An analogue of Amitsur's property for the ring of pseudo-differential operators

H. Melis Tekin Akcin

Abstract

Let $R$ be a ring with a derivation $\delta$. In this paper, we prove that an analogue of Amitsur's property holds for left T-nilpotent radideals of pseudo-differential operator rings $R((x^{-1}; \delta))$, where $R$ is a $\delta$-compatible ring. As a direct consequence of this fact, we obtain an alternative characterization of the prime radical of $R((x^{-1}; \delta))$.

Key Words: Amitsur’s property, $\delta$-compatible rings, left T-nilpotent, prime radical, pseudo-differential operator ring.

Mathematics subject classification 2010: 16N40, 16N60, 16S32.

1 Introduction

In this paper, we study rings of pseudo-differential operators, which can be seen as noncommutative generalizations of commutative Laurent series rings. The idea of using the algebra of pseudo-differential operators $R((\delta^{-1}))$ is started with Schur (see [13]), later works on these algebras have done by Goodearl [7] and Tuganbaev [16]. In [16], Tuganbaev has studied the ring theoretical properties of pseudo-differential operator rings. Besides being used to construct new examples in ring theory, these rings also have some applications in different fields of mathematics, see [5] and [15] for more information.

Throughout this paper, $R$ denotes an associative ring with identity (unless otherwise stated), an ideal means a two-sided ideal and the notation $\leq$ is used to denote ideals. Let $R$ be a ring equipped with a derivation $\delta$ (i.e., $\delta$ is an additive map on $R$ satisfying the product rule $\delta(ab) = \delta(a)b + a\delta(b)$, for each $a, b \in R$). The pseudo-differential operator ring over the coefficient ring $R$ formed by formal series $\sum_{i=-\infty}^{n} a_{i}x^{i}$, where $x$ is a variable, $n$ is an integer (maybe negative), and the coefficients $a_{i}$ belong to the ring $R$ and is denoted by the notation $R((x^{-1}; \delta))$. In [16] Proposition 7.2, it is verified that $R((x^{-1}; \delta))$ satisfies all the ring axioms, where the addition is defined as usual and multiplication is defined with respect to the relations

$$xa = ax + \delta(a), \quad x^{-1}a = \sum_{i=0}^{\infty}(-1)^{i}\delta^{i}(a)x^{-i-1},$$

for all $a \in R$. If $\delta$ is the zero derivation, then there exists an isomorphism of the ring $R((x^{-1}; \delta))$ onto the ordinary Laurent series ring $R((x))$ (This isomorphism
maps $x^{-1}$ onto $x$). The Amitsur’s property of a radical says that the radical of a polynomial ring is again a polynomial ring. This nomenclature is used since it was Amitsur who initially proved that many classical radicals such as the prime, Levitzki, Jacobson, and BrownMcCoy have this property. Moreover, in [8] Proposition 4.10, it is proved that the left T-nilpotent radideal of a polynomial ring also satisfies the Amitsur’s property. It is a natural question to extend the Amitsur’s property for other ring extensions. As a generalization of Amitsur’s property, in [9], the concept of $\delta$-Amitsur property is introduced for the ring of differential operators. In [4], Ferrero, Kishimoto and Motose have proved that the Jacobson, prime and Wedderburn radicals again possess $\delta$-Amitsur’s property. Also, in [9, Theorem 3.3], it is showed that the left T-nilpotent radideal of the ring of differential operators satisfies the $\delta$-Amitsur property.

In their seminal papers [8] and [9], the authors have studied how to characterize the left T-nilpotent radideals of skew Laurent polynomial rings and the rings of differential operators. Our primary motivation in this paper is to give a description of the left T-nilpotent radideals of pseudo-differential operator rings. Before proceeding the main results, we need to recall some concepts and definitions which will be useful while discussing the left T-nilpotent radideals of pseudo-differential operator rings.

Let $R$ be a ring and $\delta$ be a derivation of $R$, we say that a subset $S \subseteq R$ is a $\delta$-subset if $\delta(S) \subseteq S$. Let $I$ be an ideal of $R$. If $I$ is a $\delta$-subset of $R$, then $I$ is called a $\delta$-ideal of $R$. According to [10], an ideal $I$ is called a $\delta$-compatible ideal if for each $a, b \in R$, $ab \in I$ implies $a\delta(b) \in I$. If the zero ideal is $\delta$-compatible, then the ring $R$ is called $\delta$-compatible.

