Non-bipartite distance-regular graphs with a small smallest eigenvalue

Zhi Qiao\textsuperscript{a}, Yifan Jing\textsuperscript{b}, and Jack Koolen\textsuperscript{a,c}

\textsuperscript{a}School of Mathematical Sciences, Sichuan Normal University, 610068, Sichuan, PR China
\textsuperscript{b}Department of Mathematics, University of Illinois at Urbana Champaign, Urbana, IL, 61801, USA
\textsuperscript{c}School of Mathematical Sciences, University of Science and Technology of China, Wen-Tsun Wu Key Laboratory of the Chinese Academy of Sciences, 230026, Anhui, PR China

Abstract

In 2017, Qiao and Koolen showed that for any fixed integer $D \geq 3$, there are only finitely many such graphs with $\theta_{\text{min}} \leq -\alpha k$, where $0 < \alpha < 1$ is any fixed number. In this paper, we will study non-bipartite distance-regular graphs with relatively small $\theta_{\text{min}}$ compared with $k$. In particular, we will show that if $\theta_{\text{min}}$ is relatively close to $-k$, then the odd girth $g$ must be large. Also we will classify the non-bipartite distance-regular graphs with $\theta_{\text{min}} \leq \frac{D-1}{D}$ for $D = 4, 5$.

Key words: Distance-regular graphs, Smallest eigenvalue, Odd girth

AMS classification: 05C75, 05E30, 05C50

1 Introduction

The odd girth of a non-bipartite graph is the length of its shortest odd cycle. Let $\Gamma$ be a non-bipartite distance-regular graph with valency $k$, diameter $D$, odd girth $g$ and smallest eigenvalue $\theta_{\text{min}}$. In \cite{6}, Qiao and Koolen showed that for any fixed integer $D \geq 3$, there are only finitely many such graphs with $\theta_{\text{min}} \leq -\alpha k$, where $0 < \alpha < 1$ is any fixed number. In this paper, we will study non-bipartite distance-regular graphs with relatively small $\theta_{\text{min}}$ compared with $k$. In the next result, we will show that if $\theta_{\text{min}}$ is relatively close to $-k$, then the odd girth $g$ must be large.

\textsuperscript{*}Corresponding author
E-mail addresses: zhiqiao@sicnu.edu.cn (Z. Qiao), yifanjing17@gmail.com (Y. Jing), koolen@ustc.edu.cn (J. Koolen)
Theorem 1.1. Let $\Gamma$ be a non-bipartite distance-regular graph with valency $k$ and odd girth $g$, having smallest eigenvalue $\theta_{\text{min}}$. Then there exists a constant $\varepsilon(g) > 0$ such that $\theta_{\text{min}} \geq -(1 - \varepsilon(g))k$.

Remark 1.2. The positive constant $\varepsilon(g)$ goes to 0 as the odd girth $g$ goes to $\infty$. For example, the $(2t + 1)$-gon has valency $k = 2$, odd girth $g = 2t + 1$ and smallest eigenvalue $\theta_{\text{min}} = 2 \cos \left( \frac{2t\pi}{2t+1} \right)$. Thus, $\varepsilon(g) \leq 1 + \frac{\theta_{\text{min}}}{k} = 2 \cos \left( \frac{t\pi}{2t+1} \right)$.

In [6], Qiao and Koolen classified non-bipartite distance-regular graphs with valency $k$, diameter $D \leq 3$ and smallest eigenvalue $\theta_{\text{min}} \leq -k/2$. Using Theorem 1.1, we will classify non-bipartite distance-regular graphs with valency $k$, diameter $D$ and smallest eigenvalue $\theta_{\text{min}} \leq -\frac{D-1}{D}k$, when $D = 4$ or 5.

Theorem 1.3. Let $\Gamma$ be a non-bipartite distance-regular graph with valency $k$, diameter $D$ and smallest eigenvalue $\theta_{\text{min}} \leq -\frac{D-1}{D}k$.

i) If $D = 4$, then $\Gamma$ is one of the following graph

a) the Coxeter graph with intersection array $\{3, 2, 2, 1; 1, 1, 1, 2\}$,

b) the 9-gon with intersection array $\{2, 1, 1, 1; 1, 1, 1, 1\}$,

c) the Odd graph $O_5$ with intersection array $\{5, 4, 4, 3; 1, 1, 2, 2\}$,

d) the folded 9-cube with intersection array $\{9, 8, 7, 6; 1, 2, 3, 4\}$.

ii) If $D = 5$, then $\Gamma$ is one of the following graph

a) the 11-gon with intersection array $\{2, 1, 1, 1, 1; 1, 1, 1, 1\}$,

b) the Odd graph $O_6$ with intersection array $\{6, 5, 5, 4, 4; 1, 1, 2, 2, 3\}$,

c) the folded 11-cube with intersection array $\{11, 10, 9, 8, 7; 1, 2, 3, 4, 5\}$.

