HYPERBOLIC BRUNNIAN LINKS

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ABSTRACT. We present practical methods to show a Brunnian link is hyperbolic. The methods, belonging to geometric topology, are based on observation, worked by hand and generally applicable. Typical examples illustrate their efficiency. We thus determine hyperbolicity for all of the known Brunnian links in literatures except two series. Especially, we discover 6 infinite series of hyperbolic Brunnian links.

FIGURE 1. A hyperbolic Brunnian link: Cirrus in [5].

1. INTRODUCTION

Every minimal nontrivial sublink of a link is either a knot or a Brunnian link. [4] shows that hyperbolic Brunnian links and $(2, 2n)$–torus links form building blocks
of Brunnian links in that they generate all Brunnian links by satellite operations. In this paper, we give practical methods to show a Brunnian link is hyperbolic.

A classical theorem by Alexander states that every smooth torus bounds at least one solid torus in $S^3$. We say a torus in $S^3$ is knotted if it bounds only one solid torus in $S^3$; otherwise, unknotted. In [4], A Brunnian link, other than Hopf link, is called s-prime/untied if there is no essential unknotted/knotted torus in its complement space. [4] further states that if a Brunnian link $L$ is s-prime, untied and not a $(2, 2n)$–torus link, then $L$ is a hyperbolic link. So the two main tasks in this paper are to discriminate whether a Brunnian link is untied and s-prime.

Extending [5], we will cope with these two tasks using geometric topology methods. [5] provides two intuitive methods, generally effective and performed by handwork, for detecting Brunnian property of a link if each component is known to be unknot. In Section 4, we will prove a universal criterion to detect whether a Brunnian link is untied.

**Theorem 1.1. (Untiedness Criterion)**
Let $L = \bigcup_{i=1}^{n} C_i$ be an $n$-component Brunnian link and $k \in \{1, \ldots, n\}$. For each $1 \leq i \leq k$, let $D_i$ be a disk bounded by $C_i$ so that $U = \bigcup_{i=1}^{k} D_i \cup L$ is a spanning complex. Suppose for every $1 \leq i \leq k$, $D_i$ is (s7). Then $L$ is untied if and only if there is no incompressible knotted torus in the complement of $U$.

To show that a Brunnian link is s-prime, instead of giving a criterion, we will first prove a universal intermediate theorem in Section 5, based on which we will provide four methods to show s-primeness in Section 6.

**Theorem 1.2. (Simple Intersection Pattern Theorem)**
Let $L = \bigcup_{i=1}^{n} C_i$ be an $n$-component Brunnian link. Let $T$ be an essential torus in the complement of $L$, splitting $S^3$ into two solid tori $V_I$ and $V_J$ so that $L \cap V_I = L_I$ and $L \cap V_J = L_J$, where $L_I = \bigcup_{i \in I} C_i$ and $L_J = \bigcup_{j \in J} C_j$.

Suppose $Q \cup R \cup S \subset I$ and $J_0 \subset J$. Let $D_Q^I(q \in Q)$, $D_J^J(j \in J_0)$, $D_R^0(r \in R)$ and $D_S^I(s \in S)$ be $L_I$-interior disks, $L_J$-interior disks, exterior cross disks, and free cross disks respectively, so that the cross disks are mutually disjoint and each disk is stable. If the union of these disks and $L$ forms a spanning complex, then after an isotopy of $T$, every disk intersects $T$ in simple intersection pattern.

We will define (s7), stable and spanning complex in Section 3 and $L_I$-interior disks, $L_J$-interior disks, exterior cross disks, free cross disks and simple intersection pattern in Subsection 3.1. In fact, the two corresponding theorems in the text will be stronger than the two presented here, but the conditions will be more complicated. We will illustrate our methods by typical examples.

Then we are interested in how many Brunnian links in history, which, to our best knowledge, all appear in [6, 8, 20, 13, 2, 15, 12, 11, 18, 16, 5], are hyperbolic. Our methods work to detect s-primeness for all of them and untiedness for all except Fountains and Jade-pendants in [5]. Especially, we list some infinite series of Brunnian links:

1. Lamp($n_1, n_2, \ldots, n_{2k}$), where $k \in \mathbb{N}_+$ and each index is an odd integer indicating the number of half twists as shown in Fig. 2.
2. deBrunner($n$), where $n \leq 5$, as shown in Fig. 3 (c.f. 20)
3. $W(n)$, where $n > 1$, as shown in Fig. 4 (c.f. 20)
4. Torus($m, n$), where $m, n \in \mathbb{N}_+$, as shown in Fig. 5 (c.f. 2-3)
(5) Tube\((m, n)\), where \(m > 0, n > 1\), as shown in Fig. 6

(6) Carpet\((m, n, p)\), where \(m, n, p \in \mathbb{N}_+\) and \(m < n\), as shown in Fig. 7

**Theorem 1.3.** The six series of links above are hyperbolic.

Our methods are worked by hand. An alternative to showing a link is hyperbolic is to use SnapPy. However, a program cannot show hyperbolicity for infinite series of links. By the theorem of W. Menasco\cite{17}, it can be verified that every alternating Brunnian link is hyperbolic. Nevertheless, these newly discovered series are not alternating at least seen from the diagrams, and our methods provide alternative proofs for hyperbolicity of alternating Brunnian links in literatures.

In the last section, we will generalize our criteria and framework to wider classes of links.

**Figure 2.** Lamp\((n_1, n_2, ..., n_{2k})\)

**Figure 3.** deBrunner(5).

**Figure 4.** W(5), constructed by H. Brunn in \cite{6}.

**Figure 5.** Torus\((m, n)\) has \(2m\) rows and each row has \(n\) components.
**Figure 6.** Tube(3,4), a tube with 3 rows and 4 columns.

**Figure 7.** Carpet(1,3,4), see [5] for notations.

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2. Preliminaries

In this paper, all objects and maps are smooth. Without special explanation, we always consider links in $S^3$. All intersections are compact and transverse. Each arc is assumed to be simple. The notations $C, L, D$ (maybe with subscripts) are always used to denote an unknot, a link and an embedded disk respectively. We use $N(\cdot)$ to denote a regular neighborhood.

For a compact set $K$ in a solid torus, $K$ is geometrically essential in it if any meridian disk intersects $K$. If $V$ is a regular neighborhood of a knot $K$ in $S^3$, the longitude of $V$ is the essential curve in $\partial V$ that is null-homologous in $S^3 - intV$, oriented similarly to $K$; the meridian is the essential curve in $\partial V$ bounding a disk in $V$ and having link number $+1$ with $K$. (c.f. [7])

Definition 2.1. [4] Let $L$ and $L'$ be links in $S^3$, and $C \subset L$ and $C' \subset L'$ be oriented unknotted components. A homeomorphism $h : S^3 - intN(C') \to N(C)$ maps the oriented meridian of $N(C')$ to the oriented longitude of $N(C)$ and maps the oriented longitude of $N(C')$ to the oriented meridian of $N(C)$, where $N(C)$ is the regular neighborhood of $C$. Then the link $(L - C) \sqcup h(L' - C')$ is the s-sum of $L(C)$ and $L'(C')$.

Definition 2.2. [4] Let $L_0 \sqcup L^n$ be a link in $S^3$ where $L^n = \sqcup_{i=1}^{n} C^i$ is an oriented unlink, and $k_i, i = 1, ... , n$, be nontrivial knot types. Let $h : U_n = S^3 - intN(L^n) \to S^3$ be an orientation-preserving embedding so that

$$S^3 - h(U_n) \cong \sqcup_{i=1}^{n} (S^3 - N(k_i)),$$

where $h(\partial N(C^i))$ corresponds to $\partial N(k_i)$, and the oriented meridian of $N(C^i)$ maps to the oriented null-homologous curve in $S^3 - h(U_n)$ corresponding to the oriented longitude of $N(k_i)$. Then $L = h(L_0)$ is an s-tie.

For Brunnian links, we have

Theorem 2.3. [4] If a torus $T$ in $S^3$ splits a Brunnian link $L$, then $T$ is incompressible, and $L$ is decomposed by $T$ as an s-sum of two Brunnian links.

Theorem 2.4. [4] Every knotted essential torus in the complement of a Brunnian link bounds the whole link in the solid torus side.

3. Credible disks and stable disks

In this section, we introduce our main tools and some notions for links with some unknotted components, and confirm these tools are always available.
3.1. Credible disks. Our basic tool is credible disk.

Definition 3.1. Let $C_i$ be a component of a link $L$ and $D_i$ be a disk bounded by $C_i$. $D_i$ is credible if there is no circle $C$ in the interior of $D_i$ such that

(i) the disk in $D_i$ bounded by $C$ intersects $L$;
(ii) there is a disk $D_C$ bounded by $C$ with $D_C \subset S^3 - L$.

To simplify the task of showing a disk is credible, we give a definition used for the proposition following it.

Definition 3.2. Let $L = \cup_{i=1}^n C_i$ be an $n$-component link and $I \subset \{1, ..., n\}$. Take disjoint disks $D_i(i \in I)$ bounded by $C_i(i \in I)$ and set $U = \cup_{i\in I}D_i \cup L$. A circle $C$ in the interior of $D_i$ is incredible for $U$ if

(i) the disk in $D_i$ bounded by $C$ intersects $L$;
(ii) there is a disk $D_C$ bounded by $C$ with $\text{int} D_C \subset S^3 - U$.

Proposition 3.3. Let $L = \cup_{i=1}^n C_i$ be an $n$-component link and $I \subset \{1, ..., n\}$. Take disjoint disks $D_i(i \in I)$ bounded by $C_i(i \in I)$ and set $U = \cup_{i\in I}D_i \cup L$. Then $D_i$ is credible for each $i \in I$ if and only if there is no incredible circle for $U$.

Proof. Since the “only if” implication is obvious, we need only proof the “if” implication. Suppose for an $i \in I$, there is a circle $C \subset \text{int} D_i$ such that the disk in $D_i$ bounded by $C$ intersects $L$ and there is a disk $D_C$ bounded by $C$ with $D_C \subset S^3 - L$. We wish to find an incredible circle for $U$.

After a perturbation near $C$, we may assume $D_C \cap \cup_{i\in I}D_i$ is a disjoint union of circles. Choose an innermost circle in $D_C$, which bounds a disk, say $D_0$, in it. Suppose $C_0 = \partial D_0$ is in $D_j$ for some $j \in I$. If $C_0$ encloses some intersection points with $L$ in $D_j$, then $C_0$ is an incredible circle for $U$. Otherwise, $D_0$ caps a 3-ball with $D_j$. Replacing $D_0$ in $D_C$ by the disk in $D_j$ bounded by $C_0$, a little beyond, we eliminate $C_0$. Step by step, we find an incredible circle for $U$. \qed

Noting that the complement of $U$ is smaller than the component of $L$, this proposition reduces the task of showing a disk is credible in application. In fact, the main content of [5] is to show disks are credible, which proposes two widely applicable methods, including several criteria and a general procedure.

3.2. Stable disks. Sometimes we will use a special kind of credible disks, called stable disks. Let $C_i$ be a component of a link $L$ and $D_i$ be a disk bounded by $C_i$. Let $L_0$ consist of the components of $L$ which intersects the interior of $D_i$, and $L_J$ be a sublink of $L - C_i$ containing $L_0$.

