Mathematical analysis of a one-dimensional model for an aging fluid

David Benoît(1,2), Lingbing He(3), Claude Le Bris(1,2) and Tony Lelièvre(1,2)

(1) CERMICS, Ecole des Ponts (ParisTech), 6 & 8 Av. B. Pascal, 77455 Marne-la-Vallée, France.
(2) INRIA Rocquencourt, MICMAC team, B.P. 105, 78153 Le Chesnay Cedex, France.
(3) Department of Mathematical Sciences, Tsinghua University Beijing 100084, P. R. China.
{benoitd,lebris,lelievre}@cermics.enpc.fr lbhe@math.tsinghua.edu.cn

May 5, 2014

Abstract

We study mathematically a system of partial differential equations arising in the modelling of an aging fluid, a particular class of non Newtonian fluids. We prove well-posedness of the equations in appropriate functional spaces and investigate the longtime behaviour of the solutions.

1 Introduction

Our purpose is to study mathematically a system of partial differential equations arising in the modelling of some particular non Newtonian fluids. These fluids are often called aging fluids. Two physical phenomena are indeed permanently competing within the flow of such fluids. On the one hand, the fluid ages in the sense that it solidifies. On the other hand, aging is counterbalanced by a flow-induced rejuvenation.

The specific modelling we consider has been proposed in [4] on the basis of phenomenological arguments and experimental observations. A coefficient $f$, called the fluidity encodes aging for all times and at every location within the fluid. The fluid is solid where $f = 0$, and behaves all the more as a liquid when $f$ grows. Our mathematical study aims to contribute to better understand how well such a model captures the essential phenomena at play in fluid aging.

For our study, we proceed in a one-dimensional setting corresponding physically to the consideration of a laminar Couette flow. Our three unknown fields, the velocity $u$, the shear stress $\tau$ and the fluidity $f$ are defined as functions of a space variable $y$ varying in the interval $[0, 1]$. They are also, of course, functions of the time $t \geq 0$. The specific system we choose for our study reads

$$\begin{cases}
\rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, \\
\lambda \frac{\partial \tau}{\partial t} = G \frac{\partial u}{\partial y} - f \tau, \\
\frac{\partial f}{\partial t} = (-1 + \xi |\tau|) f^2 - \nu f^3.
\end{cases}$$

(1.1a) (1.1b) (1.1c)
Six dimensionless coefficients, all positive, constant in time and throughout the domain, are present in the system: the density $\rho$, the viscosity $\eta$, the characteristic relaxation time $\lambda$, the elastic modulus $G$, and two coefficients $\xi$ and $\nu$ specifically related to the equation for the evolution of the fluidity $f$. System (1.1) is a fully coupled system of three equations. The first two equations are classical in nature. The first one is the equation of conservation of momentum for $u$. The second equation rules the evolution of the shear stress $\tau$. The non-classical ingredient therein (as opposed, say, to an Oldroyd-B type equation) is the presence of an extra parameter, the fluidity $f$, the role of which is formally similar to that of an inverse time in a relaxation phenomenon. We note that when $f$ is a constant in time and throughout the domain, the equation agrees with the one-dimensional Oldroyd-B equation considered e.g. in [2, 5]. The third equation is of the form of one of the many such evolution equations suggested in [4]. We hope it is, in this respect, prototypical of a general class. It models the evolution of the fluidity $f$ in function of the stress tensor. The right-hand side of (1.1c) may differ from one model to another. The important ingredient is the presence of two competitive terms: a negative term modelling aging and a positive term modelling rejuvenation. For mathematical convenience, we have taken two particular instances of these two terms.

We examine well-posedness and longtime behaviour for system (1.1). Because we provide self-contained proofs, our study is rather long. Similar questions on a different, although related model for a viscoelastic fluid, have been examined in [9, 10].

Our article is articulated as follows.

To start with, we prove in Section 2 that the system under consideration admits a global-in-time solution in appropriate functional spaces. The solution is shown to be unique, and indeed strong. System (1.1) is thus satisfied in a classical sense. Our precise statement is the object of Theorem 2.1. The bulk of Section 2 consists of our proof. The arguments are standard arguments of mathematical fluid dynamics: formal a priori estimates, approximation, rigorous a priori estimates, convergence. The many nonlinearities present in system (1.1) however prevent us, in the current state of our understanding, from extending our analysis to settings in dimensions higher than or equal to two. Technically, this is related to the fact we repeatedly use, in our arguments, that $H^1$ functions are $L^\infty$ functions, a specificity of the one-dimensional setting of course.

In Sections 3 and 4, we study the long time behaviour of the solution. Section 3 deals with return to equilibrium. We supply the system with homogeneous Dirichlet boundary conditions for the velocity and investigate whether the flow converges to a steady state. For homogeneous Dirichlet boundary conditions, the steady states are $(u \equiv 0, \tau \equiv c, f \equiv 0)$, where $c$ is a constant throughout the domain. The long-time convergence to these steady-states sensitively depends, in system (1.1), of the fluidity $f$. The situation is qualitatively different depending on the fluidity $f_0$ at initial time. The more delicate, but of course more interesting, case mathematically is the case where the fluidity $f_0$ at initial time does not vanish everywhere: a part of the material, possibly the whole of it, is originally fluid. Section 3.1 addresses this case. We show (and the proof is quite substantial even in the one-dimensional setting we consider) that the flow converges to the null steady-state in suitable functional norms. The precise statement is the purpose of Theorem 3.1. The convergence is then shown to be polynomial in time, for all three fields $u$, $\tau$ and $f$. The rates of convergence are made precise in Theorem 3.2. Numerical simulations we perform in
Section 5 will show these rates are indeed sharp. It is interesting to emphasize the physical signification of our mathematical results. With regard to modelling, the convergence of the fluidity $f$ to zero that we establish, under homogeneous Dirichlet boundary conditions, means that when left at rest, the fluid progressively solidifies, a certainly intuitive fact. In addition, for $u$ and $\tau$, the rate of convergence sensitively depends on the size of the region where, originally, the material is liquid (a size measured by our parameter $\beta$ defined in (3.17) and present in the right-hand sides of the estimates of Theorem 3.2). The larger the liquid region the quicker the convergence of both the velocity and the shear stress to zero. It is not completely clear to us whether the latter qualitative behaviour is or not compatible with experimental observations or physical intuition.

If the material is entirely solid at initial time, that is $f_0 \equiv 0$ everywhere, the behaviour is quite different. Then the material stays solid for all times, while the velocity and shear stress vanish exponentially fast. We present the simple analysis of this behaviour in Section 3.2. Note that the result agrees with simple physical intuition.

Non-homogeneous boundary conditions, studied throughout Section 4, are, as always for questions related to long-time behaviours, significantly more intricate to address. We adopt constant boundary conditions, respectively $u = 0$ and $u = a > 0$ at $y = 0$ and $y = 1$. We begin by showing in Section 4.1 that, when we impose that the fluidity is strictly positive everywhere, there exists a unique steady state. We next show in Section 4.2 that this steady state is stable under small perturbations. Our precise result is stated in Theorem 4.1. When the perturbations of the state are not small, analyzing return to equilibrium is, in general, beyond our reach. We are however able to show that, when we assume a particular form of the initial condition (namely linear velocity, constant shear stress, constant positive fluidity), then return to equilibrium does hold true even if the initial condition is not close to the steady state. Some suitable assumptions relating the size of the parameters in system (1.1) and the non-zero boundary condition $a$ are also needed (see condition (4.21)). Our precise result is Theorem 4.2. The reason why we have to assume this specific form of the initial conditions is purely technical (and our numerical simulations will actually show that these restrictions are, in practice, unnecessary). In that case, system (1.1) reduces to a two-dimensional system of ordinary differential equations, for which Poincaré-Bendixson Theory allows us to understand the longtime behaviour. Our study is performed in Section 4.3.

As briefly mentioned above, Section 5 presents some numerical simulations. We first show that the rates of convergence estimated by our various mathematical arguments in the various regimes considered in Sections 3 and 4 are indeed sharp. We also investigate numerically the stability of the steady state. Our simulations show that, irrespective of the size of the initial perturbation (and thus in a more general regime than that for our mathematical arguments), the fluid returns to equilibrium, or more generally converges to the suitable steady state. The rates of convergence are also examined.

We conclude this introduction by mentioning that, despite their limitations, our results show that the model derived in [4] does adequately account for aging and rejuvenation. However, two shortcomings need to be emphasized. Both originate from the mathematical nature of equation (1.1c) (and are actually related to the fact that the Cauchy Lipschitz theory applies to this equation). First, when $f$ vanishes, then $f$ remains zero for all subsequent times. This property, present everywhere in our mathematical study, pre-
vents fluidification to occur after solidification. This clearly limits the range of materials covered by the modelling (compare muds and concrete, say). Second, \( f \) can only vanish asymptotically and never in finite time unless it is already zero before. Otherwise stated, solidification can occur, but never in finite time: again a modelling limitation. The one usefulness, if any, of our study, is therefore to point out that a mathematically well founded model where fluidification and solidification compete on an equal footing is still to be derived. Our study implicitly points to suitable directions to this end.

Further mathematical investigations on models for aging fluids will be presented in [1].

2 Global existence and uniqueness

In this section, we establish the following global existence and uniqueness result for system (1.1) supplied with initial conditions \( u_0, \tau_0, f_0 \) and the boundary conditions \( u(t, 0) = 0 \) and \( u(t, 1) = a \geq 0 \) for all time \( t \in [0, T] \) (where \( a \) is a constant scalar).

**Theorem 2.1** Recall that \( \Omega \) is the one-dimensional domain \([0, 1]\) and that \( T > 0 \) is fixed. Consider the initial data

\[
(u_0, \tau_0, f_0) \in H^1(\Omega)^3 \quad \text{with} \quad f_0 \geq 0.
\]

Then there exists a unique global solution \((u, \tau, f)\) to system (1.1) such that for any \( T > 0 \),

\[
(u, \tau, f) \in \left( C([0, T]; H^1) \cap L^2([0, T]; H^2) \right) \times C([0, T]; H^1) \times C([0, T]; H^1)
\]

and \( f \geq 0 \) for all \( x \in \Omega \) and \( t \in [0, T] \).

In addition, we have

\[
\left( \frac{\partial u}{\partial t}, \frac{\partial \tau}{\partial t}, \frac{\partial f}{\partial t} \right) \in L^2([0, T]; L^2) \times C([0, T]; L^2) \times C([0, T]; L^1),
\]

so that the equations in (1.1) are all satisfied in the strong sense in time.

Before we get to the proof, we eliminate the non-homogeneous Dirichlet boundary condition, introducing the auxiliary velocity field

\[
u(t, y) - ay.
\]

This velocity field, which we still denote by \( u \), solves the system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, \\
\frac{\lambda}{\partial t} \frac{\partial \tau}{\partial t} &= G \frac{\partial u}{\partial y} - f \tau + Ga, \\
\frac{\partial f}{\partial t} &= \left( -1 + \xi |\tau| \right) f^2 - \nu f^3,
\end{align*}
\]

supplied with the homogeneous Dirichlet boundary conditions \( u(t, 0) = 0 \) and \( u(t, 1) = 0 \) for all time \( t \in [0, T] \) and initial conditions \( u_0, \tau_0, f_0 \in H^1(\Omega) \). The proof of Theorem 2.1 will actually be completed on system (2.4). The result on (1.1) then immediately follows.

**Proof.** The proof falls in eight steps. The first five steps consist in deriving formal a priori estimates. These estimates are next made rigorous for a sequence of approximate solution in Step 6. The convergence of this sequence is proven in Step 7 thereby establishing existence of a solution to (2.4). Step 8 addresses uniqueness.
Step 1: Non-negativity of the fluidity. Let us first formally prove that \( f \geq 0 \).

Fix \( y \in \Omega \) and introduce

\[
E_0 = \{ y \in \Omega, f_0(y) > 0 \}.
\]

For \( y \in \Omega \setminus E_0 \), we have \( f_0(y) = 0 \) and thus \( f(t, y) = 0 \) for all time \( t \in [0, T] \) because of (2.4c). On the other hand, for \( y \in E_0 \), we now show that \( f(t, y) > 0 \) for all time \( t \in [0, T] \). We argue by contradiction and suppose, by continuity of \( f(\cdot, y) \), that

\[
t_m = \inf \{ t \in (0, T), f(t, y) = 0 \} < T.
\]

The Cauchy-Lipschitz Theorem applied to (2.4c) with zero as initial condition at time \( t_m \) implies that \( f(t, y) = 0 \) for \( t \in (t_m - \varepsilon, t_m + \varepsilon) \) for \( \varepsilon > 0 \), which contradicts the definition of \( t_m \).

We have therefore shown that \( f \) stays zero where it is zero, and stays positive where it is positive, which in particular implies non-negativity everywhere.

Step 2: Formal first energy estimates. We again argue formally. We first multiply the evolution equation (2.4a) on \( u \) by \( u \) itself and integrate over the domain. This gives a first estimate

\[
\frac{1}{2} \rho \frac{d}{dt} \| u(t, \cdot) \|^2_{L^2} + \eta \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|^2_{L^2} = \int_{\Omega} \left( \frac{\partial \tau}{\partial y} u \right)(t, \cdot).
\]

(2.5)

Similarly, we multiply the evolution equation (2.4b) by \( \tau \) and integrate over \( \Omega \) to find

\[
\frac{1}{2} \lambda \frac{d}{dt} \| \tau(t, \cdot) \|^2_{L^2} + \left\| \sqrt{f} \tau(t, \cdot) \right\|^2_{L^2} = G \int_{\Omega} \left( \frac{\partial u}{\partial y} \right)(t, \cdot) + G a \bar{\tau}(t),
\]

(2.6)

where we denote by

\[
\bar{q}(t) = \int_{\Omega} q(t, y) dy
\]

(2.7)

the average over \( \Omega \) of a function \( q : (t, y) \in [0, T] \times \Omega \to \mathbb{R} \).

Combining estimates (2.5) and (2.6) and using integration by parts and the fact that \( u \) vanishes on the boundary, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( G \rho \| u(t, \cdot) \|^2_{L^2} + \lambda \| \tau(t, \cdot) \|^2_{L^2} \right) + G \eta \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|^2_{L^2} + \left\| \sqrt{f} \tau(t, \cdot) \right\|^2_{L^2} = G a \bar{\tau}(t).
\]

(2.8)

We now turn to (2.4c). Integrating (2.4c) over \( \Omega \) yields

\[
\frac{d}{dt} \| f(t, \cdot) \|_{L^1} + \| f(t, \cdot) \|^2_{L^2} + \nu \| f(t, \cdot) \|^3_{L^3} = \xi \int_{\Omega} (|\tau|f^2)(t, \cdot).
\]

(2.9)

The Young inequality

\[
\xi |\tau|^2 = \sqrt{\nu} f^2 \cdot \frac{\xi}{\sqrt{\nu}} |\tau|f^2 \leq \frac{\nu}{2} f^3 + \frac{\xi^2}{2\nu} f^2
\]
then yields
\[
\frac{d}{dt} \| f(t, \cdot) \|_{L^1} + \| f(t, \cdot) \|_{L^2}^2 + \frac{\nu}{2} \| f(t, \cdot) \|_{L^3}^3 \leq \frac{\xi^2}{2\nu} \left\| \sqrt{f} \tau \right\|_{L^2}^2. \tag{2.10}
\]
Collecting (2.8) and (2.10), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( G\rho \| u(t, \cdot) \|_{L^2}^2 + \lambda \| \tau(t, \cdot) \|_{L^2}^2 + \frac{2\nu}{\xi^2} \| f(t, \cdot) \|_{L^1} \right) + G\eta \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{1}{2} \left\| \left( \sqrt{f} \tau \right)(t, \cdot) \right\|_{L^2}^2 \leq C a \| \tau(t, \cdot) \|_{L^2}^2 \tag{2.11}
\]
where \( C \), here and throughout our text, denotes a constant, the actual value of which is independent from \( T \) and only depends on the domain \( \Omega \) and the coefficients \( \rho, \eta, \lambda, G, \xi, \nu \) in (2.4).

