Optimal Verification of Two-Qubit Pure States

Kun Wang\textsuperscript{1,2} and Masahito Hayashi\textsuperscript{3,2,4,*}

\textsuperscript{1}Department of Computer Science and Technology, State Key Laboratory for Novel Software Technology, Nanjing University, Nanjing, 210093, China
\textsuperscript{2}Shenzhen Institute for Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen, 518055, China
\textsuperscript{3}Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan
\textsuperscript{4}Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117542, Singapore

(Dated: March 12, 2022)

In a recent work [Phys. Rev. Lett. 120, 170502 (2018)], Pallister et al. proposed an optimal strategy to verify non-maximally entangled two-qubit pure states under the constraint that the accessible measurements being locally projective and non-adaptive. Their nice result leads naturally to the question: What is the optimal strategy among general LOCC measurements? In this Letter, we answer this problem completely for two-qubit pure states. To be specific, we give the optimal strategy for each of the following available classes of measurements: (i) local operations and one-way classical communication (one-way LOCC) measurements; (ii) local operations and two-way classical communication (two-way LOCC) measurements; and (iii) separable measurements. Surprisingly, our results reveal that for the two-qubit pure state verification problem, two-way LOCC measurements remarkably outperforms one-way LOCC measurements and has the same power as the separable measurements.

Introduction.—On the way to quantum era, quantum devices for generating particular states have been extensively studied and widely used [1–4]. As so, it becomes necessary to verify that these devices truly work as they are expected reliably and efficiently with measurements that are accessible. A standard approach is to estimate the output states with quantum state tomography [5–12]. However, this method is both time-consuming and computationally difficult, even verifying a few-qubit photonic state is already experimentally challenging [13, 14]. Various non-tomographic approaches have been designed for this task [15–22], using only local measurements. Though these methods achieve considerable efficiency, no optimal method except for the maximally entangled state [23–26] is known so far.

Given the intrinsic difficulty in state verification, in this Letter we focus on verifying the non-maximally entangled two-qubit pure states, in hopes of gaining deeper understanding on the verification problem. These states find wide applications in quantum information theory [27, Section 3.3], making their verification important both from the theoretical and the experimental points of view. Specially, we construct optimal strategies when different classes of measurements are available: one-way LOCC measurements, two-way LOCC measurements, and separable measurements. We find that for the problem under consideration, two-way LOCC measurements achieves the same performance as that of separable measurements, while outperforms one-way LOCC measurements dramatically.

Before presenting the results, we review the notations. We denote by $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle \equiv (|0\rangle - |1\rangle)/\sqrt{2}$ the eigenstates of the Pauli $X$ operator, by $|T\rangle \equiv (|0\rangle + i|1\rangle)/\sqrt{2}$ and $|\perp\rangle \equiv (|0\rangle - i|1\rangle)/\sqrt{2}$ the eigenstates of the Pauli $Y$ operator. When measuring a qubit with a Pauli operator, the outcome is written as $(-1)^i$ where $i \in \{0,1\}$. We denote by $\mathcal{H}$ the two-qubit composite system and by $\mathbb{I}$ the identity operator on $\mathcal{H}$. We say a positive operator $T$ with $0 \leq T \leq \mathbb{I}$ a one-way LOCC (local operations and only one-way classical communication) POVM element on $\mathcal{H}$ if the two-outcome POVM $\{T, \mathbb{I} - T\}$ can be implemented by one-way LOCC. We also define a two-way LOCC (local operations and two-way classical communication) POVM element and a separable POVM element in the same way by using the two-way LOCC and the separable operations, respectively. Interested readers might refer to [28] for details on these operations. We write the set of one-way LOCC from Alice to Bob, one-way LOCC from Bob to Alice, two-way LOCC, and separable POVM elements as $\mathcal{T}_c$, $\mathcal{T}_{nc}$, $\mathcal{T}_s$, and $\mathcal{T}_{sep}$. These classes satisfy the relation $\mathcal{T}_c \subseteq \mathcal{T}_{nc} \subseteq \mathcal{T}_s \subseteq \mathcal{T}_{sep}$. The condition $T \in \mathcal{T}_c$ is equivalent to the condition $\mathbb{I} - T \in \mathcal{T}_s$, where $c \in \{\rightarrow, \leftarrow, \leftrightarrow, \text{sep}\}$. For a positive operator $\Omega$ on $\mathcal{H}$, $\lambda_i(\Omega)$ denotes the $i$-th eigenvalue of $\Omega$ and $\lambda^i_1(\Omega)$ denotes the $i$-th largest eigenvalue of $\Omega$, where $i = 1, 2, 3, 4$.

Two-qubit pure state verification.—Consider a quantum device that is designed to produce the two-qubit pure state

$$|\Psi\rangle = \sqrt{1 - \lambda} |00\rangle + \sqrt{\lambda} |11\rangle,$$

where $\lambda \in [0, 1/2]$. However, it might work incorrectly and actually outputs states $\sigma_1, \sigma_2, \ldots, \sigma_N$ in $N$ runs. It is promised the fidelity $\langle \Psi | \sigma_j | \Psi \rangle$ is either 1 or satisfies $\langle \Psi | \sigma_j | \Psi \rangle \leq 1 - \epsilon$ for all $j$ for some $\epsilon > 0$. The task is to determine which is the case. The conclusion is useful if we assume the next state $\sigma_{N+1}$ has the same behavior as the previous ones.