Definition 3.4. The disk $D_i$ is stable for $L_J$ if $\sharp(D_i \cap L_J)$ is minimal among all disks bounded by $C_i$ whose interior only intersect $L_J$, and $D_i$ is stable if it is stable for $L_0$.

Clearly a stable disk is credible and the converse is not true in general. To reduce the task of showing a disk is stable, we introduce the following definition.

Definition 3.5. A circle $C$ in the interior of $D_i$ is unstable for $L_J$ if $C$ bounds a disk $D_{C_1}$ such that

(i) $D_{C_1} \cap L = D_{C_1} \cap L_J$;
(ii) $\sharp(D_{C_1} \cap L) < \sharp(D_{C_0} \cap L)$, where $D_{C_0}$ is the disk in $D_i$ bounded by $C$. 
Obviously that $D_i$ is stable for $L_J$ implies that there is no unstable circle for $L_J$. The following lemma shows the converse is also true.

**Lemma 3.6.** There is no unstable circle for $L_J$ in $D_i$ if and only if $D_i$ is stable for $L_J$.

**Proof.** Suppose there is no unstable circle for $L_J$ on $D_i$. Assume for contradiction that $D'_i$ is a disk bounded by $C_i$, having less number of intersection points with $L_J$ than $D_i$. After perturbation, $D'_i$ intersects $D_i$ transversely in their interiors. Choose an innermost circle in $D'_i$. Using the same argument as in the proof of Lemma 3.8(2), we get a contradiction. \hfill $\square$

The task of showing a disk is stable can be further reduced by considering mutually disjoint disks. Let $L = \bigcup_{i = 1}^{n} C_i$ be an $n$-component link and $I \subset \{1, \ldots, n\}$. Take disjoint disks $D_i(i \in I)$ bounded by $C_i(i \in I)$.

**Definition 3.7.** For any $k \in I$, a circle $C$ in the interior of $D_k$ is unstable avoiding $\cup_{i \in I} D_i$ if $C$ bounds a disk $D_{C1}$ such that

1. $D_{C1} \cap \cup_{i \in I} D_i = C$;
2. $\sharp(D_{C1} \cap L) < \sharp(D_{C0} \cap L)$, where $D_{C0}$ is the disk in $D_i$ bounded by $C$.

**Lemma 3.8.** (1) Fix $k \in I$. Then there is no unstable circle in $D_k$ avoiding $\cup_{i \in I} D_i$ if and only if there is no disk $D$ bounded by $C_k$ such that

1. $D \cap \cup_{i \in I} D_i = C_k$, and
2. $\sharp(D \cap (L - C_k)) < \sharp(D_k \cap (L - C_k))$.

(2) For any $k \in I$ there is no unstable circle in $D_k$ avoiding $\cup_{i \in I} D_i$ if and only if for any $k \in I$ there is no unstable circle in $D_k$ for $L - \cup_{i \in I} C_i$.

**Proof.** (1). The “only if” implication is trivial. For the “if” implication, suppose there is an unstable circle $C$ bounding $D_{C0}$ in $D_k$ and $D_{C1}$ as in Definition 3.7. Then the disk $(D_k - D_{C0}) \cup D_{C1}$ has less intersection points with $L - C_k$, and after perturbation, its interior is disjoint from $D_k$.

(2). Since the “if” implication is obvious, we need only proof the “only if” implication. Set $L_J = L - \cup_{i \in I} C_i$. Suppose there is a circle $C$ in the interior of some $D_k$, bounding $D_{C0}$ in $D_k$ and bounding $D_{C1}$ as in Definition 3.7. Then $D_{C1} \cap \cup_{i \in I} D_i$ is disjoint union of circles. Let $C_0$ be an innermost circle in $D_{C1}$, which bounds a disk $D_0$ in it and bounds a disk $D_{j0}$ in some $D_j$. Since $D_j$ is stable avoiding $\cup_{i \in I} D_i$, $D_{j0}$ has no more number of intersection points with $L_J$ than $D_0$. Take the immersed disk $(D_{C1} - D_0) \cup D_{j0}$ and push the $D_{j0}$ in it a little downward $D_j$ to eliminate $C_0$. We denote the obtained immersed disk by $E_{C2}$.

![Figure 8](image.png)

**Figure 8.** Two kinds of surgeries to eliminate double curves.
Generally $E_{C_2}$ may not be embedded since $C_0$ may not be innermost in $D_j$. Nevertheless, its singularities consists of only double points, forming disjoint circles. This is an easy case of Dehn’s Lemma(21, 9), so we can revise it to get an embedding disk. For each self-intersection circle, change the immersion map either on two disks or in an annulus region, as illustrated in Figure 8, and then smooth the corners. We will get an embedding disk $D_{C_2}$ having the same intersection points with $L_J$ as $E_{C_2}$.

Step by step we eventually get a disk $D_{CN}$, which intersects $\bigcup_{i \in I} D_i$ in $C$ and has no more number of intersection points with $L_J$ than $D_{C_1}$. Then $D_{CN}$ has less number of intersection points with $L_J$ than $D_{C_0}$. Thus $C$ is unstable avoiding $\bigcup_{i \in I} D_i$, a contradiction. □

In summary, the following proposition simplifies significantly the task of showing a disk is stable.

**Proposition 3.9.** Let $L = \bigcup_{i=1}^n C_i$ be an $n$-component link and $I \subset \{1, \ldots, n\}$. Let $D_i (i \in I)$ be disjoint disks bounded by $C_i (i \in I)$. Then every $D_i (i \in I)$ is stable for $L - \bigcup_{i \in I} C_i$ if and only if for any $i \in I$ there is no disk $D$ bounded by $C_i$ such that

(i) $D \cap \bigcup_{k \in I} D_k = C_i$;
(ii) $\sharp(D \cap (L - C_i)) < \sharp(D \cap (L - C_i))$.

**Proof.** This follows immediately from the previous two lemmas. □

Applying this proposition, we present how to prove a disk is stable by two examples.

**Example 3.10.** Consider the Milnor link in Fig. 9(1). The components $C_1, C_2, C_3$ bound mutually disjoint oriented disks $D_1, D_2, D_3$ as Fig. 9(2), and the component $C_0$ is homotopically nontrivial in $S^3 - \bigcup_{i=1}^3 C_i$, whose fundamental group is freely generated by $g_1, g_2, g_3$ as Fig. 9(2). To obtain the representation of a loop in $\pi_1(S^3 - \bigcup_{i=1}^3 C_i)$, just record the intersections of the loop and the disks in order with orientations. Notice that $C_0$ intersects $D_1$ and $D_2$ both in 4 points, and intersects $D_3$ in two points. With the base point $P \in C_0$, we see $C_0$ represents $[g_1, g_2, g_3]$ in $\pi_1(S^3 - \bigcup_{i=1}^3 C_i)$. We show $D_1, D_2, D_3$ are all stable.

![Figure 9. Stable disks for Milnor link.](image)

**Proof.** Since $[g_1, g_2, g_3]$ is a reduced word, unique when generators of the free group are fixed, any isotopy of $D_1 \cup D_2 \cup D_3$ cannot give less intersection points with $C_0$ for each disk. To show $D_1$ is stable, it suffices to show there is no disk bounded by $C_1$, whose interior disjoint from $D_1 \cup D_2 \cup D_3$, has less intersection.
points with $C_0$ than $C_1$. Suppose such a disk $D'_1$ exists, then $S_1 = D_1 \cup D'_1$ is a sphere separating $D_2$ and $D_3$. There are two cases.

**CASE 1:** The positive side of $D_1$ and $C_2$ belong to one side of $S_1$. Consider the element $C_0$ represents in $\pi_1(S^3 - \cup_{i=1}^3 C_i)$, that is $g_1 g_2 g_1^{-1} g_2^{-1} g_3 g_2 g_1^{-1} g_2^{-1} g_1 g_3^{-1}$. We can assume the base point $P$ is in the negative side of $S_1$ and find an arc from $P$ representing $g_1 g_2 g_1^{-1}$, which avoids $D'_1$. Then the fourth letter $g_2^{-1}$ means we have to go to the positive side of $S_1$ to have an arc representing $g_2^{-1}$. But the third letter $g_3^{-1}$ means the arc representing the fourth letter $g_2^{-1}$ avoids $D_1$, so it must intersect $D'_1$. In other words, between the third letter and the fourth letter, $C_0$ intersects $D'_1$.

In the same way, between the fourth letter $g_2^{-1}$ and the fifth letter $g_3$, between the fifth letter $g_3$ and the sixth letter $g_2$, and between the sixth letter $g_2$ and the seventh letter $g_1$, our loop intersects $D'_1$.

In short, in the representation $g_1 g_2 g_1^{-1} g_2^{-1} g_3 g_2 g_1^{-1} g_2^{-1} g_1 g_3^{-1}$, replacing $g_2$ and $g_2^{-1}$ by $2$, $g_3$ and $g_3^{-1}$ by $3$, $g_1$ by $1-1^+$, and $g_1^{-1}$ by $1+1^-$, we get a cyclic sequence

$$1-1^+ 21+1^- \mid 2 \mid 3 \mid 2 \mid 1-1^+ 21+1^- 3.$$

Since $1^+, 2$ belong to one side, and $1^-, 3$ belong to the other side, the number of how many adjacent pairs in this cyclic sequence cross through $S_1$ gives the minimum number of $D'_1 \cap C_0$, which is 4.

**CASE 2:** The positive side of $D_1$ and $C_3$ belong to one side of $S_1$. Make replacement in the representation $g_1 g_2 g_1^{-1} g_2^{-1} g_3 g_2 g_1^{-1} g_2^{-1} g_1 g_3^{-1}$ as before. Now $1^+, 3$ belong to one side, and $1^-, 2$ belong to the other side. The number of adjacent pairs in this cyclic sequence cross through $S_1$ is 8, as

$$1-1^+ \mid 2 \mid 1+1^- 2 \mid 3 \mid 21+1^- \mid 2 \mid 1+1^- \mid 3 \mid.$$

The same method applies to $D_2, D_3$. \hfill \Box

![Figure 10. Lamp(1,1,1,1,1,1,1,1).](image)

**Example 3.11.** Consider the link in Fig. 10(1). We show the grey disk $D$, bounded by $C_1$ in Fig. 10(2) is stable. Let $D'$ be a disk bounded by $C_1$ whose interior avoids $D$. It suffices to show $\sharp(D' \cap C_2) \geq \sharp(D \cap C_2)$. The disk $D$ cuts $C_2$ into 8 arcs $\alpha_1, \alpha_2, \ldots, \alpha_8$, successively indexed along $C_2$ as shown in Fig. 10(2). Notice that each $\alpha_i$ can only intersect $D'$ in even number of points, since the two endpoints of each $\alpha_i$ are on the same side of $D$. The result will be proved by showing that for any odd $i$, either $\sharp(D' \cap \alpha_i) \geq 2$ or $\sharp(D' \cap \alpha_{i+1}) \geq 2$. 
In fact, we can say $lk(\alpha_i, \alpha_{i+1}) = 1$ in the sense that connecting the endpoints of $\alpha_i$ by an arc on the positive side of $D$ and connecting the endpoints of $\alpha_{i+1}$ by an arc on the negative side of $D$ gives two curves with linking number equal to 1. See Fig. 10. If $D'$ intersects neither $\alpha_i$ nor $\alpha_{i+1}$, it would cap off $\alpha_i$ from $\alpha_{i+1}$ and thus $lk(\alpha_i, \alpha_{i+1}) = 0$, a contradiction.

