Numerical stability analysis of the local inertial equation with semi- and fully implicit friction term treatments: assessment of the maximum allowable time step

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Abstract. The local inertial equation (LIE) as a simple mathematical model has been widely used for flood simulation. So far, the maximum allowable time step of the discretized LIE with the conventional semi-implicit scheme has been believed to follow the standard Courant-Friedrichs-Lewy condition. However, we demonstrate that this is not true from the viewpoint of a numerical stability analysis considering the model non-linearity. In addition, a fully-implicit variant of the scheme with higher stability is presented, indicating its practical advantages.

Keywords: Local inertial model, fully implicit treatment, von Neumann stability analysis, model non-linearity

1. Introduction

Flooding is among the most serious natural disasters and can happen everywhere around the world. Flood-inundation modeling is one of the most essential techniques for supporting flood warning and flood risk management planning in floodplains [1–4]. For these practical purposes, the following Saint-Venant equations, which are derived through the vertical integration of the Navier-Stokes equations, is most commonly used when water depth is much smaller than the horizontal scale of floodplains because the hydrostatic assumption is valid under this condition [5].

\[
\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q^2}{h} \right) + gh \frac{\partial h}{\partial x} + gh \frac{\partial z}{\partial x} + \frac{g n^2}{h^{7/3}} |q| = 0,
\]

(1)

\[
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0,
\]

(2)
where $t$ is the time, $x$ is the space coordinate, $h$ is the water depth, $q$ is the line discharge, $g$ is the gravitational acceleration, $z$ is the bed elevation, and $n$ is the Manning’s roughness coefficient. (1) is the momentum equation and (2) is the mass balance equation. The above particular equations are for the one-dimensional rectangular channel flow with a uniform channel width. In addition to this simplification, when the horizontal scale of flood flow is far larger than vertical scale of water depth, the advective acceleration term in the Saint-Venant equations (the second term in (1)) is less dominant than other terms, i.e. the local inertial, pressure, gravitation, and friction terms (the first, third, forth, and fifth term, respectively); then, the following local inertial formulation of the momentum equation serves as an effective candidate of the Saint-Venant equations [6–9]:

$$\frac{\partial q}{\partial t} + gh \frac{\partial h}{\partial x} + gh \frac{\partial z}{\partial x} + \frac{gn^2}{h^{7/3}} |q|^3 = 0. \tag{3}$$

This reduced momentum equation is practically of importance because flood-inundation modelers can avoid using the complicated discretization of the advective acceleration term that requires sophisticated numerical techniques [10–13]. The Local Inertial Equation (hereinafter, referred to as LIE), which consists of the mass balance equation (2) and the reduced momentum equation (3), can be seen as a nonlinear relaxed wave equation. On the other hand, the conventional diffusive wave equations, which further omit the local inertial term in the reduced momentum equation is a parabolic equation [14]. Due to the parabolic nature, the maximum allowable time step (hereinafter, referred to as MATS) for the diffusive wave equation is practically very small compared with that of the LIE [14]. This remarkable property implies another advantage of using the LIE. Recall that the Courant-Friedrichs-Lewy (CFL) condition is given as

$$\Delta t \leq \frac{\Delta x}{\sqrt{gh}}, \tag{4}$$

where $\Delta t$ and $\Delta x$ are time and space steps of finite difference method, respectively.

However, it is well known that the simulation of LIE with the time step following the CFL condition sometimes collapses [7, 15]. Although the reason of this reduced computational stability has been considered as the existence of the friction term, the last term of (3), it has yet to be clearly described from a theoretical viewpoint. To investigate the impact of the treatment of source terms on the numerical stability of the LIE, Tanaka et al. [4] applied the von Neumann stability analysis to the LIE with explicit, semi-implicit, and fully implicit treatments of the discharges in the friction term. The semi-implicit treatment is among the most widely utilized methods in applications owing to its practical stability and simplicity in implementations [16, 18]. Their stability analysis was based on linearized gravitation and friction terms of the momentum (3). The linearization was performed for tractability of the analysis. The MATS derived from the theoretical stability analysis clearly demonstrated that the explicit treatment is the least stable and that the CFL condition (4) is the stability condition for the semi-implicit treatment; however, comparisons between the theoretical results and numerical benchmark results indicate that the CFL condition is actually too optimistic, which is a popular empirical phenomenon shared among flood modelers [7, 18]. The reason of this phenomenon should be clarified so that flood modelers can facilitate their analysis.
with the LIE and that better mathematical and numerical models are developed for more effective flood simulation.

