Chiral photocurrent in parity-violating magnet and enhanced response in topological antiferromagnet

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Rectified electric current induced by irradiating light, so-called photocurrent, is an established phenomenon in optoelectronic physics. In this paper, we present a comprehensive classification of the photocurrent response arising from the parity violation in bulk systems. We clarify the contrasting role of $T$- and $PT$-symmetries and consequently find a new type of photocurrent phenomena characteristic of parity-violating magnets, magnetic rectification current and gyration current. Especially, the gyration current is induced by the circularly-polarized light and it is the counterpart of the shift current caused by the linearly-polarized light. This photocurrent adds a new functionality of materials studied in various fields of condensed matter physics such as multiferroics and spintronics. A list of materials is provided. Furthermore, we show that the gyration current is strongly enhanced by topologically nontrivial band dispersion. On the basis of the microscopic analysis of Dirac models, we demonstrate the divergent photocurrent response and elucidate the importance of tilting of Dirac cones.

I. INTRODUCTION

Optical responses have been providing a lot of interests in condensed matter physics. The optical probes are extensively implemented in the spectroscopy such as the angle-resolved photo-emission spectroscopy and real-space imaging of material phases. Recent studies have clarified exotic phenomena where light and electron are strongly coupled to each other; for instance, photo-induced phase transitions and higher harmonic generations in solids [1–3]. Among the nonlinear optical responses, the photocurrent response is constantly offering renewed interests.

The photocurrent phenomenon was historically attributed to the internal field and surface effects of ferroelectric materials [4–7] or to the heterostructure whose prototypical example is the p-n junction device [8, 9]. On the other hand, the photocurrent response originating from the bulk electronic structure has also been clarified. The discovery of the bulk photocurrent can be traced back to the study of a well-known ferroelectric system, BaTiO$_3$ [10]. The bulk photocurrent has been theoretically investigated by perturbative calculations [11–14]. Subsequently, a first-principles calculation has successfully explained the photocurrent response in ferroelectric materials [15, 16].

Whereas the basic formalism [14, 17–19] and first-principles calculations [15, 16, 20, 21] have been established, recent developments in topological science have provided us with new insights into the photocurrent response. The system hosting a topologically nontrivial electronic structure shows enhanced photoelectric responses due to diverging geometric quantities [22–28]. Importantly, robustness of the nontrivial band dispersion may be ensured by its topological property, and it is beneficial for invulnerable and high-performance optoelectronic devices. Recent experiments have actually supported the enhanced photoelectric responses in various topological materials [29–33].

In general, the photocurrent response is allowed when the parity symmetry ($P$-symmetry) is violated. This symmetry requirement was satisfied by noncentrosymmetric crystal structures in the previous studies. On the other hand, we have overlooked the other type of parity violation, that is, the magnetic parity violation [34–36]. In the case of the magnetic parity violation, the magnetic order breaks not only the time-reversal symmetry ($T$-symmetry) but also the parity symmetry. In a class of such parity-violating magnet the combined symmetry, namely, $PT$-symmetry is preserved [34–36]. This symmetry is a striking property of the parity-violating magnets distinct from conventional noncentrosymmetric systems where the $T$-symmetry is preserved. According to the group-theoretical classification combined with model studies [37–41], the $T$ and $PT$ are fundamental symmetries characterizing quantum phases, and essentially distinguish the electronic structure and physical responses unique to the parity violation [35, 36]. The magnetic parity violation has already been discussed in the contexts of multiferroics and spintronics. The candidate materials actually exist in a broad range of magnetic compounds [35, 42, 43]. In spite of these findings, there is few studies focusing on the photocurrent in magnetic systems except for a few recent theoretical works [44–46]. Thus, it is highly desirable for promoting the functionality of matter to understand a role of the magnetic parity violation in the photoelectric phenomena.

This work mainly consists of two parts. Firstly, we present a systematic classification of the photocurrent responses from the viewpoint of $T$- and $PT$-symmetries. Following the established perturbative treatment based on the spinless free fermions, we clarify the contrast-
ing roles of these fundamental symmetries and complete all the photocurrent responses. It is shown that the photocurrent is clearly classified on the basis of these symmetries. Furthermore, the classification result leads us to discovery of new linearly- and circularly-polarized photo-induced currents which we name magnetic rectification current and gyration current, respectively. These photocur- rents are unique to the magnetically-parity-violating systems and show different properties from the known photocurrent arising from the magnetic parity violation [44, 45]. We also generalize our classification scheme to spinful systems. Especially, owing to the Kramers degeneracy, careful treatment is required to obtain gauge-invariant formulas for \(\mathcal{PT}\)-symmetric systems.

Secondly, we clarify basic properties of the gyration current. The gyration current is the counterpart of the shift current and closely related to quantum geometry of the electronic structure. Using the spinful Hamiltonian having the magnetic parity violation, we present microscopic calculations of the gyration current, and compare it with the attenuation coefficient and joint density of states which contribute to the optoelectronic phenomena. Moreover, we show that the gyration current is strongly enhanced by topologically nontrivial electronic structures. We introduce a model Hamiltonian mimicking a real topological antiferromagnet CuMnAs, and show analytically expressions for the gyration current coefficient. A divergent behavior in the low-frequency regime results from the nontrivial quantum geometry. We also show numerical calculations indicating that slightly-massive Dirac electrons also realize an enhanced gyration current. Note that CuMnAs is a promising material for antiferromagnetic spintronics [47]. Thus, our results may motivate interdisciplinary investigations between topological science, optoelectronics, and antiferromagnetic spintronics.

The outline of the paper is as follows. In Sec. II, we introduce the formalism based on the perturbative calculation in terms of the electric field. Sec. III presents the classification of photocurrent responses in spinless systems by making use of the \(\mathcal{T}\) - and \(\mathcal{PT}\)-symmetries. In Secs. IIIA and IIIB, we describe the photocurrent unique to metals, and Secs. IIICC and IIIDD are devoted to the formulation of the photocurrent allowed in both metals and insulators. Table I summarizes the classification result of Sec. III. The formulation is generalized to the spinful case in Sec. IV. In Sec. V, we study the gyration current in details. We first discuss basic properties [Sec. VA] and next study a simple model [Sec. VBB]. Furthermore, divergent enhancement of the gyration current response in topological antiferromagnet is proposed in Sec. VCC. Finally, we summarize this work in Sec. VI.

II. FORMULATION

This section shows the formalism of perturbative calculations of nonlinear optical responses within the free particle approximation. Although the calculation has been done in previous theoretical studies [12–14, 17–19], the derivation is shown below for completeness. The noninteracting Hamiltonian is given by

\[
H_0 = \int \frac{dk}{(2\pi)^2} \sum_a \epsilon_{ka} c^\dagger_k a c_{ka}.
\]

(1)

where we define the annihilation and creation operators \(c_{ka}, c^\dagger_{ka}\) of the Bloch state \(|\psi_{ka}\rangle = \exp(ik \cdot \hat{r}) |u_a(k)\rangle\) labeled by the crystal momentum \(k\) and band index \(a\). The periodic part of the Bloch state satisfies a Bloch equation,

\[
H_0(k) |u_a(k)\rangle = \epsilon_{ka} |u_a(k)\rangle.
\]

(2)

Next, we consider interaction between electrons and electromagnetic fields. Since the illuminating light is spatially uniform in the length scale of a lattice constant and photo-electric field is much more strongly coupled to electrons than photo-magnetic field, the effect of electromagnetic field is approximated by an uniform electric field, that is written as \(\mathbf{E}(t)\). This is the so-called electric-dipole approximation [8]. The applied electric field can be introduced to the Hamiltonian by two approaches; length gauge and velocity gauge approaches [17–19].

In the velocity gauge approach [11, 18, 19], the electric field modifies the kinetic part of the noninteracting Hamiltonian. The canonical momentum \(\mathbf{p}\) is replaced as

\[
\mathbf{p} \rightarrow \mathbf{p} - q \mathbf{A}(t),
\]

(3)

where \(\mathbf{E}(t) = -\partial_t \mathbf{A}(t)\) and \(q\) is the charge of carriers. In this framework, the electric field gives rise to a shift of the momentum. Thus, we can make use of well-established diagrammatic techniques to calculate the nonlinear optical responses [19, 45, 48]. On the other hand, with the length gauge approach, the electric field is taken into account by the dipole Hamiltonian written as

\[
H_{RE} = -qr \cdot \mathbf{E}(t).
\]

(4)

In a general sense, the position operator breaks the translation symmetry of solids and may make the Bloch representation less convenient to describe the Hamiltonian under the electric field. In the infinite volume limit, however, the position operator is written in the Bloch representation as \(49, 50\)

\[
[r]_{ab} = i \partial_k \delta_{ab} + \xi_{ab}.
\]

(5)

The position operator consists of the derivative of crystal momentun \(\partial_\mu = \partial / \partial k_\mu\) and the Berry connection \(\xi_{ab} = i \langle u_a(k) | \partial_k u_b(k) \rangle\) defined in the manifold of the Brillouin zone. Especially, the Berry connection is a characteristic term of crystalline systems. Although the position operator obtained in Eq. (5) is not diagonal in the band index, we can proceed to the perturbative calculations without discarding the Bloch basis. These two gauge choices should be identical to respect the gauge
TABLE I. Classification of photocurrent responses in terms of $\mathcal{T}$- and $\mathcal{PT}$-symmetries and of linearly-polarized ($\downarrow$) and circularly-polarized ($\bigcirc$) lights. Note that the responses with the superscript ‘*$'$ are allowed in metals. The bold-faced class is clarified by this work.

| $\mathcal{T}$ | $\mathcal{PT}$ |
|--------------|-----------------|
| (\downarrow) Shift current | Drude term* |
| (\bigcirc) Berry curvature dipole term* | Magnetic injection current |
| Electric injection current | Magnetic rectification current |
| Electric rectification current | Gyration current |

Invariance. The equivalence has been confirmed in non-interacting systems by explicitly carrying out the time-dependent gauge transformation\[13, 17\]. In the following, we adopt the length gauge. In fact, by using the length gauge approach, various contributions to the nonlinear optical responses are clearly divided in terms of intraband and interband transitions.

To obtain the expectation value of the nonlinear electric current, we derive the current density operator $qv$ where $v$ is the velocity operator. In the framework of the first quantization with the Heisenberg picture, the velocity operator in the length gauge is given by

$$v^{(E)}(t) = \frac{1}{\hbar} [\rho^{\mu}(t), H(t)],$$

where the Hamiltonian $H(t)$ consists of Eqs. (1) and (4) in the length gauge. Because of the commutative property between the dipole Hamiltonian and the position operator, the electric field does not make any correction to the velocity operator of the unperturbed Hamiltonian [Eq. (1)]. Thus, the velocity operator in the Bloch representation is obtained as

$$v_{ab} = v^{(E)} = \hbar^{-1} \nabla_k \epsilon_{ab} \delta_{ab} + i \hbar^{-1} \epsilon_{ab} \xi_{ab}.$$  (7)

We note that the velocity operator in the length gauge is expressed in a modified form since the perturbative part arising from Eq. (3) does not commute with the position operator [17–19].

The perturbative calculations are straightforwardly conducted in the same way as the linear response theory [51]. Here, we derive the nonlinear optical conductivity by following the density matrix approach [12–14, 17]. Introducing the density matrix operator $P = \sum_n e^{-iH(t) / (k_B T)} |n\rangle \langle n|$, we obtain the time-evolution as

$$i\hbar \partial_t P(t) = [H(t), P(t)].$$

(8)

Note that we adopt the Schrödinger picture in the following calculations. When the perturbative calculations are conducted in the Bloch representation, it is convenient to use the reduced density matrix defined by

$$\rho_{k,ab}(t) = \text{Tr}_r \{ c_{kq}^\dagger c_{kq} P(t) \}.$$  (9)

In the following, the momentum dependence of the reduced density matrix $\rho_{k}$ is implicit unless otherwise mentioned. Equation (8) in the frequency domain is obtained as

$$\langle \hbar \omega - \epsilon_{ab} \rangle \rho_{ab} (\omega) = -q \int \frac{d\Omega}{2\pi} \frac{E^\mu (\Omega)}{\omega} \{ r_k^\mu, \rho (\omega - \Omega) \}_{ab},$$

where $\epsilon_{ab} = \epsilon_{ka} - \epsilon_{kb}$ and we adopt a convention for the Fourier transformation given by

$$\rho_{ab} (t) = \frac{1}{\hbar} \int \frac{d\omega}{2\pi} e^{-i\omega t} \rho_{ab} (\omega).$$

(11)

Regarding the magnitude of the electric field $|E|$ as the perturbation parameter, the reduced density matrix $\rho = \sum_n \rho^{(n)}$ is expanded by powers of the electric field $\rho^{(n)} = O(|E|^n)$. Thus, we obtain the recursive equation,

$$\langle \hbar \omega - \epsilon_{ab} \rangle \rho^{(n+1)} (\omega) = -q \int \frac{d\Omega}{2\pi} \frac{E^\mu (\Omega)}{\omega} \{ r_k^\mu, \rho^{(n)} (\omega - \Omega) \}_{ab},$$

where the zeroth component is given by $\rho^{(0)} (\omega) = 2\pi \delta (\omega) f (\epsilon) \delta_{ab}$ with the Fermi distribution function $f (\epsilon) = [1 + \exp ((\epsilon - \mu) / (k_B T))]^{-1}$ and the chemical potential $\mu$. Following Refs. [17, 18], we introduce the matrix $d^\omega$ defined by

$$d^\omega_{ab} = \frac{1}{\hbar \omega + i0 - \epsilon_{ab}},$$

(13)

where $+i0$ is the infinitesimal and positive scalar derived from the adiabatic application of the external field [51]. Then, Eq. (12) is recast as

$$\rho^{(n+1)} (\omega) = -q \int \frac{d\Omega}{2\pi} d^\omega \| E^\mu (\Omega) \{ r_k^\mu, \rho^{(n)} (\omega - \Omega) \}_{ab},$$

(14)

For classification of contributions to nonlinear optical conductivity, we make use of the intraband-derband decomposition of the position operator [13, 14]. The position operator in the Bloch representation $r_k^\mu$ [Eq. (5)] is divided into the diagonal and off-diagonal components in the band index as $r_1$ and $r_2$. The perturbation by the electric field is classified into the intraband effect $-qr_1 \cdot E$ and interband effect $-qr_2 \cdot E$. Sequentially calculating the corrections to the reduced density matrix $\rho^{(n)} (n > 0)$, we obtain the second-order correction $\rho^{(2)}$ as

$$\rho^{(2)}_{ab} (\omega) = \rho^{(ii)}_{ab} (\omega) + \rho^{(ei)}_{ab} (\omega) + \rho^{(ie)}_{ab} (\omega) + \rho^{(ee)}_{ab} (\omega).$$

(15)
where we classify the components by intraband (i) and interband (e) effects. Each term is explicitly given by

\[
\rho^{(ii)}_{ab}(\omega) = (-i\text{q})^2 \int \frac{d\Omega d\Omega'}{(2\pi)^2} E^\mu(\Omega) E^{\nu}(\Omega') d^\nu_{ab} d^\omega_{-\Omega} \partial_\mu \partial_\nu f(\epsilon_{ka}) \times 2\pi \delta_{ab}(\omega - \Omega - \Omega'), \tag{16}
\]

\[
\rho^{(ei)}_{ab}(\omega) = -i\text{q}^2 \int \frac{d\Omega d\Omega'}{(2\pi)^2} E^\mu(\Omega) E^{\nu}(\Omega') d^\omega_{ab} d^\omega_{-\Omega} \xi^\mu_{ab} \partial_\mu f_{ab} \times 2\pi \delta(\omega - \Omega - \Omega'), \tag{17}
\]

\[
\rho^{(ie)}_{ab}(\omega) = -i\text{q}^2 \int \frac{d\Omega d\Omega'}{(2\pi)^2} E^\mu(\Omega) E^{\nu}(\Omega') d^\omega_{ab} [\partial_\mu (d^\omega_{\nu,\Omega} f_{ab} \xi^\mu_{ab}) - i (\xi^\mu_{\nu,ab} - \xi^\mu_{eb}) d^\omega_{\mu,ab} f_{ab} \xi^\mu_{eb}] \times 2\pi \delta(\omega - \Omega - \Omega'), \tag{18}
\]

\[
\rho^{(ee)}_{ab}(\omega) = \text{q}^2 \sum_c \int \frac{d\Omega d\Omega'}{(2\pi)^2} E^\mu(\Omega) E^{\nu}(\Omega') d^\omega_{ab} [d^{\omega,\Omega}_{\nu,\mu} \xi^\nu_{ac,bc} f_{ac} - d^{\omega,\Omega}_{\mu,\nu} \xi^\nu_{cb,ac} f_{cb}] \times 2\pi \delta(\omega - \Omega - \Omega'). \tag{19}
\]