Applying the Gronwall Lemma to (2.11), we obtain
\[
\sup_{t \in [0, T]} \left( \| u(t, \cdot) \|_{L^2}^2 + \| \tau(t, \cdot) \|_{L^2}^2 + \| f(t, \cdot) \|_{L^1} \right) + \int_0^T \left( \| u(t, \cdot) \|_{H^1}^2 + \left\| \left( \sqrt{f} \tau \right)(t, \cdot) \right\|_{L^2}^2 \right) dt \leq C_{0,T}, \tag{2.12}
\]
where \( C_{0,T} \) is a constant depending not only on \( \Omega, \rho, \eta, \lambda, G, \xi, \nu \), but also on the boundary condition \( a \), the initial data \( u_0, \tau_0, f_0 \) and the time \( T \).

**Remark 2.1** For homogeneous boundary conditions, that is \( a = 0 \), we mention that the right-hand sides of (2.8) and thus (2.11) vanish. The constant \( C_{0,T} \) in (2.12) therefore does not depend on \( T \) and we get a bound uniform in time.

**Step 3: A priori estimates on an auxiliary function.** Denote
\[
g(t, y) = \int_0^y (\tau(t, x) - \bar{\tau}(t)) dx.
\]
This function \( g \) satisfies homogeneous Dirichlet boundary conditions and formally solves
\[
\frac{\partial^2 g}{\partial y^2} = \frac{\partial \tau}{\partial y}.
\]
Using (2.4a) and (2.4b), which respectively imply
\[
\rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2}{\partial y^2} \left( u + \frac{1}{\eta} g \right),
\]
and
\[
\lambda \frac{\partial g}{\partial t} = - \int_0^y (f \tau - \bar{f} \tau) dx + Gu,
\]
we remark that the auxiliary function
\[ U = u + \frac{1}{\eta} \int_0^y (\tau - \bar{\tau}) \]  
\[ = u + \frac{1}{\eta} g \]  
(2.13)
solves:
\[ \frac{\partial U}{\partial t} = \frac{\eta}{\rho} \frac{\partial^2 U}{\partial y^2} - \frac{1}{\lambda \eta} \int_0^y (f \tau - \bar{f} \bar{\tau}) \, dx + \frac{G}{\lambda \eta} u. \]  
(2.14)

Multiplying equation (2.14) by \( \frac{\partial^2 U}{\partial y^2} \) and integrating over \( \Omega \) yields
\[ \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial U}{\partial y} (t, \cdot) \right\|_{L^2}^2 + \frac{\eta}{\rho} \left\| \frac{\partial^2 U}{\partial y^2} (t, \cdot) \right\|_{L^2}^2 \leq C \left( \left\| (f \tau)(t, \cdot) \right\|_{L^1} \int_\Omega \left\| \frac{\partial^2 U}{\partial y^2} \right\|_{(t, \cdot)} + \int_\Omega \left\| u \frac{\partial^2 U}{\partial y^2} \right\|_{(t, \cdot)} \right), \]  
(2.15)
after elementary manipulations in the right-hand side. Then using the Young and the Cauchy-Schwartz inequalities, we obtain
\[ \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial U}{\partial y} (t, \cdot) \right\|_{L^2}^2 + \frac{\eta}{2 \rho} \left\| \frac{\partial^2 U}{\partial y^2} (t, \cdot) \right\|_{L^2}^2 \leq C \left( \left\| (f \tau)(t, \cdot) \right\|_{L^1} \left\| (\sqrt{f} \tau)(t, \cdot) \right\|_{L^2}^2 + \left\| u(t, \cdot) \right\|_{L^2}^2 \right). \]  
(2.15)

Using (2.12), we know that the right-hand side is \( L^1(0, T) \). In view of our regularity assumptions on the initial conditions, we have \( \frac{\partial U}{\partial y} \big|_{t=0} = \frac{\partial u_0}{\partial y} + \frac{1}{\eta} (\tau_0 - \bar{\tau}_0) \in L^2(\Omega) \). We therefore deduce from (2.15) that
\[ U \in L^\infty([0, T], H^1_0) \cap L^2([0, T], H^2). \]  
(2.16)

**Step 4: \( L^\infty \) estimates.** We are now in position to obtain (again formal) \( L^\infty \)-bounds on \( \tau \) and \( f \). We consider the evolution equation (2.4b), which we rewrite in terms of \( U \) defined by (2.13) and using \( \bar{\tau} \) defined for \( \tau \) as in (2.7):
\[ \lambda \frac{\partial \tau}{\partial t} = G \frac{\partial U}{\partial y} - \left( f + \frac{G}{\eta} \right) \tau + \frac{G}{\eta} \bar{\tau} + Ga. \]

Multiplying this equation by \( \tau \), we obtain
\[ \frac{\lambda}{2} \frac{d}{dt} |\tau|^2 + \left( f + \frac{G}{\eta} \right) |\tau|^2 \leq C \left( |\tau| \cdot \left\| \frac{\partial U}{\partial y} \right\| + |\tau| \cdot \left\| \tau \right\|_{L^2} + a |\tau| \right), \]
so that, repeatedly applying the Young inequality,
\[ \frac{\lambda}{2} \frac{d}{dt} |\tau|^2 + \left( f + \frac{G}{2\eta} \right) |\tau|^2 \leq C \left( \left\| \frac{\partial U}{\partial y} \right\|^2 + \left\| \tau \right\|_{L^2}^2 + a \right). \]  
(2.17)
We apply the Gronwall Lemma to (2.17) and use $\frac{\partial U}{\partial y} \in L^2([0, T], L^\infty)$ because of (2.16), estimate (2.12) and $\tau_0 \in H^1(\Omega)$ to obtain

$$\|\tau(t, \cdot)\|_{L^\infty} \leq C_{0,T} \tag{2.18}$$

that is, $\tau \in L^\infty([0, T], L^\infty)$.

As for the function $f$, using the Duhamel formula for the evolution equation (2.4c) rewritten as

$$\frac{\partial f}{\partial t} = (-f - \nu f^2) f + \xi |\tau| f^2,$$

we obtain, for almost all $y \in \Omega$,

$$f(t, y) = e^{-\int_0^t (f + \nu f^2)(s, y) ds} f_0(y) + \xi \int_0^t e^{-\int_s^t (f + \nu f^2)(s', y) ds'} |\tau| f^2 (s, y) ds$$

$$\leq f_0(y) + \frac{\xi}{\nu} \|\tau\|_{L^\infty([0, T], L^\infty)} \int_0^t e^{-\int_s^t \nu f^2(s', y) ds'} \nu f^2(s, y) ds,$$

where we have used the non-negativity of $f$ and the previously derived $L^\infty$-bound on $\tau$ to obtain the second line. The above equation leads to

$$f(t, y) \leq f_0(y) + \frac{\xi}{\nu} \|\tau\|_{L^\infty([0, T], L^\infty)} \left(1 - e^{-\int_0^t \nu f^2(s, y) ds}\right)$$

$$\leq f_0(y) + \frac{\xi}{\nu} \|\tau\|_{L^\infty([0, T], L^\infty)} \tag{2.19}.$$

Using that $f_0 \in H^1$ and that we work in a one-dimensional setting, we obtain that $f \in L^\infty([0, T], L^\infty)$.

**Remark 2.2** For homogeneous boundary conditions, the Gronwall Lemma applied to (2.17) implies

$$\|\tau(t, \cdot)\|^2_{L^\infty} \leq \|\tau_0\|^2_{L^2} e^{-\frac{C_{0,T} t}{\nu}} + \int_0^t \left\|\frac{\partial U}{\partial y}(s, \cdot)\right\|^2_{L^\infty} ds + \sup_{t \in [0, T]} \|\tau(t, \cdot)\|^2_{L^2} \int_0^t e^{-\frac{C_{0,T}}{\nu}(s-t)} ds. \tag{2.20}$$

Moreover, as explained at the end of Step 4, the constant $C_{0,T}$ in (2.12) does not depend on $T$. Hence, the right-hand side of (2.15) and the bound in $L^2([0, T]; L^\infty)$-norm for $\frac{\partial U}{\partial y}$, deduced from (2.16), also do not depend on $T$. It follows from (2.20) that the $L^\infty$-bound (2.18) on $\tau$ is uniform in time. Equation (2.19) yields a similar conclusion for the bound on $f$.

**Step 5: Second a priori estimates.** In order to get estimates on higher order derivatives, we now differentiate with respect to $y$ the evolution equation (2.4b) and obtain

$$\lambda \frac{\partial}{\partial t} \frac{\partial \tau}{\partial y} = G \frac{\partial^2 u}{\partial y^2} - f \frac{\partial \tau}{\partial y} - f \frac{\partial f}{\partial y}$$

$$= G \frac{\partial U}{\partial y^2} - \frac{G}{\eta} \frac{\partial \tau}{\partial y} - f \frac{\partial \tau}{\partial y} - f \frac{\partial f}{\partial y}. \tag{2.21}$$
Likewise, we differentiate with respect to \( y \) the evolution equation (2.4c) and get
\[
\frac{\partial}{\partial t} \left( \frac{\partial f}{\partial y} \right) = \xi |\tau| f^2 + 2(\xi |\tau| - 1) f \frac{\partial f}{\partial y} - \mu f \frac{\partial f}{\partial y}.
\] (2.22)

Multiplying equations (2.21) and (2.22) respectively by \( \frac{\partial \tau}{\partial y} \) and \( \frac{\partial f}{\partial y} \), integrating over the domain, summing up and using that both \( \tau \) and \( f \) are in \( L^\infty([0, T], L^2) \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \lambda \left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|^2_{L^2} + \left\| \frac{\partial f}{\partial y}(t, \cdot) \right\|^2_{L^2} \right) \leq C_{0,T} \int_\Omega \left( \frac{\partial^2 U}{\partial y^2} \frac{\partial \tau}{\partial y} + \left( \frac{\partial \tau}{\partial y} \right)^2 + \frac{\partial f}{\partial y} \frac{\partial \tau}{\partial y} + \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} + \left( \frac{\partial f}{\partial y} \right)^2 \right)(t, \cdot).
\] (2.23)

Repeatedly applying the Young inequality and using that \( \tau \) belongs to \( L^2([0, T], H^1) \) (which implies \( \left\| \frac{\partial \tau}{\partial y}(t, y) \right\| \leq \left\| \frac{\partial \tau}{\partial y}(t, y) \right\|_{L^2} \) for almost all \( t \in [0, T] \) and \( y \in \Omega \)), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \lambda \left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|^2_{L^2} + \left\| \frac{\partial f}{\partial y}(t, \cdot) \right\|^2_{L^2} \right) \leq C_{0,T} \left( \left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|^2_{L^2} + \left\| \frac{\partial f}{\partial y}(t, \cdot) \right\|^2_{L^2} + \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|^2_{L^2} \right).
\] (2.23)

We apply the Gronwall Lemma to (2.23), use that \( \tau_0, f_0 \in H^1(\Omega) \) and the estimate (2.16) to obtain that \( \tau, f \in L^\infty([0, T], H^1) \cap L^2([0, T], H^2) \).

**Step 6: Construction of an approximate solution.** Now that we have established all the necessary formal a priori estimates, we turn to the construction of a sequence of approximating solutions to (2.4) on which we will rigorously derive these a priori estimates. We introduce, for \( n \geq 1 \), the sequence of systems

\[
\begin{align*}
\frac{\partial u_n}{\partial t} &= \eta \frac{\partial^2 u_n}{\partial y^2} + \frac{\partial \tau_n}{\partial y}, \\
\frac{\partial \tau_n}{\partial t} &= G \frac{\partial u_n}{\partial y} - f_{n-1} \tau_n + Ga, \\
\frac{\partial f_n}{\partial t} &= (-1 + \xi |\tau_n|) f_{n-1} f_n - \nu f_{n-1} f_n^2,
\end{align*}
\] (2.24a, 2.24b, 2.24c)

supplied with the homogeneous Dirichlet boundary conditions \( u_n(t, 0) = 0 \) and \( u_n(t, 1) = 0 \) for all time \( t \in [0, T] \) and initial conditions \((u_{n0}, \tau_{n0}, f_{n0}) = (u_0, \tau_0, f_0)\). We actually use the initial condition \((u_0, \tau_0, f_0)\) also to initialize the iterations in \( n \), thus the coincidence of notation.

We argue by induction. Consider
\[
(u_{n-1}, \tau_{n-1}, f_{n-1}) \in \left( C([0, T]; H^1) \cap L^2([0, T]; H^2) \right) \times C([0, T]; H^1) \times C([0, T]; H^1)
\]
and \( f_{n-1} \geq 0 \). We first show that there exists a unique solution \((u_n, \tau_n, f_n)\) to (2.24) belonging to the same functional spaces and such that \( f_n \geq 0 \). For this purpose, we decompose (2.24) into two subsystems: the linear (Oldroyd-B) type model coupling the evolution equations (2.24a) on \( u_n \) and (2.24b) on \( \tau_n \) on the one hand and the ordinary differential equation (2.24c) on \( f_n \) satisfied for all \( y \in \Omega \) on the other hand. The existence and uniqueness of a solution \((u_n, \tau_n)\) in the space \( (C([0, T]; H^1) \cap L^2([0, T]; H^2)) \times C([0, T]; H^1) \) for the former system is obtained using a classical approach (see for instance [5] for a very close system). We now turn to \( f_n \). We show that \( f_n \) exists in \( C([0, T]; H^1) \) and \( f_n \geq 0 \). The equation (2.24c) writes

\[
\frac{\partial f_n}{\partial t} = \psi(t, f_n, y),
\]

\[
f_{n\mid t=0} = f_0
\]

where \( \psi \) is a function from \([0, T] \times \mathbb{R} \times \Omega \) to \( \mathbb{R} \).

We first fix \( y \in \Omega \) and show that the function \( f_n(\cdot, y) \) is continuous in time and non-negative. The function \( \psi \) is continuous in its first two variables and locally Lipschitz in its second variable. The Cauchy-Lipschitz Theorem shows there exists a unique local solution with \( f_0(y) \) as initial condition. Let \([0, T^*)\) be the interval of existence of the maximal solution for positive time. For all \( t \in [0, T^*) \), we have \( f_n \geq 0 \), using Step 1. In addition, since \( f_{n-1} \) and \( f_n \) are both non-negative, (2.24c) implies for all \( t \in [0, T^*) \),

\[
\frac{\partial f_n}{\partial t} \leq \xi |\tau_n| f_{n-1} f_n
\]

\[
\leq \xi \|\tau_n\|_{C_T(L^\infty)} \|f_{n-1}\|_{C_T(L^\infty)} f_n,
\]

using that both \( \tau_n \) and \( f_{n-1} \) belong to \( C([0, T]; H^1) \). The Gronwall Lemma then proves that \( f_n \) remains bounded on \([0, T^*)\) and thus we have established existence and uniqueness on \([0, T]\).

We now turn to the local property of continuity of \( f_n \) as a function of \( y \). We use that the function \( \psi \) is continuous in \( y \), because both \( \tau_n \) and \( f_{n-1} \) are continuous in \( y \) in our one-dimensional setting and the theorem on the continuous dependence on a parameter for ordinary differential equations of the form (2.25) (see e.g. [3] Theorem 1.11.1, p. 126).