On the other hand, this analysis results also indicated that the fully implicit treatment of the discharges in the friction term allowed far larger MATS than the CFL condition. This potential advantage of the fully implicit treatment was examined by Tanaka and Yoshioka [17], where its simple implementation was described and examined; however, the computational MATS was much smaller values than Tanaka et al. [4], and the cause of this discrepancy has not been clarified. The analysis performed in Tanaka et al. [4] was unique in a sense that the von Neumann stability analysis was applied to the friction term as well differential terms; however, it failed to complete the investigation because water depth non-linearity in the pressure, gravitation and friction terms were dealt with a constant quantity; namely, these terms were linearized.

Based on the above research background, the objective of this paper is set as to apply the von Neumann stability analysis to the semi- and fully implicit treatments of the friction term of the LIE without the above-mentioned linearization of the momentum equation. Although the present stability analysis is not free from some linearization techniques, it is less linearized than the previous one [4] and is therefore considered to closer to the original LIE as a system of partial differential equations. The theoretical MATSs for the semi- and fully implicit treatments are derived, and they are compared with those computed in the benchmark cases. Our results thus demonstrate that the stability issue of the LIE is due to the model non-linearity, which was not considered in the previous analysis [4], and that the fully implicit treatment of the friction term exhibits better computational stability, indicating its practical advantages.

2. Finite difference method

2.1. Stability analysis

The momentum equations in the two-dimensional formulation, which is typically applied for simulating two-dimensional flood expansion from rivers or the sea, are

$$\frac{\partial q_x}{\partial t} + gh \frac{\partial h}{\partial x} + gh \frac{\partial z}{\partial x} + \frac{gn^2 q_x q_x}{h^{7/3}} = 0.$$  \hfill (5)

$$\frac{\partial q_y}{\partial t} + gh \frac{\partial h}{\partial y} + gh \frac{\partial z}{\partial y} + \frac{gn^2 q_y q_y}{h^{7/3}} = 0.$$  \hfill (6)

where \(q_x\) and \(q_y\) are the line discharge in the \(x\) and \(y\) directions, respectively. This formulation shows that the two-dimensional equations straightforwardly applies the one-dimensional equation to each horizontal direction; therefore, the following analysis is carried out for one-dimensional formulation, i.e. (3) and (2). According to Almeida et al. [15], the finite difference schemes of (3) and (2) are shown as follows:

$$\frac{q_{x,i+1/2}^{k+1/2} - q_{x,i+1/2}^{k-1/2}}{\Delta t} + gh_i \left( h_{i+1}^k + z_{i+1} - \left( h_i^k + z_i \right) \right) \frac{\Delta x}{h_i^{7/3}} \left. \frac{gn^2 q_x q_x}{h_i^{7/3}} \right|_{t+1/2}^k = 0.$$  \hfill (7)
Figure 1: Definition of variables along time and space coordinate. Water depth $h$ and line discharge $q$ are defined on white and black circles; the mass and momentum equations are discretized at white and black rectangles, respectively.

$$\frac{h_i^{k+1} - h_i^k}{\Delta t} + \frac{q_{i+1/2}^{k+1/2} - q_{i-1/2}^{k+1/2}}{\Delta x} = 0,$$

where $\Delta t$ is the time increment, $\Delta x$ is the spatial grid space, $i$ is the spatial step, and $k$ is the temporal step. $h_i^k = \max(h_{i+1}^k + z_{i+1}, h_i^k + z_i) - \max(z_{i+1}, z_i)$ shows the flowing water depth at the upstream grid between adjacent two cells. This paper assumes that the spatial step $i$ is the upstream point having higher elevation and water level, i.e. $h_i = h_i^k$. This is satisfied in the following benchmark test. As shown in Fig. 1, variables $h$ and $q$ are defined on a staggered grid along both the temporal and spatial grid spaces as shown as white and black circles, respectively. In Fig. 1, the mass and momentum equations are solved at white and black rectangles, respectively. There exist three candidates (explicit, semi-implicit, and fully implicit) to treat the friction term (Tanaka et al. [4]):