Summation over the repeated Greek indices such as \(\mu = x, y, z\) is implicit, and \(f_{ab} = f(\epsilon_{ka}) - f(\epsilon_{kb})\). Note that the components \(\rho^{(ii)}\) and \(\rho^{(ei)}\) are finite only when the low-energy carriers are present owing to the Fermi surface or thermal excitations as implied by the Fermi surface factor \(\partial_\mu f\) [52–55]. On the other hand, the other terms \(\rho^{(ie)}\) and \(\rho^{(ee)}\) contribute to the nonlinear optical conductivity even in insulating systems at the absolute zero temperature [13]. In the perturbative calculation of the nonlinear response, the result should not be affected by an arbitrary permutation of applied external fields [19]. Thus, we symmetrize the indices and frequencies of electric fields. Exemplified by Eq. (16), the expression is modified as

\[
\rho^{(ii)}_{ab}(\omega) = \frac{(-i\text{q})^2}{2} \int \frac{d\Omega d\Omega'}{(2\pi)^2} E^\mu(\Omega) E^{\nu}(\Omega') d^\nu_{ab} d^\omega_{-\Omega} \partial_\mu \partial_\nu f(\epsilon_{ka}) \times 2\pi \delta_{ab}(\omega - \Omega - \Omega') + [(\mu, \Omega) \leftrightarrow (\nu, \Omega')]. \tag{20}
\]

Finally, we obtain the full expression

\[
J_{(2)}^\nu(\omega) = \int \frac{dk}{(2\pi)^d} \sum_{a,b} q^{\nu,\mu}_{ab} \rho^{(2)}_{ab}(\omega), \tag{21}
\]

\[
\equiv \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \tilde{\sigma}^{\mu,\nu,\lambda}(\omega_1, \omega_2) E^\nu(\omega_1) E^\lambda(\omega_2), \tag{22}
\]

for the second-order nonlinear electric current density. Considering the common factor, we take a convention for the second-order optical conductivity \(\sigma^{\mu,\nu,\lambda}(\omega_1, \omega_2)\) given by

\[
\tilde{\sigma}^{\mu,\nu,\lambda}(\omega_1, \omega_2) = 2\pi \delta(\omega - \Omega - \omega_2) \sigma^{\mu,\nu,\lambda}(\omega_1, \omega_2). \tag{23}
\]

Classifying the components by following the decomposition in Eq. (15), the conductivity tensor is divided as

\[
\sigma^{\mu,\nu,\lambda} = \sigma^{\mu,\nu,\lambda}_{ii} + \sigma^{\mu,\nu,\lambda}_{ei} + \sigma^{\mu,\nu,\lambda}_{ie} + \sigma^{\mu,\nu,\lambda}_{ee}, \tag{24}
\]

where each component is obtained as

\[
\sigma^{\mu,\nu,\lambda}_{ii}(\omega_1, \omega_2) = \text{q}^2 \int \frac{dk}{(2\pi)^d} \sum_{a} -\nu^{\mu}_{aa} d^{\nu}_{aa} d^{\omega}_{-\Omega} \partial_\nu f(\epsilon_{ka}) + [(\nu, \omega_1) \leftrightarrow (\lambda, \omega_2)], \tag{25}
\]

\[
\sigma^{\mu,\nu,\lambda}_{ei}(\omega_1, \omega_2) = \text{q}^2 \int \frac{dk}{(2\pi)^d} \sum_{a,b} -\nu^{\mu}_{ab} d^{\nu}_{aa} d^{\omega}_{-\Omega} \xi^\mu_{ab} \partial_\nu f_{ba} + [(\nu, \omega_1) \leftrightarrow (\lambda, \omega_2)], \tag{26}
\]

\[
\sigma^{\mu,\nu,\lambda}_{ie}(\omega_1, \omega_2) = \text{q}^2 \int \frac{dk}{(2\pi)^d} \sum_{a,b} -\nu^{\mu}_{ab} d^{\nu}_{aa} d^{\omega}_{-\Omega} \partial_\nu (d^{\nu}_{ab} f_{ba} \xi^\mu_{ba}) - i (\xi^{\mu}_{\nu,ab} - \xi^{\mu}_{eb}) d^{\nu}_{ba} \xi^\mu_{ba} f_{ba} \xi^\mu_{ba} + [(\nu, \omega_1) \leftrightarrow (\lambda, \omega_2)], \tag{27}
\]

\[
\sigma^{\mu,\nu,\lambda}_{ee}(\omega_1, \omega_2) = \text{q}^2 \int \frac{dk}{(2\pi)^d} \sum_{a,b,c} \nu^{\mu}_{ab} d^{\nu}_{ac} d^{\omega}_{-\Omega} \xi^\mu_{ac,bc} f_{ac} - d^{\nu}_{bc} \xi^\nu_{cb,ac} f_{cb} + [(\nu, \omega_1) \leftrightarrow (\lambda, \omega_2)]. \tag{28}
\]

The expression is consistent with the previous results [13, 14, 56]. Although the above formula is generally applicable to second-order optical responses such as second harmonic generation [57] and parametric generation pro-
cess [8], we only focus on the photocurrent response in the following sections.

III. PHOTOCURRENT FORMULA

In this section, we derive the photocurrent formulas in $T$-/PT-symmetric systems. For the photocurrent response, the frequencies are taken as

$$\omega = 0, \ \omega_1 = -\Omega, \ \omega_2 = \Omega,$$  \hspace{1cm} (29)

where we assume $\Omega > 0$ without loss of generality. In this section, we consider spinless systems to clarify the contrasting role of $T$ and PT-symmetries. Note that the classification is extended to the spinful systems later [Sec. IV].

Firstly, we present a basic symmetry consideration of the photocurrent. The photocurrent response is classified into the linearly-polarized and circularly-polarized light-induced currents which we call LP-photocurrent and CP-photocurrent, respectively. Owing to the fact that the time-domain electric field is real, the electric field in the frequency domain satisfies the relation,

$$E(\omega) = E^*(-\omega).$$  \hspace{1cm} (30)

The electric current in Eq. (22) is transformed as

$$J^{\mu}_{(2)}(\omega = 0) = \int \frac{d\omega_2}{2\pi} \sigma^\mu_{\nu;\lambda}(0; -\Omega, \Omega)E^\nu(-\Omega)E^\lambda(\Omega),$$  \hspace{1cm} (31)

$$= \int \frac{d\Omega}{2\pi} \sigma^\mu_{\nu;\lambda}(0; -\Omega, \Omega)(E^\nu(\Omega))^*E^\lambda(\Omega).$$  \hspace{1cm} (32)

In the case of the LP-photocurrent, the indices of irradiating electric fields are symmetric. Thus, the LP-photocurrent response is rewritten as

$$J^\mu_{LP} = \int \frac{d\Omega}{2\pi} \sigma^\mu_{\nu;\lambda}(0; -\Omega, \Omega)\times \frac{1}{2} \left[ (E^\nu(\Omega))^*E^\lambda(\Omega) + (E^\lambda(\Omega))^*E^\nu(\Omega) \right] ,$$  \hspace{1cm} (33)

$$= \int \frac{d\Omega}{2\pi} \sigma^\mu_{\nu;\lambda}(0; -\Omega, \Omega)L^{\nu;\lambda}(\Omega),$$  \hspace{1cm} (34)

where we define a symmetric tensor

$$L^{\nu;\lambda}(\Omega) = \text{Re} \left[ E^\nu(\Omega)(E^\lambda(\Omega))^* \right].$$  \hspace{1cm} (35)

Using the symmetrized photocurrent conductivity $\sigma^\mu_{\nu;\lambda} = \sigma^\mu_{\nu;\lambda} + \sigma^\mu_{\lambda;\nu}$, the LP-photocurrent formula is defined by

$$J^\mu_{LP} = \int \frac{d\Omega}{2\pi} \eta^\mu_{\nu;\lambda}(0; -\Omega, \Omega)L^{\nu;\lambda}(\Omega).$$  \hspace{1cm} (36)

The symmetry of the LP-photocurrent tensor $\eta^\mu_{\nu;\lambda}$ is same as that of the piezoelectric tensor. Hence, the LP-photocurrent is allowed in noncentrosymmetric systems belonging to the piezoelectric class [58].

On the other hand, the indices of irradiating electric fields are anti-symmetric for the CP-photocurrent tensor. The response formula is obtained as

$$J^\mu_{CP} = \int \frac{d\Omega}{2\pi} \sigma^\mu_{\nu;\lambda}(0; -\Omega, \Omega)$$

$$\times \frac{1}{2} \left[ (E^\nu(\Omega))^*E^\lambda(\Omega) - (E^\lambda(\Omega))^*E^\nu(\Omega) \right] ,$$  \hspace{1cm} (37)

$$= \int \frac{d\Omega}{2\pi} \sigma^\mu_{\nu;\lambda}(0; -\Omega, \Omega)i\epsilon_{\nu;\lambda\tau}F^\tau(\Omega),$$  \hspace{1cm} (38)

where we introduce

$$F(\Omega) = \frac{i}{2} E(\Omega) \times E^*(\Omega).$$  \hspace{1cm} (39)

The sign of the vector $F$ represents handedness of the circularly-polarized light; for example, taking the left-handed circularly-polarized light along the $z$-direction, $E = E_0(1, i, 0)$ leads to $F = |E_0|^2 \hat{\varepsilon}$. By introducing an axial tensor

$$\kappa^{\mu;\tau}(\Omega) = i\epsilon_{\nu;\lambda\tau}\sigma^\mu_{\nu;\lambda}(0; -\Omega, \Omega),$$  \hspace{1cm} (40)

the formula for CP-photocurrent is given by

$$J^\mu_{CP} = \int \frac{d\Omega}{2\pi} \kappa^{\mu;\tau}(\Omega)F^\tau(\Omega).$$  \hspace{1cm} (41)

The noncentrosymmetric crystallographic point groups with the non-zero $\kappa$ are called gyrotropic (optically-active) point groups [58]. Therefore, the piezoelectric and gyrotropic point groups having the $T$ or PT-symmetry are shown in Appendix C with a list of materials. Note that the introduced quantities $L^{\nu;\lambda}$ and $F^\tau$ are related to the Stokes parameters [59]. Thus, by taking the linearly-polarized light corresponding to the equator of the Poincaré sphere, $L^{\nu;\lambda} \neq 0$ and $F = 0$ are satisfied. To the contrary, in the case of the circularly-polarized light described by the north and south poles of the Poincaré sphere, $L^{\nu;\lambda} = 0$ and $F \neq 0$ are satisfied.

Now, we proceed to the derivation of photocurrent responses. As shown in seminal works, the photocurrent responses in the $T$-symmetric systems have already been clarified in both insulators [12–14] and metals [54]. On the other hand, the photocurrent phenomenon arising from the magnetic order remains unexplored except for a few recent theoretical studies [44–46]. Although we reproduce some of the known results in the following subsections, our calculation is distinct from the previous theoretical studies because of the following reasons; we systematically investigate all the photocurrent responses from the viewpoint of the $T$- and PT-symmetries, unify the reported works, and importantly clarify new photocurrents, named magnetic rectification current and gyration current. In the following, we analyze Eqs. (25)-(28) one by one. Frequency dependence of the conductivity tensor is implicit unless otherwise explicitly denoted. Table II shows the classification result of the photocurrent responses in $T$- and PT-symmetric systems.
TABLE II. Classification of photocurrent responses in $\mathcal{T}$-/
$\mathcal{PT}$-symmetric systems. The symbols $\downarrow$ and $\uparrow$ denote photocurrents induced by linearly-polarized and circularly-
polarized lights, respectively. The photocurrent denoted by
‘(our work)’ is clarified in this work. The symbols ‘$d$’ and ‘$o$’
in the term $\sigma_\text{ee}$ represent the diagonal and off-diagonal
components of the velocity matrix $v_{ab}$ in Eq. (28), while ($P$) and ($\delta$)
denote the terms consisting of the principal integration and delta function, respectively.

| $\mathcal{T}$ | $\mathcal{P}\mathcal{T}$ |
|----------------|-----------------|
| $\sigma_{ii}$  | $\times$ | $\downarrow$ [45] |
| $\sigma_{ei}$  | $\circ$ [54] | $\times$ |
| $\sigma_{\text{ee};d}$ | $\circ$ [11, 14] $\uparrow$ [44] |
| $\sigma_{\text{ee};o}(\delta)$ | $\downarrow$ [14] | $\circ$ (this work) |
| $\sigma_{\text{ee};o}(P)$ | $\circ$ [14] | $\downarrow$ (this work) |

A. Fermi surface effect I: Drude term

We first consider the intraband-only contribution
[Eq. (25)] which we call Drude term [60]. The Drude term
does not essentially require the multi-band structures and can be captured by the conventional Boltzmann’s transport
theory where the single band is treated [52]. The photocurrent response is evaluated as
given by

$$
\sigma_{ii}^{\mu,\nu,\lambda}(\omega; \omega - \omega_1, \omega_2) = \frac{q^3}{2} \int \frac{dk}{(2\pi)^3} \sum_a v_{aa}^\mu \delta_{aa}^\nu \partial_\lambda \partial_\lambda f(\epsilon_{ka})
$$

$$
+ [(\nu, \omega - \omega_2) \leftrightarrow (\lambda, \omega_2)],
$$

$$
= - \frac{q^3}{2\hbar^2 \omega} \left( \frac{1}{\omega_2} + \frac{1}{\omega - \omega_2} \right) \int \frac{dk}{(2\pi)^3} \sum_a v_{aa}^\mu \partial_\nu \partial_\lambda f(\epsilon_{ka}),
$$

$$
\omega \rightarrow 0, \omega_2 \rightarrow \Omega
$$

$$
\sigma_{ii}^{\mu,\nu,\lambda} \text{ is therefore classified as the LP-photocurrent response since we can interchange the order of partial derivatives } \partial_\nu \partial_\lambda.
$$

We hence rewrite

$$
\eta_D^{\mu,\nu,\lambda} = \frac{q^3}{2\hbar^2 \Omega^2} \int \frac{dk}{(2\pi)^3} \sum_a v_{aa}^\mu \partial_\nu \partial_\lambda f(\epsilon_{ka}).
$$

The subscript ‘$D$’ denotes ‘Drude’ term. It is noteworthy that the magnitude diverges as $\sim \Omega^{-2}$ in the low-frequency regime $\Omega \ll 1$. Owing to Eq. (7), the momentum integral in Eq. (44) is recast as

$$
\int \frac{dk}{(2\pi)^3} v_{aa}^\mu \partial_\nu \partial_\lambda f(\epsilon_{ka}) = \frac{1}{\hbar} \int \frac{dk}{(2\pi)^3} (\partial_\mu \partial_\nu \partial_\lambda f(\epsilon_{ka}) = \frac{1}{\hbar} \int \frac{dk}{(2\pi)^3} (\partial_\mu \partial_\nu \partial_\lambda f(\epsilon_{ka}),
$$

which is finite if and only if both of the $\mathcal{P}$- and $\mathcal{T}$-symmetries are broken [60]. In fact, the $\mathcal{T}$-symmetry ensures the degeneracy between $\pm \mathbf{k}$ points in the Bril-
louin zone. Thus, third derivative of the energy spectrum, $\partial_\mu \partial_\nu \partial_\lambda \epsilon_{ka}$, is canceled out by the integration over $\mathbf{k}$. On the other hand, the $\mathcal{PT}$-symmetry does not forbid the anti-symmetric band dispersion and allows the Drude term [see Table II].