We now show that \( \frac{\partial f_n}{\partial y} \) belongs to \( C([0, T]; L^2) \). We consider, for almost all \( y \in \Omega \), the following linear ordinary differential equation on \( \frac{\partial f_n}{\partial y} \):

\[
\frac{\partial}{\partial t} \left( \frac{\partial f_n}{\partial y} \right) = A \frac{\partial f_n}{\partial y} + B,
\]

where we have introduced the functions

\[
A = \xi |\tau_n| f_{n-1} - f_{n-1} - 2\nu f_{n-1} f_n \in C([0, T]; L^\infty),
\]

\[
B = \xi \frac{\partial |\tau_n|}{\partial y} f_{n-1} f_n + (\xi |\tau_n| - 1) f_n \frac{\partial f_{n-1}}{\partial y} - \nu f_n^2 \frac{\partial f_{n-1}}{\partial y} \in C([0, T]; L^2).
\]

The Cauchy-Lipschitz Theorem then guarantees the existence of \( \frac{\partial f_n}{\partial y}(\cdot, y) \) continuous in time, for almost all \( y \in \Omega \). The Duhamel formula applied to (2.27) yields, for all \( t \in [0, T] \)
and almost all \( y \in \Omega \),
\[
\frac{\partial f_n}{\partial y}(t, y) = \frac{\partial f_0}{\partial y}(y) e^{f_0 \Lambda(y)} + \int_0^t B(s, y) e^{f_0 \Lambda(y)} ds, 
\]
so that, using (2.1), (2.28) and (2.29), \( \frac{\partial f_n}{\partial y} \) belong to \( C([0, T]; L^2) \). As (2.27) is the derivative with respect to \( y \) of (2.24c), this yields \( f_n \in C([0, T]; H^1) \).

Now that we have established, for all \( n \), the existence of a solution \( (u_n, \tau_n, f_n) \) to (2.24) in the appropriate functional spaces (as in (2.2)-(2.3)), we derive, for \( (u, \tau, f) \), the a priori estimates formally established on \( (u, \tau, f) \) in the previous steps. Estimate (2.8) now reads
\[
\frac{1}{2} \frac{d}{dt} \left( G \rho \| u_n(t, \cdot) \|^2_{L^2} + \lambda \| \tau_n(t, \cdot) \|^2_{L^2} \right) + G \eta \left\| \frac{\partial u_n}{\partial y}(t, \cdot) \right\|^2_{L^2} + \left\| \left( \sqrt{f_{n-1} \tau_n} \right) (t, \cdot) \right\|^2_{L^2} = G a \tau_n(t). \tag{2.30}
\]
Likewise, (2.10) is now replaced by
\[
\frac{d}{dt} \| f_n(t, \cdot) \|_{L^1} + \int_{\Omega} (f_{n-1} f_n) (t, \cdot) + \frac{\nu}{2} \int_{\Omega} (f_{n-1} f_n^2) (t, \cdot) \leq \frac{\gamma^2}{2 \nu} \left( \sqrt{f_{n-1} \tau_n} \right) (t, \cdot) \right\|^2_{L^2}, \tag{2.31}
\]
Collecting (2.30) and (2.31) yields the following estimate, analogous to (2.11),
\[
\frac{1}{2} \frac{d}{dt} \left( G \rho \| u_n(t, \cdot) \|^2_{L^2} + \lambda \| \tau_n(t, \cdot) \|^2_{L^2} + \frac{2 \nu}{\gamma^2} \| f_n(t, \cdot) \|_{L^1} \right) 
+ G \eta \left\| \frac{\partial u_n}{\partial y}(t, \cdot) \right\|^2_{L^2} + \frac{1}{2} \left\| \left( \sqrt{f_{n-1} \tau_n} \right) (t, \cdot) \right\|^2_{L^2} \leq C a \| \tau_n(t, \cdot) \|^2_{L^2}, \tag{2.32}
\]
and therefore, (2.12) holds with \( (u_n, \tau_n, f_n) \) instead of \( (u, \tau, f) \).
The arguments given in Step 3 to derive (2.15) and in Step 4 for the \( L^\infty \) estimates can be mimicked for the approximate system in \( (u_n, \tau_n, f_{n-1}) \) instead of \( (u, \tau, f) \), and the corresponding auxiliary functions \( g_n \) and \( U_n \).

At this point, we have rigorously established on \( (u_n, \tau_n, f_n) \) and our formal estimates of steps 2 to 4
\[
\sup_n \sup_{t \in [0, T]} \left( \| u_n(t, \cdot) \|_{L^2} + \| \tau_n(t, \cdot) \|_{L^2} + \| f_n(t, \cdot) \|_{L^1} \right) \leq C_{0,T}, \tag{2.33}
\]
and
\[
\sup_n \sup_{t \in [0, T]} \left( \| U_n(t, \cdot) \|_{H^1} + \| \tau_n(t, \cdot) \|_{L^\infty} + \| f_n(t, \cdot) \|_{L^\infty} \right) + \| U_n \|_{L^2_t(H^2)} \leq C_{0,T}, \tag{2.34}
\]
where we recall that \( C_{0,T} \) denotes various constants which depend on the coefficients in system (2.4), the initial data \( u_0, \tau_0, f_0 \) and the time \( T \).
We now turn to the a priori estimates of Step 5. Using arguments similar to the formal arguments of Step 5, we obtain

$$\frac{d}{dt} \left( \lambda \left\| \frac{\partial \tau_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \leq C_{0,T} \left( \left\| \frac{\partial \tau_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial \tau_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial^2 U_n(t)}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \right).$$ (2.35)

We now observe that showing $H^1$ bounds on $\tau_n$ and $f_n$ is less straightforward than in our formal Step 5. We introduce

$$Y_n(t) = \left\| \frac{\partial \tau_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f_n}{\partial y}(t, \cdot) \right\|_{L^2}^2.$$

Integrating (2.35) from 0 to $t \leq T$, we see that $Y_n$ satisfies

$$Y_n(t) \leq C_{0,T} \int_0^t Y_n + C_{0,T} \int_0^t Y_{n-1} + C_{0,T} \|U_n\|_{L^2_s(H^2)}^2 + Y_0.$$ (2.36)

Applying the Gronwall Lemma to (2.36), we find

$$Y_n(t) \leq \left( C_{0,T} \|U_n\|_{L^2_s(H^2)}^2 + Y_0 \right) e^{C_{0,T}t} + C_{0,T} \int_0^t Y_{n-1}(s)e^{C_{0,T}(t-s)} ds$$

which we rewrite

$$Y_n(t) \leq C_{0,T} + C_{0,T} \int_0^t Y_{n-1}(s) ds.$$

Arguing by induction, one can check that this implies, for all $t \in [0, T]$ and $n$,

$$Y_n(t) \leq C_{0,T} \sum_{i=0}^{n-1} \frac{(C_{0,T}t)^i}{i!} + \frac{(C_{0,T}t)^n}{n!} Y_0.$$ (2.37)

It follows that, $C_{0,T}$ denoting various constants,

$$\sup_n \sup_{t \in [0, T]} Y_n(t) \leq C_{0,T} e^{C_{0,T}T},$$ (2.38)

Recalling that $\frac{\partial \tau_n}{\partial y} = \frac{\partial U_n}{\partial y} - \frac{1}{\eta} (\tau_n - \tau_0)$, we use inequalities (2.34) and (2.37) to derive

$$\sup_n \sup_{s \in [0, T]} \left( \left\| u_n(s, \cdot) \right\|_{H^1} + \left\| \tau_n(s, \cdot) \right\|_{H^1} + \left\| f_n(s, \cdot) \right\|_{H^1} + \left\| u_n \right\|_{L^2_s(H^2)} \right) \leq C_{0,T}.$$ (2.39)

This implies

$$\sup_n \sup_{s \in [0, T]} \left\| \frac{\partial \tau_n}{\partial t}(s, \cdot) \right\|_{L^2_s(L^2)} + \left\| \frac{\partial f_n}{\partial t}(s, \cdot) \right\|_{L^2_s(L^2)} \leq C_{0,T}.$$ (2.40)
Step 7: Convergence of the sequence of approximate solution. The bounds obtained in the previous steps, namely (2.38) and (2.39) show that, at least up to extraction of a subsequence, we have the weak convergences

\[(u_n, \tau_n, f_n) \rightarrow (u, \tau, f)\] weakly-* in \(L^\infty([0, T]; H^1)^3,\) \(u_n \rightharpoonup u\) weakly in \(L^2([0, T]; H^2),\)

\[
\begin{vmatrix}
\frac{\partial u_n}{\partial t}, \frac{\partial \tau_n}{\partial t} & \frac{\partial f_n}{\partial t}
\end{vmatrix} \rightharpoonup \begin{vmatrix}
\frac{\partial u}{\partial t}, \frac{\partial \tau}{\partial t} & \frac{\partial f}{\partial t}
\end{vmatrix}
\] weakly in \(L^2([0, T]; L^2)^3.\)

But, in order to pass to the limit in (2.24), we need the convergence of the whole sequence itself because (2.24) involves indices \(n - 1\) and \(n\) and strong convergence to establish convergence of the product terms \(\tau_n f_{n-1} f_n, f_n^2 \tau_n.\)

We now establish strong convergence of the whole sequence. We prove this convergence in \((L^\infty([0, T]; L^2(\Omega))^3.\) This will a posteriori imply that all the convergences (2.40), (2.41) and (2.42) actually hold true not only for a subsequence, but the whole sequence itself. And this will provide sufficient information to pass to the limit in our nonlinear terms.

We introduce the notation: \(\tilde{h}_n = h_n - h_{n-1}\) and derive the evolution equations for \((\tilde{u}_n, \tilde{\tau}_n, \tilde{f}_n)\)

\[
\begin{align*}
\rho \frac{\partial \tilde{u}_n}{\partial t} &= \eta \nabla^2 \tilde{u}_n + \frac{\partial \tilde{\tau}_n}{\partial y}, \\
\lambda \frac{\partial \tilde{\tau}_n}{\partial t} &= G \frac{\partial \tilde{u}_n}{\partial y} - f_{n-1} \tilde{\tau}_n - \tau_{n-1} \tilde{f}_{n-1}, \\
\frac{\partial \tilde{f}_n}{\partial t} &= (-1 + \xi |\tau_{n-1}|) (f_{n-1} \tilde{f}_n + f_{n-1} \tilde{f}_{n-1}) - \nu f_{n-1}^2 (f_n + f_{n-1}) \tilde{f}_n - \nu f_{n-1} f_{n-1} \tilde{f}_n + \xi f_{n-1} f_{n-1} \tilde{f}_n.
\end{align*}
\]

Since \((u_n, \tau_n, f_n)\) belong to the spaces that appear in (2.2) and (2.3), the same holds for \((\tilde{u}_n, \tilde{\tau}_n, \tilde{f}_n)\). We multiply equations (2.43a), (2.43b) and (2.43c), respectively by \(\tilde{u}_n, \tilde{\tau}_n\) and \(\tilde{f}_n\), integrate over \(\Omega\), sum up and use the non-negativity of \(f_{n-1}\) and \(f_n\) to find

\[
\frac{d}{dt} \left( G\rho \|\tilde{u}_n(t, \cdot)\|^2_{L^2} + \lambda \|\tilde{\tau}_n(t, \cdot)\|^2_{L^2} + \|\tilde{f}_n(t, \cdot)\|^2_{L^2} \right) \leq -\int_\Omega \tau_{n-1} \tilde{f}_{n-1} \tilde{\tau}_n(t, \cdot) + \int_\Omega (-1 + \xi |\tau_{n-1}|) (f_{n-1} \tilde{f}_n + f_{n-1} \tilde{f}_{n-1}) - \nu f_{n-1}^2 (f_n + f_{n-1}) \tilde{f}_n + \xi f_{n-1} f_{n-1} \tilde{f}_n.
\]

The presence of two indices \(n - 1\) and \(n\) again makes an additional step necessary. We introduce \(X_n(t) = \|\tilde{u}_n(t, \cdot)\|^2_{L^2} + \|\tilde{\tau}_n(t, \cdot)\|^2_{L^2} + \|\tilde{f}_n(t, \cdot)\|^2_{L^2}\). Repeatedly using the \(L^\infty\)-bounds (2.34) on \(\{\tau_n, f_n, \tau_{n-1}, f_{n-1}\}\) and the Young inequality, we see that \(X_n\) satisfies

\[
X_n(t) \leq C_{0,T}(X_n(t) + X_{n-1}(t)).
\]

Applying the Gronwall Lemma to (2.44), we find

\[
X_n(t) \leq C_{0,T} \int_0^t X_{n-1}(s)e^{C_{0,T}(t-s)} ds \leq C_{0,T}e^{C_{0,T}T} \int_0^t X_{n-1}(s) ds,
\]

13
which implies that
\[ X_n(t) \leq \frac{(C_0 T e^{C_0 T t})^{n-1}}{(n-1)!} \sup_{s \in [0,T]} X_1(s). \]

The sequence \((u_n, \tau_n, f_n)\) is therefore a Cauchy sequence in \((L^\infty([0,T]; L^2(\Omega)))^3\). The sequence converges in this space.

Now that we have strong convergence of the whole sequence, we show how to pass to the limit in all the terms of (2.24), including the nonlinear ones. We only consider \(|\tau_n| f_{n-1} f_n\). The other terms can be treated using similar arguments. We use a classical compactness result [8, Theorem 5.1, p. 58] to deduce from (2.40) and (2.42) that \(\tau_n\) and \(f_n\) strongly converge respectively to \(\tau\) and \(f\) in \(L^2([0,T]; L^4(\Omega))\). Moreover, \(f_{n-1}\) strongly converges to \(f\) in \(L^\infty([0,T]; L^2(\Omega))\). We thus have convergence for \(|\tau_n| f_{n-1} f_n\) in \(L^1([0,T]; L^1(\Omega))\).

The triple \((u, \tau, f)\) thus satisfies system (2.24), at least in the weak sense. We now derive further regularity. We have
\[ u \in L^2([0,T]; H^2) \] with \(\frac{\partial u}{\partial t} \in L^2([0,T]; L^2)\),
and therefore, by interpolation (see [12] Chapter 3, Lemma 1.2),
\[ u \in C([0,T]; H^1) \cap L^2([0,T]; H^2). \]

Moreover, we have
\[ \left(\frac{\partial \tau}{\partial t}, \frac{\partial f}{\partial t}\right) \in L^2([0,T]; L^2)^2 \]
and, using the second a priori estimate (2.23),
\[ \left(\frac{\partial \tau}{\partial t}, \frac{\partial f}{\partial t}\right) \in L^2([0,T]; L^2)^2, \]
so that,
\[ (\tau, f) \in C([0,T]; H^1)^2. \]

We have obtained (2.2) and therefore (2.3), using system (2.4). The non-negativity of the fluidity is preserved, passing to the limit. This completes the existence proof.

**Step 8: Uniqueness.** Consider \((u_1, \tau_1, f_1)\) and \((u_2, \tau_2, f_2)\) satisfying (2.2) and solutions to system (2.4) supplied with the same initial condition \((u_0, \tau_0, f_0) \in H^1(\Omega)\). We introduce \((\tilde{u} = u_2 - u_1, \tilde{\tau} = \tau_2 - \tau_1, \tilde{f} = f_2 - f_1)\) which therefore satisfies
\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial t} &= \eta \frac{\partial^2 \tilde{u}}{\partial y^2} + \frac{\partial \tilde{\tau}}{\partial y}, \\
\frac{\lambda \partial \tilde{\tau}}{\partial t} &= G \frac{\partial \tilde{u}}{\partial y} - f_2 \tilde{\tau} - \tau_1 \tilde{f}, \\
\frac{\partial \tilde{f}}{\partial t} &= -(f_1 + f_2) \tilde{f} + \xi f_1^2 \tilde{\tau} + \xi |\tau_2|(f_1 + f_2) \tilde{f} - \nu (f_1^2 + f_1 f_2 + f_2^2) \tilde{f},
\end{align*}
\]
supplied with *homogeneous* boundary conditions and \((0, 0, 0)\) as initial data. Multiplying equations \((2.45a)\), \((2.45b)\) and \((2.45c)\), respectively by \(\tilde{u}\), \(\tilde{\tau}\) and \(\tilde{f}\), integrating over \(\Omega\), summing up, using the \(L^\infty\)-bounds established in Step 4 for terms involving \(\tau_1, \tau_2, f_1, f_2\) and repeatedly applying the Young inequality, we find

\[
\frac{1}{2} \frac{d}{dt} \left( \rho G \|\tilde{u}(t, \cdot)\|_{L^2}^2 + \lambda \|\tilde{\tau}(t, \cdot)\|_{L^2}^2 + \|\tilde{f}(t, \cdot)\|_{L^2}^2 \right) \leq C_{0,T} \left( \|\tilde{\tau}(t, \cdot)\|_{L^2}^2 + \|\tilde{f}(t, \cdot)\|_{L^2}^2 \right).
\]

The Gronwall Lemma then implies uniqueness. This concludes the proof of Theorem 2.1.

