Application of Scalar Type Operators to Decomposability

Adicka Daniel Onyango

Department of Mathematics, Actuarial and Physical Sciences, University of Kabianga, P.O.Box 2030-20200, Kericho, Kenya.

Author’s contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

In this paper, we give some application of scalar type operators to Decomposability. In particular, we show that if \( H \) is of \((\alpha, \alpha + 1)\) type \( R \) and that it generates a strongly continuous group on a Banach space, then its resolvent is Decomposable hence scalar type.

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1 Introduction

Definition: Decomposable Operator A bounded operator \( H \) on a complex Banach space \( X \) is decomposable provided that whenever \( \{U_1, U_2, \ldots, U_n\} \) is an open cover of \( C \), there exists closed, \( H \)-invariant subspaces \( Y_k \) such that \( X = Y_1 + Y_2 + \ldots + Y_n \) and \( \sigma(H \mid Y_k) \subseteq U_k, k = 1, 2, \ldots, n \).

This class of operators contains all normal operators on a Hilbert space and compact Banach space operators hence they are of \((\alpha, \alpha + 1)\) type \( R \) operators [1]. The following Theorem due to Albrecht
and Eschmier [2] gives the necessary and sufficient condition for a bounded operator $H \in B(X)$ to be decomposable.

**Theorem 1 [2]:** A bounded operator $H \in B(X)$ is decomposable if and only if $H$ has Bishop’s property ($\beta$) and the decomposition property ($\delta$).

**Definition:** Let $X$ be a Banach space and $\Omega$ an open subset of the plane. Let $Hol(\Omega, X)$ denote the space of analytic functions from $\Omega$ to $X$. Then $Hol(\Omega, X)$ is a Fretchet space with respect to uniform convergence on the compact subsets of $\Omega$. The operator $H \in B(X)$ is said to possess Bishop’s property ($\beta$), provided that for every open subset $\Omega \subset \mathbb{C}$, $H_\Omega : Hol(\Omega, X) \to Hol(\Omega, X)$, $H_\Omega f(z) = (z - H)f(z)$ is injective with closed range.

**Decomposition Property ($\delta$)**

If $F$ is a closed subspace of $\mathbb{C}$, then the glocal analytic spectral subspace $X_H(F)$ is $X_H(F) = X \cap \text{ran}H_C \setminus F$, that is $x \in X_H(F)$ if there exist an analytic function $f : \mathbb{C} \setminus F \to X$ so that $(H - \lambda)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. A bounded linear operator $H \in B(X)$ has the decomposition property ($\delta$) if $X = X_H(U) + X_H(V)$ for every open cover $\{U, V\}$ of $\mathbb{C}$.

Albrecht and Escheneier [2] established the remarkable fact that the properties ($\beta$) and ($\delta$) are dual to each other. Indeed, $H \in B(X)$ has property ($\beta$) (resp ($\delta$)) if and only if $H^*$ has ($\delta$) (resp.$(\beta)$).

We shall greatly use the following formulation by Laursen and Neumann [3]

**Theorem 2 [3]:** Let $H \in B(X)$ and $D$ be a closed disk that contains $\sigma(H)$, and let $V$ be an open neighborhood of $D$. Suppose that there exist a totally disconnected compact subset $E$ of the boundary of $D$, a locally bounded function $\omega : V \setminus E \to (0, \infty)$ and an increasing function $\gamma : (0, \infty) \to (0, \infty)$ such that log of $\gamma$ has an integrable singularity at zero and $\gamma(\text{dist}(\lambda, \partial D)) \parallel x \parallel \leq \omega(\lambda) \parallel (H - \lambda)x \parallel$ for all $x \in X$ and $\lambda \in V \setminus \partial D$, then $H$ has property ($\beta$).

In particular, the above Theorem provide sufficient conditions in terms of the norms of resolvents sufficient for bishop’s property ($\beta$).

**Lemma 3:** Let $H$ be generator of arbitrarily continuous semigroup on a Banach space $X$ and let $\lambda, \mu \in \rho(H)$, then $R(\lambda, H)R(\mu, H) = R(\mu, H)R(\lambda, H)$

**Proof:** The proof follows immediately from the well known resolvent identity;

$$R(\lambda, H) - R(\mu, H) = -(\mu - \lambda)R(\lambda, H)R(\mu, H)$$

for all $\lambda, \mu \in \rho(H)$.

