Equivalence of the quantum harmonic oscillator and the classical oscillator subject to a random force

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Abstract

We show that the Schrödinger equation for the quantum harmonic oscillator can be derived as an approximation to the Newtonian mechanics of a classical harmonic oscillator subject to a random force for time intervals $O(m/\hbar)$. Conversely every solution to the Schrödinger equation arises this way. In other words the quantum harmonic oscillator is approximately nothing but the classical harmonic oscillator with the same mass and frequency subject to a random force.

1 Introduction

In this paper we continue our investigation initiated in our previous paper [1] on the question: Can a non-relativistic quantum particle in a potential in one dimension be described by Newtonian mechanics? In our previous paper we showed that when a classical harmonic oscillator is subject to a specific random force, its fluctuating motion around the classical trajectory can be described by the Schrödinger equation. We achieved this by showing that the position process of the oscillator approximately satisfies the postulates of Nelson’s stochastic formulation of quantum mechanics. However the result obtained was only valid for motions where the energy can be treated approximately as constant. In this paper we generalize this result and show that for a harmonic oscillator with mass $m$, when $\hbar/m$ is small, for time intervals $O(m/\hbar)$, all the solutions to the classical harmonic oscillator subject to a specific random force are accurately described by the Schrödinger equation. The converse is also true: given any solution to the Schrödinger equation for the quantum harmonic oscillator one can associate a classical harmonic oscillator with the same mass and frequency subject to a random force for which the Schrödinger equation is an accurate approximation.

The paper is organized as follows. At the end of this section we summarize the results mentioned above as two propositions. In section 2 we present the proof of Proposition 1. In Section 3 we present the proof of Proposition 2. Although we call these as proofs we do not make any pretense of being mathematically rigorous. In Section 4 we discuss the results. For the convenience of the reader we included the mathematical background originally provided in our previous paper [1] as appendices. See the appendices for a review of stochastic differential equations (Appendix A), review of Nelson’s stochastic formulation of quantum mechanics (Appendix B) and the method of stochastic averaging (Appendix C). In Appendix D we provide a derivation of Newton-Nelson law for a particle subject to a random force which is originally given in our previous paper.

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Proposition 1 states that Schrödinger equation gives an accurate description for the harmonic oscillator subject to the random force in eq.1. Proposition 2 states the converse: every solution to the Schrödinger equation approximately arises from such a process. In other words the quantum harmonic oscillator is approximately nothing but the classical harmonic oscillator with the same mass and frequency subject to a random force.

**Proposition 1.** Consider the harmonic oscillator with frequency $\omega$ and mass $m$ subject to a random force with position and velocity variables denoted by $(x(t), v(t))$:

\[
\begin{align*}
  \frac{dx(t)}{dt} &= v(t) \\
  \frac{dv(t)}{dt} &= -\omega^2 x(t) + \epsilon \omega dW(t)
\end{align*}
\]  

where we set $\epsilon = \sqrt{\frac{2\hbar}{m}}$ and $dW(t)$ is the Wiener process. For a review of the Wiener process and stochastic differential equations see Appendix A. Let $(x(t), v(t))$ the solution to eq.1 on the time interval $[0,T]$ with the initial probability of finding the particle at $(x, v)$ given by $\rho_0(x, v)$. Let $\rho(v|x, t)$ be the conditional probability of finding the particle with velocity $v$ given that it is at position $x$ at time $t$. Then for times $O(1/\epsilon^2)$, the stochastic process $x(t)$ can be approximated by the following stochastic process:

\[
\frac{dx(t)}{dt} = b(x(t), t)dt + \frac{\hbar}{m} dW(t)
\]  

which is in the form of Nelson’s first postulate. For a review of Nelson’s stochastic formulation of quantum mechanics see Appendix B. Here $b(x,t) = \int v \rho(v|x, t) dv$ is the mean velocity conditioned on that particle is at position $x$ at time $t$ and $dW(t)$ is the Wiener process. Moreover the Newton-Nelson law for the harmonic oscillator (Appendix D):

\[
\frac{1}{2} (D_x D_+ + D_+ D_x) x(t) = -\omega^2 x(t)
\]  

is satisfied by the $x(t)$ part of the phase space process (eq.1). Since Nelson’s two postulates are equivalent to the Schrödinger equation, the phase space process (eq.1) is approximately described by the Schrödinger equation for the quantum harmonic oscillator:

\[
\frac{i\hbar}{\partial t} \psi(x,t) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \right) \psi(x,t)
\]  

for times $O(1/\epsilon^2)$, with the initial wave function given by

\[
\psi(x,0) = \sqrt{\rho(x,0)} \exp\left( \frac{i}{\hbar} S(x,0) \right)
\]  

where

\[
\rho(x, t = 0) = \int \rho_0(x, v) dv
\]  

is the initial probability of finding the particle at $x$ and the initial phase is

\[
S(x,0) = \int_{x_0}^{x} mb(x', 0) dx' - \frac{\hbar}{2} \log \rho(x, t = 0) + c
\]  

where $x_0$ and $c$ are constants.
Proposition 2. Let

\[ \psi(x,t) = \sqrt{\rho(x,t)} \exp \left( \frac{i}{\hbar} S(x,t) \right) \] (1.8)

