THE KILLING SPINOR EQUATION WITH HIGHER ORDER POTENTIALS

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Abstract. Let \((M^n, g)\) be a Riemannian spin manifold. The basic equations in supergravity models of type IIA string theory with 4-form flux involve a 3-form \(T\), a 4-form \(F\), a spinorial covariant derivative \(\nabla\) depending on \(\nabla^g, T, F\), and a \(\nabla\)-parallel spinor field \(\Psi\). We classify and construct many explicit families of solutions to this system of spinorial field equations by means of non-integrable special geometries. The latter include \(\alpha\)-Sasakian structures in dimensions 5 and 7, almost Hermitian structures in dimension 6 and cocalibrated \(G_2\)-structures in dimension 7. We show that there are several examples also satisfying an additional constraint for the energy-momentum tensor.

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1. Introduction

In the seventies A. Gray considered non-integrable special Riemannian geometries of small dimensions \((n \leq 8)\). A decade later these structures started to play a crucial role in the investigations of T. Friedrich et al. concerning eigenvalue estimates for the Dirac operator on a Riemannian manifold.

Following a latent period, the interest in non-integrable geometries emerged once again from the developments of theoretical physics in the common sector of type II string theory. At the turn of the century T. Friedrich and his collaborators developed a new and systematic approach to non-integrable special geometries which takes up certain aspects of string theory straightforwardly. This approach relies in a natural way on the notion of characteristic connection \(\nabla^c\). This is an affine connection with totally skew-symmetric torsion \(T^c\), the so-called torsion form, which can be associated to certain \(G\)-structures. This point of view lead to show that structures with parallel torsion \((\nabla^c T^c = 0)\) are of particular interest. They provide in fact a basis to solve Strominger’s equations in a natural manner (see [20, 31]).
Since a few years now a more general system of spinorial field equations than Strominger’s has become central in type II string theory. These models of supergravity – the so-called models with fluxes – can be described geometrically by a tuple \((M^n, g, T, F, \Psi)\) consisting of a Riemannian spin manifold \((M^n, g)\), a 3-form \(T\), a 4-form \(F\) and a spinor field \(\Psi\) satisfying

\[
\nabla^g_X \Psi + \frac{1}{4} (X \cdot T) \cdot \Psi + p (X \cdot F) \cdot \Psi + q (X \wedge F) \cdot \Psi = 0,
\]

\[
\nabla_{T} \Psi = 0.
\]

where \(\nabla^g\) denotes the Levi-Civita connection of \((M^n, g)\) and \(p, q \in \mathbb{R}\) are real parameters (see article [14]). The first of these three equations – the so-called Killing spinor equation – should be satisfied for any vector field \(X \in TM\). If one introduces the new spinorial covariant derivative

\[
\nabla_X \Psi := \nabla^g_X \Psi + \frac{1}{4} (X \cdot T) \cdot \Psi + p (X \cdot F) \cdot \Psi + q (X \wedge F) \cdot \Psi,
\]

the Killing spinor equation takes the particularly simple form \(\nabla \Psi = 0\). Considering the Kaluza-Klein reduction of \(M\)-theory (see [5, 7, 32]) the relevant dimension for \(M^n\) lies between 4 and 8. Moreover, additional algebraic constraints occur, for example the algebraic intertwining between the 3-form or the 4-form and the spinor field \(\Psi\)

\[
T \cdot \Psi = \lambda \cdot \Psi, \quad F \cdot \Psi = \kappa \cdot \Psi, \quad \lambda, \kappa \in \mathbb{C}.
\]

Obviously, the system consisting of (\(\Re\)) and (\(\Re \Re\)) generalizes Strominger’s model by introducing the new degree of freedom given by a 4-form \(F\), usually called flux form.

The task of the present work is the construction of solutions to the system (\(\Re\)) & (\(\Re \Re\)).

The paper is structured as follows: In section 2 we fit the problem into the framework of special geometries. We then specialize to the structures of concern in section 3. In section 4 we present the classification and construction techniques used to solve the Killing spinor equation and we list results thus obtained in sections 5–7. Eventually we review these in the light of the entire system (\(\Re\)) & (\(\Re \Re\)).

2. Setup

Two questions underlie the whole discussion:

- Which is the ‘correct way’ to obtain solutions to the entire system?
- Can the differential forms \(T\) and \(F\) be chosen or fixed in a canonical way?

The starting point for answering the first question is to consider the Killing spinor equation, \(\nabla \Psi = 0\). Solving this is the central task of the present paper and will be addressed in the following sections in a systematic way.

Behind the second question hides the aim to control the high degree of freedom in the choice of differential forms. To tackle this we study certain classes of non-integrable \(G\)-structures with characteristic connection \(\nabla^c\) and parallel torsion form, \(\nabla^c T^c = 0\). These structures have been investigated a lot over the last years (see [18–20, 30]). This approach has the advantage that many geometric properties follow from the parallelism of \(T^c\), and a natural ansatz for the 3-form is to require \(T\) to be fixed up to a real parameter: \(T \sim T^c\). Unfortunately, there is no natural/canonical 4-form in this setup, to the effect that – at least in principle – \(F\) is completely arbitrary. To overcome this
problem we make a special assumption on $F$, and furthermore demand it to be parallel with respect to $\nabla^c$.

Precisely, we take a class $(M^n, g, \nabla^c T^c = 0)$ of non-integrable $G$-structures with parallel torsion, fix a spin structure, define $\nabla^c$-parallel 4-forms $F_i$ and assume the following for the differential forms $T$ and $F$:

$$F = \sum_i A_i \cdot F_i, \quad T = B \cdot T^c, \quad A_i, B \in \mathbb{R}.$$ 

Given this, we try to solve $\nabla \Psi = 0$ on the underlying special structure. Which are the classes of special geometries dealt with is readily explained.

3. Special geometries

Our treatise includes the dimensions 5, 6 and 7. In dimension 5 we investigate $\alpha$-Sasakian structures. As far as dimension 6 is concerned we choose almost Hermitian structures with parallel torsion, classified by N. Schoenmann (see [29, 30]). In dimension 7 we consider both $\alpha$-Sasakian structures and cocalibrated $G_2$-structures with parallel torsion, the latter exhaustively described in [18].

3.1. $\alpha$-Sasakian structures. We begin with some basic definitions of contact geometry. The book of Blair [8] and the articles [4, 13] may serve as general references. An almost contact metric structure consists of an odd-dimensional manifold $M^{2k+1}$ equipped with a Riemannian metric $g$, a vector field $\xi$ of length one, its dual 1-form $\eta$ (contact form) as well as an endomorphism $\varphi$ of the tangent bundle such that the algebraic relations

$$\varphi(\xi) = 0, \quad \varphi^2 = -\text{Id} + \eta \otimes \xi, \quad g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X) \eta(Y)$$

are satisfied. The fundamental form $\Phi$ and the Nijenhuis tensor $N$ of an almost contact metric structure are defined by

$$\Phi(X, Y) := g(X, \varphi(Y)), \quad N(X, Y) := [\varphi, \varphi](X, Y) + d\eta(X, Y) \cdot \xi.$$ 

There are many special types of almost contact metric structures in the literature. We introduce those appearing in this paper. An almost contact metric structure is called normal if its Nijenhuis tensor vanishes, $N = 0$. A quasi-Sasakian structure has additionally closed fundamental form, $d\Phi = 0$. A normal almost contact metric structure with

$$N = 0, \quad d\eta = \alpha \cdot \Phi, \quad \alpha \in \mathbb{R}\backslash\{0\}$$

is called $\alpha$-Sasakian. Taking $\alpha = 2$ above restricts to Sasakian structures.

It is known (cf. [20, Thm. 8.2]) that every quasi-Sasakian manifold $(M^{2k+1}, g, \xi, \eta, \varphi)$ admits a unique metric connection $\nabla^c$ with totally skew-symmetric torsion $T^c$,

$$g(\nabla^c_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} \cdot T^c (X, Y, Z),$$ 

preserving the quasi-Sasakian structure, $\nabla^c \xi = \nabla^c \varphi = 0$. The torsion form is given by

$$T^c = \eta \wedge d\eta.$$ 

Obviously, $T^c$ is parallel with respect to $\nabla^c$ in the case of $\alpha$-Sasakian structures.

For every almost contact metric structure $(M^{2k+1}, g, \xi, \eta, \varphi)$ there exists an oriented orthonormal frame $(e_1, \ldots, e_{2k+1})$ such that

$$\xi = e_{2k+1}, \quad \Phi = e_{12} + e_{34} + \ldots + e_{2k-1, 2k}.$$
We will call this frame an adapted frame of the corresponding structure. Here and henceforth we shall not distinguish between vectors and covectors and use the notation $e_{i_1 \ldots i_m}$ for the exterior product $e_{i_1} \wedge \ldots \wedge e_{i_m}$.

3.2. Almost Hermitian structures. An almost Hermitian structure $(M^6, g, J)$ is a six-dimensional manifold $M^6$ equipped with a Riemannian metric $g$ and an orthogonal almost complex structure $J : TM \to TM$,

$$J^2 = -\text{Id}_{TM}, \quad g(JX, JY) = g(X, Y).$$

The Kähler form $\Omega$ is defined by $\Omega(X, Y) := g(JX, Y)$. A nearly Kähler structure is an almost Hermitian structure that satisfies the condition

$$(\nabla^g_X J)(X) = 0, \quad \forall X \in TM.$$

Finally, Kähler manifolds are characterized by the $\nabla^g$-parallelism of the almost complex structure, $\nabla^g J = 0$.

A six-dimensional almost Hermitian structure can be understood as a $U(3)$-reduction of the corresponding bundle of orthonormal frames $F(M^6, g)$. The space of 3-forms

$$\Lambda^3(\mathbb{R}^6) = \Lambda^3_0(\mathbb{R}^6) \oplus \Lambda^3_1(\mathbb{R}^6) \oplus \Lambda^3_2(\mathbb{R}^6)$$

decomposes into three irreducible $U(3)$-components as described for example in [3].

Decomposing the Lie algebra $\mathfrak{so}(6)$ into $\mathfrak{u}(3)$ and its orthogonal complement $\mathfrak{m}^6$ the fundamental classes of six-dimensional almost Hermitian structures are defined by the irreducible $U(3)$-submodules of

$$\mathbb{R}^6 \otimes \mathfrak{m}^6 = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.\]$$

$\mathcal{W}_1$, $\mathcal{W}_3$ and $\mathcal{W}_4$ can be characterized in terms of the irreducible submodules of $\Lambda^3(\mathbb{R}^6)$ above (see [3]). The notation is in accordance with the Gray-Hervella classification [27].

It is known (cf. [3, Cor. 3.5]) that there exists an unique affine connection $\nabla^c$ with totally skew-symmetric torsion $T^c$ preserving the almost Hermitian structure ($\nabla^c J = 0$) if and only if this structure is of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.

We say that six-dimensional almost Hermitian structures of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ belong to the class $\mathcal{C}[G]$ for the proper subgroup $G \subset U(3)$, if the corresponding torsion form is parallel with respect to the characteristic connection, $\nabla^c T^c = 0$, and if $\text{Hol}(\nabla^c) \subset G \subset \text{Iso}(T^c)$.

Here $\text{Hol}(\nabla^c)$ is the holonomy of the characteristic connection and $\text{Iso}(T^c)$ the connected component at the identity of the isotropy group of $T^c$. The classes $\mathcal{C}[G]$ were investigated in [30]. We will discuss those with non-abelian $G$ in more detail in section 6.