B. Fermi surface effect II: Berry curvature dipole term

In this subsection, we consider the photocurrent derived from the $\sigma_{ei}$ term [Eq. (26)]. Although this component is characteristic to metals as the Drude term is, the response needs the multi-band effect. A derivation has successfully been obtained by the semiclassical theory [54, 61]. Supposing Eq. (29), the expression is rewritten by

$$
\sigma_{ei}^{\mu,\nu,\lambda} = \frac{q^3}{2\hbar^2 \Omega^2} \int \frac{dk}{(2\pi)^3} \sum_{\alpha \neq \beta} \epsilon_{aa}^{\mu,\nu} \delta_{\alpha\beta} \partial_\lambda f(\epsilon_{ka}) + [(\nu, \Omega) \leftrightarrow (\lambda, \Omega)],
$$

$$
= \frac{q^3}{2\hbar^2 \Omega^2} \int \frac{dk}{(2\pi)^3} \sum_{\alpha \neq \beta} (\epsilon_{aa}^{\mu,\nu} - \epsilon_{ab}^{\mu,\nu}) \partial_\lambda f(\epsilon_{ka}) + [(\nu, \Omega) \leftrightarrow (\lambda, \Omega)],
$$

$$
= \frac{q^3}{2\hbar^2 \Omega^2} \int \frac{dk}{(2\pi)^3} \sum_a i\epsilon_{\mu\nu\lambda} \Omega^{a}_{\alpha} \partial_\lambda f(\epsilon_{ka}) + [(\nu, \Omega) \leftrightarrow (\lambda, \Omega)],
$$

where we introduced the Berry curvature for the $\alpha$-th band as

$$
\Omega^{a}_{\alpha} = \epsilon_{\mu\nu\lambda} \partial_{\alpha} \epsilon_{\nu\mu}^a = \frac{i}{2} \sum_{b \neq a} \epsilon_{\mu\nu\lambda} \left( \xi^{b}_{ab} \xi_{\nu\mu}^a - \xi^{b}_{ab} \xi_{\mu\nu}^a \right).
$$

Conducting a partial derivative in the last line, the formula is transformed to the well-known form

$$
\sigma_{ei}^{\mu,\nu,\lambda} = - \frac{iq^3}{2\hbar^2 \Omega^2} \int \frac{dk}{(2\pi)^3} \sum_a (\epsilon_{\mu\nu\lambda} \partial_\lambda \Omega^{a}_{\alpha} - \epsilon_{\mu\nu\lambda} \partial_\lambda \Omega^{a}_{\alpha}) f(\epsilon_{ka}),
$$

which is called Berry curvature dipole term [54, 55]. Here we introduce the Berry curvature dipole defined by

$$
D^{\mu\nu} = \int \frac{dk}{(2\pi)^3} \sum_a f(\epsilon_{ka}) \partial_\mu \Omega^{a}_{\nu}.
$$
as

$$
\sigma_{\nu \lambda}^{\mu \nu} = i \epsilon_{\nu \lambda} \sigma_{\epsilon \lambda}^{\mu \lambda},
$$

(53)

$$
= - \frac{q^3}{\hbar^2 \Omega} (D_{\mu \nu} - \delta_{\mu \nu} \text{Tr}[D]),
$$

(54)

which depends on the frequency of irradiating lights as $O(\Omega^{-1})$.

The symmetry of the Berry curvature dipole is the same as that of the CP-photocurrent tensor, and hence it is allowed in the $T$-preserved gyrotrropic crystals [54, 55]. In contrast, in $PT$-symmetric systems, the Berry curvature $\Omega^\mu_\nu$ vanishes at each $k$ point since it is odd-parity under the $PT$-operation. Thus, the photocurrent response derived from $\sigma_{\epsilon \lambda}$ is regarded as the Berry curvature dipole effect which is unique to the $T$-symmetric and metallic systems, whereas it is forbidden in the $PT$-symmetric or insulating systems.

C. Fermi sea effect I: injection current

we next consider the $\sigma_{\epsilon \lambda}$ term [Eq. (28)]. Especially, in this subsection we focus on the diagonal component of the velocity operator $v^\mu_{\epsilon \lambda} (\epsilon = b)$ and denote the corresponding conductivity tensor as $\sigma_{\epsilon \epsilon \lambda}$. With the condition Eq. (29), the equation is recast as

$$
\sigma_{\epsilon \epsilon \lambda}^{\mu \nu} = \lim_{\omega \to 0} \frac{-i\pi q^3}{\hbar \omega} \int \frac{dk}{(2\pi)^d} \sum_{a \neq c} \Delta_{\epsilon \lambda}^{\mu \nu} \epsilon_{\epsilon \lambda} f_{ac} \delta(h \Omega - \epsilon_a),
$$

(57)

Eq. (57) is rewritten as

$$
\sigma_{\epsilon \epsilon \lambda}^{\mu \nu} = \lim_{\omega \to 0} \frac{-i\pi q^3}{\hbar \omega} \int \frac{dk}{(2\pi)^d} \sum_{a \neq c} \Delta_{\epsilon \lambda}^{\mu \nu} \epsilon_{\epsilon \lambda} f_{ac} \delta(h \Omega - \epsilon_a),
$$

(59)

$$
= \lim_{\omega \to 0} \frac{-i\pi q^3}{\hbar \omega} \int \frac{dk}{(2\pi)^d} \sum_{a \neq c} \Delta_{\epsilon \lambda}^{\mu \nu} \left( g_{\epsilon \lambda}^{\mu \nu} - \frac{i}{2} \Omega_{\epsilon \lambda}^{\mu \nu} \right) f_{ac} \delta(h \Omega - \epsilon_a),
$$

(60)

where we introduce the band-resolved quantum metric and Berry curvature which are respectively given by

$$
g_{\epsilon \lambda}^{\mu \nu} = \frac{1}{2} (\xi_{\epsilon \lambda \mu \nu} + \epsilon_{\epsilon \lambda \mu \nu}),
$$

(61)

$$\Omega_{\epsilon \lambda}^{\mu \nu} = i (\xi_{\epsilon \lambda \mu \nu} - \epsilon_{\epsilon \lambda \mu \nu}).
$$

(62)

These geometric quantities are related to the U(1) quantum metric and Berry curvature as $g_{\epsilon \lambda}^{\mu \nu} = \sum_{\epsilon \neq \lambda} g_{\epsilon \lambda}^{\mu \nu}$ and $\Omega_{\epsilon \lambda}^{\mu \nu} = \sum_{\epsilon \neq \lambda} \epsilon^{\mu \nu} \Omega_{\epsilon \lambda}^{\mu \nu}/2$ [63]. The band-resolved quantum metric (Berry curvature) is symmetric (anti-symmetric) under $\nu \leftrightarrow \lambda$ and contributes to the LP-photocurrent (CP-photocurrent).

Eq. (60) is the general formula for the photocurrent arising from the component $\sigma_{\epsilon \epsilon \lambda}$. Then, we proceed to the classification by the $T$- and $PT$-symmetries below. Beforehand, we investigate the transformation property of geometric quantities under those symmetry operations. As shown in Appendix A, the Berry connection is transformed by these symmetries as $\xi_{\epsilon \lambda \mu \nu} (k) = \xi_{\epsilon \lambda \mu \nu} (-k)$ for the $T$-symmetry and as $\epsilon_{\epsilon \lambda \mu \nu} (k) = -\epsilon_{\epsilon \lambda \mu \nu} (k)$ for the $PT$-symmetry. Accordingly, the band-resolved geometric quantities are transformed as

$$
g_{\epsilon \lambda}^{\mu \nu} (k) = g_{\epsilon \lambda}^{\mu \nu} (-k),
$$

(63)

$$\Omega_{\epsilon \lambda}^{\mu \nu} (k) = -\Omega_{\epsilon \lambda}^{\mu \nu} (-k),
$$

(64)

for the $T$-symmetry and as

$$
g_{\epsilon \lambda}^{\mu \nu} (k) = g_{\epsilon \lambda}^{\mu \nu} (k),
$$

(65)

$$\Omega_{\epsilon \lambda}^{\mu \nu} (k) = -\Omega_{\epsilon \lambda}^{\mu \nu} (k),
$$

(66)

for the $PT$-symmetry. Making use of the fact that the group velocity difference $\Delta_{\epsilon \lambda}^{\mu}$ is odd/even under $T/PT$-symmetry, we can show that either of the band-resolved quantum metric or Berry curvature contributes to the photocurrent response.

In the $T$-symmetric systems, the corresponding photocurrent is obtained as

$$
\sigma_{\epsilon \epsilon \lambda}^{\mu \nu} (T) = \lim_{\omega \to 0} \frac{-i\pi q^3}{\hbar \omega} \int \frac{dk}{(2\pi)^d} \sum_{a \neq c} \Delta_{\epsilon \lambda}^{\mu \nu} \epsilon_{\epsilon \lambda} f_{ac} \delta(h \Omega - \epsilon_a),
$$

(55)

which satisfies the anti-symmetric condition under the permutation $\nu \leftrightarrow \lambda$, and hence represents the CP-photocurrent. This is called 'injection current' [14]. Fol-
lowing the definition in Eq. (40), we obtain the CP-photocurrent tensor

\[ \kappa^{\mu\nu}_{\text{inj}} = i\epsilon_{\nu\lambda\tau}\sigma^{\mu\lambda}_{\text{ee}d}(T) \]

\[ = \lim_{\omega \to 0} \frac{-i\pi q^3}{2\hbar\omega} \int \frac{dk}{(2\pi)^d} \sum_{a \neq c} \epsilon_{\nu\lambda\tau}\Delta^{\mu\lambda}_{ac}\Omega^{\nu\lambda}_{ac}f_{ac}\delta(h\Omega - \epsilon_{ca}). \]  

(66)

The band-resolved Berry curvature is further simplified by the circular representation of the Berry connection given by [64]

\[ \xi^{\nu}_{ab} = \frac{1}{\sqrt{2}}(\xi^{+}_{ab} \pm i\xi^{-}_{ab}). \]  

(67)

On the basis of this representation, Eq. (62) is recast as

\[ \Omega^{\nu\lambda}_{ab} = |\xi^{+}_{ab}|^2 - |\xi^{-}_{ab}|^2, \]  

(68)

which indicates the optical transition amplitude under irradiating circularly-polarized light [64]. Accordingly, Eq. (66) with \( \nu = z \) is rewritten as

\[ \kappa^{\mu\nu}_{\text{inj}} \]

\[ = \lim_{\omega \to 0} \frac{i\pi q^3}{h\omega} \int \frac{dk}{(2\pi)^d} \sum_{a \neq c} (|\xi^{+}_{ab}|^2 - |\xi^{-}_{ab}|^2) \Delta^{\mu\nu}_{ac}f_{ac}\delta(h\Omega - \epsilon_{ca}). \]  

(69)

The injection current in the \( T \)-symmetric systems arises from the band-resolved Berry curvature. Therefore, non-magnetic Weyl semimetals hosting the divergent Berry curvature, such as TaAs, are potential candidates which show a giant injection current response [22, 65]. On the other hand, the \( PT \)-symmetry requires that the Berry curvature vanishes at each \( k \). Hence, the injection current in the \( PT \)-symmetric systems originates from the band-resolved quantum metric. The formula is written by

\[ \sigma^{\mu\nu}_{\text{ee}d}(PT) \]

\[ = \lim_{\omega \to 0} \frac{-i\pi q^3}{h\omega} \int \frac{dk}{(2\pi)^d} \sum_{a \neq c} \Delta^{\mu\nu}_{ac}\Omega^{\nu\lambda}_{ac}f_{ac}\delta(h\Omega - \epsilon_{ca}). \]  

(70)

This expression satisfies the symmetric property for the permutation \( \nu \leftrightarrow \lambda \). Thus, the photocurrent is classified as a LP-photocurrent. This result is consistent with Refs. [44–46]. The response tensor is given by \( \eta^{\mu\nu\lambda}_{\text{inj}} = \sigma^{\mu\nu\lambda}_{\text{ee}d}(PT)/2 + \sigma^{\mu\nu\lambda}_{\text{ee}d}(PT)/2 \) with Eq. (70). In contrast to the band-resolved Berry curvature, the band-resolved quantum metric represents the dipole transition amplitude under the linearly-polarized light.

As shown above, the geometric property related to the injection current is different between the \( T \)-symmetric and \( PT \)-symmetric systems. Whereas the CP-photocurrent in the former is owing to the band-resolved Berry curvature, the LP-photocurrent in the latter arises from the band-resolved quantum metric. Thus, we distinguish the injection currents allowed in the \( T \)-and \( PT \)-symmetric systems as "electric injection current" and "magnetic injection current", respectively [see Table 1].

The general formula in Eq. (60) is decomposed as

\[ \sigma^{\mu\nu\lambda}_{\text{ee}d} = \eta^{\mu\nu\lambda}_{\text{inj}} - \frac{i}{2}\epsilon_{\nu\lambda\tau}\kappa^{\mu\tau}_{\text{inj}}, \]  

(71)

and both of the electric and magnetic injection currents are allowed in the absence of the \( T \) and \( PT \)-symmetry. We will see a parallel discussion for the rectification current and shift current in Sec. III D.

In addition to the quantum geometric quantities, two factors are responsible for these injection currents; joint density of states and group-velocity difference \( \Delta^{\mu}_{ac} \). The joint density of states is defined as

\[ J(\Omega) = \sum_{a \neq c} \int \frac{dk}{(2\pi)^d} \delta(h\Omega - \epsilon_{ca}). \]  

(72)

It measures the number of electrons excited by illuminating light having the frequency \( \Omega \) and plays a crucial role in linear optical responses [66]. \( J(\Omega) \) is strongly enhanced in the presence of the generalized van Hove singularity where the following condition is satisfied

\[ \partial_{k\epsilon_{ac}} \equiv 0. \]  

(73)

The generalized van Hove singularity originates not only from a pair of usual van Hove singularities given by \( \partial_{k\epsilon_{ka}} = \partial_{k\epsilon_{kc}} \equiv 0 \) but also from the subspace in the Brillouin zone satisfying \( \partial_{k\epsilon_{ka}} = \partial_{k\epsilon_{kc}} \neq 0 \). The factor \( \Delta^{\mu}_{ac} \), however, weakens the contribution from the latter singularity points. Thus, it may be important for a sizeable injection current to make use the normal van Hove singularity satisfying

\[ \partial_{k\epsilon_{ka}} \equiv 0, \partial_{k\epsilon_{kc}} \equiv 0, \partial_{\mu\epsilon_{kc}} \cdot \partial_{\mu\epsilon_{ka}} < 0, \]  

(74)

where the coordinate \( k_{\mu} \) denotes the direction of the injection current. Such dispersion can be found in prototypical direct-gap semiconductors.

Peculiarly, response coefficients of the injection currents diverge in the limit of \( \omega \to 0 \). This seemingly unphysical behavior can be bounded by the scattering rate [18]. By introducing the scattering rate \( \gamma \), the matrix \( d^{\nu}_{ab} \) is modified as

\[ d^{\nu}_{ab} = \frac{1}{h\omega + i0 - \epsilon_{ab}} \rightarrow \frac{1}{h\omega + i\gamma - \epsilon_{ab}}. \]  

(75)

Accordingly, for instance, the formula of Eq. (66) is replaced with

\[ \kappa^{\mu\nu}_{\text{inj}} \rightarrow \pi q^3 \int \frac{dk}{(2\pi)^d} \sum_{a \neq c} \Delta^{\mu\nu}_{ac}\Omega^{\nu\lambda}_{ac}f_{ac}f_{ac} \frac{1}{(h\Omega - \epsilon_{ca})^2 + \gamma^2}. \]  

(76)

Then, the expression converges in the limit \( \omega \to 0 \).
Finally, we analyze the remaining terms, that is, the \( \sigma_{ee} \) term with the off-diagonal component of \( v_{ab} \) in Eq. (28) and the \( \sigma_{ie} \) term in Eq. (27). We denote the former contribution by \( \sigma_{ee+ie} \).