be a solution to the Schrödinger equation (eq.1.4). Let \( b(x,t) \) be the associated nonlinear velocity defined by

\[ b(x,t) = \frac{1}{m} \frac{\partial}{\partial x} S(x,t) + \frac{\hbar}{2m} \frac{\partial}{\partial x} \log \rho(x,t) \] (1.9)

and let \( x(t) \) be the Nelson process (eq.1.2) defined by \((\rho(x,t),b(x,t))\) associated to the solution to the Schrödinger equation. Then there exist a phase space process of the form (eq.1.1) with the same mass and frequency \((m,\omega)\) and with the solution \((x(t),v(t))\) described by the probability distribution \(\hat{\rho}(x,v,t)\) such that approximately

\[ \int \hat{\rho}(x,v,t) dv = \rho(x,t) \] (1.10)

and the \(x(t)\) part of the process satisfies

\[ dx(t) = b(x(t),t) dt + \frac{\hbar}{m} dW(t) \] (1.11)

on a time interval \(O(1/\epsilon^2)\).

2 Proof of Proposition 1

Note that the process (eq.1.1) is linear and therefore exactly solvable. However we will not utilize the exact solution and invoke a more general derivation. The plan is as follows. We will first apply a transformation to polar coordinates to see that the energy and phase variables are slowly varying as compared to the position and velocity variables. This enables us to use the method of stochastic averaging (Appendix C). Then we average the dynamics for the polar variables over one period of oscillation. When we go back to the position and velocity variables, this induces a random term in the position process. Then we integrate out the velocity variable to obtain a Markov process in position space satisfying Nelson’s first postulate. See Appendix B for a review of Nelson’s postulates for stochastic formulation of quantum mechanics. In our previous paper [1] we showed that the Newton-Nelson law (Nelson’s second postulate) is also satisfied by the process eq.1.1. We provide this derivation in Appendix D for the convenience of the reader. Since Nelson’s two postulates are equivalent to the Schrödinger equation we have the result: the dynamics described by eq.1.1 can be approximated by a Nelson process for a time interval \(O(1/\epsilon^2)\) and therefore can be approximately described by the Schrödinger equation. In the following we repeat parts of the derivations given in [1] for completeness.

The dynamics (eq.1.1) in phase space is not in standard form for averaging. Therefore apply the coordinate transformation

\[ x = r \cos(\omega t + \phi) \]
\[ v = -\omega r \sin(\omega t + \phi) \] (2.1)
or

\[ r = \sqrt{x^2 + \frac{v^2}{\omega^2}} \quad (2.2) \]

\[ \phi = -\arctan\left(\frac{v}{\omega x}\right) - \omega t. \]

To calculate the differential of \( r \) and \( \phi \) we use Ito’s lemma (see Appendix A) and obtain

\[
\begin{align*}
    dr &= \frac{x}{r} dx + \frac{v}{\omega^2 r} dv + \frac{1}{2} \frac{x^2}{\omega^2 r^3} (dv)^2 = \frac{(\epsilon \omega)^2}{2} \frac{x^2}{\omega^2 r^3} dt + \epsilon \omega \frac{v}{\omega^2 r} dW \\
    d\phi &= \frac{v}{\omega r^2} dx - \frac{x}{\omega r^2} dv + \frac{1}{2} \frac{2xv}{\omega^3 r^4} (dv)^2 - \omega dt = (\epsilon \omega)^2 \frac{xv}{\omega^3 r^4} dt - \epsilon \omega \frac{x}{\omega r^2} dW. 
\end{align*}
\]

We see that both \( r \) and \( \phi \) are slowly varying. Therefore we apply the method of averaging over one period \( T = \frac{2\pi}{\omega} \) of the harmonic oscillator. This amounts to fixing \( r \) and averaging over the angle variable. Denote the time average of a function \( f(x, v) \) by

\[ \langle f(x, v) \rangle_T = \frac{1}{T} \int_0^T f(x(t), v(t)) dt. \quad (2.4) \]

The evolution equations can be approximated by the following averaged equations over time intervals of order \( O(1/\epsilon^2) \):

\[ dr = \frac{(\epsilon \omega)^2}{2} \frac{1}{\omega^2 r^3} \langle x^2 \rangle_T dt + \epsilon \omega \frac{1}{\omega^2 r} \sqrt{\langle v^2 \rangle_T} dW_1 \quad (2.5) \]

\[ d\phi = (\epsilon \omega)^2 \frac{1}{\omega^3 r^4} \langle xv \rangle_T dt + \epsilon \omega \frac{1}{\omega r^2} \sqrt{\langle v^2 \rangle_T} dW_2 \]

where \( dW_1 \) and \( dW_2 \) are independent Wiener processes. The averaged quantities are calculated to be

\[ \langle x^2 \rangle_T = \frac{1}{2\pi} \int_0^{2\pi} r^2 \cos^2 \theta d\theta = \frac{r^2}{2} \]

\[ \langle v^2 \rangle_T = \frac{\omega^2}{2\pi} \int_0^{2\pi} r^2 \sin^2 \theta d\theta = \frac{\omega^2 r^2}{2} \]

\[ \langle xv \rangle_T = -\frac{\omega}{2\pi} \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta = 0. \]

