The Runge-Kutta Method in Geometric Multiplicative Calculus

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Abstract

The Runge-Kutta Method is derived in the framework of geometric multiplicative calculus in analogy to the established ordinary Runge-Kutta Method. A Butcher Schema like behavior can be observed in the multiplicative case as well. In order to remove the restriction of geometric Multiplicative Calculus of being only applicable to positive valued functions of real variable, Complex Multiplicative Calculus is used to extend the Multiplicative Runge-Kutta Method to complex valued functions of real variable, including real valued functions of real variable. Furthermore, different methods to circumvent the breakdown of the multiplicative derivative at the roots of the functions are presented and discussed, resulting in the universal applicability of the proposed method. Finally, the presented technique is applied to selected well known problems in Biology, Physics, and Mathematics.

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1. Introduction

The solutions of many problems in science and engineering are of exponential or trigonometric nature, which is apparently qualitatively the same in complex calculus. So, what is more nearby, than including the nature of the solution of the problem into the modeling process or the solution process?
Both approaches can be accomplished by using geometric multiplicative calculus \[^1\].

The invention of Multiplicative Calculus can be dated back to 1938, when Volterra \[^2\] proposed the so-called Volterra Calculus, that was later identified as a special case of multiplicative calculus. After approximately 30 years of silence in this field, Multiplicative, or as Grossmann and Katz called it, Geometric Calculus is discussed as one of the Non-Newtonian Calculi in \[^1\]. Later, Grossmann studied Bigeometric Calculus or Volterra Calculus explicitly \[^3\]. Finally the complete mathematical description of geometric multiplicative calculus, was given by Bashirov et al. \[^4\]. Based on this work various multiplicative numerical approximation techniques were proposed and discussed \[^5\] [6] [7] [8]. Multiplicative Calculus has also found its way into biomedical image analysis \[^9\] or modelling with differential equations \[^10\]. Furthermore even the Runge-Kutta method was developed in the framework of Bigeometric or Volterra calculus for applications in dynamic systems by Aniszewska et al. in \[^11\].

In order to ease the reading of this paper, we will use multiplicative calculus synonymical to geometric multiplicative calculus.

One drawback of Multiplicative Calculus, generally put forward, is that Multiplicative Calculus can only be applied to positive-valued functions of real variable. As shown in \[^12\], \[^13\], and \[^14\] this restriction can be easily removed by the usage of complex multiplicative calculus. The fact that the derivative is a local property suggests the extension to the complex domain. In the following we will focus on complex valued functions of real variable, in order to remove the restriction to deal with purely positive functions. The multiplicative Cauchy-Riemann conditions become trivial in this case.

Then in section \[^2\] the multiplicative Euler method, the 3rd and 4th order multiplicative Runge-Kutta method for positive-valued functions of real variable will be elaborated, and the extension to complex-valued functions of real variable will be presented. As everybody knows, another drawback of multiplicative calculus is the breakdown of the multiplicative derivative at the roots of the functions. Section \[^2.4\] covers the solution to that problem as well. In section \[^3\] the multiplicative Runge-Kutta Method will be applied to well known problems in Biology, Physics, and Mathematics and the results will be compared with the results of the ordinary Runge-Kutta Method. Finally, all the findings will be summarised in the section \[^4\].

Appendix \[^A\] gives a short overview of multiplicative and complex multiplicative calculus.
2. Multiplicative Runge Kutta Method for real-valued functions of real variable

In this section the Multiplicative Runge-Kutta Method, also referred as MRK-method in the following, will be derived for the 2nd order case exemplarily explicitly. Only the starting equations and the results of the third and fourth order MRK-method will be presented.

The methods being derived in the following will be used to find suitable approximations to the solution of multiplicative initial value problems of the form:

\[ y^*(x) = f(x, y), \]

with the initial condition

\[ y(x_0) = y_0. \]  

2.1. 2nd order MRK method

The simplest approach to find an approximation to the solution of the differential equation (1) with the initial value (2) is the second order Runge-Kutta Method, also known as the Euler method. In analogy to the ordinary Euler method we will derive in the following the second order Multiplicative Runge-Kutta method or the Multiplicative Euler method by making the ansatz:

\[ y(x + h) = y(x) \cdot f_0^ah \cdot f_1^{bh}, \]

where

\[ f_0 = f(x, y), \text{ and} \]
\[ f_1 = f(x + ph, y \cdot f_0^{qh}). \]

