The Finsler-like geometry of the \((t, x)\)-conformal deformation of the jet Berwald-Moór metric

Mircea Neagu
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Abstract
The aim of this paper is to develop on the 1-jet space \(J^1(\mathbb{R}, M^n)\) the Finsler-like geometry (in the sense of distinguished (d-) connection, d-torsions, d-curvatures and some gravitational-like and electromagnetic-like geometrical models) attached to the \((t, x)\)-conformal deformation of the Berwald-Moór metric.

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1 Introduction
The geometric-physical Berwald-Moór structure ([5], [12], [11]) was intensively investigated by P.K. Rashevski ([18]) and further fundamented and developed by D.G. Pavlov, G.I. Garas’ko and S.S. Kokarev ([15], [16], [8], [17]). At the same time, the physical studies of Asanov ([1]) or Garas’ko and Pavlov ([8]) emphasize the importance of the Finsler geometry characterized by the total equality in rights of all non-isotropic directions in the theory of space-time structure, gravitation and electromagnetism. For such a reason, one underlines the important role played by the Berwald-Moór metric

\[ F : TM \to \mathbb{R}, \quad F(y) = \sqrt{y^1y^2...y^n}, \quad n \geq 2, \]

whose tangent Finslerian geometry is studied by geometers as Matsumoto and Shimada ([9]) or Balan ([3]). In such a perspective, according to the recent geometric-physical ideas proposed by Garas’ko in [7] and [6], we consider that a Finsler-like geometric-physical study for the \((t, x)\)-conformal deformations of the jet Berwald-Moór structure is required. Consequently, this paper investigates on the 1-jet space \(J^1(\mathbb{R}, M^n)\) the Finsler-like geometry (together with a theoretical-geometric gravitational field-like theory) of the \((t, x)\)-conformal deformation of
the Berwald-Moőr metric

\[ F(t, x, y) = e^{\sigma(x)} \sqrt{h_{11}(t)} \cdot [y_1^1 y_1^2 \ldots y_1^n]^{1/n}, \quad (1.1) \]

where \( \sigma(x) \) is a smooth non-constant function on \( M^n \), \( h_{11}(t) \) is the dual of a Riemannian metric, \( y_1^1, y_1^2, \ldots, y_1^n \) are the coordinates of the 1-jet space \( J^1(\mathbb{R}, M^n) \), which transform by the rules (the Einstein convention of summation is assumed everywhere):

\[ \tilde{t} = \tilde{t}(t), \quad \tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{y}_i^1 = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} \cdot y_j^1, \quad (1.2) \]

where \( i, j = 1, n \), \( \text{rank } (\partial \tilde{x}^i/\partial x^j) = n \) and \( d\tilde{t}/dt \neq 0 \). Note that the particular jet Finsler-like geometries (together with their corresponding jet geometrical gravitational field-like theories) of the \((t, x)\)-conformal deformations of the Berwald-Moőr metrics of order three and four are now completely developed in the papers [13] and [14].

Based on the geometrical ideas promoted by Miron and Anastasiei in the classical Lagrangian geometry of tangent bundles ([10]), together with those used by Asanov in the geometry of 1-jet spaces ([2]), the differential geometry (in the sense of d-connections, d-torsions, d-curvatures, gravitational and electromagnetic geometrical theories) produced by an arbitrary jet rheonomic Lagrangian function \( L : J^1(\mathbb{R}, M^n) \rightarrow \mathbb{R} \) is now exposed in the monograph [4]. In what follows, we apply the general jet geometrical results from book [4] to the \((t, x)\)-conformal deformed jet Berwald-Moőr metric (1.1).