Note that as \( \langle xv \rangle_T = 0 \) we had the ease to set the \( dW_2 \) dependent term in \( dr \) and the \( dW_1 \) dependent term in \( d\phi \) to 0. In general these terms can not be made to vanish upon stochastic averaging. Substitute the averaged quantities in the averaged equations to get:

\[ dr = \frac{\epsilon^2}{4} \frac{1}{r} dt + \epsilon \frac{1}{\sqrt{2}} dW_1 \]

\[ d\phi = \epsilon \frac{1}{\sqrt{2r}} dW_2. \]

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Note that the averaged evolution of the amplitude of oscillations \( r \) is independent of \( \phi \) and the evolution of \( \phi \) is determined by the evolution of \( r \). Using averaged equations we can derive the averaged evolution of the position and velocity variables using eqs. [2.1] and [2.7]

\[
dx = dr \cos(\omega t + \phi) - r \sin(\omega t + \phi) (\omega dt + d\phi) - \frac{1}{2} r \cos(\omega t + \phi) (d\phi)^2
\]

\[
= v dt + \frac{\epsilon}{\sqrt{2r}} (xdW_1 + \frac{v}{\omega} dW_2)
\]

\[
dv = -dr \omega \sin(\omega t + \phi) - \omega r \cos(\omega t + \phi) (\omega dt + d\phi) + \frac{1}{2} \omega r \sin(\omega t + \phi) (d\phi)^2
\]

\[
= -\omega^2 x dt + \frac{1}{\sqrt{2r}} (vdW_1 - \omega xdW_2).
\]

We can simplify the stochastic terms noting that given \( x(t) \) and \( v(t) \), \( dW_1(t) \) and \( dW_2(t) \) are independent Gaussian processes. A linear combination

\[
a(x(t), v(t))dW_1(t) + b(x(t), v(t))dW_2(t)
\]

of independent zero mean Gaussian processes is again a zero mean Gaussian process with variance \( a^2(x(t), v(t)) + b^2(x(t), v(t)) \). Therefore the equation for the position and velocity variables can be written as

\[
dx = v dt + \frac{\epsilon}{\sqrt{2}} d\tilde{W}_1
\]

\[
dv = -\omega^2 x dt + \frac{\epsilon}{\sqrt{2}} \omega d\tilde{W}_2
\]

where we used eq. [2.2] and \( d\tilde{W}_i \) are Wiener processes. Note that \( d\tilde{W}_1 \) and \( d\tilde{W}_2 \) are uncorrelated: \( \langle d\tilde{W}_1 d\tilde{W}_2 \rangle = 0 \). Therefore they are independent processes as they are jointly Gaussian. The probability \( \rho(x, v, t) \) of finding the particle at the phase space point \( (x, v) \) at time \( t \) is governed by the Fokker-Planck equation which can be derived from the process eq. [2.10] [2.8]:

\[
\frac{\partial \rho(x, v, t)}{\partial t} = -\frac{\partial}{\partial x} (v \rho(x, v, t)) + \frac{\partial}{\partial v} \left( \omega^2 x \rho(x, v, t) \right) + \frac{\epsilon^2}{4} \frac{\partial^2 \rho(x, v, t)}{\partial x^2} + \frac{\epsilon^2 \omega^2}{4} \frac{\partial^2 \rho(x, v, t)}{\partial v^2}.
\]

Now we integrate out the velocity in the above equation to obtain the equation governing the probability \( \rho(x, t) = \int \rho(x, v, t) dv \) of finding the particle at \( x \) at time \( t \):

\[
\frac{\partial \rho(x, t)}{\partial t} = -\frac{\partial}{\partial x} (b(x, t) \rho(x, t)) + \frac{\epsilon^2}{4} \frac{\partial^2 \rho(x, t)}{\partial x^2}
\]

where

\[
b(x, t) = \int v \rho(v|x, t) dv
\]

and

\[
\rho(v|x, t) = \frac{\rho(x, v, t)}{\rho(x, t)}
\]
Furthermore we have the natural probability normalization constraint $\rho$ of the joint density $b$ by $\rho$ to construct the solution to the Schrödinger equation, the only free function is $\rho$ such that $\rho$ matches with the Fokker-Planck equation (eq.3). Since Nelson’s two postulates are equivalent to the Schrödinger equation we have shown that the dynamics (eq.1) is approximately described by the Schrödinger equation.

### 3 Proof of Proposition 2

Next we will show that the converse is also true: for any solution to the Schrödinger equation for the quantum harmonic oscillator we can find a process of the form eq.1 which is equivalent to Nelson’s first postulate: $\rho(x,t) = \frac{\rho(x,t)}{\sqrt{\rho(x,t)}} \exp(\frac{1}{\hbar}S(x,t))$. Then $\rho(x,t)$ and $S(x,t)$ satisfy the Madelung equations (Appendix B):

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} \left( \frac{\rho}{m} \frac{\partial S}{\partial x} \right) \tag{3.1}
\]

\[
\frac{\partial S}{\partial t} = -\frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 - U(x) + \frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial^2}{\partial x^2} \sqrt{\rho} \tag{3.2}
\]

where the potential is $U(x) = \frac{m}{2} \omega^2 x^2$ for the harmonic oscillator. We define $b(x,t)$ as

\[
b(x,t) = \frac{1}{m} \frac{\partial}{\partial x} S(x,t) + \frac{\hbar}{2m} \frac{\partial}{\partial x} \log \rho(x,t). \tag{3.3}
\]