This paper is organized as follows. In the next section, we give the definitions and some preliminary results. In section 3 we give a proof of Theorem 1.1. In the last section, we give a proof of Theorem 1.3.

2 Preliminaries

For more background, see [4] and [7].

All the graphs considered in this paper are finite, undirected and simple. Let $\Gamma$ be a graph with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. Denote $x \sim y$ if the vertices $x, y \in V$ are adjacent. The distance $d(x, y) = d_\Gamma(x, y)$ between two vertices $x, y \in V(\Gamma)$ is the length of a shortest path connecting $x$ and $y$. The maximum distance between two vertices in $\Gamma$ is the diameter $D = D(\Gamma)$. We use $\Gamma_i(x)$ for the set of vertices at distance $i$ from $x$ and write, for the sake of simplicity, $\Gamma(x) := \Gamma_1(x)$. The degree of $x$ is the number $|\Gamma(x)|$ of vertices adjacent to it. A graph is regular with valency $k$ if the degree of each of
its vertices is $k$. The *girth* and *odd girth* of a graph is the length of its shortest cycle, and shortest odd cycle, respectively. A graph is called *bipartite* if it has no odd cycle.

A connected graph $\Gamma$ with diameter $D$ is called *distance-regular* if there are integers $b_i$, $c_i$ ($i = 0, 1, \ldots, D$) such that for any two vertices $x, y \in V(\Gamma)$ with $d(x, y) = i$, there are exactly $c_i$ neighbors of $y$ in $\Gamma_{i-1}(x)$ and $b_i$ neighbors of $y$ in $\Gamma_{i+1}(x)$, where we define $b_D = c_0 = 0$. In particular, $\Gamma$ is a regular graph with valency $k := b_0$. We define $a_i := k - b_i - c_i$ ($i = 0, 1, \ldots, D$) for notational convenience. Note that $a_i = |\Gamma(y) \cap \Gamma_i(x)|$ holds for any two vertices $x, y$ with $d(x, y) = i$ ($i = 0, 1, \ldots, D$).

For a distance-regular graph $\Gamma$ and a vertex $x \in V(\Gamma)$, we denote $k_i := |\Gamma_i(x)|$ and $p_{ij}^h := |\{w \mid w \in \Gamma_i(x) \cap \Gamma_j(y)\}|$ for any $y \in \Gamma_h(x)$. It is easy to see that $a_i$, $b_i$ and $c_i$ ($i = 0, 1, \ldots, D$) are called the *intersection numbers*, and the array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the *intersection array* of $\Gamma$. The matrix $L$ is called the *intersection matrix* of $\Gamma$, where

$$L = \begin{pmatrix}
  a_0 & b_0 & 0 & 0 & \cdots & 0 \\
  c_1 & a_1 & b_1 & 0 & \cdots & 0 \\
  c_2 & a_2 & \ddots & \ddots & \ddots & 0 \\
  0 & \cdots & \cdots & c_{D-1} & a_D
\end{pmatrix}.$$

Let $\Gamma$ be a distance-regular graph with $v$ vertices and diameter $D$. Let $A_i$ ($i = 0, 1, \ldots, D$) be the $(0, 1)$-matrix whose rows and columns are indexed by the vertices of $\Gamma$ and the $(x, y)$-entry is 1 whenever $d(x, y) = i$ and 0 otherwise. We call $A_i$ the *distance-$i$ matrix* and $A := A_1$ the *adjacency matrix* of $\Gamma$. The eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$ of the graph $\Gamma$ are just the eigenvalues of its adjacency matrix $A$. We denote $m_i$ the *multiplicity* of $\theta_i$. Note that the $D + 1$ distinct eigenvalues of $\Gamma$ are precisely the eigenvalues of $L$ (see [17 Proposition 2.7]).

