This note attempts to make clear the relation between configurations of points in a space \( Y \) and those in its Cartesian product with the reals. It turns out to be a very simple relation whose proof uses nothing new.

Let \( Y \) be an unbased space. Denote by \( Y^j \) the \( j \)-fold Cartesian product of \( Y \) with itself. For present purposes we consider the circle \( S^1 \) to be the quotient of the unit interval \([0,1]/\{0,1\}\). If \( X \) is a based space then \( \Sigma X \) is defined to be \( X \wedge S^1 \) and \( \Omega X \) is defined to be the loop space of \( X \), that is, the space of based maps from \( S^1 \) to \( X \).

**Definition 1** Define \( C(Y) \) to be the subspace of \( Y^j \) consisting of \( j \)-tuples of distinct points in \( Y \). If \( \nu \) is an injective function from \( \{1,\ldots,i\} \) to \( \{1,\ldots,j\} \) then define \( \nu^*: C(Y)^j \to C(Y)_i \) by sending \( (y_1,\ldots,y_i) \) to \( (y_{\nu(1)},\ldots,y_{\nu(i)}) \). If \( X \) is a nondegenerately based space, define \( \nu_*: X^i \to X^j \) sending \( (x_1,\ldots,x_i) \) to \( (x'_1,\ldots,x'_j) \) where \( x'_k = x_l \) if \( k = \nu(l) \) and \( x'_k \) is the basepoint if \( k \) is not in the image of \( \nu \).

Note that these maps are compatible with composition; i.e. \( (\nu \circ \mu)_* = \nu_* \circ \mu_* \) and \( (\nu \circ \mu)^* = \mu^* \circ \nu^* \). In particular, the maps \( \nu^* \) define a free action of the \( j \)-fold symmetric group \( S_j \) on \( C(Y)_j \).

The spaces \( C(Y)_j \) and the maps \( \nu^* \) define a *coefficient system* in the sense of [2], and we define an equivalence relation \( \sim \) on \( \coprod_j C(Y)_j \times X^j \) generated by \( (\nu^*(\vec{y}),\vec{x}) \sim (\vec{y},\nu_*(\vec{x})) \). Define

\[
C(Y,X) = \left( \coprod_j C(Y)_j \times X^j \right) / \sim.
\]

In their recent paper [5], Cohen and Taylor deal with the space \( C(\mathbb{R} \times Y,X) \). Recall that a *weak metric space* is a space \( Y \) together with a continuous function \( d: Y \times Y \to [0,\infty) \) such that \( d^{-1}(0) \) is the diagonal in \( Y \times Y \). The main result of this note is:
**Theorem 1** Let $Y$ be a weak metric space and $X$ a nondegenerately based space. There is a space $C_1(Y, X)$ and a pair of maps

$$C(\mathbb{R} \times Y, X) \xleftarrow{\phi} C_1(Y, X) \xrightarrow{\alpha} \Omega C(Y, \Sigma X)$$

such that:

1. $C_1(-, -)$ is functorial with respect to based maps in the second variable and injective maps in the first variable, and $\phi$ and $\alpha$ are natural;
2. $\phi$ is a homotopy equivalence; and
3. $\alpha$ is a weak homotopy equivalence if $X$ is path-connected.

The proof uses the methods from [1] and [2]. The space $C_1(Y, X)$ is another space derived from a “coefficient system.” Let $\mathcal{C}_j(Y)_j$ be the subspace of $(\mathbb{R} \times \mathbb{R} \times Y)^j$ consisting of $j$-tuples of triples $((a_1, b_1, y_1), \ldots, (a_j, b_j, y_j))$ such that for all $i$, $a_i < b_i$ and for all $k \neq l$, $y_k = y_l$ implies $b_k \leq a_i$ or $b_i \leq a_k$.

We can define a coefficient system structure $\nu^*$, $\nu_*$ on $\{\mathcal{C}_j(Y)_j\}_{j \geq 0}$ by acting on triples, and define and $\sim$ on $\bigsqcup_j \mathcal{C}_j(Y)_j \times X_j$ generated by $(\nu^*(\kappa), \bar{x}) \sim (\kappa, \nu_*(\bar{x}))$. The quotient space $C_1(Y, X)$ can be thought of as consisting of configurations of line segments in $\mathbb{R} \times Y$ with disjoint interiors, labeled by points of $X$; a segment labeled by the basepoint drops out under the identification $\sim$. For compactness of notation, we will use

$$\binom{(a_i, b_i, y_i)}{1 \leq i \leq j}$$

as shorthand for $((a_1, b_1, y_1), \ldots, (a_j, b_j, y_j)) \in \mathcal{C}_j(Y)_j$, and

$$\left[a_i, b_i, y_i, x_i\right]_{1 \leq i \leq j}$$

for the image of $((a_1, b_1, y_1), \ldots, (a_j, b_j, y_j), (x_1, \ldots, x_j))$ in $C_1(Y, X)$. Similarly we will use the shorthand $[y_i, x_i]_{1 \leq i \leq j}$ for points of $C(Y, X)$.