**Lemma 4:** Let $H$ be as in Lemma [3] and let $T = R(\lambda, H)$. Then $\mu \in \rho(T)$ if and only if $\lambda - \frac{1}{\mu} \in \rho(T)$. In this case, we have

$$(\mu - T)^{-1} = \frac{1}{\mu} I + \frac{1}{\mu^2} R(\lambda - \frac{1}{\mu}, H)$$

(1.1)

From equation (1), $R(\lambda - \frac{1}{\mu}, H) - T = (\lambda - \frac{1}{\mu})T R(\lambda - \frac{1}{\mu}, H)$ which implies $\mu T = (\mu - T)R(\lambda - \frac{1}{\mu}, H)$. Multiplying by $(\lambda - H)$ and dividing through by $\mu$ yields

$$I = \frac{1}{\mu} (\lambda - \frac{1}{\mu} - H + \frac{1}{\mu}) R(\lambda - \frac{1}{\mu}, H)(\mu - T)$$

$$= \frac{1}{\mu}(I + \frac{1}{\mu} R(\lambda - \frac{1}{\mu}, H))(\mu - T)$$

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Thus
\[(\mu - T)^{-1} = -\frac{1}{\mu} I + \frac{1}{\mu^2} R(\lambda - \frac{1}{\mu}, H).\]

The next theorem which is a major result in this section indicates that the kind of resolvent we are dealing with here are decomposable.

**Theorem 5:** If \( H \) is a generator of arbitrarily strongly continuous semigroup on Banach Space \( X \) with \( \sigma(H, X) \subset \{ z : \text{Re}(z) \leq c \} \) on a Banach space \( X \), then the resolvent operator \( R(\lambda, H) \) is decomposable for all \( \lambda \in \rho(H, X) \).

**Proof:** Let \( H \) be the generator of strongly continuous semigroup with \( \sigma(H, X) \subset \{ z : \text{Re}(z) \leq c \} \) on a Banach space \( X \). Let \( \lambda, \mu \in \rho(H) \) and \( T = R(\lambda, H) \). By the Hille Yosida theorem we have
\[\| R(\lambda - \frac{1}{\mu}, H) \| \leq \frac{M}{\text{Re}(\lambda - \frac{1}{\mu}) - c}\]
where \( M > 0 \) is a constant.

Now, by the spectral mapping theorem, we get
\[\sigma(T) = \{ \frac{1}{\lambda - it} : t \in \mathbb{R} \} \cup \{0\}\]
letting \( \omega = \frac{1}{\lambda - it} \) where \( \lambda = \text{Re}(\lambda) + i\text{Im}(\lambda) \)
\[\sigma(T) = \{ \omega : |\omega - \frac{1}{2\text{Re}(\lambda)}| = \frac{1}{2\text{Re}(\lambda)}, \text{Re}(\lambda) > 0 \}\]
For any \( \mu \in \rho(T) \), we have \( |\mu - \frac{1}{2\lambda}| > \frac{1}{2\lambda} \) which implies \( \text{Re}(\lambda - \frac{1}{\mu}) > 0 \) and thus \( \text{dist}(\mu, \sigma(T)) = \text{Re}(\lambda - \frac{1}{\mu}) \). Consequently,
\[\| R(\lambda - \frac{1}{\mu}, H) \| \leq \frac{M}{\text{dist}(\mu, \sigma(T))}\]

And from Lemma 4, we obtain
\[\| R(\mu, T) \| \leq \frac{1}{|\mu|} + \frac{1}{|\mu|^2}\text{dist}(\mu, \sigma(T))\]
It follows from Theorem 2 that \( T \) has Bishop property (\( \beta \)). Moreover, the adjoint operator \( T^* \) satisfies \( \sigma(T^*) = \sigma(T) \) and thus
\[\| R(\mu, T^*) \| \leq \frac{1}{|\mu|} + \frac{1}{|\mu|^2}\text{dist}(\mu, \sigma(T))\]
which indicates that \( T^* \) has Bishop’s property (\( \beta \)). This implies that \( T \) has property (\( \delta \)). Thus by Theorem 1 it follows that \( H \) is decomposable.