2 The canonical nonlinear connection

Let us rewrite the \((t, x)\)-conformal deformed jet Berwald-Moőr metric (1.1) in the form

\[ F(t, x, y) = e^{\sigma(x)} \sqrt{h_{11}(t)} \cdot [G_{1[n]}(y)]^{1/n}, \]

where

\[ G_{1[n]}(y) = y_1^1 y_1^2 \ldots y_1^n. \]

Hereinafter, the fundamental metrical d-tensor produced by the metric (1.1) is given by the formulæ (see [4])

\[ \tilde{g}_{ij}(t, x, y) \overset{\text{def}}{=} \frac{h_{11}(t)}{2} \frac{\partial^2 F^2}{\partial y_i^1 \partial y_j^1} \Rightarrow \]

1We assume that we have \( y_1^1 y_1^2 \ldots y_1^n > 0 \). This is a domain of existence where we can \( y \)-differentiate the Finsler-like function \( F(t, x, y) \).

2Throughout this paper the Latin letters \( i, j, k, m, r, \ldots \) take values in the set \( \{1, 2, \ldots n\} \).
\[ g_{ij}(t, x, y) := g_{ij}(x, y) = e^{2\sigma(x)} \left( \frac{2}{n} - \delta_{ij} \right) \frac{G_{1[n]}^2}{y_1^1 y_1^2}. \tag{2.1} \]

where we have no sum by \( i \) or \( j \). Moreover, the matrix \( \tilde{g} = (\tilde{g}_{ij}) \) admits the inverse \( \tilde{g}^{-1} = (\tilde{g}^{jk}) \), whose entries are

\[ \tilde{g}^{jk} = e^{-2\sigma(x)}(2 - n\delta^{jk})G_{1[n]}^{-2/n} y_1^j y_1^k \] (no sum by \( j \) or \( k \)). \tag{2.2}

Let us consider that the Christoffel symbol of the Riemannian metric \( h_{11}(t) \) on \( \mathbb{R} \) is

\[ \kappa^1_{11} = \frac{h_{11}^1}{2} \frac{dh_{11}}{dt}, \]

where \( h_{11}^1 = 1/h_{11} > 0 \). Then, using a general formula from [4] and taking into account that we have

\[ \frac{\partial G_{1[n]}}{\partial y_1^1} = \frac{G_{1[n]}}{y_1^1}, \]

we find the following geometrical result:

**Proposition 2.1** For the \((t, x)\)-conformal deformed Berwald-Moór metric \((1.1)\), the energy action functional

\[ \hat{E}(t, x(t)) = \int_a^b \sqrt{h_{11}^1} dt = \int_a^b e^{2\sigma(x)} \left[ y_1^1 y_1^2 ... y_1^n \right]^{2/n} \cdot h_{11}^1 \sqrt{h_{11}^1} dt, \]

where \( y = dx/dt \), produces on the 1-jet space \( J^1(\mathbb{R}, M^n) \) the canonical non-linear connection

\[ \hat{\Gamma} = \left( M^{(i)}_{(1)1} = -\kappa^1_{11} y_1^1, \ N^{(i)}_{(1)j} = n\sigma_i y_1^j \delta^i_j \right), \tag{2.3} \]

where

\[ \sigma_i = \frac{\partial \sigma}{\partial x^i}. \]

**Proof.** For the energy action functional \( \hat{E} \), the associated Euler-Lagrange equations can be written in the equivalent form (see [4])

\[ \frac{d^2 x^i}{dt^2} + 2H^{(i)}_{(1)1} (t, x^k, y_1^k) + 2G^{(i)}_{(1)1} (t, x^k, y_1^k) = 0, \tag{2.4} \]

where the local components

\[ H^{(i)}_{(1)1} \overset{\text{def}}{=} -\frac{1}{2} \kappa^1_{11}(t) y_1^1 \]

and

\[ G^{(i)}_{(1)1} \overset{\text{def}}{=} \frac{h_{11}^1 g_{1p}^i}{4} \left[ \frac{\partial^2 F^2}{\partial x^p \partial y_1^1} y_1^1 - \frac{\partial F^2}{\partial x^p} + \frac{\partial^2 F^2}{\partial t \partial y_1^1} + \right. \]

\[ \left. + \frac{\partial F^2}{\partial y_1^1} \kappa^1_{11}(t) + 2h_{11}^1 \kappa^1_{11} y_1^1 \right] = \frac{n}{2} \sigma_i (y_1^1)^2 \]
represent, from a geometrical point of view, a spray on the 1-jet space $J^1(\mathbb{R}, M^n)$.

