On the SO(5) Effective Field
Theory of High $T_c$ Superconductors

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Abstract

We construct the low-energy effective theory for the $SO(5)$ model of high-$T_c$ superconductivity, recently proposed by S.C. Zhang (cond-mat/9610140). This permits us to develop a systematic expansion for low-energy observables in powers of the small symmetry-breaking interactions. The approximate $SO(5)$ symmetry predicts relations amongst these observables, which are model-independent consequences of Zhang’s proposed symmetry-breaking pattern.

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1. Introduction

In a recent remarkable paper [1] Zhang has suggested an elegant framework for understanding the relationship between the superconducting (SC) and antiferromagnetic (AF) properties of the high-$T_c$ superconductors. His proposal comes in two parts. First, he argues that these systems enjoy an approximate $SO(5)$ symmetry which contains as a subgroup the $SO(3) \times SO(2)$ symmetry of spin rotations and electromagnetic gauge transformations. The $SO(5)$ symmetry is only approximate, in the sense that it is explicitly broken to $SO(3) \times SO(2)$ by small interaction terms in the Hamiltonian, whose scale we denote by $E_{sb}$. It is also explicitly broken by the doping of electrons or holes away from half filling. Second, the $SO(5)$ symmetry is argued to be spontaneously broken to $SO(4)$ by the dynamics which binds the electrons into spin-singlet pairs. The energy scale, $E_5$, which characterizes the order parameter for this breaking is imagined to be comparable to the Fermi energy, and is taken much larger than $E_{sb}$. For energies much smaller than $E_5$, the symmetry-breaking dynamics is purely concerned with how the $SO(5)$-breaking order parameter aligns relative to the direction of explicit symmetry breaking. In ref. [1] Zhang argues convincingly how such a framework can synthesize a great many features of these systems, and proposes a model low-energy effective theory which qualitatively describes their properties, and does so quantitatively in the vicinity of the critical point between the antiferromagnetic (AF) and superconducting (SC) regimes.

Here we generalize Zhang’s model to incorporate the most general possible interactions which are consistent with the assumed symmetry-breaking pattern. Being the most general possible such lagrangian, it must then incorporate the low-energy limit of any particular microscopic model which realizes these symmetries. Because the low-energy behaviour involves the interactions of Goldstone (and pseudo-Goldstone\footnote{A pseudo-Goldstone boson is the Goldstone boson for a symmetry which is only approximate.} [2]) bosons, they are guaranteed to interact only weakly at low energies. Consequently, a mean-field treatment of the effective theory is justified to describe these systems for energies well below $E_5$, and away from any critical points. This description closely parallels that of chiral perturbation theory [3], which has been very successfully used to describe the low-energy interactions...
of pions and nucleons within the framework of Quantum Chromodynamics.

Eq. (5) expresses our main results, which consist of a number of relations amongst the low-energy observables of these systems which are generic consequences of, and therefore strong tests of, the approximate $SO(5)$ symmetry. For the sake of brevity we here simply state our results, with a more thorough discussion to appear elsewhere.

We start with the lagrangian density for the four would-be Goldstone bosons, $\theta^\alpha$, $\alpha = 1, \ldots, 4$, corresponding to the spontaneous breakdown $SO(5) \to SO(4)$ [5]. Writing this as a derivative expansion, and, for simplicity, ignoring the system’s spatial anisotropy gives:

$$\mathcal{L}_{\text{inv}}(\theta) = -V_0 + \frac{f_t^2}{2} \partial_t n^T \partial_t n - \frac{f_s^2}{2} \nabla n^T \cdot \nabla n + \text{higher derivatives}.$$  (1)

Here $V_0$, $f_t$ and $f_s$ are real constants which are of order unity (in units where $E_5 = 1$). The four variables, $\theta^\alpha$, parameterize the ground-state field configurations, which fill out the space $SO(5)/SO(4)$ or, equivalently, the 4-sphere. We use in (1) coordinates consisting of a five-dimensional vector — the ‘superspin’ — $n(\theta)$, having unit length, $n^T n = 1$. $n$ transforms simply with respect to $SO(5)$: $n \to g n$, for $g$ a real, orthogonal five-by-five matrix.

