Monte Carlo Study of the Critical Behavior of Random Bond Potts Models

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We present results of Monte Carlo simulations of random bond Potts models in two dimensions, for different numbers of Potts states, \( q \). We introduce a simple scheme which yields continuous self-dual distributions of the interactions. As expected, we find multifractal behavior of the correlation functions at the critical point and obtain estimates of the exponent \( \eta_n \) for several moments, \( n \), of the correlation functions, including typical \((n \to 0)\), average \((n = 1)\) and others. In addition, for \( q = 8 \), we find that there is only a single correlation length exponent \( \nu \) describing the correlation length away from criticality. This is numerically very close to the pure Ising value, \( \nu = 1 \).

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I. INTRODUCTION

In the theoretical study of phase transitions in disordered systems, it is neither practical nor useful to give a description of a single sample with a particular realization of the disorder. Rather, a detailed theory would yield a probability distribution of the possible results for, say, the free energy, obtained by sampling over all possible realizations of the disorder. However, one generally argues that important quantities, such as the free energy, are self averaging, that is to say independent of the particular realization of the disorder in the thermodynamic limit. In this case, one is generally content to calculate simply the average value. This theoretical result should agree with an experiment value, which is done on a single large sample, within an error that goes to zero in the thermodynamic limit.

In this paper, we will be particularly concerned with spin-spin correlation functions. Clearly, in a random system, one does not have self averaging of individual correlation functions (though the structure factor, the sum over all correlations, may be self averaging). Nonetheless, the tacit assumption is generally made that all reasonable ways of characterizing the distribution of correlation functions will give qualitatively similar results, and, in particular, will give the same values for critical exponents. We shall call such behavior of the correlation functions “conventional”.

We know, however, that there are certain cases where this is not true. For example, in certain models for quantum phase transitions with disorder\[1\], distributions of correlation functions are so broad that average and typical values (characterized, say, by the median) behave in qualitatively different ways both at and away from the critical point.

In this paper we discuss the question of whether a “conventional” description of correlation functions is correct for a classical model in two dimensions, the random-\( q \)-state Potts model\[2\]. This model has been extensively studied by both analytical and numerical approaches\[3\]. It is known\[3\] that the pure model has a second order transition when the number of Potts states, \( q \), is less than or equal to 4 and is first order for \( q > 4 \). The specific heat exponent of the pure system is positive for \( q > 2 \), which, from the Harris criterion\[4\], leads to the conclusion that disorder will be a relevant perturbation in this case. Furthermore, the transition is always second order in two-dimensions\[5,6\], so disorder has a particularly strong effect for \( q > 4 \), even changing the order of the transition. The resulting critical behavior is poorly understood, and its study is one of the main aims of the present work.

Ludwig\[7\] has argued that the power law decay of the correlations at the critical point is not governed by a single exponent, \( \eta \), as would be expected if “conventional” behavior occurred, but rather shows “multifractal” behavior in which there is no simple relation between the exponents, \( \eta_n \), of the various moments of the correlation function defined in Eq. (1) below. Rather, the whole set of exponents can be conveniently represented by a function \( f(\alpha) \), related to the Legendre transform of \( \eta_n \), as we shall discuss in \( \S 3 \). The \( \eta_n \) have been calculated analytically\[8\] to one-loop order for \( q = 2 \), small where the disorder is only weakly relevant. Away from the critical point, Ludwig\[7\] also claims, to one-loop order in an expansion away from \( q = 2 \), that there is only a single exponent \( \nu \), characterizing the divergence of the correlation length.

Numerical evidence that the correlations at criticality are multifractal has been provided by the transfer matrix calculations of Jacobsen and Cardy\[9\]. They showed that cumulants of the log of the correlation function stay non-zero as the lattice size is increased, which indicates multifractality. However, their values for \( \eta_n \) for \( n > 0 \) are not obtained directly but are derived from the cumulant expansion and so become inaccurate for large \( n \). In addition, Picco\[10\] has argued that, for the distribution of disorder used in Ref.\[9\], the data are in a crossover region between random and pure behavior, at least for the larger \( q \) values, and so do not give accurate values of the true exponents of the random system. To our knowledge, the perturbative claim that there is only a single correlation...
length exponent $\nu$ has not been verified by other means. Here we study the $q$-state Potts ferromagnet by Monte Carlo simulations, using the Wolff algorithm which greatly accelerates equilibration. The main features of our work are as follows.

