Flexible Particle Markov chain Monte Carlo methods with an application to a factor stochastic volatility model

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Abstract

Particle Markov Chain Monte Carlo methods are used to carry out inference in non-linear and non-Gaussian state space models, where the posterior density of the states is approximated using particles. Current approaches usually perform Bayesian inference using a particle Marginal Metropolis-Hastings algorithm, a particle Gibbs sampler, or a particle Metropolis within Gibbs sampler. This paper shows how the three ways of generating variables mentioned above can be combined in a flexible manner to give sampling schemes that converge to a desired target distribution. The advantage of our approach is that the sampling scheme can be tailored to obtain good results for different applications, for example when some parameters and the states are highly correlated. We investigate the properties of this flexible sampling scheme, including conditions for uniform convergence to the posterior. We illustrate our methods with a factor stochastic volatility state space model where one group of parameters can be generated in a straightforward manner in a particle Gibbs step by conditioning on the states, and a second group of parameters are generated without conditioning on the states because of the high dependence between such parameters and the states.

Keywords: Diffusion equation; Factor stochastic volatility model; Metropolis-Hastings; Particle Gibbs sampler.

1 Introduction

Our article deals with statistical inference for both the unobserved states and the parameters in a class of state space models. Its main goal is to give a flexible approach to constructing
sampling schemes that converge to the posterior distribution of the states and the parameters. The sampling schemes generate particles as auxiliary variables. This work extends the methods proposed by Andrieu et al. [2010], Lindsten and Schön [2012b], Lindsten et al. [2014] and Olsson and Ryden [2011].

Andrieu et al. [2010] introduce two particle Markov chain Monte Carlo (MCMC) methods for state space models. The first is particle marginal Metropolis-Hastings (PMMH), where the parameters are generated with the states integrated out. The second is particle Gibbs (PG), which generates the parameters given the states. They show that the augmented density targeted by this algorithm has the joint posterior density of the parameters and states as a marginal density. Andrieu et al. [2010] and Andrieu and Roberts [2009] show that the law of the marginal sequence of parameters and states, sampled using either PG or PMMH, converges to the true posterior as the number of iterations increase. Both particle MCMC methods are the focus of recent research. Olsson and Ryden [2011] and Lindsten and Schön [2012b] use backward simulation [Godsill et al., 2004] for sampling the state vector, instead of ancestral tracing [Kitagawa, 1996]. Lindsten and Schön [2012b] extend the PG sampler to a particle Metropolis within Gibbs (PMwG) sampler to deal with the case where the parameters cannot be generated exactly conditional on the states. Unless stated otherwise, we write PG to denote the PG and PMwG samplers that generate the parameters conditional on the states.

Our work extends the particle MCMC framework introduced in Andrieu et al. [2010], Lindsten and Schön [2012b] and Lindsten et al. [2014] to situations where using just PMMH or just PG is impossible or inefficient. We derive a particle sampler on the same augmented space as the PMMH and PG samplers, in which some parameters are sampled conditionally on the states and the remaining parameters are sampled with the states integrated out. We call this a PMMH+PG sampler. We show that the PMMH+PG sampler targets the same augmented density as the PMMH or PG samplers. We provide supplementary material showing that the Markov chain generated by the algorithm is uniformly ergodic, given regularity conditions. It implies that the marginal law of the Markov chain generated by $n^{th}$ iteration of the algorithm converges to the posterior density function geometrically fast, uniformly on its starting value, as $n \to \infty$.

We use ancestral tracing in the particle Gibbs step to make the presentation accessible and the online supplementary material shows how to modify the methods proposed in the paper to incorporate auxiliary particle filters and backward simulation in the particle Gibbs step. The proofs may also be modified using arguments found in Olsson and Ryden [2011], and the same results hold.

As a main application we propose a general algorithm for Bayesian inference on a factor stochastic volatility (SV) model. These models are used to jointly model many co-varying financial time series, as they are able to capture their common features using only a small number of latent factors (see, e.g. Chib et al. [2006] and Kastner et al. [2017]). We consider a factor SV model in which the volatilities of the factors follow a traditional SV model (as in Chib et al. [2006] and Kastner et al. [2017]) and the log-volatilities of the idiosyncratic errors follow either a continuous time Ornstein-Uhlenbeck (OU) process [Stein and Stein, 1991] or
a GARCH diffusion process [Chib et al., 2004, Kleppe et al., 2010]. The OU process admits a closed form transition density whereas the GARCH process does not. Such factor models can also apply to spatial temporal data with a large number of spatial measurements at each time point.

The paper is organized as follows. Section 2 introduces the basic concepts and notation used throughout the paper as well as the PMMH+PG sampler for estimating a single state space and its associated parameters. Section 3 applies the methods to a factor stochastic volatility model. This is more complex than the sampling scheme introduced in Section 2 because many univariate sampling schemes are involved as part of the overall sampling scheme. Section 4 gives empirical results for the factor stochastic volatility using both simulated and real data. The appendix in the paper presents some more details of the sampling scheme for the factor model introduced in Section 3. The paper has an online supplement which contains some further empirical results and technical results.

2 The PMMH+PG sampling scheme for state space models

This section introduces a sampling scheme that combines PMMH and PG steps for the Bayesian estimation of a state space model. The first three sections give preliminary results and Section 2.4 presents the sampling scheme. The methods and models introduced in this section are used in the more complex models in Section 3.

2.1 State space model

Define \( \mathbb{N} \) as the set of positive integers and let \( \{X_t\}_{t \in \mathbb{N}} \) and \( \{Y_t\}_{t \in \mathbb{N}} \) denote \( \mathcal{X} \)-valued and \( \mathcal{Y} \)-valued stochastic processes, where \( \{X_t\}_{t \in \mathbb{N}} \) is a latent Markov process with initial density \( f_{1}^{\theta}(x) \) and transition density \( f_{t}^{\theta}(x'|x) \), i.e.,

\[
X_1 \sim f_{1}^{\theta}(\cdot) \quad \text{and} \quad X_t|(X_{t-1} = x) \sim f_{t}^{\theta}(\cdot|x) \quad (t = 2, 3, \ldots).
\]

The latent process \( \{X_t\}_{t \in \mathbb{N}} \) is observed only through \( \{Y_t\}_{t \in \mathbb{N}} \), whose value at time \( t \) depends on the value of the hidden state at time \( t \), and is distributed according to \( g_{t}^{\theta}(y|x) \):

\[
Y_t|(X_t = x) \sim g_{t}^{\theta}(\cdot|x) \quad (t = 1, 2, \ldots).
\]

The densities \( f_{t}^{\theta} \) and \( g_{t}^{\theta} \) are indexed by a parameter vector \( \theta \in \Theta \), where \( \Theta \) is an open subset of \( \mathbb{R}^{d_{\theta}} \), and all densities are with respect to suitable dominating measures, denoted as \( dx \) and \( dy \). The dominating measures are frequently taken to be the Lebesgue measure if \( \mathcal{X} \in \mathcal{B}(\mathbb{R}^{d_x}) \) and \( \mathcal{Y} \in \mathcal{B}(\mathbb{R}^{d_y}) \), where \( \mathcal{B}(A) \) is the Borel \( \sigma \)-algebra generated by the set \( A \). Usually \( \mathcal{X} = \mathbb{R}^{d_x} \) and \( \mathcal{Y} = \mathbb{R}^{d_y} \).

We use the colon notation for collections of random variables, i.e., for integers \( r \leq s \), \( a_{r:s} = (a_r, \ldots, a_s) \), \( a_{t:r}^{u:s} = (a_{t}^{u}, \ldots, a_{r}^{u}) \) and for \( t \leq u \), \( a_{t:r}^{u:s} = (a_{t}^{u}, \ldots, a_{r}^{u}) \). The joint probability
density function of \((x_{1:T}, y_{1:T})\) is
\[
p(x_{1:T}, y_{1:T}|\theta) = f_1^\theta(x_1)g_1^\theta(y_1|x_1) \prod_{t=2}^T f_t^\theta(x_t|x_{t-1}) g_t^\theta(y_t|x_t).
\]
We define \(Z_1(\theta) := p(y_1|\theta)\) and \(Z_t(\theta) := p(y_t|y_{1:t-1}, \theta)\) for \(t \geq 2\), so the likelihood is \(Z_{1:T}(\theta) = Z_1(\theta) \times Z_2(\theta) \ldots Z_T(\theta)\). The joint filtering density of \(X_{1:t}\) is
\[
p(x_{1:t}, y_{1:t}|\theta) = \frac{p(x_{1:t}, y_{1:t}|\theta)}{Z_{1:t}(\theta)}.
\]
The posterior density of \(\theta\) and \(X_{1:T}\) can also be factorized as
\[
p(x_{1:T}, \theta|y_{1:T}) = \frac{p(x_{1:T}, y_{1:T}|\theta)p(\theta)}{Z_{1:T}},
\]
where the marginal likelihood \(Z_{1:T} = \int_\Theta Z_{1:T}(\theta) p(\theta) \, d\theta = p(y_{1:T})\). This factorization is used in the particle Markov chain Monte Carlo algorithms.

### 2.2 Target distribution for state space models

We first approximate the joint filtering densities \(\{p(x_t|y_{1:t}, \theta) : t = 1, 2, \ldots\}\) sequentially, using particles, i.e., weighted samples, \((x_{1:N}^t, \bar{w}_{1:N}^t)\), drawn from auxiliary distributions \(m_t^\theta\). This requires specifying importance densities \(m_t^\theta(x_1) := m_1(x_1|Y_1 = y_1, \theta)\) and \(m_t^\theta(x_t|x_{t-1}) := m_t(x_t|X_{t-1} = x_{t-1}, Y_{1:t} = y_{1:t}, \theta)\), and a resampling scheme \(M(a_{t-1}^1|\bar{w}_{t-1}^1)\), where each \(a_{t-1}^i = k\) indexes a particle in \((x_{1:N}^t, \bar{w}_{1:N}^t)\), and is sampled with probability \(\bar{w}_{t-1}^k\). We refer to [Doucet et al. 2000], [Van Der Merwe et al. 2001], and [Guo et al. 2003] for the choice of importance densities and [Douc and Cappé 2005] for a comparison between resampling schemes. Unless stated otherwise, upper case letters indicate random variables and lower case letters indicate the corresponding values of these random variables, e.g., \(A_t^i\) and \(a_t^i\), \(X_t\) and \(x_t\). We denote the vector of particles by
\[
U_{1:T} := (X_1^{1:N}, \ldots, X_T^{1:N}, A_1^{1:N}, \ldots, A_T^{1:N})
\]
where \(a_t^i\) is the value of the random variable \(A_t^i\) and its sample space by \(\mathcal{U} := \mathcal{X}^{TN} \times \mathbb{N}^{(T-1)N}\).

The Sequential Monte Carlo (SMC) algorithm used here is the same one as in Section 4.1 of [Andrieu et al. 2010], and is defined in Section S1 and Algorithm S1 in the supplementary material. The algorithm provides an unbiased estimate of the likelihood
\[
\hat{Z}_{1:T}(\theta) = Z(u_{1:T}, \theta) = \prod_{t=1}^T \left( N^{-1} \sum_{i=1}^N w_t^i \right),
\]
where
\[
w_1^i = \frac{f_1^\theta(x_1^i)g_1^\theta(y_1|x_1^i)}{m_1^\theta(x_1^i)}, \ w_t^i = \frac{g_t^\theta(y_t|x_t^i)\prod_{k=1}^{t-1} f_{k-1}^\theta(x_k^i|a_{k-1}^i)}{m_t^\theta(x_t^i|a_{t-1}^i)} \text{ for } t = 2, \ldots, T, \text{ and } \bar{w}_t = \frac{w_t^i}{\sum_{j=1}^N w_t^j}.
\]
The joint distribution of the particles given the parameters is

\[
\psi (u_{1:T}|\theta) := \prod_{i=1}^{N} m_t^\theta (x_t^i) \prod_{t=2}^{T} \mathcal{M}(a_{i-1}^{1:N}|\bar{w}_{i-1}^{1:N}) \prod_{i=1}^{N} m_t^\theta (x_t^{a_{i}^{T-1}}).
\] (1)

The key idea of particle MCMC methods is to construct a target distribution on an augmented space that includes the particles \( U_{1:T} \) and has a marginal distribution equal to \( p(x_{1:T}, \theta|y_{1:T}) \). This section describes the target distribution from Andrieu et al. [2010]. Later sections describe current MCMC methods to sample from this distribution and hence sample from \( p(x_{1:T}, \theta|y_{1:T}) \). Section S3 of the supplementary material describes other choices of target distribution and how it is straightforward to modify our results to apply to these distributions.

The simplest way of sampling from the particle approximation of \( p(x_{1:T}|y_{1:T}, \theta) \) is called ancestral tracing. It was introduced in Kitagawa [1996] and used in Andrieu et al. [2010] and consists of sampling one particle from the final particle filter. The method is equivalent to sampling an index \( J = j \) with probability \( \bar{w}_T^j \), tracing back its ancestral lineage \( b_T^j = j \) and \( b_{t-1}^j = a_{t-1}^{b_{t-1}} \) and choosing the particle \( x_{1:T}^j = (x_{1:T}^1, \ldots, x_{1:T}^T) \).

With some abuse of notation, for a vector \( a_t \), denote \( a_t^{(-k)} = (a_t^1, \ldots, a_t^{k-1}, a_t^{k+1}, \ldots, a_t^N) \), with obvious changes for \( k \in \{ 1, N \} \), and denote

\[
u_{1:T}^{(-j)} = \left\{ x_1^{(-b_1^j)}, \ldots, x_{T-1}^{(-b_{T-1}^j)}, x_T^{(-j)}, a_1^{(-b_1^j)}, \ldots, a_{T-1}^{(-b_{T-1}^j)} \right\}.
\]

It simplifies the notation to sometimes use the following one-to-one transformation

\[
(u_{1:T}, j) \leftrightarrow \left\{ x_{1:T}^j, b_{1:T-1}^j, j, u_{1:T}^{(-j)} \right\},
\]

and switch between the two representations and use whichever is more convenient. Note that the right hand expression will sometimes be written as \( \left\{ x_{1:T}, b_{1:T-1}^j, j, u_{1:T}^{(-j)} \right\} \) without ambiguity.

We now assume Assumptions S1 and S2 which we give in Section S1 of the online supplement.

The target distribution from Andrieu et al. [2010] is

\[
\tilde{\pi}^N (x_{1:T}, b_{1:T-1}, j, u_{1:T}^{(-j)}, \theta) := \frac{p(x_{1:T}, \theta|y_{1:T})}{N^T} \frac{\psi (u_{1:T}|\theta)}{m_1^\theta (x_1^1) \prod_{t=2}^{T} \bar{w}_{t-1}^{a_t^{b_t}} m_t^\theta (x_{t}^{b_t}|x_{t-1}^{a_t^{b_t-1}})} \cdot (2)
\]

Assumption S1 ensures that \( \tilde{\pi}^N (u_{1:T}|\theta) \) is absolutely continuous with respect to \( \psi (u_{1:T}|\theta) \) so \( \psi (u_{1:T}|\theta) \) can be used as a Metropolis-Hasting proposal density for generating from \( \tilde{\pi}^N (u_{1:T}|\theta) \).

From Assumption S2, equation (2) has the following marginal distribution

\[
\tilde{\pi}^N (x_{1:T}, b_{1:T-1}, j, \theta) = \frac{p(x_{1:T}, \theta|y_{1:T})}{N^T},
\]

and hence \( \tilde{\pi}^N (x_{1:T}, \theta) = p(x_{1:T}, \theta|y_{1:T}) \). The online supplement gives more detail.
2.3 Conditional sequential Monte Carlo (CSMC)

The particle Gibbs algorithm in Andrieu et al. [2010] uses exact conditional distributions to construct a Gibbs sampler. If we use the ancestral tracing augmented distribution given in (2), then this includes the conditional distribution given by \( \tilde{\pi} \left( u_{1:T}^{(-j)} | x_{1:T}^j, b_{1:T-1}^j, j, \theta \right) \), which involves constructing the particle approximation conditional on a pre-specified path. The conditional sequential Monte Carlo algorithm, introduced in Andrieu et al. [2010], is a sequential Monte Carlo algorithm in which a particle \( X_{1:T} = (X_{1}^{B_1}, \ldots, X_{T}^{B_T}) \), and the associated sequence of ancestral indices \( B_{1:T-1} \) are kept unchanged. In other words, the conditional sequential Monte Carlo algorithm is a procedure that resamples all the particles and indices except for \( U_{1:T} = (X_{1:T}^{J_i}, A_{1:T-1}^{J_i}) = (X_{1}^{B_1}, \ldots, X_{T}^{B_T}, B_1^j, \ldots, B_T^j) \). Algorithm S2 of the supplementary material describes the conditional sequential Monte Carlo algorithm (as in Andrieu et al. [2010]), consistent with \((x_{1:T}^i, a_{1:T-1}^i, j)\).

2.4 Flexible sampling scheme for state space models

This section introduces a sampling scheme that is suitable for the state space form given in Section 2.1 where some of the parameters can be generated exactly conditional on the state vectors, but other parameters must be generated using Metropolis-Hasting proposals. Let \( \theta := (\theta_1, \ldots, \theta_p) \) be a partition of the parameter vector into \( p \) components where each component may be a vector and let \( 0 \leq p_1 \leq p \). Let \( \Theta = \Theta_1 \times \ldots \times \Theta_p \) be the corresponding partition of the parameter space. We use the notation \( \theta_{-i} := (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_p) \). The following sampling scheme generates the parameters \( \theta_1, \ldots, \theta_{p_1} \) using PMMH steps and the parameters \( \theta_{p_1+1}, \ldots, \theta_p \) using PG steps. We call this a PMMH+PG sampler. To simplify the discussion, we assume that both particle marginal Metropolis-Hastings steps and particle Gibbs steps are used, i.e., \( 0 < p_1 < p \).

**Sampling Scheme 1 (PMMH+PG Sampler)**: Given initial values for \( U_{1:T}, J \) and \( \theta \), one iteration of the MCMC involves the following steps.

1. **(PMMH sampling)** For \( i = 1, \ldots, p_1 \)
   
   Step \( i \):
   
   (a) Sample \( \theta_i^* \sim q_1(|U_{1:T}, J, \theta_{-i}, \theta_i) \).
   
   (b) Sample \( U_{1:T}^* \sim \psi(|\theta_{-i}, \theta_i^*) \).
   
   (c) Sample \( J^* \sim \tilde{\pi}^N(|U_{1:T}^*, \theta_{-i}, \theta_i^*) \).
   