There is an obvious map $\phi_j$ from $\mathcal{C}_j(Y)_j$ to $\mathcal{C}(\mathbb{R} \times Y)_j$ taking each segment to its center-point. This map respects permutations and so induces a map $\phi$ from $C_1(Y, X)$ to $C(\mathbb{R} \times Y, X)$.

There is also a map $\phi_j$ from $\mathcal{C}(\mathbb{R} \times Y)_j$ to $\mathcal{C}_j(Y)_j$ which we define as follows. Use the weak metric $d$ on $Y$ to define $g : (\mathbb{R} \times Y) \times (\mathbb{R} \times Y) \to [0, \infty)$ by setting

$$g((a, y), (a', y')) = \frac{1}{2} \left( \frac{|a - a'|^2 + d(y, y')}{|a - a'| + d(y, y') + 1} \right)$$

so $g((a, y), (a', y')) \leq \frac{1}{2}|a - a'|$ if $y = y'$. Let $\kappa = ((a_1, y_1), \ldots, (a_j, y_j)) \in \mathcal{C}(\mathbb{R} \times Y)_j$ and define

$$v(\kappa) = \min_{k \neq l} \{g((a_k, y_k), (a_l, y_l))\}.$$
It’s clear that \( v(\kappa) > 0 \) and that the intervals \([a_k - v(\kappa), a_k + v(\kappa)]\) and \([a_l - v(\kappa), a_l + v(\kappa)]\) do not overlap when \( y_k = y_l \), so we can define

\[
\tilde{\phi}_j(\kappa) = (a_l - v(\kappa), a_l + v(\kappa), y_i)_{1 \leq i \leq j}.
\]

These induce a map \( \tilde{\phi} : C(\mathbb{R} \times Y, X) \to C_1(Y, X) \). Further, \( \phi_j \) and \( \tilde{\phi}_j \) are easily seen to be inverse \( S_j \)-equivariant homotopy equivalences: \( \phi_j \tilde{\phi}_j \) is the identity of \( \mathcal{C}(\mathbb{R} \times Y)_j \), and there is a deformation from the identity of \( \mathcal{C}_1(Y)_j \) to \( \tilde{\phi}_j \phi_j \) by linearly scaling the intervals around their centers. So by Lemma 2.7(ii) of [2], \( \phi \) is a homotopy equivalence.

Next we need to define \( \alpha \). For the purposes of this section it is more convenient to work with a homeomorphic copy of \( C_1(Y, X) \). Let

\[
\tilde{C}_1(Y, X) = \left\{ (a_i, b_i, y_i, x_i)_{1 \leq i \leq j} \in C_1(Y, X) | 0 < a_i < b_i < 1 \text{ for all } i \right\}
\]

This subspace is clearly homeomorphic to \( C_1(Y, X) \) via the homeomorphism of the reals \( \mathbb{R} \) with the open interval \((0, 1)\). Let \( w = (a_i, b_i, y_i, x_i)_{1 \leq i \leq j} \) be a point of \( \tilde{C}_1(Y, X) \). For a given \( t \), define

\[
\alpha(w)(t) = [y_i, [x_i, s_i]]_{1 \leq i \leq j} \text{ and } a_i \leq t \leq b_i,
\]

where \( s_i = (t - a_i)/(b_i - a_i) \). For a given \( t \) and for each \( i \) satisfying \( a_i \leq t \leq b_i \), we observe that \( 0 \leq s_i \leq 1 \) and the points \( \{y_i | 1 \leq i \leq j \text{ and } a_i \leq t \leq b_i\} \) are distinct; also \( \alpha(w)(0) = \alpha(w)(1) \) is the basepoint \( * \) of \( C(Y, \Sigma X) \). Thus \( \alpha(w) \) is a well-defined loop in \( C(Y, \Sigma X) \).

