A quantitative obstruction to collapsing surfaces

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Abstract: We provide a quantitative obstruction to collapsing surfaces of genus at least 2 under a lower curvature bound and an upper diameter bound.

Keywords: curvature, diameter, volume, filling radius, systole, Gromov-Hausdorff distance

1 Introduction

S. Alesker posed the following question at MathOverflow [1]. Let $(M_i)$ be a sequence of 2-dimensional orientable closed surfaces of genus $g \geq 2$ endowed with smooth Riemannian metrics of Gaussian curvature at least $-1$ and diameter at most $D$. By the Gromov compactness theorem, one can choose a subsequence converging in the Gromov-Hausdorff (GH) sense to a compact Alexandrov space with curvature at least $-1$ and Hausdorff dimension 0, 1, or 2. Let us assume that the limit space has dimension 1. Then it is either a circle or a segment. Can these possibilities (circle and segment) be obtained in the limit $M$ of $(M_i)$? We show that these possibilities cannot occur, and quantify this statement by providing an explicit lower bound for the filling radius of $M$. For related results see [2].

2 Impossibility of collapse

We prove the impossibility of collapse in dimension 2, in the following sense.

Theorem 2.1. The distance between a strongly isometric map from a closed orientable surface $M$ of genus $g \geq 2$ of Gaussian curvature $K \geq -1$ and diameter at most $D$ to a metric space $Z$, and a map from $M$ to a graph in $Z$, is at least $\frac{\pi(g-1)}{3 \sin D}$.

Thus we obtain a quantitative lower bound rather than merely the nonexistence of Shioya-Yamaguchi-type collapse to spaces of positive codimension (see [3, 4]).

Corollary 2.2. Let $D > 0$. GH limits of metrics on a closed orientable surface of genus $g \geq 2$ with Gaussian curvature at least $-1$ and diameter at most $D$ are necessarily 2-dimensional.

Recall that the systole of a Riemannian manifold $M$ is the least length of a noncontractible loop of $M$. For an overview of systolic geometry see [5].
The filling radius \( \text{FillRad} M \) of a closed \( n \)-dimensional manifold \( M \) is defined as the infimum of all \( c > 0 \) such that the inclusion of \( M \) in its \( c \)-neighborhood in any strongly isometric embedding of \( M \) in a Banach space sends the fundamental homology class \([M]\) of \( M \) to the zero class, by means of the induced homomorphism on \( H_n(M) \). Here the embedding can be taken to be into the space of bounded functions on \( M \) which sends a point \( p \in M \) to the distance function from \( p \). This embedding is strongly isometric (ambient distance restricted to \( M \) coincides with intrinsic distance on \( M \)) if the function space is equipped with the sup-norm.

**Lemma 2.3** (Gromov’s lemma). The systole of an aspherical manifold \( M \) is at most six times the filling radius of \( M \).

**Proof.** Consider a strongly isometric embedding of the surface \( M \) into a Banach space \( B \). The space \( B \) can be assumed finite-dimensional if the metric condition is relaxed to a requirement of being bilipschitz with to a bilipschitz factor arbitrarily close to 1; see [6]. Suppose \( M \) is “filled” (in the homological sense) by a chain \( C \) (in the sense that \( M \) is the boundary of \( C \)). Then the induced homomorphism \( H_n(M) \to H_n(C) \) sends \([M]\) to the zero class. Consider a triangulation of \( C \) into infinitesimal simplices (here the term “infinitesimal” is used informally in its meaning “sufficiently small” though this could be rendered rigorous as in [7]).

We argue by contradiction. Let \( R > 0 \) be strictly smaller than a sixth of the systole. Suppose the chain \( C \) is contained in an open \( K \)-neighborhood of \( M \) in \( B \). We will retract \( C \) back to \( M \), while fixing the subset \( M \subseteq C \), contradicting the fact that the nonvanishing fundamental class \([M]\) is sent to a zero class in \( C \).

For each vertex of the triangulation of \( C \), we choose a nearest point of \( M \). To extend the retraction to the 1-skeleton of \( C \), we map each edge (of a triangle of the triangulation) to a minimizing path joining the images of the two vertices in \( M \). The length of such a minimizing path is less than \( 2R \) (plus the infinitesimal side length of the triangle) by the triangle inequality. Hence the boundary of each 2-cell of the triangulation is sent to a loop of length at most \( 6R \) (plus an infinitesimal). Since this length is less than the systole of \( M \), the map can now be extended to the 2-skeleton of \( C \).

To extend the map to the 3-skeleton, note that the universal cover of \( M \) is contractible and hence \( \pi_2(M) = 0 \), and similarly for the higher homotopy groups. Therefore the skeletal retraction extends to all of \( C \) inductively. The contradiction completes the proof of the lemma.

**Proof of Theorem 2.1.** We exploit Gromov’s notion of the filling radius of a manifold [8]. The argument relies only on basic Jacobi field estimates and basic homotopy theory. We seek a suitable lower bound so as to rule out positive-codimension collapse. Choose a noncontractible closed geodesic \( \gamma \subseteq M \) of length equal to the systole \( \text{sys}(M) \). Consider the normal exponential map along \( \gamma \). Using the lower curvature bound, we obtain an upper bound on the total area of \( M \) as \( 2 \text{sys}(M) \sinh(D) \), where \( D \) is the diameter. The bound follows by applying Rauch bounds on Jacobi fields (this is an ingredient in the proof of Toponogov’s theorem); see e.g., Cheeger-Ebin [9, Theorem 5.8, pp. 97–98]. The bound results from comparison with the area of a hyperbolic collar of width \( D \) around a closed geodesic of the same length as \( \gamma \). Therefore, the systole is bounded below as follows:

\[
\text{sys}(M) \geq \frac{\text{area}(M)}{2 \sinh D}. \tag{2.1}
\]

Meanwhile the area is bounded below by the Gauss-Bonnet theorem:

\[
\text{area}(M) \geq \int_M K = 2\pi(2g - 2),
\]

where \( g \) is the genus. Furthermore the filling radius of \( M \) is bounded below by a sixth of the systole by Gromov’s Lemma 2.3. Therefore the bound (2.1) implies

\[
\text{FillRad}(M) \geq \frac{1}{6} \text{sys}(M) \geq \frac{\text{area}(M)}{12 \sinh D} \geq \frac{\pi(g - 1)}{3 \sinh D}. \tag{2.2}
\]

The theorem now follows from the fact the distance between a strongly isometric map from \( M \) to a metric space \( Z \) and a map from \( M \) to a graph in \( Z \) is bounded below by the filling radius; see e.g., [8, p. 127, Example].

This
proves that aspherical surfaces of curvature bounded below by $-1$ with diameter bounded above by $D$ cannot collapse, so that a GH limit is necessarily 2-dimensional as follows.

To prove Corollary 2.2, note that if a metric on $M$ is sufficiently close to a finite graph $\Gamma$ in the sense of the GH distance, then the construction of the proof of Lemma 2.3 produces a map from $M$ to $\Gamma$ which is close to the embedding of $M$ in $Z$, contradicting the lower bound (2.2).

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