The jump of the Milnor number in the $X_9$ singularity class

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December 11, 2013

Abstract

The jump of the Milnor number of an isolated singularity $f_0$ is the minimal non-zero difference between the Milnor numbers of $f_0$ and one of its deformations $(f_s)$. We prove that for the singularities in the $X_9$ singularity class their jumps are equal to 2.

1 Introduction

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an (isolated) singularity, i.e. $f_0$ is a germ at 0 of a holomorphic function having an isolated critical point at $0 \in \mathbb{C}^n$, and $0 \in \mathbb{C}$ as the corresponding critical value. More specifically, there exists a representative $\hat{f}_0 : U \to \mathbb{C}$ of $f_0$, holomorphic in an open neighborhood $U$ of the point $0 \in \mathbb{C}^n$, such that $\hat{f}_0(0) = 0$, $\nabla \hat{f}_0(0) = 0$ and $\nabla \hat{f}_0(z) \neq 0$ for $z \in U \setminus \{0\}$, where for a holomorphic function $f$ we put $\nabla f = \nabla_z f := (\partial f / \partial z_1, \ldots, \partial f / \partial z_n)$.

In the sequel we will identify germs of holomorphic functions with their representatives or the corresponding convergent power series. The ring of germs of holomorphic functions of $n$ variables will be denoted by $\mathcal{O}^n$.

A deformation of the singularity $f_0$ is the germ of a holomorphic function $f = f(s,z) : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that:

1. $f(0,z) = f_0(z)$.

\*AMS subject classification: 32S05, 14B05, 32S30, 14B07, Keywords: Milnor number, singularity, deformation of singularity

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2. \( f(s,0) = 0 \),

3. for each \( |s| \ll 1 \) it is \( \nabla_z f(s,z) \neq 0 \) for \( z \neq 0 \) in a (small) neighborhood of \( 0 \in \mathbb{C}^n \).

The deformation \( f(s,z) \) of the singularity \( f_0 \) will also be treated as a family \( (f_s) \) of germs, taking \( f_s(z) := f(s,z) \). In this context, the symbol \( \nabla f_s \) will always denote \( \nabla_z f_s \).

**Remark.** Notice that in the deformation \( (f_s) \) of \( f_0 \) there can occur smooth germs, that is germs satisfying \( \nabla f_0(0) \neq 0 \).

By the above assumptions it follows that, for every sufficiently small \( s \), one can define a (finite) number \( \mu_s \) as the Milnor number of \( f_s \), namely

\[
\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}_s^n / (\nabla f_s) = i_0 \left( \frac{\partial f_s}{\partial z_1}, \ldots, \frac{\partial f_s}{\partial z_n} \right),
\]

where the symbol \( i_0 \left( \frac{\partial f_s}{\partial z_1}, \ldots, \frac{\partial f_s}{\partial z_n} \right) \) denotes the multiplicity of the ideal \( \left( \frac{\partial f_s}{\partial z_1}, \ldots, \frac{\partial f_s}{\partial z_n} \right) \mathcal{O}_s^n \).

Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities [GLS07, Ch. I, Thm. 2.6 and Ch. II, Prop. 2.57], there exists an open neighborhood \( S \) of the point \( 0 \in \mathbb{C}^n \) such that

1. \( \mu_s = \text{const. for } s \in S \setminus \{0\} \),

2. \( \mu_0 \geq \mu_s \) for \( s \in S \).

The (constant) difference \( \mu_0 - \mu_s \) for \( s \in S \setminus \{0\} \) will be called the jump of the deformation \( (f_s) \) and denoted by \( \lambda((f_s)) \). The smallest nonzero value among all the jumps of deformations of the singularity \( f_0 \) will be called the jump (of the Milnor number) of the singularity \( f_0 \) and denoted by \( \lambda(f_0) \).

The first general result concerning the problem of computation of the jump was S. Gusein-Zade’s [Gus93], who proved that there exist singularities \( f_0 \) for which \( \lambda(f_0) > 1 \) and that for irreducible plane curve singularities \( f_0 \) it holds \( \lambda(f_0) = 1 \). He showed that generic elements in some classes of singularities (satisfying conditions concerning the Milnor numbers and modality) fulfill \( \lambda(f_0) > 1 \), but he did not give any specific example of such a singularity.

