AREA-MINIMIZING CONES OVER MINIMAL EMBEDDINGS OF R-SPACES

SHINJI OHNO AND TAKASHI SAKAI

ABSTRACT. In this paper, by constructing area-nonincreasing retractions, we prove area-minimizing properties of some cones over minimal embeddings of R-spaces.

1. Introduction

Let \( C_B \) be the cone over a submanifold \( B \) of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \). The cone \( C_B \) is minimal in \( \mathbb{R}^n \) if and only if \( B \) is minimal in \( S^{n-1} \). We call a cone \( C_B \) area-minimizing if its truncated cone \( C^1_B \) has the least area among all integral currents with the same boundary \( B \). Solutions of Plateau’s problem can have singularities as integral currents. At an isolated conical singularity, the tangent cone is area-minimizing. Hence, in order to understand such singularities, we should study area-minimizing properties of minimal cones.

Lawlor [L] gave a sufficient condition, so-called the curvature criterion, for a cone to be area-minimizing, using an area-nonincreasing retraction. With this criterion, he obtained a complete classification of area-minimizing cones over products of spheres and the first examples of area-minimizing cones over nonorientable manifolds. Kerckhove [Ke] proved that some cones over isolated orbits of the adjoint representations of \( SU(n) \) and \( SO(n) \) are area-minimizing. A symmetric R-space can be minimally embedded in the sphere in a canonical way. Hirohashi, Kanno and Tasaki [HKT] constructed area-nonincreasing retractions onto the cones over symmetric R-spaces associated with symmetric pairs of type \( B_l \). Furthermore, Kanno [Ka] proved that cones over some symmetric R-spaces are area-minimizing.

In this paper, we study area-minimizing properties of cones over minimal embeddings of R-spaces, not only symmetric R-spaces. In Theorem 2 we give a construction of retractions generalizing the method given in [HKT]. Applying this theorem we give some examples of area-minimizing cones over minimal embeddings of R-spaces. In Section 5 we discuss area-minimizing properties of cones over products of R-spaces.

2. Preliminaries

2.1. Area-minimizing cones. Let \( B \) be a submanifold of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \). We define the cone \( C_B \) and the truncated cone \( C^1_B \) over \( B \) by

\[
\begin{align*}
C_B &= \{ tx \in \mathbb{R}^n \mid 0 \leq t, x \in B \}, \\
C^1_B &= \{ tx \in \mathbb{R}^n \mid 0 \leq t \leq 1, x \in B \}.
\end{align*}
\]

Both \( C_B \) and \( C^1_B \) have an isolated singularity at the origin \( 0 \in \mathbb{R}^n \).

Definition 1. A cone \( C_B \) is called area-minimizing if \( C^1_B \) has the least area among all integral currents with boundary \( B \).

Let \( V \) and \( W \) be two vector spaces with inner products, and let \( F : V \to W \) be a linear map. Suppose \( \dim V = n \geq \dim W = m \). We define the Jacobian \( JF \) of \( F \) by

\[
JF = \sup \{ \| F(v_1) \wedge \cdots \wedge F(v_m) \| \}.
\]

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where \( \{v_1, \ldots, v_m\} \) runs over all orthonormal systems of \( V \). If \( F \) is not surjective, then \( JF = 0 \). If \( F \) is surjective, then

\[
JF = \|F(v_1) \wedge \cdots \wedge F(v_m)\|
\]

for any orthonormal basis of \((\ker F)^\perp\).

**Definition 2.** A retraction \( \Phi : \mathbb{R}^n \to C_B \) is called differentiable if \( \Phi : \mathbb{R}^n \setminus \Phi^{-1}(0) \to C_B \setminus \{0\} \) is \( C^1 \). A differentiable retraction \( \Phi \) is called area-nonincreasing if \( J(d\Phi)_x \leq 1 \) holds for all \( x \in \mathbb{R}^n \setminus \Phi^{-1}(0) \).

**Proposition 1.** Let \( B \) be a compact submanifold of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \). Suppose that there exists an area-nonincreasing retraction \( \Phi \) from \( \mathbb{R}^n \) to \( C_B \). Then \( C_B \) is area-minimizing.

**Proof.** Let \( S \) be an integral current which has the same boundary \( B \) as \( C_B^1 \). Since \( \Phi(S) \supset C_B^1 \), we have

\[
\Vol(C_B^1) \leq \Vol(\Phi(S)) = \Vol(\Phi(S) \setminus \{0\}) \leq \int_{S \setminus \Phi^{-1}(0)} \|d\Phi(e_1 \wedge \cdots \wedge e_k)\|d\mu_S
\]

\[
\leq \int_{S \setminus \Phi^{-1}(0)} J(d\Phi)_x d\mu_S \leq \int_{S \setminus \Phi^{-1}(0)} 1d\mu_S \leq \int_S 1d\mu_S = \Vol(S),
\]

where \( \{e_1, \ldots, e_k\} \) is an orthonormal frame of \( S \). \( \square \)

If \( C_B \) is area-minimizing, then \( C_B \) is minimal in \( \mathbb{R}^n \). Therefore, to find area-minimizing cones, it suffices to consider cones over minimal submanifolds of \( S^{n-1} \). For this purpose, we use \( s \)-representations, which are the linear isotropy representations of Riemannian symmetric spaces.

### 2.2. Riemannian symmetric pairs and restricted root systems.

Let \( G \) be a connected Lie group and \( \theta \) be an involutive automorphism of \( G \). We denote by \( F(\theta, G) \) the fixed point set of \( \theta \), and we denote by \( F(\theta, G)_0 \) the identity component of \( F(\theta, G) \). For a closed subgroup \( K \) of \( G \), the pair \((G, K)\) is said to be a Riemannian symmetric pair if \( F(\theta, G)_0 \subset K \subset F(\theta, G) \) and \( \Ad(K) \) is compact. Let \((G, K)\) be a Riemannian symmetric pair, and \( \mathfrak{g} \) and \( \mathfrak{t} \) be Lie algebras of \( G \) and \( K \), respectively. We immediately see that

\[
\mathfrak{t} = \{X \in \mathfrak{g} \mid d\theta(X) = X\}.
\]

We put

\[
\mathfrak{m} = \{X \in \mathfrak{g} \mid d\theta(X) = -X\}.
\]

We denote by \( \langle \cdot, \cdot \rangle \) an inner product on \( \mathfrak{g} \) which is invariant under the actions of \( \Ad(K) \) and \( d\theta \). Then \( \langle \cdot, \cdot \rangle \) induces a left-invariant metric on \( G \) and a \( G \)-invariant metric on \( M = G/K \) to be a Riemannian symmetric space, which are denoted by the same symbol \( \langle \cdot, \cdot \rangle \). Since \( d\theta \) is involutive, we have an orthogonal direct sum decomposition of \( \mathfrak{g} \):

\[
\mathfrak{g} = \mathfrak{t} + \mathfrak{m}.
\]

This decomposition is called the canonical decomposition of \((\mathfrak{g}, \mathfrak{t})\). For the origin \( o \in G/K \), we can identify the tangent space \( T_o(G/K) \) with \( \mathfrak{m} \) by the differential of the natural projection \( \pi : G \to G/K \).

