On the dislocation density tensor in the Cosserat theory of elastic shells

Mircea Bîrsan and Patrizio Neff

February 19, 2016

Abstract

We consider the Cosserat continuum in its finite strain setting and discuss the dislocation density tensor as a possible alternative curvature strain measure in three-dimensional Cosserat models and in Cosserat shell models. We establish a close relationship (one-to-one correspondence) between the new shell dislocation density tensor and the bending-curvature tensor of 6-parameter shells.

1 Introduction

The Cosserat type theories have recently seen a tremendous renewed interest for their prospective applicability to model physical effects beyond the classical ones. These comprise notably the so-called size–effects ("smaller is stiffer").

In a finite strain Cosserat-type framework the group of proper rotations $\text{SO}(3)$ has a dominant place. The original idea of the Cosserat brothers [16] to consider independent rotational degrees of freedom in addition to the macroscopic displacement was heavily motivated by their treatment of plate and shell theory. Indeed, in shell theory it is natural to attach a preferred orthogonal frame (triad) at any point of the surface, one vector of which is the normal to the midsurface, the other two vectors lying in the tangent plane. This is the notion of the "triédre caché". The idea to consider then an orthogonal frame which is not strictly linked to the surface, but constitutively coupled, leads to the notion of the "triédre mobile". And this then is already giving rise to a prototype Cosserat shell (6-parameter) theory. For an insightful review of various Cosserat-type shell models we refer to [8].

However, the Cosserat brothers have never proposed any more specific constitutive framework, apart from postulating euclidean invariance (frame-indifference) and hyperelasticity.

For specific problems it is necessary to choose a constitutive framework and to determine certain strain and curvature measures. This task is still not conclusively done, see e.g. [30].

Among the existing models for Cosserat-type shells we mention the theory of simple elastic shells [7], which has been developed by [34, 35] and [5, 6]. Later, this theory has been successfully applied to describe the mechanical behavior of laminated, functionally graded, viscoelastic or porous plates in [1, 2, 3, 4] and of multi-layered, orthotropic, thermoelastic shells in [10, 11, 14, 31]. Another remarkable approach is the general 6-parameter theory of elastic shells presented in [23, 13, 18]. Although the starting point is different, one can see
that the kinematical structure of the nonlinear 6-parameter shell theory is identical to that of a Cosserat shell model, see also [12, 13].

In this paper we would like to draw attention to alternative curvature measures, motivated by dislocation theory, which can also profitably be used in the three-dimensional Cosserat model and the Cosserat shell model. The object of interest is Nye’s dislocation density tensor \( \text{Curl} \ P \). Within the restriction to proper rotations it turns out that Nye’s tensor provides a complete control of all spatial derivatives of rotations [28] and we rederive this property for micropolar continua using general curvilinear coordinates. Then we focus on shell-curvature measures and define a new shell dislocation density tensor using the surface Curl operator. Then, we prove that a relation analogous to Nye’s formula holds also for Cosserat (6-parameter) shells.

The paper is structured as follows. In Section 2 we present the kinematics of a three-dimensional Cosserat continuum, as well as the appropriate strain measures and curvature strain measures, written in curvilinear coordinates. Here, we show the close relationship between the wryness tensor and the dislocation density tensor, including the corresponding Nye’s formula. In Section 3 we define the Curl operator on surfaces and present several representations using surface curvilinear coordinates. These relations are then used in Section 4 to introduce the new shell dislocation density tensor and to investigate its relationship to the elastic shell bending-curvature tensor of 6-parameter shells.

2 Strain measures of a three-dimensional Cosserat model in curvilinear coordinates

Let \( B \) be a Cosserat elastic body which occupies in its reference (initial) configuration the domain \( \Omega_\xi \subset \mathbb{R}^3 \). A generic point of \( \Omega_\xi \) will be denoted by \((\xi_1, \xi_2, \xi_3)\). The deformation of the Cosserat body is described by a vectorial map \( \varphi_\xi \) and a microrotation tensor \( R_\xi \),

\[
\varphi_\xi : \Omega_\xi \rightarrow \Omega_c, \quad R_\xi : \Omega_\xi \rightarrow SO(3),
\]

where \( \Omega_c \) is the deformed (current) configuration. Let \((x_1, x_2, x_3)\) be some general curvilinear coordinates system on \( \Omega_\xi \). Thus, we have a parametric representation \( \Theta \) of the domain \( \Omega_\xi \)

\[
\Theta : \Omega \rightarrow \Omega_\xi, \quad \Theta(x_1, x_2, x_3) = (\xi_1, \xi_2, \xi_3),
\]

where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with Lipschitz boundary \( \partial \Omega \). The covariant base vectors with respect to these curvilinear coordinates are denoted by \( g_i \) and the contravariant base vectors by \( g^j \) (\( i, j = 1, 2, 3 \)), i.e.

\[
g_i = \frac{\partial \Theta}{\partial x_i} = \Theta_i, \quad g^j \cdot g_i = \delta_i^j,
\]

where \( \delta_i^j \) is the Kronecker symbol. We employ the usual conventions for indices: the Latin indices \( i, j, k, \ldots \) range over the set \( \{1, 2, 3\} \), while the Greek indices \( \alpha, \beta, \gamma, \ldots \) are confined to the range \( \{1, 2\} \); the comma preceding an index \( i \) denotes partial derivatives with respect to \( x_i \); the Einstein summation convention over repeated indices is also used.

Introducing the deformation function \( \varphi \) by the composition

\[
\varphi := \varphi_\xi \circ \Theta : \Omega \rightarrow \Omega_c, \quad \varphi(x_1, x_2, x_3) := \varphi_\xi \left( \Theta(x_1, x_2, x_3) \right),
\]

we can express the (elastic) deformation gradient \( F \) as follows

\[
F := \nabla_\xi \varphi_\xi (\xi_1, \xi_2, \xi_3) = \nabla_\xi \varphi(x_1, x_2, x_3) \cdot \left[ \nabla_\xi (x_1, x_2, x_3) \right]^{-1}.
\]
Using the direct tensor notation, we can write

$$\nabla_x \varphi = \varphi, i \otimes e, \quad \nabla_x \Theta = g, i \otimes e, \quad [\nabla_x \Theta]^{-1} = e, j \otimes g',$$

where $e, i$ are the unit vectors along the coordinate axes $Ox, i$ in the parameter domain $\Omega$. Then, the deformation gradient can be expressed by

$$F = \varphi, i \otimes g'.$$

The orientation and rotation of points in Cosserat (micropolar) media can also be described by means of triads of orthonormal vectors (called directors) attached to every point. We denote by $\{d, 0, i\}$ the triad of directors ($i = 1, 2, 3$) in the reference configuration $\Omega_\xi$ and by $\{d, i\}$ the directors in the deformed configuration $\Omega_c$, see Figure 1. We introduce the elastic microrotation $Q_e$ as the composition

$$Q_e := R_\xi \circ \Theta : \Omega \to SO(3), \quad Q_e(x_1, x_2, x_3) := R_\xi(\Theta(x_1, x_2, x_3)),$$

which can be characterized with the help of the directors by the relations

$$Q_e d, 0, i = d, i, \quad i.e., \quad Q_e = d, i \otimes d, 0, i.$$

Let $Q_0$ be the initial microrotation (describing the position of the directors in the reference configuration $\Omega_\xi$)

$$Q_0 e, i = d, 0, i, \quad i.e., \quad Q_0 = d, 0, i \otimes e, i.$$

Then, the total microrotation $R$ is given by

$$R : \Omega \to SO(3), \quad R(x_i) := Q_e(x_i) Q_0(x_i) = d, i(x_i) \otimes e, i.$$

The non-symmetric Biot-type stretch tensor (the elastic first Cosserat deformation tensor, see [16], p. 123, eq. (43)) is now

$$U_e := Q_e^T F = (d, 0, i \otimes d, i) (\varphi, j \otimes g') = (\varphi, j \cdot d, i) d, 0, i \otimes g'.$$

and the non-symmetric strain tensor for nonlinear micropolar materials is defined by

$$E_e := U_e - I_3 = (\varphi, j \cdot d, i - g, j \cdot d, 0, i) d, 0, i \otimes g'.$$
where \( \mathbf{1}_3 = \mathbf{g}_i \otimes \mathbf{g}_i = \mathbf{d}_i^i \otimes \mathbf{d}_i^i \) is the unit three-dimensional tensor. As a strain measure for curvature (orientation change) one can employ the so-called wryness tensor \( \Delta \) given by:

\[
\Delta := \mathbf{Q}^\top \left( \mathbf{R}^\top \mathbf{R}_i \right) \mathbf{Q}_i \otimes \mathbf{g}_i
\]

where \( \mathbf{axl}(\mathbf{A}) \) denotes the axial vector of any skew-symmetric tensor \( \mathbf{A} \). For a detailed discussion on various strain measures of nonlinear micropolar continua we refer to the paper [30].