The two-dimensional version of the problem of computation of the jump, and more precisely – of the non-degenerate jump (i.e., all the families \( (f_s) \) being considered are to be made of Kouchnirenko non-degenerate singularities), has been studied in [Bod07], [Wal10].

The following are examples of classes of singularities that fulfill the assumptions of the Gusein-Zade theorem.

1. The class \( X_9 \), in the terminology of [AGV85]. It consists of singularities stably equivalent to the singularities of the form \( f_0^a(x,y) := x^4 + y^4 + ax^2y^2, a \in \mathbb{C}, a^2 \neq 4 \). The singularities are of modality 1 and \( \mu(f_0^a) = 9 \).
2. The class $W_{1,0}$, in the terminology of [AGV85]. It consists of singularities stably equivalent to the singularities of the form $f^{(a,b)}_0(x,y) := x^4 + y^6 + (a + by)x^3, a, b \in \mathbb{C}, a^2 \neq 4$. The singularities are of modality 2 and $\mu(f^{(a,b)}_0) = 15$.

By the Gusein-Zade result, generic elements $f$ of the classes $X_9$ and $W_{1,0}$ satisfy $\lambda(f) > 1$. However, determining the jump of any particular element of these classes is still an open and difficult problem. Gusein-Zade did not give any specific example of a singularity $f$ with $\lambda(f) > 1$. The purpose of this work is to prove (Thm. 5) that for the singularities in the $X_9$ class we have

$$\lambda(f_0^a) = 2$$

(and that therefore all the singularities of the class $X_9$ are „generic” in the family $X_9$). In the class $W_{1,0}$ we obtain only a partial result (Prop. 3). Namely, for the singularities in $W_{1,0}$ that are stably equivalent to the ones in the subclass

$$f^{(0,b)}_0(x,y) = x^4 + y^6 + bx^2y^4, \ b \in \mathbb{C},$$

we have

$$\lambda(f^{(0,b)}_0) = 1$$

therefore these singularities are not „generic” in the family $W_{1,0}$).

This implies that the jump $\lambda(f_0)$ is not a topological invariant of singularities (Cor. 2).

In the light of the above results the following problems arise:

1. Show that for the remaining singularities in the $W_{1,0}$ class, i.e. for the singularities stably equivalent to $f^{(a,b)}_0 := x^4 + y^6 + (a + by)x^3, a, b \in \mathbb{C}, 0 \neq a^2 \neq 4$, we have $\lambda(f^{(a,b)}_0) = 2$,

and more general ones (posed by Bodin in [Bod07]):

(2) Find an algorithm that computes $\lambda(f_0)$.

(3) Give the list of all possible Milnor numbers arising from deformations of $f_0$ (see [Wal10] for partial results in the non-degenerate case).

2 Preliminaries

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{R}_+$ be the set of nonnegative real numbers. Let $f_0(x,y) = \sum_{(i,j) \in \mathbb{N}^2} a_{ij}x^iy^j$ be a singularity. Put $\text{supp}(f_0) := \{(i,j) \in \mathbb{N}^2 : a_{ij} \neq 0\}$. The Newton diagram of $f_0$ is defined as the convex hull of the set

$$\bigcup_{(i,j) \in \text{supp}(f_0)} (i,j) + \mathbb{R}^2_+.$$
and is denoted by $\Gamma_+(f_0)$. It is easy to see that the boundary (in $\mathbb{R}^2$) of the diagram $\Gamma_+(f_0)$ is a sum of two half-lines and a finite number of compact line segments. The set of those line segments will be called the Newton polygon of the singularity $f_0$ and denoted by $\Gamma(f_0)$. For each segment $\gamma \in \Gamma(f_0)$ we define a weighted homogeneous polynomial

$$
(f_0)_\gamma := \sum_{(i,j) \in \gamma} a_{ij}x^iy^j.
$$

A singularity $f_0$ is called non-degenerate (in the Kouchnirenko sense) on a segment $\gamma \in \Gamma(f_0)$ iff the system

$$
\frac{\partial (f_0)_\gamma}{\partial x}(x,y) = 0, \quad \frac{\partial (f_0)_\gamma}{\partial y}(x,y) = 0
$$

has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$. $f_0$ is called non-degenerate iff it is non-degenerate on every segment $\gamma \in \Gamma(f_0)$.