Let $R$ be a ring with a derivation $\delta$ and if $I$ is a $\delta$-ideal of $R$, then $\overline{\delta} : R/I \rightarrow R/I$ is a derivation of $R/I$ induced by the derivation $\delta$.

**Lemma 1.1.** [10] Lemma 2.1] Let $R$ be a ring and $\delta$ be a derivation of $R$. Assume that $R$ is $\delta$-compatible. If $ab = 0$, then $a\delta^n(b) = \delta^m(a)b = 0$ for any non-negative integers $n, m$.

If $S$ is a subset of a ring $R$, we denote the left annihilator of $S$ in $R$ by the notation $(0 : S)$. For an arbitrary ring $R$, the ideals $R^{(\alpha)}$ are defined recursively in [5] as follows: $R^{(0)} = 0$, $R^{(\alpha+1)}/R^{(\alpha)} = (0 : R/R^{(\alpha)})$ and $R^{(\alpha)} = \bigcup_{\beta<\alpha} R^{(\beta)}$, if $\alpha$ is a limit ordinal. If $R^{(\mu)} = R$ for some ordinal $\mu$, then the series

$$0 = R^{(0)} \subseteq R^{(1)} \subseteq \ldots \subseteq R^{(\alpha)} \subseteq \ldots \subseteq R^{(\mu)} = R$$

is called the upper left annihilator series of $R$.

**Remark 1.** It can be seen easily that, by using transfinite induction [3], Proposition 9, Section 1.3], the ideals defined as above are actually $\delta$-ideals, i.e., $\delta(R^{(\alpha)}) \subseteq R^{(\alpha)}$, for each ordinal $\alpha$.  

2
2 Main Results

In radical theory, it is interesting to characterize the radicals of ring extensions in terms of the base rings. In [8] and [9], the authors have proved the analogue of this question for the left T-nilpotent radicals of skew Laurent polynomial rings and the ring of differential operators, by using König’s tree lemma. In this section, we investigate the left T-nilpotent radical of pseudo-differential operator rings $R((x^{-1};\delta))$, where $R$ is a $\delta$-compatible ring. One difficulty with extending the situation for pseudo-differential operator rings is that we no longer have a finite coefficient set. We begin this section by giving the concept of left T-nilpotent set and its properties.

Definition 2.1. [11] A set $S \subseteq R$ is called left T-nilpotent if for any countable sequence of elements $s_1, s_2, \ldots \in S$, there exists an integer $k \geq 1$ such that $s_1s_2 \ldots s_k = 0$.

Note that right T-nilpotent sets are defined in a similar way and we say that a set is T-nilpotent, if it is both left and right T-nilpotent. The terms are due to Bass [2], but the concepts were introduced by Levitzki [12]. By the very definition, it is easy to see that any subset of a left T-nilpotent set is again left T-nilpotent. Also, if an ideal is left T-nilpotent, then it is nil. Moreover, if an ideal is nilpotent, then it is left T-nilpotent.

Proposition 2.2. [11, Proposition 23.15] Let $R$ be a ring and $I$ be an ideal of $R$. If $I$ is left T-nilpotent, then $I \subseteq P(R)$, where $P(R)$ denotes the prime radical of $R$.

Lemma 2.3. [8, Lemma 4.2] Let $R$ be a ring, $I \subseteq R$ and $J$ be an ideal of $R$. If $I$ and $J$ are left T-nilpotent, then so is $I + J$.

Proposition 2.4. [8, Proposition 4.3] Let $R$ be a ring, $I \subseteq R$ and $J$ be a one-sided ideal of $R$.

1. If $J$ is left T-nilpotent, then $RJ$ is left T-nilpotent.
2. If $I$ and $J$ are left T-nilpotent, then $I + J$ is left T-nilpotent.

For a deeper knowledge and basic results about left T-nilpotency, see [8, section 4] and [11, section 23]. The left T-nilpotent radical of $R$ is denoted by $\mathcal{I}_l$ and defined as the ideal function given by

$$\mathcal{I}_l(R) = \sum\{I \subseteq R : I \text{ is left T-nilpotent}\}.$$ 

As one might expect, the ideal $\mathcal{I}_l(R)$ does not need to be left T-nilpotent itself.