As an alternative to the wryness tensor \( \Delta \) one can make use of the Curl operator to define the so-called dislocation density tensor \( \mathbf{D}_e \) by

\[
\mathbf{D}_e := \mathbf{Q}^\top \mathbf{Curl} \mathbf{Q}_e,
\]

which is another curvature measure for micropolar continua. Note that the Curl operator has various definitions in the literature, but we will make its significance clear in the next Subsection 2.1, where we present the Curl operator in curvilinear coordinates. The relationship between the wryness tensor \( \Delta \) and the dislocation density tensor \( \mathbf{D}_e \) is discussed in the Subsection 2.2 in details.

Using the strain and curvature tensors \((\mathbf{E}_e, \mathbf{D}_e)\) the elastically stored energy density \( W \) for the isotropic nonlinear Cosserat model can be expressed as [26, 22]

\[
W(\mathbf{E}_e, \mathbf{D}_e) = W_{\text{mp}}(\mathbf{E}_e) + W_{\text{curv}}(\mathbf{D}_e),
\]

where

\[
W_{\text{mp}}(\mathbf{E}_e) = \mu \left( \frac{\text{dev}_3 \text{sym} \mathbf{E}_e}{2} + \mu_c \| \text{skew} \mathbf{E}_e \|^2 + \frac{\kappa}{2} (\text{tr} \mathbf{E}_e)^2 \right),
\]

\[
W_{\text{curv}}(\mathbf{D}_e) = \mu L_c^p \left( a_1 \| \text{dev}_3 \text{sym} \mathbf{D}_e \|^2 + a_2 \| \text{skew} \mathbf{D}_e \|^2 + a_3 (\text{tr} \mathbf{D}_e)^2 \right)^{p/2},
\]

where \( \mu \) is the shear modulus, \( \kappa \) is the bulk modulus of classical isotropic elasticity, and \( \mu_c \) is called the Cosserat couple modulus, which are assumed to satisfy

\[
\mu > 0, \quad \kappa > 0, \quad \text{and} \quad \mu_c > 0.
\]

The parameter \( L_c \) introduces an internal length which is characteristic for the material, \( a_i > 0 \) are dimensionless constitutive coefficients and \( p \geq 2 \) is a constant exponent. Here, \( \text{dev}_3 \mathbf{X} := \mathbf{X} - \frac{1}{3} (\text{tr} \mathbf{X}) \mathbf{1}_3 \) is the deviatoric part of any second order tensor \( \mathbf{X} \).

Under these assumptions on the constitutive coefficients the existence of minimizers to the corresponding minimization problem of the total energy functional has been shown, e.g., in [26, 22].

\section*{2.1 The Curl operator}

For a vector field \( \mathbf{v} \), the (coordinate-free) definition of the vector \( \text{curl} \mathbf{v} \) is

\[
(\text{curl} \mathbf{v}) \cdot \mathbf{c} = \text{div} (\mathbf{v} \times \mathbf{c}) \quad \text{for all constant vectors} \ \mathbf{c},
\]

where \( \cdot \) denotes the scalar product and \( \times \) the vector product. The Curl of a tensor field \( \mathbf{T} \) is the tensor field defined by

\[
(C \text{url} \mathbf{T})^\top \mathbf{c} = \text{curl} (\mathbf{T}^\top \mathbf{c}) \quad \text{for all constant vectors} \ \mathbf{c}.
\]

\textbf{Remark 2.1.} The operator \( \text{Curl} \mathbf{T} \) given by \( 5 \) coincides with the Curl operator defined in [32, 25]. However, for other authors the Curl of \( \mathbf{T} \) is the transpose of \( \text{Curl} \mathbf{T} \) defined by \( 4 \), see e.g. [21, 27].
Then, from (4) and (5), we obtain the following formulas
\[
\text{curl} \mathbf{v} = -v_i \times g^i, \quad \text{Curl} \mathbf{T} = -T_{,i} \times g^i.
\] (6)
Indeed, the definition (4) yields
\[
(\text{curl} \mathbf{v}) \cdot c = \text{div}(\mathbf{v} \times c) = (\mathbf{v} \times c)_i \cdot g^i = (v_i \times c) \cdot g^i = (g^i \times \mathbf{v}_i) \cdot c,
\]
and the equation (6) holds. Further, from (5) we get
\[
(\text{Curl} \mathbf{T})^T c = \text{curl}(\mathbf{T}^T c) = g^i \times (T^T c)_i = g^i \times (T^T c) = (g^i \times T^T c)_j c.
\]
so it follows \( \text{Curl} \mathbf{T} = (g^i \times T^T c)_j = -T_{,i} \times g^i \) and the relations (6) are proved.

In order to write the components of \( \text{curl} \mathbf{v} \) and \( \text{Curl} \mathbf{T} \) in curvilinear coordinates, we introduce the following notations
\[
g_{ij} = g_i \otimes g_j, \quad g = \det (g_{ij})_{3 \times 3} > 0.
\]
The alternating (Ricci) third-order tensor is
\[
\epsilon = -1_{3 \times 1} = \epsilon_{ijk} g^i \otimes g^j \otimes g^k = \epsilon^{ijk} g_i \otimes g_j \otimes g_k,
\]
where
\[
\epsilon_{ijk} = \sqrt{g} \epsilon_{ijk}, \quad \epsilon^{ijk} = \begin{cases} 1, & (i,j,k) \text{ is even permutation} \\
-1, & (i,j,k) \text{ is odd permutation} \\
0, & (i,j,k) \text{ is no permutation}
\end{cases}
\]
The covariant, contravariant, and mixed components of any vector field \( \mathbf{v} \) and any tensor field \( \mathbf{T} \) are introduced by
\[
\mathbf{v} = v_k g^k = v^i g_i, \quad \mathbf{T} = T_{jk} g^j \otimes g^k = T^{jk} g_j \otimes g_k = T^j_k g_j \otimes g^k.
\]
For the partial derivatives with respect to \( x_i \) we have the well-known expressions
\[
v_{,i} = v_{k[i]} g^k, \quad \mathbf{T}_{,i} = T_{jk[i]} g^j \otimes g^k = T^{jk[i]} g_j \otimes g^k,
\] (7)
where a subscript bar preceding the index \( i \) denotes covariant derivative w.r.t. \( x_i \).

Using the relations (7) in (6) we can write the components of \( \text{curl} \mathbf{v} \) and \( \text{Curl} \mathbf{T} \) as follows
\[
\text{curl} \mathbf{v} = \epsilon^{ijk} v_{j[i]} g_k, \quad \text{Curl} \mathbf{T} = \epsilon^{ijk} T_{j[i]} g^j \otimes g_k = \epsilon^{ijk} T^j_{k[i]} g_j \otimes g_k.
\] (8)
Indeed, from (6), and (7), we find
\[
\text{curl} \mathbf{v} = -(v_{k[i]} g^k) \times g^i = -v_{k[i]} (g^k \times g^i) = -v_{k[i]} (\epsilon^{kji} g_j) = \epsilon^{ijk} v_{j[i]} g_k.
\]
Analogously, from (5), and (7), we get
\[
\text{Curl} \mathbf{T} = -(T_{k[i]} g^k \otimes g^i) \times g^j = -T_{k[i]} (g^k \times g^i) = \epsilon^{ij[k} T_{j]} g^k \otimes g_j.
\]
Thus the equations (6) are proved.

