SOME REMARKS ON THE PARTIAL REGULARITY OF A SUITABLE WEAK SOLUTION TO THE NAVIER–STOKES CAUCHY PROBLEM

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The goal of the paper is to explore some of the issues related to local regularity of a suitable weak solution to the Navier–Stokes Cauchy problem. The obtained results are in the spirit of the well-known results by Caffarelli–Kohn–Nirenberg. Bibliography: 11 titles.

1. Introduction

We deal with the Navier–Stokes Cauchy problem
\begin{align*}
  u_t + u \cdot \nabla u + \nabla \pi u &= \Delta u, \quad \nabla \cdot u = 0, \quad \text{in } (0, T) \times \mathbb{R}^3, \\
  u(0, x) &= u_0(x) \text{ on } \{0\} \times \mathbb{R}^3.
\end{align*}

(1.1)

In system (1.1), $u$ is the kinetic field, $\pi u$ is the pressure field, $u_t := \frac{\partial}{\partial t} u$, and $u \cdot \nabla u := u_k \frac{\partial}{\partial x_k} u$. We investigate the partial regularity of a suitable weak solution, and find a new sufficient condition for the existence of a regular solution. Our results are in the spirit of those obtained in [1] and, for small data, in [3]. As in [2, 3, 6], our study can be considered as an attempt to highlight what is possible to obtain without extra condition, in the setting of the $L^2$-theory. In this connection, although it is not our main goal, we would like to stress that our results could lead to a sort of structure theorem in the space-time cylinder. To be more precise, we recall the well-known Leray’s structure theorem related to a weak solution. Leray’s theorem claims that there exist an interval of regularity of the form $(\theta, \infty)$ and a sequence of intervals of regularity included in $(0, \theta)$, whose complementary set on $(0, \theta)$ is a set of zero $\frac{1}{2}$-Hausdorff measure. Mutatis mutandis, the results of [1] (see below Theorem 1.4) and of this note give a sort of structure theorem for a suitable weak solution related to the Cauchy problem. More precisely, under a suitable assumption on the initial data, we prove in Theorem 1.4 that a suitable weak solution is regular for all $t > 0$ in the exterior of a ball of radius $R_0$. In this note, we also prove that, almost everywhere, a point $(t, x) \in (0, \theta) \times B(R_0)$ is the center of a parabolic neighborhood of regularity for a suitable weak solution. Hence, the set $(0, \theta) \times B(R_0)$ contains at most a sequence of open sets of regularity, whose complementary set in $(0, \theta) \times B(R_0)$ has at most zero $1$-Hausdorff measure.

To state the details of our main results better, we split the introduction in two short subsections. In the first one, we recall some definitions and notation following those in [1]. Then we recall two fundamental regularity results obtained in [1], and, with alternative proof, in [11], and their consequences. In the second subsection, we give the statement of our results.

1.1. Suitable weak solutions. We begin by recalling the following definition.

Definition 1.1. Let $u_0 \in J^2(\mathbb{R}^3)$. A pair $(u, \pi_u)$, where $u : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\pi_u : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$, is said to be a weak solution to problem (1.1) if

i) for all $T > 0$, $u \in L^2(0, T; J^{1,2}(\mathbb{R}^3))$ and $\pi_u \in L^2((0, T) \times \mathbb{R}^3),$

ii) $\lim_{t \to 0} \|u(t) - u_0\|_2 = 0$.

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iii) for all $t,s \in (0,T)$, the pair $(u,\pi_u)$ satisfies the equation
\[
\int_{s}^{t} \left[ (u,\varphi_t) - (\nabla u, \nabla \varphi) + (u \cdot \nabla \varphi, u) + (\pi_u, \nabla \cdot \varphi) \right] d\tau + (u(s), \varphi(s)) = (u(t), \varphi(t))
\]
for all $\varphi \in C_0^1([0,T] \times \mathbb{R}^3)$.

In [1], in order to investigate the regularity of a weak solution, an energy relation having a local character is introduced.

**Definition 1.2.** A pair $(u,\pi_u)$ is said to be a suitable weak solution if it is a weak solution in the sense of Definition 1.1 and, moreover,
\[
\int_{\mathbb{R}^3} |u(t)|^2 \phi(t) dx + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\nabla u|^2 \phi dx d\tau \leq \int_{\mathbb{R}^3} |u(\sigma)|^2 \phi(\sigma) dx \]
\[
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u|^2 (\phi_t + \Delta \phi) dx d\tau + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|u|^2 + 2\pi_u) u \cdot \nabla \phi dx d\tau
\]
for all $t \geq \sigma$, for $\sigma = 0$, and a.e. in $s \geq 0$, and for all nonnegative $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$.

In [1] and [7], the following existence result is proved.

**Theorem 1.1.** For all $u_0 \in L^2(\mathbb{R}^3)$, there exists a suitable weak solution.

As a consequence of inequality (1.2) and the existence theorem, one gets the following statement.

**Corollary 1.1.** A suitable weak solution satisfies the strong energy inequality
\[
\|u(t)\|^2_2 + 2 \int_{s}^{t} \|\nabla u(\tau)\|^2_2 d\tau \leq \|u(s)\|^2_2, \text{ for all } t \geq s, s = 0, \text{ and a.e. in } s \geq 0. \quad (1.3)
\]

Moreover, for all $s$ such that (1.3) holds, we have
\[
\lim_{t \to s^+} \|u(t) - u(s)\|_2 = 0. \quad (1.4)
\]

Let us recall the definition of a singular point for a weak solution.

**Definition 1.3.** We say that $(t,x)$ is a singular point for a weak solution $(u,\pi_u)$ if $u \notin L^\infty$ in any neighborhood of $(t,x)$; the remaining points, where $u \in L^\infty(I(t,x))$ for some neighborhood $I(t,x)$, are said to be regular.

**Definition 1.4.** We say that $u$ is a regular solution in $(t_0,t_1) \times \Omega \subseteq (0,T) \times \mathbb{R}^3$ if $u$ is a weak solution, $u_t \in L^q_{loc}((t_0,t_1) \times \Omega)$ for some $q > 1$, and $u \in L^\infty((t_0 + \delta,t_1 - \delta) \times \Omega)$ for all $\delta > 0$.

It is known that a regular solution in $(t_0,t_1) \times \Omega$ is smooth on compact subsets contained in $(t_0,t_1) \times \Omega$, see, e.g., [10].

Following [1], we introduce the parabolic cylinders
\[
Q_r = Q_r(t,x) := \{(\tau,y) : t - r^2 < \tau < t \text{ and } |y-x| < r\}
\]
and
\[
Q^*_r := Q^*_r(t,x) := \{(\tau,y) : t - \frac{7}{8}r^2 < \tau < t + \frac{1}{8}r^2 \text{ and } |y-x| < r\}. \quad (1.5)
\]

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For \( r \in (0, t^{\frac{1}{2}}) \), we set

\[
M(r) = M(t, x, r) := r^{-2} \iint_{Q_r} (|u|^3 + |u||\pi_u|)dyd\tau + r^{-\frac{13}{4}} \int_{t-r^2}^{t} \left( \int_{|x-y|<r} |\pi_u|dy \right)^{\frac{5}{4}} d\tau,
\]

where \( Q_r \) is as in (1.5).

In paper \([1]\), in connection with the regularity of a suitable weak solution, the authors establish two regularity criteria. The first is Proposition\( 1 \) on p. 775 (or Corollary 1, on p. 776).

