McShane-type identities for quasifuchsian representations of nonorientable surfaces

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Abstract

We adapt Bers’ double uniformization for nonorientable surfaces and show that the space $QF(N)$ of quasifuchsian representations for a nonorientable surface $N$ is the Teichmüller space $\mathcal{T}(dN)$ of an orientable double of $N$. We then utilize the inherited complex structure of $QF(N) = \mathcal{T}(dN)$ to show that Norbury’s McShane identities for nonorientable cusped hyperbolic surfaces $N$ generalizes to quasifuchsian representations and punctured torus bundles for $N$.

1 Introduction

Disclaimer: all surfaces, whether orientable or non-orientable, in this paper have negative Euler characteristic unless stated otherwise.

The Teichmüller space $\mathcal{T}(S)$ of a finite-area hyperbolic surface $S$ is a foundational object in various subjects, ranging from moduli space theory, complex analysis, complex dynamics, low-dimensional geometry and topology and representation theory. In representation theory, the space $\mathcal{T}(S)$ manifests as the character variety of discrete faithful (i.e.: Fuchsian) representations from the surface group $\pi_1(S)$ to $PSL(2, \mathbb{R})$. Another avatar of Teichmüller space in representational theory arises when describing the character variety $QF(S)$ of characters for quasifuchsian representations of $\pi_1(S)$ into $PGL(2, \mathbb{C})$. Specifically, Bers’s uniformization theorem [Ber60a] tells us that:

**Theorem (Bers).** For $S$ an orientable surface, the space $QF(S)$ of quasifuchsian representations of $S$ is complex analytically equivalent to $\mathcal{T}(S \cup \bar{S})$.

Here, the space $QF(S)$ is rendered a complex manifold when regarded as an open subset contained in the character variety of representations from $\pi_1(S)$ to $PGL(2, \mathbb{C})$, and we compare it to the standard complex structure on $\mathcal{T}(S \cup \bar{S})$ [Ahl60, Ber60b, Ber61].

2010 Mathematics Subject Classification. Primary 57M50; Secondary 57N05.

Project funded by the China Postdoctoral Science Foundation grant (General Financial Grant No. 2016M591154, [THE OTHER GRANT]), a Lift-Off Fellowship from the Australian Mathematical Society and the AK Head Mathematical Scientists Travelling Fellowship from the Australian Academy of Science.
Akiyoshi-Miyachi-Sakuma take advantage of this complex structure and invoke the identity theorem for holomorphic functions to show that McShane’s identities [McS91, McS98] for marked hyperbolic surfaces extend to the space of quasifuchsian representations. Given a finite-volume cusped (possibly nonorientable) hyperbolic surface $F$ with a distinguished cusp $p$, let $S(F) = S_1(F) \cup S_2(F)$ denote the union of the following two (possibly empty) sets:

- let $S_1(F)$ be the collection of embedded geodesic-bordered (open) 1-holed Möbius bands on $F$ which contain cusp $p$. We denote an arbitrary 1-holed Möbius band $M$ by the unordered pair $\{\alpha_1, \beta_1\}$ of simple closed 1-sided geodesics contained in $M$; and
- let $S_2(F)$ be the collection of embedded geodesic-bordered (open) pairs of pants on $F$ which contain cusp $p$. We denote an arbitrary pair of pants $P$ in $S_2(F)$ by the unordered pair $\{\alpha_2, \beta_2\}$ of simple closed 2-sided geodesics on $F$, which, together with cusp $p$, bound $P$.

**Note 1.** We regard cusps as 2-sided geodesics of length 0, and thus allow $\alpha_2$ or $\beta_2$ to be cusps. And in the special case when $F$ is a 1-cusped torus $S_{1,1}$, the boundary geodesics $\alpha_2$ and $\beta_2$ are both the same curve.

**Note 2.** We regard pairs of pants embedded within Möbius strips $M \in S_1(F)$ as elements of $S_2(F)$. In particular, each embedded 1-holed Möbius strip contains precisely two embedded pairs of pants $P_\alpha, P_\beta \in S_2(F)$ respectively obtained by cutting $M$ along $\alpha_1$ and $\beta_1$ (see Figure 1).

![Figure 1](image.png)

Figure 1: (left to right) a 1-holed Möbius strip $M$; a pair of pants $P_\alpha \subset M$; the other pair of pants $P_\beta \subset M$.

**Theorem** (orientable quasifuchsian identity). Consider an orientable cusped hyperbolic surface $S$ with a distinguished cusp $p$. For any $\rho \in \Omega F(S)$, we have the absolutely convergent series

$$\sum_{\{\alpha_2, \beta_2\} \in S_2(S)} \left( e^{i(\ell_{\alpha_2}(\rho) + \ell_{\beta_2}(\rho))} + 1 \right)^{-1} = \frac{1}{2},$$

where $\ell_\alpha(\rho)$ here denotes the complex length of the geodesic $\alpha$ taken with respect to $\rho$.

**Note 3.** In the special case that $S$ is a 1-cusped torus, the above result may be found in Bowditch [Bow98]. His strategy of proof employs trace-based algebraic structures generalizing Penner’s $\lambda$-lengths [Pen87], instead of Akiyoshi-Miyachi-Sakuma’s holomorphicity and identity theorem based proof.
Quasifuchsian surface groups occupy a dense open subset of the set of all
Kleinian surface groups \( \Gamma \). Thus, on the \( \text{PGL}(2, \mathbb{C}) \) character variety for \( \pi_1(S) \), quasifuchsian representations continuously interpolate between hyperbolic surfaces (Fuchsian representations) and complete finite hyperbolic 3-manifolds (certain boundary points of \( \mathfrak{F}(S) \)) such as pseudo-Anosov mapping tori. Bowditch studies this interpolation so as to obtain a McShane-type identity for punctured torus bundles, and describes the cusp geometry of these mapping tori in terms of certain summands in this identity.

Akiyoshi-Miyachi-Sakuma generalize Bowditch’s work for general hyperbolic punctured orientable surface bundles [AMS04, AMS06]. First recall that any hyperbolic surface bundle may be constructed by taking \( S \times [0,1] \) and identifying \( S \times \{0\} \) with \( S \times \{1\} \) via some pseudo-Anosov map \( \phi \). We denote a hyperbolic surface bundle obtained in such a way by \( M_\phi \).

**Theorem** (punctured orientable surface bundle identity). Given a hyperbolic orientable surface bundle \( M_\phi \) with orientable fiber \( S \) and monodromy representation \( \phi \). Let \( S_\phi \) denote the collection of unordered pairs \( \{\alpha_2, \beta_2\} \) of simple closed geodesics in \( M_\phi \) homotopic to an unordered pair of simple closed geodesics in \( S_2(S) \), then we have the absolutely convergent series

\[
\sum_{\{\alpha_2, \beta_2\} \in S_\phi} \left( e^{\frac{1}{2}(\ell_{\alpha_2}(\phi) + \ell_{\beta_2}(\phi))} + 1 \right)^{-1} = 0.
\]

One of our primary goals is to obtain, using the Akiyoshi-Miyachi-Sakuma strategy, identities for non-orientable hyperbolic surfaces such as the following cusped surface version of Norbury’s Theorem 2 from [Nor08]:

**Theorem** (Norbury’s nonorientable cusped surface identity). Given a nonorientable cusped hyperbolic surface \( N \) with Fuchsian monodromy representation \( \rho \),

\[
\sum_{\{\alpha_1, \beta_1\} \in S_1(N)} \left( e^{\frac{1}{2}(\ell_{\alpha_1}(\rho) + \ell_{\beta_1}(\rho))} - 1 \right)^{-1} + \sum_{\{\alpha_2, \beta_2\} \in S_2(N)} \left( e^{\frac{1}{2}(\ell_{\alpha_2}(\rho) + \ell_{\beta_2}(\rho))} + 1 \right)^{-1} = \frac{1}{2}.
\]

**Note 4.** Denote the geometric intersection number of two geodesics \( \alpha \) and \( \beta \) by \( \alpha \cdot \beta \). Then, the summand for each of the partial sums in the above identity may be expressed as:

\[
\left( e^{\frac{1}{2}(\ell_{\alpha}(\rho) + \ell_{\beta}(\rho))} + (-1)^{\alpha \cdot \beta} \right)^{-1}.
\]

We henceforth adopt this notational convention for succinctness.

**Note 5.** Strictly speaking, the above identity does not appear in [Nor08], which has a McShane identity for bordered non-orientable surfaces. This cusped surface version of the McShane identity may be derived from Theorem 2 of Norbury’s paper by dividing both sides by \( L_1 \), taking the limit as \( L_1 \) approaches 0 and then applying a little algebraic manipulation. We give the explicit derivation in Appendix A.

There are known extensions of Norbury’s identity to quasifuchsian representations when the underlying non-orientable surface is sufficiently topologically simple:
• the twice-punctured Klein bottle in [Nor08],
• the thrice-punctured projective plane in [HN17, HTZ18] and
• the thrice-bordered projective plane in [MP15].

In each of these cases, the proof strategy is based on algebraic methods akin to Bowditch’s strategy in [Bow98].

1.1 Main results

Consider now a non-orientable cusped hyperbolic surface $N$ with an oriented double cover $dN$, and let $i : dN \to dN$ denote the orientation-reversing involution inducing the quotient map from $dN$ to $N \cong dN/(x \sim i(x))$.

**Theorem 1.** The space $\Omega \mathcal{F}(N)$ of quasifuchsian representations of $N$, as a holomorphic slice of the $\mathrm{PGL}(2, \mathbb{C})$ representation variety for $\pi_1(N)$, is complex analytically equivalent to the Teichmüller space $T(dN)$. Moreover, the Teichmüller space $T(N)$, regarded as the Fuchsian locus in $\Omega \mathcal{F}(N)$, is a connected and maximal dimensional totally real analytic submanifold of $\Omega \mathcal{F}(N)$.

We use this complex structure on $\Omega \mathcal{F}(N)$, coupled with a version of the identity theorem for multivariate holomorphic functions, to prove the following:

**Theorem 2.** Given a nonorientable cusped hyperbolic surface $N$ and a quasifuchsian representation $\rho : \pi_1(N) \to \mathrm{PGL}(2, \mathbb{C})$, define $S(N)$ to be the set of embedded pairs of pants and Möbius bands containing cusp $p$ (as per Note 4). Then,

$$\sum_{\{\alpha, \beta\} \in S(N)} \left( e^{\frac{1}{2} (\ell_\alpha(p) + \ell_\beta(p))} + (-1)^{\alpha \cdot \beta} \right)^{-1} = \frac{1}{2},$$

where $\ell_\gamma(p)$ is the complex length of $\gamma$ (see Section 2.4).

As in the orientable surface case, we consider what happens as we deform to a punctured non-orientable surface bundle $M_\varphi$ and obtain:

**Theorem 3.** Given a pseudo-Anosov map $\varphi : N \to N$ let $\phi$ denote the monodromy representation for the mapping torus $M_\varphi$. Then,

$$\sum_{\{\alpha, \beta\} \in S_\varphi} \left( e^{\frac{1}{2} (\ell_\alpha(\phi(p)) + \ell_\beta(\phi(p)))} + (-1)^{\alpha \cdot \beta} \right)^{-1} = 0,$$

(1)

where $S_\varphi = S(N)/([\alpha, \beta] \sim [\varphi, \alpha, \varphi, \beta])$ denotes the set of homotopy classes in $M_\varphi$ of of pairs of pants (containing cusp $p$) lying on a fiber $N$.

It is also possible to obtain geometric data regarding the cusp geometry of $M_\varphi$. Any embedded horospheric cross-section of cusp $p$ in $M_\varphi$ is the same Euclidean torus $T_p$ up to homothety. Given a pair of generators $[\alpha], [\beta]$ for $\pi_1(T_p)$ (i.e. a marking on $T_p$), we define the marked modulus of $[T_p, ([\alpha], [\beta])$ to be the Teichmüller space parameter for this marked torus in $\mathcal{T}_{1,0} = \mathbb{H} \subset \mathbb{C}$ (see, for example, Section 1.2.2 of [IT92]). We now introduce a little language to state the marked modulus of $T_p$ in terms of a topologically meaningful sub-series of the McShane identity for $M_\varphi$. 
Definition 1 (Pseudo-Anosov map). We call a homeomorphism \( \varphi : N \rightarrow N \), for a non-orientable hyperbolic surface \( N \), pseudo-Anosov if there is a pair \((\mathcal{F}^s, \mathcal{F}^u)\) of measured foliations such that:

- the stable measured foliation \( \mathcal{F}^u \) and the unstable measured foliation \( \mathcal{F}^s \) are transverse outside of the singular loci;
- the map \( \varphi \) preserves their underlying foliations, and acts on \( \mathcal{F}^s \) as multiplication by \( K^{-1} < 1 \) and on \( \mathcal{F}^u \) as multiplication by \( K > 1 \).

Figure 2: (left to right) the stable foliation \( \mathcal{F}^s \) around a cusp; the unstable foliation \( \mathcal{F}^u \) around a cusp; singular foliations as points in the set \( S_1^p \) of directions emanating from \( p \).

Note 6. A homeomorphism \( \varphi : N \rightarrow N \) is pseudo-Anosov if and only if it lifts to a pseudo-Anosov map \( d\varphi : dN \rightarrow dN \) which commutes with the orientation-reversing involution \( \iota : dN \rightarrow dN \). By replacing \( \varphi \) with \( \varphi \circ \iota \) if necessary, we set \( d\varphi \) to be orientation-preserving.

Definition 2 (Signature of a pseudo-Anosov map). Given a pseudo-Anosov map \( \varphi : N \rightarrow N \), the singular leaves \( \{\lambda^+_1, \ldots, \lambda^+_t\} \) of its stable foliation around cusp \( p \) and the singular leaves \( \{\lambda^-_1, \ldots, \lambda^-_{t-1}\} \) of its unstable foliation around cusp \( p \) interlace one another as illustrated in Figure 2. The action of \( \varphi \) preserves each set of singular leaves and acts on \( \{\lambda^+_1, \ldots, \lambda^+_t\} \) by cyclic permutation, shifting the index by some \( s \in \{0, \ldots, t-1\} \). We refer to the pair \((s, t)\) as the signature of the pseudo-Anosov map \( \varphi \) at \( p \). When \( s = 0 \), we say that \( \varphi \) has simple signature at \( p \).