Then $\rho(x,t)$ satisfies

\[
\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x} (b(x,t)\rho(x,t)) + \frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} \rho(x,t) \tag{3.4}
\]

which is equivalent to Nelson’s first postulate:

\[
dx(t) = b(x(t),t)dt + \sqrt{\frac{\hbar}{m}}dW(t). \tag{3.5}
\]

Therefore given a solution to the Schrödinger equation, the associated Nelson process is completely specified by $\rho(x,t)$ and $b(x,t)$. We would like to construct a solution to eq.2 to match with the Fokker-Planck equation (eq.3). Since $\rho(x,t)$ is already fixed by the solution to the Schrödinger equation, the only free function is $\rho(v|x,t)$ as these two determine the joint density $\rho(x,v,t) = \rho(v|x,t)\rho(x,t)$. Comparing to eq.2 the mean of $\rho(v|x,t)$ is fixed by $b(x,t)$ which is determined from the solution of the Schrödinger equation via eq.3. Furthermore we have the natural probability normalization constraint $\int \rho(v|x,t)dv = 1$. Therefore the problem becomes the following: Given a solution to the Schrödinger equation represented by $\rho(x,t)$ and $b(x,t)$, construct a solution $\rho(x,v,t)$ (construct $\rho(v|x,t)$) to eq.2 such that $\rho(x,v,t) = \rho(v|x,t)\rho(x,t)$, $b(x,t) = \int v\rho(v|x,t)dv$ and $\int \rho(v|x,t)dv = 1$. We will construct the solution $\rho(v|x,t)$ by finding it moments as follows. We will show that all the
moments \( \langle v^n \rangle_x = \int v^n \rho(v|x,t) dv \) where \( n = 2, 3, 4, \ldots \) can be obtained by taking velocity moments of eq(2.11) given \( \rho(x,t) \) and \( b(x,t) \). To see this multiply both sides of eq(2.11) by \( v^n \) and integrate over \( v \) to obtain the moment hierarchy equations

\[
\frac{\partial}{\partial t} \langle v^n \rangle_x \rho(x,t) = \frac{\partial}{\partial x} \langle (v^{n+1})_x \rho(x,t) \rangle - \omega^2 x \rho(x,t) \langle v^{n-1} \rangle_x \\
+ \frac{\epsilon^2}{4} \frac{\partial^2}{\partial x^2} \langle v^n \rangle_x \rho(x,t) + n(n-1) \frac{\epsilon^2}{4} \omega^2 \langle v^{n-2} \rangle_x \rho(x,t)
\]

(3.6)

where we have used integration by parts to simplify terms with derivatives with respect to \( v \). The case \( n = 0 \) corresponds to Nelson’s first postulate. For \( n = 1 \) we can solve for \( \langle v^2 \rangle_x \) given the knowledge of \( \rho(x,t) \) and \( b(x,t) \). For \( n = 2 \) we can solve for \( \langle v^3 \rangle_x \) given \( \rho(x,t) \), \( b(x,t) \) and \( \langle v^2 \rangle_x \) just obtained. The remaining \( \langle v^n \rangle_x \) can be constructed similarly. Note that the solutions are not necessarily unique as derivatives of \( \langle v^n \rangle_x \) with respect to \( x \) appear in the equations. We have thus constructed all the moments of \( \rho(v|x,t) \) from the knowledge of \( \rho(x,t) \) and \( b(x,t) \). However the moments of a distribution determine it as can be seen from the following. Introduce the Fourier transform of \( \rho(v|x,t) \):

\[
\tilde{\rho}(k|x,t) = \int \rho(v|x,t)e^{-ikv}dv.
\]

(3.7)

The moments \( \langle v^n \rangle_x \) can be expressed as

\[
\langle v^n \rangle_x = i^n \frac{d^n}{dk^n} \tilde{\rho}(k|x,t)|_{k=0}.
\]

(3.8)

We can Taylor expand \( \tilde{\rho}(k|x,t) \):

\[
\tilde{\rho}(k|x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dk^n} \tilde{\rho}(k|x,t)|_{k=0} k^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n \langle v^n \rangle_x k^n.
\]

(3.9)

Therefore the moments \( \langle v^n \rangle_x \) determine the Fourier transform \( \tilde{\rho}(k|x,t) \) which determines \( \rho(v|x,t) \) via the inverse Fourier transform:

\[
\rho(v|x,t) = \frac{1}{2\pi} \int e^{ikv} \tilde{\rho}(k|x,t) dk.
\]

(3.10)

