ON VARIETIES OF GROUPS IN WHICH ALL PERIODIC GROUPS ARE ABELIAN

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Abstract. To solve a number of problems on varieties of groups, stated by Kleiman, Kuznetsov, Ol’shanskii, Shmel’kin in the 1970’s and 1980’s, we construct continuously many varieties of groups in which all periodic groups are abelian and whose pairwise intersections are the variety of all abelian groups.

1. Introduction

The first example of a nonabelian variety of groups in which all finite groups are abelian was constructed by Ol’shanskii [7]. This example provided a positive solution to a problem of Hanna Neumann [6, Problem 5] on the existence of such varieties of group and demonstrated that a nonabelian variety of groups might have a very limited intersection with the class of finite groups. Shmel’kin [5, Problem 4.73(b)] posed a strengthened version of this Hanna Neumann’s problem by asking about the existence of a nonabelian variety of groups in which all periodic groups are abelian. A positive solution to this problem of Shmel’kin was announced by the authors in [1], [9]. The aim of this article is to present details of the following construction which is sketched in [1, Theorem 3] and which solves Shmel’kin’s problem (together with a number of other problems mentioned below; for the sake of simplicity of proofs we change the identities of [1]).

Let Ω = (Ω(1), Ω(2), . . .) be an infinite sequence of numbers Ω(k) ∈ {0, 1}, k = 1, 2, . . ., let p_k be the kth prime number, q_k = (p_1 . . . p_k)k, and set

\[ v_k(x, y) = [x^d, y^d], x^{dq_k}, \]

(1)

\[ w_{\Omega,k}(x, y) = [x, y]^{\Omega(k)+\varepsilon_1 v_k(x, y)}^{d} [x, y]^{\varepsilon_2 v_k(x, y)}^{d} \ldots \]

\[ \ldots [x, y]^{\varepsilon_{h-1} v_k(x, y)}^{d} [x, y]^{\varepsilon_{h} v_k(x, y)}^{d}, \]

(2)

where \([a, b] = aba^{-1}b^{-1}\) is the commutator of a and b, h \equiv 0 (mod 10),

\[ \varepsilon_{10\ell+1} = \varepsilon_{10\ell+2} = \varepsilon_{10\ell+3} = \varepsilon_{10\ell+5} = \varepsilon_{10\ell+6} = 1, \]

\[ \varepsilon_{10\ell+4} = \varepsilon_{10\ell+7} = \varepsilon_{10\ell+8} = \varepsilon_{10\ell+9} = \varepsilon_{10\ell+10} = -1, \]

\(\ell = 0, 1, \ldots, h/10 - 1,\) and h, d, n are sufficiently large positive integers (with n \(\gg d \gg h \gg 1\)). The following is our main result.

Theorem. Let \(\Omega = (\Omega(1), \Omega(2), \ldots)\) be an infinite sequence of numbers \(\Omega(1), \Omega(2), \ldots \in \{0, 1\}\) that contains infinitely many 1’s and \(\mathfrak{M}_\Omega\) be the variety of groups defined by identities \(w_{\Omega,k}(x, y) = 1, k = 1, 2, \ldots,\) where the words \(w_{\Omega,k}(x, y)\) are

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given by formula (2). Then $\mathcal{M}_\Omega$ is a nonabelian variety of groups in which all periodic groups are abelian. Furthermore, if $\Omega_1 \neq \Omega_2$ then the intersection $\mathcal{M}_{\Omega_1} \cap \mathcal{M}_{\Omega_2}$ is the variety $\mathfrak{A}$ of all abelian groups.