Every pseudo-Anosov map of simple signature at \( p \) has a canonical marking on the cusp torus at \( p \) by taking the pair \((m_p, l_p)\), where the meridian \( m_p \) forms the core of the cusp torus and the longitude \( l_p \) may be constructed as follows:

Definition 3 (longitude [AMS06, Definition 3.4]). Recall that \( M_\varphi \) may be constructed by taking \( F \times [0, 1] \) and identifying its two ends via \( \varphi \). Choose an arbitrary point \( x \) lying on an arbitrary singular leaf \( \lambda \subset F \), the interval \([x] \times [0, 1]\) forms a path on \( M_\varphi \) with both end points \((x, 0)\) and \((x, 1)\) on \( \lambda \). Joining these end points along \( \lambda \) results in a simple closed loop \( l_\varphi \) that is, up to homotopy, independent of our choice of \( \lambda \) and \( x \). We refer to \( l_\varphi \) as the longitude of the cusp torus \( T_\varphi \) of \( M_\varphi \) at cusp \( p \).
Finally, we require the following identification to help group the summands of our identity.

**Proposition 4.** There is a purely topological bijection between $\mathcal{S}(N)$ and the collection $\Delta$ of (simple) ideal geodesics on $N$ with both ends up $p$, such that each of the curves $\{\alpha, \beta\} \in \mathcal{S}(N)$ is freely homotopic to its corresponding ideal geodesic $\sigma \in \Delta$ when $\alpha, \beta$ and $\sigma$ are regarded as simple closed curves on $N \cup \{p\}$, that is: the surface $N$ with cusp $p$ filled in.

The “singular” leaves $\lambda^\pm_i$ are only singular at cusp $p$, and thus form simple bi-infinite paths on $N$. The geodesic representative for $\lambda^+_i$ (resp. $\lambda^-_i$) has one end at cusp $p$ and the other end spirals towards a leaf of the unstable (resp. stable) measured lamination of $\varphi$, and we endow each $\lambda^+_i$ with the orientation going from the cusp $p$ to the measured lamination. We regard the cyclically ordered set $\{\lambda^-_1, \lambda^-_2, \ldots, \lambda^-_t, \lambda^-_1\}$ of interlacing singular leaves as a cyclically ordered set of “directions” in the circle’s worth of “directions” $S^1_p$ emanating from cusp $p$ (see Figure 2). This allows us to partition $\Delta$ into:

- $\Delta^+_i$: the set of ideal geodesics $\sigma \in \Delta$ where both ends of $\sigma$ are launched within an interval $(\lambda^-_i, \lambda^-_i) \subset S^1_p$ for some $i$;
- $\Delta^-_i$: the set of ideal geodesics $\sigma$ where both ends of $\sigma$ are launched within an interval $(\lambda^+_i, \lambda^+_i) \subset S^1_p$ for some $i$;
- $\Delta^0_i$: the remaining set of ideal geodesics consisting of those with one end in each of the two interval types.

By Proposition 4, this induces a partition of $\mathcal{S}(N)$ into $S^+(N) \sqcup S^0(N) \sqcup S^-(N)$. Since $\varphi$ fixes the singular leaves $\{\lambda^\pm_i\}$, this partition is $\varphi$-invariant and descends to a partition

$$S_\varphi = S^+_\varphi \sqcup S^0_\varphi \sqcup S^-_\varphi.$$

**Theorem 5.** Given a pseudo-Anosov map $\varphi$ with simple signature $(0, t)$, the marked modulus $\text{mod}_p(\varphi)$, with respect to the marking $(m_p, t_p)$, of the cusp-$p$ torus $T_\varphi$ of a mapping torus $M_\varphi$ is given by:

$$\text{mod}_p(\varphi) = \left( \frac{2}{t} \sum_{(\alpha, \beta) \in S^+_\varphi} + \frac{1}{t} \sum_{(\alpha, \beta) \in S^0_\varphi} \right) \left( e^{\frac{t}{4}(\ell_\alpha(\phi) + \ell_\beta(\phi)) + (-1)^{\alpha \beta}} \right)^{-1}.$$

**Note 7.** For $\varphi$ with simple signature, Theorem 3 is equivalent to the symmetric expression

$$\text{mod}_p(\varphi) = -\left( \frac{2}{t} \sum_{(\alpha, \beta) \in S^-_\varphi} + \frac{1}{t} \sum_{(\alpha, \beta) \in S^0_\varphi} \right) \left( e^{\frac{t}{4}(\ell_\alpha(\phi) + \ell_\beta(\phi)) + (-1)^{\alpha \beta}} \right)^{-1}.$$

**Note 8.** For a pseudo-Anosov map $\varphi$ with signature $(s, t)$, the pseudo-Anosov map $\hat{\varphi} := \varphi^\frac{t}{s}$, and of its powers do have simple signature at $p$. This allows us to understand the cusp geometry of $M_\varphi$ via $M_\hat{\varphi}$. This is explained in Section 6.2.

As with Theorem 3, the above identity is derived by considering related decompositional identities describing the cusp geometry of quasifuchsian 3-manifolds deforming to a mapping torus $\varphi$. 

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1.2 Acknowledgments

We would like to thank Hideki Miyachi and Makoto Sakuma for teaching us the ideas of their proof of their extension of the classical Fuchsian McShane identity to the quasifuchsian context; Kenichi Ohshika for invaluable information regarding pseudo-Anosov mapping tori; Greg McShane, Paul Norbury, Athanase Papadopoulos and Ser Peow Tan for helpful discussions.

2 Character varieties for $\pi_1(N)$

2.1 Non-orientable surface group representations

We use $\text{PSL}^\pm\left(2, \mathbb{R}\right)$ to refer to projective classes of real matrices with determinant $\pm 1$. We say that a discrete faithful representation $\rho : \pi_1(N) \to \text{PGL}(2, \mathbb{C})$ is Fuchsian iff. its limit set is the ideal boundary of an embedded hyperbolic plane in $\mathbb{H}^3$.

Note 9. This is equivalent to the condition that $\rho$ may be conjugated by an element of $\text{PGL}(2, \mathbb{C})$ to a representation $\rho' : \pi_1(N) \to \text{PSL}^\pm\left(2, \mathbb{R}\right)$, where 1-sided curves are sent to $\text{PSL}^{-}\left(2, \mathbb{R}\right) \subset \text{PSL}^\pm\left(2, \mathbb{R}\right)$, that is: the subset of (real projective classes) of matrices with determinant $-1$. This in turn implies that 2-sided curves are sent to matrices with determinant $1$.

We say that a discrete faithful representation $\rho : \pi_1(N) \to \text{PGL}(2, \mathbb{C})$ is quasifuchsian iff. its limit set is a $\rho(\pi_1(N))$-invariant Jordan curve $C_\rho \subset \partial_\infty \mathbb{H}^3$. There is an alternative convention regarding quasifuchsian representations, and defines them to be those with its limit set a subset of a $\rho(\pi_1(N))$-invariant Jordan curve $C_\rho$. In this language, the representations we consider are known as type I quasifuchsian representations.

We denote the space of characters for quasifuchsian representations $\rho : \pi_1(N) \to \text{PGL}(2, \mathbb{C})$, regarded as a subset of the character variety

$$\text{Hom}(\pi_1(N), \text{PGL}(2, \mathbb{C}))/\text{PGL}(2, \mathbb{C}),$$

by $\Omega\mathcal{F}(N)$. Fuchsian representations are a special class of quasifuchsian representations, and we refer to the subset of $\Omega\mathcal{F}(N)$ occupied by Fuchsian representations as the Fuchsian locus in $\Omega\mathcal{F}(N)$.

2.2 Fuchsian representations of orientable doubles

Let $\rho_0 : \pi_1(N) \to \text{PSL}^\pm\left(2, \mathbb{R}\right)$ be a Fuchsian representation for the hyperbolic structure $N$. We regard $\pi_1(dN)$ as a index 2 subgroup of $\pi_1(N)$, and denote the restriction representation to $\pi_1(dN)$ by

$$d\rho_0 := \rho_0|_{\pi_1(dN)} : \pi_1(dN) \to \text{PSL}(2, \mathbb{R}).$$

We fix an arbitrary element $\alpha_0 \in \pi_1(N) - \pi_1(dN)$ and set $A_0 := d\rho_0(\alpha_0)$. Since $\pi_1(dN)$ is an index 2 subgroup of $\pi_1(N)$, it is necessarily a normal subgroup and hence $\pi_1(dN) = \alpha_0 \cdot \pi_1(dN) \cdot \alpha_0^{-1}$. Further observe that

$$A_0 \cdot \mathbb{H} = \mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}z < 0\}.$$
Note also that the set of homotopy classes $\pi_1(N) - \pi_1(dN)$ with 1-sided curve representatives may be regarded as the set of glide-reflection maps 

$$\mathcal{A}_0 : \mathbb{H} \to \mathbb{H} \text{ defined by } z \mapsto [A_0 \cdot z].$$

2.3 Double uniformization theorem for nonorientable surfaces

Fix three arbitrary hyperbolic elements $\gamma_0, \gamma_1, \gamma_\infty \in \pi_1(dN)$ and normalize every character $[\rho] \in \mathcal{QF}(N)$ to be the representation $\rho$ where the attracting fixed point of $\rho(\gamma_z)$ in $\partial \mathbb{H}^3 = \hat{\mathbb{C}}$ is $z$. This is an embedding of the quasifuchsian character variety $\mathcal{QF}(N)$ as a slice within the $\text{PGL}(2, \mathbb{C})$ representation variety for $\pi_1(N)$. In particular, the embedding is algebraic and hence induces a complex structure on $\mathcal{QF}(N)$. We also renormalize $\rho_0$ so as to lie on this slice.

Theorem 1. The space $\mathcal{QF}(N)$ of quasifuchsian representations of $N$, as a holomorphic slice of the $\text{PGL}(2, \mathbb{C})$ representation variety for $\pi_1(N)$, is complex analytically equivalent to the Teichmüller space $\mathcal{T}(dN)$. Moreover, the Teichmüller space $\mathcal{T}(N)$, regarded as the Fuchsian locus in $\mathcal{QF}(N)$, is a connected and maximal dimensional totally real analytic submanifold of $\mathcal{QF}(N)$.

Note 10. In specifying the complex structure on $\mathcal{T}(dN)$, we orient $dN$ as the upper-half plane conformal end $\mathbb{H} \subset \hat{\mathbb{C}}/\rho_0(\pi_1(dN))$ rather than the lower-half plane conformal end $\overline{\mathbb{H}}/\rho_0(\pi_1(dN))$.

Proof. Given an arbitrary quasifuchsian representation $\rho \in \mathcal{QF}(N)$, we denote the restriction $\rho|_{\pi_1(dN)}$ by $d\rho$, and observe that $d\rho$ is also quasifuchsian since $\rho$ and $d\rho$ share the same Jordan curve $C_\rho$ at infinity. Orient $C_\rho$ so that $0, 1, \infty \in C_\rho$ are in increasing order, the Jordan domain $\Omega_\rho$ bordered counterclockwise by $C_\rho$ gives a marked conformal structure on $dN$ given by the action of $\pi_1(dN)$ on $\Omega_\rho$ via $\rho$. This gives a well-defined map 

$$\Phi : \mathcal{QF}(N) \to \mathcal{T}(dN).$$

We first show that $\Phi$ is surjective. Given the Beltrami differential $\mu$ corresponding to an arbitrary marked conformal structure in $\mathcal{T}(dN)$, define a new Beltrami differential given by:

$$\mu_\#(z) = \begin{cases} 
\mu(z), & \text{if } z \in \mathbb{H}; \\
\mu(A_0 \cdot z), & \text{if } z \in \overline{\mathbb{H}}; \\
0 & \text{otherwise.}
\end{cases} \quad (2)$$

Since $\|\mu_\#\|_\infty < 1$, up to Möbius transformation, there is a unique homeomorphism $\psi_{\mu_\#} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfying the Beltrami equation for $\mu_\#$.

Consider an arbitrary $\gamma \in \pi_1(N)$, if $\gamma \in \pi_1(dN)$, then $\mu_\# \circ (\rho_0(\gamma)) \equiv \mu$

- on $\mathbb{H}$ because $\mu$ is $\pi_1(dN)$-invariant;
- on $\overline{\mathbb{H}}$ because $A_0 \cdot \rho_0(\gamma) \cdot A_0^{-1}$ is in $\rho_0(\pi_1(dN))$.

Similarly, if $\gamma \in \pi_1(N) - \pi_1(dN) = \alpha_0^{-1} \cdot \pi_1(dN)$, then $\mu_\# \circ (\rho_0(\gamma)) \equiv \mu$

- on $\mathbb{H}$ because $A_0 \cdot \rho_0(\gamma)$ is in $\rho_0(\pi_1(dN))$;
- on $\overline{\mathbb{H}}$ because $\rho_0(\gamma) \cdot A_0^{-1}$ is in $\rho_0(\pi_1(dN))$. 

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Thus, for any \( \gamma \in \pi_1(N) \), the maps \( \psi_{\mu} \) and \( \psi_{\mu} \circ \rho_0(\gamma) \) both satisfy the Beltrami equation. The uniqueness of solutions to the differential equation, up to Möbius transformation, tells us that there is an element \( A_\gamma \in \text{PGL}(2, \mathbb{C}) \) such that

\[
A_\gamma \circ \psi_{\mu} \equiv \psi_{\mu} \circ \rho_0(\gamma). \tag{3}
\]

Define a map \( \rho_\mu : \pi_1(N) \to \text{PGL}(2, \mathbb{C}) \) that takes \( \gamma \) to \( A_\gamma \). The fact that this is a representation is due to (3). Since \( \psi_{\mu} \) is a homeomorphism, we see that \( \rho \) is a quasifuchsian representation and hence \( \Phi \) is surjective.

