HYPERBOLIC CLUSTER ALGEBRAS

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Abstract. In this article, we set up for a new class of non-commutative algebras that carry a non-commutative cluster structure. This structure is related naturally to some Hyperbolic algebras like Weyl Algebras, classical and quantized universal enveloping algebras of $sl_2$ and the quantum coordinate algebra of $SL(2)$. The cluster structure gives rise to some combinatorial data, called cluster strings, which are used to introduce a class of representations of Weyl algebras. Irreducible and indecomposable representations are also introduced from the same data.

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1. Introduction

Cluster algebras were invented by S. Fomin, and A. Zelevinsky [1, 14, 15, 16, 34, 35]. A cluster algebra is a commutative algebra with a distinguished set of generators called cluster variables and particular type of relations called mutations. A quantum version was introduced in [10] and [2]. The original motivation was to create an algebraic framework to study total positivity and dual canonical basis in coordinate rings of certain semi simple algebraic groups. It was inspired by the discovery of a connection between total positivity and canonical basis, due to G Lusztig, [24].

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Hyperbolic algebras were first introduced by A. Rosenberg in [31], and his motivation was to find a ring theoretical framework to study the representation theory of some important "small algebras" like, the first Heisenberg, Weyl algebras, the universal enveloping algebra of the Lie algebra $sl(2)$. A complete list of "small algebras" can be found in [30]. Also, in [30] Rosenberg has obtained the representation theory of all "small algebras" using the Hyperbolic algebra as the framework. This paper works as a setting up for the cluster structure in the hyperbolic algebras, and details are given for our two running examples Weyl algebra and quantum coordinate algebra of $SL(2,k)$.

In the last decade, the age of the cluster algebras theory, the theory has witnessed a remarkable growth due to the many links that have been discovered with a wide range of subjects. Recently, D. Hernandez and B. Leclerc in [20] and [22], and Nakajima in [28] have started using the rich cluster algebra structure to solve some classical representation theory problems. This paper has to do with this trend.

In this paper we show that a partial relaxing of the commutativity relations of the frozen variables and cluster variables, extends the theory to include some essential objects in representation theory, like hyperbolic algebras.

We also, introduce and study non commutative seeds, that are different from the quantum seeds introduced in [10], and [2], we also study the cluster structure comes out from this type of seeds. We show that this type of seeds exist naturally in some hyperbolic algebras and the cluster algebras of these seeds are naturally related to some known hyperbolic algebras, like Weyl algebras. The non-commutativity is controlled by a ring (the ring of coefficients) automorphism $\varphi$, and in the case of $\varphi = id$ we get Fomin and Zelevinsky cluster algebra.

Weyl algebras (algebras of differential operators with polynomial coefficients) among others are one of the most important examples of hyperbolic algebras. They essentially relevant for the theory of infinite dimensional representations of Lie algebras. They appear as primitive quotients of universal enveloping algebras of nilpotent Lie algebras, which reduces the study of irreducible representations of nilpotent Lie algebras to study simple modules over Weyl algebras. In the case of reductive Lie algebras, Weyl algebras also appear as algebras of differential operators on (translations of) big Shubert cells. Which is used to develop methods in non-commutative geometry to reduce the study of the irreducible representations of reductive Lie algebras to study simple modules over Weyl algebras. More generale hyperbolic algebras play a similar role (via quantum D-modules on quantum flag varieties) for quantized enveloping algebras.

This article is organized as follows; First section is devoted for introduction and to recall the basic concepts of both geometric cluster algebras and quantum cluster algebras, most of the materials of this section are available in [1],[13],[14], [15] and [16].

In the second section, we introduce the hyperbolic seeds which are combinatorial data similar to geometrical seeds, however the hyperbolic seeds are defined in non-commutative fields (skew fields). The mutations of the hyperbolic seeds are also introduced. Sufficient conditions for the mutation on the hyperbolic seeds to be a periodic operation (or equivalently finite type cluster algebra) are provided. In the rest of this section, we introduce the harvest group, which is the group of the skew field automorphisms that keep the set of all harvest variables invariant. Some basic properties of these groups are studied.
In the last part of this section, we introduce the hyperbolic algebras and some construction theorems are provided. Unfoundedly the Laurent phenomenon is not generally satisfied, more conditions need to be imposed to get it hold. Sufficient conditions for Laurent phenomenon are all satisfied for the cases we are covering in this article like Weyl algebras and quantized coordinate rings of $sl_2$. Positivity conjecture is remaining conjectured in the hyperbolic cluster algebras case. However it is automatically true in our examples. Third section is entirely devoted for providing the details of some examples like Weyl algebras and quantum coordinate ring of $SL_2$.

In the last section, we introduce the cluster strings which are certain sets of linear combinations of cluster monomials encoding the harvest graphs. These sets are selected to be generators for a class of representations for Weyl algebras. Also, they provide the core of irreducible and indecomposable representations of Weyl algebras.

Throughout the paper, $K$ is a field of zero characteristic and $F = K(t_1, t_2 \ldots t_m)$, is the field of rational functions in $m$ independent (commutative) variables over $k$.

We always denote $(b_{ij})$ for the square matrix $B$, $(c_{ij})$ for $C$, etc., and $[1, m] = \{1, 2, \ldots, m\}$.

1.1. Geometric Cluster Algebra. Most of the material of this section is quoted from [1, 14, 15, 16, 34, 35].

Definition 1.1.1. (1) A geometric seed of rank $m = t + n$ in $F$ over $K$ is a pair $(\tilde{X}, \tilde{B})$ where $\tilde{X} = f \cup X \subset F$, with $f \subset \tilde{X}$ (the set of frozen variables) is of cardinal number $t$ and elements of $f$ are $K$-algebraically independent elements of $F$ such that, if $G$ is the free group generated by the frozen variables, then $X = \tilde{X} - f$ is a transcendence basis of $F$ over $K[G]$. $\tilde{B}$ is an $m \times n$ integral matrix, with the rows labeled by the elements of $\tilde{X}$, and columns are labeled by the elements of $X$, and the sub matrix $B = (b_{ij})$ with rows labeled by elements of $X$, is kew-symmetrizable, i.e. $d_i b_{ij} = d_j b_{ji}$ for some positive integers $d_1, \ldots, d_n$.

(2) The diagram of a sign-skew-symmetric matrix $\tilde{B} = (b_{ij})$ is the weighted directed graph, $\Gamma(\tilde{B})$, with set of vertices $[1, n]$, such that there is an edge from $i$ to $j$ if and only if $b_{ij} > 0$, and this edge is assigned the weight $|b_{ij} b_{ji}|$.

(3) A geometric seed $(\tilde{X}, \tilde{B})$ is called connected if $\Gamma(\tilde{B})$ consists of exactly one connected component.

Definition 1.1.2 (Geometric seed mutation). For each fixed $k \in \{1, \ldots, n\}$, and each given geometric seed $(X, B)$ we define a new pair $\mu_k(\tilde{X}, \tilde{B}) = (\tilde{X}^*, \tilde{B}^*)$ by setting $X^* = (x^*_1, \ldots, x^*_n)$ with

\[
(1.1.1) \quad x^*_i = \begin{cases} 
  x_i & \text{if } i \neq k, \\
  \frac{\prod_{b_{ji} > 0} \tau_{b_{ji}} + \prod_{b_{ji} < 0} \tau_{b_{ji}}^{-1}}{x_i} & \text{if } i = k.
\end{cases}
\]

and $\tilde{B}^*_k = (b^*_{ij})$ with
The operation $\mu_k$ is called a mutation in $k -$direction.

**Definition 1.1.3. (Distinguished seeds)** A geometric seed $p = (\tilde{X}, \tilde{B})$ is called a distinguished seed if it satisfies the following two conditions

\begin{equation}
\quad (1.1.3) \quad b_{ij} b_{ik} \geq 0, \quad \forall \ i, j, k \in [1, n],
\end{equation}

and the second condition is Cartan counterpart $A(B) = (a_{ij})$ of $B$, is of finite type as a Cartan matrix.

The type of $p$ is the same as the Cartan-Killing type of $A(B) = (a_{ij})$ is defined by

\begin{equation}
\quad (1.1.4) \quad a_{ij} = \begin{cases} 
2, & \text{if} \ i = k, \\
-|b_{ij}|, & \text{if} \ i \neq k.
\end{cases}
\end{equation}

Let $\mathcal{S}$ be the set of all geometric seeds in $F$. Fix $p = (\tilde{X}, \tilde{B}) \in \mathcal{S}$. Let $S$ denote the mutation equivalence class of $p$, and $\mathcal{X}_S$ be the set of all cluster variables in $S$.

**Definition 1.1.4. (Geometric Cluster algebra)** Let $\mathcal{X}_S$ be the set of all cluster variables in $S$ i.e. the union of all clusters in $S_C$. The cluster algebra $\mathcal{A}_n(S)$ of rank $n$, associated to the initial geometric seed $p = (\tilde{X}, \tilde{B})$ (of rank $m = n + t$), is defined to be the $\mathbb{Z}[G]$-subalgebra of $F$ generated by $\mathcal{X}_S$ i.e.

\begin{equation}
\quad (1.1.5) \quad \mathcal{A}_n(S) := \mathbb{Z}[G][\mathcal{X}_S] \subset F
\end{equation}

**Definition 1.1.5. (Cluster pattern of $\mathcal{A}_n(S)$ [16])**. The cluster pattern $\mathcal{T}_n(S)$ of the cluster algebra $\mathcal{A}_n(S)$ is an regular $n$−ary tree whose edges are labeled by the numbers $1, 2, \ldots, n$ such that the $n$ edges emanating from each vertex receive different labels. The vertices are assigned to be the elements of $S$ (the seeds) such that the endpoints of any edge are obtained from each other by seed mutation in the direction of the edge label.

One can see, the cluster pattern of $\mathcal{A}_n(S)$ can be completely determined by any seed in $S$.

**Definition 1.1.6.** A cluster algebra, $\mathcal{A}_n(S)$, is called of finite type, if $S$ is a finite set. Equivalently if $\mathcal{X}_S$ is finite.

The details for the following two theorems are available in [15].

**Theorem 1.1.7. (Finite type classification)** For a cluster algebra $\mathcal{A}_n(S)$, the following are equivalent:

- $\mathcal{A}_n(S)$ is of finite type;
- for every seed $(\tilde{X}, \tilde{B})$ in $S$, the entries of the matrix $B = (b_{ij})$ satisfy the inequalities $|b_{ij} b_{ji}| \leq 3$, for all $i, j \in [1, n]$;
- $S$ contains a distinguished seed.

**Theorem 1.1.8.** Every finite type Cartan matrix corresponds to one and only one, up to field automorphism, finite type cluster algebra. Furthermore, a cluster algebra $\mathcal{A}_n(S)$ is of finite type if and only if $S$ contains a distinguished seed and
cluster type of $\mathcal{A}_n(S)$ is the same as the Cartan-Killing type of the Cartan counter part the distinguished seed.

Remark 1.1.9. If there is a geometric seed $(\tilde{X}, \tilde{B})$ in $S$ such that, there is a linear ordering of $\{1, 2, \cdots n\}$, where $b_{ij} \geq 0$ for all $i < j$, then the geometric seed $(\tilde{X}, \tilde{B})$ is called acyclic seed, and $\mathcal{A}_n(S)$ is called acyclic cluster algebra, and in this case we have:

\[(1.1.6) \quad \mathcal{A}_n(S) = \mathbb{Z}[G][x_k, x_k^* ; k \in [1, n]],\]

and $\mathcal{A}_n(S)$ is finitely generated as an algebra.

Theorem 1.1.10. (Laurent Phenomenon). The cluster algebra $\mathcal{A}_n(S)$ is contained in the integral ring of Laurent polynomials $\mathbb{Z}[G][X^\pm]$, for any cluster $X \in S_C$, i.e.

\[(1.1.7) \quad \mathcal{A}_n(S) \subset \mathbb{Z}[G][X^\pm] = \mathbb{Z}[G][x_1^\pm, x_2^\pm, \ldots, x_n^\pm]\]

for any cluster $X = (x_1, x_2, \ldots, x_n)$.

More precisely, every non zero element $y \in \mathcal{A}_n(S)$, can be uniquely written as

\[(1.1.8) \quad y = \frac{P(x_1, x_2, \ldots, x_n)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}},\]

where $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$, and $P(x_1, x_2, \ldots, x_n)$ is an element of the ring of polynomials $\mathbb{Z}[G][x_1, x_2, \ldots, x_n]$, which is not divisible by any cluster variables $x_1, x_2, \ldots, x_n$.

1.2. Quantum Cluster Algebra. All the material of this subsection are quoted from [2]

Definition 1.2.1. Compatible pairs Let $\tilde{B}$ and $\tilde{X}$ be as in definition 1.1.1, and $\Lambda = (\lambda_{ij})$ be a skew symmetric $m \times m$ integral matrix with row and columns labeled by the elements of $\tilde{X}$. The pair $(\Lambda, B)$ is said to be a compatible pair if

\[(1.2.1) \quad \sum_{k=1}^{m} b_{kj} \lambda_{ki} = \begin{cases} d_j, & \text{if } i = j, \\ 0, & \text{if } i \neq k \end{cases}\]

Definition 1.2.2. Based quantum tours, Toric frames and quantum seeds.

(1) Let $L$ be a lattice of rank $m$, with skew symmetric bilinear form $\Lambda : L \times L \to \mathbb{Z}$. Let $q$ be a formal variable, and $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \subset \mathbb{Q}(q^{\frac{1}{2}})$ be the ring Laurent polynomials in $q^{\frac{1}{2}}$. The based quantum torus associated with $L$ is $\mathbb{Z}[q^{\frac{1}{2}}]$-algebra, $\tau = \tau(\Lambda)$ with a $\mathbb{Z}[q^{\frac{1}{2}}]$-basis $\{X^e : e \in L\}$ and multiplication given by

\[(1.2.2) \quad X^e X^f = q^{\frac{\lambda_{ef}}{2}} X^{e+f}, \quad x^0 = 1 \quad \text{and} \quad (x^e)^{-1} = x^{-e}, \quad \text{for } e, f \in L.\]

Let $\mathbb{F}$ be the skew-field of fractions of $\tau$, then one can see that $d \mapsto d \cdot 1^{-1}$ is an embedding of $\tau$ in $\mathbb{F}$.
(2) A toric frame in $F$ is a mapping $M : \mathbb{Z}^m \to F^*$, given by $M(c) = \lambda(X^\gamma(c))$, for some $\lambda$ an $F$-automorphism, and $\gamma : \mathbb{Z}^m \to \mathbb{L}$ is an isomorphism of lattices.

Note that the elements $M(c)$ form $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-basis of an isomorphic copy $\lambda(\tau)$ of the based quantum torus $\tau$.

