Canonical measure and the flatness of a FRW universe

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Abstract

We consider the claim of Hawking and Page that the canonical measure applied to Friedmann-Robertson-Walker models with a massive scalar field can solve the flatness problem i.e. $\Omega \simeq 1$, regardless of Inflation occurring or not. We point out a number of ways this prediction, which relies predominantly on post-Planckian regions of the classical phase space, could break down.

By considering a general potential $V(\phi)$ we are able to understand how the ambiguity for $\Omega$ found by Page in the $R^2$ theory, is present in general for scalar field models when the potential is bounded from above. We suggest reasons why such potentials are more realistic, which then results in the value of $\Omega$ being arbitrary.

Although the canonical measure gives an ambiguity (due to the infinite measure over arbitrary scale factors) for the possibility of inflation, the inclusion of an input from quantum cosmology could resolve this ambiguity. This could simply be that due to a “quantum event” the universe started small; and provided a suitable scalar potential is present an inflationary period could then be “near certain” to proceed in order to set $\Omega$ infinitesimally close to unity. We contrast the measure obtained in this way with the more usual ones obtained in quantum cosmology: the Hartle-Hawking and Tunneling ones.
1 Introduction

One of the reasons for the introduction of inflationary universe models was to resolve the flatness problem—(see eg. refs.[1,2] for a review). Namely that the universe is very close to its critical density \( \rho^{\text{cr}} \) i.e \( \Omega \equiv \rho/\rho^{\text{cr}} \) is unity. There is some doubt that inflation can entirely resolve this problem [3-5]: if \( \Omega \) is initially far from unity then either recollapse or insufficient inflation would instead result.

It is also the case that at present most of the experimental evidence points to an open universe \( 0 < \Omega \leq 1 \), see eg. ref.[5] for a review of this topic. It is therefore of importance to understand what values of \( \Omega \) are to be expected in FRW universes. This is where the question of measure comes in: what is a typical solution of the system of equations describing the universe, and what does it imply for parameters (eg. \( \Omega \)) that we measure experimentally?

1.1 The flatness question

Let us first consider how such a flatness problem occurs in classical cosmology. To see this consider a FRW universe with a perfect fluid equation of state: \( p = (\gamma - 1)\rho \), \( p \)=pressure. The Hamiltonian equation is given by

\[
H^2 + \frac{k}{a^2} = \rho
\]

with \( H \) the Hubble parameter and \( k \) the spatial curvature (we use Planck units throughout and set \( 8\pi G/3 \equiv 1 \)). There is also the continuity equation

\[
\dot{\rho} + 3H\gamma \rho = 0 \quad \Rightarrow \quad \rho = \frac{\rho_o}{a^{3\gamma}} \quad (\rho_o = \text{constant})
\]

Since \( \Omega = \rho/H^2 \) we can write

\[
\Omega = \frac{\rho}{\rho - k/a^2} = \frac{\rho_o}{\rho_o - k a^{3\gamma - 2}}.
\]

If the strong energy condition is satisfied i.e. \( \gamma > 2/3 \) then as the scale factor \( a \to 0 \), \( \Omega \) is set initially to one. Whereas if \( \gamma < 2/3 \) then as \( a \to 0 \), \( \Omega \) is sent towards \( \infty \) for \( k = 1 \), and zero for \( k = -1 \).

For time \( t \) increasing to the future, the value of \( \Omega \) diverges away from 1 whenever the strong energy condition is satisfied i.e. \( \gamma > 2/3 \). Specifically,

\[
\Omega - 1 \propto t^{2-4/3\gamma}.
\]
We can therefore estimate the value of Ω at the planck time assuming no inflationary phase occurred and γ ∼ 1 throughout the evolution of the universe. The age of the universe today is \(\sim 10^{90} t_{pl}\) where \(t_{pl}\) is the planck time. Using expression (4), we can relate Ω at different times as:

\[
\frac{(\Omega - 1)_{\text{now}}}{(\Omega - 1)_{\text{then}}} \approx \frac{(10^{90})^{2/3}}{1} \approx 10^{60} \tag{5}
\]

If we assume that today Ω \(\approx 1\) then at the planck time we require \((\Omega - 1) < 10^{-60}\) i.e. \(\Omega \sim 1 \pm 10^{-60}\).

In order to achieve this value of Ω at the Planck epoch \(a = 1\) requires the constant \(\rho_0 \geq 10^{60}\), since from expression (3) we have \(\rho_0 = ka^{3\gamma-2}\Omega/(\Omega - 1)\).

Therefore to explain the flatness problem from a classical standpoint we need to understand why the energy density or equivalently the Hubble parameter was so large during the initial stages of the universe.

While still on the subject of FRW universes with perfect fluids we point out a few quantities of Einstein-DeSitter (k=0) models which will later be useful. The density \(\rho\) is given by

\[
\rho = \frac{1}{6\pi\gamma t^2} \tag{6}
\]

and the scale factor behaves as—see eg.[6]

\[
a \simeq \rho_0^{1/3} t^{2/3\gamma} \tag{7}
\]

Although we are primarily interested in scalar field models we will at times use the perfect fluid as an analogy when it is useful.

