SINGULARITY OF EIGENFUNCTIONS AT THE JUNCTION OF SHRINKING TUBES. PART I.

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Abstract. Consider two domains connected by a thin tube: it can be shown that, generically, the mass of a given eigenfunction of the Dirichlet Laplacian concentrates in only one of them. The restriction to the other domain, when suitably normalized, develops a singularity at the junction of the tube, as the channel section tends to zero. Our main result states that, under a nondegeneracy condition, the normalized limiting profile has a singularity of order \( N - 1 \), where \( N \) is the space dimension. We give a precise description of the asymptotic behavior of eigenfunctions at the singular junction, which provides us with some important information about its sign near the tunnel entrance. More precisely, the solution is shown to be one-sign in a neighborhood of the singular junction. In other words, we prove that the nodal set does not enter inside the channel.

1. Introduction and statement of the main results

We are concerned with the behavior of eigenfunctions of the Dirichlet Laplacian on dumbbell domains depending on a parameter and disconnecting in some limit process. More precisely, let us consider two slightly different domains which are connected by a thin tube so that the mass of a given eigenfunction is concentrated in one of the two domains. Then the restriction of the eigenfunction to the other domain develops a singularity right at the junction of the tube, as the section of the channel shrinks to zero. The purpose of this paper is to describe the features of this singularity formation.

A strong motivation for the interest in the spectral analysis of thin branching domains comes from the theory of quantum graphs modeling waves in thin graph-like structures (narrow waveguides, quantum wires, photonic crystals, blood vessels, lungs) and having applications in nanotechnology, optics, chemistry, medicine, see e.g. [23, 12] and references therein.

The behavior of the eigenvalues and eigenfunctions of the Laplace operator in varying domains has been intensively studied in the literature starting from [7, 13, 22, 25, 26] and more recently in [4, 5, 6, 11, 14, 17], where spectral continuity is discussed under different kind of perturbations and boundary conditions (of either Dirichlet or Neumann type). The problem of rate of convergence for eigenvalues of elliptic systems was investigated in [27], while in [9] estimates of the splitting between the first two eigenvalues of elliptic operators under Dirichlet boundary conditions are provided. We also mention that some results on the behavior of eigenfunctions of the Laplace operator under singular perturbation adding a thin handle to a compact manifold have been obtained in [3]. As far as the nonlinear counterpart of the problem is concerned, the effect of the domain shape on the number of positive solutions to some nonlinear Dirichlet boundary value problems has been investigated in [15, 16], where domains constructed as connected approximations to a finite number of disjoint or touching balls have been considered, proving that the number of positive solutions which are not “large” grows with the number of the balls.

When dealing with a dumbbell domain which is going to disconnect, the spectral continuity proved e.g. in [17] implies that eigenfunctions of the approximating problem converge to the eigenfunction of some limit eigenvalue problem on a domain with two connected components, whose spectrum is therefore the union of the spectra on the two components; as a consequence, if an eigenfunction of the limit problem is supported in one of the two domains, then the corresponding
eigenfunction of the approximating problem is going to vanish on the other domain. We are going
to show that a suitable normalization of such eigenfunction develops a singularity at the junction
of the tube, whose rate is related to the order of the zero that the limit eigenfunction has at the
other junction (see Theorem 1.2). The description of the behavior of eigenfunctions at the junction
will also provide information about nodal sets; more precisely we will prove in Corollary 1.3 that
if the limit eigenfunction has at one junction of the tube a zero of order one, then the nodal regions
of the corresponding eigenfunctions on the dumbbell stay away from the other junction.

In this paper we set up a strategy to evaluate the rate to the singularity at the junction, based
upon a sharp control of the transversal frequencies along the connecting tube. To this aim, we shall
exploit the monotonicity method introduced by Almgren [2] in 1979 and then extended by Garofalo
and Lin [21] to elliptic operators with variable coefficients in order to prove unique continuation
properties. We mention that monotonicity methods were recently used in [18, 19, 20] to prove not
only unique continuation but also precise asymptotics near singularities of solutions to linear and
semilinear elliptic equations with singular potentials, by extracting such precious information from
the behavior of the quotient associated with the Lagrangian energy.

As a paradigmatic example, let us consider the following dumbbell domain in $\mathbb{R}^N = \mathbb{R} \times \mathbb{R}^{N-1}$,
$N \geq 3$,
$$
\Omega^\varepsilon = D^- \cup C_\varepsilon \cup D^+ 
$$
where $\varepsilon \in (0, 1)$,
$$
D^- = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 < 0\}, \\
C_\varepsilon = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : 0 \leq x_1 \leq 1, \frac{x'}{x_1} \in \Sigma\}, \\
D^+ = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 > 1\},
$$
and $\Sigma \subset \mathbb{R}^{N-1}$ is an open bounded set with $C^{2,\alpha}$-boundary containing 0. For simplicity of notation,
without loss of generality, we assume that $\Sigma$ satisfies
$$
\{x' \in \mathbb{R}^{N-1} : |x'| \leq \frac{1}{\sqrt{2}}\} \subset \Sigma \subset \{x' \in \mathbb{R}^{N-1} : |x'| < 1\}.
$$

![Figure 1. The domain $\Omega^\varepsilon$.](image)

We also denote, for all $t > 0$,
$$
B_t^+ := D^+ \cap B(e_1, t), \quad B_t^- := D^- \cap B(0, t),
$$
where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$ and $B(P, t) := \{x \in \mathbb{R}^N : |x - P| < t\}$ denotes the ball of radius $t$
centered at $P$. Let $p \in C^2(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ satisfying
$$
\begin{align*}
\text{(2)} & \quad p \geq 0 \text{ a.e. in } \mathbb{R}^N, \quad p \in L^{N/2}(\mathbb{R}^N), \quad \nabla p(x) : x \in L^{N/2}(\mathbb{R}^N), \quad \frac{\partial p}{\partial x_1} \in L^{N/2}(\mathbb{R}^N), \\
\text{(3)} & \quad \begin{cases}
p \not\equiv 0 \text{ in } D^-, & p \not\equiv 0 \text{ in } D^+, \\
p(x) = 0 \text{ for all } x \in \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : 1/2 \leq x_1 \leq 1, \ x' \in \Sigma\} \cup B_t^+.
\end{cases}
\end{align*}
$$
While assumption (2) makes the problem consistent with the usual spectral theory, (3) is introduced
for technical reasons; we don’t believe it is necessary: its only use is in section 2, to prove some
uniform estimates for approximating eigenfunctions close to the right junction uniformly with
respect to the parameter $\varepsilon$. Possible weakening of assumption (3) is the object of a current
elaboration.
By classical spectral theory, for every open set $\Omega \subset \mathbb{R}^N$ such that $p \not= 0$ in $\Omega$, the weighted eigenvalue problem
\[
\begin{cases}
-\Delta \varphi = \lambda p \varphi, & \text{in } \Omega, \\
\varphi = 0, & \text{on } \partial \Omega,
\end{cases}
\]
adopts a sequence of diverging eigenvalues \(\{\lambda_k(\Omega)\}_{k \geq 1}\); in the enumeration
\[
\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \cdots
\]
we repeat each eigenvalue as many times as its multiplicity. We denote $\sigma_p(\Omega) = \{\lambda_k(\Omega) : k \geq 1\}$. For all $\varepsilon \in (0, 1)$, we also denote
\[
\lambda_k^\varepsilon = \lambda_k(\Omega^\varepsilon), \quad \sigma_p^\varepsilon = \sigma_p(\Omega^\varepsilon).
\]
It is easy to verify that $\sigma_p(D^- \cup D^+) = \sigma_p(D^-) \cup \sigma_p(D^+)$. Let us assume that there exists $k_0 \geq 1$ such that
\[
\begin{align*}
\lambda_{k_0}(D^+) & \text{ is simple and the corresponding eigenfunctions have in } e_1 \text{ a zero of order } 1, \\
\lambda_{k_0}(D^+) & \not\in \sigma_p(D^-).
\end{align*}
\]
In view of [24], these non degeneracy assumptions hold generically with respect to domain (and weight) variations. We can then fix an eigenfunction $\varphi_{k_0}^+ \in D^{1,2}(D^+) \setminus \{0\}$ associated to $\lambda_{k_0}(D^+)$, i.e. solving
\[
\begin{cases}
-\Delta \varphi_{k_0}^+ = \lambda_{k_0}(D^+) p \varphi_{k_0}^+, & \text{in } D^+, \\
\varphi_{k_0}^+ = 0, & \text{on } \partial D^+,
\end{cases}
\]
such that
\[
\frac{\partial \varphi_{k_0}^+}{\partial x_1}(e_1) > 0.
\]
Here and in the sequel, for every open set $\Omega \subseteq \mathbb{R}^N$, $D^{1,2}(\Omega)$ denotes the functional space obtained as completion of $C_c^\infty(\Omega)$ with respect to the Dirichlet norm $(\int_{\Omega} |\nabla u|^2 dx)^{1/2}$.

We refer to [17, Example 8.2, Corollary 4.7, Remark 4.3] for the proof of the following lemma.

**Lemma 1.1.** Let
\[
\tilde{k} = \text{card } \{j \in \mathbb{N} \setminus \{0\} : \lambda_j(D^- \cup D^+) \leq \lambda_{k_0}(D^+)\}
\]
\[
= k_0 + \text{card } \{j \in \mathbb{N} \setminus \{0\} : \lambda_j(D^-) \leq \lambda_{k_0}(D^+)\},
\]
so that $\lambda_{k_0}(D^+) = \lambda_{\tilde{k}}(D^- \cup D^+)$. Then
\[
\lambda_{\tilde{k}}^\varepsilon \to \lambda_{k_0}(D^+) \quad \text{as } \varepsilon \to 0^+.
\]
Furthermore, for every $\varepsilon$ sufficiently small, $\lambda_{\tilde{k}}^\varepsilon$ is simple and there exists an eigenfunction $\varphi_{\tilde{k}}^\varepsilon$ associated to $\lambda_{\tilde{k}}^\varepsilon$, i.e. satisfying
\[
\begin{cases}
-\Delta \varphi_{\tilde{k}}^\varepsilon = \lambda_{\tilde{k}}^\varepsilon p \varphi_{\tilde{k}}^\varepsilon, & \text{in } \Omega^\varepsilon, \\
\varphi_{\tilde{k}}^\varepsilon = 0, & \text{on } \partial \Omega^\varepsilon,
\end{cases}
\]
such that
\[
\varphi_{\tilde{k}}^\varepsilon \to \varphi_{k_0}^+ \quad \text{in } D^{1,2}(\mathbb{R}^N) \quad \text{as } \varepsilon \to 0^+,
\]
where in the above formula we mean the functions $\varphi_{\tilde{k}}^\varepsilon, \varphi_{k_0}^+$ to be trivially extended to the whole $\mathbb{R}^N$.

We mention that uniform convergence of eigenfunctions has been established in [10, §5.2].

Henceforward, for simplicity of notation, we denote
\[
(8) \quad u_\varepsilon = \varphi_{\tilde{k}}^\varepsilon \quad \text{and} \quad u_0 = \varphi_{k_0}^+.
\]
Hence, for small $\varepsilon$, $u_\varepsilon$ solves
\[
\begin{cases}
-\Delta u_\varepsilon = \lambda_{\tilde{k}}^\varepsilon p u_\varepsilon, & \text{in } \Omega^\varepsilon, \\
u_\varepsilon = 0, & \text{on } \partial \Omega^\varepsilon.
\end{cases}
\]
Let us assume Theorem 1.2. The unit sphere that, for every sequence $\varepsilon \to 0^+$, there exists a subsequence $\{\varepsilon_n\}$ such that for every $\varepsilon_n \to 0^+$ there exists a subsequence $\{\varepsilon_{n_j}\}$, $U \in C^2(D^-) \cup \left( \bigcup_{t>0} \mathcal{H}_t^- \right)$, $U \neq 0$, and $\beta < 0$ such that

$$u_{\varepsilon_{n_j}} \to U \quad \text{as} \quad j \to +\infty \quad \text{strongly in} \quad \mathcal{H}_t^- \quad \text{for every} \quad t > 0$$

and in $C^2(B_{t_1}^- \setminus B_{t_2}^-)$ for all $0 < t_1 < t_2$;

$$\lambda^{N-1} U(\lambda x) \to \beta \frac{x_1}{|x|^N} \quad \text{as} \quad \lambda \to 0^+ \quad \text{strongly in} \quad \mathcal{H}_t^- \quad \text{for every} \quad t > 0$$

and in $C^2(B_{t_1}^- \setminus B_{t_2}^-)$ for all $0 < t_1 < t_2$;

$$\beta = -\frac{\int_{S^{N-1}} U Y_1 \, d\sigma - \frac{\lambda a(D^+)}{N} \int_{D^+} p(x) U(x) Y_1 \left( \frac{x}{|x|} \right) \left( \frac{|x|}{|x|_{S^{N-1}}} + \frac{x_{D^+ \setminus B_{t_1}^-}}{|x|_{S^{N-1}}} \right) \, dx}{\Upsilon_N}.$$

In the forthcoming paper [1], some improvements of Theorem 1.2 will be obtained; more precisely, the dependence on the subsequence will be removed and the exact asymptotic behavior of the normalization $\sqrt{\int_{S_{\beta}} u_{\varepsilon_{n_j}}^2 \, d\sigma}$ will be derived.

The description of the behavior of eigenfunctions at the junction given by Theorem 1.2 provides us with some important information about the sign of $u_\varepsilon$ near the left junction. More precisely, the nondegeneracy condition (4) on the right junction implies that the solution is one-sign in a neighborhood of the left one. In other words, the nodal set of $u_\varepsilon$ does not enter inside the channel.

**Corollary 1.3.** Let us assume (2)–(6) hold and let $u_\varepsilon$ as in (8). Then there exists $R > 0$ such that

for every $r \in (0, R)$ there exists $\varepsilon_r > 0$ such that $u_\varepsilon > 0$ in $\mathcal{H}_t^-$ for all $\varepsilon \in (0, \varepsilon_r)$.

The paper is organized as follows. In section 2 we prove some estimates from above and from below of eigenfunctions of the approximating problem close to the right junction uniformly with respect to the parameter $\varepsilon$. In section 3 we introduce a frequency function associated to the approximating problem and study its behavior at the left, in the corridor, and at the right of the domain. Sections 4 and 5 contain a blow-up analysis (at the right and at the left junction respectively) leading to some uniform bounds of the frequency function which allow describing, in section 6, the asymptotic behavior of the eigenfunctions (suitably normalized) close to the left junction of the tube, thus proving Theorem 1.2 and Corollary 1.3.
2. Estimates on $u_{\varepsilon}$ on the right

This section collects some estimates of eigenfunctions $u_{\varepsilon}$ close to the right junction, which will be crucial to control the frequency function at the right.

**Lemma 2.1.** There exist $0 < r_0 < 3$, $\varepsilon_0 \in (0, r_0/2)$, and $C_0 > 0$ such that

$$\frac{1}{C_0}(x_1 - 1) \leq u_{\varepsilon}(x) \leq C_0(x_1 - 1)$$

for all $x \in D^+ \cap \partial B_{r_0}^+$ and $\varepsilon \in (0, \varepsilon_0)$.

**Proof.** From Lemma 1.1 and classical elliptic regularity theory,

$$u_{\varepsilon} \to u_0 \text{ in } C^{2}_{\text{loc}}(D^+ \setminus \{e_1\}) \text{ and } \nabla u_{\varepsilon} \to \nabla u_0 \text{ in } C^{1}_{\text{loc}}(D^+ \setminus \{e_1\}).$$

Furthermore (6) implies that there exist $C > 0$ and $r_0 \in (0, 3)$ such that

$$\partial u_0 / \partial x_1(x) \geq C, \quad u_0(x) > 0, \quad \text{for all } x \in B_{r_0}^+.$$

Let $t_0 \in (1, 1 + r_0/4)$ such that, if $x = (x_1, x') \in A_0 := (B_{r_0}^+ \setminus B_{(3r_0)/4}^+) \cap \{1 < x_1 < t_0\}$, then $(1, x') \in B_{r_0}^+ \setminus B_{r_0/2}^+$. By (14) and continuity of $u_0$, there exist $c > 0$ such that

$$u_0(x) \geq c \quad \text{for all } x \in (B_{r_0}^+ \setminus B_{(3r_0)/4}^+) \setminus A_0.$$

From (13), there exists $\varepsilon_0 \in (0, r_0/2)$ such that equation (9) is satisfied for $\varepsilon \in (0, \varepsilon_0)$ and

$$\left| \frac{\partial u_{\varepsilon}}{\partial x_1}(x) - \frac{\partial u_0}{\partial x_1}(x) \right| \leq \frac{C}{2}$$

for all $x \in B_{r_0}^+ \setminus B_{r_0/2}^+$ and $\varepsilon \in (0, \varepsilon_0)$,

$$|u_{\varepsilon}(x) - u_0(x)| \leq \frac{C}{2}$$

for all $x \in (B_{r_0}^+ \setminus B_{(3r_0)/4}^+) \setminus A_0$ and $\varepsilon \in (0, \varepsilon_0)$.

Estimate (17) together with (15) implies that

$$u_{\varepsilon}(x) \geq \frac{c}{2}$$

for all $x \in (B_{r_0}^+ \setminus B_{(3r_0)/4}^+) \setminus A_0$ and $\varepsilon \in (0, \varepsilon_0)$.

On the other hand, (16) together with (14) implies

$$\frac{\partial u_{\varepsilon}}{\partial x_1}(x) \geq \frac{C}{2}$$

for all $x \in B_{r_0}^+ \setminus B_{r_0/2}^+$ and $\varepsilon \in (0, \varepsilon_0)$.

We notice that, if $x \in A_0$ then from (19) it follows that

$$u_{\varepsilon}(x_1, x') = u_{\varepsilon}(1, x') + \int_1^{x_1} \frac{\partial u_{\varepsilon}}{\partial x_1}(s, x') \, ds > 0.$$

Combining (18) and (20) we conclude that

$$u_{\varepsilon}(x) > 0$$

for all $x \in (B_{r_0}^+ \setminus B_{(3r_0)/4}^+) \setminus A_0$ and $\varepsilon \in (0, \varepsilon_0)$.

If $x \in D^+ \cap \partial B_{r_0}^+$ and $\varepsilon \in (0, \varepsilon_0)$, from (19) and (21) we have that

$$u_{\varepsilon}(x) = u_{\varepsilon}(x - \frac{x_1 - 1}{4} e_1) + \int_1^{x_1} \frac{\partial u_{\varepsilon}}{\partial x_1}(x - \frac{(1 - t)(x_1 - 1)}{4} e_1) \frac{x_1 - 1}{4} \, dt \geq \frac{C}{2} \frac{x_1 - 1}{4}$$

thus proving the stated lower bound. The upper bound follows combining (16), (17), and (20). \qed

The following iterative Brezis-Kato type argument yields a uniform $L^\infty$-bound for $\{u_{\varepsilon}\}$.

**Lemma 2.2.** There exists $C_1 > 0$ such that

$$|u_{\varepsilon}(x)| \leq C_1$$

for all $x \in \Omega^c$ and $\varepsilon \in (0, \varepsilon_0)$.

**Proof.** Since $u_{\varepsilon} \to u_0$ in $D^{1,2}(\mathbb{R}^N)$, we have that

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \|u_{\varepsilon}\|_{L^{2^*}(\mathbb{R}^N)} < \infty.$$

We claim that

$$\|u_{\varepsilon}\|_{L^{2^*}(\mathbb{R}^N)} \leq C_1 (q - 2)^{\frac{1}{q}} \|u_{\varepsilon}\|_{L^q(\mathbb{R}^N)}$$

for some $q > 2$ and all $\varepsilon \in (0, \varepsilon_0)$.
The claim can be proved by following the Brezis-Kato procedure \cite{8}. For every \( n \in \mathbb{N} \), we set \( u^n_0 = \min\{ n, |u_n| \} \) and test (9) with \( u_n (u^n_0)^{q-2} \) thus obtaining
\[
(q - 2) \int_{\Omega'} |\nabla u^n_0|^2 (u^n_0)^{q-2} \, dx + \int_{\Omega'} |\nabla u_n|^2 (u^n_0)^{q-2} \, dx = \lambda_n \int_{\Omega'} p u^n_0 (u^n_0)^{q-2} \, dx.
\]
Letting \( C(q) = \min \{ \frac{2}{q-2}, \frac{1}{2} \} \), we then obtain
\[
C(q) \int_{\Omega'} |\nabla ((u^n_0)^{\frac{q}{q-1}} u_n)|^2 \, dx \leq C(q) \int_{\Omega'} \left( \frac{(q - 2)^2}{2} (u^n_0)^{q-2} |\nabla u^n_0|^2 + 2(u^n_0)^{q-2} |\nabla u_n|^2 \right) \, dx
\leq \lambda_n \int_{\Omega'} p u^n_0 (u^n_0)^{q-2} \, dx \leq \text{const} \int_{\Omega'} |u_n|^q \, dx
\]
for some const > 0 independent of \( \epsilon \) and \( q \), which, letting \( n \to +\infty \), implies claim (23) by Sobolev inequality. Starting from \( q = 2^* \) and iterating the estimate of claim (23), we obtain that, for all \( n \in \mathbb{N} \), \( n \geq 1 \), letting \( q_n = 2 \left( \frac{2}{n+1} \right)^2 \), there holds
\[
\| u_n \|_{L^{2q+1}(\mathbb{R}^N)} \leq \| u_n \|_{L^{2q}^*(\mathbb{R}^N)} C_{\Sigma^2} \frac{\sqrt{n}}{2} \prod_{k=1}^{n} (q_k - 2) \frac{\sqrt{n}}{2} \leq \text{const} \| u_n \|_{L^{2^*}^r(\mathbb{R}^N)}
\]
for some const > 0 independent of \( \epsilon \) and \( n \). Letting \( n \to \infty \), (22) yields the conclusion.

We denote
\begin{align*}
T^+_1 &= \{(x_1, x') : x' \in \Sigma, x_1 \leq 1\}, \quad \tilde{D} = D^+ \cup T^+_1, \\
T^-_c &= \{(x_1, x') : x_1 \notin \Sigma, x_1 \leq 1\}, \\
T^-_1 &= \{(x_1, x') : x_1 \in \mathbb{R}, x' \in \Sigma\},
\end{align*}
and, for \( r \in \mathbb{R} \setminus \{1, 2\} \),
\begin{align*}
\tilde{\Omega}_r &= \left\{ \begin{array} {ll}
\{(x_1, x') \in T^-_1 : x_1 < r\}, & \text{if } r \leq 1, \\
T^-_1 \cup B^+_r, & \text{if } r > 2,
\end{array} \right. \\
\Gamma_r &= \left\{ \begin{array} {ll}
\{(x_1, x') \in T^-_1 : x_1 = r\}, & \text{if } r \leq 1, \\
\Gamma^+_r, & \text{if } r > 2,
\end{array} \right.
\end{align*}
where, for all \( t > 0 \), we denote
\[
\Gamma^+_t = D^+ \cap \partial \tilde{B}^+_t.
\]
Let us define
\[
f : T^-_1 \to \mathbb{R}, \quad f(x_1, x') = e^{-\sqrt{\lambda_1(\Sigma)}(x_1-1)} \psi^1_\Sigma(x'),
\]
where \( \lambda_1(\Sigma) \) is the first eigenvalue of the Laplace operator on \( \Sigma \) under null Dirichlet boundary conditions and \( \psi^1_\Sigma(x') \) is the corresponding positive \( L^2(\Sigma) \)-normalized eigenfunction, so that
\[
\begin{cases}
-\Delta_{x'} \psi^1_\Sigma(x') = \lambda_1(\Sigma) \psi^1_\Sigma(x'), & \text{in } \Sigma, \\
\psi^1_\Sigma = 0, & \text{on } \partial \Sigma,
\end{cases}
\]
being \( \Delta_{x'} = \sum_{j=2}^{N} \frac{\partial^2}{\partial x_j^2} \), \( x' = (x_2, \ldots, x_N) \). In particular \( f \in C^2(\tilde{T}_1) \) and satisfies
\[
\begin{cases}
-\Delta f = 0, & \text{in } T^-_1, \\
f = 0, & \text{on } \partial T^-_1.
\end{cases}
\]
Lemma 2.4 below shows how harmonic functions in \( D^+ \) can be extended (up to a finite energy perturbation) to harmonic functions in \( \tilde{D} \) with finite energy at \( -\infty \). In order to prove it, the following Poincaré type inequality is needed.

**Lemma 2.3.** There exists a constant \( C_P = C_P(N) \) depending only on the dimension \( N \) such that for every function \( v : D^+ \setminus B^+_1 \to \mathbb{R} \) satisfying
\[
v \in \bigcap_{R > 1} H^1(B^+_R \setminus B^+_1) \quad \text{and} \quad v = 0 \text{ on } \{ x_1 = 1, |x'| > 1 \},
\]
there holds
\[
\int_{B^+_2 \setminus B^+_1} v^2(x) \, dx \leq C_P R^2 \int_{B^+_2 \setminus B^+_1} |\nabla v(x)|^2 \, dx \quad \text{for all } R > 1.
\]
Lemma 2.4. For every $\psi \in C^2(D^+) \cap C^1(\bar{D}^+)$ such that
\[
\begin{cases}
-\Delta \psi = 0, & \text{in } D^+,
\psi = 0, & \text{on } \partial D^+,
\end{cases}
\]
there exists a unique function $u = T(\psi)$ such that
\[
\int_{\tilde{D}} \left( |\nabla u(x)|^2 + |u(x)|^2 \right) dx < +\infty \text{ for all } R > 2,
\]
\[-\Delta u = 0 \text{ in a distributional sense in } \tilde{D}, \quad u = 0 \text{ on } \partial \tilde{D},
\]
\[
\int_{D^+} |\nabla (u - \psi)(x)|^2 dx < +\infty.
\]
Furthermore
\[
\mathcal{T}(\psi) - \tilde{\psi} \in \mathcal{D}^{1,2}(\tilde{D}), \quad \text{where } \tilde{\psi} := \begin{cases}
\psi & \text{in } D^+ \\
0 & \text{in } T_1^-.
\end{cases}
\]

Proof. Let us define $J_\psi : \mathcal{D}^{1,2}(\tilde{D}) \to \mathbb{R}$ as
\[
J_\psi(\varphi) = \frac{1}{2} \int_{\tilde{D}} |\nabla \varphi(x)|^2 dx - \int_{\Sigma} \varphi(1, x') \left( \frac{\partial \psi}{\partial x_1} \right)_+(1, x') dx',
\]
where $\left( \frac{\partial \psi}{\partial x_1} \right)_+(1, x') := \lim_{t \to 0^+} \frac{\psi(t + 1, x')}{t}$. By standard minimization methods it is easy to prove that there exists $w \in \mathcal{D}^{1,2}(\tilde{D})$ such that $J_\psi(w) = \min_{\mathcal{D}^{1,2}(\tilde{D})} J_\psi$. In particular $w$ satisfies
\[
0 = dJ_\psi(w)[\varphi] = \int_{\tilde{D}} \nabla w(x) \cdot \nabla \varphi(x) dx - \int_{\Sigma} \varphi(1, x') \left( \frac{\partial \psi}{\partial x_1} \right)_+(1, x') dx'
\]
for all $\varphi \in \mathcal{D}^{1,2}(\tilde{D})$. Hence the function $u : \tilde{D} \to \mathbb{R}$,
\[
\left\{
\begin{array}{ll}
w + \psi, & \text{in } D^+, \\
w, & \text{in } T_1^-,
\end{array}
\right.
\]
satisfies (28), (30), and, for every $\varphi \in C_0^\infty(\tilde{D})$,
\[
\int_{\tilde{D}} \nabla u(x) \cdot \nabla \varphi(x) dx = \int_{\tilde{D}} \nabla w(x) \cdot \nabla \varphi(x) dx + \int_{D^+} \nabla \psi(x) \cdot \nabla \varphi(x) dx
\]
\[
eq \int_{\Sigma} \varphi(1, x') \left( \frac{\partial \psi}{\partial x_1} \right)_+(1, x') dx' - \int_{\Sigma} \varphi(1, x') \left( \frac{\partial \psi}{\partial x_1} \right)_+(1, x') dx' = 0
\]
thus implying (29). To prove uniqueness, let us assume that $u_1$ and $u_2$ both satisfy (29–30); then the difference $u = u_1 - u_2$ solves
\[
-\Delta u = 0 \text{ in a distributional sense in } \tilde{D}, \quad u = 0 \text{ on } \partial \tilde{D},
\]
and satisfies
\[
\int_{D^+} |\nabla u(x)|^2 dx = \int_{D^+} |\nabla (u_1 - \psi)(x) - \nabla (u_2 - \psi)(x)|^2 dx < +\infty.
\]
For all $R > 2$ let $\eta_R$ be a cut-off function satisfying
\[
\eta_R \in C^\infty(\tilde{D}), \quad \eta_R \equiv 1 \text{ in } \tilde{\Omega}_R, \quad \eta_R \equiv 0 \text{ in } D^+ \setminus B_{2R-1}^+, \quad |\nabla \eta_R(x)| \leq \frac{2}{R-1} \text{ in } \tilde{D}.
\]
Multiplying (33) with $\eta^2_R u$ and integrating by parts over $\tilde{D}$ we obtain
\[
\int_{\tilde{D}} |\nabla u(x)|^2 \eta^2_R dx = -2 \int_{\tilde{D}} u(x) \eta_R(x) \nabla u(x) \cdot \nabla \eta_R(x) dx
\]
\[
\leq \frac{1}{2} \int_{\tilde{D}} |\nabla u(x)|^2 \eta^2_R dx + 2 \int_{\tilde{D}} \eta^2_R(x) \nabla \eta_R(x)^2 dx
\]
thus implying, in view of Lemma 2.3,
\[
\frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx \leq \frac{1}{2} \int_{\widetilde{D}} |\nabla u(x)|^2 \eta \phi_2(x) \, dx \\
\leq 2 \int_{\widetilde{D}} u^2(x) |\nabla \eta_R(x)|^2 \, dx \leq \frac{8}{(R-1)^2} \int_{B_{2(n-1)}^+ \setminus B_{n-1}^+} u^2(x) \, dx \\
\leq 8CP \int_{B_{2(n-1)}^+ \setminus B_{n-1}^+} |\nabla u(x)|^2 \, dx.
\]

Letting \( R \to +\infty \), from (34) we deduce that \( \int_{\widetilde{D}} |\nabla u|^2 \, dx = 0 \) and hence \( u \) must be constant on \( \widetilde{D} \).