When we consider the photocurrent response by adopting Eq. (29), the formula for \( \sigma_{ie} \) is recast with use of Eq. (7) as

\[
\sigma_{ie}^{\mu,\nu,\lambda} = \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^3} \left[ \sum_{a \neq b} \left[ -\partial_{\nu} \xi^{\mu}_{ab} + i \left( \xi^{\mu}_{aa} - \xi^{\mu}_{ba} \right) \xi^{\lambda}_{ab} \xi^{\nu}_{ba} \sigma_{ab} \Omega \right] \right. \\
+ \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right]. \tag{77}
\]

It is convenient to introduce the U(1)-covariant derivative \( D \) which acts on the physical quantity in the Bloch representation as \([13, 14, 17]\)

\[
[D_{\mu} O]_{ab} = \partial_{\mu} O_{ab} - i \left( \xi^{\mu}_{aa} - \xi^{\mu}_{ba} \right) O_{ab}. \tag{78}
\]

Then, we rewrite Eq. (77) as

\[
\sigma_{ie}^{\mu,\nu,\lambda} = \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^3} \left[ \sum_{a \neq b} -\left[ D_{\nu} \xi^{\mu}_{ab} \right] \xi^{\lambda}_{ab} \xi^{\nu}_{ba} \sigma_{ab} \Omega \right] \\
+ \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right]. \tag{79}
\]

Similar expression can be found in the term \( \sigma_{ee} \) which is given by

\[
\sigma_{ee}^{\mu,\nu,\lambda} (\omega; \omega_1, \omega_2) = \frac{q^3}{2} \int \frac{dk}{(2\pi)^3} \left[ \sum_{a \neq b \neq c} v^{\omega_1}_{ab} d^{\omega_2}_{ba} \left( d^{\omega_1}_{ca} \xi^{\nu}_{bc} \xi^{\lambda}_{ca} \sigma_{ca} \Omega \right) \right] \\
+ \left[ (\nu, \omega_1) \leftrightarrow (\lambda, \omega_2) \right]. \tag{80}
\]

In the condition Eq. (29), the formula is recast as

\[
\sigma_{ee+ie}^{\mu,\nu,\lambda} = \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^3} \left[ \sum_{a \neq b \neq c} i \xi^{\mu}_{ab} \left( d^{\nu}_{ca} \xi^{\lambda}_{bc} \xi^{\nu}_{ca} \sigma_{ca} \Omega \right) \right. \\
+ \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right], \tag{81}
\]

\[
= \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^3} \left[ \sum_{a \neq b \neq c} i \left( \xi^{\mu}_{ab} \sigma_{bc} \xi^{\nu}_{ab} \sigma_{ca} \xi^{\lambda}_{ba} \right) \xi^{\lambda}_{ca} \sigma_{ca} \Omega \right] \\
+ \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right], \tag{82}
\]

where we use Eq. (7) in the first line. As for the summation over the band index \( b \), we can use the following formula \([13]\)

\[
[D_{\mu} \xi^{\nu}_{ab}]_{ac} - [D_{\nu} \xi^{\mu}_{ab}]_{ac} = \sum_{b \neq a, c} i \left( \xi^{\mu}_{ab} \sigma_{bc} \xi^{\nu}_{ac} \sigma_{ca} \right). \tag{83}
\]

The \( \sigma_{ee+ie} \) term is therefore rewritten by

\[
\sigma_{ee+ie}^{\mu,\nu,\lambda} = \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^3} \sum_{a \neq b} \left[ (D_{\mu} \xi^{\nu}_{ab})_{ac} - (D_{\nu} \xi^{\mu}_{ab})_{ac} \right] \xi^{\lambda}_{ac} \sigma_{ac} \Omega \\
+ \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right]. \tag{84}
\]

Summing up Eqs. (79) and (84), we obtain a simplified expression as

\[
\sigma_{ee+ie}^{\mu,\nu,\lambda} = \sigma_{ee}^{\mu,\nu,\lambda} + \sigma_{ie}^{\mu,\nu,\lambda} \\
= \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^3} \sum_{a \neq b} \left[ D_{\mu} \xi^{\nu}_{ab} \right]_{ab} \xi^{\lambda}_{ba} \sigma_{ab} \Omega \\
+ \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right]. \tag{85}
\]

Using Eq. (58), the formula is decomposed into

\[
\sigma_{ee+ie}^{\mu,\nu,\lambda} = \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^3} \sum_{a \neq b} f_{ab} \left[ \frac{1}{\hbar \Omega - \epsilon_{ba}} \right] \\
\times \left[ \text{Re} A_{ab}^{\mu,\nu,\lambda} \frac{1}{\hbar \Omega - \epsilon_{ba}} + \pi \text{Im} A_{ab}^{\mu,\nu,\lambda} \delta(\hbar \Omega - \epsilon_{ba}) \right]. \tag{86}
\]

Here, we introduced

\[
S_{ab}^{\mu,\nu,\lambda} = [D_{\mu} \xi^{\nu}_{ab} \xi^{\lambda}_{ab}] + [D_{\nu} \xi^{\mu}_{ba} \xi^{\lambda}_{ba}], \tag{87}
\]

\[
A_{ab}^{\mu,\nu,\lambda} = [D_{\mu} \xi^{\nu}_{ab} \xi^{\lambda}_{ab}] - [D_{\nu} \xi^{\mu}_{ba} \xi^{\lambda}_{ba}]. \tag{88}
\]

Owing to the Hermitian property of the Berry connection, general formulas for the LP and CP-photocurrent coefficients are obtained as

\[
\sigma_{ee+ie}^{\mu,\nu,\lambda} = \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^3} \sum_{a \neq b} f_{ab} \left[ \frac{1}{\hbar \Omega - \epsilon_{ba}} \right] \\
\times \left[ \text{Re} S_{ab}^{\mu,\nu,\lambda} \frac{1}{\hbar \Omega - \epsilon_{ba}} + \pi \text{Im} A_{ab}^{\mu,\nu,\lambda} \delta(\hbar \Omega - \epsilon_{ba}) \right]. \tag{89}
\]

and

\[
\mathcal{S}_{ee+ie}^{\mu,\nu,\lambda} = \epsilon_{\tau,\nu,\lambda} \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^3} \sum_{a \neq b} f_{ab} \left[ \frac{1}{\hbar \Omega - \epsilon_{ba}} \right] \\
\times \left[ \text{Im} S_{ab}^{\mu,\nu,\lambda} \frac{1}{\hbar \Omega - \epsilon_{ba}} - \pi \text{Re} A_{ab}^{\mu,\nu,\lambda} \delta(\hbar \Omega - \epsilon_{ba}) \right]. \tag{90}
\]

which do not include any imaginary component.

Now, we present a symmetry classification of the general expressions, Eqs. (89) and (90), as we did for the injection current. The \( T \)-symmetry leads to the relation \([67]\)

\[
[D_{\mu}(k) \xi^{\nu}_{ab} \xi^{\lambda}_{ba}] = - [D_{\mu}(-k) \xi^{\nu}_{ab} \xi^{\lambda}_{ba}] \xi^{\lambda}_{ab}(-k). \tag{91}
\]

Combining this relation with the relation \( \epsilon_{ka} = -\epsilon_{-ka} \) ensured by the \( T \)-symmetry, Eq. (86) is transformed as

\[
\sigma_{ee+ie}^{\mu,\nu,\lambda}(T) = \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^3} \sum_{a \neq b} f_{ab} \left[ \frac{1}{\hbar \Omega - \epsilon_{ba}} \right] \\
\times \left[ \text{Im} S_{ab}^{\mu,\nu,\lambda} \frac{1}{\hbar \Omega - \epsilon_{ba}} + \pi \text{Im} A_{ab}^{\mu,\nu,\lambda} \delta(\hbar \Omega - \epsilon_{ba}) \right]. \tag{92}
\]
This is the photoconductivity formula in the $\mathcal{T}$-symmetric systems. The integrand including the principal value and that with delta function are anti-symmetric and symmetric under the permutation $\nu \leftrightarrow \lambda$, respectively. Thus, the former corresponds to the CP-photocurrent called (electric) rectification current \[14, 68\],

$$
\kappa_{\text{ER}}^{\mu \tau} = -\frac{q^3}{\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_{a \neq b} \epsilon_{\nu \lambda \tau} \text{Im} \left( [D_{\mu} \xi_{ab}^\lambda] f_{ab} P \frac{1}{M} - \epsilon_{ba} \right).
$$

(93)

By using the band-resolved Berry curvature, $\kappa_{\text{ER}}^{\mu \tau}$ is rewritten as

$$
\kappa_{\text{ER}}^{\mu \tau} = \frac{q^3}{\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_{a \neq b} \partial_{\mu} R_{ab,\nu}^\lambda f_{ab} P \frac{1}{M} - \epsilon_{ba}.
$$

(94)

On the other hand, the latter is the LP-photocurrent called shift current \[11, 14\],

$$
\eta_{\text{shift}}^{\mu \nu \lambda} = \frac{q^3}{2\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_{a \neq b} \partial_{\mu} \delta(\hbar \Omega - \epsilon_{ba})

\times \text{Im} \left( [D_{\mu} \xi_{ab}^\lambda] f_{ab} P \frac{1}{M} - \epsilon_{ba} \right).
$$

(95)

Taking both components into account, we denote the total photoconductivity as follows

$$
\sigma_{\text{rec+ic}}^{\mu \nu \lambda}(\mathcal{T}) = \eta_{\text{shift}}^{\mu \nu \lambda} - \frac{i}{2} \epsilon_{\nu \lambda \tau} \kappa_{\text{ER}}^{\mu \tau}.
$$

(96)

The rectification current is a polarization current from the second-order nonlinear polarization and hence does not demand the light to be resonantly coupled to the electron bands \[14\]. The polarization current arises from the reversible process of the polarization dynamics, and thus, it vanishes under the $\mathcal{T}$-preserved perturbation such as the linearly-polarized light. Indeed, the rectification current in Eq. (96) is a CP-photocurrent, where the $\mathcal{T}$-symmetry is violated by the circularly-polarized light. On the other hand, the shift current is caused by the resonantly-excited band electrons and hence it can appear under the linearly-polarized light.

Here, we discuss the shift current term in details. Following the prescription presented in Ref. \[14\], we decompose the Berry connection into the magnitude and phase

$$
\xi_{ab}^\nu = |\xi_{ab}^\nu| \exp(-i\phi_{ab}^\nu).
$$

(97)

$|\xi_{ab}^\nu| = |\xi_{ba}^\nu|$ and $\phi_{ab}^\nu = -\phi_{ba}^\nu$ are satisfied by the Hermitian property. The shift current formula Eq. (95) is recast as

$$
\eta_{\text{shift}}^{\mu \nu \lambda} = -\frac{q^3}{2\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_{a \neq b} \left( R_{ab,\nu}^\mu + R_{ab,\nu}^{\mu \lambda} \right) \delta(\hbar \Omega - \epsilon_{ba})

\times \text{Im} \left( [D_{\mu} \xi_{ab}^\lambda] f_{ab} P \frac{1}{M} - \epsilon_{ba} \right). \nonumber
$$

(98)

where we introduced so-called shift vector defined by

$$
R_{ab,\nu}^\mu = \partial_{\mu} \phi_{ab}^\nu + \epsilon_{\nu \lambda a} - \xi_{bb}^\nu.
$$

(99)

This vector implies the positional shift of the excited electron along the $\mu$-direction inside a unit cell through the interband transition $a \leftrightarrow b$ \[69, 70\]. In particular, taking $\nu = \lambda$, we obtain the well-known formula for the shift current \[14\],

$$
\eta_{\text{shift}}^{\mu \nu \nu} = -\frac{q^3}{2\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_{a \neq b} R_{ab,\nu}^\mu f_{ab} \delta(\hbar \Omega - \epsilon_{ba}).
$$

(100)

Note that the shift vector and band-resolved quantum metric are individually invariant against the $U(1)$-gauge transformation. The shift current \[Eq. (100)\] is in sharp contrast to the magnetic injection current \[Eq. (70)\], another LP-photocurrent allowed in insulators. The shift current is described by the shift vector in the real-space picture, whereas the magnetic injection current current arises from the group-velocity difference $\Delta^\nu_{ab}$ which is a characteristic property in the momentum-space. The joint density of states and band-resolved quantum metric play important roles in both LP-photocurrents.

Now we move on to the photocurrent in the $\mathcal{PT}$-symmetric systems, a main topic of this paper. We can simplify Eq. (86) by making use of the $\mathcal{PT}$-symmetry. After the parallel discussion, we obtain

$$
\sigma_{\text{rec+ic}}^{\mu \nu \lambda}(\mathcal{PT}) = \frac{q^3}{2\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_{a \neq b} f_{ab}

\times \left[ \text{Re} \ S_{ab}^{\mu \nu \lambda} P \frac{1}{M} - i\pi \text{Re} \ A_{ab}^{\mu \nu \lambda} \delta(\hbar \Omega - \epsilon_{ba}) \right].
$$

(101)
term containing the delta function represent responses to the linearly-polarized light and to the circularly-polarized light, respectively. Thus, the photoconductivity consists of a LP-photocurrent arising from the polarization current and a CP-photocurrent due to the resonantly-excited electrons.

In contrast to the electric rectification current, the new LP-photocurrent, a photo-induced polarization current, does not need violation of T-symmetry since it is broken by magnetic order in equilibrium. Thus, we call the LP-photocurrent magnetic rectification current. The formula is obtained as
\[
\eta_{\text{MR}}^{\mu\nu\lambda} = \frac{q^3}{2\hbar} \int \frac{dk}{(2\pi)^d} \sum_{\alpha \neq \beta} f_{\alpha\beta} \frac{1}{\hbar \Omega - \epsilon_{\alpha\beta}} \times \text{Re} \left( [D_\mu \xi_\alpha^{\nu\lambda}]_{\alpha\beta} \xi_{\alpha\beta}^{\mu\lambda} + [D_\mu \xi_\beta^{\nu\lambda}]_{\alpha\beta} \xi_{\alpha\beta}^{\mu\lambda} \right),
\]
(102)

which is related to the derivative of the band-resolved quantum metric while the electric rectification current is given by the derivative of the band-resolved Berry curvature as shown in Eq. (94).

On the other hand, the CP-photocurrent is the counterpart of the shift current as we will discuss in Sec. V A in details. This photocurrent has properties distinguished from the shift current. It is induced by the circularly-polarized photon instead of the linearly-polarized photon, and unique to the magnetically-parity-violating system. We therefore call the response gyration current. The gyration current formula is given by
\[
\kappa_{\text{gyro}}^{\mu\nu} = \frac{\pi q^3}{h} \int \frac{dk}{(2\pi)^d} \sum_{\alpha \neq \beta} \epsilon_{\nu\lambda\tau} \text{Re} \left( [D_\mu \xi_\alpha^{\tau\nu}]_{\alpha\beta} \xi_{\alpha\beta}^{\mu\lambda} f_{\alpha\beta} \delta(\hbar \Omega - \epsilon_{\alpha\beta}) \right).
\]
(103)

It is noteworthy that only the gyration current is induced by the circularly-polarized light in the PT-symmetric systems. Therefore, we can unambiguously detect the gyration current by measuring the CP-photocurrent. This is not the case of the CP-photocurrent of T-symmetric systems because of the admixture of various CP-photocurrents such as the Berry curvature dipole term and electric injection current [22] [see Table I]. The photocurrent measurements may be useful to identify the symmetry of a parity-violating order parameter in magnetic materials. Combining the gyration current with the magnetic rectification current, we obtain the photoconductivity in the PT-symmetric systems,
\[
\sigma_{\text{ec+ee}}^{\mu\nu\lambda}(\mathbb{T}^T) = \eta_{\text{MR}}^{\mu\nu\lambda} - \frac{i}{2} \epsilon_{\nu\lambda\tau} \kappa_{\text{gyro}}^{\mu\tau\nu}.
\]
(104)

Collecting Eqs. (96) and (104), we rewrite the general formula for the \(\sigma_{\text{ec+ee}}\) term and decompose it into the LP-photocurrent and CP-photocurrent,
\[
\eta_{\text{MR}}^{\mu\nu\lambda} = \eta_{\text{shift}}^{\mu\nu\lambda} + \eta_{\text{MR}}^{\mu\nu\lambda},
\]
(105)
\[
\kappa_{\text{gyro}}^{\mu\nu} = \kappa_{\text{ER}}^{\mu\nu} + \kappa_{\text{gyro}}^{\mu\nu}.
\]
(106)

Summarizing this section, we reproduced the formulas for several known photocurrent responses, and uncovered new photocurrents, magnetic rectification current and gyration current. In Sec. V, we discuss the gyration current in details. As we have seen, the \(T\) and \(PT\)-symmetry play contrasting roles. The obtained classification completes all the photocurrent responses within the free particle approximation and provides clear decomposition of the general photoconductivity coefficients [See Eqs. (71), (105), and (106)]. When both \(T\)- and \(PT\)-symmetries are broken, the photocurrent allowed by each symmetry is admixed with each other. Similar discussion can be found in the second-order nonlinear conductivity [60].

