Local Classical Strategies vs Geometrical Quantum Constraints

R. Restrepo-Villegas,1 E. Agudelo,2 J. Castrillón,1 and B. A. Rodríguez1,†

1Grupo de Física Atómica y Molecular (GFAM), Instituto de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Antioquia UdeA, Calle 70 No. 52-21, Medellín, Colombia. 2Arbeitsgruppe Theoretische Quantenoptik, Institut für Physik, Universität Rostock, D-18051 Rostock, Germany.

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We use an alternative approach to show that quantum entanglement-like correlations cannot be reproduced for any classical protocol. In our proposal, quantum geometric restrictions are impose over the physical system related to the existence of entanglement and we demonstrate that there is no classical local strategy that can reproduce them completely. Typically, the implementations of Bell inequalities have as a starting point the expectation of classical behavior and as conclusion the violation due to the quantum character of the system. We go the other way around. For this purpose, we build a computational simulation based on the scheme of non-communicating students. In this scheme, the students cannot manipulate the quantum systems but they may set up in advance a common strategy and share some common classical data in order to try to reproduce the given quantum correlations of such systems. By thoroughly searching in the whole space of classical strategies we conclude that local operations and classical communications does not satisfy the geometrical constraints imposed by quantum entanglement.

Keywords: Quantum Entanglement, Bell inequalities, LOCC, Non-communicating students.

I. INTRODUCTION

Quantum entanglement is, without a doubt, one of the most interesting phenomena of quantum physics. It describes the event in which two (or more) systems can be intimately connected to each other, no matter how far they are. Moreover, there is no classical equivalent to quantum entanglement, that is, it is a purely quantum phenomenon. Entanglement was a term originally conceived by E. Schrödinger [1] to name non-local correlations between physical systems with quantum properties and used audaciously by A. Einstein, B. Podolsky and N. Rosen (EPR) in their 1935 famous paper, in an attempt to demonstrate that Quantum Mechanics (QM) was an incomplete theory [2]. For EPR it was inadmissible that the properties of remote systems can be intrinsically linked as predicted by the QM. Supposing that the properties of the composing parts of a physical system define the system a priori in its totality is a premise that can be classified as self-evident, at least classically speaking. During the early years of the first quantum revolution, this kind of discussions were relegated to old men and philosophers because apparently such debate had nothing to do with physics and it was merely a matter of interpretation. It was J. S. Bell who in 1964 proposed a set of inequalities that could be realized experimentally with present technology [3]. Bell’s ideas changed our classic and naive notion of reality forever. As noted by N. D. Mermin: “The point is no longer that quantum mechanics is an extraordinarily peculiar theory; but that the world is an extraordinarily peculiar place.” [4]. Soon afterwards, experiments [5][10] showed that entanglement and therefore QM, fully describe objective physical reality.

There are different ways to propose the Bell’s theorem, or his inequalities [11][10]. Generally, one assumes the suppositions of local realism and show that any prediction based on these assumptions disagrees with the predictions made by the QM, thus representing a violation of Bell’s inequalities and local realism by quantum theory. We present here an alternative approach to show that quantum entanglement-like correlations cannot be reproduced for any classical mean. We start from quantum predictions, more specifically, from geometric constraints imposed by entanglement, and we show that there is no possible classical scenario capable of satisfying such predictions completely. It is our aim to argue that protocols involving only local operations and classical communications (LOCC) cannot replicate the behavior of a quantum entangled system. We discuss our findings in particular in the case of pairs of entangled photons in polarization. The method we propose, consists on a exhaustive scan over a space of parameters that characterizes the different classic strategies that can be considered in order to simulate the probabilities of coincidences obtained in a quantum experiment with photon pairs prepared in a maximally entangled state. We show how none of the conceivable strategies can produce solutions out of the region defined by the classical conditions.

The paper is organized as follows. We start with the description of the problem and the experimental protocols to be compared in Sec. [11] the quantum and the classical one. In Sec. [11] we discuss the different initial conditions treated together with its probabilities of coincidence. We compare the predictions of quantum
theory with the classical results obtained by the non-
communicating students protocol, in the form of a nu-
merical simulation that investigates all the possible strategies
they could agree on. Finally, in Sec. [IV] we discuss our
findings and present the conclusions of this work.

II. QUANTUM EXPERIMENT VS CLASSICAL
SIMULATION

In order to address the question whether a probabil-
ity distribution obtained through the performance of an
experiment with quantically correlated particles can be
classically simulated, we consider the following two sce-
narios. Firstly, we deal with a quantum experiment based
on polarization measurements of entangled photon pairs,
and secondly, as the classical situation, we consider two
uncommunicated students with unlimited classical power
and a source of random numbers who will try to simulate
the given experimental outcome.

