Navigation in non-uniform density social networks

Yanqing Hu, Yong Li, Zengru Di, Ying Fan*
Department of Systems Science, School of Management and Center for Complexity Research, Beijing Normal University, Beijing 100875, China
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Recent empirical investigations suggest a universal scaling law for the spatial structure of social networks. It is found that the probability density distribution of an individual to have a friend at distance \(d\) scales as \(P(d) \propto d^{-1}\). Since population density is non-uniform in real social networks, a scale invariant friendship network (SIFN) based on the above empirical law is introduced to capture this phenomenon. We prove the time complexity of navigation in 2-dimensional SIFN is at most \(O(\log^4 n)\). In the real searching experiment, individuals often resort to extra information besides geography location. Thus, real-world searching process may be seen as a projection of navigation in a \(k\)-dimensional SIFN\((k > 2)\). Therefore, we also discuss the relationship between high and low dimensional SIFN. Particularly, we prove a 2-dimensional SIFN is the projection of a 3-dimensional SIFN. As a matter of fact, this result can also be generated to any \(k\)-dimensional SIFN.

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I. INTRODUCTION

To understand the structure of the social networks in which we live is a very interesting problem. As part of the recent surge of interest in networks, there have been active research about social networks [1-6]. Besides some well known common properties such as small-world and community structure [7-9], much attention has been dedicated to navigation in real social networks.

In the 1960s, Milgram and his co-workers conducted the first small-world experiment [10]. Randomly chosen individuals in the United States were asked to send a letter to a particular recipient using only friends or acquaintances. The results of the experiment reveal that the average number of intermediate steps in a successful chain is about six. Since then, “six degrees of separation” has become the subject of both experimental and theoretical research [11, 12]. Recently, Dodds et al carried out an experiment study in a global social network consisting about 60,000 email users [13]. They estimated that social navigation can reach their targets in a median of five to seven steps, which is similar to the results of Milgram’s experiment.

The first theoretical navigation model was proposed by Kleinberg [14, 15]. He introduced an \(n \times n\) lattice to model social networks. In addition to the links between nearest neighbors, each node \(u\) is connected to a random node \(v\) with a probability proportional to \(d(u, v)^{-r}\), where \(d(u, v)\) denotes the lattice distance between \(u\) and \(v\). Kleinberg has proved that the optimal navigation can be obtained when the power-law exponent \(r\) equals to \(d\), where \(d\) is the dimensionality of the lattice, and the time complexity of navigation in that case is at most \(O(\log^2 n)\). Since then, much attention has been dedicated to Kleinberg’s navigation model [14-18]. Roberson et al. studied the navigation problem in fractal networks, where they proved that \(r = d\) was also the optimal power-law exponent in the fractal case [19]. Carmi, Cartozo and their co-

operators have provided exact solutions respectively for the asymptotic behavior of Kleinberg’s navigation model [20-21]. More recently, the navigation problem with a total cost restriction has also been discussed, where the cost denotes the length of the long-range connections [22, 23].

Meanwhile, recent empirical investigations suggest a universal spatial scaling law on social networks. Liben-Nowell et al explored the role of geography alone in routing messages within the LiveJournal social network [24]. They found that the probability density function (PDF) of geographic distance \(d\) between friendship was about \(P(d) \propto d^{-1}\). Adamic and Ada also observed the \(P(d) \propto d^{-1}\) law when investigating the Hewlett-Packard Labs email network [25]. Lambiotte et al analyzed the statistical properties of a communication network constructed from the records of a mobile phone company [26]. Their empirical results showed that the probability that two people \(u\) and \(v\) living at a geographic distance \(d(u, v)\) were connected by a link was proportional to \(d(u, v)^{-2}\). Because the number of nodes having distance \(d\) to any given node is proportional to \(d\) in 2-dimensional world, so the probability for an individual to have a friend at distance \(d\) should be \(P(d) \propto d \cdot d^{-2} = d^{-1}\). More recently, Goldenberg et al studied the effect of IT revolution on social interactions [27]. Through analyzing an extensive data set of the Facebook online social network, they pointed out that social communication decrease inversely with the distance \(d\) following the scaling law \(P(d) \propto d^{-1}\) as well.

