CHARACTERIZATION OF BOUNDED SMOOTH SOLUTIONS TO THE AXIALLY
SYMMETRIC NAVIER-STOKES EQUATIONS IN AN INFINITE PIPE WITH
NAVIER-SLIP BOUNDARY

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ABSTRACT

Bounded smooth solutions of the stationary axially symmetric Navier-Stokes equations in
an infinite pipe, equipped with the Navier-slip boundary condition, are considered in this paper.
Here “smooth” means the velocity is continuous up to second-order derivatives, and “bounded”
means the velocity itself and its gradient field are bounded. It is shown that such solutions
with zero flux at one cross section, must be swirling solutions: \( u = (-C x_2, C x_1, 0) \). A slight
modification of the proof will show that for an alternative slip boundary condition, solutions
will be identically zero.

Meanwhile, if the horizontal swirl component of the axially symmetric solution, \( u_\theta \), is
independent of the vertical variable \( z \), it is proven that such solutions must be helical solutions:
\( u = (-C_1 x_2, C_1 x_1, C_2) \). In this case, boundedness assumptions on solutions can be relaxed
extensively to the following growing conditions:

With respect to the distance to the origin, the vertical component of the velocity, \( u_z \), is sub-
linearly growing, the horizontal radial component of the velocity, \( u_r \), is exponentially growing,
and the swirl component of the vorticity, \( \omega_\theta \), is polynomially growing at any order.

Also, by constructing a counterexample, we show that the growing assumption on \( u_r \) is
optimal.

KEYWORDS: Navier-Stokes system, axially symmetric, bounded smooth, swirling solutions,
helical solutions.

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1 Introduction

The 3D stationary Navier-Stokes (NS) equations which describes the motion of stationary viscous incompressible fluids follows that

\[
\begin{cases}
\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \Delta \mathbf{u} = 0, \\
\nabla \cdot \mathbf{u} = 0,
\end{cases}
\text{ in } \mathcal{D} \subset \mathbb{R}^3.
\]  

Here \( \mathbf{u}(x) \in \mathbb{R}^3, p(x) \in \mathbb{R} \) represents the velocity and the scalar pressure respectively. In this paper, we consider the domain \( \mathcal{D} \) to be an infinitely long pipe, i.e.

\[
\mathcal{D} = \{ x : |x_h| < 1, x_3 \in \mathbb{R} \},
\]  

where \( x = (x_1, x_2, x_3), x_h = (x_1, x_2) \) and \( |x_h| = \sqrt{x_1^2 + x_2^2} \). The boundary condition will be equipped with the following:

**The total Navier-slip boundary condition & impermeable boundary condition:**

\[
\begin{cases}
(\mathbb{S} \mathbf{u} \cdot \mathbf{n})_T = 0, \\
\mathbf{u} \cdot \mathbf{n} = 0,
\end{cases}
\text{ \forall x \in \partial \mathcal{D}.} \tag{NSB}
\]

Here \( \mathbb{S} = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \) is the stress tensor, where \( (\nabla \mathbf{u})^T \) is the transpose of the Jacobian matrix \( \nabla \mathbf{u} \), and \( \mathbf{n} \) is the unit outer normal vector of \( \partial \mathcal{D} \). For a vector field \( \mathbf{v}, \mathbf{v}_T \) stands for its tangential part: \( \mathbf{v}_T := \mathbf{v} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} \). The condition (NSB) is from the general Navier-slip boundary condition and impermeable boundary condition which was introduced by Claude-Louis Naiver in 1820s \cite{22}:

\[
\begin{cases}
2(\mathbb{S} \mathbf{u} \cdot \mathbf{n})_T + \alpha \mathbf{u}_T = 0, \\
\mathbf{u} \cdot \mathbf{n} = 0.
\end{cases}
\]  

Here \( \alpha \geq 0 \) stands for the friction constant which may depend on various elements, such as the property of the boundary and the viscosity of the fluid. When \( \alpha = 0 \), boundary condition (1.3) turns to the total Navier-slip boundary (NSB), and when \( \alpha \to \infty \), boundary condition (1.3) degenerates into the no-slip boundary condition \( \mathbf{u} \equiv 0 \) on the boundary.

We write \( \mathcal{D} \) to be

\[
\mathcal{D} = \Sigma \times \mathbb{R},
\]
where the cross section $\Sigma \in \mathbb{R}^2$ is a unit disc. The domain considered here is a high-degree simplification of the following “distorted cylinder”, i.e.

$$\mathcal{D} = \hat{\Sigma} \times \mathbb{R},$$

where $\hat{\Sigma} \in \mathbb{R}^2$ is a simply connected bounded domain with smooth boundary.

Let $\mathcal{D}_0$ be a simply connected bounded domain with smooth boundary in $\mathbb{R}^3$ and $\mathcal{D}_0 \cap \mathcal{D} \neq \emptyset$. Existence problem of weak solutions in domain $\mathcal{D}_{\text{Union}} := \mathcal{D} \cup \mathcal{D}_0$ with Navier-slip boundary was addressed in [12] and regularity of solutions was also implied there. On the other hand, if $\mathcal{D}_0 \subset \mathcal{D}$ is an “obstacle” in $\mathcal{D}$, then the two dimensional existence problems and asymptotic behaviors of smooth solutions in domain $\mathcal{D}_{\text{Diff}} := \mathcal{D}\setminus \mathcal{D}_0$ with Navier-slip boundary condition are obtained in [20][21].

There have also been many pieces of literature in studying the existence, uniqueness and asymptotic behavior of the Navier-Stokes equations in a distorted pipe $\mathcal{D}_{\text{Union}}$ or $\mathcal{D}_{\text{Diff}}$ with no-slip boundary and with the Poiseuille flow as the asymptotic profile at infinity (Leray’s problem: Ladyzhenskaya [15, p. 77] and [16, p. 551]). The first remarkable contribution on the solvability of Leray’s problem is due to Amick [1, 2], who reduced the solvability problem to the resolution of a variational problem related to the stability of the Poiseuille flow in a flat cylinder. However, uniqueness and existence of solutions with large flux are left open. Ladyzhenskaya and Solonnikov [17] gave a detailed analysis of this problem on existence, uniqueness and asymptotic behavior of small-flux solutions. One may refer to [3, 10, 24] and references for more details on well-posedness, decay and far-field asymptotic analysis of solutions for Leray’s problem and related topics. A systematic review and study of Leray’s problem can be found in [7, Chapter XIII]. Recently Wang-Xie in [27] studied uniform structural stability of Poiseuille flows for the 3D axially symmetric solutions in the 3D pipe $\mathcal{D}$ and for the 2D solutions in a periodic strip, where a force term appears on the right hand of equation (1.1).

Compared to the no-slip boundary condition, this model with the Navier-slip boundary condition has different physical interpretations and gives different mathematical properties. Literature [20][12] only addressed the existence and regularity problems of weak solutions, but uniqueness was left open. The purpose of this paper can be viewed as an attempt in this aspect. We focus on problems in the regular infinite pipe $\mathcal{D}$ defined in (1.2), and the solution we considered will be axially symmetric and bounded. Existence problems of axially symmetric solutions in bounded multiply connected domains and exterior domains with prescribed boundary value can be found in [13][14]. See also a recent extension to the helical invariant solutions in [19].

In this paper, a family of bounded smooth helical solutions will be found, and we mainly concern with the characterization of bounded smooth axially symmetric solutions in $\mathcal{D}$ with the boundary condition [NSB]. The existence of solutions in $\mathcal{D}$ is evident due to the simplicity and speciality of the domain.

Our proof will be carried out in the framework of cylindrical coordinates $(r, \theta, z)$ which enjoys the following relationship with 3D Euclidian coordinates:

$$x = (x_1, x_2, x_3) = (r \cos \theta, r \sin \theta, z).$$

A stationary axially symmetric solution of the incompressible Navier-Stokes equations is given as

$$u = u_r(r, z)e_r + u_\theta(r, z)e_\theta + u_z(r, z)e_z,$$
where the basis vectors \( e_r, e_\theta, e_z \) are
\[
e_r = \left( \frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \quad e_\theta = \left( -\frac{x_2}{r}, \frac{x_1}{r}, 0 \right), \quad e_z = (0, 0, 1),
\]
while the components \( u_r, u_\theta, u_z \), which are independent of \( \theta \), satisfy
\[
\begin{align*}
(u_r \partial_r + u_z \partial_z) u_r - \left( \frac{u_\theta}{r} \right)^2 + \partial_r p &= \left( \Delta - \frac{1}{r^2} \right) u_r, \\
(u_r \partial_r + u_z \partial_z) u_\theta + \frac{u_\theta u_r}{r} &= \left( \Delta - \frac{1}{r^2} \right) u_\theta, \\
(u_r \partial_r + u_z \partial_z) u_z + \partial_z p &= \Delta u_z, \\
\nabla \cdot b &= \partial_r u_r + \frac{u_r}{r} + \partial_z u_z = 0,
\end{align*}
\]
where \( b = u_r e_r + u_z e_z \).

We can also compute the axi-symmetric vorticity \( \omega = \nabla \times \mathbf{u} = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z \) as follows
\[
\begin{align*}
\omega_r &= -\partial_z u_\theta, \quad \omega_\theta = \partial_z u_r - \partial_r u_z, \quad \omega_z = \left( \partial_r + \frac{1}{r} \right) u_\theta,
\end{align*}
\]
which satisfies
\[
\begin{align*}
(u_r \partial_r + u_z \partial_z) \omega_r - \left( \Delta - \frac{1}{r^2} \right) \omega_r - (\omega_r \partial_r + \omega_z \partial_z) u_r &= 0, \\
(u_r \partial_r + u_z \partial_z) \omega_\theta - \left( \Delta - \frac{1}{r^2} \right) \omega_\theta - \frac{u_r}{r} \omega_\theta - \frac{1}{r} \partial_z (u_\theta)^2 &= 0, \\
(u_r \partial_r + u_z \partial_z) \omega_z - \Delta \omega_z - (\omega_r \partial_r + \omega_z \partial_z) u_z &= 0.
\end{align*}
\]

In the cylindrical coordinates, the total Navier-slip boundary condition (NSB) is represented as
\[
\begin{align*}
\partial_r u_\theta - \frac{u_\theta}{r} &= 0, \\
\partial_r u_z &= 0, \quad \forall x \in \partial \mathcal{D}, \\
u_r &= 0,
\end{align*}
\]
whose computation is postponed to Appendix A.

Clearly direct calculation shows that, for arbitrarily constants \( C_1 \) and \( C_2 \), the following type of helical solutions
\[
u = C_1 r e_\theta + C_2 e_z \tag{1.7}
\]
solves (1.4) with the boundary condition (1.6). We further note that helical solutions (1.7), which is smooth in \( \mathcal{D} \), enjoys the following property:

\textit{The solution itself and its gradient are uniformly bounded in} \( \mathcal{D} \). \quad (*)

Thus a natural question raises:
Are helical solutions (1.7) the only smooth solutions of system (1.4) with the boundary condition (1.6) which enjoys property [\textcircled{6}]?