Just as before, there exists always an adapted frame $(e_1, \ldots, e_6)$, that is such that

$$\Omega = e_{12} + e_{34} + e_{56}.$$

3.3. Cocalibrated $G_2$-structures. Consider the space $\mathbb{R}^7$, fix an orientation and denote a chosen oriented orthonormal basis by $(e_1, \ldots, e_7)$. The Lie group $G_2$ can be described as the isotropy group of the 3-form

$$\omega^3 = e_{127} + e_{135} - e_{146} - e_{236} - e_{245} + e_{347} + e_{567}.\]$$

A $G_2$-structure is a triple $(M^7, g, \omega^3)$ consisting of a seven-dimensional Riemannian manifold $(M^7, g)$ and a 3-form $\omega^3$ such that there exists an oriented orthonormal adapted frame $(e_1, \ldots, e_7)$ realizing $(*)$ at every point.
Obviously, a $G_2$-structure is a reduction of the structure group of orthonormal frames of the tangent bundle to $G_2$. The space of 3-forms decomposes into three irreducible $G_2$-components (see for example \[23\]),

$$\Lambda^3(\mathbb{R}^7) = \Lambda_1^3(\mathbb{R}^7) \oplus \Lambda_2^3(\mathbb{R}^7) \oplus \Lambda_3^3(\mathbb{R}^7).$$

Using these components one can describe the irreducible $G_2$-submodules $\mathcal{X}_1$, $\mathcal{X}_3$ and $\mathcal{X}_4$ of the decomposition

$$\mathbb{R}^7 \otimes m^7 = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$$

where $m^7$ is the orthogonal complement of $g_2$ inside $\mathfrak{so}(7)$ (see \[20\]). In this way we obtain the Fernández-Gray description \[16\] of non-integrable $G_2$-structures by differential equations. For example, a $G_2$-structure is of type $\mathcal{X}_1$, i.e. a \textit{nearly parallel} structure, if and only if there exists a real number $\lambda \neq 0$ such that

$$d\omega^3 = -\lambda \cdot *\omega^3.$$

$G_2$-structures of type $\mathcal{X}_1 \oplus \mathcal{X}_3$ – the so-called \textit{cocalibrated} structures – are characterized by a coclosed 3-form,

$$\delta\omega^3 = 0.$$

The following is known (cf. \[20\], Thm. 4.7]) on the existence of characteristic connection: There exists an unique affine connection $\nabla^c$ with totally skew-symmetric torsion $T^c$ preserving the $G_2$-structure ($\nabla^c\omega^3 = 0$) if and only if this structure is of type $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$.

Cocalibrated, non-nearly parallel $G_2$-structures with parallel characteristic torsion,

$$\nabla^c T^c = 0,$$

for which the holonomy algebra $\text{hol}(\nabla^c)$ of the characteristic connection is a proper subalgebra $\mathfrak{g} \subset g_2$, are said to \textit{belong to the class} $\mathcal{C}[\mathfrak{g}]$. Structures of class $\mathcal{C}[\mathfrak{g}]$ with non-abelian $\mathfrak{g}$ were investigated in \[18\]. We will return to this in section 7.

4. Techniques

To solve $\nabla \Psi = 0$ as explained we proceed in two steps:

(1) We ‘classify’ solutions, i.e. determine necessary conditions for $\nabla \Psi = 0$ to hold.

(2) We construct solutions with the help of (1).

4.1. Classification Technique. Fix a Riemannian spin manifold $(M^n, g)$ as in section 2, denote its spinor bundle by $\Sigma$ and assume there exists a solution $\Psi \in \Gamma(\Sigma)$ of $\nabla \Psi = 0$. The spinor field $\Psi$ is an element in the kernel of the spinorial curvature tensor

$$R^\nabla(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$ 

By understanding this differential operator as an endomorphism of spinors, we may define its \textit{first and second contraction} $K^\nabla(X)$ and $K^\nabla$ via Clifford multiplication relative to an adapted frame $(e_1 \ldots, e_n)$,

$$K^\nabla(X) := \sum_k e_k \cdot R^\nabla(e_k, X), \quad K^\nabla := \sum_k e_k \cdot K^\nabla(e_k).$$
We denote the curvature terms related to $\nabla = \nabla^c$ by $R^c(X,Y)$, $K^c(X)$ and $K^c$. If we compare $R^\nabla(X,Y)$ to $R^c(X,Y)$, we obtain

$$R^\nabla(X,Y) = R^c(X,Y) + \sum_{j=1}^{12} R_j(X,Y)$$

where the algebraic terms $R_j(X,Y)$ are given by $(s := (B - 1)/4)$

$$R_1(X,Y) = s \left( T^c(X,Y) \downarrow T^c \right),$$
$$R_2(X,Y) = p \left( T^c(X,Y) \downarrow F \right),$$
$$R_3(X,Y) = q \left( T^c(X,Y) \wedge F \right),$$
$$R_4(X,Y) = s^2 \left( (X \downarrow T^c) \cdot (Y \downarrow T^c) - (Y \downarrow T^c) \cdot (X \downarrow T^c) \right),$$
$$R_5(X,Y) = s p \left( (X \downarrow T^c) \cdot (Y \downarrow F) - (Y \downarrow F) \cdot (X \downarrow T^c) \right),$$
$$R_6(X,Y) = s p \left( (X \downarrow F) \cdot (Y \downarrow T^c) - (Y \downarrow T^c) \cdot (X \downarrow F) \right),$$
$$R_7(X,Y) = s q \left( (X \downarrow T^c) \cdot (Y \wedge F) - (Y \wedge F) \cdot (X \downarrow T^c) \right),$$
$$R_8(X,Y) = s q \left( (X \wedge F) \cdot (Y \downarrow T^c) - (Y \downarrow T^c) \cdot (X \wedge F) \right),$$
$$R_9(X,Y) = p^2 \left( (X \downarrow F) \cdot (Y \downarrow F) - (Y \downarrow F) \cdot (X \downarrow F) \right),$$
$$R_{10}(X,Y) = p q \left( (X \downarrow F) \cdot (Y \wedge F) - (Y \wedge F) \cdot (X \downarrow F) \right),$$
$$R_{11}(X,Y) = p q \left( (X \wedge F) \cdot (Y \downarrow F) - (Y \downarrow F) \cdot (X \wedge F) \right),$$
$$R_{12}(X,Y) = q^2 \left( (X \wedge F) \cdot (Y \wedge F) - (Y \wedge F) \cdot (X \wedge F) \right).$$

Let

$$M_j(X) := \sum_k e_k \cdot R_j(e_k, X), \quad M_j := \sum_k e_k \cdot M_j(e_k)$$

denote the contractions. We derive the following formulae for the first and second contraction of $R^\nabla(X,Y)$ using the proof of [20, Cor. 3.2]:

$$K^\nabla(X) = \frac{1}{2} \Ric^c(X) - \frac{1}{2} \cdot (X \downarrow \sigma^{T^c}) + \sum_{j=1}^{12} M_j(X), \quad K^\nabla = -\frac{1}{2} \cdot \text{Scal}^c - 2 \cdot \sigma^{T^c} + \sum_{j=1}^{12} M_j.$$

Here $\sigma^{T^c}$ denotes the 4-form

$$\sigma^{T^c} := \frac{1}{2} \cdot \sum_i \left( e_i \downarrow T^c \right) \wedge \left( e_i \downarrow T^c \right).$$

To summarize: If $T$ and $F$ are chosen parallel with respect to $\nabla^c$, we are able to compute the first and second contraction of the spinorial curvature tensor $R^\nabla(X,Y)$ algebraically, provided we know the characteristic Ricci tensor $\Ric^c$. By a careful inspection of the kernels of these spinorial endomorphism we shall obtain necessary conditions for $\nabla \Psi = 0$. We will demonstrate this technique and exhibit the corresponding contractions $K^\nabla(X)$ and $K^\nabla$ in the case of almost Hermitian structures of class $C[\text{SU}(3)]$.

For simplicity we split the problem $\nabla \Psi = 0$ in the following sections by considering the covariant derivatives

$$\nabla = \begin{cases} 
\nabla^0 & \text{for } p = (n - 4)/4, \; q = 1 \\
\nabla^1 & \text{for } p = (n - 4)/4, \; q \text{ arbitrary} \\
\nabla^2 & \text{for } p = 0, \; q = 1 
\end{cases}$$
thus
\[
\nabla^0_X \Psi = \nabla^q_X \Psi + \frac{1}{4} (X \llcorner T) \cdot \Psi + \frac{n-4}{4} (X \llcorner F) \cdot \Psi + (X \wedge F) \cdot \Psi,
\]
\[
\nabla^1_X \Psi = \nabla^q_X \Psi + \frac{1}{4} (X \llcorner T) \cdot \Psi + \frac{n-4}{4} (X \llcorner F) \cdot \Psi + q (X \wedge F) \cdot \Psi,
\]
\[
\nabla^2_X \Psi = \nabla^q_X \Psi + \frac{1}{4} (X \llcorner T) \cdot \Psi + (X \wedge F) \cdot \Psi.
\]

Up to a rescaling of $F$ the first case refers to a particular ratio between the parameters $p$ and $q$ in $\nabla$, namely $4p = (n-4)q$. This is special in several ways, so we will call the corresponding equation of special type. For example, this is justified by the fact that the Dirac operator defined by $\nabla$ does not depend on $F$ for $p/q = (n-4)/4$ (see [1]).

4.2. Construction Technique. The construction of solutions depends very specifically on the classification results, and therefore on the underlying special structure. However, there are two general ideas on how one can use these results for constructing solutions.

Firstly, we restrict further by only considering simply connected spin manifolds, and suppose that $K^\nabla(X)\Psi = 0$ is satisfied for every spinor field belonging to a one-dimensional spin subbundle $\Sigma^1$ of $\Sigma$. The methods of [21] show that if $R^\nabla(X, Y)|_{\Sigma^1} \equiv 0$ holds and if $\nabla$ preserves $\Sigma^1$, then there exists a $\nabla$-parallel spinor field in $\Sigma^1$.

Secondly, solving $K^\nabla(X)\Psi = 0$ suggests in many cases to look for a ‘special’ spinor field, for example one parallel with respect to $\nabla^c$ or a Killing spinor. If we can prove the existence of this spinor field, we have a starting point for the construction of solutions.

The presentation of results thus obtained is divided into broad parts for each class of special geometries introduced in section 3. We describe examples relative to the various subclasses and state the classification results. Eventually we discuss which necessary conditions might be sufficient for the construction of solutions. An exception is made for almost Hermitian structures of class $C[SU(3)]$. For that case we will additionally present the techniques for clarity. This also means that we omit the completely analogous but lengthy proofs for all other cases, to be found in the thesis [28].

5. $\alpha$-Sasakian structures

Relative to an adapted frame the torsion form $T^c$ of an $\alpha$-Sasakian structure is given by (see section 3)
\[
T^c = \alpha \cdot \Phi \wedge \eta = \alpha \cdot (e_{12} + e_{34} + \ldots + e_{2k-1,2k}) \wedge e_{2k+1}.
\]

Recall [20] that the characteristic Ricci tensor is symmetric,
\[
\text{Ric}^c(X, Y) = \text{Ric}^c(Y, X),
\]
due to the $\nabla^c$-parallelism of $T^c$.

**Lemma 5.1.** On an $\alpha$-Sasakian manifold $(M^{2k+1}, g, \xi, \eta, \phi)$ of dimension $2k+1$ the Riemannian Ricci curvature in the direction $\xi$ is equal to $k\alpha^2/2$. 
Proof. In analogy to [8], the relation

\[ \alpha g(X, \varphi(Y)) = d\eta(X, Y) = 2g(\nabla_X^g \xi, Y) \]

shows that \( \nabla^g \) acts on \( \xi \) by the endomorphism \( \varphi \),

\[ \nabla_X^g \xi = -\frac{\alpha}{2} \cdot \varphi(X). \]

Let \( X \) now denote an unit vector field orthogonal to \( \xi \). Then

\[ R^g(\xi, X) \xi = -\frac{\alpha}{2} \cdot \nabla^g \xi \varphi(X) + \frac{\alpha}{2} \cdot \varphi([\xi, X]) = -\frac{\alpha}{2} \cdot \nabla^g \xi \varphi(X) + \frac{\alpha}{2} \cdot [\xi, \varphi(X)] \]

and hence \( g(R^g(\xi, X) \xi) = \alpha^2/4. \)

To compare \( \text{Ric}^c \) and \( \text{Ric}^g \) we can use the following formula (see [20]):

\[ \text{Ric}^c_{ij} = \text{Ric}^g_{ij} - \frac{1}{4} \cdot T^c_{imn} T^c_{jmn} = \text{Ric}^g_{ij} - \frac{\alpha^2}{2} \cdot \text{diag}(1, \ldots, 1, k). \]

**Proposition 5.1.** The characteristic Ricci tensor \( \text{Ric}^c \) of an \( \alpha \)-Sasakian manifold vanishes in direction of the contact vector field \( \xi \), \( \text{Ric}^c(\xi) = 0 \).