$$\left. \frac{g n^2 q |q|}{h_i^k} \right|_{i+1/2} = \left\{ \frac{g n^2 q_{i+1/2}^{k-1/2} q_{i+1/2}^{k-1/2}}{(h_i^k)^{7/3}}, \frac{g n^2 q_{i+1/2}^{k-1/2} q_{i+1/2}^{k-1/2}}{(h_i^k)^{7/3}}, \frac{g n^2 q_{i+1/2}^{k-1/2} q_{i+1/2}^{k-1/2}}{(h_i^k)^{7/3}} \right\}.$$

According to von Neumann stability analysis of (3) and (2) with the linearization such that $h_i^k$ in the coefficient of pressure, gravitation, and friction terms is handled as a positive constant [4], the semi-implicit treatment admits the MATS to follow the CFL condition (see (4)) as for the typical wave equations; fully implicit treatment allows the largest time step among the three candidates [4]:

$$\frac{\Delta x}{\Delta t} \geq \frac{g h_i^k}{\sqrt{1 + g n^2 (h_i^k)^{7/3} Q \Delta t}},$$

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where $Q$ is the uniform flow discharge. (10) clearly shows that the MATS of the fully implicit treatment is larger than that of the conventional CFL condition (4) serving as the stability condition for the semi-implicit treatment.

2.2. Numerical solver

The momentum equation with semi-implicit treatment of the friction term [7] is simply rewritten as

$$
q^{k+1/2} = \frac{q^{k-1/2} - gh^k \Delta t \left( \frac{h^{k+1} + z_{i+1}}{\Delta x} - \frac{h^k + z_i}{\Delta x} \right)}{1 + g \Delta t^2 q^{k-1/2} / \left( h^k \right)^{7/3}}. 
$$

(11)

Although it is not a explicit discretization, (11) does not require any iteration methods for its resolution: its time marching can be achieved in an explicit manner. This may be the reason why the semi-implicit discretization has been utilized as the most common option for discretization of the LIE. On the other hand, the fully implicit treatment of the friction term has not been well discussed so far. This may be due to the belief that its resolution at each time step requires some iteration solvers, which may be time-consuming; however, this belief turns out to be false. Tanaka and Yoshioka [17] demonstrated that the momentum equation with the fully implicit treatment can also give a formally explicit time-marching formula as follows:

$$
q^{k+1/2} = \begin{cases} 
\frac{1}{2b_i^k \Delta t} \left( -1 + \sqrt{1 - 4c_i^k \Delta t c_i^k} \right) \geq 0 (c_i^k \leq 0) \\
\frac{1}{2b_i^k \Delta t} \left( 1 - \sqrt{1 + 4b_i^k \Delta t c_i^k} \right) < 0 (c_i^k > 0),
\end{cases} 
$$

(12)

where the coefficients $b_i^k$ and $c_i^k$ are defined as

$$
b_i^k = \frac{gn^2}{(h_i^k)^{7/3}}, \quad c_i^k = d_i^k \Delta t - q_i^{k-1/2}, \quad \text{and} \quad d_i^k = \frac{gh_i \left( h_{i+1}^k + z_{i+1} \right) - \left( h_i^k + z_i \right)}{\Delta x}. 
$$

(13)

The momentum equation with the fully implicit treatment of the friction term thus has almost the same algorithmic complexity with the semi-implicit one. This analytical result suggests that the fully implicit treatment can possibly serve as a more stable alternative to the conventional semi-implicit one. From (12), it should be also noted that computational time of the implicit and semi-implicit scheme under the same temporal resolution is comparable since both of them do not require iteration methods, like the Newton’s method, for solving nonlinear equations.