Before answering this question, we recall that the flux \( \Phi(z) \) at the cross section \( \Sigma \), which is defined by

\[
\Phi(z) := \int_\Sigma u(x_h, z) \cdot \nu dx_h,
\]

is a constant. Here \( \nu = e_z \) is the unit normal vector of \( \Sigma \) pointing to the positive \( z \) direction. Actually by using the divergence free condition of the velocity and the boundary condition (NSB)\(_2\), we have

\[
\frac{d}{dz} \Phi(z) = \int_\Sigma \frac{d}{dz} u_c(x_h, z) dx_h,
\]

\[
= - \int_\Sigma (\partial_{x_1} u_1 + \partial_{x_2} u_2)(x_h, z) dx_h
\]

\[
= - \int_{\partial \Sigma} (n_1 u_1 + n_2 u_2)(x_h, z) dS(x_h)
\]

\[
= - \int_{\partial \Sigma} (u \cdot n)(x_h, z) dS(x_h) = 0,
\]

where \( n = (n_1, n_2, 0) \) is the unit outer normal vector of \( \partial \mathcal{D} \). Then for any \( z \in \mathbb{R} \), we will denote \( \Phi(z) = \Phi \).

Our first main result in this paper gives a positive answer to the above question in the case that the flux \( \Phi \) is zero (corresponding to \( C_2 = 0 \)):

**Theorem 1.1.** Let \( u \) be a smooth solution of the axially symmetric Navier-Stokes equations (1.4) in the infinite pipe \( \mathcal{D} \) subject to the total Navier-slip boundary condition (NSB). Suppose \( u \) and its gradient are uniformly bounded in \( \mathcal{D} \), and

\[
\Phi = \int_\Sigma u_c(x_h, z) dx_h = 0. \tag{1.8}
\]

Then \( u \) must be the following type of swirling solutions:

\[
u = C_1 r e_\theta, \quad p = \frac{C_1^2 r^2}{2}.
\]
In the previous theorem, we only consider the case that the flux is zero. Besides, we observe that solutions (1.7) enjoy the following property:

Its swirl component \( u_\theta = C_1 r \) is independent of \( z \).

In the following theorem, we will conclude that if the horizontal swirl component of the axially symmetric solution is independent of \( z \), then (1.7) are the only group of smooth solutions to (1.4) subject to the boundary condition (1.6). Besides, Our boundedness assumptions on the velocity itself and gradient of which will be extensively relaxed to the following:

\[
\begin{align*}
|u_r(r, z)| &\leq C \text{Re}^{-\gamma_0}, \\
|u_z(r, z)| &\leq C|z|^{\delta_0}, \quad \text{uniformly with } r \in [0, 1], \\
|\omega_\theta(r, z)| &\leq C|z|^{M_0},
\end{align*}
\]

for any \( \gamma_0 < \alpha \approx 3.83171 \), \( \delta_0 < 1 \), and \( M_0 > 0 \). Here \( \alpha \) is the first positive root of the Bessel function \( J_1 \). We recall that \( J_\beta \) are canonical solutions of Bessel’s ordinary differential equation

\[
s^2 J_\beta''(s) + s J_\beta'(s) + (s^2 - \beta^2)J_\beta(s) = 0,
\]

which can be expressed by the following series form:

\[
J_\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \beta + 1)} \left( \frac{s}{2} \right)^{2n+\beta}.
\]

**Remark 1.2.** The reason why there is an \( r \) on the righthand of (1.9) is that for a smooth solution \( u \), in the cylindrical coordinates, \( u_r \) vanishes at \( r = 0 \). When doing Taylor expansion of \( u_r \) at \( r = 0 \) in the \( r \) direction, the zero order derivative term is missing, so it is reasonable to assume a one order \( r \) control on \( u_r \) for \( r \in [0, 1] \).

\[ \square \]

**Theorem 1.3.** Let \( u \) be a smooth solution of the axially symmetric Navier-Stokes equations (1.4) in the infinite pipe \( D \) subject to the total Navier-slip boundary condition (NSB). Suppose \( u \) satisfies (1.9) and \( u_\theta \) is independent of \( z \)-variable, then \( u \) must be of the following type of helical solutions:

\[
u = C_1 r e_\theta + C_2 e_z, \quad p(r, z) = C_1^2 r^2 / 2, \quad \forall c_1, c_2 \in \mathbb{R}.
\]

\[ \square \]

**Figure 2:** A helical solution in the infinite pipe \( D \)
Remark 1.4. We emphasize that the condition (1.9) above is sharp, because we have the following non-trivial counterexample which grows exactly as \( Ce^{\alpha z} \) when \( z \to \infty \):

\[
\begin{align*}
    u &= -\cosh(\alpha z)J_1(\alpha r)e_r + \sinh(\alpha z)J_0(\alpha r)e_z, \\
    p &= -\frac{1}{2} \left( \cosh^2(\alpha z)J_1'(\alpha r) + \sinh^2(\alpha z)J_0'(\alpha r) \right).
\end{align*}
\]

(1.12)

Here \( J_0, J_1 \) are Bessel functions defined in (1.11), while \( \alpha \approx 3.83171 \) is the smallest positive root of \( J_1 \). One can verify (1.12) being the solution of (1.4) with the boundary condition (1.6) by direct calculations. Here we leave the details to the interested reader. Unfortunately, our example here cannot reflect whether the growing assumptions in (1.9)\(_2\) and (1.9)\(_3\) are sharp.

\[\square\]

If we switch the Navier total slip condition to an alternative slip condition

\[
\begin{align*}
    (\nabla \times u) \times n &= 0, \\
    u \cdot n &= 0, \\
    \forall x \in \partial D,
\end{align*}
\]

(SB)

we still have a similar Liouville-type result (vanishing or constancy of solutions). Noting that a non-zero swirling solution does not enjoy (SB), we can conclude the following theorem:

**Theorem 1.5.** Let \( u \) be a smooth solution of the axially symmetric Navier-Stokes equations (1.4) in the infinite pipe \( D \) subject to the slip boundary condition (SB). Suppose \( u \) and its gradient are uniformly bounded in \( D \) and

\[
\Phi = \int_{\Sigma} u_z(x_h, z)dx_h = 0,
\]

then \( u \equiv 0 \).

\[\square\]

There has already been much literature studying Liouville-type results on the Navier-Stokes equations subject to various boundary conditions in various unbounded domains. Readers can refer to \([5, 6, 25, 26, 4, 23]\) and references therein for more Liouville-type results on the stationary Navier-Stokes equations. Moreover, our results in the above Theorems can be extended from the stationary case to the case of ancient solutions (backward global solutions) under suitable assumptions. However, for simplification of idea presenting, we omit this extension here and leave it to further works. See \([9]\) where the authors established a Liouville-type result for the ancient solution to the Navier-Stokes equations in the half plane with the no-slip boundary condition.

Liouville-type results of ancient solutions is connected to the regularity of solutions to the initial value problem of the non-stationary Navier-Stokes equations. Type I blow-up solutions of the Navier-Stokes initial value problem could not exist provided the Liouville-type result holds for bounded ancient solutions. See \([11, 8]\).

Before ending our introduction, we briefly outline our strategy for proofs of Theorem 1.1, Theorem 1.3 and Theorem 1.5. The most important step of proving Theorem 1.1 is to show that \( \mathbb{S}u \in L^2(D) \). In this process, \( L^\infty \) oscillation boundedness of the pressure in \( D_{2Z} \setminus D_Z \) (see (1.15) for the definition of \( D_Z \)) is essential, which will be presented in Section 2.2. Then combining the
square integrability of $Su$ and boundedness of the velocity together with its gradient, a trick of integration by parts and Poincaré inequality will indicate that $u_\varepsilon$ actually belongs to $L^2(D)$, which will result in the vanishing of $Su$. After analyzing the ingredients of $Su$, we finally conclude the validity of Theorem 1.1.

The idea for proof of Theorem 1.3 is completely different from that of Theorem 1.1. Under the assumption of Theorem 1.3 we will see that the quantity $\Omega := \omega_\theta/r$ satisfies a nice linear elliptic equation with an advection term. Under the growing assumption $(1.9)$ in domain $D$, by using the Nash-Moser iteration, we can show that actually $\Omega \equiv 0$, which indicates that $b = u_\varepsilon e_r + u_\varepsilon e_z$ must be harmonic in $D$. Then by constructing a barrier function, applying maximum principle and assumptions on $b$, one derives $u_\varepsilon \equiv 0$ and $u_\varepsilon$ must be a constant. From then on, $(1.4)_2$ is reduced to a linear ordinary differential equation of $u_\theta$, and we finally obtain $u_\theta = C_1 r$.

Proof of Theorem 1.5 shares many similarities with that of Theorem 1.1. Instead of showing $L^2$ boundedness and vanishing of $Su$, we will show that actually the vorticity belongs to $L^2(D)$ and vanishes. Then boundedness, smoothness and boundary condition will assure the vanishing of the velocity, which proves Theorem 1.5.

For the generalized Navier boundary condition $(1.3)$ in $D$, one can derive that in cylindrical coordinates, $(1.3)$ is equivalent to

$$
\begin{align*}
\partial_r u_\theta - \frac{u_\theta}{r} + \alpha u_\theta &= 0, \\
\partial_r u_\varepsilon + \alpha u_\varepsilon &= 0, \\
 u_\varepsilon &= 0,
\end{align*}
\quad \forall x \in \partial D. \tag{1.13}
$$

For given flux $\Phi := \int_{\Sigma} u_\varepsilon(x_h,z)dx_h = \text{const.}$, we can find a family of bounded smooth solutions satisfying $(1.4)$ with boundary condition $(1.13)$ as follows

$$
u = C_1 r \chi_{\{\alpha = 0\}} e_\theta + \frac{2(\alpha + 2)\Phi}{(\alpha + 4)\pi} \left(1 - \frac{\alpha}{\alpha + 2} r^2\right) e_\varepsilon, \quad p = \frac{C_1^2 r^2}{2} \chi_{\{\alpha = 0\}} - \frac{8\alpha \Phi}{(\alpha + 4)\pi} z, \tag{1.14}
$$

where $C_1$ is an arbitrary constant, and $\chi_{\{\alpha = 0\}}$ is the characteristic function on $\{\alpha = 0\}$, which means

$$
\chi_{\{\alpha = 0\}} = \begin{cases} 
1, & \alpha = 0, \\
0, & \alpha > 0.
\end{cases}
$$

When $\alpha \to +\infty$, the boundary condition $(1.13)$ becomes the no-slip boundary and the solution $(1.14)$ corresponds to the Hagen-Poiseuille flow in $D$. Uniqueness of Hagen-Poiseuille flow is still open. Our Theorem 1.1 states that in the case $\alpha = 0$ and $\Phi = 0$, we can show that $(1.14)$ is the unique bounded smooth solutions of $(1.4)$ with the boundary condition $(1.13)$. For general $0 \leq \alpha \leq +\infty$ and $\Phi$, we have the following conjecture.

**Conjecture 1.6.** Let $u$ be a smooth solution of the axially symmetric Navier-Stokes equations $(1.4)$ in the infinite pipe $D$ with the flux $\Phi$ and subject to the Navier-slip boundary condition $(1.3)$ for any $0 \leq \alpha \leq +\infty$. Suppose $u$ and its gradient are uniformly bounded, then the solution $u$ must be of the form $(1.14)$. 
Throughout this paper, $C_{a,b,c,...}$ denotes a positive constant depending on $a$, $b$, $c$, ... which may be different from line to line. For two quantities $A_1$, $A_2$, we denote $A_1 \vee A_2 = \max\{A_1, A_2\}$. Meanwhile, for $Z > 1$, we denote

$$D_Z := \{(r, \theta, z) : 0 \leq r < 1, 0 \leq \theta \leq 2\pi, -Z < z < Z\},$$

the truncated pipe with the length of $2Z$. We also apply $A \lesssim B$ to denote $A \leq CB$. Moreover, $A \simeq B$ means both $A \lesssim B$ and $B \lesssim A$.