The multiplicative taylor expansion of \( y(x + h) \) up to order 2 is given as

\[ y(x + h) = y(x) \cdot y^*(x)^h \cdot y^{**}(x)^{h^2/2} \cdot ..., \]

Remembering that,

\[ y^*(x) = f(x, y) \quad \text{and} \quad y^{**}(x) = f_x^*(x, y) \cdot f_y^*(x, y)^{y\ln f(x, y)} \]

Then the multiplicative Taylor expansion of \( y(x + h) \) becomes

\[ y(x + h) = y(x) \cdot f(x, y)^h \cdot f_x^*(x, y)^{h^2/2} \cdot f_y^*(x, y)^{y\ln f(x, y)^{h^2/2}}. \]
where \( f_\times^* (x, y) \) denotes the multiplicative partial derivative with respect to \( x \) and \( f_y^* (x, y) \) with respect to \( y \) respectively.

In order to compare (8) with (3) we need to expand also \( f_1 \) using the multiplicative taylor theorem up to order 1 as the power of the ansatz (3) also includes one \( h \). Recalling that \( y \) is a function of \( x \) the taylor expansion for \( f_1 \) becomes by the application of the chain rule of multiplicative derivative

\[
f_1 = f(x, y) \cdot f_\times^* (x, y)^{ph} \cdot f_y^* (x, y)^{qh \ln f_0}.
\]

With \( f_0 = f(x, y) \), the Taylor expansion of \( f_1 \) up to order 1 in \( h \) becomes

\[
f_1 = f(x, y) \cdot f_\times^* (x, y)^{ph} \cdot f_y^* (x, y)^{qh \ln f(x,y)}.
\]  \hspace{1cm} (9)

Then by substituting (9) and (4) in (3), we get the Multiplicative Runge-Kutta expansion for the comparison with the multiplicative Taylor expansion of (8) as

\[
y(x + h) = y(x) \cdot f(x, y)^{(a+b)h} \cdot f_\times (x, y)^{bph^2} \cdot f_y (x, y)^{bqh^2} \ln f(x,y)^{bqh^2} \]  \hspace{1cm} (10)

Comparison of the powers of \( f(x, y) \) and its partial derivatives in (10) with (8) up to order 2 in \( h \) gives:

\[
a + b = 1 \]  \hspace{1cm} (11)
\[
bp = \frac{1}{2} \]  \hspace{1cm} (12)
\[
bq = \frac{1}{2} \]  \hspace{1cm} (13)

Obviously, we have infinitely many solutions of the equations (11)-(13), as the number of unknowns is greater than the number of equations. Furthermore, we can see that \( p = q \) and \( a + b = 1 \), which can be easily reflected in analogy to the regular Butcher Tableau [15] as the multiplicative Butcher Tableau as following.

\[
0 \\
\frac{p}{a} \begin{bmatrix} q \\ b \end{bmatrix}
\]

One possible choice of the parameters \( a, b, p, \) and \( q \) is as following:

\[
a = \frac{1}{2}, b = \frac{1}{2}, p = 1, \text{ and } q = 1.
\]
Here we can see that we evaluate the function at the endpoints of the interval, and give equal weights to the contributions of $f_0$ and $f_1$, resulting in the multiplicative Euler method formulae

\[
\begin{align*}
y(x + h) & = y(x) \cdot f_0^h \cdot f_1^h \\
f_0 & = f(x, y) \\
f_1 & = f(x + h, y \cdot f_0^h).
\end{align*}
\]

Of course, depending on the problem the parameters can be also chosen differently in order to satisfy the equations (11)-(13).

2.2. 3rd order MRK method

For the derivation of the 3rd order Multiplicative Runge-Kutta method we make the ansatz.

\[
\begin{align*}
y(x + h) & = y(x) \cdot f_0^{ah} \cdot f_1^{bh} \cdot f_2^{ch} \\
f_0 & = f(x, y) \\
f_1 & = f(x + ph, y \cdot f_0^{qh}) \\
f_2 & = f(x + p_1h, y \cdot f_0^{q_1h} \cdot f_1^{q_2h})
\end{align*}
\]

