DIFFERENTIAL GEOMETRY OF SO*(2n)-STRUCTURES AND SO*(2n)Sp(1)-STRUCTURES–PART II

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ABSTRACT. In the first part of this series of articles we studied almost hypercomplex skew-Hermitian structures and almost quaternionic skew-Hermitian structures, as those underlying geometric structures on 4n-dimensional oriented manifolds appearing as 𝐺-structures for the Lie groups SO*(2n) and SO*(2n)Sp(1), respectively. There the corresponding intrinsic torsion is computed and the number of algebraic types of related geometries is derived, together with the minimal adapted connections (with respect to certain normalizations conditions). Here we use these results to present the related first-order integrability conditions in terms of the several algebraic types and other constructions. In particular, we use distinguished connections to provide a more geometric interpretation of the presented integrability conditions and highlight some features of certain classes.

The second main contribution of this note is the illustration of several specific types of such geometries via a variety of examples. We present an analogue of the notion of quaternionification of vector spaces, at the level of manifolds, which use to describe two general constructions providing examples of SO*(2n)-structures. We also use the bundle of Weyl structures and describe examples of SO*(2n)Sp(1)-structures in terms of parabolic geometries.

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References


\section*{Introduction}

This is the second part in a series of articles devoted to the study of geometric structures underlying manifolds admitting a reduction of the frame bundle to one of the Lie groups $\SO^*(2n)$ or $\SO^*(2n) \Sp(1)$, where $\SO^*(2n)$ denotes the quaternionic real form of $\SO(2n, \C)$ and $\SO^*(2n) \Sp(1) = \SO^*(2n) \times \Z_2 \Sp(1)$. Recall that such structures, called almost hypercomplex/quaternionic skew-Hermitian structures are respectively defined by pairs $(H, \omega)$ and $(Q, \omega)$, where $H$ is an almost hypercomplex structure, $Q$ is an almost quaternionic structure and $\omega$ is a non-degenerate 2-form which is $H$-Hermitian, respectively $Q$-Hermitian. We refer to such an almost symplectic form $\omega$ by the term scalar 2-form.

In the first part \cite{CGW21} we show that such pairs admit a symmetric 4-tensor $\Phi$ and a quaternionic skew-Hermitian form $h$, both stabilized by $\SO^*(2n) \Sp(1)$. These tensors allow an alternative approach to almost quaternionic skew-Hermitian geometries, since each of them can be used as a defining tensor, providing an analogy with the fundamental 4-form $\Omega$ in almost quaternionic Hermitian $(qH)$ geometries. The main achievement in the first part was the computation of the corresponding intrinsic torsion of $\SO^*(2n)$- or $\SO^*(2n) \Sp(1)$-structures, and the derivation of normalization conditions which let us realize this torsion via the the minimal adapted connections $\nabla^H, \omega$ and $\nabla^Q, \omega$, respectively. In particular, \cite{CGW21} provides the number of algebraic types of $\SO^*(2n)$- and $\SO^*(2n) \Sp(1)$-geometries. For $n > 3$ we show that there exist seven special $\Sp(1)$-invariant pure types $\lambda_1, \ldots, \lambda_7$ of $\SO^*(2n)$-structures, and ten general types, so up to $2^{10}$ algebraic types of $\SO^*(2n)$-geometries. For $\SO^*(2n) \Sp(1)$ $(n > 3)$ we obtain five pure types $\lambda_1, \ldots, \lambda_5$ and hence up to $2^5$ algebraic types of $\SO^*(2n) \Sp(1)$-geometries. For the low-dimensional cases $n = 2, 3$ we deduce that are further types of such non-integrable geometries.

In this paper we explore all these types and explain the contribution of the different intrinsic torsion components in the obstruction of the integrability of $H$, $Q$ and $\omega$. In particular, one of the main results in the present work is the description of 1st-order integrability conditions on almost hypercomplex skew-Hermitian manifolds $(M^{4n}, H, \omega)$, and almost quaternionic skew-Hermitian manifolds $(M^{4n}, Q, \omega)$. The methodology for such a procedure partially builds on the results from \cite{CGW21}, but is also based on branching rules, and on other more geometric approaches which arise by using distinguished connections. This second approach allows us to provide a more geometric interpretation of the presented integrability conditions and highlight some features of certain classes. Moreover, for $\SO^*(2n) \Sp(1)$-structures we establish a theory related to the covariant derivative $\nabla \Phi$, where $\nabla$ is any affine connection on $M$. Then we provide a specification of this theory in the case where $\nabla$ is given by an almost symplectic connection $\nabla^\omega$ (with respect to the related scalar 2-form $\omega$), or by the unimodular Oproiu connection $\nabla^{Q, \vol}$ (with respect to the pair $(Q, \vol = \omega^\otimes n)$), etc. Sections 1 and 2 are devoted to the description of these methodologies and aforementioned outcome.

The other major contribution of this paper is the presentation of a variety of examples of manifolds admitting such $G$-structures, and in particular of examples realizing some of the aforementioned algebraic types. In more details, the first class of examples is based on modifying global frames on $\R^{4n}$, or on some almost symplectic manifold. Remarkably enough, the latter method provides us with a generalization of the so-called quaternionification of a real vector space (see \cite{BrD85}) to the manifold setting. We exploit this observation by presenting two general constructions of manifolds with $\SO^*(2n)$-structures. On the other hand, based on certain Euclidean spaces we construct examples of almost hypercomplex skew-Hermitian structures with intrinsic torsion in certain proper submodules. This allows the realization of some mixed algebraic types of $\SO^*(2n)$- and $\SO^*(2n) \Sp(1)$-geometries. In addition, we provide examples of pure type $\lambda_3$ and present applications of our generalized quaternionification on specific (almost) symplectic manifolds. This description takes place in Section 3.
The second class of examples is presented in Section 4. Here we provide a general description of homogeneous almost hypercomplex skew-Hermitian manifolds, or almost quaternionic skew-Hermitian manifolds. This includes a presentation of the invariant adapted connections, in terms of Nomizu-type maps (and the soldering form). We also derive how the intrinsic torsion arises from such a description. Then we analyze some explicit examples. In particular, we examine the reductive homogeneous space \( M = K/L = \text{SL}(4, \mathbb{R})/\text{SL}(2, \mathbb{R}) \) and show the existence of a \( \text{SL}(4, \mathbb{R}) \)-invariant \( SO^*(2n) \)-structure of generic \( \text{Sp}(1) \)-invariant type \( \chi_{1234567} \), which confirms that this type is realizable. Moreover, we present examples of left-invariant \( SO^*(2n) \)-structures on a 12-dimensional unipotent Lie group.

The final class of examples is based on general functors from the categories of almost quaternionic manifolds, or quaternionic affine manifolds, and is presented in Section 5. In this direction, initially we focus on the total space of the Weyl bundle over the quaternionic projective space \( N = \mathbb{HP}^n = G/P \), or the cotangent bundle of \( N \), respectively. The related constructions are essentially motivated by the fact that for any quaternionic vector space \( (V, Q) \), the cotangent space \( T^*V \cong V \times V^* \) admits a canonical linear quaternionic skew-Hermitian structure, where the associated scalar 2-form \( \omega \) is the canonical symplectic form given by the natural pairing \( V \times V^* \to \mathbb{R} \) (see Section 5). We show that there are two possible ways to generalize this result to the manifold setting, which are essentially related each other, since the Weyl bundle and cotangent bundle are diffeomorphic. When the source is a more general quaternionic affine manifold than \( \mathbb{HP}^n \), we compare the various possible combinations of the resulting almost quaternionic and almost symplectic structures, and describe conditions for their pairwise compatibility, as almost quaternionic skew-Hermitian structures.

In the final section we pose some open problems related to the geometry of \( SO^*(2n) \)- and \( SO^*(2n) \text{Sp}(1) \)-structures. These question either emphasize further open tasks related to the local differential geometry of such \( G \)-structures (curvature invariants, twistor constructions, a metric view point of \( SO^*(2n) \)-structures, etc), or have a more general character and are about the classification of such structures under certain assumptions, or a complete clarification of the relationship between \( SO^*(2n) \text{Sp}(1) \)-structures and quaternionic geometries in terms of parabolic geometries. Some of these questions and further related tasks are treated in the third part of this series of works, see [CGW21III]. Finally, for the convenience of the reader and with the aim to make the whole series self-complete, the current article also includes an appendix where we describe some basic topological features of manifolds admitting \( SO^*(2n) \)- or \( SO^*(2n) \text{Sp}(1) \)-structures.

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1. Integrability conditions of \( SO^*(2n) \)-structures and \( SO^*(2n) \text{Sp}(1) \)-structures

1.1. Basics on almost hypercomplex/quaternionic skew-Hermitian structures. We begin by briefly recalling the concept of almost hypercomplex skew-Hermitian structures, and almost quaternionic skew-Hermitian structures, as introduced in [CGW21].

Let us consider a \( 4n \)-dimensional connected smooth manifold \( M \), and assume once and for all that \( n > 1 \). Next we will maintain the conventions appearing in Section 1 of [CGW21] and in particular use the EH-formalism. Recall that:

- An almost hypercomplex skew-Hermitian (hs-H in short) structure on \( M \) consists of a pair \( (H, \omega) \), where \( H = \{ J_a : a = 1, 2, 3 \} \) is an almost hypercomplex structure and \( \omega \) is a scalar 2-form with respect to \( H \), that is \( \omega \) is a non-degenerate 2-form which is \( H \)-Hermitian, i.e., \( \omega(J_a X, J_a Y) = \omega(X, Y) \) for any \( X, Y \in \Gamma(TM) \).
- An almost quaternionic skew-Hermitian (qs-H in short) structure on \( M \) consists of a pair \( (Q, \omega) \), where \( Q \) is an almost quaternionic structure \( Q \subset \text{End}(TM) \) on \( M \) and \( \omega \) is a scalar 2-form with
respect to $Q$, that is $\omega$ is a non-degenerate 2-form which is $Q$-Hermitian, i.e., $\omega(JX, JY) = \omega(X, Y)$ for any $J \in S(Q)$ and $X, Y \in \Gamma(TM)$.

**Convention:** Let us recall some of our conventions from [CGW21]. We will denote by $[EH]$ the 4$n$-dimensional real vector space equipped with the standard (linear) quaternionic structure $Q_0$. Also, $H_0$ will denote the standard admissible basis of $Q_0$, and we will use the notation $\omega_0$ for the standard scalar 2-form on $[EH]$ with respect to $Q_0 = \langle H_0 \rangle$.

Almost hs-H structures are in bijection with reductions $\mathcal{P}$ of the frame bundle $F(M)$ of $M$ to $SO^*(2n)$. Such reductions consist of skew-Hermitian bases of $T_x M$ with respect to $(H_x, \omega_x)$, inducing a linear hypercomplex isomorphism $u : T_x M \to [EH]$. We recall that by a skew-Hermitian basis with respect to $(H_x, \omega_x)$ we mean a symplectic basis

$$e_1, \ldots, e_{2n}, f_1, \ldots, f_{2n}$$

of the scalar 2-form $\omega_x$, i.e.,

$$\omega(e_r, e_s) = 0, \quad \omega(f_r, f_s) = 0, \quad \omega(e_r, f_s) = 1, \quad \omega(e_r, f_s) = 0, \quad (r \neq s)$$

for $1 \leq r \leq 2n$ and $1 \leq s \leq 2n$, which is adapted to the linear hypercomplex structure $H_x = \{J_1, J_2, J_3\}$ on $T_x M$ in the following sense:

$$J_1(e_c) = e_{c+n}, \quad J_2(e_c) = f_c, \quad J_3(e_c) = f_{c+n},$$

for $1 \leq c \leq n$, see also [CGW21, Definition 1.24]. With respect to ordering, we should mention that such a base differs by an adapted basis (or frame) of $H$ in terms of $[AM96, p. 209]$.

Similarly, almost qs-H structures are in bijection with reductions $\mathcal{Q}$ of the frame bundle of $M$ to $SO^*(2n)\Sp(1)$. Such reductions consist of all skew-Hermitian bases of $T_x M$ with respect to $(H_x, \omega_x)$, where $H_x$ is some admissible basis $H_x$ of $Q_x$. Such bases induce a linear quaternionic isomorphism $u : T_x M \to [EH]$.

Let $\{J_a : a = 1, 2, 3\}$ be a local admissible frame of $Q$ and let $(Q, \omega)$ be an almost qs-H structure on $M$. Then, we may attach three pseudo-Riemannian metric tensors of signature $(2n, 2n)$, given by $g_{J_a}(X, Y) = \omega(X, J_a Y)$. These tensors are globally defined only when $H$ is a global admissible frame, i.e., when $(H, \omega)$ is an almost hs-H structure on $M$. However, the quaternionic skew-Hermitian form $h$ defined by

$$h(X, Y)Z = \omega(X, Y)Z \sum_a g_{J_a}(X, Y)J_a Z$$

is independent of the local admissible frame $H$ of $Q$, and moreover has stabilizer the Lie group $SO^*(2n)\Sp(1)$. Thus, it can be viewed as a global smooth section of the associated bundle $\mathcal{Q} \times SO^*(2n)\Sp(1) \left([EH]^* \otimes_R [EH]^* \otimes_R \GL([EH])\right)$, that is, $h$ is a $(1, 3)$-tensor globally defined on $M$. Similarly, the symmetric 4-tensor

$$\Phi = \sum_a g_{J_a} \otimes g_{J_a} = \text{Sym}(\omega(\cdot, \Im(h)\cdot))$$

is independent of the local admissible frame $H$ and has stabilizer $SO^*(2n)\Sp(1)$, see [CGW21] for more details. Thus, it is a global tensor on $(M, Q, \omega)$, i.e., $\Phi \in \Gamma(S^4T^*M)$. As a consequence, these tensors provide an alternative approach to almost qs-H structures.

Let us finally recall that on an almost hypercomplex/quaternionic skew-Hermitian manifold we have introduced adapted connections $\nabla^H, \omega$ and $\nabla^Q, \omega$, respectively, and chosen normalization conditions which establish these connections as the corresponding unique minimal connections. In Section 2 we will recall further related details and provide a more geometric viewpoint of these adapted connections.
1.2. Applications of branching rules. In this section we use the results presented in the first paper [CGW21] on intrinsic torsion and minimal connections of $\SO^*(2n)\Sp(1)$- and $\SO^*(2n)$-structures, to derive 1st-order integrability conditions for such structures. In particular, by [CGW21] we shall use Proposition 3.14, Corollary 3.15, Corollary 3.17, Theorem 4.3, and Proposition 4.5. Some of our arguments are also based on “branching rules” and in particular on the following general remark from the theory of $G$-structures.

**Remark 1.1.** Let $K \subset G$ be a reductive Lie subgroup of a reductive linear Lie group $G$. Then, a $K$-structure on a manifold $M$ is also an example of a $G$-structure on $M$. Assume that we have equivalence classes of minimal $K$-connections and minimal $G$-connections. Let $\nabla^K$ and $\nabla^G$ be minimal connections for the $K$-structure and $G$-structure, respectively. Then we have

\[
\nabla^K = \nabla^G + A
\]

for some $(1, 2)$-tensor field $A \in \Gamma(T^*M \otimes \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of $G$. Therefore, the restriction to $K$ of the intrinsic torsion module of $G$ will appear as a submodule of the intrinsic torsion module of $K$, that is

\[
\mathcal{H}(\mathfrak{g})|_K \subset \mathcal{H}(\mathfrak{k}),
\]

where $\mathfrak{k} \subset \mathfrak{g}$ is the Lie algebra of $K \subset G$. Hence, one can obtain geometric interpretations of certain submodules of the $K$-intrinsic torsion by branching the intrinsic torsion modules of $G$ to $K$.

Next we shall apply this method to $\SO^*(2n)\Sp(1)$-structures, and also to $\SO^*(2n)$-structures, where in the latter case much of our focus is devoted to the types $\mathcal{X}_1, \ldots, \mathcal{X}_7$, characterized in terms of $\Sp(1)$-invariant conditions (see [CGW21, Corollary 3.17]).

1.3. 1st-order integrability conditions for $\SO^*(2n)\Sp(1)$-structures. Initially, it is convenient to derive 1st-order integrability conditions about $\SO^*(2n)\Sp(1)$-structures.

1.3.1. Quaternionic torsion. Let us consider the Lie group $G = \GL(n, \mathbb{H})\Sp(1)$. Then, $K = \SO^*(2n)\Sp(1) \subset G$ is a (closed) Lie subgroup of $G$. The intrinsic torsion module of $G$ was computed in [AM96], and there is a unique normalization condition (due to multiplicity one in the decomposition of quaternionic torsion tensors), given by

\[
D(\mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1)) = R(3\theta + \pi_1 + 2\pi_{2n}),
\]

where $\theta$ is the fundamental weight of $\mathfrak{sp}(1)$, and here the notation $\pi_i$ refers to the $i$-th fundamental weight of $\mathfrak{sl}(n, \mathbb{H})$. Hence, any almost quaternionic connection has a $G$-equivariant decomposition of its torsion tensor with one component of this isotype (although this component may vanish), and this does not require that the connection is minimal. Therefore, this applies in particular to connections which are minimal with respect to $K$. Thus we may branch to $K$, a procedure which gives rise to the following

**Proposition 1.2.** Viewed as a $\SO^*(2n)\Sp(1)$-module, the restriction $D(\mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1))|_{\SO^*(2n)\Sp(1)}$ satisfies the following

\[
D(\mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1))|_{\SO^*(2n)\Sp(1)} \cong [K S^3 H]^* + [\Lambda^3 E S^3 H]^* = X_{12} = X_1 + X_2.
\]

Therefore, the component of the intrinsic torsion of an almost qs-$H$ structure (and hence of the torsion of $\nabla^{Q,\omega}$) taking values in these submodules only depends on, and coincides with the intrinsic torsion of the underlying almost quaternionic structure. This yields the following.

**Theorem 1.3.** Let $(Q, \omega)$ be an almost qs-$H$ structure with canonical connection $\nabla^{Q,\omega}$. Then $Q$ is quaternionic, if and only if $(Q, \omega)$ is of type $\mathcal{X}_{345}$, that is the torsion components $[K S^3 H]^*$ and $[\Lambda^3 E S^3 H]^*$ of $\nabla^{Q,\omega}$ vanish.
1.3.2. Symplectic torsion. Let $G = \text{Sp}(4n, \mathbb{R})$ be the symplectic automorphism group. It is well known that (see for example [AP15, CS17])

$$\mathcal{D}(\text{sp}(4n, \mathbb{R})) \cong \Lambda^3(\mathbb{R}^{4n})^* \cong \Lambda^3(\mathbb{R}^{4n})^* \oplus \mathbb{R}^{4n},$$

and in particular the intrinsic torsion can be identified with the 3-form $d\omega$. Similarly to the previous case, we may branch this module with respect to $\text{so}^*(2n) \oplus \text{sp}(1)$.

**Proposition 1.4.** Viewed as a $\text{SO}^*(2n) \text{Sp}(1)$-module, the restriction $\mathcal{D}(\text{sp}(4n, \mathbb{R}))|_{\text{SO}^*(2n) \text{Sp}(1)}$ satisfies the following

$$\mathcal{D}(\text{sp}(4n, \mathbb{R}))|_{\text{SO}^*(2n) \text{Sp}(1)} \cong [\Lambda^3 \text{ES}^3 H]^* \oplus [KH]^* \oplus [EH]^* = \mathcal{X}_{234}.$$ 

Thus, a combination of Proposition 1.4 with [CGW21, Proposition 4.5] yields the following theorem.

**Theorem 1.5.** Let $(Q, \omega)$ be an almost qs-H structure with canonical connection $\nabla^{Q,\omega}$. Then $\omega$ is symplectic, if and only if $(Q, \omega)$ is of type $\mathcal{X}_{15}$, that is the torsion components $[\Lambda^3 \text{ES}^3 H]^*$, $[KH]^*$ and $[EH]^*$ of $\nabla^{Q,\omega}$ vanish.

**Remark 1.6.** Similarly to the almost-symplectic case, for $\text{SO}^*(2n) \text{Sp}(1)$-structures the vectorial torsion component with values in $\mathcal{X}_4 = [EH]^*$ can be modified by a conformal change

$$\omega \mapsto f \cdot \omega,$$

where $f \in C^\infty(M)$ is some non-vanishing function. Then, by [CGW21, Theorem 3.12] we deduce that the canonical connection $\nabla^{Q,\omega}$ is modified by

$$-d f \otimes \text{id} - \frac{4n}{n+1} \pi_S(\omega \otimes df^T) + \frac{n}{n+1} \sum_{a=1}^{3} df \circ J_a \otimes J_a,$$

where $df^T$ denotes the symplectic transpose of $df$ and $\pi_S$ is the projection defined by

$$\pi_S(\omega \otimes Z)(X,Y) := \text{Sym} \left( \pi_{1,1}(\omega(X,\cdot) \otimes Z)Y \right),$$

with

$$\pi_{1,1} : \text{gl}(\mathbb{H}) \rightarrow \text{gl}(n, \mathbb{H}), \quad \pi_{1,1}(\omega(X,\cdot) \otimes Z)Y = \frac{1}{4} \left( \omega(X,Y)Z - \sum_a g_{J_a}(X,Y)J_aZ \right),$$

see [CGW21, Section 3.2]. It turns out that this change does not affect the compatibility with $Q$. In this way we obtain a new $\text{SO}^*(2n) \text{Sp}(1)$-structure $(Q, f \cdot \omega)$. When the initial $\text{SO}^*(2n) \text{Sp}(1)$-structure $(Q, \omega)$ is of type $\mathcal{X}_{145}$, then (locally) there is a smooth function $f$ on $M$ such that $f \cdot \omega$ is symplectic, which is equivalent to say that $(Q, f \cdot \omega)$ is of type $\mathcal{X}_{15}$. We deduce that the type $\mathcal{X}_{145}$ of $\text{SO}^*(2n) \text{Sp}(1)$-structures characterizes a certain subclass of the so-called parabolic (locally) conformally symplectic structures, examined by Čap and Salač in [ČS17].

1.3.3. Intersection torsion and compatibility torsion. Notice that the submodule $\mathcal{X}_2 = [\Lambda^3 \text{ES}^3 H]^*$ appears in both the quaternionic and the symplectic torsion. Since $\mathcal{X}_2$ has multiplicity one in $\mathcal{D}(\text{gl}(n, \mathbb{H}) \oplus \text{sp}(1))$, $\mathcal{D}(\text{sp}(4n, \mathbb{R}))$ and $\Lambda^3[EH]^* \otimes [EH]$, this submodule must coincide, and represents a smooth compatibility condition between those almost symplectic structures and almost quaternion structures which are algebraically compatible. Thus we may call the submodule $\mathcal{X}_2$ the module corresponding to intersection torsion. On the other side, and since we have exhausted the quaternionic and symplectic torsion, we now consider the complement of their union in the intrinsic torsion. This submodule will be called the module corresponding to compatibility torsion, and it consists of the simple submodule $\mathcal{X}_5 = [S^3 EH]^*$. By [CGW21, Proposition 3.14], this submodule is isotypically unique in the space of torsion tensors, hence it is independent of our normalization condition. From this and the previous two theorems, we obtain
Theorem 1.7. Let \((Q, \omega)\) be an almost qs-H structure with canonical connection \(\nabla^{Q,\omega}\). Then \(Q\) is quaternionic and \(\omega\) symplectic, if and only if \((Q, \omega)\) is of type \(X_5\), that is the torsion of \(\nabla^{Q,\omega}\) is contained in the submodule \([S_0^3 \mathbb{E} H]\)^*.

1.4. 1st-order integrability conditions for \(SO^*(2n)\)-structures. Next we will determine 1st-order integrability conditions for \(SO^*(2n)\)-structures on some fixed 4n-dimensional smooth connected manifold \(M\). To do so we benefit from [CGW21, Corollary 3.17] and the interpretation of the intrinsic torsion module \(\mathcal{H}(so^*(2n))\) as an \(Sp(1)\)-module.

1.4.1. Hypercomplex torsion. In terms of Remark 1.1 let us now set \(G = SO^*(2n) Sp(1)\) and \(K = SO^*(2n) \subset G\). Then, by [CGW21, Theorem 4.3] we may pose the following decomposition of hypercomplex torsion

\[
\text{Im}(\pi_H) \cong \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_6 \cong 2\Lambda^3 E^* \oplus 2K^* \oplus 2E^*.
\]

Here, the first splitting should be read in terms of \(G\)-modules, while the second one in terms of \(K\)-modules.

Therefore, the component of the intrinsic torsion as an almost hs-H structure (and hence of the torsion of \(\nabla^{H,\omega}\)) taking values in these \(G\)-modules only depends on, and coincides with the intrinsic torsion of the underlying almost hypercomplex structure. This yields the following:

Theorem 1.8. Let \((H, \omega)\) be an almost hs-H structure with canonical connection \(\nabla^{H,\omega}\). Then \(H\) is hypercomplex, if and only if \((H, \omega)\) is of type \(X_{3457}\), that is the torsion components \(\mathcal{X}_1 = [K S^3 H]^*\), \(\mathcal{X}_2 = [\Lambda^3 \mathbb{E} S^3 H]^*\) and \(\mathcal{X}_6 = [E S^3 H]^*\) of \(\nabla^{H,\omega}\) vanish.

1.4.2. Symplectic torsion. Let us now apply Proposition 1.4 to obtain the following branching:

\[
\mathcal{D}(sp(4n, \mathbb{R}))|_{SO^*(2n)} \cong \mathcal{X}_{234}|_{SO^*(2n)} = ([\Lambda^3 \mathbb{E} S^3 H]^* \oplus [K H]^* \oplus [E H]^*)|_{SO^*(2n)} = 2\Lambda^3 \mathbb{E}^* \oplus K^* \oplus \mathbb{E}^*.
\]

As a consequence of [CGW21, Proposition 4.5] and Proposition 1.4 we obtain the following theorem.

Theorem 1.9. Let \((H, \omega)\) be an almost hs-H structure with canonical connection \(\nabla^{H,\omega}\). Then \(\omega\) is symplectic, if and only if \((H, \omega)\) is of type \(X_{1567}\), that is the torsion components \([\Lambda^3 \mathbb{E} S^3 H]^*\), \([K H]^*\) and \([E H]^*\) of \(\nabla^{H,\omega}\) vanish.