A. Quantum setup: Bell’s experiment with photons

Our quantum experiment is based on the polariza-
tion measurements of entangled photon pairs. Which is
a standard technique in quantum optics [17–22]. Typ-
ically, in this scenario a non-linear crystal is pumped
with a laser. Through Spontaneous Parametric Down-
Conversion (SPDC) a single input photon is split into
two output photons, called the signal and the idler. Af-
ter the pair of photons emerge simultaneously from the
common source, the crystal, they are spatially separated
and can be put to travel in opposite ways (space-like sep-
ated). Consequently, two distant observers can make
measurements of these individual systems. The two in-
dependent observers, Alice and Bob, look for correlations
between the polarization of each photon in a pair emit-
ted simultaneously. Such scheme, it is the so-called Bell’s
experiment with pairs of entangled photons [23].

Polarization measurements— Let us consider the Bell
state $|\phi^+\rangle = \frac{1}{\sqrt{2}} (|HH\rangle + |VV\rangle)$ (maximally entangled)
and a type I SPDC. Here, the polarization of the two
photons is the same, and orthogonal to the polarization
of the pump photon. It is our aim to describe the corre-
lations between the pair of photons by the measurements
of their individual polarization. For this purpose, both
Alice and Bob have a polarization analyzer (PA), and in
each run of the experiment they can select in their own
PA in some angles $\theta_A$ and $\theta_B$, respectively, and finally
observe the outcomes of each PA. There are just two
kind of possible outcomes: the photon pass ($P$) through
the PA, or well the photon is absorbed ($A$) by the PA, cf.
Fig. 1. Therefore, for many runs of the experiment,
the outcomes ($A$) and ($P$) may vary, even if the same
measurements are made. We labeled $R_A$ and $R_B$ the
outcomes of Alice and Bob measurements, respectively.
In this scenario, a possible experiment could be to set:

$\theta_A = 0^\circ$ and $\theta_B = 30^\circ$; with resulting outcomes ($A$) and
($P$), respectively. Without loss of generality and with
the aim of simplifying the experiment and the calcula-
tions, we will restrict the measurements to only three
angles. Such angles are $0^\circ$, $30^\circ$, and $60^\circ$, i.e., on each run
of the experiment, Alice and Bob will randomly pick one
of these three values, set their own PA accordingly, and
record the outcome together with the chosen angle.

In case that the photons have been prepared in the
particular state $|\phi^+\rangle$, QM predicts that the joint proba-
bility of the measurement coincidences on both sides is
determined by the square cosine of the phase shift be-
tween the angles of both PAs, ($\Delta \theta = |\theta_A - \theta_B|$). Now,
for the three angles that we chose, the phase shift could
only be $\Delta \theta = \{0^\circ, 30^\circ, 60^\circ\}$. Therefore, we have nine
possible configurations for the PA pair: $0^\circ - 0^\circ$, $0^\circ - 30^\circ$,
$0^\circ - 60^\circ$, $30^\circ - 0^\circ$, $30^\circ - 30^\circ$, $30^\circ - 60^\circ$, $60^\circ - 0^\circ$, $60^\circ - 30^\circ$, and
$60^\circ - 60^\circ$.

Quantum facts— Based on the three possible mis-
matches we can state three characteristics or quantum
facts that Alice and Bob observe about the behavior of
their photons. Under the initial condition, $|\phi^+\rangle$, these
three facts are:

$F_1$: When the two PAs are oriented at the same angle
($\Delta \theta = 0^\circ$), the joint probability of coincidence is
always equal to unity.

$F_2$: When the offset at the angle of the PAs is equal to
$30^\circ$, the joint probability of coincidence is equal to
$3/4$.

$F_3$: When the offset at the angle of the PAs is equal to
$60^\circ$, the joint probability of coincidence is equal to
$1/4$.
Could this behavior be classically simulated? Let us introduce the scenario and the strategies considered to solve this question.

B. Classical setup: Uncommunicated students

The experimental context for the classical situation is known in the literature as the problem of uncommunicated students [20]. Basically, this gedanken experiment is summarized in the job that has been assigned to two students who are uncommunicated (spatially separated), to reproduce the behavior of the pair of photons described before. Consider then two students, one of them will play the role of the photons that reach Alice, and the other will play the role of the photons that reach Bob. The rules are simple: both begin in the same room (the common source), after deciding how to reach their objective (common strategy) they must exit through different doors being henceforth uncommunicated, see Fig. 2.