Let $\mathcal{U}$ be a variety of groups. Recall that a variety of groups $\mathcal{V}$ is called just-non-$\mathcal{U}$ if $\mathcal{V}$ properly contains the variety $\mathcal{U}$, i.e. $\mathcal{U} \subset \mathcal{V}$, and there is no group variety $\mathcal{V}'$ such that $\mathcal{U} \subset \mathcal{V}' \subset \mathcal{V}$. It follows from Zorn’s lemma that every group variety $\mathcal{M}_\Omega$ of our Theorem contains a just-non-$\mathfrak{A}$ variety $\tilde{\mathcal{M}}_\Omega$, where $\mathfrak{A}$ is the variety of all abelian groups. Since the set of sequences $\Omega$ with infinitely many 1’s is continuous and $\tilde{\mathcal{M}}_{\Omega_1} \cap \tilde{\mathcal{M}}_{\Omega_2} = \mathfrak{A}$ if $\Omega_1 \neq \Omega_2$, we have the following.

Corollary. Let $\mathfrak{A}$ denote the variety of all abelian groups. Then the set of just-non-$\mathfrak{A}$ varieties of groups is continuous.

We remark that Kleiman [2] earlier found a solvable variety of groups $\mathfrak{K}$ such that the set of just-non-$\mathfrak{K}$ varieties is continuous. This Corollary provides a new insight into the structure of the lattice of group varieties and enables us to solve the following three problems posed in the 1970’s and 1980’s.

Problem 1 (Ol’shanskii, Problem 4.46(b) [5]). How many varieties are there that have no finite basis for their identities and in which any proper subvariety has a finite basis for its identities?

Problem 2 (Kuznetsov, Problem 6.16 [5]). How many varieties are there that contain only finitely many or countably many subvarieties?

Problem 3 (Kleiman, Problem 8.20 [5]). How many just-non-abelian (or just-non-nilpotent) varieties of groups are there?

The authors are grateful to A.Yu. Ol’shanskii for pointing out that Kozhevnikov [3], [4] earlier constructed continuously many just-non-$\mathfrak{A}_n$ varieties of groups, where $n \gg 1$ is odd and $\mathfrak{A}_n$ is the variety of all abelian groups of exponent $n$. In particular, this result of Kozhevnikov also implies that the sets of varieties of groups in Problems 1–3 are continuous. The authors also wish to thank the referee for useful remarks.

2. Proof of Theorem

It is fairly easy to see that all periodic groups in the variety $\mathcal{M}_\Omega$ of our Theorem are abelian. Indeed, it follows from the identity $w_{\Omega, k}(x, y) \equiv 1$, where $w_{\Omega, k}(x, y)$ is given by (2), that if $\Omega(k) = 1$ then the quasiidentity $x^{q_k} = 1 \rightarrow [x, y] = 1$ holds in the variety $\mathcal{M}_\Omega$. In particular, any element of finite order of a group $G \in \mathcal{M}_\Omega$ lies in the center of $G$. It is also clear that if $\Omega_1 \neq \Omega_2$, say $\Omega_1(k_0) \neq \Omega_2(k_0)$ for some $k_0$, then it follows from identities $w_{\Omega_1, k_0}(x, y) \equiv 1$, $w_{\Omega_2, k_0}(x, y) \equiv 1$ that $[x, y] \equiv 1$ which means that $\mathcal{M}_{\Omega_1} \cap \mathcal{M}_{\Omega_2} = \mathfrak{A}$. Therefore, the only nontrivial part of our Theorem is to show that $\mathcal{M}_\Omega$ is a nonabelian variety of groups.

To prove that $\mathcal{M}_\Omega$ is nonabelian, we will construct a presentation for a free group of rank $m > 1$ in $\mathcal{M}_\Omega$ by means of generators and defining relations and use the geometric machinery of graded diagrams, developed by Ol’shanskii [7], [8], to study this group. In particular, we will use the notation and terminology of [8] and all notions that are not defined in this paper can be found in [8].

As in [8], we will use numerical parameters
\[
\alpha \succ \beta \succ \gamma \succ \delta \succ \varepsilon \succ \zeta \succ \eta \succ \iota
\]
and \( h = \delta^{-1}, \ d = \eta^{-1}, \ n = i^{-1} \) (here \( h, d, n \) were already used in (1)-(2)) and employ the least parameter principle (LPP) (according to LPP a small positive value for, say, \( \zeta \) is chosen to satisfy all inequalities whose smallest (in terms of the relation \( \succ \)) parameter is \( \zeta \).