There exists an additional theoretical advantage of using the fully implicit treatment over the semi-implicit one. Considering the quadratic nature of (12) when adopting the fully implicit treatment, it can be rewritten to uniquely find the updated value $q_i^{k+1/2}$ as

$$
q_i^{k+1/2} = \frac{c_i^k}{2b_i^k |c_i^k| \Delta t} \left( 1 - \sqrt{1 + 4 |c_i^k| b_i^k \Delta t} \right). 
$$

(14)

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The behavior of the momentum equation for large time step \((\Delta t \to +\infty)\) is a key for assessing stability of numerical schemes for the equations of shallow water flows since it effectively characterizes qualitative properties of numerical solutions for not small \(\Delta t\) and small water depths [19]. For the large time step limit \((\Delta t \to +\infty)\) that represents a practical situation where the temporal resolution is not sufficiently fine, \(q^{k+1/2}\) with the fully implicit treatment approaches the equation

\[
\left| \frac{q^{k+1/2}}{q_{i+1/2}^{k+1/2}} \right| = \sqrt{\frac{(\Delta t)^{5/3}}{n}} \frac{\sqrt{(h_{i+1}^k + z_{i+1}^k) - (h_i^k + z_i^k)}}{\Delta x}, \tag{15}
\]

which is exactly the theoretical value of the line discharge for the uniform flow, in which \(d_i^k\) is independent from \(\Delta x\) as desired. On the other hand, the same limit for the momentum equation with the semi-implicit treatment leads to the resolution-dependent result as

\[
q_{i+1/2}^{k+1/2} = -\frac{\left(\Delta t\right)^{10/3}}{n^2 q_{i+1/2}^{k-1/2} \Delta x} \left[ (h_{i+1}^k + z_{i+1}^k) - (h_i^k + z_i^k) \right], \tag{16}
\]

which seems to be a less natural equation than (15). In addition, (16) would lead to numerical instability since \(q_{i+1/2}^{k+1/2}\) would become small (large) when \(q_{i+1/2}^{k-1/2}\) is large (small), implying a possibility that numerical solutions under such a condition generates numerical oscillations, in which large and small values of the line discharge is computed alternately at each successive time steps.

Further mathematical analysis reveals another advantage of using the fully implicit treatment. The right-hand side of (12) can be rewritten as a univariate function \(F\) as \(F(q_{i+1/2}^{k-1/2})\), which satisfies the relationship

\[
\frac{dF(q_{i+1/2}^{k-1/2})}{dq} = \begin{cases} 
(1 - 4b_i^k \Delta t c_i^k)^{-1/2} & (c_i^k < 0) \\
(1 + 4b_i^k \Delta t c_i^k)^{-1/2} & (c_i^k > 0)
\end{cases}, \tag{17}
\]

implying \(\left| \frac{dF(q_{i+1/2}^{k-1/2})}{dq} \right| \leq 1\) for any real value \(q_{i+1/2}^{k-1/2}\) and that the function \(F\) defines a contraction mapping property. This remarkable property of \(F\) complies with the unconditional stability of the fully implicit treatment in the sense of the nonlinear stability against uniform flows subject to not large perturbations [17]. It should be noted that this contraction mapping property does not follow for the semi-implicit treatment. Computational accuracy of the two schemes is first-order in both space and time. This is because of that the continuity and momentum equations are discretized with the simple one-step method and that the water depth \(h_i^k\) in the momentum equation are upwinded. (See, its definition given in (8)).

3. von Neumann stability analysis of local inertial equation

The von Neumann stability analysis to the LIE by Tanaka et al. [4] has a severe limitation that they treated \(h_i^k\) in the pressure, gravitation, and friction terms as a constant value. This section describes the stability analysis considering all the water depths \(h_i^k\) in the momentum
equation to be demonstrated as below as variable. Denote the small perturbation of the water depth and the discharge from the uniform flow depth as $\eta$ and $\epsilon$, respectively; then, they are expressed as

$$\eta = h - H,$$  \hspace{1cm} (18)

$$\epsilon = q - Q,$$  \hspace{1cm} (19)

where the uniform flow discharge $Q$ and water depth $H$ are related by the following steady flow condition:

$$Q = \frac{H^{5/3}}{n} \left( -\frac{\partial z}{\partial x} \right)^{1/2}.$$  \hspace{1cm} (20)

