A 1d Traffic Model with Threshold Parameters

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Abstract. The basic properties of traffic are analyzed using a simple deterministic one-dimensional car following model with continuous variables based on a model introduced by Nagel and Hermann [Physica A 199 254 (1993)] including a few modifications. As a first case we investigate the creation and propagation of jams in a platoon generated by a slow leading vehicle. In a second case we look at a system with the size L, periodic boundary conditions and identical vehicles. As a strong dependence on the initial configuration of the fundamental diagram's shape can be found.

1 Definition of the Model

To get a parallel update of all vehicles, the first step is to change the velocity of all vehicles taking into account the threshold parameters and according to the rules defined below. In the second step the position of all vehicles is changed using the velocity calculated in the first step.

Position, velocity and threshold parameters of the vehicles are continuous variables. t is the time step used in the simulations.

1st step - velocity: A vehicle decelerates if the headway distance $x_i(t)$ is smaller than a safety distance $\delta$. The headway distance after the deceleration step is determined by

$$x_i(t) - v_i(t) \cdot t < \delta \quad \Rightarrow \quad x_i(t+\Delta t) = x_i(t) - v_i(t) \cdot \Delta t$$ (1)

The acceleration of a vehicle depends on a threshold parameter and the maximum velocity $v_{\text{max}}$:

$$x_i(t) - v_i(t) \cdot t > \delta \quad \Rightarrow \quad v_i(t+\Delta t) = \min (v_{\text{max}}, v_i(t) + a \cdot \Delta t)$$ (2)

The acceleration coefficient is determined by the headway distance if it is greater than and the headway distance is smaller than:

$$a = a_{\text{max}} \cdot \frac{x_i(t)}{\delta}$$ (3)
2nd step - position: The positions are changed.

\[ x_i(t + \Delta t) = x_i(t) + v(t + \Delta t) \Delta t \] (4)

Using a constant acceleration coefficient \( \gamma \) we can compare this model to the deterministic case of the Nagel-Schreckenberg model [3], as the velocity is determined by de- and acceleration with the acceleration causing a discretisation of the velocity.

2 Platoon

Fig. 1. The figures on the left side show the trajectories of vehicles and jams of different sizes. On the right side the oscillations of the velocity and relative position (position in the coordinate system of the leading vehicle) of the second and third vehicle are shown including the threshold conditions for accelerating and breaking. Parameters: \( \gamma = 15; \gamma = 35; \gamma = 100; \gamma = 20; a_{\text{max}} = 1; v_{\text{max}} = 30; v_{\text{lead}} = 25 \)

A slow leading vehicle that is followed by faster ones creates a platoon of vehicles. In this platoon, jams are generated directly behind the leading vehicle. They propagate either through the entire platoon or at least through parts of it. These jams are created because a vehicle is not able to slow down exactly to the
velocity of the one ahead. In our model this is forced by the parameter \( t \), that is the headway distance after a breaking step. For \( t \) the headway distance at the beginning of the breaking step is smaller than the headway distance after the breaking. The consequence is, that the vehicle has to increase the headway distance within the breaking time step by decelerating to a slower velocity than the car ahead. This leads to an oscillation in the velocities and the headway distances of the vehicles and can add up in a constructive way generating jams with the size of the entire platoon.

3 Periodic Boundary Conditions

In our study, we work on the fundamental diagram \( \text{flux vs. density} \) of a system with the size \( L \) and periodic boundary conditions using identical vehicles. Strong dependencies on the fluctuations in the initial configuration have been found and investigations of three different cases have been made. On one side we look at a highly symmetric case without any fluctuations using equidistant start positions. On the other side we use homogeneous random start positions. Between these two cases, random fluctuations are used to move the vehicles out of their equidistant positions. The maximum of these fluctuations, however, is controlled by a parameter \( L \) that can be in the range from zero up to the system size. The idea is that with the parameter \( L \) the fluctuations or the disorder of the initial state can be characterized including the two extreme cases mentioned above representing the configuration for the minimum and maximum values of \( L \). For these extreme cases analytical solutions can also be found. The velocity of all vehicles is set to zero in the initial configuration and the same value is used for the parameters and .

3.1 Equidistant Starting Positions

![Graph showing flux vs. density](image)

**Fig. 2.** Flux for different simulation time-steps and equidistant start positions. Parameters: \( = 15; = 35; = 10; a_{\text{max}} = 1; v_{\text{m}} = 30\)
In this highly ordered case the model generates no jams and the fundamental diagram is a straight line with a slope of $v_{\text{max}}$. The maximum flux can be obtained from the accelerating condition of the model.

$$\text{crit} = \frac{1}{tv_{\text{max}}}$$  \hspace{1cm} (5)

$$v_{\text{crit}} = v_{\text{max}} \text{crit}$$  \hspace{1cm} (6)

For $t > 0$ the maximum flux jumps to zero at the maximum density $\rho_{\text{max}} = 1$, which is determined only by the acceleration threshold parameter.

