TCHEBYCHEV’S CHARACTERISTIC OF REARRANGEMENT INVARIANT SPACE.

E.Ostrovsky and L.Sirota, ISRAEL.

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan.
e-mails: eugostrovsky@list.ru; galo@list.ru; sirota@zahav.net.il

Abstract.

We introduce and investigate in this short article a new characteristic of rearrangement invariant (r.i.) (symmetric) space, namely so-called Tchebychev’s characteristic.

We reveal an important class of the r.i. spaces - so called regular r.i. spaces and show that the majority of known r.i. spaces: Lebesgue-Riesz, Grand Lebesgue Spaces, Orlicz, Lorentz and Marcinkiewicz r.i. spaces are regular. But we construct after several examples of r.i. spaces without the regular property.

Applications - Probability theory and Statistics.

Key words and phrases: Rearrangement invariant (r.i.) space, regular r.i. space, Tchebychev’s characteristic, fundamental function, Grand Lebesgue Space (GLS), measure, resonant, Probability, distribution, tail function, partial order, associate and conjugate (dual) space, relation of equivalence, Orlicz, Lorentz and Marcinkiewicz spaces, upper and lower estimates.

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1 Notations. Statement of problem.

Let $(\Omega, \mathcal{A}, \mu)$ be measure space with sigma-finite non-trivial measure $\mu$ and $(F, || \cdot || = || \cdot ||_F)$ be any rearrangement invariant (r.i.) space over $(\Omega, \mathcal{A}, \mu)$.

The detail investigate of r.i. spaces see in the classical books [1], [8].

Hereafter $C, C_j$ will denote any non-essential finite positive constants. As usually, for the measurable function $f : \Omega \to \mathbb{R}$,

$$|f|_p(\Omega, \mu) = |f|_p(\mu) = |f|_p = \left[ \int_\Omega |f(\omega)|^p \mu(d\omega) \right]^{1/p}, \quad 1 \leq p < \infty,$$

$L_p(\Omega, \mu) = L_p(\mu) = \{ f : |f|_p < \infty \}$; $m$ will denote usually Lebesgue measure, and we will write in this case $m(dx) = dx$; $|f|_\infty \overset{df}{=} \varpisup_\omega |f(\omega)|$.

We will conclude without loss of generality in the case when $\mu(\Omega) < \infty \Rightarrow \mu(\Omega) = 1$, call: ”probabilistic case” and denote $\mathbf{P} = \mu$,

$$E\xi = \int_\Omega \xi(\omega) \mathbf{P}(d\omega).$$

We presume for example construction that the source measurable space $(\Omega, \mathcal{A}, \mu)$ is sufficiently rich; it is suffices to set $\Omega = [0, 1]$ or $\Omega = [0, \infty)$ with Lebesgue measure $m$.

We denote as usually for arbitrary finite a.e.measurable function (random variable) $\xi(\omega)$ its Tail function $T_\xi(t)$ as follows:
\[ T_\xi(t) = \mu\{\omega : |\xi(\omega)| \geq t\}, \; t > 0. \]

The left inverse function to the \( T_\xi(t) \) is denoted \( \xi^*(t) \).

**Definition 1.1.** Tchebychev’s characteristic \( T_F(t), t > 0 \) of the space \((F, ||\cdot|| = ||\cdot||_F)\) is defined as follows:

\[
T^{(F)}(t) = T^{(F, ||\cdot||)}(t) \overset{\text{def}}{=} \sup_{\xi: \xi \in F, ||\xi||_F = 1} T_\xi(t). \tag{1.1}
\]

Our aim is to investigate the function \( T_F(t) \) for sufficiently greatest values \( t : t > t_0 = \text{const} > 0 \) for different classes of r.i. spaces \((F, ||\cdot|| = ||\cdot||_F)\).

A possible applications of tail estimates: Functional Analysis, see the classical books of C.Bennet and R.Sharpley [1], S.G. Krein Yu.V. Petunin and E.M. Semenov [8], and also in the articles [16], [17], [18]; Probability Theory [19], [21]; Numerical Methods Monte-Carlo [20]; Statistics [10], [11], [12], theory of random processes and fields [7], [13] etc.