To achieve this task, we perform two-outcome measurements from a set of accessible measurements to test
the state. Each two-outcome measurement \( \{ T_i, 1 - T_i \} \) is specified by an operator \( T_i \), which corresponds to passing the test, and is performed with probability \( \rho_i \). We require that the target state \( | \Psi \rangle \) always passes the test, that is, \( T_i | \Psi \rangle = | \Psi \rangle \) for all \( T_i \). In the bad case, the maximal probability that \( \sigma_j \) passes the test is given by [22, 29]
\[
\max_{\langle \sigma_j | \sigma_j \rangle \leq 1 - \epsilon} \text{Tr} ( \Omega \sigma_j ) = 1 - \left[ 1 - \lambda_j^2 ( \Omega ) \right] \epsilon,
\]
where \( \Omega = \sum_i \rho_i T_i \) is called an strategy. After \( N \) runs, \( \sigma_j \) in the bad case can pass all tests with probability at most \( 1 - \left[ 1 - \lambda_j^2 ( \Omega ) \right] \epsilon \). Hence to achieve confidence \( 1 - \delta \), it suffices to take [29]
\[
N \geq \frac{\ln \delta}{\ln \left[ 1 - \lambda_j^2 ( \Omega ) \right] \epsilon} \approx \frac{1}{\lambda_j^2 ( \Omega ) \epsilon} \ln \frac{1}{\delta}. \tag{2}
\]

The optimal strategy is obtained by minimizing the second largest eigenvalue \( \lambda_j^2 ( \Omega ) \). If there is no restriction on the accessible measurements, the optimal strategy is given by the measurement \( \{ | \Psi \rangle \langle \Psi |, 1 - | \Psi \rangle \langle \Psi | \} \), under which \( \Omega = | \Psi \rangle \langle \Psi |, \lambda_j^2 ( \Omega ) = 0 \), and \( N \approx \epsilon^{-1} \ln^{-1} \delta \). This efficiency cannot be improved if collective measurements are allowed [22]. However, it is difficult to perform such measurements experimentally when \( | \Psi \rangle \) is entangled. It is thus meaningful to devise efficient (or even optimal) strategies based on measurements satisfying reasonable constraints. Owari and Hayashi [30] studied the case where the incorrect states are the maximally mixed state, with the target to minimize the trace of \( \Omega \). They derived optimal strategies when one-way LOCC and separable measurements are available, and showed that two-way LOCC measurements remarkably improves the performance compared to one-way LOCC measurements. Recently, Pallister, Linden, and Montanaro [29] proposed an optimal strategy \( \Omega_{\text{PLM}} \) to verify \( | \Psi \rangle \), under the constraint that the accessible measurements must be locally projective and non-adaptive. The strategy \( \Omega_{\text{PLM}} \) [31] has the second largest eigenvalue
\[
\lambda_2^2 ( \Omega_{\text{PLM}} ) = \frac{2 + 2 \sqrt{\lambda (1 - \lambda)}}{4 + 2 \sqrt{\lambda (1 - \lambda)}}. \tag{3}
\]

Note that the set of accessible measurements in [29] forms a strict subset of \( T_\rightarrow \).

These interesting results lead to the question: What is the optimal strategy when general LOCC measurements, adaptive choices of local measurements are available? In this paper, we investigate this problem comprehensively. We derive optimal strategies for verifying \( | \Psi \rangle \) when the following different classes of measurements are available: \( T_\rightarrow, T_{\leftarrow}, T_{\leftrightarrow}, \text{ and } T_{\text{exp}} \). In the following, we say a strategy \( \Omega \) is in \( T_{ \rightarrow } \) and written \( \Omega \in T_{ \rightarrow } \) if \( \Omega = \sum_i \rho_i T_i \) and \( T_i \in T_{ \rightarrow } \) for all \( i \), a strategy \( \Omega \) is optimal in \( T_{ \rightarrow } \) if \( \Omega \in T_{ \rightarrow } \) and for arbitrary \( \Omega' \in T_{ \leftrightarrow } \) satisfying \( \Omega' | \Psi \rangle = | \Psi \rangle \), \( \lambda_2^2 ( \Omega') \leq \lambda_2^2 ( \Omega ) \).

Here we discuss some general properties of arbitrary strategy \( \Omega \). Consider the product of local unitaries \( U_\theta \otimes U_{- \theta} \), where \( U_\theta = | \theta \rangle \langle \theta | + e^{i \theta} | 1 \rangle \langle 1 \rangle \) and \( \theta \in [0, 2 \pi] \). Let \( | \Psi^\perp \rangle \mapsto \sqrt{\lambda (00) - \sqrt{1 - \lambda (11)} \rangle \rangle, \) then \( \{| \Psi \rangle, | \Psi^\perp \rangle, |01 \rangle, |10 \rangle \} \) are the four eigenstates of \( U_\theta \otimes U_{- \theta} \). Using this property we can simplify the form of \( \Omega \) by averaging, where the averaged strategy is defined as
\[
\Omega_\alpha := \frac{1}{2 \pi} \int_0^{2 \pi} ( U_\theta \otimes U_{- \theta} ) | \Psi \rangle \langle \Psi | ( U_\theta \otimes U_{- \theta} )^\dagger \ d \theta.
\]

Since the second largest eigenvalues of \( \Omega_\alpha \) and \( \Omega \) are the matrix norms of \( P^\perp \Omega P^\perp \) and \( P^\perp \Omega P \) with \( P^\perp := I - | \Psi \rangle \langle \Psi | \), we have \( \lambda_2^2 ( \Omega_\alpha ) \leq \lambda_2^2 ( \Omega ) \). That is, averaging over \( \theta \) cannot make the strategy worse. As \( I \geq \Omega_\alpha \geq | \Psi \rangle \langle \Psi | \) and the vectors \{01\} and \{10\} of \( U_\theta \otimes U_{- \theta} \) have different eigenvalues from that of \( | \Psi \rangle \) and \( | \Psi^\perp \rangle \), after averaging \( \Omega_\alpha \) can be expressed as
\[
\Omega_\alpha = | \Psi \rangle \langle \Psi | + \lambda_2^2 | \Psi^\perp \rangle \langle \Psi^\perp | + \lambda_3^2 (01) \langle 01 | + \lambda_4^2 (10) \langle 10 | \tag{4}
\]

for some \( \lambda_2, \lambda_3, \lambda_4 \in [0, 1) \). We also consider the fact that \( | \Psi \rangle \) is invariant under qubits swapping. Using the swapping operation \( s \) for the roles of Alice and Bob, the resulting strategy \( \Omega_{\alpha s} := \frac{1}{2} \Omega_\alpha + \frac{1}{2} s ( \Omega_\alpha ) \) has performance at least as good as that of \( \Omega_\alpha \) and admits the form
\[
\Omega_{\alpha s} = | \Psi \rangle \langle \Psi | + \lambda_2^2 | \Psi^\perp \rangle \langle \Psi^\perp | + \lambda_3^2 (01) \langle 01 | + \lambda_4^2 (10) \langle 10 | \tag{5}
\]

for some \( \lambda_2, \lambda_3 \in [0, 1) \). We should be careful when using the swapping invariance property, as to implement the strategy \( \Omega_{\alpha s} \), an extra step of messaging is required.