   (d) Set \( (\theta_i, U_{1:T}, J) \leftarrow (\theta_i^*, U_{1:T}^*, J^*) \) with probability
      
      \[
      \alpha_i (U_{1:T}, J, \theta_i; U_{1:T}^*, J^*, \theta_i^*|\theta_{-i}) = 1 \wedge \\
      \frac{\tilde{\pi}^N(U_{1:T}^*, \theta_i^*|\theta_{-i}) q_i(U_{1:T}, \theta_i|U_{1:T}^*, J^*, \theta_i^*, \theta_{-i})}{\tilde{\pi}^N(U_{1:T}, \theta_i|\theta_{-i}) q_i(U_{1:T}, \theta_i^*|U_{1:T}, J, \theta_{-i}, \theta_i)},
      \]
      
      \((4)\)
where
\[ q(U_{1:T}^*, \theta_i^*|U_{1:T}, J, \theta_{-i}, \theta_i) = q_{i,1}(\theta_i^*|U_{1:T}, J, \theta_{-i}, \theta_i)\psi(U_{1:T}^*|\theta_{-i}, \theta_i). \]

2. (PG sampling) For \( i = p_1 + 1, \ldots, p \)
   Step i:
   
   (a) Sample \( \theta_i^* \sim q_i(\cdot|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}, \theta_i). \)
   (b) Set \( \theta_i \leftarrow \theta_i^* \) with probability
   \[
   \alpha_i \left( \theta_i; \theta_i^* |X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i} \right) = \\
   1 \wedge \frac{\pi^N(\theta_i^*|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i})}{\pi^N(\theta_i|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i})} \times \frac{q_i(\theta_i|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}, \theta_i^*)}{q_i(\theta_i^*|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}, \theta_i)}. \quad (5)
   \]

3. Sample \( U_{1:T}^{(-J)} \sim \pi^N(\cdot|X_{1:T}^J, B_{1:T-1}^J, J, \theta) \) using the conditional sequential Monte Carlo algorithm (CSMC) discussed in Section 2.3.

4. Sample \( J \sim \pi^N(\cdot|U_{1:T}, \theta). \)

Note that Parts 2 to 4 are the same as the particle Gibbs sampler described in [Andrieu et al., 2010] or the particle Metropolis within Gibbs sampler described in [Lindsten and Schön, 2012a]. Part 1 differs from the particle Marginal Metropolis-Hastings approach discussed in [Andrieu et al., 2010] by generating the variable \( J \) which selects the trajectory. This is necessary since \( J \) is used in Part 2.

A major computational cost of the algorithm is generating the particles \( p_1 \) times in Part 1 and running the CSMC algorithm in Part 3. Hence there is a computational cost in using the PMMH+PG sampler compared to a particle Gibbs sampler. Similar comments apply to a blocked PMMH sampler.

Section S2 of the supplementary material discusses the convergence of Sampling Scheme 1 to its target distribution and Section S4 of the supplementary material illustrates the sampling scheme by applying it to a univariate OU model for daily stock return data.

Remark 1 [Andrieu et al., 2010] show that
\[
\frac{\pi^N(U_{1:T}, \theta_i^*|\theta_{-i})}{\psi(U_{1:T}^*|\theta_{-i}, \theta_i)} = \frac{Z(U_{1:T}, \theta)p(\theta_i|\theta_{-i})}{p(y_{1:T}|\theta_{-i})}, \quad (6)
\]
and hence the Metropolis-Hasting acceptance probability in (4) simplifies to
\[
1 \wedge \frac{Z(\theta_i^*, \theta_{-i}, U_{1:T}^*)}{Z(\theta_i, \theta_{-i}, U_{1:T})} \frac{q_i,1(\theta_i^*|U_{1:T}^*, J^*, \theta_{-i}, \theta_i)p(\theta_i^*)}{q_i,1(\theta_i|U_{1:T}, J, \theta_{-i}, \theta_i)p(\theta_i|\theta_{-i})}. \quad (7)
\]
Equation (7) shows the PMMH steps can be viewed as involving a particle approximation to an ideal sampler which we use to estimate the likelihood of the model. This version of the PMMH algorithm can also be viewed as a Metropolis-Hastings algorithm using an unbiased estimate of the likelihood.
Remark 2  Part 1 of the sampling scheme is a good choice for parameters $\theta_i$ which are highly correlated with the state vector $X_{1:T}$. Part 2 of the sampling scheme is a good choice if the parameter $\theta_i$ is not highly correlated with the states and it is possible to sample exactly from the distribution

$$\tilde{\pi}^N (\theta_i | X_{1:T}, \theta_{-i}) = p (\theta_i | X_{1:T}, y_{1:T}, \theta_{-i})$$

or a good approximation is available as a Metropolis-Hastings proposal. See Lindsten and Schön [2012a] for more discussion about the particle Metropolis-Hasting proposals in Part 2.

3 Sampling schemes for factor stochastic volatility models

3.1 The factor stochastic volatility model

Factor stochastic volatility (SV) models are a popular approach to jointly model many co-varying financial time series, as they are able to capture their common features using only a small number of latent factors (see, e.g., Chib et al. [2006] and Kastner et al. [2017]). However, estimating time-varying multivariate factor models can be very challenging because the likelihood involves calculating an integral over a very high-dimensional latent state space, and the number of parameters in the model can be large. Current approaches to estimate these models often employ MCMC samplers. We argue that our particle MCMC method can be used to estimate a larger class of SV models.

We consider a factor SV model with the volatilities of the factors following a traditional SV model [Chib et al., 2006, Kastner et al., 2017], while the log volatility of the idiosyncratic errors follow a continuous time Ornstein-Uhlenbeck (OU) process [Stein and Stein, 1991] or a GARCH diffusion process [Chib et al., 2004, Kleppe et al., 2010]. The log volatility of an OU process admits a closed form state transition density, whereas the GARCH process does not. Our estimation methods are applied to Euler approximations of the diffusion process driving the log volatilities, hence can handle diffusions that do not admit closed form transition densities [see Ignatieva et al., 2015 for other diffusions whose transition equations need an Euler approximation because they cannot be expressed in closed form]. We show on both simulated and real data that the PMMH+PG sampler works well. The sampling scheme generates the latent factors on the PG step and then, conditioning on the latent factors, estimates a series of univariate state space models. Most of the parameters of the model are generated in the PG step, whereas a few are left for the PMMH step to improve mixing. We note that our example merely illustrates our methods which can naturally handle multiple factors and most types of log-volatilities for both the factors and idiosyncratic errors.

Suppose that $P_t$ is a $S \times 1$ vector of daily stock prices and define $y_t := \log P_t - \log P_{t-1}$ as the log-return of the stocks. We model $y_t$ as a factor SV model

$$y_t = \beta f_t + V_t^j \epsilon_t \quad (t = 1, \ldots, T),$$

(8)
where \( \mathbf{f} \) is a \( K \times 1 \) vector of latent factors (with \( K \ll S \)), \( \mathbf{\beta} \) is a \( S \times K \) factor loading matrix of unknown parameters. Appendix A.1 gives further details on the restrictions on \( \mathbf{\beta} \). We model the latent factors as \( \mathbf{f}_t \sim N(0, \mathbf{D}_t) \) and \( \mathbf{e}_t \sim N(0, \mathbf{I}) \), so that \( \mathbf{y}_t|(\mathbf{f}_t, \mathbf{h}_t) \sim N(\mathbf{\beta f}_t, \mathbf{V}_t) \).

The time-varying variance matrices \( \mathbf{D}_t \) and \( \mathbf{V}_t \) depend on unobserved random variables \( \mathbf{\lambda}_t = (\lambda_{1t}, \ldots, \lambda_{Kt}) \) and \( \mathbf{h}_t = (h_{1t}, \ldots, h_{St}) \) such that

\[
\mathbf{D}_t := \text{diag}(\exp(\lambda_{1t}), \ldots, \exp(\lambda_{Kt})), \quad \mathbf{V}_t := \text{diag}(\exp(h_{1t}), \ldots, \exp(h_{St})).
\]

Each \( \lambda_{kt} \) is assumed to follow an independent stochastic volatility process

\[
\lambda_{kt} = \phi_k \lambda_{kt-1} + \tau_{fk} \eta_{kt}, \quad k = 1, \ldots, K, \tag{9}
\]

with \( \eta_{kt} \sim N(0, 1) \). The log volatilities \( h_{st} \) follow a either a Gaussian OU continuous time volatility process or a GARCH diffusion continuous time volatility process.

The continuous time Ornstein-Uhlenbeck (OU) process \( \{h_{st}\}_{t \geq 1} \), introduced by Stein (1991), satisfies

\[
dh_{st} = \alpha_s (\mu_s - h_{st}) \, dt + \tau_{es} d\mathbf{W}_{st}, \quad \text{for} \quad s = 1, \ldots, S, \tag{10}
\]

where \( \mathbf{W}_t \) is a Wiener process. The transition distribution for each \( h_{st} \) is [Lunde et al., 2015, p. 7]

\[
h_{st}|h_{st-1} \sim N \left( \mu_s + \exp(-\alpha_s)(h_{st-1} - \mu_s), \frac{1 - \exp(-2\alpha_s)}{2\alpha_s} \tau_{es}^2 \right), \quad s = 1, \ldots, S. \tag{11}
\]

with \( h_{s1} \sim N(\mu_s, \tau_{es}^2) \). The transition equations are in state space form with \( x_{1:T} = h_{1:T} \) and the parameters are \( \alpha_s > 0, \mu_s \) and \( \tau_{es}^2 > 0 \).

The Euler scheme approximates the evolution of the log-volatilities \( h_{st} \) in equation (10) by placing \( M - 1 \), evenly spaced, latent points between times \( t \) and \( t + 1 \). The intermediate volatility components are denoted by \( h_{st,1}, \ldots, h_{st,M-1} \), and it is convenient to set \( h_{st,0} = h_t \) and \( h_{st,M} = h_{t+1} \). The equation for the Euler evolution, starting at \( h_{t,0} \) is (see for example, Stramer and Bognar [2011], pg. 234)

\[
h_{st,j}|h_{st,j-1} \sim N \left( h_{st,j-1} + \alpha_s (\mu_s - h_{st,j-1}) \delta, \tau_{es}^2 \delta \right), \tag{12}
\]

for \( j = 1, \ldots, M \), where \( \delta = 1/M \).

The continuous time GARCH diffusion process \( \{h_{st}\}_{t \geq 1} \) [Chib et al., 2004, Kleppe et al., 2010] satisfies

\[
dh_{st} = \left\{ \alpha_s (\mu_s - \exp(h_{st})) \exp(-h_{st}) - \frac{\tau_{es}^2}{2} \right\} dt + \tau_{es} d\mathbf{W}_{st}, \quad \text{for} \quad s = 1, \ldots, S, \tag{13}
\]

where the \( \mathbf{W}_{st} \) are independent Wiener processes. The Euler approximation of the state transition density of equation (13) yields the transition density between steps (see for example, Wu et al. [2018], pg. 21)

\[
h_{st,j+1}|h_{st,j} \sim N \left( h_{st,j} + \left\{ \alpha_s (\mu_s - \exp(h_{st,j})) \exp(-h_{st,j}) - \frac{\tau_{es}^2}{2} \right\} \delta, \tau_{es}^2 \delta \right) \tag{14}
\]
for \( j = 0, ..., M - 1 \), where \( \delta = 1/M \).

We denote the parameter vector for the factor stochastic volatility model given by equations (8), (9) and either (11), (12) or (14) by

\[
\omega = (\beta; (\phi_k, \tau_{f_k}), k = 1, \ldots, K; (\alpha_s, \mu_s, \tau_{\epsilon_s}), s = 1, \ldots, S).
\]

Section 4 applies the methods in the paper to the stochastic volatility factor model.

### 3.2 Target density for the factor SV model

Although the factor model outlined in Section 3.1 can be written in the state space form in Section 2.1, it is more efficient to take advantage of extra structure in the model and base the sampling on multiple independent univariate state space models. This section outlines the conditional independence structure and the more complex target density and sampling schemes required for this approach to estimating the posterior distribution of the factor SV model. Appendix A gives further details.

#### 3.2.1 Conditional independence in the factor SV model

The key to making the estimation of the factor SV model tractable is that given the values of \((y_{1:T}, f_{1:T}, \omega)\) and the conditional independence of the innovations of the returns, the factor model in equation (8) separates into independent components consisting of \(K\) univariate SV models for the latent factors and \(S\) univariate state space models for the idiosyncratic errors. For \( k = 1, ..., K \), we have that

\[
f_{kt} | \lambda_{kt} \sim N(0, \exp(\lambda_{kt})),
\]

with the transition density given in equation (9). For \( s = 1, ..., S \), we have

\[
y_{st} | f_t, h_{st} \sim N(\beta_s f_t, \exp(h_{st})),
\]

with the exact and approximate transition densities given in equations (11), (12) or (14).

#### 3.2.2 The closed form density case

This section provides an appropriate target density for a factor model with the closed form density given in equation (11). The target density includes all the random variables produced by \(K + S\) univariate particle filters that generate the factor log volatilities \(\lambda_{k,1:T}\) for \(k = 1, ..., K\) and the idiosyncratic log volatilities \(h_{s,1:T}\) for \(s = 1, ..., S\), as well as the factors \(f_{1:T}\) and the parameters \(\omega\). It is convenient in the developments below to define \(\theta = (f_{1:T}, \omega)\).

To specify the univariate particle filters that generate the factor log volatilities \(\lambda_{k,1:T}\) for \(k = 1, ..., K\) we use equations (9) and (15) and to generate the idiosyncratic log volatilities \(h_{s,1:T}\) for \(s = 1, ..., S\) we use equations (11) and (16). We denote the weighted samples by \((\lambda_{k,t}^{1:N}, \omega_{f_k}^{1:N})\) and \((h_{s,t}^{1:N}, \omega_{\epsilon_s}^{1:N})\). We denote the proposal densities by \(m_{f_k}^{\theta}(\lambda_{kt})\), \(m_{f_k}^{\theta}(\lambda_{kt}|\lambda_{kt-1})\), \(m_{\epsilon_s}^{\theta}(h_{st})\) and \(m_{\epsilon_s}^{\theta}(h_{st}|h_{st-1})\) for \(t = 2, ..., T\). We denote the resampling
schemes by $\mathcal{M}_f (a_{f,k,t-1}^1|\tilde{w}_{f,k,t-1}^j)$ for $k = 1, \ldots, K$, where each $a_{f,k,t-1}^i = j$ indexes a particle in $(\lambda_{kt}^1, \tilde{w}_{fkt}^j)$ and is chosen with probability $\tilde{w}_{fkt}^j$; the resampling scheme $\mathcal{M}_e (a_{est}^1|\tilde{w}_{est}^j)$ for $s = 1, \ldots, S$ is defined similarly. We denote the vector of particles by

$$ U_{f,1:K,1:T} = (\lambda_{1:1;K,1:T}^1, A_{f,1:K,1:T-1}^1) \tag{17} $$

and

$$ U_{e,1:S,1:T} = (h_{1:1;S,1:T}^1, A_{e,1:S,1:T-1}^1) \tag{18} $$

The joint distribution of the particles given the parameters is

$$ \psi_{fk}(U_{fk,1:T}|\theta) = \prod_{i=1}^{N} m_{fk1}(\lambda_{k1}) \prod_{t=2}^{T} \left\{ \mathcal{M}_f (a_{fkt-1}^1|\tilde{w}_{fkt}^j) \prod_{i=1}^{N} m_{fk1}(\lambda_{kt}^1) \prod_{i=1}^{N} \frac{\lambda_{fkt}^i | \lambda_{kt-1}^i}{\lambda_{fkt}^j | \lambda_{kt-1}^j} \right\} \tag{19} $$

for $k = 1, \ldots, K$ and

$$ \psi_{es}(U_{es,1:T}|\theta) = \prod_{i=1}^{N} m_{es1}(h_{s1}) \prod_{t=2}^{T} \left\{ \mathcal{M}_e (a_{est-1}^1|\tilde{w}_{est}^j) \prod_{i=1}^{N} m_{es1}(h_{st}^1) \prod_{i=1}^{N} \frac{h_{st}^i | h_{st-1}^i}{h_{st}^j | h_{st-1}^j} \right\} \tag{20} $$

for $s = 1, \ldots, S$.

Next, we define indices $J_{fk} = j$ for each $k = 1, \ldots, K$, then trace back its ancestral lineage $b_{fkt}^j (b_{fkt}^j = j, b_{fkt-1}^j = a_{fkt-1}^j)$, and select the particle trajectory $\lambda_{k,1:T}^j = (\lambda_{k1}^j, \ldots, \lambda_{kT}^j)$. Similarly, we define indices $J_{es} = j$ for each $s = 1, \ldots, S$, then trace back its ancestral lineage $b_{est}^j (b_{est}^j = j, b_{est-1}^j = a_{est-1}^j)$, and select the particle trajectory $h_{s,1:T}^j = (h_{s1}^j, \ldots, h_{sT}^j)$.

The augmented target density of the factor model is defined as

$$ \tilde{\pi}^N(U_{f,1:K,1:T}, U_{e,1:S,1:T}, J_f, J_e, \theta) := \pi \left( \lambda_{1:1;K,1:T}^j, h_{1:1;S,1:T}^j, J_f, J_e, \theta \right) \prod_{k=1}^{K} m_{fk1}(\lambda_{k1}) \prod_{t=2}^{T} \left\{ \mathcal{M}_f (a_{fkt-1}^1|\tilde{w}_{fkt}^j) \prod_{i=1}^{N} m_{fk1}(\lambda_{kt}^1) \prod_{i=1}^{N} \frac{\lambda_{fkt}^i | \lambda_{kt-1}^i}{\lambda_{fkt}^j | \lambda_{kt-1}^j} \right\} \prod_{s=1}^{S} \left\{ \mathcal{M}_e (a_{est-1}^1|\tilde{w}_{est}^j) \prod_{i=1}^{N} m_{es1}(h_{st}^1) \prod_{i=1}^{N} \frac{h_{st}^i | h_{st-1}^i}{h_{st}^j | h_{st-1}^j} \right\} \tag{21} $$

The first term in equation (21) is defined using the joint distribution of the factor SV model using equation (11) for the selected trajectories and the factors $f_{1:T}$ conditional on $(y, \omega)$. 