1. We directly determine the $\eta_n$ for a wide range of values of $q$ and for several values of $n$, including $n = 0$ (typical), 1 (average) and others, as well as the whole distribution of correlation functions. We find that $\eta_0$ varies much more strongly with $q$ than does $\eta_1$. Curiously, we find that within the errorbars $\eta_2$ varies only slightly, if at all, with respect to $q$.

2. We use a different distribution for the disorder from that which is generally taken. It has the advantage that it is still self dual, so the critical point can be determined exactly, but it is continuous and very broad. As a result, we argue that it is less susceptible to crossover effects than the “bi-modal” distributions used before.

3. We verify explicitly that, at least for $q = 8$, the exponents for the divergence of the average and typical correlation lengths are equal, to within fairly small numerical errors. Furthermore, to within errors, the value is the same as that of the pure two-dimensional Ising model. Note, though, that other aspects of the critical behavior, such as the decay of the correlations at the critical point, are quite different from those of the pure two-dimensional Ising model.

The layout of the paper is as follows. In II we discuss the model and some characteristics of the simulations. In III we discuss multifractal behavior, which is expected to describe the correlations at the critical point. Our results at the critical point are discussed in IV while our results away from the critical point are given in §V. In VI we summarize our results and give some perspectives for future work.

II. THE MODEL

The Hamiltonian of the $q$-state Potts model is given by

$$\beta H = - \sum_{i,j} K_{ij} \delta_{n_i n_j}, \quad (1)$$

where each site $i$ on an $N = L \times L$ square lattice is in one of $q$-states, characterized by an integer $n_i = 1, 2, \ldots, q$. The couplings, $K_{ij}$, are positive, and include the factor of $\beta = 1/k_B T$. They are independent random variables, drawn from a probability distribution, $P(K)$.

For the pure system, the partition function with coupling $K$ is closely related to the partition function with the “dual” coupling, $K^*$, where

$$(e^K - 1)(e^{K^*} - 1) = q. \quad (2)$$

If $K$ is large (low temperature) then $K^*$ is small (high temperature) and vice-versa. Assuming that there is a single transition, this must be at the self-dual point where $K^* = K = K_c$, with $e^{K_c} = 1 + \sqrt{q}$.

For the random case, the model is still self-dual, and hence at its critical point, provided that the distribution of the dual couplings $P^*$ is equal to the distribution of the original couplings. One simple example, which has been extensively used in numerical work, is two delta functions with equal weight,

$$P(K) = \frac{1}{2} \left[ \delta(K - K_1) + \delta(K - K_2) \right], \quad (3)$$

which is self dual if $K_2 = K_1^*$. Hence Eq. (3) describes a family of self dual distributions characterized by a single parameter, $R \equiv K_1/K_2$. However, if this ratio is too close to unity, then very large sizes are needed otherwise the system is in a “crossover” region between the critical behavior of the pure and the random systems. On the other hand, if the ratio is made too large, the distribution is fairly close to that of the percolation problem at criticality, and again the system will be in a crossover regime.

To avoid these crossovers, it is useful to study the model in Eq. (1) with a self-dual continuous distribution. Although Jauslin and Swendsen [14] describe how one particular continuous self-dual distribution can be constructed, we are not aware of any results obtained with such a distribution. Here, we show that with a simple change of variables one can trivially generate any self-dual distribution. In terms of the variable $y$, defined by

$$e^y = \frac{e^K - 1}{\sqrt{q}}, \quad (4)$$

the duality condition takes the simple form

$$y^* = -y. \quad (5)$$

Hence, expressed in terms of $y$, any even distribution, $P_Y(y)$, is self-dual. Note that because Eq. (4) is so simple the Jacobian in the transformation from the distribution of $y$ to the distribution of $y^*$ is unity, unlike the situation going from the distribution of $K$ to that of $K^*$, where the Jacobian is non-trivial.