For each eigenvalue $\theta_i$ of $\Gamma$, let $U_i$ be a matrix with its columns forming an orthonormal basis for the eigenspace associated with $\theta_i$. And $E_i := U_i^*U_i^T$ is called the *minimal idempotent* associated with $\theta_i$, satisfying $E_i E_j = \delta_{ij} E_j$ and $AE_i = \theta_i E_i$, where $\delta_{ij}$ is the Kronecker delta. Note that $vE_0$ is the all-ones matrix $J$.

The set of distance matrices $\{A_0 = I, A_1, A_2, \ldots, A_D\}$ forms a basis of a commutative $\mathbb{R}$-algebra $\mathcal{A}$, known as the *Bose-Mesner algebra*. The set of minimal idempotents $\{E_0 = \frac{1}{v} J, E_1, E_2, \ldots, E_D\}$ is another basis of $\mathcal{A}$. There exist $(D + 1) \times (D + 1)$ matrices $P$ and $Q$ (see [24 p.45]), such that the following relations hold

$$A_i = \sum_{j=0}^D P_{ji} E_j \quad \text{and} \quad E_i = \frac{1}{v} \sum_{j=0}^D Q_{ji} A_j \quad (i = 0, 1, \ldots, D).$$

Note that $Q_{0i} = m_i$ (see [24 Lemma 2.2.1]).

Let $E_i = U_i^*U_i^T$ be the minimal idempotent associated with $\theta_i$, where the columns of $U_i$ form an orthonormal basis of the eigenspace associated with $\theta_i$. We denote the $x$-th row
of $\sqrt{v/m_i}U_i$ by $\hat{x}$. Note that $E_i \circ A_j = \frac{1}{m_i}Q_jA_j$, hence all the vectors $\hat{x}$ are unit vectors and the cosine of the angle between two vectors $\hat{x}$ and $\hat{y}$ is $u_j(\theta) := \frac{Q_ju_i}{Q_0u_i}$, where $d(x, y) = j$. The map $x \mapsto \hat{x}$ is called a normalized representation and the sequence $(u_j(\theta))_{j=0}^D$ is called the standard sequence of $\Gamma$, associated with $\theta_i$. As $AU_i = \theta_iU_i$, we have $\theta_i\hat{x} = \sum_{y \sim x} \hat{y}$, and hence the following holds:

$$
\begin{cases}
  c_ju_{j-1}(\theta) + a_ju_j(\theta) + b_ju_{j+1}(\theta) = \theta_iu_j(\theta) & (j = 1, 2, \ldots, D-1), \\
  c_Du_{D-1}(\theta) + a_Du_D(\theta) = \theta_iu_D(\theta),
\end{cases}
$$

(2)

with $u_0(\theta_i) = 1$ and $u_1(\theta_i) = \frac{\theta_i}{k}$.

**Lemma 2.1.** (c.f. [7, Theorem 2.8]) Let $\Gamma$ be a distance-regular graph with diameter $D$ and $v$ vertices. Let $\theta$ be an eigenvalue of $\Gamma$ and $(u_i)_{i=0}^D$ be the standard sequence associated with $\theta$. Then the multiplicity $m(\theta)$ of $\theta$ as an eigenvalue of $\Gamma$ satisfies

$$m(\theta) = \frac{v}{\sum_{i=0}^D k_iu_i^2},$$

(3)

$$\leq \max\left\{ \frac{1}{u_1^2}, \ldots, \frac{1}{u_{j-1}^2}, \frac{1}{j\sum_{i=0}^{j-1} k_iu_i^2} \right\} \ (j = 1, 2, \ldots, D).$$

(4)

**Proof.** We only give a proof of Equation (4).

$$\frac{v}{\sum_{i=0}^D k_iu_i^2} = \frac{\sum_{i=0}^D k_iu_i^2}{\sum_{i=0}^D k_iu_i^2} \leq \max\left\{ \frac{\sum_{i=0}^{j-1} k_i}{\sum_{i=0}^j k_iu_i^2}, \frac{\sum_{i=j}^D k_i}{\sum_{i=j}^D k_iu_i^2} \right\},$$

$$\frac{\sum_{i=0}^{j-1} k_i}{\sum_{i=0}^j k_iu_i^2} \leq \max\left\{ \frac{1}{u_1^2}, \ldots, \frac{1}{u_{j-1}^2} \right\},$$

$$\frac{\sum_{i=j}^D k_i}{\sum_{i=j}^D k_iu_i^2} \leq \frac{\sum_{i=j}^D k_i}{k_iu_i^2}. $$

\[\square\]