Let \( A = \{ a_1, \ldots, a_m \} \) be an alphabet, \( m > 1 \), and \( F(A) \) be the free group in \( A \). Elements of \( F(A) \) are referred to as words in \( A^{\pm 1} = A^* \cup A^{-1} \) or just words. Denote \( G(0) = F(A) \) and let the set \( R_0 \) be empty. Now consider an arbitrary infinite sequence \( \Omega = (\Omega(1), \Omega(2), \ldots) \), where \( \Omega(k) \in \{ 0, 1 \}, \ k = 1, 2, \ldots \) To define the group \( G(i) \) by induction on \( i \geq 1 \) for this given \( \Omega \), assume that the group \( G(i - 1) \) is already constructed by its presentation

\[
G(i - 1) = \langle A \mid R = 1, R \in R_{i-1} \rangle.
\]

Let \( X_i \) be a set of words (in \( A^{\pm 1} \)) of length \( i \), called periods of rank \( i \), which is maximal with respect to the following two properties:

(A1) If \( A \in X_i \) then \( A \) (that is, the image of \( A \) in \( G(i - 1) \)) is not conjugate in \( G(i - 1) \) to a power of a word of length \( < |A| = i \).

(A2) If \( A, B \) are distinct elements of \( X_i \) then \( A \) is not conjugate in \( G(i - 1) \) to \( B \) or \( B^{-1} \).

If the images of two words \( X, Y \) are equal in the group \( G(i - 1), i \geq 1 \), then we will say that \( X \) is equal in rank \( i - 1 \) to \( Y \) and write \( X \equiv_{i-1} Y \). Analogously, we will say that two words \( X, Y \) are conjugate in rank \( i - 1 \) if their images are conjugate in the group \( G(i - 1) \). As in [4], a word \( A \) is called simple in rank \( i - 1, i \geq 1 \), if \( A \) is conjugate in rank \( i - 1 \) neither to a power \( B^\ell \), where \( |B| = |A| \), nor to a power of period of some rank \( \leq i - 1 \). We will also say that two pairs \((X_1, X_2), (Y_1, Y_2)\) of words are conjugate in rank \( i - 1, i \geq 1 \), if there is a word \( W \) such that \( X_1 \equiv_{i-1} WY_1W^{-1} \) and \( X_2 \equiv_{i-1} WY_2W^{-1} \).

Consider the set of all possible pairs \((X, Y)\) of words in \( A^{\pm 1} \) and pick a positive integer \( k \). This set is partitioned by equivalence \( k \)-classes \( \mathcal{C}_A(k) \), \( \ell = 1, 2, \ldots \), of the equivalence relation \( \sim_k \) defined by \( (X_1, Y_1) \sim_k (X_2, Y_2) \) if and only if the pairs \((v_k(X_1, Y_1), w_{\Omega(k)}(X_1, Y_1))\) and \((v_k(X_2, Y_2), w_{\Omega(k)}(X_2, Y_2))\) are conjugate in rank \( i - 1 \).

It is convenient to enumerate (in some way)

\[
\mathcal{C}_{A, 1}(k), \mathcal{C}_{A, 2}(k), \ldots
\]

all \( k \)-classes of pairs \((X, Y)\) such that \( w_{\Omega(k)}(X, Y) \neq 1 \) and \( v_k(X, Y) \) is conjugate in rank \( i - 1 \) to some power \( A^f \), where \( A \in X_i \) and \( f = f(X, Y) \) are fixed.