This paper is arranged as follows: Section 2 is devoted to the proof of Theorem 1.1, and the proof of Theorem 1.3 could be found in Section 3. Proof of Theorem 1.5 will be presented in Section 4.

2 Proof of Theorem 1.1

In this section, we devote to proof of Theorem 1.1. In Section 2.1, we deduce a uniform bound of $\partial_z \omega_\theta$ by using classical energy estimate of (1.5) and the Moser’s iteration. Then it will be applied to derive the $L^\infty$ oscillation boundedness of the pressure in Section 2.2. Based on these preparations, we finish proving Theorem 1.1 in Section 2.3.

2.1 Uniform bound of $\partial_z \omega_\theta$

Denoting $g := \partial_z \omega_\theta$ and taking $z$-derivative on (1.5), one arrives

$$-\left(\Delta - \frac{1}{r^2}\right)g + b \cdot \nabla g = \nabla \cdot F,$$

(2.1)

where

$$F := -\omega_\theta \partial_z b + \left(\frac{u_r}{r} \omega_\theta + 2 \frac{u_\theta}{r} \partial_z u_\theta\right) e_z.$$

(2.2)

From (A.3), we see that $F \in L^\infty$ provided $u$ and $\nabla u$ are bounded. Meanwhile, we observe that from the boundary condition (1.6):

$$g \equiv 0, \quad \text{on} \quad \partial D.$$

Now we are ready to state the desired lemma of this section, with its proof based on the Moser’s iteration and energy estimate.

**Lemma 2.1.** Let $(u_r, u_\theta, u_z)$ be a smooth solution of (1.4) in $D$, subject to Navier total slip boundary condition (1.6) and $\omega_\theta$ be the swirl component of its vorticity. Then $\partial_z \omega_\theta$ is uniformly bounded in $D$.

**Proof.** For $q \geq 1$, we multiply (2.1) by $q g^{q-1}$ to get

$$-q g^{q-1} \Delta g + \frac{q}{r^2} g^q + b \cdot \nabla g^q = q g^{q-1} \nabla \cdot F.$$

(2.3)
Noting that
\[ \Delta g^q = \text{div} \left( q g^{q-1} \nabla g \right) = g g^{q-1} \Delta g + q(q - 1) g^{q-2} |\nabla g|^2, \]
one derives from (2.3) that
\[ - \Delta g^q + q(q - 1) g^{q-2} |\nabla g|^2 + \frac{q}{r^2} g^q b \cdot \nabla g^q = q g^{q-1} \nabla \cdot F. \quad (2.4) \]
Let \( \phi \) be a smooth cut-off function in \( z \) variable which is bounded up to its second-order derivatives, supported on \( [L-1, L+1] \) for some \( L \in \mathbb{R} \), which will be specified later. Using \( g^q \phi^2 \) as a test function to the equation (2.4) and noting that
\[ q(q - 1) \int_{\mathcal{D}} g^{2q-2} |\nabla g|^2 \phi^2 dx = \frac{q - 1}{q} \int_{\mathcal{D}} |\nabla g|^2 \phi^2 dx \geq 0, \]
one deduces
\[ \int_{\mathcal{D}} \nabla g^q \cdot (g^q \phi^2) dx + q \int_{\mathcal{D}} \frac{g^{2q} \phi^2}{r^2} dx + \int_{\mathcal{D}} b \cdot \nabla g^q (g^q \phi^2) dx \leq q \int_{\mathcal{D}} g^{2q-1} \nabla \cdot F \phi^2 dx. \quad (2.5) \]
We further denote \( f := g^q \) for convenience. First we see
\[ I_1 = \int_{\mathcal{D}} |\nabla (f \phi)|^2 dx - \int_{\mathcal{D}} f^2 |\nabla \phi|^2 dx. \]
Clearly, \( I_2 \geq 0 \). Using the divergence free property of \( b \), one finds \( I_3 \) satisfies
\[ I_3 = \frac{1}{2} \int_{\mathcal{D}} b \cdot \nabla f^2 \phi^2 dx = - \int_{\mathcal{D}} u \cdot \partial_x f^2 \phi^2 dx. \]
Applying integration by parts, one derives
\[ I_4 = - q(2q - 1) \int_{\mathcal{D}} g^{2q-2} \nabla g \cdot F \phi^2 dx - q \int_{\mathcal{D}} g^{2q-1} F \cdot \nabla \phi^2 dx \]
\[ \leq \frac{1}{2} \int_{\mathcal{D}} |\nabla (f \phi)|^2 dx + C q^2 \int_{\mathcal{D}} |F|^2 g^{2q-2} \phi^2 dx + \int_{\mathcal{D}} |g|^{2q-1} |F| \phi |\nabla \phi| dx. \]
Plugging estimates \( I_1-I_4 \) into (2.5), we conclude that
\[ \int_{\mathcal{D}} |\nabla (f \phi)|^2 dx \leq C \left( |\nabla \phi|_{L^\infty(\mathcal{D})} (||u||_{L^\infty(\mathcal{D})} + ||\nabla \phi||_{L^\infty(\mathcal{D})} + q^2) \right) \int_{\text{supp } \phi} (|g| \vee |F|_{L^\infty(\mathcal{D})})^{2q} dx. \]
Using the Sobolev imbedding and noting that \( \phi \) is supported on an interval whose length equals 2, one arrives
\[ \left( \int_{\{x: \phi = 1\}} (|g| \vee |F|_{L^\infty(\mathcal{D})})^{6q} dx \right)^{\frac{1}{6q}} \leq C \frac{1}{q^2} \left( |\nabla \phi|_{L^\infty(\mathcal{D})} (||b||_{L^\infty(\mathcal{D})} + ||\nabla \phi||_{L^\infty(\mathcal{D})} + q^2) \right)^{\frac{1}{q^2}} \times \left( \int_{\text{supp } \phi} (|g| \vee |F|_{L^\infty(\mathcal{D})})^{2q} dx \right)^{\frac{1}{2q}}. \quad (2.6) \]
Let $\frac{1}{2} \leq z_2 < z_1 \leq 1$ and assume $\phi$ is supported on the interval $[L - z_1, L + z_1]$, and $\phi \equiv 1$ on $[L - z_2, L + z_2]$. Meanwhile, the gradient of $\phi$ satisfies the following estimate:

$$\|\nabla \phi\|_{L^\infty} \leq \frac{C}{z_1 - z_2}.$$ 

Thus (2.6) indicates that

$$\left( \int_{\Sigma \cap [L - z_2, L + z_2]} (|g| \vee \|F\|_{L^\infty(D)})^{6q} \, dx \right)^{\frac{1}{6q}} \leq C \frac{1}{z_1} \left( (z_1 - z_2)^{-2} + C |\|b\|_{L^\infty(D)}^2 + q^2 \right)^{\frac{1}{z_1}} \times \left( \int_{\Sigma \cap [L - z_1, L + z_1]} (|g| \vee \|F\|_{L^\infty(D)})^{2q} \, dx \right)^{\frac{1}{2q}}. 

(2.7)$$

Now $\forall k \in \mathbb{N} \cup \{0\}$, we choose $q_k = 3^k$ and $z_{1k} = \frac{1}{2} + \left( \frac{1}{2} \right)^{k+1}$, $z_{2k} = z_{1,k+1} = \frac{1}{2} + \left( \frac{1}{2} \right)^{k+2}$, respectively. Denoting

$$\Psi_k := \left( \int_{\Sigma \cap [L - z_{1k}, L + z_{1k}]} (|g| \vee \|F\|_{L^\infty(D)})^{2q_k} \, dx \right)^{\frac{1}{2q_k}},$$

and iterating (3.14), it follows that

$$\Psi_{k+1} \leq C \frac{1}{3^k} \left( 4^{k+2} + C |\|b\|_{L^\infty(D)} + 3^{2k} \right)^{\frac{1}{3^k}} \Psi_k \leq \cdots \leq \left( C |\|b\|_{L^\infty(D)} \right)^{\frac{1}{3^k} \sum_{j=0}^{k} 3^{j/k}} \sum_{j=0}^{k} 3^j \Psi_0 \leq C |\|b\|_{L^\infty(D)} \Psi_0. $$

Performing $k \to \infty$, the above Moser’s iteration implies

$$\|g\|_{L^\infty(\Sigma \cap [L - 1/2, L + 1/2])} \leq C |\|b\|_{L^\infty(D)} \left( \|g\|_{L^2(\Sigma \cap [L - 1, L + 1])} + \|F\|_{L^\infty(D)} \right).$$

(2.8)

Finally, define another cut off function of $z$-variable $\tilde{\phi}$ who has bounded derivatives up to order 2, supported on $[L - 2, L + 2]$ and $\tilde{\phi} \equiv 1$ in $[L - 1, L + 1]$. Multiplying (1.5) by $\omega_0 \tilde{\phi}^2$ and integrating on $D$, one deduces

$$\int_D \left| \nabla (\omega_0 \tilde{\phi}) \right|^2 \, dx + \int_D \frac{\omega_0^2 \tilde{\phi}^2}{r^2} \, dx = \int_D \omega_0^2 \tilde{\phi}^2 \, dx - \int_D u_0 \partial_z \omega_0 \tilde{\phi}^2 \, dx = \int_D \frac{u_0}{r} \omega_0 \tilde{\phi}^2 \, dx - 2 \int_D \frac{u_0}{r} \partial_z u_0 \omega_0 \tilde{\phi}^2 \, dx.$$ 

By the representation of $\nabla u$ (A.2), one derives that

$$\|\nabla \omega_0\|_{L^2(\Sigma \cap [L - 1, L + 1])} \leq C |\|u, \nabla u\|_{L^\infty(D)}|.$$ 

(2.9)

Meanwhile, expression of $F$ (2.2) also indicates that

$$\|F\|_{L^\infty(D)} \leq C |\|u, \nabla u\|_{L^\infty(D)}|.$$ 

(10.10)

Substituting (2.9) and (2.10) in (2.8), one concludes that

$$\|g\|_{L^\infty(\Sigma \cap [L - 1/2, L + 1/2])} \leq C |\|u, \nabla u\|_{L^\infty(D)}|.$$ 

Noting that the right-hand side above is independent of $L$, thus we have derived the uniform boundedness of $g$ in $D$. 

□
2.2 Boundedness of the pressure

Based on the boundedness of $\partial_z \omega_0$, the $L^\infty$ oscillation bound of the pressure $p$ in $D_{2Z} \setminus D_Z$ can be obtained. The lemma is stated as follows:

**Lemma 2.2.** Under the same assumptions of Theorem 1.1, we have

$$\sup_{x \in D_{2Z} \setminus D_Z} |p(r, z) - p(0, Z)| \leq C,$$

where $C > 0$ is a uniform constant independent of $Z$.