5.1. \( \alpha \)-Sasakian structures in dimension 5. Fix the flux form by \( F = A \cdot \eta \). Using this together with \( T = B \cdot T^c \) and \( \text{Ric}^c \) leads to the first and second contractions \( K^\nabla(X), K^\nabla \) relative to \( \nabla = \nabla^1, \nabla^2 \). By proposition 5.1 these contractions assume a particularly simple form along the contact vector field \( \xi \),

\[ K^{\nabla^1}(\xi) = -1/2 \alpha^2 (B^2 - 1) \cdot \eta - 1/4 A (B + 1) \cdot d\eta, \]

\[ K^{\nabla^2}(\xi) = -1/2 \alpha^2 (B^2 - 1) \cdot \eta, \]

\[ K^{\nabla^1} = -1/2 (\text{Scal}^c - 3 \alpha^2 (B^2 - 1) + 3 A^2) + 1/2 B (B - 3) \cdot (d\eta \wedge d\eta) \]

\[ + 1/2 A (B - 3 + 4 q) \cdot (\eta \wedge d\eta), \]

\[ K^{\nabla^2} = -1/2 (\text{Scal}^c - 3 \alpha^2 (B^2 - 1)) + 1/2 B (B - 3) \cdot (d\eta \wedge d\eta) \]

\[ + 2 A \cdot (\eta \wedge d\eta). \]

As a start we consider \( \nabla^1 \)-parallel spinor fields.

**Proposition 5.2.** If there exists a \( \nabla^1 \)-parallel spinor field \( \Psi_1 \) for \( \nabla^1 \neq \nabla^c \), then the following assertions hold:

1. The component of \( \Psi_1 \) in the one-dimensional subbundles defined by \( \Phi \cdot \Psi = \pm 2i \cdot \Psi \) vanishes, and the parameters are related by \( A = \pm \alpha (B - 1) \neq 0 \) respectively.
2. The characteristic Ricci tensor is \( \text{Ric}^c = (B + 2 q (B - 1)) \alpha^2 \cdot (g - \eta \otimes \eta). \)
3. If \( \Psi_1 \) contains a non-vanishing component that satisfies the equation \( \Phi \cdot \Psi = 0 \), then \( B = -1 \) and \( q = 0 \).

As mentioned in section 4 a natural question is which of these necessary conditions are sufficient as well. We can formulate an answer for line bundles. Let us thus assume that

\[ A = \pm \alpha (B - 1), \quad \text{Ric}^c = (B + 2 q (B - 1)) \alpha^2 \cdot \text{diag}(1, 1, 1, 1, 0). \]
Then the equation $K^\nabla^i(X)\Psi = 0$ is satisfied for all spinor fields in the one-dimensional spin subbundle defined by $\Phi \cdot \Psi = \pm 2i \cdot \Psi$ (signs according), and the spinorial covariant derivative $\nabla^1$ preserves this subbundle,

$$\nabla_X^1\Psi - \nabla_X^c\Psi = \pm \frac{i}{2} \alpha (B - 1) (1 + 2q) \eta(X) \cdot \Psi \quad \text{for} \quad \Phi \cdot \Psi = \pm 2i \cdot \Psi.$$

**Theorem 5.1.** Let $(M^5, g, \xi, \eta, \varphi)$ be a five-dimensional, simply connected, $\alpha$-Sasakian spin manifold with flux form $F = A \cdot \eta$. If the parameters are related by $A = \pm \alpha (B - 1)$ and

$$\text{Ric}^c = (B + 2q(B - 1)) \alpha^2 \cdot (g - \eta \otimes \eta),$$

then there exists a $\nabla^1$-parallel spinor field in the one-dimensional subbundle defined by the relation $\Phi \cdot \Psi = \mp 2i \cdot \Psi$ respectively.

**Example 5.1.** We provide an example in the case of $\alpha = 2$, i.e. Sasakian structures. Simply connected, Sasakian spin manifolds with $\text{Ric}^c = \text{diag}(4, 4, 4, 4, 0)$ can be constructed – for instance – as bundles over four-dimensional Kähler-Einstein manifolds with positive scalar curvature: Consider a simply connected Kähler-Einstein manifold $(N^4, \bar{g}, J)$ with scalar curvature $\text{Scal}^g = 32$. Then there exists an $S^1$-bundle $M^5 \to N^4$ with $(M^5, g, \xi, \eta, \varphi)$ Sasakian, such that $\text{Ric}^g = \text{diag}(6, 6, 6, 6, 4)$ (see [24]). Using Theorem 5.1 we deduce the existence of a solution $\Psi_1$ to $\nabla^1\Psi = 0$ with $\Phi \cdot \Psi_1 = \pm 2i \cdot \Psi_1$ by choosing $T = B \cdot (\eta \wedge d\eta), F = \mp 2(B - 1) \cdot \eta, q = -1/2$ respectively.

We proceed with the case $\nabla^2\Psi = 0$.

**Proposition 5.3.** If there exists a $\nabla^2$-parallel spinor field $\Psi_2$, then $B = 1$ and one of the following occurs:

1. $\text{Ric}^c = \alpha (\alpha \pm 2A) \cdot (g - \eta \otimes \eta)$ and $\Phi \cdot \Psi_2 = \mp 2i \cdot \Psi_2$.
2. $A = 0, \text{Ric}^c = -\alpha^2 \cdot (g - \eta \otimes \eta)$ and $\Phi \cdot \Psi_2 = 0$.
3. $A \neq 0, \text{Scal}^g = -4 \alpha^2, \Phi \cdot \Psi_2 = 0$ and $(M^5, g)$ is not $\eta$-Einstein.

**Remark 5.1.** This proposition generalizes results of [20].

**Example 5.2.** A Sasakian structure ($\alpha = 2$) that admits a Ricci tensor as in (2) and a $\nabla^c$-parallel spinor field with $\Phi \cdot \Psi = 0$, is locally equivalent to the Sasakian structure arising from left invariant vector fields on the five-dimensional Heisenberg group (see [19]).

As for the converse of proposition 5.3 suppose that

$$B = 1, \quad \text{Ric}^c = \alpha (\alpha \pm 2A) \cdot \text{diag}(1, 1, 1, 1, 0).$$

Then $K^\nabla^2(X)\Psi = 0$ is satisfied for spinor fields in the subbundle defined by $\Phi \cdot \Psi = \mp 2i \cdot \Psi$ respectively. Computing the difference

$$\nabla_X^2\Psi - \nabla_X^c\Psi = -i A \eta(X) \cdot \Psi \quad \text{for} \quad \Phi \cdot \Psi = \pm 2i \cdot \Psi,$$

we obtain that $\nabla^2$ preserves these spin subbundles.

**Theorem 5.2.** Let $(M^5, g, \xi, \eta, \varphi)$ be a five-dimensional, simply connected, $\alpha$-Sasakian spin manifold with flux form $F = A \cdot \eta$. Then there exists a $\nabla^2$-parallel spinor field in the one-dimensional subbundle defined by $\Phi \cdot \Psi = \pm 2i \cdot \Psi$, if $B = 1$ and if the characteristic Ricci tensor is correspondingly given by

$$\text{Ric}^c = \alpha (\alpha \mp 2A) \cdot (g - \eta \otimes \eta).$$
Remark 5.2. There exists no solution to $\nabla^2 \Psi = 0$ for $F = A \cdot \ast \eta \neq 0$ on the Sasakian structure described in example 5.1.

5.2. \(\alpha\)-Sasakian structures in dimension 7. As in the five-dimensional case we begin with fixing the flux form, $F = A \cdot \ast (\eta \wedge \Phi)$, and then directly proceed with classification results for the equation $\nabla^1 \Psi = 0$.

Proposition 5.4. The existence of a $\nabla^1$-parallel spinor field $\Psi_1$ for $\nabla^1 \neq \nabla^c$ leads to one of the following cases:

1. The parameters are related by $A (4q - 6) = \alpha (B - 1)$ and $A \neq 0$. The spinor field is fixed by $\Psi_1 = \Psi_1^+ + \Psi_1^-$ and $\Phi \cdot \Psi_1^\pm = \pm 3 i \cdot \Psi_1^\pm$. The characteristic Ricci tensor is

$$\text{Ric}^c = \alpha (2\alpha - 9A) \cdot (g - \eta \otimes \eta).$$

2. The parameters satisfy $A = -\alpha / 3$ and $B = -4q / 3 - 1$. The spinor field is given by $\Psi_1 = \Psi_1^+ + \Psi_1^-$ and $\Phi \cdot \Psi_1^\pm = \pm 3 i \cdot \Psi_1^\pm$. The characteristic Ricci tensor has the diagonal form

$$\text{Ric}^c = \alpha^2 \cdot (g - \eta \otimes \eta).$$

Again following section 4, suppose

$$A (4q - 6) = \alpha (B - 1), \quad \text{Ric}^c = \alpha (2\alpha - 9A) \cdot \text{diag}(1, 1, 1, 1, 1, 1, 0).$$

Then $K^\nabla^1 (X) \Psi = 0$ holds for all spinor fields in the one-dimensional spin subbundles defined by $\Phi \cdot \Psi = \pm 3 i \cdot \Psi$. The simple check of

$$\nabla_X^1 \Psi - \nabla_X^\nabla^1 \Psi = \mp \frac{9}{2} i A \eta(X) \cdot \Phi \cdot \Psi = \pm 3 i \cdot \Psi$$

allows to conclude that $\nabla^1$ preserves these subbundles.

Theorem 5.3. Let $(M^7, g, \xi, \eta, \varphi)$ be a seven-dimensional, simply connected, $\alpha$-Sasakian spin manifold with flux form $F = A \cdot \ast (\eta \wedge \Phi)$. There exist two $\nabla^1$-parallel spinor fields $\Psi_1^\pm$ satisfying $\Phi \cdot \Psi_1^\pm = \pm 3 i \cdot \Psi_1^\pm$, if the system parameters are related by $A (4q - 6) = \alpha (B - 1)$ and if the characteristic Ricci tensor is given by

$$\text{Ric}^c = \alpha (2\alpha - 9A) \cdot (g - \eta \otimes \eta).$$

Example 5.3. Simply connected Sasakian manifolds which admit the characteristic Ricci tensor of theorem 5.3 can be constructed via the Tanno deformation of an arbitrary seven-dimensional Einstein-Sasakian structure $(M^7, \tilde{g}, \tilde{\xi}, \tilde{\eta}, \tilde{\varphi})$. This deformation is defined by

$$\varphi := \tilde{\varphi}, \quad \xi := a^2 \cdot \tilde{\xi}, \quad \eta := a^{-2} \cdot \tilde{\eta}, \quad g := a^{-2} \cdot \tilde{g} + (a^{-4} - a^{-2}) \cdot \tilde{\eta} \otimes \tilde{\eta}$$

with the deformation parameter $a^2 = 3/2$ (see [9]). We recommend the article [10] for further constructions of Sasakian structures of $\eta$-Einstein type.

We close this subsection by classifying the solution space of $\nabla^2 \Psi = 0$.

Proposition 5.5. Suppose there exists a $\nabla^2$-parallel spinor field $\Psi_2$. Then $\Psi_2$ is parallel with respect to $\nabla^c$, $4A = \alpha (B - 1)$ is satisfied and one of the following holds:

1. $\text{Ric}^g = \frac{a^2}{2} \cdot \text{diag}(5, 5, 5, 5, 5, 3)$, $\Psi_0 = \Psi_0^+ + \Psi_0^-$ and $\Phi \cdot \Psi_0^\pm = \pm 3 i \cdot \Psi_0^\pm$.
2. $\text{Scal}^g = \frac{a^2}{2}$, $\Psi_0 = \Psi_0^+ + \Psi_0^-$ and $\Phi \cdot \Psi_0^\pm = \pm 3 i \cdot \Psi_0^\pm$.

In case (2) $(M^7, g)$ is not $\eta$-Einstein.
Remark 5.3. The integrability condition (1) corresponds with [20, Thm. 9.2] in the case of seven-dimensional Sasakian structures. Due to the necessary $\nabla^c$-parallelism of the solving spinor field we can refer to theorem 5.3 for the construction of solutions to $\nabla^2 \Psi = 0$.

6. Almost Hermitian structures

In this section we will consider almost Hermitian structures of class $C[G]$ for a connected, non-abelian subgroup $G \subset U(3)$ that stabilizes a non-trivial 3-form. $G$ will be therefore up to conjugation one of the following groups (see [29]):

1. $SU(3) \hookrightarrow U(3)$,
2. $SO(3) \hookrightarrow U(3)$, (three-dimensional, irreducible complex representation of $SU(2)$).
3. $SU(2) \hookrightarrow \iota U(3)$, $\iota: A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$.
4. $U(2) \hookrightarrow \iota_k U(3)$, $\iota_k: A \mapsto \begin{bmatrix} A & 0 \\ 0 & \text{det}(A)^k \end{bmatrix}$, $k = -1, 0, 1$.

