Confidence surfaces for the mean of locally stationary functional time series

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July 8, 2022

Abstract

The problem of constructing a simultaneous confidence surface for the 2-dimensional mean function of a non-stationary functional time series is challenging as these bands cannot be built on classical limit theory for the maximum absolute deviation between an estimate and the time dependent regression function. In this paper we propose new bootstrap methodology to construct such a region. Our approach is based on a Gaussian approximation for the maximum norm of sparse high-dimensional vectors approximating the maximum absolute deviation. The elimination of the zero entries produces (besides the time dependence) additionally dependencies such that "classical" multiplier bootstrap is not applicable. To solve this issue we develop a novel multiplier bootstrap, where blocks of the coordinates of the vectors are multiplied with random variables, which mimic the specific structure between the vectors appearing in the Gaussian approximation. We prove the validity of our approach by asymptotic theory, demonstrate good finite sample properties by means of a simulation study and illustrate its applicability analyzing a data example.

keywords: locally stationary time series, functional data, confidence surface, Gaussian approximation, multiplier bootstrap.

1 Introduction

In the big data era data gathering technologies provide enormous amounts of data with complex structure. In many applications the observed data exhibits certain degrees of dependence and smoothness and thus may naturally be regarded as discretized functions. A major tool for the statistical analysis of such data is functional data analysis (FDA) which has found considerable attention in the statistical literature (see, for example, the monographs of [Bosq, 2000], [Ramsay and Silverman, 2005], [Ferraty and Vieu, 2010], [Horváth and Kokoszka, 2012], [Hsing and Eubank, 2015], among others). In FDA the considered parameters, such as the mean or the (auto-)covariance (operator) are functions themselves, which makes the development of statistical methodology challenging. Most of the literature considers Hilbert space-based methodology...
Figure 1: Volatility Smile at different (one minus) times to maturity ($u = 0.2$, $0.4$, $0.6$ and $0.8$). The variable $t$ refers to ‘moneyness’

for which there exists by now a well developed theory. In particular, this approach allows the application of dimension reduction techniques such as (functional) principal components (see, for example, Shang, 2014). On the other hand, in many applications data is observed on a very fine grid and it is reasonable to assume that functions are at least continuous (see also Ramsay and Silverman, 2005 for a discussion of the integral role of smoothness). In such cases fully functional methods can prove advantageous and have been recently, developed by Horváth et al. (2014), Bucchia and Wendler (2017), Aue et al. (2018), Dette et al. (2020) and Dette and Kokot (2020) among others.

In this paper we are interested in statistical inference regarding the mean functions of a not necessarily stationary functional time series $(X_{i,n})_{i=1,...,n}$ in the space $L^2[0,1]$ of square integrable functions on the interval $[0,1]$. As we do not assume stationarity, the mean function $t \to \mathbb{E}[X_{i,n}(t)]$ is changing with $i$ and we assume that it is given by $\mathbb{E}[X_{i,n}(t)] = m(\frac{i}{n},t)$, where $m$ is a smooth function on the unit square. Our goal is the construction of simultaneous confidence surfaces for the (time dependent) mean function $(u,t) \to m(u,t)$ of the locally stationary functional time series $\{X_{i,n}\}_{i=1,...,n}$. As an illustration we display in Figure 1 the implied volatility of an SP500 index as a function of moneyness ($t$) at different times to maturity ($u$, which is scaled to the interval $[0,1]$). These functions are quadratic and known as “volatility smiles” in the literature on option pricing. They seem to slightly vary in time. In practice it is important to assess whether these “smiles” are time-invariant. We refer the interested reader to Section 3.3 for a more detailed discussion (in particular we construct there a confidence surface for the function $(u,t) \to m(u,t)$).

To our best knowledge confidence bands have only been considered in the stationary case, where $m(u,t) = m(t)$. Under the assumption of stationarity they can be constructed using the sample mean $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_{i,n}$.
\[ \frac{1}{n} \sum_{i=1}^{n} X_{i,n} \] and the weak convergence of \( \sqrt{n}(\bar{X}_n - m) \) to a centered Gaussian process (see, for example, Degras, 2011; Cao et al., 2012; Degras, 2017; Dette et al., 2020) who either assume that data is observed on a dense grid or that the full trajectory can be observed. More recently, alternative simultaneous confidence (asymptotic) bands have been constructed by Liebl and Reimherr (2019); Telschow and Schwartzman (2022) using the Gaussian Kinematic formula.

On the other hand, although non-stationary functional time series have found considerable interest in the recent literature (see, for example, van Delft and Eichler, 2018; Aue and van Delft, 2020; Bücher et al., 2020; Kurisu, 2021a,b; van Delft and Dette, 2021), the problem of constructing a confidence surface for the mean function has not been considered in the literature so far. A potential explanation that a solution is still not available, consists in the fact that due to non-stationarity smoothing is required to estimate the function \( u \to m(u,t) \) (for a fixed \( t \)). This results in an estimator converging with a \( 1/\sqrt{b_n n} \) rate (here \( b_n \) denotes a bandwidth). On the other hand, in the stationary case (where \( m \) does not depend on \( u \)), the sample mean \( \bar{X}_n \) can be used, resulting in a \( 1/\sqrt{n} \) rate.

As a consequence, a weak convergence result for the sample mean in the non-stationary case is not available and the construction of simultaneous confidence surfaces for the regression function \((u,t) \to m(u,t)\) is challenging. In this paper we propose a general solution to this problem, which is not based on weak convergence results. As an alternative to “classical” limit theory (for which it is not clear if it exists in the present situation) we develop Gaussian approximations for the maximum absolute deviation between the estimate and the regression function. These results are then used to construct a non-standard multiplier bootstrap procedure for the construction of simultaneous confidence surfaces for the mean function of a locally stationary functional time series. Our approach is based on approximating the maximal absolute deviation \( \hat{\Delta} = \max_{u,t \in [0,1]} |\hat{m}(u,t) - m(u,t)| \) by a maximum taken over a discrete grid, which becomes dense with increasing sample size. We thus relate \( \hat{\Delta} \) to the maximum norm of a sparse high-dimensional vector. We then further develop Gaussian approximations for the maximum norm of sparse high-dimensional random vectors based on methodology proposed by Chernozhukov et al. (2013); Zhang and Cheng (2018). Finally, the covariance structure of this vector (which is actually a high-dimensional long-run variance) is mimicked by a multiplier bootstrap. Our approach is non-standard in the following sense: due to the sparsity the Gaussian approximations in the cited literature cannot be directly used. In order to make these applicable we reduce the dimension by deleting vanishing entries. However, this procedure produces additional spatial dependencies (beside the dependencies induced by time series), such that the common multiplier bootstrap is not applicable. Therefore we propose a novel multiplier bootstrap, where instead of the full vector, individual blocks of the vector are multiplied with independent random variables, such that for different vectors a certain amount (depending on the lag) of these multipliers coincide.

The remaining part of the paper is organized as follows. The statistical model is introduced in Section 2 where we describe our approach in an informal way and propose two confidence surfaces for the mean function \( m \). In Section 3 we demonstrate good finite sample properties of our approach by means of a simulation study and illustrate its applicability analyzing a data example. Section 4 is devoted to rigorous statements under which conditions our method provides valid (asymptotic) confidence surfaces. Section
contains technical proofs. As a by-product of our approach, we also derive in the online supplement of this paper new confidence bands for the functions \( t \to m(u,t) \) (for fixed \( u \)) and \( u \to m(u,t) \) (for fixed \( t \)), which provide efficient alternatives to the commonly used confidence bands for stationary functional data or real valued locally stationary data, respectively (see Section A for details). Finally, some concrete examples of locally stationary functional time series and all auxiliary results for proofs in Section 5 are deferred to the online supplement.

2 Confidence surfaces for the mean of non-stationary time series

Throughout this paper we consider the model

\[
X_{i,n}(t) = m(\frac{i}{n}, t) + \varepsilon_{i,n}(t), \quad i = 1, \ldots, n, \tag{2.1}
\]

where \((\varepsilon_{i,n})_{i=1,\ldots,n}\) is a centred locally stationary process in \(L^2[0,1]\) of square integrable functions on the interval \([0,1]\) (see Section 4 for a precise mathematical definition) and \(m : [0,1] \times [0,1] \to \mathbb{R}\) is a smooth mean function. This means that at each time point “\(i\)” we observe a function \(t \to X_{i,n}(t)\). We are interested in a simultaneous confidence surface

\[
C_n = \{ f : [0,1]^2 \to \mathbb{R} \mid \hat{L}_1(u,t) \leq f(u,t) \leq \hat{U}_1(u,t) \ \forall u,t \} \tag{2.2}
\]

for the mean function \((u,t) \to m(u,t)\), where \(\hat{L}_1\) and \(\hat{U}_1\) are appropriate lower and upper bounds calculated from the data. The methodology developed in this paper allows the construction of simultaneous confidence bands for the functions \(t \to m(u,t)\) (for fixed \(u\)) and \(u \to m(u,t)\) (for fixed \(t\)), which are developed in Section A of the online supplement for the sake of completeness.

Our approach is based on the maximum deviation

\[
\hat{\Delta}_n = \sup_{t,u \in [0,1]} |\hat{m}(u,t) - m(u,t)|, \tag{2.3}
\]

where

\[
\hat{m}(u,t) = \frac{1}{nb_n} \sum_{i=1}^{n} X_{i,n}(t)K\left(\frac{i}{n} - u\right) \\
= \frac{1}{nb_n} \sum_{i=1}^{n} m(\frac{i}{n}, t)K\left(\frac{i}{n} - u\right) + \frac{1}{nb_n} \sum_{i=1}^{n} \varepsilon_{i,n}(t)K\left(\frac{i}{n} - u\right) \tag{2.4}
\]

denotes the common Nadaraya-Watson estimate with kernel \(K\) (supported on the interval \([-1,1]\) such that \(\int_{-1}^{1} K(x)dx = 1\)) and bandwidth \(b_n\). Other estimates as local linear regression could be considered as well without changing our main theoretical results (note that we consider a uniform design and therefore the local linear and Nadaraya Watson estimator behave very similarly within the interval \([b_n, 1 - b_n]\)).
2.1 Confidence surfaces with fixed width

Note that under smoothness assumption the deterministic term in (2.4) approximates $m(u,t)$. Therefore, in order to obtain quantiles for the distribution of the maximum deviation estimate (2.3), we define

$$
\hat{\Delta}(u,t) := \hat{m}(u,t) - m(u,t)
$$

as the difference between the regression function $m$ and its estimate (2.4), and note that, under smoothness assumptions and for an increasing sample size $n$, we can approximate the maximum deviation on a discrete grid, i.e.

$$
\hat{\Delta}_n := \max_{b_n \leq u \leq 1-b_n, 0 \leq t \leq 1} \sqrt{nb_n}|\hat{\Delta}(u,t)| \approx \max_{[nb_n] \leq l \leq n-[nb_n]} \sqrt{nb_n}|\hat{\Delta}(\frac{l}{n}, \frac{k}{p})| \approx \max_{[nb_n] \leq l \leq n-[nb_n]} \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{n} \varepsilon_{i,n}(\frac{k}{p})K\left(\frac{i}{n} - \frac{l}{nb_n}\right),
$$

where $p$ is increasing with $n$ as well. Therefore, the bootstrap procedure will be based on a Gaussian approximation of the right hand side of (2.6), which is the maximum norm of high-dimensional sparse vector. In this section our approach will be stated in a rather informal way, rigorous statements can be found in Section 4.

To be precise, define for $1 \leq i \leq n$ the $p$-dimensional vector

$$
Z_i(u) = (Z_{i,1}(u), \ldots, Z_{i,p}(u))^\top
$$

$$
= K\left(\frac{\frac{i}{n} - u}{b_n}\right)(\varepsilon_{i,n}(\frac{1}{p}), \varepsilon_{i,n}(\frac{2}{p}), \ldots, \varepsilon_{i,n}(\frac{p-1}{p}), \varepsilon_{i,n}(1))^\top,
$$

where $K(\cdot)$ and $b_n$ are the kernel and bandwidth used in the estimate [2.4], respectively. Next we define the $p$-dimensional vector

$$
Z_{i,l} = Z_i(\frac{l}{n}) = (Z_{i,l,1}, \ldots, Z_{i,l,p})^\top,
$$

where

$$
Z_{i,l,k} = \varepsilon_{i,n}(\frac{k}{p})K\left(\frac{i}{n} - \frac{l}{nb_n}\right) \quad (1 \leq k \leq p).
$$

Note that, by (2.6),

$$
\hat{\Delta}_n \approx \max_{[nb_n] \leq l \leq n-[nb_n]} \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{n} Z_{i,l,k} \approx \max_{1 \leq k \leq p} \left| \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{n} (Z_{i,[nb_n]}^\top, \ldots, Z_{i,n-[nb_n]}^\top)^\top \right|_\infty,
$$

where $|a|_\infty$ denotes the maximum norm of a finite dimensional vector $a$ (the dimension will always be clear from context). The entries in the vector $Z_{i,l}$ are zero whenever $|i - l|/(nb_n) \geq 1$. Therefore, the high-dimensional vector $(Z_{i,[nb_n]}^\top, \ldots, Z_{i,n-[nb_n]}^\top)^\top$ is sparse and common Gaussian approximations for its maximum norm (see, for example, Chernozhukov et al., 2013, Zhang and Cheng (2018) are not applicable.
To address this issue we reconstruct high-dimensional vectors, say \( \tilde{Z}_j \), by eliminating vanishing entries in the vectors \( Z_{i,t} \) and rearranging the nonzero ones. While this approach is very natural it produces additional dependencies, which require a substantial modification of the common multiplier bootstrap as considered, for example, in Zhou and Wu (2010), Zhou (2013), Karmakar et al. (2021) or Mies (2021) for (low dimensional) locally stationary time series.

More precisely, we define the \((n-2\lceil nb_n \rceil +1)p\)-dimensional vectors \( \tilde{Z}_1, \ldots, \tilde{Z}_{2\lceil nb_n \rceil-1} \) by

\[
\tilde{Z}_i = (Z_{i,[nb_n]}, Z_{i+1,[nb_n]+1}, \ldots, Z_{n-2\lceil nb_n \rceil+i,n-[nb_n]})^\top.
\] (2.10)

We also put \( \tilde{Z}_{2\lceil nb_n \rceil} = 0 \) and note that

\[
\frac{1}{\sqrt{nb_n}} \sum_{i=1}^n Z_i \bigg|_\infty = \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{2\lceil nb_n \rceil-1} \tilde{Z}_i \bigg|_\infty,
\] (2.11)

where \( \tilde{Z}_i := (Z_{i,[nb_n]}, Z_{i,[nb_n]+1}, \ldots, Z_{i,n-[nb_n]})^\top \). Note that the right hand side of (2.11) is a sum of the \((n-2\lceil nb_n \rceil -1)p\) dimensional vectors

\[
\tilde{Z}_1 = K(1-\lceil nb_n \rceil/nb_n) \varepsilon_1, 0, 0, \ldots, 0, 0)^\top,
\]
\[
\tilde{Z}_2 = K(2-\lceil nb_n \rceil/nb_n) \varepsilon_2, 0, 0, \ldots, 0, 0)^\top,
\]
\[
\vdots
\]
\[
\tilde{Z}_{2\lceil nb_n \rceil-1} = K(\lceil nb_n \rceil-1/nb_n) \varepsilon_{2\lceil nb_n \rceil-1}, 0, 0, \ldots, 0)^\top,
\] (2.12)

where \( \varepsilon_i = (\varepsilon_{i,n}(1/n), \ldots, \varepsilon_{i,n}(n/p)) \). On the other hand the left hand side of (2.11) is a sum of the sparse vectors

\[
\tilde{Z}_1 = \left( K\left(1-\lceil nb_n \rceil/nb_n\right)\varepsilon_1, 0, 0, \ldots, 0, 0 \right)^\top,
\]
\[
\tilde{Z}_2 = \left( K\left(2-\lceil nb_n \rceil/nb_n\right)\varepsilon_2, K\left(1-\lceil nb_n \rceil/nb_n\right)\varepsilon_2, 0, 0, \ldots, 0, 0 \right)^\top,
\]
\[
\vdots
\]
\[
\tilde{Z}_{n-1} = \left( 0, 0, 0, \ldots, 0, K\left(\lceil nb_n \rceil-1/nb_n\right)\varepsilon_{n-1} \right)^\top.
\] (2.13)

Although, the vectors on both sides of (2.11) are very different, and the number of terms in the sum is different, the non-vanishing elements over which the maximum is taken on both sides coincide. We note that this transformation yields some computational advantages and, even more important, it allows the development of a Gaussian approximation and a corresponding multiplier bootstrap, which is explained next.

To be precise, observing (2.9), we see that the right hand side of (2.11) is an approximation of the maximum absolute deviation \( \max_{u,t} \sqrt{nb_n} |\hat{\Delta}(u,t)| \). In Theorem 4.1 in Section 4.2 we will show that the vectors \( \tilde{Z}_1, \ldots, \tilde{Z}_{2\lceil nb_n \rceil-1} \) in (2.11) can be replaced by Gaussian vectors. More precisely we prove
the existence of \((n - 2[nb_n] + 1)p\)-dimensional centred Gaussian vectors \(\tilde{Y}_1, \ldots, \tilde{Y}_{2[nb_n]-1}\) with the same auto-covariance structure as the vector \(\tilde{Z}_i\) in (2.10) such that

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \max_{0 \leq t \leq 1} \sqrt{n}b_n \Delta(u, t) \leq x \right) - \mathbb{P}\left( \left| \frac{1}{\sqrt{n}b_n} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_i \right|_\infty \leq x \right) \right| = o(1) \quad (2.14)
\]

if \(p\) is an appropriate sequence converging to infinity with the sample size (for example, \(p = \sqrt{n}\)).

The estimate (2.14) is the basic tool for the construction of a simultaneous confidence surface for the regression function \(m\). For its application it is necessary to generate Gaussian random vectors \(\tilde{Y}_i\) with the same auto-covariance structure as the vector \(\tilde{Z}_i\) in (2.10), which is not trivial. To see this, note that the common multiplier bootstrap approach for approximating the distribution of \(\frac{1}{\sqrt{n}b_n} \sum_{i=1}^{2[nb_n]-1} \tilde{Z}_i\) replaces the \(\tilde{Z}_i\) by block sums multiplied with independent random variables, such as \(R_i(W_i + s_1(\tilde{Z}_i - \frac{1}{2[nb_n]-1} \sum_{s=1}^{2[nb_n]-1} \tilde{Z}_i)/\sqrt{m})\) (see Zhang and Cheng, 2014) or \(R_i(W_i + \sum_{s=1}^{2[nb_n]-1} \tilde{Z}_i)/\sqrt{m}\) (see Zhou, 2013), where \(R_1, \ldots, R_{2[nb_n]}\) are independent standard normal distributed random variables. However, this would not yield to valid approximation due to the additional dependencies between \(\tilde{Z}_1, \ldots, \tilde{Z}_{2[nb_n]-1}\). As an alternative we therefore propose a multiplier bootstrap, which also mimics this dependence structure by multiplying the \(p\)-dimensional blocks of the vectors \(\tilde{Z}_i\) by standard normal distributed random variables, which reflects the specific dependencies of these vectors. In other words the vectors \(\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3, \ldots\) in (2.12) are replaced by

\[
\begin{align*}
K\left(\frac{1-\left[nb_n\right]}{nb_n}\right) & \left(\tilde{e}_{1:1+m}R_1, \tilde{e}_{2:2+m}R_2, \ldots, \tilde{e}_{n-2[nb_n]+1:n-2[nb_n]+1+m}R_{n-2[nb_n]+1}\right)^\top, \\
K\left(\frac{2-\left[nb_n\right]}{nb_n}\right) & \left(\tilde{e}_{2:2+m}R_2, \tilde{e}_{3:3+m}R_3, \ldots, \tilde{e}_{n-2[nb_n]+2:n-2[nb_n]+2+m}R_{n-2[nb_n]+2}\right)^\top, \\
K\left(\frac{3-\left[nb_n\right]}{nb_n}\right) & \left(\tilde{e}_{3:3+m}R_3, \tilde{e}_{4:4+m}R_4, \ldots, \tilde{e}_{n-2[nb_n]+3:n-2[nb_n]+3+m}R_{n-2[nb_n]+3}\right)^\top, \\
& \vdots
\end{align*}
\]

respectively, where

\[
\tilde{e}_{j:j+m} = \frac{1}{\sqrt{m}} \sum_{r=j}^{j+\lfloor m/2 \rfloor - 1} \tilde{e}_r - \frac{1}{\sqrt{m}} \sum_{r=j+\lfloor m/2 \rfloor}^{j+\lfloor m/2 \rfloor - 1} \tilde{e}_r. \quad (2.16)
\]

Here we consider local block sums (of increasing length) to mimic the dependence structure of the error process. A difference of local block sums is used to mitigate the effect of bias if the unknown errors \(\tilde{e}_i\) are replaced by nonparametric residuals from a local linear fit, which we do next.

To be precise consider (for fixed \(t\)) the local linear estimator of \(m\) with bandwidth \(d_n > 0\), that is

\[
\left(\hat{m}_{d_n}(u, t), \frac{\partial}{\partial u} \hat{m}_{d_n}(u, t)\right)^\top = \arg\min_{\beta_0, \beta_1} \sum_{i=1}^{n} \left( X_{i,n}(t) - \beta_0 - \beta_1 \left( \frac{i}{n} - u \right) \right)^2 H\left( \frac{i}{n} - u \right) \quad (2.17)
\]

where \(H(x) = 0.75(1 - x^2)1(|x| \leq 1)\) is the Epanechnikov kernel. We define the nonparametric residuals
by
\[ \hat{\epsilon}_{i,n}(t) = X_{i,n}(t) - \hat{m}_{dn}(\frac{i}{n}, t), \] (2.18)
and the \( p \)-dimensional vector
\[ \hat{Z}_i(u) = (\hat{Z}_{i,1}(u), \ldots, \hat{Z}_{i,p}(u))^\top \]
(2.19)
\[ = K\left( \frac{\hat{\epsilon}_{i,n}(\frac{1}{p}), \hat{\epsilon}_{i,n}(\frac{2}{p}), \ldots, \hat{\epsilon}_{i,n}(\frac{p-1}{p}), \hat{\epsilon}_{i,n}(1)}{b_n} \right)^\top, \]
as an analog of (2.7). Similarly, we define the analog of (2.10) by
\[ \hat{\tilde{Z}}_j = \left( \hat{Z}_{j,[nb_n]}; \hat{Z}_{j+1,[nb_n]}; \ldots; \hat{Z}_{n-2[nb_n]+j,n-[nb_n]} \right)^\top, \]
(2.20)
where \( \hat{Z}_{i,l} = \hat{Z}_{i,\left(\frac{l}{N}\right)} = (\hat{Z}_{i,l,1}, \ldots, \hat{Z}_{i,l,p})^\top \). Note that we have replaced \( Z_{i,l} \) in (2.10) by \( \hat{Z}_{i,l} \), which can be calculated from the data. These vectors will be used in Algorithm 1 to define empirical versions of the vectors in (2.15), which then mimic the dependence structure of the vectors \( \tilde{Y}_1, \ldots, \tilde{Y}_{2[nb_n]-1} \) in the Gaussian approximation (2.14) (see equations (2.23) and (2.24) in Algorithm 1 below). The simultaneous confidence surface for the mean function \( m \) is finally defined by
\[ C_n = \{ f : [0, 1]^2 \to \mathbb{R} \mid \hat{L}_1(u, t) \leq f(u, t) \leq \hat{U}_1(u, t) \ \forall u \in [b_n, 1 - b_n] \ \forall t \in [0, 1] \}, \]
(2.21)
where the definition of functions \( \hat{L}_1, \hat{U}_1 : [0, 1]^2 \to \mathbb{R} \) is given in Algorithm 1. Finally, Theorem 4.2 in Section 4 shows that \( C_n \) defines a valid asymptotic \((1 - \alpha)\) confidence surface for the regression function \( m \) in model (2.1).