IV. GENERALIZATION TO SPINFUL SYSTEMS

The formulation is straightforwardly generalized to the spinful system. Classification of the LP/CP-photocurrent [Table II] does not depend on whether the system is spinless or spinful. The photoconductivity formula in the \(PT\)-symmetric systems, however, is slightly modified due to the Kramers degeneracy appearing at each \(k\).

Owing to the double degeneracy ensured by the \(PT\)-symmetry, the Bloch states have U(2)-gauge degree of freedom at least. Note that the gyration current formula in spinless systems [Eq. (103)] is not invariant under the U(2)-gauge transformation. Thus, we modify the decomposition of nonlinear conductivity tensor in Eq. (24) so as to be U(2)-gauge invariant. Firstly, the Berry connection is divided as
\[
\xi_{\mu\nu}^A = \alpha_{\mu\nu} + A_{\mu\nu}^A,
\]
(107)
where the intraband Berry connection \(\alpha_{\mu\nu}^A\) is introduced for the degenerate bands satisfying \(\epsilon_{\alpha\beta} = \epsilon_{\beta\alpha}\). With the decomposition of the Berry connection, the intraband position operator \(r_{\mu\nu}^A\) is modified as
\[
(r_{\mu\nu}^A)_{\alpha\beta} = i \partial_{\mu\nu} \delta_{\alpha\beta} + \alpha_{\mu\nu}^A,
\]
(108)
and the interband position operator is given by \((r_{\mu\nu}^A)_{\alpha\beta} = A_{\mu\nu}^A\). Accordingly, we define the band-resolved quantum metric and Berry curvature by
\[
\gamma_{\mu\nu}^A = \frac{1}{2} (A_{\mu\nu}^A A_{\alpha\beta}^A - A_{\mu\nu}^A A_{\alpha\beta}^A),
\]
(109)
\[
\Omega_{\mu\nu}^A = i (A_{\mu\nu}^A A_{\alpha\beta}^A - A_{\mu\nu}^A A_{\alpha\beta}^A).
\]
(110)
Based on the U(2)-type position operators, we divide the nonlinear optical conductivity into four terms. The calculation can be done as in the spinless systems, and hence we give the derivation in Appendix B.

In the following, we consider formulas for the photocurrent in the \(PT\)-symmetric and spinful systems, that is, the Drude term, magnetic injection current, magnetic
rectification current, and gyration current. The Drude term is the same as Eq. (45) except for the Kramers degree of freedom included in the summation over the band indices. In the case of spinful systems, the antisymmetrically distorted band structure is realized by the coupling between the parity-violating magnetic order and the sublattice-dependent spin-orbit coupling [37, 38].

Similarly, the photocurrent formula for the magnetic injection current is obtained by replacing the Berry connection in Eq. (70) with the interband Berry connection in Eq. (70) with the interband Berry connection, whereas the formulas for the magnetic rectification current and gyration current are obtained as

\[ \kappa_{\text{gyro}}^\mu = \frac{\pi q^3}{\hbar} \int \frac{dk}{(2\pi)^d} \sum_{a \neq b} f_{ab} \delta(h\Omega - \epsilon_{ba}) \times \text{Re} \left( i \left[ D_\mu \xi^+ \right]_{ab} \xi^-_{ba} - i \left[ D_\mu \xi^- \right]_{ab} \xi^+_{ba} \right). \]  

(113)

Note that this formula can be applied to the system without \( PT \)-symmetry. We write the left/right-handed Berry connections \( \xi^\pm \) by

\[ \xi^\pm_{ab} = |\xi^\pm_{ab}| \exp(-i\phi^\pm_{ab}), \]

(114)

which satisfy the relation \( \phi^+_{ab} = -\phi^-_{ba} \) due to the Hermitian property of \( \xi^\pm_{ab} \) (\( \mu = x, y \) and \( a \neq b \)). Then, the gyration current formula is recast as

\[ \kappa_{\text{gyro}}^\mu = \frac{\pi q^3}{\hbar} \int \frac{dk}{(2\pi)^d} \sum_{a \neq b} f_{ab} \delta(h\Omega - \epsilon_{ba}) \times \left( R^\mu_{ab,+} |\xi^+_{ab}|^2 - R^\mu_{ab,-} |\xi^-_{ab}|^2 \right). \]  

(115)

Here, we introduced chiral shift vector given by

\[ R^\mu_{ab,\pm} \delta_\mu \phi^\pm_{ab} + \xi^\mu_{\pm ab} \xi^-_{ab}, \]

(116)

which is invariant under the U(1)-gauge transformation. The meaning of Eq. (115) is clear. Corresponding to the handedness of the dipole transition amplitude denoted by \( |\xi^\pm_{ab}|^2 \), the circularly-polarized light excites the electrons. Through the interband transition \( a \leftrightarrow b \), the excited electron makes positional shift determined by the chiral shift vector. The resulting electrons’ flow gives rise to the gyration current. Interestingly, a similar expression has been obtained in a recent study of a circularly-induced nonlinear polarization in a layered system [63].

The transition amplitudes, \( |\xi^\pm_{ab}|^2 \), are further decomposed into

\[ |\xi^\pm_{ab}|^2 = (g^x_{ab} + g^y_{ab}) \mp \Omega^y_{ab}, \]

(117)

which consist of the band-resolved quantum metric and Berry curvature. Although other photocurrents allowed in insulators are related to either of the band-resolved quantum metric or Berry curvature, the gyration current is derived from both geometric quantities. Using the decomposition in Eq. (117), Eq. (115) is transformed as [63]

\[ \kappa_{\text{gyro}}^\mu = \frac{\pi q^3}{\hbar} \int \frac{dk}{(2\pi)^d} \sum_{a \neq b} f_{ab} \delta(h\Omega - \epsilon_{ba}) \times \left[ R^\mu_{ab,+} - R^\mu_{ab,-} \right] \Omega^y_{ab} \]

(118)

where \( \Omega^y_{ab} \) is the U(2)-gauge invariant. We can straightforwardly show that all the obtained expressions are U(2)-gauge invariant.

In conclusion, although the photocurrent formulas for the spinful system are mostly the same as those for the spinless system, the rectification current and gyration current are modified due to the different gauge symmetry. Note that the formulation can be easily generalized to the system having \( n \)-fold degenerated bands. In particular, in a high-symmetric subspace of the Brillouin zone manifold, a high degeneracy with \( n = 4, 6 \) may exist in a symmetry-enforced way [65]. Hence, our formulation gives insights into the photocurrent responses arising from such multi-fold degenerate fermions [65, 71, 72].

We comment that the U(2)-gauge invariant formulation becomes unnecessary when the \( PT \)-symmetry is absent. Then, Kramers degeneracy is lifted, and the U(2)-covariant derivative is replaced by that for the U(1)-gauge [Eq. (78)]. In particular, calculations of \( T \)-symmetric spinful systems can be conducted as in the spinless case. Thus, the formulas of the photocurrent are the same as those for spinless systems.

V. ANALYSIS OF GYRATION CURRENT

In this section, we investigate the gyration current response [Eq. (103) for spinless systems and Eq. (112) for spinful systems] in details. After revealing basic properties in Sec. V A, we present a microscopic study based on a spinful model in Sec. V B. Furthermore, we show a giant gyration current arising from divergent geometric quantities in a topological antiferromagnet [Sec. V C].

A. Basic property

Firstly, we consider the spinless system for simplicity. Since the gyration current is induced by the circularly-polarized light, it is convenient to adopt the circular representation as in the electric injection current. With the circularly-polarized light along the \( z \)-direction, the response formula is rewritten as

\[ \kappa_{\text{gyro}}^\mu = \frac{\pi q^3}{\hbar} \int \frac{dk}{(2\pi)^d} \sum_{a \neq b} f_{ab} \delta(h\Omega - \epsilon_{ba}) \times \text{Re} \left( i \left[ D_\mu \xi^+ \right]_{ab} \xi^-_{ba} - i \left[ D_\mu \xi^- \right]_{ab} \xi^+_{ba} \right). \]

(113)

where \( \Omega \) is the U(1)-covariant derivative. We can straightforwardly show that all the obtained expressions are U(2)-gauge invariant.

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When we impose the $\mathcal{PT}$-symmetry, we have

$$
\kappa_{\text{gyro}}^{\mu z} = \frac{2\pi q^2}{\hbar} \sum_{a \neq b} f_{ab} \delta(\hbar \Omega - \epsilon_{ba}) R_{ab}^{\mu} + (g_{ab}^{xx} + g_{ab}^{yy}).
$$

(119)

where we used the relations, $\Omega_{\mu \nu} = 0$ and $\phi_{\mu \nu} = -\phi_{\mu \nu} + \pi$.

Next, we consider the gyration current in the spinful system [Eq. (112)]. For the U(2)-gauge description, we assume the $\mathcal{PT}$-symmetric system below. The formula is recast as

$$
\kappa_{\text{gyro}}^{\mu z} = \frac{2\pi q^2}{\hbar} \int \frac{dk}{(2\pi)^2} \sum_{a \neq b} f_{ab} \delta(\hbar \Omega - \epsilon_{ba})

\times \text{Re} \left( i \left[ \mathcal{D}_\mu \sigma^+ \right]_{ab} A_{ba}^{\sigma^+} - i \left[ \mathcal{D}_\mu \sigma^- \right]_{ab} A_{ba}^{\sigma^-} \right),
$$

(120)

where we used the relations, $\Omega_{\mu \nu} = 0$ and $\phi_{\mu \nu} = -\phi_{\mu \nu} + \pi$.

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\times \text{Re} \left( (R_{ab}^{\mu})^2 - (\sigma_{\mu \nu})^2 \right)

+ \alpha_{\mu \nu}^{\sigma} \left( A_{ab}^{\sigma} - A_{ba}^{\sigma} \right)

+ \alpha_{\mu \nu}^{\sigma} \left( A_{ab}^{\sigma} - A_{ba}^{\sigma} \right),
$$

(121)

where $(a, \bar{a})$ denotes the Kramers pair ensured by the $\mathcal{PT}$-symmetry and we introduced the circular representation of the Berry connection $A_{ab}^{\sigma}$ as in Eq. (67). The gyration current in spinful systems is described by not only the chiral shift vector but also the intra-Kramers Berry connection $\alpha_{\mu \nu}^{\sigma}$. Owing to the spin degree of freedom, the contribution from the band-resolved Berry curvature is not canceled out in the $\mathcal{PT}$-symmetric systems in contrast to the formula for the spinless fermions [Eq. (119)].

B. Model study of gyration current

In this section, we present a microscopic calculation of the gyration current in a spinful model. The $\mathcal{PT}$-preserved but $\mathcal{P}$-broken system is realized by the antiferromagnetic order in locally-noncentrosymmetric systems. The locally-noncentrosymmetric system hosts crystalline sublattices whose site-symmetry lacks $\mathcal{P}$-symmetry while the global $\mathcal{P}$-symmetry is preserved by interchanging the sublattice. Such peculiar crystal symmetry gives rise to the sublattice-dependent anti-symmetric spin-orbit coupling (sASOC) [37, 38, 73, 74]. In many cases, effects of the sASOC do not appear in macroscopic phenomena while the spin- and momentum-resolved spectroscopy can capture a fingerprint of the sASOC [75]. On the other hand, a sublattice-dependent order unveils the sASOC in the way that a coupling between the sASOC and order parameter gives rise to nontrivial electronic structures and cross-correlated responses [76]. For instance, the combination of sASOC with antiferromagnetic order leads to an asymmetric band dispersion, which is an essential ingredient in the Drude term. Note that such parity-breaking magnetic systems exist in a broad range of compounds [35, 42, 43].

The adopted Hamiltonian is modeled after such parity-violating magnets. A two-dimensional rectangular lattice system consists of two sublattices labeled as A and B. Owing to the locally-noncentrosymmetric property, the site-symmetry is denoted by the noncentrosymmetric point group $C_{2v}$ (mm2), while the global symmetry is centrosymmetric labeled by $D_{2h}$ (mmm). In the point group $C_{2v}$, the Rashba-type ASOC and (anisotropic) Dresselhaus-type ASOC are allowed [77]. Thus, the system hosts these types of ASOC in the sublattice-dependent way as its sASOC. With the tight-binding approximation, the Bloch Hamiltonian is given by

$$
H(k) = \left( \begin{array}{c} \epsilon_0(k) + g_A(k) \cdot \sigma \\ V_{AB}(k) \end{array} \right),
$$

(122)

where $\sigma$ and $\tau$ are Pauli matrices representing the spin and sublattice degrees of freedom, respectively. The components are defined as

$$
\epsilon_0(k) = -t (\cos k_x + \cos k_y),
$$

(123)

$$
V_{AB}(k) = -2t \cos \frac{k_x}{2} \cos \frac{k_y}{2},
$$

(124)

$$
g_A(k) = g_0(k) + h_{AF} = \left( \begin{array}{c} h_{AF}^{xx} - \alpha_R \sin k_y + \alpha_D \sin k_y \\ h_{AF}^{yy} - \alpha_R \sin k_x + \alpha_D \sin k_x \\ h_{AF}^{zz} \end{array} \right),
$$

(125)

and $g_B(k) = -g_A(k)$. The parameters $t = 1.0$ and $\tilde{t} = 0.5$ are intra-sublattice and inter-sublattice hopping parameters, respectively. Importantly, we introduce the Rashba-type and Dresselhaus-type sASOC parameterized by $\alpha_R = 0.2$ and $\alpha_D = 0.4$, respectively. We take the molecular field for the antiferromagnetic order as $h_{AF} = (1.6, 0, 0)$, which represents $x$-collinear antiferromagnetic order. The doubly-degenerate energy spectrum for Eq. (122) is obtained as

$$
\epsilon_{k \pm} = \epsilon_0(k) \pm \sqrt{V_{AB}(k)^2 + g(k)^2}. \quad (126)
$$

Mainly owing to the large molecular field $h_{AF}$, two degenerate bands are separated by the energy gap, $\delta \epsilon = 2\sqrt{V_{AB}(k)^2 + g(k)^2}$. The point group symmetry is denoted by $mmm$ lacking the $\mathcal{P}$-symmetry in the antiferromagnetic state. Indeed, the antiferromagnetic order parameter is characterized by the odd-parity irreducible representation $B_{2u}$ of the point group $D_{2h}$. According to the reduced symmetry,

$$
\kappa_{\text{gyro}}^{xx} \neq 0, \quad \kappa_{\text{gyro}}^{yy} = 0. \quad (127)
$$