In order to explicitly break $SO(5)$ down to $SO(3) \times SO(2)$ we introduce the symmetry-breaking matrix: $M = \epsilon \text{diag}(m_q, m_s, m_s, m_s, m_q)$. Here $\epsilon = E_{sb}/E_5 \sim \text{few } \%$, is the small dimensionless parameter which describes the strength of the explicit $SO(5)$ breaking. Following Zhang, we choose the spin $SO(3)$ to rotate $n_2, n_3$ and $n_4$ into one another, while the electromagnetic $SO(2)$ rotates $n_1$ into $n_5$. The explicit breaking of $SO(5)$ symmetry by doping is achieved within the microscopic theory by coupling a chemical potential, $\mu$, to the electron’s electric charge, $Q$. In units of $E_5$, we expect $\mu$ to lie in the range $|\mu| \lesssim 0.1 - 0.2$ for the doping of high-$T_c$ systems. Since electric charge is a generator of $SO(5)$, we take: $Q_{ij} = q(\delta_{1i}\delta_{5j} - \delta_{1j}\delta_{5i})$. $q = \pm 2$ is the electric charge (in units of $e$) of the superconducting order parameter.

The lagrangian for the system is then the most general local function of $n(\theta)$, $M$ and

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2 We choose units for which $\hbar = c = k_B = E_5 = 1$. 

\( \mu Q \) which satisfies \( \mathcal{L}(g_n, g M g^T, g \mu Q g^T) = \mathcal{L}(n, M, \mu Q) \). The leading terms in a derivative expansion therefore are:

\[
\mathcal{L}_{sb} = -V + f^2_t \left[ A \partial_t n_Q^T \partial_t n_Q + B \partial_t n_S^T \partial_t n_s + C (n_Q^T \partial_t n_Q)^2 \right] \\
- f^2_s \left[ D \nabla n_Q^T \cdot \nabla n_Q - E \nabla n_S^T \cdot \nabla n_s - F (n_Q^T \nabla n_Q)^2 \right] + \text{higher derivatives,}
\]

where \( n_Q \) and \( n_S \) respectively are the components of \( n \) within the electromagnetic and spin subspaces. The functions \( V, A, B, C, D, E \) and \( F \) are all functions of \( n, M \) and \( \mu Q \). For instance the scalar potential, \( V \), has the general form:

\[
V[n, M, \mu Q] = \sum_{k=1}^{\infty} 2^{-k} (v_{kQ} n_Q^T n_Q + v_{kS} n_S^T n_S)^k.
\]

Analogous expansions may also be written for the functions \( A \) through \( F \) (with coefficients denoted \( a_{kQ}, a_{kS} \) through \( f_{kQ}, f_{kS} \)). Coupling to electromagnetic fields, \( A_\mu \), is achieved by replacing ordinary derivatives of \( n_Q \) with covariant ones: \( \partial_\mu n_Q \rightarrow D_\mu n_Q = \partial_\mu n_Q - q e A_\mu Q n_Q \).

It is useful to express the conserved spin and electromagnetic currents, which are obtained from (1) and (2) using Noether’s theorem, for the effective theory, since it is to these that external probes often couple. The terms involving the fewest derivatives are:

\[
\rho_{\text{em}} = -f^2_t \left( 1 + 2A \right) n_Q^T \partial_t n_Q, \quad \mathbf{j}_{\text{em}} = f^2_s \left( 1 + 2D \right) n_Q^T Q \nabla n_Q, \\
\bar{\rho}_{\text{spin}} = f^2_t \left( 1 + 2B \right) \bar{n}_s \times \partial_t \bar{n}_s, \quad \bar{\mathbf{j}}_{\text{spin}} = -f^2_s \left( 1 + 2E \right) \bar{n}_s \times \nabla \bar{n}_s.
\]

Notice that eqs. (1), (2) and (4) also capture the form for the free energy and currents at nonzero temperature if all coefficients are understood as functions of \( T \). This is because these expressions are the most general possible consistent with the derivative expansion and the assumed symmetry-breaking pattern.

The predictive power of eq. (2) emerges once we perturb in the small symmetry-breaking parameters \( \epsilon \) and \( \mu \). This requires knowing the dependence of the coefficients
in eq. (3) (and its analogs for the functions $A$ to $F$) on these small parameters. It is an easy exercise to see that the following properties hold: $v_{ks}, a_{ks}, b_{ks}, c_{ks}, d_{ks}, e_{ks}$ and $f_{ks}$ are independent of $\mu$. These coefficients are at most $O(\epsilon)$ in size, as are all of the others when restricted to half-filling: $\mu = 0$. The antisymmetry of the matrix $Q$ further implies that $v_{kQ}, a_{kQ}, b_{kQ}, d_{kQ}$ and $e_{kQ}$ must be even under the interchange $\mu \to -\mu$. The same is not true for the coefficients $c_{kQ}$ and $f_{kQ}$. Because most low-energy observables do not appreciably depend on these last coefficients, they also enjoy a symmetry between positive and negative doping.