Then, (2) and (3) are re-written as:

$$\frac{\partial (H + \eta)}{\partial t} + \frac{\partial (Q + \epsilon)}{\partial x} = 0,$$  \hspace{1cm} (21)

$$\frac{\partial (Q + \epsilon)}{\partial t} + g(H + \eta) \frac{\partial (H + \eta)}{\partial x} + g(H + \eta) \frac{\partial z}{\partial x} + \frac{g n^2}{(H + \eta)^{7/3}}[Q + \epsilon(Q + \epsilon)]^2 = 0,$$  \hspace{1cm} (22)

respectively. As the steady flow variables $H$ and $Q$ is constant along time and space directions, and given the product of perturbation terms $\eta$ and $\epsilon$ is negligible:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \epsilon}{\partial x} = 0,$$  \hspace{1cm} (23)

$$\frac{\partial \epsilon}{\partial t} + gH \frac{\partial \eta}{\partial x} + gH \frac{\partial z}{\partial x} + g\eta \frac{\partial z}{\partial x} + g n^2 (Q|Q|H^{-7/3})(1 + \eta/H)^{-7/3} + g n^2 (|Q| \epsilon + |Q| \epsilon) H^{-7/3} = 0.$$  \hspace{1cm} (24)

As $\eta/H << 1$ is assumed, the momentum (24) becomes:

$$\frac{\partial \epsilon}{\partial t} + gH \frac{\partial \eta}{\partial x} + gH \frac{\partial z}{\partial x} + g\eta \frac{\partial z}{\partial x} + g n^2 (Q|Q|H^{-7/3} - \frac{7}{3} g n^2 (Q|Q|H^{-10/3} \eta + g n^2 (|Q| \epsilon + |Q| \epsilon) H^{-7/3} = 0.$$  \hspace{1cm} (25)

By the steady flow condition (20), (25) becomes

$$\frac{\partial \epsilon}{\partial t} + gH \frac{\partial \eta}{\partial x} + \left( \frac{7}{3} \right) g n^2 \frac{\partial z}{\partial x} + g n^2 (|Q| \epsilon + |Q| \epsilon) H^{-7/3} = 0.$$  \hspace{1cm} (26)

This is the more strictly linearized momentum equation than that handled in Tanaka et al. [4]. The coefficient 7/3 in the third term arises from the non-constant treatment of $h$, and replacing the coefficient 1+7/3 by 1 and $\eta$ by the steady water depth $H$ in this term reduces to the previous linearization [4]. The impact of this additional coefficient 7/3 turns out to be critical in the stability analysis below.

The discretization methods of the linearized LIE for the semi- and fully implicit treatments are explained as follows. For the semi-implicit treatment of the friction term, (23) and (26) are discretized as

$$\frac{\epsilon_{i+1/2}^{k+1/2} - \epsilon_{i+1/2}^{k-1/2}}{\Delta t} + gH \frac{\eta_i^k - \eta_i^k}{\Delta x} - \frac{10}{3} g n^2 \eta_i^k + g n^2 \left( \frac{\epsilon_{i+1/2}^{k-1/2} + \epsilon_{i+1/2}^{k+1/2}}{\Delta t} \right) QH^{-7/3} = 0,$$  \hspace{1cm} (27)
\[
\frac{\eta_{i}^{k+1} - \eta_{i}^{k}}{\Delta t} + \frac{\epsilon_{i+1/2}^{k+1/2} - \epsilon_{i-1/2}^{k+1/2}}{\Delta x} = 0, \tag{28}
\]

where \( I = -(z_{i+1} - z_{i}) / \Delta x \) denotes the slope gradient. For simplicity, steady flow discharge \( Q \) and its perturbation \( \epsilon \) were assumed to be positive. This assumption is justified since we assume \( |\epsilon| << Q \). These two difference equations are combined into the one equation of \( \epsilon \) as

\[
\epsilon_{i+1/2}^{k+1/2} - 2\epsilon_{i+1/2}^{k-1/2} + \epsilon_{i+1/2}^{k-3/2} - \mu (\epsilon_{i+3/2}^{k-1/2} - 2\epsilon_{i+1/2}^{k-1/2} + \epsilon_{i-1/2}^{k-1/2}) + \lambda (\epsilon_{i+1/2}^{k-1/2} - \epsilon_{i-1/2}^{k-1/2}) + \gamma (\epsilon_{i+1/2}^{k-1/2} - \epsilon_{i-1/2}^{k-3/2}) = 0, \tag{29}
\]

where denote \( \mu, \gamma, \lambda \) as

\[
\mu = gH \left( \frac{\Delta t}{\Delta x} \right)^2, \tag{30}
\]
\[
\gamma = gn^2 Q \Delta t \Delta H^{-7/3}, \tag{31}
\]
\[
\lambda = \frac{10}{3} g \eta \left( \frac{\Delta t}{\Delta x} \right)^2, \tag{32}
\]
respectively.