To show \( \alpha \) is a weak equivalence, we use the same idea as [1], namely to fit it into a comparison of quasifibration sequences. Define \( E_1(Y, X) \) to be the quotient space of \( \tilde{C}_1(Y, X) \times [0, 1] \) where we identify \((a_i, b_i, y_i, x_i)_{1 \leq i \leq j}, s \) and \((a_i', b_i', y_i', x_i')_{1 \leq i \leq j+k}, s \) if \( (a_i, b_i, y_i, x_i) = (a_i', b_i', y_i', x_i') \) for \( 1 \leq i \leq j \) and \( a_{j+1}, \ldots, a_{j+k} \geq s \). Note that all points of the form \((w, 0)\) are identified with the basepoint \(*, 0\) of \( E_1(Y, X) \), so \( E_1(Y, X) \) is contractible.

Define a map \( \bar{\alpha} \) from \( E_1(Y, X) \) to the path space \( PC(Y, \Sigma X) \) by

\[
\bar{\alpha}(w, s)(t) = \begin{cases} 
\alpha(w)(t), & \text{if } t \leq s, \\
\alpha(w)(s), & \text{if } t \geq s.
\end{cases}
\]

Defining \( \iota : \tilde{C}_1(Y, X) \to E_1(Y, X) \) by \( \iota(w) = (w, 1) \) and \( q : E_1(Y, X) \to C(Y, \Sigma X) \) by \( q(w, s) = \bar{\alpha}(w, s)(1) \), we have the following commutative diagram

\[
\begin{array}{ccc}
\tilde{C}_1(Y, X) & \xrightarrow{\alpha} & \Omega C(Y, \Sigma X) \\
| & \downarrow \iota & \\
E_1(Y, X) & \xrightarrow{\bar{\alpha}} & PC(Y, \Sigma X) \\
| & \downarrow q & \\
C(Y, \Sigma X) & \xrightarrow{p_1} & C(Y, \Sigma X)
\end{array}
\]
where $p_1$ is projection on the endpoint. Thus by comparison of the long exact sequences of homotopy groups, it is enough to show that $q$ is a \textit{quasifibration}, that is, a map $q : E \rightarrow B$ such that for all $b \in B$ the canonical map from $q^{-1}(b)$ to the homotopy fiber of $q$ over $b$ is a weak homotopy equivalence.

Recall from \cite{3} the Dold-Thom criterion for a map over a filtered base space to be a quasifibration. Let $B$ be a space with closed subspaces

$$F_0B \subseteq F_1B \subseteq \ldots F_jB \subseteq \ldots \subseteq B$$

and $B = \bigcup_{j \geq 0} F_j$, and let $q : E \rightarrow B$ be a map. A subspace $V \subseteq B$ is called \textit{distinguished} if the restriction $q : q^{-1}(V) \rightarrow V$ is a quasifibration. Then

\textbf{Theorem 2} (Dold and Thom) $B$ is distinguished provided that

1. $F_0B$ is distinguished, and for each $j > 0$ every open subset of $F_jB \setminus F_{j-1}B$ is distinguished, and

2. for each $j > 0$ there is a homotopy $h_t : U \rightarrow U$ of a neighborhood $U$ of $F_{j-1}B$ in $F_jB$, and a homotopy $H_t : q^{-1}(U) \rightarrow q^{-1}(U)$ such that:

   (a) $h_0$ is the identity map of $U$, $h_1(U) \subseteq F_{j-1}B$, and for all $t$,
   
   $h_t(F_{j-1}B) \subseteq F_{j-1}B$,

   (b) $H_0$ is the identity map of $q^{-1}(U)$ and for all $t$, $qH_t = h_tq$, and

   (c) for all $z \in U$, the map $H_1 : q^{-1}(z) \rightarrow q^{-1}(h_1(z))$ is a homotopy equivalence.

Here we give $C(Y, \Sigma X)$ the filtration of $[\Pi]$, that is $F_jC(Y, \Sigma X)$ is defined to be the image of $(\bigcup_{0 \leq k \leq j} \mathcal{E}(Y)_k \times (\Sigma X)^k)$. This has the property that $F_0C(Y, \Sigma X)$ consists of just the basepoint $*$, and $F_jC(Y, \Sigma X) \setminus F_{j-1}C(Y, \Sigma X)$ is homeomorphic to the image of $\mathcal{E}_1(Y)_j \times (X \setminus \{*\} \times (0,1))^j$.