The quantity that enters in statistical mechanics averages is $x \equiv e^{-K}$, which, for the ferromagnetic couplings discussed here, takes values in the range from 0 to 1. In order that the model has strong disorder we seek a distribution of $x$ that is non-zero everywhere, and has a finite weight at the end points. This means that $P_Y(y) \propto \exp(-|y|)$ for $y \to \pm \infty$. A convenient choice, which we use in the rest of this paper for simulations at criticality, is

$$P_Y(y) = \frac{1}{\pi} \mathrm{sech}(y), \quad (6)$$

2
for which the corresponding distribution for \( x \equiv e^{-K} \) is

\[
P_X(x) = \frac{2}{\pi} \frac{\sqrt{q}}{(1 - x)^2 + qx^2}.
\]

This is plotted in Fig. 1 for several values of \( q \). Another advantage of this distribution is that it is easy to generate random numbers with probability \( P_X(x) \) (which are needed for the Wolff algorithm [21]). If \( r \) is a random number with a uniform distribution between 0 and 1, one simply takes

\[
x = \frac{1}{1 + \sqrt{q} \tan(\pi r/2)}.
\]

We apply periodic boundary conditions. The numbers of samples used are shown in Table I, and the number of Monte Carlo sweeps are shown in Table II. Note that the number of sweeps for averaging is quite large, given that we use the Wolff algorithm which equilibrates the system quickly even for large sizes. The reason is that we wish to obtain the whole distribution of spin-spin correlation functions, so each correlation function has to be obtained with high precision. We cannot use noisy data for the individual correlation functions and rely on averaging over a large number of pairs to improve the statistics. This would be fine for the average but lead to systematic errors for other moments. For each sample we determine the correlation function for spins \( L/2 \) apart for a large number of pairs, depending on \( L \), as indicated in Table III.

### Table I. The number of samples used for each value of \( q \) and \( L \)

| \( q \) | \( L=16 \) | \( L=32 \) | \( L=64 \) | \( L=128 \) | \( L=256 \) | \( L=512 \) |
|---|---|---|---|---|---|---|
| 2  | 100  | 100  | 200  | 125  | 18  | -   |
| 3  | 3000 | 1200 | 400  | 400  | 365 | 95  |
| 4  | 2000 | 600  | 210  | 200  | 125 | 50  |
| 5  | 2000 | 600  | 200  | 200  | 100 | -   |
| 8  | 1500 | 1000 | 904  | 977  | 426 | 87  |
| 20 | 1000 | 600  | 513  | 500  | 265 | 67  |

### Table II. The number of Monte Carlo sweeps for averaging. In brackets is the number of sweeps for equilibration.

Note that for \( q = 8, L = 512, 43 \) of the 87 samples were done with 50000(500) sweeps.

### III. MULTIFRACTAL BEHAVIOR

Consider the spin-spin correlation function, \( C_{i,j} \), for a pair of sites, \( i \) and \( j \). If the spins are separated by a distance \( r \), we shall also denote this by \( C(r) \), where

\[
C(r) = \frac{q}{q - 1} \langle \delta_{n_{i,j}} - \frac{1}{q} \rangle,
\]

in which \( \langle \cdots \rangle \) denotes a thermal average for a single realization of the disorder. At the critical point, the correlations decay with a power law and we find it convenient to define a set of exponents \( \eta_n \) by

\[
[C(r)]^{1/n}_{\text{av}} \sim r^{-\eta_n},
\]

for \( n = 0, 1, 2, \ldots \), where averages over disorder are indicated by \( [\cdots]_{\text{av}} \). The \( n = 0 \) value in Eq. (10), which gives the behavior of a “typical” correlation function, is obtained as the limit \( n \to 0 \), i.e. \( \exp[\ln C(r)]_{\text{av}} \). A “typical” correlation function can also be defined as the median of the distribution, with very similar results.