Such as in the LiveJournal social network, population density is non-uniform in real social networks [24]. To deal with the navigation problem with non-uniform population density, a scale invariant friendship network (SIFN for short) model based on the above spatial scaling law \(P(d) \propto d^{-1}\) of social networks is proposed in this paper. We prove the time complexity of navigation in a 2-dimensional SIFN is at most \(O(\log^4 n)\), which indicates social networks is navigable. Dodds et al have pointed out that individuals often resort to extra information such as education and professional information besides geography location in the real searching experiment [13]. Considering this phenomenon, navigation process in real world may be seen as the projection of nav-
igation in a higher dimensional SIFN. Therefore, we further discuss the relationship between high and low dimensional SIFN. Particularly, we prove that a 2-dimensional SIFN can be seen as the projection of any k-dimensional SIFN (k > 2) through theoretical analysis.

II. NAVIGATION IN NON-UNIFORM DENSITY SOCIAL NETWORKS

To deal with the non-uniform population density in real social networks, we divide the whole population into small areas and give the following two assumptions. First, the population density is uniform in each small area. Second, the minimum population density among the areas is \( m \), while the maximum is \( M \). We set \( m > 0 \) to guarantee that a searching algorithm can always make some progress toward any target at every step of the chain.

Like Kleinberg’s network (KN for short) and Liben-Nowell’s rank-based friendship network (RFN for short), we employ an \( n \times n \) lattice to construct SIFN. Without loss of generality, we assume each node \( u \) has \( q \) directed long-range connections, where \( q \) is a constant [15]. To generate a long-range connection of node \( u \), we first randomly choose a distance \( d \) according to the observed scaling law \( P(d) \propto d^{-1} \) in social networks. Then randomly choose a node \( v \) from the node set, whose elements have the same lattice distance \( d \) to node \( u \), and create a directed long-range connection from \( u \) to \( v \). The lattice is assumed to be large enough that the long-range connections will not overlap.

For simplicity, we set \( q = 1 \). Let \( S \) denote the set of all nodes, then the probability that \( u \) chooses \( v \) as its long-range connection in SIFN can be given by eq. (1).

\[
Pr_{SIFN}(u,v) = \frac{1}{c(u,v)} \frac{d(u,v)^{-1}}{\sum_{w \in S} d(u,w)^{-1}} \quad (1)
\]

where \( c(u,v) = |\{x | d(u,x) = d(u,v), x \in S\}| \) and \( d(u,v) \) denotes the lattice distance between nodes \( u \) and \( v \). Likewise, the probability that \( u \) chooses \( v \) as its long-range connection in KN and RFN are given respectively by eq. (2) and eq. (3).

\[
Pr_{KN}(u,v,r) = \frac{d(u,v)^{-r}}{\sum_{w \in S} d(u,w)^{-r}} \quad (2)
\]

\[
Pr_{RFN}(u,v) = \frac{\text{rank}_k(v)^{-1}}{\sum_{w \in S} \text{rank}_k(w)^{-1}} \quad (3)
\]

where \( \text{rank}_k(v) = |\{w | d(u,w) < d(u,v), x \in S\}| \) denotes the number of nodes within distance \( d(u,v) \) to node \( u \) in RFN [15, 24]. Notice that, the number of nodes with a distance \( d(u,v) \) in a \( k \)-dimensional \((k > 1)\) lattice is proportional to \( d(u,v)^{k-1} \). Thus, a node \( u \) connects to node \( v \) with probability proportional to \( d(u,v)^{-\alpha} \) does not mean \( P(d) \propto d^{-\alpha} \) but \( P(d) \propto d^{-\alpha + k-1} \) instead. Therefore, \( Pr_{KN}(u,v,k) \), \( Pr_{SIFN}(u,v) \) and \( Pr_{RFN}(u,v) \) are exactly the same for any \( k \)-dimensional lattice based network when population density is uniform. However, SIFN always satisfies the empirical results \( P(d) \propto d^{-1} \) in social networks compared with KN and RFN. Further, \( Pr_{KN}(u,v,k), Pr_{SIFN}(u,v) \) and \( Pr_{RFN}(u,v) \) can be quite different when the population density is non-uniform.

Since our 2-dimensional SIFN captures the non-uniform population density property in the real social networks, we purposefully divide the navigation process into two stages for simplicity. First send messages inside a small area and then among the areas. To analyze the time complexity of navigation in a 2-dimensional SIFN, we first compare the following two searching strategies as shown in FIG.1. Strategy \( \mathcal{A} \), send the message directly to target \( t \) from the current message holder using Kleinberg’s greedy routing strategy. At each step, the message is sent to one of its neighbors who is most close to the target in the sense of lattice distance. Strategy \( \mathcal{B} \), the message is first sent to a given node \( j \) using Kleinberg’s greedy strategy and then to the target node \( t \) using the same strategy. Suppose we start from a source node \( s \), after one step, the message reaches nodes \( A_1 \) and \( B_1 \) respectively with strategies \( \mathcal{A} \) and \( \mathcal{B} \). Consider \( B_1 \) as the new source node, then we should get \( A_2 \) and \( B_2 \) respectively with strategies \( \mathcal{A} \) and \( \mathcal{B} \) in the next step.