Perturbing in $\epsilon$ and $\mu$ is powerful because all interactions amongst the $\theta^\alpha$ in eqs. (1) and (2) vanish in the limit $\epsilon = \mu = 0$. Mean field theory is therefore justified so long as three criteria hold: (i) $\mu$ and $\epsilon$ are small; (ii) the derivative expansion is justified (long wavelengths compared to $1/E_5$; and (iii) we are not in the vicinity of critical points, for which non-Goldstone modes become massless, with the associated uncontrollable fluctuations in the infrared.

We now summarize some of the predictions which follow to leading order in $\epsilon$ and $\mu$. For details concerning their derivation, and some results beyond leading order, see [4]. To leading order it suffices to take $A = B = C = D = E = F = 0$ and keep only the $SO(5)$-invariant interactions having two derivatives. The symmetry-breaking terms then enter predictions only through the potential, $V$. It suffices to keep only the terms having $k = 1, 2$ in the expansion (3), and to expand the coefficients $v_{1Q}$ and $v_{2Q}$ to linear order in $\mu^2$; e.g.: $v_{1Q} = v^0_{1Q} + v^1_{1Q}\mu^2$ etc. In this limit the phase diagram, long-wavelength dispersion relations, and electric and magnetic stiffness are all determined by five parameters, which we choose to be $m^2 \equiv v^0_{1Q} - v_{1s} = O(\epsilon), \kappa \equiv -v^1_{1Q} = O(1), \xi \equiv (v^1_{2Q})^2 = O(1), f_s = O(1)$ and $f_t = O(1)$.

Since these five parameters relate more than five low energy observables for the SC and AF phases, we obtain testable relationships amongst these observables, which we now summarize.

Writing the magnitudes of $n_s$ and $n_Q$ as $|n_s| = \sin \theta$ and $|n_Q| = \cos \theta$, and minimizing the potential with respect to $\theta$ gives two generic phases: $\theta_{\min} = 0, \pi$ (SC); and $\theta_{\min} = \frac{\pi}{2}, \frac{3\pi}{2}$.
The curvature of the potential at these extrema are $M^2_{\text{AF}} \equiv (d^2V/d\theta^2)_{\theta=\pi/4} \approx m^2 - \kappa \mu^2$, and $M^2_{\text{SC}} \equiv (d^2V/d\theta^2)_{\theta=0} \approx -m^2 + \kappa \mu^2 - \xi \mu^4$. Notice the sum rule: $M^2_{\text{AF}} + M^2_{\text{SC}} = -(v_2q)^2 \leq 0$, which guarantees that either or both of these curvatures must be negative (for all $\mu$). A third, mixed, phase (MX) — Zhang’s ‘spin-bag’ phase — exists when both $M^2_{\text{AF}}$ and $M^2_{\text{SC}}$ are negative. In this case the minima are $\cos 2\theta_{\text{MX}} = (M^2_{\text{SC}} - M^2_{\text{AF}})/v^2_{2s}$, and the curvature at this minimum is $M^2_{\text{MX}} = -2M^2_{\text{SC}}M^2_{\text{AF}}/(M^2_{\text{SC}} + M^2_{\text{AF}})$.

The critical doping which defines the boundary between these phases is found by solving the conditions $M^2_{\text{AF}}(\mu^2_{\text{AF}}) = 0$ and $M^2_{\text{SC}}(\mu^2_{\text{SC}}) = 0$, giving $\mu^2_{\text{AF}} = \mu^2_{\text{SC}+} + O(\epsilon^2) = (m^2/\kappa) + O(\epsilon^2)$, and $\mu^2_{\text{SC}+} = (\kappa/\xi) + O(\epsilon)$. We see that the AF phase is predicted to extend only for dopings which are $O(\epsilon)$, i.e. a few %. The mixed phase, whose existence requires $\mu^2_{\text{SC}+} < \mu^2_{\text{AF}}$, occurs over a range of dopings which extend to $O(\epsilon^2)$ past the AF phase boundary, if it exists at all. Then one enters the SC phase, which persists over a wide $O(1)$ range of dopings. These results break down for $\mu \sim \mu^2_{\text{SC}+}$, since there the expansion in powers of $\mu^2$ and $1/E_5$ fail.