Remark 2.2. In the special case of Cartesian coordinates, the relations (6) and (5) admit the simple form
\[
\text{curl} \mathbf{v} = -v_i \times e_i = e_{ijk} v_{j[i} e_{k]}, \quad \text{Curl} \mathbf{T} = -T_{ij} \times e_i = e_{ijk} T_{j[i} e_{j]},
\]
where \( \mathbf{v} = v_i e_i \), and \( \mathbf{T} = T_{ij} e_i \otimes e_j \) are the corresponding coordinates. Moreover, in this case one can write
\[
\text{Curl} \mathbf{T} = e_i \otimes \text{curl}(T_i) \quad \text{ for } \quad T = e_i \otimes T_i,
\] (9)
where \( T_i = T_{ij} e_j \) are the three rows of the \( 3 \times 3 \) matrix \( (T_{ij})_{3 \times 3} \). The relation (9) shows that Curl is defined row-wise [28]: the rows of the \( 3 \times 3 \) matrix Curl \( \mathbf{T} \) are respectively the three vectors curl\( (T_i) \), \( i = 1, 2, 3 \).
The relation \((11)\) is the analogue of \((9)\) for curvilinear coordinates. Similarly, by differentiating \((10)\) with respect to \(x_j\) we get
\[
T_{ij} = g^i \otimes T_{ij} \quad \text{with} \quad T_{ij} := T_{ij} - \Gamma_{ij}^r T_r = T_{ij} g^k.
\]
Taking the vector product of \((11)\) with \(g^i\) we obtain
\[
\text{Curl} \ T = -T_{ij} \times g^j = -(g^i \otimes T_{ij}) \times g^j, \quad \text{i.e.}
\]
\[
\text{Curl} \ T = g^i \otimes \text{curl}_{\text{cov}}(T_i) \quad \text{where} \quad \text{curl}_{\text{cov}}(T_i) := -T_{ij} \times g^j.
\]
The relation \((12)\) is the analogue of \((9)\) for curvilinear coordinates. Similarly, by differentiating \((10)\) with respect to \(x_j\) one can obtain the relation
\[
\text{Curl} \ T = g_i \otimes \text{curl}_{\text{cov}}(T^i) \quad \text{where we denote}
\]
\[
\text{curl}_{\text{cov}}(T^i) := -T_{ij} \times g^j \quad \text{and} \quad T_{ij} := T_{ij} + \Gamma_{ij}^r T^r = T_{ij} g^k.
\]

### Remark 2.3
In order to write the corresponding formula in curvilinear coordinates which is analogous to \((9)\), we introduce the vectors \(T_i := T_{ij} g^j\) and \(T^i := T^{ij} g_j = T_{ij} g^j\) such that it holds
\[
T = g^i \otimes T_i \quad \text{and} \quad T = g_i \otimes T^i.
\]

\[
\text{Curl} \ T = -T_{ij} \times g^j = -(g^i \otimes T_{ij}) \times g^j, \quad \text{i.e.}
\]
\[
\text{Curl} \ T = g^i \otimes \text{curl}_{\text{cov}}(T_i) \quad \text{where} \quad \text{curl}_{\text{cov}}(T_i) := -T_{ij} \times g^j.
\]

### 2.2 Relation between the wryness tensor and the dislocation density tensor

Let \(A = A_{ij} g^i \otimes g^j\) be an arbitrary skew-symmetric tensor and \(\text{axl}(A) = a_k g^k\) its axial vector. Then, the following relations hold
\[
A = \text{axl}(A) \times 1_3 = 1_3 \times \text{axl}(A),
\]
\[
\text{axl}(A) = -\frac{1}{2} \epsilon : A = -\frac{1}{2} \epsilon^{ijk} A_{ij} g_k, \quad (14)
\]
\[
A = -\epsilon \text{axl}(A) = -\epsilon^{ijk} a_k g_i \otimes g_j,
\]
where the double dot product \(\cdot \cdot\) of two tensors \(B = B^{ij} g_i \otimes g_j\) and \(T = T_{ij} g^i \otimes g^j\) is defined as \(B : T = B^{ij} T_{ij} g_i\).

Using these relations we can derive the close relationship between the wryness tensor and the dislocation density tensor; it holds
\[
\mathcal{D}_e = -\Gamma^T + (\text{tr} \Gamma) 1_3, \quad \text{or equivalently},
\]
\[
\Gamma = -\mathcal{D}_e^T + \frac{1}{2} (\text{tr} \mathcal{D}_e) 1_3. \quad (15)
\]

Indeed, in view of the equation \((14)\) and the definition \((1)\) we have
\[
Q^e_i Q_{e,k} \otimes g^k = -\epsilon \text{axl}(Q^e_i Q_{e,k}) \otimes g^k = -\epsilon \Gamma
\]
\[
= -\epsilon_{ijk} g^i \otimes g^j \otimes g^k (\Gamma^k g_i \otimes g^j) = -\epsilon_{ijk} \Gamma_k g_i \otimes g^j \otimes g^k.
\]
Hence, we deduce
\[
Q^e_i Q_{e,k} = -\epsilon_{ijk} \Gamma_k g_i \otimes g^j. \quad (17)
\]

In view of \((6)\), the definition \((2)\) can be written in the form
\[
\mathcal{D}_e = Q^e_i (g_e \times g^i) = -(Q^e_i Q_{e,k}) \times g^k. \quad (18)
\]
Inserting (17) in (23), we obtain
\[ \mathbf{D}_e = \epsilon_{ij}s_i \Gamma_{ik} (g^i \otimes g^j) \times g^k = \epsilon_{ij}s_i \Gamma_{ik} g^i \otimes (\epsilon^jkr g_r) = (\epsilon_{ij} s_i \epsilon^jkr) \Gamma_{ik} g^i \otimes g_r = (\delta^k_i - \delta^k_r) \Gamma_{ik} g^i \otimes g_r = \Gamma_{ik} g^i \otimes g_r - \Gamma_{ik} g^i \otimes g_r = (\text{tr} \Gamma) 1_k - \Gamma^T. \]

Thus, the relation (15) is proved. If we apply the trace operator and the transpose in (15) we obtain also the relation (16). For infinitesimal strains this formula is well-known under the name Nye's formula, and \((-\Gamma)\) is also called Nye's curvature tensor [29]. This relation has been first established in [28].

Let us find the components of the wryness tensor and the dislocation density tensor in curvilinear coordinates. To this aim, we write first the skew-symmetric tensor
\[ Q^T_e Q_{e,i} = (d_j^0 \otimes d_j^i) (d_k^i \otimes d_k^j) = (d_j^0 \cdot d_k^j) d_k^i \otimes d_k^j = d_j^k \cdot d_k^j \cdot d_j^i \otimes d_j^j. \]

Then, we obtain for the axial vector the equation
\[ \text{axl}(Q^T_e Q_{e,i}) = -\frac{1}{2} e_{jks} (d_j \cdot d_k,i - d_j^i \cdot d_k^i) d_j^i. \] (20)

Indeed, according to (14) and (19) we can write
\[ \text{axl}(Q^T_e Q_{e,i}) = -\frac{1}{2} e : (Q^T_e Q_{e,i}) = \frac{1}{2} (e_{jks} d_k^0 \otimes d_k^j \otimes d_k^j) : \left( (d_j \cdot d_k,i - d_j^i \cdot d_k^i) d_j^i \otimes d_j^i \right) = -\frac{1}{2} e_{jks} (d_j \cdot d_k,i - d_j^i \cdot d_k^i) d_j^i \]
and the relation (20) is proved. Using (20) in the definition (1) we find the following formula for the wryness tensor
\[ \Gamma = \frac{1}{2} e_{jks} (d_k,i \cdot d_k - d_k,i \cdot d_k) d_k^i \otimes g^i. \] (21)

To obtain an expression for the components of \(\mathbf{D}_e\) we insert (19) in (18) and we get
\[ \mathbf{D}_e = - (d_j,i \cdot d_k,i - d_j,i \cdot d_k^i) (d_k^i \otimes d_k^i) \times g^j = (d_j,i \cdot d_k,i - d_k^i \cdot d_k^i) d_k^i \otimes (d_k^i \times g^i). \] (22)