In this paper, we consider only Riemannian symmetric spaces of compact type. We suppose that \( G \) is compact and semisimple. Take and fix a maximal abelian subspace \( \mathfrak{a} \) in \( \mathfrak{m} \) and a maximal abelian subalgebra \( \mathfrak{t} \) in \( \mathfrak{g} \) including \( \mathfrak{a} \). For \( \lambda \in \mathfrak{t} \), we put

\[
\tilde{\mathfrak{g}}_\lambda = \{X \in \mathfrak{g}^C \mid [H, X] = \sqrt{-1}\langle\lambda, H\rangle X \ (H \in \mathfrak{t})\}
\]

and define the root system \( \tilde{\mathcal{R}} \) of \( \mathfrak{g} \) by

\[
\tilde{\mathcal{R}} = \{\lambda \in \mathfrak{t} \setminus \{0\} \mid \tilde{\mathfrak{g}}_\lambda \neq \{0\}\}.
\]

For \( \lambda \in \mathfrak{a} \), we put

\[
\mathfrak{g}_\lambda = \{X \in \mathfrak{g}^C \mid [H, X] = \sqrt{-1}\langle\lambda, H\rangle X \ (H \in \mathfrak{a})\}.
\]
and define the restricted root system $R$ of $(\mathfrak{g}, \mathfrak{k})$ by
\[ R = \{ \lambda \in \mathfrak{a} \setminus \{ 0 \} \mid g_\lambda \neq \{ 0 \} \}. \]

Denote the orthogonal projection from $t$ to $a$ by $H \mapsto \overline{\mathfrak{k}}$. We extend a basis of $a$ to that of $t$ and define a lexicographic orderings $> \alpha$ on $a$ and $t$ with respect to these basis. Then for $H \in \mathfrak{t}$, $\overline{\mathfrak{k}} > 0$ implies $H > 0$. We denote by $\tilde{F}$ the fundamental system of $\tilde{R}$ with respect to $\rangle$, by $F$ the fundamental system of $R$ with respect to $\rangle$. We define
\[ \tilde{R}_0 = \{ \lambda \in \tilde{R} \mid \tilde{\lambda} = 0 \}, \quad \tilde{F}_0 = \{ \alpha \in \tilde{F} \mid \alpha = 0 \}. \]

Then we have
\[ R = \{ \lambda \in \tilde{R} \setminus \tilde{R}_0 \}, \quad F = \{ \alpha \in \tilde{F} \setminus \tilde{F}_0 \}. \]

We denote the set of positive roots by
\[ \tilde{R}_+ = \{ \lambda \in \tilde{R} \mid \lambda > 0 \}, \quad R_+ = \{ \lambda \in R \mid \lambda > 0 \}. \]

We put
\[ \mathfrak{e}_0 = \{ X \in \mathfrak{e} \mid [H, X] = 0 \ (H \in a) \} \]
and for each $\lambda \in R_+$
\[ \mathfrak{e}_\lambda = \mathfrak{e} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}), \quad m_\lambda = m \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}). \]

We then have the following lemma.

Lemma 1 ([HKT]). (1) We have orthogonal direct sum decompositions:
\[ \mathfrak{e} = \mathfrak{e}_0 + \sum_{\lambda \in R_+} \mathfrak{e}_\lambda, \quad m = a + \sum_{\lambda \in R_+} m_\lambda. \]

(2) For each $\mu \in \tilde{R}_+ \setminus \tilde{R}_0$ there exist $S_\mu \in \mathfrak{e}$ and $T_\mu \in m$ such that
\[ \{ S_\mu \mid \mu \in \tilde{R}_+, \ \overline{\mu} = \lambda \}, \quad \{ T_\mu \mid \mu \in \tilde{R}_+, \ \overline{\mu} = \lambda \} \]
are, respectively, orthonormal bases of $\mathfrak{e}_\lambda$ and $m_\lambda$ and that for any $H \in a$
\[ [H, S_\mu] = \langle \mu, H \rangle T_\mu, \quad [H, T_\mu] = -\langle \mu, H \rangle S_\mu. \]

For each $\lambda \in R_+$ we put $m(\lambda) = \dim m_\lambda = \dim \mathfrak{e}_\lambda$. $m(\lambda)$ is called the multiplicity of $\lambda$. We define a subset $D$ of $a$ by
\[ D = \bigcup_{\lambda \in R_+} \{ H \in a \mid \langle \lambda, H \rangle = 0 \}. \]

Each connected component of $a \setminus D$ is called a Weyl chamber. We define the fundamental Weyl chamber $C$ by
\[ C = \{ H \in a \mid \langle \alpha, H \rangle > 0 \ (\alpha \in F) \}. \]

The closure of $C$ is given by
\[ \overline{C} = \{ H \in a \mid \langle \alpha, H \rangle \geq 0 \ (\alpha \in F) \}. \]

For each subset $\Delta \subset F$, we define a subset $C^\Delta \subset \overline{C}$ by
\[ C^\Delta = \{ H \in \overline{C} \mid \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \ \langle \beta, H \rangle = 0 \ (\beta \in F \setminus \Delta) \}. \]

Then we have the following lemma.

Lemma 2 ([HKT]). (1) For $\Delta_1 \subset F$
\[ \overline{C}^{\Delta_1} = \bigcup_{\Delta \subset \Delta_1} C^\Delta \]
is a disjoint union. In particular $\overline{C} = \bigcup_{\Delta \subset F} C^\Delta$ is a disjoint union.

(2) $\Delta_1 \subset \Delta_2$ if and only if $C^{\Delta_1} \subset \overline{C}^{\Delta_2}$, for $\Delta_1, \Delta_2 \subset F$. 

For each \( \alpha \in F \) we define \( H_\alpha \in a \) by
\[
\langle H_\alpha, \beta \rangle = \delta_{\alpha \beta} (\beta \in F),
\]
where \( \delta_{\alpha \beta} \) is Kronecker’s delta. Then for \( \Delta \subset F \) we have
\[
C^\Delta = \left\{ \sum_{\alpha \in \Delta} x_\alpha H_\alpha \mid x_\alpha > 0 \right\}.
\]

3. Construction of retractions

The notation of the preceding section will be preserved. The linear isotropy representation of a Riemannian symmetric space \( G/K \) is called an \( s \)-representation. The \( s \)-representation of \( G/K \) on \( T_0(G/K) \) and the adjoint representation \( \text{Ad}(K) \) on \( \mathfrak{m} \) are equivalent. Since an \( s \)-representation is an orthogonal representation, for a unit vector \( H \in \mathfrak{m} \), the orbit \( \text{Ad}(K)H \) is a submanifold of the unit sphere \( S \subset \mathfrak{m} \). Orbits of \( s \)-representations are called R-spaces. The orbit space of an \( R \)-space is homeomorphic to \( \mathcal{T} \), more precisely for any \( X \in \mathfrak{m} \), there exists \( k \in K \) and unique \( H \in \mathcal{T} \) such that \( X = \text{Ad}(k)H \). The decomposition of \( \mathcal{T} \) in Lemma 2 is the decomposition of the orbit type. From the following theorem, we can see that for each orbit type, there exists a unique minimal orbit.

**Theorem 1** ([HTST]). For any nonempty subset \( \Delta \subset F \), there exists a unique \( H \in S \cap C^\Delta \) such that the linear isotropy orbit \( \text{Ad}(K)H \) is a minimal orbit of \( S \).

**Corollary 1.** An isolated orbit (i.e. \( \Delta = \{\alpha\} \)) is a minimal submanifold of \( S \).

Kitagawa and Ohnita ([KO]) calculated the mean curvature vector \( m_H \) of \( \text{Ad}(K)H \) in \( \mathfrak{m} \) at \( H \):
\[
m_H = -\sum_{\lambda \in R_+ \setminus R_0^K} \frac{\lambda}{\langle \lambda, H \rangle}.
\]
This expression is used in the proof of Theorem 1. We consider cones over minimal embeddings of R-spaces that obtained in this way, and construct retractions.