It follows from definitions that every class \( \mathcal{C}_{A, j}(k) \) contains a pair

\[
(X^j_{A, j, k}, Y^j_{A, j, k})
\]

with the following properties. The word \( X^j_{A, j, k} \) is graphically equal (that is, letter-by-letter) equal to a power of \( B^j_{A, j, k} \), where \( B^j_{A, j, k} \) is simple in rank \( i - 1 \) or a period of rank \( \leq i - 1 \); \( Y^j_{A, j, k} = Z^j_{A, j, k} \), where \( \equiv \) means the graphical equality, \( Y^j_{A, j, k} \) is graphically equal to a power of \( C_{A, j, k} \), where \( C_{A, j, k} \) is simple in rank \( i - 1 \) or a period of rank \( \leq i - 1 \). We can also assume that if \( D_1 \in \{ A, B^j_{A, j, k}, C_{A, j, k} \} \) is conjugate in rank \( i - 1 \) to \( D_2 \equiv_{i-1} \), where \( D_2 \in \{ A, B^j_{A, j, k}, C_{A, j, k} \} \), then \( D_1 \equiv D_2 \). Finally, the word \( Z^j_{A, j, k} \) is picked for fixed \( X^j_{A, j, k}, Y^j_{A, j, k} \) so that the length \( |Z^j_{A, j, k}| \) is minimal (and the pair \( (X^j_{A, j, k}, Z^j_{A, j, k} Y^j_{A, j, k} Z^{-1}_{A, j, k}) \) belongs to \( \mathcal{C}_{A, j}(k) \)). Similar to [7], [8], the
triple \((X_{A^i,j,k}, Y_{A^i,j,k}, Z_{A^i,j,k})\) is called an \((A^i,j,k)\)-triple corresponding to the class \(\mathcal{C}_{A^i,j}(k)\) (in rank \(i-1\)).

Now for every class \(\mathcal{C}_{A^i,j}(k)\) we pick a corresponding \((A^i,j,k)\)-triple

\[(X_{A^i,j,k}, Y_{A^i,j,k}, Z_{A^i,j,k})\]

in rank \(i-1\) and define a defining word \(R_{A^i,j,k}\) of rank \(i\) as follows. Pick a word \(W_{A^i,j,k}\) of minimal length so that

\[v_k(X_{A^i,j,k}, Y_{A^i,j,k}) \equiv W_{A^i,j,k} A^{-1} W_{A^i,j,k}^{-1}.\]

Let \(T_{A^i,j,k}, U_{A^i,j,k}\) be words of minimal length such that

\[T_{A^i,j,k} \equiv W_{A^i,j,k}^{-1} X_{A^i,j,k} Y_{A^i,j,k} W_{A^i,j,k},\]

\[U_{A^i,j,k} \equiv W_{A^i,j,k}^{-1} X_{A^i,j,k} Y_{A^i,j,k}^2 W_{A^i,j,k}.\]

In accordance with (2), if \(\Omega(k) = 0\), then we set

\[R_{A^i,j,k} = T_{A^i,j,k}^{\varepsilon_1} A^{(d^k n + 1)} T_{A^i,j,k}^{\varepsilon_2} A^{(d^k n + 2)} \cdots T_{A^i,j,k}^{\varepsilon_h} A^{(d^k n + h)},\]  \(\text{and, if } \Omega(k) = 1, \text{ then we set}\)

\[R_{A^i,j,k} = U_{A^i,j,k} A^{(d^k n + 1)} U_{A^i,j,k} A^{(d^k n + 2)} \cdots U_{A^i,j,k} A^{(d^k n + h)},\]

where \(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_h, h, n\) are defined as in (2).

It follows from definitions that the word \(R_{A^i,j,k}\) is conjugate (by \(W_{A^i,j,k}^{-1}\)) in rank \(i-1\) to the word \(v_{0,k}(X_{A^i,j,k}, Y_{A^i,j,k}) \neq 1\).