For instance, let \( \theta_n \) be \( w(n) \), \( w(n) \to \infty \) at \( n \to \infty \), \( n \) is volume of sample, consistent statistical estimate of an unknown parameter \( \theta \) for which

\[ || w(n)|\theta_n - \theta | ||_F \leq \sigma. \]

We can construct the confidence interval for the value \( \theta \) by means of inequality

\[ P(w(n)|\theta_n - \theta | \geq u) \leq T^{(F)}(u/\sigma). \]

### 2 Simple properties of Tchebychev’s characteristic. Examples.

**A.** Note that

\[
T^{(F)}(t) = \sup_{\xi: \xi \in F, ||\xi||_F \leq 1} T_\xi(t). \tag{2.1}
\]

Moreover,

\[
\sup_{\xi: \xi \in F, ||\xi||_F = C} T_\xi(t) = \sup_{\xi: \xi \in F, ||\xi||_F \leq C} T_\xi(t) = T^{(F)}(t/C), \; C = \text{const} > 0. \tag{2.2}
\]

**B.** Let on the space \((F, ||\cdot||)\) be an other norm |||\cdot||| for which

\[ |||\xi||| \geq C_1 |||\xi|||, \; 0 < C_1 = \text{const} < \infty. \]

Then

\[
T^{(F, ||\cdot||)}(t) \leq T^{(F, |||\cdot|||)}(t/C_1). \tag{2.3}
\]

Analogously, if
\[ ||\xi|| \leq C_2 \|\|\xi\|\|, \quad 0 < C_2 = \text{const} < \infty, \]

then

\[ T^{(F,\|\cdot\|)}(t) \geq T^{(F,\|\|\cdot\|\|)}(t/C_2). \quad (2.4) \]

**C. Definition 2.1.** A two tail functions \( T_1(t) \) and \( T_2(t) \) are equivalent, write: \( T_1(\cdot) \sim T_2(\cdot) \) iff there exist three finite positive constants \( t_0 > 0, \ 0 < C_1 \leq C_2 < \infty \) for which

\[ T_2(t/C_2) \leq T_1(t) \leq T_2(t/C_1), \quad t \geq t_0. \quad (2.5) \]

We will write also \( T_1(\cdot) \ll T_2(\cdot) \), iff

\[ T_1(t) \leq T_2(t/C_1), \quad t \geq t_0. \quad (2.6) \]

Evidently, the relation ” \ll ” is partial order on the set of all tail functions an the relation ” \sim ” is the relation of equivalence. Also if \( T_1 \ll T_2 \) and \( T_2 \ll T_1 \), then \( T_1 \sim T_2 \).

**Corollary 2.1.** If two norms on the space \( F \| \cdot \| \) and \( \|\| \cdot \|\| \) are equivalent in the usually sense, then

\[ T^{(F,\|\cdot\|)}(\cdot) \sim T^{(F,\|\|\cdot\|\|)}(\cdot). \quad (2.7) \]

As we will see further, the converse proposition is'nt true.

Recall that the measure space is said to be resonant, if it is non-atomic or conversely completely atomic with all the atoms having equal measure, see [1], chapter 2, section 7.

**D. Theorem 2.1.** Let the measure \( \mu(\cdot) \) be resonant; then for any r.i. space \( (F,\|\cdot\|) \)

\[ T^{(F,\|\cdot\|)}(t) \leq \frac{C_3(F)}{t}. \quad (2.8) \]

**Proof.** It is known, see [1], chapter 2, section 2, theorem 2.2 that in the considered case

\[ ||\xi||F \geq C_4|\xi|_1. \]

We use further Tchebychev’s inequality:

\[ T^{(L_1)}(t) \leq C_5/t, \quad t > 0. \]

The assertion of the theorem 2.1 follows from the inequality (2.3).