The Gronwall Lemma then implies uniqueness. This concludes the proof of Theorem 2.1.

\[
\diamondsuit
\]

3 Longtime behaviour for *homogeneous* boundary conditions

In this section, we study the longtime behaviour of system \((1.1)\) supplied with *homogeneous* boundary conditions. We will show convergence to a steady state and establish a rate for this convergence. For *homogeneous* boundary conditions, the \(H^1\)-steady states of \((1.1)\) such that \(f \geq 0\) are exactly the states \((u \equiv 0, \tau \equiv c, f \equiv 0)\), where \(c\) is a constant throughout the domain.

Indeed, such a steady state \((u_\infty, \tau_\infty, f_\infty)\) satisfies, combining equation \((1.1a)\) integrated over the domain and \((1.1b)\),

\[
\tau_\infty \left( \frac{\eta}{G} f_\infty + 1 \right) = c,
\]

where \(c\) is a constant over the domain. We now distinguish between two cases. Either \(c = 0\), in which case \(\tau_\infty \equiv 0\). The *homogeneous* boundary conditions on \(u\) and \((1.1b)\) imply that \(u_\infty \equiv 0\) and \((1.1c)\) that \(f_\infty \equiv 0\). Or \(c \neq 0\) and it follows from \((3.1)\) that \(\tau_\infty\) is non-zero and has a constant sign and from \((1.1b)\) that \(\partial u_\infty / \partial y\) has a constant sign. Because of the *homogeneous* boundary conditions on the velocity, we obtain that \(u_\infty \equiv 0\). Therefore, \((1.1b)\) yields \(f_\infty \tau_\infty = 0\) and \(f_\infty \equiv 0\), because \(\tau_\infty\) is non-zero in this case.

We will show that the longtime behaviour differs both in terms of steady state and rate of convergence, depending whether \(f_0 \neq 0\) or \(f_0 \equiv 0\). When \(f_0 \neq 0\), a case studied in subsection 3.1, the solution \((u, \tau, f)\) converges to the steady state \((0, 0, 0)\) in the longtime and the rates of convergence are power-laws of the time. In the case \(f_0 \equiv 0\), the fluidity \(f\) vanishes for all time, as easily seen on \((1.1c)\). In subsection 3.2, we show that \((u, \tau, f)\) then converges to \((0, \overline{\tau_0}, 0)\) in the longtime at an exponential rate, \(\overline{\tau_0}\) being the average of \(\tau_0\) over \(\Omega\). Evidently, the former case \(f_0 \neq 0\) require more efforts than the latter case \(f_0 \equiv 0\) where \(f \equiv 0\) for all times.

3.1 Case \(f_0 \neq 0\)

In this subsection, we consider the case \(f_0 \geq 0\), \(f_0 \neq 0\). We first establish the convergence in the longtime.
Theorem 3.1  Supply system \[1.1\] with homogeneous boundary conditions and initial conditions that satisfy \[2.1\] and \(f_0 \neq 0\). The solution \((u, \tau, f)\), the existence and uniqueness of which have been established in Theorem \[2.1\], converges to the steady state \((0, 0, 0)\) in \(H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)\) in the longtime:

\[
\|u(t, \cdot)\|_{H^1} + \|\tau(t, \cdot)\|_{L^\infty} + \|f(t, \cdot)\|_{L^\infty} \to 0.
\]

Proof. The proof falls in three steps. In the first step, we establish a lower bound for the average of the fluidity \(f\), which, in Step 2, is useful to prove convergence in the longtime in \(L^2(\Omega)\). In the third step, we show convergence of \((u(t, \cdot), \tau(t, \cdot), f(t, \cdot))\) in \(H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)\).

In this section, \(C_0\) denotes various constants that are independent from time, while \(C_i, i=1,\ldots,4\) denote some fixed constants independent from time. These constants \(C_0\) and \(C_i\) used to be denoted \(C_{0,T}\) in the previous section. The subscript \(T\) is now omitted because, as explained in Remarks \[2.1\] and \[2.2\], the constants are independent from \(T\) for homogeneous boundary conditions.

Step 1: A lower bound for the average of \(f\). We first derive a lower bound on \(\bar{f}\), defined as in \[2.7\], and not directly on \(f\) because the latter may vanish (since \(f_0\) may vanish) on some part of the domain. Since \(f_0 \neq 0\), there exists, by continuity of \(f_0\) (assumed in \(H^1\)), a non-empty closed interval \(\Omega_0\) in \(\Omega\) where \(f_0\) does not vanish. Arguing as in Step 1 of the proof of Theorem \[2.1\], the fluidity \(f\) does not vanish for all \(t > 0\) and \(y \in \Omega_0\). The evolution equation \[1.1c\] on \(f\) rewrites, for all \(t > 0\) and \(y \in \Omega_0\),

\[
\frac{\partial}{\partial t} \frac{1}{f} = 1 - \xi|\tau| + \nu f. \tag{3.2}
\]

As explained in Remark \[2.2\], the \(L^\infty\)-bounds on \(\tau\) and \(f\) are uniform in time for homogeneous boundary conditions. The equation \[3.2\] thus implies, for all \(y \in \Omega_0\) and \(t > 0\),

\[
\frac{\partial}{\partial t} \frac{1}{f} \leq C_0,
\]

and therefore,

\[
f(t, y) \geq \frac{1}{\frac{1}{f_0(y)} + C_0 t},
\]

\[
\geq \frac{1}{\left\| \frac{1}{f_0} \right\|_{L^\infty(\Omega_0)} + C_0 t}. \tag{3.3}
\]

Since \(\bar{f} \geq \int_{\Omega_0} f\), this yields the lower bound

\[
\bar{f}(t) \geq \frac{C_1}{1 + C_0 t}. \tag{3.4}
\]
Step 2: Longtime convergence in $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. We now show the longtime convergence in $L^2$. Estimates (2.8) and (2.15) respectively rewrite, for homogeneous boundary conditions,

$$
\frac{1}{2} \frac{d}{dt} \left( G \rho \|u(t, \cdot)\|_{L^2}^2 + \lambda \|\tau(t, \cdot)\|_{L^2}^2 \right) + G \eta \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \left( \sqrt{f} \tau \right)(t, \cdot) \right\|_{L^2}^2 = 0 \tag{3.5}
$$

and

$$
\frac{1}{2} \frac{d}{dt} \left\| U(t, \cdot) \right\|_{L^2}^2 + \frac{\eta}{2\rho} \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \leq C_2 \left( \|f(t, \cdot)\|_{L^1} \left\| \left( \sqrt{f} \tau \right)(t, \cdot) \right\|_{L^2}^2 + \|u(t, \cdot)\|_{L^2}^2 \right), \tag{3.6}
$$

where $U$ is defined by (2.13). The evolution equation on $\bar{\tau}$ writes

$$
\lambda \frac{d}{dt} \bar{\tau} + \bar{f} \ddot{\tau} = -\sqrt{f} (\tau - \bar{\tau}). \tag{3.7}
$$

We introduce the positive scalar $\epsilon$, to be fixed later on. We use the Cauchy-Schwarz and Young inequalities

$$
|\bar{\tau}| \left| \sqrt{f} (\tau - \bar{\tau}) \right| \leq \sqrt{f} |\bar{\tau}| \left( \sqrt{f} (\tau - \bar{\tau}) \right)(t, \cdot) \left\|_{L^2} \leq \epsilon f |\bar{\tau}|^2 + \frac{1}{4\epsilon} \left( \left( \sqrt{f} (\tau - \bar{\tau}) \right)(t, \cdot) \right)^2, \tag{3.8}
$$

so that, multiplying evolution equation (3.7) by $\bar{\tau}$, we obtain

$$
\frac{\lambda}{2} \frac{d}{dt} |\bar{\tau}|^2 + (1 - \epsilon) f |\bar{\tau}|^2 (t) \leq \frac{1}{4\epsilon} \|f(t, \cdot)\|_{L^\infty} \|\tau(t, \cdot)\|_{L^2}^2. \tag{3.8}
$$

The evolution equation on $\tau - \bar{\tau}$ reads

$$
\lambda \frac{\partial}{\partial t} (\tau - \bar{\tau}) + \frac{G}{\eta} (\tau - \bar{\tau}) = -(f \tau - \bar{f} \tau) + G \frac{\partial U}{\partial y}. \tag{3.9}
$$

Multiplying evolution equation (3.9) by $\tau - \bar{\tau}$, integrating over $\Omega$ and repeatedly using the Young inequality, we find

$$
\frac{1}{2} \frac{d}{dt} \left( \lambda \|\tau - \bar{\tau}\|_{L^2}(t, \cdot) \right)^2 + \frac{G}{2\eta} \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2 \leq C \left( \|f(t, \cdot)\|_{L^2} + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right), \tag{3.10}
$$

so that, using the uniform in time $L^\infty$-bound on $f$,

$$
\frac{1}{2} \frac{d}{dt} \left( \lambda \|\tau - \bar{\tau}\|_{L^2}(t, \cdot) \right)^2 + \frac{G}{2\eta} \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2 \leq C_3 \left( \|f(t, \cdot)\|_{L^\infty} \left\| \left( \sqrt{f} \tau \right)(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right). \tag{3.10}
$$

We introduce some positive scalars $m_1, m_2, m_3$ and the energy function

$$
E(t) = m_1 (G \rho \|u(t, \cdot)\|_{L^2}^2 + \lambda \|\tau(t, \cdot)\|_{L^2}^2) + m_2 \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + m_3 \lambda \|\tau(t)\|_{L^2}^2 + \lambda |\bar{\tau}(t)|^2, \tag{3.11}
$$
which therefore satisfy, combining (3.5), (3.6), (3.8) and (3.10),

\[
\frac{1}{2} \frac{dE}{dt} (t) + (m_1 - C_2 m_2 \| f(t, \cdot) \|_{L^1} - C_3 m_3 \| f(t, \cdot) \|_{L^\infty}) \| (\sqrt{\tau}) (t, \cdot) \|_{L^2}^2
\]

\[+ (m_1 G_\eta - C_2 m_2 c_p) \| \partial_u (t, \cdot) \|_{L^2}^2 + \left( \frac{n}{2 \rho} m_2 - C_3 m_3 c_p \right) \| \partial^2 U (t, \cdot) \|_{L^2}^2 \]

\[+ \left( \frac{G}{2 \eta} m_3 - \frac{1}{4 \epsilon} \| f(t, \cdot) \|_{L^\infty} \right) \| (\tau - \bar{\tau}) (t, \cdot) \|_{L^2}^2 + (1 - \epsilon) \| \bar{\tau} \|_{L^2}^2 \leq 0, \tag{3.12} \]

where \( C_p \) is the Poincaré constant.

The coefficients \( m_1, m_2, m_3 \) are chosen sufficiently large so that, for all time \( t > 0 \), every term in the left-hand side of (3.12) is positive. The conditions

\[
\frac{G}{2 \eta} m_3 > \frac{1}{4 \epsilon} \sup_{t > 0} \| f(t, \cdot) \|_{L^\infty},
\]

\[
\frac{n}{2 \rho} m_2 > C_3 m_3 c_p,
\]

\[
m_1 > \max \left( \frac{C_2 m_2 c_p}{G \eta}, C_2 m_2 \sup_{t > 0} \| f(t, \cdot) \|_{L^1} + C_3 m_3 \sup_{t > 0} \| f(t, \cdot) \|_{L^\infty} \right)
\]

are sufficient. Using in addition the lower bound (3.3) and the Poincaré inequality, (3.12) becomes

\[
\frac{1}{2} \frac{dE}{dt} (t) + C_0 \left( \| u(t, \cdot) \|_{L^2}^2 + \| (\tau - \bar{\tau}) (t, \cdot) \|_{L^2}^2 + \| \partial U (t, \cdot) \|_{L^2}^2 \right) + (1 - \epsilon) \frac{C_1}{1 + C_0 t} \| \bar{\tau} \|_{L^2}^2 \leq 0.
\]

Using the triangle inequality

\[
\epsilon \frac{m_1}{2} \| \tau(t, \cdot) \|_{L^2}^2 \leq \epsilon \| \bar{\tau} \|_{L^2}^2 + \epsilon \| (\tau - \bar{\tau}) (t, \cdot) \|_{L^2}^2,
\]

we find, for \( t \) sufficiently large,

\[
\frac{1}{2} \frac{dE}{dt} + \frac{1}{\lambda} \min \left( 1 - 2 \epsilon, \frac{\epsilon}{2 m_1} \right) \frac{C_1}{1 + C_0 t} E \leq 0. \tag{3.14}
\]

We take \( \epsilon < \frac{1}{2} \) and apply the Gronwall Lemma to (3.14) to obtain that \( E \) goes to zero in the longtime limit. In particular, we have

\[
\lim_{t \to \infty} \left( \| u(t, \cdot) \|_{L^2}^2 + \| \tau(t, \cdot) \|_{L^2}^2 + \| \partial U (t, \cdot) \|_{L^2}^2 \right) = 0. \tag{3.15}
\]

**Step 3: Longtime convergence in** \( H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega) \). Combining (2.15) and (2.17), using the uniform in time \( L^\infty \)-bound on \( f \) and the Poincaré inequality on \( \partial U / \partial y \), the spatial average of which is zero, we obtain

\[
\frac{d}{dt} \left( \| \partial U (t, \cdot) \|_{L^2}^2 + \lambda |\tau(t, y)|^2 \right) + \left( \| \partial^2 U (t, \cdot) \|_{L^2}^2 + |\tau(t, y)|^2 \right)
\]

\[\leq C_0 \left( \| u(t, \cdot) \|_{L^2}^2 + \| \tau(t, \cdot) \|_{L^2}^2 + \| \partial U (t, \cdot) \|_{L^2}^2 \right).
\]
We apply the Gronwall Lemma and use the convergence (3.15) that is uniform in space to derive
\[
\lim_{t \to \infty} \| \tau(t, \cdot) \|_{L^\infty} = 0.
\]
Using the convergence of \( \| \tau(t, \cdot) \|_{L^\infty} \), the evolution equation (1.1c) on \( f \) implies, for \( t \) sufficiently large,
\[
\frac{\partial f}{\partial t} \leq -\frac{1}{2} f^2. \tag{3.16}
\]
This yields the convergence of \( \| f(t, \cdot) \|_{L^\infty} \) to zero in the longtime.

Additionally, using the definition (2.13) and (3.15), \( \frac{\partial u}{\partial y} \) converges to zero in \( L^2(\Omega) \).

This ends the proof. ⊖

We now turn to making precise the rates of convergence to the steady-state. We introduce the non-negative scalar
\[
\beta = \text{meas}\{ y \in \Omega | f_0(y) > 0 \}. \tag{3.17}
\]
By assumption in this section, we have \( \beta > 0 \). The following result establishes the convergence rates in function of \( \beta \). In Section 5.1 we will check using numerical simulations that these rates are indeed sharp.