We define some maps on $\tilde{C}_1(Y, X)$ to help elucidate the proof. If $w = [a_i, b_i, y_i, x_i]_{1 \leq i \leq j}$ and $w' = [a_i, b_i, y_i, x_i]_{j+1 \leq i \leq j+k}$ are configurations in which for all $k \neq l$ the sets

$$\{(t, y_i) \in \mathbb{R} \times Y | a_i < t < b_i\}$$

are pairwise disjoint, then let $w \cup w' = [a_i, b_i, y_i, x_i]_{1 \leq i \leq j+k}$. This is continuous on the subspace of $\tilde{C}_1(Y, X) \times \tilde{C}_1(Y, X)$ on which it is defined.

If $s$ and $t$ are real numbers with $s < t$ and $w = [a_i, b_i, y_i, x_i]_{1 \leq i \leq j}$, then define

$$\text{shrink}_{s, t}(w) = [s + (t-s)a_i, s + (t-s)b_i, y_i, x_i]_{1 \leq i \leq j},$$

which linearly compresses a configuration of segments in $(0,1) \times Y$ into the slice $(s, t) \times Y$. Note that the composition $\mu : \tilde{C}_1(Y, X) \times \tilde{C}_1(Y, X) \rightarrow \tilde{C}_1(Y, X)$ defined by

$$\mu(w, w') = \text{shrink}_{0, \frac{1}{2}}(w) \cup \text{shrink}_{\frac{1}{2}, 1}(w')$$

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defines an \( H \)-space structure on \( \tilde{C}_1(Y, X) \).

For an element \( z = [y_i, [x_i, s_i]]_{1 \leq i \leq j} \in F_j C(Y, \Sigma X) \setminus F_{j-1} C(Y, \Sigma X) \) we define

\[
\lambda(z) = \left[ \frac{1}{2} - \frac{s_i}{2}, 1 - \frac{s_i}{2}, y_i, x_i \right]_{1 \leq i \leq j}.
\]

This maps via \( \alpha \) to a loop whose value is \( z \) at \( t = \frac{1}{2} \), and is well-defined and continuous on \( F_j C(Y, \Sigma X) \setminus F_{j-1} C(Y, \Sigma X) \).

For an element \( w \in \tilde{C}_1(Y, X) \) and \( s \in [0, 1] \), we can define a function \( \text{below}_s(w) = [a_i, b_i, y_i, x_i]_{1 \leq i \leq j} \) and \( b_i \leq s \),

the segments of \( w \) contained in \([0, s] \times Y\). This is continuous on \( q^{-1}(F_j C(Y, \Sigma X) \setminus F_{j-1} C(Y, \Sigma X)) \).

For a relatively open set \( V \subseteq F_j C(Y, \Sigma X) \setminus F_{j-1} C(Y, \Sigma X) \) define \( \psi : \tilde{C}_1(Y, X) \times V \rightarrow q^{-1}(V) \) by

\[
\psi(w, z) = \left( \text{shrink}_{0, \frac{1}{2}}(w) \cup \text{shrink}_{\frac{1}{2}, 1}(\lambda(z)), \frac{3}{4} \right).
\]

If \((w, s) \in E_1(Y, X)\) define \( \bar{\psi}(w, s) = \text{below}_s(w) \). It follows that there is a commutative diagram

\[
\begin{array}{ccc}
\tilde{C}_1(Y, X) \times V & \xrightarrow{(\bar{\psi}, q)} & q^{-1}(V) \\
p_2 & \swarrow \psi & \searrow q \\
V & & V
\end{array}
\]

The left map is projection on the second factor and so is the simplest kind of quasifibration; thus the proof of part (1.) will be complete when we have shown that \( \psi \) and \((\bar{\psi}, q)\) are inverse equivalences over \( V \). But this is clear: \( \bar{\psi}\psi(w, z) \) is just \( \text{shrink}_{0, \frac{1}{2}}(w) \), and if \( q(w, s) = z \), then

\[
\psi(\bar{\psi}(w, s), q(w, s)) = \psi(\text{below}_s(w), z) = \left( \text{shrink}_{0, \frac{1}{2}}(\text{below}_s(w)) \cup \text{shrink}_{\frac{1}{2}, 1}(\lambda(z)), \frac{3}{4} \right),
\]

and linearly deforming all the segments to their original locations, and simultaneously deforming \( \frac{3}{4} \) to \( s \) linearly, describes a homotopy over \( V \) of \( \psi \circ (\bar{\psi}, q) \) to the identity.