Since \( u \) vanishes on \( \partial \widetilde{D} \), we deduce that \( u \equiv 0 \) and then \( u_1 = u_2 \) in \( \widetilde{D} \) thus proving uniqueness. \( \Box \)

Henceforward we denote
\[
\Phi_1 = \mathcal{T}(x_1 - 1).
\]

Since in the case \( \psi(x) = x_1 - 1 \) we have that \( \left( \frac{\partial \psi}{\partial x_1} \right)_+(1, x') = 1 > 0 \), the minimum of the functional \( J_{x_1 - 1} \) defined in (32) is attained by a nonnegative function \( w \). Hence we deduce that
\[
\Phi_1(x_1, x') \geq (x_1 - 1)^+ \quad \text{for all } (x_1, x') \in \widetilde{D}.
\]

Hence, from the Strong Maximum Principle we deduce that
\[
\Phi_1(x_1, x') > 0 \quad \text{for all } (x_1, x') \in \widetilde{D}.
\]

For all \( r \in \mathbb{R} \), let us denote
\[
T_{1, r} = \{(x_1, x') : x' \in \Sigma, \ x_1 \leq r\}, \quad \Gamma_r := \{(x_1, x') : x' \in \Sigma, \ x_1 = r\},
\]
and define \( \mathcal{E}_r \) as the completion of \( C_c^\infty(T_{1, r}) \) with respect to the norm \( (\int_{T_{1, r}} |\nabla v|^2 \, dx)^{1/2} \) (which is actually equivalent to the norm \( (\int_{T_{1, r}} |\nabla v|^2 \, dx + \int_{\Gamma_r} v^2 \, d\sigma)^{1/2} \), i.e. \( \mathcal{E}_r \) is the space of finite energy functions in \( T_{1, r} \) vanishing on \( \{(x_1, x') : x_1 \leq r \text{ and } x' \in \partial \Sigma\} \).

The following Lemma associate an Almgren type frequency function to harmonic functions in \( \mathcal{E}_R \) and describe its behavior at \( -\infty \).

**Lemma 2.5.** Let \( R \in \mathbb{R} \) and \( \phi \in \mathcal{E}_R \setminus \{0\} \) satisfying
\[
\begin{cases}
-\Delta \phi = 0, & \text{in } T_{1, R}, \\
\phi = 0, & \text{on } \{(x_1, x') : x_1 \leq R \text{ and } x' \in \partial \Sigma\},
\end{cases}
\]
in a weak sense, and let \( N_\phi : (-\infty, R) \to \mathbb{R} \) be defined as
\[
N_\phi(r) := \frac{\int_{T_{1, r}} |\nabla \phi(x)|^2 \, dx}{\int_{\Gamma_r} \phi^2(x) \, d\sigma}.
\]

Then
i) \( N_\phi \) is non decreasing in \( (-\infty, R) \);
ii) there exists \( K_0 \in \mathbb{N} \), \( K_0 \geq 1 \), such that
\[
\lim_{r \to -\infty} N_\phi(r) = \sqrt{\lambda_{K_0}(\Sigma)},
\]
where \( \lambda_{K_0}(\Sigma) \) is the \( K_0 \)-th eigenvalue of the Laplace operator on \( \Sigma \) under null Dirichlet boundary conditions;
iii) if \( N_\phi \equiv \gamma \) for some \( \gamma \in \mathbb{R} \) then \( \gamma = \sqrt{\lambda_{K_0}(\Sigma)} \) and \( \phi(x_1, x') = e^{\sqrt{\lambda_{K_0}(\Sigma)}x_1} \psi(x') \) for some eigenfunction \( \psi \) of \( -\Delta_{x'} \) in \( \Sigma \) associated to the eigenvalue \( \lambda_{K_0}(\Sigma) \);
iv) if \( \phi > 0 \) in \( T_{1, R} \), then \( K_0 = 1 \).

**Proof.** It is easy to prove that \( N_\phi \in C^1(-\infty, R) \) and, for all \( r \in (-\infty, R) \),
\[
N'_\phi(r) = 2 \left( \int_{\Gamma_r} \frac{\partial \phi}{\partial x_1}^2 \, d\sigma \right) \left( \int_{\Gamma_r} \phi^2 \, d\sigma \right) - 2 \left( \int_{\Gamma_r} \frac{\partial \phi}{\partial x_1} \, d\sigma \right)^2.
\]
Hence, Schwarz’s inequality implies that $N_\phi'(r) \geq 0$ for all $r < R$. Therefore $N_\phi$ is non-decreasing in $(-\infty, R)$ and statement i) is proved. By monotonicity, there exists

$$\gamma := \lim_{r \to -\infty} N_\phi(r) \in [0, +\infty).$$

For every $\lambda > 0$ let us define

$$\phi_\lambda(x_1, x') := \frac{\phi(x_1 - \lambda, x')}{\sqrt{\int_{\Gamma_{r, \lambda}} \phi^2 d\sigma}}.$$ 

We have that $\phi_\lambda \in \mathcal{E}_{R, \lambda}$,

$$\int_{\Gamma_R} \phi_\lambda^2 d\sigma = 1,$$

and $\phi_\lambda$ weakly solves

$$\begin{cases}
-\Delta \phi_\lambda = 0, & \text{in } T_{1, R, \lambda}, \\
\phi_\lambda = 0, & \text{on } \{(x_1, x') : x_1 \leq R + \lambda \text{ and } x' \in \partial \Sigma\}.
\end{cases}$$

Moreover, the change of variable $(x_1, x') = (y_1 - \lambda, y')$ yields

$$N_\phi(r - \lambda) = \frac{\int_{T_1, r} |\nabla \phi_\lambda(y)|^2 dy}{\int_{T_1, r} \phi_\lambda^2 d\sigma} \quad \text{for all } r < R + \lambda.$$

In particular we have that

$$N_\phi(R - \lambda) = \int_{T_{1, R}} |\nabla \phi_\lambda(y)|^2 dy \leq N_\phi \left(\frac{R}{2}\right) \quad \text{for every } \lambda \geq \frac{R}{2},$$

and hence $\{\phi_\lambda\}_{\lambda \geq R/2}$ is bounded in $\mathcal{E}_R$. Therefore there exist a sequence $\lambda_n \to +\infty$ and some $\tilde{\phi} \in \mathcal{E}_R$ such that $\phi_{\lambda_n} \to \tilde{\phi}$ weakly in $\mathcal{E}_R$ and a.e. in $T_{1, R}$. From compactness of the embedding $\mathcal{E}_R \hookrightarrow L^2(\Gamma_R)$ and (40) we deduce that $\int_{\Gamma_R} \tilde{\phi}^2 d\sigma = 1$; in particular $\tilde{\phi} \not\equiv 0$. Passing to the weak limit in (41) as $\lambda_n \to +\infty$ we have that

$$\begin{cases}
-\Delta \tilde{\phi} = 0, & \text{in } T_{1, R}, \\
\tilde{\phi} = 0, & \text{on } \{(x_1, x') : x_1 \leq R \text{ and } x' \in \partial \Sigma\}.
\end{cases}$$

By classical elliptic regularity estimates, we also have that $\phi_{\lambda_n} \to \tilde{\phi}$ in $C^2(T_{1, r_2} \setminus T_{1, r_1})$ for all $r_1 < r_2 < R$. Therefore, multiplying (43) by $\tilde{\phi}$ and integrating over $T_{1, r}$ with $r < R$, we obtain

$$\int_{T_{1, r}} \frac{\partial \phi_{\lambda_n}}{\partial x_1} \phi_{\lambda_n} \phi_\lambda d\sigma \to \int_{T_{1, r}} \frac{\partial \tilde{\phi}}{\partial x_1} \tilde{\phi} d\sigma = \int_{T_{1, r}} |\nabla \tilde{\phi}(x)|^2 dx.$$ 

On the other hand, multiplication of (41) by $\phi_{\lambda_n}$ and integration by parts over $T_{1, r}$ yield

$$\int_{T_{1, r}} |\nabla \phi_{\lambda_n}(x)|^2 dx = \int_{T_{1, r}} \frac{\partial \phi_{\lambda_n}}{\partial x_1} \phi_{\lambda_n} d\sigma.$$ 

From (44) and (45), we deduce that $\|\phi_{\lambda_n}\|_{\mathcal{E}_r} \to \|\tilde{\phi}\|_{\mathcal{E}_r}$ and then $\phi_{\lambda_n} \to \tilde{\phi}$ strongly in $\mathcal{E}_r$ for every $r < R$. Therefore, for every $r < R$, passing to the limit as $\lambda_n \to +\infty$ in (42) and letting $\gamma$ as in (39), we obtain that

$$N_{\tilde{\phi}}(r) = \gamma \quad \text{for all } r < R,$$

where

$$N_{\tilde{\phi}}(r) = \frac{\int_{T_{1, r}} |\nabla \tilde{\phi}(y)|^2 dy}{\int_{T_{1, r}} \tilde{\phi}^2 d\sigma}.$$ 

Then

$$N_{\tilde{\phi}}'(r) = 2 \left(\frac{\int_{T_{1, r}} |\tilde{\phi}|^2 d\sigma}{\int_{T_{1, r}} \tilde{\phi}^2 d\sigma}\right) \left(\int_{T_{1, r}} \tilde{\phi}^2 d\sigma\right) - \left(\frac{\int_{T_{1, r}} \tilde{\phi} d\sigma}{\int_{T_{1, r}} \tilde{\phi}^2 d\sigma}\right) = 0 \quad \text{for all } r < R.$$
Since equality in the Schwarz’s inequality holds only for parallel vectors, we infer that \( \frac{\partial \tilde{\phi}}{\partial x_j} \) and \( \tilde{\phi} \) must be parallel as vectors in \( L^2(\Gamma_r) \), hence there exists some function \( \eta : (-\infty, R) \to \mathbb{R} \) such that

\[
\frac{\partial \tilde{\phi}}{\partial x_1}(x_1, x') = \eta(x_1) \tilde{\phi}(x_1, x') \quad \text{for all } x_1 \in (-\infty, R) \text{ and } x' \in \Sigma.
\]

Integration with respect to \( x_1 \) yields

\[
\tilde{\phi}(x_1, x') = \varphi(x_1)\psi(x') \quad \text{for all } x_1 \in (-\infty, R) \text{ and } x' \in \Sigma,
\]

where \( \varphi(x_1) = e^{\int_{x_1}^{x_1+1} \eta(s)ds} \), \( \psi(x') = \tilde{\phi}(R, x') \). From (43) and (47), we derive

\[
\varphi''(x_1)\psi(x') + \varphi(x_1)\Delta_{x'}\psi(x') = 0.
\]

Taking \( x_1 \) fixed, we deduce that \( \psi \) is an eigenfunction of \(-\Delta_{x'}\) in \( \Sigma \) under homogeneous Dirichlet boundary conditions. If \( \lambda_{K_0}(\Sigma) \) is the corresponding eigenvalue then \( \varphi(x_1) \) solves the equation

\[
\varphi''(x_1) - \lambda_{K_0}(\Sigma)\varphi(x_1) = 0
\]

and hence \( \varphi \) is of the form

\[
\varphi(x_1) = c_1e^{\sqrt{\lambda_{K_0}(\Sigma)}}(x_1 - R) + c_2e^{-\sqrt{\lambda_{K_0}(\Sigma)}}(x_1 - R) \quad \text{for some } c_1, c_2 \in \mathbb{R}.
\]

Since the function \( e^{-\sqrt{\lambda_{K_0}(\Sigma)}}(x_1 - R) \psi(x') \notin E_R \), then \( c_2 = 0 \) and \( \varphi(x_1) = c_1e^{\sqrt{\lambda_{K_0}(\Sigma)}}(x_1 - R) \). Since \( \varphi(R) = 1 \), we obtain that \( c_1 = 1 \) and then

\[
\tilde{\phi}(x_1, x') = e^{\sqrt{\lambda_{K_0}(\Sigma)}}(x_1 - R) \psi(x'), \quad \text{for all } x_1 \in (-\infty, R) \text{ and } x' \in \Sigma.
\]

Substituting (48) into (46) we obtain that \( \gamma = \sqrt{\lambda_{K_0}(\Sigma)} \). Hence statement ii) is proved. We notice that the above argument of classification of harmonic functions \( \tilde{\phi} \) with constant frequency \( \gamma \) also proves statement iii).

In order to prove iv), let us assume that \( \phi > 0 \) in \( T_{1, R} \). Then \( \phi_\lambda > 0 \) in \( T_{1, R+\lambda} \). Hence a.e. convergence implies that \( \tilde{\phi} > 0 \) in \( T_{1, R} \). From the Strong Maximum Principle we obtain that \( \tilde{\phi} > 0 \) in \( T_{1, R} \), which necessarily implies that \( \psi > 0 \) in \( \Sigma \). Then \( \psi \) must be the eigenfunction associated to the first eigenvalue, i.e. \( \lambda_{K_0}(\Sigma) = \lambda_1(\Sigma) \).\( \square \)

The previous lemma allows describing the behavior of the Almgren type frequency quotient naturally associated to the function \( \Phi_1 \) introduced in (35). For all \( r \in \mathbb{R} \setminus (1, 2) \), let \( \tilde{N}(r) = \tilde{N}_{\Phi_1}(r) \) be the frequency function associated to \( \Phi_1 \), i.e.

\[
\tilde{N}(r) = \tilde{N}_{\Phi_1}(r) = \frac{\Lambda_N(r) \int_{\tilde{\Gamma}_r} |\nabla \Phi_1(x)|^2 dx}{\int_{\tilde{\Gamma}_r} \Phi_1(x) d\sigma},
\]

where

\[
\Lambda_N(r) = \begin{cases} 
\left( \frac{2}{\omega_{N-1}} \right)^{\frac{1}{N-1}} |\tilde{\Gamma}_r|^{\frac{1}{N-1}} = r - 1, & \text{if } r \geq 2, \\
\left( \frac{N-1}{\omega_{N-1}} \right)^{\frac{1}{N-1}} |\tilde{\Gamma}_r|^{\frac{1}{N-1}} = 1, & \text{if } r \leq 1,
\end{cases}
\]

\( |\tilde{\Gamma}_r| \) denotes the \((N-1)\)-dimensional volume of \( \tilde{\Gamma}_r \), and \( \omega_{N-1} \) is the volume of the unit sphere \( S^{N-1} \), i.e. \( \omega_{N-1} = \int_{S^{N-1}} d\sigma(\theta) \).

An immediate consequence of Lemma 2.5 and (37) is the following corollary.

**Corollary 2.6.** \( \lim_{r \to -\infty} \tilde{N}(r) = \sqrt{\lambda_1(\Sigma)} \).

As a left counterpart of Lemma 2.4, we now construct a harmonic extension to \( \tilde{D} \) of the function \( f \) defined in (27) (up to a finite energy perturbation in the tube) having finite energy at the right.

**Lemma 2.7.** There exists a unique function \( \Phi_2 : \tilde{D} \to \mathbb{R} \) such that

\[
\int_{\tilde{D}^+} \left( |\nabla \Phi_2(x)|^2 + |\Phi_2(x)|^2 \right) dx < +\infty,
\]

\[
-\Delta \Phi_2 = 0 \text{ in a distributional sense in } \tilde{D}, \quad \Phi_2 = 0 \text{ on } \partial \tilde{D},
\]

\[
\int_{T_1} |\nabla (\Phi_2 - f)(x)|^2 dx < +\infty,
\]
where \( f \) is defined in (27). Furthermore
\[
(54) \quad \Phi_2 \geq f \quad \text{in} \ T_1 \quad \text{and} \quad \Phi_2 \geq 0 \quad \text{in} \ \tilde{D}.
\]

**Proof.** Let us define \( J : D^{1,2}(\tilde{D}) \to \mathbb{R} \) as
\[
J(\varphi) = \frac{1}{2} \int_{\tilde{D}} |\nabla \varphi(x)|^2 \, dx + \int_{(1,+) \times \partial \Sigma} \frac{\partial f}{\partial \nu} \, d\sigma - \frac{1}{2} \int_{\tilde{D}} |\nabla \varphi(x)|^2 \, dx + \int_{(1,+) \times \partial \Sigma} \varphi(x_1, x') e^{-\sqrt{\lambda_1}(x_1-1)} \frac{\partial \psi_1}{\partial \nu} (x') \, d\sigma,
\]
where \( \nu \) denotes the normal external unit vector to \( \partial T_1 \) and \( \nu_{x'} \) the normal external unit vector to \( \partial \Sigma \). It is easy to prove that \( J(\varphi) \geq c_1||\varphi||^2_{D^{1,2}(\tilde{D})} - c_2 \) for some constants \( c_1, c_2 > 0 \) and all \( \varphi \in D^{1,2}(\tilde{D}) \) and that \( J \) is weakly lower semi-continuous. Hence there exists \( w \in D^{1,2}(\tilde{D}) \) such that \( J(w) = \min_{D^{1,2}(\tilde{D})} J \). Since, by the Hopf Lemma, \( \frac{\partial \psi_1}{\partial \nu} < 0 \) on \( \partial \Sigma \), we can assume that \( w \geq 0 \) (otherwise we take \( |w| \) which is still a minimizer). The minimizer \( w \) satisfies
\[
0 = dJ(w)[\varphi] = \int_{\tilde{D}} \nabla w(x) \cdot \nabla \varphi(x) \, dx + \int_{(1,+) \times \partial \Sigma} \varphi(x) \frac{\partial f}{\partial \nu} \, d\sigma
\]
for all \( \varphi \in D^{1,2}(\tilde{D}) \). Hence the function \( \Phi_2 : \tilde{D} \to \mathbb{R}, \)
\[
\Phi_2 = \begin{cases} w + f, & \text{in} \ T_1, \\ w, & \text{in} \ \tilde{D} \setminus T_1, \end{cases}
\]
satisfies (51), (53), (54), and, for every \( \varphi \in C^\infty_c(\tilde{D}) \),
\[
\int_{\tilde{D}} \nabla \Phi_2(x) \cdot \nabla \varphi(x) \, dx = \int_{\tilde{D}} \nabla w(x) \cdot \nabla \varphi(x) \, dx + \int_{T_1} \nabla f(x) \cdot \nabla \varphi(x) \, dx
\]
\[
= - \int_{(1,+) \times \partial \Sigma} \frac{\partial f}{\partial \nu} \, d\sigma + \int_{(1,+) \times \partial \Sigma} \varphi(x) \frac{\partial f}{\partial \nu} \, d\sigma = 0
\]
thus implying (52). To prove uniqueness, let us assume that \( u_1 \) and \( u_2 \) both satisfy (52–53); then the difference \( u = u_1 - u_2 \) solves
\[
(55) \quad -\Delta u = 0 \quad \text{in a distributional sense in} \ \tilde{D}, \quad u = 0 \quad \text{on} \ \partial \tilde{D},
\]
and satisfies
\[
(56) \quad \int_{T_1} |\nabla u(x)|^2 \, dx = \int_{T_1} |\nabla (u_1 - f)(x) - \nabla (u_2 - f)(x)|^2 \, dx < +\infty.
\]
For all \( t < 1 \) let \( \eta_t \) be a cut-off function satisfying
\[
\eta_t \in C^\infty(\tilde{D}), \quad \eta_t(x_1, x') = 1 \text{ if } x_1 > t, \quad \eta_t(x_1, x') = 0 \text{ if } x_1 < t, \quad |\nabla \eta_t(x)| \leq 2 \text{ in } \tilde{D}.
\]
Multiplying (55) with \( \eta_t^2 u \) and integrating by parts over \( \tilde{D} \) we obtain
\[
\int_{\tilde{D}} |\nabla u(x)|^2 \eta_t^2(x) \, dx = -2 \int_{\tilde{D}} u(x) \eta_t(x) \nabla u(x) \cdot \nabla \eta_t(x) \, dx
\]
\[
\leq \frac{1}{2} \int_{\tilde{D}} |\nabla u(x)|^2 \eta_t^2(x) \, dx + 2 \int_{\tilde{D}} u^2(x) |\nabla \eta_t(x)|^2 \, dx
\]
thus implying
\[
\frac{1}{2} \int_{\tilde{D} \cap \{x_1 > t\}} |\nabla u(x)|^2 \, dx \leq \frac{1}{2} \int_{\tilde{D}} |\nabla u(x)|^2 \eta_t^2(x) \, dx
\]
\[
\leq 2 \int_{\tilde{D}} u^2(x) |\nabla \eta_t(x)|^2 \, dx \leq 8 \int_{\tilde{D} \cap \{t-1 < x_1 < t\}} u^2(x) \, dx
\]
where the constant \( \tilde{C}_P > 0 \) depends only on the dimension and is the best constant of the Poincaré inequality for functions on \((-1,0) \times \Sigma\) vanishing on \( \partial \Sigma \). Letting \( t \to -\infty \), from (56) we deduce
that $\int_{\tilde{D}} |\nabla u|^2 = 0$ and hence $u$ must be constant on $\tilde{D}$. Since $u$ vanishes on $\partial \tilde{D}$, we deduce that $u \equiv 0$ and then $u_1 = u_2$ in $\tilde{D}$, thus proving uniqueness. \hfill \square

**Remark 2.8.** From (54) and the Strong Maximum Principle we deduce that

$$\Phi_2(x_1, x') > 0 \quad \text{for all } (x_1, x') \in \tilde{D}.$$ 

The functions $\Phi_1$ and $\Phi_2$ can be estimated as follows.

**Lemma 2.9.**

(i) For every $\delta > 0$ there exists $c(\delta) > 0$ such that

$$\left| \Phi_1(x) - (x_1 - 1)^+ \right| \leq c(\delta) \frac{x_1 - 1}{|x - e_1|^{N}} \quad \text{and} \quad \Phi_2(x) \leq c(\delta) \frac{x_1 - 1}{|x - e_1|^{N}}$$

for all $x \in D^+ \setminus B^+_{1+\delta}$.

(ii) There exists $C_2 > 0$ such that

$$\Phi_1(x) \leq C_2 e^{\sqrt{\lambda_1(\Sigma)} \frac{1-1}{x_1}} \quad \text{for all } x \in T^+_{1-1}.$$

**Proof.** Let us first prove (i) for the function $w = \Phi_1(x) - (x_1 - 1)^+ = T(x_1 - 1)(x) - (x_1 - 1)^+$ (the analogous estimate for $\Phi_2$ can be proved in a similar way). We observe that $w$ belongs to $D^{1,2}(\tilde{D})$ in view of (31) and weakly solves $-\Delta w = 0$ in $D^+ \setminus B^+_{1}$ by (29); moreover $w(x) = 0$ for all $x \in \{(x_1, x') : x_1 = 1, |x - e_1| > 1\}$. Therefore, its Kelvin transform

$$\tilde{w}(x) = |x - e_1|^{-(N-2)} w \left( \frac{x - e_1}{|x - e_1|^2} + e_1 \right)$$

belongs to $H^1(B^+_{1})$ and weakly satisfies

$$\begin{aligned}
-\Delta \tilde{w}(x) &= 0, \quad \text{in } B^+_{1}, \\
\tilde{w}(x) &= 0, \quad \text{on } \{(x_1, x') : x_1 = 1, |x - e_1| < 1\}.
\end{aligned}$$

By classical elliptic estimates, for any $\delta > 0$ there exists $c(\delta) > 0$ such that $|\frac{\partial \tilde{w}}{\partial x_1}| \leq c(\delta)$ in $B^+_{1/(\delta+1)}$, thus implying

$$|\tilde{w}(x_1, x')| = |\tilde{w}(1, 1') + \int_1^{x_1} \frac{\partial \tilde{w}}{\partial x_1} (s, x') ds| \leq \int_1^{x_1} \left| \frac{\partial \tilde{w}}{\partial x_1} (s, x') \right| ds \leq c(\delta)(x_1 - 1)$$

for all $(x_1, x') \in B^+_{1/(\delta+1)}$, which implies (i). To prove (ii), it is enough to observe that, in view of (1), the function $v(x_1, x') = e^{\sqrt{\lambda_1(\Sigma)} \frac{x_1}{x_1}} \Phi_1(x')$ is well-defined, harmonic and strictly positive in $T^+_{1}$, bounded from below away from 0 on $\{(x_1, x') \in T^+_{1} : x_1 = 1\}$, and $\int_{T^+_{1}} (|\nabla v|^2 + |v|^2) < +\infty$. Hence, from the Maximum Principle we deduce that $\Phi_1(x) \leq \text{const} v(x)$ in $T^+_{1}$, thus implying statement (ii). \hfill \square

In order to control $u_\epsilon$ with suitable sub/super-solutions and obtain the needed upper and lower estimates, let us introduce the following functions:

(57) \quad $\Phi^\epsilon : D^+ \cup T^-_\epsilon \to \mathbb{R}, \quad \Phi^\epsilon(x) = \epsilon \Phi_1 \left( e_1 + \frac{x - e_1}{\epsilon} \right) + 2\gamma_\epsilon \epsilon \Phi_2 \left( e_1 + \frac{x - e_1}{\epsilon} \right),$

(58) \quad $\tilde{\Phi}^\epsilon : D^+ \cup T^-_\epsilon \to \mathbb{R}, \quad \tilde{\Phi}^\epsilon(x) = \epsilon \Phi_1 \left( e_1 + \frac{x - e_1}{\epsilon} \right) - \sqrt{2}\gamma_\epsilon \epsilon \Phi_2 \left( e_1 + \frac{x - e_1}{\sqrt{2}\epsilon} \right),$

where $\gamma_\epsilon = \left( \frac{\epsilon}{2\epsilon} \exp \left( \frac{\sqrt{\lambda_1(\Sigma)}}{4\epsilon} \right) \right)^{-1}, \quad \tilde{\gamma}_\epsilon = \left( \sqrt{2}\epsilon \exp \left( \frac{\sqrt{\lambda_1(\Sigma)}}{2\sqrt{2}\epsilon} \right) \right)^{-1}.$

We notice that $\Phi^\epsilon, \tilde{\Phi}^\epsilon$ are well-defined in view of (1).
Let us define
\[ \text{Principle we reach the conclusion.} \]
We observe that
\[ \hat{u}_\varepsilon(x') \in \mathbb{R}^N : \frac{x'}{\varepsilon} \in \Sigma, \frac{1}{2} < x_1 \leq 1. \]

**PROOF.** Let us first observe that
\[ -\Delta \Phi^\varepsilon = 0, \text{ in } D^+ \cup T^+_{-\varepsilon}. \]
Moreover, if \( x \in \Gamma_{r_0}^+ = \partial B_{r_0}^+ \cap D^+ \) and \( \varepsilon \in (0, \varepsilon_0) \), then Lemma 2.1 implies that
\[ |u_\varepsilon(x)| \leq C_0(x_1 - 1), \]
while (36) and (54) ensure
\[ \Phi^\varepsilon(x) \geq \frac{\varepsilon(x_1 - 1)^+}{\varepsilon} = (x_1 - 1)^+ \text{ in } D^+ \cup T^+_{-\varepsilon}. \]
From (61–62) we deduce that
\[ |u_\varepsilon(x)| \leq C_0 \Phi^\varepsilon(x) \text{ for all } x \in \Gamma_{r_0}^+ \text{ and } \varepsilon \in (0, \varepsilon_0). \]
On the other hand, if \( x = (x_1, x') \in T^+_{-\varepsilon} \) and \( x_1 = \frac{1}{2} \), then from (36), (54), (27), and (1), it follows that
\[ \Phi^\varepsilon(x) \geq 2\gamma_\varepsilon \varepsilon \frac{\sqrt{\lambda_{\varepsilon}}}{} \psi_1^\varepsilon \left( \frac{x'}{2\varepsilon} \right) \geq \min_{y' \in \mathbb{R}^{N-1}} \psi_1^\varepsilon(y') = C_4 > 0 \]
thus implying, in view of Lemma 2.2, that
\[ |u_\varepsilon(x)| \leq \frac{C_1}{C_4} \Phi^\varepsilon(x) \text{ for all } x = (x_1, x') \in T^+_{-\varepsilon} \text{ such that } x_1 = \frac{1}{2}. \]
From (63) and (64) we conclude that
\[ |u_\varepsilon(x)| \leq \max \left\{ C_0, \frac{C_1}{C_4} \right\} \Phi^\varepsilon(x) \text{ for all } x \in \partial B_{-\varepsilon}. \]
Since, from (3) and Kato’s inequality, \(-\Delta |u_\varepsilon| \leq 0\) in \( B_{\varepsilon} \), from (60), (65), and the Maximum Principle we reach the conclusion.