For the sake of simplicity, we consider the case of convenient singularities $f_0$, i.e. we suppose that $\Gamma_+(f_0)$ intersects both coordinate axes in $\mathbb{R}^2$. For such singularities we denote by $S$ the area of the domain bounded by the coordinate axes and the Newton polygon $\Gamma(f_0)$. Let $a$ (resp. $b$) be the distance of the point $(0,0)$ to the intersection of $\Gamma_+(f_0)$ with the horizontal (resp. vertical) axis. The number

$$
\nu(f_0) := 2S - a - b + 1
$$

is called the Newton number of the singularity $f_0$. Let us recall Planar Kouchnirenko Theorem.

**Theorem 1.** ([Kou76]) For a convenient singularity $f_0$ we have:

1. $\mu(f_0) \geq \nu(f_0)$.
2. if $f_0$ is non-degenerate then $\mu(f_0) = \nu(f_0)$.

Theorem 1 can be completed in the following way.

**Theorem 2.** (Płoski, [Pło90, Pło99]) If for a convenient singularity $f_0$ there is $\nu(f_0) = \mu(f_0)$ then $f_0$ is non-degenerate.

We will also need a „global” result concerning projective algebraic curves.

**Theorem 3.** ([GP01 Prop. 6.3]) Let $\mathcal{C} \subset \mathbb{CP}^2$ be a projective algebraic curve of degree $d$. Suppose that $m$ irreducible components of $\mathcal{C}$ pass through a point $P \in \mathcal{C}$. Then the Milnor number $\mu_P(\mathcal{C})$ of $\mathcal{C}$ at $P$ satisfies the inequality

$$
\mu_P(\mathcal{C}) \leq (d-1)(d-2) + m - 1.
$$
The rest of the section is devoted mainly to the concept of a versal unfolding. It is based on the book by Ebeling [Ebe07].

Let \( f_0: (\mathcal{C}^n, 0) \rightarrow (\mathcal{C}, 0) \) be a germ of a holomorphic function. An unfolding of \( f_0 \) is a holomorphic germ \( F: (\mathcal{C}^n \times \mathcal{C}^k, 0) \rightarrow (\mathcal{C}, 0) \) such that \( F(z, 0) = f_0(z) \) and \( F(0, u) = 0 \).

Two unfoldings \( F: (\mathcal{C}^n \times \mathcal{C}^k, 0) \rightarrow (\mathcal{C}, 0) \) and \( G: (\mathcal{C}^n \times \mathcal{C}^k, 0) \rightarrow (\mathcal{C}, 0) \) of \( f_0 \) are said to be equivalent, if there exists a holomorphic map-germ

\[
\psi: (\mathcal{C}^n \times \mathcal{C}^k, 0) \rightarrow (\mathcal{C}^n, 0), \quad \psi(z, 0) = z, \quad \psi(0, u) = 0
\]

such that

\[
G(z,u) = F(\psi(z,u), u).
\]

It is easy to see that this notion of equivalence is in fact an equivalence relation in the set of unfoldings of \( f_0 \).

Let \( F: (\mathcal{C}^n \times \mathcal{C}^k, 0) \rightarrow (\mathcal{C}, 0) \) be an unfolding of \( f_0 \) and \( \varphi: (\mathcal{C}^l, 0) \rightarrow (\mathcal{C}^k, 0) \) - a holomorphic map-germ. The unfolding of \( f_0 \) induced from \( F \) by \( \varphi \) is defined by the formula

\[
G(z, u) = F(z, \varphi(u)).
\]

An unfolding \( F: (\mathcal{C}^n \times \mathcal{C}^k, 0) \rightarrow (\mathcal{C}, 0) \) of \( f_0 \) is called versal if any unfolding of \( f_0 \) is equivalent to one induced from \( F \).

The following proposition will be useful.

**Proposition 1.** ([Mar82, Ch. 4, Prop. 2.4]) If \( f \in \mathcal{O}^n \) is a singularity, \( m \) is the maximal ideal in \( \mathcal{O}^n \), then

\[
\dim_{\mathcal{C}} \frac{\mathcal{O}^n}{m(Vf) \mathcal{O}^n} = \dim_{\mathcal{C}} \frac{\mathcal{O}^n}{(Vf) \mathcal{O}^n} + n.
\]

The main result concerning versal unfoldings is the following.