1.4.3. Intersection torsion and compatibility torsion. Observe by Theorems 1.8 and 1.9 that similarly to the \(SO^*(2n) Sp(1)\)-case, the submodule \(\mathcal{X}_2 = [\Lambda^3 \mathbb{E} S^3 H]^*\) appears in both the hypercomplex and the symplectic torsion. As before we call \(\mathcal{X}_2\) the module corresponding to intersection torsion. Moreover, and since we have exhausted the hypercomplex and symplectic torsion, we can now consider the complement of their union in the intrinsic torsion. This submodule will be called the module corresponding to compatibility torsion, and in this case it consists of the mixed type module \(\mathcal{X}_{37} = \mathcal{X}_5 \oplus \mathcal{X}_7 = [S_0^3 \mathbb{E} H]^* \oplus [E H]^*\). Consequently by Theorems 1.8 and 1.9 we obtain

Theorem 1.10. Let \((H, \omega)\) be an almost hs-H structure with canonical connection \(\nabla^{H,\omega}\). Then \(H\) is hypercomplex and \(\omega\) symplectic, if and only if \((H, \omega)\) is of mixed type \(\mathcal{X}_{37}\), that is the torsion of \(\nabla^{H,\omega}\) is contained in the submodule \([S_0^3 \mathbb{E} H]^* \oplus [E H]^*\).

Remark 1.11. Due to the results presented in [CGW21, Corollary 3.17] and the first part of [CGW21, Theorem 4.3], one should mention that the module \(\mathcal{X}_{37}\) appearing in the last conclusion is uniquely fixed by our normalization condition for \(\nabla^{H,\omega}\).

2. Integrability conditions via distinguished connections

2.1. General theory. We recall that there are several distinguished linear connections that can be used to study \(SO^*(2n)\)-structures \((H, \omega)\), or \(SO^*(2n) Sp(1)\)-structures \((Q, \omega)\). For instance, in the first part we used the Obata connection \(\nabla^H\) and the unimodular Oproiu connection \(\nabla^{Q,\text{vol}}\), and
obviously there are many others which can facilitate an examination of the underlying geometries, such as almost symplectic connections \( \nabla^\omega \), etc. These connections naturally act on the tensors associated to such structures (see Section 2 of [CGGW21]), but in general they do not preserve them. For example, for a generic almost hs-H structure \((H, \omega)\) we have \( \nabla^H \omega \neq 0, \nabla^H h \neq 0, \nabla^H \Phi \neq 0 \), etc. Nevertheless, we can relate the values of the corresponding covariant derivatives with certain intrinsic torsion components, and hence with 1st-order integrability conditions. Before we proceed with details, let us pose a general result which can be used as a guideline for the discussion that follows. We begin with the following definition.

**Definition 2.1.** Let \( K, L \subset \text{GL}(n, \mathbb{R}) \) be reductive Lie groups. We will say that a \( K \)-structure \( \mathcal{P}_K \) on a smooth manifold \( M \) is compatible with a \( L \)-structure \( \mathcal{P}_L \) on \( M \), if \( \mathcal{P} := \mathcal{P}_K \cap \mathcal{P}_L \) is a \( G := K \cap L \)-structure on \( M \), which means that for each \( x \in M \), there exists a frame \( u \in \mathcal{F}_x(M) \) that is adapted for both the \( K \)- and the \( L \)-structure.

**Proposition 2.2.** Let \( K, L \subset \text{GL}(n, \mathbb{R}) \) be two reductive Lie groups, and assume that \( \mathcal{P}_K \) (respectively \( \mathcal{P}_L \)) is a \( K \)-structure (respectively a \( L \)-structure) on a \( n \)-dimensional manifold \( M \), which are compatible. Assume that \( \nabla^K \) is a \( K \)-connection and that the \( L \)-structure \( \mathcal{P}_L \) is defined by a tensor field \( F \) on \( M \), that is \( F \) is stabilized by \( L \). Then, the following hold for the corresponding \( G := K \cap L \)-structure \( \mathcal{P} \):

1) If \( \nabla^L \) is a \( L \)-connection, then the \((1,2)\)-tensor \( \mathcal{A} := \nabla^L - \nabla^K \) takes values in \( T^*M \otimes (1 + \mathfrak{k})_P \), where

\[
(1 + \mathfrak{k})_P := \mathcal{P} \times_G (1 + \mathfrak{k}),
\]

and \( I, \mathfrak{k} \) are the Lie algebras of \( L, K \), respectively.

2) The covariant derivative \( \nabla^K F \) is a smooth section of a bundle \( \mathcal{E} \to M \) isomorphic to the associated bundle \( T^*M \otimes (\mathfrak{k}/\mathfrak{g})_P \), where \( \mathfrak{g} \) is the Lie algebra of \( G \).

3) Let \( \mathfrak{k} = \mathfrak{g} \oplus \mathfrak{m} \) be a \( G \)-invariant direct sum decomposition, and let \( s : \mathcal{E} \to T^*M \otimes \mathfrak{m}_P \) be a splitting of the natural projection \( p : T^*M \otimes \mathfrak{k}_P \to \mathcal{E} \), that is \( p \circ s = \text{id}_\mathcal{E} \). Then the \( K \)-connection

\[
\nabla := \nabla^K - s(\nabla^K F)
\]

preserves \( F \), i.e., \( \nabla F = 0 \). So in particular, \( \nabla \) is a \( G \)-connection.

4) The intrinsic torsion \( H(I) \) of the \( L \)-structure \( \mathcal{P}_L \) and the intrinsic torsion \( H(\mathfrak{g}) \) of the \( G \)-structure \( \mathcal{P} \), are both given by the appropriate projections of the torsion

\[
T = T^K - \delta(s(\nabla^K F))
\]

of \( \nabla \) to the corresponding intrinsic torsion module, where \( \delta : T^*M \otimes \mathfrak{k}_P \to \text{Tor}(M) \) is the Spencer alternation operator related to the \( K \)-structure and \( T^K \) is the torsion of \( \nabla^K \).

**Proof.** Let us choose a (local) frame \( u \in \Gamma(\mathcal{P}) \). At \( u \) the connection 1-form of \( \nabla^L \) takes values in \( I \), and the connection 1-form of \( \nabla^K \) in \( \mathfrak{k} \). Hence, it follows that their difference tensor \( \mathcal{A} \) is a smooth section of \( T^*M \otimes (1 + \mathfrak{k})_P \). This proves the first claim. As a consequence, and since by assumption the tensor \( F \) is stabilized by \( L \), the covariant derivative \( \nabla^K F \) is

\[
\nabla^K F = \nabla^L F - \mathcal{A} \cdot F = -\mathcal{A} \cdot F,
\]

where \( \mathcal{A} \cdot F \) encodes the natural action of 1-forms with values in \( \text{End}(TM) \) on the tensor bundle over \( M \). Since the latter action is linear, with kernel given by 1-forms with values in the Lie algebra \( \mathfrak{l} \), its image \( \mathcal{A} \cdot F \) is a section of a bundle \( \mathcal{E} \) over \( M \), which corresponds to the section \( p(\mathcal{A}) \) of the following associated bundle

\[
T^*M \otimes ((\mathfrak{k} + I)/I)_P = T^*M \otimes (\mathfrak{k}/\mathfrak{g})_P.
\]

This provides the isomorphism

\[
\mathcal{A} \cdot F \in \Gamma(\mathcal{E}) \mapsto p(\mathcal{A}) \in \Gamma(T^*M \otimes (\mathfrak{k}/\mathfrak{g})_P).
\]
Next, under our assumptions and by the definition of $\nabla$ we have
\[ \nabla F := \nabla^K F - (s(\nabla^K F)) \cdot F = \nabla^K F - p(s(\nabla^K F)) = \nabla^K F - \nabla^K F = 0. \]
This yields the third assertion, which moreover shows that the connection $\nabla$ is both an $L$-connection and a $G$-connection. Thus the intrinsic torsion of the $L$-structure and the intrinsic torsion of the $G$-structure are both given by the projection of $T$ to the corresponding intrinsic torsion module. This proves the final assertion. The stated formula for the expression of $T$ follows easily by (2.1).

We may apply Proposition 2.2 to $\text{SO}^*(2n)$- and $\text{SO}^*(2n)\text{Sp}(1)$-structures. In Table 1 we summarize the Lie groups $K, L$ provided by the defining tensors $F$ of $\text{SO}^*(2n)$- and $\text{SO}^*(2n)\text{Sp}(1)$-structures. However, let us postpone that, and initially illustrate Proposition 2.2 by a more characteristic example, related to metric connections.

**Example 2.3.** Let $K = \text{GL}(n, \mathbb{R})$ and $L = \text{O}(n)$, and let $g$ be a Riemannian metric on a $n$-dimensional manifold $M$. Then $G = K \cap L = \text{O}(n)$ and we denote by $\mathcal{O}(M) \to M$ the orthonormal frame bundle. A reductive complement of $\mathfrak{o}(n)$ in $\mathfrak{gl}$ coincides with the space of symmetric 2-tensors on $\mathbb{R}^n$, that is
\[ \mathfrak{gl}(n, \mathbb{R}) = \mathfrak{o}(n) \oplus \mathfrak{m} \cong \Lambda^2(\mathbb{R}^n)^* \oplus S^2(\mathbb{R}^n)^*, \quad \mathfrak{m} \cong S^2(\mathbb{R}^n)^*. \]

Let $\nabla^K$ be a torsion-free linear connection on $M$. Then, the covariant derivative $\nabla^K g$ takes values in $T^*M \otimes \mathfrak{m}_{\mathcal{O}(M)}$. For some $(1, 2)$-tensor field $A$ on $M$, it is easy to see that the linear connection $\nabla = \nabla^K + A$ is a metric connection if and only if
\[ (\nabla^K g)(Y, Z) = g(A_X Y, Z) + g(Y, A_X Z) = A(X, Y, Z) + A(X, Z, Y) \]
where we set $A(X, Y, Z) = g(A_X Y, Z)$ and $A_X Y = A(X, Y)$. Since the quantity in the left hand side should be symmetric in the last two indices, the above relation is equivalent to say that
\[ g(A(X, Y), Z) = \frac{1}{2}(\nabla^K g)(Y, Z), \quad \forall \ X, Y, Z \in \Gamma(TM). \]
In other words, for the splitting map $s : \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathfrak{m}_{\mathcal{O}(M)})$ we get that the image $-s(\nabla^K g)$ is the $(1, 2)$-tensor field $A$ defined by (2.2) and hence
\[ \nabla := \nabla^K - s(\nabla^K g) = \nabla^K + A \]
is a metric connection with respect to $g$, that is $\nabla g = 0$. If we want to obtain the Levi-Civita connection, i.e., $\nabla = \nabla^R$, we should choose a different complement $S^2T^*M \otimes TM$. This is because in this case $\nabla$ is also a torsion-free connection, so the difference $\nabla - \nabla^K$ should take values in $S^2T^*M \otimes TM$. In addition, the related isomorphism $s : \mathcal{E} \to S^2T^*M \otimes TM$ is encoded by the well-known Koszul formula.

| $F$ | $\omega$ | $\omega$ | $H$ | $h$ | $\Phi$ |
|-----|--------|--------|-----|-----|-------|
| $K$ | $\text{GL}(n, \mathbb{H})$ | $\text{GL}(n, \mathbb{H})\text{Sp}(1)$ | $\text{Sp}(4n, \mathbb{R})$ | $\text{GL}(4n, \mathbb{R})$ | $\text{GL}(4n, \mathbb{R})$ |
| $L$ | $\text{Sp}(4n, \mathbb{R})$ | $\text{Sp}(4n, \mathbb{R})$ | $\text{GL}(n, \mathbb{H})$ | $\text{SO}^*(2n)\text{Sp}(1)$ | $\text{SO}^*(2n)\text{Sp}(1)$ |
| $G$ | $\text{SO}^*(2n)$ | $\text{SO}^*(2n)\text{Sp}(1)$ | $\text{SO}^*(2n)$ | $\text{SO}^*(2n)\text{Sp}(1)$ | $\text{SO}^*(2n)\text{Sp}(1)$ |

**Table 1.** Examples of groups $K, L, G = K \cap L$ and the tensor $F$

**Remark 2.4.** Before we proceed with a detailed investigation of most of the cases presented in Table 1, let us emphasize on the fact that the connection $\nabla$ of Proposition 2.2 does not have to be a minimal connection. However, for particular cases (as for example the connection $\nabla^H$ introduced...
in [CGW21, Theorem 3.8]), one can derive minimality with respect to certain normalization conditions, as we do in [CGW21, Theorem 4.3]. On the other hand, in general the explicit descriptions of the isomorphism $\mathcal E \cong T^*M \otimes (\mathfrak f/\mathfrak g)_P$ and of the splitting map $s$ appearing in Proposition 2.2, are both highly non-trivial tasks.

2.2. The contribution of the Obata connection. Let us consider the following situation:

$$K := \text{GL}(n, \mathbb{H}), \quad L := \text{Sp}(4n, \mathbb{R}), \quad F := \omega, \quad \nabla K = \nabla^H,$$

where $H$ is an almost hypercomplex structure on a $4n$-dimensional manifold $M$ and $\omega$ is a scalar $2$-form on $M$ with respect to $H$. This means that $G = K \cap L = \text{SO}^*(2n)$ and we write $\pi : P \to M$ for the corresponding $\text{SO}^*(2n)$-structure on $M$. In this case we have $m \cong [S^2 E]^*$ and $\nabla^H \omega$ is a smooth section of the bundle $\mathcal E \cong T^*M \otimes \mathfrak m_P = T^*M \otimes ([S^2 E]^*)_P$.

For the splitting map $s : \Gamma(\mathcal E) \to \Gamma(T^*M \otimes \mathfrak m_P)$ we get that the image $-s(\nabla^H \omega)$ is the $(1, 2)$-tensor field $A$ introduced in [CGW21, Theorem 3.8], satisfying

$$(\nabla^H \omega)(Y, Z) = \frac{1}{2} (\nabla^H_X \omega)(Y, Z), \quad \forall X, Y, Z \in \Gamma(TM).$$

Consequently, we obtain that

$$\nabla = \nabla^H - s(\nabla^H \omega) = \nabla^H + A =: \nabla^H_\omega$$

is a $\text{SO}^*(2n)$-connection, and hence $\nabla^H_\omega F = \nabla^H_\omega \omega = 0$.

Note that when the scalar $2$-form $\omega$ is $\nabla^H$-parallel, i.e.,

$$\nabla^H_\omega(Y, Z) = 0, \quad \forall X, Y, Z \in \Gamma(TM),$$

then by the non-degeneracy of $\omega$ we obtain the vanishing of $A$, and hence $\nabla^H_\omega = \nabla^H$. We see that

**Lemma 2.5.** The scalar $2$-form $\omega$ is $\nabla^H$-parallel if and only if

$$\nabla^H_\omega(Y, Z) = (\nabla^H_\omega)(X, Z),$$

for any $X, Y, Z \in \Gamma(TM)$.

*Proof.* We prove only the converse direction, which is less trivial. Assume that (2.4) holds. This means that the tensor field $A$ takes values in the first prolongation $\mathfrak g(n, \mathbb{H})^{(1)}$, which is trivial. So the claims follows. \hfill \Box

Let us now examine 1st-order integrability conditions via $\nabla^H$. In these terms we have that

**Proposition 2.6.** The torsion of $\nabla^H_\omega$ is given by

$$T^H \omega(X, Y) = T^H(X, Y) + \delta(A)(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Thus, the adapted connection $\nabla^H_\omega$ is torsion-free if and only if $T^H$ satisfies

$$(2.5) \quad T^H = 0, \quad \text{and} \quad \nabla^H_\omega = 0.$$

In other words, $T^H_\omega = 0$ if and only if $H$ is 1-integrable and $\nabla^H_\omega = \nabla^H$.

*Proof.* The first statement is easy and implies that $T^H_\omega = 0$ if and only if

$$\omega(T^H(X, Y), Z) + \frac{1}{2} \left( (\nabla^H_X \omega)(Y, Z) - (\nabla^H_Y \omega)(X, Z) \right) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$. By Theorem 1.8 we have that $T^H \in \mathcal X_{126}$ and by [CGW21, Theorem 4.3] we conclude that $\delta(A) \in \mathcal X_{3457}$. Since they belong to different components, they should vanish simultaneously and in combination with Lemma 2.5 we derive the stated conditions in (2.5). In particular, note that $T^H_\omega = 0$, if and only if $\delta(A) = 0$ identically, i.e., $A$ is symmetric and
$T^H$ vanishes. But by Lemma 2.5 the first condition is equivalent to saying that $A = 0$, i.e., $\nabla^H = \nabla^H$.

Let us now examine the closedness of the scalar 2-form $\omega$.

**Proposition 2.7.** Let $(M, H = \{J_a : a = 1, 2, 3\}, \omega)$ be an almost hs-H manifold. Then, the differential of $\omega$ satisfies

$$
(2.6) \quad d\omega(X, Y, Z) = \pi_\omega(T^H, \omega)(X, Y, Z) := \mathfrak{S}_{X,Y,Z}\omega(T^H, \omega)(X, Y, Z),
$$

for any $X, Y, Z \in \Gamma(TM)$, and hence if $T^H, \omega = 0$ then $d\omega = 0$. More general, $d\omega = 0$ or equivalently $(M, H, \omega)$ is of type $X_{1567}$, if and only if

$$
\pi_\omega(T^H) = 0, \quad \text{and} \quad \mathfrak{S}_{X,Y,Z}(\nabla^H)(Y, Z) = 0,
$$

for any $X, Y, Z \in \Gamma(TM)$.

**Proof.** The linear connection $\nabla^H, \omega$ has torsion given by $T^H, \omega$ by definition, and consequently the differential of $\omega$ is given by

$$
\begin{align*}
\text{d}(\omega(Y, Z)) & = (\nabla^H_X\omega)(Y, Z) - (\nabla^H_Y\omega)(X, Z) + (\nabla^H_Z\omega)(X, Y) + \omega(T^H, \omega)(X, Y, Z) \\
& = \mathfrak{S}_{X,Y,Z}\omega(T^H, \omega)(X, Y, Z),
\end{align*}
$$

where the second equality follows since $\nabla^H, \omega$ preserves the pair $(H, \omega)$ (and so the first line must vanish). Finally, regarding the closedness of $\omega$, note that the second condition $\mathfrak{S}_{X,Y,Z}(\nabla^H)(Y, Z) = 0$ is equivalent to $\mathfrak{S}_{X,Y,Z}\delta(A)(X, Y, Z) = 0$, for any $X, Y, Z \in \Gamma(TM)$.

So, to summarize our conclusions about the scalar 2-form $\omega$, when the torsion of $\nabla^H, \omega$ vanishes, the almost symplectic structure induced by the scalar 2-form $\omega$ is closed and hence $(\omega, \nabla^H, \omega)$ is a Fedosov structure in the sense that $\nabla^H, \omega$ is a compatible torsion-free connection of the symplectic form $\omega$. However, we should mention that the Darboux coordinate frames do not sit in the corresponding reduction to $SO^*(2n)$, so there is a smooth map

$$
f : M \to Sp(4n, \mathbb{R})/SO^*(2n)
$$

describing this difference. Such an obstruction can be viewed as a smooth section of the quotient bundle $S(M)/SO^*(2n) \cong S(M) \times Sp(4n, \mathbb{R})/(Sp(4n, \mathbb{R})/SO^*(2n))$, where $S(M)$ denotes the symplectic line bundle, i.e., the principal $Sp(4n, \mathbb{R})$-bundle over $M$ consisting of all symplectic bases with respect to $\omega$.

### 2.3. The contribution of Oproiu connections

As a second application of Proposition 2.2 we consider the following situation:

$$
K := GL(n, \mathbb{H})Sp(1), \quad L := Sp(4n, \mathbb{R}), \quad F := \omega, \quad \nabla^K = \nabla^Q,
$$

where $Q$ is an almost quaternionic structure on a 4n-dimensional manifold $M$ and $\omega$ is a scalar 2-form on $M$ with respect to $Q$. We have $G = K \cap L = SO^*(2n)Sp(1)$ and denote by $\pi : Q \to M$ the corresponding $SO^*(2n)Sp(1)$-structure on $M$. Note that an Oproiu connection $\nabla^Q$ plays the role of the $K$-connection. By [CGW21] we know that the covariant derivative $\nabla^Q, \omega$ has values in

$$
[EH]^* \otimes (\omega_0) \oplus [S^2_0 E]^* \cong [EH]^* \otimes (\mathbb{R} \cdot Id \oplus \frac{\mathfrak{sl}(n, \mathbb{H})}{\mathfrak{so}^*(2n)}).
$$

This corresponds to $m = \mathbb{R} \cdot Id \oplus \frac{\mathfrak{sl}(n, \mathbb{H})}{\mathfrak{so}^*(2n)}$ and

$$
\mathcal{E} \cong T^*M \otimes m_Q.
$$
However, in Theorem \cite[Theorem 3.12]{CGW21}, for the map \( p : T^*M \otimes (\mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1))_Q \to \mathcal{E} \) we have chosen the following splitting
\[
s : \mathcal{E} \to (\text{Ker} \delta \oplus ([E H]^* \otimes [S^2 E]^*))_Q \subset T^*M \otimes (\mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1))_Q.
\]
This is the complement of \( T^*M \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))_Q \) in \( T^*M \otimes (\mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1))_Q \), and one should observe that this is a different complement than \( T^*M \otimes \text{m}_Q \). In particular, by the proof of \cite[Theorem 3.12]{CGW21} we deduce that \( s = s_1 + s_2 \), where \( s_1 \) takes values in \( \text{Ker} \delta \) and \( s_2 \) takes values in \([E H]^* \otimes [S^2 E]^*\), respectively. These are given by
\[
s_1(\nabla^Q\omega) = -\frac{\text{Tr}_2(A)}{4(n+1)} \otimes \text{id} + \pi_A(\omega \otimes \frac{\text{Tr}_2(A)T}{(n+1)}) - \pi_s(\omega \otimes \frac{\text{Tr}_2(A)T}{(n+1)}) + \sum_{a=1}^3 \frac{\text{Tr}_2(A)}{4(n+1)} \circ J_a \otimes J_a
\]
\[
s_2(\nabla^Q\omega) = -A + \frac{\text{Tr}_2(A)}{4(n+1)} \otimes \text{id} - \pi_A(\omega \otimes \frac{\text{Tr}_2(A)T}{(n+1)}),
\]
where \( A \) is a \((1, 2)\)-tensor field satisfying \( \omega(A(X, Y), Z) = \frac{1}{2}(\nabla^Q\omega)(Y, Z) \), for any \( X, Y, Z \in \Gamma(TM) \), \( \text{Tr}_2(A)(X) := \text{Tr}(A(X, \cdot)) \), and \( \pi_A \) is the projection defined by
\[
\pi_A(\omega \otimes Z)(X, Y) := \text{Asym} \left( \pi_{1,1}(\omega(X, \cdot) \otimes Z) \right) Y,
\]
see \cite[Section 3.2]{CGW21} for details. The reason for this decomposition is that
\[
\nabla^Q,\text{vol} = \nabla^Q - s_1(\nabla^Q\omega)
\]
is the unimodular Oproiu connection with respect to \( \text{vol} = \omega^{2n} \). Moreover, we see that the tensor field \( A^{\text{vol}} \) defined by the relation
\[
\omega(A^{\text{vol}}(X, Y), Z) = \frac{1}{2}(\nabla^Q,A^{\text{vol}})(Y, Z), \quad \forall X, Y, Z \in \Gamma(TM)
\]
satisfies
\[
A^{\text{vol}} = -s_2(\nabla^Q\omega) \quad \text{and} \quad \text{Tr}_2(A^{\text{vol}}) = 0.
\]
As a conclusion, we arrive to the formula
\[
\nabla^Q,\omega = \nabla^Q - s(\nabla^Q,\omega) = \nabla^Q - s_1(\nabla^Q\omega) - s_2(\nabla^Q\omega) = \nabla^Q,\text{vol} + A^{\text{vol}}
\]
for the \( \text{SO}^*(2n) \mathfrak{Sp}(1) \)-connection \( \nabla^Q,\omega \) from \cite[Theorem 3.12]{CGW21}. Let us now pose the following

\textbf{Lemma 2.8.} \textit{The condition} \( \delta(A^{\text{vol}}) = 0 \) \textit{is equivalent to}
\[
(\nabla^Q_X,\omega)(Y, Z) = \frac{\text{Tr}_2(A)(X)}{2(n+1)} \omega(Y, Z) - \omega(\pi_A(\omega(X, \cdot) \otimes \frac{2\text{Tr}_2(A)T}{(n+1)})Y, Z) \quad (i)
\]
\[
= \frac{1}{4(n+1)} \left( \text{Tr}_2(A)(X)\omega(Y, Z) + \sum_a \text{Tr}_2(A)(J_a X) g_{J_a}(Y, Z) - \omega(X, Y) \text{Tr}_2(A)(Z) \right.
\]
\[
- \sum_a g_{J_a}(X, Y) \text{Tr}_2(A)(J_a Z) \right) \quad (ii)
\]
\[
= \frac{\text{Tr}_2(A)(h(Y, Z)X - h(X, Y)Z)}{4(n+1)} \quad (iii)
\]
\textit{for all} \( X, Y, Z \in \Gamma(TM) \), \textit{where the tensor field} \( A \) \textit{is given by}
\[
(2.7) \quad \omega(A(X, Y), Z) = \frac{1}{2}(\nabla^Q_X,\omega)(Y, Z), \quad \forall X, Y, Z \in \Gamma(TM).
\]
\textit{In particular},
\[
(2.8) \quad \omega(A^{\text{vol}}(X, Y), Z) = \frac{1}{2}(\nabla^Q_X,\omega)(Y, Z) - \frac{\text{Tr}_2(A)(h(Y, Z)X - h(X, Y)Z)}{8(n+1)}
\]
for any $X,Y,Z \in \Gamma(TM)$.

**Proof.** Since $A^{\text{vol}}$ belongs to a complement of $\text{Ker}(\delta)$, by [CGW21, Theorem 4.3] we deduce that $\delta(A^{\text{vol}}) = 0$, if and only if $s_2(\nabla^Q \omega) = 0$. Thus, the first relation (i) follows by the definition given above for $s_2$ and $A$. By [CGW21, Section 3.2] we also deduce that

$$
\omega(\pi_A(\omega(X,\cdot)) \otimes \frac{2 \text{Tr}_2(A)^T}{(n+1)} Y, Z) = \omega\left( \frac{1}{8} \omega(X,Y) \frac{2 \text{Tr}_2(A)^T}{(n+1)} Y - \omega(X,Y) \frac{2 \text{Tr}_2(A)^T}{(n+1)} J_a Y, Z \right)
$$

$$
= \frac{1}{4(n+1)} \left( \omega(X,Y) \omega(\text{Tr}_2(A)^T, Z) - \omega(X,Y) \frac{2 \text{Tr}_2(A)^T}{(n+1)} J_a Y, Z \right)
$$

$$
= \frac{1}{4(n+1)} \left( \omega(X,Y) \frac{\text{Tr}_2(A)(Z) + \text{Tr}_2(A)(Y)}{2(n+1)} \omega(Y,Z) + \sum_a g_{Ja}(X,Y) \frac{\text{Tr}_2(A)(Ja Z) - \sum_a \text{Tr}_2(A)(Ja X) g_{Ja}(Y,Z)}{2(n+1)} \right).
$$

This formula is combined with the first term $\frac{\text{Tr}_2(A)(X)}{2(n+1)} \omega(Y,Z)$ and gives the second stated formula (ii). Finally, the last equality (iii) follows by the definition of the quaternionic skew-Hermitian form $h$, while the relation (2.8) is also a simple consequence of the above. $\square$

Due to this description we are now able to proceed by specifying 1st-order integrability conditions for $SO^*(2n) Sp(1)$-structures in terms of the connections $\nabla^Q$ and $\nabla^{Q,\text{vol}}$.