Therefore, the canonical nonlinear connection associated to this spray has the local components (see [4])

\[ M^{(i)}_{(1)1} \overset{def}{=} 2H^{(i)}_{(1)1} = -K^{(i)}_{11} y^i_1, \quad N^{(i)}_{(1)j} \overset{def}{=} \frac{\partial G^{(i)}_{(1)1}}{\partial y^j_1} = n\sigma_i y^i_1 \delta^j_i. \]

\[ \Box \]

3 The Cartan canonical $^\ast\Gamma$-linear connection. Its d-torsions and d-curvatures

The nonlinear connection [2,3] produces the dual adapted bases of d-vector fields (no sum by $i$)

\[ \left\{ \frac{\delta}{\delta t} = \frac{\partial}{\partial t} + K^{1}_{11} y^p_1 \frac{\partial}{\partial y^p_1}; \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - n\sigma_i y^i_1 \frac{\partial}{\partial y^i_1}; \quad \frac{\partial}{\partial y^i_1} \right\} \subset \mathcal{X}(E) \quad (3.1) \]

and d-covector fields (no sum by $i$)

\[ \left\{ dt; \quad dx^i; \quad \delta y^i_1 = dy^i_1 - K^{1}_{11} y^i_1 dt + n\sigma_i y^i_1 dx^i \right\} \subset \mathcal{X}^*(E), \quad (3.2) \]

where $E = J^1(\mathbb{R}, M^n)$. The naturalness of the geometrical adapted bases (3.1) and (3.2) is coming from the fact that, via a transformation of coordinates (1.2), their elements transform as the classical tensors. Therefore, the description of all subsequent geometrical objects on the 1-jet space $J^1(\mathbb{R}, M^n)$ (e.g., the Cartan canonical linear connection, its torsion and curvature etc.) will be given in local adapted components. Consequently, by direct computations, we obtain the following geometrical result:

**Proposition 3.1** The Cartan canonical $^\ast\Gamma$-linear connection, produced by the $(t, x)$-conformal deformed Berwald-Moór metric (1.1), has the following adapted local components (no sum by $i, j$ or $k$):

\[ C^\ast \Gamma = \begin{pmatrix} K^{1}_{11}, \ G^k_{j1} = 0, \ L^1_{jk} = n\delta^i_j \delta^i_k \sigma_i, \ C^{(1)}_{j(k)} = C^i_{jk} \cdot \frac{y^i_1 y^k_1}{y^j_1} \end{pmatrix}, \quad (3.3) \]

where

\[ C^i_{jk} = -\frac{2}{n^2} + \frac{\delta^i_j + \delta^i_k + \delta^i_{jk}}{n} - \delta^i_j \delta^i_k. \]
The adapted components of the Cartan canonical connection are given by the formulas (see [4])

\[
G_{j1}^{k} \overset{\text{def}}{=} \frac{\delta g_{km}^{*}}{\delta x^{j}} = 0,
\]

\[
L_{ijk}^{l} \overset{\text{def}}{=} \frac{\delta g_{jm}^{*}}{\delta x^{i}} + \frac{\delta g_{km}^{*}}{\delta x^{j}} - \frac{\delta g_{jk}^{*}}{\delta x^{m}} = n\delta_{j}^{i}\delta_{k}^{l},
\]

\[
C_{ij(k)}^{l(1)} \overset{\text{def}}{=} \frac{\delta g_{jm}^{*}}{\delta y_{i}^{1}} + \frac{\delta g_{km}^{*}}{\delta y_{j}^{1}} - \frac{\delta g_{jk}^{*}}{\delta y_{m}^{1}} = \frac{\delta g_{jm}^{*}}{\delta y_{i}^{1}} = C_{ij(k)}^{l(1)} \cdot \frac{y_{i}^{1}}{y_{j}^{1}},
\]

where we also used the equality

\[
\frac{\delta g_{jm}^{*}}{\delta x^{k}} = n\delta_{jk}g_{jm}^{*} = n\delta_{mk}g_{jm}^{*}.
\]