One of the main results in this section deals with the Killing spinor equation of special type.

Theorem 6.1. Let $G$ be a connected, non-abelian subgroup of $U(3)$ that stabilizes a non-trivial 3-form and $(M^6, g, J)$ a six-dimensional, almost Hermitian spin manifold of class $C[G]$ with characteristic connection $\nabla^c$, 4-form $F = A \cdot \Omega \neq 0$, 3-form $T = B \cdot T^c$ and spinorial covariant derivative

$$\nabla^0_X \Psi = \nabla_X^g \Psi + \frac{1}{4} (X \cdot J T) \cdot \Psi + \frac{1}{2} (X \cdot J F) \cdot \Psi + (X \wedge F) \cdot \Psi.$$ 

There exists a $\nabla^0$-parallel spinor field $\Psi_0$ if and only if the following conditions are satisfied:

1. The spinor field $\Psi_0$ is parallel with respect to the characteristic connection $\nabla^c$ and satisfies $\ast \Omega \cdot \Psi = -3 \cdot \Psi$.
2. The 3-form $T$ coincides with the characteristic torsion, $T = T^c$.

The sufficiency of (1) and (2) follows from a direct computation after fixing an adapted frame $(e_1, \ldots, e_6)$ and a spin representation. Due to the complexity of the algebraic systems we have split the description into

$C[SU(3)], \ C[SO(3)], \ C[SU(2)] \text{ and } C[U(2), \iota_k]$ 

in order to prove necessity. We shall only describe in detail the first case, as the remaining are completely analogous.

6.1. Almost Hermitian structures of class $C[SU(3)]$. This class is equivalent to the class of strictly (i.e. non-Kähler) nearly Kähler structures. The torsion form of $\nabla^c$ is given by

$$T^c = a \cdot (-e_{246} + e_{136} + e_{145} + e_{235}) \in \Lambda^3_2(\mathbb{R}^6), \quad a \in \mathbb{R}, \ a > 0$$
for a chosen adapted frame \((e_1, \ldots, e_6)\). Known formulae (see [6])

\[
\| (\nabla^g X) (J)(Y) \|^2 = a^2 \left\{ \| X \| \| Y \|^2 - g(X, Y)^2 - g(JX, Y)^2 \right\},
\]

\[
(\nabla^g X) (J)(Y) = \frac{1}{2} \cdot \{ J(T^c(X, Y)) - T^c(X, JY) \}
\]

ensure that \((M^6, g, J)\) is Einstein (see [26]),

\[
\text{Ric}^g = 5 a^2 \cdot g.
\]

Then

\[
\text{Ric}^c_{ij} = \text{Ric}^g_{ij} - \frac{1}{4} \cdot T^c_{imn} T^c_{jmn} = 4 a^2 \cdot g_{ij}.
\]

**Example 6.1.** Simply connected, homogeneous examples of strictly nearly Kähler structures include the six-dimensional sphere \(S^6\), the complex projective space \(\mathbb{CP}(3)\) and the flag manifold \(F(1, 2)\) (see [12]).

We now study the spin geometry of the local model \(\mathbb{R}^6\). Denote by \(\Delta_6\) the space of complex spinors on \(\mathbb{R}^6\). Vectors and forms act on \(\Delta_6\) by Clifford multiplication, for which we choose the matrix representation of [17]. As an endomorphism of spinors the torsion form \(T^c\) then reads

\[
T^c = \begin{bmatrix}
0 & 4a & 0 \\
-4a & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Define the eigenspaces of \(T^c\) by

\[
\Delta^{1\pm} := \{ \Psi \in \Delta_6 : T^c \cdot \Psi = \pm a \cdot \Psi \}, \quad \Delta^6 := \{ \Psi \in \Delta_6 : T^c \cdot \Psi = 0 \}.
\]

Consequently, the torsion form \(T^c\) splits the spinor bundle \(\Sigma\) of \((M^6, g)\) into two one-dimensional subbundles \(\Sigma^{1\pm}\) and one six-dimensional subbundle \(\Sigma^6\). We denote the components of an arbitrary spinor field \(\Psi \in \Gamma(\Sigma)\) correspondingly by

\[
\Psi = \Psi^{1+} + \Psi^{1-} + \Psi^6.
\]

The ansatz for the family of 4-forms \(F\) chosen here is given by the Hodge dual of the Kähler form,

\[
F := A \cdot \ast \Omega, \quad A \in \mathbb{R}.
\]

Let us consider \(\nabla^1\). The spinorial covariant derivative \(\nabla^1\) can be rewritten as follows:

\[
\nabla_X^1 \Psi = \nabla_X^c \Psi + \frac{B - 1}{4} (X \ast T^c) \cdot \Psi + \frac{A}{2} (X \ast \ast \Omega) \cdot \Psi + A q (X \ast \ast \Omega) \cdot \Psi.
\]

Since \(T^c\) is parallel with respect to \(\nabla^c\), the splitting of the spinor bundle is preserved by \(\nabla^c\). With this property we derive the first necessary conditions on the existence of \(\nabla^1\)-parallel spinor fields.

**Lemma 6.1.** If there exists a \(\nabla^1\)-parallel spinor field \(\Psi_1\), the following statements hold:

1. If \(\Psi_1 \in \Gamma(\Sigma^6)\), then \(A \neq 0, B = 1\) and \(q = -1\).
2. If \(\Psi_1 \in \Gamma(\Sigma^{1+} \oplus \Sigma^{1-})\), the system parameters satisfy
   \[
   (\ast) \quad 2 (q - 1) A = \pm a (B - 1).
   \]
   If this expression is non-trivial, then \(\Psi_1^{1\pm} = 0\) respectively.
3. If \(\ast\) holds, then \(\nabla^1 \Psi = \nabla^c \Psi\) is satisfied for arbitrary \(\Psi \in \Gamma(\Sigma^{1\pm})\) respectively.
Proof. We prove (2). Fix a generic $\nabla^1$-parallel spinor field $\Psi_1 \in \Gamma(\Sigma^{1+} \oplus \Sigma^{1-})$ by

$$\Psi_1 = [(p_+ - p_-) i, (p_+ + p_-), 0, 0, 0, 0, 0, 0]^T.$$ 

We respectively have $\Psi_1 \in \Gamma(\Sigma^{1\pm})$ if and only if the complex-valued function $p_\mp$ vanishes. Due to the $\nabla^1$-parallelism of $\Psi_1$

$$-\nabla^1_{e_1} \Psi_1 = \nabla^1_{e_1} \Psi_1 - \nabla^1_{e_1} \Psi_1 = \frac{B-1}{4} (e_1 \mathcal{J}^c) \cdot \Psi_1 + \frac{A}{2} (e_1 \mathcal{J} \ast \Omega) \cdot \Psi_1 + A \, q \, (e_1 \wedge \ast \Omega) \cdot \Psi_1$$ 

$$= \begin{bmatrix}
0 \\
0 \\
A (q-1) (p_+ - p_-) + \frac{a}{2} (B-1) (p_+ + p_-) \\
0 \\
0 \\
\frac{a}{2} (B-1) (p_+ - p_-) i + A (q-1) (p_+ + p_-) i \\
0 \\
0 
\end{bmatrix}.$$ 

Since $\nabla^1_{e_1} \Psi_1 \in \Gamma(\Sigma^{1+} \oplus \Sigma^{1-})$, we deduce

$$2 (q-1) A = \pm a (B-1).$$

Writing an arbitrary vector field $X \in TM$ in components,

$$X = X^1 \cdot e_1 + \ldots + X^6 \cdot e_6,$$

we derive the following assuming $(q-1) A \neq 0$:

$$-\nabla^1_X \Psi_1 = \nabla^1_X \Psi_1 - \nabla^1_X \Psi_1 = \frac{B-1}{4} (X \mathcal{J}^c) \cdot \Psi_1 + \frac{A}{2} (X \mathcal{J} \ast \Omega) \cdot \Psi_1 + A \, q \, (X \wedge \ast \Omega) \cdot \Psi_1$$ 

$$= a (B-1) p_\pm \cdot \begin{bmatrix}
0 \\
0 \\
(X^1 - i X^2) \\
(-X^3 + i X^4) \\
(i X^5 + X^6) \\
\pm (i X^1 - X^2) \\
\pm (-i X^3 + X^4) \\
\pm (X^5 + i X^6) 
\end{bmatrix}.$$ 

We conclude that $p_\pm$ has to vanish respectively. \hfill \Box

We now start to apply the technique of section 4 to $\nabla^1 \Psi = 0$, and therefore assume that such a spinor field $\Psi_1$ exists. Let us consider the three distinct cases

1. $\Psi_1^{1+} + \Psi_1^{1-} \neq 0$ and $\Psi_1^6 \neq 0$
2. $\Psi_1^{1+} + \Psi_1^{1-} \neq 0$ and $\Psi_1^6 = 0$
3. $\Psi_1^{1+} + \Psi_1^{1-} = 0$ and $\Psi_1^6 \neq 0$

separately. Recall $\Psi_1$ is an element in the kernel of $K\nabla^1(e_i)$ and $K\nabla^1$,

$$K\nabla^1(e_i) \Psi_1 = 0, \quad K\nabla^1 \Psi_1 = 0, \quad i = 1, \ldots, 6.$$
Case (1). We use the endomorphisms $K^{\nabla^1}$, $K^{\nabla^1}(e_6)$, $K^{\nabla^1}(e_4)$ and $K^{\nabla^1}(e_2)$. In the
chosen representation these are given by

$$K^{\nabla^1} = \begin{bmatrix}
-6m_1 & 6im_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6im_2 & -6m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1
\end{bmatrix},$$

$$K^{\nabla^1}(e_6) = \begin{bmatrix}
0 & 0 & 0 & 0 & i n_1 & 0 & 0 & n_2 \\
0 & 0 & 0 & 0 & n_2 & 0 & 0 & -i n_1 \\
0 & 0 & 0 & 0 & 0 & 0 & -i c_2 & 0 \\
0 & 0 & 0 & 0 & i c_2 & 0 & 0 & 0 \\
i m_1 & m_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i c_2 & 0 & 0 & 0 \\
0 & 0 & -i c_2 & 0 & 0 & 0 & 0 & 0 \\
m_2 & -i m_1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$K^{\nabla^1}(e_4) = \begin{bmatrix}
0 & 0 & 0 & n_1 & 0 & 0 & i n_2 & 0 \\
0 & 0 & 0 & -i n_2 & 0 & 0 & n_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 \\
-6m_1 & im_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\
-im_2 & -m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$K^{\nabla^1}(e_2) = \begin{bmatrix}
0 & 0 & -n_1 & 0 & -i n_2 & 0 & 0 & 0 \\
0 & 0 & i n_2 & 0 & -n_1 & 0 & 0 & 0 \\
m_1 & -im_2 & 0 & 0 & 0 & 0 & 0 & c_2 \\
0 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 \\
im_2 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0
\end{bmatrix}.$$
Case (2). Lemma 6.1 leads to one of the following cases:
- \( A = 0, B = 1 \) and \( \Psi_1 \in \Gamma(\Sigma^{1+} \oplus \Sigma^{1-}) \).
- \( A \neq 0, B = 1, q = 1 \) and \( \Psi_1 \in \Gamma(\Sigma^{1+} \oplus \Sigma^{1-}) \).
- \( 2A(q-1) = a(B-1) \neq 0 \) and \( \Psi_1 \in \Gamma(\Sigma^{1-}) \).
- \( 2A(q-1) = -a(B-1) \neq 0 \) and \( \Psi_1 \in \Gamma(\Sigma^{1+}) \).
\( \nabla^c \Psi_1 = 0 \) holds and system (*) is satisfied for all of them.

Case (3). Lemma 6.1 reduces the problem to one of the two cases
- \( A = 0, B = 1 \).
- \( A \neq 0, B = 1, q = -1 \).

In analogy to case (1) the system \( n_1^2 = n_2^2, c_1 = 0, c_2 = 0 \) becomes inconsistent.

Summarizing the cases (1)–(3) and using lemma 6.1 proves the following theorem.

**Theorem 6.2.** Let \((M^6, g, J)\) be a six-dimensional, almost Hermitian spin manifold of class \( \mathcal{C}[SU(3)] \) with flux form \( F = A \cdot \ast \Omega \). Then there exists a \( \nabla^1 \)-parallel spinor field \( \Psi_1 \) if and only if the following conditions are satisfied:

1. The spinor field \( \Psi_1 \) is parallel with respect to \( \nabla^c \) and satisfies \( \ast \Omega \cdot \Psi = -3 \cdot \Psi \).
2. The system parameters satisfy \( 2(q-1)A = \pm a(B-1) \). If this expression is non-trivial, then \( T^c \cdot \Psi = \mp 4 a \cdot \Psi \) respectively.