The above discussed framework is non-adversarial in the sense that the malicious device produces incorrect states randomly and independently. One may also consider the adversarial scenario where the malicious device may produce an arbitrary state \( \rho \) on the whole system \( \mathcal{H}^{N+1} \) [22]. The task is then to ensure that the reduced state on one system has fidelity larger than \( 1 - \epsilon \) by performing \( N \) tests on other systems. We remark that minimizing the second largest eigenvalue leads the optimization of the strategy even in this scenario.

**Strategy using one-way LOCC measurements.**—First we propose a strategy in \( T_{ \rightarrow } \). Then we show it is optimal when only \( T_{ \rightarrow } \) are available.

Let \( | v_+ \rangle = \sqrt{1 - \lambda} | 0 \rangle \pm \sqrt{\lambda} | 1 \rangle \). Alice performs the X measurement on the target state and sends outcome \( \pm \) to Bob. If \( i = 0 \), Bob performs measurement \( \{ | v_+ \rangle \langle v_+ |, 1 - | v_+ \rangle \langle v_+ | \} \) and accepts if the outcome is \( v_+ \). If \( i = 1 \), Bob performs measurement \( \{ | v_- \rangle \langle v_- |, 1 - | v_- \rangle \langle v_- | \} \) and accepts if the outcome is \( v_- \). The corresponding POVM element \( T_1 \) (passing the test) has the form
\[
T_1 = | + \rangle \langle + | \otimes | v_+ \rangle \langle v_+ | + | - \rangle \langle - | \otimes | v_- \rangle \langle v_- |.
\]

We define other two POVM elements \( T_2 \) and \( T_3 \) similarly to \( T_1 \) but with the X measurement replaced by the Y and...
Z measurements on Alice’s side, respectively. These two elements read

$$
T_2 = |T\rangle\langle T| \otimes |w_\perp\rangle\langle w_\perp| + |\perp\rangle\langle \perp| \otimes |w_\perp\rangle\langle w_\perp|,
$$

$$
T_3 = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|,
$$

where $|w_\perp\rangle = \sqrt{1-\lambda}|0\rangle + i\sqrt{\lambda}|1\rangle$. It holds that $T_j|\Psi\rangle = |\Psi\rangle$ and $T_j \in T_\rightarrow$ for $j = 1, 2, 3$.

The one-way strategy goes as follows. In each round, Alice chooses a measurement from $\{T_1, T_2, T_3\}$ with a prior probability $\{\frac{1}{2}, \frac{1}{2}, p\}$ to test the state, where $p \in [0, 1]$ is a free parameter. The strategy has the form

$$
\Omega_\rightarrow = \frac{1-p}{2}T_1 + \frac{1-p}{2}T_2 + pT_3
= |\Psi\rangle\langle \Psi| + p|\Psi^\perp\rangle\langle \Psi^\perp| + (1-p)|0\rangle\langle 0| + (1-p)(1-\lambda)|10\rangle\langle 10|,
$$

Minimizing $\lambda_2^+|\Omega_\rightarrow\rangle$ w.r.t. $p \in [0, 1]$, we get $p = \frac{1-\lambda}{2\lambda}$ and $\Omega_\rightarrow = |\Psi\rangle\langle \Psi| + \lambda_2^+|\Psi^\perp\rangle\langle \Psi^\perp| + \lambda_3^+|0\rangle\langle 0| + \lambda_2^+|10\rangle\langle 10|$, where $\lambda_2^+ = \frac{1-\lambda}{2\lambda}$ and $\lambda_3^+ = \frac{\lambda}{2\lambda}$. Obviously, $\Omega_\rightarrow \in T_\rightarrow$.

Now we show the optimality of $\Omega_\rightarrow$. Let $[t, s] := \sqrt{t}|0\rangle + e^{i\theta}\sqrt{1-t}|1\rangle$, where $t \in [0, 1]$ and $s \in [0, 2\pi]$. When a one-way LOCC strategy $\Omega$ detects $|\Psi\rangle$ with certainty, the strategy is composed of Alice’s POVM $\int f(t, s)\{t, s|P_{T_2}(dtds)$ with some probability distribution $P_{T_2}$ and Bob’s two-outcome measurements $\{t, s, B\}$, where $|t, s, B\rangle = \tilde{\lambda}^{-1}\sqrt{t(1-\lambda)}|0\rangle + e^{-i\theta}\sqrt{1-t}\lambda|1\rangle$. Then, the strategy $\Omega$ is written as

$$
\Omega = 2 \int |t, s\rangle\langle t, s| \otimes |t, s, B\rangle\langle t, s, B|P_{T_2}(dtds).
$$

Following the averaging argument in Eq. (4), we get $\Omega_\alpha$, obtained from $\Omega$. For the analysis of $\Omega_\alpha$, we treat the variable $t$ in Eq. (7) as the random variable $T$ subject to $P_{T_2}$, and focus on the expectation $E_T$ under the marginal distribution $P_T$. To guarantee that Alice’s measurement in $\Omega_\alpha$ is a POVM, $E_T[T] = \frac{3}{2}$ needs to hold. In the Supplemental Material [32] we show that $\Omega_\alpha$ satisfies Eq. (4) with

$$
\lambda_2(\Omega_\alpha) = 1 - \Xi, \lambda_3(\Omega_\alpha) = \lambda_3, \lambda_4(\Omega_\alpha) = \Xi(1-\lambda),
$$

where $\Xi := 2E_T\left[\frac{T(1-T)}{T+\lambda-2\lambda T}\right] \geq 0$. As $\lambda_3(\Omega_\alpha) \leq \lambda_4(\Omega_\alpha), \lambda_2(\Omega_\alpha)$ is minimized when $\lambda_3(\Omega_\alpha) = \lambda_4(\Omega_\alpha)$. Solving the equation, we get $\Xi = \frac{1}{2\lambda}$ and $\lambda_2^+(\Omega_\alpha) = \frac{1}{2\lambda}$. This concludes the optimality of $\Omega_\rightarrow$.