To see that \( \Phi \) is injective, consider (equivalently normalized) quasifuchsian representations \( \rho_1, \rho_2 \) such that \( \Phi(\rho_1) \equiv \Phi(\rho_2) \). By Bers’ original arguments, the two respective conformal ends of the quasifuchsian representations \( d\rho_1 \) and \( d\rho_2 \) are equivalent, and hence \( d\rho_1 \equiv d\rho_2 \) as representations and the Jordan curves \( C_{\rho_1}, C_{\rho_2} \) are equivalent. This in turn means that the attracting and repelling fixed points of \( \rho_1(\alpha_0) \) and \( \rho_2(\alpha_0) \) must be the same. Moreover, since \( \alpha_0^2 \in \pi_1(dN) \), the real part of the translation lengths for \( \rho_1(\alpha_0) \) and \( \rho_2(\alpha_0) \) must agree and their imaginary components are equivalent up to addition by either \( 0 \) or \( \pi t \). However, we know that these two transformations exchange the two components of \( \hat{C} - C_{\rho_1} \) and this ensures that \( \rho_1(\alpha_0) = \rho_2(\alpha_0) \). Therefore, the representations \( \rho_1 \) and \( \rho_2 \) are equivalent on \( \alpha_0 \cdot \pi_1(dN) \) and hence on all of \( \pi_1(N) \).

We next show that \( \Phi^{-1} \) is a holomorphic map. Putting this with the bijectivity of \( \Phi^{-1} \) and Hartog’s theorem ensures the biholomorphicity of \( \Phi \). Let \( \mu^t(\cdot) \) be a complex analytic family of Beltrami differentials in \( \mathcal{T}(dN) \) around \( \mu = \mu^0 \). By examining equation (2), we see that \( \mu^t \) is also a complex analytic family of Beltrami differentials. Then, by the holomorphic dependence of the family \( \{\psi^t := \psi_{\mu^t}\} \) of quasiconformal mappings (see, for example, the immediate Corollary to Theorem 4.37 of [IT92]), we know that for any \( z \in \hat{C} \), the point \( \psi^t(z) \in \hat{C} \) varies holomorphically with respect to \( t \in \mathbb{C} \). We also know from the bijectivity of \( \Phi \) that

\[
\Phi^{-1}(\mu^t) = \rho_{\mu^t} = \psi_{\mu^t} \circ \rho_0 \circ \psi_{\mu^t}^{-1}. \tag{4}
\]

To show that \( \Phi^{-1} \) is holomorphic, it suffices to show that \( \rho_{\mu^t} \) varies holomorphically with respect to \( t \). Now, given any non-peripheral element \( \gamma \in \pi_1(N) \), the cross-ratio

\[
(\rho_{\mu^t}(\gamma)^+, \rho_{\mu^t}(\gamma)^-; z, \rho_{\mu^t}(\gamma) \cdot z), \text{ of } \tag{5}
\]

- the attracting fixed point \( \rho_{\mu^t}(\gamma)^+ \) of \( \rho_{\mu^t}(\gamma) \),
- the repelling fixed point \( \rho_{\mu^t}(\gamma)^- \) of \( \rho_{\mu^t}(\gamma) \),
- an arbitrary point \( z \) away from \( \rho_{\mu^t}(\gamma)^\pm \) and
- its image \( \rho_{\mu^t}(\gamma) \cdot z \) under the action of \( \rho_{\mu^t}(\gamma) \),

varies holomorphically with respect to \( t \). This cross-ratio suffices to recover the trace of \( \rho_{\mu^t}(\gamma) \) up to sign, and since \( \mathcal{T}(dN) \) is a simply connected domain, we may choose the correct sign for the trace by making the desired choice.
on the Fuchsian locus and analytically continuing over the entire character variety. By Hartog’s theorem, the composition of \( \Phi^{-1} \) and any trace function \( \text{tr} \circ \rho(\gamma) \) (for non-peripheral \( \gamma \)) is a holomorphic function on \( J(dN) \), and since trace functions give global coordinates on the character variety \( \Omega F(N) \), we obtain the desired holomorphicity of \( \Phi^{-1} \) and hence the agreement of complex analytic structure on \( \Omega F(N) \) and \( J(dN) \).

Finally, we show that the Fuchsian locus \( T(N) \subset \Omega F(N) = J(dN) \) is a maximal dimensional totally real analytic submanifold. To clarify, we need to show that \( T(N) \) is half-dimensional and that for every point \( x \in T(N) \), we have

\[ T_x T(N) \cap J(T_x T(N)) = \{ 0 \}, \]

where \( J \) denotes the almost complex structure on \( \Omega F(N) \) (see, for example, Definition 5.2 of [Lou15]). To show this, we consider the antiholomorphic involution \( \iota \) on \( J(dN) \) given by flipping the underlying orientation of \( dN \). This action, when interpreted as an action on \( \Omega F(N) = J(dN) \), is equivalent to precomposing a given Beltrami differential \( \mu \in \Omega F(N) \) by the complex conjugation map on \( \hat{\mathcal{C}} \). The fixed-point locus of \( \iota \) is precisely the Fuchsian locus \( T(N) \). By a general characterization of maximal totally real analytic submanifolds (see, for example, Prop 6.3 of [Lou15]), we conclude that \( T(N) \) is a connected half-dimensional totally real analytic submanifold of \( \Omega F(N) \).

**Note 11.** By combining Theorem 1 with the classical quasifuchsian character variety obtained from Bers’ simultaneous uniformization theorem, we see that Theorem 1 holds true even after replacing \( N \) with a (possibly disconnected) complete finite-area hyperbolic surface \( F \) and \( dN \) with an oriented double cover \( \Delta F \) of \( F \).

**Corollary 6.** The (orientation-preserving) mapping class group \( \Gamma^+(dN) \) of the oriented double cover \( dN \) is the group of biholomorphisms of \( \Omega F(N) \); except when \( N \) is Dyck’s surface (the sphere with three cross-caps), in which case the automorphism group is \( \Gamma^+(dN) \) modulo the \( \mathbb{Z}_2 \) generated by the hyperelliptic involution on \( dN \).

*Proof.* This is an immediate consequence of Theorem 1 and Royden’s theorem, which asserts that the automorphism group of \( J(dN) \) is the mapping class group \( \Gamma^+(dN) \); except when \( dN \) is the genus 2 oriented closed surface (and hence \( N \) is Dyck’s surface), in which case we need to take \( \Gamma^+(dN) \) modulo the hyperelliptic involution.

**Corollary 7.** Elements within the mapping class group \( \Gamma^\pm(N) \) act on \( \Omega F(N) \) either biholomorphically or anti-biholomorphically. In particular, the index 2 (normal) subgroup \( \Gamma^+(N) \) of \( \Gamma^\pm(N) \) which acts biholomorphically on \( \Omega F(N) \) is also known as the twist group – the subgroup generated by Dehn-twists along 2-sided curves.

*Proof.* Homeomorphisms on \( N \) lift to homeomorphisms on \( dN \) and this embeds the mapping class group \( \Gamma^\pm(N) \) as a subgroup of the (possibly orientation-reversing) mapping class group \( \Gamma^\pm(dN) \) of the oriented double cover \( dN \). First note that the action of \( \Gamma^\pm(N) \) on \( J(dN) \), regarded as a subgroup of \( \Gamma^\pm(dN) \), is precisely the action of \( \Gamma^\pm(N) \) on \( \Omega F(N) = J(dN) \). This is easy to see on the Fuchsian locus, and hence holds true in general because of the topological nature of this action. Since \( \Gamma^+(dN) \) acts holomorphically on \( J(dN) \) and \( \Gamma^-(dN) = \Gamma^+(dN) - \Gamma^+(dN) \) acts antiholomorphically, we obtain the holomorphic/antiholomorphic nature of the action of \( \Gamma^\pm(N) \).
Next note that cross-cap slides (see [Sze12]) lift to orientation-reversing mapping classes, and so \( \Gamma^\pm(N) \) does not embed as a subgroup of \( \Gamma^+(dN) \). In particular, this means that the subgroup \( \Gamma^\pm(N) \cap \Gamma^+(dN) \) of holomorphically acting mapping classes has index at least 2 in \( \Gamma^\pm(N) \). However, the twist group \( \Gamma^+(N) \) is a subgroup of \( \Gamma^+(N) \cap \Gamma^+(dN) \) because Dehn twists along 2-sided curves lift to orientation-preserving mapping classes. Since \( \Gamma^+(N) \) has index 2 in \( \Gamma^\pm(N) \), we conclude that the twist group \( \Gamma^+(N) \) is the holomorphic subgroup \( \Gamma^\pm(N) \cap \Gamma^+(dN) \).

2.4 Complex lengths

Our identities (Theorem 2, Theorem 3 and Theorem 5) are stated in terms of the complex lengths \( \ell_\gamma \) of curves \( \gamma \).

**Definition 4 ((Geometric) Complex Length).** Given a discrete faithful surface group representation \( \rho \), the real component of \( \ell_\gamma(\rho) \) is defined to be the translation length

\[
\text{Re}(\ell_\gamma(\rho)) := \min_{x \in H} d_{\mathbb{H}^3}(x, \rho(\gamma) \cdot x)
\]

of \( \rho(\gamma) \). Note that this is equivalent to the length of the unique geodesic representative of \( \gamma \) in \( \mathbb{H}^3/\rho(\pi_1(\Gamma)) \).

When \( \rho(\gamma) \) is loxodromic (including hyperbolic), the imaginary component \( \text{Im}(\ell_\gamma(\rho)) \in [\mathbb{R}/2\pi\mathbb{Z}] \) is defined in terms of the rotation angle \( \theta \) of the loxodromic transformation \( \rho(\gamma) \) around its invariant axis. If \( \gamma \) is a 2-sided curve, then \( \text{Im}(\ell_\gamma(\rho)) := \theta + 2\pi\mathbb{Z} \). If \( \gamma \) is a 1-sided curve, then \( \text{Im}(\ell_\gamma(\rho)) := \theta - \pi + 2\pi\mathbb{Z} \).

If \( \gamma \) is parabolic (this arises when \( \gamma \) is peripheral) we set its imaginary component to be 0, and hence its total complex length is 0.

**Note 12.** This normalization for the complex length of 1-sided geodesics \( \gamma \) by subtracting \( \pi \) yields the unique holomorphic function \( \ell_\gamma : \Omega\mathcal{F}(N) \to \mathbb{C}/2\pi\mathbb{Z} \) that agrees with the translation length of \( \gamma \) on the Fuchsian locus.

**Note 13.** We have defined complex geodesic length \( \ell_\gamma(\rho) \) to be functions from \( \Omega\mathcal{F}(N) \) to \( \mathbb{C}/2\pi\mathbb{Z} \). This suffices for our purposes, as we always exponentiate these lengths in our identities. However, when dealing with quasifuchsian representations, we may invoke the simply-connectedness of \( \Omega\mathcal{F}(N) \) (Theorem 1) to lift these length functions to maps of the form \( \tilde{\ell}_\gamma : \Omega\mathcal{F}(N) \to \mathbb{C} \), such that \( \ell_\gamma \) is equal to the translation length of \( \gamma \) on the Fuchsian locus.

Here is a more algebraic approach to defining complex lengths for quasifuchsian representations: fix a lift of \( \rho_0 \) to a \( SL^\times(2, \mathbb{C}) \) representation \( \tilde{\rho}_0 : \pi_1(N) \to SL^\times(2, \mathbb{C}) \) so that 2-sided curves have determinant 1 and 1-sided curves have determinant \(-1\) and use the simply connectedness of \( \Omega\mathcal{F}(N) \) to continuously extend this lift over all of \( \Omega\mathcal{F}(N) \). Having done so, we may define complex length as follows:

**Definition 5 ((Algebraic) Complex Length).** When \( \gamma \) is 1-sided, its complex length is defined to be \( 2\text{arcsinh}(|\frac{1}{2}\text{tr} \circ \tilde{\rho}(\gamma)|) \) of the trace of \( \tilde{\rho}(\gamma) \) along the Fuchsian locus, and the analytically extension of this function elsewhere on \( \Omega\mathcal{F}(N) \); when \( \gamma \) is a 2-sided geodesic, its complex length \( \ell_\gamma \) is defined to be \( 2\text{arccosh}(|\text{tr} \circ \tilde{\rho}(\gamma)|) \) along the Fuchsian locus, and the analytic extension of this function everywhere-else.
The fact that geometric definition and our algebraic description agree may be shown using the holomorphic identity theorem (see, for example, Proposition 6.5 of [Lou15]): both the geometrically defined length functions and its algebraic counterpart yield holomorphic functions on $\mathcal{O}(N)$ and agree on the Fuchsian locus – a maximal dimensional totally real analytic submanifold, and therefore must be the same function.

3 Simple geodesics on $N$

Consider the monodromy representation $\rho_0$ for $N$. The restriction of its limit curve $C_{\rho_0}$ to $\hat{\mathcal{C}} - \{\infty\}$ is precisely the real axis $\mathbb{R} \subset \hat{\mathcal{C}}$, and $\mathbb{R}/\rho_0(m_p) = \mathbb{R}/\mathbb{Z}$ canonically identifies with the set $\mathbb{S}_p^1$ of complete (oriented) geodesics in $N$ emanating from $p$. On the other hand, Theorem 1 tells us that for an arbitrary quasifuchsian representation $\rho$, there are quasiconformal maps which $\pi_1(N)$-equivariantly identify $C_{\rho_0}$ with $C_\rho$ and hence identify $C_\rho/\rho(m_p)$ with $\mathbb{S}_p^1$. We pay particular attention to two subsets of $\mathbb{S}_p^1$:

1. $\bar{\Delta}$: the set of oriented bi-infinite simple geodesics on $N$ with both source and sink based at $p$;

2. $\mathbb{G}$: the set of oriented simple complete geodesics on $N$ with source based at $p$.