Let us define
\[ \tilde{u}_\varepsilon : \mathbb{R}^N \to \mathbb{R}, \quad \tilde{u}_\varepsilon(x) = \frac{1}{\varepsilon} u_\varepsilon(e_1 + \varepsilon(x - e_1)), \]
where
\[ \tilde{\Omega}^\varepsilon := e_1 + \frac{\Omega^\varepsilon - e_1}{\varepsilon} = \left\{ x \in \mathbb{R}^N : e_1 + \varepsilon(x - e_1) \in \Omega^\varepsilon \right\}. \]
We observe that \( \tilde{u}_\varepsilon \) solves
\[ \begin{cases} -\Delta \tilde{u}_\varepsilon(x) = \varepsilon^2 \lambda_{\varepsilon} p(e_1 + \varepsilon(x - e_1)) \tilde{u}_\varepsilon(x), & \text{in } \tilde{\Omega}^\varepsilon, \\ \tilde{u}_\varepsilon = 0, & \text{on } \partial \tilde{\Omega}^\varepsilon. \end{cases} \]
From Lemma 2.10, the following uniform estimate on the gradient of \( u_\varepsilon \) on half-annuli with radius of order \( \varepsilon \) can be derived.

**Lemma 2.11.** For every \( 1 < r_1 < r_2 < \frac{r_0}{\varepsilon_0} \) there exists \( C_{r_1, r_2} > 0 \) such that
\[ |\nabla u_\varepsilon(x)| \leq C_{r_1, r_2} \text{ for all } x \in B_{r_2}^+ \setminus B_{r_1}^+ \text{ and } \varepsilon \in (0, \varepsilon_0). \]

**PROOF.** From Lemma 2.10 and (57), it follows that, letting \( \tilde{u}_\varepsilon \) as in (66–67),
\[ |\tilde{u}_\varepsilon(x)| \leq \frac{C_3}{\varepsilon} \Phi^\varepsilon(e_1 + \varepsilon(x - e_1)) \]
\[ = C_3 \left( \Phi_1(x) + 2\gamma_\varepsilon \Phi_2 \left( \frac{x + e_1}{2} \right) \right) \text{ for all } x \in B_{r_0/\varepsilon}^+, \varepsilon \in (0, \varepsilon_0). \]
Let us fix $R_1, R_2$ such that $1 < R_1 < r_1 < r_2 < R_2 < \frac{e_0}{\varepsilon_0}$. From (69) it follows that
\[
|\tilde{u}_\varepsilon(x)| \leq \text{const} \quad \text{for all } x \in B_{R_2}^+ \setminus B_{R_1}, \quad \varepsilon \in (0, \varepsilon_0),
\]
for some const $> 0$ independent of $\varepsilon$ (but depending on $R_1, R_2$). Hence, from (68) and classical elliptic estimates, we deduce that
\[
|\nabla \tilde{u}_\varepsilon(x)| \leq C_{r_1, r_2} \quad \text{for all } x \in B_{r_2}^+ \setminus B_{r_1}^+, \quad \varepsilon \in (0, \varepsilon_0),
\]
thus proving the statement. \hfill \Box

A lower bound for $u_\varepsilon$ can be given in terms of the function $\tilde{\Phi}^\varepsilon$ defined in (58).

**Lemma 2.12.** There exist $C_5 > 0$ and $\varepsilon_1 \in (0, \varepsilon_0)$ such that
\[
 u_\varepsilon(x) \geq C_5 \Phi^\varepsilon(x) \quad \text{for all } \varepsilon \in (0, \varepsilon_1) \text{ and } x \in B_\varepsilon,
\]
where $B_\varepsilon$ is defined in (59) and $\Phi^\varepsilon$ in (58).

**Proof.** Let us first observe that
\[
(70) \quad -\Delta \Phi^\varepsilon = 0, \quad \text{in } D^+ \cup T_\varepsilon^-.
\]
Moreover, if $x \in \Gamma_{r_0}^+$ and $\varepsilon \in (0, \varepsilon_0)$, then Lemma 2.1 implies that
\[
(71) \quad u_\varepsilon(x) \geq \frac{1}{C_0}(x_1 - 1).
\]
Furthermore, from (54) and (58), we have that
\[
(72) \quad \Phi^\varepsilon(x) \leq \varepsilon \Phi_1 \left( \frac{e_1}{e_0} \right) \quad \text{for all } x \in D^+ \cup T_\varepsilon^-.
\]
From Lemma 2.9, there exist $C_6, C_7 > 0$ such that
\[
(73) \quad \Phi_1(x) \leq (x_1 - 1) \left( 1 + \frac{C_6}{|x - e_1|^N} \right) \leq C_7(x_1 - 1) \quad \text{for all } x \in D^+ \setminus B_2^+.
\]
Combining (72) and (73), we obtain that
\[
\Phi^\varepsilon(x) \leq C_7(x_1 - 1) \quad \text{for all } x \in D^+ \setminus B_{2r_2},
\]
which, together with (71), yields
\[
(74) \quad \Phi^\varepsilon(x) \leq C_6 C_7 u_\varepsilon(x) \quad \text{for all } x \in \Gamma_{r_0}^+ \text{ and } 0 < \varepsilon < \varepsilon_0.
\]
On the other hand, if $x = (x_1, x') \in T_\varepsilon^-$ and $x_1 = \frac{1}{2}$, then (58), (72), Lemma 2.9(ii), (54), (27), and Lemma 2.2 yield
\[
(75) \quad \Phi^\varepsilon(x) \leq C_2 \varepsilon e^{-\frac{\sqrt{2} |x'|}{2\varepsilon}} - \min_{y' \in \mathbb{R}^{N-1}} \psi^\varepsilon_1(y') \leq -\frac{1}{2} \min_{|y'| \leq 1/\sqrt{2}} \psi^\varepsilon_1(y') \leq \frac{u_\varepsilon(x)}{2C_1} \min_{|y'| \leq 1/\sqrt{2}} \psi^\varepsilon_1(y')
\]
provided $\varepsilon$ is sufficiently small. Estimates (74) and (75) imply the existence of some $C_5 > 0$ and $\varepsilon_1 > 0$ such that
\[
 u_\varepsilon(x) \geq C_5 \Phi^\varepsilon(x) \quad \text{for all } \varepsilon \in (0, \varepsilon_1) \text{ and } x \in \partial B_\varepsilon,
\]
which, together with (70) and the Maximum Principle, yields the conclusion. \hfill \Box

**Lemma 2.13.** There exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that
\[
 u_\varepsilon(x) \geq \frac{C_5}{2}(x_1 - 1) \quad \text{for all } x \in B_{r_0}^+ \setminus B_{2r_2}, \quad \varepsilon \in (0, \varepsilon_2).
\]

**Proof.** From (58), (36), and Lemma 2.9, it follows that, for all $x \in B_{r_0}^+ \setminus B_{2r_2},$
\[
(76) \quad \Phi^\varepsilon(x) = \varepsilon \Phi_1 \left( \frac{e_1 + \frac{x - e_1}{\varepsilon}}{e_0} \right) - \sqrt{2} \gamma_\varepsilon \Phi_2 \left( \frac{e_1 + \frac{x - e_1}{\varepsilon}}{\sqrt{2} \varepsilon} \right)
\]
\[
\geq (x_1 - 1) - \text{const} \gamma_\varepsilon(x_1 - 1) \geq \frac{1}{2}(x_1 - 1)
\]
provided $\varepsilon$ is sufficiently small. The conclusion follows from Lemma 2.12 and (76). \hfill \Box
3. The frequency function

In this section we introduce an Almgren type quotient associated to problem (9) and study its monotonicity properties with the aim of uniformly controlling the transversal frequencies along the connecting tube.

For every \(\epsilon > 0\), let \(\xi_\epsilon : \mathbb{R} \setminus ((-\epsilon, 0) \cup (1, 1+\epsilon)) \rightarrow \mathbb{R}\) be such that
\[
\begin{cases}
\xi_\epsilon(r) = -r, & \text{if } r \leq -\epsilon, \\
\xi_\epsilon(r) = r, & \text{if } 0 \leq r \leq 1, \\
\xi_\epsilon(r) = r - 1, & \text{if } r \geq \epsilon + 1.
\end{cases}
\]

For \(r \in \mathbb{R} \setminus ((-\epsilon, 0) \cup (1, 1+\epsilon))\), we define
\[
\Omega_\epsilon^r = \begin{cases}
D^- \setminus \overline{B_{\xi_\epsilon(r)}}, & \text{if } r \leq -\epsilon, \\
D^- \cup \{(x_1, x') \in \mathcal{C}_r : x_1 < r\}, & \text{if } 0 \leq r \leq 1, \\
D^- \cup \mathcal{C}_r \cup B^+_{\xi_\epsilon(r)}, & \text{if } r \geq \epsilon + 1,
\end{cases}
\]
\[
\Gamma_\epsilon^r = \begin{cases}
D^- \cap \partial B^-_{\xi_\epsilon(r)}, & \text{if } r \leq -\epsilon, \\
\{(x_1, x') \in \mathcal{C}_r : x_1 = r\}, & \text{if } 0 \leq r \leq 1, \\
D^+ \cap \partial B^+_{\xi_\epsilon(r)}, & \text{if } r \geq \epsilon + 1.
\end{cases}
\]

We also denote
\[
\Omega_r := D^- \setminus \overline{B_{-r}} \quad \text{for all } r < 0
\]
and notice that
\[
\Omega_\epsilon^r = \Omega_r \quad \text{for all } r \leq -\epsilon.
\]

![Figure 2. The moving domains \(\Omega_\epsilon^r\) for different values of the parameters.](image)

A key role in the definition and in the study of the frequency associated to problem (9) is played by Lemmas 3.4 and 3.6 below, which give a Poincaré type lemma on domains \(\Omega_{-t}\), \(t > 0\), for functions in \(\mathcal{H}_t\) and, respectively, a uniform coercivity type estimate for the quadratic form associated to equation (9) in domains \(\Omega_\epsilon^r\), \(r < 1\). An important ingredient for their proof is the Kelvin transform, which is described in the following remark.

**Remark 3.1.** For all \(R > 0\), \(v \in \mathcal{H}_R^-\) if and only if its Kelvin transform \(\tilde{v}(x) = |x|^{-(N-2)}v\left(\frac{x}{|x|^2}\right)\)
belongs to \(H^1(B_{1/R}^-)\) and has null trace on \(\partial B_{1/R}^- \cap \partial D^-\); furthermore
\[
\int_{B_{1/R}^-} |\nabla \tilde{v}(x)|^2 \, dx + (N-2)R \int_{\Gamma_{1/R}^-} \tilde{v}^2 \, d\sigma = \int_{\Omega_{-R}} |\nabla v(x)|^2 \, dx,
\]
\[
\int_{B_{1/R}^-} |\tilde{v}(x)|^2 \, dx = \int_{\Omega_{-R}} |v(x)|^2 \, dx, \quad \text{and} \quad R^2 \int_{\Gamma_{1/R}^-} \tilde{v}^2(\cdot) \, d\sigma = \int_{\Gamma_{R}^-} v^2(\cdot) \, d\sigma.
\]
Functions in \(\mathcal{H}_R^-\) satisfy the following Sobolev type inequality.

**Lemma 3.2.** There exists a constant \(C_S = C_S(N)\) depending only on the dimension \(N\) such that for all \(t > 0\) and \(v \in \mathcal{H}_t^-\) there holds
\[
C_S \left( \int_{\Omega_{-t}} |v(x)|^2 \, dx \right)^{2/2} \leq \int_{\Omega_{-t}} |\nabla v(x)|^2 \, dx.
\]
PROOF. By scaling it is enough to prove the inequality for \( t = 1 \), which, in view of remark 3.1, is equivalent to prove that

\[
C_S \left( \int_{B_1^-} |w(x)|^2 \, dx \right)^{2/2'} \leq \int_{B_1^-} |\nabla w(x)|^2 \, dx + (N - 2) \int_{\Gamma_1^-} w^2 \, d\sigma
\]

for all \( w \in H^1(B_1^-) \) such that \( w \equiv 0 \) on \( \partial B_1^- \cap \partial D^- \). Such inequality follows easily from classical Sobolev embeddings by trivially extending \( w \) in \( B(0,1) \) and observing that

\[
\int_{B(0,1)} |\nabla w(x)|^2 \, dx + (N - 2) \int_{\partial B(0,1)} w^2 \, d\sigma
\]

is an equivalent norm in \( H^1(B(0,1)) \). \( \square \)

The Poincaré inequality we will state in Lemma 3.4 with its best constant is a consequence of the following lemma, which is the counterpart of Lemma 2.5 for the frequency of harmonic functions in \( H^- \).

**Lemma 3.3.** Let \( R > 0 \) and \( \phi \in H^-_R \setminus \{0\} \) satisfying

\[
\begin{cases}
-\Delta \phi = 0, & \text{in } \Omega_R, \\
\phi = 0, & \text{on } \partial \Omega_R \cap \partial D^-,
\end{cases}
\]

in a weak sense, and let \( N^-_\phi : (R, +\infty) \to \mathbb{R} \) be defined as

\[
N^-_\phi (r) := r \frac{\int_{\Omega_r} |\nabla \phi(x)|^2 \, dx}{\int_{\Gamma_r^-} \phi^2(x) \, d\sigma}.
\]

Then

i) \( N^-_\phi \) is non-increasing in \( (R, +\infty) \);

ii) there exists \( K_0 \in \mathbb{N}, K_0 \geq 1 \), such that

\[
\lim_{r \to \infty} N^-_\phi (r) = N - 2 + K_0;
\]

iii) if \( N^-_\phi \equiv \gamma \) for some \( \gamma \in \mathbb{R} \) then \( \gamma = N - 2 + K_0 \) and \( \phi(x) = |x|^{-N+2-K_0} Y(x/|x|) \) for some eigenfunction \( Y \) of \(-\Delta_{\mathbb{S}^{N-1}}\) associated to the eigenvalue \( K_0(N - 2 + K_0) \), i.e. satisfying

\[-\Delta_{\mathbb{S}^{N-1}} Y = K_0(N - 2 + K_0) Y \text{ on } \mathbb{S}^{N-1};\]

iv) if \( \phi > 0 \) in \( \Omega_R \), then \( K_0 = 1 \).

**Proof.** Let \( \tilde{\phi} \in H^1(B_{1/R}^-) \) be the Kelvin transform of \( \phi \), i.e. \( \tilde{\phi}(x) = |x|^{-(N-2)} \phi(\frac{x}{|x|^2}) \). Then \( \tilde{\phi} \) satisfies

\[
\begin{cases}
-\Delta \tilde{\phi} = 0, & \text{in } B_{1/R}^-, \\
\tilde{\phi} = 0, & \text{on } \partial B_{1/R}^- \cap \partial D^-,
\end{cases}
\]

and, by Remark 3.1, the frequency function \( N^-_\tilde{\phi} \) can be rewritten as

\[
N^-_\tilde{\phi} (r) = N - 2 + \tilde{N} \left( \frac{1}{r} \right),
\]

where

\[
\tilde{N} : \left( 0, \frac{1}{R} \right) \to \mathbb{R}, \quad \tilde{N}(t) := \frac{t \int_{B^{-}_{1/R}} |\nabla \tilde{\phi}(x)|^2 \, dx}{\int_{\Gamma^{-}_{1/R}} \tilde{\phi}^2 \, d\sigma}.
\]

Let us define

\[
\tilde{\phi}_0(x) = \begin{cases} \tilde{\phi}_0(x_1, x') = \begin{cases} \tilde{\phi}_0(x_1, x') & \text{if } x_1 \leq 0, \\
-\tilde{\phi}_0(-x_1, x') & \text{if } x_1 > 0,
\end{cases} & \text{if } x_1 \leq 0, \\
\tilde{\phi}_0(x_1, x') & \text{if } x_1 > 0,
\end{cases}
\]

and observe that \( \tilde{\phi}_0 \in H^1(B(0, 1/R)) \) satisfies \( \tilde{\phi}_0(-x_1, x') = -\tilde{\phi}_0(x_1, x') \) and weakly solves

\[-\Delta \tilde{\phi}_0 = 0, \quad \text{in } B(0, 1/R).
\]

Moreover

\[
\tilde{N}(t) = \frac{t \int_{B(0,1)} |\nabla \tilde{\phi}_0(x)|^2 \, dx}{\int_{\partial B(0,1)} \tilde{\phi}_0^2 \, d\sigma}.
\]
From the classical Almgren monotonicity formula [2]

\[ \tilde{N}'(t) = \frac{2t}{\int_{\partial B(t,0)} \left| \frac{\partial \tilde{\phi}}{\partial \nu} \right|^2 d\sigma} \left( \int_{\partial B(t,0)} \tilde{\phi}_0 \frac{\partial \tilde{\phi}}{\partial \nu} d\sigma \right) - \left( \int_{\partial B(t,0)} \tilde{\phi}_0 \frac{\partial ^2 \tilde{\phi}}{\partial \nu^2} d\sigma \right)^2, \]

for all \( t \in (0, 1/R) \), where \( \nu = \nu(x) = \frac{x}{R} \), hence from Schwarz’s inequality \( \tilde{N}' \geq 0 \) and the function \( t \in (0, 1/R) \mapsto \tilde{N}(t) \) is non-decreasing, thus implying, in view of (78), that \( N_\nu \) is non-increasing in \((R, +\infty)\) and proving statement i). Furthermore from [18, Theorem 1.3] there exist \( K_0 \in \mathbb{N} \) and an eigenfunction \( Y \) of \(-\Delta_{S^{N-1}}\) associated to the eigenvalue \( K_0(N - 2 + K_0) \), i.e. satisfying \(-\Delta_{S^{N-1}}Y = K_0(N - 2 + K_0)Y \) on \( S^{N-1} \), such that

\[ \lim_{t \to 0^+} \tilde{N}(t) = \frac{N - 2}{2} + \sqrt{\left( \frac{N - 2}{2} \right)^2 + K_0(N - 2 + K_0)} = K_0 \]

and

\[ \lambda^{-K_0} \tilde{\phi}_0(\lambda \theta) \rightarrow Y(\theta) \quad \text{in} \quad C^{1,\tau}(S^{N-1}), \]

\[ \lambda^{1-K_0} \nabla \tilde{\phi}_0(\theta) \rightarrow K_0 Y(\theta) + \nabla_{S^{N-1}} Y(\theta) \quad \text{in} \quad C^{0,\tau}(S^{N-1}), \]

as \( \lambda \to 0^+ \), for every \( \tau \in (0, 1) \). Since \( \tilde{\phi}_0 \) vanishes on \( B(0, 1) \cap \{0\} \times \mathbb{R}^{N-1} \), from (82) we infer that \( Y \) vanishes on the equator \( S^{N-1} \cap \{0\} \times \mathbb{R}^{N-1} \). Therefore, \( Y \) can not be the first eigenfunction of \(-\Delta_{S^{N-1}}\) and hence \( K_0 \geq 1 \) necessarily. Statement ii) then follows from (78) and (81).

Let us now assume that \( N_\nu \equiv \gamma \) for some \( \gamma \in \mathbb{R} \), so that \( \tilde{N}(t) \equiv \gamma - N + 2 \) in \((0, 1/R)\) and hence \( \tilde{N}'(t) = 0 \) for any \( t \in (0, 1/R) \). By (80) we obtain

\[ \left( \int_{\partial B(t,0)} \frac{\partial \tilde{\phi}_0}{\partial \nu} \right) \left( \int_{\partial B(t,0)} \tilde{\phi}_0 \frac{\partial \tilde{\phi}_0}{\partial \nu} d\sigma \right) - \left( \int_{\partial B(t,0)} \tilde{\phi}_0 \frac{\partial ^2 \tilde{\phi}_0}{\partial \nu^2} d\sigma \right)^2 = 0 \quad \text{for all} \quad t \in (0, 1/R), \]

i.e. \( \tilde{\phi}_0 \) and \( \tilde{\phi}_0 \) have the same direction as vectors in \( L^2(\partial B(0, t)) \) and hence there exists a function \( \eta = \eta(t) \) such that \( \frac{\partial \tilde{\phi}_0}{\partial t}(t, \theta) = \eta(t) \tilde{\phi}_0(t, \theta) \) for \( t \in (0, 1/R) \) and \( \theta \in S^{N-1} \). After integration we obtain

\[ \tilde{\phi}_0(t, \theta) = e^{\int_0^t \eta(s)ds} \tilde{\phi}_0 \left( \frac{1}{R}, \theta \right) = \varphi(t) \psi(\theta), \quad t \in (0, 1/R), \theta \in S^{N-1}, \]

where \( \varphi(t) = e^{\int_0^t \eta(s)ds} \) and \( \psi(\theta) = \tilde{\phi}_0 \left( \frac{1}{R}, \theta \right) \). Since \( -\Delta \tilde{\phi}_0 = 0 \) in \( B(0, 1/R) \), (84) yields

\[ -\varphi''(t) - \frac{N - 1}{t} \varphi'(t) \psi(\theta) - \frac{\varphi(t)}{t^2} \Delta_{S^{N-1}} \psi(\theta) = 0, \]

Taking \( t \) fixed we deduce that \( \psi \) is an eigenfunction of the operator \(-\Delta_{S^{N-1}}\). If \( K_0(N - 2 + K_0) \) is the corresponding eigenvalue then \( \varphi(t) \) solves the equation

\[ -\varphi''(t) - \frac{N - 1}{t} \varphi(t) + \frac{K_0(N - 2 + K_0)}{t^2} \varphi(t) = 0 \]

and hence \( \varphi(t) \) is of the form

\[ \varphi(t) = c_1 t^{K_0} + c_2 t^{-(N - 2) - K_0}, \quad \text{for some} \quad c_1, c_2 \in \mathbb{R}. \]

Since the function \( |x|^{-(N - 2) - K_0} \psi \left( \frac{y}{|x|} \right) \notin H^1(B_1/R) \), then \( c_2 = 0 \) and \( \varphi(t) = c_1 t^{K_0} \). Since \( \varphi \left( \frac{1}{R} \right) = 1 \), we obtain that \( c_1 = R^{K_0} \) and then

\[ \tilde{\phi}_0(t, \theta) = R^{K_0} t^{K_0} \psi(\theta), \quad \text{for all} \quad t \in (0, 1/R) \text{ and} \theta \in S^{N-1}. \]

Therefore \( \psi(y) = R^{K_0} |y|^{-N + 2 - K_0} \psi \left( \frac{y}{|y|} \right) \) in \( \Omega_{-R} \). Substituting (85) into (79) and taking into account that \( \tilde{N}(t) \equiv \gamma - N + 2 \), we obtain that necessarily \( \gamma - N + 2 = K_0 \), i.e. \( \gamma = N - 2 + K_0 \). Claim iii) is thereby proved.

If \( \phi > 0 \) in \( \Omega_{-R} \), then \( \phi > 0 \) in \( B_{1/R}^{+} \) and Hopf’s Lemma implies that

\[ \frac{\partial \tilde{\phi}}{\partial x_1}(0, x') < 0, \quad \text{for all} \quad x' \in \mathbb{R}^{N-1} \text{ s.t. } |x'| < \frac{1}{R}. \]
(86) and (83) imply that $K_0 \leq 1$. Hence $K_0 = 1$ and statement iv) is proved. \hfill\Box

We are now ready to prove the following Poincaré type inequality.

**Lemma 3.4.** For all $t > 0$ and $v \in \mathcal{H}_t$ there holds
\[
\frac{1}{t^{N-2}} \int_{\Omega_t} |\nabla v(x)|^2 \, dx \geq \frac{N-1}{t^{N-1}} \int_{\Gamma_t^-} v^2 \, d\sigma,
\]
being $N - 1$ the optimal constant.

**Proof.** By scaling it is enough to prove the inequality for $t = 1$, i.e. the statement of the lemma is equivalent to prove that the infimum
\[
\mathcal{I} := \inf_{w \in \mathcal{H}_1 \setminus \{0\}} \frac{\int_{\Omega_1} |\nabla w(x)|^2 \, dx}{\int_{\Gamma_1^-} w^2 \, d\sigma}
\]
is equal to $N - 1$. By standard minimization arguments and compactness of the embedding $\mathcal{H}_1 \hookrightarrow L^2(\Gamma_1^-)$, it is easy to prove that the infimum $\mathcal{I}$ is strictly positive and attained by some function $w_0 \in \mathcal{H}_1 \setminus \{0\}$ satisfying
\[
\begin{cases}
-\Delta u_0 = 0, & \text{in } \Omega_1, \\
u_0 > 0, & \text{in } \Omega_1, \\
\frac{\partial \nu_0}{\partial \nu} = -I u_0, & \text{on } \Gamma_1, \\
u_0 \equiv 0, & \text{on } \partial \Omega_1 \cap \partial D^-,
\end{cases}
\]
being $\nu = \frac{\partial \mathcal{I}}{\partial u_0}$. Then Lemma 3.3 implies that
\[
\mathcal{I} = \frac{\int_{\Omega_1} |\nabla u_0(x)|^2 \, dx}{\int_{\Gamma_1^-} u_0^2 \, d\sigma} = N^-_w(1) \geq \lim_{r \to +\infty} N^-_{w_0}(r) \geq N - 1.
\]

On the other hand the quotient $(\int_{\Omega_1} |\nabla w(x)|^2 \, dx)(\int_{\Gamma_1^-} w^2 \, d\sigma)^{-1}$ evaluated in $w(x_1, x') = \frac{x_1}{|x|}$ is equal to $N - 1$, thus implying that $\mathcal{I} \leq N - 1$. \hfill\Box

**Remark 3.5.** By remark 3.1, Lemma 3.4 is equivalent to
\[
r \int_{B_1^-} |\nabla w(x)|^2 \, dx \geq \int_{\Gamma_1^-} w^2 \, d\sigma \quad \text{for all } w \in H^1(B_1^-) \text{ such that } w \equiv 0 \text{ on } \partial B_1^- \cap \partial D^-.
\]

Lemma 3.6 below provides a uniform coercivity type estimate for the quadratic form associated to equation (9), whose validity is strongly related to the nondegeneracy condition (5).

**Lemma 3.6.**

i) For every $f \in L^{N/2}(\mathbb{R}^N)$ and $M > 0$, there exist $\hat{r}_{M, f} > 0$ and $\bar{\varepsilon}_{M, f} \in (0, \varepsilon_0)$ such that for all $\varepsilon \in (0, \bar{\varepsilon}_{M, f})$ and $r \in (\varepsilon, \hat{r}_{M, f})$
\[
\int_{\Omega_{-r}} |\nabla u_\varepsilon(x)|^2 \, dx \geq M \int_{\Omega_{-r}} |f(x)| u_\varepsilon^2(x) \, dx.
\]

ii) For every $f \in L^{N/2}(\mathbb{R}^N)$ and $M > 0$, there exists $\bar{\varepsilon}_{M, f} \in (0, \varepsilon_0)$ such that for all $r \in (0, 1)$ and $\varepsilon \in (0, \bar{\varepsilon}_{M, f})$
\[
\int_{\Omega_r^\varepsilon} |\nabla u_\varepsilon(x)|^2 \, dx \geq M \int_{\Omega_r^\varepsilon} |f(x)| u_\varepsilon^2(x) \, dx.
\]

**Proof.** To prove i), we argue by contradiction and assume that there exist $f \in L^{N/2}(\mathbb{R}^N)$, $M > 0$, and sequences $\varepsilon_n \to 0^+$, $r_n \to 0^+$, such that $r_n > \varepsilon_n$ and denoting $u_n = u_{\varepsilon_n}$,
\[
\int_{\Omega_{-r_n}} |\nabla u_n(x)|^2 \, dx < M \int_{\Omega_{-r_n}} |f(x)| u_n^2(x) \, dx.
\]

Let us define
\[
v_n(x) = \begin{cases} u_n(x), & \text{if } x \in \Omega_{-r_n}, \\
\left(\frac{r_n}{|x|}\right)^{N-2} u_n\left(\frac{r_n^2}{|x|^2}\right), & \text{if } x \in B_{n_r}^-.
\end{cases}
\]
We notice that \( v_n \in \mathcal{D}^{1,2}(D^-) \) and, by Remark 3.1,

\[
\int_{B_r^n} |\nabla v_n(x)|^2 dx + \frac{N-2}{r_n} \int_{\Gamma_{r_n}} v_n^2 d\sigma = \int_{\Omega_{-r_n}} |\nabla u_n(x)|^2 dx,
\]

thus implying

\[
\int_{D^-} |\nabla v_n(x)|^2 dx \leq \int_{D^-} |\nabla v_n(x)|^2 dx + \frac{N-2}{r_n} \int_{\Gamma_{r_n}} v_n^2 d\sigma = 2 \int_{\Omega_{-r_n}} |\nabla u_n(x)|^2 dx.
\]

From (87) and (88) it follows that, if

\[
\{H_n\} \subset D^- \quad \text{and} \quad \{w_n\} \subset D^-.
\]

then

\[
\int_{D^-} |\nabla w_n(x)|^2 dx < 2M.
\]

Hence there exists a subsequence \( \{w_{n_k}\}_k \) such that

\[
w_{n_k} \rightharpoonup w \quad \text{weakly in} \quad \mathcal{D}^{1,2}(D^-)
\]

for some \( w \in \mathcal{D}^{1,2}(D^-). \) From \( \int_{D^-} |f(x)|v_n^2(x)dx \geq \int_{\Omega_{-r_n}} |f(x)|u_n^2(x)dx \) we deduce that

\[
\int_{D^-} |f(x)|w_n^2(x)dx \geq 1
\]

which implies that \( w \neq 0. \) Since \( w_n \) solves

\[
\begin{cases}
-\Delta w_n = \lambda^+_n p w_n, & \text{in } \Omega_{-r_n}, \\
w_n = 0, & \text{on } \partial D^-,
\end{cases}
\]

and \( r_n \to 0^+ \), from (7) we conclude that \( w \) weakly solves

\[
\begin{cases}
-\Delta w = \lambda_{-k}(D^+) p w, & \text{in } D^-,
\end{cases}
\]

\[
\begin{cases}
w = 0, & \text{on } \partial D^-,
\end{cases}
\]

thus implying \( \lambda_{-k}(D^+) \in \sigma_{\text{int}}(D^-) \) and contradicting assumption (5).