**Lemma 2.2.** [4, Proposition 4.1.6] Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D$. Then the following conditions hold

i) $1 = c_1 \leq c_2 \leq \cdots \leq c_D$,

ii) $k = b_0 \geq b_1 \geq \cdots \geq b_{D-1}$,

iii) $k_i$’s ($i = 1, 2, \ldots, D$) are positive integers,

iv) the multiplicities are positive integers.
Lemma 2.3. (c.f. [2 Proposition 3.1]) Let $\Gamma$ be a non-bipartite distance-regular graph with valency $k$ and odd girth $g = 2t + 1$. Then

$$\sum_{i=0}^{t} p_i(\eta) u_i \geq 0,$$

where $(u_i)_{i=0}^{D}$ is the standard sequence associated with the smallest eigenvalue $\theta_{\min}$, $\eta$ is any eigenvalue of the $g$-gon, and $p_i(x)$ is defined as the following

\begin{align*}
p_0(x) &= 1, \\
p_1(x) &= x, \\
p_2(x) &= x^2 - 2, \\
p_i(x) &= x p_{i-1}(x) - p_{i-2}(x) \quad (i = 3, 4, \ldots, t).
\end{align*}

Proof. Let $\Delta$ be any $g$-gon in $\Gamma$. Let $B_i$ $(i = 0, 1, \ldots, t)$ be the matrix with rows and columns indexed by $V(\Delta)$, where the $(v, w)$-entry is 1 whenever $d_{\Gamma}(v, w) = i$ and 0 otherwise. Note that $d_{\Delta}(v, w) = d_{\Gamma}(v, w)$ for any two vertices $v, w \in V(\Delta)$, and we have $B_i = p_i(B_1)$ with $p_i(x)$ as Equation (6). By [2 Proposition 3.1], for any eigenvalue $\eta$ of $\Delta$, we have Equation (5). \hfill \square

Lemma 2.4. (c.f. [6 Lemma 5.2]) Let $\Gamma$ be a distance-regular graph with valency $k$ and smallest eigenvalue $\theta_{\min}$. If $a_1 = 0$ and $\theta_{\min} < \frac{12 - 5k}{7}$, then $c_2 \leq 2$.

Proof. Choose two vertices $x, y \in V(\Gamma)$ with $d(x, y) = 2$. As $a_1 = 0$, the subgraph induced on $x, y \cup \Gamma(x) \cup \Gamma(y)$ is a $K_{2, c_2}$. Let $x \mapsto \hat{x}$ be a normalized representation associated with $\theta = \theta_{\min}$. Consider the Gram matrix of the image of the $K_{2, c_2}$ with the bipartition, we see that

$$Q = \begin{pmatrix}
\frac{1}{2}(1 + u_2)
& u_1 \\
\frac{1}{c_2}(1 + (c_2 - 1) u_2)
\end{pmatrix}$$

is positive semidefinite, by [4 Proposition 3.7.1 (iii)]. Then $(1, 1)Q(1, 1)^t \geq 0$, which in turn implies $(u_1 + u_2)((2 + c_2) \frac{1 - u_2}{u_1 + u_2} + 4 c_2) \geq 0$. As $a_1 = 0$, we see $u_1 + u_2 = \frac{(\theta + k)(\theta - 1)}{k(k - 1)} < 0$, that is

$$\frac{4c_2}{2 + c_2} \leq - \frac{1 - u_2}{u_1 + u_2} = \frac{k - \theta}{1 - \theta}.$$  

When $k > 1$, we have $\theta < \frac{12 - 5k}{7} < \frac{4 - k}{3}$ and $c_2 \leq \frac{2k - 2\theta}{4 - 3\theta - k} < 3$. \hfill \square

Lemma 2.5. Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D$, having smallest eigenvalue $\theta_{\min}$ with associated standard sequence $(u_i)_{i=0}^{D}$. Then

$$|u_{i+1}| \geq \frac{|(\theta_{\min} - a_i) u_i| - c_i |u_{i-1}|}{b_i} \quad (i = 0, 1, \ldots, D - 1).$$

Proof. By Equation (2) we see that $u_i = \frac{(\theta_{\min} - a_i) u_i}{b_i}$ $(i = 1, 2, \ldots, D)$. As $\theta_{\min} < 0$ is the smallest eigenvalue, by [4 Corollary 4.1.2], we see that $u_{i+1}, -u_i$ and $u_{i-1}$ has the same sign. The result follows. \hfill \square
3 \ Main Theorem

In this section we will prove our main result.