\[
Pr_{t \rightarrow j \rightarrow t} \leq Pr_{t \rightarrow B_1 \rightarrow j \rightarrow t} \quad (4)
\]

FIG. 1: Two strategies of sending message in a 2-dimensional SIFN. Strategy \( \mathcal{A} \), send the message directly to target \( t \) from the current message holder using Kleinberg’s greedy routing strategy. At each step, the message is sent to one of its neighbors who is most close to the target in the sense of lattice distance. Strategy \( \mathcal{B} \), the message is first sent to a given node \( j \) using Kleinberg’s greedy strategy and then to the target node \( t \) using the same strategy. Suppose we start from a source node \( s \), after one step, the message reaches nodes \( A_1 \) and \( B_1 \) respectively with strategies \( \mathcal{A} \) and \( \mathcal{B} \). Consider \( B_1 \) as the new source node, then we should get \( A_2 \) and \( B_2 \) respectively with strategies \( \mathcal{A} \) and \( \mathcal{B} \) in the next step.
reach $A_2$ and $B_2$ with strategies $\mathcal{A}$ and $\mathcal{B}$ respectively in the next step. Following the same deduction, we have $T(B_1 \rightarrow A_2 \rightarrow t) \leq T(B_1 \rightarrow B_2 \rightarrow t)$. Repeat this process until the message reaches the given node $j$ with strategy $\mathcal{B}$, then we should have a monotone increasing sequence of expected delivery time $\{T(s \rightarrow B_1 \rightarrow t), T(s \rightarrow B_2 \rightarrow t), \ldots, T(s \rightarrow j \rightarrow t)\}$. Therefore, we can obtain $T(s \rightarrow t) \leq T(s \rightarrow j \rightarrow t)$, which means strategy $\mathcal{A}$ is better than strategy $\mathcal{B}$. This analysis can be extended to any $k$-dimensional SIFN.

Based on the first assumption and the fact that SIFN is identical to KN when population density is uniform, the expected steps spent in each small area using Kleinberg greedy algorithm is at most $O(\log^2 n)$. Consider each small area as a node, we will get a new 2-dimensional weighted lattice. The weight (population) of the nodes is between $m$ and $M$ based on the second assumption. Thus we have

$$
c \frac{m}{M} d^{-1} \leq Pr_{\text{SIFN}}(u, v) \leq c \frac{M}{m} d^{-1} \tag{4}
$$

where $c$ is a constant and $Pr_{\text{SIFN}}(u, v)$ represents the probability that area $u$ is connected to area $v$ in the new weighted lattice.

We say that the execution of greedy algorithm is in phase $j$ ($j > 0$) when the lattice distance from the current node to target $t$ is greater than $2^j$ and at most $2^{j+1}$. Obviously, we have

$$
\sum_{d=1}^{n} d^{-1} \leq 1 + \int_1^n x^{-1} dx = 1 + \log n < 2 \log n. \tag{5}
$$

Further, we define $B_j$ as the node set whose elements are within lattice distance $2^j + 2^{j+1} < 2^{j+2}$ to $t$. Let $|B_j|$ denote the number of nodes in set $B_j$, we should have

$$
|B_j| > 1 + \sum_{i=1}^{2^j} i > 2^{2j-1}. \tag{6}
$$

Suppose that the message holder is currently in phase $j$, then the probability that the node is connected by a long-range link to a node in phase $j-1$ is at least $(Mm^{-1}2 \log n \cdot 4 \cdot 2^{2j+4})^{-1}$. The probability $\psi(x)$ to reach the next phase $j-1$ in more than $x$ steps can be given by

$$
\psi(x) = (1 - (Mm^{-1}2 \log n \cdot 4 \cdot 2^{2j+4})^{-1})^x \tag{7}
$$

and the average number of steps required to reach phase $j-1$ is

$$
<x> = \sum_{i=1}^{\infty} \left(1 - \frac{m}{256M \log n}\right)^{i-1} = \frac{256M \log n}{m}. \tag{8}
$$

Since the initial value of $j$ is at most $\log n$, then the expected total number of steps required to reach the target is at most $O(\frac{M}{m} \log^2 n)$.