**Theorem 4.** Let \( f_0: (\mathcal{C}^n, 0) \rightarrow (\mathcal{C}, 0) \) be a singularity and put \( \mu = \mu(f_0) \). Let \( g_1, \ldots, g_{\mu+n-1} \in \mathcal{O}^n \) be any representatives of a basis of the \( \mathcal{C} \)-vector space \( \frac{m}{m(\mathcal{C}^n)} \). Then the holomorphic germ

\[
F: (\mathcal{C}^n \times \mathcal{C}^{\mu+n-1}, 0) \rightarrow (\mathcal{C}, 0)
\]

defined as

\[
F(z, u) := u_1g_1(z) + \ldots + u_{\mu+n-1}g_{\mu+n-1}(z) + f_0(z)
\]

is a versal unfolding of \( f_0 \).

**Remark.** The proof of the above theorem runs in a very similar way to that given by Ebeling ([Ebe07, Prop. 3.17]); see also [Wal81, Thm. 3.4] for a more general, but less explicit, approach to the concept of a versal unfolding and a proof of Theorem 3.

Let \( f: (\mathcal{C}^m, 0) \rightarrow (\mathcal{C}, 0) \) and \( g: (\mathcal{C}^m, 0) \rightarrow (\mathcal{C}, 0) \) be two germs of holomorphic functions. We say that \( f \) is stably equivalent to \( g \) (see [AGV85]) iff there exists \( p \in \mathbb{N}, \mu \geq \max\{m, n\} \), such that \( \tilde{f} := f(z_1, \ldots, z_\mu) + \frac{z_{\mu+1}^2}{2} + \ldots + \frac{z_p^2}{2} \) is biholomorphically
equivalent to \( \tilde{g} := g(w_1, \ldots, w_m) + w_{m+1}^2 + \ldots + w_{m'}^2 \), i.e. there exists a biholomorphism \( \Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) such that \( \tilde{f} \circ \Phi = \tilde{g} \).

It is easy to check that the Milnor number of a singularity is an invariant of the stable equivalence. The same is true for the jump of a singularity.

**Proposition 2.** The jump of a singularity is an invariant of the stable equivalence.

**Proof.** Since obviously \( \lambda (f) = \lambda (g) \) for any two biholomorphically equivalent singularities \( f \) and \( g \), it suffices to prove that for a singularity \( f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) the equality

\[
\lambda (f_0(z)) = \lambda (f_0(z) + z_{n+1}^2)
\]

holds, where \( z = (z_1, \ldots, z_n) \).

First we consider the case \( \mu (f_0) = 1 \). Clearly, \( \text{ord} f_0 = 2 \). For the deformation \( f_s(z) := f_0(z) + sz_1 \) we have \( \mu (f_0) - \mu (f_s) = 1, s \neq 0 \). Hence \( \lambda (f_0) = 1 \). Similarly, \( \lambda (f_0(z) + z_{n+1}^2) = 1 \).

Now assume that \( \mu (f_0) \geq 2 \).

First note, that if \( f_s \) is a deformation of \( f_0 \) then the family \( (f_s(z) + z_{n+1}^2) \) is a deformation of \( f_0(z) + z_{n+1}^2 \). Clearly, \( \mu (f_s(z) + z_{n+1}^2) = \mu (f_s(z)) \) so

\[
\lambda (f_0(z)) \geq \lambda (f_0(z) + z_{n+1}^2). 
\]

To prove the opposite inequality we take a deformation \( (g_s) \) of \( g_0(z, z_{n+1}) := f_0(z) + z_{n+1}^2 \) that realizes \( \lambda (g_0) \) i.e. \( \mu (g_0) - \mu (g_s) = \lambda (g_0) \) for \( s \neq 0 \). Let, by Theorem 4, \( h_1, \ldots, h_{\mu+n-1} \in \mathcal{O}^n \) constitute a basis of \( \frac{m_0}{m_\nu h_{0(V_0)} \otimes \mathcal{O}^{n+1}} \), where \( \mu := \mu (f_0) \) and \( m_\nu \) is the maximal ideal of \( \mathcal{O}^n \). Then \( h_1, \ldots, h_{\mu+n-1}, z_{n+1} \) constitute a basis of \( \frac{m_{n+1}}{m_0(V_0)} \). Hence, up to a biholomorphism, we may assume that

\[
g_s(z, z_{n+1}) = v_1(s) h_1(z) + \ldots + v_{\mu+n-1}(s) h_{\mu+n-1}(z) + v_{\mu+n}(s) z_{n+1} + f_0(z) + z_{n+1}^2,
\]

for holomorphic \( v_1, \ldots, v_{\mu+n} : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \).