(3) A quantum seed is a pair $(M, \tilde{B})$, where $M$ is a toric frame in $F$, and $\tilde{B}$ is an $m \times n$ integral matrix with rows labeled by $[1, m]$ and columns labeled by an $n$-element subset $\text{ex}$ of $[1, m]$, such that $(\Lambda_M, \tilde{B})$ is a compatible pair, where $\Lambda_M$ is the bilinear form obtained from $\Lambda$ by transferring the form $\Lambda$ from $L$ to $\Lambda$ by the lattice isomorphism $\gamma$.

**Definition 1.2.3.** Let $(M, \tilde{B})$ be a quantum seed, we generate new quantum seeds by applying mutations on $(M, \tilde{B})$ as follows. Fix $k \in \text{ex}$, and $\epsilon \in \{-1, 1\}$. We define $\mu_k(M) = M' : \mathbb{Z}^m \to F^*$ is given by, for $c = (c_1, \ldots, c_m) \in \mathbb{Z}^m_{\geq 0}$, we have

$$
(1.2.3) \quad M'(c) = \sum_{p=0}^{c_k} \binom{c_k}{p} q^{\frac{1}{2}p} M(E_{\epsilon}c + \epsilon p b^k), \quad M'(-c) = M'(c)^{-1},
$$

where $\binom{c_k}{p} q^{\frac{1}{2}p}$ is the $q^{\frac{1}{2}}$-binomial coefficients, and the matrix $E_{\epsilon} = (e_{ij})$ is an $m \times m$ matrix given by

$$
e_{ij} = \begin{cases} 
\delta_{ij}, & \text{if } j \neq k, \\
-1, & \text{if } i = j = k, \\
\max(0, -\epsilon b_{ij}) & \text{if } i \neq j = k
\end{cases}
$$

and the vector $b^k$ is the $k$th column of $\tilde{B}$. Mutation on $\tilde{B}$ is defined before.

**Definition 1.2.4.** Let $(M, \tilde{B})$ be a quantum seed, and let $\tilde{X} = (x_1, \ldots, x_m)$, where $x_i = M(e_i)$. $\{e_i\}^n_{i=1}$ is a basis of $\mathbb{Z}^m$. The set of cluster variables of $(M, \tilde{B})$ is $X = \{x_j, j \in \text{ex}\}$. Let $C = \tilde{X} - X$, and $G'$ be the free group generated by elements of $C$, written multiplicatively. The quantum cluster algebra is defined to be the $\mathbb{Z}[q^{\pm \frac{1}{2}}][G']$-subalgebra of $F$ generated by all the cluster variables in every quantum seed that obtain from $(M, \tilde{B})$ by applying some sequence of mutations.

2. Hyperbolic Cluster Algebra setting up

2.1. Seeds and Hyperbolic seeds. Let $\mathcal{P}$ be a finitely generated abelian group, written multiplicatively, with set of generators $F = \bigcup_{i=1}^{n} F_i$, where $F_i$’s are distinct sets. Denote the cardinal number of each $F_i$ by $|F_i| = m_i$, and let $m = \sum m_i$. Let $R = K[\mathcal{P}]$ be the group ring of $\mathcal{P}$ over $K$. Let $D$ be the domain generated by $R$ and the $n$ (formal) independent and commutative indeterminates $t_1, t_2, \ldots, t_n$, with the following relations

$$
t_i f = ft_i, \quad \forall f \in F_j, \forall j \neq i \in [1, n].
$$

However, the indeterminant $t_i$ is not necessarily to commute with elements of $F_i$. One can see that $D$ is an Ore domain, then $D$ is embedded in its field of fractions, denoted by $\mathcal{F}$, by the natural embedding $d \mapsto d \cdot 1^{-1}$. 
Remark 2.1.1. For more information about Ore domains, we refer to [29], and Appendix A, Ore domains and skew fields of fractions, in [2].

Definition 2.1.2. Seeds and Hyperbolic Seeds

A seed \( i \) of rank \( n \) in \( F \) is the triple \((F,X,\Gamma)\), where

- \( F \) is as described above, and is called the set of frozen variables
- \( X = (x_1, \ldots, x_n) \in F^n \) such that if \( D' \) is the domain generated by \( R \) and \( x_1, \ldots, x_n \) then the field of fractions of \( D' \) is an \( R \)-linear automorphic copy of \( F \) i.e. there is an automorphism on \( F \) that fixes the frozen variables and sends \( x_i \) to \( x_{\sigma(i)} \), for each \( i \in [1,n] \) and for some permutation \( \sigma \). Elements of \( \{x_1, \ldots, x_n\} \) are called cluster variables, and \( C := F_1 \times F_2 \times \ldots F_n \times X \) is called a cluster.
- \( \Gamma \) is an oriented graph with set of vertices \( I := I_f \cup I_a = [1,k] \), where \( k = m + n \), with no 2-cycles nor loops. Let \( \tilde{X} = F \cup \{x_1, \ldots, x_n\} \). There is a one-to-one correspondence map \( v : \tilde{X} \rightarrow I \), here, \( v(t) \in I_f \) for every frozen variable \( t \), and \( v(x_i) \in I_a \) for every cluster variable \( x_i \in X \).

Seed Mutations We need the following combinatorial data before we define the seed mutation. Let \( i = (F,X,\Gamma) \) be a seed, and \( L \) be the lattice \( \mathbb{Z}^k \), \( k = m + n \). Here, \( m = |F| = \sum m_i \), and \( n = |X| \). Consider the following arrangement (enumeration) for elements of \( F \), \( F_i = \{f_{1i}, f_{2i}, \ldots, f_{mi}\} \), \( i \in [1,n] \). For a vertex \( v(x_i), k \in [1,n] \) of \( \Gamma \) we associate two vectors of \( L \), as follows; the first vector

\[
\vec{r}_1^k = (r_{i1}, \ldots, r_{im_1}, r_{i2}, \ldots, r_{im_2}, \ldots, r_{in_1}, \ldots, r_{in_n}, l_1, \ldots, l_k-1, -1, l_k+1, \ldots, l_n)
\]

where \( r_{ij} \) is the number of arrows in \( \Gamma \) directed from the vertex \( v(f_{ij}) \) toward \( v(x_k) \), and for \( i \neq k \), \( l_i \) is the number of arrows from the vertex \( v(x_i) \) toward \( v(x_k) \), and we have \(-1\) at the place of \( l_k \). The second vector is

\[
\vec{t}_1^k = (r'_{i1}, \ldots, r'_{im_1}, r'_{i2}, \ldots, r'_{im_2}, \ldots, r'_{in_1}, \ldots, r'_{in_n}, l'_1, \ldots, l'_{k-1}, -1, l'_{k+1}, \ldots, l'_n)
\]

which encodes the number of arrows targeting other vertices from \( v(x_k) \), i.e each component of \( \vec{t}_1^k \) is the number of arrows with source as \( v(x_k) \) and target as the corresponding vertex, except for the component corresponding to the vertex \( v(x_k) \), we have \(-1\). This defines the map \( r : X \rightarrow \tilde{X} \rightarrow \mathbb{Z}^k \times \mathbb{Z}^k \), given by \( x_k \mapsto (\vec{r}_1^k, \vec{t}_1^k) \). Each of these two vectors defines a map from \( \tilde{X} \) to \( \mathbb{Z}_{\geq -1} \), given by

\[
\vec{r}_1^k(t) = \begin{cases} r_{ij}, & \text{if } t = f_{ij}, \\ -1, & \text{if } t = x_k \\ l_i & \text{if } t = x_j, j \neq k \end{cases}
\]

For a cluster \( C \), we consider the following two maps \( M^R_C, M^L_C : L \rightarrow F \) given as follows. For a vector \( a = (a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn}, b_1, \ldots, b_n) \in L \), we assign the following two monomials

\[
M^R_C(a) = t_{11}^{a_{11}} \cdots t_{m1}^{a_{m1}} \cdot t_{12}^{a_{12}} \cdots t_{m2}^{a_{m2}} \cdots t_{1n}^{a_{1n}} \cdots t_{mn}^{a_{mn}} \cdot x_1^{b_1} \cdots x_n^{b_n}
\]

\[
M^L_C(a) = x_1^{b_1} \cdots x_n^{b_n} \cdot t_{11}^{a_{11}} \cdot t_{m1}^{a_{m1}} \cdot t_{12}^{a_{12}} \cdots t_{m2}^{a_{m2}} \cdots t_{1n}^{a_{1n}} \cdots t_{mn}^{a_{mn}}
\]

Definition 2.1.3. Let \( i = (F,X,\Gamma) \) be a seed of rank \( n \) in \( F \). At each cluster variable (non-frozen) \( x_k \) we can obtain two new triples \( \mu^R_k(i) = (F', \mu^R_k(X), \Gamma') \) and \( \mu^L_k(i) = (F', \mu^L_k(X), \Gamma') \) from \( i \), by applying the following steps;
\( F' = F \)

\[ \mu^R_k(X) = (x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n), \text{ where} \]

\[ x'_k = M^R_\ell(\tilde{r}^k) + M^R_\ell(\tilde{r}^{k}), \]

and

\[ \mu^L_k(X) = (x_1, \ldots, x_{k-1}, \hat{x}_k, x_{k+1}, \ldots, x_n), \text{ where} \]

\[ x'_k = M^L_\ell(\tilde{r}^k) + M^L_\ell(\tilde{r}^{k}). \]

\( \Gamma' \) is obtained from \( \Gamma \) by applying the following six rules:

1. every arrow not incident to \( k \) stays with no change
2. every arrow incident to \( k \) change its direction
3. The following three rules are applied only to vertices corresponding to non frozen variables
4. if there are two vertices \( i \) and \( j \) in \( \Gamma \) such that there are \( r \) arrows targeting the vertex \( v(x_k) \) from \( i \) and \( r' \) arrows targeting \( j \) from \( v(x_k) \), in \( \Gamma' \) we add number of arrows from \( i \) to \( j \) equals \( rr' \)
5. remove all 2-cycles between \( i \) and \( j \) in \( \Gamma' \)
6. apply steps (4) and (5) for all vertices connected with \( v(x_k) \) in \( \Gamma \).

\( \mu^R_k \) and \( \mu^L_k \) are called right and left mutations in the \( k \)-direction respectively. \( \mu^R_k(\hat{i}) \) and \( \mu^L_k(\hat{i}) \) obtained in the above way are said to be obtained from \( \hat{i} \) by applying right and left mutations in the \( k \)-direction respectively.

**Remark 2.1.4.**

1. This definition of mutations on oriented, no loops, and 2-cycles graphs is equivalent to the definition of mutations on skew-symmetric matrices corresponding to such graphs, definition 1.1.2.
2. One can see that; both of \( \mu^R_k(X) \) and \( \mu^L_k(X) \) are not necessarily to be commutative subsets of elements of \( \mathcal{F} \), then the triples \( \mu^R_k(\hat{i}) \) and \( \mu^L_k(\hat{i}) \) are not necessarily to be seeds in general. So, we need more conditions to guarantee that right and left mutations always produce seeds if they are feded with seeds, as we will see in the following.

**Notations 2.1.5.** For a new triple obtained from \( \hat{i} = (F, X, \Gamma) \) by applying mutation in the \( k \)-direction, we will use the following notations in the rest of the article

\[ \hat{i}' = \mu^R_k(\hat{i}), \text{ (respect. } \hat{i} = \mu^L_k(\hat{i}) \) \]

For \( X' = (x'_1, \ldots, x'_n) \), we write \( x'_j := \mu^R_{k,\hat{i}}(x_j) \), for \( j \in [1, n] \). Here,

\[ \mu^R_{k,\hat{i}}(x_j) = \begin{cases} x_j, & \text{if } j \neq k, \\ M^R_\ell(\tilde{r}^k) + M^R_\ell(\tilde{r}^{k}), & \text{if } j = k \end{cases} \]

(respect \( X = (x'_1, \ldots, x'_n) \), we write \( \mu^L_{k,\hat{i}}(x_j) := \hat{x}_j \) \)

The mutation of \( \hat{i} \) at \( k \) gives rise to two \( R \)-linear \( \mathcal{F} \) automorphisms, \( T^R_{\hat{i},\hat{i}} \) and \( T^L_{\hat{i},\hat{i}} \), where \( T^R_{\hat{i},\hat{i}} : \mathcal{F} \to \mathcal{F} \) is induced by

\[ T^R_{\hat{i},\hat{i}}(t) = t, \forall t \in \mathcal{R}, \text{ and } \mu^R_{k,\hat{i}}(x_j) := T^R_{\hat{i},\hat{i}}(x_j), \forall j \in [1, n], \]

(respect to \( T^L_{\hat{i},\hat{i}} \), \( \mu^L_{k,\hat{i}}(x_j) := T^L_{\hat{i},\hat{i}}(x_j), \forall j \in [1, n] \). \( T^R_{\hat{i},\hat{i}} \) and \( T^L_{\hat{i},\hat{i}} \) are called right and left mutation automorphisms respectively.
Proof. (1) We prove it for Lemma 2.1.7. Let \( x \in F \cap N(x_k) \) where \( N_1(x_k) \) is defined to be the subset of \( X \), of all frozen variables or cluster variables \( x \), where the components corresponding to \( x \) in \( \mathbb{F}_1^k \) or \( \mathbb{F}_-^k \) is non zero, and is denoted by \( N_1(x_k) \). So, \( N(x_k) = N_1(x_k) \cup N_1(-x_k) \), where \( N_1(x_k) = \{ x \in X; \mathbb{F}_1^k(x) > 0 \} \) and \( N_1(-x_k) = \{ x \in X; \mathbb{F}_-^k(x) > 0 \} \). Let \( \mathbb{F}_1^{k+} = F_k \cap N_1(x_k) \) and \( \mathbb{F}_1^{-} = F_k \cap N_1(-x_k) \).

**Definition 2.1.6.** The quadruple \( (F, X, \Gamma, \varphi) \) is called a Hyperbolic seed of rank \( n \) in \( F \), if it satisfies the following two conditions

1. the triple \( (F, X, \Gamma) \) is a seed of rank \( n \) in \( F \) satisfying the equations

\[
N_1(x_k) \cap N_1(x_i) \cap F_k = \emptyset, \forall i, \forall k \in [1, n],
\]

(this equation guarantees that non of the frozen variables of \( F_k \) is in the neighborhood of any cluster variable \( x_i \) for \( k \neq i \)),

2. \( \varphi \) is an \( R \)-linear automorphism of \( F \) satisfies the following

\[
fx_i = \varphi(x_i)f, \forall f \in F_i, \forall i \in [1, n].
\]

The above equations induce the following equations

\[
x_i f = f \varphi^{-1}(x_i), \forall f \in F_i, \forall i \in [1, n].
\]

**Lemma 2.1.7.** Let \( \mathbb{i} = (F, X, \Gamma, \varphi) \) be a hyperbolic seed in \( F \). Then, the following are true

1. For any sequence of right mutations (respect to left) \( \mu_R^i \mu_R^i \mu_R^i \cdots \mu_R^i \), \( \mu_L^i \mu_L^i \mu_L^i \cdots \mu_L^i \) (respect to \( \mu_L^i \mu_L^i \mu_L^i \cdots \mu_L^i \)) is again a hyperbolic seed.