Analogously to the derivation of the multiplicative Euler method, presented above, the 3rd order multiplicative Runge-Kutta method is derived on the basis of the equations (14)-(17) resulting:

\[
\begin{align*}
p & = q \\
p_1 & = q_1 + q_2
\end{align*}
\]

and

\[
\begin{align*}
a + b + c & = 1 \\
bp + cp_1 & = \frac{1}{2} \\
bp^2 + cp_1^2 & = \frac{1}{3}
\end{align*}
\]
As \( p \) and \( p_1 \) are determined by the choices of \( q, q_1, \) and \( q_2 \) we have to solve the set of equations (20)-(22) with respect to \( a, b, c \) as functions of \( p \) and \( p_1 \) and get:

\[
a = -\frac{6 p p_1 + 3 p + 3 p_1 - 2}{6 p p_1} \tag{23}
\]

\[
b = -\frac{3 p_1 - 2}{6 p (p - p_1)} \tag{24}
\]

\[
c = -\frac{2 - 3 p}{6 p_1 (p - p_1)} \tag{25}
\]

Resulting in the multiplicative Butcher Tableau

\[
\begin{array}{c|ccc}
0 & p & q & p_1 \\
p & q & q_1 & q_2 \\
p_1 & a & b & c
\end{array}
\]

As before, the number of solutions are infinitely many, as the number of unknowns is larger than the number of independent equations. One approach of choosing the constants is to evaluate the functions at the beginning of the interval (for \( f_0 \)), in the mid of the interval (for \( f_1 \)), and at the end of the interval (for \( f_2 \)). Furthermore we give equal weights to the function evaluated at the left and right endpoint of the interval of length \( h \), and double the weight for the value in the mid of the interval; resulting in the constants \( a = \frac{1}{6}, b = \frac{2}{3}, c = \frac{1}{6}, p = \frac{1}{2}, p_1 = 1, q = \frac{1}{2}, q_1 = -1, q_2 = 2 \) and we get for the 3rd order MRK-method:

\[
y(x + h) = y(x) \cdot f_0^h \cdot f_1^{2h} \cdot f_2^h
\]

\[
f_0 = f(x, y)
\]

\[
f_1 = f(x + \frac{h}{2}, y \cdot f_0^h)
\]

\[
f_2 = f(x + h, y \cdot f_0^{-h} \cdot f_1^{2h})
\]

### 2.3. 4th order MRK method

In practice, mainly the 4th order Runge-Kutta method is used. In analogy to the above described 2nd and 3rd order multiplicative Runge-Kutta methods, we will now employ the 4th order multiplicative Runge-Kutta method. Therefore we make the ansatz
\[
y(x + h) = y(x) \cdot f_0^{ah} \cdot f_1^{bh} \cdot f_2^{ch} \cdot f_3^{dh}
\]
\[
f_0 = f(x, y)
\]
\[
f_1 = f(x + ph, y \cdot f_0^{qh})
\]
\[
f_2 = f(x + p_1h, y \cdot f_0^{qh} \cdot f_1^{q_1h})
\]
\[
f_3 = f(x + p_2h, y \cdot f_0^{qh} \cdot f_1^{q_1h} \cdot f_2^{q_2h})
\]

Again we need to find the Taylor expansions of \( f_0, f_1, f_2 \) and \( f_3 \) in order to substitute into the 4th order multiplicative Runge-Kutta formula, and compare it with the Taylor expansion of \( y(x + h) \) up to order 4. After a lengthy calculation we get by comparison, the following set of equations

\[
p = q \tag{26}
\]
\[
p_1 = q_1 + q_2 \tag{27}
\]
\[
p_2 = q_3 + q_4 + q_5 \tag{28}
\]

and

\[
a + b + c + d = 1 \tag{29}
\]
\[
b p + c p_1 + d p_2 = \frac{1}{2} \tag{30}
\]
\[
b p^2 + c p_1^2 + d p_2^2 = \frac{1}{3} \tag{31}
\]

As \( p, p_1, \) and \( p_2 \) are determined by the choices of \( q, q_1, q_2, q_3, q_4, \) and \( q_5 \) we have to solve the set of equations (29)-(31) with respect to \( b, c, d \) as functions of \( a, p, p_1, p_2 \) and get:

\[
b = -\frac{6a p_1 p_2 - 6 p_1 p_2 + 3 p_1 + 3 p_2 - 2}{6(p - p_1)(p - p_2)} \tag{32}
\]
\[
c = -\frac{-6a p p_2 + 6 p p_2 - 3 p - 3 p_2 + 2}{6(p - p_1)(p_1 - p_2)} \tag{33}
\]
\[
d = -\frac{6a p p_1 - 6 p p_1 + 3 p + 3 p_1 - 2}{6(p - p_2)(p_1 - p_2)} \tag{34}
\]
Resulting in the multiplicative Butcher Tableau

|   | 0 |   |   |   |
|---|---|---|---|---|
| p | q |   |   |   |
| p1| q1| q2|   |   |
| p2| q3| q4| q5|   |
| a | b | c | d |   |

We can easily see if the function \( f(x, y) \) is independent of \( y \), the result is independent from the selection of \( q_1 \) to \( q_5 \), and therefore any selection will give the same result.