The set $\bar{\Delta}$ is contained in $\mathbb{G}$, and we may topologize both of these spaces via the subspace topology on $\mathbb{S}_p^1$.

Definition 6. We say that that an ideal geodesic $\sigma$ in $\Delta$ or $\bar{\Delta}$ is a 1-sided (or 2-sided) ideal geodesic if, upon filling in the cusp $p$ on the surface $N$, $\sigma$ completes to a 1-sided (resp. 2-sided) curve. We denote the collection of 1-sided ideal geodesics by $\Delta_1$ and the collection of 2-sided geodesics by $\Delta_2$.

3.1 Fattening simple geodesics

Any (simple) ideal geodesic $\sigma \in \Delta$ may be fattened up into an (open) geodesically bordered surface as follows: any sufficiently small $\epsilon$-neighborhood of a 2-sided $\sigma$ is a pair of pants (Figure 3 – left), whereas any small $\epsilon$-neighborhood of a 1-sided $\sigma$ is topological equivalent to a punctured Möbius band (Figure 3 – right). Isotoping the boundaries of these $\epsilon$-fattened surfaces until they are geodesically bordered results in elements of $S_i(N)$ for $i$-sided ideal geodesics $\sigma \in \Delta_i$. This fattening procedure is well-defined as any two sufficiently small $\epsilon$-neighborhoods are related by a deformation retract, and this gives us an injective function Fat : $\Delta_i \rightarrow S_i(N)$.

Proposition 4. The Fat map gives a topologically defined bijection between $S(N)$ and the collection $\Delta$ of (simple) ideal arcs on $N$ with both ends up $p$, such that each of the two curves $\{\alpha, \beta\} \in S(N)$ is freely homotopic to its corresponding ideal geodesic $\sigma \in \Delta$ when $\alpha, \beta$ and $\sigma$ are regarded as simple closed curves on $N \cup \{p\}$, that is: the surface $N$ with cusp $p$ filled in.
Proof. The descriptions of $\alpha$ and $\beta$ are simple topological consequences of the fattening procedure and we only prove the statement that Fat is a bijection. The fact that Fat is a surjection is clear from Figure 3 and the existence of geodesic representatives for homotopy classes of ideal arcs on hyperbolic surfaces. The fact that Fat is an injection on $\Delta_2$ follows from the fact that for every embedded pair of pants $P \in S_2(N)$ there is a unique simple ideal geodesic, with both cusps up $p$, which lies completely on $P$. For injectivity on $\Delta_1$, we remark that for any embedded Möbius band $M$ with cusp $p$, there are precisely three (unoriented) simple ideal geodesics on $M$ with both ends going up $p$ (Lemma 8). Two of these are 2-sided and hence correspond to elements of $\Delta_2$ and only one is 1-sided.

Note 14. Thanks to the above result, we may regard elements of $\vec{\Delta}$ triples as $\{\alpha, \beta ; \epsilon\}$, where $\{\alpha, \beta\}$ is an element of $S(N)$ and $\epsilon \in \{-, +\} = \{\pm\}$ (arbitrarily) specifies the orientation of the bi-infinite ideal geodesic.

Note 15. The fattening procedure is a fundamentally topological construction, and hence every homeomorphism $\varphi : N \to N$ acts equivariantly on $\Delta$ and $S(N)$ with respect to the fattening map $\text{Fat} : \Delta \to S(N)$.

Lemma 8. There are precisely fourteen elements of $\vec{\mathcal{G}} \subset S^1_p$ on any (open) punctured Möbius strip $M$ containing cusp $p$ and one other geodesic border. Moreover,

1. these fourteen oriented geodesics are naturally grouped as seven pairs of geodesics, where each pair is related by the reflection involution on $M$.

2. three of these pairs are of elements of $\vec{\Delta}$ and each pair consists of the same ideal geodesic with its two opposing orientations. The inner pair $\{\lambda^-, \lambda^+\}$ of geodesics are oriented versions of the 1-sided geodesic $\lambda$ shown in the top left diagram in Figure 4. The outer two pairs $\{\lambda^\alpha_\alpha, \lambda^\alpha_\beta\}$ and $\{\lambda^\beta_\alpha, \lambda^\beta_\beta\}$ are oriented versions of the two 2-sided ideal geodesics $\lambda^\alpha_0$ and $\lambda^\beta_0$ depicted in the bottom left diagram in Figure 4.

3. the remaining four pairs are of elements of $\vec{\mathcal{G}} - \vec{\Delta}$ (blue geodesics in Figure 4) consisting of simple bi-infinite geodesics with one end spiraling to some simple closed geodesic. Specifically, two of the pairs $\{\mu^\alpha_\alpha, \mu^\alpha_\beta\}$ and $\{\mu^\beta_\alpha, \mu^\beta_\beta\}$ spiral to the
Figure 4: (left column) all three unoriented (hence six oriented) simple ideal geodesics on $M$ with both ends up the cusp $p$; (middle column) all the simple ideal geodesics which do not intersect $\alpha$; (right column) all the simple ideal geodesics which do not intersect $\beta$.

Figure 5: A depiction of all fourteen oriented geodesics in $\vec{G}$ which lie on $M$, as a subset of the set $S^1_p$ of all directions emanating from cusp $p$.

4. the four pairs of geodesics in $\vec{G} - \vec{\triangle}$ and the three pairs of geodesics in $\vec{\triangle}$ interlace each other as elements of $S^1_p$ as per Figure 5.

5. as per Figure 5, each of the six elements of $\{\lambda^\pm, \lambda_\alpha^\pm, \lambda_\beta^\pm\} \subset \vec{\triangle}$ is adjacent to two open intervals which constitute connected components of $S^1_p - \vec{G}$. All twelve such open intervals are distinct.

Proof: The existence of these seven pairs of oriented geodesics is due to the existence of unique geodesics representatives of curves on hyperbolic surfaces. Provided that we believe that these are all the simple ideal geodesics on $M$, all five placement properties stated in the lemma are easily deduced from Figure 4, the uniqueness of geodesic representatives for homotopy classes of

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ideal paths and the fact that these geodesics do not intersect if they have homotopy equivalent representative paths which do not intersect. Therefore, the only thing that we need to prove is that there are no other ideal geodesics on $M$. First note that since $\{\lambda^\pm, \mu^\pm\}$ lie on pairs of pants contained in $M$, the eight open intervals adjacent to these four oriented 2-sided geodesics all correspond to self-intersecting geodesics [McS98, Theorem 9]. The remaining four intervals correspond to geodesics which are launched in between $\lambda^-$ and $\mu^-\alpha$ or $\lambda^-\beta$, or $\lambda^+\alpha$ and $\mu^+\beta$. These four intervals have equivalent roles to one another, and so we only consider one of them.

Figure 6: The shaded gray region represents one of the four intervals of directions considered in the previous paragraph.

Any geodesic $\gamma$ launched within one of these four intervals (gray region in Figure 6) will necessarily hit $\alpha$ (without loss of generality) at an angle $\theta < \theta_0$, where $\theta_0$ is the angle between $\alpha$ and $\lambda$. Let us consider this configuration on the universal cover in Figure 7.

Figure 7: The geodesic $\gamma$ must self-intersect.

We see in Figure 7 that a lift $\tilde{\gamma}$ of $\gamma$ is launched from a lift $P \in \partial_\infty \mathbb{H}^2$ of cusp $p$, hits a lift $\tilde{\alpha}$ of $\alpha$ and re-emerges on $\tilde{\alpha}$ as $\tilde{\gamma}$ glide-reflected by a distance of $\ell_\alpha$ along $\tilde{\alpha}$. Denote the point of re-emergence by $Q$. A little hyperbolic
trigonometry (one may use, for example, Theorem 2.2.2 of [Bus10]) suffices to show that the angle $\theta_Q$ between the geodesic $PQ$ and $\tilde{\alpha}$ is strictly greater than $\theta_0$. Since $\tilde{\gamma}$ re-emerges from $Q$ at an angle $\tilde{\theta} < \theta_0 < \theta_Q$ within the triangle bordered by $\tilde{\alpha}, \tilde{\gamma}$ and $PQ$ it must eventually hit one of the sides of this triangle. It cannot hit $PQ$ or $\tilde{\alpha}$ as that would form hyperbolic 2-gons, and therefore must intersect $\tilde{\gamma}$. This intersection descends to a self-intersection point on $\gamma$.

3.2 The classification of simple geodesics

Theorem 9. The following three types of behaviors partition $\tilde{\mathcal{G}}$:

1. $\gamma$ is an isolated point in $\tilde{\mathcal{G}}$ iff. either $\gamma$ has both ends up cusps or if it spirals to a 1-sided geodesic;

2. $\gamma$ is a boundary point of $\tilde{\mathcal{G}}$ iff. $\gamma$ spirals towards a 2-sided simple closed geodesic;

3. $\gamma$ is neither a boundary nor an isolated point of $\tilde{\mathcal{G}}$ iff. $\gamma$ spirals toward a (minimal) geodesic lamination which is not a simple closed geodesic.

Proof. The proof of this result is fairly similar to its orientable-case counterpart and we only outline most of the necessary steps. To begin with, we know that the $\omega$-limit set of the constant speed flow along an oriented geodesic ray $\gamma$ is either a minimal geodesic lamination or empty (i.e.: $\gamma$ goes up a cusp).

Observe that when the $\omega$-limit of $\gamma \in \tilde{\mathcal{G}}$ is a 2-sided geodesic $\alpha$, the geodesic $\gamma$ fattens up to a geodesically bordered pair of pants homotopy equivalent to any sufficiently small $\epsilon$-neighborhood of $\gamma$. Thus, by Lemma 8, there is at least one open interval in $S^1_p - \tilde{\mathcal{G}}$ adjacent to $\gamma$ and hence $\gamma$ is either an isolated point or a boundary point in $\tilde{\mathcal{G}}$. Let $\sigma$ be a simple ideal geodesic which intersects $\alpha$, then the sequence of ideal geodesics obtained by Dehn-twisting $\sigma$ along $\alpha$ is a sequence in $\tilde{\mathcal{G}}$ approaching $\gamma$. Therefore, any geodesic $\gamma$ which spirals to a 2-sided geodesic is a boundary point of $\tilde{\mathcal{G}}$.

Next we consider the case when the $\omega$-limit of $\gamma$ is a geodesic lamination $\Omega(\gamma)$ which is not a simple closed geodesic. We follow Mirzakhani’s proof [Mir07, Theorem 4.6] and show that $\gamma$ is not an isolated point by approximating it by a sequence of geodesics $\{\gamma_i\}$ in $\tilde{\mathcal{G}}$ which each spiral to a distinct simple closed geodesic. Construct a sequence of quasigeodesics $\tilde{\gamma}_i \in \tilde{\mathcal{G}}$ as follows: fix a sequence of positive numbers $\{\epsilon_i\}$ converging to 0. For each $\epsilon_i$, traverse along $\gamma$ until you come to a point $\gamma(t_1)$ along $\gamma$ within distance $\epsilon_i$ of a previous point $\gamma(t_0)$ on $\gamma$ so that

- the geodesic arc $\eta$ between $\gamma(t_0)$ and $\gamma(t_1)$ does not intersect $\gamma|(-\infty, t_1]$ (except at its ends),
- the arc $\eta$ is within $\epsilon_1$ radians of being orthogonal to $\gamma$ at its two ends, and
- the unit tangent vectors $\gamma'(t_0)$ and $\gamma'(t_1)$ are almost parallel; i.e.: the parallel transport of $\gamma'(t_1)$ to $\gamma'(t_0)$ is within $\epsilon_1$ radians of $\gamma'(t_0)$.
Take the quasigeodesic $\hat{\gamma}_i$ to be the path which traverses along $\gamma$ until time $t_0$ and then indefinitely traverses the broken geodesic loop formed by joining $\eta$ and $\gamma|_{[t_0,t_1]}$ and let $\gamma_i \in \mathfrak{G}$ be the simple geodesic representative of $\hat{\gamma}_i$. The sequence $\{\gamma_i\}$ approaches $\gamma$. Moreover, depending on whether $\eta$ is chosen to turn clockwise or anticlockwise when one goes from $\gamma(t_1)$ to $\gamma(t_0)$, we may construct $\{\gamma_i\}$ to approach $\gamma$ from both sides. Therefore $\gamma$ cannot be a boundary point either.

The previous two paragraphs tell us that the only possible isolated points in $\mathfrak{G}$ are ideal geodesics $\gamma$ with both ends up cusps or geodesics $\gamma$ which spiral toward 1-sided simple closed geodesics. Conversely, any such $\gamma$ is an isolated point. If $\gamma$ is an ideal geodesic with both ends up the same cusp (we may assume cusp $p$ wlog), then it is isolated by Lemma 8. If $\gamma$ goes between different cusps, then it fattens to an embedded pair of pants and by McShane’s original proof [McS98, Theorem 9], it must be an isolated point. If $\gamma$ spirals to a 1-sided simple closed geodesic $\alpha$, then $\gamma$ fattens to an embedded cusped Möbius band, and is isolated by Lemma 8. This proves statement 1. Since geodesics $\gamma$ which spiral to a geodesic lamination which is not a closed geodesic cannot be boundary points, this proves statement 2 and hence statement 3.

**Corollary 10.** The set $\mathfrak{G} - \mathfrak{△}$ is a Cantor set of measure 0.

**Proof.** This follows as a consequence of Theorem 9 because $\mathfrak{G} - \mathfrak{△}$ is a (non-empty) perfect, compact, totally disconnected metric space. The fact that it has measure 0 is a consequence of the Birman-Series geodesic sparsity theorem [BS85].

**Note 16.** Theorem 9 tells us that every isolated point in $\mathfrak{G}$ is surrounded by two intervals (one of the left, one on the right) of “directions” in $\mathbb{S}^1$ where geodesics shot out in those directions must self-intersect. In fact, every summand in the Fuchsian McShane identity may be interpreted as the measure of some such interval-pair.