### 1.2 Inflation

We generally assume that given a sufficient amount of inflation the flatness problem can be solved. This is provided the initial conditions are not, in some sense, too extreme. This is because you can always extrapolate back any value of Ω you choose after a fixed amount of inflation, to a corresponding

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1 Assuming the universe is described by a dust model \(\gamma = 1\), the actual age is more like \(\sim H^{-1} \approx 10^{60} t_{pl}\) but this still results in a bound \(\Omega = 1 \pm 10^{-40}\). The size of the present universe is at least \(10^{60} L_{pl}\).
initial value [3]. Therefore whether inflation can solve the flatness problem
depends on the measure for initial conditions. If a set amount of inflation
say $\sim 100$ e-foldings does not resolve the flatness question we will call such
measures extreme. The measures considered in this paper: the canonical and
those from quantum cosmology, are generally found not to be extreme. If
they undergo a certain amount of inflation they should resolve the flatness
question: i.e. set $\Omega$ infinitesimally close to unity.

The question we mostly address is whether inflation occurs or not? This
depends on the form of the scalar potential $V(\phi)$ and so only a subset of
possible potential allow inflation. For this reason we first try to resolve the
flatness question without requiring inflation em per se. Even if the flatness
problem is resolved without resorting to inflation there are still a number of
other reasons why it might be required: for example, to resolve the horizon
question [1,2]. It is therefore important to know how likely inflation is to
happen, for a given form of the potential $V(\phi)$.

By the way, inflation, which has $\gamma \approx 0$ would lead to a problem if con-
tinued back to $a = 0$ and indeed defeat its own purpose if it initially has
$\Omega >> 1$ or $\Omega \sim 0$. Inflation is however not usually assumed to occur imme-
diately from the singularity but at $a \sim 10^{-30} cm$ [1,2].

2 The measure for $\Omega$

While investigating inflation people have tried to put a measure on in-
flationary solutions compared to non-inflationary ones in order to give prob-
abilities for various quantities such as $\Omega$ in a typical universe [7-9]. These
studies have usually considered equipartition of initial conditions at an initial
planck energy density, when the classical equations are first expected to be
valid. By varying the initial spatial curvature and Hubble parameter in this
way it is found that inflation is highly probable [7], with the strong proviso
that an inflationary potential has already been chosen: for example a massive
scalar field $V(\phi) = 1/2m^2\phi^2$ with $m << 1$. A different approach, related to
equipartition, is found in ref.[8], while recently a measure using the DeWitt
metric has been found [9].

Perhaps a more rigorous measure, introduced by Gibbons, Hawking and
Stewart (GHS) [10], is that using a symplectic 2-form $\omega$ which describes
an invariant volume in the phase space (of dimension $2n$) of solutions $\Omega$.

$^2$The measure is actually the symplectic form to the power $n - 1$, but we only consider
This is possible because the models under investigation are even dimensional Hamiltonian systems. For such models the Lie derivative of $\omega$ with respect to the Hamiltonian vector field $X$ is zero: $L_X \omega = 0$. There is some doubt that this would hold for a more general relativistic system but for the simplified models with few degrees of freedom it would seem appropriate [11].

Within this approach we consider the claim of Hawking and Page [12] that the GHS canonical measure predicts that the universe is spatially flat i.e. that the density parameter $\Omega = 1$, regardless of Inflation occurring or not. For the perfect fluid model this would correspond to the prediction that $\rho_0 \rightarrow \infty$, but note that this also implies $a$ and $\dot{a} \rightarrow \infty$ cf. eq.(7). We later comment on the validity of this result and find that it implies vastly post-Planckian scales.

We will essentially use the same equations and notations as presented in ref.[12]. This considers a massive scalar field model described by the following equations

$$\dot{a}^2 = a^2 \dot{\phi}^2 + m^2 a^2 \phi^2 - k$$
$$\ddot{a} = -2a\dot{\phi}^2 + m^2 a\phi^2$$
$$\ddot{\phi} = -3\frac{\dot{\alpha}}{\alpha} \dot{\phi} - m^2 \phi$$

It is helpful to define a rescaled scale factor $\alpha$ such that $\alpha = \ln a$. Using the dimensionless variables

$$x = \phi \quad y = \frac{\dot{\phi}}{m} \quad z = \frac{\dot{\alpha}}{m} \quad \eta = mt$$

the system of equations can be written in the form [12]

$$\frac{dx}{d\eta} = y$$
$$\frac{dy}{d\eta} = -x - 3yz$$
$$\frac{dz}{d\eta} = x^2 - 2y^2 - z^2$$

models with $n = 2$. 
plus a constraint
\[ \frac{k}{m^2 a^2} = x^2 + y^2 - z^2 \] (15)
k is the spatial curvature \( \pm 1, 0 \). This model or in fact that with any scalar field potential \( V(\phi) \) is characterized by an initial stiff regime as \( a \to 0 \), essentially because the kinetic term \( \dot{\phi}^2 \sim a^{-6} \) dominates as the singularity is approached. It therefore behaves like a perfect fluid with \( \gamma \simeq 2 \).

For such models, the canonical measure is given by [12]
\[ \omega = -d(a\dot{a}) \wedge da + d(a^3 \dot{\phi}) \wedge d\phi . \] (16)
In this model there are 4 phase-space variables \((a, \dot{a}, \phi, \dot{\phi})\), plus one constraint, making 3 independent variables. One of these is the parameter along the trajectory, i.e. the “time”, so there is a remaining 2 parameters describing the model. The measure is independent of the choice of 2 dimensional initial data surface chosen provided each solution hits this surface only once [10,12]. This might not be true for \( k = 1 \) in that some solutions recollapse before a given scale factor or energy density is reached - but we ignore this possible complication.