3.2 Homogeneous Random Starting Positions

![Fig. 3. The left side shows density waves in a space-time diagram. The right side shows the flux for random initial positions and various safety distances. Parameters: $a = 15; b = 35; c = 10; a_{\text{max}} = 1; v_{\text{max}} = 30$](image)

In this case we find a triangular-shaped fundamental diagram with a maximum density $\rho_{\text{max}} = 1$ and a critical density $\rho_{\text{crit}}$ with maximum flux $v_{\text{max}}$. Up to the critical density the system is free-flowing and from the critical density up to the maximum density there are both, jammed and free-flowing areas. Because of the sharp boundaries between congested and free areas, the system can be seen as a mixture of two phases like water and ice. The size of the jammed and free phase, respectively, can be obtained through the normalized (divided by the maximum flux) falling straight line of the fundamental diagram.

Looking at the trajectories of vehicles entering a congested area, it can be proven that the density of the congested area depends on the velocity of the entire jammed area itself. In the final state of the system, only the congested areas with the slowest velocity survive. Taking this into account and neglecting the acceleration and deceleration stripes, there are two phases characterized by the density and the velocity of the vehicles within each phase. In order to obtain the
The fundamental diagram for this two-phase system the mean velocity is calculated.

\[
\bar{v}(t) = \frac{1}{N} \sum_{i=1}^{N} v_i = \frac{1}{N} (N \bar{v}_j + N \bar{v}_f) = \frac{\bar{v}_j}{j} + \frac{\bar{v}_f}{f} + v_j \tag{7}
\]

The indices \( j \) and \( f \) are used for jammed and free phase, respectively.

The density of the congested area can be obtained from the braking rule of the model.

\[
\rho = \frac{1}{v_j} \tag{8}
\]

The outflow of a congested area determines the density of the free flowing area that is equal to the critical density. As a result of analyzing the trajectories, it is found that the time \( T \), that is needed to accelerate and pass the window, is an important value in this model.

\[
T = \frac{1}{2} + \frac{1}{4} + \frac{2}{a} t^2 + 1 \tag{9}
\]

The braces \( \{ \} \) operate on the given argument by rounding down to the next integer value. The result for the critical density using \( v_{\text{max}} = \bar{v}_f \) is therefore:

\[
\rho_{\text{crit}} = \frac{1}{\bar{v}_j + \frac{v_{\text{max}} - \bar{v}_j}{\bar{v}_f} T} \tag{10}
\]

From equations (7, 8, 9, 10) the exact solution for the outflow using a constant acceleration coefficient can be obtained:

\[
\sigma(x) = \frac{1}{T} \frac{1}{v_j} + \frac{v_j}{T} \tag{11}
\]

Investigations on the connection between the jam velocity and the slope of the fundamental diagram have shown an exact agreement with the results of Lighthill and Whitham [3].

\[
v_{\text{jam}} = \frac{d}{\Delta}
\tag{12}
\]

### 3.3 Equidistant Starting Positions with Random Fluctuation

Equidistant positions including small fluctuations are used for the initial configuration.

\[
x_i(t = 0) = \frac{i}{d} + \frac{p}{2} \tag{13}
\]

The random number \( p \) generates the fluctuations and characterizes their magnitude with respect to the density.
Fig. 4. The left side shows the trajectories of vehicles and jams with a velocity greater than 0 and a density of 20/km. The fundamental diagram is shown on the right side. Parameters: $a = 15; a_m = 35; v_m = 10; a_{m \text{ max}} = 1; v_{m \text{ max}} = 30; p = 0.75$

A higher maximum flux and critical density are obtained compared to the homogeneous random case. From the critical density up to the maximum equilibrium density $l = 1$, we get congested areas consisting of vehicles with a velocity $v_i > 0$ that leads to a greater mean velocity and a greater flux. From the equilibrium maximum density up to a cut-off density $cut$, the flux matches exactly the homogeneous random case. The cut-off density $cut$ can be calculated directly from the acceleration condition including the fluctuations.

$$cut = 1 + p$$

(14)

4 Outlook

A simple 1d traffic model with deterministic propagation rules has been studied. A strong dependence on the fluctuations of the initial state has been found and for the extreme cases analytical solutions can be obtained for the fundamental diagram.

It would be interesting to generalize the present model to more than one lane and to introduce more realistic velocity dependent threshold parameters. In the vicinity of the critical density investigations with respect to bistable states would be highly interesting.

References

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