Let us now prove ii). We argue by contradiction and assume that there exist \( f \in L^{N/2}(\mathbb{R}^N) \), \( M > 0 \), and sequences \( \varepsilon_n \to 0^+ \), \( r_n \in (0,1) \), such that denoting \( u_n = u_{\varepsilon_n} \),

\[
(89) \quad \int_{\Omega^+_n} |\nabla u_n(x)|^2 dx < M \int_{\Omega^+_n} |f(x)|u_n^2(x)dx.
\]

Let us define

\[
v_n(x) = \begin{cases}
u_n(x), & \text{if } x \in \Omega^+_n, \\
u_n(2r_ne_1 - x), & \text{if } 2r_ne_1 - x \in \Omega^+_n, \\
0, & \text{otherwise}.
\end{cases}
\]

We notice that \( v_n \in \mathcal{D}^{1,2}(\mathbb{R}^N) \) and, by (89),

\[
\int_{\mathbb{R}^N} |\nabla v_n(x)|^2 dx = 2 \int_{\Omega^+_n} |\nabla u_n(x)|^2 dx < 2M \int_{\Omega^+_n} |f(x)|u_n^2(x)dx,
\]

thus implying that, letting

\[
w_n = \frac{v_n}{(\int_{\Omega^+_n} |f(x)|u_n^2(x)dx)^{1/2}},
\]

then \( w_n \in \mathcal{D}^{1,2}(\mathbb{R}^N) \) and

\[
\int_{\mathbb{R}^N} |\nabla w_n(x)|^2 dx < 2M.
\]
Hence there exist a subsequence \( \{w_{n_k}\}_k \) and some \( w \in D^{1,2}(\mathbb{R}^N) \) such that \( w_{n_k} \rightharpoonup w \) weakly in \( D^{1,2}(\mathbb{R}^N) \) and a.e. in \( \mathbb{R}^N \). From
\[
1 = \int_{\Omega^+_n} |f(x)|w_{n_k}^2(x)dx = \int_{D^-} |f(x)|w_{n_k}^2(x)dx + \int_{\Omega^+_n \setminus D^-} |f(x)|w_{n_k}^2(x)dx,
\]
\[
\int_{\Omega^+_n \setminus D^-} |f(x)|w_{n_k}^2(x)dx \leq \|w_{n_k}\|_{L^p(\mathbb{R}^N)}\|f\|_{L^{N/2}(\Omega^+_n \setminus D^-)} = o(1) \quad \text{as} \quad n \to +\infty,
\]
\[
\int_{D^-} |f(x)|w_{n_k}^2(x)dx = \int_{D^-} |f(x)|w^2(x)dx + o(1) \quad \text{as} \quad k \to +\infty,
\]
we deduce that
\[
\int_{D^-} |f(x)|w^2(x)dx = 1
\]
and hence \( w \neq 0 \) in \( D^- \). On the other hand, a.e. convergence of \( w_{n_k} \) to \( w \) implies that \( w = 0 \) on \( \partial D^- \). Furthermore, passing to the weak limit in the equation \( -\Delta w_{n_k} = \lambda_k w_{n_k}^2 \) satisfied by \( w_{n_k} \) in \( D^- \), we conclude that \( w \) weakly solves
\[
\begin{cases}
-\Delta w = \lambda_{k_0}(D^+)pw, & \text{in} \ D^-,

w = 0, & \text{on} \ \partial D^-,
\end{cases}
\]
thus implying \( \lambda_{k_0}(D^+) \in \sigma_p(D^-) \) and contradicting assumption (5). \( \square \)

From Lemma 3.6 and (3), there exist \( \hat{R} \in (0, 1) \) and \( \hat{\epsilon} \in (0, \epsilon_0) \) such that, for every \( \epsilon \in (0, \hat{\epsilon}) \),
\[
\int_{\Omega^+_{\hat{R}}} \left| \nabla u_{\epsilon} \right|^2 - \lambda_{\hat{R}}^2 \| u_{\epsilon} \|^2 \right| dx \geq \frac{1}{2} \int_{\Omega^+_{\hat{R}}} |\nabla u_{\epsilon}|^2 dx \quad \text{for all} \ r \in (-\hat{R}, -\epsilon) \cup (0, 1),
\]
and
\[
\int_{\Omega^+_{1/2}} \left| \nabla u_{\epsilon} \right|^2 - \lambda_{\hat{R}}^2 \| u_{\epsilon} \|^2 \right| dx \geq \frac{1}{2} \int_{\Omega^+_{1/2}} |\nabla u_{\epsilon}|^2 dx \quad \text{for all} \ r \in (1 + \epsilon, 4).
\]
Estimates (90) and (91), together with equation (9) and classical unique continuation principle, imply that
\[
\int_{\Gamma^+_{\epsilon}} u_{\epsilon}^2(x) dx > 0 \quad \text{for all} \ \epsilon \in (0, \hat{\epsilon}) \ \text{and} \ r \in (-\hat{R}, -\epsilon) \cup (0, 1) \cup (1 + \epsilon, 4).
\]
Therefore, for all \( \epsilon \in (0, \hat{\epsilon}) \), the frequency function \( N_{\epsilon} : (-\hat{R}, -\epsilon) \cup (0, 1) \cup (1 + \epsilon, 4) \to \mathbb{R} \),
\[
N_{\epsilon}(r) = \frac{\Lambda_N(r, \epsilon) \int_{\Omega^+_{\hat{R}}} \left| \nabla u_{\epsilon}(x) \right|^2 - \lambda_{\hat{R}}^2 \| u_{\epsilon}(x) \|^2 \right| dx}{\int_{\Gamma^+_{\epsilon}} u_{\epsilon}^2(x) dx},
\]
where
\[
\Lambda_N(r, \epsilon) = \begin{cases}
\left( \frac{2}{2N-1} \right)^{\frac{1}{2N-1}} |\Gamma^+_{\epsilon}| \gamma_{\epsilon} = \xi_{\epsilon}(r), & \text{if} \ r \in (-\infty, -\epsilon) \cup (1 + \epsilon, +\infty),

\left( \frac{N-1}{2(N-2)} \right)^{\frac{1}{2N-2}} |\Gamma^+_{\epsilon}| \gamma_{\epsilon} = \epsilon, & \text{if} \ r \in [0, 1],
\end{cases}
\]
and \( |\Gamma^+_{\epsilon}| \) denotes the \((N-1)\)-dimensional volume of \( \Gamma^+_{\epsilon} \), is well defined.

3.1. The frequency function at the right. If \( \epsilon \in (0, \hat{\epsilon}) \) and \( r \in (1 + \epsilon, 4) \), then
\[
N_{\epsilon}(r) = N_{\epsilon}^+(r - 1)
\]
where, for \( t \in (\epsilon, 3) \),
\[
N_{\epsilon}^+(t) = \frac{D^+_t(t)}{H^+_t(t)},
\]
\[
D^+_t(t) = \frac{1}{t^{N-2}} \int_{D^-\cup C_t \cup B^+_t} \left| \nabla u_{\epsilon}(x) \right|^2 - \lambda_{\hat{R}}^2 \| u_{\epsilon}(x) \|^2 \right| dx,
\]
\[
H^+_t(t) = \frac{1}{t^{N-1}} \int_{\Gamma^+_t} u_{\epsilon}^2(x) dx,
\]
with \( \Gamma^+_t \) as defined in (26). The behavior of \( N_{\epsilon}^+ \) for small \( t \) and \( \epsilon \) is described by the following proposition.
Proposition 3.7. There holds

\[
\lim_{t \to 0^+} \left( \lim_{\varepsilon \to 0^+} N^+_\varepsilon(t) \right) = \lim_{r \to 1^+} \left( \lim_{\varepsilon \to 0^+} N^\varepsilon(r) \right) = 1.
\]

**Proof.** Let us first notice that the strong $D^{1,2}(\mathbb{R}^N)$ convergence of $u_\varepsilon$ to $\varphi^+_{k_0}$ ensured by Lemma 1.1 implies that, for all $t \in (0, 3)$,

\[
\lim_{\varepsilon \to 0^+} N^+_\varepsilon(t) = N^+(t)
\]

where

\[
N^+(t) = \frac{t \int_{B^+_t} \left( |\nabla \varphi^+_{k_0}(x)|^2 - \lambda_{k_0}(D^+) p(x) (\varphi^+_{k_0}(x))^2 \right) dx}{\int_{B^+_t} (\varphi^+_{k_0}(x))^2 d\sigma}.
\]

Let us define

\[
\varphi \varepsilon(x) = \varphi \varepsilon(x_1, x') = \left\{ \begin{array}{ll}
\varphi^+_{k_0}(x_1 + 1, x'), & \text{if } x_1 \geq 0,

\varphi^+_{k_0}(-x_1 + 1, x'), & \text{if } x_1 < 0,
\end{array} \right.
\]

and observe that $\varphi \varepsilon \in D^{1,2}(\mathbb{R}^N)$ satisfies $\varphi \varepsilon(-x_1, x') = -\varphi \varepsilon(x_1, x')$ and weakly solves

\[-\Delta \varphi \varepsilon(x) = \lambda_{k_0}(D^+) p_0(x) \varphi \varepsilon(x),\]

where

\[
p_0(x) = p_0(x_1, x') = \left\{ \begin{array}{ll}
p(x_1 + 1, x'), & \text{if } x_1 \geq 0,

p(-x_1 + 1, x'), & \text{if } x_1 < 0.
\end{array} \right.
\]

Moreover $N^+$ can be rewritten as

\[
N^+(t) = \frac{t \int_{B^+(0, t)} \left| \nabla \varphi \varepsilon(x_1) \right|^2 - \lambda_{k_0}(D^+) p_0(x_1) (\varphi \varepsilon_0^2(x)) dx}{\int_{\partial B^+(0, t)} \varphi \varepsilon_0^2(x) d\sigma}.
\]

Hence, from [18, Theorem 1.3] it follows that there exist $j_0 \in \mathbb{N}$ and an eigenfunction $Y$ of $-\Delta_{\mathbb{S}^{N-1}}$ associated to the eigenvalue $j_0(N - 2 + j_0)$, i.e. satisfying $-\Delta_{\mathbb{S}^{N-1}} Y = j_0(N - 2 + j_0) Y$ on $\mathbb{S}^{N-1}$, such that

\[
\lim_{t \to 0^+} N^+(t) = -\frac{N - 2}{2} + \sqrt{\left( \frac{N - 2}{2} \right)^2 + j_0(N - 2 + j_0) = j_0}
\]

and

\[
\lambda^{-j_0} \varphi \varepsilon(\lambda \theta) \to Y(\theta) \quad \text{in } C^{1, \gamma}(\mathbb{S}^{N-1}),
\]

\[
\lambda^{1-j_0} \nabla \varphi \varepsilon(\lambda \theta) \to j_0 Y'(\theta) + \nabla_{\mathbb{S}^{N-1}} Y(\theta) \quad \text{in } C^{0, \gamma}(\mathbb{S}^{N-1}),
\]

as $\lambda \to 0^+$, for every $\tau \in (0, 1)$. Since the nodal set of $\varphi \varepsilon$ is $\{0\} \times \mathbb{R}^{N-1}$, we infer that $Y$ vanishes on the equator $\mathbb{S}^{N-1} \cap \{ \{0\} \times \mathbb{R}^{N-1} \}$. Therefore, $Y$ can not be the first eigenfunction of $-\Delta_{\mathbb{S}^{N-1}}$ and hence $j_0 \geq 1$ necessarily. On the other hand, (6) and (97) imply that $j_0 \leq 1$. Hence $j_0 = 1$. The conclusion hence follows from (94) and (95). \hfill \Box

**Lemma 3.8.** For all $\varepsilon \in (0, \varepsilon)$ and $t \in (2\varepsilon, 3)$ there holds

\[
t \int_{\Gamma^+_t} |\nabla u_\varepsilon|^2 d\sigma = 2 \varepsilon \int_{\Gamma^+_t} \left( |\nabla u_\varepsilon|^2 - 2 \frac{\partial u_\varepsilon}{\partial \nu} \right)^2 + (N - 2) \int_{B^+_t \setminus B^{2\varepsilon}_t} |\nabla u_\varepsilon(x)|^2 dx + 2t \int_{\Gamma^+_t} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 d\sigma.
\]

**Proof.** The stated identity follows from multiplication of equation (9) by $(x - e_1) \cdot \nabla u_\varepsilon$ and integration by parts over $B^+_t \setminus B^{2\varepsilon}_t$. \hfill \Box

**Lemma 3.9.** For all $\varepsilon \in (0, \varepsilon), N^\varepsilon_+ \in C^1(2\varepsilon, 3)$ and

\[
(N^\varepsilon_+)(t) = \frac{2t \left[ \left( \int_{\Gamma^+_t} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 d\sigma \right) \left( \int_{\Gamma^+_t} u_\varepsilon^2 d\sigma \right) - \left( \int_{\Gamma^+_t} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} d\sigma \right)^2 \right]}{\left( \int_{\Gamma^+_t} u_\varepsilon^2 d\sigma \right)^2} + R^+_\varepsilon,
\]

for all $t \in (2\varepsilon, 3)$, where

\[
R^+_\varepsilon = \int_{2\varepsilon}^t \left( - (N - 2) u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} + 2\varepsilon |\nabla u_\varepsilon|^2 - 4\varepsilon \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 \right) d\sigma.
\]
Lemma 3.10. which yields the conclusion in view of (100) and (101).

Corollary 3.12. Proof.

From Lemma 3.8 and (99) we deduce

\[(100) \quad (D^+)(t) = \frac{d}{dt} \left( \frac{1}{N-2} \int_{D^+} \left( |\nabla u_\varepsilon|^2 - \lambda_k^2 u_\varepsilon^2(x) \right) dx + \frac{1}{t^{N-2}} \int_{\partial D^+} |\nabla u_\varepsilon|^2 dx \right)\]

for all \( t \in (2\varepsilon, 3) \). Furthermore

\[(101) \quad (H^+)(t) = -\frac{2}{t^{N-1}} \int_{\Gamma^+} \frac{\partial u_\varepsilon}{\partial \nu} u_\varepsilon \, d\sigma \]

which, in view of (99), implies

\[(102) \quad (H^+)(t) = \frac{2}{t} D^+ (t) \quad \text{for all } t \in (\varepsilon, 3).\]

From (93) and (102) it follows that

\[(103) \quad (\mathcal{N}^+)(t) = \frac{(\mathcal{D}^+)(t) H^+ (t) - \frac{t}{2} (H^+)^2 (t))}{(H^+)^2 (t)} \]

which yields the conclusion in view of (100) and (101). \( \square \)

Lemma 3.10. For \( \varepsilon \in (0, \bar{\varepsilon}) \), let \( R_\varepsilon^+ \) as in (98). There exists \( C_\varepsilon > 0 \) such that, for all \( \varepsilon \in (0, \bar{\varepsilon}) \),

\[ |R_\varepsilon^+| \leq C_\varepsilon \varepsilon N. \]

Proof. From (98), Lemmas 2.10 and 2.11, and (57), it follows that, for all \( \varepsilon \in (0, \bar{\varepsilon}) \),

\[ |R_\varepsilon^+| \leq \text{const} \int_{\Gamma^+_{2\varepsilon}} (\varepsilon + \Phi^\varepsilon) \, d\sigma = \text{const} \left( 2^{N-2} \varepsilon N \omega_{N-1} + \varepsilon^N \int_{\Gamma^+_{2\varepsilon}} \Phi_1 d\sigma + 2^N \gamma_\varepsilon \varepsilon^N \int_{\Gamma^+_{2\varepsilon}} \Phi_2 d\sigma \right) \]

thus implying the conclusion. \( \square \)

As a consequence of the above estimates, we finally obtain the following uniform control of the frequency close to the right junction of the tube.

Lemma 3.11. There exists \( C_\varepsilon > 0 \) such that, for all \( \varepsilon \in (0, \min\{\varepsilon_2, \bar{\varepsilon}\}) \) and \( t \in (2\varepsilon, \varepsilon_0) \),

\[(103) \quad (\mathcal{N}^+)(t) \geq -C_\varepsilon \frac{\varepsilon N}{t^{N+1}} \]

Proof. From Lemma 2.13, we deduce that, for all \( t \in (2\varepsilon, \varepsilon_0) \) and \( \varepsilon \in (0, \min\{\varepsilon_2, \bar{\varepsilon}\}) \),

\[(104) \quad \int_{\Gamma^+_{2\varepsilon}} u_\varepsilon^2 d\sigma \geq \frac{C_\varepsilon^2}{4} \int_{\Gamma^+_{2\varepsilon}} (x_1 - 1)^2 d\sigma = \frac{C_\varepsilon^2}{8} t^{N+1} \int_{S^{N-1}} |\theta \cdot e_1|^2 d\sigma(\theta). \]

The conclusion follows from Lemma 3.9, Schwarz’s inequality, Lemma 3.10, and (104). \( \square \)

Corollary 3.12. For all \( \varepsilon \in (0, \min\{\varepsilon_2, \bar{\varepsilon}\}) \) and \( r_1, r_2 \) such that \( 1 + 2\varepsilon < r_1 < r_2 < 1 + r_0 \) there holds

\[ \mathcal{N}_\varepsilon (r_1) \leq \mathcal{N}_\varepsilon (r_2) + C_\varepsilon \left( \frac{\varepsilon}{r_1 - 1} \right)^N \leq \mathcal{N}_\varepsilon (r_2) + C_\varepsilon \frac{\varepsilon N}{N^2} \]

in view of (100).
3.2. The frequency function at the left. If \( \varepsilon \in (0, \bar{\varepsilon}) \) and \( r \in (-\bar{R}, -\varepsilon) \), then

\[
\mathcal{N}_\varepsilon(r) = \mathcal{N}_\varepsilon^-(r)
\]

where, for \( t \in (\varepsilon, \bar{R}) \),

\[
\begin{align*}
\mathcal{N}_\varepsilon^-(t) &= \frac{D_\varepsilon^-(t)}{H_\varepsilon^-(t)}, \\
D_\varepsilon^-(t) &= \frac{1}{t^{N-2}} \int_{\Omega_{-t}} (|\nabla u_\varepsilon|^2 - \lambda_\varepsilon^2 p(x) u_\varepsilon^2(x)) dx, \\
H_\varepsilon^-(t) &= \frac{1}{t^{N-1}} \int_{\Gamma_{-t}} u_\varepsilon^2(x) d\sigma,
\end{align*}
\]

with \( \Gamma_{-t}^\varepsilon \) defined in (10).

Lemma 3.14. For \( t > \varepsilon \) there holds

\[
t \int_{\Gamma_{-t}^\varepsilon} (|\nabla u_\varepsilon|^2 - \lambda_\varepsilon^2 p(x) u_\varepsilon^2(x)) d\sigma = 2t \int_{\Gamma_{-t}^\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 d\sigma - (N - 2) \int_{\Omega_{-t}} |\nabla u_\varepsilon(x)|^2 dx + \lambda_\varepsilon^2 \int_{\Omega_{-t}} (N p(x) + x \cdot \nabla p(x)) u_\varepsilon^2(x) dx,
\]

where \( \nu = \nu(x) = \frac{x}{|x|} \).

Proof. The stated identity follows from multiplication of equation (9) by \( x \cdot \nabla u_\varepsilon \) and integration by parts over \( \Omega_{-t} \). \( \square \)

Lemma 3.15. For \( \varepsilon \in (0, \bar{\varepsilon}) \) and \( t \in (\varepsilon, \bar{R}) \) there holds

\[
\begin{align*}
\frac{d}{dt} D_\varepsilon^-(t) &= - \frac{2}{t^{N-2}} \int_{\Gamma_{-t}^\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 d\sigma - \frac{\lambda_\varepsilon^2}{t^{N-1}} \int_{\Omega_{-t}} (2p(x) + x \cdot \nabla p(x)) u_\varepsilon^2(x) dx, \\
\frac{d}{dt} H_\varepsilon^-(t) &= \frac{2}{t^{N-1}} \int_{\Gamma_{-t}^\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} d\sigma = - \frac{2}{t} D_\varepsilon^-(t), \\
\frac{d}{dt} \mathcal{N}_\varepsilon^-(t) &= - 2t \left( \int_{\Gamma_{-t}^\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 d\sigma \right) \left( \int_{\Gamma_{-t}^\varepsilon} u_\varepsilon^2(x) d\sigma \right) - \left( \int_{\Gamma_{-t}^\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} d\sigma \right)^2 \\
&\quad \left( \int_{\Gamma_{-t}^\varepsilon} u_\varepsilon^2(x) d\sigma \right)^2 - \lambda_\varepsilon^2 \int_{\Omega_{-t}} (2p(x) + x \cdot \nabla p(x)) u_\varepsilon^2(x) dx \int_{\Gamma_{-t}^\varepsilon} u_\varepsilon^2(x) d\sigma.
\end{align*}
\]
PROOF. Since
\[
\frac{d}{dt} D^{-}_\varepsilon(t) = -\frac{N-2}{t^{N-1}} \int_{\Omega_{-t}} \left( |\nabla u_\varepsilon(x)|^2 - \lambda_\varepsilon^2 p(x) u_\varepsilon^2(x) \right) dx - \frac{1}{t^{N-2}} \int_{\Gamma_{-t}} \left( |\nabla u_\varepsilon|^2 - \lambda_\varepsilon^2 p(x) \right) d\sigma,
\]
(110) follows from Lemma 3.14. From direct calculation, we obtain that
\[
\frac{d}{dt} H^{-}_\varepsilon(t) = \frac{2}{t^{N-1}} \int_{\Gamma_{-t}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} d\sigma,
\]
while testing equation (9) with $u_\varepsilon$ and integration over $\Omega_{-t}$ yield
\[
\int_{\Omega_{-t}} \left( |\nabla u_\varepsilon|^2 - \lambda_\varepsilon^2 p(x) u_\varepsilon^2(x) \right) dx = -\int_{\Gamma_{-t}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} d\sigma,
\]
thus implying (111). Finally, (112) follows from (110), (111), and $(N^{-}_\varepsilon)' = (D^{-}_\varepsilon H^{-}_\varepsilon - D^{-}_\varepsilon^2 (H^{-}_\varepsilon)^2)$. \qed

The following estimates strongly rely on Lemmas 3.4 and 3.6.

**Lemma 3.16.** For every $\delta \in (0,1)$ there exist $\bar{\varepsilon} \in (0,\bar{\varepsilon})$ and $\bar{\varepsilon} \in (0,\varepsilon)$ such that, for every $\varepsilon \in (0,\bar{\varepsilon})$,

\begin{align}
\frac{d}{dt} H^{-}_\varepsilon(t) &\leq -\frac{2(1-\delta)(N-1)}{t} \quad \text{for all } t \in (\varepsilon, \bar{\varepsilon}), \\
\frac{d}{dt} D^{-}_\varepsilon(t) &\leq -\frac{2(1-\delta)(N-1)}{t} \quad \text{for all } t \in (\varepsilon, \bar{\varepsilon}), \\
H^{-}_\varepsilon(t_1) &\geq \left( \frac{t_2}{t_1} \right)^{2(1-\delta)(N-1)} H^{-}_\varepsilon(t_2) \quad \text{for all } t_1, t_2 \in (\varepsilon, \bar{\varepsilon}) \text{ such that } t_1 < t_2, \\
D^{-}_\varepsilon(t_1) &\geq \left( \frac{t_2}{t_1} \right)^{2(1-\delta)(N-1)} D^{-}_\varepsilon(t_2) \quad \text{for all } t_1, t_2 \in (\varepsilon, \bar{\varepsilon}) \text{ such that } t_1 < t_2.
\end{align}

**Proof.** From Lemmas 3.15, 3.6, and 3.4, we deduce that, for every $\delta \in (0,1)$, there exist $\bar{\varepsilon} > 0$ and $\bar{\varepsilon} > 0$ such that, for every $\varepsilon \in (0,\bar{\varepsilon})$ and $t \in (\varepsilon, \bar{\varepsilon})$, there holds
\[
\frac{d}{dt} H^{-}_\varepsilon(t) = -\frac{2}{t^{N-1}} \int_{\Omega_{-t}} \left( |\nabla u_\varepsilon(x)|^2 - \lambda_\varepsilon^2 p(x) u_\varepsilon^2(x) \right) dx \\
\leq -\frac{2(1-\delta)}{t^{N-1}} \int_{\Omega_{-t}} |\nabla u_\varepsilon(x)|^2 dx \leq -\frac{2(1-\delta)(N-1)}{t} H^{-}_\varepsilon(t)
\]
which yields (113). From (111), we have that
\[
\int_{\Omega_{-t}} \left( |\nabla u_\varepsilon(x)|^2 - \lambda_\varepsilon^2 p(x) u_\varepsilon^2(x) \right) dx = -\int_{\Gamma_{-t}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} d\sigma
\]
which, by Schwarz’s inequality, Lemmas 3.6 and 3.4, up to shrinking $\bar{\varepsilon} > 0$ and $\bar{\varepsilon} > 0$, for every $\varepsilon \in (0,\bar{\varepsilon})$ and $t \in (\varepsilon, \bar{\varepsilon})$ yields
\begin{align}
\int_{\Gamma_{-t}} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 d\sigma \\ \geq \frac{1}{\varepsilon} \int_{\Omega_{-t}} \left( |\nabla u_\varepsilon(x)|^2 - \lambda_\varepsilon^2 p(x) u_\varepsilon^2(x) \right) dx \\
\geq \frac{1 - \delta}{2} \frac{1}{t^{N-1}} \int_{\Omega_{-t}} |\nabla u_\varepsilon(x)|^2 dx \\
\geq \frac{(1 - \delta)(N-1)}{t} \int_{\Omega_{-t}} \left( |\nabla u_\varepsilon(x)|^2 - \lambda_\varepsilon^2 p(x) u_\varepsilon^2(x) \right) dx.
\end{align}
Proof. From Lemma 3.16, letting \( \delta_0 = \frac{2N-5}{4(N-1)} \in (0,1) \), there holds

\[
D^-_\varepsilon (t_1) \geq \left( \frac{t_2}{t_1} \right)^{\frac{N+1}{2}} D^-_\varepsilon (t_2)
\]

for every \( \varepsilon \in (0, \delta_0) \) and \( t_1, t_2 \in (\varepsilon, \bar{\delta}_0) \) such that \( t_1 < t_2 \).

Let us fix \( \delta > 0 \). From (2), Lemma 3.6, (90), and (7), there exist \( \bar{\delta}_0 \in (\varepsilon, \bar{\delta}_0) \) and \( \bar{\delta}_0 \in (0, \min\{\bar{\delta}_0, \bar{\delta}_0\}) \) such that

\[
\|2p + x \cdot \nabla p\|_{L^\infty(B_{\bar{\delta}_0}^-)} \leq \left( \frac{2N}{\omega_{N-1}} \right)^{\frac{1}{N}} C_5 \delta
\]

\[
\lambda_\varepsilon^2 \leq 2\lambda_{b_0}(D^+) \text{ for all } \varepsilon \in (0, \delta_0),
\]

\[
\int_{\Omega_{-\varepsilon}} \left( |\nabla u_{\varepsilon}|^2 - \lambda_\varepsilon^2 |u_{\varepsilon}|^2 \right) dx \geq \frac{1}{2} \int_{\Omega_{-\varepsilon}} |\nabla u_{\varepsilon}|^2 dx, \text{ for all } \varepsilon \in (0, \bar{\delta}_0), \ t \in (\varepsilon, \bar{\delta}_0)
\]

\[
\int_{\Omega_{-\varepsilon}} \left( |\nabla u_{\varepsilon}|^2 - \lambda_\varepsilon^2 |u_{\varepsilon}|^2 \right) dx \geq \frac{4\lambda_{b_0}(D^+)}{\delta} \int_{\Omega_{-\varepsilon}} 2p + x \cdot \nabla p |u_{\varepsilon}|^2 dx, \text{ for all } \varepsilon \in (0, \bar{\delta}_0), \ t \in (\varepsilon, \bar{\delta}_0)
\]

Let \( \bar{\delta}_0 = \bar{\delta}_0^{5/3} \). From (112), (121), and Schwarz’s inequality, we have that, for all \( \varepsilon \in (0, \delta_0) \) and \( t \in (\varepsilon, \bar{\delta}_0) \)

\[
\frac{d}{dt} N^-_\varepsilon (t) = \frac{\varepsilon}{N^-_\varepsilon (t)} \left( I_\varepsilon (t) + II_\varepsilon (t) \right)
\]

where

\[
I_\varepsilon (t) := \frac{2\lambda_{b_0}(D^+)}{t} \frac{\int_{\Omega_{-\varepsilon}} |2p(x) + x \cdot \nabla p(x)| u_{\varepsilon}^2(x) dx}{\int_{\Omega_{-\varepsilon}} (|\nabla u_{\varepsilon}(x)|^2 - \lambda_\varepsilon^2 p(x) u_{\varepsilon}^2(x)) dx}
\]

\[
II_\varepsilon (t) := \frac{2\lambda_{b_0}(D^+)}{t} \frac{\int_{\Omega_{-\varepsilon}} |2p(x) + x \cdot \nabla p(x)| u_{\varepsilon}^2(x) dx}{\int_{\Omega_{-\varepsilon}} (|\nabla u_{\varepsilon}(x)|^2 - \lambda_\varepsilon^2 p(x) u_{\varepsilon}^2(x)) dx}
\]
By Hölder inequality, (122), Lemma 3.2, and (120), \( \mathcal{I}_\varepsilon(t) \) can be estimated as

\[
\mathcal{I}_\varepsilon(t) \leq \frac{\|2p + x \cdot \nabla p\|_{L^\infty(B_\varepsilon/3)}}{\Omega - \Omega_{\varepsilon/3}} \left( \int_{\Omega - \Omega_{\varepsilon/3}} \left( \int_{\Omega - \Omega_{\varepsilon/3}} |u_\varepsilon(x)|^2 dx \right)^{5/2} \right)^{2/5} \leq \frac{2}{C_S} \left( \frac{\omega_{N-1}}{2N} \right)^{\frac{1}{2N}} \|2p + x \cdot \nabla p\|_{L^\infty(B_\varepsilon/3)} t \leq \frac{\delta}{4\lambda_0(D^+)} \frac{\delta}{t^{\frac{N}{f_3/2}}} \delta \quad \text{for all } t \in (\varepsilon, \tilde{R}_S) \text{ and } \varepsilon \in (0, \varepsilon_0). \quad (126)
\]

for every \( \varepsilon \in (0, \varepsilon_0) \) and \( \varepsilon \in (0, \varepsilon_0) \). On the other hand, from (123) and (119)

\[
H_\varepsilon(t) = \int_{\Omega - \Omega_{\varepsilon/3}} |2p(x) + x \cdot \nabla p(x)|u_\varepsilon(x) dx \int_{\Omega - \Omega_{\varepsilon/3}} \left( |\nabla u_\varepsilon(x)|^2 - \lambda_\varepsilon^p p(x)u_\varepsilon(x)^2 \right) dx \leq \frac{\delta}{4\lambda_0(D^+)} \frac{\delta}{t^{\frac{N}{f_3/2}}} \delta \quad \text{for all } t \in (\varepsilon, \tilde{R}_S) \text{ and } \varepsilon \in (0, \varepsilon_0). \quad (127)
\]

(125), (126), and (127) imply that

\[
\mathcal{I}_\varepsilon(t) \leq \delta \quad \text{for all } t \in (\varepsilon, \tilde{R}_S) \text{ and } \varepsilon \in (0, \varepsilon_0). \quad (128)
\]

Estimate (118) follows from (128) and (124).

