Towards physically motivated proofs of the Poincaré and geometrization conjectures

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Although the Poincaré and the geometrization conjectures were recently proved by Perelman, the proof relies heavily on properties of the Ricci flow previously investigated in great detail by Hamilton. Physical realization of such a flow can be found, for instance, in the work by Friedan (Ann. Physics 163 (1985), 318-419). In his work the renormalization group flow for a nonlinear sigma model in $2 + \varepsilon$ dimensions was obtained and studied. For $\varepsilon = 0$, by approximating the $\beta-$ function for such a flow by the lowest order terms in the sigma model coupling constant, the equations for Ricci flow are obtained. In view of such an approximation, the existence of this type of flow in nature is questionable. In this work, we find totally independent justification for the existence of Ricci flows in nature. This is achieved by developing a new formalism extending the results of two dimensional conformal field theories (CFT’s) to three and higher dimensions. Equations describing critical dynamics of these CFT’s are examples of the Yamabe and Ricci flows realizable in nature. Although in the original works by Perelman some physically motivated arguments can be found, their role in his proof remain rather obscure. In this paper, steps are made toward making these arguments more explicit thus creating an opportunity for developing alternative, more physically motivated, proofs of the Poincaré and geometrization conjectures.

Keywords: Ginzburg-Landau functional, Hilbert-Einstein action, conformal field theories in 2 and 3 dimensions, critical dynamics of phase transitions, Ricci and Yamabe flows, dilaton gravity, Nash and Perelman’s entropy

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1 Introduction

The history of physics is full of examples of situations when experimental observations lead to very deep mathematical results. Beginning with Newton’s dynamics equations, which gave birth to analysis, through the heat and wave equations, which gave birth to mathematical physics, through Maxwell’s equations (experimentally discovered by Faraday), which ultimately had led to special relativity on one hand and the theory of homology and cohomology on another, etc. thus stimulating the development of topology. Methods of celestial mechanics, especially those developed by Poincaré, had led Bohr and his collaborators to quantum mechanics, on one hand, and to KAM and chaos theory, on another. A simple observation by Einstein that gravity can be locally eliminated in the freely falling elevator cabin had led to general relativity, on one hand, and to great advancements in differential geometry, on another. Based on these examples, it is only natural to expect that the recent complete proof of the Poincaré and geometrization conjectures by Perelman [1-3] should be connected with some processes taking place in the real world. The purpose of this work is to demonstrate that this is indeed the case. Since this paper is not a review, we made no efforts to list all the latest physics-related results inspired by Perelman’s works. Instead, we shall only quote those works which are immediately relevant to the content of our paper.

We begin with the observation that in the work of Friedan [4] on the nonlinear sigma model in $2 + \varepsilon$ dimensions, published in 1985, the reference was made to a talk by Bourguignon (delivered in 1979) "On Ricci curvature and Einstein metric" in which for the first time the idea of usefulness of Ricci flows to differential geometry and related disciplines was introduced, e.g. see Ref.10 in Friedan’s paper. This idea was picked up by Hamilton in 1982 [5] who began from that time on a systematic study of Ricci flows. Although written with exceptional clarity, his results summarized in Ref.[6] apparently escaped physicists attention for many years. This is caused by several factors, which are worth mentioning. The Ricci-type flow equations appeared in the physics literature in the 80’s in connection with the renormalization group analysis of nonlinear sigma models and also of strings propagating on curved backgrounds [4,7] without any reference to the parallel developments in mathematics. Because of this, the point of view was developed in the physics literature, especially that related to string theory [8], that all major results of general relativity along with quantum corrections can be deduced from the action functional whose minimization produces the Euler-Lagrange equations for the renormalization group (RG) flow for strings in curved backgrounds, e.g. see [8], Ch-r 3, page 180. In the simplest case such a functional (e.g. see Eq.(3.4.58) on page 180) is a modification of the Hilbert-Einstein functional describing gravity, leading to the action functional

\[ \text{Hilbert-Einstein functional} \]

\[ \text{modified by Perelman} \]

\[ \text{to account for quantum corrections} \]

\[ \text{leading to action functional} \]

\[ \text{discussed in this paper} \]

\[ \text{It is based on his PhD thesis completed in August of 1980.} \]
for the dilaton gravity. The only problem with such a functional lies in the fact that it is "living" in 26 dimensional space-time for a bosonic string and in 10 dimensional space-time for a fermionic (heterotic) string [9]. These facts, apparently were known to Grisha Perelman who used them in his calculations, many of which (but not those involving such functionals) are specific only for 3 dimensional Riemannian manifolds and, hence, are not readily extendable to higher dimensions and a pseudo-Riemannian-type of metric.

At this point it is essential to notice that both in the work by Friedan [4] and in the string-theoretic literature just cited, the dilaton gravity functional emerges only as a result of the lowest order approximation in RG-type calculations. This fact can be seen very clearly on page 322 of Friedan’s paper, where in his Eqs (1.5) and (1.6) one has to put $\varepsilon = 0$ and retain only terms of order $O(T^{-1})$ in the sigma model coupling constant $T$. Such a retention then straightforwardly leads to the equation for the Ricci flow. When terms of order $T$ and higher are included into these RG flow equations, such an RG flow is no longer of Ricci type. The procedure of an ad hoc truncation of an infinite asymptotic RG series expansion for the RG $\beta -$ function (e.g. see Eq.(1.5) in Friedan’s paper) in order to get the needed equation for the Ricci flow makes the physical existence of such a flow doubtful. Since both the Poincare’ and the geometrization conjectures were proved with help of the unperturbed (nonmodified by terms other than those induced by diffeomorphisms) Ricci flows [1-3], it is essential to find physical processes that employ such unperturbed Ricci-type flows. The present work provides examples of such physical processes. It enables us to reconnect recent significant mathematical advances with physical processes realizable in nature in accord with the logic of development of mathematical physics described at the beginning of this section.

In view of space limitation, before discussing the content of this paper in some detail, we would like to mention the following. The existing proofs of the Poincare’ and the geometrization conjectures are either too compressed [1] or too bulky [2,3,10]. In both cases, it may take a considerable amount of time for the nonexpert reader to learn and to understand these papers. Fortunately, there are succinct publications the present the key ideas of the proof in a language that is considerably more familiar to a much wider group of people with mathematical background. Among these reviews, we would especially recommend those written by Bessieres [11], on the proof of the Poincare’ conjecture, and by Kapovich [12] on the proof of the geometrization conjecture. This paper is written under the assumption that our readers have previously read these reviews.

The organization of the rest of this paper is the following. In Section 2 the scaling analysis of the Ginzburg-Landau (G-L) functional, widely used in the theory of critical phenomena, is discussed using both arguments known in
physics literature and those known in mathematics. In particular, we argue that in the physics literature there is a marked difference in treatments of two and the higher dimensional models exhibiting critical behavior. While in two dimensions the full conformal invariance is used (at least at criticality), in dimensions higher than two only scale invariance is taken into account at criticality. Using scaling arguments from both physics and mathematics, we demonstrate how the existing scaling treatments in higher dimensions can be improved in order to take into account the effects of full conformal invariance. This improvement is brought to completion in Section 3 where we connect the G-L functional used in the physics literature with the Yamabe functional known in mathematics. We demonstrate that, if properly interpreted, both functionals upon minimization produce the same G-L-type equations used in literature on critical phenomena. The advantage of working with Yamabe functionals lies in our observations that a) such a functional can be rewritten in terms of the Hilbert-Einstein action functional for Euclidean gravity and, b) that such a functional is manifestly conformally invariant. Such a connection between the H-E action and its pseudo-Riemannian extension is needed in applications in relativity which are studied in Section 4. Connections between the G-L and Yamabe functionals exist only for dimensionality higher than two. In two dimensions one has to find an analog of the G-L-Yamabe functional. This task is accomplished in Section 5 in which not only such an analog is found but, also, its connection with known results in string and CFT. Specifically, we were able to connect such an analog with the much earlier results obtained by Distler, Kawai and David for 2d quantum gravity in noncritical dimensions pedagogically summarized in the book by Hatfield [13]. Applications of their results to CFT are summarized in the lecture notes by Abdalla et al [14] and, more recently, in the review by Nakayama [15]. Because of such connections, the extension of these 2 dimensional CFT results to higher dimensions can be done quite systematically. This is discussed in Section 6. In Section 7 we develop an alternative method allowing us to reobtain results known already for 2 and higher dimensions. This formalism is based on the observation that results obtained in previous sections are just static solutions of more general equations describing critical dynamics for systems of the G-L-type. By noticing that mathematically such dynamics coincides with that known for the Yamabe-type flows discovered by Hamilton [16], we follow the logic of his works in order to relate the Yamabe flow to the more general Ricci flow of which the Yamabe flow is a special case. To do so we use some results by Perelman [1], which he was using in his proofs of the Poincare' and the geometrization conjectures. Our use of Perelman’s results is not mechanical however. By exposing physical arguments hidden in his work, we were able to reobtain many of his results much simpler as compared to treatments which can be found in current works by mathematicians [2,3,10] aimed at explaining and elaborating on many hidden details of Perelman’s original proof. Finally, in Section 8 we discuss other possible physical systems whose dynamics is expected to be described in terms of Ricci flows. We also discuss the conditions under which experiments on critical dynamics should be conducted in order to detect the effects of Ricci and Yamabe flows.
2 Scaling analysis of the G-L functional

Conventional scaling analysis of the G-L functional used routinely in physics literature can be found, for example, in Ref.[17]. This analysis differs somewhat from that for the \( \phi^4 \) model as described in the monograph by Itzykson and Zuber [18]. For the sake of the discussion that will follow, we would like to provide a sketch of the arguments for both cases now.

We begin with the \( \phi^4 \) model following Ref.[18]. Let \( L(x) \) be the Lagrangian of this scalar field model whose action functional in \( d \) dimensions \( S[\phi] \) is given by \( S[\phi] = \int d^d x L(x) \). Let furthermore \( \lambda \) be some nonnegative parameter. Then the requirement that \( S[\phi] \) be independent of \( \lambda \), i.e. \( \int d^d x L(x) = \int d^d x \lambda^d L(\lambda x) \), leads to the constraint

\[
\int d^d x (x \cdot \partial_x + d) L(x) = 0 \quad (2.1)
\]

obtained by differentiation of \( S[\phi] \) with respect to \( \lambda \) with \( \lambda \) being equalled to one at the end of calculation. For \( L(x) \) given by

\[
L(x) = \frac{1}{2} (\nabla \phi)^2 + m^2 \phi^2 + \frac{\hat{G}}{4!} \phi^4 \quad (2.2)
\]

the change of \( L(x) \) under the infinitesimal scale transformation is given by\[3\]

\[
\frac{\delta L}{\delta \varepsilon} = (x \cdot \partial_x + d) L(x) + (d - 4) \frac{\hat{G}}{4!} \phi^4 - m^2 \phi^2. \quad (2.3)
\]

Comparison between Eq.s(2.1) and (2.3) implies that the action \( S[\phi] \) is scale invariant if \( d = 4 \) and \( m^2 = 0 \). The result just obtained raises an immediate question: given that the massless G-L action is scale invariant for \( d = 4 \), will it also be conformally invariant under the same conditions? We provide the answer to this question in several steps.

First, let \( M \) be some Riemannian manifold whose metric is \( g \). Then, any metric \( \tilde{g} \) conformal to \( g \) can be written as \( \tilde{g} = e^f g \) with \( f \) being a smooth real valued function on \( M \) [19]. Let \( \Delta_g \) be the Laplacian associated with metric \( \tilde{g} \) and, accordingly, let \( \Delta_{\tilde{g}} \) be the Laplacian associated with metric \( \tilde{g} \). Richardson [20] demonstrated that

\[
\Delta_{\tilde{g}} = e^{-f} \Delta_g - \frac{1}{2} (d - 1) f e^{-f} \Delta_g - \frac{1}{2} \left( \frac{d}{2} - 1 \right) e^{-f} (\Delta_g f) + \frac{1}{2} \left( \frac{d}{2} - 1 \right) e^{-f} \Delta_g \circ f \quad (2.4)
\]

(here \( \Delta_g \circ f \) \( \Psi \) should be understood as \( \Delta_g (f(x) \Psi(x)) \)). Eq.s(2.3) and (2.4) imply that the conformally invariant (string-type) functional \( S[X] = \int_M d^d x \sqrt{g(\nabla g X)} \).

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\[3\]In arriving at this result we took into account Eq.(13-40) of Ref.[18] along with condition \( D = \frac{d}{2} - 1 \), with \( D \) defined in Eq.(13-38) of the same reference. Also, we changed signs (as compared to the original source) in \( L(x) \) to be in accord with the accepted conventions for the G-L functional.

\[4\]That is \( \Delta_g \Psi = -(\det g) \frac{1}{2} \partial_i (g^{ij} (\det g) \partial_j \Psi) \) for some scalar function \( \Psi(x) \).
(\nabla_g X) exists only for \( d = 2 \). For \( d > 2 \), in view of Eq.(2.4), the conformal invariance of the \( \phi^4 \) model is destroyed. Thus, using conventional field-theoretic perturbational methods one encounters a problem with conformal invariance at the zeroth order level (in the coupling constant \( \hat{G} \)), unless \( d = 2 \). This problem formally does not occur if one requires our physical model to be only scale invariant. This is undesirable however in view of the fact that in 2 dimensions an arbitrary conformal transformation of the metric tensor \( g \) of the underlying two dimensional manifold \( M \) is permissible at criticality [21]. Abandoning the requirement of general conformal invariance in 2 dimensions in favor of scale invariance in higher dimensions would destroy all known string-theoretic methods of obtaining exact results in two dimensions. Our general understanding of critical phenomena depends crucially on our ability to solve two dimensional models exactly. All critical properties for the same type of models in higher dimensions are expected to hold even in the absence of exact solvability. Fortunately, the situation can be considerably improved by reanalyzing and properly reinterpreting the scaling results for dimensions higher than 2.