Note that we can only take the index $\nu = z$ in $\kappa_{\text{gyro}}^{\nu \nu}$ because of the absence of the $k_z$-dispersion in the two-dimensional model. A lot of well-known magnetoelectric insulators such as LiTPO$_4$ ($T = Fe, Co, Ni$) [78–83] are characterized by the same irreducible representation and allow the gyration current response in Eq. (127).
In addition to the gyration current response function, we calculate the joint density of states \( J(\Omega) \) in Eq. (72) and the attenuation coefficient \( \varepsilon_{\text{att}} \) given by [14, 21, 84]

\[
\varepsilon_{\text{att}}^{\mu\nu} = i\pi q^2 \int \frac{dk}{(2\pi)^2} \sum_{a \neq b} \mathcal{A}_{ab}^{\mu\nu} A_{ba}^{\mu\nu} f_{ab} \delta(h\Omega - \epsilon_{ba}), \tag{128}
\]

\[
= i\pi q^2 \int \frac{dk}{(2\pi)^2} \sum_{a \neq b} \left( \frac{g_{ab}^{\mu\nu} - i\Omega_{ab}^{\mu\nu}}{2} \right) f_{ab} \delta(h\Omega - \epsilon_{ba}), \tag{129}
\]

which is derived from the dissipative part of the expectation value \( \text{Tr}[\text{gr}_e P^{(1)}] \) with the interband position operator \( r_e \) and the first-order perturbed density matrix \( P^{(1)} \). Under the linearly-polarized light along the \( \mu \)-direction, the attenuation coefficient is solely determined by the band-resolved quantum metric \( g_{ab}^{\mu\nu} \). Thus, the comparison between the shift current coefficient \( \sigma_{\text{shift}} \) and the symmetric component of the attenuation coefficient \( \varepsilon_{\text{att}}^{\mu\nu} \) is informative [8, 15, 16, 21]. On the other hand, the attenuation of the circularly-polarized light arises from both of the band-resolved quantum metric and Berry curvature [63, 64]. We define the attenuation coefficients of the left-handed (+) and right-handed (−) circularly-polarized lights as

\[
\varepsilon_{\text{att}}^{\pm} = \frac{1}{2}(\varepsilon_{xx}^{\pm} + \varepsilon_{yy}^{\pm}) \mp \frac{i}{2}(\varepsilon_{xx}^{\pm} - \varepsilon_{yy}^{\pm}), \tag{130}
\]

\[
= i\pi q^2 \int \frac{dk}{(2\pi)^2} \sum_{a \neq b} \left[ \frac{1}{2} g_{ab}^{xy} + g_{ab}^{yx} \right] + \frac{1}{2} \Omega^{xy}_{ab} f_{ab} \delta(h\Omega - \epsilon_{ba}), \tag{131}
\]

\[
= i\pi q^2 \int \frac{dk}{(2\pi)^2} \sum_{a \neq b} \frac{1}{2} |\mathcal{A}_{ab}|^2 f_{ab} \delta(h\Omega - \epsilon_{ba}). \tag{132}
\]

In the \( T-/PT\)-symmetric system, the band-resolved Berry curvature does not contribute to the attenuation coefficients in Eq. (131) due to the Kramers degeneracy. Thus, in the numerical calculation, we calculate \( \varepsilon_{\text{att}} = \varepsilon_{xx}/2 + \varepsilon_{yy}/2 \) and take a dimensionless value defined by \( \tilde{\varepsilon}_t = \varepsilon_{\text{att}}/\varepsilon_0 \), where \( \varepsilon_0 \) and \( l \) are the vacuum permittivity and thickness of the system, respectively [21]. We show the numerically-calculated gyration current coefficient \( \kappa_{\text{gyro}}^{\mu\nu} \), attenuation coefficient \( \varepsilon_{\text{att}} \), and joint density of states \( J(\Omega) \) [85] in Fig. 1. For numerics, we approximate the delta function in Eqs. (72), (112), and (132) by the Lorentian function. This treatment corresponds to taking into account a phenomenological scattering rate \( \gamma = 0.01 \). We assume the absolute zero temperature \( (T = 0) \) and fix the chemical potential between the two bands in Eq. (126). The system in the insulating state satisfies \( f(\epsilon_{ba}) = 0 \) for the upper band \( (a = \pm) \) and \( f(\epsilon_{ba}) = 1 \) for the lower band \( (a = -) \).

Figure 1 plots the frequency dependence. We see that the three quantities mostly share the peak positions. Thus, it is indicated that the frequency dependence of the gyration current coefficient is roughly determined by the joint density of states. This is consistent with the conventional understanding of the optical conductivity [66]. On the other hand, in the presence of a geometrically nontrivial electronic structure, the gyration current may show strong enhancement which cannot be attributed to the joint density of states. As an example, we investigate the gyration current in a topologically nontrivial antiferromagnet in the next subsection.

C. Enhanced gyration current in topological materials

Dirac and Weyl electrons with gapless band dispersions give rise to various nontrivial phenomena. For instance, geometric properties of such electronic structure lead to unconventionally large nonlinear responses such as nonlinear Hall effect [55, 62, 86], higher harmonic generations [19, 31], injection current [22–24, 26–28, 30, 71, 87], and shift current [69, 88]. Based on these findings, we investigate the possibility of the giant gyration current response in topological materials.

The model Hamiltonian is obtained by taking the parameters in Eq. (122) as

\[
t = 0.08, \quad \tilde{t} = 1, \quad \alpha_R = 0.8, \quad \alpha_D = 0, \quad \hbar_{\text{AF}} = (0.6, 0, 0) . \tag{133}
\]

This model has been proposed as an effective two-dimensional model Hamiltonian of tetragonal CuMnAs [89]. We plot the band dispersion of the Hamiltonian in Fig. 2. Interestingly, gapless points appear along the high-symmetry line \( (k_x = \pi) \). Appearance of the gapless points is due to the facts that the sASOC over-
whelm the molecular field and that the inter-sublattice hoppings are forbidden by the mirror symmetry denoted by \( \{ M_{x}[1/2, 0, 0] \} \) [40, 89–91]. The coordinates of the gapless points are analytically obtained as \((\pi, \pi/2 \pm k_0)\) with \(k_0 = \arccos(h_{AB}/c_{R}) \in (0, \pi/2)\). Here we denoted \(h_{AB} = h_{AB} \hat{x}\), and adopt the energy unit \(|t| = 1\) [eV] for a quantitative estimation.

![Image of quantum estimation](image)

**FIG. 2.** Band structure of the Hamiltonian [Eq. (122) with Eq. (133)]. (Left panel) Dispersion over all the Brillouin zone. (Right panel) Enlarged view around the gapless Dirac points.

To calculate the gyration current arising from the gapless band electrons, we analyze an effective Dirac Hamiltonian given by

\[
H(\mathbf{k}; s_z) = v_0 k_y + a_1 k_y \sigma_x - a_2 k_x \sigma_y + w k_x \tau_x + \Delta, \tag{134}
\]

where the coefficients are obtained from the microscopic parameters as

\[
v_0 = t \cos k_0, \quad a_1 = \alpha_R \sin k_0 s_z, \quad a_2 = \alpha_R, \quad w = t \cos \left( \frac{\pi/2 + s_z k_0}{2} \right), \quad \Delta = t \sin k_0 s_z. \tag{135}
\]

We introduced the label \(s_z = \uparrow, \downarrow\), representing the Dirac nodes at \((\pi, \pi/2 \pm k_0)\) and \((\pi, \pi/2 \pm k_0)\), respectively. Note that the \(v_0\) term gives rise to the energy shift of the Dirac cones along the \(y\)-axis, whereas \(\Delta\) gives the opposite energy shift to the two Dirac nodes. Below we show that the tilting is important for enhancement of the gyration current.

Here, we take one of the Dirac nodes and calculate contribution to the gyration current coefficient. Introducing the polar coordinate by \( \rho \sin \theta = |a_1| k_y \) and \( \rho \cos \theta = (a_2^2 + w^2)^{1/2} k_x \), we write the energy spectrum of Eq. (134) as

\[
\epsilon_{k\pm; s_z} = \rho \left( \frac{v_0}{|a_1|} \sin \theta \pm 1 \right) + \Delta. \tag{136}
\]

Owing to the double degeneracy, summation over the band indices can be computed with fixing the momentum \( \mathbf{k} \) in Eq. (120). When we take the frequency of light as \( \Omega > 0 \) and assume the absolute zero temperature \( T = 0 \), the summation is evaluated as

\[
\sum_{a=-} \sum_{b=+} \text{Re} \left( i [\mathbf{D}_\mu \mathbf{A}^+]_{ab} \mathbf{A}^\dagger_{ba} - i [\mathbf{D}_\mu \mathbf{A}^-]_{ab} \mathbf{A}^\dagger_{ba} \right) = \frac{1}{\rho^2} a_1^2 (a_2^2 + w^2) \sin \theta. \tag{137}
\]

The summation was taken over the lower degenerated bands for \(a\) and over the upper degenerated bands for \(b\), respectively. We notice that the gyration current is totally canceled out if the tilting parameter is zero, since the energy dispersion is symmetric under \( k_y \rightarrow -k_y \) when \( v_0 = 0 \). Thus, the tilting parameter is an essential ingredient for the gyration current response.

After some simple algebra, we obtain the analytical expression for the gyration current coefficient as

\[
\kappa^{\rm \text{gyro}}_{xy}(\Omega) = \sum_{s_z=\uparrow, \downarrow} \frac{2g^2}{\pi \hbar^4 \Omega^2} (a_2^2 + w^2)^{1/2} \text{sgn}(v_0) \times \text{Re} \left[ \sqrt{1 - \frac{v_0^2}{v_0^2} \frac{\mu + \Delta}{\hbar \Omega/2 + 1} - 1} - \sqrt{1 - \frac{a_1^2}{v_0^2} \frac{\mu + \Delta}{\hbar \Omega/2 - 1} - 1} \right]. \tag{138}
\]

Differences in contributions from the two Dirac nodes can be found in the energy shift of the Dirac nodes \(\Delta\) and in \(w(s_z = \uparrow) \neq w(s_z = \downarrow)\). Otherwise, Dirac electrons around \((\pi, \pi/2 \pm k_0)\) equally contribute to the gyration current response. As a result, the tilting parameters \(v_0\) and \(\Delta\) play two important roles as illustrated in Fig. 3. Firstly, the tilting of each Dirac cone due to \(v_0\) prevents the gyration current from compensation of the contributions from \(\pm k_y\). Secondly, cancellation between the gyration current from the two Dirac cones is suppressed when the opposite potential shift \(\Delta\) sufficiently separates the Dirac nodes. Supposing a small potential difference \(\Delta\), the gyration current is partially compensated in the low-frequency regime as shown in the lower panel of Fig. 5. Therefore, Dirac nodes separated along the energy axis are favorable for a divergent photocurrent response in the low-frequency regime. Consequently, for an enhanced gyration current response, it is important to hunt for materials hosting strongly tilted gapless dispersions such as the type II Dirac materials [92, 93].

We clarify the impact of tilting by taking the Dirac node labeled by \(s_z = \uparrow\). For a fixed frequency \(\Omega\), the gyration current appears in the region given by

\[
- \frac{\hbar \Omega}{2} \left( \frac{|v_0|}{a_1} + 1 \right) \leq \mu - t \sin k_0 \leq \frac{\hbar \Omega}{2} \left( \frac{|v_0|}{a_1} - 1 \right), \tag{139}
\]

for \(\mu < t \sin k_0\), and

\[
- \frac{\hbar \Omega}{2} \left( \frac{|v_0|}{a_1} - 1 \right) \leq \mu - t \sin k_0 \leq \frac{\hbar \Omega}{2} \left( \frac{|v_0|}{a_1} + 1 \right), \tag{140}
\]
FIG. 3. Mechanism of the enhanced gyration current response in the tilted Dirac system. A coordinate $q_y = k_y - \pi/2$ is introduced. (a) Contributions from $q_y = k_0 \pm k$ are not canceled out because of the tilting of a single cone. A dotted arrow represents the transition prohibited by Pauli blockade. (b) The opposite energy shift $\pm |\Delta|$ of the nodes prevents cancellation between two Dirac nodes.

for $\mu > t \sin k_0$. For the parameters in Eq. (133), $|v_0/a_1| = 0.11 < 1$. Thus, the chemical potential has energy windows where the gyration current response is finite. The width $\delta \Omega_1 = \hbar \Omega |v_0/a_1|$ increases in proportion to the frequency $\Omega$, while it vanishes in the non-tilted system ($v_0 = 0$). When the chemical potential lies in the window, the gyration current is extensively enhanced as $O(\Omega^{-2})$ in the low-frequency regime.

When the tilting parameter $v_0$ increases, the system changes from a type-I Dirac system ($|v_0/a_1| < 1$) to a type II Dirac system ($|v_0/a_1| > 1$). In the type-II Dirac system, the width of the energy window reaches as large as $\delta \Omega_{II} \geq \hbar \Omega$. On the other hand, the upper and lower energy windows given in Eqs. (139) and (140) overlap with each other, and hence the gyration current is partially canceled out. The tilting parameters do not influence the maximal value of the gyration current coefficient as shown in Fig. 4, because the Berry connection itself is not relevant to the trace of the Dirac Hamiltonian in Eq. (134).

On the basis of the analytical formula in Eq. (138), we plot $(\mu, \Omega)$ dependence of the gyration current coefficient by taking both of the two nodes into account (Fig. 5). It is clearly shown that the energy windows of two nodes grow from the offset energies given by $\Delta$ and overlap near $\Omega \sim 2|\Delta| \sim 0.1$ [eV]. In the overlapped region, the total gyration current coefficient is decreased by partial cancellation. Interestingly, the gyration current shows divergent behavior in the low-frequency regime $\Omega \ll 1$. Taking an available low-frequency light in the Terahertz regime $\Omega = 1$ [meV], the energy window of each node is evaluated as

$$0.44 \leq |\mu - 52| \leq 0.56 \quad \text{for } s_z = \uparrow, \quad (141)$$

$$0.44 \leq |\mu + 52| \leq 0.56 \quad \text{for } s_z = \downarrow, \quad (142)$$

where the unit of the millivolt is abbreviated. The gyration current coefficient is estimated as large as $|\kappa^{xz}_{\text{gyro}}| \sim 10$ [A·V$^{-2}$].

FIG. 4. Chemical potential dependence of the gyration current coefficient for the frequency of light $\Omega = 100$ [meV]. Several lines are shown by changing the ratio of $v_0$ to $a_1$. The ratio $v_0/a_1 = -0.1, -0.8, \text{ and } -1.5$ represent the type-I Dirac (red line), highly-tilted type-I Dirac (blue line), and type-II Dirac regimes (green line). We introduced $\Delta_+ = t \sin k_0$ for the energy shift of the Dirac node $s_z = \uparrow$.

FIG. 5. (Upper panel) Chemical potential and frequency dependences of the gyration current coefficient in the unit [A·V$^{-2}$]. (Lower panel) Chemical potential profile of the gyration current coefficient for $\Omega = 100$ [meV].
Here we move on to a slightly-gapped Dirac system which is realized when $|h_{AF}| \geq |\alpha_B|$. We numerically calculate the gyration current response and find that the massive Dirac system is also responsible for an enhanced gyration current. By increasing the molecular field so as to surpass the sASOC, the two Dirac nodes are merged at $(k_x, k_y) = (\pi, \pi/2)$ and then turn into the massive Dirac dispersion. Fig. 6 shows the numerical results of Eq. (120) with the discretized Brillouin zone mesh $N = 1500^2$ and a phenomenological scattering rate $\gamma = 0.01$ [eV]. We assume an insulating state at $T = 0$, that is, the chemical potential is positioned in the energy gap. Such electronic structure may be realized in MnBi$_2$Te$_4$ thin film consisting of the double septuple layers [94]. Interestingly, we see a large gyration current coefficient $|k_{gyro}^x| \sim 100$ [$\mu$A·V$^{-2}$] for a relatively high frequency $h\Omega \sim 100$ [meV] of light. The coefficient is therefore expected to be an order of magnitude larger than the photoconductivity of typical semiconductors such as GaAs [21].

The enhanced photocurrent response may be attributed to two reasons. One is that the quadratic band edge at $(k_x, k_y) = (\pi, \pi/2)$ forms a generalized van Hove singularity [see Eq. (73)]. The van Hove singularity gives rise to a large joint density of states $J(\Omega)$ leading to an enhanced gyration current, while this factor is absent in the linear and gapless Dirac system. The other reason is that the geometric quantity is still large in a slightly-gapped regime. As the antiferromagnetic molecular field $h_{AF}$ increases and geometric quantity becomes smaller, the maximum value of the gyration current coefficient is suppressed [inset of Fig. 6]. The exchange splitting due to the antiferromagnetic order grows as the temperature is lowered, and the gyration current is therefore expected to show a drastic temperature dependence. This nontrivial temperature dependence is a striking property of the photocurrent in magnetically-parity-violating systems.