**Proof.** We only consider $(D_{2Z} \setminus D_Z) \cap \{x : z > 0\}$ since the rest part is essentially the same. Let us start with the oscillation of the pressure along the $r$–axis. From (1.4) and the identity

$$\left( \Delta - \frac{1}{r^2} \right) u_r = \partial_z \omega_0,$$

one sees that

$$\partial_r p = \partial_z \omega_0 - (u_r \partial_r + u_z \partial_z) u_r + \frac{(u_\theta)^2}{r}.$$ (2.12)

For any $z \in \mathbb{R}$ and $r_1, r_2 \in [0, 1]$, we integrate (2.12) with $r$ on $[r_1, r_2]$ to derive

$$p(r_2, z) - p(r_1, z) = \int_{r_1}^{r_2} \partial_z \omega_0 dr - \int_{r_1}^{r_2} \left[ (u_r \partial_r + u_z \partial_z) u_r - \frac{(u_\theta)^2}{r} \right] dr$$

$$= \int_{r_1}^{r_2} \partial_z \omega_0(r, z) dr - \frac{1}{2} \left( u_r^2(r_2, z) - u_r^2(r_1, z) \right) - \int_{r_1}^{r_2} (u_r \partial_r u_r)(r, z) dr$$

$$+ \int_{r_1}^{r_2} \frac{u_\theta^2}{r^2} \partial_z \omega_0(r, z) dr.$$ (2.13)

Noting that

$$|\nabla u| \lesssim |\partial_r u_r| + |\partial_z u_r| + \frac{|u_r|}{r} + |\partial_r u_\theta| + |\partial_z u_\theta| + \frac{|u_\theta|}{r} + |\partial_r u_z| + |\partial_z u_z|,$$

which follows from (A.2), by the boundedness assumption of $u$ and $\nabla u$, together with the uniform bound of $\partial_z \omega_0$ in Section 2.1, one derives the oscillation bound from (2.13):

$$|p(r_2, z) - p(r_1, z)| \leq C(1 + \|\partial_z \omega_0\|_{L^\infty(D_{2Z})}) \leq C < \infty, \quad \forall r_1, r_2 \in [0, 1], \quad z \in \mathbb{R},$$

where $C$ is an absolute constant which is independent of $r_1, r_2$ and $z$. This finishes the oscillation estimate of $p(r, z)$ when $z$ is fixed. Now we turn to the oscillation of $p$ along the $z$–direction. (1.4) and identity

$$-\Delta u_z = \frac{1}{r} \partial_z (r \omega_0)$$

indicate that

$$\partial_z p = -\frac{1}{r} \partial_z (r \omega_0) - u_r \partial_z u_r - u_z \partial_z u_z.$$ (2.15)
Multiplying (2.15) by \( r \) and integrating it with respect to \( r \) on \((0, 1)\), one obtains
\[
\frac{d}{dz} \int_0^1 p(r, z) r dr = - \int_0^1 \partial_r (r \omega) dr - \int_0^1 (u_r \partial_r + u_z \partial_z) u_z r dr.
\]
(2.16)

Recalling the boundary condition (1.6), we find \( \omega \equiv 0 \) on \( \partial D \), which implies \( P_1 \equiv 0 \). On the other hand, using the divergence-free condition and integration by parts, we derive
\[
P_2 = - \int_0^1 \partial_r (ru_r) u_z r dr + \int_0^1 u_z \partial_z u_z r dr
\]
\[
= \int_0^1 \partial_z (ru_r) u_z r dr + \frac{1}{2} \frac{d}{dz} \int_0^1 u_z^2 r dr
\]
\[
= \frac{d}{dz} \int_0^1 u_z^2 r dr.
\]

(2.16) indicates
\[
\frac{d}{dz} \int_0^1 p(r, z) r dr = - \frac{d}{dz} \int_0^1 u_z^2(r, z) r dr.
\]
(2.17)

For any fixed \( z \in [Z, 2Z] \), we integrate the above identity from \( Z \) to \( z \). Then we have
\[
\left| \int_0^1 [p(r, z) - p(r, Z)] r dr \right| \leq \left| \int_0^1 [u_z^2(r, z) - u_z^2(r, Z)] r dr \right| \leq C.
\]
(2.18)

Recalling the mean value theorem, there exists \( r_\ast \in [0, 1] \) such that
\[
|p(r_\ast, z) - p(r_\ast, Z)| = \left| \frac{\int_0^1 [p(r, z) - p(r, Z)] r dr}{\int_0^1 r dr} \right| \leq C.
\]
(2.19)

This completes the oscillation of \( p \) parallel to the \( z \)-direction. To conclude the general oscillation of the pressure in the pipe, we apply the triangle inequality: for any \( r \in [0, 1] \), it follows that
\[
|p(r, z) - p(0, Z)| \leq |p(r, z) - p(r_\ast, z)| + |p(r_\ast, z) - p(r_\ast, Z)| + |p(r_\ast, Z) - p(r, Z)|.
\]

Plugging (2.14) and (2.19) into the above inequality, we finally arrive at
\[
|p(r, z) - p(0, Z)| \leq C,
\]
(2.20)

where \( C \) is an absolute positive constant independent of \( r, z \) and \( Z \). Thus (2.11) is proved by taking the supremum of (2.20) over \((r, z) \in [0, 1] \times ([-2Z, -Z] \cup [Z, 2Z]) \).
2.3 End of the proof

In this subsection, we will finish the proof of Theorem 1.1. Namely: If the flux $\Phi \equiv 0$, any smooth solution of (1.4) in an infinite pipe subject to the Navier total slip condition with the velocity and its first-order derivatives being bounded must be a swirling solution $u = C_1r e_\theta$.

The proof is divided into three steps: First we show the stress tensor $S = \frac{1}{2} (\nabla u + (\nabla u)^T)$ is globally $L^2$-integrable. Using a 2D Poincaré inequality and one insightful observation motivated by [29], we then find that $u_z$ also belongs to $L^2(D)$. Finally, we arrive at the vanishing of the stress tensor, which indicates the desired result in Theorem 1.1.

2.3.1 $L^2$ boundendness of stress tensor

Let $\psi : \mathbb{R} \to [0, 1]$ be a smooth cut-off function satisfying

$$\psi(l) = \begin{cases} 1, & l \in [-1, 1], \\ 0, & |l| \geq 2, \end{cases}$$

with $\psi'$ and $\psi''$ being bounded. Set

$$\psi_Z(z) := \psi \left( \frac{z}{Z} \right),$$

where $Z$ is a large positive number. Clearly the derivatives of the scaled cut-off function $\psi_Z$ enjoy

$$|\partial^n \psi_Z| \leq \frac{C}{Z^n}, \quad \text{for any } n \in \mathbb{N}. \quad (2.21)$$

Testing the equation

$$u \cdot \nabla u + \nabla p = \Delta u$$

with $u \psi_Z$, we have

$$\int_D \psi_Z u \Delta u dx = \int_D \psi_Z \left( u \cdot \nabla u + \nabla (p - p(0, Z)) \right) dx. \quad (2.22)$$

To proceed the further calculation in the cylindrical coordinates, we first note that the divergence free property of the velocity indicates

$$\sum_{i,j=1}^{3} \int_{D_{2Z}} \psi_Z u_i \partial_j u_j dx = \sum_{i,j=1}^{3} \int_{D_{2Z}} \psi_Z \partial_j (\partial_j u_i + \partial_i u_j) dx. \quad (2.23)$$

Below, we use the Einstein summation convention for repeated indexes. Using integration by parts, we further derive

$$\int_{D_{2Z}} \psi_Z u_i \partial_j (\partial_j u_i + \partial_i u_j) dx = -\int_{D_{2Z}} \partial_j \psi_Z u_i (\partial_j u_i + \partial_i u_j) dx - \int_{D_{2Z}} \psi_Z \partial_j u_i (\partial_j u_i + \partial_i u_j) dx$$

$$+ \int_{\partial D_{2Z}} \psi_Z u_i n_j (\partial_j u_i + \partial_i u_j) dS,$$

where $T_1, T_2, T_3$ are the boundaries of $D_{2Z}$. 


where \( n_j \) is the \( j \)-th component of the \( n \) – the unit outward normal vector field on \( \partial D_{2Z} \). Term \( T_1 \) could be split into two parts, the first half reads

\[
\int_{D_{2Z}} \partial_j \psi_Z u_i \partial_j u_i \, dx = \frac{1}{2} \int_{D_{2Z}} \partial_j \psi_Z \partial_j |u|^2 \, dx = \frac{1}{2} \int_{D_{2Z}} \partial_i \psi_Z \partial_i |u|^2 \, dx = -\frac{1}{2} \int_{D_{2Z}} \partial_z \psi_Z |u|^2 \, dx,
\]

where we have used the fact that \( \psi_Z \) is only \( z \)-dependent and supported in \([-2Z, 2Z]\). Similarly, the second half of \( T_1 \) follows that

\[
\int_{D_{2Z}} \partial_j \psi_Z u_i \partial_i u_j \, dx = \int_{\partial D_{2Z}} (u \cdot \nabla \psi_Z)(u \cdot n) \, dS - \int_{D_{2Z}} \partial_z \psi_Z u_z^2 \, dx.
\]

Due to the impermeable condition, one sees the first term on the right hand of the above equality is zero. Thus we conclude that

\[
T_1 = -\int_{D_{2Z}} \partial_z \psi_Z \left( \frac{1}{2} |u|^2 + u_z^2 \right) \, dx. \tag{2.24}
\]

Recalling that the stress tensor is defined by

\[
\mathbb{S} u = \frac{1}{2} (\partial_j u_i + \partial_i u_j)_{1 \leq i, j \leq 3},
\]

and using its symmetry, we arrive that

\[
T_2 = \frac{1}{2} \sum_{i,j=1}^{3} \int_{D_{2Z}} \psi_Z \left( \partial_j u_i + \partial_i u_j \right)^2 \, dx = 2 \int_{D_{2Z}} \psi_Z \mathbb{S} u^2 \, dx. \tag{2.25}
\]

Now applying the Navier-slip condition (NSB), one notes that

\[
n_j \left( \partial_j u_i + \partial_i u_j \right) = c(x)n_i,
\]

where \( c(x) \) is a scalar-valued function. Inserting this identity to \( T_3 \), we find

\[
T_3 = \int_{\partial D_{2Z}} c \psi_Z (u \cdot n) \, dS = 0. \tag{2.26}
\]

Next we come back to the right hand side of (2.22). Noting \( u \) is divergence-free, integration by parts shows

\[
\int_{D_{2Z}} u \psi_Z \left( u \cdot \nabla u + \nabla \left[ p - p(0, Z) \right] \right) \, dx = \int_{D_{2Z}} \psi_Z u_i \partial_i \left( \frac{1}{2} |u|^2 + \left[ p - p(0, Z) \right] \right) \, dx
\]

\[
= \int_{\partial D_{2Z}} \psi_Z (u \cdot n) \left( \frac{1}{2} |u|^2 + \left[ p - p(0, Z) \right] \right) \, dS - \int_{D_{2Z}} \partial_z \psi_Z u_z \left( \frac{1}{2} |u|^2 + \left[ p - p(0, Z) \right] \right) \, dx. \tag{2.27}
\]
Here $T_4$ above also vanishes by the stationary wall condition (1.6). Therefore we arrive that by plugging (2.24), (2.25), (2.26), (2.27) into (2.22)

\[
2 \int_{\mathcal{D}_{zz}} \psi_Z |\mathcal{S}u\|_x^2 \, dx = \int_{\mathcal{D}_{zz}} \partial_z^2 \psi_Z \left( \frac{1}{2} |u^2 + u_z^2| \right) \, dx + \int_{\mathcal{D}_{zz}} \partial_z \psi_Z u_z \left( \frac{1}{2} |u^2 + [p - p(0, Z)] \right) \, dx . \tag{2.28}
\]