**Lemma 3** ([HKT]). Suppose \( \phi \) is a mapping of \( \mathcal{T} \) into itself such that \( \phi(C^\Delta) \subset \overline{C^\Delta} \) for each \( \Delta \subset F \). Then \( \phi \) extends to a mapping \( \Phi \) of \( \mathfrak{m} \) as
\[
\Phi(X) = \text{Ad}(k)\phi(H)
\]
for each \( X = \text{Ad}(k)H \) (\( k \in K, H \in \mathcal{T} \)).

The following theorem is a generalization of Proposition 2.6 in [HKT].

**Theorem 2.** For \( A \in \mathcal{T} \), we put \( \Delta_0 = \{\alpha \in F \mid \langle \alpha, A \rangle > 0\} \). Let \( f : \mathcal{T} \to \mathbb{R}_{\geq 0} \) be a continuous function. Define a continuous mapping \( \phi : \mathcal{T} \to \{tA \mid t \geq 0\} \) by \( \phi(x) = f(x)A \). If \( f \) satisfies
(1) \( f(tA) = t \) (\( t \geq 0 \)),
(2) \( f|_{C^\Delta} = 0 \) (\( \Delta \subset F \) with \( \Delta_0 \not\subset \Delta \)),
then \( \phi \) extends to a retraction \( \Phi : \mathfrak{m} \to C_{\text{Ad}(K)A} \).

**Proof.** First, we show that \( \phi \) satisfies the assumption of Lemma 3. For \( \Delta \subset F \) if \( \Delta_0 \subset \Delta \), then \( C^{\Delta_0} \subset C^\Delta \). Hence
\[
\phi(C^\Delta) = \{tA \mid t \geq 0\} \subset C^{\Delta_0} \subset \overline{C^\Delta}
\]
holds. If \( \Delta_0 \not\subset \Delta \), then \( \phi(C^\Delta) = \{0\} \) since \( f|_{C^\Delta} = 0 \). Therefore, \( \phi \) satisfies the assumption of Lemma 3. We also get
\[
\Phi(\mathfrak{m}) = \{\text{Ad}(k)f(H)A \mid k \in K, H \in \mathcal{T}\}
\]
\[
= \{t\text{Ad}(k)A \mid k \in K, t \geq 0\} = C_{\text{Ad}(K)A}.
\]
Thus \( \Phi \) is a surjection from \( \mathfrak{m} \) onto \( C_{\text{Ad}(K)A} \). Next we show that \( \Phi \) is continuous. Let \( \{P_n\}_{n \in \mathbb{N}} \) be a sequence in \( \mathfrak{m} \) with limit \( P_\infty \in \mathfrak{m} \). Points \( P_n \) and \( P_\infty \) can be expressed as
Proposition 2. Let $\Phi : m \to C_{\text{Ad}(K_\Lambda)}$ be a retraction which constructed by Theorem 2. If $\Phi|_{a \setminus \Phi^{-1}(\{0\})}$ is $C^1$, then $\Phi|_{m \setminus \Phi^{-1}(\{0\})}$ is $C^1$. In this case $\Phi$ is area-nonincreasing if and only if $J(d\Phi)_x \leq 1$ holds for each $x \in C \setminus \Phi^{-1}(\{0\})$.

Proof. If $\Phi$ is $C^1$ at $H \in \overline{C}$, then $\Phi$ is $C^1$ at $\text{Ad}(k)H$ for all $k \in K$. Thus we assume $H \in \overline{C} \setminus \Phi^{-1}(\{0\})$. For $H \in \overline{C} \setminus \Phi^{-1}(\{0\})$, we put $\Delta = \{\alpha \in F \mid \langle \alpha, H \rangle > 0\}$. Since $f(H) > 0$, we get $\Delta_0 \subset \Delta$ and $C\Delta_0 \subset \overline{C}$. By Lemma 1 we have

$$m = a + \sum_{\lambda \in R_+ \setminus R_0} \mathbb{R} \cdot T\lambda.$$

Since $\Phi|_{a \setminus \Phi^{-1}(\{0\})}$ is $C^1$, we consider only $T\lambda$ direction for each $\lambda \in \tilde{R}_+ \setminus \tilde{R}_0$. If $\langle \lambda, H \rangle = 0$, then $[T\lambda, H] = \langle \lambda, H \rangle S\lambda = 0$ from Lemma 1. Thus there exists $k \in Z^H_K = \{k \in K \mid \text{Ad}(k)H = H\}$ such that $\text{Ad}(k)T\lambda \in a$. Therefore

$$\Phi(H + tT\lambda) = \text{Ad}(k)^{-1}\Phi(\text{Ad}(k)(H + tT\lambda)).$$

Since $\text{Ad}(k)(H + tT\lambda) \in a$ and $\Phi|_{a \setminus \Phi^{-1}(\{0\})}$ is $C^1$, we have the directional derivative of $\Phi$ along $T\lambda$. If $\langle \lambda, H \rangle \neq 0$, then from Lemma 1 we have that $c(t) = \text{Ad}(\exp(-tS\lambda/\langle \lambda, H \rangle))H$ is curve in $m$ with $c(0) = H$ and $c'(0) = T\lambda$. Thus

$$\left. \frac{d}{dt} \right|_{t=0} \Phi(c(t)) = \left. \frac{d}{dt} \right|_{t=0} \Phi \left( \text{Ad} \left( \exp \left( -tS\lambda \langle \lambda, H \rangle \right) \right) H \right) = \left[ -S\lambda, \phi(H) \right] \langle \lambda, A \rangle \langle \lambda, H \rangle f(H) T\lambda.$$

Therefore $\Phi$ is a differentiable retraction from $m$ into $C_{\text{Ad}(K_\Lambda)}$. Since $\Phi|_{m \setminus \Phi^{-1}(\{0\})}$ is $C^1$, the mapping $\overline{C} \setminus \Phi^{-1}(\{0\}) \to \mathbb{R}$; $x \mapsto J(d\Phi_x)$ is continuous. Hence, if $J(d\Phi_x) \leq 1$ for each $x \in C \setminus \Phi^{-1}(\{0\})$, then $J(d\Phi_x) \leq 1$ for each $x \in C \setminus \Phi^{-1}(\{0\})$. \hfill \Box

We will compute $J(d\Phi_x)$ of $\Phi$ in Theorem 2 for $x \in C \setminus \Phi^{-1}(\{0\})$.