In what we are calling “conventional behavior” all the \( \eta_n \) are equal. However, according to Ludwig [7], there is no simple relation between the \( \eta_n \) for the random Potts model in two dimensions. Instead one has multifractal behavior in which the probability distribution of the \( C(r) \) is given by

\[
\hat{P}(\alpha) = N \exp[-f(\alpha) \ln r]
\]

where

\[
\alpha = -\frac{\ln C(r)}{\ln r},
\]

\[
\hat{P}(\alpha) = \frac{1}{\pi} \frac{\sqrt{q}}{(1 - x)^2 + qx^2},
\]

\[
P_X(x) = \frac{2}{\pi} \frac{\sqrt{q}}{(1 - x)^2 + qx^2}.
\]
and \( N \) is the normalization. From general considerations, 
\( f(\alpha) \) must have a minimum at some point, \( \alpha_0 \) say, and the 
value of \( f(\alpha_0) \) can be absorbed into the normalization, 
so we set \( f(\alpha_0) = 0 \).

In the thermodynamic limit, averages can be done by 
a saddle point method, with the result that for each \( n \), 
there is a corresponding value of \( \alpha \) 
so we set \( f(\alpha_0) = 0 \).

\[
f'(\alpha) = -n \tag{13}
\]

and then
\[
m\eta_n = f(\alpha) + \alpha n. \tag{14}
\]

The error in the saddle point calculation is of order \( 1/\sqrt{\ln r} \) which falls off extremely slowly with distance. 
This will be important in the analysis of the numerical 
results in the next section.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{saddle_point}
\caption{A sketch of the function \( f(\alpha) \) which characterizes 
the multifractal nature of the distribution of the correlation functions. 
To determine the exponent \( \eta_n \), describing the power law decay of the \( n \)-th moment of the correlations 
at the critical point, according to Eq. (14), locate the point 
where the slope of the curve is \(-n\) and draw the tangent at 
that point. Where the tangent intersects the horizontal axis 
is \( \eta_n \). Clearly, \( \eta_0 = \alpha_0 \), the location of the minimum.
\end{figure}

Eqs. (13) and (14) imply a simple graphical relation
between \( f(\alpha) \) and the \( \eta_n \), illustrated in Fig. 2. One 
locates the point on the \( f(\alpha) \) curve with slope \(-n\), and 
where this intersects the horizontal axis is \( \eta_n \). Clearly 
then \( \eta_0 = \alpha_0 \), the location of the minimum. Because the 
\( \eta_n \) cannot be negative (otherwise the correlations would 
grow with distance), the function \( f(\alpha) \) will diverge as \( \alpha \) 
approaches some non-negative value.

\section*{IV. RESULTS AT CRITICALITY}

We concentrate on the correlation function between 
two spins as far apart as possible in the lattice in either the 
horizontal or vertical direction, \( e.g. \) if one site is at 
\((0,0)\) the other is at \((0, L/2) \) or \((L/2, 0) \). Fig. 3 shows 
our results for \( \langle C(L/2)^n \rangle^{1/n} \) for several values of \( n \) at 

criticality for the case of \( q = 8 \). The slopes are equal 
to \(-\eta_n \). In the “conventional” picture, they would all 
be equal. Clearly this is not the case; rather the slopes 
change with \( n \) as expected for multifractal behavior.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{exponents}
\caption{Data for the \( q = 8 \) model at criticality for sizes 
between \( L = 16 \) and \( 512 \), for \( n = 0, 1, 2, 3 \) and \( 4 \). The lines are 
fits to the data and the slopes give the values for \( \eta_n \) indicated.
\end{figure}

Fig. 4 summarizes our results for the exponents \( \eta_q \) 
which are also presented in numerical form in Table IV.

Notice, from Fig. 3, that \( \langle C^n \rangle^{1/n} \) is an increasing function 
of \( n \), as expected on general grounds, since \( \ln \langle C^n \rangle \)
is a convex function of \( n \). This implies that, for fixed \( q \), \( \eta_n \) decreases with increasing \( n \), as seen in Fig. 3 and 
Table IV. Clearly \( \eta_0 \), representing the decay of typical 
correlation functions, varies very strongly with \( q \), while the 
other \( \eta_n \) vary less strongly. For small \( n \), \( \eta_n \) increases 
as \( q \) increases, whereas for large \( n \) the converse is true. 
Within errors, \( \eta_2 \) is independent of \( q \) for the range studied.

It would be of interest to investigate the limit \( q \to \infty \), 
but we are not able to equilibrate \( q \) values very much larger than 20. However, a naive extrapolation of our data gives \( \lim_{q \to \infty} \eta_1 = 0.37 \pm 0.01 \).