If we consider the measure on an initial data hypersurface \( a = a_0 = \text{constant} \) then this simplifies to [12]
\[ \omega = a_0^3 d\dot{\phi} \wedge d\phi . \] (17)
Using the notation of ref.[12] such that \( \theta = \tan^{-1}(x/y) \) this can be written in the form \[ \omega = m a_0^3 dz \wedge d\theta = m^{-1} a_0^3 d\rho \wedge d\theta . \] (18)
where the energy density is \( \rho \equiv \dot{\phi}^2/2 + m^2 \phi^2 \). The measure, or rather its integral, therefore diverges as the Hubble parameter \( z \equiv H \equiv m^{-1} \dot{a} \) or the energy density \( \rho \) is taken to infinity. This occurs because the variables \( z \) or \( \rho \) are allowed to have unrestricted ranges, while still corresponding to regions of the classical phase space.

This divergence in the measure as \( \rho \to \infty \) is the reason for the prediction that \( \Omega = 1 \). To see this another way, note that the Hubble parameter is

\[3\text{Strictly speaking this expression does not treat the spatial curvature curvature correctly, but this only cause a difference if } z \text{ or } \rho \text{ had restricted ranges, see later eq. (23).}\]
related to $\Omega$ in the following way for $k \neq 0$\,[3,4]

$$H^2 = \frac{k}{a^2(\Omega - 1)}$$

(19)

Only for $H$ or $\rho \to \infty$ do we have $\Omega = 1$ at fixed $a$. It is important to realize that there is only one solution in the measure with exactly $k = 0$ or $\Omega = 1$. The solutions are instead infinitesimally close to $\Omega = 1$ as $H \to \infty$. Rewriting the measure in terms of $\Omega$

$$\omega = m^{-1}a_0 \frac{k}{|\Omega - 1|^2} d\Omega \wedge d\theta$$

(20)

we see the divergence as $\Omega \to 1$.

In Fig.(1) we sketch the probability distribution $P(\Omega) \equiv \omega$ for $\Omega$ which is taken from this expression (20). This shows that $\Omega$ is peaked around one regardless of inflation since there is no need to insist on $m \ll 1$ to obtain this result. It remains even for potentials too steep $m \geq 1$ to have inflationary behaviour.

The divergence occurs as $\Omega \to 1$ but actually $\Omega = 1 \pm \epsilon$ and $\epsilon > \rho_{\text{max}}^{-1}$. This argument is got from substituting $H^2 = \rho/\Omega$ into eq.(19). If there are limits on $\rho$ then $\epsilon \neq 0$ and $\Omega$ is not exactly one (except for the one solution where $k=0$).

Also if there are any upper limit of the energy density $\rho$ i.e. $0 < \rho \leq \rho_{\text{max}}$ or equivalently the Hubble parameter $0 < z \leq z_{\text{max}}$ then the measure is instead finite and the flatness of the universe is not necessarily assured. Let us estimate what this upper limit should be in order to predict that $\Omega$ be one today.

The massive scalar field model, after any possible inflationary phase, oscillates around the minimum of the potential with an equation of state like a dust model $\gamma = 1$\,[12]. If we assume that no inflationary phase occurred, then as found in section (1.1), we require $\Omega \sim 1 \pm 10^{-60}$ at the Planck epoch. From the expression (19) for $H$ this corresponds to an Hubble parameter $H^2 \geq 10^{60} M_{\text{pl}}^2$, where we would expect quantum gravity effects to dominate when $H^2 \sim M_{\text{pl}}^2$ ! We can estimate the upper bound on $H$ over which the integral of the measure, proportional to $H^2$ should be taken in order that today the probability be 99% that $\Omega \simeq 1$ . A simple consideration of areas $\propto H^2$ gives that the measure needs to be valid up to $\sim 10^{61} M_{\text{pl}}^2$ in order
to predict \( \Omega \simeq 1 \). If we can assume that \( H \) is in the range \( 0 < H < \infty \) then the measure \( \propto H^2 \) is dominated by extremely large \( H \to \infty \). Because \( \Omega = (H^2 + k/a^2)/H^2 \) and \( H^2 \gg k/a^2 \), \( \Omega \) is pushed extremely close to 1. Any subsequent phase which satisfies the strong energy condition (\( \gamma > 2/3 \)) would take an extremely long time before \( \Omega \) diverges from 1. If however we have an upper bound on \( H = H_u \) above which the theory is no longer valid and \( H_u < 10^{30} M_{pl} \) then today \( \Omega \simeq 1 \) would not be expected.

### 2.1 Reasons for \( \Omega \neq 1 \)

**a) Limits on the energy density \( \rho \)**

We have seen that the measure \( \propto H^2 \) or \( \rho \) requires an unrestricted range in order that the flatness of the universe be resolved without resorting to the subset of cases that have inflation. Note however that although we require \( \rho \geq 10^{60} \) at the Planck epoch, we do not require it to be infinite otherwise it would not appear consistent with the density of the universe today (modulo uncertainties in the age and size of the universe, plus expansion rates).

Because \( \Omega \) is not exactly one it will diverge away from 1 for arbitrary large scale factors. For \( k = 1 \) universes \( \Omega \to \infty \) before then recollapsing, while for \( k = -1 \) universes \( \Omega \to 0 \).