**Proposition 1.1.** Let \((u, \pi_u)\) be a suitable weak solution in some parabolic cylinder \( Q_r(t, x) \).

There exist \( \varepsilon_1 > 0 \) and \( c_0 > 0 \) independent of \((u, \pi_u)\) such that if

\[
M(t, x, r) \leq \varepsilon_1,
\]

then

\[
|u(\tau, y)| \leq c_1 \frac{1}{r}, \quad \text{a.e. in } (\tau, y) \in Q_{\frac{r}{2}}(t, x),
\]

where \( c_1 := c_0 \varepsilon_1^{\frac{2}{3}} \). In particular, a suitable weak solution \( u \) is regular in \( Q_{\frac{r}{2}}(t, x) \).

This result is used in \([1]\), to prove another regularity criterion, see Proposition\( 2 \) on p. 776:

**Proposition 1.2.** There is a constant \( \varepsilon_3 > 0 \) with the following property: if \((u, \pi_u)\) is a suitable weak solution in some parabolic cylinder \( Q_r^+(t, x) \) and

\[
\limsup_{r \to 0} r^{\frac{1}{2}} \iint_{Q_r^+} |\nabla u|^2 dyd\tau \leq \varepsilon_3,
\]

then \((t, x)\) is a regular point.

The above criterion is employed to get the following two main results (respectively, Theorem B on p. 772 and Theorem D on p. 774 in \([1]\)).

**Theorem 1.2.** For any suitable weak solution, the one-dimensional parabolic Hausdorff measure of the set \( S \) of singular points is equal to zero.

**Theorem 1.3.** There exists an absolute constant \( L_0 > 0 \) with the following property: if \( u_0 \in J^2(\mathbb{R}^3) \) and

\[
||u_0||_{L^2(\{|x|>R\})} = L < L_0,
\]

then there exists a suitable weak solution to (1.1), which is regular in the domain

\[
\{(t, x) : |x|^2 < t(L_0 - L)\}.
\]

There is a difference in the meaning of the above theorems. Theorem 1.2 gives a geometric measure of possible set \( S \) of singular points. Theorem 1.3 establishes the existence of a suitable weak solution to (1.1), having finite scaling invariant metric

\[
\sup_{0<\tau<t} \int_{\{|r|<\tau\} \times \mathbb{R}^3} |u|^2|x|^{-1}dx < \infty, \quad \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla u|^2|x|^{-1}dxd\tau < \infty \quad t > 0.
\]

Hence, \( x = 0 \) is regular for \( t > 0 \).

Finally, as a corollary of the latter result, we have the following statement \([1, \text{Corollary, p. 820}]\):

**Theorem 1.4.** Let \((u, \pi_u)\) be a suitable weak solution for the initial data \( u_0 \). Assume that \( \|\nabla u_0\|_{L^2(|x|>R)} < \infty \). Then there exists \( R_0 > R \) such that \( u \in L^\infty((\delta, \infty) \times \{x : |x| > R_0\}) \) for all \( \delta > 0 \).
1.2. The goals of the paper. We work in the context of Theorem 1.3 and Theorem 1.7 (below) already proved in [3]. Both of these theorems deal with a scaling invariant norm that leads to (1.11) provided that, at the initial instant, weighted norm (1.10),

$$E(u_0, x) := \int_{\mathbb{R}^3} |u_0(y)|^2 |x - y|^{-1} dy, \ x \in \mathbb{R}^3,$$

is small in a suitable sense. One consequence of the smallness is the existence of a regular solution global in time.

In the present note, we study the existence of a suitable weak solution that, at least locally in time, satisfies the regularity criterion of Proposition 1.1 and, as a consequence, is locally a regular solution. Also in this case, the result follows from the assumption that weighted norm (1.12) of the initial data is finite; however, contrary to Theorem 1.3 and Theorem 1.7, we do not require smallness. As a consequence, we are able to deduce the regularity only locally in time.

**Theorem 1.5.** Let \( u(t,x) \) be a suitable weak solution. Assume that for \( x \in \mathbb{R}^3 \), there exists \( v_0 \in J^{1,2}(\mathbb{R}^3) \) such that

$$\psi(x) := \int_{\mathbb{R}^3} \frac{|u_0(y) - v_0(y)|^2}{|x - y|} dy < \frac{1}{(4c)^2},$$

where the constant \( c \) is independent of \( u_0, x, \) and \( v_0 \). Then there exists \( t(x) > 0 \) such that

$$u \in L^\infty(Q_{\left(\frac{7}{6}\right)^{1/2} s, x}) \text{ for all } s \in (0, t(x)).$$

In particular, if \((\tau, y) \in Q_{\left(\frac{7}{6}\right)^{1/2} s, x}\) is a Lebesgue point, then

$$|u(\tau, y)| \leq c\tau^{-\frac{1}{2}}.$$  

**Corollary 1.2.** Let \( u(t,x) \) be a suitable weak solution. Then for all \( \sigma \) of validity of weighted energy inequality (1.2), there exists a set \( E \subseteq \mathbb{R}^3 \) such that \( \mathbb{R}^3 - E \) has zero Lebesgue measure and satisfies the following property: for all \( x \in E(\sigma) \), there exists \( t(x) > 0 \) such that

$$u \in L^\infty(Q_{\left(\frac{7}{6}\right)^{1/2} \left(\sigma + \frac{7}{6}s, x\right)}) \text{ for all } s \in (0, t(x)).$$

In particular, if \((\tau, y) \in Q_{\left(\frac{7}{6}\right)^{1/2} \left(\sigma + \frac{7}{6}s, x\right)}\) is a Lebesgue point, then

$$|u(\tau, y)| \leq c(\tau - \sigma)^{-\frac{1}{2}}$$

with \( c \) independent of \( \tau \).

We give some comments.

First, we observe that Theorem 1.5 seems to be similar to Theorem 1.3. The difference is in the fact that we require for the initial data not condition (1.10), but weaker condition (1.13) which is almost everywhere satisfied with \( u_0 \in J^2(\Omega) \). The theorem establishes a result of local regularity for a suitable weak solution of (1.1). The local character is expressed in (1.14), because either the solution is \( L^\infty \) just on the parabolic cylinder, or the height of the cylinder depends on \( x \) through \( t(x) \).