3.3. Spanning complex. We now introduce our main tool in this paper.

Definition 3.12. Let $L = \bigcup_{i=1}^{n} C_i$ be an $n$-component link. Fix $I \subseteq \{1, ..., n\}$ and for any $i \in I$, let $D_i$ be a credible disk bounded by $C_i$. The union $U = \bigcup_{i \in I} D_i \cup L$ is a spanning complex for $L$ if the following regularity conditions hold:

(Ri) $U$ has only generic singularities of double or triple points.
(Rii) (No trivial intersection circle.) $\forall i, j \in I$, there is no circle component $C$ of $D_i \cap D_j$ such that the disk bounded by $C$ in $D_i$ does not intersect $L$.
(Riii) (No trivial annulus region.) $\forall i, j \in I$, there is no pair of circle components of $D_i \cap D_j$ such that they bound annuli in both $D_i$ and $D_j$ and neither annulus intersects $L$.

We point out that, given the credible disks $D_i (i \in I)$, one can always modify $\bigcup_{i \in I} D_i \cup L$ to meet the above regularity conditions. In fact, general differential topology guarantees condition (Ri). For condition (Rii), since $D_j$ is also credible, if such $C$ exists, then the disk bounded in $D_j$ by $C$ also does not intersect $L$. Suppose $C$ bounds $D_{C_i}$ and $D_{C_j}$ in $D_i$ and $D_j$, respectively. Interchanging $D_{C_i}$ by $D_{C_j}$ and then smoothing the corners near $C$, we can eliminate $C$. The trouble is that the new $D_i$ and $D_j$ may have self-intersection. Notice that the singularities can only be double points. By changing the immersion map and smoothing the corners, as in the second paragraph of the proof of Lemma 3.8, we get embedded ones. For condition (Riii), if such pair of intersection circles exists, say $C_{i_1} \cup C_{i_2}$, which bounds $A_i$ in $D_i$ and bounds $A_j$ in $D_j$. Interchanging $A_j$ and $A_i$ and then smoothing the corners, we can eliminate $C_{i_1} \cup C_{i_2}$. The new $D_i$ and $D_j$ may have self-intersection, but we can get embedded new $D_i$ and $D_j$ from them as above.

We conclude this section with a definition used in the statements of our theorems.

Definition 3.13. Let $C_i$ be a component of a link $L$ and $D_i$ be a credible disk bounded by $C_i$. For a natural number $N$, we say $D_i$ is $(sN)$ if either

(i) $\sharp (D_i \cap (L - C_i)) < N$; or
(ii) $D_i$ is stable.

4. Criterion for Untiedness

In this section we give a sufficient and necessary criterion to detect untiedness for Brunnian links and illustrate our method by examples.

4.1. Decision Theorem and Example analysis.

Theorem 4.1. (Untiedness Criterion)

Let $L = \bigcup_{i=1}^{n} C_i$ be an $n$-component Brunnian link and $k \in \{1, ..., n\}$. For each $i \leq k$, let $D_i$ be a disk bounded by $C_i$ so that $U = \bigcup_{i=1}^{k} D_i \cup L$ is a spanning complex. Suppose for every $i \leq k$,

\[
\begin{dcases}
D_i \text{ is } (s7), & \text{if } n = 2 \text{ and } lk(C_1, C_2) = 1; \\
D_i \text{ is } (s8), & \text{in other cases.}
\end{dcases}
\]
Then $L$ is untied if and only if there is no incompressible knotted torus in the complement of $U$.

The proof of this theorem will be given at the end of this section.

Recall that a spanning complex is always available. The “if” implication of this theorem gives a method to show a Brunnian link is untied. The “only if” implication indicates this method is theoretically universal. Furthermore, if a Brunnian link is an $s$-tie, we can always find a companion torus by choosing a spanning complex $U$ satisfying the condition in Theorem 4.1 since an incompressible knotted torus in $S^3 - U$ is a desired companion.

Let us illustrate our method with an example.

![Figure 11. Brunn’s chain.](image)

**Example 4.2.** Consider Brunn’s chain in Fig. 11(1). Take the disks as shown in Fig. 11(2), which are all credible as demonstrated in [5]. The union of these disks forms a spanning complex, say $U$. Since $S^3 - \text{int}N(U)$ is a handlebody, where $N(U)$ is a regular neighborhood of $U$, by Theorem 4.1 Brunn’s chains are untied.

From this example, we can see that this method is intuitive and efficient. It works conveniently to show all the Brunnian links in literatures, except Fountains and Jade-pendant in [5], are untied.

4.2. **Proof of Theorem 4.1**

The “if” implication:

Suppose that $L$ is an $s$-tie. Let $T$ be an essential torus in $S^3 - L$, bounding a solid torus $V$ in $S^3$ containing $L$. We may assume $T$ is an *outermost* essential torus. More formally, there is no essential torus $T_0$ in $S^3 - L$ so that

(i) $T_0$ bounds a solid torus containing $V$,
(ii) $T_0$ is not parallel to $T$.

The intersection $\bigcup_{i=1}^k D_i \cap T$ is a union of circles, forming a graph in $T$. We wish to make $D_i$ into $V$ one by one by an isotopy of $T$. First consider $D_1$. We do this in three steps.

**Step 1.** Eliminate inessential circles in $T$.

Let $C$ be a circle component of $D_1 \cap T$, bounding a disk $D_{CT}$ in $T$ and a disk $D_{C1}$ in $D_1$. Since $D_1$ is credible, $D_{C1}$ does not intersect $L$. Thus we have the following two observations:

(i) Any circle component of $D_1 \cap T$ contained in $D_{C1}$ is inessential in $T$. This is because $T$ is essential.
(ii) For any \( i > 1 \), there is no circle component of \( D_1 \cap D_i \) contained in \( D_{C_1} \).
This is by regularity condition \((\text{Rii})\).

By (i), we may assume \( C \) is innermost in \( D_1 \) as a circle component of \( D_1 \cap T \). Then \( D_{C_1} \cup D_{CT} \) bounds a 3-ball, say \( B \). Pushing \( D_{CT} \) across \( B \) a little beyond \( D_{C_1} \), we eliminate \( C \). Do it inductively. After finitely many steps, \( D_1 \cap T \) contains no inessential circles in \( T \).

To guarantee the validity of making \( D_i(i > 1) \) into \( V \) later, we will need the following two more observations:

(iii) The singularities of \( U \) contained in \( D_{C_1} \) consists of proper arcs. This follows from that \( D_{C_1} \) does not intersect \( L \) and (ii).

(iv) For any \( i > 1 \), \( D_i \cap B \) is a proper surface, and each boundary component of it is either contained in \( D_{CT} \) or is an alternating sequence of properly embedded arcs in \( D_{CT} \) and in \( D_{C_1} \). This follows from (iii).

See Fig. 12 for an example.

![Figure 12](image-url)

**Figure 12.** The red circles in \( D_{CT} \) are intersections with \( D_1 \). The black graph is intersection with \( D_i(i > 1) \).

**Step 2.** Eliminate exterior annulus regions in \( D_1 \).
Consider a component of \( D_1 \cap T \) innermost in \( D_1 \). Since it bounds a disk in \( D_1 \), it must be the meridian of \( V \). Thus \( D_1 \cap T \) is a union of meridians of \( V \), cutting \( D_1 \) into regions alternatively in \( \text{int} V \) and in \( S^3 - V \). We call a component of \( D_1 - T \) in \( \text{int} V \) an **interior region** in \( D_1 \), and call a component of \( D_1 - T \) in \( S^3 - V \) an **exterior region** in \( D_1 \). Clearly every exterior region does not intersect \( L \).

We claim that if annulus \( A \) is an exterior region in \( D_1 \), then \( A \) is \( \partial \)-parallel to \( T \). In fact, if otherwise, then \( \partial A \) would cut \( T \) into two annuli \( A_1 \) and \( A_2 \) such that both \( A \cup A_1 \) and \( A \cup A_2 \) bound knot complements in \( S^3 - \text{int} V \). This implies that the core of \( V \) is a connected sum of two knots, contradicting that \( T \) is outermost.

So we can isotope \( T \) to make \( A \) into \( V \). Step by step, we can eliminate all the exterior annulus regions in \( D_1 \).

**Step 3.** Verify \( D_1 \subset V \).
Consider \( D_1 \) cut by \( D_1 \cap T \). We have the following three observations on the regions in \( D_1 \):

(v) The outermost region and all of the innermost regions are interior, in view of \( C_1 \subset V \) and that every intersection circle is a meridian of \( V \).

(vi) If the outermost region is an annulus, it must intersects \( L - C_1 \). In fact, if not, \( C_1 \) would be parallel to a meridian of \( T \). However, by Brunnian property, \( L - C_1 \) is trivial in \( V \). Then \( L \) would also be trivial in \( V \), a contradiction.
(vii) For any exterior region $\Omega$, $\partial \Omega$ should be one outer circle enclosing odd number($\geq 3$) of inner circles in $D_1$, and the disk bounded by each inner circle intersects $L$ in at least 2 points. In fact, since $\partial \Omega$ is null-homologic in $S^3 - \text{int} V$, $\partial \Omega$ has even number of components, and by Step 2, it can not have only 2 components.

We now show by cases that $D_1 \subset V$.

CASE 1: $D_1$ is stable. Suppose $D_1$ has exterior regions. Then the components of $D_1 \cap T$ can not be all innermost in $D_1$. By (v), each innermost region in $D_1$ is a meridian disk of $V$. There exist such a disk region, say $D_{u1}$, and a component of $D_1 \cap T$ adjacent to $\partial D_{u1}$ in $T$, say $C_{u+1,1}$, such that

(i) $\sharp(D_{u1} \cap L)$ is minimal among all the innermost regions,

(ii) $C_{u+1,1}$ is either not innermost in $D_1$, or innermost but not achieving minimal intersection points with $L$.

Let $A_T$ be the annulus in $T$ between $C_{u+1,1}$ and $\partial D_{u1}$. Then by (vii), the disk $D_{u1} \cup A_T$ has less intersection points with $L$ than the disk in $D_1$ bounded by $C_{u+1,1}$. So $C_{u+1,1}$ is an unstable circle on $D_1$, a contradiction.

CASE 2: $\sharp(D_1 \cap L) < 7$, if $n = 2$ and $lk(C_1, C_2) = 1$; $\sharp(D_1 \cap L) < 8$, in other cases. Suppose $D_1$ has exterior regions. By (vi) and (vii), $D_1$ intersects $L - C_1$ in more than 6 points. If the outermost region intersects another component, say $C_2$, in only one point, then $lk(C_1, C_2) = 1$ and $n = 2$. As shown in Fig. 13(1), we have $\sharp(D_1 \cap L) \geq 7$. Otherwise, $\sharp(D_1 \cap L) \geq 8$, as shown in Fig. 13(2). So under this condition, $D_1 \subset V$.

![Figure 13.](image)

Now we follow the three steps above to make $D_2$ into $V$ by an isotopy of $T$. The key point is that this isotopy of $T$ will keep $D_1 \cap T = \emptyset$.