**Remark 6.1.** If we fix the spinorial field equation \( \nabla^1 \Psi = 0 \) by requiring \( q = 1 \), i.e. if we consider the special type \( \nabla^0 \Psi = 0 \), the 3-form \( T \) coincides with \( T^c \) and we have proven theorem 6.1 for structures of class \( \mathcal{C}[SU(3)] \).

Solving \( \nabla^1 \Psi = 0 \) reduces to \( \nabla^c \Psi = 0 \). A direct computation proves the following.

**Proposition 6.1.** On almost Hermitian spin manifolds of class \( \mathcal{C}[SU(3)] \) there exist two \( \nabla^c \)-parallel spinor fields \( \Psi_\pm \) such that \( T^c \cdot \Psi_\pm = \mp 4 a \cdot \Psi_\pm \).

We conclude with \( \nabla^2 \Psi = 0 \).

**Proposition 6.2.** If there exists a \( \nabla^2 \)-parallel spinor field \( \Psi_2 \) with \( \nabla^2 \neq \nabla^c \), the following conditions are satisfied:

1. The system parameters fulfill \( 2A = \pm a(B-1) \) and the component of \( \Psi_2 \) in the one-dimensional spin subbundle defined by \( T^c \cdot \Psi = \pm 4 a \cdot \Psi \) respectively, vanishes.
2. If \( B \neq 2 \), the spinor field \( \Psi_2 \) is \( \nabla^c \)-parallel and fixed by \( \ast \Omega \cdot \Psi = -3 \cdot \Psi \).

6.2. Almost Hermitian structures of class \( \mathcal{C}[SO(3)] \). The characteristic torsion splits into two components,

\[
T^c = T_2^c + T_{12}^c \in \Lambda^3_2(\mathbb{R}^6) \oplus \Lambda^3_{12}(\mathbb{R}^6).
\]

We fix an adapted frame such that

\[
T_2^c = a \cdot (-e_{135} + e_{146} + e_{236} + e_{245}) \in \Lambda^3_2(\mathbb{R}^6),
\]
\[
T_{12}^c = (b + cJ) \cdot (3e_{135} + e_{146} + e_{236} + e_{245}) \in \Lambda^3_{12}(\mathbb{R}^6)
\]

for real parameters \( a, b, c \). In this the characteristic Ricci tensor is

\[
\text{Ric}^c = 4(a^2 - b^2 - c^2) \cdot g.
\]
Example 6.2. Homogeneous examples \((M^6 = G/H)\) of class \(C[SO(3)]\) can be constructed using one of the following spaces as the base manifold \(M^6\): \(SL(2, \mathbb{C})\), \(S^3 \times S^3\), \(\tilde{E}_3\), \(N\) (see [29]). Here \(\tilde{E}_3 = SU(2) \times \mathbb{R}^3\) is the universal covering of the group of Euclidean motions of \(\mathbb{R}^3\) and \(N\) a nilpotent Lie group.

We then come to results regarding \(\nabla^0 \Psi = 0\).

**Proposition 6.3.** Assuming \(F = A \cdot * \Omega \neq 0\) there exists a \(\nabla^0\)-parallel spinor field \(\Psi_0\) if and only if the following conditions are satisfied:

1. The spinor field \(\Psi_0\) is parallel with respect to \(\nabla^c\) and satisfies \(* \Omega \cdot \Psi = -3 \cdot \Psi\).
2. The parameter \(B\) equals 1.

If \(A = 0\) instead, \(\nabla^0 \Psi_0 = 0\) implies either

1. \(\text{Ric}^c \neq 0\), \(B = 1\) and \(* \Omega \cdot \Psi_0 = -3 \cdot \Psi_0\) or
2. \(\text{Ric}^c = 0\) and \(B = \pm 1\).

A direct computation leads to the existence of two solutions to \(\nabla^c \Psi = 0\) which are eigenspinors of \(T^c\),

\[ T^c \cdot \Psi_\pm = \pm 2 \|T^c\| \cdot \Psi_\pm. \]

**Proposition 6.4.** Almost Hermitian spin manifolds of class \(C[SO(3)]\) admit two \(\nabla^c\)-parallel spinor fields \(\Psi_\pm\) satisfying \(* \Omega \cdot \Psi = -3 \cdot \Psi\).

We conclude by constructing solutions to the non-special type of Killing spinor equation based on these \(\nabla^c\)-parallel spinor fields.

**Theorem 6.3.** Let \((M^6, g, J)\) be a six-dimensional, almost Hermitian spin manifold of class \(C[SO(3)]\) with flux form \(F = A \cdot * \Omega\). The equation

\[(*) \quad s (X \cdot T^c) \cdot \Psi_0 + p (X \cdot F) \cdot \Psi_0 + q (X \cdot F) \cdot \Psi_0 = 0\]

holds for the ansatz \(\Psi_0 = p_+ \cdot \Psi_+ + p_- \cdot \Psi_-\) if and only if it is solved by \(p_+ \cdot \Psi_+\) and \(p_- \cdot \Psi_-\) separately, and if \(b \cdot s = c \cdot s = 0\). The spinor field \(\Psi_\pm\) is a solution to \((*)\) if and only if

\[(2p - q) A = \pm 2 a s.\]

**Remark 6.2.** \(\nabla^c\)-parallel spinor fields \(\Psi_\pm\) solving the Killing spinor equation for \(2p \neq q\) (non-special type) and \(A \neq 0\) force the almost Hermitian structure \((M^6, g, J)\) to belong to the class \(C[SU(3)]\), i.e. \(b = c = 0\).

### 6.3. Almost Hermitian structures of class \(C[SU(2)]\).

The torsion form \(T^c\) splits into two components,

\[ T^c = T^c_{14} + T^c_6 \in \Lambda^3(\mathbb{R}^6) \oplus \Lambda^3(\mathbb{R}^6) \oplus \Lambda^3(\mathbb{R}^6), \]

whose expression in an adapted frame reads

\[
T^c_{14} = a \cdot (e_{145} + e_{235}) \in \Lambda^3(\mathbb{R}^6) \oplus \Lambda^3(\mathbb{R}^6),
\]

\[
T^c_6 = b \cdot (e_{125} + e_{345}) \in \Lambda^3(\mathbb{R}^6)
\]

for some real parameters \(a, b\). In this frame we define the 2-forms \(\Omega_1\) and \(\Omega_2\) which are \(\nabla^c\)-parallel for structures of class \(C[SU(2)]\) and \(C[U(2), \iota_k]\),

\[
\Omega_1 := e_{56}, \quad \Omega_2 := e_{12} + e_{34}.
\]

The characteristic Ricci tensor \(\text{Ric}^c\) is given by

\[
\text{Ric}^c = (a^2 + b^2) \cdot \text{diag}(1, 1, 1, 1, 0, 0).
\]
Example 6.3. The Hopf fibration $S^1 \to S^5 \to \mathbb{CP}(2)$ gives rise to a Sasakian structure on the five-dimensional sphere $S^5$. There exists an $S^1$-bundle $M^6 \to S^5$ carrying an almost Hermitian structure of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ such that the corresponding torsion form $T^c$ is parallel with respect to $\nabla^c$ and

$$\text{Hol}(\nabla^c) \subset SU(2) \subset \text{Iso}(T^c)$$

is satisfied (see [30]). Every torsion form, i.e. every combination of the parameters $a$ and $b$ can be realized using this method of construction.

We directly come to results regarding the Killing spinor equation of special type.

**Theorem 6.4.** Let $(M^6, g, J)$ be a six-dimensional, almost Hermitian spin manifold of class $C[SU(2)]$ with flux form $F = A_1 \cdot \ast \Omega_1 + A_2 \cdot \ast \Omega_2$. Assuming $B \neq 2$ there exists a $\nabla^0$-parallel spinor field $\Psi_0$ if and only if the following conditions are satisfied:

1. The spinor field $\Psi_0$ is parallel with respect to $\nabla^c$ and satisfies $\ast \Omega_1 \cdot \Psi = -\Psi$.
2. The parameter $B$ equals 1.
3. One of the following occurs:
   - $A_1 = A_2 \neq 0$ and $\ast \Omega_2 \cdot \Psi_0 = -2 \cdot \Psi_0$.
   - $A_1 = -A_2 \neq 0$ and $\ast \Omega_2 \cdot \Psi_0 = 2 \cdot \Psi_0$.
   - $A_1 = A_2 = 0$.

If $B = 2$ instead, $\nabla^0 \Psi_0 = 0$ implies $A_1^2 \neq A_2^2$ and $\ast \Omega_1 \cdot \Psi_0 = -\Psi_0$.

A direct computation yields the following.

**Proposition 6.5.** Any almost Hermitian structure of class $C[SU(2)]$ admits four $\nabla^c$-parallel spinor fields $\Psi_1^\pm, \Psi_2^\pm$ such that $\ast \Omega_1 \cdot \Psi_i^\pm = -\Psi_i^\pm$ and $\ast \Omega_2 \cdot \Psi_i^\pm = \pm 2 \cdot \Psi_i^\pm$.

We then answer the question which Killing spinor equations of non-special type can be solved by these spinor fields.

**Theorem 6.5.** Let $(M^6, g, J)$ be a six-dimensional, almost Hermitian spin manifold of class $C[SU(2)]$ with flux form $F = A_1 \cdot \ast \Omega_1 + A_2 \cdot \ast \Omega_2$. The equation

$$s(X, \Omega T^c) \cdot \Psi_0 + p(X, \Omega F) \cdot \Psi_0 + q(X \wedge F) \cdot \Psi_0 = 0$$

is solved by

$$\Psi_0 = p_1^+ \cdot \Psi_1^+ + p_2^+ \cdot \Psi_2^+ + p_1^- \cdot \Psi_1^- + p_2^- \cdot \Psi_2^-$$

if and only if the torsion term vanishes ($s = 0$) and if one of the following occurs:

1. $p_1^+ = 0, A_1 = A_2$ and $2p = q$.
2. $p_1^+ = 0, A_1 = -2A_2$ and $p = -q$.
3. $p_1^- = 0, A_1 = -A_2$ and $2p = q$.
4. $p_1^- = 0, A_1 = 2A_2$ and $p = -q$.

6.4. **Almost Hermitian structures of class $C[U(2), t_0]$.** There exists an adapted frame such that the characteristic torsion is given by

$$T^c = a \cdot (e_{125} + e_{345}) \in \Lambda^3(\mathbb{R}^6)$$
for a positive real parameter $a$. The characteristic Ricci tensor is

$$\text{Ric}^c = \begin{bmatrix}
  U_1 + U_2 & 0 & V_1 & V_2 & 0 & 0 \\
  0 & U_1 + U_2 & -V_2 & V_1 & 0 & 0 \\
  V_1 & -V_2 & U_2 & 0 & 0 & 0 \\
  V_2 & V_1 & 0 & U_2 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

where $U_i, V_i \in C^\infty(M)$ are smooth functions.

**Theorem 6.6.** Let $(M^6, g, J)$ be a six-dimensional, almost Hermitian spin manifold of class $\mathcal{C}[U(2), \iota_0]$ with flux form $F = A_1 \cdot *\Omega_1 + A_2 \cdot *\Omega_2$. Assuming $\nabla^0 \neq \nabla^c$ there exists a $\nabla^0$-parallel spinor field $\Psi_0$ if and only if the following conditions are satisfied:

1. The spinor field $\Psi_0$ is parallel with respect to $\nabla^c$.
2. The 3-form $T$ coincides with $T^c$.
3. One of the following occurs:
   - $A_1 = +A_2 \neq 0$ and $*\Omega_2 \cdot \Psi_0 = -2 \cdot \Psi_0$.
   - $A_1 = -A_2 \neq 0$ and $*\Omega_2 \cdot \Psi_0 = 2 \cdot \Psi_0$.

A direct computation now yields the following.

**Proposition 6.6.** Almost Hermitian structures of class $\mathcal{C}[U(2), \iota_0]$ that admit a $\nabla^c$-parallel spinor field belong to the class $\mathcal{C}[SU(2)]$.

**Remark 6.3.** The existence of spinor fields solving $\nabla^0 \Psi = 0$ or $\nabla^c \Psi = 0$ forces the almost Hermitian structure to belong to the class $\mathcal{C}[SU(2)]$. We can therefore refer to proposition 6.5 and theorem 6.5 for the construction of solutions to the Killing spinor equation of non-special type.