The von Neumann stability analysis investigates the numerical stability of an equation of interest using a typical term from the Fourier series of the round-off error (denoted as \( \delta \)) as follows:

\[
\delta_{i}^{k} = G^k e^{j\theta}, \tag{33}
\]

where \( G \) is the complex amplitude ratio, \( j (j^2 = -1) \) is the imaginary unit, \( \theta \) is the wave number. Given \( \delta \) as the round-off error of \( \epsilon \), the governing equation of temporal evolution of \( \delta \) is obtained by substituting \( \delta \) to \( \epsilon \) in (29) as

\[
(1 + \gamma)G^2 + [-2\{1 - \mu(1 - \cos \theta)\} + \gamma(1 - \cos \theta) + \lambda \sin \theta] G + 1 - \gamma = 0. \tag{34}
\]

Let \( \alpha = 1 + \gamma, u = 2\{1 - \mu(1 - \cos \theta)\} - \lambda (1 - \cos \theta), \) and \( v = -\lambda \sin \theta; \) then, this quadratic equation with complex coefficients are formally expressed as

\[
G = \frac{1}{2\alpha} [u + vj \pm (p + qj)], \tag{35}
\]

where \( p \) and \( q \) are real coefficients defined as

\[
p = \sqrt{\frac{\text{Re}\Delta + |\Delta|}{2}} \quad \text{and} \quad q = \frac{\text{Im}\Delta}{\sqrt{2(\text{Re}\Delta + |\Delta|)}}, \tag{36}
\]

where \( \text{Re}\Delta \) and \( \text{Im}\Delta \) are the real and imaginary part of the complex discriminant \( \Delta \) of (34), respectively. The necessary and sufficient condition of stability here is consequently derived as:

\[
|G| \leq 1 \iff |u + vj \pm (p + qj)| \leq 4\alpha^2 \quad \text{for all} \quad \theta \quad (0 \leq \theta < 2\pi). \tag{37}
\]

Because (37) cannot be handled analytically, the condition described in (37) is numerically examined for \( 0 \leq \theta < 2\pi \) where the examined values of \( \theta \) is set as \( 2m\pi/M \) \((m = 1, 2, ..., M)\).
where $M = 20,000$. We regard that the stability holds true for each couple $(\Delta t, \Delta x)$ if the condition of (37) is satisfied for all examined $\theta$.

For the fully implicit treatment, in a similar manner, (26) is discretized as follows:

\[
\frac{\epsilon_{i+1/2}^{k+1} - \epsilon_{i+1/2}^{k-1/2}}{\Delta t} + gH \frac{\eta_{i+1}^{k} - \eta_{i}^{k}}{\Delta x} - \frac{10}{3} gH \epsilon_{i+1/2}^{k} + 2g\eta_{i+1}^{k}Q^{H^{-7/3}} = 0. \tag{38}
\]

Being different from (27), $\epsilon$ in the friction term is discretized at time step $k + 1/2$. The corresponding equation of the complex amplitude ratio $G$ is derived as

\[
(1 + 2\gamma)G^2 + [-2(1 + \gamma - \mu(1 - \cos \theta)) + \lambda(1 - \cos \theta) + \lambda j \sin \theta] G + 1 = 0. \tag{39}
\]

The stability condition is also given by (37), where $\alpha = 1 + 2\gamma, u = 2(1 + \gamma - \mu(1 - \cos \theta)) - \lambda(1 - \cos \theta), \text{ and } v = -\lambda \sin \theta$.