We claim that the \( g_s \)'es are not smooth. Indeed, in the opposite case we would have for \( s \neq 0 \)

\[
\lambda (g_0) = \mu (g_0) - \mu (g_s) = \mu (g_0).
\]

On the other hand, for the deformation \( \tilde{g}_s(z, z_{n+1}) := s(z_1^2 + \ldots + z_n^2) + g_0(z, z_{n+1}) \) of \( g_0 \) we would have, for sufficiently small \( s \neq 0 \), \( \mu (\tilde{g}_s) = 1 \) and then \( \mu (g_0) - \mu (\tilde{g}_s) = \mu (f_0) - 1 > 0 \). Hence \( \lambda (g_0) \leq \mu (g_0) - \mu (\tilde{g}_s) = \mu (g_0) - 1 \), a contradiction to \( \lambda (g_0) \).

Since the \( g_s \)'es are not smooth, \( v_{\mu+n} = 0 \). Thus for the deformation \( f_s(z) := v_1(s) h_1(z) + \ldots + v_{\mu+n-1}(s) h_{\mu+n-1}(z) + f_0(z) \) of \( f_0 \) there is \( \mu (g_s) = \mu (f_s) \) and

\[
\lambda (g_0) = \mu (g_0) - \mu (g_s) = \mu (f_0) - \mu (f_s) = \lambda (f_s).
\]

This implies \( \lambda (f_0) \leq \lambda (g_0) \). \( \square \)
3 Main Results

In this section we will present the proofs of the results. We begin with the main theorem, concerning the class $X_0$.

**Theorem 5.** For the singularities
\[ f_0^n(x, y) = x^4 + y^4 + ax^2y^2, \]
where $a \in \mathbb{C}, a^2 \neq 4$, we have
\[ \lambda(f_0^n) = 2. \]
Moreover, for every singularity of type $X_0$ its jump is equal to 2.

First we state and prove a lemma.

**Lemma 1.** The (classes of the) monomials $x^iy^j$ with $0 < i + j \leq 3$ and the monomial $x^2y^2$ form a basis of the $\mathbb{C}$-vector space $m/ m(\nabla f_0^n)$.

**Proof.** We have $\nabla f_0^n(x, y) = (4x^3 + 2axy^2, 4y^3 + 2ax^2y)$. Let us note that $x^5, x^3y \in m(\nabla f_0^n)$ because
\[ x^5 = \left( \frac{x^2}{4} + \frac{2ay^2}{4(a^2 - 4)} \right) \frac{\partial f_0^n}{\partial x} + \left( \frac{-a^2xy}{4(a^2 - 4)} \right) \frac{\partial f_0^n}{\partial y} \]
and
\[ x^3y = \left( \frac{-y}{a^2 - 4} \right) \frac{\partial f_0^n}{\partial x} + \left( \frac{ax}{2(a^2 - 4)} \right) \frac{\partial f_0^n}{\partial y}. \]
Since $f_0^n$ is symmetric with respect to $x$ and $y$, also $y^5, xy^3 \in m(\nabla f_0^n)$. Hence the classes of the monomials
\[ x, y, x^2, xy, x^2y, x^3, x^2y^2, x^4, x^3y^2, xy^3 \]
generate $m/ m(\nabla f_0^n)$. Since $x^4 \equiv -\frac{3}{2}y^2, y^4 \equiv -\frac{3}{2}x^2y^2$ modulo $m(\nabla f_0^n)$, we get that the classes of the monomials $x^iy^j$ with $0 < i + j \leq 3$ and the monomial $x^2y^2$ also generate the space $m/ m(\nabla f_0^n)$. They form a basis of $m/ m(\nabla f_0^n)$ because by Proposition 1 $\dim_{\mathbb{C}} m/ m(\nabla f_0^n) = \dim_{\mathbb{C}} \partial^n / m(\nabla f_0^n) - 1 = \dim_{\mathbb{C}} \partial^n / (\nabla f_0^n) \partial^n + 1 = \mu(f_0^n) + 1 = 10$. □