2. \( \mu_R^k \mu_L^i(\mathbb{i}) = \mu_L^k \mu_R^i(\mathbb{i}) = \mathbb{i}, \forall k \in [1, n] \)

**Proof.** (1) We prove it for \( \mu_R^k(\mathbb{i}) \), (respect \( \mu_L^k(\mathbb{i}) \)) and the proof for an arbitrary sequence of right (respect to left) mutations is by induction on the length of the sequence. To prove that \( \mu_R^k(\mathbb{i}) \) (respect to \( \mu_L^k(\mathbb{i}) \)) is again a seed, for any \( k \in [1, n] \), we first show the commutativity of the elements of \( \{ x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n \} \), which is reduced to show that \( x_k x_i = x_i x_k, \forall k \in [1, n] \). By definition of mutation; \( x_k' \) (respect to \( x_k \)) is a sum of two monomials each of them is a product of elements only from the set \( \{ x_k^{-1} \} \cup N_1(x_k) \). By the definition of the seed, we have \( x_i \) must commute with all elements of \( \{ x_1, \ldots, x_{k-1}, x_k^{-1}, x_{k+1}, \ldots, x_n \} \) thanks to the commutativity of \( \{ x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n \} \). Equations 2.1.1 and 2.1.8 imply that \( x_k \) commutes with all elements of \( N_1(x_k) \) because non of the elements of \( F_i \) is in \( N_1(x_k) \). To finish the proof, It is enough to show that, there is a \( R \)-linear automorphism \( \varphi' \) (respect \( \varphi \)) of \( F \) such that \( \mu_R^k(\mathbb{i}) = (F', X', \Gamma', \varphi') \) (respect \( \mu_L^k(\mathbb{i}) \)) is a hyperbolic seed. Let \( \varphi' = T_1^{i} \varphi T_1^{i} \), and \( x_k = \mu_{1,k}(x) \). We have
\[
f_{ij}x_j' = T_{i\downarrow}(f_{ij}x_j) \\
= T_{i\downarrow}(\varphi(x_j,t_{ij})) \\
= T_{i\downarrow}\varphi T_{i\downarrow}(x_j',f_{ij}) \\
= \varphi'(x_j',f_{ij}) \\
= \varphi'(x_j')f_{ij}.
\]

Part (2) justifies proving it for sequences consists only of right mutations or only of left mutations, as of the mixed sequences are reduced to either right or left mutations sequences. This finishes the proof of part (1).

(2) One can see that the action of \(\mu^R_k\mu^L_k\) and \(\mu^L_k\mu^R_k\) coincide with the identity action on both of \(F\) and \(\Gamma\), since the right and left mutations act like Fomin-Zelevinsky mutation on \(F\) and \(\Gamma\), and Fomin-Zelevinsky mutation is involutive.

In the following we show that \(\mu^R_k\mu^L_k(X) = X\), and \(\mu^L_k\mu^R_k(X) = X\) is quite similar. Let \(F, X, \Gamma\), and \(\mu^R_k\mu^L_k(F, X, \Gamma)\).

We have \(\mu^R_k(x_k) = \hat{x}_k = x_k^{-1}(\bar{M}_C(\hat{r}^{-1}_k) + \bar{M}_C(\hat{r}^{-1}_k))\), where \(\bar{M}_C(\hat{r}^{-1}_k) = x_kM_C(\hat{r}^{-1}_k)\) and \(\bar{M}_C(\hat{r}^{-1}_k) = x_kM_C(\hat{r}^{-1}_k)\). By definition of mutation on \(\Gamma\), one has \(\hat{r}^{-1}_k = \hat{r}^{-1}_k\) and \(\hat{r}^{-1}_k = \hat{r}^{-1}_k\). Then condition 2.1.8, and commutation relations 2.1.1, guarantee \(\bar{M}_C(\bar{r}^{-1}_k) = \bar{M}_C(\bar{r}^{-1}_k)\) and \(\bar{M}_C(\bar{r}^{-1}_k) = \bar{M}_C(\bar{r}^{-1}_k)\).

Hence,
\[
\mu^R_k(x_k) = (\bar{M}_C(\hat{r}^{-1}_k) + \bar{M}_C(\hat{r}^{-1}_k)(\hat{x}_k)^{-1}) \\
= (\bar{M}_C(\hat{r}^{-1}_k) + \bar{M}_C(\hat{r}^{-1}_k)(\hat{x}_k)^{-1}) \\
= x_k.
\]

Finally, for the automorphism \((\varphi')\) of the seed \(\mu^R_k\mu^L_k(\hat{x})\). From part (1), we have \((\varphi') = T_{i\downarrow}(\varphi)T_{i\downarrow}\varphi T_{i\downarrow}T_{i\downarrow}(\varphi') = T_{i\downarrow}T_{i\downarrow}\varphi T_{i\downarrow}T_{i\downarrow} = id_\varphi\varphi id_\varphi = \varphi\).

\(\square\)

**Definition 2.1.8. The Harvest sets and (right and left) Harvest patterns**

(1) Let \(\hat{i}\) be a seed (or a hyperbolic seed) a cluster variable \(x \in F\), is said to be a right harvest element (respect to a left harvest element) of \(\hat{i}\) if \(x\) is a cluster variable in some seed \(j\), where \(j\) can be obtained from \(\hat{i}\) by applying some sequence of right mutations (respect to left mutations). The set of all harvest right elements of \(\hat{i}\) (respect to harvest left) is called the **the right harvest set** (respect to the left harvest set) of \(\hat{i}\), and is denoted by \(\chi^R(\hat{i})\) (respect to \(\chi^L(\hat{i})\)). The set of all right and left harvest elements of \(\hat{i}\) is called the **harvest set** of \(\hat{i}\) and is denoted by \(\chi(\hat{i})\). So, \(\chi(\hat{i}) = \chi^R(\hat{i}) \cup \chi^L(\hat{i})\).

(2) Let \(\hat{i}\) be a hyperbolic seed. The harvest pattern \(T(\hat{i})\) of \(\hat{i}\) is a directed graph built in the following way; label an initial vertex with \(\hat{i}\), and from \(\hat{i}\) we generate arrows as follows, every \(k \in [1,n]\) corresponds to two arrows going out from \(\hat{i}\) one for right mutation in \(k\)-direction and the other one is for the left mutation in same direction, each arrow is targeting a new hyperbolic seed, which is generated by the indicated mutation applied to \(\hat{i}\). Now repeat the process to the new vertices.
Note that; harvest patterns are not necessary to be regular graphs, and could contain cycles.

(3) Right (respect to left) harvest pattern is defined in same way with the restriction, every \(k \in [1, n]\) corresponds to only one arrow which is the right mutation at \(k\) (respect to left mutation), and is denoted by \(T^R(i)\) (respect to \(T^L(i)\)).

**Open Problem 2.1.9.** Given a hyperbolic seed \(i = (F, X, \Gamma, \varphi)\)
- What are necessary and sufficient conditions on \(\Gamma\) and \(\varphi\) that guarantee \(\chi(i)\) to be a finite set. Same question can be raised for right and left harvest sets.
- What are necessary and sufficient conditions on \(\Gamma\) and \(\varphi\) that guarantee \(T(i)\) (respect to \(T^R(i)\) and \(T^L(i)\)) to be a finite graph or a periodic graph.

In the following we will provide some sufficient conditions on a seed \(i\) that guarantees \(\chi(i)\) to be a finite set (respect to \(T(i)\)).

**Definition 2.1.10.** A hyperbolic seed \(i\) is said to be a well-connected seed, if it satisfies condition; there are \(n\) nonnegative integers \(a_1, \ldots, a_n\), such that

\[
\sum_{f_{ik} \in F^k_{1,+}} \mathcal{F}^k_{1}(f_{ik}) = \sum_{f_{ik} \in F^k_{1,-}} \mathcal{F}^k_{1}(f_{ik}) = a_k, \quad \forall k \in [1, n]
\]

In this case the \(n\)-tuples \(f_i = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}\) is called the frozen rank of \(i\).

Through the rest of the article, any statement contains \(\mu_k\) without the superscript \(R\) or \(L\), the statement is true for right and left mutations \(\mu^R_k\) and \(\mu^L_k\) respectively. However, the proofs are written for right mutations and for left mutations the proofs are quite similar in most of the cases.

**Proposition 2.1.11.** The mutation in any direction of a hyperbolic well-connected seed is again well-connected, and the frozen rank is invariant under the mutation.

**Proof.** Immediate from the definitions of well-connected seeds and the definition of mutation. \(\square\)

**Theorem 2.1.12.** Let \(i = (F, X, \Gamma, \varphi)\) be a well-connected hyperbolic seed, with \(\varphi\) unipotent automorphism. Then, \(\mu_k\) is invertible on \(i\) for each \(k \in [1, n]\). More precisely, there is a non negative integer \(r\) such that

\[
(\mu^R_k)^{2r}((F, X, \Gamma)) = (\mu^L_k)^{2r}((F, X, \Gamma)) = (F, X, \Gamma).
\]

The proof of the theorem is a consequence of the following lemma.

**Lemma 2.1.13.** Let \(i = (F, X, \Gamma, \varphi)\) be a well-connected hyperbolic seed. Then for every cluster variable \(x_k\) in the cluster \(X\), we have

\[
(\mu^R_{1,k})^2(x_k) = \varphi^{a_k}(x_k) \quad \text{for some nonnegative integer } a_k.
\]

\[
(\mu^L_{1,k})^2(x_k) = \varphi^{-a_k}(x_k) \quad \text{for some nonnegative integer } a_k.
\]
Proof. We start with proving 2.1.14. Let
\[ \bar{r}_i^k = (r_{i1}, \ldots, r_{im}, r_{i2}, \ldots, r_{im2}, \ldots, r_{in}, l_{i1}, \ldots, l_{ik-1}, -1, l_{ik+1}, \ldots, l_{in}) \]
and
\[ \hat{r}_i^k = (r_{i1}', \ldots, r_{im}', r_{i2}', \ldots, r_{im2}', \ldots, r_{in}', l_{i1}', \ldots, l_{ik-1}', -1, l_{ik+1}', \ldots, l_{in}'). \]
Then, we have, \( \mu_{1,k}(x_k) = M_C(\hat{r}_i^k) + M_C(\bar{r}_i^k) \). By definition of \( M_C(\hat{r}_i^k) \) and \( M_C(\bar{r}_i^k) \), and the commutation relations 2.1.1, one can see \( M_C(\hat{r}_i^k) \) and \( M_C(\bar{r}_i^k) \) can be written as follows
\[ M_C(\hat{r}_i^k) = \tilde{M}_C(\hat{r}_i^k)x_k^{-1} \quad \text{and} \quad M_C(\bar{r}_i^k) = \tilde{M}_C(\bar{r}_i^k)x_k^{-1}. \]
Where,
\[ \tilde{M}_C(\hat{r}_i^k) = t_{i1}^{r_{i1}'} \ldots t_{im}^{r_{im}'} \cdot t_{i1}^{r_{i1}''} \ldots t_{im}^{r_{im}''} \cdot t_{i1}^{r_{i1}'''} \ldots t_{im}^{r_{im}'''}, \]
and
\[ \tilde{M}_C(\bar{r}_i^k) = t_{i1}^{r_{i1}} \ldots t_{im}^{r_{im}} \cdot t_{i1}^{r_{i1}'} \ldots t_{im}^{r_{im}'} \cdot t_{i1}^{r_{i1}''} \ldots t_{im}^{r_{im}''} \].
If \( C' \) is the cluster of the seed \( i' = \mu_i^R(1) \), and \( \mu_{1,k}^R(x_k) = x_k' \), then by the definition of mutation on \( \Gamma \), we have \( \hat{r}_i^k = \hat{t}_{i'}^k \), and \( \bar{r}_i^k = \bar{t}_{i'}^k \). Hence, \( M_C(\hat{r}_i^k) = \tilde{M}_C(\hat{t}_{i'}^k) \), and \( M_C(\bar{r}_i^k) = \tilde{M}_C(\bar{t}_{i'}^k) \). Then one has
\[ (\mu_{1,k}^R)^2(x_k) = \mu_{1,k}^R((\tilde{M}_C(\hat{r}_i^k) + \tilde{M}_C(\bar{r}_i^k))x_k^{-1}) \]
\[ = (\tilde{M}_C(\hat{t}_{i'}^k) + \tilde{M}_C(\bar{t}_{i'}^k))x_k(\tilde{M}_C(\hat{r}_i^k) + \tilde{M}_C(\bar{r}_i^k))^{-1} \]
\[ = (\tilde{M}_C(\hat{r}_i^k) + \tilde{M}_C(\bar{r}_i^k))x_k(\tilde{M}_C(\hat{r}_i^k) + \tilde{M}_C(\bar{r}_i^k))^{-1}. \]
Since \( i \) is a well-connected seed, then there is a positive integer \( a_k \) such that
\[ \sum_{f_{ik} \in F_{1,i}^k} \hat{r}_i^k(f_{ik}) = \sum_{f_{ik} \in F_{-1,i}^k} \bar{r}_i^k(f_{ik}) = a_k. \]
In the case of \( a_k = 0 \), we have no thing to prove. Assume it’s nonzero, then, using the commutation relations 2.1.1, and 2.1.9 one can see
\[ (\tilde{M}_C(\hat{r}_i^k) + \tilde{M}_C(\bar{r}_i^k))x_k = \varphi^{a_k}(x_k)(\tilde{M}_C(\hat{r}_i^k) + \tilde{M}_C(\bar{r}_i^k)). \]
This finishes the proof of equations 2.1.14. The proof of 2.1.15 is quite similar except for using the commutation relations 2.1.10 instead of 2.1.9 in the step before the last one. \( \square \)

Proof. of theorem 2.1.12 We prove it for right mutations
Assume that \( i = (F, X, \Gamma, \varphi) \) is as in the statement of theorem 2.1.12 and \( \varphi^r = id_x \) for some non negative integer \( r \). By definition of mutations on the cluster variables, the mutation in the \( k \)-direction leaves every cluster variable with no change except for \( x_k \). Therefore the following sequence of repeated mutations in the \( k \)-direction \( (\mu_k^R)^{2r} \), will leave every cluster variable, other than \( x_k \) unchanged, and for \( x_k \), the lemma tells us
\[ (\mu_k^R)^{2r}(x_k) = \varphi^{a_k}(x_k) = x_k. \]
Also, one can see \( (\mu_k^R)^2(\Gamma) = \Gamma \). Then, \( (\mu_k^R)^{2r}(\Gamma) = \Gamma \). This finishes the proof of the theorem for the right mutation case, and the case of left mutation is quite similar. \( \square \)
Exemple 2.1.14. The simplest nontrivial well-connected seed. Consider the hyperbolic well-connected seed \( i = (F, X, \Gamma, \varphi) \) where \( F = \{ f_{11}, f_{12} \}, X = \{ x \}, \Gamma \) is the following graph

\[
\begin{array}{cc}
& 1' & \text{and let } \varphi \text{ be a } R\text{-linear automorphism of } F \text{ satisfying the conditions 2.1.9. Here } \\
I_f = \{ 1', 2' \} & \text{and } I_a = \{ 1 \}, \text{ and the frozen rank is } (1). \text{ This seed produces the following harvest set } \\
& \chi(1) = \{ x, (f_{11}+f_{12})x^{-1}, x^{-1}(f_{11}+f_{12}), \varphi(x), (f_{11}+f_{12})\varphi^{-1}(x), \varphi^{-1}(x)(f_{11}+f_{12}); k \in \mathbb{Z} \} \\
\end{array}
\]

One can see that \( \chi(1) \) is a finite set if and only if \( \varphi \) is a unipotent automorphism.