For temperatures satisfying $T_0(\mu) < T < E_5$ $SO(5)$ remains broken, but thermal transitions are possible between the SC and AF phases. An estimate for $T_0(\mu)$ is obtained by finding the height of the potential barrier in $V$ as a function of $\mu$. For $\mu$ chosen so that it is the AF phase which minimizes $V$, we find: $T_0(\text{AF}) \propto V_{\text{barrier}} = \frac{1}{4}(M^2_{\text{AF}} - M^2_{\text{SC}}) \approx \frac{1}{2}(m^2 - \kappa \mu^2)$. On the other hand, choosing $\mu$ so that $V$ favours the SC phase gives: $T_0(\text{SC}) \propto V_{\text{barrier}} = \frac{1}{4}(M^2_{\text{SC}} - M^2_{\text{AF}})$.

As discussed by Zhang, in each phase the four modes, $\theta^\alpha$, group themselves into bona fide gapless Goldstone modes (having dispersion relation $\omega^2(k) = c^2_0 k^2$ at low energies), and pseudo-Goldstone states (for which $\omega^2(k) = c^2_p k^2 + \varepsilon^2$). The lagrangian given by (1) and (2) predicts the observables $c_0$, $c_p$ and $\varepsilon$ for each phase. For brevity we record here only the lowest-order predictions for the AF and SC phases.

In the AF phase two modes are gapless (magnons), with the other two forming an electrically charged pseudo-Goldstone doublet with respect to the unbroken $SO(2)$. We find $(c_0)_{\text{AF}} = (c_p)_{\text{AF}} = c_s \equiv f_s/f_t + O(\epsilon, \mu^2)$. The gap is given as a function of doping by $\varepsilon^2_{\text{AF}}(\mu^2) = (m^2 - \kappa \mu^2)/f_t^2$. 


In the SC phase there is only one gapless mode, which is ‘eaten’ by the photon via the Anderson-Higgs mechanism, with the remaining three states forming a spin-triplet pseudo-Goldstone state. It is this state which Zhang argues persuasively [6] has been seen in neutron scattering experiments [7] in the spin-flip channel. The predictions in this case are:

\[(c_0)_{SC} = (c_p)_{SC} = c_s, \quad \varepsilon^2_{SC} = (−m^2 + \kappa\mu^2 − \xi\mu^4)/f_t^2.\]

The final observables for which we present predictions are the electric and magnetic stiffness in the SC phase. Coupling to an electromagnetic potential, and eliminating the SC phase’s Goldstone mode by an appropriate gauge choice, we find the term:

\[L' = \frac{1}{2}a_E^2A_0^2 - \frac{1}{2}a_M^2A^2, \quad \text{with} \quad a_E = qef_t \quad \text{and} \quad a_M = qef_s. \quad \text{a}_E \quad \text{and} \quad \text{a}_M \quad \text{represent} \quad \text{the medium’s} \quad \text{electric} \quad \text{and} \quad \text{magnetic} \quad \text{stiffness, normalized so that the penetration depth of electric and magnetic fields are respectively given by \(a_E^{-1}\) and \(a_M^{-1}\).}

By eliminating the parameters \(m^2, \kappa, \xi, f_t\) and \(f_s\) we obtain several parameter-independent relations among the observables described above. We summarize these here:

\[
\begin{align*}
\varepsilon^2_{AF}(\mu) &= \frac{\varepsilon^2_{AF}(0)}{\mu_{AF}^2} \left[ \mu_{AF}^2 - \mu^2 \right], \\
\varepsilon^2_{SC}(\mu) &= \frac{\varepsilon^2_{SC}(\text{max})}{\mu_{\text{max}}^2} \left( \mu^2 - \mu_{\text{SC} -}^2 \right) \left( 2\mu_{\text{max}}^2 - \mu^2 \right), \\
\varepsilon^2_{AF}(0) &= 2 \frac{\varepsilon^2_{SC}(\text{max})}{\mu_{\text{max}}^2}, \\
\mu_{AF}^2 &= \mu_{\text{SC} -}^2 + O(\epsilon^2), \\
(c_0)_{AF} &= (c_p)_{AF} = (c_p)_{SC} = (c_0)_{SC} = \frac{a_M}{a_E}.
\end{align*}
\]

In these expressions \(\varepsilon^2_{AF}(0) = m^2 / f_t^2\) denotes the AF gap energy at zero doping, and \(\varepsilon^2_{SC}(\text{max}) = (\kappa^2 / 4\xi f_t^2)\) denotes the maximum gap size in the SC phase, which occurs when \(\mu_{\text{max}}^2 = \mu_{\text{SC} +}^2 / 2 = \kappa / 2\xi\).

