KLOOSTERMAN SUMS WITH TWICE-DIFFERENTIABLE FUNCTIONS

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Abstract. We bound Kloosterman-like sums of the shape
\[ \sum_{n=1}^{N} \exp(2 \pi i (x[f(n)] + y[f(n)]^{-1})/p), \]
with integers parts of a real-valued, twice-differentiable function \( f \) is satisfying a certain limit condition on \( f'' \), and \( [f(n)]^{-1} \) is meaning inversion modulo \( p \). As an immediate application, we obtain results concerning the distribution of modular inverses inverses \( [t]^{-1} \mod p \). The results apply, in particular, to Piatetski-Shapiro sequences \( [t^c] \) with \( c \in (1, \frac{2}{3}) \). The proof is an adaptation of an argument used by Banks and the first named author in a series of papers from 2006 to 2009.

1. Introduction and main result

1.1. Background and motivation. The Piatetski-Shapiro sequence associated with \( c \in (1, 2) \) is defined by \((\lfloor n^c \rfloor)_{n \in \mathbb{N}}\), where \( \lfloor x \rfloor = \min\{m \in \mathbb{Z} : m \leq x\} \) denotes the floor function. Such sequences are named in honour of Pyateckii-Šapiro [25] who, at the suggestion of A. O. Gelfond, has proved the following prime number theorem:
\[
\#\{n \leq N : \lfloor n^c \rfloor \text{ prime}\} = (1 + o(1)) \frac{N}{c \log N} \quad \text{as } N \to \infty
\]
for \( c \) in the range \( 1 < c < \frac{12}{11} \). Such a result may be viewed as an intermediate step to tackling the problem of investigating the number of primes represented by a fixed quadratic polynomial; consider for instance Landau’s famous problem of ascertaining whether \( n^2 + 1 \) is prime for infinitely many \( n \in \mathbb{N} \). Informally speaking, the upper bound for the exponents \( c \) for which one is able to establish (1.1) measures the progress towards quadratic polynomials. To date, the largest admissible \( c \)-range seems to be \( 1 < c < \frac{2817}{2426} \) due to Rivat and Sargos [27] (see also the references to the previous record holders they give in their paper). Naturally, also lower bound sieves have been employed, and the corresponding current record is a...
version of (1.1) with a lower bound of the right order of magnitude instead of an asymptotic formula and $1 < c < \frac{443}{205}$ due to Rivat and Jie [26].

Investigations into arithmetic properties of Piatetski-Shapiro sequences are not confined to studying prime values; they have been studied with respect to various other topics, including, but not limited to, the following:

- smooth, rough, and square-free numbers [1, 2, 6],
- almost-primes [7, 9],
- additive problems [4, 20, 22],
- intersection with special sequences [3, 5, 8, 13, 19], and
- digital expansions [21, 24, 28].

For a broader picture of the scope of each topic, we refer the interested reader to the references within the cited items.

Here we study a question of distribution of modular inverses modulo a prime $p$ of Piatetski-Shapiro sequences and, in fact, more general sequences. Our motivation comes from [29], where the distribution of inverses modulo a prime $p$ of Beatty sequences is considered. Furthermore, since this question immediately leads us to a problem of estimating Kloosterman-like sums with Piatetski-Shapiro sequences, this serves as an additional motivation. Indeed, this is an additive analogue of the results from [13], which concern bounds for character sums of the form

$$\sum_{n=1}^{N} \chi([f(n)]) \quad (N \geq 1),$$

where $f$ is a real-valued, twice-differentiable function satisfying a certain limit condition on $f''$ (see (1.4) below), $\chi$ is a non-trivial multiplicative character modulo a prime $p$, which is assumed to be in a suitable range with respect to $N$. The results from [13] apply, in particular, to power functions $f(t) = t^c$ with $c \in (1, \frac{4}{3})$ and apply, amongst other things, to bounding the least quadratic non-residue in Piatetski-Shapiro sequences (see also [5, 8]).