Switching the role between Alice and Bob, we get a symmetric version $\Omega_\rightarrow$ of $\Omega_\rightarrow$. Consider the new strategy $\Omega_\rightarrow_\alpha = (\Omega_\rightarrow + \Omega_\rightarrow)/2$. Minimizing the second largest eigenvalue of $\Omega_\rightarrow_\alpha$ w.r.t. $p$ gives

$$
\hat{\Omega}_\rightarrow_\alpha = |\Psi\rangle\langle \Psi| + \frac{1}{3}(\mathbb{1} - |\Psi\rangle\langle \Psi|).
$$

This two-way two-step strategy outperforms $\Omega_\rightarrow$ in the small regime of $\lambda$. More details on $\Omega_\rightarrow_\alpha$ can be found in the Supplemental Material [32].

**Strategy using two-way LOCC measurements.**—First we describe two measurements both detecting $|\Psi\rangle$ correctly. They are inspired by the two-way LOCC test given in [30]. Then we show an appropriate convex combination of these measurements achieves optimality even if separable measurements are available. In what follows, we assume $\delta = 1 - \sqrt{\frac{1}{1+\sqrt{1-\lambda}}}$ and $p = \frac{\lambda}{1+\sqrt{1-\lambda}}$.

Consider the following measurement procedure:

1. Alice performs measurement $\{M_0 \equiv \delta|0\rangle\langle 0|, M_1 \equiv (1-\delta)|0\rangle\langle 0| + |1\rangle\langle 1|\}$ and sends the measurement outcome $M_1$ to Bob.

2. Conditioning on $i$, Bob does the following. If $i = 0$, Bob performs $Z$ measurement and accepts when the outcome is 0. If $i = 1$, Bob performs $X$ measurement and sends outcome $j \in \{0, 1\}$ to Alice.

3. Conditioning on $j$, Alice performs measurement $\{\sigma_j, \mathbb{1} - \sigma_j\}$ to check the state she holds, where $\sigma_j$ is the post-measurement state on Alice’s system when the input state is $\delta|0\rangle\langle 0| + (1-\delta)|0\rangle\langle 0| + |1\rangle\langle 1|$. If she detects $\sigma_j$, she accepts.

The corresponding POVM element $T_i^{A\rightarrow B}$ (passing the test) has the form

$$
T_i^{A\rightarrow B} = |\delta\rangle\langle 0| \otimes |0\rangle\langle 0| + |\delta\rangle\langle 0| \otimes |\delta\rangle\langle 0| + |1\rangle\langle 1|, \quad i = 0, 1
$$

where $|\delta\rangle = \sqrt{1-\lambda}|0\rangle + \sqrt{\lambda}|1\rangle$, $A = \frac{(1-\lambda)(1-\delta)}{1-\delta+\lambda\delta}$, and $B = \frac{1}{1+\sqrt{1-\lambda}}$. Note that $|\delta\rangle$ are not normalized. See Fig. 1 for illustration of this measurement. Note that $T_i^{A\rightarrow B}(|\Psi\rangle) = |\Psi\rangle$ and $T_i^{A\rightarrow B} \in T_\rightarrow$. The superscript $A \rightarrow B$ of $T_i$ indicates that $T_i$ begins with Alice sending outcome to Bob. A symmetric element $T_i^{B\rightarrow A}$ is obtained by switching the role between Alice and Bob. We define another POVM element $T_2^{A\rightarrow B}$ analogous to $T_i^{A\rightarrow B}$ but

![FIG. 1. The two-way measurement $T_i^{A\rightarrow B}$, $i = 1, 2, 3$. Alice first performs measurement $\{\delta|0\rangle\langle 0|, (1-\delta)|0\rangle\langle 0| + |1\rangle\langle 1|\}$ and sends outcome to Bob. Conditioned on the outcome, Bob adopts different measurements on his post-measurement state and sends outcome to Alice if necessary. Alice then performs the corresponding two-outcome measurement to detect the final state she holds.](image-url)
with the X measurement replaced by the Y measurement on Bob’s side, which reads
\[
T_2^{A\rightarrow B} = \delta |0\rangle \langle 0| \otimes |0\rangle \langle 0| + |\tilde{\omega}_-\rangle \langle \tilde{\omega}_-| \otimes |T\rangle \langle T| + |\tilde{\omega}_+\rangle \langle \tilde{\omega}_+| \otimes |\perp\rangle \langle \perp|,
\]
where $|\tilde{\omega}_\pm\rangle = \sqrt{(1 - \delta)}A|0\rangle \pm i\sqrt{\delta}B|1\rangle$. By construction, $T_2^{\rightarrow B}|\Psi\rangle = |\Psi\rangle$ and $T_2^{\rightarrow B} \in {\mathcal T}_s$.

Our two-way strategy is given by the following procedure. In each round, Alice chooses a measurement from $\{T_1^{A\rightarrow B}, T_2^{A\rightarrow B}, T_B^{B\rightarrow A}, T_B^{B\rightarrow A}\}$ with a prior distribution $\{1 - p, \frac{1}{2}p, \frac{1}{2}p, p\}$ to verify the state, where $T_3$ is defined in Eq. (6). If $T_1^{A\rightarrow B}$ is chosen, Alice executes the measurement; if $T_B^{B\rightarrow A}$ is chosen, Alice sends notification to Bob to ask Bob executing the measurement. The corresponding strategy is given by
\[
\Omega_{\leftrightarrow} = \frac{p - 1}{4} (T_1^{A\rightarrow B} + T_2^{A\rightarrow B} + T_B^{B\rightarrow A} + T_B^{B\rightarrow A}) + p T_3 = |\Psi\rangle \langle \Psi| + \lambda^* (1 - |\Psi\rangle \langle \Psi|),
\]
where
\[
\lambda^* = \frac{\sqrt{\lambda(1 - \lambda)}}{1 + \sqrt{\lambda(1 - \lambda)}}.
\]

By construction, $\Omega_{\leftrightarrow} |\Psi\rangle = |\Psi\rangle$ and $\Omega_{\leftrightarrow} \in {\mathcal T}_s$. In the Supplemental Material [32], we show how the magic values of $\delta$ and $p$ are chosen.