Exemple 2.1.15. The simplest non well-connected seed. Consider the hyperbolic well-connected seed \( i = (F, X, \Gamma, \varphi) \) where \( F = \{ f \}, X = \{ x \}, \Gamma \) is the following graph

\[
\begin{array}{cc}
& 1' & \text{and let } \varphi \text{ be a } R\text{-linear automorphism of } F \text{ satisfying the conditions 2.1.9. Here } \\
I_f = \{ 1' \} & \text{and } I_a = \{ 1 \}. \text{ We have the following infinite harvest set; } \\
& \chi(1) = \{ x, (1+f)^{k+1}x^{-1}(1+f), (1+f)^{k}x(1+f)^{-k}, (1+f)^{k}x^{-1}(1+f)^{k+1}, (1+f)^{-k}x^{-1}(1+f)^{k}, k \in \mathbb{Z} \}. \text{ In this case, this seed has no frozen rank, and so condition } \\
\end{array}
\]

2.2. The Groups of Harvest Automorphisms.

Definition 2.2.1. An \( R \)-linear automorphism \( \phi \) of \( F \) is called a harvest automorphism of a seed \( i \) if it leaves the harvest set of \( i \), \( \chi(1) \), invariant. The group of all such automorphisms is called the group of Harvest automorphisms of \( i \), and is denoted by \( H[i] \). Right and left harvest automorphisms groups, denoted by \( H^R[i] \) and \( H^L[i] \) respectively, can be defined in the same way, by replacing the harvest set of \( i \) by right and left harvest sets, respectively.

Open Problem 2.2.2. For a given (hyperbolic) seed \( i = (F, X, \Gamma, \varphi) \), describe the group of all harvest automorphisms of \( i \) (respect to right and left harvest automorphisms).

The following proposition and theorem provide a big class of harvest automorphisms of some seeds.

Proposition 2.2.3. If \( i = (F, X, \Gamma, \varphi) \) is a well-connected hyperbolic seed, then \( \varphi \) gives rise to an infinite set of harvest automorphisms of \( i \).

Proof. Fix a well-connected hyperbolic seed \( i = (F, X, \Gamma, \varphi) \). For every \( l \in \mathbb{Z} \), we define an \( R \)-linear automorphism on \( F \), \( \phi_l \), induced by

\[
(2.2.1) \quad \phi_l(t) = t, \quad \forall t \in R \quad \text{and} \quad \phi_l(x_k) = \varphi^{l_{ak}}(x_k), \quad \forall k \in [1, n],
\]

where \( (a_1, \ldots, a_k, \ldots, a_n) \) is the frozen rank of \( i \), (from definition 2.1.10). In the following, We prove that \( \phi_l \) is an harvest automorphism for every \( l \in \mathbb{Z} \).

First, for nonnegative integers. Let \( l = 1 \). Lemma 2.1.13 tells us that, the action of this automorphism on the cluster variables of \( i \) is identified with the action of the sequence of the mutation automorphisms \( \prod_{k=1}^{k=n}(\mu_k^R)^2 \), which corresponds to the sequence of mutations \( \prod_{k=1}^{k=n}(\mu_k^R)^2 \). So, \( \phi_1 \) sends every cluster variable in \( X \) to
a harvest element of \( i \) which is a cluster variable in the seed \( \prod_{k=1}^{k=n}(\mu_i^R)^2(1) \). By definition of \( \phi_1 \), one can see it does depend only on the frozen rank of \( i \) which is invariant under mutation, thanks to proposition 2.1.11.

Now, let \( x \) be any harvest element of the seed \( i \), then it is a cluster variable in some seed \( j \), which can be obtain from \( i \) by applying some sequence of mutations say \( \mu_i^R \cdot \mu_{i-1}^R \cdot \cdots \mu_1^R \). From theorem 2.2.13, we must have \( \phi_1(x) \) is a cluster variable in the seed \( \prod_{k=1}^{k=n}(\mu_i^R)^2(1) = \prod_{k=1}^{k=n}(\mu_i^R)^2( \mu_i^R \cdots \mu_1^R ) \), i.e. it is a harvest element of \( i \), and this proves it for \( l = 1 \).

For \( l \geq 2 \), one can see that, the action of \( \phi_l \) on the elements of \( \tilde{X} \) is the same as the action of \((\prod_{k=1}^{k=n}(\mu_i^R)^2)^l\). Then, the same argument as above we can see that \( \phi(x) \) sends every element in \( \tilde{X} \) to a cluster variable in the seed \((\prod_{k=1}^{k=n}(\mu_i^R)^2)^l(1) \), and the case of an arbitrary harvest element \( x \) is the same as \( l = 1 \) with the obvious changes.

Second, the case of \( l \) is a negative integer is similar, with using the left mutations rather than the right mutation, i.e. the superscript \( R \) will be replaced by \( L \) when it makes sense, and use the equations 2.1.14 instead of equations 2.1.13.

Remark 2.2.4. Let \( S_n \) be the set of all seeds of rank \( n \) in \( F \), that share the same set of frozen variables \( F \) and \( S_n \) be the set of all the graphs of the elements of \( S_n \) with set of vertices \( I = F \cup [1, n] \). Let \( \mathfrak{s}_n \) be the symmetric group in \( n \) letters. We have, \( \mathfrak{s}_n \) acts on \( S_n \) as follows, for \( \Gamma \in \mathfrak{s}_n \) and \( \sigma \) be a permutation in \( \mathfrak{s}_n \), \( \sigma(\Gamma) \) is obtained from \( \Gamma \) simply by permuting the vertices of \( \Gamma \).

Lemma 2.2.5. Let \( \Gamma \) be a graph as defined in 2.1.1. Then for any sequence of mutations \( \mu_{i_k}, \mu_{i_{k-1}}, \ldots, \mu_{i_1} \), we have

\[(2.2.2) \quad \sigma(\mu_{i_k} \mu_{i_{k-1}} \cdots \mu_{i_1}(\Gamma)) = \mu_{\sigma(i_k)} \mu_{\sigma(i_{k-1})} \cdots \mu_{\sigma(i_1)}(\sigma(\Gamma)), \quad \forall \sigma \in \mathfrak{s}_n.\]

Proof. Part (1) of theorem 2.8 in [32]

Theorem 2.2.6. Let \( i = (F, X, \Gamma) \) and \( i' = (F, X', \Gamma') \) be two elements of \( S_n \), such that \( \Gamma = \sigma(\Gamma) \), for some permutation \( \sigma \in \mathfrak{s}_n \). Then the automorphism \( T_{1 \Gamma'} \sigma \), induced by \( T_{1 \Gamma'} \sigma(t) = t, \forall t \in R \), and \( T_{1 \Gamma'} \sigma(x_i) = x_{\sigma(i)}', \) is a harvest automorphism of \( i \). (The theorem can be phrased for left mutations as well)

Proof. The proof will be broken into three steps.

Step one:

\[(2.2.3) \quad T_{1 \Gamma'}(\mu_{i_k}(x_k)) = \mu_{\Gamma'}(\sigma(k))(x_{\sigma(k)}'), \quad \forall k \in [1, n].\]

Since \( \Gamma' = \sigma(\Gamma) \), one can see \( \mu_{\Gamma'}(\sigma(k)) = \sigma(\mu_{\Gamma'}(\sigma(k))) \) and \( T_{1 \Gamma'}(\sigma(k)) = T_{1 \Gamma'}(\sigma(k)). \)

Then

\[T_{1 \Gamma'}(\mu_{i_k}(x_k)) = T_{1 \Gamma'}(\sigma(\mu_{\Gamma'}(\sigma(k)))) = T_{1 \Gamma'}(\sigma(\mu_{\Gamma'}(\sigma(k)))) = \mu_{\Gamma'}(\sigma(k))(x_{\sigma(k)}').\]

Second step: Lemma 2.2.5.

Third step: In this step we show,

\[(2.2.4) \quad T_{\mu_{i_1}}(\mu_{i_2}) \cdots (x_{i_k}) = T_{1 \Gamma'}(\sigma(\mu_{i_k}(x_i))), \quad \forall i, k \in [1, n].\]

For \( k \neq i \), we have \( \mu_{i_1}(i_k)(\mu_{i_1}(x_i)) = \mu_{i_1}(x_i) \), then...
Exemple 2.3.2. The hyperbolic algebra of rank \( n \) is a hyperbolic algebra of rank \( n \) with the commutation relations:

\[
T_{\mu,\mu_\sigma}(x_i) = T_{\mu,\mu_\sigma}(x_i) + M_C(T_{\mu,\mu_\sigma}(x_i))
\]

More precisely, for any seed \( \mu \), the \( \mu \)-ring generated by the indeterminates \( x_1, \ldots, x_n \) is a hyperbolic algebra of rank \( n \). For any harvest element \( y \in \chi(1) \), \( T_{\mu}(y) \) is a cluster variable in some seed. Then, \( \chi = \mu_i \mu_{i-1} \cdots \mu_i(1) = (F, Y, \chi) \) for some sequence of mutations. Step 2 implies

\[
T_{\mu}(y) = T_{\mu}(y) = T_{\mu}(y) = T_{\mu}(y)
\]

which finishes the proof.

Remark 2.2.7. The group of all Harvest automorphisms is invariant under mutation. More precisely, for any seed \( \mu \), the following equation is satisfied

\[
H[\mu] = H[\mu_j(1)] = \cdots = H[\mu_j(1)] = \cdots
\]

However, the equation is not satisfied for right and left harvest groups, and even the inclusion is not guaranteed.

2.3. Hyperbolic Cluster Algebra.

Definition 2.3.1. Hyperbolic Algebra-\([30, 31]\) Let \( \mathcal{R} \) be a commutative ring. Let \( \theta = (\theta_1, \ldots, \theta_n) \) be an \( n \)-tuple of automorphisms of \( \mathcal{R} \), and let \( \xi_1, \ldots, \xi_n \) be a fixed set of elements of \( \mathcal{R} \). The hyperbolic algebra of rank \( n \), denote by \( \mathcal{R}(\theta, \xi, n) \), is defined to be the \( \mathcal{R} \)-ring generated by the indeterminates \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) with the commutation relations:

\[
x_i r = \theta_i(r)x_i \quad \text{and} \quad y_i r = \theta_i^{-1}(r)y_i, \quad \text{for any } i \in [1, n], \quad \text{and for any } r \in \mathcal{R}
\]

\[
x_i y_j = \xi_i, \quad \forall i \in [1, n] \quad x_i y_j = y_j x_i, \quad x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i, \quad \forall i \neq j.
\]

Example 2.3.2. The \( n \)-th weyl algebra \( A_n \) is a hyperbolic algebra of rank \( n \), (cf. [30]).

Let \( A_n \) be the Weyl algebra of \( 2n \) variables \( x_1, \ldots, x_n, y_1, \ldots, y_n \), and the relations

\[
x_i y_j = y_j x_i + 1 \quad \forall i \in \{1, \ldots, n\}, \quad \text{and} \quad x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i \quad \text{for } i \neq j.
\]

Let \( \xi_i = y_i x_i, \quad \mathcal{R} = \mathcal{K} [\xi_1, \ldots, \xi_n], \) and \( \theta : \mathcal{R} \to \mathcal{R} \), induced by \( \xi_i \mapsto \xi_i + 1, \xi_j \mapsto \xi_j, j \neq i. \) One can see \( A_n = \mathcal{R}(\theta, \xi, n) \) is a hyperbolic algebra of rank \( n \).
Theorem 2.3.8. The coordinate algebra of \( SL_q(2,k) \) (c.f. \([30, 31]\)). The coordinate algebra \( A(SL_q(2,k)) \) of algebraic quantum group \( SL_q(2,k) \) is the \( K \)-algebra generated by the indeterminants \( x, y, u, \) and \( v \) subject to the following relations

\[
(2.3.4) \quad qux = xu, \quad qxv = xv, \quad qyu = uy, \quad qyv = vy, \quad uv = vu, \quad q \in K^*
\]

\[
(2.3.5) \quad xy = quv + 1, \quad \text{and} \quad yx = q^{-1}uv + 1.
\]

\( A(SL_q(2,k)) = \mathcal{R}(\xi, \theta, 1) \) is a hyperbolic algebra of rank 1, the hyperbolic structure can be seen as follows. \( \mathcal{R} = K[u,v] \) the algebra of polynomials in \( u, v \) and \( \theta \in \text{Aut.}(\mathcal{R}) \) is given by \( \theta(f(u,v)) = f(qu,qv) \) for any polynomial \( f(u,v) \), and \( \xi = 1 + q^{-1}uv \).

Definition 2.3.4. A quadruple \( h = (\mathcal{R}, X, \Gamma, \theta) \) is said to be a hyperbolic feed in \( \mathcal{F}, \) if

- \( \mathcal{R} \) is a commutative ring
- \( X = (x_1, \ldots, x_n) \in \mathcal{F}^n, \) such that the skew field of fractions of the domain \( \mathcal{R}[X] \) is an \( R \)-automorphic copy of \( \mathcal{F} \)
- \( \Gamma \) as in the definition 1.1
- \( \theta = (\theta_1, \ldots, \theta_n) \) is an \( n \)-tuple of commutative ring automorphisms of \( \mathcal{R} \) satisfies

\[
(2.3.6) \quad x_i^+/r = \theta_i(\mathcal{R})x_i^+, \quad \forall i \in [1,n], \quad \forall r \in \mathcal{R}.
\]

Definition 2.3.5. Mutations in hyperbolic feeds is defined in same way as in the case of hyperbolic seeds, with the obvious change by leaving the commutative ring \( \mathcal{R} \) invariant.