**Remark 3.2** It is important to note that the vertical d-tensor \(C_{ij(k)}^{l(1)}\) also has the properties (see also [9], [13] and [14]):

\[
C_{ij(k)}^{l(1)} = C_{kj(i)}^{l(1)}, \quad C_{ij(m)}^{l(1)} = 0, \quad C_{j(m)}^{m(1)} = 0, \quad C_{i(k)m}^{m(1)} = 0,
\]

with sum by \(m\), where

\[
C_{ij(k)}^{l(1)} \overset{\text{def}}{=} \frac{\delta C_{ij(k)}^{l(1)}}{\delta x^{m}} + C_{i(k)}^{l(1)}L_{rj}^{r} - C_{r(k)}^{l(1)}L_{ij}^{r} - C_{i(k)}^{l(1)}L_{rj}^{r}.
\]

**Proposition 3.3** The Cartan canonical connection of the \((t, x)\)-conformal deformed Berwald-Moôr metric (1.1) has two effective local torsion d-tensors:

\[
R_{(1)}^{(r)} \overset{\text{def}}{=} -\frac{2}{n^{2}} + \frac{\delta r_{i}^{r} + \delta r_{j}^{r} + \delta r_{ij}}{n} - \delta r_{i}^{r}, \quad P_{(1)}^{r} \overset{\text{def}}{=} \frac{\delta r}{\delta x^{i}} = \frac{\delta r}{\delta y_{i}^{1}},
\]

where

\[
\sigma_{pq} := \frac{\delta^{2}r}{\delta x^{i} \delta x^{j}}.
\]

**Proof.** Generally, an \(h\)-normal \(\Gamma\)-linear connection on the 1-jet space \(J^{1}(\mathbb{R}, M^{n})\) has eight effective local d-tensors of torsion (for more details, see [4]). For the Cartan canonical connection these reduce only to two (the other six are zero):

\[
R_{(1)}^{(r)} \overset{\text{def}}{=} \frac{\delta N_{(1)}^{(r)}}{\delta x^{j}} = \frac{\delta N_{(1)}^{(r)}}{\delta x^{j}}, \quad P_{(1)}^{r} \overset{\text{def}}{=} C_{i(j)}^{r(1)}.
\]
Proposition 3.4. The Cartan canonical connection of the $(t, x)$-conformal deformed Berwald-Moór metric (1.1) has three effective local curvature $d$-tensors:

\[ R_{ijk}^l = \frac{\partial L_{ij}^l}{\partial x^k} - \frac{\partial L_{ik}^l}{\partial x^j} + L_{ij}^r L_{rk}^l - L_{ik}^r L_{rj}^l + C_{(i(r)}^l C_{(j)}^{(1)} r_{(k)} - C_{(i(k)}^l C_{(j)}^{(1)} r_{(r)}\]

\[ S_{(i(j)(k)}^{(1)(1)} = \frac{\partial C_{(i(j)}^{(1)} r_{(k)}^{(1)}}{\partial y_1^k} - \frac{\partial C_{(i(k)}^{(1)} r_{(j)}^{(1)}}{\partial y_1^j} + C_{(i(j)}^{(1)} C_{(r(k)}}^{(1)} r_{(r)}^{(1)} - C_{(i(k)}^{(1)} C_{(r(j)}^{(1)} r_{(r)}^{(1)}\]

Proof. Generally, an $h$-normal $\Gamma$-linear connection on the 1-jet space $J^1(\mathbb{R}, M^n)$ has five effective local $d$-tensors of curvature (for more details, see [4]). For the Cartan canonical connection (3.3) these reduce only to three (the other two are zero); these are

\[ R_{ijk}^l = \frac{\partial L_{ij}^l}{\partial x^k} - \frac{\partial L_{ik}^l}{\partial x^j} + L_{ij}^r L_{rk}^l - L_{ik}^r L_{rj}^l + C_{(i(r)}^l C_{(j)}^{(1)} r_{(k)} - C_{(i(k)}^l C_{(j)}^{(1)} r_{(r)}\]