Proposition 3. We denote

$$R^{\Delta_0}_+ = \{\lambda \in R_+ \mid \langle \lambda, A \rangle = 0\}.$$

$$J(d\Phi_x) = \|\text{grad} f_x\| \prod_{\lambda \in R_+ \setminus R^{\Delta_0}_+} \left( \frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)} \left( x \in C \setminus \Phi^{-1}(\{0\}) \right).$$

Proof. From the proof of Proposition 2 we have

$$d\Phi_x(H) = df_x(H)A \; (H \in a), \quad d\Phi_x(T\lambda) = \frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) T\lambda \; (\lambda \in \tilde{R}_+ \setminus \tilde{R}_0)$$

for $x \in C \setminus \Phi^{-1}(\{0\})$. Thus we get

$$d\Phi_x(a) \subset Ra \subset a, \quad d\Phi_x \left( \sum_{\mu \in R_+} m_\mu \right) \subset \sum_{\mu \in R_+} m_\mu.$$
Since \( \mathfrak{a} \) and \( \sum_{\mu \in R_+} m_{\mu} \) are orthogonal, we have
\[
J(d\Phi_x) = J(d\Phi_x|_{\mathfrak{a}}) \times J(d\Phi_x|_{\sum_{\mu \in R_+} m_{\mu}}).
\]
We put \( J_1(x) = J(d\Phi_x|_{\mathfrak{a}}), J_2(x) = J(d\Phi_x|_{\sum_{\mu \in R_+} m_{\mu}}) \) and compute each of these.
\[
J_1(x) = \sup\{\|d\Phi_x(v)\| \mid v \in \mathfrak{a}, \|v\| = 1\}
\]
\[
= \sup\{\langle (\text{grad} f)_x, v \rangle \mid v \in \mathfrak{a}, \|v\| = 1\} = \| (\text{grad} f)_x \|.
\]
Since \( \ker (d\Phi_x|_{\sum_{\mu \in R_+} m_{\mu}}) = \sum_{\mu \in R^A_+} m_{\mu}, \{T_\lambda \mid \lambda \in \hat{R}_+, \langle \lambda, A \rangle > 0\} \) is an orthonormal basis of \( \ker (d\Phi_x|_{\sum_{\mu \in R_+} m_{\mu}}) \downarrow = \sum_{\mu \in R_+ \setminus R^A_+} m_{\mu} \). Hence
\[
J_2(x) = \left\| \bigwedge_{\lambda \in \hat{R}_+, \langle \lambda, A \rangle > 0} d\Phi_x(T_\lambda) \right\| = \left\| \bigwedge_{\lambda \in \hat{R}_+, \langle \lambda, A \rangle > 0} \frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) T_\lambda \right\|
\]
\[
= \prod_{\lambda \in \hat{R}_+, \langle \lambda, A \rangle > 0} \frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) = \prod_{\lambda \in R_+ \setminus R^A_+} \left( \frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)}.
\]
Therefore we get
\[
J(d\Phi)_x = J_1(x) J_2(x) = \| (\text{grad} f)_x \| \prod_{\lambda \in R_+ \setminus R^A_+} \left( \frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)}.
\]
\[
\square
\]

4. Example of area-minimizing cones over \( R \)-spaces

Using Theorem 2, Proposition 2, and Proposition 3, we investigate area-minimizing properties of cones over \( R \)-spaces. First we consider cones over isolated orbits of \( s \)-representations of irreducible symmetric pairs of compact type of rank two. Principal orbits of these representations are homogeneous hypersurfaces in the sphere. The area-minimizing properties of the cones over homogeneous minimal hypersurfaces were investigated in [HS] and [L].

We shall follow the notations of root systems in [B]. Partly we used Maxima\footnote{http://maxima.sourceforge.net/} for algebraic computations.

4.1. Type \( A_2 \),
\[
\mathfrak{a} = \{\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \mid \xi_1 + \xi_2 + \xi_3 = 0\}, \quad F = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3\}.
\]
Then we have \( R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \). For \( \lambda \in R_+ \), we put \( m = m(\lambda) \). We have
\[
H_{\alpha_1} = \frac{1}{3}(2e_1 - e_2 - e_3), \quad H_{\alpha_2} = \frac{1}{3}(e_1 + e_2 - 2e_3).
\]
We put
\[
A_1 = \frac{H_{\alpha_1}}{\|H_{\alpha_1}\|} = \frac{1}{\sqrt{6}}(2e_1 - e_2 - e_3), \quad A_2 = \frac{H_{\alpha_2}}{\|H_{\alpha_2}\|} = \frac{1}{\sqrt{6}}(e_1 + e_2 - 2e_3).
\]
Since \( \text{Ad}(K)A_1 \) and \( \text{Ad}(K)A_2 \) are isometric, we consider only the cone over \( \text{Ad}(K)A_1 \).
4.1.1. *Cones over Ad*(K)A₁. We put Δ₀ = {α₁} then RΔ₀⁺ = {α₂}. For x = x₁Hα₁ + x₂Hα₂ ∈ C, we define
\[ f(x) = \sqrt[3]{\frac{2}{3}} \left( (α₁, x)^2 \left( α₁ + \frac{3}{2} x₂, x \right) \right)^{\frac{1}{3}} = \sqrt[3]{\frac{2}{3}} \left( x₁^2 \left( x₁ + \frac{3}{2} x₂ \right) \right)^{\frac{1}{3}}. \]

Since

1. \( f(tA₁) = \sqrt[3]{\frac{2}{3}} \left( \left( \sqrt[3]{2} t \right)^{3} \right)^{\frac{1}{3}} = t, \)
2. for each \( Δ \subset F, \) if \( Δ₀ \not\subset Δ, \) then \( f|_{CΔ} = 0, \)

we can apply Theorem 2 to this case. It is clear that \( Φ|_{d \Phi^{-1}(0)} \) is \( C¹. \) Thus \( Φ \) is a differentiable retraction by Proposition 2. Since

\[ \frac{∂f}{∂x₁}(x) = \sqrt[3]{\frac{2}{3}} \left( x₁^2 \left( x₁ + \frac{3}{2} x₂ \right) \right)^{-\frac{2}{3}} (x₁² + x₁x₂), \]
\[ \frac{∂f}{∂x₂}(x) = \sqrt[3]{\frac{2}{3}} \left( x₁^2 \left( x₁ + \frac{3}{2} x₂ \right) \right)^{-\frac{2}{3}} \frac{x₂²}{2}, \]

we get
\[ J₁(x) = ∥(grad f)_{x}∥ = \sqrt[3]{\frac{2}{3}} \left( x₁^2 \left( x₁ + \frac{3}{2} x₂ \right) \right)^{-\frac{2}{3}} \frac{3}{2} \sqrt{x₁^4 + 3x₁³x₂ + 2x₁²x₂²}. \]

On the other hand,
\[ J₂(x) = \left( \frac{⟨α₁, A₁⟩}{⟨α₁, x⟩} f(x) \right)^m \left( \frac{⟨α₁ + α₂, A₁⟩}{⟨α₁ + α₂, x⟩} f(x) \right)^m = \left( \frac{x₁^2 \left( x₁ + \frac{3}{2} x₂ \right)^{2}}{x₁ + x₂} \right)^{\frac{m}{4}}. \]

Then
\[ (x₁ + x₂)^{3} - x₁ \left( x₁ + \frac{3}{2} x₂ \right)^{2} = \frac{3}{4} x₁²x₂ + x₂³ ≥ 0, \]
thus
\[ \left( \frac{x₁^2 \left( x₁ + \frac{3}{2} x₂ \right)^{2}}{x₁ + x₂} \right)^{\frac{m}{4}} ≤ 1. \]

We put
\[ D = J₁(x) × \left( \frac{x₁^2 \left( x₁ + \frac{3}{2} x₂ \right)^{2}}{x₁ + x₂} \right)^{\frac{1}{2}} = \left( \frac{3x₁² + 6x₁x₂ + 4x₂²}{3²4(x₁ + x₂)^{12}} \right)^{\frac{1}{4}}. \]