Our strategy $\Omega_{\leftrightarrow}$ can be implemented by two-way LOCC, using up to three step classical communication. This makes it possible for experimental implementation. When $\lambda = 0$, $|\Psi\rangle = |00\rangle$, whose optimal strategy is provably given by the measurement $\{|00\rangle \langle 00|, \perp - |00\rangle \langle 00|\}$. Our strategy $\Omega_{\leftrightarrow}$ reduces exactly to this optimal measurement when $\lambda = 0$, which means our two-way strategy is globally optimal for $|00\rangle$. However, all other strategies $- \Omega_{\text{PLM}}, \Omega_{\rightarrow}, \Omega_{\leftrightarrow}$, and $\Omega_{\rightarrow} - \delta$ do not share this property.

Optimality of our two-way strategy.—We show the optimality of $\Omega_{\leftrightarrow}$ among strategies using separable measurements. Since a two-way LOCC measurement is a separable measurement, this optimality also shows the optimality using two-way LOCC measurements. The proof is divided into two parts: first we prove all optimal strategies in $\mathcal{T}_{\text{sep}}$ are homogeneous, then we construct explicitly an optimal homogeneous strategy in $\mathcal{T}_{\text{sep}}$.

A strategy $\Omega$ for $|\Psi\rangle$ is homogeneous if it has the form
\[
\Omega = |\Psi\rangle \langle \Psi| + \delta (1 - |\Psi\rangle \langle \Psi|),
\]
where $\delta \in [0, 1]$. As examples, the strategies $\hat{\Omega}_{\rightarrow}$ and $\hat{\Omega}_{\leftrightarrow}$ are homogeneous. Now we prove that the optimal strategies using separable measurements are always homogeneous. Following the arguments in Eq. (5), we know optimal strategies $\Omega$ in $\mathcal{T}_{\text{sep}}$ can always be written as Eq. (5) for some $\lambda_2, \lambda_3 \in [0, 1]$. Assume on the contrary $\lambda_2 \neq \lambda_3$, we then construct homogeneous strategies with smaller second largest eigenvalues than that of $\Omega$, which in turn violates the optimality of $\Omega$. In [30, Theorem 1], the authors proposed a separable test of the form
\[
T_4 = |\Psi\rangle \langle \Psi| + \sqrt{\lambda(1 - \lambda)} (|01\rangle \langle 01| + |10\rangle \langle 10|).
\]
In case $\lambda_2 > \lambda_3$, we consider a convex combination between $\Omega$ and $T_4$ such that the combination is homogeneous. The new strategy has a smaller second largest eigenvalue than that of $\Omega$. We can show the opposite case in the same way using $T_3$ defined in Eq. (6) instead of $T_4$.

Now we derive an optimal homogeneous strategy in $\mathcal{T}_{\text{sep}}$. We are actually interested in the following optimization problem:
\[
\min \delta
\]
\[\text{s.t. } 0 \leq \delta \leq 1, \Omega = |\Psi\rangle \langle \Psi| + \delta (1 - |\Psi\rangle \langle \Psi|), \Omega \in \mathcal{T}_{\text{sep}}.
\]
As the separability condition is equivalent to the PPT (positive partial transpose) condition for two-qubit operators [33, 34], this problem can be analytically solved. Denote by $\Omega_T^{Bn}$ the partial transpose of $\Omega$ on system $B$. The eigenvalues of $\Omega_T^{Bn}$ are
\[
\lambda_1 = 1 - \lambda + \lambda \delta, \quad \lambda_2 = \lambda + \delta - \lambda \delta,
\]
\[
\lambda_3 = \delta + (1 - \delta) \sqrt{\lambda(1 - \lambda)}, \quad \lambda_4 = \delta - (1 - \delta) \sqrt{\lambda(1 - \lambda)}.
\]
As $\lambda_1, \lambda_2, \lambda_3 \geq 0$ for $\lambda \in [0, 1/2]$ and $\delta \in [0, 1]$, the condition $\Lambda T^{Bn} \geq 0$ is then equivalent to $\lambda_4 \geq 0$, resulting
\[
\delta \geq \delta^* := \frac{\sqrt{\lambda(1 - \lambda)}}{1 + \sqrt{\lambda(1 - \lambda)}}.
\]
The optimal homogeneous strategy then has the form
\[
\Omega_{\text{sep}} = |\Psi\rangle \langle \Psi| + \delta^* (1 - |\Psi\rangle \langle \Psi|).
\]
Together with the fact that optimal strategies in $\mathcal{T}_{\text{sep}}$ are always homogeneous, we completely solve the problem of verifying $|\Psi\rangle$ using separable measurements. What’s more, as $\lambda_2^2(\Omega_{\leftrightarrow}) = \lambda_2^2(\Omega_{\text{sep}})$, the optimality can be achieved by two-way LOCC measurements.

Comparison.—In Fig. 2 we plot the second largest eigenvalues for various strategies: $\Omega_{\rightarrow}$, $\Omega_{\leftrightarrow}$, $\Omega_{\leftrightarrow}$, and $\Omega_{\text{PLM}}$, as a function of $\lambda$, which is the Schmidt coefficient of state $|\Psi\rangle$. One can see that our proposed strategies give remarkable improvements over $\Omega_{\text{PLM}}$, this witnesses the advantage of adaptivity in state verification: allowing conditional measurements can markedly improve the verification efficiency. Intuitively, one might expect that the more entangled the $|\Psi\rangle$, the harder to verify it using local measurements. The two-way strategies $\Omega_{\rightarrow}$ and $\Omega_{\leftrightarrow}$ justify this intuition. However, the one-way strategy $\Omega_{\rightarrow}$, though achieves optimality when $|\Psi\rangle$ is maximally entangled, has inefficient performance in small regime of $\lambda$, where $|\Psi\rangle$ is less entangled. This due to that in the
one-way case, the symmetric role between Alice and Bob cannot be utilized. The strict gaps among \(\Omega_{→}, \Omega_{↔},\) and \(\Omega_{→}^{*}\) reveal the power of classical communication in state verification: with just an extra step of messaging, one can significantly boost the performance.