Proof of Theorem 1.1

If \( g = 3 \), then \( \theta_{\min} \geq -\frac{k}{2} \) by [7] Proposition 2.11. So we may assume \( g \geq 5 \). Let \( t = \frac{g-1}{2} \) and \( \Delta \) be a \( g \)-gon in \( \Gamma \). Let \( (u_i)_{i=0}^t \) be the standard sequence associated with the smallest eigenvalue \( \theta = \theta_{\min} \).

Assume \( c_i \leq \zeta k \) for some \( \zeta \leq \frac{1}{2} \). By Lemma 2.3 we have \( \sum_{i=0}^t p_i(\eta)u_i \geq 0 \), where \( p_i(x) \) is as Equation (6).

We claim that there exist constants \( N_i \) such that
\[
|u_i - \left( \frac{\theta}{k} \right)^i| \leq N_i \zeta \quad (i = 0, 1, \ldots, D).
\]
(8)

Note that \( u_0 = 1 \) and \( u_1 = \frac{\theta}{k} \). Assume \( |u_i - \left( \frac{\theta}{k} \right)^i| \leq N_i \zeta \) for some \( 1 \leq i \leq t-1 \). As \( c_i \leq \zeta k \), we see that \( c_i \leq c_t \leq \zeta k \) and \( b_i = 1 - c_i \geq (1 - \zeta)k \). Then
\[
|u_{i+1} - \left( \frac{\theta}{k} \right)^{i+1}| = \left| \frac{\theta}{b_i} u_i - \left( \frac{\theta}{k} \right)^i \right| + \left| \frac{\theta}{b_i} \left( \frac{\theta}{k} \right)^i - \left( \frac{\theta}{k} \right)^{i+1} \right| + \frac{c_i}{b_i} |u_{i-1}|
\]
\[
\leq \left| \frac{\theta}{b_i} u_i - \left( \frac{\theta}{k} \right)^i \right| + \frac{\theta}{b_i} \left( \frac{\theta}{k} \right)^i + \frac{c_i}{b_i} |u_{i-1}|
\]
\[
\leq \left| \frac{\theta}{b_i} N_i \zeta + \frac{c_i}{b_i} |u_{i-1}| \right|
\]
\[
\leq (2N_i + 4) \zeta,
\]
where \( \frac{k}{b_i} \leq \frac{1}{\zeta} \leq 2 \) and \( \frac{c_i}{b_i} \leq \frac{\zeta}{\zeta} \leq 2 \zeta \) (\( \zeta \geq \frac{1}{2} \)). So we may take \( N_0 = N_1 = 0 \) and \( N_i = 2N_{i-1} + 4 \) (\( i = 2, 3, \ldots, t \)).

Note that \( p_i(\eta) \) is an eigenvalue of the distance-\( i \) graph of \( \Delta \). Hence \( |p_i(\eta)| \leq 2 \) (\( i = 0, 1, \ldots, t \)), and by Equation (8), we have
\[
\sum_{i=0}^t p_i(\eta)u_i \leq \sum_{i=0}^t p_i(\eta)\left( \frac{\theta}{k} \right)^i + t |p_i(\eta)| \cdot |u_i - \left( \frac{\theta}{k} \right)^i|
\]
\[
\leq \sum_{i=0}^t p_i(\eta)\left( \frac{\theta}{k} \right)^i + M_1 \zeta,
\]
\[
(9)
\]
where \( M_1 = \sum_{i=0}^t 2N_i \).