Since
\[ J(dΦ)_{x} = D × \left( \frac{x₁^2 \left( x₁ + \frac{3}{2} x₂ \right)^{2}}{x₁ + x₂} \right)^{m-2}, \]
if \( D ≤ 1, \) then \( J(dΦ)_{x} ≤ 1 \) for \( m ≥ 2. \) Since
\[ 3²4(x₁ + x₂)^{12} - (3x₁² + 6x₁x₂ + 4x₂²)^{3} x₁² (2x₁ + 3x₂)^{4} \]
\[ = 216x₁^{10} x₂^2 + 2376x₁⁹x₂³ + 11925x₁⁸x₂⁴ + 35838x₁⁷x₂⁵ + 71120x₁⁶x₂⁶ \]
\[ + 9688x₁⁵x₂⁷ + 91152x₁⁴x₂⁸ + 57888x₁³x₂⁹ + 23328x₁²x₂¹⁰ + 5184x₁x₂¹¹ + 432x₂¹² \]
\[ ≥ 0, \]
we have \( D ≤ 1. \) Therefore, cones over Ad(K)A₁ are area-minimizing for \( m ≥ 2. \)
4.2. **Types** B₂, BC₂ and C₂. Types C₂ and B₂ are isomorphic, thus it suffices to compute the type B₂ case. Moreover setting the multiplicity of long roots to zero, the set of restricted roots of type BC₂ reduces to that of type B₂. We have

\[ F = \{ \alpha_1 = e_1 - e_2, \, \alpha_2 = e_2 \}, \]

\[ R_+ = \{ \alpha_1, \, \alpha_2, \, \alpha_1 + \alpha_2, \, 2\alpha_1 + 2\alpha_2, \, 3\alpha_1 + 3\alpha_2 \}, \]

\[ H_{\alpha_1} = e_1, \quad H_{\alpha_2} = e_1 + e_2, \]

and put

\[ m(\alpha_1) = m_1, \quad m(\alpha_2) = m_2, \quad m(2\alpha_2) = m_3. \]

4.2.1. **Cones over Ad(K)A₁.** We put \( \Delta_0 = \{ \alpha_1 \} \), then we have

\[ A_1 = \frac{H_{\alpha_1}}{\|H_{\alpha_1}\|} = e_1, \]

and

\[ R^\Delta_0 = \{ \lambda \in R_+ \mid \langle \lambda, A_1 \rangle = 0 \} = \{ \alpha_2, \alpha_1 \}. \]

For \( x = x_1H_{\alpha_1} + x_2H_{\alpha_2} \in C \), we define

\[ f(x) = \sqrt{\langle \alpha_1, x \rangle \langle 2\alpha_2, x \rangle} = \sqrt{x_1(x_1 + 2x_2)}. \]

Then we can show that \( f \) satisfies the condition of Theorem 2 and \( \Phi \) is differentiable. Moreover \( J(d\Phi_x) \leq 1 \) holds for \( m_2 + m_3 \geq 2 \).

Therefore, cones over Ad(K)A₁ are area-minimizing for \( m_2 + m_3 \geq 2 \).

4.2.2. **Cones over Ad(K)A₂.** We put \( \Delta_0 = \{ \alpha_2 \} \), then we have

\[ A_2 = \frac{H_{\alpha_2}}{\|H_{\alpha_2}\|} = \frac{e_1 + e_2}{\sqrt{2}} \]

and

\[ R^\Delta_0 = \{ \lambda \in R_+ \mid \langle \lambda, A_2 \rangle = 0 \} = \{ \alpha_1 \}. \]

For \( x = x_1H_{\alpha_1} + x_2H_{\alpha_2} \in C \), we define

\[ f(x) = \sqrt{2} \left( \langle \alpha_2, x \rangle ^2 / \left( \frac{3}{2} \alpha_1 + \alpha_2, x \right) \right)^{\frac{1}{2}} = \sqrt{2} \left( x_2^2 / \left( \frac{3}{2} x_1 + x_2 \right) \right)^{\frac{1}{2}}. \]

Then we can show that \( f \) satisfies the condition of Theorem 2 and \( \Phi \) is differentiable. Moreover \( J(d\Phi_x) \leq 1 \) holds for \( m_2 + m_3 \geq 2 \).

Therefore, cones over Ad(K)A₂ are area-minimizing for \( m_2 + m_3 \geq 2 \).

4.3. **Type** G₂. We have

\[ F = \{ \alpha_1, \, \alpha_2 \}, \]

\[ R_+ = \{ \alpha_1, \, \alpha_2, \, \alpha_1 + \alpha_2, \, 2\alpha_1 + \alpha_2, \, 3\alpha_1 + \alpha_2, \, 3\alpha_1 + 2\alpha_2 \}, \]

\[ \langle \alpha_1, \alpha_1 \rangle = 1, \quad \langle \alpha_1, \alpha_2 \rangle = -\frac{3}{2}, \quad \langle \alpha_2, \alpha_2 \rangle = 3, \]

\[ H_{\alpha_1} = 4\alpha_1 + 2\alpha_2, \quad H_{\alpha_2} = \frac{2}{3}(3\alpha_1 + 2\alpha_2), \]

and put

\[ m = m(\alpha_1) = m(\alpha_2). \]
4.3.1. Cones over $\text{Ad}(K)A_1$. We put $\Delta_0 = \{\alpha_1\}$ then we have
\[ A_1 = \frac{H_{\alpha_1}}{\|H_{\alpha_1}\|} \]
and
\[ R_{\Delta_0}^+ = \{\lambda \in R_+ \mid \langle \lambda, A_1 \rangle = 0\} = \{\alpha_2\}. \]
For $x = x_1H_{\alpha_1} + x_2H_{\alpha_2} \in \overline{C}$, we define
\[ f(x) = \sqrt{\frac{4}{3} \langle \alpha_2, x \rangle \langle 3\alpha_1 + \alpha_2, x \rangle} = \sqrt{\frac{4}{3} x_1(x_1 + x_2)}. \]
Then we can show that $f$ satisfies the condition of Theorem 2 and $\Phi$ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m \geq 2$.
Therefore cones over $\text{Ad}(K)A_1$ are area-minimizing for $m \geq 2$.

4.3.2. Cones over $\text{Ad}(K)A_2$. We put $\Delta_0 = \{\alpha_2\}$ then we have
\[ A_2 = \frac{H_{\alpha_2}}{\|H_{\alpha_2}\|}, \]
and
\[ R_{\Delta_0}^+ = \{\lambda \in R_+ \mid \langle \lambda, A_1 \rangle = 0\} = \{\alpha_1\}. \]
For $x = x_1H_{\alpha_1} + x_2H_{\alpha_2} \in \overline{C}$, we define
\[ f(x) = \sqrt{\frac{4}{3} \langle \alpha_2, x \rangle \langle 3\alpha_1 + \alpha_2, x \rangle} = \sqrt{\frac{4}{3} x_2(3x_1 + x_2)}. \]
Then we can show that $f$ satisfies the condition of Theorem 2 and $\Phi$ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m \geq 2$.
Therefore, cones over $\text{Ad}(K)A_2$ are area-minimizing for $m \geq 2$.