The proof of part (2.) rests on the fact that the inclusion \( F_{j-1} C(Y, \Sigma X) \hookrightarrow F_j C(Y, \Sigma X) \) is a cofibration, which comes from the fact that \( X \) is nondegenerately based. Let \( W \) be a neighborhood of the basepoint \( * \) in \( X \) and let \( K_t : X \rightarrow X \) be a based homotopy where \( K_0 = \text{id} \) and \( K_1(W) = \{ * \} \). Let \( L_t \) be a linear deformation of \([0, 1]\) from the identity to the map

\[
L_1(t) = \begin{cases} 
0, & \text{if } t \leq \frac{1}{4}; \\
2t - \frac{1}{4}, & \text{if } \frac{1}{4} \leq t \leq \frac{2}{4}; \text{ and} \\
1, & \text{if } t \geq \frac{3}{4}.
\end{cases}
\]
Use the same symbol $L_t$ to denote the induced homotopy on $S^1$. Then $J_t = K_t \wedge L_t$ is a deformation of $\Sigma X = X \wedge S^1$ which collapses a neighborhood $W' = W \wedge ([0, \frac{1}{4}) \cup (\frac{3}{4}, 1])$ of the basepoint. Thus let

$$U = \left\{ y_i, [x_i, s_i] \right\}_{1 \leq i \leq j} | x_i, s_i \in W' \text{ for some } i \right\},$$

and use the functoriality of $C_{1(-,-)}$ to define $h_t(z) = C(1_Y, J_t)(z)$. For any $z = [y_i, [x_i, s_i]]$ in $U$, $J_t([x_i, s_i])$ will be $*$ for at least one index $i$, and so $J_t(z) \in F_{j-1}C(Y, \Sigma X)$. It is clear that $J_t$ preserves $F_{j-1}C(Y, \Sigma X)$ and so part (2a) is complete.

If $(w, s) \in q^{-1}(U)$ and $w = [a_i, b_i, x_i, y_i]_{1 \leq i \leq j}$, define

$$H_t(w, s) = \left( [(1-t)a_i + ta'_i, (1-t)b_i + tb'_i, y_i, K_t(x_i)] \right)_{1 \leq i \leq j}, s,$$

where $a'_i = a_i + \frac{1}{4}(b_i - a_i)$ and $b'_i = b_i - \frac{1}{4}(b_i - a_i)$. It is straightforward to verify that $qH_t = h_tq$ and so (2b.) is complete.

Finally, the restriction of $H_1$ to fibers fits into a homotopy-commutative diagram

$$
\begin{array}{ccc}
q^{-1}(z) & \xrightarrow{H_1} & q^{-1}(h_1(z)) \\
\downarrow & & \downarrow \\
\tilde{C}_1(Y, X) & \xrightarrow{\xi \circ C_{1(1_Y, K_t)}} & \tilde{C}_1(Y, X)
\end{array}
$$

where we have already shown that the maps $\tilde{\psi}$ are homotopy equivalences, and where $\xi$ is multiplication by the element

$$[a'_i, b'_i, y_i, K_t(x_i)]_{1 \leq i \leq j} \text{ and } b'_s < b_i$$

in the $H$-space structure on $\tilde{C}_1(Y, X)$. Since $\tilde{C}_1(Y, X)$ is connected (because $X$ is) this is a homotopy equivalence. This completes the proof of (2c.), and hence $q$ is a quasifibration.

More can be said. By extending and iterating the definition and theorem, we can prove

**Corollary 1** Let $Y$ be a weak metric space and $X$ a nondegenerately based space. For each $n \geq 1$ there is a space $C_n(Y, X)$ and a pair of maps

$$C(\mathbb{R}^n \times Y, X) \xrightarrow{\phi_n} C_n(Y, X) \xrightarrow{\alpha_n} \Omega^n C(Y, \Sigma^n X)$$

such that:

1. $C_n(-,-)$ is functorial with respect to based maps in the second variable and injective maps in the first variable, and $\phi_n$ and $\alpha_n$ are natural;

2. $\phi_n$ is a homotopy equivalence; and
3. \( \alpha_n \) is a weak homotopy equivalence if \( X \) is path-connected.

There is an evident action of the little \( n \)-cubes operad \( C_n \) of [1] on all the spaces appearing in the Corollary, and \( \phi_n \) and \( \alpha_n \) can be seen to be \( C_n \)-maps.

It is also true (and proved in [4]) that when \( X \) is not path connected, \( \alpha_n \) is a group-completion for \( n \geq 2 \).

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