Comment on “Exact three-dimensional wave function and the on-shell $t$ matrix for the sharply cut-off Coulomb potential: Failure of the standard renormalization factor”

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Abstract

The solutions analytically derived by Glöckle et al. [Phys. Rev. C 79, 044003 (2009)] for the three-dimensional wave function and on-shell $t$ matrix in the case of scattering on a sharply cut-off Coulomb potential appear to be fallacious. And their renormalization factor lacks mathematical grounds.

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In a recent paper by Glöckle et al. [1], nonrelativistic scattering of two equally charged particles with mass $m$ interacting via potential $V(r) = \frac{e^2}{r} \Theta(R - r)$ was considered. The authors argued that they analytically derived the exact wave function and scattering amplitude for arbitrary values of a cut-off radius $R$. On this basis they obtained a renormalization factor which relates the scattering amplitude in the limit $R \to \infty$ with the physical Coulomb scattering amplitude. The purpose of this Comment is (i) to show that the analytical results of Glöckle et al. [1] are erroneous for finite values of $R$ and are mathematically ungrounded in the limit $R \to \infty$ and (ii) to point out a different renormalization approach which is free from uncertainties associated with the cut-off renormalization.

The authors of [1] made an unjustified premise that the solution of the Lippmann-Schwinger equation for $r < R$ obeys the form

$$
\Psi_R^{(+)}(\vec{r}) = Ae^{i\vec{p} \cdot \vec{r}} F_1(-i\eta, 1, i(pr - \vec{p} \cdot \vec{r})), \tag{1}
$$

where $\eta = \frac{me^2}{2p}$ is a Sommerfeld parameter. The constant [1]

$$
A = \frac{1}{F_1(-i\eta, 1, 2ipR)} \tag{2}
$$

was determined in Ref. [1] by inserting (1) into the Lippmann-Schwinger equation

$$
\Psi_R^{(+)}(\vec{r}) = e^{i\vec{p} \cdot \vec{r}} - \frac{\mu e^2}{2\pi} \int \frac{d^3 \vec{r}'}{r'} e^{ip|\vec{r} - \vec{r}'|} \Theta(R - r') \Psi_R^{(+)}(\vec{r}') \tag{3}
$$

and solving the latter at $r = 0$. In fact, the correct form of the solution in the interior region $r < R$ is

$$
\Psi_R^{(+)}(\vec{r}) = \frac{1}{4\pi} \int d^2 \hat{\vec{k}} \mathcal{A}(\hat{\vec{k}}) e^{i\hat{\vec{k}} \cdot \vec{r}} F_1(-i\eta, 1, i(pr - \hat{\vec{p}} \cdot \vec{r})), \tag{4}
$$

where the function $\mathcal{A}(\hat{\vec{k}})$ is defined on a unit sphere. To determine $\mathcal{A}(\hat{\vec{k}})$ one may employ the usual partial wave formalism (see, for instance, the textbook [2]). Consider the following expansion in Legendre polynomials:

$$
\mathcal{A}(\hat{\vec{k}}) = \sum_l (2l + 1) A_l P_l(\hat{\vec{p}} \cdot \hat{\vec{k}}). \tag{5}
$$

Matching the interior Lippmann-Schwinger solution and its derivative to the exterior ones at $r = R$ yields

$$
A_l = \frac{i(pR)^{-2}}{W(\psi_l, h_l^{(1)})(pR)}, \tag{6}
$$

2
where \( h_{l}^{(1)} \) is a spherical Hankel function of the first kind and
\[
\psi_l(pr) = e^{\sigma_l} \frac{\Gamma(l+1+i\eta)}{\Gamma(1+i\eta)} \frac{(2pr)^l}{(2l+1)!} e^{-i\eta} F_1(l+1-i\eta,2l+2,2ipr),
\]
with the Coulomb phase shift \( \sigma_l = \arg \Gamma(l+1+i\eta) \). It can be checked that \( A_0 = A \) but \( A_{l\geq 1} \neq A \), i.e. (1) is clearly invalid.

The expression for the scattering amplitude (the on-shell \( t \) matrix) in Ref. [1] is invalid as well, since it derives from the wave function (1). The limit of vanishing screening \( (R \to \infty) \) has been considered previously in the literature (see, for instance, [2, 4, 5] and references therein). Using asymptotic forms of \( \psi_l \) and \( h_{l}^{(1)} \) one readily arrives at
\[
A_l \simeq e^{-\frac{\pi \eta}{2}} \Gamma(1+i\eta) e^{-i\eta \ln(2pR)} + O \left( \frac{1}{pR} \right),
\]
provided \( l \ll pR \). When \( l \gg pR \), the phase shifts behave as \( \delta_l \to 0 \) due to the angular momentum barrier. The intermediate situation \( l \sim pR \) is very hard to handle. Thus, the convergence \( A_l \to e^{-\frac{\pi \eta}{2}} \Gamma(1+i\eta) e^{-i\eta \ln(2pR)} \) is not uniform, i.e. it depends on \( l \), and therefore taking the limit \( R \to \infty \) in (5) presents quite a challenge. Nevertheless, it can be shown that the asymptotic form for the scattering amplitude is
\[
f_R = e^{-2i\eta \ln(2pR)} f_c + f_{osc},
\]
where \( f_c \) is the physical Coulomb scattering amplitude. The first term in the right-hand side of (8) appears because of (7). The term \( f_{osc} \) oscillates rapidly like \( \cos(qR) \), where \( q \) is the momentum transfer. It integrates out to zero with the incident wave packet and, hence, makes no contribution to the cross section as measured in typical experiments (see [2] for details).