## 2 Hardy Spaces

Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the unit disk of the complex plane and \( H(D) \) denote the Frechet space of functions analytic on \( D \). For \( 0 < p < \infty \), the hardy spaces on the unit disk, \( H^p(D) \) are defined as \( H^p(D) = \{ f \in H(D) : \| f \|_{H^p(D)} = \sup_{0<r<1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \ d\theta < \infty \right) \}. \) We refer to [4] for the basic and comprehensive theory of Hardy spaces. In particular, it is important to note that every \( f \in H^p(D), 0 < p < \infty \), has non tangential boundary values almost everywhere on \( \partial D \) and
\[\| f \|_{H^p(D)} = \left( \int_0^{2\pi} |f(re^{i\theta})|^p \ d\theta \right)^{\frac{1}{p}}\]
Where we regard the boundary function as an extension of $f$. Moreover the growth condition for the functions in $H^p(D)$ is given by

$$ |f(z)|^p \leq \frac{1}{1-|z|^2} \|f\|^{p}_{H^p(D)}$$

$1 \leq p < \infty$, $f \in H^p(D)$.

We consider the following self analytic map $\varphi_t : D \to D$ given by

$$\varphi_t(z) = e^{-ct}z$$

for all $z \in D$, $t > 0$. We define the corresponding weighted composition operators on $H^p(D)$ by

$$T_tf(z) = (\varphi_t'(z))^{\gamma}f(\varphi_t(z))$$

$$= e^{-ct}\gamma f(e^{-ct}z)$$

for all $f \in H^p(D)$, $\gamma = \frac{1}{p}$.

3 Main Results

The following theorem gives both the semigroup and spectral properties of this group $\{T_t\}$ of composition operators.

**Theorem 6:** Let $H^p(D), 1 \leq p \leq \infty$ be hardy space of the unit disk $D$. Define a self analytic map $\varphi_t : D \to D$ by $\varphi_t(z) = e^{-ct}z$ and the corresponding weighted composition operator $T_t : H^p(D) \to H^p(D)$ by $T_tf(z) = e^{-ct}\gamma f(e^{-ct}z)$ where $c \in \mathbb{C}$, $t \geq 0$ and $\gamma = \frac{1}{p}$. Then the following hold:

(a) $\{T_t\}_{t \in \mathbb{R}}$ is a group of isometries on $H^p(D)$

(b) $\{T_t\}_{t \in \mathbb{R}}$ is strongly continuous.

(c) The infinitesimal generator $H$ of $T_t$ is given by $Hf(z) = -cHf(z) - czf'(z)$ with the domain $dom(H) = \{f \in H^p(D) : zf'(z) \in H^p(D)\}$

(d) $\sigma(H) = \sigma_p(H) = \{-c(n + \frac{1}{p}) : n = 0, 1, 2, ...\}$

(e) If $Re(c) = 0$, then $R(c, H)$ is compact, decomposable and a scalar type operator.

**Proof:** By definition and change of variables argument, we have

$$\|T_tf\|_{H^p(D)}^p = \int_0^{2\pi} |(T_tf)(e^{i\theta})|^p \, d\theta$$

$$= \int_0^{2\pi} |f(\varphi_t(e^{i\theta}))| \varphi_t'(e^{i\theta})ie^{i\theta}|^p \, d\theta.$$ 

Let $\omega = \varphi_t(e^{i\theta})$, then $d\omega = \varphi_t'(e^{i\theta})ie^{i\theta}d\theta$ and

$$\|T_tf\|_{H^p(D)}^p = \int_0^{2\pi} |f(\omega)|^p \, d\omega = \int_0^{2\pi} |f|^{p} \, d\omega = \|f\|_{H^p(D)}^p.$$
This means that $T_t$ is an isometry. Moreover, $T_0T_s = T_{s+t}$ for all $s, t \in \mathbb{R}$ and $T_s = I$ where $I$ is the identity operator. So $\{(T_t)\}_{t \in \mathbb{R}}$ is a group of isometries as desired.

To show that $\{(T_t)\}_{t \geq 0}$ is strongly continuous, it suffices to show that $\lim_{t \to 0} \| T_tf - f \|_p = 0$ for every $f \in H^p(D)$. Let $X(D)$ be the set containing all functions in $H^p(D)$ that are continuous on $D$. Then $X(D)$ is dense in $H^p(D)$. Thus for $f \in H^p$ and arbitrary $\epsilon > 0$, there exists $g \in X(D)$ such that $\| f - g \|_p < \epsilon$, then

$$
\| T_tf - f \|_p \leq \| T_tf - T_tg \|_p + \| T_tg - g \|_p + \| g - f \|_p
$$

$$
= 2 \| f - g \|_p + \| T_tg - g \|_p
$$

Now for all $g \in X(D)$, $T_t g(z) \to g(z)$ for all $g \in \partial D$ and by isometry of $(T_t)$, we have $\| T_t g \|_p \to \| g \|_p$. Fatou's lemma then gives $\| T_t g - g \|_p \to 0$. Thus $\| T_tf - f \|_p \leq 2\epsilon$, and hence $(T_t)$ is strongly continuous.

