Classical sampling theorem in digital holography

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Abstract. It is proposed to change the present paradigm of the sampling theorem, accepted in papers on digital holography, from ‘when recording digital hologram, Nyquist criterion should be fulfilled’ to ‘when recording on-axis digital hologram, low-pass sampling theorem should be fulfilled, and when recording off-axis one, band-pass sampling theorem should be fulfilled’. Experimental and theoretical basis for this proposal is presented.

1. Present paradigm of the sampling theorem in digital holography

According to Merriam-Webster online dictionary ‘paradigm’ may mean ‘a theory or a group of ideas about how something should be done, made, or thought about’. In the following it will be very convenient to use word ‘paradigm’ for brief denotation of a group of ideas about how the sampling theorem should be stated, applied and thought about in the field of digital holography.

It is commonly known that when recording digital hologram, requirements of the sampling theorem should be fulfilled. In many papers on digital holography sampling theorem is formulated in one and the same way, as the requirement of having at least two pixels of the digital image sensor per each period of interference fringes (for example, see [1,2]). This requirement can be written in the following form, which we will call ‘Nyquist criterion’:

\[ f_{\text{max}} \leq \frac{f_S}{2} = f_N, \]

where \( f_{\text{max}} \) is maximum spatial frequency of the holographic pattern, \( f_S \) is sampling frequency of the digital image sensor, \( f_N \) is so-called Nyquist frequency of the image sensor or Nyquist limit.

This criterion is usually used without mentioning any conditions of its applicability as something that is inherently valid for the entire digital holography by its nature. Therefore we may say that the essence of present paradigm of the sampling theorem in digital holography reduces to the proposition ‘when recording digital hologram, Nyquist criterion should be fulfilled’.

2. Experimental results, which present paradigm does not explain

However there are situations when accuracy of Nyquist criterion at first glance could be doubted. An example of such a case is presented in [3,4].

2.1. Outline of the experiments

In [3] we were experimentally investigating what will happen to reconstructed image if during the process of digital hologram recording spatial frequencies of the holographic structure exceed Nyquist limit. For this purpose we use off-axis lensless Fourier transform digital holography setup.
Numerical reconstruction of lensless Fourier digital hologram with Fresnel algorithm reduces to calculation of its Fourier transform [1]. Thus it turns out that power spectrum of lensless Fourier digital hologram contains reconstructed image of the object.

The object was made as a white paper circle with an “F” letter and was attached to a metal rod with possibility of horizontal lateral displacement along the plane of the image sensor. Shifting the object we were changing its angular displacement $\theta$ from the reference point source and thus were changing carrier frequency of the holographic structure, obtained in the sensor plane. Thus we were able to record digital holograms with increasing carrier frequency of the holographic structure.

2.2. Results of the experiments

Figure 1 presents five examples of power spectra of recorded holographic structures. From figures 1(a) and 1(b) we see, that with increase of object displacement $\theta$ reconstructed image of the object moves towards the edge of the observed part of the power spectrum, until in the figure 1(c) the carrier frequency of holographic structure reaches the Nyquist limit. After the Nyquist limit is reached, reconstructed image does not vanish, but reappears from the opposite edge of the spectrum (figure 1(d)) and then begins to repeat its previous trajectory throughout the spectrum (figure 1(e)).

![Figure 1. Power spectra of experimental holographic structures, obtained at different angular displacements $\theta$ of the object: (a) $\theta=1.13^\circ$; (b) $\theta=2.08^\circ$; (c) $\theta=2.71^\circ$; (d) $\theta=3.33^\circ$; (e) $\theta=4.27^\circ$ [3].](image)

Attempt to interpret these experimental results in frameworks of the paradigm described in the Section 1 leads to contradiction between experimental fact of image reconstruction beyond Nyquist limit and unconditional requirement of the paradigm to observe Nyquist criterion. As a result, the authors of [5], where similar experiments were made, conclude that ‘classical Shannon constraints can be relaxed without deteriorating the quality of the image reconstruction’.

3. On the way to new paradigm of the sampling theorem in digital holography

In our opinion clarification of the contradiction described above consists not in recognizing ‘incorrectness’ or ‘excessive severity’ of Nyquist criterion, but in recognizing the fact that Nyquist criterion is conditional and simply does not apply to the case under consideration.

3.1. Low-pass sampling theorem

Let’s turn to classical works of Kotelnikov [6,7] and Shannon [8] from which commonly used form of the sampling theorem called ‘Nyquist criterion’ arises. In original work of Kotelnikov [6,7] the sampling theorem is stated as follows (English translation is given according to [9]):

‘Any function $F(t)$ which consists of the frequencies from 0 to $f_1$ may be transmitted continuously to any desired degree of precision using numbers which follow on with $1/(2f_1)$ seconds’.

Thus according to its original form sampling theorem is applicable only to signals with frequencies from 0 to $f_1$, i.e. only to low-pass signals. Following Lüke [9], we will call this original form of the sampling theorem ‘low-pass sampling theorem’.

In the process of transformation from original sampling theorem to its commonly used form called ‘Nyquist criterion’ condition of its applicability has been lost. As a consequence of this loss, it has become a common practice to refer to Nyquist criterion in connection with any setup of digital holograms recording, without taking into account type of signal being recorded.
In experiments with off-axis lensless Fourier holography the spatial signal containing information required for object image reconstruction consisted of the frequencies not from 0 to \( f_1 \), but from \( f_1 \neq 0 \) to \( f_2 \), i.e. it was a band-pass signal. That is why low-pass sampling theorem, including Nyquist criterion, is not applicable there.

### 3.2. Band-pass sampling theorem

Now the question arises: if Nyquist criterion does not apply to sampling of band-pass spatial signals, then what criterion does? The answer is given in [6,7] where in addition to sampling theorem for low-pass signals Kotelnikov also stated sampling theorem for band-pass ones:

‘Any function \( F(t) \) which consists of the frequencies from \( f_1 \) to \( f_2 \) may be transmitted continuously to any desired degree of precision using numbers which follow on with \( 1/[2(f_2 - f_1)] \) seconds’.

For the purpose of distinction let’s call sampling theorem for band-pass signals ‘band-pass sampling theorem’. The criterion for band-pass signal sampling, corresponding to equation (1), will be

\[
\Delta f = \frac{f_2 - f_1}{2} = f_N.
\]  

(2)

It can be seen from the figure 1 that in all five spectra being presented condition (2) is fulfilled, so there is no surprise that image of the object was reconstructed even beyond Nyquist limit. As for gradual decay of the reconstructed image, which occurs with increase of the carrier frequency of holographic structure, it has nothing to do with exceeding the Nyquist limit and is caused only by non-zero value of the pixel fill-factor [4].

### 4. Summary

Thus from the very origin [6,7] there were two variants of the classical sampling theorem for time signals: one for low-pass signals (consisting of the frequencies from 0 to \( f_1 \)) and another for band-pass signals (consisting of the frequencies from \( f_1 \) to \( f_2 \)). The same paradigm should be used for space signals sampling in digital holography.

In digital holography the spatial signal, which contains information needed for object image reconstruction is the holographic structure. In the vicinity of each point of the image sensor spatial frequencies of the holographic structure are defined by the angle between the object beam and the reference beam.

In case of on-axis digital holography the holographic structure in general is a low-pass spatial signal and therefore sampling criterion in form of equation (1) should be used. In case of off-axis digital holography the holographic structure in general is a band-pass spatial signal and therefore sampling criterion in form of equation (2) should be applied.

### References

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