**Conclusion.**—In this Letter, we studied the two-qubit pure state verification problem in depth. We constructed optimal strategies when the following classes of measurement are accessible: (i) one-way LOCC; (ii) two-way LOCC; and (iii) separable measurements. Our proposed strategies are dramatically more efficient than all known candidates based on local measurements and are comparable to the optimal strategy when there is no restriction on the accessible measurement at all. Our results revealed that for this problem, the two-way LOCC measurements remarkably outperforms the one-way LOCC measurements and achieves the same performance as the separable measurements. In principle, the technique used here to construct strategies for verifying two-qubit pure states can be generalized to pure states of more qubits and higher dimensions. However, it might be rather difficult to get the optimal strategies.

MH is grateful to Dr. Xiao-Dong Yu and Prof. Zhu Huangjun for helpful discussions. KW was supported by the National Natural Science Foundation of China (Grant No. 61300050) and the Program B for Outstanding PhD Candidate of Nanjing University (Grant No. 201801B047). MH was supported in part by Fund for the Promotion of Joint International Research (Fostering Joint International Research) Grant No. 15KK0007, a JSPS Grant-in-Aids for Scientific Research (B) No.16KT0017 and for Scientific Research (A) No.17H01280, and Kayamori Foundation of Information Science Advancement.

---

masahito@math.nagoya-u.ac.jp

[1] T. Monz, P. Schindler, J. T. Barreiro, M. Chwalla, D. Nigg, W. A. Coish, M. Harlander, W. Hänsel, M. Hennrich, and R. Blatt, Phys. Rev. Lett. **106**, 130506 (2011).
[2] X. Wang, L. Chen, W. Li, H.-L. Huang, C. Liu, C. Chen, Y. Luo, Z. Su, D. Wu, Z. Li, et al., Phys. Rev. Lett. **117**, 210502 (2016).
[3] C. Song, K. Xu, W. Liu, C. Yang, S. Zheng, H. Deng, Q. Xie, K. Huang, Q. Guo, L. Zhang, et al., Phys. Rev. Lett. **119**, 180511 (2017).
[4] N. Friis, O. Marty, C. Maier, C. Hempel, M. Holzäpfel, P. Jurcevic, M. B. Plenio, M. Huber, C. Roos, R. Blatt, et al., Phys. Rev. X **8**, 021012 (2018).
[5] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press New York, 1976).
[6] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[7] M. Hayashi, “A linear programming approach to attainable Cramér-Rao type bounds,” in *Quantum Communication, Computing, and Measurement* (Springer, 1997) pp. 99–108.
[8] R. D. Gill and S. Massar, Phys. Rev. A **61**, 042312 (2000).
[9] D. Gross, Y.-K. Liu, S. T. Flammia, S. Becker, and J. Eisert, Phys. Rev. Lett. **105**, 150401 (2010).
[10] T. Sugiyama, P. S. Turner, and M. Murao, Phys. Rev. Lett. **111**, 160406 (2013).
[11] R. O’Donnell and J. Wright, in *Proceedings of the 48th Annual ACM Symposium on Theory of Computing* (ACM, 2016) pp. 899–912.
[12] J. Haah, A. W. Harrow, Z. Ji, X. Wu, and N. Yu, IEEE Trans. Inf. Theory **63**, 5628 (2017).
[13] H. Häffner, W. Hänsel, C. Roos, J. Benhelm, M. Chwalla, T. Körber, U. Rapol, M. Riebe, P. Schmitt, C. Becher, et al., Nature **438**, 643 (2005).
[14] J. Carolan, J. D. Meinecke, P. J. Shadbolt, N. J. Russell, N. Ismail, K. Wörhoff, T. Rudolph, M. G. Thompson, J. L. O’Brien, J. C. Matthews, et al., Nat. Photonics **8**, 621 (2014).
[15] G. Tóth and O. Gühne, Phys. Rev. Lett. **94**, 060501 (2005).
[16] S. T. Flammia and Y.-K. Liu, Phys. Rev. Lett. **106**, 230501 (2011).
[17] M. P. da Silva, O. Landon-Cardinal, and D. Poulin, Phys. Rev. Lett. **107**, 210404 (2011).
[18] M. Hayashi and T. Morimae, Phys. Rev. Lett. **115**, 220502 (2015).
[19] Y. Takeuchi and T. Morimae, Phys. Rev. X **8**, 021060 (2018).
[20] T. Morimae, Y. Takeuchi, and M. Hayashi, Phys. Rev. A **96**, 062321 (2017).
[21] Y. Takeuchi, A. Mantri, T. Morimae, A. Mizutani, and J. F. Fitzsimons, arXiv:1806.09138.
[22] H. Zhu and M. Hayashi, arXiv:1806.05565.
[23] M. Hayashi, K. Matsumoto, and Y. Tsuda, J. Phys. A. Math. Gen. **39**, 14427 (2006).
[24] M. Hayashi, B.-S. Shi, A. Tomita, K. Matsumoto, Y. Tsuda, and Y.-K. Jiang, Phys. Rev. A **74**, 062321.
(2006).

[25] M. Hayashi, A. Tomita, and K. Matsumoto, New J. Phys. 10, 043029 (2008).

[26] M. Hayashi, New J. Phys. 11, 043028 (2009).

[27] N. Gisin, “Bell inequalities: Many questions, a few answers,” in Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle (Springer Netherlands, Dordrecht, 2009) pp. 125–138.

[28] M. Hayashi, Quantum Information Theory: Mathematical Foundation, Graduate Texts in Physics (Springer Berlin Heidelberg, 2016).

[29] S. Pallister, N. Linden, and A. Montanaro, Phys. Rev. Lett. 120, 170502 (2018).

[30] M. Owari and M. Hayashi, New J. Phys. 10, 013006 (2008).

[31] See Theorem 1 (restated) in the Supplementary Material of [29] for the exact form of this strategy.

[32] See the Supplemental Material for further discussions and proofs.

[33] E. Størmer, Acta Math. 110, 233 (1963).

[34] S. L. Woronowicz, Rep. Math. Phys. 10, 165 (1976).
Supplemental Material: Optimal Verification of Two-Qubit Pure States

The supplemental material is organized as follows. In the first section, we complement the proof that the proposed strategy using one-way LOCC measurements is optimal. In the second section, we explain the details of the two-way two-step strategy \( \hat{\Omega}_{\leftrightarrow} \) given in Eq. (8) of the main text. In the last section, we show how the magic values of \( \delta \) and \( p \) are obtained when constructing the optimal strategy \( \Omega_{\leftrightarrow} \) using two-way three-step LOCC measurements.