\[ P_{(i(j)(k)}^{(1)(1)} = \frac{\partial C_{(i(j)}^{(1)} r_{(k)}^{(1)}}{\partial y_1^k} - \frac{\partial C_{(i(k)}^{(1)} r_{(j)}^{(1)}}{\partial y_1^j} + C_{(i(j)}^{(1)} C_{(r(k)}}^{(1)} r_{(r)}^{(1)} - C_{(i(k)}^{(1)} C_{(r(j)}^{(1)} r_{(r)}^{(1)}\]

where we used the equality

\[ P_{(i(j)(k)}^{(1)(1)} = \frac{\partial C_{(i(j)}^{(1)} r_{(k)}^{(1)}}{\partial y_1^k} - \frac{\partial C_{(i(k)}^{(1)} r_{(j)}^{(1)}}{\partial y_1^j} + C_{(i(j)}^{(1)} C_{(r(k)}}^{(1)} r_{(r)}^{(1)} - C_{(i(k)}^{(1)} C_{(r(j)}^{(1)} r_{(r)}^{(1)}\]

\[ h_{11} dt \otimes dt + \hat{g}_{ij} dx^i \otimes dx^j + h_{11}^* \delta y_1^i \otimes \delta y_1^j, \quad (4.1)\]

where \( \hat{g}_{ij} \) is given by (2.1), and we have

\[ \delta y_1^i = dy_1^i - K_{11}^1 y_1^i dt + n \sigma_i y_1^i dx^i \ (\text{no sum by } i).\]

From an abstract physical point of view, the metrical $d$-tensor (4.1) may be regarded as a “non-isotropic gravitational potential”. In our geometric-physical
approach, one postulates that the non-isotropic gravitational potential \( G \) is
governed by the following geometrical Einstein-like equations:

\[
\text{Ric} (\mathbf{C^*}) - \frac{\text{Sc} (\mathbf{C^*})}{2} \mathbf{G} = \mathcal{K} \mathcal{T},
\]

(4.2)

where

\[\text{Ric} (\mathbf{C^*})\]

is the Ricci \( d \)-tensor associated to the Cartan canonical linear connection (3.3);

\[\text{Sc} (\mathbf{C^*})\]

is the scalar curvature;

\[\mathcal{K}\]

is the Einstein constant and \( \mathcal{T} \) is the intrinsic non-isotropic stress-energy \( d \)-tensor of matter.

Therefore, using the adapted basis of vector fields (3.1), we can locally describe the global geometrical Einstein-like equations (4.2). Consequently, some direct computations lead to:

**Lemma 4.1** The Ricci tensor of the Cartan canonical connection \( \mathbf{C^*} \) of the \((t, x)\)-conformal deformed Berwald-Moór metric (1.1) has the following two effective local Ricci \( d \)-tensors (no sum by \( i, j, k \) or \( l \)):

\[
R_{ij} = \begin{cases} 
-\sigma_{ij} - \sum_{m=1, m \neq j}^{n} \sigma_{jm} \frac{y_{m}^{i}}{y_{1}^{i}}, & i \neq j \\
0, & i = j,
\end{cases}
\]

(4.3)

**Proof.** Generally, the Ricci tensor of the Cartan canonical connection \( \mathbf{C^*} \) produced by an arbitrary jet Lagrangian function is determined by six effective local Ricci \( d \)-tensors (for more details, see [4]). For our particular Cartan canonical connection (3.3) these reduce only to the following two (the other four are zero):

\[
R_{ij} \overset{\text{def}}{=} R_{ij}^{m} = \frac{\partial L_{ij}^{m}}{\partial x^{m}} - \frac{\partial L_{ij}^{m}}{\partial x^{j}} + L_{ij}^{r} L_{rm}^{m} - L_{im}^{r} L_{rj}^{m} + C_{i(r)}^{m(1)} R_{(r)jm}^{m},
\]

\[
S_{(i)(j)}^{(1)(1)} \overset{\text{def}}{=} S_{i(m)(1)}^{m(1)} = \frac{\partial C_{i(m)}^{m(1)}}{\partial y_{1}^{m}} - \frac{\partial C_{i(m)}^{m(1)}}{\partial y_{1}^{j}} + C_{i(j)}^{r(1)} C_{r(m)}^{m(1)} - C_{i(m)}^{r(1)} C_{r(j)}^{m(1)} =
\]