**Proposition 2.9.** The torsion of $\nabla^{Q,\omega}$ is given by

$$
T^{Q,\omega}(X,Y) = T^Q(X,Y) + \delta(A^{\text{vol}})(X,Y), \quad \forall X,Y \in \Gamma(TM),
$$

where

$$
\omega(\delta(A^{\text{vol}})(X,Y), Z) = \frac{1}{2} (\nabla^Q_X \omega)(Y,Z) - \frac{1}{2} (\nabla^Q_Y \omega)(X,Z)
$$

$$
+ \frac{\text{Tr}_2(A)(h(X,Z)Y - h(Y,Z)X + 2\omega(X,Y)Z)}{8(n+1)}
$$

with $A$ defined by (2.7). Thus, the adapted connection $\nabla^{Q,\omega}$ is torsion-free if and only if

$$
(2.9) \quad T^Q = 0, \quad \text{and} \quad (\nabla^Q_X \omega)(Y,Z) = \frac{\text{Tr}_2(A)(h(Y,Z)X - h(X,Y)Z)}{4(n+1)}.
$$

In the case that $\nabla^Q = \nabla^{Q,\text{vol}}$ the second condition in (2.5) is simplified to $(\nabla^Q_X \omega)(Y,Z) = 0$.

**Proof.** Since all of the Oproiu connections have the same torsion, i.e., $T^Q = T^{Q,\text{vol}}$ (see e.g. [AM96]), it remains for us to compute $\delta(A^{\text{vol}})$. In particular, the first relation is an immediate consequence of the relation $T^Q = T^{Q,\text{vol}}$ and the definition of $\nabla^{Q,\omega}$. For the explicit formula of $\delta(A^{\text{vol}})$ we apply Lemma 2.8, which means that we obtain the stated formula by skew-symmetrization of (2.8) in the first two indices, and using the definition of $h$. In addition, the stated integrability conditions for the vanishing of $T^{Q,\omega}$ also occur as a direct consequence of Lemma 2.8. This is because $T^Q \in X_{12}$ and $\delta(A^{\text{vol}}) \in X_{345}$, so these components vanish simultaneously. The last claim follows by the vanishing of $\text{Tr}_2(A^{\text{vol}}).$ $\square$
As for the differential of the scalar 2-form $\omega$, we have in mind a similar method as the one presented for $SO^*(2n)$-structures, and get the following.

**Proposition 2.10.** Let $(M,Q,\omega)$ be an almost qs-$H$ manifold. Then, the differential of the scalar 2-form $\omega$ satisfies

$$d\omega(X,Y,Z) = \pi_\omega(T^Q\omega)(X,Y,Z) = \mathcal{G}_{X,Y,Z}\omega(T^Q\omega(X,Y),Z),$$

for any $X,Y,Z \in \Gamma(TM)$, and hence if $T^Q\omega = 0$ then $d\omega = 0$. More generally, $d\omega = 0$ or equivalently $(M,Q,\omega)$ is of type $A_{15}$ if and only if

$$\pi_\omega(T^Q) = 0, \quad \text{and} \quad \mathcal{G}_{X,Y,Z}(\nabla_X \omega)(Y,Z) = \mathcal{G}_{X,Y,Z}\frac{Tr_2(A)(h(Y,Z)X - h(X,Y)Z)}{4(n+1)},$$

for any $X,Y,Z \in \Gamma(TM)$.

### 2.4. The use of the fundamental symmetric 4-tensor.

Let us now discuss the case

$$K := \text{GL}(4n,\mathbb{R}), \quad L := SO^*(2n)\text{Sp}(1), \quad F := \Phi, \quad \nabla^K = \nabla,$$

where $\nabla$ is any affine connection on $(M,Q,\omega)$. We have $G = K \cap L = SO^*(2n)\text{Sp}(1)$ and $\Phi \in \Gamma(S^4T^*M)$ is the fundamental 4-tensor of the corresponding $G$-structure, which we will denote again by $\pi : Q \rightarrow M$. We will work in an algebraic setting by using the 4-tensor $\Phi_0$, and we will identify the Lie algebra $\mathfrak{k} = \text{gl}(4n,\mathbb{R})$ of $K$ with the $G$-module of endomorphisms of $[E\ H]$, i.e.,

$$\mathfrak{k} = \text{gl}(4n,\mathbb{R}) \cong \text{End}([E\ H]).$$

Recall that $\text{End}([E\ H])$ admits the following $SO^*(2n)\text{Sp}(1)$-equivariant decomposition (see [CGW21, Section 1.3])

$$\text{End}([E\ H]) \cong \mathbb{R} \cdot \text{id} \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n) \oplus \frac{\mathfrak{sl}(n,\mathbb{H})}{\mathfrak{so}^*(2n)} \oplus \frac{\mathfrak{sp}(\omega_0)}{\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)} \oplus [\Lambda^2 E S^2 H]^*.$$

Here $\mathbb{R} \cdot \text{id} \cong \langle \omega_0 \rangle$ and

$$\mathfrak{sp}(1) \cong [S^2 H]^*, \quad \mathfrak{so}^*(2n) \cong [\Lambda^2 E]^*, \quad \frac{\mathfrak{sl}(n,\mathbb{H})}{\mathfrak{so}^*(2n)} \cong [S^2_0 E]^*, \quad \frac{\mathfrak{sp}(\omega_0)}{\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)} \cong [S^2_0 E S^2 H]^*.$$

By the above decomposition, we deduce that

$$\mathfrak{k} = \text{gl}(4n,\mathbb{R}) = (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)) \oplus \mathfrak{m} = \mathfrak{g} \oplus \mathfrak{m},$$

is a reductive decomposition of $\mathfrak{k}$, where the reductive complement $\mathfrak{m}$ is given by

$$\mathfrak{m} := \left(\mathbb{R} \cdot \text{id} \oplus \frac{\mathfrak{sl}(n,\mathbb{H})}{\mathfrak{so}^*(2n)} \oplus \frac{\mathfrak{sp}(\omega_0)}{\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)} \oplus [\Lambda^2 E S^2 H]^*\right).$$

We know by the first part that $\Phi_0$ is stabilized by $L = G$, hence according to Proposition 2.2 the covariant derivative $\nabla\Phi$ takes values in the associated bundle $\mathcal{E}$ with fiber isomorphic to

$$[E\ H]^* \otimes \mathfrak{m} =: \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4,$$

where we set

$$\mathcal{W}_1 := [E\ H]^* \otimes \text{id},$$

$$\mathcal{W}_2 := [E\ H]^* \otimes \frac{\mathfrak{sl}(n,\mathbb{H})}{\mathfrak{so}^*(2n)} \cong [E\ H]^* \otimes [S^2_0 E]^*,$$

$$\mathcal{W}_3 := [E\ H]^* \otimes \frac{\mathfrak{sp}(\omega_0)}{\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)} \cong [E\ H]^* \otimes [S^2_0 E S^2 H]^*,$$

$$\mathcal{W}_4 := [E\ H]^* \otimes [\Lambda^2 E S^2 H]^*. $$
respectively. Note that $\mathcal{W}_2$, $\mathcal{W}_3$ and $\mathcal{W}_4$ are reducible as $G$-modules, and below one of our goals is to provide their expressions into irreducible submodules and relate them with the algebraic types $\lambda_{i_1 \ldots i_j}$ for $1 \leq i_1 < \ldots < i_j \leq 5$. Moreover, we will explicitly specify the isomorphism

$$s : \mathcal{E} \to (\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4)_Q.$$ 

We begin with the following lemma.

**Proposition 2.11.** The projections $w_a : [E]^* \otimes \text{End}([E]) \to \mathcal{W}_a$ for $a = 1, 2, 3, 4$, given below, are equivariant and independent of the choice of an admissible basis $\{ J_a, a = 1, 2, 3 \}$ for the standard quaternionic structure $Q_0$ on $[E]$:

1) $w_1 : [E]^* \otimes \text{End}([E]) \to \mathcal{W}_1$ defined by

$$w_1(A)(X, Y) := \frac{\text{Tr}(A_X)}{4n} Y.$$ 

In particular, the elements of $\mathcal{W}_1$ take form $\alpha_1 = \xi \otimes \text{id}$ for some $\xi \in [E]^*$

2) $w_2 : [E]^* \otimes \text{End}([E]) \to \mathcal{W}_2$ defined by

$$w_2(A)(X, Y) := \pi_A(A_X) Y - \frac{\text{Tr}(A_X)}{4n} Y.$$ 

In particular, the pure elements $\alpha_2 \in \mathcal{W}_2$ are given by $\omega_0(\alpha_2, \cdot) = \xi \otimes \hat{\omega}$ for some $\xi \in [E]^*$ and a scalar 2-form $\hat{\omega} \in [S_0^2 E]^*$, i.e., $\text{Tr}_2(\alpha_2) = 0$.

3) $w_3 : [E]^* \otimes \text{End}([E]) \to \mathcal{W}_3$ defined by

$$w_3(A)(X, Y) := \text{Sym}(A_X - \pi_{1,1}(A_X)) Y + \sum_a \frac{\text{Tr}(A_X \circ J_a)}{4n} J_a Y.$$ 

In particular, the pure elements $\alpha_3 \in \mathcal{W}_3$ are given by $\omega_0(\alpha_3, \cdot) = \xi \otimes \hat{\omega}$ for some $\xi \in [E]^*$, $J_b \in \{ J_a, a = 1, 2, 3 \}$ and a scalar 2-form $\hat{\omega} \in [S_0^2 E]^*$, i.e., $\hat{\omega}(\cdot, J_b) = \hat{\omega}(\cdot, J_b) \in [S_0^2 E]^* \otimes \mathfrak{sp}(1)$ and $\text{Tr}_2(\alpha \circ J_b) = 0$.

4) $w_4 : [E]^* \otimes \text{End}([E]) \to \mathcal{W}_4$ defined by

$$w_4(A)(X, Y) := \text{Asym}(A_X - \pi_{1,1}(A_X)) Y.$$ 

In particular, the pure elements $\alpha_4 \in \mathcal{W}_4$ are given by $\omega_0(\alpha_4, \cdot) = \xi \otimes \rho(\cdot, J_b)$ for some $\xi \in [E]^*$, $J_b \in \{ J_a, a = 1, 2, 3 \}$ and quaternionic-Hermitian Euclidean metric $\rho \in [\Lambda^2 E]^*$.

**Proof.** The claimed form for the pure elements $\alpha_i \in \mathcal{W}_i$ ($i = 1, \ldots, 4$) follows by Proposition 1.7 and Remark 1.8 in [CGW21]. By [AM96, Section 1.4], we have

$$\text{Im}(\pi_{1,1}) = \mathfrak{gl}(n, \mathbb{H}) \cap \mathfrak{Ker}(\pi_{1,1}) = \frac{\mathfrak{sp}(\omega_0)}{\mathfrak{so}^*(2n)} \oplus [\Lambda^2 E S^2 H]^* \cong \mathfrak{sp}(1) \oplus [S_0^2 E S^2 H]^* \oplus [\Lambda^2 E S^2 H]^*,$$

and also the explicit form of the projections to the trace components $\mathbb{R} \cdot \text{id}$ and $\mathfrak{sp}(1)$, that is

$$\mathfrak{gl}(4n, \mathbb{R}) \ni A \mapsto \frac{\text{Tr}(A)}{4n} \mathbb{R} \mathbb{I} \mathfrak{I} \mathfrak{d}, \quad \mathfrak{gl}(4n, \mathbb{R}) \ni A \mapsto -\sum_a \frac{\text{Tr}(A \circ J_a)}{4n} J_a \in \mathfrak{sp}(1).$$

Thus, what remains is the application of the anti/symmetrization operators $\text{Asym}$ and $\text{Sym}$, which provide the correct projections according to the $SO^*(2n) \mathfrak{sp}(1)$-decompositions given in [CGW21, Proposition 1.7]. For instance, consider the case of $w_3 : [E]^* \otimes \text{End}([E]) \to \mathcal{W}_3 = [E]^* \otimes [S_0^2 E S^2 H]^*$. Let us recall the following $SO^*(2n) \mathfrak{sp}(1)$-equivariant decomposition from Part I:

$$S^2 [E]^* \cong [\Lambda^2 E]^* \oplus [S_0^2 E S^2 H]^* \oplus \mathfrak{sp}(1).$$

This means that, for an element $A_X - \pi_{1,1}(A_X)$ in the kernel of $\pi_{1,1}$, the symmetrization must belong to $[S_0^2 E S^2 H]^* \oplus \mathfrak{sp}(1)$. Thus, by subtracting the trace component $\mathfrak{sp}(1)$ we obtain the stated formula for $w_3$. The rest of the cases are treated similarly. \[\square\]
Let us now describe how the pure elements $\alpha_i \in W_i$ from Lemma 2.11 act on the fundamental 4-tensor $\Phi_0$:

**Lemma 2.12.** • *For $\alpha_1 \in W_1$ given by $\alpha_1 = \xi \otimes \text{id}$, we have*

$$\alpha_1 \cdot \Phi_0 = -4\xi \otimes \Phi_0.$$

• *For $\alpha_2 \in W_2$ given by $\omega_0(\alpha_2, \cdot) = \xi \otimes \tilde{\omega}$, we have*

$$\alpha_2 \cdot \Phi_0 = -4\xi \otimes \sum_{a=1}^{3} \tilde{g}_J a \otimes g_J a.$$

• *For $\alpha_3 \in W_3$ given by $\omega_0(\alpha_3, \cdot) = \xi \otimes \hat{g}_J b$, we have*

$$\alpha_3 \cdot \Phi_0 = -4\xi \otimes (\hat{g}_J b \otimes g_J a).$$

• *For $\alpha_4 \in W_4$ given by $\omega_0(\alpha_4, \cdot) = \xi \otimes \rho(\cdot, J_b)$, we have*

$$\alpha_4 \cdot \Phi_0 = 4\xi \otimes (\rho \otimes g_J b).$$

*Proof.* The last claim is known from [CGW21, Section 1.3], hence we only need to prove the rest of the cases. For the first one, and for any $x, y, z, w, u \in [EH]$ we obtain that

$$\alpha_1 \cdot \Phi_0(x, y, z, w)(u) = -\Phi_0(\alpha_1 x, y, z, w)(u) - \Phi_0(x, \alpha_1 y, z, w)(u) - \Phi_0(x, y, \alpha_1 z, w)(u)$$

$$= -\Phi_0(\frac{1}{4n} \text{Tr}_2(\alpha_1)(u)x, y, z, w) - \Phi_0(x, \frac{1}{4n} \text{Tr}_2(\alpha_1)(u)y, z, w)$$

$$= \frac{1}{n} \text{Tr}_2(\alpha_1)(u)\Phi_0(x, y, z, w).$$

Similarly, for any $x, y, z, w, u \in [EH]$ we get

$$(\alpha_2 \cdot g_J a)(x, y)(u) = -g_J a(\alpha_2 x, y)(u) - g_J a(\alpha_2 y)(u) = -\omega_0(\alpha_2 x, J_a y)(u) - \omega_0(\alpha_2 y, J_a x)(u)$$

$$= -\xi(u)\tilde{\omega}(x, J_a y) - \xi(u)\tilde{\omega}(y, J_a x)$$

$$= -2\xi(u)\hat{g}_J a(x, y),$$

for any $a = 1, 2, 3$. But then the definition of $\Phi_0$ provides the claimed formula. Let us now check the third case. For $a \neq b$ and for any $x, y, z, w, u \in [EH]$ we get

$$(\alpha_3 \cdot g_J a)(x, y)(u) = -g_J a(\alpha_3 x, y)(u) - g_J a(\alpha_3 y)(u) = -\omega_0(\alpha_3 x, J_a y)(u) - \omega_0(\alpha_3 y, J_a x)(u)$$

$$= -\xi(u)\hat{g}_J a(x, J_a y) - \xi(u)\hat{g}_J a(y, J_a x)$$

$$= -2\xi(u)\hat{g}_J a(x, y).$$

Finally, for $a = b$ and any $x, y, z, w, u \in [EH]$ we see that

$$(\alpha_3 \cdot g_J b)(x, y)(u) = -g_J b(\alpha_3 x, y)(u) - g_J b(\alpha_3 y)(u) = -\omega_0(\alpha_3 x, J_b y)(u) - \omega_0(\alpha_3 y, J_b x)(u)$$

$$= -\xi(u)\hat{g}_J b(x, J_b y) - \xi(u)\hat{g}_J b(y, J_b x)$$

$$= 0.$$

Thus, again our assertion follows by the definition of $\Phi_0$. \qed
In the next step we will recover the elements $\alpha_i \in \mathcal{W}_i$ from their action on $\Phi_0$. With this goal in mind it is useful to review how to produce a $(1,2)$-tensor by a $(0,5)$-tensor. For such a procedure we need to consider the inverse of the scalar 2-form $\omega$, and of the induced (local) metrics $g_{Ja}, a = 1, 2, 3$. Since there are two possible conventions how to define the inverse of an almost symplectic form, for the convenience of the reader we make this precise.

**Definition 2.13.** Let $e_1, \ldots, e_{2n}, f_1, \ldots, f_{2n}$ be a skew-Hermitian basis of $[E H]$. The inverse $\omega_0^{-1}$ of the standard scalar 2-form $\omega_0$ on $[E H]$ is expressed by

$$\omega_0^{-1} := \sum_{c=1}^{2n} e_c \otimes f_c - f_c \otimes e_c \in [S^2 E],$$

Moreover, for a $(0,2)$ tensor $A$, we define the following contractions

$$A(\omega_0^{-1}, \cdot) := \sum_{c=1}^{2n} A(e_c, \cdot) f_c - A(f_c, \cdot) e_c \in \text{End}([E H]),$$

$$A(\cdot, \omega_0^{-1}) := \sum_{c=1}^{2n} A(\cdot, f_c) e_c - A(\cdot, e_c) f_c \in \text{End}([E H]),$$

$$A(\omega_0^{-1}) := \sum_{c=1}^{2n} A(e_c, f_c) e_c - A(f_c, e_c) \in \mathbb{R}.$$

It is not hard to show that the inverse $\omega_0^{-1}$ and the above contractions are all independent of the choice of a skew-Hermitian basis. Observe also that

$$\omega_0(\cdot, \omega_0^{-1}) = \omega_0(\omega_0^{-1}, \cdot) = \text{id},$$

$$g_{Ja}(\cdot, \omega_0^{-1}) = -g_{Ja}(\omega_0^{-1}, \cdot) = -J_a,$$

$$\omega_0(\cdot, g_{Ja}^{-1}) = -\omega_0(g_{Ja}^{-1}, \cdot) = \left\{ \begin{array}{ll} J_a J_b, & a \neq b, \\ \text{id}, & a = b, \end{array} \right.$$ and

$$A(\omega_0^{-1}) = \begin{cases} 0, & A \in [S^2 H]^* \oplus [A^2 E]^* \oplus [S_0^2 E]^* \oplus [S_0^2 E S^2 H]^* \oplus [A^2 E S^2 H]^*, \\ 4n, & A = \omega_0. \end{cases}$$

$$A(g_{Ja}^{-1}) = \begin{cases} 0, & \langle \omega_0 \rangle \oplus [A^2 E]^* \oplus [S_0^2 E]^* \oplus [S_0^2 E S^2 H]^* \oplus [A^2 E S^2 H]^*, \\ 0, & A = g_{Ja}, a \neq b, \\ 4n, & A = g_{Ja}. \end{cases}$$

Based on Lemma 2.12 we can prove the following

**Proposition 2.14.** Suppose that the $(3,1)$-tensor $\hat{h}_0$ is given by

$$\hat{h}_0 := \text{id} \otimes \omega_0^{-1} + \sum_a J_a \otimes g_{Ja}^{-1},$$

for some admissible basis $H = \{J_a, a = 1, 2, 3\}$ of the standard quaternionic structure $Q_0$ on $[E H]$. Then, $\hat{h}_0$ does not depend on the choice of $H$. Moreover, the linear map

$$c : [E H]^* \otimes S^4[E H]^* \to \bigotimes^3 [E H]^*, \quad A \mapsto \sum_a A(x, J_a y, z, g_{Ja}^{-1}), \quad \forall x, y, z \in [E H],$$

...
defines a natural contraction of \( \hat{h}_0 \) with tensors of type \((0,5)\). In particular,

\[
\begin{align*}
    c(\alpha_1 \cdot \Phi_0)(x,y,z) &= -8(2n-1)\omega_0(\alpha_1(x,y),z), \\
    c(\alpha_2 \cdot \Phi_0)(x,y,z) &= -8(n-1)\omega_0(\alpha_2(x,y),z), \\
    c(\alpha_3 \cdot \Phi_0)(x,y,z) &= \frac{16n-3}{3}\omega_0(\alpha_3(x,y),z), \\
    c(\alpha_4 \cdot \Phi_0)(x,y,z) &= -\frac{8(n+1)}{3}\omega_0(\alpha_4(x,y),z).
\end{align*}
\]

Proof. The general formula for the natural contraction can be easily observed, when we express the application of the natural contraction of \( \hat{h}_0 \) with the \((0,5)\)-tensor \( \xi \otimes \omega_0 \otimes \omega_0 \) by

\[
\xi \otimes (\omega_0(\id x,y) \otimes \omega_0(z,\omega_0^{-1}) + \sum_a \omega_0(J_a x,y) \otimes g_{J_a}(z,\omega_0^{-1})),
\]

for some \( \xi \in [EH]^* \). In particular, by using this formula we can verify that

\[
\omega_0(\id x,y) \otimes \omega_0(z,\omega_0^{-1}) + \sum_a \omega_0(J_a x,y) \otimes g_{J_a}(z,\omega_0^{-1}) = \omega_0(x,y) \otimes \id + \sum_a g_{J_a}(x,y) \otimes J_0 = h_0,
\]

which proves our first assertion.

Now, for \( A \in [EH]^* \otimes S^4[EH]^* \) we can conclude that \( A(x,y,y,\omega_0^{-1}) = 0 \) for all \( x,y,z \in [EH]^* \) and the claimed formula for \( c \) easily follows. In order to prove the rest of the presented formulas, we will compute \( A(x,J_a y,z,g_{J_a}^{-1}) \) for some \( A = \xi \otimes (B \circ g_{J_a}) \in [EH]^* \otimes S^4[EH]^* \), where \( B \in S^2[EH]^* \).

