Solving the Brockett problem based on robust stabilization

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Abstract. The paper proposes a method for solving the Brockett problem based on the synthesis of robust modal control for systems with non-stationary parameters, which does not require complex adaptation algorithms that involve the introduction of a variable regulator matrix. A transition to the task of synthesizing the regulator is shown, using a certain optimality criterion with limitations, which provides similar values to those of the regulator coefficients. A practical numerical example of setting and solving this problem is presented, which confirms the effectiveness of the proposed method. To illustrate the results obtained, a root travel time curve was constructed for two systems: robust and non-robust.

1. Introduction
In dynamic control systems, internal facility parameters change over time. In the case of a modal controller, this fact significantly impairs the quality parameters of the regulation, up to and including loss of stability. There are classical adaptation techniques, which imply a change in the regulator parameters following the change in the characteristic polynomial coefficients [1-8]. The essence of the Brockett problem lies in the construction of an adaptive regulator $R(t)$, which would ensure the stabilization of the system when its parameters $B(t)$ are not stationary [9].

In this work we rejected the adaptive system and synthesized a robust controller with a matrix $R = \text{const}$, which is insensitive to changes in the internal parameters of the object.

2. Theoretical prerequisites for ensuring robustness
The theoretical prerequisites for ensuring the robustness of modal control methods will be analyzed using the example of an object $y^{(s)} + a_1 y + a_0 y = k u$, with a characteristic matrix presented in the Frobenius form:

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_{s-1} \end{bmatrix},$$

(1)

with a control matrix $N = \begin{bmatrix} 0 & 0 & \ldots & k \end{bmatrix}^T$ and with an output matrix: $A = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$:

$$\dot{x} = Bx + Nu;$$

$$y = Ax.$$

After introducing the modal controller into the feedback, the characteristic matrix of the system will take the following form:
The analysis of the matrix (3) shows that by choosing the controller coefficients in such a way that the condition: $k \cdot r_{i+1} \gg a_i$ is fulfilled, it is possible to ensure the robustness of the system to changes in the parameters $a_0, a_1, \ldots, a_{n-1}$.

The stated theoretical prerequisites only indicate the fundamental possibility of ensuring the robustness of the system, but do not indicate a way of implementation of this possibility, which is not obvious. Indeed, the typical definition of $n$ coefficients of the controller is carried out as a result of equating the $n$ coefficients of the characteristic equation to the $n$ coefficients of the desired polynomial. The solution of the resulting system of equations is unique and does not imply the fulfillment of the condition $k \cdot r_{i+1} \gg a_i$. As a result, ensuring the robustness of the system turns out to be impossible due to the absence of free variables.

3. A method for practical implementation of robust modal control

To introduce free variables into the synthesis problem, we will use the fact that the quality indicators of the system are provided by a small number of dominant roots, and there is no need to form the exact location of all $n$ roots of the characteristic equation. That means that you can set the desired location of only a part of the roots: $S_1^*, S_2^*, \ldots, S_m^* (m < n)$, and the position of the remaining $n - m$ roots will be determined by the measure of their distance $b$ from the desired roots in the form of an inequality: $S_j \leq b \cdot S_p^*(p = 1, n; j = n - m, n)$. Then the complete system of conditions for the implementation of robust modal control will take the following form:

$$c_i = d_i;$$
$$S_j < b \cdot S_p^*;$$
$$k \cdot \eta > q \cdot a_{i-1};$$

$$i = 1,n; p = 1,m; j = n - m, n,$$

where: $c_i$ signifies coefficients of the closed-loop system polynomial $|s \cdot E - B + N \cdot R|$, ($s$ is the Laplace operator); $d_i$ signifies the coefficients of the polynomial (5):

$$\left(s - S_1^*\right) \cdot \left(s - S_2^*\right) \cdot \ldots \cdot \left(s - S_m^*\right) \cdot \left(s - S_{m+1}\right) \cdot \ldots \cdot \left(s - S_n\right),$$

$S_j$ signifies free roots of the characteristic equation (5); $q$ is the coefficient that determines the degree (depth) of robustness.

System (4) contains $n$ equations and $n + (n-m)$ unknowns. Since $m < n$, then $2n - m > n$, then the number of unknowns is greater than the number of equations, and it becomes possible to fulfill conditions (4) not uniquely. Therefore, it is necessary to introduce some criterion for the solution of problem (4), which ensures its uniqueness.

Let us use the criterion:

$$F(r_1, r_2, \ldots r_n) = \sum_{i=1}^{n} \left(r_i - \sqrt{r_1 \cdot r_2 \cdot \ldots \cdot r_n}\right)^2 \rightarrow \min,$$

which provides close values of the regulator coefficients. Thus, the construction of a robust controller has been reduced to a mathematical programming problem.