1.2. Notation. Throughout the paper, the notation $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leq c|U|$ for some positive constant $c$, which, may occasionally depend on the function $f$ and on the small positive real parameter $\varepsilon$ and on the positive integer parameter $k$. We use subscripts to indicate such dependencies.

We always use $p$ to denote a prime number and then for $x, y, u \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, we define

$$\psi_{x,y,p}(u) = \begin{cases} e_p(xu + yu^{-1}) & \text{if } u \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad (1.3)$$

where $e_p(a) = \exp(2\pi ia/p)$ and the calculation inside the argument of $e_p(\cdot)$ is to be performed in $\mathbb{F}_p$. 


1.3. **Main results.** In this paper, we first outline how to adapt the arguments from [13] to bound exponential sums with $\psi_{x,y,p}(\lfloor f(n) \rfloor)$ which immediately provide some non-trivial information about the distribution of modular inverses $[f(n)]^{-1} \pmod{p}$. We remark that for the scenario of [10–12], any non-trivial bound on the character sum implies the desired result about the distribution of quadratic non-residues. We also use the opportunity to provide slightly more precise information about the dependence of our saving $\delta$ on the parameter $\kappa$, characterising the growth of $f$. In particular, the explicit formula (1.7) below shows that $\delta$ is a monotonic function of $\varepsilon$ and $\kappa$.

Our main result may be stated as follows:

**Theorem 1.1.** Let $\frac{2}{3} < \kappa < 1$, and let $f$ be a real-valued, twice-differentiable function such that

$$\lim_{t \to \infty} \frac{\log f''(t)}{\log t} = -\kappa.$$  

Then, for a sufficiently small $\varepsilon > 0$, for all $\psi_{x,y,p}$, with $p$ prime and $x, y, u \in \mathbb{F}_p$, as in (1.3) and all integers $N$ in the range

$$p^{1/(2\kappa)+\varepsilon} \leq N < p^{1/(2-\kappa)},$$  

the uniform bound

$$\sum_{n=1}^{N} \psi_{x,y,p}(\lfloor f(n) \rfloor) \ll_{\varepsilon,f} N p^{-\delta}$$  

holds with

$$\delta = 2^{-11} \varepsilon^2 \kappa^4.$$  

For an interval $\mathcal{I} = [K + 1, K + H - 1] \subseteq [0, p - 1]$ with integers $K$ and $H$, we denote by $T_f(N, \mathcal{I})$ the number of positive integers $n \leq N$ for which the smallest positive residue $[f(n)]^{-1} \pmod{p}$ falls in $\mathcal{I}$, that is

$$T_f(N, \mathcal{I}) = \{n : [f(n)]^{-1} \equiv h \pmod{p}, 1 \leq n \leq N, h \in \mathcal{I}\}.$$  

**Corollary 1.2.** On the hypotheses of Theorem 1.1, we have

$$T_f(N, \mathcal{I}) = \frac{HN}{p} + O_{\varepsilon,f}(N p^{-\delta} \log p).$$  

where $\delta$ is given by (1.7).

**Corollary 1.3.** Let $f$ be a real-valued, twice-differentiable function such that (1.4) holds. There exists a constant $\xi > 0$ which depends only on $\kappa$, such that for

$$p \geq H, N \quad \text{and} \quad HN \geq p^{2-\xi}$$  

we have $T_f(N, \mathcal{I}) > 0$.

Note that Corollary 1.3 is an analogue of [29, Theorem 5.1], where a result of this type is given for Beatty sequences.
2. OUTLINE OF THE ARGUMENT

2.1. Preliminaries. As large parts of the arguments in [13] essentially carry over verbatim to the proof of Theorem 1.1, we choose not to repeat them here in full detail. Instead, we give an informal description of the underlying argument. The argument ultimately relies on a bound for certain double sums, [13, Lemma 4.1], and it is this bound which we need to adapt to our setting. A proof of this adapted bound, which is contained in our Corollary 3.2 below, is then carried out in full detail in Section 3.1. Some further details pertaining to the explicit value of $\delta$ given in Theorem 1.1 are contained in Sections 3.2 and 3.3.