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The prior for $\omega$ is specified by the user. The conditional independence results discussed in Section 3.2.1 show that this distribution factors into separate terms for the $K + S$ univariate state space models given by equations (9), (11), (15) and (16) as shown in equation (22) below.

$$
\pi \left( \lambda^J_{1:K,1:T}, h^J_{1:S,1:T}, \theta \right) = \pi \left( \theta \right) \prod_{k=1}^{K} \pi \left( \lambda^J_{k,1:T|\theta} \right) \prod_{s=1}^{S} \pi \left( h^J_{s,1:T|\theta} \right).
$$

### 3.2.3 Approximating the transition density by an Euler scheme

This section provides an appropriate target density for a factor model with the Euler approximation given in equation (12) or equation (14). We follow the approach in Lindsten et al. (2015) and introduce state vectors for $s = 1, \ldots, S$ defined as $x_{s1} = h_{s1}$ and $x_{st} = (h_{st}, h_{s,t-1,M-1}, \ldots, h_{s,t-1,1})^T$, for $t = 2, \ldots, T$. The state transition densities are given by

$$
f^\theta_{st}(x_{s,t}|x_{s,t-1}) = \prod_{j=1}^{M} f^\theta_{s,t-1,j}(h_{s,t-1,j}|h_{s,t,j-1}) \quad (t = 2, \ldots, T),
$$

where the densities $f^\theta_{s,t,j}(h_{s,t,j}|h_{s,t,j-1})$ for $j = 1, \ldots, M$, $t = 1, \ldots, T - 1$ and $s = 1, \ldots, S$ are defined by equation (12) or equation (14). We use the proposal densities

$$
m^\theta_{est}(x_{st}|x_{s,t-1}) = f^\theta_{st}(x_{st}|x_{s,t-1}) \quad (t = 2, \ldots, T \text{ and } s = 1, \ldots, S)
$$

which can be generated using equation (12) or equation (14). With these modifications, we use the same construction as Section 3.2.2. The modifications give

$$
U_{\epsilon,1:S,1:T} = \left( x^1_{1:S,1:T}, A^1_{1:S,1:T-1} \right)
$$

$$
\psi_{\epsilon s} \left( U_{\epsilon s,1:T} | \theta \right) = \prod_{i=1}^{N} m^\theta_{es1}(x^i_{s1}) \prod_{t=2}^{T} \left\{ \mathcal{M}_e \left( a^1_{es,t-1} \right| w^1_{es,t-1} \right) \prod_{i=1}^{N} m^\theta_{est}(x^i_{st} x^a_{st-1,t-1}) \right\}
$$

$$
\tilde{\pi}^N \left( U_{f1,1:K,1:T}, U_{\epsilon,1:S,1:T}, J_f, J_{\epsilon}, \theta \right) := \frac{\pi \left( \lambda^J_{1:K,1:T}, x^J_{1:S,1:T}, \theta \right) \prod_{k=1}^{K} \psi_{fk} \left( U_{fk,1:T|\theta} \right)}{NT(K+S)} \prod_{s=1}^{S} \frac{\psi_{es} \left( U_{es,1:T|\theta} \right)}{m^\theta_{es1}(x^b_{s1}) \prod_{t=2}^{T} \left( w^b_{es,t-1} \right) m^\theta_{est}(x^b_{st} x^a_{st-1,t-1})}
$$

$$
\pi \left( \lambda^J_{1:K,1:T}, x^J_{1:S,1:T}, \theta \right) = \pi \left( \theta \right) \prod_{k=1}^{K} \pi \left( \lambda^J_{k,1:T|\theta} \right) \prod_{s=1}^{S} \pi \left( x^J_{s,1:T|\theta} \right).
$$
3.3 PMMH+PG sampling scheme for the factor SV model

We illustrate our methods using the PMMH \((\alpha, \tau_{fj}^2, \tau_e^2) + PG (\beta, f_{1:T}, \phi, \mu)\) sampler, which was found to give good performance in the Empirical studies in Section 4. It is straightforward to modify the sampling scheme for other choices of which parameters to sample with a PMMH step and which to sample with a PG step. Our procedure to determine an efficient sampling scheme is to run the PG algorithm first to identify which parameters have large IACT, or, in some cases, require a large amount of computational time to generate in the PG step. We then generate these parameters in the PMMH step. See, for example, our discussion of the univariate OU model in Section S4 of the supplement. In particular, we note that if an Euler approximation is used, then generating any parameter in the OU or GARCH model is time intensive as we need to determine, store and use the ancestor history of the entire state vector.

The sampling schemes for the factor SV model with the closed form transition density given by equation (11) and the model with the Euler scheme given by equation (12) or equation (14) have the same structure, so Sampling Scheme 2 is given below in a generic form and the appropriate state space models are used for the different cases, see Sections 3.2.2 and 3.2.3 for details. We have simplified the conditional distributions in Sampling Scheme 2 wherever possible using the conditional independence properties discussed in Section 3.2.1.

The Metropolis-Hasting proposal densities for Sampling scheme 2 are given in Section 3.2.1.

**Sampling Scheme 2** \((\text{PMMH} (\alpha, \tau_{fj}^2, \tau_e^2) + \text{PG} (\beta, f_{1:T}, \phi, \mu))\) Given initial values for \(U_{f,1:T}, U_{e,1:T}, J_f, J_e\) and \(\theta\), one iteration of the MCMC involves the following steps.

1. (PMMH sampling),
   (a) For \(k = 1, ..., K\)
      i. Sample \((\tau_{fjk}^2) \sim q_{\tau_{fjk}^2} (\cdot | U_{f,1:T}; \tau_{fjk}^2; \theta \setminus \tau_{fjk}^2)\)
      ii. Sample \(U^*_{fjk,1:T} \sim \psi_{fjk} (\cdot | \tau_{fjk}^2; \theta \setminus \tau_{fjk}^2)\)
      iii. Sample \(J_{fjk}^*\) from \(\mathcal{N} (\cdot | U^*_{f,1:T}; \tau_{fjk}^2; \theta \setminus \tau_{fjk}^2)\)
      iv. Set \((\tau_{fjk}, U_{f,1:T}, J_f) \leftarrow (\tau_{fjk}^*, U^*_{f,1:T}, J_{fjk}^*)\) with probability

\[
\alpha (U_{f,1:T}, J_f, \tau_{fjk}^2; U^*_{f,1:T}, J_{fjk}^*, \tau_{fjk}^2 | \theta \setminus \tau_{fjk}^2) = \frac{1 \land Z(U^*_{f,1:T}, \tau_{fjk}^2; \theta \setminus \tau_{fjk}^2) p(\tau_{fjk}^2 | f_{1:T})}{Z(U_{f,1:T}, \tau_{fjk}^2; \theta \setminus \tau_{fjk}^2) p(\tau_{fjk}^2 | f_{1:T})} \times \frac{q_{\tau_{fjk}^2} (\tau_{fjk}^* | U_{f,1:T}; \tau_{fjk}^2, \theta \setminus \tau_{fjk}^2)}{q_{\tau_{fjk}^2} (\tau_{fjk}^2 | U_{f,1:T}; \tau_{fjk}^2, \theta \setminus \tau_{fjk}^2)}.
\]

(b) For \(s = 1, ..., S\)
   i. Sample \((\alpha_{s, \tau_{es}^2}) \sim q_{\alpha_{s, \tau_{es}^2}} (\cdot | U_{e,1:T}; \alpha_{s, \tau_{es}^2}, \theta \setminus \alpha_{s, \tau_{es}^2})\)
   ii. Sample \(U^*_{e,1:T} \sim \psi_{es} (\cdot | \alpha_{s, \tau_{es}^2}, \theta \setminus \alpha_{s, \tau_{es}^2})\)
ii. Sample $J_{es}$ from $\tilde{\pi}^N (\cdot | U_{es,1:T}^*, \alpha^*_s, \tau^2_{es}, \theta_{\alpha_s, \tau^2_{es}})$

iv. Set $(\alpha^*_s, \tau^2_{es}, U_{es,1:T}, J_{es}) \leftarrow (\alpha^*_s, \tau^2_{es}, U_{es,1:T}, J_{es}^*)$ with probability

$$\alpha \left( U_{es,1:T}, J_s, (\alpha^*_s, \tau^2_{es}) ; U^*_{es,1:T}, J^*_{es} ; (\alpha^*_s, \tau^2_{es}) \right | \theta_{\alpha_s, \tau^2_{es}} ) =$$

$$\frac{1}{Z} \frac{Z ( U_{es,1:T}^*, \alpha^*_s, \tau^2_{es}, \theta_{\alpha_s, \tau^2_{es}} ) p ( \alpha^*_s, \tau^2_{es} ; U_{es,1:T}^*, \alpha^*_s, \tau^2_{es}, \theta_{\alpha_s, \tau^2_{es}} )}{Z ( U_{es,1:T}, \alpha_s, \tau^2_{es}, \theta_{\alpha_s, \tau^2_{es}} ) p ( \alpha_s, \tau^2_{es} ; U_{es,1:T}, \alpha_s, \tau^2_{es}, \theta_{\alpha_s, \tau^2_{es}} )}.$$

2. (PG sampling)

(a) Sample $\beta \mid \lambda_{1:T}, A_{1:T}, J_{1:T}, f, J, \theta \setminus \beta, y_{1:T}$ using equation (31) in Appendix A.1.

(b) Redraw the diagonal elements of $\beta$ through the deep interweaving procedure described in Appendix A.3. This step is necessary to improve the mixing of the factor loading matrix $\beta$.

(c) Sample $f_{1:T}, \lambda_{1:T}^J, h_{1:T}, B_{f,1:T-1}, J^f_{1:T-1}, J_{f,1:T}, \theta_{f,1:T}, y_{1:T}$ using equation (32) in Appendix A.3.

(d) For $k = 1, ..., K$

i. Sample $\phi^*_k$ from the proposal $q_{\phi_k} (\cdot | \lambda_{k,1:T}, \theta \setminus \phi_k)$ and set $\phi_k \leftarrow \phi^*_k$ with probability

$$\frac{1}{\tilde{\pi}^N ( \phi_k | \lambda_{k,1:T}, B_{k,1:T-1}, J_{k,1:T}, \theta \setminus \phi_k )} \times \frac{q_{\phi_k} ( \phi^*_k | \lambda_{k,1:T}, B_{k,1:T-1}, J_{k,1:T}, \theta \setminus \phi_k )}{q_{\phi_k} ( \phi_k | \lambda_{k,1:T}, B_{k,1:T-1}, J_{k,1:T}, \theta \setminus \phi_k )}.$$

ii. Sample $U_{k,1:T}^{(-J_{k})} \sim \tilde{\pi}^N (\cdot | \lambda_{k,1:T}, B_{k,1:T-1}, J_{k}, \theta)$ using the conditional sequential Monte Carlo algorithm (CSMC) discussed in Section S2.

iii. Sample $J_{k} \sim \pi^N (\cdot | U_{k,1:T}^{(-J_{k})}, \theta)$.

(e) For $s = 1, ..., S$

i. Sample $\mu^*_s$ from the proposal $q_{\mu_s} (\cdot | h_{s,1:T}, \theta \setminus \mu_s)$ and set $\mu_s \leftarrow \mu^*_s$ with probability

$$\frac{1}{\tilde{\pi}^N ( \mu_s | h_{s,1:T}, B_{s,1:T-1}, J_{s}, \theta \setminus \mu_s )} \times \frac{q_{\mu_s} ( \mu^*_s | h_{s,1:T}, B_{s,1:T-1}, J_{s}, \theta \setminus \mu_s )}{q_{\mu_s} ( \mu_s | h_{s,1:T}, B_{s,1:T-1}, J_{s}, \theta \setminus \mu_s )}.$$

ii. Sample $U_{s,1:T}^{(-J_{s})} \sim \tilde{\pi}^N (\cdot | h_{s,1:T}, B_{s,1:T-1}, J_{s}, \theta)$ using the conditional sequential Monte Carlo algorithm (CSMC) discussed in Section 2.3.

iii. Sample $J_{s} \sim \pi^N (\cdot | U_{s,1:T}, \theta)$. 

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3.3.1 Proposal densities

This section details the proposal densities used in Sampling Scheme 2 for the exact OU model given by equation (11). We will specify other cases such as the Euler evolution given by equation (12) and the GARCH diffusion model given by equation (14) when describing the sampling scheme.

- For $k = 1, \ldots, K$, $q_{\tau_j^2}$ is an adaptive random walk.
- For $s = 1, \ldots, S$, $q_{\alpha_s, \tau_s^2}$ is an adaptive random walk.
- For $k = 1, \ldots, K$, $q_{\phi_k} \left( \cdot | \lambda_{k,1:T}, \theta \right) = N \left( c_{\phi_k}, d_{\phi_k} \right)$, where
  
  \[ c_{\phi_k} = \frac{d_{\phi_k}}{\tau_{jk}^2} \sum_{t=2}^{T} \lambda_{kt} \lambda_{kt-1}, \quad \text{and} \quad d_{\phi_k} = \frac{\tau_{jk}^2}{\sum_{t=2}^{T-1} \lambda_{kt}}, \]

- For $s = 1, \ldots, S$, $q_{\mu_s} \left( \cdot | h_{s,1:T}, \theta \right) = N \left( c_{\mu_s}, d_{\mu_s} \right)$, where
  
  \[ c_{\mu_s} = \frac{d_{\mu_s}}{\tau_{es}^2} \left( h_{s,1} (2 \alpha_s) + \left( \frac{2 \alpha_s}{1 - \exp (-2 \alpha_s)} \right) \left( \sum_{t=2}^{T} (h_{s,t} - \exp (-\alpha_s) h_{s,t} + \exp (-2 \alpha_s) h_{s,t-1}) \right), \right) \]
  \[ d_{\mu_s} = \frac{\tau_{es}^2}{(2 \alpha_s) + \left( \frac{2 \alpha_s}{1 - \exp (-2 \alpha_s)} \right) (T - 1) (1 - 2 \exp (-\alpha_s) + \exp (-2 \alpha_s))^2} \]

3.4 The PMMH sampling scheme for the factor stochastic volatility model

The PMMH method generates the parameters by integrating out all the latent factors, so that the observation equation is given by

\[ y_t | \lambda_t, h_t, \omega \sim N \left( 0, \beta D_t \beta' + V_t \right). \] (28)

The state transition equations are given by equations (9) and either equation (11) for the closed form case or equation (23) for the Euler scheme.

4 Empirical Studies

This section presents empirical results for the factor SV model described in Section 3 to illustrate the flexibility of the sampling approach given in our article. That is, we show it is desirable to generate parameters that are highly correlated with the states using a PMMH approach.
step that does not condition on the states. Conversely, if there is a subset of parameters that is not highly correlated with the states, then it is preferable to generate them using a particle Gibbs step, or a particle Metropolis within Gibbs step, that conditions on the states, especially when the subset is large. A simple example of the methods is given in Section S4 of the supplementary material where Sampling Scheme [I] is applied to a univariate OU model for daily stock return data.

4.1 Preliminaries

To define our measure of the inefficiency of a sampler that takes computing time into account, we first define the integrated autocorrelation time (IACT) for a univariate parameter \(\theta\),

\[
\text{IACT}_\theta := 1 + 2 \sum_{j=1}^{\infty} \rho_{j,\theta}
\]

where \(\rho_{j,\theta}\) is the correlation of the iterates of \(\theta\) in the MCMC after the chain has converged. A large value of IACT for one or more of the parameters indicates that the chain does not mix well.

We estimate \(\text{IACT}_\theta\) based on \(M\) iterates \(\theta^{[1]}, ..., \theta^{[M]}\) (after convergence) as

\[
\hat{\text{IACT}}_{\theta,M} = 1 + 2 \sum_{j=1}^{L_M} \hat{\rho}_{j,\theta},
\]

where \(\hat{\rho}_{j,\theta}\) is the estimate of \(\rho_{j,\theta}\), \(L_M = \min(1000, L)\) and \(L = \min_{j \leq M} |\hat{\rho}_{j,\theta}| < \frac{2}{\sqrt{M}}\) because \(1/\sqrt{M}\) is approximately the standard error of the autocorrelation estimates when the series is white noise. Let \(\hat{\text{IACT}}_{\text{MAX}}\) and \(\hat{\text{IACT}}_{\text{MEAN}}\) be the maximum and mean of the estimated IACT values over all the parameters in the model, respectively. Our measure of the inefficiency of a sampler based on \(\hat{\text{IACT}}_{\text{MAX}}\) is the time normalized variance (TNV).

\[
\text{TNV}_{\text{MAX}} = \hat{\text{IACT}}_{\text{MAX}} \times \text{CT},
\]

where CT is the computing time in seconds per iteration; we define the inefficiency of a sampler based on \(\hat{\text{IACT}}_{\text{MEAN}}\) similarly. The relative time normalized variance (RTNV) shows the TNV relative to the most efficient method.

We use the following notation to describe the algorithms used in the examples. The basic samplers, as used in Sampling Schemes [I] or [2] are PMMH(\(\cdot\)) and PG(\(\cdot\)). These samplers can be used alone or in combination. For example, PMMH(\(\theta\)) means using a PMMH step to sample the parameter vector \(\theta\); PMMH(\(\theta_1\)) + PG(\(\theta_2\)) means sampling \(\theta_1\) in the PMMH step and \(\theta_2\) in the PG step; and PG(\(\theta\)) means sampling \(\theta\) using the PG sampler. In all the examples, the PMMH step uses the bootstrap particle filter to sample the particles and the adaptive random walk in Roberts and Rosenthal [2009] as the proposal density for the parameters. The particle filter and the parameter samplers are implemented in Matlab.
We note that efficient Metropolis-Hastings schemes for standard MCMC are possible because both the likelihood and its derivatives can be computed exactly or, in the case of derivatives, by finite differences. This is not the case for PMMH where the likelihood is estimated and its derivatives are not available, even numerically. We therefore use the adaptive random walk in all our examples to generate the parameters as it provides a general and flexible sampling solution that does not require the computation or estimation of the local gradient or hessian. The random walk proposal scales the covariance matrix by the factor \(2.38/\sqrt{d}\) in the non-particle case and \(2.56/\sqrt{d}\) in the particle case [Sherlock et al., 2015], where \(d\) is the number of parameters in the random walk. This means that for large \(d\) the scale factor can be very small and the random walk is very inefficient because it moves slowly. Moreover, and importantly, [Sherlock et al., 2015] show that in the particle case, the optimal acceptance rate of random walk proposals is 7\%, which suggests that it is quite inefficient. [Sherlock et al., 2015] derive this acceptance rate assuming that \(d \to \infty\), which also means that, at least theoretically, the step size is very small.

The use of more efficient particle MALA algorithms is discussed by [Nemeth et al., 2016] who write “Our results show that the behaviour of particle MALA depends crucially on how accurately we can estimate the gradient of the log-posterior. If the error in the estimate of the gradient is not controlled sufficiently well as we increase dimension, then asymptotically there will be no advantage in using particle MALA over a particle MCMC algorithm using a random-walk proposal.” However, as noted by [Sherlock et al., 2015], it is, in general, even more difficult to obtain accurate estimates of the gradient of the log-posterior than it is to get accurate estimates of the log-posterior.