Remark 2.1. Several authors consider (stationary) functional data models with noisy observation (see Cao et al., 2012; Chen and Song, 2015 among others) and we expect that the results presented in this section can be extended to this scenario. More precisely, consider the model
\[ Y_{ij} = X_{i,n}(\frac{j}{N}) + \sigma(\frac{j}{N}) z_{ij}, \ 1 \leq i \leq n, 1 \leq j \leq N, \]
where \( X_{i,n} \) is the functional time series defined in (2.1), \( \{z_{ij}\}_{i=1,...,n;j=1,...,N} \) is an array of centred independent identically distributed observations and \( \sigma(\cdot) \) is a positive function on the interval \([0, 1]\). This means that one can not observe the full trajectory of \( \{X_{i,n}(t) \mid t \in [0, 1]\} \), but only the function \( X_{i,n} \) evaluated at the discrete time points \( 1/N, 2/N, \ldots, (N-1)/N, 1 \) subject to some random error. If \( N \to \infty \) as \( n \to \infty \), and the regression function \( m \) in (2.1) is sufficiently smooth, we expect that we can construct simultaneous confidence bands and surfaces by applying the procedure described in this section to smoothed trajectories.

For example, we can consider the smooth estimate
\[ \hat{m}(u, \cdot) = \arg \min_{g \in S_p} \sum_{i=\left[nu - \sqrt{n}\right]}^{\left[nu + \sqrt{n}\right]} \sum_{j=1}^{N} (Y_{i,j} - g(\frac{j}{N}))^2, \]
(2.22)
where $\mathcal{S}_p$ denotes the set of splines of order $p$, which depends on the smoothness of the function $t \rightarrow m(u, t)$. We can now construct confidence bands applying the methodology to the data $\tilde{X}_{i,n}(\cdot) = \tilde{m}(\frac{i}{\sqrt{n}}, \cdot), i = 1, \ldots, \sqrt{n}$ due to the asymptotic efficiency of the spline estimate (see Proposition 3.2-3.4 in Cao et al., 2012).

Alternatively, we can also obtain smooth estimates $t \rightarrow \tilde{X}_{i,n}(t)$ of the trajectory using local polynomials, and we expect that the proposed methodology applied to the data $\tilde{X}_{1,n}, \ldots, \tilde{X}_{n,n}$ will yield valid simultaneous confidence bands and surfaces, where the range for the variable $t$ is restricted to the interval $[c_n, 1 - c_n]$ and $c_n$ denotes the bandwidth of the local polynomial estimator used in smooth estimator of the trajectory.

Algorithm 1:

(a) Calculate the $(n - 2[nb_n] + 1)p$-dimensional vectors $\hat{Z}_i$ in (2.20)

(b) For window size $m_n$, let $m'_n = 2\lceil m_n/2 \rceil$, define the vectors

$$
\hat{S}_{jm'_n} = \frac{1}{\sqrt{m'_n}} \sum_{r=j}^{j+\lceil m_n/2 \rceil - 1} \hat{Z}_r - \frac{1}{\sqrt{m'_n}} \sum_{r=j+\lceil m_n/2 \rceil}^{j+m'_n-1} \hat{Z}_r
$$

(2.23)

and denote by $\hat{\epsilon}_{j,j+m'_n,k}$ the $p$-dimensional sub-vector of the vector $\hat{S}_{jm'_n}$ in (2.23) containing its $(k - 1)p + 1$st - $kp$th components.

(c) for $r=1, \ldots, B$ do

- Generate i.i.d. $N(0, 1)$ random variables $\{R_i^{(r)}\}_{i=1,\ldots,n-m'_n}$. Calculate

$$
T^{(r)}_k = \sum_{j=1}^{2[nb_n]-m'_n} \hat{\epsilon}_{j,j+m'_n,k} R_i^{(r)} , \quad k = 1, \ldots, n - 2[nb_n] + 1, \quad (2.24)
$$

$$
T^{(r)} = \max_{1 \leq k \leq n-2[nb_n]+1} |T^{(r)}_k|_\infty.
$$

end

(d) Define $T_{(1-\alpha)B}$ as the empirical $(1 - \alpha)$-quantile of the bootstrap sample $T^{(1)}, \ldots, T^{(B)}$ and

$$
\hat{L}_1(u, t) = \tilde{m}(u, t) - \hat{r}_1, \quad \hat{U}_1(u, t) = \tilde{m}(u, t) + \hat{r}_1
$$

where

$$
\hat{r}_1 = \frac{\sqrt{2T_{(1-\alpha)B}}}{\sqrt{nb_n}\sqrt{2[nb_n] - m'_n}}.
$$

Output: Simultaneous confidence surface for the mean function $m$. 

9
2.2 Confidence surfaces with varying width

The confidence surface in Algorithm 1 has a constant width and does not reflect the variability of the estimate \( \hat{m} \) at the point \((u, t)\). In this section we will construct a simultaneous confidence surface adjusted by an estimator of the long-run variance (see equation (4.4) in Section 4.1 for the exact definition). Among others, this approach has been proposed by Degras (2011) and Zheng et al. (2014) for repeated measurement data from independent subjects where a variance estimator is used for standardization. It has also been considered by Zhou and Wu (2010) who derived a simultaneous confidence tube for the parameter of a time varying coefficients linear model with a (real-valued) locally stationary error process. In the situation of non-stationary functional data as considered here this task is challenging as an estimator of the long-run variance is required, which is uniformly consistent on the square \([0, 1]^2\).

In order to define such an estimator let \( H \) denote the Epanechnikov kernel and define for some bandwidth \( \tau_n \in (0, 1) \) the weights
\[
\tilde{\omega}(t, i) = H\left(\frac{i - n}{\tau_n}\right) / \sum_{i=1}^{n} H\left(\frac{i - n}{\tau_n}\right).
\]
Let \( S_{k,r}^X = \frac{1}{\sqrt{p}} \sum_{i=k}^{k+r-1} X_{i,n} \) denote the normalized partial sum of the data \( X_{k,n}, \ldots, X_{k+r-1,n} \) (note that these are functions) and define for \( w \geq 2 \)
\[
\Delta_j(t) = \frac{S_{j-w+1,w}^X(t) - S_{j+1,w}^X(t)}{\sqrt{w}}.
\]
An estimator of the long-run variance (where the exact definition is in (4.4)) is then defined by
\[
\hat{\sigma}^2(u, t) = \sum_{j=1}^{n} \frac{w \Delta_j^2(t)}{2} \tilde{\omega}(u, j),
\]
if \( u \in [w/n, 1 - w/n] \). For \( u \in [0, w/n) \) and \( u \in (1 - w/n, 1] \) we define it as \( \hat{\sigma}^2(u, t) = \hat{\sigma}^2(w/n, t) \) and \( \hat{\sigma}^2(u, t) = \hat{\sigma}^2(1 - w/n, t) \), respectively. We will show in Proposition 4.1 in Section 4 that this estimator is uniformly consistent.

To state the bootstrap algorithm for a simultaneous confidence surface of the form (2.2) with varying width, we introduce the following notation
\[
\hat{Z}_i^\sigma(u) = (\hat{Z}_{i,1}^\sigma(u), \ldots, \hat{Z}_{i,p}^\sigma(u))^\top
\]
\[
= K\left(\frac{n - u}{\tau_n}\right) \left(\frac{\hat{\varepsilon}_{i,n}(\frac{1}{p})}{\hat{\sigma}(\frac{1}{n}, \frac{1}{p})}, \frac{\hat{\varepsilon}_{i,n}(\frac{2}{p})}{\hat{\sigma}(\frac{1}{n}, \frac{2}{p})}, \ldots, \frac{\hat{\varepsilon}_{i,n}(\frac{p-1}{p})}{\hat{\sigma}(\frac{1}{n}, \frac{p-1}{p})}, \frac{\hat{\varepsilon}_{i,n}(1)}{\hat{\sigma}(\frac{1}{n}, 1)} \right)^\top
\]
and consider the normalized analog
\[
\hat{Z}_j^\sigma = (\hat{Z}_{j,1}^\sigma)^\top, \ldots, \hat{Z}_{j,p}^\sigma)^\top
\]
of the vector \( \hat{Z}_j \) in (2.20), where \( \hat{Z}_{i,l}^\sigma = \hat{Z}_i^\sigma(\frac{l}{n}) = (\hat{Z}_{i,l,1}^\sigma, \ldots, \hat{Z}_{i,l,p}^\sigma)^\top \). The simultaneous confidence surface
with varying width for the mean function \( m \) is then defined by

\[
\mathcal{C}_n^\hat{\sigma} = \{ f : [0, 1]^2 \to \mathbb{R} \mid \hat{L}_2(u, t) \leq f(u, t) \leq \hat{U}_2(u, t) \ \forall u \in [b_n, 1 - b_n] \ \forall t \in [0, 1] \},
\]

(2.27)

where the functions \( \hat{L}_2 \) and \( \hat{U}_2 \) are constructed in Algorithm 2. Theorem 4.4 in Section 4.2 shows that this defines a valid asymptotic \((1 - \alpha)\) confidence surface for the function \( m \) in model (2.1).

**Algorithm 2:**

(a) Calculate the the estimate of the long-run variance \( \hat{\sigma}^2 \) in (2.25).

(b) Calculate the \((n - 2 \lfloor nb_n \rfloor + 1)p\)-dimensional vectors \( \hat{Z}_i^\hat{\sigma} \) in (2.26).

(c) For window size \( m_n \), let \( m'_n = 2 \lfloor m_n/2 \rfloor \), define

\[
\hat{S}^\hat{\sigma}_{jm} = \frac{1}{\sqrt{m'_n}} \sum_{r=j}^{j+m_n'/2-1} \hat{Z}_r - \frac{1}{\sqrt{m'_n}} \sum_{r=j+m_n'/2}^{j+m_n'-1} \hat{Z}_r
\]

and denote by \( \hat{\varepsilon}^\hat{\sigma}_{j,j+m'_n,k} \) be the \( p \)-dimensional sub-vector of the vector \( \hat{S}^\hat{\sigma}_{jm'_n} \) containing its \((k - 1)p + 1st - kpth\) components.

(d) for \( r=1, \ldots, B \) do

- Generate i.i.d. \( N(0, 1) \) random variables \( \{ R_i^{(r)} \}_{i=1,\ldots,n-m'_n} \). Calculate

\[
T^{\hat{\sigma},(r)}_k = \sum_{j=1}^{2 \lfloor nb_n \rfloor - m'_n} \hat{\varepsilon}^\hat{\sigma}_{j,j+m'_n,k} R_i^{(r)} , \quad k = 1, \ldots, n - 2 \lfloor nb_n \rfloor + 1,
\]

\[
T^{\hat{\sigma},(r)} = \max_{1 \leq k \leq n - 2 \lfloor nb_n \rfloor + 1} |T^{\hat{\sigma},(r)}_k|_\infty.
\]

end

(e) Define \( T_{\lfloor (1 - \alpha)B \rfloor}^{\hat{\sigma}} \) as the empirical \((1 - \alpha)\)-quantile of the sample \( T^{\hat{\sigma},(1)}, \ldots, T^{\hat{\sigma},(B)} \) and

\[
\hat{L}_2(u, t) = \hat{m}(u, t) - \hat{r}_2(u, t), \quad \hat{U}_2(u, t) = \hat{m}(u, t) + \hat{r}_2(u, t),
\]

where

\[
\hat{r}_2(u, t) = \frac{\hat{\sigma}(u, t) \sqrt{2T_{\lfloor (1 - \alpha)B \rfloor}^{\hat{\sigma}}}}{\sqrt{nb_n} \sqrt{2 \lfloor nb_n \rfloor - m'_n}}.
\]

**Output:** Simultaneous confidence surface (2.27) with varying width for the mean function \( m \).

**Remark 2.2.** The methodology presented so far can be extended to construct a simultaneous confidence surfaces for the vector of mean functions of a multivariate locally stationary functional time series. For
simplicity we consider a 2-dimensional series of the form

\[
\begin{pmatrix}
X_{i,n}^1(t) \\
X_{i,n}^2(t)
\end{pmatrix} = \begin{pmatrix}
m_1\left(\frac{i}{n}, t\right) \\
m_2\left(\frac{i}{n}, t\right)
\end{pmatrix} + \begin{pmatrix}
\varepsilon_{i,n}^1(t) \\
\varepsilon_{i,n}^2(t)
\end{pmatrix},
\]  

(2.28)

and define for a = 1, 2

\[
\hat{Z}_i^a(\mathbf{u}) = (\hat{Z}_{i,1}^a(\mathbf{u}), \ldots, \hat{Z}_{i,p}^a(\mathbf{u}))^\top
\]

\[
= K\left(\frac{i}{n} - \mathbf{u} \bigg| b_n\right) \left(\frac{\hat{\sigma}_{m,n}(\frac{1}{n})}{\hat{\sigma}_a(\frac{1}{n}, \frac{1}{p})}, \frac{\hat{\sigma}_{m,n}(\frac{2}{n})}{\hat{\sigma}_a(\frac{2}{n}, \frac{1}{p})}, \ldots, \frac{\hat{\sigma}_{m,n}(\frac{n}{n})}{\hat{\sigma}_a(\frac{n}{n}, \frac{1}{p})}, \frac{\hat{\sigma}_{m,n}(1)}{\hat{\sigma}_a(\frac{1}{n}, 1)}\right)^\top,
\]

where \(\hat{\varepsilon}_{i,n}^a(t) = X_{i,n}^a(t) - \hat{m}_{a,d_n}\left(\frac{i}{n}, t\right)\) and \(\hat{m}_{a,d_n}\left(\frac{i}{n}, t\right)\) is the local linear estimator of the function \(m_a\left(\frac{i}{n}, t\right)\) in (2.28) with bandwidth \(d_n\) (see equation (2.17) for its definition) and \(\hat{\sigma}_a^2\left(\frac{i}{n}, t\right)\) is the estimator of long-variance of \(\hat{\varepsilon}_{i,n}^a\) defined in (2.25). Next we consider the \(2(n - 2\lceil nb_n \rceil + 1)p\)-dimensional vector

\[
\hat{Z}_j^\sigma = \left(\hat{Z}_{j,\lceil nb_n \rceil}^\sigma, \hat{Z}_{j,\lceil nb_n \rceil+1}^\sigma, \ldots, \hat{Z}_{j,n-2\lceil nb_n \rceil+1}^\sigma, \hat{Z}_{j,n-2\lceil nb_n \rceil+1}^\sigma\right)^\top,
\]

where \(\hat{Z}_{i,d}^\sigma = \hat{Z}_i^\sigma\left(\frac{i}{n}\right) = (\hat{Z}_{i,1,1}^1, \hat{Z}_{i,1,1}^2, \ldots, \hat{Z}_{i,1,p}^1, \hat{Z}_{i,1,p}^2)^\top\) contains information from both components. Define for \(a = 1, 2\)

\[
\hat{L}_{3,a}(u, t) = \hat{m}_a(u, t) - \hat{r}_{3,a}(u, t), \quad \hat{U}_{3,a}(u, t) = \hat{m}_a(u, t) + \hat{r}_{3,a}(u, t),
\]

where

\[
\hat{r}_{3,a}(u, t) = \frac{\hat{\sigma}_a(u, t) \sqrt{2T_{\lceil (1 - \alpha)B \rceil}^\sigma}}{\sqrt{nb_n} \sqrt{2\lceil nb_n \rceil - m_n'}}
\]

and \(T_{\lceil (1 - \alpha)B \rceil}^\sigma\) is generated in the same way as in step (e) of Algorithm 2 with \(p\) replaced by \(2p\), \(\hat{m}_a\) is the kernel estimator of \(m_a\) defined in (2.4). Further, define for \(a = 1, 2\) the set of functions

\[
\mathcal{C}_{a,n} = \{ f : [0, 1]^2 \to \mathbb{R} \mid \hat{L}_{3,a}(u, t) \leq f(u, t) \leq \hat{U}_{3,a}(u, t) \ \forall u \in [b_n, 1 - b_n] \ \forall t \in [0, 1]\}.
\]

Suppose that the mean functions and error processes of \(X_{i,n}^1(t)\) and \(X_{i,n}^2(t)\) satisfy the conditions of Theorem 4.4, then it can be proved that the set \(\mathcal{C}_{1,n} \times \mathcal{C}_{2,n}\) defines an asymptotic \((1 - \alpha)\) simultaneous confidence surface for the vector function \((m_1, m_2)^\top\). The details are omitted for the sake of brevity.

## 3 Finite sample properties

In this section we study the finite sample performance of the simultaneous confidence surfaces proposed in the previous sections. We start giving some more details regarding the general implementation of the algorithms. A simulation study and a data example are presented in Section 3.2 and Section 3.3 respectively.
3.1 Implementation

For the estimator of the regression function in (2.4) we use the kernel (of order 4)

\[ K(x) = (45/32 - 150x^2/32 + 105x^4/32)1(|x| \leq 1), \]

and for the bandwidth we choose \( b_n = 1.2d_n \). Here the parameter \( d_n \) is chosen as the minimizer of

\[ \text{MGCV}(b) = \max_{1 \leq s \leq p} \frac{\sum_{i=1}^{n} (\hat{m}_b(i, n, s) - X_i, n, s)^2}{(1 - \text{tr}(Q_s(b))/n)^2}, \]  

(3.1)

\( Q_s(b) \) is an \( n \times n \) matrix defining the local linear estimator in (2.17), that is

\[ \hat{m}_b(1, n, s), \hat{m}_b(2, n, s), \ldots, \hat{m}_b(1, s) \top = Q_s(b)(X_1, n, s), \ldots, X_{n, n, s}) \top. \]

The criterion (3.1) is motivated by the generalized cross validation criterion introduced by Craven and Wahba (1978) and will be called Maximal Generalized Cross Validation (MGCV) method throughout this paper.

For the estimator of the long-run variance in (2.25) we use \( w = \lfloor n^{2/7} \rfloor \) and \( \tau_n = n^{-1/7} \) as recommended in Dette and Wu (2019). The window size in the multiplier bootstrap is then selected by the minimal volatility method advocated by Politis et al. (1999). For the sake of brevity, we discuss this method only for Algorithm 2 in detail (the method for Algorithm 1 is similar). We consider a grid of window sizes \( \hat{m}_1 < \ldots < \hat{m}_M \) (for some integer \( M \)). We first calculate \( \hat{S}_{\hat{m}_s}^{\hat{S}_\sigma, \diamond} \) defined in step (c) of Algorithm 2 for each \( \hat{m}_s \). Let \( \hat{S}_{\hat{m}_s}^{\hat{S}_\sigma, \diamond} \) denote the \( (n - 2[nb_n] + 1)p \) dimensional vector with \( r \)th entry defined by

\[ \hat{S}_{\hat{m}_s, r}^{\hat{S}_\sigma, \diamond} = \frac{1}{2[nb_n] - \hat{m}_s} \sum_{j=1}^{2[nb_n] - \hat{m}_s} (\hat{S}_{\hat{m}_s, r}^{\hat{S}_\sigma, \diamond})^2, \]

and consider the standard error of \( \{\hat{S}_{\hat{m}_s, r}^{\hat{S}_\sigma, \diamond}\}_{s=k-2}^{k+2} \), that is

\[ \text{se}(\{\hat{S}_{\hat{m}_s, r}^{\hat{S}_\sigma, \diamond}\}_{s=k-2}^{k+2}) = \left( \frac{1}{4} \sum_{s=k-2}^{k+2} \left( \hat{S}_{\hat{m}_s, r}^{\hat{S}_\sigma, \diamond} - \frac{1}{5} \sum_{s=k-2}^{k+2} \hat{S}_{\hat{m}_s, r}^{\hat{S}_\sigma, \diamond} \right)^2 \right)^{1/2}. \]

Then we choose \( m'_n = \hat{m}_j \) where \( j \) is defined as the minimizer of the function

\[ MV(k) = \frac{1}{(n - 2[nb_n] + 1)p} \sum_{r=1}^{(n-2[nb_n]+1)p} \text{se}(\{\hat{S}_{\hat{m}_s, r}^{\hat{S}_\sigma, \diamond}\}_{s=k-2}^{k+2}) \]

in the set \( \{3, \ldots, M - 2\} \). Throughout this section we consider \( p = \lceil \sqrt{n} \rceil \).
\subsection{Simulated data}

We consider two regression functions

\[
\begin{align*}
m_1(u, t) &= (1 + u)(6(t - 0.5)^2 + 1), \\
m_2(u, t) &= (1 + u^2)(6(t - 0.5)^2(1 + 1(t > 0.3)) + 1)
\end{align*}
\]

(note that \(m_2\) is discontinuous at the point \(t = 0.3\)). For the definition of the error processes let \(\{\varepsilon_i\}_{i \in \mathbb{Z}}\) be a sequence of independent standard normally distributed random variables and \(\{\eta_i\}_{i \in \mathbb{Z}}\) be a sequence of independent \(t\)-distributed random variables with 8 degrees of freedom. Define the functions

\[
\begin{align*}
a(t) &= 0.5 \cos(\pi t/3), \\
b(t) &= 0.4 t, \\
c(t) &= 0.3 t^2, \\
d_1(t) &= 1 + 0.5 \sin(\pi t), \\
d_2,1(t) &= 2t - 1, \\
d_2,2(t) &= 6t^2 - 6t + 1,
\end{align*}
\]

and \(\mathcal{F}^1_i = (\ldots, \varepsilon_{i-1}, \varepsilon_i), \mathcal{F}^2_i = (\ldots, \eta_{i-1}, \eta_i)\). We consider the following two locally stationary time series models defined by

\[
\begin{align*}
G_1(t, \mathcal{F}^1_i) &= a(t)G_1(t, \mathcal{F}^1_{i-1}) + \varepsilon_i, \\
G_2(t, \mathcal{F}^2_i) &= b(t)G_2(t, \mathcal{F}^2_{i-1}) + \eta_i - c(t)\eta_{i-1}.
\end{align*}
\]

Note that \(G_1\) is a locally stationary AR(1) process (or equivalently a locally stationary MA(\(\infty\)) process), and that \(G_2\) is a locally stationary ARMA(1, 1) model. With these processes we define the following functional time series model (for \(1 \leq i \leq n, 0 \leq t \leq 1\))

\[
\begin{align*}
(\text{a}) & \quad X_{i,n}(t) = m_1(\frac{i}{n}, t) + G_1(\frac{i}{n}, \mathcal{F}^1_i)d_1(t)/3. \\
(\text{b}) & \quad X_{i,n}(t) = m_1(\frac{i}{n}, t) + G_1(\frac{i}{n}, \mathcal{F}^1_i)d_{2,1}(t)/2 + G_2(\frac{i}{n}, \mathcal{F}^2_i)d_{2,2}(t)/2. \\
(\text{c}) & \quad X_{i,n}(t) = m_2(\frac{i}{n}, t) + G_1(\frac{i}{n}, \mathcal{F}^1_i)d_1(t)/3. \\
(\text{d}) & \quad X_{i,n}(t) = m_2(\frac{i}{n}, t) + G_1(\frac{i}{n}, \mathcal{F}^1_i)d_{2,1}(t)/2 + G_2(\frac{i}{n}, \mathcal{F}^2_i)d_{2,2}(t)/2.
\end{align*}
\]

In Figure 2 we display typical 95\% simultaneous confidence surfaces of the form (2.2) from one simulation run for model (a) with sample size \(n = 800\) and \(B = 1000\) bootstrap replications, which are calculated by Algorithm 1 (constant width) and Algorithm 2 (varying width). We observe that there exist differences between the surfaces with constant and variable width, but they are not substantial.