2.2. Reduction from long sums to sums over short intervals. To get started, consider the sum

$$
\sum_{n=1}^{N} \psi_{x,y,p}([f(n)])
$$

from Theorem 1.1. We now fix some constant $c > 0$ (whose final choice depends on $\kappa$ and $\varepsilon$), to satisfy the inequalities

$$
N^{1-\kappa/2} \geq p^{3c}, \quad N^{\kappa/2-c} \geq p^{3c}, \quad N \geq p^{1/2+3c},
$$

as well as

$$
N^{\kappa-c} \geq p^{1/2+3c}.
$$

Next, for some parameter $R$, the sum (2.1) can be decomposed into $R$ sums over a small initial segment, $n \leq N p^{-c}$ and $R$ ‘short’ sums over $K_j < n \leq K_{j-1}$ with $j = 1, \ldots, R$ and numbers $N = K_0 > K_1 > \ldots > K_R = N p^{-c}$ satisfying

$$
K_{j-1} - K_j = \Delta K_j, \quad j = 1, \ldots, R.
$$

for some parameter $\Delta$.

Concerning the sum over the initial segment, already the trivial estimate is satisfactory, as it is within the bound (1.6) of Theorem 1.1. The remaining short sums are then treated with Lemma 2.1 below. Of course, to do this in the first place, the parameters $R$ and $\Delta$ have to be chosen appropriately. The specific choice used in [13], which also works in the setting of this paper, is (with the above choice of $c$)

$$
R = \left\lfloor N^{1+c-\kappa} \log^2 p \right\rfloor \quad \text{and} \quad \Delta = p^{c/R} - 1,
$$

and then

$$
K_j = p^{-j c/R} N, \quad j = 0, \ldots, R.
$$

One now verifies that (2.4) holds. We plainly refer to [13] for the technical details.
2.3. Reduction from sums over short intervals to double sums. For the short sums, one can use the following result: (This is already adapted to our setting; see [13, Lemma 5.1] for the corresponding character sum variant.)

**Lemma 2.1.** Fix $\varepsilon > 0$ and $\frac{3}{4} < \kappa < 1$. Let $f$ be a real-valued, twice-differentiable function satisfying (1.4). Then, for all $\psi_{x,y;p}$, with $p$ prime and $x,y,u \in \mathbb{F}_p$, as in (1.3), and all real numbers $K, L$ that satisfy the inequalities

$$K^{\kappa - \varepsilon} \geq L \geq K^{\kappa/2}p^{\varepsilon}, \quad K \leq p^{1/(2-\kappa)}, \quad L \geq p^{1/2 + \varepsilon},$$

the uniform bound

$$\sum_{K < n < K + L} \psi_{x,y;p}([f(n)]) \ll_{\varepsilon, f} L p^{-\delta_0}$$

holds with some constant $\delta_0 > 0$ that may only depend on $\varepsilon$ and $\kappa$.

Next, we sketch the idea of the proof of Lemma 2.1. Trivially, for every integer $h \geq 0$, one has

$$\sum_{K < n < K + L} \psi_{x,y;p}([f(n)]) = \sum_{K < n < K + L} \psi_{x,y;p}([f(n + h)]) + O(h).$$

Therefore, upon averaging over all $h \leq H$ for some parameter $H$,

$$\sum_{K < n < K + L} \psi_{x,y;p}([f(n)]) = \frac{1}{H} \sum_{h = 1}^{H} \sum_{K < n < K + L} \psi_{x,y;p}([f(n + h)]) + O(H).$$