This observation leads to the next step in our arguments. Using the book by Amit [17], we consider first the scaling of the non interacting ( free ) G-L theory whose action functional is given by

\[
S[\phi] = \int d^d x \{ (\nabla \phi)^2 + m^2 \phi^2 \}.
\]

Suppose now that, upon rescaling, the field \( \phi \) transforms according to the rule:\(^5\)

\[
\tilde{\phi}(Lx) = L^\omega \phi(x).
\]

If we require

\[
\int d^d x \{ (\nabla \phi)^2 + m^2 \phi^2 \} = \int d^d x L^d \{ (\nabla \tilde{\phi})^2 + \tilde{m}^2 \tilde{\phi}^2 \}
\]

and use Eq.(2.6) we obtain,

\[
S[\phi] = \int d^d x L^d \{ (\nabla \phi)^2 L^{2\omega - 2} + \tilde{m}^2 \phi^2 L^{2\omega} \}.
\]

In order for the functional \( S[\phi] \) to be scale invariant the mass \( m^2 \) should scale as follows: \( \tilde{m}^2 = m^2 L^{-2} \). With this requirement the exponent \( \omega \) is found to be:\(^6\)

\[
\omega = 1 - \frac{d}{2}.
\]

The above scaling can be done a bit differently following the same Ref.[17]. For this purpose we notice that although the action \( S[\phi] \) is scale invariant, there is some freedom of choice for the dimensionality of the field \( \phi \). For instance, instead of \( S[\phi] \) we can consider

\[
S[\phi] = \frac{1}{a^d} \int d^d x \{ (\nabla \phi)^2 + m^2 \phi^2 \}
\]

\(^5\)Notice that this is already a special kind of conformal transformation.
with $a^d$ being some volume, say, $a^d = \int d^d x$. Then, by repeating arguments related to Eq.(2.8), we obtain: $\omega = 1$ (instead of $\omega = 1 - \frac{d}{2}$ previously obtained) in accord with Eq.(2-66) of the book by Amit [17]. Although, from the point of view of scaling analysis both results are actually equivalent, they become quite different if we want to extend such scaling analysis by considering general conformal transformations. Even though, in view of Eq.(2.4), such a task seems impossible to accomplish, fortunately, this is not true as we would like to demonstrate. This leads us to the next step.

Noticing that the mass term scales as the scalar curvature $R$ for some Riemannian manifold $M$, i.e. the scaling $\tilde{m}^2 = m^2 L^{-2}$ is exactly the same as the scaling of $R$ given by

$$\tilde{R} = L^{-2} R.$$  \hfill (2.10)

This result can be found, for example, in the book by Wald, Ref. [22], Eq.(D.9), page 446. In general, the scalar curvature $R(g)$ changes under the conformal transformation $\hat{g} = e^{2f} g$ according to the rule [19]

$$\hat{R} = \frac{1}{\phi^{p-1}} \{ R(g) - 2(d-1) \Delta_g f - (d-1)(d-2) |\nabla_g f|^2 \}, \hfill (2.11)$$

where $\Delta_g f$ is the Laplacian of $f$ and $\nabla_g f$ is the covariant derivative defined with respect to the metric $g$. From here we see that, indeed, for constant $f$’s the scaling takes place in accord with Eq.(2.10). Now, however, we can do more. Following Lee and Parker [19], we simplify the above expression for $R$. For this purpose we introduce a substitution: $e^{2f} = \phi^{p-2}$, where $p = \frac{2d}{d-2}$, so that $\hat{g} = \phi^{p-2} g$. With such a substitution, Eq.(2.11) acquires the following form:

$$\hat{R} = \phi^{1-p} (\alpha \Delta_g \varphi + R(g) \varphi), \hfill (2.12)$$

with $\alpha = 4 \frac{d-1}{d-2}$. Clearly, such an expression makes sense only for $d \geq 3$ and breaks down for $d = 2$. But we know already the action $S[X]$ which is both scale and conformally invariant in $d = 2$. It is given after Eq.(2.4). Fortunately, the results obtained in this section can be systematically extended in order to obtain actions, which are both scale and conformally invariant in 3 dimensions and higher. This is described in the next section.

3 G-L functional and the Yamabe problem

We begin with the following observation. Let $\hat{R}(\hat{g})$ in Eq.(2.11) be some constant (that this is indeed the case, we shall demonstrate shortly). Then Eq.(2.12) acquires the following form

$$\alpha \Delta_g \varphi + R(g) \varphi = \hat{R}(\hat{g}) \varphi^{p-1}. \hfill (3.1)$$

By noticing that $p - 1 = \frac{d+2}{d-2}$ we obtain at once: $p - 1 = 3$ (for $d = 4$) and $p - 1 = 5$ (for $d = 3$). These are familiar Ginzburg-Landau values for
the exponents of interaction terms for critical and tricritical G-L theories [23].

Once we recognize these facts, the action functional producing the G-L-type Eq.(3.1) can be readily constructed. For this purpose it is sufficient to rewrite Eq.(2.9) in a manifestly covariant form. We obtain,

$$S[\varphi] = \frac{1}{(\int_M d^d x \sqrt{g} \varphi^p)^{\frac{2}{p}}} \int_M d^d x \sqrt{g} \{\alpha (\nabla g(\varphi))^2 + R(g) \varphi^2\} \equiv \frac{E[\varphi]}{||\varphi||^2_p}. \quad (3.2)$$

Minimization of this functional produces the following Euler-Lagrangian equation

$$\alpha \Delta_g \varphi + R(g) \varphi - \lambda \varphi^{p-1} = 0 \quad (3.3)$$

with constant $\lambda$ denoting the extremum value for the ratio:

$$\lambda = \frac{E[\varphi]}{||\varphi||^2_p} = \inf \{S[\varphi]: \hat{g} \text{ conformal to } g\}. \quad (3.4)$$

In accord with the Landau theory of phase transitions [24] it is expected that the conformal factor $\varphi$ is a smooth nonnegative function on $M$ achieving its extremum value $\bar{\varphi}$. Comparison between Eqs.(3.1) and (3.3) implies that $\lambda = \bar{R}(\hat{g})$ as required. These results belong to Yamabe, who obtained the Euler-Lagrange G-L-type Eq.(3.3) upon minimization of the functional $S[\varphi]$ without any knowledge of the Landau theory. The constant $\lambda$ is known in literature as the Yamabe invariant [19,25]. Its value is an invariant of the conformal class $(M, g)$. Given these facts, we obtain the following:

**Definition 3.1.** The **Yamabe problem** lies in finding a compact Riemannian manifold $(M, g)$ of dimension $n \geq 3$ whose metric is conformal to the metric $\hat{g}$ of a constant scalar curvature.

Subsequent developments, e.g. that given in Ref.[26,27], extended this problem to manifolds with boundaries and to non-compact manifolds. It is not too difficult to prove that the (Yamabe-Ginzburg-Landau-like) functional is manifestly conformally invariant. For this purpose, we need to rewrite Eq.(2.12) in the following equivalent form

$$\varphi^p \hat{R}(\hat{g}) = (\alpha \Delta_g \varphi + R(g) \varphi^2). \quad (3.5)$$

This can be used in order to rewrite $E[\varphi]$ as follows: $E[\varphi] = \int d^d x \sqrt{g} \varphi^p \hat{R}(\hat{g})$. Next, by noting that $\int d^d x \sqrt{\hat{g}} = \int d^d x \sqrt{g} \varphi^p$, we can rewrite the Yamabe functional in the Hilbert-Einstein form

$$S[\varphi] = \frac{\int d^d x \sqrt{g} \hat{R}(\hat{g})}{(\int d^d x \sqrt{g})^{\frac{2}{p}}} \quad (3.6)$$

where both the numerator and the denominator are invariant with respect to conformal changes in the metric, e.g. scale change.
Remark 3.2. In order to use these results in statistical mechanics, we have to demonstrate that the extremum of the Yamabe functional $S[\varphi]$ is realized for manifolds $M$ whose scalar curvature $R(g)$ in Eq.(3.3) is also constant. This is the essence of the Yamabe problem (as defined above) for physical applications. Fortunately, the Yamabe problem has been solved positively by several authors starting with Yamabe himself (whose proof contained some mistakes however) and culminated in the work by Schoen [28]. Details can be found in the review paper by Lee and Parker [19] and in the monograph by Aubin [29].

Remark 3.3. In view of the relation $\int d^d x \sqrt{g} = \int d^d x \sqrt{g\varphi^p}$, it is clear that for the fixed background metric $g$ Eq.(3.3) can be obtained alternatively using the following variational functional

$$\tilde{S}[\varphi] = \int d^d x \sqrt{g} \{ \alpha (\nabla g \varphi)^2 + R(g) \varphi^2 \} - \tilde{\lambda} \int d^d x \sqrt{g} \varphi^p$$

(3.7)

where the Lagrange multiplier $\tilde{\lambda}$ is responsible for the volume constraint. Such a form of the functional $\tilde{S}[\varphi]$ brings this higher dimensional result in accord with that to be developed below for two dimensions (e.g. see Section 5, Eq.(5.24), and Section 7).

Apart from the normalizing denominator, Eq.(3.6) represents the Hilbert-Einstein action for pure gravity defined for Riemannian $d$ dimensional space. The denominator, the volume $V$ taken to power $\frac{p}{2}$, serves the purpose of making $S[\varphi]$ manifestly conformally invariant, Ref.[28], page 150.

4 From G-L to Hilbert-Einstein functional

In this section we would like to analyze the significance of the cosmological constant term in the Hilbert-Einstein action for gravity from the point of view of the G-L model. For this purpose, following Dirac [30] let us consider the extended Hilbert-Einstein (H-E) action functional $S^c(g)$ for pure gravity with an extra (cosmological constant) term defined for some (pseudo) Riemannian manifold $M$ (of total space-time dimension $d$) without a boundary:

$$S^c(g) = \int_M R\sqrt{g}d^dx + C \int_M d^dx \sqrt{g}.$$  

(4.1.)

The (cosmological) constant $C$ is determined based on the following chain of arguments. First, we introduce the following.

$$6\tilde{\lambda} = \lambda \frac{2}{p} \equiv \lambda \frac{d-2}{d} \quad {\text{6}}$$
**Definition 4.1.** Let $R_{ij}$ be the Ricci curvature tensor. Then the *Einstein space* is defined as a solution of the following vacuum Einstein equation

$$ R_{ij} = \lambda g_{ij} \tag{4.2} $$

with $\lambda$ being constant.

From this definition it follows that in both the Riemannian and pseudo-Riemannian cases

$$ R = d\lambda \tag{4.3} $$

Following Dirac [30], variation of the action $S^c(g)$ produces

$$ R_{ij} - \frac{1}{2} g_{ij} R + \frac{1}{2} C g_{ij} = 0. \tag{4.4} $$

The combined use of Eq.s(4.3) and (4.4) produces:

$$ C = \lambda \left( d - \frac{2}{2} \right) \tag{4.5} $$

With these results, using Eq.s(4.3) and (4.5) we can rewrite Eq.(4.4) as follows:

$$ R_{ij} - \frac{1}{2} g_{ij} R + \frac{1}{2d} (d - 2) R g_{ij} = 0. \tag{4.6} $$

**Remark 4.2.** These observations allow us to look at possible pseudo-Riemannian extension of the results originally obtained by Yamabe, Refs.[19, 29, 31], for Riemannian manifolds. Although at first sight this might be of significance importance for high energy physics applications, in a companion publication [32] we shall argue that dealing with Riemannian manifolds is already quite sufficient so that the Yamabe results can be used in high energy physics without changes. It should be clear as well that the earlier introduced notion of the Einstein space is applicable to both Riemannian and pseudo-Riemannian manifolds.

In view of this remark, we would like to argue that Eq.(4.6) can also be obtained by varying the Yamabe functional, Eq.(3.6). Indeed, following Aubin [29] and Schoen [28], let $t$ be some small parameter labeling the family of metrics: $g_{ij}(t) = g_{ij} + th_{ij}$. Then, these authors demonstrate that

$$ \left( \frac{d R_i}{d t} \right)_{t=0} = \nabla^j h_{ij} - \nabla^j h^j_i - R^{ij} h_{ij} \tag{4.7} $$

and

$$ \left( \frac{d}{d t} \sqrt{|g_t|} \right)_{t=0} = \frac{1}{2} \sqrt{|g_t|} g^{ij} h_{ij}, \tag{4.8} $$

where, as usual, $|g| = |\det g_{ij}|$. Consider now the Yamabe functional, Eq.(3.6), but, this time, written for the family of metrics which belong to the same conformal class. We have,

$$ \mathcal{R}(g(t)) = (V(t))^{\frac{1}{d - 2}} \int_M R(g(t))DV(t), \tag{4.9} $$
where the volume is given by \( V(t) = \int_M d^d x \sqrt{g(t)} \) and, accordingly, \( DV(t) = d^d x \sqrt{g(t)} \).

Using Eqs.(4.7) and (4.8) in Eq.(4.9) and taking into account that the combination \( \nabla^i \nabla^j h_{ij} - \nabla^i \nabla_j h^i \) is the total divergence produces the following result:

\[
\left( \frac{d}{dt} R(g(t)) \right)_{t=0} = V(0) - \frac{d}{dt} \left[ \int_M (Rg^{ij}/2 - R^{(i)}h_{ij})DV(0) \right]_M + \int_M DV(0) \int_M h_{ij} g^{ij} DV(0).
\]

If the metric \( g \) is the critical point of \( R(g(t)) \), then

\[
\left[ R_{ij} - \frac{R}{2} g_{ij} \right] \int_M DV(0) + \left( \frac{1}{2} - \frac{1}{d} \right) \int_M RDV(0) g_{ij} = 0.
\]

From here, multiplication of both sides by \( g^{ij} \) and subsequent summation produces at once

\[
R - \frac{R}{2} d + \left( \frac{1}{2} - \frac{1}{d} \right) < R > = 0,
\]

where \( < R > = \frac{1}{V(0)} \int RDV(0) \) is the average scalar curvature. Eq.(4.12) can be rewritten as \( R = < R > \). But this condition is formally equivalent to the Einstein condition, Eq.(4.2), in view of Eqs.(4.3), (4.6)! Hence, under such circumstances, Eqs.(4.11) and (4.12) are equivalent.

**Remark 4.3.** It is important to mention that in general the condition \( R = < R > \) is more restrictive than the condition, Eq.(4.2), defining Einstein spaces. That is to say, the constant Ricci tensor \( R_{ij} \) always leads to the constant scalar curvature \( R \) while the Riemannian spaces of constant scalar curvature \( R \) are not necessarily spaces for which the Einstein condition, Eq.(4.2), holds. Hence, they may or may not be of Einstein type. This was demonstrated by Herglotz in 1916 (immediately after Einstein’s general relativity was completed) [33, page 148].

His results were subsequently generalized in Refs.[34,35]. We shall return to this topic in Section 7 where we discuss this problem dynamically following the works by Hamilton and Perelman. Using dynamical arguments, we demonstrate that the equation \( R = < R > \) is compatible with the Einstein condition and, hence describes the Einstein spaces.

In view of this remark, in the rest of this work we shall develop our formalism under the assumption that for the set of metrics of fixed conformal class, the variational problem for the G-L functional, Eq.(3.7), is equivalent to the variational problem for the H-E functional, Eq.(4.1), for pure gravity in the presence of the cosmological constant. Because of this noted formal equivalence, presence of the cosmological term in the H-E action amounts to volume conservation for the G-L variational problem. From here, it should be clear that Eq.(2.12) is equivalent to the condition: \( R = < R > \) in view of Eq.(3.4).