Notice that, if \( h = (F, X, \Gamma, \varphi) \) is a hyperbolic seed, then taking \( \mathcal{R} = \mathbb{Z}[\mathcal{P}], \) as defined in 2.1, we may have a hyperbolic feed with the same data of \( i \) if there is an \( \mathcal{R} \)-automorphisms \( \{\theta_i\}_{i=1}^n \) satisfies equations 2.3.6.

Definition 2.3.6. The Hyperbolic Cluster Algebra For a hyperbolic feed \( h = (\mathcal{R}, X, \Gamma, \theta) \) (respect to a hyperbolic seed \( i = (F, X, \Gamma, \varphi) \)), the hyperbolic cluster algebra \( \mathcal{H}(h) \) (respect to \( \mathcal{H}(i) \)) is defined to be the \( \mathcal{R} \)-subalgebra (respect \( \mathcal{R} \)-subalgebra, \( R = \mathbb{Z}[\mathcal{P}], \) \( \mathcal{P} \) is the free abelian group generated by elements of \( F \) of \( \mathcal{F} \)) of \( \mathcal{F} \) generated by the harvest set \( \chi(h) \) (respect to \( \chi(i) \)).

Remark 2.3.7. If \( h = (\mathcal{R}, X, \Gamma, \theta) \) is a hyperbolic feed (respect to \( i = (F, X, \Gamma, \varphi) \) is a hyperbolic seed) with \( \theta = id_{\mathcal{R}} \) (respect to \( \varphi = id_{\mathcal{F}} \)) then the hyperbolic cluster algebra \( \mathcal{H}(h) \) (respect to \( \mathcal{H}(i) \)) coincides with the geometric Fomin-Zelevinsky (commutative) cluster algebra associated to the seed \( p = (F, X, \Gamma) \), and in this case \( \mathcal{F} \) is a commutative field.

Theorem 2.3.8. Let \( h = (\mathcal{R}, X, \Gamma, \theta) \) be a hyperbolic feed of rank \( n \), with \( X = (x_1, \ldots, x_n) \), such that \( N_+(x_k) \subset \mathcal{R} \) for all \( k \in [1,n] \) and let \( x_k^r = \mu^{R_+}_{k,\pm}(x_k) \), (respect \( \mu^{L_+}_{k,\pm}(x_k) \)), \( \xi_k = x_kx_k^r \) (respect \( \xi_k = x_k\xi_k^r \) then the following are true:

1. \( R(i) := R(\xi, \theta^{-1}, n) \) is a hyperbolic algebra of rank \( n \) (respect to \( R(\xi, \theta, n) \)).
2. \( \mu^{R}_{k,\pm}(i) \) (respect \( \mu^{L}_{k,\pm}(i) \)) is again a hyperbolic feed
3. Right and left mutations on feeds define homomorphisms between hyperbolic algebras from part (1).
(4) There is a monomorphism \( \psi : \mathcal{R}(\mathfrak{i}) \rightarrow \mathcal{R}[x_1^\pm, \ldots, x_n^\pm] \), where \( \mathcal{R}[x_1^\pm, \ldots, x_n^\pm] \) is the ring of Laurent polynomials in \( x_1, \ldots, x_n \), with coefficients from \( \mathcal{R} \). More precisely, every element \( z \) of \( \mathcal{R}(\mathfrak{i}) \) can be written uniquely as linear combinations of cluster monomials of the initial cluster \( \mathcal{X} \).

(5) Let \( \mathcal{R} = \mathbb{Z}[\mathbb{P}] \), then we have

\[
\mathcal{H}(\mathfrak{i}) = \mathcal{R}(\mathfrak{i})
\]

Proof. (1) Since \( N_1(x_k) \subset \mathcal{R} \) for all \( k \in [1, n] \), then \( x'_k x_k = \xi_k \in \mathcal{R} \) (respect to \( x_k x'_k \)). Let \( r \in \mathcal{R} \). Then we have, \( x_k r = \theta_k(r)x_k, \forall k \in [1, n] \) this is because of the assumption that \( \mathfrak{i} \) is a hyperbolic feed.

Also, we have

\[
x'_k r = \xi x^{-1} r = \xi \theta_k^{-1}(r)x_k^{-1} = \theta_k^{-1}(r)\xi x^{-1} = \theta_k^{-1}(r)x'_k,
\]

other commutation relations of the hyperbolic algebra structure are immediate from the commutation relations of the hyperbolic feeds and feed mutations, (left mutation case is quite similar).

(2) Let \( \mu_{k,1}^{R}(\mathfrak{i}) = (R, X', \Gamma, \theta) \). By the definition of mutation on \( \Gamma \) it is easy to see that; since \( N_1(x_k) \subset \mathcal{R} \) for all \( k \in [1, n] \) then \( N_{\mu_{k,1}^{R}(\mathfrak{i})}(x'_k) \subset \mathcal{R} \) for all \( k \in [1, n] \), (respect to left mutation on \( \mathfrak{i} \)), which means \( x'_k x_k = \xi_k \in \mathcal{R} \) (respect to \( x_k x'_k \)). We have

\[
x''_kr = \xi_k x_k x^{-1} r = \xi_k x_k x^{-1} \xi_k^{-1} = \xi_k^{-1} \theta_k(r)x_k \xi_k = \theta_k(r)\xi_k x_k \xi_k^{-1} = \theta_k(r)x''_k,
\]

and

\[
x''_{k-1}r = \xi x^{-1} r = \xi \theta_k^{-1}(r)x_k^{-1} = \theta_k^{-1}(r)\xi x^{-1} = \theta_k^{-1}(r)x''_k.
\]

Also, since \( N_1(x_k) \cap N_1(x_k) \cap \mathcal{R} \) is an empty set for every \( i \in [1, n] \), then \( x_i \) commutes with \( \xi_k \), and hence

\[
x'_k x_i = \xi x_k x^{-1} x^{-1} = \xi \theta_k(r)x_k \xi_k = x_i x'_k, \forall i \in [1, n].
\]

This finishes the proof of part (2).

(3) Consider the \( \mathcal{R} \)-linear automorphism on \( \mathcal{F} \), denoted by \( T_{\mathfrak{i},k}^{R} : \mathcal{F} \rightarrow \mathcal{F} \), and induced by, \( x_k \mapsto \mu_{k,1}^{R}(x_k) \), and \( x_i \mapsto x_i, \forall i \neq k \in [1, n] \). The restriction of this automorphism on \( \mathcal{R}(\mathfrak{i}) \) induces the following algebra homomorphism \( T_{\mathfrak{i},k}^{R} : \mathcal{R}(\mathfrak{i}) \rightarrow \mathcal{R}(\mathfrak{i}') \), given by \( r \mapsto r, \forall r \in \mathcal{R} \), and \( x_k \mapsto \xi x_k^{-1} = x'_k, \forall k \in [1, n] \), which implies \( x'_k \mapsto \xi_k x_k \xi_k^{-1} = \mu_{k,1}^{R}(x'_k) \). Finally, it is easy to see that the hyperbolic commutation relations 3.1, and 3.2 are invariant under \( T_{\mathfrak{i},k}^{R} \). (the argument for \( T_{\mathfrak{i},k}^{L} \) and \( T_{\mathfrak{i},k}^{L'} \) is quite similar).

(4) By definition 2.3.1 and part (1) above, we have \( \mathcal{R}(\mathfrak{i}) \) is generated by \( \mathcal{R} \) and \( x_1, \ldots, x_n \) and \( x'_1, \ldots, x'_n \), with relations (2.3.1) and (2.3.2) replacing \( y_i \)'s with \( x_i \)'s. Let \( m \) be any monomial in \( x_1, \ldots, x_n \) and \( x'_1, \ldots, x'_n \), first part of relations 2.3.2 can be used to remove every possible sub monomials of the form \( x_i x'_i \) and replace them with \( \xi_i \). So, \( m \) can be written as a monomial from the following direct sum of skew rings of polynomials

\[
(2.3.8) \quad \mathcal{R} = \mathcal{R}[x_1] \oplus x^{-1}[x_1] \oplus \cdots \oplus x^{-1}[x_1] \oplus \mathcal{R}[x_1] \oplus x^{-1}[x_1] \oplus \cdots \oplus x^{-1}[x_1] \oplus \cdots
\]
where $\mathcal{R}^i = \mathcal{R}[x_1^\pm, \ldots, x_{i-1}^\pm, x_{i+1}^\pm, \ldots, x_n^\pm]$. The ring $\mathcal{R}$ inherits the multiplication and the relations of $\mathcal{R}(\mathfrak{i})$. Finally, consider the map $\psi$ that sends every monomial from $\mathcal{R}(\mathfrak{i})$ to itself after applying the relations 2.3.1, whenever possible. One can see that $\psi$ is a monomorphism.

(5) $\mathcal{H}(\mathfrak{i})$ is generated by the harvest elements of $\mathfrak{i}$, and a random harvest element $h$ can be written as $h = g_1 x_k^{\pm 1} \sum_{j=1}^t n_j f_{ij}^{\alpha_j}$, where $g_1$ and $g_2$ are elements of $\mathcal{R}$, thanks to the condition $N_i(x_k) \subset \mathcal{R}$, and $x_k$ is an initial cluster variable. Since, $g_2 \in \mathcal{R}$, then $g_2 = \sum_{j=1}^{n_j} n_j f_{ij}^{\alpha_j}$, where, $\alpha_j, n_j \in \mathbb{Z}$. Then

$$h = g_1 x_k^{\pm 1} \sum_{j=1}^t n_j f_{ij}^{\alpha_j}$$

$$= g_1 \sum_{j=1}^t n_j (\theta(f_{ij}))^{\pm \alpha_j} x_k^{\pm} \in \mathcal{R}(\mathfrak{i}).$$

Which means $\mathcal{H}(\mathfrak{i}) \subseteq \mathcal{R}(\mathfrak{i})$, and the other direction is obvious.

Corollary 2.3.9. 
(1) Every hyperbolic cluster algebra, comes from a hyperbolic feed (respect to a hyperbolic seed) satisfies the conditions of theorem 2.3.8, is a hyperbolic algebra in the sense of definition 2.3.1.

(2) Deeper interpretations for for part (3) of theorem 2.3.8
(a) every vertex of the harvest pattern of a hyperbolic feed $\mathfrak{i}$ gives raise to a hyperbolic algebra, the collection of all such hyperbolic algebras forms a scheme of hyperbolic algebras glued by the homomorphisms $\mathcal{T}^R_{\mathfrak{1} \mathfrak{1}'}$.

(b) From representation theory point of view, the homomorphism $\mathcal{T}^R_{\mathfrak{1} \mathfrak{1}'}$ induces a functor on the category of representations of the hyperbolic algebra $\mathcal{H}(\mathfrak{i})$, replacing the action of $x_i$’s by the action of $x_i'$’s. In the special case of Weyl algebra $A_n$, $\mathcal{T}^R_{\mathfrak{1} \mathfrak{1}'}$ acting as a functor on the category of representations of the weyl algebra $A_n$, exchanges the action of the deferential operators with the multiplication by the corresponding indeterminant.

(c) $\mathcal{T}^R_{\mathfrak{1} \mathfrak{1}'}$, coincides with the the canonical anti-automorphism mentioned in [30] page 62, which is plays an essential role in describing the representation theory of hyperbolic algebras.

(3) The hyperbolic algebra of rank 1, $\mathcal{R}(\mathfrak{i})$ is isomorphic to $\mathcal{R}[x] \oplus x^{-1}\mathcal{R}[x^{-1}]$

3. Examples

3.1. The Weyl Algebra.

Exemple 3.1.1. Let $A_n$ be the Weyl algebra of $2n$ variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ satisfying the relations 2.3.3. Consider the following seed $\mathfrak{i} = (F,Y,\Gamma)$, where $Y = (y_1, \ldots, y_n)$ with $F = \{\xi_i| \xi_i = y_i x_i, 1 \leq i \leq n\}$ is the set of frozen variables, with $F_i = \{\xi_i\}$ and $\Gamma$ is the graph

(3.1.1) $\begin{array}{ccccccccc}
\mathfrak{1} & \mathfrak{2} & \mathfrak{3} & \cdots & \mathfrak{n-1} & \mathfrak{n} \\
\mathfrak{1}' & \mathfrak{2}' & \mathfrak{3}' & \cdots & \mathfrak{n-1}' & \mathfrak{n}'
\end{array}$
where \([1, n]\) corresponds to the elements of \(Y\), and \([1', n']\) corresponds to the elements of \(F\). Let \(P\) be the free abelian group generated by the elements of \(F\) and written multiplicatively. The hyperbolic cluster algebra \(\mathcal{H}(\mathfrak{i})\) corresponding to \(A_n\) is the \(\mathbb{Z}[P]\)-subalgebra of the skew field of fractions \(\mathcal{F}\) of the domain \(K[P][Y]\), (we take \(\mathcal{F}\) to be the field of fractions of \(A_n\)), generated by the harvest set of the hyperbolic seed \(\mathfrak{i} = (F, Y, \Gamma, \varphi)\), where \(\varphi\) is a \(\mathbb{Z}[P]\)-linear automorphism of \(\mathcal{F}\) induced by \(\varphi(x_i) = y_i x_i y_i^{-1}, \varphi(y_i) = x_i y_i^{x_i^{-1}}\). One can see that \(\mathfrak{i}\) is not well-connected seed, and \(\varphi\) is not a nilpotent automorphism. In the following we will see that it's harvest set is an infinite set.

**A hyperbolic feed associated to \(A_n\).** Consider the following assignments:

\[
R := K[\xi_1, \ldots, \xi_n], \quad \theta = (\theta_1, \ldots, \theta_n) \quad \text{where} \quad \theta_i : R \to R \quad \text{given by} \quad \theta_i(\xi_j) = \begin{cases} 
\xi_i + 1, & \text{if } i = j, \\
\xi_j, & \text{if } i \neq j.
\end{cases}
\]

These choices make \(\mathfrak{h} = (R, Y, \Gamma, \theta)\) a hyperbolic feed in \(\mathcal{F}\). This hyperbolic feed satisfies all the conditions of theorem 3.8., then the mutation on the feed \(\mathfrak{h}\) provides us with an infinite class of Weyl algebras connected (glued) by algebra homomorphisms induced by mutations.

**The harvest set of the seed \(\mathfrak{i}\), \(\chi(\mathfrak{i})\):** Since \(\mu_{ik}^R(y_k) = (\xi_k + 1)y_k^{-1} = x_k, k \in [1, n]\), we have

\[
(y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_n) \xrightarrow{\mu_{ik}^R} (y_1, \ldots, y_{k-1}, (\xi_k + 1)y_k^{-1}, y_{k+1}, \ldots, y_n) = (y_1, \ldots, y_{k-1}, x_k, y_{k+1}, \ldots, y_n).
\]

Then, the right mutations in the directions 1, 2, \ldots, \(n\) cover all the generators of the Weyl algebra \(A_n\), which means the fact

\[
(3.1.2) \quad A_n \hookrightarrow \mathcal{H}(\mathfrak{h}) = \mathcal{R}(\mathfrak{h}).
\]

**Notice:** The same phenomenon occurs if we start with \(\mathfrak{i} = (F, X, \Gamma)\) with \(X = (x_1, \ldots, x_n)\) and apply left mutations rather than right mutations.