We next investigate the coverage probabilities of the different surfaces constructed in this paper for sample sizes \(n = 500\) and \(n = 800\). All results are based on 1000 simulation runs and \(B = 1000\) bootstrap replications. The left part of Table 1 shows the coverage probabilities of the surfaces with constant width while the results in the right part correspond to the bands with varying width. We observe that the simulated coverage probabilities are close to their nominal levels in all cases under consideration, which illustrates the validity of our methods for finite sample sizes.
Figure 2: 95% simultaneous confidence surfaces (A.7) and (2.27) for the regression function in model (a) from \( n = 800 \) observations. Left panel: constant width (Algorithm 1); Right panel: varying width (Algorithm 2).

Table 1: Simulated coverage probabilities of the simultaneous confidence bands (A.7) and (2.27) calculated by Algorithm 1 (constant width) and Algorithm 2 (varying width), respectively.

|          | constant width | varying width |
|----------|----------------|---------------|
|          | model (c)      | model (d)     | model (c) | model (d)      |
| level    | 90% 95%        | 90% 95%       | 90% 95%   | 90% 95%       |
| \( n = 500 \) | 90.1% 95.1%   | 89.1% 95.1%   | 90.4% 95.6% | 89.5% 95.3%   |
| \( n = 800 \) | 89.6% 94.8%   | 90.5% 95.0%   | 90.1% 95.5% | 89.7% 94.8%   |
| level    | 90% 95%        | 90% 95%       | 90% 95%   | 90% 95%       |
| \( n = 500 \) | 88.9% 94.8%   | 89.7% 95.4%   | 91.0% 95.7% | 88.3% 94.3%   |
| \( n = 800 \) | 90.1% 95.1%   | 90.8% 95.5%   | 90.1% 95.8% | 89.7% 95.5%   |

We conclude this section mentioning that confidence bands for the regression function \( m \) for a fixed \( u \) or a fixed \( t \) can be constructed in a similar manner and details and some additional numerical results for these bands are discussed in Section A of the supplemental material.

### 3.3 Real data

In this section we illustrate the proposed methodology analyzing the implied volatility (IV) of the European call option of SP500. These options are contracts such that their holders have the right to buy the SP500 at a specified price (strike price) on a specified date (expiration date). The implied volatility is derived from the observed SP500 option prices, directly observed parameters, such as risk-free rate and expiration date, and option pricing methods, and is widely used in the studies of quantitative finance. For more details, we refer to [Hull (2003)](Hull2003).

We collect the implied volatility and the strike price from the ‘optionmetrics’ database and the SP500 index from the CRSP database. Both databases can be accessed from Wharton Research Data Service (WRDS). We calculate the simultaneous confidence surfaces for the implied volatility surface, which is
a two variate function of time (more precisely time to maturity) and moneyness, where the moneyness is calculated using strike price divided by SP500 indices. The options are collected from December 21, 2016 to July 19, 2019, and the expiration date is December 20, 2019. Therefore the length of time series is 647. Within each day we observe the volatility curve, which is the implied volatility as a function of moneyness.

Figure 3: 95% simultaneous confidence surface of the form (2.2) for the IV surface. Left panel: constant width (Algorithm 1); Right panel: variable width (Algorithm 2).

Recently, Liu et al. (2016) models IV via functional time series. Following their perspective, we shall study the IV data via model (2.1), where $X_{i,n}(t)$ represents the observed volatility curve at a day $i$, with total sample size $n = 647$. We consider the options with moneyness in the range of $[0.8, 1.4]$, corresponding to options that have been actively traded in this period (note that, our methodology was developed for functions on the interval $[0, 1]$, but it is obvious how to extend this to an arbitrary compact interval $[a, b]$). The number of observations for each day varies from 34 to 56, and we smooth the implied volatility using linear interpolation and constant extrapolation.

In practice it is important to determine whether the volatility curve changes with time, i.e., to test $H_0 : m(u, t) \equiv m(t)$. As pointed out by Daglish et al. (2007), the volatility surface of an asset would be flat and unchanging if the assumptions of Black–Scholes (Black and Scholes, 1973) hold. In particular, Daglish et al. (2007) demonstrate that for most assets the volatility surfaces are not flat and are stochastically changing in practice. We can provide an inference tool for such a conclusion using the simultaneous confidence surfaces developed in Section 3. For example, note, that by the duality between confidence regions and hypotheses tests, an asymptotic level $\alpha$ test for the hypothesis $H_0 : m(u, t) \equiv m(t)$ is obtained by rejecting the null hypothesis, whenever the surface of the form $m(u, t) = m(t)$ is not contained in an $(1 - \alpha)$ simultaneous confidence surface of the form (2.2).

Therefore we construct the 95% simultaneous confidence surface for the regression function $m$ with constant and varying width using Algorithm 1 and Algorithm 2, respectively. The parameter chosen by the procedures described in Section 3.1 are given by $b_n = 0.12$ and $m_n = 18$. The results are depicted
in Figure 3 (for a better illustration the z-axis shows 100× implied volatility). We observe from both figures that the simultaneous confidence surfaces do not contain a surface of the form \( m(u, t) = m(t) \) and therefore reject the null hypothesis (at significance level 0.05%).

In the supplemental material, we construct the simultaneous confidence bands for fixed \( t \) and fixed \( u \) to study time to maturity, and the well documented volatility smile, see Figures A.6 and A.7 and the discussion in Section A.3 of the supplemental material.

4 Theoretical justification

In this section we first present the locally stationary functional time series model for which the theoretical results of this paper are derived (Section 4.1). We also describe under which conditions Algorithm 1 and 2 provide valid asymptotic \((1 - \alpha)\) confidence surfaces for the regression function \( m \) in model (2.1) (Section 4.2 and 4.3). Throughout this paper we use the notation

\[
\Theta(a,b) = a\sqrt{1 \vee \log((b/a))}
\]

for positive constants \( a, b \), and \( |v|_\infty = \max_{1 \leq i \leq k} |v_i| \) denotes the maximum norm of a \( k \)-dimensional vector \( v = (v_1, \ldots, v_k)^\top \) (the dimension \( k \) will always be clear from the context). The notation \( a \vee b \) denotes the maximum of the real numbers \( a \) and \( b \).

4.1 Locally stationary processes and physical dependence

We begin with an assumption for the mean function \( m \) in model (2.1).

**Assumption 4.1 (mean function).** For each fixed \( t \in [0, 1] \) the function \( u \rightarrow m(u, t) \) is four times continuously differentiable with bounded fourth order derivative, that is

\[
\sup_{t,u \in [0,1]} \left| \frac{\partial^4}{\partial u^4} m(u, t) \right| \leq M_0
\]

for some constant \( M_0 \).

Note that for the consistency of the estimate in (2.4) at a given point \( u \) it is not necessary to assume smoothness of the function \( m \) in the second argument. In fact, in the proof of Theorem 4.3 in Section 5.3 we show that the difference between \( \hat{m}(u, t) \) and \( m(u, t) \) can be uniformly approximated by a weighted sum of the random variables \( \varepsilon_{1,n}(t), \ldots, \varepsilon_{n,n}(t) \). As a consequence, an approximation of the form (2.6) for an increasing number of points \( \{t_1, \ldots, t_p\} \) is guaranteed by an appropriate smoothness condition on the error process \( \{\varepsilon_{i,n}(t)\}_{i=1,\ldots,n} \), which will be introduced next.

**Assumption 4.2 (error process).** The error process has the form

\[
\varepsilon_{i,n}(t) = G\left(\frac{1}{n}, t, F_i\right) , \quad i = 1, \ldots, n
\]
where $\mathcal{F}_i = (\ldots, \eta_{i-1}, \eta_i), (\eta_i)_{i \in \mathbb{Z}}$ is a sequence of independent identically distributed random variables in some measurable space $\mathcal{S}$ and $G : [0, 1] \times [0, 1] \times \mathcal{S}^\mathbb{Z} \to \mathbb{R}$ denotes a filter with the following properties

(1) There exists a constant $t_0 > 0$ such that

$$\sup_{u, t \in [0, 1]} \mathbb{E}(t_0 \exp(G(u, t, F_0))) < \infty. \tag{4.1}$$

(2) Let $(\eta'_i)_{i \in \mathbb{N}}$ denote a sequence of independent identically distributed random variables which is independent of but has the same distribution as $(\eta_i)_{i \in \mathbb{Z}}$. Define $F^*_i = (\ldots, \eta_{i-1}, \eta'_0, \eta_1, \ldots, \eta_i)$ and consider for some $q \geq 2$ the dependence measure

$$\delta_q(G, i) = \sup_{u, t \in [0, 1]} \|G(u, t, F_i) - G(u, t, F^*_i)\|_q. \tag{4.2}$$

There exists a constant $\chi \in (0, 1)$ such that

$$\delta_q(G, i) = O(\chi^i). \tag{4.3}$$

(3) For the same constant $q$ as in (2) there exists a positive constant $M$ such that

$$\sup_{t \in [0, 1], u_1, u_2 \in [0, 1]} \|G(u_1, t, F_i) - G(u_2, t, F_i)\|_q \leq M|u_1 - u_2|.$$

(4) The long run variance

$$\sigma^2(u, t) := \sum_{k=\infty}^{\infty} \text{Cov}(G(u, t, F_0), G(u, t, F_k)). \tag{4.4}$$

of the process $(G(u, t, F_i))_{i \in \mathbb{Z}}$ satisfies

$$\inf_{u, t \in [0, 1]} \sigma^2(u, t) > 0.$$

Assumption 4.2(2) requires that the dependence measure is geometrically decaying. Similar results as presented in this section can be obtained under summability assumptions with substantially more intensive mathematical arguments and complicated notation, see Remark 4.1(ii) below for some details. Assumption 4.2(3) means that the locally stationary functional time series is smooth in $u$, while the smoothness in $t$ is provided in the next assumption. They are crucial for constructing simultaneous confidence surfaces of the form (2.2).

**Assumption 4.3.** The filter $G$ in Assumption 4.2 is differentiable with respect to $t$. If $G_2(u, t, F_i) = \frac{\partial}{\partial t} G(u, t, F_i), G_2(u, 0, F_i) = G_2(u, 0+, F_i), G_2(u, 1, F_i) = G_2(u, 1-, F_i)$, we assume that there exists a
constant $q^* > 2$ such that for some $\chi \in (0, 1)$

$$\delta_{q^*}(G_2, i) = O(\chi^i).$$

**Assumption 4.4 (kernel).** The kernel $K(\cdot)$ is a symmetric continuous function which vanishes outside the interval $[-1, 1]$ and satisfies $\int_{\mathbb{R}} K(x) dx = 1$, $\int_{\mathbb{R}} K(v) v^2 dv = 0$. Additionally, the second order derivative $K''$ is Lipschitz continuous on the interval $(-1, 1)$.

In the online supplement we present several examples of locally stationary processes satisfying these assumptions (see Section B).

### 4.2 Theoretical analysis of the methodology in Section 2.1

The bootstrap methodology introduced in Section 2 is based on the Gaussian approximation (2.14), which will be stated rigorously in Theorem 4.1 below. Theorem 4.2 shows under which conditions the confidence surface (A.7) has asymptotic level $(1 - \alpha)$.

**Theorem 4.1 (Justification of Gaussian approximation (2.14)).** Let Assumptions 4.1 - 4.4 be satisfied and assume that

$$n^{1+a}a^{-1} = o(1), \quad n^{2b}b^{-1} = o(1)$$

for some $0 < a < 4/5$. Then there exists a sequence of centred $(n - 2[nb] + 1)p$-dimensional centred Gaussian vectors $\tilde{Y}_1, \ldots, \tilde{Y}_{2[nb]-1}$ with the same auto-covariance structure as the vector $\tilde{Z}_i$ in (2.10) such that the distance $\mathfrak{P}_n$ defined in (2.14) satisfies

$$\mathfrak{P}_n = O\left((nb)^{-1-11\epsilon}/8 + \Theta\left(\sqrt{nb}(b^{-1} + 1/n), np\right) + \Theta\left((np)^{1/q^*}(nb)^{-1} + 1/p\right)\right)$$

for any sequence $p \to \infty$ with $np = O(\exp(n^\epsilon))$ for some $0 \leq \epsilon < 1/11$. In particular, for the choice $p = n^c$ with $c > 0$ we have

$$\mathfrak{P}_n = o(1)$$

if the constant $q^*$ in Assumption 4.3 is sufficiently large.

Theorem 4.1 is the main ingredient to prove the validity of the bootstrap simultaneous confidence surface $C_n$ defined in (A.7) by Algorithm 1. More precisely, we show in Section 5 the following result.

**Theorem 4.2.** Assume that the conditions of Theorem 4.1 hold, and that $nd_3^3 \to \infty$, $nd_6^6 = o(1)$. Define

$$\vartheta_n = \frac{\log^2 n}{m_n} + \frac{m_n \log n}{nb_n} + \sqrt{\frac{m_n}{nb_n}}(np)^{4/q^*}. $$

If $p \to \infty$ such that $np = O(\exp(n^\epsilon))$ for some $0 \leq \epsilon < 1/11$ and

$$\vartheta_n^{1/3} \left(1 \vee \log \left(\frac{np}{\vartheta_n}\right)^{2/3} + \Theta\left(\sqrt{m_n \log np} \left(\frac{1}{\sqrt{nd_n}} + d_n^2\right)(np)^{1/9}q^*(q+1), np\right)\right) = o(1),$$

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then the simultaneous confidence surface \((A.7)\) constructed by Algorithm 1 satisfies

\[
\lim_{n \to \infty} \lim_{B \to 1} \mathbb{P}(m \in C_n \mid \mathcal{F}_n) = 1 - \alpha
\]

in probability.

Remark 4.1.

(i) A careful inspection of the proofs in Section 5 shows that it is possible to prove similar results under alternative moment assumptions. For example, Theorem 4.1 holds under the assumption

\[
\mathbb{E} \left[ \sup_{0 \leq u, t \leq 1} (G(u, t, F_0))^4 \right] < \infty.
\] (4.5)

The details are omitted for the sake of brevity. Note that the sup in (4.5) appears inside the expectation, while it appears outside the expectation in (4.1). Thus neither (4.1) implies (4.5) nor vice versa.

(ii) Assumption 4.2(2) requires geometric decay of the dependence measure \(\delta_q(G, i)\) and a careful inspection of the proofs in Section 5 shows that similar (but weaker) results can be obtained under less restrictive assumptions. To be precise, define \(\Delta q_{k, q} = \sum_{i=k}^{\infty} \delta_q(G, i)\), \(\Xi_M = \sum_{i=M}^{\infty} i \delta_2(G, i)\) and consider the following assumptions.

(a) \(\sum_{i=0}^{\infty} i \delta_3(G, i) < \infty\).

(b) There exist constants \(M = M(n) > 0\), \(\gamma = \gamma(n) \in (0, 1)\) and \(C_2 > 0\) such that

\[
(2[nb_n])^{3/8} M^{-1/2} l_n^{-5/8} \geq C_2 l_n^p
\]

where \(l_n = \max(\log(2[nb_n]) (n - 2[nb_n] + 1)p/\gamma), 1)\).

Then under the conditions of Theorem 4.1 with Assumption 4.2(2) replaced by (a) and (b), we have

\[
\mathbb{P}_n = O\left(\eta'_n + \Theta\left(\sqrt{nb_n} (b_n^q + \frac{1}{nb_n}), np\right) + \Theta\left(((np)^{1/q^*} ((nb_n)^{-1} + 1/p)^{q^*/q^*}, np\right)\right)
\]

with

\[
\eta'_n = (nb_n)^{-1/8} M^{1/2} l_n^{7/8} + \gamma + \left((nb_n)^{1/8} M^{-1/2} l_n^{-3/8} \right)^q/1+q) \left(np \Delta q_{M, q}\right)^{1/(1+q)}
\]

\[+\Xi_M^{1/3} (1 \vee \log(np/\Xi_M))^{2/3}.
\]

The same arguments as given in the proof of Theorem 4.2 show that (under the other conditions in this theorem) the set \(C_n\) defined by \((A.7)\) defines an (asymptotic) \((1 - \alpha)\) simultaneous confidence surface if \(\eta'_n = o(1)\). For example, if \(\delta_q(G, i) = O(i^{-1-\alpha})\) for some \(\alpha > 0\), \(p = n^\beta\) for some \(\beta > 0\) and \(b_n = n^{-\gamma}\) for some \(0 < \gamma < 1\), then \(\eta'_n = o(1)\) if \((1 + \beta) - (1 - \gamma)q\alpha/4 < 0\), which gives a lower bound on \(q\).
4.3 Theoretical analysis of the methodology in Section 2.2

In this section we will prove that the surface (2.27) defines an asymptotic $(1 - \alpha)$ confidence surface with varying width for the mean function $m$. If the long-run variance in (4.4) would be known, a confidence surface could be based on the “normalized” maximum deviation of

$$\hat{\Delta}^\sigma(u, t) = \frac{\hat{m}(u, t) - m(u, t)}{\sigma(u, t)}.$$ 

Therefore we will derive a Gaussian approximation for the vector $(\hat{\Delta}^\sigma(\frac{1}{n}, \frac{1}{p}))_{l=1, \ldots, n; k=1, \ldots, p}$ first and define for $1 \leq i \leq n$ the $p$ dimensional vector

$$Z_i^\sigma(u) = (Z_{i,1}^\sigma(u), \ldots, Z_{i,p}^\sigma(u))^\top = \frac{i}{n} - \frac{u}{b_n} \left( \varepsilon_{i,n}^\sigma(\frac{1}{p}), \varepsilon_{i,n}^\sigma(\frac{2}{p}), \ldots, \varepsilon_{i,n}^\sigma(\frac{p-1}{p}), \varepsilon_{i,n}^\sigma(1) \right)^\top,$$

where $\varepsilon_{i,n}^\sigma(t) = \varepsilon_i(t)/\sigma(i, t)$. Similarly as in Section 2.2 we consider the $p$-dimensional vector

$$Z_{i,l}^\sigma = Z_i^\sigma(\frac{1}{n}) = (Z_{i,l,1}^\sigma, \ldots, Z_{i,l,p}^\sigma)^\top,$$

where

$$Z_{i,l,k}^\sigma = \varepsilon_{i,n}^\sigma(\frac{k}{p}) K \left( \frac{i}{n} - \frac{l}{n} \right) (1 \leq k \leq p).$$

Finally, we define the $(n - 2\lceil nb_n \rceil + 1)p$-dimensional vectors $\tilde{Z}_1^\sigma, \ldots, \tilde{Z}_{2\lceil nb_n \rceil - 1}^\sigma$ by

$$\tilde{Z}_j^\sigma = \left( Z_{j,0}^{\sigma,\top}, Z_{j,1,\lceil nb_n \rceil}^{\sigma,\top}, \ldots, Z_{j,n-2\lceil nb_n \rceil+j, n-\lceil nb_n \rceil}^{\sigma,\top} \right)^\top$$

and obtain the following result.

**Theorem 4.3.** Let the Assumptions of Theorem 4.1 be satisfied and assume that the partial derivative $\frac{\partial^2 \sigma(u, t)}{\partial u \partial t}$ exists and is bounded on $[0, 1]^2$. Then there exist $(n - 2\lceil nb_n \rceil + 1)p$-dimensional centred Gaussian vectors $\bar{Y}_1^\sigma, \ldots, \bar{Y}_{2\lceil nb_n \rceil - 1}^\sigma$ with the same auto-covariance structure as the vector $\tilde{Z}_i^\sigma$ in (4.6) such that

$$\mathfrak{P}_n^\sigma \equiv \sup_{x \in \mathbb{R}} \mathbb{P} \left( \max_{b_n \leq u \leq 1 - b_n, 0 \leq t \leq 1} \sqrt{b_n} \left| \Delta^\sigma(u, t) \right| \leq x \right) - \mathbb{P} \left( \frac{1}{\sqrt{b_n}} \sum_{i=1}^{2\lceil nb_n \rceil - 1} \bar{Y}_i^\sigma \big| \infty \leq x \right)$$

$$= O \left( (nb_n)^{-1-11\alpha/8 + \Theta(\sqrt{b_n^4 + \frac{1}{nb_n}}), np) + \Theta(\sqrt{np} (\frac{q^*}{q^*+1}, np) + \Theta(b_n^q, np) \right),$$

for any sequence $p \rightarrow \infty$ with $np = O(\exp(n^c))$ for some $0 \leq \alpha < 1/11$. In particular, for the choice $p = n^c$ for any $c > 0$ we have

$$\mathfrak{P}_n^\sigma = o(1)$$
if the constant $q^*$ in Assumption 4.3 is sufficiently large such that
\[ \Theta\left(\left((np)^{1/q^*}\left((nb_n)^{-1} + 1/p\right)\right)^{q^*/q^*+1}, np\right) = o(1). \]

The next result shows that the estimator $\hat{\sigma}$ defined by (2.25) is uniformly consistent. Thus, roughly speaking, the unknown long-run variance $\sigma^2$ in Theorem 4.3 can be replaced by $\hat{\sigma}^2$ and the result can be used to prove the validity the confidence surface (2.27) defined by Algorithm 2 (see Theorem 4.4 below and its proof in Section 5).