To progress further, one would like to separate the variables $n$ and $h$ in the argument of $\psi_{x,y;p}$. This is achieved using the following formula:

$$f(n + h) = f(n) + hf'(K) + I_{n,h} + J_{n,h},$$

where

$$I_{n,h} = h \int_{K}^{n} f''(u) \, du \quad \text{and} \quad J_{n,h} = \int_{n}^{h + n} f''(u)(h + n - u) \, du.$$  

In view of (1.4), the relevant integrals can be seen to be acceptably small provided that $p$ is large enough. In particular, the last observation also crucially relies upon $n$ being not too large with respect to $K$, which is the case, because we are dealing with short sums. We now suppose that

$$0 \leq I_{n,h} + J_{n,h} < 1.$$  

However, there still is a complication with this approach arising through the floor function. Indeed, for any $\xi_0 \in [0, 1)$,

$$[f(n + h)] = [f(n) + \xi_0] + [hf'(K) - \xi_0] + \eta_{n,h,\xi_0},$$

with some undesired correction term $\eta_{n,h} \in \{0, 1, 2\}$ depending on both $n$ and $h$ (and $\xi_0$). To get around this, one restricts the averaging to only those $h \leq H$ for which the fractional part of $hf'(K) - \xi_0$ is small. By a clever choice of $\xi_0$, using the pigeonhole principle, one can ensure that the set of $h \leq H$ having the desired
property is not too sparse without actually having to know anything about the distribution of the fractional parts of \( hf'(K) \) as \( h = 1, 2, \ldots, H \). Then, in (2.6) one restricts to only those \( n \) such that the fractional part of \( f(n) + \xi_0 \) is bounded away from 1 in such a way that no carry to a next integer occurs when adding the four terms
\[
f(n) + \xi_0, \quad hf'(K) - \xi_0, \quad I_{n,h} \quad J_{n,h}.
\]
Clearly, these choices of \( n \) and \( h \) ensure that (2.7) holds with \( \eta_{n, h, \xi_0} = 0 \). Finally, one can see that the number of \( n \) which had to be discarded from (2.6) is not too large. In [13] this is accomplished by bounding the discrepancy of the sequence of fractional parts of \( f(n) \) as \( n = 1, 2, \ldots, N \), using the Erdős–Turán inequality to translate this to a problem of estimating certain exponential sums, and estimating these sums using a standard application of the van der Corput method.

The above argument, and in particular the additive split (2.7), reduces the proof of Lemma 2.1 to bounding double sums of the shape
\[
\sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} a_u b_v \psi_{x,y;p}(u + v),
\]
where \( \mathcal{U}, \mathcal{V} \) are subsets of \( \mathbb{F}_p \) and \( a_u, b_v \ (u \in \mathcal{U}, v \in \mathcal{V}) \) are certain weights. We also recall (2.6) and the subsequent discussion about separating \( n \) and \( h \); the need for including the weights arises from the potential failure of, e.g., \([f(n) + \xi_0] \) to produce only distinct values modulo \( p \) as \( n \) varies.

2.4. Concluding the proof. Given the above discussion, it is evident that, up to carrying out the technicalities which are, however, all readily found in [13], Theorem 1.1 follows from Lemma 2.1, and in turn Lemma 2.1 may be deduced from appropriate bounds for double sums (2.8). Such a suitable bound is given in Corollary 3.2 below.

3. Technical Details

3.1. Bounds for certain double sums. Here we prove a bound on the double sums (2.8) which concludes the proof of Theorem 1.1. In fact, we first give a slightly more general bound:

**Lemma 3.1.** Suppose that \( p \) is prime and \( \mathcal{U}, \mathcal{V} \) are subsets of \( \mathbb{F}_p \) of cardinalities \( U = \# \mathcal{U} \) and \( V = \# \mathcal{V} \). Then, for an arbitrary fixed integer \( k \), for any complex numbers \( a_u, b_v \ (u \in \mathcal{U}, v \in \mathcal{V}) \) and \( x, y \in \mathbb{F}_p \) with \( y \neq 0 \), we have
\[
\sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} a_u b_v \psi_{x,y;p}(u + v) \ll_k ABU^{1-1/(2k)}(V^{1/2}p^{1/(4k)} + Vp^{1/(4k)})
\]
where
\[
A = \max_{u \in \mathcal{U}} |a_u| \quad \text{and} \quad B = \max_{v \in \mathcal{V}} |b_v|.
\]
\textbf{Proof.} Denote the left hand side of (3.1) by $\mathcal{G}$. Then we apply Hölder’s inequality and subsequently extend the summation over $u \in \mathcal{V}$ to $u \in \mathbb{F}_p$, getting
\[
\mathcal{G}^{2k} \leq A^{2k} U^{2k-1} \left| \sum_{u \in \mathcal{V}} \left| \sum_{v \in \mathcal{V}} b_v \psi_{x,y,p}(u+v) \right|^{2k} \right.
\]
\[
\leq A^{2k} U^{2k-1} \left| \sum_{u \in \mathbb{F}_p} \left| \sum_{v \in \mathcal{V}} b_v \psi_{x,y,p}(u+v) \right|^{2k} \right. \]

Therefore,
\[
\mathcal{G}^{2k} \leq A^{2k} U^{2k-1} \sum_{u \in \mathbb{F}_p} u \prod_{v=(v_1, \ldots, v_k) \in \mathcal{V}^k} \prod_{s=1}^k b_v \psi_{x,y,p}(u+v) b_w \psi_{x,y,p}(u+w) \]
\[
= A^{2k} U^{2k-1} \sum_{v=(v_1, \ldots, v_k) \in \mathcal{V}^k} \prod_{s=1}^k b_v \psi_{x,y,p}(u+v) b_w \psi_{x,y,p}(u+w) \]

where
\[
R_{v,w}(X) = \sum_{r=1}^k \frac{1}{X + v_r} - \sum_{s=1}^k \frac{1}{X + w_s} \in \mathbb{F}_p(X),
\]
and $\sum^*$ restricts the summation to those $u \in \mathbb{F}_p$ such that all of the numbers
\[
u + v_1, \ldots, u + v_k, u + w_1, \ldots, u + w_k \in \mathbb{F}_p^k
\]
that is, are non-zero in $\mathbb{F}_p$. Thus,
\[
\mathcal{G}^{2k} \leq A^{2k} B^{2k} U^{2k-1} \sum_{v,w \in \mathcal{V}^k} \left| \sum^* u \right| e_p(R_{v,w}(u)) \]

We can assume that $p > k$ as otherwise there is nothing to prove. Examining the poles of the rational function $R_{v,w}(X)$, we see that it is constant, and in fact vanishes, only if the vectors $v$ and $w$ differ only by a permutation of their components. This happens only for $O_k(V^k)$ choices of $v, w \in \mathcal{V}^k$. For such choices we estimate the inner most sum in (3.2) trivially as $p$. Hence the total contribution to (3.2) from such $v, w \in \mathcal{V}^k$ is $O_k(V^k p)$.

For the remaining $O(V^{2k})$ choices we use the Weil bound of exponential sums with rational functions (see, for example, [23, Theorem 2], several more general bounds can also be found in [15]) and conclude that the total contribution to (3.2) from such $v, w \in \mathcal{V}^k$ is $O_k(V^{2k} p^{1/2})$.

Hence,
\[
\mathcal{G}^{2k} \leq k A^{2k} B^{2k} U^{2k-1} (V^k p + V^{2k} p^{1/2}),
\]
and the result follows. \hfill \Box
Corollary 3.2. For any \(\varepsilon > 0\), in the setting of Lemma 3.1, and assuming that
\[ U \geq p^{1/2+\varepsilon} \quad \text{and} \quad V \geq p^\varepsilon, \]
we have
\[ \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} a_u b_v \psi_{x,y;p}(u + v) \ll_{\varepsilon} ABUV p^{-\varepsilon^2}. \]