Since
\[
\Omega = 1 + \frac{k}{a^2 H^2} = 1 + \frac{k}{\dot{a}^2} \tag{21}
\]
the measure at fixed \( \Omega \) is equivalent to the measure at fixed \( \dot{a} \). In ref.[12] the measure for \( \dot{a} = 0 \) or \( \Omega = \infty \) was obtained and found to be finite, in contrast to the infinite total measure of solutions. This calculation only differs by some overall constant if the measure is done at some other choice of \( \dot{a} \) or \( \Omega \), essentially because eq. (15) has the form
\[
x^2 + y^2 = \frac{\dot{a}^2}{m^2 a^2} + \frac{k}{m^2 a^2} = \text{const.} \quad \text{at fixed } \dot{a} \tag{22}
\]

Note that the numerics of the calculation for \( \dot{a} = 0, k = 1 \) (\( \Omega = \infty \)) turn out the same for \( \dot{a} = \sqrt{2}, k = -1 \) (\( \Omega = 1/2 \)). If however \( \rho \) is restricted the total measure is finite and any value of \( \Omega \) is roughly equally probable for arbitrary scale factors \( a \). This statement could be made more precise but requires the initial upper bound for \( \rho \) and at what scale factor you measure \( \Omega \) to be known. The important point however is, the claim that \( \Omega = 1 \) for arbitrarily large \( a \) requires the measure be dominated by \( \rho_{\text{max}} \to \infty \). Or to
see this another way: the measure eq.(18) when the curvature is included is modified by letting $z \rightarrow z^*$, where

$$z^*^2 = \left( z^2 + \frac{k}{m^2a^2} \right).$$

(23)

If $z \leq z_{max} < \infty$ we can always reduce the scale factor so that the curvature dominates i.e. $|k|/m^2a^2 > z^2$ and so set $\Omega \neq 1$.

It has been pointed out to me [13] that the problem of requiring vastly post-Planck-epoch densities can be avoided by considering the measure not at fixed scale factor but rather at a fixed energy density (which can be independent of the scale factor as the scalar curvature becomes negligible [13] cf. eq.(6)). This is claimed to be more physically reasonable since we have no experimental evidence for an upper limit on $a$ but we do know the present energy density within certain limits.

Let us investigate this by considering the canonical measure for fixed energy density $\rho$ given by the expression [12]

$$\omega = -\dot{\phi}d\phi \wedge d(a^3) = 2m^{-1}\rho \sin^2 \theta d\theta \wedge d(a^3).$$

(24)

This is dominated by the divergence at large scale factors $a$[4] and this aspect was used in ref.[12] to suggest that the curvature $k/a^2$ is correspondingly small at fixed $\rho$ so giving the prediction that $\Omega = 1$. Does this not mean that we could consider only energies below the Planck epoch and still obtain the result that the measure predicts that the universe is flat?. The problem is that the solutions start initially at $a = 0$ [12] and then by the Planck time the scale factor has already tended to infinity. This means that $\dot{a}$ is now the quantity which is massive in units of $M_{pl}^2$, which in turn sets $\Omega$ extremely close to unity cf. eq.(21). But such extreme values of “velocity” $\dot{a}$ beyond the Planck size $\sim 1$ will induce quantum effects e.g. particle creation, and so actually requiring $\dot{a} \geq 10^{30}M_{pl}^2$ is extremely suspect. It might also cause problems for nucleosynthesis if the expansion rate is too fast. Any upper limit in $\dot{a}$ below this value again sets the total measure (in eq. (24) ) finite.

Because of this the value of $\Omega$ would no longer be expected to be 1 in the universe today, at time $10^{60}$ Planck units later.

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4We ignore doubts that this seems contradictory in that if the initial curvature $k = 1$ dominates $a$ cannot become large.
The problem of requiring the classical equations to be valid in extremely quantum gravity regions in order to claim $\Omega = 1$ is therefore still present when we consider the measure at fixed energy density, and is rather a consequence of the fact that the phase plane trajectories of the classical equations are all appearing from the initial singularity, which is surrounded by a quantum gravity region through which the solutions have to pass.

There is another reason for suggesting a limit $\rho \leq \rho_{\text{max}}$: to be consistent with a universe described by a finite value of $a$.\footnote{There is an element of philosophical prejudice in this requirement.} It is unclear how the predictions from the measure, which results in the claim that $\rho \to \infty$ or $a \to \infty$ could fit the description of our universe.\footnote{Note another way of seeing how the claim that $a \to \infty$ comes about: from eq. (19) we obtain $k/a^2 = H^2(\Omega - 1)$ so we see that as $\Omega \to 1$ then $a \to \infty$.} With the proviso that future experimental evidence (coming especially from a better knowledge of the deceleration parameter $q$) will in future give an finite upper limit to the scale factor i.e. $q \neq 1/2 \Rightarrow a \neq \infty$. Only when $k = 0$ or $q = 1/2$ does a finite age of the universe not imply a finite scale factor. Recall also that the constant $\rho_0$ for the perfect fluid model is related to $q$ as:

$$\rho_0 = \frac{2q}{H(2q - 1)^{3/2}}. \quad (25)$$

So $q = 1/2$ sets $\rho_0 \to \infty$.

The big bang model with $k = 1$, starts out with a small scale factor and expands to a possible large value but staying finite throughout its evolution. Contrast this with the measure at fixed scale factor $a$ which suggests $\rho \to \infty$, or at fixed energy density that $a \to \infty$. Although some upper limit on $\rho$ or $a$ (maximum possible scale factor) consistent with our universe could be larger that that required to solve the flatness problem it seems clear that an upper limit over which the measure can act must exist.