Another way to construct a function from its moments is to introduce an orthonormal basis of polynomials \( P_n(v) \) such that \( \int P_n(v)P_m(v) dv = \delta_{nm} \) and calculating the components of \( \rho(v|x,t) \) in this polynomial basis: \( c_n = \int P_n(v) \rho(v|x,t) dv \) using the information of moments. Then \( \rho(v|x,t) \) is determined by the components \( c_n \). The rigorous conditions under which these functional analytic results hold are beyond the scope of this paper. We therefore have shown that one can construct \( \rho(v|x,t) \) given \( \rho(x,t) \) and \( b(x,t) \) therefore construct a solution \( \rho(x,v,t) \) to the Fokker-Planck equation (eq(2.11) which is an approximation to the original dynamics \( \text{(11)} \). One can show that the Fokker Planck equation (eq(2.11)) is indeed satisfied by the constructed solution

\[
\rho(x,v,t) = \rho(x,t) \frac{1}{2\pi} \int e^{ikv} \sum_{n=0}^{\infty} \frac{1}{n!} \langle v^n \rangle_x k^n dk
\]

(3.11)

as \( \langle v^n \rangle_x \) solve the moment hierarchy equations (eq(3.6)).
4 Scholia

In this paper we showed that the quantum harmonic oscillator is approximately equivalent to the classical harmonic oscillator with the same mass and frequency subject to a random force. We achieved this by first showing that the Schrödinger equation provides an accurate description for the motion of the classical oscillator subject to a random force. Then we showed that every solution to the Schrödinger equation approximately arises this way.

For future work we would like to know if the result presented here for the harmonic oscillator case applies to more general potentials. In our previous paper [1] we could only show that this result holds for motions where the energy can be treated as constant.

The random force that we use here is white noise and its strength depends on the frequency and the mass of the harmonic oscillator as well as the Planck’s constant. In our previous paper [1] we showed that there exists a colored Gaussian random force spectrum working for all frequencies. Finding a spectrum that would work for all mass values seems not possible in the present framework. However if include friction in the system then by modifying Nelson’s postulates one can in principle include all the dependence of mass in a universal friction term. This will also be the subject of future work.

Suppose we succeeded in finding a universal force working for all potentials. Then we can suspect that if all quantum behavior arises this way in nature: a classical particle or a system of particles or classical fields subject to noise. Then we ask: what is the nature of this noise? Since we give up on quantum mechanics as fundamental but still want the quantum behavior to emerge we need a universal noise working for all matter. The only source of such noise with the universality property is the gravitational field. There is a natural candidate for this gravitational noise: the gravitational field created by the the irregular distribution of matter in the universe. From kinetic theory applied to stars in galaxies (and kinetic theory applied to structure formation in cosmology) we know that a test star (any test particle) is subject to a random force due to its gravitational interaction with the rest of the stars (matter particles) in the galaxy (universe). However equivalence principle implies that in a background gravitational field matter falls at a rate independent of its mass. Therefore even if the noise is caused by the background gravitational field generated by the mass distribution in the universe we still need to include dependence on the mass of the particle in the equations. As discussed before this is in principle possible by including a friction term in the equations. Investigation of potential sources of noise and friction will also be subject of future work.

One obstruction against the possibility for a hidden variable theory as the one we presented here for multiple particles is Bell’s theorem. We know that [4, 5, 6] Nelson’s original Markovian stochastic mechanics is manifestly non-local for multiple particles. However in principle a non-Markovian version could yield a local theory [4, 5, 6]. We expect that when we apply our logic in this paper to multiple oscillators we would get non-Markovian processes in the joint position space of the particles since we are dimensionally reducing from the joint phase space process. The investigation of whether multi-particle entangled states can be understood from the framework presented here will also be subject of future work.

We note that in our framework the Schrödinger equation is the zeroth order approximation to the classical oscillator subject to a random force. One can in principle calculate higher order corrections. This would introduce systematic correction terms to the Schrödinger equation and would cause spontaneous decoherence. Thus if quantum mechanics emerges in the way we describe here, we have a stochastic process which is more fundamental than the Nelson process described by the Schrödinger equation. We remark that this had the potential settle down the measurement problem.
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A Review of stochastic differential equations

We give a brief review of Ito stochastic calculus and stochastic differential equations. We will only state results formally which are relevant for our purposes and refer the reader to standard textbooks on the subject (e.g. [2, 3]). Let $\xi(t)$ be the Gaussian process with zero mean and unit variance (also known as white noise), i.e.

$$\langle \xi(t) \rangle = 0, \quad \forall t \quad (A.1)$$

and for times $(t_1, t_2, ..., t_n)$, $(\xi(t_1), \xi(t_2), ..., \xi(t_n))$ are Gaussian correlated random variables with co-variance

$$\langle \xi(t_1)\xi(t_2) \rangle = \delta(t_1 - t_2). \quad (A.2)$$

Note that for $t_1 \neq t_2$, $\xi(t_1)$ and $\xi(t_2)$ are independent. We define the Wiener process $W(t)$ as the formal time integral of $\xi(t)$:

$$W(t) = \int_0^t \xi(s)ds \quad (A.3)$$

where we set the initial time to $t = 0$ without loss of generality. We can also write this as $dW(t) = \xi(t)dt$. The Wiener process is again Gaussian since it is a linear combination of independent Gaussian random variables. Its mean is zero as can be directly seen from the definition. Its co-variance is calculated as

$$\langle dW(t_1)dW(t_2) \rangle = \int_0^{t_1} \int_0^{t_2} \langle \xi(s_1)\xi(s_2) \rangle ds_1ds_2 = \min(t_1, t_2). \quad (A.4)$$