This discussion again suggests why using PG may be preferred to PMMH whenever possible, because it may be easier to obtain better proposals within a PG framework.

### 4.2 Factor stochastic volatility model

#### 4.2.1 Simulation Study

We conducted a simulation study to compare several estimation approaches: PG with ancestral tracing (PGAT), PG with backward simulation (PGBS), PMMH+PG, and PMMH with exact and approximate transition densities for the Gaussian OU process. We simulated data with \(T = 1,000\), \(S = 20\), and \(K = 1\) from the factor model in equation (8). We set \(\alpha_s = 0.06\) and \(\tau^2_s = 0.1\) for all \(s\), \(\phi_1 = 0.98\), \(\tau^2_{f1} = 0.1\) and \(\beta_s = 0.8\) for all \(s\). For every unrestricted element of the factor loading matrix \(\beta\), we chose independent Gaussian distributions, i.e. \(\beta_{sk} \sim N(0, 1)\), and the priors for the state transition density parameters are \(\alpha_s \sim IG\left(\frac{v_0}{2}, \frac{\alpha_0}{2}\right)\), \(\tau^2_s \sim IG\left(\frac{v_0}{2}, \frac{\alpha_0}{2}\right)\), \(\tau^2_{fk} \sim IG\left(\frac{v_0}{2}, \frac{\alpha_0}{2}\right)\), where \(v_0 = 10\) and \(s_0 = 1\), and \(\phi_k \sim U(-1, 1)\). These prior densities cover most possible values in practice. The initial state of \(\lambda_{kt}\) is normally distributed \(N\left(0, \frac{\tau^2_{fk}}{1-\phi_k^2}\right)\) for \(k = 1, ..., K\). The initial state of \(h_{st}\) is also normally distributed \(N\left(\mu_s, \frac{\tau^2_s}{2\sigma_s^2}\right)\) for \(s = 1, ..., S\). We ran all the sampling schemes for 11,000 iterations and discarded the initial 1,000 iterates as warmup. We used \(M = 10\) latent points for the Euler approximations to the state transition densities.
Gaussian OU process with exact transition density

The PMMH method uses the observation density in equation (28), which includes all \((K + S)\) dimensional latent log-volatilities simultaneously. This becomes a high dimensional (20 dimensional) state space model. The performance of the standard PMMH sampler depends critically on the number of particles \(N\) used to estimate the likelihood. Pitt et al. [2012] suggest selecting the number of particles \(N\) such that the variance of the log of the estimated likelihood is around 1 to obtain an optimal tradeoff between computing time and statistical efficiency. Table 1 gives the variance of the log of estimated likelihood for different number of particles for the PMMH method using the bootstrap filter and shows that even with 5,000 particles, the log of the estimated likelihood still has a large variance and the Markov chain for the PMMH approach could get stuck. We therefore do not report results for the PMMH method as it is computationally very expensive and its TNV would be significantly higher than the PG and PMMH+PG methods. The correlated PMMH proposed by Deligiannidis et al. [2018] correlates the random vectors \(u\) and \(u'\), used to construct the estimators of the likelihood at the current and proposed values of the parameters \((\theta\) and \(\theta')\) to reduce the variance of the difference between \(\log \left( Z_{1:T} (\theta', u') \right) - \log \left( Z_{1:T} (\theta, u) \right) \) appearing in the PMMH acceptance ratio. We set the correlation between the individual elements of \(u\) and \(u'\) to \(corr(u_i, u'_i) = 0.999999\). We then obtained 1,000 independent estimates of \(\log \left( Z_{1:T} (\theta, u) \right)\) and computed their sample correlation. The sample correlation was 0.96, showing that it is difficult to preserve the correlation in such a high dimensional state space model and that the correlated PMMH Markov chain will still get stuck unless enough particles are used to ensure that the variance of the log of the estimator of the likelihood is close to 1.

A second problem with the PMMH approach is the large number of parameters to be estimated. Constructing proposals in high dimensions is remarkably difficult, and often requires estimating gradients and Hessian matrices. On the other hand, simpler approaches such as the adaptive random walk are very inefficient in large dimensions. Hence, it is natural to use a parameter splitting strategy and hybrid samplers.

Table S3 in Section S5 of the supplement shows the IACT values for the parameters in the factor SV model estimated using three different samplers using the exact transition density, PMMH \((\alpha, \tau^2_e, \tau^2_f) + \text{PG} (\beta, \mu, \phi)\), PGAT \((\beta, \alpha, \tau^2_e, \tau^2_f, \phi)\) and PGBS \((\beta, \alpha, \tau^2_e, \tau^2_f, \phi)\). All three samplers estimate the factor loading matrix \(\beta\) and \(\mu\) with comparable IACT values. The PMMH+PG sampler always has lower IACT values than both PG samplers for parameters \(\alpha, \tau^2_e, \tau^2_f, \phi\). There are some improvements in terms of IACT obtained by using PGBS compared to PGAT. Table 2 summarises the estimation results using the exact transition density and shows that in terms of TNV\(_{\text{MAX}}\), the PMMH+PG sampler is 9.25 and 4.19 times better than PGAT and PGBS, respectively, and in terms of TNV\(_{\text{MEAN}}\), the PMMH+PG is 2.69 and 2.55 times better than PGAT and PGBS, respectively.

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Gaussian OU process with Euler evolution transition density

Table S4 in Section S5 of the supplement shows the IACT values for all the parameters in the model for the three samplers using the Euler approximation scheme for the transition density. The table shows that the PMMH+PG samplers with the exact and approximate state transition densities have very similar IACT values for all the parameters suggesting that the inefficiency of the PMMH+PG sampler does not deteriorate when the Euler approximation is used. However, both PG samplers, PGAT and PGBS, using the Euler approximation are significantly worse than the PGAT and PGBS samplers with the exact transition density. For example, the IACT of $\tau^2$ in PGAT with the exact transition density is 283.23, compared to 977.93 for PGAT with the Euler approximation.

Table 3 summarises the estimation results with the Euler approximation of the transition density and shows that in terms of TNV$_{\text{MAX}}$, the PMMH+PG sampler is 60.57 and 50.72 times better than PGAT and PGBS, respectively, and in terms of TNV$_{\text{MEAN}}$, the PMMH+PG sampler is 14.67 and 12.95 times better than the PGAT and PGBS samplers, respectively. It is also interesting to see that when we use an Euler approximation for the diffusion then all three samplers PMMH+PG, PGAT, and PGBT take approximately the same computing time because for the PG samplers we need to store and trace back all the latent log-volatilities $h_{st}$ and the $M$ latent data points between $t$ and $t+1$ for all $s = 1, ..., S$ and $t = 1, ..., T$, whereas the PMMH+PG sampler only needs to store and trace back the latent log-volatilities $h_{st}$ for all $s = 1, ..., S$ and $t = 1, ..., T$. Therefore, the PMMH+PG sampler is also more efficient in terms of memory usage if it is necessary to use an Euler approximation.

4.2.2 Application to US stock returns

We now apply our methods to a sample of daily US industry stock returns data. The data, obtained from the Kenneth French website$^1$ consists of daily returns for $S = 20$ value-weighted industry portfolios, using a sample from January 3rd, 2001 to the 24th of December, 2003, a total of 1,000 observations. We compare several estimation approaches: PGAT, PGBS, PMMH+PG for the Gaussian OU model with the exact and approximate transition densities, as well as for the GARCH diffusion model with approximate transition densities.

Tables S5, S6, and S7 in Section S6 of the supplement show the IACT values for all the parameters in the factor model SV estimated with the exact transition density for the Gaussian OU model and the Euler approximation for the transition density for the Gaussian OU and GARCH diffusion models. As for the simulated data, all three samplers estimate the factor loading matrix $\beta$ and $\mu$ efficiently and with comparable IACT values. The performance of the PMMH+PG sampler does not deteriorate for the real data, whereas both PGAT and PGBS samplers get worse in terms of the IACT values of the parameters, especially with the Euler approximation. Overall, the PMMH+PG samplers always have smaller IACT values than both PGAT and PGBS samplers for all the state transition parameters.

$^1$ http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/datalibrary.html
Tables 4 and 5 summarise the estimation results for the Gaussian OU model and show that in terms of TNV_{MAX}, the PMMH+PG samplers is 20.87 and 13.91 times better than PGAT and PGBS samplers with the exact transition density, respectively, and the PMMH+PG sampler is 53.94 and 58.71 times, respectively, better than the PGAT and PGBS with the Euler approximation. In terms of TNV_{MEAN}, the PMMH+PG samplers is 5.61 and 4.73 times better than PGAT and PGBS samplers with the exact transition density, respectively, and the PMMH+PG sampler is 22.17 and 22.40 times, respectively, better than PGAT and PGBS samplers when using the Euler approximation.

Figures S1 and S2 in Section S6 of the supplement present the kernel density estimates of marginal posterior densities of four representative $\alpha$ and $\tau^2$ parameters, respectively, for the US stock returns data. The density estimates are for PMMH+PG using exact and approximate transition densities and PG with approximate transition densities using ancestral tracing and backward simulation for the Gaussian OU model. The figures show that both PMMH+PG samplers produce estimates that are close to each other, whereas the PG samplers are much less reliable and suggest that the PG estimators did not converge.

Table 6 summarises the estimation results for the GARCH diffusion model and show that in terms of TNV_{MAX}, the PMMH+PG is 19.56 and 22.11 times better than PGAT and PGBS samplers. In terms of TNV_{MEAN}, the PMMH+PG is 25.84 and 28.01 times better than PGAT and PGBS, respectively. This confirms the usefulness of the PMMH+PG samplers for this class of the model.

Table 1: The Variance of the log of the estimated likelihood for the PMMH method with the exact transition density for different numbers of particles for the simulated dataset with $T = 1,000$, $S = 20$, and $K = 1$ evaluated at the true values of the parameters. CPU time is the time in seconds to estimate the likelihood.

| Number of Particles | Variance of log-likelihood | CPU time |
|---------------------|-----------------------------|----------|
| 250                 | 1672.07                     | 4.39     |
| 500                 | 766.38                      | 8.57     |
| 2500                | 331.65                      | 45.03    |
| 5000                | 243.82                      | 130.53   |
Table 2: Comparing different samplers in terms of Time Normalised Variance with the exact transition density for the Gaussian OU model: Sampler I: PMMH $\left(\alpha, \tau^2_e, \tau^2_f\right) + \text{PG} \left(\beta, \mu, \phi\right)$, Sampler II: PGAT $\left(\beta, \alpha, \tau^2_e, \mu, \phi, \tau^2_f\right)$, sampler III: PGBS $\left(\beta, \alpha, \tau^2_e, \mu, \phi, \tau^2_f\right)$ for simulated data with $T = 1000$, $S = 20$, and $K = 1$, and number of particles $N = 500$. Time denotes the time taken in seconds for one iteration of the method.

|       | I  | II | III |
|-------|----|----|-----|
| $\hat{I}ACT_{\text{MAX}}$ | 18.07 | 283.23 | 101.64 |
| $TNV_{\text{MAX}}$ | 33.97 | 314.39 | 142.30 |
| $RTNV_{\text{MAX}}$ | 1 | 9.25 | 4.19 |
| $\hat{I}ACT_{\text{MEAN}}$ | 8.54 | 38.96 | 29.26 |
| $TNV_{\text{MEAN}}$ | 16.06 | 43.25 | 40.96 |
| $RTNV_{\text{MEAN}}$ | 1 | 2.69 | 2.55 |
| Time | 1.88 | 1.11 | 1.40 |

Table 3: Comparing different samplers in terms of Time Normalised Variance with the Euler approximation for the state transition density for the Gaussian OU model: Sampler I: PMMH $\left(\alpha, \tau^2_e, \mu, \tau^2_f\right) + \text{PG} \left(\beta, \phi\right)$, Sampler II: PGAT $\left(\beta, \alpha, \tau^2_e, \mu, \phi, \tau^2_f\right)$, sampler III: PGBS $\left(\beta, \alpha, \tau^2_e, \mu, \phi, \tau^2_f\right)$ for the simulated data with $T = 1,000$, $S = 20$, and $K = 1$, and number of particles $N = 1,000$. Time denotes the time taken in seconds for one iteration of the method.

|       | I  | II | III |
|-------|----|----|-----|
| $\hat{I}ACT_{\text{MAX}}$ | 17.57 | 977.93 | 792.88 |
| $TNV_{\text{MAX}}$ | 113.50 | 6874.85 | 5756.31 |
| $RTNV_{\text{MAX}}$ | 1 | 60.57 | 50.72 |
| $\hat{I}ACT_{\text{MEAN}}$ | 14.17 | 191.04 | 163.26 |
| $TNV_{\text{MEAN}}$ | 91.54 | 1343.01 | 1185.27 |
| $RTNV_{\text{MEAN}}$ | 1 | 14.67 | 12.95 |
| Time | 6.46 | 7.03 | 7.26 |
Table 4: Comparing different samplers in terms of Time Normalised Variance with the exact transition density for the Gaussian OU model: Sampler I: PMMH \((\alpha, \tau^2, \tau_f) + \text{PG} (\beta, \mu, \phi)\), Sampler II: PGAT \((\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)\), sampler III: PGBS \((\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)\) for US stock returns data with \(T = 1,000\), \(S = 20\), and \(K = 1\), and number of particles \(N = 500\). Time denotes the time taken in seconds for one iteration of the method.

|       | I     | II   | III  |
|-------|-------|------|------|
| IACT\(_{\text{MAX}}\) | 20.57 | 682.49 | 382.86 |
| TNV\(_{\text{max}}\) | 38.26 | 798.51 | 532.18 |
| RTNV\(_{\text{max}}\) | 1     | 20.87 | 13.91 |
| IACT\(_{\text{MEAN}}\) | 8.54  | 76.19 | 54.06 |
| TNV\(_{\text{MEAN}}\) | 15.88 | 89.14 | 75.14 |
| RTNV\(_{\text{MEAN}}\) | 1     | 5.61  | 4.73  |
| Time  | 1.86  | 1.17  | 1.39  |

Table 5: Comparing different samplers in terms of Time Normalised Variance with the Euler approximation for state transition density for the Gaussian OU model: Sampler I: PMMH \((\alpha, \tau^2, \mu, \tau_f) + \text{PG} (\beta, \phi)\), Sampler II: PGAT \((\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)\), sampler III: PGBS \((\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)\) with backward simulation for US stock returns data with \(T = 1,000\), \(S = 20\), and \(K = 1\), and number of particles \(N = 1,000\). Time denotes the time taken in seconds for one iteration of the method.

|       | I     | II   | III  |
|-------|-------|------|------|
| IACT\(_{\text{MAX}}\) | 23.99 | 1215.77 | 1228.99 |
| TNV\(_{\text{max}}\) | 152.82 | 8242.92 | 8971.63 |
| RTNV\(_{\text{max}}\) | 1     | 53.94  | 58.71  |
| IACT\(_{\text{MEAN}}\) | 12.99 | 270.58 | 253.90 |
| TNV\(_{\text{MEAN}}\) | 82.75 | 1834.53 | 1853.47 |
| RTNV\(_{\text{MEAN}}\) | 1     | 22.17  | 22.40  |
| Time  | 6.37  | 6.78  | 7.30   |
Table 6: Comparing different samplers in terms of Time Normalised Variance with the Euler approximation for the state transition density for the GARCH diffusion model. Sampler I: PMMH ($\alpha, \tau_v^2, \mu, \tau_f^2$) + PG ($\beta, \phi$), Sampler II: PGAT ($\beta, \alpha, \tau_v^2, \mu, \phi, \tau_f^2$), Sampler III: PGBS ($\beta, \alpha, \tau_v^2, \mu, \phi, \tau_f^2$) for US stock returns data with $T = 1000$, $S = 20$, and $K = 1$, and number of particles $N = 1000$. Time denotes the time taken in seconds for one iteration of the method.

|          | I       | II      | III     |
|----------|---------|---------|---------|
| IACT_{MAX} | 147.16  | 3098.27 | 3257.52 |
| TNV_{MAX}  | 1392.13 | 27233.79| 30783.56|
| RTNV_{MAX} | 1       | 19.56   | 22.11   |
| IACT_{MEAN}| 17.38   | 483.37  | 487.28  |
| TNV_{MEAN} | 164.41  | 4248.82 | 4604.80 |
| RTNV_{MEAN}| 1       | 25.84   | 28.01   |
| Time       | 9.46    | 8.79    | 9.45    |

5 Discussion

Our article introduces a flexible particle Markov chain Monte Carlo sampling scheme for state space models where some parameters are generated without conditioning on the states (PMMH) while other parameters are generated conditional on the states (PG). Previous schemes generated exclusively using PMMH or PG without combining both strategies. The technical contribution of our article is to set out the required particle framework for the flexible sampler and to obtain uniform ergodicity under given assumptions. Our examples demonstrate that it is advantageous to use this flexible sampling scheme to generate the parameters that are highly correlated with the states without conditioning on the states (the PMMH component) while other parameters are generated by particle Gibbs (PG).

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A Further details of the sampling scheme for the factor stochastic volatility model

A.1 Sampling the factor loading matrix $\beta$

First, to identify the parameters for the factor loading matrix $\beta$, we follow the usual convention and set the upper triangular part of $\beta$ to zero (Geweke and Zhou 1996). This parameterisation imposes an order dependence. Second, the model is also not identified without further constraining either the scale of the $k$th column of $\beta$ or the variance of $f_{kt}$. The usual solution is to set the diagonal elements of the factor loading matrix $\beta_{kk}$ to one, for $k = 1, \ldots, K$, while the level $\mu_{2k,t}$ of the factor volatility $\lambda_{k,t}$ is modeled to be unknown. However, Kastner et al. [2017] note that this approach makes the variable ordering dependence stronger. We follow Kastner et al. [2017] and leave the diagonal elements $\beta_{kk}$ unrestricted and set the level $\mu_{2k}$ of the factor volatility $\lambda_{k,t}$ to zero for $k = 1, \ldots, K$.