This would mean that at some stage the measures cannot be applied over unrestricted values of the parameters $\rho$ or $a$ so giving a finite measure for such models. We would be left with the problem of why these restrictions on $\rho$ in order to provide a realistic universe model are consistent with large enough values to also solve the flatness problem. In other words we require the value of $\rho$ at the Planck scale to be finite, say less than $\rho_{\text{max}}$ but greater
that $\rho_{\text{flat}} \sim 10^{60}$ Planck units in order to solve the flatness problem. According to the measure we should expect $\rho \to \infty$ but rather we require the prediction $10^{60} < \rho \leq \rho_{\text{max}}$.

In summary of this section, the GHS measure $\propto \rho$ can indeed explain the flatness of the universe but assumes the theory to be valid up to enormous energy densities or velocities. If the theory becomes invalid before this value is achieved then the prediction of $\Omega = 1$ cannot be sustained without resorting to mechanisms like inflation. Because only a certain subset of possible potentials can have inflation this is a more restrictive requirement.

To be consistent with our universe the measure must break down at some finite energy density $\rho_{\text{max}}$ but this could be at a value which still allows the explanation of $\Omega = 1$.

b) Limits on the scalar potential $0 \leq V(\phi) \leq V(\phi)_{\text{max}}$.

There is an alternative way in which the universe need not be spatially flat even assuming $\rho \to \infty$. We consider the case of an arbitrary potential $V(\phi)$ but with unrestricted $\rho$. In analogy with the angle $\theta$ which was introduced to parameterize the massive scalar field case, we can introduce the angle $\Psi$, such that cf. ref.[14]

$$\dot{\phi} = \rho^{1/2} \sin(\Psi) \quad (26)$$

$$V(\phi) = \rho \cos^2(\Psi) \quad (27)$$

Define $f(\phi) = (V(\phi)/\rho)^{1/2}$ so that $\Psi = \cos^{-1} f$. Then

$$\frac{d\phi}{d\Psi} = -\frac{\sqrt{1 - f^2}}{f'} \quad (\equiv df/d\phi) \quad (28)$$

Substituting for $d\phi$ into the measure for fixed energy density eq.(24), gives,

$$\omega = -\dot{\phi} \frac{\sqrt{(1 - f^2)}}{f'} d(\Psi) \wedge d(a^3) \quad (29)$$

Or in terms of $V(\phi)$ we obtain

$$\omega = \frac{\sqrt{V(\phi)}}{V'(\phi)} \rho \sin^2 \Psi d(\Psi) \wedge d(a^3) \quad (30)$$
For this arbitrary potential the measure still is dominated by the $a^3$ divergence, but in contrast to the massive scalar field case (see Fig. (2)) we can now obtain an infinite measure even for finite $a$ when 1) $df/d\phi = 0$ or equivalently $dV/d\phi = 0$ and 2) $1 > f$ which implies $\rho > V(\phi)$.

When calculating the measure at fixed $\Omega$ one obtains expressions of the form (cf. eq.(4.12) in ref. [12]),
\[
\omega = \frac{\text{const.}}{V'(\phi)} \left(1 - 3\sin^2 \Psi \right) da \wedge d\Psi
\]
where the constant is related to that on the RHS of eq.(22). When integrated over allowed values of $\Psi$ this gives a total measure for scale factor $a$
\[
\mu \sim \pi \frac{V(\phi)}{V'(\phi)} a .
\]
Because of the factor $V'$ in the denominator there is a possible infinite measure for any fixed $\Omega$ at a given scale factor $a$. Any value of $\Omega$ would have roughly the same probability of occurring and without some extra mechanism to restrict it would appear arbitrary.

For $k = -1$ the universe will expand until possibly moving onto the plateau where $V'(\phi) = 0$. Once there an inflationary epoch could result and so ensure in this case that $\Omega = 1$ is reached to sufficient precision. An example of this would be a potential of the form $V(\phi) = \tanh^2 \phi$, see Fig.(3)

However there is an ambiguity in the GHS measure for inflation to occur [12], essentially because of the $a^3$ divergence in the measure which prevents you concluding that even “inflationary looking” potentials must inflate. We suggest later how this problem might be resolved by appealing to quantum initial conditions. Other bounded potentials with non-inflationary plateaus such as $V(\phi) = \cos^2 \phi$ would also not allow the ambiguity of $\Omega$ even for $k = -1$ universes, to be resolved, see Fig.(4).
In general any potential which tends to a constant say $V_1$ (i.e bounded) and

\footnote{Provided the energy density is large enough to allow the $V'$ region to be reached, so $\Omega$ is above a small lower bound.}
the energy density $\rho$ is greater than $V_1$ would not resolve the flatness of the universe. This enables us to understand the $R^2$ model considered in ref.[14], where it was not apparent whether the non-resolution of the flatness was caused by some aspect of the $R^2$ model. We now see that this non-resolution persists for ordinary gravity together with a suitable scalar field.

The $R^2$ model is conformally related to a scalar theory with potential

$$V(\phi) = \frac{1}{2\epsilon} (1 - \exp(-2\phi))^2$$

(33)

where $\phi = 1/2 \ln(1 + 2\epsilon R)$ and we limit $\phi \geq 0$. In this case $f' = \exp(-2\phi)$ which tends to zero as $\phi \to \infty$ - so giving the infinite measure.

This fall off in the potential towards a constant is a fairly common feature as theories are pushed to their breaking points eg. the over driven pendulum becoming circular motion. It might therefore be the case that as the quantum gravity regime is reached the potential reaches a plateau. See the related discussion in ref.[15] that due to the conformal anomaly the potential is bounded.