By Equation (8), we see that \( p_i(x) = \lambda_i^1 + \lambda_i^2 \) (\( i = 1, 2, \ldots, t \)), with \( \lambda_i = \frac{1}{2}(x \pm \sqrt{x^2 - 4}) \) (\( i = 1, 2 \)). Define \( f(x, y) = \sum_{i=0}^t p_i(x)y^i = \frac{1-(\lambda_1y)^{i+1}}{1-\lambda_1y} + \frac{1-(\lambda_2y)^{i+1}}{1-\lambda_2y} - 1 \). Note that the eigenvalues of \( \Delta \) are \( 2 \cos \frac{2\pi i}{g} \) (\( i = 0, 1, \ldots, g-1 \)). Take \( \eta = 2 \cos \frac{2\pi(t-1)}{g} \), then we see
\[
f(\eta, -1) = -M_2,
\]
\[
(10)
\]
where \( M_2 = 1/\cos\left(\frac{(t-1)\pi}{g}\right) \). In fact,
\[
f(2\cos\frac{2\pi j}{g}, -1) = \frac{1 - \left(-\frac{e^{2\pi i \frac{j}{g}}}{g}\right)^{t+1}}{1 - \left(-\frac{e^{-2\pi i \frac{j}{g}}}{g}\right)} + \frac{1 - \left(-\frac{e^{-2\pi i \frac{j}{g}}}{g}\right)^{t+1}}{1 - \left(-\frac{e^{2\pi i \frac{j}{g}}}{g}\right)} - 1
\]
\[
= (-1)^t \cdot \frac{e^{2\pi i \frac{j(t+1)}{g}}} {1 + e^{2\pi i \frac{j}{g}}}
\]
\[
= (-1)^{t+j}/\cos\frac{j\pi}{g}.
\]

Take \( \zeta = \min\{\frac{M_1}{2M_1}, \frac{1}{2}\} \). Note that \( M_1, M_2 \), and hence \( \zeta \) is determined by \( g \). By Equation\( [\text{10}] \), we see \( f(\eta, -1) + M_1 \zeta \leq -\frac{M_2}{2} < 0 \). We also have \( f(\eta, 0) + M_1 \zeta = 1 + M_1 \zeta > 0 \). By Equation\( [\text{6}] \) and \( [\text{3}] \), we have \( 0 \leq f(\eta, \frac{\theta}{2}) + M_1 \zeta \). Take \( -(1 - \varepsilon_1(\zeta)) \) as the smallest root \( y \) of the equation \( f(\eta, y) + M_1 \zeta = 0 \) in the interval \((-1, 0)\). It follows that \( \theta \geq -(1 - \varepsilon_1(\zeta))k \).

Now we consider the case \( c_t > \zeta k \).

If \( c_t > 1 \), then we claim that the diameter \( D \leq \frac{4\pi}{\zeta} \) and \( \theta_{\min} \geq -(1 - \varepsilon_2(\zeta))k \) for some constant \( \varepsilon_2(\zeta) > 0 \). Without loss of generality, we may assume \( 4t \leq D \leq 4t+1 \) for some integer \( t \geq 1 \). If \( c_{2t-1+j} = c_{2t-1} \neq 1 \), by [7, Theorem 7.1], we see \( j \leq 2t - 1 \), that is \( c_{2t-1} \geq c_{2t-1+j} \). Then \( c_{2t-1} = c_{2t+j} > c_{2t-1+j} \) for some \( 0 \leq j \leq 2t - 1 \), and \( c_{2t-1} \geq 2c_t \) by [7, Proposition 7.2]. This implies \( k \geq c_{2t} \geq 2c_t \), that is \( D \leq 4t(\frac{2\pi}{\zeta})^2 \leq \frac{4\pi}{\zeta} \). Then by [6, Theorem 1.1], the set \( S \) of distance-regular graphs with valency \( k \), diameter \( D \leq \frac{4\pi}{\zeta} \), smallest eigenvalue \( \theta_{\min} \leq -(1 - \varepsilon_1(\zeta))k \) and odd girth \( g \) is finite. Take
\[
\varepsilon_2(\zeta) = \begin{cases} 
\frac{\min_{r \in S} \frac{k + \theta_{\min}}{k}}{\varepsilon_1(\zeta)}, & \text{if } S \neq \emptyset, \\
\varepsilon_1(\zeta), & \text{otherwise}.
\end{cases}
\]

If \( c_t = 1 \), then \( k < \frac{1}{2} \). The set \( S' \) of distance-regular graphs with valency \( k < \frac{1}{2} \) and odd girth \( g \) is finite, by [1, Theorem 1.1]. Take
\[
\varepsilon_3(\zeta) = \begin{cases} 
\frac{\min_{r \in S'} \frac{k + \theta_{\min}}{k}}{\varepsilon_1(\zeta)}, & \text{if } S' \neq \emptyset, \\
\varepsilon_1(\zeta), & \text{otherwise}.
\end{cases}
\]

Take \( \varepsilon = \min\{\varepsilon_1(\zeta), \varepsilon_2(\zeta), \varepsilon_3(\zeta)\} \) and the result follows.