Once outside the room, a question will be asked to each of them.

Alice and Bob have no idea of what question they will be asked every run of the experiment and once they leave the initial room they cannot communicate neither their inquiry nor her/his answer, to her/his partner. Nevertheless, while they are in the common room just before the run, they are allowed to build any common strategy in terms of coordinating their responses. Their aim is to guarantee that after many repetitions of the experiment their answers show exactly the same kind of correlations that the pair of photons that reach Alice and Bob in the quantum experiment. Furthermore, they are allowed to adopt a new strategy every time the experiment is repeated, so that, in the long run when the questions asked differs in $\Delta \theta$ the answers will match $\cos^2(\Delta \theta)$ of the times. In other words, the students should reproduce the quantum facts, i.e, when they are asked for the same angle ($\Delta \theta = 0^\circ$), they always must give the same answer; when the questions differ by $30^\circ$, their answers need to agree $3/4$ of the time; and when the answers differ by $60^\circ$, their answers must match $1/4$ of the time [25].

Strategies for coincidences—At this point, we can make two important questions: which is the best strategy for trying to reproduce the behavior of the photons? and how to know which option should they choose on each possible situation, even before both leaves the room? In order to respond these questions, let us study all the possible combinations of the students answers—considering that they choose the same answer when asked for the same question, to fulfill $F_1 = 1$—by grouping such answers on 4 couples of equivalent strategies about each question.

We will represent strategies with the following notation: $[R_{A,B}(0^\circ), R_{A,B}(30^\circ), R_{A,B}(60^\circ)]$, where the first item represent the answer, of both students, to question “$0^\circ$?”, the second to question “$30^\circ$?”, and the third to question “$60^\circ$?”. Therefore, the eight possible strategies are:
The “probability of playing” and the “probability of reproducing the behavior of the pairs” will be the “probability of playing” and the “probability of reproducing the behavior of the pairs” will be the “probability of playing” and the “probability of reproducing the behavior of the pairs”. Since, by summing the “probability of playing” and the “probability of reproducing the behavior of the pairs”, we must look for how many times the probabilities will disagree if only the question “0” or “30” is made to one of them, and they will agree in any other case, and so on. This procedure ensures the perfect and strict correlation about which questions will be asked, i.e., each machinedecision of which strategy to adopt without information about which questions “0” or “30” receive answers that disagree, then we must look for how many times the probabilities will be played.

Reasoning similarly, playing the probabilities give us the proportion of the experiments in which questions “0” or “30” will disagree, and playing the probabilities give us the proportion of the experiments in which questions “0” or “30” will disagree. Consequently, in order to reproduce the behavior of the photons the students must consider the probabilities such that

\[ \alpha + \beta + \gamma + \delta = 1. \]

We must remember that the students will make their decision of which strategy to adopt without information about which questions will be asked, i.e., each machine generates random questions to both of them, and such process is independent of the election of the strategy that students had made. So in the long run, the results of this experiment will depend only on the values of \( \alpha, \beta, \gamma, \) and \( \delta \). E.g., if we want to know how often the pair of questions “0” or “30”, and “30” or “60” will receive answers that disagree, then we must look for how many times the probabilities \( \beta + \gamma \) were played.

These simple statements apparently solve the problem of the students of reproducing the behavior of the pairs of entangled photons but here is when problems become evident. Since, by summing \( \beta + \gamma \) and \( \gamma + \delta \) we obtain

\[ (\beta + \gamma) + (\gamma + \delta) = 1/4 + 1/4 = 1/2, \]
\[ 3/4 + 2\gamma = 1/2, \]

where we have taken into account \( \beta + \gamma \). It results in \( \gamma = -1/8 \).

However, by definition \( \gamma \) must be a non-negative number. In consequence, the conclusion is that there are not possible scenarios in which the students could choose the strategies (5) and (6) or 12.5 percent of the time. All this means that it does not exist a possible long-term selection of the set of strategies that students can adopt to ensure that their answers will reproduce the same correlations that photons have when prepared originally in the maximally entangled state \( |\phi^+\rangle \).