Note that the data in Fig. 2 for \( q = 2 \) show a small 
but non-zero variation of \( \eta_n \) with \( n \). However, \( q = 2 \) is 
expected to have marginal behavior, in which \( \eta_n = 1/4 \) 
(pure value) for all \( n \) but with logarithmic corrections. 
i.e. \( \langle C(r)^n \rangle^{1/n} \sim r^{-1/4} \ln(r/r_0)^{\lambda_n} \), where, for example, \( \lambda_0 = -1/8 \) and \( \lambda_1 = 0 \). The data cannot unambiguously determine the form of the logarithmic factors (though it is consistent with them for appropriate values of \( \lambda_n \) and \( r_0 \)). The fits used to get the data in Fig. 2 therefore assume a pure power law behavior, and the apparent small change of \( \eta_n \) with \( n \) for \( q = 2 \) is presumably due to ne-
TABLE IV. The values of $\eta_n$, defined by Eq. (10), at criticality, for different values of $q$ and $n$. $n = 1$ corresponds to the average correlation function and $n = 0$ to the typical correlation function. The values for $n = 0$ are obtained by averaging results for the exponential of the average of the log and the median. These agreed within expected statistical fluctuations. The quantity in brackets is the error in the last decimal place.

| $n$ | $q$ | $\eta_n$ | $n$ | $q$ | $\eta_n$ | $n$ | $q$ | $\eta_n$ |
|-----|-----|----------|-----|-----|----------|-----|-----|----------|
| 0   | 2   | 0.274(9) | 1   | 2   | 0.252(8) | 2   | 2   | 0.235(7) |
| 0   | 3   | 0.315(5) | 1   | 3   | 0.269(4) | 2   | 3   | 0.237(4) |
| 0   | 4   | 0.356(10)| 1   | 4   | 0.287(7) | 2   | 4   | 0.242(5) |
| 0   | 8   | 0.507(11)| 1   | 8   | 0.323(6) | 2   | 8   | 0.239(4) |
| 0   | 20  | 0.746(25)| 1   | 20  | 0.347(10)| 2   | 20  | 0.231(6) |
| 3   | 2   | 0.222(6) | 4   | 2   | 0.210(6) |     |     |          |
| 3   | 3   | 0.213(3) | 4   | 3   | 0.195(3) |     |     |          |
| 3   | 4   | 0.212(5) | 4   | 4   | 0.190(4) |     |     |          |
| 3   | 8   | 0.193(3) | 4   | 8   | 0.163(3) |     |     |          |
| 3   | 20  | 0.176(5) | 4   | 20  | 0.144(4) |     |     |          |

glect of the logarithmic corrections. Note in Table IV, that for $n = 1, q = 2$, (where the logarithmic corrections are predicted to be absent) we get $\eta_1$ very close to the pure Ising value of 1/4, as expected.

![Figure 4](image)  
FIG. 4. Results for $\eta_n$, for several values of $n$, plotted against $1/q$.

![Figure 5](image)  
FIG. 5. A comparison between our (OY) estimates of $\eta_1$ (relevant for the average correlation function), with those of Jacobsen and Cardy (JC), Monte Carlo results of Chatelain and Berche (CB) and Picco (P). Ref. 8 also has data from Transfer Matrix techniques which agree with their Monte Carlo data within the errors.

In Fig. 5 we compare our results for $\eta_1$ with those of other authors. An extrapolation of results to larger values of $q$ is consistent with the result of Picco for $q = 64$. Our results also agree with Picco's for $q = 3$. For larger $q$ values they lie above those of Jacobsen and Cardy. This is not surprising in view of Picco's claim that the strength of disorder used by Jacobsen and Cardy is not large enough for $q = 8$ to be in the asymptotic critical regime. The bimodal distribution used in those studies is characterized by the ratio $R$ of the two interactions. Ref. 8 used $R = 2$ whereas Picco argues that $R \approx 10$ is needed. More surprisingly, our results also lie somewhat above those of Chatelain and Berche who used stronger disorder, $R \geq 10$ for $q \geq 8$. Perhaps in that case there is a crossover from the percolation critical point. Our results do, however, also agree with the result (not shown on the figure) of Wiseman and Domany that, for $q = 4$, $\eta_1 = 0.290 \pm 0.006$. For $q = 3$, our results also agree with the analytical three-loop calculation of Dotsenko et al., though the difference between the random and pure value is not very great in this case.