**Corollary 3.18.** For every \( \delta > 0 \), let \( \tilde{R}_S \in (0, 1) \) and \( \varepsilon_0 > 0 \) as in Lemma 3.17. Then, for every \( \varepsilon \in (0, \varepsilon_0) \) and \( r_1, r_2 \) such that \( -\tilde{R}_S < r_1 < r_2 < -\varepsilon \), there holds

\[
N_\varepsilon(r_1) \leq N_\varepsilon(r_2)e^{A(r_2-r_1)}. \quad \text{for } r \in (0, 1), \quad (129)
\]

**Proof.** It follows from integration of (118). \( \square \)

### 3.3. The frequency function in the corridor.

If \( \varepsilon \in (0, \varepsilon_0) \) and \( 0 < r < 1 \), then

\[
N_\varepsilon(r) = \frac{\varepsilon D_\varepsilon^c(r)}{H_\varepsilon^c(r)}, \quad (130)
\]

where

\[
D_\varepsilon^c(r) = \int_{\Omega^-} \left( |\nabla u(x)|^2 - \lambda_\varepsilon^p p(x)u_\varepsilon^2(x) \right) dx, \quad H_\varepsilon^c(r) = \int_{\Gamma_\varepsilon^c} u_\varepsilon^2(x) d\sigma.
\]

**Lemma 3.19.** For all \( \varepsilon \in (0, \varepsilon_0) \) and \( r \in (0, 1) \),

\[
\int_{\Gamma_\varepsilon^c} \left( |\nabla u_\varepsilon|^2 - \lambda_\varepsilon^p p u_\varepsilon^2 \right) d\sigma = 2 \int_{\Gamma_x^c} \frac{\partial u_\varepsilon}{\partial x_1} dx + \int_{S_x} \frac{\partial u_\varepsilon}{\partial x_1} dx - \lambda_\varepsilon^p \int_{\Gamma_x^c} \frac{\partial p}{\partial x_1} u_\varepsilon^2 dx,
\]

where \( S_x = \partial D^- \setminus \Gamma_\varepsilon^c = \left\{ (0, x^c) \in \mathbb{R} \times \mathbb{R}^{N-1} : \frac{x}{r} \notin \Sigma \right\} \).

**Proof.** The stated identity follows from multiplication of equation (9) by \( \frac{\partial u_\varepsilon}{\partial x_1} \) and integration by parts over \( \Omega_x^c \). \( \square \)

**Lemma 3.20.** For all \( \varepsilon \in (0, \varepsilon_0) \) and \( r \in (0, 1) \) there holds

\[
\frac{d}{dr} D_\varepsilon^c(r) = 2 \int_{\Gamma_\varepsilon^c} \frac{\partial u_\varepsilon}{\partial x_1} dx + \int_{S_x} \frac{\partial u_\varepsilon}{\partial x_1} dx - \lambda_\varepsilon^p \int_{\Gamma_x^c} \frac{\partial p}{\partial x_1} u_\varepsilon^2 dx,
\]

\[
\frac{d}{dr} H_\varepsilon^c(r) = 2 \int_{\Gamma_\varepsilon^c} u_\varepsilon \frac{\partial u_\varepsilon}{\partial x_1} dx - 2 D_\varepsilon^c(r),
\]

\[
\frac{d}{dr} N_\varepsilon(r) = \varepsilon \left[ \frac{\left( \int_{\Gamma_x^c} \frac{\partial u_\varepsilon}{\partial x_1} \right)^2 d\sigma \left( \int_{\Gamma_\varepsilon^c} u_\varepsilon^2 d\sigma \right) - \left( \int_{\Gamma_x^c} u_\varepsilon \frac{\partial u_\varepsilon}{\partial x_1} d\sigma \right)^2 \left( \int_{\Gamma_\varepsilon^c} u_\varepsilon^2 d\sigma \right) \left( \int_{\Gamma_x^c} \frac{\partial u_\varepsilon}{\partial x_1} \right)^2 d\sigma \right]}{\int_{\Gamma_\varepsilon^c} u_\varepsilon^2 d\sigma} \right] - \varepsilon \lambda_\varepsilon^p \int_{\Gamma_x^c} \frac{\partial u_\varepsilon}{\partial x_1} dx.
\]


Proof. Since
\[
\frac{d}{dr} D_k^e(r) = \int_{\Gamma_k^e} \left( |\nabla u_e|^2 - \lambda_k^e p u_e^2 \right) d\sigma,
\]
(130) follows from Lemma 3.19. From direct calculation, we obtain that
\[
\frac{d}{dr} H_k^e(r) = 2 \int_{\Gamma_k^e} u_e \frac{\partial u_e}{\partial x_1} d\sigma,
\]
while testing equation (9) with \( u_e \) and integrating over \( \Omega_r^e \), we have that
\[
\int_{\Omega_r^e} \left( |\nabla u_e(x)|^2 - \lambda_k^e p u_e^2(x) \right) dx = \int_{\Gamma_r^e} u_e \frac{\partial u_e}{\partial x_1} d\sigma,
\]
thus implying (131). Finally, \( (N_e)' = \varepsilon \frac{(\Omega_r)' H_k^e - D_k^e (H_k^e)'}{(H_k^e)^2} \), (130), and (131) yield (132). \( \square \)

Lemma 3.21. For every \( \delta > 0 \) there exist \( \varepsilon_\delta^e \in (0, \varepsilon) \) such that
\[
\begin{align*}
\frac{d}{dr} N_e(r) &\geq -\delta \quad \text{for all } \varepsilon \in (0, \varepsilon_\delta^e) \text{ and } r \in (0, 1), \\
N_e(r_1) &\leq N_e(r_2) e^{\delta (r_2 - r_1)} \quad \text{for all } \varepsilon \in (0, \varepsilon_\delta^e) \text{ and } 0 < r_1 < r_2 < 1.
\end{align*}
\]
Proof. From (132) and Schwarz’s inequality we have that, for all \( \varepsilon \in (0, \varepsilon) \) and \( r \in (0, 1) \),
\[
\frac{d}{dr} N_e(r) \geq -\varepsilon \lambda_k^e \int_{\Omega_r^e} \frac{\partial}{\partial x_1} (x) u_e^2(x) dx.
\]
By part ii) of Lemma 3.6, for every \( \delta > 0 \) there exist \( \varepsilon_\delta^e \in (0, \varepsilon) \) such that, for every \( \varepsilon \in (0, \varepsilon_\delta^e) \) and \( r \in (0, 1) \),
\[
\int_{\Omega_r^e} \left( |\nabla u_e(x)|^2 - \lambda_k^e p(x) u_e^2(x) \right) dx \geq \frac{\lambda_k^e}{\delta} \int_{\Omega_r^e} \left| \frac{\partial p}{\partial x_1} (x) \right| u_e^2(x) dx.
\]
Estimate (133) follows from (135), (136), and (129). (134) follows from integration of (133). \( \square \)

4. Blow-up at the right

Throughout this section, \( \tilde{u}_e \) will denote the scaling of \( u_e \) introduced in (66–67). For every \( R > 1 \) we define \( H_R^+ \) the completion of
\[
\mathcal{D}_R^+ := \left\{ v \in C^\infty((-\infty, 1] \times \mathbb{R}^{N-1}) : \supp v \subseteq \mathbb{R}^N \setminus \{(1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : |x'| > R\} \right\}
\]
with respect to the norm \( (\int_{(-\infty, 1] \times \mathbb{R}^{N-1}) \cup B_R^+ |\nabla v|^2 dx)^{1/2} \) (which is actually equivalent to the norm \( (\int_{(-\infty, 1] \times \mathbb{R}^{N-1}) \cup B_R^+ |\nabla v|^2 dx + \int_{B_R^+} v^2 d\sigma)^{1/2} \) by Poincaré inequality), i.e. \( H_R^+ \) is the space of functions with finite energy in \((-\infty, 1] \times \mathbb{R}^{N-1}) \cup B_R^+ \) vanishing on \{(1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : |x'| \geq R\}.

Lemma 4.1. For every sequence \( \varepsilon_n \to 0^+ \) there exist a subsequence \( \{\varepsilon_n\}_k \) and \( \tilde{u} \in \bigcup_{R > 2} H_R^+ \) such that
\begin{itemize}
  \item[i)] \( \tilde{u}_{\varepsilon_n} \to \tilde{u} \) strongly in \( H_R^+ \) for every \( R > 2 \) and a.e.;
  \item[ii)] \( \tilde{u} \equiv 0 \) in \( \mathbb{R}^N \setminus \tilde{D} \);
  \item[iii)] \( \tilde{u} \) weakly solves
  \[
  \begin{cases}
  -\Delta \tilde{u}(x) = 0, & \text{in } \tilde{D}, \\
  \tilde{u} = 0, & \text{on } \partial \tilde{D},
  \end{cases}
  \]
  with \( \tilde{D} \) as in (24);
  \item[iv)] \( \tilde{u}(x) \geq \frac{c_2}{2}(x_1 - 1) \) for all \( x \in D^+ \setminus B_2^+ \).
\end{itemize}
Let $R > 2$. From Lemma 2.10 and (57), there exists $C_R > 0$ such that

\begin{equation}
(138) \quad \int_{\Gamma_R^+} |\tilde{u}_\varepsilon|^2 d\sigma = \frac{1}{\varepsilon^2} \int_{\Gamma_R^+} u_\varepsilon^2 (e_1 + \varepsilon (x - e_1)) d\sigma \leq C^2 \int_{\Gamma_R^+} \left( \Phi_1 (x) + 2\gamma_2 \Phi_2 \left( \frac{x + e_1}{2} \right) \right)^2 d\sigma \leq C_R
\end{equation}

for all $\varepsilon \in (0, r_0/R)$. By the change of variable $x = e_1 + \varepsilon (y - e_1)$ we have that

\begin{equation}
(139) \quad \mathcal{N}_\varepsilon (1 + R\varepsilon) = \frac{R \int_{\tilde{\Omega}_{R+1}^\varepsilon} \left( |\nabla \tilde{u}_\varepsilon(y)|^2 - \lambda_k \varepsilon^2 p(e_1 + \varepsilon (y - e_1)) \tilde{u}_\varepsilon^2 (y) \right) dy}{\int_{\Gamma_R^+} \tilde{u}_\varepsilon^2 (y) d\sigma}
\end{equation}

where

\begin{equation}
\tilde{\Omega}_{R+1}^\varepsilon := \{(y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1} : y_1 < 1 - \frac{1}{\varepsilon}\} \cup \{(y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1} : 1 - \frac{1}{\varepsilon} \leq y_1 \leq 1, y' \in \Sigma \} \cup B^+_R.
\end{equation}

From Corollary 3.12

\begin{equation}
(140) \quad \mathcal{N}_\varepsilon (1 + R\varepsilon) \leq \mathcal{N}_\varepsilon (1 + r_0) + \frac{C_9}{N^2 N}
\end{equation}

for all $\varepsilon \in (0, \min \{r_0/R, \varepsilon_2\})$. From the strong $D^{1,2} (\mathbb{R}^N)$ convergence of $u_\varepsilon$ to $\varphi_0^+$ ensured by Lemma 1.1, we deduce that there exists some positive constant $C_{10} > 0$ (depending on $r_0$ but independent of $\varepsilon$) such that $\mathcal{N}_\varepsilon (1 + r_0) \leq C_{10}$ for all $\varepsilon \in (0, \varepsilon_0)$, so that (138–140) yield

\begin{equation}
(141) \quad \int_{\tilde{\Omega}_{R+1}^\varepsilon} \left( |\nabla \tilde{u}_\varepsilon(y)|^2 - \lambda_k \varepsilon^2 p(e_1 + \varepsilon (y - e_1)) \tilde{u}_\varepsilon^2 (y) \right) dy \leq \left( C_{10} + \frac{C_9}{N^2 N} \right) \frac{\int_{\Gamma_R^+} \tilde{u}_\varepsilon^2 (y) d\sigma}{R} \leq \left( C_{10} + \frac{C_9}{N^2 N} \right) \frac{C_R}{R}
\end{equation}

for all $\varepsilon \in (0, \min \{r_0/R, \varepsilon_2\})$. From (141), Lemma 3.6, and assumption (3), we obtain that

\begin{equation}
(142) \quad \int_{\tilde{\Omega}_{R+1}^\varepsilon} |\nabla \tilde{u}_\varepsilon(y)|^2 dy \leq 2 \left( C_{10} + \frac{C_9}{N^2 N} \right) \frac{C_R}{R}
\end{equation}

for all $\varepsilon \in (0, \min \{r_0/R, \varepsilon_2, 2, 2\lambda_0 (D_+)\})$. In view of (138) and (142), we have proved that for every $R > 2$ there exists $\varepsilon_R > 0$ such that

\begin{equation}
(143) \quad \{\tilde{u}_\varepsilon\}_{\varepsilon \in (0, \varepsilon_R)} \text{ is bounded in } H_{R}^+.
\end{equation}

Let $\varepsilon_n \to 0^+$. From (143) and a diagonal process, we deduce that there exist a subsequence $\varepsilon_{n_k} \to 0^+$ and some $\tilde{u} \in \bigcup_{R > 2} H_{R}^+$ such that $\tilde{u}_{\varepsilon_{n_k}} \rightharpoonup \tilde{u}$ weakly in $H_{R}^+$ for every $R > 2$. In particular $\tilde{u}_{\varepsilon_{n_k}} \rightharpoonup \tilde{u}$ a.e., so that $\tilde{u} \equiv 0$ in $\mathbb{R}^N \setminus \tilde{D}$. Passing to the weak limit in (68), we obtain that $\tilde{u}$ is a weak solution to (137). By classical elliptic estimates, we also have that $\tilde{u}_{\varepsilon_{n_k}} \to \tilde{u}$ in $C^2 (B^+_{r_1} \setminus B^+_R)$ for all $1 < r_1 < r_2$. Therefore, multiplying (137) by $\tilde{u}$ and integrating over $T_{1}^- \cup B^+_R$ with $T_1$ as in (24), we obtain

\begin{equation}
(144) \quad \int_{\Gamma_R^+} \frac{\partial \tilde{u}_{\varepsilon_{n_k}}}{\partial \nu} \tilde{u}_{\varepsilon_{n_k}} d\sigma \to \int_{\Gamma_R^+} \frac{\partial \tilde{u}}{\partial \nu} \tilde{u} d\sigma = \int_{\Gamma^-_R \cup B^+_R} |\nabla \tilde{u}(x)|^2 dx \quad \text{as } k \to +\infty.
\end{equation}

On the other hand, multiplication of (68) by $\tilde{u}_{\varepsilon_{n_k}}$ and integration by parts over $\tilde{\Omega}_{R+1}^{\varepsilon_{n_k}}$ yield

\begin{equation}
(145) \quad \int_{\tilde{\Omega}_{R+1}^{\varepsilon_{n_k}}} |\nabla \tilde{u}_{\varepsilon_{n_k}}(x)|^2 dx = \int_{\Gamma_R^+} \frac{\partial \tilde{u}_{\varepsilon_{n_k}}}{\partial \nu} \tilde{u}_{\varepsilon_{n_k}} d\sigma + \lambda_k \varepsilon_{n_k}^2 \int_{\tilde{\Omega}_{R+1}^{\varepsilon_{n_k}}} p(e_1 + \varepsilon_{n_k} (x - e_1)) \tilde{u}_{\varepsilon_{n_k}}^2 (x) dx.
\end{equation}

We claim that

\begin{equation}
(146) \quad \varepsilon_{n_k}^2 \int_{\tilde{\Omega}_{R+1}^{\varepsilon_{n_k}}} p(e_1 + \varepsilon_{n_k} (x - e_1)) \tilde{u}_{\varepsilon_{n_k}}^2 (x) dx \to 0 \quad \text{as } k \to +\infty.
\end{equation}

Indeed, from Lemma 3.6, for every $\delta > 0$ there exists $k_0$ such that for all $k \geq k_0$

\begin{equation}
\int_{\tilde{\Omega}_{1/2}^-} p(y) u_{\varepsilon_{n_k}}^2 (y) dy \leq \delta \int_{\tilde{\Omega}_{1/2}^-} |\nabla u_{\varepsilon_{n_k}}(y)|^2 dy
\end{equation}
and hence, from the change of variable $y = e_1 + k e_n(x - e_1)$, assumption (3), and (142), we deduce that
\[
\varepsilon_n e^{2 n} \int_{\Omega_{R+1}} p(e_1 + k e_n(x - e_1)) \frac{\partial u_n}{\partial n}(x) \, dx = \varepsilon_n e^{N} \int_{\Omega_{R+1}} p(y) \frac{\partial u}{\partial n}(y) \, dy
\]
\[
= \varepsilon_n e^{-N} \int_{\Omega_{R+1}} p(y) \frac{\partial u}{\partial n}(y) \, dy \leq \delta \varepsilon_n e^{-N} \int_{\Omega_{R+1}} |\nabla u_n(y)|^2 \, dy
\]
\[
= \delta \int_{\Omega_{R+1}} |\nabla u_n(x)|^2 \, dx \leq 2 \delta \left( C_0 + \frac{C_0}{N^2} \right) C_R R^2,
\]
thus proving claim (146). Combining (144), (145), and (146), we conclude that $\|\tilde{u}_n\|_{H^1_0} \to \|\tilde{u}\|_{H^1_0}$ and then $\tilde{u}_n \to \tilde{u}$ strongly in $H^1_R$ for every $R > 2$.

To prove iv), it is enough to observe that Lemma 2.13 implies that, for $k$ large,
\[
\tilde{u}_n(x) \geq \frac{C_0}{2}(x_1 - 1) \quad \text{for all } x \in B_{R_0}(\Omega_1) \setminus B_2^0,
\]
which yields iv) thanks to a.e convergence of $\tilde{u}_n$ to $\tilde{u}$.

\[\square\]

**Remark 4.2.** We notice that the function $\tilde{u}$ found in Lemma 4.1 satisfies
\[
\int_{\tilde{D}} |\nabla \tilde{u}(x)|^2 \, dx = +\infty.
\]
Indeed, $\int_{\tilde{D}} |\nabla \tilde{u}(x)|^2 \, dx < +\infty$ would imply, by testing (137) with $\tilde{u}$, that $\tilde{u} \equiv 0$ in $\tilde{D}$, thus contradicting statement iv) of Lemma 4.1.

**Lemma 4.3.** Let $\tilde{u}$ be as in Lemma 4.1 and, for $r \in [0, 1)$, let $\tilde{N}_{\tilde{u}}(r)$ be the frequency function associated to $\tilde{u}$, i.e.
\[
\tilde{N}_{\tilde{u}}(r) = \frac{\Lambda_N(r) \int_{\tilde{D}_r} |\nabla \tilde{u}(x)|^2 \, dx}{\int_{\tilde{D}_1} \tilde{u}^2(x) \, dx},
\]
with $\tilde{D}_r$ and $\tilde{D}_1$ defined in (25) and $\Lambda_N(r)$ as in (50). Then
i) $\lim_{r \to +\infty} \tilde{N}_{\tilde{u}}(r) = 1$;
ii) there exists $c > 0$ such that $\int_{\tilde{D}_r} |\nabla (\tilde{u} - \tilde{c}(x_1 - 1))(x)|^2 \, dx < +\infty$.

**Proof.** We notice that $\tilde{N}_{\tilde{u}}$ is well defined in $[0, 1)$ in view of equation (137) and classical unique continuation (in particular $\tilde{u} \not\equiv 0$ by part iv) of Lemma 4.1). Let us first prove that
\[
\lim_{r \to +\infty} \tilde{N}_{\tilde{u}}(r) \leq 1.
\]
Indeed, letting $\varepsilon_n \to 0^+$ and $\{\varepsilon_n\}_k$ as in Lemma 4.1, passing to the limit as $k \to +\infty$ in (139), and using (146), we have that
\[
\lim_{k \to +\infty} \tilde{N}_{\varepsilon_n}(1 + R \varepsilon_n) = \tilde{N}_{\tilde{u}}(1 + R) \quad \text{for every } R > 0,
\]
which, together with Corollary 3.13, implies for every $\delta > 0$ the existence of some $\tilde{R}_\delta$ such that
\[
\tilde{N}_{\tilde{u}}(1 + R) \leq 1 + \delta \quad \text{for all } R > \tilde{R}_\delta,
\]
thus proving claim (147).

It is easy to prove that there exists $g \in H^1_0(D^+)$ such that
\[
\begin{cases}
-\Delta g = 0, & \text{in } D^+, \\
g = \tilde{u}, & \text{on } \partial D^+,
\end{cases}
\]
i.e. $g$ is a finite-energy harmonic extension of $\tilde{u}|_{\partial D^+}$ in $D^+$. We observe that the Kelvin transform $\tilde{g}(x) = |x - e_1|^{-N-2} g(x - e_1) = |x - e_1|^{-N} g(x - e_1) + e_1$ belongs to $H^1_0(B_r^1)$ and weakly satisfies
\[
\begin{cases}
-\Delta \tilde{g}(x) = 0, & \text{in } B_r^1, \\
\tilde{g}(x) = 0, & \text{in } \{(x_1, x') : x_1 = 1, |x'| < 1\}.
\end{cases}
\]
By classical elliptic estimates, there exists \( c_g > 0 \) such that \( \left| \frac{\partial \tilde{g}}{\partial x^i} \right| \leq c_g \) in \( B^+_{1/2} \), thus implying

\[
|g(x_1, x')| = \left| \tilde{g}(1, x') + \int_1^{x_1} \frac{\partial \tilde{g}}{\partial x_1}(s, x') \, ds \right| \leq \int_1^{x_1} \left| \frac{\partial \tilde{g}}{\partial x_1}(s, x') \right| \, ds \leq c_g(x_1 - 1)
\]

for all \((x_1, x') \in B^+_{1/2}\). Then

\[
|g(x)| \leq c_g \frac{x_1 - 1}{|x - e_1|^N}
\]

for all \( x \in D^+ \setminus B^+_{1/2} \). Let us observe that the function \( v := \tilde{u} - g \in H^1_{\text{loc}}(D^+) \setminus \{0\} \) satisfies

\[
\begin{cases}
-\Delta v(x) = 0, & \text{in } D^+,

v = 0, & \text{on } \partial D^+,

\int_{B^+} |\nabla v(x)|^2 \, dx < +\infty, & \text{for all } r > 0.
\end{cases}
\]

Let us define

\[
\mathcal{N}_v : (0, +\infty) \to \mathbb{R}, \quad \mathcal{N}_v(t) := \frac{t \int_{B^+_r} |\nabla v(x)|^2 \, dx}{\int_{\Gamma^+_t} v^2(x) \, d\sigma}.
\]

Direct calculations yield

\[
\mathcal{N}_v(t) = \frac{2t \left[ \left( \int_{\Gamma^+_t} \left| \frac{\partial v}{\partial \nu} \right|^2 \, d\sigma \right) \left( \int_{\Gamma^+_t} v^2 \, d\sigma \right) - \left( \int_{\Gamma^+_t} \frac{\partial v}{\partial \nu} \, d\sigma \right)^2 \right]}{\left( \int_{\Gamma^+_t} v^2 \, d\sigma \right)^2}, \quad \text{for all } t > 0,
\]

where \( \nu = \nu(x) = \frac{x - e_1}{|x - e_1|^N} \). In particular, Schwarz’s inequality implies that \( \mathcal{N}_v \) is non-decreasing in \((0, +\infty)\). From Remark 3.5 it follows that

\[
\mathcal{N}_v(t) \geq \lim_{r \to 0^+} \mathcal{N}_v(r) \geq 1 \quad \text{for all } t > 0.
\]

From (148) and Lemma 4.1, it follows that, if \( x \in \Gamma^+_t \) and \( t > 2 \), then

\[
\left( 1 - \frac{2c_g}{C_5 t^N} \right) \tilde{u}(x) \leq v(x) \leq \left( 1 + \frac{2c_g}{C_5 t^N} \right) \tilde{u}(x),
\]

so that

\[
\left( 1 - \frac{2c_g}{C_5 t^N} \right)^2 \int_{\Gamma^+_t} \tilde{u}^2 \, d\sigma \leq \int_{\Gamma^+_t} v^2 \, d\sigma \leq \left( 1 + \frac{2c_g}{C_5 t^N} \right)^2 \int_{\Gamma^+_t} \tilde{u}^2 \, d\sigma
\]

for all \( t > \max \{2, (2c_g/C_5)^{1/N}\} \). Let us fix \( \delta > 0 \). For every \( R > 2 \) there holds

\[
\int_{\Gamma^+_R} |\nabla v(x)|^2 \, dx - (1 + \delta) \int_{\Omega_{R+1}^+} |\nabla \tilde{u}(x)|^2 \, dx
\]

\[
\leq \int_{\Omega_{R+1}^+} |\nabla g(x)|^2 \, dx - 2 \int_{\partial \Omega_{R+1}^+} \nabla g(x) \cdot \nabla \tilde{u}(x) \, d\sigma - \delta \int_{\Omega_{R+1}^+} |\nabla \tilde{u}(x)|^2 \, dx
\]

\[
\leq \left( 1 + \frac{2}{\delta} \right) \int_{\Omega_{R+1}^+} |\nabla g(x)|^2 \, dx + \frac{\delta}{2} \int_{\Omega_{R+1}^+} |\nabla \tilde{u}(x)|^2 \, dx - \delta \int_{\Omega_{R+1}^+} |\nabla \tilde{u}(x)|^2 \, dx
\]

and hence, for all \( R > \max \{2, (2c_g/C_5)^{1/N}\} \),

\[
\mathcal{N}_v(R) \leq \frac{\left( 1 + \frac{1}{\delta} \right) \int_{\Omega_{R+1}^+} |\nabla \tilde{u}(x)|^2 \, dx}{(1 - \frac{2c_g}{C_5 R^N})^2} \frac{R \int_{\Omega_{R+1}^+} |\nabla \tilde{u}(x)|^2 \, d\sigma}{\int_{\Omega_{R+1}^+} \tilde{u}^2 \, d\sigma} \frac{\left( 1 + \frac{1}{\delta} \right) \int_{\Omega_{R+1}^+} |\nabla g(x)|^2 \, dx}{\int_{\Omega_{R+1}^+} |\nabla \tilde{u}(x)|^2 \, dx}
\]

\[
= \frac{\left( 1 + \frac{1}{\delta} \right) \int_{\Omega_{R+1}^+} |\nabla \tilde{u}(x)|^2 \, dx}{(1 - \frac{2c_g}{C_5 R^N})^2} \tilde{\mathcal{N}}_v(R + 1) \left( 1 + \frac{1}{\delta} \right) \frac{\int_{\Omega_{R+1}^+} |\nabla g(x)|^2 \, d\sigma}{\int_{\Omega_{R+1}^+} |\nabla \tilde{u}(x)|^2 \, d\sigma}.
\]
On the other hand, for every \( R > 2 \) there holds

\[
\int_{B_R^+} |\nabla v(x)|^2 \, dx - (1 - \delta) \int_{\partial B_{R+1}} |\nabla \tilde{u}(x)|^2 \, dx
\]

\[
= - \int_{T^+} |\nabla \tilde{u}(x)|^2 \, dx + \int_{B_R^+} |\nabla g(x)|^2 \, dx - 2 \int_{B_R^+} \nabla g(x) \cdot \nabla \tilde{u}(x) \, dx + \delta \int_{\partial B_{R+1}} |\nabla \tilde{u}(x)|^2 \, dx
\]

\[
\geq - \int_{T^+} |\nabla \tilde{u}(x)|^2 \, dx + \left( 1 - \frac{2}{\delta} \right) \int_{B_R^+} |\nabla g(x)|^2 \, dx
\]

and hence, for all \( R > \max \{ 2, (2c_g/C_0)^{1/N} \} \),

\[
N_{v}(R) \geq \frac{(1 - \delta) \bar{N}_{\Omega}(R + 1)}{(1 + \frac{c_g}{C_0} R^{-N})^2} \left( 1 - \frac{1}{1 - \delta} \int_{\partial B_{R+1}} |\nabla \tilde{u}(x)|^2 \, dx \right) + \frac{1 - \frac{2}{\delta}}{1 - \frac{1}{1 - \delta} \int_{\partial B_{R+1}} |\nabla \tilde{u}(x)|^2 \, dx}.
\]

Since \( \int_{B_R^+} |\nabla g(x)|^2 \, dx = O(1) \) and \( \int_{\partial B_{R+1}} |\nabla \tilde{u}(x)|^2 \, dx \to +\infty \) as \( R \to +\infty \) (see Remark 4.2), passing to \( \limsup \) and \( \liminf \) in the in (152–153) we obtain that

\[
(1 - \delta) \limsup_{R \to \infty} \frac{\bar{N}_{\Omega}(R)}{(1 + \frac{c_g}{C_0} R^{-N})^2} \leq (1 + \delta) \limsup_{R \to \infty} \bar{N}_{\Omega}(R) \quad \text{for all } \delta > 0,
\]

\[
(1 - \delta) \liminf_{R \to \infty} \bar{N}_{\Omega}(R) \leq (1 + \delta) \liminf_{R \to \infty} \bar{N}_{\Omega}(R) \quad \text{for all } \delta > 0,
\]

thus implying, in view of from (151),

\[
\liminf_{R \to \infty} \bar{N}_{\Omega}(R) = \liminf_{R \to \infty} N_v(R) \geq 1,
\]

and, in view of (147),

\[
1 \geq \limsup_{R \to \infty} \bar{N}_{\Omega}(R) = \limsup_{R \to \infty} N_v(R).
\]

From (154) and (155) we deduce that

\[
\lim_{R \to \infty} \bar{N}_{\Omega}(R) = \lim_{R \to \infty} N_v(R) = 1,
\]

thus proving statement i). Furthermore (156), (151), and the fact that \( N_v \) is non decreasing imply that

\[
N_v(t) \equiv 1 \quad \text{in } (0, +\infty).
\]

Therefore \( N_v(t) = 0 \) for any \( t > 0 \). From (150) we obtain

\[
\left( \int_{T^+} \left| \frac{\partial v}{\partial \nu} \right|^2 \, d\sigma \right) \left( \int_{T^+} v^2 \, d\sigma \right) = \left( \int_{T^+} \frac{\partial v}{\partial \nu} \, d\sigma \right)^2
\]

for all \( t > 0 \),

which implies that \( v \) and \( \frac{\partial v}{\partial \nu} \) are linearly dependent as vectors in \( L^2(T^+) \), i.e. there exists a function \( \eta = \eta(t) \) such that \( \frac{\partial v}{\partial \nu}(e_1 + t\theta) = \eta(t)v(e_1 + t\theta) \) for \( t > 0 \). After integration we have

\[
v(e_1 + t\theta) = e^{\int_1^t \eta(s) \, ds}v(e_1 + \theta) = \varphi(t)\psi(\theta) \quad t > 0, \theta \in S^{N-1}_+,
\]

where \( S^{N-1}_+ := \{ \theta = (\theta_1, \theta_2, \ldots, \theta_N) \in S^{N-1} : \theta_1 > 0 \} \). \( \varphi(t) = e^{\int_1^t \eta(s) \, ds} \) and \( \psi(\theta) = \psi(e_1 + \theta) \).