**Proposition 4.1.** Let the assumptions of Theorem 4.1 be satisfied and assume that the partial derivative $\frac{\partial^2 \sigma(u,t)}{\partial u^2}$ exists on the square $[0, 1]^2$, is bounded and is continuous in $u \in (0, 1)$. If $w \to \infty$, $w = o(n^{2/5})$, $w = o(n\tau_n)$, $\tau_n \to 0$ and $n\tau_n \to \infty$ we have that

\[
\left\| \sup_{u \in [\gamma_n, 1]} |\sigma^2(u, t) - \sigma^2(u, t)| \right\|_q = O\left(g_n + \tau_n^2\right),
\]

\[
\left\| \sup_{u \in [0, \gamma_n) \cup (1 - \gamma_n, 1]} |\sigma^2(u, t) - \sigma^2(u, t)| \right\|_q = O\left(g_n + \tau_n\right),
\]

where
\[
g_n = \frac{w^{5/2}}{n} \tau_n^{1/q'} + w^{1/2} n^{-1/2} \tau_n^{-1/2 - 2/q'} + w^{-1},
\]

\[
\gamma_n = \tau_n + w/n, \quad q' = \min(q, q^*) \quad \text{and} \quad q, q^* \quad \text{are defined in Assumptions 4.2 and 4.3, respectively.}
\]

Proposition 4.1 and Theorem 4.3 yield that $\mathcal{P}^\delta_n = o_p(1)$ provided that $\mathcal{P}^\eta_n = o(1)$.

**Theorem 4.4.** Assume that the conditions of Theorem 4.2, Proposition 4.1 hold, that $p = n^c$ for some $c > 0$, and that $q^*$ in Theorem 4.3 satisfies $\Theta\left(\left((np)^{1/q^*}\left((nb_n)^{-1} + 1/p\right)\right)^{q^*/q^*+1}, np\right) = o(1)$. Further assume there exists a sequence $\eta_n \to \infty$ such that

\[
\Theta\left(\left(\sqrt{m_n \log np} (g_n + \tau_n) \eta_n (np)^{q^*/(q^*+1)}, np\right), \eta_n^{-q'} = o(1),
\]

where $\gamma_n$, $g_n$ and $q'$ are defined in Proposition 4.1. Then the simultaneous confidence surface (2.27) defined by Algorithm 2 satisfies

\[
\lim_{n \to \infty} \lim_{B \to \infty} \mathbb{P}(m \in \mathcal{C}_n^\eta | \mathcal{F}_n) = 1 - \alpha
\]

in probability.

## 5 Proofs

In the following proofs, for two real sequence $a_n$ and $b_n$ we write $a_n \lesssim b_n$, if there exists a universal positive constant $M$ such that $a_n \leq M b_n$. Let $1(\cdot)$ be the usual indicator function.
5.1 Proof of Theorem 4.1

5.1.1 Gaussian approximations on a finite grid

The proofs of Theorem 4.1 is based on an auxiliary result providing a Gaussian approximation for the maximum deviation of the quantity \( \sqrt{nb_n} |\Delta(u, t_v)| \) over the grid of \( \{1/n, \ldots, n/n\} \times \{t_1, \ldots, t_p\} \) where \( t_v = \frac{v}{p} \) (\( v = 1, \ldots, p \)).

Proposition 5.1. Assume that \( n^{1+a}b_n^0 = o(1) \), \( n^{-1}b_n^{-1} = o(1) \) for some \( 0 < a < 4/5 \), and let Assumptions 4.2 and 4.4 be satisfied.

(i) For a fixed \( u \in (0, 1) \), let \( Y_1(u), \ldots, Y_n(u) \) denote a sequence of centred \( p \)-dimensional Gaussian vectors such that \( Y_i(u) \) has the same auto-covariance structure as the vector \( Z_i(u) \) defined in (2.7). If \( p = O(\exp(n^t)) \) for some \( 0 \leq t < 1/11 \), then

\[
\mathbb{P}_{p,n}(u) := \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \max_{1 \leq v \leq p} \sqrt{nb_n} |\Delta(u, t_v)| \leq x \right) - \mathbb{P}\left( \left| \frac{1}{\sqrt{nb_n}} \sum_{i=1}^n Y_i(u) \right|_\infty \leq x \right) \right| = O\left( (nb_n)^{-1-11t}/8 + \Theta\left( \sqrt{nb_n} \left( b_n^4 + \frac{1}{nb_n} \right), p \right) \right)
\]

(ii) Let \( \tilde{Y}_1, \ldots, \tilde{Y}_{2[nb_n]-1} \) denote independent \( (n - 2[nb_n] + 1)p \)-dimensional centred Gaussian vectors with the same auto-covariance structure as the vector \( \tilde{Z}_i \) in (2.10). If \( np = O(\exp(n^t)) \) for some \( 0 \leq t < 1/11 \), then

\[
\mathbb{P}_{p,n} := \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \max_{|nb_n| \leq l \leq n-|nb_n|, 1 \leq v \leq p} \sqrt{nb_n} |\tilde{\Delta}(\frac{l}{n}, t_v)| \leq x \right) - \mathbb{P}\left( \left| \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_i \right|_\infty \leq x \right) \right| = O\left( (nb_n)^{-1-11t}/8 + \Theta\left( \sqrt{nb_n} \left( b_n^4 + \frac{1}{nb_n} \right), np \right) \right)
\]

Proof of Proposition 5.1. Using Assumptions 4.1 and 4.4 and a Taylor expansion we obtain

\[
\sup_{u \in [b_n, 1-b_n], t \in [0,1]} \left| \mathbb{E}(\hat{m}(u, t)) - m(u, t) - b_n^2 \int K(v)v^2 dv \frac{\partial^2}{\partial u^2} m(u, t)/2 \right| \leq M\left( \frac{1}{nb_n} + b_n^4 \right) \quad (5.1)
\]

for some constant \( M \). Notice that for \( u \in [b_n, 1-b_n] \),

\[
\hat{m}(u, t) - \mathbb{E}(\hat{m}(u, t)) = \frac{1}{nb_n} \sum_{i=1}^n G(\frac{i}{n}, t, \mathcal{F}_i) K\left( \frac{i - u}{b_n} \right) = \frac{1}{nb_n} \sum_{i=[n(u+b_n)]}^{[n(u-b_n)]} G(\frac{i}{n}, t, \mathcal{F}_i) K\left( \frac{i - u}{b_n} \right).
\]

Therefore, observing the definition of \( Z_i(u) \) in (2.7) we have (notice that \( Z_i(u) \) is a vector of zero if...
\[
\max_{1 \leq v \leq p} \sqrt{nb_n} |\hat{m}(u, t_{uv}) - E(\hat{m}(u, t_{uv}))| = \left| \frac{1}{\sqrt{nb_n}} \sum_{i=[n(u-b_n)]}^{[n(u+b_n)]} Z_i(u) \right|_\infty.
\]

We will now apply Corollary 2.2 of Zhang and Cheng (2018) and check its assumptions first. By Assumption 4.2(2) and the fact that the kernel is bounded it follows that

\[
\max_{1 \leq t \leq p} \sup_i \left| Z_{i,t}(u) - Z_{i,t}^{(i-j)}(u) \right|_2 = O(\chi^j),
\]

where for any (measurable function) \( g = g(F_i) \), we define for \( j \leq i \) the function \( g^{(j)} \) by \( g^{(j)} = g(F^{(j)}_i) \), where \( F^{(j)}_i = (\ldots, \eta_{j-1}, \eta_j', \eta_{j+1}, \ldots, \eta_i) \) and \( \{\eta_i'\}_{i \in \mathbb{Z}} \) is an independent copy of \( \{\eta_i\}_{i \in \mathbb{Z}} \) (recall that \( F_i = (\eta_{-\infty}, \ldots, \eta_i) \)). Lemma C.3 in Section C of the supplemental material shows that condition (9) in the paper of Zhang and Cheng (2018) is satisfied. Moreover Assumption 4.2(1) implies condition (13) in this reference. Observing that for random vector \( v = (v_1, \ldots, v_p) \) and all \( x \in \mathbb{R} \)

\[
\{|v|_\infty \leq x\} = \left\{ \max(v_1, \ldots, v_p, -v_1, \ldots, -v_p) \leq x \right\},
\]

we can use Corollary 2 of Zhang and Cheng (2018) and obtain

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{n} Y_i(u) \mid x \right) - \mathbb{P}\left( \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{n} Z_i(u) \mid x \right) \right| = O((nb_n)^{- (1+1')/8}). \tag{5.3}
\]

Therefore part (i) of the assertion follows from (5.1), (5.3) and Lemma C.1 in Appendix C of the supplement.

For part (ii), note that by the definition of the vector \( \tilde{Z}_i \) in (2.10) we have that

\[
\max_{1 \leq v \leq [nb_n]} \max_{[nb_n] \leq t \leq [n-b_n]} \left| W_n\left(\frac{t}{n}, t_{uv}\right) \right| = \max_{[nb_n] \leq t \leq [n-b_n]} \left| \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{n} Z_i\left(\frac{t}{n}\right) \right|_\infty = \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{2[nb_n]-1} \tilde{Z}_i \mid_\infty,
\]

where we use the notation

\[
W_n(u, t) = \sqrt{nb_n} (\hat{m}(u, t) - E(\hat{m}(u, t))) = \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{n} G\left(\frac{i}{n}, t, F_i\right) K\left(\frac{i}{n} - u \right) \tag{5.4}
\]

for the sake of brevity. Let \( \tilde{Z}_{i,s} \) denote the \( s \)-th entry of the vector \( \tilde{Z}_i \) defined in (2.10) \( (1 \leq s \leq (n - 2[nb_n] + 1)p) \). By Assumption 4.2(2) it follows that

\[
\max_{1 \leq s \leq (n - 2[nb_n] + 1)p} \sup_{i} \left\| \tilde{Z}_{i,s} - \tilde{Z}_{i,s}^{(i-j)} \right\|_2 = O(\chi^j),
\]
By Lemma [C.3] in Section C of the supplemental material we obtain the inequality

\[ c_1 \leq \min_{1 \leq j \leq (n-2[nb_n]+1)p} \tilde{\sigma}_{j,j} \leq \max_{1 \leq j \leq (n-2[nb_n]+1)p} \tilde{\sigma}_{j,j} \leq c_2 \]

for the quantities

\[ \tilde{\sigma}_{j,j} := \frac{1}{2\lceil nb_n \rceil - 1} \sum_{i,l=1}^{2\lceil nb_n \rceil - 1} \text{Cov}(\tilde{Z}_{i,j}, \tilde{Z}_{l,j}). \]

Therefore condition (9) in the paper of Zhang and Cheng (2018) holds, and condition (13) in this reference follows from Assumption 4.2(1). As a consequence, Corollary 2.2 in Zhang and Cheng (2018) yields

\[ \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \max_{\lceil nb_n \rceil \leq l_1 \leq n-\lceil nb_n \rceil} \left| W_n(\frac{l_1}{n}, \frac{t}{p}) \right| \leq x \right) - \mathbb{P}\left( \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{\lceil nb_n \rceil - 1} \tilde{Y}_i \leq x \right) \right| = O((nb_n)^{-(1-11\epsilon)/8}). \quad (5.5) \]

Consequently part (ii) follows by the same arguments given in the proof of part (i) via an application of Lemma C.1 in Section C of the supplemental material.

\[ \diamond \]

**5.1.2 Proof of Theorem 4.1**

For \( p \in \mathbb{N} \) define by \( t_v = \frac{v}{p} \), \( v = 0, \ldots, p \) an equidistant partition of the interval \([0,1]\) and let \( M \) be a sufficiently large generic constant which may vary from line to line. Recalling the notation of \( W_n(u,t) \) in \( (5.4) \) we have by triangle inequality

\[ \left| \sup_{b_n \leq u \leq 1-b_n, 0 \leq t \leq 1} |W_n(u,t)| - \max_{\lceil nb_n \rceil \leq l_1 \leq n-\lceil nb_n \rceil} |W_n(\frac{l_1}{n}, \frac{t}{p})| \right| \leq \tilde{W}_n, \]

where

\[ \tilde{W}_n = \max_{\lceil nb_n \rceil \leq l_1 \leq n-\lceil nb_n \rceil, 1 \leq s \leq p} |W_n(u,t) - W_n(\frac{l_1}{n}, \frac{s}{p})|. \]

By Assumption 4.3 Burkholder’s inequality and similar arguments as given in the proof of Proposition 1.1 of Dette and Wu (2022) we obtain

\[ \sup_{u,t \in [0,1]} \left\| \frac{\partial}{\partial u} W_n(u,t) \right\|_q^* \leq \frac{M}{b_n}, \quad \sup_{u,t \in [0,1]} \left\| \frac{\partial}{\partial t} W_n(u,t) \right\|_q^* \leq M, \]

\[ \sup_{u,t \in [0,1]} \left\| \frac{\partial^2}{\partial u \partial t} W_n(u,t) \right\|_q^* \leq \frac{M}{b_n}. \quad (5.6) \]
Note that we have for $\tau_s > 0$, $s = 1, 2$ and $x, y \in [0, 1)$,
\[
\sup_{0 \leq t_1 \leq \tau_1, 0 \leq t_2 \leq \tau_2} |W_n(t_1 + x, t_2 + y) - W_n(x, y)| \leq \int_0^{\tau_1} \left\| \frac{\partial}{\partial u} W_n(x, y) \right\|_{q^*} du + \int_0^{\tau_2} \left\| \frac{\partial}{\partial v} W_n(x, y) \right\|_{q^*} dv + \int_0^{\tau_1} \int_0^{\tau_2} \left\| \frac{\partial^2}{\partial x \partial t} W_n(x + u, y + v) \right\|_{q^*} dudv.
\]
Therefore, (5.6) and similar arguments as in the proof of Proposition B.2 of Dette et al. (2019) show
\[
\|\tilde{W}_n\|_{q^*} = O((np)^{1/q^*}((nb_n)^{-1} + 1/p)). \tag{5.7}
\]
Observing (5.5) (in the proof of Proposition 5.1), Lemma C.1 (in Section C of the supplement) and (5.7) it therefore follows that
\[
\mathfrak{P}_n \lesssim (nb_n)^{-(1-11\iota)/8} + \Theta\left(\sqrt{nb_n} \left(b_n^4 + \frac{1}{nb_n}\right), np\right) + \Theta(\delta, np) + \mathbb{P}(\tilde{W}_n > \delta)
\]
\[
\lesssim (nb_n)^{-(1-11\iota)/8} + \Theta\left(\sqrt{nb_n} \left(b_n^4 + \frac{1}{nb_n}\right), np\right) + \Theta(\delta, np)
\]
\[
+ \left(\left((np)^{1/q^*}((nb_n)^{-1} + 1/p)/\delta\right)^{q^*} \right.
\]
Solving $\delta = \left((np)^{1/q^*}((nb_n)^{-1} + 1/p)/\delta\right)^{q^*}$ we get $\delta = \left((np)^{1/q^*}((nb_n)^{-1} + 1/p)\right)^{q^*}$ and the assertion of the theorem follows.

### 5.2 Proof of Theorem 4.2

In the following discussion we use the following notation. For any vector $y_n$ indexed by $n$, let $y_{n,r}$ be its $r_{th}$ component. For example, $\hat{S}_{rm_{n,j}}$ is the $j_{th}$ entry of the vector $\hat{S}_{rm_n}$.

Let $T_k$ denote the statistic generated by (2.2) in one bootstrap iteration of Algorithm 1 and define for integers $a, b$ the quantities
\[
T^o_{ap+b} = \sum_{j=1}^{2[nb_n]-m'_{n}} \hat{S}_{jm_{n}(a-1)p+b} R_{k+j-1}, a = 1, \ldots, n - 2[nb_n] + 1, 1 \leq b \leq p
\]
\[
T^o := \left(T^o_1, \ldots, T^o_{(n-2[nb_n]+1)p}\right)^\top = \left(T_1^\top, \ldots, T_{n-2[nb_n]+1}\right)^\top
\]
\[
T = |T^o|_\infty = \max_{1 \leq k \leq n-2[nb_n]+1} |T_k|_\infty
\]
It suffices to show that the following inequality holds

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(T^\circ / \sqrt{2[nb_n]} - m_n' \leq x | \mathcal{F}_n) - \mathbb{P}\left( \frac{1}{\sqrt{2nb_n}} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_i \leq x \right) \right|$$

$$= O_p\left( \theta_n^{1/3} \left\{ 1 \vee \log \left( \frac{np}{\theta_n} \right) \right\}^{2/3} + \Theta\left( \left( \sqrt{m_n \log np} \left( \frac{1}{\sqrt{n}} + \frac{d_n}{n} \right) \right)^{1/2} q/(q+1), np \right) \right).$$

If this estimate has been established, Theorem 4.2 follows from Theorem 4.1, which shows that the probabilities $\mathbb{P}\left( \max_{b_n \leq u \leq 1 - b_n, 0 \leq t \leq 1} \sqrt{nb_n} | \Delta(u, t) | \leq x \right)$ can be approximated by the probabilities

$$\mathbb{P}\left( \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_i \leq x \right)$$

uniformly with respect to $x \in \mathbb{R}$.

For a proof of (5.8) we assume without loss of generality that $m_n$ is even so that $m_n' = m_n$. For convenience, let $\sum_{i=a}^{b} Z_i = 0$ if the indices $a$ and $b$ satisfy $a > b$. Given the data, it follows for the conditional covariance

$$((2[nb_n] - 1) - m_n + 1)\sigma^T_{(k_1-1)p+j_1,(k_2-1)p+j_2} := \mathbb{E}(T^\circ_{(k_1-1)p+j_1}T^\circ_{(k_2-1)p+j_2} | \mathcal{F}_n)$$

$$= \mathbb{E}\left( \sum_{r=1}^{2[nb_n]-m_n} \tilde{S}_{rm_n,(k_1-1)p+j_1} R_{k_1+r-1} \sum_{r=1}^{2[nb_n]-m_n} \tilde{S}_{rm_n,(k_2-1)p+j_2} R_{k_2+r-1} | \mathcal{F}_n \right)$$

$$= \sum_{r=1}^{2[nb_n]-m_n-(k_2-k_1)} \tilde{S}_{(r+k_2-k_1)m_n,(k_1-1)p+j_1} \tilde{S}_{rm_n,(k_2-1)p+j_2},$$

where $1 \leq k_1 \leq k_2 \leq (n - 2[nb_n] + 1)$, $1 \leq j_1, j_2 \leq p$. Here, without generality, we assume $k_1 \leq k_2$. Define $\tilde{T}^\circ$, and $\tilde{S}_{jm_n}$ in the same way as $T^\circ$, and $\tilde{S}_{jm_n}$ in (2.24) and (2.23), respectively, where the residuals $\tilde{Z}_i$ defined in (2.20) and used in step (a) of Algorithm 1 have been replaced by quantities $\tilde{Z}_i$ defined in (2.10). Then we obtain by similar arguments

$$((2[nb_n] - 1) - m_n + 1)\sigma^T_{(k_1-1)p+j_1,(k_2-1)p+j_2} := \mathbb{E}(\tilde{T}^\circ_{(k_1-1)p+j_1}\tilde{T}^\circ_{(k_2-1)p+j_2} | \mathcal{F}_n)$$

$$= \sum_{r=1}^{2[nb_n]-m_n-(k_2-k_1)} \tilde{S}_{(r+k_2-k_1)m_n,(k_1-1)p+j_1} \tilde{S}_{rm_n,(k_2-1)p+j_2}. \quad (5.10)$$

Recall the definition of the random variable $\tilde{Y}_j$ in Proposition 5.1 and denote by $\tilde{Z}_{j,i}, \tilde{Y}_{j,i}$ the $i$th component
of the vectors \( \hat{Z}_j \) and \( \hat{Y}_j \), respectively (1 \( \leq \) \( i \) \( \leq \) \( (n - 2[nb_n] + 1)p \), 1 \( \leq j \) \( \leq 2[nb_n] - 1 \)). Then we obtain

\[
\sigma_{(k_1-1)p+j_1,(k_2-1)p+j_2} := \mathbb{E}\left( \frac{1}{2[nb_n] - 1} \sum_{i_1=1}^{2[nb_n]-1} \hat{Y}_{i_1,(k_1-1)p+j_1} \sum_{i_2=1}^{2[nb_n]-1} \hat{Y}_{i_2,(k_2-1)p+j_2} \right)
\]

\[
= \mathbb{E}\left( \sum_{i_1=1}^{2[nb_n]-1} \hat{Z}_{i_1,(k_1-1)p+j_1} \sum_{i_2=1}^{2[nb_n]-1} \hat{Z}_{i_2,(k_2-1)p+j_2} \right)
\]

\[
= \mathbb{E}\left( \sum_{i_1=1}^{2[nb_n]-1} Y_{i_1+(k_1-1)[nb_n]+(k_1-1),j_1} \sum_{i_2=1}^{2[nb_n]-1} Z_{i_2+(k_2-1)[nb_n]+(k_2-1),j_2} \right),
\]

where \( Y_{i_1+(k_1-1)[nb_n]+(k_1-1),j_1} \) is the \( j_1 \)th entry of the \( p \)-dimensional random vector \( Z_{i_1+(k_1-1)[nb_n]+(k_1-1),j_1} \) and \( Y_{i_2+(k_2-1)[nb_n]+(k_2-1),j_2} \) is defined similarly. We will show at the end of this section that

\[
\left\| \max_{k_1,k_2,j_1,j_2} |\sigma_{(k_1-1)p+j_1,(k_2-1)p+j_2} - \sigma_{(k_1-1)p+j_1,(k_2-1)p+j_2}^\infty| \right\|_{f/2} = O(\vartheta_n).
\]

If (5.12) holds, it follows from Lemma C.3 in the supplement that there exists a constant \( \eta_0 > 0 \) such that

\[
\mathbb{P}\left( \min_{1 \leq j \leq \rho} \frac{T_{(k-1)p+j}}{\lambda_{(k-1)p+j} \geq \eta_0} \right) \geq 1 - O(\vartheta_n^{f/2}).
\]

Then, by Theorem 2 of Chernozhukov et al. (2015), we have

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{|\hat{T}|_{\infty}}{\sqrt{2[nb_n] - m_n}} \leq x \mid \mathcal{F}_n \right) - \mathbb{P}\left( \frac{1}{\sqrt{2nb_n}} \left| \sum_{i=1}^{2[nb_n]-1} \hat{Y}_i \right|_{\infty} \leq x \right) \right| = O_p(\vartheta_n^{1/3} (1 \lor \log(np)\vartheta_n^{2/3})).
\]

Since conditional on \( \mathcal{F}_n \), \( (\hat{T}, T) \) is an \( (n - 2[nb_n] + 1)p \) dimensional Gaussian random vector we obtain by the (conditional) Jensen inequality and conditional inequality for the concentration of the maximum of a Gaussian process (see Chapter 5 in Appendix A of Chatterjee, 2014 where a similar result has been derived in Lemma A.1) that

\[
\mathbb{E}(|\hat{T} - T|_{\infty}^{q} \mid \mathcal{F}_n) \leq M \sqrt{\log np} \left( \max_{m=1}^{\infty} \left( \sum_{j=1}^{2[nb_n]-m} (S_{j,\mu_{m+1}} - S_{j,\mu_{m}})^2 \right)^{1/2} \right)^{1/2}
\]

for some large constant \( M \) almost surely. Observing that

\[
\max_{1 \leq i \leq n} |Z_i|^l \leq \sum_{1 \leq i \leq n} |Z_i|^l \text{ for any } l > 0, n \in \mathbb{N}
\]
and using a similar argument as given in the proof of Proposition 1.1 in Dette and Wu (2022) yields

$$\frac{1}{\sqrt{2|nb_n| - m_n}} \left\| \left( n - 2[nb_n] + 1 \right) \left\{ \sum_{r=1}^{2[nb_n]} \left( \hat{S}_{jm_n,r} - S_{jm_n,r} \right)^2 \right\}^{1/2} \right\|_q \leq O \left( \sqrt{m_n} \left( \frac{1}{\sqrt{nd_n}} + d_n^2 \right)^{(np)^{1/2}} \right),$$

(recall that $d_n$ is the bandwidth of the local linear estimator (2.17)) and combining this result with the (conditional version) of Lemma C.1 in Appendix C of supplement and (5.14) yields

$$\sup_{x \in \mathbb{R}} \left| P \left( \frac{|T\circ|}{\sqrt{2|nb_n| - m_n}} > x \right| F_n \right) - P \left( \frac{1}{\sqrt{2|nb_n|}} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_{|i|} > x \right) \right| \leq \sup_{x \in \mathbb{R}} \left| P \left( \frac{|\tilde{T}\circ - T\circ|}{\sqrt{2|nb_n| - m_n}} > \delta \right| F_n \right) + O(\Theta(\delta, np)) \right|

$$

$$\sup_{x \in \mathbb{R}} \left| P \left( \frac{|\tilde{T}\circ|}{\sqrt{2|nb_n| - m_n}} > x \right| F_n \right) - P \left( \frac{1}{\sqrt{2|nb_n|}} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_{|i|} > x \right) \right| \leq \sup_{x \in \mathbb{R}} \left| P \left( \frac{|\tilde{T}\circ - T\circ|}{\sqrt{2|nb_n| - m_n}} > \delta \right| F_n \right) + O(\Theta(\delta, np)) \right|

$$

where we have used the Markov’s inequality. Taking $\delta = \left( \sqrt{m_n} \log np (\frac{1}{\sqrt{nd_n}} + d_n^2 (np)^{1/2}) \right)^{q/(q+1)}$ in (5.16), and combining this estimate with (5.13) yields (5.8) completes the proof.