### III. CLASSICAL STRATEGIES FOR QUANTUM COINCIDENCES

Our goal is to introduce a strategy to determine if there is a way to simulate classically a set of quantum probabilities related to given outputs of a realistic experiment. Therefore, our approach is not limited to a given initial condition although we discuss the particular case where the entangled parts are initially prepared in the quantum state \( |\phi^+\rangle \). In order to stress the applicability of the simulation we would like to review the results considering a set of states for the photon pairs: the four Bells states and the two mixtures,

\[
\hat{\rho}_{\text{Max}} = \frac{1}{4} \left( |HH\rangle \langle HH| + |HV\rangle \langle HV| + |VH\rangle \langle VH| + |VV\rangle \langle VV| \right),
\]

and

\[
\hat{\rho} = \frac{1}{2} \left( |HH\rangle \langle HH| + |VV\rangle \langle VV| \right).
\]

We want to discuss the possibility for the students to claim (wrongly) a successful simulation of quantum states of different nature—in the sense of the corresponding quantum facts, c.f. Table I.

| State         | \( F_1 \) | \( F_2 \) | \( F_3 \) |
|---------------|----------|----------|----------|
| \( |\phi^+\rangle \) | 1        | 3/4      | 1/4      |
| \( |\phi^-\rangle \) | 1/2      | 3/8      | 1/4      |
| \( |\psi^+\rangle \) | 1/2      | 5/8      | 3/4      |
| \( |\psi^-\rangle \) | 0        | 1        | 3/4      |
| \( \hat{\rho}_{\text{Max}} \) | 1/2      | 1/2      | 1/2      |
| \( \hat{\rho} \) | 3/4      | 9/16     | 1/4      |

Table I. Values of \( \{F_1, F_2, F_3\} \) for all the states.

In order to calculate the probabilities that in each case represent the facts we need to know the joint probabilities of coincidence of each state, c.f. Table I (in Appendix A) we present the necessary steps to obtain the results of Table I.

Once we know the expressions for the probability of coincidence, we can calculate the value of the probabilities for each of the quantum facts. In Table I we present a recap of the values of \( \{F_1, F_2, F_3\} \) for all the states. The previous analysis of probabilities that lead
the simulation conditioned by $\Delta p$. So, this number is the first important factor in the amount of different experiments that the simulation will perform. Therefore, each point of space $(\alpha, \beta, \gamma)$ is mapped into a $(\alpha, \beta, \gamma)$ plane. In Fig. 4, we show the space of parameters $(\alpha, \beta, \gamma)$, restricted to the planes $\alpha = 0$ (blue), $\beta = 0$ (red), and $\gamma = 0$ (green). Each point is mapped into the facts space $(F_2, F_3)$.

The parameters space is given by a regular tetrahedron, this is a geometrical imposition over the space of classical probabilities. To each point of the space $(\alpha, \beta, \gamma)$ corresponds a set of facts $(F_1, F_2, F_3)$. On account of the design of the protocol the fact 1 is always satisfied, which implies that $F_1 = 1$ independently of the set of chosen values $(\alpha, \beta, \gamma)$. Therefore, each point of space $(\alpha, \beta, \gamma)$ is mapped into a $(F_2, F_3)$ plane. In Fig. 4, we show the

### Table II. Probabilities of Coincidence for the four Bell’s states and the two statistical mixtures $\hat{\rho}_{\text{Max}}$ and $\hat{\rho}$

| State | Probability of coincidences |
|-------|----------------------------|
| $|\phi^+\rangle$ | $\cos^2(\theta_s + \theta_t)$ |
| $|\psi^+\rangle$ | $\sin^2(\theta_s + \theta_t)$ |
| $\hat{\rho}_{\text{Max}}$ | $\frac{1}{2} \left[ \cos(2\theta_s) \cos(2\theta_t) + 1 \right]$ |
| $\hat{\rho}$ | $\frac{1}{2} \left[ \cos(2\theta_s) \cos(2\theta_t) + 1 \right]$ |

Furthermore, for each selection of $\alpha$, $\beta$, $\gamma$ and $\delta$, the simulation will run the experiment a great number of times in order to verify if that selection satisfy fact 2 and fact 3 simultaneously. Then, $\alpha$ times of the experiments the simulation will choose randomly between strategy (1) and strategy (2), $\beta$ times the simulation will choose randomly between strategy (3) and strategy (4), $\delta$ between strategy (5) and strategy (6), and $\delta$ between strategy (7) and strategy (8). E.g, the selection $[1,0,0,0]$ means that the simulation choose randomly between strategies (1) and (2) 100 percent of the time, or equivalently, 50% of the time simulation will choose strategy (1), and the other 50% of the time strategy (2). Note that we cover the whole space of parameters with a distance $\Delta p$ in all directions between point and point. Therefore, reaching an accuracy of $\Delta p$.