The most naive approach in selecting the coefficients is to distribute them evenly, i.e. we split the interval into 4 equidistant regions and evaluate the functions at the beginning, at one third of the interval \( p = 1/3 \), two thirds of the interval \( p_1 = 2/3 \), and at the end of the interval \( p_2 = 1 \). Furthermore the weights of the functions should also be distributed evenly, i.e. \( a = b = c = d = 1/4 \) resulting finally in the coefficients \( q_1 = 1/3 \), \( q_2 = 1/3 \), and \( q_3 = q_4 = q_5 = 1/3 \). Although this trivial approach looks very promising, we want to compare the results for different coefficients explicitly for the following multiplicative initial value problem

\[
y^*(x) = e^{\frac{x-1}{x^2}}, \quad \text{with } y(1) = 1.
\]  

(35)

The corresponding ordinary initial value problem is given as:

\[
y'(x) = 1 - \frac{1}{x} \quad \text{with } y(1) = 1.
\]  

(36)

The analytic solution of this initial value problem can be easily obtained as

\[
y(x) = x - \ln x.
\]  

(37)

First we look at the most naive approach, i.e to distribute everything evenly, represented in the following Butcher tableau (38):

|   | 0 | 1/3 | 3/3 | 1 |
|---|---|-----|-----|---|
| 1/2| 1/3 | 1/3 | 1/3 | 1/3 |
| 1  | 1/4 | 1/4 | 1/4 | 1/4 |

(38)
In the following table we compare the results of the multiplicative Runge-Kutta Method and the ordinary Runge-Kutta Method for the coefficients in (38).

| $x$ | $y_{mult}$ | $y_{newt}$ | $y_{exact}$ | relative err $\text{mult}$ in % | relative err $\text{newt}$ in % |
|-----|------------|------------|-------------|-------------------------------|-------------------------------|
| 1   | 1          | 1          | 1           | 0                             | 0                             |
| 1.5 | 1.0922     | 1.2119     | 1.0945      | 0.2132                        | 10.72                         |
| 2   | 1.3036     | 1.4885     | 1.3069      | 0.2518                        | 13.90                         |
| 2.5 | 1.5801     | 1.8061     | 1.5837      | 0.2278                        | 14.04                         |
| 3   | 1.8977     | 2.1519     | 1.9014      | 0.1922                        | 13.17                         |

Table 1: Comparison of the results of the multiplicative and ordinary 4th order Runge-Kutta Method for the parameters in (38).

In this simple example we can observe that the results of the multiplicative Runge-Kutta Method is better by three orders in magnitude concerning the relative error for the same step size.

Nevertheless, the choice of the coefficients in analogy to the ordinary Runge-Kutta method turns out to be the most preferable in this case. If we choose the constants $a = \frac{1}{6}, b = \frac{1}{3}, c = \frac{1}{3}, d = \frac{1}{6}, p = \frac{1}{2}, p_1 = \frac{1}{2}, p_2 = 1, q = \frac{1}{2}, q_2 = \frac{1}{2}, q_5 = 1, q_1 = q_3 = q_4 = 0$ we get the 4th order multiplicative Runge-Kutta method:

$$y(x + h) = y(x) \cdot f_0^h \cdot f_1^h \cdot f_2^h \cdot f_3^h$$

$$f_0 = f(x, y)$$

$$f_1 = f(x + \frac{h}{2}, y \cdot f_0^h)$$

$$f_2 = f(x + \frac{h}{2}, y \cdot f_1^h)$$

$$f_3 = f(x + h, y \cdot f_2^h)$$

\[ (39) \]
The comparison of the results for the coefficients according to (39) is given in the table below.

| $x$ | $y_{\text{mult}}$ | $y_{\text{newt}}$ | $y_{\text{exact}}$ | $\text{err}_{\text{mult}}$ | $\text{relative err}_{\text{mult}}$ | $\text{err}_{\text{newt}}$ | $\text{relative err}_{\text{newt}}$ |
|-----|------------------|-------------------|-------------------|-----------------|-------------------------------|-----------------|-------------------------------|
| 1   | 1                | 1                 | 1                 | 0               | 0                             | 0               | 0                             |
| 1.5 | 1.0945           | 1.2123            | 1.0945            | $3.128 \times 10^{-3}$ | 10.76                        | 0               | 0                             |
| 2   | 1.3068           | 1.4892            | 1.3069            | $2.955 \times 10^{-3}$ | 13.95                        | 0               | 0                             |
| 2.5 | 1.5837           | 1.8068            | 1.5837            | $2.681 \times 10^{-3}$ | 14.09                        | 0               | 0                             |
| 3   | 1.9013           | 2.1527            | 1.9014            | $2.339 \times 10^{-3}$ | 13.22                        | 0               | 0                             |

Table 2: Comparison of the results of the multiplicative and ordinary 4th order Runge-Kutta Method for the parameters given in (39).