Let $I((x^{-1};\delta))$ be the subset of $R((x^{-1};\delta))$ whose coefficients are all contained in $I$. We begin with the following lemma which gives the relations between the ideals of $R((x^{-1};\delta))$ and $R$.

Lemma 2.5. Let $R$ be a ring and $\delta$ be a derivation of $R$. Then the following statements hold:
(1) If \( I \) is a right ideal of \( R \), then \( I((x^{-1}; \delta)) \) is a right ideal of \( R((x^{-1}; \delta)) \).

(2) Let \( I \) be an ideal of \( R \). Then \( I((x^{-1}; \delta)) \) is an ideal of \( R((x^{-1}; \delta)) \) if and only if \( I \) is a \( \delta \)-ideal of \( R \).

(3) If \( I \) is a nilpotent \( \delta \)-ideal of \( R \), then \( I((x^{-1}; \delta)) \) is a nilpotent ideal of \( R((x^{-1}; \delta)) \).

Proof. The proof can be seen easily, by using [14] Lemma 2.1. \( \Box \)

**Lemma 2.6.** Let \( R \) be a ring and \( \delta \) be a derivation of \( R \). Assume that \( R \) is \( \delta \)-compatible. If \( aR \) is left \( T \)-nilpotent, then \( \delta^i(a)R \) is left \( T \)-nilpotent for each non-negative integer \( i \).

Proof. Fix an arbitrary sequence of elements \( r_1, r_2, \ldots \in R \). By the assumption, there exists an integer \( k \geq 1 \) such that \( ar_1ar_2\ldots ar_k = 0 \). Since \( R \) is \( \delta \)-compatible, by Lemma [1,1] we have \( \delta^i(a)r_1\delta^i(a)r_2\ldots \delta^i(a)r_k = 0 \) for each non-negative integer \( i \). This means that \( \delta^i(a)R \) is left \( T \)-nilpotent. \( \Box \)

The following ring-theoretic characterization of \( T \)-nilpotence is obtained by Levitzki [12]. We state this result without the proof (the interested reader is referred to see [12] and [5, Theorem 1.3], for more information).

**Theorem 2.7.** Let \( R \) be a ring (maybe without identity). Then \( R \) is left \( T \)-nilpotent if and only if the upper left annihilator series of \( R \) exists.

This result enables us to obtain an analogue of Amitsur’s property for the left \( T \)-nilpotent radideal of pseudo-differential operator rings.

**Lemma 2.8.** Let \( R \) be a ring and \( \delta \) be a derivation of \( R \). If \( I \) is a left \( T \)-nilpotent \( \delta \)-ideal of \( R \), then \( I((x^{-1}; \delta)) \) is a left \( T \)-nilpotent ideal of \( R((x^{-1}; \delta)) \).

Proof. We will use Levitzki’s characterization to prove that \( I((x^{-1}; \delta)) \) is left \( T \)-nilpotent. Since \( I \) is left \( T \)-nilpotent, the upper left annihilator series of \( I \) exists. Let

\[
0 = I^{(0)} \subseteq I^{(1)} \subseteq \ldots \subseteq I^{(\alpha)} \subseteq \ldots \subseteq I^{(\nu)} = I,
\]

where \( I^{(\alpha+1)}/I^{(\alpha)} = (0 : I/I^{(\alpha)}) \) and \( I^{(\alpha)} = \bigcup_{\beta<\alpha} I^{(\beta)} \), if \( \alpha \) is a limit ordinal. We wish to obtain the upper left annihilator series for \( I((x^{-1}; \delta)) \). By the Remark [1] and Lemma [2.3.2], we have that \( I^{(\alpha)}((x^{-1}; \delta)) \) is an ideal of \( I((x^{-1}; \delta)) \) for any ordinal \( \alpha \). Let \( f(x) = \sum_{i=-\infty}^{m} a_ix^i \in I^{(\alpha+1)}((x^{-1}; \delta)) \), then for any \( g(x) = \sum_{i=-\infty}^{m} b_ix^i \in I((x^{-1}; \delta)) \) we have that each coefficient of the product \( f(x)g(x) \) is a \( \mathbb{Z} \)-linear combination of terms of the form

\[
a_i\delta^k(b_j),
\]