**E. Examples.** 1. Classical Lebesgue-Riesz spaces.

We have in the case \( \mu(\Omega) = \infty \) and \( 1 \leq p < \infty \):

\[ T^{(L_p)}(t) = t^{-p}, \quad t > t_0. \]

When \( \mu(\Omega) = 1 \)

\[ T^{(L_p)}(t) = \min \left(1, t^{-p}\right), \quad t > t_0. \quad (2.9) \]
Indeed, the upper estimate it follows from Tchebychev’s inequality; the lower estimate follows from the consideration of following example:

\[ P(\xi = t) = t^{-p}, \quad P(\xi = 0) = 1 - t^{-p}, \quad t > 1. \]

Obviously,

\[ T^{(L_\infty)}(t) = 0, \quad t > 1. \]

2. Generalized Lorentz space.
Let \( \mu = \mathbb{P} \) and let \( w = w(t) \), \( t > 0 \) be positive continuous strictly increasing function, \( \lim_{t \to \infty} w(t) = \infty \). A generalized Lorentz space \( L^w \) consists by definition on all the measurable functions \( \xi(\omega) \) with finite norm (more precisely, quasinorm)

\[ ||\xi||_{L^w} = \sup_{t > 0} [w(t) T(\xi)(t)]. \] (2.10)

We conclude as before

\[ T^{L^w}(t) = \min(1, 1/w(t)). \] (2.11)

**Remark 2.1.** We observe if \( w(t) = t^p \), then \( T^{L^w}(t) = T^{L^p}(t), \quad t > 1 \), but the spaces \( L^w \) and \( L^p \) are not isomorphic.

3 Tchebychev’s characteristic and fundamental functions. Regular r.i. spaces.

We study in this section the relations between Tchebychev’s characteristic and fundamental functions.

We impose on the measure \( \mu \) here the restriction that it is diffuse: for arbitrary measurable set \( B \) there is its measurable subset \( D \) such that

\[ \mu(D) = \mu(B)/2. \]

Recall that a fundamental function \( \phi_F(\delta), \delta \in (0, \infty) \) of the r.i. space \( (F, || \cdot ||) \) over the measurable space \( (\Omega, \mathcal{A}, \mu) \) may be defined as follows:

\[ \phi_F(\delta) \overset{def}{=} \sup_{D: \mu(D) \leq \delta} ||I(D)||F. \] (3.1)

Here and further \( I(D) = I(D, \omega) \) is an indicator function of the measurable set \( D \).

The application of fundamental function in the functional analysis, in particular, in the theory of interpolation of operators is described in [1], [8]; the application in the theory of approximation see in [15].

Many examples of fundamental functions for different r.i. spaces are computed in the books [1], [8]. For the so-called Grand Lebesgue Spaces the fundamental functions are investigated and calculated in [9], [16].
Let us consider for instance the case of Orlicz’s space $Or(N)$ over our measurable space, in which we assume the measure $\mu$ to be diffuse. We suppose also that the Young function $N = N(u)$ is in addition strictly monotonic on the positive semi-axes and continuous.

We will use in this article the Luxemburg norm in the space $Or(N)$:

$$||\xi||_{Or(N)} = \inf \left\{ k > 0, \int_{\Omega} N \left( \frac{|\xi(\omega)|}{k} \right) \mu(d\omega) \leq 1 \right\}. \quad (3.2)$$

The fundamental function of this space has a view

$$\phi_{Or(N)}(\delta) = \frac{1}{N^{-1}(1/\delta)}.$$

Hereafter $g^{-1}(t)$ will denote the inverse function to the function $g(\cdot)$.