Yet, the mixed sequence mutations \(\mu_{ik}^L \mu_{ik}^R\) and \(\mu_{ik}^R \mu_{ik}^L\) act like identity on every seed (feed) in the harvest pattern of \(\mathfrak{h}\), but the harvest set of \(\mathfrak{i}\) is an infinite set because the unmixed mutations sequences never reproduce the seed (feed) \(\mathfrak{i}\) again, as we will see in the following.
Let $A_1 = K < x, y : (xy = \xi + 1)$, where $\xi = yx$, consider the following seed $i = (R, y, \cdot \mapsto \cdot y)$, where $R = \mathbb{Z}[\mathbb{P}], \mathbb{P} = \{y^n; n \in \mathbb{Z}\}$ the free group generated by $y$, written multiplicatively. We have the following harvest patterns for this case.

\[
\begin{align*}
(y_1, \ldots, y_{k-1}, y_k, y_{k+1} \ldots, y_n) & \overset{\mu_i^k}{\Rightarrow} (y_1, \ldots, y_{k-1}, y_k^{-1}(\xi_k + 1), y_{k+1} \ldots, y_n) \\
& \overset{\mu_i^k}{\Rightarrow} (y_1, \ldots, y_{k-1}, (\xi_k + 1)^{-1}y_k(\xi_k + 1), y_{k+1} \ldots, y_n) \\
& \overset{\mu_i^k}{\Rightarrow} (y_1, \ldots, y_{k-1}, (\xi_k + 1)^{-1}y_k^{-1}(\xi_k + 1)^2, y_{k+1} \ldots, y_n) \\
& \overset{\mu_i^k}{\Rightarrow} (y_1, \ldots, y_{k-1}, (\xi_k + 1)^{-2}y_k(\xi_k + 1)^{-2}, y_{k+1} \ldots, y_n) \\
& \cdots \\
& \overset{\mu_i^k}{\Rightarrow} (y_1, \ldots, y_{k-1}, (\xi_k + 1)^{-j}y_k^{-1}(\xi_k + 1)^{j+1}, y_{k+1} \ldots, y_n) \\
& \overset{\mu_i^k}{\Rightarrow} (y_1, \ldots, y_{k-1}, (\xi_k + 1)^{-(j+1)}y_k(\xi_k + 1)^{+j+1}, y_{k+1} \ldots, y_n) \\
& \cdots
\end{align*}
\]

and

\[
\begin{align*}
(x_1, \ldots, x_{k-1}, x_k, x_{k+1} \ldots, x_n) & \overset{\mu_i^R}{\Rightarrow} (x_1, \ldots, x_{k-1}, (\xi_k + 1)x_k^{-1}, x_{k+1} \ldots, x_n) \\
& \overset{\mu_i^R}{\Rightarrow} (x_1, \ldots, x_{k-1}, (\xi_k + 1)x_k(\xi_k + 1)^{-1}, x_{k+1} \ldots, x_n) \\
& \overset{\mu_i^R}{\Rightarrow} (x_1, \ldots, x_{k-1}, (\xi_k + 1)^2x_k^{-1}(\xi_k + 1)^{-1}, x_{k+1} \ldots, x_n) \\
& \overset{\mu_i^R}{\Rightarrow} (x_1, \ldots, x_{k-1}, (\xi_k + 1)^{-2}x_k(\xi_k + 1)^{-2}, x_{k+1} \ldots, x_n) \\
& \cdots \\
& \overset{\mu_i^R}{\Rightarrow} (x_1, \ldots, x_{k-1}, (\xi_k + 1)^{j+1}x_k^{-1}(\xi_k + 1)^{-j}, x_{k+1} \ldots, x_n) \\
& \overset{\mu_i^R}{\Rightarrow} (x_1, \ldots, x_{k-1}, (\xi_k + 1)^{-(j+1)}x_k(\xi_k + 1)^{+j+1}, x_{k+1} \ldots, x_n) \\
& \cdots
\end{align*}
\]

Also, since $N_i(x_k)$ does not contain any non-frozen initial cluster variable, for every $k \in [1, n]$, one can see that $\mu_i^R \mu_i^R$ and $\mu_i^R \mu_i^R$ (respect to $\mu_i^R \mu_i^R$ and $\mu_i^R \mu_i^R$) act in the same way on any seed (feed) in the harvest pattern of $i$. Therefore the harvest set of $i$ can be restricted only to the following elements

\[
\begin{align*}
\chi(i) &= \{y_1, \ldots, y_n, (\xi_k + 1)^{-j-1}y_k^{-1}(\xi_k + 1)^{+j}, (\xi_k + 1)^{-j}y_k(\xi_k + 1)^{+j}; j \in \mathbb{N}, k \in [1, n]\} \\
& \cup \{(\xi_k + 1)^{+j+1}y_k^{-1}(\xi_k + 1)^{-j}, (\xi_k + 1)^{-j}y_k(\xi_k + 1)^{-j}; j \in \mathbb{N}, k \in [1, n]\}
\end{align*}
\]

\[
\begin{align*}
&= \{x_1, \ldots, x_n, (\xi_k + 1)^{+j+1}x_k^{-1}(\xi_k + 1)^{-j}, (\xi_k + 1)^{-j}x_k(\xi_k + 1)^{-j}; j \in \mathbb{N}, k \in [1, n]\} \\
& \cup \{(\xi_k + 1)^{-j}x_k^{-1}(\xi_k + 1)^{+j}, (\xi_k + 1)^{-j}x_k(\xi_k + 1)^{+j}; j \in \mathbb{N}, k \in [1, n]\}
\end{align*}
\]

The harvest patterns of the first Weyl algebra $A_1$.
3.1.3) $\cdots \rightarrow R \ rac{y}{L} \rightarrow y^{-3} \rightarrow R \ rac{y}{L} \rightarrow y^{-2} \rightarrow R \ rac{y}{L} \rightarrow y^{-1} \rightarrow R \ rac{y}{L} = y_0 \rightarrow R \ rac{y}{L} \rightarrow y_1 \rightarrow R \ rac{y}{L} \rightarrow y_2 \rightarrow R \ rac{y}{L} \rightarrow y_3 \rightarrow R \ rac{y}{L} \cdots$

(here $\rightarrow L$ is left mutation and $\rightarrow R$ is right mutation). Which can be encoded by the following equations

\[(3.1.4)\]  
\[y_{k+1}y_k = y_ky_{k+1} + 1, \quad \text{for} \quad k \in 2\mathbb{Z},\]

\[(3.1.5)\]  
\[y_ky_{k+1} = y_{k+1}y_k + 1, \quad \text{for} \quad k \in 2\mathbb{Z} + 1.\]

These equations are equivalent to say that, each arrow from the harvest pattern corresponds to a copy of the first Weyl algebra, denoted by $A^k_1 = K\langle y_k, y_{k+1} \rangle, k \in \mathbb{Z}$, and mutations define algebra maps between these Weyl algebras, given by $T_k : A^k_1 \rightarrow A^{k+1}_1$, $y_k \mapsto y_{k+1}$ for $k \in \mathbb{Z}_{\geq 0}$, and $T_k : A^k_1 \rightarrow A^{k+1}_1$, $y_k \mapsto y_{k-1}$ for $k \in \mathbb{Z}_{<0}$.

Remark 3.1.2. **Fomin-Zelevinsky finite type classification** [16] **fails in this case.** In the case of the first Weyl algebra $A_1 = K \langle x, y \rangle > / (xy - yx = 1)$ with the seed $\mathbf{i} = (\xi = yx, y, '1 ---- '1')$ here $\mathbf{i}$ is of $A_1$ type as a cluster algebra based on Fomin-Zelevinsky finite type classification however $\chi(\mathbf{i})$ is an infinite set, which means Fomin-Zelevinsky finite type classification does not work in this case.

3.2. **The coordinate algebra of** $SL_q(2, k)$. Recall definition 2.3.3. Consider the following hyperbolic feed $\mathfrak{h} = (\mathcal{R}, x, \Gamma, \theta)$, where $\mathcal{R} = K[u, v]$ and $\theta : \mathcal{R} \rightarrow \mathcal{R}$ given by $\theta(f(u, v)) = f(qu, qv)$ as in the definition 2.3.3, and $\Gamma$ is given by

\[(3.2.1)\]

\[
\begin{array}{c}
1 \\
\downarrow \quad 1 \\
\uparrow \quad 2 \\
\\
1' \\
\end{array}
\]

Consider the set of frozen variables $F$ given by $F = \{qu, v\}$. In this case we deform the mutation (right and left) as follows, mutation of $q$ is $q^{-1}$, i.e. for example $\mu_2(quv+1) = q^{-1}uv + 1$.

Remark that; $\mathfrak{h}$ satisfies the conditions of theorem 2.3.8 and the conditions of the well-connected seeds.

One can see the right mutation on $\mathfrak{h}$ will produce the following new feed $\mathfrak{h'} = (\mathcal{R}, y, \Gamma', \theta)$, since

\[(3.2.2)\]  
\[\mu^R_1(x) = (q^{-1}uv + 1)x^{-1} = \xi x^{-1} = y.\]

and $\Gamma'$ is as follows

\[(3.2.3)\]

\[
\begin{array}{c}
1 \\
\downarrow \quad 1 \\
\uparrow \quad 2 \\
\\
1' \\
\end{array}
\]

Applying left mutation on $\mathfrak{h'}$ produces the original seed $\mathfrak{h}$. Also, we have

\[(3.2.4)\]  
\[A(SL_q(2, k)) \leftrightarrow \mathcal{R}(\mathfrak{h}) = \mathcal{H}(\mathfrak{h})\]
The harvest set of $h$: Let $\zeta = quv + 1$. We have

$$
\chi(h) = \{x, \zeta^j x, \zeta^{j+1} x, \zeta^{-j}, j \in \mathbb{N}\} \cup \{y, \xi^j y, \zeta^{j+1} y, \zeta^{-j}, j \in \mathbb{N}\}.
$$

4. Irreducible representation arising from the harvest graphs of Weyl Algebras

In the following we introduce a family of indecomposable and irreducible representations arising from the harvest pattern of the Weyl algebra $A_n$, and we speculate that the same structures can be defined for any hyperbolic algebra with a hyperbolic cluster structure.

Definition 4.0.1. Let $h = (F,Y,\Gamma,\theta)$ be a hyperbolic feed of rank $n$. The harvest pattern $T(h)$ of $h$ is a directed $2n$-regular graph, defined in the following way; label an initial vertex with $h$, and from $i$ we generate arrows as follows, every $k \in [1,n]$ corresponds to two arrows going out from $i$ one for right mutation and the other one is for the left mutation in $k$-direction, each arrow is targeting a new hyperbolic feed, which is generated by the indicated mutation applied to $h$. Now repeat the process to the new vertices.

Remark 4.0.2.

- Harvest patterns contain cycles, and could be an infinite graph.
- Since $\mu^R_k \mu^L_k(i) = \mu^L_k \mu^R_k(i) = i$, then each vertex generates exactly $2n$ 2-cycles connecting it with the $n$ adjacent vertices.
- Each vertex represents a feed, which is formed from a cluster and a graph, and each cluster contains the same frozen variables and the clusters of any two adjacent seeds are different in only one harvest element.

Definition 4.0.3. Let $h = (F,Y,\Gamma,\theta)$ be the hyperbolic feed of rank $n$, introduced in example 3.1.1. Here $F = \{\xi_1, \ldots, \xi_n\}$ is the set of frozen variables, $Y = (y_1, \ldots, y_n)$ is the $n$-tuples of initial non-frozen cluster variables, and $\theta = (\theta_1, \ldots, \theta_n)$ is $n$-tuples of $K[\xi_1, \ldots, \xi_n]$-automorphisms, given by

$$
\theta_i(\xi_j) = \begin{cases} 
\xi_i + 1, & \text{if } i = j, \\
\xi_j, & \text{if } i \neq j .
\end{cases}
$$

Let $\chi(h)$ be the harvest set of $h$. In addition to the $n$-th Weyl Algebra $A_n$, we have the following algebras related to $h$.

- The Hyperbolic cluster algebra $\mathcal{H}(h)$,
- The algebra $\mathcal{B} = K(y_1, \ldots, y_n)[y'_1, \ldots, y'_n]$, the algebra of polynomials in the first generation cluster variables $\mu^R_i(y_i) = y'_i, i \in [1,n]$, (respect to $\mu^L_i(y_i) = y'_i, i \in [1,n]$), with coefficients from the field of fractions of the initial cluster variables,
- The algebra $B = K(\xi_1, \ldots, \xi_n)[\chi(h)]$ the algebra of polynomials in the harvest set of $h$ with coefficients from the field of fractions of the frozen variables.

Remark 4.0.4. We have the following obvious inclusions,

(4.0.5) $A_n \hookrightarrow \mathcal{H}(h) \hookrightarrow \mathcal{B},$

(4.0.6) $A_n \hookrightarrow \mathcal{H}(h) \hookrightarrow B.$
i.e. The hyperbolic cluster algebra is an intermediate algebra between the \( n \)-th Weyl Algebra and each of \( \mathfrak{B} \), and \( B \).

**Motivations:** The representation theory of the three algebras \( A_n \) and the algebras \( \mathfrak{B} \), and \( B \) are closely related, see for example [3].

**Definition 4.0.5. Space of Representations** \( V_n \). Let \( h = (F,Y,\Gamma,\theta) \) be a hyperbolic feed of rank \( n \), a **cluster monomial** of \( h \) is a monomial formed from harvest elements that are showing up (at least once) as cluster variables in some seed in the harvest pattern of \( h \). To visualize it, the monomial \( m = z_1^{\beta_1} \cdots z_n^{\beta_n}, \beta_i \in \mathbb{Z}_{\geq 0}, i \in [1,n] \) is a cluster monomial if \( (z_1, \ldots, z_n) \) is a cluster in a seed in the harvest pattern of \( h \). In case of \( \beta_i \in \mathbb{Z}_{> 0}, \forall i \in [1,n] \), \( m \) is called a **full cluster monomial**.

**The space of representations** \( V_n \) is defined to be the \( K(\xi_1, \ldots, \xi_n) \)-left span by the set of all cluster monomials of \( h \).

**Lemma 4.0.6.** For any hyperbolic feed \( h \) (respect to hyperbolic seed), the space of representations \( V_n \) is independent of \( h \), and depends only on \( T(h) \) the harvest pattern of \( h \).