Estimate (1.15) (respectively, (1.17)) expresses in which way the solution can be singular in \( t = 0 \) (respectively, in \( \sigma \)) provided that \( x \in E \) (respectively, \( x \in E(\sigma) \)).
In the way specified below, the set \( E \) represents a new aspect of our result of local regularity, stated with initial data in \( J^2(\mathbb{R}^3) \). Actually, if \( u_0 \in J^2(\mathbb{R}^3) \), then the Riesz potential

\[
\mathcal{E}(u_0, x) := \int_{\mathbb{R}^3} \frac{u_0^2(y)}{|x - y|} \, dy
\]

is well posed a.e. in \( x \in \mathbb{R}^3 \). This claim is a consequence of the fact that by the Hardy–Littlewood–Sobolev theorem, the following transformation is well defined:

\[
u_0^2 \in L^1(\mathbb{R}^3) \rightarrow \mathcal{E}(u_0, x) := \int_{\mathbb{R}^3} \frac{u_0^2(y)}{|x - y|} \, dy \in L(3, \infty)(\mathbb{R}^3).
\]

Hence it is almost everywhere finite. Let \( \{u_0^k\} \) be a sequence of smooth functions converging to \( u_0 \) in \( L^2(\mathbb{R}^3) \), for example the mollified of \( u_0 \). Given \( x \in \mathbb{R}^3 \) and \( k \in \mathbb{N} \), we define a sequence

\[
\psi^k(x) := \int_{\mathbb{R}^3} \frac{|u_0 - u_0^k|^2}{|x - y|} \, dy.
\]

By the Hardy–Littlewood–Sobolev theorem (see Lemma 2.6), it is easy to verify that the sequence \( \{\psi^k\} \) converges to zero almost everywhere in \( x \in E \subseteq \mathbb{R}^3 \). Therefore assumption (1.13) is satisfied almost everywhere in \( x \), and \( E \) is the set from Corollary 1.2. We prove that for any \( x \in E \), there exists \( t(x) > 0 \) such that \( M(\frac{T}{3}s, x, r) \leq \varepsilon_1 \) for suitable \( r \) and any \( s \in (0, t(x)) \). By Proposition 1.1, this result implies the regularity in \( Q_{\frac{T}{3}} \left( \frac{T}{6}s, x \right) \) for any \( s \in (0, t(x)) \). Therefore, if \( S_x \) is the projection of the set \( S \) of singular points onto \( \mathbb{R}^3 \), given in Theorem 1.2 (and whose one-dimensional Hausdorff measure is zero by this theorem), then Corollary 1.5 shows that \( S \subseteq \mathbb{R}^3 \setminus E \). This claim makes clear that we do not improve the regularity established in [1] (according with result proved in [8]); rather we investigate the existence of a possible size, as a function in \( x \) belonging to \( E \), of the parabolic neighborhood of regularity of a weak solution.\(^1\)

In Corollary 1.2, a dependence of the set \( E \) on \( \sigma \) is established: to see this one should use (1.2) and right-continuity of the weak solution in \( L^2 \)-norm.

The following results are two main consequences of Theorem 1.5.

**Theorem 1.6.** Let \( u(t, x) \) be a suitable weak solution. Assume that there exist \( \Omega \subseteq \mathbb{R}^3 \) and \( v_0 \in J^{1,2}(\mathbb{R}^3) \) such that

\[\psi(x) < \frac{1}{(4c)^2} \text{ uniformly in } x \in \Omega.\]

Then there exists \( T_0 \) such that (1.14) and (1.15) hold for all \((s, x) \in (0, T_0) \times \Omega\).

We observe that if \( \Omega \equiv \mathbb{R}^3 \), then Theorem 1.6 shows the existence of a regular solution \((u, \pi_u)\) on \((0, T_0) \times \mathbb{R}^3\).

**Corollary 1.3.** Let \( u(t, x) \) be a suitable weak solution. For any \( B(R) \) and any \( \varepsilon > 0 \), there exist a set \( \Omega_\varepsilon \subseteq B(R) \) with \( \text{meas}(B(R) \setminus \Omega_\varepsilon) < \varepsilon \), and real \( T_0(\varepsilon) > 0 \) such that (1.14) holds for all \((s, x) \in (0, T_0(\varepsilon)) \times \Omega_\varepsilon\).

**Theorem 1.7.** Let \( u(t, x) \) be a suitable weak solution. Assume that \( \text{ess sup} \, \mathcal{E}(u_0, x) \) is sufficiently small. Then \((u, \pi_u)\) is regular for all \( t > 0 \), and unique up to a function \( c(t) \) for the pressure field.

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\(^1\)Actually, \( \psi \in L(3, \infty)(\mathbb{R}^3) \) and \( \nabla \psi \in L_{\text{loc}}^\infty(\mathbb{R}^3) \), then the set \( \mathbb{R}^3 \setminus E \), where \( \psi \) is not defined a priori can have the Hausdorff measure greater than 1, which is the Hausdorff measure of the set of possible singular points, established in [1].
The last theorems can be considered as regular solutions counterparts for Theorem 1.5 and Corollary 1.2, provided that the assumptions on the data are stronger than the simple assumption $u_0 \in J^2(\mathbb{R}^3)$. The theorems work in the light of the scaling invariant weighted norm (1.18).

Theorem 1.6 establishes a local existence result under the assumption of “suitable closedness” of the initial data $u_0 \in L^2(\mathbb{R}^3)$, with respect to weighted norm (1.18) to a smooth function $v_0$. As the existence is achieved on the element $v_0$ of the approximation which is close to $u_0$ in metric (1.18), we are not able to find the size of $T_0$ with the help of $u_0$, but $(0,T_0)$ is just (a priori) a subinterval of existence of the smooth solution $(v,\pi)$ corresponding to $v_0$. In this connection we point out that the above question on the size of $T_0$ is just (a priori) a subinterval of existence of the smooth solution $(v,\pi)$ corresponding to $v_0$. In detail. We conclude that in the statement of Theorem 1.6, one can replace $J^{1,2}$ with any space $X$ which is suitable to ensure the existence of a regular solution corresponding to $v_0$.

Corollary 1.3 makes operational condition (1.21) on a suitable subdomain. Indeed the existence of the domain $\Omega \subset B(\mathbb{R})$ follows from the construction of the sequence $\{v^k\}$ almost everywhere converging to zero and the Severini–Egorov theorem.

Theorem 1.7 furnishes a global existence result just requiring a smallness condition. It is also an immediate consequence of our previous result in [3].

2. Preliminaries

Below we recall some results which are fundamental for our goals.

**Lemma 2.1.** Assume that $|x|^3 u \in L^2(\mathbb{R}^3)$, $|x|^\alpha \nabla u \in L^2(\mathbb{R}^3)$, and the following statements hold:

(i) $r \geq 2$, $\gamma + \frac{3}{r} > 0$, $\alpha + \frac{3}{r} > 0$, $\beta + \frac{3}{r} > 0$, and $a \in [\frac{1}{2},1]$;
(ii) $\gamma + \frac{3}{r} = a(\alpha + \frac{1}{2}) + (1-a)(\beta + \frac{3}{r})$ (dimensional balance),
(iii) $a(\alpha - 1) + (1-a)\beta \leq \gamma \leq a\alpha + (1-a)\beta$.

Then

$$\|x|^\gamma u\|_r \leq c\|x|^\alpha \nabla u\|_2 \|x|^\beta u\|_2^{-a}, \quad (2.1)$$

where $c$ is a constant independent of $u$.

*Proof.* See [1, Lemma 7.1].

**Lemma 2.2.** Assume that $\mathbb{K}$ is a singular bounded transformation of the Calderón–Zigmund kind from $L^p$ into $L^p$, $p \in (1,\infty)$. Then $\mathbb{K}$ is also a bounded transformation from $L^p$ into $L^p$ with respect to the measure $(\mu + |x|)^\alpha dx$, $\mu \geq 0$, provided that $\alpha \in (-n, n(p-1))$.

*Proof.* See [9, Theorem 1].