6.5. Almost Hermitian structures of class $\mathcal{C}[U(2), \iota_1]$. Using an adapted frame the torsion form can be written as

$$T^c = a \cdot (e_{135} - e_{245} + e_{236} + e_{146}) \in \Lambda_3^0(\mathbb{R}^6).$$

for some positive real parameter $a$. The characteristic Ricci tensor is proportional to the metric, $\text{Ric}^c = 4a^2 \cdot g$.

We consider the Killing spinor equation of special type.

**Theorem 6.7.** Let $(M^6, g, J)$ be a six-dimensional, almost Hermitian spin manifold of class $\mathcal{C}[U(2), \iota_1]$ with flux form $F = A_1 \cdot *\Omega_1 + A_2 \cdot *\Omega_2$. Then there exists a $\nabla^0$-parallel spinor field $\Psi_0$ if and only if the following conditions are satisfied:

1. The spinor field $\Psi_0$ is parallel with respect to $\nabla^c$ and fixed by $*\Omega_2 \cdot \Psi_0 = 2 \cdot \Psi_0$.
2. The parameter $B$ equals 1.
3. The flux form parameters satisfy $A_1 = -A_2$.

A direct computation yields the following on the existence of $\nabla^c$-parallel spinor fields.

**Proposition 6.7.** There exist no $\nabla^c$-parallel spinor fields for almost Hermitian structures of class $\mathcal{C}[U(2), \iota_1]$. 
6.6. Almost Hermitian structures of class $C[\text{U}(2), \tau_{-1}]$. This class is a subclass of $C[\text{SU}(3)]$.

Example 6.4. An almost Hermitian structure of this class is locally isomorphic to the complex projective space $\mathbb{CP}(3)$ equipped with the nearly Kähler structure coming from the twistor construction, realized by $\text{SO}(5)/\text{U}(2)$ (see [29]).

We then directly state results regarding $\nabla^0 \Psi = 0$ using a different ansatz on $F$ than for the class $C[\text{SU}(3)]$.

**Theorem 6.8.** Let $(M^6, g, J)$ be a six-dimensional, almost Hermitian spin manifold of class $C[\text{U}(2), i_{-1}]$ with flux form $F = A_1 \cdot \Omega_1 + A_2 \cdot \Omega_2$. Then there exists a $\nabla^0$-parallel spinor field $\Psi_0$ if and only if the following conditions are satisfied:

1. The spinor field $\Psi_0$ is parallel with respect to $\nabla^c$ and fixed by $\Omega_2 \cdot \Psi_0 = -2 \cdot \Psi_0$.
2. The parameter $B$ takes the value 1.
3. The flux form parameters satisfy $A_1 = A_2$.

**Remark 6.4.** The existence of $\nabla^0$-parallel spinor fields forces the 4-form to be proportional to $\Omega$.

We conclude by constructing solutions to the Killing spinor equation of non-special type using the $\nabla^c$-parallel spinor fields $\Psi_\pm$ of proposition 6.1.

**Theorem 6.9.** Let $(M^6, g, J)$ be a six-dimensional, almost Hermitian spin manifold of class $C[\text{U}(2), i_{-1}]$ with flux form $F = A_1 \cdot \Omega_1 + A_2 \cdot \Omega_2$. The equation

$$(*) \quad s (X \cdot T^c) \cdot \Psi_0 + p (X \cdot F) \cdot \Psi_0 + q (X \wedge F) \cdot \Psi_0 = 0$$

holds for ansatz $\Psi_0 = p_+ \cdot \Psi_+ + p_- \cdot \Psi_-$ if and only if it is solved by $p_+ \cdot \Psi_+$ and $p_- \cdot \Psi_-$ separately. The spinor field $\Psi_\pm$ is a solution to $(*)$ if and only if

$$(p + q) (A_1 - A_2) = 0, \quad q A_1 - 2 p A_2 \pm 2 as = 0.$$

**Remark 6.5.** Qualitatively, there exist two possible deformations of the equation $\nabla^c \Psi = 0$ leading to a Killing spinor equation with non-vanishing flux form such that either $\Psi_+$ or $\Psi_-$ is a solution.

7. Cocalibrated $G_2$-structures

In this section we study nearly parallel $G_2$-structures and cocalibrated $G_2$-structures of class $C[g]$ where $g$ is a proper, non-abelian subalgebra of $g_2$. There exist up to conjugation eight subalgebras of this type [15],

$$\text{su}(3), \text{so}(3) \subset \text{su}(3), \text{su}(2) \subset \text{su}(3), \text{u}(2) \subset \text{su}(3), \text{su}(2) \subset \text{su}(3), \text{so}(2) \subset \text{su}(2), \text{so}_{ir}(3) \subset \text{su}(3).$$

Cocalibrated $G_2$-structures of class $C[\text{so}_{ir}(3)]$ do not exist (see [18]). Let us state the main result concerning the Killing spinor equation of special type.

**Theorem 7.1.** Let $g$ be a proper, non-abelian subalgebra of $g_2$ and $(M^7, g, \omega^3)$ a cocalibrated $G_2$-manifold of class $C[g]$ with characteristic connection $\nabla^c$, characteristic torsion $T^c$, 4-form $F = A \cdot \omega^3 \neq 0$, 3-form $T = B \cdot T^c$ and spinorial covariant derivative

$$\nabla_X^0 \Psi = \nabla_X^g \Psi + \frac{1}{4} (X \cdot T) \cdot \Psi + \frac{3}{4} (X \cdot F) \cdot \Psi + (X \wedge F) \cdot \Psi.$$
In case $B \neq -7$, there exists a $\nabla^0$-parallel spinor field $\Psi_0$ if and only if the following conditions are satisfied:

1. The spinor field $\Psi_0$ is parallel with respect to $\nabla^c$ and satisfies $*\omega^3 \cdot \Psi = -7 \cdot \Psi$.
2. The 3-form $T$ coincides with the torsion form, $T = T^c$.

The sufficiency of (1) and (2) can be checked in a direct computation after fixing an adapted frame $(e_1, \ldots, e_7)$ and a spin representation. For the necessity we split the consideration into $C[\mathfrak{su}(3)] \cup C[\mathfrak{s}(3)]$, $C[\mathfrak{su}(2)] \cup C[\mathfrak{u}(2)]$ and $C[\mathfrak{su}_c(2)] \cup C[\mathbb{R} \oplus \mathfrak{su}_c(2)] \cup C[\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)]$. The latter will be called $C[\mathfrak{su}_c(2) \text{rel.}]$ henceforth.

7.1. **Nearly parallel $G_2$-structures.** Recall that the fundamental form $\omega^3$ of a given nearly parallel $G_2$-structure $(M^7, g, \omega^3)$ satisfies $d\omega^3 = -\lambda \cdot *\omega^3$ for a real parameter $\lambda \neq 0$ (see section 3). The torsion form is given by

$$T^c = -\frac{\lambda}{6} \cdot \omega^3 \in \Lambda_1^3(\mathbb{R}^7).$$

Obviously, $T^c$ is parallel with respect to $\nabla^c$. The characteristic Ricci tensor has the diagonal form

$$\text{Ric}^c = \frac{\lambda^2}{3} \cdot g.$$

Let us study the spin geometry of nearly parallel $G_2$-structures. The existence of a 3-form satisfying the differential equation $d\omega^3 = -\lambda \cdot *\omega^3$ is equivalent to the existence of a real Killing spinor (see [23]), $\nabla^g_X \Psi = \frac{\lambda}{8} X \cdot \Psi$. Denote by $KS(M^7, g)$ the space of all Killing spinors to the Killing number $\lambda/8$,

$$KS(M^7, g) := \{ \Psi \in \Gamma(\Sigma) : \nabla^g_X \Psi = (\lambda/8) X \cdot \Psi \}.$$

The dimension of $KS(M^7, g)$ is bounded by three in the case of compact, simply connected $M^7 \neq S^7$ (see [6]), $1 \leq \dim[KS(M^7, g)] \leq 3$. We say that a nearly parallel $G_2$-structure *is of type* $i$, if the dimension of $KS(M^7, g)$ equals $i$.

**Example 7.1.** Nearly parallel $G_2$-structures of type 3 are 3-Sasakian manifolds (see [22]). The only regular examples are $S^7$ and $N(1,1) = \text{SU}(3)/\text{S}^1$ (see [6]). Those of type 2 are Einstein-Sasakian manifolds (see [22]) and can be constructed as circle bundles over six-dimensional Kähler-Einstein manifolds $X^6$. A homogeneous example is $N(1,1)$ with $X^0 = F(1,2)$. Finally, an example of type 1 is $\text{SO}(5)/\text{SO}(3)$ equipped with a special Riemannian metric of Bryant (see [11]).

As an endomorphism of spinors the fundamental form $\omega^3$ reads

$$\omega^3 = \text{diag}(-7, 1, 1, 1, 1, 1, 1).$$

We split an arbitrary spinor field $\Psi$ into two components,

$$\Psi = \Psi^1 + \Psi^7, \quad \omega^3 \cdot \Psi^1 = -7 \cdot \Psi^1, \quad \omega^3 \cdot \Psi^7 = \Psi^7.$$

**Proposition 7.1.** Let $\Psi_0 \in KS(M^7, g)$ be a real Killing spinor. Then $\Psi_0, \Psi_0^7 \in KS(M^7, g)$.

**Proof.** A direct computation yields

$$\frac{\lambda}{8} X \cdot \Psi + \frac{1}{4} (X \cdot \nabla^c) \cdot \Psi = 0, \quad \left(\frac{\lambda}{8} X \cdot \Psi + \frac{1}{4} (X \cdot \nabla^c) \cdot \Psi\right)^1 = 0.$$
for arbitrary spinor fields $\Psi$ and $\hat{\Psi}$ satisfying $\omega^3 \cdot \Psi = -7 \cdot \Psi$ and $\omega^3 \cdot \hat{\Psi} = \Psi$, respectively. A real Killing spinor $\Psi_0 \in KS(M^7, g)$ therefore satisfies $(\nabla_X \Psi)^1 = 0$ and hence $\Psi_0$ is $\nabla$-parallel.

We now move on to classification results relative to $\nabla^1 \Psi = 0$.

**Proposition 7.2.** Let $F = A \cdot * \omega^3$ be the flux form. If there exists a $\nabla^1$-parallel spinor field $\Psi_1$, one of the following holds:

1. The component $\Psi_1^1$ vanishes. $\Psi_1$ is a real Killing spinor, $\Psi_1 \in KS(M^7, g)$. The parameters satisfy $-24 A (q - 1) = \lambda (B - 1)$.
2. The component $\Psi_1^1$ vanishes. $\Psi_1$ is a real Killing spinor, $\Psi_1 \in KS(M^7, g)$. The parameters satisfy $A = \lambda/6$, $B = -4q - 3$.
3. Both components of $\Psi_1$ are non-trivial, and the parameters satisfy either
   - $A = \lambda/3$ and $B = -8q + 9$ or
   - $A = 0$ and $B = 3$.

A short computation directly leads to the following.

**Theorem 7.2.** Let $(M^7, g, \omega^3)$ be a nearly parallel $G_2$-manifold with flux form $F = A \cdot * \omega^3$. A spinor field $\Psi_1$ is $\nabla^1$-parallel, if it is a real Killing spinor, $\Psi_1 \in KS(M^7, g)$, and if one of the following holds:

- $-24 A (q - 1) = \lambda (B - 1)$ and $\Psi_1^1 = 0$.
- $A = \lambda/6$, $B = -4q - 3$ and $\Psi_1^1 = 0$.

**Remark 7.1.** There exist $i$ Killing spinors for nearly parallel $G_2$-structures of type $i$. Without loss of generality, each of these satisfies either $\Psi_1^1 = 0$ or $\Psi_1^2 = 0$ (cf. proposition 7.1).

**Remark 7.2.** Let us compare the previous results to those of [1]. We therefore consider a simply connected, nearly parallel $G_2$-structure, and normalize its metric such that the scalar curvature equals 168. Given this there exists a real Killing spinor $\Psi$ ($\nabla_X^g \Psi = X \cdot \Psi$) with $\omega^3 \cdot \Psi = -7 \cdot \Psi$. The equation

$$X \cdot \Psi + \frac{r}{4} (X \lrcorner \omega^3) \cdot \Psi + s (X \lrcorner \ast \omega^3) \cdot \Psi + t (X \wedge \ast \omega^3) \cdot \Psi = 0$$

holds for this spinor field if and only if $16 s = -4 + 12 q - 3 r$ (see [1]). If we translate the latter relation into the conventions of this paper,

$$r = -\frac{4}{3} B, \quad s = \frac{3}{4} A, \quad t = q A,$$

we obtain $-24 A (q - 1) = 8 (B - 1)$, and the result above corresponds with proposition 7.2 and theorem 7.2.