**Proof of (5.12).** To simplify the notation, write

$$G_{j,i,k} = G(i+k-1, j/p, F_{i+k-1}), \quad G_{j,i,k,u} = G(i+k-1+u, j/p, F_u)$$

Without loss of generality, we consider the case $k_1 \leq k_2$. We calculate $\sigma_{(k_1-1)p+j_1,(k_2-1)p+j_2}^2$ observing the representation

$$Z_{i_1+(k_1-1),[nb_n]+(k_1-1),j_1} = G_{j_1,i_1,k_1} K \left( \frac{i_1-[nb_n]}{nb_n} \right).$$

By Lemma C.2 in the supplement it follows that

$$\mathbb{E} \left[ Z_{i_1+(k_1-1),[nb_n]+(k_1-1),j_1} Z_{i_2+(k_2-1),[nb_n]+(k_2-1),j_2} \right] = O(\chi_{i_1-i_2+k_1-k_2}).$$

uniformly for $1 \leq i_1,i_2 \leq 2[nb_n] - 1, 1 \leq j_1,j_2 \leq p, 1 \leq k_1,k_2 \leq n - 2[nb_n] + 1$. We first show that (5.12) holds whenever $k_2 - k_1 > 2[nb_n] - m_n$. On the one hand, observing and (5.9) and (5.10) that if $2[nb_n] - m_n - (k_2 - k_1) < 0$ then

$$\frac{\hat{T}\circ}{\sigma_{(k_1-1)p+j_1,(k_2-1)p+j_2}} \leq 0 \quad \text{a.s.}$$

(5.18)
Moreover, by (5.11) and (5.17), straightforward calculations show that

\[
\sigma^\ast_{(k_1-1)p+j_1,(k_2-1)p+j_2} = \frac{1}{2[nb_n]^{-1}} \sum_{i_1=1}^{2[nb_n]-1} \sum_{i_2=1}^{2[nb_n]-1} \chi_{i_2-i_1+k_2-k_1} = O\left(\frac{m_n}{nb_n}\right).
\] (5.19)

Combining (5.18), (5.19) and by applying similar argument to \(k_1 \geq k_2\), we obtain

\[
\| \max_{|k_2-k_1|>2[nb_n]-m_n} \left| \sigma^\ast_{(k_1-1)p+j_1,(k_2-1)p+j_2} - \sigma^\ast_{(k_1-1)p+j_1,(k_2-1)p+j_2} \right| \|_{q/2} = O\left(\frac{m_n}{nb_n}\right).
\] (5.20)

Now consider the case that \(k_2 - k_1 \leq 2[nb_n] - m_n\). Without losing generality we consider \(k_1 \leq k_2\). Again by (5.11)

\[
\mathbb{E}\left( \sum_{i_1=1}^{k_2-k_1} Z_{i_1+(k_1-1),[nb_n]+(k_1-1),j_1} \sum_{i_2=1}^{2[nb_n]-1} Z_{i_2+(k_2-1),[nb_n]+(k_2-1),j_2} \right)
\]

\[
= O\left( \sum_{i_1=1}^{k_2-k_1} \sum_{i_2=1}^{2[nb_n]-1} \chi_{i_2-i_1+k_2-k_1} \right) = O\left( \sum_{i_1=1}^{k_2-k_1} \sum_{i_2=1}^{2[nb_n]-1} \chi_{i_2-i_1+k_2-k_1} \right) = O(1),
\]

\[
\mathbb{E}\left( \sum_{i_1=1}^{2[nb_n]-1} Z_{i_1+(k_1-1),[nb_n]+(k_1-1),j_1} \sum_{i_2=2[nb_n]-(k_2-k_1)}^{2[nb_n]-1} Z_{i_2+(k_2-1),[nb_n]+(k_2-1),j_2} \right)
\]

\[
= O\left( \sum_{i_1=1}^{2[nb_n]-1} \sum_{i_2=2[nb_n]-(k_2-k_1)}^{2[nb_n]-1} \chi_{i_2-i_1+k_2-k_1} \right) = O\left( \sum_{i_1=1}^{2[nb_n]-1} \sum_{i_2=2[nb_n]-(k_2-k_1)}^{2[nb_n]-1} \chi_{i_2-i_1+k_2-k_1} \right) = O(1).
\]

Let \(a = \lceil M \log n \rceil\) for a sufficiently large constant \(M\). Using (5.11), it follows (considering the lags up to \(a\)) that

\[
\sigma^\ast_{(k_1-1)p+j_1,(k_2-1)p+j_2}
\]

\[
= \frac{1}{2[nb_n]-1} \mathbb{E}\left( \sum_{i_1=k_2-k_1+1}^{2[nb_n]-1} Z_{i_1+(k_1-1),[nb_n]+(k_1-1),j_1} \sum_{i_2=1}^{2[nb_n]-1} Z_{i_2+(k_2-1),[nb_n]+(k_2-1),j_2} \right)
\]

\[
+ O((nb_n)^{-1})
\]

\[
= \frac{1}{2[nb_n]-1} \mathbb{E}\left( \sum_{i_1,i_2=1}^{2[nb_n]-(k_2-k_1)-1} G_{j_1,i_1,k_2} K\left(\frac{i_1+k_2-k_1-[nb_n]}{nb_n}\right) G_{j_2,i_2,k_2} K\left(\frac{i_2-[nb_n]}{nb_n}\right) \right) + O((nb_n)^{-1})
\]

\[
= A + B + O(nb_n\chi^a + (nb_n)^{-1}),
\] (5.21)
where the terms $A$ and $B$ are defined by

\[
A := \frac{1}{2\lceil nb_n \rceil - 1} \sum_{i=1}^{2\lceil nb_n \rceil - (k_2-k_1)-1} A_i, \tag{5.22}
\]

\[
A_i = \mathbb{E}(G_{j_1,i,k_2,0}G_{j_2,i,k_2,0})K(\frac{i+k_2-k_1-\lceil nb_n \rceil}{nb_n})K(\frac{i-\lceil nb_n \rceil}{nb_n})
\]

\[
B = \frac{1}{2\lceil nb_n \rceil - 1} \sum_{u=1}^{a} (B_1,u + B_2,u),
\]

\[
B_1,u = \sum_{i=1}^{2\lceil nb_n \rceil - (k_2-k_1)-1-u} B_1,u,i, \tag{5.23}
\]

\[
B_2,u =: \sum_{i=1}^{2\lceil nb_n \rceil - (k_2-k_1)-1-u} B_2,u,i. \tag{5.24}
\]

and

\[
B_1,u,i = \mathbb{E}(G_{j_1,i,k_2,u}G_{j_1,i,k_2,0})K(\frac{i+u+k_2-k_1-\lceil nb_n \rceil}{nb_n})K(\frac{i-\lceil nb_n \rceil}{nb_n})
\]

\[
B_2,u,i = \mathbb{E}(G_{j_1,i,k_2,0}G_{j_2,i,k_2,u})K(\frac{i+k_2-k_1-\lceil nb_n \rceil}{nb_n})K(\frac{i+u-\lceil nb_n \rceil}{nb_n})
\]

Therefore, by (5.21), we have that

\[
\sigma_{(k_1-1)p+j_1,(k_2-1)p+j_2}^2 = \frac{1}{2\lceil nb_n \rceil - 1} \left( \sum_{i=1}^{2\lceil nb_n \rceil - (k_2-k_1)} A_i + \sum_{u=1}^{a} \sum_{i=1}^{2\lceil nb_n \rceil - (k_2-k_1)-u} (B_1,u,i + B_2,u,i) \right) + O(nb_n \chi^a + (nb_n)^{-1}). \tag{5.25}
\]

Now for the term in (5.10) we have

\[
m_n \tilde{S}_{(r+k_2-k_1)m_n,(k_1-1)p+j_2}^2 = \left( \sum_{i=r+k_2-k_1}^{r+k_2-k_1+m_n/2-1} - \sum_{i=r+k_2-k_1+m_n/2}^{r+k_2-k_1+m_n} \right) Z_{i+k_1-1,\lceil nb_n \rceil + k_1-1,j_1}
\]

\[
\times \left( \sum_{i=r}^{r+m_n/2-1} - \sum_{i=r+m_n/2}^{r+m_n} \right) Z_{i+k_2-1,\lceil nb_n \rceil + k_2-1,j_2}
\]

\[
= \left( \sum_{i=r}^{r+m_n/2-1} - \sum_{i=r+m_n/2}^{r+m_n} \right) G_{j_1,i,k_2}K(\frac{i+k_2-k_1-\lceil nb_n \rceil}{nb_n}) \times \left( \sum_{i=r}^{r+m_n/2-1} - \sum_{i=r+m_n/2}^{r+m_n} \right) G_{j_2,i,k_2}K(\frac{i-\lceil nb_n \rceil}{nb_n}).
\]

By Lemma [C.2] in the supplement, it follows that uniformly for $|k_2 - k_1| \leq 2\lceil nb_n \rceil - m_n$ and $1 \leq r \leq
\([2nb_n] - m_n - (k_2 - k_1)\),

\[
m_n \mathbb{E} \tilde{\mathcal{S}}_{(r+k_2-k_1)m_n,(k_1-1)p+j_1} \tilde{\mathcal{S}}_{r m_n,(k_2-1)p+j_2}
= \sum_{i=r}^{r+m_n} \mathbb{E}(G_{j_1,i,k_2} G_{j_2,i,k_2}) K\left(\frac{i+k_2-k_1 - \lfloor nb_n \rfloor}{nb_n}\right) K\left(\frac{i - \lceil nb_n \rceil}{nb_n}\right)
+ \sum_{u=1}^{m} \left( \sum_{i=r}^{r+m_n-u} \left( \mathbb{E}(G_{j_1,i,(k_2+u)} G_{j_2,i,k_2}) K\left(\frac{i+k_2-k_1 - \lfloor nb_n \rfloor}{nb_n}\right) K\left(\frac{i - \lceil nb_n \rceil}{nb_n}\right) \right) + O(m_n \chi^a + a^2), \right) = O(\chi^a)
\]

where the the term \(m_n \chi^a\) corresponds to the error of omitting terms in the sum with a large index \(a\), and the term \(a^2\) summarizes the error due to ignoring different signs in the product \(\tilde{\mathcal{S}}_{(r+k_2-k_1)m_n,(k_1-1)p+j_1} \tilde{\mathcal{S}}_{r m_n,(k_2-1)p+j_2}\) (for each index \(u\), we omit \(2u\)). Furthermore, by Assumption \ref{ass:4.4} \ref{ass:4.2} \ref{ass:4.3} it follows that uniformly for \(|u| \leq a\)

\[
\frac{1}{m_n} \sum_{i=r}^{r+m_n} \mathbb{E}(G_{j_1,i,k_2} G_{j_2,i,k_2}) K\left(\frac{i+k_2-k_1 - \lfloor nb_n \rfloor}{nb_n}\right) K\left(\frac{i - \lceil nb_n \rceil}{nb_n}\right) = A_r + O\left(\frac{m_n}{nb_n}\right), \quad (5.27)
\]

\[
\frac{1}{m_n} \sum_{i=r}^{r+m_n-u} \mathbb{E}(G_{j_1,i,(k_2+u)} G_{j_2,i,k_2}) K\left(\frac{i+k_2-k_1 - \lfloor nb_n \rfloor}{nb_n}\right) K\left(\frac{i - \lceil nb_n \rceil}{nb_n}\right) = B_{1,u,r} + O\left(\frac{m_n}{nb_n} + \frac{a}{m_n}\right), \quad (5.28)
\]

\[
\frac{1}{m_n} \sum_{i=r}^{r+m_n-u} \mathbb{E}(G_{j_1,i,(k_2+u)} G_{j_2,i,k_2}) K\left(\frac{i+k_2-k_1 - \lfloor nb_n \rfloor}{nb_n}\right) K\left(\frac{i - \lceil nb_n \rceil}{nb_n}\right) = B_{2,u,r} + O\left(\frac{m_n}{nb_n} + \frac{a}{m_n}\right), \quad (5.29)
\]

where terms \(A_r, B_{1,u,r}\) and \(B_{2,u,r}\) are defined in equations \ref{eq:5.22}, \ref{eq:5.23} and \ref{eq:5.24}, respectively. Notice that \ref{eq:5.10} \ref{eq:5.26}, \ref{eq:5.27}, \ref{eq:5.28} \ref{eq:5.29} yield that

\[
\mathbb{E} \sigma^n_{(k_1-1)p+j_1,(k_2-1)p+j_2} = \frac{1}{2[\lfloor nb_n \rfloor - m_n - (k_2-k_1)]} \left\{ \sum_{r=1}^{2[\lfloor nb_n \rfloor - m_n - (k_2-k_1)]} \left( A_r + O\left(\frac{m_n}{nb_n}\right) \right) + \sum_{u=1}^{m} \sum_{r=1}^{m_n} \left( B_{1,u,r} + B_{2,u,r} + O\left(\frac{m_n}{nb_n} + \frac{a}{m_n}\right) \right) \right\} + O\left(\chi^a + \frac{a^2}{m_n}\right). \quad (5.30)
\]

Lemma \ref{lem:3.2} in the supplement implies

\[
\max_{1 \leq r \leq 2[\lfloor nb_n \rfloor - (k_2-k_1)], 1 \leq k_1 \leq k_2 \leq (n-2[\lfloor nb_n \rfloor]), s=1,2} B_{s,u,r} = O(\chi^a),
\]

which yields in combination with equations \ref{eq:5.25}, \ref{eq:5.30} with \(a = M \log n\) for a sufficiently large constant.
$M$, and a similar argument applied to the case that $k_1 \geq k_2$,

$$\max_{1 \leq k_1, k_2 \leq (n-2) [n b_n] + 1, \ |k_2 - k_1| \leq 2 [n b_n] - m_n, 1 \leq j_1, j_2 \leq p} \left| \mathbb{E} \sigma^\circ (k_1 - 1) p + j_1, (k_2 - 1) p + j_2 - \sigma^\circ (k_1 - 1) p + j_1, (k_2 - 1) p + j_2 \right| = O \left( \frac{\log^2 n}{m_n} + \frac{m_n \log n}{n b_n} \right). \tag{5.31}$$

Furthermore, using (5.15), the Cauchy-Schwartz inequality, a similar argument as given in the proof of Lemma 1 of Zhou (2013) and Assumption 4.2(2) yield that

$$\left\| \max_{1 \leq k_1, k_2 \leq (n-2) [n b_n] + 1, \ |k_2 - k_1| \leq 2 [n b_n] - m_n} \left| \sigma^\circ (k_1 - 1) p + j_1, (k_2 - 1) p + j_2 - \sigma^\circ (k_1 - 1) p + j_1, (k_2 - 1) p + j_2 \right| \right\|_{q/2} = O \left( \sqrt{\frac{m_n}{n b_n}} (np)^{4/q} \right). \tag{5.32}$$

Combining (5.31) and (5.32), we obtain

$$\left\| \max_{k_1, k_2, j_1, j_2 \ |k_2 - k_1| \leq 2 [n b_n] - m_n} \left| \sigma^\circ (k_1 - 1) p + j_1, (k_2 - 1) p + j_2 - \sigma^\circ (k_1 - 1) p + j_1, (k_2 - 1) p + j_2 \right| \right\|_{q/2} = O \left( \frac{\log^2 n}{m_n} + \frac{m_n \log n}{n b_n} + \sqrt{\frac{m_n}{n b_n}} (np)^{4/q} \right). \tag{5.33}$$

Therefore the estimate (5.12) follows combining (5.20) and (5.33).

### 5.3 Proof of Theorem 4.3

Similarly to (5.1) and (5.2) in the proof of Theorem 5.1 we obtain

$$\sup_{u \in [b_n, 1-b_n], t \in [0,1]} \frac{1}{\sigma(u,t)} \left| \mathbb{E}(\hat{m}(u,t)) - m(u,t) \right| \leq M \left( \frac{1}{nb_n} + b_n^4 \right) \tag{5.34}$$

for some constant $M$, where we have used the fact that, by Assumption 4.4 $\int K(v) v^2 dv = 0$. Moreover, by a similar but simpler argument as given in the proof of equation (B.7) in Lemma B.3 of Dette et al. (2019) we have for the quantity

$$\frac{\hat{m}(u,t) - \mathbb{E}(\hat{m}(u,t))}{\sigma(u,t)} = \Psi^\circ(u,t) := \frac{1}{nb_n} \sum_{i=1}^{n} G\left( \frac{i}{n}, t, \mathcal{F}_i \right) \left( \frac{i}{n} - u \right) K\left( \frac{i}{b_n} \right)$$

the estimate

$$\left\| \sup_{u \in [b_n, 1-b_n], t \in [0,1]} \sqrt{nb_n} |\Psi^\circ(u,t) - \Psi^\circ(u,t)| \right\|_q = O(b_n^{1-2/q}) \tag{5.35}$$

where

$$\Phi^\circ(u,t) = \frac{1}{nb_n} \sum_{i=1}^{n} G\left( \frac{i}{n}, t, \mathcal{F}_i \right) \left( \frac{i}{n} - u \right) K\left( \frac{i}{b_n} \right).$$
Following the proof of Theorem 4.1, we find that

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \max_{b_n \leq u \leq 1-b_n, 0 \leq t \leq 1} \sqrt{nb_n} \Phi(u,t) \leq x \right) - \mathbb{P}\left( \left| \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_i^\sigma \right| \leq x \right) \right| = O\left( (nb_n)^{-11/4} + \Theta\left( ((np)^{1/q^*}((nb_n)^{-1} + 1/p) \frac{q^*}{q^*+1}, np \right) \right).
\]

Combining this result with Lemma C.1 in the supplement (with \( X = \max_{b_n \leq u \leq 1-b_n, 0 \leq t \leq 1} \sqrt{nb_n} \Phi(u,t) \), \( Y = \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_i^\sigma \), \( X' = \max_{b_n \leq u \leq 1-b_n, 0 \leq t \leq 1} \sqrt{nb_n} \Psi(u,t) \)) and (5.35) gives

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \max_{b_n \leq u \leq 1-b_n, 0 \leq t \leq 1} \sqrt{nb_n} \Psi(u,t) \leq x \right) - \mathbb{P}\left( \left| \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_i^\sigma \right| \leq x \right) \right| = O\left( (nb_n)^{-11/4} + \Theta\left( ((np)^{1/q^*}((nb_n)^{-1} + 1/p) \frac{q^*}{q^*+1}, np \right) \right)
+ \mathbb{P}\left( \sup_{u \in [b_n,1-b_n], t \in [0,1]} \sqrt{nb_n} \Phi(u,t) - \Psi(u,t) > \delta \right) + \Theta(\delta, np) = O\left( (nb_n)^{-11/4} + \Theta\left( ((np)^{1/q^*}((nb_n)^{-1} + 1/p) \frac{q^*}{q^*+1}, np \right) \right) + \Theta(\delta, np) + \frac{b_n^{q^*+2}}{\delta^q}.
\]

(5.36)

Taking \( \delta = b_n^{q^*-2} \) we obtain for the last two terms in (5.36)

\[
\Theta(\delta, np) + \frac{b_n^{q^*-2}}{\delta^q} = O\left( \Theta(b_n^{q^*-2}, np) \right).
\]