\[\square\]

**Remark 3.1.** When the odd girth \( g = 5 \) and \( c_2 \leq \zeta k \), we may take \( N_2 = \frac{2}{\zeta} \). Then
\[f(x, y) + M_1 \zeta = 1 + xy + (x^2 - 2)g + \frac{4k}{\zeta^2} \geq 0 \]. By substituting \( \eta = 2 \cos \frac{2\pi}{5} \) into \( f(\eta, \frac{\theta}{2}) + M_1 \zeta \geq 0 \), we find an inequality between \( \zeta \) and \( \frac{2}{\eta} \). For example, if \( \zeta = 0.1 \), then \( \theta \geq -0.78k \).

## 4 Distance-regular graphs with relatively small \( \theta_{\min} \)

In this section we study distance-regular graphs with relatively small \( \theta_{\min} \). In the rest of this section we will give a proof of Theorem\( [\text{13}] \).
Proof of Theorem L.3
Assume Γ has odd girth $g = 2t + 1$. Let $(u_i)_{i=0}^D$ be the standard sequence associated with the smallest eigenvalue $\theta = \theta_{\text{min}}$.

We first consider the case $D = 4$. We may assume $k \geq 5$, otherwise Γ is the 9-gon or the Coxeter graph by [3] and [5, Theorem 1.1].

As $\theta < -\frac{k}{2}$, by [7, Proposition 2.11], we have $a_1 = 0$. If $a_2 \neq 0$, that is $t = 2$, then substitute $\eta = 2 \cos \frac{2\pi(t-1)}{g}$ into Equation (5) and we get $rac{(k-t)(2k+\sqrt{3}+\sqrt{7})}{2k(k-1)} \geq 0$, which implies that $\theta \geq -\frac{2k-\sqrt{3}+\sqrt{7}}{\sqrt{3}+\sqrt{7}}$. Combine it with $\theta \leq -\frac{3}{4}k$, we see that $k \leq 2$. Hence $a_2 = 0$. Note that $\theta \leq -\frac{3}{4}k < \frac{12-5k}{7}$, by Lemma 2.4, we see $c_2 \leq 2$.

If $a_3 \neq 0$, then consider

$$
\sum_{i=0}^{t} p_i(\eta)u_i \geq 0 \\
-\frac{D-1}{D} k \geq \theta
$$

with $\eta = 2$, we obtain that $k \leq 4$ if $c_2 = 1$, and $k \leq 8$ if $c_2 = 2$. No intersection arrays satisfy Lemma 2.2 with $5 \leq k \leq 8$, $D = 4$, $a_1 = a_2 = 0 \neq a_3$, $c_2 = 2$ and $\theta_{\text{min}} \leq -\frac{3}{4}k$. Hence $a_3 = 0$.

Assume $k \geq 36$. Since $k \geq 36$, $c_2 \leq 2$ and $\theta \leq -\frac{3}{4}k$, by Equation (7), we obtain $|u_2| \geq 0.5500$ and $|u_3| \geq 0.3926$. Now we consider the intersection matrix $L$ of Γ, where

$$L = \begin{pmatrix}
0 & k & 0 & 0 & 0 \\
1 & 0 & k-1 & 0 & 0 \\
0 & c_2 & 0 & k-c_2 & 0 \\
0 & 0 & c_3 & 0 & k-c_3 \\
0 & 0 & 0 & c_4 & k-c_4
\end{pmatrix}.$$

We see that $k^2 + \theta^2 \leq \operatorname{tr}(L^2) \leq k^2 + 6k + c_4(2k - c_4)$, where $c_2 \leq 2$ and $c_3 \leq c_4$. Since $k \geq 36$ and $\theta \leq -\frac{3}{4}k$, we obtain that $\frac{\theta}{k} \geq 0.2227$. By Lemma 2.1, we see that $m \leq \max\{\frac{1}{u_1}, \frac{1}{u_2}, \frac{k_3+k_4}{k_3 u_4}\}$. Since $k_3 b_3 = k_4 c_4$, we see $\frac{k_3+k_4}{k_3 u_4} \leq \frac{1}{u_4}(1 + \frac{k}{c_4})$. With $|u_1| \geq 0.5500$, $|u_3| \geq 0.3926$ and $\frac{\theta}{k} \geq 0.2227$, we obtain $m \leq 36$. By [4, Theorem 4.4.4], we see that $k \leq m < 36$, a contradiction. It follows that $k \leq 35$. Then we check the intersection arrays satisfy Lemma 2.2 with $5 \leq k \leq 35$, $D = 4$, $a_1 = a_2 = a_3 = 0 \neq a_4$, $c_2 = 1$ or 2 and $\theta_{\text{min}} \leq -\frac{3}{4}k$, and we get the folded 9-cube and odd graph $O_5$. This shows the case $D = 4$.