### B. Uncommunicated students results

We present the results of simulation for $F_2$ and $F_3$ together with the probabilities of coincidence for all the Bells states, $|\phi^+\rangle$ and $|\psi^+\rangle$, and the mixtures, $\hat{\rho}_{\text{Max}}$ and $\hat{\rho}$ considering the space of parameters $(\alpha, \beta, \gamma)$ for a fixed $\Delta p$. Note that we only need three of the parameters since $\delta = 1 - (\alpha + \beta + \gamma)$.

![Figure 4. Parameters space $(\alpha, \beta, \gamma)$, restricted to the planes $\alpha = 0$ (blue), $\beta = 0$ (red), and $\gamma = 0$ (green). Each point is mapped into the facts space $(F_2, F_3)$](image)

The simulation was based on the mismatches, i.e., $F_2$ and $F_3$. It was obtained also imposing the strict condition of fulfilling always $F_1 = 1$ which is set by our initial condition, $|\phi^+\rangle$. When this condition is established and the analysis is done taking into account just the disagreements, what we obtain is $\gamma = -1/8, 1/2, 1/4, -1/8, 1/4, 1/16$ for the states $|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle, \hat{\rho}_{\text{Max}},$ and $\hat{\rho}$ respectively. Notice that the only states that have a negative $\gamma$ are $|\phi^+\rangle$ and $|\psi^+\rangle$. These have $F_1$ equals 1 and 0, respectively. From the point of view of the correlations they are equivalent because having total certainty or total ignorance relates exactly to the same correlation. Therefore, they are of the same nature in the sense of quantum facts.

We show with our simulation that in the subspace of disagreements, $F_2$ vs $F_3$, is impossible for the students to obtain the probabilities that characterize the states $|\phi^+\rangle$ and $|\psi^-\rangle$ but in the case of $|\phi^-\rangle, |\psi^+\rangle, \hat{\rho}_{\text{Max}},$ and $\hat{\rho}$ they could reproduce their facts and erroneously conclude that they indeed simulate a quantum state.

### A. Description of simulation

We propose a numerical simulation that runs through the entire parameter space $\alpha, \beta, \gamma$ and $\delta$, by discrete variations of a value $\Delta p$ of each parameter. We guarantee that the condition $\alpha + \beta + \gamma + \delta = 1$ is always satisfied, that is, certifying that we cover all the possible strategies that students can choose. E.g, if $\Delta p = 0.1$ and we start at configuration $[1,0,0,0,0,0,0,0]$, the next set of strategies will be $[0,9,0,1,0,0,0,0]$, and the next $[0,9,0,0,0,1,0,0]$, and so on until the last term to be evaluated is $[0,0,0,0,0,1,0]$. Varying in this way the parameters, they will form a set which has a size defined by the tetrahedral number

$$T_n = \binom{n + 2}{3},$$

where $n = (1/\Delta p) + 1$, and $\Delta p = 1/p$ with $p \in \mathbb{N}$. Consider again the case of $\Delta p = 0.1$, here we have $n = 11$ and $T_n = 286$, which is the size of the grid of the entire set of parameters. Evidently, as smaller is $\Delta p$, bigger is $T_n$, and bigger is the size of the set of probabilities that define the amount of different experiments that the simulation will perform. So, this number is the first important factor in the simulation conditioned by $\Delta p$. 

$$\begin{array}{c|c}
\text{State} & \text{Probability of coincidences} \\
|\phi^+\rangle & \cos^2(\theta_s + \theta_t) \\
|\psi^+\rangle & \sin^2(\theta_s + \theta_t) \\
\hat{\rho}_{\text{Max}} & \frac{1}{2} \left[ \cos(2\theta_s) \cos(2\theta_t) + 1 \right] \\
\hat{\rho} & \frac{1}{2} \left[ \cos(2\theta_s) \cos(2\theta_t) + 1 \right] \\
\end{array}$$
planes in the parameters space corresponding to configuration \( \alpha = 0, \beta = 0, \) and \( \gamma = 0. \) Note that, the planes \( \alpha = 0 \) and \( \gamma = 0 \) are mapped into the triangular frontier of classical facts space, c.f. Fig. 5. Furthermore, we show the point “Q” which corresponds to the quantum prediction for the probability of coincidence of the entangled state \(|\phi^+\rangle\).