We can observe that the change of the coefficients in the Butcher tableau did not result in drastic changes in the case of the ordinary 4th order Runge-Kutta Method, but in this example the relative error drops by two orders of magnitude in the case of 4th order Multiplicative Runge-Kutta Method. In this example the advantage of the Multiplicative Runge-Kutta Method over the Newtonian version can be seen quite easily. Also the sensitivity of the results to the values in the multiplicative Butcher tableau becomes evident and must be analyzed separately in detail.

2.4. Extension to complex valued functions of real variable

One of the drawbacks of Multiplicative Calculus, generally put forward, is its restriction to positive valued functions of real variable. In order to overcome this restriction the theory of Multiplicative Calculus was extended to the complex domain. The first approach was made by Uzer in [14], but Bashirov and Riza gave a more detailed analysis of the complex multiplicative derivative in [12]. As it is well known from complex analysis, the differentiation rules are a little bit more complicated for complex valued functions of complex variable as the Cauchy-Riemann conditions have to be satisfied. But here we are only interested in complex valued functions of real variable, which simplifies the issue drastically, as the multiplicative counterparts of the Cauchy-Riemann conditions have not to be taken into account and the differentiation can be carried out independently for the real and the imaginary part. As illustrated in [12] the multiplicative derivative can be calculated
everywhere except at the point $0 + 0i$ in the complex plane. So the 4th order multiplicative Runge-Kutta Method works now also for negative valued functions as the phase factor is responsible for the change of the sign. The only problem that could not be solved by extending Multiplicative Calculus to the complex domain is that the Multiplicative derivative is not defined at the roots of the function. So a switch to Newtonian Calculus becomes inevitable at these points. In every step of the Multiplicative Runge-Kutta Method we get the value of the function at this point and its multiplicative derivative at this point. The easiest approach is, to perform a linear approximation in the sense of Newtonian Calculus as following. At each step we get $f(x_i)$ and $f^*(x_i)$, if $f^*(x_{i+1})$ is not defined, i.e. the Multiplicative Derivative is not defined because $f(x_{i+1}) \approx 0$, then we can calculate $f'(x_i) = f(x_i) \ln f^*(x_i)$ and we can apply the linear approximation as following:

$$L(x, x_i) = f(x_i) + \left( f(x_i) \ln f^*(x_i) \right) (x - x_i)$$

Then the linear approximation should be used until $f^*(x)$ becomes reasonably large, and we continue the Multiplicative Runge-Kutta method from this point on. The results show that the linear approximation increases the error of the approximation significantly. Although it helps us to circumvent the problem at the roots of the function the quality of the approximation becomes worse. Of course other higher order approximations have also been tested with a similar result. But if we use the ordinary Runge-Kutta Method for a couple of steps until the multiplicative derivative becomes again reasonably large and this values are then used as input of the Multiplicative Runge-Kutta method, the results are reasonably good, and often even better than using the ordinary Runge-Kutta method alone.

If we assume that $f(x_{i-1}) > 0$, and $f(x_{i+1}) < 0$ and that the function is decreasing, then accordingly there must be a point $\xi \in [x_{i-1}, x_{i+1}]$ where $f(\xi) = 0$. In this case the multiplicative derivative of $f(x)$ is not defined at $\xi$. Therefore the Multiplicative Runge-Kutta method will be applied on the interval $[x_0, x_{i-1}]$, and on the interval $[x_{i+1}, x_n]$. On the interval $[x_{i-1}, x_{i+1}]$ we apply the ordinary Runge-Kutta Method, using the values $f(x_{i-1})$ and $f^*(x_{i-1})$ calculated by the Multiplicative Runge-Kutta Method as input for the ordinary Runge-Kutta Method, and vice versa for the point $x_{i+1}$. 


