Geometric quantum speed limits for Markovian dynamics in open quantum systems

Kang Lan, Shijie Xie, and Xiangji Cai

1 School of Physics, State Key Laboratory of Crystal Materials, Shandong University, Jinan 250100, People’s Republic of China
2 School of Science, Shandong Jianzhu University, Jinan 250101, People’s Republic of China
* Authors to whom any correspondence should be addressed.
E-mail: xsj@sdu.edu.cn and xiangjicai@foxmail.com

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Abstract
We study theoretically the geometric quantum speed limits (QSLs) of open quantum systems under Markovian dynamical evolution. Three types of QSL time bounds are introduced based on the geometric inequality associated with the dynamical evolution from an initial state to a final state. By illustrating three types of QSL bounds at the cases of presence or absence of system driving, we demonstrate that the unitary part, dominated by system Hamiltonian, supplies the internal motivation for a Markovian evolution which deviates from its geodesic. Specifically, in the case of unsaturated QSL bounds, the parameters of the system Hamiltonian serve as the eigen-frequency of the oscillations of geodesic distance in the time domain and, on the other hand, drive a further evolution of an open quantum system in a given time period due to its significant contribution in dynamical speedup. We present physical pictures of both saturated and unsaturated QSLs of Markovian dynamics by means of the dynamical evolution trajectories in the Bloch sphere which demonstrates the significant role of system Hamiltonian even in the case of initial mixed states. It is further indicated that whether the QSL bound is saturated is ruled by the commutator between the Hamiltonian and reduced density matrix of the quantum system. Our study provides a detailed description of QSL times and reveals the effects of system Hamiltonian on the unsaturation of QSL bounds under Markovian evolution.

1. Introduction
How to speed up the evolution of quantum systems, is a significant problem in quantum mechanics theory [1–6]. The concept of quantum speed limit (QSL) provides a lower bound for understanding the minimal evolution time for a quantum system to traverse a given predetermined distance [7–15]. Based on the in-depth experimental and theoretical studies of QSL time, the practical applications can be expected in quantum computation and transport [16–20]. It is important for the development of protecting quantum information and quantum optimal theory [21–24]. Particularly, the advantages of QSL have been exploited in the field of quantum battery [25–28]. With in-depth understanding of QSL, the speed limit has been extended to the classical systems [29–31]. It has demonstrated that the vanishing QSL times is traced back to reduced uncertainty in quantum observables, which is associated with the emergent classical behavior [32]. Recently, there is a research advancing the application of the QSL to quantum resource theories [33].

The minimal time for a closed quantum system under unitary time evolution evolving from an initial state to an orthogonal one was initially derived by Mandelstam and Tammm (MT) as \( \tau \geq \frac{\pi}{\Delta E} \) relying on the energy variance of the system with \( (\Delta E)^2 = \langle \psi | H^2 | \psi \rangle - (\langle \psi | H | \psi \rangle)^2 \) [34]. Margolus and Levitin (ML) then derived another bound related to the average energy, \( \tau \geq \frac{2\pi}{E} \), where \( E = \langle \psi | H | \psi \rangle \) is the mean energy and assumed the ground state energy \( E_0 = 0 \) [35]. A unified tight time bound for unitary time evolution of closed quantum systems is defined as \( \tau \geq \frac{1}{2\pi} \max \left\{ \frac{\Delta E}{2\pi}, \frac{E}{2\pi} \right\} \) by combining the MT and ML bounds [36].
Benefiting from a wide variety of metrics, such as trace distance, Bures angle, relative purity, just to name a few [37–40], a great many excellent researches have successfully generalized the MT and ML types bounds [18, 41–46]. Furthermore, tight QSL bounds can be obtained via the distance between generalized Bloch vectors in both unitary and nonunitary processes [47, 48]. The importance of choosing different metrics in nonunitary process has been demonstrated by employing an elegant information formalism [49]. Meanwhile, a theoretical analysis has been made in detail on the existence of a continuous-in-time saturation of the QSL time bounds [50]. In addition, distinct QSL bounds have been established related to the geometry phase of the quantum evolution [51, 52]. And the dynamical occurrence of the orthogonality catastrophe has been associated with the QSL [53]. Recently, an action QSL time bound has been established associated with the instantaneous speed of the dynamical evolution [54].

The ratio between QSL time and actual evolution time suggests the possibility of speeding up the dynamical evolution. When the ratio equals one, the QSL bound is saturated and with no speedup potential. In contrast, the QSL bound is unsaturated if the ratio is less than one, and with potential for dynamical evolution speedup. Non-Markovian effect has been shown to speed up quantum evolution [41]. Dynamical speedup mechanisms involving the regulation of non-Markovianity have subsequently been proposed both theoretically and experimentally [55–60], which indicates the non-Markovian effect is significant in inducing unsaturated QSL bounds in open dynamics. However, dynamical speedup in this way is very dependent on variable environmental feedback, there is a strong need to explore other sources that generate the unsaturated QSL bounds besides non-Markovianity [43, 61]. Seeing that the system Hamiltonian is decisive for regulating unitary evolution speed [34–36], the responsibility undertaken by the system Hamiltonian in open process should be clarified in depth, which may provide a self-existent accelerated way without the participation of non-Markovianity.

In this paper, we present three different definitions of the QSL time for open quantum systems, based on distinct physical motivations. We consider a Markovian dynamics of a two-level system, and give constructive answer for illustrating the unsaturated QSL bounds. To this end of clarifying the QSLs of Markovian dynamics, we examine the QSL bounds in different cases. It is shown that when open dynamics is completely governed by the environment, the actual evolution path always follows its geodesic. However, when the open dynamics contains the unitary part dominated by the system Hamiltonian, the actual evolution path deviates from its geodesic. Based on this, we can say that \[ [H_0, \rho_t] \neq 0 \] is the key for the unsaturated QSL bounds of Markovian dynamics. With both influences of system driving and environmental dissipation effect, we demonstrate that in this case, the system Hamiltonian is an intrinsic motivation for dynamical speedup and causes the oscillations of geodesic distance in time domain.

The paper is organized as follows. In section 2, we introduce three types of QSL times based on the geometric approaches. In section 3, we study analytically the QSL time bounds in a two-level quantum system under Markovian evolution. In section 4, we discuss numerically the influences of the system Hamiltonian on the three types of QSL bounds by analyzing the quantum evolution in the Bloch sphere. The conclusions are given in section 5.