We rewrite the last vector product as
\[ d_k^i \times g^i = d_k^i \times \left[ (g^j \cdot d_k^i) d_k^j \right] = (g^j \cdot d_k^i) d_k^i \times d_k^j = e_{krj} (g^j \cdot d_k^i) d_k^i \]
and we insert it in (22) to find the following expression for the dislocation density tensor
\[ \mathbf{D}_e = e_{krj} (d_j,i \cdot d_k,i - d_k^i \cdot d_k^i) (g^j \cdot d_k^i) d_k^i \otimes d_k^i. \] (23)

Remark 2.4. In the special case of Cartesian coordinates one can identify \(d_k^0 = e_i \cdot g^i = g_i = e_i\), and the relations (21) and (22) simplify to the forms
\[ \Gamma = \frac{1}{2} e_{jks} (d_k,i \cdot d_k) e_k \otimes e_j, \]
\[ \mathbf{D}_e = e_{jks} (d_k,i \cdot d_k) e_k \otimes e_k. \]
Remark 2.5. One can find various definitions of the wryness tensor in the literature, see e.g. [33], where $\Gamma$ is called the curvature strain tensor. Thus, one can alternatively define the wryness tensor by

$$\Gamma = Q^T \omega, \quad (24)$$

where $\omega$ is the second order tensor given by

$$\omega = \omega_i \otimes g^i \quad \text{with} \quad Q_{e,j} = \omega_i \times Q_e. \quad (25)$$

If we compare the definition (1) with (24), (25), we see that indeed

$$Q_e^T \omega_i = a x (Q_e^T Q_{e,i}),$$

i.e.

$$Q_{e,i} = \frac{1}{2} [Q_e (d_j \times d_{j,i}) - d_j \times d_{j,i}], \quad (26)$$

By a straightforward but lengthy calculation one can prove that the vectors $\omega_i$ are expressed in terms of the directors by

$$\omega_i = \frac{1}{2} [Q_e (d_j \times d_{j,i}) - d_j \times d_{j,i}] \otimes g^i. \quad (27)$$

Inserting (27) in (24) and (28) we obtain the expression of the wryness tensor written with the help of the directors

$$\Gamma = \frac{1}{2} [Q_e^T (d_j \times d_{j,i}) - d_j \times d_{j,i}] \otimes g^i. \quad (28)$$

3 The Curl operator on surfaces

Let $S$ be a smooth surface embedded in the Euclidean space $\mathbb{R}^3$ and let $y_0(x_1, x_2), y_0 : \omega \rightarrow \mathbb{R}^3$, be a parametrization of this surface. We denote the covariant base vectors in the tangent plane by $a^1, a^2$, and the contravariant base vectors by $a_1, a_2$:

$$a_\alpha = \frac{\partial y_0}{\partial x_\alpha} = y_{0,\alpha}, \quad a_\alpha \cdot a_\beta = \delta_\alpha^\beta$$

and let

$$a_3 = a^3 = n_0 = \frac{a_1 \times a_2}{|a_1 \times a_2|},$$

where $n_0$ is the unit normal to the surface. Further, we designate by

$$a_{\alpha\beta} = a_\alpha \cdot a_\beta, \quad a^{\alpha\beta} = a^\alpha \cdot a^\beta, \quad a = \sqrt{\det (a_{\alpha\beta})_{2 \times 2}} = |a_1 \times a_2| > 0$$

and we have

$$a^\alpha \times a^\beta = \epsilon^\alpha_\beta a_3, \quad a^3 \times a^\alpha = \epsilon^\alpha_\beta a_3, \quad a_\alpha \times a_\beta = \epsilon_{\alpha\beta} a_3, \quad a_3 \times a_\alpha = \epsilon_{\alpha\beta} a_\beta, \quad (29)$$

where $\epsilon^\alpha_\beta = \frac{1}{2} \epsilon_{\alpha\beta} \delta_\alpha^\beta \epsilon_{\alpha\beta} = \alpha \epsilon_{\alpha\beta} \epsilon_{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ is the two-dimensional alternator given by $\epsilon_{12} = -e_{21} = 1, e_{11} = e_{22} = 0$.

Then, $a = a_{\alpha\beta} a^\alpha \otimes a^\beta = a_{\alpha\beta} a_\alpha \otimes a^\beta = a_\alpha \otimes a^\alpha$ represents the first fundamental tensor of the surface $S$, while the second fundamental tensor $b$ is defined by

$$b = \text{Grad} n_0 = -n_{0,\alpha} \otimes a^\alpha = b_{\alpha\beta} a^\alpha \otimes a^\beta = b_\alpha \otimes a^\alpha, \quad \text{with}$$

$$b_{\alpha\beta} = -n_{0,\beta} \cdot a_\alpha = b_{\beta\alpha}, \quad b_\alpha = -n_{0,\alpha} \cdot a^\alpha.$$
The surface gradient \( \text{Grad} \), and surface divergence \( \text{Div}_s \) operators are defined for a vector field \( \mathbf{v} \) by

\[
\text{Grad}_s \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_\alpha} \otimes a^\alpha = v_{,\alpha} \otimes a^\alpha, \quad \text{Div}_s \mathbf{v} = \text{tr} [\text{Grad}_s \mathbf{v}] = v_{,\alpha} \cdot a^\alpha. \tag{30}
\]

We also introduce the so-called *alternator tensor* \( \mathbf{c} \) of the surface \( S \)

\[
\mathbf{c} = -n_0 \times \mathbf{a} = -\mathbf{a} \times n_0 = \epsilon^{\alpha\beta} a_{\alpha} \otimes a_{\beta} = \epsilon_{\alpha\beta} a_{\alpha} \otimes a_{\beta}. \tag{31}
\]

The tensors \( \mathbf{a} \) and \( \mathbf{b} \) are symmetric, while \( \mathbf{c} \) is skew-symmetric and satisfies \( \mathbf{c} \mathbf{c} = -\mathbf{a} \). Note that the tensors \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \) defined above are *planar*; i.e., they are tensors in the tangent plane of the surface. Moreover, \( \mathbf{a} \) is the identity tensor in the tangent plane.

We define the **surface Curl operator** \( \text{curl}_s \) for vector fields \( \mathbf{v} \) and, respectively, \( \text{Curl}_s \) for tensor fields \( \mathbf{T} \) by

\[
\begin{align*}
(\text{curl}_s \mathbf{v}) \cdot \mathbf{k} &= \text{Div}_s (\mathbf{v} \times \mathbf{k}) \quad \text{for all constant vectors } \mathbf{k}, \tag{32} \\
(\text{Curl}_s \mathbf{T})^T \mathbf{k} &= \text{curl}_s (\mathbf{T}^T \mathbf{k}) \quad \text{for all constant vectors } \mathbf{k}. \tag{33}
\end{align*}
\]

Thus, \( \text{curl}_s \mathbf{v} \) is a vector field, while \( \text{Curl}_s \mathbf{T} \) is a tensor field.

**Remark 3.1.** These definitions are analogous to the corresponding definitions \( [1] \), \( [5] \) in the three-dimensional case. Notice that the curl operator on surfaces has a different significance for other authors, see e.g. \( [9] \).

From the definitions \( [32] \) and \( [33] \) it follows

\[
\text{curl}_s \mathbf{v} = -v_{,\alpha} \times a^\alpha, \quad \text{Curl}_s \mathbf{T} = -T_{,\alpha} \times a^\alpha. \tag{34}
\]

Indeed, in view of \( [30] \) and \( [32] \) we have

\[
(\text{curl}_s \mathbf{v}) \cdot \mathbf{k} = \text{Div}_s (\mathbf{v} \times \mathbf{k}) = (\mathbf{v} \times \mathbf{k})_{,\alpha} \cdot a^\alpha = (v_{,\alpha} \times \mathbf{k}) \cdot a^\alpha = (a^\alpha \times v_{,\alpha}) \cdot \mathbf{k}
\]

for all constant vectors \( \mathbf{k} \)

and also

\[
(\text{Curl}_s \mathbf{T})^T \mathbf{k} = \text{curl}_s (\mathbf{T}^T \mathbf{k}) = a^\alpha \times (\mathbf{T}^T k),
\]

which implies \( \text{Curl}_s \mathbf{T} = (a^\alpha \times \mathbf{T}^T )^T = -T_{,\alpha} \times a^\alpha \), so the relations \( [34] \) hold true.