**Note 17.** The set $\mathfrak{G} - \mathfrak{△}$ may be obtained by iteratively process of removing open intervals surrounding $\mathfrak{△}$. In particular, no remnant (i.e.: unremoved) closed interval at any given finite step in this process remains unperturbed — it will, at some stage, have some open interval removed from its “center”. In fact, it is possible to order the removal of these open sets in much the same way as one might when constructing the usual Cantor set, and in this regard, the fact that $\mathfrak{G} - \mathfrak{△}$ is a Cantor set is unsurprising.

### 4 Identities for quasifuchsian representations

We now prove our McShane identity for quasifuchsian representations (Theorem 2) of non-orientable surface groups by first showing that the series constituting one of the sides of our McShane identity yields a holomorphic function, and then invoking a version of the identity theorem (see, for example, Proposition 6.5 of [Lou15]) for holomorphic functions on complex manifolds to assert that the identity holds over the entire quasifuchsian character variety. The first step involves showing that this series is uniformly and absolutely convergent over compact sets on the character variety.
4.1 Series holomorphicity and proof of McShane identity

The aim of this subsection is to show that:

**Proposition 11.** The series

\[ H(p) := \sum_{\{\alpha, \beta\} \in \mathcal{S}(N)} \left( e^{\frac{1}{2}(\ell_\alpha(p) + \ell_\beta(p))} + (-1)^{\alpha \beta} \right)^{-1} \]  

(6)

gives a well-defined holomorphic function on \( \mathcal{QF}(N) \).

**Note 18.** It is also possible to derive this result by combining Corollary 4.2 of [AMS06] with the identity theorem based argument used to prove Theorem 2. The key observation is that the usual McShane identity for \( dN \) is a refinement of the McShane identity for \( N \), where each of the \( S_1(N) \) terms have each been broken into infinitely many terms.

**Proof.** We use the fact that a pointwise convergent sequence of holomorphic functions that is uniformly convergent on all compact sets converges to a holomorphic function. To begin with, we specify an ordering on the summands for (6) and consider the sequence of partial sums for this series.

Let \( R \subset \mathbb{H} = \mathbb{N} \) be a fundamental domain for \( N \) such that \( R \) is a finite sided geodesic ideal polygon. The boundary \( \partial R \) of \( R \) projects to a collection of disjoint ideal geodesics \( \pi(\partial R) \) on \( N \), and every essential simple closed geodesic pair \( \{\gamma_1, \gamma_2\} \in \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \) intersects \( \pi(\partial R) \) transversely and non-trivially. Thus, to any collection of geodesics \( \{\gamma_1, \ldots, \gamma_k\} \), we may assign a positive integer \( \|\{\gamma_1, \ldots, \gamma_k\}\| \) denoting the total number of geodesic segments that \( \{\gamma_1, \ldots, \gamma_k\} \) splits into when cut along \( \pi(\partial R) \). Order the elements of \( \mathcal{S} \) as a sequence \( \{(\gamma_1, \gamma_2)_1 \in \mathbb{N} \) with nondecreasing \( \|\{\gamma_1, \gamma_2\}\| \) and consider the function \( Q_0 : \mathbb{N} \to \mathbb{N} \) counting the number of \( \{\gamma_1, \gamma_2\} \) with \( \|\{\gamma_1, \gamma_2\}\| \leq n \):

\[ Q_0(n) := \text{Card} \{\{\gamma_1, \gamma_2\} \in \mathcal{S} \mid \|\{\gamma_1, \gamma_2\}\| \leq n\} \]

(7)

It is clear that \( Q_0(n) \) is bounded above by \( P_0(|n|)^2 \), for

\[ P_0(n) := \text{Card} \{\gamma - (\text{the image of}) \text{ a simple closed geodesic on } dN \mid \|\gamma\| \leq n\} \]

The function \( P_0 \) is in turn bounded above by polynomial [BS85, Lemma 2.2], and therefore \( Q_0(n) \) is bounded above by a polynomial in \( n \).

Consider the following sequence of partial sums:

\[ H_n(p) := \sum_{\{\gamma_1, \gamma_2\}_1 \text{ for } \ell \leq n} \left( e^{\frac{1}{2}(\ell_{\gamma_1}(p) + \ell_{\gamma_2}(p))} + (-1)^{\gamma_1 \gamma_2} \right)^{-1}. \]

Since the length functions \( \ell_\gamma \) are holomorphic on \( T(dN) = \mathcal{QF}(N) \), each partial sum \( H_n \) is a holomorphic function on \( \mathcal{QF}(N) \).

It remains to show that for any compact set \( C \subset \mathcal{QF}(N) \), the sequence of functions \( \{H_n\}_{n \in \mathbb{N}} \) is uniformly absolutely convergent. We utilize the following fact [AMS06, Lemma 5.2]: let \( p_0 \) be a Fuchsian representation for \( N \), then
for every compact set $C \subset QF(N)$, there exist $C$-dependent constants $c > 0$ and $k > 0$ such that for all $\gamma \in S$ and $\rho \in C$,

$$\frac{1}{k} \|\gamma\| \leq \frac{1}{k} \ell_{\gamma}(\rho) \leq \text{Re}(\ell_{\gamma}(\rho)).$$

Therefore, we obtain the following comparisons:

$$\sum_{\{\gamma_1, \gamma_2\}_i} \left| \left( e^{\frac{1}{2}(\ell_{\gamma_1}(\rho) + \ell_{\gamma_2}(\rho))} + (-1)^{\gamma_1 \cdot \gamma_2} \right)^{-1} \right| \leq \sum_{\{\gamma_1, \gamma_2\}_i} \left( e^{\frac{1}{2}\|\gamma_1 \cdot \gamma_2\|} - 1 \right)^{-1} \leq \sum_{m=1}^{\infty} \frac{Q_0(m) - Q_0(m - 1)}{e^{\pi m} - 1}. \quad (9)$$

The fact that (9) converges ensures that $H(\rho) := \lim_{n \to \infty} H_n(\rho)$ is well-defined and that the sequence $(H_i)$ is uniformly absolutely convergent. Finally, the absolute convergence of this series ensures that this limit is independent of the ordering we placed on $S$ when summing the series.

**Theorem 12.** Given a nonorientable cusped hyperbolic surface $N$ and a lift of a quasifuchsian representation $\tilde{\rho}$ to a representation $\rho : \pi_1(N) \to \text{SL}(2, \mathbb{C})$, and define $S(N)$ as before, then,

$$\sum_{\{\alpha, \beta\} \in S(N)} \left( e^{\frac{1}{2}(\ell_{\alpha}(\rho) + \ell_{\beta}(\rho))} + (-1)^{\alpha \cdot \beta} \right)^{-1} = \frac{1}{2}.$$

**Proof.** By Proposition 11, we know that $H(\cdot)$ defines a holomorphic function on $QF(N)$. Moreover, we know that $H \equiv \frac{1}{2}$ on the Fuchsian locus of $QF(N)$, which is a totally real analytic submanifold of maximal dimension. Thus, the identity theorem [Lou15, Proposition 6.5] tells us that $H(\rho) = \frac{1}{2}$ for every $\rho \in QF(N)$, giving us the desired identity. \hfill \Box

## 5 Identity for horo-core annuli

Given a quasifuchsian representation $\rho : \pi_1(N) \to \text{PSL}(2, \mathbb{C})$, consider the convex core of its corresponding 3-manifold $H/\rho(\pi_1(N))$. Any sufficiently small horospherical cross-section of the cusp $p$ in $H/\rho(\pi_1(N))$ is a flat annulus.

**Definition 7** (horo-core annulus). The conformal structure of this annulus is independent of the chosen horosphere (given that it is sufficiently small). We refer to this flat annulus, up to homothety, as the *horo-core annulus* of $\rho$ at $p$.

Normalize every $\rho$ so that $\rho(m_p) = \pm \left[ \frac{1}{2} \right]$, since the limit curve $C_p$ is invariant under translation by 1 (i.e.: the action of $\rho(m_p)$), there must be points on the limit curve $C_p$ realizing the minimum and the maximum height (i.e.: imaginary component) of $C_p$ on $\mathbb{C}$.

**Definition 8** (width partition of $\bar{\Delta}$). Let $\bar{z}_-$ and $\bar{z}_+$ respectively be a lowest point and a highest point on $C_p$ and let $z_- , z_+$ denote their projected images on $S^1_p = C_p/\mathbb{Z}$. The points $z_\pm$ define a bipartition of $\Delta$ as follows, let:

- $\bar{\Delta}^+(\rho)$ denote the subset of $\bar{\Delta}$ composed of simple bi-infinite geodesics with launching directions in the half-open interval $[z_-, z_+]$ (oriented with respect to $m_p$);
• \( \vec{\Delta}^- (\rho) \) denote the subset of \( \vec{\Delta} \) composed of simple bi-infinite geodesics with launching directions in the half-open interval \([z_-, z_+]\) (also oriented with respect to \( m_p \)).

We call any bipartition \((\vec{\Delta}^+ (\rho), \vec{\Delta}^- (\rho))\) obtained from such a process a width partition.

The main result of this section is the following identity for the modulus of the horo-core annulus of a quasifuchsian representation:

**Theorem 13.** Given a width partition \((\vec{\Delta}^+ (\rho), \vec{\Delta}^- (\rho))\), the modulus \( \text{mod}_p (\rho) \) of the horo-core annulus at \( p \) of a quasifuchsian representation \( \rho \) is given by:

\[
\text{mod}_p (\rho) = \text{Im} \sum_{\{ \alpha, \beta, \epsilon \} \in \vec{\Delta}^+ (\rho)} (e^{\frac{1}{2} (\ell_\alpha (\rho) + \ell_\beta (\rho))} + (-1)^{\alpha \beta})^{-1}
\]

\[
= -\text{Im} \sum_{\{ \alpha, \beta, \epsilon \} \in \vec{\Delta}^- (\rho)} (e^{\frac{1}{2} (\ell_\alpha (\rho) + \ell_\beta (\rho))} + (-1)^{\alpha \beta})^{-1}.
\]

In order to prove this, we first establish a non-orientable generalization of Akiyoshi-Miyachi-Sakuma’s [AMS06, Theorem 2.3].

5.1 Width formula

Let \( \xi, \eta \in \vec{\Sigma} - \vec{\Delta} \) be two oriented simple bi-infinite geodesics emanating from the cusp \( p \), then the pair \( \{ \xi, \eta \} \) bipartitions the set \( \vec{\Delta} \), composed of all oriented simple bi-infinite geodesic arcs with both ends at \( p \), into the following subsets:

• \( \vec{\Delta}^\xi \) consisting of all the geodesic arcs in \( \vec{\Delta} \) which are launched (along the orientation of \( m_p \)) between \( \xi \) (inclusive) and \( \eta \) (exclusive), and

• \( \vec{\Delta}^\eta \) consisting of all the geodesic arcs in \( \vec{\Delta} \) which are launched (along the orientation of \( m_p \)) between \( \eta \) (inclusive) and \( \xi \) (exclusive).

**Lemma 14** (Width formula). Given a quasifuchsian representation \( \rho \) normalized so that the boundary monodromy of \( m_p \) is given by \( \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \). Then, the function \( w^{\xi}_\eta : \Omega \mathcal{F} (N) \to \mathbb{C} \) given by

\[
w^{\xi}_\eta (\rho) = \sum_{\{ \alpha, \beta, \epsilon \} \in \vec{\Delta}^\xi} (e^{\frac{1}{2} (\ell_\alpha (\rho) + \ell_\beta (\rho))} + (-1)^{\alpha \beta})^{-1}
\]

\[
= 1 - \sum_{\{ \alpha, \beta, \epsilon \} \in \vec{\Delta}^\eta} (e^{\frac{1}{2} (\ell_\alpha (\rho) + \ell_\beta (\rho))} + (-1)^{\alpha \beta})^{-1}
\]

is well-defined, holomorphic and gives the complex distance between the (non-\( \infty \)) endpoint \( x \) of \( \xi \), and the (non-\( \infty \)) endpoint \( y \) of \( \eta \).

**Proof.** First note that since \( w^{\xi}_\eta \) is a subseries of (6), our proof of Proposition 11 ensures that \( w^{\xi}_\eta \) is a well-defined holomorphicity function. To show that \( w^{\xi}_\eta \) satisfies (12) and (13) and may be interpreted as the complex distance between \( x \) and \( y \), we show that these properties are satisfied on the Fuchsian locus and invoke the identity theorem.
First, we observe that $x$ and $y$ may be regarded as holomorphic functions on $\mathcal{F}(N)$ as follows: given an arbitrary element $\rho \in \mathcal{F}(N) = \mathcal{T}(N)$, let $\mu$ be a Beltrami differential on $dN$ representing $\rho$. The canonical $\mu$-quasinormal mapping $\psi_{\mu} : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$. Even though this map is dependent on the representative $\mu$ chosen, the restriction of $\psi_{\mu}$ to $\hat{\mathcal{R}}$ is independent of $\mu$ as $\psi_{\mu}$ must take the attracting fixed-points of $\rho_{0}(\gamma)$ to the corresponding attracting fixed points of $\rho(\gamma)$ for every $\gamma \in \pi_{1}(N)$. We denote this restricted function by $\psi_{\rho}$. The holomorphic dependence of $\psi_{\mu}$ with respect to $\mu$ (see, for example, [IT92, Theorem 4.37]) ensures that the function $\psi_{\rho} : \mathcal{F}(N) \times \hat{\mathcal{R}} \to \hat{\mathcal{C}}$ that takes $(\rho, z)$ to $\psi_{\rho}(z)$ is holomorphic in the first coordinate. Take $x_{0}$ and $y_{0}$ to be the points in $\mathcal{R} = \mathcal{C}_{\rho_{0}} \cup \{\infty\}$ which constitute the respective non-infinite endpoints of $\xi$ and $\eta$ with respect to $\rho_{0}$. Then, define the holomorphic functions $x(\rho) := \psi_{\rho}(x_{0})$ and $y(\rho) := \psi_{\rho}(y_{0})$. The function $\bar{w}_{\rho}^{\xi} : \mathcal{F}(N) \to \mathbb{C}$ defined by

$$
\bar{w}_{\rho}^{\xi}(\rho) := x(\rho) - y(\rho)
$$

is therefore also holomorphic.