**Proof.** The statement of the lemma is equivalent to say ” any two hyperbolic feeds (seeds) in the harvest pattern of \( h \) have the same harvest pattern", and to see this fact: let \( f \) be any hyperbolic feed in \( T(h) \). Then, \( f \) can be obtained from \( h \) by applying some sequence of mutations, without lose of generality we may assume it is a sequence of right mutations only say \( \mu_{i_1}^R \cdots \mu_{i_l}^R \). But part (2) of lemma 2.1.7 tells us that we can obtain \( h \) from \( f \) by applying the sequence of left mutations \( \mu_{i_1}^L \cdots \mu_{i_l}^L \). This finishes the proof. However, we may realize this fact by realizing that any two vertices in the harvest pattern are connected by two oppositely directed paths. \( \square \)

**Remark 4.0.7.** In the case of \( h \) is the hyperbolic feed associated to the Weyl algebra or the coordinate algebra of \( SL_n(2, K) \), the situation is easier since in this case a cluster monomial is any monomial formed from any set of harvest elements. In order to see this fact we need to recall the following two, easy to prove, combinatorial proposition.

**Proposition 4.0.8.** If \( h \) is the hyperbolic feed associated to the Weyl algebra or the coordinate algebra of \( SL_n(2, K) \), then the following are true

1. For any set of \( n \) (or less) different harvest elements, not including two elements produced from the same initial cluster variable, there is at least one seed in the harvest pattern of \( h \) which contains all of them.
2. For \( z_1 \) and \( z_2 \) are any two harvest elements produced from the same initial cluster variable, then we have two cases for their product:
   - if \( z_2 \) can be obtained from \( z_1 \) by applying sequence of mutations of odd length, then \( z_1z_2 \in K(\xi_1, \ldots, \xi_n) \)
   - if \( z_2 \) can be obtained from \( z_1 \) by applying sequence of mutations of even length, then \( z_1z_2 \) can be written as \( g\gamma_1^2 \), for some \( g \in K(\xi_1, \ldots, \xi_n) \).

A left action of the algebras \( A_n, \mathfrak{B}, \) and \( B \) on \( V_n \).

Consider the following notation; let \( Y = (y_1, \ldots, y_n) \) be the initial cluster, for \( t \in \mathbb{Z}_{\geq 0}, y_{i+m+n} \) denotes the harvest element obtained from the initial cluster variable \( y_i \) by applying one of the following sequence of mutations \( (\mu_i^R)^t \) if \( t \geq 0 \) or \( (\mu_i^L)^t \) if \( t < 0 \).
Using the above notation, a typical element of $V_n$ can be written as a sum of monomials like the following monomial

$$(4.0.7) \quad v = f(\xi_1, \ldots, \xi_n) y_{1,m_1}^{\beta_1} \cdots y_{n,m_n}^{\beta_n},$$

where $f(\xi_1, \ldots, \xi_n) \in R$, and $(\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$, and $(m_1, \ldots, m_n) \in \mathbb{Z}^n$

A left action on the general term $v$ is defined as follows

$$(4.0.8) \quad y_i(v) = f(\xi_1, \ldots, \xi_{i-1}, \theta_i^{-1}(\xi_i), \ldots, \xi_n) y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i}^{\beta_i} y_{i+1,m_{i+1}}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n}.$$  

$$(4.0.9) \quad x_i(v) = \theta_i(\xi_i) f(\xi_1, \ldots, \xi_{i-1}, \theta_i(\xi_i), \ldots, \xi_n) y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i}^{\beta_i} y_{i+1,m_{i+1}}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n}.$$  

**Lemma 4.0.9.**  
1. The action of $x_i$ and $y_i$ is invertible. In particular, the action is compatible with mutations, and hence is defined for all harvest elements, and the action of $x_i$ can be recovered from the action of $y_i$, for every $i$.
2. $V_n$ is a left $A_n$, $\mathcal{B}$, and $B$ module with the action induced by the action of the initial clusters $y_i$, defined above.

**Proof.**  
1. If $y_i^{-1}$ and $x_i^{-1}$ denote the inverses of $y_i$ and $x_i$ respectively, then we have $y_i^{-1}$ acts like $\theta_i^{-1}(\xi_i)x_i$, and $y_i^{-1}$ acts like $\xi_i^{-1}y_i$. For the rest of the statement is an immediate calculation recalling that $\mu_i(y_i) = (1+\xi) y_i^{-1} = x_i$.
2. The action is consistent with relations 2.3.3. One can see $(x_i y_j - y_j x_i)(v)$ can be written as follows

$$= x_i( f(\xi_1, \ldots, \xi_{i-1}, \theta_i^{-1}(\xi_i), \xi_{i+1}, \ldots, \xi_n) \beta_1^{\beta_1} \cdots \beta_{i-1}^{\beta_{i-1}} \beta_i^{\beta_i} y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i}^{\beta_i} y_{i+1,m_{i+1}}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n})$$

$$- y_i(\theta_i(\xi_i) f(\xi_1, \ldots, \xi_{i-1}, \theta_i(\xi_i), \xi_{i+1}, \ldots, \xi_n) \beta_1^{\beta_1} \cdots \beta_{i-1}^{\beta_{i-1}} \beta_i^{\beta_i} y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i}^{\beta_i} y_{i+1,m_{i+1}}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n})$$

$$= \theta_i(\xi_i) f(\xi_1, \ldots, \xi_{i-1}, \theta_i^{-1}(\theta_i(\xi_i)), \xi_{i+1}, \ldots, \xi_n) \beta_1^{\beta_1} \cdots \beta_{i-1}^{\beta_{i-1}} \beta_i^{\beta_i} y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i}^{\beta_i} y_{i+1,m_{i+1}}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n})$$

$$- \theta_i(\theta_i^{-1}(\xi_i)) f(\xi_1, \ldots, \xi_{i-1}, \theta_i(\theta_i^{-1}(\xi_i)), \ldots, \xi_n) \beta_1^{\beta_1} \cdots \beta_{i-1}^{\beta_{i-1}} \beta_i^{\beta_i} y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i}^{\beta_i} y_{i+1,m_{i+1}}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n})$$

$$= (\theta_i(\xi_i) - \xi_i)v$$

$$= v.$$

In a similar way, one gets $(x_i y_j - y_j x_i)(v) = 0$, for $i \neq j$

**Example 4.0.10.** Consider the hyperbolic feed (seed) $\mathcal{I}$ introduced in example 3.1.1. The $i$-th branch of the harvest pattern $\mathcal{T}(\mathcal{I})$ is as follows:

$$(4.0.10) \quad \begin{array}{c}
\left( y_{1,m_1} \cdots y_{i,m_{i-1}} \cdots y_{n,m_n} \right) R \left( y_{1,m_1} \cdots y_{i,m_{i-1}} \cdots y_{n,m_n} \right) L \left( y_{1,m_1} \cdots y_{i,m_{i-1}} \cdots y_{n,m_n} \right) \quad \left( y_{1,m_1} \cdots y_{i,m_{i+1}} \cdots y_{n,m_n} \right) R \left( y_{1,m_1} \cdots y_{i,m_{i+1}} \cdots y_{n,m_n} \right) L \left( y_{1,m_1} \cdots y_{i,m_{i+1}} \cdots y_{n,m_n} \right).
\end{array}$$

Here, right mutations go to the right direction and left go to left. For sake of simplicity, we skipped labeling each vertex by the whole seed data and kept only the cluster variables, since the other data are all invariant under right and left mutations in the $i$-the direction.
In this case, $V_n$ is the left $\mathcal{R}$-linear span generated by the following set
(4.0.11)
$\{y_{1,m_1}^\beta \cdots y_{n,m_n}^\beta \mid m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_{\geq 0}^n\}.$

4.1. The Cluster Strings and the string submodules of $V_n$. Before introducing the cluster strings we need to develop some notations. For $b \in \mathbb{Z}$, we have

$$\theta^b(-) = \begin{cases} 
\overbrace{\theta(\theta(\cdots \theta(\cdots \theta(\cdots (-)) \cdots))}^{b\text{-times}}, & \text{if } b > 0, \\
\text{id}_R, & \text{if } b = 0, \\
\overbrace{\theta^{-1}(\theta^{-1}(\cdots \theta^{-1}(\cdots (-)) \cdots))}^{b\text{-times}}, & \text{if } b < 0.
\end{cases}$$

Consider the following two sets of monomials in elements from the set $\{\theta^b(\xi); b \in \mathbb{Z}\}$

$$M^+(\xi) := \{1, \theta^b(\xi^\pm)\theta^{t+1}(\xi^\pm)\cdots \theta^{q+t}(\xi^\pm)|q, t \in \mathbb{Z}_{\geq 0}\}$$

$$M^-(\xi) := \{1, \theta^b(\xi^\pm)\theta^{t+1}(\xi^\pm)\cdots \theta^{q+t}(\xi^\pm)|q + t, t \in \mathbb{Z}_{\leq 0}\}.$$ 

Now we are ready to introduce one more set of monomials, $M(\xi)$

(4.1.1) 

$$M(\xi) := \{m_1 m_2 \mid m_1 \in M^+(\xi) \text{ and } m_2 \in M^-(\xi)\}.$$ 

Let $R$ be any one of the following rings $K[\xi_1, \ldots, \xi_n], K(\xi_1, \ldots, \xi_n)$ or $K[\mathbb{F}]$, and $A$ be any of the algebras $A_n, \mathfrak{B}$ or $B$. Let $E = \{\xi_1^\pm, \ldots, \xi_n^\pm\}$. The set of all monomials formed from the elements of $E$ is denoted by $M(E)$.

Fix a natural number $l \in \mathbb{N}$ and a 1-1 map $\sigma : [1, l] \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n$. Let $\beta = (\beta_1, \ldots, \beta_l) \in (\mathbb{Z}^n)^l$, and $m = (m_1, \ldots, m_l) \in (\mathbb{Z}_{\geq 0}^n)^l$ be such that $\sigma(j) = (\sigma_1(j), \sigma_2(j)) = (\beta_j, m_j)$, where $\sigma_1(j) = \beta_j = (\beta_{j1}, \ldots, \beta_{jn})$ and $\sigma_2(j) = m_j = (m_{j1}, \ldots, m_{jn}), j \in [1, l]$.

For $t = (t_1, \ldots, t_n) \in \mathbb{Z}^n$ and $h \in R$, we introduce one more important subset of $R$

(4.1.2) 

$$\alpha(t, h) := \{e \alpha_1 \cdots \alpha_n h(\theta_{1}^\xi(\xi_1), \ldots, \theta_{n}^\xi(\xi_n)) | \alpha_i \in M(\xi_i), e \in M(E), i \in [1, n]\}.$$ 

**Definition 4.1.1. Cluster strings of base $l$.** Every non-negative integer $l$, $f = (f_1, \ldots, f_l) \in R^l$, and a 1-1 map $\sigma : [1, l] \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n$ corresponds to a cluster string, defined as follows

(4.1.3) 

$$S_l(\sigma, f) := \{\sum_{j=1}^{l} g_j y_{1,m_{j1}+t_{j1}}^{\beta_{1,j}} \cdots y_{n,m_{jn}+t_{jn}}^{\beta_{n,j}} \mid t_j = (t_{j1}, \ldots, t_{jn}) \in \mathbb{Z}^n, g_j \in \alpha(t_j, f_j), j \in [1, l]\}.$$ 

**Exemple 4.1.2.** Cluster string of base 2. Let $l = 2$, $\sigma_1(1) = (1, 0), \sigma_2(2) = (1, 2)$, $\sigma_2(1) = (1, 1), \sigma_2(2) = (0, 1)$, and $f = (\xi_1^2 + \xi_2, \xi_2 \xi_1)$, we have for $t = (t_1, t_2)$

$$\alpha(t, \xi_1^2 + \xi_2) = \{e \alpha_1 \alpha_2 \theta_{1}^{2\xi_1}(\xi_1) + \theta(\xi_2)) | \alpha_i \in M(\xi_i), e \in M(E), i \in [1, 2]\}.$$ 

$$\alpha(t, \xi_2 \xi_1) = \{e \alpha_1 \alpha_2 \theta_{1}^{2\xi_1}(\xi_1) \theta^{\xi_2}(\xi_2)) | \alpha_i \in M(\xi_i), e \in M(E), i \in [1, 2]\}.$$ 

With the above date we have

$$S_2(\sigma, f) = \{g_1 y_{1,t_1+1} + g_2 y_{1,0+t_2}, y_{2,1+t_2}^2 | g_1 \in \alpha(t, \xi_1^2 + \xi_2), g_2 \in \alpha(t, \xi_2 \xi_1), t_{ij} \in \mathbb{Z}, i, j \in [1, 2]\}.$$
Definition 4.1.3. Let \( S_l(\sigma, f) \) be a cluster string. The sub module of \( V_n \) generated by \( S_l(\sigma, f) \) is called a string submodule of base \( l \) associated to \( S_l(\sigma, f) \). This submodule is denoted by \( W_l(\sigma, f) \) and called a string submodule if there is no possibility of confusion.

Remark 4.1.4. Each element of \( V_n \) gives rise to a cluster string and hence a submodule of \( V_n \). To see that; every element \( v \) of \( V_n \) can be written as follows

\[
 f_1(\xi_1, \ldots, \xi_n)y_1^{\beta_{11}} \cdots y_{n,m_{1n}}^{\beta_{1n}} + \cdots + f_l(\xi_1, \ldots, \xi_n)y_1^{\beta_{l1}} \cdots y_{n,m_{ln}}^{\beta_{ln}}.
\]

Where \( f_1, \ldots, f_l \) are elements of \( R \), and a \( 1 \rightarrow 1 \) map \( \sigma : [1, l] \rightarrow \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n \) can be defined such that \( \sigma(j) = (\sigma_1(j), \sigma_2(j)) \), where \( \sigma_1(j) = (\beta_{1j}, \ldots, \beta_{nj}) \) and \( \sigma_2(j) = (m_{1j}, \ldots, m_{nj}) \), \( j \in [1, l] \). Consider the cluster string \( S_l(\sigma, f) \), with \( f = (f_1, \ldots, f_l) \). This cluster string is denoted by \( S(v) \) and the submodule of \( V_n \) generated by \( S(v) \) denoted by \( W(v) \).