Since \( v \) satisfies (149), then

\[
\varphi''(t) + \frac{N - 1}{t} \varphi'(t) \psi(\theta) + \frac{\varphi(t)}{t^2} \Delta_{S^{N-1}} \psi(\theta) = 0.
\]

Taking \( t \) fixed, we deduce that \( \psi \) is an eigenfunction of the operator \( -\Delta_{S^{N-1}} \) on \( S^{N-1}_+ \) under null Dirichlet boundary conditions on \( \partial S^{N-1}_+ \), i.e. there exists \( K_0 \in N, K_0 > 1 \), such that

\[
\begin{cases}
-\Delta_{S^{N-1}} \psi = K_0(N - 2 + K_0)\psi, & \text{in } S^{N-1}_+,
\psi = 0, & \text{on } \partial S^{N-1}_+.
\end{cases}
\]

Then \( \varphi(t) \) solves the equation

\[
\varphi''(t) + \frac{N - 1}{t} \varphi'(t) - \frac{K_0(N - 2 + K_0)}{t^2} \varphi(t) = 0.
\]
and hence \( \varphi \) is of the form
\[
\varphi(r) = c_1 t^{K_0} + c_2 t^{-(N-2)-K_0},
\]
for some \( c_1, c_2 \in \mathbb{R} \). Since, by elliptic regularity theory, \( v \) is smooth in \( \overline{\Omega}^- \), \( c_2 \) must be 0 and
\[
\varphi'(t) = c_1 t^{K_0}.
\]
Since \( \varphi(1) = 1 \), we obtain that \( c_1 = 1 \) and then
\[
(159) \quad v(e_1 + \theta t) = t^{K_0} \psi(t), \quad \text{for all } t > 0 \text{ and } \theta \in \mathbb{S}^{N-1}.
\]
Substituting (159) into (157), we find that \( 1 \equiv N_c(t) \equiv K_0 \) and therefore \( K_0 = 1 \). Being \( N-1 \) the first eigenvalue of problem (158), \( \psi \) is simple. Hence there exists \( \hat{c} \in \mathbb{R} \) such that \( \psi(\theta) = \hat{c} \theta_1^{N} \) and \( v(x) = \hat{c} (x_1 - 1)^{+} \). Lemma 4.1 part iv) and estimate (148) imply that \( \hat{c} > 0 \), thus proving ii). \( \square \)

**Corollary 4.4.** Let \( \tilde{u} \) be as in Lemma 4.1 and \( \hat{c} \) as in Lemma 4.3. Then
\[
\tilde{u} = \hat{c} T(x_1 - 1) = \hat{c} \Phi_1
\]
where \( \Phi_1 \) is defined in (35).

**Proof.** It follows from Lemmas 4.1 and 4.3, taking into account Lemma 2.4 and the fact that \( T(c\psi) = c T(\psi) \).

**Lemma 4.5.** For every \( R > 0 \)
\[
\lim_{\varepsilon \to 0^+} \mathcal{N}^c_r(1 - R \varepsilon) = \mathcal{N}(1 - R),
\]
with \( \mathcal{N} \) as in (49).

**Proof.** Fix \( R > 0 \). Let \( \varepsilon_n \to 0^+ \). From Lemma 4.1 and Corollary 4.4, there exist a subsequence \( \{\varepsilon_{n_k}\}_k \) and \( \hat{c} > 0 \) such that \( \tilde{u}_{\varepsilon_{n_k}} \to \hat{c} \Phi_1 \) strongly in \( \mathcal{H}_r^+ \) for every \( r > 2 \). By the change of variable \( x = e_1 + \varepsilon(y - e_1) \), we have that, for \( \varepsilon < \frac{1}{R} \),
\[
N_{\varepsilon}(1 - R \varepsilon) = \frac{\int_{\tilde{\Omega}_{1-R}} (|\nabla \tilde{u}_{\varepsilon}(y)|^2 - \lambda_1 \varepsilon^2 p(e_1 + \varepsilon(y - e_1)) |\tilde{u}_{\varepsilon}^2(y)|) dy}{\int_{\tilde{\Omega}_{1-R}} \tilde{u}_{\varepsilon}^2(y) d \sigma}
\]
where \( \tilde{\Omega}_{1-R} \) is defined in (25) and
\[
\tilde{\Omega}_{1-R} := \{(y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1} : y_1 < 1 - \frac{1}{2} \} \cup \{(y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1} : 1 - \frac{1}{\varepsilon} \leq y_1 \leq 1 - R, \ y' \in \Sigma \}.
\]
From strong convergence of \( \tilde{u}_{\varepsilon_{n_k}} \) to \( \hat{c} \Phi_1 \) in \( \mathcal{H}_r^+ \) for every \( r > 2 \), passing to the limit in (160) along the subsequence \( \{\varepsilon_{n_k}\}_k \) and using (146), we obtain that
\[
\lim_{k \to +\infty} \mathcal{N}^c_{\varepsilon_{n_k}}(1 - R \varepsilon_{n_k}) = \frac{\int_{\tilde{\Omega}_{1-R}} (|\nabla \hat{c} \Phi_1(y)|^2 dy}{\int_{\tilde{\Omega}_{1-R}} (\hat{c} \Phi_1)^2(y) d \sigma} = \frac{\int_{\tilde{\Omega}_{1-R}} (|\Phi_1(y)|^2 dy}{\int_{\tilde{\Omega}_{1-R}} \Phi_1^2(y) d \sigma} = \mathcal{N}(1 - R),
\]
where \( \tilde{\Omega}_{1-R} \) is defined in (25). Since the limit depends neither on the sequence \( \{\varepsilon_n\}_{n \in \mathbb{N}} \) nor on its subsequence \( \{\varepsilon_{n_k}\}_{k \in \mathbb{N}} \), we conclude that the convergence actually holds as \( \varepsilon \to 0^+ \) thus proving the lemma. \( \square \)

**Lemma 4.6.** For every \( R > 0 \) and \( \delta > 0 \), there exists \( \varepsilon_{R, \delta} \in (0, \varepsilon) \) such that
\[
\mathcal{N}(r) < (1 + \delta') \sqrt{\lambda_1(\Sigma)} \quad \text{for all } r \in (0, R \varepsilon) \text{ and } \varepsilon \in (0, \varepsilon_{R, \delta}).
\]

**Proof.** Let \( \delta > 0 \) and choose \( \delta' > 0 \) sufficiently small such that \( (1 + \delta')^2 \varepsilon^\delta < 1 + \delta \). From Corollary 2.6, there exists \( \tilde{R}_\delta > 0 \) such that
\[
(161) \quad \mathcal{N}(1 - \tilde{R}_\delta) < (1 + \delta') \sqrt{\lambda_1(\Sigma)}.
\]
From Lemma 4.5, there exists \( \varepsilon_\delta > 0 \) such that
\[
(162) \quad \mathcal{N}(1 - R \varepsilon) < (1 + \delta') \mathcal{N}(1 - \tilde{R}_\delta) \quad \text{for all } \varepsilon \in (0, \varepsilon_\delta).
\]
Let \( R > 0 \). Letting \( \varepsilon_{R, \delta} \) as in Lemma 3.21 and using (134), (161), and (162), for all \( r \in (0, R \varepsilon) \) and \( 0 < \varepsilon < \min \{ \varepsilon_\delta, \varepsilon_{R, \delta}, \frac{1}{R + R \varepsilon} \} \) we obtain
\[
\mathcal{N}(r) \leq \mathcal{N}(1 - \tilde{R}_\delta \varepsilon) e^{\delta'(1 - \tilde{R}_\delta \varepsilon - r)} < (1 + \delta')^2 \varepsilon^{\delta'} \sqrt{\lambda_1(\Sigma)} < (1 + \delta) \sqrt{\lambda_1(\Sigma)}.
\]
The lemma is thereby proved. □

5. Blow-up at the left

Let us define

\[ \hat{u}_\varepsilon : \hat{\Omega}^\varepsilon \to \mathbb{R}, \quad \hat{u}_\varepsilon(x) = \frac{u_\varepsilon(\varepsilon x)}{\sqrt{\varepsilon^{1-N} \int_{\hat{\Gamma}_\varepsilon} u_\varepsilon^2 d\sigma}} \]

where

\[ \hat{\Omega}^\varepsilon := \{ x \in \mathbb{R}^N : \varepsilon x \in \Omega^\varepsilon \} = D^- \cup \{(x_1, x') \in T_1 : 0 \leq x_1 \leq 1/\varepsilon \} \cup \{(x_1, x') : x_1 > 1/\varepsilon \}. \]

We observe that \( \hat{u}_\varepsilon \) solves

\[ \begin{cases} -\Delta \hat{u}_\varepsilon(x) = \varepsilon^2 \lambda_1^\varepsilon \rho(\varepsilon x) \hat{u}_\varepsilon(x), & \text{in } \hat{\Omega}_\varepsilon^\varepsilon, \\ \hat{u}_\varepsilon = 0, & \text{on } \partial \hat{\Omega}^\varepsilon. \end{cases} \]

We denote

\[ T_1^+ = \{(x_1, x') : x' \in \Sigma, x_1 \geq 0 \}, \quad \hat{D} = D^- \cup T_1^+. \]

For every \( R > 0 \) we define

\[ \hat{\Omega}_R = D^- \cup \{(x_1, x') \in T_1^+ : x_1 < R \}, \quad \hat{\Gamma}_R = \Gamma_R = \{(x_1, x') \in T_1^+ : x_1 = R \}, \]

and \( \mathcal{H}_R \) as the completion of

\[ \mathcal{D}_R := \{ v \in C^\infty(\hat{\Omega}_R) : \text{supp } v \subseteq \hat{D} \}. \]

with respect to the norm \( (\int_{\hat{\Omega}_R} |\nabla v|^2 dx)^{1/2} \) (which is equivalent to \( (\int_{\hat{\Omega}_R} |\nabla v|^2 dx + \int_{\hat{\Omega}_R} v^2 d\sigma)^{1/2} \)), i.e. \( \mathcal{H}_R \) is the space of functions with finite energy in \( \hat{\Omega}_R \) vanishing on \( \{(x_1, x') \in \partial \hat{\Omega}_R : x_1 < R \} \).

The change of variable \( y' = \varepsilon x' \) yields

\[ \int_{\hat{\Gamma}_R} \hat{u}_\varepsilon^2 d\sigma = 1. \]

**Lemma 5.1.** For every \( R > 1 \), there exists \( \hat{\varepsilon}_R > 0 \) such that

\[ \int_{\hat{\Gamma}_R} \hat{u}_\varepsilon^2 d\sigma \leq e^{4 \sqrt{\lambda_1(\Sigma)}}(R-1) \quad \text{for all } \varepsilon \in (0, \hat{\varepsilon}_R). \]

**Proof.** For \( R > 1 \), let \( \hat{\varepsilon}_R = \hat{\varepsilon}_{R,1} > 0 \) as in Lemma 4.6. From Lemma 4.6, (131), and (129) it follows that

\[ \frac{d}{dR} H^\varepsilon(R) \leq \frac{2}{\varepsilon} N^\varepsilon(r) \leq \frac{4}{\varepsilon} \sqrt{\lambda_1(\Sigma)} \quad \text{for all } r \in (0, R] \text{ and } \varepsilon \in (0, \hat{\varepsilon}_R), \]

which after integration between \( \varepsilon \) and \( R\varepsilon \) yields

\[ H^\varepsilon(R\varepsilon) \leq H^\varepsilon(\varepsilon)e^{4 \sqrt{\lambda_1(\Sigma)}(R-1)} \quad \text{for all } \varepsilon \in (0, \hat{\varepsilon}_R). \]

(163) and the change of variable \( y' = \varepsilon x' \) yield

\[ \int_{\hat{\Gamma}_R} \hat{u}_\varepsilon^2 d\sigma = \frac{H^\varepsilon(R\varepsilon)}{H^\varepsilon(\varepsilon)} \]

thus implying the conclusion. □

**Lemma 5.2.** For every sequence \( \varepsilon_n \to 0^+ \) there exist a subsequence \( \{\varepsilon_{n_k}\}_k \) and \( \hat{u} \in \bigcup_{R>1} \mathcal{H}_R \) such that

i) \( \hat{u}_{\varepsilon_{n_k}} \to \hat{u} \) strongly in \( \mathcal{H}_R \) for every \( R > 1 \) and a.e.;

ii) \( \hat{u} \equiv 0 \) in \( \hat{D} \);

iii) \( \hat{u} \) weakly solves

\[ \begin{cases} -\Delta \hat{u}(x) = 0, & \text{in } \hat{D}, \\ \hat{u} = 0, & \text{on } \partial \hat{D}. \end{cases} \]
PROOF. Let $R > 1$. By the change of variable $x = \varepsilon y$ we have that, for $\varepsilon \in (0, \min\{1/R, \delta\})$,

\begin{equation}
\mathcal{N}_\varepsilon(R_\varepsilon) = \frac{\int_{\hat{\Omega}_R} \big( (\nabla \hat{u}_\varepsilon(y))^2 - \lambda_1^\varepsilon \varepsilon^2 p(\varepsilon y) \hat{u}_\varepsilon^2(y) \big) dy}{\int_{\hat{\Omega}_R} \hat{u}_\varepsilon^2(y) d\sigma}.
\end{equation}

From Lemma 4.6, for every $\delta > 0$ there exists $\varepsilon_{R, \delta} > 0$ such that

\begin{equation}
\mathcal{N}_\varepsilon(R_\varepsilon) < (1 + \delta) \sqrt{\lambda_1(\Sigma)} \quad \text{for all } \varepsilon \in (0, \varepsilon_{R, \delta}).
\end{equation}

Choosing $\delta = 1$, from (168), (169), and Lemma 5.1, we have that

\begin{equation}
\int_{\hat{\Omega}_R} \big( (\nabla \hat{u}_\varepsilon(y))^2 - \lambda_1^\varepsilon \varepsilon^2 p(\varepsilon y) \hat{u}_\varepsilon^2(y) \big) dy \leq 2 \sqrt{\lambda_1(\Sigma)} \int_{\hat{\Omega}_R} \hat{u}_\varepsilon^2(y) d\sigma \leq 2 \sqrt{\lambda_1(\Sigma)} \varepsilon^{4/\sqrt{\lambda_1(\Sigma)}} (R - 1)
\end{equation}

for all $\varepsilon \in (0, \varepsilon_R)$, where $\varepsilon_R = \varepsilon_{R, 1} > 0$ (accordingly with the notation of Lemma 5.1). From (170) and Lemma 3.6, we obtain that for all $\varepsilon \in (0, \min\{\varepsilon_R, \varepsilon_{2, \lambda_\varepsilon(D^+)}\})$

\begin{equation}
\int_{\hat{\Omega}_R} |\nabla \hat{u}_\varepsilon(y)|^2 dy \leq 4 \sqrt{\lambda_1(\Sigma)} \varepsilon^{4/\sqrt{\lambda_1(\Sigma)}} (R - 1).
\end{equation}

In view of (171) and Lemma 5.1, we have that for every $R > 1$ there exists $\varepsilon_R > 0$ such that

\begin{equation}
\{ \hat{u}_\varepsilon \}_{\varepsilon \in (0, \varepsilon_R)} \text{ is bounded in } \mathcal{H}_R.
\end{equation}

Let $\varepsilon_n \to 0^+$. From (172) and a diagonal process, we deduce that there exist a subsequence $\varepsilon_{n_k} \to 0^+$ and some $\hat{u} \in \bigcup_{R > 1} \mathcal{H}_R$ such that $\hat{u}_{\varepsilon_{n_k}} \rightharpoonup \hat{u}$ weakly in $\mathcal{H}_R$ for every $R > 1$ and almost everywhere. From compactness of the embedding $\mathcal{H}_R \hookrightarrow L^2(\hat{\Gamma}_1)$ and (166) we deduce that $\int_{\hat{\Omega}_R} \hat{u}_\varepsilon^2 d\sigma = 1$; in particular $\hat{u} \neq 0$.

Passing to the weak limit in (164), we obtain that $\hat{u}$ is a weak solution to (167). By classical elliptic estimates, we also have that $\hat{u}_{\varepsilon_{n_k}} \rightharpoonup \hat{u}$ in $C^2((x_1, x_2) \in T_1^+: r_1 \leq x_1 \leq r_2)$ for all $0 < r_1 < r_2$. Therefore, multiplying (167) by $\hat{u}$ and integrating over $\hat{\Omega}_R$, we obtain

\begin{equation}
\int_{\hat{\Omega}_R} \frac{\partial \hat{u}_{\varepsilon_{n_k}}}{\partial x_1} \hat{u}_{\varepsilon_{n_k}} d\sigma \to \int_{\hat{\Omega}_R} \frac{\partial \hat{u}}{\partial x_1} \hat{u} d\sigma = \int_{\hat{\Omega}_R} |\nabla \hat{u}(x)|^2 dx.
\end{equation}

On the other hand, multiplication of (164) by $\hat{u}_{\varepsilon_{n_k}}$ and integration by parts over $\hat{\Omega}_R$ yield

\begin{equation}
\int_{\hat{\Omega}_R} |\nabla \hat{u}_{\varepsilon_{n_k}}(x)|^2 dx = \int_{\hat{\Omega}_R} \frac{\partial \hat{u}_{\varepsilon_{n_k}}}{\partial x_1} \hat{u}_{\varepsilon_{n_k}} d\sigma + \lambda_{\varepsilon_{n_k}} \hat{u}_{\varepsilon_{n_k}}^2 \int_{\hat{\Omega}_R} p(\varepsilon_{n_k} x) \hat{u}_{\varepsilon_{n_k}}^2(x) dx.
\end{equation}

We claim that, for every $R > 1$,

\begin{equation}
\varepsilon_{n_k}^2 \int_{\hat{\Omega}_R} p(\varepsilon_{n_k} x) \hat{u}_{\varepsilon_{n_k}}^2(x) dx \to 0 \quad \text{as } n \to +\infty.
\end{equation}

Indeed, from Lemma 3.6, for every $\delta > 0$ there exists $k_0$ such that for all $k \geq k_0$

\begin{equation}
\int_{\hat{\Omega}_{R_{\varepsilon_{n_k}}}} p(y) u_{\varepsilon_{n_k}}^2(y) dy \leq \delta \int_{\hat{\Omega}_{R_{\varepsilon_{n_k}}}} |\nabla u_{\varepsilon_{n_k}}(y)|^2 dy
\end{equation}

and hence, from the change of variable $y = \varepsilon_{n_k} x$ and (171), we deduce that

\begin{align*}
\varepsilon_{n_k}^2 \int_{\hat{\Omega}_R} p(\varepsilon_{n_k} x) \hat{u}_{\varepsilon_{n_k}}^2(x) dx &= \frac{\varepsilon_{n_k}}{L_{\varepsilon_{n_k}}^\varepsilon_{n_k}} \int_{\hat{\Omega}_{R_{\varepsilon_{n_k}}}} p(y) u_{\varepsilon_{n_k}}^2(y) dy \\
&\leq \frac{\varepsilon_{n_k} \delta}{L_{\varepsilon_{n_k}} u_{\varepsilon_{n_k}}^2} \int_{\hat{\Omega}_{R_{\varepsilon_{n_k}}}} |\nabla u_{\varepsilon_{n_k}}(y)|^2 dy \\
&= \delta \int_{\hat{\Omega}_R} |\nabla \hat{u}_{\varepsilon_{n_k}}(x)|^2 dx \leq 4\delta \sqrt{\lambda_1(\Sigma)} e^{4/\sqrt{\lambda_1(\Sigma)}} (R - 1),
\end{align*}

thus proving claim (173). Combining (173), (174), and (175), we conclude that $\|\hat{u}_{\varepsilon_{n_k}}\|_{\mathcal{H}_R} \to \|\hat{u}\|_{\mathcal{H}_R}$ and then $\hat{u}_{\varepsilon_{n_k}} \rightharpoonup \hat{u}$ strongly in $\mathcal{H}_R$ for every $R > 1$. 
\square
Remark 5.3. We notice that the function \( \hat{u} \) found in Lemma 5.2 satisfies
\[
\int_D |\nabla \hat{u}(x)|^2 \, dx = +\infty.
\]
Indeed, \( \int_D |\nabla \hat{u}(x)|^2 \, dx < +\infty \) would imply, by testing (167) with \( \hat{u} \), that \( \hat{u} \equiv 0 \) in \( \hat{D} \), thus contradicting statement ii) of Lemma 5.2.

We also observe that, denoting as \( \hat{H}(r) = \int_{\hat{\Gamma}_r} \hat{u}^2 \, d\sigma \) for all \( r > 0 \), multiplication of (167) by \( \hat{u} \) and integration over \( \hat{\Omega}_r \) yield
\[
\frac{d}{dr} \hat{H}(r) = 2 \int_{\hat{\Gamma}_r} \hat{\nu} \frac{\partial \hat{u}}{\partial x_1} \, d\sigma = 2 \int_{\hat{\Gamma}_r} |\nabla \hat{u}(x)|^2 \, dx = \int_D |\nabla \hat{u}(x)|^2 \, dx = +\infty \quad \text{as } r \to +\infty,
\]
thus implying that
\[
\lim_{r \to +\infty} \hat{H}(r) = \lim_{r \to +\infty} \int_{\hat{\Gamma}_r} \hat{u}^2 \, d\sigma = +\infty.
\]

Lemma 5.4. Let \( \hat{u} \) as in Lemma 5.2 and, for \( r > 0 \), let \( \hat{N}_\hat{u}(r) \) be the frequency function associated to \( \hat{u} \), i.e.
\[
\hat{N}_\hat{u}(r) = \frac{\int_{\hat{\Gamma}_r} |\nabla \hat{u}(x)|^2 \, dx}{\int_{\hat{\Gamma}_r} \hat{u}(x) \, d\sigma}, \quad r > 0,
\]
with \( \hat{\Omega}_r \) and \( \hat{\Gamma}_r \) defined in (165). Then
\[
i) \lim_{r \to +\infty} \hat{N}_\hat{u}(r) = \sqrt{\lambda_1(\Sigma)};
\]
\[
ii) \text{there exists } \hat{c} \in \mathbb{R} \setminus \{0\} \text{ such that } \int_{T_1} |\nabla (\hat{u} - \hat{c}h)(x)|^2 \, dx < +\infty,
\]
where
\[
(176) \quad h : T_1 \to \mathbb{R}, \quad h(x_1, x') = f(1 - x_1, x') = e^{\sqrt{\lambda_1(\Sigma)} x_1} \psi^F_1(x'),
\]
being \( f \) defined in (27).