Optimal strategy using one-way LOCC measurements

In this section we give more details on the proof that our proposed one-way LOCC strategy is optimal. As is shown in the main text, a general one-way LOCC strategy for verifying \( |\Psi\rangle \) can be written as

\[
\Omega = 2 \int |t, s\rangle \langle t, s| \otimes |t, s, B\rangle \langle t, s, B| P_{TS}(dtds).
\]  

(S1)

To analyze \( \Omega \), we treat the variable \( t \) in Eq. (S1) as the random variable \( T \) subject to the marginal distribution \( P_T \) ans use \( E_T \) to denote the expectation under \( P_T \). The constraint that Alice’s measurement in \( \Omega \) must be a POVM induces

\[
\Omega^A = \text{Tr}_B \Omega = 2 \left( E_T [T] E_T [1 - T] \right) = 1.
\]

(S2)

Focusing on the diagonal terms, we have the condition \( E[T] = \frac{1}{2} \). Here, we do not use the condition for the off-diagonal terms. By letting \( \Xi := 2 E_T \left[ \frac{T(1-T)}{D} \right] \) with \( D := T(1-\lambda) + (1-T)\lambda \), the condition gives the following two relations

\[
2E_T \left[ \frac{T^2}{D} \right] (1-\lambda) + \lambda \Xi = 2 E_T \left[ \frac{T(T(1-\lambda) + (1-T)\lambda)}{D} \right] = 1, \quad (S3)
\]

\[
2E_T \left[ \frac{(1-T)^2}{D} \right] \lambda + (1-\lambda) \Xi = 2 E_T \left[ \frac{T(T(1-\lambda) + (1-T)\lambda)}{D} \right] = 1. \quad (S4)
\]

We then get \( \Omega_a \) from \( \Omega \), using the averaging technique described in the main text. When expressed in the standard basis, \( \Omega_a \) satisfies

\[
\Omega_a = 2 \left( \begin{array}{cccc}
E_T \left[ \frac{T^2}{D} \right] (1-\lambda) & 0 & 0 & E_T \left[ \frac{T(1-T)}{D} \right] \sqrt{\lambda(1-\lambda)} \\
0 & E_T \left[ \frac{T(1-T)}{D} \right] \lambda & 0 & 0 \\
0 & 0 & E_T \left[ \frac{T(1-T)}{D} \right] (1-\lambda) & 0 \\
E_T \left[ \frac{T(1-T)}{D} \right] \sqrt{\lambda(1-\lambda)} & 0 & 0 & E_T \left[ \frac{(1-T)^2}{D} \right] \lambda
\end{array} \right)
\]

(S5)

\[
= |\Psi\rangle \langle \Psi| + \lambda_2 |\Psi^\perp\rangle \langle \Psi^\perp| + \lambda_3 |01\rangle \langle 01| + \lambda_4 |10\rangle \langle 10|,
\]

(S7)

with \( \lambda_3 = \langle 01|\Omega_a|01\rangle = \Xi \lambda \), \( \lambda_4 = \langle 10|\Omega_a|10\rangle = \Xi (1-\lambda) \), and

\[
\lambda_2 = \langle \Psi^\perp|\Omega_a|\Psi^\perp\rangle = \left[ 1 - \lambda \Xi \right] \lambda - 2 \lambda (1-\lambda) \Xi + \left[ 1 - (1-\lambda) \Xi \right] (1-\lambda) = 1 - \left[ \lambda^2 + 2 \lambda (1-\lambda) + (1-\lambda)^2 \right] \Xi
\]

(S8)

\[
= 1 - \Xi.
\]

(S9)

Strategy using two-way two-step LOCC measurements

Here we explain in detail the two-way two-step LOCC strategy \( \hat{\Omega}_{\leftrightarrow} \), given in Eq. (8) of the main text. We first describe its construction and then prove its optimality when only two-step classical communication are allowed.
Considering the symmetric role between Alice and Bob, we construct from \( \Omega_{\leftrightarrow} \) a new strategy which outperforms \( \Omega_{\leftrightarrow} \) in the small regime of \( \lambda \). The strategy \( \Omega_{\leftrightarrow} \) is implemented by Alice sending measurement outcomes to Bob and Bob performing conditional measurements. We then get a symmetric version \( \Omega_{\leftrightarrow} \) of \( \Omega_{\leftrightarrow} \) by switching the role between Alice and Bob. The new strategy goes as follows. In each round, Alice first tosses a fair coin, if it is head up, they use \( \Omega_{\leftrightarrow} \); if it is tail up, they use \( \Omega_{\leftrightarrow} \). The corresponding new strategy then has the form

\[
\hat{\Omega}_{\leftrightarrow} = \frac{1}{2} \Omega_{\leftrightarrow} + \frac{1}{2} \Omega_{\rightarrow} = |\Psi\rangle\langle \Psi| + p|\Psi^{\perp}\rangle\langle \Psi^{\perp}| + \frac{1-p}{2} (|01\rangle\langle 01| + |10\rangle\langle 10|).
\]

(S11)

Minimizing the second largest eigenvalue of \( \hat{\Omega}_{\leftrightarrow} \) w.r.t. \( p \in [0,1] \), we get \( p = \frac{1}{3} \) and

\[
\hat{\Omega}_{\leftrightarrow} = |\Psi\rangle\langle \Psi| + \frac{1}{3} (1 - |\Psi\rangle\langle \Psi|).
\]

(S12)

We remark that different from \( \Omega_{\rightarrow} \) and \( \Omega_{\leftrightarrow} \), \( \hat{\Omega}_{\leftrightarrow} \) must be implemented by two-way two-step LOCC. This is due to the symmetrization technique we used to construct \( \hat{\Omega}_{\leftrightarrow} \) from \( \Omega_{\rightarrow} \) and \( \Omega_{\leftrightarrow} \). Alice and Bob need an extra step of classical communication to agree on which strategy (\( \Omega_{\rightarrow} \) or \( \hat{\Omega}_{\leftrightarrow} \)) is used in current round. Comparing the performance of \( \Omega_{\rightarrow} \) (\( \hat{\Omega}_{\leftrightarrow} \)) and \( \hat{\Omega}_{\leftrightarrow} \), one sees the power of classical communication in verification: with just one extra bit of messaging, \( \hat{\Omega}_{\leftrightarrow} \) outperforms \( \Omega_{\rightarrow} \) (\( \Omega_{\leftrightarrow} \)) significantly.