2. Geometric quantum speed limits

2.1. General framework

Geometric QSL generally arises from the fact that the geodesic is the shortest path among all physical evolutions between a given initial state \( \rho_0 \) and a final state \( \rho_f \) [49, 54]. For a closed quantum system under unitary evolution, the geodesic between the initial and final states is definite and generally unique whereas it is indefinite for nonunitary dynamics of an open quantum system. To derive the geometric QSL bounds in dynamics of open quantum system, it is necessary to choose a metric \( \mathcal{D}(\rho_i, \rho_f) \) to be as the geodesic distance between two arbitrary states \( \rho_i \) and \( \rho_f \) in the dynamical evolution which satisfies the following mathematical properties (figure 1)

- non-negativity
\[ \mathcal{D}(\rho_i, \rho_f) \geq 0; \]
- identity of indiscernible
\[ \mathcal{D}(\rho_i, \rho_f) = 0, \text{ if } \rho_f = \rho_i; \]
- symmetry
\[ \mathcal{D}(\rho_i, \rho_f) = \mathcal{D}(\rho_f, \rho_i); \]
- triangle inequality
\[ \mathcal{D}(\rho_i, \rho_f) \leq \mathcal{D}(\rho_i, \rho_0) + \mathcal{D}(\rho_0, \rho_f). \]
The geometric QSL for a quantum system evolving from an initial state $\rho_0$ to a final state $\rho_f$ can be derived physically from the inequality between the lengths of the geodesic and actual path [18]

$$D(\rho_0, \rho_f) \leq \sum_{i=1}^{n} D(\rho_{i-1}, \rho_{i}) = \mathcal{L}(\rho_0, \rho_f),$$  \hspace{1cm} (1)

where the actual evolution time $\tau$ is divided into $n$ infinitesimal time $\Delta t = \tau/n$ with $n \to +\infty$. If the quantum system evolves from the initial state $\rho_0$ to final state $\rho_f$ always along the geodesic, the geometric bound in equation (1) is saturated. The instantaneous speed along the actual path in the dynamical evolution can be expressed as [21, 54]

$$v(t) = \frac{d}{dt} \mathcal{L}(\rho_0, \rho_f) = \lim_{\Delta t \to 0} \frac{D(\rho_f, \rho_f+\Delta t)}{\Delta t},$$  \hspace{1cm} (2)

where the following relation has been used

$$\lim_{\Delta t \to 0} D(\rho_f, \rho_f+\Delta t) = \lim_{\Delta t \to 0} [\mathcal{L}(\rho_0, \rho_f+\Delta t) - \mathcal{L}(\rho_0, \rho_f)].$$  \hspace{1cm} (3)

Based on the inequality between the lengths of geodesic and actual path and the instantaneous speed, there are mainly two types of definitions of the QSL time, namely, the lower bound of the actual evolution time, for a quantum system evolving from the initial state $\rho_0$ to a final state $\rho_f$. The first one corresponds to the time $\tau_{\text{min}}$ for the quantum system evolving a given predetermined distance as the same length along the actual path as that set by the geodesic distance in terms of the instantaneous speed of the actual evolution [50]

$$D(\rho_0, \rho_f) = \int_0^{\tau_{\text{min}}} v(t)dt.$$  \hspace{1cm} (4)

This type QSL time depends on the actual evolution time only implicitly through the final state. The second one corresponds to the time $\tau_{\text{ave}}$ in terms of the time-averaged speed of the actual evolution of the quantum system

$$\tau \geq \tau_{\text{ave}} = \frac{D(\rho_0, \rho_f)}{v(\tau)},$$  \hspace{1cm} (5)

where $\bar{X}(\tau) = \langle 1/\tau \rangle \int_0^\tau X(t)dt$ denotes the time average. This type QSL time depends on the actual evolution time and actual evolution path. When the actual evolution path deviates from its geodesic, it should further state that for some given dynamical processes despite the geodesic distance is fixed and finite, the QSL time $\tau_{\text{ave}}$ may diverge, i.e., the minimal time required to infinitely approach the steady state of the quantum system being infinite whereas the QSL time $\tau_{\text{min}}$ is still bounded and provides a valid way to quantify the minimal evolution time for these dynamical processes [50].

There is also another derivation of the geometric QSL for a quantum system evolving from an initial state $\rho_0$ to a final state $\rho_f$, based on the following inequality [43]

$$D(\rho_0, \rho_f) = \int_0^\tau \frac{d}{dt} D(\rho_0, \rho_f)dt \leq \int_0^\tau \left| \frac{d}{dt} D(\rho_0, \rho_f) \right| dt,$$  \hspace{1cm} (6)

where the rate of change of the length of geodesic is usually regarded as the ‘velocity’ along the geodesic in the direction from the initial state $\rho_0$ to final state $\rho_f$.

$$\sigma(t) = \frac{d}{dt} D(\rho_0, \rho_f) = \lim_{\Delta t \to 0} \frac{D(\rho_f, \rho_f+\Delta t) - D(\rho_0, \rho_f)}{\Delta t}.$$  \hspace{1cm} (7)

Physically, when the rate of change of the length of geodesic $\sigma(t)$ is positive, it means that the final state $\rho_f$ is moving away from the initial state $\rho_0$; when $\sigma(t)$ is negative, it reflects that the final state $\rho_f$ is moving...
towards to the initial state $\rho_0$ as shown in figure 2. Based on the mathematical inequality for the rate of change of the length of geodesic in equation (6), another type QSL time can be expressed as

$$\tau \geq \tau_{\text{rat}} = \frac{\mathcal{D}(\rho_0, \rho_t)}{\sigma(t)}.$$  \hspace{1cm} (8)

This type QSL time depends on the actual evolution time but not depend on the actual evolution path. If we employ inequalities for operators to the mathematical inequality in equation (6), some other types of QSL times can be further derived as lower bounds of $\tau_{\text{rat}}$ \cite{41,42,43}.