First, we see that

\[
\begin{align*}
    (\xi \otimes (B \circ g_{J_a}))(x,y,z,u,v) &= \frac{\xi(x)}{6}(B(y,z)g_{J_a}(u,v) + B(y,u)g_{J_a}(z,v) + B(y,v)g_{J_a}(z,u) \\
    &+ B(z,u)g_{J_a}(y,v) + B(z,v)g_{J_a}(y,u) + B(u,v)g_{J_a}(y,z)),
\end{align*}
\]

for any \( x,y,z,u,v \in [EH] \). Thus, by setting \( Z := (\xi \otimes (B \circ g_{J_a}))(x,J_a y,z,g_{J_a}^{-1}) \) we see that

\[
Z = \frac{\xi(x)}{6}(B(J_a y,z)g_{J_a}(g_{J_a}^{-1}) + 2B(J_a y,g_{J_a}(z,g_{J_a}^{-1}) \\
    + 2B(z,g_{J_a}(J_a y,g_{J_a}^{-1}) + B(g_{J_a}^{-1})g_{J_a}(J_a y,z))
\]

\[
= \begin{cases} 
    \frac{\xi(x)}{6}(-B(g_{J_a}^{-1})g_{J_a}(y,z) - 2B(J_a y,J_a g_{J_a}z) - 2B(z,J_a y)), & a \neq b, \\
    \frac{\xi(x)}{6}(4nB(J_a y,z) + 2B(J_a y,z) + 2B(J_a y,J_a y) + B(g_{J_a}^{-1})w_0(y,z)), & a = b.
\end{cases}
\]

Let us initially consider the case \( B = g_{J_a} \). Then we obtain the following:

\[
(\xi \otimes (B \circ g_{J_a}))(x,J_a y,z,g_{J_a}^{-1}) = \begin{cases} 
    \frac{\xi(x)}{3}(-2\omega_0(y,z)), & a \neq b, \\
    \frac{\xi(x)}{3}(4n + 2)\omega_0(y,z), & a = b.
\end{cases}
\]

Therefore, having in mind the formulas from Lemma 2.12 we result with

\[
\begin{align*}
    c(\alpha_1 \cdot \Phi_0)(x,y,z) &= -4 \sum_{a,b=1}^3 \xi(x) \otimes (g_{J_b} \circ g_{J_a})(J_a y,z,g_{J_a}^{-1}) \\
    &= -4(4n + 2 - 4)\xi(x)\omega_0(y,z) = -8(2n - 1)\omega_0(\alpha_1(x,y),z).
\end{align*}
\]

Assume now that \( B = \hat{g}_{J_a} \). For this case we compute

\[
(\xi \otimes (B \circ \hat{g}_{J_a}))(x,J_a y,z,g_{J_a}^{-1}) = \begin{cases} 
    -\frac{\xi(x)}{3}2\hat{\omega}(y,z), & a \neq b, \\
    \frac{\xi(x)}{3}(2n + 2)\hat{\omega}(y,z), & a = b
\end{cases}
\]
and consequently, in combination with Lemma 2.12 we conclude that
\[ c(\alpha_2 \cdot \Phi_0)(x, y, z) = -4\xi(x) \otimes \sum_{a,b=1}^{3} (\hat{g}_{J_b} \circ g_{J_a})(J_ay, z, g_{J_a}^{-1}) \]
\[ = -4\xi(x)(2n + 2 - 4)\hat{\omega}(y, z) = -8(n - 1)\omega_0(\alpha_2(x, y), z). \]

We should also consider the case \( B = \hat{g}_{J_c}, c \neq b. \) Then we obtain
\[ (\xi \otimes (B \circ \hat{g}_{J_b}))(x, J_ay, z, g_{J_a}^{-1}) = \begin{cases} 0, & c \neq a \neq b, \\ \frac{\xi(x)}{3}(-2)\hat{g}_{J_b}J_c(y, z), & c = a \neq b, \\ \frac{\xi(x)}{3}(-2n - 2)\hat{g}_{J_b}J_c(y, z), & a = b, \end{cases} \]
and therefore, for \( J_b = J_1 \) we compute
\[ c(\alpha_3 \cdot \Phi_0)(x, y, z) = -4\xi(x) \otimes \sum_{a,c=1,c\neq 1}^{3} (\hat{g}_{J_c} \circ g_{J_a})(J_ay, z, g_{J_a}^{-1}) \]
\[ = -4\xi(x) \otimes \sum_{a}^{3} (\hat{g}_{J_b} \circ g_{J_2} - \hat{g}_{J_2} \circ g_{J_2})(J_ay, z, g_{J_a}^{-1}) \]
\[ = 0 - 4\xi(x)\frac{3}{3}(-2n - 2)\hat{g}_{J_2}J_3(y, z) - 4\xi(x)\frac{3}{3}(-2)\hat{g}_{J_2}J_2(y, z) \]
\[ -0 + 4\xi(x)\frac{3}{3}(-2n - 2)\hat{g}_{J_2}J_2(y, z) \]
\[ = \frac{\xi(x)}{3}(8n + 8 - 8 - 8n + 8)\hat{g}_{J_1}(y, z) = \frac{16n}{3}\omega_0(\alpha_3(x, y), z). \]

In fact, the same holds for \( J_b = J_2, J_b = J_3. \) Finally, let us we consider the case \( B = \rho. \) Then we get
\[ (\xi \otimes (B \circ \hat{g}_{J_b}))(x, J_ay, z, g_{J_a}^{-1}) = \begin{cases} 0, & a \neq b, \\ \frac{\xi(x)}{3}(-2n + 2)\rho(\gamma, J_ay), & a = b, \end{cases} \]
and a direct computation proves our claim, i.e.,
\[ c(\alpha_4 \cdot \Phi_0)(x, y, z) = 4\xi(x) \otimes \sum_{a}^{3} (\rho \circ g_{J_a})(J_ay, z, g_{J_a}^{-1}) \]
\[ = 4\xi(x)\frac{3}{3}(-2n - 2)\rho(\gamma, J_ay) = -\frac{8(n + 1)}{3}\omega_0(\alpha_4(x, y), z). \]

Let us now explain how the above results become the key ingredients for an explicit construction of the map
\[ s: \mathcal{E} \rightarrow (W_1 \oplus W_2 \oplus W_3 \oplus W_4) \sigma. \]

**Theorem 2.15.** The map
\[ s(\nabla \Phi) := -\left(\frac{w_1}{8(2n - 1)} + \frac{w_2}{8(n - 1)} - \frac{3w_3}{16n} + \frac{3w_4}{8(n + 1)}\right)(c(\nabla \Phi)) \]
satisfies \( \mathcal{P} \circ s = \mathcal{L}_s. \) In particular,
\[ \nabla + \left(\frac{w_1}{8(2n - 1)} + \frac{w_2}{8(n - 1)} - \frac{3w_3}{16n} + \frac{3w_4}{8(n + 1)}\right)(c(\nabla \Phi)) \]
is a \( SO^*(2n)Sp(1) \)-connection.
Proposition 2.16. The modules $\mathcal{W}_i$ ($i = 1, 2, 3, 4$) admit the following $SO^*(2n)\ Sp(1)$-equivariant decompositions:

\[ \mathcal{W}_1 \cong [E\ H]^* , \quad \mathcal{W}_2 \cong [K\ H]^* \oplus [E\ H]^* \oplus [S_0^3\ E\ H]^* , \]

\[ \mathcal{W}_3 \cong [K\ S^3\ H]^* \oplus [E\ S^3\ H]^* \oplus [S_0^3\ E\ S^3\ H]^* \oplus [K\ H]^* \oplus [E\ H]^* \oplus [S_0^3\ E\ H]^* , \]

\[ \mathcal{W}_4 \cong [K\ S^3\ H]^* \oplus [E\ S^3\ H]^* \oplus [\Lambda^3\ E\ S^3\ H]^* \oplus [K\ H]^* \oplus [E\ H]^* \oplus [\Lambda^3\ E\ H]^* . \]

Moreover,

\[ \bigoplus_a \mathcal{W}_a \cap \ker \delta \cong [K\ S^3\ H]^* \oplus [E\ S^3\ H]^* \oplus [S_0^3\ E\ S^3\ H]^* \oplus [K\ H]^* \oplus [E\ H]^* \oplus [S_0^3\ E\ H]^* , \]

\[ \delta(\bigoplus_a \mathcal{W}_a) \cap \delta([E\ H]^* \otimes (so^*(2n) \oplus \mathfrak{sp}(1))) \cong [E\ S^3\ H]^* \oplus [K\ H]^* \oplus 2[E\ H]^* \oplus [\Lambda^3\ E\ H]^* , \]

and consequently

\[ \delta(\bigoplus_a \mathcal{W}_a)/\delta([E\ H]^* \otimes (so^*(2n) \oplus \mathfrak{sp}(1))) \cong \mathcal{X}_{12345} . \]

Proof. By [CGW21, Theorem 4.3] we know the decomposition of $\mathcal{W}_1$ and $\mathcal{W}_2$. Since

\[ \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \cong \Lambda^2[E\ H]^* \otimes [E\ H] \]

we obtain the decomposition of $\mathcal{W}_4$ by [CGW21, Proposition 3.14]. Note that if one replaces the antisymmetric tensors with trace-free symmetric tensors, then the decomposition of $\mathcal{W}_3$ is analogous to the decomposition of $\mathcal{W}_4$. Hence, by [CGW21, Proposition 3.14] and by the triviality of $\ker \left( \delta \mid_{[E\ H]^* \otimes (so^*(2n) \oplus \mathfrak{sp}(1))} \right)$ we obtain

\[ S^2[E\ H]^* \otimes [E\ H] \cong \mathcal{W}_3 \oplus \left( [E\ H]^* \otimes (so^*(2n) \oplus \mathfrak{sp}(1)) \right) \]

\[ \cong \mathcal{W}_3 \oplus \delta\left( [E\ H]^* \otimes (so^*(2n) \oplus \mathfrak{sp}(1)) \right) \]

\[ \cong [K\ S^3\ H]^* \oplus 2[E\ S^3\ H]^* \oplus [S_0^3\ E\ S^3\ H]^* \oplus 2[K\ H]^* \oplus 3[E\ H]^* \oplus [S_0^3\ E\ H]^* \oplus [\Lambda^3\ E\ H]^* \]

and thus both of the claimed intersections follow. These also imply the final assertion. □

Let us emphasize on the fact that although $\bigoplus_a \mathcal{W}_a \cap \ker \delta \cong \mathcal{W}_3$, we have only $\ker \delta \cap \mathcal{W}_3 = [S_0^3\ E\ S^3\ H]^*$ for the intersection. Nevertheless, we know that

\[ \ker \delta \cap \left( \mathcal{W}_3 \oplus \left( [E\ H]^* \otimes (so^*(2n) \oplus \mathfrak{sp}(1)) \right) \right) = S^3[E\ H]^* \cong [E\ S^3\ H]^* \oplus [K\ H]^* \oplus [S_0^3\ E\ S^3\ H]^* \oplus [E\ H]^* , \]
which is the well-known first prolongation of almost symplectic structures. Similarly,
\[ \text{Ker} \delta \cap \left( \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \left( [\mathbb{E} H]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)) \right) \right) = [\mathbb{E} H]^*, \]
which is the well-known first prolongation of almost quaternionic structures. This was described in our terms in [CGW21, Lemma 3.11] (see also [Sa86, AM96]).

Having in mind that torsion-free connections are minimal \( \text{GL}(4n, \mathbb{R}) \)-connections, based on the Proposition 2.16 we result to the following characterization of the algebraic types of the corresponding \( \text{SO}^*(2n) \)- and \( \text{SO}^*(2n) \text{Sp}(1) \)-geometries.

**Proposition 2.17.** For a torsion-free connection \( \nabla \), the following holds:

- \( \mathcal{X}_1 \) is the image of the components \( 2[\mathbb{K} S^3 H]^* \subset \mathcal{W}_3 \oplus \mathcal{W}_4 \) by \( \delta \), and vanishes when the component of \( s(\nabla \Phi) \) in these two modules belongs to the kernel \( \text{Ker}(\delta) \).
- \( \mathcal{X}_2 \) is the image of the component \( [\Lambda^3 \mathbb{E} S^3 H]^* \subset \mathcal{W}_4 \) by \( \delta \), and vanishes when this component of \( s(\nabla \Phi) \) vanishes.
- \( \mathcal{X}_3 \) is the image of the components \( 2[\mathbb{K} H]^* \subset \mathcal{W}_2 \oplus \mathcal{W}_4 \) by \( \delta \), and vanishes when the component of \( s(\nabla \Phi) \) in these two modules belongs to the kernel \( \text{Ker}(\delta) \).
- \( \mathcal{X}_4 \) is the image of the components \( 2[\mathbb{E} H]^* \subset \mathcal{W}_2 \oplus \mathcal{W}_4 \) by \( \delta \), and vanishes when the component of \( s(\nabla \Phi) \) in these two modules belongs to the kernel \( \text{Ker}(\delta) \).
- \( \mathcal{X}_5 \) is the image of the components \( 2[\mathbb{S}^3_0 \mathbb{E} H]^* \subset \mathcal{W}_2 \oplus \mathcal{W}_3 \) by \( \delta \), and vanishes when the component of \( s(\nabla \Phi) \) in these two modules belongs to the kernel \( \text{Ker}(\delta) \).
- \( \mathcal{X}_6 \) is the image of the components \( 2[\mathbb{S}^3 \mathbb{E} H]^* \subset \mathcal{W}_3 \oplus \mathcal{W}_4 \) by \( \delta \), and vanishes when the component of \( s(\nabla \Phi) \) in these two modules belongs to the kernel \( \text{Ker}(\delta) \).
- \( \mathcal{X}_7 \) is the image of the components \( 3[\mathbb{E} H]^* \subset \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4 \) by \( \delta \), and vanishes when the component of \( s(\nabla \Phi) \) in these three modules belongs to the kernel \( \text{Ker}(\delta) \).

**Proof.** Clearly, for modules isomorphic to \( \mathcal{X}_i \) that are not contained in \( [\mathbb{E} H]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)) \), the claim follows. For \( \mathcal{X}_i \) that are contained in \( [\mathbb{E} H]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)) \), the claim follows by the discussion above. In particular, for the case of \( \mathcal{X}_3 \) we can exclude \( [\mathbb{K} H]^* \subset \mathcal{W}_3 \) since \( \delta([\mathbb{K} H]^*) \) lies in \( \delta([\mathbb{E} H]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))) \). For \( \mathcal{X}_4 \), based on the same reason we exclude \( [\mathbb{E} H]^* \subset \mathcal{W}_3 \). \( \square \)

**Remark 2.18.** In general it is a hard task to derive the projections to the particular components, and moreover to find the elements of \( [\mathbb{E} H]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)) \) which remove the part of the torsion belonging to

\[ \delta(\bigoplus_a \mathcal{W}_a) \cap \delta([\mathbb{E} H]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))). \]

We will carry out this problem in a later part of this series of papers devoted to \( \text{SO}^*(2n) \)- and \( \text{SO}^*(2n) \text{Sp}(1) \)-structures. Such a “removal” would produce a minimal \( \text{SO}^*(2n) \text{Sp}(1) \)-connection for a particular normalization condition, and as we discuss below the (minimal) connections described in [CGW21, Theorem 3.12] provide a particular example of this remarkable situation.

Let us proceed with the application of Theorem 2.15 and Proposition 2.16 for an almost symplectic connection \( \nabla = \nabla^\omega \). In this case, we see that

\[ s(\nabla^\omega \Phi) := \frac{3\omega_3}{16n} (c(\nabla^\omega \Phi)) \in \mathcal{W}_3, \]
and we can conclude the following

**Corollary 2.19.** For an almost symplectic connection \( \nabla^\omega \) with torsion \( T^\omega \), the following holds:

- \( \mathcal{X}_1 \) is the image of the component \( [\mathbb{K} S^3 H]^* \subset \mathcal{W}_3 \) by \( \delta \), and vanishes when this component of \( s(\nabla^\omega \Phi) \) vanishes.
- \( \mathcal{X}_2 \) is the component \( [\Lambda^3 \mathbb{E} S^3 H]^* \) of \( T^\omega \), and vanishes when this component of \( T^\omega \) vanishes.
- \( \mathcal{X}_3 \) is the component \( [\mathbb{K} H]^* \) of \( T^\omega \), and vanishes when this component of \( T^\omega \) vanishes.
\(X_4\) is the component \([E H]^*\) of \(T^\omega\), and vanishes when this component of \(T^\omega\) vanishes.

\(X_5\) is the image of the component \([S^3_0 E H]^*\) \(\subset W_3\) by \(\delta\), and vanishes when this component of \(s(\nabla \Phi)\) vanishes.

\(X_6\) is the image of the components \([E S^3 H]^*\) \(\subset W_3\) by \(\delta\), and vanishes when this component of \(s(\nabla \Phi)\) vanishes.

\(X_7\) is the image of the components \([E H]^*\) \(\subset W_3\) by \(\delta\), and vanishes when this component of \(s(\nabla \Phi)\) vanishes.

Of course, the application of Theorem 2.15 and Proposition 2.16 for \(\nabla = \nabla^H\), and \(\nabla = \nabla^{Q,\text{vol}}\), respectively, must allow us to recover the results we obtained before (a procedure which also allows the verification of our claims in Theorem 2.15). Indeed, as an immediate consequence of Theorem 2.15 we obtain

**Corollary 2.20.** 1) On an almost hypercomplex skew-Hermitian manifold \((M, H, \omega)\) the adapted connection \(\nabla^{H,\omega}\) satisfies \(\nabla^{H,\omega} = \nabla^H - s(\nabla^H \Phi)\), which is equivalent to saying that \(\omega(-s(\nabla^H \Phi), \cdot) = \frac{1}{2} \nabla^{H,\omega}\).

2) On an almost quaternionic skew-Hermitian manifold \((M, Q, \omega)\) the adapted connection \(\nabla^{Q,\omega}\) satisfies \(\nabla^{Q,\omega} = \nabla^{Q,\text{vol}} - s(\nabla^{Q,\text{vol}} \Phi)\), i.e., \(-s(\nabla^{Q,\text{vol}} \Phi) = A^{\text{vol}}\).

**Proof.** For the first case Theorem 2.15 gives the expression

\[
s(\nabla^H \Phi) := -\left(\frac{w_1}{8(2n-1)} + \frac{w_2}{8(n-1)}\right)(c(\nabla \Phi)) \in W_1 \oplus W_2
\]

and the results follow by the uniqueness and the construction of the minimal connection \(\nabla^{H,\omega}\). Similarly, for \(\nabla = \nabla^{Q,\text{vol}}\) by Theorem 2.15 we obtain

\[
s(\nabla^{Q,\text{vol}} \Phi) := -\frac{w_2}{8(n-1)}(c(\nabla \Phi)) \in W_2
\]

and one concludes as before, by the uniqueness and the construction of the minimal connection \(\nabla^{Q,\omega}\). \(\square\)

Let us finally indicate the application of Theorem 2.15 and Proposition 2.16 for \(\nabla = \nabla^Q\). In this case we obtain

\[
s(\nabla^Q \Phi) := -\left(\frac{w_1}{8(2n-1)} + \frac{w_2}{8(n-1)}\right)(c(\nabla \Phi)) \in W_1 \oplus W_2.
\]

However, as in [CGW21, Theorem 3.12] we can remove the component \([E H]^*\) in \(\delta(W_1 \oplus W_2) \cap \delta([E H]^* \otimes (so^*(2n) \oplus sp(1)))\). Thus, the same results as for \(\nabla^{Q,\omega}\) follow.

3. A VARIETY OF EXAMPLES

3.1. Almost hypercomplex skew-Hermitian structures on \(\mathbb{R}^{4n}\). On \(\mathbb{R}^{4n}\) we can define an almost hs-H structure by a choice of frame \(u = (e_1, \ldots, e_{2n}, f_1, \ldots, f_{2n})\), which we regard as a skew-Hermitian frame. This means that \(\omega\) is given by \(\omega := \sum_{a=1}^{2n} e_a^* \wedge f_a^*\), while the almost hypercomplex structure \(H = \{I, J, K\}\) has the form

\[
I := \sum_{a=1}^{n} (e_a^* \otimes e_{n+a} - e_{n+a}^* \otimes e_a + f_a^* \otimes f_{n+a} - f_{n+a}^* \otimes f_a),
\]

\[
J := \sum_{a=1}^{n} (e_a^* \otimes f_a - f_a^* \otimes e_a - e_{n+a}^* \otimes f_{n+a} + f_{n+a}^* \otimes e_{n+a}),
\]

\[
K := \sum_{a=1}^{n} (e_a^* \otimes f_{a+n} - f_{a+n}^* \otimes e_a + e_{n+a}^* \otimes f_a - f_a^* \otimes e_{n+a}),
\]
where \( \vartheta = (e_1^*, \ldots, e_{2n}^*, f_1^*, \ldots, f_{2n}^*) \) is the dual co-frame to \( u \). As a corollary we obtain that

**Corollary 3.1.** On \((\mathbb{R}^{4n}, H, \omega)\), the tensors \( g_1, g_J, g_K, h \) and \( \Phi \) are given by

\[
g_1 := \sum_{a=1}^{n} (e_a^* \otimes f_{n+a}^* - e_{n+a}^* \otimes f_a^*),
\]

\[
g_J := \sum_{a=1}^{n} (-e_a^* \otimes e_a^* - f_a^* \otimes f_a^* + e_{n+a}^* \otimes e_{n+a}^* + f_{n+a}^* \otimes f_{n+a}^*),
\]

\[
g_K := \sum_{a=1}^{n} (-e_a^* \otimes e_{a+n}^* - f_{a+n}^* \otimes f_a^*),
\]

\[
h := (\sum_{a=1}^{2n} e_a^* \wedge f_a^*) \odot (\sum_{a=1}^{2n} (e_a^* \otimes e_a^* + f_a^* \otimes f_a^*)
\]

\[+ (\sum_{a=1}^{n} (e_a^* \otimes f_{n+a}^* - e_{n+a}^* \otimes f_a^*)) \odot (\sum_{a=1}^{n} (e_a^* \otimes e_{n+a}^* - e_{n+a}^* \otimes e_a^* + f_a^* \otimes f_{n+a}^* - f_{n+a}^* \otimes f_a^*))
\]

\[+ (\sum_{a=1}^{n} (-e_a^* \otimes e_a^* - f_a^* \otimes f_a^* + e_{n+a}^* \otimes e_{n+a}^* + f_{n+a}^* \otimes f_{n+a}^*)) \odot (\sum_{a=1}^{n} (e_a^* \otimes f_a^* - e_a^* \otimes f_{n+a}^* - e_{n+a}^* \otimes f_a^* + e_{n+a}^* \otimes f_{n+a}^*)
\]

\[+ (\sum_{a=1}^{n} (-e_a^* \otimes e_{a+n}^* - f_{a+n}^* \otimes f_a^*)) \odot (\sum_{a=1}^{n} (-e_a^* \otimes e_{a+n}^* - f_{a+n}^* \otimes f_a^*).
\]

\[
\Phi := (\sum_{a=1}^{n} e_a^* \otimes f_{n+a}^* - e_{n+a}^* \otimes f_a^*) \odot (\sum_{a=1}^{n} (e_a^* \otimes f_{n+a}^* - e_{n+a}^* \otimes f_a^*)
\]

\[+ (\sum_{a=1}^{n} (-e_a^* \otimes e_a^* - f_a^* \otimes f_a^* + e_{n+a}^* \otimes e_{n+a}^* + f_{n+a}^* \otimes f_{n+a}^*)) \odot (\sum_{a=1}^{n} (e_a^* \otimes f_a^* - e_a^* \otimes f_{n+a}^* - e_{n+a}^* \otimes f_a^* + e_{n+a}^* \otimes f_{n+a}^*)
\]

\[+ (\sum_{a=1}^{n} (-e_a^* \otimes e_{a+n}^* - f_{a+n}^* \otimes f_a^*)) \odot (\sum_{a=1}^{n} (-e_a^* \otimes e_{a+n}^* - f_{a+n}^* \otimes f_a^*))
\]

Note that the explicit form of \( \omega \) and of \( g_1, g_J, g_K \) provided above answers a problem posed in [Har90, p. 38]. Let us now recall the following fact which is very useful for applications.

**Lemma 3.2.** Consider a G-structure on \( M = \mathbb{R}^{4n} \) defined by a frame \( u \). Let \( \nabla^u \) be the G-connection with zero connection 1-form in \( u \), i.e., \( \nabla^u_M(x) \) is the directional derivative in the direction \( \sum_{a=1}^{2n} e_a^*(X(x))e_a + f_a^*(X(x))f_a \), where \( x \in M \). Then

\[
T^\nabla^u = d \vartheta, \quad R^\nabla^u = 0
\]

holds for the torsion \( T^\nabla^u \) and the curvature \( R^\nabla^u \) of \( \nabla^u \). In particular, the intrinsic torsion of the G-structure is given by \( p(d \vartheta) \), where \( p : \text{Tor}(M) \to \mathcal{H}(g) \) is the projection.

In our situation, \( G = \text{SO}^*(2n) \) and by [CGW21, Theorem 4.3] we know that the values of torsion \( T^\nabla^u \) corresponding to \( \nabla^u \) can be projected to \( \delta([E] \otimes \text{so}^*(2n)) \), along our normalization condition \( D(\text{so}^*(2n)) \). Using the inversion of \( \delta \), one can construct the minimal connection \( \nabla^H \omega \). However, since the intrinsic torsion of the corresponding \( \text{SO}^*(2n) \)-structure can be directly obtained from \( T^\nabla^u \), it is not necessary to specify a minimal connection for the discussion of integrability conditions in examples. Many of the examples presented in Section 3 occur via this approach. Next we shall use the traces \( \text{Tr}_i \) (\( i = 1, \ldots, 4 \)) introduced in [CGW21, Section 3.2], and also the maps \( \pi_H, \text{Alt} \) from [CGW21, Section 4.1].

**Example 3.3.** Let us begin with the following example that is of type \( X_{12} \), and which has the additional property that the adapted connection \( \nabla^H \omega \) described in [CGW21, Theorem 3.8] coincides with \( \nabla^u \) in the fixed frame. On \( \mathbb{R}^{12} = (x_1, \ldots, x_{12}) \) consider the following frame \( u \), and its dual
co-frame ∇:

\[
\begin{align*}
\omega &= dx_1 \wedge dx_7 + dx_2 \wedge dx_8 + dx_3 \wedge dx_9 + dx_4 \wedge dx_10 + dx_5 \wedge dx_11 \\
&\quad + (dx_6 + dx_4 dx_2 - x_1 dx_5 + x_10 dx_8 - x_7 dx_11) \wedge (dx_12 - x_10 dx_2 - x_7 dx_5 \\
&\quad + x_4 dx_8 + x_1 dx_11), \\
\vartheta &= \{0,0,0,0,0, dx_4 \wedge dx_2 - dx_1 \wedge dx_5 + dx_10 \wedge dx_8 - dx_7 \wedge dx_11, \\
&\quad 0,0,0,0,-dx_10 \wedge dx_2 - dx_7 \wedge dx_5 + dx_4 \wedge dx_8 + dx_1 \wedge dx_11\}, \\
T^\nabla &= (-dx_2 \wedge dx_4 - dx_1 \wedge dx_5 - dx_8 \wedge dx_10 - dx_7 \wedge dx_11) \otimes \partial x_6 \\
&\quad + (dx_2 \wedge dx_10 + dx_5 \wedge dx_7 + dx_4 \wedge dx_8 + dx_1 \wedge dx_11) \otimes \partial x_1.
\end{align*}
\]

It is relatively easy to see that the traces Tr_1(T^\nabla),...,Tr_4(T^\nabla) vanish, and thus by [CGW21, Lemma 3.11] in combination with [CGW21, Proposition 4.5] we deduce that T^\nabla has no intrinsic torsion contained in the components X_4,X_6,X_7. Moreover, we can prove that \pi_H(T^\nabla) = T^\nabla, hence in combination with [CGW21, Theorem 4.3] it follows that this example is of type X_12. From the same theorem we obtain the claim \nabla^H,\omega = \nabla^u.

**Example 3.4.** The following example shows that even if the torsion of \nabla^u is simple enough, the **almost hs-H structure** can have complicated mixed type (X_123567 in this example). Indeed, on \mathbb{R}^8 = (x_1,...,x_8) consider the following frame \{u\} and its dual co-frame \vartheta:

\[
\begin{align*}
u &= \{\partial x_1, \partial x_2, \partial x_3 - x_2 \partial x_1, \partial x_4, \partial x_5, \partial x_7, \partial x_8\}, \\
\vartheta &= \{dx_1 + x_2 dx_3, dx_2, dx_3, dx_4, dx_5, dx_6, dx_7, dx_8\}.
\end{align*}
\]

We compute \omega = (dx_1 + x_2 dx_3) \wedge dx_5 + dx_2 \wedge dx_6 + dx_3 \wedge dx_7 + dx_4 \wedge dx_8,

\[
\vartheta = \{dx_2 \wedge dx_3, 0,0,0,0,0,0,0\}, \quad \text{and} \quad T^\nabla = dx_2 \wedge dx_3 \otimes \partial x_1.
\]

In this case the traces Tr_1(T^\nabla),...,Tr_4(T^\nabla) are non-trivial, and again by [CGW21, Lemma 3.11] in combination with [CGW21, Proposition 4.5] we see that they provide the following components of the intrinsic torsion, namely X_4,X_6,X_7 while X_4 vanishes. Let us denote by T^\nabla_0 the part of the torsion T^\nabla with removed the components X_4,X_6,X_7. By applying the maps Alt and \pi_H to T^\nabla_0, and despite of the fact that these maps do not provide projections to the intrinsic torsion components X_1,X_2,X_3,X_5 (since Alt and \pi_H do not commute), we see that the components X_1,X_2,X_3,X_5 are non-trivial, as well. Hence all together we conclude that this example is of type X_123567.