**Theorem 3.2** Supply system (1.1) with homogeneous boundary conditions and initial conditions that satisfy (2.1) and \( f_0 \not\equiv 0 \). The solution \((u, \tau, f)\), the existence and uniqueness of which have been established in Theorem 2.1, satisfies the following convergence estimates: for any arbitrarily small \( \alpha > 0 \), there exists a constant \( \kappa_\alpha \) independent from time and there exists a time \( t_0 \), both depending on the domain, the initial data, the coefficients in the system and \( \alpha \), such that, for all \( t > t_0 \),
\[
\| u(t, \cdot) \|_{H^1} + \| \tau(t, \cdot) \|_{L^2} \leq \kappa_\alpha (1 + t)^{-\frac{\beta}{2(1-\alpha)}}, \tag{3.18}
\]
where \( \beta \) is defined by (3.17) and for all \( t > t_0 \) and \( y \in \Omega \), we have
\[
\frac{1}{f(t_0,y) + (1+\alpha)(t-t_0)} \leq f(t,y) \leq \frac{1}{f(t_0,y) + (1-\alpha)(t-t_0)}. \tag{3.19}
\]
In addition, there exists another constant \( \kappa_\alpha \), such that, for all \( t > t_0 \),
\[
\| u(t, \cdot) \|_{H^2} + \| (\tau - \bar{\tau})(t, \cdot) \|_{L^2} \leq \kappa_\alpha (1 + t)^{-1 - \frac{\beta}{2(1-\alpha)}}, \tag{3.20}
\]
\[
\left\| \left( \eta \frac{\partial u}{\partial y} + \tau - \bar{\tau} \right)(t, \cdot) \right\|_{L^2} \leq \kappa_\alpha (1 + t)^{-2 - \frac{\beta}{2(1-\alpha)}}, \tag{3.21}
\]
where the function \( \bar{\tau} \) is the spatial average of \( \tau \), defined as in (2.7).

**Proof.** The proof falls in four steps. We first consider the fluidity, then derive first convergence rates for the velocity and the stress. A study of the auxiliary function defined by (2.13) next allows to conclude on the convergence estimates (3.20) and (3.21).

We fix \( \epsilon \) an arbitrarily small positive scalar, actually equal to \( \alpha^4 \), where \( \alpha \) is the constant that appears in the statement of the Theorem. The constants \( \kappa_\epsilon \) depend on \( \epsilon \) and have value that may vary from one instance to another, the actual value being irrelevant.
Step 1: Convergence rate for the fluidity. In view of Theorem 3.1, \( \|\tau(t, \cdot)\|_{L^\infty} \) and \( \|f(t, \cdot)\|_{L^\infty} \) vanish in the longtime. This implies that there exists a time \( t_0 \), such that, for all \( y \in \Omega \) and \( t > t_0 \), the evolution equation (1.1c) on \( f \) leads to
\[
-(1 + \epsilon)f^2(t, y) \leq \frac{\partial f}{\partial t}(t, y) \leq -(1 - \epsilon)f^2(t, y). \tag{3.22}
\]
For all \( y \in \Omega \) such that \( f_0(y) > 0 \), we have \( f(t_0, y) > 0 \), as shown in Step 1 of the proof of Theorem 2.1. The equation (3.19) becomes, for such \( y \) and \( t > t_0 \),
\[
\frac{1}{f(t_0, y)} + (1 + \epsilon)(t - t_0) \leq f(t, y) \leq \frac{1}{f(t_0, y)} + (1 - \epsilon)(t - t_0). \tag{3.23}
\]
and hence (3.19). This inequality is also valid for all \( y \) such that \( f_0(y) = 0 \), that is \( f(t_0, y) = 0 \), and therefore for all \( y \in \Omega \).

Step 2: First convergence rates for the velocity and the stress. We first make more precise the lower bound on \( \bar{\tau} \). As \( f_0 \) is continuous, there exists a closed set \( \Omega_\epsilon \) such that \( f_0(\Omega_\epsilon) > 0 \) and \( \text{meas}\{\Omega_\epsilon\} = \beta(1 - \epsilon) \). As shown in Step 1 of the proof of Theorem 2.1, we also have \( f(t_0, \Omega_\epsilon) > 0 \). Furthermore, as \( f(t_0, \cdot) \) is continuous, we obtain \( f(t_0, \Omega_\epsilon) > \kappa_\epsilon \). The inequality (3.23) thus becomes, for all \( y \in \Omega_\epsilon \) and \( t > t_0 \),
\[
f(t, y) > \frac{1}{1 + \epsilon \kappa_\epsilon + t}.
\]
It follows from \( \bar{f} \geq \int_{\Omega_\epsilon} f \) that, for all \( t > t_0 \),
\[
\bar{f}(t) \geq \beta \frac{1 - \epsilon}{1 + \epsilon \kappa_\epsilon + t}. \tag{3.24}
\]
We now use the energy \( E \) introduced in (3.11). As \( \|f(t, \cdot)\|_{L^\infty} \) vanishes in the longtime, the coefficients \( m_1, m_2, m_3 \) can be chosen arbitrarily small in (3.12), independently from \( \epsilon \), for \( t > t_0 \) sufficiently large. We insert (3.24) in (3.12) so that, for sufficiently large \( t \),
\[
\frac{1}{2} \frac{dE}{dt} + C_0 \left( \|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2 + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \\
+ (1 - \epsilon)\beta \frac{1 - \epsilon}{1 + \epsilon \kappa_\epsilon + t}\|\bar{\tau}\|^2(t) \leq 0.
\]
Using the triangle inequality (3.13), we obtain, for sufficiently large \( t \),
\[
\frac{1}{2} \frac{dE}{dt} + \frac{\beta}{\lambda} \min \left( 1 - 2\epsilon, \frac{\epsilon}{2m_1} \right) \frac{1 - \epsilon}{1 + \epsilon \kappa_\epsilon + t} \|\tau\|^2(t) \leq 0.
\]
and therefore, using that \( 1 - 2\epsilon < \frac{\epsilon}{2m_1} \) as \( m_1 \) is arbitrarily small,
\[
\frac{1}{2} \frac{dE}{dt} + \frac{\beta}{\lambda} (1 - 4\epsilon) \frac{1 - \epsilon}{1 + \epsilon \kappa_\epsilon + t} E \leq 0. \tag{3.25}
\]
Applying the Gronwall Lemma to (3.25), we find,
\[
\|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2 + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon(1 + t)^{-\frac{2\beta}{\lambda}(1 - 4\epsilon)}. \tag{3.26}
\]
where we recall that \( \kappa_\epsilon \) denotes various constants. We have obtained (3.18).
Step 3: Convergence rate for the auxiliary function $U$. We recall that the function $U$ is defined by (2.13). We first prove that $U$ is more regular than claimed in (2.16). We rewrite (2.14)

$$\frac{\partial U}{\partial t} - \eta \frac{\partial^2 U}{\rho \partial y^2} = -\frac{1}{\lambda \eta} \int_0^y (f \tau - \bar{f} \tau) dx + \frac{G}{\lambda \eta} u.$$  

(3.27)

We deduce that $\frac{\partial U}{\partial y}$ satisfies the heat equation with a right-hand side in $L^2_{loc}((t_0, +\infty), L^2)$ and initial condition $\frac{\partial U}{\partial y}(t_0, \cdot) \in H^1(\Omega)$ at time $t_0$ (up to a possible modification on a set of times of measure zero). Therefore, we have $\frac{\partial U}{\partial y} \in H^1_{loc}((t_0, +\infty), L^2)$, so that

$$U \in H^1_{loc}((t_0, +\infty), H^1_0).$$  

(3.28)

We next differentiate (2.14) with respect to $t$, insert (1.1a) and find,

$$\frac{\partial^2 U}{\partial t^2} - \eta \frac{\partial^2}{\rho \partial y^2} \left( \frac{\partial U}{\partial t} \right) - \frac{G}{\lambda \rho} \frac{\partial^2 U}{\partial y^2} = I,$$  

(3.29)

where $I$ is the function defined by

$$I(t, y) = -\frac{1}{\lambda \eta} \int_0^y \left( \frac{\partial f \tau}{\partial t} - \frac{\partial \bar{f} \tau}{\partial t} \right) dx.$$  

(3.30)

We now regularize $I$ as follows. We consider a sequence of functions $I_m$ such that for all $m$, $I_m$ is infinitely differentiable from $(t_0, +\infty)$ to $L^2(\Omega)$ and as $m \to \infty$,

$$I_m \to I \text{ in } L^2_{loc}((t_0, +\infty), L^2).$$  

(3.31)

Consider a solution $U_m \in C^\infty((t_0, +\infty), H^2 \cap H^1_0)$ to

$$\frac{\partial^2 U_m}{\partial t^2} - \eta \frac{\partial^2}{\rho \partial y^2} \left( \frac{\partial U_m}{\partial t} \right) - \frac{G}{\lambda \rho} \frac{\partial^2 U_m}{\partial y^2} = I_m.$$  

(3.32)

Equation (3.32) has been studied in [6, 7]. Inspired by arguments from these references, we introduce the energy functions $H_m$ and $F_m$ depending on a constant $\delta \in (0, 1)$ to be determined later

$$H_m(t) = \left\| \frac{\partial U_m}{\partial t}(t, \cdot) \right\|_{L^2}^2 + \frac{G}{\lambda \rho} \left\| \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{\eta}{\rho} \left\| \frac{\partial U_m}{\partial t}(t, \cdot) \right\|_{L^2}^2 + \delta \int_\Omega \left( \frac{\partial U_m}{\partial t} \right) \left( \frac{\partial U_m}{\partial y} \right) \left( t, \cdot \right) + 2 \frac{\eta}{\rho} \left\| \frac{\partial \frac{\partial U_m}{\partial t}}{\partial y}(t, \cdot) \right\|_{L^2}^2 + 2 \frac{G}{\lambda \rho} \delta^2 \int_\Omega \left( \frac{\partial \frac{\partial U_m}{\partial t}}{\partial y} \right) \left( t, \cdot \right) \left( \frac{\partial U_m}{\partial y} \right) \left( t, \cdot \right).$$

and

$$F_m(t) = \left( \frac{\eta}{\rho} - \delta^2 \frac{G}{\lambda \rho} \right) \left\| \frac{\partial \frac{\partial U_m}{\partial t}}{\partial y}(t, \cdot) \right\|_{L^2}^2 - \delta \left\| \frac{\partial U_m}{\partial t}(t, \cdot) \right\|_{L^2}^2 + \delta \frac{G}{\lambda \rho} \left\| \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \delta^2 \left\| \frac{\partial^2 U_m}{\partial t^2}(t, \cdot) \right\|_{L^2}^2.$$
We multiply (3.32) by $\frac{\partial U_m}{\partial t} + \delta U_m + \delta^2 \frac{\partial^2 U_m}{\partial t^2}$, integrate over $\Omega$ and find

$$\frac{1}{2} \frac{dH_m}{dt}(t) + F_m(t) = \int_{\Omega} I_m(t, y) \left( \frac{\partial U_m}{\partial t} + \delta U_m + \delta^2 \frac{\partial^2 U_m}{\partial t^2} \right) (t, y) dy.$$ 

We use the Poincaré inequality and choose $\delta$ sufficiently small, depending on the domain and the coefficients in (1.1) such that, for suitable constants $c_1, c_2$ and $c_3$,

$$c_1 \left( \left\| \frac{\partial}{\partial t} \frac{\partial U_m}{\partial y} (t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial U_m}{\partial y} (t, \cdot) \right\|_{L^2}^2 \right) \leq H_m(t) \leq c_2 \left( \left\| \frac{\partial}{\partial t} \frac{\partial U_m}{\partial y} (t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial U_m}{\partial y} (t, \cdot) \right\|_{L^2}^2 \right),$$

(3.33)

and

$$F_m(t) \geq c_3 \left( \left\| \frac{\partial}{\partial t} \frac{\partial U_m}{\partial y} (t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial U_m}{\partial y} (t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial^2 U_m}{\partial t^2} (t, \cdot) \right\|_{L^2}^2 \right).$$

(3.34)

Using the upper bound in (3.33) and (3.34) and the Young and the Poincaré inequalities, we obtain

$$\frac{dH_m}{dt}(t) + CH_m(t) \leq \| I_m(t, \cdot) \|_{L^2}^2.$$ 

(3.35)

We multiply the above equation by $e^{Ct}$, integrate from $t_0$ to $t$ and find

$$H_m(t)e^{Ct} \leq H_m(t_0)e^{Ct_0} + \int_{t_0}^{t} \| I_m(s, \cdot) \|_{L^2}^2 e^{Cs} ds.$$ 

(3.36)

Equation (3.29) is linear so that by (3.28) and (3.31), we can pass to the limit $m \to \infty$ in (3.36) and find, for all $t > t_0$,

$$H(t) e^{Ct} \leq H(t_0) e^{Ct_0} + \int_{t_0}^{t} \| I(s, \cdot) \|_{L^2}^2 e^{Cs} ds.$$ 

(3.37)

where $H$ is defined as $H_m$ with $U$ instead of $U_m$.

The study of (3.29) reduces to the understanding of (3.37). We now make precise the behaviour of $I$ or more precisely at the one of $\frac{\partial f \tau}{\partial t}$. We combine equations (1.1b) and (1.1c) to find

$$\frac{\partial f \tau}{\partial t} = f \left( -\frac{1}{\lambda} f \tau + \frac{G}{\lambda} \frac{\partial u}{\partial y} \right) + \tau \left( -f^2 - \nu f^3 + \xi |\tau| f^2 \right).$$

(3.38)

Multiplying the evolution equation (3.38) by $f \tau$ and integrating over $\Omega$ yields

$$\frac{1}{2} \left\| \frac{\partial f \tau}{\partial t} (t, \cdot) \right\|_{L^2}^2 = \int_{\Omega} \left( -\frac{1}{\lambda} f \tau - 1 - \nu f + \xi |\tau| \right) f^3 \tau^2 + \frac{G}{\lambda} \int_{\Omega} f^2 \tau \frac{\partial u}{\partial y} \leq C_0 \| f(t, \cdot) \|_{L^\infty}^2 \left( \| \frac{\partial u}{\partial y} (t, \cdot) \|_{L^2}^2 + \| \tau (t, \cdot) \|_{L^2}^2 \right),$$

(3.39)
where we have used the $L^\infty$-bounds on both $\tau$ and $f$ and the Cauchy-Schwarz inequality to derive the second line. Inserting (3.23) which gives the convergence in $\frac{1}{t}$ of $\|f(t, \cdot)\|_{L^\infty}$ and (3.26), equation (3.39) implies

$$
\left\| \frac{\partial f}{\partial t}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1 + t)^{-2 - 2\beta\lambda (1 - 4\epsilon)}.
$$

(3.40)

Since the $L^2$-norm of $\frac{\partial f}{\partial t}$ controls the $L^2$-norm of $I$, we insert (3.40) in (3.37) so that, for all $t > t_0$,

$$
H(t)e^{Ct} \leq H(t_0)e^{Ct_0} + \kappa_\epsilon \int_{t_0}^t \frac{e^{Cs}}{(1 + s)^{2 + 2\beta\lambda (1 - 4\epsilon)}} ds.
$$

(3.41)

Moreover, for $q > 0$, for all $t > t_0$, we integrate by parts to obtain

$$
\int_{t_0}^t \frac{e^{Cs}}{(1 + s)^q} ds \leq \frac{q}{C(1 + t_0)} \int_{t_0}^t \frac{e^{Cs}}{(1 + s)^q} ds + \frac{e^{Ct}}{C(1 + t)^q}.
$$

(3.42)

We insert (3.42) with $q = 2 + 2\beta\lambda (1 - 4\epsilon)$ in (3.41), so that for $t$ sufficiently large

$$
H(t) \leq \frac{1}{(1 + t)^{2 + 2\beta\lambda (1 - 4\epsilon)}}.
$$

(3.43)

Using the lower bound in (3.33), we have therefore obtained

$$
\left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1 + t)^{-2 - 2\beta\lambda (1 - 4\epsilon)}.
$$

(3.44)

**Step 4: Convergence rates (3.20) and (3.21).** Using (3.10) rewritten as

$$
\frac{1}{2} \frac{d}{dt} \left( \lambda \|\tau - \bar{\tau}\|_{L^2}^2 + \frac{G}{2\eta} \|\tau - \bar{\tau}\|_{L^2}^2 \right) \leq C \left( \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \|f(t, \cdot)\|_{L^\infty} \|\tau(t, \cdot)\|_{L^2}^2 \right),
$$

and convergence estimates (3.23), (3.26), (3.44), we obtain

$$
\|\tau - \bar{\tau}\|_{L^2}^2 \leq \kappa_\epsilon (1 + t)^{-2 - 2\beta\lambda (1 - 4\epsilon)},
$$

and eventually

$$
\left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1 + t)^{-2 - 2\beta\lambda (1 - 4\epsilon)}.
$$

We thus obtain (3.20) with $\alpha = 4\epsilon$ and conclude establishing (3.21) as follows: we return to (3.39) and improve the convergence estimate for $\frac{\partial f}{\partial t}$, namely

$$
\left\| \frac{\partial f}{\partial t}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1 + t)^{-4 - 2\beta\lambda (1 - 4\epsilon)}.
$$
This implies, mimicking (3.41) and using (3.42) with \(q = 4 + 2\beta(1 - 4\epsilon)\), that for \(t\) sufficiently large,

\[
\left\| \frac{\partial U}{\partial y} (t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1 + t)^{-4 - \frac{2}{\lambda}(1 - 4\epsilon)},
\]

that is (3.21) with \(\alpha = 4\epsilon\).

\[\Box\]

### 3.2 Case \(f_0 \equiv 0\)

In the case \(f_0 \equiv 0\), \(f\) vanishes for all time. System (1.1) then reads

\[
\begin{align*}
\rho \frac{\partial u}{\partial t} &= \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial\tau}{\partial y}, \quad (3.46a) \\
\lambda \frac{\partial \tau}{\partial t} &= G \frac{\partial u}{\partial y}. \quad (3.46b)
\end{align*}
\]

The existence and uniqueness of a regular solution to (3.46) is easy to establish. The longtime behaviour of system (3.46) is now made precise.