**Lemma 2.3.** Assume that $(u,\pi_u)$ is a suitable weak solution. Then the pressure field admits the following representation formula

$$\pi_u(t,x) = -D_x D_y \int_{\mathbb{R}^3} \mathcal{E}(x-y)u^i(y)u^j(y)dy, \text{ a.e. in } (t,x) \in (0,\infty) \times \mathbb{R}^3, \quad (2.2)$$

and

$$\pi_u(t,x) \in L^\frac{3}{2}(0,T;L^\frac{5}{2}(\mathbb{R}^3)). \quad (2.3)$$
Lemma 2.4. For any \( v_0 \in J^{1,2}(\mathbb{R}^3) \), there exists a unique regular solution \( (v, \pi_v) \) to problem (1.1) on some interval \((0, T)\) such that

\[
v \in C([0, T); J^{1,2}(\mathbb{R}^3)), \quad v_1, D^2 v, \nabla \pi_v \in L^2(0, T; L^2(\Omega)),
\]

where \( T \geq c\|\nabla u_0\|^{-4}_2 \).

Proof. The result was proved by Leray in [4]. \(\square\)

For \( \mu \geq 0 \), we define the functionals

\[
E(v, t, x, \mu) := \int_{\mathbb{R}^3} \frac{|v(t, y)|^2}{(|x - y|^2 + \mu^2)^{1/2}} dy,
\]

\[
\mathcal{D}(v, t, x, \mu) := \int_{\mathbb{R}^3} \frac{|
abla v(t, y)|^2}{(|x - y|^2 + \mu^2)^{1/2}} dy,
\]

and set

\[
p(y) := (|x - y|^2 + \mu^2)^{-1/2}.
\]

When no confusion arises, we omit some or all dependencies on \((v, t, x, \mu)\). For \( \mu \geq 0 \), the sum

\[
E(v, t, x, \mu) + \int_0^t \mathcal{D}(v, \tau, x, \mu) d\tau
\]

is called a weighted energy.

Lemma 2.5. Let \((v, \pi_v)\) be the regular solution from Lemma 2.4. Then for all \( \mu > 0 \), the following weighted energy relation and weighted energy inequality hold for all \( t \in [0, T) \) and \( x \in \mathbb{R}^3 \):

\[
\begin{align*}
E(v, t, x, \mu) + 2 \int_0^t \mathcal{D}(v, \tau, x, \mu) d\tau + 3\mu^2 \int_0^t \int_{\mathbb{R}^3} \frac{v^2(\tau, y)}{|x - y|^2 + \mu^2}^2 dy d\tau \\
= E(v, 0, x, \mu) + \int_0^t \int v \otimes v \cdot \nabla p dy d\tau + 2 \int_0^t \pi_v v \cdot \nabla p dy d\tau,
\end{align*}
\]

\[
\begin{align*}
E(t, x, \mu) + \int_0^t \mathcal{D}(\tau, x, \mu) d\tau + 3\mu^2 \int_0^t \int_{\mathbb{R}^3} \frac{v^2(\tau, y)}{|x - y|^2 + \mu^2}^2 dy d\tau \\
\leq E(0, x, \mu) + c \int_0^t E(\tau, x, \mu) \|\nabla v(\tau)\|_2^4 d\tau.
\end{align*}
\]

Proof. Identity (2.7) can formally be obtained by multiplying equation (1.1)_1 by \( vp \) and integrating by parts on \((0, t) \times \mathbb{R}^3 \). Let us show that it is well defined for any \( \mu > 0 \). First, we remark that the hypothesis on \( v_0 \) implies that \( E(0, x, \mu) < \infty \) for all \( x \in \mathbb{R}^3 \) and \( \mu \geq 0 \).
Multiplying equation (1.1) by \(vp\) and integrating by parts on \((0, t) \times \mathbb{R}^3\), we obtain
\[
\mathcal{E}(t, x, \mu) + 2 \int_0^t \mathcal{D}(\tau, x, \mu) d\tau + 3\mu^2 \int_0^t \int_{\mathbb{R}^3} \frac{v^2(\tau, y)}{|x-y|^2 + \mu^2} dy d\tau \\
= \mathcal{E}(0, x, \mu) - 2 \int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla v) \cdot vp dy d\tau - 2 \int_0^t \int_{\mathbb{R}^3} \nabla \pi v \cdot v \cdot pdy d\tau \\
=: \mathcal{E}(0, x, \mu) + 2 \int_0^t (J_1 + J_2) d\tau.
\] (2.9)

Let us show that the right-hand side is well defined. Applying Hölder’s inequality and inequality (2.1), we get
\[
|J_1| \leq \|v||x-y|^2 + h^2\|^{-\frac{1}{4}} \|v\|_4 \|\nabla v\|_2 \leq \mathcal{E}^\frac{1}{4} \mathcal{D}^\frac{3}{4} \|\nabla v\|_2 \leq \frac{1}{4} \mathcal{D} + c\mathcal{E} \|\nabla v\|_4^4.
\]
From representation formula (2.2), after integrating by parts we obtain
\[
\nabla \pi v(t, x) = \nabla \int_{\mathbb{R}^3} D_y \mathcal{E}(x-y) \nu^i(y) D_y v^j(y) dy.
\]

Hence, applying Hölder’s inequality and employing Lemma 2.2, we deduce
\[
|J_2| \leq \|\nabla \pi v||x-y|^2 + \mu^2\|^{-\frac{1}{4}} \|v||x-y|^2 + \mu^2\|^{-\frac{1}{4}} \|v\|_4 \\
\leq c\|v \cdot \nabla v||x-y|^2 + \mu^2\|^{-\frac{1}{4}} \|v||x-y|^2 + \mu^2\|^{-\frac{1}{4}} \|v\|_4.
\]

Applying again Hölder’s inequality and subsequently (2.1), we get the estimate
\[
|J_2| \leq c\|v||x-y|^2 + \mu^2\|^{-\frac{1}{4}} \|v\|_4 \|\nabla v\|_2 \leq \mathcal{E}^\frac{1}{4} \mathcal{D}^\frac{3}{4} \|\nabla v\|_2 \leq \frac{1}{4} \mathcal{D} + c\mathcal{E} \|\nabla v\|_4.
\]

Hence from (2.9) and via estimates for the terms \(J_1\) and \(J_2\), we obtain integral inequality (2.8). Using this inequality together with the regularity of \(v\), one can easily deduce that (2.7) holds for all \(\mu > 0\) and all \(t \in [0, T)\). 

**Lemma 2.6.** Let \(u_0 \in J^2(\mathbb{R}^3)\). There exists a set \(\mathcal{E}\) such that \(\mathbb{R}^3 - \mathcal{E}\) has zero Lebesgue measure. Moreover, for all \(x \in \mathcal{E}\) and all \(\eta > 0\), there exists \(\overline{u}_0 \in J^{1,2}(\mathbb{R}^3)\) such that
\[
\int_{\mathbb{R}^3} \frac{|u_0 - \overline{u}_0|^2}{|x-y|} dy < \eta.
\] (2.10)

Furthermore, for all \(R > 0\) and \(\varepsilon > 0\), there exists \(\Omega_\varepsilon \subseteq \mathcal{E}\) such that \(\text{meas}(B(R) - \Omega_\varepsilon) < \varepsilon\) and
\[
\int_{\mathbb{R}^3} \frac{|u_0 - \overline{u}_0|^2}{|x-y|} dy < \eta \text{ uniformly in } x \in \Omega_\varepsilon.
\] (2.11)