Consider $D_2 \cap T$. For Step 1, by the same token, choose a circle $C$ innermost in $D_2$, which bounds a disk $D_{CT}$ in $T$ and a disk $D_{C2}$ in $D_2$, and then push $D_{CT}$ across the 3-ball cobounded with $D_{C2}$ to eliminate $C$. We have two similar observations for $D_2$ as clauses (iii) and (iv) for $D_1$. Notice that $D_{CT}$ and $D_1$ do not intersect. Thus $D_{C2}$ also does not intersect $D_1$. Therefore the isotopy of $D_{CT}$, replacing $D_{CT}$ by $D_{C2}$, keeps $D_1 \cap T = \emptyset$. So we can eliminate all the inessential circles in $T$ in finitely many steps. For Step 2, we can eliminate exterior annulus regions in $D_2$ just as before because $D_1$ is already contained in $V$. For Step 3, the same argument as demonstrated before shows that $D_2 \subset V$.

Performing the same procedure for $D_i$ for $i$ from 2 to $k$, we have $U \subset V$. Since $T$ is incompressible in $S^3 - L$, clearly $T$ is incompressible in $S^3 - U$.

The “only if” implication:

Suppose $T$ is an incompressible torus in the complement of $U$. Let $V$ be the solid torus bounded by $T$ containing $U$. We proof by negation that $L$ is an s-tie. If $L$ is untied, then $T$ is compressible in the complement of $L$ and thus there is
a meridian disk $D$ of $V$ not intersecting $L$. Notice that any meridian disk of $V$ intersects $U$. It follows that for any $i \in I$, $D \cap D_i$ is a disjoint union of circles. The intersection $\bigcup_{i \in I} D_i \cap D$ forms a graph.

Let $C$ be a component of $D_1 \cap D$ innermost in $D_I$, bounding $D_{C1}$ in $D_1$ and bounding $D_C$ in $D$. Since $D_1$ is credible, $D_{C1}$ does not intersect $L$. By (Rii), the singularities of $U$ contained in $D_{C1}$ consists of proper arcs. Replacing $D_C$ in $D$ by $D_{C1}$ and pushing it a little beyond $D_{C1}$, we get a new disk $D$ with less intersection components with $D_1$. After finitely many steps, $D$ is disjoint from $D_1$.

A careful examination shows that the same procedure for $D_2$ keeps $D_1 \cap D = \emptyset$.

Step by step for each $D_i$, we eventually get a meridian disk of $V$ not intersecting $U$, a contradiction.

5. SIMPLE INTERSECTION PATTERN THEOREM

To show a Brunnian link is s-prime is to prove no essential torus splits it into two sublinks. We divide our strategy in two steps. In this section, as the first step, we give a general intermediate result, stating that a spanning complex satisfying certain conditions can intersect a splitting torus in a simple form. In the next section, as the second step, we give criteria to negate the existence of such torus.

5.1. SIMPLE INTERSECTION PATTERN. We first define several disks with respect to a partition of a link.

Definition 5.1. Let $L = \bigcup_{i=1}^n C_i$ be an $n$-component link. Let $L_I = \bigcup_{i \in I} C_i$ and $L_J = \bigcup_{j \in J} C_i$ be proper sublinks of $L$ with $I \cap J = \{1, \ldots, n\}$. For any $i \in I$ and $j \in J$, the credible disks defined as follows are called $L_I$-interior disk, $L_J$-interior disk, exterior cross disk and free cross disk respectively, denoted $D_i^I$, $D_i^J$, $D_i^E$ and $D_i^F$ respectively.

- $D_i^I$ : bounded by $C_i$ such that $D_i^I \cap L_J = \emptyset$;
- $D_i^J$ : bounded by $C_j$ such that $D_i^J \cap L_I = \emptyset$;
- $D_i^E$ : bounded by $C_i$ such that $D_i^E \cap (L_i - C_i) = \emptyset$ and $D_i^E \cap L_J \neq \emptyset$;
- $D_i^F$ : bounded by $C_i$ such that $D_i^F \cap (L_i - C_i) \neq \emptyset$ and $D_i^F \cap L_J \neq \emptyset$.

$L_I$-interior disk and $L_J$-interior disk are interior disks, and exterior cross disk and free cross disk are cross disks.

Consider a torus $T$ splitting $L$ as $L_I$ and $L_J$ and the intersection of $T$ and the disks defined above.

Definition 5.2. An interior disk intersects $T$ in simple intersection pattern if it does not intersect $T$, and a cross disk intersects $T$ in simple intersection pattern if each intersection circle is innermost in the disk and is a meridian of $V_J$.

Remark 5.3. Let $\beta_I$ and $\beta_J$ be the core of $V_I$ and $V_J$ respectively. Consider a cross disk $D_i$ intersecting $T$ in simple intersection pattern.

1. Recall that cross disks are credible. It is easy to see that each inner disk in it is a credible disk for $\beta_I \cup L_J$, and the outer region in it corresponds to a credible disk for $L_I \cup \beta_J$.
2. If $D_i$ is stable, consider $\beta_I \cup L_J$. It can be verified that each inner disk in $D_i$ is a stable disk bounded by $\beta_I$, and thus all the inner disks have the same number of intersection points with $L_J$.

Now we turn to Brunnian links.
Proposition 5.4. Let \( L = \cup_{i=1}^{n} C_i \) be an \( n \)-component Brunnian link. Let \( L_I = \cup_{i \in I} C_i \) and \( L_J = \cup_{j \in J} C_j \) be proper sublinks of \( L \) with \( I \cup J = \{1, \ldots, n\} \).

(1) For any exterior cross disk \( D_i^E \), we have \( \mathcal{z}(D_i^E \cap (L - C_i)) \geq 4 \).

(2) For any free cross disk \( D_i^F \), we have \( \mathcal{z}(D_i^F \cap (L - C_i)) \geq 6 \).

Remark 5.5. If \( \mathcal{z}(D_i^E \cap (L - C_i)) = 4 \), then \( D_i^E \) is stable for \( L_J \).

The proofs of this proposition and the following main theorem will be given at the end of this section.

Theorem 5.6. (Simple Intersection Pattern Theorem)

Let \( L = \cup_{i=1}^{n} C_i \) be an \( n \)-component Brunnian link. Let \( T \) be an essential torus in the complement of \( L \), splitting \( S^3 \) into two solid tori \( V_I \) and \( V_J \) so that \( L \cap V_I = L_I \) and \( L \cap V_J = L_J \), where \( L_I = \cup_{i \in I} C_i \) and \( L_J = \cup_{j \in J} C_j \). Suppose \( Q \cup R \cup S \subset I \) and \( J_0 \subset J \). Let \( D_i^Q \) \((q \in Q)\), \( D_j^R \) \((j \in J_0)\), \( D_i^E \) \((r \in R)\) and \( D_i^S \) \((s \in S)\) be \( L_I \)-interior disks, \( L_J \)-interior disks, exterior cross disks, and free cross disks respectively, so that the cross disks are mutually disjoint and

\[
\begin{cases}
D_i^Q \text{ is } (s8), & \forall q \in Q; \\
D_j^R \text{ is } (s8), & \forall j \in J_0; \\
D_i^E \text{ is } (s10), & \forall r \in R; \\
D_i^S \text{ is stable, or contains a longitude of } V_I \text{ with } \mathcal{z}(D_i^S \cap (L - C_s)) < 8, & \forall s \in S.
\end{cases}
\]

If the union of these disks and \( L \) forms a spanning complex, then after an isotopy of \( T \), every disk intersects \( T \) in simple intersection pattern.

Remark 5.7. We explain the conditions in this theorem in more detail.

(1) The cross disks \( D_i^E \) \((r \in R)\), \( D_i^F \) \((s \in S)\) are required to be mutually disjoint. As will be seen in the proof, \( \cup_{q \in Q} D_i^Q \) and \( \cup_{j \in J_0} D_j^Q \) are actually disjoint.

(2) In (iv), the condition “\( D_i^S \) contains a longitude of \( V_I \)” seems not natural. But actually once \( R \neq \emptyset \), \( D_i^S \) has to contain a longitude of \( V_I \), since the cross disks are mutually disjoint.

5.2. Proof of the main Theorem. We first present a simple lemma on s-sum decomposition of Brunnian links.

Lemma 5.8. Let \( T \) be an essential torus in the complement of a Brunnian link \( L \), splitting \( L \) into proper sublinks \( L_1 \subset V_I \) and \( L_2 \subset V_2 \), where \( V_I \) is a solid torus bounded by \( T \), for \( i = 1, 2 \).

(1) Any meridian disk in \( V_i \) intersects \( L_i \) with at least 2 points, for \( i = 1, 2 \);

(2) If a component of \( L_1 \) bounds a disk whose intersection with \( T \) contains a meridian of \( V_2 \), then the intersection contains at least two meridians of \( V_2 \).

Proof of Theorem 5.6. Let \( U \) denote the spanning complex formed by \( L \) and all the credible disks. Then \( U \cap T \) is a union of circles, forming a graph in \( T \). We will isotope \( T \) to make the \( L_I \)-interior disks, the \( L_J \)-interior disks and the cross disks into simple intersection pattern in turn. So we divide our proof in three steps.

Step 1. Make \( L_I \)-interior disks into \( V_I \) one by one by an isotopy of \( T \).

We shall adopt the same procedure as in the proof of Theorem 4.1. Consider a disk \( D_i^I \). We make it into \( V_I \) in three steps.

Step 1.1. Eliminate inessential circles in \( T \).

Let \( C \) be a circle component of \( D_i^I \cap T \), bounding a disk \( D_{CT} \) in \( T \) and a disk \( D_{Cq} \) in \( D_i^I \). Using the same argument as in the proof of Theorem 4.1, we may assume...
$C$ is innermost in $D_{Cq}$. Then $D_{Cq} \cup D_{CT}$ bounds a 3-ball. Since no sphere splits a Brunnian link, this 3-ball does not intersect $L$. So we can eliminate $C$ by an isotope of $T$ as demonstrated before. Do it inductively. After finitely many steps, $D_q^T \cap T$ contains no inessential circle in $T$.

**Step 1.2.** Eliminate annulus regions in $V_J$.

Now $D_q^T \cap T$ is a disjoint union of circles. Consider a circle component innermost in $D_q^T$. Since the disk in $D_q^T$ bounded by this circle does not intersect $L_I$, it is the meridian disk of $V_I$. Thus all the components of $D_q^T \cap T$ are meridian of $V_I$.

The intersection $D_q^T \cap T$ cuts $D_q^T$ into regions alternatively in $V_I$ and in $V_J$. Let $A$ be an annulus region contained in $V_J$. Since the boundary components of $A$ are longitude of $V_J$, it follows that $A$ must cut $V_J$ into two solid tori. Notice that the core of each solid tori and the core of $V_I$ form a Hopf link, which is Brunnian.

By Theorem 2.3 one of the two solid tori does not intersect $L_I$. Isotope $T$ across this solid torus to make $A$ into $V_I$. Step by step, we can eliminate all the annulus regions in $V_J$.

**Step 1.3.** Verify $D_q^T \subset V_I$.

For either the case that $D_q^T$ is stable or the case $y(D_q^T \cap L_I) < 8$, an argument similar to the one used in Step 3 in the proof of Theorem 4.1 shows that $D_q^T \subset V_I$.