Recalling (2.21), the bounds on the derivatives of scaled cut-off function $\psi_Z$, and the boundedness of $u$ and pressure, one derives from (2.28) that

\[
\int_{\mathcal{D}_{zz}} \psi_Z |\mathcal{S}u\|_x^2 \, dx \leq C |\mathcal{D}_{zz}| \left( Z^2 + Z^{-1} \right) \leq C,
\]

where $C$ is a universal constant depending only on the $L^\infty$ bound of $u$ and $\nabla u$ given in the assumption. After letting $Z \to \infty$, the above inequality shows the stress tensor is globally $L^2$-integrable:

\[
\int_\mathcal{D} |\mathcal{S}u\|_x^2 \, dx \leq C < \infty. \tag{2.29}
\]

2.3.2 $L^2$ boundedness of $u_z$

First we observe that $\|u_z\|_{L^2(\mathcal{D})}$ can be controlled by $\|\partial_z u_z\|_{L^2(\mathcal{D})}$ under the assumption that the flux $\Phi = 0$. Noting that

\[
\frac{1}{|\Sigma|} \int_{\Sigma} u_z(x_h, z) \, dx_h = \frac{1}{|\Sigma|} \Phi = 0,
\]

then we apply the one dimensional Poincaré inequality to derive

\[
\int_{\Sigma} |u_z(r, z)|^2 \, dx_h = \int_{\Sigma} \left| u_z(x_h, z) - \frac{1}{|\Sigma|} \int_{\Sigma} u_z(x_h, z) \, dx_h \right|^2 \, dx_h
\leq S_0^2 \int_{\Sigma} |\nabla_h u_z(x_h, z)|^2 \, dx_h = S_0^2 \int_{\Sigma} |\partial_z u_z(r, z)|^2 \, dx_h,
\]

where $\nabla_h = (\partial_1, \partial_2)$ and $S_0$ is independent of $z \in \mathbb{R}$. Integrating with $z$-variable on $\mathbb{R}$, we arrive

\[
\|u_z\|_{L^2(\mathcal{D})} \leq S_0 \|\partial_z u_z\|_{L^2(\mathcal{D})}. \tag{2.30}
\]

However, we cannot get the $L^2$ boundedness of $\partial_z u_z$ directly from (2.29). In fact, by the expression of the stress tensor (A.4), one only has the uniform $L^2$ bound of $(\partial_z u_z + \partial_z u_z)$. Nevertheless, one observes

\[
\int_{\mathcal{D}_{zz}} (\partial_z u_z)^2 \, dx = \int_{\mathcal{D}_{zz}} (\partial_z u_z + \partial_z u_z)^2 \, dx - \int_{\mathcal{D}_{zz}} (\partial_z u_z)^2 \, dx - 2 \int_{\mathcal{D}_{zz}} \partial_z u_z \partial_z u_z \, dx
\leq C + 2 \int_{\mathcal{T}_6} \partial_z u_z \partial_z u_z \, dx.
\]
Now it remains to derive the boundedness of $T_6$. With idea motivated by [29], after using the divergence free of $u$ and integration by parts, we deduce

$$
\int_{D_{2Z}} \partial_r u_r \partial_z u_z \, dx = -2\pi \int_{-2Z}^{2Z} \int_0^1 u_z \partial_z^2 (ru_z) \, dr \, dz = 2\pi \int_{-2Z}^{2Z} \int_0^1 u_z \partial_z^2 (ru_z) \, dr \, dz
$$

(2.31)

$$
= -\int_{D_{2Z}} (\partial_z u_z)^2 \, dx + 2\pi \left( \int_0^1 u_z(r, 2Z) \partial_z u_z(r, 2Z) \, dr - \int_0^1 u_z(r, -2Z) \partial_z u_z(r, -2Z) \, dr \right)
$$

$T_7$

Here $T_7$ can be bounded by the $L^2$ norm of stress tensor (2.29), while $T_8$ is controlled by the $L^\infty$ bounds of $u$ and $\nabla u$. Noting that $T_6$ is estimated uniformly with respect to $Z$. This, together with (2.30) implies

$$
\|u_z\|_{L^2(D)} \leq C < \infty.
$$

2.3.3 Vanishing of $\int_D |\mathcal{S}u|^2$ and finishing of the proof

Based on the $L^2$ bound of $u_z$, now we can estimate $T_5$ in (2.28) in an alternative approach, by using Hölder inequality:

$$
|T_5| \leq \sup_{x \in D_{2Z} \cap D_Z} \left\{ \frac{1}{2} |u|^2 + \left[ p - p(0, Z) \right] \frac{C}{Z} \|u_z\|_{L^2(D_{2Z})} |D_{2Z}|^{1/2} \right\} \leq CZ^{-1/2}.
$$

Thus we deduce from (2.28)

$$
\int_{D_{2Z}} \psi_Z |\mathcal{S}u|^2 \, dx \leq C|D_{2Z}|Z^{-2} + CZ^{-1/2} \to 0, \quad \text{as } Z \to +\infty,
$$

which indicates that

$$
\int_D |\mathcal{S}u|^2 \, dx = 0 \quad (2.32)
$$

by letting $Z \to \infty$. By the expression of $\mathcal{S}u$ (A.4), one finds

$$
u_r \equiv \partial_z u_\theta \equiv \partial_z u_z \equiv \partial_r u_z \equiv 0, \quad \partial_r u_\theta = \frac{u_\theta}{r}.
$$

The above estimates, together with the vanishing flux ($\Phi = 0$), indicate

$$
u_z \equiv 0, \quad \text{and} \quad u_\theta = Cr.
$$

Thus we conclude that $u = Cr e_\theta$, which is a swirling solution.

Let us give some discussions of Theorem 1.1 here. Based on our previous proof in this section, we naturally believe that if condition (1.8) is abandoned, then the solution must be a helical solution:

$$
u = C_1 r e_\theta + C_2 e_z.
$$

(2.33)
However, our method in this paper fails when we handle solutions with the flux $\Phi \neq 0$, because we can no longer apply the Poincaré inequality in Section 2.3.2 to derive the $L^2$ integrability of $u_z$. Meanwhile, if we denote
\[ c_0 := \frac{1}{|\Sigma|} \int_{\Sigma} u_z(x_b, z) dx_b = \frac{1}{|\Sigma|} \Phi, \]
then $u_z - c_0$ enjoys a similar Poincaré inequality as (2.30):
\[ ||u_z - c_0||_{L^2(D)} \leq S_0 ||\partial_z u_z||_{L^2(D)}, \]
which guarantees the $L^2$ boundedness of $u_z - c_0$. However, one additional term appears in $T_5$ of (2.28), which is:
\[ T_5' := c_0 \int_{D_2^z} \partial_z \psi_z \left( \frac{1}{2} |u|^2 + [p - p(0, Z)] \right) dx. \]
Without any integrability of the head pressure $\frac{1}{2} |u|^2 + [p - p(0, Z)]$, we can only show $T_5'$ is bounded, which results in
\[ \int_D \|u\|^2 dx < C < \infty. \]
However, we are unable to conclude $T_5' \to 0$ as $Z \to \infty$, thus vanishing of $\int_D \|u\|^2 dx$ can not be obtained. In fact, using integration by parts on $z$ in $T_5'$, we have
\[ T_5' = -c_0 \int_{D_2^z} \psi_z \partial_z \left( \frac{1}{2} |u|^2 + p \right) dx. \]
By following the argument in Section 2, one derives
\[ \int_D \|u\|^2 dx = -\lim_{Z \to \infty} \frac{c_0}{2} \int_D \psi_z \partial_z \left( \frac{1}{2} |u|^2 + p \right) dx \]
instead of (2.32). Recalling (2.17), one deduces that
\[ \int_D \|u\|^2 dx = -\lim_{Z \to \infty} \frac{c_0}{4} \int_D \psi_z \partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) dx. \tag{2.34} \]
Thus if $\partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) \in L^1(D)$ (or $\partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right)$ has a fixed sign), one concludes the following identity by Lebesgue’s dominated convergence theorem (or monotone convergence theorem):
\[ \int_D \|u\|^2 dx + \frac{c_0}{4} \int_D \partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) dx = 0. \tag{2.35} \]

At the moment, even with identities (2.34) and (2.35) for bounded (up to first-order derivatives) smooth axisymmetric solutions of stationary Navier-Stokes equations in $D$ subject to the total Navier-slip boundary condition in hand, we neither show the trivialness of $\mathbb{S} u$, nor find a nontrivial bounded solution apart from (2.33) which satisfies conditions of Theorem 1.1. Indeed, we leave characterization of the non-zero flux solutions in Conjecture 1.6.

Nevertheless, a direct observation of (2.35) indicates that: If $u$ is independent of $z$, then the right hand side of (2.35) vanishes and we can conclude $\mathbb{S} u \equiv 0$, and thus conclude that $u = C_1 r e_\theta + C_2 e_z$ as we desire. In the next section, we see that only $u_\theta$ being independent of $z$ is adequate for us to derive Theorem 1.3. Besides, the asymptotic assumptions of $u$ and its derivatives can be largely loosened.
Let us outline the proof at the beginning of this section: Under the assumptions in Theorem 1.3, our first step is showing $\omega_\theta \equiv 0$, which indicates $b = u_re_r + u_ze_z$ must be harmonic in $\mathcal{D}$. Then by applying the boundary condition and the asymptotic behavior of $b$, one derives $u_r \equiv 0$ and $u_z$ must be a constant. From then on (1.4) turns to a linear ordinary differential equation of $u_\theta$, and we finally prove $u_\theta = C_1r$.

### 3.1 Vanishing of $\omega_\theta$

Noting that $u_\theta$ is independent of $z$, we find (1.5) now turns to

$$ (u_r \partial_r + u_z \partial_z)\omega_\theta - \left( \Delta - \frac{1}{r^2} \right) \omega_\theta - \frac{u_r}{r} \omega_\theta = 0. $$

From the Navier-slip boundary condition (1.6), one has

$$ \omega_\theta = \partial_z u_r - \partial_r u_z = 0, \quad \text{on } \partial \mathcal{D}. $$

Denoting $\Omega := \frac{\omega_\theta}{r}$, direct calculation shows

$$ \begin{cases} 
(u_r \partial_r + u_z \partial_z)\Omega - \left( \Delta + \frac{2}{r} \partial_r \right) \Omega = 0, & \text{in } \mathcal{D}; \\
\Omega = 0, & \text{on } \partial \mathcal{D}. 
\end{cases} \tag{3.1} $$

In the following, we first provide a mean value inequality of $\Omega$ deduced by Moser’s iteration.

**Lemma 3.1.** Assume $b = u_re_r + u_ze_z$ is a smooth divergence-free axially symmetric vector field. Then any weak solution $\Omega$ of boundary value problem (3.1) satisfies the following mean value inequality:

$$ \sup_{x \in \mathcal{D}_{\tau_2Z}} |\Omega| \leq C_q(\tau_1 - \tau_2)^{-\frac{q}{2}} \left( 1 + \|u_z\|_{L^\infty(\mathcal{D}_{\tau_1Z} \setminus \mathcal{D}_{\tau_2Z})} \right)^{\frac{q}{2}} Z^{-\frac{q}{2}} \left( \int_{\mathcal{D}_{\tau_1Z}} |\Omega|^2 \, dx \right)^{\frac{1}{2}} \tag{3.2} $$

for any $q > 2, Z > 1$, and $\frac{1}{2} \leq \tau_2 < \tau_1 \leq 1$.