The set \(S_i\) of defining words of rank \(i\) consists of all possible words \(R_{A^i,j,k}\) given by (3)–(4) (over all equivalence classes \(\mathcal{C}_{A^i,j}(k), A \in \mathcal{X}_i, k = 1, 2, \ldots\)). Finally, we put \(\mathcal{R}_i = \mathcal{R}_{i-1} \cup S_i\) and set

\[G(i) = \langle A \mid R = 1, R \in \mathcal{R}_i \rangle.\]  \(\text{The inductive definition of groups } G(i), i \geq 0, \text{ is now complete and we can consider the limit group } G(\infty) \text{ given by defining words of all ranks } j = 1, 2, \ldots\)

\[G(\infty) = \langle A \mid R = 1, R \in \bigcup_{j=0}^{\infty} \mathcal{R}_j \rangle.\]  \(\text{We will prove (in Lemma } 4) \text{ that } G(\infty) \text{ is the free group of the variety } \mathfrak{M}_1 \text{ in the alphabet } A, \text{ that is, } G(\infty) \text{ is naturally isomorphic to the quotient } F(A)/W_\infty(F(A)), \text{ where } W_\infty(F(A)) \text{ is the verbal subgroup of } F(A) \text{ defined by the set } W_\infty = \{w_{0,k}(x, y) \mid k = 1, 2, \ldots\}, \text{ and then show that } G(\infty) \text{ is not abelian. But first we need to study the presentation } 5 \text{ of } G(i). \text{ As in Sects. 29–30 } 5, \text{ following Lemmas } 1, 2 \text{ are proved by induction on } i \geq 0 (\text{whose base for } i = 0 \text{ is trivial}).\]

**Lemma 1.** The presentation 5 of \(G(i)\) satisfies the condition \(R\) of \(5\) Sect. 25].

**Proof.** This proof is quite similar to the proof of Lemma 29.4 \(5\). Inductive references to Lemmas 30.3, 30.4, 30.5 \(5\) (in rank \(i-1\)) are replaced by references to Lemma 4. Note that, by Lemma 4 and LPP, we have that

\[|f|(d^k n + h) \leq 100 \xi^{-1}(d^k n + h) < d^{k+1} n\]

(LPP: \(\delta = h^{-1} \succ \xi \succ \eta = d^{-1} \succ \iota = n^{-1}\)) which implies that, repeating the arguments of Lemma 29.3 \(5\), we can conclude that the defining relations \(R, R'\) correspond to the same value of \(k\) which enables us to finish the proof of the analogue of Lemma 29.3 as in \(5\). \(\square\)
Now suppose that $X, Y$ are some words with $[X, Y] \neq 1$ and $k$ is an arbitrary positive integer. Conjugating the pair $(X, Y)$ in rank $i$ if necessary, we can assume that $X = B^{f_a}$, $Y = ZC^{-fc}Z^{-1}$, where each of $B, C$ is either simple in rank $i$ or a period of some rank $\leq i$ and, when $B^{f_a}$, $C^{-fc}$ are fixed, the word $Z$ is picked to have minimal length. Furthermore, consider the following equalities

$$[X^d, Y^d] \doteq W_D D^{f_D} W_D^{-1},$$
$$[[X^d, Y^d], X^{d_n}] \doteq W_E E^{f_E} W_E^{-1},$$

where each of $D, E$ is either simple in rank $i$ or a period of some rank $\leq i$ and the conjugating words $W_D, W_E$ are picked (when $D, E$ are fixed) to have minimal length. Without loss of generality, we can also suppose that if $i = \Delta$ of rank $\leq i$ and, as in the proof of Lemma 25.19 [8], it follows from Lemmas 24.8 and inequalities (8)–(9), we have $|Z| < 15\zeta^{-2}|D^{f_D}|$, (7)

$$0 < |f_D| \leq 100\zeta^{-1},$$
(8)

$$\max(|B^{df_B}|, |C^{df_C}|) \leq \zeta^{-1}|D^{f_D}|,$$
(9)

$$|Z| < 31\zeta^{-2}|D^{f_D}|.$$  
(10)

**Lemma 2.** In the foregoing notation, the following inequalities hold

Proof. If $f_D = 0$, that is, $[X^d, Y^d] \doteq 1$ then, by Lemmas 1, 25.2 and 25.12 [8], we have $[X, Y] \doteq 1$, contrary to the choice of $X, Y$. Hence $f_D \neq 0$.

