SU$_X(r, L)$ IS SEPARABLY UNIRATIONAL

GEORG HEIN

Abstract. We show that the moduli space of SU$_X(r, L)$ of rank $r$ bundles of fixed determinant $L$ on a smooth projective curve $X$ is separably unirational.

1. Introduction

In a discussion V. B. Mehta pointed out to me that for certain applications about the cohomology of moduli of vector bundles on smooth projective curves over algebraically closed fields in characteristic $p$ it is necessary to have that SU$_X(r, L)$ is separably unirational. This short note provides us with a proof of this statement.

Theorem 1. Let $X$ be a projective curve of genus $g \geq 2$ over an algebraically closed field $k$ of arbitrary characteristic. We fix a line bundle $L$ on $X$. The moduli space SU$_X(r, L)$ of $S$-equivalence classes of semistable vector bundles of rank $r$ with determinant isomorphic to $L$ is separably unirational, that means there exists an open subset $U \subset \mathbb{P}^{(r^2-1)(g-1)}$, and an étale morphism $U \to SU_X(r, L)$.

For a discussion of the notion separable unirationality and typical applications see the lecture notes [2, 1.10 and 1.11]. If the characteristic of $k$ is zero, then separably unirational and unirational coincide. Thus, in this case the result is well known (see for example page 53 in Seshadri’s lecture notes [1]).

2. Proof of theorem [1]

Let $d = \deg(L)$ be the degree of $L$. Fix a line bundle $M$ such that $\deg(M) < \frac{d}{r} - 2g$. Let $E$ be any semistable vector bundle on $X$ with $\text{rk}(E) = r$, and $\text{det}(E) \cong L$. Set $M_0 := M^{\oplus(r+1)}$, and $M_1 := M^{\oplus(r+1)} \otimes L^{-1}$.

If $\text{Ext}^1(M, E) \neq 0$, then by Serre duality $\text{Hom}(E, M \otimes \omega_X) \neq 0$. $M \otimes \omega_X$ is a stable bundle of slope $\mu(M \otimes \omega_X) = \deg(M) + 2g - 2 < \mu(E)$. Thus, there can be only the zero morphism in $\text{Hom}(E, M \otimes \omega_X)$. So we have $\text{Ext}^1(M, E) = 0$.

By the same argument we conclude that $\text{Ext}^1(M, E(-P)) = 0$ for every point $P \in X(k)$. Therefore $\text{Hom}(M, E) \to \text{Hom}(M, E \otimes k(P))$ is a surjection. We conclude that $\text{Hom}(M, E) \otimes M \to E$ is surjective. Since $X$ is of dimension one, for a general subspace $V \subset \text{Hom}(M, E)$ of dimension $r + 1$ the restriction $V \otimes M \to E$ is surjective. We obtain a surjection $M_0 \xrightarrow{\pi} E$. The kernel of $\pi$ is a line bundle and its determinant
is \( \det(\ker(\pi)) \cong M_0 \otimes \det(E)^{-1} = M^{\otimes (r+1)} \otimes L^{-1} = M_1 \). Taking one isomorphism \( M_1 \xrightarrow{\sim} \ker(\pi) \) we obtain for any \( E \) as before the existence of a short exact sequence
\[
0 \rightarrow M_1 \xrightarrow{\iota} M_0 \xrightarrow{\pi} E \rightarrow 0.
\]
We use this to parameterize all bundles in \( SU_X(r, L) \) as cokernels of morphisms \( M_1 \rightarrow M_0 \).

To do so, we define \( V := \text{Hom}(M_1, M_0)^\vee \), consider \( \mathbb{P}(V) \xleftarrow{q} \mathbb{P}(V) \times X \xrightarrow{\alpha} X \), and the natural morphism \( p^*\mathcal{O}_{\mathbb{P}(V)}(-1) \otimes q^*M_1 \xrightarrow{\alpha} q^*M_0 \). We denote the cokernel of \( \alpha \) by \( \mathcal{E} \).

Let \( U_1 \) be the open subset of points \( u \in \mathbb{P}(V) \) such that \( \mathcal{E}_u := q_*(\mathcal{E} \otimes p^*k(u)) \) is a semistable bundle on \( X \). We see that the resulting morphism gives a surjection \( U_1 \xrightarrow{p} SU_X(r, L) \).

Next we show that \( \rho \) is infinitesimal surjective. We take a point \([i] \in \mathbb{P}(V)\) corresponding to a short exact sequence \([1]\). Let \( D = k[\varepsilon]/\varepsilon^2 \) be the ring of dual numbers. To give an infinitesimal deformation of \( E \) corresponds to give a flat family \( E_D \) on \( X_D = \text{Spec}(D) \times X \) which specializes to \( E \) when restricting to the reduced fiber \( X_0 \cong X \). Since we want to consider deformations with fixed determinant we have an isomorphism \( \det(E_D) \cong q_D^*L \) where \( q_D \) is the projection \( X_D \rightarrow X \). The flat deformation \( E_D \) yields a short exact sequence \( 0 \rightarrow E \rightarrow E_D \rightarrow E \rightarrow 0 \) on \( \text{Spec}(D) \times_k X \) which gives the exact sequence
\[
\text{Hom}_{X_D}(q_D^*M_0, E_D) \xrightarrow{\pi} \text{Hom}_{X_D}(q_D^*M_0, E) \xrightarrow{\text{Ext}^1_{X_D}(q_D^*M_0, E)} \text{Ext}^1_X(M_0, E)
\]

Since \( M_0 = M^{\otimes (r+1)} \) we conclude from \( \text{Ext}^1(M, E) = 0 \) that \( \text{Ext}^1(M_0, E) = 0 \). So \( \pi \) is the restriction of some \( \pi_D \in \text{Hom}_{X_D}(q_D^*M_0, E_D) \) to the reduced fiber. The morphism \( \pi_D : q_D^*M_0 \rightarrow E_D \) is surjective. Again \( \ker(\pi_D) \) is isomorphic to the line bundle \( \det(q_D^*M_0) \otimes \det(E_D)^{-1} = q_D^*(\det(M_0) \otimes L^{-1}) = q_D^*M_1 \). Fixing such an isomorphism we obtain a short exact sequence of \( \mathcal{O}_{X_D} \) bundles
\[
0 \rightarrow q_D^*M_1 \xrightarrow{\iota_D} q_D^*M_0 \xrightarrow{\pi_D} E_D \rightarrow 0.
\]

We conclude that any deformation \( E_D \) of \( E \) is induced by a deformation \( \iota_D \) of \( \iota \). Now for a general linear subspace \( L \subset \mathbb{P}(V) \) of dimension \( \dim SU_X(r, L) = (r^2-1)(g-1) \) passing through a stable \([E] \in \mathbb{P}(V)\) the composition of tangent maps is an isomorphism to \( T_{SU_X(r, L), E} \). Thus, on some Zariski open subset \( U \subset (L \cap U_1) \) containing \([E]\) the morphism \( U \rightarrow SU_X(r, L) \) is étale.

\[ \square \]

References

[1] C. S. Seshadri, Fibrés vectoriels sur les courbes algébriques, Astérisque 96, Société Mathématique de France, Paris, 1982.

[2] K. Smith with an appendix by J. Rosenberg, Rational and Non-Rational Algebraic Varieties: Lectures of János Kollár. \texttt{alg-geom/9707013}

Universität Duisburg-Essen, Fachbereich Mathematik, 45117 Essen, Germany

E-mail address: georg.hein@uni-due.de