We can actually prove that the strategy \( \hat{\Omega}_{\leftrightarrow} \) is the best we can hope when only two-step classical communication is allowed. Any two-step strategy can be written as a convex combination of one-way LOCC strategies from Alice to Bob and one-way LOCC strategies from Bob to Alice. In [30, Theorem 3] it was proved that for any one-way LOCC strategy \( \Omega \) satisfying \( \mathbb{I} \geq \Omega \geq |\Psi\rangle\langle \Psi|, \text{Tr} \Omega \geq 2 \) holds. Hence, the second largest eigenvalue of \( \Omega \) is no smaller than \( \frac{1}{3} \), concluding the optimality of \( \hat{\Omega}_{\leftrightarrow} \).

**Optimization of strategy using two-way LOCC measurements**

When constructing the strategy \( \Omega_{\leftrightarrow} \) in the main body, we prefix two magic variables \( \delta = 1 - \sqrt{\frac{\lambda}{1-\lambda}} \) and \( p = \frac{\lambda}{1+\sqrt{\lambda(1-\lambda)}} \). Here we show that they are actually chosen so that the second largest eigenvalue of \( \Omega_{\leftrightarrow} \) is minimized. From now on we assume \( \delta \in [0,1] \) and \( p \in [0,1] \) are two free parameters to be optimized. By construction, \( \Omega_{\leftrightarrow} \) is given by

\[
\Omega_{\leftrightarrow} = \frac{p-1}{4} (T_{A\rightarrow B}^{1} + T_{A\rightarrow B}^{2} + T_{1\rightarrow A}^{B} + T_{2\rightarrow A}^{B}) + pT_{3}.
\]

(S13)

It can be shown that \( \Omega_{\leftrightarrow} \) admits the following spectral decomposition:

\[
\Omega_{\leftrightarrow} = |\Psi\rangle\langle \Psi| + \lambda_{2}(\delta, p)|\Psi^{\perp}\rangle\langle \Psi^{\perp}| + \lambda_{3}(\delta, p) (|01\rangle\langle 01| + |10\rangle\langle 10|),
\]

(S14)

where

\[
\lambda_{2}(\delta, p) = \frac{p(1-\delta) + \delta}{1-\delta + \delta p}, \quad \lambda_{3}(\delta, p) = \frac{(1-p)\left[\lambda + (1-\lambda)(1-\delta)^{2}\right]}{2(1-\delta + \delta p)}.
\]

(S15)

Our target is to minimize \( \lambda_{2}^{2}(\Omega_{\leftrightarrow}) \), the second largest eigenvalue of \( \Omega_{\leftrightarrow} \), over the free parameters \( \delta \) and \( p \) for fixed \( \lambda \). This optimization problem then is given by

\[
\lambda_{2}^{2}(\Omega_{\leftrightarrow}) := \min_{\delta \in [0,1], p \in [0,1]} \max \{ \lambda_{2}(\delta, p), \lambda_{3}(\delta, p) \}.
\]

(S16)

\( \lambda_{2}^{2}(\Omega_{\leftrightarrow}) \) is minimized for fixed \( \lambda \) when the derivatives with respect to \( \delta \) and \( p \) vanish. As \( \lambda_{2}(\delta, p) \) is monotonically increasing with \( p \) while \( \lambda_{3}(\delta, p) \) is monotonically decreasing with \( p \) in the range \( p \in [0,1] \), and \( \lambda_{2}(0, \delta) < \lambda_{3}(0, \delta) \) for \( \delta \in [0,1] \), \( \lambda_{2}^{2}(\Omega_{\leftrightarrow}) \) is minimized when \( \lambda_{2}(\delta, p) = \lambda_{3}(\delta, p) \). Solving this equation w.r.t. \( p \), we get

\[
p^{*} = \frac{-\lambda \delta^{2} + \delta^{2} - 2\delta + 1}{2\lambda \delta - \lambda \delta^{2} + \delta^{2} - 4\delta + 3}, \quad \lambda_{2}(\delta) = \lambda_{3}(\delta) = \frac{2\lambda \delta - \lambda \delta^{2} + \delta^{2} - 2\delta + 1}{2\lambda \delta - \lambda \delta^{2} + \delta^{2} - 4\delta + 3}.
\]

(S17)
where $p^*$ is the solution of the equation, which is also the optimal choice of $p$. We now minimize $\lambda_2(\delta)$ w.r.t. $\delta$. The partial derivative is given by

$$\frac{\partial \lambda_2}{\partial \delta} = -\frac{2(1 - \lambda)(\delta^2 - 2\delta + \frac{1-2\lambda}{1-\lambda})}{(2\lambda\delta - \lambda\delta^2 + \delta^2 - 4\delta + 3)^2} = -\frac{2(1 - \lambda)(\delta - \delta_-)(\delta - \delta_+)}{(2\lambda\delta - \lambda\delta^2 + \delta^2 - 4\delta + 3)^2},$$

(S18)

where $\delta_{\pm} = 1 \pm \sqrt{\frac{\lambda}{1-\lambda}}$. Solving the equation $\partial \lambda_2/\partial \delta = 0$ in the range $\delta \in [0, 1]$ gives the optimal choice $\delta^* = \delta_- = 1 - \sqrt{\frac{\lambda}{1-\lambda}}$. Substituting $\delta^*$ into $p^*$, we get $p^* = \frac{\lambda}{1+\sqrt{\lambda(1-\lambda)}}$ expressed in terms of $\lambda$ solely. Substituting in the optimal choices of $\delta^*$ and $p^*$ gives the following optimal strategy

$$\Omega_{\leftrightarrow} = |\Psi\rangle\langle\Psi| + \frac{\sqrt{\lambda(1-\lambda)}}{1 + \sqrt{\lambda(1-\lambda)}} (1 - |\Psi\rangle\langle\Psi|).$$

(S19)

We are done.