As a matter of fact, it means that we are using strategy $\mathcal{B}$ to send message in 2-dimensional SIFN when the navigation process is divided into the above 2 stages. Thus, the time complexity of navigation in SIFN with strategy $\mathcal{B}$ is at most $O(\frac{M}{m} \log^2 n)$. However, actual navigation process in real world should be carried out regardless of the above two assumptions, which indicates individuals should use strategy $\mathcal{A}$. Based on the above analysis, strategy $\mathcal{A}$ performs better than strategy $\mathcal{B}$ on average. Therefore, the time complexity of navigation in 2-dimensional SIFN is at most $O(\log^2 n)$ with non-uniform population density.

### III. RELATIONSHIP BETWEEN HIGH AND LOW DIMENSIONAL SIFN

The empirical results show individuals always resort to extra information such as profession and education information besides the target’s geography location when routing messages[13]. Then, real navigation process in social networks may be modeled with a higher dimensional SIFN. In the following, we will discuss the relationship between the high and low dimensional SIFN and prove that a 2-dimensional SIFN can be obtained by any $k$-dimensional SIFN ($k > 2$). Particularly, we will provide the theoretic analysis for the case where $k = 3$. The analysis can be generated to any $k$ dimensional cases.

We employ a random variable $D_3$ to denote the friendship distance in a 3-dimensional SIFN. For simplicity, a continuous expressions is used. Since, the long-range connections in 3-dimensional SIFN satisfies the above empirical law, the PDF of $D_3$ can be expressed by

$$
P(D_3 = d) = \frac{1}{\ln d_M - \ln d_m} \frac{1}{d}, d_m \leq d \leq d_M \tag{9}
$$

where $d_m$ and $d_M$ denote the minimum and maximum distance respectively in the 3-dimensional SIFN.

We can obtain a 2-dimensional network model if we project a 3-dimensional SIFN to a 2-dimensional world. Similarly, a random variable $D_2$ is used to denote the friendship distance in the new 2-dimensional network model. It is not difficult to understand that the condition for a 2-dimensional SIFN should be the PDF of $D_2$ satisfies $P(d) \propto d^{-1}$. Since $D_2$ is the projection of $D_3$, then $D_2$ can be seen as the product of $D_3$ and $X$. Here random variable $X$ is independent on $D_3$ and its PDF can be given by eq.(10).

$$
P(X = x) = \frac{1}{\lambda}, 0 \leq x \leq \lambda \tag{10}
$$

where $0 \leq \lambda < 1$. Finally, the PDF of $D_2$ can be written as

$$
P(D_2 = d) = \begin{cases} 0, & d \leq 0 \\ \frac{d - d_m}{d_M - d_m} \ln d_M - \ln d_m, & d_m \leq d \leq d_M \\ \frac{d - d_m}{d_M - d_m} \ln d_M - \ln d_m, & d > d_M \lambda \end{cases} \tag{11}
$$

When taking account of real social networks, $d_M$ is large enough that the term $\frac{d}{d_M}$ will approach its limit of 0. Meanwhile, the term $d_m \lambda$ can be neglected when compared with
$d_M \lambda$, because $\lambda \leq 1$ and $d_m$ is relatively small. Thus the PDF of $D_2$ can be simplified into $P(d) \propto d^{-1}$, which is identical to that of $D_3$ in a 3-dimensional SIFN.

Through theoretical analysis, we have proved a 2-dimensional SIFN can be seen as the projection of a 3-dimensional SIFN. Likewise, we can get a 2-dimensional SIFN from any $k$-dimensional ($k > 2$) SIFN. Notice that individuals are always restricted on the 2-dimensional geography world even they possess extra information from other dimensions. Thus, real-world searching process may be seen as the projection of navigation in a high dimensional SIFN. Our analysis indicate that SIFN model may explain the navigability of real social networks even take account of the fact that individuals always resort to extra information in real searching experiments.

IV. CONCLUSION

Recent investigations suggest that the probability distribution of having a friend at distance $d$ scales as $P(d) \propto d^{-1}$. We propose an SIFN model based on this spatial property to deal with navigation problem with non-uniform population density. It has been proved that the time complexity of navigation in 2-dimensional SIFN is at most $O(\log^3 n)$, which corresponds to the upper bond of navigation in real social networks. Given the fact that individuals are always restricted on the 2-dimensional geography world even they possess information of the higher dimensions, actual searching process can be seen as a projection of navigation in a higher $k$-dimensional SIFN. Through theoretical analysis, we prove that the projection of a higher $k$-dimensional SIFN results in a 2-dimensional SIFN. Therefore, SIFN model may explain the navigability of real social networks even take account of the information from higher dimensions other than geography dimensions.

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