Next we isotope $T$ to make another $L_I$-interior disk $D_q^J$ into $V_I$ by the same token. We now check that the isotopy of $T$ keeps $D_q^T \cap T = \emptyset$. For Step 1.1, similarly as in the proof of Theorem 4.1 this is due to the regularity condition (Rii) and that $D_q^T$ does not intersect any disk in $T$. For Step 1.2, it is guaranteed by that $D_q^T$ is already contained in $V_I$.

Performing the same procedure for $L_J$-interior disks one by one, we have that $\cup_{q \in Q} D_q^J \subset V_I$.

**Step 2.** Make $L_J$-interior disks into $V_J$ one by one by an isotopy of $T$.

With $\cup_{q \in Q} D_q^J$ already contained in $V_I$, we proceed as in Step 1 to make $L_J$-interior disks into $V_J$. Consider a disk $D_q^T$. We only need to check that in the similar three steps, the isotopy of $T$ keeps $D_q^T \cap T = \emptyset$ for all $q \in Q$.

**Step 2.1.** Eliminate inessential circles in $T$.

In this step, if $D_{CJ}^T$ contained in $D_q^T$, corresponding to $D_{Cq}$ in Step 1.1, does not intersect any $L_I$-interior disk by (Rii). Thus the 3-ball it caps with $T$ does not intersect any $L_I$-interior disk.

**Step 2.2.** Eliminate annulus regions in $V_J$.

In such a case, $D_q^T \cap L_I = \emptyset$ implies that $D_q^T \cap T$ can only be disjoint union of meridians of $V_J$. The intersection circles cut $D_q^T$ into regions alternatively in $V_I$ and in $V_J$. Let $A$ be an annulus region contained in $V_I$. An argument similar to the one used in Step 1.2 shows that $A$ caps a solid torus with $T$, say $V_A$, which does not intersect $L_I$.

We claim that $A \cap D_q^T = \emptyset$, and thus $V_A \cap D_q^T = \emptyset$ for all $q \in Q$. In fact, by (Rii), there is no component of $D_q^T \cap A$ inessential in $A$ for any $q \in Q$. Suppose $C_A$ is a component of $D_q^T \cap A$ essential in $A$ for a $q \in Q$. Since $C_A$ is a core of $V_I$, the disk in $D_q^T$ bounded by $C_A$ must intersect $L_I$, a contradiction.

An isotopy of $T$ across $V_A$ makes $A$ into $V_J$.

**Step 2.3.** Verify $D_q^T \subset V_J$.

This step is a judgement, where everything is fixed.
Perform the same procedure for other $L_J$-interior disks one by one. We can verify that $T$ keeps avoiding $\bigcup_{q \in Q} D^I_q$ and the $L_J$-interior disks already contained in $V_J$ by the same arguments as in Step 1 and Step 2 respectively. Now we have $\bigcup_{j \in J} D^J_j \subset V_J$.

**Step 3.** Make the cross disks into simple intersection pattern.

Recall that all of the cross disks are mutually disjoint, so it is convenient to consider them together. We proceed as in the proof of Theorem 4.1.

**Step 3.1.** Eliminate inessential circles in $T$.

The intersection of the cross disks and $T$ consists of mutually disjoint circles. Let $D_i$ be one of the exterior or free cross disks. Let $C$ be a circle component of $D_i \cap T$, bounding a disk $D_{CT}$ in $T$ and a disk $D_{C_i}$ in $D_i$. Using the same argument as in Step 1 in the proof of Theorem 4.1, we may assume $C$ is innermost in $D_{CT}$. Then $D_{C_i} \cup D_{CT}$ bounds a 3-ball. As demonstrated in Step 1.1, this 3-ball does not intersect $L$, and thus we can eliminate $C$ by an isotope of $T$.

Step by step for all cross disks, we eliminate all inessential intersection circles in $T$.

**Step 3.2.** Eliminate annulus regions $\partial$-parallel to $T$.

It is easy to see that each component of $(\bigcup_{r \in R} D^E_r \cup \bigcup_{s \in S} D^F_s) \cap T$ is either the longitude of $V_I$ or the meridian of $V_I$. Let $D_i$ be one of the exterior or free cross disks. The intersection $D_i \cap T$ cuts $D_i$ into planar regions alternatively in $V_I$ and in $V_J$. In this step, we eliminate all the proper annulus regions $\partial$-parallel to $T$ without intersecting $L$. Strictly speaking, let $A$ be a proper annulus region contained in $V_I$ which caps a solid torus with $T$, say $V_A$, such that $V_A \cap L = \emptyset$. We claim that $A \cap D^I_q = \emptyset$, and thus $V_A \cap D^I_q = \emptyset$, for any $q \in Q$. Then an isotopy of $T$ across $V_A$ makes $A$ into $V_J$.

We now prove the claim. By (Rii), there is no component of $D^I_q \cap A$ inessential in $A$ for any $q \in Q$. Suppose $C_A$ is a component of $D^I_q \cap A$ essential in $A$ for a $q \in Q$. Let $P$ be a component of $D^I_q \cap V_A$ one of whose boundary components is $C_A$. Clearly the boundary components of $P$ are all parallel in $A$. By (Riii), $P$ is not an annulus. Then $P$ is compressible in $V_A$. Let $D_{P_1}$ be a compression disk for $P$ in $V_A$. Since $D^I_q$ is credible, the disk in $D^I_q$ bounded by $\partial D_{P_1}$, denoted $D_{P_0}$, does not intersect $L$. Without loss of generality, we may reselect $C_A$ so that it is contained in $D_{P_0}$. However, since $C_A$ is essential in $A$, it is parallel to either the longitude or the meridian of $V_I$. In either case, any disk bounded by $C_A$ intersects $L$, a contradiction.

We can eliminate such annulus regions contained in $V_J$ in a similar manner.

**Step 3.3.** Verify simple intersection pattern.

Consider all of the cross disks cut by $T$. For both exterior and free cross disks, the following claim investigates the remaining annulus regions. To make length short, we arrange the proof of this claim after the end of the proof of the theorem.

**Claim:** After Step 3.2,

(Ai) any proper annulus region in $V_I$ intersects $L_I$, and intersects any component of $L_I$ in even number of points;

(Aii) any annulus region in $V_J$ intersects $L_J$, and intersects any component of $L_J$ in even number of points.

In the remainder of the proof, we discuss exterior cross disks and free cross disks separately.
Exterior cross disks. For any \( r \in R \), the intersection \( D_r^E \cap T \) cuts \( D_r^E \) into regions alternatively in \( V_I \) and \( V_J \). Since \( D_r^E \) is an exterior cross disk, the intersection circles are longitudes of \( V_I \). We have the following observations on the regions in \( D_r^E \).

(Ei) Each innermost region is a meridian disk of \( V_J \). In view of linking number, each innermost disk intersects any components of \( L_J \) in even number of points unless \( L_J \) has only one component.

(Eii) The outermost region is contained in \( V_I \). It is not an annulus by Lemma 5.8. Its boundary consists of \( C_s \) and even number of intersection circles unless \( L_I = C_r \), in view of linking number.

(Eiii) No region in \( V_I \) intersects \( L - C_r \). For each region \( \Omega \) in \( V_I \), except the outermost one, \( \partial \Omega \) is one outer circle enclosing odd number (\( \geq 3 \)) of inner circles in \( D_r^E \). In fact, since \( \partial \Omega \) is null-homologic in \( V_I \), such a region has even number (\( > 2 \)) of boundary components.

Now we show by cases that \( D_r^E \) is of simple intersection pattern.

CASE 1: \( D_r^E \) is stable. Let \( D_{ur} \) be an innermost region in \( D_r^E \) so that \( \sharp (D_{ur} \cap L_J) \) is minimal among all the innermost regions in \( D_r^E \). Let \( C_{u+1,r} \) be a component of \( D_r^E \cap T \) adjacent to \( \partial D_{ur} \) in \( T \), and \( A_T \) be the annulus in \( T \) between \( \partial D_{ur} \) and \( C_{u+1,r} \). Then the disk \( D_{ur} \cup A_T \) has the same intersection points with \( L \) as \( D_{ur} \). Since \( D_r^E \) is stable, the disk bounded in \( D_r^E \) by \( C_{u+1,r} \) must be innermost and have the same number of intersection points as \( D_{ur} \). It follows that all the intersection circles are innermost in \( D_r^E \).

CASE 2: \( \sharp (D_r^E \cap (L - C_r)) < 10 \). Suppose there is an intersection circle in \( D_r^E \) which is not innermost. By (Aii), (Ei), (Eii) and (Eiii), \( D_r^E \) intersects \( L - C_r \) in at least 10 points, as shown in Fig. 14.

Figure 14. White regions are in \( V_I \), and gray regions are in \( V_J \).

Free cross disks. For any \( s \in F \), the intersection \( D_s^F \cap T \) cuts \( D_s^F \) into regions alternatively in \( V_I \) or meridians of \( V_I \). We discuss the two possibilities separately.

First suppose that the circles are longitude of \( V_I \). We have the following observations on the regions in \( D_s^F \).

(Fi) Each innermost region is a meridian disk of \( V_J \), and it intersects any components of \( L_J \) in even number of points unless \( L_J \) has only one component.

(Fii) The outermost region is contained in \( V_I \). It is not an annulus by Lemma 5.8. Its boundary consists of \( C_s \) and even number of intersection circles.

(Fiii) For each region \( \Omega \) in \( V_I \), except the outermost one, \( \partial \Omega \) is one outer circle enclosing odd number of inner circles in \( D_s^F \).

Now suppose that the intersection circles are meridian of \( V_I \). We have the following observations on the regions in \( D_s^F \).
(F'i) The outermost region and all of the innermost regions are contained in \( V_I \).
Each innermost region is a meridian disk of \( V_I \).
(F'ii) If the outermost region is an annulus, it intersects \( L_I - C_s \), and thus it intersects any component of \( L_I - C_s \) in even number of points, in view of linking number. In fact, if not, \( C_s \) would be parallel to a meridian of \( V_I \). By Brunnian property, \( L_I - C_s \) is trivial in \( V_I \). This would implies \( L_I \) is trivial in \( V_I \), a contradiction.
(F'iii) For any region \( \Omega \) in \( V_J \), \( \partial \Omega \) is one outer circle enclosing odd number of inner circles in \( D_{E_s} \), and each disk in \( D_{E_s} \) bounded by one of such inner circle intersects \( L_I \).
We now show by cases that \( D_{E_s} \) is of simple intersection pattern.
CASE 1: \( D_{E_s} \) is stable. Suppose that the intersection circles are meridian of \( V_I \). By (F'i), the components of \( D_{E_s} \cap T \) can not be all innermost in \( D_{E_s} \). Notice that each innermost region is a meridian disk of \( V_I \). There exist an innermost region in \( D_{E_s} \), say \( D_{us} \), and a component of \( D_1 \cap T \) adjacent to \( \partial D_{us} \) in \( T \), say \( C_{u+1,s} \), such that
(i) \( \sharp (D_{us} \cap L_I) \) is minimal among all the innermost regions,
(ii) \( C_{u+1,s} \) is either not innermost in \( D_{E_s} \), or innermost but not achieving minimal intersection points with \( L_I \).
Let \( A_T \) be the annulus in \( T \) between \( C_{u+1,s} \) and \( \partial D_{us} \). Then by (F'iii) and (Aii), the disk \( D_{us} \cup A_T \) has less intersection points with \( L \) than the disk in \( D_{E_s} \) bounded by \( C_{u+1,s} \). This implies that \( C_{u+1,s} \) is an unstable circle on \( D_{E_s} \), a contradiction.
Hence \( D_{E_s} \cap T \) consists of longitudes of \( V_I \). The remainder of the argument is analogous to that in Case 1 for exterior cross disks and is left to the reader.
CASE 2: \( \sharp (D_{E_s} \cap (L - C_s)) < 8 \) and \( D_{E_s} \) contains a longitude of \( V_I \). All of the intersection circles are longitude of \( V_I \). Suppose there is a circle in \( D_{E_s} \) which is not innermost. By Lemma 5.8(1), every meridian disk of \( V_J \) intersects \( L_I \) in at least 2 points. Then by (Fi), (Fii), (Fiii), (Ai) and (Aii), \( D_{E_s} \) intersects \( L - C_s \) in at least 8 points, as shown in Fig. 15.