**Proof.** We only prove (3.2) with $\tau_1 = 1, \tau_2 = \frac{1}{2}$ for simplicity, since the general case could be derived by a direct scaling strategy. For any real number $l \geq 1$, we find $h := \Omega^l$ satisfies

$$ \Delta h - l(l - 1)\Omega^{l-2} |\nabla \Omega|^2 + \frac{2}{r} \partial_r h - b \cdot \nabla h = 0. \tag{3.3} $$

Set $\frac{1}{2} \leq \sigma_2 < \sigma_1 \leq 1$ and choose $\zeta = \zeta(z)$ to be a smooth cut-off function satisfying

$$ \begin{cases} 
\text{supp } \zeta \subset [-\sigma_1, \sigma_1], \\
0 \leq \zeta \leq 1, \\
|\zeta'| \leq \frac{1}{\sigma_1 - \sigma_2}. 
\end{cases} \quad \text{in } [-\sigma_2, \sigma_2]. $$

This concludes the proof of Lemma 3.1.
Denoting \( \zeta(Z) := \zeta(\frac{z}{Z}) \) and testing (3.3) with \( \zeta^2 h \), noting that

\[
\lambda(l - 1) \int_{\mathcal{D}_{r, z}} \Omega^{2l-2} |\nabla \Omega|^2 \zeta^2 dx = \frac{l - 1}{l} \int_{\mathcal{D}_{r, z}} |\nabla \Omega|^2 \zeta^2 dx \geq 0,
\]

we arrive

\[
\int_{\mathcal{D}_{r, z}} \Delta h \zeta^2 h dx + \int_{\mathcal{D}_{r, z}} \frac{2}{r} \partial_r h \zeta^2 h dx - \int_{\mathcal{D}_{r, z}} b \cdot \nabla h \zeta^2 h dx \geq 0. \tag{3.4}
\]

Next we handle \( M_1 - M_3 \) term by term. Using integration by parts and direct calculations, we first see

\[
M_1 = -\int_{\mathcal{D}_{r, z}} \nabla h \cdot \nabla(\zeta^2 h) dx = -\int_{\mathcal{D}_{r, z}} |\nabla(h \zeta)|^2 dx + \int_{\mathcal{D}_{r, z}} h^2 |\zeta^2|^2 dx. \tag{3.5}
\]

Here the boundary term of the cylindrical surface is cancelled because \( h = 0 \) on \( \partial D \), while those coming from the cross sections \( D \cap \{ z = \pm \sigma_1 Z \} \) vanish due to the cut off function \( \zeta \) is compactly supported. On the other hand, using axisymmetry of the solution

\[
M_2 = 2\pi \int_0^1 \partial_r (h^2 \zeta^2) dr dz = -2\pi \int_\mathbb{R} h^2(0, z) \zeta^2(z) dz \leq 0. \tag{3.6}
\]

Before we bound \( M_3 \), let us introduce the stream function of axisymmetric velocity field \( b = u_r e_r + u_z e_z \). By the divergence-free property \( \partial_r (ru_r) + \partial_z (ru_z) = 0 \), there exists a scalar function \( L_\theta = L_\theta(r, z) \) such that

\[
-\partial_r L_\theta = u_r, \quad \text{and} \quad \frac{1}{r} \partial_r (r L_\theta) = u_z. \tag{3.7}
\]

Using integration by parts again together with boundary condition \( h = 0 \) on \( \partial D \), we derive that

\[
M_3 = \frac{1}{2} \int_{\mathcal{D}_{r, z}} b \cdot \nabla h^2 \zeta dx = -\int_{\mathcal{D}_{r, z}} u_z \zeta \zeta' h^2 dx = -2\pi \int_\mathbb{R} \int_0^1 \partial_r (r L_\theta) \zeta \zeta' h^2 dr dz
\]

\[
= 4\pi \int_\mathbb{R} \int_0^1 (r L_\theta) \partial_r (h \zeta) h \zeta' dr dz.
\]

By the mean value theorem and (3.7), there exists \( \tilde{r} \in (0, r) \) such that

\[
L_\theta(r, z) = \tilde{r} u_z(\tilde{r}, z)r,
\]

thus we can further bound \( M_3 \) by

\[
|M_3| \leq 4\pi ||u_z||_{L^\infty(\mathcal{D}_{r, z} \cap \mathcal{D}_{r, z})} \int_\mathbb{R} \int_0^1 |\nabla (h \zeta) h \zeta'| r dr dz
\]

\[
\leq \frac{1}{2} \int_{\mathcal{D}_{r, z}} |\nabla (h \zeta)|^2 dx + 2 ||u_z||_{L^\infty(\mathcal{D}_{r, z} \cap \mathcal{D}_{r, z})} \int_{\mathcal{D}_{r, z}} h^2 |\zeta'|^2 dx. \tag{3.8}
\]
Now substituting (3.5), (3.6), and (3.8) in (3.4), taking the maximum of $\zeta$, it follows that

$$\int_{D_{\sigma_1Z}} |\nabla(h\zeta)|^2 dx + 2\pi \int \left( \frac{\zeta}{h^2(0, z)} \right) d\zeta \leq \frac{C \left( 1 + \|u_z\|_{L^\infty(D_{\sigma_1Z})} \right)}{(\sigma_1 - \sigma_2)^2 Z^2} \int_{D_{\sigma_1Z}} h^2 dx. \quad (3.9)$$

Recalling $h = 0$ on $\partial D$, for any fixed $z \in \mathbb{R}$, the following 2D Poincaré inequality holds:

$$\|h(h, z)\zeta(z)\|_{L^2(\Sigma)}^2 \leq C \left( \|\partial_r [h(h, z)\zeta(z)]\|_{L^2(\Sigma)}^2 \right),$$

where $C > 0$ here is independent of $z$. Integrating with $z$ on $\mathbb{R}$ and taking the square root, one has the following 3D Poincaré inequality

$$\|h\zeta\|_{L^2(D_{\sigma_1Z})} \leq C \|\partial_r (h\zeta)\|_{L^2(D_{\sigma_1Z})}. \quad (3.10)$$

For any $q \in (2, 6)$, Interpolation, Sobolev inequality and (3.10) imply that

$$\|h\zeta\|_{L^q(D_{\sigma_1Z})} \leq \|h\zeta\|_{L^2(D_{\sigma_1Z})} \|h\zeta\|_{L^2(D_{\sigma_1Z})} \leq C \|\nabla(h\zeta)\|_{L^2(D_{\sigma_1Z})} \|h\zeta\|_{L^2(D_{\sigma_1Z})} \leq C \|\nabla(h\zeta)\|_{L^2(D_{\sigma_1Z})} \|\partial_r (h\zeta)\|_{L^2(D_{\sigma_1Z})} \leq C \|\nabla(h\zeta)\|_{L^2(D_{\sigma_1Z})}. \quad (3.11)$$

Here $s \in (0, 1)$ depends on $q$. Combining (3.9) and (3.11), we derive

$$\|h\|_{L^q(D_{\sigma_1Z})} \leq C \left( 1 + \|u_z\|_{L^\infty(D_{\sigma_1Z})} \right) \|h\|_{L^2(D_{\sigma_1Z})},$$

which is equivalent to

$$\left( \int_{D_{\sigma_1Z}} |\Omega|^q dx \right)^{\frac{1}{q}} \leq C \left( 1 + \|u_z\|_{L^\infty(D_{\sigma_1Z})} \right)^{\frac{1}{q}} \left( \int_{D_{\sigma_1Z}} |\Omega|^2 dx \right)^{\frac{1}{2}} \left( \int_{D_{\sigma_1Z}} |\Omega|^2 dx \right)^{\frac{1}{2}} \left( \int_{D_{\sigma_1Z}} |\Omega|^2 dx \right)^{\frac{1}{2}}. \quad (3.12)$$

Now for any $k = 0, 1, 2, \ldots$, we choose $l_k = \left( \frac{q}{2} \right)^k$ and $\sigma_{1k} = \frac{1}{2} + \left( \frac{1}{2} \right)^{k+1}$, $\sigma_{2k} = \frac{1}{2} + \left( \frac{1}{2} \right)^{k+2}$, respectively. Denoting

$$\Psi_k := \left( \int_{D_{\sigma_1Z}} |\Omega|^{2l} dx \right)^{\frac{1}{2l}},$$

and noting that

$$D_{\sigma_{1k}Z} \setminus D_{\sigma_{2k}Z} \subset D_Z \setminus D_{Z/2}, \quad \forall k = 0, 1, 2, \ldots,$$

then (3.12) follows that

$$\Psi_{k+1} \leq C \left( \frac{1}{2} \right)^k \left( \frac{1}{2} \right)^k \left( 1 + \|u_z\|_{L^\infty(D_{\sigma_kZ})} \right) \left( \frac{1}{2} \right)^k Z^{-\frac{1}{2}} \left( \frac{1}{2} \right)^k \Psi_k \leq \cdots \leq C \sum_{j=0}^{k} \left( \frac{1}{2} \right)^j \left( \frac{1}{2} \right)^j \left( 1 + \|u_z\|_{L^\infty(D_{\sigma_kZ})} \right) \sum_{j=0}^{k} \left( \frac{1}{2} \right)^j \left( \frac{1}{2} \right)^j \Psi_0. \quad (3.13)$$
Performing $k \to \infty$, then iteration (3.13) implies a mean value inequality of $\Omega$, that is

$$\sup_{x \in D_{2Z}} |\Omega| \leq C_q \left( 1 + \|u_\varepsilon\|_{L^\infty(D_Z)} \right)^{\frac{q}{2}} Z^{\frac{\alpha_k}{q-2}} \left( \int_{D_Z} |\Omega|^2 dx \right)^{\frac{1}{2}},$$

for any $q > 2$. This completes the proof of Lemma 2.2.

Since $u_\varepsilon$ satisfies (1.9) in $D_Z$, (3.2) indicates that

$$\sup_{x \in D_{2Z}} |\Omega|^2 \leq C_q (\tau_1 - \tau_2)^{\frac{2q}{\alpha_k}} Z^{\frac{2(\alpha_k-1)q}{q-2}} \left( \sup_{x \in D_{\tau_1Z}} |\Omega|^2 \right) \left( \int_{D_{\tau_1Z}} |\Omega|^2 dx \right)^{\frac{1}{2}}.$$  

(3.14)

However, due to the lack of boundedness of the second-order derivatives of $u$, we are unable to control $\|\Omega\|_{L^2(D_Z)}$ at the moment. Next we will use an algebraic trick to convert the $L^2$-norm on the right hand side of (3.14) to an $L^1$-norm. This trick comes from Li-Schoen [18]. Here goes the lemma:

**Lemma 3.2 (modified mean value inequality).** Suppose $b = u_r e_r + u_z e_z$ is a smooth divergence-free axisymmetric vector field and $\|u_\varepsilon\|_{L^\infty(D_Z)} \leq Z^\alpha$. Then any weak solution $\Omega$ of boundary value problem (3.1) satisfies the following mean value inequality for any $q > 2$, $Z > 1$:

$$\sup_{x \in D_{2Z}} |\Omega| \leq C_q Z^{\frac{2(\alpha_k-1)q}{q-2}} \left( \int_{D_Z} |\Omega|^2 dx \right)^{\frac{1}{2}}.$$  

(3.15)

**Proof.** For any $\frac{1}{2} \leq \tau_2 < \tau_1 \leq 1$, (3.14) implies that

$$\sup_{x \in D_{\tau_1Z}} |\Omega|^2 \leq C_q (\tau_1 - \tau_2)^{\frac{2q}{\alpha_k}} Z^{\frac{2(\alpha_k-1)q}{q-2}} \left( \sup_{x \in D_{\tau_1Z}} |\Omega|^2 \right) \left( \int_{D_{\tau_1Z}} |\Omega|^2 dx \right)^{\frac{1}{2}}.$$

Denoting $\tau_{1k} = \tau_{2k+1} = 1 - \left( \frac{1}{2} \right)^{k+1}$, $\tau_{2k} = 1 - \left( \frac{1}{2} \right)^{k+1}$, and $\Phi_k := \sup_{x \in D_{\tau_{1k}Z}} |\Omega|^2$, it follows that

$$\Phi_k \leq C_q Z^{\frac{2(\alpha_k-1)q}{q-2}} \Phi_{k+1} \left( \int_{D_Z} |\Omega|^2 dx \right).$$  

(3.16)

Iterating (3.16) from $k = 0$ to infinity, one arrives

$$\sup_{x \in D_{2Z}} |\Omega|^2 \leq C_q \sum_{j=0}^{\infty} 2^{-j} \left( \sum_{j=0}^{\infty} \frac{2q}{\alpha_k} \sum_{j=0}^{\infty} \frac{1}{2^j} \left( Z^{\frac{2(\alpha_k-1)q}{q-2}} \right) \left( \int_{D_Z} |\Omega|^2 dx \right)^{2^{-j}} \right) \left( \int_{D_Z} |\Omega|^2 dx \right)^{\frac{2}{2^{-j}}},$$

which follows that

$$\sup_{x \in D_{2Z}} |\Omega| \leq C_q Z^{\frac{6(\alpha_k-1)q}{q-2}} \left( \int_{D_Z} |\Omega|^2 dx \right)^{\frac{1}{2}}.$$
This completes the proof of Lemma 3.2.