The result shown in Eq.(4.12) becomes trivial for \( d = 2 \). Physically, however, the case \( d = 2 \) is important since it is relevant to all known exactly solvable models of statistical mechanics treatable by methods of conformal field theories. Hence, now we would like to discuss needed modifications of the obtained results in order to obtain a two dimensional analog of the G-L theory.
5 Ginzburg-Landau-like theory in two dimensions

5.1 Designing the two dimensional G-L-Yamabe functional

From the field-theoretic treatments of the G-L model \cite{36} we know that a straightforward analysis based on asymptotic $\varepsilon$--expansions from the critical dimension (4) to the target dimension (2) is impractical. At the same time, the results of CFT and exactly solvable models make sense thus far only in $d = 2$. The question arises: is there an analog of G-L Eq.(3.1) in two dimensions? And, if such an analog does exist, what use can be made of such a result? In this section we provide affirmative answers to these questions. We demonstrate that: a) indeed, such a two dimensional analog of the G-L equation does exist and is given by the Liouville Eq.(5.19), b) the functional, Eq.(5.14), whose minimization produces such an equation is the exact two dimensional analog of the G-L-Yamabe functional, Eq.(3.2), c) these results can be (re)obtained from the existing string-theoretic formulations of the CFT which were developed entirely independently.

To discuss topics related to items a) and b) just mentioned, we begin with the observation that in two dimensions, Eq.(2.4) acquires a very simple form

$$\Delta \hat{g} = e^{-2f} \Delta g,$$  \hspace{1cm} (5.1)

where we use a factor of 2 to be in accord with Eq.(2.11) for the scalar curvature. According to Eq.(2.11), the scalar curvature in two dimensions transforms like

$$\hat{R}(\hat{g}) = e^{-2f} \left\{ R(g) - \Delta g 2f \right\}$$  \hspace{1cm} (5.2)

while the area $dA = d^2x \sqrt{g}$ transforms like

$$d\hat{A} = e^{-2f} dA.$$  \hspace{1cm} (5.3)

These facts immediately suggest that the previously introduced action functional

$$S[X] = \int_M d^2x \sqrt{g} (\nabla g X) \cdot (\nabla g X)$$  \hspace{1cm} (5.4)

is conformally invariant. Using results by Polyakov \cite{37} and noting that $(\nabla g X) \cdot (\nabla g X) = g^{\alpha\beta} \partial_{\alpha} X^\mu \partial_{\beta} X_\mu$, we need to consider the following path integral

$$\exp(-F(g)) \equiv \int D[\phi] \exp(-\frac{1}{2} \int_M d^2x \sqrt{g} g^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi).$$  \hspace{1cm} (5.5)

Here the symbol $F(g)$ stands for the "free energy" usually defined this way in statistical mechanics.\footnote{Without loss of generality, we would like to consider the case of one component field $\phi$ only.} Fortunately, this integral was calculated by Polyakov \cite{37}.

\footnote{Usually, instead of $F(g)$ one writes $F(g)/k_B T$, where $T$ is the temperature and $k_B$ is the Boltzmann’s constant. In the present case the problems we are studying do not require specific values for these constants. For this reason they will be suppressed.}
for two dimensional manifolds $M$ without boundaries and by Alvarez [38] for manifolds with boundaries. In this work, to avoid unnecessary complications, we shall be concerned with manifolds without boundaries. Although in the original work by Polyakov, one can find the final result of calculation of the above path integral, the details of this calculation can only be found elsewhere. In particular, we shall follow pedagogically written papers by Weisberger [39,40] and Osgood, Phillips and Sarnak [41-43] (OPS).

To begin, let $\tilde{g}_{\alpha\beta}$ be some reference metric, while $g_{\alpha\beta}$ is a metric conformally related to it, i.e. $g_{\alpha\beta} = \exp(-2\varphi)\tilde{g}_{\alpha\beta}$. Should the above path integral be for the flat (i.e. $g_{\alpha\beta} = \delta_{\alpha\beta}$) two dimensional manifold, one would have at once the result:

$$F = \frac{1}{2} \ln \det \Delta_{0},$$

where the prime indicates that the zero mode is omitted. Because the flatness assumption in general is physically incorrect, it is appropriate to pose a problem: how is the path integral for the metric $g$ related to the path integral for the metric $\tilde{g}$? The paper [41] by OPS provides an answer, e.g. see Eq.(1.13) of this reference. To connect this equation with that known in the physics literature we replace it by the equivalent expression

$$\ln \left( \frac{\det' \Delta_{g}}{A_{\tilde{g}}} \right) - \ln \left( \frac{\det' \Delta_{\tilde{g}}}{A_{g}} \right) = \frac{1}{6\pi} \frac{1}{2} \int_{M} dA_{g} \left\{ |\nabla_{g} \varphi|^{2} + R(g)\varphi \right\}$$

useful in applications to strings and CFT, e.g see Ref.[13], page 637.

It is worthwhile to provide a few computational details leading to Eq.(5.6). Let, therefore, $\tilde{\psi}_{i}$ be eigenfunctions of the Laplacian $\Delta_{\tilde{g}}$ with eigenvalues $\lambda_{i}$ arranged in such a way that $0 = \lambda_{0} < \lambda_{1} \leq \lambda_{2} \leq \cdots$, i.e.

$$- \Delta_{\tilde{g}} \tilde{\psi}_{i} + \lambda_{i} \tilde{\psi}_{i} = 0.$$  \hspace{1cm} (5.7)

Then, we construct the zeta function

$$\zeta_{g}(s) = \sum_{i=1}^{\infty} \tilde{\lambda}_{i}^{-s}$$

in such a way that

$$\det' \Delta_{g} = \exp(-\zeta'_{g}(0))$$

with $\zeta'_{g}(0) = \left( \frac{d}{ds} \zeta_{g}(s) \right)_{s=0}$. Using Eq.(5.1), we obtain as well

$$- e^{-2\varphi} \Delta_{g} \psi_{i} + \lambda \psi_{i} = 0.$$  \hspace{1cm} (5.10)

In particular, for a constant $\varphi = \tilde{\varphi}$ we obtain

$$\zeta_{\tilde{g}}(s) = \sum_{i=1}^{\infty} \left( e^{-2\tilde{\varphi} \lambda_{i}} \right)^{-s} = e^{2\tilde{\varphi}} \zeta_{g}(s).$$

\footnote{In arriving at this result we took into account that the Gaussian curvature $K$ in Ref.[41] is related to Riemannian curvature $R$ as $R = 2K$.}
Use of Eq.(5.9) in Eq.(5.11) produces:

\[
\zeta'_g(0) = \zeta'_g(0) + 2\bar{\varphi}(\chi(M) - 1).
\] (5.12)

This result was obtained with help of the known fact, Ref.[41], Eq.(1.9), that

\[
\zeta_g(0) = \chi(M) - 1
\] (5.13)

with \(\chi(M)\) being Euler's characteristic of a two dimensional manifold \(M\) without boundaries. In view of the definition, Eq.(5.9), and the Gauus-Bonn et theorem, we observe that Eq.(5.6) is reduced to Eq.(5.12) for the case of constant conformal factor \(\varphi = \bar{\varphi}\), as required.

Using Eq.(5.6) and, following OPS [41], we would like to consider the related functional \(F(\varphi)\) defined by

\[
F(\varphi) = \frac{1}{2} \int_M dA_g \{ |\nabla_g \varphi|^2 + R(g) \varphi \} - \pi \chi(M) \ln \int_M dA_g e^{2\varphi}.
\] (5.14)

This functional is the exact analog of the Yamabe functional, Eq.(3.2), in dimensions 3 and higher as we shall demonstrate below and in Section 7. In the meantime, in view of Eq.(5.6), it can be rewritten in the following equivalent form\(^{10}\)

\[
F(\varphi) = -6\pi \ln \det \Delta_g + \pi (6 - \chi(M)) \ln A.
\] (5.15)

Let \(a\) be some constant, then

\[
F(\varphi + a) = F(\varphi).
\] (5.16)

Such invariance signifies the fact that the above action is scale invariant. This property is in complete accord with Eqs.(2.7) and (2.8). If we impose a constraint (fix the gauge) : \(A = 1\), then we would end up with the Liouville-like action used in CFT. We would like to explain all this in some detail now.

Following OPS, it is convenient to replace the constraint \(A = 1\) by the alternative constraint on the field \(\varphi\):

\[
\int_M \varphi dA_g = 0.
\] (5.17)

To demonstrate that such a constraint is equivalent to the requirement \(A = 1\), we note that, provided that the field \(\psi\) minimizes \(F(\psi)\) subject to the constraint Eq.(5.17), the field

\[
\varphi = \psi - \frac{1}{2} \ln \int_M \exp(2\psi) dA_g
\] (5.18)

\(^{10}\)Here one should understand the word "equivalence" in the sense that both functionals produce the same critical metrics upon minimization.
minimizes $\mathcal{F}(\varphi)$ subject to the constraint $A = 1$. Using these facts, minimization of $\mathcal{F}(\psi)$ produces the following Liouville equation

$$-\Delta_g \psi + \frac{1}{2} R(g) - \frac{2\pi \chi(M) \exp(2\psi)}{\int_M \exp(2\psi) dA_g} = 0. \quad (5.19)$$

Comparing this result with Eq.(5.2) and taking into account that

$$\int_M \exp(2\psi) dA_g = \hat{R}(\hat{g}) = \text{const} \quad (5.20)$$

we conclude that, provided that the background metric $g$ is given so that the scalar curvature $R(g)$ (not necessarily constant) can be calculated, the Liouville Eq.(5.19) is exactly analogous to that previously obtained in Eq.(3.3) (which is the same as Eq.(2.12)). In view of this analogy, use of Eq.s(3.4) and (3.6) as well as Eq.s(5.14),(5.15) causes the functional $\mathcal{F}(\varphi)$ to attain its extremum for metric $\hat{g}$ of constant scalar curvature $\hat{R}(\hat{g})$. To decide if the extremum is minimum or maximum we have to consider separately the cases $\chi(M) > 0$ and $\chi(M) \leq 0$. In the case of $\chi(M) > 0$ we have only to consider manifolds homeomorphic to $S^2$ so that $\chi(M) = 2$. Fortunately, this case was considered in detail by Onofri [44]. Using his work, the following inequality

$$\ln \int_{S^2} dA_{\hat{g}} \exp(\psi) \leq \int_{S^2} dA_{\hat{g}} \psi + \frac{1}{4} \int_{S^2} dA_{\hat{g}} |\nabla \psi|^2 \quad (5.21)$$

attributed to Aubin [29] and inspired by earlier results by Trudinger and Moser [45], is helpful for deciding whether the obtained extremum is a minimum or maximum. Here $\hat{g}$ is the metric of the unit sphere $S^2$ with a constant Gaussian curvature of 1. The metric $g$ conformal to $\hat{g}$ is given by $g = \exp(2\psi) \hat{g}$, with $\psi$ obeying the Liouville equation (just like Eq.(5.19)), where both $R(g)$ and $\hat{R}(\hat{g})$ are constant by virtue of the initial choice of $\hat{g}$. By combining Eq.s(5.6),(5.14) and (5.15) with the inequality (5.21) and taking into account that by design $\ln \det' \Delta_g = 0$ we obtain,

$$-3\pi \ln \frac{\det' \Delta_g}{\det' \Delta_{\hat{g}}} = \frac{1}{4} \int_{S^2} dA_{\hat{g}} \{|\nabla \hat{g} \psi|^2 + 2\psi\} - \ln \int_{S^2} dA_{\hat{g}} \exp(2\psi) \geq 0, \quad (5.22)$$

with equality occurring only at the extremum $\psi = \psi^*$, with function $\psi^*$ being a solution of the Liouville Eq.(5.19). It can be shown [44] that: a) such a solution involves only Möbius transformations of the sphere $S^2$ and that, b) the functional $\mathcal{F}(\varphi)$ is invariant with respect to such transformations. The case of $\chi(M) \leq 0$ is treated in Section 2.2 of the OPS paper, Ref.[41] and leads to the same conclusions about the extremality of the functional $\mathcal{F}(\varphi)$.

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11Generalization of these two dimensional results to higher dimensional manifolds of even dimensions can be found in the paper by Beckner [45].
Corollary 5.1. The two dimensional results just obtained by design are in accord with results obtained in higher dimensions (discussed in Sections 3.6 and to be discussed in Section 7). In particular, the functional $F(\varphi)$ is the exact analog of the Yamabe functional $S[\varphi]$, Eq.(3.2). Since in both cases the functionals are "translationally" (actually, "scale") invariant, e.g. compare Eq.(3.6) with Eq.(5.16), in both cases, the extremum is realized for the metric conformal to the metric of constant scalar curvature, e.g. compare Eq.(3.3) with the Liouville Eq.(5.19).

5.2 Connections with string and CFT

The obtained results allow us now to discuss topic c) listed at the beginning of this section. Using known results for string and conformal field theories, the results just obtained can be given a statistical mechanical interpretation. Following [14,15,46], it is of interest to consider averages of the vertex operators. They are defined by

$$\left\langle \prod_{i=1}^{n} \exp(\beta_i \phi(z_i)) \right\rangle \equiv \int D[\phi] \exp\{-S_L(\phi)\} \prod_{i=1}^{n} \exp(\beta_i \phi(z_i)), \quad (5.23)$$

where the Liouville action $S_L(\phi)$ is given (in notations adopted from these references) by

$$S_L(\phi) = \frac{1}{8\pi} \int_M dA_3[|\nabla \phi|^2 - QR(\hat{g})\phi + 8\pi \bar{\mu} \exp(\alpha_+ \phi)]. \quad (5.24)$$

The actual values and the meaning of the constants $Q$, $\bar{\mu}$ and $\alpha_+$ are explained in these references and are of no immediate use for us. Clearly, upon proper rescaling, we can bring $S_L(\phi)$ to the form which agrees with $F(\varphi)$, defined by Eq.(5.14), especially in the trivial case when both $\chi(M)$ and $\bar{\mu}$ are zero. When they are not zero, the situation in the present case becomes totally analogous to that discussed earlier for the Yamabe functional. In particular, in Section 3 we noted that the G-L Euler-Lagrange Eq.(3.3) can be obtained either by minimization of the Yamabe functional, Eq.(3.2), (or (4.9)) or by minimization of the G-L functional, Eq.(3.7), where the coupling constant $\lambda$ plays the role of the Lagrange multiplier keeping track of the volume constraint. In the present case, variation of the Liouville action $S_L(\phi)$ will produce the Liouville equation, e.g. see Eqs.(5.19)-(5.20), which is the two dimensional analog of the G-L Eq.(3.3). Such variation is premature, however, since we can reobtain $F(\varphi)$ exactly using the path integral, Eq.(5.23). This procedure then will lead us directly to the Liouville Eq.(5.19).