It is worth mentioning that when the evolution of the quantum system is always along the geodesic, the three types of QSL times are equivalent and equal to the actual evolution time, namely, $\tau_{\text{min}} = \tau_{\text{ave}} = \tau_{\text{rat}} = \tau$. That is, the geometric QSL bound is saturated corresponding that the dynamical evolution is optimal. In this specific case, the three types of QSL times are diverges. If the quantum system does not evolve along the geodesic, the geometric QSL bound is never saturated and the dynamical evolution is not optimal. In this case, the ratio between the QSL time and evolution time provides an estimate of how far from optimal the evolution time is with respect to the QSL time \cite{50,54}.

**2.2. QSL based on the Bures angle metric**

Based on the framework established above, there is an infinite family of QSLs time bounds due to the fact that the number of geometric metrics is infinite \cite{49}. To study the geometric QSL time of the quantum system, we can employ a widely used metric, namely, the Bures angle, which is closely related to the quantum Fisher information of dynamical evolution \cite{18}.

Based on the Bures angle metric, the length of geodesic between the initial and final states can be quantified as

$$\mathcal{D}_B(\rho_0, \rho_t) = \arccos F_B(\rho_0, \rho_t),$$  \hspace{1cm} (9)

where $F_B(\rho_0, \rho_t) = \text{Tr} \sqrt{\rho_0 \rho_t \rho_0}$ is the Bures fidelity between the initial state $\rho_0$ and final state $\rho_t$ \cite{62}.

The instantaneous speed along the actual path can be expressed as

$$v(t) = \frac{1}{2} \sqrt{F_Q(t)},$$  \hspace{1cm} (10)

where $F_Q(t) = \text{Tr}(\rho_t L_t^2)$ is the quantum fisher information of the quantum system along the actual evolution path with the symmetric logarithmic derivative operator $L_t$ defined by $\dot{\rho} = (\rho_t L_t + L_t \rho_t)/2$.

The QSL time $\tau_{\text{min}}$ for a quantum system to traverse a given predetermined distance set by the geodesic distance from initial state $\rho_0$ to final state $\rho_t$ satisfies

$$\mathcal{D}_B(\rho_0, \rho_t) = \frac{1}{2} \int_0^{\tau_{\text{min}}} \sqrt{\text{Tr}(\rho_t L_t^2)} dt.$$  \hspace{1cm} (11)

The QSL time $\tau_{\text{ave}}$ which defines the lower bound of the actual evolution time can be expressed as

$$\tau_{\text{ave}} = \frac{\mathcal{D}_B(\rho_0, \rho_t)}{\frac{1}{2} \int_0^{\tau_{\text{ave}}} \sqrt{\text{Tr}(\rho_t L_t^2)} dt}.$$  \hspace{1cm} (12)

The 'velocity' along the geodesic in direction from the initial state $\rho_0$ to final state $\rho_t$ can be written as

$$\sigma(t) = \frac{d}{dt} \mathcal{D}_B(\rho_0, \rho_t) = -\frac{\text{Tr} \frac{\dot{\rho}_0 \dot{\rho}_t}{\sqrt{\rho_0 \rho_t \rho_0}}}{2 \sqrt{1 - (\text{Tr} \sqrt{\rho_0 \rho_t \rho_0})^2}}.$$  \hspace{1cm} (13)
Correspondingly, the QSL time $\tau_{rat}$ can be written as

$$\tau_{rat} = \frac{D_B(p_0, p_r)}{\frac{1}{2} \int_0^T \frac{\Omega^2}{2 \sqrt{1 - (\epsilon / \Omega)^2}} dt} \quad (14)$$

3. QSLs bounds in a qubit system under Markovian dynamics

In this section, we mainly study the QSL time bounds of a qubit system coupled to a dissipative bosonic reservoir based on the geometric approaches introduced in section 2. We show the three types of QSL bounds for the qubit system under Markovian dissipation dynamics and further provide an explicit physical picture for the origin of the unsaturated QSL bounds.

3.1. Markovian dissipation dynamics of a qubit system

The qubit system can be described as a two-level system with the Hamiltonian written by [63]

$$H_0 = \frac{\hbar}{2}(\epsilon \sigma_z + \Omega \sigma_x), \quad (15)$$

where $\epsilon$ and $\Omega$ denote the energy level difference and tunneling coupling between the uncoupled states $|1\rangle$ and $|2\rangle$, and $\sigma_x = |1\rangle \langle 1| - |2\rangle \langle 2|$ and $\sigma_z = |1\rangle \langle 2| + |2\rangle \langle 1|$ are the Pauli matrices in the basis $\{|1\rangle, |2\rangle\}$. Physically, this model can be also used to describe a double quantum dot system composed of two well potential connected by the internal tunneling coupling of which the basis $|1\rangle$ and $|2\rangle$ denote a particle localized in the first or the second quantum dot, respectively [64, 65]. The physical meanings of the energy level difference $\epsilon$ and tunneling coupling $\Omega$ depend on the implementation of the qubit system.

In the presence of the dissipative reservoir, the dynamics of the qubit system is closely associated with the energy transfer process between the system and the environment. By means of photon or phonon emission, the dissipative reservoir generally makes the qubit system relax from the excited state $|e\rangle$ to its ground state $|g\rangle$ in terms of the eigen-basis [64]

$$|e\rangle = \cos(\theta/2)|1\rangle + \sin(\theta/2)|2\rangle,$$

$$|g\rangle = -\sin(\theta/2)|1\rangle + \cos(\theta/2)|2\rangle, \quad (16)$$

via a unitary transformation from the original basis $\{|1\rangle, |2\rangle\}$ with $\theta = \arctan(\Omega / \epsilon)$. Correspondingly, the Hamiltonian in equation (15) of the qubit system can be rewritten in the diagonalized form as

$$H_0 = \frac{\hbar}{2}\Omega_1 \sigma_z + \frac{\hbar}{2}\Omega_2 (\cos \theta \sigma_z + \sin \theta \sigma_x), \quad (17)$$

where $\Omega_1 = \sqrt{\epsilon^2 + \Omega^2}$ denotes the eigen-frequency between the excited state $|e\rangle$ and ground state $|g\rangle$, $\sigma_z = |e\rangle \langle e| - |g\rangle \langle g|$ is the Pauli matrix in the basis $\{|e\rangle, |g\rangle\}$, and the quantization axis $\sigma_z$ forms the angles $\theta$ and $\pi/2 - \theta$ with respect to $\sigma_z$ and $\sigma_x$ axes. In terms of the eigen-basis $\{|e\rangle, |g\rangle\}$, the Hamiltonian of the total system (the qubit system plus the dissipative reservoir) can be expressed as [64]