**Theorem 3.3** Supply system (3.46) with homogeneous boundary conditions. Consider a solution \((u, \tau)\) in the space

\[
(C([0, +\infty); H^1) \cap L^2_{t,loc}([0, +\infty); H^2)) \times C([0, +\infty); H^1)
\]

Then, the solution converges exponentially fast to the steady state \((0, \bar{\tau}_0)\) in \(H^1(\Omega) \times L^2(\Omega)\) in the longtime: there exist two constants \(C,\) independent from time and initial data, and \(C_0,\) independent from time, such that, for \(t\) sufficiently large,

\[
\left\| \frac{\partial u}{\partial y} (t, \cdot) \right\|_{L^2} + \|\tau(t, \cdot) - \bar{\tau}_0\|_{L^2} \leq C_0 e^{-Ct}.
\]

**Proof.** We perform the same manipulations as those used to obtain equation (3.29) in Step 3 of the proof of Theorem 3.2. Since we deal here with the case \(f \equiv 0\), we have \(I = 0\) in (3.29). We have proven that studying the longtime behaviour to (3.29) amounts to proving (3.37). We therefore find, for \(t\) sufficiently large,

\[
\left\| \frac{\partial U}{\partial y} (t, \cdot) \right\|_{L^2} \leq C_0 e^{-Ct}.
\]

We next differentiate equation (3.46a) with respect to \(t\) and insert (3.46b) to obtain

\[
\frac{\partial^2 u}{\partial t^2} - \eta \frac{\partial^2}{\partial y^2} \left( \frac{\partial u}{\partial t} \right) - \frac{G}{\rho \lambda} \frac{\partial^2 u}{\partial y^2} = 0.
\]

The function \(u\) satisfies the same equation as \(U\) and thus has the same convergence rate. Applying the Gronwall Lemma to (3.9) therefore implies, for \(t\) sufficiently large,

\[
\| (\tau - \bar{\tau}) (t, \cdot) \|_{L^2} \leq C_0 e^{-Ct}.
\]

Integrating (3.46b) over \(\Omega\), we have

\[
\lambda \frac{\partial \bar{\tau}}{\partial t} = 0,
\]

so that \(\bar{\tau}(t) = \bar{\tau}_0\) for all times. We thus have the convergence estimate (3.47).

\[\Box\]
4 Longtime behaviour for non-homogeneous boundary conditions in a simple case

In this section, we study the longtime behaviour of the system (1.1) supplied with non-homogeneous boundary conditions $u(t,0) = 0$ and $u(t,1) = a$ (where $a$ is a constant scalar different from zero and chosen positive, without loss of generality, $a > 0$).

We denote $(u_\infty, \tau_\infty, f_\infty)$ a stationary state to the system (1.1). We only consider the simplified case

$$f_\infty > 0 \text{ everywhere.} \tag{4.1}$$

The only stationary state that satisfies (4.1) is made explicit in subsection 4.1. In subsection 4.2, we show convergence in the longtime to this stationary state for small initial perturbations. In subsection 4.3, we study the longtime behaviour for initial data that satisfy $f_0 > 0$ without any smallness conditions, but only in a simplified case that reduces system (1.1) to a system of ordinary differential equations.

We do not state any result for the convergence to stationary states when fluidity vanishes on some part of $\Omega$.

4.1 Stationary state

The following lemma makes precise the stationary state that satisfies the condition (4.1).

**Lemma 4.1 (Stationary state)** Supply system (1.1) with non-homogeneous boundary conditions $u_\infty(0) = 0$ and $u_\infty(1) = a > 0$. The unique stationary solution $(u_\infty, \tau_\infty, f_\infty)$ in $(H^1(\Omega))^3$ satisfying (4.1) reads

$$(u_\infty, \tau_\infty, f_\infty)(y) = \left( ay, \frac{\sqrt{1 + 4\nu \xi Ga} + 1}{2\xi}, \frac{\sqrt{1 + 4\nu \xi Ga} - 1}{2\nu} \right). \tag{4.2}$$

**Remark 4.1** It is easy to extend the above result to stationary solutions $(u_\infty, \tau_\infty, f_\infty)$ in $H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$ that satisfy $f_\infty \neq 0$. Introducing $\Omega_\infty = \{ y \in \Omega, f_\infty(y) > 0 \}$ and $\beta_\infty = \text{meas}(\Omega_\infty)$, the set of such stationary solutions reads

$$\left( \frac{\partial u_\infty}{\partial y}, \tau_\infty, f_\infty \right)(y) = \begin{cases} \left( a, \frac{1 + \xi \tau_L}{\nu}, 0 \right) & \text{on } \Omega_\infty \\ \left( 0, \eta \frac{a}{\beta_\infty} + \tau_L, 0 \right) & \text{on } \Omega \setminus \Omega_\infty, \end{cases}$$

with $\tau_L = \frac{1}{2\xi} \left( 1 + \sqrt{1 + 4\nu \xi Ga} / \beta_\infty \right)$.

**Proof.** The stationary states $(u_\infty, \tau_\infty, f_\infty) : \Omega \rightarrow \mathbb{R}$ of the system (1.1) that satisfy (4.1) are solutions of the following system

$$\begin{cases} 0 = \eta \frac{\partial^2 u_\infty}{\partial y^2} + \frac{\partial \tau_\infty}{\partial y}, \tag{4.3a} \\ 0 = G \frac{\partial u_\infty}{\partial y} - f_\infty \tau_\infty, \tag{4.3b} \\ f_\infty = \frac{-1 + \xi |\tau_\infty|}{\nu}. \tag{4.3c} \end{cases}$$
We now show that such a steady state is unique and explicitly identify it. Since $\tau_\infty \in H^1(\Omega)$, (4.3a) shows that $u_\infty$ belongs to $H^2(\Omega)$. We integrate (4.3a) and (4.3b) over $\Omega$ and obtain

$$K = \eta \frac{\partial u_\infty}{\partial y} + \tau_\infty,$$

where $K$ is a constant and, using the boundary conditions on $u_\infty$,

$$\int_\Omega f_\infty \tau_\infty = Ga.$$  

(4.5)

We combine (4.3b) and (4.4) to obtain

$$\left(\frac{\eta}{G} f_\infty + 1\right) \tau_\infty = K$$

(4.6)

so that $\tau_\infty$ has the constant sign of $K$. Equation (4.5) then implies that $\tau_\infty$, thus $K$ are positive.

We now claim that $\tau_\infty$ is constant over $\Omega$: inserting (4.3c) in (4.6), we obtain that $\tau_\infty$ satisfies

$$\tau_\infty \left(1 + \eta \frac{-1 + \xi \tau_\infty}{G \nu}\right) = K.$$

It is easy to see that this equation has a unique positive solution $\tau_\infty$. It follows from (4.4) that $\frac{\partial u_\infty}{\partial y}$ is constant throughout $\Omega$ so that, using the boundary conditions, $u_\infty(y) = ay$.

We rewrite equation (4.3b) as

$$Ga = \frac{-1 + \xi \tau_\infty}{\nu} \tau_\infty,$$

to find the value of

$$\tau_\infty = \frac{1}{2\xi} \left(1 + \sqrt{1 + 4\nu \xi Ga}\right).$$

The stationary state reads $\left(ay, \tau_\infty, \frac{-1 + \xi \tau_\infty}{\nu}\right)$, that is (4.2).

4.2 Longtime behaviour with smallness assumption

The following theorem states the convergence in the longtime to the stationary state (4.2) for small initial perturbations.

**Theorem 4.1** Supply system (1.1) with non-homogeneous boundary conditions $u(t, 0) = 0$ and $u(t, 1) = a > 0$. Consider the solution $(u, \tau, f)$ the existence and uniqueness of which have been established in Theorem 2.1 and the associated stationary state $(u_\infty, \tau_\infty, f_\infty)$ defined by (4.2). There exists $\epsilon > 0$ (sufficiently small so that at least $\tau_0$ and $f_0$ are positive), such that, if the initial data $(u_0, \tau_0, f_0)$ for (1.1) satisfy

$$\|u_0 - u_\infty\|_{H^1}^2 + \|	au_0 - \tau_\infty\|_{L^\infty}^2 + \|f_0 - f_\infty\|_{L^\infty}^2 \leq \epsilon^2,$$

then $(u(t), \tau(t), f(t)) \rightarrow (u_\infty, \tau_\infty, f_\infty)$ as $t \rightarrow \infty$. 

\diamond
then the solution \((u, \tau, f)\) of system \((1.1)\) converges, as \(t\) goes to infinity, to \((u_\infty, \tau_\infty, f_\infty)\) in \(H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)\).

More precisely, there exist a constant \(C\) independent from \(\epsilon\), time and initial data and a constant \(\kappa_\epsilon\) independent from time such that, for \(t\) sufficiently large,

\[
\|u(t, \cdot) - u_\infty(\cdot)\|_{H^1} + \|\tau(t, \cdot) - \tau_\infty(\cdot)\|_{L^\infty} + \|f(t, \cdot) - f_\infty(\cdot)\|_{L^\infty} \leq \kappa_\epsilon e^{-(C-\epsilon)t}. \tag{4.7}
\]

**Remark 4.2** It is indeed possible, under the same assumptions, to prove that both \(\tau\) and \(f\) converge to zero in \(H^1(\Omega)\) and not only in \(L^\infty(\Omega)\). The proof is more tedious. We omit it here for brevity and refer to [1].

Before we get to the proof, we note that we will return to system \((1.1)\), and not \((2.4)\) since of course boundary conditions will play a crucial role throughout the section. We also rewrite system \((1.1)\) as

\[
\begin{align*}
\rho \frac{\partial u}{\partial t} &= \eta \frac{\partial^2 u}{\partial y^2} ((u - u_\infty) + u_\infty) + \frac{\partial}{\partial y} ((\tau - \tau_\infty) + \tau_\infty), \\
\lambda \frac{\partial \tau}{\partial t} &= G \frac{\partial}{\partial y} ((u - u_\infty) + u_\infty) - ((f - f_\infty) + f_\infty)((\tau - \tau_\infty) + \tau_\infty), \\
\frac{\partial f}{\partial t} &= (-1 + \xi((\tau - \tau_\infty) + \tau_\infty))((f - f_\infty) + f_\infty)^2 - \nu((f - f_\infty) + f_\infty)^3.
\end{align*}
\]

To lighten the notation, we henceforth denote \((u, \tau, f)\) instead of \((u - u_\infty, \tau - \tau_\infty, f - f_\infty)\) and consider

\[
\begin{align*}
\rho \frac{\partial u}{\partial t} &= \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, \tag{4.8a} \\
\lambda \frac{\partial \tau}{\partial t} &= G \frac{\partial u}{\partial y} - f_\infty \tau - \tau_\infty f - f \tau, \tag{4.8b} \\
\frac{\partial f}{\partial t} &= -\nu(f + f_\infty)^2 f + \xi(f + 2f_\infty) f \tau + \xi f^2_\infty \tau, \tag{4.8c}
\end{align*}
\]

supplied with homogeneous boundary conditions on \(u\) and initial data that satisfy

\[
\|u_0\|_{H^1}^2 + \|\tau_0\|_{L^\infty}^2 + \|f_0\|_{L^\infty}^2 \leq \epsilon^2.
\]

**Proof.** The proof is divided into three steps. The first step establishes a priori energy estimates on system \((4.8)\). In the second step, we show that the solution remains small for sufficiently small perturbations. In Step 3 we show that, still for small perturbations, the solution converges to the steady state and that the rate of convergence is exponential.

As in the previous proofs, \(C\) and \(\kappa_\epsilon\) denote various constants the value of which may vary from one instance to another, the actual value being irrelevant.

**Step 1: A priori energy estimates.** We argue as in Step 2 of the proof of Theorem 2.1. We multiply \((4.8a), (4.8b)\) and \((4.8c)\) respectively by \(u, \tau\) and \(f\), integrate over \(\Omega\)
and find

\[
\rho \frac{d}{dt} \|u(t, \cdot)\|_{L^2}^2 + \eta \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 = \int_\Omega u \frac{\partial \tau}{\partial y}(t, \cdot) dt,
\]

\[
\frac{\lambda}{2} \frac{d}{dt} \|\tau(t, \cdot)\|_{L^2}^2 + \int_\Omega (f + f_\infty) \tau^2(t, \cdot) = G \int_\Omega \frac{\partial u}{\partial y}(t, \cdot) - \tau_\infty \int_\Omega f(t, \cdot),
\]

\[
\frac{1}{2} \frac{d}{dt} \|f(t, \cdot)\|_{L^2}^2 + \nu \int_\Omega (f + f_\infty)^2 f^2(t, \cdot) = \xi f_\infty^2 \int_\Omega f(t, \cdot) + \xi \int_\Omega (f + 2f_\infty)^2 f^2(t, \cdot).
\]

Combining these estimates leads to

\[
\frac{1}{2} \frac{d}{dt} \left( \rho G \|u(t, \cdot)\|_{L^2}^2 + \lambda \|\tau(t, \cdot)\|_{L^2}^2 + \frac{\tau_\infty}{\xi f_\infty^2} \|f(t, \cdot)\|_{L^2}^2 \right) + \eta G \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 + f_\infty \|\tau(t, \cdot)\|_{L^2}^2 + \tau_\infty \left( \|f(t, \cdot)\|_{L^2}^2 + \|f(t, \cdot)\|_{L^2}^2 \right) = 0,
\]

(4.9)

using the \(L^\infty\)-estimate on \([0,T]\) on \(f\) established in the proof of Theorem 2.1 and the Young inequality. We now use Step 3 of the proof of Theorem 2.1 and more precisely the estimate (2.13) on \(U\) defined by (2.13). We have

\[
\frac{d}{dt} \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \eta \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \leq C \left( \|u(t, \cdot)\|_{L^2}^2 + \|f\|_{L^\infty(L^\infty)} \|\tau(t, \cdot)\|_{L^2}^2 \right).
\]

(4.10)