These are robust predictions of the assumed symmetry-breaking pattern regardless of the nature of the microscopic physics which is believed to be responsible, making their experimental verification particularly interesting. They receive calculable corrections in powers of \(\epsilon\), which are most easily computed using the effective lagrangian presented here [4]. More predictions are possible should the MX phase arise, since the properties of the
three Goldstone and one pseudo-Goldstone states in this phase are predictable in terms of the same parameters used here.

We close with a speculation concerning how the $SO(5)$ model might account for the anomalous electrical resistivity of the normal phase of the high-$T_c$ materials, which depend linearly on temperature (when doped with holes and near optimal doping). Within the $SO(5)$ model this resistivity arises within a ‘normal’ phase below the $SO(5)$-breaking scale, but at temperatures higher than the barrier between the AF and SC phases. The characteristic new feature of this ‘normal’ phase is the existence there of the light pseudo-Goldstone states, some of which carry electric charge. No such states are present in the normal phase of a BCS superconductor. In the rest of this section we wish to give a (naive) argument that if these light states play a significant role as carriers of electric current, then their scattering might give rise to a resistivity which is linear in $T$.

To see why this is so, we estimate the temperature dependence of the resistivity by computing the temperature-dependence of the two-body rate for scattering of the appropriate charge carriers from various quasiparticles within the material: $\rho(T) \sim \langle n \sigma v \rangle \sim n \Pi |A|^2$. Here $n$ describes the density of ‘target’ quasiparticles from which the charge carriers scatter, while $A$ describes the amplitude for this scattering and $\Pi$ represents the phase space available for the final states. The average is over the thermal distribution of scatterers, and the initial distribution of charge carriers. This kind of estimate reproduces the low-temperature $T^5$ dependence when applied to electron-phonon scattering, and the $T^2$ dependence when applied to electron-electron scattering.

The basic observation now comes in two parts. First, notice that the electrically-charged, pseudo-Goldstone states of Zhang’s $SO(5)$ model obey a ‘relativistic’ kinematics since their gap energy is smaller than the temperatures of interest. Second, unlike low-energy phonons, the pseudo-Goldstone states need not interact only through derivative couplings, and so their scattering amplitude, $A$, from other particles need not be suppressed by powers of $T$ at low temperatures. Combining these observations gives the standard estimate for the scattering rate of a ‘relativistic’ particle, which follows purely on dimensional grounds: $\rho \propto n \Pi |A|^2 \propto T$. In this, admittedly simplistic, picture a resistivity
which is linear in $T$ is seen as the signature that an appreciable part of the electromagnetic current is being carried by the light pseudo-Goldstone charge carriers. This motivates better understanding of the transport properties of these systems in this temperature range.

The results of our analysis are encouraging, with the effective theory naturally accounting for the features of the high-$T_c$ phase diagram. The predictions of eqs. (5) make an even more quantitative test of the $SO(5)$-invariant picture. Clearly much more remains to be done with this low-energy lagrangian, most notably including a systematic discussion of their transport properties, and an analysis of the response of the pseudo-Goldstone modes to various probes.

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2. References

[1] S.-C. Zhang, *SO(5) Quantum Nonlinear σ Model Theory of the High Tc Superconductivity*, Stanford preprint, cond-mat/9610140.

[2] S. Weinberg, Phys. Rev. Lett. 29, 1698 (1972)

[3] S. Weinberg, Phys. Rev. Lett. 18, 188 (1967); Phys. Rev. 166, 1568 (1968); Physica 96A, 327 (1979); C.G. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177, 2247 (1969); J. Gasser and H. Leutwyler, Ann. Phys. 158, 142 (1984); Nucl. Phys. B250, 465 (1985)

[4] C.P. Burgess and C.A. Lütken, in preparation.

[5] For an introduction to Goldstone bosons in high-energy and condensed matter physics, see: C.P. Burgess, *An Introduction to Effective Lagrangians and their Applications*, lecture notes for the Swiss Troisième Cycle, Lausanne, June 1995.

[6] E. Demler and S.C. Zhang, Phys. Rev. Lett. 75, 4126 (1995)

[7] H.A. Mook et al., Phys. Rev. Lett. 70, 3490 (1993); J. Rossat-Mignod et al., Physica (Amsterdam) 235C, 59 (1994); H.F. Fong et al., Phys. Rev. Lett. 75, 316 (1995)