**Example 3.5.** Next we present an example where a conformal change of the symplectic structure is applied. On the open set U \subset \mathbb{R}^8 = (x_1,...,x_8) of \mathbb{R}^8 given by \{x_1 \neq 0\}, consider the following frame \{u\} and its dual co-frame \vartheta:

\[
\begin{align*}
u &= \{\frac{1}{x_1} \partial x_1, \frac{1}{x_1} \partial x_2, \frac{1}{x_1} \partial x_3, \frac{1}{x_1} \partial x_4, \frac{1}{x_1} \partial x_5, \frac{1}{x_1} \partial x_7, \frac{1}{x_1} \partial x_8\}, \\
\vartheta &= \{x_1 dx_1, x_1 dx_2, x_1 dx_3, x_1 dx_4, x_1 dx_5, x_1 dx_6, x_1 dx_7, x_1 dx_8\}.
\end{align*}
\]
Therefore, the scalar 2-form is given by $\omega = x_1^2 \sum_{a=1}^4 dx_a \wedge dx_{4+a}$ and we compute
\[
\begin{align*}
d\vartheta &= \{0, dx_1 \wedge dx_2, dx_1 \wedge dx_3, dx_1 \wedge dx_4, dx_1 \wedge dx_5, dx_1 \wedge dx_6, dx_1 \wedge dx_7, dx_1 \wedge dx_8\}, \\
T^\nabla u &= \sum_{a=2}^8 \frac{1}{x_1} dx_1 \wedge dx_a \otimes \partial x_a = \delta \left( \frac{1}{2x_1} dx_1 \otimes \text{id} \right).
\end{align*}
\]

By Remark 1.6 we know that projections to $X_1$ and $X_7$ are proportional to $\frac{1}{2x_1} dx_1$, and thus in this example the corresponding $\text{SO}^*(2n)$-structure is of type $X_4$. In particular, the almost hypercomplex structure is hypercomplex, according to Theorem 1.8.

**Example 3.6.** There is also the following example of type $X_{1567}$ that carries a symplectic scalar 2-form. On $\mathbb{R}^8 = (x_1, \ldots, x_8)$ consider the following frame $u$ and its dual co-frame $\vartheta$:
\[
\begin{align*}
u &= \{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5 - x_4 \partial x_1, \partial x_7, \partial x_8\}, \\
\vartheta &= \{dx_1 + x_4 dx_5, dx_2, dx_3, dx_4, dx_5, dx_6, dx_7, dx_8\}.
\end{align*}
\]

In these terms the scalar 2-form is given by $\omega = \sum_{a=1}^4 dx_a \wedge dx_{4+a}$, which is symplectic. Moreover,
\[
d\vartheta = \{dx_4 \wedge dx_5, 0, 0, 0, 0, 0, 0\}, \quad T^\nabla u = dx_4 \wedge dx_5 \otimes \partial x_1.
\]

Since $(\partial x_1)^T = dx_5$ with respect to $\omega$, we can verify that $\text{Alt}(T^\nabla u) = 0$ and moreover that this example describes an $\text{SO}^*(2n)$-structure of type $X_{1567}$. To obtain this claim, as above we compute the traces $Tr_1$ and the image of $\pi_H$, a procedure which allows us to deduce that their components in all of $X_1, X_3, X_6, X_7$ are nonvanishing, see also Theorem 1.9.

**Example 3.7.** As the last example in this paragraph, we present an example of pure type $X_3$. On $\mathbb{R}^{12} = (x_1, \ldots, x_{12})$, let us consider the following frame $u$ and its dual co-frame $\vartheta$:
\[
\begin{align*}
u &= \{\partial x_1, \partial x_2 + x_4 \partial x_6 + x_7 \partial x_9 + x_{10} \partial x_{12}, \partial x_3, \partial x_4, \partial x_5 - x_4 \partial x_1, \partial x_7, \partial x_8 + x_{10} \partial x_6 - x_1 \partial x_6 + x_5 \partial x_{12}, \partial x_6, \\
&\qquad \partial x_7, x_1 \partial x_9 - x_4 \partial x_{12}, \partial x_9, \partial x_{10}, \partial x_{11} - x_7 \partial x_6 + x_4 \partial x_9 - x_1 \partial x_{12}, \partial x_{12}\}, \\
\vartheta &= \{dx_1, dx_2, dx_3, dx_4, dx_5, dx_6 - x_4 dx_2 + x_1 dx_5 - x_10 dx_8 + x_7 dx_{11}, dx_7, dx_8, \\
&\quad dx_9 - x_7 dx_2 + x_{10} dx_5 + x_1 dx_8 - dx_4 dx_{11}, dx_{10}, dx_{11}, dx_{12} - x_10 dx_2 - x_7 dx_5
\}
\]

Then, a computation shows that
\[
\begin{align*}
\omega &= dx_1 \wedge dx_7 + dx_2 \wedge dx_8 + dx_3 \wedge (dx_9 - x_7 dx_2 + x_{10} dx_5 + x_1 dx_8 - x_4 dx_{11}) \\
&\quad + dx_4 \wedge dx_{10} + dx_5 \wedge dx_{11} + (dx_6 - x_4 dx_2 + x_1 dx_5 - x_10 dx_8 + x_7 dx_{11}) \\
&\quad \wedge (dx_{12} - x_10 dx_2 - x_7 dx_5 + x_4 dx_8 + x_1 dx_{11}) \\
d\vartheta &= \{0, 0, 0, 0, 0, -dx_4 \wedge dx_2 + dx_1 \wedge dx_5 - dx_{10} \wedge dx_8 + dx_7 \wedge dx_{11}, \\
&\quad 0, 0, -dx_7 \wedge dx_2 + dx_{10} \wedge dx_5 + dx_1 \wedge dx_8 - dx_4 \wedge dx_{11}, 0, 0, \\
&\quad -dx_{10} \wedge dx_2 - dx_7 \wedge dx_5 + dx_4 \wedge dx_8 + dx_1 \wedge dx_{11}\}
\end{align*}
\]
\[
\begin{align*}
T^\nabla u &= (dx_2 \wedge dx_4 + dx_1 \wedge dx_5 + dx_8 \wedge dx_{10} + dx_7 \wedge dx_{11}) \otimes \partial x_6 \\
&\quad + (dx_2 \wedge dx_7 - dx_5 \wedge dx_{10} + dx_1 \wedge dx_8 - dx_4 \wedge dx_{11}) \otimes \partial x_9 \\
&\quad + (dx_2 \wedge dx_{10} + dx_5 \wedge dx_7 + dx_4 \wedge dx_8 + dx_1 \wedge dx_{11}) \otimes \partial x_{12}
\end{align*}
\]

With the help of [CGW21, Lemma 3.11] we see that the traces $Tr_1(T^\nabla u), \ldots, Tr_4(T^\nabla u)$ vanish, and the intrinsic torsion components $X_4, X_6, X_7$ are trivial. Although the maps $\text{Alt}$ and $\pi_H$ do not commute, by applying them to $T^\nabla u$ we conclude that this example is of pure type $X_3 = [KH]^*$. 

Differential Geometry of SO*(2n)-Structures and SO*(2n)Sp(1)-Structures

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3.2. Quaternionifications. Let us recall that for a real/complex vector space \( U \) we can form the (right) quaternionic vector spaces \( U \otimes_{\mathbb{R}} \mathbb{H} \) and \( U \otimes_{\mathbb{C}} \mathbb{H} \) (see [BrD85]), together with the induced flat hypercomplex structures on them. Unlike the situation with complexification of real analytic manifolds, this cannot be done with the transition maps and thus in general there is no direct generalization of complexification to quaternionification in the (real analytic) manifold setting. This is because of the fact that hypercomplex/quaternionic morphisms are determined by finite jets and form a finite dimensional Lie group. However, symplectomorphisms are not determined by some finite jet, and their pseudo-group can be infinite dimensional. In particular, there is no faithful functor from the category of almost symplectic structures to the category of almost hypercomplex/quaternionic skew-Hermitian structures.

On the other hand, if we choose a symplectic frame \( u := (e_1, \ldots, e_n, f_1, \ldots, f_n) \) on an almost symplectic manifold \((M, \omega)\), and require that the tangent bundle \( TM \) is globally trivial (such that the only transition maps that can be “quaternionified” are the constant one), then we can quaternionify \( M \) via the following two approaches which generalize the quaternionifications of a vector space mentioned above.

(a) The analogue of \( U \otimes_{\mathbb{R}} \mathbb{H} \) in the manifold setting. Let \((M, \omega)\) be a \(2n\)-dimensional almost symplectic manifold and set \( M^\alpha_{\mathbb{H}} := M \times M \times M \times M \). Consider the vector fields

\[
\{e_{i1}, f_{i1}, e_{i2}, f_{i2}, e_{i3}, f_{i3}, e_{i4}, f_{i4} : i = 1, \ldots, n \}
\]

where each of \(e_{ij}, f_{ij}\) is a section of the tangent bundle corresponding to the \(j\)-th component \(M_j \cong M\) of \(M^\alpha_{\mathbb{H}}\) \((j = 1, \ldots, 4)\), and represents a vector field \(e_i\), respectively \(f_i\), in the symplectic frame \(u\) on \(M_j\). Of course, any arbitrary non-degenerate combination of these vector fields can be regarded as a skew-Hermitian frame, i.e., frame that provides skew-Hermitian basis of \(\mathbb{R}^n\) for each \(x \in M^\alpha_{\mathbb{H}}\), in terms of \([CGW21, \text{Definition 1.24}]\). Such a frame defines an almost qs-\(H\) structure \(\{\{I, J, K\}, \omega^\alpha_{\mathbb{H}}\}\) on \(M^\alpha_{\mathbb{H}}\), so this construction provides examples of \(\text{SO}^*(4n)\)-structures on \(8n\)-dimensional manifolds. In fact, below we shall only consider frames compatible with the inclusion \(\text{Sp}(2n, \mathbb{R}) \subset \text{SU}(n, n) \subset \text{SO}^*(4n)\), because this makes the resulting scalar 2-form \(\omega^\alpha_{\mathbb{H}}\) independent of the choice of the symplectic frame \(u\). Next it is useful to interpret the inclusion \(\text{sp}(2n, \mathbb{R}) \to \text{so}^*(4n)\) as follows:

\[
\text{sp}(2n, \mathbb{R}) \to \text{so}^*(4n) : \left( \begin{array}{cccc}
\alpha + \gamma & \beta + \delta \\
-\beta + \delta & \alpha - \gamma
\end{array} \right) \mapsto \left( \begin{array}{cccc}
\alpha & i\gamma & \beta & i\delta \\
-\beta & \alpha & i\delta & \beta \\
-i\delta & \beta & \alpha & i\gamma \\
-i\gamma & \alpha & -\beta & i\delta
\end{array} \right)
\]

where \(\alpha\) is an anti-symmetric \((n \times n)\)-matrix, and \(\beta, \gamma, \delta\) are symmetric \((n \times n)\)-matrices. Then we can prove the following

**Theorem 3.8.** Let \((M, \omega)\) be a \(2n\)-dimensional almost symplectic manifold with trivial tangent bundle, defined by a global symplectic frame \(u\). Let \(\{\{I, J, K\}, \omega^\alpha_{\mathbb{H}}\}\) be the almost hs-\(H\) structure on \(M^\alpha_{\mathbb{H}} = M \times M \times M \times M\) determined by the following skew-Hermitian frame

\[
\begin{align*}
e_{i1} + f_{i2}, \ldots, e_{i1} + f_{n2}, e_{i3} + f_{i4}, \ldots, e_{n3} + f_{n4}, \\
e_{i1} - f_{i2}, \ldots, e_{n3} - f_{n4}, -e_{i1} + f_{i2}, \ldots, -e_{n3} + f_{n4}, \\
f_{i1} - e_{i2}, \ldots, f_{n1} - e_{n2}, f_{i3} - e_{i4}, \ldots, f_{n3} - e_{n4},
\end{align*}
\]

where the vector fields \(e_{ij}, f_{ij}\) with \(i = 1, \ldots, n, j = 1, \ldots, 4\) are determined by \(u\) as above. Then, \(\omega^\alpha_{\mathbb{H}}\) is independent of the symplectic frame \(u\), and moreover \(d\omega^\alpha_{\mathbb{H}} = 0\) if and only if \(d\omega = 0\).
Proof. It is easy to check that the change of the symplectic frame \( u \) is compatible with the inclusion \( \text{Sp}(2n, \mathbb{R}) \subset \text{SU}(n, n) \subset \text{SO}^*(4n) \). Moreover, we compute

\[
\omega_H^\alpha((x_1, x_2, x_3, x_4))((X_1, X_2, X_3, X_4), (Y_1, Y_2, Y_3, Y_4)) = \sum_{j=1}^4 \frac{1}{2} \omega(x_j)(X_j, Y_j)
\]

for any \( X_j, Y_j \) tangent to the \( j \)-th copy of \( M \) in \( M_\mathbb{H}^\alpha \). Thus, \( \omega_H^\alpha \) is independent of the symplectic frame \( u \) and we deduce that \( d\omega_H^\alpha = 0 \) if and only if \( d\omega = 0 \).

Let us emphasize that the almost hypercomplex structure determined by the stated skew-Hermitian frame in Theorem 3.8 depends on the symplectic frame \( u \). This is because the image in \( \text{Sp}(8n, \mathbb{R}) \) is not contained in \( \text{SO}^*(4n) \) at points \((x_1, x_2, x_3, x_4) \in M_\mathbb{H}^\alpha \) with different transition maps between frames at these points. Note also that the construction established above can be applied to Lie groups admitting left-invariant almost symplectic structures. However, be aware that the quaternionization of a Lie algebra is no longer a Lie algebra (the Lie bracket will not be antisymmetric). This means that the result will not be homogeneous, in general.

**Proposition 3.9.** Let \( K \) be a Lie group endowed with a left-invariant almost symplectic structure \( \omega \), defined by a symplectic basis \( \{e_1, \ldots, e_n, f_1, \ldots, f_n\} \) in the Lie algebra \( \mathfrak{k} \). Then, the corresponding skew-Hermitian bases introduced in Theorem 3.8 are independent of the choice of the symplectic basis and are \( K \)-invariant.

**Proof.** Any left-invariant almost symplectic structure \( \omega \) on \( K \) is uniquely determined by its value \( \omega_e \) which is a skew-symmetric bilinear form on \( \mathfrak{k} \). Obviously, \( \omega_e \) corresponds to a symplectic basis \( \{e_1, \ldots, e_n, f_1, \ldots, f_n\} \) in \( \mathfrak{k} \). Since the change of the symplectic basis is analogous to constant change of trivialization, the claim follows.

**Example 3.10.** Let us illustrate Proposition 3.9 via an explicit example, by considering the following matrix group

\[
M = \begin{pmatrix}
\exp(y_1) & 0 \\
y_2 & \exp(-y_1)
\end{pmatrix}
\]

for \( y_1, y_2 \in \mathbb{R} \). The left invariant vector fields \( e_1 = \partial_{y_1} + y_2 \partial_{y_2} \) and \( f_1 = \exp(-y_1)\partial_{y_2} \) form a symplectic basis for the following left-invariant symplectic structure on \( M \):

\[\omega = \exp(y_1)(d y_1 \wedge d y_2).\]

Let us also denote by \((x_1, \ldots, x_8)\) the corresponding coordinates on \( M_\mathbb{H}^\alpha = M^{4x4} \). Then, the specific skew-Hermitian bases introduced in Theorem 3.8 are given by

\[e_{ij} = \partial_{x_{2j-1}} + x_{2j} \partial_{x_{2j}}, \quad f_{ij} = \exp(-x_{2j-1})\partial_{x_{2j}}.\]

Consequently, the corresponding almost hs-H structure on \( M_\mathbb{H}^\alpha \) can be expressed as follows:

\[
\omega_H^\alpha = \sum_{j=1}^4 \frac{\exp(x_{2j-1})}{2} \, dx_{2j-1} \wedge dx_{2j},
\]

\[
I = (\partial_{x_5} - (x_2 \exp(x_1 - x_5) - x_6) \partial_{x_6}) \, dx_1 + \exp(x_1 - x_5) \partial_{x_6} \, dx_2
\]
\[
- (\partial_{x_7} - (x_4 \exp(x_3 - x_7) - x_8) \partial_{x_8}) \, dx_3 - \exp(x_3 - x_7) \partial_{x_8} \, dx_4
\]
\[
- (\partial_{x_1} - (x_6 \exp(x_5 - x_1) - x_2) \partial_{x_2}) \, dx_5 - \exp(x_5 - x_1) \partial_{x_2} \, dx_6
\]
\[
+ (\partial_{x_3} - (x_8 \exp(x_7 - x_3) - x_4) \partial_{x_4}) \, dx_7 + \exp(x_7 - x_3) \partial_{x_4} \, dx_8,
\]

\[
J = -(\partial_{x_3} - (x_2 \exp(x_1 - x_3) - x_4) \partial_{x_4}) \, dx_1 - \exp(x_1 - x_3) \partial_{x_4} \, dx_2
\]
\[
+ (\partial_{x_1} - (x_4 \exp(x_3 - x_1) - x_2) \partial_{x_2}) \, dx_3 + \exp(x_3 - x_1) \partial_{x_2} \, dx_4
\]
\[
- (\partial_{x_7} - (x_6 \exp(x_5 - x_7) - x_8) \partial_{x_8}) \, dx_5 - \exp(x_5 - x_7) \partial_{x_8} \, dx_6
\]
\[
+ (\partial_{x_5} - (x_8 \exp(x_7 - x_5) - x_6) \partial_{x_6}) \, dx_7 + \exp(x_7 - x_5) \partial_{x_6} \, dx_8,
\]
with $K =IJ$. Moreover, the torsion of $\nabla^u$ has the nice expression

$$ T^{\nabla^u} = 2\left( d\, x_1 \wedge d\, x_2 \otimes \partial_{x_2} + d\, x_3 \wedge d\, x_4 \otimes \partial_{x_4} + d\, x_5 \wedge d\, x_6 \otimes \partial_{x_6} + d\, x_7 \wedge d\, x_8 \otimes \partial_{x_8} \right). $$

Now, we see that $d\, \omega_\beta^a = 0$, hence by Theorem 1.9 we deduce that $M^\beta_{\mathbb{H}}$ must be of type $\mathcal{A}_{1567}$. Moreover, none of the almost complex structures $\{I, J, K\}$ are integrable, and a computation based on these traces $\partial_{x_j}$ and the map $\pi_u$, shows that $\mathcal{A}_5$ and $\mathcal{A}_6$ are trivial. Thus, $M^\beta_{\mathbb{H}}$ is of type $\mathcal{A}_{17}$.

**($\beta$) The analogue of $U \otimes \mathbb{C} \mathbb{H}$ in the manifold setting.** Next we further facilitate the approach discussed above, by considering an almost complex structure $J_g$ on a $2n$-dimensional almost symplectic manifold $(M, \omega)$, provided by fixing a Riemannian metric $g$ on $M$, that is, a $U(n)$-structure.

In this case the corresponding symplectic frames $u = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ are characterized by the property

$$ J_g(e_a) = f_a, \quad J_g(f_a) = -e_a, \quad a = 1, \ldots, n. $$

Next, again we need to assume global existence of such a frame.

So, let us consider the product $M^\beta_{\mathbb{H}} = M \times M$ and the vector fields

$$ \{e_{i1}, f_{i1}, e_{i2}, f_{i2} : i = 1, \ldots, n\} $$

where as before each of the vector fields $e_{ij}, f_{ij}$ is a section of the tangent bundle corresponding to the $j$-th component $M_j \cong M$ of $M^\beta_{\mathbb{H}}$ ($j = 1, 2$), and represents a vector field $e_i$, respectively $f_i$, in the symplectic frame $u$ on $M_j$. Any arbitrary non-degenerate combination of these vector fields can be regarded as a skew-Hermitian frame, which defines an almost hs-H structure $\{I, J, K\}$ of the symplectic manifold $(M, \omega)$. We focus only on frames compatible with the inclusion $U(n) \subset SO^*(2n)$, because this makes the resulting scalar 2-form $\omega_\beta^\mathbb{H}$ (and one of the complex structures) independent of the choice of the symplectic frame $u$. The corresponding inclusion has the form

$$ u(n) \to so^*(2n) : \alpha + i\beta \mapsto \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} $$

where $\alpha$ (respectively $\beta$) is an anti-symmetric (respectively symmetric) $(n \times n)$-matrix. Using this formula it is simple to prove the following

**Theorem 3.11.** Let $(M, \omega)$ be a $2n$-dimensional almost symplectic manifold with trivial tangent bundle, provided by a global symplectic frame $u$ defining a $U(n)$-structure. Let $\{I, J, K\}, \omega_\beta^\mathbb{H}$ be the almost hs-H structure on $M^\beta_{\mathbb{H}} = M \times M$ determined by the skew-Hermitian frame

$$ e_{11}, \ldots, e_{n1}, \quad e_{12}, \ldots, e_{n2}, \quad f_{11}, \ldots, f_{n1}, \quad f_{12}, \ldots, f_{n2}, $$

for the vector fields $e_{ij}, f_{ij}, i = 1, \ldots, n, j = 1, 2$, determined by $u$ as above. Then, $\omega_\beta^\mathbb{H}$ is independent of the symplectic frame $u$, and the almost complex structure $J$ depends only on the $U(n)$-structure defined by $u$. Moreover, $d\, \omega_\beta^\mathbb{H} = 0$ if and only if $d\, \omega = 0$.

**Proof.** By a simple computation we see that the change within $U(n)$ of the symplectic frame $u$ is compatible with the inclusion $U(n) \subset SO^*(2n)$. Moreover, it follows that

$$ \omega_\beta^\mathbb{H}(x_1, x_2)((X_1, X_2), (Y_1, Y_2)) = \sum_{j=1}^{2} \omega(x_j)(X_j, Y_j) $$

for $X_j, Y_j$ tangent to the $j$-th copy of $M$ in $M^\beta_{\mathbb{H}}$. Consequently, $\omega_\beta^\mathbb{H}$ does not depend on the choice of the symplectic frame $u$. Moreover, we see that $d\, \omega_\beta^\mathbb{H} = 0$, and only if $d\, \omega = 0$. Moreover, the restriction of $J$ to each copy of $M$ in $M^\beta_{\mathbb{H}}$ is given by $J_g = g \circ \omega^{-1}$, and thus depends only on the fixed $U(n)$-structure. $\square$
Remark 3.12. For a fixed $2n$-dimensional almost symplectic manifold $(M, \omega)$ the almost hypercomplex skew-Hermitian manifolds $(M^\beta_{\mathbb{H}_1}, \omega^\beta_{\mathbb{H}_1})$ and $(M^\beta_{\mathbb{H}_2}, \omega^\beta_{\mathbb{H}_2})$ constructed in this section have different dimensions $(8n$ and $4n)$. Another question is whether there are two almost symplectic manifold $(M, \omega)$ of dimension $2n$ and $(M', \omega')$ of dimension $4n$ such that there is a hypercomplex symplectomorphism $f : (M^\beta_{\mathbb{H}_1}, \omega^\beta_{\mathbb{H}_1}) \to ((M')^\beta_{\mathbb{H}_1}, (\omega')^\beta_{\mathbb{H}_1})$. This is indeed possible. Consider the intersection $U(n) = \text{Sp}(2n, \mathbb{R}) \cap U(2n) \subset SO^*(4n)$ and fix a frame of a $U(n) \subset \text{Sp}(2n, \mathbb{R})$-structure on $M$ and consider $M' = M \times M$ with the $U(n) \subset \text{U}(2n)$-structure given by a frame constructed from the fixed frame on $M$ in a way compatible with the inclusion $U(n) \subset U(2n)$. Then $(M^\beta_{\mathbb{H}_1}, \omega^\beta_{\mathbb{H}_1})$ and $((M')^\beta_{\mathbb{H}_1}, (\omega')^\beta_{\mathbb{H}_1})$ constructed for the corresponding frames are related by a hypercomplex symplectomorphism.