On the other hand, (5.34), (5.36) and Lemma C.1 (with \( X = \max_{b_n \leq u \leq 1-b_n, 0 \leq t \leq 1} \sqrt{nb_n} \Delta(u,t) \), \( Y = \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_i^\sigma \), \( X' = \max_{b_n \leq u \leq 1-b_n, 0 \leq t \leq 1} \sqrt{nb_n} \tilde{\Delta}(u,t) \)) and \( \delta = M \sqrt{nb_n} (\frac{1}{nb_n} + b_n^4) \) with a sufficiently large constant \( M \) yield

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \max_{b_n \leq u \leq 1-b_n, 0 \leq t \leq 1} \sqrt{nb_n} \tilde{\Delta}(u,t) \leq x \right) - \mathbb{P}\left( \left| \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{2[nb_n]-1} \tilde{Y}_i^\sigma \right| \leq x \right) \right| = O\left( (nb_n)^{-11/4} + \Theta\left( ((np)^{1/q^*}((nb_n)^{-1} + 1/p) \frac{q^*}{q^*+1}, np \right) \right)
+ \Theta\left( \sqrt{nb_n} (b_n^{q^*} + \frac{1}{nb_n}), np \right) + \Theta(b_n^{q^*-2}, np).
\]

5.4 Proof of Proposition 4.1

Define \( \bar{\tilde{S}}_{k,r}^G(t) = \frac{1}{\sqrt{T}} \sum_{i=k}^{k+r-1} G(i/n, \mathcal{F}_i) \), and define for \( u \in [w/n, 1-w/n] \)

\[
\tilde{\Delta}(t) = \frac{\bar{\tilde{S}}_{j-w+1,w}(t) - \bar{\tilde{S}}_{j+1,w}(t)}{\sqrt{w}}, \quad \bar{\sigma}^2(u,t) = \frac{n}{w} \tilde{\Delta}(t)^2 / 2 \tilde{\omega}(u,j).
\]

34
as the analogs of $\Delta_j(t)$ defined in the main article and the quantities in (2.25), respectively. We also use the convention $\hat{\sigma}^2(u, t) = \hat{\sigma}^2(w/n, t)$ and $\hat{\sigma}^2(u, t) = \hat{\sigma}^2(1 - w/n, t)$ if $u \in [0, w/n)$ and $u \in (1 - w/n, 1]$, respectively. Assumption 4.1 and the mean value theorem yield

$$
\max_{j} \sup_{w \leq j \leq n - w} \sup_{0 \leq t \leq 1} |\Delta_j(t) - \Delta_j(t)| = \max_{w \leq j \leq n - w} \sup_{0 \leq t \leq 1} \left| \sum_{r=j-w+1}^{j+w} m(r/n, t) - \sum_{r=j+1}^{j+w} m(r/n, t) \right| = O(w/n). \quad (5.37)
$$

On the other hand, Assumption 4.2 and Assumption 4.3 and similar arguments as given in the proof of Lemma 3 of Zhou and Wu (2010) give

$$
\max_j \|\tilde{\Delta}_j(t)\|_{q^*} = O(\sqrt{w}), \quad \max_j \left\| \frac{\partial}{\partial t} \tilde{\Delta}_j(t) \right\|_{q^*} = O(\sqrt{w}). \quad (5.38)
$$

Here we use the convention that $\frac{\partial}{\partial t} \tilde{\Delta}_j|_{t=0} = \frac{\partial}{\partial t} \tilde{\Delta}_j|_{t=0+}$, $\frac{\partial}{\partial t} \tilde{\Delta}_j|_{t=1} = \frac{\partial}{\partial t} \tilde{\Delta}_j|_{t=1-}$. Moreover, Proposition B.1 of Dette et al. (2019) yields

$$
\max_j \left\| \sup_t |\tilde{\Delta}_j(t)| \right\|_{q^*} = O(\sqrt{w}). \quad (5.39)
$$

Now we introduce the notation $C_j(t) = \tilde{\Delta}_j(t) - \Delta_j(t)$ (note that this quantity is not random) and obtain by (5.37) the representation

$$
\hat{\sigma}^2(u, t) - \hat{\sigma}^2(u, t) = \sum_{j=1}^{n} \frac{w(2\tilde{\Delta}_j(t) - C_j(t))C_j(t)}{2} \hat{w}(u, j) = \sum_{j=1}^{n} w\tilde{\Delta}_j(t)C_j(t)\hat{w}(u, j) + O(w^3/n^2) \quad (5.40)
$$

uniformly with respect to $u, t$. Furthermore, by (5.37) we have

$$
\sup_{t \in [0, 1]} \left| \sum_{j=1}^{n} w\tilde{\Delta}_j(t)C_j(t)\hat{w}(u, j) \right| \leq W^\circ(u) := M(w/n) \sum_{j=1}^{n} w \sup_{t \in [0, 1]} |\Delta_j(t)|\hat{w}(u, j), \quad (5.41)
$$

where $M$ is a sufficiently large constant. Notice that $W^\circ(u)$ is differentiable with respect to the variable $u$. Therefore it follows from the triangle inequality, (5.39) and Proposition B.1 of Dette et al. (2019), that

$$
\left\| \sup_{u \in [\gamma_n, 1 - \gamma_n]} |W^\circ(u)| \right\|_{q^*} = O(\frac{w^{5/2}}{n}n^{-1/q^*}). \quad (5.42)
$$

Combining (5.40) and (5.42), we obtain

$$
\left\| \sup_{u \in [\gamma_n, 1 - \gamma_n]} |\hat{\sigma}^2(u, t) - \hat{\sigma}^2(u, t)| \right\|_{q^*} = O(\frac{w^{5/2}}{n}n^{-1/q^*} + w^3/n^2). \quad (5.43)
$$
By Burkholder inequality (see for example Wu, 2005) in $L^{q/2}$ norm, (5.38) and similar arguments as given in the proof of Lemma 3 in Zhou and Wu (2010) we have

$$
\sup_{u \in [\gamma_n, 1-\gamma_n]} \left\| \frac{\partial^2}{\partial t^2} \left( \sigma^2(u, t) - \mathbb{E}(\sigma^2(u, t)) \right) \right\|_{q/2} = O\left( w^{1/2} n^{-1/2} \tau_n^{-1/2} \right),
$$

$$
\sup_{u \in [\gamma_n, 1-\gamma_n]} \left\| \frac{\partial}{\partial u} \left( \sigma^2(u, t) - \mathbb{E}(\sigma^2(u, t)) \right) \right\|_{q/2} = O\left( w^{1/2} n^{-1/2} \tau_n^{-1/2} \right),
$$

$$
\sup_{u \in [\gamma_n, 1-\gamma_n]} \left\| \frac{\partial}{\partial u} + \frac{\partial^2}{\partial u^2} \left( \sigma^2(u, t) - \mathbb{E}(\sigma^2(u, t)) \right) \right\|_{q/2} = O\left( w^{1/2} n^{-1/2} \tau_n^{-1/2} \right).
$$

(5.44)

It can be shown by similar but simpler argument as given in the proof of Proposition B.2 of Dette et al. (2019) that these estimates imply

$$
\sup_{u \in [\gamma_n, 1-\gamma_n]} \left\| \sigma^2(u, t) - \mathbb{E}(\sigma^2(u, t)) \right\|_{q/2} = O\left( w^{1/2} n^{-1/2} \tau_n^{-1/2} \right),
$$

(5.45)

Moreover, it follows from the proof of Theorem 4.4 of Dette and Wu (2019) that

$$
\sup_{u \in [\gamma_n, 1-\gamma_n]} \left| \mathbb{E} \sigma^2(u, t) - \sigma^2(u, t) \right| = O\left( \sqrt{w/n} + w^{-1} + \tau_n^2 \right),
$$

$$
\sup_{u \in [0, \gamma_n] \cup (1-\gamma_n, 1]} \left| \mathbb{E} \sigma^2(u, t) - \sigma^2(u, t) \right| = O\left( \sqrt{w/n} + w^{-1} + \tau_n \right)
$$

(5.46)

and the assertion is a consequence of (5.43), (5.45) and (5.46).

### 5.5 Proof of Theorem 4.4

Recall that $g_n = \frac{w^{1/2} \tau_n^{1/q'}}{n} + w^{1/2} n^{-1/2} \tau_n^{-1/2} - 2/q' + w^{-1}$ and let $\eta_n$ be a sequence of positive numbers such that $\eta_n \to \infty$ and $(g_n + \tau_n) \eta_n \to 0$ (note that $g_n + \tau_n$ is the convergence rate of the estimator $\hat{\sigma}^2$ in Proposition 4.1). Define the $\mathcal{F}_n$ measurable event

$$
A_n = \left\{ \sup_{u \in [0,1], t \in [0,1]} |\hat{\sigma}^2(u, t) - \sigma^2(u, t)| > (g_n + \tau_n) \eta_n \right\},
$$

then Proposition 4.1 and Markov’s inequality yield

$$
\mathbb{P}(A_n) = O\left( \eta_n^{-q'} \right).
$$

(5.47)
Then by Theorem 4.3, Proposition 4.1 and Lemma C.1 in the supplement we have

\[ P_T^\varphi = \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{2\lfloor nb_n \rfloor - 1} \tilde{Y}_i^\varphi \leq x \right) - \mathbb{P}\left( \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{\lfloor nb_n \rfloor - 1} \tilde{Y}_i \leq x \right) \right| = o_p(1). \tag{5.48} \]

Let \( T_k^\varphi \) denote the statistic \( T_k^{\hat{\varphi},(r)} \) in step (d) of Algorithm 2 generated by one bootstrap iteration and define for integers \( a, b \) the quantities

\[ T_{\alpha_p+b}^{\hat{\varphi},\tilde{\varphi}} = \sum_{j=1}^{2\lfloor nb_n \rfloor - m'_n} \hat{S}_{jm'_n,(a-1)p+b} R_{k+j-1}, \quad a = 1, \ldots, n - 2\lfloor nb_n \rfloor + 1, 1 \leq b \leq p \]

\[ T^{\hat{\varphi},\varphi} := \left( (T_1^{\hat{\varphi},\varphi})^\top, \ldots, (T_{(n-2\lfloor nb_n \rfloor+1)p}^{\hat{\varphi},\varphi})^\top \right)^\top = \left( T_1^\varphi, \ldots, T_{n-2\lfloor nb_n \rfloor+1}^\varphi \right)^\top \]

and therefore

\[ T^{\varphi} = |T^{\hat{\varphi},\varphi}|_{\infty} = \max_{1 \leq k \leq n-2\lfloor nb_n \rfloor+1} |T_k^\varphi|_{\infty} \]

We recall the notation (4.6), introduce the \( (n-2\lfloor nb_n \rfloor+1)p \)-dimensional random vectors \( \hat{S}_{jm'_n}^{\sigma,*} = \sum_{r=j}^{j+m'_n-1} \hat{Z}_r^{\sigma} \), and

\[ \hat{S}_{jm'_n} = \frac{1}{\sqrt{m'_n}} \hat{S}_{jm'_n,[m_n/2]} - \frac{1}{\sqrt{m'_n}} \hat{S}_{jm'_n,[m_n/2]+1,[m_n/2]} \]

and consider

\[ T_k^\varphi = \sum_{j=1}^{2\lfloor nb_n \rfloor - m'_n} \hat{S}_{jm'_n,(k-1)p+1:kp} R_{k+j-1}, \quad k = 1, \ldots, n - 2\lfloor nb_n \rfloor + 1, \]

\[ T^{\sigma,\varphi} := \left( (T_1^{\sigma,\varphi})^\top, \ldots, (T_{(n-2\lfloor nb_n \rfloor+1)p}^{\sigma,\varphi})^\top \right)^\top = \left( T_1^\varphi, \ldots, T_{n-2\lfloor nb_n \rfloor+1}^\varphi \right)^\top, \]

where \( T^{\sigma,\varphi} \) is obtained from \( T^{\hat{\varphi},\varphi} \) by replacing \( \hat{\varphi} \) by \( \varphi \). Similar arguments as given in the proof of Theorem 4.2 show, that it is sufficient to show the estimate

\[ \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( |T^{\sigma,\varphi}/\sqrt{2\lfloor nb_n \rfloor - m'_n}|_{\infty} \leq x | \mathcal{F}_n \right) - \mathbb{P}\left( \frac{1}{\sqrt{2nb_n}} \sum_{i=1}^{\lfloor nb_n \rfloor - 1} \tilde{Y}_i^\varphi \leq x \right) \right| = O_p \left( \vartheta_n^{1/3} \{ 1 \vee \log(np_{\varphi}) \}^{2/3} + \Theta \left( \sqrt{m_n \log np} \left( \frac{1}{\sqrt{nd_n}} + d_n^2 \right) (np)^{\frac{1}{2q}} q/(q+1), np \right) \right. \]

\[ + \Theta \left( \left( \sqrt{m_n \log np} ((g_n + \tau_n) \eta_n)(np)^{\frac{1}{2q}} \right)^{q/(q+1)}, np \right) + n_{-q}^{-q} \], \tag{5.49} \]

where \( \vartheta_n \) and \( d_n \) are defined in Theorem 4.2. The assertion of Theorem 4.4 then follows from (5.48).
Now we prove (5.49). By the first step in the proof of Theorem 4.2 it follows that

$$
\sup_{x \in \mathbb{R}} \left| \mathbb{P}(|T^{\sigma,\phi}/\sqrt{2[nb_n]} - m_n| \leq x | F_n) - \mathbb{P}\left( \frac{1}{\sqrt{2nb_n}} \sum_{i=1}^{2[nb_n]-1} \hat{Y}_i^\sigma \right) \leq x \right| \\
= O_p\left( \frac{1}{n} \{1 + \log(np)\}^{2/3} + \Theta\left( \left( \sqrt{m_n} \log np + \frac{1}{np} \right)^{\frac{1}{2}} \right) \right). \tag{5.50}
$$

By similar arguments as given in the proof of Theorem 4.2 we have

$$
\mathbb{E}(|T^{\sigma,\phi} - T^{\hat{\sigma},\hat{\phi}}|_q, 1(A_n)| F_n) \leq M \left( \sqrt{\log np} \max_{r=1}^{[2nb_n] - m_n} \left( \sum_{j=1}^{[2nb_n] - m_n'} \left( \hat{S}_{j m_n', r}^\sigma - \hat{S}_{j m_n', r}^{\hat{\phi}} \right)^2 1(A_n) \right)^{1/2} \right)^q. \tag{5.51}
$$

for some large constant $M$ almost surely, and the triangle inequality, a similar argument as given in the proof of Proposition 1.1 in Dette and Wu (2022) and (5.15) yield

$$
\frac{1}{\sqrt{2[nb_n]} - m_n} \left( \sum_{j=1}^{[2nb_n] - m_n'} \left( \hat{S}_{j m_n', r}^\sigma - \hat{S}_{j m_n', r}^{\hat{\phi}} \right)^2 1(A_n) \right)^{1/2} = O\left( \sqrt{m_n(g_n + \tau_n)} \eta_n(np)^{\frac{1}{2}} \right). \tag{5.52}
$$

This together with the (conditional version) of Lemma C.1 and (5.51) shows that

$$
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{|T^{\sigma,\phi}|_\infty}{\sqrt{2[nb_n]} - m_n} > x | F_n \right) - \mathbb{P}\left( \frac{1}{\sqrt{2nb_n}} \sum_{i=1}^{2[nb_n]-1} \hat{Y}_i^\sigma \right) > x \right| \\
\leq \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{|T^{\sigma,\phi}|_\infty}{\sqrt{2[nb_n]} - m_n} > x | F_n \right) - \mathbb{P}\left( \frac{1}{\sqrt{2nb_n}} \sum_{i=1}^{2[nb_n]-1} \hat{Y}_i^\sigma \right) > x \right| \\
+ \mathbb{P}\left( \frac{|T^{\sigma,\phi} - T^{\hat{\sigma},\hat{\phi}}|_\infty}{\sqrt{2[nb_n]} - m_n} > \delta | F_n \right) + O(\Theta(\delta, np)) \\
\leq \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{|T^{\sigma,\phi}|_\infty}{\sqrt{2[nb_n]} - m_n} > x | F_n \right) - \mathbb{P}\left( \frac{1}{\sqrt{2nb_n}} \sum_{i=1}^{2[nb_n]-1} \hat{Y}_i^\sigma \right) > x \right| \\
+ O_p\left( \delta^{-q} \left( \sqrt{m_n} \log np (g_n + \tau_n) \eta_n(np)^{\frac{1}{2}} \right)^q \right) + O(\Theta(\delta, np) + \eta_n^{-q'}),
$$

where we used Markov’s inequality and (5.47). Taking

$$
\delta = \left( \sqrt{m_n} \log np (g_n + \tau_n) \eta_n(np)^{\frac{1}{2}} \right)^{q/(q+1)}
$$

and observing (5.50) yields (5.49) and proves the assertion.

**Online Supplement** contains further results for confidence bands for the functions $t \to m(u, t)$ (fixed $u$) and $u \to m(u, t)$ (fixed $t$), additional simulation and data analysis results, examples of locally stationary functional time series, and auxiliary results for proofs.

**Acknowledgement** Holger Dette gratefully acknowledges Collaborative Research Center “Statistical modeling of nonlinear dynamic processes” (SFB 823, Project A1, C1) of the German Research Foundation.
Supplemental Material for ‘Confidence surfaces for the mean of locally stationary functional time series’

Abstract

Section [A] of the supplemental material contains some details about simultaneous confidence bands for the regression function in model (2.1), where one of the arguments is fixed (Section [A.1]) including additional numerical results for this case (see Section [A.2]). In Section [B] we provide examples of locally stationary processes, illustrating our approach of modeling non-stationary functional data. Finally, Section [C] provides auxiliary results for the proofs in Section 5 of the main article.

A Simultaneous confidence bands for fixed $u$ or $t$

A.1 Theoretical background and algorithms

In this section we present the simultaneous confidence band for the regression function $(u,t) \rightarrow m(u,t)$ in model (2.1), where one of the arguments $u$ and $t$ is fixed. More precisely, we consider

1) simultaneous confidence bands for fixed $t$, which have the form

$$C(t) = \{ f : [0,1] \rightarrow \mathbb{R} \mid \hat{L}_1(u,t) \leq f(u) \leq \hat{U}_1(u,t) \ \forall u \} ,$$  

where $\hat{L}_1$ and $\hat{U}_1$ are appropriate lower and upper bounds calculated from the data. As $t \in [0,1]$ is fixed these bounds can be derived generalizing results for confidence bands in nonparametric regression from the independent (see Konakov and Piterbarg, 1984, Xia, 1998, Proksch, 2014 among others) to the locally stationary case (see also Wu and Zhao, 2007, for results in a model with a stationary error process). An alternative approach based on multiplier bootstrap will be given below.

2) simultaneous confidence bands for fixed $u$, which have the form

$$C(u) = \{ f : [0,1] \rightarrow \mathbb{R} \mid \hat{L}_2(u,t) \leq f(t) \leq \hat{U}_2(u,t) \ \forall t \in [0,1] \} ,$$

where $\hat{L}_2$ and $\hat{U}_2$ are appropriate lower and upper bounds calculated from the data. Note that these bounds can not be directly calculated using results of Dette et al. (2020) as these authors develop their methodology under the assumption of stationarity.

Recall the definition of the residuals $\hat{\varepsilon}_{i,n}(t)$ in (2.18) and of the long run variance estimator $\hat{\sigma}$ in (2.25) in the main article. For the construction of a simultaneous confidence bands for a fixed $t \in [0,1]$ of the
form (A.1) we define

\[
\hat{Z}_i(u, t) = K \left( \frac{\hat{\varepsilon}_i - u}{b_n} \right) \hat{\varepsilon}_{i,n}(t), \quad \hat{Z}_{i,l}(t) = \hat{Z}_i \left( \frac{L}{n}, t \right),
\]

\[
\hat{\sigma}_i(u, t) = K \left( \frac{\hat{\varepsilon}_i - u}{b_n} \right) \hat{\varepsilon}_{i,n}(t), \quad \hat{\sigma}_{i,l}(t) = \hat{\sigma}_i \left( \frac{L}{n}, t \right).
\]

Next we consider the \((n - 2\lceil nb_n \rceil + 1)\)-dimensional vectors

\[
\hat{Z}_j(t) = (\hat{Z}_{j,[nb_n]}(t), \hat{Z}_{j+1,[nb_n]+1}(t), \ldots, \hat{Z}_{n-2[nb_n]+j,n-\lceil nb_n \rceil}(t))^\top, \quad (A.3)
\]

\[
\hat{\sigma}_j(t) = (\hat{\sigma}_{j,[nb_n]}(t), \hat{\sigma}_{j+1,[nb_n]+1}(t), \ldots, \hat{\sigma}_{n-2[nb_n]+j,n-\lceil nb_n \rceil}(t))^\top. \quad (A.4)
\]

\((1 \leq j \leq 2\lceil nb_n \rceil - 1)\), then a simultaneous confidence band for fixed \(t \in [0, 1]\) can be generated by the Algorithms \textbf{A.3} (constant width) and Algorithm \textbf{A.4} (varying width).
Algorithm A.3:

**Result:** simultaneous confidence band of the form (A.1) with fixed width

(a) Calculate the $(n - 2[nb_n] + 1)$-dimensional vector $\hat{Z}_j(t)$ in (A.3);

(b) For window size $m_n$, let $m'_n = 2[m_n/2]$, define

$$\hat{S}_{jm'_n}(t) = \frac{1}{\sqrt{m'_n}} \sum_{r=j}^{j+|m_n/2|-1} \hat{Z}_r(t) - \frac{1}{\sqrt{m'_n}} \sum_{r=j+|m_n/2|}^{j+m'_n-1} \hat{Z}_r(t) \quad (A.5)$$

Let $\hat{\varepsilon}_{j+m'_n,k}(t)$ be the $k$th component of $\hat{S}_{jm'_n}(t)$.

(c) for $r=1, \ldots, B$ do

- Generate independent standard normal distributed random variables $\{R_i^{(r)}\}_{i \in [1,n-m'_n]}$.