To write the components of \( \text{curl}_s \mathbf{v} \) and \( \text{Curl}_s \mathbf{T} \) we employ the covariant derivatives on the surface. Let \( \mathbf{v} = v_i a^i \) be a vector field on \( S \). Then, we have

\[
\begin{align*}
a^\alpha_{,\beta} &= -\Gamma^\alpha_{\beta\gamma} a^\gamma + b^\alpha_{\beta} a^3, \\
v_{,\alpha} &= (v_{,\beta} - b^3_{\beta} v_3) a^\beta + (v_3_{,\beta} + b^3_{\beta} v_3) a^3,
\end{align*}
\]

where \( v_{,\beta} = v_{\beta,\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma \) is the covariant derivative with respect to \( x_\alpha \). Inserting this relation in \( [34] \) and using \( [39] \), we obtain

\[
\text{curl}_s \mathbf{v} = \epsilon^{\alpha\beta} [(v_{,\beta} + b^3_{\beta} v_3) a_{\alpha} + v_{,\alpha} a_3]. \tag{36}
\]

For a tensor field \( \mathbf{T} = T_{ij} a^i \otimes a^j = T^{ij} a_i \otimes a_j = T^a_j a_i \otimes a^j \) on the surface, the derivative \( T_{,\gamma} \) can be expressed as

\[
T_{,\gamma} = (T_{a\beta\gamma} - b_\gamma a_{\beta}) a^\alpha \otimes a^\beta + (T_{a\beta\gamma} + b_\gamma a^3 - b_3 a_{\beta}) a^\alpha \otimes a^3
\]

\[
+ (T_{a\beta\gamma} + b_3 a_{\beta} - b_\gamma a_{3}) a^3 \otimes a^\alpha + (T_{a\beta\gamma} + b^i_{\beta} T_{i\alpha \gamma} + b^i_{\gamma} T_{i\alpha \beta} - b_3 T_{a\beta} a^3 \otimes a^\alpha
\]

\[
+ (T_{3a\beta \gamma} + b^i_{\beta} T_{i3 \alpha} + b^i_{\gamma} T_{i3 \beta} - b_\gamma T_{a3} a^3 \otimes a^3 \otimes a^3, \tag{37}
\]

\[
9
\]
where the covariant derivatives are

\[ T_{\alpha \beta \gamma} = T_{\alpha \beta, \gamma} - \Gamma_{\beta \gamma}^\delta T_{\alpha \delta} - \Gamma_{\alpha \gamma}^\delta T_{\delta \beta} \],
\[ T_{\alpha \alpha \gamma} = T_{\alpha \alpha, \gamma} - \Gamma_{\alpha \gamma}^\delta T_{\delta \alpha} \],
\[ T_{\alpha 0 \gamma} = T_{\alpha 0, \gamma} - \Gamma_{\alpha \gamma}^\delta T_{0 \delta} \].

Using [37] in (34) we obtain with the help of (29, 12)

\[ \text{Curl} \, T = \epsilon^{\beta \gamma} (T_{\alpha \beta \gamma} + b_\alpha \sigma b_{\alpha \gamma} T_{3 \sigma} - b_{\alpha \gamma} T_{3 3}) a^\alpha \otimes a_\beta + \epsilon^{\beta \gamma} (T_{\alpha \beta \gamma} - b_{\alpha \gamma} T_{3 \beta}) a^\alpha \otimes a_3 + \epsilon^{\gamma \beta} (T_{\alpha \beta \gamma} + b_{\alpha \gamma} T_{3 \beta}) a^3 \otimes a_\beta + \epsilon^{\gamma \beta} (T_{\alpha \beta \gamma} + b_{\alpha \gamma} T_{3 \beta}) a^3 \otimes a_3 \].

Alternatively, one can use the mixed components \( a_i \otimes a_j \)

\[ \text{Curl} \, T = \epsilon^{\beta \gamma} (T_{\alpha \beta \gamma} + b_\alpha \sigma b_{\alpha \gamma} T_{3 \sigma} - b_{\alpha \gamma} T_{3 3}) a_\alpha \otimes a_\beta + \epsilon^{\beta \gamma} (T_{\alpha \beta \gamma} - b_{\alpha \gamma} T_{3 \beta}) a_\alpha \otimes a_3 + \epsilon^{\gamma \beta} (T_{\alpha \beta \gamma} + b_{\alpha \gamma} T_{3 \beta}) a_3 \otimes a_\beta + \epsilon^{\gamma \beta} (T_{\alpha \beta \gamma} + b_{\alpha \gamma} T_{3 \beta}) a_3 \otimes a_3 \].

where

\[ T^\alpha_{\beta \gamma} = T^\alpha_{\beta, \gamma} + \Gamma^\alpha_{\beta \gamma} T^\beta_\sigma - \Gamma^\beta_{\sigma \gamma} T^\alpha_\sigma \],
\[ T^\alpha_{0 \beta \gamma} = T^\alpha_{0, \beta \gamma} + \Gamma^\alpha_{0 \beta \gamma} T^0_\sigma - \Gamma^\beta_{\sigma \gamma} T^\alpha_\sigma \].

Remark 3.2. In order to obtain a formula analogous to (9) and (12), (13) for Curl\, T we write \( T \) in the tensor basis \( \{ a_i \otimes a_j \} \)

\[ T = a^i \otimes T_i = a_i \otimes T^i \] with \( T_i = T_{ij} a^j \), \( T^i = T_{ij}^j a^i \).

By differentiating the first equation with respect to \( x_i \) we get

\[ T_{i, \gamma} = a^\alpha \otimes T_{\alpha i, \gamma} + a^\alpha \otimes T_{\alpha, \gamma i} = - (\Gamma^\alpha_{\gamma i}) a^\beta + b_\alpha \sigma b_{\alpha \gamma} T_{3 \sigma} - b_{\alpha \gamma} T_{3 3}) a^\alpha \otimes T_{\beta i} + a^i \otimes T_{1, \gamma} \]
\[ = a^\alpha \otimes (T_{\alpha, \gamma} - \Gamma^\alpha_{\gamma \beta} T_{\beta \sigma} - b_{\alpha \gamma} T_3) + a^3 \otimes (T_{3, \gamma} + b_{\alpha \gamma} T_{3 \sigma}) \].

Taking the vector product with \( a^\gamma \) and using (33) we find

\[ \text{Curl} \, T = - [a^\alpha \otimes (T_{\alpha i, \gamma} - b_{\alpha \gamma} T_3) + a^3 \otimes (T_{3, \gamma} + b_{\alpha \gamma} T_{3 \sigma})] \times a^\gamma \],

with \( T_{\alpha i, \gamma} := T_{\alpha i, \gamma} - \Gamma^\alpha_{\gamma \beta} T_{\beta \sigma} \). Similarly, we obtain

\[ \text{Curl} \, T = - [a_\alpha \otimes (T^\alpha_{\alpha, \gamma} - b^\alpha \sigma b_{\alpha \gamma} T^\sigma_3) + a_3 \otimes (T^3_{\gamma, \gamma} + b_{\alpha \gamma} T_{3 \sigma})] \times a^\gamma \],

with \( T^\alpha_{\alpha, \gamma} := T^\alpha_{\alpha, \gamma} + \Gamma^\alpha_{\beta \gamma} T_3^\beta \). The equations (40) and (41) are the counterpart of the relations (12) and, respectively, (13) in the three-dimensional theory.

4 The shell dislocation density tensor

Let us present first the kinematics of Cosserat-type shells, which coincides with the kinematics of the 6-parameter shell model, see [19, 15, 13].