$$H = H_0 + H_R + H_I$$

$$= \frac{\hbar}{2}\Omega_1 \sigma_z + \hbar \sum_k \omega_k a_k^\dagger a_k + \hbar \sum_k \left( g_0 \sigma_+ a_k + g_k^* \sigma_- a_k^\dagger \right) \quad (18)$$

where $\sigma_+ = |e\rangle \langle g|$ and $\sigma_- = |g\rangle \langle e|$ are the raising and lowering operators in the basis $\{|e\rangle, |g\rangle\}$, the index $k$ represents different modes of the reservoir with frequency $\omega_k$, $a_k^\dagger$ and $a_k$ are the creation and annihilation operators, and $g_k$ denotes the corresponding coupling constant.

In the weak coupling limit and based on the Weisskopf–Wigner approximation, the dissipation dynamics of the quantum system in the eigen-basis $\{|e\rangle, |g\rangle\}$ is described by the master equation as [64]

$$\dot{\rho}_t = -\frac{i}{\hbar}[H_0, \rho_t] + \Gamma_t \left[ \sigma_- \rho_t \sigma_+ - \frac{1}{2} \{\sigma_+ \rho_t \sigma_- + \text{h.c.} \} \right] \quad (19)$$

where $\Gamma_t > 0$ denotes the dissipation rate. The quantum master equation (19) is of Lindblad form with a positive dissipation rate generated by a dynamical semigroup of completely positive and trace-preserving maps corresponding to that the dynamics of the quantum system is Markovian [66]. The first term on the
right-hand side describes the unitary evolution, which does not disturb the populations of the excited and ground states, yet results in oscillations of the off-diagonal elements of reduced density matrix with the eigen-frequency $\Omega$. The second term describes the nonunitary process, which obviously causes an exponential decay of the quantum system from the excited state $|e\rangle$ to ground state $|g\rangle$ and, on the other hand, eliminates the off-diagonal terms to zero $[64, 67]$.

By transforming the master equation (19) back to the original basis $\{|1\rangle, |2\rangle\}$, the dynamical evolution of the system yields the quantum master equation $[64]$ 

$$ \dot{\rho}_f = L_0[\rho_f] + L_1[\rho_f] = -\frac{i}{\hbar} [H_0, \rho_f] + \Gamma_f \left[ \Sigma_+ \rho_f \Sigma_+ - \frac{1}{2} \{ \Sigma_+ \Sigma_-, \rho_f \} \right], $$

where $L_0[\rho_f]$ describes the unitary evolution of the system, $L_1[\rho_f]$ stands for the nonunitary evolution of the system induced by the dissipative reservoir, and $\Sigma_\pm = U \sigma_\pm U^\dagger$ with $U = \cos(\theta/2) I - i \sin(\theta/2) \sigma_y$. The reduced density matrices $\rho_f$ and $\rho_i$ are associated with the unitary transformation $\rho_f = U \rho_i U^\dagger$ and they describe the same quantum state in the bases $\{|1\rangle, |2\rangle\}$ and $\{|e\rangle, |g\rangle\}$, respectively. The dynamical evolution described by the quantum master equation (20) is also Markovian with the Lindblad operators $\Sigma_{\pm}$ and positive relation rate $\Gamma_f$ due to the fact that the property of the dynamical map is preserved under unitary transformations $[62]$. It is worth noting that the Lindblad operators $\Sigma_{\pm}$ in the master equation (20) is closely related to the system Hamiltonian $H_0$ and the dissipation forms the angles $\theta$ and $\pi/2 - \theta$ with respect to $\sigma_z$ and $\sigma_x$ axes, respectively. For the case $\theta = 0$, namely, the tunneling coupling $\Omega_0 = 0$, the qubit system undergoes dynamical dissipation along $\sigma_z$ axis with the Lindblad operators $\Sigma_{\pm} = \sigma_{\pm} = 1/2 (\sigma_z \pm i \sigma_y)$ whereas it undergoes dissipation along $\sigma_x$ axis in dynamical evolution with the Lindblad operators $\Sigma_{\pm} = 1/2 (\sigma_z \pm i \sigma_y)$ for the case $\theta = \pi/2$, namely, the energy level difference $\epsilon = 0$.

We should realize that the off-diagonal terms derived from equation (20) do not necessarily represent the quantum coherence between the excited and ground states, but only when $\Omega_0 = \epsilon$ and the dissipation along $\sigma_z$ axis $[67]$.

The minimal time of unitary evolution of a closed quantum system from an initial state to the final state is determined by the energy variance $\Delta E$ or average energy $\langle E \rangle$ of the system, which are closely related to the system Hamiltonian. When it is generalized to the open case, the contribution to the minimal time originating from the system Hamiltonian can not be easily ignored. To better present the influence of system Hamiltonian on the QSL bounds for Markovian dynamics, thus we derive the quantum master equation (20) in the Schrödinger picture rather than in the interaction picture. In terms of the Bloch vector representation of the reduced matrix density $\rho_f(t) = \text{tr}(\sigma_k \rho_f) (k = x, y, z)$, the solution of the quantum master equation (20) can be analytically expressed as 

$$ \rho_f = \frac{1}{2} \left( \begin{array}{cc} 1 + r_x(t) & r_x(t) - ir_y(t) \\ r_x(t) + ir_y(t) & 1 - r_x(t) \end{array} \right), $$

where the components of the Bloch vector are written as 

$$ r_x(t) = F(t) \left\{ \sin^2 \theta F(t) + \cos^2 \theta \cos(\Omega_0 t) \right\} r_x(0) - \cos \theta \sin(\Omega_0 t) r_y(0) $$

$$ + \sin \theta \cos \theta [F(t) - \cos(\Omega_0 t)] r_x(0) + \sin \theta [F^2(t) - 1], $$

$$ r_y(t) = F(t) \left[ \cos \theta \sin(\Omega_0 t) r_x(0) + \cos(\Omega_0 t) r_y(0) - \sin \theta \sin(\Omega_0 t) r_x(0) \right], $$