**Proof of Theorem 5.** By Proposition 2 it is enough to prove the first part of the theorem. Let us fix $a \in \mathbb{C}, a^2 \neq 4$ and let $f_0 := f_0^n$. We have $\mu(f_0) = 9$. Let us consider the deformation
\[ f_s(x, y) := x^4 + (y^2 + sx)^2 + ax^2(y^2 + sx) \]
of $f_0$. Let us apply the change of coordinates: $x \mapsto x - sy^2, y \mapsto sy$, for $s \neq 0$. In these coordinates the $f_s$'es take the form
\[ \tilde{f}_s(x, y) = s^2x^4 + as^3xy^2 + s^4y^8 + \left[(ax^3 + x^4 - 2a^2s^2x^2y^2 - 4sx^3y^2 + 6s^2x^2y^4 - 4s^3xy^6) \right]. \]
It is easily seen that such \( f_s \)'s are non-degenerate if \( s \neq 0 \). Thus, by Kouchnirenko theorem, we get \( \mu (f_s) = \nu (f_s) = 7 \) and so
\[
\mu (f_s) = 7 \quad \text{for} \quad s \neq 0. \tag{4}
\]
This means that \( \lambda (f_s) = 2 \) and therefore \( \lambda (f_0) \leq 2 \). By the definition of the jump of a singularity, there are only two cases: \( \lambda (f_0) = 1 \) or \( \lambda (f_0) = 2 \). We will exclude the first possibility. Suppose to the contrary, that there exists a deformation \( (f_s) \) of the singularity \( f_0 \) with the property that
\[
\mu (f_s) = 8 \quad \text{for} \quad s \neq 0. \tag{5}
\]
By Theorem\textsuperscript{4} and Lemma\textsuperscript{1} we may assume that
\[
f_s (x, y) = s_{10} (s) x + s_{91} (s) y + s_{20} (s) x^2 + s_{11} (s) xy + s_{02} (s) y^2 + s_{30} (s) x^3 + s_{21} (s) x^2 y + s_{12} (s) y^3 + s_{03} (s) \gamma + s_{22} (s) x^2 y^2 + f_0 (x, y),
\]
where \( s_{10}, \ldots, s_{22} : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \) are holomorphic. Since \( \deg f_s = 4 \) and \( \mu (f_s) = 8 \) for \( s \neq 0 \), by Theorem\textsuperscript{3} three or four of the irreducible components of the curve \( \mathcal{C} := \{(x, y) \in \mathbb{C}^2 : f_s (x, y) = 0\} \) pass through the origin. Hence \( \ord f_s = 3 \) or \( \ord f_s = 4 \), for \( 0 < |s| \ll 1 \). The latter case is impossible by Theorem\textsuperscript{1} because then \( \mu (f_s) \geq \nu (f_s) \geq 9 \). Thus, it suffices to consider the case \( \ord f_s = 3 \). So, assume \( \ord f_s = 3 \) for \( s \neq 0 \). Fix any small \( s_0 \in \mathbb{C} \setminus \{0\} \). We can write
\[
f_{s_0} (x, y) = s_{30} x^3 + s_{21} x^2 y + s_{12} xy^2 + s_{03} y^3 + (s_{22} + a) x^2 y^2 + x^4 + y^4,
\]
with \( s_{ij} = s_{ij} (s_0) \in \mathbb{C} \). Since \( \ord f_{s_0} = 3 \), \( f_0 \) has to be degenerate. Otherwise, by checking all the possible cases, we would get \( \mu (f_{s_0}) \leq 6 \) (by the Kouchnirenko theorem), which contradicts (5). Since \( \gcd (3, 4) = 1 \), the degeneracy of \( f_{s_0} \) may only happen on a segment of \( \Gamma (f_{s_0}) \) lying in the line: \( u + v = 3 \). So, we may write
\[
f_{s_0} (x, y) = (\alpha x + \beta y)^2 (\gamma x + \delta y) + (s_{22} + a) x^2 y^2 + x^4 + y^4,
\]
for some \( \alpha, \beta, \gamma, \delta \in \mathbb{C}, |\alpha| + |\beta| > 0, |\gamma| + |\delta| > 0 \). Moreover, \( \alpha \neq 0 \) and \( \beta \neq 0 \) because otherwise \( f_{s_0} \) would be non-degenerate. If we change coordinates: \( \alpha x + \beta y \mapsto x, y \mapsto y \) then \( f_{s_0} \) takes the form
\[
\widetilde{f}_{s_0} (x, y) = x^2 (e x + \zeta y) + P_4 (x, y),
\]
where \( e, \zeta \in \mathbb{C}, |e| + |\zeta| > 0 \), and \( P_4 \) is a non-zero homogeneous polynomial of degree 4. We easily check, considering all the possible cases, that \( f_{s_0} \) is non-degenerate. So, again by the Kouchnirenko theorem, we would have
\[
\mu (f_{s_0}) = \mu (\widetilde{f}_{s_0}) = \nu (\widetilde{f}_{s_0}) \leq 6,
\]
which contradicts (5).