Proof. Taking \(k = [1/(2\varepsilon)]\) in Lemma 3.1 we have \(V \geq p^{1/(2k)}\) and thus the right hand side of (3.1) can be replaced with
\[ ABU^{1-1/(2k)}V p^{1/(4k)} = ABUV (p^{1/2}U^{-1})^{1/(2k)} \leq ABUV p^{-\varepsilon/(2k)}. \]
Since \(U \leq p\), we have \(\varepsilon \leq 1/2\) and thus one verifies that \(2k \leq \varepsilon^{-1}\). The result now follows. \(\square\)

3.2. Explicit version of Lemma 2.1. Before being able to address the explicit choice of \(\delta\) given in (1.7), we need an explicit version of Lemma 2.1. We remark that in the next result the saving \(\delta_0\) is independent of \(\kappa\).

Lemma 3.3. In the setting of Lemma 2.1, assuming \(\varepsilon\) to be sufficiently small, one may take \(\delta_0 = \varepsilon^2/26\).

Proof. We extract only the relevant part of the proof of [13, Theorem 5.1] (adjusted to our setting): we have
\[ \sum_{K < n \leq K + L} \psi_{x,y;p}(\lfloor f(n) \rfloor) \ll_{\varepsilon,f} \mathcal{B} V^{-1} p^{o_f(1)} + Lp^{-\delta_1} + p^{\varepsilon/2}, \]
where
- \(\delta_1 > 0\) is from [13, Equation (12)]
- \(\mathcal{B}\) is the bound obtained from Corollary 3.2 applied with
  \[ L \geq U = p^{1/2+\varepsilon+o_f(1)} \quad \text{and} \quad V = p^{\varepsilon/4+o_f(1)}, \]
and weights \(a_u, b_u\) of size \(p^{o_f(1)}\) as \(p \to \infty\).

Therefore, by Corollary 3.2 and (2.5),
\[ \sum_{K < n \leq K + L} \psi_{x,y;p}(\lfloor f(n) \rfloor) \ll_{\varepsilon,f} Lp^{-(\varepsilon/5)^2+o_f(1)} + Lp^{-\delta_1} + Lp^{-1/2-\varepsilon/2}. \]
Now only the value of \(\delta_1\) needs closer inspection. Looking at the lines that precede [13, Equation (12)], and recalling (2.5), one may check that, for \(\varepsilon\) sufficiently small, \(\delta_1 = \varepsilon^2\) is admissible. Upon plugging this in the above bound, the result follows. \(\square\)
3.3. **Proof of Theorem 1.1.** Assume that \( \varepsilon > 0 \) is sufficiently small. Then the inequalities (2.2) are all implied by (2.3), and the latter is clearly satisfied when choosing

\[
c = \frac{\varepsilon \kappa^2}{1 + 6 \kappa + 2 \varepsilon \kappa} < 1.
\]

A close inspection of the proof of [13, Theorem 6.1] shows that

\[
\sum_{n=1}^{N} \psi_{x,y,p}([f(n)]) \ll_{\varepsilon,f} Np^{-\delta_1} + Np^{-c},
\]

where \( \delta_1 \) is now any admissible exponent \( \delta \) of \( p \) from the bound obtained from Lemma 2.1 with \( \varepsilon \) replaced by \( c \). In particular, by Lemma 3.3, we may choose \( \delta_1 = c^2/26 \). This shows that in (1.6), again assuming \( \varepsilon \) to be sufficiently small, we may take \( \delta \) as in (1.7), which concludes the proof of Theorem 1.1.

3.4. **Proofs of Corollary 1.2 and Corollary 1.3.** Corollary 1.2 follows at once if one combines Theorem 1.1 with the Erdős–Turán inequality (see, for instance, [17, Theorem 1.21]).