The following lemma provides some basic properties of the cluster strings:

Lemma 4.1.5. (1) The cluster strings are invariant under the action of every monomial formed from elements of the set \( \mathcal{E} = E \cup \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \), and we can recover any cluster string from any of its element. In particular for any cluster string \( S_l(f, \sigma) \) and for any \( v \in S_l(f, \sigma) \)

\[
(4.1.4) \quad \mathcal{M}(\mathcal{E})v = \left\{ \sum_{j=1}^{l} g_j y_1^{\beta_{1j} t_1} \cdots y_{n,m_{jn}}^{\beta_{jn} t_n} \mid t = (t_1, \ldots, t_n) \in \mathbb{Z}^n, g_j \in a(t, f_j) \right\} \subset S_l(f, \sigma).
\]

Where \( \mathcal{M}(\mathcal{E}) \) is the set of all monomials formed from the elements of the set \( \mathcal{E} \). Hence, for any string submodule \( W_l(f, \sigma) \) we have

\[
(4.1.5) \quad W_l(f, \sigma) = \sum \text{copies of } S_l(f, \sigma)
\]

(2) \( l \neq l' \), then \( S_l(\sigma, f) \neq S_{l'}(\sigma', f) \)

(3) For \( \sigma = \sigma' \), then \( S_l(\sigma, g) = S_l(\sigma, f) \), if and only if \( g \in a_m(t, f) \), for some \( t \in \mathbb{Z}^n \).

(4) Let \( g = (g_1, \ldots, g_n) \) with \( g_i \in a_m(f, t_i) \) for some \( t_i \in \mathbb{Z}^n \). Then \( S_l(\sigma', g) = S_l(\sigma, f) \) if and only if \( \sigma'(j) = \sigma(j) + (0, q_j) \forall j \in [1, l] \) for some \( q_j \in \mathbb{Z}^l \).

In the following, all the cluster strings \( S_l(\sigma, f) \) are with \( \sigma : [1, l] \rightarrow \mathbb{Z}^l_\geq 0 \times \mathbb{Z}^n \) i.e. all the cluster monomials are full.

(5) Every submodule of \( V_n \) is generated by a set of cluster strings

(6) Any two proper submodules of a string submodule \( W_l(\sigma, f) \) have non-zero intersection. In particular \( W_l(\sigma, f) \) is indecomposable module, however it is not necessarily to be irreducible.

(7) If \( W_l(\sigma, f) \) is a string module with base \( l \), then for any \( w \in W_l(\sigma, f) \) the string \( S(w) \) is of base equals a multiple of \( l \).

(8) there is a bijection between the set of all cyclic submodules of \( V_n \) and the set of all string submodules.

Proof. (1) To see that the action of any element of \( \mathcal{M}(\mathcal{E}) \) on any element of \( S_l(\sigma, f) \) is again an element of \( S_l(\sigma, f) \).
We have $x_i$ sends $f_j(\xi_1,\ldots,\xi_n) y_{1,m_1}^{\beta_{j1}} \cdots y_{n,m_n}^{\beta_{jn}}$ to
\[
\theta(\xi_i)f_j(\xi_1,\ldots,\xi_{i-1},\theta(\xi),\xi_{i+1},\ldots,\xi_n) y_{1,m_1}^{\beta_{j1}} \cdots y_{i(m_j)+1,m_{j+1}}^{\beta_{ji-1}} y_{i(m_j)+1,m_{j+1}}^{\beta_{ji}} y_{i(m_j)+1,m_{j+1}}^{\beta_{ji+1}} \cdots y_{n,m_n}^{\beta_{jn}}.
\]
while $y_i$ sends it to
\[
f_j(\xi_1,\ldots,\xi_{i-1},\theta^{-1}_i(\xi),\xi_{i+1},\ldots,\xi_n) y_{1,m_1}^{\beta_{j1}} \cdots y_{i(m_j)-1,m_{j+1}}^{\beta_{ji}} y_{i(m_j)-1,m_{j+1}}^{\beta_{ji+1}} \cdots y_{n,m_n}^{\beta_{jn}}.
\]

Which means none of the base $l$, $f$ nor the map $\sigma$ are changed under the action of $x_1,\ldots,x_n$ or $y_1,\ldots,y_n$. Then they keep every cluster string invariant, and hence same for every monomial formed from the set $E$. The same change will occur in each term of the $l$-terms of every element of $S_l(f,\sigma)$, which justify 4.1.4.

Now, a random element of $A$ is a linear combination of elements of $M(E)$ with coefficients from the field $K$. Remark that in the case of $A = \mathfrak{B}$, the inverses of $x_i$ and $y_i$ still keep the cluster strings invariant, we refer to part one of lemma 4.0.9. Therefore, any element of $A$ will send any element of $S_l(\sigma,f)$ into a sum of elements each of them is an element of $S_l(f,\sigma)$, which proves that $W_l(\sigma,f)$ is entirely included in a sum of copies of $S_l(\sigma,f)$ and obviously any sum of copies of $S_l(f,\sigma)$ is included in $W_l(\sigma,f)$ which finishes the proof of 4.1.5.

(2) This is immediate if we recall that the maps $\sigma$ and $\sigma'$ are $1 - 1$.

(3) (\Rightarrow) Obvious.

(\Leftarrow) if $g \in a(t, f)$ for some $t = (t_1,\ldots,t_n) \in \mathbb{Z}^n$, then there are $\alpha_i \in M(\xi_i)$, and $e \in M(E)$ such that $g = e \alpha_1 \cdots \alpha_n f(\theta^1_1(\xi_1),\ldots,\theta^n_n(\xi_n))$. Remark that elements of $M(\xi)$ are invertible for any choice of $R$, then $S_l(f,\sigma) \subseteq S_l(g,\sigma)$, and the other inclusion is direct from the way we write $g$ in terms of $f$.

(4) (\Rightarrow)

It is easy to see that, if $\sigma'_j(j) = \sigma_1(j), \forall j \in [1,l]$, then $\sigma' = \sigma + (0,q_j)$ for some $q_j \in \mathbb{Z}^n$.

Assume that $\sigma'_j(j_0) \neq \sigma(j_0) + (0,q_j)$ for some $j_0 \in [1,l]$ and for every $q_j \in \mathbb{Z}^n$. Then $\sigma'_j(j) \neq \sigma_1(j)$. Then, the element $\sum_{j=1}^l g_j y_{1,m_1}^{\beta_{j1}} + t_1 \cdots y_{n,m_n}^{\beta_{jn}}$, with $\sigma'_j(j) = (\beta_{j1},\ldots,\beta_{jn})$, is an element of $S_l(\sigma',g)$ but is not an element of $S_l(\sigma,f)$.

(\Leftarrow)Immediate.

(5) First notice that from parts 2 and 3 and proof of part 8, we conclude that every two cluster strings are either identical or have zero intersection. So we can introduce the following equivalence relation

\[
s \sim s' \text{ if and only if } s \text{ and } s' \text{ belong to the same string module.}
\]

Let $W$ be any submodule of $V_n$. Let $W^* = W/ \sim$. Then we have the following identity

\[
W = \oplus_{w \in W^*} W(w).
\]

(6) Let $W_1$ and $W_2$ be any two proper submodules of $W_l(\sigma,f)$. Then, there are two non zero elements $w_i \in W_i$ for $i = 1,2$. The above arguments guarantee that $S(w_1)$ and $S(w_2)$ are of bases $l_1$ and $l_2$ respectively, such that they are multiples of $l$, and not equal to $l$, as we will see in the proof of part 7. WLOG assume $l_1 < l_2$, and $l_i = d_i l, i = 1,2$, for some $d_i$ and
\(d_2\) natural numbers. Let \(l'\) be the least common multiple of \(l_1\) and \(l_2\). So, 
\(l' = n_i l_i\), for some \(n_i \in \mathbb{N}, i = 1, 2\). Consider the following element

\[
(4.1.8) \quad w' = \sum_{i=1}^{l'} s_i, \text{ where } s_i \in S(v).
\]

Here we show \(w' \in W(w_1) \cap W(w_2)\):

Write

\[
(4.1.9) \quad w_1 = \sum_{b=1}^{l_1} \sum_{j=1}^{l} e_j^{(b)} \alpha_{1j}^{(b)} \cdots \alpha_{nj}^{(b)} f_{1j}^{(b)} \cdots f_{mj}^{(b)} \beta_{1j} \cdots \beta_{nj} y_{i,mj,l+j(b)} \cdots y_{n,mj_n+l(b)}
\]

Where for \(j \in [1, l_1]\) and \(b \in [1, l_2]\) we have \(e_j^{(b)} \in \mathcal{M}(E), \alpha_{ij}^{(b)} \in \mathcal{M}(\xi_i), \) and \(f_{ij}^{(b)} = f_{ij}^{(b)}(\xi_1), \ldots, f_{ij}^{(b)}(\xi_n), \forall i \in [1, n].\)

Following remark 4.1.4, we can introduce the cluster string associated to \(w_1\) as follows

Let \(\sigma_j^{(b)}(j) = (\beta_j, t_j^{(b)})\), where \(\beta_j = (\beta_{1j}, \ldots, \beta_{nj})\), and \(t_j^{(b)} = (t_{1j}, \ldots, t_{nj})\). Let

\(\hat{\sigma} : [1, l_1] \to \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}\), given by \(\hat{\sigma}(j) = \sigma_j^{(b)}(j)\) for \(j \in [(b-1)l, bl], b \in [1, d_1]\),

and \(\hat{f} = (f_{11}^{(b)}, f_{1n}^{(b)}, f_{n1}^{(b)}, f_{n2}^{(b)}, \ldots, f_{n1}^{(b)} ) \in R^{d_1}\).

Writing \(w'\) in the same way we can see that

\[
w' \in \sum_{i=1}^{d_1 n_1} \text{ copies of } S(w) \subset \sum_{i=1}^{n_1} \text{ copies of } S(w_1).
\]

In a quite similar way we can show \(w' \in \sum_{i=1}^{n_1} \text{ copies of } S(w_2).\) But we have \(\sum_{i=1}^{n_1} \text{ copies of } S(w_i) \subset W(w_1), i = 1, 2.\) Which means

\[
(4.1.10) \quad w' \in W(w_1) \cap W(w_2).
\]

To see that; let \(w\) be an element of \(W_1(\sigma, f)\). Then \(w = av\) for some \(a \in A\) and \(v \in S_l(\sigma, f)\). Here, \(a\) can be written as \(\sum_{i=1}^{d} k_i e_i\), where \(k_i \in K^*, e_i \in \mathcal{M}(E), \forall i \in [1, n] (e_1, \ldots, e_d\) are different monomials). Each of \(e_i, \) as we saw above, does not change the superscripts of the monomials of \(v,\) however it change the second subscripts simultaneously with the coefficients, such that the action’s output is still an element of \(S_l(\sigma, f).\) Also, since \(\sigma : [1, l] \to \mathbb{Z}_{>0} \times \mathbb{Z}\) i.e. \(\beta_{ij} >, \forall i \in [1, n], j \in [1, l].\) This condition guarantees that each term of \(v\) is a product of a coefficient from the ring \(R\) times a full harvest monomial. So, the action of any element of \(\mathcal{M}(E)\) must change every term of \(v.\) Hence, in deed \(w = \sum_{i=1}^{d} s_i,\) where \(s_i\) is an element of \(S_i(\sigma, f),\) for all \(i \in [1, d].\) Therefore, the element \(w = av\) is a sum of \(dl\) different terms where each term belongs to a copy of \(S_l(\sigma, f).\) Consider the cluster string \(S(w)\) associated to \(w.\) One can see \(W(w)\) is of base \(dl\) and every cluster string contained in \(W\) consists of elements of \(W\) i.e. every cluster string contained in \(W\) is of the form \(W(w)\) which is of base equals a multiple of \(l.\)

Let \(W\) be a cyclic module generated by \(w.\) Then \(W(w) \subseteq W.\) To see the other direction. We have every \(v \in W\) is an element of a sum of copies
of the cluster string $S(w)$ which is a subset of $W(w)$. So the bijection is defined to send $W$ to $S(w)$.

Now let $S_i(\sigma, f)$ be a cluster string. Fix a generic element $w$ of $S_i(\sigma, f)$. One can see $S(w) = S_i(\sigma, f)$. So, $S_i(\sigma, f)$ is sent back to $W(w)$. Remark that, the above argument shows that, every element of $S_i(\sigma, f)$ can replace $w$, i.e. $S(w) = S(w')$ for every $w' \in S_i(\sigma, f)$.

\[ \square \]

**Corollary 4.1.6.**

(1) For every two elements $w$ and $w'$ of the string module $W_i(\sigma, f)$, if $w = \sum_{i=1}^{d} s_i$ and $w' = \sum_{i=1}^{d} s'_i$, where $s_i$ and $s'_i$ are elements of $S_i(\sigma, f)$, for all $i \in [1, d]$, then $S(w) = S(w')$ and $W_i(w) = W_i(w')$, (immediate from the definition of cluster strings and the above arguments).

(2) For any string module $W_i(\sigma, f)$, every cyclic module is of the form $W_{di}(\sigma, f)$, where $d \in \mathbb{Z}_{>0}$, $\hat{\sigma} : [1, dl] \rightarrow \mathbb{Z}_{>0} \times \mathbb{Z}^n$ with $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2)$, and $\hat{\sigma}_1(j) = (\beta_{(j-i)1}, \ldots, \beta_{(j-i)n})$, $j \in [(il + 1, (i + 1)l], i \in [a, d - 1]$, $\hat{\sigma}_2(j) = (t_{(j-1)}^{(b)}, \ldots, t_{jn}^{(b)}) \in \mathbb{Z}^n$, $\forall j \in [1, dl]$ and $f = (f_1^{(b)}, \ldots, f_{n}^{(b)}, f_1^{(b)}, \ldots, f_{n}^{(b)}), f_1^{(b)}, \ldots, f_{n}^{(b)}$, where $f_j^{(b)} = f_j(\theta_1^{j(b)}(\xi_1), \ldots, \theta_{(n)}^{j(b)}(\xi_n)), \forall j \in [1, l], b \in [1, d]$.

**Horizontal** Infinite base **Cluster strings**. Fix $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_{>0}$.

Consider the following element

\[ w(\beta) = \sum_{t= (t_1, \ldots, t_n) \in \mathbb{Z}^n} y_{1,m_1+t_1}^{\beta_1} \cdots y_{n,m_n+t_n}^{\beta_n}. \]

Denote the cluster string of $w(\beta)$ by $S(\beta)$ and the string submodule by $W(\beta)$.

**Theorem 4.1.7.** For every $\beta \in \mathbb{Z}_{>0}^n$, $W(\beta)$ is an irreducible $\mathfrak{B}$ module.

**Proof.** One can see the following

\[ y_i w(\beta) = w(\beta) \quad \text{and} \quad x_i w(\beta) = \theta_i(\xi) w(\beta), \forall i \in [1, n]. \]

Therefore, for any $b \in \mathfrak{B}$, we have $bw(\beta) = \hat{f}(x_1, \ldots, \xi_n)w(\beta)$, for some $\hat{f} \in K(\xi_1, \ldots, \xi_n)$. Which means $\mathfrak{B}w(\beta) = K(\xi_1, \ldots, \xi_n)w(\beta)$. i.e. $W(\beta)$ is in fact a one dimensional vector space. Then it is an irreducible $\mathfrak{B}$ module.

\[ \square \]

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