In Fig. 5 we show the result of a sweep over the whole space of classic parameters with a fixed distance between each point, that is, \( \Delta p = 0.04. \) Again, for each point in space \((\alpha, \beta, \gamma)\) we obtain one point in facts space \((F_2, F_3)\), see Fig. 6. It is important to note that the classical space have a geometrical restriction that is violated by the probability of coincidence of state \(|\phi^+\rangle\). Such geometrical restriction can be established from the two straight lines that confine the classical region, that is,

\[
F_2 = \frac{1}{2}(F_3 + 1), \quad (4a)
\]

\[
F_2 = \frac{1}{2}(F_3 - 1). \quad (4b)
\]

These last equations define the maximal and minimal value of \( F_2 \) in the classical region, respectively. Mathematically it is, \(-\frac{1}{2}(F_3 - 1) \leq F_2 \leq \frac{1}{2}(F_3 + 1)\) or equivalently \(-F_3 \leq 2F_2 - 1 \leq F_3\). From where we can obtain the inequality imposed by the classical restrictions, given by

\[
|2F_2 - 1| \leq F_3. \quad (5)
\]

According to our numerical simulation, any strategy, or any choice, that uncommunicated students can make it is included in such classical region. Clearly, the point "Q", corresponding to \(|\phi^+\rangle\) is outside of such region, showing thus, that there is no classical strategy capable of reproducing the behavior the pairs of entangled photons.

In Fig. 7 we also show the quantum predictions for the other states. The red points represent all the states the students can simulate with their classical strategies within the whole volume of the tetrahedron. All possible results of a classical protocol. The black points describe the given quantum states. The states \(|\phi^-\rangle, |\psi^-\rangle, \hat{\rho}_{\text{Max}}, \) and \(\hat{\rho}\) appear to be contained in the classical region, but, remember that such states do not satisfy the condition \(F_1 = 1\) imposed by the initial state \(|\phi^+\rangle\). As expected \(|\phi^+\rangle\) is not the only state that has its point out of the classical region. The equivalency between \(F_1 = 1\) and \(F_1 = 0, \) having total certainly or total ignorance, makes \(|\psi^-\rangle\) also impossible to simulate with the set of strategies used by the students that could reproduce its behavior in the \((F_2 - F_3)\) plane. The results show that the students
may believe they simulated quantum states, even entangled, but that is a mere consequence of not considering all the quantum facts. The protocol can be modified in order to satisfy each value of $F_1$ for each quantum state. States with equivalent $F_1$ will be out of the classical zone and the rest could be wrongly reported to be successfully simulated via classical strategies due to the lack of information.

### IV. DISCUSSION AND CONCLUSIONS

Quantum effects cannot be classically simulated. That said, we conclude that an entangled state have some partial correlations that can be always reproduced classically. In this case, correlations associated to fact 2 and fact 3 can be reproduced, but for the proper characterization of the entangled state more correlations are needed, given by fact 1. Therefore, when all the geometrical conditions imposed by quantum correlations are taken into account is not possible to reproduce solely with classical strategies the corresponding probabilities that characterize any quantum state. Additionally, for any entangled state it is possible to find other different directions of polarization that produce facts that can not be emulated by non-communicating students. In this sense, the key property is the entanglement and the geometric constraint imposed by Bell’s theorem on classical results.

We have carried out an exhaustive search of all the possible classical strategies that try to reproduce the correlations of a pair of entangled photons, and we have found that non of these strategies can reproduce the quantum predictions. The procedures performed by non-communicating students are a proper implementation of LOCC, i.e., students can only respond to each experiment in their own local environment, outside the common room which is the only space where they can communicate and choose an strategy.

The conclusion is that no classical procedure LOCC is capable to reproduce the quantum entangled states.

On the other hand, the geometric constraint that we have found on the classical facts space must naturally be related to some kind of Bell inequality, a relation that we are still studying.

### Appendix A: Quantum Description of Probability of Coincidences

In order to describe the probability of coincidence for a pair of entangled photons in polarization we describe the physical system in the polarization basis, $\{H, V\}$. Here we calculate the joint probability of detecting a signal photon and idler photon to have elliptical polarization states.