In view of equality $[B^{df_B}, ZC^{df_C}Z^{-1}] \doteq W_D D^{f_D} W_D^{-1}$, there is a reduced diagram $\Delta$ of rank $i$ on a thrice punctured sphere the labels of 3 cyclic sections of whose boundary $\partial \Delta$ are $B^{df_B}, B^{-df_B}, D^{f_D}$. If $|f_D| > 100\zeta^{-1}$ then $\Delta$ is a G-map (see Sect. 24.2 [8]) and, as in the proof of Lemma 25.19 [8], it follows from Lemma 24.8 [8] that $D^{f_D} \equiv 1$, contrary to $f_D \neq 0$. Hence, $|f_D| \leq 100\zeta^{-1}$ and inequalities [10] are proven.

If, say, $|D^{f_D}| < \zeta |B^{df_B}|$, then $\Delta$ is an E-map (see Sect. 24.2 [8]) and a contradiction to $f_D \neq 0$ follows from Lemma 24.6 [8] exactly as above. Hence, $|B^{df_B}| \leq \zeta^{-1}|D^{f_D}|$ and inequalities [8] are proven.

It follows from definitions and Lemma 30.2 [8] that

$$|Z| < 7\zeta^{-1}(|B^{df_B}| + |C^{df_C}| + |D^{f_D}|).$$

Then, in view of inequalities [8], we have

$$|Z| < 7\zeta^{-1}(2\zeta^{-1} + 1)|D^{f_D}| < 15\zeta^{-2}|D^{f_D}|,$$

as claimed in [9].

By estimates [8]–[10], we have

$$|[X^d, Y^d]| = 2(|B^{df_B}| + |C^{df_C}| + 2|Z|) < 2(2\zeta^{-1} + 30\zeta^{-2})|D^{f_D}| = 61\zeta^{-2}|D^{f_D}|.$$  

Hence, it follows from Lemmas [1] and 22.1 [8] that

$$|W_D| < (\gamma + \frac{1}{7})(|[X^d, Y^d]| + |D^{f_D}|) < 31\zeta^{-2}|D^{f_D}|$$

and Lemma 2 is proved. \qed
Lemma 3. In the foregoing notation, the following inequalities hold

\[ 0 < |f_E| \leq 100\zeta^{-1}, \quad (11) \]
\[ |D^{df_D}| \leq \zeta^{-1}|E^{f_E}|, \quad (12) \]
\[ |B^{dq_k,f_B}| \leq \zeta^{-1}|E^{f_E}|, \quad (13) \]
\[ |W_E| < 3\zeta^{-1}|E^{f_E}|. \quad (14) \]

Proof. Assume that \([X^d, Y^d]^i, X^{dq_k}] \leq 1. Then, by Lemmas 1, 25.2 and 25.12 \[S\], we have \([X^d, Y^d], X \leq 1\) and so \([Y^dX^dY^{-d}, X^d] \leq 1\). In view of Lemma 25.14 \[S\], we further have \([X^d, Y^d] \leq 1\). Then, as before, by Lemmas 1, 25.2, 25.12 \[S\], we obtain that \([X, Y] \leq 1\), contrary to the choice of \(X\) and \(Y\).

It follows from definitions that
\[ |W_D D^{df_D} W_D^{-1}, B^{dq_k,f_B}] \leq W_E E^{f_E} W_E^{-1} \quad (15) \]
and so there is a reduced diagram of rank \(i\) on a thrice punctured sphere the labels of 3 cyclic sections of whose boundary are \(D^{df_D}, D^{-df_D}, E^{f_E}\). Now we can repeat proofs of inequalities \((11)–(13)\) to obtain \((11)–(12)\).

In view of equality \((15)\), there is a reduced diagram of rank \(i\) on a thrice punctured sphere the labels of 3 cyclic sections of whose boundary are \(B^{dq_k,f_B}, B^{-dq_k,f_B}, E^{f_E}\). Now we can see that the proof of inequality \((13)\) is analogous to that of inequality \((12)\).