Another example of the flatness being unresolved is that of a cosmological constant and a massless scalar field (axion). If such quantities are present in the universe then the classical measure would not unambiguously give $\Omega = 1$. Other degrees of freedom cf. the Bianchi(1) model with its extra unrestricted variable (see eq.(3.7) in ref. [16]), have a similar effect to the bounded potential in that another variable has an infinite range which introduces ambiguities in predictions from the measure. If the measure was extended to inhomogeneous models we would expect such problems to remain, and the bounded potential can be taken as simulating the problems inherent when an anisotropy or inhomogeneity is present.

It has been suggested [13] that this bounded potential still predicts $\Omega = 1$. By restricting $\rho = \rho_{\text{anth}} < V_1$ the singularity at $V' = 0$ is avoided, where $\rho_{\text{anth}}$ is the maximum density from anthropic grounds consistent with life. However if we are assuming a classical description since the initial singularity, we are primarily interested in the measure applied to the relevant physics applying to the earliest stages of the universe. The evolution (trajectory of the universe) is set during this initial high density epoch and a subsequent change in the physics (at lower $\rho$) with its different measure will not change

13
this. The measure does not determine dynamics but only allows you to give probabilities amongst various solutions allowed by the dynamics, because in the early universe with its possible $R^2$ theory and large energy densities ($\rho \geq V_1$) this ambiguity in the flatness is possible. Even if later in the evolution of the universe such theories are no longer valid, the “the die has been cast” and we should not attempt to predicts things using re-calculated measures. For example, we could obtain erroneous conclusions that the universe goes from non-flat to flat suddenly as a massless axion acquires a mass after some QCD phase transition.

So far we have been interested in showing reasons why $\Omega$ might not be unity within the canonical measure framework regardless of inflation. Indeed the scheme works for any non-bounded potential and so is more general than inflation which requires a “flat” potential.

If the measure acts over an unlimited range of initial energy density then $\Omega$ is set infinitesimally close to one and would still be 1 at a time $10^{60}t_{pl}$ later. This occurs because the massive scalar field behaves initially like that of a perfect fluid with the strong energy condition satisfied, actually $\gamma \sim 2$. The analogy with the perfect fluid model is that at the initial start of the Big Bang we are choosing at random the constant $\rho_0$ which is uniform within the range $0 \leq \rho_0 \leq \infty$. A typical universe will therefore have $\rho_0 \to \infty$, see also ref.[5]. In this regard the canonical measure simply puts this argument (of choosing random constants) on a more formal footing for the scalar field case.

The energy densities (or values of $\dot{a}$) involved in dominating the measure are enormous and the classical theory will surely have been superceded by a quantum gravity regime at the relatively low energy scale $\rho \sim$ Planck size.

3 Quantum Era
Let us consider briefly the values of $\Omega$ that might occur if the universe was initially created in a quantum event with an Hubble parameter $H^2 \sim 1$ or energy density $0 < \rho < 2$ (for an initial size $a \sim 1$). We are assuming that the classical equations are valid immediately the universe is created. From expression (17) notice that $\Omega$ for both closed and open universes diverges away from one for scale factors approaching one from above, (i.e as $a \to 1$). This means that typically $\Omega$ is in the range $0 < \Omega \leq 2$. For larger initial scale factors $\Omega \to 1$ as $a \gg 1$. In order to obtain $\Omega \gg 1$ we would require
initially $H^2 \simeq 0$.

Inflation can help a much wider spread of initial values of $\Omega$ become spatially flat. In the previous example of an initial quantum event with $H \sim 1$, only when $\Omega \simeq 0$ or $\rho \simeq 0$ might we fail to achieve $\Omega = 1$ due to insufficient matter being present to give inflation. It is therefore important to know from a possible quantum gravity theory the possible value of the initial Hubble parameter.

One approach is that of the Wheeler DeWitt equation which gives a solution for the wave function of the universe $\Psi$. For a simple $k = 1$ model of a constant scalar field $V(\phi) = \text{const.}$ the probability of the initial Hubble parameter $H$ is given by

$$\Psi_H^2 \sim \exp \left( \pm 1/H^2 \right)$$

where the + and - signs correspond to the Hartle-Hawking and Tunneling boundary conditions respectively see eg. [17,18]. The Hartle-Hawking case would suggest that $H \simeq 0$ which would suggest an initial $\Omega >> 1$, and little possibility of inflation to correct this. There is however some dispute on this interpretation [17,18]. The tunneling boundary condition on the other hand would suggest a larger initial Hubble parameter and so a closer to one value for $\Omega$. This could then be set infinitesimally close to one by a reasonable period of inflation.

Very little has been done for spatially open models ($k=-1$) to see what predictions are made for $H$ and $\Omega$ and whether a period of inflation would ensue. Some exact solutions of the WDW equation when $k = -1$ have been found [19] for conformally coupled scalar fields, but these are not inflationary models.

Recently a related approach based on a classical change of signature has been investigated [20]. In the sense that euclidean regions can be described by the classical equations, but the equations can in turn be quantized. The WDW equation in this case appears only consistent with a tunneling boundary condition.

A quantum analogue of the GHS measure which predicts the probability of $\Omega$ at the start of the classical evolution is required. In the WDW scheme Gibbons, Grishchuk and Sidorov [21] have made an initial attempt to define a typical wavefunction, and find that it is similar to the tunneling one. This
usually predicts inflation provided a suitable potential is available and so this would suggest that the result $\Omega = 1$ could be obtained.