Now that we have estimates in Sobolev spaces, we turn to point wise estimates on \(\tau\) and \(f\). We refine our argument in Step 4 of the proof of Theorem 2.1. We rewrite the evolution equation (4.8b) as

\[
\lambda \frac{d}{dt} + \left( f + f_\infty + \frac{G}{\eta} \right) \tau = G \frac{\partial u}{\partial y} + \frac{G}{\eta} \bar{\tau} - \tau_\infty f,
\]

multiply it by \(\tau\), apply the Young inequality and obtain

\[
\frac{\lambda}{2} \frac{d}{dt} \|\tau\|_{L^2}^2 + \left( f + f_\infty + \frac{G}{2\eta} \right) \|\tau\|_{L^2}^2 \leq \eta \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^\infty}^2 + \|\tau(t, \cdot)\|_{L^2}^2 - \tau_\infty f \tau.
\]

(4.11)

Similarly, we multiply (4.8c) by \(f\) and find

\[
\frac{1}{2} \frac{d}{dt} \|f\|_{L^2}^2 + \nu \left( f + f_\infty \right)^2 \|f\|_{L^2}^2 = \xi (f + 2f_\infty) \|f\|_{L^2}^2 + \xi f_\infty^2 \tau f.
\]

(4.12)

We combine (4.11) and (4.12) and use the Poincaré inequality on \(\frac{\partial U}{\partial y}\), the spatial average of which is zero, to obtain

\[
\frac{\lambda}{2} \frac{d}{dt} \|\tau\|_{L^2}^2 + \frac{\tau_\infty}{2\xi f_\infty^2} \frac{d}{dt} \|f\|_{L^2}^2 + \left( f + f_\infty + \frac{G}{2\eta} \right) \|\tau\|_{L^2}^2
\]

\[
+ \frac{\tau_\infty}{\xi f_\infty^2} \left( \nu (f + f_\infty)^2 - \xi \|f\|_{L^\infty(L^\infty)} + 2f_\infty \|\tau\|_{L^\infty(L^\infty)} \right) \|f\|_{L^2}^2
\]

\[
\leq \eta \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2.
\]

(4.13)
**Step 2: Smallness of the solution for small perturbations.** We now prove that, for $\epsilon \in (0, 1)$ to be fixed later on and an initial condition satisfying

$$\|u_0\|_{H^1}^2 + \|\tau_0\|_{L^\infty}^2 + \|f_0\|_{L^\infty}^2 \leq \epsilon^2,$$  \hspace{1cm} (4.14)

we have, for all time $t > 0$,

$$\|u(t, \cdot)\|_{H^1}^2 + \|\tau(t, \cdot)\|_{L^\infty}^2 + \|f(t, \cdot)\|_{L^\infty}^2 \leq \epsilon.$$  \hspace{1cm} (4.15)

We argue by contradiction and suppose

$$T_M = \inf \{ t \in \mathbb{R}^+ \mid \left( \|u(t, \cdot)\|_{H^1}^2 + \|\tau(t, \cdot)\|_{L^\infty}^2 + \|f(t, \cdot)\|_{L^\infty}^2 \right) \geq \epsilon \}$$

is finite.

For all $t \leq T_M$, we use the estimates from the previous step. For $\epsilon$ sufficiently small such that all the terms in the left-hand side of (4.13) are positive (this gives one condition on $\epsilon$), we have, integrating (4.13) from 0 to $t$,

$$\rho G \|u(t, \cdot)\|_{L^2}^2 + \lambda \|\tau(t, \cdot)\|_{L^2}^2 + \frac{\tau_\infty}{\xi f_{\infty}^2} \|f(t, \cdot)\|_{L^2}^2 \leq C \epsilon^2.$$  \hspace{1cm} (4.16)

Integrating (4.10) from 0 to $t$ then yields

$$\left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \int_0^t \left\| \frac{\partial^2 U}{\partial y^2}(s, \cdot) \right\|_{L^2}^2 \, ds \leq C \epsilon^2.$$  \hspace{1cm} (4.17)

We now integrate (4.13) from 0 to $t$ and get

$$\frac{\lambda}{2} |\tau|^2 + \frac{\tau_\infty}{2 \xi f_{\infty}^2} |f|^2 \leq C \epsilon^2.$$  \hspace{1cm} (4.18)

Choosing $\epsilon$ sufficiently small such that $C \epsilon^2 < \epsilon$ (which gives another condition on $\epsilon$) contradicts the definition of $T_M$, and so $T_M = \infty$. It follows that (4.15) holds for all time $t > 0$, the solution remains small.

**Step 3: Convergence to the stationary state.** We now prove that, if the initial data satisfy (4.14), then the solution converges exponentially fast to the stationary state in the longtime. For $t$ sufficiently large, (4.9) implies that

$$\|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^\infty}^2 + \|f(t, \cdot)\|_{L^\infty}^2 \leq \kappa e^{-(C-\epsilon)t}.$$  \hspace{1cm} (4.18)
Adding (4.10) multiplied by \(2\rho\) to (4.13) leads to
\[
\frac{d}{dt} \left( \left\| \frac{\partial U}{\partial y} (t, \cdot) \right\|_{L^2}^2 + \frac{\lambda}{2} \tau^2 + \frac{\tau_{\infty}}{2 \xi f_{\infty}^2} |f|^2 \right) + \eta \left\| \frac{\partial^2 U}{\partial y^2} (t, \cdot) \right\|_{L^2}^2 + (f + f_{\infty} + \frac{G}{2\eta}) \tau^2 \\
+ \frac{\tau_{\infty}}{\xi f_{\infty}^2} \left( \nu(f + f_{\infty})^2 - \xi \|f\|_{L^2(\mathbb{R})}^2 - 2f_{\infty} \|\tau\|_{L^2(\mathbb{R})} |f|^2 \right) \\
\leq C \left( \|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^\infty}^2 + \|f(t, \cdot)\|_{L^2}^2 + \|f\|_{L^2(\mathbb{R})}^2 \|\tau(t, \cdot)\|_{L^2}^2 \right).
\]

We use the Poincaré inequality on \(\frac{\partial U}{\partial y}\), the spatial average of which is zero, apply the Gronwall Lemma, insert (4.18) and find
\[
\left\| \frac{\partial U}{\partial y} (t, \cdot) \right\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^\infty}^2 + \|f(t, \cdot)\|_{L^2}^2 \leq \kappa e^{-\left(C - \epsilon\right)t}.
\]

This convergence estimate is equivalent to (4.7) and we have exponential convergence.

\[\diamond\]

4.3 Longtime behaviour without smallness assumption (simplified case)

We now examine the longtime behaviour of system (1.1) supplied with \textit{not necessarily small} initial data \((u_0, \tau_0, f_0)\). We are unable to prove a general result and focus our attention to the particular case where the initial condition is \(u_0 = ay\) (a positive constant), \(\tau_0 = \text{constant} = \tau_0\), \(f_0 = \text{constant} = f_0 > 0\). In such a case, a substantial simplification occurs. Indeed, (1.1) reduces to the following system of ordinary differential equations:

\[
\begin{align*}
\lambda \frac{\partial \tau}{\partial t} &= -f\tau + Ga \\
\frac{\partial f}{\partial t} &= (-1 + \xi |\tau|) f^2 - \nu f^3,
\end{align*}
\]

supplied with initial conditions \(\tau_0, f_0 \in \mathbb{R}\) with \(f_0 > 0\).

System (4.19) has a unique steady state such that \(f_{\infty} > 0\) and it reads
\[
(\tau_{\infty}, f_{\infty}) = \left( \frac{\sqrt{1 + 4\nu \xi Ga} + 1}{2\xi}, \frac{\sqrt{1 + 4\nu \xi Ga} - 1}{2\nu} \right).
\]

Indeed, such a steady state \((\tau_{\infty}, f_{\infty})\) satisfies \(f_{\infty} \tau_{\infty} = Ga\) (so that \(\tau_{\infty} > 0\)) and \(\nu f_{\infty} = -1 + \xi \tau_{\infty}\). Combining these equations implies \((-1 + \xi \tau_{\infty}) \tau_{\infty} = \nu Ga\). This equation has a unique solution given in (4.20).

In addition, we introduce the condition
\[
-\frac{1}{\lambda} - 2 + 2\xi(1 + Ga) \left( \frac{1}{\sigma} + \frac{\lambda \xi}{2Ga} \left( \frac{\nu \sigma + 1}{\xi} + \frac{4}{\xi} \right)^2 \right) < 0,
\]
with
\[
\sigma = \min \left\{ \frac{3Ga}{Ga \nu + 4\tau_{\infty}}, \frac{\sqrt{1 + 4\nu \xi Ga} - 1}{3\nu} \right\},
\]

30
We are unable to perform our proof without this additional assumption. The numerical simulations we perform (see Figure 3) however show convergence holds even when (4.21) is not satisfied.

**Theorem 4.2** Supply system \((4.19)\) with initial conditions \(\tau_0, f_0 \in \mathbb{R}\) with \(f_0 > 0\). Then the solution \((\tau, f)\) remains bounded.

In addition, under assumption \((4.21)\), the solution \((\tau, f)\) converges to \((\tau_\infty, f_\infty)\) in the longtime and the rate of convergence is exponential: for \(t\) sufficiently large,

\[
|\tau(t) - \tau_\infty| + |f(t) - f_\infty| \leq C_0 e^{-C_r t},
\]

where \(C_0\) is a constant independent from time and \(C_r\) reads

\[
C_r = \begin{cases} 
\frac{1}{2} \left( \frac{f_\infty}{\lambda} + \nu f^2_\infty \right) - \frac{1}{2} \sqrt{\Delta}, & \text{if } \Delta \geq 0, \\
\frac{1}{2} \left( \frac{f_\infty}{\lambda} + \nu f^2_\infty \right), & \text{if } \Delta < 0,
\end{cases}
\]

with

\[
\Delta = f^2_\infty \left( \left( \frac{1}{\lambda} + \nu f_\infty \right)^2 - 4 \left( \frac{\nu}{\lambda} f_\infty + \frac{\xi}{\lambda} \tau_\infty \right) \right).
\]

**Proof.** The proof is divided into seven steps. Step 1 introduces simplifications on the initial data and the system, that are not restrictive for the longtime behaviour. Some notation is given in Step 2. A lower bound on \(f\) is derived in Step 3 and is used in Step 4 to prove that the solution is bounded. Further restrictions are made in Step 5 still without loss of generality. Step 6 establishes the convergence, which is proven to be exponential in Step 7.

We consider until Step 4 the maximal solution to \((4.19)\); although the solution a posteriori exists for all times because of boundedness.

**Step 1: Simplifications on the initial data.** We show that \(\tau\) and \(f\) solution to \((4.19)\) remain positive, possibly after some time for \(\tau\). We first remark that, since \(f_0 > 0\), \(f > 0\) for all times, arguing as in Step 1 of the proof of Theorem 2.1. On the other hand, if \(\tau \leq 0\) on some time interval, evolution equation \((4.19a)\) thus implies that \(\tau\) increases strictly on this time interval (recalling that \(a > 0\)). Hence, there exist a time \(T_0\) such that \(\tau(T_0) > 0\). Moreover, for all \(t > T_0\), \(\tau\) remains positive (since if \(\tau\) is zero at one time \(T_1 > T_0\), \(\frac{d\tau}{dt}(T_1) = Ga > 0\), which is in contradiction with \(\tau > 0\) for \(t < T_1\).)

For the purpose of studying the longtime limit, we may always consider, without loss of generality, the system

\[
\begin{aligned}
\lambda \frac{\partial \tau}{\partial t} &= -f \tau + Ga \\
\frac{\partial f}{\partial t} &= (-1 + \xi \tau) f^2 - \nu f^3,
\end{aligned}
\]

supplied with positive initial conditions \(\tau_0, f_0\).
Step 2: Some notation. We consider the three subdomains:

\[
A_1 = \left\{ (\tau, f) | f \geq \frac{\xi \tau - 1}{\nu}, f \leq \sigma \right\},
\]

\[
A_2 = \left\{ (\tau, f) | f \leq \frac{\xi \tau - 1}{\nu}, f \leq \sigma \right\},
\]

\[
A_3 = \left\{ (\tau, f) | f \geq \sigma \right\},
\]

where we recall that \( \sigma \) is defined by (4.22). We also introduce their intersections:

\[
\Gamma_{13} = \left\{ (\tau, \sigma) | \tau \leq \frac{\xi \sigma - 1}{\nu} \right\},
\]

\[
\Gamma_{12} = \left\{ (\tau, f) | f = \frac{\xi \tau - 1}{\nu}, f \leq \sigma \right\},
\]

\[
\Gamma_{23} = \left\{ (\tau, \sigma) | \tau \geq \frac{\xi \sigma - 1}{\nu} \right\}.
\]

See Figure 1 for a graphical description.

![Figure 1: Notation on \((0, +\infty) \times (0, +\infty)\)](image)

Step 3: Lower bound on \( f \). We now establish a lower bound for the fluidity \( f \) in each domain. In the cases \((\tau_0, f_0) \in A_2 \) or \((\tau_0, f_0) \in A_3 \), we have

\[
f \geq \min \{ f_0, \sigma \}.
\]

The case \((\tau_0, f_0) \in A_1 \) requires more developments. The evolution equations (4.26a) and (4.26b) respectively rewrite

\[
\frac{d}{dt} \left( \frac{1}{f} \right) = 1 - \xi \tau + \nu f,
\]

\[
\frac{\lambda}{2} \frac{d}{dt} \left( \tau - \frac{4}{\xi} \right)^2 = -f \tau \left( \tau - \frac{4}{\xi} \right) + Ga \left( \tau - \frac{4}{\xi} \right).
\]
We combine these two equations and obtain

\[
\lambda \xi \frac{d}{dt} \left( \frac{Ga}{\lambda \xi} f + \frac{1}{2} \left( \tau - \frac{4}{\xi} \right)^2 \right) = -3Ga + Gav f - \xi f^2 \tau + 4f \tau \\
\leq -3Ga + (Gav + 4\tau_\infty) \sigma \\
\leq 0,
\]

where we have used firstly that \(0 \leq f \leq \sigma\) and \(\tau \leq \tau_\infty\) in \(A_1\) and secondly (4.22). Integrating (4.27) yields

\[
\frac{Ga}{\lambda \xi} f(t) \leq \frac{Ga}{\lambda \xi} \max \left\{ \frac{1}{\tilde{f}_0}, \frac{1}{\sigma} \right\} + \frac{1}{2} \left( \frac{\nu \sigma + 1}{\xi} + \frac{4}{\xi} \right)^2,
\]

using that \(\tau \leq \frac{\nu \sigma + 1}{\xi}\). We introduce

\[
m_f = \left( \max \left\{ \frac{1}{\tilde{f}_0}, \frac{1}{\sigma} \right\} + \frac{\lambda \xi}{2Ga} \left( \frac{\nu \sigma + 1}{\xi} + \frac{4}{\xi} \right)^2 \right)^{-1},
\]

which is therefore a lower bound for \(f\) in the region \(A_1\). This lower bound also holds for initial conditions that belong to \(A_2\) and \(A_3\) and we thus have, for all \(t > 0\),

\[
f(t) \geq m_f,
\]

with \(m_f\) defined by (4.28).