The amplitude derived in Ref. [1] in the limit \( R \to \infty \) resembles the form (8), however its derivation lacks mathematical grounds because it is carried out using (1) instead of the exact wave function (4). The wave function (1) can be presented as a product \( C_R \Psi_c^{(+)} \), where \( \Psi_c^{(+)} \) is a Coulomb wave and \( C_R \) is a constant \( (C_{R \to \infty} \to e^{-i\eta \ln(2pR)}) \). The Coulomb wave satisfies a homogeneous Lippmann-Schwinger equation
\[
\Psi_c^{(+)}(\vec{r}) = -\frac{\mu e^2}{2\pi} \int \frac{d^3\vec{r}'}{|\vec{r}-\vec{r}'|} \Psi_c^{(+)}(\vec{r}') \Psi_c^{(+)}(\vec{r}).
\]
Let us introduce an auxiliary function which is a difference between the exact wave function (4) and the wave function (1) in the limit \( R \to \infty \):
\[
\psi_R(\vec{r}) = \Psi_R^{(+)}(\vec{r}) - e^{-i\eta \ln(2pR)} \Psi_c^{(+)}(\vec{r}).
\]
According to (3) and (9), this function satisfies the following equation ($r < R$):

$$\psi_R(\vec{r}) = \psi_R^{(0)}(\vec{r}) - \frac{\mu e^2}{2\pi} \int \frac{d^3r'}{r'} \frac{e^{ip|\vec{r}' - \vec{r}|}}{|\vec{r} - \vec{r}'|} \Theta(R - r') \psi_R(\vec{r}')$$, \hspace{1cm} (11)

with the inhomogeneous term

$$\psi_R^{(0)}(\vec{r}) = e^{ip\vec{r}} + \frac{\mu e^2}{2\pi} e^{-i\eta \ln(2pR)} \int \frac{d^3r'}{r'} \frac{e^{ip|\vec{r}' - \vec{r}|}}{|\vec{r} - \vec{r}'|} \Theta(r' - R) \Psi^+(\vec{r}')$$, \hspace{1cm} (12)

For $r \ll R$ one has approximately

$$\frac{e^{ip|\vec{r}' - \vec{r}|}}{|\vec{r} - \vec{r}'|} \approx \frac{e^{ipr'}}{r'} e^{-i\vec{p}' \cdot \vec{r}}$$, \hspace{1cm} \vec{p}' = \frac{p \hat{r}}{r'},$$

and it can be shown that $\psi_R^{(0)}(\vec{r}) \approx 0$. However this does not imply that $\psi_R^{(0)}(\vec{r}) \approx 0$ for any $r < R$. In fact, this is definitely not the case when $r \lesssim R$ (within the partial wave formalism this situation corresponds to the $l \lesssim pR$ terms).

Using (10), the scattering amplitude can be presented as

$$f_R = -\frac{\mu e^2}{2\pi} e^{-i\eta \ln(2pR)} \int \frac{d^3r'}{r'} \frac{e^{-i\vec{p}' \cdot \vec{r}} \Theta(R - r') \Psi^+(\vec{r}')}{r'} - \frac{\mu e^2}{2\pi} \int \frac{d^3r'}{r'} \frac{e^{-i\vec{p}' \cdot \vec{r}} \Theta(R - r') \psi_R(\vec{r})}{r'},$$ \hspace{1cm} (13)

where $\vec{p}' = p \hat{r}$. Glöckle et al. [1] have studied asymptotic behavior of the first term only. They unjustifiably have neglected the second term which, due to nontrivial properties of $\psi_R$ in the region $r \lesssim R$, potentially can provide a nonvanishing contribution to the scattering amplitude in the limit $R \to \infty$. Thus, their analysis is incomplete and the validity of their renormalization factor is questionable. In this connection, the results of the numerical calculations presented in Ref. [1] can not be a decisive argument in favour of the renormalization factor, for in this particular case one deals with divergent and rapidly oscillating quantities.

Finally, it is useful to note that the renormalization treatments involving cut-off Coulomb potentials are of doubtful value from a practical viewpoint, especially in the case of many-body Coulomb scattering. In this respect, the methods based on regularization and renormalization of the Lippmann-Schwinger equations in the on-shell limit are more efficient. The two-particle case is fully explored: (i) the Green’s function is derived analytically both in coordinate and in momentum representations [7], (ii) an off-shell amplitude is known [8], and (iii) the rules for taking the on-shell limit are formulated [9]. This allows to generalize the two-particle results to the many-particle case (see, for example, [10]).
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