Further, let $\xi \geq 0$, $||\xi||_{Or(N)} = 1$. Since the Young function $N = N(u)$ is strictly monotonic and continuous

$$\int_{\Omega} N(\xi(\omega)) \mu(d\omega) = 1,$$

therefore

$$T_\xi(t) \leq 1/N(t).$$

We conclude analogously to the Lebesgue-Riesz and Lorentz spaces considering the example

$$P(\xi_0 = t) = 1/N(t) = 1 - P(\xi_0 = 0), \ t = \text{const} : N(t) > 1,$$

for which

$$E_{N}(\xi_0) = 1,$$

that

$$T^{(Or(N))}(t) = 1/N(t), \ t > t_0. \quad (3.4)$$

**Definition 3.1.** The r.i. space $(F, || \cdot ||F)$ is said to be regular r.i. space, if

$$\left[ \frac{1}{\phi_F(1/t)} \right]^{-1} = \frac{1}{T^{(F)}(t)}. \ t > t_0. \quad (3.5)$$

The r.i. space $(F, || \cdot ||F)$ is said to be weak regular r.i. space, if

$$\left[ \frac{1}{\phi_F(1/t)} \right]^{-1} \simeq \frac{1}{T^{(F)}(t)}. \ t > t_0. \quad (3.6)$$

We have proved the following fact.

**Theorem 3.1.** The Orlicz’s space $Or(N)$ over our measurable space, in which we assume the measure $\mu$ to be diffuse and suppose also that the Young function $N = N(u)$ is in addition strictly monotonic increase and continuous, is regular r.i. space.
If we replace the Luxemburg norm on some equivalent, we obtain the weak regular space.

**Examples 3.1.** For the spaces $L_p$ over diffuse sigma-finite measure we have

$$
\phi_{L_p}(\delta) = \delta^{1/p}, \quad T^{(L_p)}(1/\delta) = \delta^p = \left[\phi_{L_p}(\delta)\right]^{-1}.
$$

Another examples of weak regular r.i. spaces are the classical Lorentz and Marcinkiewicz spaces.

## 4 Tchebychev’s characteristic of associate regular r.i. spaces.

Recall that the associate r.i. space $(F', \| \cdot \|_{F'})$ to the space $(F, \| \cdot \|_F)$ consists on all the measurable functions $g : \Omega \to \mathbb{R}$ with finite norm

$$
\|g\|_{F'} = \sup_{\xi : \|\xi\|_{F'} = 1} \left| \int_{\Omega} g(\omega) \, \xi(\omega) \, \mu(d\omega) \right|.
$$

(4.0)

Under some additional conditions (absolutely continuous norm etc.) the associate space may coincides with conjugate (dual) space $(F^*, \| \cdot \|_{F^*})$; for instance, it is true for Orlicz’s space $(\Omega, N(u))$ iff the Young function $N(u)$ satisfies the $\Delta_2$ condition.

**Theorem 4.1.** Assume again that $(\Omega, \mathcal{A}, \mu)$ is resonant measure space. Suppose also both the r.i. spaces $(F, \| \cdot \|_F)$, $(F', \| \cdot \|_{F'})$ are regular. Then

$$
\left[\frac{1}{T(F)}\right]^{-1}(t) \cdot \left[\frac{1}{T(F')}\right]^{-1}(t) = t, \quad t > 0.
$$

(4.1)

**Proof.** Since both the r.i. spaces $(F, \| \cdot \|_F)$, $(F', \| \cdot \|_{F'})$ are regular,

$$
\phi_F(\delta) = \left[\frac{1}{T(F)}\right]^{-1}\left(\frac{1}{\delta}\right), \quad \phi_{F'}(\delta) = \left[\frac{1}{T(F')}\right]^{-1}\left(\frac{1}{\delta}\right).
$$

(4.2)

We will use the known identity [1], chapter 2, section 5:

$$
\phi_F(\delta) \cdot \phi_{F'}(\delta) = \delta.
$$

(4.3)

It remains to substitute in equality (4.3) expressions (4.2) and write $t$ instead $1/\delta$.

**Corollary 4.1.** If $F' = F^*$, then

$$
\left[\frac{1}{T(F)}\right]^{-1}(t) \cdot \left[\frac{1}{T(F^*)}\right]^{-1}(t) = t, \quad t > 0.
$$

(4.4)

**Corollary 4.2.** If both the r.i. spaces $(F, \| \cdot \|_F)$, $(F', \| \cdot \|_{F'})$ are weakly regular, then

$$
\left[\frac{1}{T(F)}\right]^{-1}(t) \cdot \left[\frac{1}{T(F')}\right]^{-1}(t) \simeq t, \quad t > 0.
$$

(4.5)
Corollary 4.3. The condition of theorem 4.1 is satisfied if for example the space $F$ is Orlicz space with continuous strictly increasing Young function $N = N(u), \ u \geq 0$.