From this we see that formally $dW(t)$ is of order $\sqrt{dt}$. We will be dealing with stochastic differential equations in the rest of the paper. Suppose we would like to make sense of the following initial value problem for the scalar variable $x(t)$:

$$\frac{dx(t)}{dt} = f(x(t)) + g(x(t))\xi(t) \quad (A.5)$$

with $p(x, t = 0) = p_0(x)$ for some initial probability distribution $p_0(x)$. An ambiguity arises when we would like to make sense of the product $g(x(t))\xi(t)$. We know that since $\xi(t)$ is independent of $\xi(s)$ for $s < t$, it is independent of $g(x(s))$ for $s < t$. But the product concerns the same times. In order to remedy this difficulty we will write the equation in differential form:

$$dx(t) = f(x(t))dt + g(x(t))dW(t) \quad (A.6)$$

which is a formal way to write the integral equation:

$$x(t) = x_0 + \int_0^t f(x(s))ds + \int_0^t g(x(s))dW(s). \quad (A.7)$$

Now if we can make sense of the integral that includes $dW(s)$ term we can define the stochastic differential equation in terms of the integral equation. There are more than one ways to define
a stochastic integral. In this paper we will operate with the Ito definition. For the other famous (Stratonovich) definition see [2, 3]. We adopt the following definition:

\[
\int_0^t g(x(s))dW(s) = \lim_{\Delta s \to 0} \sum_i g(x(s_i))(dW(s_{i+1}) - dW(s_i))
\] (A.8)

where \(\Delta s = s_{i+1} - s_i, \forall i\). Therefore the increment \(dW(s_{i+1}) - dW(s_i)\) is independent of \(g(x(s_i))\). However with this definition we need to update the chain rule of calculus. Suppose that we would like to calculate the equation that is obeyed by a function of \(x\), say \(y = f(x)\). Remember that \(dW(t)\) is of order \(\sqrt{dt}\). Thus in order to correctly calculate \(dy\) we should expand it up to second order. Without proof we state the Ito’s lemma:

\[
dy = \frac{df}{dx} dx + \frac{1}{2} \frac{d^2 f}{dx^2} (dx)^2 = \frac{df}{dx} dx + \frac{1}{2} \frac{d^2 f}{dx^2} g^2(x) dt.
\] (A.9)

Note that in the expansion of \((dx)^2\) we omitted terms of order \(dt^{3/2}\) and only kept those of order \(dt\) and \(\sqrt{dt}\). We will also need the two dimensional version of this. Suppose we have two processes \(x_1\) and \(x_2\) with independent Wiener processes \(dW_1(t)\) and \(dW_2(t)\):

\[
dx_i(t) = f_i(x_1, x_2) dt + g_i(x_1, x_2) dW_i(t).
\] (A.10)

If \(y = f(x_1, x_2)\) then we can write the differential \(dy\) as

\[
dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{1}{2} \left( \frac{\partial}{\partial x_1} g_1^2 + \frac{\partial}{\partial x_2} g_2^2 \right) dt.
\] (A.11)

\section*{B  Review of Nelson’s stochastic mechanics}

We give a review of Nelson’s stochastic formulation of non-relativistic quantum mechanics in one dimension. For more details see Nelson’s original paper[7], his two books[8, 6] and Guerra’s review[9]. Consider the Schrödinger equation:

\[
i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x) \right) \psi(x,t).
\] (B.1)

Putting \(\psi(x,t) = \sqrt{\rho(x,t)} e^{iS(x,t)}\) we get the Madelung equations:

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} \left( \frac{1}{m} \frac{\partial S}{\partial x} \right)
\] (B.2)

\[
\frac{\partial S}{\partial t} = -\frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 - U(x) + \frac{\hbar^2}{2m\sqrt{\rho}} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \rho}{\partial x^2}
\] (B.3)

where \(\rho(x,t)\) is the probability of finding the particle at \((x,t)\) and \(S(x,t)\) is the phase of the wave function. We recognize first of the equations as the continuity equation with velocity \(\frac{1}{m} \frac{\partial S}{\partial x}\). The second of the equations apart from the last term (quantum potential) on the right hand side is the Hamilton-Jacobi equation. Thus if \(\hbar = 0\), we have the classical ensemble of particles. The Newton’s equations of motion are then the equations that characteristic curves obey corresponding to this set of Madelung partial differential equations. Since the quantum potential term depends on the probability \(\rho(x,t)\), giving deterministic characteristics seems not possible. However as Nelson proved[7, 8, 6], it is possible to give a Markovian stochastic
process associated to the solution of Madelung equations in position space. We start by assuming that a particle obeys the following stochastic differential equation:

\[ dx(t) = b(x(t), t) dt + \sqrt{\frac{\hbar}{m}} dW(t) \]  

(B.4)

where \( b(x(t), t) \) is a general function and \( dW \) is the Wiener process. We call this as Nelson’s first postulate. The diffusion equation associated to this is

\[ \frac{\partial \rho(x, t)}{\partial t} = -\frac{\partial}{\partial x} \left( b(x, t) \rho(x, t) \right) + \frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} \rho(x, t) \]  

(B.5)

where \( \rho(x, t) \) is the probability of finding the particle at \( x \) at time \( t \). In order to match with the continuity equation we define

\[ \frac{\partial}{\partial x} S(x, t) = m \left( b(x, t) - \frac{\hbar}{2m} \frac{\partial}{\partial x} \log \rho(x, t) \right) \]  