\[
= \frac{\partial C_{i(m)}^{m(1)}}{\partial y_{1}^{m}} - C_{i(m)}^{r(1)} C_{r(j)}^{m(1)},
\]

with sum by \( r \) and \( m \). ■

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Lemma 4.2 The scalar curvature of the Cartan canonical connection $C \Gamma^*$ of the $(t, x)$-conformal deformed Berwald-Moór metric (1.1) has the value

$$\text{Sc} (C \Gamma^*) = -e^{-2\sigma} G_{1[n]}^{-2/n} \left[ 4nY_{11} + \left( n^2 - 3n + 2 \right) h_{11} \right],$$

where

$$Y_{11} = \sum_{p < q}^{n} \sigma_{pq} y_p^q y_q^p.$$

Proof. The scalar curvature of the Cartan canonical connection (3.3) is given by the formula (for more details, see [4]):

$$\text{Sc} (C \Gamma^*) = *g^{pq} R_{pq} + h_{11} *g^{pq} S^{(1)(1)}_{(p)(q)}. $$

The local description in the adapted basis of vector fields (3.1) of the global geometrical Einstein-like equations (4.2) is given by (for more details, see [4]):

$$e^{-2\sigma} G_{1[n]}^{-2/n} \left[ 2nY_{11} + \frac{n^2 - 3n + 2}{2} h_{11} \right] h_{11} = K T_{11},$$

$$R_{ij} + e^{-2\sigma} G_{1[n]}^{-2/n} \left[ 2nY_{11} + \frac{n^2 - 3n + 2}{2} h_{11} \right] g_{ij} = K T_{ij},$$

$$S^{(1)(1)}_{(i)(j)} + e^{-2\sigma} G_{1[n]}^{-2/n} \left[ 2nY_{11} + \frac{n^2 - 3n + 2}{2} h_{11} \right] h_{11} *g_{ij} = K T^{(1)(1)}_{(i)(j)} \quad (4.4)$$

$$0 = T_{ii}, \quad 0 = T_{1i}, \quad 0 = T^{(1)}_{(i)1},$$

$$0 = T^{(1)}_{1(i)}, \quad 0 = T^{(1)}_{(1)i}, \quad 0 = T^{(1)}_{(i)(j)}. $$

Corollary 4.4 The non-isotropic stress-energy $d$-tensor of matter $T$ satisfies the following geometrical conservation laws (sum by $m$):

$$T^{1}_{1/1} + T^{m}_{1[m] + T^{(m)}_{(1)1} (1)(m) = 0}$$

$$T^{1}_{i/1} + T^{m}_{i[m] + T^{(m)}_{(1)i} (1)(m) = E^{m}_{i[m]}$$

$$T^{(1)}_{(i)/1} + T^{(n)}_{(i)(i)} + T^{(m)(1)}_{(1)(i) (m)} = 2e^{-2\sigma} G_{1[n]}^{-2/n} \left[ n \frac{\partial Y_{11}}{\partial y_i^1} - 2Y_{11} \right],$$

where (sum by $r$):

$$T^{1}_{1} = h_{11} T_{11} = K^{-1} e^{-2\sigma} G_{1[n]}^{-2/n} \left[ 2nY_{11} + \frac{n^2 - 3n + 2}{2} h_{11} \right],$$