Let $k_s$ denote the number of unrestricted elements in row $s$ of $\beta$ and define

$$F_s = \begin{bmatrix} f_{11} & \cdots & f_{k,s} \\ \vdots & \ddots & \vdots \\ f_{1T} & \cdots & f_{k,sT} \end{bmatrix}, \quad \text{and} \quad \tilde{V}_s = \begin{bmatrix} \exp(h_{s,1}) & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \exp(h_{s,T}) \end{bmatrix}.$$  

We sample the factor loadings $\beta_{s,.} = (\beta_{s1}, \ldots, \beta_{sk_s})^T$, for $s = 1, \ldots, S$, independently for each $s$ using the Gibbs-update

$$\beta_{s,.} | f, y_{s,.}, h_{s,.} \sim N_k (a_{sT}, b_{sT}), \quad (31)$$

where $b_{sT} = (F_s^T \tilde{V}_s^{-1} F_s + I_{k_s})^{-1}$ and $a_{sT} = b_{sT} F_s^T \tilde{V}_p F_{s,.}$.

A.2 Deep Interweaving

To improve the mixing in the draws of the factor loading matrix we employ the following deep interweaving strategy introduced by Kastner et al. [2017].

- Determine the vector $\beta^*_{k,.}$, where $\beta^*_{sk} = \beta_{sk}^{\text{old}} / \beta_{kk}^{\text{old}}$ in the $k$th column of the transformed factor loading matrix $\beta^*$.

- Define $\lambda^{*,}_{k,.} = \lambda^{\text{old}}_{k,.} + 2 \log |\beta^{\text{old}}_{kk}|$ and sample $\beta^{\text{new}}_{kk}$ from $p (\beta_{kk} | \beta^*_{k,.}, \lambda^{*,}_{k,.}, \phi_k, \tau_{f,k}^2)$.

- Update $\beta_{k,.} = \frac{\beta^{\text{new}}_{kk}}{\beta_{kk}} \beta^{\text{old}}_{k,.} f_{k,.} = \frac{\beta^{\text{old}}_{kk}}{\beta_{kk}} f^{\text{old}}_{k,.}$, and $\lambda^{*,}_{k,.} = \lambda^{\text{old}}_{k,.} + 2 \log |\beta^{\text{old}}_{kk}/\beta^{\text{old}}_{kk}|$.

In the deep interweaving representation the scaling parameter $\beta_{kk}$ is sampled indirectly through $\mu_k = \log \beta^2_{kk}$, $k = 1, \ldots, K$. The implied prior $p(\mu_k) \propto \exp (\mu_k / 2 - \exp (\mu_k) / 2)$ and the density $p (\beta^{*,}_{k,.} | \mu_k) \sim N_{k_1} (0, \exp (-\mu_k)) I_{k_1}$ and the likelihood yields the posterior

$$p (\mu_k | \beta^{*,}_{k,.}, \lambda^{*,}_{k,.}, \phi_k, \tau_{f,k}^2) \propto p (\lambda^{*,}_{k,.} | \mu_k, \phi_k, \tau_{f,k}^2) p (\beta^{*,}_{k,.} | \mu_k) p (\mu_k),$$
which is not in recognisable form. We draw a proposal for \( \mu_k^{\text{prop}} \) from \( N(A, B) \) where

\[
A = \frac{\sum_{t=2}^{T-1} \lambda_{k,t}^* + (\lambda_{k,T}^* - \phi_k \lambda_{k,1}^*)}{T - 1 + 1/B_0}, \quad B = \frac{\tau_{j,k}^2 / (1 - \phi_k)^2}{T - 1 + 1/B_0}.
\]

Denoting the current value \( \mu_k \) by \( \mu_k^{\text{old}} \), the new value \( \mu_k^{\text{prop}} \) gets accepted with probability

\[
\min (1, R),
\]

where

\[
R = \frac{p(\mu_k^{\text{prop}}) p(\lambda_{k,1}^* | \mu_k^{\text{prop}}, \phi_k, \tau_{j,k}^2) p(\beta_{\ast,k}^* | \mu_k^{\text{prop}}) p(\beta_{\ast,k}^* | \mu_k^{\text{old}}) p(\beta_{\ast,k}^* | \mu_k^{\text{old}})}{p(\mu_k^{\text{old}}) p(\lambda_{k,1}^* | \mu_k^{\text{old}}, \phi_k, \tau_{j,k}^2) p(\beta_{\ast,k}^* | \mu_k^{\text{old}}) p(\beta_{\ast,k}^* | \mu_k^{\text{old}})},
\]

where

\[
p_{\text{aux}}(\mu_k^{\text{old}} | \phi_k, \tau_{j,k}^2) \sim N(0, B_0 \tau_{j,k}^2 / (1 - \phi_k)^2).
\]

The constant \( B_0 \) is set to large value \( 10^5 \) as in Kastner et al. [2017].

**A.3 Sampling the Latent Factors \( f_{1:T} \)**

We obtain after some algebra that

\[
\{f_t\} \mid y, \{h_t\}, \{\lambda_t\}, \beta \sim N(a_t, b_t),
\]

where \( b_t = (\beta^T V_t^{-1} \beta + D_t^{-1})^{-1} \) and \( a_t = b_t \beta^T V_t^{-1} y_t \).

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Online Supplement for “Flexible Particle Markov chain
Monte Carlo methods with an application to a factor
stochastic volatility model”

We use the following notation in the supplement. Equation (1), Algorithm 1, and Sampling
Scheme 1, etc, refer to the main paper, while equation (S1), Algorithm S1, and Sampling
Scheme S1, etc, refer to the supplement. Section [S1] lists some of the algorithms used in
the main paper. These algorithms are used in [Andrieu et al. 2010] and are included here
for notational consistency. Section [S2] discusses the convergence of Sampling Scheme 1 to
its target distribution. Section [S3] discusses other choices of target distribution and how it
is straightforward to modify the results in the main paper to apply to these distributions.
Section [S4] illustrate the flexibility of the methods by applying them to a univariate OU
model for daily stock return data. Section [S5] presents some additional tables and plots
based on the analysis reported in Sections 4.2.1 and 4.2.2.

S1  Algorithms

The Sequential Monte Carlo algorithm used here is the same one as in [Andrieu et al. 2010]
and is defined as follows.

Algorithm S1 (Sequential Monte Carlo)

1. For $t = 1$:
   
   (a) Sample $X_1^i$ from $m_1^\theta(x)$, for $i = 1, \ldots, N$
   
   (b) Calculate the importance weights
   
   $$w_1^i = \frac{f_1^\theta(x_1^i) g_0(y_1|x_1^i)}{m_1^\theta(x_1^i)} \quad (i = 1, \ldots, N),$$

   and normalize them to obtain $\bar{w}_1^{1:N}$.

2. For $t = 2, 3, \ldots$:

   (a) Sample the ancestral indices $A_t^{1:N} \sim \mathcal{M}(a^{1:N} | \bar{w}_t^{1:N})$

   (b) Sample $X_t^i$ from $m_t^\theta(x|x_{t-1}^{a_{t-1}})$, $i = 1, \ldots, N$

   (c) Calculate the importance weights

   $$w_t^i = \frac{f_\theta(x_t^i|x_{t-1}^{a_{t-1}}) g_\theta(y_t|x_t^i)}{m_t^\theta(x_t^i|x_{t-1}^{a_{t-1}})} \quad (i = 1, \ldots, N)$$

   and normalize them to obtain $\bar{w}_t^{1:N} = w_t^{1:N} / \sum_{i=1}^N w_t^i$.

S1
Algorithm S2 is the conditional sequential Monte Carlo algorithm (as in Andrieu et al. [2010]), consistent with \((x_{i:T}, a_{1:T-1}, j)\).

**Algorithm S2 (Conditional Sequential Monte Carlo)**

1. Fix \(X_{j:1:T} = x_{j:1:T}\) and \(A_{1:T-1}^j = b_{1:T-1}^j\).

2. For \(t = 1\)
   - (a) Sample \(X_1^i\) from \(m_1^\theta(x) \, dx\), for \(i \in \{1, \ldots, N\} \setminus \{b_1^j\}\).
   - (b) Calculate the importance weights
     \[ w_1^i = \frac{f_1^\theta(x_1^i) \, g_\theta(y_1|x_1^i)}{m_1^\theta(x_1^i)} \quad (i = 1, \ldots, N), \]
     and normalize them to obtain \(\bar{w}_{1:N}^1\).

3. For \(t = 2, \ldots, T\)
   - (a) Sample the ancestral indices
     \[ A_{t-1}^{(-b_t^j)} \sim \mathcal{M}(a^{(-b_t^j)} | \bar{w}_{t-1}^{1:N}). \]
   - (b) Sample \(X_t^i\) from \(m_t^\theta \left( x_t | x_{t-1} \right) \, dx\), \(i \in \{1, \ldots, N\} \setminus \{b_t^j\}\).
   - (c) Calculate the importance weights
     \[ w_t^i = \frac{f_t^\theta \left( x_t^i | x_{t-1}^{a_{t-1}^i} \right) \, g_\theta(y_t|x_t^i)}{m_t^\theta \left( x_t^i | x_{t-1}^{a_{t-1}^i} \right)} \quad (i = 1, \ldots, N) \]
     and normalized them to obtain \(\bar{w}_{t}^{1:N}\).

**S2 Ergodicity**

This section first discusses the assumptions required for the particle filter. We then discuss convergence of Sampling Scheme 1 in total variation norm and then consider the stronger condition of uniform convergence.

For \(t \geq 1\), we define,
\[ S_t^\theta = \{ x_{1:t} \in \chi^t : \pi(x_{1:t} | \theta) > 0 \} \quad \text{and} \quad Q_t^\theta = \{ x_{1:t-1} \in \chi^{t-1} : \pi(x_{1:t-1} | \theta) m_t^\theta(x_t | x_{1:t-1}, y_{1:t}) > 0 \}. \]

Assumption [S1] ensures that the proposal densities \(\pi(x_{1:t} | \theta) m_t^\theta(x_t | x_{1:t-1}, y_{1:t})\) can be used to approximate \(\pi(x_{1:t} | \theta)\) for \(t \geq 1\).
Assumption S1 [Andrieu et al., 2010] We assume that $S^\theta_t \subseteq Q^\theta_t$ for any $\theta \in \Theta$ and $t = 1,...,T$

Assumption S1 is always satisfied in our implementation because we use the bootstrap filter with $p(\mathbf{x}_t|\mathbf{x}_{t-1}, \theta)$ as a proposal density which are positive everywhere.

We also require Assumption S2 given below.

Assumption S2 [Andrieu et al., 2010] For any $k = 1,...,N$ and $t = 1,...,T$, the resampling scheme $M(a^{k}_1|\bar{w}^1_{t-1})$ satisfies $M(a^{k}_t = j|\bar{w}^1_{t-1}) = \bar{w}^j_{t-1}$.

Assumption S2 is satisfied by the popular resampling schemes, such as multinomial, systematic, residual resampling.

Under Assumption S2 it is straightforward to show that the algorithm samples from the target density of the random variable $U^{(-J)}_{1:T} = \left(X^{(-B^1_1)},...,X^{(-B^T_1)},A^{(-B^1_2)},...,A^{(-B^T_1)}_{T-1}\right)$, conditional on $U_{1:T}$ and index $J$ given by

$$\tilde{\pi}^N \left(u^{(-j)}_{1:T}|x_{1:T},b_{1:T-1},j,\theta\right) = \frac{\psi \left(u_{1:T}|\theta\right) \prod_{t=1}^{T} \bar{w}^{a^t_{t-1}}_{t-1} m^\theta(x^b_t|x^{a^t_{t-1}}_{t-1})}{m^\theta(x^{b_1})} ;$$

see Andrieu et al. 2010 for details.

We now discuss convergence of Sampling Scheme 1 in total variation norm and then consider the stronger condition of uniform convergence. Note that, by construction, Sampling Scheme 1 has the stationary distribution $\tilde{\pi}^N \left(x_{1:T},b_{1:T-1},j,u^{(-j)}_{1:T},\theta\right)$ defined in (2). From Roberts and Rosenthal 2004 Theorem 4, irreducibility and aperiodicity are sufficient conditions for the Markov chain obtained using Sampling Scheme 1 to converge to its stationary distribution in total variation norm for $\tilde{\pi}^N$-almost all starting values. These conditions must be checked for a particular sampler and it is often straightforward to do so. We will relate Sampling Scheme 1 to the particle Metropolis within Gibbs sampling scheme defined below.

Sampling Scheme S3 (Ideal) Given initial values for $U_{1:T}$, J and $\theta$, one iteration of the MCMC sampling scheme involves the following steps

1. (PMMH sampling) For $i = 1,...,p_1$

   Step i:

   (a) Sample $\theta_i^* \sim q_i(\cdot|U_{1:T},J,\theta_{-i},\theta_i)$.
   (b) Sample $(J^*,U^*_1:T) \sim \tilde{\pi}^N (\cdot|\theta_{-i},\theta_i^*)$. 

S3
(c) Set \((\theta_i, U_{1:T}, J) \leftarrow (\theta^*_i, U^*_{1:T}, J^*)\) with probability

\[
\tilde{\alpha}_i(\theta_{1:T}, J, \theta_i; U_{1:T}, J^*, \theta^*_i|\theta_{-i}) = 1 \wedge \frac{\tilde{\pi}^N(\theta_i^*|\theta_{-i})}{\tilde{\pi}^N(\theta_i|\theta_{-i})} \frac{q_{i,1}(\theta_i^*|U_{1:T}, J, \theta_{-i}, \theta_i)}{q_i(\theta_i|X_{1:T}, B_{1:T-1}, J, \theta_{-i}, \theta_i)}
\]

(S1)

2. (PG sampling) For \(i = p_1 + 1, \ldots, p\)

Step \(i:\)

(a) Sample \(\theta^*_i \sim q_i(\cdot|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}, \theta_i).\)

(b) Set \(\theta_i \leftarrow \theta^*_i\) with probability

\[
\alpha_i\{\theta_i; \theta_i^*|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}\} = 1 \wedge \frac{\tilde{\pi}^N(\theta_i^*|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i})}{\tilde{\pi}^N(\theta_i|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i})} \frac{q_i(\theta_i^*|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}, \theta_i)}{q_i(\theta_i^*|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}, \theta_i)}
\]

(S2)

3. Sample \(U_{1:T}^{(-J)} \sim \tilde{\pi}^N(\cdot|X_{1:T}^J, B_{1:T-1}^J, J, \theta)\) using Algorithm S3.

4. Sample \(J \sim \tilde{\pi}^N(\cdot|U_{1:T}, \theta)\).

We call Sampling Scheme S3 an ideal particle sampling scheme because in Part 1 Step \(i(b)\) it generates the particles \(U^*_{1:T}\) from their conditional distribution \(\tilde{\pi}^N(\cdot|\theta_{-i}, \theta^*_i)\) instead of using a Metropolis-Hastings proposal. Thus comparing Sampling Schemes 1 and S3 allows us to concentrate on the effect of the Metropolis-Hastings proposal for the particles on the convergence of the sampler.

Remark S3 Andrieu and Roberts [2009] and Andrieu and Vihola [2015] discuss the relationship between PMMH sampling schemes with one block of parameters and an ideal Metropolis-Hastings sampling scheme not involving the particles. Sampling Schemes 1 and S3 are more general. Our approach is similar to, but generalizes, the results in [Andrieu and Roberts [2009] and Andrieu and Vihola [2015]] to more complex sampling schemes.

To develop the theory of Sampling Schemes 1 and S3 we require the following definitions. Let \(\{V^{(n)}, n = 1, 2, \ldots\}\) be the iterates of a Markov chain defined on the state space \(V := \mathcal{U} \times \mathcal{N} \times \Theta\). For \(i = 1, \ldots, p\), let \(K_{i}(v; \cdot)\) be the substochastic transition kernel of the \(i\)th step of Sampling Scheme 1 that defines the probabilities for accepted Metropolis-Hastings moves and define

\[
K := K_1K_2 \ldots K_p
\]

to be the substochastic transition kernel that defines the probabilities for accepted Metropolis-Hastings moves. Note that probabilities involving the substochastic kernels provide lower bounds on the probabilities for the transition kernel of the corresponding Markov chain.
For \( i = 1, \ldots, p_1 \)

\[
K_i (U_{1:T}, J, \theta_{-i}, \theta_i; U_{1:T}', J_i', \theta_{-i}, \theta_i') =
\pi^N (J_i' | U_{1:T}', \theta_{-i}, \theta_i') q_i (U_{1:T}', \theta_i' | U_{1:T}, J, \theta_{-i}, \theta_i) \times \alpha_i (U_{1:T}, J, \theta_i; U_{1:T}', J_i', \theta_i' | \theta_{-i}).
\]

Similarly, for \( i = 1, \ldots, p \), let \( \tilde{K}_i (v; \cdot) \) be the substochastic transition kernel of the \( i \)th step of Sampling Scheme \( S_3 \) that defines the probabilities for accepted Metropolis-Hastings moves and define

\[
\tilde{K} = \tilde{K}_1 \tilde{K}_2 \ldots \tilde{K}_p,
\]

where the kernels \( K_i \) and \( \tilde{K}_i \) only differ for \( i = 1, \ldots, p_1 \).

The next theorem gives a sufficient condition for Sampling Scheme \( 1 \) to be irreducible and aperiodic and is similar to Theorem 1 of Andrieu and Roberts [2009].

**Theorem S1** If \( \tilde{K} \) is irreducible and aperiodic then \( K \) is irreducible and aperiodic. **Proof.** For \( i = 1, \ldots, p_1 \), \( \pi^N (\cdot | \theta_{-i}, \theta_i') \ll \psi (\cdot | \theta_{-i}, \theta_i') \) and the result now follows from from Assumption 1 of Andrieu et al. [2010]. \[ \blacksquare \]

We now follow the approach in Andrieu and Roberts [2009] and show the uniform ergodicity of the sampling schemes by giving sufficient conditions for the existence of minorization conditions for Sampling Scheme \( 1 \). These minorization conditions are equivalent to uniform ergodicity by Theorem 8 of Roberts and Rosenthal [2004]. The results use the following technical lemmas.