### 1. Probabilities and Operators

First, we are going to define the general state of the $\{H, V\}$ basis which have the form:

$$|\psi\rangle = c_1 |H\rangle + c_2 |V\rangle,$$

where $c_1, c_2 \in \mathbb{C}$, must satisfy $|c_1|^2 + |c_2|^2 = 1$. This means we can write $c_1$ and $c_2$ like this:

$$c_1 = |c_1| e^{i\theta_1},$$

$$c_2 = |c_2| e^{i\theta_2},$$

with $|c_1| = \cos(\theta)$, and $|c_2| = \sin(\theta)$. Replacing these in Eq. (A1) we obtain

$$|\psi\rangle = \cos(\theta) e^{i\theta_1} |H\rangle + \sin(\theta) e^{i\theta_2} |V\rangle = \cos(\theta) |H\rangle + \sin(\theta) e^{i\varphi} |V\rangle,$$

where $\varphi = \theta_2 - \theta_1$, and the global phase $e^{i\theta_2}$ is depe- cated. Hence, we can write the general states for elliptical polarization and their respective orthogonal states as:

$$|e_{s,i}\rangle = \cos(\theta_{s,i}) |H\rangle + \sin(\theta_{s,i}) e^{i\varphi_{s,i}} |V\rangle,$$

$$|e_{s,i}^\perp\rangle = \sin(\theta_{s,i}) |H\rangle - \cos(\theta_{s,i}) e^{-i\varphi_{s,i}} |V\rangle.$$

Here, the sub-indexes $s$ and $i$ means signal photon and idler photon, respectively, and the super index $\perp$ means orthogonal. With the first two states in the Eq. (A4a) we can write the projection operator. And equivalently, with the last two states in Eq. (A4b) we can write the projection operator for the orthogonal states as

$$\hat{P}(e_s, e_i) = |e_s, e_i\rangle \langle e_s, e_i|,$$

$$\hat{P}(e_s^\perp, e_i^\perp) = |e_s^\perp, e_i^\perp\rangle \langle e_s^\perp, e_i^\perp|.$$

Note that, we can write these two operators in terms of the $\{H, V\}$ basis, but such work is excessively long and inefficient resulting in sixteen terms. These operators in Eqs. (A5) allow us to calculate the joint probability that a given state has elliptical polarization, or orthogonal to it, as the case may be. Such joint probability is defined as

$$P_{|\psi\rangle}(e_s, e_i) = |\langle \psi| e_s, e_i\rangle|^2,$$

$$P_{|\psi\rangle}(e_s^\perp, e_i^\perp) = |\langle \psi| e_s^\perp, e_i^\perp\rangle|^2.$$

However, these joint probabilities, Eq. (A6), will not give us the complete set of coincidences in an experiment as described in Section II A. To obtain the probability of coincidences, it is necessary to take into account the probability that both photons are detected, i.e., they both pass through the polarizer; as well as the probability that
they will not pass, i.e., they are not detected. Such probability can be written as the sum of the probability of measurement (or detection) the angles $\theta_s$ and $\theta_i$ plus the probability of measurement the orthogonal angles $\theta_s^\perp$ and $\theta_i^\perp$, that is

$$P^C_{|\psi\rangle}(\theta_s, \theta_i) = P^D_{|\psi\rangle}(\theta_s, \theta_i) + P^\overline{D}_{|\psi\rangle}(\theta_s, \theta_i) = P^D_{|\psi\rangle}(\theta_s, \theta_i) + P^D_{|\psi\rangle}(\theta_s^\perp, \theta_i^\perp), \quad (A7)$$

where $C$, $D$ and $\overline{D}$ means coincidences, detection, and not detection, respectively. Note that the last term in Eq. (A7) show to us that the probability of no detection of the state in angles $\theta_s$ and $\theta_i$ is the same that the probability of detection of the state in the orthogonal angles $\theta_s^\perp$ and $\theta_i^\perp$.

In the rest of this section we are going to calculate such joint probability of coincidences for two kind of states: the first group of states are the four Bell's entangled states $|\overline{27}\rangle$, the second one are two particular mixed states; which are described as

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|HH\rangle \pm |VV\rangle), \quad (A8a)$$

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|HV\rangle \pm |VH\rangle), \quad (A8b)$$

and

respectively.

**a. For Bell’s states**

First, we are going to calculate the joint measurement probabilities $P(|\psi\rangle)$ and $P(|\psi^\perp\rangle)$, Eq. (A6), for photons prepared in one each of Bell’s states, Eq. (A8), starting from $|\phi^{\pm}\rangle$, that is

$$P_{|\phi^{\pm}\rangle}(e_s, e_i) = \langle \phi^{\pm} | \langle e_s, e_i \langle e_s, e_i | \phi^{\pm} \rangle = \frac{1}{2}|\langle HH|e_s, e_i \rangle + \langle VH|e_s, e_i \rangle|^2 \quad (A10)$$

$$= \frac{1}{2}|\cos(\theta_s)\cos(\theta_i) + \sin(\theta_s)\sin(\theta_i)|^2. \quad (A11)$$