**Step 4: Boundedness.** The purpose of this step is to prove that the solution \((\tau, f)\) remains bounded. Applying the Duhamel formula on (4.26a) yields

\[
\tau(t) = e^{-\int_0^t \frac{f(s)}{\lambda} ds} \tau_0 + \frac{Ga}{\lambda} \int_0^t e^{-\int_s^t \frac{f(\tau')}{\lambda} d\tau'} ds,
\]

so that, using the lower bound (4.29) on \(f\),

\[
\tau(t) \leq e^{-\frac{m_f t}{\lambda}} \tau_0 + \frac{Ga}{m_f}.
\]

Therefore, \(\tau\) is bounded, and there exists a time \(t_0\) such that, for all \(t > t_0\),

\[
\tau(t) \leq \frac{Ga + 1}{m_f}.
\]

We now turn to the boundedness of \(f\). We introduce \(M_\tau = \frac{Ga + 1}{m_f}\) and \(M_f = \frac{2}{\nu} (-1 + \xi M_\tau)\). We will show that, for all \(t > t_0\),

\[
f(t) < \max (f(t_0), M_f).
\]
We distinguish between two cases. Let us first suppose that $\frac{\partial f}{\partial t}(t_0) \geq 0$. In this case, $f(t_0) \leq \frac{1}{\nu}(-1 + \xi \tau(t_0)) < M_f$ because of (4.30). Moreover, for all $t > t_0$, $f(t) < M_f$. Indeed, by contradiction, if

$$t_1 = \inf \{ t > t_0, f(t_1) = M_f \} < +\infty,$$

then, by continuity, $\frac{\partial f}{\partial t}(t_1) \geq 0$. On the other hand, we have

$$\frac{\partial f}{\partial t}(t_1) = f^2(t_1)(-1 + \xi \tau(t_1) - \nu f(t_1)) < f^2(t_1)(-1 + \xi M_\tau - \nu M_f) < 0,$$

hence the contradiction.

In the other case $\frac{\partial f}{\partial t}(t_0) < 0$, $f$ strictly decreases until (possibly) equality occurs at a later time $t_3 \left(\frac{\partial f}{\partial t}(t_3) = 0\right)$, which leads to the previous case with $t_3$ instead of $t_0$. In any case, we have obtained (4.31) for all $t > t_0$.

**Step 5: Further simplifications on the initial data.** Table 1 first summarizes how $(\tau, f)$ behaves when it touches an intersection line. We use Table 1 to show that the solution enters region $A_3$ at some time.

| starting line | $\frac{\partial f}{\partial t}$ | $\frac{\partial \tau}{\partial t}$ | entering region |
|---------------|-------------------------------|-------------------------------|-----------------|
| $\Gamma_{13}$ | -                             | +                             | $A_1$           |
| $\Gamma_{12}$ | 0                             | +                             | $A_2$           |
| $\Gamma_{23}$ | +                             |                               | $A_3$           |

Table 1: Motion on intersection lines

solution enters region $A_3$ at some time.

In region $A_1$, we have $\frac{\partial \tau}{\partial t} > 0$, so that there does not exist any periodic orbit inside region $A_1$. There is also no steady state in this region. Using the Poincaré-Bendixson Theorem on the bounded solution of ordinary differential equation system (4.26), the solution leaves region $A_1$ at some time. According to Table 1, it enters region $A_2$.

Applying similar arguments on region $A_2$ where $\frac{\partial f}{\partial t} > 0$, the solution enters region $A_3$ at some time.

We can therefore restrict the studying of the longtime limit to initial data $(\tau_0, f_0)$ that belongs to region $A_3$, without loss of generality.

The bounds (4.29) and (4.30) become

$$f(t) \geq \left(\frac{1}{\sigma} + \frac{\lambda \xi}{2G\alpha} \left(\frac{\nu \sigma + 1}{\xi} + \frac{4}{\xi}\right)^2\right)^{-1}.$$

(4.32)
and
\[
\tau(t) \leq (Ga + 1) \left( \frac{1}{\sigma} + \frac{\lambda \xi}{2Ga} \left( \frac{\nu \sigma + 1}{\xi} + \frac{4}{\xi} \right)^2 \right).
\] (4.33)

**Step 6: Convergence.** We introduce
\[
G(\tau, f) = -\frac{1}{\lambda} f \tau + \frac{1}{\lambda} Ga
\]
and
\[
F(\tau, f) = (-1 + \xi \tau) f^2 - \nu f^3.
\]
We have
\[
\frac{\partial G(\tau, f)}{\partial \tau} + \frac{\partial F(\tau, f)}{\partial f} = f \left( -\frac{1}{\lambda} - 2 + 2 \xi \tau - 3 \nu f \right),
\]
and
\[
< f \left( -\frac{1}{\lambda} - 2 + 2 \xi (1 + Ga) \left( \frac{1}{\sigma} + \frac{\lambda \xi}{2Ga} \left( \frac{\nu \sigma + 1}{\xi} + \frac{4}{\xi} \right)^2 \right) \right),
\]
using (4.33) and the positivity of \( f \). Because of our assumption (4.21), the right-hand sides \( F \) and \( G \) of (4.26) satisfy
\[
\frac{\partial G(\tau, f)}{\partial \tau} + \frac{\partial F(\tau, f)}{\partial f} < 0.
\]
According to the Dulac Criterion, there does not exist any periodic orbit for (4.26). Since it has only one steady state \((\tau_\infty, f_\infty)\), the solution converges to it:
\[
\lim_{t \to \infty} (|\tau(t) - \tau_\infty| + |f(t) - f_\infty|) = 0.
\]

**Step 7: Exponential convergence.** Now that we have convergence to the steady state, we can use linear stability. System (4.26) linearized around the stationary state \((\tau_\infty, f_\infty)\) reads
\[
\frac{d}{dt} \begin{pmatrix} \tau_l \\ f_l \end{pmatrix} = \begin{pmatrix} -f_\infty & -\tau_\infty \\ \xi f_\infty^2 & -\nu f_\infty^2 \end{pmatrix} \begin{pmatrix} \tau_l \\ f_l \end{pmatrix}.
\]
The eigenvalues of the associated matrix depend on the sign of \( \Delta \) defined by (4.25). If \( \Delta < 0 \), the eigenvalues are complex and their real part is \(-\frac{1}{2} \left( \frac{f_\infty}{\lambda} + \nu f_\infty^2 \right)\). If \( \Delta \geq 0 \), the eigenvalues are real negative, the smaller one in absolute value is \(-\frac{1}{2} \left( \frac{f_\infty}{\lambda} + \nu f_\infty^2 \right) + \frac{1}{2} \sqrt{\Delta} \).
The real part of the eigenvalues gives the rate of convergence and hence of values of \( C_r \) in (4.24).
5 Numerical results

In this section, we present numerical simulations that complement the theoretical results on the behaviour of the previous sections.

We simulate numerically (1.1) in the interval $\Omega = [0,1]$ and the interval $[0,T]$ for $T = 10000$. The system is supplied either with homogeneous boundary conditions or non-homogeneous boundary conditions $u(t,0) = 0$ and $u(t,1) = a$ for all time $t \in [0,T]$. In the latter case, we take $a = 1$. As for the initial conditions, we take sinusoidal functions for all three fields. The values of $u_0$ oscillate between $-0.002$ and $0.002$ for homogeneous boundary conditions and between $0$ and $a$ otherwise. The values of $\tau_0$ and $f_0$ oscillate between $-0.5$ and $0.5$.

We use the following set of physical parameters. The density $\rho = 0.001$ and the viscosity $\eta = 1$ so that the Reynolds number is low. The elastic modulus $G$ and the coefficients $\xi$ and $\nu$ are equal to one. The characteristic relaxation time $\lambda$ is $0.5$ unless otherwise stated.

System (1.1) is solved using a constant time step $\Delta t = 0.005$ with the following time scheme:

\[
\begin{align*}
\frac{\rho}{\Delta t} (u_n - u_{n-1}) &= \eta \frac{\partial^2 u_n}{\partial y^2} + \frac{\partial \tau_{n-1}}{\partial y}, \\
\frac{\lambda}{\Delta t} (\tau_n - \tau_{n-1}) &= G \frac{\partial u_n}{\partial y} - f_{n-1} \tau_{n-1}, \\
\frac{1}{\Delta t} (f_n - f_{n-1}) &= (-1 + \xi |\tau_n|) f_{n-1} f_n - \nu f_{n-1} f_n^2.
\end{align*}
\] (5.1a) (5.1b) (5.1c)

For the space variable, we use linear $P_1$ finite elements for $u$ and piecewise constant finite elements for both $\tau$ and $f$. Note that, in contrast to the approximating system (2.24) we used for our theoretical proof, we take $\tau_{n-1}$ instead of $\tau_n$ in the right-hand sides of (5.1a) and (5.1b). This allows us to solve each equation separately. This choice is made for simplicity. Other approaches could have been employed. For our tests, we use elements of constant size $h = 0.002$ and perform the computations using Scilab [11].

5.1 Homogeneous boundary conditions

We first focus on the homogeneous boundary conditions on $u$ considered in Section 3. The case $f_0 \equiv 0$, that implies $f \equiv 0$ for all times, is uninteresting numerically. We therefore only show results for $f_0 \not\equiv 0$. In this case, we have convergence to the stationary state $(0,0,0)$ as proven in Theorem 3.1. The convergence estimates are established in Theorem 3.2. We recall the parameter

\[ \beta = \text{meas} \{ y \in \Omega | f_0(y) > 0 \} \]
and these convergence rates: for $\alpha$ arbitrarily small, there exist various constants $\kappa_\alpha$ and a time $t_0$, such that, for all $t > t_0$,

$$
\|\tau(t, \cdot)\|_{L^2} \leq \kappa_\alpha (1 + t)^{-\frac{\beta}{\lambda}(1 - \alpha)},
$$

$$
\frac{1}{f(t_0, y)} + (1 + \alpha)(t - t_0) \leq f(t, y) \leq \frac{1}{f(t_0, y)} + (1 - \alpha)(t - t_0),
$$

$$
\|u(t, \cdot)\|_{H^1} + \|\tau(t, \cdot)\|_{L^2} \leq \kappa_\alpha (1 + t)^{-\frac{\beta}{\lambda}(1 - \alpha)}
$$

$$
\left\| \left( \eta \frac{\partial u}{\partial y} + \tau - \bar{\tau} \right)(t, \cdot) \right\|_{L^2} \leq \kappa_\alpha (1 + t)^{-2 \frac{\beta}{\lambda}(1 - \alpha)}.
$$

Note that the last three estimates are exactly the same as in Theorem 3.2, the first estimate is an immediate consequence of (3.18) and (3.23). We now check that these estimates are sharp. We begin with the case $f_0 > 0$ on $\Omega$ that is $\beta = 1$. The evolutions of $\|\tau(t, \cdot)\|_{L^2}, \|f(t, \cdot)\|_{L^2}, \|u(t, \cdot)\|_{H^1}, \|\tau(t, \cdot)\|_{L^2}, \left\| \left( \eta \frac{\partial u}{\partial y} + \tau - \bar{\tau} \right)(t, \cdot) \right\|_{L^2}$ are represented in Figure 2(a). We use a log-log representation. The slopes $s$, which correspond to a decrease as $t^s$, are fitted on the numerical results and indicated on Figure 2(a): the numerical convergence rates, obtained with $\lambda = 0.5$, are in good agreement with the estimates.

We next consider cases where $f_0 = 0$ on some part of the domain. In Figure 2(b) we show simulations obtained with different values of $\beta$. For each simulation, that is for each value of $\beta$ considered, the convergence rates are fitted and represented as a function of $\beta$. The numerical and theoretical convergence rates $s$ agree.

We have extended these results to the other values of $\lambda$ than $\lambda = 0.5$ and other values of the parameters $\rho, \eta, G, \xi, \nu$ to check that the convergence estimates of Theorem 3.2 depend only on $\lambda$ and $\beta$ and are indeed sharp.

Figure 2: (a) Time evolution in log-log scale for homogeneous boundary conditions; the points are the simulated trajectories; the lines and the corresponding slopes $s$ are fitted. (b) Fitted convergence rates $s$ for $\beta = 0, 0.01, 0.1, 0.6, 0.9, 0.99$; the lines are the theoretical convergence rates function of $\beta$. 

37
5.2 Non-homogeneous boundary conditions

The longtime behaviour for non-homogeneous boundary conditions has been studied in Section 4. We consider only stationary states \((u_\infty, \tau_\infty, f_\infty)\) that satisfy \(f_\infty > 0\) everywhere. We have shown that such a steady state (4.2) is unique. We established in Theorem 4.1 that we have convergence to this steady state for small perturbations. To have convergence, we of course need to assume \(f_0 > 0\) everywhere. We observe numerically that no other condition, and specifically non assumption on the smallness of the data, is required. We consider the perturbations \((u - u_\infty, \tau - \tau_\infty, f - f_\infty)\) to equilibrium and show that they vanish in the longtime, see Figure 3. The evolution is plotted in semi-logarithmic scale. The convergences of the various norms \(\|\tau(t, \cdot)\|_{L^2}, \|f(t, \cdot)\|_{L^2}, \|u(t, \cdot)\|_{H^1} + \|\eta \frac{\partial u}{\partial y} + \tau - \bar{\tau}\|_{L^2}\) are indeed exponential.

![Figure 3: Time evolution of the perturbation to equilibrium in semi-logarithmic scale for non-homogeneous boundary conditions; the points are the simulated trajectories and the line and the corresponding slope are fitted.](image)

In section 4 in order to establish a result without any smallness assumption, we have considered a particular initial data that reduces (1.1) to the ordinary differential equation system (4.19). We have obtained convergence to the stationary state (4.20) and explicit formula for the rate of convergence. Numerically, we observe convergence even when the condition (4.21), which was assumed for the proof of Theorem 4.2, is not satisfied. The time evolution is shown in the space of \((\tau, f)\) and the convergence are represented in Figure 4(a). We check that the convergence is exponential as observed numerically in the general case of (1.1) (see Figure 3). Moreover, we compute the convergence rate and compare it to the theoretical rate \(C_r\) defined by (4.24). The evolution of the perturbation function \(|\tau(t) - \tau_\infty| + |f(t) - f_\infty|\) is plotted as a function of time in semi-logarithmic scale in Figure 4(b). The first case \(\lambda = 0.5\) correspond to the case when the eigenvalues of the associated linearized system are complex, the expected value of \(C_r\) is 0.8090; the other case \(\lambda = 0.1\) is when the eigenvalues are real negative, the expected value of \(C_r\) is 1.7895. The theoretical and numerical value agree.

Acknowledgements. The authors are grateful to François Lequeux (ESPCI Paris) for many stimulating and enlightening discussions on the rheology of complex fluids.
Figure 4: For the system of ordinary differential equations (4.19), time evolution (a) in the space \((\tau, f)\) for \(\lambda = 0.5\); (b) of the perturbation to equilibrium in semi-logarithmic scale for \(\lambda = 0.5, 0.1\); the points are the simulated trajectories and the line and the corresponding slope are fitted.

References

[1] D. Benoit. *Various theoretical and numerical issues related to the simulation of non-Newtonian fluids*. PhD thesis, Université Paris Est, Ecole des Ponts ParisTech, in preparation.

[2] S. Boyaval, T. Lelièvre, and C. Mangoubi. Free-energy-dissipative schemes for the Oldroyd-B model. *Mathematical Modelling and Numerical Analysis*, 43(2):523–561, 2009.

[3] H. Cartan. *Calcul différentiel*. Hermann, Paris, 1967. (in French)

[4] C. Derec, A. Ajdari, and F. Lequeux. Rheology and aging: a simple approach. *Eur. Phys. J. E*, 4(3):355–361, 2001.

[5] C. Guilloupé and J. C. Saut. Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type. *RAIRO-Mathematical Modelling and Numerical Analysis*, 24(3):369–401, 1990.

[6] L. He and L. Xu. Global well-posedness for viscoelastic fluid system in bounded domains. *SIAM Journal on Mathematical Analysis*, 42:2610, 2010.

[7] S. Kawashima and Y. Shibata. Global existence and exponential stability of small solutions to nonlinear viscoelasticity. *Communications in mathematical physics*, 148(1):189–208, 1992.

[8] J.L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Études mathématiques. Gauthier-Villars, 1969. (in French)

[9] M. Renardy. Global existence of solutions for shear flow of certain viscoelastic fluids. *Journal of Mathematical Fluid Mechanics*, 11(1):91–99, 2009-03-01.
[10] M. Renardy. Some global stability results for shear flows of viscoelastic fluids. *Journal of Mathematical Fluid Mechanics*, 11(1):100–109, 2009-03-01.

[11] Scilab, http://www.scilab.org

[12] R. Temam. *Navier-Stokes equations: theory and numerical analysis*. North-Holland Pub. Co., 1979.