$$ r_z(t) = F(t) \left\{ \sin \theta \cos \theta [F(t) - \cos(\Omega_0 t)] r_x(0) + \sin \theta \sin(\Omega_0 t) F(t) r_y(0) + [\cos^2 \theta F(t) $$

$$ + \sin^2 \theta \cos(\Omega_0 t)] r_z(0) \right\} + \cos \theta [F^2(t) - 1], $$

with the decoherence factor $F(t) = \exp(-1/2 \Gamma_f t)$. For the case dissipation along $\sigma_z$ axis, namely, $\theta = 0$ and $\Omega_0 = \epsilon$, the components of the Bloch vector in equation (22) can be reduced to 

$$ r_x(t) = F(t) [\cos(\epsilon t) r_x(0) - \sin(\epsilon t) r_y(0)], $$

$$ r_y(t) = F(t) [\sin(\epsilon t) r_x(0) + \cos(\epsilon t) r_y(0)], $$

$$ r_z(t) = F^2(t) r_z(0) + [F^2(t) - 1]. $$

For the case dissipation along $\sigma_x$ axis, namely, $\theta = \pi/2$ and $\Omega_0 = \Omega$, the components of the Bloch vector in equation (22) can be reduced to
\[ r_x(t) = F^2(t)r_x(0) + [F^2(t) - 1], \]
\[ r_y(t) = F(t)[\cos(\Omega t)r_y(0) - \sin(\Omega t)r_z(0)], \]
\[ r_z(t) = F(t)[\sin(\Omega t)r_y(0) + \cos(\Omega t)r_z(0)]. \] (24)

3.2. QSL bounds for the qubit system

In the following, we study the influences of the system Hamiltonian and dissipation effect of the environment on the QSL bounds by means of analytical results. For convenience, we mainly analyze the case that the dissipation along \( \sigma_z \) axis (i.e., \( \theta = 0 \)). These scenarios for other dissipative directions (\( \theta \in [0, \pi/2] \)) can be analyzed in a similar way due to the rotational symmetry.

3.2.1. QSL bounds without influence of dissipation effect of the environment

We first study the case there is no dissipation effect of the environment, namely, \( \Gamma = 0 \). In this case, the dynamical evolution of the qubit system is unitary governed by the Hamiltonian \( H_0 \). For the system evolving from a pure initial state, the instantaneous speed along the actual evolution path can be expressed as

\[ v(t) = \frac{\Omega_L}{2} \sqrt{[\cos \theta r_x(0) - \sin \theta r_z(0)]^2 + r_y^2(0)} = \frac{\Delta E}{\hbar}, \] (25)

where \( \Delta E = \sqrt{tr[H_0^2] - tr[H_0^2]} \) denotes the energy variance of the system with respect to the initial state \( \rho_0 \). The QSL times \( \tau_{min} \) and \( \tau_{ave} \) are equivalent and can be written as

\[ \tau_{min} = \tau_{ave} = \frac{\hbar D_0(\rho_0, \rho_t)}{\Delta E}, \] (26)

which are consistent with the MT type of QSL bound [18]. In the unitary case, the quantum system evolves with a constant speed and the corresponding geodesic distance between the initial and final states is

\[ D_0(\rho_0, \rho_t) = \arccos \left\{ \frac{1}{2} \left[ 1 + \sum_r r_r(0)\tau_r(t) \right] \right\} = \sqrt{S}, \] (k = x, y, z).

Substituting equation (22) into equation (27) in terms of the unitary condition of \( F(t) = 1 \), we find that the energy \( \Omega_L \) serves as the eigen-frequency of the oscillations of geodesic distance over time \( t \) proved by

\[ S = [\sin \theta r_x(0) + \cos \theta r_z(0)]^2, \]
\[ P = [\cos \theta r_x(0) - \sin \theta r_z(0)]^2 + r_y^2(0). \] (28)

In contrast, the actual evolution length is quantified as \( L(\rho_0, \rho_t) = \frac{\Delta E}{\hbar} t \), which proves the inequality \( L(\rho_0, \rho_t) > D_0(\rho_0, \rho_t) \), since the actual evolution length increases linearly with time \( t \) but the geodesic distance not. Therefore, the system Hamiltonian induces two equal classes of unsaturated QSL bounds. Another QSL time \( \tau_{sat} \) can be expressed as

\[ \tau_{sat} = \frac{D_0(\rho_0, \rho_t)}{\frac{1}{t} \int_0^t D_0(\rho_0, \rho_t) dt}. \] (29)

Obviously, according to equation (27), we arrive at an inequality related to the QSL bound \( \tau_{sat}/\tau \), i.e.,

\[ D_0(\rho_0, \rho_t) \leq \left| D_0(\rho_0, \rho_t) \right| \Leftrightarrow \tau_{sat}/\tau \leq 1, \]

due to the contribution of eigen-frequency \( \Omega_L \) to the non-monotonicity of geodesic distance. If the evolution time \( t \leq \pi/\Omega_L \), the length of geodesic over time \( t \) remains monotonically increasing, which means the QSL bound \( \tau_{sat}/\tau \) is saturated whereas it is unsaturated for \( t > \pi/\Omega_L \).

3.2.2. QSL bounds with no influence of system driving

We now study the case that the system driving has no influence on the dynamical evolution of the system, namely, \( L_0[\rho] = 0 \). There is a relationship between the components of the Bloch vector and the angle \( \theta \) as

\[ |H_0, \rho_t| = 0 \Leftrightarrow \left\{ \left| r_x(t)/r_z(t) = \tan \theta \right| \Leftrightarrow r_f(t) = 0 \right\}. \] (30)

Equation (30) places a restriction on the quantum coherence between two eigenstates (i.e., \(|e\rangle \) and \(|g\rangle\)) of the system Hamiltonian \( H_0 \) by employing the \( l_1 \)-norm-based coherence measure as [68–70]

\[ C(\rho) = \sqrt{[\cos \theta r_x(t) - \sin \theta r_z(t)]^2 + r_y^2(t)} \equiv 0, \] (31)
which means that the system starts in the eigenstate and there is no quantum coherence in the open dynamical evolution. The instantaneous speed can be written as
\[ v(t) = \frac{\Gamma_r}{2} \sqrt{\frac{1 + \cos \theta_r(t) + \sin \theta_r(t)}{1 - \cos \theta_r(t) - \sin \theta_r(t)}}, \]  
(32)
and the geodesic distance between the initial state and final state is expressed as
\[ D_b(\rho_0, \rho_t) = \arccos \sqrt{\frac{1}{2} \left[ 1 + \sum_q r_q(0) r_q(t) + \lambda^q(0) \lambda^q(t) \right]} \quad (q = x, z), \]  
(33)
with \( \lambda^q(t) = \sqrt{1 - \sum_q r_q^2(t)} \) to examine if the system is pure. For a pure state, \( \lambda^q(t) = 0 \) whereas \( \lambda^q(t) > 0 \) for mixed states.