There are few studies of any other values for $\eta_n$ with $n \neq 1$. Conformal field theory has been employed by Lewis to obtain second order expansions for $\eta_n$; for $q = 3$ he calculates $\eta_0 = 0.314$, $\eta_2 = 0.236$ (which agrees
with a similar work by Dotsenko et al.\textsuperscript{28}, and \( \eta_2 = 0.220 \). These values agree quite well with ours. Jacobsen and Cardy\textsuperscript{29} present results from which \( \eta_0 \) can be inferred\textsuperscript{30} for \( q = 3 \) and 8. However, we find that any reasonable extrapolation of their data to infinite system size gives results which are significantly below ours, especially for \( q = 8 \). This may be related to Picco’s remarks\textsuperscript{30} that the ratio of interactions, \( R \), used in Ref.\textsuperscript{3} is too small, at least for \( q = 8 \). Dotsenko et al.\textsuperscript{28} perform Monte Carlo simulations estimating \( \eta_2 = 0.227(2) \) for both \( q = 3 \) and \( q = 4 \). A more detailed analysis\textsuperscript{3} of this data yields 0.23(2) for \( q = 3 \) and 0.229(2) for \( q = 4 \). Although these values are slightly smaller than ours, they support the trend that \( \eta_2 \) is barely changing with \( q \).

The exponent characterizing the relevance of weak disorder is \( y \equiv \alpha_P / \nu_P \), where \( \alpha_P \) and \( \nu_P \) are the specific heat and correlation length exponents of the pure system. This vanishes as \( q \to 2^+ \) and has the value \( y = 2/5 \) for \( q = 3 \). Ludwig\textsuperscript{31} has shown that for \( (n-1)y \ll 1 \)

\[
\eta_n - \eta_1 \simeq -\frac{y}{8}(n-1).
\]  

For \( q = 3 \) this gives \( \eta_0 - \eta_1 = 0.05 \), which agrees well with our result of 0.046 (with an uncertainty of around 0.006), see Table I. This is in contrast to Jacobsen and Cardy\textsuperscript{29} who stated that their data did not agree with this relation, though they did not quote a value for \( \eta_0 \).

From the values of the exponents \( \eta_n \), we can construct the multiscaling function \( f(\alpha) \) from (13) and (14). This is shown in Fig. 6 for \( q = 8 \).

![Fig. 6. The solid line is a reconstruction, for \( q = 8 \), of the multiscaling function, \( f(\alpha) \). It is obtained as the tangent to the dashed lines, whose formulae are \( n(\eta_n - \alpha) \), for \( n = 0, 1, 2, 3 \) and 4, see Eqs. (13) and (14). See also Fig. 2.](image)

![Fig. 7. The distribution of \( \log_{10} C(L/2) \), for \( L = 512 \) and several different values of \( q \) at criticality.](image)

So far, we have discussed results for the moments of the correlation function. Now, in Fig. 7, we show results for the whole distribution of \( \log_{10} C(L/2) \), for \( L = 512 \) and several different values of \( q \) at criticality. The vertical scale is chosen so that, for a log-normal distribution, the tails of the curves would be two straight lines symmetric about the peak. The data for \( q = 2 \) is not very different from this, but for large \( q \), especially \( q = 20 \), the curve is not only much broader but also very asymmetric. Furthermore, even in the tail to the left of peak, the data for \( q = 20 \) is significantly curved. It would be interesting to understand the behavior of the distribution in the limit \( q \to \infty \).

Similar data is shown in Fig. 8 for \( q = 8 \) and different values of \( L \) at criticality. As expected, the distribution becomes broader for increasing \( L \). For small \( L \) the data for small \( C(L/2) \) is roughly straight indicating that this tail follows close to a log-normal distribution. However, for \( L = 512 \), significant curvature in the tail can be seen. Furthermore, attempts to fit to Eq. (11) to directly determine the multiscaling function \( f(\alpha) \) (and hence to compare with the reconstruction of this function from the moments in Fig. 8) were unsuccessful. Eqs. (13) and (14) and Fig. 6 depend upon a saddle point approximation which is only valid for \( \sqrt{\ln(L/2)} > 1 \), which is not realizable in a Monte Carlo simulation.