(B.6)

where we assumed that \( \rho(x, t) \) is nowhere zero. For a discussion of what happens at zeros see [6]. We want \( S(x, t) \) just defined in this way to satisfy the quantum Hamilton-Jacobi equation. We could postulate it as a partial differential equation but Nelson found a way to write this solely in terms of the stochastic particle trajectory. The quantum Hamilton-Jacobi equation can be shown to be equivalent to the following equation:

\[ \frac{1}{2} (D_+ D_- + D_- D_+) x(t) = -\frac{1}{m} \frac{\partial U[x]}{\partial x} |_{x(t)} \]  

(B.7)

where \( D_+ \) and \( D_- \) are forward and backward derivatives which will be defined below, the right hand side is the classical acceleration of the particle evaluated on the stochastic trajectory and the left hand side is the time-symmetric stochastic acceleration. This is the stochastic analogue of Newton’s second law. Thus we call this as Newton-Nelson law or Nelson’s second postulate. The forward and backward derivatives are defined to be

\[ D_+ x(t) = \lim_{\Delta t \to 0^+} E\left[ \frac{x(t + \Delta t) - x(t)}{\Delta t} \right] \]  

(B.8)

\[ D_- x(t) = \lim_{\Delta t \to 0^+} E\left[ \frac{x(t) - x(t - \Delta t)}{\Delta t} \right] \]  

(B.9)

where \( E[f|x(t)] \) denotes the expectation of \( f \) conditioned on \( x(t) \). For any function \( F(x, t) \) we can write its forward and backward derivatives explicitly as follows

\[ (D_+ F)(x, t) = \frac{\partial}{\partial t} F(x, t) + b(x, t) \frac{\partial}{\partial x} F(x, t) + \frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} F(x, t) \]  

(B.10)

\[ (D_- F)(x, t) = \frac{\partial}{\partial t} F(x, t) + \left( b(x, t) - \frac{\hbar}{m} \frac{\partial}{\partial x} \log \rho(x, t) \right) \frac{\partial}{\partial x} F(x, t) - \frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} F(x, t). \]  

(B.11)

The derivation of the formula for \( D_+ \) is straightforward but the calculation of \( D_- \) is subtler [8, 6, 9]. Using these formulas it is straightforward to show that the Newton-Nelson law is equivalent to the \( x \) derivative of the second Madelung equation (equation (B.3)). It has been shown that for each solution of the Schrödinger equation there is an associated stochastic process satisfying Nelson’s postulates and if Nelson’s postulates are satisfied that one can
construct a wave function which satisfies the Schrödinger equation with its absolute square the probability density of the position of particle. The stochastic formulation can be generalized to particles propagating in higher dimensions, multiple particles, fields and particles with spin \[6, 9\].

\section{Method of stochastic averaging}

We introduce the method of averaging of stochastic differential equations. There are several formulations of stochastic averaging though we will only consider the theorem due to Khas’minskii\[10\,11\,12\,13\,14\] applied to two dimensional systems in Ito form. Consider the process \((x,y)\):

\[
dx(t) = \epsilon^2 f_1(x(t), y(t), t)dt + \epsilon g_1(x(t), y(t), t)dW(t) \tag{C.1}
\]

\[
dy(t) = \epsilon^2 f_2(x(t), y(t), t)dt + \epsilon g_2(x(t), y(t), t)dW(t) \tag{C.2}
\]

where \(dW\) is the Wiener process and \(0 < \epsilon \ll 1\) which means that \((x,y)\) are slowly varying in time as compared to \(f_i\) and \(g_i\). We assume that \(f_i\) and \(g_i\) are sufficiently continuously differentiable and bounded. Then for times of order \(O(1/\epsilon^2)\) the dynamics can be uniformly (over the time interval) approximated by the following averaged system\[13\]:

\[
dx(t) = \epsilon^2 \bar{f}_1(x(t), y(t))dt + \epsilon \sigma_{11}(x(t), y(t))dW_1(t) + \epsilon \sigma_{12}(x(t), y(t))dW_2(t) \tag{C.3}
\]

\[
dy(t) = \epsilon^2 \bar{f}_2(x(t), y(t))dt + \epsilon \sigma_{21}(x(t), y(t))dW_1(t) + \epsilon \sigma_{22}(x(t), y(t))dW_2(t) \tag{C.4}
\]

where \(dW_1\) and \(dW_2\) are independent Wiener processes and the averaged functions are given by

\[
\bar{f}_i(x,y) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f_i(x,y,t)dt \tag{C.5}
\]

\[
(\sigma \sigma^T)_{ij}(x,y) = \lim_{T \to \infty} \frac{1}{T} \int_0^T g_i(x,y,t)g_j(x,y,t)dt \tag{C.6}
\]

where \(\sigma^T\) denotes the matrix transpose of \(\sigma\). Note that \(\sigma\) is unique up to a time dependent orthogonal transformation \(R(t)\) as \(\sigma R(t)(\sigma R(t))^T = \sigma^T\). It can be shown that \(R(t)[dW_1(t) \, dW_2(t)]^T\) is again a Wiener process therefore replacing \(\sigma\) with \(\sigma R(t)\) does not change the diffusion process \((x,y)\). For applications below we need the periodic version of averaging. For periodic systems we can write