By definition, the infinitesimal generator $H$ of $T_t$ is given by

$$
H(f) = \lim_{t \to 0} \frac{T_tf - f}{t}, \quad f \in D(H)
$$

$$
= \lim_{n \to \infty} \frac{e^{-ct}f(e^{-ct}z) - f(z)}{t}
$$

$$
= \frac{\partial}{\partial t} (e^{-ct}f(e^{-ct}z)) \bigg|_{t=0}
$$

$$
= -c\gamma e^{-ct}f(e^{-ct}z) + e^{-ct}f'(e^{-ct}z)
$$

$$
= -c\gamma f(z) - czf'(z),
$$

which implies that $D(H) \subseteq \{f \in H^p(D) : zf'(z) \in H^p(D)\}$. Conversely, let $f \in H^p(D)$ such that $zf'(z) \in H^p(D)$. Then for $z \in D$, we have

$$
T_tf(z) - f(z) = \int_0^t \frac{\partial}{\partial s} (e^{-cs}f(\phi_s(z))) \, ds
$$

$$
= \int_0^t \left( -c\gamma e^{-cs}f(\phi_s(z)) - czf'(e^{-cs}z) \right) \, ds
$$

$$
= \int_0^t T_s(F) \, ds
$$

where $F(z) = -c\gamma f(z) - czf'(z)$. Thus $\lim_{t \to 0} \frac{T_tf - f}{t} = \lim_{t \to 0} \frac{1}{t} \int_0^t T_s(F) \, ds$ Now for $F \in H^p(D)$ the limit exists and equal to $F$. Thus $D(H) \supseteq \{f \in H^p(D) : zf'(z) \in H^p(D)\}$, as claimed.

To obtain the point spectrum of $H$, let $\lambda$ be an eigenvalue and $f$ be the corresponding eigenvector. Then the eigenvalue equation $Hf = \lambda f$ reduces to the differential equation

$$
-cHf(z) - czf'(z) = \lambda f(z)
$$

which is equivalent to

$$
-czf'(z) = (\lambda + cH)f(z)
$$

To solve the above ODE, we continue as follows;

$$
\frac{f'(z)}{f(z)} = -\frac{1}{c}(\lambda + cH) \frac{dz}{z}
$$

$$
\Leftrightarrow
$$

$$
\frac{df(z)}{f(z)} = -\frac{1}{c}(\lambda + cH) \frac{dz}{z}.
$$
Therefore
\[ \ln f(z) = -\frac{1}{c} (\lambda + cH) \ln z + C \]
and thus
\[ f(z) = z^{-\frac{1}{c}(\lambda + cH)} \]
for \( c \neq 0 \). Since \( z^{-\frac{1}{c}(\lambda + c\gamma)} \in H(D) \) if and only if \(-\frac{1}{c}(\lambda + c\gamma) \in \mathbb{Z}_+\). That is \(-\gamma + \frac{1}{c} = n, n = 0,1,2,...\) Hence \( \sigma_p(H) = \{ -c(n + \gamma) : n = 0,1,2,... \} \)

Clearly, if \( \text{Re}(c) = 0 \), then \( c \in \rho(H) \) and therefore, the resolvent operator \((c - H)^{-1}\) reduces to
\[ R(c,H)f(z) = \frac{1}{cz} \int_0^z f(\xi)d\xi. \]

As remarked by Cowen and Macluer [5], such resolvents are compact and therefore
\[ \sigma(H) = \sigma_p(H) \]

Now by Theorem 5, \( R(c,H) \) is decomposable and hence of scalar type

4 Conclusion

In this study we gave application of scalar type operators to Decomposibility. In particular, we showed that if \( H \) is of \((\alpha,\alpha + 1)\) type \( R \) and that it generates a strongly continuous group on a Banach space, then its resolvent is Decomposable and therefore it is scalar type.

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Competing Interests

Author has declared that no competing interests exist.

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