As a check on our results we also determined \( \eta_1 \) from the mean square magnetization at the critical point. In \( d = 2 \), finite size scaling predicts

\[
\langle m^2 \rangle_{\text{av}} \sim L^{-2\beta/\nu} = L^{-\eta_1}.
\]  

where \( \beta \) is the order parameter exponent, \( \nu \) the correlation length exponent.
FIG. 8. The distribution of \( \log_{10} C(L/2) \), for \( q = 8 \) and several different values of \( L \) at criticality.

\[
m^2 = \frac{1}{N^2} \sum_{i,j} C_{i,j} = \frac{q}{q-1} \left( \frac{q}{q} \rho_n^2 - \frac{1}{q} \right),
\]

and \( \rho_n \) is the fraction of sites in state \( n \).

\[
\begin{array}{|c|c|}
\hline
q & \eta_1 \\
\hline
3 & 0.259(4) \\
8 & 0.319(7) \\
20 & 0.345(9) \\
\hline
\end{array}
\]

TABLE V. The values of \( \eta_1 \), obtained from the mean square magnetization, Eq. (16), at criticality, for different values of \( q \).

For \( q = 8 \) the data is shown in Fig. 9 and the values of \( \eta_1 \) determined this way are given in Table V. The results agree with those found from the correlation function \( C(L/2) \), see Table IV, to within statistical errors.

V. RESULTS AWAY FROM CRITICALITY

Although the distribution of correlation functions at the critical point has multifractal behavior, Ludwig\(^7\) claims that only a single exponent \( \nu \) describes the divergence of the correlation length as the critical point is approached.

There are analytical predictions\(^3,10,13\) for how \( \nu \) varies with \( q \) for \( q \) near 2. The deviation of these results from the value for the pure Ising model, \( \nu = 1 \), is small, and
numerical studies have so far been unable to resolve it. Here we concentrate on \( q = 8 \), where disorder changes the order of the transition from first to second and so the perturbative analytical approach is not applicable.

We define

\[
t = \frac{(T - T_c)}{T_c},
\]

so the Boltzmann factors, \( x \equiv e^{-K} \), are first calculated at the critical point from the self-dual distribution in Eq. (7), and are then modified by the replacement \( x \to x^{1/(1+t)} \) away from criticality.

According to finite size scaling, the average correlation function away from criticality, \([C(L/2,t)]_{av}\), is related to that at criticality by

\[
\frac{[C(L/2,t)]_{av}}{[C(L/2,0)]_{av}} = f(L^{1/\nu}t),
\]

where \( f \) is a scaling function. There is an analogous expression for the typical correlation function. Figs. 10 and 11 show scaling plots, assuming this form, in which \( \nu \) has been adjusted to get the best data collapse. From the chi-squared of the fits we estimate

\[
\nu = 1.01 \pm 0.02 \quad \text{average} \quad (20)
\]

\[
\nu = 0.99 \pm 0.02 \quad \text{typical}, \quad (21)
\]

showing that the values of \( \nu \) agree to within the errors.

We believe that this is the first numerical calculation which verifies that the exponents for the average and typical correlation length are equal. This indicates that the correlations have conventional, rather than multifractal, behavior away from the critical point. It is also interesting that \( \nu \) is close to (and perhaps equal to) 1, the value for the pure two-dimensional Ising model.

VI. CONCLUSIONS

In this work we have presented results for the critical exponents of the random \( q \)-state Potts model, for various moments of the correlation functions, at and away from the critical point. We have confirmed in greater detail than before that that the correlations have multifractal behavior at the critical point. We have also verified, for the first time, that there is only a single exponent \( \nu \) describing the divergence of the correlation length.