Example 3.13. Let us consider the following 6-dimensional unipotent matrix group

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ y_1 & 1 & 0 & 0 \\ y_2 & y_3 & 1 & 0 \\ y_6 & y_5 & y_4 & 1 \end{pmatrix}$$

with $y_1, \ldots, y_6 \in \mathbb{R}$. Its Lie algebra $\mathfrak{m}$ is a 3-step nilpotent Lie algebra. The following

$$e_1 = \partial_{y_1} + y_3 \partial_{y_2} + y_5 \partial_{y_6}, \quad e_2 = \partial_{y_2} + y_5 \partial_{y_6}, \quad e_3 = \partial_{y_3} + y_4 \partial_{y_5}, \quad f_1 = \partial_{y_4}, \quad f_2 = \partial_{y_5}, \quad f_3 = \partial_{y_6}$$

are left-invariant vector fields on $M$, which form a symplectic frame for the following left-invariant almost symplectic structure:

$$\omega = y_5 \, d y_1 \wedge d y_3 + d y_1 \wedge d y_4 - d y_3 \wedge d y_5 + d y_2 \wedge d y_5 + d y_3 \wedge d y_6.$$  

Moreover, they form a $g$-orthogonal frame, where $g = \omega \circ J$ is the left-invariant metric induced via the pair $(\omega, J)$ and $J$ is given by

$$J := \left((y_5 - y_3 y_4) \partial_{y_5} + \partial_{y_4} + (y_4 y_5 - y_3 y_4^2 - y_3) \partial_{y_5}\right) d y_1 + (y_4 \partial_{y_5} + (y_3^2 + 1) \partial_{y_5}) d y_2$$

$$+ (y_4 \partial_{y_2} + (y_3^2 + 1) \partial_{y_6}) d y_3 - (\partial_{y_1} + y_3 \partial_{y_2} + y_5 \partial_{y_6}) d y_4 - (\partial_{y_2} + y_4 \partial_{y_6}) d y_5$$

$$- (\partial_{y_3} + y_4 \partial_{y_5}) d y_6.$$  

Let us denote by $(x_1, \ldots, x_{12})$ the corresponding coordinates on $M^\beta_{\mathbb{H}_1}$. Then we see that the skew-Hermitian frame of Theorem 3.11 has the form

$$e_{11} = \partial_{x_1} + x_3 \partial_{x_2} + x_5 \partial_{x_6}, \quad e_{21} = \partial_{x_2} + x_3 \partial_{x_6}, \quad e_{31} = \partial_{x_3} + x_4 \partial_{x_5}, \quad e_{12} = \partial_{x_4} + x_9 \partial_{x_8} + x_{11} \partial_{x_7},$$

$$e_{22} = \partial_{x_6} + x_{11} \partial_{x_{12}}, \quad e_{32} = \partial_{x_9} + x_{10} \partial_{x_{11}}, \quad f_{11} = \partial_{x_{1+3}}, f_{12} = \partial_{x_{1+9}}.$$  

Hence we compute

$$\omega^\beta_{\mathbb{H}_1} = x_5 \, d x_1 \wedge d x_3 + d x_1 \wedge d x_4 - x_3 \, d x_1 \wedge d x_5 + d x_2 \wedge d x_5 + d x_3 \wedge d x_6$$

$$+ x_{11} \, d x_7 \wedge d x_9 + d x_7 \wedge d x_{10} - x_9 \, d x_7 \wedge d x_{11} + d x_8 \wedge d x_{11} + d x_9 \wedge d x_{12},$$

$$I = \left((\partial_{x_1} + (x_9 - x_3) \partial_{x_4} + (x_3 x_4 - x_{10} x_3 + x_{11} - x_5) \partial_{x_5}) d x_1 + (\partial_{x_8} + (x_{10} - x_4) \partial_{x_{12}}) d x_2$$

$$+ (\partial_{x_9} + (x_{10} - x_4) \partial_{x_{11}}) d x_3 + \partial_{x_4} \partial_{x_5} d x_4 + \partial_{x_1} d x_5 + \partial_{x_{12}} d x_6$$

$$- (\partial_{x_4} - (x_9 - x_3) \partial_{x_2} - (x_3 x_4 - x_{10} x_9 + x_{11} - x_5) \partial_{x_5}) d x_7 - (\partial_{x_2} - (x_{10} - x_4) \partial_{x_6}) d x_8$$

$$- (\partial_{x_3} - (x_{10} - x_4) \partial_{x_6}) d x_9 - \partial_{x_4} d x_{10} - \partial_{x_5} d x_{11} - \partial_{x_6} d x_{12},$$

$$J = \left((x_5 - x_3 x_4) \partial_{x_2} + x_4 \partial_{x_3} - x_5 \partial_{x_4} - x_3 \partial_{x_5} \partial_{x_6}\right) d x_1 + (x_3 \partial_{x_3} + (x_3^2 + 1) \partial_{x_5}) d x_2$$

$$+ (x_4 \partial_{x_4} + x_7 \partial_{x_5} + x_7 \partial_{x_6} + x_8 \partial_{x_6}) d x_3 - (\partial_{x_1} + x_3 \partial_{x_2} + x_5 \partial_{x_6}) d x_4 - (\partial_{x_2} + x_4 \partial_{x_5}) d x_5 - (\partial_{x_3} + x_4 \partial_{x_5}) d x_6$$

$$- (\partial_{x_4} + x_9 \partial_{x_8} + x_{10} \partial_{x_{10}} + (x_{11} - x_{10} \partial_{x_{10}} - x_9 x_{11} \partial_{x_{11}}) d x_7 - (\partial_{x_2} + x_{10} \partial_{x_{12}} + (x_{10}^2 + 1) \partial_{x_{12}}) d x_8$$

$$- (\partial_{x_3} + x_{10} \partial_{x_5} + (x_{10}^2 + 1) \partial_{x_{12}}) d x_9 + (\partial_{x_7} + x_9 \partial_{x_8} + x_{11} \partial_{x_{12}}) d x_{10} + (\partial_{x_8} + x_{10} \partial_{x_{12}}) d x_{11} + (\partial_{x_9} + x_{10} \partial_{x_{11}}) d x_{12}.$$
and the pair \( (H = \{I, J, K = IJ\}, \omega_2^{\beta}) \) induces an almost hypercomplex skew-Hermitian structure on \( M_2^{\beta} \). The torsion of \( \nabla^u \) is given by
\[
T^{\nabla^u} = d x_1 \wedge d x_3 \otimes \partial_{x_2} - (x_3 d x_1 \wedge d x_4 - d x_1 \wedge d x_5 - d x_2 \wedge d x_4) \otimes \partial_{x_6} + d x_3 \wedge d x_4 \otimes \partial_{x_5} \\
+ d x_7 \wedge d x_9 \otimes \partial_{x_8} - (x_9 d x_7 \wedge d x_{10} - d x_7 \wedge d x_{11} - d x_8 \wedge d x_{10}) \otimes \partial_{x_{12}} \\
+ d x_9 \wedge d x_{10} \otimes \partial_{x_{11}}.
\]
Let us remark that neither \( d \omega \) nor \( d^2 \omega^{\beta} \) are vanishing, and the almost complex structures \( I, J, K \) are not complex. A computation shows that all the four traces \( \operatorname{Tr}_i \) vanish, so the explicit type of \( (H = \{I, J, K = IJ\}, \omega_2^{\beta}) \) is \( \chi_{1235} \).

4. Homogeneous SO\(^*(2n)\)- and SO\(^*(2n)\) Sp\((1)\)-structures

In this section we shall focus on SO\(^*(2n)\)- and SO\(^*(2n)\) Sp\((1)\)-structures which are invariant under the action of a Lie group \( K \). In other words we shall treat 4n-dimensional (connected) homogeneous manifolds \( K/L \) admitting a \( K \)-invariant SO\(^*(2n)\)-structure or a \( K \)-invariant SO\(^*(2n)\) Sp\((1)\)-structure (for details on homogeneous spaces we refer to [Hel78, ČSi09]).

4.1. A characterization of invariant scalar 2-forms. Let \( K/L \) be a homogeneous space where \( L \subset K \) is a closed Lie subgroup of \( K \), and let us denote by \( \chi : L \to \operatorname{Aut}(\mathfrak{t}/l) \) the isotropy representation of \( L \) on \( \mathfrak{t}/l \cong T_o K/L \), where \( o = eL \in K/L \) is the identity coset. Recall by [CGW21, Proposition 4.8] that hypercomplex/quaternionic symplectomorphisms are determined by their first jet. Therefore, to discuss invariant SO\(^*(2n)\)- or SO\(^*(2n)\) Sp\((1)\)-structures on a homogeneous space \( K/L \) as above, without loss of generality we may assume that the isotropy representation \( \chi \) is faithful. So we can identify the Lie algebra \( \mathfrak{l} \) of the stabilizer \( L \) with the Lie algebra \( \mathfrak{c}_s(\mathfrak{l}) \) of the linear isotropy group \( \chi(L) \subset \operatorname{Aut}(\mathfrak{t}/l) \). Recall also that a homogeneous space \( K/L \) is called reductive if there exists a \( \operatorname{Ad}(L) \)-invariant complement \( \mathfrak{m} \) of \( \mathfrak{l} \) in \( \mathfrak{t} \), that is \( \mathfrak{t} = \mathfrak{l} \oplus \mathfrak{m} \) and \( \operatorname{Ad}(L) \mathfrak{m} \subset \mathfrak{m} \), which implies \( [\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m} \). Then one can identify \( \mathfrak{m} \cong T_o K/L \) and \( \chi \) with \( \operatorname{Ad} : L \to \operatorname{Aut}(\mathfrak{m}) \), i.e., \( \chi \cong \operatorname{Ad}_K |_{\mathfrak{l} \times \mathfrak{m}} \).

As is well-known, the geometric properties of \( K/L \) can be examined by restricting our attention to the origin \( o = eL \). In particular, \( K \)-invariant tensors on \( K/L \) are in bijective correspondence with \( \chi(L) \)-invariant tensors on the tangent space of the origin \( o = eL \). Thus for example we see that

1) A \( K \)-invariant almost hypercomplex structures corresponds to a linear almost hypercomplex structure \( H = \{I, J, K\} \) on \( \mathfrak{t}/l \), such that each of the linear complex structures \( I, J, K \in \operatorname{End}(\mathfrak{t}/l) \) is commuting with the isotropy representation \( \chi \).

2) A \( K \)-invariant almost quaternionic structure corresponds to a linear quaternionic structure \( Q \subset \operatorname{End}(\mathfrak{t}/l) \) on \( \mathfrak{t}/l \), such that each \( J \in \operatorname{S}(Q) \) is normalized in \( Q \) by conjugation via the isotropy representation \( \chi \).

Consider a homogeneous space \( K/L \) with a \( K \)-invariant almost hypercomplex structure \( H \) or with a \( K \)-invariant almost quaternionic structure \( Q \). Then it makes sense to consider the corresponding space \( \Lambda_2^{\omega_o}(\mathfrak{t}/l)^* \) of scalar 2-forms on \( \mathfrak{t}/l \) with respect to \( H_o \) or \( Q_o \), respectively. Given such a 2-form \( \omega_o \), we may use it to identify \( \mathfrak{t}/l \) and its dual \( (\mathfrak{t}/l)^* \). Next we shall denote by \( \Lambda_2^{\omega_o}(\mathfrak{t}/l) \) the space of \( \chi(L) \)-invariant scalar 2-forms on \( M = K/L \) and by \( \Omega_2^{\omega_o}(M)^K \) the corresponding smooth sections, i.e., smooth \( K \)-invariant scalar 2-forms on \( M = K/L \). For any case we may give the following characterization.

**Proposition 4.1.** a) On a homogeneous spaces \( M = K/L \) with a \( K \)-invariant almost hypercomplex structure \( H = \{I, J, K\} \) the following are equivalent:

1) There is a \( K \)-invariant almost hypercomplex skew-Hermitian structure \( (H, \omega) \) on \( K/L \), that is \( \omega \in \Omega_2^{\omega_o}(M)^K \);
2) The linear scalar 2-form $\omega_\alpha$ on $\mathfrak{k}/\mathfrak{l} \cong T_o K/L$ is invariant under the isotropy representation $\chi : L \to \text{Aut}(\mathfrak{k}/\mathfrak{l})$, that is $\omega_\alpha \in \Lambda^2(\mathfrak{k}/\mathfrak{l})$;

3) There is a basis of $\mathfrak{k}/\mathfrak{l}$ adapted to $H_o = \{I_o, J_o, K_o\}$ (in terms of [CGW21, Definition 1.17]) inducing a Lie group homomorphism $i : L \to \text{GL}(n, \mathbb{H}) \subset \text{GL}([E H])$, such that $(d i)(1) \subset \mathfrak{so}^*(2n) \subset \mathfrak{gl}([E H])$.

$\beta$) On a homogeneous spaces $K/L$ with a $K$-invariant almost quaternionic structure $Q$ the following are equivalent:

1) There is a $K$-invariant almost quaternionic skew-Hermitian structure $(Q, \omega)$ on $K/L$, that is $\omega \in \Omega^2_{\mathbf{sc}}(M)^K$;

2) The linear scalar 2-form $\omega_\alpha$ on $\mathfrak{k}/\mathfrak{l} \cong T_o K/L$ is invariant under the isotropy representation $\chi : L \to \text{Aut}(\mathfrak{k}/\mathfrak{l})$, that is $\omega_\alpha \in \Lambda^2(\mathfrak{k}/\mathfrak{l})$;

3) For any admissible basis $H_o$ of $Q_o$, there is a basis of $\mathfrak{k}/\mathfrak{l}$ adapted to $H_o$ inducing a Lie group homomorphism $i : L \to \text{GL}(n, \mathbb{H}) \text{Sp}(1) \subset \text{GL}([E H])$, such that $(d i)(1) \subset \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1) \subset \mathfrak{gl}([E H])$.

Let us now pass to invariant adapted connections to such structures. Let us also record that

**Corollary 4.2.** The minimal $\text{SO}^*(2n)$-connections $\nabla^{H,\omega}$ and $\text{SO}^*(2n)\text{Sp}(1)$-connections $\nabla^{Q,\omega}$ for $K$-invariant almost $\text{hs}$-$H$/qs-$H$ structure on $K/L$, respectively, are $K$-invariant connections.

**Proof.** Since hypercomplex/quaternionic symplectomorphisms map $\text{SO}^*(2n)$- and $\text{SO}^*(2n)\text{Sp}(1)$-connections onto $\text{SO}^*(2n)$- and $\text{SO}^*(2n)\text{Sp}(1)$-connections with related torsion, respectively, the first claim follows from uniqueness of the minimal connections $\nabla^{H,\omega}$ and $\nabla^{Q,\omega}$. \hfill $\Box$

In order to capture the information about general invariant adapted connections, we need to consider the following generalization of a classical result of [W58], see also [CS09, Prop. 1.5.15].

**Proposition 4.3.** On a homogeneous space $K/L$ the following holds:

1) There is a $K$-invariant almost $\text{hs}$-$H$ structure on $K/L$ together with $K$-invariant $\text{SO}^*(2n)$-connection $\nabla$, if and only if there is Lie group homomorphism $i : L \to \text{SO}^*(2n)$ and a linear map $\alpha : \mathfrak{k} \to [E H] \oplus \mathfrak{so}^*(2n)$ that restricts to a linear isomorphism $\mathfrak{k}/\mathfrak{l} \cong [E H]$ and satisfies

$$\alpha(Y) = (d i)(Y), \quad \alpha(\text{Ad}(\ell))(X) = \text{Ad}(i(\ell))(\alpha(X)),$$

for all $X \in \mathfrak{k}$, $Y \in \mathfrak{l}$ and $\ell \in L$.

2) There is a $K$-invariant almost qs-$H$ structure on $K/L$ together with $K$-invariant $\text{SO}^*(2n)\text{Sp}(1)$-connection $\nabla$, if and only if there is Lie group homomorphism $i : L \to \text{SO}^*(2n)\text{Sp}(1)$ and a linear map $\alpha : \mathfrak{k} \to [E H] \oplus \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$ that restricts to a linear isomorphism $\mathfrak{k}/\mathfrak{l} \cong [E H]$ and satisfies

$$\alpha(Y) = (d i)(Y), \quad \alpha(\text{Ad}(\ell))(X) = \text{Ad}(i(\ell))(\alpha(X)),$$

for all $X \in \mathfrak{k}$, $Y \in \mathfrak{l}$ and $\ell \in L$.

**Proof.** These claims follow from results in [CS09]. Indeed, the tautological 1-form on the geometric structure together with the connection 1-form of the $K$-invariant connection $\nabla$, form a Cartan connection of affine type $([E H] \times \text{SO}^*(2n), \text{SO}^*(2n))$ and $([E H] \times \text{SO}^*(2n)\text{Sp}(1), \text{SO}^*(2n)\text{Sp}(1))$, respectively. Conversely, the projections to $\mathfrak{so}^*(2n)$ or $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$ (along $[E H]$) of the restriction of the map $\alpha$ to $\mathfrak{k}/\mathfrak{l}$ induces (together with the Maurer-Cartan form on $K$) the $K$-invariant $\text{SO}^*(2n)$-connection or $\text{SO}^*(2n)\text{Sp}(1)$-connection, respectively. Note that $K \times \text{SO}^*(2n)$ or $K \times \text{SO}^*(2n)\text{Sp}(1)$ are the underlying invariant $\text{SO}^*(2n)$-structures or $\text{SO}^*(2n)\text{Sp}(1)$-structures, respectively, where the tautological 1-form is induced by the projection to $[E H]$ of the restriction of the map $\alpha$ to $\mathfrak{k}/\mathfrak{l}$ (together with the Maurer-Cartan form on $K$). \hfill $\Box$

Let $\mathfrak{g}$ be one of the Lie algebras $\mathfrak{so}^*(2n)$ or $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$ and $(\alpha, i)$ a pair satisfying the conditions of the above proposition. The linear map $\alpha : \mathfrak{k} \to [E H] \oplus \mathfrak{g}$ splits into the maps
\(\alpha_{[EH]} : \mathfrak{t} \to [EH]\) providing the isomorphism \(\mathfrak{t}/l \cong [EH]\) and the so-called Nomizu map \(\alpha_\mathfrak{g} : \mathfrak{t} \to \mathfrak{g}\). Note that \(\alpha_{[EH]}|l = 0\) and \(\alpha_\mathfrak{g}|l = d\ i\). Let \(\nabla\) be a \(K\)-invariant connection on \(K/L\) corresponding to a pair \((\alpha, i)\). Then, the torsion \(T^\nabla_\alpha \in \Lambda^2 [EH]^* \otimes [EH]\) of \(\nabla\) is given by

\[
T^\nabla_\alpha(x, y) = \alpha_\mathfrak{g}(\alpha_{[EH]}^{-1}(x))y - \alpha_\mathfrak{g}(\alpha_{[EH]}^{-1}(y))x - \alpha_{[EH]}([\alpha_{[EH]}^{-1}(x), \alpha_{[EH]}^{-1}(y)]_l),
\]

for any \(x, y \in [EH]\).

Similarly, the curvature \(R^\nabla_\alpha \in \Lambda^2 [EH]^* \otimes \mathfrak{g}\) of \(\nabla\) is given by

\[
R^\nabla_\alpha(x, y) = [\alpha_\mathfrak{g}(\alpha_{[EH]}^{-1}(x)), \alpha_\mathfrak{g}(\alpha_{[EH]}^{-1}(y))]_0 - \alpha_{[EH]}([\alpha_{[EH]}^{-1}(x), \alpha_{[EH]}^{-1}(y)]_l),
\]

for any \(x, y \in [EH]\).

On reductive homogeneous spaces \(K/L\) carrying reductive decomposition \(\mathfrak{t} = \mathfrak{m} \oplus \mathfrak{l}\), the description of \(K\)-invariant \(SO^*(2n)\) and \(SO^*(2n)\)-structures in terms of Proposition 4.3 gets simplified. We can use \(\alpha_{[EH]}\) to directly identify \(\mathfrak{m}\) with \([EH]\) and thus restrict \(\alpha_\mathfrak{g}\) to a map \(\alpha_\mathfrak{v} : [EH] \to \mathfrak{g}\). In particular,

\[
T^\nabla_\alpha(x, y) = \alpha_\mathfrak{v}(x)y - \alpha_\mathfrak{v}(y)x - [x, y]_\mathfrak{m},
\]

\[
R^\nabla_\alpha(x, y) = [\alpha_\mathfrak{v}(x), \alpha_\mathfrak{v}(y)]_0 - \alpha_\mathfrak{v}([x, y]_\mathfrak{m}) - (d\ i)([x, y]_l),
\]

for any \(x, y \in [EH] = \mathfrak{m}\), where we split \([x, y]_l = [x, y]_\mathfrak{m} \oplus [x, y]_l\) according to the reductive decomposition.

**Remark 4.4.** Recall that there is a \(K\)-invariant \(G\)-connection \(\nabla^u\) with \(\alpha_\mathfrak{v} = 0\), called the canonical connection on \(K/L\), which depends on the reductive complement \(\mathfrak{m}\). The torsion and curvature of \(\nabla^u\) are given by (see \([KN69]\))

\[
T^{\nabla^u}_\alpha(x, y) = -[x, y]_\mathfrak{m}, \quad R^{\nabla^u}_\alpha(x, y) = -(d\ i)([x, y]_l)
\]

respectively, for any \(x, y \in [EH]\). The canonical connection is the closest analogue of the connections discussed in Lemma 3.2, when we interpret \(u\) as a local frame of

\[
T(\exp(m)L) \subset T(K/L)
\]

consisting of left invariant vector fields corresponding to a skew-Hermitian basis of \(\mathfrak{m} = [EH]\).

Indeed, the intrinsic torsion of such \(SO^*(2n)\)- or \(SO^*(2n)\) \(Sp(1)\)-geometries is given

\[
p(T^{\nabla^u}_\alpha) = p(-\lbrack \cdot, \cdot \rbrack_\mathfrak{m}).
\]

### 4.2. Homogeneous examples

For both \(SO^*(2n)\)- and \(SO^*(2n)\) \(Sp(1)\)-structures, a large family of reductive homogeneous examples is obtained by 4n-dimensional Lie groups \(K\), by considering linear isomorphisms \(\alpha = \alpha_{[EH]} : \mathfrak{t} \to \mathfrak{m} = [EH]\), which trivially satisfy all the conditions of Proposition 4.3. Such isomorphisms provide all left invariant \(SO^*(2n)\)- and \(SO^*(2n)\) \(Sp(1)\)-geometries on 4n-dimensional Lie groups. Let us present such an example.

**Example 4.5.** Consider the Lie group \(K\) of lower triangular matrices in \(GL(3, \mathbb{H})\) with Lie algebra

\[
\begin{pmatrix}
0 & 0 & 0 \\
x_1 + x_4 i + x_7 j + x_{10} k & 0 & 0 \\
x_2 + x_5 i + x_8 j + x_{11} k & x_3 + x_6 i + x_9 j + x_{12} k & 0
\end{pmatrix}.
\]

Let us also consider a linear isomorphism \(\alpha : \mathfrak{t} \to [EH]\), such that \((x_1, \ldots, x_{12})\) are the coordinates in a skew-Hermitian basis of \([EH]\). If \(u = (e_1, \ldots, e_6, f_1, \ldots, f_6)\) is the corresponding \(K\)-invariant frame of left invariant vector fields on \(K\) and \(\vartheta = (e^1, \ldots, e^6, f^1, \ldots, f^6)\) is the corresponding dual
co-frame, then the following holds for the torsion of the connection $\nabla^u$:

$$T_{\nabla^u} = -[\cdot, \cdot]_l$$

$$= (e_1^* \wedge e_3^* - f_4^* \wedge f_6^* + e_4^* \wedge e_6^* - f_3^* \wedge f_5^*) \otimes e_2$$

$$+ (e_1^* \wedge e_5^* - f_3^* \wedge f_4^* - e_3^* \wedge e_4^* - f_1^* \wedge f_6^*) \otimes e_5$$

$$+ (e_1^* \wedge f_5^* - f_4^* \wedge e_6^* + e_4^* \wedge f_6^* + f_1^* \wedge e_3^*) \otimes f_2$$

$$+ (e_1^* \wedge f_6^* + f_4^* \wedge e_3^* - e_4^* \wedge f_3^* + f_1^* \wedge e_5^*) \otimes f_5.$$ 

Since the torsion is neither a 3-form, nor of vectorial type, this $SO^*(6)$-structure on $K$ is neither of type $X_{234}$, nor of type $X_{47}$. It is relatively easy to see that the traces $\text{Tr}_1(T_0^u), \ldots, \text{Tr}_4(T_0^u)$ and $\pi_H(T_0^u)$ vanish. Therefore, the type of this example is actually $X_{35}$. In particular, the map $\alpha$ is hypercomplex linear and the hypercomplex structure on this example coincides with the natural hypercomplex structure on $K$.

**Example 4.6.** Consider the reductive homogeneous space $\text{SL}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})$. We may identify the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of $L = \text{SL}(2, \mathbb{R})$ with the following matrices

$$\mathfrak{l} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & l_1 & l_2 & \cdot \\ 0 & 0 & l_3 & -l_1 \end{pmatrix} : l_1, l_2, l_3 \in \mathbb{R} \right\}.$$ 

Then, a reductive complement $\mathfrak{m}$ of $\mathfrak{l}$ in $\mathfrak{k}$ is given by

$$\mathfrak{m} = \left\{ \begin{pmatrix} a_3 & a_9 & a_1 + a_5 + a_7 + a_{11} & -a_1 - a_5 \\ a_6 & a_{12} & -2a_1 - 2a_{11} & a_1 + a_5 - a_7 + a_{11} \\ a_2 - a_4 - a_8 - a_{10} & a_8 + a_{10} & -\frac{1}{2}a_3 - \frac{1}{2}a_{12} & 0 \\ 2a_2 - 2a_{10} & -a_2 - a_4 + a_8 + a_{10} & 0 & -\frac{1}{2}a_3 - \frac{1}{2}a_{12} \end{pmatrix} \right\},$$

with $a_1, \ldots, a_{12} \in \mathbb{R}$. Let us now define the map

$$\alpha|_{\mathfrak{EH}}(A) := (a_1, \ldots, a_{12}),$$

where the right hand side are coordinates in an skew-Hermitian basis $B$ of $[\mathfrak{EH}]$. It is not hard to check that $\mathfrak{l}$ is mapped into $\mathfrak{so}^*(2n)$ by the map induced on endomorphisms by $\alpha|_{\mathfrak{EH}}$. Moreover, it integrates to Lie algebra homomorphism

$$i : \text{SL}(2, \mathbb{R}) \to SO^*(6),$$

however it is not so pleasant to present its explicit form here, since is represented by a $12 \times 12$ matrix. Clearly, the map $\alpha$ with component $\alpha|_{\mathfrak{m}} = \alpha|_{\mathfrak{EH}}$ and $\alpha|_{\mathfrak{l}} = \text{d}i$, together with the map $i$, satisfy the conditions of Proposition 4.3. Therefore

**Proposition 4.7.** The homogeneous space $\text{SL}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})$ carries a $\text{SL}(4, \mathbb{R})$-invariant $SO^*(6)$-structure. In particular, after the identification $\mathfrak{m} = [\mathfrak{EH}]$ we have $H_0 = H_0, \omega_0 = \omega_0$ for the standard hypercomplex structure $H_0$ and standard symplectic form $\omega_0$ on $[\mathfrak{EH}]$.