By the above computation, we get the following table of cones over isolated orbits of the $s$-representations of irreducible symmetric spaces of rank two.
| type | symmetric pair | multiplicities | \( A_i \) | orbit | symm. or not | \( \text{dim. of orbit and sphere} \) | area-min. |
|------|----------------|---------------|----------|-------|-------------|---------------------------------|---------|
| \( A_2 \) | (SU(3), SO(3)) | (1, 1) | \( A_1 \) | \( \mathbb{R}P^2 \) | symmetric | (2, 4) |  
| | (SU(3) \times SU(3), SU(3)) | (2, 2) | \( A_1 \) | \( \mathbb{C}P^2 \) | symmetric | (4, 7) |  
| | (SU(6), Sp(3)) | (4, 4) | \( A_1 \) | \( \mathbb{H}P^2 \) | symmetric | (8, 13) |  
| | (E_6, F_4) | (8, 8) | \( A_1 \) | \( \mathbb{O}P^2 \) | symmetric | (16, 25) |  
| \( B_2 \) | (SO(5) \times SO(5), SO(5)) | (2, 2) | \( A_1 \) | \( \mathbb{G}_2(\mathbb{R}^5) \) | symmetric | (6, 9) |  
| | (SO(5), SO(2) \times SO(3)) | (1, 1) | \( A_1 \) | SO(5)/U(2) | symmetric | (6, 9) |  
| | (SO(4 + n), SO(2) \times SO(2 + n)) | (1, n) | \( A_1 \) | symmetric | (n + 2, 2n + 3) | \( n \geq 2 \) |  
| | | | \( A_2 \) | symmetric | (2n + 1, 2n + 3) | \( n \geq 2 \) |  
| \( C_2 \) | (Sp(2), U(2)) | (1, 1) | \( A_1 \) | U(2)/O(2) | symmetric | (3, 5) |  
| | (Sp(2) \times Sp(2), Sp(2)) | (2, 2) | \( A_1 \) | Sp(2)/U(2) | symmetric | (6, 9) |  
| | (Sp(4), Sp(2) \times Sp(2)) | (4, 3) | \( A_1 \) | Sp(2) | symmetric | (11, 15) |  
| | (SU(4), S(U(2) \times U(2))) | (2, 1) | \( A_1 \) | Sp(2) | symmetric | (10, 15) |  
| | (SO(8), U(4)) | (4, 1) | \( A_1 \) | Sp(2) | symmetric | (5, 7) |  
| | | | \( A_2 \) | U(2) | symmetric | (4, 7) | \( n \geq 2 \) |  

\( \mathbb{K}_c \) a [Ke] [HKT] a [Ka] a [L]
| type | symmetric pair | multiplicities | $A_i$ | orbit | symm. or not | dim. of orbit and sphere | area-min. |
|------|----------------|----------------|-------|-------|-------------|--------------------------|----------|
| BC₂  | (SU(4 + $n$), S(U(2) × U(2 + $n$))) | (2, (2$n$, 1)) | $A_1$ | U(5)/(Sp(1) × U(3)) | (2$n + 3$, 4$n + 7$) | (n ≥ 1) |   |
|      | (SO(10), U(5)) | (4, (4, 1)) | $A_2$ | U(5)/(Sp(2) × U(1)) | (4$n + 4$, 4$n + 7$) | (n ≥ 1) |   |
|      | (Sp(4 + $n$), Sp(2) × Sp(2 + $n$)) | (4, (4$n$, 3)) | $A_1$ | U(5)/(Sp(1) × U(3)) | (13, 19) |   |   |
|      | (E₀, T¹ · Spin(10)) | (6, (8, 1)) | $A_2$ | U(5)/(Sp(2) × U(1)) | (14, 19) |   |   |
| G₂   | (G₂, SO(4)) | (1, 1) | $A_1$ | (4$n + 11$, 8$n + 15$) | (8$n + 10$, 8$n + 15$) |   |   |
|      | (G₂ × G₂, G₂) | (2, 2) | $A_2$ | (21, 31) | (24, 31) |   |   |
4.4. Type $A_3$. Theorem\[2\] can be applied to cones over minimal orbits, not only isolated orbits. We demonstrate the area-minimizing property for the cone over a minimal orbit, which is not an isolated orbit, of the $s$-representation of symmetric spaces of type $A_3$.

$$\mathfrak{a} = \left\{ \sum_{i=1}^{4} \xi_i e_i \mid \sum_{i=1}^{4} \xi_i = 0 \right\},$$

$$F = \{ \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4 \}.$$  

Then $R_+ = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 \}$ and for $\lambda \in R_+$, we put $m(\lambda) = m$. We have

$$H_{\alpha_1} = \frac{1}{4}(3e_1 - e_2 - e_3 - e_4), \quad H_{\alpha_2} = \frac{1}{4}(2e_1 + 2e_2 - 2e_3 - 2e_4), \quad H_{\alpha_3} = \frac{1}{4}(e_1 + e_2 + e_3 - 3e_4).$$

We put $\Delta_0 = \{ \alpha_1, \alpha_3 \}$, and we have

$$A = \frac{H_{\alpha_1} + H_{\alpha_3}}{\sqrt{2}} = \frac{e_1 - e_4}{\sqrt{2}}.$$  

Then the orbit $\text{Ad}(K)A$ is a minimal submanifold of the sphere $S \subset \mathfrak{m}$. We get

$$R_+^{\Delta_0} = \{ \lambda \in R_+ \mid \langle \lambda, A \rangle = 0 \} = \{ \alpha_2 \}.$$  

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} + x_3 H_{\alpha_3} \in \mathbb{C}$, we define

$$f(x) = \sqrt{2}(\langle \alpha_1, x \rangle \langle \alpha_3, x \rangle \langle \alpha_1 + \alpha_2, x \rangle \langle \alpha_2 + \alpha_3, x \rangle)^{\frac{1}{2}} = \sqrt{2}(x_1 x_3 (x_1 + x_2) (x_2 + x_3))^{\frac{1}{2}}.$$  

Then we can show that $f$ satisfies the condition of Theorem\[2\] and $\Phi$ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m \geq 4$.

Therefore, cones over $\text{Ad}(K)A$ are area-minimizing for $m \geq 4$. The only symmetric pair which satisfies $m \geq 3$ is $(\text{SU}(6), \text{Sp}(3))$.

5. Reducible cases

In this section, we consider cones over products of two R-spaces. Let $(G_i, K_i) \ (i = 1, 2)$ be Riemannian symmetric pairs, and put $(G, K) = (G_1 \times G_2, K_1 \times K_2)$. We define the notation for $(G_i, K_i)$ as follows. Let

$$\mathfrak{g}_i = \mathfrak{t}_i + \mathfrak{m}_i \ (i = 1, 2)$$

be the canonical decompositions of Lie algebras $\mathfrak{g}_i$ of $G_i$. Take and fix a maximal abelian subspace $\mathfrak{a}_i$ in $\mathfrak{m}_i$. We denote by $R_i$ the restricted root system of $(\mathfrak{g}_i, \mathfrak{t}_i)$ with respect to $\mathfrak{a}_i$. We put the fundamental systems $F_i$ of $R_i$ by $F_i = \{ \alpha_{i1}, \ldots, \alpha_{il_i} \}$. $R_{i+}$ is the set of positive roots in $R_i$. We set

$$\mathcal{C}_i = \{ H \in \mathfrak{a}_i \mid \langle \alpha, H \rangle > 0 \ (\alpha \in F_i) \},$$

$$\mathcal{C}_i^{\Delta} = \{ H \in \mathfrak{a}_i \mid \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \ (\beta, H) = 0 \ (\beta \in F_i \setminus \Delta) \},$$

where $\Delta \subset F_i$. The direct sum of the $s$-representations of $(G_i, K_i)$ is the $s$-representation of $(G, K) = (G_1 \times G_2, K_1 \times K_2)$. Then, we have