![Figure 15](image)

**Figure 15.** White regions are in \( V_I \), and gray regions are in \( V_J \).

\[ \Box \]

*Proof of the claim.* We only prove (Ai), and (Aii) follows in a similar manner by interchanging \( V_I \) and \( V_J \) in the proof below.
Let \( A \) be a proper annulus region in \( V_J \). Assume for contrary that \( A \cap L_I = \emptyset \).
First suppose that all the intersection circles are longitude of \( V_I \). Then \( A \) must cut \( V_I \) into two solid tori. Since the core of each solid tori and the core of \( V_J \) form a Hopf link, by Theorem 2.3, one of the two solid tori does not intersect \( L_I \). However, we have eliminated such \( A \) in Step 3.2, a contradiction.
Now suppose that all the intersection circles are meridian of $V_I$. If $A$ is $\partial$-parallel in $V_I$, then it cuts off a compression solid torus $V_A$ in $V_I$. If $V_A$ contains some components of $L_I$, by Proposition 3.2.1 in [4], the proper sublink in the closure of $V_I - V_A$ would not be geometrically essential. A compression disk in this solid torus, after isotopy, would be a meridian disk in $V_I$. However, again by Proposition 3.2.1 in [4], $L_I$ is geometrically essential in $V_I$, a contradiction. So $V_A$ does not intersect $L_I$. Recall that we have eliminated such $A$ in Step 3.2. Hence $A$ cannot be $\partial$-parallel in $V_I$. Then $\partial A$ cuts $T$ into two annuli such that $A$ and one of the annuli cobound a knotted solid torus $V_{AI}$ in $V_I$. By Theorem 2.4, the sublink in $V_{AI}$ is not geometrically essential. A compression disk in $V_{AI}$, after isotopy, would be a meridian disk in $V_I$, again a contradiction. So $V_A$ does not intersect $L_I$. We have shown that $A$ intersects $L_I$. Since $A$ separates $V_I$, it intersects any component of $L_I$ in even number of points. □

Proof of Proposition 5.4. Let $T$ be an essential torus in the complement of $L$, splitting $S^3$ into two solid tori $V_I$ and $V_J$ so that $L \cap V_I = L_I$ and $L \cap V_J = L_J$.

(1) Applying Theorem 5.6, we may assume $D^F_i$ intersects $T$ in simple intersection pattern. By Lemma 5.8, any meridian disk of $V_I$ has more than one intersection points, and the outermost region has more than two boundary components. So if $\sharp(D^F_i \cap (L - C_i)) \leq 4$, the only possibility is as shown in Fig. 16(1).

(2) First suppose all the intersection circles in $D^F_i$ are longitude of $V_I$. Applying Theorem 5.6, we may assume $D^F_i$ is of simple intersection pattern. By Lemma 5.8(1) and (Fii) in the proof of Theorem 5.6, there are at least two inner disks each intersecting $L_J$ in at least 2 points. In view of linking number, the outer region intersects $L_I - C_i$ in even number of points. So $D^F_i$ with least number of intersection is as shown in Fig. 16(2).

Now suppose that all the intersection circles in $D^F_i$ are meridian of $V_I$. By Lemma 5.8(1), and (Aii) and (Fii) in the proof of Theorem 5.6, if $\sharp(D^F_i \cap (L - C_i)) \leq 6$, the intersections in $D^F_i$ can only be as shown in Fig. 16(3).

![Figure 16. White regions are in $V_I$, and gray regions are in $V_J$.](image)

6. Criteria for S-primeness

Based on the theorem in the previous section, we present four methods to prove a Brunnian link is s-prime and illustrate them by typical examples. In this section, we call the following paragraph Standard premise:

Let $L = \cup_{i=1}^n C_i$ be an $n$-component Brunnian link. Let $T$ be an essential torus in the complement of $L$, splitting $S^3$ into two solid tori $V_I$ and $V_J$, such that $L \cap V_I = L_I = \cup_{j \in J} C_i$ and $L \cap V_J = L_J = \cup_{j \in J} C_i$. Suppose $Q \cup R \cup S \subset I$ and $J_0 \subset J$. Let
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$D^q_i (q \in Q)$, $D^j_j (j \in J_0)$, $D^E_r (r \in R)$ and $D^F_s (s \in S)$ be $L_I$-interior disks, $L_J$-interior disks, exterior cross disks, and free cross disks respectively, such that

- the cross disks are mutually disjoint;
- the union of these disks and $L$ forms a spanning complex $U$;
- every disk intersects $T$ in simple intersection pattern.

6.1. Analysis on plausible annuli. To show a hypothetical torus cannot exist, we may estimate the shape of the torus. We thus introduce some notations.

Under Standard premise, the components of the intersection of $T$ and the cross disks are plausible circles, denoted $\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_t$, successively indexed along $T$. For any $k = 1, 2, ..., t$, the annulus in $T$ between $\tilde{C}_k$ and $\tilde{C}_{k+1}$ is a plausible annulus, denoted $\tilde{A}_k$; the disk bounded by $\tilde{C}_k$ in a cross disk is a plausible disk, denoted $\tilde{D}_k$; the connected component of $V_J - (\cup_{r \in R} D^E_r \cup \cup_{s \in S} D^F_s)$ containing $\tilde{A}_k$ is a plausible cylinder, denoted $\tilde{B}_k$.

Proposition 6.1. Under Standard premise, let $\alpha$ be an arc component of $L_I - \cup_{r \in R} D^E_r \cup \cup_{s \in S} D^F_s$.

1. $\alpha$ connects two consecutive plausible disks on the corresponding sides of cross disks unless two endpoints of $\alpha$ are on one side of a cross disk.

2. If $\alpha$ connects two plausible disks $\tilde{D}_k$ and $\tilde{D}_{k+1}$ in one cross disk $D_i$, and $\alpha$ is unknotted, i.e. forming unknot with an arc connecting $\partial \alpha$ in $D_i$, then $\alpha$ is the core of $\tilde{B}_k$.

Here, a core of a plausible cylinder is a trivial arc in the plausible cylinder connecting its two bases.

Proof. Since (1) is obvious, we need only prove (2). There are two cases depending on whether $\tilde{D}_k$ and $\tilde{D}_{k+1}$ are on the same side of $D_i$. First suppose they are on the same side of $D_i$. Let $\beta_\alpha$ be an arc in $D_i$ connecting $\partial \alpha$ so that $\beta_\alpha$ intersects both $\tilde{C}_k$ and $\tilde{C}_{k+1}$ in exactly one point. Let $D_\alpha$ be a disk bounded by $\alpha \cup \beta_\alpha$. Consider $D_\alpha$ and $D_i$. A standard innermost circle/outermost arc argument shows that we may assume $D_\alpha \cap D_i = \beta_\alpha$. Then consider $D_\alpha$ and $\tilde{A}_k$. By the choice of $\beta_\alpha$, $D_\alpha \cap \tilde{A}_k$ consists of an arc connecting $\tilde{C}_k$ and $\tilde{C}_{k+1}$ and some trivial circles in $\tilde{A}_k$. We can eliminate all the circle components by a standard innermost circle argument. Notice that the remained arc component is parallel to the core of $\tilde{B}_k$. Thus $\alpha$ is the core of $\tilde{B}_k$. See Fig. 17(1).

**Figure 17.** The arc $\alpha$ is the core of $\tilde{B}_k$. 
Now suppose two plausible disks are on different sides of $D_i$. The proof is almost identical, the major change being the substitution of $\beta_\alpha$ into a shape “T” graph for $D_\alpha \cap D_i$. See Fig. [17](2).

**Example 6.2.** Recall the link in Fig. [10](1). Using the notations in Example 3.11, we prove it is $s$-prime. Suppose there is an essential torus $T$ splitting the link. Take the stable exterior cross disk $D_1$ bounded by $C_1$ as shown in Fig. [10](2). Since $\text{lk}(C_1, C_2) = 0$ and $C_1$ and $C_2$ are symmetric, we may assume $\text{lk}(C_1, \beta_I) = 0$. Thus there is a plausible cylinder, say $\tilde{B_k}$, whose bases are on the same side of $D_1$. Without lose of generality, assume its bases are on the positive side of $D_1$. Clearly, $\tilde{B_k}$ intersects $C_2$ in some odd labeled arcs. Let $\alpha_i$ be a component of $\tilde{B_k} \cap C_2$. We claim $\alpha_i$ is the core of $\tilde{B_k}$. In fact, if not, $\partial \alpha_i$ would be in one plausible disk, say $\tilde{D_k}$. Then once delete all the other odd labeled arcs, $\tilde{A_k} \cup \tilde{D_k+1}$ would be a disk capping $\alpha_i$ with $D_1$. This implies $\text{lk}(\alpha_i, \alpha_{i+1}) = 0$, a contradiction.

Hence all the components of $\tilde{B_k} \cap C_2$ are core of $\tilde{B_k}$. By Lemma 5.8(1), $\tilde{B_k} \cap C_2$ has more than one component. Consider two of the components. Then they are parallel to each other in $\tilde{B_k}$. Thus for any even labeled arc $\alpha_j$, this two arcs have the same linking number with $\alpha_j$. This is impossible as we know $\text{lk}(\alpha_i, \alpha_{i+1}) = 1$ and otherwise the linking number is 0.

### 6.2. Components discarded.

This method is powerful for large links.

**Proposition 6.3.** Under Standard premise with $R \cup S \neq \emptyset$, if delete some connected components of $U$ contained in $V_I$, there is no connected component of $U$ contained in $V_J$ split out by a sphere.

**Proof.** By Theorem 2.3, the link $L_I \cup \beta_I$ is Brunnian. Thus no sphere splits $L_I \cup \beta_I$. By construction of $U$, no sphere splits $\beta_I \cup (U \cap V_J)$. Notice that a cross disk in simple intersection pattern contains a longitude of $V_J$, which is parallel to $\beta_I$. It follows from the construction of $U$ that in the complex $U$ with some connected components in $V_I$ discarded, no connected component in $V_J$ can be split out by a sphere.

This proposition provides a method to rule out essential torus splitting $L_I$ and $L_J$, as the hypothetical torus $T$ cannot exist if discarding some components of $U$ in $V_I$ makes some component of $U$ in $V_J$ split out.