**Lemma S2** For \( i = 1, \ldots, p_1 \),

\[
\alpha_i (U_{1:T}, J, \theta_i; U_{1:T}', J_i', \theta_i' | \theta_{-i}) \geq \left\{ 1 \wedge \frac{\pi^N (U_{1:T}' | \theta_{-i}, \theta_i') \psi (U_{1:T} | \theta_{-i}, \theta_i)}{\pi^N (U_{1:T} | \theta_{-i}, \theta_i) \psi (U_{1:T}' | \theta_{-i}, \theta_i')} \right\} \times \widetilde{\alpha}_i (U_{1:T}, J, \theta_i; U_{1:T}', J_i', \theta_i' | \theta_{-i})
\]

**Proof.** From [4],

\[
\alpha_i (U_{1:T}, J, \theta_i; U_{1:T}', J_i', \theta_i' | \theta_{-i}) \]

\[
= 1 \wedge \frac{\pi^N (U_{1:T}' | \theta_{-i}, \theta_i') q_i (U_{1:T}' | \theta_{-i}, \theta_i) \psi (U_{1:T} | \theta_{-i}, \theta_i) \times \pi^N (\theta_i' | \theta_{-i}) \psi (U_{1:T} | \theta_{-i}, \theta_i) \times \pi^N (\theta_i | \theta_{-i}) \psi (U_{1:T}' | \theta_{-i}, \theta_i)}{\pi^N (U_{1:T} | \theta_{-i}, \theta_i) \psi (U_{1:T}' | \theta_{-i}, \theta_i') \times \pi^N (\theta_i' | \theta_{-i}) \psi (U_{1:T} | \theta_{-i}, \theta_i) \times \pi^N (\theta_i | \theta_{-i}) \psi (U_{1:T}' | \theta_{-i}, \theta_i)}
\]

\[
\geq \left\{ 1 \wedge \frac{\pi^N (U_{1:T}' | \theta_{-i}, \theta_i') \psi (U_{1:T} | \theta_{-i}, \theta_i) \times \pi^N (\theta_i' | \theta_{-i}) \psi (U_{1:T}' | \theta_{-i}, \theta_i)}{\pi^N (U_{1:T} | \theta_{-i}, \theta_i) \psi (U_{1:T}' | \theta_{-i}, \theta_i') \times \pi^N (\theta_i' | \theta_{-i}) \psi (U_{1:T} | \theta_{-i}, \theta_i) \times \pi^N (\theta_i | \theta_{-i}) \psi (U_{1:T}' | \theta_{-i}, \theta_i)} \right\} \times \widetilde{\alpha}_i (U_{1:T}, J, \theta_i; U_{1:T}', J_i', \theta_i' | \theta_{-i})
\]

\[ \blacksquare \]
Lemma S3 Suppose that
\[ \frac{\tilde{\pi}^N(U_{1:T}^*|\theta)}{\psi(U_{1:T}^*|\theta)} \leq \gamma < \infty \] (S3)
for all \( U_{1:T}^* \in \mathcal{U}, \theta \in \mathcal{S} \). Then, for \( i = 1, \ldots, p_1 \), each Markov transition kernel \( K_i \) satisfies
\[ K_i \geq \gamma^{-1} \tilde{K}_i \] (S4)
and hence
\[ K \geq \gamma^{-p_1} \tilde{K} \] (S5).

Proof. Fix \( i \in \{1, \ldots, p_1\} \) and let \( A \in \mathcal{B}(\mathcal{U}) \), \( J, J^* \in \{1, \ldots, N\} \) and \( B \in \mathcal{B}(\Theta_i) \). Then
\[
K_i(U_{1:T}, J, \theta_{-i}, \theta_i; A, J^*, \theta_{-i}, B)
= \int_{A \times B} \tilde{\pi}^N(J^*|U_{1:T}^*, \theta_{-i}, \theta_i^*) \alpha_i(U_{1:T}, J, \theta_i; U_{1:T}^*, J^*, \theta_{-i}^*) dU_{1:T}^* d\theta_i^*
\geq \int_{A \times B} \tilde{\pi}^N(J^*|U_{1:T}^*, \theta_{-i}, \theta_i^*) \alpha_i(U_{1:T}, J, \theta_{-i}; U_{1:T}^*, J^*, \theta_{-i}^*) dU_{1:T}^* d\theta_i^*
\geq \gamma^{-1} \int_{A \times B} \tilde{\pi}^N(U_{1:T}^*, J^*|\theta_{-i}, \theta_i^*) \alpha_i(U_{1:T}, J, \theta_{-i}; U_{1:T}^*, J^*, \theta_{-i}^*) dU_{1:T}^* d\theta_i^*
= \gamma^{-1} \tilde{K}_i(U_{1:T}, J, \theta_{-i}, \theta_i; A, J^*, \theta_{-i}, B),
\]
which proves (S4). Apply (S4) for each \( i \) to get (S5). Lemma S3 can be used to find sufficient conditions for the existence of minorization conditions for Sampling Scheme 1 as given in the theorem below, which is similar to [Andrieu and Roberts 2009], Theorem 8. Let \( \mathcal{L}_N\{V^{(n)} \in \cdot\} \) denote the sequence of distribution functions of the random variables \( \{V^{(n)} : n = 1, 2, \ldots\} \), generated by Sampling Scheme 1 and let \( |\cdot|_{TV} \) be total variation norm.

Theorem S4 Suppose that Sampling Scheme 5 satisfies the following minorization condition: there exists a constant \( \epsilon > 0 \), a number \( n_0 \geq 1 \), and a probability measure \( \nu \) on \( \mathcal{V} \) such that \( \tilde{K}^{n_0}(v; A) \geq \epsilon \nu(A) \) for all \( v \in \mathcal{V}, A \in \mathcal{B}(\mathcal{V}) \). Suppose also that the conditions of Lemma S3 are satisfied. Then Sampling Scheme 1 satisfies the minorization condition
\[ K^{n_0}(v; A) \geq \gamma^{-p_1 n_0} \epsilon \nu(A) \]
and for all starting values for the Markov Chain
\[ |\mathcal{L}_N\{V^{(n)} \in \cdot\} - \tilde{\pi}^N\{V^{(n)} \in \cdot\}|_{TV} \leq (1 - \delta)^{|n/n_0|}, \]
where \( 0 < \delta < 1 \) and \( |n/n_0| \) is the greatest integer not exceeding \( n/n_0 \).
Proof. To show the first part, suppose \( \tilde{K}^n(v;A) \geq \epsilon \nu(A) \) for all \( v \in \mathcal{V}, A \in \mathcal{B}(\mathcal{V}) \). Fix \( v \in \mathcal{V}, A \in \mathcal{B}(\mathcal{V}) \). Applying Lemma S3 repeatedly gives

\[
K^n(v;A) \geq \gamma^{-p_1n} \tilde{K}^n(v;A) \geq \gamma^{-p_1n} \epsilon \nu(A)
\]
as required. The second part follows from the first part and Roberts and Rosenthal [2004], Theorem 8. ■

Lemma S5 gives sufficient conditions for Lemma S3 to hold. The first condition is from Andrieu et al. [2010].

**Lemma S5** Suppose

(i) There is a sequence of finite, positive constants \( \{c_t : t = 1, \ldots, T\} \) such that for any \( x_t : \tau \in \mathcal{S}(\theta) \) and all \( \theta \in \mathcal{S} \), \( f_\theta(x_t|x_{t-1}) g_\theta(y_t|x_t) \leq c_t m_\theta(x_t|x_{t-1}) \).

(ii) There exists an \( \epsilon > 0 \) such that for all \( \theta \in \mathcal{S} \), \( p(y_{1:T}|\theta) > \epsilon \).

If (i) and (ii) hold, then the conditions in Lemma S3 are satisfied.

**Proof.** Part (i) implies that for all \( \theta \in \mathcal{S} \) and all \( U_{1:T} \in \mathcal{U} \), \( Z(U_{1:T},\theta) \leq \prod_{t=1}^T c_t \). Hence Part (ii) implies that

\[
\frac{Z(U_{1:T},\theta)}{p(y_{1:T}|\theta)} < \frac{\prod_{t=1}^T c_t}{\epsilon}.
\]

From (6),

\[
\frac{\tilde{\pi}^N(U^*_T|\theta)}{\psi(U^*_T|\theta)} = \frac{Z(U_{1:T},\theta)}{p(y_{1:T}|\theta)}
\]
giving the result. ■

**Remark S4** The results above can be modified for the factor stochastic volatility model given in Section S3 in a straightforward way. Details are available from the authors on request.

**Remark S5** If the states are sampled using backward simulation, similar arguments can be applied to obtain corresponding results (see Section S3). The mathematical details of the derivation use the results in Olsson and Ryden [2011] and Lindsten and Schön [2012a].

**S3 Backward simulation**

Godsill et al. [2004] introduce the **backward simulation** algorithm which samples the indices \( J_T, J_{T-1}, \ldots, J_1 \) sequentially, and differs from ancestral tracing which samples one index \( J \) and traces back its ancestral lineage. The **backward simulation** algorithm (Algorithm S3 below) is used in the PMCMC setting by Olsson and Ryden [2011] (in the PMMH algorithm)
and Lindsten and Schön [2012a] (in the PG algorithm). Chopin and Singh [2015] studied the PG algorithm with backward simulation and found that it yields a smaller autocorrelation than the corresponding algorithm using ancestral tracing. Moreover, it is more robust to the resampling scheme (multinomial resampling, systematic resampling, residual resampling or stratified resampling) used in the resampling step of the algorithm.

**Algorithm S3 (Backward Simulation)**

1. Sample \( J_T = j_t \) conditional on \( u_{1:T} \), with probability proportional to \( w_{j_t}^{T} \), and choose \( x_{j_t}^{T} \);

2. For \( t = T - 1, \ldots, 1 \), sample \( J_t = j_t \) conditional on 
\[ (u_{1:t}, j_{t+1:T}, x_{t+1}^{j_{t+1}}, \ldots, x_{T}^{j_{T}}), \]
with probability proportional to \( w_t^{j_t} \psi_\theta(x_{t+1}^{j_{t+1}}|x_{t}^{j_t}) \), and choose \( x_{t}^{j_t} \).

We denote the particles selected and the trajectory selected by
\[ x_{1:T}^{(-j_1:T)} = \{ x_1^{(-j_1)}, \ldots, x_T^{(-j_T)} \}. \]
It will simplify the notation to sometimes use the following one-to-one transformation
\[ (u_{1:T}, j_{1:T}) \leftrightarrow \{ x_1^{j_1:T}, j_{1:T}, x_{1:T}^{(-j_1:T)}, a_{1:T-1} \}, \]
and switch between the two representations and use whichever is more convenient.

The augmented space in this case consists of the particle filter variables \( U_{1:T} \) and the sampled trajectory \( J_{1:T} \) and PMCMC methods using backward simulation target the following density

\[ \tilde{\pi}_{BSi}^N \left( x_{1:T}, j_{1:T}, x_{1:T}^{(-j_1:T)}, a_{1:T-1}, \theta \right) := \frac{p(x_{1:T}, \theta|y_{1:T}) \psi(u_{1:T}|\theta)}{N^T m_1^\theta(x_1^{j_1}) \prod_{t=2}^T \bar{w}_{t-1}^{a_{t-1}^i} m_t^\theta(x_t^{a_t}|x_{t-1}^{a_{t-1}}) \times} \]
\[ \prod_{t=2}^T \frac{w_t^{j_t} f(x_{t+1}^{j_{t+1}}|x_t^{j_t})}{\sum_{i=1}^N w_t^{a_t^i} f(x_{t}^{a_t^i}|x_{t-1}^{a_{t-1}^i})}. \]

(S6)

Olsson and Ryden [2011] show that, under Assumption 2 of Andrieu et al. [2010],
\[ \tilde{\pi}_{BSi}^N \left( x_{1:T}, j_{1:T}, x_{1:T}^{(-j_1:T)}, a_{1:T-1}, \theta \right) \]
has the following marginal distribution
\[ \tilde{\pi}_{BSi}^N (x_{1:T}, j_{1:T}, \theta) = \frac{p(x_{1:T}, \theta|y_{1:T})}{N^T}, \]

S8
The conditional sequential Monte Carlo algorithm used in the backward simulation also changes. It is given in [Lindsten et al. 2014] and generates from the full conditional distribution

$$\tilde{\pi}^N_{BSi}(x_{1:T}, \theta) = p(x_{1:T}, \theta|y_{1:T}).$$

Sampling Scheme S4 (general-BSi)  Given initial values for $U_{1:T}$, $J_{1:T}$ and $\theta$, one iteration of the MCMC involves the following steps

1. *(PMMH sampling)* For $i = 1, \ldots, p_1$

   Step i:

   (a) Sample $\theta_i^* \sim q_{BSi,i,1}(\cdot|U_{1:T}, J_{1:T}, \theta_{-i}, \theta_i)$.

   (b) Sample $U_{i1:T}^* \sim \psi(\cdot|\theta_{-i}, \theta_i^*)$.

   (c) Sample $J_{i1:T}^*$ from $\tilde{\pi}^N_{BSi}(\cdot|U_{1:T}^*, \theta_{-i}, \theta_i^*)$.

   (d) Set $(\theta_i, U_{1:T}, J_{1:T}) \leftarrow (\theta_i^*, U_{1:T}^*, J_{1:T}^*)$ with probability

   $$(S7) \quad \alpha_i(U_{1:T}, J_{1:T}, \theta_i; U_{1:T}^*, J_{1:T}^*, \theta_i^*|\theta_{-i}) = \frac{\tilde{\pi}^N_{BSi}(U_{1:T}^*, \theta_i^*|\theta_{-i}) q_{BSi,i}(U_{1:T}, \theta_i|U_{1:T}^*, J_{1:T}^*, \theta_i^*)}{\tilde{\pi}^N_{BSi}(U_{1:T}, \theta_i|\theta_{-i}) q_{BSi,i}(U_{1:T}^*, \theta_i^*|U_{1:T}, J_{1:T}, \theta_{-i}, \theta_i^*)}$$

   where

   $$q_{BSi,i}(U_{1:T}^*, \theta_i^*|U_{1:T}, J_{1:T}, \theta_{-i}, \theta_i) = q_{BSi,i,1}(\theta_i^*|U_{1:T}, J_{1:T}, \theta_{-i}, \theta_i)\psi(U_{1:T}|\theta_{-i}, \theta_i^*).$$

2. *(PG or PMwG sampling)* For $i = p_1 + 1, \ldots, p$

   Step i:

   (a) Sample $\theta_i^* \sim q_i(\cdot|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}, \theta_i)$.

   (b) Set $\theta_i \leftarrow \theta_i^*$ with probability

   $$\alpha_i(\theta_i; \theta_i^*|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}) = \frac{\tilde{\pi}^N(\theta_i^*|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}) q_i(\theta_i|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}, \theta_i^*)}{\tilde{\pi}^N(\theta_i|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}) q_i(\theta_i^*|X_{1:T}^J, B_{1:T-1}^J, J, \theta_{-i}, \theta_i^*)}.$$
The PMMH steps in Sampling Scheme S4 simplify similarly to Sampling Scheme 1. Olsson and Ryden [2011] show that
\[ \tilde{\pi}^N_{BSi}(U_{1:T}, \theta_i|\theta_{-i}) = \frac{Z(U_{1:T}, \theta)p(\theta_i|\theta_{-i})}{p(y_{1:T}|\theta_{-i})}, \]
which is the same expression as (6). Hence, the Metropolis-Hasting acceptance probability in (S7) simplifies to
\[ 1 \land \frac{Z(\theta^*_i, \theta_{-i}, U_{1:T}^*) q_{BSi,i,1}(\theta_i|U_{1:T}^*, J^*, \theta_{-i}, \theta^*_i)p(\theta_i^*|\theta_{-i})}{Z(\theta_i, \theta_{-i}, U_{1:T}) q_{BSi,i,1}(\theta^*_i|U_{1:T}, J, \theta_{-i}, \theta_i)p(\theta_i|\theta_{-i})}. \]

The results in Section S2 can be modified for the distribution \( \tilde{\pi}^N_{BSi}(\cdot) \), instead of the distribution \( \tilde{\pi}^N(\cdot) \) in a straightforward way. Details are available from the authors on request.

## S4 Empirical example: Univariate OU model

This section illustrates Sampling Schemes 1 and S4 by applying them to a univariate SV model. We simplify equation (8) by considering univariate observations and by removing the factor term. We consider a univariate OU model given by equation (11) as well as its Euler approximation given by equation (12). We give results for several estimation approaches: PG with ancestral tracing (PGAT), PG with backward simulation (PGBS), PMMH, and PMMH+PG. The aim of this section is to illustrate the flexibility of the sampling schemes for a simple example, rather than carrying out a detailed study to determine which method is preferred, but we have given performance comparisons using the measures described in Section 4.1 for completeness.

We apply the methods to a sample of daily US steel industry stock returns data obtained from the Kenneth French website\(^2\) using a sample from January 3rd, 2001 to the 24th of December, 2003, a total of 1,000 observations. The priors for the OU parameters are \( \alpha \sim IG\left(\frac{v_0}{2}, \frac{s_0}{2}\right) \), \( \tau^2 \sim IG\left(\frac{v_0}{2}, \frac{s_0}{2}\right) \), and \( p(\mu) \propto 1 \). These prior densities cover most possible values in practice. We ran all the sampling schemes for 11,000 iterations and discarded the initial 1,000 iterations as warmup. We used \( M = 10 \) latent points for the Euler approximations to the state transition densities.

Table S1 shows the IACT values for the parameters in the univariate OU model estimated using four different samplers using the exact transition densities, PMMH (\( \alpha, \tau^2, \mu \)), PGAT (\( \alpha, \tau^2, \mu \)), PGBS (\( \alpha, \tau^2, \mu \)), and PMMH (\( \alpha, \tau^2 \)) + PG (\( \mu \)). The reasons we use PMMH (\( \alpha, \tau^2 \)) + PG (\( \mu \)) is that both PG samplers have a large IACT for both parameters \( \alpha \) and \( \tau^2 \) and we show that putting those two parameters in the PMMH step improves the mixing significantly. Table S2 summarises the estimation results with an Euler approximation of the transition density. The two tables show that it is necessary to use PMMH for at least the \( \alpha \) and \( \tau^2 \) parameters. See our comments in Section 3.3 on a strategy for identifying the parameters that should be generated by PMMH.