Calculate

$$T_k^{(r)}(t) = \sum_{j=1}^{2[nb_n]-m'_n} \hat{\varepsilon}_{j+m'_n,k}(t) R_{k+j-1}^{(r)}, \quad k = 1, \ldots, n - 2[nb_n] + 1,$$

$$T^{(r)}(t) = \max_{1 \leq k \leq n - 2[nb_n] + 1} |T_k^{(r)}(t)|.$$  

end

(d) Define $T_{(1-\alpha)B}(t)$ as the empirical $(1 - \alpha)$-quantile of the sample $T^{(1)}(t), \ldots, T^{(B)}(t)$ and

$$\hat{L}_3(u,t) = \hat{m}(u,t) - \hat{r}_3(t) \quad , \quad \hat{U}_3(u,t) = \hat{m}(u,t) + \hat{r}_3(t)$$

where

$$\hat{r}_3(t) = \frac{\sqrt{2}T_{(1-\alpha)B}(t)}{\sqrt{n b_n}} \frac{1}{\sqrt{2[nb_n] - m'_n}}$$

**Output:** $C_n(t) = \{ f : [0,1]^2 \to \mathbb{R} \ | \ \hat{L}_3(u,t) \leq f(u,t) \leq \hat{U}_3(u,t) \ \forall u \in [b_n, 1-b_n] \}$
Algorithm A.4:

Result: simultaneous confidence band of the form (A.1) with varying width

(a) Calculate the estimate of the long-run variance \( \hat{\sigma}^2 \) in (2.25)

(b) Calculate the \((n - 2[nb_n] + 1)\)-dimensional vectors \( \hat{Z}_j^\phi(t) \) in (A.4)

(c) For window size \( m_n \), let \( m_n' = 2[m_n/2] \), define

\[
\hat{S}_{jm_n'}^\phi(t) = \frac{1}{\sqrt{m_n'}} \sum_{r=j}^{j+|m_n/2|-1} \hat{Z}_r^\phi(t) - \frac{1}{\sqrt{m_n'}} \sum_{r=j+[m_n/2]}^{j+m_n'-1} \hat{Z}_r^\phi(t)
\]

Let \( \hat{S}_{jm_n',k}^\phi(t) \) be the \( k \)th component of \( \hat{S}_{jm_n'}^\phi(t) \). Let \( \hat{e}_{j+m_n'+k}^\phi(t) \) be the \( k \)th component of \( \hat{S}_{jm_n'}^\phi(t) \).

(d) for \( r=1, \ldots, B \) do

- Generate independent standard normal distributed random variables \( \{R_i^{(r)}\}_{i=[1,n-m_n']} \).

- Calculate

\[
T_k^\phi,(r)(t) = \sum_{j=1}^{2[nb_n]-m_n'} \hat{e}_{j+m_n'+k}^\phi(t)R_{k+j-1}^{(r)}, \ k = 1, \ldots, n - 2[nb_n] + 1,
\]

\[
T_k^\phi,(r)(t) = \max_{1 \leq k \leq n - 2[nb_n] + 1} |T_k^\phi,(r)(t)|.
\]

end

(e) Define \( T_{(1-\alpha)B_j^\phi}(t) \) as the empirical \((1 - \alpha)\)-quantile of the sample \( T_{\phi,(1)}(t), \ldots, T_{\phi,(B)}(t) \) and

\[
\hat{L}_4^\phi(u,t) = \hat{m}(u,t) - \hat{r}_4(u,t), \quad \hat{U}_4^\phi(u,t) = \hat{m}(u,t) + \hat{r}_4(u,t)
\]

where

\[
\hat{r}_4(u,t) = \frac{\hat{\sigma}(u,t)\sqrt{2T_{(1-\alpha)B_j^\phi}(t)}}{\sqrt{nb_n} \sqrt{2[nb_n] - m_n'}}
\]

Output:

\[
C_n^\phi(t) = \{ f : [0, 1]^2 \to \mathbb{R} \mid \hat{L}_4^\phi(u,t) \leq f(u,t) \leq \hat{U}_4^\phi(u,t) \ \forall u \in [b_n, 1 - b_n] \}.
\]
The following result shows that the sets constructed by Algorithms A.3 and A.4 are asymptotic $(1-\alpha)$-confidence bands of the form [A.1]. The proof is similar to but easier than the proof of Theorems 4.2 and 4.3 of the main article and is therefore omitted for the sake of brevity.

**Theorem A.1.** Assume that the conditions of Theorem 4.1 hold. Define

$$\vartheta_n^* = \log \frac{2n}{m_n} + \sqrt{\frac{m_n b_n}{n} n^{4/q}}.$$  

(i) If $\vartheta_n^* \{1 \lor \log(\frac{n}{\vartheta_n^*})\}^{2/3} + \Theta(\sqrt{\frac{m_n \log n}{m_n} + d_n^2}(n)\frac{1}{n})^{q/(q+1)}, n = o(1)$ we have that for any $\alpha \in (0, 1)$ and any $t \in [0, 1]$

$$\lim_{n \to \infty} \lim_{B \to \infty} \mathbb{P}(m \in C_n(t) \mid \mathcal{F}_n) = 1 - \alpha$$

in probability.

(ii) If further the conditions of Theorem 4.3 and Proposition 4.1 hold, then

$$\lim_{n \to \infty} \lim_{B \to \infty} \mathbb{P}(m \in C_n^*(t) \mid \mathcal{F}_n) = 1 - \alpha$$

in probability.

The next theorem presents a Gaussian approximation in the case where $u$ is fixed. It is the basis for the construction of a confidence band for fixed $u$ and its proof follows by similar (but easier) arguments as given in the proof of Theorem 4.1 of the main article.

**Theorem A.2.** Let Assumptions 4.1 - 4.4 be satisfied and assume that the bandwidth in (2.4) satisfies $n^{1+q} b_n^9 = o(1)$, $n^{a-1} b_n^{-1} = o(1)$ for some $0 < a < 4/5$. For any fixed $u \in (0, 1)$ there exists a sequence of centred $p$-dimensional Gaussian vectors $(Y_i(u))_{i \in \mathbb{N}}$ with the same covariance structure as the vector $Z_i(u)$ in (2.7), such that

$$\mathcal{P}_n(u) := \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \max_{0 \leq t \leq 1} \sqrt{n b_n} |\Delta(u, t)| \leq x \right) - \mathbb{P}\left( \left| \frac{1}{\sqrt{n b_n}} \sum_{i=1}^n Y_i(u) \right|_\infty \leq x \right) \right| = O\left( (nb_n)^{-(1-11\iota)/8} + \Theta\left( \sqrt{nb_n} (b_n^4 + \frac{1}{nb_n})^p \right) + \Theta\left( \frac{1}{p^{1+\xi}} \right) \right)$$

for any sequence $p \to \infty$ with $p = O(\exp(n^\iota))$ for some $0 \leq \iota < 1/11$. In particular,

$$\mathcal{P}_n(u) = o(1)$$

if $p = n^c$ for some $c > 0$ and the constant $q^*$ in Assumption 4.3 is sufficiently large.
Algorithm A.5:

**Result:** simultaneous confidence band for fixed $u \in [b_n, 1-b_n]$ as defined in (A.2)

(a) Calculate the $p$-dimensional vectors $\hat{Z}_i(u)$ in (2.19)

(b) For window size $m_n$, let $m'_n = 2\lfloor m_n/2 \rfloor$, define

$$
\hat{S}_{jm_n'}(u) = \frac{1}{\sqrt{m_n'}} \sum_{r=j}^{j+\lfloor m_n/2 \rfloor-1} \hat{Z}_r(u) - \frac{1}{\sqrt{m_n'}} \sum_{r=j+\lfloor m_n/2 \rfloor}^{j+m'_n-1} \hat{Z}_r(u)
$$

(c) for $r=1, \ldots, B$ do

- Generate independent standard normal distributed random variables $\{R_{ij}\}_{i=\lfloor nu-nb_n \rfloor}^{\lfloor nu+nb_n \rfloor}$
- Calculate the bootstrap statistic

$$
T^{(r)}(u) = \left| \sum_{j=\lfloor nu-nb_n \rfloor}^{\lfloor nu+nb_n \rfloor-m'_n+1} \hat{S}_{jm_n'}(u) R_{ij}^{(r)} \right|_\infty
$$

end

(d) Define $T_{(1-\alpha)B}^{(1)}(u)$ as the empirical $(1-\alpha)$-quantile of the sample $T^{(1)}(u), \ldots, T^{(B)}(u)$ and

$$
\hat{L}_5(u,t) = \hat{m}(u,t) - \hat{r}_5(u) \quad \hat{U}_5(u,t) = \hat{m}(u,t) + \hat{r}_5(u)
$$

where

$$
\hat{r}_5(u) = \frac{\sqrt{2}T_{(1-\alpha)B}^{(1)}(u)}{\sqrt{nb_n} \sqrt{\lfloor |nu+nb_n| - |nu-nb_n| - m'_n + 2 \rfloor}}
$$

Output:

$$
C_n(u) = \{ f : [0,1]^2 \to \mathbb{R} \mid \hat{L}_5(u,t) \leq f(u,t) \leq \hat{U}_5(u,t) \ \forall t \in [0,1] \}. \quad (A.6)
$$
Algorithm A.6:

Result: simultaneous confidence band of the form \( (A.2) \) with varying width.

(a) For given \( u \in [b_n, 1 - b_n] \), calculate the the estimate of the long-run variance \( \hat{\sigma}^2(u, \cdot) \) in \( (2.25) \).

(b) Calculate the vector \( \hat{Z}_i^{\hat{\sigma}_u}(u) \) in \( (A.8) \).

(c) For window size \( m_n \), let \( m'_n = 2\lfloor m_n/2 \rfloor \) and define the \( p \)-dimensional random vectors

\[
\hat{S}_{jm_n'}^{\hat{\sigma}_u}(u) = \frac{1}{\sqrt{m'_n}} \sum_{r=j}^{j+m'_n-1} \hat{Z}_{r}^{\hat{\sigma}_u}(u) - \frac{1}{\sqrt{m'_n}} \sum_{r=j+\lfloor m_n/2 \rfloor}^{j+m'_n-1} \hat{Z}_{r}^{\hat{\sigma}_u}(u)
\]

(d) for \( r=1, \ldots, B \) do

- Generate independent standard normal distributed random variables \( \{ R_{ij}^{(r)} \}_{i=[nu-nb_n]}^{[nu+nb_n]} \).
- Calculate the bootstrap statistic

\[
T_{\hat{\sigma}_u,(r)}(u) = \left| \sum_{j=[nu-nb_n]}^{[nu+nb_n]-m'_n+1} \hat{S}_{jm_n'}^{\hat{\sigma}_u}(u)R_{ij}^{(r)} \right|_\infty
\]

end

(e) Define \( T_{\hat{\sigma}_u,(1-(1-\alpha)B)}(u) \) as the empirical \( (1 - \alpha) \)-quantile of the sample \( T_{\hat{\sigma}_u,(1)(u), \ldots, T_{\hat{\sigma}_u,(B)}(u) \) and

\[
\hat{L}^{\hat{\sigma}_u}(u, t) = \hat{m}(u, t) - \hat{r}^{\hat{\sigma}_u}(u, t), \quad \hat{U}^{\hat{\sigma}_u}(u, t) = \hat{m}(u, t) + \hat{r}^{\hat{\sigma}_u}(u, t),
\]

where

\[
\hat{r}^{\hat{\sigma}_u}(u, t) = \frac{\hat{\sigma}(u, t)\sqrt{2T_{\hat{\sigma}_u,(1-(1-\alpha)B)}(u)}}{\sqrt{nb_n}\sqrt{([nu+nb_n] - [nu-nb_n] - m'_n + 2)}}
\]

Output:

\[
C_{\hat{\sigma}_u}^{\hat{\sigma}_u}(u) = \{ f : [0, 1]^2 \to \mathbb{R} \mid \hat{L}^{\hat{\sigma}_u}(u, t) \leq f(u, t) \leq \hat{U}^{\hat{\sigma}_u}(u, t) \ \forall t \in [0, 1] \}.
\]

Next we present details of the algorithms for a simultaneous confidence band for a fixed \( u \) (of the form \( (A.2) \)) with fixed and varying width. For this purpose we define the \( p \)-dimensional vector

\[
\hat{Z}_i^{\hat{\sigma}_u}(u) = (\hat{Z}_{i,1}^{\hat{\sigma}_u}(u), \ldots, \hat{Z}_{i,p}^{\hat{\sigma}_u}(u))^\top
\]

\[
= K \left( \frac{i - u}{bn} \right) \left( \frac{\hat{\varepsilon}_{i,n}(1)}{\hat{\sigma}(u, \frac{1}{p})}, \frac{\hat{\varepsilon}_{i,n}(2)}{\hat{\sigma}(u, \frac{2}{p})}, \ldots, \frac{\hat{\varepsilon}_{i,n}(p-1)}{\hat{\sigma}(u, \frac{p-1}{p})}, \frac{\hat{\varepsilon}_{i,n}(1)}{\hat{\sigma}(u, 1)} \right)^\top,
\]

where \( \hat{\varepsilon}_{i,n} \) and \( \hat{\sigma} \) are defined by \( (2.18) \) and \( (2.25) \) in the main article, respectively. Algorithms A.5 and
A.6 provides asymptotically correct the confidence bands of type (A.2). The next Theorem A.3 yields the validity of Algorithms A.5 and A.6, which is a consequence of Theorem A.2.

**Theorem A.3.** Assume that the conditions of Theorem 4.1 hold, and that \( nd_n^3 \rightarrow \infty, nd_n^6 = o(1) \). Define
\[
\varphi_n' = \frac{\log^2 n}{m_n} + \frac{m_n \log n}{nb_n} + \sqrt{\frac{m_n}{nb_n} p^{4/q}}
\]
and assume that \( p \rightarrow \infty \) such that \( p = O(\exp(n^\iota)) \) for some \( 0 \leq \iota < 1/11 \).

(1) If \( \alpha \in (0, 1) \) and
\[
\varphi_n^{1/3} \left\{ 1 \lor \log \left( \frac{p}{\varphi_n'} \right) \right\}^{2/3} + \Theta \left( \left( \frac{\sqrt{m_n \log p}}{\sqrt{nd_n}} + d_n^2 \right)^{1/2} p^{q/(q+1)} \right) = o(1),
\]
then we have for the confidence band in (A.6)
\[
\lim_{n \rightarrow \infty} \lim_{B \rightarrow \infty} \mathbb{P}(m \in C_n(u) \mid \mathcal{F}_n) = 1 - \alpha
\]
in probability.

(ii) If further the conditions of Theorem 4.3 and Proposition 4.1 hold, then for the confidence band in (A.7)
\[
\lim_{n \rightarrow \infty} \lim_{B \rightarrow \infty} \mathbb{P}(m \in C_{\hat{\sigma}}(u) \mid \mathcal{F}_n) = 1 - \alpha
\]
in probability.

The proof of Theorem A.3 follows by similar (but easier) arguments as given in the proof of Theorem 4.2 and Theorem 4.3.

**Remark A.1.** One can prove similar results under alternative moment assumptions. In fact, Theorem A.2 remains valid if condition (4.1) is replaced by
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq 1} (G(u, t, F_0))^4 \right] < \infty. \tag{A.9}
\]
Moreover, one can prove Theorem A.2 under weaker assumptions than Assumption 4.2 (ii), which requires geometrically decaying dependence measure. More precisely, if the assumptions of Theorem A.2 hold, where Assumption 4.2 (ii) is replaced by assumption (a) in (ii) of Remark 4.1 in the main article and the following conditions

(b1) There exist constants \( M = M(n) > 0, \gamma = \gamma(n) \in (0, 1) \) and \( C_1 > 0 \) such that
\[
(2[nb_n])^{3/8} M^{-1/2} l_n^{-5/8} \geq C_1 l_n
\]
where \( l_n = \max(\log(2\lceil nb_n \rceil p/\gamma), 1) \).

Recall the quantity \( \Xi_M \) and \( \Delta_{M,q} \) defined in 4.1 of the main article. Then we have

\[
\Psi_n(u) = O\left( \eta_n + \Theta\left( \sqrt{\frac{1}{nb_n} (\frac{1}{nb_n} + \frac{1}{n})}, p \right) + \Theta\left( \frac{1}{p^{1+\frac{p}{2}}} \right) \right)
\]

with

\[
\eta_n = (nb_n)^{-1/8} M^{1/2} l_n^{7/8} + \gamma + \left( (nb_n)^{1/8} M^{-1/2} l_n^{-3/8} \right)^{q/(1+q)} \left( p \Delta_{M,q}^q \right)^{1/(1+q)}
\]

\[+ \Xi_M^{1/3} (1 \lor \log (p/\Xi_M))^{2/3}.\]

By similar arguments as given in Remark 4.1 of the main article, the sets \( C_n(u) \) and \( \hat{C}_{\sigma u} n(u) \) defined by (A.6) and (A.7), respectively, define an (asymptotic) \((1 - \alpha)\) simultaneous confidence surface if \( \eta'_n = o(1) \).

For example, if \( \delta_q(G,i) = O(i^{-1-\alpha}) \) for some \( \alpha > 0 \), \( \beta = \frac{n}{2} \) for some \( \beta > 0 \) and \( b_n = n^{-\gamma} \) for some \( 0 < \gamma < 1 \), then \( \eta_n = o(1) \) if \( \beta - (1-\gamma)qA/4 < 0 \), which gives a lower bound on \( q \).

A.2 Finite sample properties

In this section we provide numerical results for the confidence bands for the regression function \( m \) with fixed \( u \) or \( t \) derived in Algorithms A.3 - A.6. As in the main part of the paper we consider simulated and real data.

For the simultaneous confidence band for a fixed \( t \in [0,1] \) in (A.1) and a fixed \( u \in (0,1) \) in (A.2), the tuning parameters are chosen in a similar way as described in Section 3.1 of the main article. In particular for a fixed \( u \in (0,1) \) we use the bandwidth \( b_n = 1.2d_n \), where \( d_n \) is chosen as the minimizer of the loss function

\[ MGCV(b) = \max_{1 \leq s \leq p} \sum_{i = [nu-nb_n]}^{[nu+nb_n]} (\hat{m}_b(i/n, z_i) - X_{i,n}(z_i))^2 / (1 - \text{tr}(Q_s(b, u))/(2nb))^2. \quad (A.10) \]

and

\[ \left( \hat{m}_b(\lceil nu-nb_n \rceil/n, z_i), ... , \hat{m}_b(\lfloor nu+nb_n \rceil/n, z_i) \right)^\top = Q_s(b, u)(X_{\lfloor nu-nb_n \rceil,n}(z_i), ..., X_{\lfloor nu+nb_n \rceil,n}(z_i))^\top, \]

The criterion (A.10) is also motivated by the generalized cross validation criterion introduced by Craven and Wahba (1978).

A.2.1 Simulated data

For simulated data, the regression functions and locally stationary functional time series are stated in Section 3.2 of the main article. We begin displaying typical 95% simultaneous confidence bands obtained from one simulation run for model (a) with sample size \( n = 800 \). Figure A.4 shows the simultaneous band of the type (A.1) with constant width (Algorithm A.3) and variable width (Algorithm A.4), while in Figure A.5 we display the simultaneous confidence bands of the form (A.2) (for fixed \( u \)) with constant width.
(Algorithm A.5) and variable width (Algorithm A.6). We observe that in all cases there exist differences between the bands with constant and variable width, but they are not substantial.

Figure A.4: 95% simultaneous confidence bands of the form (A.1) (fixed $t = 0.5$) for the regression function in model (a) from $n = 800$ observations. Left panel: constant width (Algorithm A.3); Right panel: varying width (Algorithm A.4).

Figure A.5: 95% simultaneous confidence band of the form (A.2) (fixed $u = 0.5$) for the regression function in model (a) from $n = 800$ observations. Left panel: constant width (Algorithm A.5); Right panel: varying width (Algorithm A.6).

We next investigate the coverage probabilities of confidence bands constructed for fixed $t = 0.5$ and $u = 0.5$ for sample sizes $n = 500$ and $n = 800$. All results presented in the following discussion are based on 1000 simulation runs and $B = 1000$ bootstrap replications. In all tables the left part shows the coverage probabilities of the bands with constant width while the results in the right part correspond to the bands with varying width.

In Table A.2 we give some results for the confidence bands of the form (A.1) (for fixed $t = 0.5$) with constant and variable width (c.f. Algorithm A.3 and Algorithm A.4), while we present in Table A.3 the
Table A.2: Simulated coverage probabilities of the simultaneous confidence band of the form (A.1) for fixed \( t = 0.5 \) calculated by Algorithm A.3 (constant width) and A.4 (varying width).

|       | constant width | varying width |
|-------|----------------|---------------|
|       | model (a)      | model (b)     | model (a) | model (b) |
| level | 90%  95%       | 90%  95%      | 90%  95%  | 90%  95%  |
| \( n = 500 \) | 90.4 % 94.9% | 89.8 % 95.9% | 90.1 % 96.3% | 89.9 % 94.6% |
| \( n = 800 \) | 89.3 % 94.5% | 90.0 % 95.5% | 89.4 % 95.0% | 90.3 % 95.6% |

|       | constant width | varying width |
|-------|----------------|---------------|
|       | model (c)      | model (d)     | model (c) | model (d) |
| level | 90%  95%       | 90%  95%      | 90%  95%  | 90%  95%  |
| \( n = 500 \) | 90.1 % 95.9% | 90.4 % 96.0% | 90.4 % 95.3% | 91.0 % 96.4% |
| \( n = 800 \) | 90.6 % 95.6% | 90.0 % 95.4% | 88.9 % 94.9% | 89.3 % 95.1% |

Table A.3: Simulated coverage probabilities of the simultaneous confidence band of the form (A.2) for fixed \( u = 0.5 \) calculated by Algorithms A.5 (constant width) and A.6 (varying width).

|       | constant width | varying width |
|-------|----------------|---------------|
|       | model (a)      | model (b)     | model (a) | model (b) |
| level | 90%  95%       | 90%  95%      | 90%  95%  | 90%  95%  |
| \( n = 500 \) | 87.8 % 92.3% | 88.5 % 93.6% | 88.5 % 93.3% | 88.1 % 92.5% |
| \( n = 800 \) | 88.7 % 93.9% | 89.0 % 94.4% | 90.8 % 94.3% | 88.2 % 93.9% |

|       | constant width | varying width |
|-------|----------------|---------------|
|       | model (c)      | model (d)     | model (c) | model (d) |
| level | 90%  95%       | 90%  95%      | 90%  95%  | 90%  95%  |
| \( n = 500 \) | 87.4 % 92.4% | 88.0 % 93.1% | 88.1 % 93.3% | 89.6 % 94.6% |
| \( n = 800 \) | 88.9 % 93.6% | 88.7 % 94.5% | 89.6 % 93.8% | 89.9 % 95.2% |

Simulated coverage probabilities of the simultaneous confidence bands of the form (A.2), where \( u = 0.5 \) is fixed (c.f. Algorithm A.5 and Algorithm A.6). We observe that the simulated coverage probabilities are close to their nominal levels in all cases under consideration, which illustrates the validity of our methods for finite sample sizes.