A prediction from some measure that $\Omega = 1$ without inflation would be preferable, since inflation is dependent on there being present a suitable potential.

In summary of this section: we would expect from an initial quantum creation of the universe that $\Omega$ is not too far initially from 1, depending on boundary conditions: in the language of section 1.2 we find no evidence of extreme measures. Such a value of $\Omega$ could be then pushed extremely close to 1 by a suitable period of inflation. Ideally a quantum gravity measure, which had no need of classical inflation, analogous to the GHS classical one would be preferable.

4 Inflationary epoch

According to ref.[12] the canonical measure cannot predict for certain if an inflationary phase occurs. The measure for the massive scalar field case typically has the form [12],

$$\omega \simeq d\phi \wedge d(a^3_1)$$  \hspace{1cm} (35)

where $a_1$ and $\phi_1$ are respectively the scale factor and initial value of the scalar field when inflation starts. There is an infinite measure due to the $a^3$ term even for values of the scalar field below some value that gives sufficient inflation. Note that this assumption that the initial scale factor can be anything is the cause of this ambiguity. But if we assume from quantum cosmology that the universe had to start “small”, then this ambiguity could be corrected.

This ambiguity is not present in quantum cosmological models since the initial scale factor is assumed to be of $\sim$ Planck size. For example the WDW equation gives, in analogy with expression (35) a quantum measure of the form, see eg. [17,18]

$$\omega \simeq \exp \left( \pm \frac{1}{V(\phi)} \right) d\phi_1$$  \hspace{1cm} (36)

where the + and - signs correspond to HH and Tunneling boundary conditions. This measure is either more peaked around small values of $\phi$ (HH) or prefers larger values (Tunneling) compared to the uniform case $\sim d\phi_1$. 

16
Encouragingly, the measure derived from a classical change of signature has the form, called $dP^{CS}_V$ in ref. [20],

$$\omega \simeq d\phi_1.$$  

(37)

It is therefore uniform in $\phi$ as is the classical GHS one: but it no longer suffers from the divergence over arbitrary scale factors. In some sense this approach gives a measure “in between” the purely classical GHS one and those from the usual WDW quantum cosmology eg. HH.

Although we might have been led to this result ($\omega \simeq d\phi_1$) by assuming the universe started “small”, it is good to see this result coming from an entirely different approach and analysis to that of the GHS one.

Once this restriction on the initial scale factor is made the total measure, at least for unbounded potentials, becomes finite and the probability of inflation can be obtained. For the massive scalar field model inflation occurs for certain values of $\theta$: those corresponding to a large enough scalar field $\phi$. Following along the lines of ref.(12) the fraction of non-inflationary solutions $f_{NI}$, for at least $z$ e-foldings of inflation is

$$f_{NI} \sim \frac{m \ln(1 + z)}{\rho^{1/2}}$$  

(38)

Provided $m \sim 10^{-4}$ and we require $\sim 100$ e-folding of inflation then $f_{NI} \sim 10^{-4}$ for energy densities $\rho \sim$ Planck value. This is a considerable improvement to the classical canonical measure reason for setting $\Omega = 1$, which needs to be valid up to energy scales $\sim 10^{40}$ planck size. We are however still at energy scales at which we expect quantum gravity effects to dominate. We also have to understand why the mass $m$ is so small, if $\rho$ is reduced it needs to become correspondingly smaller to predict $f_{NI} \sim 0$. This example could result in the prediction that $\Omega = 1$, if for example the energy density $\rho$ is limited and the GHS measure cannot itself give $\Omega = 1$. This is possible because the total measure of solutions is finite and makes it possible to give the fraction of inflationary solutions among them.

In the case of a bounded potential however there is still the divergence due to the $V'(\phi)$ term in the denominator. It still appears not possible to determine the probability of inflation even for potentials like $V(\phi) = \tanh^2(\phi)$ which at first sight look very inflationary. There is still an infinite measure
for any value of $\Omega$ and certain of these cannot inflate\textsuperscript{8}. However, in the case of $k = -1$ universes it should be possible once the $a^2$ divergence is eliminated to conclude that inflation is near certain to occur. This is provided the initial energy density is above the plateau so that as the universe expands it will fall onto the plateau at random, with an almost certain initial $\phi > \phi_*$, where $\phi_*$ is the value of the field that gives sufficient inflation $\sim 60$ e-foldings \cite{1,2}. One can see that this scenario is likely in the case of the $R^2$ model see eq.(6.113) in Ref.[14].

5 Anthropic restrictions

Somehow the anthropic principle should play a role within the measure question. The measure has been used to define a probability for some quantity eg. $\Omega$ in a typical universe. This has depended on the assumption that an ensemble of universes is possible. One in effect “throws dice” at the start of the Big Bang to determine the constant(s) ($\rho_0$ in the perfect fluid case) that determines all the subsequent evolution of the universe. If the throw is a “dud” (i.e $\rho_0$ small) then the universe recollapses or is empty. Life will therefore not be around at a time $10^{60}t_{pl}$ later to question why such a case happened.