Let $\nabla$ be the $K$-invariant connection on $K/L = \text{SL}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})$ induced by the pair $(\alpha, i)$. Since $\alpha_{\nabla} = 0$, the $K$-invariant $SO^*(6)$-connection corresponding to the pair $(\alpha, i)$ is again the canonical connection $\nabla^u$. In fact it is not surprising that this example is of generic type $X_{1234567}$, since in the basis $B$ one can compute the following complicated expression for the torsion $T_{\nabla^u}$:

$$T_{\nabla^u}((a_1, \ldots, a_{12}), (b_1, \ldots, b_{12})) = -([[(a_1, \ldots, a_{12}), (b_1, \ldots, b_{12})]_\mathfrak{m}]_\mathfrak{B})$$

$$= (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}),$$
where
\[
\begin{align*}
t_1 &= a_1 \wedge b_3 + a_1 \wedge b_{12} + a_7 \wedge b_9 - a_5 \wedge b_9 + \frac{1}{2} (a_6 \wedge b_{11} - a_5 \wedge b_{12} + a_6 \wedge b_7 + a_{11} \wedge b_{12} + a_7 \wedge b_{12} + a_3 \wedge b_{11} \\
&\quad + a_3 \wedge b_7 - a_3 \wedge b_5),
t_2 &= -a_2 \wedge b_{12} + a_4 \wedge b_9 + a_8 \wedge b_9 - \frac{1}{2} a_3 \wedge b_{10} + \frac{1}{2} a_6 \wedge b_8 - a_2 \wedge b_3 - \frac{1}{2} a_{10} \wedge b_{12} - \frac{1}{2} a_3 \wedge b_4 - \frac{1}{2} a_8 \wedge b_{12} \\
&\quad + \frac{1}{2} a_6 \wedge b_{10} - \frac{1}{2} a_3 \wedge b_8 - \frac{1}{2} a_4 \wedge b_{12},
t_3 &= -a_4 \wedge b_5 + a_2 \wedge b_7 + a_7 \wedge b_{10} + a_2 \wedge b_{11} - a_8 \wedge b_{11} - a_1 \wedge b_{10} - a_5 \wedge b_{10} + a_6 \wedge b_9 + a_5 \wedge b_8 - a_4 \wedge b_7 \\
&\quad + a_1 \wedge b_9 - a_2 \wedge b_5 + a_7 \wedge b_9 - a_4 \wedge b_{11} + a_1 \wedge b_4 + a_1 \wedge b_2 - a_{10} \wedge b_{11},
t_4 &= a_9 \wedge b_{10} + a_2 \wedge b_9 - \frac{1}{2} a_2 \wedge b_{12} + \frac{1}{2} a_3 \wedge b_{10} - \frac{1}{2} a_6 \wedge b_8 + \frac{1}{2} a_2 \wedge b_3 + \frac{1}{2} a_{10} \wedge b_{12} + a_3 \wedge b_4 + \frac{1}{2} a_8 \wedge b_{12} \\
&\quad - \frac{1}{2} a_6 \wedge b_{10} + \frac{1}{2} a_3 \wedge b_8 - a_4 \wedge b_{12},
t_5 &= a_5 \wedge b_{12} - \frac{1}{2} a_6 \wedge b_{11} + a_9 \wedge b_{11} - \frac{1}{2} a_7 \wedge b_{12} + \frac{1}{2} a_1 \wedge b_3 - \frac{1}{2} a_6 \wedge b_7 - \frac{1}{2} a_{11} \wedge b_{12} - \frac{1}{2} a_3 \wedge b_{11} - \frac{1}{2} a_1 \wedge b_{12} \\
&\quad - a_1 \wedge b_9 - \frac{1}{2} a_3 \wedge b_7 - a_3 \wedge b_5, 
t_6 &= -2 a_2 \wedge b_7 - 2 a_7 \wedge b_{10} + 2 a_8 \wedge b_{11} + a_3 \wedge b_6 + 2 a_5 \wedge b_{10} - 2 a_1 \wedge b_8 + a_6 \wedge b_{12} + 2 a_2 \wedge b_5 + 2 a_4 \wedge b_{11} - 2 a_1 \wedge b_4, 
t_7 &= -\frac{1}{2} a_5 \wedge b_{12} + a_7 \wedge b_{12} + \frac{1}{2} a_1 \wedge b_3 + a_9 \wedge b_{11} - a_3 \wedge b_7 - \frac{1}{2} a_{11} \wedge b_{12} - \frac{1}{2} a_3 \wedge b_{11} + \frac{1}{2} a_1 \wedge b_6 - \frac{1}{2} a_3 \wedge b_5 \\
&\quad + \frac{1}{2} a_5 \wedge b_6 - \frac{1}{2} a_1 \wedge b_{12} - a_1 \wedge b_9, 
t_8 &= -a_9 \wedge b_{10} - a_2 \wedge b_9 + \frac{1}{2} a_2 \wedge b_6 + \frac{1}{2} a_2 \wedge b_{12} - \frac{1}{2} a_3 \wedge b_{10} - \frac{1}{2} a_2 \wedge b_3 - \frac{1}{2} a_{10} \wedge b_{12} + \frac{1}{2} a_3 \wedge b_4 - a_8 \wedge b_{12} \\
&\quad + \frac{1}{2} a_4 \wedge b_6 + a_3 \wedge b_8 + \frac{1}{2} a_4 \wedge b_{12}, 
t_9 &= a_4 \wedge b_5 - a_9 \wedge b_{12} - a_7 \wedge b_{10} + a_8 \wedge b_{11} - a_3 \wedge b_9 + a_2 \wedge b_5 - a_7 \wedge b_8 - a_1 \wedge b_4 - a_1 \wedge b_2 + a_{10} \wedge b_{11}, 
t_{10} &= -\frac{1}{2} a_2 \wedge b_6 - \frac{1}{2} a_2 \wedge b_{12} + a_4 \wedge b_9 + a_8 \wedge b_9 + a_3 \wedge b_{10} + \frac{1}{2} a_2 \wedge b_3 - a_{10} \wedge b_{12} - \frac{1}{2} a_3 \wedge b_4 - \frac{1}{2} a_8 \wedge b_{12} \\
&\quad - \frac{1}{2} a_4 \wedge b_6 - \frac{1}{2} a_3 \wedge b_8 - \frac{1}{2} a_4 \wedge b_{12} - \frac{1}{2} a_5, 
t_{11} &= b_{12} + a_5 \wedge b_9 - a_7 \wedge b_9 - \frac{1}{2} a_3 \wedge b_7 - \frac{1}{2} a_1 \wedge b_3 - \frac{1}{2} a_5 \wedge b_6 + a_{11} \wedge b_{12} + \frac{1}{2} a_1 \wedge b_{12} - a_3 \wedge b_{11} - \frac{1}{2} a_1 \wedge b_6 \\
&\quad - \frac{1}{2} a_7 \wedge b_{12} + \frac{1}{2} a_3 \wedge b_5, 
t_{12} &= -a_4 \wedge b_5 + a_2 \wedge b_7 + a_7 \wedge b_{10} - a_2 \wedge b_{11} - a_8 \wedge b_{11} + a_1 \wedge b_{10} - a_5 \wedge b_{10} - a_6 \wedge b_9 - a_5 \wedge b_8 + a_4 \wedge b_7 \\
&\quad + a_1 \wedge b_8 - a_2 \wedge b_5 + a_7 \wedge b_9 - a_4 \wedge b_{11} + a_1 \wedge b_4 + a_1 \wedge b_2 - a_{10} \wedge b_{11}.
\end{align*}
\]

5. $SO^*(2n)\text{Sp}(1)$-structures and the bundle of Weyl structures

In this section we present some constructions providing examples of $SO^*(2n)\text{Sp}(1)$-manifolds, arising within the context of cotangent bundles and Weyl structures. We begin with the following observation in the linear setting, which provides some motivation for what it follows.

**Proposition 5.1.** Let $(U, Q)$ be a quaternionic vector space. Then the cotangent space $T^* U \cong U \times U^*$ admits a canonical linear qs-H structure, where the scalar 2-form $\omega$ is the canonical symplectic form given by the natural pairing $U \times U^* \to \mathbb{R}$. Under the identification of $U$ with the right quaternionic vector space $\mathbb{H}^n$, the map
\[
\rho : \mathbb{H}^n \oplus (\mathbb{H}^n)^* \ni (u, \xi) \mapsto (u, \xi^t) \in \mathbb{H}^{2n}
\]
provides a natural quaternionic Darboux basis, see [CGW21, Definition A.7].
Proof. The linear map $\rho$ induces a Lie group homomorphism $GL(n, \mathbb{H}) \rightarrow GL(2n, \mathbb{H}) \rightarrow Sp(1)$ and it is a simple observation that $\rho_\ast(gl(n, \mathbb{H})) \subset so^\ast(4n)_0 \subset so^\ast(4n)$, where $so^\ast(4n)_0$ is the reductive part of the $[1]$-grading of $so^\ast(4n)$ presented in [CGW21, Proposition A.9]. We observe that in the bases provided by $\rho$ the natural linear quaternionic structure on $U \times U^\ast$ coincides with the standard quaternionic structure induced by the corresponding quaternionic Darboux basis on $T^\ast U$. Similarly, the natural pairing

$$((\cdot, \cdot)) : \mathbb{H}^n \oplus (\mathbb{H}^n)^\ast \rightarrow \mathbb{R}, \quad ((u, \xi)) = \xi(u), \quad \forall \ u \in \mathbb{H}^n, \xi \in (\mathbb{H}^n)^\ast,$$

corresponds to the standard symplectic form obtained by the quaternionic Darboux basis (and hence it is a scalar 2-form with respect the induced linear quaternionic structure on $U \times U^\ast$). \hfill \Box

Below we show that there are two possible ways to generalize this result to the manifold setting. One of them is based on the canonical almost symplectic structure $\omega_\mathcal{W}$ which exists on the bundle of Weyl structures over an almost quaternion manifold found in [CM19]. The other one relies on the canonical symplectic structure $\omega_\mathcal{C}$ which exists on the cotangent bundle of any manifold. Next our aim is to discuss conditions for existence of almost quaternionic structures $Q$ which are compatible with $\omega_\mathcal{C}$ and $\omega_\mathcal{W}$, in the sense that $\omega_\mathcal{C}$, respectively $\omega_\mathcal{W}$, becomes a scalar 2-form with respect to the associated almost quaternionic structure $Q$. A further aim is to proceed by comparing these two $SO^\ast(2n) \rightarrow Sp(1)$-structures, a procedure which allows us to derive a theorem in terms of the $\mathcal{P}$-tensor appearing in parabolic geometries.

5.1. Weyl structures. One of the methods used in geometry and theory of PDE’s is to introduce new ”jet variables” which make the geometric objects or differential equations under examination, linear (in these new variables). When one works in a coordinate free setting, the basic tool to relate the new variables with the initial manifold, is encoded by the notion of connections. The use of connections with torsion makes the background theory of semiholonomic jets unnecessarily complicated in applications, and one circumvents it via the use of Cartan geometries.

The basic object in the theory of Cartan geometries is the Cartan bundle, that is a principal bundle $\mathcal{G} \rightarrow N$ over a smooth manifold $N$ which should be viewed as the bundle of new ”jet coordinates”, with structure group $P$. Such a $P$ should be considered as the Lie group of ”jets of transition maps between the jet coordinates”. The actual assignment of the coordinates to points $u \in \mathcal{G}$ is provided by a Cartan connection of type $(G, P)$, that is an isomorphism $T_u \mathcal{G} \cong g$ to the Lie algebra $g$ of $G$. Of course, it is natural to require that this map is equivariant with respect to the ”change of coordinates” given by $P$ and that it reproduces the fundamental vector fields of the $P$-action on $\mathcal{G}$. In these coordinates, the tangent space is identified with $\mathcal{G} \times g/p$, where $\chi$ denotes the isotropy representation of $G/P$. Set $G_0 := G/\ker(\chi)$. Then, the quotient $\mathcal{G}_0 := \mathcal{G}/\ker(\chi)$ defines a $G_0$-structure on $N$.

Definition 5.2. The Weyl structures of the Cartan geometry are $G_0$-equivariant sections $\mathcal{G}_0 \rightarrow \mathcal{G}$.

So, the Weyl structures are the tools which provide the identification $\mathcal{G}_0 := \mathcal{G}/\ker(\chi)$. In particular, the Weyl structures are in a bijective correspondence with smooth sections of the bundle $\mathcal{G}/G_0$, which is therefore called the bundle of Weyl structures associated to the Cartan geometry (see [CS03, CS09, CM19] for Weyl structures related to parabolic geometries).

Let us assume that there is a $G_0$-invariant decomposition

$$g = g_- \oplus g_0 \oplus p_+,$$

where $g_- \cong g/p$ and $p_+$ is Lie algebra of $\ker(\chi)$. Then, the pullback on $\mathcal{G}_0$ of a Cartan connection by a section $\mathcal{G}_0 \rightarrow \mathcal{G}$ (Weyl structure), decomposes according to its image to

- the component with values in $g_-$, which is the tautological 1-form defining the underlying $G_0$-structure.
- the component with values in $g_0$, which is the connection 1-form of a $G_0$-connection.
the component with values in $p_+$, which is usually called the $P$-tensor, see \cite{CS03, CS09}.

**Remark 5.3.** In the case of almost quaternionic geometries, $G_0$ corresponds to the $GL(n, \mathbb{H}) \text{Sp}(1)$-structure and a Weyl structure is equivalent to an Oproiu connection. This is because the bundle $G$, viewed as a subbundle of the second order semiholonomic frame bundle, is spanned by all of the Oproiu connections. Moreover, the $P$-tensor carries the remaining information about the Cartan connection (which is uniquely determined by the other data as we review later).

### 5.2. The use of the bundle of Weyl structures.

To begin with and connect the above discussion with $SO^+(2n) \text{Sp}(1)$-structures, let us recall that the quaternionic projective space is the homogeneous space $N = \mathbb{HP}^n = G/P$, where $G = P \text{GL}(n + 1, \mathbb{H})$ is the projective linear group satisfying $\text{SL}(n+1, \mathbb{H})/\mathbb{Z}_2 \cong P \text{GL}(n+1, \mathbb{H})$ and $P$ is the parabolic subgroup stabilizing the quaternionic line spanned by the first basis vector in $\mathbb{H}^{n+1}$. $G/P$ serves as the flat model of the associated Cartan geometry. A Levi subgroup of $P$ is given by $G_0 = \text{GL}(n, \mathbb{H}) \text{Sp}(1)$ and we have the following $G_0$-invariant decomposition into $g_0$-modules:

$$g = \mathfrak{sl}(n+1, \mathbb{H}) = g_- \oplus g_0 \oplus p_+, \quad T_e P G/P \cong g_-, \quad p = g_0 \oplus p_+.$$ 

The Lie group $G$ coincides with the total space of the Cartan bundle $G \to N$ over $N$ and the Cartan connection is provided by the Maurer–Cartan form of $G$, that is we have a canonical trivialization of the tangent bundle

$$TG = G \times \mathfrak{sl}(n+1, \mathbb{H}).$$

Therefore, the homogeneous space $G/G_0$ coincides with the bundle of Weyl structures over $G/P = \mathbb{HP}^n$ and in particular the pseudo-Wolf space

$$\tilde{M} := \text{SL}(n+1, \mathbb{H})/(\text{GL}(1, \mathbb{H}) \text{SL}(n, \mathbb{H}))$$

covers $M = G/G_0 = P \text{GL}(n+1, \mathbb{H})/\text{SL}(n, \mathbb{H}) \text{Sp}(1)$. Note that locally, this is the analogue of the linear construction provided by Proposition 5.1, since $\exp(g_-)$ has an open orbit in $N$ which coincides with the quaternionic vector space $g_- \cong T_e \mathbb{HP}^n$. Therefore, as a corollary of Proposition 5.1 and \cite[Theorem 5.2]{CGW21} we state the following

**Proposition 5.4.** The $8n$ dimensional homogeneous space $M = G/G_0$ has a natural torsion-free $G$-invariant qs-$H$ structure. In particular, the invariant scalar 2-form $\omega_N$ is induced by the pairing between $g_-$ and $p_+$, provided by the Killing form corresponding to the Lie algebra $\mathfrak{sl}(n+1, \mathbb{H})$, that is $g_- \cong [E H]$ and $p_+ \cong [E H]^*$, respectively.

**Proof.** The $G_0$-equivariant projection $\pi : TG \to TM$ yields the decomposition

$$TM = G \times G_0 \left( \mathfrak{sl}(n+1, \mathbb{H})/g_0 \right) = G \times G_0 \left( g_- \oplus p_+ \right),$$

and so $TM$ inherits an almost quaternionic structure from the canonical linear quaternionic structures on $g_- \cong [E H]$ and $p_+ \cong [E H]^*$. In fact, as follows from \cite[Theorem 5.2]{CGW21} this is an invariant quaternionic structure, since the covering $\tilde{M} \to M$ is a quaternionic symplectomorphism. \hfill \Box

Next we shall show that by starting with a general almost quaternionic manifold $(N, Q)$, instead of the flat model $\mathbb{HP}^n$, we can generalize the above result by providing a universal construction within the category of non-integrable $SO^+(2n) \text{Sp}(1)$-structures. For this goal it is useful to construct the Cartan connection from local data first, i.e., a (local) quaternionic co-frame

$$\nu : TN \to g_- \cong [E H].$$

So, let us assign to $(N, Q)$ the Cartan bundle $G \to N$ together with a normal Cartan connection. Note that locally, a choice of a co-frame $\nu$ as above provides the local trivialization

$$G = N \times G_0 \times \exp(p_+).$$
The pullback on $N$ of the Cartan connection is the following matrix of 1-forms on $N$:

\[
\left( \begin{array}{c}
\gamma_{i} \nu^i \\
\nu
\end{array} \right) \in \left( \begin{array}{cc}
\mathfrak{sp}(1) + \mathbb{R} & \mathfrak{p}_+ \\
\mathfrak{g}_- & \mathfrak{gl}(n, \mathbb{H})
\end{array} \right).
\]

Here, the sum $\gamma_{i} + \Gamma_{i} : TN \to \mathfrak{g}_0$ can be viewed as the connection 1-form of an Oproiu connection $\nabla^Q$ and $P$ is the $P$-tensor in $T^*N \otimes T^*N$. Up to change of Oproiu connection, these are uniquely determined by the following normalization condition:

\[
\partial^* \kappa := \sum_{i} 2\{Z_i, \kappa(\cdot, X_i)\} = 0,
\]

where $\kappa : N \to \Lambda^2 \mathfrak{g}^*_+ \otimes \mathfrak{sl}(n+1, \mathbb{H})$ is the curvature of the Cartan connection, that is

\[
\kappa := d \left( \begin{array}{c}
\gamma_{i} \nu^i \\
\nu
\end{array} \right) + \{ \left( \begin{array}{c}
\gamma_{i} \nu^i \\
\nu
\end{array} \right), \left( \begin{array}{c}
\gamma_{i} \nu^i \\
\nu
\end{array} \right) \}.
\]

Here, $\{ \cdot \}$ is the Lie bracket in $\mathfrak{sl}(n+1, \mathbb{H})$ and $\{X_i\}, \{Z_i\}$ are dual bases of $\mathfrak{g}_-, \mathfrak{p}_+$, respectively. Let us also mention that the requirement of $\gamma_i$ taking values in $\mathfrak{sp}(1)$, together with the normalization condition $\partial^* \kappa = 0$, assigns to the co-frame $\nu$ a unique unimodular Oproiu connection and a unique symmetric $P$-tensor. This provides the construction of the Cartan connection starting with the co-frame $\nu$, since at the point

\[(x, g_0, \exp(Z)) \in N \times G_0 \times \exp(\mathfrak{p}_+) = \mathcal{G},\]

the Cartan connection consists of two parts: namely of the $\text{Ad}(g_0 \exp(Z))^{-1}$-action on the above pullback, and of the Maurer-Cartan form on $P = G_0 \times \exp(\mathfrak{p}_+)$.

Now, based on a local description one can observe that the choice of an Oproiu connection $\nabla^Q$ provides a global trivialization $\mathcal{G} = \mathcal{G}_0 \times \exp(\mathfrak{p}_+)$ and the pullback on $\mathcal{G}_0$ decomposes as above, which means that the $P$-tensor is globally defined. In particular,

**Corollary 5.5.** The bundle of Weyl structures $\mathcal{G}/G_0$ over $N$ trivializes to $\mathcal{G}_0 \times_{G_0} \exp(\mathfrak{p}_+)$. 

Clearly, sections of this bundle are in bijective correspondence with Oproiu connections on $N$. The formulas for how these trivializations (and the corresponding pullbacks) change with the change of the Oproiu connection, can be found in [CS09]. However we will not need them explicitly.

The main advantage of the use of the Cartan connections for the description of almost quaternionic structures is that it immediately gives rise to the following result.

**Theorem 5.6.** There is a functor from the category of almost quaternionic manifolds to the category of almost qs-H manifolds. This functor restricts to a functor from the category of quaternionic manifolds to the category of almost qs-H manifolds with symplectic scalar 2-form $\omega_N$.

**Proof.** Let $(N, Q)$ be an almost quaternionic manifold. This admits a canonical Cartan connection on the $P$-bundle $\mathcal{G} \to N$, which provides the trivialization

\[T \mathcal{G} = \mathcal{G} \times \mathfrak{sl}(n + 1, \mathbb{H}).\]

We have $M = \mathcal{G}/G_0$ and

\[TM = \mathcal{G} \times_{G_0} \mathfrak{sl}(n + 1, \mathbb{H})/\mathfrak{g}_0 = \mathcal{G} \times_{G_0} (\mathfrak{g}_- \oplus \mathfrak{p}_+),\]

where the equality does not depend on the Levi factor $G_0$. Therefore, as in Proposition 5.4, we obtain an almost qs-H structure on $M$. Since the Cartan connection and the corresponding trivialization is invariant with respect to all quaternionic automorphisms, this construction is a functor from the category of almost quaternionic manifolds to the category of almost qs-H manifolds. Then, the second claim follows from the first one and [CM19, Theorem 3.1].

Let us now illustrate Theorem 5.6 via an explicit example, by using an almost quaternionic manifold.
Example 5.7. Consider \( N = \mathbb{R}^8 = (x_1, \ldots, x_8) \) endowed with the quaternionic co-frame
\[
\nu = (d x_1, d x_2, d x_3, d x_4, d x_5 + x_1 d x_2, d x_6, d x_7, d x_8)
\]
providing an isomorphism \( T_N \mathbb{R}^8 \rightarrow [E] \), and let us denote by \( Q \) the induced almost quaternionic structure on \( N \). Following the above construction, we obtain the following matrix of 1-forms on \( N \) for the pullback of the Cartan connection along \( \nu \):
\[
C := \begin{pmatrix}
0 & 0 & 0 \\
d x_1 + d x_2 i + d x_3 j + d x_4 k & 0 & 0 \\
d x_5 + x_1 d x_2 + d x_6 i + d x_7 j + d x_8 k & \frac{1}{6}(2 d x_2 + 2 d x_1 i - d x_4 j + d x_3 k) & 0
\end{pmatrix}.
\]
Therefore, on the trivialization \( N \times \exp(p_+) = \mathbb{R}^8 \times p_+ = (x_1, \ldots, x_8, p_1, \ldots, p_8) \), there is the following co-frame provided by the pullback of the Cartan connection along \( \nu \) and the action of \( \exp(p_+) \):
\[
\begin{pmatrix}
\frac{1}{6} & - (p_1 - p_3 i - p_5 j - p_7 k) & - (p_3 - p_5 i - p_7 j - p_9 k) \\
0 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
p_1 - p_3 i - p_5 j - p_9 k & p_3 - p_5 i - p_7 j - p_9 k
\end{pmatrix} =
\begin{pmatrix}
c_1 & c_2 \\
d x_1 + d x_2 i + d x_3 j + d x_4 k & d x_5 + x_1 d x_2 + d x_6 i + d x_7 j + d x_8 k
\end{pmatrix},
\]
where
\[
c_1 = \{ (-p_1 + p_3 i + p_5 j + p_7 k) \cdot (d x_1 + d x_2 i + d x_3 j + d x_4 k) + (-p_5 + p_3 i + p_7 j + p_9 k) \cdot (d x_5 + x_1 d x_2 + d x_6 i + d x_7 j + d x_8 k) - (p_1 - p_3 i - p_5 j - p_9 k) + \frac{1}{6} (p_1 - p_3 i - p_5 j - p_9 k) \}
\]
\[
c_2 = \{ (-p_1 + p_3 i + p_5 j + p_7 k) \cdot (2 d x_2 + 2 d x_1 i - d x_4 j + d x_3 k) + (d p_1 - p_3 i - p_5 j - p_9 k) \}
\]
By using the above expressions, we can now provide the quaternionic Darboux basis (see [CGW21, Definition A.7]) via the isomorphism
\[
T(N \times \exp(p_+)) \rightarrow \mathbb{H}^4 : (d x_1 + d x_2 i + d x_3 j + d x_4 k, d x_5 + x_1 d x_2 + d x_6 i + d x_7 j + d x_8 k, \tilde{c}_1, \tilde{c}_2).
\]
In the quaternionic Darboux basis the almost quaternionic structure \( Q \) corresponds to right multiplication by \(-i, -j, -k\), and the scalar 2-form is expressed by
\[
\omega = -2 p_5 \frac{1}{3} d x_1 \wedge d x_2 + \frac{1}{6} p_8 x_1 \wedge d x_3 - \frac{1}{6} p_7 x_1 \wedge d x_4 + x_1 \wedge d p_1 + \frac{1}{6} p_7 d x_2 \wedge d x_3 + \frac{1}{6} p_8 d x_2 \wedge d x_4 + \frac{1}{6} p_7 d x_1 \wedge d x_2 \wedge d x_3
\]
\[
+ \frac{1}{6} p_8 d x_2 \wedge d x_4 + d x_2 \wedge d x_2 \wedge d p_2 + x_1 d x_2 \wedge d x_3 \wedge d p_3 - \frac{1}{3} p_5 d x_3 \wedge d x_4 + d x_3 \wedge d x_3 \wedge d p_3
\]
\[
+ d x_4 \wedge d x_4 + d x_3 \wedge d x_5 + d x_5 \wedge d x_6 + d x_6 \wedge d x_7 \wedge d x_7 + d x_7 \wedge d x_8 \wedge d x_8.
\]
Let us point out that \( d \omega \neq 0 \), and that the quaternionic structure is not torsion-free, i.e., \( T Q \neq 0 \). However, the (intrinsic) torsion of this example is too complicated to be discussed in full details here.

5.3. The use of the cotangent bundle. Let us now proceed with the second approach based on the cotangent bundle, and the observation that
\[
G_0 \times G_0 \exp(p_+)
\]
is naturally diffeomorphic to \( T^*N \). Locally, this diffeomorphism is given by the map
\[
t : N \times \exp(p_+) \rightarrow T^*N, \quad t(x, \exp(Z)) = (x, \nu^i(x)Z_i),
\]
for the local co-frame \( \nu \). Therefore,
\[
G \times G_0 (g_- \oplus p_+) = TM \cong TT^*N,
\]
where the second isomorphism above depends only on the Oprea connection \( \nabla \), corresponding to the almost quaternionic structure \( Q \) induced by \( \nu \). In other words, we have two (almost) symplectic forms on \( TT^*N \):
(1) the canonical symplectic form $\omega_\mathcal{C}$ on $T^*N$,
(2) the pullback $\omega_{W,\nabla} = t^*\omega_W$ on $T^*N$ of the almost symplectic form $\omega_W$ on $M$ provided by $\nabla$.

Moreover, we have two (almost) quaternionic structures:
(a) the quaternionic structure $Q_{\text{hor}}$ induced by the horizontal distribution of $\nabla$ on $T^*N$,
(b) the pullback $Q_{\text{pul}} = t^*Q_W$ on $T^*N$ of the almost quaternionic structure $Q_W$ on $M$ provided by $\nabla$.

We obtain the following compatibility statements.

**Theorem 5.8.** Under the above assumptions the following claims hold:

1. The (almost) symplectic forms $\omega_\mathcal{C}$ and $\omega_{W,\nabla}$ on $T^*N$ coincide, if and only if $(N,Q)$ is quaternionic and the Oprea connection $\nabla$ is unimodular. The unimodularity condition is equivalent to say that $\nabla$ preserves some volume form and the corresponding $\mathcal{P}$-tensor is symmetric.
2. The (almost) quaternionic structures $Q_{\text{hor}}$ and $Q_{\text{pul}}$ on $T^*N$ coincide, if and only if $\mathcal{P}$ has values in $[E,E]^*$. 
3. The pair $(Q_{\text{pul}},\omega_{W,\nabla})$ defines an almost quaternionic skew-Hermitian structure on $T^*N$.
4. The pair $(Q_{\text{hor}},\omega_{W,\nabla})$ defines an almost quaternionic skew-Hermitian structure on $T^*N$, if and only if
   \[
   0 = \mathcal{P}(X,Y) - \mathcal{P}(Y,X) + \mathcal{P}(JY,JX) - \mathcal{P}(JX,JY),
   \]
   for all $X,Y \in \Gamma(TN)$ and $J \in \Gamma(Z)$. This is equivalent to say that either $\nabla$ is unimodular, or $\mathcal{P}$ has values in $[E,E]^*$.
5. The pair $(Q_{\text{pul}},\omega_\mathcal{C})$ defines a smooth almost quaternionic skew-Hermitian structure on $T^*N$, if and only if $(N,Q)$ is quaternionic and either $Q_{\text{pul}} = Q_{\text{hor}}$, or $\omega_\mathcal{C} = \omega_{W,\nabla}$.
6. The pair $(Q_{\text{hor}},\omega_\mathcal{C})$ defines a smooth almost quaternionic skew-Hermitian structure on $T^*N$, if and only if $(N,Q)$ is a quaternionic manifold.