We proceed with classification results regarding $\nabla^2 \Psi = 0$.

**Proposition 7.3.** Let $F = A \cdot * \omega^3$ be the flux form. If there exists a $\nabla^2$-parallel spinor field $\Psi_2$, one of the following assertions hold:

1. The parameters satisfy $-24 A = \lambda (B - 1)$. $\Psi_2$ is a real Killing-Spinor, $\Psi_2 \in KS(M^7, g)$, and its component $\Psi_2^1$ vanishes.
2. The parameters satisfy $-24 A = \lambda (B - 3)$. Both components of $\Psi_2$ are non-trivial, i.e. $\Psi_2^1, \Psi_2^2 \neq 0$.

We conclude with results on the existence of $\nabla^2$-parallel spinor fields.
Theorem 7.3. Let \((M^7, g, \omega^3)\) a nearly parallel \(G_2\)-manifold with flux form \(F = A \cdot \ast \omega^3\). A spinor field \(\Psi_2\) is parallel with respect to \(\nabla^2\), if it is a real Killing spinor, \(\Psi_2 \in KS(M^7, g)\), with \(\nabla^2_2 = 0\), and if the relation \(-24 A = \lambda (B - 1)\) holds.

7.2. Cocalibrated \(G_2\)-structures of class \(\mathcal{C}[\mathfrak{su}(3)]\). There are two different types of admissible torsion forms. In an adapted frame these are given by

\[
T_I^c = a \cdot (e_{127} + e_{347} + e_{567}), \\
T_{II}^c = a \cdot (-2e_{123} + e_{136} - e_{145} + e_{235} + e_{246} + 2e_{356}) + b \cdot (-2e_{124} - e_{135} - e_{146} + e_{236} - e_{245} + 2e_{456}) + c \cdot (e_{135} - e_{146} - e_{236} - e_{245})
\]

where \(a, b, c\) are real parameters. The characteristic Ricci tensor has the diagonal form

\[
\text{Ric}^c = \lambda \cdot \text{diag}(1, 1, 1, 1, 1, 1, 0)
\]

for a constant \(\lambda\) depending on the torsion type of the underlying structure,

\[
\lambda_I = 2a^2, \quad \lambda_{II} = -4(a^2 + b^2 - c^2).
\]

Example 7.2. A cocalibrated \(G_2\)-manifold of class \(\mathcal{C}[\mathfrak{su}(3)]\) with torsion type I, for example, is homothetic to an \(\eta\)-Einstein-Sasakian manifold whose Riemannian Ricci tensor is given by \(\text{Ric}^g = 10 \cdot g - 4 \cdot e_7 \otimes e_7\) (see [18]). A complete, simply connected, cocalibrated \(G_2\)-manifold of the same class but with torsion type II is isometric [18] to the product of a six-dimensional strictly nearly Kähler manifold with \(\mathbb{R}\).

After defining the 3-form

\[
D_1 := (e_{127} + e_{347} + e_{567}) - \omega^3 = \text{diag}(4, -4, 0, 0, 0, 0, 0)
\]

we can state the following result on necessary conditions for \(\nabla^0 \Psi = 0\).

Theorem 7.4. Let \((M^7, g, \omega^3)\) be a cocalibrated \(G_2\)-manifold of class \(\mathcal{C}[\mathfrak{su}(3)]\) or \(\mathcal{C}[\mathfrak{so}(3)]\) with flux form \(F = A \cdot \ast \omega^3\). Assuming \(A \neq 0\) there exist a \(\nabla^0\)-parallel spinor field \(\Psi_0\) if and only if the following is satisfied:

1. The spinor field \(\Psi_0\) is \(\nabla^c\)-parallel and satisfies \(D_1 \cdot \Psi = 4 \cdot \Psi\).
2. The parameter \(B\) equals 1.

If \(A = 0\) instead, \(\nabla^0 \Psi_0 = 0\) implies either

1. \(\text{Ric}^c \neq 0\), \(B = 1\), \(\Psi_0 = \Psi_0^+ + \Psi_0^-\) and \(D_1 \cdot \Psi_0^\pm = \pm 4 \cdot \Psi_0^\pm\) or
2. \(\text{Ric}^c = 0\) and \(B = \pm 1\).

Remark 7.3. The condition \(\text{Ric}^c = 0\) can only be realized for structures of class \(\mathcal{C}[\mathfrak{so}(3)]\) with torsion type II.

A direct computation yields that two \(\nabla^c\)-parallel spinor fields exist,

\[
\Psi_+ := [1, 0, 0, 0, 0, 0, 0]^T, \quad \Psi_- := [0, 1, 0, 0, 0, 0, 0]^T.
\]

Proposition 7.4. Any cocalibrated \(G_2\)-structures of class \(\mathcal{C}[\mathfrak{su}(3)]\) or \(\mathcal{C}[\mathfrak{so}(3)]\) admits two \(\nabla^c\)-parallel spinor fields \(\Psi_\pm\) such that \(D_1 \cdot \Psi_\pm = \pm 4 \cdot \Psi_\pm\).

Let us construct solutions to the non-special type of Killing spinor equation based on these \(\nabla^c\)-parallel spinor fields but putting a more general assumption on the \(\nabla^c\)-parallel flux form,

\[
F = A_1 \cdot F_1 + A_2 \cdot (F_2 + F_3), \quad A_1, A_2 \in \mathbb{R}.
\]
Here the 4-forms $F_i$ are defined by

$$F_1 := -e_{2467} + e_{2357} + e_{1457} + e_{1367}, \quad F_2 := e_{1256} + e_{3456}, \quad F_3 := e_{1234}.$$ 

**Theorem 7.5.** Let $(M^7, g, \omega^3)$ be a cocalibrated $G_2$-manifold of class $\mathcal{C}[\mathfrak{su}(3)]$ or $\mathcal{C}[\mathfrak{so}(3)]$ with flux form $F = A_1 \cdot F_1 + A_2 \cdot (F_2 + F_3)$. The equation

$$(*) \quad s(X \cdot T^c) \cdot \Psi_0 + p (X \cdot F) \cdot \Psi_0 + q (X \cdot F) \cdot \Psi_0 = 0$$

holds for the ansatz $\Psi_0 = p_+ \cdot \Psi_+ + p_- \cdot \Psi_-$ if and only if it is solved by $p_+ \cdot \Psi_+$ and $p_- \cdot \Psi_-$ separately, and if $a \cdot s = b \cdot s = 0$ in case of torsion type II. The spinor field $\Psi_\pm$ is a solution to $(*)$ if and only if the system

$$(p + q) (\pm A_2 - A_1) = \alpha^\pm s, \quad 4 p A_1 = \pm 3 q A_2 - \beta^\pm s$$

holds for the parameters $\alpha^\pm$ and $\beta^\pm$, which depend on the torsion type,

$$\alpha_1^\pm = \pm a, \quad \alpha_II^\pm = -c, \quad \beta_1^\pm = \pm 3a, \quad \beta_II^\pm = 0.$$

**7.3. Cocalibrated $G_2$-structures of class $\mathcal{C}[\mathfrak{su}(2)] \cup \mathcal{C}[\mathfrak{u}(2)]$.** In an adapted frame the three admissible types of torsion forms read

$$T_1^c = a \cdot (e_{127} + e_{347}) + b \cdot e_{567},$$

$$T_II^c = a \cdot (e_{135} - e_{146} - e_{236} - e_{245}) + b \cdot (e_{127} + e_{347} - 2 e_{567}), \quad a \neq 0,$$

$$T_III^c = a \cdot (e_{135} - e_{245})$$

for real parameters $a, b$. For structures of class $\mathcal{C}[\mathfrak{su}(2)]$ with torsion type I or II the relation $a = 0$ or $a = -b$ is satisfied, respectively. There exists no structure of class $\mathcal{C}[\mathfrak{u}(2)]$ with torsion type III. The Ricci tensor $\text{Ric}^c$ of the characteristic connection is diagonal,

$$\text{Ric}^c = \text{diag}(\lambda, \lambda, \lambda, \lambda, \kappa, \kappa, 0).$$

Here the numbers $\lambda$ and $\kappa$ depend on the torsion type,

$$\lambda_1 = a^2 + a b, \quad \lambda_II = 4 a^2 - b^2, \quad \lambda_III = a^2,$$

$$\kappa_1 = 2 a b, \quad \kappa_II = 4 a^2 - 4 b^2, \quad \kappa_III = 0.$$

**Example 7.3.** Any complete, simply connected, cocalibrated $G_2$-manifold of class $\mathcal{C}[\mathfrak{su}(2)]$ with torsion type I – for instance – splits [18] into the product $M^7 = Y^4 \times S^3$ of the sphere $S^3$ with a four-dimensional, complete, simply connected, Ricci-flat, anti-selfdual manifold $Y^4$.

For the formulation of classification results on $\nabla^0 \Psi = 0$ we split the tangent bundle into

$$TM^7 = E_1 \oplus E_2$$

where $E_2$ is spanned by $\{e_5, e_6, e_7\}$, and define the 3-form

$$D_2 := -(e_{135} - e_{146} - e_{236} - e_{245}) - \frac{1}{2} \cdot (e_{127} + e_{347} - 2 e_{567})$$

$$= \text{diag}(4, -4, -2, -2, 1, 1, 1, 1, 1, 1).$$

**Theorem 7.6.** Let $(M^7, g, \omega^3)$ be a cocalibrated $G_2$-manifold of class $\mathcal{C}[\mathfrak{su}(2)]$ or $\mathcal{C}[\mathfrak{u}(2)]$ with flux form $F = A \cdot \omega^3$. Assuming $A \neq 0$ there exists a $\nabla^0$-parallel spinor field $\Psi_0$ if and only if the following conditions are satisfied:

1. The spinor field $\Psi_0$ is parallel with respect to $\nabla^c$ and fixed by $D_2 \cdot \Psi_0 = 4 \cdot \Psi_0$.
2. $\mathcal{T}$ coincides with the torsion form.
If \( A = 0 \) instead, \( \nabla^0 \Psi_0 = 0 \) implies either

1. \( \text{Ric}^c|_{E_1} \neq 0, \text{Ric}^c|_{E_2} \neq 0, B = 1, \Psi_0 = \Psi_0^+ + \Psi_0^- \) and \( D_2 \cdot \Psi_0^+ = \pm 4 \cdot \Psi_0^- \) or
2. \( \text{Ric}^c|_{E_1} \neq 0, \text{Ric}^c|_{E_2} = 0, B = 1, \Psi_0 = \Psi_0^+ + \Psi_0^- + \Psi_0^2 \) and \( D_2 \cdot \Psi_0^2 = -2 \cdot \Psi_0^2 \) or
3. \( \text{Ric}^c = 0 \) and \( B = \pm 1 \).

**Remark 7.4.** The condition \( \text{Ric}^c = 0 \) can only be realized on certain structures of class \( C[u(2)] \) with torsion type I.

Again, the solution of \( \nabla^0 \Psi = 0 \) reduces to \( \nabla^c \Psi = 0 \). A direct computation leads to the existence of two spinor fields solving the latter equation, \( \Psi^+ \) and \( \Psi^- \) (see subsection 7.2). There exist another two \( \nabla^c \)-parallel spinor fields for structures of class \( C[su(2)] \),

\[
\Psi_1 := [0, 0, 1, 0, 0, 0, 0]^T, \quad \Psi_2 := [0, 0, 0, 1, 0, 0, 0]^T.
\]

**Proposition 7.5.** Any cocalibrated \( G_2 \)-structures of class \( C[su(2)] \) or \( C[u(2)] \) admits two \( \nabla^c \)-parallel spinor fields \( \Psi_{\pm} \) such that \( D_2 \cdot \Psi_{\pm} = \pm 4 \cdot \Psi_{\pm} \). There exist another two \( \nabla^c \)-parallel spinor fields \( \Psi_1, \Psi_2 \) for structures of class \( C[su(2)] \) satisfying \( D_2 \cdot \Psi = -2 \cdot \Psi \).

Recall the definition of the 4-forms \( F_1, F_2 \) and \( F_3 \) of the last subsection. We proceed by constructing solutions to the non-special type of Killing spinor equation based on the \( \nabla^c \)-parallel spinor fields \( \Psi_{\pm} \).