Consider the previously mentioned case of the inflationary potential $V(\phi) = \tanh^2(\phi)$. We found that roughly speaking: closed universes are near certain to collapse while open ones are almost certain to inflate. On anthropic grounds we could then conclude that inflation did occur and that $\Omega \leq 1$ in the subsequent universe (depending on how much inflation occurred and at what time in the future we wish to measure $\Omega$)\textsuperscript{9}. In this example there was initially a roughly 50/50 chance of either recollapse or inflation occurring. The “quantum process” responsible for such a classical universe would not have to be repeated very often in order to get a universe which did inflate and could later sustain life. Even if we had concluded that recollapse was vastly more probable, (for the $V(\phi) = \tanh^2(\phi)$ potential case) than inflation then on anthropic grounds we could, if we existed in such a universe, conclude

\textsuperscript{8}This is due mostly to universes recollapsing which due to the initial data surface we have already potentially underestimated.

\textsuperscript{9}Note that this differs from my earlier criticism of an anthropic argument where I saw no reason to try and limit the possible value of $\rho$ in the early universe, only much later in its evolution do we require $\rho \leq \rho_{anth}$. The initial high value of $\rho \geq V_1$ was causing an ambiguity in all universes not just recollapsing ones.
that inflation did occur anyway. This suggests that the “quantum process” responsible for the start of such universes could repeatedly attempt many tries and only finally after many goes did inflation occur.

The eternal inflationary universe model [1,22] tries to justify this sort of argument: why an ensemble of universes is possible. Some of these universes will be long lived and so suitable for life to evolve. But this scenario still requires an initial domain to first start inflating, which also comes from a singularity.

The singularity theorems have recently been extended to include inflationary regimes, and are still found to require the existence of singularities even for open (k=-1) models [23].

The eternal inflationary model is therefore not entirely immune from the measure question. i.e. what if the possibility of the creation of the first inflating domain is minute? Would we still have to rely then on anthropic arguments? But such worries can anyway wait until the likely initial conditions from a quantum gravity theory are better known.

6 Conclusions

We have seen that the canonical measure when all regions of phase space are allowed predicts $\Omega = 1$ for monotonically increasing scalar potentials. This essentially occurs because the kinetic term is rapidly diverging to $\infty$ as $a \to 0$ and so gives an enormous initial Hubble parameter which sets $\Omega$ initially very close to one. It would then take an enormous time (during periods when the strong energy condition is satisfied $\gamma > 2/3$) if it is to eventually diverge away from unity. It therefore does solve the flatness problem without appealing to inflation (or even requiring the presence of an inflationary potential) but at the cost of assuming the classical equations are valid arbitrarily close to the initial singularity.

We have given two ways in which this prediction could break down a) If the maximum energy density (or velocity $\dot{a}$) over which the measure acts is less than $\sim 10^{40}$ Planck units or b) there is an upper limit (plateau) to the scalar potential. The first of these conditions is especially suspect in light of the massive energy densities (or derivatives of the metric $\dot{a}$) present which are still assumed to obey the classical equations. We would rather expect the metric to become “fuzzy” at around the Planck scale or quantum processes
to invalidate the dynamics.

If we assume a quantum event started the universe then a rough order of magnitude predictions is that $H \sim 1$ and so the Hubble parameter is not large enough to set $\Omega = 1$ to sufficient accuracy. Unless some quantum analogue of the GHS measure predicted an initial value of $\Omega$ exactly one, we would have to rely on a subsequent classical mechanism (inflation) to explain the spatial flatness of the universe.

In the WDW approach, depending on boundary conditions we might have sufficient inflation to set $\Omega = 1$ provided a suitable mechanism of inflation (a classical event) is present after the quantum epoch. This is more restrictive than the GHS mechanism for the purely classical case, which does not require an inflationary epoch. Inflation does however enable the flatness problem to be solved without appealing to energy scales vastly greater than the Planck scale.

Even if the flatness problem is resolved, inflation might be needed to resolve other problems eg. to dilute anisotropy or to generate perturbations for galaxy formation [1,2]. It is important to determine the chances of inflation happening.

The classical measure gives an ambiguity for the probability of inflation occurring (there are $\infty$ solutions of both inflationary and non-inflationary cases).

This ambiguity can be corrected by means of a simple input from quantum cosmology: that the initial scale factor was small. When this is done the measure ($\omega \sim d\phi_1$) is moderately inflationary compared to purely quantum cosmological measures. Moderate because unlike the tunneling case, it is not peaked at large values of $\phi$, but neither is it peaked, like the HH case, at small $\phi$. It also agrees with the measure found from a classical signature change approach.
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Figures

Fig. 1) The probability of $P(\Omega)$ against $\Omega$. This shows the divergence in the measure at $\Omega = 1$ but in order to approach exactly 1 ($\epsilon \to 0$), would require the energy density $\rho \to \infty$.

Fig. 2) A potential which the GHS measure would predict has $\Omega = 1$ without requiring inflation. Note that in this example $V(\phi) = \phi^2$ i.e. $m = 1$ and so is too steep to inflate.

Fig. 3) An example $V(\phi) = \tanh^2(\phi)$ of a bounded potential which can have any value of $\Omega$ according to the GHS measure. Even though this potential is “inflationary” the probability of inflation given by the GHS measure is ambiguous and so we cannot conclude that inflation must set $\Omega = 1$. With certain quantum initial conditions which ensured that the evolution started on the plateau, we could get a “certain” prediction that inflation occurs i.e. $\phi > \phi_\ast$.

Fig. 4) An example of a bounded potential $V(\phi) = \cos^2(\phi)$, which also does not have the possibility of inflating. The potential is too steep to inflate and the scalar field quickly rolls down to a minimum. In such cases the value of $\Omega$, at either a fixed value of the scale factor or the energy density, is arbitrary according to the GHS classical measure.