$$8$$
\[ \mathcal{T}_1^{\text{def}} = g^{mr} \mathcal{T}_{r1} = 0, \quad \mathcal{T}^{(m)}_{(1)} = h_{111} g^{mr} \mathcal{T}^{(1)}_{(r)1} = 0, \quad \mathcal{T}^1_1 = h^{11} \mathcal{T}_{111} = 0, \]
\[ \mathcal{T}_i^{\text{def}} = g^{mr} \mathcal{T}_{ri} := E_i^m = K^{-1} \left[ g^{mr} R_{ri} + \left(2nY_{11} + \frac{n^2 - 3n + 2}{2} h_{11} \right) \delta_i^m \right], \]
\[ \mathcal{T}^{(m)}_{(1)} = h_{111} g^{mr} \mathcal{T}^{(1)}_{(r)i} = 0, \quad \mathcal{T}^{(1)}_{(i)} = h^{11} \mathcal{T}^{(1)}_{1(i)} = 0, \quad \mathcal{T}^{(m)}_{(i)} = g^{mr} \mathcal{T}^{(r)}_{(i)} = 0, \]
\[ \mathcal{T}^{(m)}_{(1)(i)} = h_{111} g^{mr} \mathcal{T}^{(1)(1)}_{r(i)} = \frac{e^{-2\sigma} G^{\alpha \beta}_{(1)n}}{K} \left[ \frac{n - 2}{n} h_{11} \frac{y^m_i}{y_i} + \left(2nY_{11} + \frac{n^2 - 5n + 6}{2} h_{11} \right) \delta_i^m \right], \]

and we also have (summation by \( m \) and \( r \), but no sum by \( i \))
\[ \mathcal{T}_{1/11} = \frac{\delta \mathcal{T}_1^m}{\delta t}, \quad \mathcal{T}_{1/11} = \frac{\delta \mathcal{T}_1^m}{\delta x^m} + \mathcal{T}_r L^r_{rm}, \]
\[ \mathcal{T}_{(1)(1)}^{(m)}_{11} = \frac{\partial \mathcal{T}_{(1)}^{(m)}}{\partial y_1^m} + \mathcal{T}^{(r)}_{(r)(1)} C_{(m)}^{(1)} = \frac{\partial \mathcal{T}_{(1)}^{(m)}}{\partial y_1^m} \]
\[ \mathcal{T}_{1/11} = \frac{\delta \mathcal{T}_1^m}{\delta t}, \quad \mathcal{T}_{1/11} = \frac{\delta \mathcal{T}_1^m}{\delta x^m} + \mathcal{T}_r L^r_{rm} - \mathcal{T}_r L^r_{rm} = E_{1/11}^m := \frac{\delta E_{1/11}^m}{\delta x^m} + n E_{1/11}^m \sigma_m = n E_{1/11}^m \sigma_m, \]
\[ \mathcal{T}^{(m)}_{(1)(i)} = \frac{\partial \mathcal{T}^{(m)}_{(1)}}{\partial y_i^m} + \mathcal{T}^{(r)}_{(r)(1)} C_{(m)}^{(1)} = \frac{\partial \mathcal{T}_{(1)}^{(m)}}{\partial y_i^m} - \mathcal{T}^{(m)}_{(r)(1)} C_{(i)(m)}^{(1)}, \]
\[ \mathcal{T}_{(1)(i)}^{(1)} = \frac{\partial \mathcal{T}_{(1)(i)}}{\partial t}, \quad \mathcal{T}_{(1)(i)}^{(m)}_{11} = \frac{\delta \mathcal{T}_{(1)(i)}}{\delta x^m} + \mathcal{T}^{(r)}_{(r)(1)} C_{(m)}^{(1)} = \frac{\partial \mathcal{T}_{(1)(i)}}{\partial y_1^m} \]
\[ \mathcal{T}^{(m)}_{(1)(i)} = \frac{\partial \mathcal{T}^{(m)}_{(1)(i)}}{\partial y_i^m} + \mathcal{T}^{(r)}_{(r)(1)} C_{(m)}^{(1)} = \frac{\partial \mathcal{T}_{(1)(i)}}{\partial y_1^m} \]

**Proof.** The local Einstein-like equations (4.1), together with some direct computations, lead us to what we were looking for. Also note that we have (summation by \( m \) and \( r \))
\[ \mathcal{T}^{(m)}_{(1)(i)} C_{(i)(m)}^{(1)} = 0. \]
4.2 Electromagnetic-like geometrical model