Corollary 4.4. Without the condition of resonance we can guarantee only the inequality

$$\left[ \frac{1}{T(F)} \right]^{-1}(t) \cdot \left[ \frac{1}{T(F')} \right]^{-1}(t) \geq t, \ t > 0. \quad (4.6)$$

This fact follows immediately from the inequality

$$\phi_F(\delta) \cdot \phi_F'(\delta) \geq \delta; \quad (4.7)$$

see also [1], chapter 2, section 5.

5 Tchebychev’s characteristic of the direct sum of r.i. spaces.

Definition 5.1. We define for two tail functions $T_1(\cdot), T_2(\cdot)$ the following operation:

$$T_1 \lor T_2(t) \overset{\text{def}}{=} \inf_{x \in [0,1]} [T_1(tx) + T_2(t(1-x))]. \quad (5.0)$$

Evidently, $T_1 \lor T_2(t)$ is again the tail function and $T_1 \lor T_2(t) = T_2 \lor T_1(t)$.

Let the r.i. spaces $(F, \| \cdot \|_F)$ and $(G, \| \cdot \|_G)$ over our measurable space have Tchebychev’s characteristic functions correspondingly $T(F)(t), T(G)(t)$. Let also a third space $H$ be a (direct) sum of this spaces: $H = F + G$.

Theorem 5.1.

$$\max \left( T(F)(t), T(G)(t) \right) \leq T(H)(t) \leq T(F)(t) \lor T(G)(t). \quad (5.1)$$

Proof. The left-hand side of bilateral inequality (5.1) is proved very simple. Let $f_0$ be a function (depending on the variable $t$) from the space $F$ such that $\|f\|_F = 1$, $T_{f_0}(t) = T(F)(t)$. Then we have for the function $h_0 = f_0 + 0 \in H: \|h_0\|_H = 1$ and $T_{h_0}(t) = T(F)(t)$, therefore

$$T(H)(t) \geq T(F)(t)$$

and analogously

$$T(H)(t) \geq T(G)(t).$$

We will prove now the right-hand inequality in (5.1).

Let $h : \Omega \to R$ be any function from the space $H$ with unit norm in this space. We can suppose without loss of generality by virtue of definition of sum of two spaces that exist two functions say $f, f \in F$ and $g, g \in G$ for which $h = f + g$ and

$$1 = \|h\|_H = \|f\|_F + \|g\|_G. \quad (5.2)$$
It follows from the equality (5.2) that

\[ ||f||_{F} \leq 1, \quad ||g||_{G} \leq 1 \]

and therefore

\[ T_{f}(t) \leq T^{(F)}(t), \quad T_{g}(t) \leq T^{(G)}(t). \]  \hspace{1cm} (5.3)

Let \( x \) be arbitrary number from the set \([0, 1]\) and \( y = 1 - x \). We have:

\[ T_{h}(t) \leq T_{f}(tx) + T_{g}(ty) \leq T^{(F)}(tx) + T^{(G)}(ty). \]

Since the value \( x \) is arbitrary in the closed interval \([0, 1]\), we conclude

\[ T_{h}(t) \leq \inf_{x \in [0, 1]} [T^{(F)}(tx) + T^{(G)}(t(1 - x))] = [T^{(F)} \vee T^{(G)}](t). \]  \hspace{1cm} (5.4)

Taking the supremum over \( h : ||h||_{H} = 1 \) we obtain

\[ T^{(H)}(t) \leq [T^{(F)} \vee T^{(G)}](t). \]  \hspace{1cm} (5.5)

This completes the proof of theorem 5.1.