\(^2\)http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/datalibrary.html
Table S1: Inefficiency factors of $\alpha$, $\tau^2$, and $\mu$ for univariate OU with exact transition density: Sampler I: PMMH ($\alpha, \tau^2$) + PG ($\mu$), Sampler II: PGAT ($\alpha, \tau^2, \mu$), Sampler III: PGBS ($\alpha, \tau^2, \mu$), and Sampler IV: PMMH ($\alpha, \tau^2, \mu$) for univariate US stock returns data with $T = 1,000$, and number of particles $N = 500$. Time denotes the time taken in seconds for one iteration of the method.

|       | I     | II    | III   | IV    |
|-------|-------|-------|-------|-------|
| IACT  | $\alpha$ | 11.71 | 57.47 | 40.27 | 13.80 |
|       | $\mu$   | 1.49  | 1.56  | 1.48  | 13.44 |
|       | $\tau^2$| 12.45 | 109.90| 115.04| 14.63 |

$\hat{\text{IACT}}$ max | 12.45 | 109.90 | 115.04 | 14.63 |

$\text{TNV}$ max | 2.37  | 12.09 | 16.11 | 1.61  |

$\text{RTNV}$ max | 1     | 5.10  | 6.80  | 0.68  |

$\hat{\text{IACT}}$ mean | 8.55  | 56.31 | 52.26 | 13.96 |

$\text{TNV}$ mean | 1.62  | 6.19  | 7.32  | 1.54  |

$\text{RTNV}$ mean | 1     | 3.82  | 4.52  | 0.95  |

Time | 0.19  | 0.11  | 0.14  | 0.11  |

Table S2: Inefficiency factors of $\alpha$, $\tau^2$, and $\mu$ for univariate OU with an Euler approximation for the transition density: Sampler I: PMMH ($\alpha, \tau^2$) + PG ($\mu$), Sampler II: PGAT ($\alpha, \tau^2, \mu$), and Sampler III: PGBS ($\alpha, \tau^2, \mu$), Sampler IV: PMMH ($\alpha, \tau^2, \mu$) for univariate US stock returns data with $T = 1,000$, and number of particles $N = 500$. Time denotes the time taken in seconds for one iteration of the method.

|       | I     | II    | III   | IV    |
|-------|-------|-------|-------|-------|
| IACT  | $\alpha$ | 11.87 | 216.09| 113.74| 14.24 |
|       | $\mu$   | 9.56  | 14.55 | 15.42 | 15.55 |
|       | $\tau^2$| 12.05 | 531.20| 341.13| 14.80 |

$\hat{\text{IACT}}$ max | 12.05 | 531.20 | 341.13 | 15.55 |

$\text{TNV}$ max | 8.44  | 196.54 | 136.45 | 2.18  |

$\text{RTNV}$ max | 1     | 23.29  | 16.17  | 0.26  |

$\hat{\text{IACT}}$ mean | 11.16 | 253.95 | 156.76 | 14.86 |

$\text{TNV}$ mean | 7.81  | 93.96  | 62.70  | 2.08  |

$\text{RTNV}$ mean | 1     | 12.03  | 8.03   | 0.27  |

Time | 0.70  | 0.37  | 0.40  | 0.14  |
S5 Tables for simulated example in Section 4.2.1

Table S3: Inefficiency factor of $\beta$, $\alpha$, $\mu$, $\tau^2$, $\phi$, and $\tau_f^2$ with exact transition density for the Gaussian OU model: Sampler I: $PMMH (\alpha, \tau^2, \tau_f^2) + PG (\beta, \mu, \phi)$, Sampler II: $PGAT (\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)$, sampler III: $PGBS (\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)$ for simulated data with $T = 1000$, $S = 20$, and $K = 1$, and number of particles $N = 500$.

|      | I   | II  | III | I   | II  | III | I   | II  | III | I   | II  | III |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\beta_1$ | 12.55 | 12.92 | 13.95 | $\alpha_1$ | 12.64 | 66.69 | 39.94 | $\tau_1^2$ | 14.70 | 136.58 | 99.80 | $\mu_1$ | 1.29 | 1.47 | 1.39 |
| $\beta_2$ | 12.67 | 13.03 | 13.94 | $\alpha_2$ | 11.76 | 44.67 | 35.59 | $\tau_2^2$ | 14.36 | 72.64 | 74.03 | $\mu_2$ | 1.28 | 1.43 | 1.33 |
| $\beta_3$ | 12.69 | 13.20 | 14.17 | $\alpha_3$ | 11.89 | 64.76 | 61.08 | $\tau_3^2$ | 12.01 | 92.80 | 101.64 | $\mu_3$ | 1.56 | 1.72 | 1.59 |
| $\beta_4$ | 12.53 | 12.37 | 13.77 | $\alpha_4$ | 13.13 | 107.58 | 59.69 | $\tau_4^2$ | 14.70 | 283.23 | 93.35 | $\mu_4$ | 1.41 | 1.40 | 1.33 |
| $\beta_5$ | 12.66 | 13.08 | 13.86 | $\alpha_5$ | 15.21 | 76.45 | 35.94 | $\tau_5^2$ | 14.56 | 123.53 | 81.58 | $\mu_5$ | 1.29 | 1.37 | 1.25 |
| $\beta_6$ | 12.76 | 12.89 | 14.01 | $\alpha_6$ | 14.80 | 37.25 | 30.74 | $\tau_6^2$ | 14.84 | 76.76 | 56.96 | $\mu_6$ | 1.25 | 1.29 | 1.18 |
| $\beta_7$ | 12.65 | 12.62 | 13.72 | $\alpha_7$ | 14.11 | 27.87 | 24.29 | $\tau_7^2$ | 13.36 | 58.61 | 43.39 | $\mu_7$ | 1.23 | 1.28 | 1.18 |
| $\beta_8$ | 12.85 | 12.96 | 13.87 | $\alpha_8$ | 13.65 | 40.08 | 19.94 | $\tau_8^2$ | 13.37 | 98.49 | 42.14 | $\mu_8$ | 1.24 | 1.27 | 1.20 |
| $\beta_9$ | 12.52 | 13.11 | 13.83 | $\alpha_9$ | 13.58 | 96.90 | 47.77 | $\tau_9^2$ | 15.06 | 144.72 | 81.66 | $\mu_9$ | 1.99 | 1.86 | 1.54 |
| $\beta_{10}$ | 12.39 | 12.81 | 14.05 | $\alpha_{10}$ | 18.07 | 23.49 | 32.13 | $\tau_{10}^2$ | 16.56 | 58.06 | 57.03 | $\mu_{10}$ | 1.29 | 1.28 | 1.23 |
| $\beta_{11}$ | 12.80 | 12.94 | 14.13 | $\alpha_{11}$ | 17.31 | 41.43 | 31.13 | $\tau_{11}^2$ | 14.33 | 75.79 | 66.30 | $\mu_{11}$ | 1.33 | 1.37 | 1.27 |
| $\beta_{12}$ | 12.75 | 13.07 | 14.22 | $\alpha_{12}$ | 16.33 | 30.14 | 47.93 | $\tau_{12}^2$ | 14.18 | 53.80 | 74.84 | $\mu_{12}$ | 1.42 | 1.35 | 1.31 |
| $\beta_{13}$ | 12.78 | 12.87 | 14.16 | $\alpha_{13}$ | 16.24 | 38.37 | 27.31 | $\tau_{13}^2$ | 13.67 | 67.67 | 47.37 | $\mu_{13}$ | 1.25 | 1.31 | 1.25 |
| $\beta_{14}$ | 12.78 | 13.04 | 14.23 | $\alpha_{14}$ | 14.41 | 38.38 | 21.61 | $\tau_{14}^2$ | 15.88 | 83.16 | 46.09 | $\mu_{14}$ | 1.27 | 1.30 | 1.26 |
| $\beta_{15}$ | 12.47 | 12.82 | 13.80 | $\alpha_{15}$ | 12.72 | 34.25 | 22.16 | $\tau_{15}^2$ | 15.39 | 60.91 | 44.90 | $\mu_{15}$ | 1.22 | 1.25 | 1.19 |
| $\beta_{16}$ | 12.91 | 12.99 | 14.01 | $\alpha_{16}$ | 15.19 | 70.11 | 42.38 | $\tau_{16}^2$ | 13.60 | 110.75 | 66.36 | $\mu_{16}$ | 1.40 | 1.62 | 1.34 |
| $\beta_{17}$ | 12.74 | 13.11 | 13.86 | $\alpha_{17}$ | 11.17 | 22.16 | 27.11 | $\tau_{17}^2$ | 11.43 | 53.60 | 51.73 | $\mu_{17}$ | 1.37 | 1.31 | 1.21 |
| $\beta_{18}$ | 12.58 | 12.93 | 13.84 | $\alpha_{18}$ | 12.74 | 28.17 | 28.51 | $\tau_{18}^2$ | 15.66 | 59.10 | 75.58 | $\mu_{18}$ | 1.33 | 1.32 | 1.30 |
| $\beta_{19}$ | 12.64 | 12.81 | 13.80 | $\alpha_{19}$ | 12.67 | 40.38 | 29.96 | $\tau_{19}^2$ | 15.74 | 78.74 | 59.19 | $\mu_{19}$ | 1.44 | 1.57 | 1.41 |
| $\beta_{20}$ | 12.77 | 13.19 | 14.08 | $\alpha_{20}$ | 12.85 | 27.12 | 22.34 | $\tau_{20}^2$ | 12.84 | 73.02 | 44.80 | $\mu_{20}$ | 1.26 | 1.38 | 1.30 |
| $\phi$ | 8.03 | 20.12 | 18.62 | $\tau_{f}^2$ | 14.76 | 73.76 | 79.14 |  |  |  |  |  |  |

S6 Tables and figures for US stock return analysis in Section 4.2.2

Table S5 gives the inefficiency factors of $\beta$, $\alpha$, $\mu$, $\tau^2$, $\phi$, and $\tau_f^2$ with the exact transition density for the Gaussian OU model for the three samplers: Sampler I: $PMMH (\alpha, \tau^2, \tau_f^2) + PG (\beta, \mu, \phi)$, Sampler II: $PGAT (\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)$, sampler III: $PGBS (\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)$ for US stock returns data with $T = 1000$, $S = 20$, and $K = 1$, and with the number of particles $N = 500$. 
Table S4: Inefficiency factor of $\beta$, $\alpha$, $\mu$, $\tau^2$, $\phi$, and $\tau_f^2$ with Euler approximation for state transition density for the Gaussian OU model: Sampler I: $\text{PMMH} (\alpha, \tau^2, \mu, \tau_f^2) + \text{PG} (\beta, \phi)$, Sampler II: $\text{PGAT} (\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)$, sampler III: $\text{PGBS} (\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)$ for simulated data with $T = 1000$, $S = 20$, and $K = 1$, and number of particles $N = 1000$.

|       | I  | II  | III  | I  | II  | III  | I  | II  | III  | I  | II  | III  |
|-------|----|-----|------|----|-----|------|----|-----|------|----|-----|------|
| $\beta_1$ | 13.72 | 13.67 | 11.06 | $\alpha_1$ | 12.85 | 159.79 | 181.78 | $\tau_f^2$ | 13.10 | 374.25 | 444.82 | $\mu_1$ | 13.27 | 12.92 | 11.82 |
| $\beta_2$ | 13.93 | 13.79 | 11.23 | $\alpha_2$ | 15.49 | 92.87 | 335.05 | $\tau_f^2$ | 13.49 | 340.28 | 792.88 | $\mu_2$ | 13.42 | 11.00 | 11.98 |
| $\beta_3$ | 13.87 | 13.60 | 11.30 | $\alpha_3$ | 12.43 | 300.77 | 272.34 | $\tau_f^2$ | 12.46 | 733.43 | 682.28 | $\mu_3$ | 15.41 | 13.09 | 13.23 |
| $\beta_4$ | 14.14 | 13.48 | 10.95 | $\alpha_4$ | 13.35 | 530.99 | 303.41 | $\tau_f^2$ | 13.46 | 977.93 | 654.65 | $\mu_4$ | 13.44 | 13.87 | 14.16 |
| $\beta_5$ | 13.63 | 13.56 | 10.95 | $\alpha_5$ | 15.72 | 93.77 | 140.44 | $\tau_f^2$ | 16.23 | 514.24 | 339.73 | $\mu_5$ | 13.24 | 11.83 | 12.87 |
| $\beta_6$ | 13.84 | 13.68 | 11.30 | $\alpha_6$ | 16.81 | 190.71 | 152.17 | $\tau_f^2$ | 16.20 | 539.97 | 418.23 | $\mu_6$ | 14.00 | 13.23 | 13.45 |
| $\beta_7$ | 13.77 | 13.69 | 11.25 | $\alpha_7$ | 17.57 | 79.74 | 102.55 | $\tau_f^2$ | 13.75 | 592.05 | 352.65 | $\mu_7$ | 13.80 | 10.85 | 11.77 |
| $\beta_8$ | 13.87 | 13.52 | 11.14 | $\alpha_8$ | 13.33 | 134.56 | 136.97 | $\tau_f^2$ | 13.99 | 392.80 | 376.86 | $\mu_8$ | 16.46 | 11.48 | 11.67 |
| $\beta_9$ | 13.69 | 13.39 | 11.15 | $\alpha_9$ | 13.50 | 395.91 | 161.91 | $\tau_f^2$ | 14.65 | 803.36 | 457.15 | $\mu_9$ | 16.12 | 13.72 | 12.55 |
| $\beta_{10}$ | 13.95 | 13.66 | 11.19 | $\alpha_{10}$ | 12.46 | 128.96 | 117.10 | $\tau_f^2$ | 13.10 | 408.40 | 357.97 | $\mu_{10}$ | 14.72 | 11.70 | 11.94 |
| $\beta_{11}$ | 13.99 | 13.84 | 11.14 | $\alpha_{11}$ | 13.55 | 273.87 | 98.71 | $\tau_f^2$ | 15.56 | 667.52 | 402.61 | $\mu_{11}$ | 12.55 | 11.51 | 12.62 |
| $\beta_{12}$ | 13.85 | 13.78 | 11.32 | $\alpha_{12}$ | 16.34 | 105.64 | 204.73 | $\tau_f^2$ | 16.09 | 356.37 | 438.96 | $\mu_{12}$ | 12.56 | 13.00 | 13.25 |
| $\beta_{13}$ | 14.20 | 13.56 | 11.13 | $\alpha_{13}$ | 13.56 | 262.15 | 136.41 | $\tau_f^2$ | 12.73 | 511.17 | 378.67 | $\mu_{13}$ | 13.18 | 14.97 | 11.28 |
| $\beta_{14}$ | 14.12 | 13.92 | 11.34 | $\alpha_{14}$ | 12.60 | 188.22 | 177.73 | $\tau_f^2$ | 12.00 | 530.42 | 428.24 | $\mu_{14}$ | 16.19 | 12.18 | 11.69 |
| $\beta_{15}$ | 13.65 | 13.27 | 11.00 | $\alpha_{15}$ | 14.79 | 200.20 | 162.37 | $\tau_f^2$ | 12.79 | 574.45 | 578.06 | $\mu_{15}$ | 15.09 | 13.01 | 12.46 |
| $\beta_{16}$ | 13.89 | 13.89 | 11.07 | $\alpha_{16}$ | 14.62 | 271.96 | 337.69 | $\tau_f^2$ | 15.67 | 470.91 | 672.67 | $\mu_{16}$ | 13.51 | 15.99 | 11.88 |
| $\beta_{17}$ | 13.77 | 13.30 | 11.07 | $\alpha_{17}$ | 16.29 | 139.51 | 87.63 | $\tau_f^2$ | 13.62 | 467.94 | 330.15 | $\mu_{17}$ | 16.63 | 12.34 | 13.24 |
| $\beta_{18}$ | 13.71 | 13.40 | 10.96 | $\alpha_{18}$ | 15.69 | 55.90 | 107.32 | $\tau_f^2$ | 17.08 | 262.38 | 317.31 | $\mu_{18}$ | 15.03 | 10.81 | 11.65 |
| $\beta_{19}$ | 13.90 | 13.69 | 11.05 | $\alpha_{19}$ | 15.73 | 284.70 | 194.08 | $\tau_f^2$ | 14.97 | 649.26 | 537.12 | $\mu_{19}$ | 15.39 | 13.53 | 11.72 |
| $\beta_{20}$ | 13.86 | 13.61 | 11.21 | $\alpha_{20}$ | 13.76 | 311.20 | 125.72 | $\tau_f^2$ | 14.99 | 667.49 | 331.18 | $\mu_{20}$ | 14.64 | 16.43 | 15.96 |

$\phi$ 7.11 | 20.88 | 17.01 | $\tau_f^2_{11}$ | 12.66 | 78.23 | 67.92
Table S5: Inefficiency factors of $\beta$, $\alpha$, $\mu$, $\tau^2$, $\phi$, and $\tau_f^2$ with exact transition density for the Gaussian OU model: Sampler I: PMMH ($\alpha, \tau^2, \tau_f^2$) + PG ($\beta, \mu, \phi$), Sampler II: PGAT ($\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2$), sampler III: PGBS ($\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2$) for US stock returns data with $T = 1000$, $S = 20$, and $K = 1$, and number of particles $N = 500$.