**Proof.** Denote the mollified functions of \(u_0\) by \(\{u_0^k\}\). It is known that \(\{u_0^k\} \subset C^\infty(\mathbb{R}^3) \cap J^{1,2}(\mathbb{R}^3)\), and \(\{u_0^k\}\) converges to \(u_0\) in \(L^2\)-norm. For all \(k \in \mathbb{N}\), we define (1.20), that is
\[
\psi^k(x) := \int_{\mathbb{R}^3} \frac{|u_0 - u_0^k|^2}{|x-y|} dy < \infty.
\]
From the Hardy–Littlewood–Sobolev theorem, it follows that for \( r \in [1, 3) \),
\[
\|\psi^k\|_{L^r(K)} \leq c(r, K)\|u_0^k - u_0\|_2^2 \quad \text{for all compact set } K \subset \mathbb{R}^3.
\]
Hence, the sequence \( \{\psi^k\} \) converges to zero in \( L^r(K) \) for all \( r \in [1, 3) \). In particular, there exists a subsequence \( \{\psi^{k_j}\} \), which converges to zero almost everywhere in \( x \in K \). We denote by \( \{K_{\nu}\} \) a sequence of compact sets such that \( K_{\nu} \subset K_{\nu+1} \) and \( \bigcup_{\nu \in \mathbb{N}} K_{\nu} = \mathbb{R}^3 \). By virtue of the above convergence, one can consider the set \( E_{\nu} \subset K_{\nu} \) of the convergence of the sequence \( \{\psi^{k_j}\} \) almost everywhere. Then using Cantor’s diagonal method, we construct a sequence \( \psi^j \) such that \( \psi^j \in \{\psi^k\} \) such that \( \pi_0 := u^j_0 \) verifies (2.10). Property (2.11) is a consequence of the above construction and the Severino–Egorov theorem. The lemma is completely proved.

\[ \square \]

3. Local in time weighted energy inequality for a suitable weak solution

In this section we prove that any suitable weak solution admits, at least locally in time, a weighted energy inequality with \( \mu = 0 \). Actually, the following lemma holds.

**Lemma 3.1.** Let \((u, \pi_u)\) be a suitable weak solution. Let \( x, v_0 \), and \( c \) be as in Theorem 1.5. Then there exists \( t^*(x) > 0 \) such that
\[
E(u, t, x) + \frac{1}{2} \int_0^t \mathcal{D}(u, \tau, x) d\tau \leq N < \infty \quad \text{for all } t \in [0, t^*(x)),
\]
where \( E(u, t, x) \) and \( \mathcal{D}(u, \tau, x) \) are as in (2.5).

**Proof.** The proof of estimate (3.1) reproduces in a suitable way an idea used in [2]. This idea follows from the Leray–Serrin arguments applied in the proof of the energy inequality in the strong form. The proof is obtained in five steps. We set \( w := u - v \) and \( \pi_w := \pi_u - \pi_v \), where \((u, \pi_u)\) is a suitable weak solution and \((v, \pi_v)\) the regular solution corresponding to \( v_0 \) and established in Lemma 2.4. The first four steps are devoted to prove the inequality
\[
E(w, t, x, \mu) + \frac{1}{2} \int_0^t \mathcal{D}(w, \tau, x, \mu) d\tau \leq \frac{1}{8c^2} \quad \text{for all } t \in [0, t^*(x)) \text{ and } \mu > 0.
\]

**Step 1.** We claim that for all \( t > 0 \),
\[
E(t, x, \mu) + 2 \int_0^t \int_{\mathbb{R}^3} \mathcal{D}(\tau, x, \mu) d\tau + 3\mu^2 \int_0^t \int_{\mathbb{R}^3} \frac{|u(\tau)|^2}{(|x - y|^2 + \mu^2)^{\frac{3}{2}}} dy d\tau
\]
\[
\leq E(0, x, \mu) + \int_0^t \int_{\mathbb{R}^3} \frac{|u(\tau)|^2 u \cdot (x - y)}{(|x - y|^2 + \mu^2)^{\frac{3}{2}}} dy d\tau + 2 \int_0^t \int_{\mathbb{R}^3} \frac{\pi_u(\tau) u(\tau) \cdot (x - y)}{(|x - y|^2 + \mu^2)^{\frac{3}{2}}} dy d\tau.
\]
In the energy inequality (1.2), we set \( \phi(\tau, y) := \frac{|x - y|^2 + \mu^2}{}^{\frac{3}{2}} h_R(y) k(\tau) \in C_c^\infty(\mathbb{R} \times \mathbb{R}^3) \), where \( h_R \) and \( k \) are such that
\[
h_R(y) := \begin{cases} 
1 & \text{if } |y| \leq R, \\
(0, 1) & \text{if } |y| \in (R, 2R), \text{ and } k(\tau) := \begin{cases} 
1 & \text{if } |\tau| \leq t, \\
(0, 1) & \text{if } |\tau| \in (t, 2t), \\
0 & \text{for } |y| \geq 2R, \end{cases}
\end{cases}
\]

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We get
\[
\int_{\mathbb{R}^3} \frac{|u(t)|^2 h_R}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} \, dy + 2 \int_0^t \int_{\mathbb{R}^3} \frac{|\nabla u(\tau)|^2 h_R}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} \, dyd\tau + 3\mu^2 \int_0^t \int_{\mathbb{R}^3} \frac{|u(\tau)|^2 h_R}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} \, dyd\tau
\]
\[
\leq \int_{\mathbb{R}^3} \frac{|u_0|^2 h_R}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} \, dy + \int_0^t \int_{\mathbb{R}^3} \frac{|u(\tau)|^2 h_R u \cdot (x-y)}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} \, dyd\tau
\]
\[
+ 2 \int_0^t \int_{\mathbb{R}^3} \pi u(\tau) h_R u(\tau) \cdot (x-y) \, dyd\tau + F(t, R)
\]
\[
:= \int_{\mathbb{R}^3} \frac{|u_0|^2 h_R}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} \, dy + I_1(t, x) + I_2(t, x) + F(t, R),
\]
where
\[
F(t, R) := \int_0^t \int_{\mathbb{R}^3} |u|^2 \left[ 2\nabla h_R \cdot \nabla (|x-y|^2 + \mu^2)^{-\frac{3}{2}} + \frac{\Delta h_R}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} \right] \, dyd\tau + \int_0^t \int_{\mathbb{R}^3} \pi u \cdot \nabla h_R \, dyd\tau.
\]

Since \( \pi_u, u^2 \in L^\frac{3}{2}(0, T; L^\frac{3}{2}(\mathbb{R}^3)) \), we can apply Hölder’s inequality and use the decay of \( \nabla h_R, \Delta h_R \) for all \( t > 0 \), to obtain \( F(t, R) = o(R) \). Let us estimate the terms \( I_i, i = 1, 2 \).