As in the proof of Lemma \[2\] we have from Lemmas \[11\] and 22.1 \[S\] that
\[ |W_E| < (\gamma + \frac{1}{2}) \cdot 2|W_D| + |D^{df_D}| + |B^{dq_k,f_B}| + \frac{1}{2}|E^{f_E}|. \]
Hence, by Lemma \[2\] and estimates \((12)–(13)\), we get
\[ |W_E| < (1 + 2\gamma) (62\zeta^{-2}d^{-1} + 1)|D^{df_D}| + |B^{dq_k,f_B}| + \frac{1}{2}|E^{f_E}| \leq 3\zeta^{-1}|E^{f_E}| \]
(LPP: \(\gamma > \zeta > \eta = d^{-1}\)) and Lemma \[3\] is proved. \(\square\)

Lemma 4. Let \(R_{A^t,j,k}\) be a defining word of rank \(i + 1\) defined by \[5\] if \(\Omega(k) = 0\) or by \[4\] if \(\Omega(k) = 1\). Then \(0 < |f| \leq 100\zeta^{-1}, |A| > d\), the words \(T_{A^t,j,k}, U_{A^t,j,k}\) do not belong to the cyclic subgroup \(\langle A \rangle\) of \(G(i)\) and
\[ \max(|T_{A^t,j,k}|, |U_{A^t,j,k}|) < d|A|. \]

Proof. It follows from definitions that, in the foregoing notation, we can assume that
\[ A = E, \]
\[ T_{A^t,j,k} = W_E^{-1}[B^{f_B}, ZC^{f_c} Z^{-1}] W_E, \]
\[ U_{A^t,j,k} = W_E^{-1}[B^{f_B}, ZC^{f_c} Z^{-1}]^2 W_E, \]
and \(f = f(A^t, j, k) = f_E\). Hence, in view of Lemmas \[2\] and \[3\] we have that
\[ 0 < |f| \leq 100\zeta^{-1}, \]
\[ f_E^{-1} E^{f_E} = |A| \geq 10^{-2}\zeta^2 |D^{df_D}| \geq 10^{-2}\zeta^3 d |B^{df_B}| \geq 10^{-2}\zeta^3 d^2 > d \]
(LPP: $\zeta \succ \eta = d^{-1}$) and

$$\max(|T_{A^f,j,k}|, |U_{A^f,j,k}|) \leq 2|W_E| + 8|Z| + 4|B^{f\alpha}| + 4|C^{f\alpha}| <
\frac{1}{6\zeta - 1 + 120\zeta - 3d - 1 + 8\zeta - 2d - 2)|E^{f\alpha}| < 7\zeta - 1 |E^{f\alpha}| < 700\zeta - 2 |E| < d|A|$$

(LPP: $\zeta \succ \eta = d^{-1}$).

Assume that one of $T_{A^f,j,k}, U_{A^f,j,k}$ belongs to $\langle E \rangle \subseteq G(i)$. Then one of $[B^{f\alpha}, ZC^{f\alpha} Z^{-1}]$, $[B^{f\alpha}, ZC^{f\alpha} Z^{-1}]^2$ is conjugate in rank $i$ to a power of $E$. However, by Lemmas 2 and 3

$$\max(|[B^{f\alpha}, ZC^{f\alpha} Z^{-1}]|, |[B^{f\alpha}, ZC^{f\alpha} Z^{-1}]^2|) < (120\zeta - 3d - 1 + 8\zeta - 2d - 2)|E^{f\alpha}| <
< 100\zeta - 1 (120\zeta - 3d - 1 + 8\zeta - 2d - 2)|E| < |E|$$

(LPP: $\zeta \succ \eta = d^{-1}$), whence $|T_{A^f,j,k}|, |U_{A^f,j,k}| < |E|$ which contradicts Lemmas 2 and 25.17 [8]. Lemma 4 is proved. \(\square\)

**Lemma 5.** The group $G(\infty)$, defined by presentation (4), is naturally isomorphic to the free group $F(A) / W_\Omega (F(A))$ of the variety $\mathcal{M}_\Omega$ in the alphabet $A$.