|       | I  | II | III | I  | II | III | I  | II | III |
|-------|----|----|-----|----|----|-----|----|----|-----|
| $\beta_1$ | 2.18 | 2.05 | 1.91 | $\alpha_1$ | 14.21 | 219.61 | 113.66 | $\tau_f^2$ | 14.37 | 260.88 | 129.79 | $\mu_1$ | 2.11 | 4.50 | 2.84 |
| $\beta_2$ | 1.68 | 1.85 | 1.90 | $\alpha_2$ | 11.87 | 35.87 | 40.80 | $\tau_2^2$ | 12.29 | 68.34 | 70.17 | $\mu_2$ | 1.20 | 1.42 | 1.18 |
| $\beta_3$ | 1.80 | 1.76 | 1.70 | $\alpha_3$ | 13.04 | 62.04 | 89.69 | $\tau_3^2$ | 13.23 | 110.88 | 157.46 | $\mu_3$ | 2.39 | 2.66 | 2.36 |
| $\beta_4$ | 1.79 | 1.76 | 1.83 | $\alpha_4$ | 14.22 | 66.24 | 51.79 | $\tau_4^2$ | 14.99 | 122.26 | 88.17 | $\mu_4$ | 1.77 | 1.83 | 1.50 |
| $\beta_5$ | 1.87 | 1.76 | 1.69 | $\alpha_5$ | 18.44 | 466.48 | 136.77 | $\tau_5^2$ | 17.14 | 682.49 | 167.35 | $\mu_5$ | 2.97 | 3.57 | 1.91 |
| $\beta_6$ | 1.66 | 1.74 | 1.67 | $\alpha_6$ | 17.31 | 113.00 | 112.08 | $\tau_6^2$ | 19.29 | 202.66 | 258.42 | $\mu_6$ | 4.88 | 5.94 | 4.11 |
| $\beta_7$ | 1.61 | 1.67 | 1.66 | $\alpha_7$ | 11.41 | 52.72 | 64.09 | $\tau_7^2$ | 14.00 | 91.79 | 92.67 | $\mu_7$ | 1.87 | 1.79 | 1.86 |
| $\beta_8$ | 1.82 | 1.93 | 1.70 | $\alpha_8$ | 18.71 | 86.37 | 45.28 | $\tau_8^2$ | 20.57 | 145.71 | 76.37 | $\mu_8$ | 2.43 | 3.41 | 1.80 |
| $\beta_9$ | 1.89 | 1.96 | 1.74 | $\alpha_9$ | 12.97 | 80.73 | 136.71 | $\tau_9^2$ | 14.23 | 116.44 | 158.23 | $\mu_9$ | 2.27 | 2.77 | 3.30 |
| $\beta_{10}$ | 1.65 | 1.73 | 1.66 | $\alpha_{10}$ | 15.25 | 119.34 | 124.61 | $\tau_{10}^2$ | 12.54 | 106.68 | 128.63 | $\mu_{10}$ | 6.21 | 7.57 | 6.70 |
| $\beta_{11}$ | 1.63 | 1.74 | 1.67 | $\alpha_{11}$ | 14.66 | 65.71 | 69.71 | $\tau_{11}^2$ | 14.44 | 121.39 | 83.53 | $\mu_{11}$ | 3.24 | 5.57 | 2.84 |
| $\beta_{12}$ | 1.65 | 1.89 | 1.69 | $\alpha_{12}$ | 17.47 | 433.51 | 97.20 | $\tau_{12}^2$ | 16.20 | 545.21 | 146.63 | $\mu_{12}$ | 3.36 | 5.94 | 2.54 |
| $\beta_{13}$ | 1.94 | 2.02 | 1.92 | $\alpha_{13}$ | 13.50 | 151.20 | 112.64 | $\tau_{13}^2$ | 13.49 | 189.17 | 145.44 | $\mu_{13}$ | 2.74 | 3.19 | 2.19 |
| $\beta_{14}$ | 1.66 | 1.79 | 1.60 | $\alpha_{14}$ | 14.48 | 70.44 | 74.94 | $\tau_{14}^2$ | 14.11 | 146.32 | 121.04 | $\mu_{14}$ | 2.01 | 2.06 | 1.73 |
| $\beta_{15}$ | 1.62 | 1.82 | 1.45 | $\alpha_{15}$ | 13.08 | 126.39 | 291.78 | $\tau_{15}^2$ | 14.80 | 148.03 | 382.86 | $\mu_{15}$ | 2.20 | 2.66 | 2.11 |
| $\beta_{16}$ | 1.69 | 1.76 | 1.83 | $\alpha_{16}$ | 11.58 | 133.17 | 39.94 | $\tau_{16}^2$ | 11.64 | 210.38 | 99.40 | $\mu_{16}$ | 1.54 | 1.54 | 1.55 |
| $\beta_{17}$ | 2.12 | 2.54 | 1.95 | $\alpha_{17}$ | 14.52 | 39.97 | 30.94 | $\tau_{17}^2$ | 15.65 | 94.23 | 54.03 | $\mu_{17}$ | 1.30 | 1.25 | 1.24 |
| $\beta_{18}$ | 1.94 | 2.04 | 1.93 | $\alpha_{18}$ | 15.24 | 51.58 | 40.02 | $\tau_{18}^2$ | 17.46 | 105.41 | 70.14 | $\mu_{18}$ | 1.36 | 1.51 | 1.36 |
| $\beta_{19}$ | 1.80 | 1.92 | 1.73 | $\alpha_{19}$ | 15.14 | 36.14 | 28.02 | $\tau_{19}^2$ | 13.73 | 81.59 | 68.35 | $\mu_{19}$ | 1.28 | 1.48 | 1.37 |
| $\beta_{20}$ | 1.87 | 1.81 | 1.73 | $\alpha_{20}$ | 14.52 | 33.78 | 28.57 | $\tau_{20}^2$ | 17.10 | 72.67 | 55.28 | $\mu_{20}$ | 1.27 | 1.51 | 1.22 |
| $\phi$  | 8.77 | 25.64 | 20.05 | $\tau_f^2$ | 14.24 | 55.08 | 48.92 |

Table S6 and S7 give the inefficiency factors of $\beta$, $\alpha$, $\mu$, $\tau^2$, $\phi$, and $\tau_f^2$ with the approximate Euler based transition density for the Gaussian OU model and for the GARCH diffusion model, respectively, for the three samplers: Sampler I: PMMH ($\alpha, \tau^2, \tau_f^2$) + PG ($\beta, \mu, \phi$), Sampler II: PGAT ($\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2$), Sampler III: PGBS ($\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2$) for US stock returns data with $T = 1000$, $S = 20$, and $K = 1$, and with the number of particles $N = 1000$.

Figures S1 and S2 present the kernel density estimates of marginal posterior densities of four representative $\alpha$ and $\tau^2$ respectively for the Gaussian OU model for the US stock returns data. The density estimates are for PMMH+PG using exact and approximate transition densities and PG with approximate transition densities using ancestral tracing and backward simulation. Both figures show that both PMMH+PG samplers produce estimates that are close to each other, whereas the PG samplers are much less reliable.
Table S6: Inefficiency factor of $\beta$, $\alpha$, $\mu$, $\tau^2$, $\phi$, and $\tau^2_f$ with Euler approximation for state transition density for the Gaussian OU model: Sampler I: $\text{PMMH} (\alpha, \tau^2, \mu, \tau^2_f)$ + $\text{PG} (\beta, \phi)$, Sampler II: $\text{PGAT} (\beta, \alpha, \tau^2, \mu, \phi, \tau^2_f)$, sampler III: $\text{PGBS} (\beta, \alpha, \tau^2, \mu, \phi, \tau^2_f)$ for US stock returns data with $T = 1000$, $S = 20$, and $K = 1$, and number of particles $N = 1000$.

|       | I   | II  | III |
|-------|-----|-----|-----|
| $\beta_1$ | 2.01 | 2.18 | 1.89 |
| $\alpha_1$ | 15.52 | 559.41 | 723.77 |
| $\tau^2_1$ | 15.33 | 787.45 | 977.73 |
| $\mu_1$ | 11.99 | 23.88 | 18.17 |
| $\beta_2$ | 1.86 | 1.84 | 1.83 |
| $\alpha_2$ | 16.40 | 554.76 | 186.87 |
| $\tau^2_2$ | 14.35 | 914.39 | 475.00 |
| $\mu_2$ | 15.46 | 12.34 | 11.59 |
| $\beta_3$ | 1.70 | 1.79 | 1.73 |
| $\alpha_3$ | 18.50 | 342.35 | 210.83 |
| $\tau^2_3$ | 19.40 | 688.81 | 546.86 |
| $\mu_3$ | 13.45 | 12.56 | 12.66 |
| $\beta_4$ | 1.70 | 1.83 | 1.76 |
| $\alpha_4$ | 15.40 | 215.12 | 111.11 |
| $\tau^2_4$ | 16.44 | 455.87 | 326.75 |
| $\mu_4$ | 12.11 | 12.82 | 12.83 |
| $\beta_5$ | 1.85 | 1.75 | 1.68 |
| $\alpha_5$ | 15.82 | 308.00 | 305.18 |
| $\tau^2_5$ | 19.29 | 576.04 | 456.33 |
| $\mu_5$ | 21.39 | 16.70 | 20.62 |
| $\beta_6$ | 1.73 | 1.75 | 1.74 |
| $\alpha_6$ | 20.72 | 494.91 | 374.78 |
| $\tau^2_6$ | 20.06 | 995.03 | 797.53 |
| $\mu_6$ | 19.43 | 26.62 | 36.67 |
| $\beta_7$ | 1.78 | 1.75 | 1.77 |
| $\alpha_7$ | 16.07 | 340.91 | 464.08 |
| $\tau^2_7$ | 14.49 | 783.92 | 754.46 |
| $\mu_7$ | 13.71 | 13.56 | 14.80 |
| $\beta_8$ | 1.83 | 1.83 | 1.74 |
| $\alpha_8$ | 19.85 | 400.60 | 128.31 |
| $\tau^2_8$ | 23.81 | 928.38 | 328.45 |
| $\mu_8$ | 18.30 | 13.56 | 12.63 |
| $\beta_9$ | 1.76 | 1.96 | 1.77 |
| $\alpha_9$ | 15.19 | 909.99 | 546.01 |
| $\tau^2_9$ | 14.84 | 1215.77 | 937.28 |
| $\mu_9$ | 13.06 | 19.86 | 19.14 |
| $\beta_{10}$ | 1.74 | 1.78 | 1.77 |
| $\alpha_{10}$ | 16.96 | 385.25 | 236.04 |
| $\tau^2_{10}$ | 23.59 | 962.51 | 716.08 |
| $\mu_{10}$ | 11.92 | 50.67 | 35.06 |
| $\beta_{11}$ | 1.77 | 1.78 | 1.74 |
| $\alpha_{11}$ | 18.43 | 368.53 | 115.84 |
| $\tau^2_{11}$ | 23.99 | 811.02 | 872.32 |
| $\mu_{11}$ | 13.76 | 15.12 | 14.85 |
| $\beta_{12}$ | 1.81 | 1.82 | 1.77 |
| $\alpha_{12}$ | 20.48 | 521.58 | 460.67 |
| $\tau^2_{12}$ | 20.43 | 771.17 | 700.72 |
| $\mu_{12}$ | 16.91 | 20.80 | 19.88 |
| $\beta_{13}$ | 1.81 | 1.86 | 1.83 |
| $\alpha_{13}$ | 17.79 | 362.85 | 548.70 |
| $\tau^2_{13}$ | 18.43 | 632.95 | 707.42 |
| $\mu_{13}$ | 15.76 | 14.73 | 19.90 |
| $\beta_{14}$ | 1.77 | 1.79 | 1.64 |
| $\alpha_{14}$ | 15.48 | 195.27 | 375.87 |
| $\tau^2_{14}$ | 17.05 | 603.04 | 704.08 |
| $\mu_{14}$ | 14.14 | 14.75 | 19.37 |
| $\beta_{15}$ | 1.59 | 1.69 | 1.57 |
| $\alpha_{15}$ | 17.48 | 485.37 | 1097.26 |
| $\tau^2_{15}$ | 15.58 | 897.29 | 1228.99 |
| $\mu_{15}$ | 15.76 | 18.84 | 29.16 |
| $\beta_{16}$ | 1.80 | 1.70 | 1.74 |
| $\alpha_{16}$ | 15.94 | 240.28 | 211.86 |
| $\tau^2_{16}$ | 14.50 | 571.97 | 434.93 |
| $\mu_{16}$ | 13.40 | 13.29 | 13.19 |
| $\beta_{17}$ | 2.14 | 2.12 | 2.02 |
| $\alpha_{17}$ | 16.99 | 143.03 | 330.84 |
| $\tau^2_{17}$ | 17.16 | 496.86 | 683.20 |
| $\mu_{17}$ | 15.79 | 11.49 | 10.91 |
| $\beta_{18}$ | 1.88 | 1.96 | 1.87 |
| $\alpha_{18}$ | 18.10 | 225.30 | 184.31 |
| $\tau^2_{18}$ | 16.15 | 518.71 | 683.36 |
| $\mu_{18}$ | 18.81 | 11.63 | 12.63 |
| $\beta_{19}$ | 1.84 | 1.88 | 1.79 |
| $\alpha_{19}$ | 16.61 | 200.54 | 70.61 |
| $\tau^2_{19}$ | 16.24 | 474.26 | 276.66 |
| $\mu_{19}$ | 16.64 | 11.33 | 8.73 |
| $\beta_{20}$ | 1.91 | 1.86 | 1.77 |
| $\alpha_{20}$ | 13.97 | 94.55 | 310.15 |
| $\tau^2_{20}$ | 16.21 | 306.76 | 726.35 |
| $\phi$ | 8.22 | 22.73 | 34.20 |
| $\tau^2_f$ | 12.36 | 52.16 | 68.27 |
Table S7: Inefficiency factor of $\beta$, $\alpha$, $\mu$, $\tau^2$, $\phi$, and $\tau_f^2$ with Euler approximation for state transition density for the GARCH diffusion model: Sampler I: $PG(\beta, \phi) + PMMH(\alpha, \tau^2, \mu, \tau_f^2)$, Sampler II: $PGAT(\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)$, sampler III: $PGBS(\beta, \alpha, \tau^2, \mu, \phi, \tau_f^2)$ for US stock returns data with $T = 1000$, $P = 20$, and $K = 1$, and number of particles $N = 1000$.

|    | I   | II   | III  | I    | II   | III  | I    | II   | III  |
|----|-----|------|------|------|------|------|------|------|------|
| $\beta_1$ | 1.91 | 2.06 | 1.88 | $\alpha_1$ | 32.83 | 197.50 | 318.60 | $\tau_1^2$ | 48.24 | 1944.73 | 1079.01 | $\mu_1$ | 111.02 | 207.54 | 229.44 |
| $\beta_2$ | 1.73 | 1.77 | 1.76 | $\alpha_2$ | 13.31 | 144.70 | 135.32 | $\tau_2^2$ | 12.39 | 2186.34 | 2205.98 | $\mu_2$ | 14.57 | 227.46 | 130.33 |
| $\beta_3$ | 1.69 | 1.76 | 1.72 | $\alpha_3$ | 13.53 | 179.77 | 186.45 | $\tau_3^2$ | 23.59 | 1794.07 | 654.43 | $\mu_3$ | 19.46 | 212.72 | 143.36 |
| $\beta_4$ | 1.70 | 1.77 | 1.76 | $\alpha_4$ | 14.94 | 225.04 | 157.04 | $\tau_4^2$ | 16.84 | 3098.27 | 1208.39 | $\mu_4$ | 23.45 | 163.81 | 118.05 |
| $\beta_5$ | 1.71 | 1.74 | 1.71 | $\alpha_5$ | 16.23 | 420.88 | 421.26 | $\tau_5^2$ | 14.29 | 558.61 | 3257.52 | $\mu_5$ | 19.34 | 322.81 | 243.11 |
| $\beta_6$ | 1.66 | 1.72 | 1.68 | $\alpha_6$ | 18.66 | 875.82 | 1166.83 | $\tau_6^2$ | 21.67 | 1097.21 | 2746.64 | $\mu_6$ | 18.93 | 359.97 | 638.61 |
| $\beta_7$ | 1.64 | 1.73 | 1.66 | $\alpha_7$ | 14.81 | 488.09 | 447.91 | $\tau_7^2$ | 16.79 | 1932.45 | 2415.33 | $\mu_7$ | 35.95 | 247.08 | 205.50 |
| $\beta_8$ | 1.75 | 1.86 | 1.72 | $\alpha_8$ | 18.77 | 180.04 | 152.92 | $\tau_8^2$ | 17.51 | 681.34 | 2236.32 | $\mu_8$ | 16.08 | 140.56 | 231.74 |
| $\beta_9$ | 1.76 | 1.79 | 1.85 | $\alpha_9$ | 23.51 | 655.71 | 543.04 | $\tau_9^2$ | 23.17 | 2465.44 | 3065.63 | $\mu_9$ | 147.16 | 434.49 | 814.62 |
| $\beta_{10}$ | 1.70 | 1.77 | 1.75 | $\alpha_{10}$ | 13.04 | 1159.77 | 969.04 | $\tau_{10}^2$ | 14.04 | 2013.82 | 1638.88 | $\mu_{10}$ | 17.20 | 902.78 | 322.78 |
| $\beta_{11}$ | 1.69 | 1.77 | 1.74 | $\alpha_{11}$ | 11.05 | 298.47 | 210.95 | $\tau_{11}^2$ | 14.72 | 1224.84 | 2551.95 | $\mu_{11}$ | 17.49 | 335.21 | 216.52 |
| $\beta_{12}$ | 1.68 | 1.86 | 1.78 | $\alpha_{12}$ | 19.20 | 462.64 | 495.65 | $\tau_{12}^2$ | 22.52 | 2865.97 | 1412.81 | $\mu_{12}$ | 49.95 | 179.47 | 351.02 |
| $\beta_{13}$ | 1.78 | 1.80 | 1.78 | $\alpha_{13}$ | 14.12 | 232.22 | 270.89 | $\tau_{13}^2$ | 13.88 | 1646.83 | 2770.24 | $\mu_{13}$ | 16.15 | 230.03 | 597.87 |
| $\beta_{14}$ | 1.61 | 1.63 | 1.63 | $\alpha_{14}$ | 17.59 | 159.37 | 337.67 | $\tau_{14}^2$ | 16.22 | 2651.10 | 1083.02 | $\mu_{14}$ | 15.34 | 146.47 | 227.23 |
| $\beta_{15}$ | 1.54 | 1.54 | 1.52 | $\alpha_{15}$ | 13.93 | 330.76 | 329.03 | $\tau_{15}^2$ | 16.10 | 1551.35 | 1303.25 | $\mu_{15}$ | 30.23 | 164.37 | 182.29 |
| $\beta_{16}$ | 1.67 | 1.69 | 1.62 | $\alpha_{16}$ | 17.17 | 352.23 | 275.77 | $\tau_{16}^2$ | 15.05 | 2166.59 | 1121.20 | $\mu_{16}$ | 11.35 | 141.30 | 246.14 |
| $\beta_{17}$ | 2.04 | 2.16 | 2.03 | $\alpha_{17}$ | 16.20 | 202.76 | 198.07 | $\tau_{17}^2$ | 17.68 | 2007.36 | 3053.61 | $\mu_{17}$ | 36.97 | 728.55 | 820.53 |
| $\beta_{18}$ | 1.83 | 1.86 | 1.77 | $\alpha_{18}$ | 13.94 | 347.07 | 192.65 | $\tau_{18}^2$ | 17.27 | 1478.12 | 2899.07 | $\mu_{18}$ | 19.63 | 311.89 | 603.94 |
| $\beta_{19}$ | 1.74 | 1.80 | 1.78 | $\alpha_{19}$ | 14.17 | 398.65 | 157.60 | $\tau_{19}^2$ | 18.14 | 2896.07 | 2682.24 | $\mu_{19}$ | 19.85 | 1340.20 | 235.55 |
| $\beta_{20}$ | 1.75 | 1.81 | 1.80 | $\alpha_{20}$ | 17.59 | 130.58 | 262.31 | $\tau_{20}^2$ | 15.98 | 2096.28 | 1352.18 | $\mu_{20}$ | 17.63 | 119.10 | 148.03 |

$\phi$ | 8.74 | 21.52 | 20.91 | $\tau_{f1}^2$ | 13.65 | 47.47 | 46.10 |
Figure S1: The kernel density estimates of marginal posterior densities of four representative $\alpha$ for the US stock returns data. The density estimates are for PMMH+PG using exact and approximate transition densities and PG with approximate transition densities using ancestral tracing and backward simulation for the Gaussian OU model.
Figure S2: The kernel density estimates of marginal posterior densities of $\tau^2$ for the US stock returns data for four representative $\tau^2$. The density estimates are for PMMH+PG using exact and approximate transition densities and PG using ancestral tracing and backward simulation for the Gaussian OU model.