When $\rho$ is in the Fuchsian locus, the number $\bar{w}_{\rho}^{\xi}(\rho)$ is equal to the length of the horocyclic segment on the length 1 horocycle truncated by $\xi$ and $\eta$ (as measured in the direction along $m_{\rho}$ from $\xi$ to $\eta$). The Birman-Series theorem tells us that the length of this horocyclic segment is equal to the sum of all of the McShane identity “gaps” lying on this segment. This is precisely expressed by the following identity as a consequence of the geometric interpretation of the usual Fuchsian identity:

$$
\bar{w}_{\rho}^{\xi}(\rho) = \bar{w}_{\rho}^{\eta}(\rho) := \sum_{(\alpha, \beta, \epsilon) \in \mathcal{F}_{1}\mathcal{H}_{1}} \left( e^{i(\ell_{\alpha}(\rho) + \ell_{\beta}(\rho))} + (-1)^{\alpha \beta} \right)^{-1}. 
$$

As with the proof of Theorem 2, the identity theorem extends the above Fuchsian identity (15) over the entire quasifuchsian character variety. Replacing $\mathcal{F}(N)$ by $\mathcal{F}(N) \times \{\pm\} = \mathcal{A} = \mathcal{A}_{\mathcal{F}}^{\mathcal{F}} \cup \mathcal{A}_{\mathcal{G}}^{\mathcal{G}}$ in the expression of Theorem 2 doubles the $\frac{1}{2}$ on the right-hand side to a 1, hence giving us equation (13). The complex distance interpretation is because $\bar{w}_{\rho}^{\xi} = \bar{w}_{\rho}^{\eta}$, and the former is defined to be the complex difference between $x$ and $y$.

5.2 The horo-core annulus identity

We now prove Theorem 13.

Proof. Let $z_{-}$ and $z_{+}$ respectively be lowest and highest height points inducing the width partition $(\Delta^{\pm}(\rho), \bar{\Delta}^{\pm}(\rho))$, we first assume that $z_{\pm}$ correspond (as described in the paragraph before Definition 8) to simple bi-infinite geodesics $\xi_{\pm} \in \mathcal{G} - \Delta$. Then, we may set $x = z_{+}$, $y = z_{-}$, $\Delta_{\mathcal{F}} = \Delta^{+}(\rho)$ and $\Delta_{\mathcal{G}} = \Delta^{-}(\rho)$ in Lemma 14. By taking the imaginary component of equations (12) and (13), we see that

$$
\text{Im}(\bar{w}_{\rho}^{\xi}(\rho)) = \text{Im} \sum_{(\alpha, \beta, \epsilon) \in \Delta^{+}(\rho)} \left( e^{i(\ell_{\alpha}(\rho) + \ell_{\beta}(\rho))} + (-1)^{\alpha \beta} \right)^{-1}
$$

$$
= -\text{Im} \sum_{(\alpha, \beta, \epsilon) \in \Delta^{-}(\rho)} \left( e^{i(\ell_{\alpha}(\rho) + \ell_{\beta}(\rho))} + (-1)^{\alpha \beta} \right)^{-1}.
$$
To show that $\text{Im}(w_\rho^\gamma(\rho)) \equiv \text{mod}_p(\rho)$, observe that the horo-core annulus is bounded above by the two hyperbolic planes $P_{\pm}$ with respective ideal boundaries given by
\[
\{u + iv \mid v = \text{Im}(z_{\pm})\} \cup \{\infty\} \subset \hat{C}.
\]
Thus, it is conformally equivalent to a flat annulus obtained by gluing a rectangle of length 1 and width $\text{Im}(z_+ - z_-) = \text{Im}(w_\rho^\gamma(\rho))$. It is well-known that this width also the modulus $\text{mod}_p(\rho)$ of this flat annulus.

To complete our proof, we consider the case when at least one of $z_\pm$ is either self-intersecting or in $\overrightarrow{\triangle}$ and show that it is possible to replace them with points corresponding to simple bi-infinite geodesics which spiral to simple closed geodesics.

Let us assume without loss of generality that $\zeta_+ / \in \overrightarrow{G} - \overrightarrow{\triangle}$. Since $z_+$ is a highest point on the limit curve $C_\rho$, the geodesics $\zeta_+$ must lie on the boundary of the convex core. This in turn means that it must avoid the pleating locus. If not, curve shortening near the pleating locus would show that there is a curve homotopy equivalent to but locally shorter than the geodesic $\zeta_+$. Thus, $\zeta_+$ lies on a geodesic-bordered (smooth) hyperbolic subsurface $X_+$ within the top boundary of the convex core. In particular, the fattening of any sufficiently small $\epsilon$-neighborhood of the subsegment of $\zeta_+$ up to its first point of self-intersection (on the convex core boundary) is topologically a pair of pants (it cannot be a 1-holed Möbius band because the pleated geodesic boundary of the convex core of $H^3/\rho(\pi_1(N)$ is topologically equivalent to an orientable surface $dN$). Since $X_+$ is geodesically convex, it must therefore contain a geodesic bordered pair of pants which contains $\zeta_+$ up to its first point of self-intersection. Since $\overrightarrow{G} - \overrightarrow{\triangle}$ is a Cantor set (Corollary 10), this means that $\zeta_+$ is launched between within a gap region bounded by simple bi-infinite geodesics $\nu_1$ and $\nu_2$ (lying on $X_+$) which spiral to simple closed geodesics.

It should be noted that $P_{\pm}$ contains a lift of the universal cover of $X_+$, and therefore lifts of $\nu_1, \nu_2$ emanating from $\infty$ must have the same height as a lift of $\zeta_+$ emanating from $\infty$. This means that the summand for the gap between the $\nu_i$ is strictly real, and replacing $\zeta_+$ with $\nu_1$ (or $\nu_2$) does not affect equations (10) and (11). Therefore, we may assume without loss of generality that $\zeta_+ \in \overrightarrow{G}$, as desired.

\section{Identities for hyperbolic mapping tori}

\subsection{Simple signature case}

When $\varphi$ is a pseudo-Anosov map of simple signature, let $\phi$ denote its monodromy representation. The longitude $l\rho$ of $\varphi$ is a candidate for the stable letter $l$ for the fundamental group $\pi_1(M_{\varphi})$ as a HNN-extension. This means that the meridian $m\rho$ and the longitude $l\rho$ define a canonical $\mathbb{Z}$-basis $(m\rho, l\rho)$ for the fundamental group of the cusp torus at $p$. We use this basis as a marking basis for the cusp torus $T_{\varphi}$.

\begin{theorem}
Given a pseudo-Anosov map $\varphi$ with simple signature $(0, t)$, the marked modulus $\text{mod}_p(\varphi)$, with respect to the marking $(m\rho, l\rho)$, of the cusp-$p$ torus $T_{\varphi}$ of
a mapping torus $M_\varphi$ is given by:

\[
\text{mod}_p(\varphi) = \left(\frac{2}{t} \sum_{(\alpha, \beta) \in S_\varphi} + \frac{1}{t} \sum_{(\alpha, \beta) \in S_\varphi}^\prime\right) \left(e^{t_{\varphi}(\Phi) + t_{\varphi}(\Phi)} + (-1)^{\alpha \beta}\right)^{-1}. \tag{16}
\]

Proof. Recall that for a pseudo-Anosov map $\varphi$, there is an associated collection of oriented simple geodesics

\[
\{\lambda_1, \lambda_1^+ \lambda_2^+, \ldots, \lambda_i^+, \lambda_i^-\} \subset \mathbb{F}_p
\]

consisting of geodesic representatives for the singular leaves, at $p$, of the stable and unstable foliations of $\varphi$. For each interval $[\lambda_i^-, \lambda_i^+]$, we fix a boundary point $\xi_i^+ \in (\lambda^-_i, \lambda_i^+) \cap \mathbb{F}_p$ and for each interval $[\lambda_i^+, \lambda_{i+1}^-]$ (including the interval from $\lambda_i^+ \to \lambda_i$), we fix a boundary point $\xi_i^- \in (\lambda_i^+, \lambda_{i+1}^-) \cap \mathbb{F}_p$. Since $\xi_i^-$ (resp. $\xi_i^+$) is a boundary point of $\mathbb{F}_p$, it spirals to some oriented simple closed 2-side geodesic, which we denote by $\gamma_i$ (resp. $\gamma_i^+$).

Let $\text{Fix}^+(A)$ denote the attracting fixed point of a loxodromic Möbius transformation $A$. Since $\varphi(t\rho)$ acts on $\mathbb{C}$ via translation, it acts by addition by some number mod$_p(\varphi)$ and for $\gamma = \gamma_i^+$ we have:

\[
\text{mod}_p(\varphi) = \varphi(1) \cdot \text{Fix}^+(\varphi(\gamma)) - \text{Fix}^+(\varphi(\gamma))
\]

Now, let $\xi_i^+$ and let $\eta_i^+ := \varphi_* \xi_i^+$ be the geodesic representative of $\varphi(\xi_i^+)$. By construction, we know that $\eta_i^+$ comes after $\xi_i^+$ on the interval $[\lambda_i^-, \lambda_i^+]$, and Lemma 14 then tells us that:

\[
\text{mod}_p(\varphi) = \lim_{n \to \infty} \left(\text{Fix}^+(\rho_n(\varphi_* \gamma)) - \text{Fix}^+(\rho_n(\gamma))\right).
\]

The restriction of $\varphi$ to $\pi_1(N) \leq \pi_1(M_\varphi)$ is the strong limit of a sequence $\{\rho_n\}$ of quasifuchsian representations of $\pi_1(N)$, therefore (see Claim 3.9 in [AMS06])

\[
\text{mod}_p(\varphi) = \lim_{n \to \infty} \left(\text{Fix}^+(\rho_n(\varphi_* \gamma)) - \text{Fix}^+(\rho_n(\gamma))\right).
\]

Now, let $\xi_i^+$ and let $\eta_i^+ := \varphi_* \xi_i^+$ be the geodesic representative of $\varphi(\xi_i^+)$. By construction, we know that $\eta_i^+$ comes after $\xi_i^+$ on the interval $[\lambda_i^-, \lambda_i^+]$, and Lemma 14 then tells us that:

\[
\text{mod}_p(\varphi) = \lim_{n \to \infty} \sum_{(\alpha, \beta, \varepsilon) \in S^\prime_{\eta_i^+}} \left(e^{t_{\eta_i^+}(\rho_n) + t_{\varphi}(\rho_n)} + (-1)^{\alpha \beta}\right)^{-1}.
\]

Since $\text{Fix}^+(\rho_n(\varphi_* \gamma))$ and $\text{Fix}^+(\rho_n(\gamma))$ are the respective non-$\infty$ end-points for $\varphi_*(\xi)$ and $\xi$, the series $\text{mod}_p(\varphi)$ is precisely given by $\text{Fix}^+(\rho_n(\varphi_* \gamma)) - \text{Fix}^+(\rho_n(\gamma))$ and hence

\[
\text{mod}_p(\varphi) = \lim_{n \to \infty} \sum_{(\alpha, \beta, \varepsilon) \in S^\prime_{\eta_i^+}} \left(e^{t_{\eta_i^+}(\rho_n) + t_{\varphi}(\rho_n)} + (-1)^{\alpha \beta}\right)^{-1}. \tag{18}
\]
On the other hand, we know by construction that $\eta_i$ comes before $\xi_i$ on $[\lambda_i^-, \lambda_i^+]$ and so we have:

$$w_{\eta_i}^{\lambda_i^-}(\rho_n) = \sum_{[\alpha, \beta, \gamma] \in \Delta_{\eta_i}^{\lambda_i^-}} \left( e^{\frac{1}{2}(\ell_\alpha(\rho_n) + \ell_\beta(\rho_n)) + (-1)^{\alpha \beta}} \right)^{-1}. $$

This time, the width $w_{\eta_i}^{\lambda_i^-}(\rho_n)$ is equal to $\text{Fix}^+(\rho_n(\eta_i^-)) - \text{Fix}^+(\rho_n(\phi_*(\eta_i^-)))$, and we instead obtain:

$$\text{mod}_p(\phi) = - \lim_{n \to \infty} \sum_{[\alpha, \beta, \gamma] \in \Delta_{\eta_i}^{\lambda_i^-}} \left( e^{\frac{1}{2}(\ell_\alpha(\rho_n) + \ell_\beta(\rho_n)) + (-1)^{\alpha \beta}} \right)^{-1}. \quad (19)$$

We now turn to the summation index sets $\Delta_{\eta_i}^{\lambda_i^+}, \ldots, \Delta_{\eta_i}^{\lambda_i^-}$ and $\Delta_{\eta_i}^{\lambda_i^+}, \ldots, \Delta_{\eta_i}^{\lambda_i^-}$. Since $\lambda_i^+$ are attractive fixed points of the action of $\phi$ on $S_p$, and $\lambda_i^-$ are the repelling fixed points, the interval $[\xi_i^-, \eta_i^+]$ is a fundamental domain for the action of $\phi$ on $(\lambda_i^-, \lambda_i^+).$ This in turn means that $\phi_*$ induces a bijection between $\Delta_{\eta_i}^{\lambda_i^+}$ and

$$\left( (\lambda_i^-, \lambda_i^+) \cap \Delta \right) / \phi_* \sim \zeta.$$  

Likewise, we get a bijection between $\Delta_{\eta_i}^{\lambda_i^+}$ and $\left( (\lambda_i^+, \lambda_i^+) \cap \Delta \right) / \phi_*$ and hence the following bijection:

$$\Delta_{\eta_i}^{\lambda_i^+} \cup \ldots \cup \Delta_{\eta_i}^{\lambda_i^+} \cup \Delta_{\eta_i}^{\lambda_i^-} \cup \ldots \cup \Delta_{\eta_i}^{\lambda_i^-} \equiv \left( \Delta - [\lambda_i^+] \right) / \phi_* \equiv \Delta / \phi_* \quad (20)$$

The latter equivalence in (20) utilizes the fact that the stable and unstable leaves $\lambda_i^\pm$ cannot have both ends up $p.$ This can be demonstrated by contradiction: the fattened pair of pants or Möbius band of a stable or an unstable leaf $\lambda$ must be (topologically) fixed under the homeomorphic action of $\phi$ (see Note 15), this in turn means that $\phi$ preserves the homotopy class of one of the simple closed geodesic boundaries of the fattening of $\lambda.$ This is impossible for a pseudo-Anosov map $\phi$ according to the classification of surface homeomorphisms.