We consider a deformable surface \( \omega \subset \mathbb{R}^3 \) which is identified with the midsurface of the shell in its reference configuration and denote with \((\xi_1, \xi_2, \xi_3)\) a generic point of the surface. Each material point is assumed to have 6 degrees of freedom (3 for translations and 3 for rotations). Thus, the deformation of the Cosserat-type shell is determined by a vectorial map \( m_\xi \) and the microrotation tensor \( R_\xi \)

\[ m_\xi : \omega_\xi \rightarrow \omega_\xi \], \[ R_\xi : \omega_\xi \rightarrow SO(3) \],

10
where $\omega_c$ denotes the deformed (current) configuration of the surface. We consider a parametric representation $y_0$ of the reference configuration $\omega_0$

$$y_0 : \omega \rightarrow \omega, \quad y_0(x_1, x_2) = (\xi_1, \xi_2, \xi_3),$$

where $\omega \subset \mathbb{R}^2$ is the bounded variation domain (with Lipschitz boundary) of the parameters $(x_1, x_2)$. Using the same notations as in Section 3, we introduce the base vectors $a_i, a^j$ and the fundamental tensors $\alpha, \beta$ for the reference surface $\omega_0$.

The deformation function $m$ is then defined by the composition

$$m = m_\xi \circ y_0 : \omega \rightarrow \omega_c, \quad m(x_1, x_2) := m_\xi(y_0(x_1, x_2)).$$

According to (30), the surface gradient of the deformation has the expression

$$\text{Grad}_m m = m_{\alpha} \otimes a^\alpha.$$  \hspace{1cm} (42)

As in the three-dimensional case (see Section 2) we define the elastic microrotation $Q_e$ by the composition

$$Q_e = R_\xi \circ y_0 : \omega \rightarrow \text{SO}(3), \quad Q_e(x_1, x_2) := R_\xi(y_0(x_1, x_2)),$$

the total microrotation $R$ by

$$R : \omega \rightarrow \text{SO}(3), \quad R(x_1, x_2) = Q_e(x_1, x_2) Q_0(x_1, x_2),$$

where $Q_0 : \omega \rightarrow \text{SO}(3)$ is the initial microrotation, which describes the orientation of points in the reference configuration.

To characterize the orientation and rotation of points in Cosserat-type shells one employs (as in the three-dimensional case) a triad of orthonormal directors attached to each point. We denote by $d_i^\xi(x_1, x_2)$ the directors in the reference configuration $\omega_\xi$ and by $d_i(x_1, x_2)$ the directors in the deformed configuration $\omega_c$ ($i = 1, 2, 3$). The domain $\omega$ is referred to an orthogonal Cartesian frame $Ox_1x_2x_3$ such that $\omega \subset Ox_1x_2$ and let $e_i$ be the unit vectors along the coordinate axes $Ox_i$. Then, the microrotation tensors can be expressed as follows

$$Q_e = d_i \otimes d_i^\xi, \quad R = Q_e Q_0 = d_i \otimes e_i, \quad Q_0 = d_i^\xi \otimes e_i.$$  \hspace{1cm} (43)

Remark 4.1. The initial directors $d_i^\xi$ are usually chosen such that

$$d_3^\xi = n_0, \quad d_3^\xi \cdot n_0 = 0,$$

i.e. $d_3^\xi$ is orthogonal to $\omega_\xi$ and $d_3^\xi$ belong to the tangent plane. This assumption is not necessary in general, but it will be adopted here since it simplifies many of the subsequent expressions. In the deformed configuration, the director $d_3$ is no longer orthogonal to the surface $\omega_c$ (the Kirchhoff-Love condition is not imposed). One convenient choice of the initial microrotation tensor $Q_0 = d_i \otimes e_i$ such that the conditions (44) be satisfied is $Q_0 = \text{pole}(a_i \otimes e_i)$, as it was shown in Remark 10 of 12.

Let us present next the shell strain and curvature measures. In the 6-parameter shell theory the elastic shell strain tensor $E_c$ is defined by

$$E_c = Q_e^T \text{Grad}_m m - a.$$  \hspace{1cm} (45)

To write the components of $E_c$ we insert (42) and (43) into (45)

$$E_c = (d_i \otimes d_i^\xi)(m_{\alpha} \otimes a^\alpha) - a_\alpha \otimes a^\alpha = (m_{\alpha} \cdot d_i - a_\alpha \cdot d_i^\xi) d_i \otimes a^\alpha.$$ 

As a measure of orientation (curvature) change, the elastic shell bending-curvature tensor $K_e$ is defined by

$$K_e = \text{axl}(Q_e^T Q_{e,\alpha}) \otimes a^\alpha = Q_0 [\text{axl}(R_e^T R_{\alpha}) - \text{axl}(Q_0^T Q_{0,\alpha})].$$  \hspace{1cm} (46)
We remark the analogy to the definition of the wryness tensor $\Gamma$ in the three-dimensional theory. Following the analogy to (4), we employ next the surface curl operator $\text{Curl}$, defined in Section 3, to introduce the new shell dislocation density tensor $D_e$ by

$$D_e = Q^e \text{Curl}_s Q_e.$$  \hspace{1cm} (47)

In view of relation (34), we can write this definition in the form

$$D_e = Q^e (-Q_{\varepsilon,\alpha} \otimes a^\alpha) = -((Q^e Q_{\varepsilon,\alpha}) \otimes a^\alpha).$$  \hspace{1cm} (48)

The tensor $D_e$ given by (47) represents an alternative strain measure for orientation (curvature) change in Cosserat-type shells.

In what follows, we want to establish the relationship between the shell bending-curvature tensor $K_e$ and the shell dislocation density tensor $D_e$. We observe that this relationship is analogous to the corresponding relations (19), (20) in the three-dimensional theory. More precisely, in the shell theory it holds

$$D_e = -K_e^T + (\text{tr} K_e) I_3 \quad \text{or equivalently,} \quad K_e = -D_e^T + \frac{1}{2} (\text{tr} D_e) I_3. \hspace{1cm} (49)$$

To prove (49), we designate the components of the shell bending-curvature tensor by $K_e = K_{\varepsilon,\alpha} d^\alpha \otimes a^\alpha$ and use (16) to write

$$(Q^e Q_{\varepsilon,\alpha}) \otimes a^\alpha = -\varepsilon \text{axl} (Q^e Q_{\varepsilon,\alpha}) \otimes a^\alpha = -\varepsilon K_e.$$

As a consequence of relations (49) we deduce the relations between the norms, traces, symmetric and skew-symmetric parts of the two tensors in the forms

$$\|D_e\|^2 = \|K_e\|^2 + (\text{tr} K_e)^2, \quad \|K_e\|^2 = \|D_e\|^2 - \frac{1}{4} (\text{tr} D_e)^2, \quad \text{tr} D_e = 2 \text{tr} K_e, \quad \text{skew} D_e = \text{skew} K_e, \quad \text{dev}_3 \text{sym} D_e = -\text{dev}_3 \text{sym} K_e.$$  \hspace{1cm} (50)

Indeed the relations (50) can be easily proved if we apply the operators tr, $\| \cdot \|$, skew, dev$_3$, and sym to the equation (49). In view of (50) and $(\text{tr} K_e)^2 \leq 3 \|K_e\|^2$, we obtain the estimate

$$\|K_e\| \leq \|D_e\| \leq 2 \|K_e\|. \hspace{1cm} (51)$$
In what follows, we write the components of the tensors $K_e$ and $D_e$. To this aim, we use the relations
\[
Q^T_e Q_{e,\alpha} = (d_i^0 \otimes d_i^0)(d_{k,\alpha} \otimes d_k^0 + d_k^0 \otimes d_{k,\alpha})
= (d_i^0 \cdot d_{k,\alpha})d_i^0 \otimes d_k^0 + d_k^0 \otimes d_{k,\alpha} = (d_i^0 \cdot d_{k,\alpha} - d_k^0 \cdot d_{k,\alpha})d_i^0 \otimes d_k^0,
\] (52)
which can be proved in the same way as equation (19). We compute the axial vector of the skew-symmetric tensor (52) and find (similar to (20))
\[
\text{axl}(Q^T_e Q_{e,\alpha}) = 1/2 \varepsilon_{ijk} (d_j^0 \cdot d_{k,\alpha} - d_k^0 \cdot d_{k,\alpha}) d_i^0.
\] (53)
By virtue of (53) the definition (46) yields
\[
K_e = 1/2 \varepsilon_{ijk} (d_j^0 \cdot d_k^0 - d_k^0 \cdot d_j^0) d_i^0 \otimes a^\alpha
= (d_2^0 \cdot d_3^0 - d_3^0 \cdot d_2^0) d_i^0 \otimes a^\alpha + (d_3^0 \cdot d_1^0 - d_1^0 \cdot d_3^0) d_i^0 \otimes a^\alpha + (d_1^0 \cdot d_2^0 - d_2^0 \cdot d_1^0) d_i^0 \otimes a^\alpha,
\] (54)
which gives the components $K_{\alpha}$ of the shell bending-curvature tensor $K_e$ in the tensor basis $\{d_i^0 \otimes a^\alpha\}$.