Implicit in our discussion has been the assumption of universality i.e. that all the exponents \( \eta_n \), and hence the multifractal function, \( f(\alpha) \), do not depend on details of the model, such as the form of the distribution of disorder. While universality is well established for pure systems, to our knowledge, it has not been convincingly demonstrated in situations where there are multifractal correlations.

Another issue which merits further discussion is the form of corrections to scaling. Generally these have a power law form, in which the exponent is that of the leading irrelevant operator. However, when there are multifractal correlations with a continuous spectrum of exponents characterized by \( f(\alpha) \), is it possible that the approach to the asymptotic limit is slower? This is certainly the case for \( f(\alpha) \) itself, where corrections fall off only as \( 1/\sqrt{\ln r} \), much slower than a power law. Whether the same slow decay of corrections also applies to the exponents \( \eta_n \) is not clear to us.

We note that in spin glasses, the numerical results, particularly for \( \eta_1 \), do not seem to satisfy universality [4]. Could a resolution be that the approach to the thermodynamic limit is only logarithmic, so astronomically large sizes are needed to see universality?

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2 For a review of the Potts model, see F. Y. Wu, Rev. Mod. Phys. 54, 235 (1982).
3 A. W. W. Ludwig and J. Cardy, Nucl. Phys. B, 285, 687 (1987).
4 J. Cardy and J. J. Jacobsen, Phys. Rev. Lett. 79, 4063 (1997).
5 J. J. Jacobsen and J. Cardy, Nucl. Phys. B, 515, 701 (1998).
6 J. Cardy, cond-mat/9806355.
7 A. W. W. Ludwig, Nucl. Phys. B, 330, 639 (1990).
8 C. Chatelain and B. Berche, cond-mat/9902212.
9 C. Chatelain and B. Berche, Phys. Rev. Lett. 80, 1670 (1998).
10 V. Dotsenko, M. Picco and P. Pujol, Nucl. Phys. B, 455, 701 (1995).
11 M. Picco, Phys. B, 54, 14930 (1996); Phys. Rev. Lett. 79, 2998 (1997).
12 M. Picco, cond-mat/9802092.
13 G. Jug and B. N. Shalaev, Phys. Rev. B, 54, 342 (1996).
14 S. Chen, A. M. Ferrenberg and D. P. Landau, Phys. Rev. E, 52, 1377 (1995).
15 R. J. Baxter, J. Phys. C, 6, L445 (1973).
16 A. B. Harris, J. Phys. C, 7, 1671 (1974).
17 Y. Imry and M. Wortis, Phys. Rev. B, 19, 3580 (1979).
18 M. Aizenman and J. Wehr, Phys. Rev. Lett. 62, 2503 (1989).
19 K. Hui and A. N. Berker, Phys. Rev. Lett. 62, 2507 (1989).
20 T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A, 33, 1141 (1986), and references therein.
21 U. Wolff, Phys. Rev. Lett. 62, 361 (1989); R. H. Swendsen and J. Wang, Phys. Rev. Lett. 58, 86 (1987).
22 W. Kinzel and E. Domany, Phys. Rev. B, 23, 3421 (1981).
23 W. Selke, L. N. Shchur, and A. L. Talapov, in Annual Reviews of Computational Physics, Vol. 1, Ed. D. Stauffer (World Scientific), p.17.
24 H. R. Jauslin and R. H. Swendsen, Phys. Rev. B, 24, 313 (1981).
25 W. Feller, An Introduction to Probability Theory and its Applications, Vol. 2, p. 153, J Wiley and Sons, New York (1966).
26 S. Wiseman and E. Domany, Phys. Rev. E, 51, 3074 (1995).
27 M. Lewis, Europhys. Lett. 43, 189 (1998); (private communication).
28 V. Dotsenko, V. Dotsenko and M. Picco, Nucl. Phys. B, 520, 663 (1998).
29 In Table 6 of Ref. 6, the column marked “1. cumulant” gives a value for $\eta_0$ for a strip of width $L$, when multiplied by $L^2/\pi$. This should then be extrapolated to $L = \infty$.
30 M. Picco, private communication.
31 L. W. Bernadi, S. Prakash and I. A. Campbell, Phys. Rev. Lett. 77, 2798 (1996); L. W. Bernadi and I. A. Campbell, Phys. Rev. B, 56, 5271 (1997).