where \( a_i \) is any coefficient of \( f(x) \) for \( i \leq n \), \( b_j \) is any coefficient of \( g(x) \) for \( j \leq m \) and \( k \) is a non-negative integer. Since \( I \) is a \( \delta \)-ideal and \( a_i \in I^{(\alpha+1)} \) for each \( i \leq n \), by the construction of the upper left annihilator series we obtain \( a_i\delta^k(b_j) \in I^{(\alpha)} \) for each \( i \leq n, j \leq m \) and non-negative integer \( k \). This means that \( f(x)g(x) \in I^{(\alpha)}((x^{-1}; \delta)) \). If \( \alpha \) is a limit ordinal, then we have
\[ I^{(\alpha)}((x^{-1}; \delta)) = \left( \bigcup_{\beta < \alpha} I^{(\beta)} \right)(x^{-1}; \delta) = \bigcup_{\beta < \alpha} I^{(\beta)}((x^{-1}; \delta)). \]

Conversely, assume that \( f(x) = \sum_{i=-\infty}^{n} a_{i} x^{i} \in I((x^{-1}; \delta)) \) such that
\[ f(x)I((x^{-1}; \delta)) \subseteq I^{(\alpha)}((x^{-1}; \delta)). \]

We need to show that \( f(x) \in I^{(\alpha+1)}((x^{-1}; \delta)) \). By the assumption, we have \( f(x)a \in I^{(\alpha)}((x^{-1}; \delta)) \) for each \( a \in I \). Since the leading term of \( f(x)a \) is \( a_{n}a \), we have that \( a_{n}a \in I^{(\alpha)} \) for each \( a \in I \). So, we obtain \( a_{n} \in I^{(\alpha+1)} \).

Set \( f'(x) = f(x) - a_{n}x^{n} \). Then
\[ f'(x)I((x^{-1}; \delta)) = f(x)I((x^{-1}; \delta)) - a_{n}x^{n}I((x^{-1}; \delta)). \]

By using the assumption and the fact that \( a_{n} \in I^{(\alpha+1)} \), we get
\[ f'(x)I((x^{-1}; \delta)) \subseteq I^{(\alpha)}((x^{-1}; \delta)). \]

If we use the same argument as above, we see that the leading coefficient of \( f'(x) \) belongs to \( I^{(\alpha+1)} \). Continuing this procedure, we get \( a_{i} \in I^{(\alpha+1)} \) for each \( i \leq n \). Thus, \( f(x) \in I^{(\alpha+1)}((x^{-1}; \delta)) \). Therefore, we obtain
\[ 0 = I^{(0)}((x^{-1}; \delta)) \subseteq I^{(1)}((x^{-1}; \delta)) \subseteq \ldots \subseteq I^{(\alpha)}((x^{-1}; \delta)) \subseteq \ldots \subseteq I((x^{-1}; \delta)) \]

is the upper left annihilator series of \( I((x^{-1}; \delta)) \), as desired. \( \square \)

**Theorem 2.9.** Let \( R \) be a ring and \( \delta \) be a derivation of \( R \). Assume that \( R \) is \( \delta \)-compatible. Then
\[ I_{l}(R((x^{-1}; \delta))) = I_{l,\delta}(R)((x^{-1}; \delta)), \]
where \( I_{l,\delta}(R) = \{ a \in R : \sum_{j=0}^{\infty} \delta^{j}(a) R \text{ is left } T\text{-nilpotent} \} \).

**Proof.** By Proposition 2.8, we have \( I_{l,\delta}(R) \) is a \( \delta \)-ideal of \( R \). Let \( f(x) = \sum_{i=-\infty}^{n} a_{i} x^{i} \in I_{l,\delta}(R)((x^{-1}; \delta)) \). Since \( a_{i} \in I_{l,\delta}(R) \) for each \( i \leq n \), we have that \( \sum_{j=0}^{\infty} \delta^{j}(a_{i}) R \) is a left \( T \)-nilpotent \( \delta \)-ideal of \( R \) for each \( i \leq n \). By Lemma 2.8 we get that \( a_{i} R((x^{-1}; \delta)) \) is a subset of the left \( T \)-nilpotent ideal \( \sum_{j=0}^{\infty} \delta^{j}(a_{i}) R((x^{-1}; \delta)) \)

of \( R((x^{-1}; \delta)) \), for each \( i \leq n \). Therefore, \( a_{i} x^{i} \in I_{l}(R((x^{-1}; \delta))) \) for each \( i \leq n \). Hence, \( f(x) \in I_{l}(R((x^{-1}; \delta))) \).