By Note 15, we know that $\Delta / \phi_*$ naturally identifies with

$$S(N) / \phi_* \times \{\pm\} = S_\phi \times \{\pm\}.$$  

Thus, by summing (18) and (19) over $i$, replacing the indices and invoking Proposition 7.6 of [AMS06] to ensure term-by-term convergence as $\rho_n$ tends to $\phi$, we obtain:

$$\sum_{[\alpha, \beta, \gamma] \in S_\phi \times \{\pm\}} \left( e^{\frac{1}{2}(\ell_\alpha(\phi) + \ell_\beta(\phi)) + (-1)^{\alpha \beta}} \right)^{-1} = t \text{mod}_p(\phi) - t \text{mod}_p(\phi) = 0.$$  

Since the actual summands are independant of $\epsilon \in \{\pm\}$, we may halve the above expression and replace the index set by $S_\phi$. Note that this suffices to prove Theorem 3 when $\phi$ has simple signature.
Instead of summing over $S_\varphi \times \{\pm\}$, we may instead sum only over $\hat{\Delta}^{\ell_t^+}_{\eta_t^+} \cup \ldots \cup \hat{\Delta}^{\ell_t^+}_{\eta_t^+}$. This is equivalent to summing over the collection of all oriented ideal geodesics $\zeta \in \hat{\Delta}/\varphi_*$ which shoot out from $p$ within some interval $(\hat{\Delta} \cap \bigcup_{i=1}^n (\lambda_i^-, \lambda_i^+)) / \varphi_*$. This is tantamount to summing over $\Delta_p^+ / \varphi_*$ (and hence $S_\varphi^+$) twice and $\Delta_p^0$ (and hence $S_\varphi^0$) once, and yields

$$t \cdot \text{mod}_p(\varphi) = \left(2 \sum_{\{\alpha, \beta\} \in S_\varphi^+} + \sum_{\{\alpha, \beta\} \in S_\varphi^0} \right) \left(e^{\pm(\ell_\alpha(\Phi)+\ell_\beta(\Phi))} + (-1)^{\alpha \beta}\right)^{-1},$$

which in turn gives us (16) as desired. Equation (17) is either similarly derived by summing over $\hat{\Delta}^{\eta_t^+}_{\kappa_t^+} \cup \ldots \cup \hat{\Delta}^{\eta_t^+}_{\kappa_t^+}$ or by applying Theorem 3. Finally, since the marking generators $(m_p, l_p)$ for the cusp torus at $p$ are respectively sent to $+\left[\begin{smallmatrix}1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\pm \left[\begin{smallmatrix}1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. The marked modulus for $T_\varphi$ with this marking generator set is $\text{mod}_p(\varphi)$ as asserted.

### 6.2 General signature

We conclude this section by addressing what happens when a pseudo-Anosov maps $\varphi$ has general signature. Given a pseudo-Anosov map $\varphi : N \to N$ of signature $(s, t)$, the map

$$\hat{\varphi} := \varphi \mp_{\text{sign}} : N \to N$$

is also pseudo-Anosov but of simple signature $(0, t)$. Then $M_{\hat{\varphi}}$ is an order $\frac{1}{\gcd(s, t)}$ finite cover of $M_\varphi$ via a covering map $\Pi : M_{\hat{\varphi}} \to M_\varphi$. We denote the respective monodromy representations for these two pseudo-Anosov mapping tori $M_{\varphi}$ and $M_{\hat{\varphi}}$ by $\varphi$ and $\hat{\varphi}$. Let us now prove Theorem 3.

**Proof of Theorem 3.** We already know from the proof of Theorem 5 that for simple signed pseudo-Anosov maps such as $\hat{\varphi}$, we have

$$\sum_{\{\alpha, \beta\} \in S_{\varphi}} \left(e^{\pm(\ell_\alpha(\Phi)+\ell_\beta(\Phi))} + (-1)^{\alpha \beta}\right)^{-1} = 0. \quad (21)$$

Since $M_{\hat{\varphi}}$ is an order $\frac{1}{\gcd(s, t)}$ finite cover of $M_\varphi$, for each pair of geodesics $\{\alpha, \beta\}$ corresponding to a pair of pants or punctured Möbius strip in $\delta(\varphi)$, there are $\frac{1}{\gcd(s, t)}$ isometric configured pairs of geodesics $\{\hat{\alpha}, \hat{\beta}\}$ covering it in $S_{\hat{\varphi}}$. This means that we may simply divide (21) by $\frac{1}{\gcd(s, t)}$ and replace $\hat{\varphi}, \hat{\alpha}, \hat{\beta}$ and $\hat{\varphi}$ with $\varphi, \alpha, \beta$ and $\varphi$ to obtain the desired result.

Finally, we turn to the geometry of the cusp $p$ torus on $M_{\varphi}$. Unlike the simple signature case, the longitude $l_p$ does not pair with the meridian $m_p$ to give a $\mathbb{Z}$-basis for the fundamental group of the cusp $p$ torus of $M_{\varphi}$. Hence, we instead choose a $\mathbb{Z}$-basis $(m_p, l_p)$ for the cusp $p$ torus of $M_{\varphi}$, and let $(\hat{m}_p, \hat{l}_p)$
be the (meridian and longitude) marking generators for the cusp \( p \) torus of \( M_\phi \). Since \( (m_\phi, 1) \) is a \( \mathbb{Z} \)-basis, there is a unique integer \( K_t \) so that
\[
\Pi_t(l_p) = \frac{1}{\phi(s,t)} \cdot l + K_t \cdot m_p
\]
as homotopy classes in the fundamental group of the cusp \( p \) torus of \( M_\phi \).

**Corollary 15.** The marked modulus \( \text{mod}_p(\varphi; l) \) for the cusp \( p \) torus \( T_\varphi \) of \( M_\varphi \), with respect to the basis \( (m_p, 1) \), is given by:
\[
\text{mod}_p(\varphi; l) = \left( \frac{2}{t} \sum_{(\alpha, \beta) \in \mathbb{S}_+^p} + \frac{1}{t} \sum_{(\alpha, \beta) \in \mathbb{S}_-^p} \right) \left( e^{\frac{1}{2}(f_\alpha(\phi) + f_\beta(\phi))} + (-1)^{\alpha, \beta} \right)^{-1} - \frac{K_t \cdot \gcd(s,t)}{t}
\]

**Proof.** Thanks to our normalization condition that
\[
\phi(m_p) = \hat{\phi}(\hat{m}_p) = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]
we know that \( \phi(l) \) and \( \hat{\phi}(\hat{l}_p) \) respectively act on \( \mathbb{C} \subset \partial_{\infty} \mathbb{H}^3 \) as translation by \( \text{mod}_p(\varphi; l) \) and \( \text{mod}_p(\hat{\varphi}) \). Coupling this with (22), we obtain that:
\[
\text{mod}_p(\varphi) = \frac{1}{\gcd(s,t)} \cdot \text{mod}_p(\varphi; l) + K_t.
\]
Rearranging equation (24) to make \( \text{mod}_p(\varphi; l) \) the subject, invoking Theorem 5 to replace \( \text{mod}_p(\hat{\varphi}) \) and employing the same index replacement trick as used in the proof of Theorem 3 then yields (23). 

**A Identity for nonorientable cusped hyperbolic surfaces**

We now give the derivation for Norbury’s nonorientable cusped surface identity from Theorem 2 in [Nor08]. We begin by stating Norbury’s result:

**Theorem** (McShane identity for non-orientable surfaces with borders). Consider a non-orientable hyperbolic surface \( N \) with geodesic borders \( \beta_1, \ldots, \beta_n \). For
\[
R(x, y, z) = x - \ln \frac{\cosh \frac{x}{2} + \cosh \frac{y+z}{2}}{\cosh \frac{x}{2} + \cosh \frac{y-z}{2}}
\]

and
\[
D(x, y, z) = R(z, y, z) + R(x, z, y) - x, \quad E(x, y, z) = R(x, 2z, y) - \frac{z}{2}
\]
on a hyperbolic surface with Euler characteristic \( \neq 1 \) the following identity holds:
\[
\sum_{\alpha, \beta} D(L_1, \ell_\gamma, \ell_\gamma) + \sum_{j=2}^n \sum_{\gamma} R(L_1, L_j, \ell_\gamma) + \sum_{\mu, \nu} E(L_1, \ell_\nu, \ell_\mu) = L_1
\]
where the sums are over simple closed geodesics. The first sum is over pairs of 2-sided geodesics \( \alpha \) and \( \beta \) that bound a pair of pants with \( \beta_1 \), the second sum is over boundary components \( \beta_j \), \( j = 2, \ldots, n \) and 2-sided geodesics \( \gamma \) that bound a pair of pants with \( \beta_1 \) and \( \beta_j \), and the third sum is over 1-sided geodesics \( \mu \) and 2-sided geodesics \( \nu \) that, with \( \beta_1 \) bound a Möbius strip minus a disk containing \( \mu \).
As we deform the hyperbolic structure on \( N \) so as to approach that of a cusped hyperbolic surface, the lengths of the boundaries \( \beta_1, \beta_n \) all tend toward 0. To obtain a McShane-type identity for a cusped hyperbolic surface, it suffices to divide both sides of (27) by \( L_1 \) and take the limit as \( L_1 \) goes to 0. The are two standard approaches to showing that the resulting term-by-term limit convergences correctly to an identity (as opposed to an inequality with \( \leq 1 \) on the right hand side). The first is to use hyperbolic geometry directly to compute the cuspidal identity and then to compare with the term-by-term limit. The second is to show that (27) divided by \( L_1 \) is a uniformly convergent series along the path (on the character variety) deforming the hyperbolic structure on \( N \) to our desired cuspidal hyperbolic structure. We take this second route, and begin by verifying that Norbury’s summands take the desired form in the \( L_1 \to 0 \) limit.

We first observe the following limit:

\[
\lim_{L_1 \to 0} \frac{R(L_1, y, z)}{L_1} = \frac{\cosh \frac{y}{2} + e^{-\frac{z}{2}}}{\cosh \frac{y}{2} + \cosh \frac{z}{2}},
\]

which in turn gives us the following:

\[
\lim_{L_1, L_j \to 0} \frac{1}{L_1} R(L_1, L_j, z) = 2(e^{\frac{y}{2}z} + 1)^{-1}, \quad \text{and} \quad \lim_{L_1 \to 0} \frac{1}{L_1} D(L_1, y, z) = \frac{1}{L_1} (R(L_1, y, z) + R(L_1, z, y) - L_1)
\]

\[
= 2(e^{\frac{y}{2}z} + 1)^{-1}.
\]

The above calculations are standard and omitted. Now, since geodesic length functions are continuous over the character variety \( \text{Rep}(N) \) of all Fuchsian characters (with with parabolic or hyperbolic boundary monodromy) of \( \pi_1(N) \), the existence of the above limits tells us that the functions

\[
\hat{R}(L_1, L_j, \ell_\gamma) := \frac{1}{L_1} R(L_1, L_j, \ell_\gamma) \quad \text{and} \quad \hat{D}(L_1, \ell_\alpha, \ell_\beta) := \frac{1}{L_1} D(L_1, \ell_\alpha, \ell_\beta)
\]

extend uniquely to continuous functions on all of \( \text{Rep}(N) \). We employ the same notation to denote their extensions.

**Note 19.** Our notation differs a little from Norbury’s here for \( \hat{R}(x, y, z) \). To clarify, our \( \hat{R}(x, y, z) \) is equal to Norbury’s \( D(x, y, z) + \hat{R}(x, z, y) \).

Already we are beginning to see that summands in the first two series of Norbury’s identity are taking the form given in the cuspidal identity. For the final summand, we need to do a little rearranging first. For any embedded 1-sided geodesics \( \mu \) and \( \mu' \). Therefore, each summand in the third term of Norbury’s identity arises in a pair \( E(L_1, \ell_\nu, \ell_\mu) + E(L_1, \ell_\nu, \ell_\mu') \). Therefore, we consider the limit

\[
\lim_{L_1 \to 0} \frac{1}{L_1} (E(L_1, \ell_\nu, \ell_\mu) + E(L_1, \ell_\nu, \ell_\mu'))
\]

\[
= \lim_{L_1 \to 0} \frac{1}{L_1} (R(L_1, 2\ell_\mu, \ell_\nu) + R(L_1, 2\ell_\mu', \ell_\nu) - L_1)
\]

\[
= 1 - \frac{\sinh \frac{y}{2}(2\cosh \frac{z}{2} + \cosh \ell_\mu + \cosh \ell_\mu')}{(\cosh \frac{y}{2} + \cosh \ell_\mu)(\cosh \frac{y}{2} + \cosh \ell_\mu')}
\]
Using the following trace relation [Nor08, Equation (6)]
\[
\cosh \frac{L}{2} + \cosh \frac{L}{2} = 2 \sinh \frac{1}{2} \sinh \frac{L}{2}, \text{ with } L \to 0,
\]
we can show that
\[
(\cosh \frac{L}{2} + \cosh \ell_\mu)(\cosh \frac{L}{2} + \cosh \ell_{\mu'}) = (1 + \cosh \frac{L}{2})(2 \cosh \frac{L}{2} + \cosh \ell_\mu + \cosh \ell_{\mu'}). \tag{35}
\]
This then tells us that
\[
\lim_{L \to 0} \frac{1}{L} (E(L, \ell, \ell_\mu) + E(L, \ell, \ell_{\mu'})) = 1 - \frac{\sinh \frac{L}{2}}{\cosh \frac{L}{2} + 1} = 2(e^{\frac{1}{2} \ell} + 1)^{-1}. \tag{37}
\]

The fact that this expression should be independent of \(\ell_\mu\) and \(\ell_{\mu'}\) is, perhaps, somewhat surprising. However, there is a geometric argument for this term which involves cutting up the orientable double cover of \(M\) (which is a hyperbolic sphere with two cusps and two geodesic borders of length \(\ell_\nu\)) along the two “shortest” ideal geodesics joining its two cusps and regluing each of the two resulting connected components into a pair of pants with two cusps and one boundary of length \(\ell_\nu\) (see Figure 8, but replace geodesic boundaries \(\beta_1, \beta_1^B\) and \(\beta_1^B\) with cusps as appropriate). We leave this as an exercise for interested readers.