Finally, one notes that
\[
\int_{\mathcal{D}} |\Omega| \, dx \leq 2\pi \| \omega_2 \|_{L^\infty(D)} \int_{-Z}^{Z} \int_{0}^{1} dr \, dz \lesssim Z^{M_0 + 1}.
\]

Therefore, as long as \( \omega_2 \) is of polynomial growth (see (1.9)) when \( z \to \infty \), we can infer from (3.15) that
\[
\sup_{x \in \mathcal{D}} |\Omega| \leq C_1 Z^{(\delta_0 - 1)q - 2 + M_0}.
\]

For any fixed \( \delta_0 < 1 \) and \( M_0 > 0 \), we can always choose \( q \) which is larger than but close enough to 2 such that (3.17) indicates
\[
\sup_{x \in \mathcal{D}} |\Omega| \lesssim Z^{-\gamma}
\]
for some \( \gamma > 0 \). This proves \( \omega_2 \) vanishes in \( \mathcal{D} \) by performing \( Z \to \infty \).

### 3.2 Vanishing of \( u_r \) and constancy of \( u_z \)

Noting that \( \nabla \times b = \omega_2 e_\theta \equiv 0 \) and the divergence-free property of \( b \), we apply the Lagrange’s formula for \( \nabla \) to deduce
\[
-\Delta b = \nabla \times \nabla \times b - \nabla (\text{div } b) = 0,
\]
which indicates
\[
\left( \Delta - \frac{1}{r^2} \right) u_r = 0; \quad \Delta u_z = 0.
\]

To prove vanishing of \( u_r \), for \( \delta > 0 \) being small, we consider the auxiliary function \( \eta_\delta \) which is defined by
\[
\eta_\delta(x) := J_1((\alpha - \delta) r) \cosh ((\alpha - \delta) z).
\]
Here \( J_1 \) is the Bessel function which is defined in (1.11) and satisfies (1.10) with \( \beta = 1 \), while \( \alpha \) is the smallest positive root of \( J_1 \). Direct calculation shows
\[
\left( \Delta - \frac{1}{r^2} \right) \eta_\delta = \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) \eta_\delta = 0.
\]

Owing to \( u_r \) is growing as (1.9) \( 1 \), we choose \( \delta << 1 \) small enough such that \( \gamma_0 < \alpha - 2\delta \). Using the concavity of \( J_1((\alpha - \delta) r) \) on the subset of \( \{ r : 0 \leq r \leq 1 \} \) where \( J_1((\alpha - \delta) r) \) is increasing, one has
\[
\eta_\delta \geq J_1((\alpha - \delta) r) \cosh ((\alpha - \delta) z) \geq C_\delta r e^{(\gamma_0 + 0)|z|},
\]
where \( C_\delta > 0 \) is a constant depends only on \( \delta \). Then the condition (1.9) \( 1 \) indicates that
\[
\lim_{|z| \to \infty} \frac{|u_r(r, z)|}{\eta_\delta(r, z)} = 0, \text{ uniformly with } r = \sqrt{x_1^2 + x_2^2} \in [0, 1].
\]
Therefore, for any fixed $\varepsilon > 0$ and $\delta$, there exists an $N_{\varepsilon,\delta} > 0$ such that

\[
\begin{cases}
(\Delta - \frac{1}{r^2}) (\varepsilon \eta_\delta \pm u_r) = 0, & \forall x \in \mathcal{D}_M, \\
\varepsilon \eta_\delta \pm u_r \geq 0, & \forall x \in \partial \mathcal{D}_M = [\partial \mathcal{D} \cap \{-M \leq z \leq M\}] \cup [\mathcal{D} \cap \{z = \pm M\}],
\end{cases}
\]

for any $M > N_{\varepsilon,\delta}$. The maximum principle indicates

\[|u_r(x)| \leq \varepsilon \eta_\delta(x), \quad \forall x \in \mathcal{D}_M. \tag{3.18}\]

By performing $M \to \infty$, one finds the estimate (3.18) actually holds for all $x \in \mathcal{D}$. Thus $u_r \equiv 0$ is proved by the arbitrariness of $\varepsilon > 0$.

Finally, the divergence-free of $u$ implies $\partial_z u_z = -\frac{1}{r} \partial_r (ru_r) \equiv 0$ in $\mathcal{D}$. The vanishing of $\omega_\theta$ and $u_r$ indicates $\partial_r u_z \equiv 0$. Thus $u_z$ must be a constant. This consequently indicates

\[b = C_2 e_z \tag{3.19}\]

for some constant $C_2 \in \mathbb{R}$.

### 3.3 End of the proof

Now substituting (3.19) in (1.4) and noting that $u_\theta$ is independent of $r$, one arrives the following ODE of $u_\theta$

\[u_\theta''(r) + \frac{1}{r} u_\theta'(r) - \frac{1}{r^2} u_\theta(r) = 0.\]

This ODE, which is of Eulerian type, is solved by

\[u_\theta(r) = \frac{C_0}{r} + C_1 r,\]

for any $C_0, C_1 \in \mathbb{R}$. Smoothness of $u_\theta$ forces that $C_0 = 0$. Thus we conclude that

\[u = u_\theta e_\theta + b = C_1 r e_\theta + C_2 e_z,\]

which completes the proof of Theorem 1.3.

\[\square\]

**Remark 3.3.** Unlike Theorem 1.1, Theorem 1.3 actually needs weaker assumptions (1.9) on the boundedness of solutions. As stated in the introduction, assumption (1.9) is sharp due to the non-trivial counterexamples in (1.12) which grow no slower than $C e^{\alpha|z|}$ as $z \to \infty$. Meanwhile, the counterexample in (1.12) has zero vorticity and zero flux in the cross section $\Sigma$. Identities (2.34) and (2.35) no longer hold for the solution in (1.12) since we have no boundedness of the head pressure $H := \frac{1}{2}|u|^2 + p - p(0, Z)$ in $\mathcal{D}_Z \setminus \mathcal{D}_Z$. 

\[\square\]
4 Proof of Theorem 1.5

Proof. We only focus on the part which is different from the proof of Theorem 1.1. Noting that \((SB)\) can be represented as
\[
\begin{align*}
\omega_\theta &= 0, \\
\omega_z &= 0, \quad \text{on } \partial D. \\
u_r &= 0,
\end{align*}
\]
Following procedures in Section 2 and Section 3, we also have the validity of Lemma 2.2. After testing Navier-Stokes equations with \(u\psi_z\), we treat the left hand side of (2.23) in the following way:
\[
\int_{D_{2z}} \psi_Z u_i \partial_{jj} u_i dx = \int_{D_{2z}} \psi_Z u_i \partial_j (\partial_j u_i - \partial_i u_j) dx \\
= - \int_{D_{2z}} \partial_j \psi_Z u_i (\partial_j u_i - \partial_i u_j) dx - \int_{D_{2z}} \psi_Z \partial_j u_i (\partial_j u_i - \partial_i u_j) dx \\
+ \int_{\partial D_{2z}} \psi_Z u_i n_j (\partial_j u_i - \partial_i u_j) ds. \tag{4.1}
\]
Similarly as (2.24), one finds
\[
I_1 = - \int_{D_{2z}} \partial_z^2 \psi_Z \left( \frac{1}{2} |u|^2 - u_z^2 \right) dx. \tag{4.2}
\]
For term \(I_2\), since
\[
(\partial_j u_i - \partial_i u_j)_{1 \leq i, j \leq 3} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},
\]
we note that
\[
I_2 = \int_{D_{2z}} \psi_Z (\partial_j u_i - \partial_i u_j)^2 dx + \int_{D_{2z}} \psi_Z \partial_i u_j (\partial_j u_i - \partial_i u_j) dx \\
= 2 \int_{D_{2z}} \psi Z |\omega|^2 dx - I_2,
\]
which implies
\[
I_2 = \int_{D_{2z}} \psi Z |\omega|^2 dx. \tag{4.3}
\]
Finally, identity
\[
n_j (\partial_j u_i - \partial_i u_j) = (\nabla \times u) \times n
\]
indicates the disappearance of \(I_3\). Substituting (4.2) and (4.3) in (4.1), and using the same integration by parts as in Section 2 which deals with the right hand of (2.22), one get
\[
\int_{D_{2z}} \psi Z |\omega|^2 dx = \int_{D_{2z}} \partial_z^2 \psi_Z \left( \frac{1}{2} |u|^2 - u_z^2 \right) dx + \int_{D_{2z}} \partial_z \psi_Z u_z \left( \frac{1}{2} |u|^2 + [p - p(0, Z)] \right) dx.
\]
Estimating the head pressure term in the same way as in Section 2, one derives
\[ \int_{\mathcal{D}} |\omega|^2 \, dx < C < \infty \]
by letting \( Z \to \infty \). Then using Poincaré inequality (2.30), we still have the \( L^2 \) bound of \( u_z \) can be controlled by \( \| \partial_r u_z \|_{L^2} \). Since
\[
\int_{\mathcal{D}_{2Z}} (\partial_r u_z)^2 \, dx = \int_{\mathcal{D}_{2Z}} (\partial_r u_r - \partial_z u_z)^2 \, dx - \int_{\mathcal{D}_{2Z}} (\partial_z u_r)^2 \, dx + 2 \int_{\mathcal{D}_{2Z}} \partial_r u_z \partial_z u_r \, dx
\]
\[ \leq \int_{\mathcal{D}} |\omega|^2 \, dx + 2 \int_{\mathcal{D}_{2Z}} \partial_r u_z \partial_r u_r \, dx \]
\[ \leq \int_{\mathcal{D}} |\omega|^2 \, dx + 2\pi \left( \int_0^1 u_z(r, 2Z) \partial_z u_r(r, 2Z) r \, dr - \int_0^1 u_z(r, -2Z) \partial_z u_r(r, -2Z) r \, dr \right) . \]
Here we have estimated \( T_6 \) above as in (2.31). This implies the global \( L^2 \) boundedness of \( u_z \).

