposition shows that when \(Ha > 1\), LTFmSM is \(H\)-strongly localisable.

**Proposition 20** When \(Ha > 1\), the process \(X_{H,\alpha_0}(x),\lambda(t)\) is \(H\)-strongly localisable at \(u\) to the LFSM defined by (8.108).

**Proof.** By Theorem 3.2 of Falconer and Liu [8], it is sufficient to prove that for each bounded interval \(J\), there is a positive \(r_0\) such that for any \(r \in (0, r_0)\),

\[
\int_{-\infty}^{\infty} \left| \frac{G_{H,\alpha_0}(x,\lambda(u+rt,x)) - G_{H,\alpha_0}(x,\lambda(u+rv,x))}{r^H} \right|^{\alpha(x)} dx \leq C |t-v|^\alpha H, \quad t, v \in J,
\]

where \(C\) is a constant. Indeed, by (4.73), for any \(0 < r \leq \min\{1/|t-v|, 1\}\), we have

\[
\int_{-\infty}^{\infty} \left| \frac{G_{H,\alpha_0}(x,\lambda(u+rt,x)) - G_{H,\alpha_0}(x,\lambda(u+rv,x))}{r^H} \right|^{\alpha(x)} dx \\
\leq \frac{1}{r^{Ha}} \int_{-\infty}^{\infty} \left| G_{H,\alpha_0}(x,\lambda(u+rt,x)) - G_{H,\alpha_0}(x,\lambda(u+rv,x)) \right|^{\alpha(x)} dx \\
\leq \frac{1}{r^{Ha}} C |rt - rv|^Ha = C |t-v|^\alpha H.
\]

This completes the proof of Proposition [20].

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