For the case dissipation along \( \sigma_z \) axis, the dynamical channel is completely dominated by the population difference \( r_z(t) \) of two eigenstates, we can obtain the instantaneous speed
\[ v(t) = \frac{\Gamma_r F(t)}{2} \sqrt{\frac{1 + r_z(0)}{2 - F^2(t)[1 + r_z(0)]}} \]  
(34)
The geodesic distance between two distinguishable states is
\[ D_b(\rho_0, \rho_t) = \arccos \sqrt{\frac{1}{2} [1 + Q_1 + Q_2 - r_z(0)]}, \]  
(35)
with parameters
\[ Q_1 = F^2(t)[r_z(0) + r_z^2(0)], \]  
\[ Q_2 = F(t)[1 + r_z(0)] \sqrt{2 - 2r_z(0) - F^2(t)[1 - r_z^2(0)]}. \]  
(36)
The component \( r_z(0) \) of initial state determines the actual evolution length as well as the geodesic distance between two distinguishable states. We obtain three types of QSL times where there is no quantum coherence.

To further simplify equations (34) and (35), we choose the \( r_z(0) = 1 \) and the system starts in the excited state. A saturation behavior of QSL bounds could be well demonstrated by an explicit equality, that is
\[ \mathcal{L}(\rho_0, \rho_t) = \frac{\Gamma_r}{2} \int_0^t \! dt' (e^{\Gamma t'} - 1)^{-1/2} = \arccos(e^{-\Gamma t'/2}) = D_b(\rho_0, \rho_t), \]  
(37)
which means \( \tau_{\text{min}} = \tau_{\text{ave}} = \tau \). In addition, for the QSL time \( \tau_{\text{rat}} \), the geodesic distance \( D_b(\rho_0, \rho_t) \) with time \( t \) is monotonically increasing, namely,
\[ D_b(\rho_0, \rho_t) = \int_0^t \! dt' |\sigma(t')|, \]  
(38)
which means that the final state \( \rho_t \) is moving away from the initial state \( \rho_0 \). Therefore, it holds \( \tau_{\text{min}} = \tau_{\text{ave}} = \tau_{\text{rat}} = \tau \).

For the QSL bounds without influence of system driving, the actual evolution length as well as the geodesic distance between two states are independent of the system Hamiltonian. There is no coherence evolution between two eigenstates of the system Hamiltonian and we can obtain the three types of saturated QSL bounds. We emphasize that the premise of \( [H_0, \rho_t] = 0 \) requires an appropriate change in the initial state to satisfy equation (31) when dissipation along other directions and thus three types of saturated QSL bounds hold for any value of \( \theta \).

3.2.3. QSL bounds with both influences of system driving and dissipation effect of environment
We further study the general case with both influences of the system driving and the dissipation effect of the environment, namely, \( L_0(\rho_t) \neq 0 \) and \( \Gamma_r \neq 0 \). The unitary part of the open dynamics brings about the coherence evolution between two eigenstates \([64]\). In this case, the instantaneous speed is described as
\[ v(t) = \frac{1}{2} \sqrt{\Omega_1^2 C^2(\rho) + \frac{\Gamma_r^2 \kappa [C^2(\rho) (\kappa + 1) - \kappa]}{C^2(\rho) + \kappa (\kappa - 1)}} \]  
(39)
with \( \kappa = 1 + \sin \theta r_z(t) + \cos \theta r_z(t) \) being twice the population of excited state of the system. In contrast to equation (32), this equation reflects the influence on the dynamical speed from the coherence between two eigenstates. The corresponding geodesic distance between two distinguishable states is

\[
\mathcal{D}_b(\rho_0, \rho_1) = \arccos \sqrt{\frac{1}{2} \left[ 1 + \sum_k \rho_k(0) r_k(t) + \lambda_k(0) \lambda_k(t) \right]},
\]

(40)

with \( \lambda_k(t) = \sqrt{1 - \sum_i q_i^2(t)}(k = x, y, z) \). Similar to the case without system driving, \( \lambda_k(t) = 0 \) for pure states, whereas \( \lambda_k(t) > 0 \) for mixed states.

We show the QSL in the case of \( \theta = 0 \), the instantaneous speed can be reduced as

\[
v(t) = \frac{1}{2} \sqrt{e^{2F^2(t)} C^2(\rho_0) + \frac{\Gamma^2_2[1 + r_z(0)] \nu_1}{4C^2(\rho_0) + \nu_2}},
\]

(41)

with \( C_c(\rho_0) = \sqrt{r_z^2(0) + r_z^2(0)} \) being the initial quantum coherence, and other two parameters are

\[
\nu_1 = 2F^2(t) C^2(\rho_0) + F^2(t)[1 + r_z(0)] [F^2(t) C^2(\rho_0) - 4],
\]

\[
\nu_2 = 4[1 + r_z(0)] [F^2(t)[1 + r_z(0)] - 2].
\]

(42)

The geodesic distance between two distinguishable states is expressed as

\[
\mathcal{D}_b(\rho_0, \rho_1) = \arccos \sqrt{\mathcal{V}_1 + \mathcal{V}_2 + \lambda^4(0)} \mathcal{V}_3,
\]

(43)

with parameters

\[
\mathcal{V}_1 = \frac{F(t)}{2} \left( [F(t)[1 + r_z(0)]^2 + C^2(\rho_0) \cos(\epsilon t)] \right),
\]

\[
\mathcal{V}_2 = 1 - \frac{1 + r_z(0)}{2} [1 + F^2(t)],
\]

\[
\mathcal{V}_3 = \frac{F(t)}{2} \sqrt{2 + 2r_z(0) - C^2(\rho_0) - F^2(t)[1 + r_z(0)]^2}.
\]