Now we consider the case $D = 5$. Similar to the case $D = 4$, we may assume $k \geq 5$, otherwise Γ is the 11-gon, by [3] and [5, Theorem 1.1]. As $\theta < -\frac{k}{2}$, by [7, Proposition 2.11], we have $a_1 = 0$. Substitute $\eta = 2 \cos \frac{2\pi(t-1)}{g}$ with $t = 2$ into Equation (5), we obtain $\theta \geq -\frac{2k-\sqrt{3}+\sqrt{7}}{\sqrt{3}+\sqrt{7}}$. Together with $\theta \leq -\frac{3}{4}k$, we see $k \leq 2$, and hence $a_2 = 0$. Since $\theta \leq -\frac{3}{4}k < \frac{12-5k}{7}$, by Lemma 2.4, we have and $c_2 \leq 2$.

If $a_3 \neq 0$, then consider Equation (11) with $\eta = 2$, we obtain that $k \leq 3$ if $c_2 = 1$, and $k \leq 5$ if $c_2 = 2$. By [4, Theorem 1.13.2], no such graphs exist with $k = 5$ and $c_2 = 2$. Hence $a_3 = 0$.
We consider $a_4 \neq 0$. If $c_3 \leq 0.3750k$, combine it with Equation (11), where $\eta = -1$ ($g = 9$), we see that $k \leq 24$. Assume $k \geq 24$, then $c_3 \geq 0.3750k$. By Equation (11), we obtain $|u_2| \geq 0.6243$ and $|u_3| \geq 0.4721$. Note $k_4 \leq \frac{b_3}{c_4} \leq \frac{1-c_3}{c_3}$ and $k_3 \leq \frac{b_3 b_4}{c_4 c_5} \leq \left(\frac{1-c_3}{c_3}\right)^2$. By Lemma 2.1, we see that $m \leq \max\left\{\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_3}\left(1 + \frac{1-c_3}{c_3} + \left(\frac{1-c_3}{c_3}\right)^2\right)\right\}$, (12)
that is $k \leq m \leq 24$ (Lemma [4, Theorem 4.4.4]). No intersection arrays satisfy Lemma 2.2 with $5 \leq k \leq 24$, $D = 5$, $a_1 = a_2 = a_3 = 0 \neq a_4$, $c_2 = 1$ or 2 and $\theta \leq -\frac{4}{5}k$. Hence $a_4 = 0$.
Assume $k \geq 71$. Then by Equation (7), we see that $|u_2| \geq 0.6348$ and $|u_3| \geq 0.4994$, where $\theta_1 \leq -\frac{4}{5}k$, $c_2 = 1$ or 2. Then as $m \geq k \geq 71$, by Equation (12), we obtain $c_3 \leq 0.2166k$. It implies $|u_4| \geq 0.3344$ by Equation (7). Consider the intersection matrix $L$, and we see that $k^2 + \theta^2 \leq tr(L^2) \leq k^2 + 6k + 4c_5k - c_5^2$, which implies $c_5 \geq 0.1440k$. And we see $k \leq m \leq \min\left\{\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_3}, \frac{1}{u_4}\left(1 + \frac{k}{c_5}\right)\right\} \leq 71$. It follows that $k \leq 71$. Then we check all intersection arrays satisfy Lemma 2.2 with $5 \leq k \leq 71$, $D = 5$, $a_1 = a_2 = a_3 = a_4 = 0 \neq a_5$, $c_2 = 1$ or 2 and $\theta \leq -\frac{4}{5}k$ and we obtain the odd graph $O_6$ and the folded 11-cube. This shows the case $D = 5$.

\section*{Acknowledgments}
JHK is partially supported by the National Natural Science Foundation of China (Grant No. 11471009 and Grant No. 11671376) and by ‘Anhui Initiative in Quantum Information Technologies’ (Grant No. AHY150200). ZQ is partially supported by the National Natural Science Foundation of China (Grant No. 11801388).

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