**Proof.** Let us choose a local quaternionic frame $\nu$. Then, the isomorphism $T_{(x,\exp(Z))}M \cong \mathfrak{g}_- \oplus \mathfrak{p}_+$ has the form
\[
\text{Ad}(\exp(Z))^{-1} \begin{pmatrix} \gamma_{\nu^i}^\nu & P_{\nu^i}^\nu \\ \Gamma_{\nu^i}^\nu & \nu^i \end{pmatrix} = \begin{pmatrix} \gamma_{\nu^i}^\nu - Z_i\nu^i P_{\nu^i}^\nu + \gamma_{\nu^i}^\nu Z_i - Z_i \Gamma_{\nu^i}^\nu \nu^i - Z_k \nu^i Z_j \end{pmatrix}.
\]

Let $(x^i)$ be local coordinates on $N$ and $(x^i,p_i)$ be the induced coordinates on $T^*N$. Then $\nu^i = \nu^i_j \ dx^j$ and thus we deduce that $N \times \exp(\mathfrak{p}_+) \longrightarrow T^*N$ has the form
\[
(x^i,Z_i) \mapsto (x^i,\nu^i_j Z_j).
\]

Thus,
\[
\partial_{x^i} + \sum_j (P_{kj} \nu^k + \gamma_{k\nu^k} Z_j - Z_i \Gamma_{kj}^l \nu^k - Z_k \nu^k Z_j) \partial_{Z_j}
\]
are the generators of $G \times_{G_0} \mathfrak{g}_-$ which we want to push forward to $T^*N$. The pushforward of $\partial_{x^i}$ is given by $\partial_{x^i} + \sum_j \frac{\partial}{\partial x^i} Z_k \partial_{p_j}$ and the pushforward of $\partial_{Z_i}$ is equal to $\sum_j \nu^i_j \partial_{p_j}$. Note also that $Z_i = (\nu^{-1})^i_j p_j$. Altogether, for the pushforward of $G \times_H \mathfrak{g}_-$ we compute
\[
\partial_{x^i} + \sum_j \frac{\partial}{\partial x^i} (\nu^{-1})^i_k p_j \partial_{p_j} + \sum_l (P_{kj} \nu^k + \gamma_{k\nu^k} (\nu^{-1})^i_j p_l - (\nu^{-1})^i_{pl} p_a \Gamma_{kj}^l \nu^k - (\nu^{-1})^i_{pl} p_a \nu^l_j (\nu^{-1})^k_p \nu^p \partial_{p_l}.
\]

Let us now use the same notation for the $\mathcal{P}$-tensor in the $(x^i,p_i)$ coordinates. Then, the above expression can be rewritten as
\[
\partial_{x^i} + \sum_j P_{ij} \partial_{p_j} + \theta(\nabla^\mathfrak{g}_-^p_{\partial_{x^k}} \partial_{x^k}) \partial_{p_i},
\]
where $\nabla^\uparrow p_i$ is the Oproiu connection differing by $\frac{1}{2} p_i$ (viewed as 1-form) from $\nabla$, and $\theta = p_i \, \mathrm{d} x^i$ is the Liouville form on $TT^*N$. The precise formula for the change of Oproiu connection by a 1-form follows the conventions of [CS09, Section 5.1.6].

Let us also mention that the horizontal subspace corresponding to $\nabla$ is spanned by vector fields of the form

$$\partial_{x^i} + \theta(\nabla_{\partial_{x^j}} \partial_{x^i}) \partial_{p_k}.$$  

We are ready now to prove the claims of the theorem. For the first claim, we compare the horizontal subspaces with the canonical symplectic form

$$\omega_C = - \mathrm{d} \theta = \mathrm{d} x^i \wedge \mathrm{d} p_i.$$  

We conclude that $\omega_C = \omega_{W,\nabla}$ if and only if both $\nabla$ is torsion-free (and thus also $\nabla^\uparrow p_i$) and moreover the $\mathcal{P}$-tensor is symmetric, that is $(N, Q)$ is a unimodular quaternionic manifold.

For the second claim, one should compare the horizontal subspaces one to each other, and conclude that the $\mathcal{P}$-tensor provides all the difference between the quaternionic structures $Q_{\text{hor}}$ and $Q_{\text{pul}}$. Consequently, the second claim follows because elements of $[\mathbb{E} \mathbb{E}]^*$ are quaternionic linear. Now, the third claim clearly holds.

For the fourth claim, set

$$\tilde{J}(X) := J(X) + P(JX) - JP(X),$$

where $X \in \Gamma(TM)$ and $J \in \Gamma(Q_{\text{pul}})$ satisfies $J^2 = - \mathrm{id}_{TM}$. Then, $\tilde{J}$ is the corresponding section of the quaternionic structure $Q_{\text{hor}}$. To check that the almost symplectic form $\omega_W$ is scalar, it suffices to check that

$$\omega_W(\tilde{J}(X), \tilde{J}(Y)) = 0,$$

for all $X, Y \in \Gamma(TN)$ and $J \in \Gamma(Q_{\text{pul}}), J^2 = - \mathrm{id}$. It is simple computation that this is equivalent to the claimed condition, because we know the possible irreducible submodules of $[\mathbb{E} \mathbb{E} \mathbb{E} \mathbb{E}]^*$ (this argument follows by [CGW21, Proposition 1.7]).

The fifth claim occurs directly by the difference $\theta(T(X,Y)) + P(X,Y) - P(Y,X)$ between the (almost) symplectic structures $\omega_C$ and $\omega_{W,\nabla}$, where $T$ denotes the torsion of $\nabla$. This is because the relation

$$\theta(T(X,Y)) = \theta(T(JX, JY))$$

can be satisfied only if the torsion vanishes, i.e., $T = 0$ (since this component of torsion is not $Q$-Hermitian in these two entries). Finally, combining the fourth and the fifth claim, the $\mathcal{P}$-parts cancel each other and the sixth claim follows. This completes our proof. \hfill $\square$

Let us now illustrate the situation of the above theorem via an example.

**Example 5.9.** We begin with $N = \mathbb{R}^8 = (x_1, \ldots, x_8)$ and the quaternionic co-frame

$$\nu := (\mathrm{d} x_1, \mathrm{d} x_2, \mathrm{d} x_3, \mathrm{d} x_4, \mathrm{d} x_5, \mathrm{d} x_6, \mathrm{d} x_7, \mathrm{d} x_8).$$

This induces a flat quaternionic structure, and has the following matrix of 1-forms on $N$ as the pullback of the Cartan connection along $\nu$:

$$C := \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{d} x_1 + \mathrm{d} x_2 i + \mathrm{d} x_3 j + \mathrm{d} x_4 k & 0 & 0 \\ \mathrm{d} x_5 + \mathrm{d} x_6 i + \mathrm{d} x_7 j + \mathrm{d} x_8 k & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

This selects a flat Oproiu connection $\nabla$ and if $\psi : N \to \exp(p_+)$ is the function corresponding to the change to another Oproiu connection $\nabla^\psi$, then the pullback of the Cartan connection along $\nu \cdot \psi$ becomes

$$C_\psi := \text{Ad}(\psi)^{-1} C + \psi^* \mu.$$
Here, \( \text{Ad} : G \to \text{Aut}(\mathfrak{g}) \) is the adjoint representation of \( G = \text{PGL}(3, \mathbb{H}) \), and \( \psi^\mu \) is the left logarithmic derivative of \( \psi : N \to \exp(\mathfrak{p}_+), \) i.e., the pullback of the Maurer-Cartan form
\[
\mu := (dp_1 - dp_2i - dp_3j - dp_4k, dp_5 - dp_6i - dp_7j - dp_8k)
\]
to \( N \). Nevertheless, in such a trivialization we obtain
\[
N \times \exp(\mathfrak{p}_+) = \mathbb{R}^8 \times \mathfrak{p}_+ = (x_1, \ldots, x_8, p_1, \ldots, p_8),
\]
and in this case the coordinates \((x_1, \ldots, x_8, p_1, \ldots, p_8)\) coincide with the coordinates on \( T^*N \). Therefore, similarly to Example 5.7 it is convenient to work with
\[
\begin{pmatrix}
1 & -(p_1 - p_2i - p_3j - p_4k) & -(p_5 - p_6i - p_7j - p_8k) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
(C_0 + d \exp(\mathfrak{p}_+)) & c_0^\psi & c_0^\psi \\
(C_0 + d \exp(\mathfrak{p}_+)) & c_0^\psi & c_0^\psi \\
(C_0 + d \exp(\mathfrak{p}_+)) & c_0^\psi & c_0^\psi
\end{pmatrix}
\]
where \( c_0^\psi = c_{11} + c_{12} + c_{13} + c_{14} \) and \( c_2^\psi = c_{21} + c_{22} + c_{23} + c_{24} \) consists of several parts:

- **c_{11} :=** \((-p_1 + p_2i + p_3j + p_4k)(d\ x_1 + d\ x_2i + d\ x_3j + d\ x_4k) + (-p_5 + p_6i + p_7j + p_8k)(d\ x_5 + d\ x_6i + d\ x_7j + d\ x_8k)) + (p_1 - p_2i - p_3j - p_4k)\)
- **c_{12} :=** \((-p_1 + p_2i + p_3j + p_4k)(d\ x_1 + d\ x_2i + d\ x_3j + d\ x_4k) + (p_5 - p_6i - p_7j - p_8k)) + (p_5 - p_6i - p_7j - p_8k)\),

which are independent on \( \psi \). We can also see that

- **c_{13} :=** \((-p_1 + p_2i + p_3j + p_4k)(d\ x_1 + d\ x_2i + d\ x_3j + d\ x_4k) + (p_1 - p_2i - p_3j - p_4k)\)
- **c_{14} :=** \((-p_1 + p_2i - p_3j + p_4k)(d\ x_1 + d\ x_2i - d\ x_3j - d\ x_4k)\)

These terms, and according to Theorem 5.8, the pair \((Q_{\text{aut}}, \omega_{WV})\) provides an almost quaternionic skew-Hermitian structure on \( T^*N \), which is induced by the quaternionic Darboux basis by using the following isomorphism:
\[
T(N \times \exp(\mathfrak{p}_+)) \to \mathbb{H}^4 : (d\ x_1 + d\ x_2i + d\ x_3j + d\ x_4k, d\ x_5 + d\ x_6i + d\ x_7j + d\ x_8k, c_0^\psi, c_2^\psi)
\]
In this basis the quaternionic structure \( Q \) corresponds to right multiplication by \(-i, -j, -k,\) and the scalar 2-form takes the standard form. Since this almost quaternionic skew-Hermitian structure is locally isomorphic to the quaternionic skew-Hermitian structure from Proposition 5.4, we can conclude that this is a torsion-free quaternionic skew-Hermitian structure. Moreover, and according to Theorem 5.8, the pair \((Q_{\text{hor}}, \omega_C)\) is an almost quaternionic skew-Hermitian structure on \( T^*N \), which is induced by the quaternion Darboux basis via the isomorphism
\[
T(N \times \exp(\mathfrak{p}_+)) \to \mathbb{H}^4 : (d\ x_1 + d\ x_2i + d\ x_3j + d\ x_4k, d\ x_5 + d\ x_6i + d\ x_7j + d\ x_8k, c_0^\psi, c_2^\psi)
\]
Finally, note that \( P_1 = c_{12} \) and \( P_2 = c_{22} \). Hence, if the tensors \( P_i \) \((i = 1, 2)\) satisfy the conditions of the Theorem 5.8, then the pair \((Q_{\text{hor}}, \omega_W)\) should be an almost quaternionic skew-Hermitian structure on \( T^*N \). We avoid presenting the corresponding quaternionic Darboux basis, since it has a long and complicated expression.
6. Further directions and open problems

In this final section we pose some questions and open problems related to the geometry of $SO^*(2n)$- and $SO^*(2n)\ Sp(1)$-structures. Some of them are studied in the third part of this series of works.

I) Examine curvature invariants, Bianchi identities, Ricci-type and other curvature types of $SO^*(2n)$- and $SO^*(2n)\ Sp(1)$-structures.

II) Realization of the pure algebraic types $\mathcal{X}_1, \ldots, \mathcal{X}_5$ of $SO^*(2n)\ Sp(1)$-structures, as well of the $Sp(1)$-invariant types $\mathcal{X}_1, \ldots, \mathcal{X}_7$ of $SO^*(2n)$-structures, specified in [CGW21]. Are there any empty classes? Provide a classification of manifolds having some certain algebraic type (e.g. skew-torsion, vectorial type), or certain holonomy with respect to $\nabla^H\omega$ or $\nabla^Q\omega$, respectively.

III) Provide a metric view point of $SO^*(2n)$-structures: Define such structures in terms of the three metrics of signature $(2n, 2n)$, and application of the approach discussed in Section 2 for the Levi-Civita connection corresponding to one of these metrics. Study the associated pseudo-Riemannian Dirac operators, see also problem X).

IV) Description of adapted twistor constructions (as an analogue of the twistor constructions in almost $h\mathbb{H}/q\mathbb{H}$ geometries, see [Sa82, Sw91]). Relate the given integrability conditions to such a description.

V) Provide the characterization of the quaternionic skew-Hermitian geometries in the image of the functor from Theorem 5.6.

VI) From the parabolic view point (see [CGH14]) it makes sense to derive the explicit relations with quaternionic geometries. That is, analyze the $SO^*(2n+2)$-orbits on the quaternionic projective space $\mathbb{HP}^n = GL(n + 1, \mathbb{H})/P$ (note that the open orbit is the symmetric space $SO^*(2n+2)/SO^*(2n)\ U(1)$ discussed in [CGW21, Section 5]), and interpret the parallel tractor scalar 2-forms. Investigate the corresponding first BGG operators and interpret the normal solutions in terms of a generalized Einstein condition.

VII) Quaternionic compactification of $SO^*(2n)\ U(1)$-structures and examination of the geometry on the boundary. In particular, what is the type of the parabolic geometry on the boundary?

VIII) Examine almost quaternion skew-Hermitian manifolds with large automorphism group, and classify those for which the dimension of the group is close to the maximal dimension.

IX) Classify non-symmetric homogeneous spaces $G/L$ admitting an invariant almost hypercomplex/quaternionic skew-Hermitian structure, with G semisimple and L reductive.

X) Provide a spinorial interpretation of $SO^*(2n)$-structures and $SO^*(2n)\ Sp(1)$-structures, and introduce the associated spinorial calculus and adapted Dirac operators. Study metaplectic geometry and symplectic Dirac operators related with metaplectic structures on 8n-dimensional quaternionic skew-Hermitian manifolds.

Appendix A. Topology of $SO^*(2n)$- and $SO^*(2n)\ Sp(1)$-structures

In this appendix we study some basic topological features of $SO^*(2n)$- and $SO^*(2n)\ Sp(1)$-structures. To do so, we should recall first some covering theory related to $SO^*(2n)$. There are several distinguished Lie groups with Lie algebra $so^*(2n)$, apart from $SO^*(2n)$. Firstly, there is the universal covering $\tilde{SO}^*(2n)$ of $SO^*(2n)$. This gives rise to the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{SO}^*(2n) \longrightarrow SO^*(2n) \longrightarrow 0.$$ 

Note that any representation of $so^*(2n)$ integrates to representation of $\tilde{SO}^*(2n)$, which however for the finite dimensional case is not faithful. In other words, $\tilde{SO}^*(2n)$ has no finite-dimensional faithful representations. At this point we should however recall that there is a maximal linear group attached to the Lie algebra $so^*(2n)$, also called the linearizer of $\tilde{SO}^*(2n)$ and denoted by $Spin^*(2n)$.
Any finite dimensional linear representation $\rho : \widetilde{SO}^*(2n) \to \text{Aut}(W)$ of $\widetilde{SO}^*(2n)$ factors as $\rho = \rho_0 \circ \psi$, where $\psi : \widetilde{SO}^*(2n) \to \text{Spin}^*(2n)$ is a (covering) homomorphism and $\rho_0 : \text{Spin}^*(2n) \to \text{Aut}(W)$ is a linear representation of $\text{Spin}^*(2n)$, that is the following diagram commutes:

$$
\begin{array}{ccc}
\text{SO}^*(2n) & \xrightarrow{\psi} & \text{Spin}^*(2n) \\
\downarrow{\rho} & & \downarrow{\rho_0} \\
\text{Aut}(W) & & .
\end{array}
$$

Based on the embedding $\text{SO}^*(2n) \subset \text{SO}(2n, 2n)$ one should view $\text{Spin}^*(2n)$ as a subgroup of $\text{Spin}(2n, 2n)$. In particular, the faithful representation of $\text{Spin}^*(2n)$ is given by the direct sum

$$R(\pi_{n-1}) \oplus R(\pi_n)$$

of the (finite-dimensional) half-spin representations of $\mathfrak{so}^*(2n)$, which is the reason behind our notational convention for $\text{Spin}^*(2n)$. In particular, the group $\text{Spin}^*(2n)$ is a (double) covering of $\text{SO}^*(2n)$, which induced the following short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}^*(2n) \xrightarrow{\lambda} \text{SO}^*(2n) \longrightarrow 0,$$

such that $\lambda^{-1}_*(\text{spin}^*(2n)) = \mathfrak{so}^*(2n)$.

In small dimensions $n \leq 4$ one may describe further Lie algebras isomorphisms, which we list below and which can be easily interpreted in terms of Satake diagrams (see [Hel78, Har90, OnV90] for more details). For $n = 1$, $\text{SO}^*(2)$ is isomorphic to $\text{SO}(2) = U(1)$, hence it is compact and non-simple. For $n = 2, 3, 4$, the corresponding Satake diagrams admit the following illustration (see [On04, CGW21])

Hence:

- For $n = 2$ there is a Lie algebra isomorphism $\mathfrak{so}^*(4) \cong \mathfrak{su}(2) \oplus \mathfrak{sl}(2, \mathbb{R})$. Moreover, $\text{Spin}^*(4)$ coincides with the Lie group $\text{SU}(2) \times \text{SL}(2, \mathbb{R})$ and we have a double covering $\text{SU}(2) \times \text{SL}(2, \mathbb{R}) \to \text{SO}^*(4)$. Hence $\text{SO}^*(4)$ is semisimple and non-simple.
- For $n = 3$, there is a Lie algebra isomorphism $\mathfrak{so}^*(6) \cong \mathfrak{su}(1, 3)$. Since $Z(\text{SO}^*(6)) = \mathbb{Z}_2$ and $Z(\text{SU}(1, 3)) = \mathbb{Z}_4$, we get that $\text{Spin}^*(6)$ coincides with the Lie group $\text{SU}(1, 3)$ and the map $\text{SU}(1, 3) \to \text{SO}^*(6)$ defines a double covering.
- For $n = 4$, there is a Lie algebra isomorphism $\mathfrak{so}^*(8) \cong \mathfrak{so}(2, 6)$. The half-spin groups associated to $\text{SO}(2, 6)$ are isomorphic to $\text{SO}^*(8)$. On the other side, $\text{Spin}^*(8)$ and $\text{Spin}(2, 6)$ are both maximal linear groups of $\mathfrak{so}^*(8)$ and hence are identical. This means that there are covering homomorphisms $\text{Spin}(2, 6) \to \text{SO}^*(8) \to P\text{SO}(2, 6)$, where the latter group is the one which acts faithfully on the projectivization of $\mathbb{R}^8$. In particular, $Z(\text{Spin}^*(8)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ (see [OnV90, p. 320]). However, $\text{SO}(2, 6)$ is not isomorphic to $\text{SO}^*(8)$ as Lie groups, since for example the have different maximal compact subgroups, i.e. $\text{SO}(2, 6) \not\cong U(4)$. Another argument is coming from topology: $\text{SO}^*(8)$ is connected but $\text{SO}(2, 6)$ has two connected components.

---

1In [OnV90] the group $\text{SO}^*(2n)$ is denoted by $U^*_n(\mathbb{H})$, see page 226.
Next we discuss topological obstructions related to $\SO^*(2n)$-structures and $\SO^*(2n)\Sp(1)$-structures. We begin with the maximal compact subgroup $\U(n)$ of $\SO^*(2n)$, which can be viewed as the following block diagonal matrix
\[
\left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in \U(n) \right\} \subset \U(2n),
\]
where $\U(2n)$ is the maximal compact subgroup of $\Sp(4n, \mathbb{R})$. This shows that $\U(n)$ does not act irreducibly on $E = \mathbb{C}^{2n}$. However, $\Sp(1)$ coincides with the centralizer $\C_U(2n)(\U(n))$ and hence also lies inside $\U(2n) \subset \Sp(4n, \mathbb{R})$. Thus, we finally obtain the following necessary topological conditions, naturally arising within the theory of $U(k)$-structures (see [L55, p. 200] or [CE14, p. 359] and recall also that according to [MS74, Problem 14-B], the odd-degree Stiefel-Whitney classes of a complex vector bundle must vanish).

**Lemma A.1.** Let $M$ be a smooth $4n$-dimensional manifold admitting a $\SO^*(2n)\Sp(1)$-structure or a $\SO^*(2n)$-structure. Then all odd Stiefel-Whitney class $w_{2k+1}(M) \in H^{2k+1}(M; \mathbb{Z}_2)$ must vanish, $w_{2k+1}(M) = 0$, for any $k \in \mathbb{N}$, and the even one must have integral lifts.

Thus, for instance, such manifolds should be oriented, that is $w_1(M) = 0$, a fact which agrees with our observation in [CGW21] and the orientation constructed via the volume form $\text{vol} = \omega^{2n}$. Note however that the above condition is only necessary and not sufficient: For instance the tangent bundle $TS^n$ of $S^n$ always satisfies these conditions, but only $S^2$ and $S^6$ admit an almost complex structure, and in particular a $\U(n)$-structure. For the compact case one can pose further topological constrains related to the existence of an almost complex structure, which can be read for example in terms of the Euler characteristic of $M$, see Theorem 3.4 in the appendix of [DeG-A21].

Let us now fix an almost $qs$-$H$ manifold $(M, Q, \omega)$ $(n > 1)$, and denote by $\pi : Q \to M$ the corresponding principal $\SO^*(2n)\Sp(1)$-bundle over $M$. Then, any point in $Q$ provides an identification between $T_xM$ and $[EH]$, for any $x \in M$, where as before $E$ and $H$ are the standard representations $E$ and $H$ of $\SO^*(2n)$ and $\Sp(1)$ respectively (see also [CGW21, Section 2]). Next we study the lifting problem of the $G = \SO^*(2n)\Sp(1)$-structure to an $\tilde{G} = \SO^*(2n) \times \Sp(1)$-structure that allows to associate global bundle analogies of the modules $E$ and $H$ over $M$.

To do so, we view $Q$ as an element of the first Čech cohomology group $H^1(M; G)$ with coefficients in the sheaf of smooth $G$-valued functions on $M$. Moreover, $\tilde{G}$ is the double cover of $G$ and so the short exact sequence
\[
0 \longrightarrow \mathbb{Z}_2 \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 0
\]
induces the coboundary homomorphism
\[
\delta : H^1(M; G) \to H^2(M; \mathbb{Z}_2).
\]
By setting $\epsilon := \delta(Q) \in H^2(M; \mathbb{Z}_2)$ we obtain a canonical cohomology class on $M$, which we will call the Marchiafava-Romani class. Then, as an analogue of the almost quaternionic-Hermitian case (see [MR75, Sa82, Sa86]), we deduce that

**Lemma A.2.** The Marchiafava-Romani class $\epsilon \in H^2(M; \mathbb{Z}_2)$ is precisely the obstruction to lifting $Q$ to a principal $\tilde{G}$-bundle $\tilde{Q}$ over $M$, or equivalently to the global existence of the vector bundles $E = Q \times_G E$ and $H = Q \times_G H$ over $M$.

Thus, when $\epsilon = 0$, the bundles $E$ and $H$ are globally defined and thus
\[
T^C M \cong E \otimes H.
\]
Recall now that given a pair $(Q, \omega)$ as above, we may visualise the almost quaternionic structure $Q \subset \End(TM)$ via the coefficient bundle associated with the $\SO^*(2n)\Sp(1)$-representation $[S^2 H]^*$ (see for example [Sa86] or [CGW21, Lemma 1.2]). Thus, one can identify $\epsilon$ with the second Stiefel-Whitney class of $Q$, i.e. $\epsilon = w_2(Q)$, see [MR75, Sa82]. Moreover, in terms of the twistor bundle...
Z = \mathbb{P}(H) \to M \text{ which is naturally associated to the almost quaternionic structure } Q, \text{ one obtains the following}

**Proposition A.3.** Let \((M, Q, \omega)\) be an almost qs-H manifold. Then, the second Stiefel-Whitney class \(w_2(M) \in H^2(M; \mathbb{Z}_2)\) satisfies

\[
w_2(M) = \begin{cases} 
\epsilon, & \text{if } n = \text{odd}, \\
0, & \text{if } n = \text{even}.
\end{cases}
\]

**Proof.** The proof is the same with the one given in [Sal82], for quaternionic Kähler manifolds, although the same result and proof applies for general almost quaternion Hermitian manifolds. □

As is well-known, on an \(4n\)-dimensional oriented manifold \(M\) the vanishing of the second Stiefel-Whitney class \(w_2\) guarantees the existence of spin structures, see for example [LM89], and also of metaplectic structures, see for example [HH06]. Thus, for \(8n\)-dimensional almost qs-H manifolds, Proposition A.3 certifies that such a manifold should admit these types of structures. In particular, by the inclusion \(SO^*(2n) \cdot Sp(1) \subset Sp(4n, \mathbb{R})\) we conclude that the vanishing of \(w_2(M)\) for \(n = 2m\), guarantees the reduction of the metaplectic structure to a certain 2-fold covering of \(SO^*(2n) \cdot Sp(1)\). Such a (unique) 2-fold covering is given by \(Spin^*(2n) \cdot Sp(1)\), and the corresponding lifts of the \(G\)-structure can been seen as a generalization of the so-called \(Spin^d\)-structures, discussed in [N95, B99, AIM21]. We plan to examine \(Spin^*(2n) \cdot Sp(1)\)-structures on \(8n\)-dimensional almost qs-H manifolds \((M, Q, \omega)\) in a forthcoming paper in this series.

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