The result of Eq. (A13) is for a general elliptical polarization joint probability of pairs of single photons prepared in the state $|\phi^{\pm}\rangle$. But we are interested only in linear polarization, i.e., $\varphi_s = \varphi_i = 0$, that means that the probability only depends of angles $\theta_s$ and $\theta_i$, so we obtain

$$P_{|\phi^{\pm}\rangle}(\theta_s, \theta_i) = \frac{1}{2}|\cos(\theta_s)\cos(\theta_i) + \sin(\theta_s)\sin(\theta_i)|^2 \quad (A12)$$

$$= \frac{1}{2}|\cos(\theta_s - \theta_i)|^2. \quad (A13)$$

The result of Eq. (A13) is for a general elliptical polarization joint probability of pairs of single photons prepared in the state $|\phi^{\pm}\rangle$. But we are interested only in linear polarization, i.e., $\varphi_s = \varphi_i = 0$, that means that the probability only depends of angles $\theta_s$ and $\theta_i$, so we obtain

$$P_{|\phi^{\pm}\rangle}(\theta_s, \theta_i) = \frac{1}{2}|\cos(\theta_s)\cos(\theta_i) + \sin(\theta_s)\sin(\theta_i)|^2 \quad (A14)$$

$$+ \sin(\theta_s)\sin(\theta_i)|^2 \quad (A15)$$

$$= \frac{1}{2}|\cos(\theta_s - \theta_i)|^2. \quad (A16)$$

$$= \frac{1}{2}\cos^2(\theta_s - \theta_i). \quad (A17)$$

Similarly, for the orthogonal projection operator in Eq. (A6) we obtain the joint probability

$$P_{|\phi^{\pm}\rangle}(\theta_s^\perp, \theta_i^\perp) = \frac{1}{2}\cos^2(\theta_s - \theta_i). \quad (A18)$$

This means that the probability of coincidence, Eq. (A7), i.e, the probability of measurement (or detection) the photons in the angles $\theta_s$ and $\theta_i$ plus the probability of measurement the photons in the orthogonal angles $\theta_s^\perp$ and $\theta_i^\perp$ for the prepared state $|\phi^{\pm}\rangle$, have the form

$$P^C_{|\phi^{\pm}\rangle}(\theta_s, \theta_i) = \frac{1}{2}\cos^2(\theta_s - \theta_i) \quad (A19)$$

$$+ \frac{1}{2}\cos^2(\theta_s - \theta_i) \quad (A19)$$

$$= \cos^2(\theta_s - \theta_i). \quad (A19)$$

The Eq. (A19) show us the perfect correlation of state $|\phi^{\pm}\rangle$, i.e., when the angles of polarizations of Alice and Bob are the same, then the probability of coincidence is always equal to one. For the other three Bell’s states: $|\phi^{-}\rangle$, $|\psi^{+}\rangle$ and $|\psi^{-}\rangle$, we obtain the following probabilities of coincidence.

The Bell’s state $|\phi^{-}\rangle$ has probability of coincidence

$$P^C_{|\phi^{-}\rangle}(\theta_s, \theta_i) = \cos^2(\theta_s + \theta_i). \quad (A20)$$

The Bell’s state $|\psi^{+}\rangle$ has probability of coincidence

$$P^C_{|\psi^{+}\rangle}(\theta_s, \theta_i) = \sin^2(\theta_s + \theta_i). \quad (A21)$$

The last but not least of Bell’s states $|\psi^{-}\rangle$ has probability of coincidence

$$P^C_{|\psi^{-}\rangle}(\theta_s, \theta_i) = \sin^2(\theta_s - \theta_i). \quad (A22)$$
b. For mixed states

Additional to Bell’s states, we are interested in the two particular mixed states in Eq. (A9). Note that the first one is a maximally mixed state, and the other one is a partially mixed state.

The probabilities of coincidence for such states are

\[
P^C_{\rho_{\text{Max}}}(\theta_s, \theta_i) = \frac{1}{2}, \quad (A23)
\]

and

\[
P^C_{\rho}(\theta_s, \theta_i) = \frac{1}{2} \left[ \cos(2\theta_s) \cos(2\theta_i) + 1 \right], \quad (A24)
\]

respectively. Note that \( P^C_{\rho_{\text{Max}}}(\theta_s, \theta_i) \) in Eq. (A23) is independent of the angles \( \theta_s \) and \( \theta_i \), and is always equal to 1/2. A resume of all the joint probabilities of coincidence calculated are show in table II.

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