Proof. Letting \( \varepsilon_n \to 0^+ \) and \( \{\varepsilon_n_k\}_k \) as in Lemma 5.2, passing to the limit as \( k \to +\infty \) in (168), and using (175), we have that
\[
\lim_{k \to +\infty} N_{\varepsilon_{n_k}}(R\varepsilon_{n_k}) = \hat{N}_\hat{u}(R) \quad \text{for every } R > 0,
\]
which, together with (169), implies that, for every \( \delta > 0 \) and \( R > 0 \),
\[
\hat{N}_\hat{u}(R) \leq (1 + \delta) \sqrt{\lambda_1(\Sigma)}.
\]
Therefore
\[
\hat{N}_\hat{u}(R) \leq \sqrt{\lambda_1(\Sigma)} \quad \text{for every } R > 0.
\]

It is easy to prove that there exists \( \zeta \in H^1_{\text{loc}}(T_1) \cap L^\infty(T_1) \) such that
\[
\begin{cases}
-\Delta \zeta = 0, & \text{in } T_1, \\
\zeta = \hat{u}, & \text{on } \partial T_1,
\end{cases}
\]
\[
\int_{T_1} |\nabla \zeta(x)|^2 \, dx < +\infty,
\]
i.e. \( \zeta \) is a finite-energy harmonic extension of \( \hat{u}|_{\partial T_1} \) in \( T_1 \). Since \( w(x_1, x') = e^{-\sqrt{\lambda_1(\Sigma)} x_1} \psi^F_1(x') \) is harmonic and strictly positive in \( T_1 \), bounded from below away from 0 in \( \{(x_1, x') \in T_1 : x_1 \leq 0\} \), and \( \int_{\{(x_1, x') \in T_1 : x_1 \leq r\}} (|\nabla w|^2 + |w|^2) < +\infty \) for all \( r \), from the Maximum Principle we deduce that \( |\zeta| \leq \text{const } w \) in \( T_1 \), thus implying that, for some \( c_\zeta > 0 \),
\[
|\zeta(x)| \leq c_\zeta e^{-\sqrt{\lambda_1(\Sigma)} x_1} \quad \text{for all } x \in T_1.
\]

Let us observe that the function \( \hat{v} := \hat{u} - \zeta \in H^1_{\text{loc}}(T_1) \setminus \{0\} \) satisfies
\[
\begin{cases}
-\Delta \hat{v}(x) = 0, & \text{in } T_1, \\
\hat{v} = 0, & \text{on } \partial T_1.
\end{cases}
\]
We notice that $\hat{v} \neq 0$ in view of Remark 5.3. Let

$$N_\varphi : \mathbb{R} \to \mathbb{R}, \quad N_\varphi(r) := \frac{\int_{T_r} |\nabla \hat{u}(x)|^2 dx}{\int_{\Gamma_r} \hat{e}^2(x) d\sigma},$$

be as in Lemma 2.5, where, for all $r \in \mathbb{R}$, $T_1$ and $\Gamma_1$ are defined in (38). From Lemma 2.5 it follows that $N_\varphi$ is non decreasing in $\mathbb{R}$ and

$$N_\varphi(t) \geq \lim_{r \to -\infty} N_\varphi(r) \geq \sqrt{\lambda_1(\Sigma)} \quad \text{for all } t \in \mathbb{R}.
$$

(178)

For all $R > 0$, $\delta \in (0, 1)$,

$$\int_{\Gamma_R} \hat{v}^2 d\sigma - (1 - \delta) \int_{\Gamma_R} \hat{u}^2 d\sigma = \int_{\Gamma_R} \zeta^2 d\sigma - 2 \int_{\Gamma_R} \zeta \hat{u} d\sigma + \delta \int_{\Gamma_R} \hat{u}^2 d\sigma \geq \left(1 - \frac{2}{\delta}\right) \int_{\Gamma_R} \zeta^2 d\sigma \geq \left(1 - \frac{2}{\delta}\right) \frac{\omega_{N-2}}{N-1} \zeta^2 e^{-\sqrt{\lambda_1(\Sigma)} R}$$

and

$$\int_{T_{1,n}} |\nabla \hat{v}(x)|^2 dx - (1 + \delta) \int_{\Omega_R} |\nabla \hat{u}(x)|^2 dx \leq \left(1 + \frac{2}{\delta}\right) \int_{T_{1,n}} |\nabla \zeta(x)|^2 dx + \delta \int_{T_{1,n}} |\nabla \hat{u}(x)|^2 dx - \delta \int_{\Omega_R} |\nabla \hat{u}(x)|^2 dx \leq \left(1 + \frac{2}{\delta}\right) \int_{T_{1,n}} |\nabla \zeta(x)|^2 dx,$$

thus implying

$$N_\varphi(R) \leq \frac{1 + \frac{2}{\delta} \int_{T_{1,n}} |\nabla \zeta(x)|^2 dx}{1 + \frac{\omega_{N-2}}{N-1} \zeta^2 e^{-\sqrt{\lambda_1(\Sigma)} R}} \leq \frac{1 + \frac{2}{\delta} \int_{\Omega_R} |\nabla \hat{u}(x)|^2 dx}{1 + \frac{\omega_{N-2}}{N-1} \zeta^2 e^{-\sqrt{\lambda_1(\Sigma)} R}}.$$

(179)

On the other hand, for all $R > 0$, $\delta \in (0, 1)$,

$$\int_{\Gamma_R} \hat{v}^2 d\sigma - (1 + \delta) \int_{\Gamma_R} \hat{u}^2 d\sigma = \int_{\Gamma_R} \zeta^2 d\sigma - 2 \int_{\Gamma_R} \zeta \hat{u} d\sigma - \delta \int_{\Gamma_R} \hat{u}^2 d\sigma \leq \left(1 + \frac{2}{\delta}\right) \int_{\Gamma_R} \zeta^2 d\sigma \leq \left(1 + \frac{2}{\delta}\right) \frac{\omega_{N-2}}{N-1} \zeta^2 e^{-\sqrt{\lambda_1(\Sigma)} R}$$

and

$$\int_{T_{1,n}} |\nabla \hat{v}(x)|^2 dx - (1 - \delta) \int_{\Omega_R} |\nabla \hat{u}(x)|^2 dx$$

$$\leq - \int_{D^-} |\nabla \hat{u}(x)|^2 dx + \int_{T_{1,n}} |\nabla \zeta(x)|^2 dx - 2 \int_{T_{1,n}} \nabla \zeta(x) \cdot \nabla \hat{u}(x) dx + \delta \int_{\Omega_R} |\nabla \hat{u}(x)|^2 dx$$

$$\geq - \int_{D^-} |\nabla \hat{u}(x)|^2 dx + \left(1 - \frac{2}{\delta}\right) \int_{T_{1,n}} |\nabla \zeta(x)|^2 dx - \frac{\delta}{2} \int_{T_{1,n}} |\nabla \hat{u}(x)|^2 dx + \delta \int_{\Omega_R} |\nabla \hat{u}(x)|^2 dx$$

$$\geq - \int_{D^-} |\nabla \hat{u}(x)|^2 dx + \left(1 - \frac{2}{\delta}\right) \int_{T_{1,n}} |\nabla \zeta(x)|^2 dx,$$
thus implying

$$N_\nu(R) \geq \frac{1 - \frac{\delta}{1 + \delta}}{\hat{N}_\nu(R)} \cdot \frac{1 - \frac{\delta}{(1 - \delta) \int_{\Omega_\nu} |\nabla \hat{u}(x)|^2 dx} + 1 - \frac{\delta}{(1 - \delta) \int_{\Omega_\nu} |\nabla \zeta(x)|^2 dx}}{1 + \frac{\delta}{(1 + \delta)(N - 1) \int_{\Gamma_\nu} u^2 \rho(\zeta) d\sigma} e^{-\sqrt{\lambda_1(\Sigma)} R}}. \tag{180}$$

Since \( \int_{T_{1, R}} |\nabla \zeta(x)|^2 dx = O(1), \int_{\Omega_\nu} |\nabla \hat{u}(x)|^2 dx \to +\infty, \) and \( \int_{\Gamma_\nu} \hat{u}^2 d\sigma \to +\infty \) as \( R \to +\infty \) (see Remark 5.3), passing to lim sup and lim inf in (179–180) we obtain that

$$\frac{1 - \frac{\delta}{1 + \delta}}{1 + \delta} \limsup_{R \to \infty} \hat{N}_\nu(R) \leq \limsup_{R \to \infty} N_\nu(R) \leq \frac{1 + \delta}{1 - \delta} \limsup_{R \to \infty} \hat{N}_\nu(R) \quad \text{for all } \delta > 0,$$

$$\frac{1 - \frac{\delta}{1 + \delta}}{1 + \delta} \liminf_{R \to \infty} \hat{N}_\nu(R) \leq \liminf_{R \to \infty} N_\nu(R) \leq \frac{1 + \delta}{1 - \delta} \liminf_{R \to \infty} \hat{N}_\nu(R) \quad \text{for all } \delta > 0,$$

thus implying, in view of (178),

$$\liminf_{R \to \infty} \hat{N}_\nu(R) = \liminf_{R \to \infty} N_\nu(R) \geq \sqrt{\lambda_1(\Sigma)} \tag{181}$$

and, in view of (177),

$$\sqrt{\lambda_1(\Sigma)} \geq \limsup_{R \to \infty} \hat{N}_\nu(R) = \limsup_{R \to \infty} N_\nu(R). \tag{182}$$

From (181) and (182) we deduce that

$$\lim_{R \to \infty} \hat{N}_\nu(R) = \lim_{R \to \infty} N_\nu(R) = \sqrt{\lambda_1(\Sigma)}, \tag{183}$$

thus proving statement i). Furthermore (183), (178), and the fact that \( N_\nu \) is non decreasing imply that

$$N_\nu(t) \equiv \sqrt{\lambda_1(\Sigma)} \quad \text{in } \mathbb{R}.$$

From Lemma 2.5 iii), it follows that there exists \( \hat{c} \in \mathbb{R} \setminus \{0\} \) such that \( \hat{u}(x, x') = \hat{c}h(x, x') \) with \( h \) as in (176). Since \( \int_{T_{1, \nu}} |\nabla (\hat{u} - \hat{c}h)(x)|^2 dx = \int_{T_{1, \nu}} |\nabla \zeta(x)|^2 dx < +\infty, \) also claim ii) is proved. \( \square \)

**Corollary 5.5.** Let \( \hat{u} \) be as in Lemma 5.2 and \( \hat{c} \) as in Lemma 5.4. Then

$$\hat{u}(x_1, x') = \hat{c} \Phi_2(1 - x_1, x')$$

where \( \Phi_2 \) is as in Lemma 2.7.

**Proof.** It follows from Lemmas 5.2 and 5.4, taking into account Lemma 2.7. \( \square \)

Let us define \( \hat{\Phi}(x_1, x') := \Phi_2(1 - x_1, x') \) and, for all \( r < -1 \)

$$\hat{N}(r) = \hat{N}_\hat{\Phi}(r) = \frac{(-r) \int_{\Omega_\Phi} |\nabla \hat{\Phi}(x)|^2 dx}{\int_{\Gamma_{\nu_{-\hat{c}}}} \hat{\Phi}(x) d\sigma},$$

with \( \Omega_\nu \) as in (77) and \( \Gamma_{\nu_{-\hat{c}}} \) as in (10), so that, according to notation of Lemma 3.3, \( \hat{N}(r) = N_{\hat{\Phi}}(-r) \) for all \( r < -1. \)

**Lemma 5.6.** \( \lim_{r \to -\infty} \hat{N}(r) = N - 1. \)

**Proof.** The proof follows from Lemma 3.3 and Remark 2.8. \( \square \)

**Lemma 5.7.** For every \( R > 1 \)

$$\lim_{\epsilon \to 0^+} N_\epsilon(-R \epsilon) = \hat{N}(-R). \tag{184}$$

**Proof.** Fix \( R > 1. \) Let \( \epsilon_n \to 0^+. \) From Lemma 5.2 and Corollary 5.5, there exist a subsequence \( \{\epsilon_n\}_k \) and \( \hat{c} \neq 0 \) such that \( \hat{u}_{\epsilon_n} \to \hat{c} \hat{\Phi} \) strongly in \( \mathcal{H}_r \) for every \( r > 1. \) By the change of variable \( x = \epsilon y \) we have that, for \( \epsilon \in (0, \epsilon) \) and \( R > 1, \)

$$N_\epsilon(-R \epsilon) = \frac{R \int_{\Omega_{-R \epsilon}} (|\nabla \hat{u}_\epsilon(y)|^2 - \lambda_\epsilon \epsilon^2 p(\epsilon y) \hat{u}_\epsilon^2(y)) dy}{\int_{\Gamma_{-R \epsilon}} \hat{u}_\epsilon^2(y) d\sigma}.$$
with $\Omega - \nu$ and $\Gamma^+_\nu$ as in (77) and (10) respectively. From strong convergence of $\tilde{u}_{e_n}$ to $\tilde{\Phi}$ in $\mathcal{H}_r$ for every $r > 1$, passing to the limit in (184) along the subsequence $\{\varepsilon_{e_n}\}_k$ and using (175) we obtain that

$$\lim_{k \to +\infty} \mathcal{N}_{e_n}(-R \varepsilon_{e_n}) = \frac{R \int_{\Omega - \nu} |\nabla (\tilde{\Phi}(y))|^2 dy}{\int_{\Gamma^+_\nu} (\tilde{\Phi})^2(y) d\sigma} \to \mathcal{N}(-R).$$

Since the limit depends neither on the sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ nor on its subsequence $\{\varepsilon_{e_n}\}_{k \in \mathbb{N}}$, we conclude that the convergence actually holds as $\varepsilon \to 0^+$ thus proving the lemma. □

**Lemma 5.8.** For every $\delta > 0$ there exist $K_\delta > 1$, $k_\delta \in (0,1)$, and $\rho_\delta \in (0, \frac{k_\delta}{K_\delta})$, such that

$$\mathcal{N}_\varepsilon(r) \leq N - 1 + \delta$$

and

$$\int_{\Omega_r} (|\nabla u_\varepsilon|^2 - \lambda K^2_p) dx \geq \frac{1}{2} \int_{\Omega_r} |\nabla u_\varepsilon|^2 dx$$

for all $r \in (-k_\delta, -K_\delta)$ and $\varepsilon \in (0, \rho_\delta)$. 

**PROOF.** Let $\delta > 0$ and fix $\delta' \in (0,1)$ such that

$$(N - 1 + 2\delta')e^{\delta'} = N - 1 + \delta.$$  

From Lemma 5.6 there exists some $K_\delta > 1$ such that $\mathcal{N}(K_\delta) < N - 1 + \delta'$. From Lemma 5.7 there exists some $\varepsilon'_\delta > 0$ such that, for all $\varepsilon \in (0, \varepsilon'_\delta)$, $\mathcal{N}_\varepsilon(-K_\delta) < \mathcal{N}(K_\delta) + \delta' < N - 1 + 2\delta'$. Letting $\tilde{R}_\delta$, $\tilde{\varepsilon}_\delta$ as in Lemma 3.17 and Corollary 3.18, we have that for all $\varepsilon \in (0, \min \{\varepsilon'_\delta, \tilde{\varepsilon}_\delta, \tilde{R}_\delta/K_\delta\})$ and $r \in (-\tilde{R}_\delta, -K_\delta)$

$$\mathcal{N}_\varepsilon(r) \leq \mathcal{N}_\varepsilon(-K_\delta) e^{\delta' R_\delta'} \leq (N - 1 + 2\delta') e^{\delta' R_\delta'} \leq N - 1 + \delta$$

and

$$\int_{\Omega_r} (|\nabla u_\varepsilon|^2 - \lambda K^2_p) dx \geq \frac{1}{2} \int_{\Omega_r} |\nabla u_\varepsilon|^2 dx.$$ 

Then the lemma follows choosing $k_\delta = \tilde{R}_\delta$, and $\rho_\delta = \min \{\varepsilon'_\delta, \tilde{\varepsilon}_\delta, \tilde{R}_\delta/K_\delta\}$. □

6. **ASYMPTOTICS AT THE LEFT JUNCTION**

Throughout this section, we fix $\delta \in (0,1)$ so that $N - 1 + \delta < N$. Let us denote $K = K_\delta > 1$, $\tilde{h} = k_\delta \in (0,1)$, and $\tilde{\rho} = \rho_\delta \in (0, \frac{1}{2})$ with $K_\delta, k_\delta, \rho_\delta$ as in Lemma 5.8, so that

$$\mathcal{N}_\varepsilon(r) \leq N - 1 + \delta < N$$

for all $r \in (-\tilde{h}, -K\varepsilon)$ and $\varepsilon \in (0, \tilde{\rho})$.

Let us denote

$$U_\varepsilon(x) = \frac{u_\varepsilon(x)}{\sqrt{\int_{\Gamma^-_\tilde{h}} u_\varepsilon^2 d\sigma}}$$

with $\Gamma^-_\tilde{h}$ as in (10). Let us notice that, for $\varepsilon \in (0, \varepsilon_0)$, $U_\varepsilon$ solves

$$\begin{cases}
-\Delta U_\varepsilon = \lambda K^2_p U_\varepsilon, & \text{in } \Omega^\varepsilon, \\
U_\varepsilon = 0, & \text{on } \partial \Omega^\varepsilon,
\end{cases}$$

and

$$\int_{\Gamma^-_\tilde{h}} U_\varepsilon^2 d\sigma = 1.$$ 

**Proposition 6.1.** For every sequence $\varepsilon_n \to 0^+$ there exist a subsequence $\{\varepsilon_{e_n}\}_k$ and a function $U \in C^2(D^-) \cup (\bigcup_{t > 0} \mathcal{H}^t)$ such that

i) $U_{e_n} \to U$ strongly in $\mathcal{H}^t$ for every $t > 0$ and in $C^2(B^-_{t_2} \setminus B^-_{t_1})$ for all $0 < t_1 < t_2$;

ii) $U \not\equiv 0$ in $D^-$;

iii) $U$ solves

$$\begin{cases}
-\Delta U(x) = \lambda K^2_p(D^+) p(x) U(x), & \text{in } D^-, \\
U = 0, & \text{on } \partial D^-;
\end{cases}$$
iv) if \( \mathcal{N}_U : (-\infty, 0) \to \mathbb{R} \) is defined as
\[
\mathcal{N}_U(r) := (-r) \int_{\Omega} \left( |\nabla U(x)|^2 - \lambda_k(D^+)p(x)U^2(x) \right) dx \frac{1}{\int_{\Gamma} U^2(x) d\sigma},
\]
then
\[
\mathcal{N}_U(r) \leq N - 1 + \delta \quad \text{for all } r \in (-\hat{h}, 0).
\]

**Proof.** Letting \( H^{-}_e(t) \) as in (109), from (111), (107)–(109), and Lemma 5.8 it follows that
\[
\frac{d}{dt} H^{-}_e(t) = -\frac{2}{t} N\epsilon(-t) \geq -\frac{2N}{t}
\]
for all \( t \in (\hat{K}_\epsilon, \hat{h}) \) and \( \epsilon \in (0, \hat{\rho}) \), which after integration yields
\[
H^{-}_e(t) \leq \hat{h}^{2N} H^{-}_e(\hat{h}) t^{-2N} \quad \text{for all } t \in (\hat{K}_\epsilon, \hat{h}).
\]
From (187), (107)–(109), (193), and Lemma 5.8, we deduce that
\[
\frac{1}{2} \int_{\Omega_{t1}} |\nabla U(x)|^2 dx \leq \int_{\Omega_{t1}} \left( |\nabla U(x)|^2 - \lambda_k p(x)U^2(x) \right) dx
\]
\[
= \int_{\Omega_{t1}} \left( |\nabla U(x)|^2 - \lambda_k p(x)U^2(x) \right) dx \geq \frac{t^{N-2}}{h^{N-1}} N\epsilon(-t) \leq N\hat{h}^{N+1} t^{-N-2}
\]
for all \( t \in (\hat{K}_\epsilon, \hat{h}) \) and \( \epsilon \in (0, \hat{\rho}) \). Hence for every \( t > 0 \)
\[
\{U_\epsilon \}_{\epsilon \in (0, \min(\rho, \hat{K})]} \quad \text{is bounded in } H^{-}_e.
\]
Let \( \epsilon_n \to 0^+ \). From (194) and a diagonal process, there exist a subsequence \( \epsilon_{n_k} \to 0^+ \) and some \( U \in \bigcup_{R > 0} \mathcal{H}_e \) such that \( U_{\epsilon_{n_k}} \to U \) weakly in \( \mathcal{H}_e \) for every \( t > 0 \) and a.e. in \( D^- \). From compactness of the embedding \( \mathcal{H}_e^- \to L^2(\Gamma^-) \), passing to the limit in (189) we obtain that \( \int_{\Gamma^-} U^2 d\sigma = 1 \); in particular \( U \neq 0 \) in \( D^- \). Passing to the weak limit in (188), we obtain that \( U \) is a weak solution to (190). By classical elliptic estimates, we also have that \( U_{\epsilon_{n_k}} \to U \) in \( C^2(B_{t_1} \setminus B_{t_2}) \) for all \( 0 < t_1 < t_2 \). Therefore, multiplying (190) by \( U \) and integrating over \( \Omega_{t^1} \), we obtain
\[
\int_{\Omega_{t^1}} \frac{\partial U_{\epsilon_{n_k}}}{\partial \nu} U_{\epsilon_{n_k}} d\sigma \to \int_{\Gamma^-} \frac{\partial U}{\partial \nu} U \, d\sigma = -\int_{\Omega_{t^1}} \left( |\nabla U(x)|^2 - \lambda_k p(x)U^2(x) \right) dx,
\]
being \( \nu = \nu(x) = \frac{x}{|x|} \). On the other hand, multiplication of (188) by \( U_{\epsilon_{n_k}} \) and integration by parts over \( \Omega_{t^1} \) yield
\[
\int_{\Omega_{t^1}} \left( |\nabla U_{\epsilon_{n_k}}(x)|^2 - \lambda_k p(x)U^2_{\epsilon_{n_k}}(x) \right) dx = -\int_{\Gamma^-} \frac{\partial U_{\epsilon_{n_k}}}{\partial \nu} U_{\epsilon_{n_k}} d\sigma.
\]
Since weak \( \mathcal{H}_e^- \)-convergence of \( U_{\epsilon_{n_k}} \) to \( U \) implies that
\[
\int_{\Omega_{t^1}} p(x)U^2_{\epsilon_{n_k}}(x) dx \to \int_{\Omega_{t^1}} p(x)U^2(x) dx \quad \text{as } k \to +\infty,
\]
combining (195), (196), and (197), we conclude that \( \|U_{\epsilon_{n_k}}\|_{\mathcal{H}_e^-} \to \|U\|_{\mathcal{H}_e^-} \) and then \( U_{\epsilon_{n_k}} \to U \) strongly in \( \mathcal{H}_e^- \) for every \( t > 0 \).

Finally, we notice that strong \( \mathcal{H}_e^- \)-convergence of \( U_{\epsilon_{n_k}} \) to \( U \) implies that, for every \( r < 0 \),
\[
\mathcal{N}_{\epsilon_{n_k}}(r) = (-r) \int_{\Omega_r} \left( |\nabla U_{\epsilon_{n_k}}(x)|^2 - \lambda_k p(x)U^2_{\epsilon_{n_k}}(x) \right) dx \frac{1}{\int_{\Gamma_r} U^2_{\epsilon_{n_k}}(x) d\sigma} \to \mathcal{N}_U(r) \quad \text{as } k \to +\infty,
\]
hence, passing to the limit in (186) as \( \epsilon = \epsilon_{n_k} \to 0 \), we obtain (192) and complete the proof.

**Lemma 6.2.** Let \( U \) be as in Proposition 6.1 and let \( \mathcal{N}_U : (-\infty, 0) \to \mathbb{R} \) be the frequency function associated to \( U \) defined in (191). Then
\[
(i) \quad \lim_{r \to 0^-} \mathcal{N}_U(r) = N - 1;
\]
(ii) for every sequence \( \lambda_n \to 0^+ \) there exist a subsequence \( \{\lambda_{n_k}\}_k \) and some constant \( c \in \mathbb{R} \setminus \{0\} \) such that

\[
\frac{U(\lambda_{n_k}x)}{\sqrt{H_U(\lambda_{n_k})}} \xrightarrow{k \to +\infty} c \frac{x_1}{|x|^N}
\]

strongly in \( \mathcal{H}_e \) for every \( t > 0 \) and in \( C^2(\overline{B_{r+t}} \setminus B_{r}) \) for all \( 0 < t_1 < t_2 \), where

\[
H_U(\lambda) := \frac{1}{\lambda^{N-1}} \int_{\Gamma_\lambda} U^2(x) \, d\sigma.
\]

**Proof.** We first notice that, letting \( \varepsilon_n \to 0^+ \) and \( \{\varepsilon_{n_k}\}_k \) as in Proposition 6.1, passing to the limit as \( k \to +\infty \), from (185) and strong \( \mathcal{H}_e \)-convergence of \( U_{\varepsilon_{n_k}} \) to \( U \) we obtain that

\[
\int_{\Omega_r} (|\nabla U|^2 - \lambda_{n_k}(D^+)pU^2) \, dx \geq \frac{1}{2} \int_{\Omega_r} |\nabla U|^2 \, dx
\]

for all \( r \in (-\tilde{h}, 0) \). In particular

\[
N_U(r) \geq 0 \quad \text{for all } r \in (-\tilde{h}, 0).
\]

Arguing as in the proof of Lemma 3.15, we can prove that, for all \( r < 0 \),

\[
\frac{d}{dr} N_U(r) = \nu_1(r) + \nu_2(r),
\]

where

\[
\nu_1(r) = -2r \left( \int_{\Gamma_r^-} \frac{|\partial U|}{\partial r} \, |\partial U| \, d\sigma \right) \left( \int_{\Gamma_r^-} U^2(x) \, d\sigma \right) - \left( \int_{\Gamma_r^-} U \frac{\partial U}{\partial r} \, d\sigma \right)^2
\]
\[
\nu_2(r) = \lambda_{n_k}(D^+) \int_{\Omega_r} (2p(x) + x \cdot \nabla p(x)) U^2(x) \, dx
\]
\[
\int_{\Gamma_r^-} U^2(x) \, d\sigma.
\]

Schwarz’s inequality implies that

\[
\nu_1(r) \geq 0 \quad \text{for all } r < 0.
\]

Furthermore

\[
\frac{\nu_2(r)}{N_U(r)} \leq \lambda_{n_k}(D^+) \frac{\int_{\Omega_r} (2p(x) + x \cdot \nabla p(x)) U^2(x) \, dx}{\left( -r \int_{\Omega_r} (|\nabla U(x)|^2 - \lambda_{n_k}(D^+)p(x)U^2(x)) \, dx \right)^{\frac{1}{2}}} \leq \frac{\varepsilon}{2},
\]

where the last inequality is obtained passing to the limit as \( \varepsilon = \varepsilon_{n_k} \to 0^+ \) in (128). Hence from (192) we obtain that

\[
\nu_2(r) \leq \tilde{\varepsilon} (N - 1 + \frac{\varepsilon}{2}), \quad \text{for all } r \in (-\tilde{h}, 0),
\]

from (204) and (205) it follows that \( \frac{d}{dr} N_U \) is the sum of a nonnegative function and of a bounded function on \( (-\tilde{h}, 0) \). Therefore \( N_U(r) = N_U(-\tilde{h}) + \int_{-\tilde{h}}^r (\nu_1(s) + \nu_2(s)) \, ds \) admits a limit as \( r \to 0^+ \) which is necessarily finite in view of (192) and (200). More precisely, denoting as

\[
\gamma := \lim_{r \to 0^-} N_U(r),
\]

(192) and (200) ensure that

\[
\gamma \in [0, N - 1 + \delta] \subset [0, N).
\]

For all \( x \in D^- \) and \( \lambda > 0 \), let us consider

\[
U_\lambda(x) := \frac{U(\lambda x)}{\sqrt{H_U(\lambda)}}
\]

where \( H_U(\lambda) \) is defined in (198). We notice that

\[
\int_{\Gamma_\lambda} U_\lambda^2 \, d\sigma = 1.
\]
Furthermore, by direct calculation (see also the proof of Lemma 3.15 which is analogous), we have that
\[ H_U'(\lambda) \frac{H_U(\lambda)}{H_U(\lambda)} = -2 \frac{\lambda}{\lambda} N_U(-\lambda) \geq -2 \frac{\lambda}{\lambda} (N - 1 + \delta) \] for all \( \lambda \in (0, \tilde{h}) \), which after integration yields
\[ H_U(\lambda_1) \leq H_U(\lambda_2) \left( \frac{\lambda_2}{\lambda_1} \right)^{2(N - 1 + \delta)} \] for all \( 0 < \lambda_1 < \lambda_2 < \tilde{h} \).

From (199), (211), and (192), for every \( t \in (0, 1) \) and \( \lambda \in (0, \tilde{h}/t) \), we have that
\[ \int_{\Omega - t} |\nabla U^\lambda(x)|^2 \, dx = t^{N-2} \frac{H_U(\lambda t)}{H_U(\lambda)} \left( \frac{\lambda t}{\lambda} \right)^{2(N - 1 + \delta)} \int_{\Omega - t} |\nabla U(x)|^2 \, dx \leq 2t^{N-2} t^{-2(N - 1 + \delta)} N_U(-\lambda t) \leq 2 t^{-N-2\delta} (N - 1 + \delta). \]

Hence for every \( t \in (0, 1) \) there exists \( \lambda_t > 0 \) such that
\[ \{U^\lambda\}_{\lambda \in (0, \lambda_t)} \text{ is bounded in } \mathcal{H}^-_t. \]

Let \( \lambda_n \to 0^+ \). From (213) and a diagonal process, we deduce that there exist a subsequence \( \lambda_{n_k} \to 0^+ \) and some \( \tilde{U} \in \bigcup_{t > 0} \mathcal{H}^-_t \) such that \( U^{\lambda_{n_k}} \to \tilde{U} \) weakly in \( \mathcal{H}^-_t \) for every \( t > 0 \) and a.e. in \( D^- \). Since \( U^\lambda \) solves
\[ \begin{cases} -\Delta U^\lambda(x) = \lambda^2 \lambda k_n (D^+) p(\lambda x) U^\lambda(x), & \text{in } D^-, \\ U^\lambda = 0, & \text{on } \partial D^-, \end{cases} \]
passing to the weak limit in (214), we obtain that \( \tilde{U} \) satisfies
\[ \begin{cases} -\Delta \tilde{U}(x) = 0, & \text{in } D^-, \\ \tilde{U} = 0, & \text{on } \partial D^- \end{cases} \]

By compactness of the embedding \( \mathcal{H}^-_t \hookrightarrow L^2(\Gamma^-) \), passing to the limit in (209), we have that \( \int_{\Gamma^-} \tilde{U}^2 \, d\sigma = 1 \). In particular \( \tilde{U} \neq 0 \).

From Lemma 3.6, for every \( \alpha > 0 \) there exists \( k_n \in \mathbb{N} \) and \( t_\alpha > 0 \) such that for all \( k > k_\alpha \) and \( t \in (\varepsilon_{n_k}, t_\alpha) \)
\[ \int_{\Omega - t} |p(x)| U_{\varepsilon_{n_k}}^2(x) \, dx \leq \alpha \int_{\Omega - t} |\nabla U_{\varepsilon_{n_k}}(x)|^2 \, dx. \]

Strong \( \mathcal{H}^-_t \)-convergence of \( U_{\varepsilon_{n_k}} \) to \( U \) then implies that
\[ \int_{\Omega - t} |p(x)| U^2 \, dx \leq \alpha \int_{\Omega - t} |\nabla U(x)|^2 \, dx, \text{ for all } t \in (0, t_\alpha). \]

Hence, by the change of variable \( x = \lambda y \) and (212), we obtain that, for every \( s > 0 \),
\[ \lambda^2 \int_{\Omega_{\lambda t}} |p(\lambda y)||U^\lambda(y)|^2 \, dy \leq \alpha \int_{\Omega_{\lambda t}} |\nabla U(y)|^2 \, dy \leq 2 \alpha s^{-N-2\delta} (N - 1 + \delta), \text{ for all } \lambda < \min \left\{ \frac{t_\alpha}{s}, \frac{\tilde{h}}{s} \right\}, \]
thus implying that, for every \( s > 0 \),
\[ \lambda^2 \int_{\Omega_{\lambda t}} |p(\lambda y)||U^\lambda(y)|^2 \, dy \to 0 \quad \text{as } \lambda \to 0^+. \]

By classical elliptic estimates, we also have that \( U^{\lambda_{n_k}} \to \tilde{U} \) in \( C^2(\overline{B_{r_1} \setminus B_{r_2}}) \) for all \( 0 < r_1 < r_2 \).