For this purpose, we need to consider the path integral, Eq.(5.23), in the absence of sources, i.e. when all $\beta_i = 0$. Following ideas of Ref.s [14,15,46], we
take into account that: a)
\[ \frac{1}{4\pi} \int_M dA_\hat{g} R(\hat{g}) = \chi(M) = 2 - 2g \]  
(5.26)

with \( g \) being genus of \( M \) and, b) the field \( \phi \) can be decomposed into \( \phi = \phi_0 + \varphi \) in such a way that \( \phi_0 \) is coordinate-independent and \( \varphi \) is subject to the constraint given by Eq.(5.17). Then, use of the identity
\[ \int_{-\infty}^{\infty} dx \exp(ax) \exp\left(-b \exp(\gamma x)\right) = \frac{\Gamma(a)}{\gamma b - a} \] 
(5.27)
(with \( \Gamma(x) \) being Euler’s gamma function) in the path integral, Eq.(5.23), requires us to evaluate the following integral
\[ I = \int_{-\infty}^{\infty} d\phi_0 \exp(\phi_0 \frac{Q}{2} \chi(M)) \exp(\bar{\mu} \int_M dA_\hat{g} \exp(\alpha + \varphi)) \exp(\alpha + \phi_0) \]
\[ = \frac{\Gamma(-s)}{\alpha_+} (\bar{\mu} \int_M dA_\hat{g} \exp(\alpha + \varphi))^s \] 
(5.28)

with \( s \) being given by
\[ s = -\frac{Q}{2\alpha_+} \chi(M). \]  
(5.29)

Using this result in Eq.(5.23), we obtain the following path integral (up to a constant)
\[ Z[\varphi] = \int D[\varphi] \exp(-\hat{\mathcal{F}}(\varphi)) \] 
(5.30)

with functional \( \hat{\mathcal{F}}(\varphi) \) given by
\[ \hat{\mathcal{F}}(\varphi) = S_L(\varphi; \bar{\mu} = 0) - \frac{Q}{2\alpha_+} \chi(M) \ln[\bar{\mu} \int_M dA_\hat{g} \exp(\alpha + \varphi)]. \]  
(5.31)

This functional (up to rescaling of the field \( \varphi \)) is just the same as the Yamabe-like functional \( \mathcal{F}(\varphi) \) given by Eq.(5.14).

Define now the free energy \( \mathcal{F} \) in the usual way via \( \mathcal{F} = -\ln Z[\varphi] \) (as was done after Eq.(5.5)) and consider the saddle point approximation to the functional integral \( Z[\varphi] \). Then, for spherical topology in view of Eq.(5.22), we (re)obtain \( \mathcal{F} \geq 0 \) with equality obtained exactly when \( \varphi = \psi^* \). Inclusion of the sources (or the vertex operators) can be taken into account also, especially in view of the results of [43].

To understand better the physical significance of the obtained results, it is useful to obtain them using a somewhat different method. The results obtained with this alternative method are also helpful when we shall discuss
their higher dimensional analogs in the next section. For this purpose we write (up to a normalization constant)

\[
\int D[\phi] \exp\{-S_L(\phi)\} = \int_0^\infty dA e^{-\bar{\mu} A} Z_L(A) \tag{5.32}
\]

where

\[
Z_L(A) = \int D[\phi] \delta(\int_M dA_\beta \exp(\alpha_+ \phi) - A) \exp(S_L(\phi; \bar{\mu} = 0)). \tag{5.33}
\]

If, as before, we assume that \( \phi = \phi_0 + \varphi \), then an elementary integration over \( \phi_0 \) produces the following explicit result for \( Z_L(A) \):

\[
Z_L(A) = -\frac{1}{\alpha_+} A^\omega \int D[\varphi] \left( \int_M dA_\beta \exp(\alpha_+ \varphi) \right)^{-(\omega+1)} \exp(-S_L(\varphi; \bar{\mu} = 0)) \tag{5.34}
\]

where the exponent \( \omega \) is given by

\[
\omega = \frac{\chi(M) Q}{2\alpha_+} - 1. \tag{5.35}
\]

Finally, using Eq.(5.34) in Eq.(5.32) produces back Eqs.(5.30) and (5.31) (again, up to a constant factor). An overall "-" sign can be removed by proper normalization of the path integral. These results can be used for computation of the correlation functions of conformal field theories (CFT). Details can be found in [14,15,46].

6 Designing higher dimensional CFT(s)

6.1 General remarks

In previous section we explained a delicate interrelationship between the path integrals Eq.(5.5) and (5.23). From the literature on CFT cited earlier it is known that, actually, both are being used for the design of different CFT models in 2 dimensions. For instance, if one entirely ignores the effects of curvature in Eq.(5.5), then one ends up with the Gaussian-type path integral whose calculation for a flat torus is discussed in detail in Ref.[21], pages 340-343, and, by different methods, in our Ref.[47]. By making appropriate changes of the boundary conditions in such a path integral (or, equivalently, by considering appropriately chosen linear combinations of modular invariants) it is possible to build partition functions for all existing CFT models. For the same purpose one can use the path integral given by Eq.(5.23) but the calculation proceeds differently as explained in Section 5. Since in 2 dimensions the conformal invariance...
is crucial in obtaining exact results, use of the Gaussian-type path integrals is, strictly speaking, not permissible. Fortunately, the saddle point-type calculations made for the path integral, Eq.(5.23), produce the same results since the extremal metrics happen to be flat [47]. If one does not neglect curvature effects in Eq.(5.5), one eventually ends up with the integrand of the path integral, Eq.(5.23). This result is a consequence of Eq.(5.6) known as the conformal anomaly. If one would like to proceed in an analogous fashion in dimensions higher than two one should be aware that there is a profound difference between calculations done in odd and even dimensions. We would like to explain this circumstance in some detail now.

In 2 dimensions the conformal invariance of the action, Eq.(5.4), has been assured by the transformational properties of the 2 dimensional Laplacian given by Eq.(5.1). In higher dimensions, the Laplacian is transformed according to Eq.(2.4), so that even the simplest Gaussian model is not conformally invariant! This observation makes use of traditional string-theoretic methods in higher dimensions problematic. In two dimensions these are based on a two stage process: first one calculates the path integral, Eq.(5.5), exactly and, second, one uses the result of such a calculation (the conformal anomaly) as an input in another path integral, e.g. Eq.(5.23), which is obtained by integrating this input over all members of the conformal class. Since in odd dimensions there is no conformal anomaly as we shall demonstrate momentarily, such a two stage process cannot be used. The situation can be improved considerably if we do not rely on the use of the two stage process just described. We would like to explain this fact in some detail now.

Even though the transformational properties (with respect to conformal transformations) of the Laplacian, Eq.(2.4), in dimensions higher than 2 are rather unpleasant, fortunately, they can be considerably improved if, instead of the usual Laplacian, one uses the conformal (Yamabe) Laplacian $\Box_g$ defined by

$$
\Box_g = \Delta_g + \hat{\alpha}R(g),
$$

where $\hat{\alpha} = \alpha^{-1} = \frac{1}{4d-2}$. By construction, in 2 dimensions it becomes the usual Laplacian. In higher dimensions its transformational properties are much simpler than those for the usual Laplacian (e.g see Eq.(2.4)). Indeed, it can be shown [48] that

$$
\Box_{e^{2f}g} = e^{-(\frac{d}{2}+1)f}\Box_g e^{(\frac{d}{2}-1)f}.
$$

This result can be easily understood if we use the results of Section 2. Indeed, since $e^{2f} = \varphi^{p-2}$ and since $p = \frac{2d}{d-2}$, we obtain at once $e^{\frac{d}{2}-1)f} = \varphi$, while the factor $e^{-(\frac{d}{2}+1)f}$ is transformed into $\varphi^{1-p}$. From here, it is clear that Eq.(2.12) for scalar curvature can be equivalently rewritten as

$$
\hat{R}(\hat{g}) = \alpha\varphi^{1-p}(\Delta_g \varphi + \alpha^{-1}R(g)\varphi),
$$

where $\hat{g} = e^{2f}g$, so that

$$
\hat{R}(\hat{g}) = \alpha\Box_{e^{2f}g}.
$$
In practical applications it could be more useful to consider two successive conformal transformations made with the conformal factors $e^{2f}$ and $e^{2h}$. If $e^{2f} = \varphi^{p-2}$ and $e^{2h} = \psi^{p-2}$, then, we obtain,

$$\Delta_g \psi + \alpha^{-1} R(\hat{g}) \psi = \varphi^{1-p} [\Delta_g (\varphi \psi) + \alpha^{-1} R(g) (\varphi \psi)].$$  \hspace{1cm} (6.2b)

This result is, of course, equivalent to Eq.(6.2a).

Consider now the following path integral

$$\exp (-F(g)) = \int D[\varphi] \exp \{-S_{\Box_g}(\varphi)\}$$ \hspace{1cm} (6.5)

where

$$S_{\Box_g}(\varphi) = \alpha \int_M d^d x \sqrt{\tilde{g}} \left\{ (\nabla_g \varphi)^2 + \alpha^{-1} R(g) \varphi^2 \right\}$$

$$= \alpha \int_M d^d x \sqrt{\tilde{g}} \Box_g \varphi = \int_M d^d x \sqrt{\tilde{g}} \tilde{R}(\hat{g})$$ \hspace{1cm} (6.6)

The conformal factor $\varphi^{-p}$ in Eq.(6.3) is eliminated by the corresponding factor coming from the volume factor of $\sqrt{\tilde{g}} = \sqrt{\exp(2f \tilde{g})} = \varphi^{p} \sqrt{\tilde{g}}$.\footnote{This result should be understood as follows. We have $\tilde{g}_{ij} = \varphi^{p-2} g_{ij}$. From here, we have for determinants $\sqrt{\tilde{g}} = |\varphi^{p-2}| \sqrt{\tilde{g}} = \varphi^{p} \sqrt{\tilde{g}}$ in accord with Eq.(3.5).}

**Corollary 6.1.** Thus, Eq.(6.5) is the exact higher dimensional analog of the two dimensional path integral Eq.(5.5), and, hence, problems related to higher dimensional CFT are those of Riemannian\footnote{Sometimes called Euclidean quantum gravity.} quantum gravity [49]. In this paper we are not considering the pseudo-Riemannian case relevant for true Einsteinian gravity. It will be discussed in a companion publication [32].

The question immediately arises: If the path integral, Eq.(6.5), is such an analog, is there a higher dimensional analog of Eq.(5.6)? The answer is "yes", if the dimension of space is even, and "no" if the dimension of space is odd [48].

### 6.2 Lack of conformal anomaly in odd dimensions

In view of its importance, we would like to provide a sketch of the arguments leading to the answer "no" in dimension 3. Clearly, the same kind of arguments will apply in any other odd dimension. In doing so, although we follow the arguments of Refs. [48,50], some of our derivations are original. We begin by assuming that there is a one parameter family of metrics: $\hat{g}(x) = \exp(2xf)g$.

Next, we define the operator $\delta_f$ via

$$\delta_f \Box_g = \frac{d}{dx} \bigg|_{x=0} \Box_{\exp(2xf)g}.$$

Taking into account Eq.(6.2) we obtain explicitly

$$\delta_f \Box_g = -2f \Box_g$$ \hspace{1cm} (6.8)

The conformal factor $\varphi^{-p}$ in Eq.(6.3) is eliminated by the corresponding factor coming from the volume factor of $\sqrt{\tilde{g}} = \sqrt{\exp(2f \tilde{g})} = \varphi^{p} \sqrt{\tilde{g}}$.\footnote{This result should be understood as follows. We have $\tilde{g}_{ij} = \varphi^{p-2} g_{ij}$. From here, we have for determinants $\sqrt{\tilde{g}} = |\varphi^{p-2}| \sqrt{\tilde{g}} = \varphi^{p} \sqrt{\tilde{g}}$ in accord with Eq.(3.5).}

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Taking into account Eq.(6.2) we obtain explicitly

$$\delta_f \Box_g = -2f \Box_g$$ \hspace{1cm} (6.8)
and, accordingly,
\[ \delta_f e^{-t\Box g} = -t(\delta_f \Box g)e^{-t\Box g}. \] (6.9)

These results allow us to write for the zeta function\(^\text{14}\)
\[ \delta_f \zeta_{\Box g}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \! dt t^{s-1} \delta_f \text{Tr}(e^{-t\Box g}) \]
\[ = \frac{1}{\Gamma(s)} \int_0^\infty \! dt t^s \text{Tr}(-2f\Box_ge^{-t\Box g}) \]
\[ = -\frac{2s}{\Gamma(s)} \int_0^\infty \! dt t^{s-1} \text{Tr}(fe^{-t\Box g}). \] (6.10)

The last line was obtained by performing integration by parts. Since for small \(t's\) it is known that, provided that \(\text{dim ker } \Box_g = 0\), \(^\text{15}\)
\[ \text{Tr}(fe^{-t\Box g}) \simeq \sum_{k=0}^\infty \left( \int_M f(x) u_k(x) \text{dvol} \right) t^{k-d/2}, \] (6.11)
where \(\text{dvol} = \sqrt{g}\text{d}^d x\), as usual. Using this result in Eq.(6.10) produces
\[ \delta_f \zeta_{\Box_g}(0) = -\frac{2s}{\Gamma(s)} \int_0^\infty \! dt t^{s-1} \text{Tr}(fe^{-t\Box g}) + \int_1^\infty \! dt t^{s-1} \text{Tr}(fe^{-t\Box g}) \]
\[ = -\frac{2s}{\Gamma(s)} \left( \sum_{k=0}^\infty \int_M f u_k \text{dvol} \right) \left. + \int_1^\infty \! dt t^{s-1} \text{Tr}(fe^{-t\Box g}) \right|_{s=0}. \] (6.12)

Since for \(s \to 0^+\) we have \((1/\Gamma(s)) \sim s\), the second (regular) term in brackets will become zero when multiplied by the combination \(\frac{2s}{\Gamma(s)}\), while the first term will become zero even if it might acquire a pole (in case if \(d = 2k\)). Hence, for all dimensions \(d \geq 3\) we obtain \(\delta_f \zeta_{\Box_g}(0) = 0\) or
\[ \zeta_{\Box_g}(0) = \zeta_{\Box_{\hat{g}}}(0). \] (6.13)

This result can be used further. Indeed, if we write
\[ \delta_f \left[ \Gamma(s) \zeta_{\Box_g}(s) \right] = \Gamma(s) \left[ \delta_f \zeta_{\Box_g}(0) + s\delta_f \zeta'_{\Box_g}(0) + O(s^2) \right] \] (6.14)

\(^{14}\)E.g. see Eq.(5.8) and take into account the identity \(x^{-s}\Gamma(s,x) = \int_0^\infty \! dy y^{s-1} \exp(-xy)\).