$$\mathbb{C} = \mathbb{C}_1 \times \mathbb{C}_2.$$  

For $\Delta \subset F_i$, $\Delta$ is expressed as $\Delta = \Delta_1 \cup \Delta_2$ where $\Delta_i \subset F_i \ (i = 1, 2)$. By Theorem\[1\] for each $\Delta_i$, there exists $A_i \in \mathbb{C}_i$ such that $\text{Ad}(K_i)A_i$ is a minimal orbit of the $s$-representation of $(G_i, K_i)$. We put $k_i = \dim \text{Ad}(K_i)A_i$ and $k = k_1 + k_2$, then

$$A = \sqrt{\frac{k_1}{k}} A_1 + \sqrt{\frac{k_2}{k}} A_2 \in \mathbb{C}$$

is a base point of a minimal orbit of the $s$-representation of $(G, K)$.
Theorem 3. Let $\Delta_0 = \Delta_1 \cup \Delta_2$ ($\Delta_i \subset F_i$). We suppose that for the cone over $\text{Ad}(K_i)A_i$, there exists an area-nonincreasing retraction constructed by a function $f_i$ on $\overline{C}_i$ in Theorem 2 and that the retraction satisfies

$$\prod_{\lambda \in R_+ \setminus R_{+}^{\Delta_i}} \left( \frac{\lambda}{\lambda, x} f_i(x) \right) \leq 1 \ (x \in \mathcal{C}_i).$$

If $\dim \text{Ad}(K_i)A_i \geq 3$, then there exists an area-nonincreasing retraction $\Phi : \mathfrak{m} \to C\text{Ad}(K)A$ that constructed by some function $f$ on $\overline{C}$ in Theorem 2 and then the retraction satisfies

$$\prod_{\lambda \in R_+ \setminus R_{+}^{\Delta_0}} \left( \frac{\lambda}{\lambda, x} f(x) \right) \leq 1 \ (x \in \mathcal{C}).$$

Proof. Let $k_i = \dim \text{Ad}(K_i)A_i$, $k = k_1 + k_2$ and put $a_i = \sqrt{k_i/k}$. $A = a_1A_1 + a_2A_2$ holds. For $x = (x_1, x_2) \in \overline{C}_1 \times \overline{C}_2 = \overline{C}$ we define

$$f(x) = \begin{cases} \frac{f_1(x_1)f_2(x_2)}{a_1^2f_1(x_1) + a_2^2f_2(x_2)} & (f_1(x_1) \neq 0 \text{ or } f_2(x_2) \neq 0) \\ 0 & (f_1(x_1) = f_2(x_2) = 0) \end{cases}$$

We will show that $f$ satisfies the conditions of Theorem 2. We can check easily $f(tA) = t$ for $t \geq 0$. For $\Delta \subset F$ with $\Delta_0 \not\subset \Delta$, using $\Delta_i \subset F_i$ we can write $\Delta = \Delta_i' \cup \Delta_2'$. Then $\Delta_i \not\subset \Delta_i'$ holds $i = 1$ or $i = 2$. Thus $f_1 = 0$ or $f_2 = 0$ holds on $\mathcal{C}^\Delta$. Therefore $f|_{\mathcal{C}^\Delta} = 0$. Since $\Phi|_{\mathfrak{m}\setminus\{0\}}$ is $C^1$, $\Phi$ is a differentiable retraction by Proposition 2. We calculate $J(d\Phi_x)$ for $x \in \mathcal{C} \setminus f^{-1}(\{0\})$. We put

$$J_1(x) = \| (\text{grad}f)_x \|, \quad J_2(x) = \prod_{\lambda \in R_+ \setminus R_{+}^{\Delta}} \left( \frac{\lambda}{\lambda, x} f(x) \right)^{m(\lambda)}$$

for $x = (x_1, x_2) = (x_1^1, x_1^2, \ldots, x_2^1, x_2^2) \in \mathcal{C} \setminus f^{-1}(\{0\}) = \mathcal{C}_1 \times \mathcal{C}_2 \setminus f^{-1}(\{0\})$. Since

$$\frac{\partial f}{\partial x_1^j} = \frac{\partial f_1}{\partial x_1^j} \frac{a_1^2f_1(x_1)}{a_2^2f_2(x_2)} + \frac{\partial f_2}{\partial x_1^j} \frac{a_2^2f_1(x_1)}{a_2^2f_2(x_2)} \ (j \in \{1, \ldots, l_1\}),$$

$$\frac{\partial f}{\partial x_2^j} = \frac{\partial f_1}{\partial x_2^j} \frac{a_1^2f_2(x_2)}{a_2^2f_2(x_2)} + \frac{\partial f_2}{\partial x_2^j} \frac{a_2^2f_2(x_2)}{a_2^2f_2(x_2)} \ (j \in \{1, \ldots, l_2\}),$$

we get

$$\text{grad}f)_x = \frac{a_1^2f_2(x_2)^2(\text{grad}f_1)_x + a_2^2f_1(x_1)^2(\text{grad}f_2)_x}{(a_1^2f_1(x_1) + a_2^2f_2(x_2))^2}$$

and

$$J_1(x) = \| (\text{grad}f)_x \| = \frac{\text{grad}f_1(x_1)^2}{(a_1^2f_1(x_1) + a_2^2f_2(x_2))^2}.$$
Put
\[ J_{2i}(x_i) = \prod_{\lambda \in R_{i+} \setminus R_{i+}^{\Delta_i}} \left( \frac{\langle \lambda, A_i \rangle}{\langle \lambda, x_i \rangle} f_i(x_i) \right)^{m(\lambda)}, \quad J_1(x_i) = \| (\nabla f_i)_{x_i} \| (i = 1, 2). \]

Note that \( J_{2i}(x_i) \leq 1, J_1(x_i)J_{2i}(x_i) \leq 1 \) holds by assumption. Since
\[ \sum_{\lambda \in R_{i+} \setminus R_{i+}^{\Delta_i}} m(\lambda) = \dim \text{Ad}(K_i) A_i = k_i, \]
we can write
\[ J_2(x) = J_{21}(x_1)J_{22}(x_2) \left( \frac{a_1 f(x)}{f_1(x_1)} \right)^{k_1} \left( \frac{a_2 f(x)}{f_2(x_2)} \right)^{k_2}. \]

Since \( J_{2i}(x_i) \leq 1, \)
\[ J_2(x) \leq \left( \frac{a_1 f(x)}{f_1(x_1)} \right)^{k_1} \left( \frac{a_2 f(x)}{f_2(x_2)} \right)^{k_2}. \]

We put
\[ X_1 = \frac{f_2(x_2)}{a_2}, \quad X_2 = \frac{f_1(x_1)}{a_1}. \]

Then we have
\[ \left( \frac{a_1 f(x)}{f_1(x_1)} \right)^{k_1} \left( \frac{a_2 f(x)}{f_2(x_2)} \right)^{k_2} = \frac{X_1^{k_1} X_2^{k_2}}{(a_1^2 X_1 + a_2^2 X_2)^k}. \]

For \( X_1, X_2 > 0, \) we define
\[ \tilde{D}(X_1, X_2) = \frac{X_1^{k_1} X_2^{k_2}}{(a_1^2 X_1 + a_2^2 X_2)^k}. \]

If \( \tilde{D} \leq 1, \) then \( J_2(x) \leq 1. \) Thus we prove \( \tilde{D} \leq 1. \) Since \( \tilde{D}(X_1, X_2) = \tilde{D}(tX_1, tX_2) (t > 0), \) in order to prove \( \tilde{D} \leq 1, \) we show \( \tilde{D}|_P \leq 1 \) where
\[ P = \{ (X_1, X_2) \in \mathbb{R}^2 \mid X_1, X_2 > 0, \ a_1^2 X_1 + a_2^2 X_2 = 1 \}. \]