4. Experimentation

Let us check the efficiency of the method using the example of a third-order object with the following matrices $B$, $N$, and $A$:
The characteristic equation of an object according to Laplace transform is 
\[ s^3 + 5s^2 + 3s + 1, \]
or, in the matrix form: 
\[ |s \cdot E - B|, \] where \( E \) is the identity matrix.

If we introduce a controller 
\[ R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \]
into the system, the characteristic equation \( PS(s) \) of the system will take the following form:
\[ PS(s) = |s \cdot E - B + N \cdot R| = s^3 + (r_3 + 5) \cdot s^2 + (r_2 + 3) \cdot s + r_1 + 1. \] (8)

Let us set the desired polynomial with two known roots \( S_1 = S_2 = -3 \) and with one unknown root \( S_3 \):
\[ PG(s) = (s+3)^2 \cdot (s-S_3). \]

Let us equate the coefficients at the same degrees \( s \) of the polynomials \( PS(s) \) and \( PG(s) \):
\[ s^3 + (r_3 + 5) \cdot s^2 + (r_2 + 3) \cdot s + r_1 + 1 = s^3 + (6-S_3) \cdot s^2 + (9-6 \cdot S_3) \cdot s - 9 \cdot S_3. \] (9)

Let us take \( b = 5; q = 10. \) As a result, the system of equations (4) with criterion (6) will take the following form:
\[
\begin{align*}
F(r_1, r_2, r_3) &= \sum_{i=1}^{3} \left( r_i - \sqrt[3]{r_1 \cdot r_2 \cdot r_3} \right)^2 \rightarrow \text{min}; \\
r_1 + 1 &= -9 \cdot S_3; \\
r_2 + 3 &= 9 - 6 \cdot S_3; \\
r_3 + 5 &= 6 - S_3; \\
k \cdot r_1 &= 10 \cdot a_0; \\
k \cdot r_2 &= 10 \cdot a_0; \\
k \cdot r_3 &= 10 \cdot a_0; \\
S_3 &\leq 5 \cdot S_1.
\end{align*}
\] (10)

The found solution for (10) is \( S_3 = -49, \ r_1 = 440, \ r_2 = 300, \ r_3 = 50. \)

5. Discussion of the results
Let us analyze the robustness of the synthesized control system. Let us construct a root-locus plot for two systems: a system synthesized with and without the robustness condition. For this purpose, the parameter was changed: \( a_2 = [2 \ldots 14]. \)

Fig. 1. Root-locus plots a) sensitive system, b) robust system
Fig. 1 shows that the area of localization of significant roots of a robust system is much smaller than the area of localization of the root-locus plot of a non-robust system, that is, the robustness of the system is achieved. At the same time, the roots $S_1$ and $S_2$ in a robust system retain their dominant value (the root $S_3$ is localized by the segment [-43, -54]), and the system provides the specified control time and the desired monotonic nature of the transient process.

6. Conclusion
An approach to solving the Brockett problem based on robust modal control has been considered. A method for constructing a robust modal controller has been proposed, which is based on the rejection of the a priori assignment of the full spectrum of eigenvalues of the characteristic matrix and the subsequent solution of the controller synthesis problem as a problem of finding the extremum of a special functional with boundary conditions.

References
[1] Yue X, 2019 Adaptive control for attitude coordination of leader-following rigid spacecraft systems with inertia parameter uncertainties // Chinese Journal of Aeronautics. V. 32, no 3, P. 688-700.
[2] Wong H, 2018 Robust control of the adaptive immune system // Seminars in Immunology. V. 36, P. 17-27.
[3] Shi Z, Zhao L and Liub Z, 2020, Variational method based robust adaptive control for a guided spinning rocket // Variational method based robust adaptive control for a guided spinning rocket.
[4] Rueda-Escobedo J. G. and Morenob J. A. 2020 Strong Lyapunov functions for two classical problems in adaptive control// Automatica A. 109250
[5] Zhu M, Wang X, Pei X, Zhang S, Dan Z, Gu S, Yang S, Miao K, Chen H and Liu J 2020 Modified robust optimal adaptive control for flight environment simulation system with heat transfer uncertainty// Chinese Journal of Aeronautics
[6] Arici M and Kara T 2020 Robust adaptive fault tolerant control for a process with actuator faults// Journal of Process Control V.92, P. 169-84
[7] Elsiisiab M and Soliman M, 2020 Optimal design of robust resilient automatic voltage regulators // ISA Transactions
[8] Deutscher J, 2017 Backstepping design of robust state feedback regulators for parabolic PIDEs with in-domain outputs // IFAC-PapersOnLine. V.50, I. 1, P. 5567-73.
[9] Brockett R A, 1999 Stabilization problem // Open Problems in Mathematical Systems and Control Theory. Springer. London. P. 75-8.