Conversely, let \( f(x) = \sum_{i=-\infty}^{n} a_{i} x^{i} \in I_{l}(R((x^{-1}; \delta))) \), where \( a_{i} \in R \) for all \( i \leq n \). We want to show that \( a_{i} \in I_{l,\delta}(R) \), for all \( i \leq n \). Fix a sequence of elements \( r_{1}, r_{2}, \ldots \in R \) and a sequence of non-negative integers \( i_{1}, i_{2}, \ldots \) and also let us define the following sequence of elements
\[ g_{1}(x) = \sum_{i=-\infty}^{m_{1}} r_{1} x^{i}, \quad g_{2}(x) = \sum_{i=-\infty}^{m_{2}} r_{2} x^{i}, \ldots \in R((x^{-1}; \delta)), \]

where \( m_{1}, m_{2}, \ldots \) are integers. Since \( f(x) \in I_{l}(R((x^{-1}; \delta))) \), there exists an integer \( k \geq 1 \) such that
\[ f(x)g_{1}(x)f(x)g_{2}(x) \ldots f(x)g_{k}(x) = 0. \]

If we expand this product, we see that the leading coefficient is
Therefore, we apply the same procedure, then we obtain
\[ f = 0. \]
Thus, the leading coefficient of
\[ f \]
for any non-negative integers
\[ i_1, \ldots, i_k. \]
Hence, \( \sum_{j=0}^{\infty} \delta^j(a_n)R \) is left T-nilpotent, and this means that \( a_n \in I_{l,\delta}(R) \). By the above discussion, we have \( a_n x^n \in I_l(R((x^{-1}; \delta))). \) Set \( f'(x) = f(x) - a_n x^n. \) Then we have \( f'(x) \in I_l(R((x^{-1}; \delta))). \) Thus, the leading coefficient of \( f'(x) \), namely \( a_n - 1 \), belongs to \( I_{l,\delta}(R) \). And if we apply the same procedure, then we obtain \( a_i \in I_{l,\delta}(R) \) for each \( i \leq n. \) Therefore, \( f(x) = \sum_{i=-\infty}^{n} a_i x^i \in I_{l,\delta}(R)((x^{-1}; \delta)). \)

\[ \square \]

In \([8\), section 5\], the higher left T-nilpotent radideals are defined as follows: Set \( I_l^{(0)} = 0. \) Let \( \alpha \) be a given ordinal. If \( \alpha \) is the successor of \( \beta \), set
\[ I_l^{(\alpha)}(R) = \{ a \in R : a + I_l^{(\beta)}(R) \in I_l(R/I_l^{(\beta)}(R)) \}. \]
If \( \alpha \) is a limit ordinal, then we define
\[ I_l^{(\alpha)}(R) = \bigcup_{\beta < \alpha} I_l^{(\beta)}(R). \]

As mentioned in \([8\), one can define the prime radical of a ring \( R \) alternatively as the limit of the left T-nilpotent radideals. Now, our aim is to generalize Theorem \([2.9\) for higher left T-nilpotent radideals by using transfinite induction. Hence, we obtain a new characterization for the prime radical of pseudo-differential operator rings \( P(R((x^{-1}; \delta))), \) where \( R \) is \( \delta \)-compatible.

**Proposition 2.10.** Let \( R \) be a ring and \( \delta \) be a derivation of \( R \). Assume that \( R \) is \( \delta \)-compatible. Then the higher left T-nilpotent radideals satisfy
\[ I_l^{(\alpha)}(R((x^{-1}; \delta))) = I_l^{(\alpha)}(R)((x^{-1}; \delta)), \]
for any ordinal \( \alpha \).

**Proof.** We will use transfinite induction to prove the statement. For \( \alpha = 1 \), the result is clear. Assume that the result is true for every ordinal \( \beta < \alpha \). If \( \alpha \) is not a limit ordinal, then \( \alpha \) is a successor of some ordinal \( \beta \) and by the assumption, we have \( I_l^{(\beta)}(R((x^{-1}; \delta))) = I_l^{(\beta)}(R)((x^{-1}; \delta)) \). We consider the natural surjection
\[ R((x^{-1}; \delta)) \to R((x^{-1}; \delta))/I_l^{(\alpha)}(R)((x^{-1}; \delta)) \]
and the natural isomorphism
\[ R((x^{-1}; \delta))/I_l^{(\alpha)}(R)((x^{-1}; \delta)) \cong (R/I_l^{(\alpha)}(R))(x^{-1}; \delta), \]
where $\delta$ is the derivation of the factor ring $R/I_l(\alpha)$ induced by $\delta$. By Theorem 2.9, we have that the coefficients of the elements of $\mathcal{I}_l(\frac{R}{I_l(\alpha)}((x^{-1}; \delta)))$ are determined by the ideal $\mathcal{I}_l(\frac{R}{I_l(\beta)}((x^{-1}; \delta)))$. By using the natural isomorphism and the natural surjection, we get the result. If $\alpha$ is a limit ordinal, then by Theorem 2.9 we have