\[
\bar{f}_i(x,y) = \frac{1}{T} \int_0^T f_i(x,y,t)dt \tag{C.7}
\]

\[
(\sigma \sigma^T)_{ij}(x,y) = \frac{1}{T} \int_0^T g_i(x,y,t)g_j(x,y,t)dt \tag{C.8}
\]

\[1\] More precisely the original process converges weakly to the averaged process as \(\epsilon \to 0\).
where $T$ is the period of oscillations which correspond to $dx = dy = 0$. The way that $dx = dy = 0$ corresponds to a periodic deterministic solution will be clarified in the examples in the following sections. The stochastic averaging principle is a generalization of its deterministic version which can be found in [15]. For deterministic averaging of a one dimensional system in action-angle variables see [16]. For more on stochastic averaging see the review [13] and the books [12, 14].

D Newton-Nelson law

We will show that the Newton-Nelson law is satisfied by a particle in a potential in one dimension subject to a random force. Consider a particle of mass $m$ in a potential $U(x)$ subject to a random force:

\[
\begin{align*}
\frac{dx(t)}{dt} &= v(t) \\
\frac{dv(t)}{dt} &= a(x(t)) + \sigma dW(t)
\end{align*}
\]  

\[ (D.1) \]

where $(x, v)$ denotes the position and velocity variables, $dW$ is the Wiener process, $\sigma$ is a positive constant and

\[
a(x) = -\frac{U'(x)}{m} = -\frac{dU(x)}{dx}.
\]  

\[ (D.2) \]

We will make use of the following formulas for forward and backward derivatives conditioned on fixed $(x(t), v(t))$ of a function $G(x, v, t)$ which can be found in section 5 of Guerra’s review [9]:

\[
(D_+ G)(x, v, t)|_{(x(t), v(t))} = \lim_{\Delta t \to 0^+} E\left[\frac{G(x(t + \Delta t), v(t + \Delta t), t + \Delta t) - G(x(t), v(t), t)}{\Delta t} | x(t), v(t)\right]
\]

\[ (D.3) \]

\[
= \frac{\partial G}{\partial t} + v \frac{\partial G}{\partial x} + a(x) \frac{\partial G}{\partial v} + \frac{\sigma^2}{2} \frac{\partial^2 G}{\partial v^2}
\]

\[
(D_- G)(x, v, t)|_{(x(t), v(t))} = \lim_{\Delta t \to 0^+} E\left[\frac{G(x(t), v(t), t) - G(x(t - \Delta t), v(t - \Delta t), t - \Delta t)}{\Delta t} | x(t), v(t)\right]
\]

\[ (D.4) \]

\[
= \frac{\partial G}{\partial t} + v \frac{\partial G}{\partial x} + (a(x) - \sigma^2 \frac{\partial}{\partial v} \log \rho(x, v, t)) \frac{\partial G}{\partial v} - \frac{\sigma^2}{2} \frac{\partial^2 G}{\partial v^2}
\]

where $\rho(x, v, t)$ is the probability of finding the particle at $x$ with velocity $v$ at time $t$. We also need the following result on conditional expectations for a set of random variables $(x, y, z)$:

\[
E[F(z)|x] = \int E[F(z)|x, v]p(v|x) dv
\]  

\[ (D.5) \]

for any function $F(z)$. To derive Newton-Nelson law we will calculate the stochastic acceleration $\frac{1}{2}(D_+ D_- + D_- D_+)x$. From equations [12.10] and [12.11] we see that

\[
D_+ x(t) = D_- x(t) = v(t)
\]  

\[ (D.6) \]
where conditioning on \( v(t) \) does not matter. Next we calculate \( D_+ D_- x(t) \) and \( D_- D_+ x(t) \) conditioned on \((x(t), v(t))\) using equations \[D.3\] and \[D.4\]

\[
D_+ D_- x(t) = D_+ |_{(x(t), v(t))} v(t) = a(x(t))
\]

\[
D_- D_+ x(t) = D_- |_{(x(t), v(t))} v(t) = a(x(t)) - \sigma^2 \frac{\partial}{\partial v} \log \rho(x, v, t).
\]

Hence

\[
\frac{1}{2} (D_+ D_- + D_- D_+) x(t) |_{(x(t), v(t))} = a(x(t)) - \sigma^2 \frac{\partial}{\partial v} \log \rho(x, v, t).
\]

In order to calculate the stochastic acceleration, which is conditioned only on \( x(t) \), we use equation \[D.5\]

\[
\frac{1}{2} (D_+ D_- + D_- D_+) x(t) |_{x(t)} = \int \frac{1}{2} (D_+ D_- + D_- D_+) x(t) |_{(x(t), v(t))} p_t(v | x) dv
\]

\[
= a(x(t)) - \frac{\sigma^2}{2} \int \frac{\partial p_t(v | x)}{\partial v} dv = a(x(t)).
\]

Thus we have shown that the Newton-Nelson law is satisfied by the process given by equation \[D.1\] This result was stated without proof in [8] for the particle in a potential subject to linear friction in equilibrium.