In the book [4], a geometrical theory for electromagnetism was also created, using only a given Lagrangian function \( L \) on the 1-jet space \( J^1(\mathbb{R}, M^n) \). In the background of the jet single-time (one-parameter) Lagrange geometry from [4], one works with the following non-isotropic electromagnetic distinguished 2-form (sum by \( i \) and \( j \)):

\[
F = F^{(1)}_{(ij)} \delta y^i_1 \wedge dx^j,
\]

where (sum by \( m \) and \( r \))

\[
F^{(1)}_{(ij)} = \frac{\hbar^{11}}{2} \left[ g^{jm} N_{(1)i}^{(m)} - g^{im} N_{(1)j}^{(m)} + \left( g^{ir} L^r_{jm} - g^{jr} L^r_{im} \right) y^m_1 \right].
\]

This is characterized by some natural geometrical Maxwell-like equations (for more details, see [10] and [4]).

**Remark 4.5** The Lagrangian function that governs the movement law of a particle of mass \( m \neq 0 \) and electric charge \( e \), which is displaced concomitantly into an environment endowed both with a gravitational field and an electromagnetic one, is given by

\[
L(t, x^k, y^k_1) = m c h^{11}(t) \varphi_{ij}(x^k) y^i_1 y^j_1 + \frac{2e}{m} A^{(1)}_{(ij)}(t, x^k) y^i_1,
\]

where

- the semi-Riemannian metric \( \varphi_{ij}(x) \) represents the isotropic gravitational potential;
- \( A^{(1)}_{(ij)}(t, x) \) are the components of a d-tensor on the 1-jet space \( J^1(\mathbb{R}, M^n) \) representing the electromagnetic potential.

Note that the jet Lagrangian function (4.5) is a natural extension of the Lagrangian (defined on the tangent bundle) used in electrodynamics by Miron and Anastasiei [10]. In our jet Lagrangian formalism applied to (4.5), the electromagnetic-like components become classical ones (see [3]):

\[
F^{(1)}_{(ij)} = -\frac{e}{2m} \left( \frac{\partial A^{(1)}_{(ij)}}{\partial x^j} - \frac{\partial A^{(1)}_{(ij)}}{\partial x^i} \right).
\]

Moreover, the second set of geometrical Maxwell-like equations reduce to the classical ones too (for more details, see [10], [3]):

\[
\sum_{\{i,j,k\}} F^{(1)}_{(ij)k} = 0,
\]

where

\[
F^{(1)}_{(ij)k} = \frac{\partial F^{(1)}_{(ij)}}{\partial x^k} - F^{(1)}_{(mj)j} \gamma^{mk}_{ij} - F^{(1)}_{(im)m} \gamma^{mk}_{ij}.
\]
Also, the geometrical Einstein-like equations attached to the Lagrangian (4.5) (see [10], [4]) are the same with the famous classical ones (associated to the semi-Riemannian metric \( \varphi_{ij}(x) \)). In author’s opinion, these facts suggest some kind of naturalness for the present abstract Lagrangian non-isotropic electromagnetic and gravitational geometrical theory.

Via some direct calculations, we easily deduce that the \((t, x)\)-conformal deformed Berwald-Moór metric (1.1) produces null non-isotropic electromagnetic components:

\[
F_{(ij)}^{(1)} = 0.
\]

It follows that our \((t, x)\)-conformal deformed jet Berwald-Moór geometrical electromagnetic-like theory is trivial. This fact probably suggests that the \((t, x)\)-conformal deformed Berwald-Moór geometrical structure (1.1) has rather gravitational connotations than electromagnetic ones.

As a conclusion, it is possible for the recent Voicu-Siparov approach of the electromagnetism in spaces with anisotropic metrics (that electromagnetic approach is different from the electromagnetic theory exposed above, and it is developed in the paper [19]) to give other interesting electromagnetic-geometrical results for spaces endowed with the Berwald-Moór geometrical structure.

**Open problem.** The author of this paper believes that the finding of some possible real physical interpretations for the present non-isotropic Berwald-Moór geometrical approach of gravity and electromagnetism may be an open problem for physicists.

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Mircea Neagu
University Transilvania of Braşov,
Department of Mathematics - Informatics
Blvd. Iuliu Maniu, no. 50, Braşov 500091, Romania.
E-mail: mircea.neagu@unitbv.ro