Figure 1: Bypass the roots where the multiplicative derivative becomes undefined. The dashed line denotes the region where the ordinary Runge-Kutta method is applied to prevent the multiplicative derivate to become infinite. The multiplicative Runge-Kutta method is applied in the region of the solid line.

The handover has been tested on several examples, e.g. on the Rössler problem discussed in section 3.3, and it turned out to work properly.

3. Examples for the Multiplicative Runge Kutta Method

3.1. Solution of a first order multiplicative differential equation (biological example)

As the first example we want to discuss the bacterial growth in food modelled by Huang [16, 17, 18].

Given is the Baranyi model [19, 20] for the bacterial growth in food described using the differential equation.

\[ y'(t) = \mu_{\text{max}} \frac{1 - e^{y - y_{\text{max}}}}{1 + e^{-\alpha(t-\lambda)}} \] (40)

The multiplicative counter part of the equation (40) is:

\[ y^*(t) = \exp \left\{ \mu_{\text{max}} \frac{1 - e^{y - y_{\text{max}}}}{\frac{y}{1 + e^{-\alpha(t-\lambda)}}} \right\} \] (41)

with the initial value \( y_0 = y(0) = 7. \)
The numerical solutions of the differential equations (40) and (41) using the corresponding Runge-Kutta Methods are not distinguishable for $h = 0.1$. But, as depicted in figure 2, the 4th order MRK - method for $h = 1$ still coincides with the solution for $h = 0.1$, whereas the ordinary 4th order Runge-Kutta method is significantly different (dotted line).

### 3.2. Solution of a second order multiplicative differential equation

Second order multiplicative initial value problems can be solved using the Multiplicative Runge-Kutta method. The solution for the second order multiplicative initial value problem

$$y^{**}(x) = f(x, y, y^*)$$  \hspace{1cm} (42)

with the initial values

$$y(x_0) = y_0, \quad \text{and} \quad y^*(x_0) = y_1$$ \hspace{1cm} (43)

can be solved by solving the coupled system of first order multiplicative differential equations

$$y^*_0(x) = y_1(x)$$ \hspace{1cm} (44)

$$y^*_1(x) = f(x, y_0, y_1).$$ \hspace{1cm} (45)
Exemplarily we want to solve the initial value problem for the 2nd order multiplicative differential equation

\[ y^{**}(x) = e. \]  \hspace{1cm} (46)

The corresponding ordinary second order differential equation is

\[ y''(x) = \frac{y'(x)^2}{y(x)} + y(x). \]  \hspace{1cm} (47)

The general solution of the differential equations (46) and (47) is

\[ y(x) = \alpha \exp\left\{ \frac{x^2}{2} + \beta x \right\}. \]  \hspace{1cm} (48)

In order to be able to compare the results with the multiplicative finite difference methods solution, discussed in [7], we select \( \alpha = 1, \beta = 1, x_0 = 1, \) and \( h = 0.25, \) resulting in the initial conditions

\[ y_0 = e^{3/2} \quad \text{and} \quad y_1 = e^2, \]  \hspace{1cm} (49)

and compare the results in the following table.

| \( x \) | \( y_{MRK} \) | \( y_{exact} \) | \text{relative} \( \text{err}_{MRK} \) in % |
| --- | --- | --- | --- |
| 1.0 | 4.481689070 | 4.48168907 | 0 |
| 1.25 | 7.62360992 | 7.62360992 | 9.3 \times 10^{-15} |
| 1.50 | 13.80457419 | 13.80457419 | 1.3 \times 10^{-14} |
| 1.75 | 26.60901319 | 26.60901319 | 1.7 \times 10^{-14} |

Table 3: results of the Multiplicative Runge-Kutta Method

| \( x \) | \( y_{MFD} \) | \( y_{exact} \) | \text{relative} \( \text{err}_{MFD} \) in % |
| --- | --- | --- | --- |
| 1.0 | 4.48168907 | 4.48168907 | 0 |
| 1.25 | 7.62360992 | 7.62360992 | 3.5 \times 10^{-13} |
| 1.50 | 13.80457419 | 13.80457419 | 5.3 \times 10^{-13} |
| 1.75 | 26.60913187 | 26.60901319 | 1.8 \times 10^{-13} |