Since \( \mu > 0 \), by virtue of the integrability properties of a suitable weak solution, Lemma 2.1 implies that

\[
|I_1(t, x)| \leq \int_0^t \frac{\| u \|_{\mathcal{B}^\frac{3}{2}((|x-y|^2 + \mu^2)^{\frac{3}{2}})}}{\| \nabla u \|_{\mathcal{B}^\frac{3}{2}((|x-y|^2 + \mu^2)^{\frac{3}{2}})}} \, d\tau \leq c \int_0^t \frac{\| u \|_{\mathcal{B}^\frac{3}{2}((|x-y|^2 + \mu^2)^{\frac{3}{2}})}}{\| \nabla u \|_{\mathcal{B}^\frac{3}{2}((|x-y|^2 + \mu^2)^{\frac{3}{2}})}} \, d\tau.
\]

For \( I_2 \), we apply Hölder’s inequality and Lemma 2.2 to obtain

\[
|I_2(t, x)| \leq c \int_0^t \frac{\| \pi_u \|_{\mathcal{B}^\frac{3}{2}((|x-y|^2 + \mu^2)^{\frac{3}{2}})}}{\| \nabla u \|_{\mathcal{B}^\frac{3}{2}((|x-y|^2 + \mu^2)^{\frac{3}{2}})}} \, d\tau \leq c \int_0^t \frac{\| \pi_u \|_{\mathcal{B}^\frac{3}{2}((|x-y|^2 + \mu^2)^{\frac{3}{2}})}}{\| \nabla u \|_{\mathcal{B}^\frac{3}{2}((|x-y|^2 + \mu^2)^{\frac{3}{2}})}} \, d\tau.
\]

Hence, as in the previous case, by Lemma 2.1 we get

\[
|I_2(t, x)| \leq c \int_0^t \frac{\| \pi_u \|_{\mathcal{B}^\frac{3}{2}((|x-y|^2 + \mu^2)^{\frac{3}{2}})}}{\| \nabla u \|_{\mathcal{B}^\frac{3}{2}((|x-y|^2 + \mu^2)^{\frac{3}{2}})}} \, d\tau.
\]

Using the estimates obtained for \( I_i, i = 1, 2 \), and the Lebesgue dominated convergence theorem, we deduce inequality (3.3) for all \( t > 0 \) as the limit as \( R \to \infty \).

**Step 2.** At this step we derive a sort of Green’s identity between the solutions \( (u, \pi_u) \) and \( (v, \pi_v) \), where \( (v, \pi_v) \) is the regular solution given in Lemma 2.4, corresponding to the initial data \( v_0 \in J^{1,2}(\mathbb{R}^3) \). In what follows, \( (0, T) \) is the existence interval of \( (v, \pi_v) \). We also recall that \( (v, \pi_v) \) is smooth for \( t > 0 \). We denote by \( \lambda(\tau) \) a smooth cutoff function such that \( \lambda(\tau) = 1 \) for \( \tau \in [s, t] \) and \( \lambda(\tau) = 0 \) for \( \tau \in [0, \frac{s}{2}] \).
For all \( t, s \in (0, T) \), we write statement iii) of Definition 1.1 with \( \varphi = \lambda vp \):

\[
\int_s^t \left[ (pu, v) - (p \nabla u, \nabla v) + (pu \cdot \nabla v, u) + (\pi u, v \cdot \nabla p) \right] d\tau + (pu(s), v(s)) = (pu(t), v(t)) + \int_s^t \left[ (\nabla u, v \otimes \nabla p) + (u \otimes u, v \otimes \nabla p) \right] d\tau. \tag{3.5}
\]

We multiply equation (1.1) written for \((v, \pi_v)\) by \(up\). After integrating by parts on \((s, t) \times \mathbb{R}^3\), we get

\[
\int_s^t \left[ (pu, v) + (p \nabla u, \nabla v) - (\pi_v, u \cdot \nabla p) \right] d\tau = \int_s^t \left[ (\nabla u, v \otimes \nabla p) + (u \cdot v, \Delta p) \right] d\tau. \tag{3.6}
\]

Taking the difference of (3.5) and (3.6), we get

\[
\int_s^t \left[ -2(p \nabla u, \nabla v) + (pu \cdot \nabla v, u) - (pv \cdot \nabla v, u) + (\pi_u, v \cdot \nabla p) + (\pi_v, u \cdot \nabla p) \right] d\tau = (pu(t), v(t)) - (pu(s), v(s)) + \int_s^t \left[ (u \otimes u, v \otimes \nabla p) - (u \cdot v, \Delta p) \right] d\tau, \tag{3.7}
\]

which establishes the required Green’s identity.

**Step 3.** Set \( w := u - v \) and \( \pi_w := \pi_u - \pi_v \). Let us derive the following estimate:

\[
\mathcal{E}(w, t, x, \mu) + \int_0^t \mathcal{D}(w, \tau, x, \mu) d\tau \\
\leq \mathcal{E}(w, 0, x, \mu) + c \int_0^t \mathcal{E}_2(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau + H(v, t, x, \mu) \tag{3.8}
\]

for all \( t \in [0, T) \), \( x \in \mathbb{R}^3 \), \( \mu > 0 \),

with

\[
H(v, t, x, \mu) := c \int_0^t \|\nabla v(\tau)\|^2 d\tau + c \int_0^t \mathcal{E}(v, \tau, x, \mu) \mathcal{D}(v, \tau, x, \mu) d\tau.
\]
We remark that from representation formula (2.2) and regularity of $v$, it follows that
\[ \pi_w = \pi^1 + \pi^2, \quad \pi^1 := D_x j \int_{\mathbb{R}^3} D_y E(x - y) w^j(y) w^j(y) dy, \]
and
\[ \pi^2 := 2 \int_{\mathbb{R}^3} D_y E(x - y) w(y) \cdot \nabla x^j(y) dy. \]  
(3.9)

Let us sum estimates (2.7) and (3.3), and then add formula (3.7) for $s = 0$ twice. Recalling the definition of $(w, \pi_w)$ and formula (3.9), we get by a straightforward computation that
\[ E(w, t, x, \mu) + 2 \int_0^t D(w, \tau, x, \mu) d\tau + 3 \mu^2 \int_0^t \int_{\mathbb{R}^3} \left( \left| x - y \right|^2 + \mu^2 \right) \pi^2 d\tau \]
\[ \leq E(w, 0, x, \mu) + F_1(w, t, x, \mu) + F_2(w, v, t, x, \mu), \]
(3.10)

where
\[ F_1 := F_1(w, t, x, \mu) := \int_0^t (w \otimes w, w \otimes \nabla p) d\tau + 2 \int_0^t (\pi_1, w \cdot \nabla p) d\tau, \]
\[ F_2 := F_2(w, v, t, x, \mu) := 2 \int_0^t (\pi_2, w \cdot \nabla p) d\tau - 2 \int_0^t (w \cdot \nabla v, w p) d\tau + \int_0^t (v \cdot \nabla p, w^2). \]

The term $F_1$ admits the same estimate as for $I_1$ and $I_2$ that were given in Step 1. Hence,
\[ |F_1| \leq c \int_0^t \psi^{\frac{1}{2}}(\tau, w, x, \mu) \mathcal{D}(\tau, w, x, \mu) d\tau \]
for all $t \in (0, T), x \in \mathbb{R}^3, \mu > 0$.

For the term $F_2$, we estimate the first two terms in a different way. Taking the representation formula of $\pi_2$ into account, we get
\[ \left| \int_0^t (\pi_2, w \cdot \nabla p) d\tau - 2 \int_0^t (w \cdot \nabla v, w p) d\tau \right| = \left| \int_0^t p \nabla \pi_2 \cdot w dy d\tau + 2 \int_0^t (w \cdot \nabla v, w p) dy d\tau \right|. \]
Hence, applying the arguments used in Lemma 2.5 to estimate $J_1$ and $J_2$, we get
\[ \left| \int_0^t (\pi_2, w \cdot \nabla p) d\tau - 2 \int_0^t (w \cdot \nabla v, w p) d\tau \right| \leq \int_0^t \left\| w p^{\frac{1}{2}} \right\|_2^2 \left\| \nabla v \right\|_2 d\tau \]
\[ \leq \int_0^t \psi^{\frac{1}{2}}(\tau, w, x, \mu) \mathcal{D}(\tau, w, x, \mu) d\tau + c \int_0^t \left\| \nabla v(\tau) \right\|_2^2 d\tau \]
for all $t \in [0, T), x \in \mathbb{R}^3, \mu > 0$.