\(^{15}\)In the case of path integrals this is always assumed since zero modes of the corresponding operators are associated with some kind of translational, rotational, etc. symmetry. To eliminate the undesirable dilatational symmetry, one actually should use the Yamabe functional, Eq.(3.6), as explained in Section 3. Although this is silently assumed thus far, arguments additional to those in Section 3 will be introduced further below.
and take into account Eq.(6.13) and the fact that \( s \Gamma(s) = 1 \) we obtain,

\[
\delta \zeta'_{\Box g}(0) = \delta f \int_0^{\infty} dt e^{-t \Box g} \left|_{s=0} \right.
\]

\[
= -2s \sum_{k=0}^{\infty} \left( \int_M f_k \, d\text{vol} \right) + \int_1^{\infty} dt e^{-(s-d/2)} \left|_{s=0} \right.
\]

\[
= -2s \left( \sum_{k=0}^{\infty} \int_M f_k \, d\text{vol} \right) + \int_1^{\infty} dt e^{-(s-d/2)} \left|_{s=0} \right.
\]

(6.15)

Applying the same arguments to Eq.(6.15) as those that were used for Eq.(6.12) we conclude that, provided \( \dim \ker \Box g = 0 \), in odd dimensions, \( \delta f \zeta'_{\Box g}(0) = 0 \), that is

\[
\zeta'_{\Box g}(0) = \zeta'_{\hat{\Box} g}(0).
\]

(6.16)

In view of the Eq.(5.9) this leads also to

\[
\text{det } \Box g = \text{det } \hat{\Box} g,
\]

(6.17)

QED.

6.3 3d CFT path integrals

6.3.1 General remarks

Our previously obtained results can be further refined if we recall Eq.s(5.9)-(5.11). In particular, let \( e^{2\bar{\phi}} \) in Eq.(5.11) be rewritten as some nonnegative constant \( l \). Then we obtain

\[
\zeta_{\hat{g}}(s) = l^s \zeta_g(s).
\]

(6.18)

This result is consistent with Eq.(6.13) for \( s = 0 \). Differentiation with respect to \( s \) produces

\[
\zeta'_{\hat{g}}(0) = \zeta_g(0) \ln l + \zeta'_g(0).
\]

(6.19)

In view of Eq.(5.9), this result is equivalent to \( \ln \text{det } \Box g = \ln \text{det } \hat{\Box} g - \zeta_g(0) \ln l \). And apparently contradicts Eq.(6.17). But the contradiction is only apparent in view of the earlier footnote. The situation is easily correctable if in the path integral, Eq.(6.5), we replace the action functional \( S_{\Box}(\varphi) \) by that of Yamabe given by Eq.s(3.2) (or (3.6)). This, by the way, allows us also to fix the value of \( \zeta_g(0) \): provided that we identify the constant \( l \) with the volume \( V \), the value of \( \zeta_g(0) = \frac{2}{p} \). After this, the situation in the present case becomes very similar to that encountered in the previous section. That is, instead of the functional \( \mathcal{F}(\varphi) \) given by Eq.(5.15), we can consider now the related functional given by

\[
\mathcal{F}(\varphi) = \ln \text{det } \Box g - \zeta_g(0) \ln V \equiv \ln \text{det } \Box g - \frac{2}{p} \ln V.
\]

(6.20)
For the path integral calculations, a functional defined in such a way is not yet sufficient. To repair this deficiency we have to impose a volume constraint. That is we need to consider the path integral of the type

\[
Z_Y(V) = \int D[\phi] \delta(\int_M d^d x \sqrt{g} \phi^p - V) \exp(-S[\phi]) \tag{6.21}
\]

with \( S[\phi] \) given by Eq.(3.2) (or (3.6)).

**Remark 6.2.** The path integral \( Z_Y(V) \) (\( Y \) in honor of Yamabe) is the exact higher dimensional analog of the "stringy" path integral \( S_L(A) \) given by Eq.(5.33). In view of Eq.(3.6), it also can be viewed as the path integral for pure (Euclidean) gravity in the presence of the cosmological constant.

In complete analogy with Eq.(5.32), the standard path integral for the self interacting scalar \( \phi^4 \) (or LGW) field theory is obtainable now as follows

\[
\int D[\phi] \exp\{-S_{LGW}(\phi)\} = \int_0^\infty dV e^{-bV} Z_Y(V). \tag{6.22}
\]

But, since the variation of the Yamabe functional produces the same Eq.(3.3) as can be obtained with help of the L-G functional, \( S_{L-G}(\phi) \), (which, in view of the Remark 3.2., can be identified with \( \tilde{S}[\phi] \) defined by Eq.(3.7)) one can develop things differently but, surely, equivalently.

To this purpose, instead of the functional \( S[\phi] \) given in Eq.(3.2) we use

\[
S_V[\phi] = \frac{1}{V^2} \int_M d^d x \sqrt{g} \left\{ (\nabla_g \phi)^2 + R(g) \phi^2 \right\} \tag{6.23}
\]

and replace \( S[\phi] \) in the exponent of the path integral in Eq.(6.21) by \( S_V[\phi] \) from Eq.(6.23). Then, instead of Eq.(6.22), we obtain,

\[
\int D[\phi] \exp\{-S_{L-G}(\phi)\} = \int_0^\infty dV Z_Y(V), \tag{6.24}
\]

where the sign \( \tilde{=} \) means "supposedly". This is so, because, in view of the Remark 3.3., at the level of saddle point calculations the left hand side and the right hand side produce the same G-L equation. Beyond the saddle point, calculations are not necessarily the same. Although we plan to discuss this issue in subsequent publications, some special cases are further discussed below in this section.

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16This fact can be used (with some caution in view of Remark 4.3.) as alternative formulation of the quantum gravity problem.

17To be in accord with standard texts on statistical mechanics, e.g. see [17], the constant scalar curvature \( R(g) \) should be identified with the squared "mass", i.e. with \( m^2 = a |T - T_c| \) where \( T \) and \( T_c \) are respectively the temperature and critical temperature. Also the sign "-" in front of \( \lambda \) term should be replaced by "+".
Remark 6.3. It should be clear, that at the level of the saddle point calculation, replacement of the functional $S[\varphi]$ in Eq. (6.21) by $\mathcal{F}(\varphi)$ from Eq. (6.20) is completely adequate, so that the sequence of steps in the analysis performed for the two dimensional case in Section 5 are transferable to higher dimensions without change.

Remark 6.4. Although in 3 dimensions we have the result given by Eq. (6.17), which forbids the use of identities like that in Eq. (5.6) still, based on arguments just presented, the functional $\mathcal{F}(\varphi)$ defined by Eq. (6.20) should be used in the exponent of the corresponding path integral replacing that given in Eq. (5.15) in 2 dimensions. Since by doing so one will be confronted with the same type of minimization problems as discussed earlier in Section [18], the functional integral thus defined is an exact 3 dimensional analog of the path integral, Eq. (5.30).

6.3.2 A sketch of 3d CFT calculations

Our obtained results allow us to proceed in complete accord with Section 5. Thus, let $\hat{g}(u) = e^{\phi(u)}g$ be a 1-parameter family of metrics of fixed volume and such that $\hat{g}(0) = g$. This implies that $\phi(0) = 0$ and $\int_M e^{\phi(u)}dV_g = V$.

In two dimensions, using results of OPS [41], especially their Eq. (1.12), it is straightforward to obtain the following result

$$\frac{d}{du} (-\ln \det \Delta_{g(u)}) \big|_{u=0} = \zeta'(g)(0) = \frac{1}{12\pi} \int_M \phi K(g) dV_g$$

(6.25)

where $K(g)$ is the Gaussian curvature for the metric $g$. $\zeta'(g)(0)$ and $\phi$ represent $\frac{d}{du} \zeta(g(u))(0) \big|_{u=0}$ and $\frac{d}{du} \phi(u) \big|_{u=0}$ respectively. Volume conservation implies

$$\frac{d}{du} \int_M e^{\phi(u)}dV_g \big|_{u=0} = \int_M \zeta'(u) dV_g = 0$$

(6.26)

in accord with the earlier result, Eq. (5.17). If the Gaussian curvature $K(g)$ is constant, then Eq. (6.25) and Eq. (6.26) produce the same result. This implies that $\frac{d}{du} \zeta(g(u))(0) \big|_{u=0} = 0$, that is $g$ is the "critical" (extremal) metric. In view of Eq. (6.25) this also means that, for such a metric, the free energy (e.g. see definition given by Eq. (5.5)) attains its extremum. By combining Eq.s (6.10), (6.14) and (6.15) we obtain at once

$$\delta f \zeta'(\Box_g)(0) = \delta f \int_0^\infty dt t^{s-1} Tr(e^{-t\Box_g}) \big|_{s=0}$$

$$= \int_0^\infty dt t^{s} Tr(-2f\Box_g e^{-t\Box_g}) \big|_{s=0}.$$  

(6.27)
From here we obtain essentially the same result as the main theorem by Richardson [20], i.e.

$$
\zeta_{\Box_g}(0) |_{u=0} = 0 = \int_M dV_g \hat{\phi}(x) \Box_g \zeta(1, x, x).
$$

That is, provided that we replace $\Box_g$ by $\Delta_g$ and require that \textit{locally} $\zeta(1, x, x) = \text{const}$, with the heat kernel $\zeta(s, x, x)$ given (as usual) by

$$
\zeta(s, x, x) = \sum_{k=1}^{\infty} \frac{\psi_k^2(x)}{\lambda_k},
$$

we obtain the main result by Richardson, Ref.[20], e.g. see his Theorem 1 and Corollary 1.1. Here $\psi_k(x)$ are eigenfunctions of the Laplacian (or Yamabe Laplacian respectively) corresponding to eigenvalues $\lambda_k$ with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$. In the two dimensional case, the condition for criticality: $\zeta'(g)(0) = 0$ is \textit{local} meaning that, provided the volume is constrained, the constancy of the Gaussian curvature $K(g)$ at a given point of $M$ is caused by the metric $g$ for which $\zeta'(g)$ is extremal. In 3 and higher dimensions, constancy of the curvature at the point of $M$ is being replaced by constancy of $\zeta(1, x, x)$ under the same conditions of volume conservation. Such a condition is only necessary but is now not sufficient, since the analog of Moser-Trudinger inequality, Eq.(5.21), used to prove sufficiency in 2 dimensions does not exist. Instead, one should study locally the second variation of $\zeta'(g)$ with respect to the underlying background metric in order to decide if such a (local) extremum is a maximum or minimum. Fortunately, this task was accomplished in Ref.s[20, 51]. In particular, Richardson [20] obtained the following theorem

**Theorem 6.5.** (Richardson) \textit{The Euclidean metric on a cubic 3-torus is a local maximum of determinant of the Laplacian with respect to fixed-volume conformal variations of the metric.}

**Remark 6.6.** This Theorem is proven only for the cubic 3-torus. The word ”local” means that there could be (or, there are, as we shall demonstrate) other 3-tori also providing local maxima for determinants. In fact, according to the result by Chiu [52], all determinants of flat 3-tori possess local maxima so that the determinant for the face centered cubic (fcc) lattice has the largest determinant. Physical implications of this fact can be found in our recent work, Ref.[47], and also will be briefly considered in Section 8 below.

**Remark 6.7.** By using the same arguments as in Hawking’s paper, Ref. [53], the above theorem by Richardson can be given a physical interpretation. It is based on use of the saddle point methods applied to the Yamabe path integral, Eq.s (6.21), (6.24).
The second variation of the Yamabe functional was calculated by Muto [54] (see also Ref.[49]) with the result:

\[ \left( \frac{d^2}{dt^2} R(g(t)) \right)_{t=0} = \frac{d-2}{2} \int_M dV_g(\sigma(\nabla_g \varphi)^2 - R(g)\varphi^2), \]  

where the constant \( \sigma = d - 1 \). As in the case of quadratic actions in the flat space [17, 36] the second variation (with volume constrained to be equal to one) looks very much the same as the original quadratic Yamabe functional, except for the "wrong" sign in front of the scalar curvature. This "wrong" sign has important physical significance in both statistical mechanics and high energy physics. While the significance to high energy physics will be discussed in the companion publication, Ref.[32], here we would like to comment on the significance to statistical mechanics. For this we need to recall that for the G-L free energy functional \( F\{\varphi\} \) given by [17]

\[ F\{\varphi\} = F_0 + \frac{1}{2} \int dV \int dV' \left[ \frac{\delta^2}{\delta \varphi(x) \delta \varphi(x')} F\{\varphi\} \bigg|_{\varphi=\varphi_0} \right] \eta(x)\eta(x') + ..., \]  

the second variation is given by

\[ F\{\varphi\} = F\{\varphi_0\} + \frac{1}{2} \int dV \int dV' \left[ \frac{\delta^2}{\delta \varphi(x) \delta \varphi(x')} F\{\varphi\} \bigg|_{\varphi=\varphi_0} \right] \eta(x)\eta(x') + ..., \]  

where

\[ \frac{\delta^2}{\delta \varphi(x) \delta \varphi(x')} F\{\varphi\} \bigg|_{\varphi=\varphi_0} = (-c \nabla^2 + a + 3b\varphi_0^2) \delta(x-x'). \]  

with \( \varphi_0 \) being obtained from the extremum condition

\[ \frac{\delta F\{\varphi\}}{\delta \varphi(x)} = -c \nabla^2 \varphi + a\varphi + b\varphi^3 - h = 0 \]  

In a special case of a coordinate independent field \( \varphi_0 \) Eq.(6.34) is reduced to

\[ a\varphi_0 + b\varphi_0^3 = 0. \]  

For \( a < 0 \) we obtain : \( \varphi_0 = \pm (|a|/b)^{1/2} \) and \( \varphi_0 = 0 \). The solution \( \varphi_0 = 0 \) is not a minimum for the free energy and, hence, should be discarded. Of the two other solutions, the system chooses one of them (which is interpreted in literature as spontaneous symmetry breaking). They both have the same free energy\(^{19}\). If we let now \( \varphi(x) = \varphi_0 + \delta \varphi \equiv \varphi_0 + \eta(x) \) and Taylor series expand \( F\{\varphi\} \) the result will coincide with Eq.s(6.32) and (6.33). Notice that for \( a < 0 \) a combination \(-|a| + 3b\varphi_0^2 = 2|a| \equiv m^2 \) (as compared to \( a \) in the high temperature phase). So, if in the high temperature phase \( m^2 = R(g) \) with \( R(g) > 0 \), in the low

\[^{19}\text{Once the choice is made, the order parameter can be considered in all subsequent calculations as positive.}\]
temperature phase (i.e. below the criticality) $R(g) < 0$. Moreover, by choosing $d = 4$ in Eq.(6.30) we obtain a combination: $m^2 = \frac{1-\Delta}{2} R(g) = 2R(g)$ in accord with the standard G-L results. The constant c in Eq.(6.33) is left unspecified in G-L theory while now it is equal to 6 (for $d = 4$). The spaces of constant negative curvature are hyperbolic.