To prove Corollary 1.3, we note that \( HN > p^{2-\xi} \) implies that \( H, N > p^{1-\xi} \). In particular, assuming \( \xi < 1/4 \) we see that \( N \) satisfies the necessary lower bound in (1.5). We now define

\[
N_0 = \min\{N, \lceil p^{1/(2-\kappa)} \rceil - 1\}
\]

thus \( T_f(N, \mathcal{S}) \geq T_f(N_0, \mathcal{S}) \) and we also see that Corollary 1.2 applies to \( T_f(N_0, \mathcal{S}) \). Taking

\[
\xi < 1 - 1/(2 - \kappa)
\]

we see that we can assume that \( N_0 = \lceil p^{1/(2-\kappa)} \rceil - 1 \). Since \( H > p^{1-\xi} \) we have the desired result.

4. **Comments**

4.1. **Some predecessors of our approach.** Results of the shape of Corollary 3.2 in the case Dirichlet characters appear to have been developed by Karatsuba [18] building on earlier work of Davenport and Erdős [16, Lemma 3] and Burgess [14, Lemma 2]. Indeed, the proof of our Corollary 3.2 proceeds in a similar vein and the averaging procedure underlying (2.6) has been used extensively (again, see [18] and the references therein).

The method has been subsequently adapted in a series of works [10–12] on properties of Beatty sequences, which often amounts to studying sums of the shape (1.2) with \( f(t) = \alpha t + \beta \) (with an irrational \( \alpha > 1 \) and real \( \beta \)) and \( \chi \) potentially replaced with some other function of arithmetic interest. Here, in contrast to the situation in (1.4) which opens up the possibility of using the van der Corput method, the quality of error terms generally also depends on the Diophantine properties of \( \alpha \).
4.2. **Further problems.** We have not made any attempt at optimising the value of $\delta$ for which one can prove Theorem 1.1. It would be interesting to see how large a value of $\delta$ the method from [13] can produce if all parameters are chosen optimally. Likewise, ascertaining the sharpest form of Corollary 1.3 would also be interesting.

Moreover, any improvement on Lemma 3.1, even for narrower ranges of $U$ and $V$, would be of independent interest. As a first step in this direction we record the following result which is non-trivial whenever $UV \gg p$ and, in this range and up to the implied constants, at least as good as Lemma 3.1 and stronger when $U$ and $V$ are both large:

**Proposition 4.1.** In the setting of Lemma 3.1, we have

$$\left| \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} a_u b_v \psi_{x,y,p}(u + v) \right| \ll ABU^{1/2}V^{1/2}p^{1/2},$$

where the implied constant is absolute.

**Proof.** We keep the notation $\mathcal{G}$ from the proof of Lemma 3.1. Then

$$\mathcal{G} = \sum_{w \in \mathbb{F}_p} \psi_{x,y,p}(w) \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} a_u b_v \frac{1}{p} \sum_{\lambda \in \mathbb{F}_p} e_p(\lambda(u + v - w))$$

$$= \frac{1}{p} \sum_{\lambda \in \mathbb{F}_p} \psi_{x,y,p}(w) \sum_{u \in \mathcal{U}} a_u e_p(\lambda u) \sum_{v \in \mathcal{V}} b_v e_p(\lambda v).$$

From the Weil bound (for the sum over $w$) and using Cauchy’s inequality, we infer

$$\mathcal{G}^2 \ll \frac{1}{\sqrt{p}} \sum_{\lambda \in \mathbb{F}_p} \left| \sum_{u \in \mathcal{U}} a_u e_p(\lambda u) \right|^2 \cdot \frac{1}{\sqrt{p}} \sum_{\lambda \in \mathbb{F}_p} \left| \sum_{v \in \mathcal{V}} b_v e_p(\lambda v) \right|^2.$$

Upon expanding the square in both sums and using orthogonality of characters, this yields

$$\mathcal{G} \ll (\sqrt{p}UA^2)^{1/2} (\sqrt{p}VB^2)^{1/2} = ABU^{1/2}V^{1/2}p^{1/2},$$

which gives the result. \( \Box \)

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