For the last term in $F_2$, we apply Hőlder’s inequality to get
\[ \left| \int_0^t (v \cdot \nabla p, w^2) d\tau \right| \leq \int_0^t \left\| w p^{\frac{1}{2}} \right\|_2 \left\| vp \right\|_2^2 d\tau. \]
By virtue of estimate (2.1) and Young’s inequality, we have
\[
\left| \int_0^t (v \cdot \nabla p, w^2) d\tau \right| \leq c \int_0^t \mathcal{E}^x(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) + c \int_0^t \mathcal{E}(v, \tau, x, \mu) \mathcal{D}(v, \tau, x, \mu) d\tau.
\]

Hence,
\[
|F_2| \leq 2 \int_0^t \mathcal{E}^x(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau + c \int_0^t \|
abla v(\tau)\|^2_d d\tau
\]
\[
+ c \int_0^t \mathcal{E}(v, \tau, x, \mu) \mathcal{D}(v, \tau, x, \mu) d\tau, \quad \text{for all } t \in [0, T), x \in \mathbb{R}^3, \mu > 0.
\]

Finally, using Young’s inequality, we get
\[
|F_2| \leq t \int_0^t \mathcal{D}(w, \tau, x, \mu) d\tau + c \int_0^t \|
abla v(\tau)\|^2_d d\tau
\]
\[
+ c \int_0^t \mathcal{E}(v, \tau, x, \mu) \mathcal{D}(v, \tau, x, \mu) d\tau, \quad \text{for all } t \in [0, T), x \in \mathbb{R}^3, \mu > 0.
\]

Now estimates for \(F_1, F_2\) and (3.10) establish integral inequality (3.8).

**Step 4. Deduction of estimate (3.2).**

Under our assumptions on \(x, v_0, \) and \(c,\) we have, a fortiori,
\[
\mathcal{E}(w, 0, x, \mu) < \frac{1}{4c}^2 \quad \text{for all } \mu > 0.
\]

Moreover by the regularity of the solution \((v, \tau_v)\) (see Lemmas 2.4 and 2.5) there exists \(t^*\) such that
\[
H(t^*) < \frac{1}{4c^2} \quad \text{for all } \mu > 0.
\]

Let us deduce (3.2), i.e.,
\[
\mathcal{E}(w, t, x, \mu) + \frac{1}{2} \int_0^t \mathcal{D}(w, \tau, x, \mu) d\tau < \frac{1}{8c^2} \quad \text{for all } t \in [0, t^*), \mu > 0.
\]

Since \(w = u - v\) is right continuous in \(L^2\)-norm in \(t = 0,\) the same continuity property holds for \(\mathcal{E}(w, t, x, \mu)\) for all \(\mu > 0.\) Therefore there exists \(\delta = \delta(\mu) > 0\) such that
\[
\mathcal{E}(w, t, x, \mu) < \frac{1}{8c^2} \quad \text{for all } t \in [0, \delta).
\]
Hence the validity of estimates (3.8) and (3.11)–(3.12) yields

\[ \mathcal{E}(w, t, x, \mu) + \int_0^t \mathcal{D}(w, \tau, x, \mu) d\tau < \frac{1}{8c^2} + c \int_0^t \mathcal{E}^{\frac{1}{2}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau, \]

for any \( t \in [0, \delta] \). In view of (3.14), this proves (3.13) on \([0, \delta]\).

Let us show that estimate (3.14) holds for \( t \in [0, t^*] \). For all \( \mu > 0 \), the function

\[ f(t, \mu) := \mathcal{E}(w, 0, x, \mu) + c \int_0^t \mathcal{E}^{\frac{1}{2}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau + H(v, t, x, \mu) \]

is uniformly continuous on \([0, t^*]\). Hence there exists \( \eta = \eta(\mu) > 0 \) such that

\[ |t_1 - t_2| < \eta \Rightarrow |f(t_1) - f(t_2)| < \frac{1}{8c^2} - \mathcal{E}(w, 0, x, \mu) - H(t^*(x)). \]

We claim that estimate (3.14) and hence estimate (3.13), also holds for \( t \in [\delta, \delta + \eta] \). Assume on the contrary that there exists \( \bar{t} \in [\delta, \delta + \eta] \) such that

\[ \mathcal{E}(w, \bar{t}, x, \mu) > \frac{1}{8c^2}. \tag{3.15} \]

On the other hand, the validity of (3.8) yields

\[ \mathcal{E}(w, \bar{t}, x, \mu) + \int_0^\bar{t} \mathcal{D}(w, \tau, x, \mu) d\tau \leq (f(\bar{t}) - f(\delta)) + f(\delta) < \frac{1}{8c^2} + c \int_0^\delta \mathcal{E}^{\frac{1}{2}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau. \]

Estimate (3.14) allows us to obtain

\[ c \int_0^\delta \mathcal{E}^{\frac{1}{2}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau < \frac{1}{\sqrt{8}} \int_0^\bar{t} \mathcal{D}(w, \tau, x, \mu) d\tau. \]

Hence the last two estimates imply

\[ \mathcal{E}(w, \bar{t}, x, \mu) < \frac{1}{8c^2}, \]

which contradicts (3.15). Since the arguments are independent of \( \delta \), the result holds for any \( t \in [0, t^*(x)] \), which proves (3.13).

**Step 5.** Since \( u = w + v \), one can use estimates (2.8) and (3.13) to deduce the inequality

\[ \int_{\mathbb{R}^3} \frac{|u(t, y)|^2}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} + \left(1 - \frac{1}{\sqrt{8}}\right) \int_{\mathbb{R}^3} \frac{|\nabla u(t, y)|^2}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau \leq N \quad \text{for all} \quad t \in [0, t^*(x)]. \]

with obvious meaning of \( N \) and \( t^*(x) \) independent of \( \mu \). This is an easy consequence of estimate (3.2) and the following remark: the families of functions

\[ \left\{ \int_0^t \int_{\mathbb{R}^3} \frac{|\nabla u(t, y)|^2}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau \right\} \text{ and } \left\{ \int_{\mathbb{R}^3} \frac{|u(t, y)|^2}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy \right\} \]

are monotone in \( \mu > 0 \). Thus by virtue of Beppo–Levi’s theorem, we deduce (3.1) in the limit as \( \mu \to 0 \). \( \square \)
Corollary 3.1. Let \((u, \pi_u)\) be a suitable weak solution. Let \(\sigma \geq 0\) be such that (1.2) is verified. Then there exists a set \(E \subseteq \mathbb{R}^3\) such that \(\mathbb{R}^3 - E\) has zero Lebesgue measure and for all \(x \in E(\sigma)\) there exists a \(t^*(x) > 0\), for which

\[
\mathcal{E}(u, t, x) + (1 - \frac{1}{\sqrt{8}}) \int_0^t \mathcal{D}(u, \tau, x) d\tau \leq N < \infty \quad \text{for all} \quad t \in [\sigma, \sigma + t^*(x)].
\]  

(3.16)

Proof. By Lemma 2.6, for all \(\sigma \geq 0\) for which \(u\) verifies (1.2), one can find a set \(E\) such that for \(x \in E\) and \(\varepsilon > 0\) there exists a function \(\pi(\sigma) \in J^{1,2}(\mathbb{R}^3)\) verifying formula (1.13) of Theorem 1.5 with \(u(\sigma) - \pi(\sigma)\). As the assumptions of Lemma 3.1 are satisfied, the result follows. \(\square\)

4. PROOF OF THEOREMS 1.5, 1.6 AND COROLLARIES 1.2, 1.3.

To prove Theorem 1.5 we use the result of Proposition 1.1. To this end, in Lemma 4.1 below we prove that for a suitable \(r > 0\), estimate (3.1) of Lemma 3.1 implies condition (1.8) of Proposition 1.1.