To discuss geometric properties of the system in more details, we introduce a quantity defined by

$$G^{yz}(k_x, k_y) = \frac{\pi a^3}{\hbar} \sum_{a, b; unocc.} \text{Re} \left( i [\mathfrak{D}_\mu A^+]_{ab} A^-_{ba} - i [\mathfrak{D}_\mu A^-]_{ab} A^+_{ba} \right),$$

(143)

which is indeed a part of integrand in Eq. (120). We also consider a momentum-resolved gyration current coefficient defined by

$$\delta \kappa_{gyro}^x (k_x, k_y) = G^{yz}(k_x, k_y) \delta(h\Omega - \delta \epsilon),$$

(144)

where $\delta \epsilon$ is the energy gap. Since both of $G^{yz}$ and $\delta \kappa_{gyro}^x$ show a dipolar profile [Figs. 7 (b,c)] around the massive Dirac point at $(k_x, k_y) = (\pi, \pi/2)$, the total gyration current coefficient is seemingly canceled out by integration over $(k_x, k_y)$. However, the cancellation is actually prevented by an asymmetric energy dispersion along the $k_y$ axis. The inter-sublattice hopping $V_{AB}(k)$ gives rise to the asymmetry of the energy gap between the momentum $(k_x, k_y)$ and $(k_x, \pi - k_y)$, and hence makes the net gyration current uncompensated. Uncompensation can be seen in the distribution of the symmetrized gyration current coefficient defined by $\delta \kappa_{gyro}^x (k_x, k_y)$ [Fig. 7 (d)]. Thus, the microscopic origin of the enhanced gyration current response is different between massless and massive Dirac systems. In the former the asymmetric band gap due to the inter-sublattice hopping plays an important role, while in the latter cancellation is prevented by the combination of tilting in the Dirac nodes and Pauli blockade [see Fig. 3].

VI. SUMMARY AND DISCUSSION

In this work, we systematically investigated the second order photocurrent and uncovered new types of photocurrent, named magnetic rectification current and gyration current. Our formalism is based on the well-established perturbative calculations [12–14], and presents formulas unifying the $PT$-symmetric parity-violated system (magnetic parity-violation) and the $T$-symmetric parity-violated one. We showed that the $T$- and $PT$-symmetry play contrasting roles in the classification of photocurrent responses. The symmetry determines which the linearly-polarized light or circularly-polarized light generates the photocurrent via the injection current, rectification current, and shift current. Our formulation also identifies the geometric quantities which give rise to these photocurrent responses. Making use of the result of classification, we found a chiral photocurrent arising from the gyration current in the $PT$-symmetric systems, which is the counterpart of the shift current in the $T$-symmetric systems [8, 11, 14].

We also elucidated that the gyration current is enhanced in the topological systems. On the basis of the minimal model for the $PT$-symmetric and topologically
FIG. 7. (a) Energy dispersion in the vicinity of \((k_x, k_y) = (\pi, \pi/2)\) in the slightly-gapped regime \((\hbar_{AF} = 0.85)\). The red-colored region represents the momentum where interband transitions are allowed with the frequency \(\Omega = 120 \text{[meV]}\) and scattering rate \(\gamma = 10 \text{[meV]}\). The momentum-resolved distributions are shown for (b) \(G^{zz}(k_x, k_y)\), (c) \(\tilde{\kappa}^{zz}_{\text{gyro}}(k_x, k_y)\), and (d) \(\tilde{\kappa}^{zz}_{\text{gyro}}(k_x, k_y) + \tilde{\kappa}^{zz}_{\text{gyro}}(k_x, \pi - k_y)\).

nontrivial antiferromagnet CuMnAs, we derived analytical expressions revealing a divergent gyration current in the low-frequency regime. In particular, tilting of the gapless Dirac dispersion is an essential ingredient for the enhanced gyration current. As shown in Fig. 6, massive Dirac systems may also cause an enhanced gyration current due to relatively large joint density of states and quantum geometric quantity.

More elaborate investigations of the gyration current in various topological materials are desirable, although this work focuses on two-dimensional Dirac electron systems. Recent studies have clarified that some magnetic space groups can ensure the multi-fold degeneracy at high-symmetry points in the Brillouin zone [65]. For instance, \(\text{Cu}_3\text{TeO}_6\) undergoes the parity-violating magnetic order and may possess six-fold degenerate electrons at the Brillouin zone corner [65, 95–97]. Although this compound is insulating and the degenerated states do not lie near the Fermi energy, related compounds may be a potential candidate to realize a giant gyration current response.

This work completes all the photocurrent responses of the band electrons. On the other hand, electron correlation effect may enrich the photocurrent phenomena [98]. Indeed, it has been shown that the strong correlation influences frequency dependence of photocurrent responses [69]. Furthermore, it has been proposed that the photocurrent can be generated through the bosonic excitations such as electromagnon and exciton in correlated systems [99–101]. Thus, interplay of correlation effects and topological electronic structures in the photocurrent generation is desired to be clarified in the future work. Moreover, relaxation of photo-excited electrons should be elaborated for more accurate description of photocurrent responses. In fact, recent experiments have used ultrafast spectrometry and successfully observed dynamics of the photo-electrons [30, 101–104]. These previous experiments worked on the photocurrent responses in \(\mathcal{T}\)-symmetric systems. On the other hand, because the antiferromagnetic magnon excitations are present, the time-resolved dynamics of photo-electrons in the \(\mathcal{PT}\)-symmetric magnetic systems may show relaxation distinct from nonmagnetic systems.

Interest on the gyration current will be shared in a vast range of the field in condensed matter physics, such as optoelectronic, multiferroics, spintronics, and topological science. In particular, the gyration current coefficient is sensitive to the parity-violating magnetic order. Thus, it may enable us to observe domain states via optical probes and to realize a magnetically-switchable photocurrent response [44]. We expect that further studies of the gyration current will be beneficial not only for fundamental research clarifying magnetic compounds but also for applications to multi-functional devices where the light, spins, and electrons are closely correlated with each other.

Note added—Recently a theoretical work on the same topic is conducted by J. Ahn and N. Nagaosa [105]. They also successfully show the \(\mathcal{T}/\mathcal{PT}\) correspondence and propose the enhanced photocurrent responses in
topological materials. Their results are consistent with ours although each work has been done in a completely independent way. We sincerely thank J. Ahn and N. Nagao for sending the manuscript before submission and agreeing to the simultaneous submission.

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Appendix A: Symmetry considerations of $T$- and $PT$-symmetries

In this section, we introduce basic transformation properties under the anti-unitary operations such as $T$- and $PT$-symmetries. Let us consider an anti-unitary symmetry described by an operator $a = \theta g$, where $\theta$ and $g$ are the time-reversal operation and a unitary symmetry operation, respectively. In particular, we take the space-inversion operation $g = I$ for the $PT$-symmetry, whereas $g = 1$ for the $T$-symmetry. Bloch states at the momentum $\k$ are related to those at $-g^{-1}\k$. The transformation property is given by

$$a \lvert u_a(\k) \rangle = \lvert u_a(-g^{-1}\k) \rangle w_{ba}(\k), \quad (A1)$$

where the matrix $\tilde w(\k)$ is unitary. In the following, we describe the basic transformation properties in spinless and spinful systems.

1. spinless system

In the spinless system, the time-reversal operation is expressed by the complex conjugation operator, $\tilde = \k$. Then, the unitary matrix can be taken as the scalar $\tilde w(\k) = 1$ when $g^2 = 1$. Owing to the equation $\theta^2 = K^2 = 1$, we obtain the formula

$$\langle u_a(\k) \rvert u_b(\k) \rangle = \langle \theta u_a(\k) \rvert \theta u_b(\k) \rangle = \langle u_a(-\k) \rvert u_b(-\k) \rangle. \quad (A2)$$

Thus, the $T$-symmetry gives constraint on the Berry connection

$$\xi_{ab}^\mu(\k) = \xi_{ba}^\mu(-\k). \quad (A3)$$

Similarly, we obtain

$$\xi_{ab}^c(\k) = -\xi_{ba}^c(\k), \quad (A4)$$

for the $PT$-symmetry.

In general, the $\tilde w(\k)$ can take an arbitrary phase factor due to the U(1)-gauge degree of freedom. For instance, for a gauge $\theta \lvert u_a(\k) \rangle = \lvert u_a(-\k) \rangle \exp[-i\phi_a(\k)]$, the Berry connection satisfies the relation

$$\xi_{ab}^\mu(\k) = \xi_{ba}^\mu(-\k) e^{-i[\phi_a(\k) - \phi_b(\k)]}. \quad (A5)$$

The formulas of Eqs. (63) and (91), however, are irrelevant to the choice of the matrix $\tilde w(\k)$. This is consistent with the fact that the obtained formulas for photocurrent responses are U(1)-gauge invariant.

2. spinful system

In the presence of the spin degree of freedom, the matrix $\tilde w(\k)$ is at least two-dimensional and has no diagonal component according to the Kramers theorem. When $g^2 = 1$, the unitary matrix $\tilde w(\k)$ is written as

$$\tilde w(\k) = \begin{pmatrix} 0 & e^{-i\phi_k} \\ e^{-i\theta_k} & 0 \end{pmatrix}, \quad (A6)$$

where $\theta_k$ and $\phi_k$ denote real-valued functions of $\k$. Owing to the Kramers theorem,

$$-\lvert u_a(\k) \rangle = a^2 \lvert u_a(\k) \rangle = a \lvert u_b(-g^{-1}\k) \rangle w_{ba}(\k) \rangle, \quad (A7)$$

$$= \lvert u_c(\k) \rangle \lvert w_{ba}^*(\k)w_{cb}(g^{-1}\k) \rangle, \quad (A8)$$

leads to the relation

$$\theta(\k) = \phi(-g^{-1}\k) + \pi. \quad (A9)$$

Therefore, we obtain the unitary matrix

$$\tilde w(\k) = \begin{pmatrix} 0 & e^{-i\phi(\k)} \\ -e^{-i\theta(-g^{-1}\k)} & 0 \end{pmatrix}, \quad (A10)$$

which describes the transformation property between doubly-degenerate states. Especially, when we take the gauge so as to satisfy $\theta(\k) \equiv 0$, the corresponding unitary matrix represents the well-known transformation property between the Kramers doublet

$$a \lvert u_\pm(\k) \rangle = \pm \lvert u_\mp(-g^{-1}\k) \rangle, \quad (A11)$$

where the subscript $\pm$ denotes the Kramers degree of freedom. Although the discussion can be generalized to other anti-unitary operations satisfying $g^2 \neq 1$, the above discussion sufficiently describes the $T$- and $PT$-symmetries.

Here, we proceed to analyze the $PT$-symmetry. For the $PT$-symmetry, $\k = -g\k$, and thus, we have the Kramers doublet labeled by $\sigma = \pm |u_{a\sigma}(\k)|$, where $a$ denotes the band index. Below we abbreviate the momentum dependence unless explicitly stated. The Kramers doublet is related to each other by Eq. (A1), and the unitary matrix $\tilde w(\k)$ is

$$\tilde w(\k) = i\sigma_g e^{-i\theta}. \quad (A12)$$

Note that we take into account a band-independent phase factor $\theta$.

First, we show the proof of the formula

$$A_{ab}^\mu(\k)A_{ba}^\nu(\k) = A_{ba}^\mu(\k)A_{ab}^\nu(\k), \quad (A13)$$
where $A_{ab}^\mu$ is the interband component of the U(2) Berry connection in Eq. (107), and $(s, s)$ labels a Kramers pair. The transformation property of the Berry connection is obtained as

$$\xi_{\sigma a, b; \tau} = i \langle u_{\sigma a} | \partial_b u_{\sigma b} \rangle, \quad (A14)$$

$$= i \langle a (\partial_b u_{\sigma b}) | a (u_{\sigma a}) \rangle, \quad (A15)$$

$$= i [\partial_b (u_{\sigma a} | u_{\sigma b})] w_{\tau \sigma}, \quad (A16)$$

$$= i [\partial_b (u_{\sigma a} | u_{\sigma a}) + i \partial_b \theta (u_{\sigma a} | u_{\sigma a})] \times (i \sigma_y)^{\tau \sigma}, \quad (A17)$$

$$= (-\xi_{\sigma a, b; \tau} - \partial_b \delta_{\alpha \beta} \delta_{\tau \sigma}) (i \sigma_y)^{\tau \sigma}. \quad (A18)$$

Taking different band indices $a \neq b$ and applying Eq. (A18) to the product $A_{ab}^\mu A_{ba}^\nu$, we obtain Eq. (A13).

Similarly, we can derive the formula

$$[D_\mu (k) A^\nu (k)]_{ab} A_{ba}^\mu (k) = [D_\mu (k) A^\nu (k)]_{ba} A_{ba}^\mu (k), \quad (A19)$$

in which $D_\mu$ indicates the U(2)-gauge covariant derivative shown in Eq. (B1). For the band indices satisfying $\epsilon_{ka} \neq \epsilon_{kb}$, the covariant derivative of the Berry connection is transformed as

$$[D_\mu (k) \xi^\nu (k)]_{\sigma a, b; \tau} = \partial_\mu \xi^\nu_{\sigma a, b; \tau} - i \left(\xi^\mu_{\alpha \sigma; a} \xi^\nu_{\beta \sigma; b} - \xi^\mu_{\alpha \sigma; b} \xi^\nu_{\beta \sigma; a}\right) \xi^\nu_{\sigma a, b; \tau},$$

$$= - \partial_\mu \xi^\nu_{\sigma a, b; \tau} - i \left(\xi^\mu_{\alpha \sigma; a} + \partial_\mu \theta \xi^\mu_{\gamma \sigma; a} - \xi^\mu_{\beta \sigma; b} - \partial_\mu \theta \xi^\mu_{\gamma \sigma; b}\right) \xi^\nu_{\beta \sigma; a},$$

$$- i \xi^\mu_{\alpha \sigma; a} \xi^\nu_{\gamma \sigma; b} + i \xi^\nu_{\beta \sigma; a} \xi^\mu_{\gamma \sigma; b}) \times (i \sigma_y)^{\gamma \sigma; \tau} \times (i \sigma_y)^{\sigma' \sigma},$$

$$= - [D_\mu \xi^\nu_{\beta \sigma; a} (i \sigma_y)^{\gamma \sigma; \tau} (i \sigma_y)^{\sigma' \sigma}. \quad (A22)$$

Combining this equation with Eq. (A18), we obtain Eq. (A19). This equation is essential for the derivation of the gyration current formula in the main text. Note that a similar analysis can be conducted in the case of $\mathcal{T}$-symmetric spinful systems.

**Appendix B: U(2)-gauge description of photocurrent responses in $\mathcal{PT}$-symmetric systems**

In this section, we show derivation of the photocurrent formulas in the $\mathcal{PT}$-symmetric spinful systems. Previous theoretical studies considered non-degenerate Bloch states and characterized intraband effects by the diagonal component of the Bloch basis [13, 14]. This assumption is reasonable in the spinless system or in the $\mathcal{PT}$-violated spinful system. Indeed, much attention have been paid to the $\mathcal{P}$-broken nonmagnetic systems, and hence the U(1)-covariant formulation is sufficient to obtain gauge-invariant expressions. On the other hand, the $\mathcal{PT}$-symmetric and spinful systems have the Kramers degeneracy in the band structure at each $k$. Thus, we have to carefully proceed to the perturbative calculations with the use of the U(2)-covariant derivative as follows.

Using the U(2) intraband position operator in Eq. (108), the U(2)-gauge covariant derivative is defined by $D_\mu = -i \partial_\mu$. The derivative acts on the physical quantities in the Bloch representation $O_{ab}$ as

$$[D_\mu O]_{ab} = \partial_\mu O_{ab} - i \left(\sum_{c} O_{ac} \partial_{\mu} a_{cb} - \sum_{c} a_{ac} \partial_{\mu} O_{cb}\right). \quad (B1)$$

We can check that $[D_\mu O]_{ab}$ is U(2)-covariant by taking the U(2)-gauge transformation $|u_{ka}(k)\rangle \rightarrow |u_{kb}(k)\rangle U_{a b}$, where the summation of the band index $a$ is taken over the Kramers pair, $\epsilon_{ba} = \epsilon_{ka}$. The U(1) quantum metric and Berry curvature are defined by

$$g^\mu_{a a} = \frac{1}{2} \sum_{\beta} \sum_{b} (A^\mu_{a \sigma, b} A^\nu_{b \sigma, a} + A^\nu_{a \sigma, b} A^\mu_{b \sigma, a}), \quad (B2)$$

$$\Omega^\mu_{a a} = \frac{1}{2} \sum_{\beta} \sum_{b} \epsilon_{\mu \nu \lambda} (A^\nu_{a \sigma, b} A^\lambda_{b \sigma, a} - A^\lambda_{a \sigma, b} A^\nu_{b \sigma, a}), \quad (B3)$$

where we explicitly show the Kramers degree of freedom $\sigma$ for the $a$-th band. Accordingly, we define the band-resolved U(1) quantum metric and Berry curvature as

$$g^\mu_{a b} = \frac{1}{2} (A^\mu_{a b} A^\nu_{b a} + A^\nu_{a b} A^\mu_{b a}), \quad (B4)$$

$$\Omega^\mu_{a b} = i (A^\mu_{a b} A^\nu_{b a} - A^\nu_{a b} A^\mu_{b a}). \quad (B5)$$

Following the U(2)-covariant decomposition of the position operator, we divide the nonlinear optical conductivity into four parts [see Eq. (24)] as
Applying Eq. (A13) to Eq. (B11) and using the relation \[ \sigma \text{ Eq. (B9)} \]. The corresponding contribution is denoted by

Thus, the Berry curvature dipole term is forbidden due to the diagonal component of the velocity operator \( v \), under the photocurrent condition \[ \text{Eq. (29)} \]. The component \( \sigma_{ij} \) has the same form as that in the U(1)-covariant representation, whereas the remaining components are modified. Thus, we investigate the photocurrent responses arising from the components other than \( \sigma_{ij} \) in the following subsections. Note again that we consider systems preserving the \( \mathcal{PT} \)-symmetry.