Now we prove a partial result concerning the class \( W_{1,0} \). \qed
Proposition 3. For the singularities \( f_0^{(0,b)}(x,y) = x^4 + y^6 + bx^2y^4 \), where \( b \in \mathbb{C} \), we have
\[
\lambda(f_0^{(0,b)}) = 1.
\]
In particular, \( \lambda(x^4 + y^6) = 1 \).

Proof. Fix \( b \in \mathbb{C} \). Since \( f_0^{(0,b)} \) is Kouchnirenko non-degenerate, it follows that \( \mu(f_0^{(0,b)}) = V(f_0^{(0,b)}) = 15 \). Consider the following deformation of \( f_0^{(0,b)} \):
\[
f_s^{(0,b)}(x,y) := x^4 + (y^2 + sx)^3 + bx^2y^4.
\]
The deformation consists of degenerate singularities (for \( s \neq 0 \)). Apply the following change of coordinates: \( x \mapsto x - sy^2, y \mapsto sy \). In these coordinates the \( f_s^{(0,b)} \) take the form
\[
f_s^{(0,b)}(x,y) = s^3x^3 + (s^4 + bs^6)y^8 + \left[x^4 - 4sx^3y^2 + (6sx^2 + bs^4)x^2y^4 - (4s^3 + 2bs^5)xy^6\right].
\]
It is immediately seen that for \( s \neq 0 \) the singularities \( f_s^{(0,b)} \) are non-degenerate and so
\[
\mu(f_s^{(0,b)}) = 14.
\]
Since the Milnor number is a biholomorphic (and even a topological) invariant of a singularity, there is also
\[
\mu(f_0^{(0,b)}) = 14.
\]
It means that for this particular deformation \( (f_s^{(0,b)}) \) of \( f_0^{(0,b)} \) we have \( \lambda((f_s^{(0,b)})) = 1 \) and consequently \( \lambda(f_0^{(0,b)}) = 1 \).

Corollary 1. For every singularity \( f_0 \) stably equivalent to one of \( f_0^{(0,b)} \), \( b \in \mathbb{C} \), the jump \( \lambda(f_0) \) of \( f_0 \) is equal to 1.

Proposition 3 implies also that \( \lambda(f_0) \) is not a topological invariant of \( f_0 \). Recall that two singularities \( f \) and \( g \) in \( \mathbb{C}^n \) have the same topological type if there exist neighbourhoods \( U \) and \( V \) of \( 0 \in \mathbb{C}^n \) and a homeomorphism \( \Phi : U \to V \) such that \( \Phi(V(f)) = V(g) \), where \( V(f) \) (resp. \( V(g) \)) is the zero set of \( f \) (resp. \( g \)) in \( U \) (resp. \( V \)).

Corollary 2. The jump of the Milnor number \( \lambda(f_0) \) is not a topological invariant of \( f_0 \).

Proof. By the Gusein-Zade theorem, for generic elements \( f_0^{(a,b)} \) of the class \( W_{1,0} \) we have \( \lambda(f_0^{(a,b)}) > 1 \). Proposition 3 gives that the elements \( f_0^{(0,b)} \) of \( W_{1,0} \) satisfy \( \lambda(f_0^{(0,b)}) = 1 \). But all the singularities \( f_0^{(a,b)}, a, b \in \mathbb{C}, a^2 \neq 4 \), have the same topological type. This follows from the general Lê-Ramanujam theorem on \( \mu \)-constant families of singularities or from the (much easier) fact that all the singularities \( f_0^{(a,b)} \) have the same resolution graph.

Acknowledgement. We thank prof. A. Płoski for discussions which led to improvement of the text of the paper.
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