Lemma 4.1. Let the assumption of Lemma 3.1 be satisfied. Then there exists \(\delta > 0\) such that

\[
M(t, x, r) \leq \varepsilon_1 \quad \text{for all} \quad r \in (0, [(1 - \delta)t]^\frac{1}{2}) \quad \text{and} \quad t \in (0, t^*(x))
\]  

(4.1)

with \(t^*(x)\) as in Lemma 3.1.

Proof. By virtue of the assumption, representation formula (2.2), and Lemma 2.2, we have

\[
\|\pi_u(t)|x - y|^{-\frac{3}{2}}\|_2^2 \leq c\|u(t)||x - y|^{-\frac{3}{2}}\|_2^2
\]  

(4.2) a.e. in \(t \in (0, t^*(x))\). Applying Hölder’s inequality, inequality (4.2), and Lemma 2.1, we conclude that for all \(t \in (0, t^*(x))\) and \(t - r^2 > 0\),

\[
\begin{align*}
\int_{t - r^2}^t \int_{|x - y| < r} |u|^3 + |v||\pi_u| \, dy \, d\tau &\leq c \int_{t - r^2}^t \left( \frac{u(\tau)}{|x - y|^{\frac{3}{2}}} + \frac{u(\tau)}{|x - y|^2} \right) \left( \frac{\pi_u(\tau)}{|x - y|^{\frac{3}{2}}} \right) \, d\tau \\
&\leq c \int_{t - r^2}^t \frac{u(\tau)}{|x - y|^{\frac{3}{2}}} \frac{\pi_u(\tau)}{|x - y|^{\frac{3}{2}}} \, d\tau \\
&= c \int_{t - r^2}^t \mathcal{E}(\tau, x)^{\frac{1}{2}} \mathcal{D}(\tau, x) \, d\tau =: N_1.
\end{align*}
\]

(4.3)

Considering the second term on the right-hand side of \(M(t, x, r)\) in (1.7), applying twice Hölder’s inequality, inequality (4.2), or the condition that all the \(t\) belong to \((0, t^*(x))\) and \(t - r^2 > 0\), we get

\[
\begin{align*}
\int_{t - r^2}^t \left[ \int_{|x - y| < r} |\pi_u(\tau, y)| \, dy \right]^{\frac{5}{3}} \, d\tau &\leq cr^{-\frac{1}{4}} \int_{t - r^2}^t \left( \frac{|\pi_u(\tau)|}{|x - y|^{\frac{3}{2}}} \right)^{\frac{5}{3}} \, d\tau \\
&\leq cr^{-\frac{1}{4}} \int_{t - r^2}^t \left( \frac{u(\tau)}{|x - y|^{\frac{3}{2}}} \right)^{\frac{5}{3}} \left( \frac{\pi_u(\tau)}{|x - y|^{\frac{3}{2}}} \right)^{\frac{5}{3}} \, d\tau \\
&\leq c \int_{t - r^2}^t \mathcal{E}(\tau, x) \mathcal{D}(\tau, x) \, d\tau =: N_2.
\end{align*}
\]

(4.4)

Hence, (4.3) and (4.4) imply that

\[
M(t, x, r) \leq N_1 + N_2.
\]  

995
Using estimate (3.1), we get
\[
N_1 + N_2 \leq cN^\frac{5}{2} \int_{t-r^2}^{t} \mathcal{D}(\tau,x)d\tau + cN^\frac{5}{2} \int_{t-r^2}^{t} \mathcal{D}(\tau,x)d\tau \text{ for all } t \in (0,t^*(x)) \text{ and } pt - r^2 > 0.
\]

On the other hand, the function \( \int_t^{t^*(x)} \mathcal{D}(\tau)d\tau \) is uniformly continuous on \([0,t^*(x)]\). Hence there exists \( \delta \in (0,1) \) such that
\[
\left[ cN^\frac{5}{2} \int_{(1-\delta)t}^{t} \mathcal{D}(\tau,x)d\tau \right]^{\frac{5}{6}} + cN^\frac{5}{2} \int_{(1-\delta)t}^{t} \mathcal{D}(\tau,x)d\tau < \varepsilon_1 \text{ for all } t \in (0,t^*(x)).
\]

The lemma is proved, after setting \( \delta = 1 - \delta \).

Now we are in position to prove Theorems 1.5 and 1.6.

Proof of Theorem 1.5. By virtue of Lemma 3.1, for any \( x \) satisfying the assumptions, estimate (3.1) holds on some interval \([0,t^*(x)]\). Set \( t(x) := \frac{c}{1} t^*(x) \). By Lemma 4.1, there exists \( \delta > 0 \) such that \( M\left(\frac{1}{2}s, x, r\right) \leq \varepsilon_1 \) for all \( r \in \left(0, \left[ \frac{(1-\delta)s}{2} \right] \right) \) and \( x \in \left(0, t^*(x) \right) \). By Proposition 1.1, this implies the local regularity (1.14), provided that \( \delta \in \left(0, \frac{1}{2}\right) \). Finally, in order to prove (1.15) it suffices to note that the point \( (s,x) \) belongs to \( Q^{\frac{1}{2}} \left( \frac{1}{2}s, x \right) \) and, if \( (s,x) \) is a Lebesgue point, then (1.15) follows from (1.9). The theorem is completely proved.

Proof of Corollary 1.2. By Corollary 3.1, there exists a set \( E(\sigma) \) such that for all \( x \in E(\sigma) \) estimate (3.1) holds on some interval \([\sigma, \sigma + t^*(x)]\). Then one can conclude as in the proof of Theorem 1.5.

Proof of Theorem 1.6. Under the assumption of the theorem, Lemma 3.1 holds for any \( x \) in \( \Omega \) with \( t^*(x) \) uniform in \( \Omega \). The last claim is a consequence of the fact that in the definition of \( \psi \), the smooth function \( \psi_0 \) is independent of \( x \in \Omega \). Hence under assumption (1.21), inequalities (3.11) and (3.12) are uniform with respect to \( x \). Setting \( T_0 := t^* \), we write (3.1) for \( t \in [0,T_0) \) for all \( x \in \Omega \). As a consequence, all the arguments used in the proof of Theorem 1.5 work independently of \( x \in \Omega \). The theorem is proved.

Proof of Corollary 1.3. Given a ball \( B(R) \) and \( \varepsilon > 0 \), we can apply Lemma 2.6 establishing property (2.11). Hence the assumption of Theorem 1.6 holds for any \( x \) in \( \Omega_\varepsilon \).

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This condition ensures that one can choose \( r = \sqrt{s} \), where \( (1-\delta)\sqrt{s} > s \).
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