Table 4: results of the Multiplicative Finite Difference method from [7]
Tables 3 and 4 show that the numerical solution of the multiplicative differential equation (46) with the initial conditions (49) and the corresponding results for the multiplicative finite difference method from [7]. In this case we can see that the Multiplicative Runge-Kutta method is slightly better than the Multiplicative Finite Difference method by one order of magnitude in the relative error. On the other hand if we solve the corresponding ordinary differential equation (47) with the corresponding initial values

\[ y_0 = e^{3/2} \quad \text{and} \quad y_1 = 2e^{3/2} \quad (50) \]

we get:

| x  | \( y_{\text{newt}} \) | \( y_{\text{exact}} \) | \( \text{relative err}_{\text{newt}} \) in % |
|-----|-----------------|-----------------|-------------------------------|
| 1   | 4.48168907     | 4.48168907     | 0                             |
| 1.25| 7.61823131     | 7.62360992     | 7.1 \times 10^{-2}            |
| 1.5 | 13.77941017    | 13.80457419    | 1.8 \times 10^{-1}           |
| 1.75| 26.51619718    | 26.60901319    | 3.5 \times 10^{-1}           |

Table 5: Results of the ordinary Runge-Kutta method

Obviously the ordinary Runge-Kutta method fails drastically in this case, as the relative error differs by 13 orders in magnitude compared to its multiplicative counterpart. Whereas the multiplicative Runge-Kutta method, as well as the multiplicative finite difference method succeed to produce proper results.

3.3. Solution of a system of multiplicative differential equation (Rössler problem)

Finally we want also show that the method, developed for the universal applicability of the Multiplicative Runge-Kutta Method in section 2.4, works without major problems. Therefore we chose exemplarily the Rössler attractor [21, 22]; also to show that the MRK method can be extended to higher dimensions. Obviously in the Rössler attractor problem \( x(t), y(t) \) have roots and therefore the MRK method seems not be applicable. Using the extension proposed in section 2.4 the Rössler attractor problem becomes accessible also for the MRK method producing reasonable results.

In the following we will give the general equations for the Rössler attractor problem in ordinary calculus and its multiplicative counterpart.
\[
\dot{x}(t) = -y(t) - z(t) \quad (51)
\]
\[
\dot{y}(t) = x(t) + \alpha y(t) \quad (52)
\]
\[
\dot{z}(t) = \beta + (x(t) - \gamma)z(t) \quad (53)
\]

The corresponding multiplicative counterparts of the equations (51)-(53) are then

\[
x^*(t) = \exp\left\{-\frac{(y(t) + z(t))}{x(t)}\right\} \quad (54)
\]
\[
y^*(t) = \exp\left\{\frac{x(t) + \alpha y(t)}{y(t)}\right\} \quad (55)
\]
\[
z^*(t) = \exp\left\{\frac{\beta + (x(t) - \gamma)z(t)}{z(t)}\right\} \quad (56)
\]

Solving the system of multiplicative differential equations (54)-(56) using the 4th order MRK method for the parameters, \(\alpha = \beta = 0.2\) and \(\gamma = 8\) we get the result depicted in figure 3. The results of the ordinary Runge-Kutta and the multiplicative Runge-Kutta methods are comparable.

Figure 3: Rössler problem with the parameters \(\alpha = \beta = 0.2\) and \(\gamma = 8\).
multiplicative calculus, were tested explicitly and the usage of the ordinary Runge-Kutta method for the transition region of the function gave the best results.

4. Conclusion

After a short motivation of the problem in the introduction, we described the multiplicative Runge-Kutta method for the solution of multiplicative initial value problems of the form

\[ y^*(x) = f(x, y), \text{ with } y(x_0) = y_0, \]

where \( x_0 \) is the starting point and \( y_0 \) the initial value. The derivation of the 2nd order multiplicative Runge-Kutta method was carried out explicitly in detail. For the higher order methods the ansatzes, the solutions, as well as the corresponding Butcher tableaus are presented. Furthermore different choices for the parameters are discussed. Another important approach presented, is the strategy how to lift the obstacles caused by the definition of the geometric multiplicative derivative is presented and discussed in section 2.4.

The proposed multiplicative Runge-Kutta method is applied exemplarily to a problem in Biology describing the bacterial growth in food according to the Baranyi model, the results are compared with the results from the ordinary Runge-Kutta method, and we could observe that the multiplicative Runge-Kutta method gives the correct result for significantly fewer points than the ordinary Runge-Kutta method. For systems of coupled multiplicative differential equations, we discussed one example of a second order multiplicative initial value problem, and compared the results with the multiplicative finite difference method as well as the ordinary Runge-Kutta method. In both cases the multiplicative Runge-Kutta method showed to be superior. The comparison with the ordinary Runge-Kutta method showed significant advantages of the multiplicative derivative as well as the multiplicative Runge-Kutta method. Finally the Rössler problem was discussed in order to show the applicability of the approaches to remove the restriction caused by the multiplicative derivative. The numerical results are comparable with the ones from the ordinary Runge-Kutta method.