$$\mathcal{I}_l^{(\alpha)}(R((x^{-1}; \delta))) = \bigcup_{\beta<\alpha} \mathcal{I}_l^{(\beta)}(R((x^{-1}; \delta))) = \left( \bigcup_{\beta<\alpha} \mathcal{I}_l^{(\beta)}(R) \right)((x^{-1}; \delta)).$$

Therefore, we can take

$$\mathcal{I}_l^{(\alpha)}(R) = \left( \bigcup_{\beta<\alpha} \mathcal{I}_l^{(\beta)}(R) \right).$$

As a direct consequence of Proposition 2.10, we have the following:

**Proposition 2.11.** Let $R$ be a ring and $\delta$ be a derivation of $R$. Assume that $R$ is $\delta$-compatible. Then we have

$$P(R((x^{-1}; \delta))) = P_\delta((x^{-1}; \delta)),$$

where $P_\delta((x^{-1}; \delta))$ is the limit of the left T-nilpotent radideals $\mathcal{I}_l^{(\alpha)}(R)$.

**Acknowledgements**

The hospitalities of Vladimir Bavula and the University of Sheffield are greatly acknowledged.

**References**

[1] S. A. Amitsur, Radicals of polynomial rings, *Canad. J. Math.* 8 (1956) 355–361.

[2] H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, *Trans. Amer. Math. Soc.* 95 (1960), 466-488.

[3] N. J. Divinsky, Rings and Radicals, Univ. Toronto Press, 1964, Math. Expositions No. 14.

[4] M. Ferrero, K. Kishimoto, K. Motose, On radicals of skew polynomial rings of derivation type, *J. London Math. Soc.* 28(2) (1983), 8–16.

[5] B. J. Gardner, Some aspects of T-nilpotence, *Pac. J. Math.* 53 (1974), 117–130.

[6] I. M. Gelfand, L. A. Dikii, Fractional powers of operators and Hamiltonian systems. *Funct. Anal. Appl.* 10(4) (1976), 13-29.
[7] K. R. Goodearl, Centralizers in dierential, pseudo-dierential, and fractional dierential operator rings, *Rocky Mountain J. Math.* 13(4) (1983), 573-618.

[8] C. Y. Hong, N. K. Kim, P. P. Nielsen, Radicals in skew polynomial and skew Laurent polynomial rings, *J. Pure Appl. Algebra* 218 (2014), 1916–1931.

[9] C. Y. Hong, N. K. Kim, Y. Lee, P. P. Nielsen, Amitsur’s property for skew polynomials of derivation type, *Rocky Mountain Journal of Mathematics* 47(7) (2017), 2211–2232.

[10] E. Hashemi, A. Moussavi, Polynomial extensions of quasi-Baer rings, *Acta Math. Hungar.* 107(3) (2005), 207-224.

[11] T. Y. Lam, A first Course in Noncommutative Rings, second ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2003, MR 1838430.

[12] J. Levitzki, Contributions to the theory of nilrings, (Hebrew, with English summary), *Riveon Lematematika* 7 (1953), 50–70.

[13] I. Schur, Uber vertauschbare lineare dierentialausdrucke, sitzungsbsber, *Berliner Math. Ges.* 4 (1905), 2–8.

[14] K. Paykan, A. Moussavi, Primitivity of skew inverse Laurent series rings and related rings, *J. Algebra Appl.* (2019), 1950116 (12 pages).

[15] M. Sato, Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassman manifold, *Lect. Notes Num. Appl. Anal.* 5 (1982), 259–271.

[16] D. A. Tuganbaev, Laurent series rings and pseudo-dierential operator rings, *J. Math. Sci.* 128(3) (2005), 2843–2893.