![Figure 18. deBrunner(5).](image-url)

**Example 6.4.** deBrunner(5) is $s$-prime. Denote the link in Fig. [18] by $L$. In view of the symmetry of $L$, it suffices to show there is no essential torus splitting $C_1$ and
Assume for contrary that $T$ is a such torus. Take a cross disk $D_2$ bounded by $C_2$ as shown in Fig 18. Depending on whether $T$ splits $C_2$ and $C_3$, the following discussion is complete.

(i) $V_I$ contains only $C_2$;
(ii) $V_I$ contains exactly $C_2 \cup C_3$;
(iii) $V_I$ contains more than $C_2 \cup C_3$ and $V_J$ contains more than $C_1$;
(iv) $V_J$ contains only $C_1$;
(v) $V_J$ contains exactly $C_1 \cup C_3$;
(vi) $V_J$ contains more than $C_1 \cup C_3$ and $V_I$ contains more than $C_2$.

None above is possible. The main point of our argument is that in $L \cup D_2$, after deleting a connected component other than $C_1 \cup D_2 \cup C_3$, then all the other components split out.

For (ii) and (iv), they are impossible by Proposition 5.4(2). Clause (i) is equivalent to (iv) by the symmetry of $L$, thus also impossible. For (iii) and (iv), they are impossible by Proposition 6.3. In fact, in $L \cup D$, once delete a circle component in $V_I$, each circle component in $V_J$ splits out. Finally, (v) is contained in (iii) after a rotation of $L$.

Proposition 6.2.1 in [4], the uniqueness of s-sum decomposition for Brunnian links, can simplify the task of showing s-primeness. Specifically, as any two tori splitting $L$ are disjoint after isotopy, if a torus $T$ splits $L$ into $L_I$ and $L_J$ and a torus $T'$ splits $L$ into $L_I'$ and $L_J'$, then one of $L_I \cap L_I'$, $L_I \cap L_J'$, $L_J \cap L_I'$, and $L_I \cap L_J'$ is empty.

![Figure 19. Torus(m, n) is s-prime.](image_url)

**Example 6.5.** For any $m > 1, n > 2$, Torus$(m, n)$ is s-prime. Up to symmetry, there is only one kind of component, as $C_{ij}$ in Fig 19. It suffices to show that no essential torus splits any one of $C_{i-1,j}$, $C_{i-1,j}$ and $C_{i+1,j}$ from the other two. Take the cross disk $D_{ij}$. By Proposition 5.4(2), no essential torus splits $C_{i+1,j}$ and $C_{ij} \cup C_{i-1,j}$ or splits $C_{i-1,j}$ and $C_{ij} \cup C_{i+1,j}$.

Suppose there is an essential torus $T$ splitting $C_{ij}$ and $C_{i-1,j} \cup C_{i+1,j}$. In $L \cup D_{ij}$, once delete a circle component other than $C_{i+1,j}$, all other circle components split out. Thus by Proposition 6.3 $T$ either splits $C_{i-1,j} \cup C_{i+1,j}$ from all the other components, or splits $C_{ij} \cup C_{i+1,j}$ from all the other components. For the first case, by the symmetry of $L$, there would be a torus splits $C_{i-1,j} \cup C_{i-1,j}$ from all the other components. by Proposition 6.2.1 in [4], this is impossible. Similarly, the second case is impossible.
Definition 6.6. Let $C$ be a component of a Brunnian link $L$. If there is an essential torus in $S^3 - L$ splitting $C$ and $L - C$, then $C$ is multiple; otherwise, simple.

Example 6.7. Carpet(1,3,4) is s-prime. Denote this link by $L$. Up to symmetry, there are 6 kinds of components. We will take cross disks in Fig. 20 one by one to show $L$ is s-prime.

On account of the symmetry of this link, the following discussion is complete.

(i) No essential torus splits one of $C_{11}$, $C_{12}$ and $C_{21}$ from the other two;
(ii) No essential torus splits one of $C_{12}$, $C_{13}$ and $C_{22}$ from the other two;
(iii) No essential torus splits $C_{31}$ and $C_{12} \cup C_{21}$;
(iv) No essential torus splits $C_{21}'$ and $C_{13} \cup C_{22}$;
(v) Every component is simple.

For (iii), take cross disk $D_{21}$ and use Proposition 5.4(2). For (i), take cross disk $D_{11}$. By Proposition 5.4(2), any essential torus neither splits $C_{12}$ and $C_{11} \cup C_{21}$, nor splits $C_{21}$ and $C_{11} \cup C_{12}$. Suppose there is an essential torus $T$ splitting $C_{11}$ and $C_{12} \cup C_{21}$. In $L \cup D_{11}$, once delete a circle component, all other circle components split out. By Proposition 6.3, either $C_{11}$ is multiple or $T$ splits $C_{12} \cup C_{21}$ from all the other components. By (iii), it only remains to show $C_{11}$ is simple.

For (iv), take cross disk $D_{22}$ and use Proposition 5.4(2). For (ii), take cross disk $D_{12}$. By Proposition 5.4(2), any essential torus neither splits $C_{13}$ and $C_{12} \cup C_{22}$, nor splits $C_{22}$ and $C_{12} \cup C_{13}$. As demonstrated before, by Proposition 6.3, it remains to show $C_{12}$ is simple, which is proved in (i).

For (v), we have shown that $C_{12}$, $C_{21}$ are simple when proving (i), $C_{13}$ and $C_{22}$ are simple when proving (ii), and $C_{31}$ is simple by (iii). For the remained case that $C_{11}$ is simple, take cross disk $D_{13}$ intersecting $C_{11}$ in 4 points. This is Case (1.0.0)
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in Lemma 6.10(1). Delete the ear $E$. Then using the approach in Example 5.3 in [5], we can show the asserted incredible circle in Proposition 6.11(1) does not exist.

6.3. Interior disks only. If there is no cross disk, we can give a sufficient and necessary criterion to detect s-primeness, parallel to Theorem 4.1.

Theorem 6.8. Let $L = \cup_{i=1}^{n} C_i$ be an $n$-component Brunnian link. Let $I$ and $J$ be disjoint nonempty subset of $\{1, \ldots, n\}$, and let $D_I^i(i \in I)$ and $D_J^j(j \in J)$ be $L_I$-interior disks and $L_J$-interior disks respectively. Suppose each interior disk is (s8) and $U_I = L_I \cup \bigcup_{i \in I} D_I^i$ and $U_J = L_J \cup \bigcup_{j \in J} D_J^j$ are disjoint spanning complex. Then there is no essential torus in the complement of $L$ splitting $L_I$ and $L_J$ if and only if any torus splitting $U_I$ and $U_J$, if exists, is \(\partial\)-parallel to a component of $L$.

Proof. We first prove the “if” implication. Suppose that $T$ is an essential torus in the complement of $L$ splitting $L_I$ and $L_J$. Then by Theorem 5.6, $(U_I \cup U_J) \cap T = \emptyset$. Clearly $T$ is incompressible in the complement of $U_I \cup U_J$. Thus by the condition, $T$ is \(\partial\)-parallel to a component of $L$, contradicting that $T$ is essential in the complement of $L$.

Now we prove the “only if” implication. Suppose that $T$ is torus in the complement of $U_I \cup U_J$ splitting $U_I$ and $U_J$ and not \(\partial\)-parallel to any component of $L$. Then $T$ splits $L_I$ and $L_J$. By Theorem 2.3, $T$ is incompressible in the complement of $L$. This completes the proof. \(\Box\)

For Brunnian links with at least 3 components, the “if” implication of this theorem gives a method to prove s-primeness, while the “only if” implication indicates this method is theoretically universal. However, this method does not work for 2-component links, and will become complicated when the number of components is large.

Example 6.9. For the link $W(5)$ in Fig. 21(1), we show no torus splits $C_1 \cup C_2$ and $C_0 \cup C_3 \cup C_4$. Take interior disks $D_I^3$ and $D_I^4$ as shown in Fig. 21(1). By Theorem 6.8, we only need to prove no torus splits $C_1 \cup C_2$ and $C_0 \cup D_I^3 \cup D_I^1$. The regular neighborhood of the later complex is isotopic to the regular neighborhood of the graph $U_I$ in Fig. 21(2). The result follows from the claim that no torus splits the red link $C_{01} \cup C_{02} \cup C_{03}$ and $C_1 \cup C_2$, which will be proved in the last paragraph of this paper by a generalized version of Theorem 6.8.
6.4. Cross disk with the least intersection points. Recall Proposition 5.4. From its proof we see that if a cross disk $D_i$ intersects $V_j$ in its meridian disks, then $\sharp(D_i \cap L_j) \geq 4$. In this subsection, we analyze cross disks intersecting $L_j$ in the least number of points in detail.

Let $L = \bigcup_{i=1}^n C_i$ be an $n$-component Brunnian link. Let $T$ be an essential torus in the complement of $L$, splitting $S^3$ into two solid tori $V_I$ and $V_J$, such that $L \cap V_I = L_I = \bigcup_{j \in J} C_i$ and $L \cap V_J = L_J = \bigcup_{j \in J} C_i$. Let $D_i$ be a cross disk bounded by $C_i$ intersecting $V_J$ in its meridian disks such that $\sharp(D_i \cap L_J) = 4$. We give the following 6 cases on the intersection relationship of $D_i$, $T$ and $L_j$. See Fig. 22 for illustration.

If $D_i$ intersects only one component of $L_j$:  

| Case | $lk(C_i, \beta_j)$ | $lk(C_j, \beta_I)$, $\forall j$ | Case | $lk(C_i, \beta_j)$ | $lk(C_j, \beta_I)$, $\forall j$ |
|------|-----------------|-------------------------------|------|-----------------|-------------------------------|
| (1.0.0) | 0 | 0 | (1.2.0) | 2 ($L_I = C_i$) | 0 |
| (1.0.2) | 0 | 2 ($L_J = C_j$) | (1.2.2) | 2 ($L_I = C_i$) | 2 ($L_J = C_j$) |

If $D_i$ intersects more than one component of $L_j$:  

| Case | $lk(C_i, \beta_j)$ | $lk(C_j, \beta_I)$, $\forall j$ | Case | $lk(C_i, \beta_j)$ | $lk(C_j, \beta_I)$, $\forall j$ |
|------|-----------------|-------------------------------|------|-----------------|-------------------------------|
| (2.0.0) | 0 | 0 | (2.2.0) | 2 ($L_I = C_i$) | 0 |

**Figure 22.** A cross disk with the least intersection points with $L_j$.

**Lemma 6.10.** (1) If $D_i$ is an exterior cross disk, one of the 6 cases above happens.  
(2) If $D_i$ is a free cross disk, one of Case (1.0.0), (1.0.2) and (2.0.0) happens.

**Proof.** On account of linking number and Brunnian property, these are all the allowed cases. \[\square\]
In the following proposition, we call an arc component of $L \setminus D_i$ an ear if its endpoints are on the same side of $D_i$.

**Proposition 6.11.** Under Standard premise, suppose that there is only one cross disk, denoted $D_i$, and $\sharp(D_i \cap L) = 4$.

1. In Case (1.0.0) (resp. (1.2.0)), there are 2 ears on one side (resp. different sides) of $D_i$ such that after deleting each ear $E$, there is an incredible circle for $U - E$ on the same (resp. the other) side of $D_i$.

2. In Case (2.0.0) (resp. (2.2.0)), once deleting an ear $E$, there is an incredible circle for $U - E$ on the same (resp. the other) side of $D_i$.