For the components of $D_e$, we insert the relation (52) in the equation (49)
\[
D_e = -(d_i^0 \cdot d_{k,\alpha} - d_k^0 \cdot d_{i,\alpha}) (d_i^0 \otimes d_k^0) \times a^\alpha.
\]
Using that $d_i^0 \times a^\alpha = d_i^0 \times ([a^\alpha \cdot d_j^0] d_k^0) = (a^\alpha \cdot d_j^0) \varepsilon_{ijk} d_k^0$, we obtain
\[
D_e = \varepsilon_{ijk} (d_j^0 \cdot d_k^0 - d_k^0 \cdot d_j^0) (a^\alpha \cdot d_j^0) d_i^0 \otimes d_k^0
\] (55)
which shows the components of the shell dislocation density tensor in the tensor basis $\{d_i^0 \otimes d_j^0\}$.

5 Remarks and discussion

Herein we present some other ways to express the shell dislocation density tensor, the shell bending-curvature tensor and discuss their close relationship.

Remark 5.1. It is sometimes useful to express the components of the shell dislocation density tensor $D_e$ in the tensor basis $\{a^\alpha \otimes a_j\}$. If we multiply the relation (19) with $n_0$ and take into account that $K_e n_0 = 0$, we find $0 = -D_e^T n_0 + 1/2 (\text{tr} D_e) n_0$, which means
\[
n_0 D_e = 1/2 (\text{tr} D_e) n_0.
\]
It follows that the components of $D_e$ in the directions $n_0 \otimes a_j$ are zero, i.e. $D_e$ has the structure
\[
D_e = D_e^\parallel + D_e^\perp a^\alpha \otimes n_0 + 1/2 (\text{tr} D_e) n_0 \otimes n_0,
\] (56)
where $D_e^\parallel = D_e \cdot a = D_e^\parallel a_j \otimes a_j \otimes a_j = D_e^\parallel \alpha^\alpha \otimes a_j$ is the planar part of $D_e$ (the part in the tangent plane). If we insert (56) into (19) and use $1/2 \text{tr} D_e = \text{tr} K_e$, we get
\[
D_e^\parallel + D_e^\perp a^\alpha \otimes n_0 + (\text{tr} K_e) n_0 \otimes n_0 = -K_{\alpha} n_0 \otimes n_0 = -K_{\alpha} a^\alpha \otimes d_i^0 + (\text{tr} K_e) (a + n_0) \otimes n_0,
\]
which implies (in view of (53)) that
\[
D_e^\perp = -K_{\alpha} n_0 \otimes n_0 \quad \text{and} \quad D_e^\parallel = -(K_e)^T + (\text{tr} K_e) a^\alpha.
\] (57)
where $K_e = a K_e = K_{\alpha} a^\alpha \otimes a^\alpha$ is the planar part of $K_e$.  

Using the relations (58)-(60) we see that 
\[ T(S) = -S^T + (\text{tr} S) a. \]  
(58)

One can prove that this transformation has the properties
\[ T(T(S)) = S \quad \text{and} \quad T(S) = -c S c, \]  
(59)

where the alternator \( c \) is defined in (51). Moreover, in view of (59) and (51) we can write
\[ T(S) \]  
the tensor in the tensor basis \( \{a^\alpha \otimes a_\beta\} \) as follows
\[ T(S) = S^2_{\alpha \beta} a^\alpha \otimes a_\beta - S^1_{\alpha \beta} a^\alpha \otimes a_\beta + S^0_{\alpha \beta}, \]  
which shows that the 2 \( \times \) 2 matrix of the components of \( T(S) \) in the basis \( \{a^\alpha \otimes a_\beta\} \) is the cofactor of the matrix of components of \( S \) in the basis \( \{a_\alpha \otimes a^\beta\} \), since
\[ \left( \begin{array}{cc} S^2_{11} & S^1_{12} \\ S^2_{21} & S^2_{22} \end{array} \right) = \text{Cof} \left( \begin{array}{cc} S^1_{11} & S^1_{12} \\ S^2_{12} & S^2_{22} \end{array} \right). \]

If the tensor \( S \) is invertible, then from the Cayley-Hamilton relation \( (S^T)^2 - (\text{tr} S) S^T + \det S = 0 \) and (58) we deduce
\[ T(S) = -S^T + (\text{tr} S) a = (\det S) S^{-T} = : \text{Cof}(S). \]  
(60)

In our case, for the shell bending-curvature tensor \( K_e \) we have \( \text{tr} K_e = \text{tr}(aK_e) = \text{tr}(K_\parallel) \), in view of (54). Then, the relation (57) yields
\[ D_\parallel = -(K_\parallel)^T + (\text{tr} K_\parallel) a. \]

Using the relations (58)-(60) we see that \( D_\parallel \) is the image of \( K_\parallel \) under the transformation \( T \), so that it holds
\[ D_\parallel = T(K_\parallel) = -c(K_\parallel)c = \text{Cof}(K_\parallel), \]  
(61)

\[ K_\parallel = T(D_\parallel) = -c(D_\parallel)c = \text{Cof}(D_\parallel). \]

From (56), (57) we can write
\[ D_e = \text{Cof}(K_\parallel) - K_{3a} a^\alpha \otimes n_0 + (\text{tr} K_\parallel) n_0 \otimes n_0, \]  
(62)

which expresses once again the close relationship between the shell dislocation density tensor \( D_e \) and the shell bending-curvature tensor \( K_e \).

**Remark 5.3.** The shell bending-curvature tensor \( K_e \) can also be expressed in terms of the directors \( d_i \). In this respect, an analogous relation to the formula (28) for the wryness tensor (see Remark 2.5) holds
\[ K_e = \frac{1}{2} \left[ Q^T_i (d_i \times d_{i,\alpha}) - d_i^0 \times d_{i,\alpha} \right] \otimes a^\alpha. \]  
(63)

To prove (63), we write the two terms in the brackets in the following form
\[ Q^T_i (d_i \times d_{i,\alpha}) = [d_k^0 \otimes d_{i,\alpha} (d_i \times d_{i,\alpha}) = [d_k \cdot (d_i \times d_{i,\alpha})] d_k^0 = [d_{i,\alpha} \cdot (d_k \times d_i)] d_k^0 = \epsilon_{kij} (d_{i,\alpha} \cdot d_j) d_k^0 \]

and similarly
\[ d_i^0 \times d_{i,\alpha} = [d_k^0 \cdot (d_i \times d_{i,\alpha})] d_k^0 = [d_{i,\alpha} \cdot (d_k^0 \times d_i)] d_k^0 = \epsilon_{kij} (d_{i,\alpha} \cdot d_j) d_k^0. \]
Inserting the last two relations into (63) we obtain
\[ Ke = \frac{1}{2} \epsilon_{ijk} \left[ (d_{j,\alpha} \cdot d_k) d^0_i - (d_{j,\alpha} \cdot d^0_k) d_i^0 \right] \otimes a^\alpha, \]
which holds true, by virtue of (64). Thus, (63) is proved.

We can put the relation (63) in the form
\[ Ke = QT e \omega \]
where we define (64)
\[ \omega = \omega_\alpha \otimes a^\alpha \]
with
\[ \omega_\alpha = \frac{1}{2} \left[ d_i \times d_i,\alpha - Q_e (d_i^0 \times d_i,\alpha) \right]. \]

If we compare the relations (64) and the definition (46), we derive
\[ \omega_\alpha = Q_e axl(Q_e,\alpha) = axl(Q_e,\alpha). \]

Then, from (16) we deduce
\[ Q_{e,\alpha} Q_e^T = \omega_\alpha \times 1_3 \] and by multiplication with \( Q_e \) we find
\[ Q_{e,\alpha} = \omega_\alpha \times Q_e, \quad \alpha = 1, 2. \]

Thus, the equations (64), (65) can be employed for an alternative definition of the shell bending-curvature tensor, namely
\[ Ke = Q_e^T \omega, \quad \omega = \omega_\alpha \otimes a^\alpha \] and \( Q_{e,\alpha} = \omega_\alpha \times Q_e \).