Equation (26) allows us to break up \(E(L, \ell, \ell_\mu) + E(L, \ell, \ell_{\mu'})\) even more finely as the following summands:
\[
E(L, \ell, \ell_\mu) + E(L, \ell, \ell_{\mu'}) = D(L, \ell, 2\ell_\mu) + D(L, \ell, 2\ell_{\mu'}) + L_1 - R(L, \ell, 2\ell_\mu) - R(L, \ell, 2\ell_{\mu'}). \tag{38}
\]
\[
\text{We already know what the limit, as } L \to 0, \text{ of (38) divided by } L_1 \text{ is. Therefore, we only need to consider}
\]
\[
\lim_{L \to 0} \frac{1}{L} (L_1 - R(L, \ell, 2\ell_\mu) - R(L, \ell, 2\ell_{\mu'})) = 1 - \left(\frac{\cosh \frac{L}{2} + e^{-\ell_\mu}}{\cosh \frac{L}{2} + \cosh \ell_\mu} + \frac{\cosh \frac{L}{2} + e^{-\ell_{\mu'}}}{\cosh \frac{L}{2} + \cosh \ell_{\mu'}}\right). \tag{41}
\]
As before, the existence of this limit means that
\[
\hat{E}(L, \ell_\mu, \ell_{\mu'}) := \frac{1}{L}(L_1 - R(L, \ell, 2\ell_\mu) - R(L, \ell, 2\ell_{\mu'})) \tag{42}
\]
extends to a continuous function over all of \(\text{Rep}(N)\). We again invoke (34) to show that
\[
\cosh \frac{L}{2} + e^{-\ell_\mu} = 2 \sinh(\ell_{\mu'}) \left(-e^{-\ell_{\mu'}} + \sinh \frac{L}{2}\right), \text{ and} \tag{43}
\]
\[
\cosh \frac{L}{2} + \cosh \ell_\mu = 2 \sinh(\ell_{\mu'}) \left(\sinh \frac{L}{2} + \sinh \frac{L}{2}\right). \tag{44}
\]
Incorporating these two identities into (41), we obtain:
\[
\lim_{L \to 0} \frac{1}{L}(L_1 - R(L, \ell, 2\ell_\mu) - R(L, \ell, 2\ell_{\mu'})) = \frac{e^{-\ell_\mu} + e^{-\ell_{\mu'}}}{\sinh \frac{L}{2} + \sinh \frac{L}{2}} = 2(e^{\frac{1}{2}(\ell_\mu + \ell_{\mu'})} - 1)^{-1}. \tag{46}
\]
Therefore, we see that the limit of \( \frac{1}{L} \left[ E(L_1, \ell_\gamma, \ell_\mu) + E(L_1, \ell_\gamma, \ell_\mu') \right] \) takes the form of three distinct terms, and we get:

\[
\left( e^{2 \ell_\gamma} + 1 \right)^{-1} = \left( e^{2 (\ell_\gamma + 2 \ell_\mu)} + 1 \right)^{-1} + \left( e^{2 (\ell_\gamma + 2 \ell_\mu')} + 1 \right)^{-1} + \left( e^{2 (\ell_\mu + \ell_\mu')} - 1 \right)^{-1}. \tag{47}
\]

There are precisely two pairs of pants embedded on \( M \), and they may be obtained from \( M \) by cutting along \( \mu \) and \( \mu' \). One of these pairs of pants has \( \beta_1, \nu \) and the 2-sided double-cover of \( \mu \) as its boundary, and its corresponding gap term is \( e^{2 (\ell_\gamma + 2 \ell_\mu)} - 1 \). The other pair has \( \beta_1, \nu' \) and the 2-sided double cover of \( \mu' \) as its boundary, with corresponding gap term \( e^{2 (\ell_\gamma + 2 \ell_\mu')} - 1 \). The remaining third term is associated to the Möbius band \( M \). Putting all of this data together with our previous two expressions tells us that the term-by-term limiting identity as \( L \) tends to 0 is indeed the one we gave as Norbury’s nonorientable cusped surface identity.

Note that (47) suggests an alternative statement of the cuspidal case identity:

**Theorem (Alternative cuspidal identity).** Let \( S_1^2 (N) \) denote the set of 2-sided geodesics \( \gamma \) on \( N \) which, along with cusp \( p \), bound an embedded Möbius strip and let \( \delta_2^1 (N) \) denote the set of unordered pairs of 2-sided geodesics \( \{ \alpha, \beta \} \) which, along with \( cusp \ p \), bound an embedded pair of pants, which does not lie on an embedded 1-holed Möbius band, on \( N \). Then,

\[
\sum_{\{ \gamma \} \in \delta_1^2 (N)} \left( e^{2 \ell_\gamma} + 1 \right)^{-1} + \sum_{\{ \alpha, \beta \} \in \delta_2^1 (N)} \left( e^{2 (\ell_\alpha + \ell_\beta)} + 1 \right)^{-1} = \frac{1}{2}. \tag{48}
\]

**Note 20.** It is possible to double the cuspidal version of Norbury’s identity and give a probabilistic interpretation of the resulting series in the fuchsian case. The summand

\[
2 \left( e^{2 (\ell_\alpha + \ell_\beta)} + 1 \right)^{-1},
\]

for the index \( \{ \alpha_2, \beta_2 \} \in \delta_2^2 (N) \), is the probability that a geodesic launched from cusp \( p \) will self-intersect before intersecting either \( \alpha \) or \( \beta \). The summand

\[
2 \left( e^{2 (\ell_\alpha + \ell_\beta)} - 1 \right)^{-1},
\]

for the index \( \{ \alpha_1, \beta_1 \} \in \delta_2^1 (N) \) corresponding to a 1-holed Möbius band \( M \) containing \( \alpha_1, \beta_1 \) with boundary \( \gamma \), is the probability that a geodesic launched from \( p \) will intersect both \( \alpha \) and \( \beta \), and then self-intersect before hitting \( \gamma \). For the alternative formulation of the cuspidal case identity, the summand

\[
2 \left( e^{2 \ell_\gamma} + 1 \right)^{-1},
\]

for the index \( \{ \gamma \} \in \delta_1^2 (N) \), is the probability that a geodesic launched from \( p \) will self-intersect before hitting \( \gamma \). Summands for \( \delta_2^2 (N) \) have already been discussed as \( \delta_2^1 (N) \) is a subset of \( \delta_2^2 (N) \).
In particular, we can see from Figure \( \nu \) that for \( \hat{\beta} \) equal the sum of the two gaps on cover \( x \). So far, in taking the term-by-term limit of Norbury’s bordered surface identity gives us the following inequality:

\[
\sum_{\{\alpha_1, \beta_1\} \in S_1(N)} (e^{\hat{\beta}}(e^{\alpha_1} + e^{\beta_1}) - 1)^{-1} + \sum_{\{\alpha_2, \beta_2\} \in S_2(N)} (e^{\hat{\beta}}(e^{\alpha_2} + e^{\beta_2}) + 1)^{-1} \leq \frac{1}{2}.\]  

(49)

To show that this is in fact an equality, we study the behavior of \( \hat{D}(L_1, \ell_\alpha, \ell_\beta) \), \( \hat{R}(L_1, \ell_j, \ell_\gamma) \) and \( \hat{E}(L_1, \ell_\mu, \ell_\mu') \) as the hyperbolic structure on \( N \) deforms to a cuspidal structure. A little algebraic manipulation suffices to show that:

\[
\hat{D}(x, y, z) = \frac{2}{x} \left( 1 + \frac{2 \sinh \frac{x}{2} \sinh \frac{y}{2}}{\cosh \frac{x}{2} + \cosh \frac{y}{2}} \right) = \frac{4 \sinh \frac{x}{2}}{x(\cosh \frac{x}{2} + \cosh \frac{y}{2})}. \]

(50)

Therefore, when \( L_1 \) is sufficiently close to 0, we have

\[
\hat{D}(L_1, \ell_\alpha, \ell_\beta) < 6e^{-\frac{1}{2}(\ell_\alpha + \ell_\beta)}. \]

For \( \hat{R}(L_1, \ell_j, \ell_\gamma) \), we utilize an alternative expression (see, for example, [TWZ06, equation (1.7)]):

\[
\hat{R}(x, y, z) = 2 \tanh^{-1} \left( \frac{\sinh \frac{x}{2} \sinh \frac{y}{2}}{\cosh \frac{x}{2} + \cosh \frac{y}{2}} \right) + \hat{D}(x, y, z) \]

(51)

\[
= \frac{1}{x} \log \left( 1 + \frac{2 \sinh \frac{x}{2} \sinh \frac{y}{2}}{\cosh \frac{x}{2} + \cosh \frac{y}{2}} \right) + \hat{D}(x, y, z) \]

(52)

\[
\leq \frac{2 \sinh \frac{x}{2} \sinh \frac{y}{2}}{\cosh \frac{x}{2} + \cosh \frac{y}{2}} + \hat{D}(x, y, z). \]

(53)

Therefore, when \( x \) and \( y \) are sufficiently close to 0, we have

\[
\hat{R}(L_1, \ell_j, \ell_\gamma) < 3e^{-\frac{1}{2}\ell_\gamma} + 3e^{-\frac{1}{2}(L_1 + \ell_\gamma)} < 6e^{-\frac{1}{2}\ell_\gamma}. \]

For \( \hat{E}(L_1, \ell_\mu, \ell_\mu') \), we employ a small geometric argument. Firstly, we know by construction that \( \hat{E}(L_1, \ell_\mu, \ell_\mu') < \frac{1}{3}(\hat{E}(L_1, \ell_\gamma, \ell_\mu) + \hat{E}(L_1, \ell_\nu, \ell_\mu')) \) and we bound this larger expression instead. On the 1-holed Möbius band \( M \) bounded by \( \beta_1 \) and \( \nu \), there is a unique simple 1-sided orthogeodesic \( \sigma \) with both endpoints based on \( \beta_1 \). Cutting \( M \) along \( \sigma \) results in an annulus, and we may reglue the two sides of this annulus along \( \sigma \) in an orientation preserving way (see Figure 8) so as to obtain a pair of pants with boundaries \( \nu, \beta_1^A \) and \( \beta_1^B \) of respective lengths \( \ell_\nu, L_1^A \) and \( L_1^B \) such that \( L_1^A + L_1^B = L_1 \).

In particular, we can see from Figure 9, which lifts \( M \) to its orientable double cover \( dM \), that the gaps corresponding to \( E(L_1, \ell_\nu, \ell_\mu) + E(L_1, \ell_\nu, \ell_\mu') \) actually equal the sum of the two gaps on \( \beta_1^A \) and \( \beta_1^B \) with total measure

\[
R(L_1^A, \ell_\nu) + R(L_1^B, \ell_\nu) \]

because the cutting and regluing procedure does not affect the positions of the four (red) geodesics spiraling to \( \nu \). Thus, we see that when \( L_1 \) (hence \( L_1^A \) and \( L_1^B \)) is sufficiently close to 0,

\[
\frac{1}{L_1} \left( E(L_1, \ell_\nu, \ell_\mu) + E(L_1, \ell_\nu, \ell_\mu') \right) = \frac{1}{L_1} \left( \hat{R}(L_1^A, \ell_\nu) + L_1^A + \hat{R}(L_1^B, \ell_\nu) + L_1^B \right) \]

\[
< 6e^{-\frac{1}{2}\ell_\nu} L_1^A + 6e^{-\frac{1}{2}\ell_\nu} L_1^B = 6e^{-\frac{1}{2}\ell_\nu}. \]

(54)
Figure 8: A Möbius band $M$ (left) cut along $\sigma$ (center) and reglued to form a pair of pants (right).

Figure 9: The red shaded region on the left diagram has length $E(L_1, \ell_\nu, \ell_\mu) + E(L_1, \ell_\nu, \ell_\mu')$ along $\beta_1$; the two red shaded regions on the right half respectively have lengths $R(L^A_1, L^B_1, \ell_\nu)$ and $R(L^A_1, L^B_1, \ell_\nu')$.

At this point, we may invoke the same argument as used in the proof of Proposition 11 to obtain a similar polynomial divided by exponential type expression as (9) for the tail of Norbury’s identity (upon appropriate rearrangement of the series). This ensures the uniform convergence of the bordered case identity as the hyperbolic structure on $N$ deforms to a cusped structure, thus allowing us to conclude that the term-by-term limit is in fact an equality.

Note 21. The above arguments obviously apply when the underlying surface is orientable, thus furnishing the nitty-gritty details for the proof of Mirzakhani’s Corollary 4.3.

Note 22. Our uniform convergence arguments also apply when an interior simple closed geodesic deforms to a cusp. Therefore, starting with a McShane identity for a surface with greater topological complexity, we may take these limits to derive identities for surfaces with lower complexity simply by taking term-by-term limits. In particular, simple geodesics which intersect the shrinking geodesic(s) must tend to length $\infty$ and summands expressing their lengths therefore tend to 0 and are excluded from the identity. This was previously noted in the special case when a pair of simple closed geodesics $\alpha, \beta$, which bound a pair of pants with $\beta_1$, deform to cusps [AMS04, Example 2.2 (2)].

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