Following procedures in Section 2.3.3, one arrives that
\[ \int_{\mathcal{D}} |\omega|^2 \, dx = 0, \]
which gives \( \omega \equiv 0 \) in \( \mathcal{D} \). First we see \( \omega_r = -\partial_z u_\theta \equiv 0 \) indicates \( u_\theta = u_\theta(r) \). Then \( \omega_z = \frac{1}{r} \partial_r (ru_\theta) \equiv 0 \) leads us to the constancy of \( ru_\theta(r) \). However, this could not happen unless \( u_\theta \equiv 0 \) due to the smoothness of \( u \). Now following the same process in Section 3.2, one derives \( b = u_r \epsilon_r + u_z \epsilon_z \equiv C_0 \epsilon_z \) for some constant \( C_0 \in \mathbb{R} \). By the flux is zero in a given cross section, one concludes that \( C_0 = 0 \). Hence Theorem 1.5 is proved.

\[ \square \]

Remark 4.1. If the flux \( \Phi \) on the unit disk of the cross section \( \Sigma \) is not zero, i.e.
\[ c_0 := \frac{1}{|\Sigma|} \int_{\Sigma} u_z(x_h, z) \, dx_h = \frac{1}{|\Sigma|} \Phi \neq 0, \]
and the other conditions are identical with those of Theorem 1.5, our method can not show that \( u \equiv c_0 \epsilon_z \). Nevertheless, one can still deduce an identity as (2.34):
\[ \int_{\mathcal{D}} |\omega|^2 \, dx = - \lim_{Z \to \infty} \frac{c_0}{2} \int_{\mathcal{D}} \psi_Z \partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) \, dx. \]
Moreover, if \( \partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) \in L^1(D) \), or \( \partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) \) has a sign, we conclude that
\[ \int_{\mathcal{D}} |\omega|^2 \, dx + \frac{c_0}{2} \int_{\mathcal{D}} \partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) \, dx = 0. \]
\[ \square \]
Remark 4.2. Similarly as Theorem 1.3, if we switch the zero-flux condition in Theorem 1.5 by \( \partial_z u_0 \equiv 0 \), we can deduce

\[ u = C_0 e_z, \]

for a constant \( C_0 \). The proof is almost identical with the proof of Theorem 1.3 so that we omit it here. Also in this case, the boundedness assumption of \( u \) and \( \nabla u \) could be weakened to (1.9).

\[ \square \]

**Appendix Computation of the boundary condition**

Here we give a derivation of the boundary condition (1.6) from (NSB) for the axially symmetric solution. First, noting that

\[ 0 = u \cdot n = u_r, \tag{A.1} \]

we deduce the third equation of (1.6).

In cylindrical coordinates, the gradient operator is represented by

\[ \nabla = e_r \partial_r + e_\theta \frac{\partial_\theta}{r} + e_z \partial_z. \]

Then we can calculate the matrix \( \nabla u \) in cylindrical coordinates and write it as a form of tensor product as follows

\[ \nabla u = \partial_r u_r e_r \otimes e_r + \partial_r u_r e_r \otimes e_z + \frac{u_r}{r} e_\theta \otimes e_\theta + \partial_\theta u_\theta e_\theta \otimes e_r + \partial_z u_z e_z \otimes e_z \]

\[ + \partial_r u_\theta e_\theta \otimes e_r + \partial_\theta u_\theta e_\theta \otimes e_z - \frac{u_\theta}{r} e_r \otimes e_\theta. \tag{A.2} \]

Equivalently

\[ \nabla u = \begin{pmatrix} \partial_r u_r & -\frac{1}{r} u_\theta & \frac{1}{2} \partial_\theta u_\theta \\ \frac{1}{r} u_r & \partial_r u_\theta & \frac{1}{2} \partial_\theta u_\theta \\ 0 & \partial_z u_\theta & \partial_z u_z \end{pmatrix} : \begin{pmatrix} e_r \otimes e_r & e_r \otimes e_\theta & e_r \otimes e_z \\ e_\theta \otimes e_r & e_\theta \otimes e_\theta & e_\theta \otimes e_z \\ e_z \otimes e_r & e_z \otimes e_\theta & e_z \otimes e_z \end{pmatrix} \]

\[ = \begin{pmatrix} \partial_r u_r & -\frac{1}{r} u_\theta & \frac{1}{2} \partial_\theta u_\theta \\ \frac{1}{r} u_r & \partial_r u_\theta & \frac{1}{2} \partial_\theta u_\theta \\ 0 & \partial_z u_\theta & \partial_z u_z \end{pmatrix} : A. \tag{A.3} \]

Then \( Su \) under the base \( A \) is represented by

\[ Su = \begin{pmatrix} \partial_r u_r & \frac{1}{2} \partial_r u_\theta - \frac{1}{2} u_\theta & \frac{1}{2} \partial_\theta u_\theta + \frac{1}{2} u_\theta \\ \frac{1}{2} \partial_z u_\theta - \frac{1}{2} \frac{1}{r} u_\theta & \frac{1}{2} \partial_\theta u_\theta & \frac{1}{2} \partial_z u_\theta \\ \frac{1}{2} \partial_z u_r & \frac{1}{2} \partial_\theta u_\theta & \partial_z u_z \end{pmatrix} : A. \tag{A.4} \]

Since the outward normal vector \( n = e_r \), we have

\[ Su \cdot n = \partial_r u_r e_r + \frac{1}{2} \left( \partial_\theta u_\theta - \frac{1}{r} u_\theta \right) e_\theta + \frac{1}{2} (\partial_z u_r + \partial_z u_\theta) e_z. \]
Then in cylinder coordinates,

\[
(\mathbf{S} \mathbf{u} \cdot \mathbf{n})_{r} = \frac{1}{2} \left( \partial_{r} u_{\theta} - \frac{1}{r} u_{\theta} \right) \mathbf{e}_{\theta} + \frac{1}{2} \left( \partial_{z} u_{r} + \partial_{r} u_{z} \right) \mathbf{e}_{z} = 0.
\]

This, together with (A.1), indicates the first two equations of (1.6).

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**REFERENCES**

[1] C. J. Amick: Steady solutions of the Navier-Stokes equations in unbounded channels and pipes. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 4 (1977), no. 3, 473–513.

[2] C. J. Amick: Properties of steady Navier-Stokes solutions for certain unbounded channels and pipes. *Nonlinear Anal.* 2 (1978), no. 6, 689–720.

[3] K. A. Ames and L. E. Payne: Decay estimates in steady pipe flow. *SIAM J. Math. Anal.* 20 (1989), no. 4, 789–815.

[4] B. Carrillo, X. Pan, Q. S. Zhang and N. Zhao: Decay and vanishing of some D-solutions of the Navier-Stokes equations. *Arch. Ration. Mech. Anal.* 237 (2020), no. 3, 1383–1419.

[5] D. Chae: Liouville-type theorems for the forced Euler equations and the Navier-Stokes equations. *Comm. Math. Phys.* 326 (2014), no. 1, 37–48.

[6] D. Chae and J. Wolf: On Liouville type theorems for the steady Navier-Stokes equations in $\mathbb{R}^3$. *J. Differential Equations* 261 (2016), no. 10, 5541–5560.

[7] G. P. Galdi: *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. Second edition.* Springer Monographs in Mathematics. Springer, New York, 2011.

[8] Y. Giga and H. Miura: On vorticity directions near singularities for the Navier-Stokes flows with infinite energy. *Comm. Math. Phys.*, 303 (2011), 289–300.

[9] Y. Giga, P. Y. Hsu and Y. Maekawa: A Liouville theorem for the planer Navier-Stokes equations with the no-slip boundary condition and its application to a geometric regularity criterion. *Comm. Partial Differential Equations* 39 (2014), no. 10, 1906–1935.

[10] C. O. Horgan and L. T. Wheeler: Spatial decay estimates for the Navier-Stokes equations with application to the problem of entry flow. *SIAM J. Appl. Math.* 35 (1978), no. 1, 97–116.

[11] G. Koch, N. Nadirashvili, G. A. Seregin and V. Šverák: Liouville theorems for the Navier-Stokes equations and applications. *Acta Math.* 203 (2009), no. 1, 83–105.

[12] P. Koneczny: On a steady flow in a three-dimensional infinite pipe. *Colloq. Math.* 104 (2006), no. 1, 33–56.
[13] M. V. Korobkov, K. Pileckas and R. Russo: Solution of Leray’s problem for stationary Navier-Stokes equations in plane and axially symmetric spatial domains. Ann. of Math. (2) 181 (2015), no. 2, 769–807.
[14] M. V. Korobkov, K. Pileckas and R. Russo: The existence theorem for the steady Navier-Stokes problem in exterior axially symmetric 3D domains. Math. Ann. 370 (2018), no. 1–2, 727–784.
[15] O. A. Ladyženskaya: Investigation of the Navier-Stokes equation for stationary motion of an incompressible fluid. (Russian) Uspehi Mat. Nauk 14 (1959), no. 3, 75–97.
[16] O. A. Ladyženskaja and V. A. Solonnikov: Determination of solutions of boundary value problems for stationary Stokes and Navier-Stokes equations having an unbounded Dirichlet integral. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 96 (1980), 117–160, 308.
[17] O. A. Ladyženskaja and V. A. Solonnikov: Determination of solutions of boundary value problems for stationary Stokes and Navier-Stokes equations having an unbounded Dirichlet integral. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 96 (1980), 117–160, 308.
[18] P. Li and R. Schoen: $L^p$ and mean value properties of subharmonic functions on Riemannian manifolds. Acta Math. 153(3–4), (1884), 279–301.
[19] K. Mikhail, W. Lyu and S. Weng: On the existence of helical invariant solutions to steady Navier-Stokes equations. arXiv:2102.13341.
[20] P. B. Mucha: On Navier-Stokes equations with slip boundary conditions in an infinite pipe. Acta Appl. Math. 76 (2003), no. 1, 1–15.
[21] P. B. Mucha: Asymptotic behavior of a steady flow in a two-dimensional pipe. Studia Math. 158 (2003), no. 1, 39–58.
[22] C. Navier: Sur les lois du mouvement des fluides. Mem. Acad. R. Sci. Inst. France, 6, (1823), 389–440.
[23] X. Pan: A Liouville theorem of Navier-Stokes equations with two periodic variables. J. Math. Anal. Appl. 485 (2020), no. 2, 123854, 7 pp.
[24] K. Piletskas: On the asymptotic behavior of solutions of a stationary system of Navier-Stokes equations in a domain of layer type. (Russian) Mat. Sb. 193 (2002), no. 12, 69–104; translation in Sb. Math. 193 (2002), no. 11–12, 1801–1836.
[25] G. Seregin: Liouville type theorem for stationary Navier-Stokes equations. Nonlinearity 29 (2016), no. 8, 2191–2195.
[26] W. Wang: Remarks on Liouville type theorems for the 3D steady axially symmetric Navier-Stokes equations. J. Differential Equations 266 (2019), no. 10, 6507–6524.
[27] Y. Wang and C. Xie: Uniform structural stability of Hagen-Poiseuille flows in a pipe. arXiv: 1911.00749.
[28] Y. Wang and C. Xie: Uniform structural stability and uniqueness of Poiseuille flows in a two dimensional periodic strip. arXiv: 2011.07467.
[29] Q. S. Zhang: Bounded solutions to the axially symmetric Navier-Stokes equation in a cusp region. arXiv:2106.08509 v1.