4. Validation of derived maximum allowable time step

4.1. Numerical experiment design

The derived MATSs for the semi-implicit and the fully implicit treatments are validated for a simple one-dimensional slope uniform flow simulation used in Tanaka et al. [4]. The initial condition is a steady uniform flow on a slope with a constant gradient. The value of the Manning’s roughness coefficient is 0.03 (m$^{-1/3}$s). The one-dimensional benchmark flow is simulated with the LIE given a perturbation with the magnitude of 0.01 % of the initial water depth and the corresponding line discharge to the lower and upper boundaries, respectively. The MATS is estimated as the maximum time step among ones with which simulated water depth along a slope match the increased steady flow water depth. To examine computational efficiency, the MATS was estimated for a variety of spatial resolutions ranging from 0.1 m to 10,000 m with three cases with different water depth and slope gradient (see Table 1).

Table 1: Summary of the examined computational conditions for Cases 1 through 3.

| Case | Water depth [m] | Slope [m/m] |
|------|----------------|-------------|
| 1    | 0.1            | 0.001       |
| 2    | 1.0            | 0.005       |
| 3    | 2.0            | 0.01        |

4.2. Results and discussions

Figure 2 presents the theoretical MATS of the semi-implicit treatment of the friction term based on the von Neumann stability analysis and the corresponding numerical counterpart. The panels (a) through (c) show the results for Cases 1 through 3, respectively. For all the cases examined, each derived MATS reasonably capture the corresponding simulated one, implying validity of the present theoretical stability analysis results. The comparison results show that the von Neumann stability analysis with the proposed treatment of the model
non-linearity quite effectively works. In all the cases, the theoretical MATSs are smaller
than the simulated ones. In addition, the theoretical and simulated MATS are smaller
than the CFL condition, which agrees qualitatively with the empirical fact that the CFL condition
is an optimistic stability condition in practice. The obtained computational results clearly
suggest that the reduced stability from the CFL condition is due to the non-linearity of the
momentum equation.

Figure 3 presents the theoretical MATS of the fully implicit treatment of the friction
term based on the von Neumann stability analysis and the corresponding numerical one. As
in Fig. 2. The panels (a) through (c) show the results for Cases 1 through 3, respectively.
The stability analysis results for the fully implicit treatment derived in the previous study by
Tanaka et al. [4] are plotted in Fig. 3 as well. The theoretical and simulated MATSs are in
a good agreement, again implying validity of the theoretical results. Through comparison
between the theoretical MATS in this paper and that in Tanaka et al. [4], we found that
appropriately handling the model non-linearly of the momentum equation is an essential
key for the stability analysis of the LIE. This is the main finding of this paper, which has
not been clarified so far. A remarkable finding from Fig. 3 is that the stability condition of
Figure 3: Maximum allowable time step for the fully implicit treatment for (a) Case 1, (b) Case 2, and (c) Case 3 (see Table 1). The solid and broken lines show the derived MATSs in this paper and Tanaka et al. [4], the circles show the simulated MATSs, and the thin line shows the CFL condition.

the fully implicit treatment is far closer to the CFL condition than that of the semi-implicit one, and thus the former has higher stability than the latter. The computational results suggest potential advantages of using the fully implicit treatment over the semi-implicit one commonly used in many flood-inundation models.

5. Conclusions

Computationally efficient and stable flood modeling is an essential element for flood disaster management. The LIE has been serving as its core mathematical tool. Computational accuracy of each scheme was not the focus of this paper, but is an important topic from both theoretical and practical viewpoints. This topic is addressed elsewhere for the explicit, semi-implicit, implicit, and the exponential [20] schemes [21]. This paper examined numerical stability of the finite difference counterparts of the one-dimensional LIE, the semi- and fully implicit treatments of the friction term in particular. The theoretical MATSs reasonably agreed with the computed ones for the benchmark cases, validating the present stability
analysis results and clearly showing that the reduced numerical stability of the operational flood simulators based on the LIE is due to the model non-linearity of the momentum equation. The results also demonstrated that the fully implicit treatment theoretically allowed far larger MATS than the semi-implicit one. The fully implicit treatment can therefore be regarded as an effective alternative to the conventional semi-implicit one.

Performance comparison of the semi- and fully implicit treatments should be examined in real problems, which is currently undergoing by the authors where a huge shallow water body in South Asia region is chosen as a model application site. Field observations of hydraulic and hydrological variables have also been carried out so far. The analysis results of the performance comparison will be presented elsewhere.

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