A.3 Real data

In this section we further study the well documented volatility smile for implied volatility of the European call option of SP500 data set considered in Section 3.3 of the main article. In Figure A.6 we display 95% simultaneous confidence bands of the form (A.1) for fixed \( t = 0.5 \) (which corresponds to Moneyness=1.1) where the parameters are chosen as \( b_n = 0.12 \) and \( m_n = 18 \). We observe that the implied volatility changes with time (or precisely the time to maturity) when moneyness (or equivalently, the strike price and underlying asset price) is specified. We also calculate confidence bands of the form (A.2) for fixed \( u = 0.5 \), by Algorithm A.5 (constant width) and Algorithm A.6 (varying width). The parameter selection procedure proposed in Section 3.3 yields \( b_n = 0.216 \) and \( m_n = 18 \), and the resulting simultaneous confidence bands of the form (A.2) are presented in Figure A.7. We observe that both 95% simultaneous confidence bands indicate that the implied volatility is a quadratic function of moneyness, which supports the well documented phenomenon of 'volatility smile'. We observe that the differences between the bands with
constant and variable width are rather small.

Figure A.6: 95% simultaneous confidence bands of the form (A.1) (fixed $t = 0.5$) for the data example in Section 3.3. Left panel: constant width (Algorithm A.3); Right panel: variable width (Algorithm A.4).

Figure A.7: 95% simultaneous confidence bands of the form (A.2) (fixed $u = 0.5$) for the IV surface. Left panel: constant width (Algorithm A.5); Right panel: variable width (Algorithm A.6).
B Examples of locally stationary error processes

In this section we present several examples for the error processes, which satisfy the assumptions of the main article.

Example B.1. Let \((B_j)_{j \geq 0}\) denote a basis of \(L^2([0, 1]^2)\) and let \((\eta_{i,j})_{i \geq 0, j \geq 0}\) denote an array of independent identically distributed centred random variables with variance \(\sigma^2\). We define the error process
\[
\epsilon_i(u, v) = \sum_{j=0}^{\infty} \eta_{i,j} B_j(u, v),
\]
assume that
\[
\sup_{u \in [0,1]} \int_0^1 \mathbb{E}(\epsilon_i^2(u, v))dv = \sigma^2 \sup_{u \in [0,1]} \sum_{s=0}^{\infty} \int B_s^2(u, v)dv < \infty.
\]

Next, consider the locally stationary MA(\(\infty\)) functional linear model
\[
\epsilon_{i,n}(t) = \sum_{j=0}^{\infty} \int_0^1 a_j(t, v) \epsilon_{i-j}(\frac{i}{n}, v)dv , \tag{B.1}
\]
where \((a_j)_{j \geq 0}\) is a sequence of square integrable functions \(a_j : [0, 1]^2 \to \mathbb{R}\) satisfying
\[
\sum_{j=0}^{\infty} \sup_{u, v \in [0,1]} |a_j(u, v)| < \infty.
\]

Define \(F_i = (\ldots, \eta_{i-1}, \eta_i)\), then we obtain from \((B.1)\) the representation of the form \(\epsilon_{i,n}(t) = G(\frac{i}{n}, t, F_i)\), where
\[
G(u, t, F_i) = \sum_{j=0}^{\infty} \int_0^1 a_j(t, v) \sum_{s=0}^{\infty} \eta_{i-j,s} B_s(u, v)dv.
\]

Further, assume that \(\|\eta_{1,1}\|_q < \infty\) for some \(q > 2\), then by Burkholder’s and Cauchy’s inequality the physical dependence measure defined in \((4.2)\) satisfies
\[
\delta_q(G,i) = \sup_{u, t \in [0,1]} \left\| \sum_{s=0}^{\infty} \int_0^1 a_i(t, v) B_s(u, v)dv (\eta_{0,s} - \eta_{0,s}') \right\|_q
\]
\[
= O\left( \sup_{u, t \in [0,1]} \left( \sum_{s=0}^{\infty} \left( \int_0^1 a_i(t, v) B_s(u, v)dv \right)^2 \right)^{1/2} \right)
\]
\[
= O\left( \sup_{t \in [0,1]} \left[ \int_0^1 a_i^2(t, v)dv \right]^{1/2} \right).
\]
Therefore Assumption 4.2(2) will be satisfied if
\[
\sup_{t \in [0,1]} \left[ \int_0^1 a_i^2(t,v)dv \right]^{1/2} = O(\chi^i) .
\]
Similarly, it follows for \( q \geq 2 \) that
\[
\|G(u,t,\mathcal{F}_0)\|_q^2 \leq Mq \sum_{j=0}^\infty \sum_{s=0}^\infty \left( \int_0^1 a_j(t,v)B_s(u,v)dv \right)^2 \|\eta_{1,1}\|_q^2
\]
\[
\leq Mq \sum_{j=0}^\infty \int_0^1 a_j^2(t,v)dv \sum_{s=0}^\infty \int_0^1 B_s^2(u,v)dv \|\eta_{1,1}\|_q^2
\]
for some sufficiently large constant \( M \). Consequently, the filter \( G \) has finite moment of order \( q \), if
\[
\sum_{j=0}^\infty \int_0^1 a_j^2(t,v)dv < \infty .
\]
Furthermore, if there exists positive constants \( M_0 \) and \( \alpha \) such that \( \|\eta_{1,1}\|_q \leq M_0 q^{1/2-\alpha} \), Assumption 4.2(1) is also satisfied, because for any fixed \( t_0 \), the sequence
\[
\frac{t_0^q \|G(u,t,\mathcal{F}_0)\|_q^q}{q!} = O\left( \frac{Ct_0^q e^{-\alpha q}}{q!} \right) = O\left( \frac{1}{\sqrt{2\pi q q^\alpha}} \right)
\]
is summable, where
\[
C = \sup_{t \in [0,1], u \in [0,1]} M_0 \sqrt{M \sum_{j=0}^\infty \int_0^1 a_j^2(t,v)dv \sum_{s=0}^\infty \int_0^1 B_s^2(u,v)dv}.
\]
Moreover, if \( b_s(u,v) := \frac{\partial}{\partial u} B_s(u,v) \) exists for \( u \in (0,1), v \in [0,1] \), then it follows observing \( \text{(B.2)} \) that Assumption 4.2(3) holds under \( \text{(B.3)} \) and
\[
\sup_{u \in [0,1]} \sum_{s=0}^\infty \int b_s^2(u,v)dv < \infty .
\]
Finally, if \( \|\eta_{1,1}\|_q < \infty \) and
\[
\sup_{t \in [0,1]} \left[ \int_0^1 \left( \frac{\partial}{\partial t} a_i(t,v) \right)^2 dv \right]^{1/2} = O(\chi^i) ,
\]
it can be shown by similar arguments as given above that Assumption 4.3 is satisfied.

**Example B.2.** For a given orthonormal basis \((\varphi_k(t))_{k \geq 1}\) of \( L^2([0,1]) \) consider the functional time series
\( (G(u, t, \mathcal{F}_i))_{i \in \mathbb{Z}} \) defined by

\[
G(u, t, \mathcal{F}_i) = \sum_{k=1}^{\infty} H_k(u, \mathcal{F}_i) \phi_k(t), \tag{B.4}
\]

where for each \( k \in \mathbb{N} \) and \( u \in [0, 1] \) the random coefficients \( (H_k(u, \mathcal{F}_i))_{i \in \mathbb{Z}} \) are stationary time series. A parsimonious choice of \( \text{(B.4)} \) is to consider \( \mathcal{F}_i = \bigcup_{k=1}^{\infty} \mathcal{F}_{i,k} \) where \( \{\mathcal{F}_{i,k}\}_{k=1}^{\infty} \) are independent filtrations. In this case we obtain

\[
G(u, t, \mathcal{F}_i) = \sum_{k=1}^{\infty} H_k(u, \mathcal{F}_{i,k}) \phi_k(t), \tag{B.5}
\]

and the random coefficients \( H_k(u, \mathcal{F}_{i,k}) \) are stochastically independent. A sufficient condition for Assumption 4.2(2) in model \( \text{(B.5)} \) is

\[
\sup_{t \in [0, 1]} \sum_{k=0}^{\infty} |\phi_k(t)| \delta_q(H_k, i) = O(\chi^i),
\]

where \( \delta_q(H_k, i) := \sup_{u \in [0, 1]} \|H_k(u, \mathcal{F}_{i,k}) - H_k(u, \mathcal{F}_{i,k}^*)\|_q \). The \( q \)th moment of the process \( G \) in \( \text{(B.5)} \) exists for \( q \geq 2 \), if

\[
\Delta_q := \sup_{t \in [0, 1], u \in [0, 1]} \sum_{k=0}^{\infty} \phi_k^2(t) \|H_k(u, \mathcal{F}_{0,k})\|_q^2 < \infty.
\]

If further \( \Delta_q = O(q^{1/2-\alpha}) \) for some \( \alpha > 0 \), then similar arguments as given in Example \( \text{B.1} \) show that Assumption 4.2(1) is satisfied as well. Finally, if the inequality

\[
\sum_{k=0}^{\infty} \phi_k^2(t) \left\| \frac{\partial}{\partial u} H_k(u, \mathcal{F}_{0,k}) \right\|_q^2 < \infty
\]

holds uniformly with respect to \( t, u \in [0, 1] \), Assumption 4.2(3) is also satisfied.

On the other hand, in model \( \text{(B.4)} \) we have \( H_k(u, \mathcal{F}_i) = \int_0^1 G(u, t, \mathcal{F}_i) \phi_k(t) dt \), and consequently the magnitude of \( \|H_k\|_q \) and \( \delta_q(H_k, i) \) can be determined by Assumption 4.2. For example, if the basis of \( L^2([0, 1]) \) is given by \( \phi_k(t) = \cos(k\pi t) \) \( (k = 0, 1, \ldots) \) and the inequality

\[
\|G(u, 0, \mathcal{F}_1)\|_q + \left\| \frac{\partial}{\partial t} G(u, 0, \mathcal{F}_1) \right\|_q + \sup_{t \in [0, 1]} \left\| \frac{\partial^2}{\partial t^2} G(u, t, \mathcal{F}_1) \right\|_q < \infty,
\]

holds for \( u \in [0, 1] \), it follows by similar arguments as given in Zhou and Dette (2020) that

\[
\sup_{u \in [0, 1]} \|H_k(u, \mathcal{F}_k)\|_q = O(k^{-2}), \quad \delta_q(H_k, i) = O\left( \min(k^{-2}, \delta_G(i, q)) \right). \tag{B.6}
\]
Similarly, assume that the basis of $L^2([0, 1])$ is given by the Legendre polynomials and that

$$\sup_{u \in [0, 1]} \max_{s=1,2,3} \left\| \int_{-1}^{1} \frac{\partial_s G(u, t, F_0)}{\sqrt{1-x^2}} dx \right\|_q < \infty.$$ 

If additionally for every $\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$\sum_{s=1,2,3} \sum_k \left\| \frac{\partial_s}{\partial t^s} G(u, x_k, F_i) - \frac{\partial_s}{\partial t^s} G(u, x_{k-1}, F_i) \right\|_q < \varepsilon$$

for any finite sequence of pairwise disjoint sub-intervals $(x_{k-1}, x_k)$ of the interval $(0, 1)$ such that $\sum_k (x_k - x_{k-1}) < \delta$, it follows from Theorem 2.1 of Wang and Xiang (2012) that (B.6) holds as well.

Finally, if

$$\sup_{t \in [0, 1]} \sum_{k=0}^{\infty} |\phi'_k(t)| \delta_{q^*}(H_k, i) = O(\chi^t),$$

it can be shown by similar arguments as given above that Assumption 4.3 is also satisfied.
C Some auxiliary results

This section contains several technical lemmas, which will be used in the proofs of the main results in Section 5.

Lemma C.1. For any random vectors $X, X', Y$, and $\delta \in \mathbb{R}$, we have that

$$
\sup_{x \in \mathbb{R}} \left| \mathbb{P}(|X'| > x) - \mathbb{P}(|Y| > x) \right| \leq \sup_{x \in \mathbb{R}} \left| \mathbb{P}(|X| > x) - \mathbb{P}(|Y| > x) \right| + \mathbb{P}(|X - X'| > \delta) + 2 \sup_{x \in \mathbb{R}} \mathbb{P}(|Y - x| \leq \delta). \tag{C.1}
$$

Furthermore, if $Y = (Y_1, ..., Y_p)^\top$ is a $p$-dimensional Gaussian vector and there exist positive constants $c_1 \leq c_2$ such that for all $1 \leq j \leq p$, $c_1 \leq \mathbb{E}(Y_j^2) \leq c_2$, then

$$
\sup_{x \in \mathbb{R}} \left| \mathbb{P}(|X'| > x) - \mathbb{P}(|Y| > x) \right| \leq \sup_{x \in \mathbb{R}} \left| \mathbb{P}(|X| > x) - \mathbb{P}(|Y| > x) \right| + \mathbb{P}(|X - X'| > \delta) + C \Theta(\delta, p), \tag{C.2}
$$

where $C$ is a constant only dependent on $c_1$ and $c_2$.

Proof of Lemma C.1. By triangle inequality, we shall see that

$$
\mathbb{P}(|X'| > x) - \mathbb{P}(|Y| > x) \leq \mathbb{P}(|X' - X| > \delta) + \mathbb{P}(|X| > x - \delta) - \mathbb{P}(|Y| > x), \tag{C.3}
$$

$$
\mathbb{P}(|X'| > x) - \mathbb{P}(|Y| > x) \geq -\mathbb{P}(|X' - X| > \delta) + \mathbb{P}(|X| > x + \delta) - \mathbb{P}(|Y| > x). \tag{C.4}
$$

Notice that right-hand side of (C.3) is

$$
\mathbb{P}(|X' - X| > \delta) + \mathbb{P}(|X| > x - \delta) - \mathbb{P}(|Y| > x - \delta) + \mathbb{P}(|Y| > x - \delta) - \mathbb{P}(|Y| > x).
$$

The absolute value of the above expression is then uniformly bounded by

$$
\mathbb{P}(|X' - X| > \delta) + \sup_{x \in \mathbb{R}} \mathbb{P}(|X| > x) - \mathbb{P}(|Y| > x) + 2 \sup_{x \in \mathbb{R}} \mathbb{P}(|Y - x| \leq \delta). \tag{C.5}
$$

Similarly, the absolute value of right-hand side of (C.4) is also uniformly bounded by (C.5), which proves (C.1). Finally, (C.2) follows from (C.1) and an application of Corollary 1 in Chernozhukov et al. (2015). Note that in this result the constant $C$ is determined by $\max_{1 \leq j \leq p} \mathbb{E}(Y_j^2) \leq c_2$ and $\min_{1 \leq j \leq p} \mathbb{E}(Y_j^2) \geq c_1$.

The following result is a consequence of of Lemma 5 of Zhou and Wu (2010).
Lemma C.2. Under the assumption 4.2(2), we have that

\[ \sup_{u_1, u_2, t_1, t_2 \in [0, 1]} |E(G(u_1, t, F_i)G(u_2, t_2, F_j))| = O(\chi^{i-j}). \]

Lemma C.3. Define

\[ \sigma_{j,j}(u) = \frac{1}{nb_n} \sum_{i,l=1}^{n} \text{Cov}(Z_{i,j}(u), Z_{l,j}(u)) \]

where \( Z_{i,j} \) are the components of the vector \( Z_i(u) \) defined in (2.7). If \( b_n = o(1) \), \( \frac{\log n}{nb_n} = o(1) \) and Assumption 4.2 and Assumption 4.4 are satisfied, then there exist positive constants \( c_1 \) and \( c_2 \) such that for sufficiently large \( n \)

\[ 0 < c_1 \leq \min_{1 \leq j \leq p} \sigma_{j,j}(u) \leq \max_{1 \leq j \leq p} \sigma_{j,j}(u) \leq c_2 < \infty. \]

for all \( u \in [b_n, 1 - b_n] \). Moreover, we have for

\[ \bar{\sigma}_{j,j} := \frac{1}{2[nb_n] - 1} \sum_{i,l=1}^{2[nb_n]-1} \text{Cov}(\tilde{Z}_{i,j}, \tilde{Z}_{l,j}), \quad (C.6) \]

the estimates

\[ c_1 \leq \min_{1 \leq j \leq (n-2[nb_n]+1)p} \bar{\sigma}_{j,j} \leq \max_{1 \leq j \leq (n-2[nb_n]+1)p} \bar{\sigma}_{j,j} \leq c_2. \]

Proof of Lemma C.3. By definition,

\[ \sigma_{j,j}(u) = \frac{1}{nb_n} \sum_{i,l=1}^{n} E\left( G\left( \frac{i}{n}, t_j, F_i \right)K\left( \frac{i}{n} - \frac{u}{b_n} \right)G\left( \frac{l}{n}, t_j, F_l \right)K\left( \frac{l}{n} - \frac{u}{b_n} \right) \right). \]

Observing Assumption 4.2 and Lemma C.2, we have

\[ E(G\left( \frac{i}{n}, t_j, F_i \right)G\left( \frac{l}{n}, t_j, F_l \right) - G(u, t_j, F_i)G(u, t_j, F_l)) = O\left( \min(\chi^{i-l}, b_n) \right) \]

uniformly with respect to \( u \in [b_n, 1 - b_n], |\frac{i}{n} - u| \leq b_n \) and \( |\frac{l}{n} - u| \leq b_n \). Consequently, observing Assumption 4.4 it follows that

\[ \sigma_{j,j}(u) = \frac{1}{nb_n} \sum_{i,l=1}^{n} E\left( G(u, t_j, F_i)K\left( \frac{i}{n} - \frac{u}{b_n} \right)G(u, t_j, F_l)K\left( \frac{l}{n} - \frac{u}{b_n} \right) + O(-b_n \log b_n) \right) \quad (C.7) \]

On the other hand, if \( r_n \) is a sequence such that \( r_n = o(1) \) and \( nb_n r_n \to \infty \), \( A(u, r_n) := \{ l : |\frac{l}{n} - u| \leq 1 - r_n, u \in [b_n, 1 - b_n] \} \) we obtain by (C.7) and Lemma C.2 that
\[
\sigma_{j,j}(u) = \frac{1}{nb_{n}} \sum_{l=1}^{n} \sum_{i=1}^{n} 1(i - l \leq nb_{n}u) \mathbb{E}\left( G(u, t_j, \mathcal{F}_i) K\left( \frac{i}{n} - \frac{u}{nb_{n}} \right) G(u, t_j, \mathcal{F}_i) K\left( \frac{i}{n} - \frac{u}{nb_{n}} \right) \right) + O(-b_{n} \log b_{n} + \chi^{nb_{n}r_{n}}) \\
= \frac{1}{nb_{n}} \sum_{l=1}^{n} K^{2}\left( \frac{i}{n} - \frac{u}{nb_{n}} \right) \sum_{1 \leq i \leq n, \quad |i - l| \leq nb_{n}r_{n}} \mathbb{E}\left( G(u, t_j, \mathcal{F}_i) G(u, t_j, \mathcal{F}_i) 1\left( \left| \frac{i}{n} - \frac{u}{nb_{n}} \right| \leq 1 \right) \right) \\
+ O(-b_{n} \log b_{n} + \chi^{nb_{n}r_{n}} + r_{n}) \\
= \frac{1}{nb_{n}} \sum_{1 \leq i \leq n, \quad l \in A(u, r_{n})} K^{2}\left( \frac{i}{n} - \frac{u}{nb_{n}} \right) \sum_{1 \leq i \leq n, \quad |i - l| \leq nb_{n}r_{n}} \mathbb{E}\left( G(u, t_j, \mathcal{F}_i) G(u, t_j, \mathcal{F}_i) 1\left( \left| \frac{i}{n} - \frac{u}{nb_{n}} \right| \leq 1 \right) \right) \\
+ O(-b_{n} \log b_{n} + \chi^{nb_{n}r_{n}} + r_{n}) \\n\text{(C.8)}
\]
uniformly for \( j \in \{1, \ldots, p\} \). We obtain by the definition of the long-run variance \( \sigma^{2}(u, t) \) in Assumption 4.2(4) and Lemma C.2 that
\[
\left| \sum_{i=1}^{n} \mathbb{E}\left( G(u, t_j, \mathcal{F}_i) G(u, t_j, \mathcal{F}_i) 1\left( \left| \frac{i}{n} - \frac{u}{nb_{n}} \right| \leq 1, |i - l| \leq nb_{n}r_{n} \right) \right) - \sigma^{2}(u, t_j) \right| = O(\chi^{nb_{n}r_{n}}) \\n\text{(C.9)}
\]
uniformly with respect to \( l \in A(u, r_{n}) = \{ l : \left| \frac{l}{n} - \frac{u}{nb_{n}} \right| \leq 1-r_{n}, u \in [b_{n}, 1-b_{n}] \} \) and \( j \in \{1, \ldots, p\} \). Combining (C.8) and (C.9) and using Lemma C.2 yields
\[
\sigma_{j,j}(u) = \frac{1}{nb_{n}} \sum_{l=1}^{n} K^{2}\left( \frac{i}{n} - \frac{u}{nb_{n}} \right) \sigma^{2}(u, t_j) + O(-b_{n} \log b_{n} + \chi^{nb_{n}r_{n}} + r_{n}) \\
= \sigma^{2}(u, t_j) \int_{-1}^{1} K^{2}(t) dt + O\left( -b_{n} \log b_{n} + \chi^{nb_{n}r_{n}} + r_{n} + \frac{1}{nb_{n}} \right). \\n\text{(C.10)}
\]
Let \( r_{n} = \frac{a \log n}{nb_{n}} \) for some sufficiently large positive constant \( a \), then the assertion of the lemma follows in view of Assumption 4.2(4)).

For the second assertion, consider the case that \( j = k_{1}p + k_{2} \) for some \( 0 \leq k_{1} \leq n - 2[nb_{n}] \) and \( 1 \leq k_{2} \leq p \). Therefore by definition (2.10) in the main article,
\[
\tilde{Z}_{i,k_{1}p+k_{2}} = G(\frac{i+k_{1}}{n}, \frac{k_{2}}{p}, \mathcal{F}_{i+k_{1}}) K(\frac{i-[nb_{n}]}{nb_{n}}),
\]
which gives for the quantity in (C.6)
\[
\tilde{\sigma}_{k_{1}p+k_{2},k_{1}p+k_{2}} = \frac{1}{2[nb_{n}]} \sum_{i,l=1}^{2[nb_{n}]-1} \mathbb{E}\left( G(\frac{i+k_{1}}{n}, \frac{k_{2}}{p}, \mathcal{F}_{i+k_{1}}) K(\frac{i-[nb_{n}]}{nb_{n}}) G(\frac{i+k_{1}}{n}, \frac{k_{2}}{p}, \mathcal{F}_{i+k_{1}}) K(\frac{l-[nb_{n}]}{nb_{n}}) \right)
\]

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Consequently, putting \( i + k_1 = s_1 \) and \( l + k_1 = s_2 \) and using a change of variable, we obtain that

\[
\tilde{\sigma}_{k_1p+k_2,k_1p+k_2} = \sigma_{k_2,k_2} \left( \frac{k_1 + \lceil nb_n \rceil}{n} \right),
\]  \hspace{1cm} (C.11)

which finishes the proof.

\[\diamond\]

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