We have \( \tilde{D}|_P = X_1^{k_1} X_2^{k_2} \) and \( X_2 = \frac{1-a_1^2 x_1}{a_2^2}. \) Since
\[ \frac{d\tilde{D}|_P}{dX_1} = k_1 X_1^{k_1-1} X_2^{k_2} + \frac{a_1^2}{a_2^2} X_2^{k_2-1} = k_1 X_1^{k_1-1} X_2^{k_2-1} (X_2 - X_1), \]
a critical point of \( \tilde{D}|_P \) is only \( X_1 = 1 \) in \( P. \) Further, we get
\[ \tilde{D}|_P \to 0 \quad \text{as} \quad X_1 \to 0 \quad \text{or} \quad \frac{1}{a_1^2}. \]

Hence \( \max \{ \tilde{D}(X_1, X_2) \mid (X_1, X_2) \in P \} = \tilde{D}(1, 1) = 1. \) Therefore
\[ J_2(x) \leq 1. \]
Then we have
\[
J(d\Phi_x) = J_1(x)J_2(x) = \|(\text{grad} f)_x\|J_2(x)
\]
\[
= \sqrt{a_1^4 f_2(x)^2 J_1(x_1)^2 + a_2^4 f_1(x_1)^4 J_1(x_2)^2}
\]
\[
J_2(x_1)J_2(x_2) \frac{(a_1 f_2(x_2))^k_1 (a_2 f_1(x_1))^k_2}{(a_1^2 f_1(x_1) + a_1^2 f_2(x_2))^k}
\]
\[
= \sqrt{a_1^4 f_2(x_2)^2 J_1(x_1)^2 J_2(x_2)^2 + a_2^4 f_1(x_1)^4 J_1(x_2)^2 J_2(x_2)^2}
\]
\[
\frac{1}{(a_1^2 f_1(x_1) + a_1^2 f_2(x_2))^k + 2}
\]
\[
\leq \sqrt{a_1^4 f_2(x_2)^4 + a_2^4 f_1(x_1)^4} \frac{(a_1 f_2(x_2))^k_1 (a_2 f_1(x_1))^k_2}{(a_1^2 f_1(x_1) + a_1^2 f_2(x_2))^k + 2}
\]
\[
= \sqrt{\frac{a_1^2 X_1^4 + a_2^2 X_2^4}{a_1^2 X_1 + a_2^2 X_2} X_2^{k_1} X_2^{k_2}}
\]
\[
D(X_1, X_2) = J(d\Phi_x)^2 = \frac{(a_1^2 X_1^4 + a_2^2 X_2^4) X_1^{2k_1} X_2^{2k_2}}{(a_1^2 X_1 + a_2^2 X_2)^{2k+4}}.
\]

We have \(D(tX_1, tX_2) = D(X_1, X_2)\) \((t > 0)\). Similar to the above argument, we consider the maximum value of \(D|_P\). Since
\[
D|_P = (a_1^2 X_1^4 + a_2^2 X_2^4) X_1^{2k_1} X_2^{2k_2},
\]
we get
\[
\frac{dD|_P}{dX_1} = 4 \left( a_1^2 X_1^3 - \frac{a_1^2}{a_2^2} a_2^2 X_2^3 \right) X_2^{2k_1} X_2^{2k_2}
\]
\[
+ \left( a_1^2 X_2^4 \right) \left( 2k_1 X_1^{2k_1-1} X_2^{2k_2} - 2k_2 \frac{a_1^2}{a_2^2} X_1^{2k_1} X_2^{2k_2-1} \right)
\]
\[
= -2a_1^2 X_2^{2k_1-1} X_2^{2k_2-1} (X_1 - X_2)
\]
\[
\times \left\{ \left( (k_1 - 3) X_1^4 + (k_2 - 3) X_2^4 \right) + 3(X_1 - X_2)^4 + 10(X_1 - X_2)^2 \right\}.
\]
Hence, if \(k_1 \geq 3, k_2 \geq 3\), then a critical point of \(D|_P\) is only \(X_1 = 1\) in \(P\). Furthermore, we get
\[
\frac{dD|_P}{dX_1} \to 0 \text{ as } X_1 \to 0 \text{ or } \frac{1}{a_1^2}.
\]
Thus \(\max\{D(X_1, X_2) \mid (X_1, X_2) \in P\} = D(1, 1) = 1\). Hence \(D \leq 1\). This implies \(J(d\Phi_x) \leq 1\).

Therefore if \(k_1 \geq 3, k_2 \geq 3\), \(\Phi\) is area nonincreasing.

\[\square\]

**Remark 1.** In 1969, Bombieri, DeGiorghi and Giusti [BDG] showed that the cone over \(S^k \times S^k \subset S^{2k+1} (k \geq 3)\) is area-minimizing. On the other hand, Lawlor [L] proved that the cone over \(S^{k_1} \times S^{k_2} \subset S^{k_1+k_2+1}\) are not area-minimizing when \(k_1 + k_2 \leq 5\) or \(k_1 = 1, k_2 = 5\). Hence, we need the condition \(k_1 \geq 3, k_2 \geq 3\) in Theorem 3.

**Remark 2.** Area-nonincreasing retractions which we constructed in Section 4 satisfy the assumption of Theorem 3. Moreover, an area-nonincreasing retraction that is constructed using Theorem 3 satisfies the assumption of Theorem 3 again. Therefore, we can apply Theorem 3 inductively. This implies that the cone over a product of two or more R-spaces with “\(\circ\)” in the table in Section 4 is area-minimizing.

**References**

[BDG] E. Bombieri, E. DeGiorgi and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. 7 (1969) 243–268.

[Bo] N. Bourbaki, *Groupes et algebres de Lie*, Hermann, Paris, 1975.

[HKT] D. Hirohashi, T. Kanno and H. Tasaki, *Area-minimizing of the cone over symmetric R-space*, Tsukuba J. Math. 24 (2000), no.1, 171–188.
[Hs] W. Y. Hsiang, *Minimal cones and the spherical Bernstein problem, II.* Invent. Math. **74** (1983), no.3, 351–369.

[HTST] D. Hirohashi, H. Tasaki, H.J. Song and R. Takagi, *Minimal orbits of the isotropy groups of symmetric space of compact type*, Differential Geom. Appl. **13** (2000), no.2, 167–177.

[Ka] T. Kanno, *Area-minimizing cones over the canonical embedding of symmetric R-spaces*, Indiana Univ. Math. J. **51** (2002), no.1, 89–125.

[KO] Y. Kitagawa and Y. Ohnita, *On the mean curvature of R-spaces*, Tôhoku Math. J. **35** (1983) 499–502.

[Ke] M. Kerckhove, *Isolated orbits of the adjoint action and area-minimizing cones*, Proc. Amer. Math. Soc. **121** (1994), no.2, 497–503.

[L] G. R. Lawlor, *A sufficient criterion for a cone to be area-minimizing*, Mem. Amer. Math. Soc. **91** (1991), no. 446.

[T] M. Takeuchi, *On conjugate loci and cut loci of compact symmetric space I*, Tsukuba J. Math. **2** (1977), 35–68.

Department of Mathematics and Information Sciences, Tokyo Metropolitan University, Minami-Osawa, Hachioji-shi, Tokyo, 192-0397 Japan

E-mail address: oono-shinji@ed.tmu.ac.jp

E-mail address: sakai-t@tmu.ac.jp