This is the counterpart of the relations (24), (25) for the wryness tensor in the three-dimensional theory of Cosserat continua. The relations (67) were used to define the corresponding shell bending-curvature tensor, e.g., in [35].

Remark 5.4. As shown by relations (3) for the three-dimensional case, one can introduce the elastically stored shell energy density \( W \) as a function of the shell strain tensor and the shell dislocation density tensor
\[ W = W(E_e, D_e). \]

If (68) is assumed to be a quadratic convex and coercive function, then the existence of solutions to the minimization problem of the total energy functional for Cosserat shells can be proved in a similar manner as in Theorem 6 of [12]. In the proof, one should employ decisively the estimate (51) and the expressions of the shell dislocation density tensor \( D_e \) established in the previous sections.

References

[1] H. Altenbach. An alternative determination of transverse shear stiffnesses for sandwich and laminated plates. *Int. J. Solids Struct.*, 37:3503–3520, 2000.

[2] H. Altenbach and V.A. Eremeyev. Direct approach-based analysis of plates composed of functionally graded materials. *Arch. Appl. Mech.*, 78:775–794, 2008.

[3] H. Altenbach and V.A. Eremeyev. On the bending of viscoelastic plates made of polymer foams. *Acta Mech.*, 204:137–154, 2009.

[4] H. Altenbach and V.A. Eremeyev. On the effective stiffness of plates made of hyperelastic materials with initial stresses. *Int. J. Non-Linear Mech.*, 45:976–981, 2010.

[5] H. Altenbach and P.A. Zhilin. Eine nichtlineare Theorie dünner Dreischichtschalen und ihre Anwendung auf die Stabilitätsuntersuchung eines dreischichtigen Streifens. *Technische Mechanik*, 3:23–30, 1982.
[6] H. Altenbach and P.A. Zhilin. A general theory of elastic simple shells (in russian). Uspekhi Mekhaniki, 11:107–148, 1988.

[7] H. Altenbach and P.A. Zhilin. The theory of simple elastic shells. In R. Kienzler, H. Altenbach, and I. Ott, editors, Theories of Plates and Shells. Critical Review and New Applications, Euromech Colloquium 444, pages 1–12. Springer, Heidelberg, 2004.

[8] J. Altenbach, H. Altenbach, and V.A. Eremeyev. On generalized Cosserat-type theories of plates and shells: a short review and bibliography. Arch. Appl. Mech., 80:73–92, 2010.

[9] G. Backus, R. Parker, and C. Constable. Foundations of Geomagnetism. Cambridge University Press, Cambridge, 1996.

[10] M. Bîrsan and H. Altenbach. A mathematical study of the linear theory for orthotropic elastic simple shells. Math. Methods Appl. Sci., 33:1399–1413, 2010.

[11] M. Bîrsan and H. Altenbach. Analysis of the deformation of multi-layered orthotropic cylindrical elastic shells using the direct approach. In H. Altenbach and V.A. Eremeyev, editors, Shell-like Structures: Non-classical Theories and Applications, pages 29–52. Springer-Verlag, Berlin Heidelberg, 2011.

[12] M. Bîrsan and P. Neff. Existence of minimizers in the geometrically non-linear 6-parameter resultant shell theory with drilling rotations. Math. Mech. Solids, 19(4):376–397, 2014.

[13] M. Bîrsan and P. Neff. Shells without drilling rotations: A representation theorem in the framework of the geometrically nonlinear 6-parameter resultant shell theory. Int. J. Engng. Sci., 80:32–42, 2014.

[14] M. Bîrsan, T. Sadowski, and D. Pietras. Thermoelastic deformations of cylindrical multi-layered shells using a direct approach. J. Thermal Stresses, 36:749–789, 2013.

[15] J. Chróscielewski, J. Makowski, and W. Pietraszkiewicz. Statics and Dynamics of Multifold Shells: Nonlinear Theory and Finite Element Method (in Polish). Wydawnictwo IPPT PAN, Warsaw, 2004.

[16] E. Cosserat and F. Cosserat. Théorie des corps déformables. Hermann et Fils (reprint 2009), Paris, 1909.

[17] V.A. Eremeyev, L.P. Lebedev, and H. Altenbach. Foundations of Micropolar Mechanics. Springer, Heidelberg - New York - Dordrecht - London, 2013.

[18] V.A. Eremeyev and W. Pietraszkiewicz. The nonlinear theory of elastic shells with phase transitions. J. Elasticity, 74:67–86, 2004.

[19] V.A. Eremeyev and W. Pietraszkiewicz. Local symmetry group in the general theory of elastic shells. J. Elasticity, 85:125–152, 2006.

[20] I.D. Ghiba, P. Neff, A. Madeo, L. Placidi, and G. Rosi. The relaxed linear micromorphic continuum: Existence, uniqueness and continuous dependence in dynamics. Math. Mech. Solids, 20:1171–1197, 2015.

[21] M.E. Gurtin. An Introduction to Continuum Mechanics., volume 158 of Mathematics in Science and Engineering. Academic Press, London, 1. edition, 1981.

[22] J. Lankeit, P. Neff, and F. Osterbrink. Integrability conditions between the first and second cosserat deformation tensor in geometrically nonlinear micropolar models and existence of minimizers. ArXiv:1504.08003, 2016.
[23] A. Libai and J.G. Simmonds. *The Nonlinear Theory of Elastic Shells*. Cambridge University Press, Cambridge, 2nd edition, 1998.

[24] A. Madeo, P. Neff, I.D. Ghiba, L. Placidi, and G. Rosi. Wave propagation in relaxed linear micromorphic continua: modelling metamaterials with frequency band gaps. *Cont. Mech. Therm.*, 27:551–570, 2015.

[25] A. Mielke and S. Müller. Lower semi-continuity and existence of minimizers in incremental finite-strain elastoplasticity. *Z. Angew. Math. Mech.*, 86:233–250, 2006.

[26] P. Neff, M. Bırsan, and F. Osterbrink. Existence theorem for geometrically nonlinear Cosserat micropolar model under uniform convexity requirements. *J. Elasticity*, 121:119–141, 2015.

[27] P. Neff, I.D. Ghiba, A. Madeo, L. Placidi, and G. Rosi. A unifying perspective: the relaxed linear micromorphic continua. *Cont. Mech. Therm.*, 26:639–681, 2014.

[28] P. Neff and I. Münch. Curl bounds Grad on SO(3). *ESAIM: Control, Optimisation and Calculus of Variations*, 14:148–159, 2008.

[29] J. Nye. Some geometrical relations in dislocated crystals. *Acta Metall.*, 1:153–162, 1953.

[30] W. Pietraszkiewicz and V.A. Eremeyev. On natural strain measures of the non-linear micropolar continuum. *Int. J. Solids Struct.*, 46:774–787, 2009.

[31] T. Sadowski, M. Bırsan, and D. Pietras. Multilayered and FGM structural elements under mechanical and thermal loads. Part I: Comparison of finite elements and analytical models. *Archives of Civil and Mechanical Engineering*, 15:1180–1192, 2015.

[32] B. Svendsen. Continuum thermodynamic models for crystal plasticity including the effects of geometrically necessary dislocations. *J. Mech. Phys. Solids*, 50(25):1297–1329, 2002.

[33] J. Tambača and I. Velčić. Existence theorem for nonlinear micropolar elasticity. *ESAIM: Control, Optimisation and Calculus of Variations*, 16:92–110, 2010.

[34] P.A. Zhilin. Mechanics of deformable directed surfaces. *Int. J. Solids Struct.*, 12:635–648, 1976.

[35] P.A. Zhilin. *Applied Mechanics – Foundations of Shell Theory (in Russian)*. State Polytechnical University Publisher, Sankt Petersburg, 2006.