**Remark 6.8.** In our earlier work, Ref[55] we emphasized the major importance of hyperbolicity in statistical mechanical calculations related to the AdS/CFT correspondence. In this work we arrived at the same conclusions quite independently.

From the work by Muto [54] we know that: a) if $R(g)$ is positive, the second variation can be made positive for appropriately chosen $\phi_2$, b) if $R(g)$ is negative, the second variation is positive for the same reasons. The positivity of the second variation implies that the extremal constant curvature metric $g$ provides a *locally stable minimum* for $\mathcal{R}(g(t))$ defined by Eq.(4.9). That is, using results of Section 4, the Einstein metric obtained as solution to Eq.(4.2) is stable among nearby metrics. This conclusion will be reinforced in the next (dynamical) section.

**Remark 6.9.** It is interesting to notice that the calculation of higher order fluctuation corrections to the Yamabe path integral, Eq.(6.21), involves calculations on the moduli space of Einsteinian metrics, Ref.[56]. This observation provides a strong link between higher dimensional LGW theory and two dimensional string inspired CFT’s discussed in the previous section. Naturally, Eq.(6.24) can be used to investigate to what extent the final results of conventional field-theoretic calculations, Ref.[17,36], might differ from more sophisticated string-theoretic calculations in the style of Ref.[14,15,46]. Additional details can also be found in our work, Ref.[47].

### 7 Critical dynamics and the Yamabe and Ricci flows

#### 7.1 Physical motivation

The results obtained in previous sections are somewhat unrealistic from a purely practical (physical) point of view. The situation in our case is analogous to that known in thermodynamics. Recall, that this discipline emerged from practical needs to improve the efficiency of heat engines. Clearly, rigorously speaking, it is not applicable to such devices since, by definition, it is valid only for time-independent phenomena. Hence, it cannot be used at all because it usually takes a very long (if not infinite) time for the system to equilibrate, especially near the critical temperature $T_c$. Thus, the description of phase transitions by the G-L theory is valid only in a rather narrow range of temperatures around

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20This can be easily understood if we expand $\psi$ into Fourier series made of eigenfunctions of the Laplacian and take into account that for *any* closed Riemannian manifold the spectrum of the Laplacian is nonnegative and nondecreasing [29].
To bring a physical system into this range of temperatures (under constant pressure) requires varying the temperature of the surrounding environment in time. In addition, by definition, the "true" phase transition should take place only in the thermodynamic limit (of infinite volume with particle density kept constant). Since in the real world the systems under study are always of finite size, this requirement is implemented by imposing physically appropriate boundary conditions, e.g. periodic. Use of some appropriate boundary conditions causes systems undergoing phase transitions to actually "live" on some manifolds/orbifolds. Under such circumstances the topology and physics become intertwined. The signature of such boundary effects can be seen already in the calculation of determinants discussed in Sections 5 and 6. More on this subject is discussed in our earlier work, Ref.[55], where, in accord with Remark 6.8., we emphasized the role of hyperbolic spaces in the theory of phase transition.\footnote{Remark 6.8. is not in contradiction with the remark 6.6. This is explained in great detail in our recent work, Ref.[47] and, is also discussed in Section 8.}

The equilibration process known in the physics literature as \textit{critical dynamics} can take a very long time (infinite in the thermodynamic limit for $T \to T_c$). Since idealization of reality is typical in physics, in this work we adopt the pragmatic (physical) point of view by considering systems of finite size with appropriately chosen boundary conditions. Then, the results of the previous sections are limiting cases of more general time-dependent G-L theory considered for such systems.

Development of time-dependent G-L theory was initiated by Landau and Khalatnikov in 1954 [57] and is also phenomenological. As such, it is based on the assumption that an order parameter $\varphi$ satisfies the relaxation equation of the type given by

$$\frac{\partial \varphi}{\partial t} = -\gamma \frac{\delta F\{\varphi\}}{\delta \varphi} = -\gamma \text{grad} F\{\varphi\}$$

(7.1)

with functional $F\{\varphi\}$ defined by Eq.(6.31) and the "friction" coefficient $\gamma$ is some (assumed to be) known non-negative constant. By rescaling time it can be eliminated. Such rescaling is assumed in all calculations below. Since such an equation was postulated, its validity was checked by real experiments with excellent outcome [58]. Being armed with such results, we would like to develop the mathematical formalism of the previous sections to account for the effects of critical dynamics.

7.2 Mathematical motivation

Eq.(7.1) is an example of the gradient flow. In this subsection we would like to place the Landau-Khalatnikov theory of critical dynamics on a more rigorous mathematical basis also involving the notion of the gradient flow.

We begin with reviewing the results of OPS, Ref. [41], (obtained for two dimensional manifolds without boundaries) in the style of Hamilton’s work,
Ref.[16]. To this purpose we notice that Eq.(5.19) is an extremum of the functional $F(\psi)$, Eq.(5.14). Clearly, if we write $\varphi = \psi + \varepsilon h$ in this functional so that

$$
\frac{d}{d\varepsilon} F(\psi + \varepsilon h) |_{\varepsilon = 0} = - \int_M dA_g (\Delta_g \psi) + \frac{1}{2} \int_M dA_g R(g) h - 2\pi \chi(M) \int_M dA_g \exp(2\psi) 
$$

then, following OSP, the scalar product $\langle \cdot, \cdot \rangle$ in the tangent space to each point at which $\psi$ is defined can be introduced according to convention:

$$
\langle \text{grad} F, h \rangle \equiv (\delta F) h = \frac{d}{d\varepsilon} F(\psi + \varepsilon h) |_{\varepsilon = 0}.
$$

In accord with Eq.(3.6) by OPS, Ref.[41], this convention implies that

$$
\text{grad} F\{\psi\} = - \Delta_g \psi + \frac{1}{2} R(g) - \frac{2\pi \chi(M) \exp(2\psi)}{\int_M \exp(2\psi) dA_g}.
$$

This result is to be compared with Eq.(5.19). Naturally, Eq.(5.19) corresponds to the condition of equilibrium, i.e. $\text{grad} F\{\psi\} = 0$. Away from equilibrium the dynamical equation postulated by OPS (see below) coincides with Eq.(7.1) postulated much earlier by Landau and Khalatnikov. To see this, we need to take into account Eqs. (5.2), (5.19) and (5.20). With their help we would like to rewrite Eq.(7.4) in the following equivalent form,

$$
- \text{grad} F\{\psi\} = \frac{1}{2} (\langle \text{R} \rangle - \text{R}) e^{2\psi},
$$

where $\langle \text{R} \rangle = \int_M d\hat{A} \text{R}(\hat{g}) / \int_M d\hat{A}$ with $d\hat{A}$ given by Eq.(5.3). Accordingly, in view of Eq.(7.1), up to time rescaling, the equation describing critical dynamics can be written as follows

$$
\frac{\partial \psi}{\partial t} = \langle R(g) \rangle - R(g)
$$

coinciding with Eq.(3.9) of OPS.

Suppose that the equilibrium (the fixed point) condition,

$$
\langle R(g) \rangle - R(g) = 0,
$$

is valid in dimensions higher than two as well. Then it coincides formally with Eq.(2.12) defining scalar curvature or with Eq.(4.12) where it was obtained from the Hilbert-Einstein functional for pure gravity in the presence of the cosmological term. In view of this observation, we would like to demonstrate that equations like Eq.(7.6) whose fixed points are determined by the equations of the type given by Eq.(7.7) indeed can be obtained in higher dimensions. In this work only 3 dimensional results will be discussed in some detail.

22In making such a comparison (as in Section 5) we took into account that $R = 2K$. 29
7.3 Critical dynamics in dimensions higher than two: Yamabe vs Ricci flows

Following Hamilton [16], we begin with

**Definition 7.1.** The normalized Ricci flow is described by the dynamical equation given by

\[
\frac{\partial}{\partial t} g_{ij} = \frac{2}{d} g_{ij} < R(g) > - 2 R_{ij}(g). \tag{7.8}
\]

In writing this equation we use the notations of Section 4 where we defined
\[
<R(g)> = \left( \int_M R \sqrt{g} d^d x \right) / \int_M \sqrt{g} d^d x = \left( \int_M R d\mu \right) / \int_M d\mu.
\]

The above flow equation formally exists for \(d \geq 2\) and should be considered along with some prescribed initial condition: \(g_{ij}(t=0) = \hat{g}_{ij}\). In two dimensions it is always possible to write \(R_{ij}(g) = \frac{1}{2} R g_{ij}\) so that all two dimensional spaces are Einsteinian [16]. Because of this fact, Eq.(7.8) is converted into

\[
\frac{\partial}{\partial t} g_{ij} = \left( < R(g) > - R(g) \right) g_{ij}, \quad d = 2. \tag{7.9}
\]

**Definition 7.2.** Eq.(7.9) defines the Yamabe flow. As such, it can be considered for any \(d \geq 2\).

**Remark 7.3.** If, like in Section 2, we choose \(\hat{g}_{ij} = e^{\psi} g_{ij}(0)\) and substitute this result into Eq.(7.9), we reobtain Eq.(7.6). Hence, Eq.(7.6) does describe the Yamabe flow.

**Corollary 7.4.** Using results of previous subsection (at least in two dimensions) the Yamabe flow is the gradient flow.

**Corollary 7.5.** Since the Yamabe flow in Eq.(7.9) is defined for any \(d \geq 2\), the fixed points for such a flow should coincide with those given by Eq.(7.7).

**Remark 7.6.** Only in 2 dimensions are the Yamabe and the normalized Ricci flows (essentially) equivalent. This is explained in great detail by Hamilton [16]. Since results of this paper crucially depend on the observation that only in 2 dimensions it is *always* possible to write \(R_{ij}(g) = \frac{1}{2} R g_{ij}\), it should be clear that attempts to extend this equivalence to higher dimensions are destined for failure in those cases where the flow takes place on spaces which are not of Einstein type. Hence, in this case the task lies in finding conditions under which the Ricci flow, for which the initial metric is chosen to be of not necessarily Einstein-type, leads to an Einstein-type metric(s) as the fixed point(s) solution(s) for such a flow. This is discussed in the next subsection.

Even though Yamabe and Ricci flows are different in higher dimensions, one may still ask a related question: can one use some Ricci flows in order to
obtain results for Yamabe-type flows? We would like to demonstrate that this is indeed possible. To this purpose, using Eq.(7.9) we obtain,

\[ \frac{1}{2} g^{ij} \frac{\partial}{\partial t} g_{ij} = <R(g)> - R(g). \]  

(7.10)

In addition, since in any dimension \( \frac{\partial}{\partial t} d\mu = \frac{\partial}{\partial t} \sqrt{\det g_{ij}} d^d x = \frac{1}{2} g^{ij} \frac{\partial}{\partial t} g_{ij} d\mu \), we obtain as well

\[ \frac{\partial}{\partial t} \int_M d\mu = \int_M \frac{\partial}{\partial t} d\mu = \int_M (<R> - R(g)) d\mu = 0. \]  

(7.11)

This equation is compatible with Eq.(7.8) and valid for \( d \geq 2 \). Indeed, if we take into account that \( \frac{\partial}{\partial t} d\mu = \frac{1}{2} g^{ij} \frac{\partial}{\partial t} g_{ij} d\mu \), use Eq.(7.8) for \( \frac{\partial}{\partial t} g_{ij} \), multiply both sides of this equation by \( g^{ij} d\mu \) and integrate the obtained result over \( M \), we reobtain Eq.(7.11). Hence, after some manipulations Eq.(7.8) describing Ricci flow (and valid for \( d \geq 2 \)) can be brought into a form identical with Eq.(7.10) describing Yamabe flow (obtained originally for \( d = 2 \)). This observation does not imply that both flows are equivalent as Remark 7.6. indicates. Nevertheless, the Yamabe flow can be looked upon as some special case of the Ricci flow.

**Corollary 7.7.** The arguments just presented demonstrate that solutions for the Yamabe flow can be obtained from solutions for the normalized Ricci flow. Moreover, in both cases the fixed points for such flows are given by Eq.(7.7) thus providing justification for the compatibility of 2 and 3 dimensional CFT discussed in previous sections and, hence, for the unique pathway for extending the results of 2d CFT to 3 and higher dimensions.

**Corollary 7.8.** Since the fixed point, Eq.(7.7), is the traced form of the fixed point equation for the Ricci flow, Eq.(7.8), that is of the equation

\[ \frac{1}{d} g_{ij} <R(g)> - R_{ij}(g) = 0, \]  

(7.12)

which is equivalent to Eq.(4.2), this means: a) that the fixed point solutions of the normalized Ricci flow always produce the Einstein spaces, provided that these fixed points are stable and, b) that the arguments presented in Remark 4.3. are indeed valid.

### 7.4 Stability of the fixed points solutions

The results obtained previously in this section cannot be used until the stability analysis of these solutions is performed. Since according to the Corollary 7.5. solutions for the Yamabe flow can be obtained from those for the Ricci flow, it is sufficient, in principle, to consider only the stability of the fixed point solutions for the Ricci flows. This path is not the most physically illuminating however as we would like to explain now.

#### 7.4.1 Dynamics of the Yamabe flows

The Yamabe flow described by Eq.(7.9) should be supplemented by the initial condition, e.g. \( g_{ij}(t = 0) = \hat{g}_{ij}(0) \equiv \hat{g} \). In order to study the evolution
of this metric it is convenient to write it in the following form [59]: $g_{ij}(t) = [\varphi(t)]^{-\frac{2}{d-2}} g_{ij}(0)$. Using such a substitution in Eq.(7.9) (along with Eq.(2.12)) allows us to rewrite it in the following equivalent form

$$\frac{\partial \varphi}{\partial t} = -\varphi^{2-p}(\alpha \Delta \varphi + R(\bar{g})\varphi) + <R(g)> \varphi$$ (7.13)

to be compared with the phenomenological result, Eq(7.1), by Landau and Khalatnikov. Although these equations look somewhat different, their fixed points (if these are stable) are the same. In view of the results of previous sections, we believe that the equations of critical dynamics known in the physical literature, e.g. see Ref.[58], all having their origin in the work by Landau and Khalatnikov [57], should be replaced by Eq.(7.13) also describing critical dynamics of physical systems of the Ginzburg-Landau type. Such a replacement is in accord with the requirement of conformal invariance at criticality as described in detail in our earlier work on AdS/CFT correspondence [55].