1. Berry curvature dipole term

Under the condition Eq. (29), the component \( \sigma_{\text{ee}} \) is recast as

Using Eq. (A13), the expression is recast as

which is symmetric under the permutation \( \nu \leftrightarrow \lambda \). This corresponds to the formula for the magnetic injection current [Eq. (70)].

2. Injection current

We consider a part of the \( \sigma_{\text{ee}} \) term derived from the diagonal component of the velocity operator \( v^\mu_{ab} \) in Eq. (B9). The corresponding contribution is denoted by \( \sigma_{\text{ee}d} \). Under the photocurrent condition [Eq. (29)], the photocconductivity is rewritten as

Using Eq. (A13), the expression is recast as

3. Gyration current

Here, we calculate the remaining terms, that is, \( \sigma_{\text{ic}} \) and \( \sigma_{\text{ec}o} \). As for the component \( \sigma_{\text{ic}} \), we use Eq. (29) and...
arrange the integrand as

\[ i \nu^{\mu \nu}_{a b} d^{\Omega}_{b a} \left[ \partial_{\nu} \left( d^{\Omega}_{b a} a^{\mu}_{a b} A^{\nu}_{b a} \right) \right. \]
\[ \left. - i \left( \sum_{c} \alpha^{\nu}_{b c} d^{\Omega}_{c a} a^{\lambda}_{a c} f_{a c} - \sum_{c} \alpha^{\nu}_{c a} d^{\Omega}_{b c} A^{\lambda}_{b c} f_{c} \right) \right], \]
\[ \text{(B16)} \]

\[ = - A^{\nu}_{a b} \left[ \partial_{\nu} \left( d^{\Omega}_{b a} A^{\lambda}_{b a} \right) \right. \]
\[ \left. - i \left( \sum_{c} \alpha^{\nu}_{b c} d^{\Omega}_{c a} A^{\lambda}_{a c} f_{a c} - \sum_{c} \alpha^{\nu}_{c a} d^{\Omega}_{b c} A^{\lambda}_{b c} f_{c} \right) \right], \]
\[ \text{(B17)} \]

\[ = d^{\Omega}_{b a} f_{a b} A^{\nu}_{a b} \left[ d^{\Omega}_{\nu \lambda} \right]_{a b} - \partial_{\nu} \left( d^{\Omega}_{b a} f_{a b} A^{\mu}_{a b} \right). \]
\[ \text{(B18)} \]

Discarding the total derivative as a surface term, the \( \sigma_{\nu} \) term is simplified as

\[ \sigma_{\nu}^{\mu \nu \lambda} = \frac{q^3}{2 h} \int \frac{d k}{(2 \pi)^3} \sum_{a \neq b} \left[ [D_{\nu} A^{\nu}_{a b}]_{a b} A^{\lambda}_{b a} f_{a b} d^{\Omega}_{b a} \right. \]
\[ \left. + \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right] \right], \]
\[ \text{(B19)} \]

The component \( \sigma_{\epsilon c o} \) is written as

\[ \sigma_{\epsilon c o}^{\mu \nu \lambda} = \frac{q^3}{2} \int \frac{d k}{(2 \pi)^3} \sum_{a \neq b} \left[ v^{\mu}_{a b} d^{\Omega}_{b a} \right. \]
\[ \times \left( \sum_{c} d^{\Omega}_{c a} A^{\nu}_{b c} A^{\lambda}_{a c} f_{a c} - \sum_{c} d^{\Omega}_{b c} A^{\nu}_{c a} A^{\lambda}_{b c} f_{c} \right) \]
\[ \left. + \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right] \right], \]
\[ \text{(B20)} \]

Although the summation of the band indices includes the Kramers pair \((a, b) = (a, \bar{a})\), the matrix element of the velocity operator satisfies \( v^{\mu}_{a b} = 0 \) by taking the orthogonal Bloch states, \([\nu_{a}(k)] \) and \([\nu_{b}(k)] \). Owing to the adiabatic parameter in \( d^{\Omega}_{b a} = (+0)^{-1} \), we have \( v^{\mu}_{a b} d^{\Omega}_{b a} = 0 \). Thus, the integrand in Eq. \( \text{(B20)} \) is recast as

\[ \sum_{a \neq b} v^{\mu}_{a b} d^{\Omega}_{b a} \left( \sum_{c} d^{\Omega}_{c a} A^{\nu}_{b c} A^{\lambda}_{a c} f_{a c} - \sum_{c} d^{\Omega}_{b c} A^{\nu}_{c a} A^{\lambda}_{b c} f_{c} \right), \]
\[ \text{(B21)} \]

\[ = \sum_{a \neq b \neq c} i A^{\nu}_{a b} \left( d^{\Omega}_{c a} A^{\nu}_{b c} A^{\lambda}_{a c} f_{a c} - d^{\Omega}_{b c} A^{\nu}_{c a} A^{\lambda}_{b c} f_{c} \right), \]
\[ \text{(B22)} \]

\[ = \sum_{a \neq c} i d^{\Omega}_{c a} A^{\lambda}_{a c} f_{a c} \left[ A^{\nu}, A^{\mu} \right]_{a c}, \]
\[ \text{(B23)} \]

\[ = \sum_{a \neq c} d^{\Omega}_{c a} A^{\lambda}_{a c} f_{a c} \left( [D_{\nu} A^{\nu}]_{a c} - [D_{\nu} A^{\mu}]_{a c} \right), \]
\[ \text{(B24)} \]

where we used a formula for the \( U(2) \)-covariant derivative

\[ [D_{\nu} A^{\nu}]_{a b} - [D_{\nu} A^{\mu}]_{a b} = i \left[ A^{\nu}, A^{\mu} \right]_{a b}. \]
\[ \text{(B25)} \]

As a result, we obtain

\[ \sigma_{\epsilon c o}^{\mu \nu \lambda} = \frac{q^3}{2 h} \int \frac{d k}{(2 \pi)^3} \sum_{a \neq b} \left( \left| [D_{\nu} A^{\nu}]_{a b} - [D_{\nu} A^{\mu}]_{a b} \right| A^{\lambda}_{b a} f_{a b} d^{\Omega}_{b a} \right. \]
\[ \left. + \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right] \right], \]
\[ \text{(B26)} \]

Summing Eqs. \( \text{(B19)} \) and \( \text{(B26)} \), we obtain the photocurrent formula

\[ \sigma_{\epsilon c o}^{\mu \nu \lambda} = \frac{q^3}{2 h} \int \frac{d k}{(2 \pi)^3} \sum_{a \neq b} \left( d^{\Omega}_{b a} f_{a b} \left[ D_{\nu} A^{\nu} \right]_{a b} A^{\lambda}_{b a} \right. \]
\[ \left. + \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right] \right), \]
\[ \text{(B27)} \]

where both \( A^{\nu}_{a b} \) and \( [D_{\nu} A^{\nu}]_{a b} \) are \( U(2) \)-covariant and hence the overall expression is \( U(2) \)-invariant. Making use of Eq. \( \text{(A19)} \), we obtain the final expression as

\[ \sigma_{\epsilon c o}^{\mu \nu \lambda} = \frac{q^3}{2 h} \int \frac{d k}{(2 \pi)^3} \sum_{a \neq b} \left( d^{\Omega}_{b a} f_{a b} \left[ D_{\nu} A^{\nu} \right]_{a b} A^{\lambda}_{b a} \right. \]
\[ \left. + \left[ (\nu, -\Omega) \leftrightarrow (\lambda, \Omega) \right] \right), \]
\[ \text{(B28)} \]

with the photoconductivity coefficients

\[ \eta^{\mu \nu \lambda}_{\text{IR}} = \frac{q^3}{2 h} \int \frac{d k}{(2 \pi)^3} \sum_{a \neq b} \left( d^{\Omega}_{b a} f_{a b} \left[ D_{\nu} A^{\nu} \right]_{a b} A^{\lambda}_{b a} \right. \]
\[ \times \left. \Re \left( [D_{\nu} A^{\nu}]_{a b} A^{\lambda}_{b a} \right) \right), \]
\[ \text{(B29)} \]

for the magnetic rectification current, and

\[ \eta^{\mu \nu \lambda}_{\text{gyro}} = \frac{q^3}{h} \int \frac{d k}{(2 \pi)^3} \sum_{a \neq b} \left( d^{\Omega}_{b a} f_{a b} \delta(\hbar \Omega - \epsilon_{a b}) \right. \]
\[ \times \left. \Im \left( [D_{\nu} A^{\nu}]_{a b} A^{\lambda}_{b a} \right) \right), \]
\[ \text{(B30)} \]

for the gyration current.

Appendix C: Classification of photocurrent response based on magnetic point group

This section lists the noncentrosymmetric magnetic point groups preserving the \( T \)- or \( PT \)-symmetry. The 122 magnetic point groups are classified into three categories; the 32 \( T \)-symmetric point groups (gray group), 32 point groups whose symmetry operations are not relevant to the time-reversal operation (black group), and the others as many as 58 (black-white (BW) group). There are 21 noncentrosymmetric and \( T \)-symmetric (\( PT \)-symmetric) point groups in the gray (BW) group (see Table. III).

For instance, let us consider the noncentrosymmetric gray group. The gray group \( G \) is described by

\[ G = H + \theta H, \]
\[ \text{(C1)} \]

with a noncentrosymmetric black group \( H \). Among the 21 groups, 20 groups other than the case with \( H = m3m(O) \) are piezoelectric. On the other hand, 18 groups
other than the cases $H = 6(C_{3h})$, $6m2(D_{3h})$, $43m(T_{d})$ are gyrotropic. Similarly, we can identify the piezoelectric groups and gyrotropic groups included in the $PT$-symmetric BW group by replacing $\theta$ with $\theta f$ in Eq. (C1).

As we mentioned in the main text, piezoelectric groups allow the LP-photocurrent while gyrotropic groups allow the CP-photocurrent. Thus, by referring to Tables I and III, we can systematically identify which photocurrent is allowed in a given noncentrosymmetric system. We also show candidate materials in Table III. Many other candidates for the $PT$-symmetric compounds can be found in Refs. [35, 42, 43].

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| Table III | List of the photocurrent responses allowed in the $\mathcal{T}$-symmetric point groups (gray group) and $\mathcal{PT}$-symmetric BW point groups. ‘MPG’ denotes magnetic point group and ‘$\mathcal{H}$’ is the maximal unitary subgroup of ‘MPG’. The symbols ‘\(\dagger\)’ and ‘\(\circ\)’ denote photocurrents induced by linearly-polarized and circularly-polarized lights, respectively. The allowed response is indicated by ‘\(\check{\,}\)’. The low of ‘CM’ shows candidate materials where ‘(ML)’/‘(BL)’ means monolayer/bilayer. The classification of the point groups with $H = C_2$, $C_{2v}$ is further divided according to the relation between the incident direction of light ($k$) and the primary axis of the point group. |
|---|---|---|
| | $\mathcal{H}$ | $\dagger$ | $\circ$ | CM |
| | (\(\mathcal{T}\)-symmetric groups) | | | | |
| 11’ | $C_1$ | ✓ | ✓ | | |
| 21’ | $C_2 \parallel k$ | (✓) | ✓ | | |
| | $C_2 \perp k$ | ✓ | ✓ | | |
| $m\dagger$ | $C_{2v} \parallel k$ | ✓ | ✓ | WTe$_2$(BL) |
| | $C_{2v} \perp k$ | ✓ | ✓ | TaIrTe$_4$, MoTe$_2$(Td), WP2 |
| 41’ | $C_4$ | ✓ | ✓ | | |
| 41’ | $S_4$ | (✓) | ✓ | | |
| 4221’ | $D_4$ | (✓) | ✓ | (TaSe$_3$)$_2$I |
| 4mm1’ | $C_{4v}$ | ✓ | ✓ | BiTeI, TaP, TaAs, NbP |
| 42m1’ | $D_{2d}$ | ✓ | ✓ | | |
| 31’ | $C_3$ | ✓ | ✓ | | |
| 3m1’ | $C_{3v}$ | ✓ | ✓ | Bi$_2$Se$_3$ (001), LiOsO$_3$ |
| 321’ | $D_3$ | (✓) | ✓ | Bi |
| 61’ | $C_6$ | ✓ | ✓ | | |
| 61’ | $C_{6v}$ | (✓) | ✓ | | |
| 6221’ | $D_6$ | (✓) | ✓ | | |
| 6m21’ | $C_{6h}$ | ✓ | ✓ | MoS$_2$(ML) |
| 231’ | $T$ | ✓ | ✓ | RhSi |
| 43m1’ | $T_d$ | ✓ | ✓ | Ce$_3$Bi$_4$(Pt,Pd)$_3$ |
| 4321’ | $O$ | ✓ | ✓ | Li$_2$BP$_{T_h}$ |
| | (\(\mathcal{PT}\)-symmetric groups) | | | | |
| 1’ | $C_1$ | ✓ | ✓ | CaMn$_2$Bi$_2$ |
| 2’/m | $C_s$ | ✓ | ✓ | SrMn$_2$As$_2$ |
| 2/m | $C_2 \parallel k$ | (✓) | ✓ | Na$_5$RuO$_4$ |
| | $C_2 \perp k$ | ✓ | ✓ | | |
| m/m’m | $D_2$ | (✓) | ✓ | LiMnPO$_4$ |
| mmm | $C_{2v} \parallel k$ | ✓ | ✓ | CuMn$_2$As, Mn$_2$Au |
| | $C_{2v} \perp k$ | ✓ | ✓ | | |
| 4/m | $C_4$ | (✓) | ✓ | | |
| 4’/m | $S_4$ | (✓) | ✓ | | |
| 4/m’m | $D_4$ | (✓) | ✓ | Fe$_2$TeO$_6$ |
| 4/m’m | $C_{4v}$ | (✓) | ✓ | Mn$_2$As$_2$B$_{12}$ |
| 4/m’m | $D_{2d}$ | (✓) | ✓ | BaMn$_2$As$_2$, EuMnBi$_2$ |
| 3’ | $C_3$ | ✓ | ✓ | | |
| 3’ | $C_{3v}$ | ✓ | ✓ | MnPS$_3$(ML) |
| 3’ | $D_3$ | ✓ | ✓ | Cr$_2$O$_3$, MnBi$_2$Te$_4$(BL) |
| 6/m | $C_6$ | (✓) | ✓ | | |
| 6’/m | $C_{6h}$ | ✓ | ✓ | | |
| 6/m’m | $D_6$ | (✓) | ✓ | | |
| 6/m’m | $C_{6v}$ | (✓) | ✓ | | |
| 6/m’m | $D_{3h}$ | ✓ | ✓ | | |
| 6’/m’m | $D_{3h}$ | ✓ | ✓ | | |
| 2’/m | $T$ | ✓ | ✓ | Cu$_3$TeO$_6$ |
| m’3 | $T_d$ | ✓ | ✓ | | |
| m’3 | $O$ | ✓ | ✓ | | |

[Refs. 35, 36], and references therein.
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