(44)

Remarkable studies have shown that the quantum coherence is a significant quantum resource related to the QSL time [42, 71–73]. Based on equations (41) and (43), we also exhibit a dependence of the QSL on the coherence \( C(\rho_0) \) and provide that the coherence is important for the QSL times in Markovian dynamics only when \([H_0, \rho_1] \neq 0\). Now by considering an initial pure state as \(|\psi_0\rangle \equiv (|1\rangle + |2\rangle)/\sqrt{2} \) [i.e., \( r_z(0) = 1, r_y(0) = 0 \)] with the maximum initial coherence, the instantaneous speed and geodesic distance are reduced respectively as

\[
v(t) = \frac{1}{4} \sqrt{4e^{-\Gamma r_z(t)^2} + \frac{\Gamma^2_2(e^{-\Gamma r_z(t)^2} - 2)}{1 - e^{\Gamma r_z(t)^2}}},
\]

\[
\mathcal{D}_b(\rho_0, \rho_1) = \arccos \sqrt{\frac{1}{2} \left[ 1 + e^{-\frac{\Delta t}{\tau}} \cos(\epsilon t) \right]}.
\]

(45)

It can be found that the parameter \( \epsilon \) of system Hamiltonian enhances the instantaneous speed at time \( t \) as well as induces the oscillations of geodesic distance over time \( t \). The presence of parameter \( \epsilon \) of system Hamiltonian in the QSL leads to a consequence of

\[
\mathcal{L}(\rho_0, \rho_1) = \int_0^t dt' v(t') > \int_0^t dt' \sigma(t') = \mathcal{D}_b(\rho_0, \rho_1),
\]

(46)

which shows that two types of QSL bounds \( \tau_{\text{min}} / \tau \) and \( \tau_{\text{ave}} / \tau \) are unsaturated. In addition, for the QSL time \( \tau_{\text{rat}} \), the geodesic distance \( \mathcal{D}_b(\rho_0, \rho_1) \) with time \( t \) is non-monotonically evolving, namely,

\[
\mathcal{D}_b(\rho_0, \rho_1) \leq \int_0^t dt' |\sigma(t')|.
\]

(47)

Noteworthy, the equality only holds for \( t \leq \pi / \Omega_L \) (\( \Omega_L = \epsilon \) if \( \theta = 0 \)), which means that the final state \( \rho_1 \) is moving away from the initial state \( \rho_0 \). In contrast, when \( t > \pi / \Omega_L \), the dynamical evolution has a behavior of approaching the initial state \( \rho_0 \). Therefore, the QSL bound \( \tau_{\text{rat}} / \tau \) is also unsaturated if the evolution time
limit and associated with the time translations generated by the system Hamiltonian \( [72] \). Our results of quantum system are relevant to the coherence in the case of \[72\].

suggest that if the scenario is extended to Markovian dynamics, the actual path as well as geodesic distance exhibited in section 3 apply to initial mixed states, not just the initial pure states. For simplicity, we still discuss in detail the origin of the unsaturated QSL bounds in the Markovian dissipation by analyzing the non-monotonicity of the geodesic distance over time \( t \).

In the unitary process, the asymmetry is interpreted as coherence, which is a characterization of speed limit and associated with the time translations generated by the system Hamiltonian \[72\]. Our results suggest that if the scenario is extended to Markovian dynamics, the actual path as well as geodesic distance of quantum system are relevant to the coherence in the case of \([H_0, \rho] \neq 0 \). In addition, we also note that a recent work showed that the initial coherence and the trade-off between coherence and mixedness play a significant role in the QSL time \[74\].

4. Results and discussions

In this section, we show the numerical results of the three types of QSL bounds in different situations. We discuss in detail the origin of the unsaturated QSL bounds in the Markovian dissipation by analyzing the quantum evolution trajectories in the Bloch sphere. Meanwhile, we will demonstrate that above conclusions exhibited in section 3 apply to initial mixed states, not just the initial pure states. For simplicity, we still display the situation that the actual dissipation along \( \sigma_z \) axis (i.e., \( \theta = 0 \)). We highlight that following analysis holds for any value of \( \theta \in [0, \pi/2] \). The QSL behaviors of other dissipative directions can be similarly illustrated as long as the initial state is changed accordingly.

4.1. QSL bounds without influence of system driving

The blue line in figure 3(a) shows the dynamical trajectory in the Bloch sphere without influence of system driving. Due to the fact \( F(t)|_{t=\infty} = 0 \), the open quantum evolution possesses a stationary pure state of locating at the south pole of the Bloch sphere, for all values of the parameters \( \epsilon \) and \( \Gamma \). Since the \( r_z(t) \) decreases monotonically and \( \lim_{t \to \infty} r_z(t) = -1 \) for arbitrary \( r_z(0) \), the open quantum evolution always along the geodesic at a Hamiltonian-independent instantaneous speed, of which its trajectory forms a line in the Bloch sphere. In figure 3(b), we show three types of saturated QSL bounds where system starts in a mixed state when the system Hamiltonian commutes with the density matrix. The physical meanings of the three types of saturated QSL bounds are different. Equality \( \tau_{\min} = \tau_{\text{ave}} = \tau \) prove that the actual path is following the geodesic. While for the equality \( \tau_{\text{rat}} = \tau \), it only means that the distance between the initial state and the current state is monotonically increasing for a short period. Figure 4(a) shows the difference \( \delta \mathcal{L}(\rho_0, \rho_t) = \mathcal{L}(\rho_0, \rho_t) - D(\rho_0, \rho_t) \) as functions of the actual evolution time \( \tau \) and the \( r_z(0) \) without influence of system driving. We find similarly that the actual paths always follow their geodesics even system starts in arbitrary mixed states, proved by \( \delta \mathcal{L}(\rho_0, \rho_t) \equiv 0 \). Three types of saturated QSL bounds are not restricted by the choice of initial state as long as it is guaranteed that \([H_0, \rho] = 0 \).
Figure 4. The actual dynamical dissipation along $\sigma_z$ axis and the fixed parameters are $\epsilon = 5\Gamma$ and $\Omega = 0$. The difference $\delta L(\rho_0, \rho_t)$ between actual evolution length and geodesic distance (a) without influence of system driving $[r_x(0) = 0], \delta L \equiv 0$; (b) with influence of system driving $[r_y(0) = 0], \delta L \geq 0$. We emphasize that the difference $\delta L = 0$ in (b) only holds for $r_z(0) = 0$.