**Example 5.1.** Let \( F, G \) be Orlicz’s spaces over probabilistic space with diffuse measure and with the Young functions correspondingly

\[ N_{F}(u) = |u|^{p_1} \log^{q_1}(e + |u|), \quad N_{G}(u) = |u|^{p_2} \log^{q_2}(e + |u|), \]

\( p_1, p_2 = \text{const} > 1, q_1, q_2 = \text{const} \). Then the space \( H = F + G \) is also the Orlicz’s space relative the Young function \( N_{h}(u) = \max(N_{F}(u), N_{G}(u)) \) and with the Tchebychev’s function

\[ T^{(H)}(t) \asymp \max(T^{(F)}(t), T^{(G)}(t)), \quad t > 1. \]

6 Tchebychev’s characteristic of Grand Lebesgue-Riesz spaces (GLS).

We recall first of all in this section for reader conventions some definitions and facts from the theory of GLS spaces.

Recently, see [2], [3], [4], [5], [6], [7], [9], [10], [11], etc. appears the so-called Grand Lebesgue Spaces \( GLS = G(\psi) = G_{\psi} = G(\psi; A, B), \quad A, B = \text{const}, A \geq 1, A < B \leq \infty, \)

spaces consisting on all the measurable functions \( f : X \to R \) with finite norms

\[ ||f||_{G(\psi)} \overset{\text{def}}{=} \sup_{p \in (A, B)} [||f||_{p}/\psi(p)]. \]  \hspace{1cm} (6.1)

Here \( \psi(\cdot) \) is some continuous positive on the open interval \((A, B)\) function such that
\[
\inf_{p \in (A, B)} \psi(p) > 0, \; \psi(p) = \infty, \; p \notin (A, B).
\]

We will denote
\[
\text{supp}(\psi) \overset{\text{def}}{=} (A, B) = \{ p : \psi(p) < \infty, \}
\]

The set of all \( \psi \) functions with support \( \text{supp}(\psi) = (A, B) \) will be denoted by \( \Psi(A, B) \).

This spaces are rearrangement invariant, see [1], and are used, for example, in the theory of probability [7], [10], [11]; theory of Partial Differential Equations [3], [6]; functional analysis [4], [5], [9], [11]; theory of Fourier series [10], theory of martingales [11], mathematical statistics [22], [23]; theory of approximation [15] etc.

Notice that in the case when \( \psi(\cdot) \in \Psi(A, \infty) \) and a function \( p \to p \cdot \log \psi(p) \) is convex, then the space \( G\psi \) coincides with some exponential Orlicz space.

Conversely, if \( B < \infty \), then the space \( G\psi(A, B) \) does not coincides with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc.

**Remark 6.1** If we introduce the discontinuous function
\[
\psi_r(p) = 1, \; p = r; \psi_r(p) = \infty, \; p \neq r, \; p, r \in (A, B)
\]
and define formally \( C/\infty = 0, \; C = \text{const} \in \mathbb{R}^1 \), then the norm in the space \( G(\psi_r) \) coincides with the \( L_r \) norm:
\[
\|f\|_{G(\psi_r)} = |f|_r.
\]

Thus, the Grand Lebesgue Spaces are direct generalization of the classical exponential Orlicz’s spaces and Lebesgue spaces \( L_r \).

**Remark 6.2** The function \( \psi(\cdot) \) may be generated as follows. Let \( \xi = \xi(x) \) be some measurable function: \( \xi : X \to \mathbb{R} \) such that \( \exists(A, B) : 1 \leq A < B \leq \infty, \forall p \in (A, B) |\xi|_p < \infty. \) Then we can choose
\[
\psi(p) = \psi_{\xi}(p) = |\xi|_p.
\]

Analogously let \( \xi(t, \cdot) = \xi(t, x), t \in T, \; T \text{ is arbitrary set, be some family } F = \{\xi(t, \cdot)\} \) of the measurable functions: \( \forall t \in T \; \xi(t, \cdot) : X \to \mathbb{R} \) such that
\[
\exists(A, B) : 1 \leq A < B \leq \infty, \sup_{t \in T} |\xi(t, \cdot)|_p < \infty.
\]
Then we can choose
\[
\psi(p) = \psi_{F}(p) = \sup_{t \in T} |\xi(t, \cdot)|_p.
\]

The function \( \psi_F(p) \) may be called as a natural function for the family \( F \). This method was used in the probability theory, more exactly, in the theory of random fields, see [10].