Appendix A. Review Multiplicative Calculus

To ensure a comprehensive understanding of this paper the important cornerstones of Multiplicative Calculus as in [4] are as following.
The definition of the multiplicative derivative, also known as *-derivative is

\[ f^*(x) = \lim_{h \to 0} \left( \frac{f(x + h)}{f(x)} \right)^{1/h} = e^{(\ln \circ f)'(x)}. \]  

(A.1)

Where ”*” denotes the multiplicative derivative. The higher order multiplicative derivatives can be obtained analogously

\[ f^{*(n)} = e^{(\ln \circ f)^{(n)}(x)}, \quad \text{for } n = 1, 2, \ldots \]  

(A.2)

Based on the definition of the multiplicative derivative (A.1), the *-differentiation rules can be obtained. Let \( f(x), g(x) \) and \( h(x) \) be *-differentiable functions and \( c \) a real constant, then

\[ (cf)^*(x) = f^*(x) \]  

(A.3)

\[ (fg)^*(x) = f^*(x)g^*(x) \]  

(A.4)

\[ (f/g)^*(x) = \frac{f^*(x)g^*(x)}{g^*(x)} \]  

(A.5)

\[ (f^h)^*(x) = f^*(x)h(x) f(x)^{h'(x)} \]  

(A.6)

\[ (f \circ g)^*(x) = f^*(g(x))g'(x) \]  

(A.7)

For the derivation of the Multiplicative Runge Kutta Method we are relying primarily on the Multiplicative Taylor Theorem and the Multiplicative Chain Rule, given according to [4] as:

**Theorem 1** (Multiplicative Taylor Theorem for one variable). Let \( A \) be an open interval and let \( f : A \to \mathbb{R} \) be \( n + 1 \) times *differentiable on \( A \). Then for any \( x, x + h \in A \), there exists a number \( \theta \in (0, 1) \) such that

\[ f(x + h) = \prod_{m=0}^{n} \left( f^*(m)(x) \right)^{h^m} \cdot \left( f^{*(n+1)}(x + \theta h) \right)^{\frac{h^{n+1}}{(n+1)!}} \]  

(A.8)

**Theorem 2** (Multiplicative Chain Rule). Let \( f \) be a function of two variables \( y \) and \( z \) with continuous partial *derivatives. If \( y \) and \( z \) are differentiable functions on \((a, b)\), such that \( f(y(x), z(x)) \) is defined for every \( x \in (a, b) \), then

\[ \frac{d^* f(y(x), z(x))}{dx} = f'_y(y(x), z(x))y'(x)f'_z(y(x), z(x))z'(x) \]  

(A.9)

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Although the complex multiplicative calculus is not used explicitly in this paper, the properties of the derivative are presented in the following as it is mentioned throughout the text. With (A.12) one can easily verify, that the Cauchy-Riemann conditions become obsolete for real variables.

Assume that \( f \) is a nowhere-vanishing differentiable complex-valued function on an open connected set \( D \) of the complex plane. Then the definition of the complex multiplicative derivative of \( f \) at \( z \in D \) is

\[
    f^*(z) = e^{f'(z)/f(z)} \tag{A.10}
\]

Where the higher order complex multiplicative derivatives can be obtained by induction, as

\[
    f^{(n)}(z) = e^{(\ln \circ f)^{(n)}(z)}, \quad \text{for } n = 1, 2, \ldots \tag{A.11}
\]

Multiplicative Cauchy-Riemann conditions can also be derived by assuming \( z = x + iy = re^{i\theta} \) and \( f(z) = u(z) + iv(z) = R(z)e^{i(\Theta(z)+2\pi n)} \) for \( z \in U \), where \( \Theta \) is any suitable branch of \( \arg f \) and \( U \subseteq D \) is a sufficiently small neighborhood of \( z \in D \).

**Theorem 3** (Multiplicative Cauchy-Riemann Conditions).

\[
    R^*_x(z) = [e^{\Theta}]^*_y(z) \quad \text{and} \quad R^*_y(z) = [e^{-\Theta}]^*_x(z) \tag{A.12}
\]

where \( g^*_x \) and \( g^*_y \) refer to the partial *derivatives in \( x \) and in \( y \), respectively, of the positive function \( g \).

Based on this, all necessary tools to derive the Multiplicative Runge Kutta Method for real valued functions of real variable are available.

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