**Proof.** It follows from the definition of defining words of the group $G(\infty)$ that each of them is in $W_\Omega (F(A))$ and so there is a natural epimorphism $G(\infty) \to F(A) / W_\Omega (F(A))$.

Suppose that $\tilde{X}, \tilde{Y}$ are some words in $A^{\pm 1}$ and

$$w_{\Omega,k}(\tilde{X}, \tilde{Y}) \neq 1$$

in $G(\infty)$ for some integer $k > 0$. Let $A$ be a period of some rank such that $A^f$ for some $f$ is conjugate in $G(\infty)$ to $\nu_k(\tilde{X}, \tilde{Y})$. (The existence of such an $A$ follows from definitions; see also Lemma 18.1 [8].) Note that, in view of (10), $\tilde{X}, \tilde{Y} \neq 1$ in $G(\infty)$. Hence, by Lemmas 2 and 3 we can replace the pair $(\tilde{X}, \tilde{Y})$ by a conjugate in the group $G(\infty)$ pair $(X, Y)$ such that $X \equiv B^{k\alpha}$, $Y \equiv ZC^{k\alpha} Z^{-1}$, and $\nu_k(X, Y) = W_A A^f W_A^{-1}$ in $G(\infty)$, where $B, C$ are some periods, $|f| > 0$,

$$|X^d| + |Y^d| = |B^{d\alpha}| + |C^{dk\alpha}| + 2|Z| < (2\zeta - 3 + 30\zeta - 2)|D^{d\alpha}| < 31\zeta - 3d - 1 |A^f|,$n

and $|X^{d\alpha}| \leq \zeta - 1 |A^f|$.

Hence,

$$|\nu_k(X, Y)| \leq 2(2d(|X^d| + |Y^d|) + |X^{d\alpha}|) <
\frac{1}{2} (2d \zeta - 3 + \zeta - 1) |A^f| < 10^3 \zeta - 3 |A^f| \leq 10^5 \zeta - 4 |A|$$

for $0 < |f| \leq 100\zeta - 1$ by Lemma 3. Consider a reduced annular diagram $\Delta$ of some rank $i$ for conjugacy of $\nu_k(X, Y)$ and $A^f$. By Lemmas 4 and 22.1 [8], $\Delta$ can be cut into a simply connected diagram $\Delta_1$ along a simple path $t$ which connects points on distinct components of $\partial \Delta$ with $|t| < (\zeta - 1)|\partial \Delta|$. Therefore,

$$|\partial \Delta_1| < (1 + 2\zeta) |\partial \Delta| < (1 + 2\zeta)(10^5 \zeta - 4 + 100 \zeta - 1)|A| \leq \frac{1}{2} |\partial \Delta| |A|$$

(LPP: $\gamma \succ \zeta \succ \eta = d^{-1} \succ \tau = n^{-1}$). Then, by Lemmas 4 and 23.16 [8] applied to $\Delta_1$, the diagram $\Delta_1$ contains no 2-cells of rank $> |A|$, whence $\Delta_1, \Delta$ are diagrams of rank $|A| - 1$. Since $A \in X_{|A|}$, it follows from the construction of defining words of rank $|A|$ that there will be a defining word in $S_{|A|}$ which guarantees that
A contradiction to assumption (16) proves that $G(\infty)$ is in $\mathfrak{M}_{\Omega}$ and Lemma 5 is proved.

Proof of Theorem. By Lemma 6, $G(\infty)$ is the free group of $\mathfrak{M}_{\Omega}$ in $A$. Assume that $[a_1, a_2] = 1$ in $G(\infty)$. Then there exists an $i > 0$ such that $[a_1, a_2]^i = 1$. This, however, contradicts Lemmas 1 and 23.16 [8]. Thus, $G(\infty)$ is not abelian and Theorem is proved.

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