Therefore, multiplying (215) by \( \tilde{U} \) and integrating over \( \Omega_{\lambda t} \), we obtain
\[ \int_{\Gamma^-} \frac{\partial U^{\lambda_{n_k}}}{\partial \nu} U^{\lambda_{n_k}} \, d\sigma \to \int_{\Gamma^-} \frac{\partial \tilde{U}}{\partial \nu} \tilde{U} \, d\sigma = - \int_{\Omega_{\lambda t}} |\nabla \tilde{U}(x)|^2 \, dx, \]
while multiplication of (214) by \( U^{\lambda_{n_k}} \) and integration by parts over \( \Omega_{\lambda t} \) yield
\[ \int_{\Omega_{\lambda t}} |\nabla U^{\lambda_{n_k}}|^2 \, dx = - \int_{\Gamma^-} \frac{\partial U^{\lambda_{n_k}}}{\partial \nu} U^{\lambda_{n_k}} \, d\sigma + \lambda_{n_k}^2 \lambda k_n (D^+) \int_{\Omega_{\lambda t}} p(\lambda_{n_k} x)|U^{\lambda_{n_k}}(x)|^2 \, dx. \]
Combining (217), (218), and (216), we conclude that \( \|U^{\lambda_{n_k}}\|_{\mathcal{H}_1^-} \to \|\tilde{U}\|_{\mathcal{H}_1^-} \) and then \( U^{\lambda_{n_k}} \to \tilde{U} \) strongly in \( \mathcal{H}_1^- \) for every \( t > 0 \).

From (191), strong convergence \( U^{\lambda_{n_k}} \to \tilde{U} \) in \( \mathcal{H}_1^- \), and (216), we have that, for every \( t > 0 \),

\[
N_U(-t\lambda_{n_k}) = \frac{t\lambda_{n_k} \int_{\Omega \setminus \|\lambda_{n_k}\|} \left( |\nabla U(x)|^2 - \lambda_{n_k}(D^+)p(x)U^2(x) \right) dx}{\int_{\Gamma^-_{\|\lambda_{n_k}\|}} U^2(x) d\sigma} \]

\[
= \frac{t \int_{\Omega^-} \left( |\nabla U^{\lambda_{n_k}}(x)|^2 - \lambda_{n_k}^2\lambda_{n_k}(D^+)p(\lambda_{n_k}x)|U^{\lambda_{n_k}}(x)|^2 \right) dx}{\int_{\Gamma^-} |\lambda_{n_k}|^2 dx} \]

\[
\to \frac{t \int_{\Omega^-} |\nabla \tilde{U}(x)|^2 dx}{\int_{\Gamma^-} \tilde{U}^2(x) d\sigma} \quad \text{as} \quad k \to +\infty.
\]

Combining (206) and (219) we conclude that

\[
\frac{t \int_{\Omega^-} |\nabla \tilde{U}(x)|^2 dx}{\int_{\Gamma^-} \tilde{U}^2(x) d\sigma} = \gamma \quad \text{for all} \quad t > 0.
\]

From Lemma 3.3 there exists \( K_0 \in \mathbb{N}, K_0 \geq 1 \), such that

\[
\gamma = N - 2 + K_0
\]

and \( \tilde{U}(x) = |x|^{-N+2-K_0}Y(x/|x|) \) for some eigenfunction \( Y \) of \( -\Delta_{S^{N-1}} \) associated to the eigenvalue \( K_0(N-2+K_0) \), i.e. satisfying \( -\Delta_{S^{N-1}}Y = K_0(N-2+K_0)Y \) on \( S^{N-1} \). From (207) and (220) we infer that necessarily \( K_0 = 1 \), so that

\[
\gamma = N - 1 \quad \text{and} \quad \tilde{U}(x) = |x|^{-N+1}Y(x/|x|).
\]

From \( \tilde{U} = 0 \) on \( \partial D^- \), we deduce that \( Y \equiv 0 \) on \( \{ \theta = (\theta_1, \theta_2, \ldots, \theta_N) \in S^{N-1} : \theta_1 = 0 \} \), hence \( Y \) is an eigenfunction of \( -\Delta_{S^{N-1}} \) on \( S^{N-1} = \{ \theta = (\theta_1, \theta_2, \ldots, \theta_N) \in S^{N-1} : \theta_1 < 0 \} \) under null Dirichlet boundary conditions associated to the eigenvalue \( N-1 \). It is easy to verify that \( N-1 \) is the first eigenvalue of such eigenvalue problem and hence it is simple; furthermore an eigenfunction associated to the eigenvalue \( N-1 \) is \( \theta = (\theta_1, \theta_2, \ldots, \theta_N) \in S^{N-1} \mapsto \theta_1 \). Therefore we conclude that there exists some constant \( c \in \mathbb{R} \setminus \{0\} \) such that \( Y(\theta) = c\theta_1 \) and then

\[
\tilde{U}(x) = c \frac{x_1}{|x|^N}.
\]

The proof is thereby completed. \( \square \)

**Lemma 6.3.** Let \( U \) as in Proposition 6.1 and let \( H_U : (0, +\infty) \to \mathbb{R} \) be defined in (198). Then

(i) \( H_U(\lambda) \leq e^{2(N-1-\delta)}\hat{k}_1^{2(N-1)}H_U(\hat{k})\lambda^{-2(N-1)} \) for all \( \lambda \in (0, \hat{k}) \);

(ii) for every \( \varrho > 0 \) there exists \( \lambda_0 > 0 \) such that \( H_U(\lambda) \geq H_U(\lambda_0)\lambda^{2(N-1-\varrho)}\lambda^{-2(N-1-\varrho)} \) for all \( \lambda \in (0, \lambda_0) \);

(iii) \( \lim_{\lambda \to 0^+} \lambda^{2(N-1)}H_U(\lambda) \) exists and is finite.

**Proof.** From Lemma 6.2 (i), (201), (204), (205), we obtain that

\[
N - 1 - N_U(-\lambda) = \int_{-\lambda}^{0} N_U'(s) ds \geq \int_{-\lambda}^{0} \nu_2(s) ds \geq -\delta(N-1+\delta)\lambda \quad \text{for all} \quad \lambda \in (0, \hat{k}),
\]

where \( \nu_2 \) is defined in (203), and then

\[
N_U(-\lambda) \leq N - 1 + \delta(N-1+\delta)\lambda \quad \text{for all} \quad \lambda \in (0, \hat{k}),
\]

which, together with (210), yields

\[
\frac{H_U'(\lambda)}{H_U(\lambda)} = -\frac{2}{\lambda}N_U(-\lambda) \geq -\frac{2(N-1)}{\lambda} - 2\delta(N-1+\delta) \quad \text{for all} \quad \lambda \in (0, \hat{k}).
\]

Integration of the above inequality between \( \lambda \) and \( \hat{k} \) proves estimate (i).
From Lemma 6.2 (i), for any \( \rho > 0 \) there exists \( \lambda_\rho > 0 \) such that \( N_U(r) > N - 1 - \rho \) for any \( r \in (-\lambda_\rho, 0) \) and hence
\[ H_U'(\lambda) \over H_U(\lambda) = -\frac{2}{\lambda} N_U(-\lambda) < -\frac{2(N - 1 - \rho)}{\lambda} \quad \text{for all} \ \lambda \in (0, \lambda_\rho). \]
Integration over the interval \((\lambda, \lambda_\rho)\) yields (ii).

In view of (i), to prove (iii) it is sufficient to show that the limit exists. From (210), Lemma 6.2 (i), and (201) it follows that
\[
\frac{d}{d\lambda} \left( \lambda^{2(N-1)} H_U(\lambda) \right) = 2\lambda^{2N-3} H_U(\lambda) \left( N - 1 + \frac{\lambda}{2} H_U'(\lambda) \right) = 2\lambda^{2N-3} H_U(\lambda) (N - 1 - N_U(-\lambda))
\]
\[= 2\lambda^{2N-3} H_U(\lambda) \int_{-\lambda}^0 N_U^0(s) ds = 2\lambda^{2N-3} H_U(\lambda) \int_{-\lambda}^0 (\nu_1(s) + \nu_2(s)) ds \]
where \( \nu_1 \) and \( \nu_2 \) are defined in (202) and (203) respectively. By integration of the above identity we obtain that, for all \( \lambda \in (0, \bar{\lambda}) \),
\[\lambda^{2(N-1)} H_U(\lambda) - \bar{\lambda}^{2(N-1)} H_U(\bar{\lambda}) = -2 \int_{\lambda}^{\bar{\lambda}} s^{2N-3} H_U(s) \left( \int_{-s}^0 \nu_1(t) \, dt \right) ds \]
\[-2 \int_{\lambda}^{\bar{\lambda}} s^{2N-3} H_U(s) \left( \int_{-s}^0 \nu_2(t) \, dt \right) ds. \]
From (204) the limit
\[\lim_{\lambda \to 0^+} \int_{\lambda}^{\bar{\lambda}} s^{2N-3} H_U(s) \left( \int_{-s}^0 \nu_1(t) \, dt \right) ds \]
exists. On the other hand from (i) and (205) it follows that
\[s^{2N-3} H_U(s) \left( \int_{-s}^0 \nu_2(t) \, dt \right) = O(1) \quad \text{as} \ s \to 0^+ \]
thus proving in particular that \( s \to s^{2N-3} H_U(s) \left( \int_{-s}^0 \nu_2(t) \, dt \right) \in L^1(0, \bar{\lambda}) \). We conclude that both terms at the right hand side of (221) admit a limit as \( \lambda \to 0^+ \), the second one being finite in view of (222), thus completing the proof of the lemma. \( \square \)

**Lemma 6.4.** Let \( U \) be as in Proposition 6.1, \( Y_1 \) as in (11), and let \( H_U : (0, +\infty) \to \mathbb{R} \) be defined in (198). Then

(i) \[\int_{\Sigma_{N-1}^-} U(\lambda \theta) Y_1(\theta) \, d\sigma(\theta) = \lambda^{1-N} \left[ \int_{\Gamma_1^-} U(x) Y_1(\frac{x}{|x|}) \, dx - \frac{\lambda_0(D^+)}{N} \int_{D^-} p(x) U(x) Y_1(\frac{x}{|x|}) \left( |x| \lambda_{B_1}(x) + \frac{\lambda_0(D^+)}{|x|^{N-1}} \right) \, dx \right] + O(\lambda^{3-N}) \quad \text{as} \ \lambda \to 0^+, \]

(ii) \[\lim_{\lambda \to 0^+} \lambda^{2(N-1)} H_U(\lambda) > 0. \]

**Proof.** Let us define, for all \( \lambda > 0 \),
\[\mu(\lambda) = \int_{\Sigma_{N-1}^-} U(\lambda \theta) Y_1(\theta) \, d\sigma(\theta), \quad \varsigma(\lambda) = \lambda_{00}(D^+) \int_{\Sigma_{N-1}^-} p(\lambda \theta) U(\lambda \theta) Y_1(\theta) \, d\sigma(\theta). \]
From (190) \( \mu \) satisfies
\[-\mu''(\lambda) - \frac{N - 1}{\lambda} \mu'(\lambda) + \frac{N - 1}{\lambda^2} \mu(\lambda) = \varsigma(\lambda), \quad \text{in} \ (0, +\infty). \]
Hence there exist \( c_1, c_2 \in \mathbb{R} \) such that
\[\mu(\lambda) = \lambda \left( c_1 \frac{1}{N} \int_{\lambda}^{1} \varsigma(t) \, dt \right) + \lambda^{1-N} \left( c_2 - \frac{1}{N} \int_{\lambda}^{1} t^N \varsigma(t) \, dt \right) \quad \text{for all} \ \lambda \in (0, +\infty). \]
Since \( p \in L^{N/2}(\mathbb{R}^N) \) and \( U \in H_0^1 \) ensure that \( \frac{p(x)U(x)Y_1(x/|x|)}{|x|^{N+\alpha}} \in L^1(\Omega_{-1}) \) for all \( \alpha \in (0, \frac{N}{2}) \) and, for all \( \lambda > 1 \),
\[
\int_\lambda^1 t^\alpha \zeta(t) dt = -\lambda k_0(D^+) \int_{B_\lambda^c \setminus B_1^c} \frac{p(x)U(x)Y_1(x/|x|)}{|x|^{N+\alpha}} dx,
\]
\[
\int_1^\lambda t^\alpha |\zeta(t)| dt \leq \lambda k_0(D^+) \int_{B_\lambda^c \setminus B_1^c} \frac{p(x)U(x)Y_1(x/|x|)}{|x|^{N+\alpha}} dx,
\]
we deduce that \( \int_\lambda^1 t^\alpha \zeta(t) dt \) admits a finite limit and \( \int_1^\lambda t^\alpha |\zeta(t)| dt = O(1) \) as \( \lambda \to +\infty \) for every \( \alpha \in (0, \frac{N}{2}) \). In particular \( \int_\lambda^1 t^\alpha \zeta(t) dt \) admits a finite limit as \( \lambda \to +\infty \) and
\[
\left| \int_1^\lambda t^N \zeta(t) dt \right| \leq \int_1^\lambda t^{N-1} |\zeta(t)| dt \leq \lambda^{N-1} \int_1^\lambda t |\zeta(t)| dt = O(\lambda^{-N}) \quad \text{as} \quad \lambda \to +\infty.
\]
Hence from (223) we deduce
\[
\mu(\lambda) = \lambda \left( c_1 - \frac{1}{N} \int_1^\lambda t |\zeta(t)| dt + o(1) \right) + O(1) \quad \text{as} \quad \lambda \to +\infty.
\]
Since \( U \in H_0^1 \) yields \( \int_1^\infty t^{-N-1} |\mu(t)|^2 dt < +\infty \), (224) necessarily implies that \( c_1 = \frac{1}{N} \int_1^\infty \zeta(t) dt \). Then (223) becomes
\[
\mu(\lambda) = \frac{\lambda}{N} \int_1^\lambda \zeta(t) dt + \lambda^{1-N} \left( c_2 - \frac{1}{N} \int_1^\lambda t^{N} \zeta(t) dt \right) \quad \text{for all} \quad \lambda \in (0, +\infty).
\]
The above formula at \( \lambda = 1 \) yields
\[
c_2 = \mu(1) - \frac{1}{N} \int_1^\infty \zeta(t) dt = \int_{\mathbb{R}^N \setminus \Omega} U(\theta) Y_1(\theta) d\sigma(\theta) - \frac{\lambda k_0(D^+)}{N} \int_{\Omega_{-1}} \frac{p(x)U(x)Y_1(x)}{|x|^{N-1}} dx.
\]
Since
\[
|\zeta(\lambda)| \leq \lambda k_0(D^+) \sup_{B_1^c} |p| \sqrt{H_U(\lambda)} \quad \text{for all} \quad \lambda \in (0, 1),
\]
from Lemma 6.3 (i) we deduce that
\[
\zeta(\lambda) = O(\lambda^{1-N}) \quad \text{as} \quad \lambda \to 0^+.
\]
Hence
\[
\frac{\lambda}{N} \int_1^\lambda \zeta(t) dt = O(\lambda^{3-N}) \quad \text{as} \quad \lambda \to 0^+,
\]
t\( ^N \zeta(t) \in L^1(0, 1) \), and \( t^N \zeta(t) = O(t) \) as \( t \to 0^+ \), so that
\[
-\frac{1}{N} \int_\lambda^1 t^{N} \zeta(t) dt = -\frac{1}{N} \int_0^1 t^{N} \zeta(t) dt + \frac{1}{N} \int_0^\lambda t^{N} \zeta(t) dt
\]
\[
= -\frac{\lambda k_0(D^+)}{N} \int_{B_1^c} |x| p(x)U(x)Y_1(\frac{x}{|x|}) dx + O(\lambda^2) \quad \text{as} \quad \lambda \to 0^+.
\]
Combining (225–228) we obtain statement (i).

To prove (ii), let us assume by contradiction that \( \lim_{\lambda \to 0^+} \lambda^{2(N-1)} H_U(\lambda) = 0 \). Since, by Schwarz’s inequality, \( H_U(\lambda) = \int_{\mathbb{R}^N \setminus \Omega} U^2(\theta) d\sigma(\theta) \geq |\mu(\lambda)|^2 \), it would follow that
\[
\lim_{\lambda \to 0^+} \lambda^{N-1} \mu(\lambda) = 0.
\]
Hence (i) would imply that
\[
\int_{\mathbb{R}^N \setminus \Omega} U(\lambda \theta) Y_1(\theta) d\sigma(\theta) = O(\lambda^{3-N}) \quad \text{as} \quad \lambda \to 0^+.
\]
Therefore, letting $U^\lambda$ as in (208) and using Lemma 6.3 (ii) with $q < 2$, we obtain that
\begin{equation}
\int_{S^{N-1}} U^\lambda(\theta)Y_1(\theta)\,d\sigma(\theta) = O(\lambda^{2-q}) \quad \text{as } \lambda \to 0^+.
\end{equation}

From Lemma 6.2 (ii), for every sequence $\lambda_n \to 0^+$ there exist a subsequence $\{\lambda_{n_k}\}_k$ and some constant $c \in \mathbb{R} \setminus \{0\}$ such that
\begin{equation}
U^{\lambda_{n_k}} \to cY_1 \text{ in } L^2(S^{N-1}).
\end{equation}

From (229) and (230) we infer that
\begin{equation*}
0 = \lim_{k \to +\infty} \int_{S^{N-1}} U^{\lambda_{n_k}}(\theta)Y_1(\theta)\,d\sigma(\theta) = c \int_{S^{N-1}} Y_1^2(\theta)\,d\sigma(\theta) = c
\end{equation*}
thus reaching a contradiction and proving statement (ii).

**Proposition 6.5.** Let $U$ be as in Proposition 6.1. Then
\begin{equation}
\lambda^{N-1}U(\lambda x) \xrightarrow{\lambda \to 0^+} \beta \frac{x_1}{|x|^N}
\end{equation}

strongly in $H^1_0$ for every $t > 0$ and in $C^2(B_{t_2}^+ \setminus B_{t_1}^-)$ for all $0 < t_1 < t_2$, where
\begin{equation}
\beta = -\int_{\Gamma_1^-} U(x)Y_1(\frac{x}{|x|})\,dx - \frac{\lambda_{n_k}(D^+)}{N} \int_{D^-} p(x)U(x)Y_1(\frac{x}{|x|})\left(|x|\chi_{B_{t_1}^-}(x) + \frac{\chi_{N-1}(x)}{|x|^{N-1}}\right)\,dx
\end{equation}
and $\Upsilon_N$ is defined in (12).

**Proof.** Let $\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, +\infty)$ such that $\lim_{n \to +\infty} \lambda_n = 0$. Then, from part (ii) of Lemma 6.2 and part (ii) of Lemma 6.4, there exist a subsequence $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$ and some constant $\beta \in \mathbb{R} \setminus \{0\}$ such that
\begin{equation}
\lambda_{n_k}^{N-1}U(\lambda_{n_k} \theta) \xrightarrow{k \to +\infty} \beta \frac{x_1}{|x|^N}
\end{equation}
strongly in $H^1_0$ for every $t > 0$ and in $C^2(B_{t_2}^- \setminus B_{t_1}^+)$ for all $0 < t_1 < t_2$. In particular
\begin{equation*}
\lambda_{n_k}^{N-1}U(\lambda_{n_k} \theta) \xrightarrow{k \to +\infty} \beta \theta_1 \quad \text{in } C^2(S^{N-1}_-)
\end{equation*}
as $k \to +\infty$.

From Lemma 6.4
\begin{align*}
\lim_{k \to +\infty} \lambda_{n_k}^{N-1} \int_{S^{N-1}} U(\lambda_{n_k} \theta)Y_1(\theta)\,d\sigma(\theta) \\
= \int_{\Gamma_1^-} U(x)Y_1(\frac{x}{|x|})\,dx - \frac{\lambda_{n_k}(D^+)}{N} \int_{D^-} p(x)U(x)Y_1(\frac{x}{|x|})\left(|x|\chi_{B_{t_1}^-}(x) + \frac{\chi_{N-1}(x)}{|x|^{N-1}}\right)\,dx,
\end{align*}
thus implying that
\begin{align*}
\beta &= -\int_{\Gamma_1^-} U(x)Y_1(\frac{x}{|x|})\,dx - \frac{\lambda_{n_k}(D^+)}{N} \int_{D^-} p(x)U(x)Y_1(\frac{x}{|x|})\left(|x|\chi_{B_{t_1}^-}(x) + \frac{\chi_{N-1}(x)}{|x|^{N-1}}\right)\,dx \\
&= -\int_{\Gamma_1^-} U(x)Y_1(\frac{x}{|x|})\,dx - \frac{\lambda_{n_k}(D^+)}{N} \int_{D^-} p(x)U(x)Y_1(\frac{x}{|x|})\left(|x|\chi_{B_{t_1}^-}(x) + \frac{\chi_{N-1}(x)}{|x|^{N-1}}\right)\,dx
\end{align*}
\begin{equation*}
\frac{\Upsilon_N}{\int_{\Gamma_1^-} U(x)Y_1(\frac{x}{|x|})\,dx - \frac{\lambda_{n_k}(D^+)}{N} \int_{D^-} p(x)U(x)Y_1(\frac{x}{|x|})\left(|x|\chi_{B_{t_1}^-}(x) + \frac{\chi_{N-1}(x)}{|x|^{N-1}}\right)\,dx}.
\end{equation*}

Hence we have proved that $\beta$ depends neither on the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$, thus implying that the convergence in (232) actually holds as $\lambda \to 0^+$ and proving the proposition.

The following lemmas investigate the sign of the $\beta$ in (231), thus allowing the study of the nodal properties of $u_\varepsilon$ close to the left junction.

**Lemma 6.6.** Let $U$ be as in Proposition 6.1 and $\beta \neq 0$ as in (231). If $\beta > 0$ (respectively $\beta < 0$) then there exists $R > 0$ such that
\begin{equation*}
\text{for every } r \in (0, R) \text{ there exists } \varepsilon_r > 0 \text{ such that } u_\varepsilon < 0 \text{ (respectively } u_\varepsilon > 0) \text{ in } \Gamma_r^- \text{ for all } \varepsilon \in (0, \varepsilon_r).
\end{equation*}
Proof. Let us prove the lemma under the assumption $\beta > 0$ (under the assumption $\beta < 0$ the argument is exactly the same). We claim that

\begin{equation}
(233)
\end{equation}

there exists $R > 0$ such that $U < 0$ in $B_R^{-}$. To prove (233), let us assume by contradiction that there exist $\lambda_n \to 0^+, \theta_n \in S^{N-1}, \bar{\theta} \in \overline{S}^{N-1}$ such that $\theta_n \to \bar{\theta}$ and $U(\lambda_n \theta_n) \geq 0$. If $\bar{\theta} \in S^{N-1}$ then from Proposition 6.5 we obtain that

\begin{equation}
0 \leq \lambda^{N-1} U(\lambda_n \theta_n) = \left( \lambda^{N-1} U(\lambda_n \theta_n) - \beta(\theta_n)_1 \right) + \beta(\theta_n)_1 = \beta \bar{\theta}_1 + o(1) \quad \text{as } n \to +\infty
\end{equation}

which yields a contradiction. On the other hand, if $\bar{\theta} \notin S^{N-1}$, i.e. if $\bar{\theta}_1 = 0$, then, letting $s > 0$ sufficiently small to have $|x|^N - N|x|^{N-2}x^2_1 > c > 0$ for all $x \in A_s := \{x \in B_1^+ \setminus B_1^2 : x_1 > -s\}$, we have that $(t_n, \theta_n) \in A_s$ for all $t \in (\lambda_n(\theta_n)_1, 0)$ and $n$ large. Since from Proposition 6.5

\begin{equation}
\lambda \frac{\partial U}{\partial x_1}(\lambda x) \to \beta \frac{|x|^{N-2} - x_1^2}{|x|^N} \quad \text{in } C^1(A_s)
\end{equation}

we deduce that $\frac{\partial U}{\partial x_1}(\lambda_n x) > 0$ for all $x \in A_s$ and $n$ large. Hence

\begin{equation}
U(\lambda_n \theta_n) = - \int_0^1 \frac{\partial U}{\partial x_1}(t, \lambda_n \theta_n') dt < 0
\end{equation}

thus giving a contradiction. Claim (233) is thereby proved. It remains to prove that

\begin{equation}
(234)
\end{equation}

for every $r \in (0, R)$ there exists $\varepsilon_r > 0$ such that $u_\varepsilon < 0$ in $\Gamma_r^-$ for all $\varepsilon \in (0, \varepsilon_r)$.

To prove (234), let us assume by contradiction that there exist $r \in (0, R)$, $\varepsilon_r \to 0^+$, $\theta_n \in S^{N-1}, \bar{\theta} \in \overline{S}^{N-1}$ such that $\theta_n \to \bar{\theta}$ and $u_\varepsilon(r \theta_n) \geq 0$ (and hence $U_\varepsilon(r \theta_n) \geq 0$). If $\bar{\theta} \notin S^{N-1}$ then from Proposition 6.1 it follows that

\begin{equation}
0 \leq U_\varepsilon(r \theta_n) = \left( U_\varepsilon(r \theta_n) - U(r \theta_n) \right) + U(r \theta_n) = U(r \theta) + o(1) \quad \text{as } n \to +\infty
\end{equation}

which contradicts (233). On the other hand, if $\bar{\theta} \in S^{N-1}$, then by Hopf’s Lemma $\frac{\partial U_\varepsilon}{\partial x_1}(r \bar{\theta}) > 0$. If $t \in (r(\theta_n)_1, 0)$, Proposition 6.1 yields

\begin{equation}
\frac{\partial U_\varepsilon}{\partial x_1}(t, r \theta_n') = \left( \frac{\partial U_\varepsilon}{\partial x_1}(t, r \theta_n') - \frac{\partial U}{\partial x_1}(t, r \theta_n') \right) + \frac{\partial U}{\partial x_1}(t, r \theta_n') = \frac{\partial U}{\partial x_1}(r \theta) + o(1)
\end{equation}

as $n \to +\infty$, so that

\begin{equation}
\frac{\partial U_\varepsilon}{\partial x_1}(t, r \theta_n') > 0
\end{equation}

provide $n$ is sufficiently large. Therefore

\begin{equation}
U_\varepsilon(r \theta_n) = - \int_{r(\theta_n)_1}^0 \frac{\partial U_\varepsilon}{\partial x_1}(t, r \theta_n') dt < 0
\end{equation}

leads to a contradiction proving claim (234). \hfill \Box

In fact, condition (6) forces the sign of $\beta$ to be negative, as we show below.

**Lemma 6.7.** Let $U$ be as in Proposition 6.1 and $\beta \neq 0$ as in (231). Then $\beta < 0$.

Proof. Let us assume by contradiction that $\beta > 0$. From Lemma 6.6, for every $n$ (sufficiently large), there exists $\varepsilon_n \in (0, 1/n)$ such that

\begin{equation}
(235)
\end{equation}

$u_\varepsilon < 0$ on $\Gamma_{1/n}^-$. Let us denote $u_{\varepsilon_n} := \max\{0, -u_\varepsilon\}$. From Lemma 2.13, $u_{\varepsilon_n} = 0$ on $\partial \Omega_1^{1+2e_n}$. Therefore, letting

\begin{equation}
v_n := \begin{cases} u_{\varepsilon_n}, & \text{in } \Omega_{-1/n}, \\ -u_{\varepsilon_n}, & \text{in } \Omega_1^{1+2e_n} \setminus \Omega_{-1/n}, \\ 0, & \text{in } \mathbb{R}^N \setminus \Omega_1^{1+2e_n}, \end{cases}
\end{equation}

(235) ensures that $v_n \in D^{1,2}(\Omega_1^{1+2e_n}) \subset D^{1,2}(\mathbb{R}^N), v_n \not\equiv 0$ in $D^\text{c}$. 

Testing equation $-\Delta u_n = \lambda_k^o n u_n$ with $v_n$, we obtain $\int_{\Omega_{1+2n}} |\nabla v_n|^2 dx = \lambda_k^o \int_{\Omega_{1+2n}} p v_n^2 dx$, hence defining

$$w_n := \frac{v_n}{\sqrt{\int_{\Omega_{1+2n}} p v_n^2 dx}},$$

we have that $w_n \in D^{1,2}(\Omega_{1+2n}) \subset D^{1,2}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} p w_n^2 dx = \int_{\Omega_{1+2n}} p w_n^2 dx = 1, \quad \int_{\mathbb{R}^N} |\nabla w_n|^2 dx = \int_{\Omega_{1+2n}} |\nabla w_n|^2 dx = \lambda_k^o.$$

Hence $\{w_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and there exists a subsequence $\{w_{n_k}\}_k$ such that $w_{n_k} \rightharpoonup w$ weakly in $D^{1,2}(\mathbb{R}^N)$ and $w_{n_k} \to w$ a.e. in $\mathbb{R}^N$, for some $w \in D^{1,2}(\mathbb{R}^N)$. Since $\text{supp } w_n \subset \Omega_{1+2n}$, a.e. convergence implies that $\text{supp } w \subset D^-$ so that $w \in D^{1,2}(D^-)$. From $\int_{\mathbb{R}^N} p w_n^2 dx = 1$ we deduce that $\int_{D^-} p w^2 dx = 1$ which implies that $w \neq 0$. Since $w_n$ solves

$$\begin{cases}
-\Delta w_n = \lambda_k^o n u_n, & \text{in } \Omega_{-1/n}, \\
 w_n = 0, & \text{on } \partial\Omega_{-1/n} \cap \partial D^-,
\end{cases}$$

weak convergence and (7) imply that $w$ weakly solves

$$\begin{cases}
-\Delta w = \lambda_k (D^+) p w, & \text{in } D^-, \\
 w = 0, & \text{on } \partial D^-,
\end{cases}$$

thus implying $\lambda_k (D^+) \in \sigma_p(D^-)$ and contradicting assumption (5).

The proofs of the main results of the paper follow by combining the previous results.

**Proof of Theorem 1.2.** It follows by combining Propositions 6.1, 6.5 and Lemma 6.7.

**Proof of Corollary 1.3.** It follows from Lemmas 6.6 and 6.7.

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