More detail investigations of tail and fundamental functions of GLS see in [10], [11], [9].

*We consider in this section only the cases \( \mu = P \) and \( B < \infty. \)
An important

**Example 6.1.** Let \( B = \text{const} > 1, \ \beta = \text{const} > 0 \) and let

\[
\psi_{B,\beta}(p) = (B - p)^{-\beta}, \ 1 \leq p < B
\]

and \( \psi_{B,\beta}(p) = \infty \) otherwise.

For instance: if \( \Omega = (0, 1), \ P = m \) and \( \xi_2(\omega) = \omega^{-1/2} \), then \( \xi_2(\cdot) \in G_{\psi_{2,1/2}}(\cdot) \).

Notice that for all positive values \( \epsilon < 0.5 \)

\[
\xi_2(\cdot) \notin G_{\psi_{2+\epsilon,1/2}}(\cdot) \cup G_{\psi_{2,1/2-\epsilon}}(\cdot)
\]

and that the function \( \psi_{2,1/2}(p) \) is equivalent to the natural function for the random variable \( \xi_2(\cdot) \).

**Lemma 6.1.** Denote

\[
\tilde{\psi} = p \cdot \log \psi(p), \ p \in [1, B).
\]

Proposition:

A. \( T^{(G(\psi))}(t) \leq \exp \left( -\tilde{\psi}^*(\log t) \right), \ t > 2. \) \hfill (6.5)

where \( h^*(\cdot) \) denotes the classical Young-Fenchel, or Legendre transform:

\[
h^*(x) = \sup_y (xy - h(y)).
\]

B. For the spaces \( G_{\psi_{B,\beta}}(\cdot) \) it true also the converse inequality up to dilation:

\[
T^{(G_{\psi_{B,\beta}})}(t) \geq \exp \left( -\tilde{\psi}^*(\log t / C(B, \beta)) \right), \ t > 2. \) \hfill (6.6)

**Proof.** A. Let \( ||\xi||_{G\psi} = 1; \) then \( ||\xi||_{p} \leq \psi(p), \ E|\xi|^p \leq \psi^p(p) \). We obtain using the Tchebychev’s inequality:

\[
T_\xi(t) \leq \exp \left( (p \log t - p \log \psi(p)) \right).
\]

The assertion (6.5) it follows after an optimization over \( p. \)

The proposition (6.6) is proved in the article [11]; see also [9].

**Example 6.2.** Denote \( \psi_m(p) = p^{1/m}, \ 1 \leq p < \infty, \ m = \text{const} > 0 \). Proposition:

\[
\xi \in G_{\psi_m}, \ \xi \not= 0 \Leftrightarrow \exists C = \text{const} \in (0, \infty), \ T_\xi(t) \leq \exp (-C t^m).
\]

We will formulate the main result of this section, which may be obtained after simple calculations basing on the lemma 6.1.

**Theorem 6.1.** There exists a non-regular r.i. space over the probabilistic space with diffuse measure, namely the space \( G_{\psi_{B,\beta}} \) with \( B > 1, \ \beta > 0. \)

**Proof.** Let us consider the space \( G_{\psi_{B,\beta}} \). In detail:

\[
T^{(G_{\psi_{B,\beta}})}(t) \asymp t^{-B} (\log t)^{\beta B}, \ t \to \infty,
\]
\[ \phi_{G_{\psi B,\beta}}(\delta) \asymp \delta^{1/B} |\log \delta|^{\beta}, \ \delta \to 0+, \]

so that at \( t \to \infty \)

\[ \left[ \frac{1}{\phi_{G_{\psi B,\beta}}(1/t)} \right]^{-1} \asymp t^{B} (\log t)^{\beta B}, \]

\[ \frac{1}{T^{(G_{\psi B,\beta})}(t)} \asymp t^{B} (\log t)^{-\beta B}. \]

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