**Remark 7.9.** Evidently, physical realization of the Yamabe and Ricci flows depends crucially on the possibility of such a replacement and, in view of results of this work, it can be considered as proven.

Using Eq.(7.10) we would like to introduce the following quantity

$$\eta = 4 d^{-2}(<R(g) > - R(g)).$$ (7.14)

In view of Eq.(7.11), integrating Eq.(7.14) we obtain,

$$\int_M \eta d\mu = 0.$$ (7.15)

By combining Eq.s(7.11) and (7.13), after some calculations which can be found in Ref.[59], we obtain,

$$\frac{\partial}{\partial t} <R(g(t))> = \frac{d-2}{2} \int_M R(g)\eta d\mu.$$ (7.16)

Next, by combining Eq.s(7.14) and (7.16) we further obtain,

$$\frac{\partial}{\partial t} <R(g(t))> = 2 \int_M R(g)(<R(g) > - R(g)) = -2 \int_M (<R(g) > - R(g))^2,$$ (7.17)

where the second equality comes from the constraint, Eq.(7.15). The last expression possesses an entropic meaning. Indeed, following Boltzmann [60], we introduce the $H$-function (the entropy) for some gas made of hard spheres elastically scattering from each other. Explicitly, it is given by

$$H = -\int f \ln f d\nu,$$ (7.18)

---

23 In two and three dimensions these fixed points were discussed in previous sections.
24 And, ultimately, the physically assisted proof of the Poincare' and geometrization conjectures depend crucially on such an assumption.
where the distribution function \( f(v,t) \) obeys the Boltzmann’s equation

\[
\frac{\partial}{\partial t} f = I(\theta, |v_1 - v|) \int |v_1 - v| I(\theta, |v_1 - v|) d\Omega d v_1 d\Omega (7.19)
\]

with \( I(\theta, |v_1 - v|) \) being some known function, \( d\Omega = \sin \theta d\theta d\phi \), and \( v' = v + n(n \cdot g) \), \( v'_1 = v_1 - n(n \cdot g) \) being respective velocities of the colliding particles after (primed) and before (nonprimed) scattering on each other, here \( g = v_1 - v \), and the vector \( n \) is a unit normal. Based on these results, it can be shown that \( dv dv_1 = d v' d v'_1 \). Using these results, let us consider calculation of \( \frac{\partial}{\partial t} H \). By combining Eqs. (7.18) and (7.19) one obtains after some calculation \( \frac{\partial}{\partial t} H \geq 0 \) with an equality only for the equilibrium case for which \( f^*(v',t) f^*(v'_1, t) - f^*(v, t) f^*(v_1, t) = 0 \). This condition is leading to the Maxwell’s distribution of particle velocities, which produce a well tested equation of state for the ideal gas. In accord with thermodynamics, such defined entropy riches its maximum at equilibrium.

**Remark 7.10.** The obtained equilibrium result for Maxwell’s distribution depends crucially on the assumption of occurrence of only binary collisions in the gas. Boltzmann’s equation, Eq.(7.19), reflects just this fact. The obtained results become much less tractable if one would like to account for ternary and higher order collisions. The same is true if one uses the full renormalization group flow equations [4] instead of just the leading order terms in these equations. The proof of the Boltzmann’s H-theorem as well as the proof of the Poincaré’ and geometrization conjectures are based on the validity of such approximations. The validity is assured by excellent agreement with experiment in Boltzmann’s case and with equally good agreement with experiment in the case of Landau-Ginzburg critical dynamics described by the Yamabe-type flow.

Based on these physical arguments, we may choose \(- < R(g(t)) > \) as our entropy (provided that the volume of the system is normalized, say, to unity). Then, we can identify Eq.(7.9) (or (7.13)) with a Boltzmann-type equation. Our previously obtained result, Eq.(4.12), then provides the physically meaningful equilibrium solution. Moreover, using these results we arrive at the conclusion that the fixed point solutions for the Yamabe flow are described by the Einstein equation, Eq.(4.2), leading to the Einstein manifolds of constant scalar curvature in accord with the Remark 4.3.

### 7.4.2 Dynamics of the Ricci flows

In this subsection we would like to demonstrate in some detail that, indeed, some Ricci flows produce results that are in agreement with those obtained for the Yamabe flows. For this purpose, following the logic of the previous subsection we have to find an analog of the entropy for the Ricci flows. This task was accomplished by Perelman in the first of his 3 papers, Ref.[1]. He noticed, albeit indirectly, that, unlike the Yamabe flow (which is the gradient-type flow), the Ricci flow is not gradient. The proof can be found in Ref.[61].

\footnote{See [60] for details.}
Accordingly, one cannot find an entropy for such a flow and, hence, to study its stability. Perelman found ingenious way around these difficulties. In view of the existing literature [2, 3, 10, 61-63], our exposition will be brief emphasizing the physical aspects of the results he had obtained.

**The Boltzmann-Nash entropy and the dilaton gravity.** In the case of the Yamabe flow our choice of $- < R(g(t)) >$ as an entropy seemed somewhat artificial in the sense that it was not motivated by any systematic procedure involving entropy as such. We would like to correct this deficiency in this subsection.

**Definition 7.11.** The heat operator $\square := \partial_t - \Delta_g$ has an operator $\square^* = -\partial_t - \Delta_g + R(g)$ as its adjoint with respect to the Ricci flow. E.g. see Ref. [61], page 22.

**Remark 7.12.** The proof is based on the observation that upon some rescaling [5] it is possible to suppress the normalization in the Ricci flow, Eq. (7.8), that is to eliminate the factor $\frac{1}{2} g^{ij} \partial_t g_{ij} > R(g) >$. As result of this elimination, Eq. (7.10) acquires the form

$$\frac{1}{2} g^{ij} \partial_t g_{ij} = -R(g)$$

(7.20)

used in the proof of the conjugacy of operators $\square$ and $\square^*$.

Let low $g_{ij}$ be a solution of the non normalized Ricci flow equation and $u = \exp(-f)$ be the solution of the equation $\square^* u = 0$ for some function $f$ such that $\int_M e^{-f} \, dV = 1$. By analogy with Boltzmann’s entropy, Eq. (7.18), following Ref. [61] we define the Nash entropy $N(u)$ as follows:

$$N(u) = \int_M u \ln u \, dV.$$  

(7.21)

The time derivative $\partial_t N(u)$ can be calculated now with help of the adjoint equation $\square^* u = 0$. Indeed, we obtain:

$$\partial_t N(u) = \int_M (\partial_t u \ln u \, dV + \partial_t u \, dV + u \ln u \, \partial_t dV)$$

$$= \int_M (R(g) u \ln u + \partial_t u \, dV)$$

$$= \int_M (-\Delta_g u \ln u + (R(g) - \Delta_g) u) \, dV.$$  

(7.22a)

On a closed manifold, the integral of $\Delta_g u$ vanishes. Because of this, the last result acquires the form

$$\partial_t N(u) = \int_M (|\nabla_g u|^2 u) + R(g) u \, dV = \int_M (|\nabla_g f| + R(g)) e^{-f} \, dV \equiv F(g_{ij}, f).$$  

(7.22b)

The functional $F(g_{ij}, f)$ is Perelman’s entropy [1] and is also an action for the dilaton gravity [8, 64]. When $f$ is a constant, taking into account the sign differences between the Boltzmann’s entropy, Eq. (7.18), and the Nash entropy,

\[26\text{ Naturally, only small part of Perelman’s results is discussed in this paper.}\]
Eq.(7.21), we obtain back our earlier result for the entropy \(- < R(g(t)) >\) as required.

**Remark 7.13.** The entropic nature of the Einstein-Hilbert and dilaton gravity actions just described might be especially useful for applications to black hole thermodynamics. Indeed, the attempts to use the Ricci flow for the description of black hole dynamics were recently made in Ref.[65]. We plan to return to this issue in subsequent publications.

**Remark 7.14.** If one starts with the functional \(\mathcal{F}(g_{ij}, f)\) without prior remarks on its entropic origin, one might employ the dilaton field \(f\) which is time-independent. This leads to some technical simplifications to be discussed below.

Having defined the entropy for the Ricci flow, the task now lies in obtaining the analog of Eq.(7.17). We would like to do so in a way consistent with our results for the static case.

**Steady Solitons.** In particular, following the logic of Section 7.2., we should begin with the replacement of the two dimensional functional \(\mathcal{F}(\psi)\) by its three dimensional analog, e.g. by \(\bar{S}[\phi]\) defined by Eq.(3.7), or, equivalently, by \(S[\phi]\) defined by Eq.(3.2). To make a connection with Perelman’s work, we need temporarily to make a short but important detour. To this purpose we set \(\phi = e^{-f/2}\) in Eq.(7.22). This causes the functional \(\mathcal{F}(g_{ij}, f)\) to acquire the following new look

\[
\mathcal{F}(g_{ij}, \phi) = \int_M (4|\nabla \phi|^2 + R(g)\phi^2) dV \tag{7.23}
\]

to be compared with the functional \(E[\phi]\) defined in Eq.(3.2). By analogy with Eq.(3.4), now we can define the constant \(\lambda_g\) in terms of the Raleigh quotient, i.e.

\[
\lambda_g = \inf_{\phi} \frac{\mathcal{F}(g_{ij}, \phi)}{\int_M \phi^2 dV} \tag{7.24}
\]

In accord with Eq.(3.3), the constant \(\lambda_g\) serves as an eigenvalue in the equation

\[
4\Delta_g \bar{\phi} + R(g)\bar{\phi} - \lambda_g \bar{\phi} = 0, \tag{7.25}
\]

where \(\bar{\phi}\) is the minimizer for the Raleigh quotient, Eq.(7.24). Equivalently,

\[
\lambda_g = \inf\{\int_M (4|\nabla \phi|^2 + R(g)\phi^2) dV, \int_M \phi^2 dV = 1\}. \tag{7.26}
\]

By analogy with Eq.s (4.7)-(4.10), we introduce the family of metrics \(g_{ij}(s) = g_{ij} + s\ h_{ij}\) in order to consider the Raleigh quotient under such a variation. Then, instead of Eq.(4.10), we obtain

\[
\frac{d}{ds} \lambda(g_{ij}(s)) = \int_M (-h_{ij})(R_{ij} + \nabla_i \nabla_j f)e^{-f} dV. \tag{7.27}
\]

\(\text{27See the second footnote for the sign convention for the Laplacian.}\)

\(\text{28For details of such calculations, please, consult [10] and [66].}\)
Since this result is analogous to that given in Eq.(7.16), the course of action is going to be the same. In particular, following Perelman, and also [10], we notice that the variation of the functional $\mathcal{F}(g_{ij}, f)$ leads to the following coupled equations for the modified Ricci flow:

$$\frac{\partial}{\partial t} g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f), \quad (7.28a)$$

$$\frac{\partial}{\partial t} f = -R + \Delta f. \quad (4.28b)$$

**Definition 7.15.** The flow defined by these coupled equations is called the *generalized Ricci flow*.

Finally, after making the identification $\frac{\partial}{\partial t} g_{ij} = h_{ij}$ and using Eq.(7.28a) in Eq.(7.27) we obtain the monotonicity result of Perelman

$$\frac{d}{ds} \lambda(g_{ij}(s)) = 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} dV \quad (7.29)$$

to be compared with Eq.(7.17). In view of this comparison, we conclude that $\frac{d}{ds} \lambda(g_{ij}(s)) = 0$ only in the case when

$$R_{ij} + \nabla_i \nabla_j f = 0. \quad (7.30)$$

**Definition 7.16.** In the existing terminology Eq.(7.30) is known as the equation for the *gradient steady soliton* [1-3, 66].

Following Perelman [part I] we can extend this result to the case of the so called breathers.

**Definition 7.17.** A metric $g_{ij}(t)$ evolving by the Ricci flow is called a *breather*, if for some $t_1 < t_2$ and $\alpha > 0$ the metric $\alpha g_{ij}(t_1)$ and $g_{ij}(t_2)$ differ only by a diffeomorphism. The cases when $\alpha = 1, \alpha < 1$ and $\alpha > 1$ correspond to *steady, shrinking and expanding* breathers respectively. *Solitons are trivial breathers* for which the above relationship between metrics holds for any pair of $t_1$ and $t_2$.

**Corollary 7.18.** If one considers the Ricci flow as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms, then the breathers and the solitons are respectively the periodic orbits and fixed points for such a system.

**Corollary 7.19.** In view of the result, Eq.(7.29), one can write $\lambda(g_{ij}(s_1)) \leq \lambda(g_{ij}(s_2))$. Because of Eq.(7.30) we obtain as well: $\lambda(g_{ij}(s_1)) = \lambda(g_{ij}(s_2)) \forall s \in [s_1, s_2]$. Hence, Eq.(7.30) providing a minimum for $\lambda(g_{ij}(s))$ is a steady breather and a steady soliton at the same time. That is, there are no breathers for the Ricci flows in accord with Perelman [1].

Going back to Eq.(7.30), multiplying it by $g^{ij}$ and making a summation over repeated indices we obtain,

$$R = \Delta g f, \quad (7.31)$$

$^{29}$Please, see again the sign convention for the Laplacian in the second footnote.
where, as before, we used the fact that \(-\Delta_g f = \nabla^i \nabla_i f\). Taking into account arguments which lead from Eq.(7.22a) to (7.22b) we conclude that for compact manifolds \(f = \text{const}\) and, hence, \(R = 0\).

**Expanding and shrinking solitons.** The obtained result is a special case of more general result by Ivey [67] who extended the work by Hamilton [16] for surface to Ricci flows on compact 3- manifolds\(^{30}\). It is instructive to conclude this subsection by discussing the physical significance of their results, especially in connection with results obtained in the previous section.