**Proof.** We only give proof for Case (1.0.0), and other cases are similar. Using labels indicated in Fig. 22, after deleting $E$, the disk $\bar{D}_1 \cup \bar{A}_2$, after a perturbation, shows that $\bar{C}_2$ is an incredible circle for $U - E$. □

In the following example, we use this proposition in Step 2, and then use Proposition 6.1 in the first paragraph of Step 3. Finally we reduce the problem of showing s-primeness into showing two knotoids are not isotopic.

**Example 6.12.** The link $W(5)$ is s-prime. Suppose $T$ is an essential splitting torus.

**Step 1.** Take the disk $D_4$ bounded by $C_4$ as depicted in Fig. 23. By Proposition 5.4, $C_0 \cup C_4$, also $C_0 \cup D_4$, are at the same side of $T$.

The red arc $\gamma$ in $D_4$ and the left arc in $C_0$ form a circle, say $C_{0l}$, and $C_{0l} \cup C_1 \cup C_2 \cup C_3$ is just $W(4)$. Notice that $W(4)$ is a Milnor link, an s-sum of two Borromean rings, as there is a torus splits $C_{0l} \cup C_3$ and $C_1 \cup C_2$. Thus either $T$ splits $C_0 \cup D_4$ and $C_1 \cup C_2 \cup C_3$, or splits $C_0 \cup D_4 \cup C_3$ and $C_1 \cup C_2$. As we have ruled out the second possibility in Example 6.9, $T$ splits $C_0 \cup C_4$ and $C_1 \cup C_2 \cup C_3$.

**Step 2.** Take $L_J$-interior disk $D^E_J$ and exterior cross disk $D^E_J$ as depicted in Fig. 23. Then $\sharp(D^E_J \cap C_0) = 4$ and this is Case (1.0.0) in Lemma 6.10. We claim $p_{1l} \cup p_{1r}$ are in one plausible disk and $p_{6l} \cup p_{16r}$ are in another plausible disk.

**Figure 23.** $W(5)$ and spanning complex.

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The red arc $\gamma$ in $D_4$ and the left arc in $C_0$ form a circle, say $C_{0l}$, and $C_{0l} \cup C_1 \cup C_2 \cup C_3$ is just $W(4)$. Notice that $W(4)$ is a Milnor link, an s-sum of two Borromean rings, as there is a torus splits $C_{0l} \cup C_3$ and $C_1 \cup C_2$. Thus either $T$ splits $C_0 \cup D_4$ and $C_1 \cup C_2 \cup C_3$, or splits $C_0 \cup D_4 \cup C_3$ and $C_1 \cup C_2$. As we have ruled out the second possibility in Example 6.9, $T$ splits $C_0 \cup C_4$ and $C_1 \cup C_2 \cup C_3$.

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In fact, otherwise once delete $\gamma_l$, by Proposition 6.11(1), $\gamma_r$ would be capped by a disk whose boundary is the asserted incredible circle. Let $\beta_r$ be an arbitrary arc in the plausible disk connecting $\partial \gamma_r$. Then $\gamma_r \cup \beta_r$ is a circle capped by that disk. However, $\gamma_r \cup \beta_r$ is homotopically nontrivial in the complement of $C_1 \cup C_2$, a contradiction.

**Step 3.** Take $L_J$-interior disk $D_l^J$ and exterior cross disk $D^E_2, D^E_3$ as depicted in Fig. 23. Since $p_{i_l}$ and $p_{i_r}$ are in one plausible disk, by Proposition 6.1(1), $\alpha_{i_l}$ and $\alpha_{i_r}$ are in one plausible cylinder and $p_{i_2}$ and $p_{i_2}$ are in one plausible disk. Notice that by Example 3.10 the disks are all stable. It follows that $\alpha_{i_l}$ and $\alpha_{i_r}$ are in one plausible cylinder, for any $i = 1, 2, \ldots, 8$.

Consider $D^E_1 \cup D^E_3 \cup \alpha_{i_l} \cup \alpha_{i_r}$, then $\alpha_{i_l}$ and $\alpha_{i_r}$ are parallel. A standard innermost circle argument shows that $\alpha_{i_l}$ and $\alpha_{i_r}$ are parallel as well in the plausible cylinder. The shape in other plausible cylinders are similar. Consequently, there is a disk $D_0$ bounded by $C_0$ which is contained in $V_J$, intersects each plausible disk in a red arc and intersects $D^J_4$ in two green arcs as shown in Fig. 24(1).

We see that $D^E_1, D^E_2, D^E_3$ and $L$ are all symmetric about the middle sphere $\Pi$, and we may assume $D^J_4 \subset \Pi$. Clearly $D_0 \cap \Pi$ consists of circles and an arc connecting $\partial \gamma$. We can eliminate all the circle components by a standard innermost circle argument so that $D_0 \cap \Pi$ is an arc, say $\gamma_\Pi$, and then isotope $D_0$ so that the shape on $D_0$ of $\gamma_\Pi$ and the red arcs in Fig. 24(1) are as shown in Fig. 24(2).

![Figure 24](image_url)

**Figure 24.** (1) $D^J_4 \cup D_0$ in $V_J$. (2) The green arc is $\gamma_\Pi$.

**Step 4.** The left half of $D_0$ cut by $\gamma_\Pi$ indicates that $C_{0l}$ can be projected to $\Pi$ as a planar curve $\gamma_{\cup_\Pi}$, that is, parallel to $\gamma_{\cup_\Pi}$ and intersecting the three cross disks in the left halves of red arcs. To complete the proof, it remains to show this is impossible. In fact, on account of how $C_{0l}$ pass through the three cross disks, $\gamma_\cap_\Pi$ can only be as shown in Fig. 25(1). On the other hand, $C_{0l}$, projected to $\Pi$, is as shown in Fig. 25(2). Our problem reduces to prove these two knotoids are not isotopic. In fact, connecting $a_1, b_2, b_3$ form two Brunnian links. One can verify that the left one is an s-sum of Whitehead link and Borromean ring, while the right one is s-prime, which can be proven by Proposition 6.11(1).

7. **Conclusions**

1. **Series of Hyperbolic Brunnian links.**

Based on the examples, we are now confident that Theorem 1.3 can be proved.
We have pointed out that all these links are untied in Subsection 4.1. We now prove they are s-prime.

1. The proof in Example 6.2 is without loss of generality to show the links in this series are all s-prime.

2. The proof in Example 6.4 is without loss of generality for $n \geq 5$. For $n = 3$, the proof is just simpler since it suffices to prove every component is simple. For deBrunner(2), use Case (1.2.0) in Proposition 6.11(1). For deBrunner(4), the case (ii) in the proof of Example 6.4 need to be modified by using Proposition 6.11(1) and then the approaches in [5] to show there is no incredible circle.

3. A proof of s-primeness will follow Example 6.12 by induction on $n$. We know $W(5)$ is s-prime. For any $n > 5$, $W(n-1)$ is s-prime, an argument similar to the one used in the first paragraph in Example 6.12 shows that the only possibility is that $T$ splits $C_0 \cup D_n$ from all the other components of the link. The remainder of the proof is identical, except that in the last paragraph we distinguish two knotoids in a simpler manner as follows. For each thick arc, connecting the red arcs from below as shown in Fig. 25, we get two links. The left one will be a Milnor link, which is an s-sum[4], while the right one will be $W(n-1)$, which is s-prime by induction.

4. For any $m > 1$, $n > 2$, we have shown that Torus($m, n$) is s-prime in Example 6.5. For $m > 1$, $n = 2$, the proof is analogues to that when $m > 1$, $n > 2$ and is left to the reader. For $n = 1$, the link Torus($m, 1$) is isotopic to deBrunner($m$). For $m = 1$, $n > 2$, Torus($1, n$) is Brun’s chain (See [6, 5] or recall Example 4.2). The proof of that Brun’s chains are s-prime is quite similar to that for deBrunner($n$) and so is omitted.

5. By suitable modification to the proof of Torus($m, n$) and Carpet ($m, n, p$) are s-prime, we can show Tube($m, n$) is s-prime for $m > 0, n > 1$. We leave the details to the reader.

6. When $n = 1$, Carpet($m, n, p$) is the $p$-component Brun’s chain. When $n > 1$, the proof in Example 6.7 is without loss of generality to show Carpet($m, n, p$) is s-prime.

2. More Hyperbolic Brunnian links.

By our methods, we discriminate s-primeness for all the Brunnian links in literatures. Recall Subsection 4.1 and the geometric classification theorem for Brunnian links in [4]. We thus get much more hyperbolic Brunnian links. Some complicated
series include Brunnian solids (Fig. 16 in [2]), Snakes, Cirrus and Wheels [5]. We
only do not know whether Fountains and Jade-pendant [5] are hyperbolic.

3. Generalizations.
While we have considered only Brunnian links in this paper, the methods can
be extended to links with some unknotted components. Replacing the conditions
of disks all by “stable”, Theorem 4.1 detects whether there is a knotted torus con-
taining the whole link in the solid torus side, Theorem 6.8 detects whether there
is an unknotted essential torus splitting the link, and Theorem 5.6 holds as well.
The proofs are almost identical, the major change being using “stable” condition
instead of Brunnian property in Step 1.2 and Step 2.2 in the proof of Theorem 5.6.
As an application, we can use the modified theorems to prove the links in [14]
Section 5 (Fig. 2) are hyperbolic.

Our methods can be further generalized by using surfaces instead of disks. For
instance, to show no torus splits $C_{01} \cup C_{02} \cup C_{03}$ and $C_1 \cup C_2$ in Fig 26, we may take
the annulus $A$ bounded by $C_0 \cup C_2$, intersecting $C_{03}$ in 4 points. Firstly, we can
prove there is no incredible circle on $A$. Then if such a torus exists, say $T$, which
bounds a solid torus $V$ containing $C_{01} \cup C_{02} \cup C_{03}$, we can prove $A \subset V$ after
isotopy. In fact, we can eliminate inessential circle in $T$ as demonstrated before. To
eliminate annulus regions in $A$ outside $V$, the main point of our argument is that
for any $i = 1, 2, 3$, $C_{0i} \cup C_1 \cup C_2$ is a Borromean ring, which is atoroidal, and thus
$C_{01}$ and $C_{02}$ can only be the core of $V$, and such annulus regions in $A$ can only be
$n$-parallel to $T$. Finally, it is easy to see no torus splitting $C_1 \cup C_2$ and $A \cup C_{03}$ by
considering a deformation retraction of $A \cup C_{03}$.

![Figure 26. No torus splits $C_{01} \cup C_{02} \cup C_{03}$ and $C_1 \cup C_2$.](image)

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8. APPENDIX: AN ALTERNATING EXAMPLE

We provide an alternative proof that a series of alternating Brunnian links are s-prime to illustrate the method in Subsection 6.3.

![Figure 27. A link in one of Tait series in [12].](image)

Example 8.1. Fig. (1) shows a link in a Tait series. In view of symmetry, we only need to prove no essential torus splits C_3 from C_1 and C_2. Take the disk D_1 as shown in Fig. (2), which is stable by an argument similar to the one used in Example 3.11. Then a deformation retraction of D_1 \cup C_2 is as shown in Fig. (3). In view of the complement space of the red part, we see that the only splitting torus is \partial-parallel to C_3.

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