**Theorem 7.7.** Let \( (M^7, g, \omega^3) \) be a cocalibrated \( G_2 \)-manifold of class \( C[su(2)] \) or \( C[u(2)] \) with flux form \( F = A_1 \cdot F_1 + A_2 \cdot F_2 + A_3 \cdot F_3 \). The equation

\[
(s) \quad s (X \lrcorner T^c) \cdot \Psi_0 + p (X \lrcorner F) \cdot \Psi_0 + q (X \wedge F) \cdot \Psi_0 = 0
\]

holds for the ansatz \( \Psi_0 = p_+ \cdot \Psi_+ + p_- \cdot \Psi_- \) if and only if it is solved by \( p_+ \cdot \Psi_+ \) and \( p_- \cdot \Psi_- \) separately, and if \( s = 0 \) in case of torsion type III. \( \Psi_{\pm} \) is a solution to \((s)\) if and only if

\[
(p+q) (\pm A_3 - A_1) = \alpha_{\pm} s, \quad (p+q) (\pm A_2 - A_1) = \beta_{\pm} s, \quad 4pA_1 = \pm q (2A_2 + A_3) - \gamma_{\pm} s.
\]

The parameters \( \alpha_{\pm}, \beta_{\pm} \) and \( \gamma_{\pm} \) depend on the type of torsion form,

\[
\alpha_{\pm} = \mp b, \quad \alpha_{II} = - (\pm 2b + a), \quad \beta_{II} = \pm a, \quad \beta_{II} = \mp b - a, \quad \gamma_{II} = \pm (2a + b) s, \quad \gamma_{II} = 0.
\]

We then proceed with construction results based on \( \Psi_1 \) and \( \Psi_2 \).

**Theorem 7.8.** Let \( (M^7, g, \omega^3) \) be a cocalibrated \( G_2 \)-manifold of class \( C[su(2)] \) with flux form \( F = A_1 \cdot F_1 + A_2 \cdot F_2 + A_3 \cdot F_3 \). The equation

\[
(s) \quad s (X \lrcorner T^c) \cdot \Psi_0 + p (X \lrcorner F) \cdot \Psi_0 + q (X \wedge F) \cdot \Psi_0 = 0
\]

holds for the ansatz \( \Psi_0 = p_1 \cdot \Psi_1 + p_2 \cdot \Psi_2 \) if and only if it is solved by \( p_1 \cdot \Psi_1 \) and \( p_2 \cdot \Psi_2 \) separately. The spinor field \( \Psi_i \) is a solution to \((s)\) if and only if

\[
2 (p + q) A_1 = \alpha^i s, \quad 2 (p + q) A_2 = \beta^i s, \quad 2 (p + q) A_3 = \gamma^i s, \quad 2 p A_2 + q A_3 = \delta^i s.
\]

The numbers \( \alpha^i, \beta^i, \gamma^i, \delta^i \) depend on the torsion type,

\[
\alpha^i = 0, \quad \alpha_{II} = 2a, \quad \alpha_{III} = a, \quad \beta^i = 2a, \quad \beta_{II} = 2b, \quad \beta_{III} = (-1)^{i-1} a, \quad \gamma^i = 2b, \quad \gamma_{II} = -4b, \quad \gamma_{III} = (-1)^{i-1} a, \quad \delta^i = b, \quad \delta_{II} = -2b, \quad \delta_{III} = (-1)^i a.
\]

**Remark 7.5.** This construction shows that for cocalibrated \( G_2 \)-structures of class \( C[su(2)] \) with torsion type I or II there exist at most four linearly independent \( \nabla \)-parallel spinor fields \( (F \neq 0) \), with type III at most three.
Example 7.4. Take a structure of class $C[\mathfrak{su}(2)]$ with torsion type II. The four spinor fields $\Psi_+, \Psi_-, \Psi_1$ and $\Psi_2$ solve ($\ast$) if and only if
\[ p = 0, \quad q \neq 0, \quad q A_1 = a s, \quad q A_2 = b s, \quad A_3 = -2 A_2. \]

7.4. Cocalibrated $G_2$-structures of class $C[\mathfrak{su}_c(2) \text{ rel.}]$. Using an adapted frame the torsion form is given by
\[ T^c = a \cdot \omega^3 + b \cdot e_{567}, \quad b \neq 0 \]
for real parameters $a, b$. The characteristic Ricci tensor has diagonal form,
\[ \text{Ric}^c = \text{diag}(\lambda, \lambda, \lambda, \kappa, \kappa, \kappa), \]
and the numbers $\lambda$ and $\kappa$ are
\[ \lambda = 12 a^2 + 3 a b, \quad \kappa = 12 a^2 + 4 a b. \]

Example 7.5. Any complete, simply connected, cocalibrated $G_2$-manifold of class $C[\mathfrak{su}_c(2)]$ or $C[\mathbb{R} \oplus \mathfrak{su}_c(2)]$ is a naturally reductive homogeneous space (see [18]).

We recall the splitting of the tangent bundle $TM^7 = E_1 \oplus E_2$ introduced in the last subsection, define the 3-form
\[ D_3 := e_{567} - \omega^3 = \text{diag}(6, -2, -2, 0, 0, 0, 0) \]
and state the classification results regarding $\nabla^0 \Psi = 0$.

Theorem 7.9. Let $(M^7, g, \omega^3)$ be a cocalibrated $G_2$-manifold of class $C[\mathfrak{su}_c(2) \text{ rel.}]$ with flux form $F = A \cdot \ast \omega^3$. If there exists a $\nabla^0$-parallel spinor field $\Psi_0$ for $A \neq 0$, one of the following holds:
1. The spinor field is $\nabla^c$-parallel and satisfies $D_3 \cdot \Psi = 6 \cdot \Psi$. The parameter $B$ equals 1.
2. The spinor field is fixed by $D_3 \cdot \Psi = -2 \cdot \Psi$. The system parameters satisfy $A = -2 a, B = -7, b = 3 a$.

If $A = 0$ instead, $\nabla^0 \Psi_0 = 0$ implies one of the following:
1. $\text{Ric}^c|_{E_1} \neq 0, \text{Ric}^c|_{E_2} \neq 0, B = 1$ and $D_3 \cdot \Psi_0 = 6 \cdot \Psi_0$.
2. $\text{Ric}^c|_{E_1} \neq 0, \text{Ric}^c|_{E_2} = 0, B = 1, \Psi_0 = \Psi_0^1 + \Psi_0^3$ and $D_3 \cdot \Psi_0^3 = -2 \cdot \Psi_0^3$.
3. $\text{Ric}^c = 0$ and $B = \pm 1$.

A direct computation leads to the existence of a $\nabla^c$-parallel spinor field, namely $\Psi_+$ (see subsection 7.2).

Proposition 7.6. Cocalibrated $G_2$-structures of class $C[\mathfrak{su}_c(2) \text{ rel.}]$ admit a $\nabla^c$-parallel spinor field $\Psi_+$ satisfying $D_3 \cdot \Psi = 6 \cdot \Psi$.

We conclude with construction results regarding the non-special type of Killing spinor equation.

Theorem 7.10. Let $(M^7, g, \omega^3)$ be a cocalibrated $G_2$-manifold of class $C[\mathfrak{su}_c(2) \text{ rel.}]$ with flux form $F = A_1 \cdot (F_1 + F_2) + A_2 \cdot F_3$. The equation
\[ s (X \perp T^c) \cdot \Psi_+ + p (X \perp F) \cdot \Psi_+ + q (X \wedge F) \cdot \Psi_+ = 0 \]
holds if and only if
\[ (p + q) (A_1 - A_2) = -b s, \quad 3 (p - q) A_1 + p A_2 = -3 a s. \]
8. Conclusions

Let us go back to the full system (ℵ) & (ℵℵ). If we define

\[ \text{Ric}^T := \text{Ric}^g_{ij} - \frac{1}{4} T_{imn} T_{jmn}, \]

the relations

\[ \delta T = 0, \quad F \cdot \Psi = \kappa \cdot \Psi, \quad \text{div}^c(\text{Ric}^T) = \text{div}(\text{Ric}^T) = 0 \]

hold in all cases discussed here. If we replace – as suggested in [2] – the equation \( \text{Ric}^T = 0 \) in (ℵ) by \( \text{div}(\text{Ric}^T) = 0 \), then every constructed solution satisfies the new system, provided furthermore that the spinor field \( \Psi \) is an eigenspinor of \( T \).

Table 1 contains a summary of our results. For instance, cocalibrated G2-structures of class \( C[\text{su}(2)] \) with torsion type I will admit at most four distinct (i.e. linearly independent) spinor fields parallel with respect to a certain family of spinorial covariant derivatives with non-vanishing 4-form. Should we further consider \( T = T^c \), there exists a family \( \nabla \) rendering three distinct spinor fields parallel. At the same time \( \text{Ric}^{T^c} = \text{Ric}^c = 0 \) is fulfilled. There exist other structures of the same type and a family \( \nabla \) with now four distinct spinor fields such that \( \nabla \Psi = 0 \), \( \text{Ric}^T = 0 \) but \( T \neq T^c \). The ‘check’ mark indicates that all these spinor fields are eigenspinors relative to the 3-form \( T \). Finally, there exist four distinct \( \nabla^c \)-parallel spinor fields.

To conclude, a few comments on possible generalizations of our spinorial field equations. Solving the following Killing spinor equation is the main concern of supergravity models in type II string theory (see [25]):

\[ \nabla^g_X \Psi + \frac{1}{4} (X \lrcorner T) \cdot \Psi + \sum_i p_i (X \lrcorner F^i) \cdot \Psi + \sum_i q_i (X \wedge F^i) \cdot \Psi = 0. \]

The differential forms \( F^i \) are of degree 2 \( i \) (type IIa) or of degree 2 \( i + 1 \) (type IIb). Due to the complexity of the algebraic systems the approach of this paper is unlikely to be suitable for this general kind of equation. However, \( \nabla^c \)-parallel spinor fields may represent natural candidates to begin with when constructing solutions. If we start from one of the structures considered in this work and set \( T = B \cdot T^c \), then the above equation will read

\[ \frac{B - 1}{4} (X \lrcorner T^c) \cdot \Psi_0 + \sum_i p_i (X \lrcorner F^i) \cdot \Psi_0 + \sum_i q_i (X \wedge F^i) \cdot \Psi_0 = 0 \]

for a \( \nabla^c \)-parallel spinor field \( \Psi_0 \). The last column of table 1 tells us how many such spinor fields exist. We conjecture that this purely algebraic equation could be solved with an appropriate ansatz for the differential forms \( F^i \).

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Table 1. Existence of solutions to $\nabla \Psi = 0$. $N$ denotes the maximum number of constructed, linearly independent spinor fields which are parallel with respect to a certain family of spinorial covariant derivatives $\nabla$. The superscript $c$ refers to the characteristic connection $\nabla^c$. In the second last column we determine whether all constructed solutions are eigenspinors of the differential form $T$.

| Dim. | Structure | $F \neq 0$ | $F = 0$ |
|------|-----------|------------|---------|
|      |           | $N$ | $N(\text{Ric}^c = 0)$ | $T \cdot \Psi = \lambda \cdot \Psi$ | $N^c$ |
| $n = 5$ | $\alpha$-Sasakian structure | 1 | 1 | 1 | ✔ | 2 |
| $n = 6$ | almost Hermitian structure | 2 | 1 | 2 | ✔ | 2 |
| of class | SU(2) | 2 | 1 | 2 | ✔ | 4 |
| $C[G]$ | U(2)₀ | 2 | 1 | 2 | ✔ | 4 |
|        | U(2)₁ | no solutions | ✔ | 2 |
| $n = 7$ | nearly parallel $G_2$-structure | 2 | 2 | 1 | ✔ | 1 |
| of class | $\mathfrak{su}(3)$ | I | 2 | 2 | 1 | ✔ | 2 |
| $C[g]$ | $\mathfrak{su}(2)$ | I | 4 | 4 | 3 | ✔ | 4 |
|        | $\mathfrak{so}(3)$ | I | 2 | 2 | 1 | ✔ | 2 |
|        | $\mathfrak{su}(2)$ | II | 4 | 2 | 1 | ✔ | 4 |
|        | $\mathfrak{u}(2)$ | III | 3 | 2 | 1 | ✔ | 4 |
|        | $\mathfrak{su}_c(2)$ rel. | 1 | 1 | 1 | ✔ | 1 |

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