To this purpose, let us recall that the fundamental solution of the heat equation in \(d\)-dimensional Euclidean space can be written as \(^{63}\)

\[
\begin{align*}
    u(x, y, \tau) &= (4\pi \tau)^{-\frac{d}{2}} \exp(-|x - y|^2 / 4\tau), \quad (7.32)
\end{align*}
\]

where \(\tau = t - T\) or \(T - t = -\tau \equiv \sigma\) depending upon whether we are dealing with the forward or the backward heat equation\(^{31}\). If we let \(y = 0\), then such a solution can be interpreted as probability\(^{32}\) since by design we have \(\int_M u dV = 1\). Using this fact, one can look for a solution of the heat equation on some Riemannian manifold \(M\) by employing the ansatz

\[
\begin{align*}
    u &= (4\pi \tau)^{-\frac{d}{2}} e^{-f}.
\end{align*}
\]

This amounts to redefining the earlier introduced function \(f = \tilde{f} - \frac{d}{2} \ln(4\pi \tau)\) so that \(e^{-\tilde{f}} = (4\pi \tau)^{-\frac{d}{2}} e^{-f}\). Evidently, \(\nabla f = \nabla \tilde{f}\) and, therefore, \(\Delta_g f = \Delta_g \tilde{f}\) as well. To simplify matters, we would like to consider how things are done in the flat case. Using Eq.(7.32) we begin with the calculation of the Nash entropy. A simple calculation produces:

\[
\begin{align*}
    N_{\text{flat}} &= -\frac{d}{2} - \frac{d}{2} \ln(4\pi \tau). \quad \text{Using this result, we now define the properly normalized Nash entropy, i.e. } N(u) - N_{\text{flat}}. \quad \text{Explicitly, we obtain,}
\end{align*}
\]

\[
\tilde{N}(u) = N(u) - N_{\text{flat}} = \int_M (-f + \frac{d}{2}) u dV. \quad (7.33)
\]

Perelman does not explain how he had obtained his result in his work. He calls such a normalized Nash entropy a "partition function", e.g. see Section 5 of Part I of Ref.[1]. This is a bit misleading however because in the same section he earlier defines the partition function correctly, i.e. in accord with the accepted rules of statistical mechanics. According to these rules one defines the free energy \(F\) via \(\ln \tilde{N}(u) \equiv -\beta F\) with \(\beta\) being subsequently identified with the inverse temperature (provided that the system of units is used in which the Boltzmann’s constant, \(k_B = 1\)). In the present case the role of temperature is played by \(\tau\). Using these definitions, one can define an "energy" \(U = <E >\) in a familiar way as \(U = -\frac{d}{\beta} \ln \tilde{N}(u) \equiv \tau \frac{d}{\beta} \ln \tilde{N}(u)\). In our case, explicitly, we obtain:

\[
\begin{align*}
    U &= \tau^2 \frac{\partial}{\partial \tau} [\int_M u \ln u dV - N_{\text{flat}}] = \int_M \tau^2 \frac{\partial}{\partial \tau} (u \ln u dV) + \frac{d}{2} \tau \int_M u dV. \quad (7.34a)
\end{align*}
\]

\(^{30}\)Later, in Ref.[68], Hamilton came up with yet another proof of Ivey’s results.

\(^{31}\)Here \(T\) is some pre assigned time.

\(^{32}\)The fundamental solution, Eq.(7.32), also has the meaning of probability in polymer physics [69]. This fact is briefly discussed in Section 8.

37
To proceed, we need to employ Eqs.(7.22a) and (7.22b). This leads us to

$$U = \tau^2 \frac{\partial}{\partial \tau} [\int_M u \ln u \, dV - N_{\text{flat}}] = \int_M [\tau^2 (R + (\nabla f)^2) + \frac{d}{2 \tau}] \, u \, dV. \tag{7.34b}$$

This result differs slightly from that obtained by Perelman. If we are interested in calculation of the entropy, such a difference is useful. Indeed, since thermodynamically the relationship

$$\beta U - \beta F = S \tag{7.35}$$

determines the entropy $S$, using already obtained results for $U$ and $F$ we obtain,

$$S_+ = \int_M [\tau (R + (\nabla f)^2) - f + d] \, u \, dV. \tag{7.36}$$

This result coincides with the result for the entropy of the Ricci expanders obtained in Ref.[61] where a considerably more cumbersome and lengthy pathway was chosen in order to obtain it. In order to obtain the entropy for the Ricci shrinkers, it is sufficient to change signs when taking time derivatives. Hence, we obtain at once:

$$S_- = \int_M [\sigma (R + (\nabla f)^2) + f - d] \, u \, dV, \tag{7.37}$$

again in accord with Ref.[61]. At this point, one can proceed either by computing

the heat capacity $C_v = \left( \frac{\partial}{\partial \beta} U \right)_V$ under the constant volume or repeat the

arguments for the steady solitons adapted to the present case. The last pathway is discussed in the paper by Cao, Ref.[66]. More physically attractive, however, is to follow the logic of Perelman and to take into account that

$$C_v = \left( \frac{\partial}{\partial \beta} U \right)_V \frac{\partial}{\partial \tau} \left( \frac{\partial^2}{\partial \beta^2} \ln \tilde{N}(u) \right) \frac{\partial \beta}{\partial \tau},$$

or

$$\tau^2 C_v = \frac{\partial^2}{\partial \beta^2} \ln \tilde{N}(u). \tag{7.38}$$

A straightforward but lengthy calculation analogous to that used in Eq.(7.34) finally leads to

$$\tau^2 C_v = \tau^4 \int_M \left| R_{ij} + \nabla_i \nabla_j f - \frac{1}{2 \tau} g_{ij} \right|^2 \, dV \tag{7.39}$$

in accord with earlier obtained result, Eq.(7.29), for steady soliton (obtained from this expression in the limit $|\tau| \to \infty$). The gradient shrinking (or expanding) solitons are respectively solutions to

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2 \tau} g_{ij} = 0 \tag{7.40a}$$

33Here we took into account that $\int_M udV = 1$ as required.
34As is well known from statistical mechanics,

$$C_v = \beta^2 (< E^2 > - < E >^2),$$

where, as before, we put $k_B = 1$. Hence, what Perelman is simply calling a fluctuation in Section 5 of Part I of his work, Ref.[1], is actually a heat capacity.
or

$$R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\tau} g_{ij} = 0 \quad (7.40b)$$

equations. By analogy with Eq.(7.30) we multiply both of these equations by $g^{ij}$ and sum over repeated indices in order to obtain

$$R - \Delta g f - \frac{d^2}{2\sigma} = 0 \quad (7.41a)$$

and

$$R - \Delta g f + \frac{d^2}{2\tau} = 0 \quad (7.41b)$$

For compact manifolds $\Delta_g f = 0$ as before. As a result, in both cases we obtain back Eq.(4.3a). By combining these results with those which follow from Eq.(7.31) we thus have rederived the following result by Ivey.

**Theorem 7.20.** There are no three-dimensional solitons or breathers on a compact connected 3-manifold other than those of constant curvature metrics.

**Remark 7.21.** This theorem was originally proved by Ivey, Ref.[67], who was inspired by the earlier obtained result by Hamilton [16] for surfaces. Our derivation, however, is inspired by Perelman, Ref.[1], who uses physical arguments. Because these arguments are not presented in sufficient detail in his papers, they were left unappreciated in the works by other mathematicians [2,3,10,61-63].

**Corollary 7.22.** In view of the obtained results it appears that, at least for compact connected 3-manifolds, one can extract the needed physical information for the Yamabe flow from that for the Ricci flow. We refer our readers to works by Hamilton [5,16,70] where many illuminating additional details can be found. The obtained result provide needed support to claims made in Remark 7.6.

**Corollary 7.23.** The generality of arguments used for reproving the Theorem 7.20. are such that they can be applied, in principle, to manifolds of any dimension $d \geq 2$. Because of this, for compact manifolds, the Euclidean dilaton gravity described by the Eq.(7.22b) for any $d \geq 2$ is reduced to the more familiar Euclidean gravity.

8 Discussion

**8.1 Other physical processes whose dynamics is described by Ricci flows**

The goal of this paper is to find some processes taking place in the real world providing a physical justification for Ricci flows. In the previous sections such processes were found, e.g. in the form of critical dynamics. The question arises whether there are other physical processes which also can be used for justifying the existence of Ricci flows. Earlier, in the text, in Ref.[65], we mentioned the dynamics of black holes. In addition, we would like to notice that the fundamental solution, Eq.(7.32), of the heat equation can be interpreted as an end-to-end
distancedistribution function used in polymer physics for the computation of various averages, e.g. the mean square end-to-end distance for the flexible polymer being modelled by a random walk [69]. Under such circumstances, the time $\tau$ is interpreted as polymer’s contour length. The curvature effects can be interpreted in terms of the polymer’s backbone rigidity [71], etc. Upon Fourier-Laplace transform of the distribution function, Eq.(7.3.2), the propagator for the Klein-Gordon quantum field can be obtained [7, 72]. Hence, it is possible to provide an interpretation of the Ricci flow processes in terms of some dynamics of quantum fields, etc. Whatever processes we may discuss, it should be clear that a specific physical process can be used only as some specific realization of the Ricci flow. Since such situation is common for all real life processes described by the equations of mathematical physics, the task lies in finding different physical situations in which this flow can be realized. For instance, since to conduct experiments on black holes [65] is unrealistic, one can think of analogous processes in liquid helium as described in the monograph by Volovik, Ref[73]. A great deal of additional condensed matter analogs of gravity-related phenomena can be found in Refs.[74] and [75] and references therein.

8.2 Interplay between topology, geometry and physics

To make our discussion complete, let us return back to the processes involving critical dynamics studied in this paper. In view of Theorem 7.20., the obtained scalar curvatures either go to zero (for expanding solitons) or blow up (for shrinking solitons) when time $t \to \infty$ [2]. Physically, the case of curvatures blowing up is not acceptable, however, e.g. see Eq.(3.1) and comments related to this equation\[35\]. Hence, physically acceptable manifolds must either be flat or hyperbolic in accord with Eq.(6.30) and Remark 6.8. Although this fact was discussed in our earlier works, [47,55], we would like to provide a sketch of relevant arguments in this subsection.

We begin with two dimensional CFT models at criticality following Ref[21], pages 340-344, and Ref.[47], Section 5, where additional details can be found. Many of these models can be obtained by some straightforward modifications of the simplest Gaussian model defined on the torus $T^2$. If $\omega_1$ and $\omega_2$ are two toral periods, one can define their ratio $\tau$ as $\tau = \frac{\omega_2}{\omega_1}$. The number $\tau$ is necessarily complex with Im$\tau = \left| \frac{1}{\omega_1} \right|^2 \equiv y$. In terms of these notations, the dimensionless free energy $F$ for the Gaussian model is obtained as follows:

$$F = \frac{1}{2} \ln \left| \eta(\tau) \right|^2,$$

(8.1)

where the function $\eta(\tau)$ is the Dedekind eta function. Such a function is known as the modular function for the once punctured torus [76]. Thus, even though

\[35\] It may be of some relevance to black holes, however. This topic requires further study within the context of liquid helium and other type of condensed matter experiments just mentioned.
originally the torus topology for the Gaussian model was used, the actual computations for this model involving the use of the first Kronecker limit formula [47] leads to the final manifestly modular invariant result for $\mathcal{F}$. The price for modular invariance is the switch in topology: from that for a flat torus (before the Kronecker limit is taken) to that for a punctured torus (after the limit is taken). The punctured torus can be obtained from some parallelogram of periods in the complex plane $\mathbb{C}$ whose vertices are removed and whose sides are identified pair-wise. Since Euler's characteristic $\chi$ for torus is 0 while that for the punctured torus is $-1$, the punctured torus represents an example of the hyperbolic surface (actually an orbifold [77]) as discussed in great detail in our work, Ref.[78]. Such a surface is not compact, however, while the results of the previous section were developed for compact surfaces. In two dimensions the situation can be repaired by using the Schottky double construction widely used in designing of CFT models [79]. Use of this construction causes us to replace the punctured torus by the double torus, i.e. by the Riemann surface of genus 2 which is also hyperbolic.

As we had demonstrated in [47] (see also Remark 6.6.) even if one begins with the standard Euclidean cube (or any parallelepiped for that matter whose sides are pair-wise identified thus making it a $T^3$) use of the analog of the first Kronecker limit formula in 3d leads to a result similar to Eq.(8.1) indicating that in complete accordance with 2d case, the limiting manifold/orbifold is hyperbolic. This fact is consistent with results of our earlier work Ref.[55] on AdS-CFT correspondence where totally independent arguments were used to arrive at such a conclusion.

We conclude this work with a brief discussion of how one can actually design compact hyperbolic 3 manifolds. A typical 3d hyperbolic manifold in which a 3d CFT lives and evolves can be built using a 2d punctured torus. To do so, we would like to use the results of our earlier work, Ref.[78], in which we noticed that such a punctured torus is the Seifert surface for the figure eight (hyperbolic) knot. The evolution of surface automorphisms in fictitious time create a 3 manifold. A specific hyperbolic 3 manifold known as a 3-manifold fibered over the circle (the puncture on the torus can be opened up so that it is homeomorphic to a circle $S^1$) as discussed in our work, Ref.[78]. The 3 manifold created in such a way will be cusped [55] and, hence, noncompact. To make such a manifold compact one can also use a hyperbolic double (analogous to a Schottky double as described in detail in Ref.[80]). Obviously, the same is achieved if one starts with a Schottky double torus and fibers it over the circle [80]. The direct 3d analog of the two dimensional punctured torus can be constructed as a product of the trice punctured sphere and an open interval (0,1) [80]. The obtained manifold is homeomorphic to a cube whose upper and lower faces along with all edges are removed. Such a designed 3 manifold is hyperbolic but unfortunately non compact. Clearly, Perelman’s work is not limited to compact manifolds and, hence, his results still can be used. The only trouble with noncompact-

\[\text{\footnotesize{\textsuperscript{36}}This restriction is not essential however. It is possible to extend the theory of \textsuperscript{Ricci and Yamabe flows to the non compact surfaces}}\textsuperscript{1,2}.\textsuperscript{\textsuperscript{\textsuperscript{Such an extension requires use of more sophisticated mathematical methods than those used in our paper.}}}\]
ness lies in the fact that the Yamabe flow considered in our work leads to the Einstein-type spaces so that compactness is synonymous with being of Einstein type. We mentioned this already Ref.s [26,27] in which Yamabe functional for manifolds with boundary were discussed. More relevant to physical applications is the work by Mazzeo et al [81] and, also Ref.[82], in which the Yamabe problem was studied for noncompact manifolds. It would be very interesting and challenging to study the Yamabe flow for noncompact manifolds discussed in Refs[81,82]. Many details of the construction of 3-manifolds and orbifolds are discussed in the exceptionally well written Ref.[77]. We conclude our paper by urging our readers to read this reference, which is also helpful for developing a solid understanding of the Poincare’ and geometrization conjectures.

**Note added in proof.** After this work was completed several important papers supporting and clarifying the results of our Section 8.2 came to our attention. In particular, in the paper by Long and Reid (Algebraic and Geometric Topology 2 (2002) 285-296) it is shown that all flat manifolds can be looked upon as cusps of hyperbolic orbifolds, e.g. see our works, Ref.s[55],[78] for a quick introduction into cusps and Ref.[77] for introduction to hyperbolic orbifolds. Many additional details elaborating on the work by Long and Reid can be found in the PhD thesis by David Ben McReinolds (arxiv:math.GT/0606571). Some related material can also be found in the paper by X.Dai (arxiv:math.DG/0106172).
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