Physically, an open quantum evolution is driven only by the nonunitary operator $L_r$, and independent of system Hamiltonian. There is no quantum coherence between eigenstates over time $t$, i.e., $C(\rho) \equiv 0$. The above results further prove that the three types of saturated QSL bounds hold for all cases without initial coherence. At this time, system Hamiltonian $H_0$ has no effect on the open dynamical evolution. Therefore, it is suitable in this special situation to employ the interaction picture, to only focus on the influence of the interaction between the system and environment on the QSL times [41, 50, 60, 61]. All quantum states are trapped on the quantization axis and the non-Markovian effect is a key factor at which the QSL bounds become unsaturated, since it makes the population flip from the excited state to ground state along the quantization axis is no longer monotonic. This is demonstrated in [41] where the two-level system starts in the excited state (i.e., satisfying equation (30)), and is driven only by the bath.

4.2. QSL bounds with both influences of system driving and environmental dissipation effect

The red and black curves in figure 3(a) show the dynamical trajectories in the Bloch sphere with both influences of system driving and dissipation effect. As shown in the figure, because of the participation of unitary part in the open dynamics, the trajectories display conical helix configuration along the $\hat{z}$-axis. This means that the coherence evolution changes the monotonicity of geodesic distance over time $t$. In figure 3(c), two types of QSL bounds $\tau_{\text{min}}/\tau$ and $\tau_{\text{ave}}/\tau$ exhibit synchronous unsaturations from time $t = 0$, which demonstrate the actual path is not following its geodesic towards the final state when system starts in a mixed state. However, the black solid line in figure 3(c) shows that the bound $\tau_{\text{rat}}/\tau$ is still saturated if $t \leq \pi/\epsilon$, despite the fact that the open quantum evolution has deviated from its geodesic. This suggests that the path to saturate the bound $\tau_{\text{rat}}/\tau$ is not unique with geodesic being one of them. Even if $\tau_{\text{rat}} = \tau$, the association between the actual evolution length and geodesic distance cannot be easily determined. Figure 4(b) shows the difference $\delta L(\rho_0, \rho_t)$ as functions of the actual evolution time $t$ and the $r_z(0)$ with both influences of system driving and environmental dissipation effect. In contrast to figure 4(a), we find that the difference $\delta L(\rho_0, \rho_t) \geq 0$ holds for arbitrary $r_z(0)$. This imparts that the actual paths will also deviate its geodesic if system starts in a mixed state. Meanwhile, we can find that as the initial coherence $[C(\rho_0) = r_z(0)]$ increases, the open quantum system has more speedup possibility. This is because the difference $\delta L(\rho_0, \rho_t)$ enhances with increasing $r_z(0)$, the actual evolution path is further away from its geodesic.

By assessing both the dynamical trajectories displayed in the Bloch sphere when $[H_0, \rho_t] \neq 0$, it is shown that three types of unsaturated QSL bounds hold for all cases with initial coherence. The parameter $\epsilon$ will present in both instantaneous speed and geodesic distance, which indicates the irreplaceability of system Hamiltonian in inducing unsaturated QSL bounds for Markovian dynamics. Our conclusions further demonstrate that this unsaturated mechanism originates from the system Hamiltonian-dominated unitary evolution. Meanwhile, some literatures have also shown the Hamiltonian plays a vital role in the minimal time [75, 76]. We should realize that now taking an interaction picture is likely to overlook the loosening effect of system Hamiltonian on the QSL times, especially for the case where there is no non-Markovianity, which induces ostensibly ‘saturated’ QSL bounds.
Figure 5. The fixed parameters are $\Omega = 0$ and $\Gamma_r = 1$. (a) Instantaneous speed along the actual path and (b) ‘velocity’ along the geodesic in the direction from the initial state $\rho_0$ to final state $\rho_\tau$ of the dynamical evolution as a function of the actual evolution time $\tau$ for different energy level difference $\epsilon$. The initial state is $|\psi_0\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$.

5. Conclusions

In conclusion, we have presented three definitions of the geometric QSL for distinct physical motivations and based on this, studied theoretically the QSLs of an open two-level system under Markovian dissipation. We have exhibited the saturated QSL time of Markovian dynamics only holds for specific classes of initial states situating on the quantization axis, which is caused by the $[H_0, \rho] = 0$. In this case of open quantum evolution completely governed by the environment, non-Markovianity can induce the unsaturated QSLs [41, 61]. By means of analytical and numerical results, we have illustrated that three types of unsaturated QSL bounds under Markovian dynamics are obtained whenever the Hamiltonian part of the master equation influences the open quantum evolution. Specifically, the system Hamiltonian serves as an intrinsic motivation for open dynamical speedup as well as the eigen-frequency of the oscillations of geodesic distance over time $t$. Meanwhile, the coherence between eigenstates has an effect on the QSL times.

It should be noted that we use the Schrödinger picture to study the contribution of system Hamiltonian $H_0$ on the QSLs. We have also shown the loosening effect of QSL bounds caused by the Hamiltonian term can be ignored only if the system Hamiltonian commutes with the reduced density matrix, namely, the system starts in the eigenstate of Hamiltonian $H_0$. In this case, it is appropriate to use interaction picture to focus only on the interaction of the system with environment. If not, it is appropriate to employ Schrödinger picture to obtain the dynamical evolution for avoiding omission of the contribution of system Hamiltonian on the unsaturated QSL bounds. Our study has demonstrated the role played by the system Hamiltonian in generating unsaturated QSL bounds of open dynamical evolutions, that has been commonly assigned by non-Markovianity in the past.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

ORCID iDs

Kang Lan https://orcid.org/0000-0001-7796-1529
Shijie Xie https://orcid.org/0000-0001-7694-9755
Xiangji Cai https://orcid.org/0000-0001-6655-5736

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